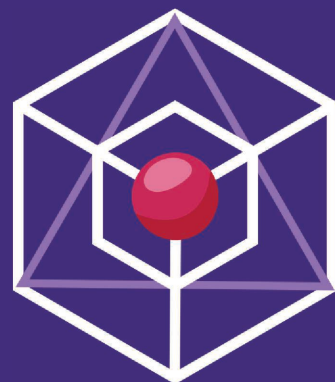


COMPLETE STUDY PACK FOR ENGINEERING ENTRANCES

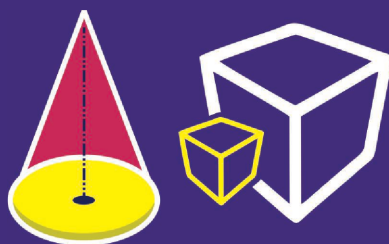
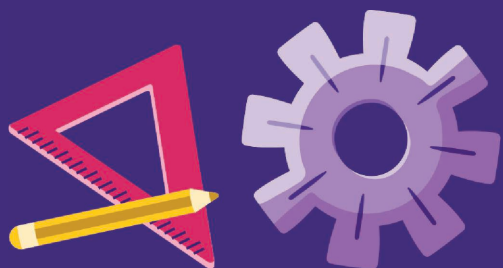
 arihant



OBJECTIVE MATHEMATICS

USEFUL FOR

JEE Main & Advanced,
BITSAT, MHT-CET,
Andhra Pradesh &
Telangana State (EAMCET),
Karnataka CET, VIT,
Kerala CEE & WB JEE etc.



More than
5000+
Questions
of All Types

Volume 2



Amit M. Agarwal

For More Material Join: @JEEAdvanced_2024

COMPLETE STUDY PACK FOR
ENGINEERING
ENTRANCES

OBJECTIVE
MATHEMATICS

Volume 2

Amit M. Agarwal



ARIHANT PRAKASHAN (SERIES), MEERUT



Arihant Prakashan (Series), Meerut

All Rights Reserved

✎ © **AUTHOR**

No part of this publication may be re-produced, stored in a retrieval system or by any means, electronic, mechanical, photocopying, recording, scanning, web or otherwise without the written permission of the publisher. Arihant has obtained all the information in this book from the sources believed to be reliable and true. However, Arihant or its editors or authors or illustrators don't take any responsibility for the absolute accuracy of any information published and the damage or loss suffered thereupon.

All disputes subject to Meerut (UP) jurisdiction only.

✎ **Administrative & Production Offices**

Regd. Office

'Ramchhaya' 4577/15, Agarwal Road, Darya Ganj, New Delhi - 110002

Tele: 011- 47630600, 43518550

Head Office

Kalindi, TP Nagar, Meerut (UP) - 250002

Tel: 0121-7156203, 7156204

✎ **Sales & Support Offices**

Agra, Ahmedabad, Bengaluru, Bareilly, Chennai, Delhi, Guwahati, Hyderabad, Jaipur, Jhansi, Kolkata, Lucknow, Nagpur & Pune.

✎ **ISBN** 978-93-25299-13-9

✎ **PO No :** TXT-XX-XXXXXXX-X-XX

Published by Arihant Publications (India) Ltd.

For further information about the books published by Arihant, log on to www.arihantbooks.com or e-mail at info@arihantbooks.com

Follow us on



PREFACE

Engineering offers the most exciting and fulfilling of careers. As a Engineer you can find satisfaction by serving the society through your knowledge of technology. Although the number of Engineering colleges imparting quality education and training has significantly increased after independence in the country, but simultaneous increase in the number of serious aspirants has made the competition difficult, it is no longer easy to get a seat in a prestigious Engineering college today.

For success, you require an objective approach of the study. This does not mean you 'prepare' yourself for just 'objective questions'. Objective Approach means more than that. It could be defined as that approach through which a student is able to master the concepts of the subject and also the skills required to tackle the questions asked in different entrances such as JEE Main & Advanced, as well other regional Engineering entrances. These two-volume books on Mathematics 'Objective Mathematics (Vol.1 & 2)' fill the needs of such books in the market in Mathematics and are borne out of my experience of teaching Mathematics to Engineering aspirants.

The plan of the presentation of the subject matter in the books is as follows

- The whole chapter has been divided under logical topic heads to cover the syllabi of JEE Main & Advanced and various Engineering entrances in India.
- The Text develops the concepts in an easy going manner, taking the help of the examples from the day-to-day life.
- Important points of the topics have been highlighted in the text. Under Notes, some extra points regarding the topics have been given to enrich the students.
- The Solved Examples make the students learn the basic problem solving skills in Mathematics. Very detailed explanations have been provided to make the students skilled in systematically tackling the problems.
- The answers / solutions to all the questions have been provided.
- The Objective Questions have been divided according to their types Single correct option, More than One, Assertion-Reason, Matching Type, Integer Type, Passage Based, etc. which can take the students to a level required for various Engineering entrances in the present scenario.
- Entrance Corner includes the Previous Years' Questions asked in JEE Main & Advanced and other various Engineering entrances. At the end of the book, JEE Main & Advanced & Other Regional Entrances Solved Papers have been given.

I would open-heartedly welcome the suggestions for the further improvements of this book (Vol.2) from the students and teachers.

Amit M. Agarwal

CONTENTS

1. MATRIX

1-44

- Introduction
- Types of Matrices
- Equality of Two Matrices
- Trace of a Matrix
- Algebra of Matrices
- Scalar Multiplication
- Transpose of a Matrix
- Symmetric Matrix
- Skew-symmetric Matrix
- Orthogonal Matrix
- Idempotent Matrix
- Involutory Matrix
- Nilpotent Matrix
- Singular Matrix
- Conjugate of a Matrix
- Transpose of Conjugate of a Matrix
- Hermitian Matrix
- Skew-Hermitian Matrix
- Elementary Transformation of a Matrix
- Invertible Matrices
- Inverse of a Matrix using Elementary Row or Column Operations

- Composition of Relations
- Congruence Modulo m
- Partial Order Relation
- Total Order Relation
- Equivalence Classes of an Equivalence Relation
- Constants and Variables
- Function
- Vertical Parallel Line Test (VPL Test)
- Domain and Range of a Real Function
- Algebra of Functions
- Explicit and Implicit Functions
- Basic Functions
- Real Valued Functions
- Bounded Functions
- Equality of Functions
- Odd and Even Functions
- Even and Odd Extensions
- Periodic Function
- Types of Functions
- Number of Functions
- Composite Functions
- Inverse Function
- Definite Function

2. DETERMINANT

45-124

- Introduction
- Definition of Determinant
- Applications of Determinant in Geometry
- Properties of Determinant
- Minors and Cofactors of Elements of a Determinant
- Adjoint of a Square Matrix
- Inverse of a Square Matrix
- Applications of Determinants and Matrices
- Rank of a Matrix
- Product of Determinants of Same Order
- Differentiation of Determinants
- Integration of Determinants
- Maximum and Minimum Values of Determinants, when Elements are Known

4. INVERSE TRIGONOMETRIC FUNCTIONS 215-274

- Inverse Function
- Inverse of Trigonometric Functions
- Graphs of Inverse Trigonometric Functions
- Domain and Range of Inverse Trigonometric Functions
- Properties of Inverse Trigonometric Functions
- Solution of the Inverse Equations
- Summation of Series Containing Inverse Trigonometric Terms
- Some Useful Graphs

3. RELATIONS AND FUNCTIONS

125-214

- Relation
- Domain and Range of a Relation
- Types of Relations

5. CONTINUITY AND DIFFERENTIABILITY 275-339

- Continuity
- Cauchy's Definition of Continuity
- Heine's Definition of Continuity
- Continuity at a Point
- Geometrical Interpretation of Continuity at a Point
- Continuity in an Open Interval
- Continuity in a Closed Interval
- Properties of Continuous Functions

- Discontinuity
- Different Kinds of Discontinuity
- Continuity of Composition of Function
- Differentiability
- Differentiability of a Function at a Point
- Geometrical Interpretation of Differentiability
- Differentiability in a Set
- Relation between Derivability and Continuity
- Properties of Differentiable Functions
- Problems of Functional Equations

6. DIFFERENTIATION 340-395

- Geometrical Meaning of Derivative
- Differential Coefficients of Standard Functions
- Rules for Differentiation
- Differentiation of Implicit Function
- Differentiation by Substitution
- Important Trigonometrical Formulae
- Differential Coefficient using Inverse
- Trigonometrical Substitutions
- Derivative of Inverse Trigonometric Functions
- Logarithmic Differentiation
- Differentiation of Parametric Functions
- Differentiation of a Function with respect to a Function
- Differentiation of Determinants
- Higher Order Derivatives

7. APPLICATION OF DERIVATIVES 396-472

- Derivatives as the Rate of Change
- Approximation and Differentials
- Errors
- Tangent and Normal
- Equations of Tangent and Normal
- Tangent and Normal in Parametric Form
- Angle of Intersection of Two Curves
- Length of Tangent, Sub-tangent, Normal and Sub-normal
- Rolle's Theorem
- Lagrange's Mean Value Theorem
- Increasing Function
- Decreasing Function
- Monotonic Function
- Operation of Monotonic Function
- Comparison of Functions using Calculus

8. MAXIMA AND MINIMA 473-521

- Introduction
- Maximum and Minimum Values of a Function
- Critical Points
- Point of Inflection
- Local Maxima and Local Minima
- Test for Local Maximum/Minimum
- Concept of Global Maximum/Minimum
- Application of Maxima and Minima to Problems

9. INDEFINITE INTEGRALS 522-609

- Integration as Reverse Process of Differentiation
- Fundamental Formulae on Integration
- Some Standard Results on Integration
- Method of Integration
- Integration by Substitution
- Integration using Trigonometric Identities
- Some Standard Integrals
- Evaluation of the Various Forms of Integrals by use of Standard Results
- Integration by Parts
- Integration by Partial Fraction
- Integration of Some Special Irrational Algebraic Functions
- Reduction Formulae

10. DEFINITE INTEGRALS 610-688

- Introduction
- Geometrical Interpretation of Definite Integral
- Fundamental Theorems of Calculus
- Evaluation of Definite Integrals by Substitution
- Properties of Definite Integrals
- Definite Integral as a Limit of Sum
- Gamma Function
- Reduction Formulae in Definite Integrals

11. AREA BOUNDED BY CURVES 689-752

- Introduction
- Curve Sketching
- Area Bounded by Curves
- Formulae for Finding the Area under the Curves
- Area of Some Important Curves
- Variable Area, Greatest and Least Values

12. DIFFERENTIAL EQUATIONS 753-828

- Introduction
- Ordinary and Partial Differential Equation

- Order and Degree of a Differential Equation
- Linear and Non-linear Differential Equation
- Solution of Differential Equation
- Formation of Differential Equation
- Differential Equation of First Order and First Degree
- Methods of Solving First Order and First Degree Differential Equations
- Orthogonal Trajectory
- Applications of First Order Differential Equation

13. VECTOR ALGEBRA 829-906

- Scalar and Vector Quantities
- Types of Vectors
- Addition of Vectors
- Subtraction of Vectors
- Multiplication of a Vector by a Scalar
- Linear Combination of Vectors
- Linearly Independent and Dependent Vectors
- Component of a Vector
- Position Vector
- Section Formulae
- Scalar (Dot) Product of Two Vectors
- Vector (Cross) Product of Two Vectors
- Scalar Triple Product
- Vector Triple Product
- Reciprocal System of Vectors
- Tetrahedron

14. THREE DIMENSIONAL GEOMETRY 907-982

- Direction Cosines
- Direction Ratios
- Angle between Two Vectors
- Straight Line in Space
- Angle between Two Lines
- Shortest Distance between Two Skew-lines
- Perpendicular Distance of a Point from a Line
- Reflection or Image of a Point in a Straight Line
- Plane

- Planes Bisecting the Angle between Two Given Planes
- Line and Plane
- Sphere

15. LINEAR PROGRAMMING 983-1014

- Introduction
- Linear Programming
- Some Definitions and Concepts Related to LPP
- General Linear Programming Problems
- Graphical Solution of Linear Inequalities
- Graph of Linear Inequalities
- Formulation of a Linear Programming Problem
- Graphical Method for Solution of Linear Programming Problems
- Solution of Linear Equations by Determinant

16. ADVANCED PROBABILITY 1015-1076

- Introduction
- Definitions of Various Terms
- Probability
- Important Results Related to Probability
- Conditional Probability
- Multiplication Theorems on Probability
- Independent Events
- More on Independent Events
- Law of Total Probability
- Baye's Theorem
- Random Variable and its Distribution
- Mean and Variance of a Random Variable
- Binomial Distribution
- Mean and Variance of Binomial Distribution
- Relation between Mean and Variance
- Mode of Binomial Distribution
- Poisson Distribution
- Geometrical Probability
- Application based Results

• JEE Advanced Solved Paper 2015	1077-1089
• JEE Main & Advanced Solved Papers 2016	1-12
• JEE Main & Advanced/ BITSAT/Kerala CEE/ KCET/AP & TS EAMCET/ VIT/MHT CET Solved Papers 2017	1-32
• JEE Main & Advanced/ BITSAT/ KCET/AP & TS EAMCET/ VIT/MHT CET Solved Papers 2018	1-40
• JEE Main & Advanced/ BITSAT/ AP & TS EAMCET/ MHT CET/WB JEE Solved Papers 2019-20	1-36

1

Matrix

Introduction

A rectangular array of mn numbers in the form of m horizontal lines (called rows) and n vertical lines (called columns), is called a matrix of order m by n , written as $m \times n$ matrix.

Such an array is enclosed by $[]$ or $()$ or $\| \|$. An $m \times n$ matrix is usually written as

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}$$

In compact form, the above matrix is represented by $A = [a_{ij}]_{m \times n}$. The numbers a_{11} , a_{12} , ..., etc., are known as elements of the matrix A , a_{ij} belongs to i th row and j th column and is called (i, j) th element of the matrix $A = [a_{ij}]$.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 9 \end{bmatrix}$ is a matrix having 2 rows and 3 columns. Its order is 2×3 and it has 6 elements:

$$a_{11} = 1, a_{12} = 2, a_{13} = 3, a_{21} = 0, a_{22} = -1, a_{23} = 9.$$

- A matrix is only an arrangement of numbers and functions, it has no definite value.
e.g. $[3] \neq 3$
- The elements of matrix may be scalar or vector quantity.

Types of Matrices

Row Matrix

A matrix which has only one row and any number of columns, is called a row matrix.

e.g. $A = [27 \ 85 \ 1 \ 4]_{1 \times 4}$ is a row matrix.

Column Matrix

A matrix is said to be a column matrix, if it has only one column and any number of rows.

e.g. $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3 \times 1}$ is a column matrix.

Chapter Snapshot

- Introduction
- Types of Matrices
- Equality of Two Matrices
- Trace of a Matrix
- Algebra of Matrices
- Scalar Multiplication
- Transpose of a Matrix
- Symmetric Matrix
- Skew-symmetric Matrix
- Orthogonal Matrix
- Idempotent Matrix
- Involutory Matrix
- Nilpotent Matrix
- Singular Matrix
- Conjugate of a Matrix
- Transpose of Conjugate of a Matrix
- Hermitian Matrix
- Skew-Hermitian Matrix
- Elementary Transformation of a Matrix
- Invertible Matrices
- Inverse of a Matrix using Elementary Row or Column Operations

1 Rectangular Matrix

A matrix in which number of rows is not equal to the number of columns or *vice-versa*, is called a rectangular matrix.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a rectangular matrix of order 2×3 .

Square Matrix

A matrix in which number of rows is equal to the number of columns, is called a square matrix. The elements a_{ij} of a square matrix $A = [a_{ij}]_{m \times m}$ for which $i = j$, i.e. the elements $a_{11}, a_{22}, \dots, a_{mm}$ are called the diagonal elements and the line along which they lie is called the principal diagonal or leading diagonal of the matrix.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}_{3 \times 3}$ is a square matrix of order

3 in which diagonal elements are 1, 2, 1.

Null Matrix or Zero Matrix

A matrix of order $m \times n$ whose all elements are zero, is called a null matrix or zero matrix of order $m \times n$. It is denoted by O .

e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ are two null matrices of orders 2×2 and 2×3 , respectively.

Diagonal Matrix

A square matrix is called a diagonal matrix, if all its non-diagonal elements are zero and diagonal elements are non-zero.

If $d_1, d_2, \dots, d_n$ are elements of principal diagonal of a diagonal matrix of order $n \times n$, then matrix is denoted as $\text{diag} [d_1, d_2, \dots, d_n]$.

e.g. $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is a diagonal matrix which is

denoted by $A = \text{diag} [a, b, c]$.

✎ The number of zeroes in a diagonal matrix is given by $n^2 - n$, where n is an order of the matrix.

Scalar Matrix

A square matrix $A = [a_{ij}]$ is said to be scalar matrix, if

- (a) $a_{ij} = 0, \forall i \neq j$
 (b) $a_{ij} = k, \forall i = j$,
 where $k \neq 0$

i.e. a diagonal matrix is said to be a scalar matrix, if the elements of principal diagonal are same.

e.g. $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ is a scalar matrix.

Unit Matrix or Identity Matrix

A square matrix $A = [a_{ij}]$ is said to be a unit matrix or identity matrix, if

- (a) $a_{ij} = 0, \forall i \neq j$ (b) $a_{ij} = 1, \forall i = j$

i.e. a diagonal matrix, whose elements of principal diagonal are equal to 1 and all remaining elements are zero, is known as unit or identity matrix. It is denoted by I_n or I .

e.g. $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are the unit

matrices of orders 2 and 3, respectively.

Upper Triangular Matrix

A square matrix $A = [a_{ij}]$ is known as upper triangular matrix, if $a_{ij} = 0, \forall i > j$.

e.g. $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 2 & 6 \\ 0 & 0 & 3 \end{bmatrix}$ is an upper triangular matrix.

Lower Triangular Matrix

A square matrix $A = [a_{ij}]$ is known as lower triangular matrix, if $a_{ij} = 0, \forall i < j$.

e.g. $A = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 2 & 0 \\ 5 & 6 & 3 \end{bmatrix}$ is a lower triangular matrix.

Horizontal Matrix

Any matrix in which the number of columns are more than the number of rows, is called a horizontal matrix.

e.g. $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 8 & 9 & 7 & -2 \\ 2 & -2 & -3 & 4 \end{bmatrix}$ is a horizontal matrix.

Vertical Matrix

Any matrix in which the number of rows are more than the number of columns, is called a vertical matrix.

e.g. $\begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$ is a vertical matrix.

Submatrices of a Matrix

A matrix B obtained by deleting the row(s) or column(s) or both of a matrix A is said to be a submatrix of A , i.e. the matrix B constituted by the array of elements which are left after deleting some rows or columns or both of matrix A , is called submatrix of A .

- (a) **Principal submatrix** A square submatrix B of a square matrix A is called a principal submatrix, if the diagonal elements of B are also diagonal elements of A .
- (b) **Leading submatrix** A principal square submatrix B is said to be a leading submatrix of a square matrix A , if it is obtained by deleting only some of the last rows and the corresponding columns such that the leading elements (i.e. a_{11}) is not lost.

e.g. $\begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ is a submatrix of $\begin{bmatrix} 2 & 3 & 4 \\ 6 & 1 & -1 \\ 5 & 2 & 4 \end{bmatrix}$.

Equality of Two Matrices

Two matrices $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are said to be equal, if

- i. Number of rows in A is equal to number of rows in B .
 - ii. Number of columns in A is equal to number of columns in B .
 - iii. $a_{ij} = b_{ij}, \forall i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.
- e.g. $A = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3-1 & 1+2 \\ -1 & 2+2 \end{bmatrix}$ are equal matrices, whereas $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$ are not equal because their orders are not same.

➤ **Example 1.** If $3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix}$

$+ \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$, then the value of z is

- (a) 1 (b) 2 (c) 3 (d) 4

Sol. (a) Given,

$$3 \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & 6 \\ -1 & 2w \end{bmatrix} + \begin{bmatrix} 4 & x+y \\ z+w & 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 3x & 3y \\ 3z & 3w \end{bmatrix} = \begin{bmatrix} x+4 & 6+x+y \\ -1+z+w & 2w+3 \end{bmatrix}$$

By definition of equality of matrix as the given matrices are equal, their corresponding elements are equal. On comparing the corresponding elements, we get

$$\begin{aligned} 3x &= x+4 \Rightarrow 2x=4 \Rightarrow x=2 \\ \text{and} \quad 3y &= 6+x+y \\ \Rightarrow 2y &= 6+x \\ \Rightarrow y &= \frac{6+x}{2} \end{aligned} \quad \dots(i)$$

On putting the value of x in Eq. (i), we get

$$y = \frac{6+2}{2} = \frac{8}{2} = 4$$

Now,

$$\begin{aligned} 3z &= -1+z+w \\ \Rightarrow 2z &= -1+w \\ \Rightarrow z &= \frac{-1+w}{2} \end{aligned} \quad \dots(ii)$$

Now,

$$\begin{aligned} 3w &= 2w+3 \\ \Rightarrow w &= 3 \end{aligned}$$

On putting the value of w in Eq. (ii), we get

$$z = \frac{-1+3}{2} = \frac{2}{2} = 1$$

Hence, the values of x, y, z and w are respectively 2, 4, 1 and 3.

Trace of a Matrix

Let $A = [a_{ij}]_{n \times n}$ be a square matrix. Then, the sum of all diagonal elements of A is called the trace of A and is denoted by $\text{tr}(A)$.

$$\text{Thus, } \text{tr}(A) = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

e.g. If $A = \begin{bmatrix} 2 & -7 & 9 \\ 0 & 3 & 2 \\ 8 & 9 & 4 \end{bmatrix}$, then $\text{tr}(A) = 2 + 3 + 4 = 9$

Properties of Trace of a Matrix

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two square matrices of order n , then

- i. $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$
- ii. $\text{tr}(AB) = \text{tr}(BA)$
- iii. $\text{tr}(\lambda A) = \lambda \text{tr}(A)$, where λ is a scalar.
- iv. $\text{tr}(A') = \text{tr}(A)$

➤ **Example 2.** If $A = [a_{ij}]$ is a scalar matrix of order n such that $a_{ij} = k, \forall i = j$, then trace of A is equal to

- (a) k^n (b) nk (c) $\frac{k}{n}$ (d) $\frac{n}{k}$

Sol. (b) Trace of $A = \sum_{i=1}^n a_{ii} = a_{11} + a_{22} + a_{33} + \dots + a_{nn}$
 $= k + k + k + \dots + k = nk$

Algebra of Matrices

Addition of Matrices

Let A and B be two matrices each of order $m \times n$. Then, the sum matrix $A + B$ is defined only if matrices A and B are of same order. Thus, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices of same order, then the sum $A + B$ is defined to be the matrix of order $m \times n$ such that $A + B = [a_{ij} + b_{ij}]$ for all i and j .

e.g. If $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 & 3 \\ 1 & 1 & 4 \end{bmatrix}$, then

$$C = A + B = \begin{bmatrix} 2+4 & 3-2 & 4+3 \\ 1+1 & 0+1 & 5+4 \end{bmatrix} = \begin{bmatrix} 6 & 1 & 7 \\ 2 & 1 & 9 \end{bmatrix}$$

whereas addition of $A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ is not defined, since the two matrices are not of same order. If A is any matrix, the negative of A , denoted by $-A$, is the matrix obtained by replacing each entry in A by its negative.

e.g. If $A = \begin{bmatrix} 2 & -1 \\ 5 & 4 \\ -6 & 0 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & 1 \\ -5 & -4 \\ 6 & 0 \end{bmatrix}$.

Properties of Addition of Matrices

- i. **Matrix addition is commutative** If A and B are two matrices of the same order, then

$$A + B = B + A$$
- ii. **Matrix addition is associative** If A , B and C are three matrices of the same order, then

$$(A + B) + C = A + (B + C)$$
- iii. **Existence of additive identity** If O is the zero matrix of the same order as that of the matrix A , then

$$A + O = A = O + A$$
- iv. **Existence of additive inverse** If A is any matrix, then

$$A + (-A) = O = (-A) + A$$
- v. **Cancellation laws hold good in case of addition of matrices** If A , B and C are matrices of the same order, then

$$A + B = A + C$$

$$\Rightarrow B = C \quad [\text{left cancellation law}]$$

and
$$B + A = C + A$$

$$\Rightarrow B = C \quad [\text{right cancellation law}]$$

Subtraction of Matrices

Let A and B be two matrices of the same order. Then, by $A - B$, we mean $A + (-B)$. In other words, to find $A - B$, we subtract each element of B from the corresponding element of A .

e.g. If $A = \begin{bmatrix} 2 & 3 \\ 6 & 1 \\ 7 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 \\ 2 & 5 \\ 6 & 3 \end{bmatrix}$

Then, $A - B = \begin{bmatrix} 2+3 & 3-4 \\ 6-2 & 1-5 \\ 7-6 & -2-3 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 4 & -4 \\ 1 & -5 \end{bmatrix}$

➤ **Example 3.** If $A = \begin{bmatrix} a^2 + b^2 & b^2 + c^2 \\ a^2 + c^2 & a^2 + b^2 \end{bmatrix}$ and

$B = \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$, then the value of $A - B$ is

(a) $\begin{bmatrix} (a+b)^2 & (b+c)^2 \\ (a-c)^2 & (a-b)^2 \end{bmatrix}$ (b) $\begin{bmatrix} (a-b)^2 & (b-c)^2 \\ (a+c)^2 & (a+b)^2 \end{bmatrix}$

(c) $\begin{bmatrix} (a-b)^2 & (b+c)^2 \\ (a-c)^2 & (a+b)^2 \end{bmatrix}$ (d) None of these

Sol. (b) Since, $A = \begin{bmatrix} a^2 + b^2 & b^2 + c^2 \\ a^2 + c^2 & a^2 + b^2 \end{bmatrix}$

and $B = \begin{bmatrix} 2ab & 2bc \\ -2ac & -2ab \end{bmatrix}$

$\therefore A - B = \begin{bmatrix} a^2 + b^2 - 2ab & b^2 + c^2 - 2bc \\ a^2 + c^2 + 2ac & a^2 + b^2 + 2ab \end{bmatrix}$

$$= \begin{bmatrix} (a-b)^2 & (b-c)^2 \\ (a+c)^2 & (a+b)^2 \end{bmatrix}$$

Multiplication of Matrices

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{n \times p}$ are two matrices such that the number of columns of A is equal to the number of rows of B , then matrix $C = [c_{ij}]_{m \times p}$ of order $m \times p$ is known as product of matrices A and B , where

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}$$

Multiplication of matrices is denoted by $C = AB$.

e.g. If $A = \begin{bmatrix} 2 & 1 \\ 3 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 2 \\ 1 & 7 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2 \times 5 + 1 \times 1 & 2 \times 2 + 1 \times 7 \\ 3 \times 5 + 5 \times 1 & 3 \times 2 + 5 \times 7 \end{bmatrix} = \begin{bmatrix} 11 & 11 \\ 20 & 41 \end{bmatrix}$$

➤ In multiplication of two matrices A and B , the order roles an important role.

Properties of Multiplication of Matrices

Let $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{n \times p}$ and $C = [c_{ij}]_{m \times p}$ be three matrices, then

- i. Generally, matrix multiplication is not commutative.
i.e. $AB \neq BA$
- ii. Matrix multiplication is associative.
i.e. $A(BC) = (AB)C$
- iii. Matrix multiplication is distributive over matrix addition.
i.e. $A(B + C) = AB + AC$
- iv. If A is an $m \times n$ matrix, I_n is an identity matrix of order $n \times n$ and I_m is an identity matrix of order $m \times m$, then

$$I_m A = A$$
and
$$A I_n = A$$
In particular, if A is a square matrix of order n , then

$$A I_n = I_n A = A$$
- v. $AB = O$ does not necessarily imply that $A = O$ or $B = O$ or both A and B are O .
e.g. $A = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \neq O$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \neq O$
But $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$

Scalar Multiplication

Let $A = [a_{ij}]$ be any $m \times n$ matrix and k be any scalar. Then, the matrix obtained by multiplying each element of A by k is called the scalar multiple of A by k and is denoted by kA .

Thus, if $A = [a_{ij}]_{m \times n}$, then $kA = [ka_{ij}]_{m \times n}$.
e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$, then $2A = \begin{bmatrix} 2 & 4 & 6 \\ 6 & 4 & 2 \\ 2 & 6 & 2 \end{bmatrix}$

Properties of Scalar Multiplication

If $A = [a_{ij}]_{m \times n}$, $B = [b_{ij}]_{m \times n}$ are two matrices and λ, μ are two scalars, then

- i. $\lambda(A + B) = \lambda A + \lambda B$
- ii. $(\lambda + \mu)A = \lambda A + \mu A$
- iii. $(\lambda)\mu A = \mu(\lambda A) = \lambda(\mu A)$
- iv. $(-\lambda)A = -(\lambda A) = \lambda(-A)$

➤ **Example 4.** The value of $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} [4 \ 5 \ 2] \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$

is

(a) $\begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$

(b) $[3 \ -6 \ -9]$

(c) $[0 \ 0 \ 0]$

(d) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Sol. (a) We have, $\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} [4 \ 5 \ 2] \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$
 $= \begin{bmatrix} 4 & 5 & 2 \\ -8 & -10 & -4 \\ 12 & 15 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$
 $= \begin{bmatrix} 8 - 15 + 10 \\ -16 + 30 - 20 \\ 24 - 45 + 30 \end{bmatrix}$
 $= \begin{bmatrix} 3 \\ -6 \\ 9 \end{bmatrix}$

➤ **Example 5.** If $[1 \ x \ 1] \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$,

then x is equal to

(a) 1

(b) -1

(c) $\frac{-9 \pm \sqrt{53}}{2}$

(d) None of the above

Sol. (c) We have, $[1 \ x \ 1] \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$\Rightarrow [1 \ 3 + 5x + 3 \ 2 + x + 2] \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$\Rightarrow [1 \ 5x + 6 \ 4 + x] \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$

$\Rightarrow 1 + 5x + 6 + x^2 + 4x = 0$

$\Rightarrow x^2 + 9x + 7 = 0$

$\therefore x = \frac{-9 \pm \sqrt{81 - 28}}{2}$

$\Rightarrow x = \frac{-9 \pm \sqrt{53}}{2}$



➤ **Example 6.** Let $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and X be a matrix such that $A = BX$, then X is equal to

- (a) $\frac{1}{2} \begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (b) $\frac{1}{2} \begin{bmatrix} -2 & 4 \\ 3 & 5 \end{bmatrix}$
 (c) $\begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$ (d) None of these

Sol. (a) Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Since, $A = BX$

$$\therefore \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 2c & 2d \end{bmatrix}$$

Here, $a = 1$, $b = 2$, $c = \frac{3}{2}$ and $d = -\frac{5}{2}$

$$\therefore X = \begin{bmatrix} 1 & 2 \\ 3/2 & -5/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & 4 \\ 3 & -5 \end{bmatrix}$$

➤ **Example 7.** If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, $n \in N$, then A^{4n}

equals

- (a) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$
 (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
 (c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 (d) $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$

Sol. (c) We have, $A^2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} = \begin{bmatrix} i^2 & 0 \\ 0 & i^2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Now, $A^4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^{4n} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{n \text{ times}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

➤ **Example 8.** If $AB = O$ for the matrices

$$A = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}, \text{ then } \theta - \phi \text{ is}$$

- (a) an odd multiple of $\frac{\pi}{2}$ (b) an odd multiple of π
 (c) an even multiple of $\pi/2$ (d) 0

Sol. (a) We have,

$$\begin{aligned} AB &= \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix} \\ &= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \cos \theta \cos \phi \sin \theta \sin \phi & \cos^2 \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin^2 \phi \\ \cos \theta \sin \theta \cos^2 \phi + \sin^2 \theta \cos \phi \sin \phi & \cos \theta \cos \phi \sin \theta \sin \phi + \sin^2 \theta \sin^2 \phi \end{bmatrix} \\ &= \cos(\theta - \phi) \begin{bmatrix} \cos \theta \cos \phi & \cos \theta \sin \phi \\ \sin \theta \cos \phi & \sin \theta \sin \phi \end{bmatrix} \end{aligned}$$

Since, $AB = O$

$$\therefore \cos(\theta - \phi) = 0$$

Hence, $\theta - \phi$ is an odd multiple of $\pi/2$.

Work Book Exercise 1.1

1 A matrix $A = [a_{ij}]$ is an upper triangular matrix, if

- a it is a square matrix and $a_{ij} = 0$, $i < j$
 b it is a square matrix and $a_{ij} = 0$, $i > j$
 c it is not a square matrix and $a_{ij} = 0$, $i > j$
 d it is not a square matrix and $a_{ij} = 0$, $i < j$

2 If A is any $m \times n$ matrix such that AB and BA are both defined, then B is an

- a $m \times n$ matrix b $n \times m$ matrix
 c $n \times n$ matrix d $m \times m$ matrix

3 If $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 4 & 0 \\ 0 & 2 & -1 \\ 1 & -3 & 2 \end{bmatrix}$, then

- a $AB = \begin{bmatrix} -5 & 8 & 0 \\ 0 & 4 & -2 \\ 3 & -9 & 6 \end{bmatrix}$ b $AB = [-2 \ -1 \ 4]$
 c $AB = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ d AB does not exist

4 If $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ and $B = \begin{bmatrix} a^2 & ab & ac \\ ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix}$,

then AB is equal to

- a B b A c O d I

5 If $U = \begin{bmatrix} 2 & -3 & 4 \end{bmatrix}$, $V = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$, $X = \begin{bmatrix} 0 & 2 & 3 \end{bmatrix}$ and

$Y = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}$, then the value of $UV + XY$ is

- a 20 b $[-20]$ c -20 d $[20]$

6 If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, then A^2 is equal to

- a A b $-A$ c $2A$ d $-2A$

- 7 Matrix A has m rows and $(n + 5)$ columns, matrix B has m rows and $(11 - n)$ columns. If both AB and BA exist, then
- AB and BA are square matrices
 - AB and BA are of orders 8×8 and 3×13 respectively
 - $AB = BA$
 - None of the above

- 8 If $A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then A^{100} is equal to
- $\begin{bmatrix} 1 & 0 \\ 25 & 0 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$
 - $\begin{bmatrix} 1 & 0 \\ (1/2)^{100} & 1 \end{bmatrix}$
 - None of these



Transpose of a Matrix

Let A be an $m \times n$ matrix. Then, $n \times m$ matrix obtained by interchanging the rows and columns of A , is called the transpose of A and is denoted by A' or A^T . Thus,

- if order of A is $m \times n$, then order of A' is $n \times m$.
- (i, j) th element of $A = (j, i)$ th element of A' .

e.g. If $A = \begin{bmatrix} 2 & -3 & -1 \\ 4 & 2 & 3 \end{bmatrix}_{2 \times 3}$, then $A' = \begin{bmatrix} 2 & 4 \\ -3 & 2 \\ -1 & 3 \end{bmatrix}_{3 \times 2}$.

Properties of Transpose of a Matrix

If A and B are two matrices, then

- $(A')' = A$
- $(A + B)' = A' + B'$
- $(AB)' = B' A'$
- $(kA)' = kA'$
- $(A^n)' = (A')^n$
- Trace $A' = \text{Trace } A$
- Trace $AA' \geq 0$

➤ **Example 9.** If $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix}$, then the value of $(A + 2B)'$ is

(a) $\begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & -4 \\ 6 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 6 \\ -4 & 5 \end{bmatrix}$

Sol. (b) Given, $A' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}$

$$\Rightarrow A = (A')' = \begin{bmatrix} -2 & 3 \\ 1 & 2 \end{bmatrix}' = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} [\because (A')' = A]$$

and $2B = 2 \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix}$

$$\therefore A + 2B = \begin{bmatrix} -2 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -2 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}$$

$$\Rightarrow (A + 2B)' = \begin{bmatrix} -4 & 1 \\ 5 & 6 \end{bmatrix}' = \begin{bmatrix} -4 & 5 \\ 1 & 6 \end{bmatrix}$$

Symmetric Matrix

A square matrix A is said to be symmetric, if $A' = A$, i.e. the matrix $A = [a_{ij}]_{n \times n}$ is said to be symmetric, provided $a_{ij} = a_{ji}$ for all i and j .

e.g. $A = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{bmatrix}$ is symmetric, since

$$A' = \begin{bmatrix} 2 & 1 & 5 \\ 1 & 0 & -3 \\ 5 & -3 & 6 \end{bmatrix} = A$$

Skew-symmetric Matrix

A square matrix A is said to be skew-symmetric, if $A' = -A$, i.e. the matrix $A = [a_{ij}]_{n \times n}$ is skew-symmetric, if $a_{ij} = -a_{ji}$ for all i and j .

➤ Elements of main diagonal of a skew-symmetric matrix are all zero, because by definition,

$$a_{ii} = -a_{ii} \Rightarrow 2a_{ii} = 0 \text{ or } a_{ii} = 0 \text{ for all values of } i.$$

Properties of Symmetric and Skew-symmetric Matrices

- If A is a square matrix, then
 - $A + A'$ is symmetric.
 - $A - A'$ is skew-symmetric.
- If A and B are two symmetric or skew-symmetric matrices of the same order, then $A + B$ is symmetric or skew-symmetric.
- If A is symmetric or skew-symmetric matrix and k is a scalar, then kA is also symmetric or skew-symmetric.
- If A and B are symmetric matrices of the same order, then the product AB is symmetric if and only if $AB = BA$.
- Every square matrix can be expressed uniquely as the sum of symmetric and skew-symmetric matrices.
- The matrix $B'AB$ is symmetric or skew-symmetric according as A is symmetric or skew-symmetric.

- vii. All positive integral powers of a symmetric matrix are symmetric.
- viii. All positive odd integral powers of a skew-symmetric matrix are skew-symmetric and positive even integral powers of a skew-symmetric matrix are symmetric.
- ix. If A and B are symmetric matrices of the same order, then
 (a) $AB - BA$ is a skew-symmetric matrix.
 (b) $AB + BA$ is a symmetric matrix.

➤ **Example 10.** If A is a skew-symmetric, then trace of A is

- (a) 1
 (b) -1
 (c) 0
 (d) None of the above

Sol. (c) Since, the diagonal elements of a skew-symmetric matrix are always zero.
 $\therefore$ Trace of $A = 0$

Orthogonal Matrix

A square matrix of order $n \times n$ is said to be orthogonal, if $AA' = I_n = A'A$.

e.g. If $A = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}$, then $A' = \frac{1}{2\sqrt{2}} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}$

$$\begin{aligned} \text{Also, } AA' &= \frac{1}{8} \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 4+4 & 4-4 \\ 4-4 & 4+4 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned}$$

Similarly, $A'A = I$.

Hence, A is orthogonal.

Properties of Orthogonal Matrix

- i. If A is an orthogonal matrix, then A' is also orthogonal.
- ii. For any two orthogonal matrices A and B , AB is an orthogonal matrix.
- iii. For any two orthogonal matrices A and B , BA is an orthogonal matrix.
- iv. If A is an orthogonal matrix, then A^{-1} is also orthogonal.

➤ **Example 11.** If the matrix $\begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$ is

orthogonal, then

- (a) $\alpha = \pm \frac{1}{\sqrt{2}}$ (b) $\beta = \pm \frac{1}{\sqrt{6}}$
 (c) $\gamma = \pm \frac{1}{\sqrt{3}}$ (d) All of these

Sol. (d) Let $A = \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix}$, then $A' = \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix}$

Since, A is orthogonal.

$$\begin{aligned} \therefore AA' &= I \\ \Rightarrow \begin{bmatrix} 0 & 2\beta & \gamma \\ \alpha & \beta & -\gamma \\ \alpha & -\beta & \gamma \end{bmatrix} \begin{bmatrix} 0 & \alpha & \alpha \\ 2\beta & \beta & -\beta \\ \gamma & -\gamma & \gamma \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 4\beta^2 + \gamma^2 & 2\beta^2 - \gamma^2 & -2\beta^2 + \gamma^2 \\ 2\beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 \\ -2\beta^2 + \gamma^2 & \alpha^2 - \beta^2 - \gamma^2 & \alpha^2 + \beta^2 + \gamma^2 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

On equating the corresponding elements, we get

$$4\beta^2 + \gamma^2 = 1 \quad \dots(i)$$

$$\text{and } 2\beta^2 - \gamma^2 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\beta = \pm \frac{1}{\sqrt{6}}, \gamma = \pm \frac{1}{\sqrt{3}}$$

Now, $\alpha^2 + \beta^2 + \gamma^2 = 1$

$$\Rightarrow \alpha^2 + \frac{1}{6} + \frac{1}{3} = 1 \Rightarrow \alpha = \pm \frac{1}{\sqrt{2}}$$

$$\text{Similarly, } \beta = \pm \frac{1}{\sqrt{6}}, \gamma = \pm \frac{1}{\sqrt{3}}$$

➤ **Example 12.** If $A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$, then AA' is

- (a) symmetric matrix
 (b) skew-symmetric matrix
 (c) orthogonal matrix
 (d) None of the above

$$\begin{aligned} \text{Sol. (a) We have, } AA' &= \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 1 & 1 \\ -1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 9+1+1 & 0+1-2 \\ 0+1-2 & 0+1+4 \end{bmatrix} \\ &= \begin{bmatrix} 11 & -1 \\ -1 & 5 \end{bmatrix} \end{aligned}$$

Clearly, AA' is symmetric as $(AA')' = AA'$.

Idempotent Matrix

A square matrix A is said to be idempotent, if $A^2 = A$.

e.g. If $A = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$

Then, $A^2 = AA$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-2 & 4-2 \\ -2+1 & -2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix} = A$$

Hence, A is idempotent.

Properties of Idempotent Matrix

- i. If A and B are idempotent matrices, then AB is an idempotent iff $AB = BA$.
- ii. If A and B are idempotent matrices, then $A + B$ is an idempotent iff $AB = BA = O$.
- iii. If A is an idempotent and $A + B = I$, then B is an idempotent and $AB = BA = O$.
- iv. Diagonal $(1, 1, 1, \dots, 1)$ is an idempotent matrix.
- v. If $AB = A$ and $BA = B$, then $A^2 = A$, $B^2 = B$.
- vi. If l_1, l_2, l_3 are direction cosines, then

$$\begin{bmatrix} l_1^2 & l_1 l_2 & l_1 l_3 \\ l_1 l_2 & l_2^2 & l_2 l_3 \\ l_3 l_1 & l_3 l_2 & l_3^2 \end{bmatrix}$$

is an idempotent as $|\Delta|^2 = 1$.

Involutory Matrix

A square matrix A is said to be involutory matrix, if $A^2 = I$.

e.g. If $A = \begin{bmatrix} -5 & -8 \\ 3 & 5 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} -5 & -8 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -5 & -8 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 25-24 & 40-40 \\ -15+15 & 25-24 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Hence, A is involutory.

Nilpotent Matrix

A square matrix A is said to be nilpotent matrix, if there exists positive integer m such that $A^m = O$. If m is the least positive integer such that $A^m = O$, then m is called the index of the nilpotent matrix A .

e.g. If $A = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$, then

$$A^2 = AA = \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix} \begin{bmatrix} ab & b^2 \\ -a^2 & -ab \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

Hence, the matrix A is nilpotent of the index 2.

Singular Matrix

A square matrix A is said to be singular matrix, if determinant of A denoted by $\det(A)$ or $|A|$ is zero.

i.e. $|A| = 0$, otherwise, A is a non-singular matrix.

e.g. The matrix $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 4 & -3 & 4 \end{bmatrix}$ is singular as

$$\begin{vmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 4 & -3 & 4 \end{vmatrix} = 0.$$

➤ **Example 13.** If $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$, then A is

a/an

- (a) idempotent matrix (b) nilpotent matrix
(c) involutory matrix (d) None of these

Sol. (a) Since, $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

Now, $A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Hence, A is an idempotent matrix.

➤ **Example 14.** If $A = \begin{bmatrix} xy & y^2 \\ -x^2 & xy \end{bmatrix}$, then A is

a/an

- (a) idempotent matrix (b) nilpotent matrix
(c) involutory matrix (d) None of these

Sol. (b) Since, $A = \begin{bmatrix} xy & y^2 \\ -x^2 & xy \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} xy & y^2 \\ -x^2 & xy \end{bmatrix} \begin{bmatrix} xy & y^2 \\ -x^2 & xy \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

$$\Rightarrow A^2 = O$$

Thus, A is a nilpotent matrix of order 2.

➤ **Example 15.** The number of values of x for

which the matrix $A = \begin{bmatrix} 3-2x & 2 & 2 \\ 2 & 4-2x & 1 \\ -2 & -4 & -1-2x \end{bmatrix}$

is singular, is

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. (c) Let $|A| = \begin{vmatrix} 3-2x & 2 & 2 \\ 2 & 4-2x & 1 \\ -2 & -4 & -1-2x \end{vmatrix} = 0$

$$\Rightarrow -2x(3-2x)^2 = 0$$

$$\Rightarrow x = 0, \frac{3}{2}$$

$\therefore$ Number of solutions is 2.

➤ **Example 16.** If A is symmetric as well as skew-symmetric matrices, then A is

- (a) diagonal matrix (b) null matrix
(c) triangular matrix (d) None of these

Sol. (b) Let $A = [a_{ij}]$

Since, A is skew-symmetric.

$$\therefore a_{ij} = 0 \text{ and } a_{ij} = -a_{ji} (i \neq j)$$

A is symmetric as well, so $a_{ij} = a_{ji}$ for all i and j .

$$\therefore a_{ij} = 0 \text{ for all } i \neq j$$

Hence, $a_{ij} = 0$ for all i and j , i.e. A is a null matrix.

➤ **Example 17.** The matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

is

- (a) idempotent matrix (b) involutory matrix
(c) nilpotent matrix (d) None of these

Sol. (b) We have, $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25-24+0 & 40-40+0 & 0+0+0 \\ -15+15+0 & -24+25+0 & 0+0+0 \\ -5+6-1 & -8+10-2 & 0+0+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, the matrix A is involutory.

Conjugate of a Matrix

The matrix obtained from any given matrix A containing complex numbers as its elements, on replacing its elements by the corresponding conjugate complex numbers, is called conjugate of A and is denoted by $\bar{A}$.

e.g. If $A = \begin{bmatrix} 1-i & 2 & 3+i \\ 7-3i & 1+i & 2 \\ 5 & 6 & 3-4i \end{bmatrix}$, then

$$\bar{A} = \begin{bmatrix} 1+i & 2 & 3-i \\ 7+3i & 1-i & 2 \\ 5 & 6 & 3+4i \end{bmatrix}$$

Properties of Conjugate of a Matrix

- i. If A is any matrix, then $\overline{(\bar{A})} = A$.
- ii. If A and B are matrices of same order, then $\overline{(A+B)} = \bar{A} + \bar{B}$
- iii. If A is any $m \times n$ matrix and B is any $n \times p$ matrix, then $\overline{(AB)} = \bar{A}\bar{B}$.
- iv. If A is any matrix and k is any scalar, then $\overline{(kA)} = k\bar{A}$
- v. If A is any square matrix, then $\overline{(A^n)} = (\bar{A})^n$.

Transpose of Conjugate of a Matrix

The transpose of the conjugate of a matrix A is called transpose conjugate of A and is denoted by A^θ

or A^* or $\bar{A}$. e.g. If $A = \begin{bmatrix} 1-i & 2 \\ 3+2i & 1+i \end{bmatrix}$, then

$$A^* \text{ or } A^\theta = \begin{bmatrix} 1+i & 3-2i \\ 2 & 1-i \end{bmatrix}$$

➤ The conjugate of the transpose of A is the same as the transpose of the conjugate of A , i.e. $\overline{(A^*)} = (\bar{A})^t = A^\theta$.

If $A = [a_{ij}]_{m \times n}$, then $A^\theta = [b_{ji}]_{n \times m}$, where $b_{ji} = \bar{a}_{ij}$, i.e. (j, i) th element of A^θ = the conjugate of (i, j) th element of A .

Properties of Transpose of Conjugate of a Matrix

- i. $(A^\theta)^\theta = A$
- ii. $(A+B)^\theta = A^\theta + B^\theta$
- iii. $(kA)^\theta = \bar{k}A^\theta$, k being any number.
- iv. $(AB)^\theta = B^\theta A^\theta$
- v. $(A^n)^\theta = (A^\theta)^n$



Hermitian Matrix

A square matrix A such that $A^\theta = \bar{A}' = A$, then A is known as Hermitian matrix.

e.g. Let $A = \begin{bmatrix} \alpha & \lambda + i\mu \\ \lambda - i\mu & \beta \end{bmatrix}$

$\therefore A' = \begin{bmatrix} \alpha & \lambda - i\mu \\ \lambda + i\mu & \beta \end{bmatrix}$

Now, $A^\theta = (\bar{A}') = \begin{bmatrix} \alpha & \lambda + i\mu \\ \lambda - i\mu & \beta \end{bmatrix} = A$

$\therefore A^\theta = A$

Thus, A is a Hermitian matrix.

- Every diagonal element of a Hermitian matrix must be real.
- A Hermitian matrix over the set of real numbers is actually a real symmetric matrix.

Properties of Hermitian Matrix

- If A is Hermitian matrix, then kA is also Hermitian for any real number k .
- If A and B are Hermitian matrices of same order, then $\lambda_1 A + \lambda_2 B$ is also Hermitian for any real number as λ_1, λ_2 .
- If A is any square matrix, then AA^* and A^*A are also Hermitian.
- If A and B are Hermitian, then AB is also Hermitian iff $AB = BA$.
- If A is a Hermitian matrix, then $\bar{A}$ is also Hermitian.
- If A and B are Hermitian matrices of same order, then $AB + BA$ is also Hermitian.
- If A is a square matrix, then $A + A^*$ is a Hermitian matrix.
- Any square matrix can be uniquely expressed as $A + iB$, where A and B are Hermitian matrices.

Skew-Hermitian Matrix

A square matrix A such that $A^\theta = -A$, then A is known as skew-Hermitian matrix.

e.g. Let $A = \begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix}$

$\therefore A' = \begin{bmatrix} 2i & 2-3i \\ -2-3i & -i \end{bmatrix}$

$$\Rightarrow A^\theta = (\bar{A}') = \begin{bmatrix} -2i & 2+3i \\ -2+3i & i \end{bmatrix} = -\begin{bmatrix} 2i & -2-3i \\ 2-3i & -i \end{bmatrix} = -A$$

$\therefore A^\theta = -A$

Thus, A is a skew-Hermitian matrix.

- If A is a skew-Hermitian matrix, then $a_{ii} = -\bar{a}_{ii} \Rightarrow a_{ii} + \bar{a}_{ii} = 0$, i.e. a_{ii} must be purely imaginary or zero.
- A skew-Hermitian matrix over the set of real numbers is actually a real symmetric matrix.

Properties of Skew-Hermitian Matrix

- If A is a skew-Hermitian matrix, then kA is also skew-Hermitian for any real number k .
- If A and B are skew-Hermitian matrices of same orders, then $\lambda_1 A + \lambda_2 B$ is also skew-Hermitian for any real number as λ_1, λ_2 , etc.
- If A and B are Hermitian matrices of same order, then $AB - BA$ is skew-Hermitian.
- If A is any square matrix, then $A - A^*$ is a skew-Hermitian matrix.
- Every square matrix can be uniquely represented as the sum of a Hermitian and a skew-Hermitian matrices.
- If A is a skew-Hermitian matrix, then iA is a Hermitian matrix.
- If A is a skew-Hermitian matrix, then $\bar{A}$ is also skew-Hermitian matrix.

Elementary Transformation of a Matrix

Anyone of the following operations on a matrix is called an elementary row or column transformation.

- Interchanging any two rows or columns. This transformation is indicated by

$$R_i \leftrightarrow R_j \text{ or } C_i \leftrightarrow C_j$$

e.g. $R_1 \leftrightarrow R_3$ to $A = \begin{bmatrix} a & b & c \\ p & q & r \\ m & n & o \end{bmatrix}$, we get

$$\begin{bmatrix} m & n & o \\ p & q & r \\ a & b & c \end{bmatrix}$$

ii. Multiplication of the elements of any row or column by a non-zero scalar quantity. This transformation is indicated as

$$R_i \leftrightarrow kR_i \text{ or } C_i \leftrightarrow kC_i$$

e.g. $C_3 \rightarrow \frac{1}{2}C_3$ to $B = \begin{bmatrix} 1 & -1 & 4 \\ 7 & 2 & 6 \end{bmatrix}$, we get

$$\begin{bmatrix} 1 & -1 & 2 \\ 7 & 2 & 3 \end{bmatrix}$$

iii. Addition of a constant multiple of the elements of any row to the corresponding element of any other row. This transformation is indicated as

$$R_i \rightarrow R_i + kR_j \text{ or } C_i \rightarrow C_i + kC_j$$

e.g. $R_2 \rightarrow R_2 - 4R_1$ to $C = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 0 & 1 \\ 2 & 0 & -3 \end{bmatrix}$, we get

$$\begin{bmatrix} 2 & 3 & 4 \\ -9 & -12 & -15 \\ 2 & 0 & -3 \end{bmatrix}$$

Invertible Matrices

If A is a square matrix of order $m \times m$ and there exists another square matrix B of the same order $m \times m$, such that $AB = BA = I_m$, then A is said to be invertible matrix and B is called the inverse matrix of A and it is denoted by A^{-1} .

e.g. Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$ be two

matrices.

Then, $AB = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and $BA = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

$$= \begin{bmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

Now, $AB = BA = I_2$. Thus, B is the inverse of A or $B = A^{-1}$ and A is the inverse of B , i.e. $A = B^{-1}$.

• A rectangular matrix does not possess its inverse, since for the products BA and AB to be defined and to be equal, it is necessary that matrices A and B should be square matrices of the same order.

• If B is the inverse of A , then A is also inverse of B .

• Inverse of a square matrix, if it exists, is unique.

• If A and B are invertible matrices of same order, then $(AB)^{-1} = B^{-1}A^{-1}$.

Inverse of a Matrix using Elementary Row or Column Operations

To find A^{-1} using elementary row operations, write $A = IA$ and apply a sequence of row operations on $A = IA$, till we get $I = BA$. The matrix B will be the inverse of A . Similarly, if we wish to find A^{-1} using column operations, then write $A = AI$ and apply a sequence of column operation on $A = AI$, till we get $I = AB$.

✎ In case, after applying one or more elementary row or column operations on $A = IA$ or $A = AI$, if we obtain all zeroes in one or more rows of the matrix A on LHS, then A^{-1} does not exist.

➤ **Example 18.** The inverse of $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

by elementary row transformation is

(a) $\begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ (b) $\frac{1}{5} \begin{bmatrix} 2 & 0 & 3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$

(c) $-\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix}$ (d) None of these

Sol. (c) Let $A = \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix}$

We know that, $A = IA$

$$\therefore \begin{bmatrix} 2 & -3 & 3 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 2 & 2 & 3 \\ 3 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

[using $R_1 \rightarrow R_1 + R_2 - R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 0 & -5 \\ 0 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -3 & -3 & 4 \end{bmatrix} A$$

[using $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$]

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & -5 & -10 \\ 0 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 4 \\ -2 & -1 & 2 \end{bmatrix} A$$

[using $R_2 \leftrightarrow R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 3/5 & 3/5 & -4/5 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

[using $R_2 \rightarrow -\frac{1}{5}R_2$ and $R_3 \rightarrow -\frac{1}{5}R_3$]

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -3/5 & 1/5 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A$$

$$\begin{aligned} & \text{[using } R_2 \rightarrow R_2 - 2R_3 \text{ and } R_1 \rightarrow R_1 - 4R_3] \\ \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} -2/5 & 0 & 3/5 \\ -1/5 & 1/5 & 0 \\ 2/5 & 1/5 & -2/5 \end{bmatrix} A \text{ [using } R_1 \rightarrow R_1 - R_2] \end{aligned}$$

$$\Rightarrow I_3 = -\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} A$$

$$\therefore A^{-1} = -\frac{1}{5} \begin{bmatrix} 2 & 0 & -3 \\ 1 & -1 & 0 \\ -2 & -1 & 2 \end{bmatrix} \quad [\because AA^{-1} = I_2]$$



Work Book Exercise 1.2

- If A is symmetric as well as skew-symmetric matrices, then A is
 - diagonal matrix
 - null matrix
 - triangular matrix
 - None of the above
- If A is a square matrix, then
 - $A + A'$ is symmetric
 - $A + A'$ is skew-symmetric
 - $A - A'$ is symmetric
 - None of the above
- If A is 3×4 matrix and B is a matrix such that $A'B$ and BA' are both defined. Then, B is of the type

a 3×4	b 3×3
c 4×4	d 4×3
- If A is a 3×3 skew-symmetric matrix, then trace of A is equal to

a -1	b 7
c $ A $	d None of these
- If A and B are two $n \times n$ non-singular matrices, then
 - AB is non-singular
 - AB is singular
 - $(AB)^{-1} = A^{-1}B^{-1}$
 - $(AB)^{-1}$ does not exist

- If $AB = A$ and $BA = B$, then B^2 is equal to

a B	b A	c 1	d 0

- The matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is

a idempotent matrix	b involutory matrix
c nilpotent matrix	d None of these

- In which of the following type of matrix there always exists an inverse?

a Idempotent matrix	b Orthogonal matrix
c Involutory matrix	d None of these

- If a 3×3 matrix A has its A , then A^2 is equal to

a $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$	b $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$
c $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	d $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

- The inverse of a symmetric matrix is

- | |
|-------------------------|
| a diagonal matrix |
| b skew-symmetric matrix |
| c symmetric matrix |
| d None of the above |

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Matrix A has x rows and $(x+5)$ columns. Matrix B has y rows and $(11-y)$ columns. If both AB and BA exist, then the respective values of x and y are
(a) 3, 9 (b) 8, 3 (c) 3, 8 (d) 4, 8

Sol. Since, AB is defined.

$\therefore$ Number of columns in A = Number of rows in B
 $\Rightarrow x + 5 = y$ or $x - y = -5$... (i)

Also, BA is defined.

$\therefore$ Number of columns in B = Number of rows in A
 $\Rightarrow x = 11 - y$ or $x + y = 11$... (ii)

From Eqs. (i) and (ii), we get $x = 3, y = 8$

Hence, (c) is the correct answer.

Ex 2. A matrix $A = [a_{ij}]_{m \times n}$ is

- (a) a horizontal matrix, if $m > n$
- (b) a horizontal matrix, if $m < n$
- (c) a vertical matrix, if $m > n$
- (d) a vertical matrix, if $m < n$

Sol. If number of rows > number of columns
or $m > n$

then vertical.

And if number of rows < number of columns

or $m < n$

then horizontal.

Hence, (b) is the correct answer.

Ex 3. If $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & c \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$, then the values of a

and c are

- (a) 1, 1 (b) 1, -1 (c) 1, 2 (d) -1, 1

Sol. We have, $I = AA^{-1}$

$$= \frac{1}{2} \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & a & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ -8 & 6 & 2c \\ 5 & -3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & c+1 \\ 0 & 1 & 2(c+1) \\ 4(1-a) & 3(a-1) & 2+ac \end{bmatrix}$$

On comparing the elements of AA^{-1} with those of I , we get

$$c + 1 = 0 \Rightarrow c = -1$$

$$\text{and } a - 1 = 0 \Rightarrow a = 1$$

Hence, (b) is the correct answer.

Ex 4. If square matrices A and B are such that $AA^{\theta} = A^{\theta}A$, $BB^{\theta} = B^{\theta}B$, $AB^{\theta} = B^{\theta}A$, then $AB(AB)^{\theta}$ is equal to

- (a) $B^{\theta}A^{\theta}AB$
- (b) $BA^{\theta}AB$
- (c) $BA^{\theta}A^{\theta}B$
- (d) None of the above

Sol. It is given that $AB^{\theta} = B^{\theta}A$, so that

$$(AB^{\theta})^{\theta} = (B^{\theta}A)^{\theta}$$

$$\Rightarrow (B^{\theta})^{\theta}A^{\theta} = A^{\theta}(B^{\theta})^{\theta}$$

$$\Rightarrow BA^{\theta} = A^{\theta}B$$

$$\begin{aligned} \text{Now, } AB(AB)^{\theta} &= ABB^{\theta}A^{\theta} \\ &= AB^{\theta}BA^{\theta} \\ &= B^{\theta}AA^{\theta}B \\ &= B^{\theta}A^{\theta}AB \end{aligned}$$

Hence, (a) is the correct answer.

Ex 5. If A and B are symmetric matrices and $AB = BA$, then $A^{-1}B$ is a

- (a) symmetric matrix
- (b) skew-symmetric matrix
- (c) unit matrix
- (d) None of the above

Sol. We have,

$$AB = BA = B'A' = (AB)'$$

$\Rightarrow AB$ is symmetric.

$$\text{Also, } ABA^{-1} = BAA^{-1} = B$$

$$\Rightarrow A^{-1}ABA^{-1} = A^{-1}B$$

$$\Rightarrow BA^{-1} = A^{-1}B$$

$$\Rightarrow (A^{-1}B)' = (BA^{-1})' = (A^{-1})'B' = A^{-1}B$$

[since, A^{-1} is symmetric]

Hence, (a) is the correct answer.

Ex 6. If $A = \begin{bmatrix} \alpha & 0 \\ 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$ and $A^2 = B$, then

the value of α is

- (a) 1
- (b) -1
- (c) 4
- (d) None of the above

Sol. Here, $A^2 = \begin{bmatrix} \alpha^2 & 0 \\ \alpha + 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$

No value of α satisfies this.

Hence, (d) is the correct answer.



Ex 7. If the matrices $A, B, (A+B)$ are non-singular, then $[A(A+B)^{-1}B]^{-1}$ is equal to

- (a) $A+B$
(b) $A^{-1}+B^{-1}$
(c) $A(A+B)^{-1}$
(d) None of the above

Sol. We have,

$$\begin{aligned}[A(A+B)^{-1}B]^{-1} &= B^{-1}[(A+B)^{-1}]^{-1}A^{-1} \\ &= B^{-1}(A+B)A^{-1} = (B^{-1}A+I)A^{-1} \\ &= B^{-1}I+IA^{-1} = B^{-1}+A^{-1}\end{aligned}$$

Hence, (b) is the correct answer.

Ex 8. For every square matrix A , $AA'-A'A$ is

- (a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a null matrix
(d) None of the above

Sol. We have, $(AA'-A'A)' = (AA')' - (A'A)'$

$$= (A')'A' - A'(A')' = AA' - A'A$$

Thus, $AA' - A'A$ is symmetric matrix.

Hence, (a) is the correct answer.

Ex 9. If $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$ is a square root of the two rowed

unit matrix, then δ is equal to

- (a) α
(b) β
(c) γ
(d) None of these

Sol. We have, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$

$$= \begin{bmatrix} \alpha^2 + \beta\gamma & \alpha\beta + \beta\delta \\ \alpha\gamma + \delta\gamma & \beta\gamma + \delta^2 \end{bmatrix}$$

$$\Rightarrow \alpha^2 + \beta\gamma = 1, \beta(\alpha + \delta) = 0,$$

$$\gamma(\alpha + \delta) = 0, \beta\gamma + \delta^2 = 1$$

$$\Rightarrow \beta = 0 = \gamma, \alpha \neq -\delta \text{ and } \alpha^2 = \delta^2$$

$$\Rightarrow \delta = \alpha$$

Hence, (a) is the correct answer.

Ex 10. If $\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-2}$, then the value of x is

- (a) $\frac{a}{125}$
(b) $\frac{2a}{25}$
(c) $\frac{2a}{125}$
(d) None of these

Sol. We have, $\begin{bmatrix} 1/25 & 0 \\ x & 1/25 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-2}$

Pre-multiply both sides by $\begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}$ we get

$$\begin{bmatrix} 1/5 & 0 \\ -a/25 + 5x & 1/5 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}^{-1}$$

Again, pre-multiply by $\begin{bmatrix} 5 & 0 \\ -a & 5 \end{bmatrix}$,

$$\text{so that } \frac{-2a}{5} + 25x = 0 \Rightarrow x = \frac{2a}{125}$$

Hence, (c) is the correct answer.

Ex 11. If C is skew-symmetric matrix of order n and X is $n \times 1$ column matrix, then $X'CX$ is a

- (a) scalar matrix
(b) unit matrix
(c) null matrix
(d) None of these

Sol. Here, X is $n \times 1$, C is $n \times n$ and X' is $1 \times n$.

So, $X'CX$ is 1×1 matrix. Let $X'CX = k$

$$\begin{aligned}\text{Then, } (X'CX)' &= X'C'(X')' = X'C'X = X'(-C)X \\ &= -X'CX = -k\end{aligned}$$

$$\Rightarrow k = -k \Rightarrow k = 0$$

$\therefore X'CX$ is null matrix.

Hence, (c) is the correct answer.

Ex 12. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $f(x) = \frac{1+x}{1-x}$, then $f(A)$ is

- (a) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$
(b) $\begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}$
(d) None of these

Sol. Here, $f(x) = \frac{1+x}{1-x}$

$$\therefore f(A) = (I+A)(I-A)^{-1}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1/2 \\ -1/2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix}$$

Hence, (b) is the correct answer.

Ex 13. If A is a skew-symmetric matrix of order 3, then matrix A^3 is

- (a) skew-symmetric matrix
(b) symmetric matrix
(c) diagonal matrix
(d) None of the above

Sol. We have, $A' = -A$, so that

$$A^3 = A \cdot A \cdot A \quad \text{and} \quad (A^3)' = A' A' A' = -A^3$$

Hence, (a) is the correct answer.

Ex 14. If P is a two rowed matrix satisfying $P' = P^{-1}$, then P is

- (a) $\begin{bmatrix} \cos \theta & -\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$
(b) $\begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix}$
(c) $\begin{bmatrix} -\cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$
(d) None of these

Sol. Let $P = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$

$$\text{Also, } P' = P^{-1} \Rightarrow PP' = PP^{-1} = I$$

$$\therefore \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix} = \begin{bmatrix} a_1^2 + a_2^2 & a_1 a_3 + a_2 a_4 \\ a_1 a_3 + a_2 a_4 & a_3^2 + a_4^2 \end{bmatrix} \\ = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a_1 = \pm \cos \theta, a_2 = \pm \sin \theta, a_3 = \pm \sin \theta, a_4 = \pm \cos \theta$$

In view of the fact that $a_1 a_3 + a_2 a_4 = 0$, we take

$$a_1 = \cos \theta = a_4, a_2 = \sin \theta \text{ and } a_3 = -\sin \theta$$

Hence, (b) is the correct answer.

Ex 15. If $A^2 = A$, then $(I + A)^4$ is equal to

- (a) $I + A$ (b) $I + 4A$
(c) $I + 15A$ (d) None of these

Sol. Here,

$$(I + A)^4 = (I + A)^2 (I + A)^2 = (I + IA + AI + A^2)^2 \\ = (I + 2A + A^2)^2 = (I + 2A + A)^2 = (I + 3A)^2 \\ = I + 3IA + 3AI + 9A^2 = I + 6A + 9A \\ = I + 15A$$

Hence, (c) is the correct answer.

Ex 16. If B, C are square matrices of order n and if $A = B + C$, $BC = CB$, $C^2 = 0$, then for any positive integer p , $A^{p+1} = B^k [B + (p+1)C]$, k is

- (a) $p+1$ (b) p (c) $p-1$ (d) $p+2$

Sol. We have, $BC = CB$, so that

$$A^{p+1} = (B + C)^{p+1} \\ = {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C + {}^{p+1}C_2 B^{p-1} C^2 \\ + \dots + {}^{p+1}C_r B^{p+1-r} C^r + \dots$$

$$\text{But } C^2 = 0 \Rightarrow C^3 = C^4 = \dots = C^r = 0$$

$$\text{Hence, } A^{p+1} = {}^{p+1}C_0 B^{p+1} + {}^{p+1}C_1 B^p C$$

$$= B^{p+1} + (p+1)B^p C$$

$$= B^p [B + (p+1)C]$$

$$\text{Thus, } k = p$$

Hence, (b) is the correct answer.

Ex 17. If $A^k = 0$, for some value of k ,

$$(I - A)^p = I + A + A^2 + \dots + A^{k-1}, \text{ where}$$

A is nilpotent with index k , then p is

- (a) -1 (b) -2
(c) -3 (d) None of these

Sol. Let $B = I + A + A^2 + \dots + A^{k-1}$

Post-multiply both sides by $(I - A)$, we get

$$B(I - A) = (I + A + A^2 + \dots + A^{k-1})(I - A)$$

$$= I - A + A - A^2 + A^2 - A^3 \\ + \dots - A^{k-1} + A^{k-1} - A^k \\ = I - A^k = I \quad [\because A^k = 0]$$

$$\Rightarrow B = (I - A)^{-1}$$

$$\text{Thus, } (I - A)^{-1} = I + A + A^2 + \dots + A^{k-1}$$

$$\therefore p = -1$$

Hence, (a) is the correct answer.

Ex 18. If $AB = A$ and $BA = B$, then A^2, B^2 is equal to

- (a) A, B (b) $-A, -B$
(c) $2A, B$ (d) None of these

Sol. We have, $AB = A$ and $BA = B$, so that

$$ABA = A^2$$

$$\Rightarrow A(BA) = A^2$$

$$\Rightarrow A \cdot B = A^2 \text{ or } A = A^2$$

$$\text{Also, } BA = B \Rightarrow BAB = B^2$$

$$\text{or } B(A) = B^2 \Rightarrow BA = B^2$$

$$\Rightarrow B = B^2$$

Hence, (a) is the correct answer.

Ex 19. If $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$, then $A^3 - A^2 - 4A$ is

equal to

- (a) -4 (b) $-4I$ (c) $2I$ (d) $-2I$

$$\text{Sol. We have, } A^2 = A \cdot A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

$$\text{and } A^3 = A^2 \cdot A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix}$$

$$\text{Thus, } A^3 - A^2 - 4A$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = -4I$$

Hence, (b) is the correct answer.

Ex 20. Given the matrix $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$.

The constants p, q, r such that $A^3 = pA^2 + qA + rI$, then

- (a) $p = 2, q = 2, r = -1$ (b) $p = 1, q = 2, r = -1$
(c) $p = -1, q = 2, r = 1$ (d) None of these

$$\text{Sol. Here, } A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix}$$

$$\therefore A^2 = A \cdot A$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix}$$

$$\text{and } A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \\ = \begin{bmatrix} 1 & 2 & -1 \\ -1 & -1 & 3 \\ 3 & 5 & -4 \end{bmatrix}$$

$$\therefore A^3 = pA^2 + qA + rI \\ = \begin{bmatrix} 0 & 0 & p \\ p & 2p & -p \\ -p & -p & 3p \end{bmatrix} + \begin{bmatrix} 0 & q & 0 \\ 0 & 0 & q \\ q & 2q & -q \end{bmatrix} + \begin{bmatrix} r & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r \end{bmatrix} \\ = \begin{bmatrix} r & q & p \\ p & 2p+r & -p+q \\ -p+q & -p+2q & 3p+r \end{bmatrix}$$

$$\Rightarrow r=1, q=2, p=-1$$

Hence, (c) is the correct answer.

Ex 21. For two unimodular complex numbers z_1 and

$$z_2, \begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \text{ is equal to}$$

- (a) $\begin{bmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (d) None of these

$$\text{Sol. Here, } \begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \\ = \left(\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix} \right)^{-1} \\ = \begin{bmatrix} z_1\bar{z}_1 + z_2\bar{z}_2 & -z_1z_2 + z_1\bar{z}_2 \\ -\bar{z}_1\bar{z}_2 + \bar{z}_1z_2 & \bar{z}_2z_2 + z_1\bar{z}_1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} \\ = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence, (c) is the correct answer.

Ex 22. If $abc = p$, $A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}$ and $AA' = I$ if and

only if a, b, c are the roots of the equation

- (a) $x^3 + p = 0$ (b) $x^3 + x^2 = 0$
(c) $x^3 - 2x^2 + p = 0$ (d) $x^3 \pm x^2 + p = 0$

$$\text{Sol. Here, } AA' = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix} \\ = \begin{bmatrix} a^2 + b^2 + c^2 & ac + ab + bc & ab + bc + ca \\ ca + ab + bc & a^2 + b^2 + c^2 & cb + ba + ac \\ ab + cb + ac & bc + ca + ab & a^2 + b^2 + c^2 \end{bmatrix} = I$$

$$\therefore a^2 + b^2 + c^2 = 1 \text{ and } ab + bc + ca = 0$$

$$\Rightarrow a + b + c = \pm 1$$

Also, $abc = p$, so that a, b, c are the roots of the equation $x^3 \pm x^2 + p = 0$

Hence, (d) is the correct answer.

Ex 23. If a skew-symmetric matrix S satisfies the relation $S^2 + I = 0$, where I is a unit matrix.

Then, SS' is equal to

- (a) I (b) $2I$
(c) $-I$ (d) None of these

Sol. Since, S is skew-symmetric matrix.

$$\therefore S' = -S$$

$$\text{We have, } S^2 + I = 0 \Rightarrow S^2 = -I + 0 = -I$$

$$\Rightarrow S \cdot S = -I$$

$$\Rightarrow S \cdot S \cdot S' = -IS' = I(-S') = IS = S$$

$$\Rightarrow S^{-1}SSS' = S^{-1}S$$

$$\text{i.e. } ISS' = I$$

$$\Rightarrow SS' = I$$

Hence, (a) is the correct answer.

Ex 24. If A is an $n \times n$ matrix whose elements are all 1 and B is an $n \times n$ matrix whose diagonal elements are all n and other elements are $n-r$. Then, A^2 is a scalar multiple of A and hence $(B-rI)[B-(n^2-nr+r)I]$ is

- (a) I (b) $-I$
(c) 0 (d) None of these

$$\text{Sol. Here, } A = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$$\therefore A^2 = A \cdot A = \begin{bmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{bmatrix} = nA \quad \dots(i)$$

$$\text{Now, } B = \begin{bmatrix} n & n-r & \dots & n-r \\ n-r & n & \dots & n-r \\ \vdots & \vdots & \ddots & \vdots \\ n-r & n-r & \dots & n \end{bmatrix}$$

$$\therefore B-rI = \begin{bmatrix} n-r & n-r & \dots & n-r \\ n-r & n-r & \dots & n-r \\ \vdots & \vdots & \ddots & \vdots \\ n-r & n-r & \dots & n-r \end{bmatrix} = (n-r)A \quad \dots(ii)$$

$$\text{Now, } (B-rI)[B-(n^2-nr+r)I] \\ = (B-rI)[B-rI - n(n-r)I] \\ = (n-r)A[(n-r)A - n(n-r)I] \quad [\text{from Eq. (ii)}] \\ = (n-r)^2 A^2 - n(n-r)^2 AI \\ = (n-r)^2 [nA - nA] = 0 \quad [\text{from Eq. (i)}]$$

Hence, (c) is the correct answer.

Ex 25. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, where a, b, c are

real positive numbers, $abc=1$ and $A^T A = I$, then the value of $a^3 + b^3 + c^3$ is

- (a) 3 (b) 2
(c) 4 (d) 1

Sol. $\therefore A^T A = I$

$$\therefore \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2 + b^2 + c^2 = 1 \quad \dots(i)$$

$$\text{and } ab + bc + ca = 0 \quad \dots(ii)$$

$$\text{Now, } a^3 + b^3 + c^3 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) + 3abc$$

$$= (a + b + c) + 3 \quad \dots(iii)$$

$$\text{and } (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$= 1 + 2 \cdot 0 = 1$$

$$\Rightarrow a + b + c = 1$$

$$[\text{since, } a, b, c \text{ are real positive numbers}]$$

$$\text{From Eq. (iii), } a^3 + b^3 + c^3 = 1 + 3 = 4$$

Hence, (c) is the correct answer.

Aliter

$$A^T A = I \Rightarrow |A^T A| = |I| \Rightarrow |A|^2 = 1$$

$$\Rightarrow (a^3 + b^3 + c^3 - 3abc)^2 = 1$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 1$$

$$[\text{since, } a, b, c \text{ are positive real numbers}]$$

$$\Rightarrow a^3 + b^3 + c^3 \geq 3abc \quad [\text{using AM} \geq \text{GM}]$$

$$\Rightarrow a^3 + b^3 + c^3 = 4$$

Ex 26. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}; a, b \in N$.

Then,

(a) there exists exactly one B such that

$$AB = BA$$

(b) there exist infinitely many B 's such that

$$AB = BA$$

(c) there cannot exist any B such that $AB = BA$

(d) there exist more than one but finite number of B 's such that $AB = BA$

Type 2. More than One Correct Option

Ex 28. If $AB = A$ and $BA = B$, then

(a) $A^2 B = A^2$

(b) $B^2 A = B^2$

(c) $ABA = A$

(d) $BAB = B$

Sol. We have, $A^2 B = A(AB) = AA = A^2$

$$B^2 A = B(BA) = BB = B^2$$

$$ABA = A(BA) = AB = A$$

$$\text{and } BAB = B(AB) = BA = B$$

Hence, (a), (b), (c) and (d) are the correct answers.

Sol. We have, $AB = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}, BA = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

If $AB = BA$, then it is clear that $a = b$, which is possible for infinitely many values of a and b .

Hence, (b) is the correct answer.

Ex 27. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}, A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Q = PAP^T$

and $X = P^T Q^{2005} P$, then X is equal to

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 4 + 2005\sqrt{3} & 6015 \\ 2005 & 4 - 2005\sqrt{3} \end{bmatrix}$

(c) $\frac{1}{4} \begin{bmatrix} 2 + \sqrt{3} & 1 \\ -1 & 2 - \sqrt{3} \end{bmatrix}$

(d) $\frac{1}{4} \begin{bmatrix} 2005 & 2 - \sqrt{3} \\ 2 + \sqrt{3} & 2005 \end{bmatrix}$

Sol. It is easy to see that $P^T P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$$\Rightarrow P^T = P^{-1}$$

Now,

$$Q^2 = Q \cdot Q = (PAP^T)(PAP^T) = PA(P^T P)AP^T$$

$$= PAIP^T = PA^2 P^T$$

$$Q^3 = Q^2 \cdot Q = (PA^2 P^T)(PAP^T)$$

$$= PA^2 (P^T P) AP^T = PA^3 P^T$$

and so on.

$$\Rightarrow Q^{2005} = PA^{2005} P^{-1}$$

$$\Rightarrow X = P^T Q^{2005} P = P^{-1} (PA^{2005} P^{-1}) P = A^{2005}$$

$$\Rightarrow A^{2005} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \dots 2005 \text{ times}$$

$$= \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$$

Hence, (a) is the correct answer.

Ex 29. Let $\alpha = \frac{\pi}{5}$ and $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, then

$$B = A + A^2 + A^3 + A^4 \text{ is}$$

(a) singular (b) non-singular

(c) skew-symmetric (d) $|B| = 1$

Sol. We have,

$$A^2 = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix}, A^3 = \begin{bmatrix} \cos 3\alpha & \sin 3\alpha \\ -\sin 3\alpha & \cos 3\alpha \end{bmatrix}$$

$$\text{and } A^4 = \begin{bmatrix} \cos 4\alpha & \sin 4\alpha \\ -\sin 4\alpha & \cos 4\alpha \end{bmatrix}$$



Now, $\cos \alpha + \cos 2\alpha + \cos 3\alpha + \cos 4\alpha$
 $= \cos \alpha + \cos 2\alpha + \cos(\pi - 2\alpha) + \cos(\pi - \alpha)$
 $\quad [\because 5\alpha = \pi]$
 $= \cos \alpha + \cos 2\alpha - \cos 2\alpha - \cos \alpha = 0$
 and $\sin \alpha + \sin 2\alpha + \sin 3\alpha + \sin 4\alpha$
 $= \sin \alpha + \sin 2\alpha + \sin(\pi - 2\alpha) + \sin(\pi - \alpha)$
 $= 2(\sin \alpha + \sin 2\alpha)$
 $= 2 \left\{ 2 \sin \left(\frac{3\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right) \right\}$
 $= 4 \sin \left(\frac{3\pi}{10} \right) \cos \left(\frac{\pi}{10} \right)$
 $= 4 \sin \left(\frac{\pi}{2} - \frac{\pi}{5} \right) \cos \left(\frac{\pi}{10} \right)$
 $= 4 \cos \left(\frac{\pi}{5} \right) \cos \left(\frac{\pi}{10} \right) = a \quad [\text{say}]$

Thus, $B = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$

$\therefore B$ is skew-symmetric.

Also, $|B| = a^2 = 16 \cos^2 \left(\frac{\pi}{5} \right) \cos^2 \left(\frac{\pi}{10} \right) > 0$

$\therefore B$ is non-singular.

Hence, (b) and (c) are the correct answers.

Ex 30. If $A = \begin{bmatrix} 0 & -\tan \frac{\alpha}{2} \\ \tan \frac{\alpha}{2} & 0 \end{bmatrix}$ and I is identity

matrix of order 2, then

(a) $I + A = (I - A) \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$

(b) $I - A = (I + A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$

(c) $I + A = (I - A) \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

(d) Both (a) and (b) are not true

Sol. If $\alpha = \pi/2$, then $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$I + A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

(a) $\text{RHS} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \neq I - A$

$\Rightarrow$ Choice (a) is false.

(b) Choice (b) is also incorrect.

(c) $\text{RHS} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = I + A = \text{LHS}$

If choice (c) is coming out to be true as a special case, then only one choice is true which is (d). Since, more than one answer should be correct, so (c) should also be true. We can prove (c) in general.

Hence, (c) and (d) are the correct answers.

Ex 31. For all x in $(-1, 1)$, let $A(x)$ be a matrix

defined by $A(x) = \frac{1}{\sqrt{1-x^2}} \begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix}$, then

(a) $A(x)A(y) = A(xy)$

(b) $A(x)A(y) = A \left(\frac{x+y}{1+xy} \right)$

(c) $A(x)A(y) = A \left(\frac{x-y}{1+xy} \right)$

(d) $(A(x))^{-1} = A(-x)$

Sol. $A(x)A(y) = \frac{1}{(\sqrt{1-x^2}\sqrt{1-y^2})} \begin{bmatrix} 1 & -x \\ -x & 1 \end{bmatrix} \begin{bmatrix} 1 & -y \\ -y & 1 \end{bmatrix}$
 $= \frac{1}{(\sqrt{1-x^2}\sqrt{1-y^2})} \begin{bmatrix} 1+xy & -(x+y) \\ -(x+y) & 1+xy \end{bmatrix}$
 $= \frac{1+xy}{(\sqrt{1-x^2}\sqrt{1-y^2})} \begin{bmatrix} 1 & \frac{x+y}{1+xy} \\ -\frac{x+y}{1+xy} & 1 \end{bmatrix}$
 $= \frac{1}{\sqrt{1-\left(\frac{x+y}{1+xy}\right)^2}} \begin{bmatrix} 1 & \frac{x+y}{1+xy} \\ -\frac{x+y}{1+xy} & 1 \end{bmatrix}$
 $= A \left(\frac{x+y}{1+xy} \right)$

$\therefore$ Choice (b) is correct.

Since, $A(0) = I_2$

Putting $y = -x$ in choice (b), we get

$A(x)A(-x) = A(0) = I_2$

$\therefore$ Choice (d) follows.

Hence, (b) and (d) are the correct answers.

Ex 32. If a square matrix A satisfies $A^2 = A$, then

(a) $(I + A)^2 = I + 3A$

(b) $(I + A)^3 = I + 4A$

(c) $(I + A)^n = I + (2^n - 1)A, \forall n \in \mathbb{N}$

(d) $(I + A)^n = I + nA, \forall n \in \mathbb{A}$

Sol. (a) $(I + A)^2 = (I + A)(I + A)$
 $= I^2 + IA + IA + A^2$
 $= I + A + A + A \quad [\because A^2 = A]$
 $= I + 3A$

$\therefore$ Choice (a) is true.

(b) $(I + A)^3 = (I + A)^2(I + A)$
 $= (I + 3A)(I + A)$
 $= I^2 + IA + 3AI + 3A^2$
 $= I + 7A$

$\therefore$ Choice (b) is false.

(c) Choice (c) is turning out to be true for $n = 2, n = 3$.

Let us try proving choice (c) for all n .

If this is true for n , then

$(I + A)^{n+1} = (I + A)^n(I + A)$

$$\begin{aligned} &= [I + (2^n - 1)A][I + A] \\ &= I^2 + IA + (2^n - 1)AI + (2^n - 1)A^2 \\ &= I + A + (2^n - 1)A + (2^n - 1)A \\ &= I + (2^{n+1} - 1)A \end{aligned}$$

$\Rightarrow$ Result is true for $n + 1$ also.

$\therefore$ Choice (c) is true for all n .

(d) Choice (d) is clearly wrong for $n = 2$.

Hence, (a) and (c) are the correct answers.

Ex 33. For what value of x , the matrix

$$\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{bmatrix} \text{ is singular?}$$

(a) $x = 1$

(b) $x = 2$

(c) $x = 0$

(d) $x = 3$

Type 3. Assertion and Reason

Directions (Ex. Nos. 34-38) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 34. Statement I The inverse of the matrix

$$A = [a_{ij}]_{n \times n}, \text{ where } a_{ij} = 0, i \geq j, \text{ is}$$

$$B = [a_{ij}^{-1}]_{n \times n}$$

Statement II The inverse of singular matrix does not exist.

Sol. Statement I $\because A = [a_{ij}]_{n \times n}$, where $a_{ij} = 0, i \geq j$

$\therefore |A| = 0$, so A is singular.

Thus, inverse of A is not defined.

Therefore, Statement I is false.

Statement II $|A| = 0$

$\therefore$ Inverse of A is not defined.

Hence, (d) is the correct answer.

Ex 35. Statement I For a singular square matrix A , if $AB = AC \Rightarrow B = C$.

Statement II If $|A| = 0$, then A^{-1} does not exist.

Sol. A^{-1} exists only for non-singular matrix.

$$AB = AC \Rightarrow B = C$$

$\therefore A^{-1}$ exists.

Hence, (d) is the correct answer.

Sol. Since, the given matrix is singular.

Therefore, its determinant is 0.

$$\Rightarrow \begin{vmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-x & 2 & 2 \\ 0 & -x & -x \\ -2 & -4 & -1-x \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 3-x & 2 & 2 \\ -x & 0 & 1 \\ 2 & 4 & 1+x \end{vmatrix} = 0 \quad [\text{applying } R_2 \rightarrow R_2 + R_3]$$

$$\Rightarrow -x\{(3-x)(1+x-4)-2(0-2)+2(0-2)\} = 0$$

$$\Rightarrow -x\{(3-x)(x-3)+4-4\} = 0 \Rightarrow x = 0, 3$$

Hence, (c) and (d) are the correct answers.

Ex 36. Statement I The product of two diagonal matrices of order 3×3 is also a diagonal matrix.

Statement II Matrix multiplication is non-commutative.

$$\begin{aligned} \text{Sol. Since, } A \text{ is } & \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} d'_1 & 0 & 0 \\ 0 & d'_2 & 0 \\ 0 & 0 & d'_3 \end{bmatrix} \\ &= \begin{bmatrix} d_1 d'_1 & 0 & 0 \\ 0 & d_2 d'_2 & 0 \\ 0 & 0 & d_3 d'_3 \end{bmatrix} \end{aligned}$$

which is also a diagonal matrix.

Thus, Statement II is true but it is not applicable in proving Statement I.

Hence, (b) is the correct answer.

Ex 37. Let A be a square matrix of order 3 satisfying $AA' = I$.

Statement I $A' = A^{-1}$

Statement II $(AB)' = B'A'$

Sol. Statement I is true because

$$AA' = I$$

$$\Rightarrow A^2 = I$$

$\therefore A$ is an invertible matrix.

$$\Rightarrow A' = A^{-1}$$

Statement II is true which is a known fact.

Obviously, Statement II is not a correct reason for Statement I.

Hence, (b) is the correct answer.



Ex 38. Statement I The inverse of the matrix

$$\begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{bmatrix} \text{ does not exist.}$$

Statement II The matrix $\begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{bmatrix}$ is singular.

Sol. Here, Statement I and Statement II are both true.

$$\therefore \begin{bmatrix} 1 & 3 & 5 \\ 2 & 6 & 10 \\ 9 & 8 & 7 \end{bmatrix} \text{ is singular.}$$

$\therefore$ Inverse does not exist.

Thus, Statement II is a correct explanation for Statement I.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 39-41) A square matrix is said to be idempotent, if $A^2 = A$.

Ex 39. If $i = \sqrt{-1}$, $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$, then which of the following matrix is idempotent?

- (a) $\begin{bmatrix} a & i \\ i & -b \end{bmatrix}$ (b) $\begin{bmatrix} b & i \\ i & a \end{bmatrix}$
(c) $\begin{bmatrix} a & i \\ i & b \end{bmatrix}$ (d) None of these

Sol. $\begin{bmatrix} a & i \\ i & b \end{bmatrix} \begin{bmatrix} a & i \\ i & b \end{bmatrix} = \begin{bmatrix} a^2 - 1 & ai + bi \\ ai + bi & -1 + b^2 \end{bmatrix} = \begin{bmatrix} a & i \\ i & b \end{bmatrix}$
 $\therefore a^2 - 1 = a \Rightarrow b^2 - 1 = b \Rightarrow a + b = 1$

$\therefore$ Other choices do not satisfy by $A^2 = A$.

Hence, (c) is the correct answer.

Ex 40. The number of 2×2 idempotent matrices with integer entries are

- (a) 1 (b) 2
(c) 3 (d) infinite

Sol. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

Now, $A^2 = A$
 $\Rightarrow a^2 + bc = a \dots(i)$

$ab + bd = b \dots(ii)$

$ac + cd = c \dots(iii)$

and $bc + d^2 = d \dots(iv)$

Let $b \neq 0$, $c \neq 0$, then from Eq. (ii), $a = -d$

Eqs. (iii) and (iv) become $a^2 - a - bc = 0$

$\therefore a = \frac{1 \pm \sqrt{1 + 4bc}}{2}$

whenever $1 + 4bc$ is a perfect square given that a is an integer, i.e. $4bc = 8, 24, 48, \dots$

Then, we get infinite values of a .

Thus, number of idempotent matrix become infinite.

Hence, (d) is the correct answer.

Ex 41. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix for arbitrary values of b and c , then
 (a) $a^2 = a + bc$ (b) $a^2 = a - bc$
 (c) $a^2 + b^2 = c^2$ (d) None of these

Sol. Since, given matrix is idempotent, therefore $A = A^2$.

$$\therefore \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a^2 + bc & b \\ c & a^2 + 1 - 2a + bc \end{bmatrix}$$

$$\Rightarrow a = a^2 + bc$$

$$\Rightarrow a^2 = a - bc$$

Hence, (b) is the correct answer.

Passage II (Ex. Nos. 42-46) If $abc = p$ and

$$A = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix}, \text{ where } A \text{ is orthogonal matrix.}$$

Ex 42. The value of $a + b + c$ is

- (a) 2 (b) p
(c) $2p$ (d) ± 1

Ex 43. The value of $ab + bc + ca$ is

- (a) 0 (b) p
(c) $2p$ (d) $3p$

Ex 44. The value of $a^2 + b^2 + c^2$ is

- (a) 1 (b) p (c) $2p$ (d) $3p$

Ex 45. The value of $a^3 + b^3 + c^3$ is

- (a) p
(b) $2p$
(c) $3p$
(d) None of the above

Ex 46. The equation whose roots are a, b, c , is

- (a) $x^3 - 2x^2 + p = 0$
(b) $x^3 - px^2 + px + p = 0$
(c) $x^3 - 2x^2 + 2px + p = 0$
(d) $x^3 \pm x^2 - p = 0$

Sol. (Ex. Nos. 42-46)

Since, A is orthogonal.

$$\therefore AA' = I$$

$$\Rightarrow A' = \begin{bmatrix} a & b & c \\ c & a & b \\ b & c & a \end{bmatrix} \begin{bmatrix} a & c & b \\ b & a & c \\ c & b & a \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + b^2 + c^2 & ac + ab + bc & ab + bc + ca \\ ca + ab + bc & a^2 + b^2 + c^2 & cb + ba + ac \\ ab + bc + ca & bc + ca + ab & a^2 + b^2 + c^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore a^2 + b^2 + c^2 = 1 \text{ and } ab + bc + ca = 0$$

$$\text{Now, } (a + b + c)^2 = a^2 + b^2 + c^2 + 2(ab + bc + ca)$$

$$\Rightarrow (a + b + c)^2 = 1$$

$$\Rightarrow a + b + c = \pm 1, ab + bc + ca = 0$$

$$\text{Also, } abc = p \therefore a^3 + b^3 + c^3 - 3abc$$

$$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

$$\Rightarrow a^3 + b^3 + c^3 = 3p \pm (1 - 0)$$

$$\Rightarrow a^3 + b^3 + c^3 = 3p \pm 1$$

Now, if a, b, c are roots of cubic, then

$$x^3 - (a + b + c)x^2 + (ab + bc + ca)x - abc = 0$$

$$\Rightarrow x^3 \pm x^2 - p = 0$$

42. (d) **43.** (a) **44.** (a) **45.** (d) **46.** (d)

Type 5. Match the Columns

Ex 47. Match the statements of Column I with values of Column II.

Column I	Column II
A. If A is real skew-symmetric matrix such that $A^2 + I = 0$. Then,	p. $BA - AB$
B. A is a matrix such that $A^2 = A$. If $(I + A)^n = I + \lambda A$, then λ equals	q. A is of even order
C. If for a matrix A , $A^2 = A$ and $B = I - A$, then $AB + BA + I - (I - A)^2$ equals	r. A
D. A is matrix with complex entries and A' stands for transpose of complex conjugate of A . If $A' = A$ and $B' = B$, then $(AB - BA)'$ equals	s. $2^n - 1$

Sol. A. $\therefore A^2 = -I$

$\therefore A$ is of even order.

$$\text{B. } (I + A)^n = C_0 I^n + C_1 I A + C_2 I A^2 + \dots + C_n I A^n$$

$$= C_0 I + C_1 A + C_2 A^2 + \dots + C_n A^n$$

$$= I + (2^n - 1)A$$

$$\therefore \lambda = 2^n - 1$$

C. $\therefore A^2 = A$ and $B = I - A$

$$\therefore AB + BA + I - (I + A^2 - 2A)$$

$$= AB + BA - A + 2A$$

$$= AB + BA + A$$

$$= A(I - A) + (I - A)A + A$$

$$= A - A + A - A + A = A$$

D. $\therefore A' = A, B' = B$

$$\therefore (AB - BA)' = B' A' - A' B' = BA - AB$$

$$A \rightarrow q; B \rightarrow s; C \rightarrow r; D \rightarrow p$$

Ex 48. If A and B are 3×3 symmetric matrices, then match the Column I with values of Column II.

Column I	Column II
A. $AB + BA$	p. Symmetric
B. $A + B$	q. Anti-symmetric
C. $AB - BA$	r. Neither symmetric nor skew-symmetric
D. $A - B$	s. Null matrix

Sol. A. $(AB + BA)' = (AB)' + (BA)'$

$$= B' A' + A' B'$$

$$= BA + AB \quad [\because B' = B, A' = A]$$

$$= AB + BA$$

$$\therefore (AB + BA) \text{ is symmetric.}$$

B. $(A + B)' = A' + B' = A + B$

$$\therefore (A + B) \text{ is symmetric.}$$

C. $(AB - BA)' = (AB)' - (BA)'$

$$= B' A' - A' B'$$

$$= BA - AB = -(AB - BA)$$

$$\therefore AB - BA \text{ is anti-symmetric.}$$

D. $(A - B)' = A' - B'$

$$= A - B$$

$$\therefore (A - B) \text{ is symmetric.}$$

$$A \rightarrow p; B \rightarrow p; C \rightarrow q; D \rightarrow p$$

Ex 49. Using n distinct real numbers, matrices each having distinct elements and of all possible orders are to be made, then match the statements of Column I with values of Column II.

Column I	Column II
A. $n = 3$	p. 72 possible matrices
B. $n = 4$	q. 12 possible matrices
C. $n = 5$	r. 240 possible matrices

Sol. A. If $n = 3$, then the matrices formed are of order $3 \times 1, 1 \times 3$.

$$\therefore \text{Possible matrices} = 3! + 3! = 12$$

$$[\text{since, } \begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ will give rise to } 3! \text{ matrices}]$$

B. For $n = 4$, the possible orders are

$$1 \times 4, 4 \times 1, 2 \times 2$$

$$\therefore \text{Total possible matrices} = 4! + 4! + 4! = 72$$

C. For $n = 5$, we have number of matrices

$$= 5! + 5! = 240$$

$$A \rightarrow q; B \rightarrow p; C \rightarrow r$$

Type 6. Single Integer Answer Type Questions



Ex 50. Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$. If $A^6 = kA - 205I$, then the numerical quantity of $k - 40$ should be _____.

Sol. (4) We know that characteristic matrix of A is

$$A - \lambda = \begin{bmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{bmatrix}$$

$\therefore$ Characteristic polynomial is

$$(1-\lambda)(3-\lambda) + 2 = 0$$

$$\Rightarrow \lambda^2 - 4\lambda + 5 = 0$$

$$\Rightarrow A^2 - 4A + 5I = 0$$

$$\Rightarrow A^2 = 4A - 5I$$

We will find higher powers of A as a linear polynomial in A .

$$\text{We have, } A^3 = A^2 A = (4A - 5I)A$$

$$= 4A^2 - 5A = 4(4A - 5I) - 5A$$

$$= 11A - 20I$$

$$\text{Now, } A^6 = A^3 \cdot A^3 = (11A - 20I)(11A - 20I)$$

$$= 121A^2 - 440A + 400I$$

$$\text{Substituting for } A^2, \text{ we get } A^6 = 44A - 205I$$

$$\Rightarrow k = 44$$

$$\therefore k - 40 = 44 - 40 = 4$$

Ex 51. If $A = \begin{bmatrix} t & t+1 \\ t-1 & t \end{bmatrix}$ is a matrix such that $AA' = I_2$, then trace of the matrix must be _____.

Sol. (0) $AA^T = \begin{bmatrix} t^2 + (t+1)^2 & t(t-1) + t(t+1) \\ t(t-1) + t(t+1) & t^2 + (t-1)^2 \end{bmatrix}$

$$\Rightarrow AA^T = I_2$$

$$\Rightarrow t^2 - t + t^2 + t = 0$$

$$\Rightarrow 2t^2 = 0 \Rightarrow t = 0$$

$$\therefore \text{Trace of } A = 0$$

Ex 52. If $A = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} = B^3 + C^3$, where B and C are 2×2 matrices with integer elements, then $-\text{tr}(B) - \text{tr}(C)$ must be equal to _____.

Sol. (3) We have, $A^2 = \begin{bmatrix} 1 & -3 \\ 0 & 4 \end{bmatrix}$

$$\text{Now, } A^2 + 3A + 2I = 0$$

$$\Rightarrow A^3 + 3A^2 + 2A = 0$$

$$\Rightarrow (A + I)^3 - A = I^3$$

$$\Rightarrow A = (A + I)^3 - I^3$$

$$= (A + I)^3 + (-I)^3$$

$$\Rightarrow B = A + I, C = -I$$

$$\Rightarrow B = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\therefore \text{tr}(B) = -1$$

$$\text{and } \text{tr}(C) = -2$$

$$\text{Hence, } -\text{tr}(B) - \text{tr}(C) = 3$$

Ex 53. Let A be a square matrix which commutes with square matrix B of the same order r and $AA' = I$ and $AB' - B'A = kI$, where k is a scalar, then k must be _____.

Sol. (0) $(AB')' = (BA)'$ [$\because AB = BA$]

$$\Rightarrow B'A' = A'B'$$

$$\Rightarrow B'A'A = A'B'A \quad [\text{post-multiplying by } A]$$

$$\Rightarrow B'I = A'B'A$$

$$\Rightarrow AB'I = AA'B'A$$

$$\Rightarrow AB' = B'A$$

$$\Rightarrow AB' - B'A \text{ is a zero matrix.}$$

$$\therefore k = 0$$

Target Exercises

Type 1. Only One Correct Option

1. If $A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$, then trace of matrix A is
 - (a) 17
 - (b) 25
 - (c) 3
 - (d) 12
2. If $2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix}$ and $A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix}$, then B is
 - (a) $\begin{bmatrix} 8 & -1 & 2 \\ -1 & 10 & -1 \end{bmatrix}$
 - (b) $\begin{bmatrix} 8 & 1 & 2 \\ -1 & 10 & -1 \end{bmatrix}$
 - (c) $\begin{bmatrix} 8 & 1 & -2 \\ -1 & 10 & -1 \end{bmatrix}$
 - (d) $\begin{bmatrix} 8 & 1 & 2 \\ 1 & 10 & 1 \end{bmatrix}$
3. What must be the matrix X , if $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$?
 - (a) $\begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$
 - (b) $\begin{bmatrix} 1 & -3 \\ 2 & -1 \end{bmatrix}$
 - (c) $\begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix}$
 - (d) $\begin{bmatrix} 2 & -6 \\ 4 & -2 \end{bmatrix}$
4. If $A = \begin{bmatrix} 1 & -3 \\ 2 & k \end{bmatrix}$ and $A^2 - 4A + 10I = A$, then k is equal to
 - (a) 0
 - (b) -4
 - (c) 4 and not 1
 - (d) 1 or 4
5. If $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$, then A^4 is equal to
 - (a) $27A$
 - (b) $81A$
 - (c) $243A$
 - (d) $729A$
6. The number of diagonal matrix A of order n for which $A^3 = A$, is
 - (a) 1
 - (b) 0
 - (c) 2^n
 - (d) 3^n
7. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and I is the unit matrix of order 2, then A^2 equals
 - (a) $4A - 3I$
 - (b) $3A - 4I$
 - (c) $A - I$
 - (d) $A + I$
8. If the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is commutative with the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, then
 - (a) $a = 0, b = c$
 - (b) $b = 0, c = d$
 - (c) $c = 0, d = a$
 - (d) $d = 0, a = b$
9. If $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$ and $f(t) = t^2 - 3t + 7$, then $f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix}$ is equal to
 - (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 - (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
 - (c) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
 - (d) $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$
10. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then $A^2 - 4A$ is equal to
 - (a) $2I_3$
 - (b) $3I_3$
 - (c) $4I_3$
 - (d) $5I_3$
11. If $[1 \ x \ 1] \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = O$ (null matrix), then the value of x is
 - (a) 0
 - (b) $\frac{2}{3}$
 - (c) $\frac{5}{4}$
 - (d) $-\frac{4}{5}$
12. If $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, then A^{100} is equal to
 - (a) $2^{100}A$
 - (b) $2^{99}A$
 - (c) $100A$
 - (d) $299A$
13. $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [2 \ 1 \ -1]$ is equal to
 - (a) $\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$
 - (b) $\begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$
 - (c) $[-1]$
 - (d) Not defined
14. If A and B are square matrices of the same order and $AB = 3I$, then A^{-1} is equal to
 - (a) $3B$
 - (b) $\frac{1}{3}B$
 - (c) $3B^{-1}$
 - (d) $\frac{1}{3}B^{-1}$
15. If A and B are square matrices of the same order such that $(A + B)(A - B) = A^2 - B^2$, then $(ABA^{-1})^2$ is equal to
 - (a) B^2
 - (b) I
 - (c) A^2B^2
 - (d) A^2
16. If A and B are two square matrices such that $AB = A$ and $BA = B$, then A and B are
 - (a) nilpotent
 - (b) idempotent
 - (c) periodic
 - (d) involutory

17. For $k = \frac{1}{\sqrt{50}}$, the values of a, b, c such that $PP' = I$,

$$\text{where } P = \begin{bmatrix} \frac{2}{3} & 3k & a \\ -\frac{1}{3} & -4k & b \\ \frac{2}{3} & -5k & c \end{bmatrix}, \text{ are}$$

- (a) $\pm \frac{16}{5\sqrt{2}}, \pm \frac{13}{5\sqrt{2}}, \pm \frac{1}{3\sqrt{2}}$
 (b) $\pm \frac{1}{3\sqrt{2}}, \pm \frac{13}{5\sqrt{2}}, \pm \frac{16}{15\sqrt{2}}$
 (c) $\pm \frac{13}{5\sqrt{2}}, \pm \frac{16}{15\sqrt{2}}, \pm \frac{1}{3\sqrt{2}}$
 (d) None of the above

18. If $A^2 = A$, then $(I + A)^4$ is equal to

- (a) $I + A$ (b) $I + 4A$
 (c) $I + 15A$ (d) None of these

19. If A is a 2×2 matrix such that $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ and

$$A^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \text{ Then, the sum of the elements of } A \text{ is}$$

- (a) -1 (b) 0 (c) 2 (d) 5

20. If $A = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$ is n th root of I_2 , then choose the correct statements.

- I. If n is odd, then $a = 1, b = 0$
 II. If n is odd, then $a = -1, b = 0$
 III. If n is even, then $a = 1, b = 0$
 IV. If n is even, then $a = -1, b = 0$
 (a) I, II, III (b) II, III, IV (c) I, II, III, IV (d) I, III, IV

21. For $A = \begin{bmatrix} 4 & 2i \\ i & 1 \end{bmatrix}$, $(A - 2I)(A - 3I)$ is a

- (a) null matrix (b) Hermitian matrix
 (c) unit matrix (d) None of these

22. If $E(\theta) = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$ and θ, ϕ differ by an odd multiple of $\frac{\pi}{2}$, then $E(\theta)E(\phi)$ is a

- (a) unit matrix (b) null matrix
 (c) diagonal matrix (d) None of these

23. If $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^4 = I$, then

- (a) $a = 1 = 2b$ (b) $a = b$ (c) $a = b^2$ (d) $ab = 1$

24. An orthogonal matrix is

- (a) $\begin{bmatrix} \cos \alpha & 2 \sin \alpha \\ -2 \sin \alpha & \cos \alpha \end{bmatrix}$ (b) $\begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$
 (c) $\begin{bmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

25. If A is a skew-symmetric matrix, then trace of A is

- (a) 1 (b) -1
 (c) 0 (d) None of these

26. Let A and B be two symmetric matrices of same order. Then, the matrix $AB - BA$ is

- (a) a symmetric matrix (b) a skew-symmetric matrix
 (c) a null matrix (d) the identity matrix

27. If $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$ is the sum of a symmetric matrix

B and skew-symmetric matrix C , then B is

- (a) $\begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 2 & -2 \\ -2 & 5 & -2 \\ 2 & 2 & 0 \end{bmatrix}$
 (c) $\begin{bmatrix} 6 & 6 & 7 \\ -6 & 2 & -5 \\ -7 & 5 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 6 & -2 \\ 2 & 0 & -2 \\ -2 & -2 & 0 \end{bmatrix}$

28. If $A = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then $I + A + A^2 + \dots$ is equal to

- (a) $\begin{bmatrix} 1 & -3 \\ 1 & 4 \end{bmatrix}$ (b) $\frac{2}{7} \begin{bmatrix} 1 & -3 \\ -1 & 4 \end{bmatrix}$
 (c) $\frac{2}{7} \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$ (d) Not define

29. If A is a square matrix, then

- (a) $A + A^T$ is symmetric matrix
 (b) AA^T is skew-symmetric
 (c) $A^T + A$ is skew-symmetric
 (d) $A^T A$ is skew-symmetric

30. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B

is the inverse of matrix A , then α is equal to

- (a) -2 (b) 1 (c) 2 (d) 5

31. If A and B are square matrices of size $n \times n$ such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true?

- (a) Either A or B is a zero matrix
 (b) Either A or B is an identity matrix
 (c) $A = B$
 (d) $AB = BA$

32. If $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$, then a non-singular matrix P such

that $P^{-1}AP = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$, is

- (a) $\begin{bmatrix} 3 & 0 \\ 0 & -1/2 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & 5 \\ 1 & 1 \end{bmatrix}$
 (c) $\begin{bmatrix} 2 & 1 \\ 5 & -1 \end{bmatrix}$ (d) None of these

Type 2. More than One Correct Option

33. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where $bc \neq 0$, satisfies the equation $x^2 + k = 0$, then

- (a) $a + d = 0$ (b) $k = -|A|$
(c) $k = |A|$ (d) None of these

34. Let $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$, then

- (a) $A^2 - 4A - 5I_3 = O$ (b) $A^{-1} = \frac{1}{5}(A - 4I_3)$
(c) A^3 is not invertible (d) A^2 is invertible

35. If B is an idempotent matrix and $A = I - B$, then

- (a) $A^2 = A$ (b) $A^2 = I$
(c) $AB = O$ (d) $BA = O$

36. If α, β and γ are three real numbers and

$$A = \begin{bmatrix} 1 & \cos(\alpha - \beta) & \cos(\alpha - \gamma) \\ \cos(\beta - \alpha) & 1 & \cos(\beta - \gamma) \\ \cos(\gamma - \alpha) & \cos(\gamma - \beta) & 1 \end{bmatrix}, \text{ then}$$

which of the following is/are true?

- (a) A is singular (b) A is symmetric
(c) A is orthogonal (d) A is not invertible

37. If D_1 and D_2 are two 3×3 diagonal matrices, then

- (a) $D_1 D_2$ is a diagonal matrix
(b) $D_1 D_2 = D_2 D_1$
(c) $D_1^2 + D_2^2$ is a diagonal matrix
(d) None of the above

38. If $A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$,

then $A_i A_k + A_k A_i$ is equal to

- (a) $2I$, if $i = k$ (b) O , if $i \neq k$
(c) $2I$, if $i \neq k$ (d) O , always

39. If A is a 3×3 Hermitian matrix such that $A^2 = O$, then

- (a) $A = O$ (b) $A \neq I$
(c) $A^3 = O$ (d) $A^2 = O$

40. If A is a 3×3 Hermitian matrix with complex elements while P and Q are two 3×3 matrices with real elements. If $A = P + iQ$, then

- (a) P must be symmetric
(b) Q must be symmetric
(c) P must be skew-symmetric
(d) Q must be skew-symmetric

41. If A and B are symmetric and non-singular matrices and $AB = BA$, then

- (a) $A^{-1}B$ is symmetric (b) AB^{-1} is symmetric
(c) $A^{-1}B^{-1}$ is symmetric (d) AB is symmetric

42. If $A(\alpha) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$$B(\beta) = \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix}, \text{ then}$$

- (a) $[A(\alpha)]^{-1} = A(-\alpha)$ (b) $A(\alpha)A(\beta) = A(\alpha + \beta)$
(c) $B(\alpha)B(\beta) = B(\alpha + \beta)$ (d) $[B(\beta)]^{-1} = B(-\beta)$

43. If A and B are two 3×3 Hermitian matrices such that $A^2 + B^2 = O$, then

- (a) $A = O$ (b) $A \neq I$ (c) $B = O$ (d) $B \neq I$

44. Let X be an unknown matrix of order 2 and I be an identity matrix of order 2, then the equation $X^2 = I$

- (a) is satisfied, if $X = I$
(b) is satisfied, if $X = \begin{bmatrix} 7i & 10 \\ 5 & -7i \end{bmatrix}$
(c) is satisfied, if $X = \begin{bmatrix} \pm \sqrt{1 - \lambda\mu} & \lambda \\ \mu & \mp \sqrt{1 - \lambda\mu} \end{bmatrix}$
(d) has no solution except identity matrix

Type 3. Assertion and Reason

Directions (Q. Nos. 45-47) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

45. **Statement I** Matrix 3×3 , $a_{ij} = \frac{i-j}{i+2j}$ cannot be expressed as a sum of symmetric and skew-symmetric matrices.

Statement II Matrix 3×3 , $a_{ij} = \frac{i-j}{i+2j}$ is neither symmetric nor skew-symmetric.

46. **Statement I** If A is orthogonal matrix of order 2, then $|A| = \pm 1$.

Statement II Every two rowed real orthogonal matrix is of any one of the form $\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$ or $\begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$.

47. Statement I Let A and B be two square matrices of the same order such that $AB = BA$, $A^m = O$ and $B^n = O$ for some positive integers m, n , then there exists a positive integer r such that $(A + B)^r = O$.

Statement II If $AB = BA$, then $(A + B)^r$ can be expanded as binomial expansion.

Matrix

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 48-50) If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$.

48. Which of the following is true?

- (a) If matrix A is singular, then matrix B is non-singular
- (b) If matrix A is non-singular, then matrix B is singular
- (c) If matrix A is singular, then matrix B is also singular
- (d) Cannot say anything

49. $(A + B)^7$ is equal to

- (a) $7(A + B)$
- (b) $7 \times I_{3 \times 3}$
- (c) $64(A + B)$
- (d) $8I$

50. $(A + I)^5$ is equal to (where, I is identity matrix)

- (a) $I + 31A$
- (b) $I + 16A$
- (c) $I + 21A$
- (d) None of the above

Passage II (Q. Nos. 51-53) Let A be an $m \times n$ matrix. If there exists a matrix L of type $n \times m$ such that $LA = I_n$, then L is called left inverse of A . Similarly, if there exists a matrix R of type $n \times m$ such that $AR = I_m$, then R is called right inverse of A . e.g. To find right inverse of matrix

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}, \text{ we take } R = \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} \text{ and solve } AR = I_3,$$

$$\text{i.e. } \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x & y & z \\ u & v & w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} x - u &= 1 & y - v &= 0 & z - w &= 0 \\ x + u &= 0 & y + v &= 1 & z + w &= 0 \\ 2x + 3u &= 0 & 2y + 3v &= 0 & 2z + 3w &= 1 \end{aligned}$$

As this system of equations is inconsistent, we say there is no right inverse for matrix A .

51. Which of the following matrices is not left inverse

of matrix $\begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix}$?

$$\begin{aligned} \text{(a)} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} & \text{(b)} & \begin{bmatrix} 2 & -7 & 3 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \\ \text{(c)} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} & \text{(d)} & \begin{bmatrix} 0 & 3 & -1 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \end{aligned}$$

52. The number of right inverses for the matrix

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \text{ is}$$

- (a) 0
- (b) 1
- (c) 2
- (d) infinite

53. For which of the following matrices, the number of left inverses is greater than the number of right inverses?

$$\begin{aligned} \text{(a)} & \begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix} & \text{(b)} & \begin{bmatrix} 3 & 2 & 1 \\ 3 & 2 & 1 \end{bmatrix} \\ \text{(c)} & \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} & \text{(d)} & \begin{bmatrix} 3 & 3 \\ 1 & 1 \\ 4 & 4 \end{bmatrix} \end{aligned}$$

Passage III (Q. Nos. 54-56) A is a matrix of order 2×2 such that $A^2 = O$, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

54. $A^2 - (a + d)A + (ad - bc)I$ is equal to

- (a) I
- (b) O
- (c) $-I$
- (d) None of the above

55. $\text{tr}(A)$ is equal to

- (a) 1
- (b) 0
- (c) -1
- (d) None of the above

56. $(I + A)^{100}$ is equal to

- (a) $100A$
- (b) $100(I + A)$
- (c) $100I + A$
- (d) $I + 100A$

Target Exercises

Type 5. Match the Columns

57. Match the statements of Column I with values of Column II.

Column I	Column II
A. If A is idempotent, then $(I - A)^n$ is	p. $2^{n-1}(I - A)$
B. If A is involutory, then $(I - A)^n$ is	q. $I - nA$
C. If A is nilpotent of index 2, then $(I - A)^n$ is	r. A
D. If A is orthogonal, then $(A^T)^{-1}$ is	s. $I - A$

58. Match the statements of Column I with values of Column II.

Column I	Column II
A. If A is an idempotent matrix and I is an identity matrix of the same order, then the value of n , such that $(A + I)^n = I + 127I$, is	p. 9
B. If $(I - A)^{-1} = I + A + A^2 + \dots + A^7$, then $A^n = O$, where n is	q. 10
C. If A is matrix such that $a_{ij} = (i + j)(i - j)$, then A is singular, if order of matrix is	r. 7
D. If a non-singular matrix A is symmetric, then A^{-1} is also symmetric, where order of A can be	s. 8

Type 6. Single Integer Answer Type Questions

59. If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix and $f(x) = x - x^2$ and $bc = 1/4$, then the value of $1/f(a)$ is _____.

60. Let X be the solution set of the equation $A^x = I$, where $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$ and I is the corresponding unit matrix and $x \subseteq N$, then the minimum value of $\Sigma(\cos^x \theta + \sin^x \theta)$, $\theta \in R$, is _____.

61. Let $A = \begin{bmatrix} 3x^2 \\ 1 \\ 6x \end{bmatrix}$, $B = [a \ b \ c]$ and $C = \begin{bmatrix} (x+2)^2 & 5x^2 & 2x \\ 5x^2 & 2x & (x+2)^2 \\ 2x & (x+2)^2 & 5x^2 \end{bmatrix}$ be three given

matrices; where a, b, c and $x \in R$. Given that $\text{tr}(AB) = \text{tr}(C)$; $x \in R$, where $\text{tr}(A)$ denotes trace of A . If $f(x) = ax^2 + bx + c$, then the value of $f(1)$ is _____.

62. Let S be the set which contains all possible values of l, m, n, p, q, r for which

$$A = \begin{bmatrix} l^2 - 3 & p & 0 \\ 0 & m^2 - 8 & q \\ r & 0 & n^2 - 15 \end{bmatrix}$$

is a non-singular idempotent matrix. Then, the sum of all the elements of the set S is _____.

63. If $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$ and

$(A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$, where I is the 2×2 identity matrix, then the product of all elements of matrix V is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Let M be a 2×2 symmetric matrix with integer entries. Then, M is invertible, if [2014]

- (a) the first column of M is the transpose of the second row of M
 (b) the second row of M is the transpose of first column of M
 (c) M is a diagonal matrix with non-zero entries in the main diagonal
 (d) the product of entries in the main diagonal of M is not the square of an integer

2. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then, $P^2 \neq 0$, when n is equal to [2013]

- (a) 57 (b) 55 (c) 58 (d) 56

3. Let $P = [a_{ij}]$ be a 3×3 matrix and $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of P is 2, then the determinant of the matrix Q is [2012]

- (a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

4. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix,

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \text{ such that [2012]}$$

- (a) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) $PX = X$ (c) $PX = 2X$ (d) $PX = -X$

5. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P , then $M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$ is equal to [2011]

- (a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN

6. Let M be a 3×3 matrix satisfying $M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$,

$$M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}. \text{ Then, the sum of}$$

the diagonal entries of M is [2011]

Directions (Q. Nos. 7-9) Let p be an odd prime number and T_p be the following set of 2×2 matrices;

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}; a, b, c \in \{0, 1, 2, \dots, p-1\} \right\}. \quad [2010]$$

7. The number of A in T_p such that A is either symmetric or skew-symmetric or both and $\det(A)$ is divisible by p , is

- (a) $(p-1)^2$
 (b) $2(p-1)$
 (c) $(p-1)^2 + 1$
 (d) $2p-1$

8. The number of A in T_p such that the trace of A is not divisible by p but $\det(A)$ is divisible by p , is

- (a) $(p-1)(p^2 - p + 1)$
 (b) $p^3 - (p-1)^2$
 (c) $(p-1)^2$
 (d) $(p-1)(p^2 - 2)$

9. The number of A in T_p such that $\text{tr}(A)$ is not divisible by p , is

- (a) $2p^2$
 (b) $p^3 - 5p$
 (c) $p^3 - 3p$
 (d) $p^3 - p^2$

JEE Main/AIEEE

10. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the

equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to [2014]

- (a) $(2, -1)$ (b) $(-2, 1)$ (c) $(2, 1)$ (d) $(-2, -1)$

11. If A is a 3×3 non-singular matrix such that $AA^T = A^T A$ and $B = A^{-1} A^T$, then BB^T is equal to [2014]

- (a) $I + B$ (b) I (c) B^{-1} (d) $(B^{-1})^T$

12. If p, q and r are 3 real numbers satisfying the matrix equation,

$$\begin{bmatrix} p & q & r \end{bmatrix} \begin{bmatrix} 3 & 4 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 1 \end{bmatrix}, \text{ then } 2p + q - r \text{ is}$$

equal to

[2013]

- (a) -3
 (b) -1
 (c) 4
 (d) 2

13. Let A , other than I or $-I$, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. If $\text{tr}(A)$ is the sum of diagonal elements of A , then [2013]

Statement I $\text{tr}(A) = 0$

Statement II $\det(A) = -1$

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

14. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ is equal to [2013]

- (a) $4 \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (b) $4 \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$ (c) $32 \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (d) $32 \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

15. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$, if u_1 and u_2 are column

matrices such that $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, then

$u_1 + u_2$ is equal to [2012]

- (a) $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ (b) $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$

16. The number of 3×3 non-singular matrices, with four entries as 1 and all other entries as 0, is [2010]

- (a) less than 4 (b) 5
(c) 6 (d) atleast 7

17. Let A be 2×2 matrix with real entries and I be the 2×2 identity matrix. Denote by $\text{tr}(A)$, the sum of diagonal entries of A . Assume that $A^2 = I$

Statement I If $A \neq I$ and $A \neq -I$, then $\det(A) = -1$

Statement II If $A \neq I$ and $A \neq -I$, then $\text{tr}(A) \neq 0$.

[2008]

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

18. If A and B are square matrices of size $n \times n$ such that $A^2 - B^2 = (A - B)(A + B)$, then which of the following will be always true? [2006]

- (a) $AB = BA$
(b) Either $A = O$ or $B = O$
(c) Either $A = I$ or $B = I$
(d) $A = B$

19. Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$; $a, b \in N$. Then, [2006]

- (a) there exists more than one but finite number of B 's such that $AB = BA$
(b) there exists exactly one B such that $AB = BA$
(c) there exist infinitely many B 's such that $AB = BA$
(d) there cannot exist any B such that $AB = BA$

20. Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$.

If B is the inverse of matrix A , then α is [2004]

- (a) -2 (b) 1 (c) 2 (d) 5

21. If $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$ and $A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$, then [2003]

- (a) $\alpha = a^2 + b^2, \beta = ab$
(b) $\alpha = a^2 + b^2, \beta = 2ab$
(c) $\alpha = a^2 + b^2, \beta = a^2 - b^2$
(d) $\alpha = 2ab, \beta = a^2 + b^2$

Other Engineering Entrances

22. If A is 3×4 matrix and B is a matrix such that $A'B$ and BA' are both defined, then B is of the type

[Karnataka CET 2014]

- (a) 4×4 (b) 3×4 (c) 4×3 (d) 3×3

23. The symmetric part of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix}$ is

[Karnataka CET 2014]

- equal to
(a) $\begin{bmatrix} 0 & -2 & -1 \\ -2 & 0 & -2 \\ -1 & -2 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 8 & 0 \\ 3 & 0 & 7 \end{bmatrix}$
(c) $\begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & 2 \\ -1 & 2 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{bmatrix}$

24. If $A = \begin{bmatrix} 2-k & 2 \\ 1 & 3-k \end{bmatrix}$ is a singular matrix, then the value of $5k - k^2$ is [Kerala CEE 2014]

- (a) 0 (b) 6 (c) -6
(d) -4 (e) 4

25. If $n \geq 2$ is an integer,

$A = \begin{bmatrix} \cos(2\pi/n) & \sin(2\pi/n) & 0 \\ -\sin(2\pi/n) & \cos(2\pi/n) & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and I is the

identity matrix of order 3. Then, [WB JEE 2014]

- (a) $A^n = I$ and $A^{n-1} \neq I$
(b) $A^m \neq I$ for any positive integer m
(c) A is not invertible
(d) $A^m = O$ for a positive integer m



26. If $AB = A$ and $BA = B$, then B^2 is equal to [RPET 2014]
 (a) B (b) A
 (c) $-B$ (d) B^3

27. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$, then A^2 is equal to [RPET 2014]
 (a) 0 (b) $-A$
 (c) I (d) $2A$

28. If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then [GGSIPU 2013]
 (a) $A^2 + 7A - 5I = O$
 (b) $A^2 - 7A + 5I = O$
 (c) $A^2 + 5A - 7I = O$
 (d) $A^2 - 5A + 7I = O$

29. Find the values of x , y and z from the following equation [J&K CET 2013]
 $\begin{bmatrix} 4 & x-z \\ 2+y & xz \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 10 \end{bmatrix}$
 (a) $x = -5, y = 3, z = 2$
 (b) $x = 5, y = -3, z = 2$
 (c) $x = 5, y = 3, z = -2$
 (d) $x = 5, y = -3, z = -2$

30. If $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$, then $A^2 + 2A - 3I$ is equal to [J&K CET 2013]
 (a) $\begin{bmatrix} 4 & -6 \\ 6 & 4 \end{bmatrix}$ (b) O
 (c) $\begin{bmatrix} -6 & 2 \\ -2 & 6 \end{bmatrix}$ (d) $5I$

31. Compute $\begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -2 & -1 \end{bmatrix}$. [J&K CET 2012]
 (a) $\begin{bmatrix} 1 & 3 \\ 6 & -4 \end{bmatrix}$ (b) $\begin{bmatrix} -1 & -3 \\ 6 & 4 \end{bmatrix}$
 (c) $\begin{bmatrix} -1 & 3 \\ -6 & 4 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & 3 \\ -6 & -4 \end{bmatrix}$

32. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$ and I is the unit matrix of order 3, then $A^2 + 2A^4 + 4A^6$ is equal to [Kerala CEE 2011]
 (a) $7A^8$ (b) $7A^7$ (c) $8I$
 (d) $6I$ (e) None of these

33. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying $AA^T = 9I_3$, then the values of a and b are respectively [Kerala CEE 2011]
 (a) 1, 2 (b) -1, 2 (c) -1, -2
 (d) 2, 1 (e) -2, -1

34. If $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix, then the value of x is [WB JEE 2011]
 (a) 4 (b) 3 (c) -4 (d) -3

35. Let A and B be 3×3 matrices. Then, $(AB)^T = BA$, if [J&K CET 2011]
 (a) A is skew-symmetric and B is symmetric
 (b) B is skew-symmetric and A is symmetric
 (c) A and B are skew-symmetric
 (d) None of the above

36. If $\begin{bmatrix} x-y-z \\ -y+z \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \\ 3 \end{bmatrix}$, then the values of x , y and z are respectively [Kerala CEE 2010]
 (a) 5, 2, 2 (b) 1, -2, 3 (c) 0, -3, 3
 (d) 11, 8, 3 (e) 4, 1, 3

37. If the matrices $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$, then AB will be [WB JEE 2010]
 (a) $\begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$ (b) $\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ (c) $\begin{bmatrix} 17 & 4 \\ 0 & -2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

Answers

Work Book Exercise 1.1

1. (b) 2. (b) 3. (d) 4. (c) 5. (d) 6. (a) 7. (a) 8. (b)

Work Book Exercise 1.2

1. (b) 2. (a) 3. (a) 4. (d) 5. (a) 6. (a) 7. (c) 8. (c) 9. (c) 10. (c)

Target Exercises

1. (a) 2. (b) 3. (a) 4. (c) 5. (d) 6. (d) 7. (a) 8. (c) 9. (b) 10. (d)
 11. (c) 12. (b) 13. (b) 14. (b) 15. (a) 16. (b) 17. (c) 18. (c) 19. (d) 20. (d)
 21. (a) 22. (b) 23. (d) 24. (b) 25. (c) 26. (a) 27. (a) 28. (c) 29. (a) 30. (d)
 31. (d) 32. (c) 33. (a,c) 34. (a,b,d) 35. (a,c,d) 36. (a,b,d) 37. (a,b,c) 38. (a,b) 39. (a,b,c) 40. (a,d)
 41. (a,b,c) 42. (all) 43. (all) 44. (a,b,c) 45. (d) 46. (a) 47. (b) 48. (c) 49. (c) 50. (a)
 51. (c) 52. (d) 53. (c) 54. (b) 55. (d) 56. (d) 57. (*) 58. (*) 59. (4) 60. (2)
 61. (7) 62. (0) 63. (0)

*

57. $A \rightarrow s; B \rightarrow p; C \rightarrow q; D \rightarrow r$

58. $A \rightarrow r; B \rightarrow s; C \rightarrow p, r; D \rightarrow p, q, r, s$

Entrances Gallery

1. (c,d) 2. (b,c,d) 3. (d) 4. (d) 5. (c) 6. (9) 7. (d) 8. (c) 9. (d) 10. (d)
 11. (b) 12. (a) 13. (a) 14. (a) 15. (d) 16. (d) 17. (c) 18. (a) 19. (c) 20. (d)
 21. (b) 22. (b) 23. (d) 24. (e) 25. (a) 26. (a) 27. (c) 28. (d) 29. (b) 30. (a)
 31. (c) 32. (a) 33. (e) 34. (c) 35. (c) 36. (b) 37. (a)

Explanations

Target Exercises

1. Given, $A = \begin{bmatrix} 1 & -5 & 7 \\ 0 & 7 & 9 \\ 11 & 8 & 9 \end{bmatrix}$

$\therefore \text{tr}(A) = 1 + 7 + 9 = 17$
[$\therefore$ trace of matrix = sum of principal diagonal elements]

2. We have,

$$2A + 3B = \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix} \quad \dots(i)$$

and $A + 2B = \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix} \quad \dots(ii)$

Multiplying Eq. (ii) by 2 and subtracting Eq. (i) from Eq. (ii), we get

$$B = 2 \begin{bmatrix} 5 & 0 & 3 \\ 1 & 6 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 4 \\ 3 & 2 & 5 \end{bmatrix} = \begin{bmatrix} 8 & 1 & 2 \\ -1 & 10 & -1 \end{bmatrix}$$

3. Given that, $2X + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix}$

$$\Rightarrow 2X = \begin{bmatrix} 3 & 8 \\ 7 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 3-1 & 8-2 \\ 7-3 & 2-4 \end{bmatrix}$$

$$\Rightarrow 2X = \begin{bmatrix} 2 & 6 \\ 4 & -2 \end{bmatrix} = 2 \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\Rightarrow X = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

4. Given, $A = \begin{bmatrix} 1 & -3 \\ 2 & k \end{bmatrix}$

and $A^2 - 4A + 10I = A$

Now, $A^2 - 4A + 10I = \begin{bmatrix} 1 & 9-3k \\ 2k-6 & 4+k^2-4k \end{bmatrix}$

But $\begin{bmatrix} 1 & 9-3k \\ 2k-6 & 4+k^2-4k \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 2 & k \end{bmatrix}$

$\Rightarrow 9-3k = -3 \Rightarrow k = 4$

5. Given, $A = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$

$\therefore A = 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

and $A^2 = 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \cdot 3 \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$= 9 \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 9A$$

$\therefore A^4 = A^2 \cdot A^2 = 9A \cdot 9A = 81A^2 = 81 \cdot 9A = 729A$

6. $A = \text{diag}(d_1, d_2, \dots, d_n)$

Given, $A^3 = A$

$\Rightarrow \text{diag}(d_1^3, d_2^3, \dots, d_n^3) = \text{diag}(d_1, d_2, \dots, d_n)$

$\Rightarrow d_1^3 = d_1, d_2^3 = d_2, \dots, d_n^3 = d_n$

Hence, all $d_1, d_2, d_3, \dots, d_n$ have three possible values ± 1 and 0. Each diagonal element can be selected in three ways. Hence, the number of different matrices is 3^n .

7. We have, $A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$
 $= \begin{bmatrix} 4+1 & -2-2 \\ -2-2 & 1+4 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$
 Now, $4A - 3I = 4 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$
 $= \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} = A^2$

Aliter

We know that, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^2 - (a+b)A + |A|I = 0$$

$$\Rightarrow A^2 - 4A + 3I = 0$$

$$\Rightarrow A^2 = 4A - 3I$$

8. Since, matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is commutative with the matrix

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

$$\Rightarrow a = a+c, a+b = b+d$$

$$\Rightarrow c = 0, a = d$$

9. Given that, $A = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$ and $f(t) = t^2 - 3t + 7$

$$\therefore A^2 = \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix}$$

$$f(A) = A^2 - 3A + 7I$$

$$= \begin{bmatrix} -7 & -12 \\ 24 & 17 \end{bmatrix} - 3 \begin{bmatrix} 1 & -2 \\ 4 & 5 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix}$$

$$\therefore f(A) + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix} = \begin{bmatrix} -3 & -6 \\ 12 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 6 \\ -12 & -9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

10. Given that, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Now, $A^2 - 4A = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = 5I_3$$

11. Given, $[1 \ x \ 1] \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ 1 \\ -2 \end{bmatrix} = 0$

$$\Rightarrow [1 \ x \ 1] \begin{bmatrix} x+2-6 \\ 0+5-2 \\ 0+3-4 \end{bmatrix} = 0$$

$$\Rightarrow [1 \ x \ 1] \begin{bmatrix} x-4 \\ 3 \\ -1 \end{bmatrix} = 0$$

$$\Rightarrow [x-4+3x-1] = 0$$

$$\Rightarrow 4x-5=0$$

$$\Rightarrow x = \frac{5}{4}$$

12. We have, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$= 2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 2A$$

$\therefore A^4 = 2A \cdot 2A = 4A^2 = 4 \times 2A = 8A = 2^3 A$ [$\because A^2 = 2A$]

Similarly, $A^8 = 2^7 A$

$\therefore A^{100} = 2^{99} A$

13. If A is a matrix of order 3×1 and B is a matrix of order 1×3 , then the matrix AB is of the order 3×3 .

Now, $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} [2 \ 1 \ -1] = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -1 & 1 \\ 4 & 2 & -2 \end{bmatrix}$

14. Given, $AB = 3I$

$$\Rightarrow A^{-1}(AB) = A^{-1}(3I) \quad [\text{pre-multiplying by } A^{-1}]$$

$$\Rightarrow B = 3A^{-1}$$

$$\Rightarrow A^{-1} = \frac{B}{3}$$

15. Given, $(A+B)(A-B) = A^2 - B^2$

$$\Rightarrow A^2 - AB + BA - B^2 = A^2 - B^2$$

$$\Rightarrow AB = BA$$

Now, $(ABA^{-1})^2 = (BAA^{-1})^2 = B^2$

16. We have, $AB = A$ and $BA = B$

Now, $AB = A \quad \dots(i)$

$$\Rightarrow (AB)A = A^2 \quad [\text{post-multiplying Eq. (i) by } A]$$

$$\Rightarrow A(BA) = A^2 \quad [\text{by associative law}]$$

$$\Rightarrow AB = A^2 \quad [\because BA = B, \text{ given}] \dots(ii)$$

From Eqs. (i) and (ii), we get $A^2 = A$

Similarly, $B^2 = B$

So, A and B are idempotent matrices.

17. For $PP' = I$,

$$\begin{bmatrix} \frac{2}{3} & 3k & a \\ -1 & -4k & b \\ \frac{2}{3} & -5k & c \end{bmatrix} \begin{bmatrix} \frac{2}{3} & -1 & \frac{2}{3} \\ 3k & -4k & -5k \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Performing matrix multiplication, we have

$$\frac{4}{9} + 9k^2 + a^2 = 1,$$

$$\frac{1}{9} + 16k^2 + b^2 = 1$$

and $\frac{4}{9} + 25k^2 + c^2 = 1$

$$\Rightarrow a^2 = \frac{169}{450}, b^2 = \frac{256}{450}, c^2 = \frac{25}{450}$$

Also, $\frac{4}{9} - 15k^2 + ac = 0, -\frac{2}{9} + 20k^2 + bc = 0$

and $-\frac{2}{9} - 12k^2 + ab = 0$

$$\Rightarrow ab = \frac{208}{450}, bc = \frac{80}{450}, ac = \frac{-65}{450}$$

Hence, $a = \pm \frac{13}{5\sqrt{2}}, b = \pm \frac{16}{15\sqrt{2}}, c = \pm \frac{1}{3\sqrt{2}}$

18. Here, $(I+A)^4 = (I+IA+A^2+A^3)^2$

$$= (I+2A+A^2)^2$$

$$= (I+2A+A^2)(I+2A+A^2)$$

$$= I + 3IA + 3AI + 9A^2$$

$$= I + 6A + 9A$$

$$= I + 15A$$

19. Given, $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \quad \dots(i)$

and $A^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \dots(ii)$

Let A be given by $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, Eq. (i) gives

$$a - b = -1 \quad \dots(iii)$$

$$c - d = 2 \quad \dots(iv)$$

Now, Eq. (ii) gives

$$A^2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = A \left(A \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = A \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

which gives

$$-a + 2b = 1 \quad \dots(v)$$

$$-c + 2d = 0 \quad \dots(vi)$$

On adding Eqs. (iii) and (v), we get $b = 0$ and $a = -1$.

On adding Eqs. (iv) and (vi), we get $d = 2$ and $c = 4$.

$\therefore$ Sum $= a + b + c + d = 5$



20. If A is n th root of I_2 , then $A^n = I_2$.

$$\text{Now, } A^2 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} a^2 & 2ab \\ 0 & a^2 \end{bmatrix} \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^3 & 3a^2b \\ 0 & a^3 \end{bmatrix}$$

$$\text{Thus, } A^n = \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix}$$

$$\text{Now, } A^n = I \Rightarrow \begin{bmatrix} a^n & na^{n-1}b \\ 0 & a^n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^n = 1, b = 0$$

21. Given, $A = \begin{bmatrix} 4 & 2i \\ i & 1 \end{bmatrix}$

$$\therefore (A - 2I)(A - 3I) = \begin{bmatrix} 2 & 2i \\ i & -1 \end{bmatrix} \begin{bmatrix} 1 & 2i \\ i & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

22. Given, $E(\theta) = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix}$

$$\therefore E(\theta) \times E(\phi) = \begin{bmatrix} \cos^2 \theta & \cos \theta \sin \theta \\ \cos \theta \sin \theta & \sin^2 \theta \end{bmatrix} \times \begin{bmatrix} \cos^2 \phi & \cos \phi \sin \phi \\ \cos \phi \sin \phi & \sin^2 \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta \cos^2 \phi + \cos \theta \sin \theta \cos \phi \sin \phi & \cos^2 \theta \cos \phi \sin \phi + \cos \theta \sin \theta \sin^2 \phi \\ \cos \theta \sin \theta \cos^2 \phi + \sin^2 \theta \cos \phi \sin \phi & \cos \theta \sin \theta \cos \phi \sin \phi + \sin^2 \theta \sin^2 \phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi \cos(\theta - \phi) & \cos \theta \sin \phi \cos(\theta - \phi) \\ \cos \phi \sin \theta \cos(\theta - \phi) & \sin \theta \sin \phi \cos(\theta - \phi) \end{bmatrix}$$

$$= \begin{bmatrix} \cos \theta \cos \phi \cos(2n+1)\frac{\pi}{2} & \cos \theta \sin \phi \cos(2n+1)\frac{\pi}{2} \\ \cos \phi \sin \theta \cos(2n+1)\frac{\pi}{2} & \sin \theta \sin \phi \cos(2n+1)\frac{\pi}{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \left[\because \cos(2n+1)\frac{\pi}{2} = 0 \right]$$

23. $\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^4 = I$

$$\text{Now, } \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} = \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}$$

$$\text{and } \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^4 = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^2 \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^2$$

$$= \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix} \begin{bmatrix} ab & 0 \\ 0 & ab \end{bmatrix}$$

$$= \begin{bmatrix} a^2b^2 & 0 \\ 0 & a^2b^2 \end{bmatrix}$$

$$\text{But } \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad [\text{given}]$$

$$\Rightarrow \begin{bmatrix} a^2b^2 & 0 \\ 0 & a^2b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a^2b^2 = 1$$

$$\Rightarrow ab = 1$$

24. Let $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$

$$\text{Then, } A' = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$\therefore AA' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & -\cos \alpha \sin \alpha + \sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha + \cos \alpha \sin \alpha & \sin^2 \alpha + \cos^2 \alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Aliter

For orthogonal matrix A , $|A| = \pm 1$.

$$\therefore \begin{vmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{vmatrix} = 1$$

25. Since, diagonal element of a skew-symmetric matrix are all zero.

$$\therefore \text{tr}(A) = \sum_{i=1}^n a_{ii} = 0$$

26. Given, $A = A', B = B'$

$$\text{Now, } (AB - BA)' = (AB)' - (BA)'$$

$$= B' A' - A' B'$$

$$= BA - AB$$

27. We have, $A = \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix}$

$$\therefore \text{Symmetric matrix, } B = \frac{A + A'}{2}$$

$$\Rightarrow B = \frac{1}{2} \left\{ \begin{bmatrix} 6 & 8 & 5 \\ 4 & 2 & 3 \\ 9 & 7 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 4 & 9 \\ 8 & 2 & 7 \\ 5 & 3 & 1 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 6 & 6 & 7 \\ 6 & 2 & 5 \\ 7 & 5 & 1 \end{bmatrix}$$

28. $I + A + A^2 + \dots = (I - A)^{-1}$

$$= \begin{bmatrix} 2 & -\frac{3}{2} \\ \frac{1}{2} & 1 \end{bmatrix}^{-1} = \frac{4}{7} \begin{bmatrix} 1 & 3 \\ \frac{1}{2} & 2 \end{bmatrix}$$

$$= \frac{2}{7} \begin{bmatrix} 1 & 3 \\ -1 & 4 \end{bmatrix}$$

29. Now, $(A + A^T)' = A^T + (A^T)'$

$$= A^T + A$$

$\therefore (A + A^T)$ is symmetric matrix.

30. Since, B is the inverse of matrix A .

$$\therefore 10A^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$$

$$\begin{aligned} \text{Now, } 10A^{-1} \cdot A &= \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \\ \Rightarrow 10I &= \begin{bmatrix} 4+4+2 & -4+2+2 & 4-6+2 \\ -5+0+\alpha & 5+0+\alpha & -5+0+\alpha \\ 1-4+3 & -1-2+3 & 1+6+3 \end{bmatrix} \\ \Rightarrow \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} &= \begin{bmatrix} 10 & 0 & 0 \\ -5+\alpha & 5+\alpha & -5+\alpha \\ 0 & 0 & 10 \end{bmatrix} \\ \Rightarrow -5+\alpha &= 0 \\ \Rightarrow \alpha &= 5 \end{aligned}$$

$$\begin{aligned} 31. (A-B)(A+B) &= A^2 + AB - BA - B^2 \\ &= A^2 - B^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow AB - BA &= 0 \\ \Rightarrow AB &= BA \end{aligned}$$

$$32. \text{ Let } P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \text{ where } ad - bc \neq 0$$

$$\text{Now, } P^{-1}AP = \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix}$$

[pre-multiplying by P and post-multiplying by P^{-1}]

$$\Rightarrow A = P \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} P^{-1}$$

$$\Rightarrow (ad - bc) \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 6 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\left[\text{using } P^{-1} = \frac{\text{adj}(P)}{|P|} \right]$$

$$\begin{aligned} \therefore \begin{bmatrix} ad - bc & 2(ad - bc) \\ 5(ad - bc) & 4(ad - bc) \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 6d & -6b \\ c & a \end{bmatrix} \\ &= \begin{bmatrix} 6ad + bc & -5ab \\ 7dc & -6bc + ad \end{bmatrix} \end{aligned}$$

The two sides are equal for

$$a=2, b=1, c=5, d=-1$$

[on proceeding choicewise]

$\therefore$ Choice (c) is correct.

Similarly, we can check for choices (a) and (b), which do not follow.

$$\begin{aligned} 33. \text{ We have, } A^2 &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} a^2 + bc & ab + db \\ ac + cd & bc + d^2 \end{bmatrix} \end{aligned}$$

As A satisfies $x^2 + k = 0$, $A^2 + kI = 0$.

$$\therefore \begin{bmatrix} a^2 + bc + k & (a+d)b \\ (a+d)c & bc + d^2 + k \end{bmatrix} = 0$$

$$\begin{aligned} \Rightarrow a^2 + bc + k &= 0 \\ bc + d^2 + k &= 0 \end{aligned}$$

$$\text{and } (a+d)b = (a+d)c = 0$$

As $bc \neq 0, b \neq 0, c \neq 0$

$$\Rightarrow a + d = 0$$

$$\Rightarrow a = -d$$

$$\text{Also, } k = -(a^2 + bc)$$

$$= -(d^2 + bc)$$

$$= -[(-ad) + bc] = |A|$$

$$34. \text{ We have, } A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$\text{Now, } A^2 - 4A - 5I_3$$

$$= \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

$$\Rightarrow 5I_3 = A^2 - 4A = A(A - 4I_3)$$

$$\Rightarrow A^{-1} = \frac{1}{5}(A - 4I_3)$$

Note that $|A| = 5$. Since, $|A^3| = |A|^3 = 5^3 \neq 0$, so A^3 is invertible. Similarly, A^2 is also invertible.

35. Since, B is an idempotent matrix.

$$\therefore B^2 = B$$

$$\text{Now, } A^2 = (I - B)^2$$

$$= (I - B)(I - B)$$

$$= I - IB - IB + B^2$$

$$= I - B - B + B^2$$

$$= I - 2B + B^2$$

$$= I - 2B + B$$

$$= I - B = A$$

$$[\because B^2 = B]$$

Therefore, A is idempotent.

$$\text{Again, } AB = (I - B)B = IB - B^2 = B - B^2$$

$$= B^2 - B^2 = 0$$

$$\text{Similarly, } BA = B(I - B) = BI - B^2$$

$$= B - B^2 = 0$$

$$36. A = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ \cos \beta & \sin \beta & 0 \\ \cos \gamma & \sin \gamma & 0 \end{bmatrix} \begin{bmatrix} \cos \alpha & \cos \beta & \cos \gamma \\ \sin \alpha & \sin \beta & \sin \gamma \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, A is symmetric and $|A| = 0$, hence singular and not invertible.

$$\text{Also, } AA^T \neq I$$

$$37. \text{ Let } D_1 = \begin{bmatrix} x_1 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & z_1 \end{bmatrix} \text{ and } D_2 = \begin{bmatrix} x_2 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & 0 & z_2 \end{bmatrix}, \text{ where}$$

$$x_1, y_1, z_1, x_2, y_2, z_2 \neq 0$$

$$\text{Then, } D_1 D_2 = D_2 D_1$$

Clearly, $D_1 D_2$ is a diagonal matrix and $D_1^2 + D_2^2$ is also a diagonal matrix.

38. Let $I = k = 1$

[say]

$$\text{Then, } A_i A_k = A_k A_i = A_1 A_1$$

$$\therefore A_i A_k = A_1 A_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$A_2 A_2 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore A_i A_k + A_k A_i = I + I = 2I$$

If $i \neq k$, let $i = 3$ and $k = 2$, then

$$A_i A_k = A_1 A_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{bmatrix}$$

$$A_2 A_1 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{bmatrix}$$

$$\Rightarrow A_1 A_2 + A_2 A_1 = O$$

39. Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

where, a_{ij} are complex numbers.

Then, $A^\theta = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \\ \bar{a}_{13} & \bar{a}_{23} & \bar{a}_{33} \end{bmatrix}$

Now, $A^2 = AA = AA^\theta$

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \\ \bar{a}_{13} & \bar{a}_{23} & \bar{a}_{33} \end{bmatrix}$$

Since, $A^2 = O$, so we must have following 3 relations (diagonal elements of product matrix).

$$|a_{11}|^2 + |a_{12}|^2 + |a_{13}|^2 = 0$$

$$|a_{21}|^2 + |a_{22}|^2 + |a_{23}|^2 = 0$$

$$|a_{31}|^2 + |a_{32}|^2 + |a_{33}|^2 = 0$$

[though we can move relations among a_{ij} , we can conclude that all elements of A are zero from these three relations only]

$$\Rightarrow A = O$$

$\therefore$ Choice (a) is correct.

Choice (b) is also correct, since $A \neq I$.

$$\therefore A^3 = A^2 \cdot A = O \cdot A = O$$

Choice (c) is also correct.

40. Since, P, Q are matrices with real elements, so we have

$$\bar{A} = P - iQ$$

Also, $A = P + iQ$

$$\Rightarrow P = \frac{A + \bar{A}}{2}$$

$$\Rightarrow P' = \frac{(A + \bar{A})'}{2} = \frac{A' + (\bar{A})'}{2}$$

$$= \frac{A' + A}{2} = \frac{\bar{A} + A}{2}$$

[since, A is Hermitian, so P is symmetric]

Similarly, we can show that Q is skew-symmetric. Thus, choices (a) and (d) are true, while choices (b) and (c) are false.

41. $\therefore$ (a) $(A^{-1}B)' = (B')(A^{-1})'$ [reversal law of transpose]

$$= B'(A')^{-1} = BA^{-1} \quad [\because B' = B, A' = A]$$

$$= A^{-1}ABA^{-1}$$

$$= A^{-1}BAA^{-1} = A^{-1}BI \quad [\because AB = BA]$$

$$= A^{-1}B$$

$\Rightarrow A^{-1}B$ is symmetric.

(b) Similarly, choice (b) can be proved.

(c) $(A^{-1}B^{-1})' = (B^{-1})'(A^{-1})'$ [reversal law of transpose]

$$= (B')^{-1}(A')^{-1}$$

$$= B^{-1}A^{-1} \quad [\because B' = B, A' = A]$$

$$= (AB)^{-1} = (BA)^{-1} \quad [\because AB = BA]$$

$$= A^{-1}B^{-1}$$

$\Rightarrow A^{-1}B^{-1}$ is symmetric.

42. Choices (b) and (c) are easily verified by simple matrix multiplication.

If they are true, then put $\beta = -\alpha$, we get

$$A(\alpha)A(-\alpha) = A(O) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$$\Rightarrow [A(\alpha)]^{-1} = A(-\alpha)$$

Similarly, we get $[B(\beta)]^{-1} = B(-\beta)$

43. As, $A^2 + B^2 = O$

$$\Rightarrow a_{ij} = O \text{ for all } i \text{ and } j, b_{ij} = O \text{ for all } i \text{ and } j.$$

$$\Rightarrow A = O, B = O$$

Hence, all choices become correct.

44. We will first test the choice (c).

Indeed the square of the matrix in choice (c)

$$= \begin{bmatrix} \sqrt{1-\lambda\mu} & \lambda \\ \mu & -\sqrt{1-\lambda\mu} \end{bmatrix} \begin{bmatrix} \sqrt{1-\lambda\mu} & \lambda \\ \mu & -\sqrt{1-\lambda\mu} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\Rightarrow$ Choice (c) is correct.

Choice (b) is also true, since it corresponds to

$\lambda = 10, \mu = 5$ or else, we can test it independently.

Choice (a) is obviously true.

45. Matrix $a_{ij} = \frac{i-j}{i+2j}$ is $A = \begin{bmatrix} 0 & -\frac{1}{5} & -\frac{2}{7} \\ \frac{1}{4} & 0 & -\frac{1}{8} \\ \frac{2}{5} & \frac{1}{7} & 0 \end{bmatrix}$ which is neither

symmetric nor skew-symmetric. But this is not the reason for which A cannot be matrices as sum of symmetric and skew-symmetric matrices. Infact, any matrix can be expressed as a sum of symmetric and skew-symmetric matrices. Hence, Statement I is false but Statement II is true.

46. $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} a^2 + b^2 &= 1 & \dots(i) \\ c^2 + d^2 &= 1 & \dots(ii) \\ \text{and } ac + bd &= 0 & \dots(iii) \end{aligned}$$

Now, $\frac{a}{d} = \frac{-b}{c} = k$ [let]

$$\Rightarrow c^2 + d^2 = 1/k^2 \text{ or } k^2 = 1$$

$$\Rightarrow k = \pm 1$$

$$\Rightarrow \frac{a}{d} = \frac{-b}{c} = \pm 1$$

Also, we must have $a, b, c, d \in [-1, 1]$ for Eqs. (i) and (ii) to get defined. Hence, without loss of generality, we can assume $a = \cos \theta$ and $b = \sin \theta$.

So, for $\frac{a}{d} = \frac{-b}{c} = 1$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \text{ and}$$

for $\frac{a}{d} = \frac{-b}{c} = -1$, we have

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{bmatrix}$$

47. Since, $AB = BA$ so we have

$$(A+B)^r = {}^rC_0 A^r + {}^rC_1 A^{r-1} B + {}^rC_2 A^{r-2} B^2 + \dots + {}^rC_s A^{r-s} B^s + \dots + {}^rC_{r-1} A B^{r-1} + {}^rC_r B^r$$

If $r = m + n$, then

$$A^{r-p} B^p = A^m B^{r-p-m} B^p = O, \text{ if } p \leq n$$

$$\text{and } A^{r-p} B^p = A^{r-p} B^n B^{p-n} = O, \text{ if } p > n$$

Then, $(A+B)^r = O$, for $r = m + n$

Thus, both statements are correct but Statement II is not correctly explaining Statement I.

48. $\therefore AB = A$

$$\therefore |AB| = |A| \quad \dots(i)$$

$$\Rightarrow |A||B| = |A|$$

$$\text{and } BA = B \Rightarrow |BA| = |B| \quad \dots(ii)$$

$$\Rightarrow |B||A| = |B|$$

If $|A| = 0$, then from Eq. (ii), $|B| = 0$

If $|B| = 0$, then from Eq. (i), $|A| = 0$

49. $\therefore AB = A, BA = B$

$$\therefore ABA = A^2$$

$$\Rightarrow A(BA) = A^2$$

$$\Rightarrow AB = A^2$$

$$\Rightarrow A = A^2$$

$$\text{Similarly, } B^2 = B$$

$$(A+B)^2 = A^2 + B^2 + AB + BA$$

$$\text{Now, } = A + B + A + B = 2(A+B)$$

$$(A+B)^3 = (A+B)^2(A+B)$$

$$= 2(A+B)^2 = 2^2(A+B)$$

$$\Rightarrow (A+B)^7 = 2^6(A+B) = 64(A+B)$$

50. $(A+I)^5 = I + 5A + 10A^2 + 10A^3 + 5A^4 + A^5$

$$= I + 5A + 10A + 10A + 5A + A$$

$$[\because A^2 = A \Rightarrow A^3 = A^4 = A^5 = \dots = A]$$

$$= I + 31A$$

51. As second row of all the options is same, we are to look at the elements of the first row. Let the left inverse

$$\text{be } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$$

$$\text{Then, } \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore a + b + 2c = 1$$

$$\Rightarrow -a + b + 3c = 1, \text{ i.e. } b = \frac{1-5c}{2}, a = \frac{1+c}{2}$$

Thus, matrices in the options (a), (b) and (d) are the inverses and matrix in option (c) is not the left inverse.

52. Let right inverse be $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$. Then,

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\text{Now, } \begin{aligned} a - c + 2e &= 1 \\ b - d + 2f &= 0 \\ 2a - c + e &= 0 \\ 2b - d + f &= 1 \end{aligned}$$

This system of equations has infinite solutions.

53. By observation, there cannot be any left inverse for options (b) and (d). So, we will check for options (a) and (c) only.

$$\text{For option (a), let the left inverse be } \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$\text{Then, } \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ -3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $a - 3b = 1, 2a + 2b = 0$ and $4a + b = 0$ which is not possible.

For option (c),

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow a + 2b + 5c = 1, 4a - 3b + 4c = 0,$$

$$d + 2e + 5f = 0, 4d - 3e + 4f = 1$$

Therefore, there are infinite number of left inverses.

$$\therefore \begin{bmatrix} 1 & 4 \\ 2 & -3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow a + 4d = 1, 2a - 3d = 0$$

$$\text{and } 5a + 4d = 0$$

which is not possible.

Therefore, there is no right inverse.

54. We have, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\therefore A^2 - (a+d)A + (ad-bc)I$$

$$= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} - (a+d) \begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix} + \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

$$= 0$$

55. If $A = O$, then $\text{tr}(A) = 0$.

Suppose $A \neq O$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Then, $|A| = 0$ and $A^2 - (a+d)A + (ad-bc)I = 0$
 $\Rightarrow a+d = 0$

56. $(I + A)^{100} = {}^{100}C_0 I^{100} + {}^{100}C_1 I^{99} A + {}^{100}C_2 I^{98} A^2 + \dots + {}^{100}C_{100} A^{100}$
 $= I + 100A + O + O + \dots + O$
 $= I + 100A$

57. A. Since, A is idempotent.

$\therefore A^2 = A$
 $\Rightarrow A^3 = AA^2 = AA = A^2 = A$
 $A^4 = AA^3 = AA = A^2 = A$
 $\Rightarrow A^n = A$
 $\therefore (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots$
 $= I + (-{}^nC_1 + {}^nC_2 - {}^nC_3 + \dots) A$
 $= I + [({}^nC_0 - {}^nC_1 + {}^nC_2 - {}^nC_3 + \dots) - {}^nC_0] A$
 $= I - A$

B. Since, A is involutory.

$\therefore A^2 = I$
 $\Rightarrow A^3 = A^5 = \dots = A$
 and $A^2 = A^4 = A^6 = \dots = I$
 $\therefore (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots$
 $= {}^nC_0 I - {}^nC_1 A + {}^nC_2 I - {}^nC_3 A + {}^nC_4 I - \dots$
 $= ({}^nC_0 + {}^nC_2 + {}^nC_4 + \dots) - ({}^nC_1 + {}^nC_3 + {}^nC_5 + \dots) A$
 $= 2^{n-1} (I - A)$

C. If A is nilpotent of index 2, then

$A^2 = A^3 = A^4 = \dots = A^n = O$
 $\therefore (I - A)^n = {}^nC_0 I - {}^nC_1 A + {}^nC_2 A^2 - {}^nC_3 A^3 + \dots$
 $= I - nA + O + O + \dots = I - nA$

D. Since, A is orthogonal.

Hence, $AA^T = I = AA^{-1} \Rightarrow (A^T)^{-1} = A$

58. A. Since, A is idempotent.

$\therefore A^2 = A^3 = A^4 = \dots = A$
 Now, $(A + I)^n = I + {}^nC_1 A + {}^nC_2 A^2 + \dots + {}^nC_n A^n$
 $= I + {}^nC_1 A + {}^nC_2 A + \dots + {}^nC_n A$
 $= I + ({}^nC_1 + {}^nC_2 + \dots + {}^nC_n) A$
 $= I + (2^n - 1) A$
 $\therefore 2^n - 1 = 127 \Rightarrow n = 7$

B. We have, $I = (I - A)(I + A + A^2 + \dots + A^7)$

$= I + A + A^2 + \dots + A^7 + (-A - A^2 - A^3 - A^4 - \dots - A^8)$
 $= I - A^8 = I \Rightarrow n = 8$

C. Here, matrix A is skew-symmetric as $a_{ij} = -a_{ji}$

and $|A| = |A^T| = (-1)^n |A|$.

So, $|A|(1 - (-1)^n) = 0$, when n is odd.

$\Rightarrow |A| = 0$

So, A is singular.

D. If A is symmetric, then A^{-1} is also symmetric matrix of any order.

59. Since, $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix.

$\therefore \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$
 $\Rightarrow \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$
 $\Rightarrow \begin{bmatrix} a^2 + bc & ab + b - ab \\ ac + c - ac & bc + (1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$
 $\Rightarrow \begin{bmatrix} a^2 + bc & b \\ c & bc + (1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$
 $\Rightarrow a^2 + bc = a \Rightarrow a - a^2 = bc = \frac{1}{4}$ [given]
 $\therefore f(a) = \frac{1}{4} \Rightarrow \frac{1}{f(a)} = 4$

60. Given, $A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$

Now, $A^2 = A \cdot A = \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 4 & -3 & 4 \\ 3 & -3 & 4 \end{bmatrix}$
 $= \begin{bmatrix} 0+4-3 & 0-3+3 & 0+4-4 \\ 0-12+12 & 4+9-12 & -4-12+16 \\ 0-12+12 & 3+9-12 & -3-12+16 \end{bmatrix}$
 $= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow A^2 = I \Rightarrow A^4 = A^6 = A^8 = \dots = I$

Now, $A^x = I \Rightarrow x = 2, 4, 6, 8, \dots$

$\therefore \Sigma(\cos^x \theta + \sin^x \theta)$
 $= (\cos^2 \theta + \sin^2 \theta) + (\cos^4 \theta + \sin^4 \theta) + (\cos^6 \theta + \sin^6 \theta) + \dots$
 $= (\cos^2 \theta + \cos^4 \theta + \cos^6 \theta + \dots) + (\sin^2 \theta + \sin^4 \theta + \sin^6 \theta + \dots)$
 $= \frac{\cos^2 \theta}{1 - \cos^2 \theta} + \frac{\sin^2 \theta}{1 - \sin^2 \theta} = \cot^2 \theta + \tan^2 \theta$

which has minimum value 2.

61. We have, $AB = \begin{bmatrix} 3ax^2 & 3bx^2 & 3cx^2 \\ a & b & c \\ 6ax & 6bx & 6cx \end{bmatrix}$

Now, $\text{tr}(AB) = \text{tr}(C)$

$\Rightarrow 3ax^2 + b + 6cx = (x+2)^2 + 2x + 5x^2, \forall x \in R$

$\Rightarrow 3ax^2 + 6cx + b = 6x^2 + 6x + 4$

$\Rightarrow a = 2, c = 1, b = 4$

$\therefore f(1) = 2 \times (1)^2 + 4 \times 1 + 1 = 7$

62. For idempotent matrix, $A^2 = A$

$\Rightarrow A^{-1}A^2 = A^{-1}A$ [$\because A$ is non-singular]

$\Rightarrow A = I$

Thus, non-singular idempotent matrix is always a unit matrix.

$\therefore I^2 - 3 = 1 \Rightarrow I = \pm 2$

$m^2 - 8 = 1 \Rightarrow m = \pm 3$

$n^2 - 15 = 1 \Rightarrow n = \pm 4$

and $p = q = r = 0$

$\therefore$ Required sum is 0.

63. Given, $A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$

$$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\Rightarrow A^4 = A^2 \cdot A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix}$$

$$\Rightarrow A^8 = \begin{bmatrix} 3^4 & 0 \\ 0 & 3^4 \end{bmatrix}$$

and $A^6 = A^4 \cdot A^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & 3^2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 3^3 & 0 \\ 0 & 3^3 \end{bmatrix}$

Let $V = \begin{bmatrix} x \\ y \end{bmatrix}$

Now, $A^8 + A^6 + A^4 + A^2 + I$

$$= \begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix} + \begin{bmatrix} 27 & 0 \\ 0 & 27 \end{bmatrix} + \begin{bmatrix} 9 & 0 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix}$$

$$\Rightarrow (A^8 + A^6 + A^4 + A^2 + I)V = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 121 & 0 \\ 0 & 121 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix} \Rightarrow \begin{bmatrix} 121x \\ 121y \end{bmatrix} = \begin{bmatrix} 0 \\ 11 \end{bmatrix}$$

$$\therefore x = 0 \text{ and } y = 1/11$$

$$\therefore V = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1/11 \end{bmatrix}$$

Hence, product of all the elements of V is 0.

Entrances Gallery

1. Let $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

(a) $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} b & c \end{bmatrix}$ are transpose.

So, $\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} b \\ c \end{bmatrix}$ is given.

$$\therefore a = b = c$$

$$\Rightarrow M = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$$

$$\Rightarrow |M| = 0$$

$\therefore$ Choice (a) is incorrect.

(b) $\begin{bmatrix} b & c \end{bmatrix}$ and $\begin{bmatrix} a \\ b \end{bmatrix}$ are transpose.

So, $a = b = c$

$\therefore$ Choice (b) is incorrect.

(c) $M = \begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix} \Rightarrow |M| = ac \neq 0$

$\therefore$ Choice (c) is correct.

(d) $M = \begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow ac \neq \lambda^2$

$\therefore$ Choice (d) is correct.

2. $P = \begin{bmatrix} \omega^2 & \omega^3 & \omega^4 & \dots & \omega^{n+2} \\ \omega^3 & \omega^4 & \omega^5 & \dots & \omega^{n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega^{n+2} & \omega^{n+3} & \omega^{n+4} & \dots & \omega^{2n+4} \end{bmatrix}$

$$P^2 = \begin{bmatrix} \omega^4 + \omega^5 + \dots & \omega^5 + \omega^7 + \omega^9 + \dots & \dots \\ \omega^5 + \omega^7 + \omega^9 + \dots & \dots & \dots \\ \vdots & \vdots & \vdots \\ \omega^{n+4} + \omega^{n+6} + \dots & \dots & \dots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ \omega^{2n+4} + \omega^{2n+6} + \dots & \dots & \dots \end{bmatrix}$$

$\therefore P^2 = \text{null matrix, if } n \text{ is a multiple of } 3.$

$\therefore P^2 \neq 0$, when n is not a multiple of 3.

3. Here, $P = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$Q = [b_{ij}]_{3 \times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

where, $b_{ij} = 2^{i+j} a_{ij}$

$$\therefore |Q| = \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix}$$

$$= 4 \times 8 \times 16 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 4a_{31} & 4a_{32} & 4a_{33} \end{vmatrix}$$

$$= 2^9 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} |P| = 2^{12} \cdot 2 = 2^{13}$$

4. Given, $P^T = 2P + I$

$$\Rightarrow (P^T)^T = (2P + I)^T = 2P^T + I$$

$$\Rightarrow P = 2P^T + I$$

$$\Rightarrow P = 2(2P + I) + I$$

$$\Rightarrow P = 4P + 3I$$

or $3P = -3I$

$$\Rightarrow PX = -IX = -X$$

5. Given, $M^T = -M$, $N^T = -N$ and $MN = NM$

$$\therefore M^2 N^2 (M^T N)^{-1} (MN^{-1})^T$$

$$= M^2 N^2 N^{-1} (M^T)^{-1} (N^{-1})^T \cdot M^T$$

$$= M^2 N (NN^{-1}) (-M)^{-1} (N^T)^{-1} (-M)$$

$$= M^2 NI (-M^{-1}) (-N)^{-1} (-M)$$

$$= -M^2 NM^{-1} N^{-1} M$$

$$= -M(MN)M^{-1} N^{-1} M$$

$$= -M(NM)M^{-1} N^{-1} M$$

$$= -MN(MM^{-1})N^{-1} M$$

$$= -M(NN^{-1})M = -M^2$$



6. Let $M = \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$

Then, $M \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $M \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

$\therefore \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} a_1 - a_2 \\ b_1 - b_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

and $\begin{bmatrix} a_1 + a_2 + a_3 \\ b_1 + b_2 + b_3 \\ c_1 + c_2 + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 12 \end{bmatrix}$

$\Rightarrow a_2 = -1, b_2 = 2, c_2 = 3, a_1 - a_2 = 1$

$b_1 - b_2 = 1, c_1 - c_2 = -1$

$a_1 + a_2 + a_3 = 0, b_1 + b_2 + b_3 = 0$

and $c_1 + c_2 + c_3 = 12$

$\therefore a_1 = 0, b_2 = 2, c_3 = 7$

$\therefore$ Sum of diagonal elements
 $= 0 + 2 + 7 = 9$

7. Given, $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$; $a, b, c \in \{0, 1, 2, \dots, (p-1)\}$

If A is skew-symmetric matrix, then $a = 0, b = -c$.

$\therefore |A| = -b^2$

Thus, p divides $|A|$, if only when $b = 0$ (i)

Again, if A is symmetric matrix, then $b = c$ and $|A| = a^2 - b^2$.

Thus, p divides $|A|$, if either p divides $(a-b)$ or p divides $(a+b)$. p divides $(a-b)$, only when $a = b$.

i.e. $a = b \in \{0, 1, 2, \dots, (p-1)\}$

i.e. p choice ... (ii)

$\therefore p$ divides $(a+b)$.

$\Rightarrow p$ choices, including $a = b = 0$ included in Eq. (i).

So, total number of choices are $(p+p-1)$, i.e. $(2p-1)$.

8. Trace of $A = 2a$, will be divisible by p iff $a = 0$.

$|A| = a^2 - bc$, for $(a^2 - bc)$ to be divisible by p .

There are exactly $(p-1)$ ordered pairs (b, c) for any value of a .

Hence, required number is $(p-1)^2$.

9. The number of matrices for which p does not divide $\text{tr}(A) = (p-1)p^2$ of these $(p-1)^2$ are such that p divides $|A|$. The number of matrices for which p divides $\text{tr}(A)$ and p does not divide $|A|$, is $(p-1)p^2$.

10. Given, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

$\therefore AA^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$

It is given that

$AA^T = 9I$

$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

On comparing, we get

$a+4+2b = 0$

$\Rightarrow a+2b = -4$... (i)

$2a+2-2b = 0$

$\Rightarrow a-b = -1$... (ii)

and $a^2+4+b^2 = 9$... (iii)

On solving Eqs. (i) and (ii), we get

$a = -2, b = -1$

which satisfies Eq. (iii).

Hence, $(a, b) \equiv (-2, -1)$

11. Given, $AA^T = A^T A$ and $B = A^{-1}A^T$

Now, $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$
 $= A^{-1}A^T A(A^{-1})^T$ [$\because (AB)^T = B^T A^T$]
 $= A^{-1}AA^T(A^{-1})^T$ [$\because AA^T = A^T A$]
 $= I(A^{-1})^T$ [$\because A^{-1}A = I$]
 $= A^T(A^{-1})^T$
 $= (A^{-1}A)^T$ [$\because (AB)^T = B^T A^T$]
 $= I^T = I$

12. $[3p+3q+2r \quad 4p+2q+0 \quad p+3q+2r] = [3 \quad 0 \quad 1]$

$\Rightarrow 3p+3q+2r = 3, 4p+2q = 0, p+3q+2r = 1$

$\Rightarrow p = 1, q = -2, r = 3$

$\therefore 2p+q-r = 2-2-3 = -3$

13. $\therefore A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $A^2 = I$ [given]

$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow b(a+d) = 0, c(a+d) = 0$

and $a^2+bc = 1, bc+d^2 = 1$

$\Rightarrow a = 1, d = -1, b = c = 0$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then, $A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$A \neq I, A \neq -I$

$\therefore \det(A) = -1$ and $\text{tr}(A) = 1-1 = 0$

14. Given, $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$

Then, $A^2 = A \cdot A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$

$\therefore A^2 + 4A - 5I = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix} + \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} + \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$

$= \begin{bmatrix} 8 & 4 \\ 8 & 0 \end{bmatrix} = 4 \begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$

15. Let $u_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$

Also, $Au_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $Au_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ [given]

We have, $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

and $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\therefore \begin{bmatrix} a_1 \\ 2a_1 + b_1 \\ 3a_1 + 2b_1 + c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} a_2 \\ 2a_2 + b_2 \\ 3a_2 + 2b_2 + c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

which gives $a_1 = 1, b_1 = -2$ and $c_1 = 1$
 $a_2 = 0, b_2 = 1$ and $c_2 = -2$

$$u_1 + u_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

16. Consider $\begin{bmatrix} 1 & * & * \\ * & 1 & * \\ * & * & 1 \end{bmatrix}$. By placing 1 in anyone of the 6* position and 0 elsewhere, we get 6 non-singular matrices. Similarly, $\begin{bmatrix} * & * & 1 \\ * & 1 & * \\ 1 & * & * \end{bmatrix}$ gives atleast one non-singular.

17. **Statement I** If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, then

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$A \neq I, A \neq -I$

$\therefore \det(A) = -1$
 So, Statement I is true.

Statement II $\text{tr}(A) = 1 - 1 = 0$
 So, Statement II is false.

18. Given, $A^2 - B^2 = (A - B)(A + B)$
 $\Rightarrow A^2 - B^2 = A^2 + AB - BA - B^2$
 So, above relation will hold always, if $AB = BA$.

19. Since, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$

Now, $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 2b \\ 3a & 4b \end{bmatrix}$

and $BA = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a & 2a \\ 3b & 4b \end{bmatrix}$

If $AB = BA$, then $a = b$.

Hence, $AB = BA$ is possible for infinitely many B 's.

20. Since, B is the inverse of matrix A , i.e. $B = A^{-1}$.

$$\therefore 10A^{-1} = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$$

$$\Rightarrow 10A^{-1}A = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} A$$

$$\Rightarrow 10I = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix} \quad [\because A^{-1}A = I]$$

$$\Rightarrow \begin{bmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{bmatrix} = \begin{bmatrix} 10 & 0 & 0 \\ -5 + \alpha & 5 + \alpha & -5 + \alpha \\ 0 & 0 & 10 \end{bmatrix}$$

$$\Rightarrow -5 + \alpha = 0 \text{ and } 5 + \alpha = 10$$

$$\Rightarrow \alpha = 5$$

21. Given that, $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}, A^2 = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix}$

$$\therefore A^2 = AA = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} a^2 + b^2 & ab + ba \\ ba + ab & b^2 + a^2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} = \begin{bmatrix} a^2 + b^2 & 2ab \\ 2ab & b^2 + a^2 \end{bmatrix}$$

$$\Rightarrow \alpha = a^2 + b^2, \beta = 2ab$$

22. Given, A is 3×4 matrix. So, A' is 4×3 matrix. Since, $A'B$ is defined, therefore B should be 3×4 . Also, BA' is defined, so B should be 3×4 . Hence, B is of the type 3×4 .

23. Given, matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix}$

$$\therefore \text{Symmetric part of } A = \frac{1}{2}(A + A')$$

$$= \frac{1}{2} \left\{ \begin{bmatrix} 1 & 2 & 4 \\ 6 & 8 & 2 \\ 2 & -2 & 7 \end{bmatrix} + \begin{bmatrix} 1 & 6 & 2 \\ 2 & 8 & -2 \\ 4 & 2 & 7 \end{bmatrix} \right\}$$

$$= \frac{1}{2} \begin{bmatrix} 2 & 8 & 6 \\ 8 & 16 & 0 \\ 6 & 0 & 14 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 \\ 4 & 8 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

24. Given, $A = \begin{bmatrix} 2-k & 2 \\ 1 & 3-k \end{bmatrix}$

Since, the matrix A is singular.

$$\therefore \begin{vmatrix} 2-k & 2 \\ 1 & 3-k \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2-k & 2 \\ 1 & 3-k \end{vmatrix} = 0$$

$$\Rightarrow (2-k)(3-k) - 2 = 0$$

$$\Rightarrow 6 - 5k + k^2 - 2 = 0$$

$$\Rightarrow 4 - 5k + k^2 = 0$$

$$\Rightarrow 5k - k^2 = 4$$



25. Given, $A = \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $A \times A = \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} \cos^2\left(\frac{2\pi}{n}\right) - \sin^2\left(\frac{2\pi}{n}\right) & -\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(2 \times \frac{2\pi}{n}\right) & 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & 0 \\ -2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & \cos\left(2 \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(2 \times \frac{2\pi}{n}\right) & \sin\left(2 \times \frac{2\pi}{n}\right) & 0 \\ -\sin\left(2 \times \frac{2\pi}{n}\right) & \cos\left(2 \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^n = \begin{bmatrix} \cos\left(2^{n-1} \times \frac{2\pi}{n}\right) & \sin\left(2^{n-1} \times \frac{2\pi}{n}\right) & 0 \\ -\sin\left(2^{n-1} \times \frac{2\pi}{n}\right) & \cos\left(2^{n-1} \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

and

$$A^{n-1} = \begin{bmatrix} \cos\left(2^{n-2} \times \frac{2\pi}{n}\right) & \sin\left(2^{n-2} \times \frac{2\pi}{n}\right) & 0 \\ -\sin\left(2^{n-2} \times \frac{2\pi}{n}\right) & \cos\left(2^{n-2} \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I$$

26. Given, $AB = A$ and $BA = B$

Now, $BA = B$

$\Rightarrow BAB = BB$ [post-multiplying both sides by B]

$\Rightarrow B(A) = B^2 \Rightarrow B = B^2$

27. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & 0+0+0 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ a+0-a & 0+b-b & 0+0+1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

28. We have, $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

and $5A = \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix}$

Now, $A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O$$

29. Given, $\begin{bmatrix} 4 & x-z \\ 2+y & xz \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ -1 & 10 \end{bmatrix}$

On comparing the corresponding elements, we get

$$x - z = 3 \quad \dots(i)$$

$$2 + y = -1$$

$$\Rightarrow y = -1 - 2 = -3 \quad \dots(ii)$$

$$\text{and } xz = 10 \quad \dots(iii)$$

$$\text{Now, } (x+z)^2 = (x-z)^2 + 4xz = (3)^2 + 4 \times 10 = 9 + 40$$

$$\Rightarrow (x+z)^2 = 49$$

$$\Rightarrow x + z = 7 \quad \dots(iv)$$

On adding Eqs. (i) and (iv), we get

$$2x = 10 \Rightarrow x = 5$$

From Eq. (i), we get

$$5 - z = 3 \Rightarrow -z = 3 - 5$$

$$z = 2$$

Hence, $x = 5$, $y = -3$ and $z = 2$.

30. Given, $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$
 $= \begin{bmatrix} 4-1 & -2-2 \\ 2+2 & -1+4 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$

Now, $A^2 + 2A - 3I = \begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix} + 2 \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
 $= \begin{bmatrix} 3+4-3 & -4-2-0 \\ 4+2-0 & 3+4-3 \end{bmatrix}$
 $= \begin{bmatrix} 4 & -6 \\ 6 & 4 \end{bmatrix}$

31. We have, $\begin{bmatrix} -1 & 3 & 0 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -2 & -1 \end{bmatrix}$
 $= \begin{bmatrix} -1+0+0 & -3+6+0 \\ 2+0-8 & 6+2-4 \end{bmatrix}$
 $= \begin{bmatrix} -1 & 3 \\ -6 & 4 \end{bmatrix}$

32. Given, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

Now, $A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & -1 \end{bmatrix}$

$\Rightarrow A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$

$A^4 = A^2 \cdot A^2 = I_3 \cdot I_3 = I_3$

$A^6 = A^2 \cdot A^4 = I_3 \cdot I_3 = I_3$

$\therefore A^2 + 2A^4 + 4A^6 = I_3 + 2I_3 + 4I_3 = 7I_3 = 7(I_3)^4$
 $= 7(A^2)^4$
 $= 7A^8$

33. We have, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ and $AA^T = 9I_3$

$\Rightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix} \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix} = 9 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 9 & 0 & a+4+2b \\ 0 & 9 & 2a+2-2b \\ a+4+2b & 2a+2-2b & a^2+4+b^2 \end{bmatrix}$

$= \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$

$\Rightarrow a+2b = -4 \quad \dots(i)$

and $2a-2b = -2 \quad \dots(ii)$

On solving Eqs. (i) and (ii), we get

$a = -2$ and $b = -1$

34. We have, $A = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$ is a symmetric matrix.

i.e. $A^T = A$

$\Rightarrow \begin{bmatrix} 3 & 2x+3 \\ x-1 & x+2 \end{bmatrix} = \begin{bmatrix} 3 & x-1 \\ 2x+3 & x+2 \end{bmatrix}$

$\Rightarrow 2x+3 = x-1$

$\therefore x = -4$

35. We have, $(AB)^T = BA$

$\Rightarrow B^T A^T = BA$

So, A and B either both symmetric or skew-symmetric matrices.

36. From given matrix equation, we have

$x - y - z = 0$

$-y + z = 5$ and $z = 3$

$\therefore x = 1, y = -2$ and $z = 3$

37. $AB = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 5 & 0 \end{bmatrix}$

$= \begin{bmatrix} 2+0+15 & -2+2+0 \\ 4+0+0 & -4+2+0 \end{bmatrix} = \begin{bmatrix} 17 & 0 \\ 4 & -2 \end{bmatrix}$

2

Determinant

Introduction

Determinants arise in connection with simultaneous linear algebraic equations, whose solutions are very complicated and lengthy. This can be easily handled, if they are expressed as determinants.

Definition of Determinant

Every square matrix $A = [a_{ij}]$ of order n can be associated to a number (real or a complex number) called determinant of square matrix, where $a_{ij} = (i, j)$ th element of A .

If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$, then determinant of A is denoted by $\det(A)$, Δ or $|A|$ and

written as

$$\det(A) = |A| = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

- For matrix A , $|A|$ is read as determinant of A and not modulus of A .
- An arrangement of numbers in the form of rows and columns is known as matrix and a matrix having equal number of rows and columns is known as square matrix.
- Matrix has no definite value, while each determinant has a definite value.
- Determinant of a matrix which is not a square matrix, cannot be found out.

Value of a Determinant of Order 1

If $A = [a_{11}]$ is a square matrix of order 1×1 , then $|A| = a_{11}$.

Value of a Determinant of Order 2

If $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is a square matrix of order 2×2 , then corresponding determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

Chapter Snapshot

- Introduction
- Definition of Determinant
- Applications of Determinant in Geometry
- Properties of Determinant
- Minors and Cofactors of Elements of a Determinant
- Adjoint of a Square Matrix
- Inverse of a Square Matrix
- Applications of Determinants and Matrices
- Rank of a Matrix
- Product of Determinants of Same Order
- Differentiation of Determinants
- Integration of Determinants
- Maximum and Minimum Values of Determinants, when Elements are Known

➤ **Example 1.** The value of $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix}$ is

- (a) 0 (b) -2 (c) 2 (d) 3

Sol. (b) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 2(5) - 3(4) = 10 - 12 = -2$

Value of a Determinant of Order 3

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of

order 3×3 , then the corresponding determinant is

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\begin{aligned} \text{and } |A| &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) \\ &\quad + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \end{aligned}$$

$$\begin{aligned} \text{e.g. If } A &= \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \text{ then } |A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 1 & 1 & 1 \end{vmatrix} \\ &= 1 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} \\ &= (2-1) - 2(3-1) + 3(3-2) \\ &= 1 - 2(2) + 3 = 4 - 4 = 0 \end{aligned}$$

➤ The determinant of order greater than 2 can be expanded along any row or column.

Applications of Determinant in Geometry

(i) Area of Triangle

If (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are the vertices of a triangle, then

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \\ &= \frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \end{aligned}$$

➤ **Example 2.** The area of the triangle with vertices $(-2, -3)$, $(3, 2)$ and $(-1, -8)$, is

- (a) $\frac{15}{2}$ sq units (b) $\frac{47}{2}$ sq units
(c) 15 sq units (d) None of these

$$\begin{aligned} \text{Sol. (c) Required area} &= \frac{1}{2} \begin{vmatrix} -2 & -3 & 1 \\ 3 & 2 & 1 \\ -1 & -8 & 1 \end{vmatrix} \\ &= \frac{1}{2} [-2(2+8) + 3(3+1) + 1(-24+2)] \\ &= \frac{1}{2} [-20 + 12 - 22] = \frac{1}{2} [-30] = 15 \text{ sq units} \end{aligned}$$

(ii) Condition of Collinearity of Three Points

Let three points be $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then these points will be collinear, if

$$\text{Area of } \triangle ABC = 0 \Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

➤ **Example 3.** If the points $(2, -3)$, $(\lambda, -1)$ and $(0, 4)$ are collinear, then the value of λ is

- (a) $\frac{1}{2}$ (b) $\frac{5}{7}$ (c) $\frac{10}{7}$ (d) $\frac{15}{7}$

Sol. (c) Since, the points $(2, -3)$, $(\lambda, -1)$ and $(0, 4)$ are collinear.

$$\begin{aligned} \therefore \begin{vmatrix} 2 & -3 & 1 \\ \lambda & -1 & 1 \\ 0 & 4 & 1 \end{vmatrix} &= 0 \\ \Rightarrow 2(-1-4) + 3(\lambda-0) + (4\lambda-0) &= 0 \\ \Rightarrow -10 + 3\lambda + 4\lambda &= 0 \\ \Rightarrow 7\lambda = 10 \Rightarrow \lambda &= \frac{10}{7} \end{aligned}$$

(iii) Equation of Straight Line Passing through Two Points

Let two points be $A(x_1, y_1)$, $B(x_2, y_2)$ and $P(x, y)$ be a point on the line joining points A and B , then the equation of line is given by

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

➤ **Example 4.** The equation of straight line passing through the points $A(3, 1)$ and $B(9, 3)$ is
(a) $x - 2 = 0$ (b) $x - 3y = 0$ (c) $y + 2 = 0$ (d) $y - 2 = 0$

Sol. (b) Let $P(x, y)$ be any point on the line joining $A(3, 1)$ and $B(9, 3)$. Then, the points A , B and P are collinear. Therefore, the area of $\triangle ABP$ will be zero.

$$\begin{aligned} \text{i.e. } \frac{1}{2} \begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} &= 0 \\ \Rightarrow \frac{1}{2} [3(3-y) - 1(9-x) + 1(9y-3x)] &= 0 \\ \Rightarrow 9 - 3y - 9 + x + 9y - 3x &= 0 \\ \Rightarrow 6y - 2x = 0 \Rightarrow x - 3y &= 0 \\ \text{Hence, the equation of the line joining the given} & \\ \text{points is } x - 3y = 0. & \end{aligned}$$

Properties of Determinant

Properties of determinants of order three only are stated below. However, these properties hold for determinants of any order. These properties help in the evaluation of determinants.

- i. The value of the determinant remains unchanged, if rows are changed into columns and columns are changed into rows, i.e. if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then interchanging}$$

$$\text{rows and columns, we get } \Delta = \begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$$

$$\text{e.g. } \begin{vmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 2 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$$

↘ If R_i = ith row and C_j = ith column, then for interchange of rows and columns, we will symbolically write $R_i \leftrightarrow C_j$.

- ii. If two rows or columns of a determinant are interchanged, then the value of the determinant so obtained is the negative of the value of the original determinant, i.e. if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then on interchanging } R_1$$

$$\text{and } R_2, \text{ we get } \Delta = - \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{e.g. } \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = - \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$$

↘ We can denote the interchanging of any two rows by $R_i \leftrightarrow R_j$ and interchanging of any two columns by $C_i \leftrightarrow C_j$.

- iii. If two rows or columns of a determinant are identical, then its value is zero. e.g.

$$\begin{vmatrix} a & a & c \\ b & b & a \\ c & c & b \end{vmatrix} = 0$$

Here, first and second columns are same.

- iv. If each element of a row (or a column) of a determinant is multiplied by a constant k , then the value of the new determinant is k times the value of the original determinant, i.e. if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then on multiplying } R_1$$

by k , we get

$$\Delta = \frac{1}{k} \begin{vmatrix} ka_{11} & ka_{12} & ka_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\text{e.g. } \begin{vmatrix} 2 & 4 & 6 \\ 4 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix}$$

[taking common 2 from R_1]

↘ By this property, we can take out any common factor from any one row or any one column of a given determinant.

- v. If any two rows or columns of a determinant are proportional, then its value is zero, i.e. if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ ka_{11} & ka_{12} & ka_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$\Delta = k \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = 0$$

$$\text{e.g. } \begin{vmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 2 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

- vi. If each element of a row or a column of a determinant is the sum of two or more terms, then the determinant can be expressed as the sum of two or more determinants, i.e. if

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + c_1 & a_{22} + c_2 & a_{23} + c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then}$$

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ c_1 & c_2 & c_3 \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

e.g.

$$\begin{vmatrix} a & b & c \\ d+e & f+g & h+i \\ j & k & l \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & f & h \\ j & k & l \end{vmatrix} + \begin{vmatrix} a & b & c \\ e & g & i \\ j & k & l \end{vmatrix}$$

- vii. If each element of a row or a column of a determinant is multiplied by a constant k and then added to (or subtracted from) the corresponding elements of any other row or column, then the value of the determinant remains the same.

i.e. if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then on applying

$R_3 \rightarrow R_3 + k R_2$, we get

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} + ka_{21} & a_{32} + ka_{22} & a_{33} + ka_{23} \end{vmatrix}$$

e.g. $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = \begin{vmatrix} a & b & c \\ d & e & f \\ g + ak & h + bk & i + ck \end{vmatrix}$

viii. If each element of a row or a column of a determinant is zero, then its value is zero.

ix. If a determinant having polynomials as its elements and if r rows or r columns become identical when a is substituted for x , then $(x - a)^{r-1}$ is a factor of given determinant.

x. The sum of the products of elements of any row or column of a determinant with the cofactors of the corresponding elements of any other row or column is zero.

i.e. if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$$a_{11}C_{31} + a_{12}C_{32} + a_{13}C_{33} = 0 \text{ and so on.}$$

xi. The sum of the products of elements of any row or column of a determinant with the cofactors of the corresponding elements of same row or column is equal to Δ .

i.e. if $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$, then

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \Delta$$

xii. If $\Delta(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ a & b & c \end{vmatrix}$, then

$$(a) \sum_{x=1}^n \Delta(x) = \begin{vmatrix} \sum_{x=1}^n f_1(x) & \sum_{x=1}^n f_2(x) & \sum_{x=1}^n f_3(x) \\ \sum_{x=1}^n g_1(x) & \sum_{x=1}^n g_2(x) & \sum_{x=1}^n g_3(x) \\ a & b & c \end{vmatrix}$$

$$(b) \prod_{x=1}^n \Delta(x) = \begin{vmatrix} \prod_{x=1}^n f_1(x) & \prod_{x=1}^n f_2(x) & \prod_{x=1}^n f_3(x) \\ \prod_{x=1}^n g_1(x) & \prod_{x=1}^n g_2(x) & \prod_{x=1}^n g_3(x) \\ a & b & c \end{vmatrix}$$

- If $\Delta = |a_{ij}|$ is a determinant of order n , then the value of the determinant $|A_{ij}|$, where A_{ij} is the cofactor of a_{ij} , is Δ^{n-1} .
- If $A = B + C$, then it is not necessary that $\det(A) = \det(B) + \det(C)$
- If A is a square matrix of order n , then $\det(kA) = k^n \det(A)$
- Determinant of a Hermitian matrix is purely real.
- Determinant of an orthogonal matrix is 1 or -1.
- If A, B and C are three square matrices of the same order such that i th row or column of A is equal to the sum of i th row or column of B and C and remaining rows or columns of A, B and C are same, then $\det(A) = \det(B) + \det(C)$
- $\det(A^n) = (\det A)^n$, where n is a positive integer.
- $\det(A^{-1}) = \frac{1}{\det(A)}$
- If the rows and columns are interchange, then Δ is unchanged, i.e. $|A^T| = |A|$.
- Product of two determinants is $|AB| = |A||B| = |BA| = |AB^T| = |A^T B| = |A^T B^T|$

➤ **Example 5.** The value of

$$\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix} \text{ is}$$

(a) 0 (b) 1 (c) -1 (d) 2

Sol. (a) We have, $\Delta = \begin{vmatrix} a & b & c \\ a+2x & b+2y & c+2z \\ x & y & z \end{vmatrix}$

$$= \begin{vmatrix} a & b & c \\ a & b & c \\ x & y & z \end{vmatrix} + \begin{vmatrix} a & b & c \\ 2x & 2y & 2z \\ x & y & z \end{vmatrix}$$

$$= 0 + 0 = 0$$

➤ **Example 6.** The value of $\Delta = \begin{vmatrix} 3 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 2 & 3 \end{vmatrix}$ is

(a) 0 (b) 1 (c) 2 (d) 3

Sol. (a) Here, R_1 and R_3 are identical.
 $\therefore$ Value of determinant is zero.

➤ **Example 7.** The value of the determinant

$$\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix} \text{ is}$$

(a) 1 (b) 0
 (c) 2 (d) None of these

Sol. (b) $\begin{vmatrix} 1 & 4 & 9 & 16 \\ 4 & 9 & 16 & 25 \\ 9 & 16 & 25 & 36 \\ 16 & 25 & 36 & 49 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & 7 \\ 4 & 5 & 7 & 9 \\ 9 & 7 & 9 & 11 \\ 16 & 9 & 11 & 13 \end{vmatrix}$

[applying $C_4 \rightarrow C_4 - C_3, C_3 \rightarrow C_3 - C_2, C_2 \rightarrow C_2 - C_1$]

$$= \begin{vmatrix} 1 & 3 & 2 & 2 \\ 4 & 5 & 2 & 2 \\ 9 & 7 & 2 & 2 \\ 16 & 9 & 2 & 2 \end{vmatrix} \quad [\text{applying } C_4 \rightarrow C_4 - C_3, C_3 \rightarrow C_3 - C_2]$$

$$= 0 \quad [\because C_3 \text{ and } C_4 \text{ are identical}]$$

➤ **Example 8.** If A, B and C are the angles of a triangle, then the value of the determinant

$$\begin{vmatrix} -1 + \cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1 + \cos A & \cos A \\ -1 + \cos B & -1 + \cos A & -1 \end{vmatrix} \text{ is}$$

- (a) 0 (b) 1
(c) -1 (d) 2

Sol. (a) $\begin{vmatrix} -1 + \cos B & \cos C + \cos B & \cos B \\ \cos C + \cos A & -1 + \cos A & \cos A \\ -1 + \cos B & -1 + \cos A & -1 \end{vmatrix}$

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

[applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$]

$$= \frac{1}{a} \begin{vmatrix} -a & \cos C & \cos B \\ a \cos C & -1 & \cos A \\ a \cos B & \cos A & -1 \end{vmatrix} = \frac{1}{a} \begin{vmatrix} 0 & \cos C & \cos B \\ 0 & -1 & \cos A \\ 0 & \cos A & -1 \end{vmatrix}$$

[applying $C_1 \rightarrow C_1 + bC_2 + cC_3$ and using projection rule]

$$= 0$$

➤ **Example 9.**

If $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = k(a+b+c)^3$,

then k is equal to

- (a) 1
(b) 2
(c) 4
(d) None of the above

Sol. (b) $\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

[applying $C_1 \rightarrow C_1 + C_2 + C_3$ and taking $2(a+b+c)$ common from C_1]

$$= 2(a+b+c) \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

[applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$]

$$= 2(a+b+c)^3 \quad [\text{expanding along } C_1]$$

$\therefore k = 2$

➤ **Example 10.** If $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)f(x)$,

then $f(x)$ is equal to

- (a) $(x-a)^2$ (b) $(x-a)^3$
(c) $(x+a)^2$ (d) $(x+a)^3$

Sol. (b) We have, $\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a) \begin{vmatrix} 1 & a & a & a \\ 1 & x & a & a \\ 1 & a & x & a \\ 1 & a & a & x \end{vmatrix}$

[applying $C_1 \rightarrow C_1 + C_2 + C_3 + C_4$ and taking $(x+3a)$ common from C_1]

$$= (x+3a) \begin{vmatrix} 1 & a & a & a \\ 0 & x-a & 0 & 0 \\ 0 & 0 & x-a & 0 \\ 0 & 0 & 0 & x-a \end{vmatrix}$$

[applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$ and $R_4 \rightarrow R_4 - R_1$]

$$= (x+3a)(x-a)^3 \quad [\text{expanding along } C_1]$$

$\therefore f(x) = (x-a)^3$

➤ **Example 11.** If $\begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix} = 0$ and

$a \neq b \neq c$, then x is equal to

- (a) $\frac{abc}{(ab+bc+ca)}$
(b) $-\frac{abc}{(ab+bc+ca)}$
(c) $\frac{(ab+bc+ca)}{abc}$
(d) $-\frac{(ab+bc+ca)}{abc}$

Sol. (b) We have, $\begin{vmatrix} x+a & a^2 & a^3 \\ x+b & b^2 & b^3 \\ x+c & c^2 & c^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x & a^2 & a^3 \\ x & b^2 & b^3 \\ x & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow x \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} + abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\Rightarrow x(a-b)(b-c)(c-a)(ab+bc+ca) + abc(a-b)(b-c)(c-a) = 0$$

$$\Rightarrow x = -\frac{abc}{(ab+bc+ca)} \quad [\because a \neq b \neq c]$$

➤ **Example 12.** The value of the determinant

$$\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} \text{ is}$$

- (a) 0 (b) $(p-q)(q-r)(r-p)$
(c) pqr (d) $3pqr$

Sol. (a) $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix}$

$$= \begin{vmatrix} p-q & p-q & p-q \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} \quad [\text{applying } R_1 \rightarrow R_1 - R_2]$$

$$= \begin{vmatrix} p-q & 0 & 0 \\ q-p & p-q & p-r \\ r-p & p-q & p-r \end{vmatrix}$$

[applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$]

$$= (p-q)(p-r) \begin{vmatrix} p-q & 0 & 0 \\ q-p & 1 & 1 \\ r-p & 1 & 1 \end{vmatrix} = 0$$

[$\because C_1$ and C_2 are identical]

➤ **Example 13.** For positive numbers x, y, z , the numerical value of the determinant

$$\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix} \text{ is}$$

- (a) 0 (b) 1
(c) 2 (d) None of these

Sol. (a) $\begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$

$$= \begin{vmatrix} 1 & \frac{\log y}{\log x} & \frac{\log z}{\log x} \\ \frac{\log y}{\log x} & 1 & \frac{\log z}{\log x} \\ \frac{\log y}{\log z} & \frac{\log y}{\log z} & 1 \end{vmatrix}$$

$$= \frac{1}{\log x \cdot \log y \cdot \log z} \begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix} = 0$$

[$\because$ all rows are identical]

➤ **Example 14.** If $D_1 = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x & y & z \end{vmatrix}$ and

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ yz & xz & xy \\ x & y & z \end{vmatrix}, \text{ then}$$

- (a) $D_1 = D_2$ (b) $D_1 = -D_2$
(c) $D_1 = -2D_2$ (d) $D_2 = 2D_1$

Sol. (a) We have, $D_2 = \begin{vmatrix} 1 & 1 & 1 \\ yz & xz & xy \\ x & y & z \end{vmatrix} = \frac{1}{xyz} \begin{vmatrix} x & y & z \\ xyz & xyz & xyz \\ x^2 & y^2 & z^2 \end{vmatrix}$

$$= \frac{xyz}{xyz} \begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = D_1$$

➤ **Example 15.** The value of the determinant

$$\begin{vmatrix} 1 & (5^{2x} - 5^{-2x})^2 & (5^{2x} + 5^{-2x})^2 \\ 1 & (6^{2x} - 6^{-2x})^2 & (6^{2x} + 6^{-2x})^2 \\ 1 & (7^{2x} - 7^{-2x})^2 & (7^{2x} + 7^{-2x})^2 \end{vmatrix} \text{ is}$$

- (a) 210^{2x} (b) 210^{-2x}
(c) 0 (d) None of these

Sol. (c) We have, $\begin{vmatrix} 1 & (5^{2x} - 5^{-2x})^2 & (5^{2x} + 5^{-2x})^2 \\ 1 & (6^{2x} - 6^{-2x})^2 & (6^{2x} + 6^{-2x})^2 \\ 1 & (7^{2x} - 7^{-2x})^2 & (7^{2x} + 7^{-2x})^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & (5^{2x} - 5^{-2x})^2 & 4 \cdot 5^{2x} \cdot 5^{-2x} \\ 1 & (6^{2x} - 6^{-2x})^2 & 4 \cdot 6^{2x} \cdot 6^{-2x} \\ 1 & (7^{2x} - 7^{-2x})^2 & 4 \cdot 7^{2x} \cdot 7^{-2x} \end{vmatrix}$$

[applying $C_3 \rightarrow C_3 - C_2$]

$$= 4 \begin{vmatrix} 1 & (5^{2x} - 5^{-2x})^2 & 1 \\ 1 & (6^{2x} - 6^{-2x})^2 & 1 \\ 1 & (7^{2x} - 7^{-2x})^2 & 1 \end{vmatrix} = 0$$

[$\because C_1$ and C_3 are identical]

➤ **Example 16.** The roots of the equation

$$\begin{vmatrix} x & \alpha & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = 0 \text{ are independent of}$$

- (a) α (b) β
(c) γ (d) All of these

Sol. (a) We have, $\begin{vmatrix} x & \alpha & 1 \\ \beta & x & 1 \\ \beta & \gamma & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x-\beta & \alpha-x & 0 \\ 0 & x-\gamma & 0 \\ \beta & \gamma & 1 \end{vmatrix} = 0$$

[applying $R_1 \rightarrow R_1 - R_2$, $R_2 \rightarrow R_2 - R_3$]

$$\Rightarrow (x-\beta)(x-\gamma) = 0 \quad [\text{expanding along } C_3]$$

Clearly, the roots are independent of α .

➤ **Example 17.** If $\sqrt{-1} = i$ and ω is a non-real cube root of unity, then the value of the

$$\text{determinant } \begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} \text{ is}$$

- (a) x^3 (b) $x^3 - 1$
(c) $x^3 + 1$ (d) None of these

Sol. (a) The given determinant

$$\begin{aligned}
 &= (x+1+\omega+\omega^2) \begin{vmatrix} 1 & 1 & 1 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix} \\
 &\quad \left[\text{applying } R_1 \rightarrow R_1 + R_2 + R_3 \text{ and taking } \right. \\
 &\quad \left. (x+1+\omega+\omega^2) \text{ common from } R_1 \right] \\
 &= x \begin{vmatrix} 1 & 0 & 0 \\ \omega & x+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-\omega^2 & x+\omega-\omega^2 \end{vmatrix} \\
 &\quad \left[\text{applying } C_2 \rightarrow C_2 - C_1 \text{ and } C_3 \rightarrow C_3 - C_1 \right] \\
 &= x \{ x^2 - (\omega - \omega^2)^2 \} - (1-\omega)(1-\omega^2) \\
 &= x^3 \quad \left[\text{using } 1+\omega+\omega^2=0 \text{ and } \omega^3=1 \right]
 \end{aligned}$$

➤ **Example 18.** If $a_i, b_i, c_i \in R, (i=1, 2, 3)$ and

$$x \in R \text{ and } \begin{vmatrix} a_1 + b_1x & a_1x + b_1 & c_1 \\ a_2 + b_2x & a_2x + b_2 & c_2 \\ a_3 + b_3x & a_3x + b_3 & c_3 \end{vmatrix} = 0, \text{ then}$$

$$(a) \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 4 \quad (b) x = \pm 1$$

$$(c) x = 2 \quad (d) \text{None of these}$$

Sol. (b) Clearly, $x = \pm 1$ satisfies the given equation.

➤ **Example 19.** The value of the determinant

$$\begin{vmatrix} \sqrt{x} + \sqrt{y} & 2\sqrt{z} & \sqrt{z} \\ \sqrt{yz} + \sqrt{2x} & z & \sqrt{2z} \\ y + \sqrt{xz} & \sqrt{yz} & z \end{vmatrix}; \text{ where } x, y, z \text{ are}$$

positive real numbers, is

$$(a) z(\sqrt{2y} - z\sqrt{y}) \quad (b) y(\sqrt{2z} - y\sqrt{z})$$

$$(c) x(\sqrt{2y} - z\sqrt{y}) \quad (d) \text{None of these}$$

$$\text{Sol. (a) We have, } \begin{vmatrix} \sqrt{x} + \sqrt{y} & 2\sqrt{z} & \sqrt{z} \\ \sqrt{yz} + \sqrt{2x} & z & \sqrt{2z} \\ y + \sqrt{xz} & \sqrt{yz} & z \end{vmatrix}$$

$$= z \begin{vmatrix} \sqrt{x} + \sqrt{y} & 2 & 1 \\ \sqrt{yz} + \sqrt{2x} & \sqrt{z} & \sqrt{2} \\ y + \sqrt{xz} & \sqrt{y} & \sqrt{z} \end{vmatrix}$$

$$\quad \left[\text{taking } \sqrt{z} \text{ common from } C_2 \text{ and } C_3 \right]$$

$$= z \begin{vmatrix} -\sqrt{y} & 2 & 1 \\ 0 & \sqrt{z} & \sqrt{2} \\ 0 & \sqrt{y} & \sqrt{z} \end{vmatrix} \quad \left[\text{applying } C_1 \rightarrow C_1 - \sqrt{y}C_2 - \sqrt{x}C_3 \right]$$

$$= \sqrt{y} \cdot z(z - \sqrt{2y}) = z(\sqrt{2y} - z\sqrt{y})$$

➤ **Example 20.** If $\begin{vmatrix} \alpha & -\beta & 0 \\ 0 & \alpha & \beta \\ \beta & 0 & \alpha \end{vmatrix} = 0$, then

$$(a) \frac{\alpha}{\beta} \text{ is one of the cube roots of unity}$$

$$(b) \alpha \text{ is one of the cube roots of unity}$$

$$(c) \beta \text{ is one of the cube roots of unity}$$

$$(d) \text{None of the above}$$

$$\text{Sol. (a) We have, } \begin{vmatrix} \alpha & -\beta & 0 \\ 0 & \alpha & \beta \\ \beta & 0 & \alpha \end{vmatrix} = 0 \Rightarrow \alpha^3 - \beta^3 = 0$$

$$\Rightarrow \left(\frac{\alpha}{\beta} \right)^3 = 1 \Rightarrow \frac{\alpha}{\beta} \text{ is one of the cube roots of unity.}$$

$$\text{➤ Example 21. If } f(x) = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix},$$

then $f(3x) - f(x)$ is equal to

$$(a) 3x\lambda^2$$

$$(b) 6x\lambda^2$$

$$(c) x\lambda^2$$

$$(d) \text{None of these}$$

Sol. (b) We have,

$$f(x) = \begin{vmatrix} x+\lambda & x & x \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix} = (3x+\lambda) \begin{vmatrix} 1 & 1 & 1 \\ x & x+\lambda & x \\ x & x & x+\lambda \end{vmatrix}$$

$$\quad \left[\text{applying } R_1 \rightarrow R_1 + R_2 + R_3 \text{ and} \right. \\
 \quad \left. \text{taking } (3x+\lambda) \text{ common from } R_1 \right]$$

$$= (3x+\lambda) \begin{vmatrix} 1 & 0 & 0 \\ x & \lambda & 0 \\ x & 0 & \lambda \end{vmatrix}$$

$$\quad \left[\text{applying } C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1 \right] \\
 = (3x+\lambda)\lambda^2 \quad \left[\text{expanding along } R_1 \right]$$

$$\therefore f(3x) - f(x) = \lambda^2[(9x+\lambda) - (3x+\lambda)] = 6x\lambda^2$$

➤ **Example 22.** If the determinant

$$\begin{vmatrix} b-c & c-a & a-b \\ b'-c' & c'-a' & a'-b' \\ b''-c'' & c''-a'' & a''-b'' \end{vmatrix} = m \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix},$$

then the value of m is

$$(a) 0$$

$$(b) 2$$

$$(c) -1$$

$$(d) 1$$

Sol. (a) Applying $C_1 \rightarrow C_1 + C_2 + C_3$ on LHS, we get

$$\begin{vmatrix} 0 & c-a & a-b \\ 0 & c'-a' & a'-b' \\ 0 & c''-a'' & a''-b'' \end{vmatrix} = m \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

$$\Rightarrow m = 0$$

➤ **Example 23.** If $|A| = \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix}$ and

$$|B| = \begin{vmatrix} q & -b & y \\ -p & a & -x \\ r & -c & z \end{vmatrix}, \text{ then}$$

$$(a) |A| = 2|B|$$

$$(b) |A| = |B|$$

$$(c) |A| = -|B|$$

$$(d) \text{None of these}$$

$$\text{Sol. (c) } |B| = - \begin{vmatrix} q & -b & y \\ p & -a & x \\ r & -c & z \end{vmatrix} = - \begin{vmatrix} q & b & y \\ p & a & x \\ r & c & z \end{vmatrix} = -|A|$$

$$\quad \left[\text{applying } R_1 \leftrightarrow R_2 \right]$$

$$= \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \quad [\text{applying } C_1 \leftrightarrow C_2]$$

$$= \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = - \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} \quad [\text{applying } R_2 \leftrightarrow R_3]$$

$$= -|A|$$

➤ **Example 24.** Solution set of the equation

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix} = 0 \text{ is}$$

- (a) $\{2, 0, 1\}$ (b) $\{2, -3, 1\}$
(c) $\{2, 1, 5\}$ (d) $\{-3, 1, 5\}$

Sol. (b) We have, $x(-3x^2 - 6x - 2x^2 + 6x) + 6(2x + 4) + 3x - 9 - 1(4x - 9x) = 0$
 $\Rightarrow -5x^3 + 30x - 30 + 5x = 0$
 $\Rightarrow 5x^3 - 35x + 30 = 0 \Rightarrow x^3 - 7x + 6 = 0$
 $\Rightarrow (x-1)(x^2 + x - 6) = 0 \Rightarrow (x-1)(x+3)(x-2) = 0$
 $\Rightarrow x = 1, 2, -3$

➤ **Example 25.** If x, y, z are in AP, then the value of the determinant of A , where

$$A = \begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}, \text{ is}$$

- (a) 2 (b) 1
(c) 0 (d) None of these

Sol. (c) We have,
 $|A| = \begin{vmatrix} 4 & 5 & 6 & x \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & x-2y+z \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$
 [applying $R_1 \rightarrow R_1 - 2R_2 + R_3$]
 $= \begin{vmatrix} 0 & 0 & 0 & 0 \\ 5 & 6 & 7 & y \\ 6 & 7 & 8 & z \\ x & y & z & 0 \end{vmatrix}$
 [since, x, y, z are in AP, so $x - 2y + z = 0$]
 $= 0$

➤ **Example 26.** If A is an orthogonal matrix, then

- (a) $\det A = 1$ (b) $\det A = 0$
(c) $\det A = -1$ (d) None of these

Sol. (a, c) Since, A is an orthogonal matrix, therefore
 $AA' = I \Rightarrow |A \cdot A'| = 1$
 $\Rightarrow |A| \cdot |A'| = 1$
 $\Rightarrow |A| \cdot |A| = 1 \quad [\because |A'| = |A|]$
 $\Rightarrow |A|^2 = 1$
 $\Rightarrow |A| = \pm 1$

➤ **Example 27.** The value of the determinant

$$\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos (n+1)x & \cos (n+2)x \\ \sin nx & \sin (n+1)x & \sin (n+2)x \end{vmatrix} \text{ is}$$

- (a) independent of n (b) independent of a
(c) independent of x (d) None of these

Sol. (a) We have, $\begin{vmatrix} a^2 & a & 1 \\ \cos nx & \cos (n+1)x & \cos (n+2)x \\ \sin nx & \sin (n+1)x & \sin (n+2)x \end{vmatrix}$
 $= \begin{vmatrix} a^2 - 2a \cos x + 1 & a & 1 \\ 0 & \cos (n+1)x & \cos (n+2)x \\ 0 & \sin (n+1)x & \sin (n+2)x \end{vmatrix}$
 [applying $C_1 \rightarrow C_1 + C_3 - 2 \cos x C_2$]
 $= (a^2 - 2a \cos x + 1) \sin x$ [expanding along C_1]
 which is independent of n .

➤ **Example 28.** If $A_1B_1C_1, A_2B_2C_2$ and $A_3B_3C_3$ are three digit numbers, each of which is

$$\text{divisible by } k, \text{ then } \Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \text{ is}$$

- (a) divisible by k (b) divisible by k^2
(c) divisible by k^3 (d) None of these

Sol. (a) Since, $A_1B_1C_1, A_2B_2C_2$ and $A_3B_3C_3$ are divisible by k , therefore
 $100A_1 + 10B_1 + C_1 = n_1k$
 $100A_2 + 10B_2 + C_2 = n_2k$
 $100A_3 + 10B_3 + C_3 = n_3k$
 where, n_1, n_2 and n_3 are integers.
 Now, $\Delta = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & 100A_1 + 10B_1 + C_1 \\ A_2 & B_2 & 100A_2 + 10B_2 + C_2 \\ A_3 & B_3 & 100A_3 + 10B_3 + C_3 \end{vmatrix}$
 [applying $C_3 \rightarrow C_3 + 10C_2 + 100C_1$]
 $= \begin{vmatrix} A_1 & B_1 & n_1k \\ A_2 & B_2 & n_2k \\ A_3 & B_3 & n_3k \end{vmatrix} = k \begin{vmatrix} A_1 & B_1 & n_1 \\ A_2 & B_2 & n_2 \\ A_3 & B_3 & n_3 \end{vmatrix} = k\Delta_1$
 Hence, Δ is divisible by k .
 [since, elements of Δ_1 are integers, so Δ_1 is an integer]

➤ **Example 29.** The determinant

$$\begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix} \text{ is}$$

- (a) > 0 , if $a > 1$
(b) $= 0$, if $a = 1$
(c) < 0 , if $a < 1$
(d) All of the above

Sol. (d) Let $\Delta = \begin{vmatrix} a^2 + 2a & 2a + 1 & 1 \\ 2a + 1 & a + 2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$

$$= \begin{vmatrix} a^2 + 2a - 3 & 2a - 2 & 0 \\ 2a - 2 & a - 1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 - R_3$ and $R_2 \rightarrow R_2 - R_3$]

$$= \begin{vmatrix} a^2 + 2a - 3 & 2a - 2 \\ 2a - 2 & a - 1 \end{vmatrix} \quad [\text{expanding along } C_3]$$

$$= \begin{vmatrix} (a+3)(a-1) & 2(a-1) \\ 2(a-1) & a-1 \end{vmatrix} = (a-1)^2 \begin{vmatrix} a+3 & 2 \\ 2 & 1 \end{vmatrix}$$

$$= (a-1)^2 \cdot (a+3-4) = (a-1)^3$$

Clearly, $\Delta > 0$, if $a > 1$; $\Delta = 0$, if $a = 1$ and $\Delta < 0$, if $a < 1$

➤ **Example 30.** If a, b, c are positive and not all

equal, then the value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

is

- (a) non-negative (b) non-positive
(c) negative (d) positive

Sol. (c) We have, $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c) \begin{vmatrix} 1 & 1 & 1 \\ b & c & a \\ c & a & b \end{vmatrix}$

[applying $R_1 \rightarrow R_1 + R_2 + R_3$ and
[taking $(a+b+c)$ common from R_1]]

$$= (a+b+c) \begin{vmatrix} 1 & 0 & 0 \\ b & c-b & a-b \\ c & a-c & b-c \end{vmatrix}$$

[applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$]

$$= (a+b+c) [-(b-c)^2 - (a-c)(a-b)]$$

[expanding along R_1]

$$= -(a+b+c)(a^2 + b^2 + c^2 - ab - ac - bc)$$

$$= -\frac{1}{2}(a+b+c)[(a-b)^2 + (b-c)^2 + (c-a)^2]$$

which is clearly negative.

[$\because a, b, c$ are positive and not all equal]

➤ **Example 31.** If $a \neq b \neq c$, then one value of x which satisfies the equation

$$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0, \text{ is given by}$$

- (a) $x = a$ (b) $x = b$ (c) $x = c$ (d) $x = 0$

Sol. (d) Let $\Delta = \begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix}$

On putting $x = a$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & a-b \\ 2a & 0 & a-c \\ a+b & a+c & 0 \end{vmatrix} = (a+c) \cdot 2a(a-b)$$

Clearly, $\Delta \neq 0$, on expanding along second column, so $x = a$ does not satisfy the equation $\Delta = 0$. Similarly, $x = b$ and $x = c$ also do not satisfy. Now on putting $x = 0$, we get

$$\Delta = \begin{vmatrix} 0 & -a & -b \\ a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Hence, $x = 0$ satisfies the equation $\Delta = 0$.

➤ **Example 32.** If

$$f(x) = \begin{vmatrix} 1 & 2(x-1) & 3(x-1)(x-2) \\ (x-1) & (x-1)(x-2) & (x-1)(x-2)(x-3) \\ x & (x-1)x & (x-1)(x-2)x \end{vmatrix},$$

then $f(41)$ is equal to

- (a) 41 (b) -41
(c) 0 (d) None of these

Sol. (c) We have, $f(x) = (x-1)(x-1)(x-2)$

$$= \begin{vmatrix} 1 & 2 & 3 \\ x-1 & x-2 & x-3 \\ x & x & x \end{vmatrix}$$

[taking $(x-1)$ and $(x-1)(x-2)$
[common from C_2 and C_3 , respectively]]

$$= (x-1)^2 \cdot (x-2) \begin{vmatrix} 1 & 2 & 3 \\ x-1 & x-2 & x-3 \\ 1 & 2 & 3 \end{vmatrix}$$

[applying $R_3 \rightarrow R_3 - R_2$
[$\because R_1$ and R_3 are identical]]

$$= 0$$

$\therefore f(41) = 0$

➤ **Example 33.** If

$$\begin{vmatrix} x^3 + 4x & x+3 & x-2 \\ x-2 & 5x & x-1 \\ x-3 & x+2 & 4x \end{vmatrix}$$

$= ax^5 + bx^4 + cx^3 + dx^2 + ex + f$ is an identity in x , where a, b, c, d, e and f are independent of x , then the value of f is

- (a) 0 (b) 15
(c) 17 (d) None of these

Sol. (c) We have, $\begin{vmatrix} 0 & 3 & -2 \\ -2 & 0 & -1 \\ -3 & 2 & 0 \end{vmatrix} = f$

[putting $x = 0$ on both sides]

$\therefore f = 17$

➤ **Example 34.** The value of the determinant

$$\begin{vmatrix} 1 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \\ a & \sin \alpha \theta & \cos \alpha \theta \\ a^2 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \end{vmatrix} \text{ is independent of}$$

- (a) α (b) β (c) θ (d) a

Sol. (a) We have, $\begin{vmatrix} 1 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \\ a & \sin \alpha \theta & \cos \alpha \theta \\ a^2 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \end{vmatrix}$

$$= \begin{vmatrix} 1 & \sin(\alpha - \beta)\theta & \cos(\alpha - \beta)\theta \\ a & \sin\alpha\theta & \cos\alpha\theta \\ a^2 - 1 & 0 & 0 \end{vmatrix}$$

[applying $R_3 \rightarrow R_3 - R_1$]

$$= (a^2 - 1)[\sin(\alpha - \beta)\theta\cos\alpha\theta - \cos(\alpha - \beta)\theta\sin\alpha\theta]$$

$$= -(a^2 - 1)\sin\beta\theta, \text{ which is independent of } \alpha.$$

➤ **Example 35.** If $b^2 - ac < 0$ and $a > 0$, then the value of the determinant

$$\begin{vmatrix} a & b & ax + by \\ b & c & bx + cy \\ ax + by & bx + cy & 0 \end{vmatrix} \text{ is}$$

- (a) positive (b) negative
(c) zero (d) $b^2 + ac$

Sol. (b) We have,

$$\begin{vmatrix} a & b & ax + by \\ b & c & bx + cy \\ ax + by & bx + cy & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax + by & bx + cy & -(ax^2 + 2bxy + cy^2) \end{vmatrix}$$

[applying $C_3 \rightarrow C_3 - xC_1 - yC_2$]

$$= -(ax^2 + 2bxy + cy^2)(ac - b^2)$$

$$= -\frac{1}{a}(b^2 - ac)[(ax + by)^2 + y^2(ac - b^2)]$$

$$< 0 \quad [\because b^2 - ac < 0 \text{ and } a > 0]$$

➤ **Example 36.** If

$$U_n = \begin{vmatrix} 1 & k & k \\ 2n & k^2 + k + 1 & k^2 + k \\ 2n - 1 & k^2 & k^2 + k + 1 \end{vmatrix} \text{ and}$$

$$\sum_{n=1}^k U_n = 72, \text{ then } k \text{ is equal to}$$

- (a) 8
(b) 9
(c) 6
(d) None of the above

Sol. (a) We have,

$$\sum_{n=1}^k U_n = \begin{vmatrix} \sum_{n=1}^k 1 & k & k \\ 2 \sum_{n=1}^k n & k^2 + k + 1 & k^2 + k \\ 2 \sum_{n=1}^k n - \sum_{n=1}^k 1 & k^2 & k^2 + k + 1 \end{vmatrix}$$

$$= \begin{vmatrix} k & k & k \\ k(k+1) & k^2 + k + 1 & k^2 + k \\ k^2 & k^2 & k^2 + k + 1 \end{vmatrix}$$

$$= \begin{vmatrix} k & 0 & k \\ k^2 + k & 1 & k^2 + k \\ k^2 & 0 & k^2 + k + 1 \end{vmatrix} \quad [\text{applying } C_2 \rightarrow C_2 - C_1]$$

$$= k(k^2 + k + 1) - k^3 = k(k + 1) = 72 \quad [\text{given}]$$

$$\Rightarrow k = 8$$

➤ **Example 37.** If α, β and γ are the roots of the equation $x^3 + ax + b = 0$, then the value of the

determinant $\begin{vmatrix} \alpha - \beta - \gamma & 2\alpha & 2\alpha \\ 2\beta & \beta - \gamma - \alpha & 2\beta \\ 2\gamma & 2\gamma & \gamma - \alpha - \beta \end{vmatrix}$ is

- (a) 0
(b) a
(c) $-b$
(d) None of these

Sol. (a) We have,

$$\begin{vmatrix} \alpha - \beta - \gamma & 2\alpha & 2\alpha \\ 2\beta & \beta - \gamma - \alpha & 2\beta \\ 2\gamma & 2\gamma & \gamma - \alpha - \beta \end{vmatrix}$$

$$= \begin{vmatrix} \alpha + \beta + \gamma & \alpha + \beta + \gamma & \alpha + \beta + \gamma \\ 2\beta & \beta - \gamma - \alpha & 2\beta \\ 2\gamma & 2\gamma & \gamma - \alpha - \beta \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 + R_2 + R_3$]

$$= \begin{vmatrix} 0 & 0 & 0 \\ 2\beta & \beta - \gamma - \alpha & 2\beta \\ 2\gamma & 2\gamma & \gamma - \alpha - \beta \end{vmatrix} = 0$$

[since, α, β and γ are the roots of $x^3 + ax + b = 0 \therefore \alpha + \beta + \gamma = 0$]

Work Book Exercise 2.1

1 The value of determinant $\begin{vmatrix} 3 & 2 & 5 \\ 9 & -1 & 4 \\ 2 & 3 & -5 \end{vmatrix}$ is

- a 230
b 166
c 64
d None of the above

2 If every element of third order determinant of value Δ is multiplied by 5, then value of determinant is

- a Δ b 5Δ c 25Δ d 125Δ

3 The value of the determinant $\begin{vmatrix} x & x+1 \\ x-1 & x \end{vmatrix}$ is

- a 1 b -1
c 2 d 0

4 The area of the triangle formed by 3 collinear points is

- a one b two c zero d four

5 If the area of a ΔABD is 3 sq units with vertices $A(1, 3)$, $B(0, 0)$ and $D(k, 0)$, then k is equal to

- a 3 b ± 3
c 2 d ± 2

- 6 The equation of the line joining $A(1, 3)$ and $B(0, 0)$ is
 a $y = 3x$ b $y = -3x$ c $y = 2x$ d $y = -2x$

- 7 If a, b, c, d, e and f are in GP, then the value of

$$\begin{vmatrix} a^2 & d^2 & x \\ b^2 & e^2 & y \\ c^2 & f^2 & z \end{vmatrix} \text{ depends on}$$

- a x and y
 b x and z
 c y and z
 d independent of x, y and z

- 8 If $A = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ and $B = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$, then

- a $A + B = 1$ b $A = B$
 c $A + B = 0$ d $A = B + 1$

- 9 If $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = k \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$, then k equals

- a 1 b -1 c 2 d -2

- 10 If the value of the determinant $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$ is

positive, then

- a $abc > 1$ b $abc > -8$
 c $abc < -8$ d $abc > -2$

- 11 In a $\triangle ABC$, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 + \sin A & 1 + \sin B & 1 + \sin C \\ \sin A + \sin^2 B & \sin B + \sin^2 C & \sin C + \sin^2 A \end{vmatrix} = 0,$$

then $\triangle ABC$ is a/an

- a equilateral triangle b isosceles triangle
 c right angled triangle d None of these

- 12 The determinant

$$\Delta = \begin{vmatrix} a^2 + x^2 & ab & ac \\ ab & b^2 + x^2 & bc \\ ac & bc & c^2 + x^2 \end{vmatrix} \text{ is divisible by}$$

- a x^4 b x^7 c x^8 d x^6

- 13 If $a^{-1} + b^{-1} + c^{-1} = 0$ such that

$$\begin{vmatrix} a+1 & 1 & 1 \\ 1 & b+1 & 1 \\ 1 & 1 & c+1 \end{vmatrix} = \Delta, \text{ then the value of } \Delta \text{ is}$$

- a 0
 b abc
 c $-abc$
 d None of the above

- 14 If $x, y \in R$, then the determinant

$$\begin{vmatrix} \cos x & -\sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x+y) & -\sin(x+y) & 0 \end{vmatrix} \text{ lies in the interval}$$

- a $[-\sqrt{2}, \sqrt{2}]$ b $[-1, 1]$
 c $[-\sqrt{2}, 1]$ d $[-1, -\sqrt{2}]$

Minors and Cofactors of Elements of a Determinant

Minors

Let $A = [a_{ij}]$ be a square matrix of order n , then minor of a_{ij} is a determinant of a square matrix of order $(n-1)$ which is denoted by M_{ij} . M_{ij} is obtained on leaving i th row and j th column of A .

e.g. Minors of elements a_1, a_2 and a_3 of determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ are } \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} \text{ and } \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix},$$

respectively.

➤ **Example 38.** If $A = \begin{vmatrix} 1 & 2 & 3 \\ -3 & 2 & -1 \\ 2 & -4 & 3 \end{vmatrix}$, then find

the minors of elements of $|A|$.

Sol. $M_{11} = \begin{vmatrix} 2 & -1 \\ -4 & 3 \end{vmatrix} = 6 - 4 = 2$

$M_{12} = \begin{vmatrix} -3 & -1 \\ 2 & 3 \end{vmatrix} = -9 + 2 = -7$

$M_{13} = \begin{vmatrix} -3 & 2 \\ 2 & -4 \end{vmatrix} = 12 - 4 = 8$

$M_{21} = \begin{vmatrix} 2 & 3 \\ -4 & 3 \end{vmatrix} = 6 + 12 = 18$

$M_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3 - 6 = -3$

$M_{23} = \begin{vmatrix} 1 & 2 \\ 2 & -4 \end{vmatrix} = -4 - 4 = -8$

$M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = -2 - 6 = -8$

$M_{32} = \begin{vmatrix} 1 & 3 \\ -3 & -1 \end{vmatrix} = -1 + 9 = 8$

and $M_{33} = \begin{vmatrix} 1 & 2 \\ -3 & 2 \end{vmatrix} = 2 + 6 = 8$

➤ **Example 39.** The minor of element 6 in the

determinant $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ is

- (a) -6 (b) 6
 (c) 22 (d) -22

Sol. (a) Since, 6 lies in the second row and third column.

∴ Its minor is $M_{23} = \begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 8 - 14 = -6$

Cofactors

The cofactor C_{ij} of a_{ij} in Δ is equal to $(-1)^{i+j}$ times the determinant of order obtained by leaving i th row and j th column of Δ .

It follows that, cofactor of an element a_{ij} of determinant

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ is } C_{ij} = (-1)^{i+j} M_{ij}, \text{ where } M_{ij}$$

is a minor of a_{ij} .

e.g. (i) Cofactor of

$$a_{12} = C_{12} = (-1)^{1+2} M_{12} = - \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$(ii) \text{ If } \Delta = \begin{vmatrix} 4 & -7 \\ -3 & 2 \end{vmatrix}, \text{ then cofactors are}$$

$$C_{11} = (-1)^{1+1} M_{11} = M_{11} = 2$$

$$C_{12} = (-1)^{1+2} M_{12} = -M_{12} = -(-3) = 3$$

$$C_{21} = (-1)^{2+1} M_{21} = -M_{21} = 7$$

$$\text{and } C_{22} = (-1)^{2+2} M_{22} = M_{22} = 4$$

- Determinant of any square matrix is equal to the sum of the product of any row (or column) with their corresponding cofactors,

$$\text{i.e. } \Delta = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

$$\text{or } \Delta = a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$$

$$\text{or } \Delta = a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$$

- If elements of a row (or column) are multiplied with cofactors of any other row (or column), then their sum is zero.

➤ **Example 40.** If $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$, then cofactor

of a_{21} is

$$(a) -(hc + fg)$$

$$(b) fg - hc$$

$$(c) fg + hc$$

$$(d) hc - fg$$

Sol. (b) We have, $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$\text{Now, cofactor of } a_{21} = (-1)^{2+1} \begin{vmatrix} h & g \\ f & c \end{vmatrix} = -(hc - fg) = fg - hc$$

Adjoint of a Square Matrix

Let $A = [a_{ij}]$ be a square matrix of order n and let C_{ij} be the cofactor of a_{ij} in the determinant $|A|$. Then, the adjoint of A , denoted by $\text{adj } A$, is defined as the transpose of the cofactor matrix.

The adjoint of a square matrix A is obtained on replacing each (i, j) th element of A by the cofactor of the (j, i) th element in $|A|$.

$$\text{e.g. If } A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 1 \\ 4 & 3 & 2 \end{bmatrix}, \text{ then we have}$$

$$C_{11} = -3, \quad C_{12} = 6, \quad C_{13} = -3$$

$$C_{21} = 5, \quad C_{22} = -10, \quad C_{23} = 5$$

$$C_{31} = 2, \quad C_{32} = -4, \quad C_{33} = 2$$

$$\text{Thus, } \text{adj}(A) = \begin{bmatrix} -3 & 5 & 2 \\ 6 & -10 & -4 \\ -3 & 5 & 2 \end{bmatrix}$$

➤ Adjoint of a square matrix of order 2 can be easily obtained by interchanging the diagonal elements and changing the signs of the non-diagonal elements.

Properties of Adjoint of a Square Matrix

If A, B are square matrices of order n and I_n is corresponding unit matrix, then

- $(\text{adj } A)A = A(\text{adj } A) = |A| \cdot I_n$
- $|\text{adj } A| = |A|^{n-1}$, if $|A| \neq 0$
- $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- If $|A| = 0$, then $(\text{adj } A)A = A(\text{adj } A) = O$
- $\text{adj}(\text{adj } A) = |A|^{n-2} A$, where A is a non-singular matrix
- $\text{adj}(A^T) = (\text{adj } A)^T$
- Adjoint of a diagonal matrix is a diagonal matrix.
- $|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$
- $\text{adj}(A^m) = (\text{adj } A)^m$, $m \in \mathbb{N}$
- $\text{adj}(kA) = k^{n-1}(\text{adj } A)$, $k \in \mathbb{R}$
- $\text{adj}(I_n) = I_n$
- $\text{adj } O = O$
- Adjoint of symmetric matrix is a symmetric matrix.
- A is singular $\Rightarrow |\text{adj } A| = 0$
- Adjoint of triangular matrix is a triangular matrix.

➤ **Example 41.** The adjoint of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix} \text{ is}$$

$$(a) \begin{bmatrix} 6 & 15 & -15 \\ -3 & 0 & 0 \\ 0 & -16 & 5 \end{bmatrix} \quad (b) \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -16 & 0 & 5 \end{bmatrix}$$

$$(c) \begin{bmatrix} -15 & 6 & 15 \\ 0 & -3 & 0 \\ 5 & 0 & -16 \end{bmatrix} \quad (d) \text{ None of these}$$

Sol. (d) If C is the matrix of cofactors of the element in $|A|$.

$$\text{Then, } C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$\therefore C_{11} = \begin{vmatrix} 5 & 0 \\ 4 & 3 \end{vmatrix} = 15, C_{12} = -\begin{vmatrix} 0 & 0 \\ 2 & 3 \end{vmatrix} = 0,$$

$$C_{13} = \begin{vmatrix} 0 & 5 \\ 2 & 4 \end{vmatrix} = -10,$$

$$C_{21} = -\begin{vmatrix} 2 & 3 \\ 4 & 3 \end{vmatrix} = 6, C_{22} = \begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = -3$$

$$C_{23} = -\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = 0$$

$$C_{31} = \begin{vmatrix} 2 & 3 \\ 5 & 0 \end{vmatrix} = -15, C_{32} = -\begin{vmatrix} 1 & 3 \\ 0 & 0 \end{vmatrix} = 0$$

$$C_{33} = \begin{vmatrix} 1 & 2 \\ 0 & 5 \end{vmatrix} = 5$$

$$\therefore C = \begin{bmatrix} 15 & 0 & -10 \\ 6 & -3 & 0 \\ -15 & 0 & 5 \end{bmatrix}$$

$$\therefore \text{adj}(A) = \text{Transpose of } C = C' = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

➤ **Example 42.** If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, then

$\text{adj}(\text{adj } A)$ is equal to

$$(a) A \quad (b) I$$

$$(c) O \quad (d) \text{ None of these}$$

Sol. (a) We know that,

$$\text{adj}(\text{adj } A) = |A|^{n-2} A, \text{ if } |A| \neq 0$$

$$\text{Since, } A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

Here, $n = 3$

$$|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$$

$$= 3(-3 + 4) + 3(2 - 0) + 4(-2 - 0) = 1 \neq 0$$

$\therefore A$ is non-singular and hence

$$\text{adj}(\text{adj } A) = |A|^{3-2} \cdot A = A$$

➤ **Example 43.** If $A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, then

$\det(\text{adj}(\text{adj } A))$ is

$$(a) (14)^4 \quad (b) (14)^3 \quad (c) (14)^2 \quad (d) (14)^1$$

Sol. (a) We know that, A is non-singular matrix of

$$\text{adj}(\text{adj } A) = |A|^{n-2} A.$$

$$\therefore \text{adj}(\text{adj } A) = |A| A \quad [\text{as } n = 3]$$

$$\det(\text{adj}(\text{adj } A)) = |A|^3 \det A = |A|^4$$

$$\text{But } |A| = \begin{vmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{vmatrix} = 14$$

$$\therefore \det(\text{adj}(\text{adj } A)) = (14)^4$$

Inverse of a Square Matrix

Let A be any n -rowed square matrix. Then, a square matrix B , such that $AB = BA = I$, is called inverse of A . The inverse of A is denoted by A^{-1} and is determined by the formula

$$A^{-1} = \frac{1}{|A|} (\text{adj } A)$$

It may be noted that,

$$AA^{-1} = A^{-1}A = I$$

Properties of the Inverse of a Matrix

i. A square matrix is invertible if and only if it is non-singular.

ii. The inverse of the inverse is the original matrix itself, i.e. $(A^{-1})^{-1} = A$.

iii. The inverse of the transpose of a matrix is the transpose of its inverse, i.e. $(A')^{-1} = (A^{-1})'$.

iv. If A and B are two invertible matrices of the same order, then AB is also invertible and moreover

$$(AB)^{-1} = B^{-1}A^{-1}$$

v. Let A, B and C be square matrices of the same order n . If A is a non-singular matrix, then

$$(a) AB = AC \Rightarrow B = C \text{ [left cancellation law]}$$

$$(b) BA = CA \Rightarrow B = C \text{ [right cancellation law]}$$

➤ These cancellation laws hold only, if the matrix A is non-singular.

vi. If A is a non-singular matrix such that A is symmetric, then A^{-1} is also symmetric.

vii. If A is a non-singular matrix, then $|A^{-1}| = |A|^{-1}$

➤ **Example 44.** If $A = \begin{bmatrix} 1 & 3 \\ 2 & -2 \end{bmatrix}$, then A^{-1}

equals

(a) $-\frac{1}{8} \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$

(b) $-\frac{1}{8} \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$

(c) $\frac{1}{8} \begin{bmatrix} -1 & -3 \\ -2 & 2 \end{bmatrix}$

(d) None of the above

Sol. (a) We have, $\text{adj}(A) = \begin{bmatrix} -2 & -2 \\ -3 & 1 \end{bmatrix}^T = \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$

Also, $|A| = \begin{vmatrix} 1 & 3 \\ 2 & -2 \end{vmatrix} = -8$

$\therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = -\frac{1}{8} \begin{bmatrix} -2 & -3 \\ -2 & 1 \end{bmatrix}$

➤ **Example 45.** For any 2×2 matrix A , if

$A(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A|$ is equal to

(a) 20 (b) 100 (c) 10 (d) 0

Sol. (c) Since, $A(\text{adj } A) = |A|I$

$\therefore \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = |A| \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} |A| & 0 \\ 0 & |A| \end{bmatrix}$

$\Rightarrow |A| = 10$

➤ **Example 46.** Inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$ is

(a) $\begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & 2 \\ 1 & -2 & 1 \end{bmatrix}$

(b) $\begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

(d) None of these

Sol. (d) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{bmatrix}$

Then, $\text{adj}(A) = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}^T = \begin{bmatrix} 2 & 0 & -1 \\ 0 & -3 & 2 \\ -1 & 2 & -1 \end{bmatrix}$

Also, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 6 \end{vmatrix} = 1(2) - 2(0) + 3(-1) = -1$

$\therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} -2 & 0 & 1 \\ 0 & 3 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

➤ **Example 47.** If $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,

then the matrix A is equal to

(a) $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Sol. (c) We have, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$

$A = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & -3 \end{bmatrix}^{-1}$

$= \frac{1}{1} \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \frac{1}{-1} \begin{bmatrix} -3 & -2 \\ -5 & -3 \end{bmatrix}$

$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 5 & 3 \end{bmatrix}$

$= \begin{bmatrix} 6-5 & 4-3 \\ -9+10 & -6+6 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

Applications of Determinants and Matrices

In this section, we shall discuss the applications of determinants and matrices for solving the system of linear equations in two or three variables and checking the consistency or inconsistency of system of linear equations.

- Consistent system** A system of linear equations is said to be consistent, if one or more than one solution exists.
- Inconsistent system** A system of linear equations is said to be inconsistent, if its solution does not exist.

Solution of a System of Linear Equations by Matrix Method

Consider a system of linear equations

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

$$a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n = b_n$$

We can express these equations as a single matrix equation

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

i.e. $AX = B$

Let $|A| \neq 0$, so that A^{-1} exists uniquely.

Pre-multiplying both sides of $AX = B$ by A^{-1} , we get

$$A^{-1}(AX) = A^{-1}B \text{ or } (A^{-1}A)X = A^{-1}B$$

$$\text{or } IX = A^{-1}B \text{ or } X = A^{-1}B$$

Hence, $X = A^{-1}B$ is the unique solution of $AX = B$, provided $|A| \neq 0$.

Criterion of Consistency

Let $AX = B$ be a system of n linear equations in n variables.

- If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$.
- If $|A| = 0$ and $(\text{adj } A)B \neq O$, then the system of equations is inconsistent, i.e. it has no solution.
- If $|A| = 0$ and $(\text{adj } A)B = O$, then the system of equations may be either consistent or inconsistent.

✎ In case (iii), to decide whether the system is consistent or inconsistent, take any two equations out of three given equations and shift one of the variables, say z , on the right hand side to obtain two equations in x, y . Solve these two equations to obtain x, y in terms of z . If these values of x and y satisfy the third equation, then the system is consistent and the values of x, y and z constitute a solution. If the values of x and y do not satisfy the third equation, then the system is inconsistent.

Homogeneous System of Equations

The system of equations $AX = B$ is said to be homogeneous, if the constants $b_1, b_2, \dots, b_n$ are all zero, i.e. if the matrix B is a zero matrix and the system is of the form $AX = O$, where O is a null matrix of order $n \times 1$.

- If $|A| \neq 0$, then its only solution $X = O$ is called the trivial solution.
- If $|A| = 0$, then $AX = O$ will have infinite solutions, i.e. it has a non-trivial solution.

✎ **Example 48.** The equations $x + 2y + 3z = 1$, $x - y + 4z = 0$ and $2x + y + 7z = 1$ have

- only two solutions
- only one solution
- no solution
- infinitely many solutions

Sol. (d) We have, $\begin{vmatrix} 1 & 2 & 3 \\ 1 & -1 & 4 \\ 2 & 1 & 7 \end{vmatrix} = 1(-11) - 2(-1) + 3(3) = 0$

$$\text{adj } (A) = \begin{bmatrix} -11 & 1 & 3 \\ -11 & 1 & 3 \\ 11 & -1 & -3 \end{bmatrix}' = \begin{bmatrix} -11 & -11 & 11 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix}$$

$$\therefore (\text{adj } A)B = \begin{bmatrix} -11 & -11 & 11 \\ 1 & 1 & -1 \\ 3 & 3 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = O$$

Now, nothing can be said about consistency as yet. Let us consider the first two equations

$$x + 2y = 1 - 3z \quad \dots (i)$$

$$x - y = -4z \quad \dots (ii)$$

On putting the value of x in Eq. (i), we get

$$3y = 1 + z \Rightarrow y = \frac{1+z}{3}$$

On putting the value of y in Eq. (ii), we get

$$x = -4z + \frac{1+z}{3} \\ = \frac{-12z + 1 + z}{3} = \frac{-11z + 1}{3}$$

On substituting the values of x and y in LHS of Eq. (iii), we get

$$2\left(\frac{-11z+1}{3}\right) + \left(\frac{1+z}{3}\right) + 7z = \frac{-22z+2+1+z+21z}{3} \\ = 1$$

Hence, the given system is consistent and has infinitely many solutions.

✎ **Example 49.** The system of linear equations $3x - y - 2z = 2$, $2y - z = -1$ and $3x - 5y = 3$ has

- a unique solution
- infinitely many solutions
- no solution
- None of the above

Sol. (c) The given system is $3x - y - 2z = 2$, $0x + 2y - z = -1$ and $3x - 5y + 0z = 3$, which can be written as $AX = B$, where

$$A = \begin{bmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

$$\text{Here, } |A| = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{vmatrix} \\ = 3(0-5) + 1(0+3) - 2(0-6) \\ = -15 + 3 + 12 = 0$$

$\therefore A$ is a singular matrix.

Therefore, nothing can be said about consistency as yet. So, we compute $(\text{adj } A)B$.

Cofactors of elements of $|A|$ are

$$A_{11} = -5, A_{12} = -3, A_{13} = -6, A_{21} = 10, A_{22} = 6, A_{23} = 12, A_{31} = 5, A_{32} = 3, A_{33} = 6$$

$$\text{adj } (A) = \begin{bmatrix} -5 & -3 & -6 \\ 10 & 6 & 12 \\ 5 & 3 & 6 \end{bmatrix}' = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix}$$

$$\therefore (\text{adj } A)B = \begin{bmatrix} -5 & 10 & 5 \\ -3 & 6 & 3 \\ -6 & 12 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} \\ = \begin{bmatrix} -10-10+15 \\ -6-6+9 \\ -12-12+18 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \\ -6 \end{bmatrix} \\ \neq 0$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent, i.e. it has no solution.

Solution of a System of Linear Equations by Determinants (Cramer's Rule)

1. Cramer's Rule : Solution of System of Linear Equations in Two Unknowns

The solution of the system of equations

$$a_1x + b_1y = c_1, \quad a_2x + b_2y = c_2$$

is given by $x = \frac{D_1}{D}$ and $y = \frac{D_2}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \quad D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\text{and } D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}, \text{ provided } D \neq 0.$$

2. Cramer's Rule : Solution of System of Linear Equations in Three Unknowns

The solution of the system of equations

$$a_1x + b_1y + c_1z = d_1$$

$$a_2x + b_2y + c_2z = d_2$$

$$a_3x + b_3y + c_3z = d_3$$

is given by $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$, where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}; \quad D_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}; \quad D_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix},$$

provided $D \neq 0$.

Conditions for Consistency

The following cases may arise:

- If $D \neq 0$, then the system is consistent and has a unique solution, which is given by Cramer's rule: $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$ and $z = \frac{D_3}{D}$
- If $D = 0$ and at least one of the determinants D_1 , D_2 and D_3 is non-zero, then the given system is inconsistent, i.e. it has no solution.
- If $D = 0$ and $D_1 = D_2 = D_3 = 0$, then the system may or may not be consistent. However, if it is consistent, then it has infinitely many solutions.

➤ In case (iii), to decide whether the system is consistent or inconsistent, take any two equations out of three given equations and shift one of the variables, say z , on the right hand side to obtain two equations in x, y . Solve these two equations to obtain x, y in terms of z . If these values of x and y satisfy the third equation, then the system is consistent and the values of x, y and z constitute a solution. If the values of x and y do not satisfy the third equation, then the system is inconsistent.

Homogeneous and Non-homogeneous Systems

If $d_1 = d_2 = d_3 = 0$, then the system is said to be homogeneous, otherwise it is called non-homogeneous.

If the system of equations is homogeneous, then $D_1 = D_2 = D_3 = 0$ (value of the determinant is zero, if one column has all elements = 0). Thus,

- if $D \neq 0$, then the system has only trivial solution ($x = y = z = 0$)
- if $D = 0$, then the system has infinitely many solutions.

Condition for Consistency of Three Linear Equations in two Unknowns

System of three linear equations in x and y

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$$a_3x + b_3y + c_3 = 0$$

will be consistent, if the values of x and y obtained from any two equations satisfy the third equation.

$$\text{or } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

➤ **Example 50.** The following system of linear equations $x - 3y + 5z = 4$

$$2x - 6y + 10z = 11$$

$$\text{and } 3x - 9y + 15z = 12$$

(a) has unique solution

(b) no solution

(c) infinite number of solutions

(d) None of the above

Sol. (b) We have,

$$D = \begin{vmatrix} 1 & -3 & 5 \\ 2 & -6 & 10 \\ 3 & -9 & 15 \end{vmatrix} = 0 \quad [\because C_2 \text{ is proportional to } C_1]$$

$$D_1 = \begin{vmatrix} 4 & -3 & 5 \\ 11 & -6 & 10 \\ 12 & -9 & 15 \end{vmatrix} = -15 \begin{vmatrix} 4 & 1 & 1 \\ 11 & 2 & 2 \\ 12 & 3 & 3 \end{vmatrix} = 0$$

$$D_2 = \begin{vmatrix} 1 & 4 & 5 \\ 2 & 11 & 10 \\ 3 & 12 & 15 \end{vmatrix} = 5 \begin{vmatrix} 1 & 4 & 1 \\ 2 & 11 & 2 \\ 3 & 12 & 3 \end{vmatrix} = 0 \quad [\because C_2 \text{ and } C_3 \text{ are identical}]$$

$$\text{and } D_3 = \begin{vmatrix} 1 & -3 & 4 \\ 2 & -6 & 11 \\ 3 & -9 & 12 \end{vmatrix} = -3 \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 11 \\ 3 & 3 & 12 \end{vmatrix} = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$[\because C_1 \text{ and } C_2 \text{ are identical}]$

$$\therefore D = D_1 = D_2 = D_3 = 0$$

So, the given system of equations may or may not be consistent.

If we now put $z = k$ in any two of three equations, we find that the two equations obtained are inconsistent as they represent a pair of parallel lines.

Hence, the given system of equations is inconsistent.

➤ **Example 51.** The system of linear equations $x + y + z = 2$, $2x + y - z = 3$ and $3x + 2y + kz = 4$ has a unique solution, if

- (a) $k \neq 0$ (b) $-1 < k < 1$
(c) $-2 < k < 2$ (d) $k = 0$

Sol. (a) The given system of equations has unique solution, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & k \end{vmatrix} \neq 0 \Rightarrow k \neq 0$$

➤ **Example 52.** The values of λ for which the equations $x + y - 3 = 0$, $(1 + \lambda)x + (2 + \lambda)y - 8 = 0$ and $x - (1 + \lambda)y + (2 + \lambda) = 0$ are consistent, are

- (a) $1, -\frac{5}{3}$ (b) $\frac{5}{3}$
(c) $-\frac{2}{3}$ (d) None of these

Sol. (a) Here, the equations are in two variables x and y . If they are consistent, then the values of x and y obtained from first two equations should satisfy the third equation and hence $D = 0$,

$$\text{i.e. } \begin{vmatrix} 1 & 1 & -3 \\ 1 + \lambda & 2 + \lambda & -8 \\ 1 & -1 - \lambda & 2 + \lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 0 & 0 \\ 1 + \lambda & 1 & -5 + 3\lambda \\ 1 & -2 - \lambda & 5 + \lambda \end{vmatrix} = 0$$

[applying $C_2 \rightarrow C_2 - C_1$, $C_3 \rightarrow C_3 + 3C_1$]

$$\Rightarrow (5 + \lambda) + (2 + \lambda)(-5 + 3\lambda) = 0$$

$$\Rightarrow 3\lambda^2 + 2\lambda - 5 = 0$$

$$\Rightarrow (\lambda - 1)(3\lambda + 5) = 0$$

$$\Rightarrow \lambda = 1, -\frac{5}{3}$$

➤ **Example 53.** Let α_1, α_2 and β_1, β_2 be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$, respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has a non-trivial solution, then

- (a) $\frac{b^2}{q^2} = \frac{ac}{pr}$ (b) $\frac{c^2}{r^2} = \frac{ab}{pq}$
(c) $\frac{a^2}{p^2} = \frac{bc}{qr}$ (d) None of these

Sol. (a) Since, α_1, α_2 and β_1, β_2 are the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ respectively, therefore

$$\alpha_1 + \alpha_2 = -\frac{b}{a}, \alpha_1 \alpha_2 = \frac{c}{a}$$

$$\text{and } \beta_1 + \beta_2 = -\frac{q}{p}, \beta_1 \beta_2 = \frac{r}{p}$$

Since, the given system of equations has a non-trivial solution.

$$\therefore \begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix} = 0, \text{ i.e. } \alpha_1 \beta_2 - \alpha_2 \beta_1 = 0$$

$$\text{or } \frac{\alpha_1}{\beta_1} = \frac{\alpha_2}{\beta_2} = \frac{\alpha_1 + \alpha_2}{\beta_1 + \beta_2} = \frac{\alpha_1 \alpha_2}{\beta_1 \beta_2}$$

$$\Rightarrow \frac{pb}{qa} = \sqrt{\frac{pc}{ra}} \Rightarrow \frac{b^2}{q^2} = \frac{ac}{pr}$$

Rank of a Matrix

If A is a matrix of order $m \times n$ and if we retain any k rows and k columns of A , then we have a square sub-matrix of order k . The determinant of the square sub-matrix of order k is called minor of A of order k .

A positive integer r is said to be the rank of a non-zero matrix A , if

- there exists atleast one minor in A of order r , which is not zero.
- every minor in A of order greater than r is zero. It is written as $\rho(A) = r$.

The rank of a zero matrix is defined to be zero.

e.g. Rank of $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$ is 2 because the minor

$$\begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \neq 0 \text{ and rank of } \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ is 3 as the}$$

$$\text{minor } \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

Properties of Rank of a Matrix

- The rank of a null matrix is defined as zero, i.e. $\rho(O) = 0$.
- If I_n is a unit matrix of order n , then its rank is n , i.e. $\rho(I_n) = n$.
- From the definition of the rank of a matrix, we concluded that
 - If a matrix A does not possess any minor of order r , then $\rho(A) < r$.
 - If atleast one minor of order r of the matrix is not equal to zero, then $\rho(A) \geq r$.

- iv. If every $(r+1)$ th order minor of A is zero, then any higher order minor will also be zero.
- v. If A is $n \times n$ non-singular matrix, then $\rho(A) = n$.
- vi. Elementary operations do not change the rank of a matrix.
- vii. If A' is a transpose of A , then $\rho(A') = \rho(A)$.
- viii. If A^* is the transpose conjugate of A , then $\rho(A^*) = \rho(A)$.
- ix. $\rho(A+B) \leq \rho(A) + \rho(B)$.
- x. If A and B are two matrices such that the product AB is defined, then rank (AB) cannot exceed the rank of the either matrix.

☞ **Example 54.** If A is a non-zero column matrix of order $m \times 1$ and B is a non-zero row matrix of order $1 \times n$, then rank of AB is equal to

- (a) n (b) m (c) 1 (d) None

Sol. (c) Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$ and $B = [b_{11} \ b_{12} \ b_{13} \ \dots \ b_{1n}]$ be

two non-zero column and row matrices, respectively.

$$\therefore AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \dots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & \dots & a_{21}b_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \dots & a_{m1}b_{1n} \end{bmatrix}$$

Since, A and B are non-zero matrices.

$\therefore$ Matrix AB will be a non-zero matrix. The matrix AB will have atleast one non-zero element obtained by multiplying corresponding non-zero elements of A and B . All the two rowed minors of AB clearly vanish.
 $\therefore$ Rank of $AB = 1$

☞ **Example 55.** If the matrix

$$A = \begin{bmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{bmatrix} \text{ has rank 3, then}$$

- (a) $y \neq (a+b+c)$
(b) $y \neq 1$
(c) $y = 0$
(d) $y \neq -(a+b+c)$ and $y \neq 0$

Sol. (d) Here, the rank of A is 3.

Therefore, the minor of order 3 of $A \neq 0$

$$\Rightarrow \begin{vmatrix} y+a & b & c \\ a & y+b & c \\ a & b & y+c \end{vmatrix} \neq 0$$

$$\Rightarrow (y+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & y+b & c \\ 1 & b & y+c \end{vmatrix} \neq 0$$

[applying $C_1 \rightarrow C_1 + C_2 + C_3$ and taking $(y+a+b+c)$ common from C_1]

$$\Rightarrow (y+a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & y & 0 \\ 0 & 0 & y \end{vmatrix} \neq 0$$

[applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$]

$$\Rightarrow (y+a+b+c)(y^2) \neq 0 \quad [\text{expanding along } C_1]$$

$$\Rightarrow y \neq 0 \text{ and } y \neq -(a+b+c).$$

Solution of a System of Linear Equations by Rank Method

Let $AX = B$ be a system of n linear equations in n variables.

1. Write the augmented matrix $[A : B]$.
2. Reduce the augmented matrix to echelon form using elementary row operations.
3. Determine the rank of coefficient of matrix A and augmented matrix $[A : B]$ by counting the number of non-zero rows in A and $[A : B]$.

- ☞
- If $\rho(A) < \rho([A : B])$, then the system of equations is inconsistent.
 - If $\rho(A) = \rho([A : B]) = \text{number of unknowns}$, then the system of equations is consistent and has a unique solution.
 - If $\rho(A) = \rho([A : B]) < \text{the number of unknowns}$, then the system of equations is consistent and has infinitely many solutions.
 - Let $AX = O$ be a homogeneous system of linear equations.
 - > If $\rho(A) = \text{number of variables}$, then $AX = O$ has a trivial solution, i.e. $X = O$.
 - > If $\rho(A) < \text{number of variables}$, then $AX = O$ has a non-trivial solution. It will have infinitely many solutions.

☞ **Example 56.** The system of equations

$$x + y + z = 8, \quad x - y + 2z = 6 \text{ and } 3x + 5y - 7z = 14$$

has

- (a) a unique solution
(b) infinite number of solutions
(c) no solution
(d) None of the above

Sol. (a) The augmented matrix $C = [A : B]$

$$\begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 1 & -1 & 2 & : & 6 \\ 3 & 5 & -7 & : & 14 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 2 & -10 & : & -10 \end{bmatrix}$$

[$\therefore R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - 3R_1$]

$$= \begin{bmatrix} 1 & 1 & 1 & : & 8 \\ 0 & -2 & 1 & : & -2 \\ 0 & 0 & -9 & : & -12 \end{bmatrix} \quad [\therefore R_3 \rightarrow R_3 + R_2]$$

Rank of $A = 3$ and rank of $C = 3$

So, rank of $A = \text{rank of } C = 3 = \text{number of unknowns}$
Hence, the equations are consistent with unique solution.

Product of Determinants of Same Order

$$\text{Let } \Delta_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Then, product of Δ_1 and Δ_2 is given by multiplying row-by-row, i.e.

$$\Delta_1 \cdot \Delta_2 = \begin{vmatrix} a_1\alpha_1 + b_1\beta_1 + c_1\gamma_1 & a_1\alpha_2 + b_1\beta_2 + c_1\gamma_2 & a_1\alpha_3 + b_1\beta_3 + c_1\gamma_3 \\ a_2\alpha_1 + b_2\beta_1 + c_2\gamma_1 & a_2\alpha_2 + b_2\beta_2 + c_2\gamma_2 & a_2\alpha_3 + b_2\beta_3 + c_2\gamma_3 \\ a_3\alpha_1 + b_3\beta_1 + c_3\gamma_1 & a_3\alpha_2 + b_3\beta_2 + c_3\gamma_2 & a_3\alpha_3 + b_3\beta_3 + c_3\gamma_3 \end{vmatrix}$$

Multiplication of determinant can also be performed by row by column, column by row or column by column as required in the problem.

➤ **Example 57.** If $\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$

$= (a^3 + b^3 + c^3 + kabc)^2$, then k is equal to

- (a) 2 (b) -2
(c) 3 (d) -3

Sol. (d) We have,

$$\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ca - b^2 & a^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= [a(bc - a^2) + b(ac - b^2) + c(ab - c^2)]^2$$

$$= [a^3 + b^3 + c^3 - 3abc]^2$$

∴ $k = -3$

➤ **Example 58.** For all values of A, B, C and P, Q, R , the value of the determinant

$$\begin{vmatrix} \cos(A - P) & \cos(A - Q) & \cos(A - R) \\ \cos(B - P) & \cos(B - Q) & \cos(B - R) \\ \cos(C - P) & \cos(C - Q) & \cos(C - R) \end{vmatrix} \text{ is}$$

- (a) 1
(b) 0
(c) 2
(d) None of the above

Sol. (b) We have,

$$\begin{vmatrix} \cos(A - P) & \cos(A - Q) & \cos(A - R) \\ \cos(B - P) & \cos(B - Q) & \cos(B - R) \\ \cos(C - P) & \cos(C - Q) & \cos(C - R) \end{vmatrix} = \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$$= 0 \cdot 0 = 0$$

➤ **Example 59.** The value of the determinant

$$\begin{vmatrix} 1 + \alpha & 1 + \alpha z & 1 + \alpha z^2 \\ 1 + \beta & 1 + \beta z & 1 + \beta z^2 \\ 1 + \gamma & 1 + \gamma z & 1 + \gamma z^2 \end{vmatrix} \text{ is}$$

- (a) $(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha)$
(b) 0
(c) $\alpha\beta\gamma$
(d) None of the above

Sol. (b)

$$\begin{vmatrix} 1 + \alpha & 1 + \alpha z & 1 + \alpha z^2 \\ 1 + \beta & 1 + \beta z & 1 + \beta z^2 \\ 1 + \gamma & 1 + \gamma z & 1 + \gamma z^2 \end{vmatrix} = \begin{vmatrix} 1 & \alpha & 0 \\ 1 & \beta & 0 \\ 1 & \gamma & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ 1 & z & 0 \\ 1 & z^2 & 0 \end{vmatrix} = 0$$

➤ **Example 60.** The value of the determinant

$$\begin{vmatrix} (a_1 - b_1)^2 & (a_1 - b_2)^2 & (a_1 - b_3)^2 & (a_1 - b_4)^2 \\ (a_2 - b_1)^2 & (a_2 - b_2)^2 & (a_2 - b_3)^2 & (a_2 - b_4)^2 \\ (a_3 - b_1)^2 & (a_3 - b_2)^2 & (a_3 - b_3)^2 & (a_3 - b_4)^2 \\ (a_4 - b_1)^2 & (a_4 - b_2)^2 & (a_4 - b_3)^2 & (a_4 - b_4)^2 \end{vmatrix}$$

is

- (a) dependent on $a_i, i = 1, 2, 3, 4$
(b) dependent on $b_i, i = 1, 2, 3, 4$
(c) dependent on $a_i, b_i, i = 1, 2, 3, 4$
(d) 0

Sol. (d) We have,

$$\begin{vmatrix} a_1^2 - 2a_1b_1 + b_1^2 & a_1^2 - 2a_1b_2 + b_2^2 & a_1^2 - 2a_1b_3 + b_3^2 & a_1^2 - 2a_1b_4 + b_4^2 \\ a_2^2 - 2a_2b_1 + b_1^2 & a_2^2 - 2a_2b_2 + b_2^2 & a_2^2 - 2a_2b_3 + b_3^2 & a_2^2 - 2a_2b_4 + b_4^2 \\ a_3^2 - 2a_3b_1 + b_1^2 & a_3^2 - 2a_3b_2 + b_2^2 & a_3^2 - 2a_3b_3 + b_3^2 & a_3^2 - 2a_3b_4 + b_4^2 \\ a_4^2 - 2a_4b_1 + b_1^2 & a_4^2 - 2a_4b_2 + b_2^2 & a_4^2 - 2a_4b_3 + b_3^2 & a_4^2 - 2a_4b_4 + b_4^2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 & -2a_1 & 1 & 0 \\ a_2^2 & -2a_2 & 1 & 0 \\ a_3^2 & -2a_3 & 1 & 0 \\ a_4^2 & -2a_4 & 1 & 0 \end{vmatrix} \times \begin{vmatrix} 1 & b_1 & b_1^2 & 0 \\ 1 & b_2 & b_2^2 & 0 \\ 1 & b_3 & b_3^2 & 0 \\ 1 & b_4 & b_4^2 & 0 \end{vmatrix}$$

$$= 0 \times 0 = 0$$

$$\begin{aligned} & \bullet \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)0 \\ & \bullet \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c) \\ & \bullet \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)[(a^2+b^2+c^2) + (ab+bc+ca)] \\ & \bullet \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca) \\ & \bullet \begin{vmatrix} x+a & b & c & d \\ a & x+b & c & d \\ a & b & x+c & d \\ a & b & c & x+d \end{vmatrix} = x^3(x+a+b+c+d) \\ & \bullet \begin{vmatrix} x^2 & (x+a)^2 & (x-a)^2 \\ y^2 & (y+a)^2 & (y-a)^2 \\ z^2 & (z+a)^2 & (z-a)^2 \end{vmatrix} = -4a^3(x-y)(y-z)(z-x) \\ & \bullet \begin{vmatrix} (y+z)^2 & xy & zx \\ xy & (x+z)^2 & yz \\ xz & yz & (x+y)^2 \end{vmatrix} = 2xyz(x+y+z)^3 \end{aligned}$$

Differentiation of Determinants

Let $\Delta(x) = \begin{vmatrix} f_1(x) & g_1(x) \\ f_2(x) & g_2(x) \end{vmatrix}$, where

$f_1(x), f_2(x), g_1(x)$ and $g_2(x)$ are functions of x , then

$$\Delta'(x) = \begin{vmatrix} f_1'(x) & g_1'(x) \\ f_2(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1(x) \\ f_2'(x) & g_2'(x) \end{vmatrix}$$

$$\text{Also, } \Delta'(x) = \begin{vmatrix} f_1'(x) & g_1(x) \\ f_2'(x) & g_2(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & g_1'(x) \\ f_2(x) & g_2'(x) \end{vmatrix}$$

Thus, to differentiate a determinant, we differentiate one row (or column) at a time, keeping others unchanged.

➤ **Example 61.** If $\Delta_1 = \begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and

$$\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix} \text{ are the given determinants, then}$$

$$(a) \Delta_1 = 3(\Delta_2)^2 \quad (b) \frac{d}{dx} \Delta_1 = 3\Delta_2$$

$$(c) \frac{d}{dx} \Delta_1 = 3\Delta_2^2 \quad (d) \Delta_1 = 3(\Delta_2)^{\frac{3}{2}}$$

Sol. (b) $\frac{d}{dx} \Delta_1 = \begin{vmatrix} 1 & 0 & 0 \\ a & x & b \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ 0 & 1 & 0 \\ a & a & x \end{vmatrix} + \begin{vmatrix} x & b & b \\ a & x & b \\ 0 & 0 & 1 \end{vmatrix}$
 $= \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} = 3\Delta_2$

➤ **Example 62.** If $f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2 \sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$,

then $\lim_{x \rightarrow 0} \left[\frac{f'(x)}{x} \right]$ is

(a) 2 (b) -2 (c) 1 (d) -1

Sol. (b) We have,

$$f(x) = \begin{vmatrix} \cos x & x & 1 \\ 2 \sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix} = \begin{vmatrix} \cos x - \tan x & 0 & 0 \\ 2 \sin x & x^2 & 2x \\ \tan x & x & 1 \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 - R_3$]
 $= (\cos x - \tan x)(x^2 - 2x^2)$ [expanding along R_1]
 $= -x^2(\cos x - \tan x)$

Now, $f'(x) = -2x(\cos x - \tan x) - x^2(-\sin x - \sec^2 x)$

$$\therefore \lim_{x \rightarrow 0} \frac{f'(x)}{x} = \lim_{x \rightarrow 0} -2(\cos x - \tan x) + x(\sin x + \sec^2 x)$$

$$= -2 \times 1 = -2$$

➤ **Example 63.** If $f(x), g(x)$ and $h(x)$ are three polynomials of degree 2 and

$$\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}, \text{ then } \Delta(x) \text{ is}$$

polynomial of degree

(a) 2 (b) 3 (c) 0 (d) at most 3

Sol. (c) $\Delta'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$
 $+ \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \end{vmatrix}$
 $+ \begin{vmatrix} f''(x) & g''(x) & h''(x) \\ 0 & 0 & 0 \end{vmatrix}$

$$\Rightarrow \Delta'(x) = 0$$

$\therefore \Delta(x)$ is constant.

Integration of Determinants

If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ \lambda_1 & \lambda_2 \end{vmatrix}$, then

$$\int_a^b \Delta(x) dx = \begin{vmatrix} \int_a^b f(x) dx & \int_a^b g(x) dx \\ \lambda_1 & \lambda_2 \end{vmatrix}$$

Here, $f(x)$ and $g(x)$ are functions of x and λ_1, λ_2 are constants.

➤ This formula is only applicable, if there is a variable only in one row or column, otherwise expand the determinant and integrate.

➤ **Example 64.** If $f(x) = \begin{vmatrix} \cos x & \sin x \\ 1 & 2 \end{vmatrix}$, then

$\int_{-0}^{\pi/2} f(x) dx$ is

- (a) 0 (b) 1 (c) 3 (d) -1

Sol. (b) $\int_0^{\pi/2} f(x) dx = \int_0^{\pi/2} \begin{vmatrix} \cos x & \sin x \\ 1 & 2 \end{vmatrix} dx$
 $= \int_0^{\pi/2} (\cos x - \sin x) dx = \left[\sin x + \cos x \right]_0^{\pi/2} = (1 + 0) - (0 + 1) = 0$

➤ **Example 65.** If $f(x) = \begin{vmatrix} x^3 & \cos^2 x & 2x^4 \\ \tan^5 x & 1 & \sec 2x \\ \sin^3 x & x^4 & 5 \end{vmatrix}$,

then $\int_{-\pi/2}^{\pi/2} f(x) dx$ is equal to

- (a) 2 (b) -2
(c) 0 (d) None of these

Sol. (c) We have, $f(-x) = \begin{vmatrix} -x^3 & \cos^2 x & 2x^4 \\ -\tan^5 x & 1 & \sec 2x \\ -\sin^3 x & x^4 & 5 \end{vmatrix} = -f(x)$
 $\therefore \int_{-\pi/2}^{\pi/2} f(x) dx = 0$

Maximum and Minimum Values of Determinants, when Elements are Known

If $|A| = \begin{vmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{vmatrix}$,

where a_i 's $\in \{\alpha_1, \alpha_2, \dots, \alpha_n\}$, then $|A|_{\max}$ when diagonal elements are $\{\min \{\alpha_1, \alpha_2, \dots, \alpha_n\}\}$ and non-diagonal elements are $\{\max \{\alpha_1, \alpha_2, \dots, \alpha_n\}\}$.

Also, $|A|_{\min} = -|A|_{\max}$

➤ **Example 66.** The maximum and minimum values of (3×3) determinant where elements belongs to $\{0, 1, 2, 3\}$, are

- (a) 9, -9 (b) 0
(c) 54, -54 (d) None of these

Sol. (c) We have, $|A|_{\max} = \begin{vmatrix} 0 & 3 & 3 \\ 3 & 0 & 3 \\ 3 & 3 & 0 \end{vmatrix}$, i.e. diagonal

elements are $\{\min \{0, 1, 2, 3\}\} = 0$ and non-diagonal elements are $\{\max \{0, 1, 2, 3\}\} = 3$.

$\therefore |A|_{\max} = 0 - 3(-9) + 3(9) = 54$
 and $|A|_{\min} = -|A|_{\max} = -54$

Work Book Exercise 2.2

1 The minor of element 4 in the determinant

$\begin{vmatrix} 1 & 3 & -2 \\ 4 & -5 & 6 \\ 3 & 5 & 2 \end{vmatrix}$ is

- a 8 b 16
c -8 d None of these

2 If $\Delta = \begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$, then

- a $C_{11} = -11$ b $C_{12} = -6$
c $C_{31} = 20$ d None of these

3 For $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, if $|A| = a_{11} C_{21} + a_{12} C_{22} + a_{13} C_{23}$, then $|A|$ is equal to

- a 1 b -1 c 0 d 2

4 If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$, then $\text{adj}(A)$ is

- a $\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}$ b $\begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$
c $\begin{bmatrix} 4 & -3 \\ 1 & 2 \end{bmatrix}$ d $\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

5 If $A = \begin{bmatrix} 2 & \lambda & -3 \\ 0 & 2 & 5 \\ 1 & 1 & 3 \end{bmatrix}$, then A^{-1} exists, where

- a $\lambda = 2$ b $\lambda = -2$
c $\lambda \neq -2$ d None of these

6 If A is a 3×3 non-singular matrix, then $|A^{-1} \text{adj}(A)|$ is

- a $|A|$ b 1 c $|A|^2$ d $|A|^{-1}$

7 If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 40 & 3 & i \end{vmatrix} = x + iy$, then

- a $x = 3, y = 1$ b $x = 1, y = 3$
c $x = 0, y = 3$ d $x = 0, y = 0$

8 The system of linear equations $x + y + z = 2$, $2x + y - z = 3$ and $3x + 2y + kz = 4$ has a unique solution, if k is not equal to

- a 4 b -4 c 0 d 3

9 The system of equations $2x - y + z = 0$, $x - 2y + z = 0$ and $\lambda x - y + 2z = 0$ has infinite number of non-trivial solution for

- a $\lambda = 1$ b $\lambda = 5$
c $\lambda = -5$ d no real value of λ

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If the rank of the matrix $\begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & a-4 \\ 1 & -2 & a+1 \end{bmatrix}$ is 1, then the value of a is
(a) -1 (b) 2 (c) -6 (d) 4

Sol. If the rank of given matrix is 1, then

$$\begin{vmatrix} 2 & 5 \\ -4 & a-4 \end{vmatrix} = 0 \Rightarrow 2a - 8 + 20 = 0$$

$$\therefore a = -6$$

Hence, (c) is the correct answer.

Ex 2. The system of equations $2x + y - 5 = 0$, $x - 2y + 1 = 0$ and $2x - 14y - a = 0$ is consistent. Then, a is equal to
(a) 1 (b) 2
(c) 5 (d) None of these

Sol. The given system of equations is

$$2x + y - 5 = 0 \quad \dots(i)$$

$$x - 2y + 1 = 0 \quad \dots(ii)$$

$$\text{and } 2x - 14y - a = 0 \quad \dots(iii)$$

This system is consistent.

$$\therefore \begin{vmatrix} 2 & 1 & -5 \\ 1 & -2 & 1 \\ 2 & -14 & -a \end{vmatrix} = 0$$

$$\Rightarrow 2(2a + 14) - 1(-a - 2) - 5(-14 + 4) = 0$$

$$\Rightarrow 4a + 28 + a + 2 + 50 = 0$$

$$\Rightarrow 5a = -80 \Rightarrow a = -16$$

Hence, (d) is the correct answer.

Ex 3. If B is an invertible matrix and A is a matrix, then

$$(a) \text{rank}(BA) = \text{rank}(A)$$

$$(b) \text{rank}(BA) \geq \text{rank}(B)$$

$$(c) \text{rank}(BA) > \text{rank}(A)$$

$$(d) \text{rank}(BA) > \text{rank}(B)$$

Sol. Since, B is invertible, therefore B^{-1} exists.

$$\text{Now, rank}(A) = \text{rank}(B^{-1}B) \leq \text{rank}(BA)$$

$$\text{But rank}(BA) \leq \text{rank}(A)$$

$$\therefore \text{Rank}(BA) = \text{Rank}(A)$$

Hence, (a) is the correct answer.

Ex 4. The system of equations $x + y + z = 0$, $2x + 3y + z = 0$ and $x + 2y = 0$ has

$$(a) \text{a unique solution; } x = 0, y = 0 \text{ and } z = 0$$

$$(b) \text{infinite solutions}$$

$$(c) \text{no solution}$$

$$(d) \text{finite number of non-zero solutions}$$

Sol. The given system of equations is $x + y + z = 0$, $2x + 3y + z = 0$ and $x + 2y = 0$

$$\text{Consider, } \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 0 \end{vmatrix} = 1(0 - 2) - 1(0 - 1) + 1(4 - 3) = -2 + 1 + 1 = 0$$

So, this system has infinite solutions.

Hence, (b) is the correct answer.

Ex 5. The system of equations $3x - y + 4z = 3$, $x + 2y - 3z = -2$ and $6x + 5y + \lambda z = -3$ has atleast one solution, if

$$(a) \lambda = -5 \quad (b) \lambda = 5$$

$$(c) \lambda = 3 \quad (d) \lambda = -13$$

Sol. The given system of equations has atleast one solution,

$$\text{if } \begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 3(2\lambda + 15) + 1(\lambda + 18) + 4(5 - 12) = 0$$

$$\Rightarrow 7\lambda = -35 \Rightarrow \lambda = -5$$

$$\text{and } (\text{adj } A)B = O, \text{ where } A = \begin{bmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & -5 \end{bmatrix}$$

$$\text{and } B = \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix}$$

$$\text{Clearly, } \text{adj}(A) = \begin{bmatrix} 5 & 15 & -5 \\ -13 & -39 & 13 \\ -7 & -21 & 7 \end{bmatrix}$$

$$\text{Now, } (\text{adj } A)B = \begin{bmatrix} 5 & 15 & -5 \\ -13 & -39 & 13 \\ -7 & -21 & 7 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So, system will have atleast one solution, if $\lambda = -5$.

Hence, (a) is the correct answer.

Ex 6. If a, b and c are positive real numbers. The following system of equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \text{ in } x, y \text{ and } z \text{ has}$$

$$(a) \text{infinite solutions}$$

$$(b) \text{unique solution}$$

$$(c) \text{no solution}$$

$$(d) \text{finite number of solutions}$$

Sol. Let $\frac{x^2}{a^2} = X$, $\frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$, then given equation will be $X + Y - Z = 1$, $X - Y + Z = 1$ and $-X + Y + Z = 1$

$$\text{Here, } A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

Now, $|A| = -4 \neq 0$

Therefore, the given system of equation has unique solution.

Hence, (b) is the correct answer.

Ex 7. The system of equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + \lambda z = \mu$ has no solution, if

- (a) $\lambda = 3, \mu = 10$ (b) $\lambda \neq 3, \mu = 10$
(c) $\lambda \neq 3, \mu \neq 10$ (d) $\lambda = 3, \mu \neq 10$

Sol. Given, $x + y + z = 6$, $x + 2y + 3z = 10$

and $x + 2y + \lambda z = \mu$

For no solution,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$$\Rightarrow 1(2\lambda - 6) - 1(\lambda - 3) + 1(2 - 2) = 0$$

$$\Rightarrow \lambda - 3 = 0 \Rightarrow \lambda = 3$$

$$\text{and } \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ \mu & 2 & 3 \end{vmatrix} \neq 0$$

$$\Rightarrow 6(6 - 6) - 1(30 - 3\mu) + 1(20 - 2\mu) \neq 0$$

$$\Rightarrow \mu - 10 \neq 0 \text{ or } \mu \neq 10$$

Hence, (d) is the correct answer.

Ex 8. Consider the system of equations in x , y and z as $x \sin 3\theta - y + z = 0$, $x \cos 2\theta + 4y + 3z = 0$ and $2x + 7y + 7z = 0$. If this system has a non-trivial solution, then for integer n , values of θ are given by

- (a) $\pi \left(n + \frac{(-1)^n}{3} \right)$ (b) $\pi \left(n + \frac{(-1)^n}{4} \right)$
(c) $\pi \left(n + \frac{(-1)^n}{6} \right)$ (d) $\frac{n\pi}{2}$

Sol. Given, $x \sin 3\theta - y + z = 0$

$x \cos 2\theta + 4y + 3z = 0$ and $2x + 7y + 7z = 0$

$$\text{For non-trivial solution, } \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \sin 3\theta(28 - 21) - \cos 2\theta(-7 - 7) + 2(-3 - 4) = 0$$

$$\Rightarrow 7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$$

$$\Rightarrow 7(3 \sin \theta - 4 \sin^3 \theta) + 14(1 - 2 \sin^2 \theta) - 14 = 0$$

$$\left[\begin{array}{l} \therefore \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta \\ \text{and } \cos 2\theta = 1 - 2 \sin^2 \theta \end{array} \right]$$

$$\Rightarrow -28 \sin^3 \theta - 28 \sin^2 \theta + 21 \sin \theta = 0$$

$$\Rightarrow -7 \sin \theta (4 \sin^2 \theta + 4 \sin \theta - 3) = 0$$

$$\Rightarrow \sin \theta (2 \sin \theta + 3)(2 \sin \theta - 1) = 0$$

$$\Rightarrow \sin \theta = 0, \sin \theta = \frac{1}{2} \quad \left[\because \sin \theta \neq -\frac{3}{2} \right]$$

$$\therefore \theta = n\pi, n\pi + (-1)^n \frac{\pi}{6}$$

Hence, (c) is the correct answer.

Ex 9. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{bmatrix}$, $B = (\text{adj } A)$ and $C = 5A$,

then $\frac{|\text{adj } B|}{|C|}$ is equal to

- (a) 5 (b) 25
(c) -1 (d) 1

$$\text{Sol. } B = \text{adj } (A) = \begin{bmatrix} 3 & 1 & 1 \\ -6 & -2 & 3 \\ -4 & -3 & 2 \end{bmatrix}$$

$$\text{Therefore, } \text{adj } (B) = \begin{bmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{bmatrix}$$

$$\text{Now, } |\text{adj } B| = \begin{vmatrix} 5 & -5 & 5 \\ 0 & 10 & -15 \\ 10 & 5 & 0 \end{vmatrix} = 625$$

$$\text{and } |C| = 125 |A| = 125 \begin{vmatrix} 1 & -1 & 1 \\ 0 & 2 & -3 \\ 2 & 1 & 0 \end{vmatrix} = 625$$

$$\therefore \frac{|\text{adj } (B)|}{|C|} = \frac{625}{625} = 1$$

Hence, (d) is the correct answer.

Aliter

Since, $|A| = 1(0 + 3) + 1(0 + 6) + 1(0 - 4) = 5$

Now, $\text{adj } B = \text{adj } (\text{adj } A) = |A| \cdot A = 5A$

$$\therefore \frac{|\text{adj } B|}{|C|} = \frac{|5A|}{|5A|} = 1$$

Ex 10. If $\text{adj} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & a & -2 \\ 1 & 1 & 0 \\ -2 & -2 & b \end{bmatrix}$, then

$[a \ b]$ is equal to

- (a) $[-4 \ 1]$ (b) $[-4 \ -1]$
(c) $[4 \ 1]$ (d) $[4 \ -1]$

$$\text{Sol. Given, } \text{adj} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & a & -2 \\ 1 & 1 & 0 \\ -2 & -2 & b \end{bmatrix} \quad \dots (i)$$

$$\text{Cofactors of elements of } \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 2 & 1 \end{bmatrix} \text{ are}$$

$$C_{11} = 5, C_{12} = 1, C_{13} = -2$$

$$C_{21} = 4, C_{22} = 1, C_{23} = -2$$

$$C_{31} = -2, C_{32} = 0 \text{ and } C_{33} = 1$$

$$\Rightarrow \begin{bmatrix} 5 & 4 & -2 \\ 1 & 1 & 0 \\ -2 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & a & -2 \\ 1 & 1 & 0 \\ -2 & -2 & b \end{bmatrix}$$

On comparing the corresponding elements, we get

$$a = 4 \quad \text{and} \quad b = 1$$

$$\therefore [a \ b] = [4 \ 1]$$

Hence, (c) is the correct answer.

Ex 11. Let λ and α be real. The system of equations $\lambda x + \sin \alpha y + \cos \alpha z = 0$, $x + \cos \alpha y + \sin \alpha z = 0$, $-x + \sin \alpha y - \cos \alpha z = 0$ has a non-trivial solution. If $\lambda = 1$, then all the values of α are

- (a) $n\pi + \frac{\pi}{2}$ (b) $n\pi + \frac{\pi}{4}, n\pi$
(c) $n\pi \pm \frac{\pi}{2}$ (d) $n\pi + \frac{\pi}{4} + \frac{\pi}{4}$

Sol. The given system has a non-trivial solution, if

$$\begin{vmatrix} \lambda & \sin \alpha & \cos \alpha \\ 1 & \cos \alpha & \sin \alpha \\ -1 & \sin \alpha & -\cos \alpha \end{vmatrix} = 0$$

On expanding the determinant, we get

$$\lambda (-\cos^2 \alpha - \sin^2 \alpha) - \sin \alpha (-\cos \alpha + \sin \alpha) + \cos \alpha (\sin \alpha + \cos \alpha) = 0$$

$$\Rightarrow \lambda = \sin 2\alpha + \cos 2\alpha$$

For $\lambda = 1$, $\sin 2\alpha + \cos 2\alpha = 1$

$$\Rightarrow \frac{1}{\sqrt{2}} \sin 2\alpha + \frac{1}{\sqrt{2}} \cos 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos \left(2\alpha - \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}} = \cos \left(2n\pi \pm \frac{\pi}{4} \right)$$

$$\Rightarrow 2\alpha = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{4}, n \text{ being an integer.}$$

$$\Rightarrow 2\alpha = 2n\pi \quad \text{or} \quad 2n\pi + \frac{\pi}{2}$$

$$\Rightarrow \alpha = n\pi \quad \text{or} \quad n\pi + \frac{\pi}{4}$$

Hence, (b) is the correct answer.

Ex 12. The system of equations $x + y + z = 3$,

$$x + 2y + 3z = 4, x + 4y + 9z = 6$$

- (a) $x = 1, z = 0$ (b) $x = 2, z = 0$
(c) $x = 0, y = 2$ (d) None of these

Sol. The system of equations can be written as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$$

$$\text{Let } A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \Rightarrow |A| = 2$$

$$\text{and } \text{adj}(A) = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix}$$

Thus, the solution of the given system is obtained from

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -5/2 & 1/2 \\ -3 & 4 & -1 \\ 1 & -3/2 & 1/2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\Rightarrow x = 2, y = 1, z = 0$$

Hence, (b) is the correct answer.

Ex 13. For two unimodular complex numbers z_1 and

$$z_2, \begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1} \text{ is equal to}$$

- (a) $\begin{bmatrix} z_1 & z_2 \\ \bar{z}_1 & \bar{z}_2 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$
(c) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (d) None of these

Sol. Here, $\begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix}^{-1} \begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix}^{-1}$
 $= \left(\begin{bmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{bmatrix} \begin{bmatrix} \bar{z}_1 & -z_2 \\ \bar{z}_2 & z_1 \end{bmatrix} \right)^{-1}$
 $= \begin{bmatrix} z_1 \bar{z}_1 + z_2 \bar{z}_2 & -z_1 z_2 + z_1 \bar{z}_2 \\ -\bar{z}_1 \bar{z}_2 + \bar{z}_1 z_1 & \bar{z}_2 z_2 + z_1 \bar{z}_1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1}$
 $= \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence, (c) is the correct answer.

Ex 14. If the system of equations $\lambda x + (b-a)y + (c-a)z = 0$, $(a-b)x + \lambda y + (c-b)z = 0$ and $(a-c)x + (b-c)y + \lambda z = 0$ has a non-trivial solution, then the value of λ is

- (a) $\lambda = 0$ (b) $\lambda = 1$
(c) $\lambda = -1$ (d) None of these

Sol. As the given system of equations has non-trivial solution

$$\Rightarrow \begin{vmatrix} \lambda & b-a & c-a \\ a-b & \lambda & c-b \\ a-c & b-c & \lambda \end{vmatrix} = 0$$

When $\lambda = 0$, then determinants becomes skew-symmetric determinants of odd order, which is equal to zero. Thus, $\lambda = 0$.

Hence, (a) is the correct answer.

Ex 15. If $[x]$ stands for the greatest integer less or equal to x , then in order that the set of equations $x - 3y = 4$; $5x + y = 2$; $[2\pi]x - [e]y = [2a]$ may be consistent, then a should lie in

- (a) $\left[3, \frac{7}{2} \right)$ (b) $\left(3, \frac{7}{3} \right)$
(c) $\left(3, \frac{7}{3} \right]$ (d) None of these

Sol. On solving $x - 3y = 4$ and $5x + y = 2$, we get

$$x = \frac{5}{8}, y = -\frac{9}{8}$$

As $[2\pi] = 6$ and $[e] = 2$

So the three equations are consistent, if

$$[2\pi] \cdot \frac{5}{8} + [e] \frac{9}{8} = [2a] \quad \text{or} \quad [2a] = 6$$

$$\text{This gives } 6 \leq 2a < 7 \quad \text{or} \quad 3 \leq a < \frac{7}{2}$$

Hence, (a) is the correct answer.

Ex 16. If $\begin{vmatrix} x & 3 & 6 \\ 3 & 6 & x \\ 6 & x & 3 \end{vmatrix} = \begin{vmatrix} 2 & x & 7 \\ x & 7 & 2 \\ 7 & 2 & x \end{vmatrix} = \begin{vmatrix} 4 & 5 & x \\ 5 & x & 3 \\ x & 4 & 5 \end{vmatrix} = 0$,

then x is equal to

- (a) 9 (b) -9
(c) 0 (d) None of these

Sol. By circulate determinant property,

$$\begin{aligned} a + b + c &= 0 \\ \Rightarrow x + 3 + 6 &= x + 2 + 7 \\ &= x + 4 + 5 = 0 \\ \Rightarrow x &= -9 \end{aligned}$$

Hence, (b) is the correct answer.

Aliter

In each determinant, applying $R_1 \rightarrow R_1 + R_2 + R_3$ and then taking out $(x+9)$ common, we get $x+9=0 \Rightarrow x=-9$

Ex 17. If $f(\theta) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & e^{i\theta} & 1 \\ 1 & -1 & -e^{i\theta} \end{vmatrix}$, then

- (a) $\int_{-\pi/2}^{\pi/2} f(\theta) d\theta = 2 \int_0^{\pi/2} f(\theta) d\theta$
(b) $f(\theta)$ is purely real
(c) $f\left(\frac{\pi}{2}\right) = 2$
(d) None of the above

Sol. On applying $R_1 \rightarrow R_1 - R_2$ and $R_3 \rightarrow R_3 - R_2$, we get

$$\begin{aligned} f(\theta) &= \begin{vmatrix} 0 & 1 - e^{i\theta} & -2 \\ 1 & e^{i\theta} & 1 \\ 0 & -1 - e^{i\theta} & -1 - e^{i\theta} \end{vmatrix} \\ &= (-1)[(1 - e^{i\theta})(-1 - e^{i\theta}) + 2(-1 - e^{i\theta})] \neq 2, \end{aligned}$$

$$\text{when } \theta = \frac{\pi}{2}$$

$$\begin{aligned} \text{Also, } f(\theta) &= -1[-(1 - e^{2i\theta}) - 2 - 2e^{i\theta}] \\ &= -1[-1 + e^{2i\theta} - 2 - 2e^{i\theta}] \\ &= -1[-3 - 2e^{i\theta} + e^{2i\theta}] = 3 + 2e^{i\theta} - e^{2i\theta} \end{aligned}$$

Hence, (d) is the correct answer.

Ex 18. Let $\mathbf{a}_r = x_r \hat{\mathbf{i}} + y_r \hat{\mathbf{j}} + z_r \hat{\mathbf{k}}$; $r=1, 2, 3$ be three mutually perpendicular unit vectors, then the

value of $\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}$ is

- (a) 0 (b) ± 1
(c) ± 2 (d) None of these

Sol. Let $\Delta = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

$$\Delta^2 = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= \begin{vmatrix} x_1^2 & y_1^2 & z_1^2 & x_1x_2 & y_1y_2 & z_1z_2 \\ x_2x_1 & y_2y_1 & z_2z_1 & x_2^2 & y_2^2 & z_2^2 \\ x_3x_1 & y_3y_1 & z_3z_1 & x_3x_2 & y_3y_2 & z_3z_2 \\ x_1x_3 & y_1y_3 & z_1z_3 & x_2x_3 & y_2y_3 & z_2z_3 \\ x_3^2 & y_3^2 & z_3^2 & & & \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \Rightarrow \Delta = \pm 1$$

Hence, (b) is the correct answer.

Ex 19. If a, b and c are sides of a ΔABC and

$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 0, \text{ then}$$

- (a) ΔABC is an equilateral triangle
(b) ΔABC is a right angled triangle
(c) ΔABC is an isosceles triangle
(d) None of the above

Sol. When $a=b$ or $b=c$ or $c=a$, then determinant reduces to zero. It is not necessary that $a=b=c$ for determinant to be zero.

Therefore, triangle is isosceles.

Hence, (c) is the correct answer.

Ex 20. Let $ax^6 + bx^5 + cx^4 + dx^3 + ex^2 + fx + g$

$$= \begin{vmatrix} (x+1) & (x^2+2) & (x^2+x) \\ (x^2+x) & (x^2+1) & (x^2+2) \\ (x^2+2) & (x^2+x) & (x+1) \end{vmatrix}, \text{ then}$$

- (a) $f=3$ and $g=-5$
(b) $f=-3$ and $g=-5$
(c) $f=-3$ and $g=-9$
(d) None of the above

Sol. On putting $x=0$ both sides of the equation, we have

$$g = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} = 9$$

Now, on differentiating both sides and then putting $x=0$, we get

$$f = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 0 \\ 1 & 0 & 0 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow f = -1 - 2 - 1 = -4$$

Hence, (d) is the correct answer.

Ex 21. If $\Delta_r = \begin{vmatrix} (2r) & x & N(N+1) \\ (6r^2-1) & y & N^2(2N+3) \\ (4r^3-2Nr) & z & N^3(N+1) \end{vmatrix}$, where

$N \in \text{natural numbers}$. Then, $\sum_{r=1}^N \Delta_r$ is equal to

- (a) N (b) N^2
(c) zero (d) None of these

Sol. $\sum_{r=1}^N 2r = 2 \left[\frac{N(N+1)}{2} \right] = N(N+1)$

$$\sum_{r=1}^N (6r^2-1) = 6N \left[\frac{(N+1)(2N+1)}{6} \right] - N$$

$$= N(2N^2 + 2N + N + 1) - N$$

$$= 2N^3 + 3N^2$$

$$\sum_{r=1}^N (4r^3 - 2Nr) = \frac{4N^2(N+1)^2}{4} - \frac{2N \cdot N(N+1)}{2}$$

$$= N^2(N+1)^2 - N^2(N+1)$$

$$= N^3 \cdot (N+1)$$

$$\Rightarrow \sum_{r=1}^N \Delta_r = \begin{vmatrix} N(N+1) & x & N(N+1) \\ 2N^3 + 3N^2 & y & N^2(2N+3) \\ N^3(N+1) & z & N^3(N+1) \end{vmatrix} = 0$$

Hence, (c) is the correct answer.

Ex 22. $\begin{vmatrix} 109 & 102 & 95 \\ 6 & 13 & 20 \\ 1 & -6 & -13 \end{vmatrix}$ is equal to

- (a) constant other than zero
(b) zero
(c) 100
(d) -1997

Sol. Since, elements of the rows are in AP with common difference 7, so given determinant is zero.

Hence, (b) is the correct answer.

Ex 23. $\Delta = \begin{vmatrix} p & 2-i & i+1 \\ 2+i & q & 3+i \\ 1-i & 3-i & r \end{vmatrix}$ is always

- (a) real (b) imaginary
(c) complex (d) None of these

Sol. Since, $\Delta = \bar{\Delta}$ as we know Δ of hermitian matrix is always real.

$\therefore \Delta$ is real only.

Hence, (a) is the correct answer.

Ex 24. If a, b and c are p th, q th and r th terms of an

HP, then $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

- (a) term containing a, b, c, p, q, r
(b) a constant
(c) zero
(d) None of the above

Sol. If A is the first term and D is the common difference of the corresponding AP, then

$$\frac{1}{a} = A + (p-1)D, \quad \frac{1}{b} = A + (q-1)D$$

and $\frac{1}{c} = A + (r-1)D$

Now, $\Delta = abc \begin{vmatrix} 1/a & 1/b & 1/c \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$

On applying $R_1 \rightarrow R_1 - D(R_2) - (A-D)R_3$, we get

$$\Delta = abc \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Hence, (c) is the correct answer.

Ex 25. $\Delta = \begin{vmatrix} my+nz & mq+nr & mb+nc \\ kz-mx & kr-mp & kc-ma \\ -nx-ky & -np-kq & -na-kb \end{vmatrix}$ is equal

to

- (a) $\Delta = 0$ (b) $\Delta \neq 0$
(c) $\Delta = f(x, y, z)$ (d) None of these

Sol. $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix} \begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix}$

where, $\begin{vmatrix} 0 & m & n \\ -m & 0 & k \\ -n & -k & 0 \end{vmatrix}$ is skew-symmetric.

$\therefore \Delta = 0$

Hence, (a) is the correct answer.

Ex 26. If $\Delta = \begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^p & p \\ x^{n+5} & x^{p+6} & x^{2n+5} \end{vmatrix} = 0$, then p is

given by

- (a) x^n
(b) $(n+1)$
(c) either (a) or (b)
(d) both (a) and (b)

Sol. If $p = x^n$, then $\Delta = 0$, since C_1 and C_3 are same and if $p = n+1$, then $\Delta = 0$, since R_1 and R_3 are same while x^5 taken common for R_3 .

Hence, (d) is the correct answer.

Ex 27. The system of equations $x+2y+3z=4$, $2x+3y+4z=5$ and $3x+4y+5z=6$ has

- (a) many solutions
(b) no solution
(c) unique solution
(d) None of the above

Sol. The given system of equation can be written in matrix

form as $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$

On applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$, we

$$\text{get } \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -6 \end{bmatrix}$$

On applying $R_1 \rightarrow (-)R_2$ and $R_3 \rightarrow R_3 - 3R_1$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 6 \end{bmatrix}$$

Now, on applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$$

$\therefore$ Rank (coefficient matrix) = 2

and rank (augmented matrix) = 2

$\therefore$ The system has infinite number of solutions.

Hence, (a) is the correct answer.

Ex 28. If the system of equations $2x + 5y + 8z = 0$, $x + 4y + 7z = 0$ and $6x + 4y - \lambda z = 0$ has a non-trivial solution, then λ is equal to

- (a) 12 (b) -2
(c) 0 (d) None of these

Sol. $\begin{vmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 6 & 4 & -\lambda \end{vmatrix} = 0$

$$\Rightarrow 2[-4\lambda - 28] - 5[-\lambda - 42] + 8[4 - 24] = 0$$

$$\Rightarrow -8\lambda - 56 + 5\lambda + 210 - 160 = 0 \Rightarrow \lambda = -2$$

Hence, (b) is the correct answer.

Ex 29. $\Delta = \begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$ is equal to

- (a) 1 (b) -1
(c) zero (d) 2

Sol. $\Delta = \begin{vmatrix} 18 & 40 & 89 \\ 4 & 9 & 20 \\ -1 & -2 & -5 \end{vmatrix}$

[applying $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 5R_1$]

$$= \begin{vmatrix} 18 & 40 & 89 \\ 0 & 1 & 0 \\ -1 & -2 & -5 \end{vmatrix} \quad [\text{applying } R_2 \rightarrow R_2 + 4R_3]$$

$$\therefore \Delta = [1(-90 + 89)] = -1$$

Hence, (b) is the correct answer.

Ex 30. $\Delta = \begin{vmatrix} 0 & i-100 & i-500 \\ 100-i & 0 & 1000-i \\ 500-i & i-1000 & 0 \end{vmatrix}$ is equal to

- (a) 100 (b) 500
(c) 1000 (d) 0

Sol. The determinant is skew-symmetric of odd order, hence it is equal to 0.

Hence, (d) is the correct answer.

Ex 31. For a fixed positive integer n , if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}, \text{ then}$$

$$\left[\frac{D}{(n!)^3} - 4 \right] \text{ is divisible by}$$

- (a) $3n$ (b) n^2
(c) n (d) None of these

Sol. We take $n!$ as a common factor from each of the rows.

$$\text{We get } \frac{D}{(n!)^3} = \begin{vmatrix} 1 & (n+1) & (n+2) \\ (n+1) & (n+1)(n+2) & (n+2)(n+3) \\ (n+2)(n+1) & (n+1)(n+2)(n+3) & (n+2)(n+1)(n+3)(n+4) \end{vmatrix}$$

On taking $(n+1)$ as common factor from C_2 and $(n+1)(n+2)$ as common factor from C_3 , we get

$$\frac{D}{(n!)^3} = (n+1)^2(n+2)$$

$$\begin{vmatrix} 1 & 1 & 1 \\ (n+1) & (n+2) & (n+3) \\ (n+2)(n+1) & (n+2)(n+3) & (n+4)(n+3) \end{vmatrix}$$

On applying $C_3 \rightarrow C_3 - C_2$ and $C_2 \rightarrow C_2 - C_1$, we get

$$\frac{D}{(n!)^3} = (n+1)^2(n+2)$$

$$\begin{vmatrix} 1 & 0 & 0 \\ (n+1) & 1 & 1 \\ (n+1)(n+2) & 2(n+2) & 2(n+3) \end{vmatrix}$$

$$= (n+1)^2(n+2) \cdot 1$$

$$\Rightarrow \frac{D}{(n!)^3} = 2(n^3 + 4n^2 + 5n + 2)$$

$$\Rightarrow \frac{D}{(n!)^3} - 4 = 2n(n^2 + 4n + 5)$$

$$\Rightarrow \left(\frac{D}{(n!)^3} - 4 \right) \text{ is divisible by } n.$$

Hence, (c) is the correct answer.

Ex 32. Without expanding the determinant at any

stage, $\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$ is equal

to

- (a) $Ax^2 - B$ (b) $Ax + B$
(c) Ax (d) None of these

(where, A and B are determinants of order 3 not involving x)

$$\text{Sol. LHS} = \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_3 - R_2$, we get

$$\text{LHS} = \begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$, we get

$$\text{LHS} = \begin{vmatrix} 4 & 0 & 0 \\ 2x^2 + 2 & 3 & 3x - 3 \\ x^2 + 4 & 0 & 2x - 1 \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - \frac{x^2}{2} R_1$ and $R_3 \rightarrow R_3 - \frac{x^2}{4} R_1$,

we get

$$\begin{aligned} \text{LHS} &= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x - 3 \\ 4 & 0 & 2x - 1 \end{vmatrix} \\ &= \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3x \\ 4 & 0 & 2x \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix} \\ &= x \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & 3 \\ 4 & 0 & 2 \end{vmatrix} + \begin{vmatrix} 4 & 0 & 0 \\ 2 & 3 & -3 \\ 4 & 0 & -1 \end{vmatrix} = Ax + B \end{aligned}$$

Hence, (b) is the correct answer.

Ex 33. The value(s) of m for which the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfying the conditions $x > 0$, $y > 0$, is/are

- (a) $m \in (0, \infty)$
 (b) $m \in \left(-\infty, -\frac{15}{2}\right) \cup (30, \infty)$
 (c) $m \in \left(-\frac{15}{2}, \infty\right)$
 (d) None of the above

Sol. By using Cramer's rule, the solution of the system is

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta}, \text{ where } \Delta = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix} = -(15 + 2m)$$

$$\Delta_x = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix} = -25m, \Delta_y = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix} = 60 - 2m$$

$$\Rightarrow x = \frac{-25m}{-(15 + 2m)} = \frac{25m(15 + 2m)}{(15 + 2m)^2} > 0$$

$$\Rightarrow m > 0 \text{ or } m < -\frac{15}{2}$$

$$\text{Also, } y = \frac{60 - 2m}{-(15 + 2m)} = \frac{2(m - 30)(15 + 2m)}{(15 + 2m)^2} > 0$$

$$\Rightarrow m > 30 \text{ or } m < -\frac{15}{2}$$

$$\Rightarrow x > 0, y > 0, \text{ if } m > 30 \text{ or } m < -\frac{15}{2}$$

For $m = -\frac{15}{2}$, the system has no solution.

Hence, (b) is the correct answer.

Ex 34. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar vectors, then

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix} \text{ is}$$

- (a) 1
 (b) 0
 (c) -1
 (d) None of the above

Sol. Since, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar vectors, so there must exist k_1, k_2, k_3 not all zero (say $k_1 \neq 0$) such that $k_1\mathbf{a} + k_2\mathbf{b} + k_3\mathbf{c} = \mathbf{0}$

Now, on applying $C_1 \rightarrow k_1 C_1 + k_2 C_2 + k_3 C_3$, we get

$$\Delta = \frac{1}{k_1} \begin{vmatrix} k_1 \mathbf{a} + k_2 \mathbf{b} + k_3 \mathbf{c} & \mathbf{b} & \mathbf{c} \\ k_1 \mathbf{a} \cdot \mathbf{a} + k_2 \mathbf{a} \cdot \mathbf{b} + k_3 \mathbf{a} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ k_1 \mathbf{b} \cdot \mathbf{a} + k_2 \mathbf{b} \cdot \mathbf{b} + k_3 \mathbf{b} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix} \quad [\because k_1 \neq 0]$$

$$= \frac{1}{k_1} \begin{vmatrix} k_1 \mathbf{a} + k_2 \mathbf{b} + k_3 \mathbf{c} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot [k_1 \mathbf{a} + k_2 \mathbf{b} + k_3 \mathbf{c}] & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot [k_1 \mathbf{a} + k_2 \mathbf{b} + k_3 \mathbf{c}] & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix}$$

$$= \frac{1}{k_1} \begin{vmatrix} \mathbf{0} & \mathbf{b} & \mathbf{c} \\ \mathbf{0} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{0} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \end{vmatrix} = 0$$

as $k_1 \mathbf{a} + k_2 \mathbf{b} + k_3 \mathbf{c} = \mathbf{0}$, since coplanar.

Hence, (b) is the correct answer.

Ex 35. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$, then

the value of $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$ is

- (a) 2
 (b) -2
 (c) 0
 (d) 3

Sol. On applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$, we get

$$0 = \begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix}$$

$$\Rightarrow 0 = (p-a) \{r(q-b) - b(c-r)\} + a(b-q)(c-r)$$

$$\Rightarrow 0 = (p-a)(rq-rb) + a(b-q)(c-r) + b(p-a)(r-c)$$

$$\Rightarrow 0 = \frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a}$$

[dividing by $(p-a)(r-c)(q-b)$]

$$\Rightarrow \frac{r}{r-c} + \left(\frac{b}{q-b} + 1\right) + \left(\frac{a}{p-a} + 1\right) = 2$$

$$\therefore \frac{r}{r-c} + \frac{q}{q-b} + \frac{p}{p-a} = 2$$

Hence, (a) is the correct answer.

Ex 36. The value of

$$\Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix} \text{ is}$$

- (a) 0
(b) 1
(c) $2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$
(d) None of the above

Sol. $\Delta = \begin{vmatrix} (a^2 - 2ax + x^2) & (a^2 - 2ay + y^2) & (a^2 - 2az + z^2) \\ (b^2 - 2bx + x^2) & (b^2 - 2by + y^2) & (b^2 - 2bz + z^2) \\ (c^2 - 2cx + x^2) & (c^2 - 2cy + y^2) & (c^2 - 2cz + z^2) \end{vmatrix}$

$$= \begin{vmatrix} a^2 & -2a & 1 \\ b^2 & -2b & 1 \\ c^2 & -2c & 1 \end{vmatrix} \times \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$= 2(a-b)(b-c)(c-a)(x-y)(y-z)(z-x)$$

Hence, (c) is the correct answer.

Ex 37. If f, g, h are differentiable functions of x and

$$\Delta(x) = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2 f)'' & (x^2 g)'' & (x^2 h)'' \end{vmatrix}, \text{ then}$$

$\Delta'(x)$ is

- (a) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$
(b) $\begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ (x^3 f'') & (x^3 g'') & (x^3 h'') \end{vmatrix}$
(c) $\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')'' & (x^3 g'')'' & (x^3 h'')'' \end{vmatrix}$
(d) None of the above

Sol. $\Delta(x) = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ 4xf'' + 2f + x^2 f''' & 4xg'' + 2g + x^2 g''' & 4xh'' + 2h + x^2 h''' \end{vmatrix}$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - 4R_2 + 2R_1$ and shifting x of R_2 to R_3 , we get

$$\Delta(x) = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$$

$$\Rightarrow \Delta'(x) = 0 + 0 + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & (x^3 g'')' & (x^3 h'')' \end{vmatrix}$$

Hence, (a) is the correct answer.

Ex 38. If A, B and C are angles of a triangle, then

$$\begin{vmatrix} e^{2iA} & e^{-iC} & e^{-iB} \\ e^{-iC} & e^{2iB} & e^{-iA} \\ e^{-iB} & e^{-iA} & e^{2iC} \end{vmatrix} \text{ is equal to}$$

- (a) 4 (b) 0 (c) -4 (d) -1

Sol. Since, $A + B + C = \pi$ and $e^{i\pi} = \cos \pi + i \sin \pi = -1 = e^{i(\pi)}$

$$\therefore e^{i(B+C)} = e^{i(\pi-A)} = -e^{-iA},$$

$$\text{and } e^{-i(B+C)} = -e^{iA}$$

On taking e^{iA}, e^{iB}, e^{iC} common from R_1, R_2 and R_3 respectively, we get

$$\Delta = - \begin{vmatrix} e^{iA} & e^{-i(A+C)} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{iB} & e^{-i(A+B)} \\ e^{-i(B+C)} & e^{-i(A+C)} & e^{iC} \end{vmatrix}$$

$$= - \begin{vmatrix} e^{iA} & -e^{iB} & -e^{iC} \\ -e^{iA} & e^{iB} & -e^{iC} \\ -e^{iA} & -e^{iB} & e^{iC} \end{vmatrix}$$

On taking e^{iA}, e^{iB}, e^{iC} common from C_1, C_2 and C_3 respectively, we get

$$\Delta = (+1) \begin{vmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{vmatrix} = -4$$

Hence, (c) is the correct answer.

Ex 39. Let a, b and c be real numbers with

$$\begin{vmatrix} a^2 + b^2 + c^2 = 1. \text{ Then, the equation} \\ ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$

represents a

- (a) parabola (b) circle
(c) straight line (d) Cannot be discussed

Sol. Given,

$$\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix} = 0$$

On applying $C_1 \rightarrow aC_1 + bC_2 + cC_3$, we get

$$\frac{1}{a} \begin{vmatrix} (a^2 + b^2 + c^2)x & bx + ay & cx + a \\ (a^2 + b^2 + c^2)y & -ax + by - c & cy + b \\ (a^2 + b^2 + c^2) & cy + b & -x - by + c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & bx + ay & cx + a \\ y & -ax + by - c & cy + b \\ 1 & cy + b & -x - by + c \end{vmatrix} = 0$$

On applying $C_2 \rightarrow C_2 - bC_1$ and $C_3 \rightarrow C_3 - cC_1$, we get

$$\begin{vmatrix} x & ay & a \\ y & -ax - c & b \\ 1 & cy & -ax - by \end{vmatrix} = 0$$

On applying $R_3 \rightarrow R_3 + xR_1 + yR_2$, we get

$$\begin{vmatrix} x & ay & a \\ y & -ax - c & b \\ x^2 + y^2 + 1 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x^2 + y^2 + 1)(aby + a^2x + ac) = 0$$

$\Rightarrow ax + by + c = 0$, which represents a straight line.

Hence, (c) is the correct answer.

Ex 40. Let α be a repeated root of the quadratic equation $f(x) = 0$ and $A(x), B(x), C(x)$ be polynomials of degree 3, 4 and 5 respectively. Then,

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

is divisible by

- (a) $2f(x)$ (b) $3f(x)$
(c) $f(x)$ (d) None of these

Sol. Let $g(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$

$$\Rightarrow g'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

We find that $g(\alpha) = 0$ and $g'(\alpha) = 0$

$\Rightarrow g(x) = (x - \alpha)^2 P(x)$, where $P(x)$ is a polynomial of degree 3. Also, $f(x) = 0$ is a quadratic equation having repeated root $\alpha \Rightarrow g(x)$ is divisible by $f(x)$.

Hence, (c) is the correct answer.

Ex 41. $\Delta = \begin{vmatrix} x^2 + x + 2 & 2x^2 + 3x + 1 \\ 20x^2 + 20x + 59 & 40x^2 + 60x + 20 \\ 2x^2 + 2x + 6 & 4x^2 + 6x + 2 \end{vmatrix}$

$$= \begin{vmatrix} 3x^2 + 5x + 3 \\ 60x^2 + 100x + 70 \\ 6x^2 + 10x + 7 \end{vmatrix}$$

$= Ax^2 + Bx + C$, where A and B are determinants of order 3 not involving x , then A, B and C are

- (a) $A = 2, B = 3, C = 1$
(b) $A = 3, B = 2, C = -1$
(c) $A = 4, B = 3, C = -1$
(d) None of the above

Sol. On applying $R_2 \rightarrow R_2 - 2R_1 - 9R_3, R_3 \rightarrow R_3 - 2R_1$, we get

$$\Delta = \begin{vmatrix} x^2 + x + 2 & 2x^2 + 3x + 1 & 3x^2 + 5x + 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2 + x & 2x^2 + 3x & 3x^2 + 5x \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + x \begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$$

where, $A = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$; $B = \begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

and $C = \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

$$\therefore A = 2, B = 3, C = 1$$

Hence, (a) is the correct answer.

Ex 42. The value of the determinant

$$\begin{vmatrix} 2 & \tan A \cot B & \tan A \cot C \\ \tan B \cot A & 2 & \tan B \cot C \\ \tan C \cot A & \tan C \cot B & 2 \end{vmatrix}$$

- (a) 1 (b) 2
(c) 0 (d) None of these

Sol. The given determinant breaks in the product of two determinants

$$\begin{vmatrix} \tan A & \cot A & 0 \\ \tan B & \cot B & 0 \\ \tan C & \cot C & 0 \end{vmatrix} \times \begin{vmatrix} \cot A & \tan A & 0 \\ \cot B & \tan B & 0 \\ \cot C & \tan C & 0 \end{vmatrix} = 0$$

Hence, (c) is the correct answer.

Ex 43. If $f(x) = \begin{vmatrix} 5 + \sin^2 x & \cos^2 x & 4 \sin 2x \\ \sin^2 x & 5 + \cos^2 x & 4 \sin 2x \\ \sin^2 x & \cos^2 x & 5 + 4 \sin 2x \end{vmatrix}$,

then

- (a) domain of function $f(x) \in (0, \infty)$
(b) range of function $f(x) \in (0, \infty)$
(c) period of function $f(x)$ is 2π
(d) $\lim_{x \rightarrow 0} \frac{f(x) - 150}{x} = 200$

Sol. On applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} 5 & -5 & 0 \\ 0 & 5 & -5 \\ \sin^2 x & \cos^2 x & 5 + 4 \sin 2x \end{vmatrix}$$

$$= 25 \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ \sin^2 x & \cos^2 x & 5 + 4 \sin 2x \end{vmatrix}$$

$$\Rightarrow f(x) = 150 + 100 \sin 2x$$

Clearly,

- (i) domain is $(-\infty, \infty)$
(ii) range is $[50, 250]$
(iii) period π

(iv) $\lim_{x \rightarrow 0} \frac{f(x) - 150}{x} = 200$

Hence, (d) is the correct answer.

- Ex 44.** The values of λ and μ for which the equations $x + y + z = 3$, $x + 3y + 2z = 6$ and $x + \lambda y + 3z = \mu$ have
- a unique solution, if $\lambda = 5, \mu \in R$
 - no solution, if $\lambda \neq 5, \mu = 9$
 - infinite many solution, if $\lambda = 5, \mu \neq 9$
 - None of the above

Sol. Consider the augmented matrix

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 1 & 3 & 2 & : & 6 \\ 1 & \lambda & 3 & : & \mu \end{bmatrix}$$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 2 & 1 & : & 3 \\ 0 & \lambda - 1 & 2 & : & \mu - 3 \end{bmatrix}$$

On applying $R_3 \rightarrow R_3 - 2R_2$, we get

$$[A : B] = \begin{bmatrix} 1 & 1 & 1 & : & 3 \\ 0 & 2 & 1 & : & 3 \\ 0 & \lambda - 5 & 0 & : & \mu - 9 \end{bmatrix}$$

Clearly,

- System has unique solution, if $\lambda \neq 5$
- System has no solution, if $\lambda = 5$ and $\mu \neq 9$
- System has infinite number of solutions, if $\lambda = 5$ and $\mu = 9$

Hence, (d) is the correct answer.

Ex 45. If $\Delta(x) =$

$$\begin{vmatrix} \sin 2x & e^x \sin x + x \cos x & \sin x + x^2 \cos x \\ \cos x + \sin x & e^x + x & 1 + x^2 \\ e^x \cos x & e^{2x} & e^x \end{vmatrix},$$

then

- $\Delta'(0) = 0$
- $\Delta'\left(\frac{\pi}{2}\right) = 0$
- $\Delta'\left(\frac{\pi}{4}\right) = 0$
- All of these

Sol. We can write $\Delta(x)$ as product of two determinants as follows:

$$\Delta(x) = \begin{vmatrix} \sin x & \cos x & 1 \\ 1 & 1 & 1 \\ e^x & 0 & 0 \end{vmatrix} \times \begin{vmatrix} \cos x & \sin x & 0 \\ e^x & x & 0 \\ 1 & x^2 & 0 \end{vmatrix} = 0$$

which is independent of x

$$\Rightarrow \Delta'(x) = 0, \forall x$$

Hence, (d) is the correct answer.

Ex 46. If $\Delta(x) = \begin{vmatrix} 4x - 4 & (x - 2)^2 & x^3 \\ 8x - 4\sqrt{2} & (x - 2\sqrt{2})^2 & (x + 1)^3 \\ 12x - 4\sqrt{3} & (x - 2\sqrt{3})^2 & (x - 1)^3 \end{vmatrix}$,

then the coefficient of x in $\Delta(x)$ is

- $64(5 - \sqrt{2} - \sqrt{3})$
- $64(5 + \sqrt{2} - \sqrt{3})$
- $64(5 + \sqrt{2} + \sqrt{3})$
- None of these

Sol. Note that $\Delta(x)$ is polynomial of degree atmost 6 in x .

If $\Delta(x) = a_0 + a_1x + a_2x^2 + \dots + a_6x^6$, then

$$\Delta'(x) = a_1 + 2a_2x + \dots + 6a_6x^5 \Rightarrow a_1 = \Delta'(0)$$

$$\text{Now, } \Delta'(x) = \begin{vmatrix} 4 & (x-2)^2 & x^3 \\ 8 & (x-2\sqrt{2})^2 & (x+1)^3 \\ 12 & (x-2\sqrt{3})^2 & (x-1)^3 \end{vmatrix}$$

$$+ \begin{vmatrix} 4x-4 & 2(x-2) & x^3 \\ 8x-4\sqrt{2} & 2(x-2\sqrt{2}) & (x+1)^3 \\ 12x-4\sqrt{3} & 2(x-2\sqrt{3}) & (x-1)^3 \end{vmatrix}$$

$$+ \begin{vmatrix} 4x-4 & (x-2)^2 & 3x^2 \\ 8x-4\sqrt{2} & (x-2\sqrt{2})^2 & 3(x+1)^2 \\ 12x-4\sqrt{3} & (x-2\sqrt{3})^2 & 3(x-1)^2 \end{vmatrix}$$

$$\Rightarrow \Delta'(0) = 0 + 0 + \begin{vmatrix} -4 & 4 & 0 \\ -4\sqrt{2} & 8 & 3 \\ -4\sqrt{3} & 12 & 3 \end{vmatrix}$$

$$\Rightarrow \Delta'(0) = 48(1 + \sqrt{2} - \sqrt{3})$$

Hence, (d) is the correct answer.

Ex 47. If $\Delta_1 = \begin{vmatrix} y^5 z^6 (z^3 - y^3) & x^4 z^6 (x^3 - z^3) \\ y^2 z^3 (y^6 - z^6) & xz^3 (z^6 - x^6) \\ y^2 z^3 (z^3 - y^3) & xz^3 (x^3 - z^3) \end{vmatrix}$

$$\begin{vmatrix} x^4 y^5 (y^3 - x^3) \\ xy^2 (x^6 - y^6) \\ xy^2 (y^3 - x^3) \end{vmatrix}$$

and $\Delta_2 = \begin{vmatrix} x & y^2 & z^3 \\ x^4 & y^5 & z^6 \\ x^7 & y^8 & z^9 \end{vmatrix}$, then $\Delta_1 \Delta_2$ is equal to

- Δ_2^3
- Δ_2^2
- Δ_2^4
- None of these

Sol. The given determinant Δ_1 is obtained by corresponding cofactors of elements of determinant Δ_2 , hence $\Delta_1 = \Delta_2^2$. Now, $\Delta_1 \Delta_2 = \Delta_2^2 \Delta_2 = \Delta_2^3$

Hence, (a) is the correct answer.

Ex 48. If $a, b > 0$ and $\Delta(x) = \begin{vmatrix} x & a & a \\ b & x & a \\ b & b & x \end{vmatrix}$, then

- $\Delta(x)$ is increasing in $(-\sqrt{ab}, \sqrt{ab})$
- $\Delta(x)$ is decreasing in $(\sqrt{ab}, \infty)$
- $\Delta(x)$ has a local minimum at $x = \sqrt{ab}$
- None of the above

Sol. Given, $\Delta(x) = \begin{vmatrix} x & a & a \\ b & x & a \\ b & b & x \end{vmatrix}$

Then, $\Delta'(x) = \begin{vmatrix} 1 & a & a \\ 0 & x & a \\ 0 & b & x \end{vmatrix} + \begin{vmatrix} x & 0 & a \\ b & 1 & a \\ b & 0 & x \end{vmatrix} + \begin{vmatrix} x & a & 0 \\ b & x & 0 \\ b & b & 1 \end{vmatrix}$

$$\Rightarrow \Delta'(x) = 3(x^2 - ab)$$

Now, sign scheme for $\Delta'(x)$,

$$\begin{array}{c|c|c} + & - & + \\ \hline -\sqrt{ab} & \sqrt{ab} & \end{array}$$

$\therefore \Delta(x)$ is increasing in $(-\infty, -\sqrt{ab}) \cup (\sqrt{ab}, \infty)$

$\Delta(x)$ is decreasing in $(-\sqrt{ab}, \sqrt{ab})$

$\Delta(x)$ has a local minimum at $x = \sqrt{ab}$

and $\Delta(x)$ has a local maximum at $x = -\sqrt{ab}$

Hence, (c) is the correct answer.

Ex 49. The number of positive integral solutions of

the equation $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 30$ is

- (a) 0 (b) 3 (c) 6 (d) 9

Sol. $\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix}$

$$= \begin{vmatrix} x^3 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} + \begin{vmatrix} 1 & x^2y & x^2z \\ 0 & y^3+1 & y^2z \\ 0 & yz^2 & z^3+1 \end{vmatrix}$$

$$= x \begin{vmatrix} x^2 & x^2y & x^2z \\ y^2 & y^3+1 & y^2z \\ z^2 & yz^2 & z^3+1 \end{vmatrix} + (y^3+1)(z^3+1) - y^3z^3$$

$$= x \begin{vmatrix} x^2 & 0 & 0 \\ y^2 & 1 & 0 \\ z^2 & 0 & 1 \end{vmatrix} + y^3z^3 + z^3 + y^3 + 1 - y^3z^3$$

[applying $C_2 \rightarrow C_2 - yC_1$ and $C_3 \rightarrow C_3 - zC_1$]
 $= x^3 + y^3 + z^3 + 1$

Given, $x^3 + y^3 + z^3 + 1 = 30 \Rightarrow x^3 + y^3 + z^3 = 29$

As $29 = 3^3 + 1^3 + 1^3$, then solutions are (3, 1, 1), (1, 3, 1), (1, 1, 3)

Hence, (b) is the correct answer.

Ex 50. If a point (x, y) moves on a curve and satisfies the equation

$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = 0$. Then,

- (a) a, b, c form an AP
 (b) a, b, c form an HP
 (c) the point (x, y) lies on a curve (pair of straight lines) through the origin
 (d) None of the above

Sol. On applying $C_3 \rightarrow C_3 - xC_1 - yC_2$, we get

$$\Delta = \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ ax+by & bx+cy & -(ax^2+ay^2+2bxy) \end{vmatrix} = 0$$

$$\Rightarrow (b^2 - ac)(ax^2 + 2bxy + ay^2) = 0$$

Either $b^2 = ac$ or $ax^2 + 2bxy + ay^2 = 0$

Thus, the point (x, y) lies on a curve (pair of straight lines) through the origin.

Hence, (c) is the correct answer.

Ex 51. If $ax^3 + bx^2 + cx + d$

$$= \begin{vmatrix} x^2 & (x-1)^2 & (x-2)^2 \\ (x-1)^2 & (x-2)^2 & (x-3)^2 \\ (x-2)^2 & (x-3)^2 & (x-4)^2 \end{vmatrix}, \text{ then}$$

- (a) $a = 1, b = 2, c = 3, d = -8$
 (b) $a = -1, b = 2, c = 3, d = -8$
 (c) $a = 0, b = 0, c = 0, d = 8$
 (d) $a = 0, b = 0, c = 0, d = -8$

Sol. On applying $C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$, we get

$$ax^3 + bx^2 + cx + d = \begin{vmatrix} (2x-1) & (2x-3) & (x-2)^2 \\ (2x-3) & (2x-5) & (x-3)^2 \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 2 & (2x-5) \\ 2 & 2 & (2x-7) \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$]

$$= \begin{vmatrix} 0 & 0 & 2 \\ 2 & 2 & (2x-7) \\ (2x-5) & (2x-7) & (x-4)^2 \end{vmatrix} = -8$$

[applying $R_1 \rightarrow R_1 - R_2$]

So, value of determinant is independent of x .

$\therefore a = b = c = 0$

and $d = -8$

Hence, (d) is the correct answer.

Ex 52. If $a, b, c, d > 0, x \in R$ and $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \leq 0$,

then $\begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$ is equal to

- (a) 1 (b) -1
 (c) 2 (d) 0

Sol. Given, $(a^2 + b^2 + c^2)x^2 - 2(ab + bc + cd)x + b^2 + c^2 + d^2 \leq 0$

$$\Rightarrow (ax - b)^2 + (bx - c)^2 + (cx - d)^2 \leq 0$$

$$\Rightarrow (ax - b)^2 + (bx - c)^2 + (cx - d)^2 = 0$$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c} = x$$

$$\Rightarrow b^2 = ac \text{ or } 2 \log b = \log a + \log c$$

Now, $\Delta = \begin{vmatrix} 33 & 14 & \log a \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$

On applying $R_1 \rightarrow R_1 + R_3$, we get

$$\Delta = \begin{vmatrix} 130 & 54 & \log a + \log c \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix}$$

Now, on applying $R_1 \rightarrow R_1 - 2R_2$,

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ 65 & 27 & \log b \\ 97 & 40 & \log c \end{vmatrix} = 0$$

Hence, (d) is the correct answer.

Ex 53. If $f(x) = \begin{vmatrix} x^2 - 4x + 6 & 2x^2 + 4x + 10 \\ x - 2 & 2x + 2 \\ 1 & 2 \end{vmatrix}$

$$\begin{vmatrix} 3x^2 - 2x + 16 \\ 3x - 1 \\ 3 \end{vmatrix},$$

then the value of $\int_{-3}^3 \frac{x^2 \sin x}{1+x^6} \cdot f(x) dx$ is

- (a) 6 (b) 3
(c) 0 (d) $\frac{\pi}{2}$

Sol. Observe that the elements of R_3 are the derivatives of the elements of R_2 and they are in turn proportional to the derivatives of the elements of R_1 . Therefore,

$$f'(x) = \begin{vmatrix} R_1' \\ R_2' \\ R_3' \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2' \\ R_3' \end{vmatrix} + \begin{vmatrix} R_1 \\ R_2 \\ R_3' \end{vmatrix} = 0, \forall x \in \mathbb{R}$$

and $f(x) = \text{constant}$

$$\text{As } f(0) = \begin{vmatrix} 6 & 10 & 16 \\ -2 & 2 & -1 \\ 1 & 2 & 3 \end{vmatrix} = 2; f(x) = 2$$

$$\text{Therefore, } \int_{-3}^3 \frac{x^2 \sin x}{1+x^6} \cdot f(x) dx = 2 \int_{-3}^3 \frac{x^2 \sin x}{1+x^6} dx$$

$$\text{Let } g(x) = \frac{x^2 \sin x}{1+x^6}$$

$$\Rightarrow g(-x) = \frac{-x^2 \sin x}{1+x^6} = -g(x)$$

So, $g(x)$ is an odd function.

$$\therefore \int_{-3}^3 \frac{x^2 \sin x}{1+x^6} \cdot f(x) dx = 0$$

Hence, (c) is the correct answer.

Ex 54. The determinant

$$\Delta = \begin{vmatrix} -1+x \sin \alpha & \cos(x+\alpha) & \sin(x+\alpha) \\ 13+x \sin \beta & \cos(x+\beta) & \sin(x+\beta) \\ -12+x \sin \gamma & \cos(x+\gamma) & \sin(x+\gamma) \end{vmatrix}$$

is independent of

- (a) x
(b) α, β
(c) β, γ
(d) None of the above

Sol. Let $f(x) = \begin{vmatrix} -1+x \sin \alpha & \cos(x+\alpha) & \sin(x+\alpha) \\ 13+x \sin \beta & \cos(x+\beta) & \sin(x+\beta) \\ -12+x \sin \gamma & \cos(x+\gamma) & \sin(x+\gamma) \end{vmatrix}$

$$\begin{aligned} \therefore f'(x) &= \begin{vmatrix} \sin \alpha & \cos(x+\alpha) & \sin(x+\alpha) \\ \sin \beta & \cos(x+\beta) & \sin(x+\beta) \\ \sin \gamma & \cos(x+\gamma) & \sin(x+\gamma) \end{vmatrix} \\ &+ \begin{vmatrix} -1+x \sin \alpha & -\sin(x+\alpha) & \sin(x+\alpha) \\ 13+x \sin \beta & -\sin(x+\beta) & \sin(x+\beta) \\ -12+x \sin \gamma & -\sin(x+\gamma) & \sin(x+\gamma) \end{vmatrix} \\ &+ \begin{vmatrix} -1+x \sin \alpha & \cos(x+\alpha) & \cos(x+\alpha) \\ 13+x \sin \beta & \cos(x+\beta) & \cos(x+\beta) \\ -12+x \sin \gamma & \cos(x+\gamma) & \cos(x+\gamma) \end{vmatrix} \\ \Rightarrow f'(x) &= \begin{vmatrix} \sin \alpha & \cos x \cos \alpha - \sin x \sin \alpha \\ \sin \beta & \cos x \cos \beta - \sin x \sin \beta \\ \sin \gamma & \cos x \cos \gamma - \sin x \sin \gamma \end{vmatrix} \\ &+ \begin{vmatrix} \sin x \cos \alpha + \cos x \sin \alpha \\ \sin x \cos \beta + \cos x \sin \beta \\ \sin x \cos \gamma + \cos x \sin \gamma \end{vmatrix} + 0 + 0 \end{aligned}$$

This determinant can be expressed as the sum of four determinants in each of which two columns coincide after taking $\cos x, \sin x$ common.

$\therefore$ The value of the determinant = 0

i.e. $f'(x) = 0 \Rightarrow f(x) = \text{a constant function, i.e. independent of } x$.

Hence, (a) is the correct answer.

Ex 55. If a, b and c are real, then the interval in

$$\text{which } f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix} \text{ is}$$

decreasing, is

- (a) $(0, \infty)$
(b) $\left(0, -\frac{2}{3}(a^2 + b^2 + c^2)\right)$
(c) $(-\infty, 0)$
(d) $\left(-\frac{2}{3}(a^2 + b^2 + c^2), 0\right)$

Sol. $f(x) = \begin{vmatrix} x+a^2 & ab & ac \\ ab & x+b^2 & bc \\ ac & bc & x+c^2 \end{vmatrix}$

$$= abc \begin{vmatrix} \frac{x+a^2}{a} & a & a \\ b & \frac{x+b^2}{b} & b \\ c & c & \frac{x+c^2}{c} \end{vmatrix}$$

$$= \begin{vmatrix} x+a^2 & a^2 & a^2 \\ b^2 & x+b^2 & b^2 \\ c^2 & c^2 & x+c^2 \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, then take common $x + a^2 + b^2 + c^2$, we get

$$f(x) = (x + a^2 + b^2 + c^2) \begin{vmatrix} 1 & 1 & 1 \\ b^2 & x+b^2 & b^2 \\ c^2 & c^2 & x+c^2 \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$, we get

$$f(x) = (x + a^2 + b^2 + c^2) \begin{vmatrix} 1 & 0 & 0 \\ b^2 & x & 0 \\ c^2 & 0 & x \end{vmatrix}$$

$$\therefore f(x) = (x + a^2 + b^2 + c^2) x^2$$

$$\Rightarrow f'(x) = 3x \left(x + \frac{2}{3} (a^2 + b^2 + c^2) \right)$$

$$f'(x) < 0, \text{ when } x \in \left(-\frac{2}{3} (a^2 + b^2 + c^2), 0 \right)$$

Hence, (d) is the correct answer.

Ex 56. If $\begin{vmatrix} a & \cot A/2 & \lambda \\ b & \cot B/2 & \mu \\ c & \cot C/2 & \gamma \end{vmatrix} = 0$, where a, b, c, A, B

and C are elements of a ΔABC with usual meaning.

Then, the value of $a(\mu - \gamma) + b(\gamma - \lambda) + c(\lambda - \mu)$ is

- (a) 0 (b) abc
(c) $ab + bc + ca$ (d) $2abc$

Sol. Given, $\begin{vmatrix} a & \cot A/2 & \lambda \\ b & \cot B/2 & \mu \\ c & \cot C/2 & \gamma \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & \frac{s(s-a)}{\Delta} & \lambda \\ b & \frac{s(s-b)}{\Delta} & \mu \\ c & \frac{s(s-c)}{\Delta} & \gamma \end{vmatrix} = 0$

$$\Rightarrow \frac{1}{r} \begin{vmatrix} a & s-a & \lambda \\ b & s-b & \mu \\ c & s-c & \gamma \end{vmatrix} = 0, \text{ where } r = \frac{\Delta}{s}$$

On applying $C_2 \rightarrow C_2 + C_1$, we get

$$\frac{1}{r} \begin{vmatrix} a & s & \lambda \\ b & s & \mu \\ c & s & \gamma \end{vmatrix} = 0 \Rightarrow \frac{\Delta}{r^2} \begin{vmatrix} a & 1 & \lambda \\ b & 1 & \mu \\ c & 1 & \gamma \end{vmatrix} = 0$$

On applying $R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$, we get

$$\frac{\Delta}{r^2} \begin{vmatrix} a-b & 0 & \lambda-\mu \\ b-c & 0 & \mu-\gamma \\ c & 1 & \gamma \end{vmatrix} = 0$$

$$\Rightarrow \frac{\Delta}{r^2} [(b-c)(\lambda-\mu) - (\mu-\gamma)(a-b)] = 0$$

$$\Rightarrow b(\lambda-\mu) - c(\lambda-\mu) - a(\mu-\gamma) + b(\mu-\gamma) = 0$$

$$\Rightarrow -a(\mu-\gamma) + b(\lambda-\mu + \mu-\gamma) - c(\lambda-\mu) = 0$$

$$\Rightarrow -a(\mu-\gamma) + b(\lambda-\gamma) - c(\lambda-\mu) = 0$$

$$\Rightarrow a(\mu-\gamma) + b(\gamma-\lambda) + c(\lambda-\mu) = 0$$

Hence, (a) is the correct answer.

Ex 57. Let p be the sum of all possible determinants of order 2 having 0, 1, 2 and 3 as their four elements. Then, the common root α of the equations

$$x^2 + ax + [m+1] = 0$$

$$x^2 + bx + [m+4] = 0$$

$$x^2 - cx + [m+15] = 0$$

such that $\alpha > p$, where $a + b + c = 0$ and

$$m = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}, \text{ is}$$

(where, $[]$ denotes the greatest integer function)

- (a) ± 3 (b) 3
(c) -3 (d) None of these

Sol. Since, α is the common root.

$$\therefore \alpha^2 + a\alpha + [m+1] = 0 \quad \dots(i)$$

$$\alpha^2 + \alpha b + [m+4] = 0 \quad \dots(ii)$$

$$\text{and } \alpha^2 - c\alpha + [m+15] = 0 \quad \dots(iii)$$

On adding Eqs. (i) and (ii) and subtracting from Eq. (iii), we get

$$\alpha^2 + [m] - 10 = 0 \quad \dots(iv)$$

But $m = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r/n}{\sqrt{1 + (r/n)^2}} = \int_0^2 \frac{x}{\sqrt{1+x^2}} dx = (\sqrt{1+x^2})_0^2 = \sqrt{5} - 1$$

$$\text{Now, } [m] = [\sqrt{5} - 1] = 1$$

From Eq. (iv), we get

$$\alpha^2 + 1 - 10 = 0 \Rightarrow \alpha = \pm 3 \quad \dots(v)$$

Now, number of determinants of order 2 having 0, 1, 2, 3 as their four elements = $4! = 24$

Let $\Delta_1 = \begin{vmatrix} a_1 & a_2 \\ a_3 & a_4 \end{vmatrix}$ be one such determinant and there exists another determinant.

$$\Delta_2 = \begin{vmatrix} a_3 & a_4 \\ a_1 & a_2 \end{vmatrix} \text{ (obtained on interchanging } R_1 \text{ and } R_2)$$

such that $\Delta_1 + \Delta_2 = 0$

p = sum of all the 24 determinants = 0

Since, $\alpha > p \Rightarrow \alpha > 0$

From Eq. (v), $\alpha = 3$

Hence, (b) is the correct answer.

Ex 58. Let $[a]$ denotes the integral part of a and $x = a_3y + a_2z, y = a_1z + a_3x, z = a_2x + a_1y$, where x, y, z are not all zero. If $a_1 = m - [m]$, m being a non-integral constant, then $a_1a_2a_3$ is

- (a) > 1 (b) > -1
(c) < 1 (d) < -1

Sol. Given,

$$x = a_3y + a_2z \quad \dots(i)$$

$$y = a_1z + a_3x \quad \dots(ii)$$

$$z = a_2x + a_1y \quad \dots(iii)$$

Since, x, y, z are not all zero, therefore given system of equations has non-trivial solution.

$$\therefore \begin{vmatrix} 1 & -a_3 & -a_2 \\ a_3 & -1 & a_1 \\ a_2 & a_1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow a_1^2 + a_2^2 + a_3^2 + 2a_1a_2a_3 = 1 \quad \dots(iv)$$

Since, $a_1 = m - [m]$ and m is not an integer.

$$\therefore 0 < a_1 < 1 \Rightarrow 0 < 1 - a_1^2 < 1 \quad \dots(v)$$

From Eq. (iv),

$$\begin{aligned} 1 - a_2^2 - a_3^2 &= a_1^2 + 2a_1a_2a_3 \\ \Rightarrow 1 - a_2^2 - a_3^2 + a_2^2 a_3^2 &= a_1^2 + 2a_1a_2a_3 + a_2^2 a_3^2 \\ \Rightarrow (1 - a_2^2)(1 - a_3^2) &= (a_1 + a_2a_3)^2 \quad \dots(vi) \end{aligned}$$

Similarly,

$$\begin{aligned} (1 - a_1^2)(1 - a_3^2) &= (a_2 + a_1a_3)^2 \quad \dots(vii) \\ (1 - a_1^2)(1 - a_2^2) &= (a_3 + a_1a_2)^2 \quad \dots(viii) \end{aligned}$$

$$\text{From Eq. (viii), } 1 - a_2^2 = \frac{(a_3 + a_1a_2)^2}{1 - a_1^2} > 0$$

From Eq. (vii),

$$\begin{aligned} 1 - a_3^2 &> 0 \\ \Rightarrow 3 - (a_1^2 + a_2^2 + a_3^2) &> 0 \\ \Rightarrow a_1^2 + a_2^2 + a_3^2 &< 3 \\ \Rightarrow 1 - 2a_1a_2a_3 &< 3 \quad [\text{from Eq. (iv)}] \\ \Rightarrow a_1a_2a_3 &> -1 \end{aligned}$$

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 59. The values of θ lying between $-\frac{\pi}{4}$ and $\frac{\pi}{2}$ and

$$0 \leq A \leq \frac{\pi}{2} \text{ and satisfying the equation}$$

$$\begin{vmatrix} 1 + \sin^2 A & \cos^2 A & 2 \sin 4\theta \\ \sin^2 A & 1 + \cos^2 A & 2 \sin 4\theta \\ \sin^2 A & \cos^2 A & 1 + 2 \sin 4\theta \end{vmatrix} = 0 \text{ are}$$

$$\begin{aligned} \text{(a) } A = \frac{\pi}{4}, \theta = -\frac{\pi}{8} & \quad \text{(b) } A = \frac{3\pi}{8}, \theta = \frac{\pi}{8} \\ \text{(c) } A = \frac{\pi}{5}, \theta = -\frac{\pi}{8} & \quad \text{(d) } A = \frac{\pi}{6}, \theta = \frac{3\pi}{8} \end{aligned}$$

Sol. Use $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$ and expand to get

$$\Delta = 2(1 + \sin 4\theta) = 0$$

$$\Rightarrow \theta = \frac{n\pi}{4} - (-1)^n \frac{\pi}{8} \Rightarrow \text{Independent of } A.$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 60. If $\begin{vmatrix} 1 & a & a^2 \\ 1 & x & x^2 \\ b^2 & ab & a^2 \end{vmatrix} = 0$, then

$$\begin{aligned} \text{(a) } x = a & \quad \text{(b) } x = b \\ \text{(c) } x = \frac{1}{a} & \quad \text{(d) } x = \frac{a}{b} \end{aligned}$$

Sol. $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$ gives

$$\Delta = (x - a)(b - 1) \begin{vmatrix} 1 & a & a^2 \\ 0 & 1 & x + a \\ b + 1 & a & 0 \end{vmatrix}$$

On expanding along C_1 , we get the values of $x = a/b$ and $x = a$.

Hence, (a) and (d) are the correct answers.

Ex 61. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (where, $bc \neq 0$) satisfies the equation $x^2 + k = 0$, then

$$\begin{aligned} \text{(a) } a + d = 0 & \quad \text{(b) } k = -|A| \\ \text{(c) } k = |A| & \quad \text{(d) None of these} \end{aligned}$$

Sol. We have, $A^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + bc & ab + db \\ ac + cd & bc + d^2 \end{bmatrix}$

As A satisfies $x^2 + k = 0$, $A^2 + kI = O$

$$\Rightarrow \begin{bmatrix} a^2 + bc + k & (a + d)b \\ (a + d)c & bc + d^2 + k \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\Rightarrow a^2 + bc + k = 0, bc + d^2 + k = 0$$

$$\text{and } (a + d)b = (a + d)c = 0$$

$$\text{As } bc \neq 0, b \neq 0, c \neq 0 \Rightarrow a + d = 0 \Rightarrow a = -d$$

$$\text{Also, } k = -(a^2 + bc) = -(d^2 + bc) = -((-ad) + bc) = |A|$$

Hence, (a) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 62-64) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Ex 62. Statement I If $f_1(x), f_2(x), \dots, f_9(x)$ are polynomials whose degree ≥ 1 , where

$$f_1(\alpha) = f_2(\alpha) = f_3(\alpha) = \dots = f_9(\alpha) = 0 \text{ and}$$

$$A(x) = \begin{bmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_4(x) & f_5(x) & f_6(x) \\ f_7(x) & f_8(x) & f_9(x) \end{bmatrix}, \text{ then } \frac{A(x)}{x - \alpha} \text{ is}$$

also a matrix of order 3×3 , whose entries are also polynomials.

Statement II $x - \alpha$ is a factor of polynomial $f(x)$, if $f(\alpha) = 0$.

Sol. $A(\alpha) = \begin{bmatrix} f_1(\alpha) & f_2(\alpha) & f_3(\alpha) \\ f_4(\alpha) & f_5(\alpha) & f_6(\alpha) \\ f_7(\alpha) & f_8(\alpha) & f_9(\alpha) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

As $x - \alpha$ is a factor of $f_1(x), f_2(x), \dots, f_9(x)$

$$f(x) = (x - \alpha) \phi(x)$$

$f(\alpha) = 0 \Rightarrow x - \alpha$ is a factor of $f(x)$

Hence, (a) is the correct answer.

Ex 63. Statement I If $(a_{11}, a_{22}, \dots, a_{nn})$ is a diagonal matrix, then $A^{-1} = \text{diag}(a_{11}^{-1}, a_{22}^{-1}, \dots, a_{nn}^{-1})$

Statement II If $A = \text{diag}(2, 1, -3)$ and $B = \text{diag}(1, 1, 2)$, then $\det(AB^{-1}) = 3$.

Sol. $\det(AB^{-1}) = \det(A) \cdot \det(B^{-1}) = \frac{\det(A)}{\det(B)} = \frac{-6}{2} = -3$

Hence, (c) is the correct answer.

Ex 64. Consider the system of equations $x - 2y + 3z = -1$, $x - 3y + 4z = 1$ and $-x + y - 2z = k$

Statement I The system of equations has no solution for $k \neq 3$.

Statement II The determinant

$$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0 \text{ for } k \neq 3.$$

Sol. Given system of equations is

$$x - 2y + 3z = -1$$

$$x - 3y + 4z = 1$$

$$-x + y - 2z = k$$

$$\therefore D = \begin{vmatrix} 1 & -2 & 3 \\ 1 & -3 & 4 \\ -1 & 1 & -2 \end{vmatrix}$$

$$= 1(6 - 4) + 2(-2 + 4) + 3(1 - 3) = 2 + 4 - 6 = 0$$

As per given determinant in Statement II, we find D_2

$$D_2 = \begin{vmatrix} 1 & -1 & 3 \\ 1 & 1 & 4 \\ -1 & k & -2 \end{vmatrix}$$

$$= - \begin{vmatrix} 1 & 3 & -1 \\ 1 & 4 & 1 \\ -1 & -2 & k \end{vmatrix} \quad [\because C_2 \leftrightarrow C_3]$$

$$= - \begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \quad [\because R_2 \leftrightarrow R_3]$$

$$= 1(-2 - 4k) - 3(-1 - k) - 1(-4 + 2) = 3 - k$$

$$\therefore D_2 \neq 0 \text{ for } k \neq 3$$

Hence, Statement II is true and also for $k \neq 3$, $D_2 \neq 0$ but $D = 0$

$\therefore$ The system of equations has no solution.

$\Rightarrow$ Statement I is true and justified by Statement II.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 65-67) Consider the determinant

$$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ d_1 & d_2 & d_3 \end{vmatrix}$$

M_{ij} = Minor of the element of i th row and j th column.

C_{ij} = Cofactor of the element of i th row and j th column.

Ex 65. Value of $b_1C_{31} + b_2C_{32} + b_3C_{33}$ is

- (a) 0 (b) Δ
(c) 2Δ (d) Δ^2

Sol. $b_1C_{31} + b_2C_{32} + b_3C_{33}$
 $= b_1 \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - b_2 \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 0$

Hence, (a) is the correct answer.

Ex 66. If all the elements of the determinant are multiplied by 2, then the value of new determinant is

- (a) 0 (b) 8Δ
(c) 2Δ (d) $2^9 \Delta$

Sol. Value of new determinant $= 2^3 \Delta = 8\Delta$

Hence, (b) is the correct answer.

Ex 67. $a_3M_{13} - b_3M_{23} + d_3M_{33}$ is equal to

- (a) 0
(b) 4Δ
(c) 2Δ
(d) Δ

Sol. $a_3M_{13} - b_3M_{23} + d_3M_{33}$
 $= a_3C_{13} + b_3C_{23} + d_3C_{33} = \Delta$ [by definition]

Hence, (d) is the correct answer.

Passage II (Ex. Nos. 68-70) Consider the determinant, $\Delta = \begin{vmatrix} p & q & r \\ x & y & z \\ l & m & n \end{vmatrix}$. M_{ij} denotes the minor of

an element in i th row and j th column. C_{ij} denotes the cofactor of an element in i th row and j th column.

Ex 68. The value of $pC_{21} + qC_{22} + rC_{23}$ is

- (a) 0 (b) $-\Delta$
(c) Δ (d) Δ^2

Sol. p, q, r are the entries of first row and C_{21}, C_{22}, C_{23} are cofactors of elements of second row.

$$\therefore pC_{21} + qC_{22} + rC_{23} = 0$$

Hence, (a) is the correct answer.

Ex 69. The value of $xC_{21} + yC_{22} + zC_{23}$ is

- (a) 0
(b) $-\Delta$
(c) Δ
(d) Δ^2

Sol. x, y, z are the entries of second row and C_{21}, C_{22}, C_{23} are cofactors of elements of second row.

$$\therefore xC_{21} + yC_{22} + zC_{23} = \Delta$$

Hence, (c) is the correct answer.

Ex 70. The value of $qM_{12} - yM_{22} + mM_{32}$ is

- (a) 0
(b) $-\Delta$
(c) Δ
(d) Δ^2

Sol. $qM_{12} - yM_{22} + mM_{32}$
 $= -qC_{12} - yC_{22} - mC_{32}$
 $= -(qC_{12} + yC_{22} + mC_{32}) = -\Delta$
 [$\because q, y, m$ are entries of second column and C_{12}, C_{22}, C_{32} are cofactors of elements of second column]
 Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 71. Let x and y be real numbers. Match the determinants on the left with values on the right.

	Column I	Column II
A.	$\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$	p. $4xy$
B.	$\begin{vmatrix} 1 & 2x & 2y \\ 1 & 2x+2y & 2y \\ 1 & 2x & 2x+2y \end{vmatrix}$	q. $x+y+2$
C.	$\begin{vmatrix} 2 & x & y \\ 1 & 1+x & y \\ 1 & x & 1+y \end{vmatrix}$	r. $-2(x^3 + y^3)$
D.	$\begin{vmatrix} 1+x & y & y \\ 1 & x+y & 1 \\ x & x & 1+y \end{vmatrix}$	s. $-4(x+y)$

Sol. A. Denote the determinant by Δ .

Using $C_1 \rightarrow C_1 + C_2 + C_3$, we can write

$$\Delta = 2(x+y)\Delta_1$$

$$\text{where, } \Delta_1 = \begin{vmatrix} 1 & y & x+y \\ 1 & x+y & x \\ 1 & x & y \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1 & y & x+y \\ 0 & x & -y \\ 0 & x-y & -x \end{vmatrix}$$

$$= -x^2 + y(x-y) = -(x^2 - xy + y^2)$$

$$\therefore \Delta = -2(x^3 + y^3)$$

B. Denote the determinant by Δ .

applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = 4 \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix} = 4xy$$

C. Denote the determinant by Δ .

Using $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$\Delta = (x+y+2)\Delta_1$, where

$$\Delta_1 = \begin{vmatrix} 1 & x & y \\ 1 & 1+x & y \\ 1 & x & 1+y \end{vmatrix} = \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

[applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$]

$$\therefore \Delta = (x+y+2)$$

D. Denote the determinant by Δ .

Using $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = 2\Delta_1$$

$$\text{where, } \Delta_1 = \begin{vmatrix} 1+x & x+y & 1+y \\ 1 & x+y & 1 \\ x & x & 1+y \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$, we get

$$\Delta_1 = \begin{vmatrix} 1+x & x+y & 1+y \\ -x & 0 & -y \\ -1 & -y & 0 \end{vmatrix} = \begin{vmatrix} 0 & x & 1 \\ -x & 0 & -y \\ -1 & -y & 0 \end{vmatrix}$$

$$= 2xy$$

[applying $R_1 \rightarrow R_1 + R_2 + R_3$]

$$\therefore \Delta = 4xy$$

$A \rightarrow r; B \rightarrow p; C \rightarrow q; D \rightarrow p$

Ex 72. Let $p(\theta) = \begin{vmatrix} -\sqrt{2} & \sin \theta & \cos \theta \\ 1 & \cos \theta & \sin \theta \\ -1 & \sin \theta & -\cos \theta \end{vmatrix}$,

$$q(\theta) = \sqrt{2} \begin{vmatrix} \sin 2\theta & 1 & 1 \\ \cos 2\theta & 2 & 3 \\ \cos 2\theta & 3 & 5 \end{vmatrix}$$

$$r(\theta) = \begin{vmatrix} \cos \theta & \sin \theta & \cos \theta \\ -\sin \theta & \cos \theta & \sin \theta \\ -\cos \theta & -\sin \theta & \cos \theta \end{vmatrix}$$

$$\text{and } s(\theta) = \begin{vmatrix} \sec^2 \theta & 1 & 1 \\ \cos^2 \theta & \cos^2 \theta & \operatorname{cosec}^2 \theta \\ 1 & \cos^2 \theta & \cot^2 \theta \end{vmatrix}$$

Match the functions on the left with their range on the right.

	Column I	Column II
A.	$p(\theta)$	p. $[0, 1]$
B.	$q(\theta)$	q. $[0, 2\sqrt{2}]$
C.	$r(\theta)$	r. $[-2, 2]$
D.	$s(\theta)$	s. $[-\sqrt{5}-2, -\sqrt{5}+2]$

Sol. A. Expand along C_1 to obtain

$$p(\theta) = (-\sqrt{2})(-1) + (-1)(-2 \sin \theta \cos \theta)$$

$$+ (-1)(\sin^2 \theta - \cos^2 \theta)$$

$$= \sqrt{2} + \sin 2\theta + \cos 2\theta = \sqrt{2} + \sqrt{2} \sin(2\theta + \pi/4)$$

$$\therefore \text{Range of } p(\theta) \text{ is } [0, 2\sqrt{2}].$$

B. On expanding along C_1 , we get

$$q(\theta) = \sqrt{2} (\sin 2\theta - \cos 2\theta)$$

Thus, range of $q(\theta)$ is $[-2, 2]$.

C. On applying $C_1 \rightarrow C_1 + C_3$, we get

$$r(\theta) = 2 \cos \theta \begin{vmatrix} 1 & \sin \theta & \cos \theta \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{vmatrix} = 2 \cos \theta$$

$\therefore$ Range of $r(\theta)$ is $[-2, 2]$.

D. On taking $\sec^2 \theta$ common from R_1 , we get

$$s(\theta) = \sec^2 \theta \begin{vmatrix} 1 & \cos^2 \theta & \cos^2 \theta \\ \cos^2 \theta & \cos^2 \theta & \operatorname{cosec}^2 \theta \\ 1 & \cos^2 \theta & \cot^2 \theta \end{vmatrix}$$

On applying $R_3 \rightarrow R_3 - R_1$, we get

$$\begin{aligned} s(\theta) &= \sec^2 \theta \begin{vmatrix} 1 & \cos^2 \theta & \cos^2 \theta \\ \cos^2 \theta & \cos^2 \theta & \operatorname{cosec}^2 \theta \\ 0 & 0 & \cot^2 \theta - \cos^2 \theta \end{vmatrix} \\ &= \sec^2 \theta (\cot^2 \theta - \cos^2 \theta) (\cos^2 \theta - \cos^4 \theta) \\ &= (\cot^2 \theta - \cos^2 \theta) \sin^2 \theta \\ &= \cos^2 \theta - \cos^2 \theta \sin^2 \theta = \cos^4 \theta \end{aligned}$$

$\therefore$ Range of $s(\theta)$ is $[0, 1]$.

$A \rightarrow q$; $B \rightarrow r$; $C \rightarrow r$; $D \rightarrow p$

Ex 73. Match the statements of Column I with values of Column II.

Column I	Column II
A. Let $ A = a_{ij} _{3 \times 3} \neq 0$. Each element is multiplied by k^{i-j} . Let $ B $ be the resulting determinant, where $k_1 A + k_2 B = 0$. Then, $k_1 + k_2$ is equal to	p. 0
B. The maximum value of a third order determinant each of its entries are ± 1 , equals	q. 4
C. $\begin{vmatrix} 1 & \cos \alpha & \cos \beta \\ \cos \alpha & 1 & \cos \gamma \\ \cos \beta & \cos \gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos \alpha & \cos \beta \\ \cos \alpha & 0 & \cos \gamma \\ \cos \beta & \cos \gamma & 0 \end{vmatrix}$, if $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma$ is equal to	r. 1
D. $\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = Ax + B$, where $A + 2B$ is equal to	s. 2

Sol. A. $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$|B| = \begin{vmatrix} a_{11} & k^{-1} a_{12} & k^{-2} a_{13} \\ ka_{21} & a_{22} & k^{-1} a_{23} \\ k^2 a_{31} & ka_{32} & a_{33} \end{vmatrix}$$

$$= \frac{1}{k^3} \begin{vmatrix} k^2 a_{11} & ka_{12} & a_{13} \\ k^2 a_{21} & ka_{22} & a_{23} \\ k^2 a_{31} & ka_{32} & a_{33} \end{vmatrix} = |A|$$

$$k_1 |A| + k_2 |B| = 0$$

$$k_1 + k_2 = 0$$

B. $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 4$

C. $\begin{vmatrix} 1 & \cos \alpha & \cos \beta \\ \cos \alpha & 1 & \cos \gamma \\ \cos \beta & \cos \gamma & 1 \end{vmatrix} = \begin{vmatrix} 0 & \cos \alpha & \cos \beta \\ \cos \alpha & 0 & \cos \gamma \\ \cos \beta & \cos \gamma & 0 \end{vmatrix}$

$$\Rightarrow \sin^2 \gamma - \cos \alpha (\cos \alpha - \cos \beta \cos \gamma) + \cos \beta (\cos \alpha \cos \gamma - \cos \beta)$$

$$= -\cos \alpha (-\cos \beta \cos \gamma) + \cos \beta (\cos \alpha \cos \gamma)$$

$$\Rightarrow \sin^2 \gamma - \cos^2 \alpha + 2 \cos \alpha \cos \beta \cos \gamma - \cos^2 \beta$$

$$= 2 \cos \alpha \cos \beta \cos \gamma$$

$$\Rightarrow \sin^2 \gamma = \cos^2 \alpha + \cos^2 \beta$$

$$\Rightarrow \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

D. $\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$

$$= \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ -4 & 0 & 0 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix}$$

[applying $R_2 \rightarrow R_2 - (R_1 + R_3)$]

$$= 4 \begin{vmatrix} x + 1 & x - 2 \\ 2x - 1 & 2x - 1 \end{vmatrix}$$

$$= 4 \begin{vmatrix} x + 1 & -3 \\ 2x - 1 & 0 \end{vmatrix} = (24x - 12)$$

$$\Rightarrow A = 24, B = -12$$

$$\therefore A + 2B = 0$$

$A \rightarrow p$; $B \rightarrow q$; $C \rightarrow r$; $D \rightarrow p$

Type 6. Single Integer Answer Type Questions

Ex 74. If $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$, then $(x + y + z) - (u + v + w)$ is _____.

Sol. (7) We have, $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$

or

$$AX = B$$

or $X = A^{-1}B$... (i)

where,

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} \text{ and } B = \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$\begin{aligned} |A| &= 1(-5-7) - 1(-2-14) + 1(2-10) \\ &= -12 + 16 - 8 \\ &= -4 \neq 0 \end{aligned}$$

Let C be the matrix of cofactors of elements of $|A|$.

$$\therefore C = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}$$

$$= \begin{bmatrix} \begin{vmatrix} 5 & 7 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & 7 \\ 2 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 5 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & 1 \\ 5 & 7 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 7 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -12 & 16 & -8 \\ 2 & -3 & 1 \\ 2 & -5 & 3 \end{bmatrix}$$

$$\therefore \text{adj}(A) = C' = \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$

$$= -\frac{1}{4} \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix}$$

$$\text{Now, } A^{-1}B = -\frac{1}{4} \begin{bmatrix} -12 & 2 & 2 \\ 16 & -3 & -5 \\ -8 & 1 & 3 \end{bmatrix} \times \begin{bmatrix} 9 & 2 \\ 52 & 15 \\ 0 & -1 \end{bmatrix}$$

$$= -\frac{1}{4} \begin{bmatrix} -4 & 4 \\ -12 & -8 \\ -20 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

From Eq. (i), $X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x & u \\ y & v \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 3 & 2 \\ 5 & 1 \end{bmatrix}$$

On equating the corresponding elements, we have

$$x = 1, u = -1; y = 3, v = 2; z = 5, w = 1$$

$$\Rightarrow (x + y + z) - (u + v + w) = 9 - 2 = 7$$

$$\text{Ex 75. If } \begin{vmatrix} -2a & a+b & a+c \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = k(b+c)(c+a)$$

$(a+b)$, then the value of k is _____.

Sol. (4) If we choose $a = 0, b = 1, c = 2$, then we get

$$0 - 1(-4 - 6) + 2(3 + 4) = 6k \Rightarrow k = 4$$

Ex 76. If A, B and C are the angles of a triangle, then

$$\begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix} = \lambda \sin A \sin B \sin C,$$

where λ is _____.

$$\text{Sol. (0) LHS} = \begin{vmatrix} \sin 2A & \sin C & \sin B \\ \sin C & \sin 2B & \sin A \\ \sin B & \sin A & \sin 2C \end{vmatrix}$$

$$= \begin{vmatrix} 2ka \cos A & kc & kb \\ kc & 2kb \cos B & ka \\ kb & ka & 2kc \cos C \end{vmatrix}$$

[from sine rule]

$$= k^3 \begin{vmatrix} 2a \cos A & c & b \\ c & 2b \cos B & a \\ b & a & 2c \cos C \end{vmatrix}$$

$$= k^3 \begin{vmatrix} a \cos A + a \cos A & a \cos B + b \cos A \\ a \cos B + b \cos A & b \cos B + b \cos B \\ a \cos C + c \cos A & b \cos C + c \cos B \end{vmatrix}$$

$$a \cos C + c \cos A$$

$$b \cos C + c \cos B$$

$$c \cos C + c \cos C$$

$$= k^3 \begin{vmatrix} \cos A & a & 0 \\ \cos B & b & 0 \\ \cos C & c & 0 \end{vmatrix} \times \begin{vmatrix} a & \cos A & 0 \\ b & \cos B & 0 \\ c & \cos C & 0 \end{vmatrix} = 0 \times 0 = 0$$

$$\Rightarrow \lambda = 0$$

Ex 77. Maximum value of

$$\begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 2x \\ \sin^2 x & 1 + \cos^2 x & 4 \sin 2x \\ \sin^2 x & \cos^2 x & 1 + 4 \sin 2x \end{vmatrix} \text{ is } \underline{\hspace{2cm}}.$$

Sol. (6) On applying $C_1 \rightarrow C_1 + C_2$, we get

$$\Delta = \begin{vmatrix} 2 & \cos^2 x & 4 \sin 2x \\ 2 & 1 + \cos^2 x & 4 \sin 2x \\ 1 & \cos^2 x & 1 + 4 \sin 2x \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = \begin{vmatrix} 2 & \cos^2 x & 4 \sin 2x \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= 2 + 4 \sin 2x \leq 6$$

Hence, maximum value is 6.

Target Exercises

Type 1. Only One Correct Option

1. If A is a square matrix such that $A^2 = A$, then $|A|$ equals

(a) 0 or 1 (b) -2 or 2
(c) -3 or 3 (d) None of these

2. If A is a square matrix such that $|A| = 2$, then for any positive integer n , $|A|^n$ is equal to

(a) 0 (b) $2n$ (c) 2^n (d) n^2

3. If each element of 3×3 matrix is multiplied by 3, then the determinant of the newly formed matrix is

(a) $3|A|$ (b) $9|A|$ (c) $27|A|$ (d) $|A|^3$

4. The value of determinant $\begin{vmatrix} 1 & 1 & 1 \\ e & \pi & \sqrt{2} \\ 2 & 2 & 2 \end{vmatrix}$ is equal to

(a) 0 (b) e
(c) π (d) $2(e - \pi + \sqrt{2})$

5. The value of determinant $\begin{vmatrix} 1 & x & y+z \\ 1 & y & z+x \\ 1 & z & x+y \end{vmatrix}$ is

(a) 0 (b) $x + y + z$
(c) $1 + x + y + z$ (d) $(x - y)(y - z)(z - x)$

6. The value of determinant $\begin{vmatrix} a_1 & la_1 + mb_1 & b_1 \\ a_2 & la_2 + mb_2 & b_2 \\ a_3 & la_3 + mb_3 & b_3 \end{vmatrix}$ is

(a) 0 (b) l
(c) m (d) lm

7. The value of determinant $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$ is

(a) $a^3\left(1 - \frac{2}{a}\right)$ (b) $a^3\left(1 + \frac{3}{a}\right)$
(c) $a^3\left(1 - \frac{3}{a}\right)$ (d) $a^3\left(1 + \frac{2}{a}\right)$

8. The value of the determinant

$$\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$$
 is

(a) 0
(b) $abc + pqr + xyz$
(c) $(a-x)(y-z)(r-p)$
(d) None of the above

9. If $D = \begin{vmatrix} 1 & a & b \\ 1 & b & c \\ 1 & c & a \end{vmatrix}$, then $\begin{vmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix}$ equals

(a) 0 (b) D
(c) $-D$ (d) None of these

10. If ω is a complex cube root of unity, then the value

of the determinant $\begin{vmatrix} 1 & \omega & \omega+1 \\ \omega+1 & 1 & \omega \\ \omega & \omega+1 & 1 \end{vmatrix}$ is

(a) 0 (b) ω (c) 2 (d) 4

11. If a, b and c are in AP, then the value of

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$
 is

(a) 3 (b) -3
(c) 0 (d) None of these

12. The solution set of the equation $\begin{vmatrix} 2 & 3 & x \\ 2 & 1 & x^2 \\ 6 & 7 & 3 \end{vmatrix} = 0$ is

(a) ϕ (b) $\{0, 1\}$ (c) $\{1, -1\}$ (d) $\{1, -3\}$

13. If $\Delta = \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 0 & 4 & 2 \end{vmatrix}$, then the value of $\begin{vmatrix} 4 & 12 & 4 \\ 8 & -4 & 4 \\ 0 & 16 & 8 \end{vmatrix}$ is

(a) 12Δ (b) 64Δ (c) 4Δ (d) 42Δ

14. The value of $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$ is

(a) 0 (b) 679 (c) 779 (d) 1000

15. If $\begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$, then the line

$ax + by + c = 0$ passes through the fixed point, which is

(a) (1, 2) (b) (1, 1) (c) (-2, 1) (d) (1, 0)

16. If $\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = \lambda a^2 b^2 c^2$, then the value of

λ is
(a) 1 (b) 2 (c) 4 (d) 3

17. The value of the determinant $\Delta = \begin{vmatrix} 1! & 2! & 3! \\ 2! & 3! & 4! \\ 3! & 4! & 5! \end{vmatrix}$ is

- (a) $2!$ (b) $3!$
(c) $4!$ (d) $5!$

18. One root of the equation $\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$ is which of the following?

- (a) $\frac{8}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$ (d) $\frac{16}{3}$

19. Let $a, b, c \in R$ such that no two of them are equal

and satisfy $\begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0$, then equation

$24ax^2 + 4bx + c = 0$ has

- (a) atleast one root in $[0, 1]$
(b) atleast one root in $\left[\frac{-1}{2}, \frac{1}{2}\right]$
(c) atleast one root in $[-1, 0]$
(d) atleast two roots in $[0, 2]$

20. If $f(x) = \begin{vmatrix} 2^{-x} & e^{x \log_e 2} & x^2 \\ 2^{-3x} & e^{3x \log_e 2} & x^4 \\ 2^{-5x} & e^{5x \log_e 2} & 1 \end{vmatrix}$, then

- (a) $f(x) + f(-x) = 0$ (b) $f(x) - f(-x) = 0$
(c) $f(x) + f(-x) = 2$ (d) None of these

21. The value of the determinant

$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ {}^m C_1 & {}^{m+1} C_1 & {}^{m+2} C_1 \\ {}^m C_2 & {}^{m+1} C_2 & {}^{m+2} C_2 \end{vmatrix}$ is equal to

- (a) 1 (b) -1
(c) 0 (d) None of these

22. If $[]$ denotes the greatest integer less than or equal to the real number under consideration and $-1 \leq x < 0$; $0 \leq y < 1$; $1 \leq z < 2$, then the value of the

determinant $\begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$ is

- (a) $[x]$ (b) $[y]$
(c) $[z]$ (d) None of these

23. The determinant $\begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$ is

equal to zero, if

- (a) a, b, c are in AP
(b) $b^3 = ac$
(c) a, b, c are in HP
(d) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

24. Value of θ lying between $\theta = 0$ and $\theta = \frac{\pi}{2}$ and

satisfying $\begin{vmatrix} 1 + \sin^2 \theta & \cos^2 \theta & 4 \sin 4\theta \\ \sin^2 \theta & 1 + \cos^2 \theta & 4 \sin 4\theta \\ \sin^2 \theta & \cos^2 \theta & 1 + 4 \sin 4\theta \end{vmatrix} = 0$

is

- (a) $\frac{7\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{9\pi}{24}$ (d) $\frac{\pi}{24}$

25. If one of the roots of the equation $\begin{vmatrix} 7 & 6 & x \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$ is

$x = -9$, then other two roots are

- (a) $\{2, 6\}$ (b) $\{3, 6\}$ (c) $\{2, 7\}$ (d) $\{3, 7\}$

26. If $D_r = \begin{vmatrix} 2^{r-1} & 2 \cdot 3^{r-1} & 4 \cdot 5^{r-1} \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix}$, then the value

of $\sum_{r=1}^n D_r$ is

- (a) 0 (b) $\alpha\beta\gamma$
(c) $\alpha + \beta + \gamma$ (d) $\alpha \cdot 2^n + \beta \cdot 3^n + \gamma \cdot 4^n$

27. The determinant $\begin{vmatrix} \cos(\alpha + \beta) & -\sin(\alpha + \beta) & \cos 2\beta \\ \sin \alpha & \cos \alpha & \sin \beta \\ -\cos \alpha & \sin \alpha & \cos \beta \end{vmatrix}$

is independent of

- (a) α (b) β
(c) α and β (d) Neither α nor β

28. $\begin{vmatrix} a \sin^2 x & a^2 + \cos^2 x & 1 \\ a \sin^2 y & a^2 + \cos^2 y & 1 \\ a \sin^2 z & a^2 + \cos^2 z & 1 \end{vmatrix}$ is equal to

- (a) a^3 (b) $a \sin x \sin y \sin z$
(c) 0 (d) None of these

29. If $\alpha, \beta, \gamma \in R$, then the determinant

$\Delta = \begin{vmatrix} (e^{i\alpha} + e^{-i\alpha})^2 & (e^{i\alpha} - e^{-i\alpha})^2 & 4 \\ (e^{i\beta} + e^{-i\beta})^2 & (e^{i\beta} - e^{-i\beta})^2 & 4 \\ (e^{i\gamma} + e^{-i\gamma})^2 & (e^{i\gamma} - e^{-i\gamma})^2 & 4 \end{vmatrix}$ is

- (a) independent of α, β and γ
(b) dependent on α, β and γ
(c) independent of α and β only
(d) independent of α and γ only

30. The value of the determinant

$\begin{vmatrix} \sqrt{6} & \sqrt{3} - i\sqrt{2} & 2 + i\sqrt{6} \\ \sqrt{3} + i\sqrt{2} & 0 & 3 - i\sqrt{8} \\ 2 - i\sqrt{6} & 3 + i\sqrt{8} & \sqrt{11} \end{vmatrix}$ is

- (a) a complex number Z , where $\text{Re}(Z) \neq 0$
(b) a purely imaginary number
(c) a real number
(d) a complex number Z , where $\text{Re}(Z) \neq 0$ and $\text{Im}(Z) \neq 0$

31. The equation $\begin{vmatrix} 1+x^4 & 1+x^8 & -1 \\ x^7+1 & 1+x^{11} & -1 \\ x^2+x+1 & x-1 & x+1 \end{vmatrix} = 0$ will

have

- (a) no real root (b) atleast one real root
(c) all roots real (d) None of these

32. If

$$f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & (x+1)x \\ 3x(x-1) & x(x-1)(x-2) & (x+1)x(x-1) \end{vmatrix},$$

then $f(100)$ is equal to

- (a) 0 (b) 1 (c) 100 (d) -100

33. The number of distinct real roots of

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$$
 in the interval $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

is

- (a) 0 (b) 2 (c) 1 (d) 3

34. If a, b and c are the roots of $x^3 - x^2 - 2004 = 0$.

Then, the value of $\begin{vmatrix} a+c & a+b & b+c \\ a(b+c) & b(c+a) & c(a+b) \\ a^2+1 & b^2+1 & c^2+1 \end{vmatrix}$

is equal to

- (a) $-2ac^3 + 2ac - 4a^2c + 2ab^3 - 2ab + 4abc$
(b) $2ac^3 + 2ac - 4a^2c + 2ab^3 - 2ab + 4abc$
(c) $-2ac^3 - ac - 4a^2c + 2ab^3 + 2ab + 4abc$
(d) None of the above

35. If $\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & z & x+y \end{vmatrix} = k(xyz)$, then k is equal to

- (a) 4 (b) -4
(c) zero (d) None of these

36. If A and B are any 2×2 matrices, then $\det(A+B) = 0$ implies

- (a) $\det A = 0$ and $\det B = 0$ (b) $\det A + \det B \neq 0$
(c) $\det A = 0$ or $\det B = 0$ (d) None of these

37. If A is an invertible matrix, then

- (a) $\text{adj } A' = (\text{adj } A)'$ (b) $\text{adj } A' = \text{adj } A$
(c) $\text{adj } A' = A'$ (d) None of these

38. If n th order square matrix A is an orthogonal, then $|\text{adj}(\text{adj } A)|$ is

- (a) always -1, if n is even (b) always 1, if n is odd
(c) always 1 (d) None of these

39. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then the value of $|A||\text{adj } A|$ is

- (a) a^3 (b) a^6 (c) a^9 (d) a^{27}

40. If $A = [a_{ij}]$ is a 4×4 matrix and c_{ij} is the cofactor of the element a_{ij} in $|A|$, then the expression $a_{11}c_{11} + a_{12}c_{12} + a_{13}c_{13} + a_{14}c_{14}$ equals

- (a) 0 (b) -1 (c) 1 (d) $|A|$

41. If A is singular matrix, then $A \cdot (\text{adj } A)$ is

- (a) singular (b) non-singular
(c) symmetric (d) not defined

42. If A' is the transpose of a square matrix A , then

- (a) $|A| \neq |A'|$
(b) $|A| = |A'|$
(c) $|A| + |A'| = 0$
(d) $|A| = |A'|$ only, when A is symmetric

43. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then the value of $|\text{adj } A|$ is

- (a) a^3 (b) a^6 (c) a^9 (d) a^{27}

44. If the system of equations

$$ax + y + z = 0, x + by + z = 0 \text{ and } x + y + cz = 0$$

($a, b, c \neq 1$) has a non-trivial solution, then the

value of $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is

- (a) -1 (b) 0
(c) 1 (d) None of these

45. In a third order of determinant, each element of the first column consists of sum of two terms, each element of the second column consists of sum of three terms and each element of the third column consists of sum of four terms. Then, it can be decomposed into n determinants, where n has value

- (a) 1 (b) 9 (c) 16 (d) 24

46. If $pqr \neq 0$ and the system of equations

$$\begin{aligned} (p+a)x + by + cz &= 0 \\ ax + (q+b)y + cz &= 0 \\ ax + by + (r+c)z &= 0 \end{aligned}$$

has a non-trivial solution, then value of $\frac{a}{p} + \frac{b}{q} + \frac{c}{r}$

- is
(a) -1 (b) 0 (c) 1 (d) 2

47. The number of solutions of

$$2x + y = 4, x - 2y = 2, 3x + 5y = 6$$
 is

- (a) 0 (b) 1
(c) 2 (d) infinitely many

48. If $0 < \theta < \pi$ and the system of equations

$$\begin{aligned} (\sin \theta)x + y + z &= 0 \\ x + (\cos \theta)y + z &= 0 \\ (\sin \theta)x + (\cos \theta)y + z &= 0 \end{aligned}$$

has a non-trivial solution, then

- (a) $\theta = \frac{\pi}{6}$ (b) $\theta = \frac{\pi}{4}$
(c) $\theta = \frac{\pi}{3}$ (d) $\theta = \frac{\pi}{2}$

49. The value of a for which the system of equations

$$a^3x + (a+1)^3y + (a+2)^3z = 0$$

$$ax + (a+1)y + (a+2)z = 0$$

$$x + y + z = 0$$

has a non-zero solution, is

- (a) 1 (b) 0
(c) -1 (d) None of these

50. If $C = 2\cos\theta$, then the value of the determinant

$$\Delta = \begin{vmatrix} C & 1 & 0 \\ 1 & C & 1 \\ 0 & 1 & C \end{vmatrix} \text{ is}$$

- (a) $\frac{\sin 4\theta}{\sin \theta}$
(b) $\frac{2\sin^2 2\theta}{\sin \theta}$
(c) $4\cos^2\theta(2\cos\theta - 1)$
(d) None of the above

51. Consider the system of equations

$$a_1x + b_1y + c_1z = 0, a_2x + b_2y + c_2z = 0,$$

$$a_3x + b_3y + c_3z = 0. \text{ If } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0, \text{ then the}$$

system has

- (a) more than two solutions
(b) one trivial and one non-trivial solution
(c) no solution
(d) only trivial solution (0, 0, 0)

52. If $a > b > c$ and the system of equations

$$ax + by + cz = 0, \quad bx + cy + az = 0 \quad \text{and}$$

$$cx + ay + bz = 0 \text{ has a non-trivial solution, then}$$

both the roots of the quadratic equation $at^2 + bt + c = 0$ are

- (a) atleast one positive real root
(b) opposite in sign
(c) positive
(d) imaginary or real

53. The multiplicative inverse of $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ is

- (a) $\begin{bmatrix} -\cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta \end{bmatrix}$
(b) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$
(c) $\begin{bmatrix} -\cos\theta & -\sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$
(d) $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$

54. If a, b and c are non-zero real numbers, then the

inverse of the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ is

- (a) $\begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}$ (b) $abc \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}$
(c) $\frac{1}{abc} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\frac{1}{abc} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

55. The multiplicative inverse of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \text{ is}$$

- (a) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

56. The value of third order determinant is 11, then the value of the square of the determinant formed by the cofactors will be

- (a) 11 (b) 121
(c) 1331 (d) 14641

57. If a system of the equations

$$(\alpha+1)^3x + (\alpha+2)^3y - (\alpha+3)^3z = 0,$$

$$(\alpha+1)x + (\alpha+2)y - (\alpha+3)z = 0 \text{ and } x + y - z = 0$$

is consistent. Then, what is the value of α ?

- (a) 1 (b) 0
(c) -3 (d) -2

58. The rank of the matrix $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 \end{bmatrix}$ is

- (a) 4 (b) 3
(c) 2 (d) 1

59. If $f(x) = \begin{vmatrix} x & \sin x & \cos x \\ x^2 & \tan x & -x^3 \\ 2x & \sin 2x & 5x \end{vmatrix}$, then $\lim_{x \rightarrow 0} \frac{f'(x)}{x}$

equals

- (a) 1
(b) 4
(c) 3
(d) -4

Type 2. More than One Correct Option

60. If $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$ are the vertices of a ΔABC , then the line

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \text{ represents}$$

- (a) the median through A
(b) the perpendicular bisector of BC
(c) the altitude through A
(d) the line joining the centroid with vertex A

61. Let $a, b > 0$ and $\Delta = \begin{vmatrix} -x & a & b \\ b & -x & a \\ a & b & -x \end{vmatrix}$, then

- (a) $a + b - x$ is a factor of Δ
(b) $x^2 + (a + b)x + a^2 + b^2 - ab$ is a factor of Δ
(c) $\Delta = 0$ has two real roots, if $a = b$
(d) None of the above

62. Let $A = \begin{bmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ -1 & -\sin \theta & 1 \end{bmatrix}$, where $0 \leq \theta \leq 2\pi$,

then

- (a) $|A| = 0$ (b) $|A| \in (0, \infty)$
(c) $|A|$ may be zero (d) $|A| \in [2, 4]$

63. Let $D(x) = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$, then

- (a) $1 - x^3$ is a factor of $D(x)$
(b) $(1 - x^3)^2$ is a factor of $D(x)$
(c) $D(x) = 0$ has 4 real roots
(d) $D'(1) = 0$

64. Let $D = \begin{vmatrix} a & a^2 & 0 \\ 1 & 2a + b & (a + b)^2 \\ 0 & 1 & 2a + 3b \end{vmatrix}$, then

- (a) $a + b$ is a factor of D (b) $a + 2b$ is a factor of D
(c) $2a + 3b$ is a factor of D (d) a^2 is a factor of D

65. The value of the determinant $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$

is zero, if

- (a) $a = -3$ (b) $a = 0$
(c) $a = 1$ (d) $a = 3$

66. If $A(\alpha, \beta) = \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & e^\beta \end{bmatrix}$, then

- (a) $A(\alpha, \beta)^T = A(-\alpha, \beta)$
(b) $A(\alpha, \beta)^{-1} = A(-\alpha, -\beta)$
(c) $\text{adj}(A(\alpha, \beta)) = e^\beta A(-\alpha, -\beta)$
(d) $A(\alpha, \beta)^T = A(\alpha, -\beta)$

67. If

$$f(x) = \begin{vmatrix} 3 & 3x & 3x^2 + 2a^2 \\ 3x & 3x^2 + 2a^2 & 3x^3 + 6a^2x \\ 3x^2 + 2a^2 & 3x^3 + 6a^2x & 3x^4 + 12a^2x^2 + 2a^4 \end{vmatrix},$$

then

- (a) $f'(x) = 0$
(b) $y = f(x)$ is a straight line parallel to X -axis
(c) $\int_0^2 f(x) dx = 32a^4$
(d) None of the above

68. If A and B are two invertible matrices of the same order, then $\text{adj}(AB)$ is equal to

- (a) $\text{adj } B \text{ adj } A$
(b) $|B| |A| B^{-1} A^{-1}$
(c) $|B| |A| A^{-1} B^{-1}$
(d) $|A| |B| (AB)^{-1}$

69. If $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$, then

- (a) constant term of $f(x)$ is 0
(b) the coefficient of x in $f(x)$ is zero
(c) constant term is $a + b$
(d) None of the above

70. The adjoint of the matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$ is

- (a) $\begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 5 & 2 \\ 3 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} 5 & -2 \\ 3 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & -2 \\ -3 & -1 \end{bmatrix}$

71. The system $AX = B$ of n equations in n unknowns has infinitely many solutions, if

- (a) $\det A \neq 0$
(b) $(\text{adj } A) B \neq 0$
(c) $\det A = 0$
(d) $(\text{adj } A) B = 0$

72. The system of equations $ax + 4y + z = 0, 2y + 3z - 1 = 0, 3x - bz + 2 = 0$ has

- (a) unique solutions, if $ab \neq 15$
(b) has infinite solutions, if $a = 3, b = 5$
(c) has no solutions, if $ab = 15, a = 1, b = 15$
(d) has no solutions, if $ab = 15, a \neq 3$

73. If $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, then

- (a) $\text{adj } A$ is a zero matrix
(b) $\text{adj } A = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$
(c) $A^{-1} = A$
(d) $A^2 = I$

74. Consider the system $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$, $a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$, $a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = 0$, where $a_{ii} > 0$ for i , $a_{ij} < 0$ for $i \neq j$ and $\sum_{j=1}^3 a_{ij} > 0$, $\sum_{j=1}^3 a_{2j} > 0$, $\sum_{j=1}^3 a_{3j} > 0$, then
- the system cannot have non-zero solutions
 - the system can have non-zero solutions
 - the system cannot be satisfied by trio (x_1, x_2, x_3) , if all x_1, x_2, x_3 are positive
 - the system cannot be satisfied by trio (x_1, x_2, x_3) , if all x_1, x_2, x_3 are negative

75. Let A_n be an $n \times n$ matrix whose (i, j) th element is

$$a_{ij} = \begin{cases} 2, & \text{if } i = j \\ -1, & \text{if } |i - j| = 1, \text{ then} \\ 0, & \text{elsewhere} \end{cases}$$

- $\det A_n = \det A_{n-1} - \det A_{n-2}$
- $\det A_n = 2 \det A_{n-1} - \det A_{n-2}$
- $\det A_n = n + 1$
- $\det A_n = n^2 - 1$

76. A 2×2 non-singular matrix with integer entries will be called an inverse special matrix, if all entries in A^{-1} are also integers. Then,

$$(a) \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ is an inverse special matrix}$$

- The number of inverse special matrix is finite
- The number of inverse special matrix is infinite
- The number of inverse special matrix is eight

77. The value of the η for which the equations $x + y + z = 1$, $x + 2y + 4z = \eta$, $x + 4y + 10z = \eta^2$ have a solution, is

- 2
- 0
- 1
- 3

78. If $f(x) = \begin{vmatrix} 5^{-x} & e^{x \log_e 5} & x^2 \\ 5^{-3x} & e^{3x \log_e 5} & x^4 \\ 5^{-5x} & e^{5x \log_e 5} & 1 \end{vmatrix}$, then

- graph of $f(x)$ is symmetrical about origin
- graph of $f(x)$ is symmetrical about Y-axis

$$(c) f_4(0) = \left[\frac{d^4 f(x)}{dx^4} \right]_{x=0} = 0$$

$$(d) g(x) = f(x) \times \log \left(\frac{a-x}{a+x} \right)$$

79. Let $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. If n is a positive integer, then

$$(a) A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$$

$$(b) \lim_{n \rightarrow \infty} \frac{1}{n} A^{-n} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$(c) \lim_{n \rightarrow \infty} \frac{1}{n^2} A^{-n} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

- None of the above

80. Let $Y = sX$, $Z = tX$, where all variables are differentiable functions of X . If

$$X_1 = \frac{dX}{dx}, X_2 = \frac{d^2 X}{dx^2} \text{ etc, then } \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$$

must be equal to

- zero

$$(b) X^2 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

$$(c) X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

- zero, if s and t are linear functions of x

81. If $f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$,

then

$$(a) f(x) = -\sin^2 x - \cos^5 x$$

$$(b) f(x) = -\cos^2 x - \sin^5 x$$

$$(c) \int_0^{\pi/2} f(x) dx = -\frac{8}{15} + \frac{\pi}{4}$$

$$(d) \int_0^{\pi/2} f(x) dx = -\frac{8}{15} - \frac{\pi}{4}$$

82. Let $f(x) = \begin{vmatrix} \frac{1}{x} & \log x & x^n \\ 1 & -\frac{1}{n} & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$, then

- n th derivative of $f(x)$ at $x = 1$ is independent of a
- n th derivative of $f(x)$ at $x = 1$ is independent of n
- n th derivative of $f(x)$ at $x = 1$ depends upon a and n
- All above are correct

Type 3. Assertion and Reason

Directions (Q. Nos. 83-93) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

83. **Statement I** If a, b, c are distinct and x, y, z are not all zero, given that $ax + by + cz = 0$, $bx + cy + az = 0$, $cx + ay + bz = 0$, then $a + b + c \neq 0$.

Statement II $a^2 + b^2 + c^2 > ab + bc + ca$, if a, b and c are distinct.

84. Statement I If $D(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$,

where f_1, f_2, f_3 are differentiable functions and $a_2, b_2, c_2, a_3, b_3, c_3$ are constants,

then $\int D(x) dx$

$$= \begin{vmatrix} \int f_1(x) dx & \int f_2(x) dx & \int f_3(x) dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + C$$

Statement II Integration of sum of several functions is equal to sum of integration of individual function.

85. Statement I There are only finitely many 2×2 matrices which commute with the matrix $\begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$.

Statement II If A is non-singular, then it commutes with $I, \text{adj } A$ and A^{-1} .

86. Statement I If A is skew-symmetric matrix of order 3, then its determinant should be zero.

Statement II If A is a square matrix, then $\det A = \det A' = \det (-A')$

87. Let α, β, γ be the roots of the equation $x^3 + ax + b = 0$; $a, b \in R$.

Statement I $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0$

Statement II Any cubic equation over reals has atleast one real root.

88. Statement I If a, b and c are different and

$$\begin{vmatrix} a & a^3 & a^4 - 1 \\ b & b^3 & b^4 - 1 \\ c & c^3 & c^4 - 1 \end{vmatrix} = 0, \text{ then}$$

$$abc(ab + bc + ac) = a + b + c.$$

Statement II

$$\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = (ab + bc + ac) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

89. Let $f(x) = \begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$

Statement I If $\sin 2x = 1$, then $f(x) = 2/3$

Statement II $f(x) = 0$, if $\sin x = \cos x$

90. Statement I If $bc + qr = ca + rp = ab + pq = -1$,

then $\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix} = 0$, where $abc, pqr \neq 0$

Statement II If the system of equations

$$a_1x + b_1y + c_1 = 0$$

$$a_2x + b_2y + c_2 = 0$$

$$a_3x + b_3y + c_3 = 0$$

has non-trivial solution, then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

91. Suppose $x > 0, y > 0, z > 0$ and

$$\Delta(a, b, c) = \begin{vmatrix} x \log 2 & 3 & 15 + \log(a^x) \\ y \log 3 & 5 & 25 + \log(b^y) \\ z \log 5 & 7 & 35 + \log(c^z) \end{vmatrix}$$

Statement I $\Delta(8, 27, 125) = 0$

Statement II $\Delta\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{5}\right) = 0$

92. Statement I The determinant of a matrix

$$\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} \text{ is zero.}$$

Statement II The determinant of a skew-symmetric matrix of odd order is zero.

93. Statement I If $A_r = \begin{bmatrix} r & r-1 \\ r-1 & r \end{bmatrix}$, where r is a natural number, then

$$|A_1| + |A_2| + \dots + |A_{2006}| = (2006)^2.$$

Statement II If A is a matrix of order 3 and $|A| = 2$, then $|\text{adj } A| = 2^2$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 94-96) Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ if U_1, U_2

and U_3 are column matrices satisfying

$$AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \text{ and } U \text{ is } 3 \times 3 \text{ matrix}$$

whose columns are U_1, U_2, U_3 .

94. The value of $|U|$ is

- (a) 3 (b) -3 (c) $\frac{3}{2}$ (d) 2

95. The sum of the elements of U^{-1} is

- (a) -1 (b) 0 (c) 1 (d) 3

96. The value of $[3 \ 2 \ 0]U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is

- (a) [5] (b) $\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ (c) [4] (d) $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Passage II (Q. Nos. 97-99) Let A be the set of all 3×3 symmetric matrices, all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are 0.

97. The number of matrices in A is

- (a) 12 (b) 6 (c) 9 (d) 3

98. The number of matrices A in A for which the

system of linear equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has unique

solution, is

- (a) less than 4 (b) at least 4 but less than 7
(c) at least 7 but less than 10 (d) at least 10

99. The number of matrices A in A for which the

system of linear equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is

inconsistent, is

- (a) 0 (b) more than 2 (c) 2 (d) 1

Passage III (Q. Nos. 100-102) A determinant is called cyclic, if it follows the arrangement symmetrically with a, b, c

e.g. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$. Now, if we increase the degree of

any row in this determinant symmetrically, its value will be increased by over expression which is also cyclic and increases the degree of the value of determinant, then

100. The value of $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$ is equal to

- (a) $(a-b)(b-c)(c-a)$
(b) $(a-b)(b-c)(c-a)(a+b+c)$
(c) $(a-b)(b-c)(c-a)(ab+bc+ca)$
(d) $(a-b)(b-c)(c-a)abc$

101. The value of $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$ is equal to

- (a) $(a-b)(b-c)(c-a)$
(b) $(a-b)(b-c)(c-a)(a+b+c)$
(c) $(a-b)(b-c)(c-a)(ab+bc+ca)$
(d) $(a-b)(b-c)(c-a)abc$

102. The value of $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$ is equal to

- (a) $(a-b)(b-c)(c-a)$
(b) $(a-b)(b-c)(c-a)(a+b+c)$
(c) $(a-b)(b-c)(c-a)(ab+bc+ca)$
(d) $(a-b)(b-c)(c-a)abc$

Passage IV (Q. Nos. 103-105) Let A be a square matrix of order 2 or 3 and I be the matrix of the same order, then the matrix $A - \lambda I$ is called characteristic matrix of the matrix A , where λ is some complex number. The determinant of the characteristic matrix is called characteristic determinant of the matrix A which will of course be a polynomial of degree 3 in λ .

The equation $\det(A - \lambda I) = 0$ is called characteristic equation of the matrix A and its roots are called characteristic roots or eigenvalues. It is known that every square matrix satisfies its characteristic equation.

103. If one of the eigen values of a square matrix A of order 3×3 is zero, then

- (a) $\det(A)$ must be non-zero
(b) $\det(A)$ must be zero
(c) $\text{adj}(A)$ must be a zero matrix
(d) None of the above

104. If A' denotes transpose of matrix A , $A'A = I$, then $\det(A - I)$ must be equal to

- (a) 0 (b) -1
(c) 1 (d) None of these

105. If $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and

$A^{-1} = \frac{1}{6}(A^2 + cA + dI)$, then the values of c and d

- are
(a) -6, -11 (b) 6, 11
(c) -6, 11 (d) 6, -11

Passage V (Q. Nos. 106-108)

$$\text{Let } A = \begin{bmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{bmatrix}; B = \begin{bmatrix} a & 1 & 1 \\ 1 & d & c \\ f & g & h \end{bmatrix}; U = \begin{bmatrix} f \\ g \\ h \end{bmatrix}, V = \begin{bmatrix} a^2 \\ 0 \\ 0 \end{bmatrix}$$

106. Suppose the equation $AX = U$, where X is a 3×1 column vector has infinite solutions, then we must have

- (a) $a = b$ or $f = h$
- (b) $c = d$ or $f = h$
- (c) $c = d$ and $g = h$
- (d) None of the above

107. The equation $BX = V$ has

- (a) no solution
- (b) unique solution
- (c) infinite solutions or no solution
- (d) None of the above

108. If $af \neq 0$, then the system $BX = V$ has

- (a) unique solution
- (b) infinite solutions
- (c) atleast one solution
- (d) no solution

Passage VI (Q. Nos. 109-111) A 3×3 determinant has its entries as either 1 or -1. The number of such determinants is $2^9 = 512$. We will call a 3×3 determinant with entries 1 or -1 as minus special, if product of element of any rows or any columns is -1

e.g. $\begin{vmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{vmatrix}$ is a minus special determinant.

109. The number of 3×3 minus special determinants must be

- (a) 10
- (b) 12
- (c) 16
- (d) 18

110. The number of $n \times n$ minus special determinants must be

- (a) 2^{n-1}
- (b) $2^{(n-1)^2}$
- (c) $\frac{13n^2 - 37n + 26}{2}$
- (d) None of the above

111. The minimum value of a 3×3 minus special determinant is

- (a) -6
- (b) -4
- (c) -2
- (d) 0

Passage VII (Q. Nos. 112-114) Note the following facts

- (1) $A(\text{adj } A) = (\text{adj } A)A = (\det A)I$
- (2) $\text{adj}(AB) = (\text{adj } B)(\text{adj } A)$
- (3) $\det(AB) = (\det A)(\det B)$, where A and B are the square matrices of order n .

112. If P is a square matrix whose determinant is unity, then $\text{adj } P^{-1}$ is equal to

- (a) $\text{adj } P$
- (b) $(\text{adj } P)^{-1}$
- (c) P^{-1}
- (d) None of the above

113. If $\text{adj } B = A$ and $\det Q = 1$, then $\text{adj } Q^{-1} B P^{-1}$ must be equal to

- (a) APQ
- (b) PAQ
- (c) B
- (d) A

114. $\det(\text{adj } AB)$ must be equal to

- (a) $(\det A \cdot \det B)^n$
- (b) $(\det A)^{n-1} (\det B)^{n-1}$
- (c) $(\det A \cdot \det B)^{n+1}$
- (d) None of the above

Passage VIII (Q. Nos. 115-117) Let $A = (a_{ij})$ be $m \times n$ matrix ($m \leq n$). The permanent of matrix A is defined as $\text{per } A = \sum a_{1i_1} a_{2i_2} a_{3i_3} \dots a_{mi_m}$, where the summation is over all m permutation of the set $N = \{1, 2, 3, \dots, n\}$

e.g. If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}$, then

$\text{per } A = (1)(2) + 2(1) + (1)(3) + (3) + (3)(2) = 22$

115. If $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$, then $\text{per } A$ must be equal to

- (a) 50
- (b) 60
- (c) 70
- (d) 80

116. If I_n is the identity matrix of order $n \times n$ and J_n is the $n \times n$ matrix in which all the elements are unity, then $\text{per} \left(\frac{1}{n} J_n \right)$ must be equal to

- (a) $n!$
- (b) $\frac{n!}{n^n}$
- (c) $\frac{n!}{n^2}$
- (d) None of the above

117. If $D_n = n! \left(1 + \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!} \right)$, then

$\text{per}(J_n - I_n)$ must be equal to

- (a) D_n
- (b) $\frac{D_n}{n!}$
- (c) $(n-1)(D_{n-1} + D_n)$
- (d) None of the above

Type 5. Match the Columns

118. Match the following for the system of linear equations

$$\lambda x + y + z = 1, x + \lambda y + z = \lambda, x + y + \lambda z = \lambda^2$$

Column I	Column II
A. $\lambda = 1$	p. unique solution
B. $\lambda \neq 1$	q. infinite solutions
C. $\lambda \neq 1, \lambda \neq -2$	r. no solution
D. $\lambda = -2$	s. consistent
	t. inconsistent

119. If $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{pmatrix}$ and $A^{-1} = pA^2 + qA + rI_3$, then

match the following:

Column I	Column II
A. p	p. $-7/2$
B. q	q. $-1/2$
C. r	r. 3
	s. negative fraction

120. Let $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}$ and

$$\text{adj}(A) = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \text{ Then, match the}$$

following.

Column I	Column II
A. ab	p. a_{12}
B. bc	q. a_{21}
C. ac	r. a_{13}
	s. a_{31}
	t. a_{23}
	u. a_{32}

121. Match the with statements of Column I with values of Column II.

Column I	Column II
A. The matrix $\begin{bmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{bmatrix}$ is a singular matrix, if λ is	p. 2
B. If $A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 0 \\ 1 & 3 & 0 \end{bmatrix}$, then the value of $ \text{adj } A $ is	q. $\frac{2}{5}$
C. The rank of the matrix $\begin{bmatrix} -5 & 3 & 2 \\ 3 & 2 & -5 \\ 4 & -1 & -3 \end{bmatrix}$ is	r. 3
D. The rank of the matrix $\begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 5 & 0 & 0 & 1 \end{bmatrix}$ is	s. 1

122. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 5$, then the value of $\Delta = \begin{vmatrix} b_2c_3 - b_3c_2 & a_3c_2 - a_2c_3 \\ b_3c_1 - b_1c_3 & a_1c_3 - a_3c_1 \\ b_1c_2 - b_2c_1 & a_2c_1 - a_1c_2 \end{vmatrix}$ is	p. 2
B. Coefficient of x in $f(x) = \begin{vmatrix} x & (1 + \sin x)^3 & \cos x \\ 1 & \log(1 + x) & 2 \\ x^2 & (1 + x)^2 & 0 \end{vmatrix}$ is	q. 0
C. If, $a_i^2 + b_i^2 + c_i^2 = 1, i = 1, 2, 3$ and $a_1a_2 + b_1b_2 + c_1c_2 = 0$, then value of the determinant $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ is	r. -2
D. The largest value of a third order determinant whose elements are 0 and 1, is	s. 25
E. If $f(x), g(x)$ and $h(x)$ are three polynomials of degree 2 and $\Delta(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \end{vmatrix}$, then $\Delta(x)$ is a polynomial of degree	t. 1

123. Match the statements of Column I with values of Column II.

Column I	Column II
A. If ω is a cube root of unity, then a root of the following equation $\begin{vmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{vmatrix}$ is	p. 0
B. If $a = i, b = \omega$ and $c = \omega^2$, then the value of determinant $\begin{vmatrix} a & a+b & a+b+c \\ 3a & 4a+3b & 5a+4b+3c \\ 6a & 9a+6b & 11a+9a+6c \end{vmatrix}$ is	q. 2
C. The value of determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix}$ is	r. i
D. Let $\Delta = \begin{vmatrix} \Sigma a^2 & \Sigma ab & \Sigma ab \\ \Sigma ab & \Sigma a^2 & \Sigma ab \\ \Sigma ab & \Sigma ab & \Sigma a^2 \end{vmatrix}$, if $\Delta = 0$, then the value of $\Sigma a^2 + 2\Sigma ab$ is	s. $3\sqrt{3}i$

Type 6. Single Integer Answer Types Questions

124. If A is a square matrix of order n such that $|\text{adj}(\text{adj } A)| = |A|^9$, then the value of n can be _____.

125. $\frac{1}{x} + \frac{2}{y} + \frac{1}{z} = 1, \frac{3}{x} - \frac{4}{y} - \frac{2}{z} = 2, \frac{2}{x} + \frac{5}{y} - \frac{2}{z} = 3$,
then the value of y is _____.

126. If $\Delta_r = \begin{vmatrix} x & y & z \\ 2^r & 2 \times 3^r & 3 \times 4^r \\ 2(2^n - 1) & 3(3^n - 1) & 4(4^n - 1) \end{vmatrix}$, then the
value of $\sum_{r=1}^n \Delta_r$ is _____.

127. If $a_1, a_2, a_3, \dots, a_{12}$ are in an AP and
 $\Delta_1 = \begin{vmatrix} a_1 a_5 & a_1 & a_2 \\ a_2 a_6 & a_2 & a_3 \\ a_3 a_7 & a_3 & a_4 \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} a_2 a_{10} & a_2 & a_3 \\ a_3 a_{11} & a_3 & a_4 \\ a_4 a_{12} & a_4 & a_5 \end{vmatrix}$,
then $\Delta_1 : \Delta_2$ is equal to _____.

128. If $2s = a + b + c$ and

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = ks^3 (s-a)(s-b)(s-c),$$

then the numerical quantity of k should be _____.

129. If x, y, z are cube roots of unity and
 $D = \begin{vmatrix} x^2 + y^2 & z^2 & z^2 \\ x^2 & y^2 + z^2 & x^2 \\ y^2 & y^2 & z^2 + x^2 \end{vmatrix}$, then real part
of D must be _____.

130. The value of θ for which the system
 $(\sin 3\theta)x - y + z = 0$,
 $(\cos 2\theta)x + 4y + 3z = 0, 2x + 7y + 7z = 0$ has
non-trivial solution is $\frac{5\pi}{\lambda}$, where λ is a positive
integer, then λ must be equal to _____.

131. Let A be an $n \times n$ square matrix given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & 1 & \dots & 1 \\ 1 & 1 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \dots & 0 \end{bmatrix}, \text{ and in the inverse of } A$$

each diagonal element is equal to $\frac{k-n}{n-1}$, where k is
a numerical quantity, then k must be equal to _____.

132. Three distinct points $P(3u^2, 2u^3), Q(3v^2, 2v^3)$ and
 $R(3w^2, 2w^3)$ are collinear, then $uv + vw + wu$ is
equal to _____.

133. The system $2ax - 23y + 29z = 0, 7x + ay + 4z = 7$,
 $5x + 2y + az = 5$ has no solution, if the value of a is
_____.

134. Let

$$p\lambda^4 + q\lambda^3 + r\lambda^2 + 8\lambda + t = \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$$

be an identity in λ , where p, q, r, s and t are
constants. Then, find t .

Entrances Gallery

JEE Advanced/IIT JEE

1. Let M and N be two 3×3 matrices such that $MN = NM$. Further, if $M \neq N^2$ and $M^2 = N^4$, then [2014]

- (a) determinant of $(M^2 + MN^2)$ is 0
 (b) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
 (c) determinant of $(M^2 + MN^2) \geq 1$
 (d) for a 3×3 matrix U , if $(M^2 + MN^2)U$ equals the zero matrix, then U is the zero matrix

2. For 3×3 matrices M and N , which of the following statement(s) is(are) not correct? [2013]

- (a) $N^T MN$ is symmetric or skew-symmetric, according as M is symmetric or skew-symmetric
 (b) $MN - NM$ is skew-symmetric for all symmetric matrices M and N
 (c) MN is symmetric for all symmetric matrices M and N
 (d) $(\text{adj } M)(\text{adj } N) = \text{adj } (MN)$ for all invertible matrices M and N

3. If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then

the possible value(s) of the determinant of P is/are [2012]

- (a) -2
 (b) -1
 (c) 1
 (d) 2

4. Let $P = [a_{ij}]$ be a 3×3 matrix and $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \leq i, j \leq 3$. If the determinant of P is 2, then the determinant of the matrix Q is [2012]

- (a) 2^{10}
 (b) 2^{11}
 (c) 2^{12}
 (d) 2^{13}

5. If $\omega \neq 1$ is a cube root of unity and S is the set of all

non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$,

where each of a, b and c is either ω or ω^2 . Then, the number of distinct matrices in the set S is [2011]

- (a) 2
 (b) 6
 (c) 4
 (d) 8

Passage (Q. Nos. 6-8) Let p be an odd prime number and T_p be the following set of 2×2 matrices

$$T_p = \left\{ A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}; a, b, c \in \{0, 1, 2, \dots, p-1\} \right\} \quad [2010]$$

6. The number of A in T_p such that A is either symmetric or skew-symmetric or both and $\det(A)$ is divisible by p , is

- (a) $(p-1)^2$ (b) $2(p-1)$ (c) $(p-1)^2 + 1$ (d) $2p-1$

7. The number of A in T_p such that the trace of A is not divisible by p but $\det(A)$ is divisible by p , is

- (a) $(p-1)(p^2 - p + 1)$ (b) $p^3 - (p-1)^2$
 (c) $(p-1)^2$ (d) $(p-1)(p^2 - 2)$

8. The number of A in T_p such that $\det(A)$ is not divisible by p , is

- (a) $2p^2$ (b) $p^3 - 5p$ (c) $p^3 - 3p$ (d) $p^3 - p^2$

9. The number of 3×3 matrices A , whose entries are

either 0 or 1 and for which the system $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

has exactly two distinct solutions, is [2010]

- (a) 0 (b) $2^9 - 1$ (c) 168 (d) 2

10. Let ω be the complex number $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$.

Then, the number of distinct complex number z

satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to [2010]

11. Let k be a positive real number and

$$A = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$

$$\text{and } B = \begin{vmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{vmatrix}$$

If $\det(\text{adj } A) + \det(\text{adj } B) = 10^6$, then $[k]$ is equal to [2010]

12. The set of all values of λ for which the system of linear equations $2x_1 - 2x_2 + x_3 = \lambda x_1$, $2x_1 - 3x_2 + 2x_3 = \lambda x_2$ and $-x_1 + 2x_2 = \lambda x_3$ has a non-trivial solution, [2015]

(a) is an empty set
(b) is a singleton set
(c) contains two elements
(d) contains more than two elements

13. If $\alpha, \beta \neq 0$, $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$$

, then k is equal to [2014]

(a) $\alpha\beta$ (b) $\frac{1}{\alpha\beta}$ (c) 1 (d) -1

14. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A

and $|A| = 4$, then α is equal to [2013]

(a) 4 (b) 11 (c) 5 (d) 0

15. The number of values of k , for which the system of equations $(k+1)x + 8y = 4k$ and $kx + (k+3)y = 3k-1$ has no solution, is [2013]

(a) infinite (b) 1 (c) 2 (d) 3

16. Let P and Q be 3×3 matrices, $P \neq Q$. If $P^3 = Q^3$ and $P^2Q = Q^2P$, then determinant of $(P^2 + Q^2)$ is equal to [2012]

(a) -2 (b) 1 (c) 0 (d) -1

17. The number of values of k for which the linear equations $4x + ky + 2z = 0$, $kx + 4y + z = 0$ and $2x + 2y + z = 0$ possess a non-zero solution, is [2011]

(a) 2 (b) 1 (c) zero (d) 3

18. If the trivial solution is the only solution of the system of equations

$$\begin{aligned} x - ky + z &= 0 \\ kx + 3y - kz &= 0 \\ 3x + y - z &= 0 \end{aligned}$$

Then, the set of all values of k is [2011]

(a) $\{2, -3\}$ (b) $R - \{2, -3\}$ (c) $R - \{2\}$ (d) $R - \{-3\}$

19. **Statement I** Determinant of a skew-symmetric matrix of order 3 is zero.

Statement II For any matrix A ,

$$\det(A^T) = \det(A) \text{ and } \det(-A) = -\det(A)$$

where, $\det(B)$ denotes the determinant of matrix B . Then, [2011]

(a) Statement I is true and Statement II is false
(b) Both statements are true
(c) Both statements are false
(d) Statement I is false and Statement II is true

20. Consider the system of linear equations

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 3 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ 3x_1 + 5x_2 + 2x_3 &= 1 \end{aligned}$$

The system has [2010]

(a) infinite number of solutions
(b) exactly 3 solutions
(c) a unique solution
(d) no solution

21. Let a, b, c be such that $(b+c) \neq 0$.

$$\text{If } \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix}$$

$$+ \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ (-1)^{n+2}a & (-1)^{n+1}b & (-1)^n c \end{vmatrix} = 0$$

then the value of n is [2009]

(a) zero (b) any even integer
(c) any odd integer (d) any integer

22. Let A be 2×2 matrix.

Statement I $\text{adj}(\text{adj } A) = A$

Statement II $|\text{adj } A| = |A|$ [2009]

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

23. Let a, b, c be any real numbers. Suppose that there are real numbers x, y, z not all zero such that $x = cy + bz$, $y = az + cx$ and $z = bx + ay$. Then, $a^2 + b^2 + c^2 + 2abc$ is equal to [2008]

(a) 1 (b) 2 (c) -1 (d) 0

24. Let A be a square matrix, all of whose entries are integers. Then, which one of the following is true? [2008]

(a) If $\det(A) = \pm 1$, then A^{-1} need not exist
(b) If $\det(A) = \pm 1$, then A^{-1} exists but all its entries are not necessarily integers
(c) If $\det(A) \neq \pm 1$, then A^{-1} exists and all its entries are non-integers
(d) If $\det(A) = \pm 1$, then A^{-1} exists and all its entries are integers

25. If $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$ for $x \neq 0, y \neq 0$, then D is [2007]

(a) divisible by neither x nor y
(b) divisible by both x and y
(c) divisible by x but not y
(d) divisible by y but not x

26. Let $A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$. If $|A^2| = 25$, then $|\alpha|$ equals [2007]

(a) 5^2 (b) 1 (c) $\frac{1}{5}$ (d) 5

27. If $a^2 + b^2 + c^2 = -2$ and

$$f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix},$$

then $f(x)$ is a polynomial of degree [2005]

(a) 2 (b) 3 (c) 0 (d) 1

28. The system of equations

$$\begin{aligned} \alpha x + y + z &= \alpha - 1 \\ x + \alpha y + z &= \alpha - 1 \\ x + y + \alpha z &= \alpha - 1 \end{aligned}$$

has no solution, if α is [2005]

(a) 1 (b) not -2 (c) either -2 or 1 (d) -2

29. If $a_1, a_2, \dots, a_n, \dots$ are in GP, then the determinant

$$\Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$

is equal to [2004, 05]

(a) 2 (b) 4 (c) 0 (d) 1

Other Engineering Entrances

34. If $\Delta_r = \begin{vmatrix} 2r-1 & {}^m C_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$, then the value of $\sum_{r=0}^m \Delta_r$ is [BITSAT 2014]

(a) 1 (b) 0 (c) 2 (d) None of the above

35. If the area of the triangle on the complex plane formed by the points $z, z+iz$ and iz is 200, then the value of $3|z|$ must be equal to [VITEEE 2014]

(a) 20 (b) 40 (c) 60 (d) 80

36. If $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = k(x+y+z)^3$, then the value of k is [Manipal 2014]

(a) 1 (b) 2 (c) 4 (d) 8

30. If the system of linear equations

$$\begin{aligned} x + 2ay + az &= 0, \quad x + 3by + bz = 0 \\ \text{and } x + 4cy + cz &= 0 \end{aligned}$$

has a non-zero solution, then a, b, c [2003]

(a) are in AP (b) are in GP (c) are in HP (d) satisfy $a + 2b + 3c = 0$

31. If 1, ω, ω^2 are the cube roots of unity, then

$$\Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$

is equal to [2003]

(a) 0 (b) 1 (c) ω (d) ω^2

32. If l, m, n are the p th, q th and r th terms of an GP and all positive, then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ equals [2002]

(a) 3 (b) 2 (c) 1 (d) 0

33. If $\omega \neq 1$ is a cube root of unity, then

$$\begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega-i & -1 \end{vmatrix}$$

equals [2002]

(a) 0 (b) 1 (c) i (d) ω

37. The value of the determinant

$$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 & 1 \\ (c^x + c^{-x})^2 & (c^x - c^{-x})^2 & 1 \end{vmatrix}$$

is [Manipal 2014]

(a) 0 (b) $2abc$ (c) $a^2 + b^2 + c^2$ (d) None of the above

38. If $\Delta_r = \begin{vmatrix} 1 & n & n \\ 2r & n^2 + n + 1 & n^2 + n \\ 2r-1 & n^2 & n^2 + n + 1 \end{vmatrix}$ and $\sum_{r=0}^n \Delta_r = 56$, then n is equal to [Manipal 2014]

(a) 4 (b) 6 (c) 7 (d) 8

39. If a, b and c are cube roots of unity, then

$$\begin{vmatrix} e^a & e^{2a} & e^{3a} - 1 \\ e^b & e^{2b} & e^{3b} - 1 \\ e^c & e^{2c} & e^{3c} - 1 \end{vmatrix}$$

is equal to [Manipal 2014]

(a) 0 (b) e (c) e^2 (d) e^3

40. If a, b and c are in AP, then the value of

$$\begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix} \text{ is } \quad [\text{Karnataka CET 2014}]$$

- (a) 0
(b) $x - (a + b + c)$
(c) $a + b + c$
(d) $9x^2 + a + b + c$

41. Consider the following statements:

- I. If any two rows or columns of a determinant are identical. Then, the value of the determinant is zero.
II. If the corresponding rows and columns of a determinant are interchanged, then the value of the determinant does not change.
III. If any two rows (or columns) of a determinant are interchanged, then the value of the determinant changes in sign.

Which of the following is correct?

[Karnataka CET 2014]

- (a) I and II
(b) I and III
(c) I, II and III
(d) I and III

42. If I denotes the 3×3 identity matrix and P is a matrix obtained by rearranging the columns of I . Then,

[WB JEE 2014]

- (a) there are six distinct choices for P and $\det(P) = 1$
(b) there are six distinct choices for P and $\det(P) = \pm 1$
(c) there are more than one choices for P and some of them are not invertible
(d) there are more than one choices for P and $P^{-1} = I$ in each choice

43. If $a_1, a_2, a_3, \dots$ are in an AP, then the value of

$$\begin{vmatrix} a_1 & a_2 & 1 \\ a_2 & a_3 & 1 \\ a_3 & a_4 & 1 \end{vmatrix} \text{ is } \quad [\text{Kerala CEE 2014}]$$

- (a) $a_4 - a_1$ (b) $\frac{a_1 - a_4}{2}$ (c) 1
(d) $\frac{a_2 + a_3}{2}$ (e) 0

44. If $\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2} \neq 0$, then the area of the

triangle whose vertices are $\left(\frac{x_1}{a}, \frac{y_1}{a}\right), \left(\frac{x_2}{b}, \frac{y_2}{b}\right)$

and $\left(\frac{x_3}{c}, \frac{y_3}{c}\right)$, is [Kerala CEE 2014]

- (a) $\frac{1}{4} abc$ (b) $\frac{1}{8} abc$ (c) $\frac{1}{4}$
(d) $\frac{1}{8}$ (e) $\frac{1}{12}$

45. If a, b and c are non-zero and different from 1, then

$$\text{the value of } \begin{vmatrix} \log_a 1 & \log_a b & \log_a c \\ \log_a \left(\frac{1}{b}\right) & \log_b 1 & \log_a \left(\frac{1}{c}\right) \\ \log_a \left(\frac{1}{c}\right) & \log_a c & \log_c 1 \end{vmatrix} \text{ is}$$

[Kerala CEE 2014]

- (a) 0
(b) $1 + \log_a(a + b + c)$
(c) $\log_a(ab + bc + ca)$
(d) 1
(e) $\log_a(a + b + c)$

46. If x, y, z are all different and $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$,

then

[AMU 2014]

- (a) $xyz = -1$ (b) $xyz = 1$
(c) $xyz = -2$ (d) $xyz = 2$

47. If three digit numbers $A28, 3B9$ and $62C$, where A, B and C are integers between 0 and 9, are divisible by a fixed integer k , then the determinant

$$\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix} \text{ is}$$

[AMU 2014]

- (a) divisible by k
(b) divisible by k^2
(c) divisible by $2k$
(d) None of these

48. If $k > 1$ and the determinant of the matrix A^2 , where

$$A = \begin{bmatrix} k & k\alpha & \alpha \\ 0 & \alpha & k\alpha \\ 0 & 0 & k \end{bmatrix}, \text{ is } k^2, \text{ then } |\alpha| \text{ is equal to}$$

[EAMCET 2014]

- (a) $\frac{1}{k^2}$ (b) k (c) k^2 (d) $\frac{1}{k}$

49. The determinant $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$ is divisible

by

[GGSIPU 2014]

- (a) $x - y$ (b) $x^2 - y^2 + xy$
(c) $x^2 + xy + y^2$ (d) $x^2 - xy + y^2$

50. Find the area of the triangle with vertices $(2, 3)$, $(0, 1)$ and $(1, 2)$.

[J&K CET 2014]

- (a) $\frac{1}{2}$ sq unit
(b) 0 sq unit
(c) 2 sq units
(d) $2\frac{1}{2}$ sq units

51. If $\omega \neq 1$ is a cube root of unity and the [BITSAT 2014]

$$A = \begin{bmatrix} 1 + 2\omega^{100} + \omega^{200} & \omega^2 & \\ & 1 + 2\omega^{100} + \omega^{200} & \\ \omega & \omega^2 & \end{bmatrix}, \text{ then}$$

- (a) A is singular (b) $|A| \neq 0$
(c) A is symmetric (d) None of these

52. If $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$, then $A \cdot (\text{adj } A)$ is equal to

[VITEEE 2014]

- (a) A (b) $|A|$
(c) $|A| \cdot I$ (d) None of these

53. The inverse of the matrix $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ is

[Karnataka CET 2014]

- (a) $\frac{1}{24} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$
(c) $\frac{1}{24} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix}$

54. If A is a matrix of order 3 such that $A(\text{adj } A) = 10I$, then $|\text{adj } A|$ is equal to [Karnataka CET 2014]

- (a) 1 (b) 10
(c) 100 (d) 10I

55. If $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$ is such that $A^{-1} = kA$, then k is equal to [AMU 2014]

- (a) 19 (b) $\frac{1}{19}$ (c) -19 (d) $-\frac{1}{19}$

56. If A is a square matrix of order $n \times n$, then $\text{adj}(\text{adj } A)$ is equal to [AMU 2014]

- (a) $|A|^n A$
(b) $|A|^{n-1} A$
(c) $|A|^{n-2} A$
(d) $|A|^{n-3} A$

57. If a, b, c and d are real numbers such that $a^2 + b^2 + c^2 + d^2 = 1$ and $A = \begin{bmatrix} a + ib & c + id \\ -c + id & a - ib \end{bmatrix}$, then A^{-1} equals [VITEEE 2014]

- (a) $\begin{bmatrix} a + ib & -c - id \\ c - id & a - ib \end{bmatrix}$ (b) $\begin{bmatrix} a - ib & c + id \\ -c + id & a + ib \end{bmatrix}$
(c) $\begin{bmatrix} a - ib & -c - id \\ c - id & a + ib \end{bmatrix}$ (d) $\begin{bmatrix} a + ib & c + id \\ c - id & a - ib \end{bmatrix}$

58. If $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$, then A^{-1} is equal to [RPET 2014]

- (a) $-\frac{1}{13} \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$ (b) $\frac{1}{13} \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$
(c) $-\begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$ (d) None of these

59. The value of k such that the lines $2x - 3y + k = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$ are concurrent, is [BITSAT 2014]

- (a) 20 (b) -7 (c) 7 (d) -20

60. If the system of equations $x + ky - z = 0$, $3x - ky - z = 0$ and $x - 3y + z = 0$ has non-zero solution, then k is equal to [VITEEE 2014]

- (a) -1 (b) 0 (c) 1 (d) 2

61. The system of linear equations $3x + y - z = 2$, $x - z = 1$ and $2x + 2y + az = 5$ has unique solution, when [Kerala CEE 2014]

- (a) $a \neq 3$ (b) $a \neq 4$ (c) $a \neq 5$
(d) $a \neq 2$ (e) $a \neq 1$

62. If the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & \alpha \end{bmatrix}$ is of rank 3, then α

equals [EAMCET 2014]

- (a) -5 (b) 5 (c) 4 (d) 1

63. The existence of the unique solution of the system of equations $x + y + z = \beta$; $5x - y + \alpha z = 10$ and $2x + 3y - z = 6$ depends on [AMU 2014]

- (a) α only (b) β only
(c) both α and β (d) neither β nor α

64. The value of $\theta \in [0, \pi/2]$ and satisfying the equation

$$\begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4 \sin 4\theta \\ \cos^2 \theta & 1 + \sin^2 \theta & 4 \sin 4\theta \\ \cos^2 \theta & \sin^2 \theta & 1 + 4 \sin 4\theta \end{vmatrix} = 0, \text{ is}$$

[BITSAT 2013]

- (a) $\frac{11\pi}{24}$ (b) $\frac{17\pi}{24}$ (c) $\frac{5\pi}{24}$ (d) $\frac{\pi}{24}$

65. The value of $\begin{vmatrix} \sin \alpha & \cos \alpha & \sin(\alpha + \gamma) \\ \sin \beta & \cos \beta & \sin(\beta + \gamma) \\ \sin \delta & \cos \delta & \sin(\gamma + \delta) \end{vmatrix}$ is

[AMU 2013]

- (a) $\sin \alpha \sin \beta \sin \delta$ (b) $\cos \alpha \cos \beta \cos \delta$
(c) 1 (d) 0

66. The value of the determinant $\begin{vmatrix} a^2 + 1 & ab & ac \\ ba & b^2 + 1 & bc \\ ca & cb & c^2 + 1 \end{vmatrix}$ is [MP PET 2012]

- (a) $(a + b + c)^2$ (b) $a^2 + b^2 + c^2$
(c) $a^2 + b^2 + c^2 + 1$ (d) 1

67. If P, Q and R are angles of ΔPQR , then the value of

$$\begin{vmatrix} -1 & \cos R & \cos Q \\ \cos R & -1 & \cos P \\ \cos Q & \cos P & -1 \end{vmatrix}$$
 is equal to [WB JEE 2012]

- (a) -1 (b) 0 (c) $\frac{1}{2}$ (d) 1

68. If $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ and $Q = PP^T$, then the value of $|Q|$

- is [WB JEE 2012]
(a) 2 (b) -2 (c) 1 (d) 0

69. The system of equations $-2x + y + z = a$, $x - 2y + z = b$ and $x + y - 2z = c$ is consistent, if

- [MP PET 2012]
(a) $a + b - c = 0$ (b) $a - b + c = 0$
(c) $a + b + c \neq 0$ (d) $a + b + c = 0$

70. The system of linear equations $x - y - 2z = 6$, $-x + y + z = \mu$ and $\lambda x + y + z = 3$ has [WB JEE 2012]

- (a) infinite number of solutions for $\lambda \neq -1$ and all μ
(b) infinite number of solutions for $\lambda = -1$ and $\mu = 3$
(c) no solution for $\lambda \neq -1$
(d) unique solution for $\lambda = -1$ and $\mu = 3$

71. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then $|3AB|$ is equal to [MP PET 2011]

- (a) -9 (b) -81 (c) -27 (d) 81

72. If $z = \begin{vmatrix} 1 & 1+2i & -5i \\ 1-2i & -3 & 5+3i \\ 5i & 5-3i & 7 \end{vmatrix}$, then $(i = \sqrt{-1})$

- [WB JEE 2011]
(a) z is purely real
(b) z is purely imaginary
(c) $z + \bar{z} = 0$
(d) $(z - \bar{z})i$ is purely imaginary

73. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 - bc & b^2 - ca & c^2 - ab \end{vmatrix}$ is equal to [GGSIPU 2011]

- (a) 0 (b) 1
(c) abc (d) $(a-b)(b-c)(c-a)$

74. If X is any matrix of order $n \times p$ (n and p are integers) and I is an identity matrix of order $n \times n$, then the matrix $M = I - X(X'X)^{-1}X'$ is

- [UP SEE 2011]
I. Idempotent matrix II. $MX = O$
(a) I is correct (b) II is correct
(c) I is incorrect (d) II is incorrect

75. If $A_{3 \times 3}$ and $|A| = 6$, then $|2 \text{adj } A|$ is equal to [GGSIPU 2011]

- (a) 48 (b) 8 (c) 288 (d) 12

76. If A is a 2×2 matrix and $|A| = 2$, then the matrix represented by $|A(\text{adj } A)|$ is equal to [J&K CET 2011]

- (a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ (c) $\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$

77. If the system of linear equations $x + 2y - 3z = 1$, $(p+2)z = 3$ and $(2p+1)y + z = 2$ has no solution, then [AMU 2011]

- (a) $p = 2$ (b) $p = -2$ (c) $p = -\frac{1}{2}$ (d) $p = 3$

78. The value of the determinant $\begin{vmatrix} 15! & 16! & 17! \\ 16! & 17! & 18! \\ 17! & 18! & 19! \end{vmatrix}$ is

- [Kerala CEE 2010]
(a) $15! + 16!$ (b) $2(15!)(16!)(17!)$
(c) $15! + 16! + 17!$ (d) $16! + 17!$
(e) None of these

79. If $\begin{vmatrix} x^2 + x & 3x - 1 & -x + 3 \\ 2x + 1 & 2 + x^2 & x^3 - 3 \\ x - 3 & x^2 + 4 & 3x \end{vmatrix}$

$$= a_0 + a_1x + a_2x^2 + \dots + a_7x^7,$$

- then the value of a_0 is
(a) 25 (b) 24 (c) 23
(d) 22 (e) 21

80. If ω is an imaginary cube root of unity and

$$\begin{vmatrix} x + \omega^2 & \omega & 1 \\ \omega & \omega^2 & 1 + x \\ 1 & x + \omega & \omega^2 \end{vmatrix} = 0,$$
 then one of the value of

- x is [WB JEE 2010]
(a) 1 (b) 0 (c) -1 (d) 2

81. If A is a non-singular matrix of order 3, then $\text{adj}(\text{adj } A)$ is equal to [BITSAT 2010]

- (a) A (b) A^{-1} (c) $\frac{1}{|A|}A$ (d) $|A|A$

82. Which one of the following is correct always for any two non-singular matrices A and B of same order?

- [VITEEE 2010]
(a) $AB = BA$ (b) $(AB)' = A'B'$
(c) $(A+B)(A-B) = A^2 - B^2$ (d) $(AB)^{-1} = B^{-1}A^{-1}$

83. If $A = \begin{bmatrix} x & -2 \\ 3 & 7 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 7/34 & 1/17 \\ -3/34 & 2/17 \end{bmatrix}$ then the

- value of x is [Kerala CEE 2010]
(a) 2 (b) 3 (c) -4
(d) 4 (e) -2

84. If $A = \begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix}$, then A^{-1} is equal to [WB JEE 2010]

- (a) $\frac{1}{7} \begin{bmatrix} -1 & -2 \\ 4 & 1 \end{bmatrix}$
(b) $\frac{1}{7} \begin{bmatrix} 1 & 2 \\ -4 & -1 \end{bmatrix}$
(c) $\frac{1}{7} \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$
(d) Does not exist

Answers

Work Book Exercise 2.1

1. (a)	2. (d)	3. (a)	4. (c)	5. (d)	6. (a)	7. (d)	8. (c)	9. (a)	10. (b)
11. (a)	12. (a)	13. (b)	14. (a)						

Work Book Exercise 2.2

1. (b)	2. (b)	3. (c)	4. (d)	5. (d)	6. (a)	7. (d)	8. (c)	9. (b)

Target Exercises

1. (a)	2. (c)	3. (c)	4. (a)	5. (a)	6. (a)	7. (b)	8. (a)	9. (b)	10. (d)
11. (c)	12. (d)	13. (b)	14. (a)	15. (b)	16. (c)	17. (c)	18. (b)	19. (a)	20. (a)
21. (a)	22. (c)	23. (d)	24. (a)	25. (c)	26. (a)	27. (a)	28. (c)	29. (a)	30. (c)
31. (b)	32. (a)	33. (c)	34. (a)	35. (a)	36. (b)	37. (a)	38. (b)	39. (c)	40. (d)
41. (a)	42. (b)	43. (b)	44. (c)	45. (d)	46. (a)	47. (b)	48. (d)	49. (c)	50. (a)
51. (a)	52. (d)	53. (b)	54. (a)	55. (d)	56. (d)	57. (d)	58. (c)	59. (d)	60. (a,d)
61. (a,b)	62. (b,d)	63. (a,b,d)	64. (a,b)	65. (a,b)	66. (a,b,c)	67. (a,b)	68. (a,b,d)	69. (a,b)	70. (a,b)
71. (c,d)	72. (a,b,d)	73. (b,c,d)	74. (b,c,d)	75. (b,c)	76. (a,c)	77. (a,c)	78. (a,c)	79. (a)	80. (c,d)
81. (a,d)	82. (a,b)	83. (d)	84. (a)	85. (d)	86. (c)	87. (b)	88. (a)	89. (d)	90. (a)
91. (b)	92. (a)	93. (b)	94. (a)	95. (b)	96. (a)	97. (a)	98. (b)	99. (b)	100. (b)
101. (c)	102. (d)	103. (b)	104. (a)	105. (c)	106. (c)	107. (c)	108. (d)	109. (c)	110. (b)
111. (b)	112. (b)	113. (b)	114. (b)	115. (b)	116. (b)	117. (a)	118. (*)	119. (*)	120. (*)
121. (*)	122. (*)	123. (*)	124. (4)	125. (5)	126. (0)	127. (1)	128. (2)	129. (4)	130. (6)
131. (2)	132. (0)	133. (3)	134. (0)						

*

118. $A \rightarrow q, s; B \rightarrow p, r; C \rightarrow p, s; D \rightarrow t, r$

119. $A \rightarrow q, s; B \rightarrow r; C \rightarrow p, s$

120. $A \rightarrow p, q; B \rightarrow t, u; C \rightarrow r, s$

121. $A \rightarrow q; B \rightarrow s; C \rightarrow p; D \rightarrow r$

122. $A \rightarrow s; B \rightarrow r; C \rightarrow t; D \rightarrow p; E \rightarrow q$

123. $A \rightarrow p; B \rightarrow r; C \rightarrow s; D \rightarrow p$

Entrances Gallery

1. (a,b)	2. (c,d)	3. (a,d)	4. (d)	5. (a)	6. (d)	7. (c)	8. (d)	9. (a)	10. (1)
11. (4)	12. (c)	13. (c)	14. (b)	15. (b)	16. (c)	17. (a)	18. (b)	19. (a)	20. (d)
21. (c)	22. (d)	23. (a)	24. (d)	25. (b)	26. (c)	27. (a)	28. (d)	29. (c)	30. (c)
31. (a)	32. (d)	33. (a)	34. (b)	35. (c)	36. (b)	37. (a)	38. (c)	39. (a)	40. (a)
41. (c)	42. (b)	43. (e)	44. (d)	45. (a)	46. (a)	47. (a)	48. (d)	49. (d)	50. (b)
51. (a)	52. (c)	53. (d)	54. (c)	55. (b)	56. (c)	57. (c)	58. (a)	59. (b)	60. (c)
61. (d)	62. (b)	63. (a)	64. (a)	65. (d)	66. (c)	67. (b)	68. (a)	69. (d)	70. (b)
71. (b)	72. (a)	73. (a)	74. (a,b)	75. (c)	76. (b)	77. (b)	78. (b)	79. (e)	80. (b)
81. (d)	82. (d)	83. (d)	84. (a)						

Explanations

Target Exercises

1. $|A|^2 = |A| \Rightarrow |A|^2 - |A| = 0$
 $|A|(|A| - 1) = 0$
 $\therefore |A| = 0 \text{ or } 1$
2. $|A|^n = 2^n$ as $|A| = 2$
3. $|3A| = 27|A|$
4. The result of given determinant = 0
 $[\because R_1 \text{ and } R_3 \text{ are proportional}]$
5. On applying $C_3 \rightarrow C_3 + C_2$ and taking common $(x + y + z)$ from C_3 , we get $\Delta = 0$ $[\because C_1, C_3 \text{ are identical}]$
6. On applying $C_2 \rightarrow C_2 - IC_1 - mC_3$, we get the value of the determinant = 0

7. On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get the given determinant

$$= \begin{vmatrix} 3+a & 3+a & 3+a \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix} = (3+a) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+a & 1 \\ 1 & 1 & 1+a \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$

$$= (3+a) \begin{vmatrix} 1 & 0 & 0 \\ 1 & a & 0 \\ 1 & 0 & a \end{vmatrix} = (3+a)a^2 = a^3 \left(1 + \frac{3}{a}\right)$$

8. On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get
 $\Delta = 0$

$$9. \begin{vmatrix} a & b & c \\ b & c & a \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} = D$$

10. On applying $C_3 \rightarrow C_3 - (C_1 + C_2)$, we get

$$\Delta = \begin{vmatrix} 1 & \omega & 0 \\ 1+\omega & 1 & -2 \\ \omega & \omega+1 & -2\omega \end{vmatrix} = 4$$

11. On applying $R_1 \rightarrow R_1 + R_3 - 2R_2$ and using $2b - a - c = 0$, we get the given determinant having all elements in the first row as zero.

$[\because a, b \text{ and } c \text{ are in AP}]$

$\therefore$ The value of determinant = 0

12. On applying $R_3 \rightarrow R_3 - 3R_1$ and $R_2 \rightarrow R_2 - R_1$, we get

$$\begin{vmatrix} 2 & 3 & x \\ 0 & -2 & x^2 - x \\ 0 & -2 & 3 - 3x \end{vmatrix} = 0$$

$$\Rightarrow 2 \begin{vmatrix} -2 & x^2 - x \\ -2 & 3 - 3x \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & x^2 - x \\ 1 & 3 - 3x \end{vmatrix} = 0$$

$$\Rightarrow 3 - 3x - x^2 + x = 0 \Rightarrow (x+3)(x-1) = 0$$

$$\Rightarrow x = 1, -3$$

$$13. \begin{vmatrix} 4 & 12 & 4 \\ 8 & -4 & 4 \\ 0 & 16 & 8 \end{vmatrix} = 4^3 \begin{vmatrix} 1 & 3 & 1 \\ 2 & -1 & 1 \\ 0 & 4 & 2 \end{vmatrix} = 64\Delta$$

$$14. \text{ Let } \Delta = \begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - C_2$ and $C_2 \rightarrow C_2 - C_3$, we get

$$\Delta = \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}$$

Now, on applying $C_1 \rightarrow C_1 - C_2$ and $C_3 \rightarrow C_3 - 10C_2$, we get

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 4 & 17 & 11 \end{vmatrix}$$

On applying $R_3 \rightarrow R_1 - R_1$ we get

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ -12 & 27 & -72 \\ 0 & -4 & 2 \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 + 3R_1$, we get

$$\Delta = \begin{vmatrix} 4 & 21 & 9 \\ 0 & 90 & -45 \\ 0 & -4 & 2 \end{vmatrix} = 4(90 \times 2 - 4 \times 45) = 0$$

$$15. \text{ We have, } \begin{vmatrix} a & b-c & c+b \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = 0$$

On applying $C_1 \rightarrow aC_1$, we get

$$\frac{1}{a} \begin{vmatrix} a^2 & b-c & c+b \\ a^2+ac & b & c-a \\ a^2-ab & a+b & c \end{vmatrix} = 0$$

On applying $C_1 \rightarrow C_1 + bC_2 + cC_3$, we get

$$\frac{1}{a} \begin{vmatrix} a^2+b^2+c^2 & b-c & c+b \\ a^2+b^2+c^2 & b & c-a \\ a^2+b^2+c^2 & a+b & c \end{vmatrix} = 0$$

$$\Rightarrow \frac{(a^2+b^2+c^2)}{a} \begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & a+b & c \end{vmatrix} = 0$$

Now, on applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\frac{(a^2+b^2+c^2)}{a} \begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -(a+b) \\ 0 & a+c & -b \end{vmatrix} = 0$$

$$\Rightarrow \frac{(a^2+b^2+c^2)}{a} (-bc + a^2 + ab + ca + bc) = 0$$

$$\Rightarrow (a^2+b^2+c^2)(a+b+c) = 0$$

$$\Rightarrow a+b+c = 0$$

$\therefore$ Line $ax + by + c = 0$ passes through the fixed point (1, 1).

$$16. \text{LHS} = a^2 b^2 c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} = 4a^2 b^2 c^2 = \lambda a^2 b^2 c^2$$

$$\therefore \lambda = 4$$

$$17. \Delta = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 6 & 24 \\ 6 & 24 & 120 \end{vmatrix} = 4!$$

$$18. \text{On applying } R_1 \rightarrow R_1 + R_2 + R_3, \text{ we get}$$

$$\begin{vmatrix} 3x-2 & 3x-2 & 3x-2 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

$$\Rightarrow (3x-2) \begin{vmatrix} 1 & 1 & 1 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

$$\therefore x = \frac{2}{3} \text{ is one of the roots.}$$

$$19. \text{We have, } \begin{vmatrix} 2a & b & c \\ b & c & 2a \\ c & 2a & b \end{vmatrix} = 0$$

$$\Rightarrow 2a(bc - 4a^2) + b(2ac - b^2) + c(2ab - c^2) = 0$$

$$\Rightarrow 6abc - 8a^3 - b^3 - c^3 = 0$$

$$\Rightarrow (2a + b + c)[(2a - b)^2 + (b - c)^2 + (c - 2a)^2] = 0$$

$$\Rightarrow 2a + b + c = 0$$

$$[\because \text{no two of } a, b \text{ and } c \text{ are equal}]$$

$$\text{Let } f(x) = 8ax^3 + 2bx^2 + cx$$

$$\text{Clearly, } f(0) = 0 \text{ and } f\left(\frac{1}{2}\right) = a + \frac{b}{2} + \frac{c}{2} = \frac{2a + b + c}{2} = 0$$

$$\text{So, } f(x) \text{ satisfies the Rolle's theorem and hence}$$

$$f'(x) = 0 \text{ has atleast one root in } \left[0, \frac{1}{2}\right].$$

$$20. \text{We have, } f(x) = \begin{vmatrix} 2^{-x} & 2^x & x^2 \\ 2^{-3x} & 2^{3x} & x^4 \\ 2^{-5x} & 2^{5x} & 1 \end{vmatrix}$$

$$[\because e^{x \log_e 2} = e^{\log_e (2^x)} = 2^x \text{ etc.}]$$

$$\text{Now, consider } f(-x) = \begin{vmatrix} 2^x & 2^{-x} & x^2 \\ 2^{3x} & 2^{-3x} & x^4 \\ 2^{5x} & 2^{-5x} & 1 \end{vmatrix}$$

$$= - \begin{vmatrix} 2^{-x} & 2^x & x^2 \\ 2^{-3x} & 2^{3x} & x^4 \\ 2^{-5x} & 2^{5x} & 1 \end{vmatrix}$$

$$= -f(x) \Rightarrow f(-x) + f(x) = 0$$

$$21. \begin{vmatrix} 1 & 1 & 1 \\ m & m+1 & m+2 \\ \frac{m(m-1)}{2} & \frac{(m+1)m}{2} & \frac{(m+2)(m+1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ m & 1 & 1 \\ \frac{m(m-1)}{2} & m & m+1 \end{vmatrix} = 1$$

$$22. \text{Since, } -1 \leq x < 0$$

$$\therefore \begin{matrix} [x] = -1 \\ \text{Also, } 0 \leq y < 1 \\ \therefore [y] = 0 \text{ and } 1 \leq z < 2 \\ \therefore [z] = 1 \end{matrix}$$

$$\text{Hence, } \begin{vmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{vmatrix} = 1 = [z]$$

$$23. \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$$

$$= (b^2 - ac)(a\alpha^2 + 2b\alpha + c) = 0$$

$$\text{Either } b^2 = ac \text{ or } a\alpha^2 + 2b\alpha + c = 0$$

$$\Rightarrow a, b, c \text{ are in GP or } x - \alpha \text{ is a factor of } ax^2 + 2bx + c.$$

$$24. \text{On applying } C_1 \rightarrow C_1 + C_2, \text{ we get}$$

$$\Delta = \begin{vmatrix} 2 & \cos^2 \theta & 4\sin 4\theta \\ 2 & 1 + \cos^2 \theta & 4\sin 4\theta \\ 1 & \cos^2 \theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$$\text{On applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1, \text{ we get}$$

$$\Delta = \begin{vmatrix} 2 & \cos^2 \theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 2 + 4\sin 4\theta = 0$$

$$\sin 4\theta = -\frac{1}{2}$$

$$\Rightarrow 4\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\Rightarrow \theta = \frac{7\pi}{24}, \frac{11\pi}{24}$$

$$25. \text{On applying } R_1 \rightarrow R_1 + R_2 + R_3, \text{ we get}$$

$$\begin{vmatrix} x+9 & x+9 & x+9 \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x+9) \begin{vmatrix} 1 & 1 & 1 \\ 2 & x & 2 \\ x & 3 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (x+9) \begin{vmatrix} 1 & 0 & 0 \\ 2 & x-2 & 0 \\ x & 3-x & 7-x \end{vmatrix} = 0$$

$$\Rightarrow (x+9)(x-2)(7-x) = 0$$

$$\Rightarrow x = 2, 7 \text{ are other roots.}$$

$$26. \sum_{r=1}^n 2^{r-1} = 1 + 2 + 2^2 + \dots + 2^{n-1} = \frac{2^n - 1}{2 - 1} = 2^n - 1$$

$$\sum_{r=1}^n 2 \cdot 3^{r-1} = \frac{2(3^n - 1)}{3 - 1} = 3^n - 1$$

$$\sum_{r=1}^n 4 \cdot 5^{r-1} = \frac{4(5^n - 1)}{5 - 1} = 5^n - 1$$

$$\therefore \sum_{r=1}^n D_r = \begin{vmatrix} 2^n - 1 & 3^n - 1 & 5^n - 1 \\ \alpha & \beta & \gamma \\ 2^n - 1 & 3^n - 1 & 5^n - 1 \end{vmatrix} = 0$$

27. On applying $R_1 \rightarrow R_1 + (\sin \beta)R_2 + (\cos \beta)R_3$, we get

$$\Delta = \begin{vmatrix} 0 & 0 & 1 + \cos 2\beta \\ \sin \alpha & \cos \alpha & \sin \beta \\ -\cos \alpha & \sin \alpha & \cos \beta \end{vmatrix}$$

$$= (1 + \cos 2\beta)(\sin^2 \alpha + \cos^2 \alpha) = 1 + \cos 2\beta$$

which is independent of α .

28. Let $\Delta = \begin{vmatrix} a \sin^2 x & a^2 + \cos^2 x & 1 \\ a \sin^2 y & a^2 + \cos^2 y & 1 \\ a \sin^2 z & a^2 + \cos^2 z & 1 \end{vmatrix}$

$$= a \begin{vmatrix} \sin^2 x & a^2 + \cos^2 x & 1 \\ \sin^2 y & a^2 + \cos^2 y & 1 \\ \sin^2 z & a^2 + \cos^2 z & 1 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + C_2$, we get

$$\Delta = a \begin{vmatrix} a^2 + 1 & a^2 + \cos^2 x & 1 \\ a^2 + 1 & a^2 + \cos^2 y & 1 \\ a^2 + 1 & a^2 + \cos^2 z & 1 \end{vmatrix}$$

$$= a(a^2 + 1) \begin{vmatrix} 1 & a^2 + \cos^2 x & 1 \\ 1 & a^2 + \cos^2 y & 1 \\ 1 & a^2 + \cos^2 z & 1 \end{vmatrix} = 0$$

29. On applying $C_1 \rightarrow C_1 - C_2$, we get

$$\begin{vmatrix} 4 & (e^{i\alpha} - e^{-i\alpha})^2 & 4 \\ 4 & (e^{i\beta} - e^{-i\beta})^2 & 4 \\ 4 & (e^{i\gamma} - e^{-i\gamma})^2 & 4 \end{vmatrix} = 0$$

30. $A^0 = A$

$\Rightarrow A$ is Hermitian matrix.

Thus, $|A|$ = purely real

31. On expanding, we get an odd degree equation.

$\therefore$ Atleast one real root.

32. On applying $C_3 \rightarrow C_3 - C_2$, we get

$$f(x) = \begin{vmatrix} 1 & x & 1 \\ 2x & x(x-1) & 2x \\ 3x(x-1) & x(x-1)(x-2) & 3x(x-1) \end{vmatrix} = 0$$

$$\Rightarrow f(100) = 0$$

33. On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} \sin x + 2 \cos x & \cos x & \cos x \\ \sin x + 2 \cos x & \sin x & \cos x \\ \sin x + 2 \cos x & \cos x & \sin x \end{vmatrix} = 0$$

$$= (\sin x + 2 \cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix} = 0$$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$\Delta = (\sin x + 2 \cos x) \begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x - \cos x & 0 \\ 0 & 0 & \sin x - \cos x \end{vmatrix} = 0$$

$$= (\sin x + 2 \cos x)(\sin x - \cos x)^2 = 0$$

Thus, $\Delta = 0 \Rightarrow \tan x = -2$ or $\tan x = 1$

Since, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$, we get $-1 \leq \tan x \leq 1$

$$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

34. Since, $x^3 - x^2 - 2004 = 0$ has three roots a, b, c .

$$\therefore a + b + c = 1 \quad \dots(i)$$

$$ab + bc + ca = 0 \quad \dots(ii)$$

$$abc = 2004 \quad \dots(iii)$$

Now, given determinant,

$$\begin{vmatrix} a+c & a+b & b+c \\ a(b+c) & b(c+a) & c(a+b) \\ a^2+1 & b^2+1 & c^2+1 \end{vmatrix} = \begin{vmatrix} 1-b & 1-c & 1-a \\ -bc & -ca & -ab \\ a^2+1 & b^2+1 & c^2+1 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} 2 & 1-c & 1-a \\ 0 & -ca & -ab \\ 4 & b^2+1 & c^2+1 \end{vmatrix} = -2a \begin{vmatrix} 1 & 1-c & 1-a \\ 0 & c & b \\ 0 & b^2+2c-1 & c^2+2a-1 \end{vmatrix}$$

[applying $R_3 \rightarrow R_3 - 2R_1$]

$$= -2a \begin{vmatrix} 1 & 1-c & 1-a \\ 0 & c & b \\ 0 & b^2+2c-1 & c^2+2a-1 \end{vmatrix}$$

$$= (-2a)(c^3 - c + 2ac - b^3 + b - 2bc)$$

$$= -2ac^3 + 2ac - 4a^2c + 2ab^3 - 2ab + 4abc$$

35. Putting $x = 1, y = 1$ and $z = 1$ on both sides, we get

$$k = 4$$

36. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

$$\Rightarrow A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \det(A + B) = 0$$

But $\det A = 1, \det B = 1$

$$\therefore \det A \neq 0, \det B \neq 0$$

$$\text{and } \det(A + B) \neq \det A + \det B \Rightarrow \det A + \det B \neq 0$$

37. Since, A is an invertible matrix, therefore $|A| \neq 0$

$$\text{Since, } |A| = |A'|$$

$$\therefore |A'| \neq 0, \text{ i.e. } A' \text{ is invertible.}$$

$$\text{Now, } A(\text{adj } A)' = |A|I_n$$

$$\Rightarrow (A \text{ adj } A)' = (|A|I_n)'$$

$$\Rightarrow (\text{adj } A)' A' = |A|I_n \quad \dots(i)$$

$$\text{Also, } (\text{adj } A') A' = |A'|I_n$$

$$\Rightarrow (\text{adj } A') A' = |A|I_n \quad [\because |A| = |A'|] \dots(ii)$$

From Eqs. (i) and (ii), we have

$$(\text{adj } A') A' = (\text{adj } A)' A'$$

$$\Rightarrow (\text{adj } A') = (\text{adj } A)' \quad [\because |A'| \neq 0]$$

38. Since, A is an orthogonal matrix, therefore we have,

$$A A^T = I$$

$$\Rightarrow |A A^T| = |I|$$

$$\Rightarrow |A| |A^T| = 1$$

$$\Rightarrow |A|^2 = 1 \quad [\because |A| = |A^T|]$$

$$\Rightarrow |A| = \pm 1$$

Now consider,

$$|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$$

$$\text{Thus, } |\text{adj}(\text{adj } A)| = 1, \text{ if } n \text{ is odd}$$

39. $|A| = a^3, |\text{adj } A| = (a^3)^2$

$$\Rightarrow |A| |\text{adj } A| = a^9$$

$$= \text{Product of } |A| |\text{adj } A| = a^9$$

40. $a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} + a_{14}C_{14}$ = Sum of product of the elements of a row by its cofactors = $|A|$

41. As $A \cdot \text{adj}(A) = |A| \cdot I_n \Rightarrow A \cdot (\text{adj } A) = |A| \cdot I_n$
 $= |A| \cdot I_n = 0$

$\Rightarrow A \cdot (\text{adj } A)$ is singular matrix.

42. $|A| = |A'|$ for all square matrix A .

43. $|A| = \begin{vmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{vmatrix} = a^3$

$|\text{adj } A| = |A|^2 = (a^3)^2 = a^6$

44. Given, homogeneous equations have non-zero

solution, if $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1-a & b-1 & 0 \\ 1-a & 0 & c-1 \end{vmatrix} = 0$

$\Rightarrow a(b-1)(c-1) - (1-a)(c-1) - (1-a)(b-1) = 0$

On dividing by $(1-a)(1-b)(1-c)$, we get

$\frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$

$\Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$

45. Number of decomposed determinants $= 2 \times 3 \times 4 = 24$

46. Since, the system has a non-trivial solution

$\therefore \begin{vmatrix} p+a & b & c \\ a & q+b & c \\ a & b & r+c \end{vmatrix} = 0$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$\begin{vmatrix} p+a & b & c \\ -p & q & 0 \\ -p & 0 & r \end{vmatrix}$

$\Rightarrow (p+a)(q)r - b(-pr) + c(pq) = 0$

$\Rightarrow pqr + [q(p+a) + bp]r = 0$

On dividing by pqr , we get

$\frac{c}{r} + \frac{b}{q} + 1 + \frac{a}{p} = 0$

$\Rightarrow \frac{a}{p} + \frac{b}{q} + \frac{c}{r} = -1$

47. On solving $2x + y = 4$, $x - 2y = 2$, we get $x = 2$, $y = 0$

This also satisfies $3x + 5y = 6$.

48. For non-trivial solution,

$\begin{vmatrix} \sin \theta & 1 & 1 \\ 1 & \cos \theta & 1 \\ \sin \theta & \cos \theta & 1 \end{vmatrix} = 0$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$\begin{vmatrix} \sin \theta & 1 & 1 \\ 1-\sin \theta & \cos \theta - 1 & 0 \\ \sin \theta & \cos \theta - 1 & 0 \end{vmatrix} = 0$

$\Rightarrow (1-\sin \theta)(\cos \theta - 1) = 0$

$\Rightarrow \sin \theta = 1$ or $\cos \theta = 1$

Now, $\cos \theta = 1$ gives $\theta = 0$, which cannot be possible

and $\sin \theta = 1$ gives $\theta = \frac{\pi}{2}$

49. For non-zero solution, $\begin{vmatrix} a^3 & (a+1)^3 & (a+2)^3 \\ a & a+1 & a+2 \\ 1 & 1 & 1 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ a & a+1 & a+2 \\ a^3 & (a+1)^3 & (a+2)^3 \end{vmatrix} = 0$

$\Rightarrow -(a-a-1)(a+1-a-2)(a+2-a) \times (a+a+1+a+2) = 0$

$\left[\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z) \right]$

$\Rightarrow -2(3a+3) = 0 \Rightarrow a = -1$

50. We have, $\Delta = \begin{vmatrix} 2 \cos \theta & 1 & 0 \\ 1 & 2 \cos \theta & 1 \\ 0 & 1 & 2 \cos \theta \end{vmatrix}$

$= 2 \cos \theta (4 \cos^2 \theta - 1) - 1(2 \cos \theta - 0)$

$= 8 \cos^3 \theta - 2 \cos \theta - 2 \cos \theta$

$= 8 \cos^3 \theta - 4 \cos \theta$

$= 4 \cos \theta (2 \cos^2 \theta - 1)$

$= 4 \cos \theta \cdot \cos 2\theta = \frac{\sin 4\theta}{\sin \theta}$

51. By definition, $|A| = 0$

$\therefore$ Infinitely many solutions.

52. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$
 $= 0$

$\Rightarrow at^2 + bt + c = 0$ has both imaginary roots ω, ω^2 or both equal roots; $t = 1, 1$.

53. $|A| = \cos^2 \theta + \sin^2 \theta = 1$

$\text{adj}(A) = \begin{bmatrix} \cos \theta & -\sin \theta \\ -(-\sin \theta) & \cos \theta \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

$A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} \cos \theta & \sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$

54. $|A| = abc$

$\text{adj}(A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix} = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

$A^{-1} = \frac{\text{adj}(A)}{|A|} = \frac{1}{abc} \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}$

55. $|A| = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$

$\text{adj}(A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

$\therefore A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

56. $|\text{adj } A|^2 = (11^2)^2 = 14641$

57. Given equations are

$$(\alpha + 1)^3 x + (\alpha + 2)^3 y - (\alpha + 3)^3 = 0,$$

$$(\alpha + 1)x + (\alpha + 2)y - (\alpha + 3) = 0 \text{ and } x + y - 1 = 0$$

Since, this system of equations is consistent.

$$\therefore \begin{vmatrix} (\alpha + 1)^3 & (\alpha + 2)^3 & -(\alpha + 3)^3 \\ (\alpha + 1) & (\alpha + 2) & -(\alpha + 3) \\ 1 & 1 & -1 \end{vmatrix} = 0$$

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 + C_1$, we get

$$\begin{vmatrix} (\alpha + 1)^3 & (\alpha + 2)^3 - (\alpha + 1)^3 & (\alpha + 1)^3 - (\alpha + 3)^3 \\ (\alpha + 1) & (\alpha + 2) - (\alpha + 1) & -(\alpha + 3) + (\alpha + 1) \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} (\alpha + 1)^3 & 3\alpha^2 + 9\alpha + 7 & -6\alpha^2 - 24\alpha - 26 \\ (\alpha + 1) & 1 & -2 \\ 1 & 0 & 0 \end{vmatrix} = 0$$

$$\Rightarrow -2(3\alpha^2 + 9\alpha + 7) + 6\alpha^2 + 24\alpha + 26 = 0$$

$$\Rightarrow 6\alpha + 12 = 0 \Rightarrow \alpha = -2$$

58. Let $A = \begin{bmatrix} 4 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 \end{bmatrix}$

Now, we take a sub-matrix of order 3×3 .

$$B = \begin{bmatrix} 4 & 1 & 0 \\ 3 & 0 & 1 \\ 6 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow |B| = -1(6 - 6) = 0$$

Now, we take a sub-matrix of order 2×2 .

$$C = \begin{bmatrix} 4 & 1 \\ 3 & 0 \end{bmatrix} \Rightarrow |C| = 0 - 3 \neq 0$$

So, the rank of matrix A is 2.

59. $\lim_{x \rightarrow 0} \frac{f'(x)}{x} = \lim_{x \rightarrow 0} \frac{f''(x)}{1} = f''(0)$

where, $f(x) = \begin{vmatrix} x & \sin x & \cos x \\ x^2 & \tan x & -x^3 \\ 2x & \sin 2x & 5x \end{vmatrix}$

$$\therefore f''(0) = -4$$

60. Let D be the mid-point of BC.

$$\therefore \text{Coordinates of } D \text{ are } \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2} \right).$$

$$\text{Now, equation of } AD = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2 + x_3}{2} & \frac{y_2 + y_3}{2} & 1 \end{vmatrix} = 0$$

By properties of determinant,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 + x_3 & y_2 + y_3 & 2 \end{vmatrix} = 0$$

Equation of median

$$AD = \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} + \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

A line joining centroid with vertex A will be same as AD.

61. On expanding, we get

$$\Delta = \frac{(a+b-x)}{I} \frac{[x^2 + (a+b)x + (a^2 + b^2 - ab)]}{II}$$

If $a = b$, then the discriminant of $x^2 + (a+b)x + (a^2 + b^2 - ab) = 0$ is $4a^2 - 4a^2 = 0$ and II part has two real and equal roots. Thus, total number of real roots for $a = b$ is three.

62. On applying $R_3 \rightarrow R_3 + R_1$, we get

$$|A| = \begin{vmatrix} 1 & \sin \theta & 1 \\ -\sin \theta & 1 & \sin \theta \\ 0 & 0 & 2 \end{vmatrix} = 2(1 + \sin^2 \theta)$$

which lies between 2 and 4.

63. On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$D(x) = (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 1 & 1 & x \\ 1 & x^2 & 1 \end{vmatrix}$$

$$= (1+x+x^2) \begin{vmatrix} 1 & x & x^2 \\ 0 & 1-x & x-x^2 \\ 0 & x^2-x & 1-x^2 \end{vmatrix} = (1-x^3)^2$$

$$\Rightarrow D'(x) = 2(1-x^3)(-3x^2)$$

$$\therefore D'(1) = 0$$

64. On applying $C_2 \rightarrow C_2 - aC_1$ and simplifying, we get

$$D = a(a+b)(a+2b)$$

65. $\Delta = (1+a)[(1+a)^2 - 1] - 1[1+a-1] + 1[1-(1+a)]$
 $= 3a^2 + a^3$

which is zero for $a = 0$ and $a = -3$

66. We have, $A(\alpha, \beta) = \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & e^\beta \end{bmatrix}$

$$= \begin{bmatrix} \cos(-\alpha) & \sin(-\alpha) & 0 \\ -\sin(-\alpha) & \cos(-\alpha) & 0 \\ 0 & 0 & e^\beta \end{bmatrix}$$

$$= A(-\alpha, \beta)$$

Also, $A(\alpha, \beta)A(-\alpha, -\beta)$

$$= \begin{bmatrix} \cos \alpha & \sin \alpha & 0 \\ -\sin \alpha & \cos \alpha & 0 \\ 0 & 0 & e^\beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & e^{-\beta} \end{bmatrix} = I$$

$$\Rightarrow A(\alpha, \beta)^{-1} = A(-\alpha, -\beta)$$

$$\text{Next, } \text{adj } A(\alpha, \beta) = |A(\alpha, \beta)| A(\alpha, \beta)^{-1} = e^\beta A(-\alpha, -\beta)$$

67. We have,

$$f(x) = \begin{vmatrix} 3 & 3x & 3x^2 + 2a^2 \\ 3x & 3x^2 + 2a^2 & 3x^3 + 6a^2x \\ 3x^2 + 2a^2 & 3x^3 + 6a^2x & 3x^4 + 12a^2x^2 + 2a^4 \end{vmatrix}$$

On applying $C_3 \rightarrow C_3 - xC_2$ and $C_2 \rightarrow C_2 - xC_1$, we get

$$f(x) = \begin{vmatrix} 3 & 0 & 2a^2 \\ 3x & 2a^2 & 4a^2x \\ 3x^2 + 2a^2 & 4a^2x & 6a^2x^2 + 2a^4 \end{vmatrix}$$

$$= 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & 2x \\ 3x^2 + 2a^2 & 2x & 3x^2 + a^2 \end{vmatrix}$$

On applying $C_3 \rightarrow C_3 - x C_2$, we get

$$f(x) = 4a^4 \begin{vmatrix} 3 & 0 & 1 \\ 3x & 1 & x \\ 3x^2 + 2a^2 & 2x & x^2 + a^2 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - 3C_3$, we get

$$f(x) = 4a^4 \begin{vmatrix} 0 & 0 & 1 \\ 0 & 1 & x \\ -a^2 & 2x & x^2 + a^2 \end{vmatrix} = 4a^4 (-a^2)(-1) = 4a^6 \Rightarrow f'(x) = 0$$

68. We have, $\text{adj } A = |A| A^{-1}$

$$\begin{aligned} \text{adj}(AB) &= |AB| (AB)^{-1} = |A| |B| (AB)^{-1} \\ &= |A| |B| (B^{-1}A^{-1}) = (|B| B^{-1}) (|A| A^{-1}) \\ &= \text{adj } B \text{ adj } A \end{aligned}$$

69. Let $f(x) = \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix}$
 $= A + Bx + Cx^2 + \dots$ (i)

On putting $x = 0$ to get the value of A , we get

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = A + 0 + \dots + A = 0$$

On differentiating both sides of Eq(i) w.r.t. x , we get

$$\begin{aligned} f'(x) &= \begin{vmatrix} a(1+x)^{a-1} & 2b(1+2x)^{b-1} & 0 \\ 1 & (1+x)^a & (1+2x)^b \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} \\ &+ \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 0 & a(1+x)^{a-1} & 2b(1+2x)^{b-1} \\ (1+2x)^b & 1 & (1+x)^a \end{vmatrix} \\ &+ \begin{vmatrix} (1+x)^a & (1+2x)^b & 1 \\ 1 & (1+x)^a & (1+2x)^b \\ 2b(1+2x)^{b-1} & 0 & a(1+x)^{a-1} \end{vmatrix} \\ &= B + 2Cx + \dots \end{aligned}$$

On putting $x = 0$, we get

$$\begin{vmatrix} a & 2b & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 0 & a & 2b \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 2b & 0 & a \end{vmatrix} = B$$

$\Rightarrow B = 0$ or coefficient $x = 0$

70. $A = \begin{bmatrix} 1 & 2 \\ 3 & -5 \end{bmatrix}$

The cofactors are given by

$$C_{11} = -5, C_{12} = -3, C_{21} = -2, C_{22} = 1$$

Let $B = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} -5 & -3 \\ -2 & 1 \end{bmatrix}$

$$\text{adj } (A) = B' = \begin{bmatrix} -5 & -2 \\ -3 & 1 \end{bmatrix}$$

71. $X = A^{-1}B, X = \frac{(\text{adj } A)B}{|A|}$

Clearly, the system has infinite solutions, if $|A| = 0$ and $(\text{adj } A)B = 0$

72. $\Delta = \begin{vmatrix} a & 4 & 1 \\ 0 & 2 & 3 \\ 3 & 0 & -b \end{vmatrix} = 30 - 2ab$

$$\Delta_x = \begin{vmatrix} 0 & 4 & 1 \\ 1 & 2 & 3 \\ -2 & 0 & -b \end{vmatrix} = 4b - 20$$

$$\Delta_y = \begin{vmatrix} a & 0 & 1 \\ 0 & 1 & 3 \\ 3 & -2 & -b \end{vmatrix} = -ab + 6a - 3$$

$$\Delta_z = \begin{vmatrix} a & 4 & 0 \\ 0 & 2 & 1 \\ 3 & 0 & -2 \end{vmatrix} = 12 - 4a$$

For unique solution, $D \neq 0, ab \neq 15$

If $ab = 15$, then

$$\Delta_x = \frac{60}{a} - 20, \Delta_y = 6a - 18, \Delta_z = 12 - 4a$$

For $a = 3$, the value of $\Delta_x, \Delta_y, \Delta_z$ vanish simultaneously. For $a \neq 3$, all the three determinants become non-zero. Thus, system has infinite solutions for $ab = 15$ and $a = 3$ and no solution for $ab = 15, a \neq 3$.

73. We know that, $\text{adj } (A) = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$

Now, $A^{-1} = \frac{\text{adj } (A)}{|A|} = -\text{adj } (A) \quad [\because |A| = -1]$

Now, $(\text{adj } A) = -A$

$\Rightarrow A^{-1} = -(\text{adj } A) = -(-A) = A$

$\therefore A^2 = A \cdot A = A \cdot A^{-1} = I$

74. It is sufficient to show that under the given conditions the determinant D of the system cannot vanish.

On applying $C_3 \rightarrow C_3 + C_2 + C_1$, we get

$$D = \begin{vmatrix} a_{11} & a_{12} & p_1 \\ a_{21} & a_{22} & p_2 \\ a_{31} & a_{32} & p_3 \end{vmatrix}$$

where, $p_i = a_{i1} + a_{i2} + a_{i3}$ and $i = 1, 2, 3$

Now, on expanding by C_3 ,

$$D = p_1(a_{21}a_{32} - a_{22}a_{31}) + p_2(a_{12}a_{31} - a_{11}a_{32}) + p_3(a_{11}a_{22} - a_{12}a_{21}) \dots (i)$$

Now, first two terms of Eq. (i) are positive because $p_1, p_2 > 0, a_{21}a_{32} > 0, a_{22}a_{31} < 0, a_{12}a_{31} > 0, a_{11}a_{32} < 0$
 Using condition, $a_{ij} < 0, \forall i \neq j$

$$a_{11} + a_{12} > a_{11} + a_{12} + a_{13} > 0$$

$$a_{11} > -a_{12} = |a_{12}|$$

and $a_{22} + a_{21} > a_{22} + a_{21} + a_{23} > 0$

and so $a_{22} > -a_{21} = |a_{21}|$

Thus, $a_{11}a_{22} > a_{12}a_{21}$

Hence, third term in Eq. (i) becomes positive.

75. Let us evaluate A_4 by expanding the determinant using the 1st row

We have, $\det A_4 = \begin{vmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{vmatrix}$

$$= 2 \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix}$$

The 1st determinant on the right side of this equation is $\det A_3$. The 2nd determinant on the right may itself be expanded from first row.

$$\Rightarrow \begin{vmatrix} -1 & -1 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 2 \end{vmatrix} = (-1) \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} + 1 \begin{vmatrix} 0 & -1 \\ 0 & 2 \end{vmatrix}$$

$$= -\det A_2 + 0 = \det A_2$$

Therefore, $\det A_4 = 2 \det A_3 - \det A_2$. Exactly in the same way we can show that

$$\det A_n = 2 \det A_{n-1} - \det A_{n-2}$$

Since, $\det A_1 = 2$ and $\det A_2 = 4 - 1 = 3$, it follows by the use of mathematical induction that $\det A_n = n + 1$.

76. Inverse of matrix given in choice (a) is $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ whose

entries are integers, therefore it is called an inverse special matrix. Choice (c) is true as such matrices can be constructed infinitely.

77. $D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = 0$

System of equations has a solution only, if

$$D_1 = D_2 = D_3 = 0$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ \eta & 2 & 4 \\ \eta^2 & 4 & 10 \end{vmatrix} = 2(\eta - 1)(\eta - 2)$$

$$D_2 = -3(\eta - 1)(\eta - 2)$$

$$D_3 = (\eta - 1)(\eta - 2) \Rightarrow \eta = 1, 2$$

78. We have,

$$f(x) = \begin{vmatrix} 5^{-x} & e^{x \log_e 5} & x^2 \\ 5^{-3x} & e^{3x \log_e 5} & x^4 \\ 5^{-5x} & e^{5x \log_e 5} & 1 \end{vmatrix} = \begin{vmatrix} 5^{-x} & 5^x & x^2 \\ 5^{-3x} & 5^{3x} & x^4 \\ 5^{-5x} & 5^{5x} & 1 \end{vmatrix}$$

$$\Rightarrow f(-x) = \begin{vmatrix} 5^x & 5^{-x} & x^2 \\ 5^{3x} & 5^{-3x} & x^4 \\ 5^{5x} & 5^{-5x} & 1 \end{vmatrix} = - \begin{vmatrix} 5^{-x} & 5^x & x^2 \\ 5^{-3x} & 5^{3x} & x^4 \\ 5^{-5x} & 5^{5x} & 1 \end{vmatrix}$$

$$= -f(x) \Rightarrow f(-x) + f(x) = 0$$

$f(x)$ is an odd function.

Hence, the graph is symmetrical about origin.

Also, $f_4(x)$ is an odd function.

$$\text{Hence, } f_4(x) = -f_4(-x) \Rightarrow f_4(0) = -f_4(0) \Rightarrow f_4(0) = 0$$

79. $A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$A^{-n} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$ can be proved by principle of mathematical induction.

$$\text{Now, } \lim_{n \rightarrow \infty} \frac{A^{-n}}{n} = \lim_{n \rightarrow \infty} \begin{bmatrix} \frac{1}{n} & 0 \\ -1 & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$$

$$\text{Again, } \lim_{n \rightarrow \infty} \frac{A^{-n}}{n^2} = \lim_{n \rightarrow \infty} \begin{bmatrix} \frac{1}{n^2} & 0 \\ -\frac{1}{n} & \frac{1}{n^2} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

80. $D = \begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix}$

$$= \begin{vmatrix} X & sX & tX \\ X_1 & sX_1 + s_1X & tX_1 + t_1X \\ X_2 & sX_2 + 2s_1X_1 + s_2X & tX_2 + 2t_1X_1 + t_2X \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - sC_1, C_3 \rightarrow C_3 - tC_1$, we get

$$D = \begin{vmatrix} X & 0 & 0 \\ X_1 & s_1X & t_1X \\ X_2 & 2s_1X_1 + s_2X & 2t_1X_1 + t_2X \end{vmatrix}$$

$$= X^2 \begin{vmatrix} s_1 & t_1 \\ 2s_1X_1 + s_2X & 2t_1X_1 + t_2X \end{vmatrix} = X^2 \begin{vmatrix} s_1 & t_1 \\ s_2X & t_2X \end{vmatrix} \quad [R_2 \rightarrow R_2 - 2X_1 R_1]$$

$$= X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}$$

If s and t are linear functions, then $s_2 = 0, t_2 = 0$

81. On applying $R_1 \rightarrow R_1 - \sec x R_3$, we get

$$f(x) = \begin{vmatrix} 0 & 0 & \sec^2 x + \cot x \operatorname{cosec} x - \cos x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

After expanding and simplifying, we get

$$f(x) = -\cos^5 x - \sin^2 x$$

$$\text{Now, } \int_0^{\pi/2} f(x) dx = \int_0^{\pi/2} -\cos^5 x dx - \int_0^{\pi/2} \sin^2 x dx$$

$$= -\frac{4 \times 2}{5 \times 3} - \frac{1}{2} \times \frac{\pi}{2} \quad [\text{using Walli's formula}]$$

$$= -\frac{8}{15} - \frac{\pi}{4}$$

82. We have, $\frac{d^n}{dx^n} \log x = \frac{(-1)^{n-1}(n-1)!}{x^n}$

and $\frac{d^n}{dx^n} x^n = n!, \frac{d^n}{dx^n} x^{-1} = \frac{(-1)^n n!}{x^{n+1}}$

$$\text{Now, } f^n(x) = \begin{vmatrix} \frac{(-1)^n \cdot n!}{x^{n+1}} & \frac{(-1)^{n-1}(n-1)!}{x^n} & n! \\ 1 & -\frac{1}{n} & (-1)^n \\ 1 & a & a^2 \end{vmatrix}$$

$$\Rightarrow f^n(1) = (-1)^n n! \begin{vmatrix} 1 & -\frac{1}{n} & (-1)^n \\ 1 & -\frac{1}{n} & (-1)^n \\ 1 & a & a^2 \end{vmatrix} = 0$$

83. $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$

$$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

Now, the other bracket cannot vanish in the light of the correct Statement II.

The system can have non-zero solutions (x, y, z are not all zero).

If $D = 0$, then $a + b + c = 0$

$\therefore$ Statement I is false.

84. $D(x) = \lambda_1 f_1(x) - \lambda_2 f_2(x) + \lambda_3 f_3(x)$

where, $\lambda_1 = (b_2c_3 - b_3c_2)$, $\lambda_2 = a_2c_3 - a_3c_2$,
 $\lambda_3 = a_2b_3 - a_3b_2$

$$\int D(x) dx = \int \lambda_1 f_1(x) dx - \int \lambda_2 f_2(x) dx + \int \lambda_3 f_3(x) dx + C \dots (i)$$

$$= \begin{vmatrix} \int f_1(x) dx & \int f_2(x) dx & \int f_3(x) dx \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + C$$

Thus, Statement I is true and follows from Statement II which we have applied at Eq. (i).

85. Statement II is true, since

$$AI = IA, AA^{-1} = A^{-1}A = I, A(\text{adj } A) = (\text{adj } A)A$$

But a matrix can commute with general order matrices which may be infinite in number.

Let $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a matrix which commute with

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}, \text{ then}$$

$$\begin{aligned} AB &= BA \\ \Rightarrow \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} a+2c & b+2d \\ -a-c & -b-d \end{bmatrix} = \begin{bmatrix} a-b & 2a-b \\ c-d & 2c-d \end{bmatrix} \end{aligned}$$

$$\Rightarrow a+2c = a-b, b+2d = 2a-b, \\ -a-c = c-d, -b-d = 2c-d$$

The above four relations are equivalent to only two independent relations

$$a-d = b, b+2c = 0$$

If $d = \lambda$, then $a = b + \lambda = -2c + \lambda$

Thus, the matrices of the form $\begin{bmatrix} \lambda-2c & -2c \\ c & \lambda \end{bmatrix}$ are all

possible 2×2 matrices which commute with given matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$, where λ and c being any

arbitrary complex numbers. Thus, Statement I is false.

86. The Statement II is false, since $\det A' = \det (-A')$ is not true indeed

$$\det (-A') = (-1)^3 \det A'$$

Now, as $A = -A'$ [$\because A$ is skew-symmetric]

$$\Rightarrow \det A = \det (-A') \\ = -\det (A') = -\det A$$

$$\Rightarrow \det A = 0$$

Thus, Statement I is true.

87. Statement I is true, because

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = -(\alpha + \beta + \gamma)$$

$$(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = 0$$

$$\text{as } \alpha + \beta + \gamma = 0$$

Statement II is also true because any polynomial equation over reals has imaginary roots in pair and hence any cubic equation has atleast one real root.

However, Statement II is not a correct reason of Statement I.

$$88. D = \begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} + \begin{vmatrix} a & a^3 & -1 \\ b & b^3 & -1 \\ c & c^3 & -1 \end{vmatrix}$$

$$D = abc \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - 1 \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

$$D = abc (D_1) - 1 (D_2) = 0 \dots (i)$$

$$\text{Now, } D_1 = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} = abc \begin{vmatrix} 1/a & a & a^2 \\ 1/b & b & b^2 \\ 1/c & c & c^2 \end{vmatrix} = \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - C_3$, we get

$$D_1 = \begin{vmatrix} bc-a^2 & a & a^2 \\ ca-b^2 & b & b^2 \\ ab-c^2 & c & c^2 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + cC_2$, we get

$$D_1 = \begin{vmatrix} bc+ac-a^2 & a & a^2 \\ bc+ac-b^2 & b & b^2 \\ ab & c & c^2 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + aC_2$, we get

$$D_1 = \begin{vmatrix} bc+ac & a & a^2 \\ bc+ac+ab-b^2 & b & b^2 \\ ab+ac & c & c^2 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + bC_2$, we get

$$D_1 = \begin{vmatrix} ab+bc+ca & a & a^2 \\ ab+bc+ca & b & b^2 \\ ab+bc+ca & c & c^2 \end{vmatrix} = (ab+bc+ca) \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

After applying $R_2 \rightarrow R_2 - R_3$, $R_3 \rightarrow R_3 - R_1$, we get

$$D_1 = (ab+bc+ca)(a-b)(b-c)(c-a) \dots (ii)$$

$$D_2 = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

After applying $R_2 \rightarrow R_2 - R_3$, $R_3 \rightarrow R_3 - R_1$, we get

$$D_2 = (a-b)(b-c)(c-a)(a+b+c) \dots (iii)$$

From Eqs. (i), (ii) and (iii), we get

$$abc(ab+bc+ca) = a+b+c \quad [\because a, b \text{ and } c \text{ are different}]$$

89. Multiplication of two determinants leads us

$$f(x) = \begin{vmatrix} 1 & -y & y \\ -y & 1 & y \\ y & y & 1 \end{vmatrix}$$

where, $y = \sin x \cos x$

On applying $C_1 \rightarrow C_1 - C_2$, $C_2 \rightarrow C_2 + C_3$, we get

$$f(x) = (1+y)^2 \begin{vmatrix} 1 & 0 & y \\ -1 & 1 & y \\ 0 & 1 & 1 \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 + R_1$, we get

$$f(x) = (1+y)^2 \begin{vmatrix} 1 & 0 & y \\ 0 & 1 & 2y \\ 0 & 1 & 1 \end{vmatrix} \\ = (1+y)^2 (1-2y)$$

When $\sin 2x = 1$, then $y = 1/2$ and $f(x) = 0$

When $\sin x = \cos x$, then $\tan x = 1$

$$\Rightarrow x = \frac{\pi}{4}$$

$$\Rightarrow y = \frac{1}{2}$$

$$\Rightarrow f(x) = 0$$

90. It is given that

$$bc + qr + 1 = 0 \quad \dots(i)$$

$$ca + rp + 1 = 0 \quad \dots(ii)$$

$$ab + pq + 1 = 0 \quad \dots(iii)$$

The determinant in the question involves a column consisting the elements ap , bq and cr . So, on multiplying Eqs. (i), (ii) and (iii) by ap , bq and cr respectively, we get

$$apbc + apqr + ap = 0$$

$$bqca + bqrp + bq = 0$$

$$abcr + pqcr + cr = 0$$

Let $x = abc$ and $y = pqr$, then the above equations becomes

$$\left. \begin{aligned} px + ay + ap &= 0 \\ qx + by + bq &= 0 \\ rx + cy + cr &= 0 \end{aligned} \right\} \quad \dots(iv)$$

Since, system (iv) has a common solution.

$$\therefore \begin{vmatrix} p & a & ap \\ q & b & bq \\ r & c & cr \end{vmatrix} = 0$$

91. Using $\log(b^c) = c \log b$ and applying $C_3 \rightarrow C_3 - 5C_2$, we get

$$\Delta a, b, c = \begin{vmatrix} x \log 2 & 3 & x \log a \\ y \log 3 & 5 & y \log b \\ z \log 5 & 7 & z \log c \end{vmatrix}$$

$\Delta(8, 27, 125) = \Delta 2^3, 3^3, 5^3 = 0$ as in this case C_1 and C_3 are proportional. Similarly,

$$\Delta(1/2, 1/3, 1/5) = \Delta(2^{-1}, 3^{-1}, 5^{-1}) = 0$$

92. Statement I is true, since $\begin{vmatrix} 0 & p-q & p-r \\ q-p & 0 & q-r \\ r-p & r-q & 0 \end{vmatrix} = 0$ and

Statement II is true.

93. $|A_r| = r^2 - (r-1)^2 = 2r - 1$

$$\Rightarrow |A_1| + |A_2| + |A_3| + \dots + |A_{2006}| = \sum_{r=1}^{2006} (2r - 1) = (2006)^2$$

Now, for Statement II, using properties

$$|A| |\text{adj } A| = |A|^3$$

$$\Rightarrow |\text{adj } A| = |A|^2$$

$$\Rightarrow |\text{adj } A| = 2^2$$

94. Let $U_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ so that $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow x = 1, y = -2, z = 1$$

$$\therefore U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Similarly, } U_2 = \begin{bmatrix} 2 \\ -1 \\ -4 \end{bmatrix} U_3 = \begin{bmatrix} 2 \\ -1 \\ -3 \end{bmatrix}$$

$$\text{Thus, } |U| = 3$$

$$95. \text{adj}(U) = \begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$$

$$\therefore U^{-1} = \frac{\text{adj}(U)}{3} \text{ and sum of the elements of } U^{-1} = 0$$

$$96. \text{The value of } [3 \ 2 \ 0]U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \\ = [3 \ 2 \ 0] \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} \\ = [-1 \ 4 \ 4] \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = [5]$$

97. **Case I** All three diagonal elements are 1.

$$\text{Number of matrices} = {}^3C_1 = 3$$

Case II Two diagonal elements are zero and one element is one.

$$\text{Number of matrices} = {}^3C_1 \cdot {}^3C_1 = 9$$

$$\therefore \text{Total matrices} = 3 + 9 = 12$$

$$98. \text{Given, } A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For unique solution, $\det(A) \neq 0$

$$\text{Case I } \det(A) = \begin{vmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{vmatrix} = 2abc \neq 0$$

Here, a, b, c is selected from 1, 0, 0. (No case is possible)

Case II

$$(i) \det(A) = \begin{vmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{vmatrix} = 2abc - c^2 \neq 0$$

Here a, b, c are selected from 1, 1, 0 (2 cases are possible)

$$(ii) \det(A) = \begin{vmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{vmatrix} = 2abc - b^2 \neq 0$$

Here a, b, c are selected from 1, 1, 0 (2 cases are possible)

$$(iii) \det(A) = \begin{vmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 1 \end{vmatrix} = 2abc - a^2 \neq 0$$

Here a, b, c are selected from 1, 1, 0 (2 cases are possible)

Hence, there are exactly 6 matrices for unique solution.

99. Given, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Case I $\begin{bmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

a, b and c are selected from 1, 0, 0.
 $\Rightarrow$
 $x + ay + bz = 1$
 $ax + y + cz = 0$
 $bx + cy + z = 0$

(i) If $a = 1, b = c = 0$, then $x + y = 1, x + y = 0$ and $z = 0$
Inconsistent system of equation.

(ii) If $a = 0 = c, b = 1$, then $x + z = 1, y = 0$ and $x + z = 0$
Inconsistent system of equation

(iii) If $c = 1, a = b = 0$, then $x = 1, z = 0, y = 0$

Case II

(i) $\begin{bmatrix} 1 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

a, b, c are selected from 1, 1, 0.
 $\Rightarrow$
 $x + ay + bz = 1$
 $ax + cz = 0$
 $bx + cy = 0$

Clearly, in all three cases, solutions are possible so system is consistent.

(ii) $\begin{bmatrix} 0 & a & b \\ a & 1 & c \\ b & c & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$
 $ay + bz = 1$
 $ax + y + cz = 0$
 $bx + cy = 0$

Clearly, $b = 0, a = c = 1$ [given]
 $y = 1, x + y + z = 0$ and $y = 0$

Inconsistent system.

More than 2 matrices are possible.

100. $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2 - a^2 & c^2 - a^2 \\ ab & -(b-a)c & -b(c-a) \end{vmatrix}$

$(a-b)(c-a) \begin{vmatrix} 1 & 0 & 0 \\ a^2 & -(b+a) & c+a \\ ab & c & -b \end{vmatrix}$
 $= (a-b)(c-a)(b-c)(a+b+c)$

101. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix} = \frac{1}{abc} \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$

$= \begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

102. $\begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = abc \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$
 $= abc(a-b)(b-c)(c-a)$

103. Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

then $A - \lambda I = \begin{bmatrix} a_1 - \lambda & b_1 & c_1 \\ a_2 & b_2 - \lambda & c_2 \\ a_3 & b_3 & c_3 - \lambda \end{bmatrix}$

$\Rightarrow \det(A - \lambda I) = (a_1 - \lambda)[(b_2 - \lambda)(c_3 - \lambda) - b_3 c_2]$
 $- b_1[a_2(c_3 - \lambda) - c_2 a_3] + c_1[a_2 b_3 - a_3(b_2 - \lambda)]$

Now, if one of the eigenvalues is zero, then one value of λ should be zero.

Constant term in the above polynomial is zero.

$\Rightarrow a_1 b_2 c_3 - a_1 b_3 c_2 - b_1 a_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3 - c_1 a_3 b_2 = 0$
 [collecting constant terms]

$\Rightarrow \det A = 0$

$\Rightarrow A$ should be singular.

104. $\det(A - I) = \det(A - I) \cdot \det A' \quad [\because \det A = \det A' = 1]$
 $= \det(AA' - A') \quad [\det AB = \det A \det B]$
 $= \det(I - A') = -\det(A' - I)$
 $= -\det(A - I)' = -\det(A - I)$
 $\Rightarrow \det(A - I) = 0$

105. The characteristic equation of given matrix A is

$\det \begin{bmatrix} 1 - \lambda & 0 & 0 \\ 0 & 1 - \lambda & 1 \\ 0 & -2 & 4 - \lambda \end{bmatrix} = 0$

$\Rightarrow (1 - \lambda)[(1 - \lambda)(4 - \lambda) + 2] = 0$

$\Rightarrow (1 - \lambda)[\lambda^2 - 5\lambda + 6] = 0$
 $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$

Since, every square matrix satisfies its characteristic equation, therefore we must have

$A^3 - 6A^2 + 11A - 6I = O$

$\Rightarrow A^2 - 6A + 11I - 6A^{-1} = O$

$\Rightarrow A^{-1} = \frac{1}{6}[A^2 - 6A + 11I]$
 $c = -6, d = 11$

106. If $AX = U$ has infinite solutions.

Then, $\begin{vmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{vmatrix} = 0$

$\Rightarrow a(bc - bd) + 1(d - c) = 0$

$\Rightarrow (ab - 1)(c - d) = 0$

$\Rightarrow ab = 1$ or $c = d$

Again, as $AX = U$ has infinite solutions, then

$D_3 = D_2 = D_1 = 0$
 $\Rightarrow \begin{vmatrix} a & 0 & f \\ 1 & c & g \\ 1 & d & h \end{vmatrix} = \begin{vmatrix} a & f & 1 \\ 1 & g & b \\ 1 & h & b \end{vmatrix} = \begin{vmatrix} f & 0 & 1 \\ g & c & b \\ h & d & b \end{vmatrix} = 0$

This yields only two independent relations namely $c = d$ and $g = h$

107. Consider the system $BX = V$

We have, $|B| = \begin{vmatrix} a & 1 & 1 \\ 1 & d & c \\ f & g & h \end{vmatrix} = 0$

$\Rightarrow BX = V$ cannot have unique solution.

108. Now, the other three determinants D_1', D_2', D_3' of the system $BX = V$ are given by

$$D_1' = \begin{vmatrix} a^2 & 1 & 1 \\ 1 & d & c \\ 0 & g & h \end{vmatrix} = 0 \quad [\because c = d, g = h]$$

$$D_2' = \begin{vmatrix} a & a^2 & 1 \\ 1 & 0 & c \\ f & 0 & h \end{vmatrix} = a^2cf - a^2df \neq 0$$

Hence, $BX = V$ has no solution.

109. The first row can either be filled in ${}^3C_3$ ways (–1 at all the places) or ${}^3C_1$ ways (–1 at one place). First row can be filled in ${}^3C_3 + {}^3C_1$ ways corresponding to each such case the second row can also be filled in ${}^3C_3 + {}^3C_1$ ways. But third row elements have got to be fixed automatically.

$\therefore$ Number of minus special determinants
 $= ({}^3C_3 + {}^3C_1)({}^3C_3 + {}^3C_1) = 16$

110. From the solution of above question, the number of $n \times n$ special determinants
 $= ({}^nC_1 + {}^nC_3 + {}^nC_5 + \dots)^{n-1}$
 [choose first $(n-1)$ rows, the last row has automatically fixed]
 $= (2^{n-1})^{n-1} = 2^{(n-1)^2}$

111. Since on expanding, determinant = sum of six terms each of which is 1 or –1.

$\Rightarrow$ Minimum cannot be less than –6.

But –6 is possible only in one case

$$\begin{aligned} a_1b_2c_3 &= -1 & a_2b_3c_1 &= -1 \\ a_3b_1c_2 &= -1 & a_1b_3c_2 &= +1 \\ a_3b_2c_1 &= +1 & a_2b_1c_3 &= +1 \end{aligned}$$

These equations cannot exist simultaneously because if we multiply all of them, we get LHS positive and RHS negative. (The value cannot be odd, since the determinant is equal to some of the three terms each of which is 0, 2, –2.)

Now, we observe that there is a determinant whose value is –4.

Take $D = \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$

$\therefore$ Minimum value is –4.

112. Given, $A(\text{adj } A) = (\text{adj } A)A$

On putting $A = P^{-1}$ and pre-multiplying by P on both sides, we get

$$PP^{-1}(\text{adj } P^{-1}) = P(\text{adj } P^{-1})P^{-1}$$

$$\text{adj } P^{-1} = P | P^{-1} |$$

$$\text{adj } P^{-1} = P \quad [\because | P^{-1} |] = | P | \dots (i)$$

Now, $P(\text{adj } P) = (\text{adj } P)P = | P | I = I$

$\therefore (\text{adj } P)^{-1} = P \quad \dots (ii)$

From Eqs. (i) and (ii), $\text{adj } P^{-1} = (\text{adj } P)^{-1}$

113. $\text{adj } Q^{-1}BP^{-1} = \text{adj } (Q^{-1}B)P^{-1} = \text{adj } (P^{-1}) \cdot \text{adj } (Q^{-1}B)$
 $= \text{adj } (P^{-1}) \cdot \text{adj } (B) \cdot \text{adj } (Q^{-1})$
 $= (\text{adj } P)^{-1} \cdot A(\text{adj } Q)^{-1} = PAQ$

114. $|\text{adj } AB| = |\text{adj } B \cdot \text{adj } A| = |\text{adj } B| |\text{adj } A|$
 $= |B|^{n-1} |A|^{n-1} \quad [\because |\text{adj } A| = |A|^{n-1}]$

115. By its very definition, the permanent of an $m \times m$ matrices of A . For the given matrix C ,
 Columns 1, 2 and 3 :

$$(1)(2)(3) + (1)(3)(1) + (1)(1)(1) + (1)(3)(1) + (1)(1)(1) + (1)(2)(1) = 2 + 3 + 1 + 3 + 2 = 12$$

Columns 1, 2 and 4 :

$$(1)(2)(1) + (1)(4)(1) + (1)(1)(1) + (1)(4)(1) + (1)(1)(1) + (1)(2)(1) = 2 + 4 + 1 + 4 + 1 + 2 = 14$$

Columns 1, 3 and 4 :

$$(1)(3)(1) + (1)(4)(1) + (1)(1)(1) + (1)(4)(1) + (1)(1)(1) + (1)(3)(1) = 3 + 4 + 1 + 4 + 1 + 3 = 16$$

Columns 2, 3 and 4 :

$$(1)(3)(1) + (1)(4)(1) + (1)(2)(1) + (1)(4)(1) + (1)(2)(1) + (1)(3)(1) = 3 + 4 + 2 + 4 + 2 + 3 = 18$$

Thus, permanent $(C) = 12 + 14 + 16 + 18 = 60$

116. The required permanent is the sum of $P(n, n) = n!$ terms,

where each of the terms is the product $\frac{1}{n} \cdot \frac{1}{n} \cdot \frac{1}{n} \dots n$

$$\text{times} = \left(\frac{1}{n}\right)^n$$

$$\text{Thus, permanent} \left(\frac{1}{n} I_n\right) = \frac{n!}{n^n}$$

117. Note that D_n is number of dearrangement of n distinct objects.

The $n \times n$ matrix $J_n - I_n$ has zeroes on the main diagonal and unity elsewhere. Hence, each dearrangement (permutation of 1, 2, 3, ..., n in such a manner that none of the elements occupies original position) of 1, 2, 3, ..., n contributes 1 to per $(J_n - I_n)$ while every other permutation of the set contributes 0.

Thus, permanent $(J_n - I_n) = D_n$

118. $D = (\lambda - 1)^2 (\lambda + 2)$

Now, $D = 0$, if $\lambda = 1$ or $\lambda = -2$

If $\lambda \neq 1, \lambda \neq -2$, then the system has a unique solution.

If $\lambda = 1$, the system becomes $x + y + z = 1$ which has infinite solutions. If $\lambda = -2$, the system is inconsistent.

119. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$

$$\text{LHS} = A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ -1 & \frac{1}{2} & \frac{1}{2} \\ 2 & 0 & -1 \end{bmatrix}$$

$$\text{RHS} = pA^2 + qA + rI_3$$

$$= \begin{bmatrix} 5p + q + r & 0 & 8p + 2q \\ 2p & 4p + 2q + r & 5p + q \\ 8p + 2q & 0 & 13p + 3q + r \end{bmatrix}$$

On comparing, we get

$$p = \frac{-1}{2}, q = 3, r = \frac{-7}{2}$$

120. $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$

$$\text{adj}(A) = \begin{bmatrix} a^2 & ab & ac \\ +ab & b^2 & bc \\ ac & bc & c^2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

On comparing,

$$ab = a_{12} \text{ or } a_{21}$$

$$bc = a_{23} \text{ or } a_{32}$$

$$ca = a_{13} \text{ or } a_{31}$$

121. A. $\begin{vmatrix} \lambda & 7 & -2 \\ 4 & 1 & 3 \\ 2 & -1 & 2 \end{vmatrix} = 0 \Rightarrow \lambda = \frac{2}{5}$

B. Now, $|A| = 1(0 - 0) + 1(0 - 0) + 1(3 - 2) = 1$

$$\therefore |\text{adj } A| = |A|^{3-1} = (1)^2 = 1$$

C. $\begin{vmatrix} -5 & 3 & 2 \\ 3 & 2 & -5 \\ 4 & -1 & -3 \end{vmatrix} = 0$ and $\begin{vmatrix} -5 & 3 \\ 3 & 2 \end{vmatrix} \neq 0$. So, the only

possible third order minor is 0, while one second order minor is non-zero. So, the rank is 2.

D. $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$

$$\therefore \text{Rank} = 3$$

122. A. We know that, if A is a square matrix of order n and B is the matrix of cofactors of elements of A.

$$\text{Then, } |B| = |A|^{n-1}$$

Here, Δ is the determinant of cofactors of

$$\text{elements of } |A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\therefore \Delta = |A|^{3-1} = |A|^2 = 5^2 = 25$$

B. We have, coefficient of x in $\begin{vmatrix} x & (1 + \sin x)^3 & \cos x \\ 1 & \log(1+x) & 2 \\ x^2 & (1+x)^2 & 0 \end{vmatrix}$

= Coefficient of x in

$$\begin{vmatrix} x & \left(1 + x - \frac{x^3}{3!} + \dots\right)^3 & 1 - \frac{x^2}{2!} + \dots \\ 1 & x - \frac{x^2}{2} + \frac{x^3}{3} - \dots & 2 \\ x^2 & 1 + 2x + x^2 & 0 \end{vmatrix}$$

$$= \text{Coefficient of x in } \begin{vmatrix} x & 1 & 1 \\ 1 & x & 2 \\ x^2 & 1 & 0 \end{vmatrix} = \text{Coefficient of x in}$$

$$[x(0 - 2) - 1(0 - 2x^2) + 1(1 - x^3)] = -2$$

C. We have, $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$= \begin{vmatrix} \Sigma a_i^2 & \Sigma a_i b_j & \Sigma a_i c_j \\ \Sigma a_i b_j & \Sigma b_i^2 & \Sigma b_i c_j \\ \Sigma a_i c_j & \Sigma b_i c_j & \Sigma c_i^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Hence, $\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \pm 1$

D. Let $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3.

$$\text{Then, } \Delta = a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

$$\Rightarrow \Delta = (a_1 b_2 c_3 + a_3 b_1 c_2 + a_2 b_3 c_1) - (a_1 b_3 c_2 + a_2 b_1 c_3 + a_3 b_2 c_1)$$

Since, each element of Δ is either 1 or 0. Therefore, the value of the determinant cannot exceed 3.

Clearly, the value of Δ is maximum when the value of each term in first bracket is 1 and the value of each term in the second bracket is zero. But

$$a_1 b_2 c_3 = a_3 b_1 c_2 = a_2 b_3 c_1 = 1$$

It implies that every element of the determinant Δ is 1 and in that case $\Delta = 0$. Thus, we may have

$$\Delta = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 2$$

E. $\Delta'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix}$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ f''(x) & g''(x) & h''(x) \\ f'''(x) & g'''(x) & h'''(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ f'(x) & g'(x) & h'(x) \\ 0 & 0 & 0 \end{vmatrix}$$

$$\Rightarrow \Delta'(x) = 0 \Rightarrow \Delta(x) \text{ is constant.}$$

123. A. On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = x \begin{vmatrix} 1 & 1 & 1 \\ \omega & x + \omega^2 & 1 \\ \omega^2 & 1 & x + \omega \end{vmatrix} \Rightarrow x = 0$$

B. $\Delta = \begin{vmatrix} a & a & a \\ 3a & 4a & 5a \\ 6a & 9a & 11a \end{vmatrix} = a^3 \begin{vmatrix} 1 & 1 & 1 \\ 3 & 4 & 5 \\ 6 & 9 & 11 \end{vmatrix} = -a^3 = -i^3 = i$

C. On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{vmatrix} = 3(\omega - \omega^2) = 3\sqrt{3}i$$

D. $\Delta = [(\Sigma a^2)^3 + (\Sigma ab)^3 + (\Sigma ab)^3 - 3(\Sigma a^2)(\Sigma ab)^2]$

which is always non-negative.

$$\text{Now, for } \Delta = 0, \Sigma a^2 + \Sigma ab + \Sigma ab = 0$$

$$\Rightarrow \Sigma a^2 + 2\Sigma ab = 0$$

124. We know that, in a square matrix of order n

$$|\text{adj } A| = |A|^{n-1}$$

$$\Rightarrow |\text{adj}(\text{adj } A)| = |\text{adj } A|^{n-1} = |A|^{(n-1)^2}$$

$$\Rightarrow n^2 - 2n - 8 = 0$$

$$\Rightarrow n = 4 \text{ as } n = -2 \text{ is not possible.}$$

Hence, value of n = 4

125. $D = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & -2 \\ 2 & 5 & -2 \end{vmatrix} = 45, D_{1/x} = \begin{vmatrix} 1 & 2 & 1 \\ 2 & -4 & -2 \\ 3 & 5 & -2 \end{vmatrix} = 36$

$$D_{1/y} = \begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & -2 \\ 2 & 3 & -2 \end{vmatrix} = 9, D_{1/z} = \begin{vmatrix} 1 & 2 & 1 \\ 3 & -4 & 2 \\ 2 & 5 & 3 \end{vmatrix} = -9$$

$$\therefore x = \frac{5}{4}, y = 5, z = -5$$

Hence, value of y is 5.

$$126. \sum_{r=1}^n \Delta_r = \begin{vmatrix} x & y & z \\ 2(2^n-1) & 3(3^n-1) & 4(4^n-1) \\ 2(2^n-1) & 3(3^n-1) & 4(4^n-1) \end{vmatrix} \Rightarrow \sum_{r=1}^n \Delta_r = 0$$

$$127. \text{Consider } \Delta_1 = \begin{vmatrix} a_1 a_5 & a_1 & a_2 \\ a_2 a_6 & a_2 & a_3 \\ a_3 a_7 & a_3 & a_4 \end{vmatrix}$$

On applying $C_3 \rightarrow C_3 - C_2$, we get

$$\Delta_1 = \begin{vmatrix} a_1 a_5 & a_1 & d \\ a_2 a_6 & a_2 & d \\ a_3 a_7 & a_3 & d \end{vmatrix}, \text{ where } d \text{ is the common}$$

difference of an AP.

$$\begin{vmatrix} a_1(a_1+4d) & a_1 & d \\ a_2(a_1+5d) & a_2 & d \\ a_3(a_1+6d) & a_3 & d \end{vmatrix} = \begin{vmatrix} a_1(a_1+4d) & a_1 & d \\ a_2(a_2+4d) & a_2 & d \\ a_3(a_3+4d) & a_3 & d \end{vmatrix}$$

$$= \begin{vmatrix} a_1^2 + 4a_1 d & a_1 & d \\ a_2^2 + 4a_2 d & a_2 & d \\ a_3^2 + 4a_3 d & a_3 & d \end{vmatrix} = \begin{vmatrix} a_1^2 & a_1 & d \\ a_2^2 & a_2 & d \\ a_3^2 & a_3 & d \end{vmatrix} + 4a_1 \begin{vmatrix} d & a_1 & d \\ d & a_2 & d \\ d & a_3 & d \end{vmatrix}$$

$$= d \begin{vmatrix} a_1^2 & a_1 & 1 \\ a_2^2 & a_2 & 1 \\ a_3^2 & a_3 & 1 \end{vmatrix} + 0$$

$$= -d(a_1 - a_2)(a_2 - a_3)(a_3 - a_1) = -2d^4$$

$$\text{Similarly, } \Delta_2 = -2d^4$$

$$\text{Thus, } \Delta_1 : \Delta_2 = 1$$

$$128. \text{Let } \alpha = s - a, \beta = s - b, \gamma = s - c \Rightarrow \alpha + \beta + \gamma = s$$

$$\therefore \text{Determinant} = \begin{vmatrix} (\beta + \gamma)^2 & \alpha^2 & \alpha^2 \\ \beta^2 & (\gamma + \alpha)^2 & \beta^2 \\ \gamma^2 & \gamma^2 & (\alpha + \beta)^2 \end{vmatrix}$$

$$= 2\alpha\beta\gamma(\alpha + \beta + \gamma)^3$$

$$= 2(s - a)(s - b)(s - c)s^3 \Rightarrow k = 2$$

$$129. \text{On applying } R_1 \rightarrow R_1 - R_2 - R_3, \text{ we get}$$

$$D = \begin{vmatrix} 0 & -2y^2 & -2x^2 \\ x^2 & y^2 + z^2 & x^2 \\ y^2 & y^2 & z^2 + x^2 \end{vmatrix} = -2 \begin{vmatrix} 0 & y^2 & x^2 \\ x^2 & z^2 & 0 \\ y^2 & 0 & z^2 \end{vmatrix}$$

$$= 4x^2 y^2 z^2 = 4(1)^2 (\omega)^2 (\omega^2)^2 = 4$$

$$\therefore \text{Real part of } D = 4$$

$$130. \text{For non-trivial solution, } \Delta = \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \sin 3\theta + 2\cos 2\theta - 2 = 0$$

$$\Rightarrow 4\sin^3 \theta + 4\sin^2 \theta - 3\sin \theta = 0$$

$$[\text{using } \sin 3\theta = 3\sin \theta - 4\sin^3 \theta \text{ and } \cos 2\theta = 1 - 2\sin^2 \theta]$$

$$\Rightarrow \sin \theta = 0 \text{ or } \sin \theta = \frac{1}{2} \left[\because \sin \theta \neq \frac{-3}{2} \right]$$

$$\Rightarrow \sin \theta = \sin \frac{\pi}{6} \Rightarrow \theta = \frac{5\pi}{6} \Rightarrow \lambda = 6$$

$$131. \text{Take } I_n \text{ as } n \times n \text{ identity matrix and } B \text{ as another } n \times n \text{ matrix, whose all elements are equal to 1, then } A = B - I \text{ and } B^2 = nB$$

$$\text{Note that } (B - I)(cB - I) = cB^2 - (c + 1)B + I$$

$$= cnB - (c + 1)B + I$$

$$\text{If we choose } c = \frac{1}{n-1}, \text{ then } (B - I) \left(\frac{1}{n-1} B - I \right) = I$$

$$\therefore (B - I)^{-1} = \left(\frac{1}{n-1} B - I \right)$$

$$\Rightarrow A^{-1} = (B - I)^{-1} = \frac{1}{n-1} B - I$$

The diagonal elements in the matrix

$$\frac{1}{n-1} B - I \text{ will be } \frac{1}{n-1} - 1 = \frac{2-n}{n-1} \Rightarrow k = 2$$

$$132. \text{Since, the points } P(3u^2, 2u^3), Q(3v^2, 2v^3) \text{ and } R(3w^2, 2w^3) \text{ are collinear.}$$

$$\therefore \begin{vmatrix} 3u^2 & 2u^3 & 1 \\ 3v^2 & 2v^3 & 1 \\ 3w^2 & 2w^3 & 1 \end{vmatrix} = 0 \Rightarrow 6 \begin{vmatrix} u^2 & u^3 & 1 \\ v^2 & v^3 & 1 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

On applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$, we get

$$\begin{vmatrix} u^2 - v^2 & u^3 - v^3 & 0 \\ v^2 - w^2 & v^3 - w^3 & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (u - v)(v - w) \begin{vmatrix} u + v & u^2 + v^2 + uv & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} u + v & u^2 + v^2 + uv & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

On applying $R_1 \rightarrow R_1 - R_2$, we get

$$\begin{vmatrix} u - w & (u^2 - w^2) + v(u - w) & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (u - w) \begin{vmatrix} 1 & u + v + w & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & u + v + w & 0 \\ v + w & v^2 + w^2 + vw & 0 \\ w^2 & w^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow v^2 + w^2 + vw - (v + w)(u + v + w) = 0$$

$$\Rightarrow v^2 + w^2 + vw - (v + w)[(v + w) + u] = 0$$

$$\Rightarrow v^2 + w^2 + vw - (v + w)^2 - u(v + w) = 0$$

$$\Rightarrow v + w^2 + vw - v^2 - w^2 - 2vw - uv - uw = 0$$

$$\Rightarrow uv + vw + uw = 0$$

$$133. \text{For no solution, } |A| = 2a^3 - 54 = 0 \Rightarrow a = 3$$

$$\text{Also, when } a = 3, (\text{adj } A) \cdot B \neq 0$$

Therefore, the system of equations has no solution, when value of $a = 3$.

$$134. \text{On putting } \lambda = 0 \text{ both sides, we get constant term will be zero.}$$

$$\Rightarrow t = 0$$

Entrances Gallery

1. $MN = NM$ and $M^2 - N^4 = O$
 $\Rightarrow (M - N^2)(M + N^2) = O \quad \dots(i)$
 $[\because MN^2 = N^2M \text{ as } MN = NM]$

$\Rightarrow |M - N^2| |M + N^2| = 0 \quad [\because \det(AB) = \det(A) \cdot \det(B)]$

Now, if $|M + N^2| \neq 0$, then the matrix $M + N^2$ is invertible and therefore, from Eq. (i), we get

$M - N^2 = O \Rightarrow M = N^2$, which is a contradiction.

Thus, $|M + N^2| = 0$

(a) $|M^2 + MN^2| = |M| |M + N^2| = 0$

$\therefore$ Option (a) is correct.

We know that if $|A| = 0$, then $AU = O$ will have solution.

Thus, $(M^2 + MN^2)U = O$ will have many U .

Hence, option (b) is correct.

(c) Obvious wrong.

(d) If $AX = O$ and $|A| = 0$, then X can be non-zero.

$\therefore$ Option (d) is wrong.

2. (a) $(N^T MN)^T = N^T M^T (N^T)^T = N^T M^T N$ is symmetric, if M is symmetric and skew-symmetric.

(b) $(MN - NM)^T = (MN)^T - (NM)^T$
 $= NM - MN = -(MN - NM)$

$\therefore$ Skew-symmetric, when M and N are symmetric.

(c) $(MN)^T = N^T M^T = NM \neq MN$

$\therefore$ Option (c) is not correct.

(d) $(\text{adj } MN) = (\text{adj } N) \cdot (\text{adj } M)$

$\therefore$ Option (d) is not correct.

3. We have, $(\text{adj } P) = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$
 $\Rightarrow |\text{adj } P| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix}$
 $= 1(3 - 7) - 4(6 - 7) + 4(2 - 1)$
 $= -4 + 4 + 4 = 4$
 $\Rightarrow |P|^2 = 4 \Rightarrow |P| = \pm 2$

4. Here, $P = [a_{ij}]_{3 \times 3} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

and $Q = [b_{ij}]_{3 \times 3} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$

where, $b_{ij} = 2^{i+j} a_{ij}$

$\therefore |Q| = \begin{vmatrix} 4a_{11} & 8a_{12} & 16a_{13} \\ 8a_{21} & 16a_{22} & 32a_{23} \\ 16a_{31} & 32a_{32} & 64a_{33} \end{vmatrix}$
 $= 2^9 \times 2 \times 4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$
 $= 2^{12} \cdot |P| = 2^{12} \cdot 2 = 2^{13}$

5. Let $A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$

Clearly, $|A| \neq 0$ as non-singular.

$\therefore \begin{vmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{vmatrix} \neq 0$

$\Rightarrow 1(1 - c\omega) - a(\omega - c\omega^2) + b(\omega^2 - \omega^2) \neq 0$

$\Rightarrow 1 - c\omega - a\omega + ac\omega^2 \neq 0$

$\Rightarrow (1 - c\omega)(1 - a\omega) \neq 0$

$\Rightarrow a \neq \frac{1}{\omega}, c \neq \frac{1}{\omega} \Rightarrow a = \omega, c = \omega \text{ and } b \in \{\omega, \omega^2\}$

Hence, the number of distinct matrices in the set is 2.

6. Given, $A = \begin{bmatrix} a & b \\ c & a \end{bmatrix}$, $a, b, c \in \{0, 1, 2, \dots, p-1\}$

If A is skew-symmetric matrix, then $a = 0, b = -c \dots(i)$

$\therefore |A| = -b^2$

Thus, p divides $|A|$, only when $b = 0$.

Again, if A is symmetric matrix, then $b = c$ and $|A| = a^2 - b^2$. Thus, p divides $|A|$, if either p divides $(a - b)$ or p divides $(a + b)$.

p divides $(a - b)$, only when $a = b$

i.e. $a = b \in \{0, 1, 2, \dots, (p-1)\} \dots(ii)$

i.e. p choices, p divides $(a + b)$.

$\Rightarrow p$ choices, including $a = b = 0$

$\therefore$ Total number of choices are $(p + p - 1) = 2p - 1$

7. Trace of $A = 2a$, which will be divisible by p , iff $a = 0$.

$|A| = a^2 - bc$, for $a^2 - bc$ to be divisible by p , a can be chosen in $(p-1)$ ways $[\because a \neq 0]$.

Now, bc should be chosen such that $a^2 - bc$ is divisible by p . Now, b can be chosen in $(p-1)$ ways and c in only 1. Thus, a can be chosen in $p-1$ ways, b can be chosen in $(p-1)$ ways and c can be chosen in 1 way.

$\therefore$ Total number of ways $= (p-1)^2$.

Example If $p = 5$ and $a = 4$, then $a^2 = 16$. Now, bc can be chosen in such a way that $a^2 - bc$ is divisible by p .

If $b = 1 \Rightarrow c = 1; b = 2 \Rightarrow c = 3; b = 3 \Rightarrow c = 2$ and $b = 4 \Rightarrow c = 4$.

8. Required number of matrices

$=$ Total cases $- (a \neq 0 \text{ and } |A| \text{ is divisible by } p)$
 $- (a = 0 \text{ and } |A| \text{ is divisible by } p)$

$= p^3 - (p-1)^2 - (2p-1)$

$= p^3 - p^2 - 1 + 2p - 2p + 1$

$= p^3 - p^2$

9. Since, $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ is linear equation in three variables

and that could have only unique, no solution or infinitely many solutions.

$\therefore$ It is not possible to have exactly two solutions.

Hence, number of matrices A is zero.

16. Given,

(i) two matrices P and Q of order 3×3 such that

$$P \neq Q$$

(ii) $P^3 = Q^3$ and $P^2Q = Q^2P$

To find the value of determinant of $P^2 + Q^2$.

On subtracting the given equation, we get

$$P^3 - P^2Q = Q^3 - Q^2P$$

$$\Rightarrow P^2(P - Q) = Q^2(Q - P)$$

$$\Rightarrow (P - Q)(P^2 + Q^2) = 0 \quad \dots(i)$$

$$\Rightarrow |P - Q| \cdot |P^2 + Q^2| = 0$$

Now, if $|P^2 + Q^2| \neq 0$, then $P^2 + Q^2$ is invertible and therefore, from Eq. (i), we get $P = Q$, which is a contradiction. Hence, $|P^2 + Q^2| = 0$.

17. Since, equation has non-zero solution

$$\Rightarrow \Delta = 0$$

$$\therefore \begin{vmatrix} 4 & k & 2 \\ k & 4 & 1 \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 4(4 - 2) - k(k - 2) + 2(2k - 8) = 0$$

$$\Rightarrow 8 - k^2 + 2k + 4k - 16 = 0$$

$$\Rightarrow k^2 - 6k + 8 = 0$$

$$\Rightarrow (k - 2)(k - 4) = 0$$

$$\Rightarrow k = 2, 4$$

$\therefore$ Number of values of k is two.

18. $x - ky + z = 0$, $kx + 3y - kz = 0$

$3x + y - z = 0$ has trivial solution.

$$\Rightarrow \begin{vmatrix} 1 & -k & 1 \\ k & 3 & -k \\ 3 & 1 & -1 \end{vmatrix} \neq 0$$

$$\Rightarrow 1(-3 + k) + k(-k + 3k) + 1(k - 9) \neq 0$$

$$\Rightarrow k - 3 + 2k^2 + k - 9 \neq 0$$

$$\Rightarrow 2k^2 + 2k - 12 \neq 0$$

$$\Rightarrow k^2 + k - 6 \neq 0$$

$$\Rightarrow (k + 3)(k - 2) \neq 0$$

$$\therefore k \neq -3 \text{ or } k \in \mathbb{R} - \{-3, 2\}$$

19. Determinant of skew-symmetric matrix of odd order is zero and of even order is perfect square.

$$\det(A^T) = \det(A) \text{ and}$$

$$\det(-A) = (-1)^n \det(A) \Rightarrow \text{Statement II is false.}$$

20. The given system of linear equations can be put in the matrix form as

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -8 \end{bmatrix}$$

[applying $R_2 \rightarrow R_2 - 2R_1$
and $R_3 \rightarrow R_3 - 3R_1$]

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 8 \end{bmatrix} \quad \left[\begin{array}{l} \text{applying } R_2 \rightarrow (-1)R_2 \\ \text{and } R_3 \rightarrow (-1)R_3 \end{array} \right]$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 5 \end{bmatrix} \quad [\text{applying } R_3 \rightarrow R_3 - R_2]$$

Clearly, the given system of equations has no solution, as rank (coefficient matrix) $= 2 \neq$ rank (augmented matrix) $= 3$

Aliter

Subtracting the addition of first two equations from third equation, we get

$0 = -5$, which is an absurd result.

Hence, the given system of equations has no solution.

$$\begin{aligned} 21. & \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & b+1 & c-1 \\ a-1 & b-1 & c+1 \\ a & -b & c \end{vmatrix} \\ &= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^n \begin{vmatrix} a+1 & a-1 & a \\ b+1 & b-1 & -b \\ c-1 & c+1 & c \end{vmatrix} \\ &= \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} + (-1)^{n+1} \begin{vmatrix} a+1 & a & a-1 \\ b+1 & -b & b-1 \\ c-1 & c & c+1 \end{vmatrix} [C_2 \leftrightarrow C_3] \\ &= (1 + (-1)^{n+2}) \begin{vmatrix} a & a+1 & a-1 \\ -b & b+1 & b-1 \\ c & c-1 & c+1 \end{vmatrix} \end{aligned}$$

This is equal to zero only, if $n + 2$ is odd, i.e. n is odd integer.

$$22. |\text{adj } A| = |A|^{n-1} = |A|^{2-1} = |A|$$

$$\text{adj}(\text{adj } A) = |A|^{n-2} A = |A|^0 A = A$$

23. Given equations are $x - cy - bz = 0$,

$$cx - y + az = 0 \text{ and } bx + ay - z = 0.$$

$$\text{For non-zero solution, } \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$$

$$\Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$$

24. As $\det(A) = \pm 1$, A^{-1} exists.

$$\text{and } A^{-1} = \frac{1}{\det(A)} (\text{adj } A) = \pm (\text{adj } A)$$

All entries in $\text{adj}(A)$ are integers.

$\therefore A^{-1}$ has integer entries.

$$25. \text{ Given, } D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$D = \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} = xy$$

Hence, D is divisible by both x and y .

$$\begin{aligned} 26. & A = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \\ \therefore A^2 &= \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 10\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 5\alpha^2 + 25\alpha \\ 0 & 0 & 25 \end{bmatrix} \end{aligned}$$

$$|A^2| = \begin{vmatrix} 25 & 25\alpha + 5\alpha^2 & 10\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 5\alpha^2 + 25\alpha \\ 0 & 0 & 25 \end{vmatrix}$$

$$= 25 \begin{vmatrix} \alpha^2 & 25\alpha + 5\alpha^2 \\ 0 & 25 \end{vmatrix} = 625\alpha^2$$

$$\therefore 625\alpha^2 = 25 \quad [\text{given}]$$

$$\Rightarrow |\alpha| = \frac{1}{5}$$

$$27. \text{ Given, } f(x) = \begin{vmatrix} 1+a^2x & (1+b^2)x & (1+c^2)x \\ (1+a^2)x & 1+b^2x & (1+c^2)x \\ (1+a^2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$f(x) = \begin{vmatrix} 1+a^2x+x+b^2x+x+c^2x & (1+b^2)x & (1+c^2)x \\ x+a^2x+1+b^2x+x+c^2x & 1+b^2x & (1+c^2)x \\ x+a^2x+x+b^2x+1+c^2x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1+(a^2+b^2+c^2+2)x & (1+b^2)x & (1+c^2)x \\ 1+(a^2+b^2+c^2+2)x & 1+b^2x & (1+c^2)x \\ 1+(a^2+b^2+c^2+2)x & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 1 & (1+b^2)x & (1+c^2)x \\ 1 & 1+b^2x & (1+c^2)x \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$[\because a^2 + b^2 + c^2 = -2, \text{ given}]$$

On applying $R_1 \rightarrow R_1 - R_3, R_2 \rightarrow R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} 0 & 0 & x-1 \\ 0 & 1-x & x-1 \\ 1 & (1+b^2)x & 1+c^2x \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x-1 \\ 1-x & x-1 \end{vmatrix} = (x-1)^2$$

Hence, $f(x)$ is of degree 2.

28. The system of given equations has no solution, if

$$\begin{vmatrix} \alpha & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} \alpha+2 & 1 & 1 \\ \alpha+2 & \alpha & 1 \\ \alpha+2 & 1 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+2) \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & 1 \\ 1 & 1 & \alpha \end{vmatrix} = 0$$

On applying $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$, we get

$$(\alpha+2) \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & 0 \\ 0 & 0 & \alpha-1 \end{vmatrix} = 0$$

$$\Rightarrow (\alpha+2)(\alpha-1)^2 = 0$$

$$\Rightarrow \alpha = 1, -2$$

But $\alpha = 1$ makes given three equations same. So, the system of equations have infinite solutions. Hence, answer is $\alpha = -2$ for which the system of equations has no solution.

29. Since, $a_1, a_2, \dots, a_n, \dots$ are in GP, then $\log a_n, \log a_{n+1}, \log a_{n+2}, \dots, \log a_{n+8}, \dots$ are in AP.

$$\text{Given, } \Delta = \begin{vmatrix} \log a_n & \log a_{n+1} & \log a_{n+2} \\ \log a_{n+3} & \log a_{n+4} & \log a_{n+5} \\ \log a_{n+6} & \log a_{n+7} & \log a_{n+8} \end{vmatrix}$$

$$\therefore \Delta = \begin{vmatrix} a & a+d & a+2d \\ a+3d & a+4d & a+5d \\ a+6d & a+7d & a+8d \end{vmatrix}$$

where, a and d are the first term and common difference of above AP.

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = \begin{vmatrix} a & d & 2d \\ a+3d & d & 2d \\ a+6d & d & 2d \end{vmatrix}$$

$$= 0 \quad [\because \text{two columns are proportional}]$$

30. Since, the system of linear equations has a non-zero

$$\text{solution, then } \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

On applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$, we get

$$\begin{vmatrix} 1 & 2a & a \\ 0 & 3b-2a & b-a \\ 0 & 4c-2a & c-a \end{vmatrix} = 0$$

$$\Rightarrow (3b-2a)(c-a) - (4c-2a)(b-a) = 0$$

$$\Rightarrow 3bc - 3ba - 2ac + 2a^2 - (4bc - 2ab - 4ac + 2a^2) = 0$$

$$\Rightarrow -4bc + 2ab + 4ac + 2a^2 = 0$$

On dividing by abc both sides, we get

$$\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

$\therefore a, b$ and c are in HP.

$$31. \text{ Given, } \Delta = \begin{vmatrix} 1 & \omega^n & \omega^{2n} \\ \omega^n & \omega^{2n} & 1 \\ \omega^{2n} & 1 & \omega^n \end{vmatrix}$$

$$= 1(\omega^{3n} - 1) - \omega^n(\omega^{2n} - \omega^{2n}) + \omega^{2n}(\omega^n - \omega^{4n})$$

$$= 1(1-1) - 0 + \omega^{2n}(\omega^n - \omega^n) \quad [\because \omega^3 = 1]$$

$$= 0$$

32. Let l, m and n be the p th, q th and r th terms of a GP whose first term is A and common ratio is R .

$$\therefore l = AR^{p-1}$$

$$\Rightarrow \log l = \log A + (p-1)\log R$$

$$\text{Similarly, } \log m = \log A + (q-1)\log R$$

$$\text{and } \log n = \log A + (r-1)\log R$$

$$\text{Now, } \begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix} = \begin{vmatrix} \log A + (p-1)\log R & p & 1 \\ \log A + (q-1)\log R & q & 1 \\ \log A + (r-1)\log R & r & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log A & p & 1 \\ \log A & q & 1 \\ \log A & r & 1 \end{vmatrix} + \begin{vmatrix} (p-1)\log R & p & 1 \\ (q-1)\log R & q & 1 \\ (r-1)\log R & r & 1 \end{vmatrix}$$

$$\log A \begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix} + \log R \begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

$$= 0 + \log R \begin{vmatrix} p & p & 1 \\ q & q & 1 \\ r & r & 1 \end{vmatrix} \quad [\text{applying } C_1 \rightarrow C_1 + C_3]$$

$$= 0 + \log R \times 0 = 0$$

$$33. \text{ Let } \Delta = \begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega-i & -1 \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_3$, we get

$$\Delta = \begin{vmatrix} 1-i & -1 & \omega^2-1 \\ 1-i & -1 & \omega^2-1 \\ -i & -1+\omega-i & -1 \end{vmatrix} \quad [\because \omega + \omega^2 = -1]$$

$$= 0 \quad [\because \text{two rows are identical}]$$

$$34. \Delta_r = \begin{vmatrix} 2r-1 & {}^m C_r & 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

$$\therefore \sum_{r=0}^m \Delta_r = \begin{vmatrix} \sum_{r=0}^m (2r-1) & \sum_{r=0}^m {}^m C_r & \sum_{r=0}^m 1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix}$$

$$= \begin{vmatrix} m^2-1 & 2^m & m+1 \\ m^2-1 & 2^m & m+1 \\ \sin^2(m^2) & \sin^2(m) & \sin^2(m+1) \end{vmatrix} = 0$$

$$[\because \text{two rows are identical}]$$

35. Let $z = x + iy$, then

$$z + iz = x + iy + i(x + iy)$$

$$= (x - y) + i(x + y)$$

and $iz = i(x + iy) = -y + ix$

Thus, the area of the triangle formed by these lines is

$$\Delta = \frac{1}{2} \begin{vmatrix} x & y & 1 \\ (x-y) & (x+y) & 1 \\ -y & x & 1 \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - (R_1 + R_3)$, we get

$$\Delta = \frac{1}{2} \begin{vmatrix} x & y & 1 \\ 0 & 0 & -1 \\ -y & x & 1 \end{vmatrix} = \frac{1}{2} (x^2 + y^2)$$

$$\Rightarrow \frac{1}{2} |z|^2 = 200 \quad [\text{given}]$$

$$|z|^2 = 400$$

$$\Rightarrow |z| = 20$$

$$\therefore 3|z| = 3 \times 20 = 60$$

$$36. \Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

$$= \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$[\text{applying } C_1 \rightarrow C_1 + C_2 + C_3]$$

On taking out $2(x + y + z)$ common from C_1 , we get

$$\Delta = 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z) \begin{vmatrix} 1 & x & y \\ 0 & y+z+x & 0 \\ 0 & 0 & z+x+y \end{vmatrix}$$

$$[\text{applying } R_2 \rightarrow R_2 - R_1 \text{ and } R_3 \rightarrow R_3 - R_1]$$

On taking out $(x + y + z)$ common from R_2 and R_3 , we get

$$\Delta = 2(x+y+z)(x+y+z)(x+y+z) \times \begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

On expanding along R_3 , we get

$$\Delta = 2(x+y+z)^3 [(1)(1-0)]$$

$$= 2(x+y+z)^3 = k(x+y+z)^3 \quad [\text{given}]$$

$$\therefore k = 2$$

$$37. \text{ Let } \Delta = \begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 & 1 \\ (c^x + c^{-x})^2 & (c^x - c^{-x})^2 & 1 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - C_2$, we get

$$\Delta = \begin{vmatrix} 4 & (a^x - a^{-x})^2 & 1 \\ 4 & (b^x - b^{-x})^2 & 1 \\ 4 & (c^x - c^{-x})^2 & 1 \end{vmatrix} = 4 \begin{vmatrix} 1 & (a^x - a^{-x})^2 & 1 \\ 1 & (b^x - b^{-x})^2 & 1 \\ 1 & (c^x - c^{-x})^2 & 1 \end{vmatrix} = 0$$

$$[\because \text{two columns are identical}]$$

38. On putting $r = 1, 2, 3, \dots, n$ and using the formula,

$$\Sigma 1 = n \text{ and } \Sigma r = \frac{n(n+1)}{2}$$

$$\Sigma (2r-1) = 1 + 3 + 5 + \dots = n^2$$

$$\therefore \sum_{r=1}^n \Delta_r = \begin{vmatrix} n & n & n \\ n(n+1) & n^2 + n + 1 & n^2 + n \\ n^2 & n^2 & n^2 + n + 1 \end{vmatrix} = 56$$

On applying $C_1 \rightarrow C_1 - C_3$ and $C_2 \rightarrow C_2 - C_3$, we get

$$\begin{vmatrix} 0 & 0 & n \\ 0 & 1 & n^2 + n \\ -n-1 & -n-1 & n^2 + n + 1 \end{vmatrix} = 56$$

$$\Rightarrow n(n+1) = 56 \Rightarrow n^2 + n - 56 = 0$$

$$\Rightarrow (n+8)(n-7) = 0$$

$$\Rightarrow n = 7, n \neq -8$$

$$39. \Delta = \begin{vmatrix} e^a & e^{2a} & e^{3a} \\ e^b & e^{2b} & e^{3b} \\ e^c & e^{2c} & e^{3c} \end{vmatrix} - \begin{vmatrix} e^a & e^{2a} & 1 \\ e^b & e^{2b} & 1 \\ e^c & e^{2c} & 1 \end{vmatrix}$$

$$= e^a \cdot e^b \cdot e^c \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} + \begin{vmatrix} e^a & 1 & e^{2a} \\ e^b & 1 & e^{2b} \\ e^c & 1 & e^{2c} \end{vmatrix}$$

$$= e^{(a+b+c)} \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} - \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix}$$

$$= (e^{a+b+c} - 1) \begin{vmatrix} 1 & e^a & e^{2a} \\ 1 & e^b & e^{2b} \\ 1 & e^c & e^{2c} \end{vmatrix} = 0 \quad [\because a + b + c = 0]$$

40. Let $\Delta = \begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix}$

On applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$, we get

$$\Delta = \begin{vmatrix} -2 & -2 & a-b \\ -2 & -2 & b-c \\ x+6 & x+7 & x+c \end{vmatrix}$$

Since, a, b and c are in AP.

$$\therefore a - b = b - c$$

Thus, rows R_1 and R_2 are identical.

Hence, $\Delta = 0$

41. By the properties of determinant, all the three statements are correct.

42. Given, $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then, $|I| = 1$
If we take I as $I_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Then, $|I_1| = -1$

Similarly, there are four other possibilities

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which will give a determinant either -1 or 1 .

Hence, there are six distinct choices for P and $\det(P) = \pm 1$

43. Given, $a_1, a_2, a_3, \dots$ are in AP.

Let $a_2 - a_1 = a_3 - a_2 = \dots = k \quad \dots(i)$

Now, let $\Delta = \begin{vmatrix} a_1 & a_2 & 1 \\ a_2 & a_3 & 1 \\ a_3 & a_4 & 1 \end{vmatrix}$

On applying $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$, we get

$$\Delta = \begin{vmatrix} a_1 & a_2 & 1 \\ a_2 - a_1 & a_3 - a_2 & 0 \\ a_3 - a_2 & a_4 - a_3 & 0 \end{vmatrix} = \begin{vmatrix} k & k & 0 \\ k & k & 0 \end{vmatrix} \quad [\text{from Eq. (i)}]$$

$$= 0 \quad [\because R_2 \text{ and } R_3 \text{ are identical}]$$

44. Given, $\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2}$

$$\Rightarrow 2 \begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2} \Rightarrow \begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix} = \frac{abc}{4} \quad \dots(ii)$$

Given vertices of a triangle are

$$\left(\frac{x_1}{a}, \frac{y_1}{a}\right), \left(\frac{x_2}{b}, \frac{y_2}{b}\right) \text{ and } \left(\frac{x_3}{c}, \frac{y_3}{c}\right).$$

$\therefore$ Area of triangle

$$= \frac{1}{2} \begin{vmatrix} \frac{x_1}{a} & \frac{y_1}{a} & 1 \\ \frac{x_2}{b} & \frac{y_2}{b} & 1 \\ \frac{x_3}{c} & \frac{y_3}{c} & 1 \end{vmatrix} = \frac{1}{2} \cdot \frac{1}{a} \cdot \frac{1}{b} \cdot \frac{1}{c} \begin{vmatrix} x_1 & y_1 & a \\ x_2 & y_2 & b \\ x_3 & y_3 & c \end{vmatrix}$$

$$= \frac{1}{2abc} \begin{vmatrix} a & x_1 & y_1 \\ b & x_2 & y_2 \\ c & x_3 & y_3 \end{vmatrix} = \frac{1}{2abc} \left(\frac{abc}{4}\right) \quad [\text{from Eq. (i)}]$$

$$= \frac{1}{8}$$

45. Let $\Delta = \begin{vmatrix} \log_a 1 & \log_a b & \log_a c \\ \log_a \left(\frac{1}{b}\right) & \log_b 1 & \log_a \left(\frac{1}{c}\right) \\ \log_a \left(\frac{1}{c}\right) & \log_a c & \log_c (1) \end{vmatrix}$

$$= \begin{vmatrix} 0 & \log_a b & \log_a c \\ -\log_a b & 0 & -\log_a c \\ -\log_a c & \log_a c & 0 \end{vmatrix}$$

Since, above determinant is a skew-symmetric of odd order, so the determinant should be zero.

$$\therefore \Delta = 0$$

46. Given, $\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ y & y^2 & y^3 \\ z & z^2 & z^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} + xyz \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (1 + xyz) \begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} = 0 \Rightarrow 1 + xyz = 0$$

as $\begin{vmatrix} x & x^2 & 1 \\ y & y^2 & 1 \\ z & z^2 & 1 \end{vmatrix} \neq 0$

$$\therefore xyz = -1$$

47. Given, $A28, 3B9$ and $62C$ are divisible by k .

$$\therefore A28 = n_1 k \Rightarrow 100A + 20 + 8 = n_1 k;$$

$$3B9 = n_2 k \Rightarrow 300 + 10B + 9 = n_2 k$$

$$\text{and } 62C = n_3 k \Rightarrow 600 + 20 + C = n_3 k,$$

where n_1, n_2 and n_3 are integers.

Let $\Delta = \begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix} = \frac{1}{100} \times \frac{1}{10} \begin{vmatrix} 100A & 300 & 600 \\ 8 & 9 & C \\ 20 & 10B & 20 \end{vmatrix}$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = \frac{1}{1000} \begin{vmatrix} 100A + 20 + 8 & 300 + 10B + 9 & 600 + 20 + C \\ 8 & 9 & C \\ 20 & 10B & 20 \end{vmatrix}$$

$$= \frac{1}{1000} \begin{vmatrix} n_1 k & n_2 k & n_3 k \\ 8 & 9 & C \\ 20 & 10B & 20 \end{vmatrix} = \frac{k}{1000} \begin{vmatrix} n_1 & n_2 & n_3 \\ 8 & 9 & C \\ 20 & 10B & 20 \end{vmatrix}$$

Hence, it is always divisible by k .

48. Given, $A = \begin{bmatrix} k & k\alpha & \alpha \\ 0 & \alpha & k\alpha \\ 0 & 0 & k \end{bmatrix}$

$\therefore |A| = \alpha k^2$

Now, $|A^2| = |A|^2 = (\alpha k^2)^2 = \alpha^2 k^4$ $[\because |A^n| = |A|^n]$

According to the given condition,

$|A^2| = k^2$

$\Rightarrow \alpha^2 k^4 = k^2$

$\Rightarrow \alpha^2 = \frac{1}{k^2}$

$\therefore |\alpha| = \frac{1}{k}$

49. Let $\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y) \begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = 2(x+y) \begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

On expanding with respect to R_1 , we get

$$\Delta = 2(x+y) [1(-x^2) - (-xy + y^2)]$$

$$= 2(x+y) (-x^2 + xy - y^2) = -2(x+y)(x^2 - xy + y^2)$$

Hence, the determinant is divisible by $x^2 - xy + y^2$.

50. Given vertices of triangle are (2, 3), (0, 1) and (1, 2).

Thus, area of triangle = $\frac{1}{2} \begin{vmatrix} 2 & 3 & 1 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$

$= \frac{1}{2} [2(1-2) - 3(0-1) + 1(0-1)]$

$= \frac{1}{2} [2(-1) + 3 - 1]$

$= \frac{1}{2} [-2 + 3 - 1] = 0$

51. Given matrix can be rewritten as,

$$A = \begin{bmatrix} 1+2\omega+\omega^2 & \omega^2 & 1 \\ 1 & 1+\omega^2+2\omega & \omega \\ \omega & \omega^2 & 2+\omega+2\omega^2 \end{bmatrix}$$

$\therefore |A| = \begin{vmatrix} \omega & \omega^2 & 1 \\ 1 & \omega & \omega \\ \omega & \omega^2 & 1+\omega^2 \end{vmatrix}$ $[\because 1+\omega+\omega^2=0]$

$= \omega \begin{vmatrix} \omega & 1 \\ 1 & \omega \end{vmatrix}$

$= \omega(0) = 0$ $[\because C_1 \text{ and } C_2 \text{ are identical}]$

52. Given, $A = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}$

Then, $|A| = 21 - 20 = 1$

$\therefore A(\text{adj } A) = \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = |A| I$$

53. Given, $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Now, $|A| = 2(12-0) = 24$

Cofactors of A are

$C_{11} = (12-0) = 12, C_{12} = 0, C_{13} = 0, C_{21} = 0,$

$C_{22} = +8, C_{23} = 0, C_{31} = 0, C_{32} = 0 \text{ and } C_{33} = 6$

$\therefore \text{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T = \begin{bmatrix} 12 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}^T$

$$= \begin{bmatrix} 12 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

$\therefore A^{-1} = \frac{1}{|A|} \text{adj}(A) = \frac{1}{24} \begin{bmatrix} 12 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 6 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$

54. Given, $A(\text{adj } A) = 10I$

We know that, $A(\text{adj } A) = |A| I$

$\Rightarrow 10I = |A| I \Rightarrow |A| = 10$

$\therefore |\text{adj } A| = |A|^{3-1} = (10)^2 = 100$

55. Given, $A = \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{-4-15} \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix}$

$$= \frac{-1}{19} \begin{bmatrix} -2 & -3 \\ -5 & 2 \end{bmatrix} = \frac{1}{19} \begin{bmatrix} 2 & 3 \\ 5 & -2 \end{bmatrix}$$

[multiplying by -1 to each element of matrix]

$= \frac{1}{19} A$

$\therefore A^{-1} = kA$

$\therefore k = \frac{1}{19}$

56. By the property of adjoint of a matrix,
 $\text{adj}(\text{adj } A) = |A|^{n-2} A$, if A is non-singular.

57. Given, $a^2 + b^2 + c^2 + d^2 = 1$... (i)

and $A = \begin{bmatrix} a+ib & c+id \\ -c+id & a-ib \end{bmatrix}$

Now, $|A| = (a+ib)(a-ib) - (c+id)(-c+id)$

$$= a^2 - (ib)^2 - [(id)^2 - (c)^2]$$

$$= a^2 + b^2 - [-d^2 - c^2]$$

$$= a^2 + b^2 + d^2 + c^2 = 1$$
 [from Eq. (i)]

$\therefore A^{-1} = \frac{1}{|A|} \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$

$$= \frac{1}{1} \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix} = \begin{bmatrix} a-ib & -c-id \\ c-id & a+ib \end{bmatrix}$$

58. Given, $A = \begin{bmatrix} 2 & 3 \\ 5 & 1 \end{bmatrix}$
 $\therefore A^{-1} = \frac{1}{2-15} \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix} = -\frac{1}{13} \begin{bmatrix} 1 & -3 \\ -5 & 2 \end{bmatrix}$

59. Given lines are concurrent.

$$\therefore \begin{vmatrix} 2 & -3 & k \\ 3 & -4 & -13 \\ 8 & -11 & -33 \end{vmatrix} = 0$$

$$\Rightarrow 2(132 - 143) + 3(-99 + 104) + k(-33 + 32) = 0$$

$$\Rightarrow -22 + 15 - k = 0 \Rightarrow k = -7$$

60. The system has non-zero solution, if

$$\begin{vmatrix} 1 & k & -1 \\ 3 & -k & -1 \\ 1 & -3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1(-k - 3) - k(3 + 1) - 1(-9 + k) = 0$$

$$\Rightarrow -6k + 6 = 0$$

$$\therefore k = 1$$

61. Given system of equations is

$$3x + y - z = 2, x - z = 1$$

and $2x + 2y + az = 5$

Condition for unique solution, $\begin{vmatrix} 3 & 1 & -1 \\ 1 & 0 & -1 \\ 2 & 2 & a \end{vmatrix} \neq 0$

$$\Rightarrow 3(0 + 2) - 1(a + 2) - 1(2 - 0) \neq 0$$

$$\Rightarrow 6 - a - 2 - 2 \neq 0$$

$$\therefore a \neq 2$$

62. Given, $A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & \alpha \end{bmatrix}$

On applying $R_2 \rightarrow R_2 - 2R_1, R_4 \rightarrow R_4 - 2R_3$
 and $R_3 \rightarrow R_3 - 3R_1$, we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 4 & 5 & \alpha - 6 \end{bmatrix}$$

On applying $R_4 \rightarrow R_4 + R_3$, we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & -3 & \alpha - 3 \end{bmatrix}$$

On applying $R_4 \rightarrow R_4 - R_3$, we get

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & -4 & -8 & 3 \\ 0 & 0 & 0 & \alpha - 5 \end{bmatrix}$$

Since, the matrix A is of rank 3.

$$\alpha - 5 = 0 \Rightarrow \alpha = 5$$

63. Given system of equations is $x + y + z = \beta$,

$$5x - y + \alpha z = 10$$

and

$$2x + 3y - z = 6$$

For unique solution, $\begin{vmatrix} 1 & 1 & 1 \\ 5 & -1 & \alpha \\ 2 & 3 & -1 \end{vmatrix} \neq 0$

$$\Rightarrow 1(1 - 3\alpha) - 1(-5 - 2\alpha) + 1(15 + 2) \neq 0$$

$$\Rightarrow 1 - 3\alpha + 5 + 2\alpha + 17 \neq 0$$

$$\Rightarrow -\alpha + 23 \neq 0, \alpha \neq 23$$

Hence, for the existence of the unique solution, system of equations depends on α only.

64. Given, $\begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4 \sin 4\theta \\ \cos^2 \theta & 1 + \sin^2 \theta & 4 \sin 4\theta \\ \cos^2 \theta & \sin^2 \theta & 1 + 4 \sin 4\theta \end{vmatrix} = 0$

On applying $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$, we get

$$\begin{vmatrix} 1 + \cos^2 \theta & \sin^2 \theta & 4 \sin 4\theta \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{vmatrix} = 0$$

On applying $C_1 \rightarrow C_1 + C_2$, we get

$$\begin{vmatrix} 2 & \sin^2 \theta & 4 \sin 4\theta \\ 0 & 1 & 0 \\ -1 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 2 + 4 \sin 4\theta = 0$$

$$\Rightarrow \sin 4\theta = -\frac{1}{2}$$

$$= -\sin \frac{\pi}{6} = \sin \left(-\frac{\pi}{6} \right)$$

$$\Rightarrow 4\theta = n\pi + (-1)^n \left(-\frac{\pi}{6} \right)$$

Hence, the values of θ between 0 and $\frac{\pi}{2}$ will be $\frac{7\pi}{24}$

and $\frac{11\pi}{24}$.

65. Let $\Delta = \begin{vmatrix} \sin \alpha & \cos \alpha & \sin(\alpha + \gamma) \\ \sin \beta & \cos \beta & \sin(\beta + \gamma) \\ \sin \delta & \cos \delta & \sin(\gamma + \delta) \end{vmatrix}$

On applying $C_3 \rightarrow C_3 - \cos \gamma C_1$, we get

$$\Delta = \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \sin \gamma \\ \sin \beta & \cos \beta & \cos \beta \sin \gamma \\ \sin \delta & \cos \delta & \cos \delta \sin \gamma \end{vmatrix}$$

$$= \sin \gamma \begin{vmatrix} \sin \alpha & \cos \alpha & \cos \alpha \\ \sin \beta & \cos \beta & \cos \beta \\ \sin \delta & \cos \delta & \cos \delta \end{vmatrix}$$

$$= \sin \gamma \cdot 0 = 0$$

66. Let $\Delta = \begin{vmatrix} a^2 + 1 & ab & ac \\ ba & b^2 + 1 & bc \\ ca & bc & c^2 + 1 \end{vmatrix}$

On applying $R_1 \rightarrow aR_1, R_2 \rightarrow bR_2$ and $R_3 \rightarrow cR_3$, we get

$$\Delta = \frac{1}{abc} \begin{vmatrix} a(a^2 + 1) & a^2b & a^2c \\ b^2a & b(b^2 + 1) & b^2c \\ c^2a & c^2b & c(c^2 + 1) \end{vmatrix}$$

On taking a common from C_1 , b from C_2 and c from C_3 , we get

$$\Delta = \begin{vmatrix} a^2+1 & a^2 & a^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\Delta = \begin{vmatrix} 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 & 1+a^2+b^2+c^2 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

Now, on taking common $(1+a^2+b^2+c^2)$ from R_1 , we get

$$\Delta = (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 1 & 1 \\ b^2 & b^2+1 & b^2 \\ c^2 & c^2 & c^2+1 \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = (1+a^2+b^2+c^2) \begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} = (1+a^2+b^2+c^2)$$

67. Let $\Delta = \begin{vmatrix} -1 & \cos R & \cos Q \\ \cos R & -1 & \cos P \\ \cos Q & \cos P & -1 \end{vmatrix}$

On multiplying C_1 by p , we get

$$\Delta = \frac{1}{p} \begin{vmatrix} -p & \cos R & \cos Q \\ p \cos R & -1 & \cos P \\ p \cos Q & \cos P & -1 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 + qC_2 + rC_3$, we get

$$\Delta = \frac{1}{p} \begin{vmatrix} -p+q \cos R+r \cos Q & \cos R & \cos Q \\ p \cos R & -1 & \cos P \\ p \cos Q & \cos P & -1 \end{vmatrix}$$

$$= \frac{1}{p} \begin{vmatrix} 0 & \cos R & \cos Q \\ 0 & -1 & \cos P \\ 0 & \cos P & -1 \end{vmatrix} = 0 \quad [\text{using projection rule}]$$

68. We have, $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix}$ and $Q = PP^T$

Clearly, $Q = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 8 & 11 \end{bmatrix}$

$\therefore |Q| = \begin{vmatrix} 6 & 8 \\ 8 & 11 \end{vmatrix} = 66 - 64 = 2$

69. Given system of equations is

$-2x + y + z = a$, $x - 2y + z = b$ and $x + y - 2z = c$

Here, $A = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Now, $|A| = \begin{vmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = 0$

For consistent, $(\text{adj } A)B = O$

Now, $\text{adj } (A) = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix}$

$\therefore (\text{adj } A)B = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = O$

$\Rightarrow 3 \begin{bmatrix} a+b+c \\ a+b+c \\ a+b+c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow a+b+c = 0$

70. Augmented matrix

$$[A:B] = \begin{bmatrix} 1 & -1 & -2 & 6 \\ -1 & 1 & 1 & \mu \\ \lambda & 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -2 & 6 \\ 0 & 0 & -1 & \mu+6 \\ \lambda+1 & 0 & -1 & 9 \end{bmatrix}$$

[applying $R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 + R_1$]

$$= \begin{bmatrix} 1 & -1 & -2 & 6 \\ 0 & 0 & -1 & \mu+6 \\ \lambda+1 & 0 & -1 & 3-\mu \end{bmatrix} \quad [\text{applying } R_3 \rightarrow R_3 - R_2]$$

So, infinite number of solutions for $\lambda = -1$ and $\mu = 3$.

71. We have, $|A| = -1$, $|B| = 3$ and A, B are matrices of order 3.

$\therefore |3AB| = 3^3 |AB| = 27 |A| |B|$

$[\because \det(AB) = \det A \cdot \det B]$
 $= 27(-1)(3) = -81$

72. Given, $z = \begin{vmatrix} 1 & 1+2i & -5i \\ 1-2i & -3 & 5+3i \\ 5i & 5-3i & 7 \end{vmatrix}$

$= 1[-21 - (25+9)] - (1+2i)7 - 14i - (25i-15)$
 $[-5i \{(-1-13i)+15i\}]$
 $= -55 - (1+2i)(22-39i) - 5i(-1+2i)$
 $= -55 - 100 - 5i + 5i + 10 = -145$

73. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ca & c^2-ab \end{vmatrix}$

On applying $C_3 \rightarrow C_3 - C_2$ and $C_2 \rightarrow C_2 - C_1$, we get

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-b \\ a^2-bc & (b^2-a^2) & (c^2-b^2) \\ & +c(b-a) & +a(c-b) \end{vmatrix}$$

$$= 1[(b-a)\{(c-b)(c+b+a)\} - (c-b)\{(b-a)(b+a+c)\}] = 0$$

74. We have, $M = I - X(X'X)^{-1}X'$

$= I - X(X^{-1}(X')^{-1})X' [\because (AB)^{-1} = B^{-1}A^{-1}]$
 $= I - (X X^{-1})(X')^{-1}X' = I - I[(X')^{-1}X']$
 $= I - I \times I [\because AA^{-1} = I = A^{-1}A]$
 $= I - I = O \quad [I^2 = I]$

Clearly, $M^2 = O = M$

So, M is an idempotent matrix. Also, product of $MX = O$.

75. We have, $|A| = 6$

Now, $|2 \text{ adj } A| = 2^3 |\text{adj } A| = 8 |A|^2 = 8 \times 36 = 288$

76. We have, $|A| = 2$

Consider, $|A(\text{adj } A)| = |A| \cdot |\text{adj } A| \quad [\because |AB| = |A| \cdot |B|]$
 $= |A| \cdot |A| = |A|^2 = 4$

$[\because |\text{adj } A| = |A|^{n-1}, \text{ if } A \text{ is a square matrix of order } n]$

Now, the matrix having determinant 4 is required matrix.

Note that, $\begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} = 4$.

77. For no solution, $\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & p+2 \\ 0 & (2p+1) & 1 \end{vmatrix} = 0$

$\Rightarrow -(p+2)(2p+1) = 0$
 $\therefore p = -2, -\frac{1}{2}$

Let $A = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & p+2 \\ 0 & (2p+1) & 1 \end{bmatrix}$

$C_{11} = -(p+2)(2p+1) = -(2p^2 + 5p + 2)$

$C_{12} = 0, C_{13} = 0$

$C_{21} = -(6p+5), C_{22} = 1, C_{23} = -(2p+1)$

$C_{31} = 2p+4, C_{32} = -(p+2), C_{33} = 0$

$\therefore \text{adj}(A) = \begin{bmatrix} -2p^2 - 5p - 2 & -5 - 6p & 2p + 4 \\ 0 & 1 & -(p+2) \\ 0 & -(2p+1) & 0 \end{bmatrix}$

Now, $(\text{adj } A)B = \begin{bmatrix} -2p^2 - 19p - 9 \\ 3 - 2p - 4 \\ -6p - 3 \end{bmatrix}$

When $p = -2$, then

$(\text{adj } A)B = \begin{bmatrix} -8 + 38 - 9 \\ 3 + 4 - 4 \\ 12 - 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 3 \\ 9 \end{bmatrix} \neq 0$

$\therefore$ No solution exist.

When $p = -\frac{1}{2}$, then

$(\text{adj } A)B = \begin{bmatrix} -2\left(\frac{1}{4}\right) + \frac{19}{2} - 9 \\ 3 - 2\left(-\frac{1}{2}\right) - 4 \\ -6\left(-\frac{1}{2}\right) - 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\therefore$ Infinite solutions may exist.

Aliter

The augmented matrix is $C = [A : B]$

$= \begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 0 & 0 & p+2 & : & 3 \\ 0 & 2p+1 & 1 & : & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & : & 1 \\ 0 & 2p+1 & 1 & : & 2 \\ 0 & 0 & p+2 & : & 3 \end{bmatrix}$
 $[R_2 \leftrightarrow R_3]$

The system of equations is inconsistent, if $r(A) = 2$ and $r(C) = 3$

Now, $r(A) = 2$, if $p+2 = 0$, i.e. $p = -2$

78. Let $\Delta = \begin{vmatrix} 15! & 16! & 17! \\ 16! & 17! & 18! \\ 17! & 18! & 19! \end{vmatrix}$

On applying $C_3 \rightarrow C_3 - C_2$ and $C_2 \rightarrow C_2 - C_1$, we get

$\Delta = \begin{vmatrix} 15! & 15 \times 15! & 16 \times 16! \\ 16! & 16 \times 16! & 17 \times 17! \\ 17! & 17 \times 17! & 18 \times 18! \end{vmatrix}$
 $= (15!)(16!)(17!) \begin{vmatrix} 1 & 15 & 16 \times 16 \\ 1 & 16 & 17 \times 17 \\ 1 & 17 & 18 \times 18 \end{vmatrix}$

On applying $R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$, we get

$\Delta = (15!)(16!)(17!) \begin{vmatrix} 0 & -1 & -33 \\ 0 & -1 & -35 \\ 1 & 17 & 18 \times 18 \end{vmatrix}$
 $= 2 \times (15!)(16!)(17!)$

79. On putting $x = 0$ in given determinant equation, we get

$a_0 = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 2 & -3 \\ -3 & 4 & 0 \end{vmatrix}$
 $= 1(0 - 9) + 3(4 + 6) = 30 - 9 = 21$

80. Given, $\begin{vmatrix} x + \omega^2 & \omega & 1 \\ \omega & \omega^2 & 1 + x \\ 1 & x + \omega & \omega^2 \end{vmatrix} = 0$

On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$\begin{vmatrix} x & \omega & 1 \\ x & \omega^2 & 1 + x \\ x & x + \omega & \omega^2 \end{vmatrix} = 0$ [$\because 1 + \omega + \omega^2 = 0$]

Hence, $x = 0$ is one of the value of x , which satisfy the above determinant equation.

81. $\therefore \text{adj}(\text{adj } A) = |A|^{n-2} A$. Here, $n = 3$

$\therefore \text{adj}(\text{adj } A) = |A| A$

82. Clearly, $(AB)^{-1} = B^{-1} A^{-1}$

83. Given, $A = \begin{bmatrix} x & -2 \\ 3 & 7 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} x & -2 \\ 3 & 7 \end{vmatrix} = 7x + 6$

$\therefore A^{-1} = \frac{1}{7x+6} \begin{bmatrix} 7 & 2 \\ -3 & x \end{bmatrix}$

But $A^{-1} = \begin{bmatrix} 7/34 & 1/17 \\ -3/34 & 2/17 \end{bmatrix}$

$\therefore \frac{7}{7x+6} = \frac{7}{34} \Rightarrow 7x + 6 = 34$

$\Rightarrow 7x = 28 \Rightarrow x = 4$

84. $|A| = \begin{vmatrix} 1 & 2 \\ -4 & -1 \end{vmatrix} = -1 + 8 = 7$ and $\text{adj}(A) = \begin{bmatrix} -1 & -2 \\ 4 & 1 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{7} \begin{bmatrix} -1 & -2 \\ 4 & 1 \end{bmatrix}$

3

Relations and Functions

Relation

Let A and B be two non-empty sets, then relation R from A to B is a subset of $A \times B$.

Let $R \subseteq A \times B$ and $(a, b) \in R$, then we say that a is related to b by the relation R , i.e. aRb .

If $(a, b) \in R$, then we write it as aRb .

If $(a, b) \notin R$, then we write $a \not R b$ and we say that a is not related to b by the relation R .

➤ **Example 1.** The relation $R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$ in roster form is

(a) $\{(2, 8), (3, 27), (5, 125), (7, 343), (9, 729)\}$

(b) $\{(3, 27), (5, 125), (7, 343)\}$

(c) $\{(2, 8), (3, 27), (5, 125), (7, 343)\}$

(d) $\{(1, 1), (2, 8), (3, 27), (5, 125), (7, 343)\}$

Sol. (c) Here, prime numbers less than 10 are 2, 3, 5 and 7.

$\therefore$ Relation in the roster form is $\{(2, 8), (3, 27), (5, 125), (7, 343)\}$.

Total number of relations Let A and B be two non-empty finite sets consisting of m and n elements, respectively. Then, $A \times B$ consists of mn ordered pairs. So, total number of subsets of $A \times B$ is 2^{mn} .

➤ **Example 2.** Let $A = \{x, y, z\}$ and $B = \{1, 2\}$, then the number of relations from A to B is

(a) 32

(b) 64

(c) 128

(d) 8

Sol. (b) $n(A) = 3, n(B) = 2$

$\Rightarrow n(A \times B) = 6$

$\therefore$ Total number of relations from A to B is $2^{2 \times 3} = 2^6 = 64$.

Domain and Range of a Relation

Let R be a relation from a set A to set B . Then, the set of all first components or coordinates of the ordered pairs belonging to R is called the domain of R while the set of all second components or coordinates of the ordered pairs in R is called the range of R .

Chapter Snapshot

- Relation
- Domain and Range of a Relation
- Types of Relations
- Composition of Relations
- Congruence Modulo m
- Partial Order Relation
- Total Order Relation
- Equivalence Classes of an Equivalence Relation
- Constants and Variables
- Function
- Vertical Parallel Line Test (VPL Test)
- Domain and Range of a Real Function
- Algebra of Functions
- Explicit and Implicit Functions
- Basic Functions
- Real Valued Functions
- Bounded Functions
- Equality of Functions
- Odd and Even Functions
- Even and Odd Extensions
- Periodic Function
- Types of Functions
- Number of Functions
- Composite Functions
- Inverse Function
- Definite Function

Thus, domain of $R = \{a : (a, b) \in R\}$
and range of $R = \{b : (a, b) \in R\}$

✎ Domain of a relation from A to B is a subset of A and its range is a subset of B .

e.g. If R is a relation from set $A = \{2, 4, 5\}$ to set $B = \{1, 2, 3, 4, 6, 8\}$ defined by xRy such that x divides y , then

i. $2R2, 2R4, 2R6, 2R8, 4R4, 4R8$ are the possible relations from set A to set B .

Also,
 $R = \{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$ is the set of ordered pairs.

ii. Domain $(R) = \{2, 2, 2, 2, 4, 4\} \equiv \{2, 4\}$

Range $(R) = \{2, 4, 6, 8, 4, 8\} \equiv \{2, 4, 6, 8\}$

Types of Relations

There are some types of relations which are given below:

Void Relation

Since, $\phi \subset A \times A$, it follows that f is a relation on A , called the empty or void relation.

Universal Relation

Since, $A \times A \subseteq A \times A$, it follows that $A \times A$ is a relation on A , called the universal relation.

Identity Relation

The relation $I_A = \{(a, a) : a \in A\}$ is called the identity relation to A .

e.g. If $A = \{1, 2, 3\}$, then the identity relation on A is given by $I_A = \{(1, 1), (2, 2), (3, 3)\}$.

Inverse Relation

If R is a relation on A , then the relation R^{-1} on A , defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$ is called an inverse relation to A .

Clearly, domain $(R^{-1}) = \text{range } (R)$

and range $(R^{-1}) = \text{domain } (R)$

e.g. Let $A = \{1, 2, 3\}$

and $R = \{(1, 2), (2, 2), (3, 1), (3, 2)\}$

Then, R being a subset of $A \times A$, is a relation on A .

Clearly, $1R2; 2R2; 3R1$ and $3R2$

Domain $(R) = \{1, 2, 3\}$ and range $(R) = \{2, 1\}$

Also, $R^{-1} = \{(2, 1), (2, 2), (1, 3), (2, 3)\}$

Domain $(R^{-1}) = \{2, 1\}$ and range $(R^{-1}) = \{1, 2, 3\}$

Reflexive Relation

A relation R on a set A is said to be reflexive, if every element of A is related to itself.

Thus, if R is reflexive, then $(a, a) \in R, \forall a \in A$.

A relation R on a set A is not reflexive, if there exists an element $a \in A$ such that $(a, a) \notin R$.

e.g.

(a) If $A = \{1, 2, 3\}$ is a set.

Then, $R = \{(1, 1), (2, 2), (3, 3), (1, 3), (2, 1)\}$ is a reflexive relation on A .

But $R' = \{(1, 1), (3, 3), (2, 1), (3, 2)\}$ is not a reflexive, because for $2 \in A, (2, 2) \notin R$

(b) Let L be the set of all lines in a plane. Then, relation R on L defined by $(l_1, l_2) \in R$ such that l_1 is parallel to l_2 is reflexive, since every line is parallel to itself.

✎ The universal relation on a non-void set A is reflexive.

Symmetric Relation

A relation R on a set A is said to be a symmetric relation iff

$$(a, b) \in R \Rightarrow (b, a) \in R, \forall a, b \in A$$

$$\text{i.e. } aRb \Rightarrow bRa, \forall a, b \in A$$

Also, a relation R on a set A is not a symmetric relation, if there are atleast two elements $a, b \in A$ such that $(a, b) \in R$ but $(b, a) \notin R$.

e.g.

(a) Let $A = \{1, 2, 3, 4\}$ be any set and R_1, R_2 be relations on A_1 such that

$$R_1 = \{(1, 3), (1, 4), (3, 1), (2, 2), (4, 1)\}$$

$$\text{and } R_2 = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$$

Clearly, R_1 is a symmetric relation on A .

However, R_2 is not so, because $(1, 3) \in R_2$ but $(3, 1) \notin R_2$.

(b) Let L be the set of all lines in a plane and R be a relation defined on L by the rule $(x, y) \in R$, x is perpendicular to y . Then, R is a symmetric on L , because $L_1 \perp L_2 \Rightarrow L_2 \perp L_1$. e.g. If $(L_2, L_1) \in R$ automatically implies $(L_2, L_1) \in R$.

- ✎ The identity and the universal relations on a non-void set are symmetric relations.
- A reflexive relation on a set A is not necessarily symmetric.
- The relation R on a set A is symmetric iff $R = R^{-1}$

Transitive Relation

Let A be any set. A relation R on A is said to be a transitive relation iff $(a, b) \in R$ and $(b, c) \in R$.

$$\Rightarrow (a, c) \in R, \forall a, b, c \in R$$

$$\text{i.e. } aRb \text{ and } bRc \Rightarrow aRc, \forall a, b, c \in A$$

e.g.

- (a) On the set N of natural numbers, the relation R defined by xRy such that x is less than y , is transitive, because for any $x, y, z \in N$,

$$x < y \text{ and } y < z \Rightarrow x < z$$

$$\text{i.e. } xRy \text{ and } yRz \Rightarrow xRz$$

- (b) Let L be the set of all straight lines in a plane. Then, the relation which is parallel to L , is a transitive relation, because for any $l_1, l_2, l_3 \in L$, $l_1 \parallel l_2$ and $l_2 \parallel l_3 \Rightarrow l_1 \parallel l_3$

✎ The identity and the universal relations, on a non-void set are always transitive.

Anti-symmetric Relation

Let A be any set. A relation R on set A is said to be an anti-symmetric relation iff $(a, b) \in R$ and $(b, a) \in R \Rightarrow a = b$ for all $a, b \in A$

- ✎ • By definition, if $(a, b) \in R$ and $(b, a) \notin R$, then R is also an anti-symmetric relation.
• The identity relation on a set A is an anti-symmetric relation.

Equivalence Relation

A relation R on a set A is said to be an equivalence relation on A iff

- (a) it is reflexive, i.e. $(a, a) \in R, \forall a \in A$.
(b) it is symmetric, i.e. $(a, b) \in R$.
 $\Rightarrow (b, a) \in R, \forall a, b \in A$
(c) it is transitive, i.e. $(a, b) \in R$ and $(b, c) \in R$.
 $\Rightarrow (a, c) \in R, \forall a, b, c \in A$

Properties of Equivalence Relation

- i. The inverse of an equivalence relation is also an equivalence relation, i.e. if R is an equivalence relation on a set A , then R^{-1} is also an equivalence relation on A .
- ii. The union of two equivalence relations on a set is not necessarily an equivalence relation on a set.
- iii. The intersection of two equivalence relations on a set is an equivalence relation on the set.

Composition of Relations

Let $R \subseteq A \times B, S \subseteq B \times C$ be two relations. Then, composition of the relation R and S denoted by $SoR \subseteq A \times C$ and is defined by

$(a, c) \in (SoR)$ iff $\exists b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$.

e.g. Let $A = \{1, 2, 3\}, B = \{a, b, c, d\}, C = \{\alpha, \beta, \gamma\}$
 $R \subseteq (A \times B) = \{(1, a), (1, c), (2, d)\}$

$$S \subseteq (B \times C) = \{(a, \alpha), (a, \gamma), (c, \beta)\}$$

$$\text{Then, } SoR \subseteq (A \times C) = \{(1, \alpha), (1, \gamma), (2, \beta)\}$$

One should be careful in computing the relation RoS . Actually, SoR starts with R and RoS starts with S .

In general, $SoR \neq RoS$

Also, $(SoR)^{-1} = R^{-1}oS^{-1}$, known as reversal rule.

✎ **Example 3.** Let R be any relation in the set A of human beings in a town at a particular time. If $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$, then R is

- (a) reflexive (b) symmetric
(c) transitive (d) an equivalence relation

Sol. (d) Here, $R = \{(x, y) : x \text{ and } y \text{ work at the same place}\}$

Then, R is reflexive as x works at same place of x .

$$\Rightarrow (x, x) \in R, \forall x \in A$$

If x and y work at same place, then y and x work at same place, i.e. if $(x, y) \in R \Rightarrow (y, x) \in R$

$\therefore R$ is symmetric.

$$\text{Let } (x, y) \in R, (y, z) \in R$$

$\Rightarrow x$ and y work at same place and y and z work at same place.

Since, all three work at same place.

$\therefore x$ and z work at same place.

$$\Rightarrow (x, z) \in R$$

$\therefore R$ is transitive.

Hence, R is an equivalence relation.

✎ **Example 4.** The relation R defined in the set $A = \{1, 2, 3, 4, 5, 6, 7\}$ by $R = \{(a, b) : \text{both } a \text{ and } b \text{ are either odd or even}\}$. Then, R is

- (a) symmetric
(b) transitive
(c) an equivalence relation
(d) reflexive

Sol. (c) Given, any element a in A , both a and a must be either odd or even, so that $(a, a) \in R$.

Further, $(a, b) \in R \Rightarrow$ both a and b must be either odd or even $\Rightarrow (b, a) \in R$. Similarly, $(a, b) \in R$ and $(b, c) \in R \Rightarrow$ All elements a, b, c must be either even or odd simultaneously.

$\Rightarrow (a, c) \in R$. Hence, R is an equivalence relation.

✎ **Example 5.** Let R be the relation in the set A of all books in a library of a college given by $R = \{(x, y) : x \text{ and } y \text{ have same number of pages}\}$. Then, R is

- (a) not reflexive (b) not symmetric
(c) not transitive (d) an equivalence relation

Sol. (d) Here, A is the set of all books in the library of a college and $R = \{(x, y) : x \text{ and } y \text{ have the same number of pages}\}$.

Now, R is reflexive, since $(x, x) \in R, \forall x \in A$, as x and x has the same number of pages.

Let $(x, y) \in R$

$\Rightarrow$ x and y have the same number of pages.
 $\Rightarrow$ y and x have the same number of pages.
 $\Rightarrow$ $(y, x) \in R$
 So, R is symmetric.
 Now, let $(x, y) \in R$ and $(y, z) \in R$.
 $\Rightarrow$ x, y have the same number of pages and y, z have the same number of pages.
 $\Rightarrow$ x and z have the same number of pages.
 $\Rightarrow$ $(x, z) \in R$
 Therefore, R is transitive. Hence, R is an equivalence relation.

Congruence Modulo m

Let m be an arbitrary but fixed integer. Two integers a and b are said to be congruence modulo m , if $a - b$ is divisible by m and we write $a \equiv b \pmod{m}$.

Thus, $a \equiv b \pmod{m} \Leftrightarrow a - b$ is divisible by m .

e.g. $18 \equiv 3 \pmod{5}$ because $18 - 3 = -15$, which is divisible by 5.

Similarly, $3 \equiv 13 \pmod{2}$ because $3 - 13 = -10$, which is divisible by 2.

But $25 \not\equiv 2 \pmod{4}$ because 4 is not a divisor of $25 - 2 = 23$

Example 6. The congruent solution of $8x \equiv 6 \pmod{14}$ is

- (a) $x = 6, 7$ (b) $x = 6, 13$
 (c) $x = 2, 13$ (d) None of these

Sol. (b) Given, $8x \equiv 6 \pmod{14}$

$$\begin{aligned}
 \therefore \quad & \lambda = \frac{8x - 6}{14}, \text{ where } \lambda \in I^+ \\
 \therefore \quad & 8x = 14\lambda + 6 \\
 \Rightarrow \quad & x = \frac{14\lambda + 6}{8} = \frac{7\lambda + 3}{4} \\
 & = \frac{4\lambda + 3(\lambda + 1)}{4} \\
 & = \lambda + \frac{3}{4}(\lambda + 1), \text{ where } \lambda \in I^+
 \end{aligned}$$

Here, greatest common divisor of 8 and 14 is 2.

So, there are two required solutions for $\lambda = 3$ and $\lambda = 7$.

$\therefore x = 6, 13$

Partial Order Relation

A relation R defined on a set A is said to be a partial order relation on A , if it is simultaneously reflexive, transitive and anti-symmetric on A .

e.g. Let N be the set of all natural numbers. Let a relation R be defined on N by $R = \{(a, b) : a, b \in N \text{ and } a \leq b\}$.

Then, R is reflexive because $a \leq a, \forall a, b, c \in N$.
 R is transitive because $a \leq b$ and $b \leq c$.

$$\Rightarrow a \leq c, \forall a, b, c \in N$$

R is anti-symmetric because $a \leq b$ and $b \leq a$.

$$\Rightarrow a = b, \forall a, b \in N.$$

Thus, R is a partial order relation.

- This relation is not equivalence relation on A because $a \leq b$ does not imply $b \leq a$.
- A non-empty set together with a partial order relation (also called partial ordering) on it is said to be a partially ordered set.

Total Order Relation

A relation R on a set A is said to be a total order relation on A , if R is a partial order relation on A such that $x, y \in A$, we must have either xRy or yRx .

e.g. Let S be a given non-empty set. We define a relation R on power set $P(S)$ by $XY \Leftrightarrow X \in Y, \forall X, Y \in P(S)$.

Clearly, R is reflexive, anti-symmetric and transitive. So, R is a partial order relation. But it is not a total order relation because if $S = \{1, 2, 3\}$. Then, $\{2\}, \{3\} \in P(S)$ but neither $\{2\} \in \{3\}$ nor $\{3\} \in \{2\}$.

A non-empty set together with a total order relation on it is called a totally ordered set or chain.

Equivalence Classes of an Equivalence Relation

Let R be equivalence relation in A ($\neq \emptyset$). Let $a \in A$. Then, the equivalence class of a denoted by $[a]$ or $\{a\}$ is defined as the set of all those points of A which are related to a under the relation R . Thus, $[a] = \{x \in A : xRa\}$. It is easy to see that

- $b \in [a] \Rightarrow a \in [b]$
- $b \in [a] \Rightarrow [a] = [b]$
- Two equivalence classes are either disjoint or identical as an example we consider a very important relation $x \equiv y \pmod{n}$, if n divides $(x - y)$, is fixed positive integer. Consider $n = 5$, then

$$\begin{aligned}
 [0] &= \{x : x \equiv (\text{mod } 5)\} = \{5p : p \in I\} \\
 &= \{\pm 5, \pm 10, \pm 15, \dots\}
 \end{aligned}$$

$$\begin{aligned}
 [1] &= \{x : x \equiv 1 \pmod{5}\} \\
 &= \{x : x - 1 = 5k, k \in I\} = \{5k + 1 : k \in I\} \\
 &= \{1, 6, 11, \dots, -4, -9, \dots\}
 \end{aligned}$$

One can easily see that there are only 5 distinct equivalence classes viz. $[0], [1], [2], [3]$ and $[4]$ when $n = 5$

Example 7. N is the set of natural numbers. The relation R is defined on $N \times N$ as follows
 $(a, b) R (c, d) \Leftrightarrow a + d = b + c$

- (a) reflexive (b) symmetric
 (c) transitive (d) All of these

Sol. (d) $(a, b) R (a, b) \Leftrightarrow a + b = b + a$

$\therefore R$ is reflexive.

$$(a, b) R (c, d) \Leftrightarrow a + d = b + c \\ \Rightarrow c + b = d + a \Rightarrow (c, d) R (a, b)$$

$\therefore R$ is symmetric.

Now, $(a, b) R (c, d)$ and $(c, d) R (e, f)$

$$\Rightarrow a + d = b + c \text{ and } c + f = d + e$$

$$\Rightarrow a + d + c + f = b + c + d + e$$

$$\Rightarrow a + f = b + e$$

$$\Rightarrow (a, b) R (e, f)$$

$\therefore R$ is transitive.

Thus, R is an equivalence relation in $N \times N$.

☞ **Example 8.** Let S be the set of all points in a plane and R be a relation on S defined as $R = \{(P, Q) : \text{distance between } P \text{ and } Q \text{ is less than } 2 \text{ units}\}$. Then, R is

(a) reflexive but not symmetric

(b) symmetric and transitive

(c) reflexive and transitive

(d) reflexive and symmetric but not transitive

Sol. (d) Given that, S is the set of all points in a plane and R is the relation on S defined as

$$R = \{(P, Q) : \text{distance between } P \text{ and } Q \text{ is less than } 2 \text{ units}\}$$

Now, for any point P in set S , the distance between P and P is 0 which is less than 2 units.

$$\therefore (P, P) \in R \Rightarrow (P, P) \in R, \forall P \in S$$

So, R is reflexive in S .

Let P and Q be two points in S such that $(P, Q) \in R$.

$$\therefore (P, Q) \in R$$

Then, distance between P and Q is less than 2 units.

$\Rightarrow$ Distance between Q and P is less than 2 units.

$$\Rightarrow (Q, P) \in R$$

$\therefore R$ is symmetric.

Now, consider points P' , Q' and R' having coordinates $(0, 0)$, $(1.5, 0)$ and $(3.2, 0)$. The distance between P' and Q' is 1.5 units which is less than 2 units and the distance between Q' and R' is 1.7 units which is also less than 2 units.

But the distance between P' and R' is 3.2, which is not less than 2 units.

$$\Rightarrow (P', Q') \in R \text{ and } (Q', R') \in R \text{ but } (P', R') \notin R$$

$\therefore R$ is not transitive on S .

☞ **Example 9.** Let $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 3)\}$ be a relation in A . Then, the minimum number of ordered pairs may be added, so that R becomes an equivalence relation, is

(a) 7

(b) 5

(c) 1

(d) 4

Sol. (a) The given relation is $R = \{(1, 2), (2, 3)\}$ in the set $A = \{1, 2, 3\}$

Now, R is reflexive, if $(1, 1), (2, 2), (3, 3) \in R$.

R is symmetric, if $(2, 1), (3, 2) \in R$.

R is transitive, if $(1, 3)$ and $(3, 1) \in R$.

Thus, the minimum number of ordered pairs which are to be added, so that R becomes an equivalence relation, is 7.

☞ **Example 10.** Let R be a relation in the set N of natural numbers defined by nRm , if n divides m . Then, R is

(a) reflexive and symmetric

(b) transitive and symmetric

(c) equivalence

(d) reflexive, transitive but not symmetric

Sol. (d) Since, n divides n , $\forall n \in N$, R is reflexive. R is not symmetric for 3, $6 \in N$, $3R6$, $6 \nR 3$. R is transitive for n, m, r , whenever $\frac{n}{m}$ and $\frac{m}{r} \Rightarrow \frac{n}{r}$, i.e. whenever n divides m and m divides r , then n will divide r .

☞ **Example 11.** Consider the non-empty set consisting of children in a family and a relation R defined as aRb , if a is brother of b . Then, R is

(a) symmetric but not transitive

(b) transitive but not symmetric

(c) neither symmetric nor transitive

(d) both symmetric and transitive

Sol. (b) Given that, $aRb \Rightarrow a$ is brother of b .

$\therefore aRa \Rightarrow a$ is brother of a , which is not true.

$\therefore R$ is not reflexive.

$aRb \Rightarrow a$ is brother of b .

This does not mean b is also brother of a as b can be a sister of a .

$\therefore R$ is not symmetric.

Now, $aRb \Rightarrow a$ is brother of b and $bRc \Rightarrow b$ is brother of c .

$\Rightarrow a$ is brother of c .

$\therefore R$ is transitive.

☞ **Example 12.** Given a non-empty set X , consider $P(X)$ which is the set of all subsets of X .

Define the relation R in $P(X)$ as follows:

For subsets A and B in $P(X)$, ARB if and only if $A \subset B$. Then, R is

(a) reflexive

(b) transitive

(c) not symmetric

(d) an equivalence relation

Sol. (a, b, c) Since, every set is a subset of itself, ARA for $A \in P(X)$. Therefore, R is reflexive.

$$\text{Let } ARB \Rightarrow A \subset B$$

This cannot be implied to $B \subset A$.

For instance, if $A = \{1, 2\}$ and $B = \{1, 2, 3\}$, then it cannot be implied that B is related to A . Therefore, R is not symmetric.

Further, if ARB and BRC , then $A \subset B$ and $B \subset C$.

$$\Rightarrow A \subset C \Rightarrow ARC$$

Therefore, R is transitive.

Hence, R is not an equivalence relation, since it is not symmetric.

➤ **Example 13.** A relation R on the set of complex numbers is defined by $z_1 R z_2 \Leftrightarrow$

$$\frac{z_1 - z_2}{z_1 + z_2} \text{ is real, then } R \text{ is}$$

- (a) equivalence relation
(b) only reflexive
(c) only transitive
(d) None of the above

Sol. (a) $z_1 R z_1 \Leftrightarrow \frac{z_1 - z_1}{z_1 + z_1}, \forall z_1 \in \mathbb{C}$

$\Rightarrow 0$ is real.

$\therefore R$ is reflexive.

$$z_1 R z_2 \Leftrightarrow \frac{z_1 - z_2}{z_1 + z_2} \text{ is real.}$$

$$\Rightarrow -\left(\frac{z_1 - z_2}{z_1 + z_2}\right) \text{ is real.}$$

$$\Rightarrow \left(\frac{z_2 - z_1}{z_2 + z_1}\right) \text{ is real.}$$

$$\Rightarrow z_2 R z_1, \forall z_1, z_2 \in \mathbb{C}$$

$\therefore R$ is symmetric.

Let $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$ and $z_3 = a_3 + ib_3$,

where $a_1, b_1, a_2, b_2, a_3, b_3 \in \mathbb{R}$

$$\text{Now, } z_1 R z_2 \Rightarrow \frac{z_1 - z_2}{z_1 + z_2} \text{ is real.}$$

$$\Rightarrow \frac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \text{ is real.}$$

$$\Rightarrow \frac{(a_1 - a_2) + i(b_1 - b_2)}{(a_1 + a_2) + i(b_1 + b_2)} \times \frac{(a_1 + a_2) - i(b_1 + b_2)}{(a_1 + a_2) - i(b_1 + b_2)} \text{ is real.}$$

$$\Rightarrow \frac{(a_1 - a_2)(a_1 + a_2) + (b_1 - b_2)(b_1 + b_2) + i[(b_1 - b_2)(a_1 + a_2) - (a_1 - a_2)(b_1 + b_2)]}{(a_1 + a_2)^2 + (b_1 + b_2)^2} \text{ is real.}$$

$$\Rightarrow (a_1 + a_2)(b_1 - b_2) - (a_1 - a_2)(b_1 + b_2) = 0$$

[for purely real, imaginary part = 0]

$$\Rightarrow 2a_2b_1 - 2b_2a_1 = 0$$

$$\Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2}$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\text{Similarly, } z_2 R z_3 \Rightarrow \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$z_1 R z_2 \text{ and } z_2 R z_3 \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2}$$

$$\text{and } \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\Rightarrow \frac{a_1}{b_1} = \frac{a_3}{b_3}$$

$$\Rightarrow z_1 R z_3$$

$\therefore R$ is transitive.

Hence, R is equivalence relation.

Work Book Exercise 3.1

1 A relation from P to Q is

a universal set of $P \times Q$

b $P \times Q$

an equivalent set of $P \times Q$

a subset of $P \times Q$

2 Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. If relation R from A to B is given by $R = \{(1, 3), (2, 5), (3, 3)\}$.

Then, R^{-1} is

$\{(3, 3), (3, 1), (5, 2)\}$

$\{(1, 3), (2, 5), (3, 3)\}$

$\{(1, 3), (5, 2)\}$

None of the above

3 Let $R = \{(a, a)\}$ be a relation on a set A . Then,

R is

symmetric

anti-symmetric

symmetric and anti-symmetric

neither symmetric nor anti-symmetric

4 The relation R defined on the set

$A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by

$\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$

$\{(2, 2), (3, 2), (4, 2), (2, 4)\}$

$\{(3, 3), (3, 4), (5, 4), (4, 3), (3, 1)\}$

All of the above

5 For real numbers x and y , we write xRy

$\Leftrightarrow x - y + \sqrt{2}$ is an irrational number. Then, the relation R is

reflexive

transitive

symmetric

None of these

6 Let $R = \{(x, y) : x^2 + y^2 = 1, x, y \in \mathbb{R}\}$ be a relation in $\mathbb{R}$. The relation R is

reflexive

transitive

symmetric

anti-symmetric

7 For the set $A = \{1, 2, 3\}$, define a relation R in the set A as

$$R = \{(1, 1), (2, 2), (3, 3), (1, 3)\}$$

Then, the ordered pair to be added to R to make it the smallest equivalence relation, is

$(1, 3)$

$(3, 1)$

$(2, 1)$

$(1, 2)$

8 Let $A = \{1, 2, 3\}$. Then, number of relations containing $(1, 2)$ and $(1, 3)$, which are reflexive and symmetric but not transitive, is

1

2

3

4

9 Let R be the relation in the set $\{1, 2, 3, 4\}$ given by $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$. Choose the correct answer.

R is reflexive and symmetric but not transitive

R is reflexive and transitive but not symmetric

R is symmetric and transitive but not reflexive

R is an equivalence relation

Constants and Variables

Those quantities which are unaltered under any mathematical operation, are called constants.

Those quantities which are altered under any mathematical operation and can be opted any value, are known as variables.

Variables are of two types.

(i) Independent Variable

Those variables which can be opted any value, are called independent variables.

(ii) Dependent Variable

Those variables whose values depend upon the other variables, are called dependent variables.

e.g. Let $y = x^2 + 7x + 2$

Here, x is an independent variable whereas y is a dependent variable. Since, the value of y depends upon the value of x .

Function

Let A and B be two non-empty sets. Then, a function f from set A to set B is a rule or method or correspondence which associates elements of set A to elements of set B such that

- all elements of set A are associated to elements in set B .
- an element of set A is associated to a unique element in set B .

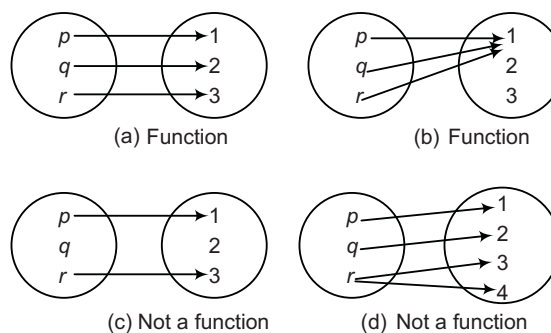
In other words, a function f from a set A to a set B associates each element of set A to a unique element of set B .

Terms such as 'map' (or mapping), 'correspondence' are used as synonyms for 'function'. If f is a function from a set A to set B , then we write $f : A \rightarrow B$ or $A \rightarrow B$.

In other words, a function f from a set A to set B associates each element of set A to a unique element of set B .

If an element $a \in A$ is associated to an element $b \in B$, then b is called the ' f -image of a ' or 'image of a under f ' or 'the value of the function f at a '. Also, a is called the pre-image of b under the function f . We write it as $b = f(a)$.

- All the independent elements (inputs) in A must have their images in set B .
- For each and every independent element in A , there corresponds one and only one image in B .
- Every function is a relation but every relation may or may not be a function.

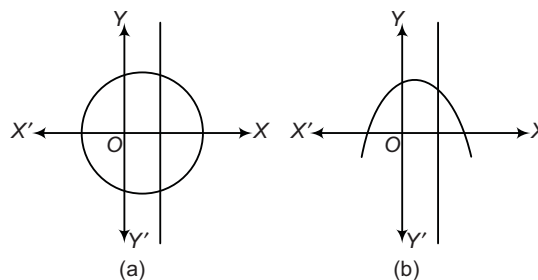


- Function cannot be multivalued.
- If A and B have m and n elements respectively, then the number of functions from A to B is n^m .

Vertical Parallel Line Test (VPL Test)

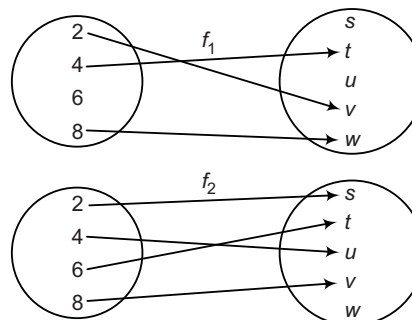
This is a geometrical test to check whether a relation is a function or not.

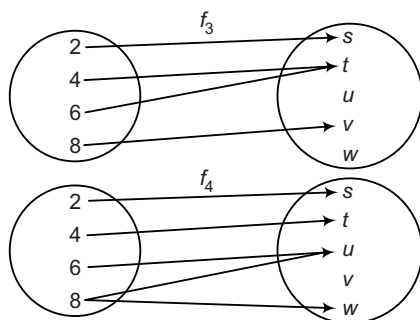
'If we draw a vertical parallel line, i.e. any line parallel to Y -axis, then if this line intersects the graph of the expression in more than one point, then the expression is a relation else, if it intersects at only one point, the expression is a function.'



In figure (a), the vertical parallel line intersects the curve at two points. Thus, the expression is a relation whereas in figure (b), the vertical parallel line intersects the curve at one point. Thus, the expression is a function.

Example 14. Let $A = \{2, 4, 6, 8\}$ and $B = \{s, t, u, v, w\}$ be two sets and f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in the following figures:





Then, which of the following is correct?

- (a) f_1 and f_2 are functions
 (b) f_3 and f_4 are functions
 (c) f_1 and f_4 are functions
 (d) f_2 and f_3 are functions

Sol. (d) Here, we see that f_1 is not a function from set A to set B , since there is an element $6 \in A$ which is not associated to any element of B but f_2 and f_3 are the functions from A to B , because under f_2 and f_3 each elements in A is associated to a unique element in B . But f_4 is not function from A to B because an element $8 \in A$ is associated to two elements u and w in B .

Domain and Range of a Real Function

Domain

Generally, real functions in calculus are described by some formulae and their domains are not explicitly stated. In such cases, in order to find the domain of a function f (say), we find all real numbers x for which $f(x)$ is a real valued.

Rules for Finding Domain

- i. Expression under even root (i.e. square root, fourth root) ≥ 0 .
- ii. Denominator $\neq 0$
- iii. If domains of $y = f(x)$ and $y = g(x)$ are D_1 and D_2 respectively, then the domain of $f(x) \pm g(x)$ or $f(x) \cdot g(x)$ is $D_1 \cap D_2$, while domain of $\frac{f(x)}{g(x)}$ is $D_1 \cap D_2 - [g(x) = 0]$.

➤ **Example 15.** The domain of the function

$$f(x) = \sqrt{\frac{(x+1)(x-3)}{x-2}} \text{ is}$$

- (a) $[-1, 2) \cup [3, \infty)$ (b) $(-1, 2) \cup [3, \infty)$
 (c) $[-1, 2] \cup [3, \infty)$ (d) None of these

Sol. (a) $f(x) = \sqrt{\frac{(x+1)(x-3)}{x-2}} = \sqrt{\frac{(x+1)(x-3)(x-2)}{x-2}}$

$f(x)$ is defined, if $(x+1)(x-3)(x-2) \geq 0$
 or $\{x - (-1)\} (x - 2) (x - 3) \geq 0$

$$\Rightarrow -1 \leq x \leq 2 \text{ or } x \geq 3$$

Also, $x - 2 \neq 0 \Rightarrow x \neq 2$
 $\therefore$ Domain of $f = \{[-1, 2) \cup [3, \infty)\} - \{2\}$
 $= [-1, 2) \cup [3, \infty)$

Codomain

The set of all possible outcomes for the function f is called the codomain of the function f .

In other words, if f is a function from A to B , i.e. $f: A \rightarrow B$, then set B is called codomain of f .

e.g. Let $f: [a, b, c] \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a function, then the codomain of f is $\{\alpha, \beta, \gamma, \delta\}$.

Range

The range of a function $f(x)$ is the set of values of $f(x)$ which attains at points in its domain. For a real function, the codomain is always a subset of R . So, range of a real function f is the set of all points y such that $y = f(x)$, where $x \in \text{dom } f(x)$.

Rules for Finding Range

First of all, find the domain of $y = f(x)$.

- i. If domain $\in$ finite number of points
Then, range $\in$ set of corresponding $f(x)$ values.
- ii. If domain $\in R$ or $R - \{\text{some finite points}\}$
Then, express x in terms of y .
From this, find y for x to be defined, i.e. find the value of y for which x exists.
- iii. If domain $\in$ a finite interval, then find the least and greatest values for range using monotonicity.

- ✎ • Range is the subset of codomain.
- Range can never exceed codomain for a given function.

➤ **Example 16.** The range of the function

$$y = \frac{x}{1+x^2} \text{ is}$$

- (a) $\left[0, \frac{1}{2}\right]$ (b) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
 (c) $\left[-\frac{1}{2}, 0\right]$ (d) None of these

Sol. (b) $y = \frac{x}{1+x^2}$, which is defined for all real x .

Domain of $y = (-\infty, \infty)$

We have, $y = \frac{x}{1+x^2}$

$$\Rightarrow y + yx^2 = x \Rightarrow x^2y - x + y = 0$$

For x to be real, $1 - 4y^2 \geq 0, y \neq 0$

$$\Rightarrow (1 - 2y)(1 + 2y) \geq 0, y \neq 0$$

$$\Rightarrow (2y - 1)(2y + 1) \leq 0, y \neq 0$$

$$\Rightarrow -\frac{1}{2} \leq y \leq \frac{1}{2}, y \neq 0$$

Also, if $y = 0$, then $x = 0$ so that $y = 0$ is also in the range.

$$\therefore \text{Range of } y = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

➤ **Example 17.** Let $f = \left\{ \left(x, \frac{x^2}{1+x^2} \right) : x \in R \right\}$ be

a function from R to R . Then, the range of f is

(a) $(0, 1)$ (b) $R - \{0, 1\}$ (c) $[0, 1)$ (d) $[0, 1]$

Sol. (c) We have, $f(x) = \frac{x^2}{1+x^2}$

$$\text{Let } f(x) = y, \text{ then } y = \frac{x^2}{1+x^2} \Rightarrow x^2 y + y = x^2$$

$$\Rightarrow y = x^2(1-y) \Rightarrow x = \sqrt{\frac{y}{1-y}}, x \in R$$

$$\text{Now, } \frac{y}{1-y} \geq 0 \Rightarrow \frac{y}{y-1} \leq 0$$

$$\Rightarrow 0 \leq y < 1 \Rightarrow y \in [0, 1)$$

Hence, range of $f = [0, 1)$

Algebra of Functions

Given functions $f : D_1 \rightarrow R$ and $g : D_2 \rightarrow R$, we describe functions $f + g$, $f - g$, fg and f/g as follows:

$f + g : D \rightarrow R$ is a function defined by

$$(f + g)(x) = f(x) + g(x)$$

$f - g : D \rightarrow R$ is a function defined by

$$(f - g)(x) = f(x) - g(x)$$

$fg : D \rightarrow R$ is a function defined by

$$(fg)(x) = f(x)g(x)$$

$\frac{f}{g} : C \rightarrow R$ is a function defined by

$$\left(\frac{f}{g} \right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

where $D = D_1 \cap D_2$ and $C = \{x \in D : g(x) \neq 0\}$

➤ **Example 18.** If $f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & x \geq 3 \end{cases}$ and

$g(x) = \begin{cases} x - 3, & x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$, then the function $\frac{f}{g}$ is

$$(a) \begin{cases} \frac{x-1}{x-3}, & x < 3 \\ \frac{x-4}{x^2+2x+2}, & 3 < x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$$

$$(b) \begin{cases} \frac{x-1}{x-3}, & x < 3 \\ \frac{x-4}{x-3}, & 3 \leq x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$$

$$(c) \begin{cases} \frac{x-1}{x-2}, & x < 3 \\ x-4, & 3 < x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$$

(d) None of the above

Sol. (a) We define the functions $f(x)$ and $g(x)$ in the intervals as shown below:

$$f(x) = \begin{cases} x^2 - 4x + 3, & x < 3 \\ x - 4, & 3 \leq x < 4 \\ x - 4, & x \geq 4 \end{cases}$$

$$\text{and } g(x) = \begin{cases} x - 3, & x < 3 \\ x - 3, & 3 \leq x < 4 \\ x^2 + 2x + 2, & x \geq 4 \end{cases}$$

$$\therefore \frac{f(x)}{g(x)} = \begin{cases} \frac{x-1}{x-3}, & x < 3 \\ \frac{x-4}{x-3}, & 3 < x < 4 \\ \frac{x-4}{x^2+2x+2}, & x \geq 4 \end{cases}$$

Explicit and Implicit Functions

A function is said to be an explicit function, if the dependent variable y is expressible completely in terms of independent variable.

A function is said to be an implicit function, if the dependent variable is not directly expressible in terms of the independent variable x .

e.g. $y = 3e^x - x^2 + 2$, $x \in R$ is an explicit function while $ye^{x^2} + xe^{-y^2} = e^{a^2}$ is an implicit function.

Basic Functions

(i) Constant Function

A function which does not change as its parameters vary, i.e. the function whose rate of change is zero. In short, we can say that a constant function is a function that always gives or returns the same value.

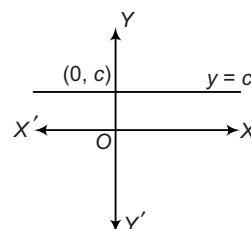
Domain $x \in (-\infty, \infty)$ i.e. $x \in R$

Range $y \in \{c\}$

Nature (a) Many-one

(b) Into function for codomain as a set of real number, i.e. R .

(c) Neither monotonically increasing nor monotonically decreasing in its domain.



➤ The function graph of one-dimensional constant function is a straight line.

(ii) Algebraic Functions

Polynomial Function

A polynomial function is the function of the form

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

The value of n must be a non-negative integer, i.e. it must be a whole number and

$a_0, a_1, a_2, \dots, a_n \in R$. The degree of the polynomial is the highest value for n , where $a_n \neq 0$, i.e. n .

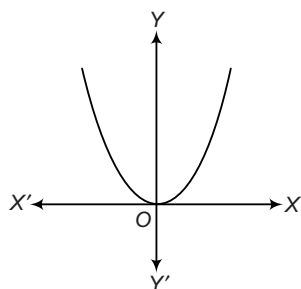
Domain The entire number line, i.e. $x \in R$.

Range Varies from function to function

Special Cases

Case I If n is even

Let $y = x^2$

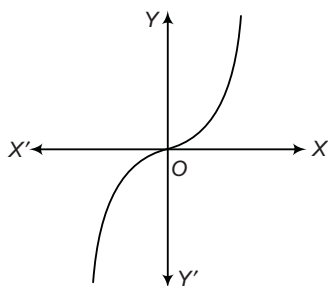


Range $y \in [0, \infty)$

- Nature**
- (a) Many-one
 - (b) Onto function for codomain as a set of non-negative real numbers.
 - (c) Monotonically decreasing for all $x \in (-\infty, 0)$.
 - (d) Monotonically increasing for all $x \in (0, \infty)$.
 - (e) Even function

Case II If n is odd

Let $y = x^3$



Range $y \in R$

- Nature**
- (a) One-one
 - (b) Onto function for codomain as a set of real numbers, i.e. R .
 - (c) Monotonically increasing throughout its domain.
 - (d) Odd function

Rational Function

A rational function is the function of the form

$$y = \frac{f(x)}{g(x)} = \frac{a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n}{b_0x^m + b_1x^{m-1} + b_2x^{m-2} + \dots + b_m}$$

The value of n must be a non-negative integer, i.e. it must be a whole number and $a_i \in R, b_j \in R, \forall 1 \leq i \leq n$ and $1 \leq j \leq m$, respectively.

Also, $g(x) \neq 0$

Then, $g(x) = 0$ will give all those points, where the function is discontinuous.

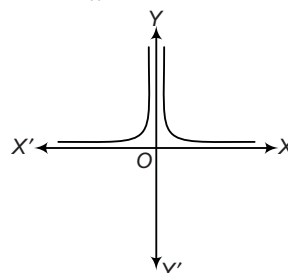
Domain $x \in R - \{x : x = \alpha, \forall (x) = 0\}$

Range Varies from function to function

Special Cases

Case I If n is even

Let $y = \frac{1}{x^2}$



Domain $x \in R - \{0\}$

Range $y \in (0, \infty)$

- Nature**
- (a) Many-one
 - (b) Onto function for codomain as a set of positive real numbers.
 - (c) Monotonically increasing for all $x \in (-\infty, 0)$.
 - (d) Monotonically decreasing for all $x \in (0, \infty)$.
 - (e) Even function

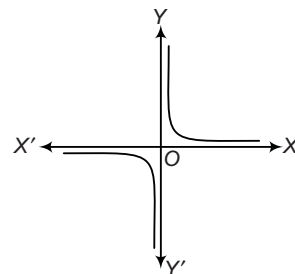
Case II If n is odd

Let $y = \frac{1}{x^3}$

Domain $x \in R - \{0\}$

Range $y \in R - \{0\}$

- Nature**
- (a) One-one
 - (b) Into function for codomain as a set of real numbers, i.e. R .
 - (c) Monotonically decreasing throughout its domain.
 - (d) Odd function



Irrational Function

An irrational function is the function of the form

$$y = \{f(x)\}^{p/q}$$

where, p and q are integers, have no common factor and also $q \neq 0, q > 0$.

Domain Varies from function to function.

Special Cases

Case I If q is even

For q being an even number, the function

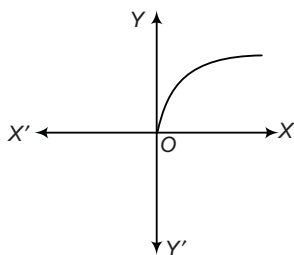
$$y = \{f(x)\}^{p/q}$$

is defined if and only if $f(x) \geq 0$

provided $\left(\frac{p}{q}\right)$ is positive else $f(x) > 0$

provided $\left(\frac{p}{q}\right)$ is negative.

Let $y = \sqrt{x}$



Domain $x \in [0, \infty)$

Range $y \in [0, \infty)$

Nature (a) One-one
(b) Into function for codomain as a set of real numbers
(c) Monotonically increasing throughout its domain

Case II If q is odd

For q being an odd number, the function

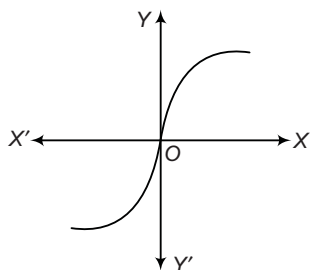
$$y = \{f(x)\}^{p/q}$$

is defined if and only if $x \in R$ provided

$\left(\frac{p}{q}\right)$ is positive else

$x \in R - \{x : x = \alpha, \forall f(\alpha) = 0\}$

Let $y = \sqrt[3]{x}$



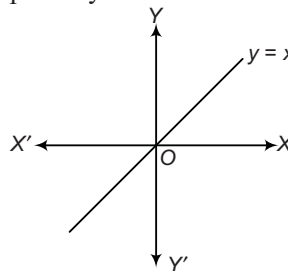
Domain $x \in R$

Range $y \in R$

Nature (a) One-one
(b) Onto function for codomain as a set of real numbers, i.e. R .
(c) Monotonically increasing throughout its domain.

Identity Function

Identity function is of the form $y = f(x) = x$ which is a straight line passing through origin and having slope unity.



Domain $x \in (-\infty, \infty)$, i.e. $x \in R$

Range $y \in (-\infty, \infty)$, i.e. $y \in R$

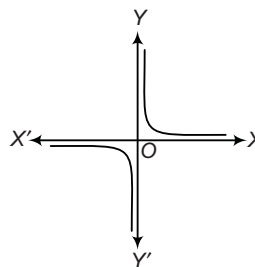
Nature (a) One-one
(b) Onto function for codomain as a set of real numbers, i.e. R .
(c) Monotonically increasing throughout its domain.
(d) Odd function

Reciprocal Function

The reciprocal function is the inverse of an identity function such that

$$y = f(x) = \frac{1}{x}$$

which is an equation of a rectangular hyperbola whose asymptotes (here X -axis and Y -axis) intersect at right angles.



Domain $x \in R - \{0\}$

Range $y \in R - \{0\}$

Nature (a) One-one
(b) Into function for codomain as a set of real numbers, i.e. R .
(c) Monotonically decreasing throughout its domain.
(d) Odd function

➤ **Example 19.** The domain of the function

$$f(x) = \sqrt{x-1} + \sqrt{5-x} \text{ is}$$

- (a) $[1, \infty)$
 (b) $(-\infty, 5)$
 (c) $(1, 5)$
 (d) $[1, 5]$

Sol. (d) $\sqrt{x-1}$ is defined for $x-1 \geq 0$

$$\text{i.e. } x \geq 1 \quad \dots(i)$$

$$\sqrt{5-x} \text{ is defined for } 5-x \geq 0, \text{ for } x \leq 5 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\text{Domain of } f = (-\infty, 5] \cap [1, \infty) = [1, 5]$$

➤ **Example 20.** The range of the function

$$f(x) = \frac{2x-2}{x^2-2x+3} \text{ is}$$

- (a) $\left[-\frac{1}{2}, \frac{1}{2}\right]$
 (b) $\left[-\frac{1}{\sqrt{2}}, 0\right]$
 (c) $\left[-\frac{1}{2}, 0\right]$
 (d) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Sol. (d) Here, $f(x) = \frac{2x-2}{x^2-2x+3}$

$$\text{Let } y = f(x)$$

$$\text{i.e. } y = \frac{2x-2}{x^2-2x+3}$$

$$\Rightarrow yx^2 - 2(y+1)x + 3y+2 = 0$$

which is a quadratic equation in x . For above quadratic equation to have real roots, $\Delta > 0$.

$$\therefore (y+1)^2 - y(3y+2) \geq 0$$

$$\Rightarrow y^2 + 1 + 2y - 3y^2 - 2y \geq 0$$

$$\Rightarrow y^2 \leq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \leq y \leq \frac{1}{\sqrt{2}}$$

$$\text{Hence, the range of } f(x) \text{ is } \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right].$$

➤ **Example 21.** The domain of the function

$$f(x) = \sqrt{1-\sqrt{1-\sqrt{1-x^2}}} \text{ is}$$

- (a) $(-\infty, 1)$ (b) $(-1, \infty)$ (c) $[0, 1]$ (d) $[-1, 1]$

Sol. (d) $f(x)$ is defined, if

$$1-\sqrt{1-\sqrt{1-x^2}} \geq 0, 1-\sqrt{1-x^2} \geq 0 \text{ and } 1-x^2 \geq 0$$

$$\Rightarrow 1-x^2 \geq 0$$

$$\Rightarrow (x+1)(x-1) \leq 0$$

$$\Rightarrow -1 \leq x \leq 1$$

Clearly, for these values, the other two inequalities hold.

Thus, domain of $f = [-1, 1]$

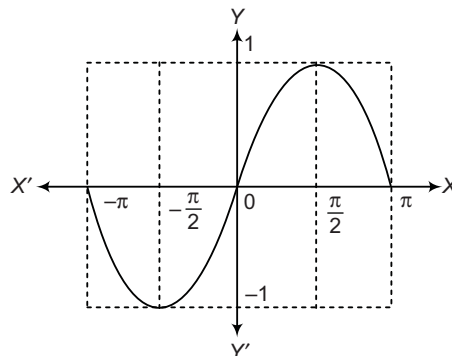
(iii) Trigonometric Function

Sine Function

Since, function is of the form $y = \sin x$.

Domain $x \in R$

Range $y \in [-1, 1]$



Nature (a) Many-one

One-one in $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) Onto function for codomain $[-1, 1]$

(c) Monotonically increasing for all

$$x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right).$$

(d) Monotonically decreasing for all

$$x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{5\pi}{2}\right).$$

(e) Odd function

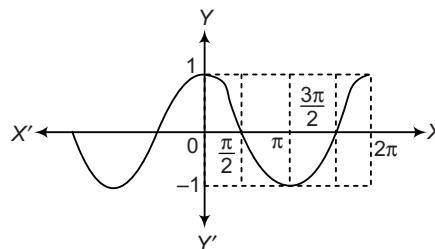
(f) Periodic with period 2π .

Cosine Function

Cosine function is of the form $y = \cos x$.

Domain $x \in R$

Range $y \in [-1, 1]$



Nature (a) Many-one

One-one in $x \in [0, \pi]$

(b) Onto function for codomain $[-1, 1]$

(c) Monotonically increasing for all

$$x \in (2n\pi - \pi, 2n\pi).$$

(d) Monotonically decreasing for all

$$x \in (2n\pi, 2n\pi + \pi).$$

(e) Even function

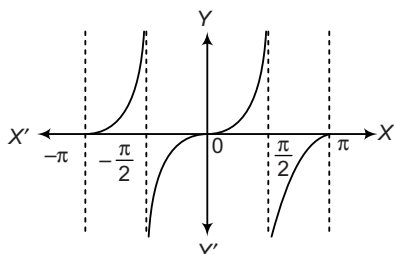
(f) Periodic with period 2π .

Tangent Function

Tangent function is of the form $y = \tan x$.

By definition, for any right angled triangle, $\tan x$ represents the ratio of perpendicular to that of its base.

Geometrically, $\tan x$ means the slope of the curve at any point (x, y) on the curve $y = f(x)$.



Domain $x \in R - \left\{ (2n+1)\frac{\pi}{2} \right\}$, where $n \in I$

Range $y \in R$

Nature (a) Many-one

One-one in $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

(b) Onto function for codomain as set of real numbers.

(c) Monotonically increasing for all $x \in R$.

(d) Odd function

(e) Periodic with period π .

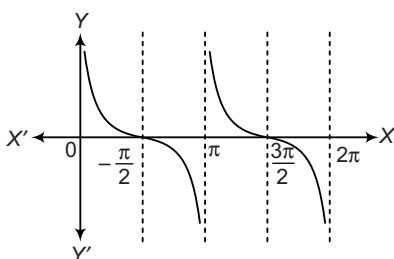
Cotangent Function

Cotangent function is of the form $y = \cot x$.

By definition, cotangent function is the reciprocal of tangent function.

Domain $x \in R - \{n\pi\}$, where $n \in I$

Range $y \in R$



Nature (a) Many-one

One-one in $x \in (0, \pi)$

(b) Onto function for codomain as set of real numbers.

(c) Monotonically decreasing for all $x \in R$.

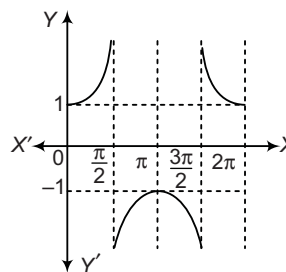
(d) Odd function

(e) Periodic with period π .

Secant Function

Secant function is of the form $y = \sec x$.

By definition, secant function is the reciprocal of cosine function.



Domain $x \in R - \left\{ (2n+1)\frac{\pi}{2} \right\}$, where $n \in I$

Range $y \in (-\infty, -1] \cup [1, \infty)$

Nature (a) Many-one

(b) Into function for codomain as a set of real numbers.

(c) Monotonically decreasing for all $x \in (2n\pi - \pi, 2n\pi)$.

(d) Monotonically increasing for all $x \in (2n\pi, 2n\pi + \pi)$.

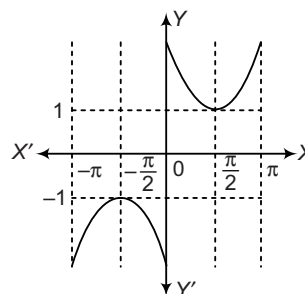
(e) Even function

(f) Periodic with period 2π .

Cosecant Function

Cosecant function is of the form $y = \operatorname{cosec} x$.

By definition, cosecant function is the reciprocal of sine function.



Domain $x \in R - \{n\pi\}$, where $n \in I$

Range $y \in (-\infty, -1] \cup [1, \infty)$

Nature (a) Many-one

One-one in $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

(b) Into function for codomain as a set of real numbers.

(c) Monotonically decreasing for all $x \in \left(2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2} \right)$.

(d) Monotonically increasing for all $x \in \left(2n\pi + \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2} \right)$

(e) Odd function

(f) Periodic with period 2π .

➤ **Example 22.** The set of all real values of x for which the function

$f(x) = \sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$ takes real values, is

- (a) $[1, 3\pi/4] \cup [7\pi/6, 6]$
 (b) $[1, 3\pi/4] \cup [6 - \pi/4, 6]$
 (c) $[1, 6]$
 (d) None of the above

Sol. (a) We have, $f(x) = \sqrt{\sin x + \cos x} + \sqrt{7x - x^2 - 6}$

$$\Rightarrow f(x) = 2^{1/4} \sqrt{\sin(x + \pi/4)} + \sqrt{7x - x^2 - 6}$$

For $f(x)$ to be real, we must have

$$\sin\left(x + \frac{\pi}{4}\right) \geq 0 \text{ and } 7x - x^2 - 6 \geq 0$$

$$\Rightarrow 2n\pi \leq x + \frac{\pi}{4} \leq (2n+1)\pi$$

and $1 \leq x \leq 6$, where $n \in \mathbb{Z}$

$$\therefore x \in \left[1, \frac{3\pi}{4}\right] \cup \left[\frac{7\pi}{6}, 6\right]$$

➤ **Example 23.** If $f(x)$ is defined on $(0, 1)$, then the domain of definition of $f(\sin x)$ is

- (a) $(2n\pi, (2n+1)\pi), n \in \mathbb{Z}$
 (b) $\left((2n+1)\frac{\pi}{2}, (2n+3)\frac{\pi}{2}\right), n \in \mathbb{Z}$
 (c) $((n-1)\pi, (n+1)\pi), n \in \mathbb{Z}$
 (d) None of the above

Sol. (a) Since, $f(x)$ is defined for all $x \in (0, 1)$. Therefore, $f(\sin x)$ is defined for

$$0 < \sin x < 1$$

$$\Rightarrow 2n\pi < x < (2n+1)\pi, n \in \mathbb{Z}$$

$$\Rightarrow x \in (2n\pi, (2n+1)\pi), n \in \mathbb{Z}$$

➤ **Example 24.** The domain of the function

$$f(x) = \frac{1}{\sqrt{|\sin x| + \sin x}} \text{ is}$$

- (a) $(-2n\pi, 2n\pi)$
 (b) $(2n\pi, (2n+1)\pi)$
 (c) $\left((4n-1)\frac{\pi}{2}, (4n+1)\frac{\pi}{2}\right)$
 (d) None of the above

Sol. (b) $f(x)$ is defined, if $|\sin x| + \sin x > 0$

$$\Rightarrow \sin x > 0 \Rightarrow 2n\pi < x < 2n\pi + \pi$$

Domain of f is $(2n\pi, (2n+1)\pi)$.

➤ **Example 25.** The range of the function

$$y = \frac{1}{2 - \sin 3x} \text{ is}$$

- (a) $\left(\frac{1}{3}, 1\right)$ (b) $\left[\frac{1}{3}, 1\right]$
 (c) $\left[\frac{1}{3}, 1\right]$ (d) None of these

Sol. (c) We have, $y = \frac{1}{2 - \sin 3x}$... (i)

Domain of $y = (-\infty, \infty)$

From Eq. (i), we have $2 - \sin 3x = \frac{1}{y}$

$$\Rightarrow \sin 3x = \frac{2y-1}{y} \Rightarrow x = \frac{1}{3} \sin^{-1}\left(\frac{2y-1}{y}\right)$$

$$\left[\begin{array}{l} \because y = \frac{1}{2 - \sin 3x} \\ \therefore y > 0 \\ \text{as } -1 \leq \sin 3x \leq 1 \end{array} \right]$$

For x to be real,

$$-1 \leq \frac{2y-1}{y} \leq 1$$

$$\Rightarrow -y \leq 2y-1 \leq y \quad [\because y > 0]$$

$$\Rightarrow 2y-1 \geq -y \text{ and } 2y-1 \leq y$$

$$\Rightarrow y \geq \frac{1}{3} \text{ and } y \leq 1$$

$$\therefore \text{Range of } y = \left[\frac{1}{3}, 1\right]$$

➤ **Example 26.** The range of the function

$$f(x) = \tan \sqrt{\frac{\pi^2}{9} - x^2} \text{ is}$$

- (a) $[0, \sqrt{3}]$ (b) $(0, \sqrt{3})$ (c) $[0, \sqrt{3})$ (d) $(0, \sqrt{3}]$

Sol. (a) Clearly, $f(x)$ is defined for $x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$.

Since, $\tan x$ is an increasing function in $\left[0, \frac{\pi}{2}\right)$

$$\text{and } 0 \leq \frac{\pi^2}{9} - x^2 \leq \frac{\pi^2}{9} \text{ for } x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$$

$$\Rightarrow 0 \leq \sqrt{\frac{\pi^2}{9} - x^2} \leq \frac{\pi}{3} \text{ for } x \in \left[-\frac{\pi}{3}, \frac{\pi}{3}\right]$$

Also, $f(-x) = f(x), \forall x \in [-\pi/3, \pi/3]$

$$\therefore \text{Range of } f = \left[f\left(-\frac{\pi}{3}\right), f(0)\right]$$

$$= \left[\tan 0, \tan \frac{\pi}{3}\right] = [0, \sqrt{3}]$$

(iv) Inverse Circular/Trigonometric Function

Arc Sine Function

Arc sine function is of the form $y = \sin^{-1} x$.

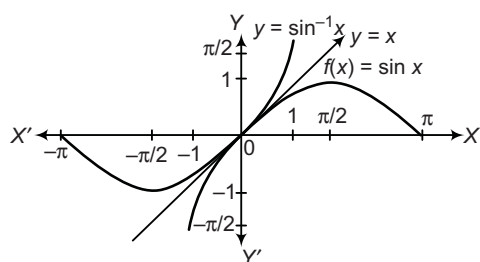
The inverse sine is the multi valued function.

Because inverse sine of 0.5 will mean either $30^\circ, 150^\circ, 390^\circ, 510^\circ, \dots$ etc., and all are valid.

But as we know that, for any function to possess an inverse, it must be bijective, so if we see the graph of sine function, it is one-one function for all

$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is the principal range for the sine function to possess an inverse.

Hence, hereafter if it is asked for inverse of sine 0.5, our answer will be 30° unless asked for more general values. Note that in the notation of $\sin^{-1} x$, is the superscript -1 denotes the inverse function, not the multiplicative inverse.



Domain $x \in [-1, 1]$

Range $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

- Nature**
- (a) One-one
 - (b) Onto function
 - (c) Monotonically increasing throughout its domain.
 - (d) Odd function.

✎ Inverse functions are symmetric in the line $y = x$.

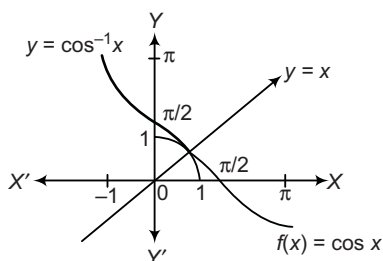
Arc Cosine Function

Arc cosine function is of the form $y = \cos^{-1} x$.

The inverse cosine is the multi valued function.

Because inverse cosine of 0.5 will mean either $-60^\circ, 60^\circ, 300^\circ, 420^\circ, \dots$, etc., and all are valid.

But as we know that, for any function to possess an inverse, it must be bijective, so if we see the graph of cosine function, then it is one-one function for all $x \in [0, \pi]$, which is the principal range for the cosine function to possess an inverse. Hence, hereafter if it is asked for inverse of cosine 0.5. our answer will be 60° unless asked for more general values.



Domain $x \in [-1, 1]$

Range $x \in [0, \pi]$

- Nature**
- (a) One-one
 - (b) Onto function
 - (c) Monotonically decreasing throughout its domain.
 - (d) Even function
- $\therefore \cos^{-1}(-x) = \pi - \cos^{-1}(x)$

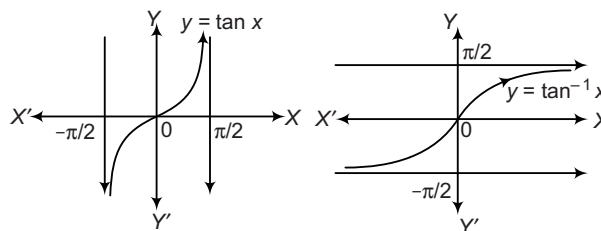
Arc Tangent Function

Arc tangent function is of the form $y = \tan^{-1} x$.

The inverse tangent is the multi valued function.

Because inverse tangent of 1 will mean either $45^\circ, 225^\circ, 405^\circ, \dots$ etc., and all are valid. But as we know that, for any function to possess an inverse, it must be bijective, so if we see the graph of tangent function, it is one-one function, for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

which is the principal range for the tangent function to possess an inverse. Hence, hereafter if it is asked for inverse of tangent 1, our answer will be only 45° unless asked for more general values.



Domain $x \in R$

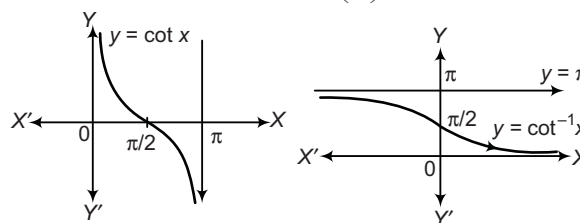
Range $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

- Nature**
- (a) One-one
 - (b) Onto function
 - (c) Monotonically increasing throughout its domain.
 - (d) Neither even nor odd

Arc Cotangent Function

Arc cotangent function is of the form $y = \cot^{-1} x$.

By definition, $\cot^{-1} x = \tan^{-1} \left(\frac{1}{x}\right)$



Domain $x \in R$

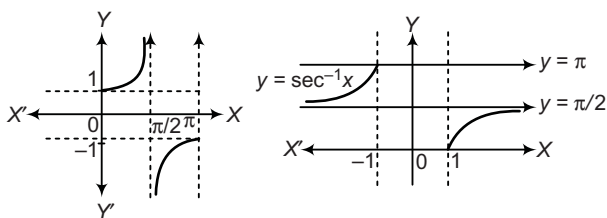
Range $y \in (0, \pi)$

- Nature**
- (a) One-one
 - (b) Onto function
 - (c) Monotonically decreasing throughout its domain.
 - (d) Neither even nor an odd function

Arc Secant Function

Arc secant function is of the form $y = \sec^{-1} x$.

By definition, $\sec^{-1} x = \cos^{-1} \left(\frac{1}{x}\right)$



Domain $x \in (-\infty, -1] \cup [1, \infty)$

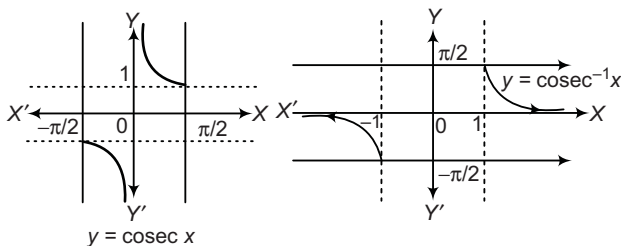
Range $y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$

Nature (a) One-one
(b) Onto function
(c) Monotonically increasing throughout its domain.
(d) Even function

Arc Cosecant Function

Arc cosecant function is of the form $y = \text{cosec}^{-1}x$.

By definition, $\text{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$



Domain $x \in (-\infty, -1] \cup [1, \infty)$

Range $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

Nature (a) One-one
(b) Onto function
(c) Monotonically decreasing throughout its domain.
(d) Odd function

Example 27. The function

$f(x) = \cot^{-1} \sqrt{x(x+3)} + \cos^{-1} \sqrt{x^2 + 3x + 1}$ is defined on the set S , where S is equal to

- (a) $\{-3, 0\}$
(b) $[-3, 0]$
(c) $[0, 3]$
(d) ϕ

Sol. (d) We know that, $\cot^{-1}x$ is defined for all $x \in \mathbb{R}$ and $\cos^{-1}x$ is defined for all $x \in [-1, 1]$. Therefore, $f(x)$ is defined, if

$$\begin{aligned} & x(x+3) > 0 \\ \text{and} & 0 \leq x^2 + 3x + 1 \leq 1 \\ \Rightarrow & x^2 + 3x > 0 \\ \text{and} & -1 \leq x^2 + 3x \leq 0 \\ \Rightarrow & S = \phi \end{aligned}$$

Example 28. The domain of the function

$$f(x) = \frac{\sqrt{9-x^2}}{\sin^{-1}(3-x)} \text{ is}$$

- (a) $(2, 3)$ (b) $[2, 3)$
(c) $(2, 3]$ (d) None of these

Sol. (b) $\sqrt{9-x^2}$ is defined for $9-x^2 \geq 0$

$$\begin{aligned} \Rightarrow & (3-x)(3+x) \geq 0 \\ \Rightarrow & (x-3)(x+3) \leq 0 \\ \Rightarrow & -3 \leq x \leq 3 \quad \dots(i) \end{aligned}$$

$\sin^{-1}(3-x)$ defined for $-1 \leq 3-x \leq 1$

$$\begin{aligned} \Rightarrow & -4 \leq -x \leq -2 \\ \Rightarrow & 2 \leq x \leq 4 \quad \dots(ii) \end{aligned}$$

$$\text{Also, } \sin^{-1}(3-x) \neq 0 \Rightarrow 3-x \neq 0 \text{ or } x \neq 3 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\text{Domain of } f = ([-3, 3] \cap [2, 4]) - \{3\} = [2, 3)$$

Example 29. The domain of the function

$$f(x) = \sin^{-1}\left(\frac{4}{3+2\cos x}\right) \text{ is}$$

- (a) $2n\pi - \frac{\pi}{3} \leq x \leq 2n\pi + \frac{\pi}{3}, n \in I$
(b) $0 \leq x \leq 2n\pi + \frac{\pi}{6}, n \in I$
(c) $2n\pi - \frac{\pi}{6} < x < 2n\pi + \frac{\pi}{6}, n \in I$
(d) $2n\pi - \frac{\pi}{6} \leq x \leq 0, n \in I$

Sol. (a) $f(x)$ is defined, if $-1 \leq \frac{4}{3+2\cos x} \leq 1$

Since, $3+2\cos x > 0$ for all real x .

$$\therefore \frac{4}{3+2\cos x} \leq 1$$

$$\Rightarrow \cos x \geq \frac{1}{2}$$

$$\Rightarrow 2n\pi - \frac{\pi}{3} \leq x \leq 2n\pi + \frac{\pi}{3}, n \in I$$

Example 30. Let $f(x) = \cos^{-1}\left(\frac{x^2}{x^2+1}\right)$.

Then, the range of f is

- (a) $(0, \pi/2]$
(b) $[-\pi/2, \pi/2]$
(c) $[-\pi/2, 0]$
(d) None of the above

Sol. (a) Since, $0 \leq \frac{x^2}{x^2+1} < 1, \forall x \in \mathbb{R}$

$$\text{Therefore, } 0 < \cos^{-1}\left(\frac{x^2}{x^2+1}\right) \leq \frac{\pi}{2}$$

$$\Rightarrow 0 < f(x) \leq \frac{\pi}{2}$$

$$\therefore \text{Range of } f(x) = (0, \pi/2]$$

➤ **Example 31.** Range of

$$f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x \text{ is}$$

- (a) $\left(\frac{\pi}{4}, \frac{3\pi}{2}\right)$ (b) $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
 (c) $\left\{\frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (d) None of these

Sol. (c) $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$, clearly domain, if $f(x)$ is $x = \pm 1$. Thus, range is $\{f(1), f(-1)\}$, i.e. $\left\{\frac{\pi}{4}, \frac{3\pi}{4}\right\}$.

(v) **Exponential Function**

A function in the form of $f(x) = a^x$ is an exponential function. It is the inverse of logarithmic function. The value of the function depends upon the value of a .

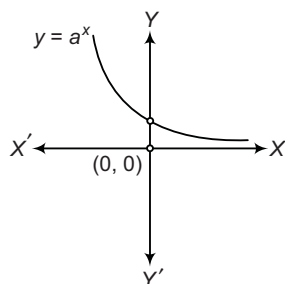
For $0 < a < 1$, the function is decreasing and for $a > 1$, the function is increasing.

Domain $x \in R$

Range $y \in R^+$

Special Cases

Case I If $0 < a < 1$



- Nature** (a) One-one
 (b) Onto function for codomain as a set of positive real numbers.
 (c) Monotonically decreasing throughout its domain.

Case II If $a > 1$

- Nature** (a) One-one
 (b) Onto function for codomain as a set of positive real numbers.
 (c) Monotonically increasing throughout its domain.

➤ **Example 32.** Let $f : R \rightarrow R$ be such that

$$f(x) = 2^x, \text{ then}$$

- (a) range of f is R (b) $\{x : f(x) = 1\} = \{0, 1\}$
 (c) $f(x + y) = f(x) \cdot f(y)$ (d) None of these

Sol. (c) Since, 2^x is positive for every $x \in R$.

So, $f(x) = 2^x$ is a positive real number for every $x \in R$.

$\therefore$ Range of $f = R^+$, i.e. $(0, \infty)$

$$\text{Now, } 2^x = 1 \Rightarrow 2^x = 2^0 \Rightarrow x = 0$$

$$\therefore \{x : f(x) = 1\} = \{0\}$$

$$\text{We have, } f(x + y) = 2^{x+y} = 2^x \cdot 2^y = f(x) \cdot f(y)$$

$$\therefore f(x + y) = f(x) \cdot f(y), \forall x, y \in R$$

(vi) **Logarithmic Function**

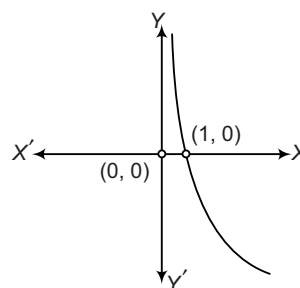
A function in the form of $f(x) = \log_a x$ is a logarithmic function. The value of the function depends upon the value of a . For $0 < a < 1$, the function is decreasing and for $a > 1$, the function is increasing.

Domain $x > 0$, i.e. $x \in (0, \infty)$

Range $y \in R$

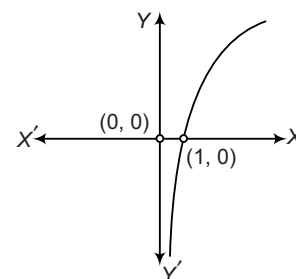
Special Cases

Case I If $0 < a < 1$



- Nature** (a) One-one
 (b) Onto function for codomain as a set of real numbers.
 (c) Monotonically decreasing throughout its domain.

Case II If $a > 1$



- Nature** (a) One-one
 (b) Onto function for codomain as a set of real numbers.
 (c) Monotonically increasing throughout its domain.

Properties of Logarithmic Function

i. $\log_e(ab) = \log_e a + \log_e b; a, b > 0$

ii. $\log_e\left(\frac{a}{b}\right) = \log_e a - \log_e b; a, b > 0$

iii. $\log_e a^m = m \log_e a; a > 0, m \in R$

iv. $\log_a a = 1; a > 0, a \neq 1$

- v. $\log_{b^m} a = \frac{1}{m} \log_b a$; $a, b > 0, b \neq 1$ and $m \in \mathbb{R} - \{0\}$
- vi. $\log_b a = \frac{1}{\log_a b}$; $a, b > 0$ and $a, b \neq \{1\}$
- vii. $\log_b a = \frac{\log_m a}{\log_m b}$; $a, b, m > 0$ and $m \neq 1$
- viii. $a^{\log_a m} = m$; $a > 0 \neq \{1\}, m > 0$
- ix. $a^{\log_c b} = b^{\log_c a}$; $a, b, c > 0$ and $c \neq 1$
- x. If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, & \text{if } m > 1 \\ x < y, & \text{if } 0 < m < 1 \end{cases}$
 $m, x, y > 0, m \neq 1$
- xi. $\log_m a = b \Rightarrow a = m^b$; $m, a > 0, m \neq 1$,
 $b \in \text{real number}$
- xii. $\log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if } m > 1 \\ a < m^b, & \text{if } 0 < m < 1 \end{cases}$
- xiii. $\log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if } m > 1 \\ a > m^b, & \text{if } 0 < m < 1 \end{cases}$
- xiv. $\log_{x^a} y^b = \frac{b}{a} \log_x y$

➤ **Example 33.** The domain of definition of the

function $y = \sqrt{\log_{10} \left(\frac{5x - x^2}{4} \right)}$ is

- (a) $[1, 4]$ (b) $(1, 4)$
(c) $[1, 4)$ (d) $(1, 4]$

Sol. (a) For y to be defined, we must have

$$\log_{10} \left(\frac{5x - x^2}{4} \right) \geq 0 \Rightarrow \frac{5x - x^2}{4} \geq 10^0$$

$$\Rightarrow 5x - x^2 \geq 4 \Rightarrow x^2 - 5x + 4 \leq 0$$

$$\Rightarrow (x - 1)(x - 4) \leq 0$$

$$\Rightarrow 1 \leq x \leq 4 \quad \dots(i)$$

$$\text{Now, } \frac{5x - x^2}{4} > 0 \Rightarrow 5x - x^2 > 0$$

$$\Rightarrow x(x - 5) < 0$$

$$\Rightarrow 0 < x < 5 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\text{Domain of } f = [1, 4] \cap (0, 5) = [1, 4]$$

➤ **Example 34.** The domain of the function

$$f(x) = \log_2 \log_3 \log_4 x \text{ is}$$

- (a) $[4, \infty)$ (b) $(4, \infty)$
(c) $(-\infty, 4)$ (d) None of these

Sol. (b) $f(x)$ is defined, if $\log_3 \log_4 x > 0, \log_4 x > 0, x > 0$

$$\Rightarrow \log_4 x > 3^0 = 1, x > 4^0 \text{ and } x > 0$$

$$\Rightarrow x > 4^1, x > 1 \text{ and } x > 0 \Rightarrow x > 4$$

$$\therefore \text{Domain of } f = (4, \infty)$$

➤ **Example 35.** The range of the function $f(x) = \log_e (3x^2 - 4x + 5)$ is

(a) $\left(-\infty, \log_e \frac{11}{3}\right]$ (b) $\left[\log_e \frac{11}{3}, \infty\right)$

(c) $\left[-\log_e \frac{11}{3}, \log_e \frac{11}{3}\right]$ (d) None of these

Sol. (b) $f(x)$ is defined, if $3x^2 - 4x + 5 > 0$

$$\Rightarrow 3 \left[x^2 - \frac{4}{3}x + \frac{5}{3} \right] > 0$$

$$\Rightarrow 3 \left[\left(x - \frac{2}{3} \right)^2 + \frac{11}{9} \right] > 0,$$

which is true for all real x .

$\therefore$ Domain of $f = (-\infty, \infty)$

$$\text{Let } y = \log_e (3x^2 - 4x + 5)$$

$$\Rightarrow e^y = 3x^2 - 4x + 5$$

$$\Rightarrow 3x^2 - 4x + (5 - e^y) = 0$$

For x to be real,

$$16 - 12(5 - e^y) \geq 0$$

$$\Rightarrow 12e^y \geq 44 \Rightarrow e^y \geq \frac{11}{3}$$

$$\Rightarrow y \geq \log_e \frac{11}{3}$$

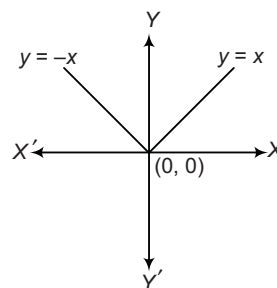
$$\therefore \text{Range of } f = \left[\log_e \frac{11}{3}, \infty \right)$$

(vii) Some Special Functions

Absolute Valued Function or Modulus Function

A function is of the form

$$y = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$



No part of the curve line lies below X -axis.

Domain $x \in \mathbb{R}$

Range $y \in \mathbb{R}^+ + \{0\} = [0, \infty)$

Nature (a) Many-one

(b) Onto function for codomain
 $= \mathbb{R}^+ + \{0\}$

(c) Monotonically increasing for all $x > 0$

(d) Monotonically decreasing for all $x < 0$

(e) Even function

(f) Continuous throughout $\mathbb{R}$.

(g) Non-differentiable at $x = 0$

Properties of Modulus Function

- i. For any real number x , $\sqrt{x^2} = |x|$.
- ii. If a and b are positive real numbers, then
 - (a) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$
 - (b) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a$ or $x \geq a$
 - (c) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b$
 $\Leftrightarrow x \in [-b, -a] \cup [a, b]$
- iii. $|x \pm y| \leq |x| + |y|$
- iv. $|x \pm y| \geq ||x| - |y||$

➤ **Example 36.** The domain of definition of

$$f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$$

- (a) $(-\infty, \infty) - [-1, 1]$
- (b) $(-\infty, \infty) - [-2, 2]$
- (c) $[-1, 1] \cup (-\infty, -2) \cup (2, \infty)$
- (d) None of the above

Sol. (c) $f(x)$ is defined, if $\frac{1-|x|}{2-|x|} \geq 0$ and $2-|x| \neq 0$

$$\Rightarrow \frac{(1-|x|)(2-|x|)}{(2-|x|)^2} \geq 0 \text{ and } x \neq -2, 2$$

$$\Rightarrow (|x| - 1)(|x| - 2) \geq 0$$

$$\text{and } x \neq -2, 2$$

$$\Rightarrow |x| \leq 1 \text{ or } |x| > 2$$

$$\Rightarrow -1 \leq x \leq 1 \text{ or } (x < -2 \text{ or } x > 2)$$

$$\therefore \text{Domain of } f = [-1, 1] \cup (-\infty, -2) \cup (2, \infty)$$

➤ **Example 37.** The domain of the function

$$f(x) = \frac{1}{\sqrt{|x| - x}}$$

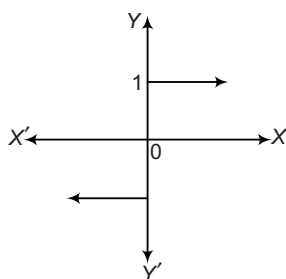
- (a) $(0, \infty)$
- (b) $(-\infty, 0)$
- (c) $(-\infty, \infty)$
- (d) None of these

Sol. (b) $f(x)$ is defined for $|x| > x$, which is true for $x < 0$ only. Thus, domain of $f = (-\infty, 0)$

Signum Function $y = \text{sgn}(x)$

It is defined by

$$y = \text{sgn}(x) = \begin{cases} \frac{|x|}{x} \text{ or } \frac{x}{|x|}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$



Domain $x \in \mathbb{R}$

Range $y \in \{-1, 0, 1\}$

- Nature**
- (a) Many-one
 - (b) Into function for codomain as the set of real numbers.
 - (c) Neither monotonically increasing nor monotonically decreasing.
 - (d) Odd function
 - (e) Discontinuous at $x = 0$

➤ Such functions also come in the category of piecewise constant functions.

Greatest Integer Function (Floor Function)

A function is of the form $y = f(x) = [x]$.

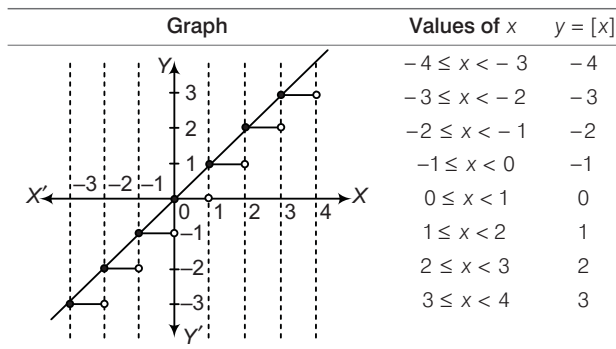
By definition, $[x]$ also known as integral part function, is equal to greatest integer less than or equal to x .

It is also known as floor function.

In general, $n \leq x < n+1$ ($n \in \text{integer}$)

$$\text{i.e. } [x] = n$$

Here, $f(x) = [x]$ could be expressed graphically as given below:



Domain $x \in \mathbb{R}$

Range $y \in \mathbb{I}$

- Nature**
- (a) Many-one
 - (b) Into function for codomain as the set of real numbers.
 - (c) Discontinuous at all integral values of x .

Properties of Greatest Integer Function

If n is an integer and x is any real number between n and $n+1$, then

- i. $[-n] = -[n]$
- ii. $[x + n] = [x] + n$
- iii. $[-x] = -[x] - 1$, x is not an integer.
- iv. $[x + y] \geq [x] + [y]$
- v. $[x] > n \Rightarrow x \geq n+1$
- vi. $[x] < n \Rightarrow x < n$

vii. $[x + y] = [x] + [y + x - [x]],$
 $\forall x, y \in R$

viii. $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right]$
 $+ \dots + \left[x + \frac{n-1}{n}\right] = [nx], n \in N$

ix. $[x] + [-x] = \begin{cases} -1, & \text{if } x \notin \text{integer} \\ 0, & \text{if } x \in \text{integer} \end{cases}$

x. $[x] - [-x] = \begin{cases} 2[x] + 1, & \text{if } x \notin \text{integer} \\ 2[x], & \text{if } x \in \text{integer} \end{cases}$

xi. $[x] = \left[\frac{x}{2}\right] + \left[\frac{x+1}{2}\right]$

Least Integer Function/Ceiling Function

A function is of the form

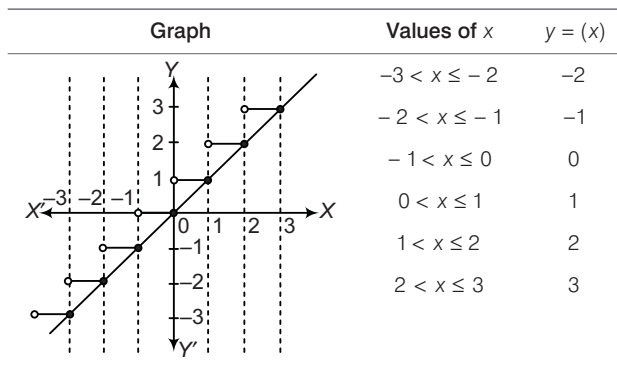
$$y = f(x) = (x) \text{ or } |x| \text{ or } [x]$$

By definition, $[x]$ is equal to least integer greater than or equal to x . It is also known as ceiling function.

In general, $n < x \leq n+1$ ($n \in \text{integer}$)

i.e. $[x] = (x) = n+1$

Here, $f(x) = (x) = [x]$ can be expressed graphically as given below:



Domain $x \in R$

Range $y \in I$

- Nature** (a) Many-one
 (b) Into function for codomain as the set of real numbers.
 (c) Discontinuous at all integral values of x .

Properties of Least Integer Function

- i. $(x) = x = [x]$ holds, if x is integer
- ii. $[x + n] = (x + n) = (x) + n, n \in \text{integer}$
- iii. $[-n] = -[n],$ where $n \in \text{integer}$
- iv. $[-x] = -[x] + 1$

v. $[x] + [-x] = \begin{cases} 1, & \text{if } x \notin \text{integer} \\ 0, & \text{if } x \in \text{integer} \end{cases}$

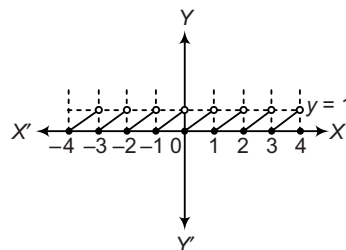
vi. $[x] + [-x] = \begin{cases} 2[x] - 1, & \text{if } x \notin \text{integer} \\ 2[x], & \text{if } x \in \text{integer} \end{cases}$

Fractional Part Function

A function is of the form $y = f(x) = \{x\}$

By definition, mathematically, $\{x\} = x - [x]$

It always returns the fractional part of the variable and is always positive.



Domain $x \in R$

Range $y \in [0, 1)$

- Nature** (a) Many-one
 (b) Into function for codomain as the set of real numbers
 (c) Monotonically increasing throughout R .
 (d) Discontinuous at all integral values of x
 (e) Periodic with period unity

Properties of Fractional Part Function

- i. If $0 \leq x < 1$, then $\{x\} = x$
- ii. If $x \in I$, then $\{x\} = 0$
- iii. If $x \notin I$ and $x > 0$, then $\{-x\} = 1 - \{x\}$

Dirichlet Function

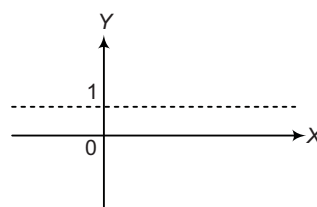
Let c and d be real numbers (usually) taken as $c = 1$ and $d = 0$. The Dirichlet function is defined by

$$y = f(x) = \begin{cases} c, & \text{for } x \text{ rational} \\ d, & \text{for } x \text{ irrational} \end{cases}$$

and is discontinuous everywhere.

The Dirichlet function can be written analytically as

$$f(x) = \lim_{m \rightarrow \infty} \left\{ \lim_{n \rightarrow \infty} \cos^{2n} (m! \pi x) \right\}$$



Domain $x \in \mathbb{R}$

Range $y \in \{c, d\}$ usually $\{0, 1\}$

Nature (a) Many-one
(b) Into function for codomain as the set of real numbers
(c) Neither increasing nor decreasing
(d) Not a periodic function
(e) Discontinuous everywhere

➤ **Example 38.** The domain of the function $f(x) = \sqrt{x^2 - [x]^2}$, where $[x]$ denotes the greatest integer less than or equal to x , is

- (a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty)$ (d) None of these

Sol. (d) $f(x)$ is defined, if $x^2 - [x]^2 \geq 0 \Rightarrow x^2 \geq [x]^2$, which is true for all positive real x and for all negative integers x .

➤ **Example 39.** Let $f(x) = (-1)^{[x]}$, where $[]$ denotes the greatest integer function, then

- (a) range of f is $\{-1, 1\}$
(b) f is an even function
(c) f is an odd function
(d) $\lim_{x \rightarrow n} f(x)$ exists, for every integer n

Sol. (a, c) $f(x) = (-1)^{[x]} = \{-1, 1\}$, since $[x] \in \mathbb{Z}$

$$\text{and } f(-x) = (-1)^{[-x]} = (-1)^{-1-[x]} \\ = \frac{(-1)^{-1}}{(-1)^{[x]}} \cdot \frac{(-1)^{[x]}}{(-1)^{[x]}} = -f(x)$$

$\therefore f(x)$ is odd.

➤ **Example 40.** The domain of the function

$$f(x) = \frac{1}{\sqrt{[x]^2 - [x] - 6}}$$

- (a) $(-\infty, -2) \cup [4, \infty)$ (b) $(-\infty, -2] \cup [4, \infty)$
(c) $(-\infty, -2) \cup (4, \infty)$ (d) None of these

Sol. (a) $f(x)$ is defined for $[x]^2 - [x] - 6 > 0$

$$\Rightarrow ([x] - 3)([x] + 2) > 0$$

$$\Rightarrow [x] < -2 \text{ or } [x] > 3$$

$$\text{But } [x] < -2 \Rightarrow [x] = -3, -4, -5, \dots$$

$$\therefore x < -2$$

$$\text{Also, } [x] > 3 \Rightarrow [x] = 4, 5, 6, \dots$$

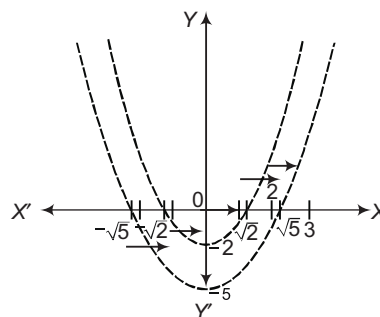
$$\Rightarrow x \geq 4$$

$$\therefore \text{Domain of } f = (-\infty, -2) \cup [4, \infty)$$

➤ **Example 41.** If domain of $f(x)$ is $[-1, 2]$, then domain of $f([x] - x^2 + 4)$, where $[]$ denotes the greatest integer function, is

- (a) $[-1, \sqrt{7}]$
(b) $[-\sqrt{3}, -1] \cup [\sqrt{3}, \sqrt{7}]$
(c) $(-1, \sqrt{7}]$
(d) $[-\sqrt{3}, -1) \cup [\sqrt{3}, \sqrt{7}]$

Sol. (b) We must have, $-1 \leq [x] - x^2 + 4 \leq 2$



$$\Rightarrow x^2 - 5 \leq [x] \leq x^2 - 2$$

$$\text{If } x^2 - 5 = \pm 2 \Rightarrow x = -\sqrt{3}, \sqrt{7}$$

$$\text{If } x^2 - 2 = 1 \Rightarrow x = \sqrt{3}$$

$$\therefore \text{Required values of } x \text{ are } [-\sqrt{3}, -1] \cup [\sqrt{3}, \sqrt{7}]$$

➤ **Example 42.** Which of the following is a function, where $[]$ denotes the greatest integer function and $\{ \}$ denotes the fractional part function?

- (a) $\frac{1}{\log[1 - |x|]}$ (b) $\frac{x!}{\{x\}}$
(c) $x! \{x\}$, $x \in \mathbb{N}$ (d) $\frac{\log(x-1)}{\sqrt{1-x^2}}$

Sol. (c) Since, only $x! \{x\}$ is defined. So, $x! \{x\}$ represents a function.

➤ **Example 43.** The range of the function f

$$\text{defined by } f(x) = \left[\frac{1}{\sin\{x\}} \right], \text{ where } [] \text{ and } \{ \}$$

respectively denote the greatest integer and the fractional part function, is

- (a) $\mathbb{I}$, the set of integers
(b) $\mathbb{N}$, the set of natural numbers
(c) $\mathbb{W}$, the set of whole numbers
(d) $\{2, 3, 4, \dots\}$

Sol. (b) $\because \{x\} \in [0, 1)$

$$\therefore \sin\{x\} \in [0, \sin 1) \text{ but } f(x) \text{ is defined, if } \sin\{x\} \neq 0$$

$$\therefore \frac{1}{\sin\{x\}} \in \left(\frac{1}{\sin 1}, \infty \right)$$

$$\Rightarrow \left[\frac{1}{\sin\{x\}} \right] \in \{1, 2, 3, \dots\}$$

➤ **Example 44.** The solutions of the equation $\{x+1\} + 2x = 4[x+1] - 6$, where $[]$ and $\{ \}$ denote integral and fractional part functions respectively, are

- (a) $1, \frac{8}{3}$ (b) $1, 2$
(c) $2, \frac{8}{3}$ (d) None of these

Sol. (a) $\{x + 1\} + 2x = 4[x + 1] - 6$
 $\Rightarrow x + 1 - [x + 1] + 2x = 4[x + 1] - 6$
 $\Rightarrow 3x + 1 = 5[x + 1] - 6$
 $\Rightarrow 3x + 1 = 5([x] + 1) - 6$
 $\Rightarrow 3x = 5[x] - 2 \quad \dots(i)$
 $\Rightarrow 3([x] + \{x\}) = 5[x] - 2$
 $\Rightarrow 3\{x\} = 2[x] - 2 \quad \dots(ii)$
 Now, $0 \leq \{x\} < 1$

$$\Rightarrow 0 \leq 3\{x\} < 3$$

$$\Rightarrow 0 \leq 2[x] - 2 < 3$$

$$\Rightarrow 2 \leq 2[x] < 5$$

$$\Rightarrow 1 \leq [x] < \frac{5}{2}$$

$$\Rightarrow [x] = 1, 2$$

From Eq. (i), we get $x = 1$, when $[x] = 1$
 and $x = 8/3$, when $[x] = 2$

Hence, $x = 1, 8/3$ are the required solutions.

Work Book Exercise 3.2

1 The domain of the function $f(x) = \sqrt{2 - 2x - x^2}$ is
 $[-\sqrt{3}, \sqrt{3}]$ $[-1 - \sqrt{3}, -1 + \sqrt{3}]$
 $[-2, 2]$ $[-2 - \sqrt{3}, -2 + \sqrt{3}]$

2 The domain of definition of the function
 $f(x) = \sin^{-1} \{\log_2 (x^2 + 3x + 4)\}$ is
 $[-4, -\frac{1}{4}]$ $[-3, -\frac{1}{3}]$
 $[-2, -1]$ None of these

3 Range of $f(x) = \sin^{-1} (\sqrt{x^2 + x + 1})$ is
 $[0, \frac{\pi}{2}]$ $[0, \frac{\pi}{3}]$ $[\frac{\pi}{3}, \frac{\pi}{2}]$ $[\frac{\pi}{6}, \frac{\pi}{3}]$

4 Range of $f(x) = [|\sin x| + |\cos x|]$, where $[]$
 denotes the greatest integer function, is
 $\{0\}$ $\{0, 1\}$
 $\{1\}$ None of these

5 Range of $\sin^{-1} \left[x^2 + \frac{1}{2} \right] + \cos^{-1} \left[x^2 - \frac{1}{2} \right]$,
 where $[]$ denotes the greatest integer function,
 is
 $\left\{ \frac{\pi}{2}, \pi \right\}$ $\{\pi\}$
 $\left\{ \frac{\pi}{2} \right\}$ None of these

6 The range of the function $f(x) = \sqrt{3x^2 - 4x + 5}$ is
 $[-\infty, \sqrt{\frac{11}{3}}]$ $(-\infty, \sqrt{\frac{11}{3}})$
 $[\sqrt{\frac{11}{3}}, \infty)$ $(\sqrt{\frac{11}{3}}, \infty)$

7 The range of the function $y = \sin^{-1} \left(\frac{x^2}{1 + x^2} \right)$ is
 $(0, \frac{\pi}{2})$ $[0, \frac{\pi}{2})$
 $[0, \frac{\pi}{2}]$ None of these

8 The range of the function $y = 3 \sin \sqrt{\frac{\pi^2}{16} - x^2}$ is
 $[0, \frac{3}{\sqrt{2}}]$ $[-\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}]$
 $[-\frac{3}{\sqrt{2}}, 0]$ None of these

9 The domain of the function
 $f(x) = \frac{1}{\sqrt{x^{12} - x^9 + x^4 - x + 1}}$ is
 $(-\infty, -1)$ $(1, \infty)$
 $(-1, 1)$ $(-\infty, \infty)$

10 Let $f(x) = 4 \cos \sqrt{x^2 - \frac{\pi^2}{9}}$, then the range of
 $f(x)$ is
 $[-1, 1]$ $[-4, 4]$
 $[0, 1]$ None of these

11 The domain of definition of the function
 $y = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$ is
 $[-2, 1]$ $[-2, 1[$
 $[-2, 0[\cup]0, 1[$ None of these

12 The range of the function
 $f(x) = 3 \sin x + 8 \cos \left(x - \frac{\pi}{3} \right) + 5$ is
 $[-5, 5]$
 $[5 - \sqrt{73 + 24\sqrt{3}}, 5 + \sqrt{73 + 24\sqrt{3}}]$
 $[5 - \sqrt{73 + \sqrt{3}}, 5 + \sqrt{5 + \sqrt{3}}]$
 None of the above

13 The domain of definition of the function
 $f(x) = \sin^{-1} [2 - 4x^2]$, where $[]$ denotes the
 greatest integer function, is
 $[-\sqrt{3}/2, \sqrt{3}/2]$ $[-\frac{\sqrt{3}}{2}, 0]$
 $[-\frac{\sqrt{3}}{2}, 0) \cup (0, \frac{\sqrt{3}}{2}]$ $[-\frac{\sqrt{3}}{2}, \infty)$

14 The domain of definition of the function
 $f(x) = \sqrt{3 - 2^x - 2^{1-x}} + \sqrt{\sin^{-1} x}$ is
 $[0, 1]$ $(0, 1]$
 $[0, 1)$ None of these

15 Let $f(x) = [9^x - 3^x + 1]$, $\forall x \in (-\infty, 1)$, then the
 range of $f(x)$, where $[]$ denotes the greatest
 integer function, is
 $[0, 1, 2, 3, 4, 5, 6, 7]$
 $\{0, 1, 2, 3, 4, 5, 6\}$
 $\{1, 2, 3, 4, 5, 6, 7\}$
 $\{1, 2, 3, 4, 5, 6\}$

Real Valued Function

All those functions of which domain and codomain are subsets of R , are called real valued functions.

Bounded Function

A function f is said to be bounded, if $|f(x)| \leq m$ for some finite m for every x in domain of f .

Equality of Functions

Two functions f and g are said to be equal iff

- domain of f = domain of g
- codomain of f = codomain of g
- $f(x) = g(x)$ for every x belonging to their common domain.

If two functions f and g are equal, then we write $f = g$.

e.g. If $A = \{1, 2\}$, $B = \{10, 13\}$, $f : A \rightarrow B$, $f(x) = x^2 + 9$ and $g : A \rightarrow B$, $g(x) = 3x + 7$, then $f = g$ because domains and codomains of both f and g are same also.

$$\begin{aligned} \text{i.e.} \quad & f(1) = 10 = g(1) \\ \text{and} \quad & f(2) = 13 = g(2) \end{aligned}$$

➤ **Example 45.** The functions

$f(x) = \log(x-1) - \log(x-2)$ and $g(x) = \log\left(\frac{x-1}{x-2}\right)$ are identical when x lies in the interval

- $[1, 2]$
- $[2, \infty]$
- $(2, \infty)$
- $(-\infty, \infty)$

Sol. (c) Since, $f(x) = \log(x-1) - \log(x-2)$

Domain of $f(x)$ is $x > 2$
and $x \in (2, \infty)$... (i)

$g(x) = \log\left(\frac{x-1}{x-2}\right)$ is defined, if $\frac{x-1}{x-2} > 0$

$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$... (ii)

From Eqs. (i) and (ii), $x \in (2, \infty)$

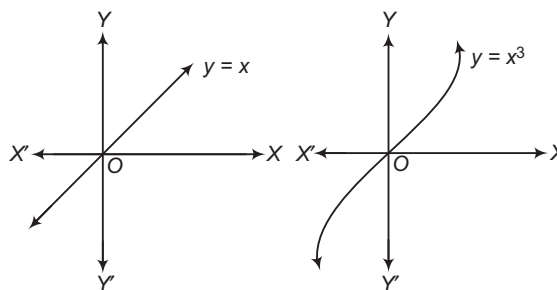
Odd and Even Functions

Odd Function

A function $f(x)$ is said to be an odd function, if $f(-x) = -f(x)$, $\forall x$.

Graph of odd function is symmetrical in opposite quadrants, i.e. the curve in first quadrant is identical to the curve in third quadrant and the curve in second quadrant is identical to the curve in fourth quadrant.

Some graphs which are symmetrical in opposite quadrants (or about origin) are given below :

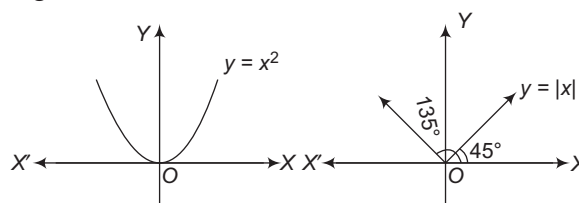


Even Function

A function $f(x)$ is said to be an even function, if $f(-x) = f(x)$, $\forall x$.

The graph is always symmetrical about Y -axis, i.e. the graph on left hand side of Y -axis is the mirror image of the curve on its right hand side.

Some graphs which are symmetrical about Y -axis, are given below:



Properties of Odd and Even Functions

- $f(x) - f(-x) = 0$
 $\Rightarrow f(x)$ is even and $f(x) + f(-x) = 0$
 $\Rightarrow f(x)$ is odd.
- A function may neither be odd nor even.
- Inverse of an even function is not defined and an even function cannot be strictly monotonic.
- Every even function is symmetric about the Y -axis and every odd function is symmetric about the origin.
- Every function (not necessarily even or odd) can be expressed as the sum of an even and an odd function.

$$\text{e.g. } f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{even}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{odd}}$$

$$\text{Let } g(x) = \left[\frac{f(x) + f(-x)}{2} \right]$$

$$h(x) = \left[\frac{f(x) - f(-x)}{2} \right], \text{ it can now easily be}$$

show that $g(x)$ is even and $h(x)$ is odd.

vi. The only function which is defined on the entire number line and is even and odd at the same time, is $f(x) = 0$.

vii. If f and g both are even or both are odd, then the function $f \cdot g$ will be even but if anyone of them is odd, then $f \cdot g$ will be odd.

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x) \cdot g(x)$	$f(x)/g(x)$	$(g \circ f)(x)$	$(f \circ g)(x)$
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

➤ **Example 46.** The function

$f(x) = \log(x + x^2 + 1)$ is

(a) an even function

(b) an odd function

(c) a periodic function

(d) neither an even nor an odd function

Sol. (b) $f(x) = \log(x + \sqrt{x^2 + 1})$

$$\Rightarrow f(-x) = \log(-x + \sqrt{x^2 + 1})$$

$$\Rightarrow f(x) + f(-x) = \log(x + \sqrt{x^2 + 1}) + \log(-x + \sqrt{x^2 + 1})$$

$$= \log(1) = 0$$

Hence, $f(x)$ is an odd function.

➤ **Example 47.** A function whose graph is symmetrical about the Y-axis, is given by

(a) $f(x) = \sin[\log(x + \sqrt{x^2 + 1})]$

(b) $f(x) = \frac{\sec^4 x + \operatorname{cosec}^4 x}{x^3 + x^4 \cot x}$

(c) $f(x + y) = f(x) + f(y), \forall x, y \in R$

(d) None of the above

Sol. (d) A function whose graph is symmetrical about the Y-axis must be even.

Since, $\sin x$ and $\log(x + \sqrt{x^2 + 1})$ are odd functions.

Therefore, $\sin(\log(x + \sqrt{x^2 + 1}))$ must be odd.

Also, $\frac{\sec^4 x + \operatorname{cosec}^4 x}{x^3 + x^4 \cot x}$ is an odd function.

Now, let $f(x + y) = f(x) + f(y), \forall x, y \in R$

$$\therefore f(0 + 0) = f(0) + f(0) \quad [\because f(0) = 0]$$

$$\Rightarrow f(x - x) = f(x) + f(-x)$$

$$\Rightarrow 0 = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$$

$\therefore f(x)$ is odd.

➤ **Example 48.** If f is an even function and g is an odd function, then the function $f \circ g$ is

(a) an even function (b) an odd function

(c) neither even nor odd (d) a periodic function

Sol. (a) We have, $f \circ g(-x) = f[g(-x)] = f[-g(x)]$

$$= f[g(x)] \quad [\because g \text{ is an odd}]$$

$$= f \circ g(x), \forall x \in R \quad [\because f \text{ is an even}]$$

$\therefore f \circ g$ is an even function.

Even and Odd Extensions

A function $f(x)$ defined on the interval $[0, a]$ can be extended to $[-a, a]$, so that $f(x)$ becomes an even or odd function in the interval $[-a, a]$. If this extension is an even function, it is called even extension and if this extension is an odd function, it is called odd extension.

Let g be the extension. Then, for even extension, we define

$$g(x) = \begin{cases} f(x), & \text{if } x \in [0, a] \\ f(-x), & \text{if } x \in [-a, 0] \end{cases}$$

and for odd extension, we define

$$g(x) = \begin{cases} f(x), & \text{if } x \in [0, a] \\ -f(-x), & \text{if } x \in [-a, 0] \end{cases}$$

➤ **Example 49.** Let the function

$f(x) = x^2 + x + \sin x - \cos x$ be defined on the interval $[0, 1]$, then odd and even extensions of $f(x)$ in the interval $[-1, 1]$, are respectively

$$(a) \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ -x^2 + x + \sin x + \cos x, & -1 \leq x < 0 \end{cases}$$

$$\text{and } \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ x^2 - x - \sin x - \cos x, & -1 \leq x < 0 \end{cases}$$

$$(b) \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ x^2 - x - \sin x - \cos x, & -1 \leq x < 0 \end{cases}$$

$$\text{and } \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ -x^2 + x + \sin x + \cos x, & -1 < x < 0 \end{cases}$$

(c) Does not exist

(d) None of the above

Sol. (a) For odd extension, make $f(x)$ an odd function in the interval $[-1, 1]$, we define $f(x)$ as follows:

$$f(x) = \begin{cases} f(x), & 0 \leq x \leq 1 \\ -f(-x), & -1 \leq x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ -(x^2 - x - \sin x - \cos x), & -1 \leq x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ -x^2 + x + \sin x + \cos x, & -1 \leq x < 0 \end{cases}$$

For even extension, make $f(x)$ an even function in the interval $[-1, 1]$, we define $f(x)$ as follows:

$$f(x) = \begin{cases} f(x), & 0 \leq x \leq 1 \\ f(-x), & -1 \leq x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x + \sin x - \cos x, & 0 \leq x \leq 1 \\ x^2 - x - \sin x - \cos x, & -1 \leq x < 0 \end{cases}$$

Periodic Function

A function $f(x)$ is said to be periodic function, if there exists a positive real number, such that

$$f(x + T) = f(x), \quad \forall x \in R$$

Then, $f(x)$ is a periodic function with period T , where T is least positive value.

Graphically

If the graph repeats at fixed interval, then function is said to be periodic and its period is the width of that interval.

Some standard results on periodic functions are given below :

Functions	Periods
(i) $\sin x, \cos x, \operatorname{cosec} x, \sec x$	2π
(ii) $\tan x, \cot x$	π
(iii) $\sin^n x, \cos^n x$ $\sec^n x, \operatorname{cosec}^n x$	π , if n is even. 2π , if n is odd or fraction.
(iv) $\tan^n x, \cot^n x$	π , if n is even or odd.
(v) $ \sin x , \cos x , \tan x , \cot x , \sec x , \operatorname{cosec} x $	π
(vi) $x - [x]$	1
(vii) Algebraic functions e.g. $\sqrt{x}, x^2, x^3 + 5, \dots$, etc.	Period does not exist.

Properties of Periodic Function

- i. If $f(x)$ is periodic with period T , then
 - (a) $c \cdot f(x)$ is periodic with period T .
 - (b) $f(x + c)$ is periodic with period T .
 - (c) $f(x) \pm c$ is periodic with period T , where c is any constant.
- ii. If $f(x)$ is periodic with period T , then $k f(cx + d)$ has period $\frac{T}{|c|}$,
i.e. period is only affected by coefficient of x , where $k, c, d \in \text{constant}$
- iii. If $f_1(x), f_2(x)$ are periodic functions with periods T_1, T_2 respectively, then we have, $h(x) = f_1(x) + f_2(x)$ has period as,

$$f(x) = \begin{cases} \frac{1}{2} \text{ LCM of } \{T_1, T_2\}, & \text{if } f_1(x) \text{ and } f_2(x) \text{ are complementary pairwise comparable even functions.} \\ \text{Or} \\ \text{LCM of } \{T_1, T_2\}, & \text{otherwise} \end{cases}$$

While taking LCM, we should always remember

 - (a) $\text{LCM of } \left(\frac{a}{b}, \frac{c}{d}, \frac{e}{f}\right) = \frac{\text{LCM of } (a, c, e)}{\text{HCF of } (b, d, f)}$

(b) LCM of rational with rational is possible.
LCM of irrational with irrational is possible. But LCM of rational with irrational is not possible.

➤ **Example 50.** Which of the following functions has period π , where $\{x\}$ denotes the fractional part of x ?

- (a) $\tan 3x + \cos \frac{5x}{2}$
- (b) $f(x) = \cos x + \{x\}$
- (c) $f(x) = \cos(\cos x) + \cos(\sin x)$
- (d) None of the above

Sol. (d) (a) Let $f(x) = \tan 3x + \cos \frac{5x}{2}$

$$\text{Period of } \tan 3x = \frac{\pi}{3} \text{ and period of } \cos \frac{5x}{2} = \frac{2\pi}{5/2} = \frac{4\pi}{5}$$

$$\therefore \text{Period of } f(x) = \frac{\text{LCM of } \pi \text{ and } 4\pi}{\text{HCF of } 3 \text{ and } 5} = 4\pi$$

(b) $f(x) = \cos x + \{x\} = \cos x + x - [x]$
 $\cos x$ is periodic with period 2π and $x - [x]$ is periodic with period 1. But LCM of 2π and 1 does not exist. Hence, $f(x)$ is not periodic.

(c) $f(x) = \cos(\cos x) + \cos(\sin x)$
 $\therefore \cos\{\cos(\pi + x)\} = \cos(-\cos x) = \cos(\cos x)$
and $\cos\{\sin(\pi + x)\} = \cos(-\sin x) = \cos(\sin x)$
 $\therefore \cos(\cos x)$ and $\cos(\sin x)$ are periodic functions with period π .

$$\text{But } \cos\left\{\cos\left(\frac{\pi}{2} + x\right)\right\} = \cos(-\sin x) = \cos(\sin x)$$

$$\text{and } \cos\left\{\sin\left(\frac{\pi}{2} + x\right)\right\} = \cos(\cos x)$$

$$\therefore f(x) \text{ is periodic with period } \frac{\pi}{2}.$$

Hence, none of the above functions have period π .

➤ **Example 51.** If $f(x) = \sin x + \cos ax$ is a periodic function, then a is

- (a) irrational number
- (b) rational number
- (c) natural number
- (d) None of these

Sol. (b) Period of $\sin x = 2\pi$ and period of $\cos ax = \frac{2\pi}{a}$

$$\therefore \text{Period of } \sin x + \cos ax = \text{LCM of } \frac{2\pi}{1} \text{ and } \frac{2\pi}{a} = \frac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of } 1 \text{ and } a} = \frac{2\pi}{\lambda}$$

where, λ is the HCF of 1 and a .

Since, λ is the HCF of 1 and a , $\frac{1}{\lambda}$ and $\frac{a}{\lambda}$ should be both integers.

$$\text{Suppose } \frac{1}{\lambda} = m \text{ and } \frac{a}{\lambda} = n, \text{ then } \frac{a}{1} = \frac{n}{m}$$

where $n, m \in I$ i.e. $a = \frac{n}{m}$

Hence, a is rational number.

- **Example 52.** The period of the function $f(x) = x[x]$ is
 (a) 1 (b) 2
 (c) non-periodic (d) None of these

Sol. (c) Let $n \leq x < n+1$

Then, $f(x) = x \cdot n$, where n changes with x .

Clearly, no constant $k > 0$ is possible for which

$f(x) = f(x+k)$ corresponding to all x .

$\therefore f(x)$ is a non-periodic function.

- **Example 53.** The period of the function $f(x) = \sin^4 x + \cos^4 x$ is
 (a) π (b) $\pi/2$
 (c) 2π (d) None of these

Sol. (b) $f(x) = \sin^4 x + \cos^4 x$

$$= (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$$

$$= 1 - \frac{1}{2}(\sin 2x)^2 = 1 - \frac{1}{2} \left[\frac{1 - \cos 4x}{2} \right]$$

$$= \frac{3}{4} + \frac{1}{4} \cos 4x$$

Since, $\cos x$ is periodic with period 2π .

$$\therefore \text{Period of } f(x) = \frac{2\pi}{4} = \frac{\pi}{2}$$

- **Example 54.** The period of the function $f(x) = a \sin kx + b \cos kx$ is
 (a) $\frac{2\pi}{k}$ (b) $\frac{2\pi}{|k|}$ (c) $\frac{\pi}{|k|}$ (d) None of these

Sol. (b) We have, $f(x) = a \sin kx + b \cos kx$

$$= \sqrt{a^2 + b^2} \left(\frac{a}{\sqrt{a^2 + b^2}} \sin kx + \frac{b}{\sqrt{a^2 + b^2}} \cos kx \right)$$

$$= \sqrt{a^2 + b^2} (\cos \theta \sin kx + \sin \theta \cos kx)$$

$$\left[\text{where, } \cos \theta = \frac{a}{\sqrt{a^2 + b^2}} \right]$$

$$= \sqrt{a^2 + b^2} \sin(kx + \theta)$$

which is a periodic function of period $\frac{2\pi}{|k|}$.

- **Example 55.** The period of the function $\log \sin^{-1}(x - [x])$, where $[]$ denotes the greatest integer function, is

- (a) 1
 (b) 2π
 (c) $\frac{\pi}{2}$
 (d) None of the above

Sol. (a) We have, $f(x) = \log(\sin^{-1}\{x\})$

$$\Rightarrow f(x+1) = f(x)$$

$\therefore$ Period of $f = 1$

- **Example 56.** If $f: R \rightarrow R$ is a function satisfying the property $f(2x+3) + f(2x+7) = 2$, $\forall x \in R$, then the period of $f(x)$ is
 (a) 2 (b) 4 (c) 8 (d) 12

Sol. (b) We have,

$$f(2x+3) + f(2x+7) = 2 \quad \dots(i)$$

On replacing x by $x+1$, we get

$$f(2x+5) + f(2x+9) = 2 \quad \dots(ii)$$

Again replacing x by $x+2$, we get

$$f(2x+7) + f(2x+11) = 2 \quad \dots(iii)$$

From Eqs. (i) and (iii), we get

$$f(2x+3) - f(2x+11) = 0$$

$$\Rightarrow f(2x+3) = f(2x+11)$$

$\therefore$ Period of $f(x) = 4$

- **Example 57.** The period of the function $f(x) = |\sin 4x| + |\cos 4x|$ is

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{8}$
 (c) $\frac{\pi}{4}$ (d) None of these

Sol. (b) Since, $|\sin x| + |\cos x|$ is a periodic function with period $\frac{\pi}{2}$.

Therefore, $f(x) = |\sin 4x| + |\cos 4x|$ is a periodic function with period $\frac{1}{4} \cdot \frac{\pi}{2}$, i.e. $\frac{\pi}{8}$.

Work Book Exercise 3.3

- 1 If f is an odd function and g is an even function, then the function fog is
 an even function
 an odd function
 neither even nor odd
 a periodic function

- 2 Let f and g be two odd functions, then the function fog is
 an even function
 an odd function
 neither even nor odd
 a periodic function

- 3 The period of the function $f(x) = 7 \cos(3x+5)$ is

- 2π $\frac{2\pi}{3}$
 $\frac{2\pi}{4}$ None of these

- 4 Which of the following functions has period π ?

- $2\cos\left(\frac{2\pi x}{3}\right) + 3\sin\left(\frac{\pi x}{3}\right)$
 $|\tan x| + \cos 2x$
 $4\cos\left(2\pi x + \frac{\pi}{2}\right) + 2\sin\left(\pi x + \frac{\pi}{4}\right)$
 None of the above

5 Let $f(x) = \left[\frac{f(x) + f(-x)}{g(x) - g(-x)} \right]^n$, then $F(x)$ is even and

odd according as n is

- even and odd odd and even
even and even odd and odd

6 The functions $f_1(x) = \log(x + \sqrt{x^2 + 1})$ and

$f_2(x) = x \left(\frac{a^x - 1}{a^x + 1} \right)$ are respectively

- odd and even functions
even and odd functions
odd and odd functions
even and even functions

7 The period of the function

$$f(x) = \left| \cos \frac{x}{2} \right| + |\sin x|$$

- $\frac{2\pi}{4\pi}$ $\frac{\pi}{\text{None of these}}$

8 The period of the function $f(x) = a \sin \lambda x + b \cos \lambda x$ is

$$\frac{2\pi}{\lambda} \quad \frac{\pi}{\lambda}$$

9 If $f(x) = \sin(\sqrt{[\lambda]} x)$ is a periodic function with period π , where $[\lambda]$ denotes the greatest integer less than or equal to λ , is π , then

- a $\lambda \in [4, 5)$
 $\lambda \in [4, 5]$
 $\lambda = 4, 5$
None of the above

10 The function $f(x) = \min \{ x - [x], -x - [-x] \}$ is a

- periodic function with period 1
periodic function with period $\frac{1}{2}$
non-periodic function
periodic function with period 2

Types of Functions

(i) One-One Function

A function $f : A \rightarrow B$ is said to one-one, if different elements in A have different images in B . This type of functions are known as injective functions.

e.g. Let $A = \{1, 2, 3\}$ and $B = \{2, 4, 6\}$. Consider $f : A \rightarrow B$, $f(x) = 2x$. Then, $f(1) = 2$; $f(2) = 4$ and $f(3) = 6$. Clearly, f is one-one function from A to B such that different elements in A have different images in B .

Methods to Check One-One Mapping

- Theoretically** If $f(x) = f(y) \Rightarrow x = y$, then $f(x)$ is one-one.
- Graphically** A function is one-one iff no line parallel to X -axis meets the graph of function at more than one point.
- By calculus** For checking whether $f(x)$ is one-one, find whether function is only increasing or only decreasing in their domain. If yes, then function is one-one, i.e. if $f'(x) \geq 0 \forall x \in \text{domain}$ or if $f'(x) \leq 0, \forall x \in \text{domain}$, then function is one-one.

➤ **Example 58.** Let $f : R \rightarrow R$ defined by $f(x) = x^3 + ax^2 + 3x + 100$, then the values of a for which f is one-one function, are

- (a) $-2 \leq a \leq 3$ (b) $-2 \leq a \leq 2$
(c) $-3 \leq a \leq 3$ (d) None of these

Sol. (c) $f(x) = x^3 + ax^2 + 3x + 100$

$$\Rightarrow f'(x) = 3x^2 + 2ax + 3$$

For $f(x)$ to be one-one, $f'(x) \geq 0$ or ≤ 0

But $f'(x)$ is a quadratic expression and coefficient of $x^2 > 0$, so that

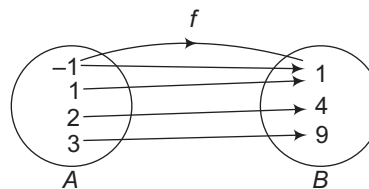
$$\begin{aligned} f'(x) &\geq 0 \Rightarrow D \leq 0 \\ \Rightarrow 4a^2 - 36 &\leq 0 \Rightarrow a^2 \leq 9 \\ \Rightarrow -3 &\leq a \leq 3 \end{aligned}$$

(ii) Many-One Function

Let $f : A \rightarrow B$. If two or more than two elements of set A have the same images in B , then f is said to be many-one.

e.g. Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$

where $f : A \rightarrow B$ given by $f(x) = x^2$



Here, $f(-1) = 1$; $f(1) = 1$; $f(2) = 4$ and $f(3) = 9$

Clearly, two elements 1 and -1 have the same image $1 \in B$.

So, f is many-one function.

➤ **Example 59.** If $f : R \rightarrow R$ is defined as

$$f(x) = \frac{x^2 + 4x + 7}{x^2 + x + 1}, \text{ then } f(x) \text{ is}$$

- (a) one-one (b) many-one
(c) onto (d) None of these

Sol. (b) For the given function, $f(x_1) = f(x_2)$

$$\Rightarrow \frac{x_1^2 + 4x_1 + 7}{x_1^2 + x_1 + 1} = \frac{x_2^2 + 4x_2 + 7}{x_2^2 + x_2 + 1}$$

$$\Rightarrow (x_1 - x_2)(2x_1 + 2x_2 + 1 + x_1x_2) = 0$$

Here, one of the solutions of $2x_1 + 2x_2 + 1 + x_1x_2 = 0$ is $x_1 = 0$ and $x_2 = -1/2$.

Hence, $f(0) = f\left(-\frac{1}{2}\right) = 7$, so that $f(x)$ is many-one.

Any function will be either one-one or many-one.

(iii) Onto Function

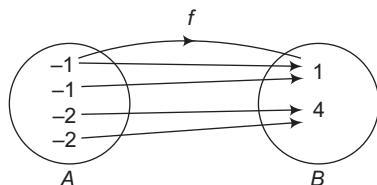
If the function $f : A \rightarrow B$ is such that each and every element in B is the f -image of atleast one element in A , then f is a function of A onto B . It is also called surjective function.

Or

If each element in codomain have atleast one pre-image in the set of domain, then function is onto.

e.g. Let $A = \{-1, 1, 2, -2\}$, $B = \{1, 4\}$ and $f : A \rightarrow B$ be a function defined by $f(x) = x^2$. Then, f is onto because

$$f(A) = \{f(-1), f(1), f(2), f(-2)\} = \{1, 4\} = B$$



If range is same as codomain, then function is onto.

Example 60. Let $f : (-\infty, \infty) \rightarrow [2, \infty)$ be a function defined by $f(x) = \sqrt{x^2 - 2a + a^2}$, $a \in R$, then the value of 'a' for which f is onto, is
(a) 1 (b) $\sqrt{3}$ (c) $1 \pm \sqrt{3}$ (d) $1 \pm \sqrt{5}$

Sol. (d) For f to be onto, range of the function should be $[2, \infty)$. So, $a^2 - 2a = 4$

$$\Rightarrow a = 1 \pm \sqrt{5}$$

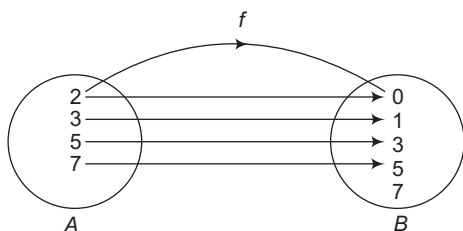
(iv) Into Function

If the function $f : A \rightarrow B$ is such that there is atleast one element in B which is not f -image of any element in A , then f is called an into function.

Or

If there is even single element in the set of codomain which does not have its pre-image, then it is known as into function.

e.g. Let $A = \{2, 3, 5, 7\}$ and $B = \{0, 1, 3, 5, 7\}$. Consider $f : A \rightarrow B$; $f(x) = x - 2$. Then, $f(2) = 0$, $f(3) = 1$, $f(5) = 3$ and $f(7) = 5$.



Clearly, f is a function from A to B . Now, there exists an element $7 \in B$, having no pre-image in A . So, f is an into function.

If range of a function f is proper subset of codomain, then function f is into.

Example 61. Let $f : N \rightarrow I$ be a function defined as $f(x) = x - 1000$, then $f(x)$ is
(a) one-one function (b) onto function
(c) into function (d) None of these

Sol. (c) Let $f(x) = y = x - 1000$

$$\Rightarrow x = y + 1000 = g(y)$$

[say]

Here, $g(y)$ is defined for each $y \in I$

But $g(y) \notin N$ for $y \leq -1000$

Hence, f is into function.

(v) One-One and Onto Function

A function which is one-one function and onto function, is called one-one and onto function.

It is also called bijective function.

Example 62. A function f from the set of natural numbers to integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{where } n \text{ is odd} \\ -\frac{n}{2}, & \text{where } n \text{ is even} \end{cases}, \text{ is}$$

- (a) one-one but not onto
- (b) onto but not one-one
- (c) both one-one and onto
- (d) neither one-one nor onto

Sol. (c) $f(n) = \begin{cases} \frac{n-1}{2}, & \text{where } n \text{ is odd} \\ -\frac{n}{2}, & \text{where } n \text{ is even} \end{cases}$ and $f : N \rightarrow I$, where

N is the set of natural numbers and I is the set of integers.

Let $x, y \in N$ and both are even, then $f(x) = f(y)$.

$$\Rightarrow -\frac{x}{2} = -\frac{y}{2} \Rightarrow x = y$$

Again, $x, y \in N$ and both are odd, then $f(x) = f(y)$

$$\Rightarrow \frac{x-1}{2} = \frac{y-1}{2} \Rightarrow x = y$$

i.e. mapping is one-one.

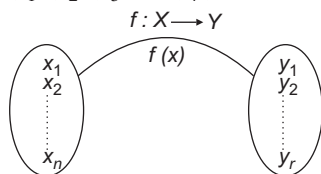
Since, each negative integer is an image of even natural number and positive integer is an image of odd natural number. So, mapping is onto. Hence, mapping is both one-one and onto.

- If domain of $f(x)$ is continuous and $dy/dx > 0, \forall x$ in domain, then f is one-one.
- If domain of $f(x)$ is continuous and $dy/dx < 0, \forall x$ in domain, then f is one-one.
- If a continuous function $f(x)$ which has either local minima or local maxima or both, then $f(x)$ will be many-one.
- Every even function is many-one.
- Every periodic function is many-one.

Number of Functions

Let $X = \{x_1, x_2, x_3, \dots, x_n\}$ (i.e. n elements)

and $Y = \{y_1, y_2, y_3, \dots, y_r\}$ (i.e. r elements)



(a) Total number of functions $= r^n = (\text{Number of elements in codomain})^{\text{number of elements in domain}}$

(b) Total number of one-one function

$$= \begin{cases} {}^r C_n \cdot n!, & r \geq n \\ 0, & r < n \end{cases}$$

(c) Total number of many-one function

$$= \begin{cases} r^n - {}^r C_n \cdot n!, & r \geq n \\ r^n, & r < n \end{cases}$$

(d) Total number of constant functions $= r$

(e) Total number of onto functions

$$= \begin{cases} r^n - {}^r C_1 (r-1)^n + {}^r C_2 (r-2)^n - {}^r C_3 (r-3)^n + \dots, & r < n \\ r!, & r = n \\ 0, & r > n \end{cases}$$

(f) Total number of into functions

$$= \begin{cases} {}^r C_1 (r-1)^n - {}^r C_2 (r-2)^n + {}^r C_3 (r-3)^n - \dots, & r \leq n \\ r^n, & r > n \end{cases}$$

➤ **Example 63.** The number of bijective functions from set A to itself when A contains 106 elements, is

- (a) 106
(b) $(106)^2$
(c) $(106)!$
(d) 2^{106}

Sol. (c) The total number of bijective functions from a set A containing 106 elements to itself is $(106)!$.

➤ **Example 64.** Set A has 3 elements and set B has 4 elements. The number of injections that can be defined from A to B , is

- (a) 144 (b) 12 (c) 24 (d) 64

Sol. (c) We know that, if X and Y are any two finite sets having m and n elements respectively, then the number of one-one functions from X to Y is given by

$$\begin{cases} {}^n P_m, & \text{if } n \geq m \\ 0, & \text{if } n < m \end{cases}$$

Thus, the number of injective mappings is ${}^4 P_3 = 24$.

➤ **Example 65.** The number of surjections from $A = \{1, 2, \dots, n\}$, $n \geq 2$ onto $B = \{a, b\}$, is

- (a) ${}^n P_2$ (b) $2^n - 2$
(c) $2^n - 1$ (d) None of these

Sol. (b) We know that, if X and Y are any two finite sets having m and n elements respectively, where $1 \leq n \leq m$, then the number of onto functions from X to Y is given by

$$\sum_{r=1}^n (-1)^{n-r} \cdot {}^n C_r \cdot r^m$$

Thus, the number of surjective mappings is

$$\sum_{r=1}^2 (-1)^{2-r} \cdot {}^2 C_r \cdot r^n = (2^n - 2)$$

Composite Functions

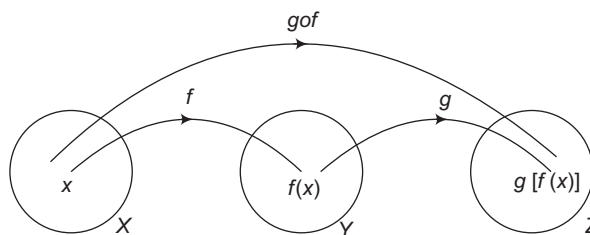
Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions. Then, a function $gof : A \rightarrow C$ defined by

$$(gof)(x) = g[f(x)], \forall x \in A$$

is called the composition of f and g .

Let A, B and C be three non-void sets and $f : A \rightarrow B, g : B \rightarrow C$ be two functions. Since, f is a function from A to B , therefore for each $x \in A$, there exists a unique element $f(x) \in B$. Again, g is a function from B to C , therefore corresponding to $f(x) \in B$, there exists a unique element $g[f(x)] \in C$. Thus, for each $x \in A$, there exists a unique element $g[f(x)] \in C$.

It follows from the above discussion that f and g , when considered together define a new function from A to C . This function is called the composition of f and g and is denoted by gof . We define it formally as follows:



- gof is defined only if for each $x \in A$, $f(x)$ is an element of g , so that we can take its g -image. Hence, for the composition gof to exist, the range of f must be a subset of the domain of g .
- gof exists iff the range of f is a subset of domain of g . Similarly, fog exists, if range of g is a subset of domain of f .

Properties of Composite Functions

- If f and g are even functions, then fog is an even function.
- If f is an odd and g is an even function, then fog is an odd function.

- iii. If f is an even and g is an odd function, then fog is an even function.
- iv. The composition of functions is not commutative.
i.e. $fog \neq gof$
- v. The composition of functions is associative,
i.e. if f, g, h are any three functions such that $(fog)oh$ and $fo(goh)$ both exist, then
 $(fog)oh = fo(goh)$
- vi. The composition of two bijective functions is also a bijective function.
- vii. The composition of any function with the identity function is the function itself,
i.e. if $f : A \rightarrow B$, then $foI_A = I_B of = f$
- viii. If f, g and h are functions from R to R , then
 $(f + g)oh = foh + goh$
and $(f \cdot g)oh = (foh) \cdot (goh)$

➤ **Example 66.** If $f(x) = \begin{cases} 2+x, & \text{if } x \geq 0 \\ 2-x, & \text{if } x < 0 \end{cases}$ then

$(f \circ f)(x)$ is

- (a) $|x|$ (b) $4 - |x|$
(c) $4 + |x|$ (d) None of these

Sol. (c) $\because f(x) = 2 + |x|$

$$\therefore (f \circ f)(x) = 2 + |f(x)| = 2 + 2 + |x| = 4 + |x|$$

➤ **Example 67.** If $f(x) = 1 - x$, $x \in [-3, 3]$, then the domain of $f[f(x)]$ is

- (a) $[-2, 3]$ (b) $(-2, 3)$ (c) $[-2, 3)$ (d) $(-2, 3]$

Sol. (a) Since, $f(x) = 1 - x$, $x \in [-3, 3]$

$$\therefore f[f(x)] = \begin{cases} f(1-x), & \text{if } -2 \leq 1-x \leq 3 \\ 1-(1-x), & \text{if } -4 \leq -x \leq 2 \\ x, & \text{if } -2 \leq x \leq 4 \end{cases}$$

i.e. $-2 \leq 1-x \leq 3$

Hence, domain of $f[f(x)]$ is $[-2, 3]$.

➤ **Example 68.** If $f(x) = \frac{1-x}{1+x}$, $x \neq 0$, then

$$f[f(x)] + f\left[f\left(\frac{1}{x}\right)\right]$$

- (a) < 2 (b) ≥ 2
(c) $= 2$ (d) None of these

Sol. (b) $f[f(x)] + f\left[f\left(\frac{1}{x}\right)\right]$

$$= f\left(\frac{1-x}{1+x}\right) + f\left(\frac{1-\frac{1}{x}}{1+\frac{1}{x}}\right), x \neq 0$$

$$\begin{aligned} &= \frac{1 - \frac{(1-x)}{1+x}}{1 + \frac{(1-x)}{1+x}} + \frac{1 - \frac{(x-1)}{(x+1)}}{1 + \frac{(x-1)}{(x+1)}} = x + \frac{1}{x} \\ &= \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 \geq 2 \end{aligned}$$

➤ **Example 69.** If $f(x) = \begin{cases} |x|, & x \leq 1 \\ 2-x, & x > 1 \end{cases}$, then

$f(f(x))$ is equal to

- (a) $\begin{cases} 2-|x|, & x < -1 \\ |x|, & -1 \leq x \leq 1 \\ |2-x|, & x > 1 \end{cases}$ (b) $\begin{cases} |x|, & x < -1 \\ 2-|x|, & -1 \leq x \leq 1 \\ |2-x|, & x > 1 \end{cases}$
(c) $\begin{cases} |2-x|, & x < -1 \\ |x|, & -1 \leq x \leq 1 \\ 2-|x|, & x > 1 \end{cases}$ (d) None of these

Sol. (a) $f(f(x)) = \begin{cases} f(|x|), & x \leq 1 \\ f(2-x), & x > 1 \end{cases}$

$$f(|x|) = \begin{cases} |x|, & x \leq 1 \text{ and } |x| \leq 1 \\ 2-|x|, & x \leq 1 \text{ and } |x| > 1 \\ |x|, & -1 \leq x \leq 1 \\ 2-|x|, & x < -1 \end{cases}$$

$$f(2-x) = \begin{cases} |2-x|, & x > 1 \text{ and } 2-x \leq 1 \\ 2-(2-x), & x > 1 \text{ and } 2-x > 1 \end{cases}$$

[impossible]

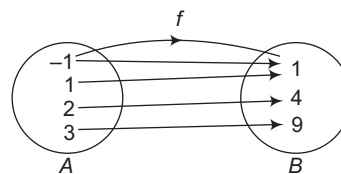
$$= |2-x|, x > 1$$

$$f(f(x)) = \begin{cases} 2-|x|, & x < -1 \\ |x|, & -1 \leq x \leq 1 \\ |2-x|, & x > 1 \end{cases}$$

Inverse Function

Let $f : A \rightarrow B$ is a bijective function, then there exists a unique function $g : B \rightarrow A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall x \in A$ and $y \in B$, then g is called inverse function of f .

Hence, $g = f^{-1} : B \rightarrow A$



Properties of Inverse Function

- i. Inverse of bijective function is unique.
- ii. Inverse of bijective function is also bijective function.
- iii. If $f : A \rightarrow B$ is bijective function and $g : B \rightarrow A$ is inverse of f , then $f \circ g = I_B$ and $g \circ f = I_A$, where I_A and I_B are identity functions of sets A and B , respectively.

- iv.** If $f : A \rightarrow B$ and $g : B \rightarrow A$ are two bijective functions, then $gof : A \rightarrow C$ is also bijective function and $(gof)^{-1} = f^{-1}og^{-1}$.
- v.** $fog \neq gof$, but if $fog = gof$, then either $f^{-1} = g$ or $g^{-1} = f$ and $(fog)(x) = (gof)(x) = x$.

✎ If the inverse of f exists, then f is called an invertible function.

Methods of Finding Inverse of a Function

- If you are asked to check whether the given function $y = f(x)$ is invertible, you need to check that $y = f(x)$ is one-one and onto.
- If you are asked to find the inverse of a bijective function $f(x)$, you do the following:
- If f^{-1} is the inverse of f , then $f(f^{-1}(x)) = x$. Apply the formula of f on $f^{-1}(x)$ and use of the above identity to solve for $f^{-1}(x)$.

✎ **Example 70.** Let $f : (4, 6) \rightarrow (6, 8)$ be a

function defined by $f(x) = x + \left\lfloor \frac{x}{2} \right\rfloor$, where $\lfloor \cdot \rfloor$

denotes the greatest integer function, then $f^{-1}(x)$ is equal to

- (a) $x - \lfloor x/2 \rfloor$ (b) $-x - 2$
 (c) $x - 2$ (d) $\frac{1}{x + \left\lfloor \frac{x}{2} \right\rfloor}$

Sol. (c) Since, $f : (4, 6) \rightarrow (6, 8)$

$$\Rightarrow f(x) = x + 2$$

$$\therefore f^{-1}(x) = x - 2$$

✎ **Example 71.** The value of the parameter α , for which the function $f(x) = 1 + \alpha x$, $\alpha \neq 0$ is the inverse of itself, is

- (a) -2
 (b) -1
 (c) 1
 (d) 2

Sol. (b) Let $y = f(x) \Rightarrow y = 1 + \alpha x \Rightarrow x = \frac{y-1}{\alpha}$

$$\Rightarrow f^{-1}(y) = \frac{y-1}{\alpha}$$

$$\text{Now, } f(x) = f^{-1}(x)$$

$$\Rightarrow 1 + \alpha x = \frac{x-1}{\alpha}$$

$$\Rightarrow 1 + \alpha x = \frac{1}{\alpha}x - \frac{1}{\alpha}$$

$$\Rightarrow \alpha = \frac{1}{\alpha}$$

$$\text{and } 1 = -\frac{1}{\alpha} \Rightarrow \alpha = -1$$

✎ **Example 72.** The inverse of the function

$$f(x) = \frac{a^x - a^{-x}}{a^x + a^{-x}} \text{ is}$$

- (a) $\frac{1}{2} \log_a \left(\frac{1-x}{1+x} \right)$ (b) $\frac{1}{2} \log_a \left(\frac{1+x}{1-x} \right)$
 (c) $\log_a \left(\frac{1+x}{1-x} \right)$ (d) None of these

Sol. (b) Let $y = \frac{a^x - a^{-x}}{a^x + a^{-x}} = \frac{a^{2x} - 1}{a^{2x} + 1}$

$$\Rightarrow \frac{y-1}{y+1} = \frac{(a^{2x}-1)-(a^{2x}+1)}{(a^{2x}-1)+(a^{2x}+1)}$$

[using componendo and dividendo rule]

$$\Rightarrow \frac{y-1}{y+1} = \frac{-2}{2a^{2x}}$$

$$\Rightarrow a^{2x} = \frac{1+y}{1-y}$$

$$\Rightarrow 2x \log_a a = \log_a \left(\frac{1+y}{1-y} \right)$$

$$\Rightarrow x = \frac{1}{2} \log_a \left(\frac{1+y}{1-y} \right)$$

✎ **Example 73.** If the function $f : [1, \infty) \rightarrow [1, \infty)$

is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is

- (a) $\left(\frac{1}{2} \right)^{x(x-1)}$ (b) $\left(\frac{1}{2} \right) [1 + \sqrt{1 + 4 \log_2 x}]$
 (c) $\left(\frac{1}{2} \right) [1 - \sqrt{1 + 4 \log_2 x}]$ (d) not defined

Sol. (b) Let $y = f(x)$

$$\Rightarrow y = 2^{x(x-1)}$$

$$\Rightarrow x(x-1) \log_2 2 = \log_2 y$$

$$\Rightarrow x^2 - x - \log_2 y = 0$$

$$\Rightarrow x = \frac{1}{2} (1 \pm \sqrt{1 + 4 \log_2 y})$$

$$\text{But } x \geq 1$$

$$\therefore x = \frac{1}{2} (1 + \sqrt{1 + 4 \log_2 y})$$

$$\Rightarrow f^{-1}(y) = \frac{1}{2} (1 + \sqrt{1 + 4 \log_2 y})$$

$$\text{or } f^{-1}(x) = \frac{1}{2} (1 + \sqrt{1 + 4 \log_2 x})$$

✎ **Example 74.** The inverse of the function

$$y = [1 - (x-3)^4]^{1/7} \text{ is}$$

- (a) $3 + (1-x^7)^{1/4}$ (b) $3 - (1-x^7)^{1/4}$
 (c) $3 - (1+x^7)^{1/4}$ (d) None of these

Sol. (a) Clearly, y is one-one and onto.

$$\text{We have, } y = [1 - (x-3)^4]^{1/7}$$

$$\Rightarrow (x-3)^4 = 1 - y^7$$

$$\Rightarrow x - 3 = (1 - y^7)^{1/4}$$

$$\therefore f^{-1}(y) = 3 + (1 - y^7)^{1/4}$$

Definite Function

The functional relation between x and y , i.e. when function is not known, is called definite function such as

$$f(x) + f(y) = f(x) \cdot f(y)$$

$$\frac{f(x)}{f(y)} = f(x) - f(y)$$

and so on.

➤ **Example 75.** If $f(n+1) = \frac{2f(n)+1}{2}$;

$n=1, 2, \dots$ and $f(1)=2$, then $f(101)$ equals

- (a) 52 (b) 49 (c) 48 (d) 51

Sol. (a) We have, $f(n+1) = \frac{2f(n)+1}{2}$ and $f(1)=2$

$$\therefore f(101) = f(1) + 100 \times \frac{1}{2} = 2 + 50$$

$$\therefore f(101) = 52$$

➤ **Example 76.** The number of solutions of the equation $a^{f(x)} + g(x) = 0$, $a > 0$, $g(x) \neq 0$ and has minimum value of $\frac{1}{2}$, is

- (a) one (b) two
(c) zero (d) infinitely

Sol. (c) We have, $a^{f(x)} + g(x) = 0$, $a > 0$

Since, minimum values of $g(x)$ is $\frac{1}{2}$.

$$\therefore g(x) > 0 \text{ and } a^{f(x)} > 0$$

$$\therefore a^{f(x)} + g(x) > 0, \forall x$$

Hence, number of solutions is zero.

➤ **Example 77.** If $f: R \rightarrow R$, $g: R \rightarrow R$ be two given functions, then

$f(x) = 2 \min \{|f(x) - g(x)|, 0\}$ equals

- (a) $f(x) + g(x) - |g(x) - f(x)|$
(b) $f(x) + g(x) + |g(x) - f(x)|$
(c) $f(x) - g(x) + |g(x) - f(x)|$
(d) $f(x) - g(x) - |g(x) - f(x)|$

Sol. (d) We have, $f(x) = 2 \min \{f(x) - g(x), 0\}$

$$= \begin{cases} 0, & f(x) > g(x) \\ 2(f(x) - g(x)), & f(x) \leq g(x) \end{cases}$$

$$= \begin{cases} f(x) - g(x) - |f(x) - g(x)|, & f(x) > g(x) \\ f(x) - g(x) - |f(x) - g(x)|, & f(x) \leq g(x) \end{cases}$$

$$\therefore f(x) = f(x) - g(x) - |g(x) - f(x)|$$

➤ **Example 78.** If $f: R \rightarrow R$ satisfies

$f(x+y) = f(x) + f(y)$, $\forall x, y \in R$ and $f(1) = 7$,

then $\sum_{r=1}^n f(r)$ is

- (a) $\frac{7n}{2}$ (b) $\frac{7(n+1)}{2}$
(c) $7n(n+1)$ (d) $\frac{7n(n+1)}{2}$

Sol. (d) $\sum_{r=1}^n f(r) = f(1) + f(2) + f(3) + \dots + f(n)$

$$= f(1) + 2f(1) + 3f(1) + \dots + nf(1)$$

$$[\because f(x+y) = f(x) + f(y)]$$

$$= (1 + 2 + 3 + \dots + n) f(1)$$

$$= \frac{n(n+1)}{2} \cdot 7$$

$$= \frac{7n(n+1)}{2}$$

Work Book Exercise 3.4

1 If $f(x) = x^n$, $n \in N$ and $gof(x) = ng(x)$, then $g(x)$ can be

- a $n|x|$
 $3x^{1/3}$
 e^x
 $\log|x|$

2 The function $f: (-\infty, -1] \rightarrow (0, e^5]$ defined by $f(x) = e^{x^3 - 3x + 2}$ is

- one-one and onto
one-one and into
many-one and into
many-one and onto

3 The function $f: N \rightarrow N$ given by $f(n) = n - (-1)^n$ is

- one-one and onto
one-one and into
many-one and onto
many-one and into

4 $f: R \rightarrow R$ given by $f(x) = x + \sqrt{x^2}$ is

- injective
bijective
surjective
None of these

5 If $f: A \rightarrow B$ given by $3^{f(x)} + 2^{-x} = 4$ is a bijection, then

- $A = \{x \in R : -1 < x < \infty\}$, $B = \{x \in R : 2 < x < 4\}$
 $A = \{x \in R : -3 < x < \infty\}$, $B = \{x \in R : 0 < x < 4\}$
 $A = \{x \in R : -2 < x < \infty\}$, $B = \{x \in R : 0 < x < 4\}$
None of the above

6 Let the function $f: R - \{-b\} \rightarrow R - \{1\}$ be

defined by $f(x) = \frac{x+a}{x+b}$, $a \neq b$, then

- f is one-one but not onto
 f is onto but not one-one
 f is both one-one and onto
None of the above

7 Let $f: R \rightarrow R$ be given by $f(x) = [x]^2 + [x+1] - 3$, where $[x]$ denotes the greatest integer less than or equal to x . Then, $f(x)$ is

- many-one and onto
many-one and into
one-one and into
one-one and onto

- 8 The function $f : A \rightarrow B$ defined by $f(x) = x^2 + 6x - 8$ is a bijection, if
- $A \in (-\infty, \infty), B \in [-17, \infty)$
 $A \in (-\infty, -3], B \in [-17, \infty)$
 $A = (-\infty, 3]$ and $B = (1, \infty)$
 $A = (3, \infty)$ and $B = (1, \infty)$
- 9 The values of a and b for which the map $f : R \rightarrow R$, given by $f(x) = ax + b (a, b \in R)$, is a bijection with $f \circ f$ as identity function, are
- $a = 1, b \in R$
 $a = 1, b = 0$ or $a = -1, b \in R$
 $a = \pm 1, b \in R$
 $a = \pm 1, b = 0$
- 10 Let M be the set of all 2×2 matrices with entries from the set R of real numbers. Then, the function $f : M \rightarrow R$ defined by $f(A) = |A|$ for every $A \in M$, is
- one-one and onto
 neither one-one nor onto
 one-one but not onto
 onto but not one-one
- 11 If $f(x) = x^2$ and $g(x) = 2^x$, then what is the solution set of $f \circ g(x) = g \circ f(x)$?
- $\{0\}$ $\{2\}$
 $\{0, 2\}$ None of these
- 12 If $f(x) = 2[x] + \cos x$, then $f : R \rightarrow R$ is
- one-one and onto
 one-one and into
 many-one and into
 many-one and onto
- 13 The inverse of the function $f(x) = \log(x^2 + 3x + 1)$; $x \in [1, 3]$ assuming it to be an onto function, is
- $\frac{-3 - \sqrt{5 + 4e^x}}{2}$
 $\frac{-3 \pm \sqrt{5 + 4e^x}}{2}$
 $\frac{-3 + \sqrt{5 + 4e^x}}{2}$
 None of the above
- 14 Suppose that $g(x) = 1 + \sqrt{x}$ and $f\{g(x)\} = 3 + 2\sqrt{x} + x$, then $f(x)$ is
- $1 + 2x^2$ $2 + x^2$
 $1 + x$ $2 + x$
- 15 Let $f(x) = ax + b$ and $g(x) = cx + d$; $a \neq 0, c \neq 0$. Assume $a = 1, b = 2$, if $(f \circ g)(x) = (g \circ f)(x), \forall x$, What can you say about c and d ?
- Both c and d are arbitrary
 $c = 1$ and d is arbitrary
 c is arbitrary and $d = 1$
 $c = 1, d = 1$
- 16 Let a, b and c be rational numbers and $f : Z \rightarrow Z$ be a function given by $f(x) = ax^2 + bx + c$. Then, $a + b$ is
- a negative integer
 an integer
 non-integral rational number
 None of the above
- 17 Let $f : R \rightarrow R$ be a function defined by $f(x) = |x|$, $\forall x \in R$ and $A = [0, 1]$, then $f^{-1}(A)$ equals
- $(-1, 1)$
 $(0, 1)$
 $(-1, 0)$
 None of the above
- 18 If the function $f : R \rightarrow R$ is such that $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x , then $f^{-1}(x)$ is
- $\frac{1}{x - [x]}$ $[x] - x$
 not defined None of these
- 19 Let $f : A \rightarrow B$ be a function defined by $f(x) = \sqrt{3} \sin x + \cos x + 4$. If f is invertible, then
- $A = \left[\frac{-2\pi}{3}, \frac{\pi}{3} \right], B = [2, 6]$
 $A = \left[\frac{\pi}{6}, \frac{5\pi}{6} \right], B = [-2, 2]$
 $A = \left[\frac{-\pi}{2}, \frac{\pi}{2} \right], B = [2, 6]$
 $A = \left[\frac{-\pi}{3}, \frac{\pi}{3} \right], B = [2, 6]$
- 20 Let $A = \left\{ x \in R : x \geq \frac{1}{2} \right\}$ and $B = \left\{ x \in R : x \geq \frac{3}{4} \right\}$. If $f : A \rightarrow B$ is defined as $f(x) = x^2 - x + 1$, then the solution set of the equation $f(x) = f^{-1}(x)$ is
- $\{1\}$
 $\{2\}$
 $\left\{ \frac{1}{2} \right\}$
 $\left\{ \frac{1}{2} \right\}$
 None of the above

WorkedOut Examples

Type 1. Only One Correct Option

- Ex 1.** If $[x]^2 - 5[x] + 6 = 0$, where $[]$ denotes the greatest integer function, then
 (a) $x \in [3, 4]$ (b) $x \in [2, 3]$
 (c) $x \in \{2, 3\}$ (d) $x \in [2, 4]$

Sol. $[x]^2 - 5[x] + 6 = 0$
 $\Rightarrow ([x] - 2)([x] - 3) = 0$
 $\Rightarrow [x] = 2, 3 \Rightarrow x \in [2, 4]$
 Hence, (d) is the correct answer.

- Ex 2.** If $\left\lceil \log_2 \left(\frac{x}{[x]} \right) \right\rceil \geq 0$, where $[]$ denotes the greatest integer function, then
 (a) $x \in (-\infty, \infty) - [0, 1]$ (b) $(-\infty, 0)$
 (c) $[1, \infty)$ (d) None of these

Sol. $\left\lceil \log_2 \left(\frac{x}{[x]} \right) \right\rceil \geq 0$
 $\Rightarrow \log_2 \left(\frac{x}{[x]} \right) \geq 0, [x] \neq 0$
 $\Rightarrow \frac{x}{[x]} \geq 1$
 $\Rightarrow \frac{x - [x]}{[x]} \geq 0, [x] \neq 0$
 $\Rightarrow \frac{\{x\}}{[x]} \geq 0, [x] \neq 0$

It implies that x is any positive real number greater than or equal to one or x is any non-zero integer.
 Hence, (d) is the correct answer.

- Ex 3.** Total number of positive real values of x satisfying $2[x] = x + \{x\}$, where $[]$ and $\{\}$ denote the greatest integer function and fractional part respectively, is
 (a) 2 (b) 1 (c) 0 (d) 3

Sol. $2[x] = x + \{x\}$
 $\Rightarrow 2[x] = [x] + 2\{x\} \quad [\because x = [x] + \{x\}]$
 $\Rightarrow \{x\} = \frac{[x]}{2} \Rightarrow 0 \leq \frac{[x]}{2} < 1$
 $\Rightarrow 0 \leq [x] < 2 \Rightarrow [x] = 0, 1$
 For $[x] = 0$, we get $\{x\} = 0 \Rightarrow x = 0$
 For $[x] = 1$, we get $\{x\} = \frac{1}{2} \Rightarrow x = \frac{3}{2}$
 Hence, (b) is the correct answer.

- Ex 4.** Total number of solutions of $[x]^2 = x + 2\{x\}$, where $[]$ and $\{\}$ denote the greatest integer function and fractional part respectively, is
 (a) 2 (b) 4
 (c) 6 (d) None of these

Sol. $[x]^2 = x + 2\{x\}$
 $\Rightarrow [x]^2 = [x] + 3\{x\} \quad [\because x = [x] + \{x\}]$
 $\Rightarrow \{x\} = \frac{[x]^2 - [x]}{3}$
 $\Rightarrow 0 \leq \frac{[x]^2 - [x]}{3} < 1$
 $\Rightarrow [x] \in \left(\frac{1 - \sqrt{13}}{2}, 0 \right] \cup \left[1, \frac{1 + \sqrt{13}}{2} \right)$
 $\Rightarrow [x] = -1, 0, 1, 2$
 $\Rightarrow \{x\} = \frac{2}{3}, 0, 0, \frac{2}{3}$
 $\Rightarrow x = -\frac{1}{3}, 0, 1, \frac{8}{3}$
 Hence, (b) is the correct answer.

- Ex 5.** Domain of $f(x) = \sqrt{\frac{x-1}{x-2\{x\}}}$, where $\{\}$ denotes the fractional part of x , is
 (a) $(-\infty, 0) \cup (0, 2]$
 (b) $[1, 2)$
 (c) $(-\infty, \infty) - [0, 2)$
 (d) $(-\infty, 0) \cup (0, 1] \cup [2, \infty)$

Sol. We must have $\frac{x-1}{x-2\{x\}} \geq 0$. We have two cases

Case I $x \geq 1$ and $x > 2\{x\}$

$\Rightarrow x \geq 2$

Common part is $[2, \infty)$.

Case II $x \leq 1$ and $x < 2\{x\}$

$\Rightarrow x < 1$ and $x \neq 0$

$\therefore$ Common part is $x \in (-\infty, 0) \cup (0, 1)$.

Finally, $x = 1$ is also a point of the domain.

Hence, (d) is the correct answer.

- Ex 6.** Domain of $f(x) = \sin^{-1} \left(\frac{[x]}{\{x\}} \right)$, where $[]$ and $\{\}$ denote the greatest integer function and fractional part respectively, is
 (a) $(0, 1)$ (b) $(-1, 1) - \{0\}$
 (c) $(-2, 2) - \{-1, 0, 1\}$ (d) None of these

Sol. We must have $-1 \leq \frac{[x]}{\{x\}} \leq 1$

From $-1 \leq \frac{[x]}{\{x\}}$, we get $x \in (0, \infty) - I^+$

From $\frac{[x]}{\{x\}} \leq 1$, we get

$[x] \leq \{x\} \Rightarrow x \in (-\infty, 0) \cup (0, 1) - I^-$

Thus, domain is $(0, 1)$.

Hence, (a) is the correct answer.

Ex 7. Domain of $f(x) = \sqrt{2\{x\}^2 - 3\{x\} + 1}$, where $\{ \}$ denotes the fractional part, in $[-1, 1]$, is

- (a) $[-1, 1] - \left(\frac{1}{2}, 1\right)$
 (b) $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right]$
 (c) $\left[-1, \frac{1}{2}\right]$
 (d) $\left[-\frac{1}{2}, 1\right]$

Sol. We must have $2\{x\}^2 - 3\{x\} + 1 \geq 0$

$$\Rightarrow \{x\} \geq 1 \text{ or } \{x\} \leq \frac{1}{2}$$

Thus, we have $0 \leq \{x\} \leq \frac{1}{2}$

$$\Rightarrow x \in \left[n, n + \frac{1}{2}\right], n \in I$$

Hence, (b) is the correct answer.

Ex 8. Domain of $f(x) = \sqrt{\cos(\sin x)} + \sqrt{\log_x \{x\}}$, where $\{ \}$ denotes the fractional part, is

- (a) $[1, \pi)$ (b) $(0, 2\pi) - [1, \pi)$
 (c) $\left(0, \frac{\pi}{2}\right) - \{ \}$ (d) $(0, 1)$

Sol. We must have $\cos(\sin x) \geq 0$

$$\Rightarrow \frac{\pi}{2} \geq \sin x \geq -\frac{\pi}{2}, \text{ which is true for all real } x.$$

Also, $\log_x \{x\} \geq 0$

If $0 < x < 1$, then

$$0 < \{x\} \leq 1$$

$$\Rightarrow 0 < x < 1$$

If $x > 1$, then $\{x\} \geq 1$

which is not possible.

Thus, domain is $(0, 1)$.

Hence, (d) is the correct answer.

Ex 9. If $[x^2] + x - a = 0$ has a solution, where $a \in N$ and $a \leq 20$, then total number of different values of a can be

- (a) 2 (b) 3
 (c) 4 (d) 6

Sol. $[x^2] + x - a = 0$

$\Rightarrow x$ has to be an integer.

$$\Rightarrow a = x^2 + x = x(x+1)$$

Thus, a can be 2, 6, 12, 20.

Hence, (c) is the correct answer.

Ex 10. If $[x + [2x]] < 3$, where $[\]$ denotes the greatest integer function, then

- (a) $x \in [0, 1)$ (b) $x \in \left(-\infty, \frac{3}{2}\right]$
 (c) $x \in \left[0, \frac{3}{2}\right)$ (d) $x \in (-\infty, 1)$

Sol. $[x + [2x]] < 3 \Rightarrow [x] + [2x] \leq 2$

Any non-positive real number will satisfy this inequality.

Now, if $x \in \left[0, \frac{1}{2}\right)$, then

$$[x] = 0, [2x] = 0$$

$\therefore$ Inequality is again satisfied.

For $x \in \left[\frac{1}{2}, 1\right)$, $[x] = 0, [2x] = 1$

$\therefore$ Inequality is still satisfied.

For $x \in \left[1, \frac{3}{2}\right)$, $[x] = 1, [2x] = 2$

$\therefore$ Inequality does not hold true.

Thus, $x \in (-\infty, 1)$

Hence, (d) is the correct answer.

Ex 11. Domain of $f(x) = \sin^{-1}[2x^2 - 3]$, where $[\]$ denotes the greatest integer function, is

- (a) $\left(-\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\right)$
 (b) $\left[-\sqrt{\frac{3}{2}}, -1\right] \cup \left[1, \sqrt{\frac{3}{2}}\right] \cup \{ \}$
 (c) $\left(-\sqrt{\frac{5}{2}}, \sqrt{\frac{5}{2}}\right)$
 (d) $\left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right]$

Sol. We must have $-1 \leq [2x^2 - 3] \leq 1$

$$\Rightarrow -1 \leq 2x^2 - 3 < 2$$

$$\Rightarrow 1 \leq x^2 < \frac{5}{2}$$

$$\Rightarrow x \in \left(-\sqrt{\frac{5}{2}}, -1\right] \cup \left[1, \sqrt{\frac{5}{2}}\right]$$

Hence, (d) is the correct answer.

Ex 12. If $y = 3[x] + 1 = 4[x - 1] - 10$, then $[x + 2y]$ is equal to

- (a) 76 (b) 61
 (c) 107 (d) 67

Sol. $y = 3[x] + 1 = 4[x - 1] - 10 = 4[x] - 14$

$$\Rightarrow [x] = 15 \text{ and } y = 3 \cdot 15 + 1 = 46$$

$$\Rightarrow [x + 2y] = 2y + [x] = 2 \times 46 + 15 = 107$$

Hence, (c) is the correct answer.

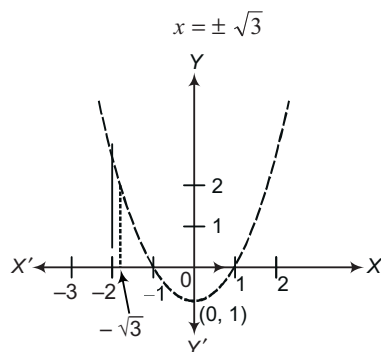
Ex 13. Domain of $f(x) = \sqrt{[x] - 1 + x^2}$, where $[\]$ denotes the greatest integer function, is

- (a) $(-\infty, -2) \cup [1, \infty)$
 (b) $(-\infty, -\sqrt{2}] \cup [1, \infty)$
 (c) $(-\infty, -3) \cup [1, \infty)$
 (d) $(-\infty, -\sqrt{3}] \cup [1, \infty)$

Sol. We must have $[x] - 1 + x^2 \geq 0$

$$\Rightarrow x^2 - 1 \geq -[x]$$

If $x^2 - 1 = 2$, then



From the adjacent graph, we get domain as $(-\infty, -\sqrt{3}] \cup [1, \infty)$

Hence, (d) is the correct answer.

Ex 14. The domain of

$f(x) = \sqrt{\sin^{-1}(3x - 4x^3)} + \sqrt{\cos^{-1}x}$ is equal to

- (a) $\left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right]$ (b) $\left[-1, -\frac{1}{2}\right] \cup \left[0, \frac{1}{2}\right]$
(c) $\left[0, \frac{1}{2}\right]$ (d) None of these

Sol. Here, $\sqrt{\cos^{-1}x}$ is defined for $x \in [-1, 1]$ and $\sqrt{\sin^{-1}(3x - 4x^3)}$ is defined for x when $0 \leq 3x - 4x^3 \leq 1$.

$$\begin{aligned} \therefore 3x - 4x^3 &\geq 0 \text{ and } 3x - 4x^3 \leq 1 \\ \Rightarrow x &\in \left(-\infty, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right] \text{ and } x \in [-1, \infty) \\ \Rightarrow x &\in \left[-1, -\frac{\sqrt{3}}{2}\right] \cup \left[0, \frac{\sqrt{3}}{2}\right] \end{aligned}$$

Hence, (a) is the correct answer.

Ex 15. Consider a function $f(n)$ defined for all $n \in \mathbb{N}$. The function satisfies the following two conditions

- (i) $f(1) + f(2) + f(3) + \dots = 1$
(ii) $f(n) = \{(1-p)p^{-1}\} \{f(n+1) + f(n+2) + \dots\}$, where $0 < p < 1$

Then, $f(2)$ is equal to

- (a) $p(1-p)$ (b) $1-p$
(c) $1+p$ (d) None of these

Sol. We have,

$$\begin{aligned} f(n) &= \{(1-p)p^{-1}\} \{f(n+1) + f(n+2) + \dots\} \\ \text{Put } n &= 1 \\ \Rightarrow f(1) &= \{(1-p)p^{-1}\} \{f(2) + f(3) + \dots\} \\ &= \{(1-p)p^{-1}\} \{1 - f(1)\} \\ &= (1-p)p^{-1} - (1-p)p^{-1}f(1) \\ \Rightarrow f(1) \{(1-p)p^{-1}\} &= (1-p)p^{-1} \\ \Rightarrow f(1) \cdot p^{-1} &= (1-p)p^{-1} \end{aligned}$$

$$\begin{aligned} \Rightarrow f(1) &= 1-p \\ \text{Put } n &= 2 \\ \Rightarrow f(2) &= \{(1-p)p^{-1}\} \{f(3) + f(4) + \dots\} \\ &= \{(1-p)p^{-1}\} \{1 - 1 + p - f(2)\} \\ &= \{(1-p)p^{-1}\} (p - f(2)) \\ &= (1-p) - (1-p)p^{-1}f(2) \\ \Rightarrow f(2) \cdot \{1 + (1-p)p^{-1}\} &= 1-p \\ \Rightarrow f(2) &= p(1-p) \end{aligned}$$

Hence, (a) is the correct answer.

Ex 16. Range of $f(x) = [\sin x] + [\cos x]$, where $[\]$ denotes the greatest integer function, is

- (a) $\{0\}$ (b) $\{0, 1\}$
(c) $\{1\}$ (d) None of these

Sol. $y = [\sin x] + [\cos x]$
 $\Rightarrow y^2 = 1 + |\sin 2x|$
 $\Rightarrow 1 \leq y^2 \leq 2$
 $\Rightarrow y \in [1, \sqrt{2}]$
 $\Rightarrow f(x) = 1, \forall x \in \mathbb{R}$
Hence, (c) is the correct answer.

Ex 17. Range of $f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x$ is

- (a) $\left(0, \frac{3\pi}{2}\right)$ (b) $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
(c) $\left(-\frac{\pi}{2}, \frac{3\pi}{4}\right]$ (d) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Sol. Clearly, the domain of the function is $[-1, 1]$.

We know that, $\tan^{-1}x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ for $x \in [-1, 1]$.

$$\begin{aligned} \therefore f(x) &= \frac{\pi}{2} + \tan^{-1}x \\ \Rightarrow \text{Range} &\in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right] \end{aligned}$$

Hence, (d) is the correct answer.

Ex 18. Range of $\sin^{-1}\left[x^2 + \frac{1}{2}\right] + \cos^{-1}\left[x^2 - \frac{1}{2}\right]$,

where $[\]$ denotes the greatest integer function, is

- (a) $\left\{\frac{\pi}{2}, \pi\right\}$ (b) $\{\pi\}$
(c) $\left\{\frac{\pi}{2}\right\}$ (d) None of these

Sol. $\left[x^2 + \frac{1}{2}\right] = \left[x^2 - \frac{1}{2} + 1\right] = 1 + \left[x^2 - \frac{1}{2}\right]$
Thus, for domain,
 $\left[x^2 - \frac{1}{2}\right] = 0, -1 \Rightarrow \left[x^2 + \frac{1}{2}\right] = 1, 0$
 $\Rightarrow f(x) = \sin^{-1}(1) + \cos^{-1}(0)$
or $f(x) = \sin^{-1}(0) + \cos^{-1}(-1)$
 $\therefore$ Range of $f(x) = \{\pi\}$
Hence, (b) is the correct answer.

Ex 19. Range of $f(x) = \sin^{-1}(\sqrt{x^2 + x + 1})$ is

- (a) $\left(0, \frac{\pi}{2}\right]$ (b) $\left(0, \frac{\pi}{3}\right]$ (c) $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

Sol. For domain, $0 \leq x^2 + x + 1 \leq 1$, but $x^2 + x + 1 > \frac{3}{4}$

$$\Rightarrow \frac{\sqrt{3}}{2} \leq \sqrt{x^2 + x + 1} \leq 1$$

$$\Rightarrow \frac{\pi}{3} \leq \sin^{-1}(\sqrt{x^2 + x + 1}) \leq \frac{\pi}{2}$$

Hence, (c) is the correct answer.

Ex 20. The minimum value of

$$f(x) = |x - 1| + |x - 2| + |x - 3| \text{ is}$$

- (a) 1 (b) 2 (c) 3 (d) 0

Sol. We have, $f(x) = \begin{cases} 6 - 3x, & x < 1 \\ 4 - x, & 1 \leq x < 2 \\ x, & 2 \leq x < 3 \\ 3x - 6, & x \geq 3 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} -3, & x < 1 \\ -1, & 1 < x < 2 \\ 1, & 2 \leq x < 3 \\ 3, & x > 3 \end{cases}$$

Thus, $f(x)$ is decreasing for $x < 2$ and increasing for $x > 2$.

$$\text{So, } \{f(x)\}_{\min} = f(2) = 2$$

Hence, (b) is the correct answer.

Ex 21. Range of $f(x) = \frac{e^x}{[x+1]}$, $x \geq 0$ is

- (a) $(0, \infty)$
(b) $[0, \infty)$
(c) $(1, \infty)$
(d) $[1, \infty)$

Sol. $f(x) = \frac{e^x}{1 + [x]}, x \geq 0$

$$\Rightarrow f(x) = \frac{e^x}{1 + I}, I \leq x < I + 1$$

where, $I \in \text{integers and } I \geq 0$

Minimum value of $f(x)$ when $x = 0$

$$\text{i.e. } f(x) \geq e^0 \text{ is } 1$$

$$\Rightarrow f(x) \geq \frac{e^0}{1 + 0}$$

$$\Rightarrow f(x) \geq 1$$

Thus, range of $f(x)$ is $[1, \infty)$.

Hence, (d) is the correct answer.

Ex 22. Range of $f(x) = [1 + \sin x] + [\cos x - 1] + [\tan^{-1} x]$, $\forall x \in [0, 2\pi]$, where $[]$ denotes the greatest integer function, is

- (a) $\{-1, 1, 2\}$
(b) $\{-1, 0, 1\}$
(c) $\{-1, 0, 1, 2\}$
(d) $\{-1, 0, 2\}$

Sol. $f(x) = [\sin x] + [\cos x] + [\tan^{-1} x]$

Then,

$$f(x) = \begin{cases} 1, & x = 0 \\ 0, & 0 < x < \tan^{-1} 1 \\ 1, & \tan^{-1} 1 \leq x < \frac{\pi}{2} \\ 2, & x = \frac{\pi}{2} \\ -1, & \frac{\pi}{2} \leq x < \frac{3\pi}{2} \\ 0, & \frac{3\pi}{2} \leq x < 2\pi \\ 2, & x = 2\pi \end{cases}$$

Thus, range of $f(x) = \{-1, 0, 1, 2\}$

Hence, (c) is the correct answer.

Ex 23. Range of $f(x) = [1 + \sin x] + \left[2 + \sin \frac{x}{2}\right]$

$$+ \left[3 + \sin \frac{x}{3}\right] + \dots + \left[n + \sin \frac{x}{n}\right], \forall x \in [0, \pi],$$

where $[]$ denotes the greatest integer function, is

(a) $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}\right\}$

(b) $\left\{\frac{n(n+1)}{2}\right\}$

(c) $\left\{\frac{n^2 + n - 2}{2}, \frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$

(d) $\left\{\frac{n(n+1)}{2}, \frac{n^2 + n + 2}{2}\right\}$

Sol. $f(x) = \frac{n(n+1)}{2} + [\sin x] + \left[\sin \frac{x}{2}\right] + \dots + \left[\sin \frac{x}{n}\right]$

Thus, range of $f(x) = \left\{\frac{n(n+1)}{2}, \frac{n(n+1)}{2} + 1\right\}$ as

$x \in [0, \pi]$.

Hence, (d) is the correct answer.

Ex 24. $f : [-4, 4] - \{-\pi, 0, \pi\} \rightarrow R$, where

$$f(x) = \cot(\sin x) + \left[\frac{x^2}{|a|}\right] \text{ and } [] \text{ denotes the}$$

greatest integer function, is an odd function. Complete set of values of a is

- (a) $(-16, 16) - \{0\}$ (b) $(-\infty, -16) \cup (16, \infty)$
(c) $[-16, 16] - \{0\}$ (d) $(-\infty, -16] \cup [16, \infty)$

Sol. For $f(x)$ to be odd, $\left[\frac{x^2}{|a|}\right]$ should not depend upon the

value of x .

$$\text{Since, } x \in [-4, 4] \Rightarrow 0 \leq x^2 \leq 16$$

$$\Rightarrow \left[\frac{x^2}{|a|}\right] = 0 \text{ if } |a| > 16$$

$$\therefore a \in (-\infty, -16) \cup (16, \infty)$$

Hence, (b) is the correct answer.

Ex 25. Period of $f(x) = \sin^{-1}(\sin x) + e^{\tan x}$ is

- (a) 2π (b) π
(c) $\frac{\pi}{2}$ (d) None of these

Sol. Period of $\sin^{-1}(\sin x)$ is 2π . Period of $e^{\tan x}$ is π .

Thus, period of $f(x) = \text{LCM}(2\pi, \pi) = 2\pi$

Hence, (a) is the correct answer.

Ex 26. Which of the following functions is non-periodic?

- (a) $\frac{2^x}{2^{[x]}}$ (b) $\sin^{-1}(\{x\})$
(c) $\sin^{-1}(\sqrt{\cos x})$ (d) $\sin^{-1}(\cos x^2)$

Sol. Period of $\frac{2^x}{2^{[x]}}$ is 1. Period of $\sin^{-1}\{x\}$ is 1. Period of

$\sin^{-1}(\sqrt{\cos x})$ is 2π , whereas $\sin^{-1}(\cos x^2)$ is non-periodic.

Hence, (d) is the correct answer.

Ex 27. If f is a periodic function, then

- (a) f has to be bounded
(b) f has to be onto
(c) f has to be into
(d) f may not be bounded

Sol. Periodic functions may not be bounded. e.g. $\tan x, \cot x$

Hence, (d) is the correct answer.

Ex 28. Period of $f(x) = x - [x + a] - b$, where $a, b \in \mathbb{R}^+$ and $[]$ denotes the greatest integer function, is

- (a) a (b) b (c) $|a - b|$ (d) 1

Sol. $f(x) = x - [x + a] - b$

$$\Rightarrow f(x) = x + a - [x + a] - (a + b)$$

$$\Rightarrow f(x) = \{x + a\} - (a + b)$$

Thus, period of $f(x)$ is 1.

Hence, (d) is the correct answer.

Ex 29. Period of $f(x) = \sin 3\pi \{x\} + \tan \pi [x]$, where $[]$ and $\{ \}$ denote the greatest integer function and fractional part respectively, is

- (a) 1 (b) 2
(c) 3 (d) None of these

Sol. Clearly, $\tan \pi [x] = 0, \forall x \in \mathbb{R}$ and period of $\sin 3\pi \{x\}$ is equal to 1.

Hence, (a) is the correct answer.

Ex 30. Period of $f(x) = [x] + [2x] + [3x] + [4x] + \dots + [nx] - \frac{n(n+1)}{2}x$, where $n \in \mathbb{N}$, is

- (a) n
(b) 1
(c) $\frac{1}{n}$
(d) None of the above

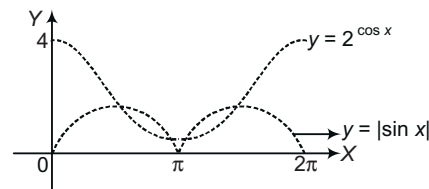
Sol. $f(x) = [x] + [2x] + [3x] + [4x] + \dots + [nx]$
 $\quad \quad \quad -(x + 2x + 3x + \dots + nx)$
 $\Rightarrow f(x) = -(\{x\} + \{2x\} + \{3x\} + \dots + \{nx\})$
Period of $f(x) = \text{LCM}\left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}\right) = 1$

Hence, (b) is the correct answer.

Ex 31. Total number of solutions of $2^{\cos x} = |\sin x|$ in $[-2\pi, 5\pi]$ is

- (a) 12 (b) 14 (c) 16 (d) 15

Sol. Graph of $y = 2^{\cos x}$ and $y = |\sin x|$ meet four times in $[0, 2\pi]$.



Thus, total number of solutions

$$= 4 + 4 + 4 + 2 = 14$$

Hence, (b) is the correct answer.

Ex 32. Let $A = \{x_1, x_2, \dots, x_m\}$, $B = \{y_1, y_2, \dots, y_n\}$, then total number of non-empty relations that can be defined from A to B , is

- (a) $m^n - 1$ (b) $n^m - 1$ (c) $mn - 1$ (d) $2^{mn} - 1$

Sol. Any relation from A to B will be a subset of $A \times B$. Now, total elements in $A \times B$ is equal to mn . This implies that number of non-empty subsets of $A \times B$ is equal to $2^{mn} - 1$. Thus, there are $2^{mn} - 1$ relations.

Hence, (d) is the correct answer.

Ex 33. If $f: \mathbb{R} \rightarrow \mathbb{R}$, where $f(x) = \frac{x^2 + 3x + 6}{x^2 + x + 1}$, then

$f(x)$ is

- (a) many-one and into (b) many-one and onto
(c) one-one and into (d) one-one and onto

Sol. Clearly, domain of $f(x)$ is all real numbers.

$$f'(x) = \frac{-2x^2 + 10x - 3}{(x^2 + x + 1)^2} = \frac{-(2x^2 - 10x + 3)}{(x^2 + x + 1)^2}$$

Now, discriminant of $2x^2 - 10x + 3$ is positive, thus

$f'(x)$ will not retain its sign throughout the domain.

Hence, $f(x)$ is many-one. Also, $\lim_{x \rightarrow \pm\infty} f(x) = 1$, thus $f(x)$ is also into.

Hence (a) is the correct answer.

Ex 34. If $f: \mathbb{N} \rightarrow \mathbb{N}$, where $f(x) = x - (-1)^x$, then f is

- (a) one-one and into (b) many-one and into
(c) one-one and onto (d) many-one and onto

Sol. $f(x) = \begin{cases} x - 1, & x \in \text{even} \\ x + 1, & x \in \text{odd} \end{cases}$

which is clearly one-one and onto.

Hence, (c) is the correct answer.

Ex 35. Let $f: R \rightarrow R$, where $f(x) = \frac{x^2 + ax + 1}{x^2 + x + 1}$.

Complete set of values of a such that $f(x)$ is onto, is

- (a) $(-\infty, \infty)$ (b) $(-\infty, 0)$
(c) $(0, \infty)$ (d) None of these

Sol. $y = \frac{x^2 + ax + 1}{x^2 + x + 1} \Rightarrow x^2(1 - y) + x(a - y) + (1 - y) = 0$

Since, x is real.

$$\therefore (a - y)^2 - 4(1 - y)^2 \geq 0$$

$$\Rightarrow -3y^2 + 2y(4 - a) + a^2 - 4 \geq 0, \forall y \in R$$

[for f to be onto]

But this is not possible.

Hence, (d) is the correct answer.

Ex 36. Let $A = \{1, 2, 3\}$. We define

- (i) $R_1 = \{(1, 1), (2, 2), (3, 3)\}$
(ii) $R_2 = \{(1, 2), (2, 1), (2, 2), (1, 1)\}$
(iii) $R_3 = \{(1, 2), (1, 3)\}$
(iv) $R_4 = \{(1, 1), (3, 3)\}$
(v) $R_5 = A \times A$, then

- (a) R_1, R_2, R_3, R_5 are all equivalence
(b) Only R_1 is equivalence and partially ordered relation
(c) R_1 and R_3 are partial order relations
(d) None of the above

Sol. (i) R_1 is reflexive and symmetric. Also, for any two pairs (x, y) and (y, z) of R_1 (in which second coordinate of one is equal to the first coordinate of other), we have necessarily $(x = y = z)$, so $(x, z) = (x, x) \in R_1, \forall x \in A$.

Therefore, R_1 is transitive also. Hence, R_1 is an equivalence relation on A . Again, for any two pairs (x, y) and (y, x) of R_1 , we have $x = y$. So, R_1 is anti-symmetric. Hence, R_1 is also partially ordered relation on A .

This is example of a relation which is both equivalence relation and partial order relation.

- (ii) R_2 is symmetric and transitive both, but R_2 is not reflexive $[(3, 3) \notin R_2]$. Also, R_2 is not anti-symmetric $[(1, 2) \in R_2 \text{ and } (2, 1) \in R_2 \text{ but } 1 \neq 2]$.
(iii) Clearly, R_3 is neither reflexive nor symmetric. But it is transitive. For $(1, 2) \in R_3$, if $(2, x) \in R_3$ such that $(1, x) \notin R_3$, then transitivity fails but that does not happen because $(2, x) \notin R_3$. So, our claim that $(1, 2) \in R_3$ and $(2, x) \in R_3 \Rightarrow (1, x) \in R_4$ remains true. Similar, in the case with $(1, 3) \in R_3$.
Moreover, R_3 is not anti-symmetric.

- (iv) R_4 is symmetric and transitive but not reflexive $[(2, 2) \notin R_4]$ on A . R_4 is anti-symmetric. For (x, y) and $(y, x) \in R_4$, $x = y$ cannot be false because (x, y) and $(y, x) \in R_4$ is true for $x = y = 2$ or $x = y = 3$ only.

- (v) R_5 is clearly an equivalence relation. Infact, it is universal relation on A .

Hence, (b) is the correct answer.

Ex 37. The relation R defined on the set N of natural numbers by $xRy \Leftrightarrow 2x^2 - 3xy + y^2 = 0$ is

- (a) symmetric but not reflexive
(b) only symmetric
(c) not symmetric but reflexive
(d) None of the above

Sol. $xRx \Leftrightarrow 2x^2 - 3x \cdot x + x^2 = 0; \forall x \in N$

$\therefore R$ is reflexive.

For $x = 1, y = 2; 2x^2 - 3xy + y^2 = 0$

$\therefore 1R2$ but for $2R1, 2 \cdot 2^2 - 3 \cdot 2 \cdot 1 + 1^2 = 3 \neq 0$

So, 2 is not related to 1.

$\therefore R$ is not symmetric.

Hence, (c) is the correct answer.

Ex 38. Let R be set of real numbers. If $f: R \rightarrow R$ is defined by $f(x) = e^x$, then f is

- (a) surjective but not injective
(b) injective but not surjective
(c) bijective
(d) neither surjective nor injective

Sol. f is injective (i.e. one-one), since $x_1, x_2 \in R$ and $x_1 \neq x_2$

$$\Rightarrow e^{x_1} \neq e^{x_2} \Rightarrow f(x_1) \neq f(x_2)$$

f is not surjective, since $e^x > 0, \forall x$ and so negative real number can be the image of any real number.

e.g. There is no real x such that $f(x) = -2$.

Hence, (b) is the correct answer.

Ex 39. Let S be any set and $P(S)$ be its power set. We define a relation R on $P(S)$ by ARB to mean $A \subseteq B, \forall A, B \in P(S)$. Then, r is

- (a) equivalence relation
(b) not an equivalence but partial order relation
(c) both equivalence and partial order relation
(d) None of the above

Sol. $A \subseteq A$, i.e. $ARA, \forall A \in P(S)$

$\therefore R$ is reflexive.

$$A \subseteq B \not\Rightarrow B \subseteq A$$

$$\therefore ARB \not\Rightarrow BRA$$

So, R is not symmetric.

$$ARB \text{ and } BRA \Rightarrow A \subseteq B \text{ and } B \subseteq A \Rightarrow A = B$$

Thus, R is anti-symmetric.

ARB and BRC

$$\Rightarrow A \subseteq B \text{ and } B \subseteq C$$

$$\Rightarrow A \subseteq C \Rightarrow ARC$$

$\therefore R$ is transitive relation.

Thus, R is partially ordered relation but not equivalence relation.

Hence, (b) is the correct answer.

Ex 40. If R and R' are symmetric relations (not disjoint) on a set A , then the relation $R \cap R'$ is

- (a) reflexive (b) symmetric
(c) transitive (d) None of these

Sol. Since, $R \cap R'$ is not disjoint, there is atleast one ordered pair say (a, b) in $R \cap R'$.

But $(a, b) \in R \cap R' \Rightarrow (a, b) \in R$ and $(a, b) \in R'$

Since, R and R' are symmetric relations.

$\therefore (b, a) \in R$ and $(b, a) \in R'$

and consequently $(b, a) \in R \cap R'$

Similarly, any other ordered pairs $(c, d) \in R \cap R'$, then we must also have $(d, c) \in R \cap R'$.

So, $R \cap R'$ is symmetric.

Hence, (b) is the correct answer.

Ex 41. Let ρ be the relation on the set R of all real numbers defined by apb iff $|a - b| \leq \frac{1}{2}$.

Then, ρ is

- (a) reflexive and symmetric but not transitive
- (b) symmetric and transitive but not reflexive
- (c) transitive but neither reflexive nor symmetric
- (d) None of the above

Sol. ρ is reflexive, since $|a - a| = 0 < \frac{1}{2}, \forall a \in R$.

ρ is symmetric, since $|a - b| < \frac{1}{2}$

$\Rightarrow |b - a| < \frac{1}{2}$

ρ is not transitive.

e.g. If we take three numbers $\frac{3}{4}, \frac{1}{3}, \frac{1}{8}$, then

$$\left| \frac{3}{4} - \frac{1}{3} \right| = \frac{5}{12} < \frac{1}{2} \text{ and } \left| \frac{1}{3} - \frac{1}{8} \right| = \frac{5}{24} < \frac{1}{2}$$

$$\text{But } \left| \frac{3}{4} - \frac{1}{8} \right| = \frac{5}{8} > \frac{1}{2}$$

$$\text{Thus, } \frac{3}{4} \rho \frac{1}{3} \text{ and } \frac{1}{3} \rho \frac{1}{8} \text{ but } \frac{3}{4} (\rho) \frac{1}{8}$$

Hence, (a) is the correct answer.

Ex 42. If $f(x) = \max \left\{ x^3, x^2, \frac{1}{64} \right\}, \forall x \in [0, \infty)$, then

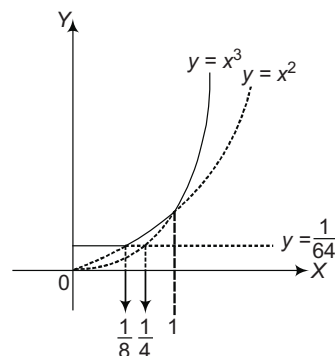
$$(a) f(x) = \begin{cases} x^2, & 0 \leq x \leq 1 \\ x^3, & x > 1 \end{cases}$$

$$(b) f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{4} \\ x^2, & \frac{1}{4} < x \leq 1 \\ x^3, & x > 1 \end{cases}$$

$$(c) f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^2, & \frac{1}{8} < x \leq 1 \\ x^3, & x > 1 \end{cases}$$

$$(d) f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^3, & x > \frac{1}{8} \end{cases}$$

$$\text{Sol. Clearly, } f(x) = \begin{cases} \frac{1}{64}, & 0 \leq x \leq \frac{1}{8} \\ x^2, & \frac{1}{8} < x \leq 1 \\ x^3, & x > 1 \end{cases}$$



Hence, (c) is the correct answer.

Ex 43. Let $f_1(n) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, then

$f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n)$ is equal to

- (a) $n f_1(n) - 1$
- (b) $(n+1) f_1(n) - n$
- (c) $(n+1) f_1(n) + n$
- (d) $n f_1(n) + n$

Sol. In the sum, $f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n)$,

1 occurs n times, $\frac{1}{2}$ occurs $(n-1)$ times, $\frac{1}{3}$ occurs $(n-2)$ times and so on.

$$\therefore f_1(1) + f_1(2) + f_1(3) + \dots + f_1(n)$$

$$= n \cdot 1 + (n-1) \cdot \frac{1}{2} + (n-2) \cdot \frac{1}{3} + \dots + 1 \cdot \frac{1}{n}$$

$$= n \left(1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{n-1}{n} \right)$$

$$= n f_1(n) - \left[\left(1 - \frac{1}{2} \right) + \left(1 - \frac{1}{3} \right) + \left(1 - \frac{1}{4} \right) \right.$$

$$\left. + \dots + \left(1 - \frac{1}{n} \right) \right]$$

$$= n f_1(n) - [n - f_1(n)]$$

$$= (n+1) f_1(n) - n$$

Hence, (b) is the correct answer.

Ex 44. Let $f(x) = x^2 + 3x - 3, x \geq 0$. If n points $x_1, x_2, x_3, \dots, x_n$ are so chosen on X -axis such that

$$(i) \frac{1}{n} \sum_{i=1}^n f^{-1}(x_i) = f \left(\frac{1}{n} \sum_{i=1}^n x_i \right)$$

$$(ii) \sum_{i=1}^n f^{-1}(x_i) = \sum_{i=1}^n x_i, \text{ where } f^{-1} \text{ denotes the inverse of } f.$$

Then, AM of x_i 's is

- (a) 1
- (b) 2
- (c) 3
- (d) 4

Sol. $\frac{f^{-1}(x_1) + f^{-1}(x_2) + \dots + f^{-1}(x_n)}{n}$
 $= f\left(\frac{x_1 + x_2 + \dots + x_n}{n}\right)$
 and $\frac{f^{-1}(x_1) + f^{-1}(x_2) + \dots + f^{-1}(x_n)}{m}$
 $= \left(\frac{x_1 + x_2 + \dots + x_n}{m}\right)$

$\Rightarrow f(\bar{x}) = \bar{x}$, where $\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n}$
 $\Rightarrow \bar{x}^2 + 3\bar{x} - 3 = \bar{x}$
 $\Rightarrow \bar{x}^2 + 2\bar{x} - 3 = 0$
 $\Rightarrow \bar{x} = -3, 1$
 $\therefore \bar{x} = 1$ [$\because x \geq 0$]
 Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 45. The function $f : N \rightarrow N$ (N is set of natural numbers) defined by $f(n) = 2n + 3$, is
 (a) surjective (b) not surjective
 (c) injective (d) None of these

Sol. f is injective, i.e. one-one.
 $n, m \in N, n \neq m$
 $\Rightarrow 2n + 3 \neq 2m + 3 \Rightarrow f(n) \neq f(m)$
 f is not surjective, i.e. f is not onto.
 e.g. $f(n) = 1$ cannot be the image of any natural number.
 Hence, (b) and (c) are the correct answer.

Ex 46. Let $f : R \rightarrow R$ be a function defined by $f(x+1) = \frac{f(x)-5}{f(x)-3}, \forall x \in R$. Then, which of the following statement(s) is/are true?
 (a) $f(2008) = f(2004)$
 (b) $f(2006) = f(2010)$
 (c) $f(2006) = f(2002)$
 (d) $f(2006) = f(2018)$

Sol. We have, $f(x+1) = \frac{f(x)-5}{f(x)-3}$... (i)

$f(x) \cdot f(x+1) - 3f(x+1) = f(x) - 5$
 $\Rightarrow f(x) = \frac{3f(x+1)-5}{f(x+1)-1}$

On replacing x by $(x-1)$, we get

$f(x-1) = \frac{3f(x)-5}{f(x)-1}$... (ii)

From Eq. (i), we get

$f(x+2) = \frac{f(x+1)-5}{f(x+1)-3}$
 $= \frac{\frac{f(x)-5}{f(x)-3} - 5}{\frac{f(x)-5}{f(x)-3} - 3} = \frac{2f(x)-5}{f(x)-2}$... (iii)

From Eq. (ii), we get

$f(x-2) = \frac{3f(x-1)-5}{f(x-1)-1}$
 $= \frac{3\left(\frac{3f(x)-5}{f(x)-1}\right) - 5}{\frac{3f(x)-5}{f(x)-1} - 1} = \frac{2f(x)-5}{f(x)-2}$... (iv)

From Eqs. (iii) and (iv), we get

$f(x+2) = f(x-2)$
 $\Rightarrow f(x+4) = f(x)$
 $\therefore f(x)$ is periodic with period 4.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 47. Let $f(x) = 1 - x - x^3$. Then, the real values of x satisfying the inequality $1 - f(x) - f^3(x) > f(1 - 5x)$ are
 (a) $(-2, 0)$ (b) $(0, 2)$
 (c) $(2, \infty)$ (d) $(-\infty, -2)$

Sol. $f(x) = 1 - x - x^3$,

On replacing x by $f(x)$ we get

$f\{f(x)\} = 1 - f(x) - f^3(x)$

Hence, the given equation is

$f\{f(x)\} > f(1 - 5x)$
 $\Rightarrow f(x) < 1 - 5x$ [$\because f'(x) = -1 - 3x^2 < 0$]
 $\Rightarrow f(x) = 1 - x - x^3$
 $\Rightarrow 1 - x - x^3 < 1 - 5x \Rightarrow x^3 - 4x > 0$
 $\Rightarrow x(x-2)(x+2) > 0$
 $\therefore x \in (-2, 0) \cup (2, \infty)$

Hence, (a) and (c) are the correct answers.

Ex 48. If a function satisfies

$(x-y)f(x+y) - (x+y)f(x-y)$
 $= 2(x^2y - y^3), \forall x, y \in R$ and $f(1) = 2$, then

- (a) $f(x)$ must be polynomial function
 (b) $f(3) = 12$
 (c) $f(0) = 0$
 (d) $f(x)$ may not be differentiable

Sol. $(x-y)f(x+y) - (x+y)f(x-y)$
 $= 2y(x-y)(x+y)$

Let $x - y = u$ and $x + y = v$

$\Rightarrow uv(v) - v f(u) = uv(v-u)$

$\Rightarrow \frac{f(v)}{v} - \frac{f(u)}{u} = v - u$

$\Rightarrow \left(\frac{f(v)}{v} - v\right) = \left(\frac{f(u)}{u} - u\right) = \text{constant}$

Let $\frac{f(x)}{x} - x = \lambda$

$\Rightarrow f(x) = (\lambda x + x^2)$

But $f(1) = 2$

$$\therefore \lambda + 1 = 2 \Rightarrow \lambda = 1$$

$$\Rightarrow f(x) = x^2 + x$$

Hence, (a), (b) and (c) are the correct answers.

Ex 49. If the fundamental period of function $f(x) = \sin x + \cos(\sqrt{4-a^2})x$ is 4π , then the values of a is/are

(a) $\frac{\sqrt{15}}{2}$ (b) $-\frac{\sqrt{15}}{2}$ (c) $\frac{\sqrt{7}}{2}$ (d) $-\frac{\sqrt{7}}{2}$

Sol. Period of $\sin x = 2\pi$

and period of $\cos(\sqrt{4-a^2})x = \frac{2\pi}{\sqrt{4-a^2}}$

$$\Rightarrow \text{LCM}\left(2\pi, \frac{2\pi}{\sqrt{4-a^2}}\right) = 4\pi \quad [\text{given}]$$

i.e. $\sqrt{4-a^2} = \frac{p}{2}$, where $p = 1, 3$

Thus, $a^2 = \frac{15}{4}, \frac{7}{4}$

$$\Rightarrow a = \pm \frac{\sqrt{15}}{2}, \pm \frac{\sqrt{7}}{2}$$

Thus, (a), (b), (c) and (d) are the correct answers.

Ex 50. Let $f(x)$ be a real valued function such that

$$f(0) = \frac{1}{2} \quad \text{and} \quad f(x+y) = f(x)f(a-y)$$

$+ f(y)f(a-x), \forall x, y \in R$, then for some real a ,

(a) $f(x)$ is a periodic function

(b) $f(x)$ is a constant function

(c) $f(x) = \frac{1}{2}$

(d) $f(x) = \frac{\cos x}{2}$

Sol. $f(x+y) = f(x)f(a-y) + f(y)f(a-x) \dots (i)$

On putting $x = y = 0$, we get $f(a) = \frac{1}{2}$

$$\Rightarrow f(x) = f(a-x)$$

On putting $y = a-x$ in Eq. (i), we get

$$f(a) = \{f(x)\}^2 + \{f(a-x)\}^2$$

$$\Rightarrow f(x) = \pm \frac{1}{2}$$

$$\therefore f(x) = 1/2 \quad \left[\because f(x) \neq -\frac{1}{2} \right]$$

Hence, (a), (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 51-57) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 51. Statement I The functions $f(x) = x \sin x$ and $f'(x) = x \cos x + \sin x$ are both non-periodic.

Statement II The derivative of differentiable function is non-periodic function.

Sol. If we take example of $x + \sin x$, it is non-periodic whereas its derivative $1 + \cos x$ is periodic.

Hence, (c) is the correct answer.

Ex 52. Statement I The maximum value of $\sin \sqrt{2}x + \sin ax$ cannot be 2, where a is positive rational number.

Statement II $\frac{\sqrt{2}}{a}$ is irrational.

Sol. The value of $\sin \sqrt{2}x + \sin ax$ can be equal to 2, if $\sin \sqrt{2}x$ and $\sin ax$ both are equal to one but both are not equal to one for any common value of x .

Hence, (a) is the correct answer.

Ex 53. Let $f: R \rightarrow R$ be a function such that

$$f(x) = \frac{e^{|x|} - e^{-x}}{e^x + e^{-x}}.$$

Statement I $f(x)$ is into function.

Statement II $f(x)$ is many-one function and many-one function is not onto.

Sol. Clearly, $f(x)$ is many-one and into function.

Hence, (c) is the correct answer.

Ex 54. Statement I The range of

$$f(x) = \sin\left(\frac{\pi}{5} + x\right) - \sin\left(\frac{\pi}{5} - x\right) - \sin\left(\frac{2\pi}{5} + x\right) + \sin\left(\frac{2\pi}{5} - x\right) \text{ is } [-1, 1].$$

Statement II $\cos \frac{\pi}{5} - \cos \frac{2\pi}{5} = \frac{1}{2}$

$$\begin{aligned} \text{Sol. } f(x) &= 2 \cos \frac{\pi}{5} \sin x - 2 \cos \frac{2\pi}{5} \sin x \\ &= 2 \sin x \left[\cos \frac{\pi}{5} - \cos \frac{2\pi}{5} \right] \\ &= 2 \sin x [\cos 36^\circ - \cos 72^\circ] \\ &= 2 \sin x [\cos 36^\circ - \sin 18^\circ] \\ &= 2 \sin x \left[\frac{\sqrt{5}+1}{4} - \left(\frac{\sqrt{5}-1}{4} \right) \right] = \sin x \end{aligned}$$

$\therefore$ Range is $[-1, 1]$.

Hence, (a) is the correct answer.

Ex 55. Statement I The period of

$$f(x) = 2\cos\frac{1}{3}(x - \pi) + 4\sin\frac{1}{3}(x - \pi) \text{ is } 3\pi.$$

Statement II If T is the period of $f(x)$, then the period of $f(ax + b)$ is $\frac{T}{|a|}$.

Sol. Periods of $2\cos\frac{1}{3}(x - \pi)$ and

$$4\sin\frac{1}{3}(x - \pi) \text{ are } \frac{2\pi}{1/3}, \frac{2\pi}{1/3} \text{ or } 6\pi, 6\pi.$$

$\therefore$ Period of their sum = 6π

Hence, (d) is the correct answer.

Ex 56. Statement I If $f(x) = \frac{1}{1-x}$, $x \neq 0, 1$, then the graph of the function $y = f(f(f(x)))$; $x > 1$, is a straight line.

Statement II If $f(x) = \frac{1}{1-x}$, then $f(f(f(x))) = x$, $\forall x \in R$

$$\text{Sol. } f(f(x)) = \frac{1}{1-f(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}, \forall x \in R - \{0, 1\}$$

$$\therefore f(f(f(x))) = \frac{1}{1-f(f(x))} = \frac{1}{1-\frac{x-1}{x}} = \frac{1}{\frac{x-x+1}{x}} = \frac{1}{\frac{1}{x}} = x, \forall x \in R - \{0, 1\}$$

So, Statement I is true and Statement II is false. Hence, (c) is the correct answer.

Ex 57. Statement I Domain of $f(x) = \frac{1}{\sqrt{|x| - x}}$ is

$(-\infty, 0)$.

Statement II $|x| = x > 0$ for $x \in R^-$.

$$\text{Sol. } f(x) = \frac{1}{\sqrt{|x| - x}}, \text{ for domain } |x| = x > 0$$

$\Rightarrow |x| > x$, which is true for $x \in R^-$, i.e., $x < 0$.

So, domain of $f(x) \in (-\infty, 0)$.

$\therefore$ Statement I is true and Statement II is the correct explanation of Statement I.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 58-60) Let $f: R \rightarrow R$ be a continuous function such that

$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2.$$

Ex 58. $f(3)$ is equal to

- (a) $f(0)$
- (b) $4 + f(0)$
- (c) $9 + f(0)$
- (d) $16 + f(0)$

$$\text{Sol. } f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2$$

$$\therefore f\left(\frac{x}{2}\right) - 2f\left(\frac{x}{4}\right) + f\left(\frac{x}{8}\right) = \left(\frac{x}{2}\right)^2$$

$$f\left(\frac{x}{4}\right) - 2f\left(\frac{x}{8}\right) + f\left(\frac{x}{16}\right) = \left(\frac{x}{4}\right)^2$$

$$\vdots$$

$$f\left(\frac{x}{2^n}\right) - 2f\left(\frac{x}{2^{n+1}}\right) + f\left(\frac{x}{2^{n+2}}\right) = \left(\frac{x}{2^n}\right)^2$$

On adding, we get

$$f(x) + f\left(\frac{x}{2}\right) + \dots + f\left(\frac{x}{2^{n+2}}\right) = x^2 \left(1 + \frac{1}{2^2} + \dots + \frac{1}{2^{2n}}\right)$$

$$\text{As } n \rightarrow \infty, \text{ we get } f(x) - f\left(\frac{x}{2}\right) = \frac{4x^2}{3}$$

Repeating the same procedure again, we get

$$f(x) - f(0) = \frac{16x^2}{9}$$

$$\therefore f(3) = 16 + f(0)$$

Hence, (d) is the correct answer.

Ex 59. The equation $f(x) - x - f(0) = 0$ has exactly

- (a) no solution
- (b) one solution
- (c) two solutions
- (d) infinite solutions

Sol. Clearly, the equation has two solutions.

Hence, (c) is the correct answer.

Ex 60. $f'(0)$ is equal to

- (a) 0
- (b) 1
- (c) $f(0)$
- (d) $-f(0)$

$$\text{Sol. } f'(x) = \frac{32x}{9}$$

$$\therefore f'(0) = 0$$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 61-63) Let

$$f(x) = \frac{1}{2} \left[f(xy) + f\left(\frac{x}{y}\right) \right] \text{ for } x, y \in R^+ \text{ such that}$$

$$f(1) = 0, f'(1) = 2.$$

Ex 61. $f(x) - f(y)$ is equal to

- (a) $f\left(\frac{y}{x}\right)$
- (b) $f\left(\frac{x}{y}\right)$
- (c) $f(2x)$
- (d) $f(2y)$

$$\text{Sol. } f(x) = \frac{1}{2} \left[f(xy) + f\left(\frac{x}{y}\right) \right] \quad \dots(i)$$

On interchanging x and y , we get

$$f(y) = \frac{1}{2} \left[f(xy) + f\left(\frac{y}{x}\right) \right] \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$f(x) - f(y) = \frac{1}{2} \left\{ f\left(\frac{x}{y}\right) - f\left(\frac{y}{x}\right) \right\} \quad \dots(iii)$$

On putting $x = 1$ in Eq. (i), we get

$$f(1) = \frac{1}{2} \left[f(y) + f\left(\frac{1}{y}\right) \right] \quad \dots(iv)$$

$$\Rightarrow f(1) = 0 \Rightarrow f(y) = -f\left(\frac{1}{y}\right)$$

$$\text{So, } f\left(\frac{x}{y}\right) = -f\left(\frac{y}{x}\right)$$

$$\therefore \text{Eq. (iii) becomes } f(x) - f(y) = f\left(\frac{x}{y}\right)$$

Hence, (b) is the correct answer.

Ex 62. $f'(3)$ is equal to

- (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Sol. $f'(x) \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

Using the result in question no. 62,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f\left(\frac{x+h}{x}\right)}{\frac{h}{x}} = \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} \\ &= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x} \cdot x} = \frac{1}{x} f'(1) = \frac{1}{x} \cdot 2 = \frac{2}{x} \\ \therefore f'(3) &= \frac{2}{3} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 63. $f(e)$ is equal to

- (a) 2 (b) 1 (c) 3 (d) 4

Sol. From question no. 63, $f'(x) = \frac{2}{x}$

$$\therefore f(x) = 2 \log x + C$$

$$\Rightarrow f(1) = 0 \Rightarrow C = 0$$

$$\therefore f(x) = 2 \log x$$

$$\therefore f(e) = 2 \log e = 2$$

Hence, (a) is the correct answer.

Type 5. Match the Columns

Ex 67. Match the statements of Column I with values of Column II.

Column I (Function)	Column II (Domain)
A. $\sqrt{\sin(\cos x)}$	p. $x \in R$
B. $\sqrt{\cos(\sin x)}^{-1}$	q. $x \in R - \left\{ n\pi \pm \frac{\pi}{6} \right\}$
C. $\tan(\pi \sin x)$	r. $x \in \left(n\pi, n\pi + \frac{\pi}{2} \right)$
D. $\ln(\tan x)$	s. $x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2} \right]$

Passage III (Ex. Nos. 64-66) Let

$g(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ and $f(x) = \sqrt{g(x)}$, $f(x)$ has its non-zero local minimum and maximum values at -3 and 3 , respectively. If a_3 belongs to the domain of the function $h(x) = \sin^{-1}\left(\frac{1+x^2}{2x}\right)$.

Ex 64. The value of $a_1 + a_2$ is

- (a) 30 (b) -30 (c) 27 (d) -27

Ex 65. The value of a_0 is

- (a) equal to 50
(b) greater than 54
(c) less than 54
(d) less than 50

Ex 66. $f(-10)$ is defined for

- (a) $a_0 > 830$
(b) $a_0 < 830$
(c) $a_0 = 830$
(d) None of the above

Sol. (Ex. Nos. 64-66)

$D_h = \{-1, 1\}$, as minimum occurs before maxima.

$$\therefore a_3 = -1$$

$$\text{Now, } g(x) = a_0 + a_1x + a_2x^2 - x^3$$

$$\begin{aligned} \Rightarrow g'(x) &= a_1 + 2a_2x - 3x^2 \\ &= -3(x-3)(x+3) = -3x^2 + 27 \end{aligned}$$

$$\therefore a_1 = 27, a_2 = 0$$

$$\therefore a_1 + a_2 = 27$$

Also, $g(-3) > 0$ and $g(3) > 0$

$$\Rightarrow a_0 > 54$$

$$\text{and } a_0 < -54$$

$$\therefore a_0 > 54$$

$$\text{Now, } g(x) = a_0 + 27x - x^3$$

$$\text{and } f(x) = \sqrt{a_0 + 27x - x^3}$$

$$\therefore f(-10) = \sqrt{a_0 - 270 - 1000}$$

Clearly, $f(-10)$ is defined for $a_0 > 1270$.

- 64.** (c) **65.** (b) **66.** (d)

Sol. A. For $\sqrt{\sin(\cos x)}$, $\sin(\cos x) \geq 0$

$$\therefore 2n\pi \leq \cos x \leq (2n+1)\pi$$

$$\Rightarrow 0 \leq \cos x \leq 1$$

$$\therefore x \in \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2} \right]$$

B. For $(\sqrt{\cos(\sin x)})^{-1}$,

$$\cos(\sin x) > 0$$

$$\Rightarrow -\frac{\pi}{2} < \sin x < \frac{\pi}{2}$$

$$\therefore x \in R$$

C. For $\tan(\pi \sin x)$,

$$\begin{aligned}\pi \sin x &\neq \pm \frac{\pi}{2} \\ \Rightarrow \sin x &\neq \pm \frac{1}{2} \\ \Rightarrow x &\neq n\pi \pm \frac{\pi}{6} \\ \therefore x &\in R - \left\{ n\pi \pm \frac{\pi}{6} \right\}\end{aligned}$$

D. For $\ln(\tan x)$, $\tan x > 0$

$$\begin{aligned}\Rightarrow 0 < x &< \frac{\pi}{2} \\ \Rightarrow n\pi < x &< n\pi + \frac{\pi}{2} \\ \therefore x &\in \left(n\pi, n\pi + \frac{\pi}{2} \right)\end{aligned}$$

$A \rightarrow s$; $B \rightarrow p$; $C \rightarrow q$; $D \rightarrow r$

Ex 68. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $ 4\sin x - 1 < \sqrt{5}$, $x \in [0, \pi]$, then domain is	p. $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
B. If $4\sin^2 x - 8\sin x + 3 \leq 0$, $x \in [0, 2\pi]$, then domain is	q. $\left[\frac{3\pi}{4}, \pi\right] \cup \{0\}$
C. If $ \tan x \leq 1$ and $x \in [-\pi, \pi]$, then domain is	r. $\left[0, \frac{3\pi}{10}\right]$
D. If $\cos x - \sin x \geq 1$ and $x \in [0, 2\pi]$, then domain is	s. $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

Sol. A. $|4\sin x - 1| < \sqrt{5}$

$$\begin{aligned}\Rightarrow -\sqrt{5} &< 4\sin x - 1 < \sqrt{5} \\ \Rightarrow 1 - \sqrt{5} &< 4\sin x < 1 + \sqrt{5} \\ \Rightarrow \frac{1 - \sqrt{5}}{4} &< \sin x < \frac{1 + \sqrt{5}}{4} \\ \Rightarrow -\frac{\pi}{10} &< x < \frac{3\pi}{10}\end{aligned}$$

But from the domain, $x \in \left(0, \frac{3\pi}{10}\right]$

B. $4\sin^2 x - 8\sin x + 3 \leq 0$, $0 \leq x < 2\pi$

$$\begin{aligned}\Rightarrow (2\sin x - 1)(2\sin x - 3) &\leq 0 \\ \Rightarrow 2\sin x - 1 &\geq 0 \\ \Rightarrow \sin x &\geq \frac{1}{2} \\ \therefore x &\in \left[\frac{\pi}{6}, \frac{5\pi}{6}\right]\end{aligned}$$

C. $|\tan x| \leq 1$, $x \in [-\pi, \pi]$

$$\begin{aligned}\Rightarrow -1 &\leq \tan x \leq 1 \\ \Rightarrow n\pi - \frac{\pi}{4} &\leq x \leq n\pi + \frac{\pi}{4}\end{aligned}$$

On putting $n = 0$, we get

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$$

On putting $n = 1$, we get

$$\begin{aligned}\pi - \frac{\pi}{4} &\leq x \leq \pi + \frac{\pi}{4} \\ \Rightarrow \frac{3\pi}{4} &\leq x \leq \frac{5\pi}{4}\end{aligned}$$

But from domain, $\frac{3\pi}{4} \leq x \leq \pi$

On putting $x = -1$, we get

$$\begin{aligned}-\pi - \frac{\pi}{4} &\leq x \leq -\pi + \frac{\pi}{4} \\ \Rightarrow -\frac{5\pi}{4} &\leq x \leq -\frac{3\pi}{4}\end{aligned}$$

But from domain, $-\pi \leq x \leq -\frac{3\pi}{4}$

$$\text{Finally, } \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$$

D. $\cos x - \sin x \geq 1$, $0 \leq x \leq 2\pi$

$$\begin{aligned}\Rightarrow \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos x - \frac{\sin x}{\sqrt{2}} \right) &\geq 1 \\ \Rightarrow \left(\cos x \cdot \cos \frac{\pi}{4} - \sin x \cdot \sin \frac{\pi}{4} \right) &\geq \frac{1}{\sqrt{2}} \\ \Rightarrow \cos \left(x + \frac{\pi}{4} \right) &\geq \frac{1}{\sqrt{2}} \\ \Rightarrow 2n\pi - \frac{\pi}{4} \leq x + \frac{\pi}{4} &\leq 2n\pi + \frac{\pi}{4} \\ \Rightarrow 2n\pi - \frac{\pi}{4} \leq x &\leq 2n\pi\end{aligned}$$

On substituting suitable values of n according to domain, $x \in \left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$.

$A \rightarrow r$; $B \rightarrow s$; $C \rightarrow p$; $D \rightarrow q$

Ex 69. Let $f(x) = ax^2 + bx + c$. If $f'(1) = 8$, $f(2) + f''(2) = 33$ and $2a + 3b + 6c = 14$, then match the statements of Column I with values of Column II.

Column I	Column II
A. Global maximum value of $f(x)$ is	p. not defined
B. If global minimum value of $f(x) = k$, then $28k$ is equal to	q. 48
C. The number of real roots of $f(x) = 0$ are	r. 0
D. The number of real roots of $f(x) = 3$ are	s. 2

Sol. $y'' = \text{constant}$

$$2a + b = 8 \quad \dots(i)$$

$$6a + 2b + c = 33 \quad \dots(ii)$$

$$2a + 3b + 6c = 14 \quad \dots(iii)$$

On solving, we get $a = 7$, $b = -6$, $c = 3$

A. Global maximum does not exist.

B. Global minimum value = $f\left(\frac{3}{7}\right) = \frac{12}{7}$

$$\therefore 28k = 48$$

C. The number of real roots of $f(x) = 0$ is 0.

D. Clearly, two real roots.

$A \rightarrow p$; $B \rightarrow q$; $C \rightarrow r$; $D \rightarrow s$

Type 6. Single Integer Answer Type Questions

Ex 70. If $f(x)$ is a function such that
 $f(x-1) + f(x+1) = \sqrt{3}f(x)$ and $f(5) = 10$,
 then the sum of digits of the value of
 $\sum_{r=0}^{19} f(5+12r)$ is _____.

Sol. (2) $f(x-1) + f(x+1) = \sqrt{3}f(x)$
 $\Rightarrow f(x) + f(x+2) = \sqrt{3}f(x+1)$
 On putting $x = x+2$, we get
 $f(x+1) + f(x+3) = \sqrt{3}f(x+2)$
 $\Rightarrow f(x-1) + 2f(x+2) + f(x+3)$
 $\quad = \sqrt{3}[\sqrt{3}f(x+1)]$
 $\Rightarrow f(x-1) + f(x+3) = f(x+1)$
 Again, putting $x = x+2$, we get
 $f(x+1) + f(x+5) = f(x+3)$
 $\Rightarrow f(x-1) + f(x+5) = 0$
 $\Rightarrow f(x+5) = -f(x-1)$
 $\Rightarrow f(x) = -f(x+6)$
 $\Rightarrow f(x+12) = f(x)$
 $\therefore \sum_{r=0}^{19} f(5+12r) = 20f(5)$
 $\quad = 20 \times 10 = 200$

Hence, sum of digits = $2 + 0 + 0 = 2$

Ex 71. If $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$ for all positive
 values of x and y , $f(1) = 0$ and $f'(1) = 1$, then
 $f(e)$ is _____.

Sol. (1) We have
 $2f(x) = f(xy) + f\left(\frac{x}{y}\right)$... (i)
 On putting $x = 1$ in Eq. (i), we get
 $2f(1) = f(y) + f\left(\frac{1}{y}\right)$... (ii)
 On replacing x by y and y by x in Eq. (i), we get
 $2f(y) = f(yx) + f\left(\frac{y}{x}\right)$... (iii)
 From Eqs. (i) and (iii), we get
 $2[f(x) - f(y)] = f\left(\frac{x}{y}\right) - \left\{ -f\left(\frac{x}{y}\right) \right\} = 2f\left(\frac{x}{y}\right)$... (iv)

$$\Rightarrow f(x) - f(y) = f\left(\frac{x}{y}\right) \quad \dots (v)$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} = f'(1) = 1$$

$$\text{and } \lim_{h \rightarrow 0} \frac{f(1+h)}{h} = 1 \quad [\because f(1) = 0]$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f\left(1 + \frac{h}{x}\right)}{\frac{h}{x}} = \frac{1}{x}$$

$$\Rightarrow f(x) = \log|x| + C$$

$$\therefore f(1) = 0 \Rightarrow C = 0$$

$$\text{Hence, } f(e) = 1$$

Ex 72. Let f be a function from the set of positive
 integers to the set of real number such that

$$(i) f(1) = 1$$

$$(ii) \sum_{r=1}^n rf(r) = n(n+1)f(n), \forall n \geq 2$$

Then, find the value of $2126f(1063)$.

Sol. (1) $f(1) + 2f(2) + 3f(3) + \dots + n(f(n))$
 $\quad = n(n+1)f(n)$
 $\Rightarrow f(1) + 2(2) + 3(3) + \dots + (n+1)f(n+1)$
 $\quad = (n+1)(n+2)f(n+1)$
 $\Rightarrow nf(n) = (n+1)f(n+1)$
 $\therefore 2f(2) = 3f(3) = \dots = n(f(n))$
 i.e. $f(n) = \frac{1}{2n}$
 $\therefore 2126f(1063) = 1$

Ex 73. If $f(x) = \frac{x^4 + x^2 + 1}{x^2 - x + 1}$, then find the value of
 $f(\omega^n)$, where ω is the non-real root of the
 equation $\omega^3 = 1$ and n is a multiple of 3.

Sol. (3) We have, $f(x) = \frac{x^4 + x^2 + 1}{x^2 - x + 1}$
 $\Rightarrow f(x) = x^2 + x + 1$
 Now, $f(\omega^n) = \omega^{2n} + \omega^n + 1 = 3$
 $\therefore \omega^n = 1$, when n is a multiple of 3.

Target Exercises

Type 1. Only one correct option

- The relation R defined in $A = \{1, 2, 3\}$ by aRb , if $|a^2 - b^2| \leq 5$. Which of the following is false?
 - $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$
 - $R^{-1} = R$
 - Domain of $R = \{1, 2, 3\}$
 - Range of $R = \{5\}$
- Let A be the set of first ten natural numbers and R be a relation in A defined by $(x, y) \in R$ if and only if $x + 2y = 10$. Which of the following is false?
 - $R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$
 - Domain of $R = \{2, 4, 6, 8\}$
 - Range of $R = \{1, 2, 3, 4\}$
 - Atleast one is false
- A relation R is defined in the set Z of integers as $(x, y) \in R$ iff $x^2 + y^2 = 9$. Which of the following is false?
 - $R = \{(0, 3), (0, -3), (3, 0), (-3, 0)\}$
 - Domain of $R = \{-3, 0, 3\}$
 - Range of $R = \{-3, 0, 3\}$
 - Atleast one is false
- The range of the function $f(x) = \frac{x^2 - x}{x^2 + 2x}$ is
 - $R / \left\{1, -\frac{1}{2}\right\}$
 - R
 - $R / \{1\}$
 - None of these
- Let $A = \{1, 2, 3, 4\}$ and $R = \{(2, 2), (3, 3), (4, 4), (1, 2)\}$ be a relation in A . Then, R is
 - reflexive
 - symmetric
 - transitive
 - None of these
- For real numbers x and y , we define xRy iff $x - y + \sqrt{5}$ is an irrational number. Then, relation R is
 - reflexive
 - symmetric
 - transitive
 - None of these
- Let X be a family of sets and R be a relation in X defined by ' A is disjoint from B '. Then, relation R is
 - reflexive
 - symmetric
 - transitive
 - anti-symmetrictic
- If $n|m$ means that n is a factor of m , then relation ' $|$ ' in $Z - \{0\}$ is
 - reflexive and symmetric
 - symmetric and transitive
 - reflexive, symmetric and transitive
 - reflexive, transitive and not symmetric
- R is a relation defined in Z by aRb if and only if $ab \geq 0$, then R is
 - reflexive
 - symmetric
 - transitive
 - equivalence
- Let n be a fixed positive integer. Define a relation R in the set Z of integers by aRb if and only if $n | a - b$. Then, relation R is
 - reflexive
 - symmetric
 - transitive
 - an equivalence relation
- If $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations, then $(SoR)^{-1}$ is equal to
 - $S^{-1}oR^{-1}$
 - RoS
 - $R^{-1}oS^{-1}$
 - None of these
- The relation 'congruence modulo m ' is
 - reflexive only
 - transitive only
 - symmetric only
 - an equivalence relation
- Let R be a relation on the set N of natural numbers defined by nRm if n divides m . Then, R is
 - reflexive and symmetric
 - transitive and symmetric
 - equivalence
 - reflexive, transitive but not symmetric
- The domain of the function $f(x) = {}^{24-x}C_{3x-1} + {}^{40-6x}C_{8x-10}$ is
 - $\{2, 3\}$
 - $\{1, 2, 3\}$
 - $\{1, 2, 3, 4\}$
 - None of the above
- The domain of the function $f(x) = \cos^{-1}\left(\frac{1-|x|}{2}\right)$ is
 - $(-\infty, -3) \cup (3, \infty)$
 - $[-3, 3]$
 - $(-\infty, -3] \cup [3, \infty)$
 - ϕ
- The domain of the function $f(x) = \sqrt{e^{\sin^{-1}(\log_{16} x^2)}}$ is
 - $\left[\frac{1}{4}, 4\right]$
 - $\left[-4, -\frac{1}{4}\right] \cup \left[\frac{1}{4}, 4\right]$
 - $\left[-4, -\frac{1}{4}\right]$
 - None of the above

- 17.** The domain of the function $f(x) = \log \left[x^2 - 5x + 6 \right]$ is
- (a) $\left[\frac{3}{2}, 2 \right) \cup (2, 3) \cup (3, \infty)$
 (b) $\left[\frac{3}{2}, \infty \right)$
 (c) $\left[\frac{1}{2}, \infty \right)$
 (d) None of the above
- 18.** The domain of the function $f(x) = \cos^{-1}(x + [x])$, where $[]$ denotes the greatest integer function, is
- (a) $[-1, 1]$ (b) $[0, 1]$
 (c) $(-1, 0)$ (d) None of these
- 19.** The domain of the function $f(x) = \underbrace{\log_2 \log_2 \dots \log_2 x}_{n \text{ times}}$ is
- (a) $(2^{n-1}, \infty)$ (b) $[2^n, \infty)$
 (c) $(2^{n-2}, \infty)$ (d) None of these
- 20.** The domain of the function $f(x) = \sqrt{x-3-2\sqrt{x-4}} - \sqrt{x-3+2\sqrt{x-4}}$ is
- (a) $[4, \infty)$ (b) $(-\infty, 4]$ (c) $(4, \infty)$ (d) $(-\infty, 4)$
- 21.** The domain of the function $f(x) = \cos \left[\log \left(\frac{\sqrt{16-x^2}}{3-x} \right) \right]$ is
- (a) $(-4, 4)$ (b) $(-4, 3)$
 (c) $(-\infty, -4) \cup (3, \infty)$ (d) None of these
- 22.** The domain of the function $f(x) = \frac{3+2(x+25)^{-0.5}}{5-(x+25)^{0.5}} + (x-5)^{0.5} + 5(x-3)^{0.5}$ is
- (a) $[5, \infty)$ (b) $[3, \infty)$
 (c) $(-25, 5)$ (d) None of these
- 23.** The domain of the function $f(x) = 4 \sqrt{\log_3 \left(\frac{1}{|\cos x|} \right)}$ is
- (a) $(-\infty, \infty)$
 (b) $(-\infty, \infty) - \{n\pi, n \in I\}$
 (c) $(-\infty, \infty) - \left\{ (2n+1)\frac{\pi}{2}, n \in I \right\}$
 (d) None of the above
- 24.** The domain of the function $f(x) = \log_3 \left[-\log_{1/2} \left(1 + \frac{1}{x^{1/5}} \right) - 1 \right]$ is
- (a) $(-\infty, 1)$ (b) $(0, 1)$
 (c) $(1, \infty)$ (d) None of the above
- 25.** The domain of the function $f(x) = \log_3 [-(\log_3 x)^2 + 5 \log_3 x - 6]$ is
- (a) $(0, 9) \cup (27, \infty)$ (b) $[9, 27]$
 (c) $(9, 27)$ (d) None of these
- 26.** The domain of the function $f(x) = x^{1/\log x}$ is
- (a) $(0, \infty) - \{1\}$ (b) $(0, \infty)$
 (c) $[0, \infty)$ (d) $[0, \infty) - \{1\}$
- 27.** The domain of the function $f(x) = \sqrt{\frac{4-x^2}{[x]+2}}$, where $[x]$ denotes the greatest integer less than or equal to x , is
- (a) $[-1, 2]$
 (b) $(-\infty, -2)$
 (c) $(-\infty, -2) \cup [-1, 2]$
 (d) None of the above
- 28.** The domain of the function $f(x) = \log \left[x^2 - x - 6 \right]$, where $[]$ denotes the greatest integer function, is
- (a) $\left(\frac{3}{2}, 3 \right) \cup (3, \infty)$ (b) $\left[\frac{3}{2}, 3 \right) \cup (3, \infty)$
 (c) $\left[\frac{3}{2}, 3 \right)$ (d) None of these
- 29.** If $f(x)$ is defined on $(0, 1)$, then the domain of definition of $f(e^x) + f(\ln |x|)$ is
- (a) $(-e, -1)$ (b) $(-e, 1)$
 (c) $(-\infty, -1) \cup (1, \infty)$ (d) $(-e, e)$
- 30.** $f(x) = \frac{\sqrt{|\tan x| + \tan x}}{\sqrt{3x}}$ is defined for
- (a) R (b) $R - \left\{ \frac{1}{3} \right\}$
 (c) $R^+ - \left\{ n\pi + \frac{\pi}{2}, n \in I^+ \right\}$ (d) None of these
- 31.** The function $f(x) = \sqrt{\log_{10} \left(\frac{5x-x^2}{4} \right)}$ exists for
- (a) $[1, 4]$ (b) $[1, 0]$ (c) $[0, 5]$ (d) $[5, 0]$
- 32.** The domain of $f(x) = \sqrt{x-4-2\sqrt{x-5}} - \sqrt{x-4+2\sqrt{x-5}}$ is
- (a) $[3, \infty)$
 (b) $[5, \infty)$
 (c) $(4, \infty)$
 (d) None of the above
- 33.** Let $f(x) = [x]^2 + [x+1] - 3$, where $[]$ denotes the greatest integer function, then
- (a) $f(x) \neq 0$ for all real x
 (b) $f(x) = 0$ for only two real values
 (c) $f(x) = 0$ for infinite number of values of x
 (d) None of the above
- 34.** The domain of definition of the function $y(x)$ given by the equation $2^x + 2^y = 2$, is
- (a) $0 < x \leq 1$
 (b) $0 \leq x \leq 1$
 (c) $-\infty < x \leq 0$
 (d) $-\infty < x < 1$

35. The domain of definition of the function

$$f(x) = \sqrt[3]{\frac{2x+1}{x^2-10x-11}}, \text{ is given by}$$

- (a) $x > 0$
(b) $R - \{-1, 11\}$
(c) $x \in R - \{-1\}$
(d) $-\infty < x < \infty$

36. The domain of the function

$$f(x) = \cot^{-1} \left(\frac{x}{\sqrt{x^2 - [x^2]}} \right), x \in R, \text{ is}$$

- (a) $R - \{\pm\sqrt{n}, n \in N\}$
(b) $R - \{\sqrt{n} : n \geq 0, n \in I\}$
(c) R
(d) $R - \{0\}$

37. The domain of the function

$$f(x) = \frac{1}{\sqrt{10C_{x-1} - 3 \times 10C_x}} \text{ contains the points}$$

- (a) 9, 10, 11 (b) 9, 10, 12
(c) All natural numbers (d) None of these

38. The domain of the function

$$f(x) = \frac{x}{\sqrt{\sin(\ln x) - \cos(\ln x)}}, n \in Z, \text{ is}$$

- (a) $(e^{2n\pi}, e^{(3n+1/2)\pi})$
(b) $(e^{(2n+1/4)\pi}, e^{(2n+5/4)\pi})$
(c) $(e^{(2n+1/4)\pi}, e^{(3n-3/4)\pi})$
(d) None of the above

39. The range of the function

$$f(x) = \sin \left[\log \left(\frac{\sqrt{4-x^2}}{1-x} \right) \right] \text{ is}$$

- (a) $[0, 1]$ (b) $(-1, 0)$ (c) $[-1, 1]$ (d) $(-1, 1)$

40. The range of the function $f(x) = \frac{5}{3-x^2}$ is

- (a) $(-\infty, 0) \cup \left[\frac{5}{3}, \infty\right)$ (b) $(-\infty, 0) \cup \left(\frac{5}{3}, \infty\right)$
(c) $(-\infty, 0] \cup \left[\frac{5}{3}, \infty\right)$ (d) None of these

41. The image of the interval $[1, 3]$ under the mapping $f: R \rightarrow R$, given by $f(x) = 2x^3 - 24x + 107$, is

- (a) $[0, 89]$
(b) $[75, 89]$
(c) $[0, 75]$
(d) None of the above

42. If $f(x) = \frac{x^2-1}{x^2+1}$, for every real number x , then the

minimum value of f

- (a) does not exist because f is unbounded
(b) is not attained even though f is bounded
(c) is equal to 1
(d) is equal to -1

43. Let f be a function defined on $[-2, 2]$ and is given

$$\text{by } f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases} \text{ and}$$

$g(x) = f(|x|) + |f(x)|$. Then, $g(x)$ is equal to

- (a) $\begin{cases} -x, & -2 \leq x < 0 \\ 0, & 0 \leq x \leq 1 \\ x-1, & 1 \leq x \leq 2 \end{cases}$ (b) $\begin{cases} -x, & -2 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 2(x-1), & 1 \leq x \leq 2 \end{cases}$
(c) $\begin{cases} -x, & -2 \leq x < 0 \\ x-1, & 0 \leq x \leq 2 \end{cases}$ (d) None of these

44. The number of real solutions of the equation $\log_{0.5} |x| = 2|x|$ is

- (a) 1
(b) 2
(c) 0
(d) None of these

45. If $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$, where $[\]$ denotes the greatest integer function, then

- (a) $f\left(\frac{\pi}{2}\right) = 1$
(b) $f(\pi) = 2$
(c) $f\left(\frac{\pi}{4}\right) = 1$
(d) None of the above

46. For a real number x , $[x]$ denotes the integral part of x . Then, the value of

$$\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \dots + \left[\frac{1}{2} + \frac{99}{100}\right] \text{ is}$$

- (a) 49 (b) 50 (c) 48 (d) 51

47. Find the domain of $\sin^{-1} \left\{ \log_2 \left(\frac{x^2}{2} \right) \right\}$.

- (a) $[-2, -1]$
(b) $[-4, 0]$
(c) $[0, 2]$
(d) None of the above

48. If $g: [-2, 2] \rightarrow R$, where $f(x) = x^3 + \tan x + \left[\frac{x^2+1}{P}\right]$

is an odd function, then the value of parametric P , where $[\]$ denotes the greatest integer function, is

- (a) $-5 < P < 5$
(b) $P < 5$
(c) $P > 5$
(d) None of the above

49. $f(x) = \frac{\cos x}{\left[\frac{2x}{\pi}\right] + \frac{1}{2}}$, where x is not an integral multiple

of π and $[\]$ denotes the greatest integer function, is

- (a) an odd function
(b) an even function
(c) Neither odd nor even
(d) None of the above

- 50.** If f is an even function defined on the interval $[-5, 5]$, then the real values of x satisfying the equation $f(x) = f\left(\frac{x+1}{x+2}\right)$, are
 (a) $\frac{-1 \pm \sqrt{5}}{2}$ (b) $\frac{-4 \pm \sqrt{5}}{2}$
 (c) $\frac{-2 \pm \sqrt{5}}{2}$ (d) None of these
- 51.** The function $f(x) = \sec[\log(x + \sqrt{1+x^2})]$ is
 (a) even (b) odd
 (c) constant (d) None of these
- 52.** Which of the following functions is odd?
 (a) $\operatorname{sgn} x + x^{2000}$ (b) $|x| - \tan x$
 (c) $x^3 \cot x$ (d) $\operatorname{cosec} x^{55}$
- 53.** Let the function $f(x) = 3x^2 - 4x + 8 \log(1 + |x|)$ be defined on the interval $[0, 1]$. The even extension of $f(x)$ to the interval $[-1, 1]$ is
 (a) $3x^2 + 4x + 8 \log(1 + |x|)$
 (b) $3x^2 - 4x + 8 \log(1 + |x|)$
 (c) $3x^2 + 4x - 8 \log(1 + |x|)$
 (d) None of the above
- 54.** If $[x]$ and $\{x\}$ represent the integral and fractional parts of x respectively, then the value of $\sum_{r=1}^{2000} \frac{\{x+r\}}{2000}$ is
 (a) x (b) $[x]$
 (c) $\{x\}$ (d) $x + 2001$
- 55.** If $f(x) = \sqrt[n]{x^m}$, $n \in \mathbb{N}$, is an even function, then m is
 (a) even integer (b) odd integer
 (c) any integer (d) not possible
- 56.** The period of the function $f(x) = \sqrt{\tan x}$ is
 (a) π (b) 2π
 (c) $\frac{\pi}{2}$ (d) None of these
- 57.** The period of the function $f(x) = \cos\left(\frac{8x+5}{4\pi}\right)$ is
 (a) 2π (b) π
 (c) π^2 (d) None of these
- 58.** The period of the function $f(x) = x - [x]$, where $[x]$ denotes the greatest integer less than or equal to x , is
 (a) 2 (b) 1
 (c) 4 (d) None of these
- 59.** If T_1 is the period of the function $y = e^{3(x-[x])}$ and T_2 is the period of the function $y = e^{3x-[3x]}$, where $[]$ denotes the greatest integer function, then
 (a) $T_1 = T_2$ (b) $T_1 = \frac{T_2}{3}$
 (c) $T_1 = 3T_2$ (d) None of these
- 60.** If the period of the function $f(x) = \sin(\sqrt{[n]}x)$, where $[n]$ denotes the greatest integer less than or equal to n , is 2π , then
 (a) $1 \leq n < 2$ (b) $1 < n < 2$
 (c) $1 \leq n \leq 2$ (d) None of these
- 61.** The period of the function $f(x) = \cos\left(\frac{\pi x}{n!}\right) - \sin\left(\frac{\pi x}{(n+1)!}\right)$ is
 (a) $2(n+1)!$
 (b) $2(n)!$
 (c) $(n+1)$
 (d) non-periodic
- 62.** The function $f(x) = k|\cos x| + k^2|\sin x| + \phi(k)$ has period $\frac{\pi}{2}$, if k is equal to
 (a) 1 (b) 2
 (c) 3 (d) None of these
- 63.** The period of the function $3^{(\sin^2 \pi x + x - [x] + \sin^4 \pi x)}$, where $[]$ denotes the greatest integer function, is
 (a) $\frac{1}{2}$ (b) 1
 (c) 2 (d) non-periodic
- 64.** The period of the function $f(x) = 2\sin x + 3\cos 2x$ is
 (a) π (b) 2π
 (c) $\frac{\pi}{2}$ (d) None of these
- 65.** The period of the function $f(x) = \begin{cases} 1, & \text{when } x \text{ is a rational} \\ 0, & \text{when } x \text{ is an irrational} \end{cases}$ is
 (a) 1 (b) 2
 (c) non-periodic (d) None of these
- 66.** The period of the function $f(x) = \cos \frac{2\pi x}{5} + \sin \frac{\pi x}{4}$ is
 (a) 5 (b) 8 (c) 12 (d) 40
- 67.** Let f be a real valued function with domain $\mathbb{R}$ satisfying $f(x+k) = 1 + [2 - 5f(x) + 10\{f(x)\}^2 - 10\{f(x)\}^3 + 5\{f(x)\}^4 - \{f(x)\}^5]^{1/5}$ for all real x and some positive constant k , then the period of the function $f(x)$ is
 (a) k (b) $2k$
 (c) non-periodic (d) None of these
- 68.** If the period of the function $f(x) = \tan(\sqrt{[k]}x)$, where $[]$ denotes the greatest integer function, is π , then
 (a) $1 < k < 2$
 (b) $1 \leq k < 2$
 (c) $k = 1, 2$
 (d) None of the above

69. The period of the function $f(x) = \sin 5x + \cos \sqrt{3}x$ is

- (a) $\sqrt{3}\pi$ (b) π
(c) non-periodic (d) None of these

70. The period of the function

$$f(x) = 3x + 3 - [3x + 3] + \sin \frac{\pi x}{2}, \text{ where } [x] \text{ denotes}$$

the greatest integer $\leq x$, is

- (a) 4 (b) 1
(c) 2 (d) non-periodic

71. The value of $n \in I$ for which the function

$$f(x) = \frac{\sin nx}{\sin\left(\frac{x}{n}\right)} \text{ has } 4\pi \text{ as its period, is}$$

- (a) 2 (b) 3
(c) 4 (d) 5

72. π is the period of the function

- (a) $|\sin x| + |\cos x|$
(b) $\sin^4 x + \cos^4 x$
(c) $\sin(\sin x) + \sin(\cos x)$
(d) $\frac{1 + 2\cos x}{\sin x(2 + \sec x)}$

73. Which of the following functions is non-periodic?

- (a) $f(x) = \tan(3x - 2)$
(b) $f(x) = \{x\}$, the fractional part of the number x
(c) $f(x) = x + \cos x$
(d) $f(x) = 1 - \frac{\cos^2 x}{1 + \tan x} - \frac{\sin^2 x}{1 + \cot x}$

74. The period of the function

$$f(x) = \frac{\sin 8x \cos x - \sin 6x \cos 3x}{\cos 2x \cos x - \sin 3x \sin 4x} \text{ is}$$

- (a) π (b) $\frac{\pi}{2}$
(c) 2π (d) None of these

75. The period of the function

$$f(x) = \frac{\sin x + \sin 2x + \sin 4x + \sin 5x}{\cos x + \cos 2x + \cos 4x + \cos 5x} \text{ is}$$

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) π (d) None of these

76. If $f(x)$ is an odd periodic function with period 2, then $f(4)$ equals

- (a) -4 (b) 4
(c) 2 (d) 0

77. If $f(x)$ and $g(x)$ are periodic functions with period 7 and 11, respectively. Then, the period of

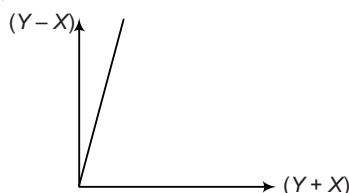
$$F(x) = f(x)g\left(\frac{x}{5}\right) - g(x)f\left(\frac{x}{3}\right) \text{ is}$$

- (a) 177
(b) 222
(c) 433
(d) 1155

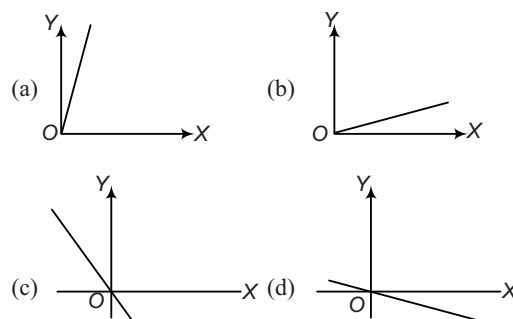
78. If the graph of $y = f(x)$ is symmetrical about lines $x = 1$ and $x = 2$, then which of the following is true?

- (a) $f(x + 1) = f(x)$
(b) $f(x + 3) = f(x)$
(c) $f(x + 2) = f(x)$
(d) None of the above

79. The graph of $(Y - X)$ against $(Y + X)$ is shown below



Which one of the following shows the graph of Y against X ?



80. If the graph of the function $f(x) = \frac{a^x - 1}{x^n (a^x + 1)}$ is

symmetrical about Y -axis, then n equals

- (a) 2 (b) $\frac{2}{3}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{3}$

81. Function $f: (-\infty, -1] \rightarrow (0, e^5]$ defined by

$$f(x) = e^{x^3 - 3x + 2}, \text{ is}$$

- (a) many-one and onto (b) many-one and into
(c) one-one and onto (d) one-one and into

82. Let $f: N \rightarrow N$ defined by $f(x) = x^2 + x + 1, x \in N$, then f is

- (a) one-one and onto (b) many-one and onto
(c) one-one but not onto (d) None of these

83. Let S be the set of all triangles and R^+ be the set of positive real numbers. Then, the function $f: S \rightarrow R^+, f(\Delta) = \text{area of } \Delta$, where $\Delta \in S$, is

- (a) injective but not surjective
(b) surjective but not injective
(c) injective as well as surjective
(d) neither injective nor surjective

84. If $A = \left\{x: -\frac{2}{5} \leq x \leq \frac{\pi - 2}{5}\right\}$, $B = \{y: -1 \leq y \leq 1\}$ and

$$f(x) = \cos(5x + 2), \text{ then the mapping } f: A \rightarrow B \text{ is}$$

- (a) one-one but not onto (b) onto but not one-one
(c) both one-one and onto (d) neither one-one nor onto

- 85.** The function $f: R \rightarrow [0, 1]$ defined by $f(x) = x - [x], \forall x \in R$, is
 (a) one-one (b) onto
 (c) both one-one and onto (d) neither one-one nor onto
- 86.** The function $f: R \rightarrow [0, \infty)$ defined by $f(x) = 4^x + 4^{|x|}$ is
 (a) one-one and into (b) many-one and into
 (c) one-one and onto (d) many-one and onto
- 87.** Let $f: R \rightarrow R$ be a function defined by $f(x) = \frac{x^2 - 8}{x^2 + 2}$, then f is
 (a) one-one but not onto (b) one-one and onto
 (c) onto but not one-one (d) neither one-one nor onto
- 88.** Let $f: R \rightarrow R$ be a function defined by $f(x) = x + \sqrt{x^2}$, then f is
 (a) injective (b) surjective
 (c) bijective (d) None of these
- 89.** If $f(x + y, x - y) = xy$, then the arithmetic mean of $f(x, y)$ and $f(y, x)$ is
 (a) y (b) x
 (c) 0 (d) None of these
- 90.** If $f(x) = 64x^3 + \frac{1}{x^3}$ and a, b are the roots of $4x + \frac{1}{x} = 3$, then
 (a) $f(a) = 12$ (b) $f(b) = 11$
 (c) $f(a) = f(b)$ (d) None of these
- 91.** If $f(x + y) = f(x) + f(y) - xy - 1, \forall x, y \in R$ and $f(1) = 1$, then the number of solutions of $f(n) = n, n \in N$, is
 (a) one (b) two
 (c) no solution (d) None of these
- 92.** If $3f(x) + 5f\left(\frac{1}{x}\right) = \frac{1}{x} - 3, \forall x (\neq 0) \in R$, then $f(x)$ is equal to
 (a) $\frac{1}{14}\left(\frac{3}{x} + 5x - 6\right)$ (b) $\frac{1}{16}\left(-\frac{3}{x} + 5x - 6\right)$
 (c) $\frac{1}{14}\left(-\frac{3}{x} + 5x + 6\right)$ (d) None of these
- 93.** The distinct linear function(s) which map(s) $[-1, 1]$ onto $[0, 2]$ is/are
 (a) $x + 1, -x + 1$
 (b) $x - 1, x + 1$
 (c) $-x + 1$
 (d) None of the above
- 94.** If $f(x) = \frac{4^x}{4^x + 2}$, then $f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \dots + f\left(\frac{1996}{1997}\right)$ is equal to
 (a) 1997 (b) 998
 (c) 0 (d) None of these
- 95.** The value of the natural number a for which $\sum_{k=1}^n f(a + k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x + y) = f(x) \cdot f(y)$ for all natural numbers x, y and further $f(1) = 2$, is
 (a) 3 (b) 4
 (c) 2 (d) None of these
- 96.** Let $f(x) = \max\{(1 - x), (1 + x), 2\}, \forall x \in R$, then
 (a) $f(x) = \begin{cases} 1 + x, & x \leq -1 \\ 2, & -1 < x < 1 \\ 1 - x, & x \geq 1 \end{cases}$
 (b) $f(x) = \begin{cases} 1 - x, & x \leq -1 \\ 1, & -1 < x < 1 \\ 1 + x, & x \geq 1 \end{cases}$
 (c) $f(x) = \begin{cases} 1 - x, & x \leq -1 \\ 2, & -1 < x < 1 \\ 1 + x, & x \geq 1 \end{cases}$
 (d) None of the above
- 97.** If $f(x) = |x - 2|$ and $g(x) = f[f(x)]$, then for $x > 20, g'(x)$ is equal to
 (a) -1 (b) 1
 (c) 2 (d) None of these
- 98.** If $f(x) = \sqrt{2 - x}$ and $g(x) = \sqrt{1 - 2x}$, then the domain of $f[g(x)]$ is
 (a) $\left(-\infty, \frac{1}{2}\right]$ (b) $\left[\frac{1}{2}, \infty\right)$
 (c) $\left(-\infty, -\frac{3}{2}\right]$ (d) None of these
- 99.** Let f be a function with domain $[-3, 5]$ and $g(x) = |3x + 4|$. Then, the domain of $(f \circ g)(x)$ is
 (a) $\left(-3, \frac{1}{3}\right)$ (b) $\left[-3, \frac{1}{3}\right]$
 (c) $\left[-3, \frac{1}{3}\right)$ (d) None of these
- 100.** If $f(x) = \frac{x - 1}{x + 1}$, then $f\{f(ax)\}$ in terms of $f(x)$ is equal to
 (a) $\frac{f(x) - 1}{a\{f(x) + 1\}}$ (b) $\frac{f(x) + 1}{a\{f(x) - 1\}}$
 (c) $\frac{f(x) - 1}{a\{f(x) - 1\}}$ (d) $\frac{f(x) + 1}{a\{f(x) + 1\}}$
- 101.** If f, g and h are functions from R to R such that $f(x) = x^2 - 1, g(x) = \sqrt{x^2 + 1}, \forall x \in R$ and $h(x) = \begin{cases} 0, & x \leq 0 \\ x, & x \geq 0 \end{cases}$, then the composition function $h \circ (f \circ g)$ is given by
 (a) x^2 (b) 0
 (c) x (d) None of these

102. If $f(x) = \cos(\log x)$, then

$$f(x) \cdot f(y) - \frac{1}{2} \left\{ f\left(\frac{x}{y}\right) + f(xy) \right\} \text{ is equal to}$$

- (a) 2 (b) 1
(c) 0 (d) None of these

103. If $f(x) = (3 - x^7)^{1/7}$, $\forall x \in R$, then $(f \circ f)(x)$ is equal to

- (a) x (b) 0
(c) $3 - x^7$ (d) None of these

104. If $f(x) = \frac{1}{1-x}$, $g(x) = f[f(x)]$ and

$$h(x) = f\{f(f(x))\}, \text{ then the value of } f(x) \cdot g(x) \cdot h(x) \text{ is}$$

- (a) 1 (b) -1
(c) 0 (d) None of these

105. If $g[f(x)] = |\sin x|$ and $f[g(x)] = (\sin \sqrt{x})^2$, then

- (a) $f(x) = \sin^2 x$, $g(x) = \sqrt{x}$
(b) $f(x) = \sin x$, $g(x) = |x|$
(c) $f(x) = x^2$, $g(x) = \sin \sqrt{x}$
(d) f and g cannot be determined

106. If $f(x) = \log \frac{1+x}{1-x}$ and $g(x) = \frac{3x+x^3}{1+3x^2}$, then

$$(f \circ g)(x) \text{ equals}$$

- (a) $-f(x)$ (b) $3f(x)$
(c) $[f(x)]^3$ (d) None of these

107. If $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$ and $h(x) = x^2$, then

- (a) $f \circ g(x) = x^2$, $x \neq 0$, $h[g(x)] = \frac{1}{x^2}$
(b) $h[g(x)] = \frac{1}{x^2}$, $x \neq 0$, $f \circ g(x) = x^2$
(c) $f \circ g(x) = x^2$, $x \neq 0$, $h[g(x)] = [g(x)]^2$, $x \neq 0$
(d) None of the above

108. If the functions f , g and h are defined from the set of real numbers R to R such that

$$f(x) = x^2 - 1, g(x) = \sqrt{x^2 + 1} \text{ and}$$

$$h(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Then, the composite function $(h \circ f \circ g)(x)$ equals

- (a) $\begin{cases} 0, & x = 0 \\ x^2, & x > 0 \\ -x^2, & x < 0 \end{cases}$ (b) $\begin{cases} 0, & x = 0 \\ x^2, & x \neq 0 \end{cases}$
(c) $\begin{cases} 0, & x \leq 0 \\ x^2, & x > 0 \end{cases}$ (d) None of these

109. Let $f: (-\infty, 2] \rightarrow (-\infty, 4]$ be a function defined by

$$f(x) = 4x - x^2, \text{ then } f^{-1}(x) \text{ is equal to}$$

- (a) $2 - \sqrt{4-x}$ (b) $2 + \sqrt{4-x}$
(c) $\sqrt{4-x}$ (d) None of these

110. Let $f: [4, \infty) \rightarrow [1, \infty)$ be a function defined by

$$f(x) = 5^{x(x-4)}, \text{ then } f^{-1}(x) \text{ equals}$$

- (a) $2 - \sqrt{4 + \log_5 x}$ (b) $2 + \sqrt{4 + \log_5 x}$
(c) $\left(\frac{1}{5}\right)^{x(x-4)}$ (d) None of these

111. If $f: R \rightarrow R$ is given by $f(x) = 3x - 5$, then

$$f^{-1}(x)$$

- (a) is given by $\frac{1}{3x-5}$
(b) is given by $\frac{x+5}{3}$
(c) does not exist because f is not one-one
(d) does not exist because f is not onto

112. Let $f: R \rightarrow R$ be given by

$$f(x) = (x+1)^2 - 1, x \geq -1, \text{ then } f^{-1}(x) \text{ is}$$

- (a) $-1 + \sqrt{x+1}$
(b) $-1 - \sqrt{x+1}$
(c) does not exist because f is not one-one
(d) does not exist because f is not onto

113. Let $f: R \rightarrow R$ be given by

$$f(x) = (x+1)^2 - 1, x \geq -1. \text{ Then, the set of values of } x \text{ for which } f(x) = f^{-1}(x), \text{ is given by}$$

- (a) $\{0\}$
(b) $\{0, -1\}$
(c) $\{-1\}$
(d) None of the above

114. Let $f: R \rightarrow R$ be a function defined by

$$f(x) = x - [x], \text{ where } [] \text{ denotes the greatest integer function, then } f^{-1}(x) \text{ is}$$

- (a) $[x] - x$
(b) $\frac{1}{x - [x]}$
(c) not defined
(d) None of the above

115. The inverse of the function $y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}} + 1$ is

- (a) $y = \frac{1}{2} \log_{10} \frac{x}{2-x}$
(b) $y = \log_{10} \frac{x}{2-x}$
(c) $y = \frac{1}{2} \log_{10} \frac{x}{1-x}$
(d) None of the above

116. The inverse of the function $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$ is

given by

- (a) $\log_e \left(\frac{x-1}{x+1} \right)^{-2}$ (b) $\log_e \left(\frac{x-2}{x-1} \right)^{1/2}$
(c) $\log_e \left(\frac{x}{2-x} \right)^{1/2}$ (d) $\log_e \left(\frac{x+1}{3-x} \right)^{1/2}$

117. Let $f: R \rightarrow R$ and $g: R \rightarrow R$ be two functions given by $f(x) = 2x - 3$ and $g(x) = x^3 + 5$. Then, $(f \circ g)^{-1}(x)$ is equal to

- (a) $\left(\frac{x-7}{2}\right)^{1/3}$ (b) $\left(\frac{x+7}{2}\right)^{1/2}$
(c) $\left(x - \frac{7}{2}\right)^{1/3}$ (d) $\left(\frac{x-2}{7}\right)^{1/3}$

118. Let $f: [2, \infty) \rightarrow X$ be defined by $f(x) = 4x - x^2$. Then, f is invertible, if X is equal to
(a) $[2, \infty)$ (b) $(-\infty, 2]$ (c) $(-\infty, 4]$ (d) $[4, \infty)$

119. If $f: R \rightarrow R$ is given by $f(x) = x^3 + 3$, then $f^{-1}(x)$ is equal to

- (a) $x^{1/3} - 3$ (b) $x^{1/3} + 3$
(c) $(x - 3)^{1/3}$ (d) $x + 3^{1/3}$

120. If $f: [1, \infty) \rightarrow [2, \infty)$ is given by $f(x) = x + \frac{1}{x}$, then $f^{-1}(x)$ is equal to

- (a) $\frac{x + \sqrt{x^2 - 4}}{2}$ (b) $\frac{x}{1 + x^2}$
(c) $\frac{x - \sqrt{x^2 - 4}}{2}$ (d) $1 + \sqrt{x^{-4}}$

Type 2. More than One Correct Option

121. Consider the real valued function satisfying $2f(\sin x) + f(\cos x) = x$. Then,

- (a) domain of $f(x)$ is R
(b) domain of $f(x)$ is $[-1, 1]$
(c) range of $f(x)$ is $\left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$
(d) range of $f(x)$ is R

122. The domain of the function

$$f(x) = \log_e \left\{ \log_{|\sin x|} (x^2 - 8x + 23) - \frac{3}{\log_2 |\sin x|} \right\}$$

contains which of the following interval(s)?

- (a) $(3, \pi)$
(b) $\left(\pi, \frac{3}{2}\right)$
(c) $\left(\frac{3\pi}{2}, 5\right)$

(d) None of the above

123. Which of the following functions have their range equal to R (the set of real numbers)?

- (a) $x \sin x$
(b) $\frac{[x]}{\tan 2x}$, $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) - \{0\}$, $[]$ denotes the greatest integer function
(c) $\frac{x}{\sin x}$
(d) $[x] + \{x\}$, $[]$ and $\{ \}$ respectively denote the greatest integer function and fractional part function

124. If $f(x)$ satisfies the relation $f(x + y) = f(x) + f(y)$, $\forall x, y \in R$ and $f(1) = 5$, then

- (a) $f(x)$ is an odd function
(b) $f(x)$ is an even function
(c) $\sum_{r=1}^m f(r) = 5 \cdot m + 1 \cdot C_2$
(d) $\sum_{r=1}^m f(r) = \frac{5m(m+2)}{3}$

125. Which of the following is/are not a function, where $[]$ and $\{ \}$ denote the greatest integer and fractional part functions respectively?

- (a) $\frac{1}{\ln[1 - |x|]}$
(b) $\frac{x!}{\{x\}}$
(c) $x! \{x\}$
(d) $\frac{\ln(x-1)}{\sqrt{(1-x^2)}}$

126. If $f(g(x))$ is one-one function, then

- (a) $g(x)$ must be one-one
(b) $f(x)$ must be one-one
(c) $f(x)$ may not be one-one
(d) $g(x)$ may not be one-one

127. If $f: R \rightarrow R$ is a periodic function such that $\{f(x) | x \in N\}$ is an infinite set, then period of the function cannot be

- (a) rational (b) irrational
(c) integer (d) π

128. Which of the following pairs of function(s) is/are identical?

- (a) $f(x) = e^{\ln \sec^{-1} x}$ and $g(x) = \sec^{-1} x$
(b) $f(x) = \tan(\tan^{-1} x)$ and $g(x) = \cot(\cot^{-1} x)$
(c) $f(x) = \operatorname{sgn}(x)$ and $g(x) = \operatorname{sgn}(\operatorname{sgn}(x))$
(d) $f(x) = \cot^2 x \cdot \cos^2 x$ and $g(x) = \cot^2 x - \cos^2 x$

129. Let $f(x) = \operatorname{sgn}(\cot^{-1} x) + \tan\left(\frac{\pi}{2}[x]\right)$, where $[x]$ is

the greatest integer function less than or equal to x . Then, which of the following alternatives is/are true?

- (a) $f(x)$ is many-one but not even function
(b) $f(x)$ is periodic function
(c) $f(x)$ is bounded function
(d) Graph of $f(x)$ remains above the X -axis

Type 3. Assertion and Reason

Directions (Q. Nos. 130-153) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

130. Consider the function satisfying the relation, if $f\left(\frac{2 \tan x}{1 + \tan^2 x}\right) = \frac{(1 + \cos 2x)(\sec^2 x + 2 \tan x)}{2}$

Statement I Range of $y = f(x)$ is R .

Statement II Linear function has range R , if domain is R .

131. Statement I The range of $\log \left(\frac{1}{1+x^2}\right)$ is $(-\infty, \infty)$.

Statement II When $0 < x \leq 1$, $\log x \in (-\infty, 0]$.

132. Statement I Range of $f(x) = \sqrt{4-x^2}$ is $[0, 2]$.

Statement II $f(x)$ is increasing for $0 \leq x \leq 2$ and decreasing for $-2 \leq x \leq 0$.

133. Let $a, b \in R, a \neq b$ and $f(x) = \frac{a+x}{b+x}$.

Statement I f is a one-one function.

Statement II Range of f is $R - \{1\}$.

134. Statement I The domain of the function $f(x) = \cos^{-1} x + \tan^{-1} x + \sin^{-1} x$ is $[-1, 1]$.

Statement II $\sin^{-1} x$ and $\cos^{-1} x$ are defined for $|x| \leq 1$ and $\tan^{-1} x$ is defined for all x .

135. Statement I The domain of the function $f(x) = \sqrt{\log_2 \sin x}$ is $(4n+1)\frac{\pi}{2}, n \in N$.

Statement II Expression under even root should be greater than or equal to 0.

136. Consider the function, if $f(x) = \sin(kx) + \{x\}$, where $\{x\}$ represents the fractional part function.

Statement I $f(x)$ is periodic for $k = m\pi$, where m is a rational number.

Statement II The sum of two periodic functions is always periodic.

137. Statement I Function $f(x) = x^2 + \tan^{-1} x$ is a non-periodic function.

Statement II The sum of two non-periodic functions is always non-periodic.

138. Statement I $f(x) = \cos(x^2 - \tan x)$ is a non-periodic function.

Statement II $x^2 - \tan x$ is a non-periodic function.

139. Let $f: X \rightarrow Y$ be a function defined by $f(x) = 2\sin\left(x + \frac{\pi}{4}\right) - 2\cos x + c$.

Statement I For set $X, x \in \left[0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right]$, $f(x)$ is one-one function.

Statement II $f'(x) \geq 0, x \in \left[0, \frac{\pi}{2}\right]$.

140. Consider the functions, $f(x) = \log_e xa$ and $g(x) = 2x + 3$.

Statement I $f(g(x))$ is one-one function.

Statement II $g(x)$ is one-one function.

141. Statement I If $f(x) = \cos x$ and $g(x) = x^2$, then $f(g(x))$ is an even functions.

Statement II If $f\{g(x)\}$ is an even function, then both $f(x)$ and $g(x)$ must be even functions.

142. Statement I A continuous surjective function $f: R \rightarrow R, f(x)$ can never be a periodic function.

Statement II For a surjective function $f: R \rightarrow R, f(x)$ to be periodic, it should necessarily be a discontinuous function.

143. Statement I The solution of equation $||x^2 - 5x + 4| - |2x - 3|| = |x^2 - 3x + 1|$ is $x \in (-\infty, 1] \cup \left[\frac{3}{2}, 4\right]$.

Statement II If $|x + y| = |x| + |y|$, then $x, y \geq 0$.

144. f is a function defined on the interval $[-1, 1]$ such that $f(\sin 2x) = \sin x + \cos x$.

Statement I If $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, then

$f(\tan^2 x) = \sec x$.

Statement II $f(x) = \sqrt{1+x}, \forall x \in [-1, 1]$.

145. Statement I The equation $f(x) = 4x^5 + 20x - 9 = 0$ has only one real root.

Statement II $f'(x) = 20x^4 + 20 = 0$ has no real root.

146. Statement I $\phi(x) = \sin(\cos x)$, $x \in \left[0, \frac{\pi}{2}\right]$ is one-one function.

Statement II $\phi'(x) \leq 0$, $\forall x \in \left[0, \frac{\pi}{2}\right]$.

147. Statement I If $f(x) = x$ and $F(x) = \frac{x^2}{x}$, then $F(x)$ and $f(x)$ are not equal functions.

Statement II At $x = 0$, $F(x)$ is not defined.

148. Statement I For the equation $kx^2 + (2 - k)x + 1 = 0$, $k \in \mathbb{R} - \{0\}$ exactly one root lies in $(0, 1)$.

Statement II If $f(k_1)f(k_2) < 0$, $f(x)$ is a polynomial, then exactly one root of $f(x) = 0$ lies in (k_1, k_2) .

149. Statement I If the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is such that $f(x) = x - [x]$, where $[]$ denotes the greatest integer less than or equal to x , then $f^{-1}(x)$ is equal to $[x] + x$.

Statement II Function f is invertible iff it is one-one and onto.

150. Statement I $f(x) = \frac{x+1}{x-1}$ is one-one function.

Statement II $\frac{x+1}{x-1}$ is monotonically decreasing function and every decreasing function is one-one.

151. Statement I $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ -x, & x \in \mathbb{Q}^c \end{cases}$ is one-one and non-monotonic function.

Statement II Every one-one function is monotonic.

152. Statement I The function $f: \mathbb{R} \rightarrow \mathbb{R}$ given $f(x) = \log_a(x + \sqrt{x^2 + 1})$, $a > 0$, $a \neq 1$ is invertible.

Statement II f is many-one and into.

153. Statement I If $f: A \rightarrow B$ and $g: B \rightarrow C$ are two functions, then $(gof)^{-1} = f^{-1}og^{-1}$.

Statement II If $f: A \rightarrow B$ and $g: B \rightarrow C$ are bijections, then f^{-1} and g^{-1} are also bijections.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 154-156) Consider the functions

$$f(x) = \begin{cases} x+1, & x \leq 1 \\ 2x+1, & 1 < x \leq 2 \end{cases}$$

and

$$g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ x+2, & 2 \leq x \leq 3 \end{cases}$$

154. The domain of the function $f(g(x))$ is

- (a) $[0, \sqrt{2}]$
- (b) $[-1, 2]$
- (c) $[-1, \sqrt{2}]$
- (d) None of the above

155. The range of the function $f(g(x))$ is

- (a) $[1, 5]$
- (b) $[2, 3]$
- (c) $[1, 2] \cup (3, 5]$
- (d) None of the above

156. The number of roots of the equation $f(g(x)) = 2$ is

- (a) 1
- (b) 2
- (c) 4
- (d) None of the above

Passage II (Q. Nos. 157-159) Consider the function $f(x)$ satisfying the identity

$$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x, \forall x \in \mathbb{R} - \{0, 1\}$$

and

$$g(x) = 2f(x) - x + 1.$$

157. The domain of $y = \sqrt{g(x)}$ is

- (a) $\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup \left[1, \frac{1+\sqrt{5}}{2}\right]$
- (b) $\left(-\infty, \frac{1-\sqrt{5}}{2}\right] \cup (0, 1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$
- (c) $\left[\frac{-1-\sqrt{5}}{2}, 0\right] \cup \left[\frac{-1+\sqrt{5}}{2}, 1\right]$
- (d) None of the above

158. The range of $y = g(x)$ is

- (a) $(-\infty, 5]$
- (b) $[1, \infty)$
- (c) $(-\infty, 1] \cup [5, \infty)$
- (d) None of the above

159. The number of roots of the equation $g(x) = 1$ is

- (a) 2
- (b) 1
- (c) 3
- (d) 0

Passage III (Q. Nos. 160-162)

Let $f(x) = f_1(x) - 2f_2(x)$

where, $f_1(x) = \begin{cases} \min\{x^2, |x|\}, & |x| \leq 1 \\ \max\{x^2, |x|\}, & |x| > 1 \end{cases}$

and $f_2(x) = \begin{cases} \min\{x^2, |x|\}, & |x| \leq 1 \\ \max\{x^2, |x|\}, & |x| > 1 \end{cases}$

and $g(x) = \begin{cases} \min\{f(t) : -3 \leq t \leq x, & -3 \leq x < 0\} \\ \max\{f(t) : 0 \leq t \leq x, & 0 \leq x \leq 3\} \end{cases}$

160. For $-3 \leq x \leq -1$, the range of $g(x)$ is

- (a) $[-1, 3]$
(b) $[-1, -15]$
(c) $[-1, 9]$
(d) None of the above

161. For $x \in (-1, 0)$, $f(x) + g(x)$ is

- (a) $x^2 - 2x + 1$
(b) $x^2 + 2x - 1$
(c) $x^2 + 2x + 1$
(d) $x^2 - 2x - 1$

162. The graph of $y = g(x)$ in its domain is broken at

- (a) 1 point
(b) 2 points
(c) 3 points
(d) None of these

Passage IV (Q. Nos. 163-165) Let $f : N \rightarrow N$, N being the set of positive integers, be a function defined by $f(x) =$ the biggest positive integer obtained by reshuffling the digits of x . e.g. $f(296) = 962$.

163. The function f is

- (a) one-one and onto
(b) one-one and into
(c) many-one and onto
(d) many-one and into

164. The biggest positive integer, which divides $f(n) - n, \forall n \in N$, is

- (a) 3
(b) 9
(c) 18
(d) 27

165. The range of f is

- (a) N
(b) set of positive integers whose digits are non-increasing from left to right
(c) set of positive integers whose digits are non-decreasing from left to right
(d) set of positive integers whose digits decrease from left to right

Passage V (Q. Nos. 166-168) Let $f(x) = ax^2 + bx + c$, $a, b, c \in R$. It is given that $|f(x)| \leq 1, \forall |x| \leq 1$.

166. The possible value of $|a + c|$, if $\frac{8}{3}a^2 + 2b^2$ is

- maximum, is given by
(a) 1
(b) 0
(c) 2
(d) 3

167. The possible value of $|a + b|$, if $\frac{8}{3}a^2 + 2b^2$ is maximum, is given by

- (a) 1
(b) 0
(c) 2
(d) 3

168. The possible maximum value of $\frac{8}{3}a^2 + 2b^2$ is given by

- (a) 32
(b) $\frac{32}{3}$
(c) $\frac{2}{3}$
(d) $\frac{16}{3}$

Passage VI (Q. Nos. 169-171) Let $f : [2, \infty) \rightarrow [1, \infty)$

defined by $f(x) = 2^{x^4 - 4x^2}$ and $g : \left[\frac{\pi}{2}, \pi\right] \rightarrow A$ defined

by $g(x) = \frac{\sin x + 4}{\sin x - 2}$ be two invertible functions.

169. $f^{-1}(x)$ is equal to

- (a) $\sqrt{2 + \sqrt{4 - \log_2 x}}$
(b) $\sqrt{2 + \sqrt{4 + \log_2 x}}$
(c) $\sqrt{2 - \sqrt{4 + \log_2 x}}$
(d) None of the above

170. The set A is equal to

- (a) $[-5, -2]$
(b) $[2, 5]$
(c) $[-5, 2]$
(d) $[-3, -2]$

171. The domain of $f^{-1}g^{-1}(x)$ is

- (a) $[-5, \sin 1]$
(b) $\left[-5, \frac{\sin 1}{2 - \sin 1}\right]$
(c) $\left[-5, -\frac{(4 + \sin 1)}{2 - \sin 1}\right]$
(d) $\left[-\frac{(4 + \sin 1)}{2 - \sin 1}, -2\right]$

Passage VII (Q. Nos. 172-174) Let $P(x)$ be polynomial of degree at most 5 which leaves remainders -1 and 1 upon division by $(x - 1)^3$ and $(x + 1)^3$, respectively.

172. The number of real roots of $P(x) = 0$ is

- (a) 1
(b) 3
(c) 5
(d) 2

173. The maximum value of $y = P'(x)$ can be obtained at x is

- (a) $-\frac{1}{\sqrt{3}}$
(b) 0
(c) $\frac{1}{\sqrt{3}}$
(d) 1

174. The sum of pairwise product of all roots (real and complex) of $P(x) = 0$ is

- (a) $-\frac{5}{3}$
(b) $-\frac{10}{3}$
(c) 2
(d) -5

Type 5. Match the Columns

175. Match the statements of Column I with the values of Column II.

Column I	Column II
A. $f(x) = \{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n$; $x \neq 0$, n p. odd function is an odd integer, is	
B. $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$ is	q. even function
C. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$ is	r. neither odd nor even function
D. $f(x) = \max\{\tan x, \cot x\}$ is	s. periodic function

176. If $\{ \}$ denotes the fractional part function and $[]$ denotes the greatest integer function, then Match the statements of Column I with the values of Column II.

Column I	Column II
A. If $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$, then the period of $f(x)$ is	p. $\frac{1}{3}$
B. If $f(x) = \cos 2\pi \{2x\} + \sin 2\pi \{2x\}$, then the period of $f(x)$ is	q. $\frac{1}{4}$
C. If $f(x) = \sin 3\pi \{x\} + \tan \pi [x]$, then the period of $\sin 3\pi (x)$ is	r. $\frac{1}{2}$
D. If $f(x) = 3x - [3x + a] - b$, where $a, b \in R^+$, then the period of $f(x)$ is	s. 1

177. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $f(x) = \sin^2 x + \sin^2 \left(x + \frac{\pi}{3}\right) + \cos x \cos \left(x + \frac{\pi}{3}\right)$, then $f(x)$ is	p. defined for all real x
B. If $g(x) = \tan(e^{(x)}) + [x + \alpha] - 5 - x$, where $[]$ denotes greatest integer less than or equal to x , then $g(x)$ is	q. even function
C. If $h(x) = \frac{x}{5^x - 1} + \frac{x}{2} + 5$, then $h(x)$ is	r. odd function
D. If $k(x) = 2 \sin^2 x - \cos 2\alpha + 4 \sin \alpha \sin x \cos(x + \alpha) + \cos 2(x + \alpha)$, $\alpha \in R$, then $k(x)$ is	s. periodic function

178. Match the statements of Column I with values of Column II.

Column I	Column II
A. Fundamental period of $f(x) = \sec^2 x - \tan^2 x$ is	p. no fundamental period
B. Fundamental period of $f(x) = \sin^2 x + \cos^2 x$ is	q. π

Column I	Column II
C. Fundamental period of $f(x) = \tan x \cdot \cot x$ is	r. $\frac{\pi}{2}$
D. Fundamental period of $f(x) = \operatorname{cosec}^2 x \{x\}$ is	s. non-periodic

179. Match the statements of Column I with values of Column II.

Column I	Column II
A. $f: R \rightarrow \left[\frac{3\pi}{4}, \pi\right)$ and $f(x) = \cot^{-1}(2x - x^2 - 2)$, then $f(x)$ is	p. one-one
B. $f: R \rightarrow R$ and $f(x) = e^{px} \sin qx$, where $p, q \in R^2$, then $f(x)$ is	q. into
C. $f: R^+ \rightarrow [4, \infty]$ and $f(x) = 4 + 3x^2$, then $f(x)$ is	r. many-one
D. $f: X \rightarrow X$ and $f(f(x)) = x, \forall x \in X$, then $f(x)$ is	s. onto

180. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $x^2 \tan x = 1, x \in [0, 2\pi]$, then the number of roots is	p. 5
B. If $2^{\cos x} = \sin x , x \in [0, 2\pi]$, then the number of roots is	q. 2
C. If $f(x)$ is a polynomial of degree 5 with real coefficients such that $f(x) = 0$ has 8 real roots, then the number of roots of $f(x) = 0$ is	r. 3
D. If $7^{ x }(5 - x) = 1$, then number of roots is	s. 4

181. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $f(x) = x + 1$ when $x < 0 = x^2 - 1$ for $x \geq 0$, then $f \circ f(x)$ for $-1 \leq x < 0$ is	p. $\frac{x-3}{2}$
B. If $f\left(\frac{2 \tan x}{1 + \tan^2 x}\right) = \frac{(\cos 2x + 1)(\sec^2 x + 2 \tan x)}{2}$, then $f(x)$ is	q. $x^2 + 2x$
C. If $f(x + y + 1) = (\sqrt{f(x)} + \sqrt{f(y)})^2, \forall x, y \in R$ and $f(0) = 1$, then $f(x)$ is	r. $1 + x$
D. If $4 < x < 5$ and $f(x) = \left[\frac{x}{4}\right] + 2x + 2$, where $[y]$ is the greatest integer $\leq y$, then $f^{-1}(x)$ is	s. $(x+1)^2$

Type 6. Single Integer Answer Type Questions

- 182.** An odd function is symmetric about the vertical line $x = a$ ($a > 0$) and if $\sum_{r=0}^{\infty} [f(1+4r)]^r = 8$, then the numerical value of $8f(1)$ is _____.
- 183.** Let $\frac{e^x - e^{-x}}{e^x + e^{-x}} = \ln \sqrt{\frac{1+x}{1-x}}$, then value of x is _____.
- 184.** The period of the function $f(x)$ which satisfies the relation $f(x) + f(x+4) = f(x+2) + f(x+6)$, is _____.
- 185.** If the maximum value of $f(x) = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is $5k + 1$, then the value of k is _____.
- 186.** If $f(x+y) = f(x) \cdot f(y)$, $\forall x, y \in R$, $f(5) = 2$, $f(0) \neq 0$, $f'(0) = 3$, then $f'(5)$ equals _____.
- 187.** Let f be a real valued invertible function such that $f\left(\frac{2x-3}{x-2}\right) = 5x-2$, $x \neq 2$. Then, the value of $f^{-1}(13)$ is _____.
- 188.** The number of integral values of x for which $\frac{\left(\frac{\pi}{2^{\tan^{-1}x}} - 4\right)(x-4)(x-10)}{x! - (x-1)!} < 0$, is _____.
- 189.** Let $f: R^+ \rightarrow R$ be a function which satisfies $f(x) \cdot f(y) = f(xy) + 2\left(\frac{1}{x} + \frac{1}{y} + 1\right)$ for $x, y > 0$, then possible value of $f\left(\frac{1}{2}\right)$ is _____.
- 190.** If the function f is continuous and has the property $f\{f(x)\} = 1-x$, then the value of $f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right)$ is _____.
- 191.** Let $a > 2$ be a constant. If there are just 18 positive integers satisfying the inequality $(x-a)(x-2a)(x-a^2) < 0$, then the value of a is _____.
- 192.** If θ is the fundamental period of function $f(x) = \sin^{99} x + \sin^{99}\left(x + \frac{2\pi}{3}\right) + \sin^{99}\left(x + \frac{4\pi}{3}\right)$, then complex number $z = |z|(\cos \theta + i \sin \theta)$ lies in the quadrant number _____.
- 193.** Suppose that $f(x)$ is a function of the form $f(x) = \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x}$ ($x \neq 0$). If $f(5) = 2$, then the value of $|f(-5)|/4$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

- 1.** Let $f_1: R \rightarrow R$, $f_2: [0, \infty) \rightarrow R$, $f_3: R \rightarrow R$ and $f_4: R \rightarrow [0, \infty)$, be defined by $f_1(x) = \begin{cases} |x|, & \text{if } x < 0 \\ e^x, & \text{if } x \geq 0 \end{cases}$

$$f_2(x) = x^2; f_3(x) = \begin{cases} \sin x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

$$\text{and } f_4(x) = \begin{cases} f_2(f_1(x)), & \text{if } x < 0 \\ f_2(f_1(x)) - 1, & \text{if } x \geq 0 \end{cases}$$

Column I	Column II
A. f_4 is	p. onto but not one-one
B. f_3 is	q. neither continuous nor one-one
C. $f_2 \circ f_1$ is	r. differentiable but not one-one
D. f_2 is	s. continuous and one-one

[2014]

Codes

A	B	C	D	A	B	C	D
(a) r	p	s	q	(b) r	p	q	s
(c) p	r	s	q	(d) p	r	q	s

- 2.** Let $f: \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$ be given by

$$f(x) = [\log(\sec x + \tan x)]^3. \text{ Then, [2014]}$$

- (a) $f(x)$ is an odd function
(b) $f(x)$ is a one-one function
(c) $f(x)$ is an onto function
(d) $f(x)$ is an even function

- 3.** The function $f: [0, 3] \rightarrow [1, 29]$ defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is [2012]

- (a) one-one and onto (b) onto but not one-one
(c) one-one but not onto (d) neither one-one nor onto

- 4.** Let $f: (-1, 1) \rightarrow R$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2 \theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$.

Then, the value(s) of $f\left(\frac{1}{3}\right)$ is/are [2012]

- (a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$ (c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$

5. Let $f(x) = x^2$ and $g(x) = \sin x$, $\forall x \in \mathbb{R}$. Then, the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f(g(x))$, is [2011]

AIEEE

6. Let R be the set of real numbers.

Statement I $A = \{(x, y) \in R \times R : y - x \text{ is an integer}\}$ is an equivalence relation on R .

Statement II $B = \{(x, y) \in R \times R : x = \alpha y \text{ for some rational number } \alpha\}$ is an equivalence relation on R .

[2011]

- (a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true, Statement II is false
(c) Statement I is false, Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

7. The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is [2011]

- (a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $(-\infty, \infty) - \{0\}$ (d) $(-\infty, \infty)$

8. Consider the relation $R = \{(A, B) | A = P^{-1}BP \text{ for some invertible matrix } P\}$ on the set of real square matrices of order 3.

Statement I R is an equivalence relation.

Statement II For any two invertible 3×3 matrices M and N , $(MN)^{-1} = N^{-1}M^{-1}$. [2011]

- (a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

9. Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of $p(2)$ is [2011]

- (a) 18 (b) 3
(c) 9 (d) 6

10. Let f be a function defined by $f(x) = (x-1)^2 + 1$, $(x \geq 1)$.

Statement I The set $\{x : f(x) = f^{-1}(x)\} = \{1, 2\}$

Statement II f is bijection and $f^{-1}(x) = 1 + \sqrt{x-1}$, $x \geq 1$. [2011]

- (a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

- (a) $\pm \sqrt{n\pi}$, $n \in \{0, 1, 2, \dots\}$
(b) $\pm \sqrt{n\pi}$, $n \in \{1, 2, \dots\}$
(c) $\pi/2 + 2n\pi$, $n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$
(d) $2n\pi$, $n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$

11. Consider the following relations

$R = \{(x, y) | x, y \text{ are real numbers and } x = wy \text{ for some rational number } w\}$;

$S = \left\{ \left(\frac{m}{n}, \frac{p}{q} \right) \mid m, n, p \text{ and } q \text{ are integers such that } n, q \neq 0 \text{ and } qm = pm \right\}$. Then, [2010]

- (a) R is an equivalence relation but S is not an equivalence relation
(b) Neither R nor S is an equivalence relation
(c) S is an equivalence relation but R is not an equivalence relation
(d) R and S both are equivalence relations

12. For real x , let $f(x) = x^3 + 5x + 1$, then [2009]

- (a) f is one-one but not onto R
(b) f is onto R but not one-one
(c) f is one-one and onto R
(d) f is neither one-one nor onto R

13. Let $f(x) = (x+1)^2 - 1$, $x \geq -1$

Statement I The set $\{x : f(x) = f^{-1}(x)\} = \{0, -1\}$.

Statement II f is a bijection. [2009]

- (a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

14. Let R be the real line. Consider the following subsets of the plane $R \times R$

$S = \{(x, y) : y = x + 1 \text{ and } 0 < x < 2\}$

$T = \{(x, y) : x - y \text{ is an integer}\}$

Which one of the following is true? [2008]

- (a) T is an equivalence relation on R but S is not
(b) Neither S nor T is an equivalence relation on R
(c) Both S and T are equivalence relations on R
(d) S is an equivalence relation on R but T is not

15. Let $f : N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$, where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$.

Show that f is invertible and its inverse is [2008]

- (a) $g(y) = \frac{y-3}{4}$
(b) $g(y) = \frac{3y+4}{3}$
(c) $g(y) = 4 + \frac{y+3}{4}$
(d) $g(y) = \frac{y+3}{4}$

16. The largest interval lying in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ for which the function $f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$ is defined, is [2007]

- (a) $[0, \pi]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(c) $\left[-\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$

17. Let W denotes the words in the English dictionary. Define the relation R by

$R = \{(x, y) \in W \times W : \text{the words } x \text{ and } y \text{ have atleast one letter in common}\}$. Then, R is [2006]

- (a) reflexive, symmetric and not transitive
(b) reflexive, symmetric and transitive
(c) reflexive, not symmetric and transitive
(d) not reflexive, symmetric and transitive

18. Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set

$A = \{3, 6, 9, 12\}$. Then, the relation is [2005]

- (a) reflexive and symmetric
(b) an equivalence relation
(c) reflexive only
(d) reflexive and transitive

19. Let $f: (-1, 1) \rightarrow B$ be a function defined by

$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, then f is both one-one and

onto when B is in the interval [2005]

- (a) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(c) $\left[0, \frac{\pi}{2}\right)$ (d) $\left[0, \frac{\pi}{2}\right)$

20. If a real valued function $f(x)$ satisfies the functional equation

$$f(x-y) = f(x)f(y) - f(a-x)f(a+y),$$

where a is a given constant and $f(0) = 1$, then $f(2a-x)$ is equal to [2005]

- (a) $f(-x)$
(b) $f(a) + f(a-x)$
(c) $f(x)$
(d) $-f(x)$

21. Let $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$ be a relation on the set $A = \{1, 2, 3, 4\}$. Then, relation R is [2004]

- (a) a function (b) transitive
(c) not symmetric (d) reflexive

22. If the graph of the function $y = f(x)$ is symmetrical about the line $x = 2$, then [2004]

- (a) $f(x+2) = f(x-2)$
(b) $f(2+x) = f(2-x)$
(c) $f(x) = f(-x)$
(d) $f(x) = -f(-x)$

23. The domain of the function $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ is [2004]

- (a) $[2, 3]$ (b) $[2, 3)$ (c) $[1, 2]$ (d) $[1, 2)$

24. A function f from the set of natural numbers to integers defined by

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases} \quad \text{is} \quad [2003]$$

- (a) one-one but not onto
(b) onto but not one-one
(c) one-one and onto both
(d) neither one-one nor onto

25. If $f: R \rightarrow R$ satisfies $f(x+y) = f(x) + f(y)$,

$\forall x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^n f(r)$ is [2003]

- (a) $\frac{7n}{2}$ (b) $\frac{7(n+1)}{2}$
(c) $7n(n+1)$ (d) $\frac{7n(n+1)}{2}$

26. The domain of definition of the function

$$f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x) \quad \text{is} \quad [2003]$$

- (a) $(1, 2)$ (b) $(-1, 0) \cup (1, 2)$
(c) $(1, 2) \cup (2, \infty)$ (d) $(-1, 0) \cup (1, 2) \cup (2, \infty)$

27. The function $f(x) = \log(x + \sqrt{x^2 + 1})$ is [2003]

- (a) an even function
(b) an odd function
(c) a periodic function
(d) neither an even nor an odd function

28. The period of $\sin^2 \theta$ is [2002]

- (a) π^2 (b) π (c) 2π (d) $\frac{\pi}{2}$

29. The domain of $\sin^{-1}\left[\log_3\left(\frac{x}{3}\right)\right]$ is [2002]

- (a) $[1, 9]$ (b) $[-1, 9]$
(c) $[-9, 1]$ (d) $[-9, -1]$

30. The period of the function $f(x) = \sin^4 x + \cos^4 x$ is [2002]

- (a) π (b) $\frac{\pi}{2}$
(c) 2π (d) None of these

31. The domain of definition of the function

$$f(x) = \sqrt{\log_{10}\left(\frac{5x-x^2}{4}\right)} \quad \text{is} \quad [2002]$$

- (a) $[1, 4]$
(b) $[1, 0]$
(c) $[0, 5]$
(d) $[5, 0]$

Other Engineering Entrances

- 32.** Let $R = \{(3, 3), (6, 6), (9, 9), (12, 12), (6, 12), (3, 9), (3, 12), (3, 6)\}$ be a relation on the set $A = \{3, 6, 9, 12\}$. Then, relation R is [BITSAT 2014]
 (a) an equivalence relation
 (b) reflexive and symmetric
 (c) reflexive and transitive
 (d) only reflexive
- 33.** Let S be the set of all real numbers. A relation R has been defined on S by $aRb \Leftrightarrow |a - b| \leq 1$, then R is [Karnataka CET 2014]
 (a) symmetric and transitive but not reflexive
 (b) reflexive and transitive but not symmetric
 (c) reflexive and symmetric but not transitive
 (d) an equivalence relation
- 34.** For any two real numbers θ and ϕ , we define $\theta R \phi$ if and only if $\sec^2 \theta - \tan^2 \phi = 1$. Then, relation R is [WB JEE 2014]
 (a) reflexive but not transitive
 (b) symmetric but not reflexive
 (c) both reflexive and symmetric but not transitive
 (d) an equivalence relation
- 35.** Let $f: N \rightarrow N$ be defined by

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}, \text{ then } f \text{ is}$$
 [Karnataka CET 2014]
 (a) onto but not one-one
 (b) one-one and onto
 (c) neither one-one nor onto
 (d) one-one but not onto
- 36.** If $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$, then the number of one-one functions from A into B is [Kerala CEE 2014]
 (a) 1340 (b) 1860 (c) 1430
 (d) 1880 (e) 1680
- 37.** The function $f(x) = x^2 + bx + c$, where b and c are real constants, describes [WB JEE 2014]
 (a) one-one mapping
 (b) onto mapping
 (c) not one-one but onto mapping
 (d) neither one-one nor onto mapping
- 38.** If $f(x)$ is an odd periodic function with period 2, then $f(4)$ is equal to [BITSAT 2014]
 (a) -4 (b) 4 (c) 2 (d) 0
- 39.** Which one of the following is not correct for the features of exponential function given by $f(x) = b^x$, where $b > 1$? [Karnataka CET 2014]
 (a) For very large negative values of x , the function is very close to 0
 (b) The domain of the function is R , the set of real numbers
 (c) The point $(1, 0)$ is always on the graph of the function
 (d) The range of the function is the set of all positive real numbers
- 40.** The domain of the function $f(x) = \sqrt{\cos x}$ is [Karnataka CET 2014]
 (a) $\left[\frac{3\pi}{2}, 2\pi\right]$ (b) $\left[0, \frac{\pi}{2}\right]$
 (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$
- 41.** Let $f(x) = |x - 2|$, where x is a real number. Which one of the following is true? [Kerala CEE 2014]
 (a) f is periodic
 (b) $f(x + y) = f(x) + f(y)$
 (c) f is an odd function
 (d) f is not one-one function
 (e) f is an even function
- 42.** The range of the function $f(x) = x^2 + 2x + 2$ is [Kerala CEE 2014]
 (a) $(1, \infty)$ (b) $(2, \infty)$ (c) $(0, \infty)$
 (d) $[1, \infty)$ (e) $(-\infty, \infty)$
- 43.** The range of the function $y = 3 \sin \left(\sqrt{\frac{\pi^2}{16} - x^2} \right)$ is [WB JEE 2014]
 (a) $[0, \sqrt{3/2}]$ (b) $[0, 1]$
 (c) $[0, 3/\sqrt{2}]$ (d) $[0, \infty)$
- 44.** If $A = \{1, 2, 3, 4\}$ and R is the relation on A defined by $\{(a, b) : a, b \in A, a \times b \text{ is an even number}\}$, then find the range of R . [J&K CET 2014]
 (a) $\{1, 2, 3, 4\}$ (b) $\{2, 4\}$
 (c) $\{2, 3, 4\}$ (d) $\{1, 2, 4\}$
- 45.** Find the domain of the function $f(x) = (x^2 + 1) / (x^2 - 3x + 3)$. [J&K CET 2014]
 (a) $R - \{1, 2\}$ (b) $R - \{1, 4\}$
 (c) R (d) $R - \{1\}$
- 46.** Find the range of the function $f: [0, 1] \rightarrow R, f(x) = x^3 - x^2 + 4x + 2 \sin^{-1} x$. [J&K CET 2014]
 (a) $[-(\pi + 2), 0]$ (b) $[0, 4 + \pi]$
 (c) $[2, 3]$ (d) $(0, 2 + \pi]$
- 47.** If $A = \{1, 2, 3, 4, 5\}$, then find the domain in the relation from A to A by $R = \{(x, y) : y = 2x - 1\}$. [J&K CET 2014]
 (a) $\{1, 2, 3\}$ (b) $\{1, 2\}$
 (c) $\{1, 3, 5\}$ (d) $\{2, 4\}$
- 48.** Let $f(x) = (x + 1)^2$ for $x \geq -1$. If $g(x)$ is a function whose graph is the reflection of the graph of $f(x)$ in the line $y = x$, then $g(x)$ is equal to [Karnataka CET 2014]
 (a) $\frac{1}{(x + 1)^2}, x > -1$
 (b) $-\sqrt{x} - 1$
 (c) $\sqrt{x} + 1$
 (d) $\sqrt{x} - 1$

- 49.** If $f(x) = \sqrt{x}$ and $g(x) = 2x - 3$, then domain of $f \circ g(x)$ is [Kerala CEE 2014]
 (a) $(-\infty, -3)$ (b) $(-\infty, -\frac{3}{2})$
 (c) $[-\frac{3}{2}, 0]$ (d) $[0, \frac{3}{2}]$
 (e) $[\frac{3}{2}, \infty)$
- 50.** If $f(x) = \frac{x+2}{3x-1}$, then $f\{f(x)\}$ is equal to [Kerala CEE 2014]
 (a) x (b) $-x$ (c) $\frac{1}{x}$
 (d) $-\frac{1}{x}$ (e) 0
- 51.** If R is the set of all real numbers and $f: R \rightarrow R$ is given by $f(x) = 3x^2 + 1$. Then, the set $f^{-1}([1, 6])$ is [WB JEE 2014]
 (a) $\left\{-\sqrt{\frac{5}{3}}, 0, \sqrt{\frac{5}{3}}\right\}$ (b) $\left[-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right]$
 (c) $\left[-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right]$ (d) $\left(-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right)$
- 52.** If $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, then $f(x)$ will be [RPET 2014]
 (a) $x^2 - 1$ (b) $x^2 - 2$ (c) x^2 (d) x^4
- 53.** Let z be the set of integers. Then, the relation $R = \{(a, b) : 1 + ab > 0\}$ on z is [AMU 2013]
 (a) reflexive and transitive but not symmetric
 (b) symmetric and transitive but not reflexive
 (c) reflexive and symmetric but not transitive
 (d) an equivalence relation
- 54.** If $f(x) = \cos ax + \sin x$ is periodic, then a must be [OJEE 2013]
 (a) irrational (b) rational
 (c) positive real number (d) None of these
- 55.** If the function $f(x) = \frac{a^x + a^{-x}}{2}$, where $a > 2$. Then, $f(x+y) + f(x-y)$ is equal to [Manipal 2013]
 (a) $f(x) - f(y)$
 (b) $f(y)$
 (c) $2f(x)f(y)$
 (d) None of the above
- 56.** If $f(x) = (x+2)^2 - 2, x \geq -2$, then $f^{-1}(x)$ is equal to [AMU 2013]
 (a) $-\sqrt{2+x} - 2$ (b) $\sqrt{2+x} + 2$
 (c) $\sqrt{2+x} - 2$ (d) $-\sqrt{2+x} + 2$
- 57.** If R is a relation on the set N , defined by $\{(x, y) : 2x - y = 10\}$, then R is [AMU 2012]
 (a) reflexive
 (b) symmetric
 (c) transitive
 (d) None of the above
- 58.** The total number of injections (one-one and into mappings) from $\{a_1, a_2, a_3, a_4\}$ to $\{b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ is [WB JEE 2012]
 (a) 400 (b) 420
 (c) 800 (d) 840
- 59.** If $f(x) = \sin \sqrt{x}$, then period of $f(x)$ is [OJEE 2012]
 (a) π
 (b) $\frac{\pi}{2}$
 (c) 2π
 (d) None of the above
- 60.** If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min f(x) > \max g(x)$, then relation between b and c is [Manipal 2012]
 (a) no relation
 (b) $0 < c < b/2$
 (c) $|c| < \sqrt{2}|b|$
 (d) $|c| > \sqrt{2}|b|$
- 61.** If $f: R \rightarrow R$ is defined by $f(x) = 2x + 3$, then $f^{-1}(x)$ [Karnataka CET 2012]
 (a) is given by $\frac{x-3}{2}$
 (b) is given by $\frac{1}{2x+3}$
 (c) does not exist because f is not injective
 (d) does not exist because f is not surjective
- 62.** If R is the set of real numbers and the functions $f: R \rightarrow R$ and $g: R \rightarrow R$ are defined by $f(x) = x^2 + 2x - 3$ and $g(x) = x + 1$. Then, the value of x for which $f[g(x)] = g[f(x)]$ is [WB JEE 2012]
 (a) -1 (b) 0 (c) 1 (d) 2
- 63.** If A and B are two equivalence relations defined on set C , then [UP SEE 2011]
 (a) $A \cap B$ is an equivalence relation
 (b) $A \cap B$ is not an equivalence relation
 (c) $A \cup B$ is an equivalence relation
 (d) $A \cup B$ is not an equivalence relation
- 64.** The function $f: [0, \infty) \rightarrow [0, \infty)$ defined by $f(x) = \frac{2x}{1+2x}$ is [J&K CET 2011]
 (a) one-one and onto (b) one-one but not onto
 (c) not one-one but onto (d) neither one-one nor onto
- 65.** The domain of $f(x) = \sin^{-1}\left[\log_2\left(\frac{x}{2}\right)\right]$ is [Karnataka CET 2011]
 (a) $0 \leq x \leq 1$ (b) $0 \leq x \leq 4$
 (c) $1 \leq x \leq 4$ (d) $4 \leq x \leq 6$
- 66.** The range of the function $f(x) = \log_e(3x^2 + 4)$ is [Kerala CEE 2011]
 (a) $[\log_e 2, \infty)$ (b) $[\log_e 3, \infty)$
 (c) $[2\log_e 3, \infty)$ (d) $[0, \infty)$
 (e) $[2\log_e 2, \infty)$

- 67.** The period of the function $f(x) = |\sin 2x| + |\cos 8x|$ is [Kerala CEE 2011]
 (a) 2π (b) π (c) $\frac{2\pi}{3}$
 (d) $\frac{\pi}{2}$ (e) $\frac{\pi}{4}$
- 68.** The even function is [WB JEE 2011]
 (a) $f(x) = \frac{a^x + a^{-x}}{a^x - a^{-x}}$
 (b) $f(x) = \frac{a^x + 1}{a^x - 1}$
 (c) $f(x) = x \cdot \frac{a^x - 1}{a^x + 1}$
 (d) $f(x) = \log_2(x + \sqrt{x^2 + 1})$
- 69.** The domain of $\cos^{-1} \frac{x-3}{2} - \log_{10} (4-x)$ is [GGSIPU 2011]
 (a) (1, 4) (b) [1, 4] (c) (1, 4] (d) [1, 4]
- 70.** If $f(x) = 3 - x$, where $-4 \leq x \leq 4$, then the domain of $\log_e [f(x)]$ is [J&K CET 2011]
 (a) $[-4, 4]$ (b) $(-\infty, 3]$ (c) $(-\infty, 3)$ (d) $[-4, 3]$
- 71.** If $f(x) = 4x^3 + 3x^2 + 3x + 4$, then $x^3 f\left(\frac{1}{x}\right)$ is equal to [MP PET 2011]
 (a) $f(-x)$ (b) $\frac{1}{f(x)}$ (c) $\left[f\left(\frac{1}{x}\right)\right]^2$ (d) $f(x)$
- 72.** If the function $f: [1, \infty) \rightarrow [1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$, then $f^{-1}(x)$ is equal to [Kerala CEE 2011]
 (a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}(1 - \sqrt{1 + 4 \log_2 x})$
 (c) $\frac{1}{2}\sqrt{1 + 4 \log_2 x}$ (d) $\frac{1}{2}(1 + \sqrt{1 + 4 \log_2 x})$
 (e) Not defined
- 73.** If $f = \{(0, -1), (-1, -3), (2, 3), (3, 5)\}$ is a function from z to z defined by $f(x) = ax + b$. Then, [AMU 2011]
 (a) $a = 1, b = -2$ (b) $a = 2, b = 1$
 (c) $a = 2, b = -1$ (d) $a = 1, b = 2$
- 74.** If $f(x)$ is a polynomial function of the second degree such that $f(-3) = 6, f(0) = 6$ and $f(2) = 11$, then the graph of the function, $f(x)$ cuts the ordinate $x = 1$ at the point [GGSIPU 2011]
 (a) (1, 8) (b) (1, 4)
 (c) (1, -2) (d) None of these
- 75.** If $g(x)$ is the inverse of $f(x)$ and $f'(x) = \cos x$, then $g'(x)$ is equal to [J&K CET 2011]
 (a) $\sec x$
 (b) $\sec [g(x)]$
 (c) $\cos [g(x)]$
 (d) $-\sin [g(x)]$
- 76.** If $A = \{1, 2, 3, 4\}$ and $B = \{1, 2, 3, 4, 5, 6\}$ are two sets and function $f: A \rightarrow B$ is defined by $f(x) = x + 2, \forall x \in A$, then the function f is [WB JEE 2010]
 (a) bijective (b) into (c) one-one (d) many-one
- 77.** The period of the function $f(\theta) = 4 + 4\sin^3 \theta - 3\sin \theta$ is [BITSAT 2010]
 (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$
 (c) $\frac{\pi}{2}$ (d) π
- 78.** The function $f(x) = \sec[\log(x + \sqrt{1+x^2})]$ is [BITSAT 2010]
 (a) odd
 (b) even
 (c) neither odd nor even
 (d) constant
- 79.** The domain of $\sin^{-1} \left[\log_2 \left(\frac{x}{12} \right) \right]$ is [Kerala CEE 2010]
 (a) [2, 12] (b) $[-1, 1]$
 (c) (6, 24) (d) $\left[\frac{2}{3}, 24 \right]$
 (e) [6, 24]
- 80.** The domain of the function $f(x) = \sqrt{\cos^{-1} \left(\frac{1-|x|}{2} \right)}$ is [WB JEE 2010]
 (a) $(-3, 3)$
 (b) $[-3, 3]$
 (c) $(-\infty, -3) \cup (3, \infty)$
 (d) $(-\infty, -3] \cup [3, \infty)$
- 81.** If $f(x) = \frac{\alpha x^2}{x+1}, x \neq -1$, then the value of α for which $f(a) = a$, where $a \neq 0$, is [BITSAT 2010]
 (a) $1 - \frac{1}{a}$
 (b) $\frac{1}{a}$
 (c) $1 + \frac{1}{a}$
 (d) $\frac{1}{a} - 1$
- 82.** If $f(x) = x^2 - 1$ and $g(x) = (x+1)^2$, then $gof(x)$ is equal to [Kerala CEE 2010]
 (a) $(x+1)^4 - 1$
 (b) $x^4 - 1$
 (c) x^4
 (d) $(x+1)^4$
 (e) $(x-1)^4 - 1$
- 83.** If f is a real valued function such that $f(x+y) = f(x) + f(y)$ and $f(1) = 5$, then the value of $f(100)$ is [Kerala CEE 2010]
 (a) 200 (b) 300 (c) 350
 (d) 400 (e) 500

Answers

Work Book Exercise 3.1

1. (d) 2. (a) 3. (b) 4. (d) 5. (a) 6. (b) 7. (b) 8. (a) 9. (b)

Work Book Exercise 3.2

1. (b) 2. (c) 3. (c) 4. (c) 5. (b) 6. (c) 7. (b) 8. (a) 9. (d) 10. (b)
11. (c) 12. (b) 13. (c) 14. (a) 15. (b)

Work Book Exercise 3.3

1. (a) 2. (b) 3. (b) 4. (b) 5. (a) 6. (d) 7. (a) 8. (c) 9. (a) 10. (a)

Work Book Exercise 3.4

1. (d) 2. (d) 3. (a) 4. (d) 5. (d) 6. (a) 7. (b) 8. (a) 9. (b) 10. (d)
11. (c) 12. (c) 13. (c) 14. (b) 15. (b) 16. (b) 17. (a) 18. (c) 19. (a) 20. (a)

Target Exercises

1. (c) 2. (d) 3. (d) 4. (a) 5. (c) 6. (a) 7. (b) 8. (d) 9. (d) 10. (d)
11. (c) 12. (d) 13. (d) 14. (a) 15. (b) 16. (b) 17. (a) 18. (b) 19. (d) 20. (a)
21. (b) 22. (a) 23. (c) 24. (b) 25. (c) 26. (a) 27. (c) 28. (b) 29. (a) 30. (c)
31. (a) 32. (b) 33. (c) 34. (d) 35. (b) 36. (b) 37. (d) 38. (b) 39. (c) 40. (a)
41. (b) 42. (d) 43. (b) 44. (b) 45. (a) 46. (b) 47. (a) 48. (c) 49. (a) 50. (a)
51. (a) 52. (d) 53. (a) 54. (c) 55. (a) 56. (a) 57. (c) 58. (b) 59. (c) 60. (c)
61. (a) 62. (a) 63. (b) 64. (b) 65. (d) 66. (d) 67. (b) 68. (b) 69. (c) 70. (a)
71. (a) 72. (d) 73. (c) 74. (b) 75. (a) 76. (d) 77. (d) 78. (c) 79. (d) 80. (d)
81. (d) 82. (d) 83. (b) 84. (c) 85. (b) 86. (a) 87. (d) 88. (d) 89. (c) 90. (c)
91. (a) 92. (b) 93. (a) 94. (b) 95. (a) 96. (c) 97. (b) 98. (d) 99. (b) 100. (a)
101. (a) 102. (c) 103. (a) 104. (b) 105. (a) 106. (b) 107. (c) 108. (b) 109. (a) 110. (b)
111. (b) 112. (a) 113. (b) 114. (c) 115. (a) 116. (d) 117. (a) 118. (c) 119. (c) 120. (a)
121. (b,c) 122. (a,b,c) 123. (a,d) 124. (a,c) 125. (a,b,d) 126. (a,c) 127. (a,c) 128. (b,c,d) 129. (all) 130. (d)
131. (d) 132. (c) 133. (b) 134. (a) 135. (a) 136. (c) 137. (c) 138. (b) 139. (d) 140. (b)
141. (c) 142. (a) 143. (b) 144. (a) 145. (a) 146. (a) 147. (a) 148. (c) 149. (d) 150. (a)
151. (c) 152. (d) 153. (d) 154. (c) 155. (c) 156. (b) 157. (b) 158. (c) 159. (d) 160. (a)
161. (b) 162. (a) 163. (d) 164. (b) 165. (b) 166. (a) 167. (c) 168. (b) 169. (b) 170. (a)
171. (c) 172. (d) 173. (d) 174. (b) 175. (*) 176. (*) 177. (*) 178. (*) 179. (*) 180. (*)
181. (*) 182. (7) 183. (0) 184. (8) 185. (8) 186. (6) 187. (3) 188. (5) 189. (4) 190. (1)
191. (5) 192. (3) 193. (7)

- *
175. $A \rightarrow p$; $B \rightarrow q$; $C \rightarrow q$; $D \rightarrow r, s$
176. $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow s$; $D \rightarrow s$
177. $A \rightarrow p, s$; $B \rightarrow s$; $C \rightarrow q$; $D \rightarrow p, s$
178. $A \rightarrow q$; $B \rightarrow p$; $C \rightarrow r$; $D \rightarrow s$
179. $A \rightarrow r, s$; $B \rightarrow r, s$; $C \rightarrow p, q$; $D \rightarrow p, s$
180. $A \rightarrow q$; $B \rightarrow s$; $C \rightarrow p$; $D \rightarrow s$
181. $A \rightarrow q$; $B \rightarrow r$; $C \rightarrow s$; $D \rightarrow p$

Entrances Gallery

1. (d) 2. (a,b,c) 3. (b) 4. (a,b) 5. (b) 6. (a) 7. (b) 8. (c) 9. (a) 10. (b)
11. (c) 12. (c) 13. (d) 14. (a) 15. (a) 16. (d) 17. (a) 18. (d) 19. (a) 20. (d)
21. (c) 22. (b) 23. (b) 24. (c) 25. (d) 26. (d) 27. (b) 28. (b) 29. (a) 30. (b)
31. (a) 32. (c) 33. (c) 34. (d) 35. (a) 36. (e) 37. (d) 38. (d) 39. (c) 40. (c,d)
41. (d) 42. (d) 43. (c) 44. (b) 45. (c) 46. (b) 47. (a) 48. (d) 49. (e) 50. (a)
51. (b) 52. (b) 53. (c) 54. (b) 55. (b) 56. (c) 57. (d) 58. (d) 59. (d) 60. (d)
61. (a) 62. (a) 63. (a) 64. (b) 65. (c) 66. (e) 67. (d) 68. (c) 69. (b) 70. (d)
71. (d) 72. (d) 73. (c) 74. (a) 75. (b) 76. (b,c) 77. (a) 78. (b) 79. (e) 80. (b)
81. (c) 82. (c) 83. (e)

Explanations

Target Exercises

- $R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$
 Now, $R^{-1} = \{(y, x) : (x, y) \in R\}$
 $= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\}$
 $= R$
 $\therefore$ Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$
 Range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$
- We have, $R = \{(x, y) : x, y \in N, x + 2y = 10, 1 \leq x, y \leq 10\}$
 Now, $x + 2y = 10$
 $\Rightarrow y = \frac{10 - x}{2}$
 $\therefore R = \{(2, 4), (4, 3), (6, 2), (8, 1)\}$
 Domain of $R = \{x : (x, y) \in R\} = \{2, 4, 6, 8\}$
 Range of $R = \{y : (x, y) \in R\} = \{4, 3, 2, 1\}$
- $x^2 + y^2 = 9 \Rightarrow y^2 = 9 - x^2$
 $\Rightarrow y = \pm \sqrt{9 - x^2}$
 $\therefore R = \{(0, 3), (0, -3), (3, 0), (-3, 0)\}$
 Domain of $R = \{x : (x, y) \in R\} = \{0, 3, -3\}$
 Range of $R = \{y : (x, y) \in R\} = \{3, -3, 0\}$
- For $f(x)$ to be defined,
 $x^2 + 2x \neq 0 \Rightarrow x(x + 2) \neq 0$
 i.e. $x \neq 0, -2$
 $\therefore$ Domain of $f = \frac{R}{\{0, -2\}}$
 For range, put $y = \frac{x^2 - x}{x^2 + 2x}; x \neq 0, -2$
 $\Rightarrow y = \frac{x - 1}{x + 2}$
 $\Rightarrow yx + 2y = x - 1$ or $x = \frac{2y + 1}{1 - y}$
 which is not defined for $y = 1$.
 Also, $x \neq 0 \Rightarrow y \neq -\frac{1}{2}$ and $x \neq -2$ for any real value of y .
 $\therefore$ Range of $f = \frac{R}{\left\{1, -\frac{1}{2}\right\}}$
- We have, $R = \{(2, 2), (3, 3), (4, 4), (1, 2)\}, \forall 1 \in A$ and $(1, 1) \notin R$
 $\therefore R$ is not reflexive.
 $(1, 2) \in R$ but $(2, 1) \notin R$
 $\therefore R$ is not symmetric.
 Hence, the relation R is transitive.
- $x \in R \Rightarrow x - x + \sqrt{5} = \sqrt{5}$ is an irrational number.
 $\therefore (x, x) \in R$
 So, R is reflexive.
 $(\sqrt{5}, 1) \in R$ because $\sqrt{5} - 1 + \sqrt{5} = 2\sqrt{5} - 1$
 which is an irrational number.
 But $(1, \sqrt{5}) \notin R$
 $\therefore R$ is not symmetric.

We have, $(\sqrt{5}, 1), (1, 2\sqrt{5}) \in R$ because

$$\sqrt{5} - 1 + \sqrt{5} = 2\sqrt{5} - 1$$

if $1 - 2\sqrt{5} + \sqrt{5} = 1 - \sqrt{5}$ are irrational numbers.

Also, $(\sqrt{5}, 2\sqrt{5}) \in R$ and $\sqrt{5} - 2\sqrt{5} + \sqrt{5} = 0$
 which is not an irrational number.

$$\therefore (\sqrt{5}, 2\sqrt{5}) \notin R$$

So, R is not transitive.

- Here, the relation R is defined as
 $R = \{(A, B) : A, B \in X \text{ and } A \cap B = \phi\}$
 Now, as $A \cap A = A$, therefore R is not reflexive.
 Let $A, B \in X$ such that
 $ARB \Rightarrow A \cap B = \phi$
 $\Rightarrow B \cap A = \phi \Rightarrow BRA$
 $\Rightarrow R$ is symmetric.

Clearly, R is not transitive as if we consider

$$A = \{1, 2, 3\}, C = \{2, 3\}$$

$$\text{and } B = \{4, 5, 6\}$$

Then, ARB, BRC but ARC .

- We have relation ' $|$ ' defined as $n|m, \forall n, m \in Z - \{0\}$.

$\therefore$ The relation ' $|$ ' is reflexive.

We have, $4|12$ but $12 \nmid 4$ is not true.

$\therefore$ The relation ' $|$ ' is not symmetric.

Let $a|b$ and $b|c$

Let $b = ak_1$ and $c = bk_2$ for integers k_1 and k_2 .

$$\therefore c = (ak_1)k_2 = a(k_1k_2) \text{ or } a|c$$

Hence, the relation is transitive.

- For $a \in Z$, we have $a \cdot a = a^2 \geq 0$
 $\therefore aRa, \forall a \in Z$
 $\therefore R$ is reflexive.
 Let aRb
 Then, $ab \geq 0$ or $ba \geq 0$ or bRa .
 $\therefore R$ is symmetric.
 Let aRb and bRc
 Then, $ab \geq 0, bc \geq 0$
 $\Rightarrow (ab)(bc) \geq 0$
 $\Rightarrow b^2(ac) \geq 0$
 $\Rightarrow ac \geq 0 \quad [\because b^2 \geq 0]$
 $\therefore R$ is transitive.
 Hence, R is an equivalence relation.

- We have, $R = \{(a, b) : n|(a - b) : a, b \in Z\}$
 Let $a \in Z$
 $\therefore a - a = 0 = n \times 0$ i.e. $n|(a - a)$
 $\therefore (a, a) \in R$ i.e. R is reflexive.
 Let $(a, b) \in R$
 $\therefore n|(a - b) \Rightarrow n|(b - a) \Rightarrow (b, a) \in R$
 $\therefore R$ is symmetric.
 Let $(a, b), (b, c) \in R$.
 $\therefore n|(a - b) \text{ and } n|(b - c) \Rightarrow n|(a - b) + (b - c)$
 $\Rightarrow n|(a - c) \Rightarrow (a, c) \in R$
 $\therefore R$ is transitive.
 Hence, R is an equivalence relation.

- 191

20. We have, $f(x) = \sqrt{x-3-2\sqrt{x-4}} - \sqrt{x-3+2\sqrt{x-4}}$
 $= |\sqrt{x-4}-1| - |\sqrt{x-4}+1|$
 $f(x)$ is defined, if $x-4 \geq 0$ or $x \geq 4$
 $\therefore$ Domain of $f = [4, \infty)$

21. $f(x)$ is defined, if $\frac{\sqrt{16-x^2}}{3-x} > 0$
 $\Rightarrow 16-x^2 > 0$ and $3-x > 0$
 $\Rightarrow (x-4)(x+4) < 0$ and $x < 3$
 $\Rightarrow -4 < x < 4$ and $x < 3$ or $-4 < x < 3$
 $\therefore$ Domain of $f = (-4, 3)$

22. $f(x)$ is defined, if $x+25 \geq 0, x \neq 0, x-5 \geq 0, x-3 \geq 0$.
 $\Rightarrow x \geq -25, x \neq 0, x \geq 5, x \geq 3$
 $\Rightarrow x \geq 5$
 $\therefore$ Domain of $f = [5, \infty)$

23. $f(x)$ is defined, if
 $\log_3\left(\frac{1}{|\cos x|}\right) \geq 0, \frac{1}{|\cos x|} > 0$ and $|\cos x| \neq 0$
 $\Rightarrow \frac{1}{|\cos x|} \geq 3^0 = 1$ and $|\cos x| > 0$
 $\Rightarrow |\cos x| \neq 0$
 $\Rightarrow x \neq (2n-1)\frac{\pi}{2}, n \in I$
 $\therefore$ Domain of $f = R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

24. $f(x)$ is defined, if
 $-\log_{1/2}\left(1+\frac{1}{x^{1/5}}\right) - 1 > 0, 1+\frac{1}{x^{1/5}} > 0, x \neq 0$
 $\Rightarrow \log_{1/2}\left(1+\frac{1}{x^{1/5}}\right) < -1, x^{1/5} + 1 > 0, x \neq 0$
 $\Rightarrow \frac{1}{x^{1/5}} > 1, x > -1$ and $x \neq 0$
 $\Rightarrow 0 < x < 1$ and $x > -1$
 $\Rightarrow 0 < x < 1$
 $\therefore$ Domain of $f = (0, 1)$

25. $f(x)$ is defined, if
 $-(\log_3 x)^2 + 5\log_3 x - 6 > 0$ and $x > 0$
 $\Rightarrow (\log_3 x - 3)(2 - \log_3 x) > 0$ and $x > 0$
 $\Rightarrow (\log_3 x - 2)(\log_3 x - 3) < 0$ and $x > 0$
 $\Rightarrow 2 < \log_3 x < 3$ and $x > 0$
 $\Rightarrow 3^2 < x < 3^3 \Rightarrow 9 < x < 27$
 $\therefore$ Domain of $f = (9, 27)$

26. $f(x)$ is defined, if $\log x \neq 0$ and $x > 0$
 $\Rightarrow x \neq 10^0 = 1$ and $x > 0$
 $\therefore$ Domain of $f = (0, 1) \cup (1, \infty) = (0, \infty) - \{1\}$

27. $f(x)$ is defined, if $\frac{4-x^2}{[x]+2} \geq 0$ and $[x]+2 \neq 0$

Case I $4-x^2 \geq 0$ and $[x]+2 > 0$
 $\Rightarrow (x-2)(x+2) \leq 0$ and $[x] > -2$
 $\Rightarrow -2 \leq x \leq 2$ and $[x] > -2$
 $\Rightarrow -1 \leq x \leq 2$

...(i)

Case II $4-x^2 \leq 0, [x]+2 < 0$
 $\Rightarrow (x-2)(x+2) \geq 0$ and $[x] < -2$
 $\Rightarrow (x \leq -2 \text{ or } x \geq 2)$ and $x < -2$
 $\therefore x < -2$... (ii)

From Eqs. (i) and (ii), we get
 Domain of $f = (-\infty, -2) \cup [-1, 2]$

28. For $f(x)$ to be defined, $x^2 - x - 6 \neq 0$,
 $\left[x + \frac{1}{2}\right] > 0, \left[x + \frac{1}{2}\right] \neq 1$
 $\Rightarrow (x-3)(x+2) \neq 0, x + \frac{1}{2} \geq 1, x + \frac{1}{2} \notin [1, 2]$
 $\Rightarrow x \neq 3, -2; x \geq \frac{1}{2}; x \notin \left[\frac{3}{2}, \frac{5}{2}\right]$
 $\therefore x \in \left[\frac{3}{2}, 3\right) \cup (3, \infty)$

29. Since, the domain of f is $(0, 1)$.
 $\therefore 0 < e^x < 1$ and $0 < \ln|x| < 1$
 $\Rightarrow \log 0 < x < \log 1$ and $e^0 < |x| < e^1$
 $\Rightarrow -\infty < x < 0$ and $1 < |x| < e$
 $\Rightarrow x \in (-\infty, 0)$ and $x \in \{(-\infty, 1) \cup (1, \infty)\} \cap (-e, e)$
 $\Rightarrow x \in (-\infty, 0)$ and $x \in (-e, -1) \cup (1, e)$
 $\therefore x \in (-e, -1)$

30. $\tan x$ is defined, if $x \neq n\pi + \frac{\pi}{2}$... (i)
 If $\tan x > 0$, then $|\tan x| + \tan x > 0$... (ii)
 If $\tan x \leq 0$, then $|\tan x| + \tan x = 0$... (iii)
 $\therefore$ Numerator is defined for both Eqs. (ii) and (iii) and non-zero.
 $\sqrt{3}x$ is defined for all $x > 0$... (iv)
 On combining Eqs. (i), (ii), (iii) and (iv), we get

$$D_f = R^+ - \left\{n\pi + \frac{\pi}{2}, n \in I^+\right\}$$

31. For y to be defined, we must have
 $\log_{10}\left(\frac{5x-x^2}{4}\right) \geq 0 \Rightarrow \frac{5x-x^2}{4} \geq 10^0$
 $\Rightarrow 5x-x^2 \geq 4 \Rightarrow x^2-5x+4 \leq 0$
 $\Rightarrow (x-1)(x-4) \leq 0$
 $\Rightarrow 1 \leq x \leq 4$

32. $f(x) = \sqrt{x-4-2\sqrt{(x-5)}} - \sqrt{x-4+2\sqrt{(x-5)}}$
 $= \sqrt{x-5-2\sqrt{(x-5)}} + 1 - \sqrt{x-4+2\sqrt{(x-5)}} + 1$
 $= \sqrt{(\sqrt{x-5}-1)^2} - \sqrt{(\sqrt{x-5}+1)^2}$
 $= |\sqrt{x-5}-1| - |\sqrt{x-5}+1|$
 Hence, domain is $[5, \infty)$.

33. We have, $f(x) = [x]^2 + [x+1] - 3$
 $= [x]^2 + [x] + 1 - 3$
 $= [x]^2 + [x] - 2$
 $= ([x]+2)([x]-1)$
 $\therefore f(x) = 0$
 $\therefore [x]+2 = 0$ or $[x]-1 = 0$
 $\Rightarrow [x] = -2$ or $[x] = 1$
 $\Rightarrow [x] = -2$
 $\Rightarrow -2 \leq x < -1$ or $1 \leq x < 2$

34. We have, $2^x + 2^y = 2$

$$\Rightarrow 2^y = 2 - 2^x$$

$$\Rightarrow y = \frac{\log(2 - 2^x)}{\log 2}$$

For y to be obtained, $2 - 2^x > 0$

$$\Rightarrow 2^x < 2$$

Since, 2^x is an increasing function, therefore we get

$$-\infty < x < 1$$

35. For $f(x)$ to be defined, we must have

$$x^2 - 10x - 11 \neq 0$$

$$\Rightarrow (x - 11)(x + 1) \neq 0$$

$$\Rightarrow x \neq 11, x \neq -1$$

$\therefore$ Domain of $f = (-\infty, \infty) - \{-1, 11\}$

36. Domain of $\cot^{-1} x$ is R and $\frac{x}{\sqrt{x^2 - [x^2]}}$ is defined, if

$$x^2 \neq [x^2] \quad [\because x^2 \geq [x^2]]$$

$\Rightarrow x^2 \neq 0$ or positive integer

Hence, domain of $f = R - \{\sqrt{n} : n \geq 0, n \in I\}$

37. Given function is defined, if $^{10}C_{x-1} > 3 \cdot ^{10}C_x$

$$\Rightarrow \frac{1}{11-x} > \frac{3}{x} \Rightarrow 4x > 33$$

$$\Rightarrow x \geq 9 \text{ but } x \leq 10$$

$$\therefore x = 9, 10$$

38. For the domain, $\sin(\ln x) > \cos(\ln x)$ and $x > 0$.

$$\therefore 2n\pi + \frac{\pi}{4} < \ln x < 2n\pi + \frac{5\pi}{4}, n \in N \cup \{0\}$$

$$\Rightarrow e^{(2n+1/4)\pi} < x < e^{(2n+5/4)\pi}$$

39. For $f(x)$ to be defined,

$$\frac{\sqrt{4-x^2}}{1-x} > 0, 4-x^2 > 0$$

and $1-x \neq 0$

Since, $\sqrt{4-x^2} \neq 0$

$$\therefore 1-x > 0 \text{ and } 4-x^2 > 0$$

$$\Rightarrow x < 1 \text{ and } (x-2)(x+2) < 0$$

$$\Rightarrow x < 1 \text{ and } -2 < x < 2$$

$$\Rightarrow -2 < x < 1$$

$\therefore$ Domain of $f = (-2, 1)$

$$\text{Since, } -\infty < \log\left(\frac{\sqrt{4-x^2}}{1-x}\right) < \infty$$

$$\therefore -1 \leq \sin\left[\log\left(\frac{\sqrt{4-x^2}}{1-x}\right)\right] \leq 1$$

Hence, range of $f = [-1, 1]$.

40. For $f(x)$ to be defined, $3 - x^2 \neq 0$ i.e. $x \neq \pm\sqrt{3}$

$$\therefore \text{Domain of } f = \frac{R}{\{\pm\sqrt{3}\}}$$

$$\text{Now, let } y = \frac{5}{3-x^2}$$

$$\Rightarrow x^2 = \frac{3y-5}{y} = \frac{y(3y-5)}{y^2}$$

For x to be real,

$$x^2 \geq 0$$

$$\Rightarrow y(3y-5) \geq 0$$

$$\text{Also, } y \neq 0$$

$$\therefore y < 0$$

$$\text{or } y \geq \frac{5}{3}$$

$$\text{Hence, range of } f = (-\infty, 0) \cup \left[\frac{5}{3}, \infty\right).$$

41. Since, the given function has minimum value 75, which is attained at $x=2$ and maximum value 89, which is attained at $x=3$.

Hence, the range of f is $[75, 89]$.

$$42. \text{ We have, } f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$$

$f(x)$ will attain its minimum value, when $\frac{2}{x^2+1}$ is

maximum i.e. when x^2+1 is maximum i.e. at $x=0$.

$\therefore$ Minimum value of $f(x)$ is $f(0) = -1$.

$$43. \text{ We have, } f(x) = \begin{cases} -1, & -2 \leq x \leq 0 \\ x-1, & 0 < x \leq 2 \end{cases}$$

Since, $x \in [-2, 2]$, so $|x| \in [0, 2]$

$$\text{Therefore, } f(|x|) = \begin{cases} -x-1, & x \in [-2, 0] \\ x-1, & x \in [0, 2] \end{cases} \quad \dots(i)$$

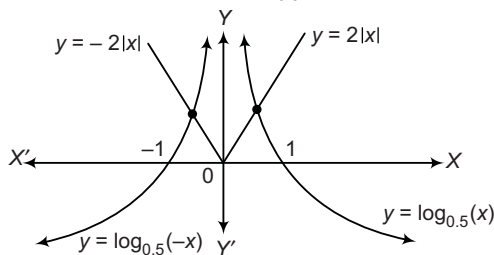
$$\Rightarrow |f(x)| = \begin{cases} 1, & -2 \leq x < 0 \\ 1-x, & 0 \leq x < 1 \\ x-1, & 1 \leq x \leq 2 \end{cases} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$g(x) = f(|x|) + |f(x)| = \begin{cases} -x-1+1, & -2 \leq x < 0 \\ x-1+1-x, & 0 \leq x < 1 \\ x-1+x-1, & 1 \leq x \leq 2 \end{cases}$$

$$= \begin{cases} -x, & -2 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 2(x-1), & 1 \leq x \leq 2 \end{cases}$$

44. Draw the graph of $y = \log_{0.5}|x|$ and $y = 2|x|$.



Clearly, from the graph, there are two solutions.

45. We have,

$$f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$$

$$= \sin 9x + \sin(-10)x$$

$$= \sin 9x - \sin 10x$$

$$\therefore f\left(\frac{\pi}{2}\right) = \sin \frac{9\pi}{2} - \sin 5\pi = 1 - 0 = 1$$

$$f(\pi) = \sin 9\pi - \sin 10\pi = 0$$

$$\text{and } f\left(\frac{\pi}{4}\right) = \sin \frac{9\pi}{4} - \sin \frac{10\pi}{4} = \frac{1}{\sqrt{2}} - 1$$

46. We have, $\left[\frac{1}{2} + \frac{x}{100}\right] = \begin{cases} 0, & \text{if } 0 < x < 49 \\ 1, & \text{if } 50 \leq x \leq 99 \end{cases}$

Thus,

$$\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \dots + \left[\frac{1}{2} + \frac{99}{100}\right] = 50$$

47. For y to be defined, $\frac{x^2}{2} > 0$... (i)

$-1 \leq \log_2 \left(\frac{x^2}{2}\right) \leq 1$... (ii)

From Eq. (i), we get $x \in \mathbb{R} - \{0\}$... (iii)

From Eq. (ii), we get

$2^{-1} \leq \frac{x^2}{2} \leq 2^1 \Rightarrow 1 \leq x^2 \leq 4$

$\Rightarrow -2 \leq x \leq -1$ or $1 \leq x \leq 2$... (iv)

From Eqs. (iii) and (iv), we get

$x \in [-2, -1] \cup [1, 2]$

48. $g(x) = x^3 + \tan x + \left[\frac{x^2 + 1}{P}\right]$

$\Rightarrow g(-x) = (-x)^3 + \tan(-x) + \left[\frac{(-x)^2 + 1}{P}\right]$

$\Rightarrow g(-x) = -x^3 - \tan x + \left[\frac{x^2 + 1}{P}\right]$

$\Rightarrow g(x) + g(-x) = 0$

Since, $g(x)$ is an odd function.

$\therefore \left(-x^3 - \tan x + \left[\frac{x^2 + 1}{P}\right]\right) + \left(-x^3 - \tan x + \left[\frac{x^2 + 1}{P}\right]\right) = 0$

$\Rightarrow 2\left[\frac{(x^2 + 1)}{P}\right] = 0 \Rightarrow 0 \leq \frac{x^2 + 1}{P} < 1$

Now, $x \in [-2, 2]$

$\Rightarrow 0 \leq \frac{5}{P} < 1 \Rightarrow P > 5$

49. $f(-x) = \frac{\cos(-x)}{\left[-\frac{2x}{\pi}\right] + \frac{1}{2}} = \frac{\cos x}{-1 - \left[\frac{2x}{\pi}\right] + \frac{1}{2}}$

[since, x is not an integral multiple of π]

$\Rightarrow f(-x) = -\frac{\cos x}{\left[\frac{2x}{\pi}\right] + \frac{1}{2}} = -f(x)$

Hence, $f(x)$ is an odd function.

50. Since, $f(x) = f\left(\frac{x+1}{x+2}\right)$

$\therefore x = \frac{x+1}{x+2} \Rightarrow x^2 + x - 1 = 0$

$\Rightarrow x = \frac{-1 \pm \sqrt{5}}{2}$... (i)

and $f(x)$ is an even function defined on $[-5, 5]$.

$\therefore f(-x) = f(x), \forall x \in [-5, 5] \Rightarrow x = -\left(\frac{x+1}{x+2}\right)$

$\Rightarrow x^2 + 3x + 1 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{5}}{2}$... (ii)

From Eqs. (i) and (ii), we get the values of x as

$x = \frac{-1 \pm \sqrt{5}}{2}$ and $x = \frac{-3 \pm \sqrt{5}}{2}$

51. Since, the function $\sec x$ is an even function and $\log(x + \sqrt{1+x^2})$ is an odd function, therefore the function $\sec[\log(x + \sqrt{1+x^2})]$ is an even function.

52. Let's some of the functions of the parts (a), (b), (c) and (d) as $f(x)$, $g(x)$, $h(x)$ and $\phi(x)$, respectively.

(a) $f(-x) = \operatorname{sgn}(-x) + (-x)^{2000} = -\operatorname{sgn} x + x^{2000}$
 $\neq f(x)$ and $\neq -f(x)$

$\therefore f$ is neither even nor odd.

(b) $g(-x) = |-x| - \tan(-x)$
 $= |x| + \tan x$

$\therefore g$ is neither even nor odd.

(c) $h(-x) = (-x)^3 \cot(-x)$
 $= -x^3(-\cot x) = x^3 \cot x = h(x)$

$\therefore h$ is an even function.

(d) $\phi(-x) = \operatorname{cosec}(-x)^{55} = \operatorname{cosec}(-x^{55})$
 $= -\operatorname{cosec} x^{55} = -\phi(x)$

$\therefore \phi$ is an odd function.

53. To make $f(x)$ an even function in the interval $[-1, 1]$, we redefined $f(x)$ as

$$f(x) = \begin{cases} f(x), & 0 \leq x \leq 1 \\ f(-x), & -1 \leq x < 0 \end{cases}$$

$$= \begin{cases} 3x^2 - 4x + 8 \log(1 + |x|), & 0 \leq x \leq 1 \\ 3x^2 + 4x + 8 \log(1 + |x|), & -1 \leq x < 0 \end{cases}$$

Thus, the even extension of $f(x)$ in the interval $[-1, 1]$ is $3x^2 + 4x + 8 \log(1 + |x|)$.

54. $\sum_{r=1}^{2000} \frac{\{x+r\}}{2000} = \sum_{r=1}^{2000} \frac{\{x\}}{2000} = 2000 \frac{\{x\}}{2000} = \{x\}$

55. Given, $f(x) = \sqrt[n]{x^m}$, $n \in \mathbb{N}$, is an even function, where $m \in \mathbb{I}$.

$\Rightarrow f(x) = f(-x) \Rightarrow \sqrt[n]{x^m} = \sqrt[n]{(-x)^m}$

$\Rightarrow x^m = (-x)^m$

So, m is an even integer.

$\therefore m = 2k, k \in \mathbb{I}$

56. Let $f(T+x) = f(x)$

$\Rightarrow \sqrt{\tan(T+x)} = \sqrt{\tan x}$

$\Rightarrow \tan(T+x) = \tan x$

$\Rightarrow T+x = n\pi + x, n \in \mathbb{I}$

Clearly, the least positive value of T which is independent of x , is π . Therefore, $f(x)$ is a periodic function of period π .

57. Since, $\cos x$ is a periodic function with period 2π , therefore $f(x) = \cos\left(\frac{8x+5}{4\pi}\right)$ will be a periodic function with period $\frac{2\pi}{\frac{8}{4\pi}}$, i.e. π^2 .

58. Let $f(T+x) = f(x)$

$\Rightarrow (T+x) - [T+x] = x - [x]$

$\Rightarrow T = [T+x] - [x] = \text{an integer.}$

Clearly, the least positive value of T which is independent of x , is 1. Hence, $f(x)$ is a periodic function of period 1.

59. Let $g(x) = e^{3\{x\}} \Rightarrow T_1 = 1$

and $f(x) = e^{\{3x\}} \Rightarrow T_2 = \frac{1}{3}$

$\therefore T_1 = 3T_2$

60. Since, $\sin x$ is a periodic function with period 2π , therefore $\sin(\sqrt{[n]}x)$ is periodic function with period $\frac{2\pi}{\sqrt{[n]}}$.

But the period of $f(x)$ is 2π .

$\therefore \frac{2\pi}{\sqrt{[n]}} = 2\pi \Rightarrow \sqrt{[n]} = 1 \Rightarrow [n] = 1 \Rightarrow 1 \leq n \leq 2$

61. Since, $\sin x$ and $\cos x$ are periodic functions with period 2π .

$\therefore$ Period of $\cos\left(\frac{\pi x}{n!}\right) = \frac{2\pi}{\frac{\pi}{n!}} = 2(n!)$

and period of $\sin\left(\frac{\pi x}{(n+1)!}\right) = \frac{2\pi}{\frac{\pi}{(n+1)!}} = 2(n+1)!$

$\therefore$ Period of $f(x) = \text{LCM of } \{2(n!), 2(n+1)!\} = 2(n+1)!$

62. Since, $|\sin x| + |\cos x|$ is a periodic function with period $\pi/2$, therefore period of $f(x)$ will be $\pi/2$, when $k = 1$.

63. $\sin^2 \pi x = \frac{1 - \cos 2\pi x}{2}$

Since, $\cos 2\pi x$ is a periodic function with period $\frac{2\pi}{2\pi} = 1$, therefore $\sin^2 \pi x$ is periodic with period 1. ... (i)

Also, $x - [x]$ is a periodic function with period 1. ... (ii)

$\therefore \sin^4 \pi x = (\sin^2 \pi x)^2 = \frac{1}{4}(1 - \cos 2\pi x)^2$
 $= \frac{1}{4}(1 + \cos^2 2\pi x - 2\cos 2\pi x)$
 $= \frac{1}{8}(3 + \cos 4\pi x - 4\cos 2\pi x)$

Since, $\cos 4\pi x$ is a periodic function with period $\frac{2\pi}{4\pi} = \frac{1}{2}$ and $\cos 2\pi x$ is a periodic function with period $\frac{2\pi}{2\pi} = 1$.

Therefore, period of $\sin^4 \pi x$

$= \text{LCM}\left(1, \frac{1}{2}\right) = \frac{\text{LCM}(1, 1)}{\text{HCF}(1, 2)} = \frac{1}{1} = 1 \quad \dots (iii)$

From Eqs. (i), (ii) and (iii), we get

Period of $3(\sin^2 \pi x + x - [x] + \sin^4 \pi x) = 1$

64. Since, $2\sin x$ is a periodic function of period 2π and $3\cos 2x$ is a periodic function of π .

$\therefore f(x) = 2\sin x + 3\cos 2x$ has period $\text{LCM of } \{\pi, 2\pi\} = 2\pi$

65. For every rational number T , we have

$f(T + x) = \begin{cases} 1, & \text{when } x \text{ is a rational} \\ 0, & \text{when } x \text{ is an irrational} \end{cases}$

But there is no least positive value of T for which $f(T + x) = f(x)$ because there are infinite number of rational numbers between any two rational numbers. Therefore, $f(x)$ is a periodic function having no fundamental period.

66. Since, $\cos \frac{2\pi x}{5}$ is a periodic function with period

$\frac{2\pi}{\frac{2\pi}{5}} = 5$ and $\sin \frac{\pi x}{4}$ is a periodic function with period $\frac{2\pi}{\frac{\pi}{4}} = 8$.

$\therefore f(x)$ is a periodic with period $= \text{LCM}(5, 8) = 40$

67. We have, $f(x + k) = 1 + [1 + \{1 - f(x)\}^5]^{1/5}$

$\Rightarrow f(x + k) - 1 = [1 - (f(x) - 1)^5]^{1/5}$

$\Rightarrow \phi(x + k) = [1 - \{\phi(x)\}^5]^{1/5}$

where, $\phi(x) = f(x) - 1$

$\Rightarrow \phi(x + 2k) = [1 - \{\phi(x + k)\}^5]^{1/5}$

$\Rightarrow \phi(x + 2k) = [1 - \{1 - \{\phi(x)\}^5\}^5]^{1/5}$
 $= \phi(x), \forall x \in R$

$\Rightarrow f(x + 2k) - 1 = f(x) - 1$

$\Rightarrow f(x + 2k) = f(x), \forall x \in R$

$\therefore f(x)$ is periodic with period $2k$.

68. The period of $f(x) = \frac{\pi}{\sqrt{[k]}}$

But the period of $f(x) = \pi$ [given]

$\therefore \frac{\pi}{\sqrt{[k]}} = \pi$

$\Rightarrow \sqrt{[k]} = 1$

$\Rightarrow 1 \leq k < 2$

69. The period of $\sin 5x$ is $\frac{2\pi}{5}$ and that of $\cos \sqrt{3}x$ is $\frac{2\pi}{\sqrt{3}}$.

As, $\frac{2\pi}{5}$ and $\frac{2\pi}{\sqrt{3}}$ do not have a common multiple, so $f(x)$ is non-periodic.

70. $3x + 3 - [3x + 3]$ has the period 1 and $\sin \frac{\pi x}{2}$ has the period $\frac{2\pi}{\frac{\pi}{2}}$ i.e. 4. Hence, period of $f(x)$ is 4.

71. For $n = 2$, we have

$$f(x) = \frac{\sin 2x}{\sin\left(\frac{x}{2}\right)} = \frac{2 \sin x \cos x}{\sin\left(\frac{x}{2}\right)}$$

$$= \frac{4 \sin \frac{x}{2} \cos \frac{x}{2} \cos x}{\sin\left(\frac{x}{2}\right)} = 4 \cos \frac{x}{2} \cos x$$

The period of $\cos x$ is 2π and that of $\cos \frac{x}{2}$ is $\frac{2\pi}{1/2} = 4\pi$.

Hence, the period of $f(x)$ is 4π .

72. The period of $|\sin x| + |\cos x|$ and $\sin^4 x + \cos^4 x$ is $\frac{\pi}{2}$.

$\sin(\sin x) + \sin(\cos x)$ has period 2π . The function $\frac{1 + 2 \cos x}{\sin x (2 + \sec x)}$ can be written in a simplified form as

$\frac{\cos x}{\sin x} = \cot x$, so it has period π .

73. $\tan(3x - 2)$ is a periodic function with period $\pi/3$. The function $f(x) = \{x\}$ is periodic with period 1. The function in option (d) can be written as

$$\begin{aligned} f(x) &= 1 - \frac{\cos^3 x}{\sin x + \cos x} - \frac{\sin^3 x}{\sin x + \cos x} \\ &= 1 - \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} \\ &= 1 - \frac{(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)}{(\sin x + \cos x)} \\ &= 1 - \left(1 - \frac{1}{2} \sin 2x\right) = \frac{1}{2} \sin 2x \end{aligned}$$

which is periodic with period π .

The function $x + \cos x$ is non-periodic as x is non-periodic.

74. We have, $f(x) = \frac{2 \sin 8x \cos x - 2 \sin 6x \cos 3x}{2 \cos 2x \cos x - 2 \sin 3x \sin 4x}$

$$\begin{aligned} &= \frac{(\sin 9x + \sin 7x) - (\sin 9x + \sin 3x)}{(\cos 3x + \cos x) + (\cos 7x - \cos x)} \\ &= \frac{\sin 7x - \sin 3x}{\cos 7x + \cos 3x} = \frac{2 \cos 5x \sin 2x}{2 \cos 5x \cos 2x} = \tan 2x \end{aligned}$$

which is periodic with period $\frac{\pi}{2}$.

75. We have,

$$\begin{aligned} f(x) &= \frac{(\sin 5x + \sin x) + (\sin 4x + \sin 2x)}{(\cos 5x + \cos x) + (\cos 4x + \cos 2x)} \\ &= \frac{2 \sin 3x \cos 2x + 2 \sin 3x \cos x}{2 \cos 3x \cos 2x + 2 \cos 3x \cos x} \\ &= \frac{2 \sin 3x (\cos 2x + \cos x)}{2 \cos 3x (\cos 2x + \cos x)} = \tan 3x \end{aligned}$$

which is periodic with period $\frac{\pi}{3}$.

76. Since, $f(x)$ is an odd periodic function with period 2.

$$\begin{aligned} \therefore f(-x) &= -f(x) \text{ and } f(x+2) = f(x) \\ \therefore f(2) &= f(0+2) = f(0), \text{ where } f(0) = 0 \\ \text{and } f(4) &= f(2+2) = f(2) = f(0) = 0 \\ \text{Thus, } f(4) &= 0 \end{aligned}$$

77. The period of $f(x)$ is 7 $\Rightarrow$ The period of $f\left(\frac{x}{3}\right)$ is $\frac{7}{1/3} = 21$

The period of $g(x)$ is 11 $\Rightarrow$ The period of $g\left(\frac{x}{5}\right)$ is

$$\frac{11}{1/5} = 55$$

$$\text{Hence, } T_1 = \text{period of } f(x) g\left(\frac{x}{5}\right) = 7 \times 55 = 385$$

$$\text{and } T_2 = \text{period of } g(x) f\left(\frac{x}{3}\right) = 11 \times 21 = 231$$

$$\therefore \text{Period of } F(x) = \text{LCM}\{T_1, T_2\} = \text{LCM}\{385, 231\} = 7 \times 11 \times 3 \times 5 = 1155$$

78. From the given data,

$$f(1-x) = f(1+x) \quad \dots (i)$$

$$\text{and } f(2-x) = f(2+x) \quad \dots (ii)$$

On replacing x by $1+x$ in Eq. (ii), we get

$$f(1-x) = f(3+x)$$

$$\Rightarrow f(1+x) = f(3+x) \quad [\text{from Eq. (i)}]$$

$$\Rightarrow f(x) = f(2+x)$$

$$79. \frac{Y-X}{Y+X} = k \Rightarrow Y-X = k(Y+X)$$

$$\Rightarrow Y(1-k) = X(1+k)$$

$$\Rightarrow Y = \left(\frac{1+k}{1-k}\right)X$$

$$\text{where, } \frac{1+k}{1-k} < -1 \text{ and } k > 1$$

$$80. f(x) = \frac{a^x - 1}{x^n(a^x + 1)}$$

Since, $f(x)$ is symmetrical about Y-axis.

$$\therefore f(x) = f(-x)$$

$$\Rightarrow \frac{a^x - 1}{x^n(a^x + 1)} = \frac{a^{-x} - 1}{(-x)^n(a^{-x} + 1)}$$

$$\Rightarrow \frac{a^x - 1}{x^n(a^x + 1)} = \frac{1 - a^x}{(-x)^n(1 + a^x)}$$

$$\Rightarrow x^n = -(-x)^n$$

Hence, the value of n which satisfy this relation, is $-\frac{1}{3}$.

$$81. f(x) = e^{x^3 - 3x + 2}$$

$$\text{Let } g(x) = x^3 - 3x + 2$$

$$\therefore g'(x) = 3x^2 - 3 = 3(x^2 - 1)$$

$$\Rightarrow g'(x) \geq 0 \text{ for } x \in (-\infty, -1]$$

So, $f(x)$ is increasing function.

$\therefore f(x)$ is one-one.

Now, the range of $f(x) = (0, e^4]$

But codomain is $(0, e^5]$.

Hence, $f(x)$ is an into function.

82. Let $x, y \in N$ such that $f(x) = f(y)$.

$$\text{Then, } f(x) = f(y)$$

$$\Rightarrow x^2 + x + 1 = y^2 + y + 1$$

$$\Rightarrow (x-y)(x+y+1) = 0$$

$$\Rightarrow x = y$$

$$\text{or } x = (-y-1) \notin N$$

$\therefore f$ is one-one.

Also, $f(x)$ does not take all positive integral values. Hence, f is into.

83. Two triangles may have equal areas.

$\therefore f$ is not one-one.

Since, each positive real number can represent area of a triangle.

$\therefore f$ is onto.

84. Let $t = 5x + 2$, then $A = \{t : 0 \leq t \leq \pi\}$

$\therefore f(t) = \cos t + x$, which is bijective in $[0, \pi]$.

Hence, $f(x)$ is bijective.

85. Let $f(x_1) = f(x_2) \Rightarrow x_1 - [x_1] = x_2 - [x_2] \Rightarrow x_1 = x_2$

If $x = 2.7$ and $x_2 = 3.7$, then

$$f(x_1) = f(x_2) = 0.7 \text{ but } x_1 \neq x_2$$

$\therefore f$ is not one-one. Clearly, f is onto.

86. Since, for different x , 4^x and $4^{|x|}$ are different positive numbers.

$\therefore f$ is one-one. Also, f is not onto as its range is $(0, \infty)$, which is proper subset of its codomain $[0, \infty)$.

87. Since, $f(x) = f(-x)$

$\therefore f$ is not one-one.

Let $y \in R$. Then, $f(x) = y$

$$\Rightarrow y = \frac{x^2 - 8}{x^2 + 2} \Rightarrow x^2 = \frac{8 + 2y}{1 - y}$$

For x to be real, $(8 + 2y)(1 - y) \geq 0$ and $1 - y \neq 0$

$$\Rightarrow (y + 4)(y - 1) \leq 0 \text{ and } y \neq 1$$

$$\Rightarrow -4 \leq y < 1$$

$\therefore$ Range of $f = [-4, 1) \subset R$

Hence, f is not onto.

88. We have, $f(x) = x + \sqrt{x^2} = x + |x|$

Clearly, f is not onto as

$$f(-1) = f(-2) = 0 \text{ but } -1 \neq -2$$

Also, f is not onto as $f(x) \geq 0, \forall x \in R$.

$\therefore$ Range of $f = (0, \infty) \subset R$.

89. Let $x + y = a$ and $x - y = b$

$$\Rightarrow x = \frac{a+b}{2} \text{ and } y = \frac{a-b}{2}$$

$$\therefore f(x+y, x-y) = xy$$

$$\Rightarrow f(a, b) = \frac{a+b}{2} \cdot \frac{a-b}{2} = \frac{a^2 - b^2}{4}$$

$$\therefore f(x, y) = \frac{x^2 - y^2}{4} \text{ and } f(y, x) = \frac{y^2 - x^2}{4}$$

Hence, arithmetic mean of $f(x, y)$ and $f(y, x)$ is zero.

90. We have, $f(a) = 64a^3 + \frac{1}{a^3} = (4a)^3 + \frac{1}{a^3}$

$$= \left(4a + \frac{1}{a}\right)^3 - 3 \cdot 4a \cdot \frac{1}{a} \left(4a + \frac{1}{a}\right)$$

$$= (3)^3 - 12 \cdot 3 = 27 - 36 = -9$$

$$\left[\begin{array}{l} \text{since, } a, b \text{ are roots of } 4x + \frac{1}{x} = 3 \\ \therefore 4a + \frac{1}{a} = 3 \end{array} \right]$$

Similarly, $f(b) = -9$

$$\therefore f(a) = f(b) = -9$$

91. Since, $f(x+y) = f(x) + f(y) - xy - 1, \forall x, y \in R$

$$\therefore f(x+1) = f(x) + f(1) - x - 1 \quad [\text{putting } y = 1]$$

$$\Rightarrow f(x+1) = f(x) - x \quad [\because f(1) = 1]$$

$$\therefore f(n+1) = f(n) - n < f(n)$$

$$\Rightarrow f(n+1) < f(n)$$

So, $f(n) < f(n-1) < f(n-2) < \dots < f(3) < f(2) < f(1) = 1$

$\therefore f(n) = n$ holds only for $n = 1$.

92. We have,

$$3f(x) + 5f\left(\frac{1}{x}\right) = \frac{1}{x} - 3, \forall x \in R \quad \dots(i)$$

$$\Rightarrow 3f\left(\frac{1}{x}\right) + 5f(x) = x - 3 \quad [\text{replacing } x \text{ by } \frac{1}{x}] \dots(ii)$$

Multiplying Eq. (i) by 3 and Eq. (ii) by 5 and then subtracting, we get

$$9f(x) - 25f(x) = \left(\frac{3}{x} - 9\right) - (5x - 15)$$

$$\Rightarrow -16f(x) = \frac{3}{x} - 5x + 6$$

$$\Rightarrow f(x) = \frac{1}{16} \left(-\frac{3}{x} + 5x - 6\right), \forall x \in R$$

93. Let the required function be $f(x) = ax + b$.

If $a > 0$, then $f(-1) = 0$ and $f(1) = 2$

$$\Rightarrow -a + b = 0 \text{ and } a + b = 2$$

$$\Rightarrow a = 1 \text{ and } b = 1$$

If $a < 0$, then $f(-1) = 2$ and $f(1) = 0$

$$\Rightarrow -a + b = 2 \text{ and } a + b = 0$$

$$\Rightarrow a = -1 \text{ and } b = 1$$

Hence, $f(x) = x + 1$ or $f(x) = -x + 1$

94. We have, $f(x) = \frac{4^x}{4^x + 2}$

$$\Rightarrow f(1-x) = \frac{4^{1-x}}{4^{1-x} + 2} = \frac{4 \cdot 4^{-x}}{4 \cdot 4^{-x} + 2}$$

$$= \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{2 + 4^x}$$

$$\therefore f(x) + f(1-x) = \frac{4^x}{4^x + 2} + \frac{2}{4^x + 2} = 1$$

$$\text{Now, } f\left(\frac{1}{1997}\right) + f\left(\frac{2}{1997}\right) + \dots + f\left(\frac{1996}{1997}\right)$$

$$= \sum_{n=1}^{1996} f\left(\frac{n}{1997}\right) = \sum_{n=1}^{998} f\left(\frac{n}{1997}\right) + \sum_{n=999}^{1996} f\left(\frac{n}{1997}\right)$$

$$= \sum_{n=1}^{998} f\left(\frac{n}{1997}\right) + \sum_{n=1}^{998} f\left(\frac{1997-n}{1997}\right)$$

$$= \sum_{n=1}^{998} \left[f\left(\frac{n}{1997}\right) + f\left(1 - \frac{n}{1997}\right) \right] = \sum_{n=1}^{998} 1 = 998$$

95. Since, $f(x+y) = f(x) \cdot f(y)$ for all natural numbers x and y and $f(1) = 2$.

$$\text{So, } f(2) = f(1+1) = [f(1)]^2 = 2^2$$

$$\text{and } f(3) = f(2+1) = f(2) \cdot f(1) = 2^3$$

Continuing this process, we find that $f(n) = 2^n$.

If $f(k) = 2^k$, then

$$f(k+1) = f(k) \cdot f(1) = 2^k \cdot 2 = 2^{k+1}$$

$$\text{Now, } \sum_{k=1}^n f(a+k) = \sum_{k=1}^n f(a) f(k) = f(a) \sum_{k=1}^n f(k)$$

$$= f(a) \sum_{k=1}^n 2^k = 2f(a)(2^n - 1) \quad \dots(i)$$

$$\text{It is given that, } \sum_{k=1}^n f(a+k) = 16(2^n - 1) \quad \dots(ii)$$

From Eqs. (i) and (ii), we get $f(a) = 8$

$$\text{So, } f(a) = 2^3 \Rightarrow 2^a = 2^3$$

$$\therefore a = 3$$

96. For $x \leq -1, 1-x \geq 2$ and $1-x \geq 1+x$

$$\therefore \max\{(1-x), 2, (1+x)\} = 1-x$$

For $-1 < x < 1, 0 < 1-x < 2$ and $0 < 1+x < 2$

$$\therefore \max\{(1+x), 2, (1+x)\} = 2$$

For $x \geq 1, 1+x \geq 2, 1+x > 1-x$

$$\therefore \max\{(1-x), 2, (1+x)\} = 1+x$$

$$\text{Hence, } f(x) = \begin{cases} 1-x, & x \leq -1 \\ 2, & -1 < x < 1 \\ 1+x, & x \geq 1 \end{cases}$$

97. $g(x) = f[f(x)] = f[|x-2|]$

$$= f(x-2) \text{ as } x > 20 = |x-2-2| = x-4$$

$$\therefore g'(x) = 1$$

98. $f[g(x)] = f(\sqrt{1-2x}) = \sqrt{2-\sqrt{1-2x}}$

For $f[g(x)]$ to be defined, $1-2x \geq 0$

and $2-\sqrt{1-2x} \geq 0$

$\Rightarrow x \leq \frac{1}{2}$

and $x \geq -\frac{3}{2}$

$\therefore -\frac{3}{2} \leq x \leq \frac{1}{2}$

Hence, the domain of $f[g(x)]$ is $\left[-\frac{3}{2}, \frac{1}{2}\right]$

99. $(fog)(x) = f[g(x)] = f(|3x+4|)$

Since, the domain of f is $[-3, 5]$.

$\therefore -3 \leq |3x+4| \leq 5$

$\Rightarrow -5 \leq 3x+4 \leq 5$

$\Rightarrow -9 \leq 3x \leq 1$

$\Rightarrow -3 \leq x \leq \frac{1}{3}$

$\therefore$ Domain of fog is $\left[-3, \frac{1}{3}\right]$

100. Let $\frac{x-1}{x+1} = f(x) = y$

$\therefore x-1 = yx+y$

$\Rightarrow x(y-1) = -1-y$

$\Rightarrow x = \frac{y+1}{1-y}$

Now, $f\{f(x)\} = f\left(\frac{x-1}{x+1}\right) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}$

$= \frac{(x-1)-(x+1)}{(x-1)+(x+1)}$

$= -\frac{2}{2x} = -\frac{1}{x}$

$\therefore f\{f(ax)\} = -\frac{1}{ax} = -\frac{1}{a\left(\frac{y+1}{1-y}\right)}$
 $= \frac{y-1}{a(y+1)} = \frac{f(x)-1}{a\{f(x)+1\}}$

101. Since, $f(x) = x^2 - 1$ and $g(x) = \sqrt{x^2 + 1}$, $\forall x \in R$

$\therefore f[g(x)] = [g(x)]^2 - 1 = x^2 + 1 - 1 = x^2, \forall x \in R$

$\therefore [ho(fog)](x) = h[f\{g(x)\}]$
 $= h(x^2) = x^2 \quad [\because x^2 \geq 0]$

102. We have, $f(x) \cdot f(y) = \frac{1}{2} \left\{ f\left(\frac{x}{y}\right) + f(xy) \right\}$
 $= \cos(\log x) \cdot \cos(\log y)$
 $= \cos(\log x) \cdot \cos(\log y) - \frac{1}{2} \left\{ \cos\left(\log \frac{x}{y}\right) + \cos[\log(xy)] \right\}$

$= \cos(\log x) \cdot \cos(\log y) - \frac{1}{2} \{ \cos(\log x - \log y) + \cos(\log x + \log y) \}$

$= \cos(\log x) \cdot \cos(\log y) - \frac{1}{2} \cdot 2 \cos(\log x) \cos(\log y)$

$= \cos(\log x) \cdot \cos(\log y) - \cos(\log x) \cdot \cos(\log y)$
 $= 0$

103. $(f \circ f)(x) = f[f(x)] = [3 - \{f(x)\}^7]^{1/7}$
 $= [3 - \{(3 - x^7)^{1/7}\}^7]^{1/7}$
 $= [3 - (3 - x^7)]^{1/7} = x$

104. We have,

$g(x) = f[f(x)] = f\left(\frac{1}{1-x}\right) = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}$

Also, $h(x) = f[f\{f(x)\}] = f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = x$

$\therefore f(x) \cdot g(x) \cdot h(x) = \frac{1}{1-x} \cdot \frac{x-1}{x} \cdot x = -1$

105. When $f(x) = \sin^2 x$ and $g(x) = \sqrt{x}$, then

$(fog)(x) = f[g(x)] = f(\sqrt{x}) = (\sin \sqrt{x})^2$

and $(gof)(x) = g[f(x)] = g(\sin^2 x) = |\sin x|$

When $f(x) = \sin x$ and $g(x) = |x|$, then

$(fog)(x) = f(g(x)) = f(|x|) = \sin |x| \neq (\sin \sqrt{x})^2$

When $f(x) = x^2$ and $g(x) = \sin \sqrt{x}$, then

$(fog)(x) = f[g(x)] = f(\sin \sqrt{x}) = (\sin \sqrt{x})^2$

and $(gof)(x) = g[f(x)] = g(x^2) = \sin \sqrt{x^2}$

$= \sin |x| \neq |\sin x|$

106. $(fog)(x) = f[g(x)] = f\left(\frac{3x+x^3}{1+3x^2}\right)$
 $= \log \frac{1+3x^2+3x+x^3}{1+3x^2-3x-x^3}$
 $= \log \frac{(1+x)^3}{(1-x)^3} = 3f(x)$

107. $f(x) = \frac{1}{x}$, $g(x) = \frac{1}{x^2}$ and $h(x) = x^2$

$\therefore f[g(x)] = x^2, x \neq 0$

Also, $h[g(x)] = \frac{1}{x^4} = [g(x)]^2, x \neq 0$

108. We have, $(fog)(x) = f[g(x)]$

$= f(\sqrt{x^2+1}) = (\sqrt{x^2+1})^2 - 1 = x^2$

$\therefore (hofog)(x) = h[(fog)(x)] = h(x^2) = \begin{cases} 0, & \text{if } x = 0 \\ x^2, & \text{if } x \neq 0 \end{cases}$

109. Let $y = 4x - x^2$

$\Rightarrow x^2 - 4x + y = 0$

$\Rightarrow x = \frac{4 \pm \sqrt{16-4y}}{2}$

$\Rightarrow x = 2 \pm \sqrt{4-y}$

But $x \leq 2$, so $x = 2 - \sqrt{4-y}$

$\therefore f^{-1}(x) = 2 - \sqrt{4-x}$

110. Let $y = 5^{x(x-4)} \Rightarrow x(x-4) = \log_5 y$

$\Rightarrow x^2 - 4x - \log_5 y = 0$

$\therefore x = \frac{4 \pm \sqrt{16+4\log_5 y}}{2}$

$= 2 \pm \sqrt{4+\log_5 y}$

But $x \geq 4$, so $x = 2 + \sqrt{4+\log_5 y}$

$\therefore f^{-1}(y) = 2 + \sqrt{4+\log_5 y}$

111. Clearly, the function $f(x) = 3x - 5$ is one-one as well as onto.

$\therefore f$ is invertible.

$$\begin{aligned} \text{Let } f(x) = y &\Rightarrow y = 3x - 5 \\ \Rightarrow x = \frac{y+5}{3} &\Rightarrow f^{-1}(x) = \frac{x+5}{3} \end{aligned}$$

112. Let $x, y \in R$ such that $x \geq -1, y \geq -1$.

$$\begin{aligned} \text{Then, } f(x) &= f(y) \\ \Rightarrow (x+1)^2 - 1 &= (y+1)^2 - 1 \\ \Rightarrow x^2 + 2x &= y^2 + 2y \\ \Rightarrow x^2 - y^2 &= -2(x-y) \\ \Rightarrow (x-y)(x+y+2) &= 0 \\ \Rightarrow x-y=0 \text{ or } x+y+2 &= 0 \\ \Rightarrow x=y \text{ or } x=y &= -1 \end{aligned}$$

$\therefore f$ is one-one.

Also, f is onto as for all $y \geq -1$, there exists

$$x = -1 + \sqrt{y+1} \geq -1 \text{ such that } f(x) = y.$$

So, f is invertible.

$$\text{Let } f(x) = y \Rightarrow (x+1)^2 - 1 = y$$

$$\Rightarrow x = -1 \pm \sqrt{y+1}$$

$$\text{But } x \geq -1 \Rightarrow x = -1 + \sqrt{y+1}$$

$$\Rightarrow f^{-1}(y) = -1 + \sqrt{y+1}$$

$$\text{Hence, } f^{-1}(x) = -1 + \sqrt{x+1}$$

113. Clearly, f is invertible and $f^{-1}(x) = -1 + \sqrt{x+1}$.

$$\begin{aligned} \text{Now, } f(x) &= f^{-1}(x) \\ \Rightarrow (x+1)^2 - 1 &= -1 + \sqrt{x+1} \\ \Rightarrow (x+1)^2 &= \sqrt{x+1} \\ \Rightarrow \sqrt{x+1}[(x+1)^{3/2} - 1] &= 0 \\ \Rightarrow \sqrt{x+1} = 0 \text{ or } (x+1)^{3/2} &= 1 \\ \Rightarrow x+1 = 0 \text{ or } x+1 &= 1 \\ \Rightarrow x = -1 \text{ or } x &= 0 \end{aligned}$$

Therefore, the set of values of x for which $f(x) = f^{-1}(x)$, is $\{-1, 0\}$.

114. Since, $x - [x] = 0$ for all integral values of x , therefore the function is not one-one. Hence, $f^{-1}(x)$ does not exist.

$$\textbf{115.} \text{ We have, } y = \frac{10^x - 10^{-x}}{10^x + 10^{-x}} + 1$$

$$\Rightarrow y - 1 = \frac{10^{2x} - 1}{10^{2x} + 1}$$

$$\Rightarrow \frac{y-1+1}{y-1-1} = \frac{10^{2x} - 1 + 10^{2x} + 1}{10^{2x} - 1 - 10^{2x} - 1}$$

[using componendo and dividendo]

$$\Rightarrow \frac{y}{y-2} = \frac{2 \cdot 10^{2x}}{-2} \Rightarrow \frac{y}{2-y} = 10^{2x}$$

$$\Rightarrow 2x = \log_{10} \left(\frac{y}{2-y} \right)$$

$$\Rightarrow x = \frac{1}{2} \log_{10} \left(\frac{y}{2-y} \right) = f^{-1}(y)$$

Hence, the inverse of the given function is

$$y = \frac{1}{2} \log_{10} \left(\frac{x}{2-x} \right)$$

$$\textbf{116.} \text{ Let } y = f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} + 2$$

$$\Rightarrow y - 2 = \frac{e^{2x} - 1}{e^{2x} + 1}$$

$$\Rightarrow (y-2)e^{2x} + y - 2 = e^{2x} - 1$$

$$\Rightarrow (y-3)e^{2x} = -1 - y$$

$$\Rightarrow e^{2x} = \frac{1+y}{3-y} = \frac{y+1}{3-y}$$

$$\Rightarrow 2x = \log \left(\frac{y+1}{3-y} \right)$$

$$\Rightarrow x = \frac{1}{2} \log \left(\frac{y+1}{3-y} \right)$$

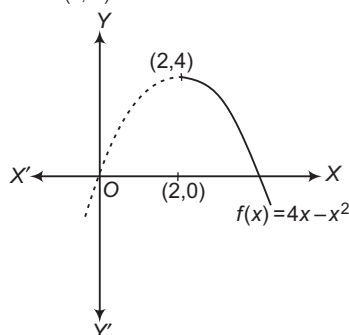
$$\therefore f^{-1}(x) = \log_e \left(\frac{x+1}{3-x} \right)^{1/2}$$

$$\textbf{117.} \text{ Let } y = (fog)(x) = f[g(x)] = f(x^3 + 5) = 2(x^3 + 5) - 3 = 2x^3 + 7$$

$$\therefore x^3 = \frac{y-7}{2} \Rightarrow x = \left(\frac{y-7}{2} \right)^{1/3}$$

$$\Rightarrow (fog)^{-1}(y) = \left(\frac{y-7}{2} \right)^{1/3} \Rightarrow (fog)^{-1}(x) = \left(\frac{x-7}{2} \right)^{1/3}$$

118. It is evident from the graph of $f(x)$ that it represent an arc of the parabola $y = 4x - x^2$ lying on the right side of its vertex $(2, 4)$.



Clearly, range of $f = (-\infty, 4]$

Hence, $x = (-\infty, 4]$

$$\textbf{119.} f(x) = x^3 + 3, x \in R$$

Since, $f(x)$ is one-one and onto.

$$\therefore f(x) = k \Rightarrow x^3 + 3 = k$$

$$\Rightarrow x = (k-3)^{1/3}$$

$$\therefore f^{-1}(x) = (x-3)^{1/3}, x \in R$$

$$\textbf{120.} \text{ We have, } f(x) = x + \frac{1}{x}$$

For $f^{-1}(x)$, put $f(x) = y$

$$\Rightarrow y = \frac{x^2 + 1}{x} \text{ or } x^2 - xy + 1 = 0$$

$$\therefore x = \frac{y \pm \sqrt{y^2 - 4}}{2}$$

$$\text{Hence, } f^{-1}(y) = \frac{y + \sqrt{y^2 - 4}}{2} \text{ [neglect -ve, as } x > 1]$$

$$\text{or } f^{-1}(x) = \frac{x + \sqrt{x^2 - 4}}{2}$$

121. Given, $2f(\sin x) + f(\cos x) = x$... (i)

Replacing x by $\frac{\pi}{2} - x$, we get

$$2f(\cos x) + f(\sin x) = \frac{\pi}{2} - x \quad \dots (ii)$$

Eliminating $f(\cos x)$ from Eqs. (i) and (ii), we get

$$3f(\sin x) = 3x - \frac{\pi}{2}$$

$$\Rightarrow f(\sin x) = x - \frac{\pi}{6}$$

$$\Rightarrow f(x) = \sin^{-1} x - \frac{\pi}{6}$$

So, $f(x)$ has the domain $[-1, 1]$.

Also, $\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \sin^{-1} x - \frac{\pi}{6} \in \left[-\frac{2\pi}{3}, \frac{\pi}{3}\right]$$

122. $f(x)$ is defined, if

$$\log_{|\sin x|} (x^2 - 8x + 23) - \frac{3}{\log_2 |\sin x|} > 0$$

$$\Rightarrow \log_{|\sin x|} \left(\frac{x^2 - 8x + 23}{8} \right) > 0$$

This is true, if $|\sin x| \neq 0, 1$ and $\frac{x^2 - 8x + 23}{8} < 1$

Now, $\frac{x^2 - 8x + 23}{8} < 1 \Rightarrow x^2 - 8x + 15 < 0$

$$\Rightarrow x \in (3, 5) - \left\{ \pi, \frac{3\pi}{2} \right\}$$

$$\therefore \text{Domain} = (3, \pi) \cup \left(\pi, \frac{3}{2} \right) \cup \left(\frac{3\pi}{2}, 5 \right)$$

123. Since, $x \sin x$ is a continuous function, so the value of $|x \sin x|$ can be made as large as we like for sufficiently large values of x .

Hence, range of $x \sin x = R$

$$\therefore \frac{[x]}{\tan 2x} = 0 \text{ for } x \in \left(0, \frac{\pi}{4} \right) \quad [\because [x] = 0]$$

For $x \in \left(-\frac{\pi}{4}, 0 \right), \frac{[x]}{\tan 2x} > 0$

Hence, values of $\frac{[x]}{\tan 2x}$ are never negative.

Thus, range of $\frac{[x]}{\tan 2x} \neq R$

$$\left| \frac{x}{\sin x} \right| > 1, \text{ whenever defined}$$

Thus, range of $\frac{x}{\sin x}$ is not R .

$|x| + \sqrt{\{x\}}$ is continuous function

$$\text{and } \lim_{x \rightarrow \infty} ([x] + \sqrt{\{x\}}) = \infty, \lim_{x \rightarrow -\infty} ([x] + \sqrt{\{x\}}) = -\infty$$

Thus, range of $[x] + \{x\} = R$

124. $f(2) = f(1+1) = 2f(1) = 10$

$$f(3) = f(2+1) = f(2) + f(1) = 10 + 5 = 15$$

Then, $f(n) = 5n$

$$\Rightarrow \sum_{r=1}^m f(r) = 5 \sum_{r=1}^m r = \frac{5m(m+1)}{2}$$

Replacing y by $-x$, we get $f(0) = f(x) + f(-x)$

Also, put $x = y = 0$

$$\Rightarrow f(0) = f(0) + f(0)$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow f(x) + f(-x) = 0$$

Hence, the function is odd.

125. (a) $f(x) = \frac{1}{\ln[1-|x|]}$ is defined, if $[1-|x|] > 0$

$$\text{and } 1 - [x] \neq 1$$

$$\Rightarrow [1-|x|] \geq 2$$

$$\Rightarrow 1 - |x| \geq 2$$

$$\Rightarrow |x| \leq -1, \text{ which is not possible.}$$

(b) Here, $x!$ is defined only when x is natural number, but $\{x\}$ becomes zero for these values of x .

Hence, $f(x)$ is not defined in this case.

(c) $f(x) = x!\{x\}$ is defined for x being a natural number. Hence, $f(x)$ is a function whose domain is $x \in N$.

(d) Here, $\ln(x-1)$ is defined only when $x-1 > 0$ i.e. $x > 1$, also $1-x^2 > 0$ for denominator i.e. $-1 < x < 1$. Hence, $f(x)$ is not defined for any value of x .

126. Since, $f(g(x))$ is one-one function.

$$\therefore f(g(x_1)) = f(g(x_2)), \text{ whenever } x_1 \neq x_2$$

$$\Rightarrow g(x_1) = g(x_2), \text{ whenever } x_1 \neq x_2$$

$\Rightarrow g(x)$ must be one-one.

Let $f(x) = y$ be satisfied by $x = x_1$ and x_2 .

If $g(x)$ is such that its range has only one of x_1 and x_2 , then $g(g(x))$ can be one-one even, if $f(x)$ is many-one.

127. Let $f : (T+x) = f(x)$ and $T \in R$

Suppose, $T = \frac{p}{q}$

$$\Rightarrow f(qT+x) = f(x)$$

$$\Rightarrow f(p+x) = f(x)$$

So, $f(n), n \in N$ cannot contain infinite elements. Thus, period of the function should be irrational.

128. (a) $f(x) = e^{\ln \sec^{-1} x}$

$g(1) = 0$ by $f(1)$ is not defined

(b) $f(x) = \tan(\tan^{-1} x) = x, \forall x \in R$

and $g(x) = \cot(\cot^{-1} x) = x, \forall x \in R$

$$(c) f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$g(x) = \text{sgn}(\text{sgn } x) = \text{sgn} \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

$$\therefore g(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

(d) $g(x) = \cot^2 x - \cos^2 x$

$$\begin{aligned} &= \frac{\cos^2 x}{\sin^2 x} - \cos^2 x \\ &= \cot^2 x \cos^2 x \\ &= f(x), \forall x \neq \pi \end{aligned}$$

129. $f(x) = \operatorname{sgn}(\cot^{-1} x) + \tan\left(\frac{x}{2}\right)$

$\operatorname{sgn}(\cot^{-1} x)$ is defined when $\cot^{-1} x$ is defined, for all $x \in \mathbb{R}$.

$\tan\left(\frac{\pi}{2}[x]\right)$ is defined when $\frac{\pi}{2}[x] \neq \frac{(2n+1)\pi}{2}$, where $n \in \mathbb{Z}$.

$$\Rightarrow [x] \neq 2n + 1$$

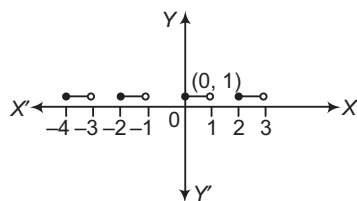
$$\Rightarrow x \notin [2n+1, 2n+2)$$

Hence, domain of $f(x)$ is $\bigcup_{n \in \mathbb{Z}} [2n, 2n+1)$

Also, $\cot^{-1} x > 0, \forall x \in \mathbb{R}$

$$\text{Then, } f(x) = 1 + \tan\left(\frac{\pi}{2}[x]\right) = 1$$

$$\Rightarrow f(x) = 1, x \in D_f$$



From graph, $f(x)$ is periodic with period 2.

130. $f\left(\frac{2 \tan x}{1 + \tan^2 x}\right) = \frac{(1 + \cos 2x)(\sec^2 x + 2 \tan x)}{2}$

$$\Rightarrow f(\tan 2x) = \frac{2 \cos^2 x (\sec^2 x + 2 \tan x)}{2}$$

$$= 1 + 2 \sin x \cos x = 1 + \sin 2x$$

$$\Rightarrow f(y) = 1 + y, \text{ where } y = \sin 2x,$$

$$\text{Now } \sin 2x \in [-1, 1]$$

$$\therefore f(y) \in [0, 2]$$

Hence, Statement I is false but Statement II is true.

131. Range of $\frac{1}{1+x^2}$ is $(0, 1]$.

$$\therefore \log\left(\frac{1}{1+x^2}\right) \in (-\infty, 0]$$

Hence, Statement I is false.

132. $f'(x) = \frac{-x}{\sqrt{4-x^2}}$

$f(x)$ is increasing for $-2 \leq x \leq 0$ and decreasing for $0 \leq x \leq 2$.

$$\Rightarrow \max\{f(x)\} = f(0) = 2$$

$$\min\{f(x)\} = f(2) = f(-2) = 0$$

$$\therefore \text{Range} \in [0, 2]$$

133. Suppose, $a > b$. Statement II is true as $f'(x) = \frac{b-a}{(b+x)^2}$

which is always negative and hence monotonic in its continuous part. Also, $\lim_{x \rightarrow -b^+} f(x) = \infty$ and

$$\lim_{x \rightarrow -b^-} f(x) = -\infty, \text{ moreover } \lim_{x \rightarrow \infty} f(x) = 1 \text{ and}$$

$$\lim_{x \rightarrow -\infty} f(x) = -1. \text{ Hence, range of } f \text{ is } \mathbb{R} - \{1\}.$$

F is obviously one-one as $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

However, Statement II is not a correct explanation for Statement I.

134. Statement I and Statement II are obviously correct.

135. For $f(x)$ to be real, $\log_2(\sin x) \geq 0$

$$\Rightarrow \sin x \geq 2^0 \Rightarrow \sin x = 1$$

$$\therefore x = (4n+1)\frac{\pi}{2}, n \in \mathbb{N}$$

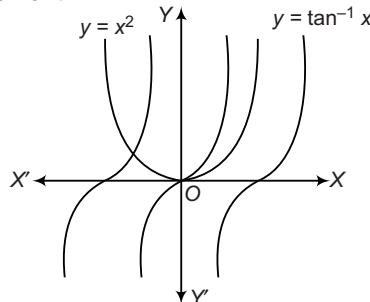
136. $\sin(kx)$ has period π/k and period of $\{x\}$ is 1.

Now, LCM of π/k and 1 exists only if k is a rational multiple of π as LCM of rational and irrational numbers does not exist. Hence, Statement I is true but Statement II is false as sum of two periodic functions is not necessarily periodic.

137. Obviously, $f(x) = x^2 + \tan^{-1} x$ is non-periodic, but sum of two non-periodic functions is not always non-periodic, as $f(x) = x$ and $g(x) = -[x]$, where $[]$ represents the greatest integer function.

$f(x) + g(x) = x - [x] = \{x\}$ is a periodic function, where $\{ \}$ represents the fractional part function.

138. Statement II



It is clear that $x^2 - \tan x$ is non-periodic function.

Statement I $\cos(x^2 - \tan x)$ is non-periodic function.

139. $f(x) = \sqrt{2} \sin x + \sqrt{2} \cos x - \sqrt{2} \cos x + c$

$$= \sqrt{2} \sin x + c$$

$$\Rightarrow f(0) = f(\pi) = c$$

Hence, it has many-one function.

140. Both statements are true, but Statement II is not a correct explanation of Statement I as $f(g(x))$ is one-one when $g(x)$ is one-one and $f(x)$ is many-one.

141. $f \circ g(x)$ can be even also when one of them is even and other is odd.

142. Consider $f(x) = \tan x$, which is surjective, periodic but discontinuous.

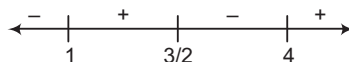
143. $||x^2 - 5x + 4| - |2x - 3|| = |x^2 - 3x + 1|$

$$\Rightarrow ||x^2 - 5x + 4| - |2x - 3|| = |(x^2 - 5x + 4) + (2x - 3)|$$

$$\Rightarrow (x^2 - 5x + 4)(2x - 3) \leq 0$$

$$\Rightarrow (x - 1)(2x - 3)(x - 4) \leq 0$$

$$\Rightarrow x \in (-\infty, 1] \cup \left[\frac{3}{2}, 4\right]$$



Hence, Statement I is true. Statement II is true as it is the property of modulus function but is not a correct explanation of Statement I.

144. $(f(\sin 2x))^2 = \sin^2 x + \cos^2 x + 2 \sin x \cos x$
 $= 1 + \sin 2x$

$\Rightarrow f(x) = \sqrt{1+x}, \forall x \in [-1, 1]$

If $x \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, then

$f(\tan^2 x) = \sqrt{1 + \tan^2 x} = \sec x$

145. Fifth degree equation must have atleast one real root.
 If it had two real roots, then $f'(x) = 0$ must have one real root.

$\therefore f'(x) = 20x^4 + 20 > 0, \forall x$

Hence, $f(x)$ has one real root and 4 complex roots.

146. $\phi'(x) = \cos(\cos x) \cdot (-\sin x)$

$\therefore \phi'(x) \leq 0$ for $0 \leq x \leq \frac{\pi}{2}$

Hence, $\phi(x)$ is one-one, as $\phi(x)$ is decreasing function.

147. $f(x) = x$ has domain $\in R$

$F(x) = \frac{x^2}{x}$ has domain $\in R - \{0\}$

$\therefore F(x)$ and $f(x)$ are not equal functions.

148. $f(k_1) \cdot f(k_2) = 0$

So, $f(x)$ has atleast one root of $f(x) = 0$ lies in $(0, 1)$.

$\therefore$ Statement II is false.

149. $f(1) = 1 - 1 = 0$

$f(0) = 0$

So, f is not one-one.

Hence, $f^{-1}(x)$ is not defined.

150. $f(x) = \frac{x+1}{x-1}$

$\Rightarrow f'(x) = \frac{(x-1) - (x+1)}{(x-1)^2}$
 $= \frac{-2}{(x-1)^2} < 0$

So, $f(x)$ is monotonically decreasing and every monotonic function is one-one.

151. For one-one function, $x_1 \neq x_2$

$\Rightarrow f(x_1) \neq f(x_2), \forall x_1, x_2 \in D_f, \sqrt{3} > 1$

But $f(\sqrt{3}) < f(1)$ and $3 > 1$

$\therefore f(5) > f(1)$

Hence, $f(x)$ is one-one but not monotonic.

152. f is injective, since $x \neq y; x, y \in R$

$\Rightarrow \log_a(x + \sqrt{x^2 + 1}) \neq \log_a(y + \sqrt{y^2 + 1})$

$\Rightarrow f(x) \neq f(y)$

f is onto because $\log_a(x + \sqrt{x^2 + 1}) = y$

$\Rightarrow x = \frac{a^y - a^{-y}}{2}$

Hence, $f(x)$ is many-one and into.

153. **Statement I** If $f: A \rightarrow B, g: B \rightarrow C$ are two functions, then $(gof)^{-1} \neq f^{-1}og^{-1}$, since functions need not posses inverse.

Statement II Bijective functions are invertibles.

Solutions (Q. Nos. 154-156)

$f(x) = \begin{cases} x+1, & x \leq 1 \\ 2x+1, & 1 < x \leq 2 \end{cases}$

$g(x) = \begin{cases} x^2, & -1 \leq x < 2 \\ x+2, & 2 \leq x \leq 3 \end{cases}$

$\Rightarrow f(x) = \begin{cases} g(x)+1, & g(x) \leq 1 \\ 2g(x)+1, & 1 < g(x) \leq 2 \end{cases}$

$\Rightarrow f(g(x)) = \begin{cases} x^2+1, & x^2 \leq 1, & -1 \leq x < 2 \\ x+2+1, & x+2 \leq 1, & 2 \leq x \leq 3 \\ 2x^2+1, & 1 < x^2 \leq 2, & -1 \leq x < 2 \\ 2(x+2)+1, & 1 < x+2 \leq 2, & 2 \leq x \leq 3 \end{cases}$

$\Rightarrow f(g(x)) = \begin{cases} x^2+1, & -1 \leq x \leq 1 \\ 2x^2+1, & 1 < x \leq \sqrt{2} \end{cases}$

154. The domain of $f(x)$ is $[-1, \sqrt{2}]$.

155. For $-1 \leq x \leq 1$, we have $x^2 \in [0, 1] \Rightarrow x^2 + 1 \in [1, 2]$

For $1 < x \leq \sqrt{2}$, we have $x^2 \in (1, 2] \Rightarrow 2x^2 + 1 \in (3, 5]$

Hence, the range is $[1, 2] \cup (3, 5]$

156. For $f(g(x)) = 2, x^2 + 1 = 2 \Rightarrow$ and $2x^2 + 1 = 2$

$\Rightarrow x = \pm 1$ or $x = \pm \frac{1}{\sqrt{2}} \Rightarrow x = \pm 1$

Hence, there are two roots.

Solutions (Q. Nos. 157-159)

$f(x) + f\left(\frac{x-1}{x}\right) = 1 + x \quad \dots(i)$

On replacing x by $\frac{x-1}{x}$ in Eq. (i), we get

$f\left(\frac{x-1}{x}\right) + f\left(\frac{\frac{x-1}{x}-1}{\frac{x-1}{x}}\right) = 1 + \frac{x-1}{x}$

$\Rightarrow f\left(\frac{x-1}{x}\right) + f\left(\frac{1}{1-x}\right) = 1 + \frac{x-1}{x} \quad \dots(ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$f(x) - f\left(\frac{1}{1-x}\right) = x - \frac{x-1}{x} \quad \dots(iii)$

On replacing x by $\frac{1}{1-x}$ in Eq. (iii), we get

$f\left(\frac{1}{1-x}\right) - f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} - \frac{1}{1-x} - \frac{1}{1-x}$

or $f\left(\frac{1}{1-x}\right) - f\left(\frac{x-1}{x}\right) = \frac{1}{1-x} - x$

157. $g(x) = \frac{x^3 - x^2 - 1}{x(x-1)} - x + 1 = \frac{x^2 - x - 1}{x(x-1)}$

Now, for $y = \sqrt{g(x)}$, we must have $\frac{x^2 - x - 1}{x(x-1)} \geq 0$

or $\frac{\left(x - \frac{1-\sqrt{5}}{2}\right)\left(x - \frac{1+\sqrt{5}}{2}\right)}{x(x-1)} \geq 0$

$\Rightarrow x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right) \cup (0, 1) \cup \left[\frac{1+\sqrt{5}}{2}, \infty\right)$

$$158. y = g(x) = \frac{x^2 - x - 1}{x(x-1)}$$

$$\Rightarrow (y-1)x^2 + (1-y)x + 1 = 0$$

Since, x is real.

$$\therefore D \geq 0$$

$$\Rightarrow (1-y)^2 - 4(y-1) \geq 0$$

$$\Rightarrow (y-1)(y-5) \geq 0$$

$$\Rightarrow y \in (-\infty, 1] \cup [5, \infty)$$

$$159. g(x) = 1 \Rightarrow \frac{x^2 - x - 1}{x(x-1)} = 1$$

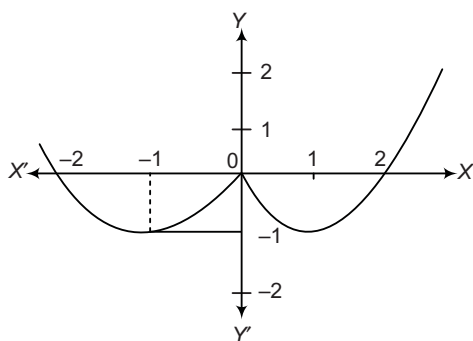
$$\Rightarrow -x - 1 = -x, \text{ which has no solution.}$$

Solutions (Q. Nos. 160-162)

$$f_1(x) = x^2 \text{ and } f_2(x) = |x|$$

$$\Rightarrow f(x) = f_1(x) - 2f_2(x) = x^2 - 2|x|$$

Graph of $f(x)$



$$g(x) = \begin{cases} f(x), & -3 \leq x < -1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x \leq 2 \\ f(x), & 2 < x \leq 3 \end{cases}$$

$$= \begin{cases} x^2 + 2x, & -3 \leq x < -1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x \leq 2 \\ x^2 - 2x, & 2 < x \leq 3 \end{cases}$$

160. The range of $g(x)$ for $[-3, -1]$ is $[-1, 3]$.

161. For $x \in (-1, 0)$, $f(x) + g(x) = x^2 + 2x - 1$

162. Obviously, the graph is broken at $x = 0$.

163. As, $f(296) = f(926)$, so f is many-one.

Further, the numbers whose digits increase from left to right have no pre-image. Hence, f is into.

164. It is easy to see that the remainder, when a positive integer is divided by 9, is the same as the sum of the digit of the number (until the sum becomes a one-digit number). Thus, $f(n)$ and n leave the same remainder, when divided by 9. Hence, 9 divides $f(n) - n$. Further, there is no reason to expect that the number is divisible by 27. The number $f(n) - n$ is not divisible by 18 also, in case $f(n) - n$ is odd.

Hence, 9 is the biggest number.

165. By the definition of f , digits of $f(x)$ are non-increasing from left to right.

Solutions (Q. Nos. 166-168)

We know that, for $|u| \leq 1, |v| \leq 1$, then $|u - v| \leq 2$

Now, $|f(1) - f(0)| \leq 2$

$$\Rightarrow |a + b| \leq 2 \Rightarrow (a + b)^2 \leq 4 \quad \dots(i)$$

Also, $|f(-1) - f(0)| \leq 2 \Rightarrow |a - b| \leq 2$

$$\Rightarrow (a - b)^2 \leq 4$$

$\dots(ii)$

Now, $4a^2 + 3b^2 = 2(a + b)^2 + 2(a - b)^2 - b^2 \leq 16$

$$\Rightarrow (4a^2 + 3b^2)_{\max} = 16, \text{ where } b = 0$$

$$\Rightarrow |a + b| = |a - b| = |a| = 2$$

Also, $|f(1) - f(0)| = |(a + c) - c| = |a| = 2$

$$\Rightarrow |a + c| = |c| = 1$$

The possible ordered triplet (a, b, c) are $(2, 0, -1)$ or $(-2, 0, 1)$.

$$\text{Also, } \frac{8}{3}a^2 + 2b^2 = \frac{2}{3}(4a^2 + 3b^2) \leq \frac{2}{3} \times 16$$

$$169. \therefore ff^{-1}(x) = x$$

$$\therefore 2^{(f^{-1}(x))^4 - 4(f^{-1}(x))^2} = x$$

$$\Rightarrow (f^{-1}(x))^4 - 4(f^{-1}(x))^2 - \log_2 x = 0$$

$$\Rightarrow (f^{-1}(x))^2 = 2 + \sqrt{4 + \log_2 x}$$

$\therefore$ Range of $f^{-1}(x)$ is $[2, \infty)$.

$$\therefore f^{-1}(x) = \sqrt{2 + \sqrt{4 + \log_2 x}} \Rightarrow f^{-1}(x) > 0$$

$$170. g(x) = \frac{\sin x + 4}{\sin x - 2}$$

$$\Rightarrow g'(x) = \frac{-\cos x}{(\sin x - 2)} \geq 0$$

$$\therefore x \in \left[\frac{\pi}{2}, \pi\right]$$

$\therefore g(x)$ is increasing function, so it is one-one function.

Hence, range is $\left[g\left(\frac{\pi}{2}\right), g(\pi)\right]$ i.e. $[-5, -2]$.

$$171. x : x \text{ in domain of } g^{-1}(x), g^{-1}(x) \text{ in domain of } f^{-1}(x)$$

$$\therefore g^{-1}(x) = \sin^{-1} \frac{2(x+2)}{x-1}, \forall x \in [5, -2] \quad \dots(i)$$

$$\Rightarrow \sin^{-1} \frac{2(x+2)}{x-2} \geq 1$$

$$\text{i.e. } 1 \leq \sin^{-1} \frac{2(x+2)}{x-1} \leq \frac{\pi}{2}$$

$$\Rightarrow \sin 1 \leq \frac{2(x+2)}{x-1} \leq 1$$

On solving this, we get

$$x \leq -\frac{(4 - \sin 1)}{2 - \sin 1} \text{ or } x > 1 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$x \in \left[-5, -\frac{(4 + \sin 1)}{2 - \sin 1}\right]$$

Solutions (Q. Nos. 172-174)

$P(x) + 1 = 0$ has a thrice repeated root at $x = 1$.

$P'(x) = 0$ has a twice repeated root at $x = 1$.

Similarly, $P'(x) = 0$ has a twice repeated root at $x = -1$.

$$\Rightarrow P'(x) \text{ is divisible by } (x-1)^2(x+1)^2.$$

$\therefore P'(x)$ has degree at most 4.

$$\therefore P'(x) = \alpha(x-1)^2(x+1)^2$$

$$\Rightarrow P(x) = \alpha \left(\frac{x^5}{5} - \frac{2}{3}x^3 + x \right) + C$$

$$\text{Now, } P(1) = -1 \text{ and } P(-1) = 1$$

$$\Rightarrow \alpha = -\frac{15}{8} \text{ and } C = 0$$

$$\therefore P(x) = -\frac{3}{8}x^5 + \frac{5}{4}x^3 - \frac{15}{8}x$$

175. A. $f(x) = \{(\operatorname{sgn} x)^{\operatorname{sgn} x}\}^n = \begin{cases} [(1)^1]^n, & x > 0 \\ [(-1)^{-1}]^n, & x < 0 \end{cases} = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$

Hence, $f(x)$ is an odd function.

B. $f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$

$$\Rightarrow f(-x) = \frac{-x}{e^{-x} - 1} - \frac{x}{2} + 1 = \frac{xe^x}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{xe^x - x + x}{e^x - 1} - \frac{x}{2} + 1$$

$$= x + \frac{x}{e^x - 1} - \frac{x}{2} + 1$$

$$= \frac{x}{e^x - 1} + \frac{x}{2} + 1 = f(x)$$

Hence, $f(x)$ is an even function.

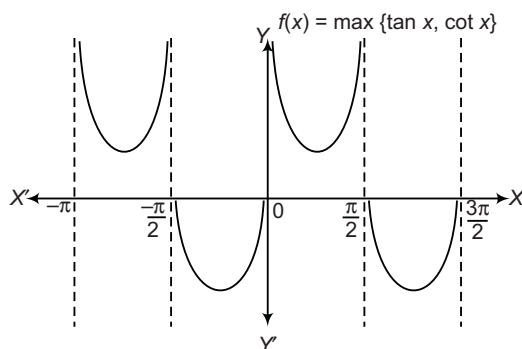
C. $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$

$$\Rightarrow f(-x) = \begin{cases} 0, & \text{if } -x \text{ is rational} \\ 1, & \text{if } -x \text{ is irrational} \end{cases} = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases} = f(x)$$

Hence, $f(x)$ is an even function.

D. $f(x) = \max\{\tan x, \cot x\}$

From the graph of function, it can be verified that $f(x)$ is neither odd nor even.



Hence, $f(x)$ is an odd function.

$$\text{Also, } f(x + \pi) = \max\{\tan(x + \pi), \cot(x + \pi)\} = \max\{\tan x, \cot x\}$$

Hence, $f(x)$ is periodic with period π .

176. A. $f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$

$\cos^2 \pi x$ and $\cos^4 \pi x$ has period 1.

$x - [x] = \{x\}$ has period 1.

Hence, the period of $f(x)$ is 1.

B. $f(x) = \cos 2\pi\{2x\} + \sin 2\pi\{2x\}$

The period $\{2x\}$ is $\frac{1}{2}$, then the period of $f(x)$ is $\frac{1}{2}$.

C. Clearly, $\tan \pi[x] = 0, \forall x \in \mathbb{R}$ and the period of $\sin 3\pi\{x\}$ is equal to 1.

D. $f(x) = 3x - [3x + a] - b$
 $= 3x + a - [3x + a] - (a + b)$
 $= \{3x + a\} - (a + b)$

Thus, the period of $f(x)$ is 1.

177. A. $f(0) = \sin^2 0 + \sin^2\left(0 + \frac{\pi}{3}\right) + \cos 0 \cos\left(0 + \frac{\pi}{3}\right)$
 $= 0 + \left(\frac{\sqrt{3}}{2}\right)^2 + \frac{1}{2} = \frac{5}{4}$

B. If period of $\tan(e^x) = 1$, then period of $[x + \alpha] - [x + \alpha] + \alpha - 5$ is also 1, hence 1 is the period.

C. $h(x) - h(-x) = 0$, hence it is even function.

D. $k(0) = 2 \sin^2 0 - \cos 2\alpha + 4 \sin \alpha \sin 0 \cos(0 + \alpha)$
 $= 0 - \cos 2\alpha + 0 + \cos 2\alpha = 0$

178. A. $f(x) = \begin{cases} 1, & x \notin (2n+1)\frac{\pi}{2} \\ \text{undefined}, & x \in (2n+1)\frac{\pi}{2} \end{cases}$

B. $f(x) = 1, x \in \mathbb{R}$

Constant functions are periodic having no fundamental period.

C. $f(x) = \begin{cases} 1, & x \in \frac{n\pi}{2} \\ \text{undefined}, & x \in \frac{n\pi}{2} \end{cases}$

So, fundamental period is $\frac{\pi}{2}$.

D. $f(x) = \begin{cases} 1 + \{x\}, & x \in n\pi \\ \text{undefined}, & x \in n \end{cases}$

179. A. $f(x) = \cot^{-1}(2x - x^2 - 2)$
 $= \cot^{-1}(-1 - (x-1)^2) - 1 - (x-1)^2 \leq -1$
 $\therefore f(0) = f(2)$

Hence, $f(x)$ is many-one.

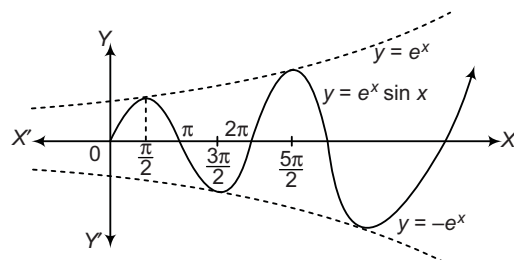
$$\therefore \cot^{-1}(2x - x^2 - 2) \in \left[-\frac{3\pi}{4}, \pi\right)$$

So, $f(x)$ is onto.

Also, $f(3) = f(-1)$, hence function is many-one.

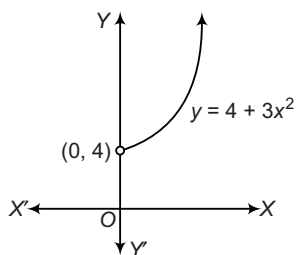
$$\therefore -1 - (x-1)^2 = -5$$

B.



Clearly, from the graph, $f(x)$ is many-one and onto.

C.



Clearly, $f(x)$ is one-one and into.

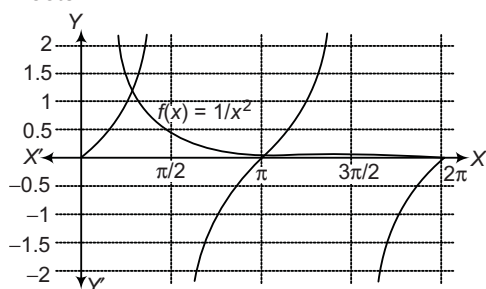
D. Let $X = \{x_1, x_2, \dots, x_n\}$

Let $f(x_1) = x_2 \Rightarrow f(f(x_1)) = f(x_2)$

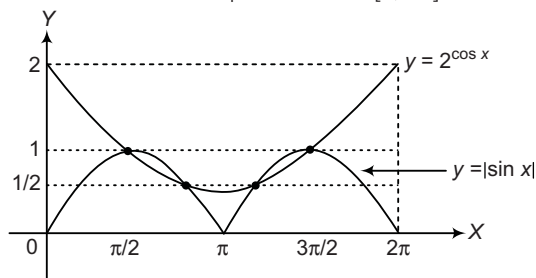
Thus, $f(x)$ is one-one and onto.

180. A. $y = \tan x = \frac{1}{x^2}$

From the graph, it is clear that it will have two real roots.



B. See the graphs of $y = 2^{\cos x}$ and $y = |\sin x|$, two curves meet at four points for $x \in [0, 2\pi]$.



So, the equation $2^{\cos x} = |\sin x|$ has four solutions.

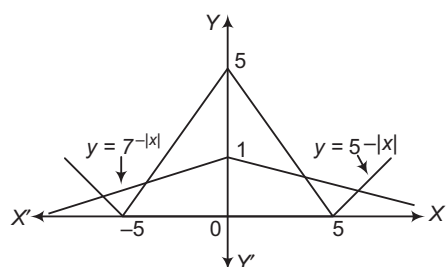
C. Given that, $f(|x|) = 0$ has 8 real roots, so $f(x) = 0$ has 4 positive roots.

Since, $f(x)$ is a polynomial of degree 5, so $f(x)$ cannot have even number of real roots.

Hence, $f(x)$ has all the five roots real and one root is negative.

D. $7^{|x|} (|5 - |x||) = 1$

$\Rightarrow |5 - |x|| = 7^{-|x|}$



It is clear from the graph that the number of roots is 4.

181. A. When $-1 \leq x < 0$, then

$$f(x) \in [0, 1) \text{ and } f(x) = x + 1$$

$$\therefore f(f(x)) = (x + 1)^2 - 1 = x^2 + 2x$$

B. $\frac{(\cos 2x + 1)(\sec^2 x + 2 \tan x)}{2}$

$$= \left(\frac{1 - \tan^2 x}{1 + \tan^2 x} + 1 \right) \left(\frac{1 + \tan^2 x + 2 \tan x}{2} \right)$$

$$= \frac{1 + \tan^2 x + 2 \tan x}{1 + \tan^2 x}$$

i.e. $f\left(\frac{2 \tan x}{1 + \tan^2 x}\right) = 1 + \frac{2 \tan x}{1 + \tan^2 x}$

$$\therefore f(x) = 1 + x$$

C. Put $x = 0, y = 0$, we get $f(1) = (1 + 1)^2 = 2^2$

Put $x = 0, y = 1$, we get $f(2) = (1 + 2)^2 = 3^2$

By induction, $f(x) = (x + 1)^2$

D. When $4 < x < 5$, $\left[\frac{x}{4}\right] = 1$

$$\Rightarrow f(x) = 2x + 3$$

$$\Rightarrow y = 2x + 3$$

$$\Rightarrow x = \frac{y - 3}{2}$$

$$\therefore f^{-1}(x) = \frac{x - 3}{2}$$

182. $f(2a - x) = f(x)$

$$\Rightarrow f(2a - x) = -f(x)$$

$\therefore f$ is odd.

$$\therefore f(x + 4a) = f(x)$$

So, f is periodic with period $4a$.

$$\Rightarrow f(1 + 4r) = f(1)$$

Now, $\sum_{r=0}^{\infty} f(1)^r = 8$

$$\Rightarrow \frac{1}{1 - f(1)} = 8$$

$$\Rightarrow f(1) = \frac{7}{8}$$

$$\Rightarrow 8f(1) = 7$$

183. Consider, $f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$= \frac{e^{2x} - 1}{e^{2x} + 1} = 1 - \frac{2}{e^{2x} + 1}$$

$$\Rightarrow f'(x) = \frac{4e^{2x}}{(e^{2x} + 1)^2} > 0, \forall x \in R$$

$\Rightarrow f(x)$ is an increasing function.

$\therefore$ Domain: R , Range: $(-1, 1)$

For $f: R \rightarrow (-1, 1)$,

$$f(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, f^{-1}: (-1, 1) \rightarrow R$$

$$\Rightarrow x = \frac{e^y - e^{-y}}{e^y + e^{-y}} \Rightarrow e^y = \sqrt{\frac{1+x}{1-x}}$$

$$\Rightarrow f^{-1}(x) = \ln \sqrt{\frac{1+x}{1-x}}$$

Hence, given equation is equivalent to $f(x) = f^{-1}(x)$.

$\Leftrightarrow f(x) = x$ as f is an increasing function.

$$\Rightarrow \ln \sqrt{\frac{1+x}{1-x}} = x \Rightarrow \frac{1+x}{1-x} = e^{2x}$$

$$\therefore x = 0$$

184. Given, $f(x) + f(x+4) = f(x+2) + f(x+6)$... (i)

Replacing x by $x+2$, we get

$$f(x+2) + f(x+6) = f(x+4) + f(x+8) \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$f(x) + f(x+4) = f(x+4) + f(x+8)$$

$$\Rightarrow f(x) = f(x+8)$$

Hence, 8 is the period.

185. Let $y = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7} = 1 + \frac{10}{3x^2 + 9x - 7}$

$$\text{Now, } 3x^2 + 9x + 7 = 3(x^2 + 3x) + 7$$

$$= 3\left(x + \frac{3}{2}\right)^2 + \frac{1}{4} \geq \frac{1}{4}, \forall x \in R$$

$$\text{Maximum value of } \frac{10}{3x^2 + 9x + 7} = 40$$

and maximum value of y is $1 + 40 = 41$.

$$\therefore 5k + 1 = 41 \Rightarrow k = 8$$

186. Put $x = y = 0$

$$\therefore f(0+0) = \{f(0)\}^2$$

$$\Rightarrow f(0) = 1 \text{ as } f(0) \neq 0$$

By definition,

$$\begin{aligned} f'(5) &= \lim_{h \rightarrow 0} \frac{f(5+h) - f(5)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(5) \cdot f(h) - f(5)}{h} \quad [\because f(x+y) = f(x) \cdot f(y)] \end{aligned}$$

$$\begin{aligned} \text{i.e. } f'(5) &= f(5) \cdot \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(5) \cdot f'(0) = 2 \times 3 = 6 \quad [\because f'(0) = 3] \end{aligned}$$

187. We have, $f\left(\frac{2x-3}{x-2}\right) = 5x-2$

$$\Rightarrow f^{-1}(5x-2) = \frac{2x-3}{x-2}$$

Let $5x-2 = 13$, then $x = 3$

$$\text{Hence, } f^{-1}(13) = \frac{2(3)-3}{3-2} = 3$$

188. $x! - x(x-1)! \neq 0 \Rightarrow x \in I^+ - \{1\}$

$$\Rightarrow \frac{\pi}{2^{\tan^{-1} x}} > 4 \quad \left[\because \tan^{-1} x < \frac{\pi}{2} \right]$$

$$\Rightarrow \frac{(x-4)(x-10)}{(x-1)!(x-1)} < 0$$

$$\therefore x \in \{5, 6, \dots, 9\}$$

189. Put $x = 1$ and $y = 1 \Rightarrow f^2(1) - f(1) - 6 = 0$

$$\Rightarrow f(1) = 3 \text{ or } f(1) = -2$$

Now, put $y = 1$

$$\Rightarrow f(x) \cdot f(1) = f(x) + 2\left(\frac{1}{x} + 2\right)$$

$$\Rightarrow f(x) \cdot f(1) = f(x) + 2\left(\frac{2x+1}{x}\right)$$

$$\Rightarrow f(x)[f(1)-1] = \frac{2(2x+1)}{x}$$

$$\Rightarrow f(x) = \frac{2(2x+1)}{x[f(1)-1]}$$

$$\text{For } f(1) = 3, \quad f(x) = \frac{2x+1}{x} \quad \dots (i)$$

$$\text{and for } x = -2, f(x) = \frac{2(2x+1)}{-3x} \quad \dots (ii)$$

$$\therefore f\left(\frac{1}{2}\right) = 4$$

190. Given, $f(f(x)) = -x + 1$

Replacing x by $f(x)$, we get

$$f(f(f(x))) = -f(x) + 1$$

$$\Rightarrow f(1-x) = -f(x) + 1$$

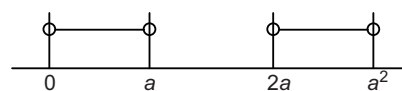
$$\Rightarrow f(x) + f(1-x) = 1$$

$$\therefore f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) = 1$$

191. As $a > 2$, hence $a^2 > 2a > a > 2$

$$\text{Now, } (x-a)(x-2a)(x-a^2) < 0$$

The solution set is shown below:



Between $(0, a)$, there are $(a-1)$ positive integers and between $(2a, a^2)$, there are $(a^2 - 2a - 1)$ integers.

$$\therefore a^2 - 2a - 1 + a - 1 = 18$$

$$\Rightarrow a^2 - a - 20 = 0$$

$$\Rightarrow (a-5)(a+4) = 0$$

$$\therefore a = 5$$

192. Clearly, fundamental period is $\frac{4\pi}{3}$, then z lies in the third quadrant.

$$\begin{aligned} 193. f(x) &= \frac{ax^8 + bx^6 + cx^4 + dx^2 + 15x + 1}{x} \\ &= \underbrace{ax^7 + bx^5 + cx^3 + dx}_{\text{odd function}} + \frac{1}{x} + 15 \end{aligned}$$

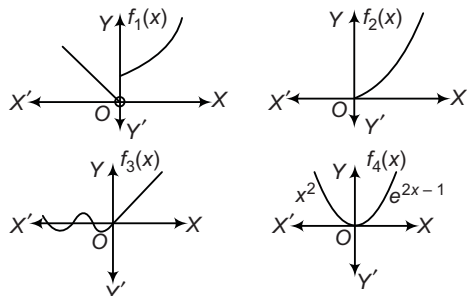
$$\text{Now, } f(x) + f(-x) = 30$$

$$\Rightarrow f(-5) = 30 - f(5) = 28$$

Entrances Gallery

1. $f_2(f_1(x)) = (f_1(x))^2 - \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \geq 0 \end{cases}$

and $f_4(x) = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \geq 0 \end{cases}$



$f_4(x)$ is many-one, onto, continuous and non-derivable.

$f_3(x)$ is many-one, into, continuous and derivable.

$f_2(x)$ is one-one, into and differentiable.

Hence, $R \leftrightarrow 2$

2. $f(x) = [\ln(\sec x + x \tan x)]^3$

$$f'(x) = \frac{3[\ln(\sec x + x \tan x)]^2 (\sec x \tan x + \sec^2 x)}{(\sec x + \tan x)}$$

$$f'(x) = \sec x [\ln(\sec x + x \tan x)]^2 > 0, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$f(x)$ is an increasing function.

$\therefore f(x)$ is one-one function.

$$\Rightarrow (\sec x + \tan x) = \tan\left(\frac{\pi}{4} + \frac{\pi}{2}\right) \quad \left[\because x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\right]$$

$$\Rightarrow 0 < \tan\left(\frac{\pi}{4} + \frac{\pi}{2}\right) < \infty$$

$$\Rightarrow 0 < \sec x + \tan x < \infty$$

$$\Rightarrow -\infty < \ln(\sec x + \tan x) < \infty$$

$$\Rightarrow -\infty < [\ln(\sec x + \tan x)]^3 < \infty$$

$$\Rightarrow -\infty < f(x) < \infty$$

$\therefore$ Range of $f(x)$ is R and thus $f(x)$ is an onto function.

$$\Rightarrow f(-x) = [\ln(\sec x - \tan x)]^3 = \left[\ln\left(\frac{1}{\sec x + \tan x}\right)\right]^3$$

$$\Rightarrow f(-x) = -[\ln(\sec x + \tan x)]^3$$

$$\Rightarrow f(x) + f(-x) = 0$$

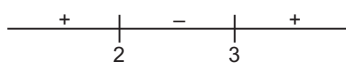
$\therefore f(x)$ is an odd function.

3. $f: [0, 3] \rightarrow [1, 29]$

$$f(x) = 2x^3 - 15x^2 + 36x + 1$$

$$\therefore f'(x) = 6x^2 - 30x + 36$$

$$= 6(x^2 - 5x + 6) = 6(x - 2)(x - 3)$$



Clearly, $f'(x) > 0$ for $[0, 2)$ and $f'(x) < 0$ for $(2, 3]$.

Hence, it is many-one.

$$\text{Now, } f'(x) = 0 \Rightarrow x = 2, 3$$

$$\text{For range, } f(0) = 1, f(2) = 29, f(3) = 28$$

$$\therefore \text{Range} \in [1, 29]$$

Hence, it is onto.

4. $f(\cos 4\theta) = \frac{2}{2 - \sec^2 \theta}$... (i)

$$\text{At } \cos 4\theta = \frac{1}{3}, 2\cos^2 2\theta - 1 = \frac{1}{3}$$

$$\Rightarrow \cos^2 2\theta = \frac{2}{3}$$

$$\Rightarrow \cos 2\theta = \pm \sqrt{\frac{2}{3}}$$

... (ii)

$$\therefore f\left(\frac{1}{3}\right) = \frac{2 \cdot \cos^2 \theta}{2\cos^2 \theta - 1} = \frac{1 + \cos 2\theta}{\cos 2\theta}$$
 [from Eq. (ii)]

$$= 1 \pm \sqrt{\frac{3}{2}}$$

5. $f(x) = x^2$ and $g(x) = \sin x$

$$\Rightarrow (g \circ f)(x) = \sin x^2$$

$$\Rightarrow go(g \circ f)(x) = \sin(\sin x^2)$$

$$\Rightarrow (f \circ g \circ g \circ f)(x) = (\sin(\sin x^2))^2$$
 ... (i)

$$\text{Again, } (g \circ f)(x) = \sin x^2$$

$$\Rightarrow (g \circ g \circ f)(x) = \sin(\sin x^2)$$

... (ii)

$$\text{Given, } (f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$$

$$\Rightarrow (\sin(\sin x^2))^2 = \sin(\sin x^2)$$

$$\Rightarrow \sin(\sin x^2) \{\sin(\sin x^2) - 1\} = 0$$

$$\Rightarrow \sin(\sin x^2) = 0$$

$$\text{or } \sin(\sin x^2) = 1$$

$$\Rightarrow \sin x^2 = 0$$

$$\text{or } \sin x^2 = \frac{\pi}{2}$$

$$\therefore x^2 = n\pi$$

which is not possible as $-1 \leq \sin \theta \leq 1$.

$$\text{Hence, } x = \pm \sqrt{n\pi}$$

6. **Statement I** $A = \{(x, y) \in R \times R : y - x \text{ is an integer}\}$

(i) **Reflexive** $xRx : (x - x) \text{ is an integer.}$

i.e. true

$\therefore$ The relation is reflexive.

(ii) **Symmetric** $xRy : (x - y) \text{ is an integer.}$

$$\Rightarrow -(y - x) \text{ is an integer.}$$

$$\Rightarrow (y - x) \text{ is an integer.}$$

$$\Rightarrow yRx$$

$\therefore$ The relation is symmetric.

(iii) **Transitive** xRy and yRz

$$\Rightarrow (x - y) \text{ is an integer and } (y - z) \text{ is an integer.}$$

$$\Rightarrow (x - y) + (y - z) \text{ is an integer.}$$

$$\Rightarrow (x - z) \text{ is an integer.}$$

$$\Rightarrow xRz$$

$\therefore$ The relation is transitive.

Hence, it is an equivalence relation.

$$B = \{(x, y) \in R \times R : x = \alpha y \text{ for some rational number } \alpha\}$$

$$\text{If } \alpha = 1$$

$$xRy : x = y \text{ (to check equivalence)}$$

(i) **Reflexive** $xRx : x = x$

$\therefore$ The relation is reflexive.

(ii) **Symmetric** $xRy : x = y \Rightarrow y = x \Rightarrow yRx$

$\therefore$ The relation is symmetric.

- (iii) **Transitive** xRy and yRz
 $\Rightarrow x = y$ and $y = z \Rightarrow x = z$
 $\therefore xRz$
 So, it is an equivalence relation.
 Hence, both statements are true but Statement II is not a correct explanation for Statement I.

7. $y = \frac{1}{\sqrt{|x| - x}}$

For domain, $|x| - x > 0$
 $\Rightarrow |x| > x$
 i.e. only possible, if $x < 0$.
 $\therefore x \in (-\infty, 0)$

8. Given, $R = \{(A, B) | A = P^{-1}BP \text{ for some invertible matrix } P\}$

To check equivalence

- (i) **Reflexive** $ARA \Rightarrow A = P^{-1}AP$, which is true only, if $P = I$.

$$\Rightarrow A = P^{-1}BP$$

For some invertible matrix P , we can let $P = I$

$$\Rightarrow ARA \Rightarrow A = I^{-1}AI$$

$$A = A$$

$\therefore$ The relation is reflexive.

- (ii) **Symmetric** $ARB \Rightarrow A = P^{-1}BP$

$$\Rightarrow PAP^{-1} = P(P^{-1}BP)P^{-1}$$

$$\Rightarrow PAP^{-1} = (PP^{-1})B(PP^{-1})$$

$$\therefore B = PAP^{-1}$$

For some invertible matrix P , we can let $R = P^{-1}$

$$\Rightarrow B = (P^{-1})AP \Rightarrow BRA$$

$\therefore$ The relation is symmetric.

- (iii) **Transitive** ARB and BRC

$$\Rightarrow A = P^{-1}BP$$

$$\text{and } B = P^{-1}CP$$

$$\Rightarrow A = P^{-1}(P^{-1}CP)P = (P^{-1})^2C(P)^2$$

$$\Rightarrow ARC, \text{ for some } P^2 = P$$

$\therefore$ The relation is transitive.

Thus, R is equivalence.

Hence, both statements are true but Statement II is not the correct explanation for Statement I.

9. Given, $p(x) = f(x) - g(x)$ has only one root -1 .

$$\therefore p(x) = (a - a_1)x^2 + (b - b_1)x + (c - c_1) \text{ has}$$

$$\frac{-(b - b_1)}{(a - a_1)} = -2$$

$$\frac{(c - c_1)}{(a - a_1)} = 1 \quad \dots(i)$$

$$\text{and } p(-2) = 2$$

$$\Rightarrow 4(a - a_1) - 2(b - b_1) + (c - c_1) = 2 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$4(a - a_1) - 4(a - a_1) + (a - a_1) = 2$$

$$\Rightarrow (a - a_1) = 2$$

On substituting $(a - a_1) = 2$ in Eq. (i), we get

$$(b - b_1) = 4 \text{ and } (c - c_1) = 2$$

$$\begin{aligned} \text{Now, } p(2) &= 4(a - a_1) + 2(b - b_1) + (c - c_1) \\ &= 8 + 8 + 2 \\ &= 18 \end{aligned}$$

10. $f(x) = (x - 1)^2 + 1$, as $x \geq 1$

$$\text{When } f(x) = f^{-1}(x) \Rightarrow f(x) = x$$

$$\Rightarrow (x - 1)^2 + 1 = x$$

$$\Rightarrow (x - 1)^2 = x - 1$$

$$\Rightarrow (x - 1)^2 - (x - 1) = 0$$

$$\Rightarrow (x - 1)(x - 1 - 1) = 0$$

$$\Rightarrow x = 1, 2$$

$$\therefore \{x : f(x) = f^{-1}(x)\} = \{1, 2\}$$

$$\text{Also, let } f(x) = y$$

$$\therefore y = (x - 1)^2 + 1$$

$$\Rightarrow (x - 1) = \pm \sqrt{y - 1}$$

$$\therefore x = 1 \pm \sqrt{y - 1}, \text{ as } x \geq 1$$

$$\text{Neglecting } 1 - \sqrt{y - 1}$$

$$\Rightarrow x = 1 + \sqrt{y - 1}$$

$$\text{or } f^{-1}(y) = 1 + \sqrt{y - 1}$$

$$\therefore f^{-1}(x) = 1 + \sqrt{x - 1}$$

Hence, both statements are correct and Statement II is the correct explanation for Statement I.

11. Since, the relation R is defined as $R = \{(x, y) | x, y \text{ are real numbers and } x = wy \text{ for some rational number } w\}$

- (i) **Reflexive** $xRx \Rightarrow x = wx$

$$\therefore w = 1 \in \text{Rational number}$$

$\Rightarrow$ The relation R is reflexive.

- (ii) **Symmetric** $xRy \Rightarrow yRx$ as $0R1$

$$\Rightarrow 0 \cdot (1) \text{ but } 1R0 \Rightarrow 1 = w \cdot (0)$$

which is not true for any rational number.

So, the relation R is not symmetric.

Thus, R is not equivalence relation.

Now, for the relation S is defined as

$$S = \left\{ \left(\frac{m}{n}, \frac{p}{q} \right) \mid m, n, p \text{ and } q \in \text{integers} \right.$$

such that $n, q \neq 0$ and $qm = pn\}$

- (i) **Reflexive** $\frac{m}{n} R \frac{m}{n} \Rightarrow mn = mn$

$\therefore$ The relation S is reflexive.

- (ii) **Symmetric** $\frac{m}{n} R \frac{p}{q} \Rightarrow mq = np$

$$\Rightarrow np = mq \Rightarrow \frac{p}{q} R \frac{m}{n}$$

$\therefore$ The relation S is symmetric.

- (iii) **Transitive** $\frac{m}{n} R \frac{p}{q}$ and $\frac{p}{q} R \frac{r}{s}$

$$\Rightarrow mq = np \text{ and } ps = rq$$

$$\Rightarrow mq \cdot ps = np \cdot rq$$

$$\Rightarrow ms = nr$$

$$\Rightarrow \frac{m}{n} = \frac{r}{s}$$

$$\Rightarrow \frac{m}{n} R \frac{r}{s}$$

$\therefore$ The relation S is transitive.

Hence, the relation S is equivalence relation.

12. Given, $f(x) = x^3 + 5x + 1$

$$\text{Now, } f'(x) = 3x^2 + 5 > 0, \forall x \in \mathbb{R}$$

$\therefore f(x)$ is strictly increasing function.

So, $f(x)$ is one-one function.

Clearly, $f(x)$ is a continuous function and also increasing on R .

$$\lim_{x \rightarrow -\infty} f(x) = -\infty \text{ and } \lim_{x \rightarrow \infty} f(x) = \infty$$

$\therefore f(x)$ takes every value between $-\infty$ and ∞ .

Thus, $f(x)$ is onto function.

13. Given, $f(x) = (x+1)^2 - 1$, $x \geq -1$

$$\Rightarrow f'(x) = 2(x+1) \geq 0 \text{ for } x \geq -1$$

$\Rightarrow f(x)$ is one-one.

Since, codomain of the given function is not given, so it can be considered as R , the set of real and consequently R is not onto.

Hence, f is not bijective.

So, Statement II is false.

$$\text{Also, } f(x) = (x+1)^2 - 1 \geq -1 \text{ for } x \geq -1$$

$$\Rightarrow R_f = [-1, \infty)$$

$$\text{Clearly, } f(x) = f^{-1}(x) \text{ at } x = 0 \text{ and } x = -1$$

$\therefore$ Statement I is true.

14. Since, $(1, 2) \in S$ but $(2, 1) \notin S$

$\therefore S$ is not symmetric.

So, S is not an equivalence relation.

$$\text{Given, } T = \{(x, y) : (x - y) \in I\}$$

Now, $x - x = 0 \in I$, it is reflexive relation.

Again now, $(x - y) \in I$

$\Rightarrow y - x \in I$, it is symmetric relation.

$$\text{Let } x - y = I_1 \text{ and } y - z = I_2$$

$$\text{Now, } x - z = (x - y) + (y - z) = I_1 + I_2 \in I$$

$\therefore T$ is also transitive.

Hence, T is an equivalence relation.

15. Since, $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$

$$\therefore Y = \{7, 11, 15, \dots\}$$

$$\text{Let } y = 4x + 3$$

$$\Rightarrow x = \frac{y-3}{4}$$

$$\therefore \text{Inverse of } f(x) \text{ is } g(y) = \frac{y-3}{4}$$

16. Given that,

$$f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$$

Here, 4^{-x^2} is defined for $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$\cos^{-1}\left(\frac{x}{2} - 1\right)$ is defined, if $-1 \leq \frac{x}{2} - 1 \leq 1$

$$\Rightarrow 0 \leq \frac{x}{2} \leq 2 \Rightarrow 0 \leq x \leq 4$$

and $\log(\cos x)$ is defined, if $\cos x > 0$.

$$\Rightarrow -\frac{\pi}{2} < x < \frac{\pi}{2}$$

$$\text{Hence, } f(x) = 4^{-x^2} + \cos^{-1}\left(\frac{x}{2} - 1\right) + \log(\cos x)$$

is defined, if $x \in \left[0, \frac{\pi}{2}\right)$.

17. Let $W = \{\text{CAT, TOY, YOU}, \dots\}$

Clearly, R is reflexive and symmetric but not transitive. [since, ${}_{\text{CAT}}R_{\text{TOY}}, {}_{\text{TOY}}R_{\text{YOU}} \not\Rightarrow {}_{\text{CAT}}R_{\text{YOU}}$]

18. Since, for every element of A , there exist elements $(3, 3), (6, 6), (9, 9), (12, 12) \in R$.

$\Rightarrow R$ is reflexive relation.

Now, $(6, 12) \in R$ but $(12, 6) \notin R$, so it is not a symmetric relation.

$$\text{Also, } (3, 6), (6, 12) \in R \Rightarrow (3, 12) \in R$$

$\therefore R$ is transitive relation.

19. Since, $x \in (-1, 1)$

$$\Rightarrow \tan^{-1} x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$$

$$\Rightarrow 2 \tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Given that,

$$f(x) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = 2 \tan^{-1} x \quad [\because x^2 < 1]$$

$$\text{So, } f(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

Hence, function is one-one and onto.

20. Given, $f(x - y) = f(x)f(y) - f(a - x)f(a + y)$

$$\text{Let } x = 0 = y$$

$$\Rightarrow f(0) = (f(0))^2 - (f(a))^2$$

$$\Rightarrow 1 = 1 - (f(a))^2 \quad [\because f(0) = 1, \text{ given}]$$

$$\Rightarrow f(a) = 0$$

$$\begin{aligned} \therefore f(2a - x) &= f\{a - (x - a)\} \\ &= f(a)f(x - a) - f(a - a)f(x) \\ &= 0 - f(x) \cdot 1 = -f(x) \end{aligned}$$

21. Given, $R = \{(1, 3), (4, 2), (2, 4), (2, 3), (3, 1)\}$

is a relation on the set $A = \{1, 2, 3, 4\}$.

(a) Since, $(2, 4) \in R$ and $(2, 3) \in R$.

So, R is not a function.

(b) Since, $(1, 3) \in R$ and $(3, 1) \in R$ but $(1, 1) \notin R$.

So, R is not transitive.

(c) Since, $(2, 3) \in R$ but $(3, 2) \notin R$.

So, R is not symmetric.

(d) Since, $(1, 1), (2, 2), (3, 3), (4, 4) \notin R$.

So, R is not reflexive.

22. Since, graph is symmetrical about the line $x = 2$.

$$\therefore f(2 + x) = f(2 - x)$$

23. Given, $f(x) = \frac{\sin^{-1}(x-3)}{\sqrt{9-x^2}}$ will be defined, if

$$-1 \leq (x-3) \leq 1 \Rightarrow 2 \leq x \leq 4 \quad \dots(i)$$

$$\text{and } 9 - x^2 > 0 \Rightarrow -3 < x < 3 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$2 \leq x < 3$$

Hence, domain of the given function is $[2, 3)$.

24. Given that,

$$f(n) = \begin{cases} \frac{n-1}{2}, & \text{when } n \text{ is odd} \\ -\frac{n}{2}, & \text{when } n \text{ is even} \end{cases}$$

and $f : N \rightarrow I$, where N is the set of natural numbers and I is the set of integers.

Let $x, y \in N$ and both are even.

Then, $f(x) = f(y)$
 $\Rightarrow -\frac{x}{2} = -\frac{y}{2}$
 $\Rightarrow x = y$

Again, $x, y \in N$ and both are odd.

Then, $f(x) = f(y)$
 $\Rightarrow \frac{x-1}{2} = \frac{y-1}{2}$
 $\Rightarrow x = y$

So, mapping is one-one.

Since, each negative integer is an image of even natural number and positive integer is an image of odd natural number. So, mapping is onto. Hence, mapping is one-one and onto.

25. $\sum_{r=1}^n f(r) = f(1) + f(2) + f(3) + \dots + f(n)$
 $= f(1) + 2f(1) + 3f(1) + \dots + nf(1)$
 $[\because f(x+y) = f(x) + f(y)]$
 $= (1+2+3+\dots+n)f(1) = f(1) \sum n$
 $= \frac{7n(n+1)}{2} \quad [\because f(1) = 7, \text{ given}]$

26. Since, $f(x) = \frac{3}{4-x^2} + \log_{10}(x^3 - x)$

For domain of $f(x)$,

$x^3 - x > 0$
 $\Rightarrow x(x-1)(x+1) > 0$

$\Rightarrow x \in (-1, 0) \cup (1, \infty)$
 and $4 - x^2 \neq 0$
 $\Rightarrow x \neq \pm 2$
 $\Rightarrow x \in (-\infty, -2) \cup (-2, 2) \cup (2, \infty)$
 $\therefore$ Common region is $(-1, 0) \cup (1, 2) \cup (2, \infty)$

27. Given that, $f(x) = \log(x + \sqrt{x^2 + 1})$
 Now, $f(-x) = \log(-x + \sqrt{x^2 + 1})$
 $\therefore f(x) + f(-x) = \log(x + \sqrt{x^2 + 1}) + \log(-x + \sqrt{x^2 + 1})$
 $= \log(1) = 0$

Hence, $f(x)$ is an odd function.

28. Since, $\sin^2 \theta = \frac{1 - \cos 2\theta}{2} = \frac{1}{2} - \frac{1}{2} \cos 2\theta$

$\therefore$ Period of $\sin^2 \theta = \frac{2\pi}{2} = \pi$

29. Since, domain of $\sin^{-1} x$ is $[-1, 1]$

$\therefore -1 \leq \log_3 \left(\frac{x}{3} \right) \leq 1$

$\Rightarrow 3^{-1} \leq \frac{x}{3} \leq 3$

$\Rightarrow 1 \leq x \leq 9$

Hence, domain of $\sin^{-1} \left[\log_3 \left(\frac{x}{3} \right) \right]$ is $[1, 9]$.

30. Given that, $f(x) = \sin^4 x + \cos^4 x$
 $\therefore f(x) = (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x$
 $= 1 - \frac{1}{2} (2\sin x \cos x)^2$
 $= 1 - \frac{1}{2} (\sin 2x)^2 = 1 - \frac{1}{2} \left(\frac{1 - \cos 4x}{2} \right)$
 $= \frac{3}{4} + \frac{1}{4} \cos 4x$

$\therefore$ The period of $f(x) = \frac{2\pi}{4} = \frac{\pi}{2}$

$[\because \cos x$ is periodic with period $2\pi]$

31. Given that, $f(x) = \sqrt{\log_{10} \left(\frac{5x - x^2}{4} \right)}$

For domain of $f(x)$,

$\log_{10} \left(\frac{5x - x^2}{4} \right) \geq 0 \Rightarrow \frac{5x - x^2}{4} \geq 1$

$\Rightarrow x^2 - 5x + 4 \leq 0$

$\Rightarrow (x-1)(x-4) \leq 0$

$\therefore x \in [1, 4]$

32. R is reflexive as $(3, 3), (6, 6), (9, 9), (12, 12) \in R$. R is not symmetric as $(6, 12) \in R$ but $(12, 6) \notin R$. R is transitive as the only pair which needs verification, is $(3, 6)$ and $(6, 12) \in R$.

$\Rightarrow (3, 12) \in R$

33. Given, $aRb \Leftrightarrow |a - b| \leq 1$

Reflexive $aRa = |a - a| = 0 \leq 1$

So, it is reflexive.

Symmetric $aRb \Leftrightarrow |a - b| \leq 1$

$\Rightarrow |b - a| \leq 1$ i.e. $aRb \Rightarrow bRa$

So, it is symmetric.

Transitive Take $a = 1, b = 2$ and $c = 3$.

Now, $|a - b| = |1 - 2| = 1$

and $|b - c| = |2 - 3| = 1$

But $|a - c| = |1 - 3| = 2 > 1$

which is not true.

i.e. $aRb, bRc \not\Rightarrow aRc$

So, it is not transitive.

34. Given relation is defined as $\theta R \phi$ such that $\sec^2 \theta - \tan^2 \phi = 1$.

Reflexive When $\theta R \theta$, then $\sec^2 \theta - \tan^2 \theta = 1$

$\Rightarrow 1 = 1$, which is true.

Thus, it is reflexive.

Symmetric When $\theta R \phi$, then $\sec^2 \theta - \tan^2 \phi = 1$

$\Rightarrow (1 + \tan^2 \theta) - (\sec^2 \phi - 1) = 1$

$\Rightarrow 2 + \tan^2 \theta - \sec^2 \phi = 1$

$\Rightarrow \sec^2 \phi - \tan^2 \theta = 1 \Rightarrow \phi R \theta$

Thus, it is symmetric.

Transitive When $\theta R \phi$ and $\phi R \psi$, then

$\sec^2 \theta - \tan^2 \phi = 1$ and $\sec^2 \phi - \tan^2 \psi = 1$

If $\theta R \psi$, then $\sec^2 \theta - \tan^2 \psi + 1 = 1 + 1$

$\Rightarrow \sec^2 \theta - \tan^2 \psi + \sec^2 \phi - \tan^2 \phi = 1 + 1$

$\Rightarrow \theta R \phi$ and $\phi R \psi$

Thus, it is transitive.

Hence, it is an equivalence relation.

35. Given,

$$f: N \rightarrow N, f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

$$\text{Now, for } n=1, f(1) = \frac{1+1}{2} = 1$$

$$\text{For } n=2, f(2) = \frac{2}{2} = 1$$

$$\therefore f(1) = f(2) = 1 \text{ but } 1 \neq 2$$

So, $f(x)$ is not one-one mapping.

$$\text{When } n \text{ is odd, then } f(n) = \frac{n+1}{2} = y \quad [\text{say}]$$

$$\Rightarrow y = \frac{n+1}{2} \Rightarrow n = 2y - 1, \forall y$$

$$\text{When } n \text{ is even, then } f(n) = \frac{n}{2} = y \quad [\text{say}]$$

$$\Rightarrow n = 2y, \forall y$$

So, $f(x)$ is onto mapping.

Hence, $f(x)$ is onto but not one-one mapping.

36. Given, $A = \{1, 3, 5, 7\}$ and $B = \{1, 2, 3, 4, 5, 6, 7, 8\}$

$$\text{Here, } n(A) = 4 \text{ and } n(B) = 8$$

$\therefore$ Number of one-one function from A into B

$$= {}^8P_4 = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$$

37. Given function is $f(x) = x^2 + bx + c$.

It is quadratic equation in x .

So, we will get a parabola either downward or upward.

So, it is many-one mapping and not onto mapping.

Hence, it is neither one-one nor onto mapping.

38. Since, $f(x)$ is an odd periodic function with period 2.

$$\therefore f(-x) = -f(x) \text{ and } f(x+2) = f(x)$$

$$\therefore f(2) = f(0+2) = f(0) \text{ and } f(-2) = f(-2+2) = f(0)$$

$$\text{Now, } f(0) = f(-2) = -f(2) = -f(0)$$

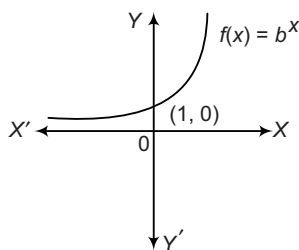
$$\Rightarrow 2f(0) = 0$$

$$\text{i.e. } f(0) = 0$$

$$\therefore f(4) = f(2+2) = f(2) = f(0) = 0$$

$$\text{Thus, } f(4) = 0$$

39. It is clear from the graph $f(x) = b^x$ that options (a), (b) and (d) are correct.



$$\text{When } x=1, \text{ then } f(1) = b^1 = b \neq 0$$

40. Given, $f(x) = \sqrt{\cos x}$, i.e. $\cos x \geq 0$

$$\text{But } -1 \leq \cos x \leq 1$$

$$\therefore 0 \leq \cos x \leq 1$$

i.e. x lies in Ist and IVth quadrants.

$$\Rightarrow 0 \leq x \leq \frac{\pi}{2}$$

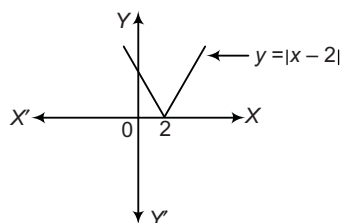
$$\text{or } \frac{3\pi}{2} \leq x \leq 2\pi$$

$$\therefore x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$$

$$\text{Also, } \cos(-x) = \cos x$$

$$\text{Hence, } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \text{ is also the domain of the function.}$$

41. Given, $f(x) = |x-2|$



When we take a horizontal line parallel to X -axis, it intersects two points. It means $f(x)$ is not one-one function.

42. Given, $f(x) = x^2 + 2x + 2$

$$= x^2 + 2x + 1 + 1 = (x+1)^2 + 1 \geq 1$$

So, the range of $f(x)$ is $[1, \infty)$.

43. Given function is $y = 3\sin\left(\sqrt{\frac{\pi^2}{16} - x^2}\right)$.

$$\text{Here, domain is } \left[0, \frac{\pi}{4}\right]$$

When $x=0$, then

$$\begin{aligned} y &= 3\sin\left(\sqrt{\frac{\pi^2}{16} - 0}\right) = 3\sin\frac{\pi}{4} \\ &= 3 \times \frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} \text{When } x = \frac{\pi}{4}, \text{ then } y &= 3\sin\left(\sqrt{\frac{\pi^2}{16} - \frac{\pi^2}{16}}\right) \\ &= 3\sin(0) = 0 \end{aligned}$$

$$\text{Hence, range of given function is } \left[0, \frac{3}{\sqrt{2}}\right]$$

44. Since, $A = \{1, 2, 3, 4\}$ and R is a relation on A defined by $\{(a, b): a, b \in A, a \times b \text{ is an even number}\}$.

Thus, R is the set of even numbers.

$$\therefore \text{Range of } R = \{2, 4\}$$

$$45. \therefore f(x) = \frac{(x^2+1)}{(x^2-3x+3)}$$

So, (x^2+1) is well-defined for all $x \in R$.

Also, x^2-3x+3 has imaginary roots.

So, x^2-3x+3 is also defined for all real numbers i.e. $x \in R$.

Hence, domain of the function is R .

46. $f: [0, 1] \rightarrow R$ such that

$$f(x) = x^3 - x^2 + 4x + 2\sin^{-1}x$$

$$\therefore f(0) = 0 \text{ and } f(1) = 4 + \pi$$

Since, it is defined for all values of R .

$$\therefore \text{Range} = [0, 4 + \pi]$$

47. Since, $A = \{1, 2, 3, 4, 5\}$
 $\therefore$ Relation from A to A by R
 $= \{(x, y) : y = 2x - 1\} = \{(1, 1), (2, 3), (3, 5)\}$
 $\therefore$ Domain $= \{1, 2, 3\}$
48. Given, $f(x) = (x + 1)^2$, $x \geq -1$ and $g(x)$ is reflection of graph $f(x)$ in the line $y = x$, then $g(x)$ is the inverse of $f(x)$.
 Let $y = (x + 1)^2$, then
 $\sqrt{y} - 1 = x$
 $\therefore \sqrt{x} - 1 = f^{-1}(x)$
49. Given, $f(x) = \sqrt{x}$ and $g(x) = 2x - 3$
 $\therefore fog(x) = f\{g(x)\} = f(2x - 3) = \sqrt{2x - 3}$
 For domain of $fog(x)$,
 $2x - 3 \geq 0 \Rightarrow 2x \geq 3 \Rightarrow x \geq \frac{3}{2}$
 $\therefore x \in \left[\frac{3}{2}, \infty\right)$
50. Given, $f(x) = \frac{x+2}{3x-1}$
 $\therefore f\{f(x)\} = f\left(\frac{x+2}{3x-1}\right) = \frac{\frac{x+2}{3x-1} + 2}{3\left(\frac{x+2}{3x-1}\right) - 1}$
 $= \frac{x+2+6x-2}{3x+6-(3x-1)} = \frac{7x}{7} = x$
51. Given, $f(x) = 3x^2 + 1$
 Let $y = 3x^2 + 1$, then
 $3x^2 = y - 1 \Rightarrow x^2 = \frac{y-1}{3}$
 $\Rightarrow x = \pm \sqrt{\frac{y-1}{3}}$
 $\therefore f^{-1}(x) = \pm \sqrt{\frac{x-1}{3}}$
 When $x \in [1, 6]$, then
 $f^{-1}(x) \in \left[-\sqrt{\frac{5}{3}}, \sqrt{\frac{5}{3}}\right]$
52. Given, $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$
 $\therefore f\left(x + \frac{1}{x}\right) = \left(x + \frac{1}{x}\right)^2 - 2$
 On putting $y = x + \frac{1}{x}$, we get $f(y) = y^2 - 2$
 $\therefore f(x) = x^2 - 2$
53. Given relation is $R = \{(a, b) : 1 + ab > 0\}$
Reflexive A relation $R \subseteq A \times A$ is reflexive, if $(a, a) \in R$.
 Now, $R = \{(a, b) : 1 + ab > 0\}$ on $\mathbb{Z}$
 $\Rightarrow (a, a) \in R \Rightarrow 1 + a^2 > 0$
 So, it is reflexive.
Symmetric If $(a, b) \in R$, then $(b, a) \in R$
 Since, $1 + ab > 0 \Rightarrow 1 + ba > 0 \Rightarrow (b, a) \in R$
 So, it is symmetric.
Transitive If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$
 Now, $(a, b) \in R$ and $(b, c) \in R$

$$\Rightarrow 1 + ab > 0 \text{ and } 1 + bc > 0$$

$$\Rightarrow 2 + b(a + c) > 0$$

i.e. $(a, c) \in R$
 So, it is not transitive.

54. Let λ be the period of $\sin x + \cos ax$.
 Then, $\sin(\lambda + x) + \cos a(\lambda + x) = \sin x + \cos ax$
 For all x in this identity, putting $x = 0$ and $x = -\lambda$, we get
 $\sin \lambda + \cos a\lambda = 1$ and $1 = -\sin \lambda + \cos a\lambda$
 On solving above equations, we get
 $\sin \lambda = 0$ and $\cos a\lambda = 1$
 Hence, $\lambda = n\pi$ and $a\lambda = 2m\pi$, where m and n are non-zero integers.
 Here, $\frac{a\lambda}{\lambda} = \frac{2m\pi}{n\pi} \Rightarrow a = \frac{2m}{n}$ [$\because \lambda \neq 0$]
 which is a rational number.

55. Given, $f(x) = \frac{a^x + a^{-x}}{2}$, where $a > 2$

$$\therefore f(x + y) + f(x - y)$$

$$= \frac{1}{2} [a^{x+y} + a^{-x-y} + a^{x-y} + a^{-x+y}]$$

$$= \frac{1}{2} [a^x(a^y + a^{-y}) + a^{-x}(a^y + a^{-y})]$$

$$= \frac{1}{2} [(a^x + a^{-x})(a^y + a^{-y})] = 2f(x)f(y)$$

56. Let $y = (x + 2)^2 - 2$, then
 $(x + 2)^2 = y + 2 \Rightarrow x + 2 = \sqrt{y + 2}$ [$\because x \geq -2$]
 $\Rightarrow x = \sqrt{y + 2} - 2$
 $\therefore f^{-1}(x) = \sqrt{x + 2} - 2$
57. **Reflexive** As $20 \in N$ but $(20, 20) \notin R$
 So, it is not reflexive.
Symmetric As $(20, 30) \in R$ but $(30, 20) \notin R$
 So, it is not symmetric.
Transitive As $(20, 30) \in R$, $(30, 50) \in R$
 but $(20, 50) \notin R$
 So, it is not transitive.

58. We know that, the total number of injective maps from a set P to a set Q , where $n(P) = M$, $n(Q) = N$ and $N \geq M$, is ${}^N P_M$.
 $\therefore$ Required number of injections $= {}^7 P_4 = 840$

59. Period of $\sin \sqrt{x}$ is not defined.

60. Given, $f(x) = x^2 + 2bx + 2c^2$
 and $g(x) = -x^2 - 2cx + b^2$
 $\therefore \min \{f(x)\} = -\frac{[4b^2 - 4(2c^2)]}{4} = -[b^2 - 2c^2]$
 $\left[\because \min y = ax^2 + bx + c, a > 0 \right]$
 $\left[\text{is } \frac{-D}{4a}, \text{ where } D = b^2 - 4ac \right]$
 $= 2c^2 - b^2$
 and $\max \{g(x)\} = -\frac{[4c^2 + 4b^2]}{4(-1)} = c^2 + b^2$
 $\left[\because \max y = ax^2 + bx + c, a < 0 \right]$
 $\left[\text{is } \frac{-D}{4a}, \text{ where } D = b^2 - 4ac \right]$

$$\begin{aligned} \therefore \min \{f(x)\} &> \max \{g(x)\} \\ \Rightarrow 2c^2 - b^2 &> c^2 + b^2 \\ \Rightarrow c^2 &> 2b^2 \\ \therefore |c| &> \sqrt{2}|b| \end{aligned}$$

61. Let $y = 2x + 3$, then

$$\begin{aligned} y - 3 &= 2x \Rightarrow x = \frac{y-3}{2} \\ \therefore f^{-1}(x) &= \frac{x-3}{2} \end{aligned}$$

62. Given, $f(x) = x^2 + 2x - 3$

$$\begin{aligned} \text{and } g(x) &= x + 1 \\ \therefore f[g(x)] &= g[f(x)] \\ \therefore f(x+1) &= g(x^2 + 2x - 3) \\ \Rightarrow (x+1)^2 + 2(x+1) - 3 &= x^2 + 2x - 3 + 1 \\ \Rightarrow x^2 + 2x + 1 + 2x - 1 &= x^2 + 2x - 2 \\ \Rightarrow 2x &= -2 \\ \therefore x &= -1 \end{aligned}$$

63. If A and B are equivalence relations, then $A \cap B$ is also an equivalence relation.

64. Since, f is one-one.

$$\begin{aligned} \therefore f(x_1) &= f(x_2) \\ \Rightarrow \frac{2x_1}{1+2x_1} &= \frac{2x_2}{1+2x_2} \\ \Rightarrow x_1 + 2x_1x_2 &= x_2 + 2x_1x_2 \\ \Rightarrow x_1 &= x_2 \end{aligned}$$

Also, f is not onto as 1 does not have pre-image.

65. Given, $f(x) = \sin^{-1}\left[\log_2\left(\frac{x}{2}\right)\right]$

$$\begin{aligned} \Rightarrow -1 &\leq \log_2\left(\frac{x}{2}\right) \leq 1 \\ \Rightarrow 2^{-1} &\leq \frac{x}{2} \leq 2^1 \\ \Rightarrow \frac{1}{2} &\leq \frac{x}{2} \leq 2 \\ \Rightarrow 1 &\leq x \leq 4 \end{aligned}$$

66. Given, $f(x) = \log_e(3x^2 + 4)$

$$\text{Let } y = \log_e(3x^2 + 4) \Rightarrow 3x^2 + 4 = e^y$$

$$\Rightarrow x = \sqrt{\frac{e^y - 4}{3}}$$

$$\text{Here, } \frac{e^y - 4}{3} \geq 0 \Rightarrow e^y \geq 4$$

$$\Rightarrow y \geq 2 \log_e 2$$

67. $\therefore$ Period of $|\sin 2x| = \frac{\pi}{2}$

$$\text{and period of } |\cos 8x| = \frac{\pi}{8}$$

$$\therefore \text{Period of } f(x) = \text{LCM of } \left\{\frac{\pi}{2}, \frac{\pi}{8}\right\} = \frac{\pi}{2}$$

68. On taking option (c),

$$\begin{aligned} f(-x) &= (-x) \frac{a^{-x} - 1}{a^{-x} + 1} \\ &= -\frac{x(1 - a^x)}{(a^x + 1)} = \frac{x(a^x - 1)}{(a^x + 1)} = f(x) \end{aligned}$$

69. Let $D_1 = \text{Domain of } \cos^{-1}\left(\frac{x-3}{2}\right)$

$$\Rightarrow -1 \leq \frac{x-3}{2} \leq 1$$

$$\Rightarrow 1 \leq x \leq 5 \quad \dots(i)$$

and $D_2 = \text{Domain of } \log_{10}(4-x)$

$$\Rightarrow 4-x > 0 \Rightarrow x < 4 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$D_1 \cap D_2 \in [1, 4)$$

70. Given, $f(x) = 3 - x$, where $-4 \leq x \leq 4$

To find domain, $\log_e[f(x)] = \log_e(3-x)$

For $\log_e[f(x)]$ to be defined,

$$3-x > 0$$

$$\Rightarrow 3 > x$$

$$\Rightarrow x < 3$$

$$\therefore \text{Domain} = [-4, 3)$$

71. Given, $f(x) = 4x^3 + 3x^2 + 3x + 4$

$$\begin{aligned} \therefore x^3 f\left(\frac{1}{x}\right) &= x^3 \left(\frac{4}{x^3} + \frac{3}{x^2} + \frac{3}{x} + 4\right) \\ &= 4 + 3x + 3x^2 + 4x^3 = f(x) \end{aligned}$$

72. Let $y = 2^{x(x-1)}$, then

$$\log_2 y = x(x-1)$$

$$\Rightarrow x^2 - x - \log_2 y = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{1 + 4 \log_2 y}}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{1 \pm \sqrt{1 + 4 \log_2 x}}{2}$$

$$\therefore f^{-1}(x) = \frac{1 + \sqrt{1 + 4 \log_2 x}}{2} \quad [\because x \geq 1]$$

73. Here, $f(0) = 0 + b \Rightarrow -1 = b$

$$\text{and } f(-1) = -a + b$$

$$\Rightarrow -3 = -a + b$$

$$\Rightarrow -a = -3 + 1$$

$$\Rightarrow a = 2$$

74. Let $f(x) = ax^2 + bx + c$

$$\therefore f(0) = 6 \Rightarrow c = 6$$

$$\therefore f(x) = ax^2 + bx + 6$$

$$\therefore f(-3) = 6$$

$$\Rightarrow 9a - 3b + 6 = 6 \Rightarrow 3a = b \quad \dots(i)$$

$$f(2) = 11$$

$$\Rightarrow 4a + 2b + 6 = 11$$

$$\Rightarrow 4a + 6a = 5 \Rightarrow a = \frac{1}{2} \Rightarrow b = \frac{3}{2} \quad [\text{from Eq. (i)}]$$

$$\therefore f(x) = \frac{x^2}{2} + \frac{3}{2}x + 6$$

$$\text{Now, } f(1) = \frac{1}{2} + \frac{3}{2} + 6 = 8$$

75. Since, $g(x)$ is the inverse of $f(x)$.

$$\therefore fog(x) = x \Rightarrow f[g(x)] = x$$

On differentiating both sides w.r.t. x , we get

$$f'[g(x)] \cdot g'(x) = 1$$

$$\begin{aligned} \therefore g'(x) &= \frac{1}{f'[g(x)]} = \frac{1}{\cos[g(x)]} \\ &= \sec[g(x)] \end{aligned}$$

76. $f(1) = 3, f(2) = 4, f(3) = 5$ and $f(4) = 6$
 $\Rightarrow 1 \in B, 2 \in B$ do not have any pre-image in A .
 $\therefore f$ is one-one and into.

77. Given, $f(\theta) = 4 + 4\sin^3 \theta - 3\sin \theta$
 $= 4 - (3\sin \theta - 4\sin^3 \theta)$
 $= 4 - \sin 3\theta$
 $\therefore$ Period of $f(\theta) = \frac{2\pi}{3}$

78. $f(x) = \sec [\log (x + \sqrt{1 + x^2})]$
 Now, $f(-x) = \sec [\log (-x + \sqrt{1 + x^2})]$
 $= \sec \left[\log \left(\frac{1}{x + \sqrt{1 + x^2}} \right) \right]$
 $= \sec [-\log (x + \sqrt{1 + x^2})]$
 $= \sec [\log (x + \sqrt{1 + x^2})] = f(x)$
 $\therefore f(x)$ is even.

79. For domain of given function,
 $-1 \leq \log_2 \left(\frac{x}{12} \right) \leq 1 \Rightarrow 2^{-1} \leq \frac{x}{12} \leq 2$
 $\Rightarrow 6 \leq x \leq 24 \Rightarrow x \in [6, 24]$

$$80. \therefore f(x) = \sqrt{\cos^{-1} \left(\frac{1 - |x|}{2} \right)}$$

$$\begin{aligned} \therefore -1 &\leq \frac{1 - |x|}{2} \leq 1 \\ \Rightarrow -2 - 1 &\leq -|x| \leq 2 - 1 \\ \Rightarrow -3 &\leq -|x| \leq 1 \\ \Rightarrow -1 &\leq |x| \leq 3 \\ \Rightarrow x &\in [-3, 3] \end{aligned}$$

81. Since, $f(a) = a \Rightarrow \frac{\alpha a^2}{a + 1} = a$

$$\Rightarrow \alpha a^2 = a^2 + a \Rightarrow \alpha = 1 + \frac{1}{a} \quad [\because a \neq 0]$$

82. Given, $f(x) = x^2 - 1$ and $g(x) = (x + 1)^2$
 $\therefore g \circ f(x) = g\{f(x)\} = g(x^2 - 1)$
 $= (x^2 - 1 + 1)^2 = x^4$

83. Given, $f(x + y) = f(x) + f(y)$

For $x = 1$ and $y = 1$,

$$f(2) = f(1) + f(1) = 2 \cdot f(1) = 10$$

Also, $f(3) = f(2) + f(1) = 15$

$$\Rightarrow f(n) = 5n$$

$$\therefore f(100) = 500$$

4

Inverse Trigonometric Functions

Inverse Function

If a function is one-one and onto from A to B , then function g which associates each element $y \in B$ to one and only one element $x \in A$, such that $y = f(x)$, then g is called the inverse function of f , defined by $x = g(y)$.

Usually, we denote $g = f^{-1}$ (read as f inverse).

$$\therefore x = f^{-1}(y)$$

↘ If $y = f(x)$ and $x = g(y)$ are two functions such that $f\{g(y)\} = y$ and $g\{f(x)\} = x$, then f and g are said to be inverse function of each other.

Inverse of Trigonometric Functions

Consider the sine function with domain R and range $[-1, 1]$. It is clear that this is not one-one, i.e. not injective, so it is not invertible. So, we have to restrict the domain of it in such a way that it becomes one-one, then it would become invertible.

Suppose we consider sine as a function with domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and codomain $[-1, 1]$, then

it is a bijective and therefore invertible. Such types of trigonometric functions are called inverse trigonometric functions.

Consider $\sin \frac{\pi}{6} = \frac{1}{2}$, which can also be written as

$$\frac{\pi}{6} = \sin^{-1} \frac{1}{2} \quad \text{or} \quad \arcsin \left(\frac{1}{2} \right)$$

Also,

$$\sin \frac{5\pi}{6} = \frac{1}{2} \quad \text{but} \quad \frac{5\pi}{6} \neq \sin^{-1} \left(\frac{1}{2} \right)$$

Chapter Snapshot

- Inverse Function
- Inverse of Trigonometric Functions
- Graphs of Inverse Trigonometric Functions
- Domain and Range of Inverse Trigonometric Functions
- Properties of Inverse Trigonometric Functions
- Solution of the Inverse Equations
- Summation of Series Containing Inverse Trigonometric Terms
- Some Useful Graphs

Clearly, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ denote angles or real numbers whose sine is x , whose cosine is x and whose tangent is x respectively, provided that the answers given are numerically smallest available.

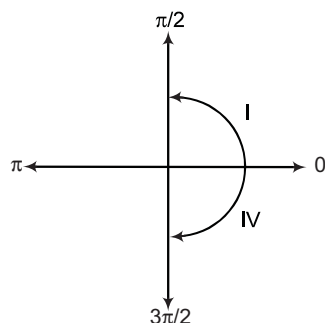
If there are two angles, one positive and the other negative having same numerical value. Then, we shall take the positive value.

e.g. $\cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$ and $\cos \left(\frac{-\pi}{4} \right) = \frac{1}{\sqrt{2}}$

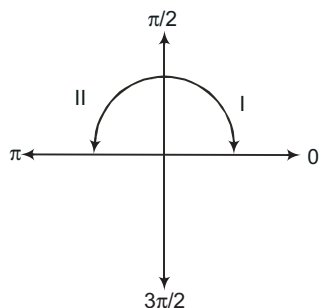
But we write $\cos^{-1} \left(\frac{1}{\sqrt{2}} \right) = \frac{\pi}{4}$

and $\cos^{-1} \left(\frac{1}{\sqrt{2}} \right) \neq \frac{-\pi}{4}$.

Intervals for Inverse Functions



Here, $\sin^{-1} x$, $\operatorname{cosec}^{-1} x$ and $\tan^{-1} x$ belong to I and IV quadrants.



Here, $\cos^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ belong to I and II quadrants.

- I quadrant is common to all the inverse functions.
 - III quadrant is not used in inverse function.
 - IV quadrant is used in the clockwise direction.
- i.e. $-\frac{\pi}{2} \leq y \leq 0$

Principal Value Branch of Inverse Trigonometric Functions

Since, sine is a many valued function, so for defining the inverse function, we restrict it as one-one and onto function, i.e. we consider domain

$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, which is called the principal value branch of sine inverse function.

The principal value branch of some trigonometric functions are as follow:

Functions	Interval of principal value
$\sin^{-1} x$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, where $y = \sin^{-1} x$
$\cos^{-1} x$	$0 \leq y \leq \pi$, where $y = \cos^{-1} x$
$\tan^{-1} x$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$, where $y = \tan^{-1} x$
$\operatorname{cosec}^{-1} x$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, where $y = \operatorname{cosec}^{-1} x$, $y \neq 0$
$\sec^{-1} x$	$0 \leq y \leq \pi$, where $y = \sec^{-1} x$, $y \neq \pi/2$
$\cot^{-1} x$	$0 < y < \pi$, where $y = \cot^{-1} x$

- If no branch of an inverse trigonometric function is mentioned, then we consider the principal value branch of the function.
- The value which lies in the principal value branch is called the principal value.

Example 1. The principal value of

$\sin^{-1} \left(\frac{\sqrt{3}}{2} \right)$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

Sol. (b) Let $\sin^{-1} \left(\frac{\sqrt{3}}{2} \right) = x$

$\Rightarrow \sin x = \frac{\sqrt{3}}{2}$

$\Rightarrow \sin x = \sin \left(\frac{\pi}{3} \right)$

$\Rightarrow x = \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

Example 2. The principal value of

$\tan^{-1}(1) + \cos^{-1} \left(-\frac{1}{2} \right) + \sin^{-1} \left(-\frac{1}{2} \right)$ is equal to

(a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $-\frac{3\pi}{4}$

Sol. (a) Let $\tan^{-1}(1) = x$

$\Rightarrow \tan x = 1 = \tan \frac{\pi}{4}$

$\Rightarrow x = \frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$

$\therefore \tan^{-1} 1 = \frac{\pi}{4}$

Let $\cos^{-1} \left(-\frac{1}{2} \right) = y$

$\Rightarrow \cos y = -\frac{1}{2} = \cos \left(\pi - \frac{\pi}{3} \right) = \cos \frac{2\pi}{3}$

$\Rightarrow y = \frac{2\pi}{3} \in [0, \pi]$

$$\therefore \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\text{Let } \sin^{-1}\left(-\frac{1}{2}\right) = z \Rightarrow \sin z = -\frac{1}{2} = \sin\left(-\frac{\pi}{6}\right)$$

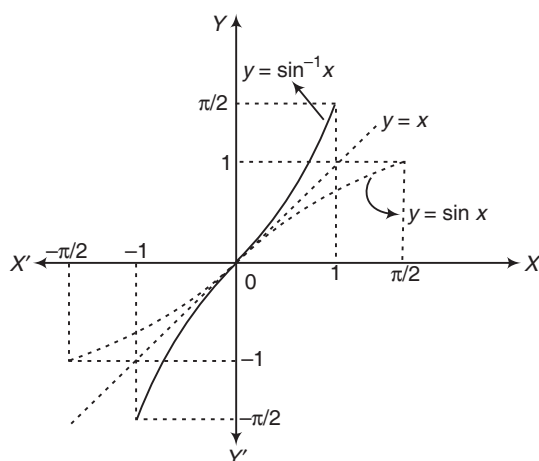
$$\Rightarrow z = -\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\therefore \sin^{-1}\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

$$\begin{aligned} \text{Hence, } \tan^{-1}(1) + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right) \\ = \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6} = \frac{3\pi + 8\pi - 2\pi}{12} = \frac{9\pi}{12} = \frac{3\pi}{4} \end{aligned}$$

Graphs of Inverse Trigonometric Functions

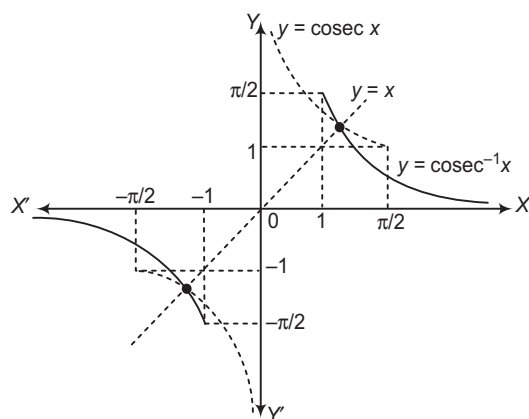
- (i) $\sin \theta = x \Rightarrow \theta = \sin^{-1} x$, where $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $x \in [-1, 1]$



➤ $\sin^{-1} x$ is an increasing function in $[-1, 1]$.

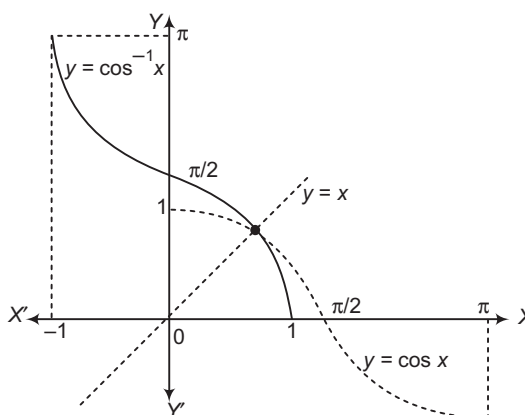
- (ii) $\operatorname{cosec} \theta = x \Rightarrow \theta = \operatorname{cosec}^{-1} x$, where

$$\theta \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right] \text{ and } x \in (-\infty, -1] \cup [1, \infty)$$



➤ $\operatorname{cosec}^{-1} x$ is a decreasing function in $(-\infty, -1]$. It also decreases in $[1, \infty)$.

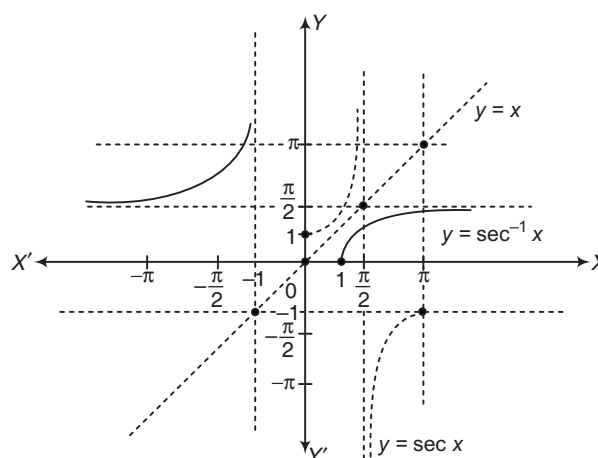
- (iii) $\cos \theta = x \Rightarrow \theta = \cos^{-1} x$, where $\theta \in [0, \pi]$ and $x \in [-1, 1]$



➤ $\cos^{-1} x$ is a decreasing function in $[-1, 1]$.

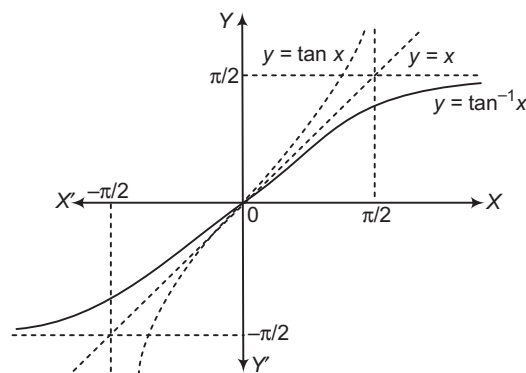
- (iv) $\sec \theta = x \Rightarrow \theta = \sec^{-1} x$, where

$$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right] \text{ and } x \in (-\infty, -1] \cup [1, \infty)$$



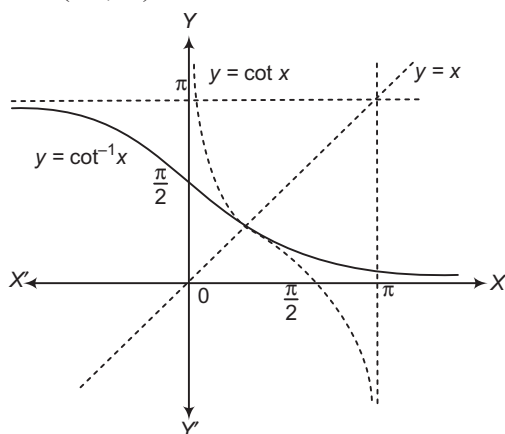
➤ $\sec^{-1} x$ is an increasing function in $(-\infty, -1]$. It also increases in $[1, \infty)$.

- (v) $\tan \theta = x \Rightarrow \theta = \tan^{-1} x$, where $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $x \in (-\infty, \infty)$



➤ $\tan^{-1} x$ is an increasing function in $\mathbb{R}$.

(vi) $\cot \theta = x \Rightarrow \theta = \cot^{-1} x$, where $\theta \in (0, \pi)$ and $x \in (-\infty, \infty)$



$\cot^{-1} x$ is a decreasing function in R .

Domain and Range of Inverse Trigonometric Functions

Suppose we defined a sine inverse trigonometric function as $\sin^{-1} \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \rightarrow [-1, 1]$. Here, $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ and $[-1, 1]$ are called domain and range of sine inverse trigonometric function respectively.

The domain and range of some inverse trigonometric functions are as follow:

Functions	Domain	Range
$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2} \right] - \{0\}$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \{\pi/2\}$
$\cot^{-1} x$	R	$(0, \pi)$

Example 3. Let $f(x) = \sec^{-1} x + \tan^{-1} x$.

Then, $f(x)$ is real for

- (a) $x \in [-1, 1]$ (b) $x \in R$
(c) $x \in (-\infty - 1] \cup [1, \infty)$ (d) None of these

Sol. (c) Since, $\sec^{-1} x$ is real, when $x \leq -1$ or $x \geq 1$ and $\tan^{-1} x$ exists for all $x \in R$.

$$\therefore x \in (-\infty, -1] \cup [1, \infty)$$

Example 4. If

$\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$, then $xy + yz + zx$ is equal to

- (a) -3 (b) 0
(c) 3 (d) -1

Sol. (c) We know that, $0 \leq \cos^{-1} x \leq \pi$

$$\therefore \cos^{-1} x + \cos^{-1} y + \cos^{-1} z = 3\pi$$

$$\therefore \cos^{-1} x = \cos^{-1} y = \cos^{-1} z = \pi$$

$$\Rightarrow x = y = z = -1$$

$$\therefore xy + yz + zx = 3$$

Properties of Inverse Trigonometric Functions

The following properties of inverse trigonometric functions are given below:

Property I

- $\sin^{-1}(\sin \theta) = \theta, \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$
- $\cos^{-1}(\cos \theta) = \theta, \forall \theta \in [0, \pi]$
- $\tan^{-1}(\tan \theta) = \theta, \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
- $\operatorname{cosec}^{-1}(\operatorname{cosec} \theta) = \theta, \forall \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \theta \neq 0$
- $\sec^{-1}(\sec \theta) = \theta, \forall \theta \in [0, \pi], \theta \neq \frac{\pi}{2}$
- $\cot^{-1}(\cot \theta) = \theta, \forall \theta \in (0, \pi)$

Example 5. The value of $\cos^{-1} \left(\cos \frac{7\pi}{6} \right)$ is

- (a) $\frac{\pi}{6}$ (b) $-\frac{\pi}{6}$
(c) $\frac{7\pi}{6}$ (d) $\frac{5\pi}{6}$

Sol. (d) Consider

$$\begin{aligned} \cos^{-1} \left(\cos \left(\frac{7\pi}{6} \right) \right) &= \cos^{-1} \left(\cos \left(2\pi - \frac{5\pi}{6} \right) \right) \\ &= \cos^{-1} \left(\cos \frac{5\pi}{6} \right) \\ &= \frac{5\pi}{6} \quad \left[\because \frac{5\pi}{6} \in [0, \pi] \right] \end{aligned}$$

Example 6. The value of $\tan^{-1} \left[\tan \left(\frac{3\pi}{4} \right) \right]$ is

- (a) $\frac{\pi}{3}$ (b) $-\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) $-\frac{\pi}{4}$

$$\begin{aligned} \text{Sol. (d)} \quad \tan^{-1} \left[\tan \left(\frac{3\pi}{4} \right) \right] &= \tan^{-1} \left[\tan \left(\pi - \frac{\pi}{4} \right) \right] \\ &= \tan^{-1} \left(-\tan \frac{\pi}{4} \right) = \tan^{-1} \left(-\frac{\pi}{4} \right) \\ &= -\frac{\pi}{4} \quad \left[\because \tan(-x) = -\tan x \right] \\ &\quad \left[\because -\frac{\pi}{4} \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right] \end{aligned}$$

➤ **Example 7.** The principal value of

$$\sin^{-1} \left[\sin \left(\frac{4\pi}{3} \right) \right] \text{ is}$$

(a) $-\frac{2\pi}{3}$

(b) $\frac{2\pi}{3}$

(c) $\frac{4\pi}{3}$

(d) None of these

Sol. (d) $\sin^{-1} \left[\sin \left(\frac{4\pi}{3} \right) \right] = \sin^{-1} \left[\sin \left(\pi + \frac{\pi}{3} \right) \right]$
 $= \sin^{-1} \left(-\sin \left(\frac{\pi}{3} \right) \right) = \sin^{-1} \left(\sin \left(-\frac{\pi}{3} \right) \right) = -\frac{\pi}{3}$
 $\left[\because -\frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right]$

➤ **Example 8.** The principal value of

$$\sin^{-1} \left\{ \sin \frac{5\pi}{6} \right\} \text{ is}$$

(a) $\frac{\pi}{6}$

(b) $\frac{5\pi}{6}$

(c) $\frac{7\pi}{6}$

(d) None of these

Sol. (a) $\sin \frac{5\pi}{6}$ is to be written as $\sin \alpha$
 where, $-\frac{\pi}{2} \leq \alpha \leq \frac{\pi}{2}$
 $\therefore \sin^{-1} \left(\sin \frac{5\pi}{6} \right) = \sin^{-1} \left\{ \sin \left(\pi - \frac{\pi}{6} \right) \right\}$
 $= \sin^{-1} \left(\sin \frac{\pi}{6} \right) = \frac{\pi}{6} \left[\because \frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right]$

➤ **Example 9.** The value of $\sin^{-1} \left(\sin \frac{5\pi}{3} \right)$ is

(a) $\frac{4\pi}{3}$

(b) $-\frac{\pi}{3}$

(c) $-\frac{5\pi}{3}$

(d) $\frac{5\pi}{3}$

Sol. (b) $\sin^{-1} \left(\sin \frac{5\pi}{3} \right) = \sin^{-1} \left[\sin \left(2\pi - \frac{\pi}{3} \right) \right]$
 $= \sin^{-1} \left(-\sin \frac{\pi}{3} \right) = \sin^{-1} \left[\sin \left(-\frac{\pi}{3} \right) \right] = -\frac{\pi}{3}$

➤ **Example 10.** Evaluate $\tan^{-1} \{ \tan (-6) \}$.

Sol. We know that,

$$\tan^{-1}(\tan \theta) = \theta, \text{ if } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$

Here, $\theta = -6$ radian, which does not lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$.

We find that,

$$2\pi - 6 \text{ lie between } -\frac{\pi}{2} \text{ and } \frac{\pi}{2} \text{ such that}$$

$$\tan(2\pi - 6) = -\tan 6 = \tan(-6)$$

$$\therefore \tan^{-1} \{ \tan(-6) \} = \tan^{-1} \{ \tan(2\pi - 6) \}$$

$$= (2\pi - 6)$$

➤ **Example 11.** The value of

$$\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right] \text{ is}$$

(a) $-\frac{\pi}{4}$

(b) $\frac{\pi}{4}$

(c) $\frac{\pi}{2}$

(d) None of these

Sol. (b) $\tan^{-1} \left[2 \cos \left(2 \sin^{-1} \frac{1}{2} \right) \right]$
 $= \tan^{-1} \left[2 \cos \left\{ 2 \sin^{-1} \left(\sin \frac{\pi}{6} \right) \right\} \right] \left[\because \sin \frac{\pi}{6} = \frac{1}{2} \right]$
 $= \tan^{-1} \left[2 \cos \left(2 \times \frac{\pi}{6} \right) \right] = \tan^{-1} \left[2 \cos \frac{\pi}{3} \right]$
 $= \tan^{-1} \left[2 \times \frac{1}{2} \right] = \tan^{-1}(1) \left[\because \cot \frac{\pi}{3} = \frac{1}{2} \right]$
 $= \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4} \left[\because \tan \frac{\pi}{4} = 1 \right]$

Property II

i.	$\sin(\sin^{-1} x) = x, \forall x \in [-1, 1]$
ii.	$\cos(\cos^{-1} x) = x, \forall x \in [-1, 1]$
iii.	$\tan(\tan^{-1} x) = x, \forall x \in R$
iv.	$\operatorname{cosec}(\operatorname{cosec}^{-1} x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$
v.	$\sec(\sec^{-1} x) = x, \forall x \in (-\infty, -1] \cup [1, \infty)$
vi.	$\cot(\cot^{-1} x) = x, \forall x \in R$

➤ **Example 12.** If

$$\tan(\sec^{-1} x) = \sin \left(\cos^{-1} \frac{1}{\sqrt{5}} \right), \text{ then } x \text{ is equal to}$$

(a) $\pm \frac{3}{\sqrt{5}}$

(b) $\pm \frac{\sqrt{5}}{3}$

(c) $\pm \frac{\sqrt{3}}{5}$

(d) None of the above

Sol. (a) We have,

$$\tan(\sec^{-1} x) = \sin \left(\cos^{-1} \frac{1}{\sqrt{5}} \right)$$

$$\Rightarrow \sqrt{\sec^2(\sec^{-1} x) - 1} = \sqrt{1 - \cos^2 \left(\cos^{-1} \frac{1}{\sqrt{5}} \right)}$$

$$\Rightarrow \sqrt{x^2 - 1} = \sqrt{1 - \frac{1}{5}}$$

$$\Rightarrow x^2 - 1 = \frac{4}{5}$$

$$\Rightarrow x = \pm \frac{3}{\sqrt{5}}$$

➤ **Example 13.** $\sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$ is equal to

- (a) 5 (b) 10
(c) 15 (d) 20

Sol. (c) We have,

$$\begin{aligned} \sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3) &= \{1 + \tan^2(\tan^{-1} 2)\} + \{1 + \cot^2(\cot^{-1} 3)\} \\ &= [1 + \{\tan(\tan^{-1} 2)\}^2] + [1 + \{\cot(\cot^{-1} 3)\}^2] \\ &= (1 + 2^2) + (1 + 3^2) = 15 \end{aligned}$$

Property III

- i. $\sin^{-1}(-x) = -\sin^{-1} x, \forall x \in [-1, 1]$
- ii. $\cos^{-1}(-x) = \pi - \cos^{-1} x, \forall x \in [-1, 1]$
- iii. $\tan^{-1}(-x) = -\tan^{-1} x, \forall x \in R$
- iv. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$
- v. $\sec^{-1}(-x) = \pi - \sec^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$
- vi. $\cot^{-1}(-x) = \pi - \cot^{-1} x, \forall x \in R$

➤ **Example 14.** The principal value of

$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \text{ is}$$

- (a) $-\frac{2\pi}{3}$ (b) $-\frac{\pi}{3}$
(c) $\frac{4\pi}{3}$ (d) $\frac{5\pi}{3}$

Sol. (b) Clearly, $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$
 $= -\sin^{-1}\left(\sin \frac{\pi}{3}\right)$
 $= -\frac{\pi}{3} \quad \left[\because \frac{\pi}{3} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]\right]$

➤ **Example 15.** The principal value of $\cot^{-1}(-1)$ is

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4}$
(c) $\frac{3\pi}{4}$ (d) None of these

Sol. (c) Clearly, $\cot^{-1}(-1) = \pi - \cot^{-1}(1)$
 $= \pi - \cot^{-1}\left(\cot \frac{\pi}{4}\right)$
 $= \pi - \frac{\pi}{4} \quad \left[\because \frac{\pi}{4} \in (0, \pi)\right]$
 $= \frac{3\pi}{4}$

➤ **Example 16.** The value of

$$\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] \text{ is}$$

- (a) 1 (b) -1 (c) 0 (d) $\frac{1}{2}$

Sol. (a) We have, $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right] = \sin\left[\frac{\pi}{3} + \sin^{-1}\frac{1}{2}\right]$
 $[\because \sin^{-1}(-x) = -\sin^{-1} x, x \in [-1, 1]]$
 $= \sin\left(\frac{\pi}{3} + \frac{\pi}{6}\right) = \sin \frac{\pi}{2} = 1$

➤ **Example 17.** The value of

$$\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right] \text{ is}$$

- (a) 1 (b) -1
(c) 0 (d) None of these

Sol. (b) $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$
 $= \cos\left[\pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$
 $[\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$
 $= \cos\left(\pi - \frac{\pi}{6} + \frac{\pi}{6}\right) = \cos \pi = -1$

➤ **Example 18.** The value of

$$\sin\left[\tan^{-1}(-\sqrt{3}) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right] \text{ is}$$

- (a) 1 (b) -1
(c) 0 (d) None of these

Sol. (a) $\sin\left[\tan^{-1}(-\sqrt{3}) + \cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
 $= \sin\left[-\tan^{-1}\sqrt{3} + \pi - \cos^{-1}\frac{\sqrt{3}}{2}\right]$
 $[\because \tan^{-1}(-x) = -\tan^{-1} x \text{ and } \cos^{-1}(-x) = \pi - \cos^{-1} x]$
 $= \sin\left[-\frac{\pi}{3} + \pi - \frac{\pi}{6}\right] = \sin \frac{\pi}{2} = 1$

➤ **Example 19.** The value of $\sin\left[\arccos\left(-\frac{1}{2}\right)\right]$

is

- (a) $\frac{1}{\sqrt{2}}$ (b) 1
(c) $\frac{\sqrt{3}}{2}$ (d) None of these

Sol. (c) $\sin\left[\arccos\left(-\frac{1}{2}\right)\right] = \sin\left(\pi - \cos^{-1}\frac{1}{2}\right)$
 $[\because \cos^{-1}(-x) = \pi - \cos^{-1} x]$
 $= \sin\left(\pi - \frac{\pi}{3}\right) = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$

➤ **Example 20.** The value of

$$\cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \text{ is}$$

- (a) $\frac{\pi}{3}$ (b) 0
(c) $\frac{2\pi}{3}$ (d) None of these

Sol. (a) $\cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$

Property IV

- i. $\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$
ii. $\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1} x, \forall x \in (-\infty, -1] \cup [1, \infty)$
iii. $\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1} x, & \text{if } x > 0 \\ -\pi + \cot^{-1} x, & \text{if } x < 0 \end{cases}$

Property V

- i. $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \forall x \in [-1, 1]$
ii. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}, \forall x \in \mathbb{R}$
iii. $\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}, \forall x \in (-\infty, -1] \cup [1, \infty)$

➤ **Example 21.** If $\cos^{-1} x - \sin^{-1} x = 0$, then x is equal to

- (a) $\pm \frac{1}{\sqrt{2}}$ (b) 1 (c) $\pm \frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

Sol. (d) $\cos^{-1} x - \sin^{-1} x = 0$

$$\begin{aligned} \Rightarrow \frac{\pi}{2} - \sin^{-1} x - \sin^{-1} x &= 0 \\ \Rightarrow \frac{\pi}{2} - 2\sin^{-1} x &= 0 \\ \Rightarrow \sin^{-1} x &= \frac{\pi}{4} \\ \therefore x &= \frac{1}{\sqrt{2}} \end{aligned}$$

➤ **Example 22.** If $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$, then

$\cos^{-1} x + \cos^{-1} y$ is equal to

- (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) π

Sol. (b) We have, $\sin^{-1} x + \sin^{-1} y = \frac{2\pi}{3}$

$$\begin{aligned} \Rightarrow \frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \cos^{-1} y &= \frac{2\pi}{3} \\ \therefore \cos^{-1} x + \cos^{-1} y &= \pi - \frac{2\pi}{3} = \frac{\pi}{3} \end{aligned}$$

➤ **Example 23.** If $4\sin^{-1} x + \cos^{-1} x = \pi$, then x equals

- (a) $\frac{1}{2}$
(b) $\frac{\sqrt{3}}{2}$
(c) $-\frac{1}{2}$
(d) None of the above

Sol. (a) We have, $4\sin^{-1} x + \cos^{-1} x = \pi$

$$\begin{aligned} \Rightarrow 4\sin^{-1} x + \frac{\pi}{2} - \sin^{-1} x &= \pi \\ \Rightarrow 3\sin^{-1} x &= \frac{\pi}{2} \\ \Rightarrow x &= \sin\left(\frac{\pi}{6}\right) = \frac{1}{2} \end{aligned}$$

➤ **Example 24.** $\tan(\cot^{-1} x)$ is equal to

- (a) $\frac{\pi}{2} - x$
(b) $\cot(\tan^{-1} x)$
(c) $\tan x$
(d) None of the above

Sol. (b) Consider

$$\tan(\cot^{-1} x) = \tan\left(\frac{\pi}{2} - \tan^{-1} x\right) = \cot(\tan^{-1} x)$$

➤ **Example 25.** If $\tan^{-1} x + 2\cot^{-1} x = \frac{2\pi}{3}$, then

x is equal to

- (a) 3 (b) $\sqrt{3}$
(c) $\sqrt{2}$ (d) $\frac{\sqrt{3}-1}{\sqrt{3}+1}$

Sol. (b) We have,

$$\begin{aligned} \tan^{-1} x + 2\cot^{-1} x &= \frac{2\pi}{3} \\ \Rightarrow \tan^{-1} x + 2\left(\frac{\pi}{2} - \tan^{-1} x\right) &= \frac{2\pi}{3} \\ \Rightarrow \tan^{-1} x &= \frac{\pi}{3} \\ \Rightarrow x &= \sqrt{3} \end{aligned}$$

➤ **Example 26.** $\sin^{-1}\{\cos(\sin^{-1} x)\} + \cos^{-1}\{\sin(\cos^{-1} x)\}$ is equal to

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) 0

Sol. (b) Consider

$$\begin{aligned} \sin^{-1}\{\cos(\sin^{-1} x)\} + \cos^{-1}\{\sin(\cos^{-1} x)\} \\ = \sin^{-1}\left\{\cos\left(\frac{\pi}{2} - \cos^{-1} x\right)\right\} + \cos^{-1}\{\sin(\cos^{-1} x)\} \\ = \sin^{-1}\{\sin(\cos^{-1} x)\} + \cos^{-1}\{\sin(\cos^{-1} x)\} = \frac{\pi}{2} \end{aligned}$$

➤ **Example 27.** If $(\sin^{-1} x)^2 + (\cos^{-1} x)^2 = \frac{5\pi^2}{8}$,

then x is equal to

- (a) ± 1 (b) $\pm \frac{1}{2}$
(c) $-\frac{1}{\sqrt{2}}$ (d) None of these

Sol. (c) We have,

$$\begin{aligned} (\sin^{-1} x)^2 + (\cos^{-1} x)^2 &= \frac{5\pi^2}{8} \\ \Rightarrow (\sin^{-1} x)^2 + \left(\frac{\pi}{2} - \sin^{-1} x\right)^2 &= \frac{5\pi^2}{8} \\ \Rightarrow 2(\sin^{-1} x)^2 - \pi \sin^{-1} x - \frac{3\pi^2}{8} &= 0 \\ \Rightarrow \sin^{-1} x &= \frac{\pi \pm 2\pi}{4} \\ \Rightarrow \sin^{-1} x &= -\frac{\pi}{4} \\ \Rightarrow x &= -\frac{1}{\sqrt{2}} \end{aligned}$$

Property VI

i. $\tan^{-1} x + \tan^{-1} y$

$$= \begin{cases} \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } xy < 1 \\ \pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x > 0, y > 0 \text{ and } xy > 1 \\ -\pi + \tan^{-1} \left(\frac{x+y}{1-xy} \right), & \text{if } x < 0, y < 0 \text{ and } xy > 1 \end{cases}$$

ii. $\tan^{-1} x - \tan^{-1} y$

$$= \begin{cases} \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } xy > -1 \\ \pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } x > 0, y < 0 \text{ and } xy < -1 \\ -\pi + \tan^{-1} \left(\frac{x-y}{1+xy} \right), & \text{if } x < 0, y > 0 \text{ and } xy < -1 \end{cases}$$

Remark

- (i) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z$
 $= \tan^{-1} \left[\frac{x+y+z-xyz}{1-xy-yz-zx} \right]$
 if $x > 0, y > 0, z > 0$ and $(xy+yz+zx) < 1$
 (ii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then
 $x+y+z=xyz$
 (iii) If $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \frac{\pi}{2}$, then
 $xy+yz+zx=1$
 (iv) $\tan^{-1} 1 + \tan^{-1} 2 + \tan^{-1} 3 = \pi$

(v) $\tan^{-1} 1 + \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \frac{\pi}{2}$

(vi) If $x_1, x_2, x_3, \dots, x_n \in R$, then
 $\tan^{-1} x_1 + \tan^{-1} x_2 + \dots + \tan^{-1} x_n$
 $= \tan^{-1} \left(\frac{s_1 - s_3 + s_5 - \dots}{1 - s_2 + s_4 - s_6 + \dots} \right)$

where, $s_1 = x_1 + x_2 + \dots + x_n = \sum x_1$

$s_2 = x_1 x_2 + x_1 x_3 + \dots + x_{n-1} x_n = \sum x_1 x_2$

$s_3 = \sum x_1 x_2 x_3, \dots$ and so on.

➤ **Example 28.** $\cot^{-1} 9 + \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4}$ is equal

to

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) None of these

Sol. (b) $\cot^{-1} 9 + \operatorname{cosec}^{-1} \frac{\sqrt{41}}{4} = \cot^{-1} 9 + \cot^{-1} \sqrt{\frac{41}{16} - 1}$

$[\because \operatorname{cosec}^{-1} x = \cot^{-1} \sqrt{x^2 - 1}]$

$= \cot^{-1} 9 + \cot^{-1} \frac{5}{4} = \tan^{-1} \frac{1}{9} + \tan^{-1} \frac{4}{5}$

$= \tan^{-1} \left(\frac{\frac{1}{9} + \frac{4}{5}}{1 - \frac{1}{9} \cdot \frac{4}{5}} \right) = \tan^{-1} \left(\frac{41}{41} \right) = \tan^{-1} 1 = \frac{\pi}{4}$

➤ **Example 29.** $\tan^{-1} \left(\frac{x}{y} \right) - \tan^{-1} \left(\frac{x-y}{x+y} \right)$ is

equal to

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) None of these

Sol. (c) $\tan^{-1} \left(\frac{x}{y} \right) - \tan^{-1} \left(\frac{x-y}{x+y} \right)$

$= \tan^{-1} \left(\frac{\frac{x}{y} - \frac{x-y}{x+y}}{1 + \frac{x}{y} \cdot \frac{x-y}{x+y}} \right) = \tan^{-1} 1 = \frac{\pi}{4}$

➤ **Example 30.** If $\tan^{-1} 3 + \tan^{-1} x = \tan^{-1} 8$, then x equals

- (a) 5 (b) $\frac{1}{5}$ (c) $\frac{5}{14}$ (d) $\frac{14}{5}$

Sol. (b) We have,

$\tan^{-1} 3 + \tan^{-1} x = \tan^{-1} 8$

$\Rightarrow \tan^{-1} x = \tan^{-1} 8 - \tan^{-1} 3$

$\Rightarrow \tan^{-1} x = \tan^{-1} \left(\frac{8-3}{1+24} \right)$

$\Rightarrow x = \frac{1}{5}$

➤ **Example 31.** If $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$, then

the value of x is

- (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) 2

Sol. (b) We have,

$$\begin{aligned}\tan^{-1}\left(\frac{1-x}{1+x}\right) &= \frac{1}{2}\tan^{-1}x \\ \Rightarrow \tan^{-1}1 - \tan^{-1}x &= \frac{1}{2}\tan^{-1}x \\ \Rightarrow \frac{\pi}{4} &= \frac{3}{2}\tan^{-1}x \\ \Rightarrow \tan^{-1}x &= \frac{\pi}{6} \\ \Rightarrow x &= \frac{1}{\sqrt{3}}\end{aligned}$$

➤ **Example 32.** Show that

$$(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3) = \pi.$$

Sol. $\tan^{-1}2 + \tan^{-1}3 = \pi + \tan^{-1}\left(\frac{2+3}{1-2 \times 3}\right)$
[as (2) (3) > 1]
 $= \pi + \tan^{-1}(-1)$

$$\therefore \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \tan^{-1}1 + \pi - \tan^{-1}1$$

$$\Rightarrow \tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$$

➤ **Example 33.** The value of $\sum_{n=1}^3 \tan^{-1} \frac{1}{n}$ is

- (a) 0 (b) π
 (c) $\frac{\pi}{2}$ (d) None of these

Sol. (c) We have,

$$\begin{aligned}\sum_{n=1}^3 \tan^{-1} \frac{1}{n} &= \tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} \\ &= \frac{\pi}{4} + \tan^{-1}\left(\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \times \frac{1}{3}}\right) \\ &= \frac{\pi}{4} + \tan^{-1}1 = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}\end{aligned}$$

➤ **Example 34.** If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$ or $\frac{\pi}{2}$, then

- (a) $x + y + z = 3xyz$ (b) $x + y + z = 2xyz$
 (c) $xy + zx + yz = 1$ (d) None of these

Sol. (c) $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\frac{x+y}{1-xy} + \tan^{-1}z$

$$= \tan^{-1}\left(\frac{\frac{x+y}{1-xy} + z}{1 - \frac{x+y}{1-xy} \cdot z}\right) = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-xz-yz}\right)$$

$$\therefore \tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi \quad \text{or} \quad \frac{\pi}{2} \quad [\text{given}]$$

$$\therefore \frac{x+y+z-xyz}{1-(xy+xz+yz)} = \tan \pi$$

$$\text{or} \quad = \tan \frac{\pi}{2}$$

Hence, $x+y+z-xyz=0$
 or $(xy+xz+yz)=1$
 i.e. $x+y+z=xyz$
 or $xy+xz+yz=1$

Property VII

$$\begin{aligned}i. \quad & \sin^{-1}x + \sin^{-1}y \\ &= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \leq x, y \leq 1 \\ & \text{and } x^2 + y^2 \leq 1 \text{ or if } xy < 0 \\ & \text{and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } 0 < x, y \leq 1 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} + y\sqrt{1-x^2}\}, & \text{if } -1 \leq x, y < 0 \text{ and } x^2 + y^2 > 1 \end{cases} \\ ii. \quad & \sin^{-1}x - \sin^{-1}y \\ &= \begin{cases} \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \leq x, y \leq 1 \\ & \text{and } x^2 + y^2 \leq 1 \text{ or if } xy > 0 \\ & \text{and } x^2 + y^2 > 1 \\ \pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } 0 < x \leq 1, \\ & -1 \leq y \leq 0 \text{ and } x^2 + y^2 > 1 \\ -\pi - \sin^{-1}\{x\sqrt{1-y^2} - y\sqrt{1-x^2}\}, & \text{if } -1 \leq x < 0, 0 < y \leq 1 \text{ and } x^2 + y^2 > 1 \end{cases}\end{aligned}$$

➤ **Example 35.** If $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{-1}x$,

then the value of x is

- (a) 0 (b) $\frac{(\sqrt{5}-4\sqrt{2})}{9}$
 (c) $\frac{(\sqrt{5}+4\sqrt{2})}{9}$ (d) $\frac{\pi}{2}$

Sol. (c) We have, $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{-1}x$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right) = \sin^{-1}x$$

$$\Rightarrow \sin^{-1}\left(\frac{1}{3} \cdot \frac{\sqrt{5}}{3} + \frac{2}{3} \cdot \frac{\sqrt{8}}{3}\right) = \sin^{-1}x$$

$$\Rightarrow \sin^{-1}\left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right) = \sin^{-1}x$$

$$\therefore x = \frac{(\sqrt{5}+4\sqrt{2})}{9}$$

Property VIII

$$\begin{aligned}
 i. \quad & \cos^{-1} x + \cos^{-1} y \\
 &= \begin{cases} \cos^{-1} \{xy - \sqrt{1-x^2} \sqrt{1-y^2}\}, & \text{if } -1 \leq x, y \leq 1 \\ & \text{and } x+y \geq 0 \\ 2\pi - \cos^{-1} \{xy - \sqrt{1-x^2} \sqrt{1-y^2}\}, & \text{if } -1 \leq x, y \leq 1 \\ & \text{and } x+y \leq 0 \end{cases} \\
 ii. \quad & \cos^{-1} x - \cos^{-1} y \\
 &= \begin{cases} \cos^{-1} \{xy + \sqrt{1-x^2} \sqrt{1-y^2}\}, & \text{if } -1 \leq x, y \leq 1 \\ & \text{and } x \geq y \\ -\cos^{-1} \{xy + \sqrt{1-x^2} \sqrt{1-y^2}\}, & \text{if } -1 \leq y \leq 0, \\ & 0 < x \leq 1 \\ & \text{and } x \geq y \end{cases}
 \end{aligned}$$

➤ **Example 36.** If $\cos^{-1} \left(\frac{p}{a} \right) + \cos^{-1} \left(\frac{q}{b} \right) = \alpha$,

then $\frac{p^2}{a^2} + k \cos \alpha + \frac{q^2}{b^2} = \sin^2 \alpha$, where k is equal to

- (a) $\frac{2pq}{ab}$ (b) $-\frac{2pq}{ab}$ (c) $\frac{pq}{ab}$ (d) $-\frac{pq}{ab}$

Sol. (b) $\cos^{-1} \left[\frac{p}{a} \cdot \frac{q}{b} - \sqrt{1 - \frac{p^2}{a^2}} \sqrt{1 - \frac{q^2}{b^2}} \right] = \alpha$

$$\Rightarrow \frac{pq}{ab} - \sqrt{1 - \frac{p^2}{a^2}} \sqrt{1 - \frac{q^2}{b^2}} = \cos \alpha$$

$$\Rightarrow \left(\frac{pq}{ab} - \cos \alpha \right)^2 = 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2 q^2}{a^2 b^2}$$

$$\Rightarrow \frac{p^2 q^2}{a^2 b^2} + \cos^2 \alpha - \frac{2pq}{ab} \cos \alpha = 1 - \frac{p^2}{a^2} - \frac{q^2}{b^2} + \frac{p^2 q^2}{a^2 b^2}$$

$$\Rightarrow \frac{p^2}{a^2} - \frac{2pq}{ab} \cos \alpha + \frac{q^2}{b^2} = 1 - \cos^2 \alpha = \sin^2 \alpha$$

$$\therefore k = -\frac{2pq}{ab}$$

➤ **Example 37.** If $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \theta$, then

$9x^2 - 12xy \cos \theta + 4y^2$ is equal to

- (a) 36 (b) $-36 \sin^2 \theta$
(c) $36 \sin^2 \theta$ (d) $36 \cos^2 \theta$

Sol. (c) We have, $\cos^{-1} \frac{x}{2} + \cos^{-1} \frac{y}{3} = \theta$

$$\Rightarrow \cos^{-1} \left\{ \frac{xy}{6} - \sqrt{1 - \frac{x^2}{4}} \sqrt{1 - \frac{y^2}{9}} \right\} = \theta$$

$$\Rightarrow xy - \sqrt{4 - x^2} \sqrt{9 - y^2} = 6 \cos \theta$$

$$\Rightarrow (xy - 6 \cos \theta)^2 = (4 - x^2)(9 - y^2)$$

$$\Rightarrow x^2 y^2 - 12xy \cos \theta + 36 \cos^2 \theta = 36 - 4y^2 - 9x^2 + x^2 y^2$$

$$\Rightarrow 9x^2 + 4y^2 - 12xy \cos \theta = 36 \sin^2 \theta$$

Property IX

$$\begin{aligned}
 i. \quad & \sin^{-1} x = \cos^{-1} \sqrt{1-x^2} = \tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) \\
 &= \cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \sec^{-1} \left(\frac{1}{\sqrt{1-x^2}} \right) \\
 &= \operatorname{cosec}^{-1} \left(\frac{1}{x} \right), x \in (0, 1) \\
 ii. \quad & \cos^{-1} x = \sin^{-1} \sqrt{1-x^2} = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) \\
 &= \cot^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) = \sec^{-1} \left(\frac{1}{x} \right) \\
 &= \operatorname{cosec}^{-1} \left(\frac{1}{\sqrt{1-x^2}} \right), x \in (0, 1) \\
 iii. \quad & \tan^{-1} x = \sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) = \cos^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) \\
 &= \cot^{-1} \left(\frac{1}{x} \right) = \sec^{-1} (\sqrt{1+x^2}) \\
 &= \operatorname{cosec}^{-1} \left(\frac{\sqrt{1+x^2}}{x} \right), x \in (0, \infty)
 \end{aligned}$$

➤ **Example 38.** $\tan(\cos^{-1} x)$, where

$|x| \leq 1 - \{0\}$, is equal to

- (a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{\sqrt{1+x^2}}$ (c) $\frac{\sqrt{1+x^2}}{x}$ (d) $\frac{x}{\sqrt{1-x^2}}$

Sol. (a) Let $\cos^{-1} x = \theta \Rightarrow x = \cos \theta$

$$\begin{aligned}
 \therefore \tan(\cos^{-1} x) &= \tan \theta = \sqrt{\sec^2 \theta - 1} \\
 &= \sqrt{\frac{1}{x^2} - 1} = \frac{\sqrt{1-x^2}}{x}
 \end{aligned}$$

➤ **Example 39.** The value of $\sin \left(\cos^{-1} \frac{3}{5} \right)$ is

- (a) $\frac{3}{5}$ (b) $\frac{4}{5}$
(c) $\frac{1}{5}$ (d) None of these

Sol. (b) $\sin \left(\cos^{-1} \frac{3}{5} \right) = \sin \left(\sin^{-1} \sqrt{1 - \frac{9}{25}} \right)$

$$\begin{aligned}
 & [\because \cos^{-1} x = \sin^{-1} \sqrt{1-x^2}] \\
 &= \sin \left(\sin^{-1} \frac{4}{5} \right) = \frac{4}{5}
 \end{aligned}$$

➤ **Example 40.** The value of

$\sin [\cot^{-1} \{ \tan (\cos^{-1} x) \}]$ is

- (a) $\frac{\sqrt{1-x^2}}{x}$ (b) $\frac{x}{\sqrt{1-x^2}}$
 (c) x (d) None of these

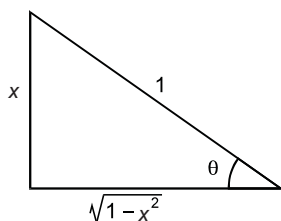
Sol. (c) Let $y = \sin [\cot^{-1} \{ \tan (\cos^{-1} x) \}]$

$$= \sin \left[\cot^{-1} \left\{ \tan \left(\tan^{-1} \frac{\sqrt{1-x^2}}{x} \right) \right\} \right]$$

$$\left[\because \cos^{-1} x = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) \right]$$

$$= \sin \left[\cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) \right] \quad \dots(i)$$

Let $\cot^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right) = \theta$



Then, $\cot \theta = \frac{\sqrt{1-x^2}}{x}$

$\Rightarrow \sin \theta = x$

From Eq. (i), $y = \sin \theta = x$

➤ **Example 41.**

$2 \tan^{-1} (\operatorname{cosec} \tan^{-1} x - \tan \cot^{-1} x)$ is equal to

- (a) $\cot^{-1} x$
 (b) $\cot^{-1} \frac{1}{x}$
 (c) $\tan^{-1} x$
 (d) None of the above

Sol. (c) The given expression

$$= 2 \tan^{-1} [\operatorname{cosec} \tan^{-1} x - \tan \cot^{-1} x]$$

$$= 2 \tan^{-1} \left[\operatorname{cosec} \left\{ \operatorname{cosec}^{-1} \left(\frac{\sqrt{1+x^2}}{x} \right) \right\} - \tan \left\{ \tan^{-1} \left(\frac{1}{x} \right) \right\} \right]$$

$$= 2 \tan^{-1} \left[\frac{\sqrt{1+x^2}}{x} - \frac{1}{x} \right] = 2 \tan^{-1} \left[\frac{\sqrt{1+x^2}-1}{x} \right]$$

$$= 2 \tan^{-1} \left[\frac{\sec \theta - 1}{\tan \theta} \right] \quad [\text{putting } x = \tan \theta]$$

$$= 2 \tan^{-1} \left[\frac{1 - \cos \theta}{\sin \theta} \right] = 2 \tan^{-1} \left[\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}} \right]$$

$$= 2 \tan^{-1} \tan \frac{\theta}{2} = 2 \cdot \frac{\theta}{2} = \theta = \tan^{-1} x$$

Property X

i. $2 \sin^{-1} x = \begin{cases} \sin^{-1} (2x \sqrt{1-x^2}), & \text{if } -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}} \\ \pi - \sin^{-1} (2x \sqrt{1-x^2}), & \text{if } \frac{1}{\sqrt{2}} \leq x \leq 1 \\ -\pi - \sin^{-1} (2x \sqrt{1-x^2}), & \text{if } -1 \leq x \leq -\frac{1}{\sqrt{2}} \end{cases}$

ii. $3 \sin^{-1} x = \begin{cases} \sin^{-1} (3x - 4x^3), & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ \pi - \sin^{-1} (3x - 4x^3), & \text{if } \frac{1}{2} < x \leq 1 \\ -\pi - \sin^{-1} (3x - 4x^3), & \text{if } -1 \leq x < -\frac{1}{2} \end{cases}$

Property XI

i. $2 \cos^{-1} x = \begin{cases} \cos^{-1} (2x^2 - 1), & \text{if } 0 \leq x \leq 1 \\ 2\pi - \cos^{-1} (2x^2 - 1), & \text{if } -1 \leq x \leq 0 \end{cases}$

ii. $3 \cos^{-1} x = \begin{cases} \cos^{-1} (4x^3 - 3x), & \text{if } \frac{1}{2} \leq x \leq 1 \\ 2\pi - \cos^{-1} (4x^3 - 3x), & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 2\pi + \cos^{-1} (4x^3 - 3x), & \text{if } -1 \leq x \leq -\frac{1}{2} \end{cases}$

Property XII

i. $2 \tan^{-1} x = \begin{cases} \tan^{-1} \left(\frac{2x}{1-x^2} \right), & \text{if } -1 < x \leq 1 \\ \pi + \tan^{-1} \left(\frac{2x}{1-x^2} \right), & \text{if } x > 1 \\ -\pi + \tan^{-1} \left(\frac{2x}{1-x^2} \right), & \text{if } x < -1 \end{cases}$

ii. $3 \tan^{-1} x = \begin{cases} \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right), & \text{if } -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}} \\ \pi + \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right), & \text{if } x > \frac{1}{\sqrt{3}} \\ -\pi + \tan^{-1} \left(\frac{3x-x^3}{1-3x^2} \right), & \text{if } x < -\frac{1}{\sqrt{3}} \end{cases}$

➤ **Example 42.** The value of $\tan \left[2 \tan^{-1} \left(\frac{1}{5} \right) - \frac{\pi}{4} \right]$

is

- (a) $\frac{7}{17}$ (b) $-\frac{7}{17}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

Sol. (b) $\tan \left[2 \tan^{-1} \left(\frac{1}{5} \right) - \frac{\pi}{4} \right]$

$$= \tan \left[\tan^{-1} \left(\frac{2 \cdot \frac{1}{5}}{1 - \left(\frac{1}{5} \right)^2} \right) - \tan^{-1} 1 \right]$$

$$\left[\because 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right) \text{ and } \tan^{-1} 1 = \frac{\pi}{4} \right]$$

$$= \tan \left[\tan^{-1} \frac{5}{12} - \tan^{-1} 1 \right] = \tan \left[\tan^{-1} \left(\frac{\frac{5}{12} - 1}{1 + \frac{5}{12} \cdot 1} \right) \right]$$

$$\left[\because \tan^{-1} x - \tan^{-1} y = \tan^{-1} \left(\frac{x-y}{1+xy} \right) \right]$$

$$= \tan \left[\tan^{-1} \left(-\frac{7}{17} \right) \right] = -\frac{7}{17}$$

➤ **Example 43.** $\sin^{-1} \frac{4}{5} + 2 \tan^{-1} \frac{1}{3}$ is equal to

- (a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) None of these

Sol. (c) $\sin^{-1} \frac{4}{5} + 2 \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{4}{3} + \tan^{-1} \left(\frac{2/3}{1-1/9} \right)$

$$\left[\because \sin^{-1} x = \tan^{-1} \left(\frac{x}{\sqrt{1-x^2}} \right) \right]$$

$$\text{and } 2 \tan^{-1} x = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$$

$$= \tan^{-1} \frac{4}{3} + \tan^{-1} \frac{3}{4} = \tan^{-1} \frac{4}{3} + \cot^{-1} \frac{4}{3} = \frac{\pi}{2}$$

➤ **Example 44.** $\cot \left(\frac{\pi}{4} - 2 \cot^{-1} 3 \right)$ is equal to

- (a) 1 (b) 7 (c) 4 (d) None of these

Sol. (b) $\cot \left(\frac{\pi}{4} - 2 \cot^{-1} 3 \right) = \cot \left[\frac{\pi}{4} - \cot^{-1} \left(\frac{3^2 - 1}{2 \times 3} \right) \right]$

$$= \cot \left(\frac{\pi}{4} - \cot^{-1} \frac{4}{3} \right) = \frac{\cot \frac{\pi}{4} \cdot \frac{4}{3} + 1}{\frac{4}{3} - \cot \frac{\pi}{4}} = \frac{\frac{4}{3} + 1}{\frac{4}{3} - 1} = 7$$

➤ **Example 45.** Solution of the equation $2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$ is

- (a) $x = 2n\pi + \frac{\pi}{4}$ (b) $x = n\pi + \frac{\pi}{4}$
 (c) $x = n\pi + \frac{\pi}{2}$ (d) None of these

Sol. (b) We have, $2 \tan^{-1} (\cos x) = \tan^{-1} (2 \operatorname{cosec} x)$

$$\Rightarrow \tan^{-1} \left(\frac{2 \cos x}{1 - \cos^2 x} \right) = \tan^{-1} (2 \operatorname{cosec} x)$$

$$\Rightarrow 2 \cos x \cdot \operatorname{cosec}^2 x = 2 \operatorname{cosec} x$$

$$\Rightarrow \operatorname{cosec} x (\cot x - 1) = 0$$

$$\Rightarrow \cot x = 1 \quad [\because \operatorname{cosec} x \neq 0]$$

$$\therefore x = n\pi + \frac{\pi}{4}, n \in \mathbb{I}$$

➤ **Example 46.** The simplified form of

$$\tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right); a > 0, \frac{-a}{\sqrt{3}} \leq x \leq \frac{a}{\sqrt{3}} \text{ is}$$

- (a) $3 \tan^{-1} \left(\frac{x}{a} \right)$ (b) $2 \tan^{-1} \left(\frac{x}{a} \right)$
 (c) $3 \tan^{-1} \left(\frac{a}{x} \right)$ (d) None of these

Sol. (a) Let $x = a \tan \theta \Rightarrow \frac{x}{a} = \tan \theta \Rightarrow \theta = \tan^{-1} \left(\frac{x}{a} \right)$

$$\therefore \tan^{-1} \left(\frac{3a^2x - x^3}{a^3 - 3ax^2} \right) = \tan^{-1} \left[\frac{3a^2(a \tan \theta) - (a \tan \theta)^3}{a^3 - 3a(a \tan \theta)^2} \right]$$

$$= \tan^{-1} \left[\frac{a^3(3 \tan \theta - \tan^3 \theta)}{a^3(1 - 3 \tan^2 \theta)} \right]$$

$$= \tan^{-1} \left(\frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right)$$

$$= \tan^{-1} (\tan 3\theta)$$

$$\left[\because \tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} \right]$$

$$= 3\theta = 3 \tan^{-1} \left(\frac{x}{a} \right)$$

Property XIII

$$i. \quad 2 \tan^{-1} x = \begin{cases} \sin^{-1} \left(\frac{2x}{1+x^2} \right), & \text{if } -1 \leq x \leq 1 \\ \pi - \sin^{-1} \left(\frac{2x}{1+x^2} \right), & \text{if } x > 1 \\ -\pi - \sin^{-1} \left(\frac{2x}{1+x^2} \right), & \text{if } x < -1 \end{cases}$$

$$ii. \quad 2 \tan^{-1} x = \begin{cases} \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), & \text{if } 0 \leq x < \infty \\ -\cos^{-1} \left(\frac{1-x^2}{1+x^2} \right), & \text{if } -\infty < x \leq 0 \end{cases}$$

➤ **Example 47.** If

$$\sin^{-1} \left(\frac{2a}{1+a^2} \right) + \sin^{-1} \left(\frac{2b}{1+b^2} \right) = 2 \tan^{-1} x,$$

then

- (a) $x = \frac{a+b}{1-ab}$ (b) $x = \frac{a-b}{1+ab}$
 (c) $x = \frac{b-a}{1+ab}$ (d) None of these

Sol. (a) We have,

$$\sin^{-1} \left(\frac{2a}{1+a^2} \right) + \sin^{-1} \left(\frac{2b}{1+b^2} \right) = 2 \tan^{-1} x$$

$$\Rightarrow 2 \tan^{-1} a + 2 \tan^{-1} b = 2 \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \left(\frac{a+b}{1-ab} \right) = \tan^{-1} x \Rightarrow x = \frac{a+b}{1-ab}$$

➤ **Example 48.**

$$\tan \left\{ \frac{1}{2} \sin^{-1} \left(\frac{2x}{1+x^2} \right) + \frac{1}{2} \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \right\} \text{ is equal}$$

to

$$(a) \infty \quad (b) 1 \quad (c) \frac{2x}{1-x^2} \quad (d) \frac{2x}{1+x^2}$$

Sol. (c) The given expression

$$= \tan \left(\frac{1}{2} \cdot 2 \tan^{-1} x + \frac{1}{2} \cdot 2 \tan^{-1} x \right) = \tan (2 \tan^{-1} x)$$

$$= \tan \left\{ \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right\} = \frac{2x}{1-x^2}$$

➤ **Example 49.** If

$$\cos^{-1} \left(\frac{1-a^2}{1+a^2} \right) - \cos^{-1} \left(\frac{1-b^2}{1+b^2} \right) = 2 \tan^{-1} x, \text{ then}$$

x is equal to

$$(a) \frac{a-b}{1+ab} \quad (b) \frac{b-a}{1+ab} \\ (c) \frac{a+b}{1-ab} \quad (d) \text{None of these}$$

Sol. (a) $\because 2 \tan^{-1} x = \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$

$$\therefore 2 \tan^{-1} a - 2 \tan^{-1} b = 2 \tan^{-1} x$$

$$\Rightarrow \tan^{-1} a - \tan^{-1} b = \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \left(\frac{a-b}{1+ab} \right) = \tan^{-1} x$$

$$\therefore x = \frac{a-b}{1+ab}$$

Solution of the Inverse Equations

Consider the following examples:

➤ **Example 50.** Solve the equation

$$2 \sin^{-1} x^2 - \sin^{-1} x - 6 = 0.$$

Sol. Let $\sin^{-1} x = y$, then

$$2y^2 - y - 6 = 0 \Rightarrow 2y^2 - 4y + 3y - 6 = 0$$

$$\Rightarrow y = 2 \text{ and } y = -1.5$$

$$\Rightarrow \sin^{-1} x = 2 \text{ and } \sin^{-1} x = -1.5$$

Since, $2 > \frac{\pi}{2}$ and $|-1.5| < \frac{\pi}{2}$, so the only solution is

$$x = -\sin 1.5.$$

➤ **Example 51.** Solve the equation

$$\sin^{-1} 6x + \sin^{-1} 6\sqrt{3}x = -\frac{\pi}{2}.$$

Sol. Let us transfer $\sin^{-1} (6\sqrt{3}x)$ into right hand side of the equation and calculate the sine of both the sides of the resulting equation.

$$\therefore \sin (\sin^{-1} 6x) = \sin \left(-\sin^{-1} 6\sqrt{3}x - \frac{\pi}{2} \right)$$

$$\Rightarrow 6x = -\sin (\sin^{-1} 6\sqrt{3}x + \sin^{-1} 1)$$

$$[\because \sin(-x) = -\sin(x)]$$

$$\Rightarrow 6x = -\sin (\sin^{-1} \sqrt{1-108x^2})$$

$$\left[\because \sin^{-1} x + \sin^{-1} y = \sin^{-1} \{x\sqrt{1-y^2} + y\sqrt{1-x^2}\} \right]$$

$$\Rightarrow 6x = -\sqrt{1-108x^2} \quad \dots(i)$$

On squaring both sides, we get

$$36x^2 = 1 - 108x^2$$

$$\Rightarrow 144x^2 = 1$$

$$\text{whose roots are } x = \frac{1}{12} \text{ and } x = -\frac{1}{12}.$$

On substituting $x = -\frac{1}{12}$ in Eq. (i) and inverse

trigonometric equation satisfying the equation but

$x = \frac{1}{12}$ is not satisfying Eq.(i), so only solution is

$$x = -\frac{1}{12}.$$

➤ **Example 52.** Solve $2 \cos^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$.

Sol. Let $x = \cos y$, where $0 \leq y \leq \pi, |x| \leq 1$

$$\therefore 2 \cos^{-1} x = \sin^{-1} (2x\sqrt{1-x^2}) \quad \dots(i)$$

$$\Rightarrow 2 \cos^{-1} (\cos y) = \sin^{-1} (2 \cos y \cdot \sqrt{1-\cos^2 y})$$

$$\Rightarrow 2 \cos^{-1} (\cos y) = \sin^{-1} (2 \cos y \cdot \sin y)$$

$$\Rightarrow 2 \cos^{-1} (\cos y) = \sin^{-1} (\sin 2y)$$

$$\Rightarrow \sin^{-1} (\sin 2y) = 2y \text{ for } -\frac{\pi}{4} \leq y \leq \frac{\pi}{4}$$

$$\text{and } 2 \cos^{-1} (\cos y) = 2y \text{ for } 0 \leq y \leq \pi$$

Thus, Eq. (i) holds only when

$$y \in [0, \pi/4]$$

$$\Rightarrow x \in [\sqrt{2}/2, 1]$$

Summation of Series Containing Inverse Trigonometric Terms

Consider the following examples:

➤ **Example 53.** Find the sum to n terms of the

$$\text{series } \operatorname{cosec}^{-1} \sqrt{10} + \operatorname{cosec}^{-1} \sqrt{50} + \operatorname{cosec}^{-1} \sqrt{170} \\ + \dots + \operatorname{cosec}^{-1} \sqrt{(n^2+1)(n^2+2n+2)}.$$

Sol. Let $\theta = \operatorname{cosec}^{-1} \sqrt{(n^2+1)(n^2+2n+2)}$

$$\text{Then, } \operatorname{cosec}^2 \theta = (n^2+1)(n^2+2n+2)$$

$$= (n^2+1)^2 + 2n(n^2+1) + n^2+1$$

$$= (n^2+n+1)^2+1$$

$$\Rightarrow \cot^2 \theta = (n^2+n+1)^2$$

$$\Rightarrow \tan \theta = \frac{1}{n^2+n+1}$$

$$= \frac{(n+1)-n}{1+(n+1)n}$$

$$\Rightarrow \theta = \tan^{-1} \left[\frac{(n+1)-n}{1+(n+1)n} \right]$$

$$= \tan^{-1} (n+1) - \tan^{-1} n$$

Thus, sum of n terms of the given series
 $= (\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2)$
 $+ (\tan^{-1} 4 - \tan^{-1} 3) + \dots + \{\tan^{-1} (n+1) - \tan^{-1} n\}$
 $= \tan^{-1} (n+1) - \frac{\pi}{4}$

➤ **Example 54.** Find the sum of the first n terms of the series
 $\cot^{-1} (3) + \cot^{-1} (7) + \cot^{-1} (13) + \cot^{-1} (21) + \dots$

Sol. Let t_r denotes the r th term of the series 3, 7, 13, 21, ...

and $S = 3 + 7 + 13 + 21 + \dots + t_n$
 Also, $S = 3 + 7 + 13 + \dots + t_{n-1} + t_n$

On subtracting, we get

$$0 = 3 + 4 + 6 + \dots + 2n - t_n$$

$$\Rightarrow t_n = 3 + 4 + 6 + \dots + 2n$$

$$= 3 + \frac{1}{2} (n-1) (4 + 2n)$$

$$= n^2 + n + 1$$

Let $T_r = \cot^{-1} (r^2 + r + 1) = \tan^{-1} \left(\frac{1}{r^2 + r + 1} \right)$

$$= \tan^{-1} \left(\frac{r+1-r}{1+r(r+1)} \right)$$

$$= \tan^{-1} (r+1) - \tan^{-1} r$$

Thus, the sum of the first n terms of the given series is

$$\tan^{-1} (n+1) - \tan^{-1} (1) = \tan^{-1} \left[\frac{n+1-1}{1+1(n+1)} \right]$$

$$= \tan^{-1} \left(\frac{n}{n+2} \right)$$

➤ **Example 55.** Total number of positive integral values of n so that the equations

$$\cos^{-1} x + (\sin^{-1} y)^2 = \frac{n\pi^2}{4} \text{ and}$$

$$(\sin^{-1} y)^2 - \cos^{-1} x = \frac{\pi^2}{16} \text{ are consistent, is}$$

equal to

- (a) 1
 (b) 4
 (c) 3
 (d) 2

Sol. (a) We have, $2 (\sin^{-1} y)^2 = \frac{4n+1}{16} \pi^2$

$$\Rightarrow 0 \leq \frac{4n+1}{32} \pi^2 \leq \frac{\pi^2}{4}$$

$$\Rightarrow -\frac{1}{4} \leq n \leq \frac{7}{4}$$

Now, $2 (\cos^{-1} x) = \frac{4n-1}{16} \pi^2$

$$\Rightarrow 0 \leq \frac{4n-1}{32} \pi^2 \leq \pi$$

$$\Rightarrow \frac{1}{4} \leq n \leq \frac{8}{\pi} + \frac{1}{4}$$

$$\therefore n = 1$$

➤ **Example 56.** Define $y = \cos^{-1} (4x^3 - 3x)$ in terms of $\cos^{-1} x$ and also draw its graph.

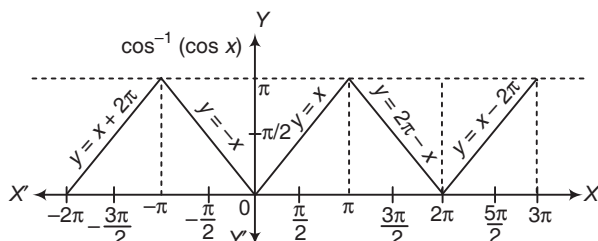
Sol. Let $y = \cos^{-1} (4x^3 - 3x)$

and $\cos^{-1} x = \theta$

$$\Rightarrow \theta \in [0, \pi] \text{ and } x = \cos \theta$$

$$\therefore y = \cos^{-1} (4 \cos^3 \theta - 3 \cos \theta)$$

$$\Rightarrow y = \cos^{-1} (\cos 3\theta) \quad \dots(i)$$



$$\therefore \theta \in [0, \pi] \quad \therefore 3\theta \in [0, 3\pi]$$

To define $y = \cos^{-1} (\cos 3\theta)$, we consider the graph of $\cos^{-1} (\cos x)$ in the interval $[0, 3\pi]$. Now, from the above graph, we can see that

(i) If $0 \leq 3\theta \leq \pi$, then
 $\cos^{-1} (\cos 3\theta) = 3\theta$

From Eq. (i), we get

$$y = 3\theta, \text{ if } \theta \leq 3\theta \leq \pi$$

$$\Rightarrow y = 3\theta, \text{ if } 0 \leq \theta \leq \frac{\pi}{3}$$

$$\Rightarrow y = 3 \cos^{-1} x, \text{ if } \frac{1}{2} \leq x \leq 1 \quad \dots(ii)$$

(ii) If $\pi < 3\theta \leq 2\pi$, then
 $\cos^{-1} (\cos 3\theta) = 2\pi - 3\theta$

From Eq. (i), we get

$$y = 2\pi - 3\theta, \text{ if } \pi < 3\theta \leq 2\pi$$

$$\Rightarrow y = 2\pi - 3\theta, \text{ if } \frac{\pi}{3} < \theta \leq \frac{2\pi}{3}$$

$$y = 2\pi - 3 \cos^{-1} x, \text{ if } -\frac{1}{2} \leq x < \frac{1}{2} \quad \dots(iii)$$

(iii) If $2\pi < 3\theta \leq 3\pi$, then
 $\cos^{-1} (\cos 3\theta) = -2\pi + 3\theta$

From Eq. (i), we get

$$\Rightarrow y = -2\pi + 3\theta, \text{ if } 2\pi < 3\theta \leq 3\pi$$

$$\Rightarrow y = -2\pi + 3\theta, \text{ if } \frac{2\pi}{3} < \theta \leq \pi$$

$$\Rightarrow y = -2\pi + 3 \cos^{-1} x, \text{ if } -1 \leq x < -\frac{1}{2} \quad \dots(iv)$$

From Eqs. (ii), (iii) and (iv), we get

$$y = \cos^{-1} (4x^3 - 3x) = \begin{cases} 3 \cos^{-1} x, & \text{if } \frac{1}{2} \leq x \leq 1 \\ 2\pi - 3 \cos^{-1} x, & \text{if } -\frac{1}{2} \leq x < \frac{1}{2} \\ -2\pi + 3 \cos^{-1} x, & \text{if } -1 \leq x < -\frac{1}{2} \end{cases}$$

Graph

For $y = \cos^{-1} (4x^3 - 3x)$,

Domain = $[-1, 1]$ and range = $[0, \pi]$

(i) If $\frac{1}{2} \leq x \leq 1$, $y = 3 \cos^{-1} x$

$$\Rightarrow \frac{dy}{dx} = \frac{-3}{\sqrt{1-x^2}} = -3(1-x^2)^{-1/2} \quad \dots(v)$$

$$\Rightarrow \frac{dy}{dx} < 0, \text{ if } x \in \left[\frac{1}{2}, 1 \right)$$

$$\Rightarrow \text{Decreasing, if } x \in \left[\frac{1}{2}, 1 \right)$$

Again, if we differentiate Eq. (v) w.r.t. x , we get

$$\frac{d^2y}{dx^2} = -\frac{3x}{(1-x^2)^{3/2}}$$

$$\Rightarrow \frac{d^2y}{dx^2} < 0, \text{ if } x \in \left[\frac{1}{2}, 1 \right)$$

$$\Rightarrow \text{Concavity downwards, if } x \in \left[\frac{1}{2}, 1 \right)$$

$$(ii) \text{ If } -\frac{1}{2} \leq x < \frac{1}{2}, \text{ then } y = 2\pi - 3\cos^{-1} x$$

$$\therefore \frac{dy}{dx} = \frac{3}{\sqrt{1-x^2}} \Rightarrow \frac{dy}{dx} > 0, \text{ if } x \in \left[-\frac{1}{2}, \frac{1}{2} \right)$$

$$\Rightarrow \text{Increasing, if } x \in \left[-\frac{1}{2}, \frac{1}{2} \right) \text{ and } \frac{d^2y}{dx^2} = \frac{3x}{(1-x^2)^{3/2}}$$

$$(a) \text{ If } x \in \left[-\frac{1}{2}, 0 \right), \text{ then } \frac{d^2y}{dx^2} < 0$$

$\Rightarrow$ Concavity downwards

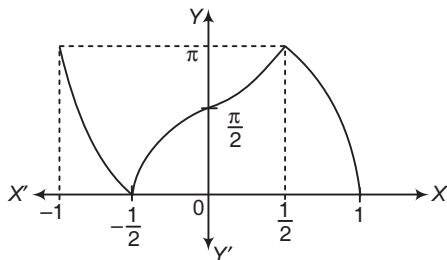
$$(b) \text{ If } x \in \left(0, \frac{1}{2} \right), \text{ then } \frac{d^2y}{dx^2} > 0$$

$\Rightarrow$ Concavity upwards

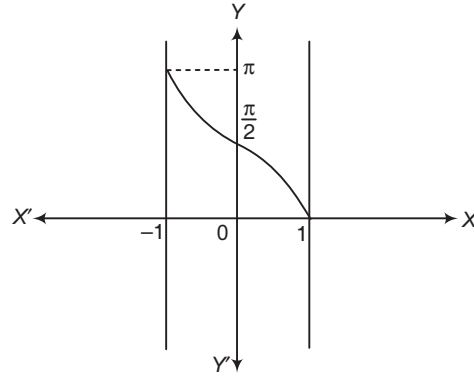
Similarly, if $-1 \leq x < -\frac{1}{2}$, then $\frac{dy}{dx} < 0$

and $\frac{d^2y}{dx^2} > 0$.

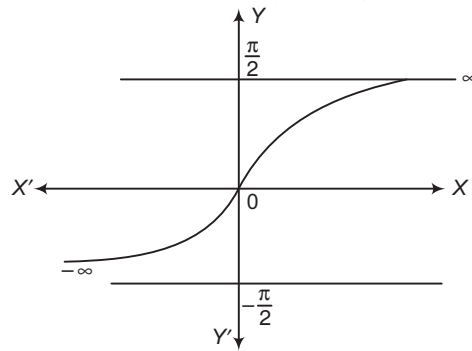
So, the graph of $y = \cos^{-1}(4x^3 - 3x)$ is given below:



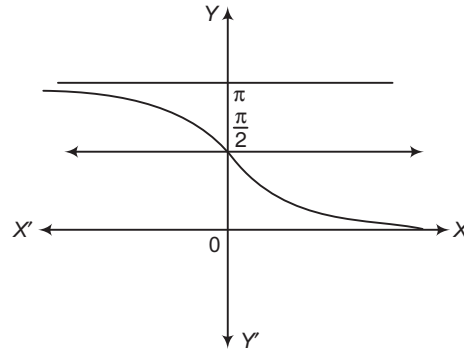
$$(ii) y = \cos^{-1} x, |x| \leq 1, y \in [0, \pi]$$



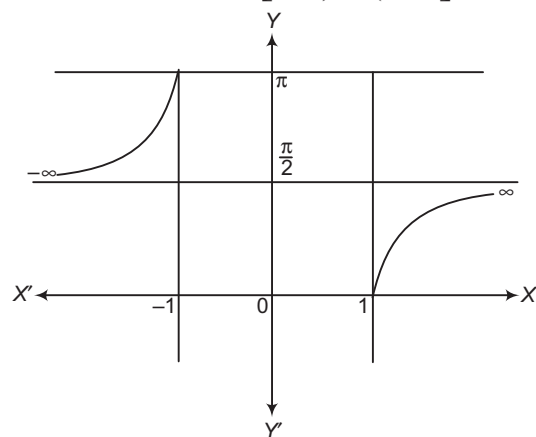
$$(iii) y = \tan^{-1} x, x \in R, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$



$$(iv) y = \cot^{-1} x, x \in R, y \in (0, \pi)$$



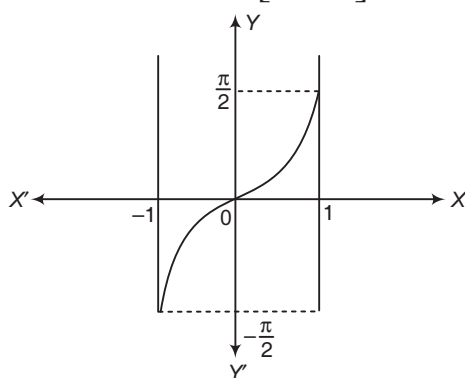
$$(v) y = \sec^{-1} x, |x| \geq 1, y \in \left[0, \frac{\pi}{2} \right) \cup \left(\frac{\pi}{2}, \pi \right]$$



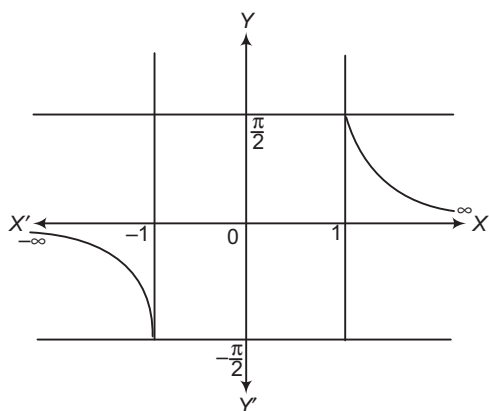
Some Useful Graphs

Part 1

$$(i) y = \sin^{-1} x, |x| \leq 1, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

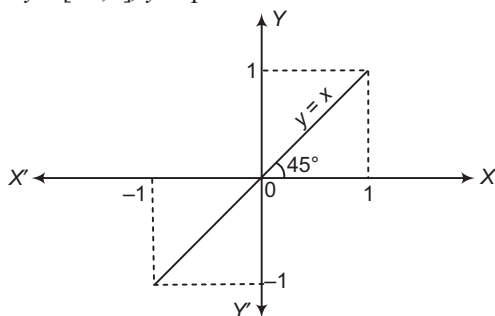


(vi) $y = \operatorname{cosec}^{-1} x, |x| \geq 1, y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$

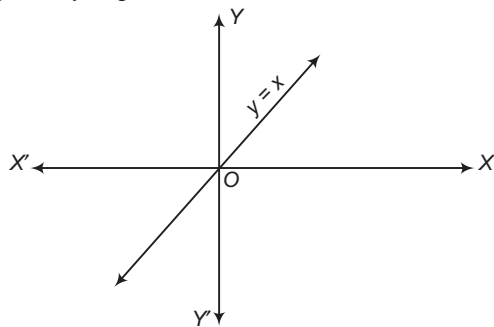


Part 2(A)

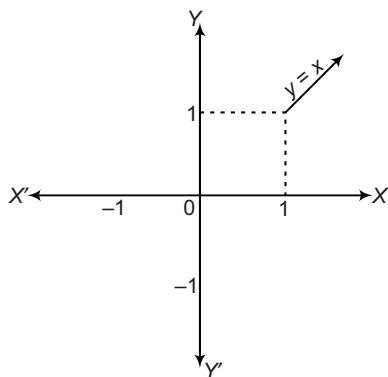
(i) $y = \sin(\sin^{-1} x) = \cos(\cos^{-1} x) = x, x \in [-1, 1], y \in [-1, 1]; y$ is periodic.



(ii) $y = \tan(\tan^{-1} x) = \cot(\cot^{-1} x) = x, x \in R, y \in R; y$ is periodic.

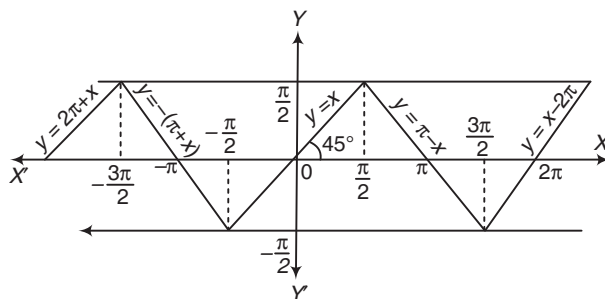


(iii) $y = \operatorname{cosec}(\operatorname{cosec}^{-1} x) = \sec(\sec^{-1} x) = x, |x| \geq 1, |y| \geq 1; y$ is periodic.

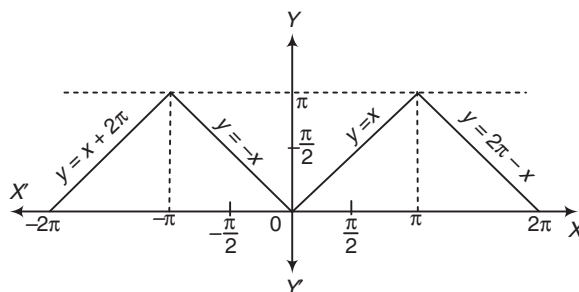


Part 2(B)

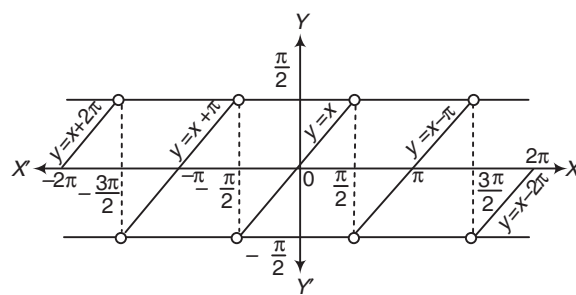
(i) $y = \sin^{-1}(\sin x), x \in R, y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]; y$ is periodic with period 2π .



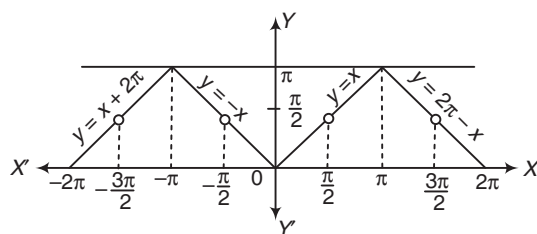
(ii) $y = \cos^{-1}(\cos x), x \in R, y \in [0, \pi]; y$ is periodic with period 2π .



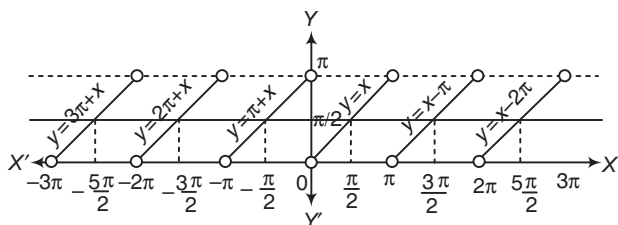
(iii) $y = \tan^{-1}(\tan x), x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}, y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right); y$ is periodic with period π .



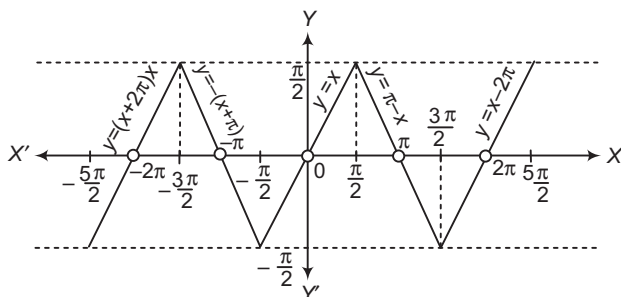
(iv) $y = \sec^{-1}(\sec x), x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}, y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]; y$ is periodic with period 2π .



- (v) $y = \cot^{-1}(\cot x)$, $x \in \mathbb{R} - \{n\pi, n \in \mathbb{I}\}$, $y \in (0, \pi)$;
 y is periodic with period π .

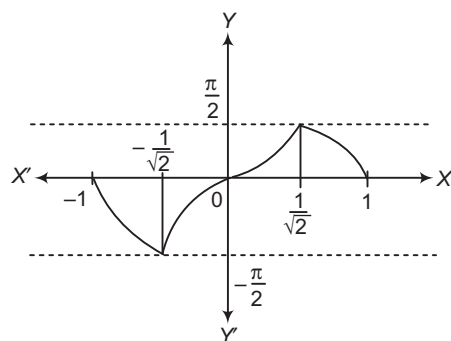


- (vi) $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$, $x \in \mathbb{R} - \{n\pi, n \in \mathbb{I}\}$,
 $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$; y is periodic with period 2π .

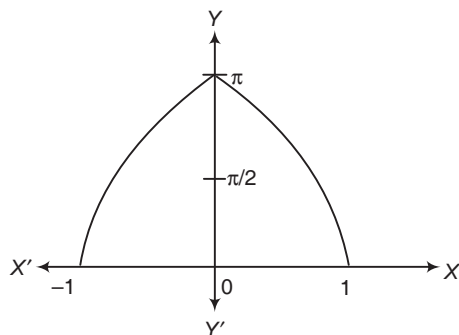


Part 2(C)

- (i) Graph of $y = \sin^{-1}(2x\sqrt{1-x^2})$

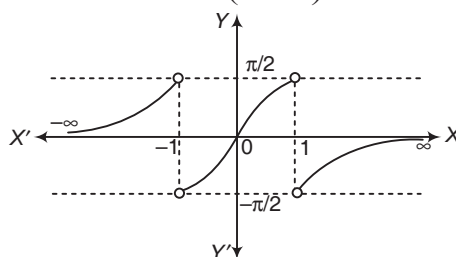


- (ii) Graph of $y = \cos^{-1}(2x^2 - 1)$

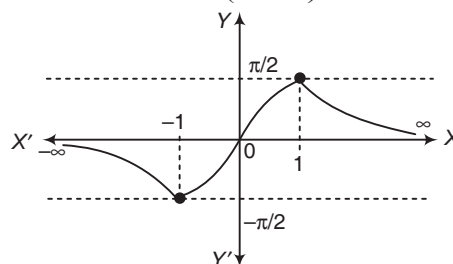


➤ In this graph, it is advisable not to check its derivability just by the inspection of the graph because it is difficult to judge from the graph that at $x = 0$, there is a sharp corner or not.

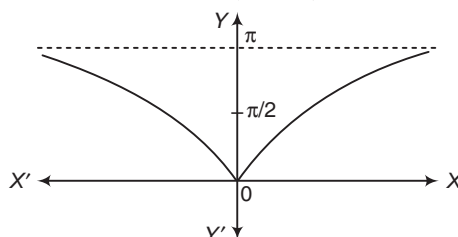
- (iii) Graph of $y = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$



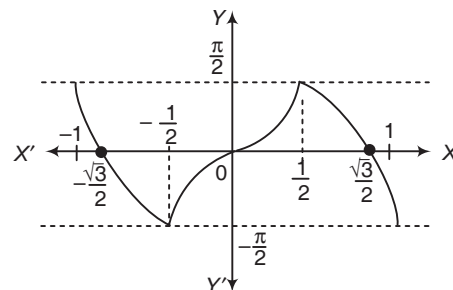
- (iv) Graph of $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$



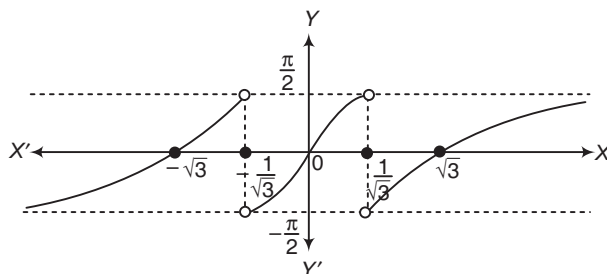
- (v) Graph of $y = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$



- (vi) Graph of $y = \sin^{-1}(3x - 4x^3)$



- (vii) Graph of $y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$





1 $\tan^{-1} \sqrt{3} - \sec^{-1}(-2)$ is equal to

- a π b $-\frac{\pi}{3}$ c $\frac{\pi}{3}$ d $\frac{2\pi}{3}$

2 The principal value of $\sin^{-1} \left\{ \tan \left(-\frac{5\pi}{4} \right) \right\}$ is

- a $\frac{\pi}{4}$ b $-\frac{\pi}{4}$ c $\frac{\pi}{2}$ d $-\frac{\pi}{2}$

3 $\tan \left[\cos^{-1} \frac{4}{5} + \tan^{-1} \frac{2}{3} \right]$ is equal to

- a $\frac{13}{6}$ b $\frac{17}{6}$ c $-\frac{13}{6}$ d $-\frac{17}{6}$

4 The value of the expression

$$\sin^{-1} \left(\sin \frac{22\pi}{7} \right) + \cos^{-1} \left(\cos \frac{5\pi}{3} \right) + \tan^{-1} \left(\tan \frac{5\pi}{7} \right) + \sin^{-1}(\cos 2)$$

- a -2 b $-\frac{\pi}{21} - 2$ c $-\frac{\pi}{21} + 2$ d $\frac{17\pi}{42} - 2$

5 If $\sin^{-1} x = \theta + \beta$ and $\sin^{-1} y + \theta = \beta$, then $1 + xy$ is equal to

- a $\cos^2 \theta + \cos^2 \beta$ b $\cos^2 \theta + \sin^2 \beta$
c $\sin^2 \theta + \sin^2 \beta$ d $\sin^2 \theta + \cos^2 \beta$

6 $\tan^{-1} \left[\frac{\cos x}{1 + \sin x} \right]$ is equal to

- a $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{3\pi}{2} \right)$
b $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
c $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(\frac{3\pi}{2}, \frac{5\pi}{2} \right)$
d $\frac{\pi}{4} - \frac{x}{2}$, for $x \in \left(-\frac{3\pi}{2}, -\frac{\pi}{2} \right)$

7 If $\tan(x + y) = 33$ and $x = \tan^{-1} 3$, then y will be

- a 0.3 b $\tan^{-1}(1.3)$
c $\tan^{-1}(0.3)$ d $\tan^{-1} \left(\frac{1}{18} \right)$

8 If $\cot^{-1} \frac{n}{\pi} > \frac{\pi}{6}$, $n \in \mathbb{N}$, then maximum value of n is

- a 6 b 7
c 5 d None of these

9 If α satisfies the inequation $x^2 - x - 2 > 0$, then a value exists for

- a $\sin^{-1} \alpha$ b $\sec^{-1} \alpha$
c $\cos^{-1} \alpha$ d None of these

10 If $f(x) = \sin^{-1} \left\{ \frac{\sqrt{3}}{2} x - \frac{1}{2} \sqrt{1-x^2} \right\}$, $-\frac{1}{2} \leq x \leq 1$,

then $f(x)$ is equal to

- a $\sin^{-1} \frac{1}{2} - \sin^{-1} x$ b $\sin^{-1} x - \frac{\pi}{6}$
c $\sin^{-1} x + \frac{\pi}{6}$ d None of these

11 If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$ and

$$f(1) = 2, f(x+y) = f(x)f(y), \text{ for all } x, y \in \mathbb{R}.$$

$$\text{Then, } x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{x+y+z}{x^{f(1)} + y^{f(2)} + z^{f(3)}} \text{ is}$$

equal to

- a 0 b 1
c 2 d 3

12 In $\triangle ABC$, if $A = \tan^{-1} 2$ and $B = \tan^{-1} 3$, then C is equal to

- a $\frac{\pi}{3}$ b $\frac{\pi}{4}$
c $\frac{\pi}{6}$ d None of these

13 $\cot^{-1} \left[\frac{\sqrt{1-\sin x} + \sqrt{1+\sin x}}{\sqrt{1-\sin x} - \sqrt{1+\sin x}} \right]$, where $x \in \left[0, \frac{\pi}{2} \right]$, is equal to

- a $\pi - x$ b $2\pi - x$ c $\frac{x}{2}$ d $\pi - \frac{x}{2}$

14 If $f(x) = x^{11} + x^9 - x^7 + x^3 + 1$ and $f[\sin^{-1}(\sin 8)] = \alpha$, α is a constant, then $f[\tan^{-1}(\tan 8)]$ is equal to

- a α b $\alpha - 2$
c $\alpha + 2$ d $2 - \alpha$

15 The value of x , where $x > 0$ and

$$\tan \left\{ \sec^{-1} \left(\frac{1}{x} \right) \right\} = \sin(\tan^{-1} 2),$$
 is

- a $\sqrt{5}$ b $\frac{\sqrt{5}}{3}$ c 1 d $\frac{2}{3}$

16 If $\sin^{-1}(x-1) + \cos^{-1}(x-3) + \tan^{-1} \left(\frac{x}{2-x^2} \right) = \cos^{-1} k + \pi$, then the value of k is

- a 1 b $-\frac{1}{\sqrt{2}}$
c $\frac{1}{\sqrt{2}}$ d None of these

17 If α, β are the roots of the equation $6x^2 + 11x + 3 = 0$, then

- a both $\cos^{-1} \alpha$ and $\cos^{-1} \beta$ are real
b both $\operatorname{cosec}^{-1} \alpha$ and $\operatorname{cosec}^{-1} \beta$ are real
c both $\cot^{-1} \alpha$ and $\cot^{-1} \beta$ are real
d None of the above

18 If $\tan^{-1}(\sin^2 \theta - 2 \sin \theta + 3) + \cot^{-1}(5^{\sec^2 \theta} + 1) = \frac{\pi}{2}$, then the value of $\cos^2 \theta - \sin \theta$ is

- a 0 b -1
c 1 d None of these

19 $\frac{\alpha^3}{2} \operatorname{cosec}^2 \left(\frac{1}{2} \tan^{-1} \frac{\alpha}{\beta} \right) + \frac{\beta^3}{2} \sec^2 \left(\frac{1}{2} \tan^{-1} \frac{\beta}{\alpha} \right)$

is equal to

- a $(\alpha - \beta)(\alpha^2 + \beta^2)$ b $(\alpha + \beta)(\alpha^2 - \beta^2)$
c $(\alpha + \beta)(\alpha^2 + \beta^2)$ d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $\alpha = \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) + \sin^{-1} \left(\frac{1}{3} \right)$ and

$$\beta = \cos^{-1} \left(\frac{\sqrt{3}}{2} \right) + \cos^{-1} \left(\frac{1}{3} \right), \text{ then}$$

- (a) $\alpha > \beta$ (b) $\alpha = \beta$
(c) $\alpha < \beta$ (d) $\alpha + \beta = 2\pi$

Sol. $\alpha + \beta = \sin^{-1} \left(\frac{\sqrt{3}}{2} \right) + \cos^{-1} \left(\frac{\sqrt{3}}{2} \right)$
 $+ \sin^{-1} \left(\frac{1}{3} \right) + \cos^{-1} \left(\frac{1}{3} \right)$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Also, $\alpha = \frac{\pi}{3} + \sin^{-1} \left(\frac{1}{3} \right) < \frac{\pi}{3} + \sin^{-1} \left(\frac{1}{2} \right)$

As $\sin \theta$ is increasing in $\left[0, \frac{\pi}{2} \right]$

$$\therefore \alpha < \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$$

Similarly, $\beta > \pi/2$

$$\left[\text{since, } \cos \theta \text{ is decreasing in } \left[0, \frac{\pi}{2} \right] \right]$$

$$\Rightarrow \beta > \frac{\pi}{2} > \alpha$$

$$\Rightarrow \alpha < \beta$$

Hence, (c) is the correct answer.

Ex 2. If $0 < x < 1$, then $\sqrt{1+x^2} [\{x \cos (\cot^{-1} x) + \sin (\cot^{-1} x)\}^2 - 1]^{1/2}$ is equal to

- (a) $\frac{x}{\sqrt{1+x^2}}$ (b) x
(c) $x\sqrt{1+x^2}$ (d) $\sqrt{1+x^2}$

Sol. $\sqrt{1+x^2} [\{x \cos (\cot^{-1} x) + \sin (\cot^{-1} x)\}^2 - 1]^{1/2}$
 $= \sqrt{1+x^2} \left[\left\{ x \cos \left(\cos^{-1} \frac{x}{\sqrt{1+x^2}} \right) \right. \right.$
 $\left. \left. + \sin \left(\sin^{-1} \frac{1}{\sqrt{1+x^2}} \right) \right\}^2 - 1 \right]^{1/2}$

$$= \sqrt{1+x^2} \left[\left\{ x \cdot \frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right\}^2 - 1 \right]^{1/2}$$

$$= \sqrt{1+x^2} [1+x^2-1]^{1/2} = x\sqrt{1+x^2}$$

Hence, (c) is the correct answer.

Ex 3. $\cos^{-1} \left(\frac{-1}{2} \right) - 2 \sin^{-1} \left(\frac{1}{2} \right) + 3 \cos^{-1} \left(-\frac{1}{\sqrt{2}} \right)$

$- 4 \tan^{-1} (-1)$ is equal to

- (a) $\frac{19\pi}{12}$ (b) $\frac{35\pi}{12}$ (c) $\frac{47\pi}{12}$ (d) $\frac{43\pi}{12}$

Sol. $\cos^{-1} \left(-\frac{1}{2} \right) - 2 \sin^{-1} \left(\frac{1}{2} \right) + 3 \cos^{-1} \left(-\frac{1}{\sqrt{2}} \right)$
 $- 4 \tan^{-1} (-1)$
 $= \pi - \cos^{-1} \left(\frac{1}{2} \right) - 2 \left(\frac{\pi}{6} \right) + 3 \left[\pi - \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) \right]$
 $+ 4 \tan^{-1} (1)$

$$= \pi - \frac{\pi}{3} - \frac{\pi}{3} + 3 \left(\pi - \frac{\pi}{4} \right) + 4 \cdot \frac{\pi}{4}$$

$$= \frac{\pi}{3} + 3 \cdot \frac{3\pi}{4} + \pi = \frac{43\pi}{12}$$

Hence, (d) is the correct answer.

Ex 4. $\tan \left[\frac{\pi}{4} + \frac{1}{2} \cos^{-1} \left(\frac{a}{b} \right) \right]$

$$+ \tan \left[\frac{\pi}{4} - \frac{1}{2} \cos^{-1} \left(\frac{a}{b} \right) \right] \text{ is equal to}$$

- (a) $\frac{2a}{b}$ (b) $\frac{2b}{a}$ (c) $\frac{a}{b}$ (d) $\frac{b}{a}$

Sol. $\tan \left[\frac{\pi}{4} + \frac{1}{2} \cos^{-1} \left(\frac{a}{b} \right) \right] + \tan \left[\frac{\pi}{4} - \frac{1}{2} \cos^{-1} \left(\frac{a}{b} \right) \right]$
 $= \tan \left[\frac{\pi}{4} + \phi \right] + \tan \left[\frac{\pi}{4} - \phi \right]$
 $\left[\text{put } \frac{1}{2} \cos^{-1} \left(\frac{a}{b} \right) = \phi \Rightarrow \cos 2\phi = \frac{a}{b} \right]$

$$= \frac{1 + \tan \phi}{1 - \tan \phi} + \frac{1 - \tan \phi}{1 + \tan \phi}$$

$$= \frac{2(1 + \tan^2 \phi)}{1 - \tan^2 \phi} = \frac{2}{\cos 2\phi} = \frac{2b}{a}$$

Hence, (b) is the correct answer.

Ex 5. $\cot^{-1} (2 \cdot 1^2) + \cot^{-1} (2 \cdot 2^2) + \cot^{-1} (2 \cdot 3^2) + \dots$ is equal to

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{5}$

Sol. $\cot^{-1} (2 \cdot 1^2) + \cot^{-1} (2 \cdot 2^2) + \cot^{-1} (2 \cdot 3^2) + \dots$

$$= \sum_{r=1}^{\infty} \cot^{-1} (2 \cdot r^2) = \sum_{r=1}^{\infty} \tan^{-1} \left(\frac{1}{2r^2} \right)$$

$$= \sum_{r=1}^{\infty} \tan^{-1} \left[\frac{(1+2r) + (1-2r)}{1 - (1+2r)(1-2r)} \right]$$

$$= \sum_{r=1}^{\infty} [\tan^{-1}(1+2r) + \tan^{-1}(1-2r)]$$

$$= \tan^{-1} 3 - \tan^{-1} 1 + \tan^{-1} 5 - \tan^{-1} 3 + \tan^{-1} 7 - \tan^{-1} 5 + \dots + \tan^{-1} \infty$$

$$= -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}$$

Hence, (a) is the correct answer.

Ex 6. $\sin \left(2 \sin^{-1} \sqrt{\frac{63}{65}} \right)$ is equal to

- (a) $\frac{2\sqrt{126}}{65}$ (b) $\frac{4\sqrt{65}}{65}$
(c) $\frac{8\sqrt{63}}{65}$ (d) $\frac{\sqrt{63}}{65}$

Sol. $\sin \left(2 \sin^{-1} \sqrt{\frac{63}{65}} \right) = \sin \left(\pi - \sin^{-1} 2 \sqrt{\frac{63}{65}} \sqrt{1 - \frac{63}{65}} \right)$
 $= \sin \left(\sin^{-1} \frac{2\sqrt{126}}{65} \right) = \frac{2\sqrt{126}}{65}$

Hence, (a) is the correct answer.

Ex 7. The value of $\lim_{|x| \rightarrow \infty} \cos(\tan^{-1}(\sin(\tan^{-1} x)))$ is

- (a) -1 (b) $\sqrt{2}$ (c) $-\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

Sol. $\lim_{|x| \rightarrow \infty} \cos(\tan^{-1}(\sin(\tan^{-1} x)))$
 $= \cos(\tan^{-1}(\sin(\tan^{-1} \infty)))$
 $= \cos(\tan^{-1}(\sin(\pi/2)))$
 $= \cos(\tan^{-1}(1)) = \cos\left(\frac{\pi}{4}\right)$
 $= \frac{1}{\sqrt{2}}$

Hence, (d) is the correct answer.

Ex 8. The value of $\tan \left\{ \cos^{-1} \left(-\frac{2}{7} \right) - \frac{\pi}{2} \right\}$ is

- (a) $\frac{2}{3\sqrt{5}}$ (b) $\frac{2}{3}$
(c) $\frac{1}{\sqrt{5}}$ (d) $\frac{4}{\sqrt{5}}$

Sol. $\tan \left\{ \cos^{-1} \left(-\frac{2}{7} \right) - \frac{\pi}{2} \right\}$
 $= \tan \left\{ \pi - \cos^{-1} \left(\frac{2}{7} \right) - \frac{\pi}{2} \right\}$
 $= \tan \left\{ \frac{\pi}{2} - \cos^{-1} \left(\frac{2}{7} \right) \right\}$
 $= \tan \left\{ \sin^{-1} \left(\frac{2}{7} \right) \right\}$
 $= \tan \left\{ \tan^{-1} \left(\frac{2}{3\sqrt{5}} \right) \right\}$
 $= \frac{2}{3\sqrt{5}}$

Hence, (a) is the correct answer.

Ex 9. The value of $\cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$ is

- (a) 2 (b) $8\pi - 26$
(c) $4\pi + 2$ (d) None of these

Sol. Since, $1 \text{ rad} = \frac{7\pi}{22}$
 $\therefore 12 \text{ rad} = \frac{7\pi}{22} \times 12 = \frac{42\pi}{11} = 4\pi - \frac{2\pi}{11} \dots(i)$
 and $14 \text{ rad} = \frac{7\pi}{22} \times 14 = \frac{49\pi}{11} = 4\pi + \frac{5\pi}{11} \dots(ii)$
 $\therefore \cos^{-1}(\cos 12) - \sin^{-1}(\sin 14)$
 $= \cos^{-1} \left[\cos \left(4\pi - \frac{2\pi}{11} \right) \right] - \sin^{-1} \left[\sin \left(4\pi + \frac{5\pi}{11} \right) \right]$
 $= \cos^{-1} \left[\cos \left(\frac{2\pi}{11} \right) \right] - \sin^{-1} \left(\sin \frac{5\pi}{11} \right) = \frac{2\pi}{11} - \frac{5\pi}{11}$
 $= 4\pi - 12 - (14 - 4\pi) = 8\pi - 26$
 [using Eqs. (i) and (ii)]
 Hence, (b) is the correct answer.

Ex 10. The principal value of $\cos^{-1}(\cos 5)$ is

- (a) 5 (b) $\pi - 5$ (c) $5 - \pi$ (d) $2\pi - 5$

Sol. As $\cos^{-1}(\cos 5)$
 $= \cos^{-1}[\cos(2\pi - 5)] = 2\pi - 5$
 Hence, (d) is the correct answer.

Ex 11. Complete solution set of $\sin^{-1} x \leq \cos^{-1} x$ is

- (a) $x \in \left[\frac{1}{\sqrt{2}}, 1 \right]$ (b) $x \in \left[-\frac{1}{\sqrt{2}}, 1 \right]$
(c) $x \in \left[-1, \frac{1}{\sqrt{2}} \right]$ (d) $x \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$

Sol. $\cos^{-1} x \geq \sin^{-1} x \Rightarrow \frac{\pi}{2} \geq 2 \sin^{-1} x$
 $\Rightarrow \sin^{-1} x \leq \frac{\pi}{4}$
 $\Rightarrow -1 \leq x \leq \sin\left(\frac{\pi}{4}\right)$
 $\therefore x \in \left[-1, \frac{1}{\sqrt{2}} \right]$

Hence, (c) is the correct answer.

Ex 12. The range of $\tan^{-1} \left(\frac{2x}{1+x^2} \right)$ is

- (a) $\left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$
(c) $\left(-\frac{\pi}{2}, \frac{\pi}{4} \right]$ (d) $\left[\frac{\pi}{4}, \frac{\pi}{2} \right]$

Sol. Since, $1 + x^2 \geq 2|x| \Rightarrow \frac{2|x|}{1+x^2} \leq 1$
 $\Rightarrow -1 \leq \frac{2x}{1+x^2} \leq 1$
 $\therefore \tan^{-1} \left(\frac{2x}{1+x^2} \right) \in \left[-\frac{\pi}{4}, \frac{\pi}{4} \right]$

Hence, (a) is the correct answer.

- Ex 13.** The value of $\sum_{r=0}^{\infty} \tan^{-1} \left(\frac{1}{1+r+r^2} \right)$ is
- (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{4}$
 (c) $\frac{\pi}{4}$ (d) None of these

Sol. $\tan^{-1} \left(\frac{1}{1+r+r^2} \right) = \tan^{-1} \left(\frac{r+1-r}{1+r(r+1)} \right)$
 $= \tan^{-1} (r+1) - \tan^{-1} r$
 $\therefore \sum_{r=0}^n [\tan^{-1} (r+1) - \tan^{-1} r]$
 $= \tan^{-1} (n+1) - \tan^{-1} 0 = \tan^{-1} (n+1)$
 $\therefore \sum_{r=0}^{\infty} \tan^{-1} \left(\frac{1}{1+r+r^2} \right) = \tan^{-1} \infty = \frac{\pi}{2}$

Hence, (a) is the correct answer.

- Ex 14.** The solution set of the equation

$$\sin^{-1} \sqrt{1-x^2} + \cos^{-1} x = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$$

– $\sin^{-1} x$ is

(a) $[-1, 1] - \{0\}$
 (b) $(0, 1] \cup \{-1\}$
 (c) $[-1, 0] \cup \{1\}$
 (d) $[-1, 1]$

Sol. $\sin^{-1} \sqrt{1-x^2} + \cos^{-1} x = \cot^{-1} \frac{\sqrt{1-x^2}}{x} - \sin^{-1} x$
 $\Rightarrow \frac{\pi}{2} + \sin^{-1} \sqrt{1-x^2} = \cot^{-1} \frac{\sqrt{1-x^2}}{x}$
 $\Rightarrow \tan^{-1} \frac{\sqrt{1-x^2}}{x} + \sin^{-1} \sqrt{1-x^2} = 0$
 $\therefore x \in [-1, 0) \cup \{1\}$
 Hence, (c) is the correct answer.

- Ex 15.** If $2 \sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$, then

- (a) $[-1, 1]$
 (b) $\left[-\frac{1}{\sqrt{2}}, 1\right]$
 (c) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$
 (d) None of the above

Sol. $2 \sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$
 Range of right hand side is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
 $\Rightarrow -\frac{\pi}{2} \leq 2 \sin^{-1} x \leq \frac{\pi}{2}$
 $\Rightarrow -\frac{\pi}{4} \leq \sin^{-1} x \leq \frac{\pi}{4}$
 $\therefore x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

Hence, (c) is the correct answer.

- Ex 16.** Let $\begin{vmatrix} \tan^{-1} x & \tan^{-1} 2x & \tan^{-1} 3x \\ \tan^{-1} 3x & \tan^{-1} x & \tan^{-1} 2x \\ \tan^{-1} 2x & \tan^{-1} 3x & \tan^{-1} x \end{vmatrix} = 0$, then the number of values of x satisfying the equation is
- (a) 1 (b) 2
 (c) 3 (d) 4

Sol. On expanding, we have
 $(\tan^{-1} x)^3 + (\tan^{-1} 2x)^3 + (\tan^{-1} 3x)^3$
 $= 3 \tan^{-1} x \tan^{-1} 2x \tan^{-1} 3x$
 $\therefore x = 0$
 Hence, (a) is the correct answer.

- Ex 17.** If $2 \tan^{-1} x = \pi + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$, then
- (a) $x > 1$ (b) $|x| < 1$
 (c) $|x| > 2$ (d) $|x| < 2$

Sol. $2 \tan^{-1} x = \pi + \tan^{-1} \left(\frac{2x}{1-x^2} \right), \forall x \in (1, \infty)$
 Hence, (a) is the correct answer.

- Ex 18.** The value of $\cos \left[\frac{\pi}{6} + \cos^{-1} \left(-\frac{1}{2} \right) \right]$ is equal to
- (a) $\frac{\sqrt{3}}{2}$ (b) $-\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Sol. $\cos \left[\frac{\pi}{6} + \cos^{-1} \left(-\frac{1}{2} \right) \right] = \cos \left(\frac{\pi}{6} + \pi - \frac{\pi}{3} \right)$
 $= \cos \left(\pi - \frac{\pi}{6} \right) = -\cos \left(\frac{\pi}{6} \right) = -\frac{\sqrt{3}}{2}$
 Hence, (b) is the correct answer.

- Ex 19.** The value of $\cos^{-1} \sqrt{\frac{2}{3}} - \cos^{-1} \left(\frac{\sqrt{6}+1}{2\sqrt{3}} \right)$ is equal to
- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{6}$

Sol. $\cos^{-1} \sqrt{\frac{2}{3}} - \cos^{-1} \left(\frac{\sqrt{6}+1}{2\sqrt{3}} \right)$
 $= \tan^{-1} \left(\frac{1}{\sqrt{2}} \right) - \tan^{-1} \left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}} \right)$
 $= \tan^{-1} \left(\frac{\frac{1}{\sqrt{2}} - \frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{6}}}{1 + \frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}(1+\sqrt{6})}} \right)$
 $= \tan^{-1} \left(\frac{1 + \sqrt{6} - \sqrt{6} + 2}{\sqrt{2} + 2\sqrt{3} + \sqrt{3} - \sqrt{2}} \right)$
 $= \tan^{-1} \left(\frac{3}{3\sqrt{3}} \right) = \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) = \frac{\pi}{6}$

Hence, (d) is the correct answer.

Ex 20. If $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$, then

$x + y + z$ is equal to

- (a) $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$ (b) xyz
(c) $xy + yz + zx$ (d) None of these

Sol. $\cot^{-1} x + \cot^{-1} y + \cot^{-1} z = \frac{\pi}{2}$

$\Rightarrow \Sigma \left(\frac{\pi}{2} - \tan^{-1} x \right) = \frac{\pi}{2}$

$\Rightarrow \Sigma \tan^{-1} x = \pi$

$\Rightarrow \tan (\Sigma \tan^{-1} x) = 0$

$\Rightarrow \Sigma \tan (\tan^{-1} x) = \pi \tan (\tan^{-1} x)$

$\therefore x + y + z = xyz$

Hence, (b) is the correct answer.

Ex 21. If $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, then the value of

$\tan^{-1} \left(\frac{\tan x}{4} \right) + \tan^{-1} \left(\frac{3 \sin 2x}{5 + 3 \cos 2x} \right)$ is

- (a) $\frac{x}{2}$ (b) $2x$
(c) $3x$ (d) x

Sol. $\tan^{-1} \left(\frac{\tan x}{4} \right) + \tan^{-1} \left(\frac{3 \sin 2x}{5 + 3 \cos 2x} \right)$

$= \tan^{-1} \left(\frac{\tan x}{4} \right) + \tan^{-1} \left(\frac{\frac{6 \tan x}{1 + \tan^2 x}}{5 + \frac{3(1 - \tan^2 x)}{1 + \tan^2 x}} \right)$

$= \tan^{-1} \left(\frac{\tan x}{4} \right) + \tan^{-1} \left(\frac{6 \tan x}{8 + 2 \tan^2 x} \right)$

$= \tan^{-1} \left(\frac{\tan x}{4} \right) + \tan^{-1} \left(\frac{3 \tan x}{4 + \tan^2 x} \right)$

$= \tan^{-1} \left(\frac{\frac{\tan x}{4} + \frac{3 \tan x}{4 + \tan^2 x}}{1 - \frac{3 \tan^2 x}{4(4 + \tan^2 x)}} \right)$

$\left[\because \left(\frac{\tan x}{4} \cdot \frac{3 \tan x}{4 + \tan^2 x} \right) < 1 \right]$

$= \tan^{-1} \left(\frac{16 \tan x + \tan^3 x}{16 + \tan^2 x} \right) = \tan^{-1} (\tan x) = x$

Hence, (d) is the correct answer.

Ex 22. The product of all values of x satisfying the

equation $\sin^{-1} \cos \left(\frac{2x^2 + 10|x| + 4}{x^2 + 5|x| + 3} \right)$
 $= \cot \left[\cot^{-1} \left(\frac{2 - 18|x|}{9|x|} \right) \right] + \frac{\pi}{2}$, is

- (a) 9 (b) -9
(c) -3 (d) -1

Sol. $\frac{\pi}{2} - \cos^{-1} \cos \left(\frac{2(x^2 + 5|x| + 3) - 2}{x^2 + 5|x| + 3} \right)$
 $= \cot \cot^{-1} \left(\frac{2}{9|x|} - 2 \right) + \frac{\pi}{2}$

$\Rightarrow \frac{\pi}{2} - 2 + \frac{2}{x^2 + 5|x| + 3} = \frac{2}{9|x|} - 2 + \frac{\pi}{2}$

$\Rightarrow |x|^2 - 4|x| + 3 = 0$

$\therefore |x| = 1, 3 \Rightarrow x = \pm 1, \pm 3$

Hence, (a) is the correct answer.

Ex 23. If $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2}$, then $\frac{1 + x^4 + y^4}{x^2 - x^2 y^2 + y^2}$

is equal to

- (a) 1 (b) 2
(c) $1/2$ (d) None of these

Sol. $\sin^{-1} x + \sin^{-1} y = \frac{\pi}{2} \Rightarrow \sin^{-1} x = \frac{\pi}{2} - \sin^{-1} y$

$\Rightarrow \sin^{-1} x = \sin^{-1} \sqrt{1 - y^2} \Rightarrow x^2 + y^2 = 1$

$\therefore \frac{1 + x^4 + y^4}{x^2 - x^2 y^2 + y^2} = \frac{1 + (x^2 + y^2)^2 - 2x^2 y^2}{1 - x^2 y^2}$
 $= \frac{1 + 1 - 2x^2 y^2}{1 - x^2 y^2} = 2$

Hence, (b) is the correct answer.

Ex 24. The range of $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$ is

- (a) $\left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$ (b) $\left[\frac{\pi}{4}, \frac{3\pi}{4} \right]$
(c) $\left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\}$ (d) None of these

Sol. Given, $f(x) = \sin^{-1} x + \tan^{-1} x + \sec^{-1} x$. Clearly, domain of $f(x)$ is $x = \pm 1$. Thus, range is $\{ f(1), f(-1) \}$.

i.e. $\left\{ \frac{\pi}{4}, \frac{3\pi}{4} \right\}$

Hence, (c) is the correct answer.

Ex 25. The value of

$\tan^{-1} \sqrt{\frac{a\lambda}{bc}} + \tan^{-1} \sqrt{\frac{b\lambda}{ca}} + \tan^{-1} \sqrt{\frac{c\lambda}{ab}}$,

where $a, b, c \in \mathbb{R}^+$ and $\lambda = \frac{abc}{ab + bc + ca}$, is

equal to

- (a) $\frac{\pi}{4}$ (b) π
(c) $\frac{\pi}{2}$ (d) None of these

Sol. Since, $\Sigma \sqrt{\frac{a\lambda}{bc}} \cdot \sqrt{\frac{b\lambda}{ca}} = \Sigma \frac{\lambda}{c} = 1$

$\therefore \tan^{-1} \sqrt{\frac{a\lambda}{bc}} + \tan^{-1} \sqrt{\frac{b\lambda}{ca}} + \tan^{-1} \sqrt{\frac{c\lambda}{ab}} = \frac{\pi}{2}$

Hence, (c) is the correct answer.

Ex 26. The range of $f(x) = \sin^{-1} x + \cot^{-1} x + \tan^{-1} x$ is

- (a) $[0, \pi]$ (b) $\left[\frac{\pi}{2}, \pi\right]$
(c) $\left[\frac{\pi}{4}, \pi\right]$ (d) $[-\pi, \pi]$

Sol. $f(x) = \sin^{-1} x + \cot^{-1} x + \tan^{-1} x$
 $= \frac{\pi}{2} + \sin^{-1} x, |x| \leq 1$
 $\therefore$ Range of $f(x)$ is $[0, \pi]$.
Hence, (a) is the correct answer.

Ex 27. The value of $\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$ is equal to

- (a) zero (b) $24 - 2\pi$
(c) $4\pi - 24$ (d) None of these

Sol. $\sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$
 $= \sin^{-1}\{\sin(12 - 4\pi)\} + \cos^{-1}\{\cos(4\pi - 12)\}$
 $= 12 - 4\pi + 4\pi - 12 = 0$
Hence, (a) is the correct answer.

Ex 28. The value of $\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$,

where $x \in \left(\frac{\pi}{2}, \pi\right)$, is

- (a) $\frac{\pi}{2}$ (b) $-\pi$
(c) π (d) $-\frac{\pi}{2}$

Sol. $\sin^{-1}[\cos\{\cos^{-1}(\cos x) + \sin^{-1}(\sin x)\}]$
 $= \sin^{-1}\{\cos(x + \pi - x)\} \quad \left[\because x \in \left(\frac{\pi}{2}, \pi\right)\right]$
 $= \sin^{-1}(\cos \pi)$
 $= \sin^{-1}(-1) = -\frac{\pi}{2}$
Hence, (d) is the correct answer.

Ex 29. Complete solution set of $\tan^2(\sin^{-1} x) > 1$ is

- (a) $\left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$
(b) $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - \{0\}$
(c) $(-1, 1) - \{0\}$
(d) None of the above

Sol. $\tan^2(\sin^{-1} x) > 1$
 $\Rightarrow \frac{\pi}{4} < \sin^{-1} x < \frac{\pi}{2} \quad \text{or} \quad -\frac{\pi}{2} < \sin^{-1} x < -\frac{\pi}{4}$
 $\Rightarrow x \in \left(\frac{1}{\sqrt{2}}, 1\right) \quad \text{or} \quad x \in \left(-1, -\frac{1}{\sqrt{2}}\right)$
 $\Rightarrow x \in \left(-1, -\frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$
Hence, (a) is the correct answer.

Ex 30. If $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$, then x equals

- (a) -1 (b) 1
(c) 0 (d) None of these

Sol. We have, $(\tan^{-1} x)^2 + (\cot^{-1} x)^2 = \frac{5\pi^2}{8}$
 $\Rightarrow (\tan^{-1} x + \cot^{-1} x)^2 - 2\tan^{-1} x \left(\frac{\pi}{2} - \tan^{-1} x\right) = \frac{5\pi^2}{8}$
 $\Rightarrow \frac{\pi^2}{4} - 2 \cdot \frac{\pi}{2} \tan^{-1} x + 2(\tan^{-1} x)^2 = \frac{5\pi^2}{8}$
 $\Rightarrow 2(\tan^{-1} x)^2 - \pi \tan^{-1} x - \frac{3\pi^2}{8} = 0$
 $\Rightarrow \tan^{-1} x = -\frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow \tan^{-1} x = -\frac{\pi}{4}$
 $\Rightarrow x = -1$
Hence, (a) is the correct answer.

Ex 31. If $-1 < x < 0$, then $\sin^{-1} x$ equals

- (a) $\pi - \cos^{-1}(\sqrt{1-x^2})$ (b) $\tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$
(c) $-\cot^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)$ (d) $\operatorname{cosec}^{-1} x$

Sol. Since, $-1 < x < 0$, $-\frac{\pi}{2} < \sin^{-1} x < 0$

Let $\sin^{-1} x = \alpha$, i.e. $\sin \alpha = x$

Then, $\tan \alpha = \frac{x}{\sqrt{1-x^2}}$

$$\Rightarrow \alpha = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

$$\therefore \sin^{-1} x = \tan^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)$$

Hence, (b) is the correct answer.

Ex 32. The value of $\sin^{-1}\left[\cos\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right]$ is

- (a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Sol. We have,

$$\begin{aligned} & \sin^{-1}\left[\cos\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}} + \cos^{-1}\frac{\sqrt{12}}{4} + \sec^{-1}\sqrt{2}\right)\right] \\ &= \sin^{-1}\left[\cos\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \cos^{-1}\frac{\sqrt{3}}{2} + \cos^{-1}\frac{1}{\sqrt{2}}\right)\right] \\ &= \sin^{-1}\left[\cos\left(\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right)\right] = \sin^{-1}\left[\cos\frac{\pi}{2}\right] \\ &= \sin^{-1}0 = 0 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 33. If $A = 2 \tan^{-1}(2\sqrt{2} - 1)$ and

$$B = 3 \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right), \text{ then}$$

- (a) $A = B$ (b) $A < B$
(c) $A > B$ (d) None of these

Sol. We have, $A = 2 \tan^{-1}(2\sqrt{2} - 1) = 2 \tan^{-1}(1.828)$

$$\Rightarrow A > 2 \tan^{-1} \sqrt{3} \Rightarrow A > \frac{2\pi}{3}$$

$$\begin{aligned} \text{Also, } 3 \sin^{-1}\left(\frac{1}{3}\right) &= \sin^{-1}\left[3 \cdot \frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right] \\ &= \sin^{-1}\left(\frac{23}{27}\right) = \sin^{-1}(0.852) \end{aligned}$$

$$\Rightarrow 3 \sin^{-1}\left(\frac{1}{3}\right) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow 3 \sin^{-1}\left(\frac{1}{3}\right) < \frac{\pi}{3}$$

$$\text{Now, } \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}(0.6) < \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3}$$

$$\Rightarrow B = 3 \sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

Thus, $A > B$

Hence, (c) is the correct answer.

Ex 34. In $\triangle ABC$, if $\angle C = \frac{\pi}{2}$ and

$$\sin^{-1} x = \sin^{-1}\left(\frac{ax}{c}\right) + \sin^{-1}\left(\frac{bx}{c}\right), \text{ where } a,$$

b, c are the sides of triangle, then total number of different values of x is

- (a) 2 (b) 3
(c) 1 (d) None of these

Sol. $\angle C = \frac{\pi}{2} \Rightarrow a^2 + b^2 = c^2$

Clearly, $|x| \leq 1$

$$\text{Also, } \frac{a^2 x^2}{c^2} + \frac{b^2 x^2}{c^2} = x^2 \leq 1$$

$$\text{Now consider, } \sin^{-1} x = \sin^{-1}\left(\frac{ax}{c}\right) + \sin^{-1}\left(\frac{bx}{c}\right)$$

$$\Rightarrow \sin^{-1} x = \sin^{-1}\left(\frac{ax}{c} \sqrt{\frac{1-b^2 x^2}{c^2}} + \frac{bx}{c} \sqrt{1-\frac{a^2 x^2}{c^2}}\right)$$

$$\Rightarrow x = \frac{ax}{c^2} \sqrt{c^2 - b^2 x^2} + \frac{bx}{c^2} \sqrt{c^2 - a^2 x^2}$$

$$\Rightarrow x = 0$$

$$\text{or } c^2 = a\sqrt{c^2 - b^2 x^2} + b\sqrt{c^2 - a^2 x^2}$$

$$\begin{aligned} \Rightarrow c^4 &= a^2(c^2 - b^2 x^2) + b^2(c^2 - a^2 x^2) \\ &\quad + 2ab\sqrt{(c^2 - b^2 x^2)(c^2 - a^2 x^2)} \end{aligned}$$

$$\begin{aligned} \Rightarrow c^4 &= a^2 c^2 + b^2 c^2 - 2a^2 b^2 x^2 \\ &\quad + 2ab\sqrt{c^4 - c^2 a^2 x^2 - c^2 b^2 x^2 + a^2 x^4 b^2} \end{aligned}$$

$$\begin{aligned} \Rightarrow c^2(c^2 - a^2 - b^2) + 2a^2 b^2 x^2 \\ = 2ab\sqrt{c^4 - c^2(a^2 + b^2)x^2 + a^2 b^2 x^4} \end{aligned}$$

$$\Rightarrow abx^2 = \sqrt{c^4 - c^2(a^2 + b^2)x^2 + a^2 b^2 x^4}$$

$$\Rightarrow a^2 x^4 b^2 = c^4 - c^4 x^2 + a^2 b^2 x^4$$

$$\Rightarrow x^2 = 1 \Rightarrow x = 1, -1$$

$\therefore$ Total number of different values of x are three, i.e. $x = 0, 1, -1$

Hence, (b) is the correct answer.

Ex 35. The equation $\sin^{-1} x = |x - a|$ will have atleast one solution, if

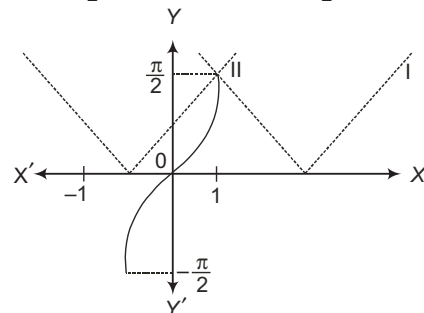
- (a) $a \in [-1, 1]$ (b) $a \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(c) $a \in \left[1, 1 + \frac{\pi}{2}\right]$ (d) $a \in \left(1 - \frac{\pi}{2}, 1 + \frac{\pi}{2}\right)$

Sol. $\sin^{-1} x = |x - a| \Rightarrow |a - x|$

When $a > 0$

In this case, $y = (a - x)$ must pass through $\left(1, \frac{\pi}{2}\right)$.

$$\Rightarrow \frac{\pi}{2} = a - 1 \Rightarrow a = 1 + \frac{\pi}{2}$$



When $a < 0$

In this case, $y = (x - a)$ must pass through $\left(1, \frac{\pi}{2}\right)$.

$$\Rightarrow \frac{\pi}{2} = 1 - a \Rightarrow a = 1 - \frac{\pi}{2}$$

Thus, for atleast one solution,

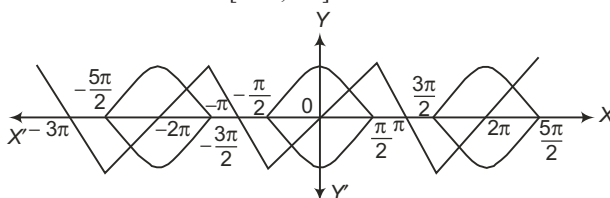
$$a \in \left[1 - \frac{\pi}{2}, 1 + \frac{\pi}{2}\right]$$

Hence, (d) is the correct answer.

Ex 36. Total number of ordered pairs (x, y) satisfying $|y| = \cos x$ and $y = \sin^{-1}(\sin x)$, where $|x| \leq 3\pi$, is equal to

- (a) 2 (b) 4 (c) 6 (d) 8

Sol. Graphs of $y = \sin^{-1}(\sin x)$ and $|y| = \cos x$ meet exactly six times in $[-3\pi, 3\pi]$.



Hence, (c) is the correct answer.

Ex 37. $\sum_{r=1}^n \tan^{-1} \left(\frac{2^{r-1}}{1+2^{2r-1}} \right)$ is equal to

- (a) $\tan^{-1}(2)^n$ (b) $\tan^{-1}(2)^n - \frac{\pi}{4}$
 (c) $\tan^{-1}(2^{n+1})$ (d) $\tan^{-1}(2^{n+1}) - \frac{\pi}{4}$

Sol.
$$\begin{aligned} \sum_{r=1}^n \tan^{-1} \left(\frac{2^{r-1}}{1+2^{2r-1}} \right) &= \sum_{r=1}^n \tan^{-1} \left(\frac{2^{r-1}}{1+2^r \cdot 2^{r-1}} \right) = \sum_{r=1}^n \tan^{-1} \left(\frac{2^r - 2^{r-1}}{1+2^r \cdot 2^{r-1}} \right) \\ &= \sum_{r=1}^n [\tan^{-1}(2^r) - \tan^{-1}(2^{r-1})] \\ &= \tan^{-1}(2^n) - \tan^{-1}1 = \tan^{-1}(2^n) - \frac{\pi}{4} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 38. $\sum_{r=1}^n \sin^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}} \right)$ is equal to

- (a) $\tan^{-1}(\sqrt{n}) - \frac{\pi}{4}$ (b) $\tan^{-1}(\sqrt{n+1}) - \frac{\pi}{4}$
 (c) $\tan^{-1}(\sqrt{n})$ (d) $\tan^{-1}(\sqrt{n+1})$

Sol.
$$\begin{aligned} \sin^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}} \right) &= \tan^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{1 + \sqrt{r}\sqrt{r-1}} \right) \\ &= \tan^{-1} \sqrt{r} - \tan^{-1}(\sqrt{r-1}) \\ \therefore \sum_{r=1}^n \sin^{-1} \left(\frac{\sqrt{r} - \sqrt{r-1}}{\sqrt{r(r+1)}} \right) &= \sum_{r=1}^n (\tan^{-1} \sqrt{r} - \tan^{-1} \sqrt{r-1}) = \tan^{-1} \sqrt{n} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 39. If $x_1 = 2 \tan^{-1} \left(\frac{1+x}{1-x} \right)$, $x_2 = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right)$,

where $x \in (0, 1)$, then $x_1 + x_2$ is equal to

- (a) 0 (b) 2π
 (c) π (d) None of these

Sol. Given, $x_1 = 2 \tan^{-1} \left(\frac{1+x}{1-x} \right)$
 and $x_2 = \sin^{-1} \left(\frac{1-x^2}{1+x^2} \right) = \tan^{-1} \left(\frac{1-x^2}{2x} \right)$
 Now, $x_1 = \pi + \tan^{-1} \left(\frac{2 \left(\frac{1+x}{1-x} \right)}{1 - \left(\frac{1+x}{1-x} \right)^2} \right)$

$$= \pi + \tan^{-1} \left(\frac{1-x^2}{-2x} \right)$$

$$= \pi - \tan^{-1} \left(\frac{1-x^2}{2x} \right)$$

$\Rightarrow x_1 + x_2 = \pi$

Hence, (c) is the correct answer.

Ex 40. The minimum value of $(\sin^{-1} x)^3 + (\cos^{-1} x)^3$ is equal to

- (a) $\frac{\pi^3}{32}$ (b) $\frac{5\pi^3}{32}$
 (c) $\frac{9\pi^3}{32}$ (d) $\frac{11\pi^3}{32}$

Sol. Let $I = (\sin^{-1} x)^3 + (\cos^{-1} x)^3$

$$= (\sin^{-1} x + \cos^{-1} x)[(\sin^{-1} x)^2 + (\cos^{-1} x)^2 - (\sin^{-1} x)(\cos^{-1} x)]$$

$$= \frac{\pi}{2} \left[(\sin^{-1} x + \cos^{-1} x)^2 - 3 \sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x \right) \right]$$

$$= \frac{\pi}{2} \left(\frac{\pi^2}{4} - \frac{3\pi}{2} \sin^{-1} x + 3(\sin^{-1} x)^2 \right)$$

$$= \frac{\pi}{2} \left[3 \left((\sin^{-1} x)^2 - \frac{\pi}{2} (\sin^{-1} x) + \frac{\pi^2}{16} - \frac{\pi^2}{16} \right) + \frac{\pi^2}{4} \right]$$

$$= \frac{\pi}{2} \left(\frac{\pi^2}{16} + 3 \left(\sin^{-1} x - \frac{\pi}{4} \right)^2 \right)$$

Now, $\frac{\pi}{4} \geq \sin^{-1} x - \frac{\pi}{4} \geq -\frac{3\pi}{4}$

$\Rightarrow \frac{9\pi^2}{16} \geq \left(\sin^{-1} x - \frac{\pi}{4} \right)^2 \geq 0$

$\Rightarrow I \geq \frac{\pi}{2} \cdot \frac{\pi^2}{16} = \frac{\pi^3}{32}$

Hence, (a) is the correct answer.

Ex 41. The maximum value of $(\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$ is equal to

- (a) $\frac{\pi^2}{2}$ (b) $\frac{5\pi^2}{4}$
 (c) π^2 (d) None of these

Sol. Let $I = (\sec^{-1} x)^2 + (\operatorname{cosec}^{-1} x)^2$

$$= (\sec^{-1} x + \operatorname{cosec}^{-1} x)^2 - 2 \sec^{-1} x \cdot \operatorname{cosec}^{-1} x$$

$$= \frac{\pi^2}{4} - 2 \sec^{-1} x \left(\frac{\pi}{2} - \sec^{-1} x \right)$$

$$= \frac{\pi^2}{4} + 2 \left((\sec^{-1} x)^2 - \frac{\pi}{2} (\sec^{-1} x) + \frac{\pi^2}{16} - \frac{\pi^2}{16} \right)$$

$$= \frac{\pi^2}{4} + 2 \left(\sec^{-1} x - \frac{\pi}{4} \right)^2$$

$$\Rightarrow I_{\max} = \frac{\pi^2}{4} + 2 \left[\frac{9\pi^2}{16} \right] = \frac{\pi^2}{8} + \frac{9\pi^2}{8}$$

$$\Rightarrow I_{\max} = \frac{5\pi^2}{4}$$

Hence, (b) is the correct answer.

Ex 42. The complete solution set of $[\cot^{-1} x]^2 - 6[\cot^{-1} x] + 9 \leq 0$, where $[\]$

denotes the greatest integer function, is

- (a) $(-\infty, \cot 3]$
 (b) $[\cot 3, \cot 2)$
 (c) $[\cot 3, \infty)$
 (d) None of the above

Sol. $[\cot^{-1} x]^2 - 6[\cot^{-1} x] + 9 \leq 0$
 $\Rightarrow ([\cot^{-1} x] - 3)^2 \leq 0$
 $\Rightarrow [\cot^{-1} x] = 3$
 $\Rightarrow 3 \leq \cot^{-1} x < 4$
 $\Rightarrow x \in (-\infty, \cot 3]$
Hence, (a) is the correct answer.

Ex 43. If $[\sin^{-1} x] + [\cos^{-1} x] = 0$, where x is a non-negative real number and $[\]$ denotes the greatest integer function, then complete set of values of x is

- (a) $(\cos 1, 1)$
(b) $(-1, \cos 1)$
(c) $(\sin 1, 1)$
(d) $(\cos 1, \sin 1)$

Sol. $[\sin^{-1} x] + [\cos^{-1} x] = 0$ and $x \geq 0$
 $\Rightarrow [\sin^{-1} x] = 0, [\cos^{-1} x] = 0$
 $\Rightarrow x \in [0, \sin 1), x \in (\cos 1, 1]$
 $\Rightarrow x \in (\cos 1, \sin 1)$
Hence, (d) is the correct answer.

Ex 44. Complete set of values of x satisfying $[\tan^{-1} x] + [\cot^{-1} x] = 2$, where $[\]$ denotes the greatest integer function, is

- (a) $(\cot 3, \cot 2)$ (b) $(\cot 3, -\tan 1]$
(c) $(\cot 3, 0)$ (d) None of these

Sol. $[\cot^{-1} x] = 0, 1, 2, 3, [\tan^{-1} x] = -2, -1, 0, 1$
Then, we have the following cases:

Case I $[\cot^{-1} x] = [\tan^{-1} x] = 1$

$\Rightarrow x \in (\cot 2, \cot 1], x \in [\tan 1, \infty)$
 $\Rightarrow$ No such x exists (as $\cot 1 < \tan 1$).

Case II $[\cot^{-1} x] = 3, [\tan^{-1} x] = -1$

$\Rightarrow x \in (-\infty, \cot 3], x \in [-\tan 1, 0) = -1$
 $\Rightarrow$ No such x exists (as $\cot 3 < -\tan 1$).

Case III $[\cot^{-1} x] = 2, [\tan^{-1} x] = 0$

$\Rightarrow x \in (\cot 3, \cot 2], x \in [0, \tan 1)$
 $\Rightarrow$ No such x exists (as $\cot 2 < 0$).

Thus, $[\cot^{-1} x] + [\tan^{-1} x]$ can't be equal to 2 for any x .

Hence, (d) is the correct answer.

Ex 45. If $[\sin^{-1} \cos^{-1} \sin^{-1} \tan^{-1} x] = 1$, where $[\]$ denotes the greatest integer function, then x belongs to the interval

- (a) $[\tan \sin \cos 1, \tan \sin \cos \sin 1]$
(b) $[\tan \cos \sin 1, \tan \sin \cos \sin 1]$
(c) $[1, -1]$
(d) $[\sin \cos \tan 1, \sin \cos \sin \tan 1]$

Sol. We have, $[\sin^{-1} \cos^{-1} \sin^{-1} \tan^{-1} x] = 1$

$$\Rightarrow 1 \leq \sin^{-1} \cos^{-1} \sin^{-1} \tan^{-1} x \leq \frac{\pi}{2}$$

$$\Rightarrow \sin 1 \leq \cos^{-1} \sin^{-1} \tan^{-1} x \leq 1$$

$$\Rightarrow \cos \sin 1 \geq \sin^{-1} \tan^{-1} x \geq \cos 1$$

$$\Rightarrow \sin \cos \sin 1 \geq \tan^{-1} x \geq \sin \cos 1$$

$$\Rightarrow \tan \sin \cos \sin 1 \geq x \geq \tan \sin \cos 1$$

Thus, $x \in [\tan \sin \cos 1, \tan \sin \cos \sin 1]$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 46. An integral solution of the equation

$$\tan^{-1} x + \tan^{-1} \left(\frac{1}{y} \right) = \tan^{-1} 3 \text{ is}$$

- (a) $(2, 7)$ (b) $(4, -13)$
(c) $(5, -8)$ (d) $(1, 2)$

Sol. $\tan^{-1} x + \tan^{-1} \left(\frac{1}{y} \right) = \tan^{-1} 3$

$$\Rightarrow \tan^{-1} \left(\frac{x + 1/y}{1 - x/y} \right) = \tan^{-1} 3 \text{ if } 1 - \frac{x}{y} > 0$$

$$\Rightarrow x + \frac{1}{y} = 3 \left(1 - \frac{x}{y} \right) \Rightarrow y = \frac{3x + 1}{3 - x}$$

which is satisfied by options (a), (b), (c) and (d).

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 47. The value of $\cos \left[\frac{1}{2} \cos^{-1} \cos \left(-\frac{14\pi}{5} \right) \right]$ is

- (a) $\cos \left(-\frac{7\pi}{5} \right)$ (b) $\sin \left(\frac{\pi}{10} \right)$
(c) $\sin \left(\frac{2\pi}{5} \right)$ (d) $-\cos \left(\frac{3\pi}{5} \right)$

Sol. We have, $\cos \left(-\frac{14\pi}{5} \right) = \cos \frac{14\pi}{5} = \cos \frac{4\pi}{5}$

$$\therefore \cos \frac{1}{2} \cos^{-1} \left(\cos \frac{4\pi}{5} \right) = \cos \frac{4\pi}{10} = \cos \frac{2\pi}{5}$$

$$= \sin \left(\frac{\pi}{2} - \frac{2\pi}{5} \right) = \sin \left(\frac{\pi}{10} \right)$$

$$\text{Also, } \cos \frac{2\pi}{5} = -\cos \left(\pi - \frac{2\pi}{5} \right) = -\cos \left(\frac{3\pi}{5} \right)$$

Hence, (b) and (d) are the correct answers.

Ex 48. For the equation $2x = \tan(2\tan^{-1} a) + 2\tan(\tan^{-1} a + \tan^{-1} a^3)$, which of the following is invalid?

- (a) $a^2 x + 2a = x$ (b) $a^2 + 2ax + 1 = 0$
(c) $a \neq 0$ (d) $a \neq -1, 1$

Sol. Given, $2x = \tan(2\tan^{-1} a) + 2\tan(\tan^{-1} a + \tan^{-1} a^3)$

$$\therefore 2x = \frac{2a}{1 - a^2} + \frac{2(a + a^3)}{1 - a^4}$$

$$\Rightarrow a \neq \pm 1 \quad \left[\because \tan 2\theta = \frac{2\tan \theta}{1 - \tan^2 \theta} \text{ and } \tan(\theta + \phi) = \frac{\tan \theta + \tan \phi}{1 - \tan \theta \tan \phi} \right]$$

$$\text{and } x = \frac{a}{1-a^2} + \frac{a}{1-a^2} = \frac{2a}{1-a^2}$$

$$\Rightarrow x(1-a^2) = 2a$$

$$\Rightarrow a^2x + 2a = x$$

Hence, (b) and (c) are the correct answers.

Ex 49. Identify the pairs(s) of functions which are identical.

(a) $y = \tan(\cos^{-1} x); y = \frac{\sqrt{1-x^2}}{x}$

(b) $y = \tan(\cot^{-1} x); y = \frac{1}{x}$

(c) $y = \sin(\arctan x); y = \frac{x}{\sqrt{1+x^2}}$

(d) $y = \cos(\arctan x); y = \sin(\operatorname{arccot} x)$

Sol. (a) $y = \tan(\cos^{-1} x)$

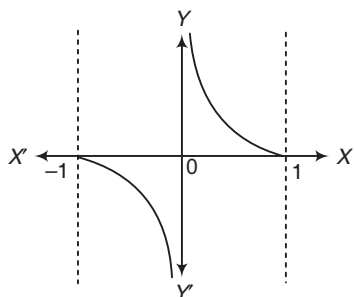
Note that $0 \leq \cos^{-1} x \leq \pi$ but $\tan \frac{\pi}{2}$ is not defined.

So, $\cos^{-1} x \neq \frac{\pi}{2}$

$\Rightarrow x \neq 0$

Hence, domain is $[-1, 1] - \{0\}$ and range is R .

The graph is as shown below:



(b) $y = \tan(\cot^{-1} x)$

Note that $0 < \cot^{-1} x < \pi$ but $\tan \frac{\pi}{2}$ is not defined.

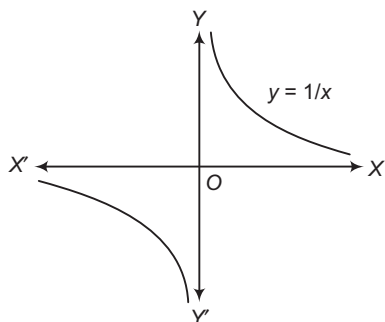
So, $\cot^{-1} x \neq \frac{\pi}{2} \Rightarrow x \neq 0$

Hence, domain = $R - \{0\}$

Since, $\cot^{-1} x \neq 0, \pi \Rightarrow y \neq 0$

Hence, range is $R - \{0\}$.

The graph is as shown below:



(c) $y = \sin(\tan^{-1} x)$

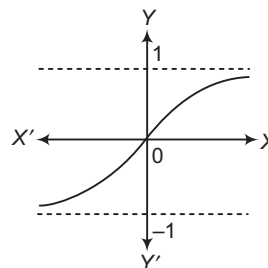
$\Rightarrow -\frac{\pi}{2} < \tan^{-1} x < \frac{\pi}{2} \Rightarrow y \in (-1, 1)$

Since, $\tan^{-1} x$ is defined for all $x \in R$ and $\sin x$ is also defined for all $x \in R$.

$\Rightarrow$ Domain is R

and $y = \frac{x}{\sqrt{1+x^2}}$

The graph is as shown below:



(d) $y = \cos(\tan^{-1} x)$

$\Rightarrow \tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

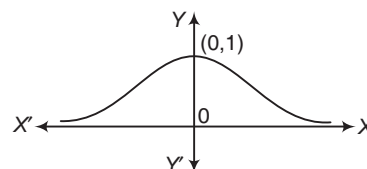
$\Rightarrow y \in (0, 1)$

Domain is R

and $y = \frac{1}{\sqrt{1+x^2}}$

Note that $\sin^{-1}(\cot^{-1} x)$ also has the same graph.

The graph is as shown below:



Hence, (a), (b), (c) and (d) are the correct answers.

Ex 50. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then

(a) $x^2 + y^2 + z^2 + 2xyz = 1$

(b) $2(\sin^{-1} x + \sin^{-1} y + \sin^{-1} z) = \cos^{-1} x + \cos^{-1} y + \cos^{-1} z$

(c) $xy + yz + zx = x + y + z - 1$

(d) $\left(x + \frac{1}{x}\right) + \left(y + \frac{1}{y}\right) + \left(z + \frac{1}{z}\right) \geq 6$

Sol. $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$ (i)

$\Rightarrow \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{\pi}{2}$ (ii)

Now, $\cos^{-1} x + \cos^{-1} y = \cos^{-1}(-z)$

$\Rightarrow xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

From Eqs. (i) and (ii), we get

$2(\sin^{-1} x + \sin^{-1} y + \sin^{-1} z) = \cos^{-1} x + \cos^{-1} y + \cos^{-1} z$

Hence, (a) and (b) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 51-53) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true, Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 51. Statement I

$$\operatorname{cosec}^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) > \sec^{-1}\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$$

Statement II $\operatorname{cosec}^{-1}x > \sec^{-1}x$, if $1 \leq x < \sqrt{2}$

Sol. $\operatorname{cosec}^{-1}x > \sec^{-1}x$

$$\begin{aligned} \Rightarrow \operatorname{cosec}^{-1}x &> \frac{\pi}{2} - \operatorname{cosec}^{-1}x \\ \Rightarrow \operatorname{cosec}^{-1}x &> \frac{\pi}{4} \\ \Rightarrow 1 \leq x < \sqrt{2} \quad \text{and} \quad \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) &\in [1, \sqrt{2}] \end{aligned}$$

Statement II is true and explain Statement I.

Hence, (a) is the correct answer.

Ex 52. Let $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Statement I $f'(2) = -2/5$

Statement II $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2 \tan^{-1}x$, $\forall x > 1$

Sol. $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2 \tan^{-1}x$, $x \geq 1$
 $\Rightarrow f'(x) = -\frac{2}{1+x^2} \Rightarrow f'(2) = -\frac{2}{5}$

Statement I is true, Statement II is true and Statement II is a correct explanation for Statement I.

Hence, (a) is the correct answer.

Ex 53. Statement I $\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

Statement II $\sin^{-1}x > \tan^{-1}y$ for $x > y$, $\forall x, y \in (0, 1)$

Sol. $\sin^{-1}x = \tan^{-1}\frac{x}{\sqrt{1-x^2}} > \tan^{-1}x > \tan^{-1}y$
 $\left[\because x > y \text{ and } \frac{x}{\sqrt{1-x^2}} > x, \text{ for } x \in (0, 1) \right]$

$\Rightarrow \sin^{-1}x > \tan^{-1}y$

$\therefore$ Statement II is true.

$e < \pi, \frac{1}{\sqrt{e}} > \frac{1}{\sqrt{\pi}}$

By Statement II,

$\sin^{-1}\left(\frac{1}{\sqrt{e}}\right) > \tan^{-1}\left(\frac{1}{\sqrt{\pi}}\right)$

$\therefore$ Statement I is true.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 54-56) It is given that $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$, where $x > 0$ and

$B = (\cos^{-1}t)^2 + (\sin^{-1}t)^2$, where $t \in \left[0, \frac{1}{\sqrt{2}}\right]$ and

$\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$ for $-1 \leq x \leq 1$ and

$\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \forall x \in \mathbb{R}.$

Ex 54. The interval in which A lies, is

- (a) $\left[\frac{\pi^3}{7}, \frac{\pi^3}{2}\right]$ (b) $\left[\frac{\pi^3}{32}, \frac{\pi^3}{8}\right]$
(c) $\left[\frac{\pi^3}{40}, \frac{\pi^3}{10}\right]$ (d) None of these

Sol. $A = (\tan^{-1}x)^3 + (\cot^{-1}x)^3$
 $\Rightarrow A = (\tan^{-1}x + \cot^{-1}x)^3 - 3 \tan^{-1}x \cot^{-1}x (\tan^{-1}x + \cot^{-1}x)$

$\Rightarrow A = \left(\frac{\pi}{2}\right)^3 - 3 \tan^{-1}x \cot^{-1}x \cdot \frac{\pi}{2}$
 $\Rightarrow A = \frac{\pi^3}{8} - \frac{3\pi}{2} \tan^{-1}x \left(\frac{\pi}{2} - \tan^{-1}x\right)$
 $\Rightarrow A = \frac{\pi^3}{32} + \frac{3\pi}{2} \left(\tan^{-1}x - \frac{\pi}{4}\right)^2$

As $x > 0, \frac{\pi^3}{32} \leq A < \frac{\pi^3}{8}$

Hence, (b) is the correct answer.

Ex 55. The maximum value of B is

- (a) $\frac{\pi^2}{8}$
(b) $\frac{\pi^2}{16}$
(c) $\frac{\pi^2}{4}$
(d) None of the above

Sol. $B = (\sin^{-1} t)^2 + (\cos^{-1} t)^2$
 $= (\sin^{-1} t + \cos^{-1} t)^2 - 2 \sin^{-1} t \cos^{-1} t$
 $= \frac{\pi^2}{4} - 2 \sin^{-1} t \left(\frac{\pi}{2} - \sin^{-1} t \right)$
 $= \frac{\pi^2}{8} + 2 \left(\sin^{-1} t - \frac{\pi}{4} \right)^2$
 $\therefore B_{\max} = \frac{\pi^2}{8} + 2 \cdot \frac{\pi^2}{16} = \frac{\pi^2}{4}$

Hence, (c) is the correct answer.

Ex 56. If least value of A is λ and maximum value of

B is μ , then $\cot^{-1} \cot \left(\frac{\lambda - \mu r}{\mu} \right)$ is equal to

- (a) $\frac{\pi}{8}$ (b) $-\frac{\pi}{8}$
 (c) $\frac{7\pi}{8}$ (d) $-\frac{7\pi}{8}$

Sol. $\lambda = \frac{\pi^3}{32}, \mu = \frac{\pi^2}{4} \Rightarrow \frac{\lambda}{\mu} = \frac{\pi}{8}$
 $\Rightarrow \frac{\lambda - \mu\pi}{\mu} = \frac{\pi}{8} - \pi = -\frac{7\pi}{8}$
 $\cot^{-1} \cot \left(\frac{\lambda - \mu\pi}{\mu} \right) = \cot^{-1} \cot \left(-\frac{7\pi}{8} \right) = \frac{\pi}{8}$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 57-59) Given that,

$$\tan^{-1} \left(\frac{2x}{1-x^2} \right) = \begin{cases} 2 \tan^{-1} x, & \text{if } |x| \leq 1 \\ -\pi + 2 \tan^{-1} x, & \text{if } x > 1 \\ \pi + 2 \tan^{-1} x, & \text{if } x < -1 \end{cases}$$

$$\sin^{-1} \left(\frac{2x}{1+x^2} \right) = \begin{cases} 2 \tan^{-1} x, & \text{if } |x| \leq 1 \\ \pi - 2 \tan^{-1} x, & \text{if } x > 1 \\ -(\pi + 2 \tan^{-1} x), & \text{if } x < -1 \end{cases}$$

and $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$, for $-1 \leq x \leq 1$

Ex 57. $\sin^{-1} \left(\frac{4x}{x^2 + 4} \right) + 2 \tan^{-1} \left(-\frac{x}{2} \right)$ is independent

of x , then

- (a) $x \in [-3, 4]$
 (b) $x \in [-2, 2]$
 (c) $x \in [-1, 1]$
 (d) $x \in [1, \infty]$

Sol. $\sin^{-1} \left(\frac{4x}{x^2 + 4} \right) + 2 \tan^{-1} \left(-\frac{x}{2} \right)$
 $= \sin^{-1} \left(\frac{2 \cdot \frac{x}{2}}{\left(\frac{x}{2} \right)^2 + 1} \right) - 2 \tan^{-1} \frac{x}{2}$

This will independent of x , if

$$\sin^{-1} \left(\frac{2 \cdot \frac{x}{2}}{\left(\frac{x}{2} \right)^2 + 1} \right) = 2 \tan^{-1} \left(\frac{x}{2} \right)$$

which is true, if $\left| \frac{x}{2} \right| \leq 1$, i.e. $|x| \leq 2$.

Hence, (b) is the correct answer.

Ex 58. If $\cos^{-1} \left(\frac{6x}{1+9x^2} \right) = -\frac{\pi}{2} + 2 \tan^{-1} 3x$, then

$x \in$

- (a) $\left(\frac{1}{3}, \infty \right)$
 (b) $(-1, \infty)$
 (c) $(-\infty, m-1)$
 (d) None of the above

Sol. $\cos^{-1} \left(\frac{6x}{1+9x^2} \right) = -\frac{\pi}{2} + 2 \tan^{-1} 3x$
 $\Rightarrow \frac{\pi}{2} - \sin^{-1} \left(\frac{6x}{1+9x^2} \right) = \frac{\pi}{2} + 2 \tan^{-1} 3x$
 $\Rightarrow \sin^{-1} \left(\frac{6x}{1+9x^2} \right) = \pi - 2 \tan^{-1} 3x$
 $\Rightarrow \sin^{-1} \left(\frac{2 \cdot 3x}{1+(3x)^2} \right) = \pi - 2 \tan^{-1} 3x$

Above is true, when $3x > 1$

$$\Rightarrow x > \frac{1}{3} \Rightarrow x \in \left(\frac{1}{3}, \infty \right)$$

Hence, (a) is the correct answer.

Ex 59. If $(x-1)(x^2+1) > 0$, then

$\sin \left(\frac{1}{2} \tan^{-1} \left(\frac{2x}{1-x^2} \right) - \tan^{-1} x \right)$ is equal to

- (a) 1
 (b) $\frac{1}{\sqrt{2}}$
 (c) -1
 (d) None of the above

Sol. Given, $(x-1)(x^2+1) > 0 \Rightarrow x > 1$

$$\therefore \sin \left[\frac{1}{2} \tan^{-1} \left(\frac{2x}{1-x^2} \right) - \tan^{-1} x \right]$$

$$= \sin \left[\frac{1}{2} (-\pi + 2 \tan^{-1} x) - \tan^{-1} x \right]$$

$$= \sin \left(-\frac{\pi}{2} \right) = -1$$

Hence, (c) is the correct answer.

Type 5. Match the Columns

Ex 60. Match the statements of Column I with values of Column II, where $[]$ represents the greatest integer function.

Column I	Column II
A. If $f(x) = \sin^{-1} x$ and $\lim_{x \rightarrow 1/2^+} f(3x - 4x^3) = a - 3$ $\lim_{x \rightarrow 1/2^+} f(x)$, then $[a]$ equals	p. 2
B. If $f(x) = \tan^{-1} g(x)$, where $g(x) = \frac{3x - x^3}{1 - 3x^2}$ and $\lim_{h \rightarrow 0} \frac{f(a + 3h) - f(a)}{3h} = \frac{3}{1 + a^2}$, when $-\frac{1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$, then $\left[\lim_{h \rightarrow 0} \frac{f\left(\frac{1}{2} + 6h\right) - f\left(\frac{1}{2}\right)}{6h} \right]$ is equal to	q. 3
C. If $\cos^{-1}(4x^3 - 3x) = a + b \cos^{-1} x$ for $-1 < x < \frac{-1}{2}$, then $[a + b + 2]$ is equal to	r. 4
D. If $f(x) = \cos^{-1}(4x^3 - 3x)$ and $\lim_{x \rightarrow 1/2^+} f'(x) = a$ and $\lim_{x \rightarrow 1/2^-} f'(x) = b$, then $a + b + 3$ equals	s. -2

Sol. A. $\sin^{-1}(3x - 4x^3) = \pi - 3 \sin^{-1} x$, if $\frac{1}{2} < x \leq 1$
 $\therefore \lim_{x \rightarrow 1/2^+} f(3x - 4x^3) = \lim_{x \rightarrow 1/2^+} (\pi - 3 \sin^{-1} x)$
 $= \pi - 3 \lim_{x \rightarrow 1/2^+} \sin^{-1} x$

$\therefore a = \pi \Rightarrow [a] = 3$

B. We know that,

$f(x) = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right) = 3 \tan^{-1} x$,

when $-\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$

We have, if $-\frac{1}{\sqrt{3}} < a < \frac{1}{\sqrt{3}}$, then

$\lim_{h \rightarrow 0} \frac{f(a + 3h) - f(a)}{3h} = \frac{3}{1 + a^2}$

$\Rightarrow f'(a) = \frac{3}{1 + a^2}$

$= \lim_{h \rightarrow 0} \frac{f\left(\frac{1}{2} + 6h\right) - f\left(\frac{1}{2}\right)}{6h}$
 $= f'\left(\frac{1}{2}\right) = \frac{12}{5}$

$\therefore$ Required value = 2

C. $\cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos 3\theta) = 3\theta - 2\pi$
 $\left[\because \frac{2\pi}{3} < \theta < \pi \right]$

$= -2\pi + 3 \cos^{-1} x$

$\therefore [a + b + 2] = [-2\pi + 3 + 2] = -2$

D. $f(x) = \cos^{-1}(4x^3 - 3x) = \cos^{-1}(\cos 3\theta)$

$= \begin{cases} 3\theta, & 0 \leq \theta \leq \frac{\pi}{3} \\ 2\pi - 3\theta, & \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2} \end{cases}$

$= \begin{cases} 3 \cos^{-1} x, & \frac{1}{2} \leq x \leq 1 \\ 2\pi - 3 \cos^{-1} x, & 0 \leq x \leq \frac{1}{2} \end{cases}$

$\therefore f'(x) = \begin{cases} \frac{-3}{\sqrt{1-x^2}}, & \frac{1}{2} \leq x \leq 1 \\ \frac{3}{\sqrt{1-x^2}}, & 0 \leq x \leq \frac{1}{2} \end{cases}$

$a = \lim_{x \rightarrow 1/2^+} f'(x) = -2\sqrt{3}$

$b = \lim_{x \rightarrow 1/2^-} f'(x) = 2\sqrt{3}$

$\therefore a + b + 3 = 3$

$A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow q$

Ex 61. Match the statements of Column I with values of Column II.

Column I	Column II
A. Absolute difference of greatest and least values of $\sqrt{2}(\sin 2x - \cos 2x)$ is	p. $\frac{\pi}{4}$
B. Absolute difference of greatest and least values of $x^2 - 4x + 3$, $x \in [1, 3]$, is	q. $\frac{\pi}{6}$
C. Greatest value of $\tan^{-1} \left(\frac{1-x}{1+x} \right)$, $x \in [0, 1]$, is	r. 4
D. Absolute difference of greatest and least values of $\cos^{-1} x^2$, $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$, is	s. 1

Sol. A. Required difference = $2 - (-2) = 4$

B. Let $f(x) = x^2 - 4x + 3$

$f'(x) = 2x - 4 = 0$

$\Rightarrow x = 2$

$f(1) = 0, f(2) = -1, f(3) = 0$

$\therefore |\text{Greatest value} - \text{Least value}| = 1$

C. $\tan^{-1} \left(\frac{1-x}{1+x} \right) = \tan^{-1} 1 - \tan^{-1} x$

$\therefore$ Greatest value = $\frac{\pi}{4}$

D. Greatest value = $\frac{\pi}{2}$

Least value = $\frac{\pi}{3}$

$\therefore$ Required difference = $\frac{\pi}{6}$

$A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$

Type 6. Single Integer Answer Type Questions

Ex 62. If $\sin^{-1} x \in \left(0, \frac{\pi}{2}\right)$, then find the value of

$$\tan \left[\frac{\cos^{-1}(\sin(\cos^{-1} x)) + \sin^{-1}(\cos(\sin^{-1} x))}{2} \right].$$

Sol. (1) As $\sin^{-1} x \in \left(0, \frac{\pi}{2}\right)$

$$\text{and } \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x \Rightarrow \cos^{-1} x \in \left(0, \frac{\pi}{2}\right)$$

$$\Rightarrow \sin(\cos^{-1} x) = \cos(\sin^{-1} x) = \sqrt{1-x^2}$$

$$\text{Thus, } \cos^{-1}(\sin(\cos^{-1} x)) + \sin^{-1}(\cos(\sin^{-1} x)) = \frac{\pi}{2}$$

$$\therefore \text{Required value} = \tan \frac{\pi}{4} = 1$$

Ex 63. If $x = \operatorname{cosec}[\tan^{-1}\{\cos(\cot^{-1}(\sec(\sin^{-1} a)))\}]$ and $y = \sec[\cot^{-1}\{\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))\}]$, where $a \in [0, 1]$. Then, for what value of λ , is $x^2 = y^2 = \lambda - a^2$?

Sol. (3) We have, $x = \operatorname{cosec}[\tan^{-1}\{\cos(\cot^{-1}(\sec(\sin^{-1} a)))\}]$

$$= \operatorname{cosec} \left[\tan^{-1} \left\{ \cos \left(\cot^{-1} \left(\sec \left(\sec^{-1} \frac{1}{\sqrt{1-a^2}} \right) \right) \right) \right\} \right]$$

$$\left[\because \sin^{-1} a = \sec^{-1} \frac{1}{\sqrt{1-a^2}} \right]$$

$$= \operatorname{cosec} \left[\tan^{-1} \left\{ \cos \left(\cot^{-1} \frac{1}{\sqrt{1-a^2}} \right) \right\} \right]$$

$$= \operatorname{cosec} \left[\tan^{-1} \left\{ \cos \left(\cos^{-1} \frac{1}{\sqrt{2-a^2}} \right) \right\} \right]$$

$$= \operatorname{cosec} \left(\tan^{-1} \frac{1}{\sqrt{2-a^2}} \right)$$

$$= \operatorname{cosec} \left(\operatorname{cosec}^{-1} \sqrt{3-a^2} \right) = \sqrt{3-a^2}$$

$$\text{and } y = \sec[\cot^{-1}\{\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1} a)))\}]$$

$$= \sec \left[\cot^{-1} \left\{ \sin \left(\tan^{-1} \left(\operatorname{cosec} \left(\operatorname{cosec}^{-1} \frac{1}{\sqrt{1-a^2}} \right) \right) \right) \right\} \right]$$

$$= \sec \left[\cot^{-1} \left\{ \sin \left(\tan^{-1} \frac{1}{\sqrt{1-a^2}} \right) \right\} \right]$$

$$= \sec \left[\cot^{-1} \left\{ \sin \left(\sin^{-1} \frac{1}{\sqrt{2-a^2}} \right) \right\} \right]$$

$$= \sec \left(\cot^{-1} \frac{1}{\sqrt{2-a^2}} \right) = \sec(\sec^{-1} \sqrt{3-a^2}) = \sqrt{3-a^2}$$

$$\therefore x^2 = y^2 = 3 - a^2 \Rightarrow \lambda = 3$$

Ex 64. Find the value of λ for which,

$$\tan^{-1} \left\{ \frac{\cos 2\alpha \sec 2\beta + \cos 2\beta \sec 2\alpha}{\lambda} \right\}$$

$$= \tan^{-1} \{ \tan^2(\alpha + \beta) \tan^2(\alpha - \beta) \} + \tan^{-1} 1.$$

Sol. (2) We have,

$$\text{RHS} = \tan^{-1} \{ \tan^2(\alpha + \beta) \tan^2(\alpha - \beta) \} + \tan^{-1} 1$$

$$= \tan^{-1} \left\{ \frac{\tan^2(\alpha + \beta) \tan^2(\alpha - \beta) + 1}{1 - \tan^2(\alpha + \beta) \tan^2(\alpha - \beta)} \right\}$$

$$= \tan^{-1} \left\{ \frac{\sin^2(\alpha + \beta) \sin^2(\alpha - \beta) + \cos^2(\alpha + \beta) \cos^2(\alpha - \beta)}{\cos^2(\alpha + \beta) \cos^2(\alpha - \beta) - \sin^2(\alpha + \beta) \sin^2(\alpha - \beta)} \right\}$$

$$= \tan^{-1} \left\{ \frac{\begin{bmatrix} \{2 \sin(\alpha + \beta) \sin(\alpha - \beta)\}^2 \\ + \{2 \cos(\alpha + \beta) \cos(\alpha - \beta)\}^2 \end{bmatrix}}{\begin{bmatrix} \{2 \cos(\alpha + \beta) \cos(\alpha - \beta)\}^2 \\ + \{2 \sin(\alpha + \beta) \sin(\alpha - \beta)\}^2 \end{bmatrix}} \right\}$$

$$= \tan^{-1} \left\{ \frac{(\cos 2\beta - \cos 2\alpha)^2 + (\cos 2\alpha + \cos 2\beta)^2}{(\cos 2\alpha + \cos 2\beta)^2 - (\cos 2\beta - \cos 2\alpha)^2} \right\}$$

$$= \tan^{-1} \left\{ \frac{\cos^2 2\alpha + \cos^2 2\beta}{2 \cos 2\alpha \cos 2\beta} \right\}$$

$$= \tan^{-1} \left\{ \frac{\cos 2\alpha \sec 2\beta + \cos 2\beta \sec 2\alpha}{2} \right\}$$

Target Exercises

Type 1. Only One Correct Option

- The value of $\cos^{-1}\left(\cos \frac{5\pi}{3}\right) + \sin^{-1}\left(\sin \frac{5\pi}{3}\right)$ is
 (a) $\frac{\pi}{2}$ (b) $\frac{5\pi}{3}$
 (c) $\frac{10\pi}{3}$ (d) 0
- The principal value of $\cos^{-1}\left(-\sin \frac{7\pi}{6}\right)$ is
 (a) $\frac{5\pi}{3}$ (b) $\frac{7\pi}{6}$
 (c) $\frac{\pi}{3}$ (d) None of these
- The principal value of $\sin^{-1}\left\{\cos\left(\sin^{-1}\frac{\sqrt{3}}{2}\right)\right\}$ is
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
 (c) $-\frac{\pi}{3}$ (d) None of these
- The value of $\cos\left\{\tan^{-1}\left(\tan \frac{15\pi}{4}\right)\right\}$ is
 (a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{2}}$
 (c) 1 (d) None of these
- $-\frac{2\pi}{5}$ is the principal value of
 (a) $\cos^{-1}\left(\cos \frac{7\pi}{5}\right)$ (b) $\sin^{-1}\left(\sin \frac{7\pi}{5}\right)$
 (c) $\sec^{-1}\left(\sec \frac{7\pi}{5}\right)$ (d) None of these
- The principal value of $\cos^{-1}\left\{\frac{1}{\sqrt{2}}\left(\cos \frac{9\pi}{10} - \sin \frac{9\pi}{10}\right)\right\}$ is
 (a) $\frac{3\pi}{20}$ (b) $\frac{17\pi}{20}$
 (c) $\frac{7\pi}{10}$ (d) None of the above
- $\operatorname{cosec}^{-1}(\cos x)$ is real, if
 (a) $x \in [-1, 1]$
 (b) $x \in R$
 (c) x is an odd multiple of $\frac{\pi}{2}$
 (d) x is a multiple of π
- If $\sin^{-1} x - \cos^{-1} x = \frac{\pi}{6}$, then x is
 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$
 (c) $-\frac{1}{2}$ (d) None of these
- If $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1$, then x is equal to
 (a) 1 (b) 0 (c) $\frac{4}{5}$ (d) $\frac{1}{5}$
- The principal value of $\sin^{-1}\left(\sin \frac{3\pi}{4}\right)$ is
 (a) $\frac{5\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{4}$ (d) $-\frac{\pi}{4}$
- $\cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right)$ is equal to
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{4}$
- $\cos^{-1}\left(\frac{15}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right)$ is equal to
 (a) $\frac{\pi}{2}$ (b) $\cos^{-1}\left(\frac{17}{221}\right)$
 (c) $\frac{\pi}{4}$ (d) None of these
- The value of $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$ is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) 0
- The value of $\cot^{-1} 3 + \sec^{-1}\frac{\sqrt{5}}{2}$ is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
 (c) $\frac{\pi}{2}$ (d) None of these
- The value of $\sin^2\left(\cos^{-1}\frac{1}{2}\right) + \cos^2\left(\sin^{-1}\frac{1}{3}\right)$ is
 (a) $\frac{17}{36}$ (b) $\frac{59}{36}$
 (c) $\frac{36}{59}$ (d) None of these
- The value of $\sin^{-1}(\sin 10)$ is
 (a) $3\pi - 10$ (b) $10 - 3\pi$
 (c) 10 (d) None of these

- 17.** The value of $\sin\left[\cot^{-1}\left(\cot\frac{17\pi}{3}\right)\right]$ is
 (a) $-\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{2}$
 (c) $\frac{1}{\sqrt{2}}$ (d) None of these
- 18.** $\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]$ is equal to
 (a) $\sqrt{\frac{x^2+2}{x^2+3}}$ (b) $\sqrt{\frac{x^2+2}{x^2+1}}$
 (c) $\sqrt{\frac{x^2+1}{x^2+2}}$ (d) None of these
- 19.** $4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}$ is equal to
 (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$
- 20.** $\sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right]$ is equal to
 (a) $\frac{6}{25}$ (b) $\frac{24}{25}$ (c) $\frac{4}{5}$ (d) $-\frac{24}{25}$
- 21.** Consider the following statements:
 $\tan^{-1}x + \tan^{-1}\frac{1}{x} =$
 I. $\frac{\pi}{2}$ for all $x \neq 0$ II. $-\frac{\pi}{2}$ for all $x \neq 0$
 III. $\frac{\pi}{2}$ for all $x > 0$ IV. $-\frac{\pi}{2}$ for all $x < 0$
 Then, correct statements are
 (a) I (b) II
 (c) III and IV (d) None of these
- 22.** If $-1 \leq x, y, z \leq 1$ such that
 $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2}$, then the value of
 $x^{2000} + y^{2001} + z^{2002} - \frac{9}{x^{2000} + y^{2001} + z^{2002}}$ is
 (a) 1 (b) 2
 (c) 3 (d) None of these
- 23.** $\tan^{-1}\frac{1}{5} + \tan^{-1}\frac{1}{7} + \tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{8}$ is equal to
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) π
- 24.** The value of $3\tan^{-1}\frac{1}{2} + 2\tan^{-1}\frac{1}{5} + \sin^{-1}\frac{142}{65\sqrt{5}}$ is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
 (c) π (d) None of these
- 25.** If $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$, then x is
 equal to
 (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{4}$ (d) 1
- 26.** $\tan^{-1}a + \tan^{-1}b$, where $a > 0, b > 0, ab > 1$, is
 equal to
 (a) $\tan^{-1}\frac{a+b}{1-ab}$ (b) $\tan^{-1}\frac{a+b}{1-ab} - \pi$
 (c) $\pi + \tan^{-1}\frac{a+b}{1-ab}$ (d) None of these
- 27.** If $\sin\theta$ and $\cos\theta$ are the roots of the equation
 $ax^2 + bx + c = 0$, then $\cos^{-1}(a^2 - b^2 + 2ac)$ is
 equal to
 (a) π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) 0
- 28.** $\cos^{-1}\left\{\frac{1}{2}x^2 + \sqrt{1-x^2} \cdot \sqrt{1-\frac{x^2}{4}}\right\} = \cos^{-1}\frac{x}{2} - \cos^{-1}x$
 holds for
 (a) $|x| \leq 1$ (b) $x \in R$
 (c) $0 \leq x \leq 1$ (d) $-1 \leq x \leq 0$
- 29.** The formula $\cos^{-1}\frac{1-x^2}{1+x^2} = 2\tan^{-1}x$ holds only for
 (a) $x \in R$ (b) $|x| \leq 1$
 (c) $x \in (-1, 1]$ (d) $x \in [1, \infty)$
- 30.** $\tan\left(\frac{\pi}{4} + \frac{1}{2}\cos^{-1}x\right) + \tan\left(\frac{\pi}{4} - \frac{1}{2}\cos^{-1}x\right)$, $x \neq 0$ is
 equal to
 (a) x (b) $2x$
 (c) $\frac{2}{x}$ (d) None of these
- 31.** If $\cos^{-1}\frac{3}{5} - \sin^{-1}\frac{4}{5} = \cos^{-1}x$, then x is equal to
 (a) 0 (b) 1
 (c) -1 (d) None of these
- 32.** $\tan^{-1}n + \cot^{-1}(n+1)$ is equal to
 (a) $\cot^{-1}(n^2 + n + 1)$ (b) $\cot^{-1}(n^2 - n + 1)$
 (c) $\tan^{-1}(n^2 + n + 1)$ (d) None of these
- 33.** If $\tan^{-1}\frac{1}{1+2x} + \tan^{-1}\frac{1}{4x+1} = \tan^{-1}\frac{2}{x^2}$, when
 $x > 0$, then x is equal to
 (a) 0 (b) 3
 (c) $-\frac{2}{3}$ (d) All of these
- 34.** Solution of the equation
 $\tan^{-1}(x-1) + \tan^{-1}x + \tan^{-1}(x+1) = \tan^{-1}3x$ is
 (a) $x = 0$ (b) $x = \pm\frac{1}{2}, 0$
 (c) $x = \pm\frac{1}{3}$ (d) None of these
- 35.** If $\tan^{-1}\frac{1}{a-1} = \tan^{-1}\frac{1}{x} + \tan^{-1}\frac{1}{a^2-x+1}$, then x is
 (a) $\frac{a}{2}$ (b) a^3
 (c) $a^2 - a + 1$ (d) $a^2 + a - 1$

- 36.** Solution of the equation $\cot^{-1} x + \sin^{-1} \frac{1}{\sqrt{5}} = \frac{\pi}{4}$ is
 (a) $x = 3$ (b) $x = \frac{1}{\sqrt{5}}$
 (c) $x = 0$ (d) None of these
- 37.** Solution of equation $\tan(\cos^{-1} x) = \sin\left(\cot^{-1} \frac{1}{2}\right)$ is
 (a) $x = \pm \frac{\sqrt{7}}{3}$ (b) $x = \pm \frac{\sqrt{5}}{3}$
 (c) $x = \pm \frac{3\sqrt{5}}{2}$ (d) None of these
- 38.** $\cot^{-1} \frac{xy+1}{x-y} + \cot^{-1} \frac{yz+1}{y-z} + \cot^{-1} \frac{zx+1}{z-x}$ is equal to
 (a) 1 (b) -1
 (c) 0 (d) None of these
- 39.** Solution of the equation $\tan^{-1} 2x + \tan^{-1} 3x = n\pi + \frac{3\pi}{4}; n \in N$ is
 (a) $x = 1, -\frac{1}{6}$ (b) $x = \frac{1}{6}$
 (c) $x = -\frac{1}{6}$ (d) None of these
- 40.** $2 \tan^{-1} \left[\tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right]$ is equal to
 (a) $\tan^{-1} \left(\frac{\sin \beta \cos \alpha}{\sin \beta + \cos \alpha} \right)$
 (b) $\tan^{-1} \left(\frac{\sin \beta \cos \alpha}{\sin \alpha + \cos \beta} \right)$
 (c) $\tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\sin \alpha + \cos \beta} \right)$
 (d) $\tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\sin \beta + \cos \alpha} \right)$
- 41.** The value of $\sin \left[\sin^{-1} \frac{\sqrt{5}}{4} + \tan^{-1} \sqrt{\frac{5}{11}} \right]$ is
 (a) $\frac{\sqrt{5}}{4\sqrt{11}}$ (b) $\frac{4}{\sqrt{35}}$
 (c) $\frac{\sqrt{55}}{8}$ (d) None of these
- 42.** If $\tan \theta + \tan \left(\frac{\pi}{3} + \theta \right) + \tan \left(-\frac{\pi}{3} + \theta \right) = a \tan 3\theta$, then a is equal to
 (a) $\frac{1}{3}$ (b) 1
 (c) 3 (d) None of the above
- 43.** If $\sin^{-1} \alpha + \sin^{-1} \beta + \sin^{-1} \gamma = \frac{3\pi}{2}$, then $\alpha\beta + \alpha\gamma + \beta\gamma$ is equal to
 (a) 1 (b) 0 (c) 3 (d) -3
- 44.** If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then the value of $\Sigma \frac{(x^{101} + y^{101})(x^{202} + y^{202})}{(x^{303} + y^{303})(x^{404} + y^{404})}$ is
 (a) 1 (b) 2
 (c) 3 (d) None of these
- 45.** The sum of infinite series $\sin^{-1} \left(\frac{1}{\sqrt{2}} \right) + \sin^{-1} \left(\frac{\sqrt{2}-1}{\sqrt{6}} \right) + \sin^{-1} \left(\frac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}} \right) + \dots + \sin^{-1} \left(\frac{\sqrt{n}-\sqrt{n-1}}{\sqrt{n(n+1)}} \right) + \dots$ is
 (a) $\frac{\pi}{2}$ (b) π (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$
- 46.** The value of $\cos^{-1} x + \cos^{-1} \left(\frac{x}{2} + \frac{1}{2} \sqrt{3-3x^2} \right)$, where $\frac{1}{2} \leq x \leq 1$, is
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) π (d) 0
- 47.** If $\tan^{-1} \frac{a}{x} + \tan^{-1} \frac{b}{x} + \tan^{-1} \frac{c}{x} + \tan^{-1} \frac{d}{x} = \frac{\pi}{2}$, then $x^4 - x^2 \Sigma ab + abcd$ is equal to
 (a) -1 (b) 0 (c) 1 (d) 2
- 48.** $\tan^{-1} \left(\frac{a_1 x - y}{a_1 y + x} \right) + \tan^{-1} \left(\frac{a_2 - a_1}{a_1 a_2 + 1} \right) + \tan^{-1} \left(\frac{a_3 - a_2}{a_2 a_3 + 1} \right) + \dots + \tan^{-1} \left(\frac{a_n - a_{n-1}}{a_{n-1} a_n + 1} \right) + \tan^{-1} \frac{1}{a_n}$ is equal to
 (a) $\tan^{-1} xy$ (b) $\tan^{-1} \frac{x}{y}$
 (c) $\tan^{-1} \frac{y}{x}$ (d) None of these
- 49.** If $x = \tan^{-1} \frac{1}{7}$ and $y = \tan^{-1} \frac{1}{3}$, then
 (a) $\cos 2x = \sin 4y$ (b) $\cos 4y = \cos 2x$
 (c) $\cos 2y = \sin 4x$ (d) None of these
- 50.** If $a \leq \tan^{-1} x + \cot^{-1} x + \sin^{-1} x \leq b$, then
 (a) $a = 0, b = \pi$ (b) $b = \frac{\pi}{2}$
 (c) $a = \frac{\pi}{4}$ (d) None of these
- 51.** If $\cot^{-1} \left(\frac{n^2 - 10n + 21.6}{\pi} \right) > \frac{\pi}{6}, n \in N$, then the minimum value of n is
 (a) 6 (b) 5
 (c) 3 (d) None of these
- 52.** The number of positive integral solutions of $\tan^{-1} x + \cot^{-1} y = \tan^{-1} 3$ is
 (a) 1 (b) 2
 (c) 3 (d) 4

53. If $\sin^{-1} \sqrt{x^2 + 2x + 1} + \sec^{-1} \sqrt{x^2 + 2x + 1} = \frac{\pi}{2}$, $x \neq 0$, then the value of $2\sec^{-1}\left(\frac{x}{2}\right) + \sin^{-1}\left(\frac{x}{2}\right)$ is
- (a) $-\frac{3\pi}{2}$ (b) $\frac{3\pi}{2}$
(c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$
54. The inequality $\sin^{-1}(\sin 5) > x^2 - 4x$ holds, if
- (a) $x = 2 - \sqrt{9 - 2\pi}$
(b) $x = 2 + \sqrt{9 - 2\pi}$
(c) $x \in (2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi})$
(d) $x > 2 + \sqrt{9 - 2\pi}$
55. $\sin^{-1}\left(\sin\left(\frac{2x^2 + 4}{1 + x^2}\right)\right) < \pi - 3$, if
- (a) $-1 \leq x \leq 0$ (b) $0 \leq x \leq 1$ (c) $-1 < x < 1$ (d) $x > 1$
56. If $\left|\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)\right| < \frac{\pi}{3}$, then
- (a) $x \in \left[-\frac{1}{3}, \frac{1}{3}\right]$ (b) $x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$
(c) $x \in \left(0, \frac{1}{\sqrt{3}}\right)$ (d) None of these
57. The value of x for which $\cos^{-1}(\cos 4) > 3x^2 - 4x$, is
- (a) $\left(0, \frac{2 + \sqrt{6\pi - 8}}{3}\right)$
(b) $\left(\frac{2 - \sqrt{6\pi - 8}}{3}, 0\right)$
(c) $(-2, 2)$
(d) $\left(\frac{2 - \sqrt{6\pi - 8}}{3}, \frac{2 + \sqrt{6\pi - 8}}{3}\right)$
58. $\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{3}{5} - \tan^{-1} \frac{8}{19}$ is equal to
- (a) $\frac{\pi}{2}$ (b) π
(c) $\frac{\pi}{4}$ (d) None of these
59. $\tan^{-1} x > \cot^{-1} x$ holds for
- (a) $x > 1$ (b) $x < 1$
(c) $x = 1$ (d) all values of x
60. If θ and ϕ are the roots of the equation $8x^2 + 22x + 5 = 0$, then
- (a) both $\sin^{-1}\theta$ and $\sin^{-1}\phi$ are real
(b) both $\sec^{-1}\theta$ and $\sec^{-1}\phi$ are real
(c) both $\tan^{-1}\theta$ and $\tan^{-1}\phi$ are real
(d) None of the above
61. Let $f(x) = \cot^{-1} x + \operatorname{cosec}^{-1} x$. Then, $f(x)$ is real for
- (a) $x \in [-1, 1]$ (b) $x \in (-\infty, -1] \cup [1, \infty)$
(c) $x \in (-\infty, \infty)$ (d) None of these
62. $\sec^{-1}(\sin x)$ is real, if
- (a) $x \in (-\infty, \infty)$ (b) $x \in [-1, 1]$
(c) $x = (2n + 1)\frac{\pi}{2}, n \in I$ (d) $x = n\pi, n \in I$
63. If $\tan^{-1} \frac{x}{\pi} < \frac{\pi}{3}$, $x \in N$, then the maximum value of x is
- (a) 2 (b) 5
(c) 7 (d) None of these
64. The solution set of the equation $\tan^{-1} x - \cot^{-1} x = \cos^{-1}(2 - x)$ is
- (a) $[0, 1]$ (b) $[-1, 1]$
(c) $[1, 3]$ (d) None of these
65. The number of real solutions of the equation $\sqrt{1 + \cos 2x} = \sqrt{2} \sin^{-1}(\sin x)$, $-\pi \leq x \leq \pi$ is
- (a) 0 (b) 1
(c) 2 (d) infinite
66. The number of real solutions of the equation $\tan^{-1} \sqrt{x(x+1)} + \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$ is
- (a) 0 (b) 1 (c) 2 (d) infinite
67. The number of real solutions (x, y) , where $|y| = \sin x, y = \cos^{-1}(\cos x)$, $-2\pi \leq x \leq 2\pi$, is
- (a) 2 (b) 1
(c) 3 (d) 4
68. The set of values of k for which $x^2 - kx + \sin^{-1}(\sin 4) > 0$ for all real x , is
- (a) ϕ (b) $(-2, 2)$
(c) R (d) None of these
69. The sum of the infinite series $\cot^{-1}(2) + \cot^{-1} 8 + \cot^{-1} 18 + \cot^{-1} 32 + \dots$ is
- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) $\frac{\pi}{8}$
70. If x_1, x_2, x_3 and x_4 are roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$, then $\sum_{i=1}^4 \tan^{-1} x_i$ is equal to
- (a) $\pi - \beta$ (b) $\pi - 2\beta$
(c) $\left(\frac{\pi}{2}\right) - \beta$ (d) $\left(\frac{\pi}{2}\right) - 2\beta$
71. If $\sum_{i=1}^{2n} \cos^{-1} x_i = 0$, then $\sum_{i=1}^{2n} x_i$ is
- (a) n (b) $2n$
(c) $\frac{n(n+1)}{2}$ (d) None of these
72. If $\sum_{i=1}^{20} \sin^{-1} x_i = 10\pi$, then $\sum_{i=1}^{20} x_i$ is equal to
- (a) 20 (b) 10
(c) 0 (d) None of these

Type 2. More than One Correct Option

73. $2 \tan^{-1}(-3)$ is equal to

- (a) $-\cos^{-1}\left(-\frac{4}{5}\right)$ (b) $-\pi + \cos^{-1}\left(\frac{4}{5}\right)$
(c) $-\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right)$ (d) $\cot^{-1}\left(\frac{4}{3}\right)$

74. If $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$, $x > 0$, then

- (a) $x = 1$ (b) $x = \sqrt{2} + 1$
(c) $x = \sqrt{2} - 1$ (d) $x = 3$

75. $\cos^{-1}x = \tan^{-1}x$, then

- (a) $x^2 = \frac{\sqrt{5}-1}{2}$ (b) $x^2 = \frac{\sqrt{5}+1}{2}$
(c) $\sin(\cos^{-1}x) = \frac{\sqrt{5}-1}{2}$ (d) $\tan(\cos^{-1}x) = \frac{\sqrt{5}-1}{2}$

76. $\cos^{-1}\sqrt{\frac{a-x}{a-b}} = \sin^{-1}\sqrt{\frac{x-b}{a-b}}$ is possible, if

- (a) $a > x > b$
(b) $a < x < b$
(c) $a = x = b$
(d) $a > b$ and x takes any value

77. If $\theta = \tan^{-1}(2\tan^2\theta) - \tan^{-1}\left(\frac{1}{3}\tan\theta\right)$, then

- (a) $\tan\theta = -2$ (b) $\tan\theta = 0$
(c) $\tan\theta = 1$ (d) $\tan\theta = 2$

78. Which of the following is a rational number?

- (a) $\sin\left(\tan^{-1}3 + \tan^{-1}\frac{1}{3}\right)$ (b) $\cos\left(\frac{\pi}{2} - \sin^{-1}\frac{3}{4}\right)$
(c) $\log_2\left[\sin\left(\frac{1}{4}\sin^{-1}\frac{\sqrt{63}}{8}\right)\right]$ (d) $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

79. If $\tan^{-1}y = 4\tan^{-1}x$, then y is infinite, if

- (a) $x^2 = 3 + 2\sqrt{2}$
(b) $x^2 = 3 - 2\sqrt{2}$
(c) $x^4 = 6x^2 - 1$
(d) $x^4 = 6x^2 + 1$

80. $3\cos^{-1}x - \pi x - \frac{\pi}{2} = 0$ has

- (a) one solution
(b) one and only one solution
(c) no solution
(d) more than one solutions

Type 3. Assertion and Reason

Directions (Q. Nos. 81-88) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

81. **Statement I** Range of expression

$$f(x) = \sin^{-1}x + \cos^{-1}x + 2\tan^{-1}x \text{ is } \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right).$$

Statement II Range should be calculated for $x \in [-1, 1]$.

82. **Statement I** If $\alpha \in \left(-\frac{\pi}{2}, 0\right)$, then

$$\sin^{-1}(\sin\alpha) + \cos^{-1}(\cos\alpha) = 0.$$

Statement II If $\alpha \in \left(-\frac{\pi}{2}, 0\right)$, then

$$\sin^{-1}(\sin\alpha) = \alpha \text{ and } \cos^{-1}(\cos\alpha) = -\alpha.$$

83. **Statement I** $3\cos^{-1}x = \cos^{-1}(4x^3 - 3x)$ is an identity for all $x \in [-1, 1]$.

Statement II $x = \cos\theta$ has a range $[-1, 1]$ for all real θ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

84. **Statement I** If

$$x = \operatorname{cosec}(\tan^{-1}(\cos(\cot^{-1}(\sec(\sin^{-1}a)))))) \text{ and } y = \sec(\cot^{-1}(\sin(\tan^{-1}(\operatorname{cosec}(\cos^{-1}a))))), \text{ where } a \in [0, 1], \text{ then } x = y.$$

Statement II

$$\sin^{-1}\left(\frac{1}{x}\right) = \operatorname{cosec}^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$$

$$\cos^{-1}\left(\frac{1}{x}\right) = \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$$

$$\tan^{-1}\left(\frac{1}{x}\right) = \begin{cases} \cot^{-1}x, & x > 0 \\ -\pi + \cot^{-1}x, & x < 0 \end{cases}$$

85. **Statement I** $\cos^{-1}x = 2\sin^{-1}\sqrt{\frac{1-x}{2}}$
 $= 2\cos^{-1}\sqrt{\frac{1+x}{2}}$

Statement II $\sin^{-1}(-x) = -\sin^{-1}x$,
 $\cos^{-1}(-x) = \pi - \cos^{-1}x$ ($-1 \leq x \leq 1$)

86. **Statement I** $\sin^{-1}\left[x - \frac{x^2}{2} + \frac{x^3}{4} - \dots\right]$

$= \frac{\pi}{2} - \cos^{-1}\left[x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots\right]$ for $0 < |x| < \sqrt{2}$ has a unique solution.

Statement II $\tan^{-1} \sqrt{x(x+1)}$

$+ \sin^{-1} \sqrt{x^2 + x + 1} = \frac{\pi}{2}$ has no solution for
 $-\sqrt{2} < x < 0$.

87. Statement I $3 \sin^{-1} \left(\frac{1}{3} \right) + \sin^{-1} \left(\frac{3}{5} \right) < \frac{2\pi}{3}$ and

$\tan^{-1} (2\sqrt{2} - 1) > \frac{\pi}{3}$

Statement II If $3 \sin^{-1} x = \frac{\pi}{6}$, then
 $6x - 8x^3 - 1 = 0$.

88. Statement I $y = \tan^{-1}(\tan x)$ and $y = \cos^{-1}(\cos x)$
 does not have any solution, if $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right)$.

Statement II $y = \tan^{-1}(\tan x) = x - \pi$, $x \in \left(\frac{\pi}{2}, \frac{3\pi}{2} \right)$

and $y = \cos^{-1}(\cos x) = \begin{cases} 2\pi - x, & x \in \left[\pi, \frac{3\pi}{2} \right] \\ x, & x \in \left[\frac{\pi}{2}, \pi \right] \end{cases}$

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 89-90) Given that,

$$f(x) = \sin\{\cot^{-1}(x+1)\} - \cos(\tan^{-1} x)$$

$$a = \cos \tan^{-1} \sin \cot^{-1} x$$

$$b = \cos(2 \cos^{-1} x + \sin^{-1} x)$$

89. The value of x for which $f(x) = 0$ is

- (a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) 1

90. If $f(x) = 0$, then a^2 is equal to

- (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{5}{9}$ (d) $\frac{9}{5}$

Passage II (Q. Nos. 91-93) Given that,

$$\alpha = \cos^{-1} \left(\frac{4}{5} \right), \beta = \tan^{-1} \left(\frac{2}{3} \right); 0 < \alpha, \beta < \frac{\pi}{2}$$

91. $\alpha + \beta$ is equal to

- (a) $\tan^{-1} \left(\frac{17}{6} \right)$ (b) $\sin^{-1} \left(\frac{17}{5\sqrt{13}} \right)$
 (c) $\sin^{-1} \left(\frac{3}{5} \right)$ (d) $\cos^{-1} \left(\frac{3}{\sqrt{13}} \right)$

92. $\cos^{-1} \left(\frac{44}{125} \right)$ is equal to

- (a) 2α (b) 3α
 (c) $\pi - 3\alpha$ (d) $\pi - 2\alpha$

93. $\sin^{-1} (117/125)$ is equal to

- (a) 2α
 (b) 3α
 (c) $\frac{\pi}{2} - 2\alpha$
 (d) $\pi - 3\alpha$

Passage III (Q. Nos. 94-95) Given that,

$$a = \tan^{-1}(3 \tan^3 \theta), b = \frac{1}{2} \cos^{-1} \left(\frac{5 \cos 2\theta + 4}{5 + 4 \cos 2\theta} \right)$$

94. $b = \phi$, where

- (a) $\tan \phi = 3 \tan \theta$ (b) $\tan \phi = \left(\frac{1}{3} \right) \tan \theta$
 (c) $\sin \phi = 3 \sin \theta$ (d) $\cos \phi = \left(\frac{1}{3} \right) \cos \theta$

95. $\theta = a - b$, if

- (a) $\tan \theta = 0$ (b) $\tan^2 \theta = \frac{9 + \sqrt{33}}{6}$
 (c) $\tan^2 \theta = \frac{9 - \sqrt{33}}{6}$ (d) $\tan \theta = 1$

Passage IV (Q. Nos. 96-98) Suppose

$f(x) = 3x^3 - 13x^2 + 14x - 2$. It is assumed that $f(x) = 0$ will have 3 roots say α, β and γ , where $\alpha < \beta < \gamma$.

96. $[\alpha], [\beta]$ and $[\gamma]$, where $[]$ denotes the greatest integer function, will be in

- (a) AP (b) GP
 (c) HP (d) None of these

97. $\lim_{n \rightarrow \infty} (\alpha)^n (\beta)^{1/n}$ will be equal to

- (a) 1 (b) e
 (c) 0 (d) None of these

98. The value of $\tan^{-1} \alpha + \tan^{-1} \beta + \tan^{-1} \gamma$ is

- (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

Passage V (Q. Nos. 99-101) The function $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\cot^{-1} x$, $\operatorname{cosec}^{-1} x$ and $\sec^{-1} x$ are called inverse circular functions. Each of the inverse circular function is multivalued. To make each inverse circular function single valued, let us define the principal value as follows :

$\sin^{-1} x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, $\cos^{-1} x \in [0, \pi]$ (in both cases

$x \in [-1, 1]$) and $\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$, $x \in (-\infty, \infty)$.

99. Number of possible solutions for the equation

$$\sin^{-1} x + \cos^{-1} y = \frac{11\pi}{2} \text{ is/are}$$

- (a) 0 (b) 1 (c) 2 (d) infinite

100. $\sin^{-1} \left(\cos \frac{5\pi}{4} \right)$ is equal to

- (a) $-\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $\frac{5\pi}{4}$ (d) $\frac{7\pi}{4}$

101. If, $2\sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$, then range of x is

- (a) all real numbers (b) $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$
(c) $[-1, 1]$ (d) $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$

Passage VI (Q. Nos. 102-105) While define inverse trigonometric functions, a new system is followed where codomains have been redefined as follows :

Function	Domain	Codomain
$\sin^{-1} x$	$[-1, 1]$	$\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$
$\tan^{-1} x$	R	$\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
$\cos^{-1} x$	$[-1, 1]$	$[\pi, 2\pi]$
$\cot^{-1} x$	R	$(\pi, 2\pi)$

Type 5. Match the Columns

106. Let $t_1 = (\sin^{-1} x)^{\sin^{-1} x}$, $t_2 = (\sin^{-1} x)^{\cos^{-1} x}$,
 $t_3 = (\cos^{-1} x)^{\sin^{-1} x}$, $t_4 = (\cos^{-1} x)^{\cos^{-1} x}$.

Match the following items of Column I with Column II.

Column I	Column II
A. $x \in (0, \cos 1)$	p. $t_1 > t_2 > t_4 > t_3$
B. $x \in \left(\cos 1, \frac{1}{\sqrt{2}}\right)$	q. $t_4 > t_3 > t_1 > t_2$
C. $x \in \left(\frac{1}{\sqrt{2}}, \sin 1\right)$	r. $t_2 > t_1 > t_4 > t_3$
D. $x \in (\sin 1, 1)$	s. $t_3 > t_4 > t_1 > t_2$

107. Match the following items of Column I with Column II.

Column I	Column II
A. $x \in [\pi, 2\pi] \Rightarrow \tan^{-1}(\tan x) $ can be	p. $ x - 2\pi $
B. $x \in [\pi, 2\pi] \Rightarrow \cot^{-1}(\cot x) $ can be	q. $ x - \pi $
C. $x \in [-\pi, \pi] \Rightarrow \sin^{-1}(\sin x) $ can be	r. $ x $
D. $x \in [-\pi, \pi] \Rightarrow \cos^{-1}(\cos x) $ can be	s. $ x + \pi $

108. Match the following items of Column I with Column II.

Column I	Column II
A. $\tan^{-1} 3 + \tan^{-1} 4$	p. $\frac{\pi}{2}$
B. $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{4}\right)$	q. $\pi - \tan^{-1}\left(\frac{7}{11}\right)$
C. $\sin^{-1}\left(\frac{1}{3}\right) + \cos^{-1}\left(\frac{1}{3}\right)$	r. $\tan^{-1}\left(\frac{7}{11}\right)$
D. $\tan^{-1}(3) + \cot^{-1}(4)$	s. $\tan^{-1}(13)$

102. $\sin^{-1}(-x)$ is equal to

- (a) $-\sin^{-1} x$ (b) $\pi + \sin^{-1} x$
(c) $2\pi - \sin^{-1} x$ (d) $3\pi - \cos^{-1}\sqrt{1-x^2}, x > 0$

103. If $f(x) = 3\sin^{-1} x - 2\cos^{-1} x$, then $f(x)$ is

- (a) even function
(b) odd function
(c) neither even nor odd
(d) even as well as odd function

104. The minimum of $(\sin^{-1} x)^3 - (\cos^{-1} x)^3$ is equal to

- (a) $-\frac{63}{8}\pi^3$ (b) $\frac{63}{8}\pi^3$ (c) $\frac{125\pi^3}{32}$ (d) $-\frac{125\pi^3}{32}$

105. The value of $\sin^{-1} x + \cos^{-1} x$ is equal to

- (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{5\pi}{2}$ (d) $\frac{7\pi}{2}$

109. Match the following items of Column I with Column II.

A. If $\tan\left[\frac{\pi}{4} + \left(\frac{1}{2}\right)\cos^{-1} x\right] + \tan\left[\frac{\pi}{4} - \left(\frac{1}{2}\right)\cos^{-1} x\right] = 1$, then the number of real solutions is	p. 0
B. If $\tan^{-1}\left(\frac{1}{2x+1}\right) + \tan^{-1}\left(\frac{1}{4x+1}\right) = \tan^{-1}\left(\frac{2}{x^2}\right)$, then the number of real solutions is	q. 2
C. If $\tan^{-1}\left[x + \left(\frac{2}{x}\right)\right] - \tan^{-1}\left(\frac{4}{x}\right) - \tan^{-1}\left[x - \left(\frac{2}{x}\right)\right] = 0$, then the number of real solutions is	r. 3
D. If $\tan^{-1}(1-x) + \tan^{-1}(1+x) = \tan^{-1} 2x$, then the number of real solutions is	s. 1

110. Match the following items of Column I with Column II.

Column I	Column II
A. If $\sin^{-1} \cos x - \cos^{-1} \sin x = a$ has at least one solution, then a belongs to	p. $\{0\}$
B. If equation $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x^2-1} + \tan^{-1}\tan y = a$ has at least one solution, then a belongs to	q. $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$
C. If $\sqrt{\sin^{-1} \sin x } + \sqrt{ \tan^{-1} \tan x } = 2\sqrt{a}$ has at least one solution, then a belongs to	r. $\left[0, \frac{\pi}{2}\right)$
D. $\sin^{-1}(x^2 + y^2) + \tan^{-1}\sqrt{4y^2-1} + \sec^{-1} x = a$ has no solution, then a may be	s. R

111. Match the following items of Column I with Column II.

Column I	Column II
A. If $\cos^{-1} \lambda + \cos^{-1} \mu + \cos^{-1} \nu = 3\pi$, then $\lambda\mu + \mu\nu + \nu\lambda$ is	p. $2n$
B. If $\sum_{i=1}^{10} \cos^{-1} x_i = 0$, then $\sum_{i=1}^{10} x_i$ is	q. $\sin^{-1} x - \frac{\pi}{6}$

Column I	Column II
C. If $\sum_{i=1}^{2n} \sin^{-1} x_i = n\pi$, then $\sum_{i=1}^{2n} x_i$ is	r. 10
D. If $f(x) = \sin^{-1} \left\{ \frac{\sqrt{3}}{2} x - \frac{1}{2} \sqrt{1-x^2} \right\}$, $-\frac{1}{2} \leq x \leq 1$, then $f(x)$ is equal to	s. 3

Type 6. Single Integer Answer Type Questions

112. If $x = \tan \left[\cos^{-1} \frac{1}{5\sqrt{2}} - \sin^{-1} \frac{4}{\sqrt{17}} \right]$, then $29x$ is equal to _____.
113. If $x = \sec^2(\tan^{-1} 2) + \operatorname{cosec}^2(\cot^{-1} 3)$ and $y = \tan^{-1} \frac{\sqrt{1+\tan^2 2} - 1}{\tan 2}$, then $x^2 - 220y^2$ is equal to _____.
114. If maximum value of $2997\sin x + 3996\cos x = k$, then $\frac{k}{555}$ is equal to _____.
115. If $S = \sum_{r=1}^{90} \tan^{-1} \left(\frac{2r}{2+r^2+r^4} \right)$, then $\frac{8190}{8192} \cot S$ is equal to _____.
116. The value of $a = -\lambda - \frac{\pi}{\mu}$, such that $x^2 + ax + \sin^{-1}(x^2 - 4x + 5) + \cos^{-1}(x^2 - 4x + 5) = 0$ has atleast one solution, then $\lambda + \mu$ is equal to _____.

117. The least value of n for which $(n-2)x^2 + 8x + n + 4 > \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$, $\forall x \in R$, where $n \in N$, is _____.
118. The number of real solutions for x, y , if $y = |\sin x|$ and $y = \sin^{-1}(\sin x)$, where $x \in [-2\pi, 2\pi]$ is _____.
119. If x_1, x_2, x_3 and x_4 are the roots of the equation $x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$ such that $\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 = n\pi + \frac{\pi}{\lambda} + \mu\beta$, then $\lambda\mu$ is _____.
120. The number of all positive integral solutions of the equation $\tan^{-1} x + \cos^{-1} \frac{y}{\sqrt{1+y^2}} = \sin^{-1} \left(\frac{3}{\sqrt{10}} \right)$, is _____.
121. If the domain of the function $f(x) = \sqrt{3 \cos^{-1}(4x) - \pi}$ is $[a, b]$, then the value of $4a + 64b$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Let $f: [0, 4\pi] \rightarrow [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0, 4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is _____ [2014]
2. The value of $\cot \left\{ \sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right\}$ is _____ [2013]
- (a) $\frac{23}{25}$ (b) $\frac{25}{23}$ (c) $\frac{23}{24}$ (d) $\frac{24}{23}$
3. Match the columns.

Column I	Column II
A. $\left(\frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4 \right)^{1/2}$ p. $\frac{1}{2} \sqrt{\frac{5}{3}}$	
takes value	

Column I	Column II
B. If $\cos x + \cos y + \cos z = 0$ $= \sin x + \sin y + \sin z$, then possible value of $\cos \frac{x-y}{2}$ is	q. $\sqrt{2}$
C. If $\cos \left(\frac{\pi}{4} - x \right) \cos 2x$ $+ \sin x \sin 2x \sec x$ $= \cos x \sin 2x \sec x + \cos \left(\frac{\pi}{4} + x \right) \cos 2x$, then possible value of $\sec x$ is	r. $\frac{1}{2}$
D. If $\cot(\sin^{-1} \sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$, $x \neq 0$, then possible value of x is	s. 1

Codes

- A B C D A B C D
(a) s r p q (b) s r q p
(c) r s q p (d) r s p q

[2013]

4. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$, where

$|x| < \frac{1}{\sqrt{3}}$. Then, the value of y is [2015]

- (a) $\frac{3x-x^3}{1-3x^2}$ (b) $\frac{3x+x^3}{1-3x^2}$
(c) $\frac{3x-x^3}{1+3x^2}$ (d) $\frac{3x+x^3}{1+3x^2}$

5. If x , y and z are in AP and $\tan^{-1} x$, $\tan^{-1} y$ and $\tan^{-1} z$ are also in AP, then [2013]

- (a) $x = y = z$ (b) $2x = 3y = 6z$
(c) $6x = 3y = 2z$ (d) $6x = 4y = 3z$

6. The value of $\cot \left(\operatorname{cosec}^{-1} \frac{5}{3} + \tan^{-1} \frac{2}{3} \right)$ is [2008]

- (a) $\frac{5}{17}$ (b) $\frac{6}{17}$ (c) $\frac{3}{17}$ (d) $\frac{4}{17}$

7. If $\sin^{-1} \left(\frac{x}{5} \right) + \operatorname{cosec}^{-1} \left(\frac{5}{4} \right) = \frac{\pi}{2}$, then a value of x is [2007]

- (a) 1 (b) 4 (c) 3 (d) 5

8. If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, then $4x^2 - 4xy \cos \alpha + y^2$ is equal to [2005]

- (a) $-4 \sin^2 \alpha$ (b) $4 \sin^2 \alpha$ (c) 4 (d) $2 \sin 2\alpha$

9. The trigonometric equation $\sin^{-1} x = 2 \sin^{-1} a$, has a solution for [2003]

- (a) $\frac{1}{\sqrt{2}} < |a| < \frac{1}{\sqrt{2}}$ (b) all real value of a
(c) $|a| \leq \frac{1}{\sqrt{2}}$ (d) $|a| \geq \frac{1}{\sqrt{2}}$

10. $\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x$, then $\sin x$ is equal to [2002]

- (a) $\tan^2 \left(\frac{\alpha}{2} \right)$ (b) $\cot^2 \left(\frac{\alpha}{2} \right)$ (c) $\tan \alpha$ (d) $\cot \left(\frac{\alpha}{2} \right)$

11. $\tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right)$ is equal to [2002]

- (a) $\frac{1}{2} \cos^{-1} \left(\frac{3}{4} \right)$ (b) $\frac{1}{2} \sin^{-1} \left(\frac{3}{5} \right)$
(c) $\frac{1}{2} \tan^{-1} \left(\frac{3}{5} \right)$ (d) $\tan^{-1} \left(\frac{1}{2} \right)$

Other Engineering Entrances

12. The greatest and least values of $(\sin^{-1} x)^2 + (\cos^{-1} x)^2$ are, respectively [BITSAT 2014]

- (a) $\frac{\pi^4}{4}$ and 0 (b) $\frac{\pi}{2}$ and $-\frac{\pi}{2}$
(c) $\frac{5\pi^2}{4}$ and $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{4}$ and $-\frac{\pi^4}{4}$

13. If $\cos^{-1} x > \sin^{-1} x$, then [BITSAT 2014]

- (a) $\frac{1}{\sqrt{2}} < x \leq 1$ (b) $0 \leq x < \frac{1}{\sqrt{2}}$
(c) $-1 \leq x < \frac{1}{\sqrt{2}}$ (d) $x > 0$

14. If $\tan^{-1} \left(\frac{a}{x} \right) + \tan^{-1} \left(\frac{b}{x} \right) = \frac{\pi}{2}$, then x is equal to [Manipal 2014]

- (a) $\sqrt{ab}$ (b) $\sqrt{2ab}$
(c) $2ab$ (d) ab

15. $\cos^{-1} [\cos \{2 \cot^{-1} (\sqrt{2} - 1)\}]$ is equal to [Manipal 2014]

- (a) $\sqrt{2} - 1$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{4}$ (d) 0

16. $\sum_{m=1}^n \tan^{-1} \left(\frac{2m}{m^4 + m^2 + 2} \right)$ is equal to [Manipal 2014]

- (a) $\tan^{-1} \left(\frac{n^2 + n}{n^2 + n + 2} \right)$ (b) $\tan^{-1} \left(\frac{n^2 - n}{n^2 - n + 2} \right)$
(c) $\tan^{-1} \left(\frac{n^2 + n + 2}{n^2 + n} \right)$ (d) None of these

17. If $\sin^{-1} a + \sin^{-1} b + \sin^{-1} c = \pi$, then the value of $a\sqrt{1-a^2} + b\sqrt{1-b^2} + c\sqrt{1-c^2}$ will be [Manipal 2014]

- (a) $2abc$ (b) abc (c) $\frac{1}{2}abc$ (d) $\frac{1}{3}abc$

18. If $0 \leq x \leq \frac{1}{2}$, then the value of

$$\tan \left[\sin^{-1} \left\{ \frac{x}{\sqrt{2}} + \frac{\sqrt{1-x^2}}{\sqrt{2}} \right\} - \sin^{-1} x \right] \text{ is}$$

- (a) 1 (b) $\sqrt{3}$ (c) -1 (d) $\frac{1}{\sqrt{3}}$ [Karnataka CET 2014]

19. The value of $\sin(2 \sin^{-1} 0.8)$ is [Karnataka CET 2014]

- (a) 0.48 (b) $\sin 1.2^\circ$
(c) $\sin 1.6^\circ$ (d) 0.96

20. If $\sin^{-1} \left(\frac{x}{13} \right) + \operatorname{cosec}^{-1} \left(\frac{13}{12} \right) = \frac{\pi}{2}$, then the value of x

- is [WB JEE 2014]
(a) 5 (b) 4 (c) 12 (d) 11

21. The principal value of $\sin^{-1} \left[\sin \left(\frac{2\pi}{3} \right) \right]$ is [AMU 2014]

- (a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$
(c) $\frac{4\pi}{3}$ (d) None of these

22. If $x > 0$, $y > 0$, $z > 0$, $xy + yx + zx < 1$ and if $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$, then $x + y + z$ is equal to [EAMCET 2014]

- (a) 0 (b) $\frac{xyz}{\sqrt{xyz}}$
(c) $3xyz$ (d) $\sqrt{xyz}$

23. $\operatorname{sech}^{-1}\left(\frac{1}{2}\right) - \operatorname{cosech}^{-1}\left(\frac{3}{4}\right)$ is equal to [EAMCET 2014]

- (a) $\log_e[3(2 + \sqrt{3})]$ (b) $\log_e\left(\frac{1 + \sqrt{3}}{3}\right)$
(c) $\log_e\left(\frac{2 + \sqrt{3}}{3}\right)$ (d) $\log_e\left(\frac{2 - \sqrt{3}}{3}\right)$

24. $\tan^{-1}(\sqrt{3}) - \cot^{-1}(-\sqrt{3})$ is equal to [GGSIU 2014]

- (a) 0 (b) $2\sqrt{3}$
(c) $-\frac{\pi}{2}$ (d) π

25. The number of value(s) of x , in which $\tan^{-1}\left(\frac{1}{x}\right) = \pi + \tan^{-1} x$, $0 < x < 1$, is [J&K CET 2014]

- (a) not defined (b) $\sqrt{3}$
(c) ± 1 (d) 0

26. The value of $\sin(2 \tan^{-1} x)$, $|x| \leq 1$ is [J&K CET 2014]

- (a) $\frac{1}{x}$ (b) x
(c) $\frac{1}{x^2}$ (d) $\frac{2x}{(1 + x^2)}$

27. If $\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) + \tan^{-1} x = \frac{\pi}{2}$, then x is equal to [J&K CET 2014]

- (a) 0 (b) 1
(c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

28. The value of $4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}$ is

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) ∞

29. If $\tan^{-1}\left(\tan \frac{5\pi}{4}\right) = \alpha$, $\tan^{-1}\left(-\tan \frac{2\pi}{3}\right) = \beta$, then [UP SEE 2013]

- (a) $4\alpha - 4\beta = 0$
(b) $4\alpha - 3\beta = 0$
(c) $\alpha > \beta$
(d) None of the above

30. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$, then the value of $x^9 + y^9 + z^9 - \frac{1}{x^9 y^9 z^9}$ is [WB JEE 2012]

- (a) 0 (b) 1
(c) 2 (d) 3

31. If $\tan^{-1} x = \frac{\pi}{4} - \tan^{-1}\left(\frac{1}{3}\right)$, then x is equal to [Karnataka CET 2011]

- (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

32. If $a \sin^{-1} x - b \cos^{-1} x = c$, then $a \sin^{-1} x + b \cos^{-1} x$ is equal to [AMU 2011]

- (a) $\frac{\pi ab + c(a - b)}{a + b}$ (b) 0
(c) $\frac{\pi ab - c(a - b)}{a + b}$ (d) $\frac{\pi}{2}$

33. If $\tan^{-1}\left(\frac{1}{1+2}\right) + \tan^{-1}\left(\frac{1}{1+(2)(3)}\right) + \tan^{-1}\left(\frac{1}{1+(3)(4)}\right) + \dots + \tan^{-1}\left(\frac{1}{1+n(n+1)}\right) = \tan^{-1} \theta$, then θ is equal to [GGSIU 2011]

- (a) $\frac{n}{n+1}$ (b) $\frac{n+1}{n+2}$
(c) $\frac{n+2}{n+1}$ (d) $\frac{n}{n+2}$

34. The value of $\sin\left[\tan^{-1}\left(\frac{1-x^2}{2x}\right) + \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)\right]$ is [J&K CET 2011]

- (a) 1 (b) 0
(c) -1 (d) $\frac{\pi}{2}$

35. The value of $\sin^{-1}\{\cos(4095^\circ)\}$ is [Kerala CEE 2010]

- (a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $-\frac{\pi}{4}$
(d) $\frac{\pi}{4}$ (e) $\frac{\pi}{2}$

36. If $a_1, a_2, a_3, \dots, a_n$ are in AP with common difference 5 and if $a_i a_j \neq -1$ for $i, j = 1, 2, \dots, n$, then $\tan^{-1}\left(\frac{5}{1+a_1 a_2}\right) + \tan^{-1}\left(\frac{5}{1+a_2 a_3}\right) + \dots + \tan^{-1}\left(\frac{5}{1+a_n a_{n+1}}\right)$ is equal to [Kerala CEE 2010]

- (a) $\tan^{-1}\left(\frac{5}{1+a_n a_{n+1}}\right)$ (b) $\tan^{-1}\left(\frac{5a_1}{1+a_n a_1}\right)$
(c) $\tan^{-1}\left(\frac{5n-5}{1+a_n a_1}\right)$ (d) $\tan^{-1}\left(\frac{5n-5}{1+a_1 a_{n+1}}\right)$
(e) $\tan^{-1}\left(\frac{5n}{1+a_1 a_n}\right)$

37. If $\tan^{-1}(x+2) + \tan^{-1}(x-2) - \tan^{-1}\left(\frac{1}{2}\right) = 0$, then one of the values of x is [Kerala CEE 2010]

- (a) -1 (b) 5 (c) $\frac{1}{2}$
(d) 1 (e) $-\frac{1}{2}$

38. The value of $\tan^{-1}\left(\frac{\sin 2 - 1}{\cos 2}\right)$ is [WB JEE 2010]

- (a) $\frac{\pi}{2} - 1$ (b) $1 - \frac{\pi}{4}$ (c) $2 - \frac{\pi}{2}$ (d) $\frac{\pi}{4} - 1$

Answers

Work Book Exercise

1. (b)	2. (d)	3. (b)	4. (d)	5. (d)	6. (a)	7. (c)	8. (c)	9. (b)	10. (b)
11. (c)	12. (b)	13. (d)	14. (d)	15. (b)	16. (c)	17. (c)	18. (c)	19. (c)	

Target Exercises

1. (d)	2. (c)	3. (a)	4. (a)	5. (b)	6. (b)	7. (d)	8. (b)	9. (d)	10. (c)
11. (c)	12. (d)	13. (a)	14. (a)	15. (b)	16. (a)	17. (b)	18. (c)	19. (d)	20. (d)
21. (c)	22. (d)	23. (b)	24. (c)	25. (c)	26. (c)	27. (b)	28. (a)	29. (d)	30. (c)
31. (b)	32. (c)	33. (b)	34. (b)	35. (c)	36. (a)	37. (b)	38. (c)	39. (a)	40. (d)
41. (c)	42. (c)	43. (c)	44. (c)	45. (a)	46. (b)	47. (b)	48. (b)	49. (a)	50. (a)
51. (c)	52. (b)	53. (b)	54. (c)	55. (c)	56. (b)	57. (d)	58. (c)	59. (a)	60. (c)
61. (b)	62. (c)	63. (b)	64. (c)	65. (c)	66. (c)	67. (a)	68. (a)	69. (b)	70. (c)
71. (b)	72. (a)	73. (a,b,c)	74. (a,b,c)	75. (a,c)	76. (a,b)	77. (a,b,c)	78. (a,b,c)	79. (a)	80. (a,b)
81. (d)	82. (a)	83. (d)	84. (b)	85. (b)	86. (c)	87. (b)	88. (a)	89. (a)	90. (c)
91. (a,b)	92. (c)	93. (b)	94. (b)	95. (a,b,c)	96. (a)	97. (c)	98. (d)	99. (b)	100. (d)
101. (b)	102. (c)	103. (b)	104. (a)	105. (c)	106. (*)	107. (*)	108. (*)	109. (*)	110. (*)
111. (*)	112. (3)	113. (5)	114. (9)	115. (1)	116. (6)	117. (5)	118. (5)	119. (2)	120. (2)
121. (7)									

*

106. $A \rightarrow q; B \rightarrow s; C \rightarrow r; D \rightarrow p$
 107. $A \rightarrow p, q; B \rightarrow q; C \rightarrow q, r, s; D \rightarrow r$
 108. $A \rightarrow q; B \rightarrow r; C \rightarrow p; D \rightarrow s$
 109. $A \rightarrow p; B \rightarrow r; C \rightarrow q; D \rightarrow s$
 110. $A \rightarrow p; B \rightarrow q; C \rightarrow p, r; D \rightarrow p, q, r, s$
 111. $A \rightarrow s; B \rightarrow r; C \rightarrow p; D \rightarrow q$

Entrances Gallery

1. (3)	2. (b)	3. (b)	4. (a)	5. (a)	6. (b)	7. (c)	8. (b)	9. (c)	10. (a)
11. (d)	12. (c)	13. (c)	14. (a)	15. (c)	16. (a)	17. (a)	18. (a)	19. (d)	20. (a)
21. (d)	22. (b)	23. (c)	24. (c)	25. (a)	26. (d)	27. (c)	28. (a)	29. (b)	30. (c)
31. (b)	32. (a)	33. (d)	34. (a)	35. (c)	36. (c)	37. (d)	38. (b)		

Explanations

Target Exercises

$$\begin{aligned} 1. \cos^{-1}\left(\cos \frac{5\pi}{3}\right) + \sin^{-1}\left(\sin \frac{5\pi}{3}\right) \\ = \cos^{-1}(\cos 300^\circ) + \sin^{-1}(\sin 300^\circ) \\ = \cos^{-1}\left(\frac{1}{2}\right) + \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) \\ = \cos^{-1} \frac{1}{2} - \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3} - \frac{\pi}{3} = 0 \end{aligned}$$

$$\begin{aligned} 2. \cos^{-1}\left(-\sin \frac{7\pi}{6}\right) &= \cos^{-1}\left\{\cos \left(\frac{\pi}{2} + \frac{7\pi}{6}\right)\right\} \\ &= \cos^{-1}\left(\cos \frac{5\pi}{3}\right) = \cos^{-1}\left\{\cos \left(2\pi - \frac{\pi}{3}\right)\right\} \\ &= \cos^{-1}\left(\cos \frac{\pi}{3}\right) = \frac{\pi}{3} \end{aligned}$$

Remember, $\cos^{-1}(\cos x) = x$, if $0 \leq x \leq \pi$

$$\begin{aligned} 3. \sin^{-1}\left(\cos \left(\sin^{-1} \frac{\sqrt{3}}{2}\right)\right) &= \sin^{-1}\left(\cos \left(\frac{\pi}{3}\right)\right) \\ &= \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} \end{aligned}$$

$$4. \tan^{-1}(\tan x) = x, \text{ if } -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Consider $\tan \frac{15\pi}{4} = \tan \left(4\pi - \frac{\pi}{4}\right) = \tan \left(-\frac{\pi}{4}\right)$

$$\begin{aligned} \therefore \cos \left\{\tan^{-1}\left(\tan \frac{15\pi}{4}\right)\right\} \\ = \cos \left\{\tan^{-1}\left(\tan \left(-\frac{\pi}{4}\right)\right)\right\} \\ = \cos \left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} \end{aligned}$$

$$\begin{aligned} 5. (a) \cos^{-1}\left(\cos \frac{7\pi}{5}\right) &= \cos^{-1}\left\{\cos \left(2\pi - \frac{3\pi}{5}\right)\right\} \\ &= \cos^{-1} \cos \left(\frac{3\pi}{5}\right) = \frac{3\pi}{5} \end{aligned}$$

$$\begin{aligned} (b) \sin^{-1}\left(\sin \frac{7\pi}{5}\right) &= \sin^{-1}\left\{\sin \left(\pi + \frac{2\pi}{5}\right)\right\} \\ &= \sin^{-1}\left\{-\sin \frac{2\pi}{5}\right\} = \sin^{-1}\left\{\sin \left(-\frac{2\pi}{5}\right)\right\} = -\frac{2\pi}{5} \end{aligned}$$

$$\begin{aligned} (c) \sec^{-1}\left(\sec \frac{7\pi}{5}\right) &= \sec^{-1}\left\{\sec \left(2\pi - \frac{3\pi}{5}\right)\right\} \\ &= \sec^{-1}\left(\sec \frac{3\pi}{5}\right) = \frac{3\pi}{5} \end{aligned}$$

$$\begin{aligned} 6. \text{ Since, } \frac{1}{\sqrt{2}}\left(\cos \frac{9\pi}{10} - \sin \frac{9\pi}{10}\right) &= \cos \left(\frac{\pi}{4} + \frac{9\pi}{10}\right) \\ &= \cos \frac{23\pi}{20} = \cos \left(2\pi - \frac{17\pi}{20}\right) = \cos \frac{17\pi}{20} \\ \therefore \cos^{-1}\left\{\frac{1}{\sqrt{2}}\left(\cos \frac{9\pi}{10} - \sin \frac{9\pi}{10}\right)\right\} \\ &= \cos^{-1}\left(\cos \frac{17\pi}{20}\right) = \frac{17\pi}{20} \end{aligned}$$

$$7. \operatorname{cosec}^{-1}(\cos x) \text{ exists, if } \cos x \leq -1 \text{ or } \cos x \geq 1. \text{ But } -1 \leq \cos x \leq 1$$

$$\therefore \operatorname{cosec}^{-1}(\cos x) \text{ exists, if } \cos x = 1 \text{ or } -1$$

$$\Rightarrow x = n\pi, n \in I$$

$$8. \left(\frac{\pi}{2} - \cos^{-1} x\right) - \cos^{-1} x = \frac{\pi}{6}$$

$$\text{or } 2\cos^{-1} x = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

$$\text{or } \cos^{-1} x = \frac{\pi}{6}$$

$$\therefore x = \frac{\sqrt{3}}{2}$$

$$9. \text{ We have, } \sin \left(\sin^{-1} \frac{1}{5} + \cos^{-1} x\right) = 1$$

$$\Rightarrow \sin^{-1} \frac{1}{5} + \cos^{-1} x = \sin^{-1} 1$$

$$\Rightarrow \sin^{-1} \frac{1}{5} + \cos^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow \cos^{-1} x = \frac{\pi}{2} - \sin^{-1} \frac{1}{5}$$

$$\Rightarrow \cos^{-1} x = \cos^{-1} \frac{1}{5}$$

$$\therefore x = \frac{1}{5}$$

$$10. \sin^{-1}\left(\sin \frac{3\pi}{4}\right) = \sin^{-1}\left[\sin \left(\pi - \frac{\pi}{4}\right)\right] = \sin^{-1}\left(\sin \frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$11. \cos^{-1}\left(\frac{1}{2}\right) + 2\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3} + 2 \cdot \frac{\pi}{6} = \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$$

$$12. \cos^{-1}\left(\frac{15}{17}\right) + 2\tan^{-1}\left(\frac{1}{5}\right) = \tan^{-1}\left(\frac{8}{15}\right) + \tan^{-1}\left(\frac{5}{12}\right)$$

$$= \tan^{-1}\left(\frac{\frac{8}{15} + \frac{5}{12}}{1 - \frac{40}{180}}\right)$$

$$= \tan^{-1}\left(\frac{171}{140}\right)$$

$$= \cos^{-1}\left(\frac{140}{221}\right)$$

$$13. \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3} = \tan^{-1} \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2} \cdot \frac{1}{3}} = \tan^{-1} \left(\frac{\frac{5}{6}}{\frac{5}{6}}\right)$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

$$14. \cot^{-1} 3 + \sec^{-1} \frac{\sqrt{5}}{2} = \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{2}$$

$$= \tan^{-1} \frac{\frac{1}{3} + \frac{1}{2}}{1 - \frac{1}{3} \cdot \frac{1}{2}}$$

$$= \tan^{-1} 1 = \frac{\pi}{4}$$

$$\begin{aligned}
 15. \sin^2\left(\cos^{-1}\frac{1}{2}\right) + \cos^2\left(\sin^{-1}\frac{1}{3}\right) \\
 = 1 - \cos^2\left(\cos^{-1}\frac{1}{2}\right) + 1 - \sin^2\left(\sin^{-1}\frac{1}{3}\right) \\
 = 1 - \left(\frac{1}{2}\right)^2 + 1 - \left(\frac{1}{3}\right)^2 = 2 - \frac{1}{4} - \frac{1}{9} = \frac{59}{36}
 \end{aligned}$$

$$\begin{aligned}
 16. \text{Let } \sin^{-1}(\sin 10) = \theta \Rightarrow \sin \theta = \sin 10 \\
 = \sin\left(\frac{35\pi}{11}\right) \left[\begin{array}{l} \because \frac{22}{7} \text{ radian} = \pi \\ \therefore 1 \text{ radian} = \frac{7\pi}{22} \\ 10 \text{ radian} = \frac{7\pi}{22} \times 10 = \frac{35\pi}{11} \end{array} \right] \\
 = \sin\left(3\pi + \frac{2\pi}{11}\right) \\
 = -\sin\frac{2\pi}{11} = \sin\left(-\frac{2\pi}{11}\right) \quad [\because \sin(-\theta) = -\sin \theta] \\
 \Rightarrow \theta = -\frac{2\pi}{11} = 3\pi - 10 \left[\because 3\pi - 10 = 3\pi - \frac{35\pi}{11} = -\frac{2\pi}{11} \right]
 \end{aligned}$$

$$\begin{aligned}
 17. \sin\left[\cot^{-1}\left(\cot\frac{17\pi}{3}\right)\right] &= \sin\left[\cot^{-1}\left(\cot\left(5\pi + \frac{2\pi}{3}\right)\right)\right] \\
 &= \sin\left[\cot^{-1}\left(\cot\left(\frac{2\pi}{3}\right)\right)\right] = \sin\left(\frac{2\pi}{3}\right) \\
 &\quad \left[\because \cot^{-1}(\cot x) = x \text{ as } 0 < \frac{2\pi}{3} < \pi \right] \\
 &= \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}
 \end{aligned}$$

$$\begin{aligned}
 18. \text{Let } \cot^{-1} x = \theta \Rightarrow \cot \theta = x \\
 \therefore \sin(\cot^{-1} x) = \sin \theta = \frac{1}{\operatorname{cosec} \theta} = \frac{1}{\sqrt{1 + \cot^2 \theta}} = \frac{1}{\sqrt{1 + x^2}} \\
 \text{Thus, } \cos[\tan^{-1}\{\sin(\cot^{-1} x)\}] = \cos\left[\tan^{-1}\frac{1}{\sqrt{1 + x^2}}\right] \\
 = \cos \phi \left[\text{put } \tan^{-1}\left(\frac{1}{\sqrt{1 + x^2}}\right) = \phi \Rightarrow \tan \phi = \frac{1}{\sqrt{1 + x^2}} \right] \\
 = \frac{1}{\sec \phi} = \frac{1}{\sqrt{1 + \tan^2 \phi}} = \frac{1}{\sqrt{1 + \frac{1}{1 + x^2}}} = \frac{1}{\sqrt{2 + x^2}}
 \end{aligned}$$

$$\begin{aligned}
 19. 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239} &= 2\tan^{-1}\frac{\frac{2}{5}}{1 - \frac{1}{25}} - \tan^{-1}\frac{1}{239} \\
 &= 2\tan^{-1}\left(\frac{5}{12}\right) - \tan^{-1}\left(\frac{1}{239}\right) \\
 &= \tan^{-1}\left(\frac{\frac{10}{12}}{1 - \frac{25}{144}}\right) - \tan^{-1}\left(\frac{1}{239}\right) \\
 &= \tan^{-1}\left(\frac{120}{119}\right) - \tan^{-1}\left(\frac{1}{239}\right) \\
 &= \tan^{-1}\left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \cdot \frac{1}{239}}\right) = \tan^{-1} 1 = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 20. \sin\left[2\cos^{-1}\left(-\frac{3}{5}\right)\right] &= \sin 2\theta, \text{ where } \cos \theta = -\frac{3}{5} \\
 &= 2\sin \theta \cos \theta = -\frac{6}{5}\sqrt{1 - \cos^2 \theta} = -\frac{6}{5}\sqrt{1 - \frac{9}{25}} \\
 &= -\frac{6}{5} \cdot \frac{4}{5} = -\frac{24}{25}
 \end{aligned}$$

$$\begin{aligned}
 21. \tan^{-1}\frac{1}{x} &= \begin{cases} \cot^{-1} x, & \text{if } x > 0 \\ -\pi + \cot^{-1} x, & \text{if } x < 0 \end{cases} \\
 \therefore \tan^{-1} x + \tan^{-1}\frac{1}{x} &= \begin{cases} \tan^{-1} x + \cot^{-1} x, & x > 0 \\ \tan^{-1} x - \pi + \cot^{-1} x, & x < 0 \end{cases} \\
 &= \begin{cases} \frac{\pi}{2}, & x > 0 \\ -\frac{\pi}{2}, & x < 0 \end{cases}
 \end{aligned}$$

$$\begin{aligned}
 22. \text{We know that,} \\
 -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}, \text{ for all } x \in [-1, 1]
 \end{aligned}$$

$$\Rightarrow \text{Maximum value of } \sin^{-1} x \text{ is } \frac{\pi}{2}.$$

$$\text{Now, } \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

$$\Rightarrow \sin^{-1} x = \frac{\pi}{2}, \sin^{-1} y = \frac{\pi}{2}, \sin^{-1} z = \frac{\pi}{2}$$

$$\begin{aligned}
 \therefore x^{2000} + y^{2001} + z^{2002} &= \frac{9}{x^{2000} + y^{2001} + z^{2002}} \\
 &= 1 + 1 + 1 - \frac{9}{1 + 1 + 1} \\
 &= 3 - 3 = 0
 \end{aligned}$$

$$23. \text{Combining 1st, 3rd and 2nd, 4th, we get given expression value}$$

$$\begin{aligned}
 &\tan^{-1}\left(\frac{\frac{1}{5} + \frac{1}{3}}{1 - \frac{1}{5} \cdot \frac{1}{3}}\right) + \tan^{-1}\left(\frac{\frac{1}{7} + \frac{1}{8}}{1 - \frac{1}{7} \cdot \frac{1}{8}}\right) \\
 &= \tan^{-1}\frac{\frac{8}{15}}{\frac{15}{15}} + \tan^{-1}\frac{\frac{15}{56}}{\frac{56}{56}} = \tan^{-1}\frac{4}{7} + \tan^{-1}\frac{3}{11} \\
 &= \tan^{-1}\frac{\frac{4}{7} + \frac{3}{11}}{1 - \frac{4}{7} \cdot \frac{3}{11}} = \tan^{-1}\frac{\frac{65}{77}}{\frac{65}{77}} = \tan^{-1} 1 = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 24. 3\tan^{-1}\frac{1}{2} + 2\tan^{-1}\frac{1}{5} + \sin^{-1}\frac{142}{65\sqrt{5}} \\
 = 2\left(\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{5}\right) + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{142}{31} \\
 = 2\tan^{-1}\frac{7}{9} + \pi + \tan^{-1}\frac{\frac{1}{2} + \frac{142}{31}}{1 - \left(\frac{1}{2}\right)\left(\frac{142}{31}\right)} \\
 = \tan^{-1}\frac{\frac{14}{9}}{1 - \frac{49}{81}} + \pi - \tan^{-1}\frac{315}{80} \\
 = \pi - \tan^{-1}\frac{63}{16} + \tan^{-1}\frac{63}{16} = \pi
 \end{aligned}$$



25. We have, $\tan^{-1}(x+1) + \tan^{-1}(x-1) = \tan^{-1}\left(\frac{8}{31}\right)$

$$\Rightarrow \tan^{-1}\left[\frac{x+1+x-1}{1-(x+1)(x-1)}\right] = \tan^{-1} \frac{8}{31}$$

$$\Rightarrow \frac{2x}{1-(x^2-1)} = \frac{8}{31}$$

$$\Rightarrow 62x = 16 - 8x^2$$

$$\Rightarrow 8x^2 + 62x - 16 = 0$$

$$\Rightarrow 4x^2 + 31x - 8 = 0$$

$$\Rightarrow 4x^2 + 32x - x - 8 = 0$$

$$\Rightarrow 4x(x+8) - 1(x+8) = 0$$

$$\Rightarrow (x+8)(4x-1) = 0$$

$$\Rightarrow x = -8, \frac{1}{4}$$

$$\therefore x = \frac{1}{4}$$

[neglecting $x = -8$ as $x^2 - 1 < 1$]

26. $\tan^{-1} a + \tan^{-1} b = \pi + \tan^{-1}\left(\frac{a+b}{1-ab}\right)$, as $ab > 1$

27. Since, $\sin \theta$ and $\cos \theta$ are the roots of the equation $ax^2 + bx + c = 0$.

We have,

$$\cos \theta + \sin \theta = \frac{-b}{a}$$

and $\cos \theta \cdot \sin \theta = \frac{c}{a}$

$$\Rightarrow \frac{b^2}{a^2} = 1 + \frac{2c}{a}$$

$$\Rightarrow b^2 = a^2 + 2ac$$

$$\Rightarrow a^2 - b^2 + 2ac = 0$$

$$\therefore \cos^{-1}(a^2 - b^2 + 2ac) = \cos^{-1} 0 = \frac{\pi}{2}$$

28. Clearly, $0 \leq \text{LHS} \leq \frac{\pi}{2}$ because $\cos^{-1} x$ is in the first quadrant, when x is positive.

$$\therefore \text{RHS} \geq 0. \text{ So, } \cos^{-1} \frac{x}{2} \geq \cos^{-1} x$$

Also, $\left|\frac{x}{2}\right| \leq 1, |x| \leq 1$. This means $|x| \leq 1$.

29. Let $x = \tan \theta$

$$\Rightarrow \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \cos^{-1}(\cos 2\theta)$$

$$= 2\theta, 0 \leq 2\theta \leq \pi$$

$$= 2 \tan^{-1} x, 0 \leq x < \infty$$

30. Let $\cos^{-1} x = \theta$

Then, the expression

$$= \frac{1 + \tan \frac{\theta}{2}}{1 - \tan \frac{\theta}{2}} + \frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}}$$

$$= \frac{2\left(1 + \tan^2 \frac{\theta}{2}\right)}{1 - \tan^2 \frac{\theta}{2}}$$

$$= \frac{2}{\cos \theta} = \frac{2}{x}$$

31. We have, $\cos^{-1} \frac{3}{5} - \sin^{-1} \frac{4}{5} = \cos^{-1} x$

$$\Rightarrow \sin^{-1}\left(\sqrt{1 - \frac{9}{25}}\right) - \sin^{-1} \frac{4}{5} = \cos^{-1} x$$

$$\Rightarrow \sin^{-1} \frac{4}{5} - \sin^{-1} \frac{4}{5} = \cos^{-1} x$$

$$\Rightarrow \cos^{-1} x = 0 \Rightarrow x = \cos 0 = 1$$

$$\therefore x = 1$$

32. $\tan^{-1} n + \cot^{-1}(n+1)$

$$= \tan^{-1} n + \tan^{-1} \frac{1}{n+1}$$

$$= \tan^{-1}\left(\frac{n + \frac{1}{n+1}}{1 - n \cdot \frac{1}{n+1}}\right) = \tan^{-1}\left(\frac{n^2 + n + 1}{n + 1 - n}\right)$$

$$= \tan^{-1}(n^2 + n + 1)$$

33. We have, $\tan^{-1} \frac{1}{1+2x} + \tan^{-1} \frac{1}{4x+1} = \tan^{-1} \frac{2}{x^2}$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{1}{1+2x} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)} \times \frac{1}{4x+1}}\right] = \tan^{-1} \frac{2}{x^2}$$

$$\Rightarrow \frac{(4x+1) + (1+2x)}{(1+2x)(4x+1) - 1} = \frac{2}{x^2}$$

$$\Rightarrow \frac{6x+2}{8x^2+6x-1} = \frac{2}{x^2}$$

$$\Rightarrow 2(3x+1)x^2 = 2(8x^2+6x)$$

$$\Rightarrow 3x^3 + x^2 = 8x^2 + 6x$$

$$\Rightarrow 3x^3 - 7x^2 - 6x = 0$$

$$\Rightarrow x(3x^2 - 7x - 6) = 0$$

$$\therefore x(x-3)(3x+2) = 0$$

$$\text{Hence, } x = 0, 3, -\frac{2}{3} \text{ but } x > 0 \Rightarrow x = 3$$

34. We have, $\tan^{-1}(x-1) + \tan^{-1} x + \tan^{-1}(x+1) = \tan^{-1} 3x$

$$\Rightarrow \tan^{-1}(x-1) + \tan^{-1}(x+1) = \tan^{-1} 3x - \tan^{-1} x$$

$$\Rightarrow \tan^{-1} \frac{(x-1) + (x+1)}{1 - (x^2-1)} = \tan^{-1} \frac{3x-x}{1+3x \cdot x}$$

$$\Rightarrow \frac{2x}{2-x^2} = \frac{2x}{1+3x^2}$$

$$\Rightarrow x + 3x^3 = 2x - x^3 \Rightarrow 4x^3 - x = 0$$

$$\Rightarrow x(4x^2 - 1) = 0$$

$$\therefore x = 0, \pm \frac{1}{2}$$

35. We have, $\tan^{-1} \frac{1}{a-1} = \tan^{-1} \frac{1}{x} + \tan^{-1} \frac{1}{a^2-x+1}$

$$\Rightarrow \tan^{-1} \frac{1}{a-1} = \tan^{-1} \frac{\frac{1}{x} + \frac{1}{a^2-x+1}}{1 - \frac{1}{x(a^2-x+1)}}$$

$$\Rightarrow \frac{1}{a-1} = \frac{a^2+1}{(a^2+1) \cdot x - x^2 - 1}$$

$$\Rightarrow x^2 - x(a^2+1) + 1 + (a-1)(a^2+1) = 0$$

$$\Rightarrow (x^2 - a^2) - (a^2+1)(x-a) = 0$$

$$\Rightarrow (x-a)(x+a-a^2-1) = 0$$

$$\therefore x = a \text{ or } a^2 - a + 1$$

$$\begin{aligned}
 36. \text{ We have, } \cot^{-1} x + \sin^{-1} \frac{1}{\sqrt{5}} &= \frac{\pi}{4} \\
 \Rightarrow \tan^{-1} \frac{1}{x} + \tan^{-1} \frac{1}{2} &= \tan^{-1} 1 \\
 \Rightarrow \tan^{-1} \frac{1}{x} &= \tan^{-1} 1 - \tan^{-1} \frac{1}{2} \\
 \Rightarrow \tan^{-1} \frac{1}{x} &= \tan^{-1} \left(\frac{1 - \frac{1}{2}}{1 + 1 \cdot \frac{1}{2}} \right) \\
 \Rightarrow \tan^{-1} \frac{1}{x} &= \tan^{-1} \left(\frac{1}{3} \right) \Rightarrow x = 3
 \end{aligned}$$

$$\begin{aligned}
 37. \text{ We have, } \tan(\cos^{-1} x) &= \sin\left(\cot^{-1} \frac{1}{2}\right) \\
 \Rightarrow \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] &= \sin(\tan^{-1} 2) \\
 \Rightarrow \tan\left[\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)\right] &= \sin\left(\sin^{-1} \frac{2}{\sqrt{5}}\right) \\
 \Rightarrow \frac{\sqrt{1-x^2}}{x} &= \frac{2}{\sqrt{5}} \\
 \Rightarrow 5(1-x^2) &= 4x^2 \Rightarrow 9x^2 = 5 \\
 \therefore x &= \pm \left(\frac{\sqrt{5}}{3}\right)
 \end{aligned}$$

$$\begin{aligned}
 38. \cot^{-1} \frac{xy+1}{x-y} + \cot^{-1} \frac{yz+1}{y-z} + \cot^{-1} \frac{xz+1}{z-x} \\
 = \tan^{-1} \frac{x-y}{1+xy} + \tan^{-1} \frac{y-z}{1+yz} + \tan^{-1} \frac{z-x}{1+xz} \\
 = (\tan^{-1} x - \tan^{-1} y) + (\tan^{-1} y - \tan^{-1} z) \\
 + (\tan^{-1} z - \tan^{-1} x) \\
 = 0
 \end{aligned}$$

$$\begin{aligned}
 39. \text{ We have, } \tan^{-1} 2x + \tan^{-1} 3x &= n\pi + \frac{3\pi}{4} \\
 \Rightarrow \tan^{-1} \left(\frac{2x+3x}{1-2x \cdot 3x} \right) &= n\pi + \frac{3\pi}{4} \\
 \Rightarrow \frac{5x}{1-6x^2} &= \tan \frac{3\pi}{4} = -1 \\
 \Rightarrow 6x^2 - 5x - 1 &= 0 \\
 \Rightarrow (x-1)(6x+1) &= 0 \\
 \therefore x &= 1, -\frac{1}{6}
 \end{aligned}$$

$$\begin{aligned}
 40. 2 \tan^{-1} \left[\tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \right] \\
 = \tan^{-1} \frac{2 \tan \frac{\alpha}{2} \tan \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{1 - \tan^2 \frac{\alpha}{2} \tan^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)} \\
 = \tan^{-1} \frac{2 \sin \left(\frac{\alpha}{2} \right) \cos \left(\frac{\alpha}{2} \right) \sin \left(\frac{\pi}{4} - \frac{\beta}{2} \right) \cos \left(\frac{\pi}{4} - \frac{\beta}{2} \right)}{\cos^2 \left(\frac{\alpha}{2} \right) \cos^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right) - \sin^2 \left(\frac{\alpha}{2} \right) \sin^2 \left(\frac{\pi}{4} - \frac{\beta}{2} \right)} \\
 = \tan^{-1} \frac{\sin \alpha \cdot \sin \left(\frac{\pi}{2} - \beta \right)}{2 \left[\cos \left(\frac{\alpha}{2} - \frac{\beta}{2} + \frac{\pi}{4} \right) \cos \left(\frac{\alpha}{2} + \frac{\beta}{2} - \frac{\pi}{4} \right) \right]}
 \end{aligned}$$

$$\begin{aligned}
 &= \tan^{-1} \frac{\sin \alpha \cos \beta}{\cos \alpha + \cos \left(\frac{\pi}{2} - \beta \right)} \\
 &= \tan^{-1} \left(\frac{\sin \alpha \cos \beta}{\cos \alpha + \sin \beta} \right)
 \end{aligned}$$

$$\begin{aligned}
 41. \text{ Consider } \sin \left[\sin^{-1} \frac{\sqrt{5}}{4} + \tan^{-1} \sqrt{\frac{5}{11}} \right] \\
 = \sin \left[\sin^{-1} \frac{\sqrt{5}}{4} + \sin^{-1} \frac{\sqrt{\frac{5}{11}}}{\sqrt{1 + \frac{5}{11}}} \right] \\
 = \sin \left[\sin^{-1} \frac{\sqrt{5}}{4} + \sin^{-1} \frac{\sqrt{5}}{4} \right] \\
 = \sin \left[2 \sin^{-1} \frac{\sqrt{5}}{4} \right] \\
 = \left[\sin \left\{ \sin^{-1} 2 \frac{\sqrt{5}}{4} \sqrt{1 - \frac{5}{16}} \right\} \right] \\
 = \frac{\sqrt{55}}{8}
 \end{aligned}$$

$$\begin{aligned}
 42. \text{ We have, } \tan \theta + \tan \left(\frac{\pi}{3} + \theta \right) + \tan \left(-\frac{\pi}{3} + \theta \right) &= a \tan 3\theta \\
 \Rightarrow \tan \theta + \frac{\sqrt{3} + \tan \theta}{1 - \sqrt{3} \tan \theta} + \frac{\tan \theta - \sqrt{3}}{1 + \sqrt{3} \tan \theta} &= a \tan 3\theta \\
 \Rightarrow \tan \theta + \frac{8 \tan \theta}{1 - 3 \tan^2 \theta} &= a \tan 3\theta \\
 \Rightarrow \frac{3(3 \tan \theta - \tan^3 \theta)}{(1 - 3 \tan^2 \theta)} &= a \tan 3\theta \\
 \Rightarrow 3 \tan 3\theta &= a \tan 3\theta \\
 \therefore a &= 3
 \end{aligned}$$

$$\begin{aligned}
 43. \text{ Since, } -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2} \\
 \Rightarrow \sin^{-1} \alpha = \frac{\pi}{2}, \sin^{-1} \beta = \frac{\pi}{2}, \sin^{-1} \gamma = \frac{\pi}{2} \\
 \therefore \alpha = \beta = \gamma = 1 \\
 \text{Thus, } \alpha\beta + \alpha\gamma + \beta\gamma &= 3
 \end{aligned}$$

$$44. \text{ We have, } \sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

We know that, $\sin^{-1} x \leq \frac{\pi}{2}$ for $x \in [-1, 1]$

Hence, above results is possible only when

$$\sin^{-1} x = \frac{\pi}{2}, \sin^{-1} y = \frac{\pi}{2}, \sin^{-1} z = \frac{\pi}{2}$$

$$\Rightarrow x = 1, y = 1, z = 1 \quad \dots (i)$$

$$\begin{aligned}
 \therefore \sum \frac{(x^{101} + y^{101})(x^{202} + y^{202})}{(x^{303} + y^{303})(x^{404} + y^{404})} \\
 = \frac{(x^{101} + y^{101})(x^{202} + y^{202})}{(x^{303} + y^{303})(x^{404} + y^{404})} + \frac{(y^{101} + z^{101})(y^{202} + z^{202})}{(y^{303} + z^{303})(y^{404} + z^{404})} \\
 + \frac{(z^{101} + x^{101})(z^{202} + x^{202})}{(z^{303} + x^{303})(z^{404} + x^{404})} \text{ [expanding summation]} \\
 = \frac{(1+1)(1+1)}{(1+1)(1+1)} + \frac{(1+1)(1+1)}{(1+1)(1+1)} + \frac{(1+1)(1+1)}{(1+1)(1+1)} \\
 = 1 + 1 + 1 = 3 \quad \text{[using Eq. (i)]}
 \end{aligned}$$

$$\begin{aligned}
 45. \text{ Let } T_n &= \sin^{-1} \left[\frac{\sqrt{n} - \sqrt{n-1}}{\sqrt{n(n+1)}} \right] \\
 &= \sin^{-1} \left[\frac{1}{\sqrt{n}} \sqrt{1 - \frac{1}{n+1}} - \frac{1}{\sqrt{n+1}} \sqrt{1 - \frac{1}{n}} \right] \\
 T_n &= \sin^{-1} \left(\frac{1}{\sqrt{n}} \right) - \sin^{-1} \left(\frac{1}{\sqrt{n+1}} \right) \\
 \therefore S &= \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) + T_2 + T_3 + T_4 + \dots \\
 &= \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) + \left\{ \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) - \sin^{-1} \left(\frac{1}{\sqrt{3}} \right) \right\} \\
 &\quad + \left\{ \sin^{-1} \left(\frac{1}{\sqrt{3}} \right) - \sin^{-1} \left(\frac{1}{\sqrt{4}} \right) \right\} \\
 &\quad + \left\{ \sin^{-1} \left(\frac{1}{\sqrt{4}} \right) - \sin^{-1} \left(\frac{1}{\sqrt{5}} \right) \right\} + \dots \\
 &= 2 \sin^{-1} \left(\frac{1}{\sqrt{2}} \right) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 46. \text{ Let } \cos^{-1} x &= y. \text{ Then, } x = \cos y, \text{ such that } \frac{1}{2} \leq x \leq 1 \text{ or} \\
 0 \leq y &\leq \frac{\pi}{3} \text{ and } \frac{x}{2} + \frac{1}{2} \sqrt{3-3x^2} = \frac{1}{2} \cos y + \frac{\sqrt{3}}{2} \sin y \\
 &= \cos \frac{\pi}{3} \cos y + \sin \frac{\pi}{3} \sin y = \cos \left(\frac{\pi}{3} - y \right) \\
 \Rightarrow \cos^{-1} \left(\frac{x}{2} + \frac{1}{2} \sqrt{3-3x^2} \right) &= \frac{\pi}{3} - y \\
 \text{because } \cos^{-1}(\cos x) &= x, \text{ if } 0 \leq x \leq \pi. \text{ Therefore, the} \\
 \text{given expression is equal to } y + \frac{\pi}{3} - y &= \frac{\pi}{3}.
 \end{aligned}$$

$$\begin{aligned}
 47. \text{ We have from the given equation,} \\
 \tan^{-1} \frac{(a+b)x}{x^2-ab} &= \frac{\pi}{2} - \tan^{-1} \frac{(c+d)x}{x^2-cd} \\
 \Rightarrow \tan^{-1} \frac{(a+b)x}{x^2-ab} &= \cot^{-1} \frac{(c+d)x}{x^2-cd} \\
 &= \tan^{-1} \frac{x^2-cd}{(c+d)x}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow (x^2-ab)(x^2-cd) &= (a+b)(c+d)x^2 \\
 \Rightarrow x^4 - x^2 \sum ab + abcd &= 0
 \end{aligned}$$

$$\begin{aligned}
 48. \text{ The given expression} \\
 &= \tan^{-1} \frac{a_1 - \frac{y}{x}}{1 + a_1 \cdot \frac{y}{x}} + (\tan^{-1} a_2 - \tan^{-1} a_1) \\
 &\quad + (\tan^{-1} a_3 - \tan^{-1} a_2) + \dots + (\tan^{-1} a_n - \tan^{-1} a_{n-1}) \\
 &\quad + \tan^{-1} \frac{1}{a_n} \\
 &= \tan^{-1} a_1 - \tan^{-1} \frac{y}{x} + \tan^{-1} a_n - \tan^{-1} a_1 + \tan^{-1} \frac{1}{a_n} \\
 &= (\tan^{-1} a_n + \cot^{-1} a_n) - \tan^{-1} \frac{y}{x} \\
 &= \frac{\pi}{2} - \tan^{-1} \frac{y}{x} = \cot^{-1} \frac{y}{x} = \tan^{-1} \frac{x}{y}
 \end{aligned}$$

$$\begin{aligned}
 49. x &= \tan^{-1} \frac{1}{7} \Rightarrow \tan x = \frac{1}{7} \\
 \text{and } y &= \tan^{-1} \frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow \tan y &= \frac{1}{3} \\
 \text{Now, } \cos 2x &= \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{24}{25} \\
 \sin 4y &= 2 \sin 2y \cos 2y = \frac{2 \cdot 2 \tan y (1 - \tan^2 y)}{(1 + \tan^2 y)^2} = \frac{24}{25} \\
 \therefore \cos 2x &= \sin 4y
 \end{aligned}$$

$$\begin{aligned}
 50. \text{ We have,} \\
 \tan^{-1} x + \cot^{-1} x + \sin^{-1} x &= \frac{\pi}{2} + \sin^{-1} x \quad \dots (i) \\
 \text{Since, } -\frac{\pi}{2} \leq \sin^{-1} x &\leq \frac{\pi}{2} \\
 \Rightarrow 0 \leq \frac{\pi}{2} + \sin^{-1} x &\leq \pi \\
 \Rightarrow 0 \leq \tan^{-1} x + \cot^{-1} x + \sin^{-1} x &\leq \pi \quad [\text{from Eq. (i)}] \\
 \therefore a = 0 \text{ and } b = \pi
 \end{aligned}$$

$$\begin{aligned}
 51. \text{ We have,} \\
 \cot^{-1} \left(\frac{n^2 - 10n + 21.6}{\pi} \right) &> \frac{\pi}{6} \\
 \Rightarrow \frac{n^2 - 10n + 21.6}{\pi} &< \cot \frac{\pi}{6} \\
 &[\text{as } \cot x \text{ is decreasing for } 0 < x < \pi] \\
 \Rightarrow n^2 - 10n + 21.6 &< \pi\sqrt{3} \\
 \Rightarrow n^2 - 10n + 25 + 21.6 - 25 &< \pi\sqrt{3} \\
 \Rightarrow (n-5)^2 &< \pi\sqrt{3} + 3.4 \\
 \Rightarrow -\sqrt{\pi\sqrt{3} + 3.4} &< n - 5 < \sqrt{\pi\sqrt{3} + 3.4} \\
 \Rightarrow 5 - \sqrt{\pi\sqrt{3} + 3.4} &< n < 5 + \sqrt{\pi\sqrt{3} + 3.4} \\
 \text{Since, } \sqrt{3}\pi &\approx 5.5 \text{ nearly} \\
 \therefore \sqrt{\pi\sqrt{3} + 3.4} &\sim \sqrt{8.9} \sim 2.9 \\
 \Rightarrow 2.1 &< n < 7.9 \\
 \therefore n = 3, 4, 5, 6, 7 & \quad [\text{as } n \in \mathbb{N}] \\
 \text{or minimum value of } n &= 3.
 \end{aligned}$$

$$52. \text{ We have, } \tan^{-1} x + \frac{\pi}{2} - \tan^{-1} y = \tan^{-1} 3$$

$$\begin{aligned}
 \Rightarrow \tan^{-1} x - \tan^{-1} y &= \tan^{-1} 3 - \frac{\pi}{2} \\
 \Rightarrow \tan^{-1} \frac{x-y}{1+xy} &= -\left(\frac{\pi}{2} - \tan^{-1} 3 \right) \\
 \Rightarrow \frac{x-y}{1+xy} &= -\tan \left(\frac{\pi}{2} - \tan^{-1} 3 \right) \\
 &= -\cot(\tan^{-1} 3) = -\frac{1}{3} \\
 \Rightarrow \frac{x-y}{1+xy} &= -\frac{1}{3}
 \end{aligned}$$

$$\begin{aligned}
 \text{If } y - x = \lambda, \text{ then } 1 + xy &= 3\lambda \\
 \Rightarrow 1 + x(x + \lambda) = 3\lambda &\Rightarrow \lambda = \frac{x^2 + 1}{3 - x}
 \end{aligned}$$

Since, x and y are positive integers, λ is an integer and $xy > 0$.

$$3\lambda - 1 > 0 \Rightarrow \lambda > \frac{1}{3}$$

For $x = 1, \lambda = 1, y = 2$

For $x = 2, \lambda = 5, y = 7$

For $x > 3, \lambda < 0$

Hence, the required solutions are
 $(x, y) = (1, 2), (2, 7)$

53. We have, $\sin^{-1}(x+1) + \sec^{-1}(x+1) = \frac{\pi}{2}$

$$\Rightarrow \sec^{-1}(x+1) = \frac{\pi}{2} - \sin^{-1}(x+1) = \cos^{-1}(x+1)$$

$$= y \quad [\text{say}]$$

Then, $x+1 = \sec y = \cos y$

$$\Rightarrow \cos^2 y = 1 \Rightarrow \cos y = \pm 1$$

$$\Rightarrow x+1 = \pm 1 \text{ but } x \neq 0 \Rightarrow x = -2$$

So that $2\sec^{-1}\left(\frac{x}{2}\right) + \sin^{-1}\left(\frac{x}{2}\right)$

$$= 2\sec^{-1}(-1) + \sin^{-1}(-1) = 2\pi - \frac{\pi}{2} = \frac{3\pi}{2}$$

54. Since, $\frac{3\pi}{2} < 5 < 2\pi$, we have $\sin 5 < 0$, so

$\sin^{-1}(\sin 5) = 5 - 2\pi$. Therefore, the given inequality can be written as

$$5 - 2\pi > x^2 - 4x \text{ or } x^2 - 4x + (2\pi - 5) < 0$$

$$\Rightarrow \left[x - \frac{4 - \sqrt{16 - 4(2\pi - 5)}}{2} \right]$$

$$\left[x - \frac{4 + \sqrt{16 - 4(2\pi - 5)}}{2} \right] < 0$$

$$\Rightarrow [x - (2 - \sqrt{9 - 2\pi})][x - (2 + \sqrt{9 - 2\pi})] < 0$$

$$\Rightarrow x \in (2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi})$$

55. Let $t = \frac{2x^2 + 4}{1 + x^2} \Rightarrow x^2 = \frac{4 - t}{t - 2}$

Since, $x^2 \geq 0$, we get $2 < t \leq 4$

So, we have to solve $\sin^{-1}(\sin t) < \pi - 3$

and $2 < t \leq 4$ simultaneously.

Now, $\frac{\pi}{2} < 2 < t \leq 4 < \frac{3\pi}{2}$

So, $\sin^{-1}(\sin t) = \pi - t$

and $\pi - t < \pi - 3 \Rightarrow t > 3$

We get, $3 < t \leq 4$

$$\Rightarrow 3 < \frac{2x^2 + 4}{x^2 + 1} \leq 4$$

$$\Rightarrow -1 < x < 1, \text{ which is the required solution.}$$

Aliter

$$\sin^{-1}\left\{\sin\left(\frac{2x^2 + 4}{1 + x^2}\right)\right\} < \pi - 3$$

$$\Rightarrow \pi - \left(2 + \frac{2}{1 + x^2}\right) < \pi - 3$$

$$\Rightarrow -\frac{2}{1 + x^2} < -1$$

$$\Rightarrow 2 > 1 + x^2 \Rightarrow x^2 - 1 < 0$$

$$\Rightarrow x \in (-1, 1)$$

56. We have, $\left|\cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right)\right| < \frac{\pi}{3}$

$$\Rightarrow -\frac{\pi}{3} < \cos^{-1}\left(\frac{1 - x^2}{1 + x^2}\right) < \frac{\pi}{3}$$

$$\Rightarrow 0 \leq \cos^{-1}\frac{1 - x^2}{1 + x^2} < \frac{\pi}{3} \Rightarrow \frac{1}{2} < \frac{1 - x^2}{1 + x^2} \leq 1$$

$$\Rightarrow 1 + x^2 < 2(1 - x^2) \leq 2(1 + x^2)$$

$$\Rightarrow 0 \leq x^2 < \frac{1}{3} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

57. $\cos^{-1}(\cos 4) = \cos^{-1}\{\cos(2\pi - 4)\} = 2\pi - 4$

$$\therefore \cos^{-1}(\cos 4) > 3x^2 - 4x$$

$$\Rightarrow 2\pi - 4 > 3x^2 - 4x$$

$$\Rightarrow 3x^2 - 4x - (2\pi - 4) < 0$$

$$\Rightarrow \frac{2 - \sqrt{6\pi - 8}}{3} < x < \frac{2 + \sqrt{6\pi - 8}}{3}$$

58. $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19}$

$$= \left(\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5}\right) - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \times \frac{3}{5}}\right) - \tan^{-1}\left(\frac{8}{19}\right)$$

$$= \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right)$$

$$= \tan^{-1}\left(\frac{\frac{27}{11} - \frac{8}{19}}{1 + \frac{27}{11} \times \frac{8}{19}}\right) = \tan^{-1}\left(\frac{425}{425}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

59. We have, $\tan^{-1}x > \cot^{-1}x$

$$\Rightarrow \tan^{-1}x > \frac{\pi}{2} - \tan^{-1}x$$

$$\Rightarrow \tan^{-1}x > \frac{\pi}{4}$$

$$\Rightarrow x > \tan\frac{\pi}{4}$$

$$\therefore x > 1$$

60. $8x^2 + 22x + 5 = 0 \Rightarrow x = -\frac{1}{4}, -\frac{5}{2}$

$$\therefore -1 < -\frac{1}{4} < 1 \text{ and } -\frac{5}{2} < -1$$

$$\therefore \sin^{-1}\left(-\frac{1}{4}\right) \text{ exists but } \sin^{-1}\left(-\frac{5}{2}\right) \text{ does not exist,}$$

$$\sec^{-1}\left(-\frac{5}{2}\right) \text{ exists but } \sec^{-1}\left(-\frac{1}{4}\right) \text{ does not exist,}$$

$$\tan^{-1}\left(-\frac{1}{4}\right) \text{ and } \tan^{-1}\left(-\frac{5}{2}\right) \text{ both exist.}$$

61. Since, $\cot^{-1}x$ exists for all $x \in \mathbb{R}$ and $\operatorname{cosec}^{-1}x$ is real when $x \leq -1$ or $x \geq 1$

$$\therefore f(x) \text{ is real for}$$

$$x \in (-\infty, -1] \cup [1, \infty)$$

62. $\sec^{-1}(\sin x)$ is real, if $\sin x \leq -1$ or $\sin x \geq 1$

But $-1 \leq \sin x \leq 1$

$$\therefore \sec^{-1}(\sin x) \text{ is real only, if}$$

$$\sin x = -1 \text{ or } 1$$

$$\Rightarrow x = (2n+1)\frac{\pi}{2}, n \in \mathbb{I}$$

63. We have, $\tan^{-1}\frac{x}{\pi} < \frac{\pi}{3} \Rightarrow \tan\left(\tan^{-1}\frac{x}{\pi}\right) < \tan\frac{\pi}{3}$

$$\Rightarrow \frac{x}{\pi} < \sqrt{3} \Rightarrow x < \sqrt{3}\pi = 5.5 (\text{approx})$$

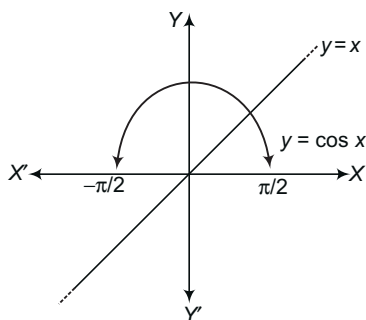
$$\therefore \text{The maximum value of } x \text{ is } 5.$$

64. $\tan^{-1}x$ and $\cot^{-1}x$ exist for all $x \in \mathbb{R}$

$$\cos^{-1}(2-x) \text{ exists, if } -1 \leq 2-x \leq 1, \text{ i.e. } 1 \leq x \leq 3$$

65. Here, $|\cos x| = \sin^{-1}(\sin x)$

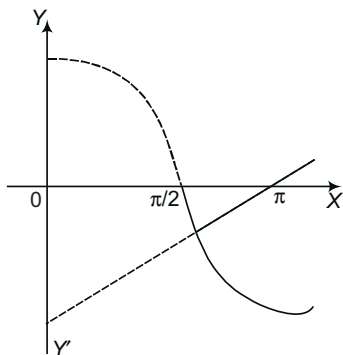
If $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, then $\cos x = x$



In this case, there is one solution obtained graphically.

If $\frac{\pi}{2} < x \leq \pi$, then $-\cos x = \sin^{-1}\{\sin(\pi - x)\} = \pi - x$

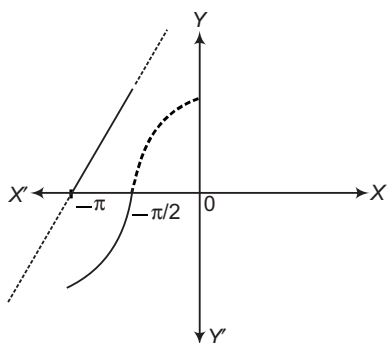
$\therefore \cos x = x - \pi$



In this case, there is one solution obtained graphically.

If $-\pi \leq x < -\frac{\pi}{2}$, then $-\cos x = \sin^{-1}\{\sin(-\pi - x)\}$

$= -x - \pi$, i.e. $\cos x = x + \pi$



$\therefore$ No solution

Hence, there are 2 solutions, when $-\pi \leq x \leq \pi$.

66. Clearly, $x(x+1) \geq 0$ and $x^2 + x + 1 \leq 1$. Together they imply $x(x+1) = 0$

$\therefore x = 0, -1$

When $x = 0$

LHS = $\tan^{-1} 0 + \sin^{-1} 1 = \frac{\pi}{2}$

When $x = -1$

LHS = $\tan^{-1} 0 + \sin^{-1} \sqrt{1-1+1} = 0 + \sin^{-1} 1 = \frac{\pi}{2}$

Thus, there are 2 solutions.

67. In $[0, \pi]$, $|y| = \sin x$, $y = \cos^{-1}(\cos x) = x$

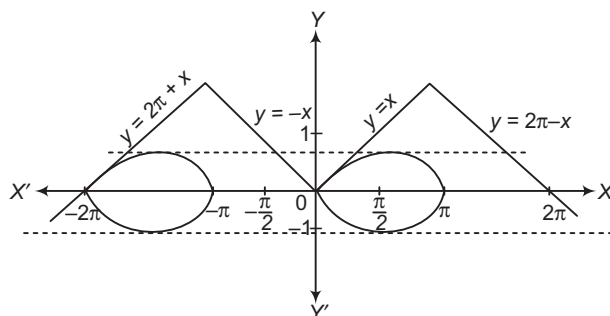
In $[\pi, 2\pi]$, $|y| = \sin x$, $y = \cos^{-1}\{\cos(2\pi - x)\} = 2\pi - x$

In $[-\pi, 0]$, $|y| = \sin x$, $y = \cos^{-1}\{\cos(-x)\} = -x$

In $[-2\pi, -\pi]$, $|y| = \sin x$,

$y = \cos^{-1}\{\cos(2\pi + x)\} = 2\pi + x$

Plotting the graphs, we have



There are 2 solutions, i.e. $(0, 0)$ and $(-2\pi, 0)$.

68. $\sin^{-1}(\sin 4) = \sin^{-1}\{\sin(\pi - 4)\} = \pi - 4$

$$\left[\because -\frac{\pi}{2} < \pi - 4 < \frac{\pi}{2} \right]$$

We have, $x^2 - kx + \pi - 4 > 0$, $\forall x \in \mathbb{R}$

$\therefore D < 0$, i.e. $k^2 - 4(\pi - 4) < 0$

or $k^2 + 4(4 - \pi) < 0$, which is not true for any real k .

69. The n th term of the series can be written as

$$\begin{aligned} u_n &= \cot^{-1}(2n^2) = \tan^{-1}\left(\frac{1}{2n^2}\right) \\ &= \tan^{-1}\left(\frac{(2n+1) - (2n-1)}{1 + (2n+1)(2n-1)}\right) \\ &= \tan^{-1}(2n+1) - \tan^{-1}(2n-1) \end{aligned}$$

so that, $u_1 = \tan^{-1} 3 - \tan^{-1} 1$

$u_2 = \tan^{-1} 5 - \tan^{-1} 3$

$u_3 = \tan^{-1} 7 - \tan^{-1} 5$

and the sum to n terms is therefore written as

$$S_n = \tan^{-1}(2n+1) - \tan^{-1} 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} S_n = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

70. We have,

$$S_1 = \sum x_1 = \sin 2\beta$$

$$S_2 = \sum x_1 x_2 = \cos 2\beta$$

$$S_3 = \sum x_1 x_2 x_3 = \cos \beta$$

$$S_4 = x_1 x_2 x_3 x_4 = -\sin \beta$$

$$\text{Now, } \sum_{i=1}^4 \tan^{-1} x_i = \tan^{-1} \frac{S_1 - S_3}{1 - S_2 + S_4}$$

$$= \tan^{-1} \frac{\sin 2\beta - \cos \beta}{1 - \cos 2\beta - \sin \beta}$$

$$= \tan^{-1} \frac{\cos \beta (2 \sin \beta - 1)}{\sin \beta (2 \sin \beta - 1)}$$

$$= \tan^{-1} \cot \beta = \tan^{-1} \left(\tan \left(\frac{\pi}{2} - \beta \right) \right) = \frac{\pi}{2} - \beta$$

71. Since, $0 \leq \cos^{-1} x_i \leq \pi$

$$\therefore \cos^{-1} x_i = 0, \forall i$$

$$\Rightarrow x_i = 1, \forall i$$

$$\therefore \sum_{i=1}^{2n} x_i = 2n$$

72. Since, $-\frac{\pi}{2} \leq \sin^{-1} x_i \leq \frac{\pi}{2}$
 $\therefore \sin^{-1} x_i = \frac{\pi}{2}, 1 \leq i \leq 20$
 $\therefore x_i = 1, 1 \leq i \leq 20$. Thus, $\sum_{i=1}^{20} x_i = 20$

73. Let $\tan^{-1}(-3) = \alpha \Rightarrow \tan \alpha = -3$
 $\Rightarrow -\frac{\pi}{2} < \alpha < 0 \Rightarrow -\pi < 2\alpha < 0$
 $\Rightarrow \cos(-2\alpha) = \cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha} = \frac{1 - 9}{1 + 9} = -\frac{4}{5}$
 $\Rightarrow -2\alpha = \cos^{-1}\left(-\frac{4}{5}\right)$
 $\Rightarrow 2 \tan^{-1}(-3) = -\cos^{-1}\left(-\frac{4}{5}\right)$
 Again, $-\pi < 2\alpha < 0 \Rightarrow 0 < 2\alpha + \pi < \pi$
 So, $\cos(\pi + 2\alpha) = -\cos 2\alpha = \frac{4}{5}$
 $\Rightarrow \pi + 2\alpha = \cos^{-1} \frac{4}{5}$
 $\Rightarrow 2 \tan^{-1}(-3) = -\pi + \cos^{-1} \frac{4}{5}$
 Finally, $-\pi < 2\alpha < 0$
 $\Rightarrow -\frac{\pi}{2} < 2\alpha + \frac{\pi}{2} < \frac{\pi}{2}$
 So, $\tan\left(\frac{\pi}{2} + 2\alpha\right) = -\cot 2\alpha = \frac{-1}{\tan 2\alpha}$
 $= \frac{-(1 - \tan^2 \alpha)}{2 \tan \alpha} = \frac{-4}{3}$
 $\therefore \frac{\pi}{2} + 2\alpha = \tan^{-1}\left(-\frac{4}{3}\right)$
 $\Rightarrow 2 \tan^{-1}(-3) = -\frac{\pi}{2} + \tan^{-1}\left(-\frac{4}{3}\right)$

74. The given equation can be written as
 $2 \cos^{-1}[\cot(2 \tan^{-1} x)] = n\pi$
 $\Rightarrow \cos^{-1}[\cot(2 \tan^{-1} x)] = \frac{n\pi}{2} \quad [n \in I]$
 Note that, n can take only three values 0, 1, 2.
 $\Rightarrow \cot(2 \tan^{-1} x) = \cos \frac{n\pi}{2} = \begin{cases} 0, & \text{if } n = 1 \\ \pm 1, & \text{if } n = 0, 2 \end{cases}$
 Now, from $\cot(2 \tan^{-1} x) = 0$, we get $2 \tan^{-1} x = k\pi + \frac{\pi}{2}$,
 where $k = -1$ or 0 .
 $\Rightarrow \tan^{-1} x = \frac{\pi}{4}$ or $-\frac{\pi}{4} \Rightarrow x = \pm 1$
 and $\cot(2 \tan^{-1} x) = \pm 1$
 $\Rightarrow \tan(2 \tan^{-1} x) = \pm 1$
 $\Rightarrow \frac{2x}{1 - x^2} = \pm 1$
 $\Rightarrow 1 - x^2 = \pm 2x$
 $\Rightarrow x^2 \pm 2x - 1 = 0$
 $\Rightarrow (x \pm 1)^2 = 2$
 $\Rightarrow x = \pm \sqrt{2} \pm 1$
 Thus, $x = \pm 1, \pm \sqrt{2} \pm 1$

Aliter

We have,
 $\sin[2 \cos^{-1}\{\cot(2 \tan^{-1} x)\}] = 0$

$$\Rightarrow \sin\left[2 \cos^{-1}\left\{\cot\left(\tan^{-1}\left(\frac{2x}{1-x^2}\right)\right)\right\}\right] = 0$$

$$\left[\because \tan^{-1} x = \tan^{-1}\left(\frac{2x}{1-x^2}\right)\right]$$

$$\Rightarrow \sin\left[2 \cos^{-1}\left\{\cot\left(\cot^{-1}\left(\frac{1-x^2}{2x}\right)\right)\right\}\right] = 0$$

$$\left[\because \cot^{-1} x = \tan^{-1} \frac{1}{x}\right]$$

$$\Rightarrow \sin\left[2 \cos^{-1}\left(\frac{1-x^2}{2x}\right)\right] = 0$$

$$\Rightarrow \sin\left[\sin^{-1}\left\{2\left(\frac{1-x^2}{2x}\right)\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right\}\right] = 0$$

$$\left[\because 2 \cos^{-1} x = \sin^{-1}(2x\sqrt{1-x^2})\right]$$

$$\Rightarrow \left(\frac{1-x^2}{2x}\right)\left[\sqrt{1-\left(\frac{1-x^2}{2x}\right)^2}\right] = 0$$

$$\Rightarrow \frac{1-x^2}{2x} = 0 \quad \text{or} \quad 1 - \left(\frac{1-x^2}{2x}\right)^2 = 0$$

$$\Rightarrow 1 - x^2 = 0 \quad \text{or} \quad \left(\frac{1-x^2}{2x}\right)^2 = 1$$

$$\Rightarrow x = \pm 1 \quad \text{or} \quad (1-x^2)^2 = 4x^2$$
 Now,
 $\Rightarrow (1-x^2)^2 - (2x)^2 = 0$
 $\Rightarrow (1-x^2-2x)(1-x^2+2x) = 0$
 $\Rightarrow x^2 + 2x - 1 = 0 \quad \text{or} \quad x^2 - 2x - 1 = 0$
 $\Rightarrow x = -1 \pm \sqrt{2} \quad \text{or} \quad x = 1 \pm \sqrt{2}$
 Hence, $x = \pm 1, -1 \pm \sqrt{2}, 1 \pm \sqrt{2}$ are the roots of the given equation.

75. Given, $\cos^{-1} x = \tan^{-1} x \Rightarrow x = \cos \theta = \tan \theta$
 $\Rightarrow \cos^2 \theta = \sin \theta$
 $\Rightarrow \sin^2 \theta + \sin \theta - 1 = 0$
 $\Rightarrow \sin \theta = \frac{-1 \pm \sqrt{1+4}}{2}$
 $\Rightarrow \sin \theta = \frac{\sqrt{5}-1}{2}$
 $\Rightarrow x^2 = \cos^2 \theta = \frac{\sqrt{5}-1}{2}$
 and $\sin(\cos^{-1} x) = \sin \theta = \frac{1}{2}(\sqrt{5}-1)$

76. LHS and RHS of the given equation are defined, if
 $\frac{a-x}{a-b} > 0 \quad \text{and} \quad \frac{x-b}{a-b} > 0$

Either $a > x > b$ or $a < x < b$.

For $a = x = b$, LHS and RHS are not defined.

77. $\theta = \tan^{-1}(2 \tan^2 \theta) - \tan^{-1}\left(\frac{1}{3} \tan \theta\right)$
 $\Rightarrow \tan \theta = \frac{2 \tan^2 \theta - \frac{1}{3} \tan \theta}{1 + \frac{2}{3} \tan^3 \theta}$
 $\Rightarrow \tan \theta \left[\frac{2 \tan \theta - \frac{1}{3}}{1 + \frac{2}{3} \tan^3 \theta} - 1 \right] = 0$

which is true, if $\tan \theta = 0$

$$\text{or } \frac{2 \tan \theta - \frac{1}{3}}{1 + \frac{2}{3} \tan^3 \theta} = 1$$

$$\Rightarrow \tan^3 \theta - 3 \tan \theta + 2 = 0$$

$$\Rightarrow (\tan \theta - 1)^2 (\tan \theta + 2) = 0$$

which holds, if $\tan \theta = 1$ or $\tan \theta = -2$

78. (a) $\sin \left(\tan^{-1} 3 + \tan^{-1} \frac{1}{3} \right) = \sin \frac{\pi}{2} = 1$

(b) $\cos \left(\frac{\pi}{2} - \sin^{-1} \frac{3}{4} \right) = \cos \left(\cos^{-1} \frac{3}{4} \right) = \frac{3}{4}$

(c) $\sin \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right)$

Let $\sin^{-1} \frac{\sqrt{63}}{8} = \theta$

So, $\sin \theta = \frac{\sqrt{63}}{8}$, if $\cos \theta = \frac{1}{8}$

We have, $\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}} = \frac{3}{4}$

$$\sin \frac{\theta}{4} = \sqrt{\frac{1 - \cos \frac{\theta}{2}}{2}} = \frac{1}{2\sqrt{2}}$$

Now, $\log_2 \sin \left(\frac{1}{4} \sin^{-1} \frac{\sqrt{63}}{8} \right) = \log_2 \frac{1}{2\sqrt{2}} = -\frac{3}{2}$

(d) $\tan \left(\frac{1}{2} \cos^{-1} \left(\frac{\sqrt{5}}{3} \right) \right) = \frac{3 - \sqrt{5}}{2}$

79. If we put $x = \tan \theta$, the given equality becomes $\tan^{-1} y = 4\theta$

$$\Rightarrow y = \tan 4\theta = \frac{2 \tan 2\theta}{1 - \tan^2 2\theta} = \frac{2 \left[\frac{2 \tan \theta}{1 - \tan^2 \theta} \right]}{1 - \left(\frac{2 \tan \theta}{1 - \tan^2 \theta} \right)^2}$$

$$= \frac{2 \cdot 2x(1 - x^2)}{(1 - x^2)^2 - 4x^2} = \frac{4x(1 - x^2)}{1 - 6x^2 + x^4}$$

Now, y is infinite, if $x^4 - 6x^2 + 1 = 0$

$$\Rightarrow x^2 = \frac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm 2\sqrt{2}$$

80. It is clear that, $x = \frac{1}{2}$ is a solution of the given equation

which can be obtained by trial and error method. The given equation can be written as

$$3 \cos^{-1} x = \pi x + \frac{\pi}{2} \quad \dots (i)$$

Since, the LHS of Eq. (i) is a decreasing function and RHS of Eq. (i) is an increasing function of x .

Therefore, Eq. (i) has only one solution. So, $x = \frac{1}{2}$ is

one and only one solution of the given equation.

81. $f(x) = \sin^{-1} x + \cos^{-1} x + 2 \tan^{-1} x$

For domain of $\sin^{-1} x$ and $\cos^{-1} x$, $-1 \leq x \leq 1$

and domain of $\tan^{-1} x$, $x \in \mathbb{R}$

$\Rightarrow$ Domain of $f(x) = [-1, 1]$

Clearly, $f(x) = \frac{\pi}{2} + 2 \tan^{-1} x$

Now, as $x \in [-1, 1]$

$$-\frac{\pi}{4} \leq \tan^{-1} x \leq \frac{\pi}{4} \Rightarrow 0 \leq f(x) \leq \pi$$

$\therefore$ Range of $f(x) = [0, \pi]$

Hence, Statement I is false and Statement II is true.

82. $\sin^{-1}(\sin x) = x$, $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

$$\cos^{-1}(\cos x) = -x, \pi \leq x \leq 0$$

If $\alpha \in \left(-\frac{\pi}{2}, 0 \right)$, then $\sin^{-1}(\sin \alpha) + \cos^{-1}(\cos \alpha)$

$$= \alpha - \alpha = 0$$

$\therefore$ Statement I is true and Statement II is correct explanation of Statement I.

83. As $3 \cos^{-1} x = \begin{cases} \cos^{-1}(4x^3 - 3x), & \text{if } \frac{1}{2} \leq x \leq 1 \\ 2\pi - \cos^{-1}(4x^3 - 3x), & \text{if } -\frac{1}{2} \leq x \leq \frac{1}{2} \\ 2\pi + \cos^{-1}(4x^3 - 3x), & \text{if } -1 \leq x \leq -\frac{1}{2} \end{cases}$

$\therefore$ Statement I is false.

84. $x = \operatorname{cosec} \left(\tan^{-1} \left(\cos \left(\cot^{-1} \left(\sec \left(\cos^{-1} \sqrt{1 - a^2} \right) \right) \right) \right) \right)$

$$= \operatorname{cosec} \left(\tan^{-1} \left(\cos \left(\cot^{-1} \frac{1}{\sqrt{1 - a^2}} \right) \right) \right)$$

$$= \operatorname{cosec} \left(\tan^{-1} \left(\cos \cos \frac{1}{\sqrt{2 - a^2}} \right) \right)$$

$$= \operatorname{cosec} \left(\tan^{-1} \sqrt{2 - a^2} \right)$$

$$= \operatorname{cosec} \left(\operatorname{cosec}^{-1} \sqrt{3 - a^2} \right) = \sqrt{3 - a^2}$$

Also, $y = \sec \left(\cot^{-1} \left(\sin \left(\tan^{-1} \left(\operatorname{cosec} \left(\cos^{-1} a \right) \right) \right) \right) \right)$

$$= \sec \left(\cot^{-1} \left(\sin \tan^{-1} \frac{1}{\sqrt{1 - a^2}} \right) \right)$$

$$= \sec \left(\cot^{-1} \left(\sin \left(\sin^{-1} \frac{1}{\sqrt{2 - a^2}} \right) \right) \right)$$

$$= \sec \left(\cot^{-1} \frac{1}{\sqrt{2 - a^2}} \right)$$

$$= \sec \left(\sec^{-1} \sqrt{3 - a^2} \right) = \sqrt{3 - a^2}$$

$\therefore$ Statement I is true, also Statement II is true but not the correct explanation of Statement I.

85. Put $x = \cos y$, then

$$\cos^{-1} x = y, 2 \sin^{-1} \sqrt{\frac{1 - \cos y}{2}} = y,$$

$$2 \cos^{-1} \sqrt{\frac{1 + \cos y}{2}} = y$$

So, Statement I is correct.

Also, Statement II is correct but does not lead to Statement I.

86. Since, $\sin^{-1} \alpha + \cos^{-1} \alpha = \frac{\pi}{2}$, using this result,

$$\text{we get } x - \frac{x^2}{2} + \frac{x^3}{4} - \dots = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \dots$$

$$\Rightarrow \frac{x}{1 + \left(\frac{x}{2} \right)} = \frac{x^2}{1 + \left(\frac{x^2}{2} \right)}$$

$\Rightarrow 2x(2+x^2) = 2x^2(2+x)$
 $\Rightarrow x[4+2x^2-4x-2x^2] = 0$
 $\Rightarrow x = 1$ as $0 < |x| < \sqrt{2}$
 So, Statement I is true.
 For Statement II, $x(x+1) \geq 0$
 and $0 \leq x^2 + x + 1 \leq 1$
 $\Rightarrow x(x+1) = 0$
 $\Rightarrow x = -1$ as $-\sqrt{2} < x < 0$
 and $x = -1$ satisfies the given equation
 So, Statement II is false.

87. $3\sin^{-1}\left(\frac{1}{3}\right) + \sin^{-1}\left(\frac{3}{5}\right)$
 $= \sin^{-1}\left(3 \times \frac{1}{3} - 4\left(\frac{1}{3}\right)^3\right) + \sin^{-1}\left(\frac{3}{5}\right)$
 $= \sin^{-1}\left(1 - \frac{4}{27}\right) + \sin^{-1}\left(\frac{3}{5}\right)$
 $= \sin^{-1}\frac{23}{27} + \sin^{-1}\frac{3}{5} < \sin^{-1}\frac{\sqrt{3}}{2} + \sin^{-1}\frac{\sqrt{3}}{2}$
 $\left[\because \frac{23}{27} = 0.85, \frac{3}{5} = 0.6 \text{ and } \frac{\sqrt{3}}{2} = 0.86\right]$
 $= \frac{\pi}{3} + \frac{\pi}{3} = \frac{2\pi}{3}$

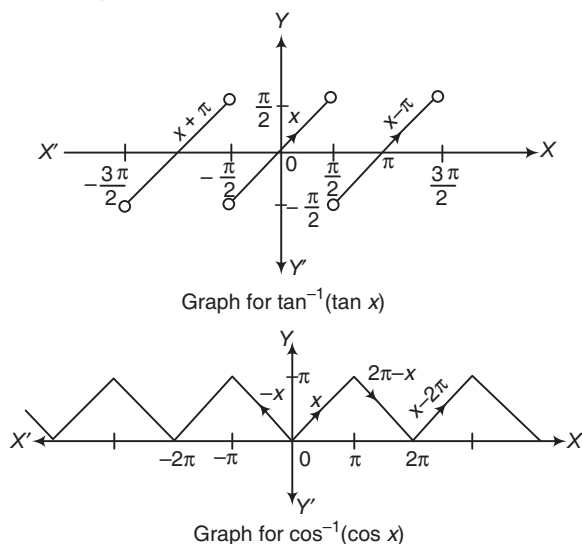
and $\tan^{-1}(2\sqrt{2}-1) > \tan^{-1}\sqrt{3} = \frac{\pi}{3}$
 $[\because 2\sqrt{2}-1 = 1.8, \sqrt{3} = 1.7]$

So, Statement I is true.

In Statement II, $3\sin^{-1}x = \frac{\pi}{6}$
 $\Rightarrow \sin(3\sin^{-1}x) = \frac{1}{2}$
 $\Rightarrow 3x - 4x^3 = \frac{1}{2}$
 $\Rightarrow 6x - 8x^3 - 1 = 0$

Showing that Statement II is also true but does not lead to Statement I.

88. From graph,



Statement II is true and is correct explanation for Statement I.

Solutions (Q. Nos. 89-90)

$f(x) = 0 \Rightarrow \sin\{\cot^{-1}(x+1)\} = \cos(\tan^{-1}x)$
 $\Rightarrow \sin \sin^{-1} \frac{1}{\sqrt{1+(x+1)^2}} = \cos \cos^{-1} \frac{1}{\sqrt{1+x^2}}$
 $\Rightarrow \frac{1}{\sqrt{1+(x+1)^2}} = \frac{1}{\sqrt{1+x^2}}$
 $\Rightarrow 1+x^2 = 2+x^2+2x$
 $\Rightarrow x = -\frac{1}{2}$, so $f(x) = 0$ for $x = -\frac{1}{2}$

Now consider,

$a = \cos \tan^{-1} \sin \cot^{-1} x = \cos \tan^{-1} \sin \alpha$
 [where, $x = \cot \alpha$]
 $\Rightarrow a = \cos \tan^{-1} \frac{1}{\sqrt{1+x^2}} = \cos \beta$
 [where, $\tan \beta = \frac{1}{\sqrt{1+x^2}}$]
 $= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} = \sqrt{\frac{x^2+1}{x^2+2}} \Rightarrow a^2 = \frac{x^2+1}{x^2+2} = \frac{5}{9}$
 for $x = -\frac{1}{2}$

Solutions (Q. Nos. 91-93)

$\cos \alpha = \frac{4}{5}, \sin \alpha = \frac{3}{5}, \tan \alpha = \frac{3}{4}$
 $\tan \beta = \frac{2}{3}, \sin \beta = \frac{2}{\sqrt{13}}, \cos \beta = \frac{3}{\sqrt{13}}$
 $\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{3}{4} \times \frac{2}{3}} = \frac{17}{6}$
 $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta = \frac{17}{5\sqrt{13}}$
 $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta = \frac{6}{5\sqrt{13}}$
 $\cos 2\alpha = 2\cos^2 \alpha - 1 = \frac{7}{25}$
 $\cos 3\alpha = 4\cos^3 \alpha - 3\cos \alpha$
 $= 4 \times \frac{64}{125} - 3 \times \frac{4}{5} = -\frac{44}{125}$
 $\cos(\pi - 3\alpha) = -\cos 3\alpha = \frac{44}{125}$
 $\Rightarrow \cos^{-1} \frac{44}{125} = \pi - 3\alpha$
 $\sin 2\alpha = 2\sin \alpha \cos \alpha = 2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$
 $\sin 3\alpha = 3\sin \alpha - 4\sin^3 \alpha = 3 \times \frac{3}{5} - 4 \times \frac{27}{125}$
 $= \frac{225 - 108}{125} = \frac{117}{125}$
 $\Rightarrow \sin^{-1} \frac{117}{125} = 3\alpha$

Solutions (Q. Nos. 94-95)

$\frac{5\cos 2\theta + 4}{5 + 4\cos 2\theta} = \frac{5(\cos^2 \theta - \sin^2 \theta) + 4(\cos^2 \theta + \sin^2 \theta)}{5 + 4(2\cos^2 \theta - 1)}$
 $= \frac{9\cos^2 \theta - \sin^2 \theta}{1 + 8\cos^2 \theta} = \frac{9 - \tan^2 \theta}{9 + \tan^2 \theta}$

$$\begin{aligned}
 &= \frac{1 - \tan^2 \phi}{1 + \tan^2 \phi}, \text{ where } \tan \phi = \frac{1}{3} \tan \theta \\
 &= \cos 2\phi \\
 \Rightarrow b &= \frac{1}{2} \cos^{-1}(\cos 2\phi) = \phi \\
 \theta &= a - b = \tan^{-1}(3 \tan^3 \theta) - \phi \\
 &= \tan^{-1}(3 \tan^3 \theta) - \tan^{-1}\left[\left(\frac{1}{3}\right) \tan \theta\right] \\
 \Rightarrow \tan \theta &= \frac{3 \tan^3 \theta - \left(\frac{1}{3}\right) \tan \theta}{1 + \tan^4 \theta} \\
 \Rightarrow \tan \theta [9 \tan^2 \theta - 1 - 3 - 3 \tan^4 \theta] &= 0 \\
 \Rightarrow \tan \theta &= 0 \\
 \text{or } 3 \tan^4 \theta - 9 \tan^2 \theta + 4 &= 0 \\
 \Rightarrow \tan^2 \theta &= \frac{9 \pm \sqrt{33}}{6}
 \end{aligned}$$

96. $f(x) = 0$ gives $3x^3 - 13x^2 + 14x - 2 = 0$

By inspection,

$$f(0) = -2, f(1) = 2, f(2) = -2, f(3) = 4$$

$$f(0)f(1) < 0$$

$$f(1)f(2) < 0$$

$$f(2)f(3) < 0$$

Root α will be lying between 0 and 1.

β lying between 1 and 2.

γ lying between 2 and 3.

$$[\alpha] = 0, [\beta] = 1, [\gamma] = 2$$

97. $\therefore 0 < \alpha < 1$

$$\therefore \lim_{n \rightarrow \infty} (\alpha)^n (\beta)^{1/n} = 0$$

98. $\tan(x + y + z)$

$$= \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - (\tan x \tan y + \tan y \tan z + \tan z \tan x)}$$

On putting $x = \tan^{-1} \alpha, y = \tan^{-1} \beta, z = \tan^{-1} \gamma$, we get

$$\tan(x + y + z) = \frac{\alpha + \beta + \gamma - \alpha\beta\gamma}{1 - (\alpha\beta + \beta\gamma + \gamma\alpha)}$$

But $\alpha + \beta + \gamma = \frac{13}{3}$

$$\alpha\beta\gamma = \frac{2}{3}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{14}{3}$$

$$\therefore \tan(x + y + z) = \frac{\left(\frac{13}{3} - \frac{2}{3}\right)}{\left(1 - \frac{14}{3}\right)}$$

$$\Rightarrow \tan(x + y + z) = -1$$

$$\Rightarrow x + y + z = -\frac{\pi}{4} + k\pi, \forall k \in \mathbb{I}$$

As α, β and γ are positive and real.

$$\Rightarrow x + y + z \text{ lies in the range } 0 \text{ to } \frac{3\pi}{2}.$$

Hence, $x + y + z = \frac{3\pi}{4}$

99. Extreme value of $\sin^{-1} x = \frac{5\pi}{2}$ and $\cos^{-1} y = 3\pi$ which gives only 1 possible ordered pair $(1, -1)$.

100. Clearly, we need $\sin^{-1}\left(-\frac{1}{\sqrt{2}}\right)$

The value of θ from the interval $\left[\frac{3\pi}{2}, \frac{5\pi}{2}\right]$ for which

$$\sin \theta = -\frac{1}{\sqrt{2}} \text{ is } \frac{7\pi}{4}.$$

101. We have, $2\theta = \sin^{-1}(\sin 2\theta)$ [let $\sin^{-1} x = \theta$]

$$\Rightarrow \frac{3\pi}{2} \leq 2\theta \leq \frac{5\pi}{2}$$

$$\Rightarrow \frac{3\pi}{4} \leq \theta \leq \frac{5\pi}{4}$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$$

102. Let $\sin^{-1}(-x) = \theta, \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\Rightarrow x = \sin(-\theta)$$

$$\Rightarrow x = \sin(2\pi - \theta)$$

where, $2\pi - \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\Rightarrow (2\pi - \theta) = \sin^{-1}(x)$$

$$\Rightarrow \sin^{-1}(-x) = 2\pi - \sin^{-1}(x)$$

103. Let $f(x) = 3\sin^{-1} x - 2\cos^{-1} x$

$$\therefore \sin^{-1}(-x) = 2\pi - \sin^{-1} x$$

$$\text{Also, } \cos^{-1}(-x) = 3\pi - \cos^{-1} x$$

$$\therefore 3\sin^{-1}(-x) - 2\cos^{-1}(-x) = -[3\sin^{-1} x - 2\cos^{-1} x] = -f(x)$$

$$\Rightarrow f(x) \text{ is odd.}$$

104. In $[-1, 1]$, $\sin^{-1} x$ is decreasing and $\cos^{-1} x$ is increasing.

$$\Rightarrow (\sin^{-1} x)^3 - (\cos^{-1} x)^3 \text{ is minimum at } x = 1.$$

$$\therefore (\sin^{-1} x)^3 - (\cos^{-1} x)^3 = \left(\frac{\pi}{2}\right)^3 - (2\pi)^3 = \frac{\pi^3}{8} - 8\pi^3 = -\frac{63}{8}\pi^3$$

105. Let $\sin^{-1} x = \theta$, where $\theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$

$$\Rightarrow x = \sin \theta = \cos \left[\frac{5\pi}{2} - \theta\right]$$

where, $\frac{5\pi}{2} - \theta \in [\pi, 2\pi]$

$$\Rightarrow \cos^{-1} x = \frac{5\pi}{2} - \theta$$

$$\Rightarrow \sin^{-1} x + \cos^{-1} x = \frac{5\pi}{2}$$

106. A. For $(0, \cos 1)$, we have $\cos^{-1} x > \sin^{-1} x$

$$\text{Also, } \cos^{-1} x > 1 \text{ and } \sin^{-1} x < 1$$

$$\Rightarrow \text{The greatest is } (\cos^{-1} x)^{\cos^{-1} x} = t_4 \text{ and the least}$$

$$\text{is } (\sin^{-1} x)^{\cos^{-1} x} = t_2$$

$$\text{and } (\sin^{-1} x)^{\sin^{-1} x} < (\cos^{-1} x)^{\sin^{-1} x}$$

$$\Rightarrow t_1 < t_3$$

$$\text{So, } t_4 > t_3 > t_1 > t_2$$

B. Similarly, in $\cos 1 < x < \frac{1}{\sqrt{2}}$

$$\cos^{-1} x > \sin^{-1} x \text{ and both are less than } 1. \text{ So, greatest is } t_3 \text{ and least is } t_2 \text{ and } t_4 > t_1.$$

$$\text{Hence, } t_3 > t_4 > t_1 > t_2$$

C. For $\frac{1}{\sqrt{2}} < x < \sin 1$, we have $1 > \sin^{-1} x > \cos^{-1} x$

So, greatest is t_2 and least is t_3 , also $t_1 > t_4$.

Hence, $t_2 > t_1 > t_4 > t_3$

D. For $\sin 1 < x < 1$, we have

$$\sin^{-1} x > 1 > \cos^{-1} x$$

So, the greatest is t_1 and least is t_3 and $t_2 > t_4$.

Hence, $t_1 > t_2 > t_4 > t_3$

107. Refer to the graphs on Page 230 of $y = \sin^{-1}(\sin x)$;
 $y = \cos^{-1}(\cos x)$; $y = \tan^{-1}(\tan x)$ and $y = \cot^{-1}(\cot x)$

108. A. $\tan^{-1} 3 + \tan^{-1} 4 = \pi + \tan^{-1} \frac{3+4}{1-3 \times 4}$
[$\because 3 > 0, 4 > 0, 1 - 3 \times 4 < 0$]

$$= \pi - \tan^{-1} \left(\frac{7}{11} \right)$$

$$\begin{aligned} \text{B. } \tan^{-1} \left(\frac{1}{3} \right) + \tan^{-1} \left(\frac{1}{4} \right) &= \tan^{-1} \frac{\frac{1}{3} + \frac{1}{4}}{1 - \left(\frac{1}{3} \right) \left(\frac{1}{4} \right)} \\ &= \tan^{-1} \left(\frac{7}{11} \right) \end{aligned}$$

$$\text{C. } \sin^{-1} \left(\frac{1}{3} \right) + \cos^{-1} \left(\frac{1}{3} \right) = \frac{\pi}{2}$$

$$\text{D. } \tan^{-1}(3) + \cot^{-1}(4) = \tan^{-1} 3 + \tan^{-1} \left(\frac{1}{4} \right) = \tan^{-1} 13$$

109. A. Taking $\left(\frac{1}{2} \right) \cos^{-1} x = \alpha$ in (A), we have

$$\frac{1 + \tan \alpha}{1 - \tan \alpha} + \frac{1 - \tan \alpha}{1 + \tan \alpha} = 1$$

$$\Rightarrow \frac{2(1 + \tan^2 \alpha)}{1 - \tan^2 \alpha} = 1 \Rightarrow \frac{2}{\cos 2\alpha} = 1 \Rightarrow \frac{2}{x} = 1$$

$\Rightarrow x = 2$ but for $x = 2$, $\cos x$ is not defined. So, it has no solution.

$$\text{B. Now, } \frac{\frac{1}{2x+1} + \frac{1}{4x+1}}{1 - \frac{1}{(2x+1)(4x+1)}} = \frac{2}{x^2} \Rightarrow \frac{6x+2}{8x^2+6x} = \frac{2}{x^2}$$

$$\text{Either } x = 0 \text{ or } 6x^2 - 14x - 12 = 0$$

$$\Rightarrow x = 0, 3, -\frac{2}{3}$$

$\therefore$ It has 3 solutions.

C. It can be written as

$$\tan^{-1} \left[x + \left(\frac{2}{x} \right) \right] - \tan^{-1} \left[x - \left(\frac{2}{x} \right) \right] = \tan^{-1} \frac{4}{x}$$

$$\Rightarrow \frac{x + \left(\frac{2}{x} \right) - \left[x - \left(\frac{2}{x} \right) \right]}{1 + \left[x^2 - \left(\frac{4}{x^2} \right) \right]} = \frac{4}{x}$$

$$\Rightarrow 1 + x^2 - \left(\frac{4}{x^2} \right) = 1$$

$$\Rightarrow x^4 = 4 \Rightarrow x = \pm \sqrt{2}$$

$\therefore$ It has only 2 real solutions.

$$\text{D. } \tan^{-1} \left(\frac{2}{x^2} \right) = \tan^{-1}(2x) \Rightarrow x = 1 \text{ only one solution.}$$

$$110. \text{ A. } \frac{\pi}{2} - \cos^{-1} |\cos x| - \frac{\pi}{2} + \sin^{-1} |\sin x| = a$$

$$\sin^{-1} |\sin x| - \cos^{-1} |\cos x| = a$$

$$[\because \sin^{-1} |\sin x| = \cos^{-1} |\cos x|]$$

$$\Rightarrow a = 0$$

$$\text{B. } \sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x^2 - 1} + \tan^{-1} \tan y = a$$

$$\text{Clearly, } 0 \leq x \leq 1$$

$$\text{and } 0 \leq x^2 - 1 \leq 1$$

$$\Rightarrow 1 \leq x^2 \leq 2$$

$$\Rightarrow x = 1$$

$$\Rightarrow \frac{\pi}{2} + \frac{\pi}{2} + \tan^{-1} \tan y = a$$

$$-\frac{\pi}{2} < \tan^{-1} \tan y < \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{2} < a < \frac{3\pi}{2}$$

$$\text{C. } |\sin^{-1} |\sin x|| = |\tan^{-1} |\tan x||,$$

$$\forall x \in \mathbb{R} - (2n+1)\frac{\pi}{2}, n \in \mathbb{I}$$

$$\sqrt{|\sin^{-1} |\sin x||} = \sqrt{a}, (a \geq 0)$$

$$|\sin^{-1} |\sin x|| = a$$

$$\Rightarrow a \in \left[0, \frac{\pi}{2} \right)$$

$$\text{D. } \sin^{-1}(x^2 + y^2) + \tan^{-1} \sqrt{4y^2 - 1} + \sec^{-1} x = a$$

$$\text{For } \sin^{-1}(x^2 + y^2), 0 \leq (x^2 + y^2) \leq 1 \quad \dots(i)$$

$$\text{For } \tan^{-1} \sqrt{4y^2 - 1}, 4y^2 - 1 \geq 0$$

$$\Rightarrow y \in \left(-\infty, -\frac{1}{2} \right] \cup \left[\frac{1}{2}, \infty \right) \quad \dots(ii)$$

$$\text{For } \sec^{-1} x, x \geq 1 \text{ or } x \leq -1 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$x, y \in \phi \Rightarrow a \in \mathbb{R}$$

111. A. Since, $0 \leq \cos^{-1} x \leq \pi$

Hence, from the question,

$$\cos^{-1} \lambda = \pi, \cos^{-1} \mu = \pi, \cos^{-1} v = \pi$$

$$\therefore \lambda = \mu = v = -1 \Rightarrow \lambda\mu + \mu v + \lambda v = 3$$

B. Since, $0 \leq \cos^{-1} x \leq \pi$

Hence, from the question,

$$\cos^{-1} x_i = 0 \text{ for all } i$$

$$\therefore x_i = 1 \text{ for all } i \Rightarrow \sum_{i=1}^{10} x_i = 10$$

$$\text{C. Since, } -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$$

$$\text{Hence, from the question, } \sin^{-1} x_i = \frac{\pi}{2}, \forall i$$

$$\therefore x_i = 1, \forall i \Rightarrow \sum_{i=1}^{2n} x_i = 2n$$

D. Let $x = \sin \theta$. Then,

$$f(x) = \sin^{-1} \left\{ \sin \left(\theta - \frac{\pi}{6} \right) \right\}, -\frac{1}{2} \leq x \leq 1$$

$$\Rightarrow -\frac{1}{2} \leq \sin \theta \leq 1$$

$$\Rightarrow -\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$$

$$\therefore f(x) = \sin^{-1} x - \frac{\pi}{6}$$

$$112. \cos^{-1} \frac{1}{5\sqrt{2}} = \tan^{-1} 7, \sin^{-1} \frac{4}{\sqrt{17}} = \tan^{-1} 4$$

$$\therefore x = \frac{7-4}{1+7 \times 4} = \frac{3}{29}$$

$$\Rightarrow 29x = 3$$

$$113. x = 1 + \tan^2(\tan^{-1} 2) + 1 + \cot^2(\cot^{-1} 3)$$

$$= 1 + 4 + 1 + 9 = 15$$

$$y = \tan^{-1} \frac{1 - \cos 2}{\sin 2} = \tan^{-1} \tan 1 = 1$$

$$\therefore x^2 - 220y^2 = 225 - 220 \times 1 = 5$$

$$114. k = 2997 \sin x + 3996 \cos x$$

$$= 999(3 \sin x + 4 \cos x)$$

$$= 999r \sin(x + \alpha), \text{ where } r \sin \alpha = 4,$$

$$\Rightarrow r \cos \alpha = 3 \text{ and } r = 5$$

$$\therefore \text{Maximum value is } 999r$$

$$= 999 \times 5 = 4995 = k$$

$$\therefore \frac{k}{555} = 9$$

$$115. \text{ Since, } \frac{2r}{2 + r^2 + r^4} = \frac{(1 + r + r^2) - (1 - r + r^2)}{1 + (1 + r + r^2)(1 - r + r^2)}$$

$$S = \sum_{r=1}^{90} [\tan^{-1}(1 + r + r^2) - \tan^{-1}(1 - r + r^2)]$$

$$= \sum_{r=1}^{90} [\tan^{-1}(1 + r + r^2) - \tan^{-1}(1 + (r-1) + (r-1)^2)]$$

$$= \tan^{-1}(1 + 90 + 90^2) - \tan^{-1} 1$$

$$= \tan^{-1} \frac{90 + 90^2}{90 + 90^2 + 2}$$

$$\Rightarrow \tan S = \frac{8190}{8192}$$

$$\Rightarrow 8190 \cot S = 8192$$

$$\Rightarrow \frac{8190}{8192} \cot S = 1$$

$$116. \text{ Since, } \sin^{-1} y \text{ is defined for } -1 \leq y \leq 1,$$

$$-1 \leq x^2 - 4x + 5 \leq 1$$

$$\Rightarrow x^2 - 4x + 4 \leq 0 \text{ and } x^2 - 4x + 6 \geq 0$$

$$\Rightarrow (x-2)^2 \leq 0 \text{ and } (x-2)^2 + 2 \geq 0$$

$$\therefore x = 2 \text{ is the only solution.}$$

On putting $x = 2$ in the original equation, we get

$$4 + 2a + \frac{\pi}{2} = 0$$

$$\Rightarrow 2a = -4 - \frac{\pi}{2}$$

$$\Rightarrow a = -2 - \frac{\pi}{4}$$

$$\Rightarrow \lambda = 2 \text{ and } \mu = 4$$

$$\text{Hence, } \lambda + \mu = 2 + 4 = 6$$

$$117. \text{ We have, } \sin^{-1}(\sin 12) + \cos^{-1}(\cos 12)$$

$$= \sin^{-1}(\sin(4\pi - (4\pi - 12)))$$

$$+ \cos^{-1}(\cos(4\pi - (4\pi - 12)))$$

$$= -(4\pi - 12) + (4\pi - 12) = 0$$

$$\text{Now, } (n-2)x^2 + 8x + n + 4 > 0, \forall x \in \mathbb{R}$$

$$\Rightarrow n - 2 > 0 \Rightarrow n \geq 3$$

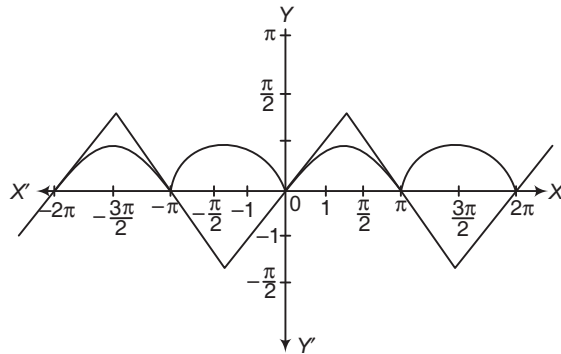
$$\text{and } 8^2 - 4(n-2)(n+4) < 0$$

$$\text{or } n^2 + 2n - 24 > 0$$

$$\Rightarrow n > 4$$

$$\Rightarrow n \geq 5 \Rightarrow n = 5$$

118. From the figure, it is clear that number of solutions is 5.



119. Since, x_1, x_2, x_3 and x_4 are the roots of the equation.

$$\therefore x^4 - x^3 \sin 2\beta + x^2 \cos 2\beta - x \cos \beta - \sin \beta = 0$$

$$\Rightarrow \sum x_i = x_1 + x_2 + x_3 + x_4 = -\frac{(-\sin 2\beta)}{1} = \sin 2\beta,$$

$$\sum x_i x_j = \cos 2\beta$$

$$\sum x_i x_j x_k = \cos \beta \text{ and } x_1 x_2 x_3 x_4 = -\sin \beta$$

$$\text{Now, } \tan[\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4]$$

$$= \frac{\sum x_i - \sum x_i x_j x_k}{1 - \sum x_i x_j + x_1 x_2 x_3 x_4}$$

$$= \frac{\sin 2\beta - \cos \beta}{1 - \cos 2\beta - \sin \beta} = \frac{2 \sin \beta \cos \beta - \cos \beta}{2 \sin^2 \beta - \sin \beta}$$

$$= \frac{\cos \beta (2 \sin \beta - 1)}{\sin \beta (2 \sin \beta - 1)} = \cot \beta$$

$$\text{or } \tan[\tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4]$$

$$= \tan\left(\frac{\pi}{2} - \beta\right)$$

$$\Rightarrow \tan^{-1} x_1 + \tan^{-1} x_2 + \tan^{-1} x_3 + \tan^{-1} x_4 = n\pi + \frac{\pi}{2} - \beta, \text{ where } n = 0, \pm 1, \pm 2, \dots, \text{ i.e. } n \text{ is an integer.}$$

$$= n\pi + \frac{\pi}{\lambda} + \mu\beta \Rightarrow \lambda\mu = 2$$

$$120. \text{ We have, } \tan^{-1} x + \cos^{-1} \frac{y}{\sqrt{1+y^2}} = \sin^{-1} \frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} \frac{1}{y} = \tan^{-1} 3$$

$$\Rightarrow \tan^{-1} \left(\frac{x + \frac{1}{y}}{1 - \frac{x}{y}} \right) = \tan^{-1} 3$$

$$\Rightarrow \frac{xy + 1}{y - x} = 3 \Rightarrow y = \frac{3x + 1}{3 - x} \quad \dots(i)$$

Now, $y > 0$

$$\Rightarrow \frac{3x + 1}{3 - x} > 0 \Rightarrow \frac{3x + 1}{x - 3} < 0$$

$$\Rightarrow -\frac{1}{3} < x < 3 \Rightarrow x = 1, 2$$

On substituting the values of x in Eq. (i), we get

$$y = 2, 7$$

Clearly, y is an integer for $x = 1, 2$.

Hence, the required solutions are (1, 2) and (2, 7).

Hence, the number of all positive integral solutions is 2.

121. We have, $f(x) = \sqrt{3 \cos^{-1}(4x) - \pi}$, which is defined, if

$$3 \cos^{-1}(4x) - \pi \geq 0$$

$$\Rightarrow \cos^{-1}(4x) \geq \frac{\pi}{3} \Rightarrow 4x \leq \frac{1}{2}$$

$$\Rightarrow x \leq \frac{1}{8} \quad \dots(i)$$

Also, $-1 \leq 4x \leq 1 \Rightarrow -\frac{1}{4} \leq x \leq \frac{1}{4} \quad \dots(ii)$

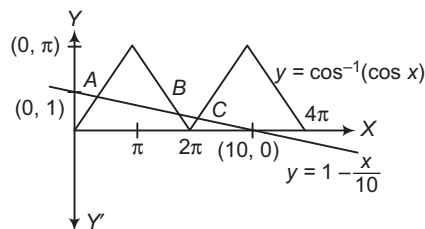
From Eqs. (i) and (ii), we have domain $\left[-\frac{1}{4}, \frac{1}{8}\right]$

$$\Rightarrow 4a + 64b = 7$$

Entrances Gallery

1. $f(x) = \cos^{-1}(\cos x) \in [0, 4\pi]$

and $f(x) = \frac{10-x}{10} = 1 - \frac{x}{10}$



So, it has 3 solutions.

2. $\cot \left\{ \sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right\}$

$$= \cot \left\{ \sum_{n=1}^{23} \cot^{-1} (1 + 2 + 4 + 6 + 8 + \dots + 2n) \right\}$$

$$= \cot \left[\sum_{n=1}^{23} \cot^{-1} \{1 + n(n+1)\} \right]$$

$$= \cot \left[\sum_{n=1}^{23} \tan^{-1} \frac{1}{1 + n(n+1)} \right]$$

$$= \cot \left[\sum_{n=1}^{23} \tan^{-1} \left\{ \frac{(n+1) - n}{1 + n(n+1)} \right\} \right]$$

$$= \cot \left[\sum_{n=1}^{23} \{ \tan^{-1}(n+1) - \tan^{-1} n \} \right]$$

$$= \cot [\tan^{-1} 24 - \tan^{-1} 1]$$

$$= \cot \left[\tan^{-1} \left(\frac{23}{25} \right) \right]$$

$$= \cot \left[\cot^{-1} \left(\frac{25}{23} \right) \right] = \frac{25}{23}$$

3. A. $\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}$

$$= \frac{\frac{1}{\sqrt{1+y^2}} + \frac{y^2}{\sqrt{1+y^2}}}{\frac{1}{\sqrt{1-y^2}} + \frac{y}{\sqrt{1-y^2}}} = \frac{\sqrt{1+y^2}}{\frac{1}{y\sqrt{1-y^2}}} = y\sqrt{1-y^4}$$

$$\Rightarrow \frac{1}{y^2} \left(\frac{\cos(\tan^{-1} y) + y \sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)} \right)^2 + y^4$$

$$= \frac{1}{y^2} (y^2(1-y^4)) + y^4 = 1 - y^4 + y^4 = 1$$

B. We have, $\cos x + \cos y + \cos z = 0$
 $\sin x + \sin y + \sin z = 0$

$$\Rightarrow \cos x + \cos y = -\cos z \quad \dots(i)$$

$$\sin x + \sin y = -\sin z \quad \dots(ii)$$

On adding and squaring Eqs. (i) and (ii), we get

$$1 + 1 + 2(\cos x \cos y + \sin x \sin y) = 1$$

$$\Rightarrow 2 + 2 \cos(x-y) = 1$$

$$\Rightarrow 2 \cos(x-y) = -1$$

$$\Rightarrow \cos(x-y) = -\frac{1}{2}$$

$$\Rightarrow 2 \cos^2 \left(\frac{x-y}{2} \right) - 1 = -\frac{1}{2}$$

$$\Rightarrow 2 \cos^2 \left(\frac{x-y}{2} \right) = \frac{1}{2}$$

$$\Rightarrow \cos^2 \left(\frac{x-y}{2} \right) = \frac{1}{4}$$

$$\Rightarrow \cos \left(\frac{x-y}{2} \right) = \frac{1}{2}$$

C. $\cos \left(\frac{\pi}{4} - x \right) \cos 2x + \sin x \sin 2x \sec x$

$$= \cos x \sin 2x \sec x + \cos \left(\frac{\pi}{4} + x \right) \cos 2x$$

$$\Rightarrow \left[\cos \left(\frac{\pi}{4} - x \right) - \cos \left(\frac{\pi}{4} + x \right) \right] \cos 2x$$

$$= (\cos x \sin 2x - \sin x \sin 2x) \sec x$$

$$\Rightarrow \frac{2}{\sqrt{2}} \sin x \cos 2x = (\cos x - \sin x) \sin 2x \sec x$$

$$\Rightarrow \sqrt{2} \sin x \cos 2x = (\cos x - \sin x) 2 \sin x$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{1}{\cos x + \sin x}$$

$$\Rightarrow x = \frac{\pi}{4}$$

$$\Rightarrow \sec x = \sec \frac{\pi}{4} = \sqrt{2}$$

D. Let $\alpha = \sin^{-1} \sqrt{1-x^2}$. Then,

$$\cot(\sin^{-1} \sqrt{1-x^2}) = \cot \alpha, \text{ where } \sin \alpha = \sqrt{1-x^2}$$

$$\Rightarrow \cot \alpha = \frac{x}{\sqrt{1-x^2}}$$

and let $\tan^{-1}(x\sqrt{6}) = \phi$

$$\therefore \sin \phi = \frac{x\sqrt{6}}{\sqrt{6x^2+1}}$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \frac{x\sqrt{6}}{\sqrt{6x^2+1}}$$

$$6x^2 + 1 = 6 - 6x^2$$

$$12x^2 = 5$$

$$x = \sqrt{\frac{5}{12}} = \frac{1}{2} \sqrt{\frac{5}{3}}$$

4. Given, $\tan^{-1} y = \tan^{-1} x + \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

where, $|x| < \frac{1}{\sqrt{3}}$

$$\Rightarrow \tan^{-1} y = \tan^{-1} \left\{ \frac{x + \frac{2x}{1-x^2}}{1 - x \left(\frac{2x}{1-x^2} \right)} \right\}$$

$$\left[\because \tan^{-1} x + \tan^{-1} y = \tan^{-1} \left(\frac{x+y}{1-xy} \right), \right]$$

$$\left[x > 0, y > 0, xy < 1 \right]$$

$$= \tan^{-1} \left(\frac{x - x^3 + 2x}{1 - x^2 - 2x^2} \right)$$

$$\tan^{-1} y = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$$

$$\Rightarrow y = \frac{3x - x^3}{1 - 3x^2}$$

Aliter

$$|x| < \frac{1}{\sqrt{3}} \Rightarrow -\frac{1}{\sqrt{3}} < x < \frac{1}{\sqrt{3}}$$

Let $x = \tan \theta$

$$\Rightarrow -\frac{\pi}{6} < \theta < \frac{\pi}{6}$$

$$\therefore \tan^{-1} y = \theta + \tan^{-1}(\tan 2\theta) = \theta + 2\theta = 3\theta$$

$$\Rightarrow y = \tan 3\theta$$

$$\Rightarrow y = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\Rightarrow y = \frac{3x - x^3}{1 - 3x^2}$$

5. Since, x, y and z are in an AP.

$$\therefore 2y = x + z$$

Also, $\tan^{-1} x, \tan^{-1} y$ and $\tan^{-1} z$ are in an AP.

$$\therefore 2 \tan^{-1} y = \tan^{-1} x + \tan^{-1} z$$

$$\Rightarrow \tan^{-1} \left(\frac{2y}{1-y^2} \right) = \tan^{-1} \left(\frac{x+z}{1-xz} \right)$$

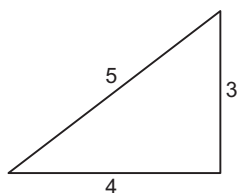
$$\Rightarrow \frac{x+z}{1-y^2} = \frac{x+z}{1-xz}$$

$$\Rightarrow y^2 = xz$$

Since, x, y and z are in AP as well as in a GP.

$$\therefore x = y = z$$

6. Since, $\operatorname{cosec}^{-1} \left(\frac{5}{3} \right) = \tan^{-1} \left(\frac{3}{4} \right)$



$$\therefore \cot \left(\tan^{-1} \frac{3}{4} + \tan^{-1} \frac{2}{3} \right)$$

$$= \cot \left(\tan^{-1} \left[\frac{\frac{3}{4} + \frac{2}{3}}{1 - \frac{1}{2}} \right] \right) = \cot \left(\tan^{-1} \left[\frac{\left(\frac{17}{12} \right)}{\left(\frac{1}{2} \right)} \right] \right)$$

$$= \cot \left[\tan^{-1} \left(\frac{17}{6} \right) \right] = \frac{6}{17}$$

$$7. \text{ Since, } \sin^{-1} \left(\frac{x}{5} \right) + \operatorname{cosec}^{-1} \left(\frac{5}{4} \right) = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) + \sin^{-1} \left(\frac{4}{5} \right) = \frac{\pi}{2}$$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) = \frac{\pi}{2} - \sin^{-1} \left(\frac{4}{5} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) = \cos^{-1} \left(\frac{4}{5} \right)$$

$$\Rightarrow \sin^{-1} \left(\frac{x}{5} \right) = \sin^{-1} \left(\frac{3}{5} \right)$$

$$\Rightarrow x = 3$$

8. Given, $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$

$$\Rightarrow \cos^{-1} \left(\frac{xy}{2} + \sqrt{1-x^2} \sqrt{1-\frac{y^2}{4}} \right) = \alpha$$

$$\Rightarrow \frac{xy}{2} + \sqrt{1-x^2} \sqrt{1-\frac{y^2}{4}} = \cos \alpha$$

$$\Rightarrow 2\sqrt{1-x^2} \sqrt{1-\frac{y^2}{4}} = 2\cos \alpha - xy$$

On squaring both sides, we get

$$\frac{4(1-x^2)(4-y^2)}{4} = 4\cos^2 \alpha + x^2y^2 - 4xy \cos \alpha$$

$$\Rightarrow 4 - 4x^2 - y^2 + x^2y^2 = 4\cos^2 \alpha + x^2y^2 - 4xy \cos \alpha$$

$$\Rightarrow 4x^2 - 4xy \cos \alpha + y^2 = 4\sin^2 \alpha$$

9. Given, $\sin^{-1} x = 2 \sin^{-1} a$

$$\text{Since, } -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{2} \leq 2 \sin^{-1} a \leq \frac{\pi}{2}$$

$$\Rightarrow -\frac{\pi}{4} \leq \sin^{-1} a \leq \frac{\pi}{4}$$

$$\Rightarrow \sin \left(-\frac{\pi}{4} \right) \leq a \leq \sin \frac{\pi}{4}$$

$$\Rightarrow -\frac{1}{\sqrt{2}} \leq a \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow |a| \leq \frac{1}{\sqrt{2}}$$

10. Given

$$\cot^{-1}(\sqrt{\cos \alpha}) - \tan^{-1}(\sqrt{\cos \alpha}) = x \quad \dots(i)$$

We know that,

$$\cot^{-1}(\sqrt{\cos \alpha}) + \tan^{-1}(\sqrt{\cos \alpha}) = \frac{\pi}{2} \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2 \cot^{-1}(\sqrt{\cos \alpha}) = \frac{\pi}{2} + x$$

$$\Rightarrow \sqrt{\cos \alpha} = \cot \left(\frac{\pi}{4} + \frac{x}{2} \right)$$

$$\Rightarrow \sqrt{\cos \alpha} = \frac{\cot \frac{x}{2} - 1}{1 + \cot \frac{x}{2}}$$

$$\Rightarrow \sqrt{\cos \alpha} = \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}}$$

On squaring both sides, we get

$$\cos \alpha = \frac{1 - \sin x}{1 + \sin x} \Rightarrow \frac{1 - \tan^2 \frac{\alpha}{2}}{1 + \tan^2 \frac{\alpha}{2}} = \frac{1 - \sin x}{1 + \sin x}$$

On applying componendo and dividendo rule, we get

$$\sin x = \tan^2 \left(\frac{\alpha}{2} \right)$$

$$\begin{aligned} 11. \tan^{-1} \left(\frac{1}{4} \right) + \tan^{-1} \left(\frac{2}{9} \right) &= \tan^{-1} \left(\frac{\frac{1}{4} + \frac{2}{9}}{1 - \frac{1}{4} \times \frac{2}{9}} \right) \\ &= \tan^{-1} \left(\frac{17}{34} \right) = \tan^{-1} \left(\frac{1}{2} \right) \end{aligned}$$

$$\begin{aligned} 12. \text{ We have, } (\sin^{-1} x)^2 + (\cos^{-1} x)^2 &= (\sin^{-1} x + \cos^{-1} x)^2 - 2 \sin^{-1} x \cdot \cos^{-1} x \\ &= \frac{\pi^2}{4} - 2 \sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x \right) \\ &= \frac{\pi^2}{4} - \pi \sin^{-1} x + 2 (\sin^{-1} x)^2 \\ &= 2 \left[(\sin^{-1} x)^2 - \frac{\pi}{2} \sin^{-1} x + \frac{\pi^2}{8} \right] \\ &= 2 \left[\left(\sin^{-1} x - \frac{\pi}{4} \right)^2 + \frac{\pi^2}{16} \right] \end{aligned}$$

Thus, the least value is $2 \left(\frac{\pi^2}{16} \right)$, i.e. $\frac{\pi^2}{8}$ and the greatest value is

$$2 \left[\left(-\frac{\pi}{2} - \frac{\pi}{4} \right)^2 + \frac{\pi^2}{16} \right], \text{ i.e. } \frac{5\pi^2}{4}.$$

$$13. \text{ We know that, } \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}, \forall x \in [-1, 1]$$

$$\begin{aligned} \therefore \cos^{-1} x > \sin^{-1} x &\Rightarrow \frac{\pi}{2} - \sin^{-1} x > \sin^{-1} x \\ \Rightarrow \frac{\pi}{2} > 2 \sin^{-1} x &\Rightarrow \sin^{-1} x < \frac{\pi}{4} \Rightarrow x < \frac{1}{\sqrt{2}} \\ \therefore -1 \leq x < \frac{1}{\sqrt{2}} \end{aligned}$$

$$14. \text{ Given, } \tan^{-1} \left(\frac{a}{x} \right) + \tan^{-1} \left(\frac{b}{x} \right) = \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} \left[\frac{\left(\frac{a}{x} + \frac{b}{x} \right)}{1 - \frac{ab}{x^2}} \right] = \frac{\pi}{2} \Rightarrow \frac{\frac{a+b}{x}}{\frac{x^2 - ab}{x^2}} = \tan \frac{\pi}{2} \Rightarrow x^2 = ab$$

$$\therefore x = \sqrt{ab}$$

$$\begin{aligned} 15. \cos^{-1} [\cos \{2 \cot^{-1} (\sqrt{2} - 1)\}] &= \cos^{-1} [\cos \{2 \times 67.50^\circ\}] \\ &= \cos^{-1} [\cos 135^\circ] = 135^\circ = 3\pi/4 \end{aligned}$$

Aliter

$$\begin{aligned} \cos^{-1} [\cos \{2 \cot^{-1} (\sqrt{2} - 1)\}] &= \cos^{-1} \left[\cos \left\{ 2 \cos^{-1} \left(\frac{\sqrt{2} - 1}{\sqrt{4 - 2\sqrt{2}}} \right) \right\} \right] \\ &= \cos^{-1} \left[\cos \left\{ \cos^{-1} \left(\frac{2(\sqrt{2} - 1)^2}{4 - 2\sqrt{2}} - 1 \right) \right\} \right] \\ &= \cos^{-1} [\cos \{2 \cos^{-1} x\}] \end{aligned}$$

$$[\because 2 \cos^{-1} x = \cos^{-1} (2x^2 - 1)]$$

$$\begin{aligned} &= \cos^{-1} \left[\cos \left\{ \cos^{-1} \left(\frac{6 - 4\sqrt{2} - 4 + 2\sqrt{2}}{4 - 2\sqrt{2}} \right) \right\} \right] \\ &= \cos^{-1} \left[\cos \left\{ \cos^{-1} \left(\frac{2 - 2\sqrt{2}}{4 - 2\sqrt{2}} \right) \right\} \right] \\ &= \cos^{-1} \left[\cos \left\{ \cos^{-1} \left(-\frac{1}{\sqrt{2}} \right) \right\} \right] \\ &= \cos^{-1} \left[\cos \left[\pi - \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) \right] \right] \\ &= \cos^{-1} \left[-\cos \left\{ \cos^{-1} \left(\frac{1}{\sqrt{2}} \right) \right\} \right] \\ &= \cos^{-1} \left(-\frac{1}{\sqrt{2}} \right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \end{aligned}$$

16. We first try to put $\tan^{-1} [(x - y)/(2 + m^2 + m^4)]$ in the form $\tan^{-1} [(x - y)/(1 + xy)]$.

$$\begin{aligned} \text{Let } x - y &= 2m \text{ and } xy = 1 + m^2 + m^4 \\ \Rightarrow x(x - 2m) &= 1 + m^2 + m^4 \\ \Rightarrow x^2 - 2mx + m^2 &= 1 + 2m^2 + m^4 \\ \Rightarrow (x - m)^2 &= (m^2 + 1)^2 \\ \Rightarrow x - m &= (m^2 + 1) \\ \Rightarrow x &= m^2 + m + 1 \\ \text{and } y &= m^2 - m + 1 \end{aligned}$$

$$\text{Consider, } \sum_{m=1}^n \tan^{-1} \left(\frac{2m}{m^4 + m^2 + 2} \right)$$

$$\begin{aligned} &= \sum_{m=1}^n \tan^{-1} \left[\frac{2m}{1 + (m^2 + m + 1)(m^2 - m + 1)} \right] \\ &= \sum_{m=1}^n \tan^{-1} \left[\frac{(m^2 + m + 1) - (m^2 - m + 1)}{1 + (m^2 + m + 1)(m^2 - m + 1)} \right] \\ &= \sum_{m=1}^n \{ \tan^{-1}(m^2 + m + 1) - \tan^{-1}(m^2 - m + 1) \} \\ &= (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 7 - \tan^{-1} 3) + (\tan^{-1} 13 \\ &\quad - \tan^{-1} 7) + \dots + \tan^{-1}(n^2 + n + 1) - \tan^{-1}(n^2 - n + 1) \\ &= \tan^{-1} \left[\frac{n^2 + n + 1 - 1}{1 + (n^2 + n + 1) \cdot 1} \right] = \tan^{-1} \left(\frac{n^2 + n}{2 + n^2 + n} \right) \end{aligned}$$

17. Let $\sin^{-1} a = A, \sin^{-1} b = B$ and $\sin^{-1} c = C$

$$\begin{aligned} \therefore \sin A &= a, \sin B = b, \sin C = c \\ \text{and } A + B + C &= \pi, \text{ then} \\ \sin 2A + \sin 2B + \sin 2C &= 4 \sin A \sin B \sin C \\ \Rightarrow \sin A \cos A + \sin B \cos B + \sin C \cos C &= 2 \sin A \sin B \sin C \\ \Rightarrow \sin A \sqrt{1 - \sin^2 A} + \sin B \sqrt{1 - \sin^2 B} &+ \sin C \sqrt{1 - \sin^2 C} = 2 \sin A \sin B \sin C \\ \Rightarrow a \sqrt{1 - a^2} + b \sqrt{1 - b^2} + c \sqrt{1 - c^2} &= 2abc \\ \text{while } \sin^{-1} a + \sin^{-1} b + \sin^{-1} c &= \pi \end{aligned}$$

18. Consider for $0 \leq x \leq \frac{1}{2}$,

$$\begin{aligned} \tan \left[\sin^{-1} \left\{ \frac{x}{\sqrt{2}} + \frac{\sqrt{1 - x^2}}{\sqrt{2}} \right\} - \sin^{-1} x \right] \\ = \tan \left[\sin^{-1} \left\{ \frac{x + \sqrt{1 - x^2}}{\sqrt{2}} \right\} - \sin^{-1} x \right] \end{aligned}$$

$$\text{Put } \sin^{-1} x = \theta \Rightarrow x = \sin \theta$$



$$\begin{aligned} \therefore \tan \left[\sin^{-1} \left\{ \frac{\sin \theta + \sqrt{1 - \sin^2 \theta}}{\sqrt{2}} \right\} - \theta \right] \\ = \tan \left[\sin^{-1} \left\{ \frac{1}{\sqrt{2}} \sin \theta + \frac{1}{\sqrt{2}} \cos \theta \right\} - \theta \right] \\ = \tan \left[\sin^{-1} \left\{ \sin \left(\theta + \frac{\pi}{4} \right) \right\} - \theta \right] = \tan \left[\theta + \frac{\pi}{4} - \theta \right] \\ = \tan \frac{\pi}{4} = 1 \end{aligned}$$

19. Let $E = \sin(2 \sin^{-1} 0.8)$

$$\text{Put } \sin^{-1} 0.8 = \theta \Rightarrow \sin \theta = 0.8$$

$$\therefore \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - 0.64} = \sqrt{0.36} = 0.6$$

$$E = \sin(2\theta) = 2 \sin \theta \cos \theta = 2 \times 0.8 \times 0.6 = 0.96$$

20. Given, $\sin^{-1} \left(\frac{x}{13} \right) + \operatorname{cosec}^{-1} \left(\frac{13}{12} \right) = \frac{\pi}{2}$... (i)

$$\text{Let } \operatorname{cosec}^{-1} \frac{13}{12} = y, \text{ then, } \operatorname{cosec} y = \frac{13}{12} \Rightarrow \sin y = \frac{12}{13}$$

$$\therefore \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - \left(\frac{12}{13} \right)^2} = \sqrt{1 - \frac{144}{169}}$$

$$= \sqrt{\frac{25}{169}} = \frac{5}{13} \Rightarrow y = \cos^{-1} \frac{5}{13}$$

$$\text{Eq. (i) becomes } \sin^{-1} \left(\frac{x}{13} \right) + \cos^{-1} \left(\frac{5}{13} \right) = \frac{\pi}{2}$$

$$\text{We know that, } \sin^{-1} \theta + \cos^{-1} \theta = \frac{\pi}{2}$$

Hence, both angles of Eq. (i) should be same.

$$\therefore x = 5$$

21. $\sin^{-1} \left(\sin \frac{2\pi}{3} \right) = \sin^{-1} \left[\sin \left(\pi - \frac{\pi}{3} \right) \right] = \sin^{-1} \left(\sin \frac{\pi}{3} \right) = \frac{\pi}{3}$

22. Given, $\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \pi$

$$\Rightarrow \tan^{-1} \left(\frac{x + y + z - xyz}{1 - xy - yz - zx} \right) = \pi$$

$$\Rightarrow \frac{x + y + z - xyz}{1 - xy - yz - zx} = \tan \pi = 0$$

$$\Rightarrow x + y + z = xyz$$

23. $\therefore \operatorname{sech}^{-1} x = \log_e \left(\frac{1 + \sqrt{1 - x^2}}{x} \right), x \in (0, 1]$

$$\text{and } \operatorname{cosech}^{-1} x = \log_e \left(\frac{1 + \sqrt{1 + x^2}}{x} \right), x > 0$$

$$\therefore \operatorname{sech}^{-1} \left(\frac{1}{2} \right) - \operatorname{cosech}^{-1} \left(\frac{3}{4} \right)$$

$$= \log_e \left[\frac{1 + \sqrt{1 - \left(\frac{1}{2} \right)^2}}{\frac{1}{2}} \right] - \log_e \left(\frac{1 + \sqrt{1 + \frac{9}{16}}}{\frac{3}{4}} \right)$$

$$= \log_e \left(\frac{1 + \sqrt{\frac{3}{4}}}{\frac{1}{2}} \right) - \log_e \left(\frac{1 + \frac{5}{4}}{\frac{3}{4}} \right)$$

$$= \log_e (2 + \sqrt{3}) - \log_e \left(\frac{9}{3} \right)$$

$$= \log_e (2 + \sqrt{3}) - \log_e (3) = \log_e \left(\frac{2 + \sqrt{3}}{3} \right)$$

24. $\tan^{-1} \sqrt{3} - \cot^{-1} (-\sqrt{3})$

$$= \tan^{-1} \sqrt{3} - [\pi - \cot^{-1} \sqrt{3}] \quad [\because \cot^{-1}(-x) = \pi - \cot^{-1} x]$$

$$= \tan^{-1} \sqrt{3} + \cot^{-1} \sqrt{3} - \pi$$

$$= \frac{\pi}{2} - \pi = -\frac{\pi}{2} \quad \left[\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \right]$$

25. We have, $\tan^{-1} \left(\frac{1}{x} \right) = \pi + \tan^{-1} x, 0 < x < 1$

$$\Rightarrow \tan^{-1} \left(\frac{1}{x} \right) - \tan^{-1} x = \pi$$

$$\Rightarrow \tan^{-1} \left(\frac{\frac{1}{x} - x}{1 + \frac{1}{x} \cdot x} \right) = \pi \Rightarrow \tan^{-1} \left(\frac{1 - x^2}{x(1 + 1)} \right) = \pi$$

$$\Rightarrow \frac{1 - x^2}{2x} = \tan \pi \Rightarrow \frac{1 - x^2}{x} = 0$$

$$\Rightarrow 1 - x^2 = 0 \text{ and } x \neq 0$$

$$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

Hence, no values of x exists.

26. $\sin(2 \tan^{-1} x) = \sin \left(\sin^{-1} \frac{2x}{1 + x^2} \right) = \frac{2x}{1 + x^2}, -1 \leq x \leq 1$

27. $\cos^{-1} \frac{(1 - x^2)}{(1 + x^2)} + \tan^{-1} x = \frac{\pi}{2}$

$$\Rightarrow 2 \tan^{-1} x + \tan^{-1} x = \frac{\pi}{2}$$

$$\Rightarrow 3 \tan^{-1} x = \frac{\pi}{2} \Rightarrow \tan^{-1} x = \frac{\pi}{6}$$

$$\therefore x = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\begin{aligned} 28. \text{ Clearly, } 2 \tan^{-1} \frac{1}{5} &= \tan^{-1} \left[\frac{2 \left(\frac{1}{5} \right)}{1 - \left(\frac{1}{5} \right)^2} \right] \\ &= \tan^{-1} \left(\frac{10}{24} \right) = \tan^{-1} \left(\frac{5}{12} \right) \end{aligned}$$

$$\text{and } 4 \tan^{-1} \frac{1}{5} = 2 \tan^{-1} \frac{5}{12}$$

$$= \tan^{-1} \left[\frac{2 \left(\frac{5}{12} \right)}{1 - \left(\frac{5}{12} \right)^2} \right] = \tan^{-1} \frac{120}{119}$$

$$\text{Also, } \tan^{-1} \frac{1}{99} - \tan^{-1} \frac{1}{70} = \tan^{-1} \left(\frac{\frac{1}{99} - \frac{1}{70}}{1 + \frac{1}{99 \times 70}} \right)$$

$$= \tan^{-1} \left(\frac{-29}{6931} \right) = -\tan^{-1} \left(\frac{1}{239} \right)$$

$$\therefore 4 \tan^{-1} \frac{1}{5} - \tan^{-1} \frac{1}{70} + \tan^{-1} \frac{1}{99}$$

$$= \tan^{-1} \left(\frac{120}{119} \right) - \tan^{-1} \left(\frac{1}{239} \right) = \tan^{-1} \left(\frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119} \times \frac{1}{239}} \right)$$

$$= \tan^{-1} \left(\frac{28680 - 119}{28561} \right) = \tan^{-1} \left(\frac{28561}{28561} \right)$$

$$= \tan^{-1}(1) = \frac{\pi}{4}$$

$$29. \text{ Clearly, } \tan^{-1} \left[\tan \frac{5\pi}{4} \right] = \tan^{-1} \left[\tan \left(\pi + \frac{\pi}{4} \right) \right] \\ = \tan^{-1} \left(\tan \frac{\pi}{4} \right) = \frac{\pi}{4}$$

$$\Rightarrow \alpha = \frac{\pi}{4}$$

$$\text{and } \tan^{-1} \left[-\tan \left(\frac{2\pi}{3} \right) \right] = \tan^{-1} \left[-\tan \left(\pi - \frac{\pi}{3} \right) \right] \\ = \tan^{-1} \left(\tan \frac{\pi}{3} \right) = \frac{\pi}{3} \Rightarrow \beta = \frac{\pi}{3}$$

$$\text{Now, consider } 4\alpha - 3\beta = 4 \left(\frac{\pi}{4} \right) - 3 \left(\frac{\pi}{3} \right) = 0$$

$$30. \text{ We know that, } |\sin^{-1} x| \leq \frac{\pi}{2}. \text{ From the given relation,} \\ \text{we observe that, each of } \sin^{-1} x, \sin^{-1} y \text{ and } \sin^{-1} z \text{ will} \\ \text{be } \frac{\pi}{2}.$$

$$\Rightarrow x = y = z = \sin \frac{\pi}{2} = 1$$

$$\therefore x^9 + y^9 + z^9 - \frac{1}{x^9 y^9 z^9} = 1 + 1 + 1 - \frac{1}{1} = 3 - 1 = 2$$

$$31. \text{ We have, } \tan^{-1} x = \frac{\pi}{4} - \tan^{-1} \left(\frac{1}{3} \right)$$

$$\Rightarrow \tan^{-1} x + \tan^{-1} \left(\frac{1}{3} \right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} \left[\frac{x + \frac{1}{3}}{1 - x \left(\frac{1}{3} \right)} \right] = \frac{\pi}{4}$$

$$\Rightarrow \frac{3x + 1}{3 - x} = 1$$

$$\Rightarrow 3x + 1 = 3 - x$$

$$\Rightarrow 4x = 2$$

$$\therefore x = \frac{1}{2}$$

$$32. \text{ Given, } a \sin^{-1} x - b \cos^{-1} x = c$$

$$\Rightarrow a \left(\frac{\pi}{2} - \cos^{-1} x \right) - b \cos^{-1} x = c$$

$$\Rightarrow (a + b) \cos^{-1} x = \frac{a\pi}{2} - c$$

$$\Rightarrow \cos^{-1} x = \frac{\frac{a\pi}{2} - c}{a + b}$$

$$\text{Again from Eq. (i), } a \sin^{-1} x - b \left[\frac{\pi}{2} - \sin^{-1} x \right] = c$$

$$\Rightarrow (\sin^{-1} x)(a + b) = c + \frac{b\pi}{2}$$

$$\Rightarrow \sin^{-1} x = \frac{c + \frac{b\pi}{2}}{a + b}$$

$$\therefore a \sin^{-1} x + b \cos^{-1} x = \frac{a \left(c + \frac{b\pi}{2} \right)}{a + b}$$

$$+ \frac{b \left(\frac{a\pi}{2} - c \right)}{a + b} = \frac{c(a - b) + ab\pi}{a + b}$$

$$33. \tan^{-1}(2) - \tan^{-1}(1) + \tan^{-1}(3) - \tan^{-1}(2) \\ + \dots + \tan^{-1}(n+1) - \tan^{-1}(n) = \tan^{-1} \theta \\ \Rightarrow \tan^{-1}(n+1) - \tan^{-1} 1 = \tan^{-1} \theta \\ \therefore \frac{n}{n+2} = \theta$$

$$34. \text{ On substituting } x = \tan \theta, \text{ we get}$$

$$\sin \left[\tan^{-1} \left(\frac{1 - \tan^2 \theta}{2 \tan \theta} \right) + \cos^{-1} \left(\frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right) \right] \\ = \sin \left[\tan^{-1}(\cot 2\theta) + \cos^{-1}(\cos 2\theta) \right] \\ = \sin \left[\tan^{-1} \left\{ \tan \left(\frac{\pi}{2} - 2\theta \right) \right\} + 2\theta \right] \\ = \sin \left(\frac{\pi}{2} - 2\theta + 2\theta \right) = 1$$

$$35. \cos(4095^\circ) = \cos(45 \times 90^\circ + 45^\circ) = -\sin 45^\circ \\ = -\sin \frac{\pi}{4} = \sin \left(-\frac{\pi}{4} \right)$$

$$\therefore \sin^{-1} \{ \cos(4095^\circ) \} = \sin^{-1} \left\{ \sin \left(-\frac{\pi}{4} \right) \right\} = -\frac{\pi}{4}$$

$$36. \text{ Since, } a_1, a_2, a_3, \dots, a_n \text{ are in an AP with common} \\ \text{difference 5.}$$

$$\Rightarrow a_2 - a_1 = a_3 - a_2 = \dots = a_n - a_{n-1} = 5$$

$$\text{Let } T_1 = \tan^{-1} \frac{5}{1 + a_1 a_2} = \tan^{-1} \frac{a_2 - a_1}{1 + a_1 a_2}$$

$$= \tan^{-1} a_2 - \tan^{-1} a_1$$

$$\text{Similarly, } T_2 = \tan^{-1} a_3 - \tan^{-1} a_2$$

$$T_3 = \tan^{-1} a_4 - \tan^{-1} a_3$$

$$\vdots$$

$$\vdots$$

$$T_{n-1} = \tan^{-1} a_n - \tan^{-1} a_{n-1}$$

$$\text{On adding all, we get}$$

$$\text{Required sum} = \tan^{-1} a_n - \tan^{-1} a_1 = \tan^{-1} \frac{a_n - a_1}{1 + a_1 a_n} \\ = \tan^{-1} \frac{a_1 + 5(n-1) - a_1}{1 + a_1 a_n} = \tan^{-1} \frac{5(n-1)}{1 + a_1 a_n}$$

$$37. \tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1} \left(\frac{1}{2} \right)$$

$$\Rightarrow \tan^{-1} \left[\frac{x+2+x-2}{1-(x+2)(x-2)} \right] = \tan^{-1} \left(\frac{1}{2} \right)$$

$$\Rightarrow \frac{2x}{1-x^2+4} = \frac{1}{2}$$

$$\Rightarrow 4x = 5 - x^2$$

$$\Rightarrow x^2 + 4x - 5 = 0$$

$$\Rightarrow (x-1)(x+5) = 0 \Rightarrow x = 1, -5$$

$$\therefore x = 1$$

$$[\text{since, } -5 \text{ does not satisfy the given equation}]$$

$$38. \frac{\sin 2 - 1}{\cos 2} = -\frac{1 - \sin 2}{\cos 2} = -\frac{(\cos 1 - \sin 1)^2}{(\cos 1 + \sin 1)(\cos 1 - \sin 1)}$$

$$= -\frac{\cos 1 - \sin 1}{\cos 1 + \sin 1} = -\frac{1 - \tan 1}{1 + \tan 1}$$

$$= -\tan \left(\frac{\pi}{4} - 1 \right) = \tan \left(1 - \frac{\pi}{4} \right)$$

$$\Rightarrow \tan^{-1} \left(\frac{\sin 2 - 1}{\cos 2} \right) = \tan^{-1} \left[\tan \left(1 - \frac{\pi}{4} \right) \right] \\ = 1 - \frac{\pi}{4}$$

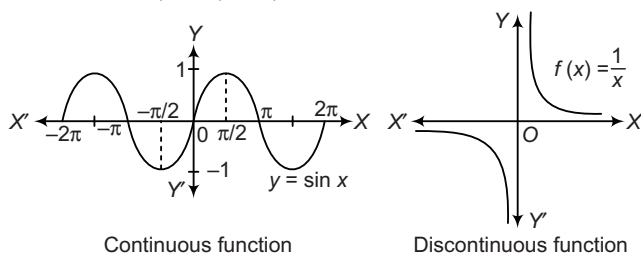
5

Continuity and Differentiability

Continuity

The geometrical significance of continuity is that if the function is continuous, its graph does not bear a break, otherwise it is discontinuous. The point where the graph of the function breaks, is called the point of discontinuity. Thus, a function is continuous at a point, if limit exist at this point and equal to the value of the function at this point.

e.g. Graphs of functions $\sin x$, $x e^x$, etc., are continuous.



Cauchy's Definition of Continuity

A function f is said to be continuous at a point a of its domain if for every $\epsilon > 0$, there exists $\delta > 0$ (dependent on ϵ) such that

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$$

Heine's Definition of Continuity

A function f is said to be continuous at a point a of its domain D , if for every sequence $\langle a_n \rangle$ of the points in D converging to a , then the sequence $\langle f(a_n) \rangle$ converges to $f(a)$, i.e. $\lim_{n \rightarrow \infty} a_n = a \Rightarrow \lim_{n \rightarrow \infty} f(a_n) = f(a)$

Continuity at a Point

A function $f(x)$ is said to be continuous at $x = a$, if

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

i.e. LHL = RHL = value of the function at a

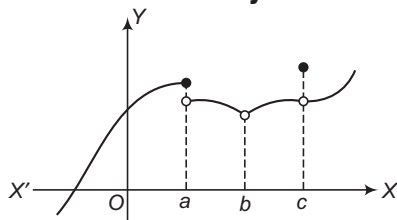
i.e. $\lim_{x \rightarrow a} f(x) = f(a)$

If $f(x)$ is not continuous at $x = a$, then we say that $f(x)$ is discontinuous at $x = a$.

Chapter Snapshot

- Continuity
- Cauchy's Definition of Continuity
- Heine's Definition of Continuity
- Continuity at a Point
- Geometrical Interpretation of Continuity at a Point
- Continuity in an Open Interval
- Continuity in a Closed Interval
- Properties of Continuous Functions
- Discontinuity
- Different Kinds of Discontinuity
- Continuity of Composition of Function
- Differentiability
- Differentiability of a Function at a Point
- Geometrical Interpretation of Differentiability
- Differentiability in a Set
- Relation between Derivability and Continuity
- Properties of Differentiable Functions
- Problems of Functional Equations

Reasons of Discontinuity

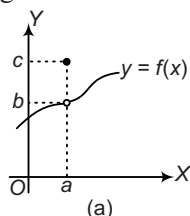


- (i) $\lim_{x \rightarrow a} f(x)$ does not exist.
- (ii) $f(x)$ is not defined at $x=b$.
- (iii) $\lim_{x \rightarrow c} f(x) \neq f(c)$

In all the above cases, geometrically, the graph of the function will exhibit a break at the point of discussion. The graph as shown is discontinuous at $x=a$, b and c .

Geometrical Interpretation of Continuity at a Point

As we can see in the figure (a) that the graph of the function suffers a gap at $x=a$, i.e. while drawing the graph of this function $y=f(x)$ with a pen, you will have to pick up your pen a bit at $x=a$ and then start plotting all points greater than a .

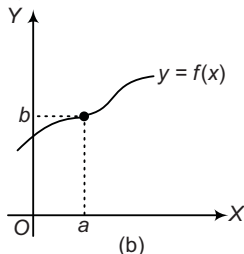


Now, it can be easily seen from the graph that the limit of $f(x)$ as x approaches a from left equals the limit of $f(x)$ as x approaches a from the right.

$$\text{i.e. } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$$

Since, the value of $f(x)$ at $x=a$ is c , so we can say that the function is discontinuous at $x=a$ and $x=a$ is the point of discontinuity.

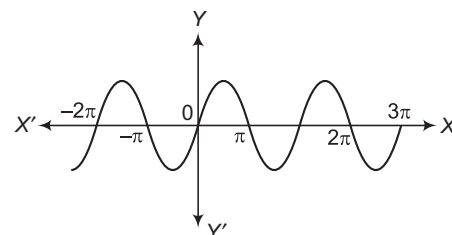
Now, if this point (a, c) is moved in such a way that it is now positioned at (a, b) , as shown in figure (b), then the gap at (a, b) will be filled so as to maintain the continuity of the graph.



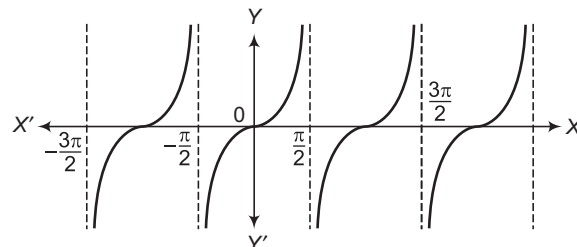
i.e. for the graph of the function to be continuous, we must have

$$\text{Limit} = \text{Value} \Rightarrow \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

- (i) $f(x) = \sin x$ is continuous in its entire domain

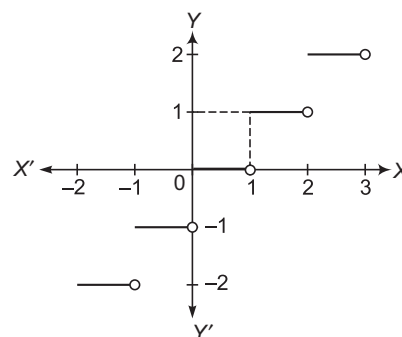


- (ii) $f(x) = \tan x$ is discontinuous at $x = (2n+1)\frac{\pi}{2}$, where $n \in I$.



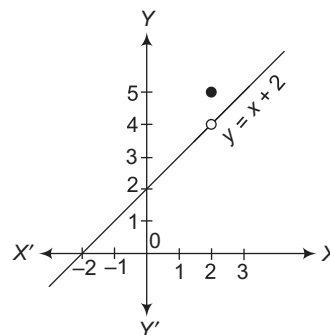
$f(x)$ will be discontinuous at $x=a$, in any of the following cases:

- (a) $\lim_{x \rightarrow a^-} f(x)$ and $\lim_{x \rightarrow a^+} f(x)$ exist but are not equal.
e.g. $y=[x]$ at $x \in I$ is discontinuous.



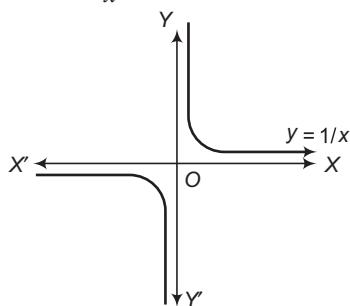
- (b) $\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x)$ but not equal to $f(a)$.

$$\text{e.g. } f(x) = \begin{cases} \frac{x^2 - 4}{x - 2}, & x \neq 2 \\ 5, & x = 2 \end{cases} \text{ at } x=2$$



(c) $f(a)$ is not defined.

e.g. $y = \frac{1}{x}$ at $x = 0$



Continuity in an Open Interval

A function $f(x)$ is said to be continuous in an open interval (a, b) , if it is continuous at each and every point of (a, b) , i.e. $y = [x]$ is continuous in $(1, 2)$.

Continuity from Left and from Right

Let $f(x)$ be a function defined in an open interval I and $a \in I$, then f is continuous from the left at a , if $\lim_{x \rightarrow a^-} f(x)$ exists and is equal to $f(a)$. Similarly, $f(x)$

is said to be continuous from the right at a , if $\lim_{x \rightarrow a^+} f(x)$ exists and is equal to $f(a)$.

Continuity in a Closed Interval

A function $f(x)$ is said to be continuous in a closed interval $[a, b]$, if

- it is continuous in (a, b) .
- value of the function at b is equal to left hand limit at b , i.e. $f(b) = \lim_{x \rightarrow b^-} f(x)$.
- Value of the function at a is equal to right hand limit at a , i.e. $f(a) = \lim_{x \rightarrow a^+} f(x)$.

Properties of Continuous Functions

- If f and g are continuous functions, then
 - $f \pm g$ and fg are continuous.
 - cf is continuous, where c is a constant.
 - f/g is continuous at those points, where $g(x) \neq 0$.

- If f is continuous in $[a, b]$, then it is bounded in $[a, b]$ i.e. there exist m and M such that $m \leq f(x) \leq M, \forall x \in [a, b]$ where, m and M are called minimum and maximum values of $f(x)$ respectively in the interval $[a, b]$.

- If f is continuous in $[a, b]$, then f assumes atleast one value between minimum and maximum values of $f(x)$.

Thus, $a \leq x \leq b \Rightarrow m \leq f(x) \leq M$ or range of $f(x) = [m, M]$, if $x \in [a, b]$.

- If f is continuous in its domain, then $|f|$ is also continuous in its domain.

- If f is continuous at a and $f(a) \neq 0$, then there exists an open interval $(a - \delta, a + \delta)$ such that for all $x \in (a - \delta, a + \delta)$, $f(x)$ has the same sign as $f(a)$.

- If f is a continuous function defined on $[a, b]$ such that $f(a)$ and $f(b)$ are of opposite signs, then there exists atleast one solution of the equation $f(x) = 0$ in the open interval (a, b) .

- If f is continuous on $[a, b]$ and maps $[a, b]$ into $[a, b]$, then for some $x \in [a, b]$, we have $f(x) = x$.

- If f is continuous in domain D , then $\frac{1}{f}$ is also continuous in $D - [x : f(x) = 0]$.

- A function $f(x)$ is said to be everywhere continuous, if it is continuous on the entire real line $(-\infty, \infty)$.

Integral function of a continuous function is a continuous function.

➤ **Example 1.** The function $f(x)$ defined by

$$f(x) = \begin{cases} \left\{ x^2 + e^{\frac{1}{2-x}} \right\}^{-1}, & x \neq 2 \\ k, & x = 2 \end{cases} \text{ is continuous}$$

from right at $x = 2$, then k is equal to

- 0
- $\frac{1}{4}$
- $-\frac{1}{2}$
- None of these

Sol. (b) It is given that $f(x)$ is right continuous at $x = 2$.

$$\therefore \lim_{x \rightarrow 2^+} f(x) = f(2) \Rightarrow \lim_{h \rightarrow 0} f(2 + h) = k$$

$$\Rightarrow \lim_{h \rightarrow 0} \left\{ (2 + h)^2 + e^{\frac{1}{2 - (2 + h)}} \right\}^{-1} = k$$

$$\Rightarrow \lim_{h \rightarrow 0} \left\{ (2 + h)^2 + e^{-1/h} \right\}^{-1} = k$$

$$\Rightarrow (4 + 0)^{-1} = k \Rightarrow k = \frac{1}{4}$$

➤ **Example 2.** If $f(x) = \frac{1 - \sin x}{(\pi - 2x)^2}$, when $x \neq \frac{\pi}{2}$

and $f\left(\frac{\pi}{2}\right) = \lambda$, then $f(x)$ will be continuous

function at $x = \frac{\pi}{2}$, when λ is equal to

(a) $\frac{1}{8}$

(b) $\frac{1}{4}$

(c) $\frac{1}{2}$

(d) None of the above

Sol. (a) $f(x)$ will be continuous at $x = \frac{\pi}{2}$, if

$$\begin{aligned} \lim_{x \rightarrow \pi/2} f(x) &= f\left(\frac{\pi}{2}\right) \\ \Rightarrow \lim_{x \rightarrow \pi/2} \frac{1 - \sin x}{(\pi - 2x)^2} &= \lambda \\ \Rightarrow \frac{1}{4} \lim_{x \rightarrow \pi/2} \frac{1 - \cos\left(\frac{\pi}{2} - x\right)}{\left(\frac{\pi}{2} - x\right)^2} &= \lambda \\ \Rightarrow \frac{1}{4} \times \frac{1}{2} &= \lambda \left[\because \lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2} \right] \\ \therefore \lambda &= \frac{1}{8} \end{aligned}$$

➤ **Example 3.** If the function $f(x)$ defined by

$$f(x) = \begin{cases} \frac{\log(1+3x) - \log(1-2x)}{x}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

is continuous at $x = 0$, then the value of a is

(a) 5

(b) 1

(c) -1

(d) None of these

Sol. (a) Since, $f(x)$ is continuous at $x = 0$.

$$\begin{aligned} \therefore \lim_{h \rightarrow 0} f(0+h) &= \lim_{h \rightarrow 0} f(0-h) = f(0) \\ \Rightarrow a &= \lim_{h \rightarrow 0} \frac{\log(1+3h) - \log(1-2h)}{h} \\ &= \lim_{h \rightarrow 0} \left\{ 3 \frac{\log(1+3h)}{3h} + 2 \frac{\log(1-2h)}{-2h} \right\} \\ &= 3 \times 1 + 2 \times 1 = 5 \end{aligned}$$

➤ **Example 4.** Let $f(x) = \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$, $x \neq \frac{\pi}{4}$.

Then, the value which should be assigned to $f(x)$ at $x = \frac{\pi}{4}$, so that it is continuous, is

(a) 1

(b) $\frac{1}{2}$

(c) 2

(d) None of these

Sol. (b) For $f(x)$ to be continuous at $x = \frac{\pi}{4}$, we must have

$$\begin{aligned} \lim_{x \rightarrow \pi/4} f(x) &= f\left(\frac{\pi}{4}\right) \\ \Rightarrow \lim_{x \rightarrow \pi/4} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x} &= f\left(\frac{\pi}{4}\right) \\ \Rightarrow \lim_{x \rightarrow \pi/4} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\tan\left(\frac{\pi}{2} - 2x\right)} &= f\left(\frac{\pi}{4}\right) \\ \Rightarrow \frac{1}{2} \lim_{x \rightarrow \pi/4} \left\{ \frac{\tan\left(\frac{\pi}{4} - x\right)}{\tan 2\left(\frac{\pi}{4} - x\right)} \right\} &= f\left(\frac{\pi}{4}\right) \\ \Rightarrow \frac{1}{2} &= f\left(\frac{\pi}{4}\right) \end{aligned}$$

➤ **Example 5.** If

$$f(x) = \begin{cases} \frac{(4^x - 1)^3}{\sin(x/4) \log(1 + x^2/3)}, & x \neq 0 \\ k, & x = 0 \end{cases}$$

is continuous at $x = 0$, then k is equal to

(a) $12 (\log_e 4)^2$

(b) $96 (\log_e 2)^3$

(c) $(\log_e 4)^3$

(d) None of these

Sol. (b) Since, $f(x)$ is continuous at $x = 0$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} f(x) &= f(0) \\ \Rightarrow \lim_{x \rightarrow 0} \frac{(4^x - 1)^3}{\sin \frac{x}{4} \log\left(1 + \frac{x^2}{3}\right)} &= k \\ \Rightarrow \lim_{x \rightarrow 0} \frac{12 \left(\frac{4^x - 1^3}{x}\right)^3}{\left(\frac{\sin \frac{x}{4}}{\frac{x}{4}}\right) \left(\frac{\log\left(1 + \frac{x^2}{3}\right)}{\frac{x^2}{3}}\right)} &= k \\ \Rightarrow \frac{12 (\log_e 4)^3}{12 (\log_e 4)^3} &= k \\ \Rightarrow k &= 96 (\log_e 2)^3 \end{aligned}$$

➤ **Example 6.** If $f(x)$ is continuous in $[0, 1]$ and

$$f\left(\frac{1}{2}\right) = 2, \text{ then } \lim_{n \rightarrow \infty} f\left(\frac{\sqrt{n}}{2\sqrt{n}+1}\right) \text{ is equal to}$$

(a) 0

(b) ∞

(c) 2

(d) None of these

Sol. (c) Since, $f(x)$ is continuous in $[0, 1]$.

$$\therefore \lim_{n \rightarrow \infty} f\left[\frac{\sqrt{n}}{2\sqrt{n}+1}\right] = f\left(\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{2\sqrt{n}+1}\right) = f\left(\frac{1}{2}\right) = 2$$

➤ **Example 7.** For the continuity of the function $f(x) = [x^2] - [x]^2$, $\forall x \in \mathbb{R}$ at the end points of the interval $[-1, 0]$, where $[\]$ denotes the greatest integer function. Which of the following is correct?

- (a) It is continuous at the end points of the interval $[-1, 0]$
 (b) It is not continuous at the end points of the interval $[-1, 0]$
 (c) $f(-1) = 1$
 (d) None of the above

Sol. (d) Continuity at $x = -1$,

$$f(-1) = [(-1)^2] - [-1]^2 = [1] - (-1)^2 = 1 - 1 = 0$$

$$\text{RHL} = \lim_{x \rightarrow -1^+} \{[x^2] - [x]^2\} = 0 - 1 = -1$$

So, $f(-1) \neq \text{RHL}$

Continuity at $x = 0$,

$$f(0) = [(0)^2] - [0]^2 = 0 - 0 = 0$$

$$\text{LHL} = \lim_{x \rightarrow 0^-} \{[x^2] - [x]^2\} = 0 - 1 = -1$$

So, $f(0) \neq \text{LHL}$

Hence, the function is not continuous at the end points of the interval $[-1, 0]$.

➤ **Example 8.** Let

$$f(x) = \begin{cases} \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{1/x}, & x \neq 0 \\ k, & x = 0 \end{cases} \text{ For what value}$$

of k is $f(x)$ continuous at $x = 0$?

- (a) 1 (b) e (c) $\frac{1}{e}$ (d) e^2

Sol. (d) $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{1/x}$

$$= \lim_{x \rightarrow 0} \left(\frac{1 + \tan x}{1 - \tan x} \right)^{1/x}$$

$$= \left[\lim_{x \rightarrow 0} \left\{ (1 + \tan x)^{\frac{1}{\tan x}} \right\} \right]^{\lim_{x \rightarrow 0} \frac{\tan x}{x}} \left[\lim_{x \rightarrow 0} \left\{ (1 - \tan x)^{\frac{-1}{\tan x}} \right\} \right]^{\lim_{x \rightarrow 0} \frac{\tan x}{x}}$$

$$= e \times e = e^2$$

Since, $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$$\Rightarrow e^2 = k$$

Hence, $f(x)$ is continuous at $x = 0$, when $k = e^2$.

➤ **Example 9.** $f(x) = x^n$, where n is a positive integer or any positive rational, is

- (a) continuous (b) discontinuous
 (c) Cannot be determined (d) None of these

Sol. (a) Let $x = c$ be any value of x .

$$\text{Then, } f(c + 0) = \lim_{n \rightarrow 0} (c + h)^n = c^n$$

$$\text{and } f(c - 0) = \lim_{n \rightarrow 0} (c - h)^n = c^n$$

Also,

$$f(c) = c^n$$

$\therefore$

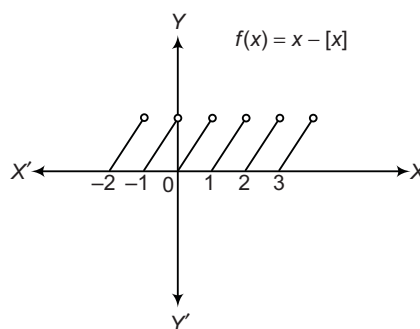
$$f(c + 0) = f(c - 0) = f(c)$$

Hence, $f(x) = x^n$ is continuous for $x = c$.

But c is any value of x . Therefore, $f(x) = x^n$ is continuous for all values of x .

➤ **Example 10.** The function $f(x) = x - [x]$, where $[\]$ denotes the greatest integer function, is
 (a) continuous everywhere
 (b) continuous at integer points only
 (c) continuous at non-integer points only
 (d) nowhere continuous

Sol. (c) It is evident from the graph of $f(x)$ that it is discontinuous at integer points and is continuous at non-integer points only.



➤ **Example 11.** Let

$$f(x) = \begin{cases} \tan x - \cot x, & x \neq \frac{\pi}{4} \\ x - \frac{\pi}{4}, & x = \frac{\pi}{4} \end{cases} \text{ Then, the}$$

value of a so that $f(x)$ is continuous at $x = \frac{\pi}{4}$, is

- (a) 2 (b) 4
 (c) 3 (d) 1

Sol. (b) If $f(x)$ is continuous at $x = \frac{\pi}{4}$, then

$$f\left(\frac{\pi}{4}\right) = \lim_{x \rightarrow \pi/4} f(x)$$

$\therefore$

$$a = \lim_{x \rightarrow \pi/4} \frac{\tan x - \cot x}{x - \frac{\pi}{4}}$$

$$= \lim_{x \rightarrow \pi/4} \frac{\sin^2 x - \cos^2 x}{\left(x - \frac{\pi}{4}\right) \sin x \cos x}$$

$$= - \lim_{x \rightarrow \pi/4} \frac{\cos 2x}{\left(x - \frac{\pi}{4}\right) \sin x \cos x}$$

$$= 2 \lim_{x \rightarrow \pi/4} \frac{\sin 2\left(\frac{\pi}{4} - x\right)}{2\left(\frac{\pi}{4} - x\right)} \times \frac{1}{\sin x \cos x}$$

$$= 2 \times 1 \times 2 = 4$$

➤ **Example 12.** If the function $f(x) = a[x+1] + b[x-1]$, where $[x]$ is the greatest integer function, then the condition for which $f(x)$ is continuous at $x=1$, is

- (a) $a+b=0$ (b) $a-b=0$
(c) $a=2b$ (d) None of these

Sol. (a) Since, $f(x)$ is continuous at $x=1$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^+} f(x) = f(1) \\ \Rightarrow \lim_{x \rightarrow 1^-} a[x+1] + b[x-1] &= \lim_{x \rightarrow 1^+} a[x+1] + b[x-1] \\ \Rightarrow a-b &= 2a+0 \cdot b \\ \Rightarrow a+b &= 0 \end{aligned}$$

➤ **Example 13.** If the function

$$f(x) = \begin{cases} \frac{x^2}{a}, & 0 \leq x < 1 \\ a, & 1 \leq x < \sqrt{2} \\ \frac{2b^2-4b}{x^2}, & \sqrt{2} \leq x < \infty \end{cases} \text{ is continuous for}$$

$0 \leq x < \infty$, then the most suitable values of a and b are

- (a) $a=1, b=-1$ (b) $a=-1, b=1+\sqrt{2}$
(c) $a=-1, b=1$ (d) None of these

Sol. (c) It is given that $f(x)$ is continuous for $0 \leq x < \infty$.

So, it is continuous at $x=1$ and $x=\sqrt{2}$.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^+} f(x) = f(1) \\ \text{and } \lim_{x \rightarrow (\sqrt{2})^-} f(x) &= \lim_{x \rightarrow (\sqrt{2})^+} f(x) = f(\sqrt{2}) \\ \Rightarrow \lim_{x \rightarrow 1^-} \frac{x^2}{a} &= \lim_{x \rightarrow 1^+} a = a \end{aligned}$$

and

$$\begin{aligned} \lim_{x \rightarrow (\sqrt{2})^-} a &= \lim_{x \rightarrow (\sqrt{2})^+} \frac{2b^2-4b}{x^2} = \frac{2b^2-4b}{2} \\ \Rightarrow \frac{1}{a} &= a \text{ and } a = \frac{2b^2-4b}{2} \end{aligned}$$

$$\Rightarrow a = \pm 1 \text{ and } b^2 - 2b = a$$

$$\text{Now, } a = 1 \text{ and } b^2 - 2b = a$$

$$\Rightarrow a = 1, b = 1 \pm \sqrt{2}$$

$$\text{and } a = -1, b^2 - 2b = a$$

$$\Rightarrow a = -1, b = 1$$

Discontinuity

A function f is said to be discontinuous at a point a in its domain, if there is a break in the graph of the function at $x=a$. Then, the point $x=a$ is called a point of discontinuity.

Since, for the function to be continuous, we must have

$$\begin{aligned} L &= R = V \text{ at } x=a \\ \text{i.e. } \lim_{x \rightarrow a^-} f(x) &= \lim_{x \rightarrow a^+} f(x) = f(a) \end{aligned}$$

But if anyone of the conditions is false, then the function will not be continuous at $x=a$. Hence, the discontinuity may arise due to any of the following situations:

(i) $L \neq R$

$$\text{i.e. } \lim_{x \rightarrow a^-} f(x) \neq \lim_{x \rightarrow a^+} f(x)$$

So, the limit of function at $x=a$ does not exist.

(ii) $L = R \neq V$

$$\text{i.e. } \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = f(a)$$

So, the limit of the function exists but is not equal to the value of the function at $x=a$.

Different Kinds of Discontinuity

Removable Discontinuity

A function $f(x)$ is said to have a removable discontinuity at a point α , if $f(\alpha+0)$, $f(\alpha-0)$ and $f(\alpha)$ exist but $f(\alpha+0) = f(\alpha-0) \neq f(\alpha)$. The function can be made continuous by defining it in such a way that $\lim_{x \rightarrow a} f(x) = f(a)$.

e.g. $f(x) = [\sin x]$, where $x \in (0, \pi)$ has a removable discontinuity at $x = \frac{\pi}{2}$.

$$\therefore \lim_{x \rightarrow \pi/2} [\sin x] = 0$$

$$\text{But } f\left(\frac{\pi}{2}\right) = 1$$

To remove this, we redefine $f(x)$ as follows:

$$f(x) = \begin{cases} [\sin x], & x \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \\ 0, & x = \frac{\pi}{2} \end{cases}$$

Now, $f(x)$ is continuous for $x \in (0, \pi)$.

Discontinuity of First Kind

A function $f(x)$ is said to have a discontinuity of the first kind or ordinary discontinuity at α , if $f(\alpha+0)$ and $f(\alpha-0)$ both exist but are not equal.

Discontinuity of Second Kind

A function $f(x)$ is said to have a discontinuity of the second kind at α , if none of the limits $f(\alpha+0)$ and $f(\alpha-0)$ exist.

Infinite Discontinuity

If either or both of $f(\alpha+0)$ and $f(\alpha-0)$ are infinite ($+\infty$ or $-\infty$), then $f(x)$ has infinite discontinuity at $x=\alpha$.

- If $f(x)$ is continuous and $g(x)$ is discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

e.g. $f(x) = x$ and $g(x) = \begin{cases} \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

- If $f(x)$ and $g(x)$ both are discontinuous at $x = a$, then the product function $\phi(x) = f(x) \cdot g(x)$ is not necessarily be discontinuous at $x = a$.

e.g. $f(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} -1, & x \geq 0 \\ 1, & x < 0 \end{cases}$

➤ **Example 14.** $f + g$ may be a continuous function, if

- f is continuous and g is discontinuous
- f is discontinuous and g is continuous
- f and g both are discontinuous
- None of the above

Sol. (c) Consider, $h(x) = f(x) + g(x)$

If $f(x)$ is continuous and $g(x)$ is discontinuous, then let us assume that $h(x)$ is continuous.

Now, $g(x) = \underbrace{h(x)}_{\text{Continuous function}} - \underbrace{f(x)}_{\text{Continuous function}}$

$\Rightarrow g(x)$ is a continuous function.

Which is contradictory to the given fact that $g(x)$ is a discontinuous function.

Hence, our assumption that $h(x)$ is continuous is wrong, i.e. if f is continuous and g is discontinuous, then $f + g$ cannot be a continuous function.

i.e. choice (a) is wrong.

Similarly, choice (b) is wrong.

From choice (c), we take

$f(x) = x - [x]$, i.e. discontinuous function

$g(x) = x + [x]$, i.e. discontinuous function

$(f + g)(x) = (x - [x]) + (x + [x]) = 2x$, i.e. continuous function

➤ **Example 15.** Continuity of the function $[\cos x]$

at $x = \frac{\pi}{2}$, where $[]$ denotes the greatest integer function, is

- continuous at $x = \frac{\pi}{2}$
- discontinuous at $x = \frac{\pi}{2}$
- Does not exist
- None of the above

Sol. (b) LHL = $\lim_{x \rightarrow \pi/2^-} [\cos x] = 0$

RHL = $\lim_{x \rightarrow \pi/2^+} [\cos x] = -1$

and $f\left(\frac{\pi}{2}\right) = \left[\cos \frac{\pi}{2}\right] = 0$

Clearly, LHL $\neq$ RHL

So, the function is discontinuous at $x = \frac{\pi}{2}$.

➤ **Example 16.** The points of discontinuity of the function $f(x) = \lim_{x \rightarrow \infty} (\sin x)^{2n}$, $n \in I$ are

- $n\pi$
- $n\pi \pm \frac{\pi}{2}$
- $n\pi \pm \frac{\pi}{4}$
- continuous everywhere

Sol. (b) We have, $f(x) = \lim_{x \rightarrow \infty} (\sin x)^{2n}$

$$= \begin{cases} 0, & \text{if } |\sin x| < 1, \text{ i.e. } -\frac{\pi}{2} + n\pi < x < \frac{\pi}{2} + n\pi \\ 1, & \text{if } |\sin x| = 1, \text{ i.e. } x = n\pi + \frac{\pi}{2}, n \in I \end{cases}$$

Here, the definition of $f(x)$ changes at $x = n\pi \pm \frac{\pi}{2}$, $n \in I$, which are clearly, points of discontinuities of $f(x)$.

➤ **Example 17.** The set of points of discontinuity of the function

$f(x) = \lim_{n \rightarrow \infty} \frac{(2 \sin x)^{2n}}{3^n - (2 \cos x)^{2n}}$ is given by

- R
- $\left\{n\pi \mp \frac{\pi}{3}, n \in I\right\}$
- $\left\{2n\pi \pm \frac{\pi}{6}, n \in I\right\}$
- None of the above

Sol. (c) We have,

$$f(x) = \lim_{n \rightarrow \infty} \frac{(2 \sin x)^{2n}}{3^n - (2 \cos x)^{2n}} = \lim_{n \rightarrow \infty} \frac{(2 \sin x)^{2n}}{(\sqrt{3})^{2n} - (2 \cos x)^{2n}}$$

$f(x)$ is discontinuous, when $(\sqrt{3})^{2n} - (2 \cos x)^{2n} = 0$

$$\Rightarrow \cos x = \frac{\sqrt{3}}{2}$$

$$\therefore x = 2n\pi \pm \frac{\pi}{6}, n \in I$$

Continuity of Composition of Function

If f is continuous at $x = c$ and g is continuous at $x = f(c)$, then the composite $g[f(x)]$ is continuous at

$x = c$. e.g. $f(x) = \frac{x \sin x}{x^2 + 2}$ and $g(x) = |x|$ are continuous

at $x = 0$, hence the composite $(g \circ f)(x) = \left| \frac{x \sin x}{x^2 + 2} \right|$ will

also be continuous at $x = 0$.

➤ **Example 18.** If $f(x) = \frac{1}{1-x}$, then the points

of discontinuity of the function $f^{3n}(x)$, where,
 $f^n = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}$, is/are

- (a) $x = 2$ (b) $x = \{0, 1\}$
 (c) $x = -1$ (d) continuous everywhere

Sol. (b) Clearly, $x = 1$ is a point of discontinuity of the function $f(x) = \frac{1}{1-x}$.

If $x \neq 1$, then

$$(f \circ f)(x) = f[f(x)] = f\left(\frac{1}{1-x}\right) = \frac{x-1}{x}$$

which is discontinuous at $x = 0$.

If $x \neq 0$ and $x \neq 1$, then

$$(f \circ f \circ f)(x) = f[(f \circ f)(x)] = f\left(\frac{x-1}{x}\right) = x$$

which is continuous everywhere.

So, the only points of discontinuity are $x = 0$ and $x = 1$.

➤ **Example 19.** If $f(x) = \frac{1}{x^2 - 17x + 66}$, then

$f\left(\frac{2}{x-2}\right)$ is discontinuous at x , is equal to

- (a) $2, \frac{7}{3}, \frac{25}{11}$ (b) $2, \frac{8}{3}, \frac{24}{11}$
 (c) $2, \frac{7}{3}, \frac{24}{11}$ (d) None of these

Sol. (c) Let $u = \frac{2}{x-2}$

$$\therefore f(u) = \frac{1}{(u-6)(u-11)}$$

$\therefore f(u)$ is undefined when u is undefined, i.e. at $x = 2$,

$$u = 6, u = 11$$

$\therefore f(u)$ is discontinuous at $x = 2, \frac{2}{x-2} = 6, \frac{2}{x-2} = 11$

$$\text{i.e. } x = 2, x = \frac{7}{3}, x = \frac{24}{11}$$

➤ **Example 20.** If $y = \frac{1}{t^2 - t - 6}$ and $t = \frac{1}{x-2}$,

then the values of x which make the function discontinuous, are

- (a) $2, \frac{2}{3}, \frac{7}{3}$ (b) $2, \frac{3}{2}, \frac{7}{3}$
 (c) $2, \frac{3}{2}, \frac{3}{7}$ (d) None of these

Sol. (b) Clearly, the function $t = \frac{1}{x-2}$ is discontinuous at

$x = 2$ and the function $y = \frac{1}{t^2 - t - 6}$ is discontinuous at

the points, where

$$t^2 - t - 6 = 0 \Rightarrow (t+2)(t-3) = 0 \Rightarrow t = -2, 3$$

$$\text{For } t = -2, \frac{1}{x-2} = -2 \Rightarrow x = \frac{3}{2}$$

$$\text{and for } t = 3, \frac{1}{x-2} = 3 \Rightarrow x = \frac{7}{3}$$

Therefore, the values of x which make the function y discontinuous, are $x = 2, 3/2$ and $7/3$.

Work Book Exercise 5.1

1 The set of points of discontinuity of the greatest integer function $[x]$ is

- a N b Z c R d ϕ

2 If $f(x) = \begin{cases} x^m \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ is continuous at

$x = 0$, then

- a $m \in (0, \infty)$ b $m \in (-\infty, 0)$
 c $m \in (1, \infty)$ d $m \in (-\infty, 1)$

3 Let $f(x) = \frac{\sin x}{x}$, $x \neq 0$. Then, $f(x)$ can be continuous at $x = 0$, if

- a $f(0) = 0$ b $f(0) = 1$
 c $f(0) = 2$ d $f(0) = -2$

4 If $f(x) = \begin{cases} ax^2 + b, & 0 \leq x < 1 \\ 4, & x = 1 \\ x + 3, & 1 < x \leq 2 \end{cases}$, then the value of (a, b)

for which $f(x)$ cannot be continuous at $x = 1$, is

- a $(2, 2)$ b $(3, 1)$ c $(4, 0)$ d $(5, 12)$

5 If $f(x) = \begin{cases} \frac{1 - \sqrt{2} \sin x}{\pi - 4x}, & x \neq \frac{\pi}{4} \\ a, & x = \frac{\pi}{4} \end{cases}$ is continuous at

$x = \frac{\pi}{4}$, then a is equal to

- a 4
 b 2
 c 1
 d $\frac{1}{4}$

6 If $f(x) = \begin{cases} \frac{x^2 - (a+2)x + a}{x-2}, & \text{for } x \neq 2 \\ 2, & \text{for } x = 2 \end{cases}$ is

continuous at $x = 2$, then a is equal to

- a 0
 b 1
 c -1
 d None of the above

- 7 If $f(x) = \frac{a \cos x - \cos bx}{x^2}$, $x \neq 0$ and $f(0) = 4$ is continuous at $x = 0$, then the ordered pair (a, b) is
- a $(\pm 1, 3)$ b $(1, \pm 3)$
c $(-1, -3)$ d $(1, 3)$

- 8 The set of points of discontinuity of the function $f(x) = |\sin x|$ is
- a $\{n\pi, n \in I\}$
b $\left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$
c ϕ
d None of the above

- 9 The set of points of discontinuity of the function $f(x) = \frac{|\sin x|}{\sin x}$ is
- a $\{0\}$ b $\{n\pi, n \in I\}$
c ϕ d None of these

- 10 If the function $f(x) = (\sin 2x)^{\tan^2 2x}$ is not defined at $x = \frac{\pi}{4}$. Then, the value of $f\left(\frac{\pi}{4}\right)$, so that f is continuous at $x = \frac{\pi}{4}$, is
- a $\sqrt{e}$ b $\frac{1}{\sqrt{e}}$
c 2 d None of these

Differentiability

Let $f(x)$ be a real-valued function defined on an open interval (a, b) , where $c \in (a, b)$. Then, $f(x)$ is said to be differentiable or derivable at $x = c$ iff $\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists finitely.

This limit is called the derivative or differential coefficient of the function $f(x)$ at $x = c$ and is denoted by $f'(c)$ or $Df(c)$ or $\frac{d}{dx}[f(x)]_{x=c}$.

Thus, $f(x)$ is differentiable at $x = c$.

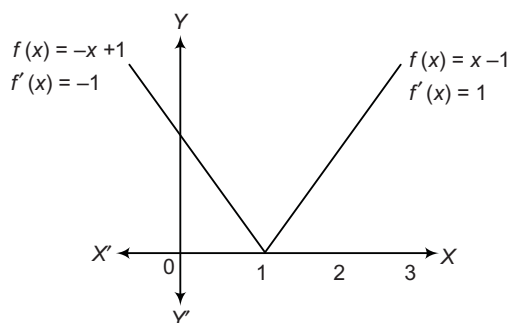
$$\Leftrightarrow \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ exists finitely.}$$

Differentiability of a Function at a Point

The function $f(x)$ is differentiable at a point P iff there exists a unique tangent at point P .

In other words, $f(x)$ is differentiable at a point P iff the curve does not have P as a corner point, i.e. the function is not differentiable at those points on which function has holes or sharp edges.

Let us consider the function $f(x) = |x - 1|$.



It is not differentiable at $x = 1$, since $f(x)$ has sharp edges at $x = 1$.

Progressive Derivative/Right Hand Derivative

Let $f(x)$ be a function and a be a point in the domain of f . Then, right hand derivative is denoted by $Rf'(a)$ and defined as

$$\begin{aligned} \text{RHD} &= Rf'(a) \\ &= \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h} \end{aligned}$$

Regressive Derivative/Left Hand Derivative

Let $f(x)$ be a function and a be a point in the domain of f . Then, left hand derivative is denoted by $Lf'(a)$ and defined as

$$\begin{aligned} \text{LHD} &= Lf'(a) \\ &= \lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = \lim_{h \rightarrow 0} \frac{f(a - h) - f(a)}{-h} \end{aligned}$$

Mathematically, a function $f(x)$ is said to be differentiable at a point a in its domain, if $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ exists finitely or if and only if

$$\lim_{x \rightarrow a^-} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}.$$

i.e. left hand derivative = right hand derivative

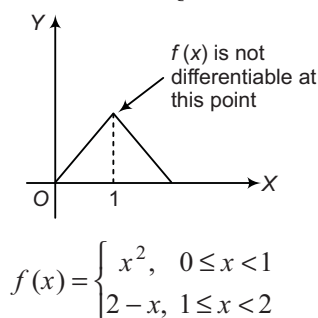
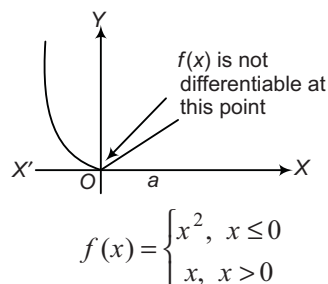
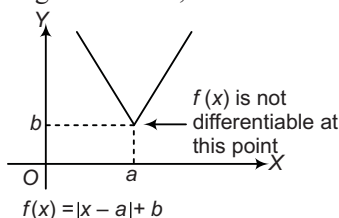
$$\text{or } Lf'(a) = Rf'(a)$$

If $Lf'(a) \neq Rf'(a)$, then function $f(x)$ is not differentiable at $x = a$.

Geometrical Interpretation of Differentiability

Geometrically, we interpret $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$ as the slope of the tangent to the graph of $y = f(x)$ at the point $[x, f(x)]$. Thus, if there is no tangent line at a certain point, the function is not differentiable at that point.

In other words, a function is not differentiable at a corner point of a curve, i.e. a point, where the curve suddenly changes direction, see the following graphs:



➤ If a function is discontinuous at the point $x = a$ then there is necessarily a break at that point in the graph of function. So, every differentiable function is continuous but the converse is not true. That is a continuous function need not be differentiable.

Differentiability in a Set

- (i) A function $f(x)$ defined on an open interval (a, b) is said to be differentiable or derivable in open interval (a, b) , if it is differentiable at each point of (a, b) .
- (ii) A function $f(x)$ defined on $[a, b]$ is said to be differentiable or derivable at the end points a and b , if it is differentiable from the right at a and from the left at b . In other words,

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} \text{ and } \lim_{x \rightarrow b^-} \frac{f(x) - f(b)}{x - b} \text{ both exist.}$$

‘If f is derivable in an open interval (a, b) and also at the end points a and b , then f is said to be derivable in the closed interval $[a, b]$.’

‘A function f is said to be a differentiable function, if it is differentiable at every point of its domain.’

Relation between Derivability and Continuity

- (i) If $f'(a)$ exists, then $f(x)$ is derivable at $x = a \Rightarrow f(x)$ is continuous at $x = a$. In general, a function f is derivable at x , then f is continuous at x , i.e. if $f(x)$ is derivable for every point of its domain of definition, then it is continuous in that domain. The converse of the above result need not to be true.
e.g. The functions $f(x) = |x|$ and $g(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ x, & x = 0 \end{cases}$ both are continuous at $x = 0$ but not derivable at $x = 0$.
- (ii) Let $f_+'(a) = \lambda$, $f_-'(a) = \mu$, where λ, μ are finite, then
 - (a) $\lambda = \mu \Rightarrow f$ is derivable at $x = a \Rightarrow f$ is continuous at $x = a$.
 - (b) $\lambda \neq \mu \Rightarrow f$ is not derivable at $x = a$ but f is continuous at $x = a$. If a function f is not differentiable but is continuous at $x = a$, it geometrically implies a sharp corner or kink at $x = a$.
- (c) If f is not continuous at $x = a$, then it is not differentiable at $x = a$.

Properties of Differentiable Functions

- i. Every polynomial function is differentiable at each $x \in \mathbb{R}$.
- ii. The exponential function a^x , $a > 0$, is differentiable at each $x \in \mathbb{R}$.
- iii. Every constant function is differentiable at each $x \in \mathbb{R}$.
- iv. The logarithmic function is differentiable at each point in its domain.
- v. Trigonometric and inverse trigonometric functions are differentiable in their respective domains.
- vi. The sum, difference, product and quotient of two differentiable functions is differentiable.
- vii. The composition of differentiable function is a differentiable function.
- viii. Absolute functions are always continuous throughout but not differentiable at their critical point.
e.g. $|x - a|$ is continuous throughout but not differentiable at $x = a$.

➤ **Example 21.** If a function $f(x)$ is defined as

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1, \text{ then} \\ x^2 - x + 1, & x > 1 \end{cases}$$

- (a) $f(x)$ is differentiable at $x=0$ and $x=1$
 (b) $f(x)$ is differentiable at $x=0$ but not at $x=1$
 (c) $f(x)$ is differentiable at $x=1$ but not at $x=0$
 (d) $f(x)$ is not differentiable at $x=0$ and $x=1$

Sol. (d) At $x=0$,

$$Rf'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^2}{h} = 0$$

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{-(-h) - 0}{-h} = -1$$

Since, $Rf'(0) \neq Lf'(0)$

$\therefore f'(0)$ does not exist.

At $x=1$,

$$Rf'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[(1+h)^2 - (1+h) + 1] - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + h}{h} = \lim_{h \rightarrow 0} (h+1) = 1$$

$$Lf'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{(1-h) - 1} = \lim_{h \rightarrow 0} \frac{(1-h)^2 - 1}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 - 2h}{-h} = \lim_{h \rightarrow 0} (2-h) = 2$$

Since, $Rf'(1) \neq Lf'(1)$

So, $f'(1)$ does not exist.

Hence, $f(x)$ is not differentiable at $x=0$ and $x=1$.

➤ **Example 22.** Let $f(x) = a + b|x| + c|x|^4$, where a, b and c are real constants. Then, $f(x)$ is differentiable at $x=0$, if

- (a) $a=0$ (b) $b=0$
 (c) $c=0$ (d) None of these

Sol. (b) Since, $f(x)$ is differentiable at $x=0$.

$$\therefore Lf'(0) = Rf'(0)$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{(a + b|-h| + c|-h|^4) - a}{-h} = \lim_{h \rightarrow 0} \frac{(a + b|h| + c|h|^4) - a}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{bh + ch^4}{-h} = \lim_{h \rightarrow 0} \frac{bh + ch^4}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} (-b - ch^3) = \lim_{h \rightarrow 0} (b + ch^3)$$

$$\Rightarrow -b = b \Rightarrow b = 0$$

➤ **Example 23.** The set of points, where the function $f(x) = |x-2|\cos x$ is differentiable, is

- (a) $(-\infty, \infty)$
 (b) $(-\infty, \infty) - \{2\}$
 (c) $(0, \infty)$
 (d) None of the above

Sol. (b) Since, $|x-2|$ is not differentiable at $x=2$, therefore $f(x) = |x-2|\cos x$ is not differentiable at $x=2$. Hence, $f(x)$ is differentiable in $(-\infty, \infty) - \{2\}$.

➤ **Example 24.** If $f(x) = \begin{cases} \frac{x}{1+|x|}, & |x| \geq 1 \\ \frac{x}{1-|x|}, & |x| < 1 \end{cases}$, then

$f(x)$ is

- (a) discontinuous and non-differentiable at $x=-1, 1$ and 0
 (b) discontinuous and non-differentiable at $x=-1$, whereas continuous and differentiable at $x=0$ and $x=1$
 (c) discontinuous and non-differentiable at $x=-1$ and $x=1$, whereas continuous and differentiable at $x=0$
 (d) None of the above

Sol. (c) We have, $f(x) = \begin{cases} \frac{x}{1-x}, & x \leq -1 \\ \frac{x}{1+x}, & -1 < x < 0 \\ \frac{x}{1-x}, & 0 \leq x < 1 \\ \frac{x}{1+x}, & x \geq 1 \end{cases}$

It can be easily seen that $f(x)$ is discontinuous and non-differentiable at $x=-1$ and $x=1$, whereas it is continuous and differentiable at $x=0$.

➤ **Example 25.** Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function and $f(1) = 4$. Then, the

value of $\lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} dt$ is

- (a) $8f'(1)$ (b) $4f'(1)$
 (c) $2f'(1)$ (d) $f'(1)$

Sol. (a) $\lim_{x \rightarrow 1} \int_4^{f(x)} \frac{2t}{x-1} dt = \lim_{x \rightarrow 1} \left[\frac{t^2}{x-1} \right]_4^{f(x)}$

$$= \lim_{x \rightarrow 1} \frac{[f(x)]^2 - 16}{x-1} = \lim_{x \rightarrow 1} \frac{[f(x) - 4][f(x) + 4]}{x-1}$$

$$= \lim_{x \rightarrow 1} \frac{f(x) - f(1)}{x-1} \cdot \lim_{x \rightarrow 1} [f(x) + 4]$$

$$= f'(1) [f(1) + 4] = f'(1) (4 + 4)$$

$$= 8f'(1) \quad [\because f(1) = 4]$$

➤ **Example 26.** The left hand derivative of $f(x) = [x]\sin(\pi x)$ at $x=k$, an integer and $[x] = \text{greatest integer } \leq x$, is

- (a) $(-1)^k (k-1)\pi$
 (b) $(-1)^{k-1} \cdot (k-1)\pi$
 (c) $(-1)^k \cdot k\pi$
 (d) $(-1)^{k-1} \cdot k\pi$

$$\begin{aligned} \text{Sol. (a) } f'(k-0) &= \lim_{h \rightarrow 0} \frac{[k-h] \sin \pi(k-h) - [k] \sin \pi k}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(-1)^{k-1}(k-1) \sin \pi h - k \times 0}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(-1)^{k-1} \cdot (k-1) \sin \pi h}{-h} \\ &= (-1)^k \cdot (k-1) \pi \end{aligned}$$

➤ **Example 27.** If $f(x) = \frac{x}{1+x} + \frac{x}{(x+1)(2x+1)}$
 $+ \frac{x}{(2x+1)(3x+1)} + \dots$, then at $x=0$,

- (a) $f(x)$ has no limit
 (b) $f(x)$ is discontinuous
 (c) $f(x)$ is continuous but not differentiable
 (d) $f(x)$ is differentiable

Sol. (b) We have, for $x \neq 0$,

$$\begin{aligned} f(x) &= \frac{x}{1+x} + \frac{x}{(x+1)(2x+1)} + \frac{x}{(2x+1)(3x+1)} + \dots \\ &= \sum_{n=1}^{\infty} \frac{x}{[(n-1)x+1][nx+1]} \end{aligned}$$

$$= \sum_{n=1}^{\infty} \left[\frac{1}{(n-1)x+1} - \frac{1}{nx+1} \right] = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{nx+1} \right] = 1$$

For $x=0$, $f(x)=0$. Thus, $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Clearly, $\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 1 \neq f(0)$

So, $f(x)$ is discontinuous at $x=0$.

➤ **Example 28.** If $f(x) = \sqrt{1-\sqrt{1-x^2}}$, then at $x=0$,

- (a) $f(x)$ is differentiable as well as continuous
 (b) $f(x)$ is differentiable but not continuous
 (c) $f(x)$ is continuous but not differentiable
 (d) $f(x)$ is neither continuous nor differentiable

$$\begin{aligned} \text{Sol. (c) } Lf'(0) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{\sqrt{1-\sqrt{1-h^2}}}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{1-(1-h^2)}}{-h} \times \frac{1}{\sqrt{1+\sqrt{1-h^2}}} \\ &= \lim_{h \rightarrow 0} \frac{-1}{\sqrt{1+\sqrt{1-h^2}}} = \frac{-1}{\sqrt{2}} \\ Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\sqrt{1-\sqrt{1-h^2}}}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{\sqrt{1+\sqrt{1-h^2}}} = \frac{1}{\sqrt{2}} \end{aligned}$$

Therefore, $f(x)$ is not differentiable at $x=0$.

Since, $Lf'(0)$ and $Rf'(0)$ are finite, therefore $f(x)$ is continuous at $x=0$.

Hence, $f(x)$ is continuous but not differentiable at $x=0$.

➤ **Example 29.** If $f(x) = \sqrt{1-e^{-x^2}}$, then at $x=0$, $f(x)$ is

- (a) differentiable as well as continuous
 (b) continuous but not differentiable
 (c) differentiable but not continuous
 (d) neither differentiable nor continuous

$$\begin{aligned} \text{Sol. (b) } Lf'(0) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{1-e^{-h^2}}}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\left[1 - \left(1 - h^2 + \frac{h^4}{2!} - \dots \right) \right]^{1/2}}{-h} \\ &= \lim_{h \rightarrow 0} \frac{h \left[1 - \frac{h^2}{2!} + \dots \right]^{1/2}}{-h} = -1 \\ Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{1-e^{-h^2}}}{h} = 1 \end{aligned}$$

So, $f(x)$ is not differentiable at $x=0$.

Since, $Lf'(0)$ and $Rf'(0)$ are finite, therefore $f(x)$ is continuous at $x=0$.

Hence, $f(x)$ is continuous but not differentiable at $x=0$.

➤ **Example 30.** If $f(x) = \begin{cases} x[x], & 0 \leq x < 2 \\ (x-1)[x], & 2 \leq x \leq 3 \end{cases}$

where $[]$ denotes the greatest integer function, then

- (a) both $f'(1)$ and $f'(2)$ do not exist
 (b) $f'(1)$ exists but $f'(2)$ does not exist
 (c) $f'(2)$ exists but $f'(1)$ does not exist
 (d) both $f'(1)$ and $f'(2)$ exist

$$\begin{aligned} \text{Sol. (a) } \text{We have, } Lf'(1) &= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(1-h)[1-h] - 1[1]}{-h} \\ &= \lim_{h \rightarrow 0} \frac{0-1}{-h} = \infty \end{aligned}$$

$\therefore f'(1)$ does not exist.

$$\begin{aligned} \text{Also, } Lf'(2) &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)[2-h] - (2-1)[2]}{-h} \\ &= \lim_{h \rightarrow 0} \frac{2-h-2}{-h} = 1 \end{aligned}$$

$$\begin{aligned} \text{and } Rf'(2) &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)[2+h] - (2-1)[2]}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(1+h)-2}{h} = \lim_{h \rightarrow 0} \frac{2h}{h} = 2 \end{aligned}$$

$\therefore f'(2)$ does not exist.

➤ **Example 31.** If $f(x) = \begin{cases} 3, & x < 0 \\ 2x + 1, & x \geq 0 \end{cases}$, then

- (a) both $f(x)$ and $f(|x|)$ are differentiable at $x = 0$
 (b) $f(x)$ is differentiable but $f(|x|)$ is not differentiable at $x = 0$
 (c) $f(|x|)$ is differentiable but $f(x)$ is not differentiable at $x = 0$
 (d) both $f(x)$ and $f(|x|)$ are not differentiable at $x = 0$

Sol. (c) We have,

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{3-1}{-h} = \infty$$

Hence, $f(x)$ is not differentiable at $x = 0$.

Also, if $x < 0$ or $x \geq 0$, then $|x| \geq 0$

$$\therefore f(|x|) = 2|x| + 1, \forall x$$

$$\therefore Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{2h+1-1}{h} = 2$$

$$\text{and } Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{2|-h|+1-1}{-h} = 2$$

Hence, $f(|x|)$ is differentiable at $x = 0$.

➤ **Example 32.** If $f(x) = \begin{cases} 4, & -3 < x < -1 \\ 5+x, & -1 \leq x < 0 \\ 5-x, & 0 \leq x < 2 \\ x^2+x-3, & 2 \leq x < 3 \end{cases}$,

then $f(|x|)$ is

- (a) differentiable but not continuous in $(-3, 3)$
 (b) continuous but not differentiable in $(-3, 3)$
 (c) continuous as well as differentiable in $(-3, 3)$
 (d) neither continuous nor differentiable in $(-3, 3)$

Sol. (b) If $-3 < x < -1$, then $1 < |x| < 3$

and if $-1 \leq x < 0$, then $0 < |x| \leq 1$.

$$\therefore f(|x|) = \begin{cases} 5-|x|, & 0 < |x| \leq 1 \\ 5-|x|, & 1 < |x| < 2 \\ |x|^2 + |x| - 3, & 2 \leq |x| < 3 \end{cases}$$

The nature of $f(|x|)$ changes only at $x = 2$.

$$Lf'(2) = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} = \lim_{h \rightarrow 0} \frac{5-(2-h) - (4+2-3)}{-h} = -1$$

$$Rf'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{[(2+h)^2 + (2+h) - 3] - (4+2-3)}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 5h}{h} = 5$$

So, $f(|x|)$ is not differentiable at $x = 2$ but $f(|x|)$ is continuous at $x = 2$, as $Lf'(2)$ and $Rf'(2)$ are finite.

Hence, $f(|x|)$ is continuous but not differentiable in $(-3, 3)$.

➤ **Example 33.** The set of all points, where

$f(x) = \sqrt[3]{x^2|x|} - |x| - 1$ is not differentiable, is

- (a) $\{0\}$ (b) $\{-1, 0\}$
 (c) $\{-1, 0, 1\}$ (d) None of these

Sol. (d) We have, $f(x) = \sqrt[3]{x^2|x|} - |x| - 1$
 $= |x| - |x| - 1 = -1$

Hence, it is differentiable for all x .

➤ **Example 34.** If

$f(x) = 2x + |x - x^2|$, $-1 \leq x \leq 1$, then $f(x)$ is

- (a) continuous but not differentiable in $[-1, 1]$
 (b) continuous as well as differentiable in $[-1, 1]$
 (c) differentiable but not continuous in $[-1, 1]$
 (d) neither differentiable nor continuous in $[-1, 1]$

Sol. (a) We have, $f(x) = 2x + |x - x^2|$, $-1 \leq x \leq 1$

$$= 2x + |x|(1-x), -1 \leq x \leq 1$$

$$= \begin{cases} 2x + (-x)(1-x), & -1 \leq x < 0 \\ 2x + x(1-x), & 0 \leq x \leq 1 \end{cases}$$

$$= \begin{cases} x + x^2, & -1 \leq x < 0 \\ 3x - x^2, & 0 \leq x \leq 1 \end{cases}$$

Since, the polynomial functions are continuous as well as differentiable everywhere, so the only doubtful point is the point $x = 0$.

$$\text{Now, } Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} = \lim_{h \rightarrow 0} \frac{(-h + h^2) - 0}{-h} = \lim_{h \rightarrow 0} (1-h) = 1$$

$$\text{and } Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{(3h - h^2) - 0}{h} = \lim_{h \rightarrow 0} (3-h) = 3$$

Since, $Lf'(0) \neq Rf'(0)$, therefore $f(x)$ is not differentiable at $x = 0$ but $f(x)$ is continuous at $x = 0$, as $Lf'(0)$ and $Rf'(0)$ are finite.

Hence, $f(x)$ is continuous but not differentiable in the interval $[-1, 1]$.

➤ **Example 35.** Let $[x]$ denotes the greatest integer less than or equal to x . If

$f(x) = [x \sin \pi x]$, then $f(x)$ is

- (a) not continuous at $x = 0$
 (b) not continuous in $(-1, 1)$
 (c) not differentiable at $x = 1$
 (d) differentiable in $(-1, 1)$

Sol. (c, d) We have, $\lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} [-h \sin(-\pi h)]$
 $= \lim_{h \rightarrow 0} [h \sin \pi h] = 0$

$$\lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} [h \sin \pi h] = 0$$

$$\text{and } f(0) = 0$$

So, $f(x)$ is continuous at $x = 0$.

It can be easily seen that $f(x)$ is continuous in $(-1, 0)$ and $f(x)$ is not differentiable at $x = 1$ but it is differentiable in $(-1, 1)$.

➤ **Example 36.** The function $f(x) = \frac{x}{1+|x|}$ is

- (a) continuous for all x but not differentiable at $x = 0$
 (b) continuous as well as differentiable for all x
 (c) neither continuous nor differentiable at $x = 0$
 (d) differentiable for all x but not continuous at $x = 0$

Sol. (b) We have, $f(x) = \begin{cases} \frac{x}{1+x}, & x > 0 \\ \frac{x}{1-x}, & x < 0 \\ 0, & x = 0 \end{cases}$

If $x > 0$ or $x < 0$, then $f(x)$ is a rational function of x having non-zero denominator. Hence, $f(x)$ is continuous and differentiable for $x > 0$ or $x < 0$. So, the only doubtful point is $x = 0$.

$$\begin{aligned} \text{At } x = 0, \quad Rf'(0) &= \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\frac{h}{1+h} - 0}{h} = 1 \\ Lf'(0) &= \lim_{h \rightarrow 0^-} \frac{f(-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0^-} \frac{\frac{-h}{1-(-h)} - 0}{-h} = 1 \end{aligned}$$

Since, $Rf'(0) = Lf'(0) = 1$

$\therefore f'(0) = 1$

$\Rightarrow f(x)$ is differentiable at $x = 0$. Since, it is differentiable so it is a continuity.

$\therefore f(x)$ is also continuous at $x = 0$.

Hence, $f(x)$ is continuous as well as differentiable for all real x .

➤ **Example 37.** The function

$f(x) = x - |x - x^2|$, $-1 \leq x \leq 1$ is continuous on the interval

- (a) $[-1, 1]$ (b) $(-1, 1)$
(c) $[-1, 1] - \{0\}$ (d) $(-1, 1) - \{0\}$

Sol. (a) We have, $f(x) = x - |x - x^2| = x - |x(1 - x)|$
 $= x - |x| |1 - x|$

$\therefore$ Continuity is to be checked at $x = 0$ and $x = 1$.

At $x = 0$,

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0^-} f(0 - h) = \lim_{h \rightarrow 0^-} -h - |-h| |1 + h| \\ &= \lim_{h \rightarrow 0^-} -h - h(1 + h) = 0 \\ \text{RHL} &= \lim_{h \rightarrow 0^+} f(0 + h) = \lim_{h \rightarrow 0^+} h - |h| |1 - h| \\ &= \lim_{h \rightarrow 0^+} h - h(1 - h) = 0 \end{aligned}$$

and $f(0) = 0$

Since, $\text{LHL} = \text{RHL} = f(0)$

$\therefore f(x)$ is continuous at $x = 0$.

At $x = 1$,

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0^-} f(1 - h) = \lim_{h \rightarrow 0^-} (1 - h) - |1 - h| |1 - (1 - h)| \\ &= \lim_{h \rightarrow 0^-} (1 - h) - h(1 - h) = 1 \end{aligned}$$

$$\begin{aligned} \text{RHL} &= \lim_{h \rightarrow 0^+} f(1 + h) \\ &= \lim_{h \rightarrow 0^+} (1 + h) - |1 + h| |1 + h| |1 - (1 + h)| = 1 \end{aligned}$$

and $f(1) = 1 - |1| |1 - 1| = 1$

$\therefore f(x)$ is continuous at $x = 1$.

Hence, $f(x)$ is continuous for all $x \in [-1, 1]$.

➤ **Example 38.** If f is an even function such that $\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$ has some finite non-zero value, then

- (a) f is continuous and derivable at $x = 0$
(b) f is continuous but not derivable at $x = 0$
(c) f may be discontinuous at $x = 0$
(d) None of the above

Sol. (b) Let $f'(0^+) = \lim_{h \rightarrow 0^+} \frac{f(h) - f(0)}{h} = k$ [say]

$$\begin{aligned} \therefore f'(0^-) &= \lim_{h \rightarrow 0^-} \frac{f(0) - f(0 - h)}{h} \\ &= \lim_{h \rightarrow 0^-} \frac{f(0) - f(h)}{h} = -k \end{aligned}$$

$\therefore f'(0^+) \neq f'(0^-)$, but both are finite.

So, $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$.

➤ **Example 39.** Let $f(x + y) = f(x) \cdot f(y)$ for all x, y , where $f(0) \neq 0$. If $f(5) = 2$ and $f'(0) = 3$, then $f'(5)$ is equal to

- (a) 6
(b) 0
(c) 1
(d) None of the above

$$\begin{aligned} \text{Sol. (a)} \quad f'(5) &= \lim_{h \rightarrow 0} \frac{f(5 + h) - f(5)}{h} = \lim_{h \rightarrow 0} \frac{f(5 + h) - f(5 + 0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(5) \cdot f(h) - f(5) \cdot f(0)}{h} \\ &= \left(\lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \right) \cdot f(5) \quad [\because f(x + y) = f(x) \cdot f(y) \text{ for all } x, y] \\ &= f'(0) \cdot f(5) = 3 \cdot 2 = 6 \end{aligned}$$

➤ **Example 40.** If $f(x) = \log_{10} x$, then at $x = 1$,

- (a) $f(x)$ is continuous as well as differentiable
(b) $f(x)$ is continuous but not differentiable
(c) $f(x)$ is differentiable but not continuous
(d) $f(x)$ is neither continuous nor differentiable

Sol. (a) We have,

$$\begin{aligned} Lf'(1) &= \lim_{h \rightarrow 0^-} \frac{f(1 - h) - f(1)}{-h} = \lim_{h \rightarrow 0^-} \frac{\log_{10}(1 - h) - 0}{-h} \\ &= \lim_{h \rightarrow 0^-} \frac{\log(1 - h)}{-h} \times \frac{1}{\log_e 10} \\ &= 1 \times \frac{1}{\log_e 10} = \log_{10} e \end{aligned}$$

$$\begin{aligned} Rf'(1) &= \lim_{h \rightarrow 0^+} \frac{f(1 + h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\log_{10}(1 + h) - 0}{h} \\ &= \lim_{h \rightarrow 0^+} \frac{\log(1 + h)}{h} \times \frac{1}{\log_e 10} = 1 \times \frac{1}{\log_e 10} = \log_{10} e \end{aligned}$$

Since, $Lf'(1) = Rf'(1)$, therefore $f(x)$ is differentiable and hence continuous at $x = 1$.

➤ **Example 41.** Let $f(x) = \cos x$ and $g(x) = [x + 2]$, where $[]$ denotes the greatest integer function. Then, the value of $(g \circ f)' \left(\frac{\pi}{2} \right)$ is

- (a) 1 (b) 0
(c) -1 (d) Does not exist

Sol. (d) $L(g \circ f)' \left(\frac{\pi}{2} \right) = \lim_{h \rightarrow 0} \frac{(g \circ f) \left(\frac{\pi}{2} - h \right) - (g \circ f) \left(\frac{\pi}{2} \right)}{-h}$

$$= \lim_{h \rightarrow 0} \frac{\left[\cos \left(\frac{\pi}{2} - h \right) + 2 \right] - \left[\cos \frac{\pi}{2} + 2 \right]}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{[\sin h + 2] - [2]}{-h} = \lim_{h \rightarrow 0} \frac{2 - 2}{h} = 0$$

$R(f \circ g)' \left(\frac{\pi}{2} \right) = \lim_{h \rightarrow 0} \frac{(f \circ g) \left(\frac{\pi}{2} + h \right) - (f \circ g) \left(\frac{\pi}{2} \right)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\left[\cos \left(\frac{\pi}{2} + h \right) + 2 \right] - \left[\cos \frac{\pi}{2} + 2 \right]}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1 - 2}{h} = \infty$$

Hence, $(g \circ f)$ is not differentiable at $x = \frac{\pi}{2}$.

➤ **Example 42.** If $f(x) = \sum_{k=0}^n a_k |x - 1|^k$, where

$a_i \in \mathbb{R}$, then

- (a) $f(x)$ is continuous at $x = 1$, $\forall a_k \in \mathbb{R}$
(b) $f(x)$ is differentiable at $x = 1$, $\forall a_k \in \mathbb{R}$
(c) $f(x)$ is differentiable at $x = 1$, provided $a_{2k+1} = 0$
(d) $f(x)$ is continuous at $x = 1$, provided $a_{2k} = 0$

Sol. (a, c) Since, $|x - 1|$, $|x - 1|^2$, etc., are continuous at $x = 1$.

$\therefore f(x)$ is continuous at $x = 1$, $\forall a_k \in \mathbb{R}$.

Also, $|x - 1|^2$, $|x - 1|^4$, etc., are all differentiable at $x = 1$ whereas $|x - 1|$, $|x - 1|^3$, etc., are not differentiable at $x = 1$.

Therefore, $f(x)$ is differentiable at $x = 1$, $\forall a_{2k+1} = 0$.

➤ **Example 43.** The values of constants a and b , so as to make the function

$$f(x) = \begin{cases} \frac{1}{|x|}, & |x| \geq 1 \\ ax^2 + b, & |x| < 1 \end{cases}, \text{ continuous as well as}$$

differentiable for all x , are

(a) $a = -\frac{1}{2}$, $b = \frac{3}{2}$

(b) $a = \frac{1}{2}$, $b = \frac{3}{2}$

(c) $a = -\frac{1}{2}$, $b = -\frac{3}{2}$

(d) None of the above

Sol. (a) Since, $f(x)$ is continuous for all x , therefore it is continuous at $x = 1$ also.

$$\therefore f(1) = \lim_{h \rightarrow 0} f(1 - h)$$

$$\Rightarrow 1 = \lim_{h \rightarrow 0} [a(1 - h)^2 + b]$$

$$\Rightarrow a + b = 1$$

Also, $f(x)$ is differentiable at $x = 1$.

$$\lim_{h \rightarrow 0} \frac{f(1 - h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{f(1 + h) - f(1)}{h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{a(1 - h)^2 + b - 1}{-h} = \lim_{h \rightarrow 0} \frac{1}{h} - 1$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{(a + b - 1) + (h^2 - 2h)a}{-h} = \lim_{h \rightarrow 0} \frac{1 - 1 - h}{h(1 + h)} \quad [\because a + b = 1]$$

$$\Rightarrow 2a = -1$$

$$\therefore a = -\frac{1}{2}$$

Hence, $a + b = 1 \Rightarrow b = 1 - a = \frac{3}{2}$

➤ **Example 44.** Let $f(x)$ be a continuous function defined for $1 \leq x \leq 3$. If $f(x)$ takes rational values for all x and $f(2) = 10$, then $f(1.5)$ is equal to

- (a) 0
(b) 10
(c) not defined
(d) any constant

Sol. (b) A continuous function $f(x)$ defined on $1 \leq x \leq 3$, having only rational values must be a constant function.

$$\therefore f(2) = 10$$

$$\Rightarrow f(x) = 10, \text{ for all } x$$

$$\therefore f(1.5) = 10$$

➤ **Example 45.** Let $f(x)$ and $g(x)$ be defined

$$\text{by } f(x) = [x] \text{ and } g(x) = \begin{cases} 0, & x \in \text{integer} \\ x^2, & \text{otherwise} \end{cases}, \text{ where}$$

$[x]$ denotes the greatest integer less than or equal to x . Then,

- (a) $g \circ f$ is continuous for all $x \in \mathbb{R}$
(b) $g \circ f$ is differentiable for all $x \in \mathbb{R}$
(c) $f \circ g$ is continuous for all $x \in \mathbb{R}$
(d) $f \circ g$ is differentiable at all x except, when $x = \pm \sqrt{n}$, $n \in \mathbb{N}$

Sol. (a, b, c, d) We have, $f(x) = \begin{cases} x, & \text{if } x \text{ is an integer} \\ n, & \text{if } x \text{ is not an integer} \\ \text{i.e. } n < x < n + 1 \end{cases}$

$$\text{and } g(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ x^2, & \text{otherwise} \end{cases}$$

$$\text{Now, } (g \circ f)(x) = g[f(x)] = g(\text{integer})$$

$$[\because f(x) \text{ is an integer, } \forall x \in \mathbb{R}]$$

$$= 0$$

$\therefore (g \circ f)(x)$ is a constant function, so it is continuous and differentiable at all $x \in R$.

$$\text{Also, } (f \circ g)(x) = f[g(x)] = \begin{cases} 0, & \text{if } x \in I \\ n, & \text{if } \sqrt{n} < x < \sqrt{n+1} \\ & \text{or } -\sqrt{n+1} < x < -\sqrt{n}; n = 0, 1, 2, \dots \end{cases}$$

Thus, $(f \circ g)(x)$ is continuous and differentiable at all x except, when $x = \pm \sqrt{n}; n = 1, 2, 3, \dots$

➤ **Example 46.** Let $f: R \rightarrow R$ be any function.

Define $g: R \rightarrow R$ by $g(x) = |f(x)|$ for all x . Then, g is

- (a) onto, if f is onto
- (b) one-one, if f is one-one
- (c) continuous, if f is continuous
- (d) differentiable, if f is differentiable

Sol. (c) Let $h(x) = |x|$, $\forall x$, clearly $h(x)$ is continuous for all x .

Then, $g(x) = |f(x)| = h(f(x)) = (h \circ f)(x)$ for all x .

Since, composition of two continuous functions is continuous, therefore g is continuous, if f is continuous.

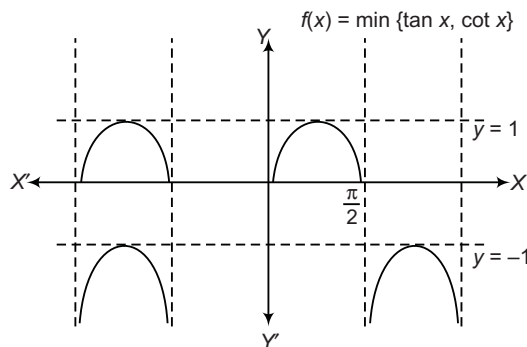
➤ **Example 47.** Let

$f(x) = \min\{\tan x, \cot x\}$, $\forall x \in R$. Find

- (i) range of $f(x)$.
- (ii) period (if periodic).
- (iii) points of discontinuity of $f(x)$.
- (iv) points of non-differentiability of $f(x)$.

Sol. We know that, $f(x) = \min\{\tan x, \cot x\}$ can be plotted in two steps:

- (a) We should plot the graph of $\tan x$ and $\cot x$.
- (b) We should find their points of intersection and neglect the area above their point of intersection.



It can seen from the graph that,

- (i) range of $f(x) = (-\infty, -1] \cup [0, 1]$
- (ii) period of $f(x) = \pi$
- (iii) points of discontinuity are $\pm \pi, \pm \frac{\pi}{2}, 0, \dots$ which can be put in the form of $\frac{n\pi}{2}, n \in I$.

- (iv) Also, the points of non-differentiability are $\pm \pi, \pm \frac{3\pi}{4}, \pm \frac{\pi}{2}, 0, \dots$ which can be put in the form of $n \frac{\pi}{4}, n \in I$.

Problems on Functional Equations

While solving problems based on this topic, always keep in mind the concept of

- (i) Limits
- (ii) Continuity
- (iii) Differentiability
- (iv) Function

In this type of problems, first we simplify the given condition and then use the derivative formula

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Further, if it is needed, then we use the integration and simplify it.

➤ **Example 48.** Let $f: R \rightarrow R$ be a function satisfies condition $f(x+y^3) = f(x) + [f(y)]^3$ for all $x, y \in R$. If $f'(0) \geq 0$, then $f(10)$ is

- (a) 8
- (b) 9
- (c) 0 or 10
- (d) 2 or 4

Sol. (c) Given, $f(x+y^3) = f(x) + [f(y)]^3$... (i)

and $f'(0) \geq 0$... (ii)

Replacing x, y by 0, we get

$$f(0) = f(0) + f(0)^3 \Rightarrow f(0) = 0 \quad \dots (iii)$$

$$\text{Also, } f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h} \quad \dots (iv)$$

$$\text{Let } I = f'(0) = \lim_{h \rightarrow 0} \frac{f[0 + (h^{1/3})^3] - f(0)}{(h^{1/3})^3}$$

$$= \lim_{h \rightarrow 0} \frac{f[(h^{1/3})^3]}{(h^{1/3})^3} = \lim_{h \rightarrow 0} \left[\frac{f(h^{1/3})}{(h^{1/3})} \right]^3 = I^3$$

$$I = I^3 \text{ or } I = 0, 1, -1 \text{ as } f'(0) \geq 0$$

$$\therefore f'(0) = 0, 1 \quad \dots (v)$$

$$\text{Thus, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f[x + (h^{1/3})^3] - f(x)}{(h^{1/3})^3}$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x) - [f(h^{1/3})]^3 - f(x)}{(h^{1/3})^3} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \left[\frac{f(h^{1/3})}{(h^{1/3})} \right]^3 = [f'(0)]^3$$

$$\Rightarrow f'(x) = 0, 1 \quad [\because f'(0) = 0, 1, \text{ from Eq. (v)}]$$

On integrating both sides, $f(x) = c$ or $x + c$ as $f(0) = 0$

$$\Rightarrow f(x) = 0 \text{ or } x$$

Thus, $f(10) = 0$ or 10

➤ **Example 49.** If $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$ for

all $x, y \in (-1, 1)$ and $\lim_{h \rightarrow 0} \frac{f(x)}{x} = 2$, then $f\left(\frac{1}{\sqrt{3}}\right)$

and $f'(1)$ are

- (a) $\frac{\pi}{3}$ and 1
- (b) $\frac{\pi}{6}$ and $\frac{1}{2}$
- (c) $\frac{\pi}{2}$ and 0
- (d) $\frac{\pi}{4}$ and $\sqrt{3}$

Sol. (a) $f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$... (i)

Putting $x = y = 0$

we get $f(0) = 0$

Putting $y = -x$,

we get $f(x) + f(-x) = f(0)$

$\Rightarrow f(-x) = -f(x)$... (ii)

Also, $\lim_{h \rightarrow 0} \frac{f(x)}{x} = 2$

Now, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$... (iii)

$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) + f(-x)}{h}$ [from Eq. (ii)]

$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f\left(\frac{x+h-x}{1-(x+h)(-x)}\right)}{h}$ [from Eq. (i)]

$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \left[\frac{f\left(\frac{h}{1+x(x+h)}\right)}{\frac{h}{1+x(x+h)}} \right]$

$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)} \times \left(\frac{1}{1+xh+x^2}\right)$

$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{f\left(\frac{h}{1+xh+x^2}\right)}{\left(\frac{h}{1+xh+x^2}\right)} \times \lim_{h \rightarrow 0} \frac{1}{1+xh+x^2}$

$\Rightarrow f'(x) = 2 \times \frac{1}{1+x^2} \quad \left[\because \lim_{h \rightarrow 0} \frac{f(x)}{x} = 2 \right]$

$\Rightarrow f'(x) = \frac{2}{1+x^2}$

On integrating both sides, $f(x) = 2 \tan^{-1}(x) + C$

where, $f(0) = 0 \Rightarrow C = 0$.

Thus, $f(x) = 2 \tan^{-1} x$

Hence, $f\left(\frac{1}{\sqrt{3}}\right) = 2 \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}$

and $f'(1) = \frac{2}{1+1^2} = \frac{2}{2} = 1$

Work Book Exercise 5.2

- 1 Which of the following functions is differentiable at $x = 0$?

- a $\cos(|x|) + |x|$ b $\cos(|x|) - |x|$
c $\sin(|x|) + |x|$ d $\sin(|x|) - |x|$

- 2 The set of all points, where the function $f(x) = \sqrt[3]{x^2|x|}$ is differentiable, is

- a $[0, \infty)$ b $(0, \infty)$
c $(-\infty, \infty)$ d $(-\infty, 0) \cup (0, \infty)$

- 3 If $f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $f(x)$ is differentiable on

- a $[-1, 1]$ b $R - \{-1, 1\}$
c $R - (-1, 1)$ d None of these

- 4 The set of points, where the function $f(x) = \sqrt{1-e^{-x^2}}$ is differentiable, is

- a $(-\infty, \infty)$
b $(-\infty, 0) \cup (0, \infty)$
c $(-1, \infty)$
d None of the above

- 5 If $f(x) = a|\sin x| + b e^{|x|} + c|x|^3$ and $f(x)$ is differentiable at $x = 0$, then

- a $a = b = c = 0$
b $a = 0, b = 0, c \in R$
c $b = c = 0, a \in R$
d $c = 0, a = 0, b \in R$

- 6 The function $f(x) = \tan^{-1}(\tan x)$ is

- a everywhere continuous
b discontinuous at $x = \frac{n\pi}{2}, n \in I$
c not differentiable at $x = (2n+1)\frac{\pi}{2}, n \in I$ such that
 $f'(x) = 1, \forall x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$
d everywhere continuous and differentiable such that $f'(x) = 1, \forall x \in R$

- 7 The number of points $(1, 3)$, where

$f(x) = a^{[x^2]}, a > 1$, is not differentiable, is

- a 0 b 3
c 5 d 7

- 8 Let $f(x) = [x]$, where $[]$ denotes the greatest integer function, then $f'(-1)$ is

- a 0 b 1
c non-existent d None of these

- 9 Let $f(x+y) = f(x)f(y), \forall x, y \in R, f(5) = 2$ and $f'(0) = 3$, then $f'(5)$ equals

- a 6 b 3
c 5 d None of these

- 10 If $f\left(\frac{x}{y}\right) = \frac{f(x)}{f(y)}, \forall x, y \in R, y \neq 0$ and $f(t) = 0$, if

$t \neq 0$ and $f'(1) = 3$, then $f(x)$ is

- a x b x^2
c x^3 d constant

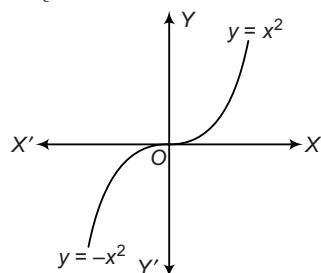
WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Exhaustive set of points, where $f(x) = x|x|$ is differentiable, is

- (a) $(-\infty, \infty) \sim \{0\}$ (b) $(-\infty, \infty) \sim \{1\}$
(c) $\{-\infty, \infty\} \sim \{-1\}$ (d) $(-\infty, \infty)$

Sol. $f(x) = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$



Graph of $f(x)$ indicates that $f(x)$ is differentiable for $(-\infty, \infty)$.

Hence, (d) is the correct answer.

Ex 2. If $f(x) = \begin{cases} \tan^{-1} x, & \text{for } |x| < 1 \\ \frac{x^2 - 1}{4}, & \text{for } |x| \geq 1 \end{cases}$, then $f(x)$ is

differentiable at

- (a) $(-\infty, \infty) - \{1\}$ (b) $(-\infty, \infty) - \{-1\}$
(c) $(-\infty, \infty) - \{1, -1\}$ (d) $(-\infty, \infty) - \{-1, 0, 1\}$

Sol. Clearly, $f(x)$ is discontinuous at $x = 1, -1$.

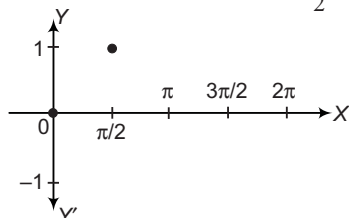
So, it is non-differentiable.

Hence, (c) is the correct answer.

Ex 3. $f(x) = [\sin x]$, where $[\]$ denotes the greatest integer function, is continuous at

- (a) $\frac{\pi}{2}$ (b) π (c) $\frac{3\pi}{2}$ (d) 2π

Sol. Clearly, $f(x)$ is continuous at $x = \frac{3\pi}{2}$.



Hence, (c) is the correct answer.

Ex 4. $f(x) = [\tan^{-1} x]$, where $[\]$ denotes the greatest integer function, is discontinuous at

- (a) $\frac{\pi}{4}, -\frac{\pi}{4}$ and 0 (b) $\frac{\pi}{3}, -\frac{\pi}{3}$ and 0
(c) $\tan 1, -\tan 1$ and 0 (d) None of these

Sol. $f(x)$ will be discontinuous when $\tan^{-1} x = 0, 1, -1$

As, $\tan^{-1} x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \forall x \in (-\infty, \infty)$

Thus, $f(x)$ is discontinuous at $x = 0, \tan 1, -\tan 1$.
Hence, (c) is the correct answer.

Ex 5. $f(x) = \begin{cases} \sin \{x\}, & \text{if } x < 1 \\ \cos x + a, & \text{if } x \geq 1 \end{cases}$, where $\{ \}$ denotes

the fractional part. If $f(x)$ is continuous at $x = 1$, then

- (a) $a = \sin 1$ (b) $a = \cos 1 - \sin 1$
(c) $a = \cos 1$ (d) $a = \sin 1 - \cos 1$

Sol. $\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (\cos x + a) = a + \cos 1$

$f(1) = a + \cos 1$
and $\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} \sin \{x\} = \sin 1$

For $f(x)$ to be continuous at $x = 1$, $a = \sin 1 - \cos 1$.

Hence, (d) is the correct answer.

Ex 6. If $f(x) = |x^{2n+1}|$, $n \in \mathbb{N}$, then

- (a) $f(x)$ is continuous but non-differentiable at $x = 0$
(b) $f(x)$ is differentiable at $x = 0$
(c) $f(x)$ is discontinuous at $x = 0$
(d) None of the above

Sol. $f(x) = x^{2n}|x|$, $f(0) = 0$

$\therefore f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h^{2n}|h|}{h} = 0$

Thus, $f(x)$ is differentiable at $x = 0$ and hence also continuous at $x = 0$.

Hence, (b) is the correct answer.

Ex 7. The value of $f(0)$, so that

$f(x) = \frac{1}{x^2} [1 - \cos(\sin x)]$ can be made

continuous at $x = 0$, is

- (a) $1/2$ (b) 2
(c) 8 (d) 4

Sol. $\lim_{x \rightarrow 0} \frac{1 - \cos(\sin x)}{x^2} = \lim_{x \rightarrow 0} \frac{2 \sin^2 \left(\frac{\sin x}{2} \right)}{x^2}$

$= \lim_{x \rightarrow 0} 2 \cdot \left(\frac{\sin \left(\frac{\sin x}{2} \right)}{\left(\frac{\sin x}{2} \right)} \right)^2 \cdot \left(\frac{\sin x}{2x} \right)^2 = \frac{1}{2}$

$\Rightarrow f(0) = \frac{1}{2}$

Hence, (a) is the correct answer.

Ex 8. If $f(x) = \begin{cases} x, & \text{for } x \text{ is rational} \\ 1-x, & \text{for } x \text{ is irrational} \end{cases}$, then at $x = \frac{1}{2}$, $f(x)$ is

- (a) continuous but not differentiable
(b) discontinuous
(c) differentiable
(d) None of the above

Sol. $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $\lim_{x \rightarrow 1/2} f(x) = \frac{1}{2}$

But $f'\left(\frac{1}{2}\right)$ is not well defined.

Hence, (a) is the correct answer.

Ex 9. If $f(x) = \begin{cases} \frac{x}{2x^2 + |x|}, & \text{for } x \neq 0 \\ 1, & \text{for } x = 0 \end{cases}$, then $f(x)$ is

- (a) continuous but not differentiable at $x = 0$
(b) differentiable at $x = 0$
(c) discontinuous at $x = 0$
(d) None of the above

Sol. $f(0+h) = \lim_{h \rightarrow 0} \frac{f(h)}{h} = \lim_{h \rightarrow 0} \frac{1}{2h^2 + h} = \lim_{h \rightarrow 0} \frac{1}{2h+1} = 1$

and $f(0-h) = \lim_{h \rightarrow 0} \frac{f(-h)}{-h} = \lim_{h \rightarrow 0} \frac{-h}{2h^2 + |-h|} = \lim_{h \rightarrow 0} \frac{-h}{2h^2 + h} = \lim_{h \rightarrow 0} \frac{-1}{2h+1} = -1$

As $f(0+h) \neq f(0-h)$, thus $f(x)$ is discontinuous at $x = 0$.

Hence, (c) is the correct answer.

Ex 10. If $f(x) = \begin{cases} x + a\sqrt{2} \sin x, & \text{for } 0 \leq x < \frac{\pi}{4} \\ 2x \cot x + b, & \text{for } \frac{\pi}{4} \leq x < \frac{\pi}{2} \\ a \cos 2x - b \sin x, & \text{for } \frac{\pi}{2} \leq x \leq \pi \end{cases}$ is

continuous in $[0, \pi]$, then

- (a) $a = -\frac{\pi}{6}$, $b = -\frac{\pi}{12}$ (b) $a = \frac{\pi}{6}$, $b = \frac{\pi}{12}$
(c) $a = \frac{\pi}{6}$, $b = -\frac{\pi}{12}$ (d) $a = -\frac{\pi}{16}$, $b = \frac{\pi}{12}$

Sol. $f\left(\frac{\pi}{4}\right) = b + \frac{\pi}{2}$, $f\left(\frac{\pi}{4} + 0\right) = b + \frac{\pi}{2}$,

$f\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4} + a$

$f\left(\frac{\pi}{2}\right) = -a - b$, $f\left(\frac{\pi}{2} + 0\right) = -a - b$, $f\left(\frac{\pi}{2} - 0\right) = b$

$\Rightarrow a + \frac{\pi}{2} = a + \frac{\pi}{4}, -a - b = b$

$\Rightarrow a = \frac{\pi}{6}, b = -\frac{\pi}{12}$

Hence, (c) is the correct answer.

Ex 11. If $f(x) = [x] + \sqrt{\{x\}}$, where $[]$ and $\{ \}$ denote the greatest integer function and fractional part respectively, then

- (a) $f(x)$ is continuous but not differentiable at $x = 1$
(b) $f(x)$ is differentiable at $x = 1$
(c) $f(x)$ is discontinuous at $x = 1$
(d) None of the above

Sol. $f(1) = 1$, $f(1-0) = 1 - 1 + 1 = 1$,

$f(1+0) = 1 + 0 = 1$

$\therefore f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$

$\Rightarrow f'(1-0) = \lim_{h \rightarrow 0} \frac{(1-1) + \sqrt{1-h} - 1}{-h}$

$= \lim_{h \rightarrow 0} -\frac{1-h-1}{h(1+\sqrt{1-h})} = \frac{1}{2}$

and $f'(1+0) = \lim_{h \rightarrow 0} \frac{1 + \sqrt{h} - 1}{h} = \infty$

Thus, $f(x)$ is continuous for all x but not differentiable at all integral values of x .

Hence, (a) is the correct answer.

Ex 12. If the first derivative of $f(x) = \log_2(\log_3(\log_{1/2}(\cos x + a)))$ exists for all real values of x , then

- (a) $a \in \left(1, \frac{3}{2}\right)$ (b) $a \in \left(\frac{1}{2}, 1\right)$
(c) $a \in \left(1, \frac{5}{2}\right)$ (d) None of these

Sol. Clearly, domain of $f(x)$ must be all real numbers.

$\therefore \log_3(\log_{1/2}(\cos x + a)) > 0, \forall x \in R$

$\Rightarrow \log_{1/2}(\cos x + a) > 1, \forall x \in R$

$\Rightarrow -0 < (\cos x + a) < \frac{1}{2}, \forall x \in R$

$\Rightarrow -\cos x < a < \frac{1}{2} - \cos x, \forall x \in R$

$\Rightarrow 1 < a < -\frac{1}{2}$

which is not possible.

Thus, first derivative of $f(x)$ cannot exist for all values of x .

Hence, (d) is the correct answer.

Ex 13. If $f(x) = \begin{cases} ax^2 + 1, & \text{for } x \leq 1 \\ x^2 + ax + b, & \text{for } x > 1 \end{cases}$ is

differentiable at $x = 1$, then

- (a) $a = 1, b = 1$
(b) $a = 1, b = 0$
(c) $a = 2, b = 0$
(d) $a = 2, b = 1$

Sol. We have, $f'(1-0) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$
 $= \lim_{h \rightarrow 0} \frac{a(1-h)^2 + 1 - (1+a)}{-h}$
 $= \lim_{h \rightarrow 0} \frac{a(h^2 - 2h)}{-h} = 2a$
 and $f'(1+0) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$
 $= \lim_{h \rightarrow 0} \frac{(1+h)^2 + a(1+h) + b - (1+a)}{h}$
 $= \lim_{h \rightarrow 0} \frac{h^2 + 2h + ah + b}{h}$
 $= 2 + a, \text{ if } b = 0$
 Thus, $2a = 2 + a, b = 0$
 $\Rightarrow a = 2, b = 0$
 Hence, (c) is the correct answer.

Ex 14. Which of the following statements is incorrect?

- (a) If $f'(x) > 0, \forall x \in \text{domain}$, then $f(x)$ must be one-one
- (b) If $f'(x) < 0, \forall x \in \text{domain}$, then $f(x)$ must be one-one
- (c) If $|f(x)|$ is continuous at $x = a$, then $f(x)$ is also continuous at $x = a$
- (d) If $f(x)$ is continuous at $x = a, f(a) = 2$ and $x = a$ is the point of local minima of $f(x)$, where $[]$ denotes the greatest integer function, $[f(x)]$ is also continuous at $x = a$

Sol. Clearly, option (d) is incorrect as
 $f(a-0) - f(a+0) = 2 = f(a)$
 Hence, (d) is the correct answer.

Ex 15. If $f(x) = [\sin x] + [\cos x], x \in [0, 2\pi]$, where $[]$ denotes the greatest integer function, then total number of points, where $f(x)$ is not differentiable, is equal to

- (a) 2
- (b) 3
- (c) 5
- (d) 4

Sol. $[\sin x]$ is not differentiable at $x = \frac{\pi}{2}, \pi, 2\pi$ and $[\cos x]$ is not differentiable at $x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$. Thus, $f(x)$ is definitely not differentiable at $x = \pi, \frac{3\pi}{2}, 0$.
 Also, $f\left(\frac{\pi}{2}\right) = 1, f\left(\frac{\pi}{2} - 0\right) = 0, f(2\pi) = 1$
 and $f(2\pi - 0) = -1$
 Thus, $f(x)$ is also not differentiable at $x = \frac{\pi}{2}$ and 2π .
 Hence, (c) is the correct answer.

Ex 16. The value of $f(0)$, so that the function

$$f(x) = \frac{\cos(\sin x) - \cos x}{x^4}$$

is continuous at each point in its domain, is
 (a) 2 (b) $\frac{1}{6}$ (c) $\frac{2}{3}$ (d) $-\frac{1}{3}$

Sol. For f to be continuous, we must have
 $f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$
 $= \lim_{x \rightarrow 0} \frac{-2 \sin\left(\frac{\sin x + x}{2}\right) \sin\left(\frac{\sin x - x}{2}\right)}{x^4}$
 $= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\sin\left(\frac{\sin x + x}{2}\right) \sin\left(\frac{\sin x - x}{2}\right)}{\left(\frac{\sin x + x}{2}\right) \left(\frac{\sin x - x}{2}\right)}$
 $\times \frac{\sin x + x}{2} \times \frac{\sin x - x}{x^3}$
 $= -\frac{1}{2} \lim_{u \rightarrow 0} \frac{\sin u}{u} \lim_{v \rightarrow 0} \frac{\sin v}{v} \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} + 1\right)$
 $\times \lim_{x \rightarrow 0} \frac{x^3 - x^5 + \dots}{x^3}$

$$\left[\because u = \frac{\sin x + x}{2}, v = \frac{\sin x - x}{2} \right]$$

$$= -\frac{1}{2} \times 1 \times 1 \times 2 \times \frac{1}{3!} = \frac{1}{6}$$

Hence, (b) is the correct answer.

Ex 17. Let $f(x) = \begin{cases} -2 \sin x, & \text{if } x \leq -\pi/2 \\ A \sin x + B, & \text{if } -\pi/2 < x < \pi/2 \\ \cos x, & \text{if } x \geq \pi/2 \end{cases}$

Then, the values of A and B so that $f(x)$ is continuous everywhere, are

- (a) $A = 0, B = 1$
- (b) $A = 1, B = 1$
- (c) $A = -1, B = 1$
- (d) $A = -1, B = 0$

Sol. Since, $\sin x$ and $\cos x$ are continuous functions, so $f(x)$ is continuous except possibly at $x = -\pi/2$ and $x = \pi/2$. For f to be continuous at $x = -\pi/2$, we must have

$$f(-\pi/2) = \lim_{x \rightarrow -\pi/2^+} f(x) = \lim_{x \rightarrow -\pi/2^+} (A \sin x + B)$$

$$= -A + B$$

$$\therefore -A + B = f(-\pi/2) = -2 \sin(-\pi/2) = 2$$

For f to be continuous at $x = \pi/2$, we must have

$$0 = \cos \pi/2 = f(\pi/2) = \lim_{x \rightarrow \pi/2^-} f(x)$$

$$= \lim_{x \rightarrow \pi/2^-} (A \sin x + B) = A + B$$

$$\therefore B = 1 \text{ and } A = -1$$

Hence, (c) is the correct answer.

Ex 18. Let $f(x) = \frac{\tan[e^2]x^3 - \tan[-e^2]x^3}{\sin^3 x}, x \neq 0$.

Then, the value of $f(0)$ for which $f(x)$ is continuous, is

- (a) 15
- (b) 12
- (c) -12
- (d) 14

Sol. Since, $7 < e^2 < 8$, so $[e^2] = 7$ and $[-e^2] = -8$

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\tan 7x^3 - \tan(-8x^3)}{\sin^3 x}$$

$$= \lim_{x \rightarrow 0} \left(7 \frac{\tan 7x^3}{7x^3} \cdot \frac{x^3}{\sin^3 x} + 8 \frac{\tan 8x^3}{8x^3} \cdot \frac{x^3}{\sin^3 x} \right)$$

$$= 7 + 8 = 15$$

Hence, (a) is the correct answer.

Ex 19. The value of $f(0)$, so that the function

$$f(x) = \frac{\sqrt{1+x} - \sqrt[3]{1+x}}{x} \text{ becomes continuous,}$$

is

- (a) $\frac{1}{6}$ (b) $\frac{1}{4}$ (c) 2 (d) $\frac{1}{3}$

Sol. The function $f(x)$ is continuous except possibly at $x = 0$. For f to be continuous at $x = 0$, we must have

$$f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{(1+x)^{1/2} - (1+x)^{1/3}}{x}$$

$$f(0) = \lim_{x \rightarrow 0} \frac{\left[1 + \frac{1}{2}x + \frac{(1/2)(-1/2)}{2}x^2 + \dots\right] - \left[1 + \frac{1}{3}x + \frac{(1/3)(-2/3)}{2}x^2 + \dots\right]}{x}$$

Therefore,

$$f(0) = \lim_{x \rightarrow 0} \frac{\left\{x \left[\frac{1}{2} + \frac{(1/2)(-1/2)}{2}x + \dots \right] - x \left[\frac{1}{3} + \frac{(1/3)(-2/3)}{2}x + \dots \right] \right\}}{x}$$

$$= \lim_{x \rightarrow 0} \left[\left(\frac{1}{2} - \frac{1}{3} \right) + \text{terms containing } x \right]$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6} \quad [\text{using L'Hospital's rule}]$$

Hence, (a) is the correct answer.

Ex 20. Let $f(x) = \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$, $x \neq a$. Then, the

value which should be assigned to f at $x = a$, so that it is continuous everywhere, is

- (a) $2/\pi$ (b) $e^{-2/\pi}$ (c) 2 (d) $e^{2/\pi}$

Sol. For f to be continuous, we must have

$$f(a) = \lim_{x \rightarrow a} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

$$\therefore \log f(x) = \tan\left(\frac{\pi x}{2a}\right) \log\left(2 - \frac{x}{a}\right)$$

$$= \frac{\log\left(2 - \frac{x}{a}\right)}{\cot\left(\frac{\pi x}{2a}\right)} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$\therefore \lim_{x \rightarrow a} \log f(x) = \lim_{x \rightarrow a} \frac{-1/a}{2 - \frac{x}{a}} \cdot \frac{-\pi/2a \operatorname{cosec}^2\left(\frac{\pi x}{2a}\right)}{\left(\frac{\pi}{2} \cdot \frac{x}{a}\right)}$$

[using L'Hospital's rule]

$$= \frac{2}{\pi} \lim_{x \rightarrow a} \frac{1}{2 - \frac{x}{a}} \times \lim_{x \rightarrow a} \sin^2\left(\frac{\pi x}{2a}\right) = \frac{2}{\pi}$$

Therefore, $\lim_{x \rightarrow a} f(x) = e^{2/\pi}$

Hence, (d) is the correct answer.

Ex 21. If $f(x) = (x/2) - 1$, then on the interval $[0, \pi]$,

- (a) both $\tan(f(x))$ and $\frac{1}{f(x)}$ are continuous
 (b) both $\tan(f(x))$ and $\frac{1}{f(x)}$ are discontinuous
 (c) both $\tan(f(x))$ and $f^{-1}(x)$ are continuous
 (d) neither $\tan(f(x))$ nor $f^{-1}(x)$ is continuous

Sol. $f(x) = (x/2) - 1$ is continuous on $[0, \pi]$ and $y = \tan x$ is continuous on $[-1, (\pi/2) - 1] = (\text{range of } f)$. Therefore, $\tan(f(x))$ is continuous on $[0, \pi]$ as $1/f(x)$ is not defined at $x = 2$, so $1/f(x)$ is not continuous on $[0, \pi]$. Now, $f^{-1}(x) = 2(x+1)$ which is clearly continuous on $[0, \pi]$.

Hence, (c) is the correct answer.

Ex 22. If the function $f(x) = 1/(1-x)$, then the number of points of discontinuity of the composite function $y = f^{3n}(x)$, where $f^n(x) = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ times}}$, $n \in N$, are

- (a) 0, 1 (b) $2n$ (c) $3n$ (d) $2n + 1$

Sol. The point $x = 1$ is a discontinuity of the function $f(x) = 1/(1-x)$. If $x \neq 1$, then

$$u(x) = f(f(x)) = \frac{x-1}{x}$$

Thus, $x = 0$ is a point of discontinuity of the function u . If $x \neq 0$ and $x \neq 1$, then $f \circ f \circ f(x) = x$. So, $y = f^{3n}(x) = (f^3(x))^n = x$ is continuous everywhere. Therefore, 0 and 1 are the only points of discontinuities of y .

Hence, (a) is the correct answer.

Ex 23. Let f be a continuous function of R such that

$$f\left(\frac{1}{4n}\right) = (\sin e^n) e^{-n^2} + \frac{n^2}{n^2 + 1}. \text{ Then, the}$$

value of $f(0)$ is

- (a) 1 (b) $1/2$
 (c) 0 (d) None of these

Sol. As f is continuous, so $f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{n \rightarrow \infty} f(x_n)$ for any sequence x_n such that $\lim_{n \rightarrow \infty} x_n = 0$. Thus,

$$f(0) = \lim_{n \rightarrow \infty} f\left(\frac{1}{4n}\right)$$

$$= \lim_{n \rightarrow \infty} \left((\sin e^n) e^{-n^2} + \frac{1}{1 + \frac{1}{n^2}} \right) = 0 + 1 = 1$$

$$\left[\because \lim_{n \rightarrow \infty} (\sin e^n) e^{-n^2} = 0 \text{ as } |(\sin e^n) e^{-n^2}| \leq e^{-n^2} \text{ and } e^{-n^2} \rightarrow 0 \text{ as } n \rightarrow \infty \right]$$

Hence, (a) is the correct answer.

Ex 24. If $\lim_{x \rightarrow a} (f(x) g(x))$ exists for any function f and g , then

- (a) $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist
(b) $\lim_{x \rightarrow a} f(x)$ exists but $\lim_{x \rightarrow a} g(x)$ may not exist
(c) $\lim_{x \rightarrow a} f(x)$ may not exist but $\lim_{x \rightarrow a} g(x)$ exists
(d) $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ may not exist

Sol. Take $f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$ and $g(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$. Then,

$f(x) g(x) = 1$ for $x \neq 0$. So, $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ does not exist but $\lim_{x \rightarrow 0} f(x) g(x) = 1$.

Hence, (d) is the correct answer.

Ex 25. If $f(x) = \begin{cases} x^p \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

- continuous but not differentiable at $x = 0$, if
(a) $0 < p \leq 1$
(b) $1 \leq p < \infty$
(c) $-\infty < p < 0$
(d) $p = 0$

Sol. Given, $f(x)$ is continuous at $x = 0$.
So, limit must exist.

i.e. $\lim_{x \rightarrow a} x^p \sin \frac{1}{x} = (0)^p \sin \infty = 0$, when

$$\text{Now, RHD} = \lim_{h \rightarrow 0} \frac{h^p \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h^{p-1} \sin \frac{1}{h} \quad \dots (i)$$

$$\begin{aligned} \text{LHD} &= \lim_{h \rightarrow 0} \frac{(-h)^p \sin \left(-\frac{1}{h}\right) - 0}{-h} \\ &= \lim_{h \rightarrow 0} (-1)^p h^{p-1} \sin \frac{1}{h} \end{aligned}$$

Since, $f(x)$ is not differentiable at $x = 0$.

$$\therefore p \leq 1 \quad \dots (ii)$$

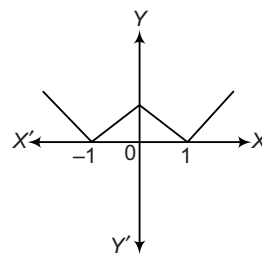
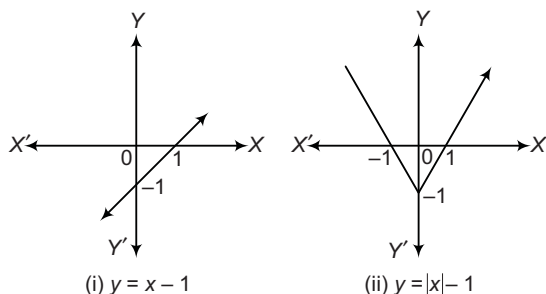
From Eqs. (i) and (ii), $0 < p \leq 1$

Hence, (a) is the correct answer.

Ex 26. If $f(x) = ||x| - 1|$, then points, where $f(x)$ is not differentiable, is/are

- (a) $0, \pm 1$ (b) ± 1 (c) 0 (d) 1

Sol. Using graphical transformation,



$$(iii) y = ||x| - 1|$$

We know that, function is not differentiable at sharp edges and in figure (iii), we have 3 sharp edges at $x = -1, 0, 1$.

So, $f(x)$ is not differentiable at $\{0, \pm 1\}$.

Hence, (a) is the correct answer.

Ex 27. If $f(x) = x(\sqrt{x} + \sqrt{x+1})$, then

- (a) $f(x)$ is continuous but differentiable at $x = 0$
(b) $f(x)$ is differentiable at $x = 0$
(c) $f(x)$ is not differentiable at $x = 0$
(d) None of the above

Sol. Given, $f(x) = x(\sqrt{x} + \sqrt{x+1})$.

At $x = 0$, LHL of $\sqrt{x}$ is not defined, therefore it is not continuous at $x = 0$.

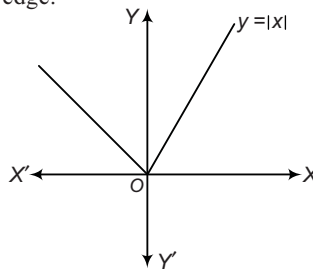
So, it is not differentiable at $x = 0$.

Hence, (c) is the correct answer.

Ex 28. The function $f(x) = e^{-|x|}$ is

- (a) continuous everywhere but not differentiable at $x = 0$
(b) continuous and differentiable everywhere
(c) not continuous at $x = 0$
(d) None of the above

Sol. It is clear from the figure that $f(x)$ is continuous everywhere and not differentiable at $x = 0$ due to sharp edge.



Hence, (a) is the correct answer.

Ex 29. If $f(x) = \sin^{-1} \{2x / (1 + x^2)\}$, then $f(x)$ is differentiable on

- (a) $[-1, 1]$ (b) $R - \{-1, 1\}$
(c) $R - (-1, 1)$ (d) None of these

Sol. Given, $f(x) = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$

$$\Rightarrow f'(x) = \frac{1}{\sqrt{1 - \left(\frac{2x}{1+x^2} \right)^2}} \times \frac{d}{dx} \left(\frac{2x}{1+x^2} \right)$$

$$= \frac{1+x^2}{\sqrt{(1-x^2)^2}} \times \frac{2(1-x^2)}{(1+x^2)^2}$$

$$= \frac{2}{1+x^2} \times \frac{1-x^2}{|1-x^2|} = \begin{cases} \frac{2}{1+x^2}, & \text{if } |x| < 1 \\ -\frac{2}{1+x^2}, & \text{if } |x| > 1 \end{cases}$$

So, $f'(x)$ does not exist for $|x| = 1$, i.e. $x = \pm 1$.
Therefore, $f(x)$ is differentiable on $R - \{-1, 1\}$.
Hence, (b) is the correct answer.

Ex 30. If $g(x) = \frac{(x-1)^n}{\log \cos^m(x-1)}$, $0 < x < 2$, m and n

are integers, $m \neq 0$, $n > 0$ and let p be the left hand derivative of $|x-1|$ at $x=1$. If $\lim_{x \rightarrow 1^+} g(x) = p$, then

- (a) $n = 1, m = 1$
(b) $n = 1, m = -1$
(c) $n = 2, m = 2$
(d) $n > 2, m = n$

Sol. Given, $g(x) = \frac{(x-1)^n}{\log \cos^m(x-1)}$; $0 < x < 2$, $m \neq 0$, n

are integers and $|x-1| = \begin{cases} x-1, & x \geq 1 \\ 1-x, & x < 1 \end{cases}$

The left hand derivative of $|x-1|$ at $x=1$ is $p = -1$.
Also, $\lim_{x \rightarrow 1^+} g(x) = p = -1$

$$\therefore \lim_{h \rightarrow 0} \frac{(1+h-1)^n}{\log \cos^m(1+h-1)} = -1$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{h^n}{m \log \cos h} = -1$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{n \cdot h^{n-1}}{m \frac{1}{\cos h} (-\sin h)} = -1$$

[using L'Hospital's rule]

$$\Rightarrow \left(\frac{n}{m}\right) \lim_{h \rightarrow 0} \frac{h^{n-2}}{\left(\frac{\tan h}{h}\right)} = 1$$

$$\Rightarrow n = 2 \quad \text{and} \quad \frac{n}{m} = 1$$

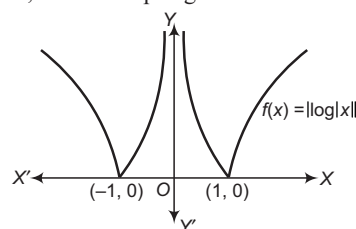
$$\therefore m = n = 2$$

Hence, (c) is the correct answer.

Ex 31. If $f(x) = |\log|x||$, then

- (a) $f(x)$ is continuous and differentiable for all x in its domain
(b) $f(x)$ is continuous for all x in its domain but not differentiable at $x = \pm 1$
(c) $f(x)$ is neither continuous nor differentiable at $x = \pm 1$
(d) None of the above

Sol. For the graph of $f(x) = |\log|x||$, it is clear that $f(x)$ is continuous everywhere but not differentiable at $x = \pm 1$, due to sharp edge.



Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 32. If $f(x) = x \frac{e^{[x]+|x|} - 2}{[x] + |x|}$, then

- (a) $\lim_{x \rightarrow 0^+} f(x) = -1$
(b) $\lim_{x \rightarrow 0^-} f(x) = 0$
(c) $\lim_{x \rightarrow 0^+} f(x)$ does not exist
(d) $\lim_{x \rightarrow 0} f(x)$ does not exist

Sol. For $-1 < x < 0$, $f(x) = x \cdot \frac{e^{-1-x} - 2}{-1-x}$

$$\text{So, } \lim_{x \rightarrow 0^-} f(x) = 0 \cdot \frac{e^{-1} - 2}{-1} = 0$$

$$\text{For } 0 < x < 1, f(x) = x \cdot \frac{e^x - 2}{x} = e^x - 2$$

$$\text{So, } \lim_{x \rightarrow 0^+} f(x) = 1 - 2 = -1$$

Thus, $\lim_{x \rightarrow 0} f(x)$ does not exist.

Hence, (a), (b) and (d) are the correct answers.

Ex 33. If $f(x) = \lim_{n \rightarrow \infty} (\sin x)^{2n}$, then f is

- (a) continuous at $x = \pi$
(b) discontinuous at $x = \pi/2$
(c) discontinuous at $x = -\pi/2$
(d) discontinuous at an infinite number of points

$$\text{Sol. } \lim_{n \rightarrow \infty} x^{2n} = \begin{cases} 0, & \text{if } |x| < 1 \\ 1, & \text{if } |x| = 1 \end{cases}$$

$$\Rightarrow f(x) = \lim_{n \rightarrow \infty} (\sin x)^{2n} = \begin{cases} 0, & \text{if } |\sin x| < 1 \\ 1, & \text{if } |\sin x| = 1 \end{cases}$$

which shows that f is continuous for all x , except possibly when $|\sin x| = 1$, i.e. when $x = (2k+1)\pi/2$, $k \in I$. For these points, we have

$$\lim_{x \rightarrow (2k+1)\pi/2} f(x) = 0 \neq 1 = f\left((2k+1)\frac{\pi}{2}\right)$$

So, $f(x)$ is discontinuous at these points.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 34. Let $f(x)$ be defined on $[0, \pi]$ by

$$f(x) = \begin{cases} x + a\sqrt{2} \sin x, & \text{for } 0 \leq x \leq \pi/4 \\ 2x \cot x + b, & \text{for } \pi/4 < x \leq \pi/2 \\ a \cos 2x - b \sin x, & \text{for } \pi/2 < x \leq \pi \end{cases}$$

If f is continuous on $[0, \pi]$, then

$$\begin{aligned} \text{(a) } a &= \frac{\pi}{6} & \text{(b) } b &= -\frac{\pi}{12} \\ \text{(c) } a &= \frac{\pi}{6} \text{ and } b = -\frac{\pi}{12} & \text{(d) } a &= \frac{\pi}{6} \text{ or } b = -\frac{\pi}{12} \end{aligned}$$

Sol. $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4} + a$

Also, $f\left(\frac{\pi}{4}\right) = \lim_{x \rightarrow \pi/4^+} = \lim_{x \rightarrow \pi/4^+} (2x \cot x + b)$

$$= \frac{\pi}{2} + b$$

So, $a - b = \frac{\pi}{4}$

Also, $f\left(\frac{\pi}{2}\right) = b$

and $f\left(\frac{\pi}{2}\right) = \lim_{x \rightarrow \pi/2^+} f(x) = \lim_{x \rightarrow \pi/2^+} (a \cos 2x - b \sin x)$

$$= -a - b$$

$\therefore 2b = -a$

Thus, $a = \frac{\pi}{6}$

and $b = -\frac{\pi}{12}$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 35. Let $f(x) = [x] + [-x]$. Then, for any integer n

and $k \in \mathbb{R} \sim I$,

(a) $\lim_{x \rightarrow n} f(x)$ exists

(b) $\lim_{x \rightarrow k} f(x)$ exists

(c) f is continuous at $x = n$

(d) f is continuous at $x = k$

Sol. $f(n) = [n] + [-n] = n - n = 0$

$$\lim_{x \rightarrow n^-} f(x) = \lim_{h \rightarrow 0^+} [n - h] + [-n + h]$$

$$= n - 1 + (-n) = -1$$

$$\lim_{x \rightarrow n^+} f(x) = \lim_{h \rightarrow 0^+} [n + h] + [-n - h]$$

$$= n + (-n - 1) = -1$$

If $k \notin I$, the functions $[x]$ and $[-x]$ are continuous at k . Thus, $\lim_{x \rightarrow k} f(x)$ exists.

Hence, (a), (b) and (d) are the correct answers.

Ex 36. Let $f(x) = g'(x) \frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}}$, where g' is

the derivative of g and is a continuous function and $a > 0$, then $\lim_{x \rightarrow 0} f(x)$ exists, if

(a) $g(x)$ is polynomial

(b) $g(x) = x$

(c) $g(x) = x^2$

(d) $g(x) = x^3 h(x)$, where $h(x)$ is a polynomial

Sol. $\lim_{x \rightarrow 0^+} \frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}} = \lim_{x \rightarrow 0^+} \frac{1 - e^{-2a/x}}{1 + e^{-2a/x}} = 1$

and $\lim_{x \rightarrow 0^-} \frac{e^{a/x} - e^{-a/x}}{e^{a/x} + e^{-a/x}} = \lim_{x \rightarrow 0^-} \frac{e^{2a/x} - 1}{e^{2a/x} + 1} = -1$

Therefore, $\lim_{x \rightarrow 0} f(x)$ exists, if $g'(0) = 0$.

If $g(x) = Ax + B$, $A \neq 0$, then $\lim_{x \rightarrow 0} f(x) = A$ and

$\lim_{x \rightarrow 0^-} f(x) = -A$. So, $\lim_{x \rightarrow 0} f(x)$ will exist, if $g(x) = x^2$

or $g(x) = x^3 h(x)$, where $h(x)$ is a polynomial.

Hence, (c) and (d) are the correct answers.

Ex 37. If $f(x) = \begin{cases} ae^{1/|x+2|} - 1, & -3 < x < -2 \\ b, & x = -2 \\ \sin\left(\frac{x^4 - 16}{x^5 + 32}\right), & -2 < x < 0 \end{cases}$ is

continuous at $x = -2$, then

(a) $a = \sin(3/5)$

(b) $b = -\sin(2/5)$

(c) $a = \sin(2/5)$

(d) $b = -a$

Sol. $\lim_{x \rightarrow -2^-} f(x) = \lim_{x \rightarrow -2^-} \frac{ae^{1/|x+2|} - 1}{2 - e^{1/|x+2|}}$

$$= \lim_{x \rightarrow -2^-} \frac{ae^{-1/|x+2|} - 1}{2e^{-1/|x+2|} - 1} = -a$$

$$\left[\because \frac{1}{|x+2|} \rightarrow \infty \text{ as } x \rightarrow -2^- \right]$$

$$\begin{aligned} \therefore \lim_{x \rightarrow -2^+} f(x) &= \lim_{x \rightarrow -2^+} \sin\left(\frac{x^4 - 16}{x^5 + 32}\right) \\ &= \lim_{x \rightarrow -2^+} \sin \frac{(x-2)(x^2+4)}{(x^4 - 2x^3 + 4x^2 - 8x + 16)} \\ &= \sin \frac{(-4)(8)}{16 + 16 + 16 + 16 + 16} = \sin\left(-\frac{2}{5}\right) \end{aligned}$$

Thus, $b = \sin(-2/5)$ and $-a = b = \sin(-2/5)$, so $a = \sin(2/5)$.

Hence, (b), (c) and (d) are the correct answers.

Ex 38. If $f(x) = \sec 2x + \operatorname{cosec} 2x$, then $f(x)$ is discontinuous at all points in

(a) $\{n\pi, n \in \mathbb{N}\}$ (b) $\left\{(2n \pm 1)\frac{\pi}{4}, n \in I\right\}$

(c) $\left\{\frac{n\pi}{4}, n \in I\right\}$ (d) $\left\{(2n \pm 1)\frac{\pi}{8}, n \in I\right\}$

Sol. $f(x) = \sec 2x + \operatorname{cosec} 2x = \frac{2(\sin 2x + \cos 2x)}{2 \cos 2x \sin 2x}$

$$= \frac{2(\sin 2x + \cos 2x)}{\sin 4x} \text{ is discontinuous, where}$$

$$4x = n\pi, n \in I \text{ or } x = \frac{n\pi}{4}$$

Options (a) and (b) also satisfy the condition, since they are subsets of option (c).

Hence, (a), (b) and (c) are the correct answers.

Ex 39. Let $f(x) = \begin{cases} x^n \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, $n \in I$, then

- (a) $\lim_{x \rightarrow 0} f(x)$ exists for every $n > 1$
 (b) f is continuous at $x = 0$ for $n > 1$
 (c) f is differentiable at $x = 0$ for every $n > 1$
 (d) None of the above

Sol. $\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x^n \cdot \sin\left(\frac{1}{x^2}\right) = 0$

If $n > 0$ and hence, true for $n > 1$

Since, $f(0) = 0$, $f(x)$ is continuous at $x = 0$.

Now, $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x^n \sin\left(\frac{1}{x^2}\right)}{x}$

$\Rightarrow \lim_{x \rightarrow 0} x^{n-1} \sin\left(\frac{1}{x^2}\right) = 0$ if $n > 1$

Thus, $f(x)$ is differentiable at $x = 0$, if $n > 1$.

Hence, (a), (b) and (c) are the correct answers.

Ex 40. If $f(x) = e^{[\cot x]}$, where $[y]$ denotes the greatest integer less than or equal to y , then

- (a) $\lim_{x \rightarrow \pi/2^+} f(x) = 1$ (b) $\lim_{x \rightarrow \pi/2^+} f(x) = \frac{1}{e}$
 (c) $\lim_{x \rightarrow \pi/2^-} f(x) = \frac{1}{e}$ (d) $\lim_{x \rightarrow \pi/2^-} f(x) = 1$

Sol. $f(x) = e^{[\cot x]}$, as $\cot x$ is negative in the II quadrant and $\cot \frac{\pi}{2} = 0$.

$\Rightarrow [\cot x] = -1$

$\therefore \lim_{x \rightarrow \pi/2^+} f(x) = e^{-1} = \frac{1}{e}$

As $x \rightarrow \frac{\pi}{2^-}$, $\cot x$ is positive (being in I quadrant)

Thus, $[\cot x] = 0$

$\therefore \lim_{x \rightarrow \pi/2^-} f(x) = e^0 = 1$

Hence, (b) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 41-42) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation of Statement I
 (b) Statement I is true, Statement II is true; Statement II is not the correct explanation of Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

Ex 41. Statement I $f(x) = |\log x|$ is differentiable at $x = 1$.

Statement II Both $\log x$ and $-\log x$ are differentiable at $x = 1$.

Sol. $f(x) = -\log x, x < 1; \log x, x \geq 1 \Rightarrow f'(x) = \begin{cases} -\frac{1}{x}, & x < 1 \\ \frac{1}{x}, & x \geq 1 \end{cases}$

$\therefore f'(1^-) = -1$, and $f'(1^+) = 1$

Hence, $f(x)$ is not differentiable.

Hence, (d) is the correct answer.

Ex 42. Statement I If $f(x)$ and $g(x)$ are continuous and differentiable functions, then there exist $c, c \in [a, b]$ such that $\frac{f'(c)}{f(a) - f(c)} + \frac{g'(c)}{g(b) - g(c)} = 1$.

Statement II If $f(x)$ and $g(x)$ are continuous and differentiable functions, then $h(x) = [f(a) - f(x)][g(b) - g(x)] \cdot e^x$ is continuous and differentiable.

Sol. Let $h(x) = [f(a) - f(x)][g(b) - g(x)] \cdot e^x$
 $h(a) = h(b) = 0$, then there exists $c \in (a, b)$ in such a way that $h'(c) = 0$.

$h'(x) = f'(x)[g(b) - g(x)]e^x - g'(x)[f(a) - f(x)]e^x + [f(a) - f(x)][g(b) - g(x)]e^x = 0$

or $\frac{f'(x)}{f(a) - f(x)} + \frac{g'(x)}{g(b) - g(x)} = 1$

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 43-46) Among various properties of continuous, we have if f is continuous function on $[a, b]$ and $f(a)f(b) < 0$, then there exists a point c in (a, b) such that $f(x) = 0$ equivalently, if f is continuous on $[a, b]$ and $x \in R$ is such that $f(a) < x < f(b)$, then there is $c \in (a, b)$ such that $x = f(c)$. It follows from the above result that the image of a

closed interval under a continuous function is a closed interval.

Ex 43. The number of continuous functions on R which satisfy $(f(x))^2 - x^2$ for all $x \in R$, is

- (a) 1 (b) 2
 (c) 4 (d) 8

Sol. Clearly, $f(0) = 0$, let $I = (0, \infty)$. Since, f is continuous on I and $f(x) \neq 0$ for any $x \in I$, so $f(x) = x, \forall x \in I$ or $f(x) = -x, \forall x \in I$. Similarly, if $J = (-\infty, 0)$ and $f(x) = x, \forall x \in J$ and $f(x) = -x$. Combining these possibilities, we have $f(x) = x, \forall x \in \mathbb{R}$; $f(x) = -x, \forall x \in \mathbb{R}$; $f(x) = |x|$; $f(x) = -|x|$. Hence, (c) is the correct answer.

Ex 44. Suppose that $f(1/2) = 1$ and f is continuous on $[0, 1]$ assuming only rational value in the entire interval. The number of such function is
(a) infinite
(b) 2
(c) 4
(d) 1

Sol. We claim that, $f(x) = 1, \forall x \in [0, 1]$. Let if possible there be $c \in [0, 1]$ such that $f(c) \neq 1$, so $f(c) < 1$ or $f(c) > 0$. Suppose that $c < 1/2$. Consider the interval $[c, 1/2]$, by above result (comprehension) f takes all values between $f(c)$ and 1 but f takes only rational values. This contradiction shows there is no $c \in [0, 1]$ such that $f(c) \neq 1$. Hence, (d) is the correct answer.

Ex 45. Let f be a continuous function on $[-1, 1]$ satisfying $(f(x))^2 + x^2 = 1, \forall x \in [-1, 1]$. The number of such functions is
(a) 2
(b) 1
(c) 4
(d) infinitely many

Sol. $f(x) = \sqrt{1-x^2}$ or $f(x) = -\sqrt{1-x^2}$ satisfy the given condition. Apply the result (comprehension) to show that these are the only possibilities. Hence, (a) is the correct answer.

Ex 46. Let $f(x) = x, x \neq 0$ and $f(0) = 1$. Then,
(a) f is a continuous function
(b) range of f is an interval
(c) range of f is $\mathbb{R}$
(d) hypothesis of the result in the comprehension is violated

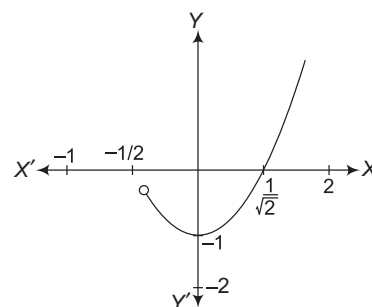
Sol. f is not continuous at $x = 0$. Range of f is not an interval for $1, -1 \in \text{Range } f$ but $0 \in \text{Range } f$. Hence, (d) is the correct answer.

Passage II (Ex. Nos. 47-49) Let a function be defined as $f(x) = \begin{cases} [x], & -2 \leq x \leq \frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \leq 2 \end{cases}$,

where $[]$ denotes the greatest integer function.

Ex 47. The number of points of discontinuity of $f(x)$ is
(a) 1
(b) 2
(c) 3
(d) None of the above

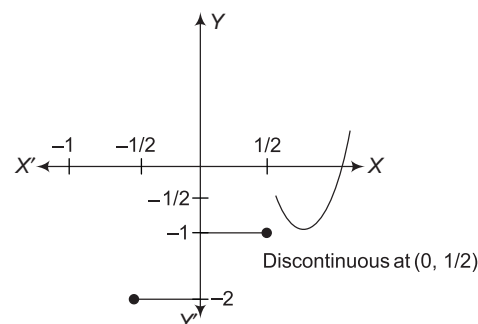
Sol.



Two points of discontinuity are $-1, -1/2$. Hence, (b) is the correct answer.

Ex 48. The function $f(x-1)$ is discontinuous at the points
(a) $-1, -\frac{1}{2}$ (b) $-\frac{1}{2}, 1$ (c) $0, \frac{1}{2}$ (d) $0, 1$

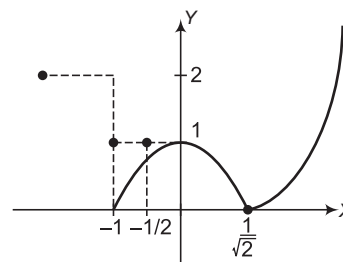
Sol.



Given function is discontinuous at $0, 1/2$. Hence, (c) is the correct answer.

Ex 49. Number of points, where $|f(x)|$ is not differentiable, is
(a) 1 (b) 2 (c) 3 (d) 4

Sol.



At $-1, -1/2, 1/\sqrt{2}$, the function is not differentiable. Hence, (c) is the correct answer.

Passage III (Ex. Nos. 50-52) Left hand derivative and right hand derivative of a function $f(x)$ at a point $x = a$ are defined as

$$f'(a^-) = \lim_{h \rightarrow 0^+} \frac{f(a) - f(a-h)}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h}$$

and

$$f'(a^+) = \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{f(a) - f(a-h)}{h}$$

$$= \lim_{x \rightarrow a^+} \frac{f(a) - f(x)}{a-x} \text{ respectively.}$$

Let f be a twice differentiable function.

Ex 50. If f is odd, then which of the following is left hand derivative of f at $x = -a$?

- (a) $\lim_{h \rightarrow 0^-} \frac{f(a-h) - f(a)}{-h}$
- (b) $\lim_{h \rightarrow 0^-} \frac{f(h-a) - f(a)}{-h}$
- (c) $\lim_{h \rightarrow 0^+} \frac{f(a) + f(a+h)}{-h}$
- (d) $\lim_{h \rightarrow 0^-} \frac{f(-a) - f(-a-h)}{-h}$

Sol. LHD = $\lim_{h \rightarrow 0^-} \frac{f(-a+h) - f(-a)}{h}$

$$= \lim_{h \rightarrow 0^-} \frac{-f(a-h) + f(a)}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{f(a-h) - f(a)}{-h}$$

Hence, (a) is the correct answer.

Ex 51. If f is even, then which of the following is right hand derivative of f' at $x = a$?

- (a) $\lim_{h \rightarrow 0^-} \frac{f'(a) + f'(-a+h)}{h}$
- (b) $\lim_{h \rightarrow 0^+} \frac{f'(a) + f'(-a-h)}{h}$

(c) $\lim_{h \rightarrow 0^-} \frac{-f'(-a) + f'(-a-h)}{-h}$

(d) $\lim_{h \rightarrow 0^+} \frac{f'(a) + f'(-a+h)}{-h}$

Sol. If f is even, then $f'(-x) = -f'(x)$

$$\therefore f''(a^+) = \lim_{h \rightarrow 0^-} \frac{f'(a-h) - f'(a)}{-h}$$

$$= \lim_{h \rightarrow 0^-} \frac{f'(a) - f'(a-h)}{h}$$

$$= \lim_{h \rightarrow 0^-} \frac{f'(a) + f'(h-a)}{h}$$

Hence, (a) is the correct answer.

Ex 52. The statement $\lim_{h \rightarrow 0} \frac{f(-x) - f(-x-h)}{h}$

$$= \lim_{h \rightarrow 0} \frac{f(x) - f(x-h)}{-h}$$
 implies that

- (a) f is odd
- (b) f is even
- (c) f is neither odd nor even
- (d) nothing can be concluded

Sol. $\lim_{h \rightarrow 0} \frac{f(-x) - f(-x-h)}{h} = f' \{-x\}$

and $\lim_{h \rightarrow 0} \frac{f(x) - f(x-h)}{-h} = -f'(x)$

$\Rightarrow f'(-x) = -f'(x)$

$\therefore f'(x)$ is an odd function.

Thus, f is an even function.

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 53. Match the following:

Column I	Column II
A. $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \sqrt{\frac{n+r}{n-r}}$	p. $\frac{\pi}{2}$
B. $\lim_{x \rightarrow \infty} \left[\frac{1}{\sqrt{n^2-1}} + \frac{1}{\sqrt{n^2-2^2}} + \dots + \frac{1}{\sqrt{n^2-(n-1)^2}} \right]$	q. $\frac{\pi}{2} + 1$
C. $\lim_{x \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{1/n}$	r. e
D. In order that the function $f(x) = (x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as	s. $\frac{1}{e}$

Sol. A. It can be reduced to

$$\int_0^1 \sqrt{\frac{1+x}{1-x}} dx = \int_0^1 \frac{1+x}{\sqrt{1-x^2}} dx$$

$$= [\sin^{-1} x - \sqrt{1-x^2}]_0^1$$

$$= \frac{\pi}{2} - [-1] = \frac{\pi}{2} + 1$$

B. $\lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{\sqrt{1-\frac{1}{n^2}}} + \frac{1}{\sqrt{1-\left(\frac{2}{n}\right)^2}} + \dots + \frac{1}{\sqrt{1-\left(\frac{n-1}{n}\right)^2}} \right]$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{n-1} \frac{1}{\sqrt{1-\left(\frac{r}{n}\right)^2}}$$

Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx also, when $r=1$,

$x = \frac{1}{n} = 0$, when $r = n-1$

$\Rightarrow \int_0^1 \frac{1}{\sqrt{1-x^2}} dx = (\sin^{-1} x)_0^1$

$= \sin^{-1} 1 - \sin^{-1} 0 = \frac{\pi}{2}$

C. $A = \lim_{n \rightarrow \infty} \left[\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \dots \frac{n}{n} \right]^{1/n}$

$\log A = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \frac{1}{n} + \log \frac{2}{n} + \log \frac{3}{n} + \dots + \log 1 \right]$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left(\frac{r}{n} \right)$$

Put $\frac{r}{n} = x$, $\frac{1}{n} = dx$ and limits 0 to 1.

$$\begin{aligned} \log A &= \int_0^1 \log x \, dx \\ &= [x \log x]_0^1 - \int_0^1 x \cdot \frac{1}{x} \, dx \end{aligned}$$

$$= [x \log x - x]_0^1 = -1$$

$$\Rightarrow A = e^{-1}$$

$$D. \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} [(1+x)^{1/x}]^{x/\tan x} = e^1$$

$$\text{So, } f(0) = e$$

$$A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow r$$

Type 6. Single Integer Answer Type Questions

Ex 54. A function $f(x)$ is defined as

$$f(x) = \begin{cases} x^m \sin \frac{1}{x}, & \text{if } x \neq 0, m \in N \\ 0, & \text{if } x = 0 \end{cases}$$

The least value of m for which $f'(x)$ is continuous at $x=0$ is _____.

$$\text{Sol. (3)} f'(0^+) = \lim_{h \rightarrow 0} \frac{h^m \sin \frac{1}{h}}{h} \text{ must exist } \Rightarrow m > 1$$

For $m > 1$,

$$f'(x) = \begin{cases} mx^{m-1} \sin \frac{1}{x} - x^{m-2} \cos \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\text{Now, } \lim_{h \rightarrow 0} h(x) = \lim_{h \rightarrow 0} mh^{m-1} \sin \frac{1}{h} - h^{m-2} \cos \frac{1}{h}$$

limit exists if $m > 2$.

$$\therefore m \in N \Rightarrow m = 3$$

Ex 55. If $f(x)$ is a differentiable function and $f'(2)=6$, $f'(1)=4$, $f'(x)$ represents the differentiation of $f(x)$ w.r.t. x , then

$$\lim_{h \rightarrow 0} \frac{f(2+2h+h^2) - f(2)}{f(1+h^2+h) - f(1)} \text{ is equal to } \underline{\hspace{2cm}}.$$

$$\begin{aligned} \text{Sol. (3)} \lim_{h \rightarrow 0} \frac{f(2+2h+h^2) - f(2)}{(2h+h^2)} \\ &= \lim_{h \rightarrow 0} \frac{(h^2+h)}{(1+h^2+h) - f(1)} \cdot \lim_{h \rightarrow 0} \frac{2h+h^2}{h^2+h} \\ &= \frac{f'(2)}{f'(1)} \cdot 2 = \frac{6 \times 2}{4} = 3 \quad [\text{using L' Hospital's rule}] \end{aligned}$$

Ex 56. The graph of function f contains the point

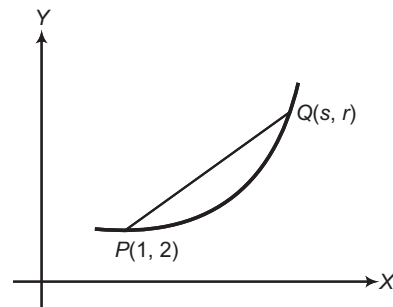
$P(1, 2)$ and $Q(s, r)$. The equation of the secant line through P and Q is

$$Y = \left(\frac{s^2 + 2s - 3}{s - 1} \right) x - 1 - s. \text{ The value of } f'(1)$$

is _____.

Sol. (4) By definition, $f'(1)$ is the limit of the slope of the secant line when $s \rightarrow 1$.

$$\begin{aligned} \text{Thus, } f'(1) &= \lim_{s \rightarrow 1} \frac{s^2 + 2s - 3}{s - 1} \\ &= \lim_{s \rightarrow 1} \frac{(s-1)(s+3)}{s-1} \\ &= \lim_{s \rightarrow 1} (s+3) = 4 \end{aligned}$$



Aliter

By substituting $x = s$ into the equation of the secant line and cancelling by $s-1$ again, we get $y = s^2 + 2s - 1$. This is $f(s)$ and its derivatives is $f'(s) = 2s + 2$, so $f'(1) = 4$.

Target Exercises

Type 1. Only One Correct Option

1. If $f(x) = \frac{x - e^x + \cos 2x}{x^2}$, $x \neq 0$ is continuous at

$x = 0$, where $[x]$ and $\{x\}$ denote the greatest integer and fractional part functions respectively, then

- (a) $f(0) = \frac{5}{2}$
 (b) $[f(0)] = -2$
 (c) $\{f(0)\} = -0.5$
 (d) $[f(0)]\{f(0)\} = -1.5$

2. Which of the following functions have finite number of points of discontinuity in R , where $[]$ represents the greatest integer function?

- (a) $\tan x$ (b) $x[x]$
 (c) $\frac{|x|}{x}$ (d) $\sin[n\pi x]$

3. If $f(x) = \frac{\tan\left(\frac{\pi - x}{4}\right)}{\cot 2x}$, $x \neq \frac{\pi}{4}$, is continuous at $x = \frac{\pi}{4}$,

then the value of $f\left(\frac{\pi}{4}\right)$ is

- (a) 1 (b) $\frac{1}{2}$
 (c) $\frac{1}{3}$ (d) -1

4. If $f(x) = \begin{cases} \frac{1 - |x|}{1 + x}, & x \neq -1 \\ 1, & x = -1 \end{cases}$, then $f([2x])$, where $[]$

represents the greatest integer function, is

- (a) discontinuous at $x = -1$
 (b) continuous at $x = 0$
 (c) continuous at $x = \frac{1}{2}$
 (d) continuous at $x = 1$

5. The function $f(x) = |2\operatorname{sgn} 2x| + 2$ has

- (a) non-removable discontinuity
 (b) removable discontinuity
 (c) infinite discontinuity
 (d) no discontinuity at $x = 0$

6. The function $f(x) = \{x\}\sin(\pi[x])$, where $[]$ denotes the greatest integer function and $\{ \}$ is the fractional part function, is discontinuous at

- (a) all x
 (b) all integer points
 (c) no x
 (d) x , which is not an integer

7. Given that, $\prod_{n=1}^n \cos \frac{x}{2^n} = \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)}$ and

$$f(x) = \begin{cases} \lim_{n \rightarrow \infty} \sum_{n=1}^n \frac{1}{2^n} \tan\left(\frac{x}{2^n}\right), & x \in (0, \pi) - \left\{\frac{\pi}{2}\right\} \\ \frac{2}{\pi}, & x = \frac{\pi}{2} \end{cases}$$

Then, which one of the following is true?

- (a) $f(x)$ has non-removable discontinuity of finite type at $x = \frac{\pi}{2}$
 (b) $f(x)$ has removable discontinuity at $x = \frac{\pi}{2}$
 (c) $f(x)$ is continuous at $x = \frac{\pi}{2}$
 (d) $f(x)$ has non-removable discontinuity of infinite type at $x = \frac{\pi}{2}$

8. The points of discontinuity of $\tan x$ are

- (a) $n\pi$, $n \in I$ (b) $2n\pi$, $n \in I$
 (c) $(2n + 1)\frac{\pi}{2}$, $n \in I$ (d) None of these

9. The set of point(s) of discontinuity of the function $\frac{1}{\log |x|}$ is

- (a) $\{-1, 0, 1\}$
 (b) $\{0\}$
 (c) $\{0, 1\}$
 (d) None of the above

10. The point(s) of discontinuity of the function

$$f(x) = \begin{cases} \frac{1}{5}(2x^2 + 3), & x \leq 1 \\ 6 - 5x, & 1 < x < 3 \\ x - 3, & x \geq 3 \end{cases}$$

is/are

- (a) $x = 1$ (b) $x = 3$
 (c) $x = 1, 3$ (d) None of these

11. For the function $f(x) = \begin{cases} x - 1, & x < 0 \\ \frac{1}{4}, & x = 0 \\ x^2, & x > 0 \end{cases}$, $\lim_{x \rightarrow 0^+} f(x)$

and $\lim_{x \rightarrow 0^-} f(x)$ are

- (a) 0, 1 (b) 0, -1
 (c) 1, -1 (d) None of these

12. If f is a periodic function, then

- (a) f' and f'' are also periodic
(b) f' is periodic but f'' is not periodic
(c) f'' is periodic but f' is not periodic
(d) None of the above

13. If $f(x) = \begin{cases} \frac{\sin(\cos x) - \cos x}{(\pi - 2x)^2}, & x \neq \frac{\pi}{2} \\ k, & x = \frac{\pi}{2} \end{cases}$ is continuous

at $x = \frac{\pi}{2}$, then k is equal to

- (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) -1

14. Let $f(x) = \begin{cases} \frac{x^4 - 5x^2 + 4}{|(x-1)(x-2)|}, & x \neq 1, 2 \\ 6, & x = 1 \\ 12, & x = 2 \end{cases}$. Then, $f(x)$

is continuous on the set

- (a) $R - \{2\}$
(b) $R - \{1, 2\}$
(c) R
(d) $R - \{1\}$

15. If $\lim_{x \rightarrow a^+} f(x) = l = \lim_{x \rightarrow a^-} g(x)$ and $\lim_{x \rightarrow a^-} f(x) = m = \lim_{x \rightarrow a^+} g(x)$, then the function $f(x) \cdot g(x)$

- (a) is not continuous at $x = a$
(b) has a limit when $x \rightarrow a$ and it is equal to lm
(c) is continuous at $x = a$
(d) has a limit when $x \rightarrow a$ but it is not equal to lm

16. The function $f(x) = \begin{cases} (x+1)^{2 - \left(\frac{1}{|x|} + \frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

- (a) discontinuous at only one point
(b) discontinuous exactly at two points
(c) continuous everywhere
(d) None of the above

17. If $f(x) = \begin{cases} \frac{1 - \sin^2 x}{3 \cos^2 x}, & x < \frac{\pi}{2} \\ a, & x = \frac{\pi}{2} \\ \frac{b(1 - \sin x)}{(\pi - 2x)^2}, & x > \frac{\pi}{2} \end{cases}$. Then, $f(x)$ is

continuous at $x = \frac{\pi}{2}$, if

- (a) $a = \frac{1}{3}, b = 2$
(b) $a = \frac{1}{3}, b = \frac{8}{3}$
(c) $a = \frac{2}{3}, b = \frac{8}{3}$
(d) None of the above

18. If $f(x) = \begin{cases} \frac{e^{[x]+x} - 1}{[x] + x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$, then

- (a) $\lim_{x \rightarrow 0^+} f(x) = 1$
(b) $\lim_{x \rightarrow 0^-} f(x) = \frac{1}{e} - 1$
(c) $f(x)$ is continuous at $x = 0$
(d) None of the above

19. The function $f(x) = \arctan \frac{1}{x-5}$ has

- (a) discontinuity of the first kind at $x = 5$
(b) discontinuity of the second kind at $x = 5$
(c) removable discontinuity at $x = 5$
(d) continuous at $x = 5$

20. If $f(x) = \lim_{n \rightarrow \infty} \frac{x}{(2 \sin x)^{2n} + 1}$ for $x \in R$, then $f(x)$ is

continuous in

- (a) $(-\infty, \infty)$
(b) $(-\infty, \infty) - \left\{x : x = n\pi \pm \frac{\pi}{6}, n \in I\right\}$
(c) $(-\infty, \infty) - \left\{x : x = n\pi \pm \frac{\pi}{3}, n \in I\right\}$
(d) None of the above

21. If $f(x) = \left[\frac{1}{\sqrt{2}} (\cos x + \sin x) \right], 0 < x < 2\pi$, where $[]$

denotes the greatest integer function, then the number of points of discontinuity of $f(x)$ is

- (a) 5
(b) 4
(c) 3
(d) None of the above

22. If $f(x) = \lim_{n \rightarrow \infty} \frac{(1 + \cos \pi x)^n + 1}{(1 + \cos \pi x)^n - 1}$, then

- (a) $f(1+0) = 1$
(b) $f(1-0) = 2$
(c) $f(x)$ is continuous at $x = 1$
(d) $f(x)$ is not continuous at $x = 1$

23. Let f be a function defined and continuous on $[2, 5]$. If $f(x)$ takes rational values for all x and $f(4) = 8$, then the value of $f(3, 7)$ is

- (a) 0
(b) 8
(c) -1
(d) None of the above

24. If $f(x) = [x] \sin \left(\frac{\pi}{[x+1]} \right)$, where $[]$ denotes the

greatest integer function, then the points of discontinuity of f in the domain are

- (a) I
(b) $I - \{0\}$
(c) $R - [-1, 0)$
(d) None of the above

25. If $f(x) = \frac{1}{[\sin x]}$, where $[]$ denotes the greatest function, then

- (a) domain of $f(x)$ is $(2n\pi + \pi, 2n\pi + 2\pi) \cup \left\{2n\pi + \frac{\pi}{2}\right\}$, where $n \in I$
 (b) $f(x)$ is not continuous when $x \in (2n\pi + \pi, 2n\pi + 2\pi)$
 (c) $f(x)$ is differentiable at $x = \frac{\pi}{2}$
 (d) All of the above

26. The function $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[]$ denotes the greatest integer function, is discontinuous at

- (a) all x
 (b) all integer points
 (c) no x
 (d) x which is not an integer

27. Let a function $f: R \rightarrow R$ satisfy the equation $f(x+y) = f(x) + f(y)$ for all x, y . If the function $f(x)$ is continuous at $x = 0$, then

- (a) $f(x)$ is discontinuous for all x
 (b) $f(x)$ is continuous for all positive real x
 (c) $f(x)$ is continuous for all x
 (d) None of the above

28. Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and $f \circ g = I$ (identity function). Then, $f'(b)$ is equal to

- (a) 2
 (b) $\frac{2}{3}$
 (c) $\frac{1}{2}$
 (d) None of these

29. If $f(x) = \begin{cases} \frac{1}{e^{1/x} + 1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is

- (a) continuous as well as differentiable at $x = 0$
 (b) continuous but not differentiable at $x = 0$
 (c) differentiable but not continuous at $x = 0$
 (d) None of the above

30. The function $f(x) = \begin{cases} \frac{x(e^{1/x} - e^{-1/x})}{e^{1/x} + e^{-1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is

- (a) continuous everywhere but not differentiable at $x = 0$
 (b) continuous and differentiable everywhere
 (c) not continuous at $x = 0$
 (d) None of the above

31. If $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\log a)^n$, then at $x = 0$, $f(x)$

- (a) has no limit
 (b) is discontinuous
 (c) is continuous but not differentiable
 (d) is differentiable

32. If $f(x) = x + \frac{x}{1+x} + \frac{x}{(1+x)^2} + \dots$, then at $x = 0$,

- $f(x)$
 (a) has no limit
 (b) is discontinuous
 (c) is continuous but not differentiable
 (d) is differentiable

33. The function

$$f(x) = \begin{cases} 1 - 2x + 3x^2 - 4x^3 + \dots, & x \neq -1 \\ 1, & x = -1 \end{cases}$$

- (a) continuous and derivable at $x = -1$
 (b) neither continuous nor derivable at $x = -1$
 (c) continuous but not derivable at $x = -1$
 (d) None of the above

34. If $f(x) = [\tan x]$, $x \in \left(0, \frac{\pi}{3}\right)$, then $f'\left(\frac{\pi}{4}\right)$ is equal to

- (a) 1
 (b) 0
 (c) Does not exist
 (d) None of the above

35. If $f(x) = \begin{cases} x^p \cos \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then at $x = 0$, $f(x)$ is

- (a) continuous, if $p > 0$ and differentiable, if $p > 1$
 (b) continuous, if $p > 1$ and differentiable, if $p > 0$
 (c) continuous and differentiable, if $p > 0$
 (d) None of the above

36. If $\operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then the function

- $f(x) = \operatorname{sgn}(\operatorname{sgn}(x))$ is
 (a) continuous as well as differentiable at $x = 0$
 (b) continuous but not differentiable at $x = 0$
 (c) differentiable but not continuous at $x = 0$
 (d) neither differentiable nor continuous at $x = 0$

37. If $f(x) = |x - 2|$ and $g(x) = f[f(x)]$, then $g'(x)$ for $x > 2$ is

- (a) 1
 (b) 2
 (c) -1
 (d) None of the above

38. Let $f(x) = a[x] + be^{|x|} + c|x|^2$, where a, b and c are real constants. If $f(x)$ is differentiable at $x = 0$, then

- (a) $b = 0, c = 0, a \in R$
 (b) $a = 0, c = 0, b \in R$
 (c) $a = 0, b = 0, c \in R$
 (d) None of the above

39. If $f(x) = \begin{cases} \frac{[x]-1}{x-1}, & x \neq 1 \\ 0, & x = 1 \end{cases}$, then $f(x)$ is

- (a) continuous as well as differentiable at $x = 1$
 (b) differentiable but not continuous at $x = 1$
 (c) continuous but not differentiable at $x = 1$
 (d) neither continuous nor differentiable at $x = 1$

40. If $4x + 3|y| = 5y$, then y as a function of x is
- (a) differentiable at $x = 0$
 - (b) continuous at $x = 0$
 - (c) $\frac{dy}{dx} = 2$ for all x
 - (d) None of the above
41. If $f(x) = |3 - x| + (3 + x)$, where (x) denotes the least integer greater than or equal to x , then
- (a) $f(x)$ is continuous as well as differentiable at $x = 3$
 - (b) $f(x)$ is continuous but not differentiable at $x = 3$
 - (c) $f(x)$ is differentiable but not continuous at $x = 3$
 - (d) $f(x)$ is neither differentiable nor continuous at $x = 3$
42. The function $f(x) = (1 + x)^{\cot x}$ is not defined at $x = 0$. The value of $f(0)$ so that $f(x)$ becomes continuous at $x = 0$, is
- (a) 1
 - (b) 0
 - (c) e
 - (d) None of the above
43. If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is
- (a) continuous as well as differentiable for all x
 - (b) continuous for all x but not differentiable at $x = 0$
 - (c) differentiable for all x but not continuous at $x = 0$
 - (d) None of the above
44. If $f(x) = \begin{cases} x(3e^{1/x} + 4), & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is
- (a) continuous as well as differentiable at $x = 0$
 - (b) continuous but not differentiable at $x = 0$
 - (c) differentiable but not continuous at $x = 0$
 - (d) None of the above
45. If the function f is defined by $y = f(x)$, where $x = 2t - |t|$, $y = t^2 + t|t|$, $t \in R$, then
- (a) $f(x)$ is differentiable as well as continuous at $x = 0$
 - (b) $f(x)$ is continuous but not differentiable at $x = 0$
 - (c) $f(x)$ is differentiable but not continuous at $x = 0$
 - (d) None of the above
46. If $f(x) = \begin{cases} 3^x, & -1 \leq x \leq 1 \\ 4 - x, & 1 < x \leq 4 \end{cases}$, then $f(x)$ is
- (a) continuous as well as differentiable at $x = 1$
 - (b) continuous but not differentiable at $x = 1$
 - (c) differentiable but not continuous at $x = 1$
 - (d) None of the above
47. If $f(x) = \begin{cases} x^3, & x^2 < 1 \\ x, & x^2 \geq 1 \end{cases}$, then $f(x)$ is
- (a) continuous as well as differentiable at $x = 1$
 - (b) differentiable but not continuous at $x = 1$
 - (c) continuous but not differentiable at $x = 1$
 - (d) None of the above
48. If $x + 4|y| = 6y$, then y as a function of x is
- (a) not continuous at $x = 0$
 - (b) not defined for all real x
 - (c) $\frac{dy}{dx} = \frac{1}{2}$ for $x > 0$
 - (d) derivable at $x = 0$
49. If $f(x) = [\tan^2 x]$, where $[]$ denotes the greatest integer function, then
- (a) $\lim_{x \rightarrow 0} f(x)$ does not exist
 - (b) $f(x)$ is continuous at $x = 0$
 - (c) $f(x)$ is non-differentiable at $x = 0$
 - (d) $f(0) = 1$
50. The function $f(x) = [x^2] + [-x]^2$, where $[]$ denotes the greatest integer function, is
- (a) continuous and derivable at $x = 0$
 - (b) neither continuous nor derivable at $x = 2$
 - (c) continuous but not derivable at $x = 2$
 - (d) None of the above
51. Let $f(x) = \frac{1}{[\sin x]}$, where $[]$ denotes the greatest integer function, then
- (a) $f(x)$ is not continuous, where $x \in (2n\pi, 2n\pi + \pi)$, $n \in I$
 - (b) $f(x)$ is differentiable at $x = \frac{\pi}{4}$
 - (c) $f(x)$ is differentiable at $x = \frac{\pi}{2}$
 - (d) None of the above
52. The points, where the function $f(x) = [x] + |1 - x|$, $1 \leq x \leq 3$, (where $[]$ denotes the greatest integer function) is not differentiable, are
- (a) $x = -1, 0, 1, 2, 3$
 - (b) $x = -1, 0, 2$
 - (c) $x = 1, 2, 3$
 - (d) $x = -1, 0, 1, 2$
53. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in I$, p is prime number and $[x]$ denotes the greatest integer less than or equal to x . Then, the number of points at which $f(x)$ is not differentiable, is
- (a) $p - 1$
 - (b) p
 - (c) $2p + 1$
 - (d) $2p - 1$
54. If $f(x) = -(-1)^{[x^3]}$, where $[]$ denotes the greatest integer function, then
- (a) $f(x)$ is discontinuous for $x = n^{1/3}$, $n \in I$
 - (b) $f\left(\frac{3}{2}\right) = 1$
 - (c) $f'(x) = 0$ for $-1 < x < 1$
 - (d) None of the above
55. If $f(x) = \cos^{-1}\left(\frac{2x}{1+x^2}\right)$, then $f(x)$ is differentiable on
- (a) $(-\infty, \infty)$
 - (b) $(-\infty, \infty) - \{0\}$
 - (c) $(-\infty, \infty) - \{-1, 1\}$
 - (d) None of these

- 56.** Let $f(x)$ be a function, satisfying the condition $f(-x) = f(x)$, for all real x . If $f'(0)$ exists, then its value is
(a) 0 (b) 1
(c) -1 (d) None of these
- 57.** Let $f: R \rightarrow R$ be a function such that $f\left(\frac{x+y}{3}\right) = \frac{f(x)+f(y)}{3}$, $f(0) = 0$ and $f'(0) = 3$. Then,
(a) $f(x)$ is a quadratic function
(b) $f(x)$ is continuous but not differentiable
(c) $f(x)$ is differentiable in R
(d) $f(x)$ is bounded in R
- 58.** If f is differentiable function satisfying $f(0) = 0$ and $g(x) = \frac{f(x)}{x}$, then the value that should be assigned to $g(0)$, so that g is continuous at 0 is
(a) 1 (b) 0
(c) $f(0)$ (d) $f'(0)$
- 59.** If $f(x+y) = 2f(x) \cdot f(y)$ for all x, y , where $f'(0) = 3$ and $f(4) = 2$, then $f'(4)$ is equal to
(a) 6 (b) 12
(c) 4 (d) None of these
- 60.** If a function $f: R \rightarrow R$ be such that $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$, where $f(x) = 1 + x\phi(x)$ and $\lim_{x \rightarrow 0} \phi(x) = 1$, then
(a) $f(x)$ does not exist
(b) $f'(x) = 2f(x)$ for all x
(c) $f'(x) = f(x)$ for all x
(d) None of these
- 61.** Let $f(x)$ be a function satisfying $f(x+y) = f(x)f(y)$ for all $x, y \in R$. If $f(x) = 1 + x\phi(x) + x^2\psi(x)$, where $\lim_{x \rightarrow 0} \phi(x) = a$ and $\lim_{x \rightarrow 0} \psi(x) = b$, then $f'(x)$ is equal to
(a) $(a+b)f(x)$
(b) $af(x)$
(c) $bf(x)$
(d) None of these
- 62.** Let f be a function satisfying $f(x+y) = f(x) + f(y)$ and $f(x) = x^3\phi(x)$ for all x and y , where $\phi(x)$ is a continuous function, then $f'(x)$ is equal to
(a) $g(0)$
(b) $g'(x)$
(c) 0
(d) None of the above
- 63.** Let $f(x+y) = f(x) \cdot f(y)$ for all $x, y \in R$ and $f(x) = 1 + x\phi(x)\log 2$, where $\lim_{x \rightarrow 0} \phi(x) = 1$. Then, $f'(x)$ is equal to
(a) $\log 2^{f(x)}$ (b) $\log[f(x)]^2$
(c) $\log 2$ (d) None of these
- 64.** Let $f(x)$ be a real function not identically zero in Z , such that $f(x+y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$, $n \in N$ and $x, y \in R$. If $f'(0) \geq 0$, then $f'(6)$ is equal to
(a) 0
(b) 1
(c) -1
(d) None of the above
- 65.** Let $f(x+y) = f(x) \cdot f(y)$ and $f(x) = 1 + xg(x)G(x)$, where $\lim_{x \rightarrow 0} g(x) = a$ and $\lim_{x \rightarrow 0} G(x) = b$. Then, $f'(x) = kf(x)$, where k is equal to
(a) $\frac{a}{b}$
(b) $1 + ab$
(c) ab
(d) None of the above
- 66.** If $f(x) = [\log_e x] + \sqrt{\{\log_e x\}}$, $x > 1$, where $[]$ and $\{ \}$ denote the greatest integer function and the fractional part function respectively, then
(a) $f(x)$ is continuous but not differentiable at $x = e$
(b) $f(x)$ is differentiable at $x = e$
(c) $f(x)$ is discontinuous at $x = e$
(d) None of the above
- 67.** If $f(2+x) = f(-x)$ for all $x \in R$, then differentiability at $x = 4$ implies differentiability at
(a) $x = 1$
(b) $x = -1$
(c) $x = -2$
(d) Cannot say anything
- 68.** The number of points of non-differentiability for $f(x) = \max\left\{|x| - 1, \frac{1}{2}\right\}$, is
(a) 4
(b) 3
(c) 2
(d) 5
- 69.** If $f(x) = x^3 \operatorname{sgn} x$, then
(a) f is derivable at $x = 0$
(b) f is continuous but not derivable at $x = 0$
(c) LHD at $x = 0$ is 1
(d) RHD at $x = 0$ is 1
- 70.** If $f(x) = \begin{cases} |1 - 4x^2|, & 0 \leq x < 1 \\ [x^2 - 2x], & 1 \leq x < 2 \end{cases}$, where $[]$ denotes the greatest integer function, then $f(x)$ is
(a) differentiable for all x
(b) continuous at $x = 1$
(c) $f(x)$ is not differentiable at $x = 1$
(d) None of the above
- 71.** If $f(x) = \begin{cases} x - 1, & x < 0 \\ x^2 - 2x, & x \geq 0 \end{cases}$, then
(a) $f(|x|)$ is discontinuous at $x = 0$
(b) $f(|x|)$ is differentiable at $x = 0$
(c) $|f(x)|$ is not differentiable at $x = 0, 2$
(d) $|f(x)|$ is continuous at $x = 0$

Type 2. More than One Correct Option

72. A function is defined as $f(x) = \begin{cases} e^x, & x \leq 0 \\ |x-1|, & x > 0 \end{cases}$

then $f(x)$ is

- (a) continuous at $x = 0$
- (b) continuous at $x = 1$
- (c) differentiable at $x = 0$
- (d) differentiable at $x = 1$

73. Let $f(x) = \int_{-2}^x |t+1| dt$, then

- (a) $f(x)$ is continuous in $[-1, 1]$
- (b) $f(x)$ is differentiable in $[-1, 1]$
- (c) $f'(x)$ is continuous in $[-1, 1]$
- (d) $f'(x)$ is differentiable in $[-1, 1]$

74. $f(x) = \begin{cases} x+a, & x \geq 0 \\ 2-x, & x < 0 \end{cases}$ and $g(x) = \begin{cases} \{x\}, & x < 0 \\ \sin x + b, & x \geq 0 \end{cases}$

and if $f(g(x))$ is continuous at $x = 0$, then which of the following is/are true (where $\{x\}$ represents the fractional part function)?

- (a) If $b = 1$, then a can take any real value
- (b) If $b < -1$, then $a + b = 1$
- (c) No values of a and b are possible
- (d) There exist finite ordered pairs (a, b)

75. The function defined as,

$$f(x) = \lim_{n \rightarrow \infty} \begin{cases} (\cos^2 x)^n & \text{if } x < 0 \\ \sqrt[n]{1+x^n} & \text{if } 0 \leq x \leq 1 \\ \frac{1}{1+x^n} & \text{if } x > 1 \end{cases}$$

following does not hold good?

- (a) continuous at $x = 0$ but discontinuous at $x = 1$
- (b) continuous at $x = 1$ but discontinuous at $x = 0$
- (c) continuous both at $x = 1$ and $x = 0$
- (d) discontinuous both $x = 1$ and $x = 0$

76. The function

$$f(x) = \begin{cases} 1, & |x| \geq 1 \\ \frac{1}{n^2}, & \frac{1}{n} < |x| < \frac{1}{n-1}, n=2, 3, \dots \\ 0, & x = 0 \end{cases}$$

- (a) is discontinuous at infinite points
- (b) is continuous everywhere
- (c) is discontinuous only at $x = \frac{1}{n}, n \in \mathbb{Z} - \{0\}$
- (d) None of the above

77. If $f(x) = \begin{cases} x^2(\operatorname{sgn}[x]) + \{x\}, & 0 \leq x < 2 \\ \sin x + |x-3|, & 2 \leq x < 4 \end{cases}$ where $[]$

and $\{ \}$ represent the greatest integer and the fractional part function, respectively.

- (a) $f(x)$ is differentiable at $x = 1$
- (b) $f(x)$ is continuous but not differentiable at $x = 1$
- (c) $f(x)$ is not differentiable at $x = 2$
- (d) $f(x)$ is discontinuous at $x = 2$

78. A function $f(x)$ satisfies the relation

$$f(x+y) = f(x) + f(y) + xy(x+y), \forall x, y \in \mathbb{R}. \quad \text{If } f'(0) = 1, \text{ then}$$

- (a) $f(x)$ is a polynomial function
- (b) $f(x)$ is an exponential function
- (c) $f(x)$ is twice differentiable for all $x \in \mathbb{R}$
- (d) $f'(3) = 8$

79. If $f(x) = \begin{cases} 3x^2 + 12x + 1, & -1 \leq x < 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$, then

- (a) $f(x)$ is increasing in $[-1, 2]$
- (b) $f(x)$ is continuous on $[-1, 3]$
- (c) $f'(2)$ does not exist
- (d) $f(x)$ has the maximum value at $x = 2$

80. If $f(x) = 0$ for $x < 0$ and $f(x)$ is differentiable at $x = 0$, then for $x > 0$, $f(x)$ may be

- (a) x^2
- (b) x
- (c) $-x$
- (d) $-x^{3/2}$

Type 3. Assertion and Reason

Directions (Q. Nos. 81-88) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

81. **Statement I** If $f(x)$ is discontinuous at $x = e$ and $\lim_{x \rightarrow a} g(x) = e$, then $\lim_{x \rightarrow a} f(g(x))$ cannot be equal to $f\left(\lim_{x \rightarrow a} g(x)\right)$.

Statement II If $f(x)$ is continuous at $x = e$ and

$$\lim_{x \rightarrow a} g(x) = e, \text{ then } \lim_{x \rightarrow a} f(g(x)) = f\left(\lim_{x \rightarrow a} g(x)\right).$$

82. **Statement I** If $|f(x)| \leq |x|$ for all $x \in \mathbb{R}$, then $|f(x)|$ is continuous at 0.

Statement II If $f(x)$ is continuous, then $|f(x)|$ is also continuous.

83. **Statement I** Let $f(x) = \lim_{m \rightarrow \infty} \{ \lim_{n \rightarrow \infty} \cos^{2m}(n! \pi x) \}$

$$\text{and } g(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases} \quad \text{Then,}$$

$h(x) = f(x) + g(x)$ is continuous for all x .

Statement II $f(x)$ and $g(x)$ are discontinuous for all $x \in R$.

84. Statement I $f(x) = \min\{\sin x, \cos x\}$ is not differentiable at $x = \frac{\pi}{2}$.

Statement II Non-differentiability of $\max\{g(x), h(x)\} \Rightarrow$ not differentiability of $\min\{g(x), h(x)\}$.

85. Statement I $f(x) = (\sin \pi x)(x-1)^{1/5}$ is differentiable at $x = 1$.

Statement II Product of two differentiable function is always differentiable.

86. Statement I The function $f(x) = (3x-1)4x^2 - 12x + 5|\cos \pi x|$ is differentiable at $x = 1/2, 5/2$.

Statement II $\cos(2n+1)\pi/2 = 0, \forall n \in I$.

87. Statement I If $f(x)$ is discontinuous at $x = e$ and $\lim_{x \rightarrow a} g(x) = e$, then $\lim_{x \rightarrow a} f(g(x))$ cannot be equal to $f(\lim_{x \rightarrow a} g(x))$.

Statement II If $f(x)$ is continuous at $x = e$ and $\lim_{x \rightarrow a} g(x) = e$, then $\lim_{x \rightarrow a} f(g(x)) = f(\lim_{x \rightarrow a} g(x))$.

88. Let $f(0) = 0, f\left(\frac{\pi}{2}\right) = 1, f\left(\frac{3\pi}{2}\right) = -1$ be a continuous and twice differentiable function.

Statement I $|f''(x)| \leq 1$ for at least one $x \in \left(0, \frac{3\pi}{2}\right)$.

Statement II According to Rolle's theorem, if $y = g(x)$ is continuous and differentiable, $\forall x \in [a, b]$ and $g(a) = g(b)$, then there exists at least one c such that $g'(c) = 0$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 89-91) Let $y = f(x)$ be defined in $[a, b]$, then

- (i) Test of continuity at $x = c, a < c < b$
- (ii) Test of continuity at $x = a$
- (iii) Test of continuity at $x = b$

Case I Test of continuity at $x = c, a < c < b$
If $y = f(x)$ is defined at $x = c$ and its value $f(c)$ is equal to limit of $f(x)$ as $x \rightarrow c$ i.e. $f(c) = \lim_{x \rightarrow c} f(x)$

or $\lim_{x \rightarrow c^-} f(x) = f(c) = \lim_{x \rightarrow c^+} f(x)$

or $LHL = f(c) = RHL$

Then, $y = f(x)$ is continuous at $x = c$

Case II Test of continuity at $x = a$

If $RHL = f(a)$

Then, $f(x)$ is said to be continuous at the end point $x = a$.

Case III Test of continuity at $x = b$, if $LHL = f(b)$

Then, $f(x)$ is continuous at right end $x = b$.

89. $f(x) = \begin{cases} \sin x, & x \leq 0 \\ \tan x, & 0 < x < 2\pi \\ \cos x, & 2\pi \leq x < 3\pi \\ 3\pi, & x = 3\pi \end{cases}$, then $f(x)$ is

discontinuous at

- (a) $\frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$
- (b) $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 3\pi$
- (c) $\frac{\pi}{2}, 2\pi$
- (d) None of these

90. Number of points of discontinuity of $[2x^3 - 5]$ in $[1, 2)$, where $[]$ denotes the greatest integer function, is

- (a) 14
- (b) 13
- (c) 10
- (d) None of these

91. The maximum value of $([x], |x|)$ is discontinuous at

- (a) $x = 0$
- (b) ϕ
- (c) $x = n, n \in I$
- (d) None of these

Passage II (Q. Nos. 92-94) Consider

$f(x) = x^2 + ax + 3$ and $g(x) = x + b$ and $F(x) = \lim_{n \rightarrow \infty} \frac{f(x) + x^{2n}g(x)}{1 + x^{2n}}$.

92. If $F(x)$ is continuous at $x = 1$, then

- (a) $b = a + 3$
- (b) $b = a - 1$
- (c) $a = b - 2$
- (d) None of these

93. If $F(x)$ is continuous at $x = -1$, then

- (a) $a + b = -2$
- (b) $a - b = 3$
- (c) $a + b = 5$
- (d) None of these

94. If $F(x)$ is continuous at $x = \pm 1$, then $f(x) = g(x)$ has

- (a) imaginary roots
- (b) both the roots positive
- (c) both the roots negative
- (d) roots of opposite signs

Passage III (Q. Nos. 95-97) Let

$f(x) = \begin{cases} x + 2, & 0 \leq x < 2 \\ 6 - x, & x \geq 2 \end{cases}, g(x) = \begin{cases} 1 + \tan x, & 0 \leq x < \frac{\pi}{4} \\ 3 - \cot x, & \frac{\pi}{4} \leq x < \pi \end{cases}$

95. $f(g(x))$ is

- (a) discontinuous at $x = \frac{\pi}{4}$
- (b) differentiable at $x = \frac{\pi}{4}$
- (c) continuous but not differentiable at $x = \frac{\pi}{4}$
- (d) differentiable at $x = \frac{\pi}{4}$, but derivative is not continuous

96. The number of points of not differentiability of $h(x) = |f(g(x))|$ is
- 1
 - 2
 - 3
 - 4

97. The range of $h(x) = f(g(x))$ is
- $(-\infty, \infty)$
 - $(4, \infty)$
 - $(-\infty, 4]$
 - None of the above

Passage IV (Q. Nos. 98-100) Let

$$f(x) = \begin{cases} [x], & -2 \leq x \leq -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \leq 2 \end{cases}$$

and $g(x) = f(|x|) + |f(x)|$, where $[]$ represents the greatest integer function.

98. The number of points where $|f(x)|$ is non-differentiable, is
- 3
 - 2
 - 4
 - 5

99. The number of points where $g(x)$ is non-differentiable, is
- 4
 - 5
 - 2
 - 3

100. The number of points where $g(x)$ is discontinuous, is
- 1
 - 2
 - 3
 - None of the above

Passage V (Q. Nos. 101-103) $f(x) = \cos x$ and

$$H_1(x) = \begin{cases} \min \{f(t); 0 \leq t < x\}, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} - x, & \frac{\pi}{2} < x \leq \pi \end{cases}$$

$f(x) = \cos x$ and

$$H_2(x) = \begin{cases} \max \{f(t); 0 < t < x\}, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} - x, & \frac{\pi}{2} < x \leq \pi \end{cases}$$

$g(x) = \sin x$

$$\text{and } H_3(x) = \begin{cases} \min \{g(t); 0 \leq t < x\}, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} - x, & \frac{\pi}{2} < x \leq \pi \end{cases}$$

$g(x) = \sin x$ and

$$H_4(x) = \begin{cases} \max \{g(t); 0 \leq t < x\}, & 0 \leq x \leq \frac{\pi}{2} \\ \frac{\pi}{2} - x, & \frac{\pi}{2} < x \leq \pi \end{cases}$$

Now,

101. Which one of the following is true?

- $H_2(x)$ is continuous and derivable in $[0, \pi]$
- $H_2(x)$ is continuous but not derivable at $x = \frac{\pi}{2}$
- $H_2(x)$ is neither continuous nor derivable at $x = \frac{\pi}{2}$
- None of the above

102. Which one of the following is true?

- $H_3(x)$ is continuous and derivable in $[0, \pi]$
- $H_3(x)$ is continuous but not derivable at $x = \frac{\pi}{2}$
- $H_3(x)$ is neither continuous nor derivable at $x = \frac{\pi}{2}$
- None of the above

103. Which one of the following is true?

- $H_4(x)$ is continuous and derivable in $[0, \pi]$
- $H_4(x)$ is continuous but not derivable at $x = \frac{\pi}{2}$
- $H_4(x)$ is neither continuous nor derivable at $x = \pi/2$
- None of the above

Type 5. Match the Columns

104. Let $f(x) = \begin{cases} \frac{5e^{1/x} + 2}{3 - e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then match the

statements of Column I with the values of Column II.

Column I	Column II
A. $y = f(x)$ is	p. continuous at $x = 0$
B. $y = xf(x)$ is	q. discontinuous at $x = 0$
C. $y = x^2f(x)$ is	r. differentiable at $x = 0$
D. $y = x^{-1}f(x)$ is	s. non-differentiable at $x = 0$

105. Suppose a function $f(x)$ satisfies the following conditions $f(x+y) = \frac{f(x)+f(y)}{1+f(x)f(y)}$, $\forall x, y$ and $f'(0) = 1$. Also, $-1 < f(x) < 1, \forall x \in R$.

Match the statements of Column I with the values of Column II.

Column I	Column II
A. $f(x)$ is differentiable over the set	p. $R - (-1, 0, 1)$
B. $f(x)$ increases in the interval	q. R
C. Number of the solutions of $f(x) = 0$ is	r. 0
D. The value of the limit $\lim_{x \rightarrow \infty} [f(x)]^x$ is	s. 1

106. Let $f(x) = \begin{cases} [x], & -2 \leq x < 0 \\ |x|, & 0 \leq x \leq 2 \end{cases}$ (where $[]$ denotes the greatest integer function)

$$g(x) = \sec x, x \in R - (2n+1)\frac{\pi}{2}.$$

Match the following statements in Column I with their values in Column II in the interval $\left(-\frac{3\pi}{2}, \frac{3\pi}{2}\right)$.

Column I		Column II
A. Limit of $f \circ g$ exist at	p.	-1
B. Limit of $g \circ f$ does not exist at	q.	π
C. Points of discontinuity of $f \circ g$ is/are	r.	$\frac{5\pi}{6}$
D. Points of differentiability of $f \circ g$ is/are	s.	$-\pi$

107. Match the statements of Column I with the values of Column II.

Column I		Column II
A. Number of points, where the function $f(x) = \begin{cases} 1 + \left[\cos \frac{\pi x}{2}\right], & 1 < x \leq 2 \\ 1 - \{x\}, & 0 \leq x < 1 \text{ and } f(1) = 0 \\ \sin \pi x , & -1 \leq x < 0 \end{cases}$ is continuous but non-differentiable	p.	0
B. $f(x) = \begin{cases} x^2 e^{1/x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ then $f(0^-)$ is equal to	q.	1
C. The number of points at which $g(x) = \frac{1}{1 + \frac{2}{f(x)}}$ is not differentiable, where $f(x) = \frac{1}{1 + \frac{1}{x}}$, is	r.	2
D. Number of points, where tangent does not exist for the curve $y = \operatorname{sgn}(x^2 - 1)$, is	s.	3

Type 6. Single Integer Answer Type Questions

108. Number of points of discontinuity of $f(x) = \tan^2 x - \sec^2 x$ in $(0, 2\pi)$ is _____.
109. If the function $f(x) = \frac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x}$ is continuous at $x = 0$, then the value of $f(0)$ is _____.
110. Number of points of discontinuity of the function $f(x) = [x^{1/x}]$, $x > 0$, where $[]$ represents the greatest integer function is _____.
111. Let $f(x) = x + \cos x + 2$ and $g(x)$ be the inverse function of $f(x)$, then $g'(3)$ equals _____.
112. Let $f(x) = x \tan^{-1}(x^2) + x^4$. Let $f^k(x)$ denotes k th derivative of $f(x)$ w.r.t. x , $k \in N$. If $f^{2m}(0) \neq 0$, $m \in N$, then m equals _____.

113. Given $\frac{\int_{f(y)}^{f(x)} e^t dt}{\int_y^x \left(\frac{1}{t}\right) dt} = 1, \forall x, y \in \left(\frac{1}{e^2}, \infty\right)$, where $f(x)$ is

continuous and differentiable function and $f\left(\frac{1}{e}\right) = 0$. If $g(x) = \begin{cases} e^x, & x \geq k \\ e^{x^2}, & 0 < x < k \end{cases}$, then the value of k for which $f(g(x))$ is continuous, $\forall x \in R^+$, is _____.

114. A differentiable function f satisfying a relation $f(x+y) = f(x) + f(y) + 2xy(x+y) - 1/3, \forall x, y \in R$ and $\lim_{h \rightarrow 0} \frac{3f(h) - 1}{6h} = \frac{2}{3}$. Then, the approx value of $[f(2)]$, where $[]$ represents the greatest integer function, is _____.
115. The number of points, where $f(x) = \operatorname{sgn}(x^2 - 3x + 2) + [x - 3]$, $x \in [0, 4]$ is discontinuous, where $[]$ denotes the greatest integer function, is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Let $f_1 : R \rightarrow R$, $f_2 : [0, \infty) \rightarrow R$, $f_3 : R \rightarrow R$ and $f_4 : R \rightarrow [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x|, & \text{if } x < 0 \\ e^x, & \text{if } x \geq 0 \end{cases}, f_2(x) = x^2;$$

$$f_3(x) = \begin{cases} \sin x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases}$$

$$\text{and } f_4(x) = \begin{cases} f_2(f_1(x)), & \text{if } x < 0 \\ f_2(f_1(x)) - 1, & \text{if } x \geq 0 \end{cases} \quad [2014]$$

	List I	List II
P.	f_4 is	1. onto but not one-one
Q.	f_3 is	2. neither continuous nor one-one
R.	$f_2 \circ f_1$ is	3. differentiable but not one-one
S.	f_2 is	4. continuous and one-one

	P	Q	R	S
(a)	3	1	4	2
(b)	1	3	4	2
(c)	3	1	2	4
(d)	1	3	2	4

Passage (Q.Nos. 2-3) Given that for each $a \in (0, 1)$

$\lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1} dt$ exists. Let this limit be $g(a)$.

In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. [2014]

2. The value of $g\left(\frac{1}{2}\right)$ is

- (a) π (b) 2π
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

3. The value of $g'\left(\frac{1}{2}\right)$ is

- (a) $\frac{\pi}{2}$ (b) π
(c) $-\frac{\pi}{2}$ (d) 0

4. Let $f : [0, 2] \rightarrow R$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$.

Let $F(x) = \int_0^{x^2} f(\sqrt{t}) dt$ for $x \in [0, 2]$. If

$F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals [2014]

- (a) $e^2 - 1$
(b) $e^4 - 1$
(c) $e - 1$
(d) e^4

5. Let $f : R \rightarrow R$ and $g : R \rightarrow R$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define

$$h : R \rightarrow R \text{ by } h(x) = \begin{cases} \max \{f(x), g(x)\}, & \text{if } x \leq 0 \\ \min \{f(x), g(x)\}, & \text{if } x > 0 \end{cases}$$

The number of points at which $h(x)$ is not differentiable, is [2014]

6. For every pair of continuous functions $f, g : [0, 1] \rightarrow R$ such that $\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0, 1]\}$, the correct statement(s) is (are) [2014]

- (a) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(b) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0, 1]$
(c) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0, 1]$
(d) $(f(c))^2 = (g(c))^2$ for some $c \in [0, 1]$

7. Let $f : [a, b] \rightarrow [1, \infty)$ be a continuous function and let $g : R \rightarrow R$ be defined as

$$g(x) = \begin{cases} 0, & \text{if } x < a, \\ \int_a^x f(t) dt, & \text{if } a \leq x \leq b, \\ \int_a^b f(t) dt, & \text{if } x > b \end{cases} \quad [2014]$$

- (a) $g(x)$ is continuous but not differentiable at a
(b) $g(x)$ is differentiable on R
(c) $g(x)$ is continuous but not differentiable at b
(d) $g(x)$ is continuous and differentiable at either a or b but not both

8. Let $f : \left[\frac{1}{2}, 1\right] \rightarrow R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then, the

value of $\int_{1/2}^1 f(x) dx$ lies in the interval [2013]

- (a) $(2e - 1, 2e)$
(b) $(e - 1, 2e - 1)$
(c) $\left(\frac{e-1}{2}, e-1\right)$
(d) $\left(0, \frac{e-1}{2}\right)$

9. If $f(x) = \begin{cases} x^2 \left| \cos \frac{\pi}{x} \right|, & x \neq 0, x \in R \\ 0, & x = 0 \end{cases}$, then f is [2012]

- (a) differentiable both at $x = 0$ and at $x = 2$
(b) differentiable at $x = 0$ but not differentiable at $x = 2$
(c) not differentiable at $x = 0$ but differentiable at $x = 2$
(d) differentiable neither at $x = 0$ nor at $x = 2$

10. For every integer n , let a_n and b_n be real numbers.

If function $f: R \rightarrow R$ is given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases} \text{ for all}$$

integers n . If f is continuous, then which of the following hold(s) for all n ? [2012]

- (a) $a_{n-1} - b_{n-1} = 0$
(b) $a_n - b_n = 1$
(c) $a_n - b_{n+1} = 1$
(d) $a_{n-1} - b_n = -1$

11. If $f: R \rightarrow R$ is a function such that $f(x+y) = f(x) + f(y)$, $\forall x, y \in R$. If $f(x)$ is differentiable at $x=0$, then [2011]

- (a) $f(x)$ is differentiable only in a finite interval containing zero

- (b) $f(x)$ is continuous, $\forall x \in R$
(c) $f'(x)$ is constant, $\forall x \in R$
(d) $f(x)$ is differentiable except at finitely many points

12. If $f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \leq 0 \\ x-1, & 0 < x \leq 1 \\ \log x, & x > 1 \end{cases}$ then [2011]

- (a) $f(x)$ is continuous at $x = -\frac{\pi}{2}$
(b) $f(x)$ is differentiable at $x = 1$
(c) $f(x)$ is differentiable at $x = -\frac{3}{2}$
(d) All of the above

JEE Main/AIEEE

13. If the function $g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx+2, & 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k+m$ is [2015]

- (a) 2
(b) $\frac{16}{5}$
(c) $\frac{10}{3}$
(d) 4

14. If $f: R \rightarrow R$ is a function defined by $f(x) = [x] \cos\left(\frac{2x-1}{2}\right)\pi$, where $[]$ denotes the

greatest integer function, then f is [2012]

- (a) continuous for every real x
(b) discontinuous only at $x=0$
(c) discontinuous only at non-zero integral values of x
(d) continuous only at $x=0$

15. Consider the function, $f(x) = |x-2| + |x-5|$, $x \in R$.
Statement I $f'(4) = 0$.

Statement II f is continuous in $[2, 5]$, differentiable in $(2, 5)$ and $f(2) = f(5)$. [2012]

- (a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

16. The values of p and q for which the function

$$f(x) = \begin{cases} \frac{\sin(p+1)x + \sin x}{x}, & x < 0 \\ q, & x = 0 \\ \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases} \text{ is continuous}$$

for all x in R , are [2011]

- (a) $p = \frac{5}{2}$, $q = \frac{1}{2}$ (b) $p = -\frac{3}{2}$, $q = \frac{1}{2}$
(c) $p = \frac{1}{2}$, $q = \frac{3}{2}$ (d) $p = \frac{1}{2}$, $q = -\frac{3}{2}$

17. If function $f(x)$ is differentiable at $x=a$, then

$$\lim_{x \rightarrow a} \frac{x^2 f(a) - a^2 f(x)}{x-a} \text{ is equal to } [2011]$$

- (a) $2a f(a) + a^2 f'(a)$
(b) $-a^2 f'(a)$
(c) $af(a) - a^2 f'(a)$
(d) $2af(a) - a^2 f'(a)$

18. Define $F(x)$ as the product of two real functions

$$f_1(x) = x, x \in IR, \text{ and } f_2(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases} \text{ as}$$

$$\text{follows, } F(x) = \begin{cases} f_1(x) \cdot f_2(x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Statement I $F(x)$ is continuous on IR .

Statement II $f_1(x)$ and $f_2(x)$ are continuous on IR . [2011]

- (a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

19. If $f: (-1, 1) \rightarrow R$ is a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then, $g'(0)$ is equal to [2010]

- (a) 4
(b) -4
(c) 0
(d) -2

20. Let $f(x) = x|x|$ and $g(x) = \sin x$

Statement I $g \circ f$ is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement II $g \circ f$ is twice differentiable at $x = 0$.

[2009]

- (a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

21. Let $f(x) = \begin{cases} (x-1)\sin \frac{1}{x-1}, & \text{if } x \neq 1 \\ 0, & \text{if } x = 1 \end{cases}$

Then, which one of the following is true? [2008]

- (a) f is differentiable at $x = 1$ but not at $x = 0$
(b) f is neither differentiable at $x = 0$ nor at $x = 1$
(c) f is differentiable at $x = 0$ and at $x = 1$
(d) f is differentiable at $x = 0$ but not at $x = 1$

22. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function defined by $f(x) = \min\{x+1, |x|+1\}$. Then, which of the following is true? [2007]

- (a) $f(x) \geq 1$, for all $x \in \mathbb{R}$
(b) $f(x)$ is not differentiable at $x = 1$
(c) $f(x)$ is differentiable everywhere
(d) $f(x)$ is not differentiable at $x = 0$

23. The function $f: \frac{\mathbb{R}}{\{0\}} \rightarrow \mathbb{R}$ given by

$$f(x) = \frac{1}{x} - \frac{2}{e^{2x} - 1} \text{ can be made continuous at } x = 0$$

by defining $f(0)$ as [2007]

- (a) 2 (b) -1
(c) 0 (d) 1

24. The set of points, where $f(x) = \frac{x}{1+|x|}$ is

differentiable, is [2006]

- (a) $(-\infty, -1) \cup (-1, \infty)$ (b) $(-\infty, \infty)$
(c) $(0, \infty)$ (d) $(-\infty, 0) \cup (0, \infty)$

25. Suppose $f(x)$ is differentiable at $x = 1$ and

$$\lim_{h \rightarrow 0} \frac{1}{h} f(1+h) = 5, \text{ then } f'(1) \text{ equals [2005]}$$

- (a) 6 (b) 5
(c) 4 (d) 3

26. Let f be differentiable for all x . If $f(1) = -2$ and $f'(x) \geq 2$ for $x \in [1, 6]$, then [2005]

- (a) $f(6) = 5$ (b) $f(6) < 5$
(c) $f(6) < 8$ (d) $f(6) \geq 8$

27. If f is a real-valued differentiable function satisfying $|f(x) - f(y)| \leq (x-y)^2$, $x, y \in \mathbb{R}$ and $f(0) = 0$, then $f(1)$ equals [2005]

- (a) 1 (b) 2 (c) 0 (d) -1

28. Let $f(x) = \frac{1 - \tan x}{4x - \pi}$, $x \neq \frac{\pi}{4}$, $x \in \left[0, \frac{\pi}{2}\right]$. If $f(x)$ is

continuous in $\left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{\pi}{4}\right)$ is [2004]

- (a) 1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) -1

29. If $f(x) = \begin{cases} xe^{-\left(\frac{1}{|x|} + \frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is [2003]

- (a) continuous as well as differentiable for all x
(b) continuous for all x but not differentiable at $x = 0$
(c) neither differentiable nor continuous at $x = 0$
(d) discontinuous everywhere

Other Engineering Questions

30. If $f(x) = \begin{cases} \left[\tan\left(\frac{\pi}{4} + x\right)\right]^{1/x}, & x \neq 0 \\ k, & x = 0 \end{cases}$, then for what

value of k , $f(x)$ is continuous at $x = 0$?

[BITSAT 2014]

- (a) 1 (b) 0 (c) e (d) e^2

31. The function $f(x) = [x] \cos \left[\frac{2x-1}{2} \right] \pi$, where $[]$

denotes the greatest integer function, is discontinuous at [Manipal 2014]

- (a) all x (b) no x
(c) all integral points (d) x which is not an integer

32. If $f(x) = \begin{cases} \frac{\sin^3(\sqrt{x}) \cdot \log(1+3x)}{(\tan^{-1} \sqrt{x})^2 (e^{5\sqrt{x}} - 1)x}, & x \neq 0 \\ a, & x = 0 \end{cases}$ is

continuous in $[0, 1]$, then a equals [Manipal 2014]

- (a) 0 (b) $\frac{3}{5}$
(c) 2 (d) $\frac{5}{3}$

33. If $f(x) = \begin{cases} \frac{3 \sin \pi x}{5x}, & x \neq 0 \\ 2k, & x = 0 \end{cases}$ is continuous at $x = 0$,

then the value of k is [Karnataka CET 2014]

- (a) $\frac{\pi}{10}$ (b) $\frac{3\pi}{10}$
(c) $\frac{3\pi}{2}$ (d) $\frac{3\pi}{5}$

34. If $f(x) = \begin{cases} ax+3, & x \leq 2 \\ a^2x-1, & x > 2 \end{cases}$, then the values of a for

which f is continuous for all x are [Kerala CEE 2014]

- (a) 1 and -2 (b) 1 and 2 (c) -1 and 2
(d) -1 and -2 (e) 0 and 3

35. Discuss the continuity of the function $f(x) = \sin 2x - 1$ at the point $x = 0$ and $x = \pi$.
[J&K CET 2014]

(a) Continuous at $x = 0, \pi$
(b) Discontinuous at $x = 0$ but continuous at $x = \pi$
(c) Continuous at $x = 0$ but discontinuous at $x = \pi$
(d) Discontinuous at $x = 0, \pi$

36. The function $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos|x|$ is non-differentiable at [Manipal 2014]
(a) -1 (b) 0 (c) 1 (d) 2

37. The function represented by the following graph is [Karnataka CET 2014]
(a) continuous but not differentiable at $x = 1$
(b) differentiable but not continuous at $x = 1$
(c) continuous and differentiable at $x = 1$
(d) neither continuous nor differentiable at $x = 1$

38. Let R be the set of all real numbers. If $f: R \rightarrow R$ is a function such that $|f(x) - f(y)|^2 \leq |x - y|^3$, $\forall x, y \in R$, then $f'(x)$ is equal to [Kerala CEE 2014]
(a) $f(x)$ (b) 1
(c) 0 (d) x^2
(e) x

39. If $f(x) = \int_1^x \sin^2\left(\frac{t}{2}\right) dt$, then the value of $\lim_{x \rightarrow 0} \frac{f(\pi + x) - f(x)}{x}$ is equal to [Kerala CEE 2014]
(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$
(d) 1 (e) 0

40. The function $f(x) = \frac{\tan\left[\pi\left\{x - \frac{\pi}{2}\right\}\right]}{2 + x^2}$, where $\{ \}$ denotes the greatest integer $\leq x$, is [WB JEE 2014]
(a) continuous for all values of x
(b) discontinuous at $x = \frac{\pi}{2}$
(c) not differentiable for some values of x
(d) discontinuous at $x = -2$

41. If $f(x + y) = f(x)f(y)$ and $f(x) = 1 + \sin(3x)g(x)$, where $g(x)$ is continuous, then $f'(x)$ is equal to [Kerala CEE 2013]
(a) $f(x)g(0)$ (b) $3g(0)$
(c) $f(x)\cos 3x$ (d) $3f(x)g(0)$
(e) None of these

42. The function $f(x) = a \sin|x| + be^{|x|}$ is differentiable at $x = 0$, when [WB JEE 2013]
(a) $3a + b = 0$ (b) $3a - b = 0$
(c) $a + b = 0$ (d) $a - b = 0$

43. If $f(x)$ is a differentiable function and $f'(4) = 5$. Then, $\lim_{x \rightarrow 2} \frac{f(4) - f(x^2)}{x - 2}$ equals [WB JEE 2013]
(a) 0 (b) 5
(c) 10 (d) -20

44. If $f(x) = \begin{cases} \int_0^x |1 - t| dt, & x > 1 \\ x - \frac{1}{2}, & x \leq 1 \end{cases}$, then [WB JEE 2013]

(a) $f(x)$ is continuous at $x = 1$
(b) $f(x)$ is not continuous at $x = 1$
(c) $f(x)$ is differentiable at $x = 1$
(d) $f(x)$ is not differentiable at $x = 1$

45. The value of f at $x = 0$, so that function $f(x) = \frac{2^x - 2^{-x}}{x}$, $x \neq 0$ is continuous at $x = 0$, is [Manipal 2013]
(a) 0 (b) $\log 4$
(c) 4 (d) e^4

46. If $f(x) = \frac{\log(1 + ax) - \log(1 - bx)}{x}$, for $x \neq 0$ and $f(0) = k$ and $f(x)$ is continuous at $x = 0$, then k is equal to [BITSAT 2012]
(a) $a + b$ (b) $a - b$
(c) a (d) b

47. If $f(x) = \begin{cases} \frac{x^2 - (a + 2)x + a}{x - 2}, & x \neq 2 \\ 2, & x = 2 \end{cases}$ is continuous at $x = 2$, then the value of a is [Karnataka CET 2012]
(a) -6 (b) 0
(c) 1 (d) -1

48. If $f(x) = x(\sqrt{x} - \sqrt{x + 1})$, then $f(x)$ is continuous in [OJEE 2012]
(a) $[0, \infty)$ (b) $(0, \infty)$
(c) $(-\infty, \infty)$ (d) $(3, \infty)$

49. A function f is defined by $f(x) = 2 + (x - 1)^{\frac{2}{3}}$ in $[0, 2]$. Which of the following is not correct? [MP PET 2012]
(a) f is continuous in $[0, 2]$
(b) $f(0) = f(2)$
(c) f is not derivable in $[0, 2]$
(d) Rolle's theorem is true in $[0, 2]$

50. The function $f(x) = |x - 1|$ is [MP PET 2012]
(a) continuous everywhere
(b) continuous everywhere, except at $x = 1$
(c) differentiable everywhere
(d) differentiable nowhere

51. The number of points of $f(x) = |x - 1| + |x - 3| + \sin x$, $x \in [0, 4]$, where $f(x)$ is not differentiable, is [OJEE 2012]
(a) 0 (b) 1
(c) 2 (d) 3

52. If $f(x) = \begin{cases} \frac{\log x}{x - 1}, & \text{if } x \neq 1 \\ k, & \text{if } x = 1 \end{cases}$ is continuous at $x = 1$, then the value of k is [Karnataka CET 2011]
(a) 0 (b) -1
(c) 1 (d) e

53. If $f(x) = \begin{cases} [x] + [-x], & \text{when } x \neq 2 \\ \lambda, & \text{when } x = 2 \end{cases}$ and it is continuous at $x = 2$, then the value of λ will be [WB JEE 2011]

(a) -1 (b) 1
(c) 0 (d) 2

54. If $f(x) = \begin{cases} -2\sin x, & -\pi \leq x \leq -\frac{\pi}{2} \\ a\sin x + b, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ \cos x, & \frac{\pi}{2} \leq x \leq \pi \end{cases}$ and it is continuous on $[-\pi, \pi]$, then [UP SEE 2011]

(a) $a = 1$ and $b = 1$ (b) $a = -1$ and $b = -1$
(c) $a = -1$ and $b = 1$ (d) $a = 1$ and $b = -1$

55. If $f(x) = \begin{cases} cx + 1, & x \leq 3 \\ cx^2 - 1, & x > 3 \end{cases}$ is continuous at $x = 3$, then c is equal to [Guj CET 2011]

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 3

56. If $f(x) = p|\sin x| + qe^{|x|} + r|x|^3$ and it is differentiable at $x = 0$, then [AMU 2011]

(a) $p = 0, q = 0$ and $r = 0$
(b) $p + q = 0$ and r is any real number
(c) $p + q + r = 0$
(d) $-p + q - r = 0$

57. $f(x) = x \sin \frac{1}{x}$ [GGSIPU 2011]

(a) is continuous but not differentiable at $x = 0$
(b) is discontinuous but differentiable at $x = 0$
(c) is differentiable at $x = 0$
(d) Cannot be determined

58. If $f(x) = \frac{(e^x - 1)^2}{\sin\left(\frac{x}{a}\right) \log\left(1 + \frac{x}{4}\right)}$ for $x \neq 0$ and $f(0) = 12$

If f is continuous at $x = 0$, then the value of a is [Kerala CEE 2010]

(a) 1 (b) -1
(c) 2 (d) -2
(e) 3

Answers

Work Book Exercise 5.1

1. (b) 2. (a) 3. (b) 4. (d) 5. (d) 6. (a) 7. (b) 8. (c) 9. (b) 10. (b)

Work Book Exercise 5.2

1. (d) 2. (d) 3. (b) 4. (b) 5. (b) 6. (c) 7. (d) 8. (c) 9. (a) 10. (d)

Target Exercises

1. (d) 2. (c) 3. (b) 4. (b) 5. (a) 6. (c) 7. (c) 8. (c) 9. (a) 10. (b)
 11. (b) 12. (a) 13. (a) 14. (b) 15. (b) 16. (a) 17. (b) 18. (a) 19. (a) 20. (b)
 21. (b) 22. (c) 23. (b) 24. (b) 25. (a) 26. (c) 27. (c) 28. (c) 29. (d) 30. (a)
 31. (d) 32. (b) 33. (b) 34. (c) 35. (a) 36. (d) 37. (a) 38. (c) 39. (d) 40. (b)
 41. (d) 42. (c) 43. (b) 44. (b) 45. (a) 46. (b) 47. (c) 48. (c) 49. (b) 50. (b)
 51. (a) 52. (c) 53. (d) 54. (a) 55. (c) 56. (a) 57. (c) 58. (d) 59. (b) 60. (c)
 61. (b) 62. (c) 63. (a) 64. (b) 65. (c) 66. (a) 67. (c) 68. (d) 69. (a) 70. (c)
 71. (c) 72. (a,b) 73. (all) 74. (a,b) 75. (a,b,c) 76. (a,c) 77. (a,c,d) 78. (a,c,d) 79. (a,b,d) 80. (a,d)
 81. (d) 82. (b) 83. (b) 84. (c) 85. (b) 86. (a) 87. (d) 88. (a) 89. (a) 90. (b)
 91. (b) 92. (a) 93. (c) 94. (d) 95. (c) 96. (b) 97. (c) 98. (a) 99. (d) 100. (b)
 101. (c) 102. (b) 103. (c) 104. (*) 105. (*) 106. (*) 107. (*) 108. (2) 109. (2) 110. (1)
 111. (1) 112. (2) 113. (1) 114. (8) 115. (4)

*

104. $A \rightarrow q, s; B \rightarrow p, s; C \rightarrow p, r; D \rightarrow q, s$

105. $A \rightarrow q; B \rightarrow q; C \rightarrow s; D \rightarrow s$

106. $A \rightarrow p, q, s; B \rightarrow p; C \rightarrow q, s; D \rightarrow p, r$

107. $A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow p$

Entrances Gallery

1. (d) 2. (a) 3. (d) 4. (b) 5. (3) 6. (a,d) 7. (a,c) 8. (d) 9. (b) 10. (d)
 11. (b,c) 12. (d) 13. (a) 14. (a) 15. (c) 16. (b) 17. (d) 18. (d) 19. (b) 20. (d)
 21. (d) 22. (c) 23. (d) 24. (b) 25. (b) 26. (d) 27. (c) 28. (c) 29. (b) 30. (d)
 31. (c) 32. (b) 33. (b) 34. (c) 35. (a) 36. (d) 37. (a) 38. (c) 39. (d) 40. (a)
 41. (d) 42. (c) 43. (d) 44. (a,d) 45. (b) 46. (a) 47. (b) 48. (b) 49. (d) 50. (a)
 51. (c) 52. (c) 53. (a) 54. (c) 55. (a) 56. (b) 57. (a) 58. (e)

Explanations

Target Exercises

$$\begin{aligned}
 1. \lim_{x \rightarrow 0} \frac{x - e^x + 1 - (1 - \cos 2x)}{x^2} &= \lim_{x \rightarrow 0} \left[\frac{x - e^x + 1}{x^2} - \frac{(1 - \cos 2x)}{x^2} \right] \\
 &= \lim_{x \rightarrow 0} \left[\frac{x + 1 - \left(1 + x + \frac{x^2}{2}\right)}{x^2} - \frac{2 \sin^2 x}{x^2} \right] \\
 &= \lim_{x \rightarrow 0} \left[\frac{-\frac{x^2}{2}}{x^2} - \frac{2 \sin^2 x}{x^2} \right] \\
 &= -\frac{1}{2} - 2 = -\frac{5}{2}
 \end{aligned}$$

[neglecting higher power of x in the expansion of e^x]

So, for continuity, $f(0) = -\frac{5}{2}$

Now, $[f(0)] = -3; \{f(0)\} = \left\{-\frac{5}{2}\right\} = \frac{1}{2}$

Hence, $[f(0)]\{f(0)\} = -\frac{3}{2} = -1.5$

2. $f(x) = \tan x$ is discontinuous when $x = (2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$

$f(x) = x[x]$ is discontinuous when $x = k, k \in \mathbb{Z}$

$f(x) = \sin[n\pi x]$ is discontinuous when $n\pi x = k, k \in \mathbb{Z}$

Thus, all the above functions have infinite number of points of discontinuity.

But $f(x) = \frac{|x|}{x}$ is discontinuous when $x = 0$ only.

3. $f(x) = \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}, x \neq \frac{\pi}{4}$, is continuous at $x = \frac{\pi}{4}$.

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \lim_{x \rightarrow \pi/4} f(x) = \lim_{x \rightarrow \pi/4} \frac{\tan\left(\frac{\pi}{4} - x\right)}{\cot 2x}$$

Now, by applying L' Hospital's rule,

$$f\left(\frac{\pi}{4}\right) = \lim_{x \rightarrow \pi/4} \frac{-\sec^2\left(\frac{\pi}{4} - x\right)}{-2 \operatorname{cosec}^2(2x)} = \frac{1}{2}$$

4. We have, $f(x) = \begin{cases} \frac{1-|x|}{1+x}, & x \neq -1 \\ 1, & x = -1 \end{cases}$

$= \begin{cases} 1, & x < 0 \\ \frac{1-x}{1+x}, & x \geq 0 \end{cases} \quad [\because f(-1) = 1, \text{ given}]$

$\Rightarrow f([2x]) = \begin{cases} 1, & [2x] < 0 \\ \frac{1-[2x]}{1+[2x]}, & [2x] \geq 0 \end{cases}$

$= \begin{cases} 1, & x < 0 \\ 1, & 0 \leq x < 1/2 \\ 0, & 1/2 \leq x < 1 \\ -1/3, & 1 \leq x < 3/2 \end{cases}$

Clearly, $f(x)$ is continuous for all $x < \frac{1}{2}$ and discontinuous at $x = \frac{1}{2}, 1$.

$$5. f(x) = 2|\operatorname{sgn}(2x)| + 2 = \begin{cases} 4, & x > 0 \\ 2, & x = 0 \\ 0, & x < 0 \end{cases}$$

Thus, $f(x)$ has non-removable discontinuity at $x = 0$.

6. $f(x) = \{x\} \sin(\pi[x])$

$= \{x\} \sin(\text{integral multiple of } \pi) = 0$

Hence, $f(x)$ is continuous for all x .

7. Given that, $\cos \frac{x}{2} \cos \frac{x}{2^2} \cos \frac{x}{2^3} \dots \cos \frac{x}{2^n}$

$$= \frac{\sin x}{2^n \sin\left(\frac{x}{2^n}\right)} \quad \dots(i)$$

Taking logarithm to the base e on both sides of Eq. (i) and then differentiating w.r.t. x , we get

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \left(\frac{1}{2^n} \cot \frac{x}{2^n} - \cot x \right)$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{n=1}^n \frac{1}{2^n} \tan \frac{x}{2^n} = \lim_{n \rightarrow \infty} \left(\frac{1}{x} \times \frac{x}{2^n} - \cot x \right)$$

$$= \left(\frac{1}{x} - \cot x \right)$$

We have, $f(x) = \begin{cases} \frac{1}{x} - \cot x, & x \in (0, \pi) \\ 2/\pi, & x = \pi/2 \end{cases}$

Clearly, $\lim_{x \rightarrow \pi/2} f(x) = \lim_{x \rightarrow \pi/2} \left(\frac{1}{x} - \cot x \right)$

$$= \frac{2}{\pi} = f\left(\frac{\pi}{2}\right)$$

Hence, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

8. Let $f(x) = \tan x$

The points of discontinuity of $f(x)$ are those points, where

$\tan x$ is infinite. This gives $\tan x = \infty$, i.e. $\tan x = \tan \frac{\pi}{2}$.

$$\therefore x = (2n+1)\frac{\pi}{2}, n \in \mathbb{I}$$

9. Let $f(x) = \frac{1}{\log |x|}$. The points of discontinuity of $f(x)$ are those points, where $f(x)$ is undefined or infinite. It is undefined when $x = 0$ and is infinite when $\log |x| = 0$, $|x| = 1$, i.e. $x = \pm 1$

$\therefore$ Set of points of discontinuity $= \{-1, 0, 1\}$.

10. At $x = 1$,

$$\text{LHL} = \lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} \frac{1}{5} [2(1-h)^2 + 3] = 1$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(1+h) = \lim_{h \rightarrow 0} [6 - 5(1+h)] = 6 - 5 = 1$$

and $f(1) = \frac{1}{5} (2 + 3) = 1$

Since, $\text{LHL} = \text{RHL} = f(1)$

$\therefore f(x)$ is continuous at $x = 1$.

At $x = 3$,

$$\text{LHL} = \lim_{h \rightarrow 0} (3 - h) = \lim_{h \rightarrow 0} [6 - 5(3 - h)] = -9$$

$$\text{and } f(3) = 3 - 3 = 0$$

Since, $\text{LHL} \neq f(3)$

$\therefore f(x)$ is discontinuous at $x = 3$.

Therefore, the only point of discontinuity is $x = 3$.

$$\begin{aligned} 11. \text{ We have, } \lim_{x \rightarrow 0^+} f(x) &= \lim_{h \rightarrow 0} f(0 + h) \\ &= \lim_{h \rightarrow 0} h^2 = 0 \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{x \rightarrow 0^-} f(x) &= \lim_{h \rightarrow 0} f(0 - h) \\ &= \lim_{h \rightarrow 0} (-h - 1) = -1 \end{aligned}$$

12. Let $f(x)$ be a periodic function with period T .

$$\Rightarrow f(x + T) = f(x)$$

$$\text{Now, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

$$\begin{aligned} \therefore f'(x + T) &= \lim_{h \rightarrow 0} \frac{f(x + T + h) - f(x + T)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = f'(x) \end{aligned}$$

Similarly, $f''(x + T) = f''(x)$

Hence, f' and f'' are also periodic functions with the same period.

13. Since, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$$\therefore \lim_{x \rightarrow \pi/2} f(x) = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow k = \lim_{h \rightarrow 0} \frac{\sin\left[\cos\left(\frac{\pi}{2} - h\right)\right] - \cos\left(\frac{\pi}{2} - h\right)}{\left[\pi - 2\left(\frac{\pi}{2} - h\right)\right]^2}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(\sin h) - \sin h}{4h^2} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{h \rightarrow 0} \frac{\cos(\sin h) \cos h - \cos h}{8h}$$

$$\begin{aligned} &\quad \text{[using L' Hospital's rule]} \\ &= \lim_{h \rightarrow 0} \frac{-\sin(\sin h) \cos^2 h - \sin h \cos(\sin h) + \sin h}{8} \end{aligned}$$

$$= \frac{0}{8} = 0$$

$$\therefore k = 0$$

14. Since, $\lim_{x \rightarrow 1} \frac{x-1}{|x-1|}$ and $\lim_{x \rightarrow 2} \frac{x-2}{|x-2|}$ do not exist.

Therefore, $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow 2} f(x)$ do not exist.

$\therefore f(x)$ is not continuous at $x = 1, 2$.

At all other points, $f(x)$ is continuous.

$\therefore f(x)$ is continuous on $R - \{1, 2\}$.

$$15. \lim_{x \rightarrow a^-} f(x) \cdot g(x) = \lim_{x \rightarrow a^-} f(x) \cdot \lim_{x \rightarrow a^-} g(x) = lm$$

$$\text{and } \lim_{x \rightarrow a^+} f(x) \cdot g(x) = \lim_{x \rightarrow a^+} f(x) \cdot \lim_{x \rightarrow a^+} g(x) = lm$$

$$\therefore \lim_{x \rightarrow a} f(x) \cdot g(x) = lm$$

16. The only doubtful point is $x = 0$.

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0} f(0 - h) = \lim_{h \rightarrow 0} (-h + 1)^{2 - \left(\frac{1}{h} - \frac{1}{h}\right)} \\ &= \lim_{h \rightarrow 0} (1 - h)^2 = 1 \end{aligned}$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(0 + h)$$

$$= \lim_{h \rightarrow 0} (h + 1)^{2 - \left(\frac{1}{h} + \frac{1}{h}\right)}$$

$$= \lim_{h \rightarrow 0} (1 + h)^{2 - \frac{2}{h}}$$

$$= \lim_{h \rightarrow 0} (1 + h)^2 [(1 + h)^{1/h}]^{-2} = e^{-2}$$

Since, $\text{LHL} \neq \text{RHL}$

$\therefore f(x)$ is not continuous at $x = 0$.

17. Since, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$$\therefore \text{LHL} = \text{RHL} = f\left(\frac{\pi}{2}\right)$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} -\frac{1 - \sin^2 x}{3 \cos^2 x} = \lim_{x \rightarrow \pi/2} \frac{b(1 - \sin x)}{(\pi - 2x)^2} = a$$

$$\Rightarrow \lim_{x \rightarrow \pi/2} -\frac{\cos^2 x}{3 \cos^2 x} = \lim_{h \rightarrow 0} \frac{b \left[1 - \sin\left(\frac{\pi}{2} + h\right)\right]}{\left[\pi - 2\left(\frac{\pi}{2} + h\right)\right]^2} = a$$

$$\Rightarrow \frac{1}{3} = \lim_{h \rightarrow 0} \frac{b(1 - \cos h)}{4h^2} = a$$

$$\Rightarrow \frac{1}{3} = \lim_{h \rightarrow 0} \frac{b}{8} \left(\frac{\sin h/2}{h/2}\right)^2 = a$$

$$\Rightarrow \frac{1}{3} = \frac{b}{8} = a$$

$$\therefore a = \frac{1}{3} \text{ and } b = \frac{8}{3}$$

$$18. \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(0 + h)$$

$$= \lim_{h \rightarrow 0} \frac{e^{[h] + h} - 1}{[h] + h}$$

$$= \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

$$\text{and } \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0 - h)$$

$$= \lim_{h \rightarrow 0} \frac{e^{[-h] - h} - 1}{[-h] - h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{-1-h} - 1}{(-1 - h)} = \frac{e^{-1} - 1}{-1}$$

Since, $\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$, therefore $f(x)$ is not continuous at $x = 0$.

19. We have,

$$\lim_{h \rightarrow 0} f(5 - h) = \lim_{h \rightarrow 0} \tan^{-1} \frac{1}{(5 - h) - 5}$$

$$= \lim_{h \rightarrow 0} \tan^{-1} \left(\frac{-1}{h}\right)$$

$$= \tan^{-1}(-\infty) = -\frac{\pi}{2}$$

$$\begin{aligned}\text{and } \lim_{h \rightarrow 0} f(5+h) &= \lim_{h \rightarrow 0} \tan^{-1} \frac{1}{(5+h)-5} \\ &= \lim_{h \rightarrow 0} \tan^{-1} \left(\frac{1}{h} \right) \\ &= \tan^{-1}(\infty) = \frac{\pi}{2}\end{aligned}$$

Since, $\lim_{h \rightarrow 0} f(5-h) \neq \lim_{h \rightarrow 0} f(5+h)$, therefore $f(x)$ has discontinuity of the first kind at $x = 5$.

20. If $|2 \sin x| < 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x}{(2 \sin x)^{2n} + 1} = \frac{x}{0+1} = x$

If $|2 \sin x| = 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x}{(2 \sin x)^{2n} + 1} = \frac{x}{1+1} = \frac{1}{2}x$

If $|2 \sin x| > 1$, $f(x) = \lim_{n \rightarrow \infty} \frac{x}{(2 \sin x)^{2n} + 1} = \frac{x}{\infty+1} = 0$

Also, $|2 \sin x| < 1 \Rightarrow |\sin x| < \frac{1}{2}$

$$\Rightarrow -\frac{1}{2} < \sin x < \frac{1}{2}$$

$$\Rightarrow n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6}$$

$$|2 \sin x| = 1 \Rightarrow |\sin x| = \frac{1}{2}$$

$$\Rightarrow \sin x = \pm \frac{1}{2} \Rightarrow x = n\pi \pm \frac{\pi}{6}$$

and $|2 \sin x| > 1 \Rightarrow |\sin x| > \frac{1}{2}$

$$\Rightarrow \sin x > \frac{1}{2} \text{ or } \sin x < -\frac{1}{2}$$

$$\Rightarrow x = n\pi \pm \frac{\pi}{6}$$

$$\text{Thus, we have } f(x) = \begin{cases} x, & n\pi - \frac{\pi}{6} < x < n\pi + \frac{\pi}{6} \\ \frac{1}{2}x, & x = n\pi \pm \frac{\pi}{6} \\ 0, & n\pi + \frac{\pi}{6} < x < n\pi + \frac{5\pi}{6} \end{cases}$$

The only doubtful points are $x = n\pi \pm \frac{\pi}{6}$.

Clearly, $f(x)$ is not continuous at $x = n\pi \pm \frac{\pi}{6}$.

$\therefore f(x)$ is continuous in

$$(-\infty, \infty) - \left\{ x : x = n\pi \pm \frac{\pi}{6}, n \in I \right\}$$

21. We have, $f(x) = \left[\frac{1}{\sqrt{2}} (\cos x + \sin x) \right] = \left[\cos \left(x - \frac{\pi}{4} \right) \right]$

Clearly, $f(x)$ is discontinuous at all those points, where $\cos \left(x - \frac{\pi}{4} \right)$ is an integer, i.e. $x - \frac{\pi}{4} = 0$,

$$\frac{\pi}{2}, \pi, \frac{3\pi}{2}$$

i.e. $x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ and $\frac{7\pi}{4}$

22. $f(1+0) = \lim_{h \rightarrow 0} f(1+h)$

$$= \lim_{h \rightarrow 0} \left[\lim_{n \rightarrow \infty} \frac{[1 + \cos(\pi + \pi h)]^n + 1}{[1 + \cos(\pi + \pi h)]^n - 1} \right]$$

$$= \lim_{h \rightarrow 0} \left[\lim_{n \rightarrow \infty} \frac{(1 - \cos \pi h)^n + 1}{(1 - \cos \pi h)^n - 1} \right] = \lim_{h \rightarrow 0} \left(\frac{0+1}{0-1} \right) = -1$$

$$\begin{aligned}f(1-0) &= \lim_{h \rightarrow 0} f(1-h) \\ &= \lim_{h \rightarrow 0} \left[\lim_{n \rightarrow \infty} \frac{[1 + \cos(\pi - \pi h)]^n + 1}{[1 + \cos(\pi - \pi h)]^n - 1} \right] \\ &= \lim_{h \rightarrow 0} \left[\lim_{n \rightarrow \infty} \frac{(1 - \cos \pi h)^n + 1}{(1 - \cos \pi h)^n - 1} \right] \\ &= \lim_{h \rightarrow 0} \left(\frac{0+1}{0-1} \right) = -1\end{aligned}$$

Also, $f(1) = \lim_{n \rightarrow \infty} \frac{(1 + \cos \pi)^n + 1}{(1 + \cos \pi)^n - 1} = \frac{0+1}{0-1} = -1$

Since, $f(1+0) = f(1-0) = f(1)$, therefore $f(x)$ is continuous at $x = 1$.

23. Since, f is continuous on $[2, 5]$, therefore f assumes atleast once, every values between $f(2)$ and $f(5)$. But it is given that $f(x)$ takes only rational values for all x and there are irrational values between $f(2)$ and $f(5)$, this is possible only, if $f(x)$ has a constant rational value at all points between $x = 2$ and $x = 5$. Since, $f(4) = 8$

$$\therefore f(3, 7) = 8$$

24. $[x+1] = 0$, if $0 \leq x+1 < 1$, i.e. $-1 \leq x < 0$

Thus, domain of $f = R - [-1, 0)$.

We have, $\sin \left(\frac{\pi}{[x+1]} \right)$ continuous at all points of

$R - [-1, 0)$ and $[x]$ continuous on $R - I$, where I denotes the set of integers. Thus, the points, where f can possibly be discontinuous, are $\dots, -3, -2, 0, 1, 2, \dots$

For $0 \leq x < 1$, $[x] = 0$ and $\sin \left(\frac{\pi}{[x+1]} \right)$ is defined.

$$\therefore f(x) = 0 \text{ for } 0 \leq x < 1$$

Also, f is not defined on $[-1, 0)$, so the continuity of f and 0 means continuity of f from right at 0. Since, f is continuous from right at 0, so f is continuous at 0. Hence, the set of points of discontinuity of f is $I - \{0\}$.

25. The function $f(x)$ is defined, when $-1 \leq \sin x < 0$ or $\sin x = 1$.

$$\Rightarrow x \in ((2n+1)\pi, (2n+2)\pi) \cup \left\{ 2n\pi + \frac{\pi}{2} \right\}, n \in I$$

When $x \in (2n\pi + \pi, 2n\pi + 2\pi)$, $f(x) = -1$

$\therefore f(x)$ is a constant function.

Hence, $f(x)$ is continuous when

$$x \in (2n\pi + \pi, 2n\pi + 2\pi).$$

$$\begin{aligned}\text{Now, } Lf' \left(\frac{\pi}{2} \right) &= \lim_{h \rightarrow 0} \frac{f \left(\frac{\pi}{2} - h \right) - f \left(\frac{\pi}{2} \right)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{1}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\left[\sin \left(\frac{\pi}{2} - h \right) \right]^{-1}}{-h} \\ &= \lim_{h \rightarrow 0} \frac{1}{[\cos h]^{-1} - 1}\end{aligned}$$

which does not exist.

Hence, $f(x)$ is not differentiable at $x = \frac{\pi}{2}$.

$$26. \text{ For } n \in I, \lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} [x] \cos\left(\frac{2x-1}{2}\right)\pi$$

$$= n \cos\left(\frac{2n-1}{2}\right)\pi = 0$$

$$\text{and } \lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} [x] \cos\left(\frac{2x-1}{2}\right)\pi$$

$$= (n-1) \cos\left(\frac{2n-1}{2}\right)\pi = 0$$

Hence, f is continuous for $x = n \in I$. Since, the functions $g(x) = [x]$ and $h(x) = \cos\left(\frac{2x-1}{2}\right)\pi$ are continuous on $x \in R - I$, so f is continuous everywhere.

27. Since, $f(x)$ is continuous at $x = 0$.

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

Take any point $x = a$, then at $x = a$,

$$\lim_{x \rightarrow a} f(x) = \lim_{h \rightarrow 0} f(a+h)$$

$$= \lim_{h \rightarrow 0} [f(a) + f(h)] \quad [\because f(x+y) = f(x) + f(y)]$$

$$= f(a) + \lim_{h \rightarrow 0} f(h) = f(a) + f(0) = f(a+0) = f(a)$$

$\therefore f(x)$ is continuous at $x = a$. Since, $x = a$ is any arbitrary point, therefore $f(x)$ is continuous for all x .

28. We have, $f \circ g = I \Rightarrow (f \circ g)(x) = x$ for all x

$$\Rightarrow f[g(x)] = x$$

$$\Rightarrow f'[g(x)] \cdot g'(x) = 1$$

$$\Rightarrow f'[g(a)] \cdot g'(a) = 1$$

$$\Rightarrow f'[g(a)] = \frac{1}{g'(a)} = \frac{1}{2} \quad [\because g'(a) = 2]$$

$$\Rightarrow f'(b) = \frac{1}{2} \quad [\because g(a) = b]$$

$$29. \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+0} = 1$$

$$\text{and } \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{1}{1+e^{1/h}}$$

$$= \lim_{h \rightarrow 0} \frac{e^{-1/h}}{e^{-1/h} + 1} = \frac{0}{1} = 0$$

Since, $\lim_{h \rightarrow 0} f(0-h) \neq \lim_{h \rightarrow 0} f(0+h)$, therefore $f(x)$ is not continuous and hence not differentiable at $x = 0$.

30. Differentiability at $x = 0$,

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-h(e^{-1/h} - e^{1/h})}{-h(e^{-1/h} + e^{1/h})}$$

$$= \lim_{h \rightarrow 0} \frac{e^{-2h} - 1}{e^{-2h} + 1} = -1$$

$$Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{h(e^{1/h} - e^{-1/h})}{h(e^{1/h} + e^{-1/h})}$$

$$= \lim_{h \rightarrow 0} \frac{1 - e^{-2h}}{1 + e^{-2h}} = 1$$

Since, $Lf'(0) \neq Rf'(0)$

$\therefore f(x)$ is not differentiable at $x = 0$. But $Lf'(0)$ and $Rf'(0)$ are finite, therefore $f(x)$ is continuous everywhere but not differentiable at $x = 0$.

$$31. \text{ We have, } f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} (\log a)^n = \sum_{n=0}^{\infty} \frac{(x \log a)^n}{n!}$$

$$= e^{x \log a} = e^{\log a^x} = a^x$$

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{a^{-h} - 1}{-h} = \log_e a$$

$$Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \log_e a$$

Since, $Lf'(0) = Rf'(0)$

$\therefore f(x)$ is differentiable at $x = 0$.

Since, every differentiable function is continuous, therefore $f(x)$ is continuous at $x = 0$.

32. For $x \neq 0$, we have

$$f(x) = x + \frac{\frac{x}{1+x}}{1 - \frac{1}{1+x}} = x + \frac{\frac{x}{1+x}}{\frac{x}{1+x}} = x + 1$$

For $x = 0$, $f(x) = 0$

$$\text{Thus, } f(x) = \begin{cases} x+1, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{Clearly, } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = 1 \neq f(0)$$

So, $f(x)$ is discontinuous and hence not differentiable at $x = 0$.

33. For $x \neq -1$, we have

$$f(x) = 1 - 2x + 3x^2 - 4x^3 + \dots$$

$$= (1+x)^{-2} = \frac{1}{(1+x)^2}$$

$$\lim_{h \rightarrow 0} f(-1-h) = \lim_{h \rightarrow 0} \frac{1}{(1-1-h)^2} = \infty$$

So, $f(x)$ is not continuous at $x = -1$.

Hence, $f(x)$ is neither continuous nor derivable at $x = -1$.

$$34. Lf'\left(\frac{\pi}{4}\right) = \lim_{h \rightarrow 0} \frac{f\left(\frac{\pi}{4} - h\right) - f\left(\frac{\pi}{4}\right)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\left[\tan\left(\frac{\pi}{4} - h\right)\right] - \left[\tan\frac{\pi}{4}\right]}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{0-1}{-h} = \infty$$

$\therefore f'\left(\frac{\pi}{4}\right)$ does not exist.

35. Continuity at $x = 0$

$$\text{LHL} = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} (-h)^p \cos \frac{1}{h} = 0, \text{ if } p > 0$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} h^p \cos \frac{1}{h} = 0, \text{ if } p > 0$$

and $f(0) = 0$

$\therefore f(x)$ is continuous at $x = 0$, if $p > 0$.

Differentiability at $x = 0$

$$\begin{aligned} Lf'(0) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(-h)^p \cos \frac{1}{h} - 0}{-h} \\ &= \lim_{h \rightarrow 0} (-h)^{p-1} \cos \frac{1}{h} = 0, \text{ if } p-1 > 0, \text{ i.e. } p > 1 \\ Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^p \cos \frac{1}{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} h^{p-1} \cos \frac{1}{h} = 0, \text{ if } p > 1 \end{aligned}$$

$\therefore f(x)$ is differentiable at $x = 0$, if $p > 1$.

36. We have,

$$\begin{aligned} \operatorname{sgn}[\operatorname{sgn}(x)] &= \begin{cases} \operatorname{sgn}\left(\frac{|x|}{x}\right), & x \neq 0 \\ \operatorname{sgn}(0), & x = 0 \end{cases} \\ &= \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \\ \therefore f(x) &= \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases} \end{aligned}$$

Continuity at $x = 0$,

$$\text{LHL} = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{-h}{|-h|} = -1 \text{ and } f(0) = 0$$

$\therefore f(x)$ is not continuous and hence not differentiable at $x = 0$.

37. For $x > 2$, $f(x) = x - 2$

$$\begin{aligned} \text{and } g(x) &= f[f(x)] = f(x-2) \\ &= x-2-2 = x-4 \end{aligned}$$

$$\therefore g'(x) = 1 \text{ for } x > 2$$

38. Since, $[x]$ and $e^{|x|}$ are not differentiable at $x = 0$, therefore for $f(x)$ to be differentiable at $x = 0$, we must have $a = 0$, $b = 0$ and c can be any real number.

$$\text{39. We have, } f(x) = \begin{cases} \frac{-1}{x-1}, & 0 < x < 1 \\ \frac{1-1}{x-1} = 0, & 1 < x < 2 \\ 0, & x = 1 \end{cases}$$

$$\lim_{h \rightarrow 0} f(1-h) = \lim_{h \rightarrow 0} \frac{-1}{(1-h)-1} = \lim_{h \rightarrow 0} \frac{1}{h} = \infty$$

$\therefore f(x)$ is not continuous and hence not differentiable at $x = 1$.

40. We have,

$$\begin{aligned} \Rightarrow 4x + 3|y| &= 5y \\ \Rightarrow 4x + 3y &= 5y, \text{ if } y \geq 0 \\ \text{and } 4x - 3y &= 5y, \text{ if } y < 0 \\ \Rightarrow y &= \begin{cases} 2x, & x \geq 0 \\ \frac{1}{2}x, & x < 0 \end{cases} \end{aligned}$$

Clearly, y is continuous at $x = 0$ but not differentiable at $x = 0$.

$$\text{Also, } \frac{dy}{dx} = \begin{cases} 2, & x \geq 0 \\ \frac{1}{2}, & x < 0 \end{cases}$$

41. We have,

$$\begin{aligned} \lim_{h \rightarrow 0} f(3-h) &= \lim_{h \rightarrow 0} |3 - (3-h)| + (3+3-h) \\ &= \lim_{h \rightarrow 0} (h+6) = 6 \end{aligned}$$

$$\begin{aligned} \text{and } \lim_{h \rightarrow 0} f(3+h) &= \lim_{h \rightarrow 0} |3 - (3+h)| + (3+3+h) \\ &= \lim_{h \rightarrow 0} (h+7) = 7 \end{aligned}$$

Since, $\lim_{h \rightarrow 0} f(3-h) \neq \lim_{h \rightarrow 0} f(3+h)$, therefore $f(x)$ is not continuous and hence not differentiable at $x = 3$.

42. For $f(x)$ to be continuous at $x = 0$, we must have

$$\begin{aligned} f(0) &= \lim_{h \rightarrow 0} (1+x)^{\cot x} \\ \Rightarrow \log f(0) &= \lim_{x \rightarrow 0} \log(1+x)^{\cot x} \\ &= \lim_{x \rightarrow 0} \cot x \log(1+x) \\ &= \lim_{x \rightarrow 0} \frac{\log(1+x)}{\tan x} \\ &= \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} \cdot \frac{x}{\tan x} = 1 \cdot 1 = 1 \\ \therefore f(0) &= e^1 = e \end{aligned}$$

43. We have,

$$f(x) = \begin{cases} xe^{-\left(\frac{-1}{x} + \frac{1}{x}\right)}, & x < 0 \\ xe^{-\left(\frac{1}{x} + \frac{1}{x}\right)}, & x > 0 \\ 0, & x = 0 \end{cases} = \begin{cases} x, & x < 0 \\ xe^{-2/x}, & x > 0 \\ 0, & x = 0 \end{cases}$$

$f(x)$ is differentiable as well as continuous everywhere except possibly at $x = 0$.

$$\text{Now, } Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{he^{-2/h} - 0}{h} \\ &= \lim_{h \rightarrow 0} e^{-2/h} = 0 \end{aligned}$$

$$\text{and } Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{-h - 0}{-h} = 1 \end{aligned}$$

Since, $Lf'(0) \neq Rf'(0)$, therefore $f(x)$ is not differentiable at $x = 0$ but $f(x)$ is continuous at $x = 0$ as $Lf'(0)$ and $Rf'(0)$ are finite.

44. We have,

$$\begin{aligned} Lf'(0) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \left[\frac{-h(3e^{-1/h} + 4)}{2 - e^{-1/h}} - 0 \right] \cdot \left(\frac{-1}{h} \right) \\ &= \frac{0+4}{2-0} = 2 \end{aligned}$$

$$\text{and } Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \left[\frac{h(3e^{1/h} + 4)}{2 - e^{1/h}} - 0 \right] \frac{1}{h} \\ &= \lim_{h \rightarrow 0} \left(\frac{3 + 4e^{-1/h}}{2e^{-1/h} - 1} \right) \\ &= \frac{3 + 0}{0 - 1} = -3 \end{aligned}$$

Since, $Lf'(0) \neq Rf'(0)$

$\therefore f(x)$ is not differentiable at $x = 0$. But $f(x)$ is continuous at $x = 0$, as $Lf'(0)$ and $Rf'(0)$ are finite.

45. When $t \geq 0, |t| = t$

$$\begin{aligned} \therefore x &= 2t - t = t \quad \text{and} \quad y = t^2 + t \cdot t = 2t^2 \\ \Rightarrow y &= 2x^2, \text{ when } x \geq 0 \quad [\text{as } t \geq 0 \Rightarrow x \geq 0] \end{aligned}$$

Also, for $t \leq 0, |t| = -t$

$$\begin{aligned} \therefore x &= 2t + t = 3t \\ \text{and} \quad y &= t^2 + t(-t) = 0 \\ \Rightarrow y &= 0, \text{ when } x \leq 0 \quad [\text{as } t \leq 0 \Rightarrow x \leq 0] \end{aligned}$$

Hence, the function is defined as

$$y = f(x) = \begin{cases} 2x^2, & x \geq 0 \\ 0, & x \leq 0 \end{cases}$$

$$\begin{aligned} \text{Now, } Lf'(0) &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{0}{-h} = 0 \end{aligned}$$

$$\begin{aligned} \text{and } Rf'(0) &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h^2}{h} = 0 \end{aligned}$$

Since, $Lf'(0) = Rf'(0)$, therefore $f(x)$ is differentiable and hence continuous at $x = 0$.

$$\begin{aligned} 46. \text{ We have, } Lf'(1) &= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{3^{1-h} - 3}{-h} \\ &= 3 \lim_{h \rightarrow 0} \left(\frac{3^{-h} - 1}{-h} \right) = 3 \log 3 \end{aligned}$$

$$\begin{aligned} \text{and } Rf'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{4 - (1+h) - 3}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} = -1 \end{aligned}$$

Since, $Lf'(1) \neq Rf'(1)$, therefore $f(x)$ is not differentiable at $x = 1$ but $f(x)$ is continuous at $x = 1$ as $Lf'(1)$ and $Rf'(1)$ are finite.

47. The function $f(x)$ can be rewritten as

$$f(x) = \begin{cases} x^3, & -1 < x < 1 \\ x, & x \leq -1 \text{ or } x \geq 1 \end{cases}$$

$$\begin{aligned} \text{Now, } Lf'(1) &= \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(1-h)^3 - 1}{-h} \\ &= \lim_{h \rightarrow 0} (3 - 3h + h^2) = 3 \end{aligned}$$

$$\begin{aligned} \text{and } Rf'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h} = 1 \end{aligned}$$

Since, $Lf'(1) \neq Rf'(1)$, therefore $f(x)$ is not differentiable at $x = 1$ but $f(x)$ is continuous at $x = 1$.

$$48. x + 4y = 6y; y \geq 0 \Rightarrow y = \frac{x}{2}, x \geq 0$$

$$\begin{aligned} \text{and } x - 4y &= 6y, y < 0 \\ y &= \frac{1}{10}x, x < 0 \end{aligned}$$

$$\therefore f(x) = \begin{cases} \frac{x}{2}, & x \geq 0 \\ \frac{x}{10}, & x < 0 \end{cases} \Rightarrow \frac{dy}{dx} = \frac{1}{2}, x > 0$$

49. If $x \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$, then $f(x) = [\tan^2 x] = 0$, which is continuous and differentiable at $x = 0$.

50. Continuity at $x = 2$

$$\begin{aligned} \text{LHL} &= \lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} [(2-h)^2] + [-2+h]^2 \\ &= \lim_{h \rightarrow 0} \{3 + (-2)^2\} = 7 \\ \text{RHL} &= \lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} [(2+h)^2] + [-2-h]^2 \\ &= \lim_{h \rightarrow 0} \{4 + (-3)^2\} = 13 \end{aligned}$$

Since, LHL $\neq$ RHL

$\therefore f(x)$ is not continuous at $x = 2$.

Differentiability at $x = 2$

$$\text{LHD} = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{(2-h) - 2} = \lim_{h \rightarrow 0} \frac{7-8}{-h} = \infty$$

So, $f(x)$ is not differentiable at $x = 2$.

Hence, the function $f(x)$ is neither continuous nor derivable at $x = 2$.

$$51. \text{ Since, } f(x) = \frac{1}{[\sin x]} \quad [\because \sin x \notin (0, 1)]$$

$$\Rightarrow x \notin [2n\pi, (2n+1)\pi] - (4n+1)\pi/2, n \in I$$

$\therefore f(x)$ is not continuous, if $x \in (2n\pi, 2n\pi + \pi), n \in I$.

52. We have, $f(x) = [x] + |1-x|, -1 \leq x \leq 3$

$$f(x) = \begin{cases} -x, & -1 \leq x < 0 \\ 1-x, & 0 \leq x < 1 \\ x, & 1 \leq x < 2 \\ 1+x, & 2 \leq x < 3 \\ 5, & x = 3 \end{cases}$$

The only doubtful points are $x = -1, 0, 1, 2$ and 3 . It can be easily seen that $f(x)$ is differentiable at $x = -1$ but not differentiable at $x = 0, 1, 2$ and 3 .

Hence, the required points are $0, 1, 2$ and 3 .

53. Since, $[x]$ is not differentiable at the integral values of x , therefore $[n + p \sin x]$ is not differentiable at the points, where $n + p \sin x$ is an integer. As p is a prime number, $n + p \sin x$ is an integer, if

$$\sin x = 1, -1, \frac{r}{p}, \text{ i.e. } x = \frac{\pi}{2}, \frac{-\pi}{2}, \sin^{-1}\left(\frac{r}{p}\right), \pi - \sin^{-1}\left(\frac{r}{p}\right),$$

where, $0 \leq r \leq (p-1)$

But $x \neq \frac{-\pi}{2}$

$\therefore f(x)$ is not differentiable at $x = \frac{\pi}{2}, \sin^{-1}\left(\frac{r}{p}\right)$,

$p - \sin^{-1}\left(\frac{r}{p}\right)$, where $0 < r \leq (p-1)$. So, the required

number of points $= 1 + 2(p-1) = 2p-1$.

54. Let $x^3 = n, n \in I \Rightarrow x = n^{1/3}$

$\therefore f(x) = (-1)^n = \pm 1$

Hence, $f(x)$ is discontinuous for $x = n^{1/3}, n \in I$.

$$\begin{aligned} 55. f'(x) &= \frac{-1}{\sqrt{1 - \left(\frac{2x}{1+x^2}\right)^2}} \times \frac{d}{dx} \left(\frac{2x}{1+x^2} \right) \\ &= \frac{-(1+x^2)}{\sqrt{(1+x^2)^2 - 4x^2}} \times \frac{2(1-x^2)}{(1+x^2)^2} \\ &= \frac{-2}{1+x^2} \cdot \frac{1-x^2}{|1-x^2|} = \begin{cases} \frac{-2}{1+x^2}, & \text{if } |x| < 1 \\ \frac{2}{1+x^2}, & \text{if } |x| > 1 \end{cases} \end{aligned}$$

Clearly, $f(x)$ is differentiable everywhere except at the points, where $|x| = 1$, i.e. $x = \pm 1$.

Hence, $f(x)$ is differentiable on $(-\infty, \infty) - \{-1, 1\}$.

56. Since, $f'(0)$ exists.

$$\begin{aligned} \therefore Rf'(0) &= Lf'(0) \\ \Rightarrow \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= - \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{h} \\ &= - \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \quad [\because f(-h) = f(h)] \end{aligned}$$

$$\Rightarrow 2 \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = 0 \Rightarrow 2f'(0) = 0$$

$$\therefore f'(0) = 0$$

57. We have,

$$f\left(\frac{x+y}{3}\right) = \frac{f(x) + f(y)}{3}, f(0) = 0 \text{ and } f'(0) = 3$$

$$\begin{aligned} \therefore f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f\left(\frac{3x+3h}{3}\right) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(3x) + f(3h)}{3} - \frac{f(3x) + f(0)}{3}}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(3h) - f(0)}{3h} = 3 \end{aligned}$$

$$\therefore f(x) = 3x + C$$

$$\therefore f(0) = 0 \Rightarrow C = 0$$

$$\therefore f(x) = 3x$$

58. We have, $f(0) = 0, g(x) = \frac{f(x)}{x}$

$$\text{Then, } \lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{f(x)}{x} = \lim_{x \rightarrow 0} f'(x) = f''(0)$$

Thus, the required value is $f''(0)$.

$$59. Rf'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} \therefore Rf'(4) &= \lim_{h \rightarrow 0} \frac{f(4+h) - f(4+0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2f(4) \cdot f(h) - 2f(4) \cdot f(0)}{h} \quad [\because f(x+y) = 2f(x) \cdot f(y)] \\ &= 2f(4) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = 2f(4) \times f'(0) \\ &= 2 \times 2 \times 3 = 12 \end{aligned}$$

$$60. Rf'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{1 + h\phi(h) - 1}{h} = f(x) \lim_{h \rightarrow 0} \phi(h) \\ &= f(x) \cdot 1 = f(x) \end{aligned}$$

$$\begin{aligned} \text{and } Lf'(x) &= \lim_{h \rightarrow 0} \frac{f(x-h) - f(x)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{f(x)f(-h) - f(x)}{-h} = f(x) \lim_{h \rightarrow 0} \frac{f(-h) - 1}{-h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{1 - h\phi(-h) - 1}{-h} \\ &= f(x) \cdot 1 = f(x) \end{aligned}$$

Hence, $f'(x)$ exists and is equal to $f(x)$.

61. We have, $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h} \quad [\because f(x+y) = f(x) \cdot f(y)] \\ &= f(x) \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} \frac{1 + h\phi(h) + h^2\phi(h)\psi(h) - 1}{h} \\ &= f(x) \lim_{h \rightarrow 0} [\phi(h) + h\phi(h)\psi(h)] \\ &= f(x)(a + 0 \cdot a \cdot b) = a \cdot f(x) \end{aligned}$$

$$62. f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(x) + f(h) - f(x)}{h} \quad [\because f(x+y) = f(x) + f(y)] \\ &= \lim_{h \rightarrow 0} \frac{h^3\phi(h)}{h} \quad [\because f(x) = x^3\phi(x)] \\ &= \lim_{h \rightarrow 0} h^2\phi(h) = 0 \times \phi(0) \\ &[\because \phi(x) \text{ is continuous at } x = 0, \text{ so } \lim_{h \rightarrow 0} \phi(h) = \phi(0)] \\ &= 0 \end{aligned}$$

$$63. f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h} \quad [\because f(x+y) = f(x) \cdot f(y)] \\ &= f(x) \lim_{h \rightarrow 0} \left(\frac{f(h) - 1}{h} \right) \\ &= f(x) \lim_{h \rightarrow 0} \frac{1 + h\phi(h)\log 2 - 1}{h} \\ &[\because f(x) = 1 + x\phi(x)\log 2] \\ &= f(x)\log 2 \lim_{h \rightarrow 0} \phi(h) \\ &= f(x) \cdot \log 2 \cdot 1 = \log 2^{f(x)} \quad [\because \lim_{h \rightarrow 0} \phi(h) = 1] \end{aligned}$$

64. We have,
 $f(x + y^{2n+1}) = f(x) + \{f(y)\}^{2n+1}$... (i)
 $\Rightarrow f(0) = f(0) + \{f(0)\}^{2n+1}$ [putting $x = y = 0$]
 $\Rightarrow f(0) = 0$

From Eq. (i), we get

$$f(1) = f(0) + \{f(1)\}^{2n+1}$$

$$= 0 + \{f(1)\}^{2n+1} \quad [\because f(0) = 0]$$

$$\Rightarrow f(1)[\{f(1)\}^{2n} - 1] = 0$$

$$\Rightarrow f(1) = 0 \text{ or } 1 \text{ or } -1$$

$$\text{Now, } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \frac{f(x)}{x} \quad [\because f(0) = 0]$$

$$\therefore f'(0) \geq 0$$

$$\therefore \lim_{x \rightarrow 0} \frac{f(x)}{0} \geq 0$$

$$\Rightarrow f(x) \geq 0 \text{ for } x > 0 \Rightarrow f(1) \neq -1$$

$$\text{Also, } f(1) = 0$$

$$\Rightarrow f(x+1) = f(x) + \{f(1)\}^{2n+1}$$

$$= f(x) \quad [\text{putting } y = 1 \text{ in Eq. (i)}]$$

$$= f(x)$$

$$= f(2) = f(3) = \dots = 0$$

which is not true. Thus, $f(1) = 1$

$$\therefore f(x+1) = f(x) + 1$$

$\Rightarrow f(x)$ is a function whose value is increased by 1 when x is increased by 1, therefore it is a linear function whose graph is a straight line passing through origin [$\because f(0) = 0$] having slope 1.

$$\text{Hence, } f(x) = x$$

$$\therefore f'(6) = 1$$

65. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x) \cdot f(h) - f(x)}{h}$

$$[\because f(x+y) = f(x) \cdot f(y)]$$

$$= f(x) \cdot \lim_{h \rightarrow 0} \left(\frac{f(h) - 1}{h} \right)$$

$$= f(x) \lim_{h \rightarrow 0} \frac{1 + hg(h)G(h) - 1}{h}$$

$$= f(x) \lim_{h \rightarrow 0} g(h)G(h)$$

$$= f(x) \lim_{h \rightarrow 0} g(h) \lim_{h \rightarrow 0} G(h) = ab f(x)$$

$$\therefore k = ab$$

66. $f(e) = [\log_e e] + \sqrt{\{\log_e e\}}$

$$= [1] + \sqrt{\{1\}} = 1 + 0 = 1$$

$$f(e^+) = [\log_e e^+] + \sqrt{\{\log_e e^+\}}$$

$$= \lim_{h \rightarrow 0} [1+h] + \sqrt{\{1+h\}} = 1 + 0 = 1$$

$$f(e^-) = [\log_e e^-] + \sqrt{\{\log_e e^-\}}$$

$$= \lim_{h \rightarrow 0} [1-h] + \sqrt{\{1-h\}}$$

$$= 0 + 1 = 1$$

Thus, $f(x)$ is continuous at $x = e$.

$$\text{Now, } f'(e^+) = \lim_{h \rightarrow 0} \frac{f(e+h) - f(e)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{[1+h] + \sqrt{\{1+h\}} - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{1 + \sqrt{h} - 1}{h} = \lim_{h \rightarrow 0} \frac{1}{\sqrt{h}} = \infty$$

Hence, $f(x)$ is not differentiable at $x = 0$.

67. $f(2+x) = f(-x)$... (i)

Replacing x by $x-1$, we have

$$f(2+x-1) = f(-x+1) \text{ or } f(1+x) = f(1-x)$$

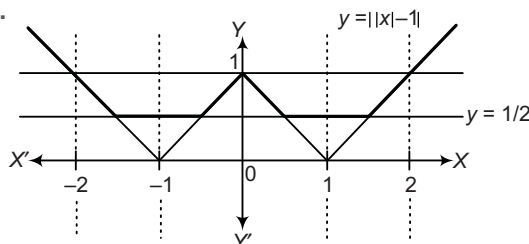
So, $f(x)$ is symmetrical about line $x = 1$.

Now, put $x = 2$ in Eq. (i), we get,

$$f(4) = f(-2)$$

Hence, differentiability at $x = 4$ implies differentiability at $x = -2$.

68.



Clearly, from the graph, $f(x)$ is not differentiable at five points, $x = -2, -1, 0, 1, 2$.

69. We have, $f(x) = \begin{cases} x^3, & x > 0 \\ 0, & x = 0 \\ -x^3, & x < 0 \end{cases}$

Clearly, $f(x)$ is continuous at $x = 0$.

$$(\text{LHD at } x = 0) = \left[\frac{d}{dx} (-x^3) \right]_{x=0} = [-3x^2]_{x=0} = 0$$

Similarly, $(\text{RHD at } x = 0) = 0$

So, $f(x)$ is differentiable at $x = 0$.

70. Since, $1 \leq x < 2$

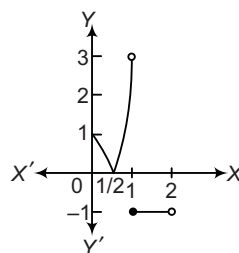
$$\therefore 0 \leq (x-1) < 1$$

$$\Rightarrow [x^2 - 2x] = [(x-1)^2 - 1]$$

$$= [(x-1)^2] - 1 = 0 - 1 = -1$$

$$\therefore f(x) = \begin{cases} 1 - 4x^2, & 0 \leq x < \frac{1}{2} \\ 4x^2 - 1, & \frac{1}{2} \leq x < 1 \\ -1, & 1 \leq x < 2 \end{cases}$$

Graph of $f(x)$ is given below:



Hence, it is clear from graph that $f(x)$ is discontinuous at $x = 1$ and not differentiable at $x = \frac{1}{2}$ and $x = 1$.

71. $f(x) = \begin{cases} |x| - 1, & |x| < 0 \\ |x|^2 - 2|x|, & |x| \geq 0 \end{cases}$

where, $|x| < 0$ is not possible.

Thus, neglecting it we get

$$f(|x|) = |x|^2 - 2|x|, |x| \geq 0$$

$$\therefore f(|x|) = \begin{cases} x^2 + 2x, & x < 0 \\ x^2 - 2x, & x \geq 0 \end{cases} \quad \dots(i)$$

$$\Rightarrow f'(|x|) = \begin{cases} 2x + 2, & x < 0 \\ 2x - 2, & x > 0 \end{cases}$$

Clearly, $f(|x|)$ is continuous at $x = 0$, but not differentiable at $x = 0$.

$$\therefore f(|x|) = \begin{cases} |x| - 1, & |x| < 0 \\ |x|^2 - 2|x|, & |x| \geq 0 \end{cases}$$

$$\text{Now, } g(x) = |f(x)| = \begin{cases} 1 - x, & x < 0 \\ -x^2 + 2x, & 0 \leq x < 2 \\ x^2 - 2x, & x \geq 2 \end{cases} \quad \dots(ii)$$

Clearly, $|f(x)|$ is discontinuous at $x = 0$, but continuous at $x = 2$.

$$\text{Also, } g'(x) = \begin{cases} -1, & x < 0 \\ -2x + 2, & 0 < x < 2 \\ 2x - 2, & x > 2 \end{cases}$$

Hence, $|f(x)|$ is not differentiable at $x = 0$ and $x = 2$.

$$72. f(x) = \begin{cases} e^x, & x \leq 0 \\ 1 - x, & 0 < x < 1 \\ x - 1, & x \geq 1 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = 1, \quad \lim_{x \rightarrow 0^+} f(x) = 1$$

$$\text{and } \lim_{x \rightarrow 1^-} f(x) = 0, \quad \lim_{x \rightarrow 1^+} f(x) = 0$$

Hence, $f(x)$ is continuous at $x = 0$ and 1 , $f'(0^-) = 1$ and $f'(0^+) = -1$.

Also, $f(x)$ is not differentiable at $x = 1$, $f'(x) = 6x + 12$.

$$73. f(x) = \int_{-2}^x |t + 1| dt = -\int_{-2}^{-1} (t + 1) dt + \int_{-1}^x (t + 1) dt$$

$$= \frac{1}{2} + \left[\frac{t^2}{2} + t \right]_{-1}^x = \frac{x^2}{2} + x + 1 \text{ for } x \geq -1$$

So, $f(x)$ is a quadratic polynomial.

$\therefore f(x)$ is continuous as well as differentiable in $[-1, 1]$.

Also, $f'(x)$ is continuous as well as differentiable in $[-1, 1]$.

74. For $b = 1$, we have

$$f(g(0)) = f(\sin(0) + 1) = f(1) = 1 + a$$

$$\text{Also, } f(g(0^+)) = \lim_{x \rightarrow 0^+} f(\sin x + 1) = f(1) = 1 + a$$

$$\text{and } f(g(0^-)) = \lim_{x \rightarrow 0^-} f(\{x\}) = f(1^-) = 1 + a$$

Hence, $f(g(x))$ is continuous for $b = 1$.

For $b < 0$,

$$f(g(0)) = f(\sin(0) + b) = f(b) = 2 - b$$

$$f(g(0^+)) = \lim_{x \rightarrow 0^+} f(\sin x + b) = f(b) = 2 - b$$

$$\text{and } f(g(0^-)) = \lim_{x \rightarrow 0^-} f(\{x\}) = f(1) = 1 + a$$

For continuity at $x = 0$, we must have $2 - b = 1 + a$ or $a + b = 1$.

$$75. \text{ We have, } f(0^-) = \lim_{n \rightarrow \infty} \left[\lim_{x \rightarrow 0^-} (\cos^2 x)^n \right]$$

$$= (\text{a value lesser than one})^\infty = 0$$

$$\text{and } f(0^+) = \lim_{n \rightarrow \infty} \left[\lim_{x \rightarrow 0^+} (1 + x^n)^{1/n} \right] = 1$$

$$\text{Also, } f(0) = 1$$

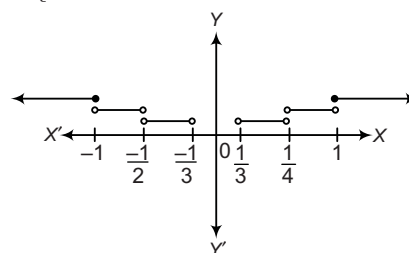
$\therefore$ Discontinuous at $x = 0$.

$$\text{Further, } f(1^-) = 1; f(1^+) = 0; f(1) = 1$$

$\therefore$ Discontinuous at $x = 1$.

$$76. f(x) = \begin{cases} 1, & |x| \geq 1 \\ \frac{1}{n^2}, & \frac{1}{n} < |x| < \frac{1}{n-1}; n = 2, 3, \dots \\ 0, & x = 0 \end{cases}$$

$$= \begin{cases} 1, & x \leq -1 \text{ or } x \geq 1 \\ \frac{1}{4}, & x \in \left(-1, -\frac{1}{2}\right) \cup \left(\frac{1}{2}, 1\right) \\ \frac{1}{9}, & x \in \left(-\frac{1}{2}, -\frac{1}{3}\right) \cup \left(\frac{1}{3}, \frac{1}{2}\right) \\ \vdots & \end{cases}$$



The function f is clearly continuous for $|x| > 1$.

We observe that

$$\lim_{x \rightarrow -1^+} f(x) = 1, \quad \lim_{x \rightarrow -1^-} f(x) = \frac{1}{4}$$

$$\text{Also, } \lim_{x \rightarrow 1/n^+} f(x) = \frac{1}{n^2} \text{ and } \lim_{x \rightarrow 1/n^-} f(x) = \frac{1}{(n+1)^2}$$

Thus, f is discontinuous for $x = \pm \frac{1}{n}; n = 1, 2, 3, \dots$

77. For continuity at $x = 1$,

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (x^2 \operatorname{sgn}[x] + \{x\}) = 1 + 0 = 1$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 \operatorname{sgn}[x] + \{x\}) = 1 \operatorname{sgn}(0) + 1 = 1$$

$$\text{Also, } f(1) = 1$$

$$\therefore \text{ LHL} = \text{RHL} = f(1)$$

Hence, $f(x)$ is continuous at $x = 1$.

Now, for differentiability,

$$f'(1^+) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^2 \operatorname{sgn}[1+h] + \{1+h\} - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h)^2 + h - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 + 3h}{h} = 3$$

$$\text{and } f'(1^-) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{(1-h)^2 \operatorname{sgn}[1-h] + \{1-h\} - 1}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{(1-h)^2 + 1 - h - 1}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{h^2 - 3h}{-h} = 3$$

$$\therefore f'(1^+) = f'(1^-)$$

Therefore, $f(x)$ is differentiable at $x = 1$.

Now, at $x = 2$,

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (x^2 \operatorname{sgn}[x] + \{x\}) = 4 \times 0 + 1$$

$$\lim_{h \rightarrow 2^+} f(x) = \lim_{h \rightarrow 2^+} (\sin x + |x - 3|) = 1 + \sin 2$$

$\therefore \text{LHL} \neq \text{RHL}$

Hence, $f(x)$ is discontinuous at $x = 2$ and then $f(x)$ is also not differentiable at $x = 2$.

78. $f(x + y) = f(x) + f(y) + xy(x + y)$

$$f(0) = 0$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(h)}{h} = -1$$

$$\therefore \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + xh(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h)}{h} + \lim_{h \rightarrow 0} x(x+h) = -1 + x^2$$

$$\therefore f'(x) = -1 + x^2$$

$$\text{Now, } f(x) = \frac{x^3}{3} - x + C$$

Hence, $f(x)$ is a polynomial function, $f(x)$ is twice differentiable for all $x \in \mathbb{R}$ and $f'(3) = 3^2 - 1 = 8$.

79. $f'(x) = 6x + 12$

For $f(x)$ is increasing, $f'(x) \geq 0 \Rightarrow x \geq -2$

So, $f(x)$ is increasing in $[-1, 2]$.

$$\therefore \lim_{x \rightarrow 2^+} f(x) = 35, \lim_{x \rightarrow 2^-} f(x) = 35$$

$$\text{and } \lim_{x \rightarrow 2} f(x) = 35$$

Thus, $f(x)$ is continuous on $[-1, 3]$.

Also, $f'(2^-) = 24$ and $f'(2^+) = -1$, therefore $f'(2)$ does not exist for maximum, $f(2) = 35$, $f(-1) = -10$ and $f(3) = 34$.

Hence, $f(x)$ has maximum value at $x = 2$.

80. Since, $f(x) = 0$, $x < 0$ and differentiable at

$$x = 0 = \text{LHD} = 0 \text{ [function is X-axis for } x < 0 \text{]}$$

If $f(x)$,

(a) x^2 , $x > 0$,

RHD at $x = 0$,

$$f'(0) = 2 \times 0 = 0 \text{ which is possible.}$$

(d) $-x^{3/2}$, $x > 0$

RHD at $x = 0$,

$$f'(0) = -\frac{3}{2} x^{1/2} = -\frac{3}{2} \times 0$$

$$= 0, \text{ which is possible.}$$

81. Statement I is incorrect because, if $\lim_{x \rightarrow a} g(x)$ and

$\lim_{x \rightarrow a} f(g(x))$ approach e from the same side of e (say right side) and $\lim_{x \rightarrow e} f(x) = f(e) \neq \lim_{x \rightarrow e} f(x)$, then

$$\lim_{x \rightarrow a} f(g(x)) = f(e^+) = f(e).$$

Statement II is correct.

82. $|f(x)| \leq |x|$

$$\Rightarrow 0 \leq |f(x)| \leq |x|$$

So, graph of $y = |f(x)|$ lies between the graph of $y = 0$ and $y = |x|$.

$$\text{Also, } |f(0)| \leq 0 \Rightarrow f(0) = 0$$

Also, from Sandwich theorem,

$$\lim_{x \rightarrow 0} 0 \leq \lim_{x \rightarrow 0} |f(x)| \leq \lim_{x \rightarrow 0} |x|$$

$$\Rightarrow \lim_{x \rightarrow 0} |f(x)| = 0$$

$\therefore y = f(x)$ is continuous at $x = 0$.

Also, Statement II is correct but it is not the correct explanation for Statement I.

83. We know that, $0 \leq \cos^2(n! \pi x) \leq 1$

$$\therefore \lim_{m \rightarrow \infty} \cos^{2m}(n! \pi x) = 0 \text{ or } 1, \text{ as}$$

$$0 \leq \cos^2(n! \pi x) < 1 \text{ or } \cos^2(n! \pi x) = 1$$

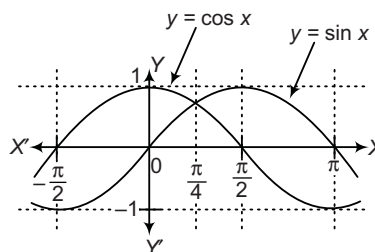
Also, $n \rightarrow \infty$, then $n! x = \text{integer}$, if $x \in \mathbb{Q}$

and $n! x \neq \text{integer}$, if $x \in \text{irrational}$

$$\text{Hence, } f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

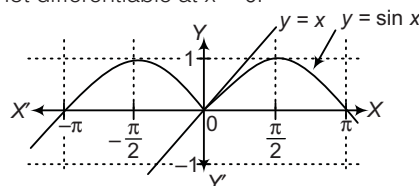
Thus, $h(x) = 1, \forall x \in \mathbb{R}$ which is continuous for all x ; however, Statement II does not correctly explain Statement I as the addition of discontinuous functions may be continuous.

84.

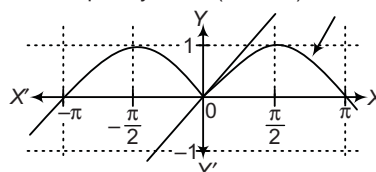


From the graph, Statement I is true.

Statement II Consider, $f(x) = \min\{x, \sin|x|\}$ is differentiable at $x = 0$, though $g(x) = \max\{x, \sin|x|\}$ is not differentiable at $x = 0$.



Graph of $y = \min(x, \sin|x|)$



Graph of $y = \max(x, \sin|x|)$

85. $f(x) = (\sin \pi x)(x - 1)^{1/5}$ is continuous function as both $(\sin \pi x)$ and $(x - 1)^{1/5}$ are continuous.

But $(x - 1)^{1/5}$ is not differentiable at $x = 1$.

$$\text{However, } f'(1^-) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin[\pi(1-h)](1-h-1)^{1/5} - 0}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\sin(\pi h)(-h)^{1/5}}{h} = 0$$

and $f'(1^+) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$

$$= \lim_{h \rightarrow 0} \frac{\sin[\pi(1+h)](1+h-1)^{1/5} - 0}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-\sin(\pi h)(h)^{1/5}}{h} = 0$$

Hence, $f(x)$ is differentiable at $x = 1$, though $(x-1)^{1/5}$ is not differentiable at $x = 1$.

However, Statement II is correct but it is not a correct explanation of Statement I.

86. Statement I is correct as though $|4x^2 - 12x + 5|$ is not differentiable at $x = \frac{1}{2}, \frac{5}{2}$ but $\cos \pi x = 0$, those points. So, $f'\left(\frac{1}{2}\right)$ and $f'\left(\frac{5}{2}\right)$ exist.

87. Statement I is incorrect because, if $\lim_{x \rightarrow a^-} g(x)$ and $\lim_{x \rightarrow a^+} g(x)$ approach e from the same side of e (say right side) and $\lim_{x \rightarrow e^+} f(x) = f(e) \neq \lim_{x \rightarrow e^-} f(x)$, then $\lim_{x \rightarrow a} f(g(x)) = f(e^+) = f(e)$.
 $\therefore$ Statement II is correct.

88. Let $h(x) = f(x) - \sin x$

$$\Rightarrow x \in \left[0, \frac{\pi}{2}\right]$$

$$\Rightarrow h(0) = h\left(\frac{\pi}{2}\right) = 0$$

According to Rolle's theorem for atleast one $c \in \left(0, \frac{\pi}{2}\right)$,

$$h'(c) = f'(c) - \sin c = 0$$

for $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$, $h\left(\frac{\pi}{2}\right) = h\left(\frac{3\pi}{2}\right) = 0$

According to Rolle's theorem, for atleast one $d \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$,

$$h'(d) = f'(d) - \cos d = 0$$

for $x \in [c, d]$, $h'(c) = h'(d) = 0$

According to Rolle's theorem, for atleast one $x \in (c, d)$,

$$h'(x) = f''(x) + \sin x = 0$$

$\therefore |f''(x)| \leq 1$ for atleast one $x \in \left(0, \frac{3\pi}{2}\right)$.

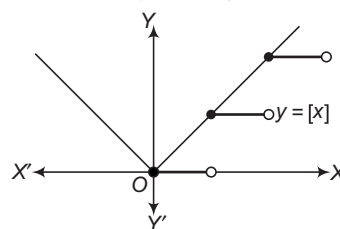
89. $f(x) = \begin{cases} \sin x, & x \leq 0 \\ \tan x, & 0 < x < 2\pi \\ \cos x, & 2\pi \leq x < 3\pi \\ 3\pi, & x = 3\pi \end{cases}$

$\therefore f(x)$ is discontinuous at $\frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$.

90. $f(x) = [2x^3 - 5]$
 $2x^2 = 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15$
Hence, number of points of discontinuity = 13.

91. $\max([x], |x|) = |x|$

Hence, discontinuity at $x = \phi$



Solutions (Q. Nos. 92-94)

$$F(x) = \lim_{n \rightarrow \infty} \frac{f(x) + x^{2n}g(x)}{1 + x^{2n}}$$

$$= \begin{cases} f(x), & 0 \leq x^2 < 1 \\ \frac{f(x) + g(x)}{2}, & x^2 = 1 \\ g(x), & x^2 > 1 \end{cases}$$

Since, $F(x)$ is continuous at $x = 1$.

$$\therefore \lim_{x \rightarrow 1^+} F(x) = \lim_{x \rightarrow 1^-} F(x) = F(1)$$

$$\Rightarrow g(1^+) = f(1^-) = \frac{f(1) + g(1)}{2}$$

$$\Rightarrow b + 1 = a + 4 = \frac{a + 4 + b + 1}{2}$$

$$\Rightarrow b + 1 = a + a = \frac{a + b + 5}{2}$$

$$\therefore a + 3 = b \quad \dots(i)$$

Again, $F(x)$ is continuous at $x = -1$.

$$\therefore \lim_{x \rightarrow -1^-} F(x) = \lim_{x \rightarrow -1^+} F(x) = F(-1)$$

$$\Rightarrow g(-1^-) = f(1^+) = \frac{f(-1) + g(-1)}{2}$$

$$\Rightarrow b - 1 = 4 - a = \frac{b - 1 + 4 - a}{2}$$

$$\Rightarrow b - 1 = 4 - a = \frac{b - a + 3}{2}$$

$$\text{or } a + b = 5 \quad \dots(ii)$$

If $F(x)$ is continuous at $x = \pm 1$ and $f(x) = g(x)$

$$\Rightarrow x^2 + ax + 3 = x + b, \text{ where } a = 1, b = 4$$

[on solving Eqs. (i) and (ii)]

$$\text{or } x^2 + (a-1)x + (3-b) = 0$$

$$\Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$$

Solutions (Q. Nos. 95-97)

For $0 \leq x < \frac{\pi}{4}$, $g(x) = 1 + \tan x$

$$x \in \left[0, \frac{\pi}{4}\right) \Rightarrow 1 + \tan x \in [1, 2)$$

So, $f(g(x)) = f(1 + \tan x) = 1 + \tan x + 2$

and for $x \in \left[\frac{\pi}{4}, \pi\right)$, $g(x) = -\cot x$

$$x \in \left[\frac{\pi}{4}, \pi\right) \Rightarrow 3 - \cot x \in [2, \infty)$$

So, $f(g(x)) = f(3 - \cot x) = 6 - (3 - \cot x)$

Let $h(x) = f(g(x)) = \begin{cases} 3 + \tan x, & 0 \leq x < \frac{\pi}{4} \\ 3 + \cot x, & \frac{\pi}{4} \leq x < \pi \end{cases}$

Clearly, $f(g(x))$ is continuous in $[0, \pi)$.

Now, $h'\left(\frac{\pi^+}{4}\right) = \lim_{x \rightarrow \pi/4^+} (-\operatorname{cosec}^2 x) = -2$

and $h'\left(\frac{\pi^-}{4}\right) = \lim_{x \rightarrow \pi/4^-} (\sec^2 x) = 2$

So, $f(g(x))$ is differentiable everywhere in $[0, \pi)$ other than at $x = \frac{\pi}{4}$.

$$|f(g(x))| = \begin{cases} |3 + \tan x|, & 0 \leq x < \frac{\pi}{4} \\ |3 + \cot x|, & \frac{\pi}{4} \leq x < \pi \end{cases}$$

which is not differentiable at $x = \frac{\pi}{4}$ and where

$$3 + \cot x = 0 \quad \text{or} \quad x = \cot^{-1}(-3)$$

For $x \in \left[0, \frac{\pi}{4}\right)$, $3 + \tan x \in [3, 4]$

For $x \in \left[\frac{\pi}{4}, \pi\right)$, $3 + \cot x \in (-\infty, 4]$

Hence, the range is $(-\infty, 4]$.

Solutions (Q. Nos. 98-100)

$$f(x) = \begin{cases} [x], & -2 \leq x \leq -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \leq 2 \end{cases} = \begin{cases} -2, & -2 \leq x < -1 \\ -1, & -1 \leq x \leq -\frac{1}{2} \\ 2x^2 - 1, & -\frac{1}{2} < x \leq 2 \end{cases}$$

$$|f(x)| = \begin{cases} 2, & -2 \leq x < -1 \\ 1, & -1 \leq x \leq -\frac{1}{2} \\ |2x^2 - 1|, & -\frac{1}{2} < x \leq 2 \end{cases} = \begin{cases} 2, & -2 \leq x < -1 \\ 1, & -1 \leq x \leq -\frac{1}{2} \\ 1 - 2x^2, & -\frac{1}{2} < x \leq \frac{1}{\sqrt{2}} \\ 2x^2 - 1, & \frac{1}{\sqrt{2}} < x \leq 2 \end{cases}$$

$$f(|x|) = \begin{cases} -2, & -2 \leq |x| < -1 \\ -1, & -1 \leq |x| \leq -\frac{1}{2} \\ 2|x|^2 - 1, & -\frac{1}{2} < |x| \leq 2 \end{cases} = \begin{cases} -2, & -2 \leq |x| < -1 \\ -1, & -1 \leq |x| \leq -\frac{1}{2} \\ 2x^2 - 1, & -2 \leq x \leq 2 \end{cases}$$

$$\therefore g(x) = f(|x|) + f(x) = \begin{cases} 2x^2 + 1, & -2 \leq x < -1 \\ 2x^2, & -1 \leq x \leq -\frac{1}{2} \\ 0, & -\frac{1}{2} < x < \frac{1}{\sqrt{2}} \\ 4x^2 - 2, & \frac{1}{\sqrt{2}} \leq x \leq 2 \end{cases}$$

$$g(-1^-) = \lim_{x \rightarrow -1^-} (2x^2 + 1) = 3, g(-1^+) = \lim_{x \rightarrow -1^+} 2x^2 = 2$$

$$g\left(-\frac{1}{2}\right) = \lim_{x \rightarrow -1/2} 2x^2 = \frac{1}{2}, g\left(-\frac{1}{2}\right) = \lim_{x \rightarrow -1/2} 0 = 0$$

$$g\left(\frac{1}{\sqrt{2}}\right) = \lim_{x \rightarrow 1/\sqrt{2}} 0 = 0,$$

$$g\left(\frac{1}{\sqrt{2}}\right) = \lim_{x \rightarrow 1/\sqrt{2}} (4x^2 - 2) = 0$$

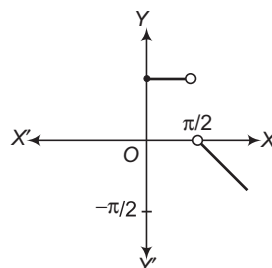
So, $g(x)$ is discontinuous at $x = -1, -\frac{1}{2}$.

Also, $g(x)$ is continuous at $x = \frac{1}{\sqrt{2}}$.

Now, $g'\left(\frac{1^-}{\sqrt{2}}\right) = 0, g'\left(\frac{1^+}{\sqrt{2}}\right) = 8\left(\frac{1}{\sqrt{2}}\right) = \frac{8}{\sqrt{2}}$

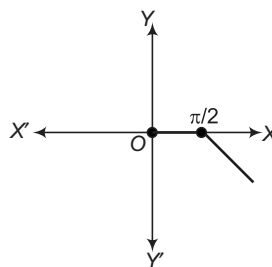
Hence, $g(x)$ is not differentiable at $x = \frac{1}{\sqrt{2}}$.

101. From figure, option (c) is correct.



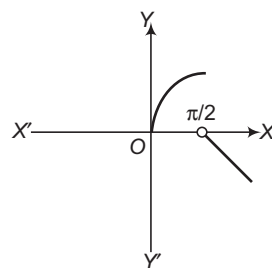
Graph for $H_2(x)$

102. From figure, option (b) is correct.



Graph for $H_3(x)$

103. From figure, option (c) is correct.



Graph for $H_4(x)$

104. A. $f(x) = \begin{cases} \frac{5e^{1/x} + 2}{3 - e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f(0^+) = \lim_{h \rightarrow 0} \frac{5e^{1/h} + 2}{3 - e^{1/h}} = \lim_{h \rightarrow 0} \frac{5 + 2e^{-1/h}}{3e^{-1/h} - 1} = -5.$$

Hence, $f(x)$ is discontinuous and not differentiable at $x = 0$.

B. $g(x) = xf(x) = \begin{cases} x \frac{5e^{1/x} + 2}{3 - e^{1/x}}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

$$f(0^+) = \lim_{h \rightarrow 0} h \frac{5e^{1/h} + 2}{3 - e^{1/h}} = \lim_{h \rightarrow 0} h \frac{5 + 2e^{-1/h}}{3e^{-1/h} - 1} = 0 \times (-5) = 0$$

$$f(0^-) = \lim_{h \rightarrow 0} h \frac{5e^{-1/h} + 2}{3 - e^{-1/h}} = 0 \times (2/3) = 0$$

Hence, $f(x)$ is continuous at $x = 0$.

$$\begin{aligned} LF'(0) &= \lim_{h \rightarrow 0} \frac{g(0-h) - g(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{-hf(-h) - 0}{-h} = \lim_{h \rightarrow 0} f(-h) \\ &= \lim_{h \rightarrow 0} \frac{5e^{-1/h} + 2}{3 - e^{-1/h}} = \frac{0 + 2}{3 - 0} = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} RF'(0) &= \lim_{h \rightarrow 0} \frac{g(0+h) - g(0)}{h} = \lim_{h \rightarrow 0} \frac{g(h) - 0}{h} \\ &= \lim_{h \rightarrow 0} f(h) = \lim_{h \rightarrow 0} \frac{5e^{1/h} + 2}{3 - e^{1/h}} \\ &= \lim_{h \rightarrow 0} \frac{5 + 2e^{-1/h}}{3e^{-1/h} - 1} = \frac{5 + 0}{0 - 1} = -5 \end{aligned}$$

$$\therefore LF'(0) \neq RF'(0)$$

Hence, $f(x)$ is not differentiable but continuous at $x = 0$.

C. For $x^2 f(x)$,

$$\text{Let } F(x) = x^2 f(x)$$

$$\begin{aligned} \therefore LF'(0) &= \lim_{h \rightarrow 0} \frac{F(0-h) - F(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 f(-h) - 0}{-h} = 0 \end{aligned}$$

$$\begin{aligned} RF'(0) &= \lim_{h \rightarrow 0} \frac{F(0+h) - F(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 f(h) - 0}{h} = 0 \end{aligned}$$

$$\therefore LF'(0) = RF'(0)$$

Hence, $F(x)$ is differentiable at $x = 0$, then it is always continuous at $x = 0$.

D. Clearly, from the above discussion $y = x^{-1} f(x)$ is discontinuous and hence not differentiable at $x = 0$.

105. Put $x = y = 0 \Rightarrow f(0) = 0$

$$\begin{aligned} \text{Now, } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{f(x) + f(h)}{1 + f(x)f(h)} - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h)[1 - \{f(x)\}^2]}{h[1 + f(x)f(h)]} \\ &= \lim_{h \rightarrow 0} \left\{ \frac{f(h) - f(0)}{h - 0} \right\} \left[\frac{1 - \{f(x)\}^2}{1 + f(x)f(h)} \right] \\ &= f'(0)[1 - \{f(x)\}^2] = 1 - \{f^2(x)\} \quad \dots(i) \end{aligned}$$

On Integrating, we get

$$\frac{1}{2} \ln \left[\frac{1 + f(x)}{1 - f(x)} \right] = x + C \text{ or } \frac{1 + f(x)}{1 - f(x)} = ke^{2x}$$

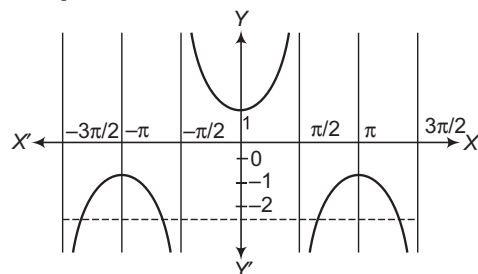
$$\text{Now, } f(0) = 0 \Rightarrow k = 1$$

$$\therefore f(x) = \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Clearly, $f(x)$ is differentiable for all $x \in R$ and from Eq. (i), $f'(x) > 0$ for all $x \in R$. Again, $f(x)$ is an odd function, so $\int_{-1}^1 f(x) dx = 0$.

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow \infty} [f(x)]^x &= \lim_{x \rightarrow \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right)^x \\ &= e^{\lim_{x \rightarrow \infty} \left(\frac{e^x - e^{-x}}{e^x + e^{-x}} \right) x} = e^{-2 \lim_{x \rightarrow \infty} \left(\frac{xe^{-x}}{e^x + e^{-x}} \right)} \\ &= e^{-2 \lim_{x \rightarrow \infty} \left(\frac{x}{e^{2x} + 1} \right)} \\ &= e^{-2 \lim_{x \rightarrow \infty} \left(\frac{1}{2e^{2x}} \right)} = e^0 = 1 \end{aligned}$$

$$106. f(x) = \begin{cases} [x], & -2 \leq x < 0 \\ |x|, & 0 \leq x \leq 2 \end{cases}$$



$$\Rightarrow f(x) = \begin{cases} -2, & -2 \leq x < -1 \\ -1, & -1 \leq x < 0 \\ x, & 0 \leq x \leq 2 \end{cases}$$

$$g(x) = \sec x, x \in R - (2n+1)\frac{\pi}{2}$$

$$fog(x) = \begin{cases} -2, & -2 \leq \sec x < -1 \\ -1, & -1 \leq \sec x < 0 \\ \sec x, & 0 \leq \sec x \leq 2 \end{cases}$$

$$\therefore fog(x) = \begin{cases} -2, & x \in \left[-\frac{4\pi}{3}, -\frac{2\pi}{3} \right] \cup \left[\frac{2\pi}{3}, \frac{4\pi}{3} \right] \\ -1, & x = -\pi, \pi \\ \sec x, & x \in \left[-\frac{\pi}{3}, \frac{\pi}{3} \right] \end{cases}$$

limit of fog exist at $x = \pi, -\pi, -1$, points of discontinuity of fog are $-\pi, \pi$.

Points of differentiability of fog are $-1, \frac{5\pi}{6}$

$$gof(x) = \begin{cases} \sec(-2), & x \in [-2, -1] - \left\{ -\frac{\pi}{2} \right\} \\ \sec(-1), & x \in [-1, 0] \\ \sec x, & x \in [0, 2] - \left\{ \frac{\pi}{2} \right\} \end{cases}$$

limit of gof does not exist at $x = -1$.

$$107. A. f(x) = \begin{cases} 0, & 1 < x \leq 2 \\ 1-x, & 1 \leq x \leq 2 \\ -\sin \pi x, & -1 \leq x < 0 \end{cases}$$

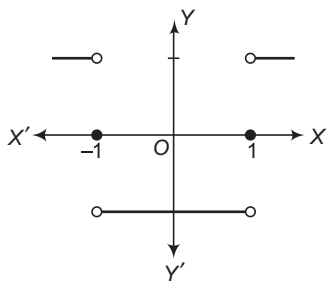
continuous at $x = 1$ but not differentiable.

$$B. f'(0^-) = \lim_{h \rightarrow 0^-} \frac{h^2 e^{-1/h} - 0}{-h} = \lim_{h \rightarrow 0^-} (-he^{-1/h}) = 0$$

$$C. g(x) = \frac{1}{1 + \frac{1}{x}(2+2x)} = \frac{x}{3x+2}. \text{ Thus, the points,}$$

where $g(x)$ is not differentiable, are $x = 0, -1, -\frac{2}{3}$.

- D. Vertical tangents exist at $x = 1$ and $x = -1$ else where horizontal tangents exist.
 $\therefore$ Number of points where tangent does not exist is 0.



108. $\tan^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$
 $\sec^2 x$ is discontinuous at $x = \frac{\pi}{2}, \frac{3\pi}{2}$
 $\therefore$ Number of discontinuities = 2

109. $f(0) = \lim_{x \rightarrow 0} \frac{\tan(\tan x) - \sin(\sin x)}{\tan x - \sin x}$
 $= \lim_{x \rightarrow 0} \frac{\tan(\tan x) - \sin(\sin x)}{\frac{\tan x}{x} \left(\frac{1 - \cos x}{x^2} \right) x^3}$
 $= 2 \lim_{x \rightarrow 0} \frac{\tan(\tan x) - \sin(\sin x)}{x^3}$
 $= 2 \lim_{x \rightarrow 0} \left[\frac{\left(\tan x + \frac{\tan^3 x}{3} + \frac{2}{15} \tan^5 x + \dots \right) - \left(\sin x + \frac{\sin^3 x}{3!} + \frac{\sin^5 x}{5} \dots \right)}{x^3} \right]$
 $= 2 \lim_{x \rightarrow 0} \left[\left(\frac{\tan x - \sin x}{x^3} \right) + \left(\frac{\tan^3 x}{3! x^3} + \frac{\sin^3 x}{3! x^3} \right) + \dots \right]$
 $= 2 \lim_{x \rightarrow 0} \left[\left(\frac{\tan x}{x} \right) \left(\frac{1 - \cos x}{x^2} \right) + \frac{1}{3} + \frac{1}{6} \right]$
 $= 2 \left[\frac{1}{2} + \frac{1}{2} \right] = 2$

110. Let $g(x) = x^{1/x}$,
 $g'(x) = x^{1/x} \frac{1 - \ln x}{x^2}$

$g_{\max} = e^{1/e} \in (1, 2), \lim_{x \rightarrow 0} x^{1/x} = 0, \lim_{x \rightarrow \infty} x^{1/x} = 1$

So, $f(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 \leq x < \infty \end{cases}$

111. $f(x) = x + \cos x + 2, f(0) = 3 \Rightarrow g(3) = 0$
 $g(f(x)) = x$
 $\Rightarrow g'(f(x)) \cdot f'(x) = 1$
 Putting $x = 0$, we get
 $f'(3) \cdot f'(0) = 1$
 Now, $f'(x) = 1 + \sin x$
 $\Rightarrow f'(0) = 1 \Rightarrow g'(3) = 1$

112. Let $g(x) = x \tan^{-1}(x^2)$. It is an odd function.

So, $g^{2m}(0) = 0$

Let $h(x) = x^4$

So, $f(x) = g(x) + h(x)$

$\Rightarrow f^{2m}(0) = g^{2m}(0) + h^{2m}(0) = h^{2m}(0) \neq 0$

It happens when $2m = 4 \Rightarrow m = 2$

113. Given, $\frac{\int_{f(y)}^{f(x)} e^t dt}{\int_y^x (1/t) dt} = 1$

$\Rightarrow e^{f(x)} - e^{f(y)} = \ln x - \ln y$

$\Rightarrow e^{f(x)} - \ln x = C$

$\Rightarrow f(x) = \ln(\ln x + C)$

Since, $f\left(\frac{1}{e}\right) = 0$

$\Rightarrow C = 2$

Now, $f(g(x)) = \begin{cases} \ln(x+2); & x \geq k \\ \ln(2+x^2); & 0 < x < k \end{cases}$

For continuity at $x = k$,

$\ln(k+C) = \ln(k^2+C) \Rightarrow$ either $k = 0$ or $k = 1$

$\therefore k > 0 \Rightarrow k = 1$

114. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + 2xh(x+h) - \frac{1}{3} - \left(f(x) + f(0) - \frac{1}{3} \right)}{h}$

$= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} + 2x^2 = f'(0) + 2x^2$

$\lim_{h \rightarrow 0} \frac{3f(h) - 1}{6h} = \lim_{h \rightarrow 0} \frac{f(h) - \frac{1}{3}}{2h}$

$= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{2h}$

$= \frac{f'(0)}{2} = \frac{2}{3} \Rightarrow f'(0) = \frac{4}{3}$

$\therefore f'(x) = \frac{4}{3} + 2x^2$

$f(x) = \lambda + \frac{4}{3}x + \frac{2x^3}{3}$

$\Rightarrow f(0) = \lambda = \frac{1}{3}$

$\therefore f(x) = \frac{2x^3}{3} + \frac{4}{3}x + \frac{1}{3} \Rightarrow f(2) = \frac{25}{3}$

115. $\text{sgn}(x^2 - 3x + 2)$ is discontinuous when $x^2 - 3x + 2 = 0$ or $x = 1, 2$.

$[x - 3] = [x] - 3$ is discontinuous at $x = 1, 2, 3, 4$.

Thus, $f(x)$ is discontinuous at $x = 3, 4$.

Now, both $\text{sgn}(x^2 - 3x + 2)$ and $[x - 3]$ are discontinuous at $x = 1$ and 2 .

Then, $f(x)$ may be continuous at $x = 1$ and 2 .

But $f(1) = -2$ and $f(1^+) = -1 + 0 - 3 = -4$

Thus, $f(x)$ is discontinuous at $x = 1$.

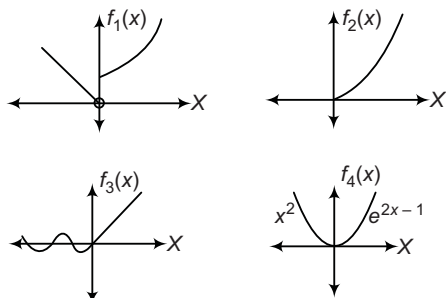
Also, $f(2) = -1$ and $f(2^+) = 1 - 1 = 0$

Hence, $f(x)$ is discontinuous at $x = 2$ also.

Entrances Gallery

$$1. f_2(f_1(x)) = (f_1(x))^2 - \begin{cases} x^2, & x < 0 \\ e^{2x}, & x \geq 0 \end{cases}$$

$$f_4(x) = \begin{cases} x^2, & x < 0 \\ e^{2x} - 1, & x \geq 0 \end{cases}$$



$\therefore f_4(x)$ is many-one, onto, continuous and non-differentiable,

$f_3(x)$ is many-one, into, continuous and derivable and $f_2(x)$ is one-one, into and differentiable.

$$2. g'(a) = \int_h^{1-h} \frac{\partial}{\partial a} t^{-a}(1-t)^{a-1} dt$$

$$= - \int_h^{1-h} t^{-a}(1-t)^{a-1} dt + t^{-a}(1-t)^{a-1} dt = 0$$

$$g(a) = \text{constant} \Rightarrow g(a) = \lambda$$

$$g(a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} \frac{1}{\sqrt{t(1-t)}} dt$$

$$g\left(\frac{1}{2}\right) = \int_h^{1-h} \frac{dt}{\sqrt{\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}} = \left[\sin^{-1} \frac{t - \frac{1}{2}}{\frac{1}{2}} \right]_h^{1-h}$$

$$= [\sin^{-1}(2t - 1)]_h^{1-h} = \sin^{-1}(1 - 2h) - \sin^{-1}(2h - 1) = \pi$$

$$3. g(a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1} dt$$

$$g(1-a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1} dt$$

$$= \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a} dt$$

$$= \lim_{h \rightarrow 0^+} \int_h^{1-h} (1-t)^{a-1} [1 - (1-t)]^{-a} dt$$

$$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$= \lim_{h \rightarrow 0^+} \int_h^{1-h} (1-t)^{a-1} t^{-a} dt$$

$$\Rightarrow g(1-a) = g(a)$$

$$\Rightarrow -g'(1-a) = g'(a)$$

$$\text{At } a = \frac{1}{2}, -g'\left(\frac{1}{2}\right) = g'\left(\frac{1}{2}\right)$$

$$\therefore g'\left(\frac{1}{2}\right) = 0$$

$$4. f'(x) = 2x f(x)$$

$$\frac{f'(x)}{f(x)} = 2x$$

$$\ln(f(x)) = x^2 + C$$

$$x = 0, f(0) = 1$$

$$C = 0$$

$$\therefore \ln(f(x)) = x^2$$

$$\Rightarrow f(x) = e^{x^2}$$

$$\therefore F(x) = f(x) + C$$

$$\Rightarrow F(x) = e^{x^2} + C \Rightarrow F(0) = 0$$

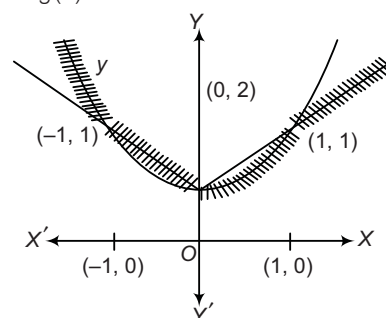
$$\text{So, } C = -1$$

$$\therefore F(x) = e^{x^2} - 1$$

$$\text{Hence, } F(2) = e^4 - 1$$

$$5. f(x) = |x| + 1 = \begin{cases} x + 1 & x \geq 0 \\ -x + 1 & x < 0 \end{cases}$$

$$\Rightarrow g(x) = x^2 = 1$$



Number of non-differentiable points are 3.

$$6. \text{ Consider } h(x) = f(x) - g(x). \text{ Assume } a < b$$

$$\Rightarrow h(a) = \lambda - g(a) > 0$$

$$\Rightarrow h(b) = f(b) - \lambda < 0$$

Else if $a > b$, $h(a) < 0$ and $h(b) > 0$.

By intermediate value theorem, $h(c) = 0$... (i)

$$(a) (f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$$

$$(f(c) - g(c))(f(c) + g(c) + 3) = 0$$

So, there exist c such that : $f(c) - g(c)$

[from Eq. (i)]

Hence, option (a) is correct.

$$(d) \text{ Similarly, } (f(c))^2 = (g(c))^2$$

$$(f(c) - g(c))(f(c) + g(c)) = 0$$

Therefore, option (d) is correct.

Options (b) and (c) are wrong as by counter

e.g. If $f(x) = g(x) = \lambda \neq 0$, then

$$(b) \lambda^2 + \lambda = \lambda^2 + 3\lambda, \text{ which is not possible.}$$

$$(c) \lambda^2 + 3\lambda = \lambda^2 + \lambda, \text{ which is not possible.}$$

$$7. \text{ It may be discontinuous at } x = a \text{ or } x = b.$$

$$\text{So, } \lim_{x \rightarrow a^-} g(x) = 0$$

$$\therefore \lim_{x \rightarrow a^+} g(x) = \lim_{x \rightarrow a^+} \int_a^x f(t) dt = \int_a^x f(t) dt = 0$$

$$\Rightarrow g(a) = \int_a^a f(t) dt = 0$$

Similarly at $x = b$, we will get $g(x)$ continuous.

So, $g(x)$ is continuous, $\forall x \in R$.

$$\Rightarrow g'(x) = \begin{cases} 0, & x < a \\ f(x), & a \leq x \leq b \\ 0, & x > b \end{cases}$$

$$\Rightarrow g'(a^-) = 0 \text{ and } g'(b^-) = f(b)$$

$$\Rightarrow g'(a^+) = f(a) \text{ and } g'(b^+) = 0$$

Since, $f(x)$ codomain is $[1, \infty)$.

$\therefore f(a)$ and $f(b)$ can never be zero.

Hence, it is not derivable at $x = a$ and $x = b$.

8. Here, $f'(x) < 2f(x)$, ... (i)

On multiplying by $e^{-\int 2dx}$ both sides of Eq. (i), we get
 $f'(x) \cdot e^{-2x} - 2e^{-2x}f(x) < 0$

$$\Rightarrow \frac{d}{dx}(f(x) \cdot e^{-2x}) < 0$$

$\therefore \phi(x) = f(x)e^{-2x}$ is decreasing for $x \in \left[\frac{1}{2}, 1\right]$.

Thus, when $x > \frac{1}{2}$

$$\phi(x) < \phi\left(\frac{1}{2}\right)$$

$$\Rightarrow e^{-2x}f(x) < e^{-1} \cdot f\left(\frac{1}{2}\right)$$

$$\Rightarrow f(x) < e^{2x-1} \cdot 1, \text{ given } f\left(\frac{1}{2}\right) = 1$$

$$\Rightarrow 0 < \int_{1/2}^1 f(x)dx < \int_{1/2}^1 e^{2x-1}dx$$

$$\Rightarrow 0 < \int_{1/2}^1 f(x)dx < \left(\frac{e^{2x-1}}{2}\right)_{1/2}^1$$

$$\therefore 0 < \int_{1/2}^1 f(x)dx < \frac{e-1}{2}$$

9. To check differentiability at $x = 0$.

$$\begin{aligned} R\{f'(0)\} &= \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 \cos \frac{\pi}{h} - 0}{h} \\ &= \lim_{h \rightarrow 0} h \cos \frac{\pi}{h} = 0 \\ L\{f'(0)\} &= \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{h^2 \cos\left(-\frac{\pi}{h}\right) - 0}{-h} = 0 \end{aligned}$$

So, $f(x)$ is differentiable at $x = 0$.

To check differentiability at $x = 2$,

$$\begin{aligned} R\{f'(2)\} &= \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2 \cos\left(\frac{\pi}{2+h}\right) - 0}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2 \cdot \cos\left(\frac{\pi}{2+h}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2+h}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2+h)^2 \cdot \sin\left(\frac{\pi h}{2(2+h)}\right)}{h \cdot \frac{\pi}{2(2+h)}} \cdot \frac{\pi}{2(2+h)} \end{aligned}$$

$$\Rightarrow R\{f'(2)\} = \pi$$

$$\begin{aligned} L\{f'(2)\} &= \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 \cos \frac{\pi}{2-h} - 2^2 \cos \frac{\pi}{2}}{-h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 \left(\cos \frac{\pi}{2-h}\right) - 0}{-h} \\ &= \lim_{h \rightarrow 0} \frac{-(2-h)^2 \cdot \sin\left(\frac{\pi}{2} - \frac{\pi}{2-h}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{(2-h)^2 \cdot \sin\left(\frac{\pi h}{2(2-h)}\right)}{h \times \frac{-\pi}{2(2-h)}} \times \frac{-\pi}{2(2-h)} \end{aligned}$$

$$\Rightarrow L\{f'(2)\} = -\pi$$

Thus, $f(x)$ is differentiable at $x = 0$ but not at $x = 2$.

10. $f(2n) = a_n$, $f(2n^+) = a_n$ and $f(2n^-) = b_n + 1$

$$\Rightarrow a_n - b_n = 1$$

$$\Rightarrow f(2n+1) = a_n \text{ and } f[(2n+1)^-] = a_n$$

$$\Rightarrow f[(2n+1)^+] = b_{n+1} - 1$$

$$\Rightarrow a_n = b_{n+1} - 1$$

$$\Rightarrow a_n - b_{n+1} = -1$$

$$\text{or } a_{n-1} - b_n = -1$$

11. $f(x+y) = f(x) + f(y)$, as $f(x)$ is differentiable at $x = 0$.

$$\Rightarrow f'(0) = k \quad \dots(i)$$

$$\begin{aligned} \text{Now, } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(h)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f'(h)}{1} = f'(0) = k \quad \dots(ii) \end{aligned}$$

$$\Rightarrow f'(x) = k$$

On integrating both sides, we get

$$f(x) = kx + C, \text{ as } f(0) = 0$$

$$\Rightarrow C = 0, f(x) = kx$$

So, $f(x)$ is continuous for all $x \in R$ and $f'(x) = k$,

i.e. constant for all $x \in R$.

$$12. f(x) = \begin{cases} -x - \frac{\pi}{2}, & x \leq -\frac{\pi}{2} \\ -\cos x, & -\frac{\pi}{2} < x \leq 0 \\ x - 1, & 0 < x \leq 1 \\ \log x, & x > 1 \end{cases}$$

Continuity at $x = -\frac{\pi}{2}$,

$$f\left(-\frac{\pi}{2}\right) = -\left(-\frac{\pi}{2}\right) - \frac{\pi}{2} = 0$$

$$\text{RHL} = \lim_{h \rightarrow 0} -\cos\left(-\frac{\pi}{2} + h\right) = 0$$

$\therefore$ Continuous at $x = -\frac{\pi}{2}$.

Continuity at $x = 0$, $f(0) = -1$

$$\text{RHL} = \lim_{h \rightarrow 0} (0+h) - 1 = -1$$

$\therefore$ Continuous at $x = 0$.

Continuity at $x = 1$, $f(1) = 0$

$$\text{RHL} = \lim_{h \rightarrow 0} \log(1 + h) = 0$$

$\therefore$ Continuous at $x = 1$.

$$\text{Here, } f'(x) = \begin{cases} -1, & x \leq -\frac{\pi}{2} \\ \sin x, & -\frac{\pi}{2} < x \leq 0 \\ 1, & 0 < x \leq 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

Differentiable at $x = 0$, LHD = 0 and RHD = 1

$\therefore$ Not differentiable at $x = 0$.

Differentiable at $x = 1$, LHD = 1 and RHD = 1

$\therefore$ Differentiable at $x = 1$.

$$\text{Also, for } x = -\frac{3}{2},$$

$$f(x) = -x - \frac{\pi}{2}$$

$\therefore$ Differentiable at $x = -\frac{3}{2}$.

So, all are correct.

13. Since, $g(x)$ is differentiable $\Rightarrow g(x)$ must be continuous.

$$\therefore g(x) = \begin{cases} k\sqrt{x+1}, & 0 \leq x \leq 3 \\ mx + 2, & 3 < x \leq 5 \end{cases}$$

$$\text{At } x = 3, \text{ RHL} = 3m + 2$$

$$\text{and at } x = 3 \text{ LHL} = 2k$$

$$\therefore 2k = 3m + 2 \quad \dots(i)$$

$$\text{Also, } g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}}, & 0 \leq x < 3 \\ m, & 3 < x \leq 5 \end{cases}$$

$$\therefore L\{g'(3)\} = \frac{k}{4} \text{ and } R\{g'(3)\} = m$$

$$\Rightarrow \frac{k}{4} = m \text{ and } k = 4m \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$k = \frac{8}{5}, m = \frac{2}{5}$$

$$\Rightarrow k + m = 2$$

14. Given, a function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$f(x) = [x] \cos \pi \left(x - \frac{1}{2} \right), \text{ where } [] \text{ denotes the greatest}$$

integer function.

To discuss, the continuity of function f .

Now, $\cos x$ is continuous, $\forall x \in \mathbb{R}$

$$\Rightarrow \cos \pi \left(x - \frac{1}{2} \right) \text{ is also continuous, } \forall x \in \mathbb{R}.$$

Hence, the continuity of f depends upon the continuity of $[x]$.

$[x]$ is discontinuous, $\forall x \in \mathbb{I}$.

So, we should check the continuity of f at $x = n$, $\forall n \in \mathbb{I}$.

LHL at $x = n$, is given by

$$\begin{aligned} f(n^-) &= \lim_{x \rightarrow n^-} f(x) = \lim_{x \rightarrow n^-} [x] \cos \pi \left(x - \frac{1}{2} \right) \\ &= (n-1) \cos \frac{(2n-1)\pi}{2} = 0 \end{aligned}$$

RHL at $x = n$, is given by

$$\begin{aligned} f(n^+) &= \lim_{x \rightarrow n^+} f(x) = \lim_{x \rightarrow n^+} [x] \cos \pi \left(x - \frac{1}{2} \right) \\ &= (n) \cos \frac{(2n-1)\pi}{2} = 0 \end{aligned}$$

Also, value of the function at $x = n$ is

$$\begin{aligned} f(n) &= [n] \cos \pi \left(n - \frac{1}{2} \right) \\ &= (n) \cos \frac{(2n-1)\pi}{2} = 0 \end{aligned}$$

$$\text{Hence, } f(n^+) = f(n^-) = f(n)$$

$\Rightarrow f$ is continuous at $x = n$, $\forall n \in \mathbb{I}$

15. Given a function f such that $f(x) = |x-2| + |x-5|$

To discuss, continuity and differentiability of f in interval $(2, 5)$.

$$\text{We know that, } |x-2| = \begin{cases} x-2, & x \geq 2 \\ 2-x, & x < 2 \end{cases}$$

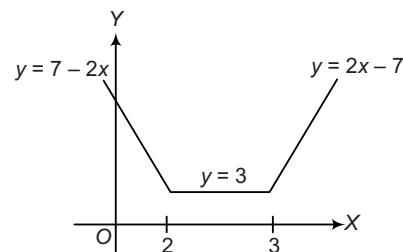
$$\text{and } |x-5| = \begin{cases} x-5, & x \geq 5 \\ 5-x, & x < 5 \end{cases}$$

$$\Rightarrow |x-2| = \begin{cases} 2-x, & x < 2 \\ x-2, & 2 \leq x \leq 5 \\ x-2, & x > 5 \end{cases}$$

$$\text{and } |x-5| = \begin{cases} 5-x, & x < 5 \\ 5-x, & 2 \leq x \leq 5 \\ x-5, & x > 5 \end{cases}$$

$$\begin{aligned} \Rightarrow f(x) &= |x-2| + |x-5| \\ &= \begin{cases} (2-x) + (5-x), & x < 2 \\ (x-2) + (5-x), & 2 \leq x \leq 5 \\ (x-2) + (x-5), & x > 5 \end{cases} = \begin{cases} 7-2x, & x < 2 \\ 3, & 2 \leq x \leq 5 \\ 2x-7, & x > 5 \end{cases} \end{aligned}$$

Now, we can draw the graph of f very easily.



From the above graph, we can analysis all the required things.

Statement I $f'(4) = 0$.

It is obviously clear that f is constant around $x = 4$.

$\therefore f'(4) = 0$.

Hence, Statement I is correct.

Statement II

It can be clearly seen that,

- (i) f is continuous, $\forall x \in [2, 5]$
- (ii) f is differentiable, $\forall x \in (2, 5)$
- (iii) $f(2) = f(5) = 3$

Hence, Statement II is also correct but obviously not a correct explanation of Statement I.

16. Here, $f(x) = \begin{cases} \sin(p+1)x + \sin x, & x < 0 \\ \frac{x}{q}, & x = 0 \\ \frac{\sqrt{x+x^2} - \sqrt{x}}{x^{3/2}}, & x > 0 \end{cases}$

Since, $f(x)$ is continuous for $x \in R$.

$\therefore$ Continuous at $x = 0$.

RHL at $x = 0$,

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\sqrt{h+h^2} - \sqrt{h}}{h^{3/2}} \\ &= \lim_{h \rightarrow 0} \frac{\sqrt{h}\{\sqrt{h+1} - 1\}}{h\sqrt{h}} \\ &= \lim_{h \rightarrow 0} \left(\frac{\sqrt{h+1} - 1}{h} \times \frac{\sqrt{h+1} + 1}{\sqrt{h+1} + 1} \right) \\ &= \lim_{h \rightarrow 0} \frac{(h+1) - 1}{h\{\sqrt{h+1} + 1\}} \end{aligned}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{1}{\sqrt{h+1} + 1} = \frac{1}{2} \quad \dots(i)$$

LHL at $x = 0$,

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{\sin(p+1)(-h) + \sin(-h)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{\sin(p+1)h}{h} + \frac{\sin h}{h} \end{aligned}$$

$$\Rightarrow (p+1) + 1 = (p+2) \quad \dots(ii)$$

$$f(0) = q \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{1}{2} = q = p + 2$$

$$\therefore p = -\frac{3}{2}, q = \frac{1}{2}$$

17. Here, $\lim_{x \rightarrow a} \frac{x^2 f(a) - a^2 f(x)}{x - a}$

$$= \lim_{x \rightarrow a} \frac{2xf(a) - a^2 f'(x)}{1 - 0} \quad \left[\frac{0}{0} \text{ form} \right]$$

[using L'Hospital's rule]

$$= 2af(a) - a^2 f'(a)$$

18. Here, $f(x) = \begin{cases} x \cdot \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

To check continuity at $x = 0$

$$\text{LHL} = \lim_{h \rightarrow 0} \left\{ (-h) \sin\left(-\frac{1}{h}\right) \right\} = 0$$

$$\text{RHL} = \lim_{h \rightarrow 0} \left\{ h \sin\left(\frac{1}{h}\right) \right\} = 0$$

$$f(0) = 0$$

$\therefore$ Continuous at $x = 0$.

$\Rightarrow$ Statement I is true.

Now, $f_2(x) = \begin{cases} \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Here, $\lim_{x \rightarrow 0} f_2(x) = \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$

= Does not exist.

$\therefore f_2(x)$ is not continuous at $x = 0$.

$\Rightarrow$ Statement II is false.

19. We have, $f : (-1, 1) \rightarrow R$

$$f(0) = -1, f'(0) = 1$$

$$g(x) = [f(2f(x) + 2)]^2$$

$$\Rightarrow g'(x) = 2[f(2f(x) + 2)] \times f'(2f(x) + 2) \times 2f'(x)$$

$$\therefore g'(0) = -4$$

20. $f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x \geq 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x \cos x^2, & x < 0 \\ 2x \cos x^2, & x \geq 0 \end{cases}$$

Clearly, $L(gof)'(0) = 0 = R(gof)'(0)$

$\therefore gof$ is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$.

$$\text{Now, } (gof)'(x) = \begin{cases} -2 \cos x^2 + 4x^2 \sin x^2, & x < 0 \\ 2 \cos x^2 - 4x^2 \sin x^2, & x \geq 0 \end{cases}$$

$$\therefore L(gof)'(0) = -2 \text{ and } R(gof)'(0) = 2$$

$$\therefore L(gof)'(0) \neq R(gof)'(0)$$

Hence, $gof(x)$ is not twice differentiable at $x = 0$.

21. $f'(1) = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h}$

$$= \lim_{h \rightarrow 0} \frac{(1-h-1) \cdot \sin\left(\frac{1}{1-h-1}\right) - 0}{-h}$$

$$= \lim_{h \rightarrow 0} \sin\left(-\frac{1}{h}\right) = -\lim_{h \rightarrow 0} \sin \frac{1}{h}$$

$$\text{and } f'(1^+) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(1+h-1) \sin\left(\frac{1}{1+h-1}\right) - 0}{h}$$

$$= \lim_{h \rightarrow 0} \sin \frac{1}{h}$$

$$\therefore f'(1^-) \neq f'(1^+)$$

$f(x)$ is not differentiable at $x = 1$.

Again,

$$f'(0) = \lim_{h \rightarrow 0} \frac{(0+h+1) \sin\left(\frac{1}{0+h+1}\right) - \sin 1}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{\left[-\left\{ (h+1) \cos\left(\frac{1}{h+1}\right) \times \left(\frac{1}{(h+1)^2}\right) \right\} + \sin\left(\frac{1}{h+1}\right) \right]}{-1}$$

[using L'Hospital's rule]

$$= \cos 1 - \sin 1$$

$$\text{and } f'(0^+) = \lim_{h \rightarrow 0} \frac{(0+h-1) \sin\left(\frac{1}{0+h-1}\right) - \sin 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(h-1) \cos\left(\frac{1}{h-1}\right) \left(\frac{-1}{(h-1)^2}\right) + \sin\left(\frac{1}{h-1}\right)}{1}$$

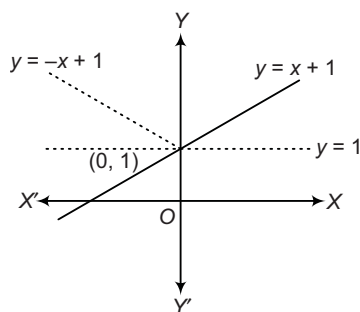
[using L'Hospital's rule]

$$= \cos 1 - \sin 1$$

$$\Rightarrow f'(0^-) = f'(0^+)$$

$\therefore f(x)$ is differentiable at $x = 0$.

22. $f(x) = \min\{x + 1, |x| + 1\}$
 $\Rightarrow f(x) = x + 1, \forall x \in R$



It is clear from the figure that, $f(x)$ is differentiable everywhere.

23. $\lim_{x \rightarrow 0} \left(\frac{1}{x} - \frac{2}{e^{2x} - 1} \right)$
 $= \lim_{x \rightarrow 0} \frac{e^{2x} - 1 - 2x}{x(e^{2x} - 1)}$
 $= \lim_{x \rightarrow 0} \frac{2e^{2x} - 2}{(e^{2x} - 1) + 2xe^{2x}}$
 [using L'Hospital's rule]
 $= \lim_{x \rightarrow 0} \frac{4e^{2x}}{4e^{2x} + 4xe^{2x}} = 1$
 [using L'Hospital's rule]

$\therefore f(x)$ is continuous at $x = 0$, then

$$\lim_{x \rightarrow 0} f(x) = f(0)$$

$$\therefore 1 = f(0)$$

24. Since, $f(x) = \frac{x}{1 + |x|}$

Let $f(x) = \frac{g(x)}{h(x)} = \frac{x}{1 + |x|}$

It is clear that, $g(x) = x$ and $h(x) = 1 + |x|$ are differentiable on $(-\infty, \infty)$ and $(-\infty, 0) \cup (0, \infty)$, respectively.

Thus, $f(x)$ is differentiable on $(-\infty, 0) \cup (0, \infty)$.

Now, we have to check the differentiability at $x = 0$.

$$\therefore \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} \frac{\frac{x}{1 + |x|} - 0}{x}$$

$$= \lim_{x \rightarrow 0} \frac{1}{1 + |x|} = 1$$

Hence, $f(x)$ is differentiable on $(-\infty, \infty)$.

25. $f'(1) = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(1+h)}{h} - \lim_{h \rightarrow 0} \frac{f(1)}{h}$

Since, $\lim_{h \rightarrow 0} \frac{f(1+h)}{h} = 5$

So, $\lim_{h \rightarrow 0} \frac{f(1)}{h}$ must be finite as $f'(1)$ exists and $\lim_{h \rightarrow 0} \frac{f(1)}{h}$

can be finite only, if $f(1) = 0$ and $\lim_{h \rightarrow 0} \frac{f(1)}{h} = 0$.

$$\therefore f'(1) = \lim_{h \rightarrow 0} \frac{(1+h)}{h} = 5$$

26. Given, $f(1) = -2$ and $f'(x) \geq 2$

$$\Rightarrow \frac{dy}{dx} \geq 2$$

$$\Rightarrow dy \geq 2dx$$

$$\Rightarrow \int_{f(1)}^{f(6)} dy \geq 2 \int_1^6 dx$$

$$\Rightarrow [y]_{f(1)}^{f(6)} \geq 2[x]_1^6$$

$$\Rightarrow f(6) - f(1) \geq 10$$

$$\Rightarrow f(6) \geq 10 + f(1)$$

$$\Rightarrow f(6) \geq 8 \quad [\because f(1) = -2]$$

Aliter

$$\frac{f(6) - f(1)}{6 - 1} \geq 2$$

$$\Rightarrow f(6) - f(1) \geq 10$$

$$\Rightarrow f(6) + 2 \geq 10$$

$$\Rightarrow f(6) \geq 8$$

27. Since, $|f(x) - f(y)| \leq (x - y)^2$

$$\lim_{x \rightarrow y} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{x \rightarrow y} |x - y|$$

$$\Rightarrow |f'(y)| \leq 0$$

$$\Rightarrow f'(y) = 0$$

$$\Rightarrow f(y) = \text{Constant}$$

$$\Rightarrow f(y) = 0 \quad [\because f(0) = 0, \text{ given}]$$

$$\Rightarrow f(1) = 0$$

28. Since, $f(x) = \frac{1 - \tan x}{4x - \pi}$

$$\therefore \lim_{x \rightarrow \frac{\pi}{4}} = \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{1 - \tan x}{4x - \pi} \right)$$

Using L'Hospital's rule,

$$= \lim_{x \rightarrow \frac{\pi}{4}} \left(\frac{-\sec^2 x}{4} \right)$$

$$= \frac{-\sec^2\left(\frac{\pi}{4}\right)}{4} = -\frac{2}{4}$$

$$\Rightarrow \lim_{x \rightarrow \frac{\pi}{4}} f(x) = -\frac{1}{2}$$

Also, $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$ so $f(x)$ will be continuous at $\frac{\pi}{4}$.

$$\therefore f\left(\frac{\pi}{4}\right) = \lim_{x \rightarrow \frac{\pi}{4}} f(x) = -\frac{1}{2}$$

29. Continuity at $x = 0$,

$$\text{LHL} = \lim_{x \rightarrow 0^-} f(x)$$

$$= \lim_{h \rightarrow 0} (0 - h)e^{-\left(\frac{1}{|h|} + \frac{1}{(-h)}\right)}$$

$$= \lim_{h \rightarrow 0} (-h)e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} = \lim_{h \rightarrow 0} (-h) = 0$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} f(x)$$

$$= \lim_{h \rightarrow 0} (0 + h)e^{-\left(\frac{1}{|h|} + \frac{1}{h}\right)}$$

$$= \lim_{h \rightarrow 0} h e^{-\left(\frac{1}{h} + \frac{1}{h}\right)} = \lim_{h \rightarrow 0} \frac{h}{e^{2/h}} = 0$$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

Therefore, $f(x)$ is continuous for all x .
Differentiability at $x = 0$,

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{(-h)e^{-\left(\frac{1}{h} - \frac{1}{h}\right)} - 0}{(-h) - 0}$$

$$= \lim_{h \rightarrow 0} e^0 = 1$$

$$Rf'(0) = \lim_{h \rightarrow 0} \frac{he^{-\left(\frac{1}{h} + \frac{1}{h}\right)} - 0}{h - 0}$$

$$= \lim_{h \rightarrow 0} \frac{1}{e^{2/h}} = 0$$

$$\Rightarrow Rf'(0) \neq Lf'(0)$$

Therefore, $f(x)$ is not differentiable at $x = 0$.

$$30. \lim_{x \rightarrow 0} \left[\tan\left(\frac{\pi}{4} + x\right) \right]^{\frac{1}{x}} = \lim_{x \rightarrow 0} \left[\frac{1 + \tan x}{1 - \tan x} \right]^{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0} \left[(1 + \tan x)^{\frac{1}{\tan x}} \right]^{\frac{\tan x}{x}} \times \lim_{x \rightarrow 0} \left[(1 - \tan x)^{\frac{1}{\tan x}} \right]^{\frac{\tan x}{x}}$$

$$= e \cdot e = e^2$$

$$31. \text{ We have, } f(x) = [x] \cos \left[\frac{2x-1}{2} \right] \pi$$

Since, $g(x) = [x]$ is always discontinuous at all integral values of points. Hence, $f(x)$ is discontinuous for all integral points.

$$32. \text{ We have, } f(x) = \begin{cases} \frac{\sin^3(\sqrt{x}) \cdot \log(1+3x)}{x \cdot (\tan^{-1} \sqrt{x})^2 \cdot (e^{5\sqrt{x}} - 1)}, & x \neq 0 \\ a, & x = 0 \end{cases}$$

For continuity in $[0, 1]$, $f(0) = f(0^+)$, otherwise it is discontinuous.

$$\text{Now, } a = \lim_{x \rightarrow 0^+} \frac{\sin^3(\sqrt{x}) \log(1+3x)}{x(\tan^{-1} \sqrt{x})^2 \cdot (e^{5\sqrt{x}} - 1)}$$

$$= \lim_{x \rightarrow 0^+} \left[\frac{3 \sin^3 \sqrt{x}}{5 \cdot (\sqrt{x})^3} \cdot \frac{(\sqrt{x})^2}{(\tan^{-1} \sqrt{x})^2} \times \frac{\log(1+3x)}{3x} \cdot \frac{5\sqrt{x}}{e^{5\sqrt{x}} - 1} \right]$$

$$= \frac{3}{5} \lim_{x \rightarrow 0^+} \frac{\sin^3 \sqrt{x}}{(\sqrt{x})^3} \cdot \frac{(\sqrt{x})^2}{\tan^{-1} \sqrt{x}} \times \frac{\log(1+3x)}{3x} \cdot \frac{5\sqrt{x}}{e^{5\sqrt{x}} - 1} = \frac{3}{5}$$

$$\therefore a = \frac{3}{5}$$

$$33. \text{ Given, } f(x) = \begin{cases} \frac{3 \sin \pi x}{5x}, & x \neq 0 \\ \frac{2k}{2k}, & x = 0 \end{cases}$$

$$\text{Now, } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \left(\frac{3 \sin \pi x}{5x} \right) = \frac{3}{5} \lim_{x \rightarrow 0} \left(\sin \frac{\pi x}{\pi x} \right) \times \pi$$

$$= \frac{3}{5} \times 1 \times \pi = \frac{3}{5} \pi$$

Also, $f(0) = 2k$

Since, $f(x)$ is continuous at $x = 0$.

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x)$$

$$\Rightarrow 2k = \frac{3}{5} \pi$$

$$\Rightarrow k = \frac{3\pi}{10}$$

$$34. \text{ Given, } f(x) = \begin{cases} ax + 3, & x \leq 2 \\ a^2x - 1, & x > 2 \end{cases}$$

Continuity at $x = 2$,

$$\text{LHL} = \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (ax + 3) = 2a + 3$$

$$\text{RHL} = \lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} (a^2x - 1) = 2a^2 - 1$$

Since, $f(x)$ is continuous for all values of x .

$$\therefore \text{LHL} = \text{RHL}$$

$$\Rightarrow 2a + 3 = 2a^2 - 1$$

$$\Rightarrow 2a^2 - 2a - 4 = 0$$

$$\Rightarrow a^2 - a - 2 = 0$$

$$\Rightarrow a^2 - 2a + a - 2 = 0$$

$$\Rightarrow a(a - 2) + 1(a - 2) = 0$$

$$\Rightarrow (a + 1)(a - 2) = 0$$

$$\therefore a = -1, 2$$

$$35. f(x) = \sin 2x - 1$$

Being sine function, $f(x)$ is defined and continuous at $x = 0, \pi$.

$$36. \text{ We know that, function } |x| \text{ is not differentiable at } x = 0.$$

$$\therefore |x^2 - 3x + 2| = |(x - 1)(x - 2)|$$

Hence, it is not differentiable at $x = 1$ and 2 .

Now, $f(x) = (x^2 - 1)|x^2 - 3x + 2| + \cos |x|$ is not differentiable at $x = 2$.

$$\text{For } 1 < x < 2, f(x) = -(x^2 - 1)(x^2 - 3x + 2) + \cos x$$

$$\text{For } 2 < x < 3, f(x) = (x^2 - 1)(x^2 - 3x + 2) + \cos x$$

$$Lf'(x) = -(x^2 - 1)(2x - 3) - 2x(x^2 - 3x + 2) - \sin x$$

$$Lf'(2) = -3 - \sin 2$$

$$Rf'(x) = (x^2 - 1)(2x - 3) + 2x(x^2 - 3x + 2) - \sin x$$

$$Rf'(2) = (4 - 1)(4 - 3) + 0 - \sin 2 = 3 - \sin 2$$

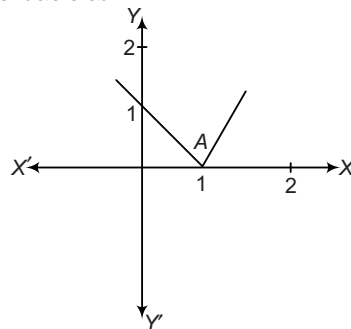
$$\text{Hence, } Lf'(2) \neq Rf'(2)$$

So, $f(x)$ is not differentiable at $x = 2$.

$$37. \text{ Given, graph is a function of } f(x) = |x - 1|.$$

It is clear from the graph that, there is no break in the graph, it shows that it is continuous.

Also, there is a sharp edge at point A, so it is not differentiable at $x = 1$.



$$38. \text{ Given, } f: R \rightarrow R \text{ and } |f(x) - f(y)|^2 \leq |x - y|^3$$

$$\Rightarrow \frac{|f(x) - f(y)|^2}{|x - y|^2} \leq |x - y|$$

$$\text{Let } y = x + h, h \rightarrow 0$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{|f(x) - f(x+h)|^2}{|x - (x+h)|^2} \leq \lim_{h \rightarrow 0} |x - (x+h)|$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{|f(x+h) - f(x)|^2}{|h|^2} \leq \lim_{h \rightarrow 0} |h|$$

$$\Rightarrow |f'(x)|^2 \leq 0 \Rightarrow |f'(x)|^2 = 0$$

$$\therefore f'(x) = 0$$

39. Given, $f(x) = \int_1^x \sin^2\left(\frac{t}{2}\right) dt$

On differentiating both sides, we get

$$f'(x) = \sin^2\left(\frac{x}{2}\right) \left[\frac{d}{dx}(x) - \frac{d}{dx}(\pi) \right]$$

[using Leibnitz's rule]

$$\Rightarrow f'(x) = \sin^2 \frac{x}{2}$$

$$\therefore \lim_{x \rightarrow 0} \frac{f'(\pi + x) - f'(\pi)}{x} = f''(\pi) = \sin^2 \frac{\pi}{2} = 1$$

40. Given, $f(x) = \frac{\tan\left\{\pi\left[x - \frac{\pi}{2}\right]\right\}}{2 + [x]^2}$

Since, $\left[x - \frac{\pi}{2}\right]$ is an integer for all x , therefore

$\pi\left[x - \frac{\pi}{2}\right]$ is an integral multiple of π for all x .

Hence, $\tan\left\{\pi\left[x - \frac{\pi}{2}\right]\right\} = 0$, for all x .

Also, $2 + [x]^2 \neq 0$ for all x . Therefore, $f(x) = 0$ for all x .

Hence, $f(x)$ is continuous and derivable for all x .

41. $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h}$

$$= f(x) \lim_{h \rightarrow 0} \frac{1 + \sin 3h [g(h)] - 1}{h}$$

$$= 3f(x) \left(\lim_{h \rightarrow 0} \frac{\sin 3h}{3h} \right) \left(\lim_{h \rightarrow 0} g(h) \right) = 3f(x)g(0)$$

42. Given, $f(x) = a \sin |x| + b e^{|x|}$

We know that, $\sin |x|$ and $e^{|x|}$ are not differentiable at $x = 0$.

Therefore, for $f(x)$ to be differentiable at $x = 0$, we must have

$$a + b = 0$$

43. Given, $f'(4) = 5$

Now, $\lim_{x \rightarrow 2} \frac{f(4) - f(x^2)}{x - 2}$ [0/0 form]

$$= \lim_{x \rightarrow 2} \frac{0 - f'(x^2) \cdot 2x}{1} = \frac{f'(4) \cdot 2 \times 2}{1}$$

$$= -(5) \times 4$$

$$= -20$$

44. Given, $f(x) = \begin{cases} \int_0^x |1-t| dt, & x > 1 \\ x - \frac{1}{2}, & x \leq 1 \end{cases}$... (i)

Now, for $x > 1$, $\int_0^x |1-t| dt = \int_0^1 |1-t| dt + \int_1^x |t-1| dt$

$$= \left[t - \frac{t^2}{2} \right]_0^1 + \left[\frac{t^2}{2} - t \right]_1^x = \left[1 - \frac{1}{2} - 0 \right] + \left[\frac{x^2}{2} - x - \frac{1}{2} + 1 \right]$$

$$= \frac{1}{2} + \frac{x^2}{2} - x - \frac{1}{2} + 1 = \frac{x^2}{2} - x + 1$$

$$\therefore \text{Eq. (i) becomes, } f(x) = \begin{cases} \frac{x^2}{2} - x + 1, & x > 1 \\ x - \frac{1}{2}, & x \leq 1 \end{cases}$$

Continuity at $x = 1$, LHL = $\lim_{x \rightarrow 1^-} f(x)$

$$= \lim_{x \rightarrow 1^-} \left(x - \frac{1}{2} \right) = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\text{RHL} = \lim_{x \rightarrow 1^+} f(x)$$

$$= \lim_{x \rightarrow 1^+} \left(\frac{x^2}{2} - x + 1 \right)$$

$$= \frac{1}{2} - 1 + 1 = \frac{1}{2}$$

and $f(1) = 1 - \frac{1}{2} = \frac{1}{2}$

$$\therefore f(1) = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x)$$

Hence, $f(x)$ is continuous at $x = 1$.

Differentiability at $x = 1$,

$$\text{LHD} = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} = \lim_{h \rightarrow 0} \frac{(1-h) - \frac{1}{2} - \left(1 - \frac{1}{2}\right)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{-h}{-h} = 1$$

$$\text{RHD} = \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{(1+h)^2}{2} - (1+h) + 1 - \left(1 - \frac{1}{2}\right)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1+h^2+2h}{2} - 1 - h + 1 - \frac{1}{2}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\frac{1}{2} + \frac{h^2}{2} + h - h - \frac{1}{2}}{h} = \lim_{h \rightarrow 0} \frac{h^2}{2h} = 0$$

$\therefore \text{LHD} \neq \text{RHD}$

Hence, $f(x)$ is not differentiable at $x = 1$.

45. $\lim_{x \rightarrow 0} \frac{2^x - 2^{-x}}{x} = \lim_{x \rightarrow 0} 2^x \log 2 + 2^{-x} \log 2$

[using L'Hospital's rule]

$$= \log 2 + \log 2 = \log 4$$

Since, function is continuous at $x = 0$.

$$\therefore f(0) = \lim_{x \rightarrow 0} \frac{2^x - 2^{-x}}{x} = \log 4$$

46. Given, $f(x) = \frac{\log(1+ax) - \log(1-bx)}{x}$

$f(x)$ is continuous at $x = k$ and $f(0) = k$.

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\log(1+ax) - \log(1-bx)}{x} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{1+ax} \cdot a + \frac{b}{1-bx} \right) = a + b$$

$$\therefore a + b = f(0) = k$$

47. Given, $f(x) = \begin{cases} \frac{x^2 - (a+2)x + a}{x-2}, & x \neq 2 \\ 2, & x = 2 \end{cases}$

$$= \lim_{x \rightarrow 2} \frac{x^2 - (a+2)x + a}{x-2} = \lim_{x \rightarrow 2} \frac{2x - (a+2)}{1}$$

[using L' Hospital's rule]

$$= 2 \times 2 - a - 2 = 2 - a$$

Since, $f(x)$ is continuous at $x=2$, therefore the limiting value $(2-a)$ must be equal to 2.
i.e. $2-a=2 \Rightarrow a=0$

48. $f(x) = x\sqrt{x} - \sqrt{x+1}$ is not defined for $x < 0$.

So, $f(x)$ is not continuous $[-\infty, 0]$.
Hence, $f(x)$ is continuous, $\forall x \in (0, \infty)$.

49. f is not derivable in $[0, 2]$. Rolle's theorem is not true in $[0, 2]$.

50. It is quite obvious that modulus function $f(x) = |x-1|$ is always continuous.

51. Since, function $|x-1|$ is not differentiable at $x=1$ and function $|x-3|$ is not differentiable at $x=3$.
Hence, $f(x)$ is not differentiable at $x=1, 3$.

52. At $x=1$,

$$\text{LHL} = \lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} \frac{\log(1+h)}{1+h-1}$$

$$= \lim_{h \rightarrow 0} \frac{\log(1+h)}{h} = 1$$

As $f(x)$ is continuous at $x=1$.

$$\therefore \lim_{x \rightarrow 1} f(x) = f(1)$$

$$\Rightarrow 1 = k$$

53. At $x=2$,

$$\text{LHL} = \lim_{x \rightarrow 2^-} [x] + [-x] = \lim_{h \rightarrow 0} [2-h] + [-2+h]$$

$$= 1 - 2 = -1$$

As it is continuous at $x=2$.

$$\therefore \lim_{x \rightarrow 2} f(x) = f(2)$$

$$\Rightarrow -1 = \lambda$$

54. At $x = \frac{\pi}{2}$,

$$\text{LHL} = \lim_{x \rightarrow \pi/2^+} [a \sin x + b] = a + b$$

$$\text{RHL} = \lim_{x \rightarrow \pi/2^-} [\cos x] = 0$$

Since, $f(x)$ is continuous at $x = \frac{\pi}{2}$.

$$\therefore a + b = 0 \quad \dots(i)$$

At $x = \frac{-\pi}{2}$,

$$\text{LHL} = \lim_{x \rightarrow -\pi/2^-} [-2 \sin x] = 2 \sin\left(\frac{-\pi}{2}\right) = 2$$

$$\text{RHL} = \lim_{x \rightarrow -\pi/2^+} [a \sin x + b] = a \sin\left(\frac{-\pi}{2}\right) + b$$

$$= -a \sin \frac{\pi}{2} + b = b - a$$

Since, $f(x)$ is continuous at $x = \frac{-\pi}{2}$.

$$\therefore b - a = 2 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get $a = -1$ and $b = 1$.

55. At $x=3$,

$$\text{LHL} = \lim_{x \rightarrow 3^-} (cx + 1) = \lim_{h \rightarrow 0} [c(3-h) + 1] = 3c + 1$$

$$\text{RHL} = \lim_{x \rightarrow 3^+} [cx^2 - 1] = \lim_{h \rightarrow 0} [c(3+h)^2 - 1] = 9c - 1$$

As, $f(x)$ is continuous at $x=3$.

$$\therefore 3c + 1 = 9c - 1$$

$$\Rightarrow 6c = 1 + 1$$

$$\Rightarrow c = \frac{2}{6} = \frac{1}{3}$$

$$\therefore c = \frac{1}{3}$$

56. Given, $f(x) = p \cdot |\sin x| + q \cdot e^{|x|} + r \cdot |x|^3$

$$f'(x) = p \cdot \cos x + q \cdot e^x + 3r \cdot x^2$$

When $x=0$,

$$f'(0) = p \cos 0 + q e^0 + r \cdot 0 = p + q = 0$$

$\Rightarrow p + q = 0$ and r is any real number.

57. LHL = $\lim_{x \rightarrow 0^-} \left\{ -h \cdot \sin\left(-\frac{1}{h}\right) \right\} = 0$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \left\{ h \cdot \sin\left(\frac{1}{h}\right) \right\} = 0$$

Also, $f(0) = 0 \times [t \in R, -1 \leq t \leq 1] = 0$

So, $f(x)$ is continuous for all $x \in R$.

To check differentiability at $x=0$,

$$\text{LHD} = \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{h \cdot \sin\left(-\frac{1}{h}\right) - 0}{-h}$$

$$= -\sin\left(\frac{1}{h}\right) = -K \quad [K \in [-1, 1]]$$

$$\text{RHD} = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h \cdot \sin\left(\frac{1}{h}\right)}{h} = \sin \frac{1}{h} = K$$

$$\therefore \text{LHD} \neq \text{RHD}$$

So, $f(x)$ is not differentiable at $x=0$.

58. $f(0) = \lim_{x \rightarrow 0} \frac{(e^x - 1)^2}{\sin\left(\frac{x}{a}\right) \log\left(1 + \frac{x}{4}\right)}$

$$= \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right)^2 \cdot \frac{\frac{x}{a} \cdot a}{\sin \frac{x}{a}} \cdot \frac{\frac{x}{4} \cdot 4}{\log\left(1 + \frac{x}{4}\right)} = 12$$

$$\Rightarrow 1^2 \cdot a \cdot 4 = 12$$

$$\therefore a = 3$$

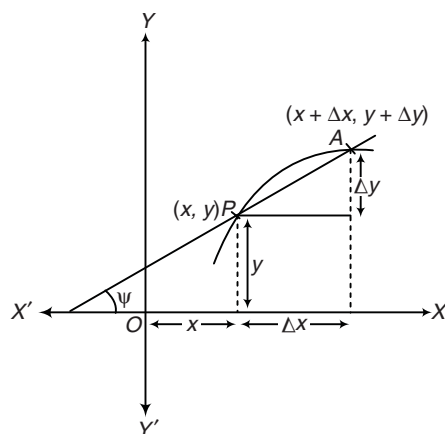
6

Differentiation

Geometrical Meaning of Derivative

Let us consider a function $y = f(x)$ in a rectangular coordinate system. We also consider a point $P(x, y)$ on the curve.

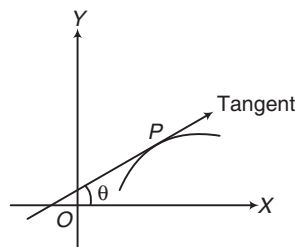
If a point corresponding to an increased value of the argument $(x + \Delta x)$ is considered, its ordinate value is given by $y + \Delta y = f(x + \Delta x)$.



The point $(x + \Delta x, y + \Delta y)$ is represented by A .

Here, PA is the secant to the curve.

Now, as $\Delta x, \Delta y \rightarrow 0 \Rightarrow PA \rightarrow 0$
i.e. the distance PA tends to zero or to a single point P .



$\Rightarrow \lim_{\Delta x \rightarrow 0} (\text{slope of chord } PA) \rightarrow (\text{slope of tangent at } P)$

or $\tan \psi = \lim_{\Delta x \rightarrow 0} \tan \theta$

$\Rightarrow \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \frac{dy}{dx} \text{ or } f'(x)$

Chapter Snapshot

- Geometrical Meaning of Derivative
- Differential Coefficients of Standard Functions
- Rules for Differentiation
- Differentiation of Implicit Function
- Differentiation by Substitution
- Important Trigonometrical Formulae
- Differential Coefficient using Inverse Trigonometrical Substitutions
- Derivative of Inverse Trigonometric Functions
- Logarithmic Differentiation
- Differentiation of Parametric Functions
- Differentiation of a Function with Respect to a Function
- Differentiation of Determinants
- Higher Order Derivatives

which means that the value of the derivative $f'(x)$ for a given value of x is equal to the tangent of the angle formed by the line tangent to the graph of the function $y = f(x)$ at the point $P(x, y)$ with the positive X -axis.

The meaning of the term 'rate of change of y with respect to x ' is that if x is increased by an additional unit, then change in y is given by $\frac{dy}{dx}$. e.g. The rate of change of displacement of a particle is defined as its velocity, so if we say that a particle is moving with velocity v km/h, then it means that time is increased by one hour, the displacement changes by v km.

Differential Coefficients of Standard Functions

(i) Algebraic Functions

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(x^n) &= nx^{n-1}, \\ \frac{d}{dx}\{f(x)\}^n &= n\{f(x)\}^{n-1} \frac{d}{dx}f(x) \\ \text{(b)} \quad \frac{d}{dx}\left(\frac{1}{x^n}\right) &= -nx^{-n-1}, \\ \frac{d}{dx}\left\{\frac{1}{f(x)}\right\}^n &= \frac{-n}{\{f(x)\}^{n+1}} \cdot \frac{d}{dx}f(x) \\ \text{(c)} \quad \frac{d}{dx}\sqrt{x} &= \frac{1}{2\sqrt{x}}, \quad \frac{d}{dx}\sqrt{f(x)} = \frac{1}{2\sqrt{f(x)}} \cdot \frac{d}{dx}f(x) \end{aligned}$$

(ii) Exponential Functions

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(e^x) &= e^x, \quad \frac{d}{dx}e^{\{f(x)\}} = [e^{\{f(x)\}}] \frac{d}{dx}f(x) \\ \text{(b)} \quad \frac{d}{dx}(a^x) &= a^x \log_e a, \\ \frac{d}{dx}a^{\{f(x)\}} &= [a^{\{f(x)\}} \log_e a] \frac{d}{dx}f(x) \end{aligned}$$

(iii) Logarithmic Functions

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(\log_e x) &= \frac{1}{x}, \\ \frac{d}{dx}\{\log_e f(x)\} &= \frac{1}{f(x)} \cdot \frac{d}{dx}f(x) \\ \text{(b)} \quad \frac{d}{dx}(\log_a x) &= \frac{1}{x \log_e a}, \\ \frac{d}{dx}\{\log_a f(x)\} &= \frac{1}{f(x) \log_e a} \cdot \frac{d}{dx}f(x) \end{aligned}$$

(iv) Trigonometric Functions

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(\sin x) &= \cos x, \\ \frac{d}{dx}\sin f(x) &= \cos f(x) \frac{d}{dx}f(x) \\ \text{(b)} \quad \frac{d}{dx}(\cos x) &= -\sin x, \\ \frac{d}{dx}\cos f(x) &= -\sin f(x) \frac{d}{dx}f(x) \\ \text{(c)} \quad \frac{d}{dx}(\tan x) &= \sec^2 x, \\ \frac{d}{dx}\tan f(x) &= \sec^2 f(x) \frac{d}{dx}f(x) \\ \text{(d)} \quad \frac{d}{dx}(\cot x) &= -\operatorname{cosec}^2 x, \\ \frac{d}{dx}\cot f(x) &= -\operatorname{cosec}^2 f(x) \frac{d}{dx}f(x) \\ \text{(e)} \quad \frac{d}{dx}(\sec x) &= \sec x \tan x, \\ \frac{d}{dx}\sec f(x) &= \sec f(x) \tan f(x) \frac{d}{dx}f(x) \\ \text{(f)} \quad \frac{d}{dx}(\operatorname{cosec} x) &= -\operatorname{cosec} x \cot x, \\ \frac{d}{dx}\operatorname{cosec} f(x) &= -\operatorname{cosec} f(x) \cot f(x) \frac{d}{dx}f(x) \end{aligned}$$

(v) Inverse Circular Functions

$$\begin{aligned} \text{(a)} \quad \frac{d}{dx}(\sin^{-1} x) &= \frac{1}{\sqrt{1-x^2}}; |x| < 1 \\ \frac{d}{dx}[\sin^{-1} f(x)] &= \frac{1}{\sqrt{1-\{f(x)\}^2}} \cdot \frac{d}{dx}f(x); |f(x)| < 1 \\ \text{(b)} \quad \frac{d}{dx}(\cos^{-1} x) &= -\frac{1}{\sqrt{1-x^2}}; |x| < 1 \\ \frac{d}{dx}[\cos^{-1} f(x)] &= -\frac{1}{\sqrt{1-\{f(x)\}^2}} \cdot \frac{d}{dx}f(x); |f(x)| < 1 \\ \text{(c)} \quad \frac{d}{dx}\tan^{-1} x &= \left(\frac{1}{1+x^2}\right); x \in R \\ \frac{d}{dx}[\tan^{-1} f(x)] &= \frac{1}{1+\{f(x)\}^2} \cdot \frac{d}{dx}f(x); f(x) \in R \\ \text{(d)} \quad \frac{d}{dx}\cot^{-1} x &= -\left(\frac{1}{1+x^2}\right); x \in R \\ \frac{d}{dx}[\cot^{-1} f(x)] &= -\frac{1}{1+\{f(x)\}^2} \cdot \frac{d}{dx}f(x); f(x) \in R \end{aligned}$$

$$(e) \frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}; |x| > 1$$

$$\frac{d}{dx}[\sec^{-1} f(x)] = \frac{1}{|f(x)|\sqrt{\{f(x)\}^2-1}} \cdot \frac{d}{dx} f(x); |f(x)| > 1$$

$$(f) \frac{d}{dx}(\operatorname{cosec}^{-1} x) = -\left(\frac{1}{|x|\sqrt{x^2-1}}\right); |x| > 1$$

$$\frac{d}{dx}[\operatorname{cosec}^{-1} f(x)] = -\frac{1}{|f(x)|\sqrt{\{f(x)\}^2-1}} \cdot \frac{d}{dx} f(x); |f(x)| > 1$$

Rules for Differentiation

Term-by-Term Differentiation

If $y = u(x) \pm v(x) \pm w(x) \pm \dots$, then

$$\frac{dy}{dx} = \frac{du}{dx}(x) \pm \frac{dv}{dx}(x) \pm \frac{dw}{dx}(x) \pm \dots$$

which is known as term-by-term differentiation.

Differentiation of Term

If $y = kf(x)$, then on differentiating, we get

$$\frac{dy}{dx} = k \frac{d}{dx} f(x)$$

which is known as differentiation of term.

Product Rule

If $y = u(x) \cdot v(x)$, then

$$\frac{dy}{dx} = \left\{ \frac{d}{dx} u(x) \right\} v(x) + u(x) \left\{ \frac{d}{dx} v(x) \right\}$$

which is known as product rule.

Quotient Rule

If $y = \frac{u(x)}{v(x)}$, then

$$\frac{dy}{dx} = \frac{\left\{ \frac{d}{dx} u(x) \right\} v(x) - \left\{ \frac{d}{dx} v(x) \right\} u(x)}{\{v(x)\}^2}$$

which is known as quotient rule.

Chain Rule

If $y = [u\{v(x)\}]$, then

$$\frac{dy}{dx} = \frac{du\{v(x)\}}{dv\{v(x)\}} \times \frac{d}{dx} v(x)$$

which is known as chain rule.

If $y = u[v\{w(x)\}]$, then

$$\frac{dy}{dx} = \frac{du[v\{w(x)\}]}{dv\{w(x)\}} \times \frac{dv\{w(x)\}}{dw(x)} \times \frac{dw(x)}{dx}$$

➤ **Example 1.** If $y = (1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$, then $\frac{dy}{dx}$ is equal to

- (a) 1 (b) -1 (c) x (d) $\sqrt{x}$

Sol. (b) We have, $y = (1+x^{1/4})(1+x^{1/2})(1-x^{1/4})$
 $= (1-x^{1/2})(1+x^{1/2}) = 1-x$
 $\therefore \frac{dy}{dx} = -1$

➤ **Example 2.** Derivative of $\cos x^3 \sin^2(x^5)$ with respect to x is

- (a) $10x^4(\sin x^5)(\cos x^5)(\cos x^3) - 3x^2 \sin x^3 \sin^2 x^5$
 (b) $5x^3(\sin x^5)(\cos x^5)(\cos x^3) - 3x^2 \sin x^3 \sin^2 x^5$
 (c) $5x^3(\sin x^5)(\cos x^5)(\cos x^3) - 6x^2 \sin x^3 \sin^2 x^5$
 (d) None of the above

Sol. (a) Let $y = \cos x^3 \sin^2(x^5)$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \{\cos x^3 \sin^2(x^5)\} \\ &= \cos x^3 \frac{d}{dx} \sin^2(x^5) + \sin^2(x^5) \frac{d}{dx} (\cos x^3) \\ &= \cos x^3 \left\{ \therefore \frac{d}{dx} (uv) = u \frac{d}{dx} v + v \frac{d}{dx} u \right\} \\ &= (\cos x^3) (2 \sin x^5) \frac{d}{dx} (\sin x^5) + \sin^2(x^5) (-\sin x^3) \frac{d}{dx} (x^3) \\ &= (\cos x^3) (2 \sin x^5) (\cos x^5) \frac{d}{dx} (x^5) + \sin^2(x^5) (-\sin x^3) (3x^2) \\ &= (\cos x^3) (2 \sin x^5) (\cos x^5) (5x^4) - \sin^2(x^5) (\sin x^3) (3x^2) \\ &= 10x^4 (\sin x^5) (\cos x^5) (\cos x^3) - 3x^2 \sin x^3 \sin^2 x^5 \end{aligned}$$

➤ **Example 3.** Derivative of $\frac{\sin(ax+b)}{\cos(cx+d)}$ with respect to x is

- (a) $a \cos(ax+b) \sec(cx+d) + c \sin(ax+b) \tan(cx+d) \sec(cx+d)$
 (b) $a \cos(ax+b) \sin(cx+d) + c \sin(ax+b) \tan(cx+d) \sec(cx+d)$
 (c) $a \cos(ax+b) \sec(cx+d) + c \sin(ax+b) \tan(cx+d) \cos(cx+d)$
 (d) None of the above

Sol. (a) Let $y = \frac{\sin(ax+b)}{\cos(cx+d)}$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{d}{dx} \left[\frac{\sin(ax+b)}{\cos(cx+d)} \right] \\ &= \frac{\cos(cx+d) \frac{d}{dx} \{\sin(ax+b)\} - \sin(ax+b) \frac{d}{dx} \{\cos(cx+d)\}}{\{\cos(cx+d)\}^2} \end{aligned}$$

[by quotient rule]

$$\begin{aligned}
 &= \frac{\cos(cx+d)\cos(ax+b) \cdot (a+0) + \sin(ax+b)}{\sin(cx+d)(c+0)} \\
 &= \frac{\cos^2(cx+d)}{a \cos(cx+d) \cos(ax+b) + c \sin(ax+b) \sin(cx+d)} \\
 &= \frac{\cos^2(cx+d)}{\cos^2(cx+d)} + \frac{c \sin(ax+b) \sin(cx+d)}{\cos^2(cx+d)} \\
 &= \frac{a \cos(ax+b)}{\cos(cx+d)} + \frac{c \sin(ax+b) \sin(cx+d)}{\cos(cx+d) \cos(cx+d)} \\
 &= a \cos(ax+b) \sec(cx+d) + c \sin(ax+b) \tan(cx+d) \\
 &\quad \sec(cx+d)
 \end{aligned}$$

➤ **Example 4.** If

$$f(x) = \left(\frac{x^a}{x^b}\right)^{a+b} \left(\frac{x^b}{x^c}\right)^{b+c} \left(\frac{x^c}{x^a}\right)^{c+a},$$

then $f'(x)$ is equal to

- (a) 1 (b) 0
(c) x^{a+b+c} (d) None of these

Sol. (b) We have, $f(x) = (x^{a-b})^{a+b} \cdot (x^{b-c})^{b+c} \cdot (x^{c-a})^{c+a}$
 $= x^{a^2-b^2} \cdot x^{b^2-c^2} \cdot x^{c^2-a^2}$
 $= x^{a^2-b^2+b^2-c^2+c^2-a^2}$
 $= x^0 = 1$

$\therefore f'(x) = 0$

➤ **Example 5.** If

$$y = \frac{1}{1+x^{\beta-\alpha} + x^{\gamma-\alpha}} + \frac{1}{1+x^{\alpha-\beta} + x^{\gamma-\beta}} + \frac{1}{1+x^{\alpha-\gamma} + x^{\beta-\gamma}}, \text{ then } \frac{dy}{dx} \text{ is equal to}$$

- (a) 0 (b) 1
(c) $(\alpha + \beta + \gamma)x^{\alpha+\beta+\gamma-1}$ (d) None of these

Sol. (a) We have,

$$\begin{aligned}
 y &= \frac{1}{1 + \frac{x^\beta}{x^\alpha} + \frac{x^\gamma}{x^\alpha}} + \frac{1}{1 + \frac{x^\alpha}{x^\beta} + \frac{x^\gamma}{x^\beta}} + \frac{1}{1 + \frac{x^\alpha}{x^\gamma} + \frac{x^\beta}{x^\gamma}} \\
 &= \frac{x^\alpha}{x^\alpha + x^\beta + x^\gamma} + \frac{x^\beta}{x^\alpha + x^\beta + x^\gamma} + \frac{x^\gamma}{x^\alpha + x^\beta + x^\gamma} \\
 &= \frac{x^\alpha + x^\beta + x^\gamma}{x^\alpha + x^\beta + x^\gamma} = 1
 \end{aligned}$$

$\therefore \frac{dy}{dx} = 0$

➤ **Example 6.** If $f(x) = \frac{1 - \cos x}{1 - \sin x}$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

- (a) 1
(b) 0
(c) ∞
(d) Does not exist

Sol. (d) We have, $f(x) = \frac{1 - \cos x}{1 - \sin x}$
 $\Rightarrow f'(x) = \frac{(1 - \sin x)(\sin x) - (1 - \cos x)(-\cos x)}{(1 - \sin x)^2}$

$$= \frac{\sin x - \sin^2 x + \cos x - \cos^2 x}{(1 - \sin x)^2} = \frac{\sin x + \cos x - 1}{(1 - \sin x)^2}$$

$$\therefore f'\left(\frac{\pi}{2}\right) = \frac{1 + 0 - 1}{(1 - 1)^2} = \frac{0}{0}$$

Therefore, $f'\left(\frac{\pi}{2}\right)$ does not exist.

➤ **Example 7.** If $y = \log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is

equal to

- (a) $\frac{4x^3}{1-x^4}$
(b) $-\frac{4x}{1-x^4}$
(c) $\frac{1}{4-x^4}$
(d) $-\frac{4x^3}{1-x^4}$

Sol. (b) We have, $y = \log(1-x^2) - \log(1+x^2)$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= \frac{-2x}{1-x^2} - \frac{2x}{1+x^2} \\
 &= -2x \cdot \frac{(1+x^2+1-x^2)}{1-x^4} = -\frac{4x}{1-x^4}
 \end{aligned}$$

➤ **Example 8.** If $y = \log \sqrt{\tan x}$, then the value of $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is given by

- (a) ∞ (b) 1 (c) 0 (d) $\frac{1}{2}$

Sol. (b) We have, $y = \log \sqrt{\tan x}$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= \frac{1}{\sqrt{\tan x}} \cdot \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x = \frac{\sec^2 x}{2 \tan x} \\
 \therefore \text{At } x = \frac{\pi}{4}, \frac{dy}{dx} &= \frac{\sec^2 \frac{\pi}{4}}{2 \tan \frac{\pi}{4}} = \frac{2}{2} = 1
 \end{aligned}$$

➤ **Example 9.** If $\phi(x) = \log_5 \log_3 x$, then $\phi'(e)$ is equal to

- (a) $e \log 5$ (b) $-e \log 5$
(c) $\frac{1}{e \log 5}$ (d) None of these

Sol. (c) We have, $\phi(x) = \log_5 \log_3 x$

$$\begin{aligned}
 &= \log_5 \left(\frac{\log x}{\log 3} \right) \\
 &= \log_5 (\log x) - \log_5 (\log 3) \\
 &= \frac{\log (\log x)}{\log 5} - \log_5 (\log 3) \\
 \Rightarrow \phi'(x) &= \frac{1}{\log 5} \cdot \frac{1}{\log x} \cdot \frac{1}{x} - 0 \\
 \therefore \phi'(e) &= \frac{1}{\log 5} \cdot \frac{1}{\log e} \cdot \frac{1}{e} = \frac{1}{e \log 5}
 \end{aligned}$$

➤ **Example 10.** If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and

$f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is equal to

(a) $\sin\left(\frac{2x-1}{x^2+1}\right)^2 \left[\frac{2+2x+x^2}{(x^2+1)^2}\right]$

(b) $\sin\left(\frac{2x-1}{x^2+1}\right)^2 \left[\frac{2+2x-2x^2}{(x^2+1)^2}\right]$

(c) $\sin\left(\frac{2x-1}{x^2+1}\right)^2 \left[\frac{2+2x-x^2}{(x^2+1)^2}\right]$

(d) None of the above

Sol. (b) We have, $y = f\left(\frac{2x-1}{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = f'\left(\frac{2x-1}{x^2+1}\right) \left[\frac{(x^2+1)2 - (2x-1)2x}{(x^2+1)^2}\right]$$

$$= \sin\left(\frac{2x-1}{x^2+1}\right)^2 \left[\frac{2+2x-2x^2}{(x^2+1)^2}\right]$$

$$\left[\because f'(x) = \sin x^2 \therefore f'\left(\frac{2x-1}{x^2+1}\right) = \sin\left(\frac{2x-1}{x^2+1}\right)^2\right]$$

Differentiation of Implicit Function

In previous rules, we have studied functions in which y was solely expressed in terms of x without complication (called explicit functions).

But if the relation between the variables x and y are given by an equation containing both and this equation is not immediately solvable for y , then y is called an implicit function of x .

Implicit functions are given by $\phi(x, y) = 0$.

Shortcut Method for Finding Implicit Function

For implicit function, put $\frac{d}{dx}\{f(x, y)\} = \frac{-\partial f / \partial x}{\partial f / \partial y}$,

where $\frac{\partial f}{\partial x}$ is partial differential of given function with respect to x , i.e. differentiating f with respect to x ,

keeping y as constant and $\frac{\partial f}{\partial y}$ means partial

differential of given function with respect to y , i.e. differentiating f with respect to y , keeping x as constant.

➤ **Example 11.** If $y \cos x + x \cos y = \pi$, then the value of $\frac{dy}{dx}$ at $x = 0$ is

- (a) 0 (b) 1 (c) -1 (d) 2

Sol. (b) Given that,

$$y \cos x + x \cos y = \pi \quad \dots (i)$$

Put $x = 0$ in Eq. (i) we get

$$y \cos 0 + 0 = \pi \Rightarrow y = \pi$$

Now, differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} \cos x - y \sin x - x \sin y \frac{dy}{dx} + \cos y = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \sin x - \cos y}{\cos x - x \sin y}$$

$$\therefore \left[\frac{dy}{dx}\right]_{x=0} = \frac{\pi \cdot \sin 0 - \cos \pi}{\cos 0 - 0 \cdot \sin \pi} = \frac{0 - (-1)}{1 - 0} = \frac{1}{1} = 1$$

➤ **Example 12.** If $\sin^2 y + \cos(xy) = k$, then $\frac{dy}{dx}$

is equal to

(a) $\frac{y \sin(xy)}{\sin 2y - x \sin(xy)}$ (b) $\frac{y \cos(xy)}{\sin 2y - x \sin(xy)}$

(c) $\frac{y \sin(xy)}{\sin 2y - x \cos(xy)}$ (d) None of these

Sol. (a) Given, $\sin^2 y + \cos xy = k$

On differentiating both sides w.r.t. x , we get

$$\frac{d}{dx}(\sin^2 y + \cos xy) = \frac{d}{dx}(k)$$

$$\Rightarrow \frac{d}{dx}(\sin^2 y) + \frac{d}{dx}(\cos xy) = 0$$

$$\Rightarrow 2 \sin y \cos y \frac{dy}{dx} + (-\sin xy) \frac{d}{dx}(xy) = 0$$

$$\left[\because \frac{d}{dx}f(g(x)) = f'(g(x)) \frac{d}{dx}g(x)\right]$$

$$\Rightarrow \sin 2y \frac{dy}{dx} - \sin xy \left(x \frac{dy}{dx} + y \cdot 1\right) = 0$$

$$[\because \sin 2x = 2 \sin x \cos x]$$

$$\Rightarrow \sin 2y \frac{dy}{dx} - x \sin xy \frac{dy}{dx} = y \sin xy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \sin(xy)}{\sin 2y - x \sin(xy)}$$

➤ **Example 13.** If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$,

then $\frac{dy}{dx}$ is equal to

(a) $\frac{x\sqrt{1-x^3}}{y\sqrt{1-y^3}}$ (b) $\frac{x^2\sqrt{1-x^3}}{y^2\sqrt{1-y^3}}$

(c) $\frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$ (d) $\frac{x^2\sqrt{1-x^6}}{y^2\sqrt{1-y^6}}$

Sol. (c) Here, $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3 - y^3)$

Let $x^3 = \sin \theta$, $y^3 = \sin \phi$, then

$$\sqrt{1-\sin^2 \theta} + \sqrt{1-\sin^2 \phi} = a^3(\sin \theta - \sin \phi)$$

$$\Rightarrow \cos \theta + \cos \phi = a^3(\sin \theta - \sin \phi)$$

$$\Rightarrow 2 \cos\left(\frac{\theta+\phi}{2}\right) \cos\left(\frac{\theta-\phi}{2}\right)$$

$$= a^3 \left[2 \cos\left(\frac{\theta+\phi}{2}\right) \sin\left(\frac{\theta-\phi}{2}\right)\right]$$

$$\Rightarrow \cos\left(\frac{\theta - \phi}{2}\right) = a^3 \sin\left(\frac{\theta - \phi}{2}\right) = \cot\left(\frac{\theta - \phi}{2}\right) = a^3$$

$$\Rightarrow \theta - \phi = 2 \cot^{-1}(a^3) \text{ or } \sin^{-1} x^3 - \sin^{-1} y^3 = 2 \cot^{-1}(a^3)$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{\sqrt{1-x^6}} \cdot 3x^2 - \frac{1}{\sqrt{1-y^6}} \cdot 3y^2 \cdot \frac{dy}{dx} = 0$$

Hence,

$$\frac{dy}{dx} = \frac{x^2 \sqrt{1-y^6}}{y^2 \sqrt{1-x^6}}$$

➤ **Example 14.** If $\sqrt{x} + \sqrt{y} = 4$, then $\frac{dx}{dy}$ at $y = 1$

is

(a) -1

(b) -3

(c) 3

(d) None of the above

Sol. (b) We have, $\sqrt{x} + \sqrt{y} = 4$... (i)

On differentiating Eq. (i) w.r.t. y , we get

$$\frac{1}{2\sqrt{x}} \cdot \frac{dx}{dy} + \frac{1}{2\sqrt{y}} = 0$$

$$\Rightarrow \frac{dx}{dy} = -\frac{\sqrt{x}}{\sqrt{y}} = \frac{\sqrt{y} - 4}{\sqrt{y}}$$

$$\therefore \left[\frac{dx}{dy}\right]_{y=1} = \frac{1-4}{1} = -3$$

➤ **Example 15.** If $y = \sqrt{\sin x + y}$, then $\frac{dy}{dx}$

equals

(a) $\frac{\cos x}{2y-1}$ (b) $\frac{\cos x}{1-2y}$ (c) $\frac{\sin x}{1-2y}$ (d) $\frac{\sin x}{2y-1}$

Sol. (a) We have, $y = \sqrt{\sin x + y}$

$$\Rightarrow y^2 = \sin x + y$$

$$\Rightarrow 2y \frac{dy}{dx} = \cos x + \frac{dy}{dx}$$

$$\Rightarrow (2y-1) \frac{dy}{dx} = \cos x$$

$$\therefore \frac{dy}{dx} = \frac{\cos x}{(2y-1)}$$

➤ **Example 16.** If $xe^{xy} = y + \sin^2 x$, then at $x = 0$, dy/dx is equal to

(a) -1

(b) 1

(c) 0

(d) None of these

Sol. (b) We have, $x e^{xy} = y + \sin^2 x$... (i)

On differentiating Eq. (i) w.r.t. x , we get

$$e^{xy} + x e^{xy} \left(x \frac{dy}{dx} + y \right) = \frac{dy}{dx} + 2 \sin x \cos x$$

On putting $x = 0$, $y = 0$, we get

$$e^0 + 0 \cdot e^0 \left(0 \cdot \left[\frac{dy}{dx} \right]_{(0,0)} + 0 \right) = \left[\frac{dy}{dx} \right]_{(0,0)} + 2 \sin 0 \cdot \cos 0$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(0,0)} = 1$$

Differentiation by Substitution

Substitution is useful to reduce the function into simple form. For problems involving inverse trigonometric functions, first try for a suitable substitution to simplify it and then differentiate it. If no such substitution is found, then differentiate directly. Some standard substitutions are given below:

S. No.	Expressions	Substitutions
1.	$\sqrt{a^2 + x^2}$	$x = a \tan \theta$ or $x = a \cot \theta$
2.	$\sqrt{a^2 - x^2}$	$x = a \sin \theta$ or $x = a \cos \theta$
3.	$\sqrt{x^2 - a^2}$	$x = a \sec \theta$ or $x = a \csc \theta$
4.	$\frac{a+x}{a-x}$ or $\frac{a-x}{a+x}$	$x = a \tan \theta$
5.	$\sqrt{\frac{a+x}{a-x}}$ or $\sqrt{\frac{a-x}{a+x}}$	$x = a \cos 2\theta$
6.	$\frac{2x}{1+x^2}$ or $\frac{2x}{1-x^2}$	$x = \tan \theta$
7.	$a \sin x + b \cos x$	$a = r \cos \alpha$, $b = r \sin \alpha$
8.	$\sqrt{x-\alpha}$ and $\sqrt{\beta-x}$	$x = \alpha \sin^2 \theta + \beta \cos^2 \theta$
9.	$\sqrt{2ax - x^2}$	$x = a(1 - \cos \theta)$

Important Trigonometrical Formulae

The following formulae will be helpful to solve the problems involving inverse trigonometric function.

i.	$\sin 2x = 2 \sin x \cos x = \frac{2 \tan x}{1 + \tan^2 x}$
ii.	$\cos 2x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x$ $= \cos^2 x - \sin^2 x = \frac{1 - \tan^2 x}{1 + \tan^2 x}$
iii.	$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$
iv.	$\frac{1 \pm \tan x}{1 \mp \tan x} = \tan \left(\frac{\pi}{4} \pm x \right)$
v.	$\sin 3x = 3 \sin x - 4 \sin^3 x$ $= 4 \sin(60^\circ - x) \cdot \sin x \cdot \sin(60^\circ + x)$
vi.	$\cos 3x = 4 \cos^3 x - 3 \cos x$ $= 4 \cos(60^\circ - x) \cdot \cos x \cdot \cos(60^\circ + x)$
vii.	$\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$ $= \tan(60^\circ - x) \cdot \tan x \cdot \tan(60^\circ + x)$
viii.	$\cos x \cos 2x \cos 2^2 x \dots \cos 2^{n-1} x = \frac{\sin 2^n x}{2^n \sin x}$

ix.

S. No.	Function	Domain	Range (Principal value)
1.	$\sin^{-1} x$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2.	$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
3.	$\tan^{-1} x$	R	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
4.	$\operatorname{cosec}^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
5.	$\sec^{-1} x$	$(-\infty, -1] \cup [1, \infty)$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$
6.	$\cot^{-1} x$	R	$(0, \pi)$

✎ If no branch of an inverse trigonometric function is mentioned, then it means the principal value branch of the function should be taken.

➤ **Example 17.** If $y = \tan^{-1} \left(\frac{\sqrt{1+x^2} - 1}{x} \right)$, then

$\frac{dy}{dx}$ is equal to

- (a) $\frac{1}{1+x^2}$ (b) $\frac{2}{1+x^2}$
 (c) $\frac{1}{2(1+x^2)}$ (d) None of these

Sol. (c) Put $x = \tan \theta$

$$\Rightarrow \theta = \tan^{-1} x, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

$$\therefore y = \tan^{-1} \left(\frac{|\sec \theta| - 1}{\tan \theta} \right) \quad \left[\because |\sec \theta| = \sec \theta, \forall \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \right]$$

$$\Rightarrow y = \tan^{-1} \left(\frac{1 - \cos \theta}{\sin \theta} \right) = \tan^{-1} \left(\tan \frac{\theta}{2} \right)$$

$$\Rightarrow y = \frac{\theta}{2} \Rightarrow y = \frac{1}{2} \tan^{-1} x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

➤ **Example 18.** Find $\frac{dy}{dx}$, where

$$y = \tan^{-1} \left(\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}} \right)$$

- (a) $\frac{1}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{2-x^2}}$
 (c) $\frac{2}{\sqrt{1-x^2}}$ (d) None of these

Sol. (d) Put $x = \cos \theta$

$$\Rightarrow \theta = \cos^{-1} x, \theta \in [0, \pi]$$

$$\therefore y = \tan^{-1} \left(\frac{\sqrt{1+\cos \theta} - \sqrt{1-\cos \theta}}{\sqrt{1+\cos \theta} + \sqrt{1-\cos \theta}} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{\sqrt{2} \cos \frac{\theta}{2} - \sqrt{2} \sin \frac{\theta}{2}}{\sqrt{2} \cos \frac{\theta}{2} + \sqrt{2} \sin \frac{\theta}{2}} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{1 - \tan \frac{\theta}{2}}{1 + \tan \frac{\theta}{2}} \right) \Rightarrow y = \frac{\pi}{4} - \frac{\theta}{2}$$

$$\Rightarrow y = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x \Rightarrow \frac{dy}{dx} = \frac{1}{2\sqrt{1-x^2}}$$

➤ **Example 19.** If $y = \cos^{-1} \sqrt{\frac{\sqrt{1+x^2} + 1}{2\sqrt{1+x^2}}}$, then

$\frac{dy}{dx}$ is equal to

- (a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1-x^2}$
 (c) $\frac{1}{2(1+x^2)}$ (d) None of these

Sol. (c) Put $x = \tan \theta$

$$\Rightarrow \theta = \tan^{-1} x$$

$$\therefore y = \cos^{-1} \sqrt{\frac{\sqrt{1+\tan^2 \theta} + 1}{2\sqrt{1+\tan^2 \theta}}} = \cos^{-1} \sqrt{\frac{\sec \theta + 1}{2\sec \theta}}$$

$$= \cos^{-1} \sqrt{\frac{1 + \cos \theta}{2}} = \cos^{-1} \sqrt{\frac{2\cos^2 \theta / 2}{2}}$$

$$= \cos^{-1} \left(\cos \frac{\theta}{2} \right) = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} x$$

$$\therefore \frac{dy}{dx} = \frac{1}{2(1+x^2)}$$

Differential Coefficient using Inverse Trigonometrical Substitutions

Sometimes, the given function can be deduced with the help of inverse trigonometrical substitution and then to find the differential coefficient, it is very easy.

i. $2\sin^{-1} x = \sin^{-1} (2x\sqrt{1-x^2})$

ii. $2\cos^{-1} x = \cos^{-1} (2x^2 - 1)$
 or $\cos^{-1} (1 - 2x^2)$

iii. $2\tan^{-1} x = \begin{cases} \sin^{-1} \left(\frac{2x}{1+x^2} \right) \\ \tan^{-1} \left(\frac{2x}{1-x^2} \right) \\ \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right) \end{cases}$

$$\begin{aligned}
 \text{iv. } & 3 \sin^{-1} x = \sin^{-1} (3x - 4x^3) \\
 \text{v. } & 3 \cos^{-1} x = \cos^{-1} (4x^3 - 3x) \\
 \text{vi. } & 3 \tan^{-1} x = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right) \\
 \text{vii. } & \cos^{-1} x + \sin^{-1} x = \pi / 2 \\
 \text{viii. } & \tan^{-1} x + \cot^{-1} x = \pi / 2 \\
 \text{ix. } & \sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2} \\
 \text{x. } & \sin^{-1} x \pm \sin^{-1} y \\
 & \quad = \sin^{-1} [x\sqrt{1-y^2} \pm y\sqrt{1-x^2}] \\
 \text{xi. } & \cos^{-1} x \pm \cos^{-1} y \\
 & \quad = \cos^{-1} [xy \mp \sqrt{(1-x^2)(1-y^2)}] \\
 \text{xii. } & \tan^{-1} x \pm \tan^{-1} y = \tan^{-1} \left[\frac{x \pm y}{1 \mp xy} \right]
 \end{aligned}$$

Derivative of Inverse Trigonometric Functions

$$\text{If } y = \sin^{-1} x \Rightarrow -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$$

$$\Rightarrow x = \sin y$$

$$\Rightarrow \frac{dx}{dy} = \cos y$$

$$\text{or } \frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}, -1 < x < 1$$

Here, $\cos y \neq -\sqrt{1-\sin^2 y}$, rather

$\cos y = \pm \sqrt{1-\sin^2 y}$ but for values of $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $\cos y$ is always positive and hence the result.

Similarly, let us find derivative of other inverse trigonometric functions.

$$\text{Let } y = \tan^{-1} x$$

$$\Rightarrow x = \tan y$$

$$\Rightarrow \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y$$

$$\Rightarrow \frac{dx}{dy} = 1 + x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+x^2}, x \in \mathbb{R}$$

$$\text{Also, if } y = \sec^{-1} x, y \in [0, \pi] - \left\{ \frac{\pi}{2} \right\}$$

$$\Rightarrow x = \sec y$$

$$\Rightarrow \frac{dx}{dy} = \sec y \tan y$$

$$\text{or } \frac{dy}{dx} = \frac{1}{x \tan y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\pm x \sqrt{\sec^2 y - 1}}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & \sec y > 1 \\ -\frac{1}{x\sqrt{x^2-1}}, & \sec y < -1 \end{cases}$$

$$\therefore \frac{dy}{dx} = \frac{1}{|x|\sqrt{x^2-1}}, x \in (-\infty, -1) \cup (1, \infty)$$

Results for the derivative of inverse trigonometric functions can be summarised as given below :

$f(x)$	$f'(x)$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}, x < 1$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}, x < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}, x \in \mathbb{R}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}, x \in \mathbb{R}$
$\sec^{-1} x$	$\frac{1}{ x \sqrt{x^2-1}}, x > 1$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{ x \sqrt{x^2-1}}, x > 1$

Example 20. If $f(x) = x \tan^{-1} x$, then $f'(1)$ is equal to

- (a) $\frac{1}{2} + \frac{\pi}{4}$ (b) $-\frac{1}{2} + \frac{\pi}{4}$
 (c) $-\frac{1}{2} - \frac{\pi}{4}$ (d) $\frac{1}{2} - \frac{\pi}{4}$

Sol. (a) We have,

$$f(x) = x \tan^{-1} x$$

$$\Rightarrow f'(x) = \tan^{-1} x + \frac{x}{1+x^2}$$

$$\therefore f'(1) = \tan^{-1} 1 + \frac{1}{2} = \frac{\pi}{4} + \frac{1}{2}$$

Example 21. If $f(x) = (1+x)^n$, then the value of $f(0) + f'(0) + \frac{f''(0)}{2!} + \dots + \frac{f^{(n)}(0)}{n!}$ is

- (a) n (b) 2^n
 (c) 2^{n-1} (d) None of these

Sol. (b) $f(0) = 1, f'(x) = n(1+x)^{n-1}, f''(x) = n(n-1)(1+x)^{n-2}, \dots, f^n(x) = n(n-1) \dots 1(1+x)^{n-n}$.

So, $f'(0) = n, f''(0) = n(n-1), \dots, f^n(0) = n!$

Hence, the given expression

$$= 1 + n + \frac{n(n-1)}{2!} + \dots + \frac{n!}{n!} \\ = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$$

➔ **Example 22.** If

$$y = \sec^{-1} \left(\frac{\sqrt{x}-1}{x+\sqrt{x}} \right) + \sin^{-1} \left(\frac{x+\sqrt{x}}{\sqrt{x}-1} \right), \text{ then } \frac{dy}{dx} \text{ is}$$

equal to

- (a) x (b) 1
(c) 0 (d) None of these

Sol. (c) We have,

$$y = \sec^{-1} \left(\frac{\sqrt{x}-1}{x+\sqrt{x}} \right) + \sin^{-1} \left(\frac{x+\sqrt{x}}{\sqrt{x}-1} \right) \\ \Rightarrow y = \cos^{-1} \frac{x+\sqrt{x}}{\sqrt{x}-1} + \sin^{-1} \frac{x+\sqrt{x}}{\sqrt{x}-1} \left[\because \sec^{-1} x = \cos^{-1} \frac{1}{x} \right] \\ = \frac{\pi}{2} \left[\because \sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \right] \\ \therefore \frac{dy}{dx} = 0$$

➔ **Example 23.** If

$$y = \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1} \left| \frac{\sqrt{a-b}}{\sqrt{a+b}} \tan \frac{x}{2} \right|, a > b > 0, \text{ then}$$

$$(a) \frac{dy}{dx} = \frac{1}{a+b\cos x} \quad (b) \frac{d^2y}{dx^2} = \frac{b\sin x}{(a+b\cos x)^2} \\ (c) \frac{dy}{dx} = \frac{1}{a-b\cos x} \quad (d) \frac{d^2y}{dx^2} = \frac{-b\sin x}{(a-b\cos x)^2}$$

Sol. (a, b) We have, $y = \frac{2}{\sqrt{a^2-b^2}} \cdot \tan^{-1} \left| \frac{\sqrt{a-b}}{\sqrt{a+b}} \tan \frac{x}{2} \right|$

$$\Rightarrow \frac{dy}{dx} = \frac{2}{\sqrt{a^2-b^2}} \cdot \frac{1}{1 + \left(\frac{\sqrt{a-b}}{\sqrt{a+b}} \tan \frac{x}{2} \right)^2} \\ \times \sqrt{\frac{a-b}{a+b}} \sec^2 \frac{x}{2} \cdot \frac{1}{2} \\ = \frac{1}{\sqrt{a-b}\sqrt{a+b}} \cdot \frac{1}{1 + \frac{a-b}{a+b} \cdot \frac{\sin^2 \frac{x}{2}}{\cos^2 \frac{x}{2}}} \cdot \sqrt{\frac{a-b}{a+b}} \sec^2 \frac{x}{2} \\ = \frac{1}{a+b} \cdot \frac{(a+b)\cos^2 \frac{x}{2}}{(a+b)\cos^2 \frac{x}{2} + (a-b)\sin^2 \frac{x}{2}} \sec^2 \frac{x}{2} \\ = \frac{1}{a \left(\cos^2 \frac{x}{2} + \sin^2 \frac{x}{2} \right) + b \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)} \\ = \frac{1}{a+b\cos x} \\ \therefore \frac{d^2y}{dx^2} = \frac{-1}{(a+b\cos x)^2} (-b\sin x) = \frac{b\sin x}{(a+b\cos x)^2}$$

Work Book Exercise 6.1

1 If $f(x) = \left(\frac{\sin^m x}{\sin^n x} \right)^{m+n} \left(\frac{\sin^n x}{\sin^p x} \right)^{n+p} \left(\frac{\sin^p x}{\sin^m x} \right)^{p+m}$,

then $f'(x)$ is equal to

- a 0 b 1
c $\cos^{m+n+p} x$ d None of these

2 If $f(x) = e^x g(x)$, $g(0) = 2$ and $g'(0) = 1$, then $f'(0)$ is equal to

- a 1 b 3
c 2 d 0

3 If $f(x) = \log |2x|$, $x \neq 0$, then $f'(x)$ is equal to

- a $\frac{1}{x}$
b $-\frac{1}{x}$
c $\frac{1}{|x|}$
d None of these

4 If $e^x + e^y = e^{x+y}$, then the value of $\frac{dy}{dx}$ at $(1, 1)$

- is
a 0
b -1
c 1
d None of the above

5 The derivative of $\tan^{-1} \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{1-x^2}}$ is

- a $\frac{x}{\sqrt{1-x^4}}$ b $-\frac{x}{\sqrt{1-x^4}}$
c $-\frac{1}{2\sqrt{1-x^2}}$ d None of these

6 If $y = \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\}$, then the value of $\frac{dy}{dx}$ is

- a $\frac{x}{\sqrt{1-x^2}}$ b $-\frac{x}{\sqrt{1-x}}$
c $-\frac{x}{\sqrt{1-x^2}}$ d None of these

7 If $y = \frac{e^{2x} \cos x}{x \sin x}$, then $\frac{dy}{dx}$ is equal to

- a $\frac{e^{2x} \{(2x-1) \cot x - x \operatorname{cosec}^2 x\}}{x^2}$
b $\frac{e^{2x} \{(2x+1) \cot x - x \operatorname{cosec}^2 x\}}{x^2}$
c $\frac{e^{2x} \{(2x-1) \cot x + x \operatorname{cosec}^2 x\}}{x^2}$
d None of the above

- 8 If $y = \tan^{-1} \sqrt{\frac{x+1}{x-1}}$, then $\frac{dy}{dx}$ is equal to
- a $-\frac{1}{2|x|\sqrt{x^2-1}}$ b $-\frac{1}{2x\sqrt{x^2-1}}$
 c $\frac{1}{2x\sqrt{x^2-1}}$ d None of these

- 9 If $y = \cos^{-1} \left(\frac{5 \cos x - 12 \sin x}{13} \right)$, $x \in \left(0, \frac{\pi}{2} \right)$, then $\frac{dy}{dx}$ is equal to
- a 1 b -1
 c 0 d None of these

Logarithmic Differentiation

So far, we have discussed derivatives of the functions of the form $(f(x))^n$, $n^{f(x)}$ and n^x , where $f(x)$ is a function of x and n is a constant. In this section, we will mainly discuss the derivatives of the functions of the form $(f(x))^{g(x)}$, where $f(x)$ and $g(x)$ are functions of x . To find the derivatives of this type of functions, we proceed as follows:

Let $y = (f(x))^{g(x)}$

Taking logarithm on both sides, we have

$$\log y = g(x) \log \{f(x)\}$$

On differentiating w.r.t. x , we get

$$\begin{aligned} \frac{1}{y} \cdot \frac{dy}{dx} &= g(x) \cdot \frac{1}{f(x)} \cdot \frac{d}{dx} f(x) + \log \{f(x)\} \cdot \frac{d}{dx} g(x) \\ \therefore \frac{dy}{dx} &= y \left[\frac{g(x)}{f(x)} \cdot \frac{d}{dx} f(x) + \log \{f(x)\} \cdot \frac{d}{dx} g(x) \right] \\ \Rightarrow \frac{dy}{dx} &= (f(x))^{g(x)} \left[\frac{g(x)}{f(x)} \cdot \frac{d}{dx} f(x) + \log \{f(x)\} \cdot \frac{d}{dx} g(x) \right] \end{aligned}$$

➤ **Example 24.** If $y = x^x$, then the value of $\frac{dy}{dx}$ is

- (a) $x^x (x + \log x)$
 (b) $x^x (1 + \log x)$
 (c) x^{x-1}
 (d) None of the above

Sol. (b) We have, $y = x^x$

Taking log on both sides, we get

$$\log y = \log x^x$$

$$\Rightarrow \log y = x \log x$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = x \frac{d(\log x)}{dx} + \log x \left(\frac{dx}{dx} \right)$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{x}{x} + \log x$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = 1 + \log x$$

$$\Rightarrow \frac{dy}{dx} = y (1 + \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^x (1 + \log x) \quad [\because y = x^x]$$

➤ **Example 25.** If $y = (\cot x)^{\sin x}$, then the value of $\frac{dy}{dx}$ is

- (a) $-\operatorname{cosec} x (\cot x)^{\sin x-1} + (\cot x)^{\sin x} \log(\cot x) \cos x$
 (b) $-\operatorname{cosec} x (\cot x)^{\sin x} + (\cot x)^{\sin x-1} \log(\cot x) \cos x$
 (c) $(\operatorname{cosec} x) (\cot x)^{\sin x-1} - (\cot x)^{\sin x} \log(\cot x) \cos x$
 (d) None of the above

Sol. (a) We have, $y = (\cot x)^{\sin x}$

Taking log on both sides, we get

$$\log y = \log(\cot x)^{\sin x}$$

$$\Rightarrow \log y = \sin x \log(\cot x)$$

On differentiating w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = \sin x \frac{d(\log \cot x)}{dx} + \log \cot x \frac{d(\sin x)}{dx}$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \sin x \times \frac{1}{\cot x} (-\operatorname{cosec}^2 x) + \log \cot x \cos x$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = -\frac{\operatorname{cosec} x}{\cot x} + \log \cot x \cos x$$

$$\Rightarrow \frac{dy}{dx} = y \left(-\frac{\operatorname{cosec} x}{\cot x} + \log \cot x \cos x \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\cot x)^{\sin x} (-\operatorname{cosec} x)}{\cot x} + (\cot x)^{\sin x} \log \cot x \cos x$$

$$\Rightarrow \frac{dy}{dx} = -\operatorname{cosec} x (\cot x)^{\sin x-1} + (\cot x)^{\sin x} \log \cot x \cos x$$

➤ **Example 26.** If $y = x^{x^{x^{x^{\dots}}}}$, then $\frac{dy}{dx}$ is equal to

- (a) yx^{y-1} (b) $\frac{y^2}{x(1-y \log x)}$
 (c) $\frac{y}{x(1+y \log x)}$ (d) None of these

Sol. (b) We have, $y = x^{x^{x^{x^{\dots}}}}$

$$\Rightarrow y = x^y \Rightarrow y = e^{y \log x}$$

$$\Rightarrow \frac{dy}{dx} = e^{y \log x} \cdot \frac{d}{dx} (y \log x)$$

$$\Rightarrow \frac{dy}{dx} = x^y \left(\frac{dy}{dx} \log x + \frac{y}{x} \right)$$

$$\Rightarrow \frac{dy}{dx} = y \left(\frac{dy}{dx} \log x + \frac{y}{x} \right) \quad [\because y = x^y]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2}{x(1-y \log x)}$$

➤ **Example 27.** If $y = x^{(\log x)^{\log \log x}}$, then $\frac{dy}{dx}$ is

equal to

(a) $\frac{y \log y}{x \log x} (2 \log \log x + 1)$

(b) $\frac{x \log x}{y \log y} (2 \log \log x + 1)$

(c) $\frac{2y \log y}{x \log x} (\log \log x + 1)$

(d) None of the above

Sol. (a) We have, $y = x^{(\log x)^{\log \log x}}$

Taking log on both sides, we get

$$\log y = (\log x)^{\log \log x} \cdot \log x$$

Again, taking log on both sides, we get

$$\log \log y = (\log \log x) \log \log x + \log \log x$$

On differentiating w.r.t. x , we get

$$\frac{1}{\log y} \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 2 \log \log x \cdot \frac{1}{\log x} \cdot \frac{1}{x} + \frac{1}{\log x} \cdot \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y \log y}{x \log x} (2 \log \log x + 1)$$

➤ **Example 28.** If $x^y \cdot y^x = 1$, then the value of

$\frac{dy}{dx}$ is

(a) $-\frac{(y+x \log y)}{(x+y \log x)} \cdot \frac{y}{x}$ (b) $\frac{(y+x \log y)}{(x+y \log x)} \cdot \frac{y}{x}$

(c) $-\frac{(x+y \log y)}{(y+x \log x)} \cdot \frac{y}{x}$ (d) None of these

Sol. (a) We have, $x^y \cdot y^x = 1$

Taking log on both sides, we get

$$y \log x + x \log y = \log 1$$

On differentiating both sides, we get

$$y \cdot \frac{d}{dx}(\log x) + \left\{ \frac{d}{dx} y \right\} \cdot \log x + \left\{ \frac{d}{dx} x \right\} \cdot \log y + x \cdot \left\{ \frac{d}{dx} \log y \right\} = 0$$

$$\Rightarrow y \cdot \frac{1}{x} + \log x \cdot \frac{dy}{dx} + 1 \cdot \log y + x \cdot \frac{1}{y} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \left[\log x + \frac{x}{y} \right] \cdot \frac{dy}{dx} = - \left[\frac{y}{x} + \log y \right]$$

$$\Rightarrow \frac{dy}{dx} = - \frac{(y+x \log y)}{(x+y \log x)} \cdot \frac{y}{x}$$

Differentiation of Parametric Functions

Sometimes, x and y are given functions of a single variable. e.g. $x = \phi(t)$, $y = \psi(t)$ are two functions and t is a variable. In such a case, x and y are called parametric functions or parametric equations and t is called the parameter.

To find $\frac{dy}{dx}$ in case of parametric functions, we first

obtain the relationship between x and y by eliminating the parameter t and then differentiate it with respect to x . But every time, it is not convenient to eliminate the parameter. Therefore, $\frac{dy}{dx}$ can also be obtained by the following formula

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

➤ **Example 29.** If $x = a \left\{ \cos \theta + \log \tan \frac{\theta}{2} \right\}$ and

$y = a \sin \theta$, then $\frac{dy}{dx}$ is equal to

(a) $\cot \theta$

(b) $\tan \theta$

(c) $\sin \theta$

(d) $\cos \theta$

Sol. (b) We have, $x = a \left\{ \cos \theta + \log \tan \frac{\theta}{2} \right\}$ and $y = a \sin \theta$

$$\Rightarrow \frac{dx}{d\theta} = a \left\{ -\sin \theta + \frac{\sec^2 \frac{\theta}{2}}{2 \tan \frac{\theta}{2}} \right\} \text{ and } \frac{dy}{d\theta} = a \cos \theta$$

$$\Rightarrow \frac{dx}{d\theta} = a \left\{ -\sin \theta + \frac{1}{\sin \theta} \right\} \text{ and } \frac{dy}{d\theta} = a \cos \theta$$

$$\Rightarrow \frac{dx}{d\theta} = a \frac{\cos^2 \theta}{\sin \theta} \text{ and } \frac{dy}{d\theta} = a \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \cos \theta}{a \frac{\cos^2 \theta}{\sin \theta}} = \tan \theta$$

➤ **Example 30.** If $x = a(\theta + \sin \theta)$ and

$y = a(1 - \cos \theta)$, then dy/dx is equal to

(a) $\tan \frac{\theta}{2}$

(b) $\cot \frac{\theta}{2}$

(c) $\tan \theta$

(d) $\cot \theta$

Sol. (a) Given, $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$

$$\therefore \frac{dx}{d\theta} = a(1 + \cos \theta) \text{ and } \frac{dy}{d\theta} = a(\sin \theta)$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a \sin \theta}{a(1 + \cos \theta)}$$

$$= \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \frac{\sin \frac{\theta}{2}}{\cos \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

➤ **Example 31.** If $x = \sqrt{a \sin^{-1} t}$ and

$y = \sqrt{a \cos^{-1} t}$, then the value of $\frac{dy}{dx}$ is

(a) $\frac{y}{x}$

(b) $\frac{x}{y}$

(c) $-\frac{y}{x}$

(d) $-\frac{x}{y}$

Sol. (c) Given, $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$
 $\Rightarrow x = a^{\frac{1}{2} \sin^{-1} t}$ and $y = a^{\frac{1}{2} \cos^{-1} t}$ [$\because (a^b)^c = a^{bc}$]

On differentiating w.r.t. t , we get

$$\frac{dx}{dt} = a^{\frac{1}{2} \sin^{-1} t} \log a \frac{d}{dt} \left(\frac{1}{2} \sin^{-1} t \right) \left[\because \frac{d}{dx} a^x = a^x \log a \right]$$

$$= a^{\frac{1}{2} \sin^{-1} t} \log a \left(\frac{1}{2\sqrt{1-t^2}} \right) = \frac{a^{\frac{1}{2} \sin^{-1} t} \log a}{2\sqrt{1-t^2}}$$

and $\frac{dy}{dt} = a^{\frac{1}{2} \cos^{-1} t} \log a \frac{d}{dt} \left(\frac{1}{2} \cos^{-1} t \right)$ [using chain rule]

$$= a^{\frac{1}{2} \cos^{-1} t} \log a \left(\frac{-1}{2\sqrt{1-t^2}} \right) = -\frac{a^{\frac{1}{2} \cos^{-1} t} \log a}{2\sqrt{1-t^2}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\frac{a^{\frac{1}{2} \cos^{-1} t} \log a}{2\sqrt{1-t^2}}}{\frac{a^{\frac{1}{2} \sin^{-1} t} \log a}{2\sqrt{1-t^2}}} = -\frac{\sqrt{a^{\cos^{-1} t}}}{\sqrt{a^{\sin^{-1} t}}} = -\frac{y}{x}$$

➤ **Example 32.** If $x = t^t$ and $y = t^{t^t}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{t^{t^t} \left[(1 + \log t) \log t + \frac{1}{t} \right]}{(1 + \log t)}$ (b) $\frac{t^{t^t} \left[1 + \log t + \frac{1}{t} \right]}{(1 + \log t)}$

(c) $\frac{t^{t^t} \left[(1 + \log t) \log t + \frac{1}{t} \right]}{(1 + \log t)^2}$ (d) None of these

Sol. (a) We have, $x = t^t = e^{t \log t}$

$$\Rightarrow \frac{dx}{dt} = e^{t \log t} (1 + \log t) = t^t (1 + \log t)$$

Also, $y = t^{t^t}$

$$\Rightarrow \log y = t^t \log t = e^{t \log t} \cdot \log t$$

$$\Rightarrow \frac{1}{y} \cdot \frac{dy}{dt} = e^{t \log t} (1 + \log t) \log t + e^{t \log t} \cdot \frac{1}{t}$$

$$\Rightarrow \frac{dy}{dt} = t^{t^t} \left\{ t^t (1 + \log t) \log t + \frac{t^t}{t} \right\}$$

$$\therefore \frac{dy}{dx} = \frac{t^{t^t} \left[(1 + \log t) \log t + \frac{1}{t} \right]}{(1 + \log t)}$$

Differentiation of a Function with Respect to a Function

Let $u = f(x)$ and $v = g(x)$ be two functions of x . Then, to find the derivative of $f(x)$ w.r.t. $g(x)$, i.e.

$\frac{du}{dv}$, we use the following formula

$$\frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

Thus, to find the derivative of $f(x)$ w.r.t. $g(x)$, we first differentiate both w.r.t. x and then divide the derivative of $f(x)$ w.r.t. x by the derivative of $g(x)$ w.r.t. x .

➤ **Example 33.** The derivative of $\sin x^3$ w.r.t. $\cos x^3$ is equal to

(a) $-\tan x^3$ (b) $-\cot x^3$
 (c) $\cot x^3$ (d) $\tan x^3$

Sol. (b) Let $y = \sin x^3 \Rightarrow \frac{dy}{dx} = \cos x^3 \cdot 3x^2$

and $z = \cos x^3 \Rightarrow \frac{dz}{dx} = -\sin x^3 \cdot 3x^2$

$$\therefore \frac{dy}{dz} = \frac{dy/dx}{dz/dx} = -\frac{\cos x^3 \cdot 3x^2}{\sin x^3 \cdot 3x^2} = -\cot x^3$$

➤ **Example 34.** The differential coefficient of $\log_{10} x$ w.r.t. $\log_x 10$ is

(a) $-\frac{(\log 10)^2}{(\log x)^2}$ (b) $\frac{(\log_x 10)^2}{(\log 10)^2}$
 (c) $\frac{(\log_{10} x)^2}{(\log 10)^2}$ (d) $-\frac{(\log x)^2}{(\log 10)^2}$

Sol. (d) Let $y = \log_{10} x$ and $z = \log_x 10 = \frac{1}{\log_{10} x}$

$$\therefore y = \frac{1}{z} \Rightarrow \frac{dy}{dz} = -\frac{1}{z^2} = -\frac{1}{(\log_x 10)^2} = -\frac{(\log x)^2}{(\log 10)^2}$$

➤ **Example 35.** The derivative of

$$\sec^{-1} \left(\frac{1}{2x^2 - 1} \right) \text{ w.r.t. } \sqrt{1-x^2} \text{ at } x = \frac{1}{2}, \text{ is}$$

(a) -4 (b) 4 (c) 2 (d) -2

Sol. (b) Let $y = \sec^{-1} \left(\frac{1}{2x^2 - 1} \right)$ and $z = \sqrt{1-x^2}$

Then, $y = \sec^{-1} \left(\frac{1}{2x^2 - 1} \right) = \cos^{-1} (2x^2 - 1)$

$$\Rightarrow y = \begin{cases} 2 \cos^{-1} x, & \text{if } 0 \leq x \leq 1 \\ 2\pi - 2 \cos^{-1} x, & \text{if } -1 \leq x < 0 \end{cases}$$

$$\therefore \frac{dy}{dx} = \begin{cases} -\frac{2}{\sqrt{1-x^2}}, & \text{if } 0 < x < 1 \\ \frac{2}{\sqrt{1-x^2}}, & \text{if } -1 < x < 0 \end{cases}$$

$$z = \sqrt{1-x^2} \Rightarrow \frac{dz}{dx} = \frac{-x}{\sqrt{1-x^2}}, \forall x \in (-1, 1)$$

$$\therefore \frac{dy}{dz} = \begin{cases} \frac{2}{x}, & \text{if } 0 < x < 1 \\ -\frac{2}{x}, & \text{if } -1 < x < 0 \end{cases}$$

Hence, $\left[\frac{dy}{dz} \right]_{x=1/2} = 4$

➤ **Example 36.** If $y = x - x^2$, then the derivative of y^2 w.r.t. x^2 is

- (a) $2x^2 + 3x - 1$ (b) $2x^2 - 3x + 1$
(c) $2x^2 + 3x + 1$ (d) None of these

Sol. (b) Let $u = y^2$ and $v = x^2$

$$\text{Then, } \frac{du}{dx} = 2y \frac{dy}{dx} = 2y(1 - 2x) \quad \left[\because \frac{dy}{dx} = 1 - 2x \right]$$

$$\text{and } \frac{dv}{dx} = 2x$$

$$\therefore \frac{du}{dv} = \frac{\frac{du}{dx}}{\frac{dv}{dx}} = \frac{2y(1 - 2x)}{2x} = \frac{(x - x^2)(1 - 2x)}{x}$$

$$[\because y = x - x^2]$$

$$= 2x^2 - 3x + 1$$

➤ **Example 37.** The differential coefficient of

$$\tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right) \text{ w.r.t. } \sec^{-1} \left(\frac{1}{2x^2-1} \right) \text{ at } x = \frac{1}{2},$$

is equal to

- (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) -1 (d) 1

Sol. (c) Let $y = \tan^{-1} \left(\frac{2x\sqrt{1-x^2}}{1-2x^2} \right)$ and $t = \sec^{-1} \left(\frac{1}{2x^2-1} \right)$

Put $x = \cos \theta$

$$\text{Then, } y = \tan^{-1} \left(\frac{2 \cos \theta \sin \theta}{1 - 2 \cos^2 \theta} \right) = \tan^{-1} (-\tan 2\theta) = -2\theta$$

$$\Rightarrow \frac{dy}{d\theta} = -2$$

$$\text{Also, } t = \sec^{-1} \left(\frac{1}{2 \cos^2 \theta - 1} \right) = \sec^{-1} (\sec 2\theta) = 2\theta$$

$$\therefore \frac{dy}{dt} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dt} = -1$$

$$\Rightarrow \left[\frac{dy}{dt} \right]_{x=\frac{1}{2}} = -1$$

➤ **Example 38.** The derivative of

$$\tan^{-1} \left(\frac{2x}{1-x^2} \right) \text{ w.r.t. } \sin^{-1} \left(\frac{2x}{1+x^2} \right), \text{ is}$$

- (a) $1, \forall x$
(b) 1 for $|x| > 1$ and -1 for $|x| < 1$
(c) 1 for $|x| < 1$ and -1 for $|x| > 1$
(d) 1 for $|x| \leq 1$ and -1 for $|x| > 1$

Sol. (c) Let $y = \tan^{-1} \left(\frac{2x}{1-x^2} \right)$

$$\text{and } z = \sin^{-1} \left(\frac{2x}{1+x^2} \right)$$

$$\text{Then, } y = \begin{cases} 2 \tan^{-1} x, & \text{if } -1 < x < 1 \\ 2 \tan^{-1} x - \pi, & \text{if } x > 1 \\ 2 \tan^{-1} x + \pi, & \text{if } x < -1 \end{cases}$$

$$\text{and, } z = \begin{cases} 2 \tan^{-1} x, & \text{if } -1 \leq x \leq 1 \\ \pi - 2 \tan^{-1} x, & \text{if } x > 1 \\ -\pi - 2 \tan^{-1} x, & \text{if } x < -1 \end{cases}$$

$$\therefore \frac{dy}{dz} = \begin{cases} 1, & \text{if } -1 < x < 1 \\ -1, & \text{if } |x| > 1 \end{cases}$$

Work Book Exercise 6.2

1 If $y = x^{1/x}$, then the value of $\frac{dy}{dx}$ at $x = e$, is

- a 1
b 0
c -1
d None of the above

2 If $x = e^{y+e^y+\dots}$, then $\frac{dy}{dx}$ is

- a $\frac{x}{1+x}$
b $\frac{1}{x}$
c $\frac{1-x}{x}$
d None of the above

3 If $xe^{xy} - y = \sin^2 x$, then $\frac{dy}{dx}$ at $x = 0$, is

- a 0
b 1
c -1
d None of the above

4 If $x^y \cdot y^x = 16$, then $\frac{dy}{dx}$ at $(2, 2)$, is

- a 1 b -1
c 0 d None of these

5 Let the function $y = f(x)$ be given by

$$x = t^5 - 5t^3 - 20t + 2 \text{ and } y = 4t^3 - 3t^2 - 18t + 3, \text{ where } t \in (-2, 2). \text{ Then, } f'(x) \text{ at } t = 1 \text{ is}$$

- a $\frac{5}{2}$
b $\frac{2}{5}$
c $\frac{7}{5}$
d None of the above

6 If $t(1+x^2) = x$ and $x^2+t^2=y$, then $\frac{dy}{dx}$ at $x=2$, is

- a $\frac{88}{125}$ b $\frac{488}{125}$
c 1 d None of these

- 7 If $y = \sin x$ and $z = \cos x$, then dy/dz is equal to
 a $-\operatorname{cosec} x \cos x$
 b $\frac{\pi}{180} \operatorname{cosec} \frac{\pi x}{180} \cos x$
 c $-\frac{\pi}{180} \operatorname{cosec} x \cos \frac{\pi x}{180}$
 d None of the above

- 8 The derivative of $\tan^{-1} \left(\frac{\sqrt{1+x^2}-1}{x} \right)$ w.r.t.

$$\cos^{-1} \left(\sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}} \right) \text{ is}$$

- a 1
 c $1/2$
 b -1
 d None of these

Differentiation of Determinants

To differentiate a determinant, we differentiate one row or column at a time, keeping others unchanged.

e.g. If $\Delta(x) = \begin{vmatrix} f(x) & g(x) \\ u(x) & v(x) \end{vmatrix}$, then

$$\frac{d}{dx} [\Delta(x)] = \begin{vmatrix} f'(x) & g'(x) \\ u(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ u'(x) & v'(x) \end{vmatrix}$$

$$\text{Also, } \frac{d}{dx} [\Delta(x)] = \begin{vmatrix} f'(x) & g(x) \\ u'(x) & v(x) \end{vmatrix} + \begin{vmatrix} f(x) & g'(x) \\ u(x) & v'(x) \end{vmatrix}$$

Similar results hold for the differentiation of determinants of higher order.

If $y = \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix}$, then

$$\frac{dy}{dx} = \begin{vmatrix} u'(x) & v'(x) & w'(x) \\ p(x) & q(x) & r(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p'(x) & q'(x) & r'(x) \\ \lambda(x) & \mu(x) & \gamma(x) \end{vmatrix} + \begin{vmatrix} u(x) & v(x) & w(x) \\ p(x) & q(x) & r(x) \\ \lambda'(x) & \mu'(x) & \gamma'(x) \end{vmatrix}$$

The differentiation can also be done by columnwise.

Example 39. If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, then

$f'(x)$ is equal to

- (a) $6x^2$ (b) $6x$ (c) 0 (d) $-6x$

$$\text{Sol. (a) } \because f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$\begin{aligned} \therefore f'(x) &= \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & 2x & x^3 \\ 1 & 2 & 3x^2 \\ 0 & 0 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & 3x^2 \\ 1 & 2x & 6x \\ 0 & 2 & 6 \end{vmatrix} \\ &= 1(12x^2 - 6x^2) + 6x(2x - 2x) - 2(6x^2 - 3x^2) \\ &\quad + 6(2x^2 - x^2) \\ &= 6x^2 + 0 - 6x^2 + 6x^2 = 6x^2 \end{aligned}$$

Example 40. If

$$f(x) = \begin{vmatrix} \sec \theta & \tan^2 \theta & 1 \\ \theta \sec x & \tan x & x \\ 1 & \tan x - \tan \theta & 0 \end{vmatrix}, \text{ then } f'(\theta) \text{ is}$$

- (a) 0 (b) -1
 (c) independent of θ (d) None of these

Sol. (b) We have,

$$\begin{aligned} f(x) &= \begin{vmatrix} \sec \theta & \tan^2 \theta & 1 \\ \theta \sec x & \tan x & x \\ 1 & \tan x - \tan \theta & 0 \end{vmatrix} \\ \Rightarrow f'(x) &= \begin{vmatrix} 0 & 0 & 0 \\ \theta \sec x & \tan x & x \\ 1 & \tan x - \tan \theta & 0 \end{vmatrix} \\ &\quad + \begin{vmatrix} \sec \theta & \tan^2 \theta & 1 \\ \theta \sec x \tan x & \sec^2 x & 1 \\ 1 & \tan x - \tan \theta & 0 \end{vmatrix} \\ &\quad + \begin{vmatrix} \sec \theta & \tan^2 \theta & 1 \\ \theta \sec x & \tan x & x \\ 0 & \sec^2 x & 0 \end{vmatrix} \\ \therefore f'(\theta) &= 0 + \begin{vmatrix} \sec \theta & \tan^2 \theta & 1 \\ \theta \sec \theta \tan \theta & \sec^2 \theta & 1 \\ 1 & 0 & 0 \end{vmatrix} + \begin{vmatrix} \sec \theta & \tan^2 \theta & 1 \\ \theta \sec \theta & \tan \theta & \theta \\ 0 & \sec^2 \theta & 0 \end{vmatrix} \\ \Rightarrow f'(\theta) &= \tan^2 \theta - \sec^2 \theta - \sec^2 \theta (\theta \sec \theta - \theta \sec \theta) = -1 \end{aligned}$$

Higher Order Derivatives

If $y = f(x)$, then $\frac{dy}{dx}$, the derivative of y with respect to x , is itself a function of x in general and can be differentiated again. To fix up the idea, we shall call $\frac{dy}{dx}$ as the first order derivative of y with respect to x and the derivative of $\frac{dy}{dx}$ w.r.t. x as the second order derivative of y w.r.t. x and will be denoted by $\frac{d^2 y}{dx^2}$.

Similarly, the derivative of $\frac{d^2 y}{dx^2}$ w.r.t. x will be termed as the third order derivative of y w.r.t. x and will be denoted by $\frac{d^3 y}{dx^3}$ and so on. The n th order derivative of y w.r.t. x will be denoted by $\frac{d^n y}{dx^n}$.

If $y = f(x)$, then the other alternative notations for $\frac{dy}{dx}, \frac{d^2 y}{dx^2}, \frac{d^3 y}{dx^3}, \dots, \frac{d^n y}{dx^n}$ are $y_1, y_2, y_3, \dots, y_n$

or $y', y'', y''' \dots, y^{(n)}$

or $Dy, D^2 y, D^3 y, \dots, D^n y$

or $f'(x), f''(x), f'''(x), \dots, f^n(x)$

The values of n th order derivatives at $x = a$ are denoted by

$$y_n(a), y^n(a), D^n y(a), f^n(a) \text{ or } \left(\frac{d^n y}{dx^n} \right)_{x=a}$$

➤ If y is a function of x given parametrically by $y = \psi(t)$ then

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{\phi'(t)}{\psi'(t)} \right) = \frac{\frac{d}{dt} \left(\frac{\phi'(t)}{\psi'(t)} \right)}{\frac{dx}{dt}}$$

Relation between $\frac{d^2 y}{dx^2}$ and $\frac{d^2 x}{dy^2}$

$$\begin{aligned} \frac{d^2 x}{dy^2} &= \frac{d}{dy} \left(\frac{dx}{dy} \right) = \frac{d}{dx} \left(\frac{1}{\left(\frac{dy}{dx} \right)} \right) \div \frac{dy}{dx} \\ &= \frac{-1}{\left(\frac{dy}{dx} \right)^2} \times \frac{d}{dx} \left(\frac{dy}{dx} \right) \div \frac{dy}{dx} \\ &= -\frac{d^2 y}{dx^2} \div \left(\frac{dy}{dx} \right)^3 \\ &= -\frac{(y'')}{(y')^3} \end{aligned}$$

Leibnitz Theorem

If u and v are functions of x such that their n th derivative exist, then

$$\begin{aligned} D^n(u \cdot v) &= {}^n C_0 (D^n u) v + {}^n C_1 (D^{n-1} u) (Dv) \\ &\quad + {}^n C_2 (D^{n-2} u) (D^2 v) + {}^n C_3 (D^{n-3} u) (D^3 v) \\ &\quad + \dots + {}^n C_r (D^{n-r} u) (D^r v) + \dots + {}^n C_n (D^n v) \end{aligned}$$

n th Derivative of Some Functions

- i. $\frac{d^n}{dx^n} [\sin(ax + b)] = a^n \sin\left(\frac{n\pi}{2} + ax + b\right)$
- ii. $\frac{d^n}{dx^n} [\cos(ax + b)] = a^n \cos\left(\frac{n\pi}{2} + ax + b\right)$
- iii. $\frac{d^n}{dx^n} (ax + b)^m = \frac{m!}{(m-n)!} a^n (ax + b)^{m-n}$
- iv. $\frac{d^n}{dx^n} [\log(ax + b)] = \frac{(-1)^{n-1} (n-1)! a^n}{(ax + b)^n}$
- v. $\frac{d^n}{dx^n} (e^{ax}) = a^n e^{ax}$
- vi. $\frac{d^n}{dx^n} (a^x) = a^x (\log a)^n$
- vii. (a) $\frac{d^n}{dx^n} [e^{ax} \sin(bx + c)]$

$$= r^n e^{ax} \sin(bx + c + n\phi)$$
 (b) $\frac{d^n}{dx^n} [e^{ax} \cos(bx + c)]$

$$= r^n e^{ax} \cos(bx + c + n\phi)$$
 where, $r = \sqrt{a^2 + b^2}$ and $\phi = \tan^{-1} \left(\frac{b}{a} \right)$

Derivative of Special Types of Function

- (i) If $y = f(x)^{\{f(x)\}^{\dots}}$, then

$$\frac{dy}{dx} = \frac{y^2 f'(x)}{f(x) [1 - y \log f(x)]}$$
- (ii) If $y = \sqrt{\frac{1+g(x)}{1-g(x)}}$, then

$$\frac{dy}{dx} = \frac{g'(x)}{[1-g(x)]^2} \sqrt{\frac{1-g(x)}{1+g(x)}}$$
- (iii) If $y = \sqrt{f(x)} + \sqrt{f(x)} + \sqrt{f(x)} + \dots$, then

$$\frac{dy}{dx} = \frac{f'(x)}{2y-1}$$
- (iv) If $\{f(x)\}^{g(y)} = e^{f(x)-g(y)}$, then

$$\frac{dy}{dx} = \frac{-f'(x)}{g'(y) + [1 + \log f(x)]^2}$$
- (v) If $\{f(x)\}^{g(y)} = \{g(y)\}^{f(x)}$, then

$$\frac{dy}{dx} = \frac{g(y)}{f(x)} \cdot \frac{f'(x)}{g'(y)} \left[\frac{f(x) \log g(y) - g(y)}{g(y) \log f(x) - f(x)} \right]$$

➤ **Example 41.** If $y = a \cos(\log x) + b \sin(\log x)$,

then $x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx}$ is equal to

- (a) 0
(b) y
(c) $-y$
(d) None of the above

Sol. (c) We have,

$$y = a \cos(\log x) + b \sin(\log x) \quad \dots(i)$$

On differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= -a \sin(\log x) \cdot \frac{1}{x} + b \cos(\log x) \cdot \frac{1}{x} \\ \Rightarrow x \frac{dy}{dx} &= -a \sin(\log x) + b \cos(\log x) \\ \Rightarrow x \frac{d^2 y}{dx^2} + \frac{dy}{dx} &= -a \cos(\log x) \cdot \frac{1}{x} - b \sin(\log x) \cdot \frac{1}{x} \\ \Rightarrow x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} &= -[a \cos(\log x) + b \sin(\log x)] \\ &= -y \quad [\text{from Eq. (i)}] \end{aligned}$$

➤ **Example 42.** If $y = \sin(m \sin^{-1} x)$, then

$(1-x^2)y'' - xy'$ is equal to

- (a) $m^2 y$ (b) my
(c) $-m^2 y$ (d) None of these

Sol. (c) We have, $y = \sin(m \sin^{-1} x)$

On differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= \cos(m \sin^{-1} x) \cdot \frac{m}{\sqrt{1-x^2}} \\ \Rightarrow (1-x^2) \left(\frac{dy}{dx} \right)^2 &= m^2 \cos^2(m \sin^{-1} x) = m^2 (1-y^2) \\ \text{Again, differentiating w.r.t. } x, \text{ we get} \\ (1-x^2) 2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 \cdot (-2x) &= -2m^2 y \frac{dy}{dx} \\ \Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} &= -m^2 y \quad \left[\text{dividing by } 2 \frac{dy}{dx} \right] \end{aligned}$$

➤ **Example 43.** If $y = x + e^x$, then $\frac{d^2 x}{dy^2}$ is

equal to

- (a) $\frac{1}{(1+e^x)^2}$ (b) $-\frac{e^x}{(1+e^x)^2}$
(c) $-\frac{e^x}{(1+e^x)^3}$ (d) e^x

Sol. (c) We have, $y = x + e^x$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= 1 + e^x \Rightarrow \frac{dx}{dy} = \frac{1}{1+e^x} \\ \Rightarrow \frac{d^2 x}{dy^2} &= \frac{d}{dy} \left(\frac{1}{1+e^x} \right) = \frac{d^2 x}{dy^2} = -\frac{1}{(1+e^x)^2} \cdot \frac{d}{dy} (1+e^x) \\ \Rightarrow \frac{d^2 x}{dy^2} &= -\frac{1}{(1+e^x)^2} \cdot e^x \cdot \frac{dx}{dy} = \frac{-e^x}{(1+e^x)^3} \end{aligned}$$

➤ **Example 44.** If $y^{1/m} = (x + \sqrt{1+x^2})$, then

$(1+x^2)y_2 + xy_1$ is equal to

- (a) $m^2 y$
(b) $m y^2$
(c) $m^2 y^2$
(d) None of the above

Sol. (a) We have, $y^{1/m} = (x + \sqrt{1+x^2})$

$$\begin{aligned} \Rightarrow y &= (x + \sqrt{1+x^2})^m \quad \dots(i) \\ \Rightarrow \frac{dy}{dx} &= m(x + \sqrt{1+x^2})^{m-1} \left(1 + \frac{x}{\sqrt{1+x^2}} \right) \\ \Rightarrow \frac{dy}{dx} &= m \frac{(x + \sqrt{1+x^2})^m}{\sqrt{1+x^2}} \\ \Rightarrow \frac{dy}{dx} &= \frac{my}{\sqrt{1+x^2}} \\ \Rightarrow y_1^2 (1+x^2) &= m^2 y^2 \\ \Rightarrow 2y_1 y_2 (1+x^2) + 2xy_1^2 &= 2m^2 y y_1 \\ \Rightarrow y_2 (1+x^2) + xy_1 &= m^2 y \end{aligned}$$

➤ **Example 45.** If $f(x)$ is given by

$f(x) = (\cos x + i \sin x)(\cos 3x + i \sin 3x) \dots$
 $\{\cos(2n-1)x + i \sin(2n-1)x\}$, then $f''(x)$ is

- (a) $n^2 f(x)$ (b) $-n^4 f(x)$
(c) $-n^2 f(x)$ (d) $n^4 f(x)$

Sol. (b) We have, $f(x) = \cos\{x + 3x + \dots + (2n-1)x\}$

$$\begin{aligned} &+ i \sin\{x + 3x + 5x + \dots + (2n-1)x\} \\ \Rightarrow f(x) &= \cos n^2 x + i \sin n^2 x \\ \Rightarrow f'(x) &= -n^2 (\sin n^2 x) + n^2 (i \cos n^2 x) \\ \Rightarrow f''(x) &= -n^4 \cos n^2 x - n^4 i \sin n^2 x \\ \Rightarrow f''(x) &= -n^4 (\cos n^2 x + i \sin n^2 x) \\ \Rightarrow f''(x) &= -n^4 f(x) \end{aligned}$$

➤ **Example 46.** If $y = \sin x$, then $\frac{d^2}{dy^2} (\cos^7 x)$ is

equal to

- (a) $35 \cos^3 x - 42 \cos^5 x$ (b) $35 \cos^3 x + 42 \cos^5 x$
(c) $42 \cos^3 x - 35 \cos^5 x$ (d) None of these

Sol. (a) We have, $y = \sin x$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \cos x \\ \text{Now, } \frac{d^2}{dy^2} (\cos^7 x) &= \frac{d}{dy} \left(\frac{d}{dy} \cos^7 x \right) \\ &= \frac{d}{dy} \left(7 \cos^6 x \cdot (-\sin x) \frac{dx}{dy} \right) \\ &= \frac{d}{dy} (-7 \sin x \cos^5 x) \\ &= -7 [\cos x \cdot \cos^5 x - 5 \cos^4 x \sin^2 x] \frac{dx}{dy} \\ &= -42 \cos^5 x + 35 \cos^3 x \end{aligned}$$

Work Book Exercise 6.3

- If $y = A \cos nx + B \sin nx$, then $\frac{d^2y}{dx^2}$ is equal to
 - n^2y
 - $-y$
 - $-n^2y$
 - None of the above
- If $x = \log p$ and $y = \frac{1}{p}$, then
 - $\frac{d^2y}{dx^2} - 2p = 0$
 - $\frac{d^2y}{dx^2} + y = 0$
 - $\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
 - $\frac{d^2y}{dx^2} - y = 0$
- If $y = (\sin^{-1} x)^2 + k \sin^{-1} x$, then $(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx}$ is equal to
 - 0
 - 1
 - 2
 - y
- If $f_n(x), g_n(x), h_n(x); n = 1, 2, 3$ are polynomials in x such that $f_n(a) = g_n(a) = h_n(a); n = 1, 2, 3$ and

$$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$
 Then, $F'(a)$ is equal to
 - 0
 - $f_1(a)g_2(a)h_3(a)$
 - 1
 - None of these
- If $x = \phi(t)$ and $y = \psi(t)$, then $\frac{d^2y}{dx^2}$ is equal to
 - $\frac{\phi' \psi'' - \psi' \phi''}{(\phi')^2}$
 - $\frac{\phi' \psi'' - \psi' \phi''}{(\phi')^3}$
 - $\frac{\phi''}{\psi''}$
 - $\frac{\psi''}{\phi''}$
- The second order derivative of $a \sin^3 t$ w.r.t. $a \cos^3 t$ at $t = \frac{\pi}{4}$, is
 - $\frac{4\sqrt{2}}{3a}$
 - 2
 - $\frac{1}{12a}$
 - 0
- Let $f(x)$ and $g(x)$ be two functions having finite non-zero 3rd order derivatives $f'''(x)$ and $g'''(x)$, $\forall x \in R$. If $f(x)g(x) = 1, \forall x \in R$, then $\frac{f'''}{f'} - \frac{g'''}{g'}$ is equal to
 - $3 \left(\frac{f''}{g} - \frac{g''}{f} \right)$
 - $3 \left(\frac{f''}{f} - \frac{g''}{g'} \right)$
 - $3 \left(\frac{g''}{g'} - \frac{f''}{g} \right)$
 - $3 \left(\frac{f''}{g} - \frac{g''}{f} \right)$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $f'(x)$ is differentiable function and $f''(x)$ is continuous at $x=0$ and $f''(0)=a$, then the value of $\lim_{x \rightarrow 0} \frac{2f(x)-3f(2x)+f(4x)}{x^2}$ is

- (a) a
(b) $2a$
(c) $3a$
(d) None of the above

Sol. We know that,

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)}, \text{ if } f(0)=g(0)=0$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 0} \frac{2f(x)-3f(2x)+f(4x)}{x^2} &= \lim_{x \rightarrow 0} \frac{2f'(x)-6f'(2x)+4f'(4x)}{2x} \\ &= \lim_{x \rightarrow 0} \frac{f'(x)-3f'(2x)+2f'(4x)}{x} \\ &= \lim_{x \rightarrow 0} \{f''(x)-6f''(2x)+8f''(4x)\} \\ &= f''(0)-6f''(0)+8f''(0) \\ &= a-6a+8a=3a \end{aligned}$$

Hence, (c) is the correct answer.

Ex 2. Let $f:R \rightarrow R$ be such that $f(1)=3$ and

$f'(1)=6$. Then, $\lim_{x \rightarrow 0} \left(\frac{f(1+x)}{f(1)} \right)^{1/x}$ equals

- (a) 1
(b) $e^{1/2}$
(c) e^2
(d) e^3

Sol. Here, $\lim_{x \rightarrow 0} \left[\frac{f(1+x)}{f(1)} \right]^{1/x}$ [1^∞ form]

$$\begin{aligned} &= \lim_{x \rightarrow 0} \left[1 + \left(\frac{f(1+x)}{f(1)} - 1 \right) \right]^{1/x} \\ &= e^{\lim_{x \rightarrow 0} \frac{f(1+x)-f(1)}{f(1)} \cdot \frac{1}{x}} \\ &= e^{\lim_{x \rightarrow 0} \frac{f(1+x)-f(1)}{x} \cdot \frac{1}{f(1)}} \left[\begin{array}{l} \text{using } \lim_{x \rightarrow a} \{1+f(x)\}^{g(x)} \\ = e^{\lim_{x \rightarrow a} f(x) \cdot g(x)} \\ \text{where, } \lim_{x \rightarrow a} f(x) \rightarrow 0 \\ \text{and } \lim_{x \rightarrow a} g(x) \rightarrow \infty \end{array} \right] \\ &= e^{f'(1)/f(1)} = e^{6/3} = e^2 \end{aligned}$$

Hence, (c) is the correct answer.

Ex 3. If $f(0)=0$ and $f'(0)=2$, then the derivative of $y=f(f(f(f(x))))$ at $x=0$, is

- (a) 2
(b) 8
(c) 16
(d) 4

Sol. $y'(x) = f'(f(f(f(x))))f'(f(f(x)))f'(f(x))f'(x)$
So,
 $y'(0) = f'(f(f(f(0))))f'(f(f(0)))f'(f(0))f'(0)$
 $= f'(f(f(0)))f'(f(0))f'(0)f'(0)$
 $= (f'(0))^4 = 2^4 = 16$

Hence, (c) is the correct answer.

Ex 4. If $y^2 = P(x)$ is a polynomial of degree 3, then

$2 \frac{d}{dx} \left(y^3 \frac{d^2 y}{dx^2} \right)$ is equal to

- (a) $P(x) + P'(x)$
(b) $P(x)P'(x)$
(c) $P(x)P''(x)$
(d) a constant

Sol. From $y^2 = P(x)$, we have $2yy_1 = P'(x)$

$$\begin{aligned} \text{i.e. } 2y_1 &= \frac{P'(x)}{y} \\ \Rightarrow 2y_2 &= \frac{yP''(x) - P'(x)y_1}{y^2} \\ &= \frac{yP''(x) - P'(x) \cdot P'(x)/2y}{y^2} \\ &= \frac{2y^2P''(x) - (P'(x))^2}{2y^3} \\ \Rightarrow 2y_2y^3 &= \frac{1}{2} [2P(x)P''(x) - (P'(x))^2] \\ \Rightarrow 2 \frac{d}{dx} \left(y^3 \frac{d^2 y}{dx^2} \right) &= \frac{1}{2} [2P'(x)P''(x) \\ &\quad + 2P(x)P'''(x) - 2P'(x)P''(x)] \\ &= \frac{1}{2} \cdot 2P(x)P'''(x) = P(x)P'''(x) \end{aligned}$$

Hence, (c) is the correct answer.

Ex 5. The value of $y''(1)$, if $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ when $y(1)=1$, is equal to

- (a) $\frac{22}{7}$
(b) $-7\frac{21}{28}$
(c) 8
(d) $-8\frac{22}{27}$

Sol. We have, $x^3 - 2x^2y^2 + 5x + y - 5 = 0$

On differentiating the given expression, we get

$$3x^2 - 4xy^2 - 4x^2yy' + 5 + y' = 0$$

On putting $x=1$, we have

$$3 - 4 \cdot 1 \cdot 1 - 4 \cdot 1 \cdot 1 \cdot y'(1) + 5 + y'(1) = 0$$

$$\Rightarrow 4 - 3y'(1) = 0$$

$$\Rightarrow y'(1) = \frac{4}{3}$$

Again differentiating, we have

$$6x - 4y^2 - 8xyy' - 8xyy' - 4x^2y'^2$$

$$- 4x^2yy'' + y'' = 0$$

On putting $x = 1$, $y = 1$ and $y'(1) = 4/2$, we get
 $6 - 4 - 8\left(\frac{4}{3}\right) - 8\left(\frac{4}{3}\right) - 4\left(\frac{16}{9}\right) - 3y''(1) = 0$
 $\Rightarrow y''(1) = -8\frac{22}{27}$

Hence, (d) is the correct answer.

Ex 6. If $y = \tan^{-1}\left(\frac{1}{1+x+x^2}\right) + \tan^{-1}\left(\frac{1}{x^2+3x+3}\right)$
 $+ \tan^{-1}\left(\frac{1}{x^2+5x+7}\right) + \dots n \text{ terms}$

then $y'(0)$ is equal to

- (a) $-\frac{1}{1+n^2}$ (b) $-\frac{n^2}{1+n^2}$
 (c) $\frac{n}{1+n^2}$ (d) None of these

Sol. $y = \tan^{-1}\left(\frac{1}{1+x+x^2}\right) + \tan^{-1}\left(\frac{1}{x^2+3x+3}\right)$
 $+ \dots + n \text{ terms}$
 $= \tan^{-1}\left[\frac{(x+1)-x}{1+x(1+x)}\right] + \tan^{-1}\left[\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}\right]$
 $+ \dots + n \text{ terms}$
 $= \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1)$
 $+ \dots + \tan^{-1}(x+n) - \tan^{-1}(x+(n-1))$
 $\Rightarrow y(x) = \tan^{-1}(x+n) - \tan^{-1}x$
 $\Rightarrow y'(x) = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$
 $\Rightarrow y'(0) = \frac{1}{1+n^2} - 1 = \frac{-n^2}{1+n^2}$

Hence, (b) is the correct answer.

Ex 7. The solution set of $f'(x) > g'(x)$, where
 $f(x) = \left(\frac{1}{2}\right)5^{2x+1}$ and $g(x) = 5^x + 4x \log 5$, is

- (a) $(1, \infty)$
 (b) $(0, 1)$
 (c) $[0, \infty)$
 (d) $(0, \infty)$

Sol. We have, $f(x) = \left(\frac{1}{2}\right)5^{2x+1}$
 $\Rightarrow f'(x) = \left(\frac{1}{2}\right)5^{2x+1}(\log 5) \times 2 = (\log 5) \cdot 5^{2x+1}$

Also, $g'(x) = 5^x \log 5 + 4 \log 5$

Since, $\{x : f'(x) > g'(x)\}$
 $= \{x : \log 5 \cdot 5^{2x+1} > (\log 5)5^x + 4 \log 5\}$
 $= \{x : 5^{2x+1} > 5^x + 4\} = \{t = 5^x : 5t^2 - t - 4 > 0\}$
 $= \{t = 5^x : (5t+4)(t-1) > 0\}$
 $= \{t = 5^x : t > 1 \text{ or } t < -\frac{4}{5}\} = \{t = 5^x : t > 1\}$
 $= (0, \infty)$

Hence, (d) is the correct answer.

Ex 8. If f is a differentiable function at a point a and $f'(a) \neq 0$, then which of the following is true?

- (a) $-f'(a) = \lim_{h \rightarrow 0} \frac{f(a) - f(a-h)}{h}$
 (b) $\frac{1}{2}f'(a) = \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a+h)}{2h}$
 (c) $f'(a) = \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a)}{h}$
 (d) None of the above

Sol. (a) $\lim_{h \rightarrow 0} \frac{f(a) - f(a-h)}{h}$
 $= \lim_{h \rightarrow 0} \frac{f(a-h) - f(a)}{-h} = f'(a)$
 (b) $\lim_{h \rightarrow 0} \frac{f(a+2h) - f(a) - \{f(a+h) - f(a)\}}{2h}$
 $= \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a)}{2h} - \frac{1}{2} \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$
 $= f'(a) - \frac{1}{2}f'(a) = \frac{1}{2}f'(a)$
 (c) $\lim_{h \rightarrow 0} \frac{f(a+2h) - f(a)}{h} = 2 \lim_{h \rightarrow 0} \frac{f(a+2h) - f(a)}{2h}$
 $= 2f'(a)$

Hence, (b) is the correct answer.

Ex 9. Which of the following could be not true, if $f''(x) = x^{-1/3}$?

- (a) $f(x) = \frac{3}{2}x^{2/3} - 3$
 (b) $f(x) = \frac{9}{10}x^{5/3} - 7$
 (c) $f'''(x) = -\frac{1}{3}x^{-4/3}$
 (d) $f'(x) = \frac{3}{2}x^{2/3} + 6$

Sol. We have, $f''(x) = x^{-1/3} = \frac{d}{dx}\left(\frac{3}{2}x^{2/3} + C\right)$

So, $f'(x) = \frac{3}{2}x^{2/3} + \text{Constant}$
 $= \frac{d}{dx}\left(\frac{9}{10}x^{5/3} + \text{Constant}\right)$
 $\Rightarrow f(x) = \frac{9}{10}x^{5/3} + \text{Constant}$

Hence, (a) is the correct answer.

Ex 10. If $y(n) = e^x e^{x^2} \dots e^{x^n}$, $0 < x < 1$. Then,
 $\lim_{n \rightarrow \infty} \frac{dy(n)}{dx}$ at $x = \frac{1}{2}$, is

- (a) e (b) $4e$
 (c) $2e$ (d) $3e$

Sol. We have, $y(n) = e^x + x^2 + \dots + x^n = e^{\frac{x(1-x^n)}{1-x}}$

So, $\frac{dy(n)}{dx} = e^{\frac{x(1-x^n)}{1-x}} \times \frac{d}{dx}\left(\frac{x(1-x^n)}{1-x}\right)$

$$\begin{aligned}\therefore \lim_{n \rightarrow \infty} \frac{dy(n)}{dx} &= \lim_{n \rightarrow \infty} e^{\frac{x}{1-x} - \frac{x^{n+1}}{1-x}} \cdot \frac{d}{dx} \left(\lim_{n \rightarrow \infty} \frac{x}{1-x} - \frac{x^{n+1}}{1-x} \right) \\ &= e^{\frac{x}{1-x}} \cdot \frac{d}{dx} \left(-1 + \frac{1}{1-x} \right) = e^{\frac{x}{1-x}} \cdot \frac{1}{(1-x)^2}\end{aligned}$$

$$\text{Thus, } \lim_{n \rightarrow \infty} \left[\frac{dy(n)}{dx} \right]_{x=\frac{1}{2}} = 4e$$

Hence, (b) is the correct answer.

Ex 11. If $f(x) = |\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to

- (a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$
(c) 1 (d) None of these

Sol. When $\frac{\pi}{2} < x < \pi$, $\cos x < 0$

$$\begin{aligned}\therefore |\cos x| &= -\cos x \\ \text{So, } f(x) &= -\cos x \\ \Rightarrow f'(x) &= \sin x \\ \therefore f'\left(\frac{3\pi}{4}\right) &= \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}}\end{aligned}$$

Hence, (b) is the correct answer.

Ex 12. If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{2}\right)$ is equal to

- (a) 1 (b) -1
(c) 0 (d) None of these

Sol. When $0 < x < \frac{\pi}{4}$, $\cos x > \sin x$

$$\therefore \cos x - \sin x > 0$$

$$\text{Also, when } \frac{\pi}{4} < x < \pi, \cos x < \sin x$$

$$\text{When } \frac{\pi}{4} < x < \pi, \cos x - \sin x < 0$$

$$\therefore |\cos x - \sin x| = -(\cos x - \sin x), \text{ when } \frac{\pi}{4} < x < \pi$$

$$\Rightarrow f'(x) = \sin x + \cos x$$

$$\Rightarrow f'\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} + \cos \frac{\pi}{2} = 1 + 0 = 1$$

Hence, (a) is the correct answer.

Ex 13. If $f(x) = \sqrt{x^2 - 10x + 25}$, then the derivative of $f(x)$ on the interval $[0, 7]$ is

- (a) 1 (b) -1
(c) 0 (d) None

Sol. We have,

$$\begin{aligned}f(x) &= \sqrt{x^2 - 10x + 25} = \sqrt{(x-5)^2} = |x-5| \\ &= \begin{cases} x-5, & x \geq 5 \\ 5-x, & x < 5 \end{cases}\end{aligned}$$

Clearly, $f(x)$ is differentiable at all points on the interval $[0, 7]$ except at $x = 5$.

So, the derivative of $f(x)$ on the interval $[0, 7]$ does not exist.

Hence, (d) is the correct answer.

Ex 14. If $f(x) = \log |2x|$, $x \neq 0$, then $f'(x)$ is equal to

- (a) $\frac{1}{x}$ (b) $-\frac{1}{x}$
(c) $\frac{1}{|x|}$ (d) None of these

Sol. We have, $f(x) = \log |2x|$

$$\begin{aligned}&= \begin{cases} \log 2x, & 2x > 0, \text{ i.e. } x > 0 \\ \log(-2x), & 2x < 0, \text{ i.e. } x < 0 \end{cases} \\ \Rightarrow f'(x) &= \begin{cases} \frac{1}{2x} \times 2, & x > 0 \\ \frac{1}{-2x} \times -2, & x < 0 \end{cases} \\ &= \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x}, & x < 0 \end{cases}\end{aligned}$$

$$\therefore f'(x) = \frac{1}{x}, x \neq 0$$

Hence, (a) is the correct answer.

Ex 15. If $f(x) = |x-3|$ and $\phi(x) = (f \circ f)(x)$, when $x > 10$, then $\phi'(x)$ is equal to

- (a) 1 (b) 0
(c) -1 (d) None of these

Sol. Since, $x > 10$, therefore $f(x) = x-3$

$$\begin{aligned}\therefore \phi(x) &= (f \circ f)(x) = f[f(x)] = f(x-3) \\ &= x-3-3 = x-6\end{aligned}$$

$$\Rightarrow \phi'(x) = 1$$

Hence, (a) is the correct answer.

Ex 16. If $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to

- (a) $\frac{2}{1+x^2}$, when $-1 < x < 1$
(b) $\frac{2}{1+x^2}$, when $x < -1$ or $x > 1$
(c) $-\frac{2}{1+x^2}$, when $-1 < x < 1$
(d) None of the above

$$\begin{aligned}\text{Sol. } \frac{dy}{dx} &= \frac{1}{\sqrt{1 - \frac{4x^2}{(1+x^2)^2}}} \times \left[\frac{(1+x^2) \cdot 2 - 2x \cdot 2x}{(1+x^2)^2} \right] \\ &= 2 \cdot \frac{1+x^2}{\sqrt{(1-x^2)^2}} \cdot \frac{1-x^2}{(1+x^2)^2} \\ &= 2 \cdot \frac{1-x^2}{|1-x^2|} \cdot \frac{1}{1+x^2} \\ \therefore \frac{dy}{dx} &= \begin{cases} \frac{2}{1+x^2}, & -1 < x < 1 \\ \frac{-2}{1+x^2}, & x < -1 \text{ or } x > 1 \end{cases}\end{aligned}$$

Hence, (a) is the correct answer.

Ex 17. If $f(x) = \sqrt{x^2 + 6x + 9}$, then $f'(x)$ is equal to

- (a) 1 for $x < -3$ (b) -1 for $x < -3$
(c) $1, \forall x \in R$ (d) None of these

Sol. We have, $f(x) = \sqrt{x^2 + 6x + 9} = \sqrt{(x+3)^2} = |x+3|$

$$= \begin{cases} x+3, & x+3 \geq 0 \text{ i.e. } x \geq -3 \\ -x-3, & x+3 < 0 \text{ i.e. } x < -3 \end{cases}$$

$$\therefore f'(x) = \begin{cases} 1, & \text{if } x \geq -3 \\ -1, & \text{if } x < -3 \end{cases}$$

Hence, (b) is the correct answer.

Ex 18. If $f(x) = |(x-4)(x-5)|$, then $f'(x)$ is equal to

- (a) $-2x+9, \forall x \in R$ (b) $2x-9$, if $4 < x < 5$
(c) $-2x+9$, if $4 < x < 5$ (d) None of these

Sol. $f(x) = |(x-4)(x-5)|$

$$= \begin{cases} (x-4)(x-5), & \text{if } (x-4)(x-5) \geq 0 \\ -(x-4)(x-5), & \text{if } (x-4)(x-5) < 0 \end{cases}$$

$$= \begin{cases} x^2 - 9x + 20, & \text{if } x \leq 4 \text{ or } x \geq 5 \\ -(x^2 - 9x + 20), & \text{if } 4 < x < 5 \end{cases}$$

$$\therefore f'(x) = \begin{cases} 2x-9, & \text{if } x \leq 4 \text{ or } x \geq 5 \\ -2x+9, & \text{if } 4 < x < 5 \end{cases}$$

Hence, (c) is the correct answer.

Ex 19. If $f(x) = (ax+b)\sin x + (cx+d)\cos x$, then the values of a, b, c and d such that $f'(x) = x \cos x, \forall x$, are

- (a) $b=c=0, a=d=1$ (b) $b=d=0, a=c=1$
(c) $c=d=0, a=b=1$ (d) None of these

Sol. $f'(x) = a \sin x + (ax+b)\cos x + c \cos x - (cx+d)\sin x$
 But $f'(x) = x \cos x, \forall x$ [given]
 $\therefore x \cos x = (a-d)\sin x + (b+c)\cos x$

$$+ ax \cos x - cx \sin x$$

On equating the coefficients of $\sin x, \cos x, x \cos x$ and $x \sin x$, we get

$$a-d=0, b+c=0, a=1, c=0$$

$$\therefore b=c=0 \text{ and } a=d=1$$

Hence, (a) is the correct answer.

Ex 20. Let $f\left(\frac{x_1+x_2+\dots+x_n}{n}\right)$

$$= \frac{f(x_1)+f(x_2)+\dots+f(x_n)}{n}, \text{ where all}$$

$x_i \in R$, are independent to each other and $n \in N$. If $f(x)$ is differentiable, $f'(0) = a$ and $f(0) = b$, then $f'(x)$ is equal to

- (a) a (b) 0
(c) b (d) None of these

Sol. On differentiating the given equation w.r.t. x , we have

$$\frac{1}{n} f'\left(\frac{x_1+x_2+\dots+x_n}{n}\right) = \frac{f'(x_1)}{n}$$

$$\left[\begin{array}{l} \text{since, all } x_i \text{'s are independent to each other,} \\ \text{so } \frac{dx_i}{dx_j} = \begin{cases} 0, & \text{if } i \neq j \\ 1, & \text{if } i = j \end{cases} \end{array} \right]$$

On putting $x_1 = x_2 = \dots = x_{n-1} = 0$ and $x_n = x$, we get

$$f'\left(\frac{x}{n}\right) = f'(0) = a \Rightarrow n f'\left(\frac{x}{n}\right) = ax + C$$

[integrating the above equation]

Since, $f(0) = b$, we have $C = nb$

$$\therefore n f'\left(\frac{x}{n}\right) = ax + nb$$

$$\Rightarrow n f'(x) = nax + nb$$

$$\Rightarrow f'(x) = ax + b$$

$$\therefore f'(x) = a, \forall x \in R$$

Hence, (a) is the correct answer.

Ex 21. If $f'(x) = \phi(x)$ and $\phi'(x) = f(x), \forall x$. Also, $f(3) = 5$ and $f'(3) = 4$. Then, the value of $[f(10)]^2 - [\phi(10)]^2$ is

- (a) 0 (b) 9
(c) 41 (d) None of these

Sol. $\frac{d}{dx} \{ [f(x)]^2 - [\phi(x)]^2 \}$

$$= 2[f(x) \cdot f'(x) - \phi(x) \cdot \phi'(x)]$$

$$= 2[f(x) \cdot \phi(x) - \phi(x) \cdot f(x)] = 0$$

$$[\because f'(x) = \phi(x) \text{ and } \phi'(x) = f(x)]$$

$$\Rightarrow [f(x)]^2 - [\phi(x)]^2 = \text{Constant}$$

$$\therefore [f(10)]^2 - [\phi(10)]^2 = [f(3)]^2 - [\phi(3)]^2$$

$$= [f(3)]^2 - [f'(3)]^2$$

$$= 25 - 16 = 9$$

Hence, (b) is the correct answer.

Ex 22. If $8f(x) + 6f\left(\frac{1}{x}\right) = x+5$ and $y = x^2 f(x)$,

then $\frac{dy}{dx}$ at $x = -1$, is equal to

- (a) 0
(b) $\frac{1}{14}$
(c) $-\frac{1}{14}$
(d) None of the above

Sol. We have,

$$8f(x) + 6f\left(\frac{1}{x}\right) = x+5 \quad \dots(i)$$

$$\text{Therefore, } 8f\left(\frac{1}{x}\right) + 6f(x) = \frac{1}{x} + 5 \quad \left[\text{putting } x = \frac{1}{x} \right] \quad \dots(ii)$$

From Eqs. (i) and (ii), we have

$$f(x) = \frac{1}{28} \left(8x - \frac{6}{x} + 10 \right)$$

$$\text{Also, } y = x^2 f(x) = \frac{1}{28} (8x^3 - 6x + 10x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{28} (24x^2 + 20x - 6)$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=-1} = \frac{1}{28} (24 - 20 - 6) = -\frac{1}{14}$$

Hence, (c) is the correct answer.

Ex 23. Let f be a function defined for all $x \in \mathbb{R}$. If f is differentiable and $f(x^3) = x^5$, $\forall x \in \mathbb{R}$, $x \neq 0$, then the value of $f'(27)$ is

- (a) 15
(b) 45
(c) 0
(d) None of the above

Sol. We have, $f(x^3) = x^5$, $\forall x (x \neq 0)$

On differentiating w.r.t. x , we get

$$f'(x^3) \cdot 3x^2 = 5x^4 \Rightarrow f'(x^3) = \frac{5}{3}x^2$$

$$\therefore f'(27) = f'(3^3) = \frac{5}{3}(3)^2 = 15$$

Hence, (a) is the correct answer.

Ex 24. Let $f(x)$ be a polynomial function satisfying

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right). \text{ If } f(4) = 65 \text{ and}$$

I_1, I_2, I_3 are in GP, then $f'(I_1), f'(I_2), f'(I_3)$ are in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. Since, $f(x)$ is a polynomial function satisfying

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right).$$

$$\therefore f(x) = x^n + 1 \text{ or } f(x) = -x^n + 1$$

$$\text{If } f(x) = -x^n + 1, \text{ then } f(4) = -4^n + 1 \neq 65$$

$$\text{So, } f(x) = x^n + 1$$

$$\text{Since, } f(4) = 65$$

$$\therefore 4^n + 1 = 65$$

$$\Rightarrow n = 3$$

$$\therefore f(x) = x^3 + 1$$

$$\Rightarrow f'(x) = 3x^2$$

$$\therefore f'(I_1) = 3I_1^2, f'(I_2) = 3I_2^2, f'(I_3) = 3I_3^2$$

Since, I_1, I_2 and I_3 are in GP.

$\therefore f'(I_1), f'(I_2)$ and $f'(I_3)$ are also in GP.

Hence, (b) is the correct answer.

Ex 25. If $f(x) = (\cos x + i \sin x)(\cos 2x + i \sin 2x) \dots (\cos nx + i \sin nx)$ and $f(1) = 1$, then $f''(1)$ is equal to

- (a) $\frac{n(n+1)}{2}$ (b) $\left[\frac{n(n+1)}{2}\right]^2$
(c) $-\left[\frac{n(n+1)}{2}\right]^2$ (d) None of these

Sol. We have, $f(x) = (\cos x + i \sin x)(\cos 2x + i \sin 2x) \dots (\cos nx + i \sin nx)$

$$\Rightarrow f(x) = \cos(x + 2x + 3x + \dots + nx)$$

$$+ i \sin(x + 2x + 3x + \dots + nx)$$

$$\Rightarrow f(x) = \cos \frac{n(n+1)}{2}x + i \sin \frac{n(n+1)}{2}x$$

$$\Rightarrow f'(x) = \frac{n(n+1)}{2} \left[-\sin \frac{n(n+1)}{2}x + i \cos \frac{n(n+1)}{2}x \right]$$

$$\Rightarrow f''(x) = -\left[\frac{n(n+1)}{2}\right]^2 \left[\cos \frac{n(n+1)}{2}x + i \sin \frac{n(n+1)}{2}x \right]$$

$$= -\left[\frac{n(n+1)}{2}\right]^2 f(x)$$

$$\therefore f''(1) = -\left[\frac{n(n+1)}{2}\right]^2 f(1) = -\left[\frac{n(n+1)}{2}\right]^2$$

Hence, (c) is the correct answer.

Ex 26. If $f(x) = x^m$, m being a non-negative integer, then the value of m for which $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$, $\forall \alpha, \beta > 0$, is

- (a) 1
(b) 2
(c) -2
(d) None of the above

Sol. We have, $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$
 $\Rightarrow (\alpha + \beta)^{m-1} = \alpha^{m-1} + \beta^{m-1}$

Since, for $m > 2$, the above equality is not valid.

$\therefore$ We must have, $m = 2$. Also, for $m = 0$, $f'(x) = 0$, $\forall x$.

So, the equality is trivially true.

Hence, (b) is the correct answer.

Ex 27. If $f(x)$ is a polynomial of degree $n (> 2)$ and $f(x) = f(k - x)$, where k is a fixed real number and n is even, then degree of $f'(x)$ is

- (a) n
(b) $n - 1$
(c) $n - 2$
(d) None of the above

Sol. Clearly, $f(x)$ must be of the form
 $f(x) = a_0[x^n + (k - x)^n] + a_1[x^{n-1} + (k - x)^{n-1}]$
 $+ \dots + a_{n-1}[x + (k - x)] + a_n$

Clearly, $f'(x)$ is a polynomial of degree $n - 1$.

Hence, (b) is the correct answer.

Ex 28. If $y = f(x)$ is an odd differentiable function defined on $(-\infty, \infty)$ such that $f'(3) = -2$, then $f'(-3)$ equals

- (a) 4 (b) 2 (c) -2 (d) 0

Sol. Since, $f(x)$ is odd.

$$\therefore f(-x) = -f(x)$$

$$\Rightarrow f'(-x)(-1) = -f'(x)$$

$$\Rightarrow f'(-x) = f'(x)$$

$$\therefore f'(-3) = f'(3) = -2 \quad [\text{given}]$$

Hence, (c) is the correct answer.

Ex 29. Let f and g be differentiable functions satisfying $g'(a) = 2$, $g(a) = b$ and $f \circ g = I$ (identity function). Then, $f'(b)$ is equal to

- (a) 2 (b) $\frac{2}{3}$
(c) $\frac{1}{2}$ (d) None of these

Sol. We have, $f \circ g = I$

$$\Rightarrow (f \circ g)(x) = x, \forall x$$

$$\Rightarrow f'[g(x)] \cdot g'(x) = 1, \forall x$$

$$\Rightarrow f'[g(a)] \cdot g'(a) = 1$$

$$\Rightarrow f'[g(a)] = \frac{1}{g'(a)} = \frac{1}{2} \quad [\because g'(a) = 2]$$

$$\Rightarrow f'(b) = \frac{1}{2} \quad [\because g(a) = b]$$

Hence, (c) is the correct answer.

Ex 30. The derivative of an odd function is always

- (a) an even function
- (b) an odd function
- (c) Does not exist
- (d) None of the above

Sol. Let f be an odd function.

$$\text{Then, } f(-x) = -f(x)$$

$$\therefore f'(-x) \cdot (-1) = -f'(x)$$

$$\Rightarrow f'(-x) = f'(x)$$

$\therefore f'$ is an even function.

Hence, (a) is the correct answer.

Ex 31. If $x^2 + y^2 = 1$, then

- (a) $yy'' - 2(y')^2 + 1 = 0$
- (b) $yy'' + (y')^2 + 1 = 0$
- (c) $yy'' + (y')^2 - 1 = 0$
- (d) $yy'' + 2(y')^2 + 1 = 0$

Sol. We have, $x^2 + y^2 = 1$... (i)

On differentiating Eq. (i) w.r.t. x , we get

$$2x + 2yy' = 0 \text{ or } x + yy' = 0$$

Again, differentiating w.r.t. x , we get

$$1 + y'y' + yy'' = 0$$

$$\Rightarrow 1 + (y')^2 + yy'' = 0$$

Hence, (b) is the correct answer.

Ex 32. If $x = a \cos t \sqrt{\cos 2t}$ and

$$y = a \sin t \sqrt{\cos 2t} \quad (a > 0), \text{ then } \left| \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} \right|$$

at $\frac{\pi}{6}$, is given by

- (a) $\frac{a}{3}$
- (b) $a\sqrt{2}$
- (c) $\frac{\sqrt{2}}{3a}$
- (d) $\frac{\sqrt{2}a}{3}$

Sol. We have,

$$\frac{dx}{dt} = a \left[-\sin t \sqrt{\cos 2t} - \frac{\cos t \sin 2t}{\sqrt{\cos 2t}} \right] = \frac{-a \sin 3t}{\sqrt{\cos 2t}}$$

$$\text{and } \frac{dy}{dt} = a \left[\cos t \sqrt{\cos 2t} - \frac{\sin t \sin 2t}{\sqrt{\cos 2t}} \right] = \frac{a \cos 3t}{\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\cot 3t$$

$$\Rightarrow \frac{d^2y}{dx^2} = 3 \operatorname{cosec}^2 3t \frac{dt}{dx} = \frac{-3 \operatorname{cosec}^2 3t \sqrt{\cos 2t}}{a \sin 3t} = -\left(\frac{3}{a}\right) \operatorname{cosec}^3 3t \sqrt{\cos 2t}$$

$$\therefore \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(1 + \cot^2 3t)^{3/2}}{\left(-\frac{3}{a}\right) \operatorname{cosec}^3 3t \sqrt{\cos 2t}} = \frac{a}{3 \sqrt{\cos \frac{\pi}{3}}} = \frac{\sqrt{2}a}{3}$$

Hence, (d) is the correct answer.

Ex 33. If $\sqrt{x^2 + y^2} = ae^{\tan^{-1}\left(\frac{y}{x}\right)}$, $a > 0$, assuming $y > 0$, then $y''(0)$ is equal to

- (a) $\frac{2}{a} e^{-\pi/2}$
- (b) $-\frac{2}{a} e^{\pi/2}$
- (c) $-\frac{2}{a} e^{-\pi/2}$
- (d) None of the above

Sol. We have, $\sqrt{x^2 + y^2} = ae^{\tan^{-1}\left(\frac{y}{x}\right)}$... (i)

When $x = 0$, $y > 0$

$$\Rightarrow y = ae^{\pi/2}$$

Taking log on both sides of Eq. (i), we get

$$\frac{1}{2} \log(x^2 + y^2) = \log a + \tan^{-1}\left(\frac{y}{x}\right)$$

On differentiating both sides w.r.t. x , we get

$$\frac{1}{2} \cdot \frac{2x + 2yy'}{x^2 + y^2} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{xy' - y}{x^2}$$

$$\Rightarrow x + yy' = xy' - y \quad \dots (ii)$$

Again, differentiating w.r.t. x , we get

$$1 + (y')^2 + yy'' = xy'' + y' - y'$$

$$\Rightarrow 1 + (y')^2 = (x - y)y''$$

$$\Rightarrow y'' = \frac{1 + (y')^2}{x - y}$$

When $x = 0$, then from Eq. (ii), $y' = -1$

$$\therefore y''(0) = \frac{2}{-ae^{\pi/2}} = -\frac{2}{a} e^{-\pi/2}$$

Hence, (c) is the correct answer.

- Ex 34.** The function y defined by the equation $xy - \log y = 1$ satisfies $x(yy'' + y'^2) - y'' + kyy' = 0$. The value of k is
 (a) -3 (b) 3
 (c) 1 (d) None of these

Sol. Given, $xy - \log y = 1$... (i)

On differentiating Eq. (i) w.r.t. x , we get

$$xy' + y \cdot 1 - \frac{1}{y} y' = 0$$

$$\Rightarrow xyy' - y' + y^2 = 0$$

$$\Rightarrow (xy - 1)y' + y^2 = 0$$

Again, differentiating w.r.t. x , we get

$$(xy - 1)y'' + y'(xy' + y \cdot 1) + 2yy' = 0$$

$$\Rightarrow x(yy'' + y'^2) - y'' + 3yy' = 0$$

$$\therefore k = 3$$

Hence, (b) is the correct answer.

- Ex 35.** If the function $y(x)$ represented by $x = \sin t$, $y = ae^{t\sqrt{2}} + be^{t\sqrt{2}}$, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ satisfies the equation $(1 + x^2)y'' - xy' = ky$, then k is equal to
 (a) 1 (b) -2
 (c) 2 (d) None of these

Sol. We have, $x = \sin t$ and $y = ae^{t\sqrt{2}} + be^{t\sqrt{2}}$

$$\Rightarrow \frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \sqrt{2}(a + b)e^{t\sqrt{2}}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sqrt{2}(a + b)e^{t\sqrt{2}}}{\cos t}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{2}y}{\sqrt{1 - x^2}}$$

$$\Rightarrow (1 - x^2)\left(\frac{dy}{dx}\right)^2 = 2y^2$$

On differentiating w.r.t. x , we get

$$(1 - x^2)2y' \cdot y'' - 2x(y')^2 = 4yy'$$

$$\Rightarrow (1 - x^2)y'' - xy' = 2y \quad [\text{dividing by } 2y']$$

$$\therefore k = 2$$

Hence, (c) is the correct answer.

- Ex 36.** If y and z are the functions of x and $y^2 + z^2 = \lambda^2$, then $y \cdot \frac{d}{dx}\left(\frac{y}{\lambda}\right) + \frac{d}{dx}\left(\frac{z^2}{\lambda}\right)$ is

equal to

(a) $\frac{z}{\lambda} \cdot \frac{dz}{dx}$

(b) $\frac{z}{\lambda} \cdot \frac{dz}{dx}$

(c) $\frac{\lambda}{z} \cdot \frac{dz}{dx}$

(d) None of the above

Sol. Given, $\lambda^2 = y^2 + z^2$

$$\Rightarrow \lambda = \sqrt{y^2 + z^2}$$

$$\text{Now, } y \frac{d}{dx}\left(\frac{y}{\lambda}\right) + \frac{d}{dx}\left(\frac{z^2}{\lambda}\right)$$

$$= y \frac{d}{dx}\left(\frac{y}{\sqrt{y^2 + z^2}}\right) + \frac{d}{dx}\left(\frac{z^2}{\sqrt{y^2 + z^2}}\right)$$

$$= y \left[\frac{\sqrt{(y^2 + z^2)} \frac{dy}{dx} - y \left(y \frac{dy}{dx} + z \frac{dz}{dx} \right)}{(y^2 + z^2)^{3/2}} \right]$$

$$+ \left[\frac{(\sqrt{y^2 + z^2}) 2z \frac{dz}{dx} - z^2 \left(y \frac{dy}{dx} + z \frac{dz}{dx} \right)}{(y^2 + z^2)^{3/2}} \right]$$

$$= \frac{1}{(y^2 + z^2)^{3/2}} \left[y \frac{dy}{dx} (y^2 + z^2) - y^2 \left(y \frac{dy}{dx} + z \frac{dz}{dx} \right) + 2z \frac{dz}{dx} (y^2 + z^2) - z^2 \left(y \frac{dy}{dx} + z \frac{dz}{dx} \right) \right]$$

$$= \frac{1}{(y^2 + z^2)^{3/2}} \left[z(y^2 + z^2) \frac{dz}{dx} \right]$$

$$= \frac{z}{\sqrt{y^2 + z^2}} \cdot \frac{dz}{dx} = \frac{z}{\lambda} \cdot \frac{dz}{dx}$$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

- Ex 37.** For the function $y = f(x) = (x^2 + bx + c)e^x$, which of the following holds?
 (a) If $f(x) > 0$ for all real $x \Rightarrow f'(x) > 0$
 (b) If $f(x) > 0$ for all real $x \Rightarrow f'(x) > 0$
 (c) If $f'(x) > 0$ for all real $x \Rightarrow f(x) > 0$
 (d) If $f'(x) > 0$ for all real $x \Rightarrow f(x) > 0$

Sol. We have, $f(x) = (x^2 + bx + c)e^x$

$$\Rightarrow f'(x) = \{x^2 + (b + 2)x + (b + c)\} e^x$$

$$f(x) > 0 \text{ iff } D = b^2 - 4c < 0$$

$$\text{Now, } f'(x) > 0 \text{ iff } D' = (b + 2)^2 - 4(b + c) = D + 4 < 0$$

Thus, for $f'(x) > 0$

$$D + 4 < 0 \text{ holds}$$

$$\Rightarrow D < 0 \Rightarrow f(x) > 0$$

Hence, (a) and (c) are the correct answers.

Ex 38. Let $f(x) = \frac{\sqrt{x-2}\sqrt{x-1}}{\sqrt{x-1}-1} \cdot x$, then

(a) $f'(10) = 1$ (b) $f'\left(\frac{3}{2}\right) = 1$

(c) domain of $f(x)$ is $x \geq 1$ (d) None

Sol. $f(x) = \frac{\sqrt{(\sqrt{x-1})^2 + 1 - 2\sqrt{x-1}}}{\sqrt{x-1}-1} \cdot x$
 $= \frac{|\sqrt{x-1}-1|}{\sqrt{x-1}-1} \cdot x = \begin{cases} -x, & \text{if } x \in [1, 2) \\ x, & \text{if } x \in (2, \infty) \end{cases}$

Hence, (a) and (b) are the correct answers.

Ex 39. If $y = x^{(\ln x)^{\ln(\ln x)}}$, then $\frac{dy}{dx}$ is equal to

(a) $\frac{y}{x} \{\ln x^{\ln x - 1} + 2 \ln x \ln(\ln x)\}$

(b) $\frac{y}{x} (\ln x)^{\ln(\ln x)} \{2 \ln(\ln x) + 1\}$

Type 3. Assertion and Reason

Directions (Ex. Nos. 40-42) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 40. Statement I For $x < 0$, $\frac{d}{dx} (\ln|x|) = \frac{1}{x}$

Statement II For $x < 0$, $|x| = -x$

$\Rightarrow \frac{d}{dx} |x| = -1$

Sol. $\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x)$
 $= \frac{1}{-x} (-1) = \frac{1}{x}$

Hence, (a) is the correct answer.

Ex 41. Statement I $\frac{d}{dx} \{\tan^{-1}(\sec x + \tan x)\}$

$= \frac{d}{dx} \{\cot^{-1}(\operatorname{cosec} x + \cot x)\}, x \in \left(0, \frac{\pi}{4}\right)$

(c) $\frac{y}{x \ln x} \{(\ln x)^2 + 2 \ln x(\ln x)\}$

(d) $\frac{y \ln y}{x \ln x} \{2 \ln(\ln x) + 1\}$

Sol. We have, $y = x^{(\ln x)^{\ln(\ln x)}}$

$\Rightarrow \ln y = (\ln x)^{\ln(\ln x)} \cdot \ln x \quad \dots(i)$

$\ln(\ln y) = \ln(\ln x) \cdot \ln(\ln x) + \ln(\ln x)$

$\Rightarrow \frac{1}{\ln y} \cdot \frac{1}{y} \cdot \frac{dy}{dx} = \frac{2 \ln(\ln x)}{\ln x} \cdot \frac{1}{x} + \frac{1}{x \ln x}$
 $= \frac{2 \ln(\ln x) + 1}{x \ln x}$

$\therefore \frac{dy}{dx} = \frac{y}{x} \cdot \frac{\ln y}{\ln x} \{2 \ln(\ln x) + 1\}$

On substituting the value of y from Eq. (i), we get

$\frac{dy}{dx} = \frac{y}{x} (\ln x)^{\ln(\ln x)} \{2 \ln(\ln x) + 1\}$

Hence, (b) and (d) are the correct answers.

Statement II $\sec^2 x - \tan^2 x$
 $= \operatorname{cosec}^2 x - \cot^2 x$

Sol. $\frac{d}{dx} \{\tan^{-1}(\sec x + \tan x)\} = \frac{d}{dx} \left\{ \tan^{-1} \left(\frac{1 + \sin x}{\cos x} \right) \right\}$
 $= \frac{d}{dx} \left\{ \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] \right\}$
 $= \frac{d}{dx} \left(\frac{\pi}{4} + \frac{x}{2} \right) = \frac{1}{2}$

and $\frac{d}{dx} \{\cot^{-1}(\operatorname{cosec} x + \cot x)\}$
 $= \frac{d}{dx} \left\{ \cot^{-1} \left(\frac{1 + \cos x}{\sin x} \right) \right\}$
 $= \frac{d}{dx} \left\{ \cot^{-1} \left(\cot \frac{x}{2} \right) \right\}$
 $= \frac{d}{dx} \left(\frac{x}{2} \right)$
 $= \frac{1}{2}$

Hence, (b) is the correct answer.

Ex 42. Statement I $f(x) = x + \sin x$ and $g(x) = 1 + \cos x$ are periodic functions.

Statement II Derivative of a differentiable of a periodic function is a periodic function.

Sol. Consider $f(x) = x + \sin x$, which is a periodic but $f'(x) = 1 + \cos x$ which is periodic with period 2π . Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage (Ex. Nos. 43-45) Let $f(x)$ be a real valued function not identically zero, such that

$$f(x + y^n) = f(x) + (f(y))^n, \forall x, y \in R$$

where, $n \in N(n \neq 1)$ and $f'(0) \geq 0$.

We may get an explicit form of the function $f(x)$.

Ex 43. The value of $f'(0)$ is

- (a) 1 (b) n
(c) $1 + n$ (d) 2

Ex 44. The value of $f(5)$ is

- (a) 2 (b) 3
(c) $5n$ (d) 5

Ex 45. $\int_0^1 f(x) dx$ is equal to

- (a) $\frac{1}{2n}$ (b) $2n$ (c) $\frac{1}{2}$ (d) 2

Sol. (Ex. Nos. 43-45)

Putting $x = y = 0$ in the given functional equation, we have

$$\begin{aligned} f(0) &= f(0 + 0) = f(0) + (f(0))^n \\ \Rightarrow f(0) &= 0 \\ \Rightarrow f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(h)}{h} = l \quad [\text{say}] \end{aligned}$$

$$\begin{aligned} \text{Again, } f'(0) &= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(0) + \{f(h^{1/n})\}^n - f(0)}{(h^{1/n})^n} \\ &= \lim_{h \rightarrow 0} \left(\frac{f(h^{1/n})}{h^{1/n}} \right)^n = l^n \end{aligned}$$

$$\text{So, } l^n = l$$

$$\Rightarrow l = 0, 1, -1$$

But $l = -1$ is ruled out as $f'(0) \geq 0$.

$$\begin{aligned} \text{If } l = 0, \text{ then } f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{f(x) + \{f(h^{1/n})\}^n - f(x)}{h} = l^n = 0 \end{aligned}$$

$\Rightarrow f'(x) = 0$, i.e. f is a constant function.

But $f(0) = 0$, so f is identically zero which is not the case.

Hence, $l = 1$

$$\Rightarrow f(x) = x + C \text{ but } f(0) = 0$$

$$\text{So, } f(x) = x$$

$$\text{Therefore, } f'(0) = 1, f(5) = 5$$

$$\begin{aligned} \text{and } \int_0^1 f(x) dx &= \int_0^1 x dx \\ &= \left[\frac{x^2}{2} \right]_0^1 = \frac{1}{2} \end{aligned}$$

43. (a) **44.** (d) **45.** (c)

Type 5. Match the Columns

Ex 46. Match the statements of Column I with values of Column II.

Column I	Column II
A. $\lim_{x \rightarrow 1/\sqrt{2}} \frac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)}$ is equal to	p. does not exist
B. If $f(x) = \log_{x^2}(\log x)$, then $f'\left(\frac{1}{2}\right)$ is equal to	q. 0
C. For the function $f(x) = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$, if $r. -\frac{1}{\sqrt{2}}$ $\frac{dy}{dx} = \sec x + p$, then p is equal to	
D. $\lim_{x \rightarrow 0} \frac{1}{x} \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$ is equal to	s. 4

$$\begin{aligned} \text{Sol. A. } \lim_{x \rightarrow 1/\sqrt{2}} \frac{x - \cos(\sin^{-1} x)}{1 - \tan(\sin^{-1} x)} \\ &= \lim_{x \rightarrow 1/\sqrt{2}} \frac{x - \sqrt{1-x^2}}{1 - \frac{x}{\sqrt{1-x^2}}} \\ &= \lim_{x \rightarrow 1/\sqrt{2}} \frac{-\sqrt{1-x^2}}{-\sqrt{1-x^2}} \\ &= -\frac{1}{\sqrt{2}} \end{aligned}$$

B. $f(x) = \ln_{x^2}(\log x)$ and $x = \frac{1}{2}$ is not in the domain.

$$\therefore f'\left(\frac{1}{2}\right) \text{ does not exist.}$$

$$\text{C. } y = f(x) = \ln \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{\sec^2\left(\frac{\pi}{4} + \frac{x}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)} \cdot \frac{1}{2} \\ &= \frac{1}{2} \cdot \frac{1}{\sin\left(\frac{\pi}{4} + \frac{x}{2}\right) \cos\left(\frac{\pi}{4} + \frac{x}{2}\right)} \\ &= \frac{1}{\sin\left(\frac{\pi}{2} + x\right)} = \sec x \end{aligned}$$

$$\therefore p = 0$$

$$\text{D. } \lim_{x \rightarrow 0} \frac{1}{x} \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}$$

$$= \lim_{x \rightarrow 0} \frac{|\tan x|}{x} \text{ does not exist.}$$

A $\rightarrow$ r; B $\rightarrow$ p; C $\rightarrow$ q; D $\rightarrow$ p

Type 6. Single Integer Answer Type Questions

Ex 47. If $f(x) = |x - 2|$ and $g(x) = f(f(x))$, then for $x > 20$, $g'(x)$ is equal to _____.

Sol. (1) Since, $f(x) = |x - 2| = x - 2$ for $x > 2$, we have
 $g(x) = f(f(x)) = f(x - 2) = x - 4$ for $x > 4$.
 Hence, $g'(x) = 1$ for $x > 20$.

Ex 48. If $y = \left(1 + \frac{1}{x}\right)^x$, then $\frac{2\sqrt{y_2(2) + \frac{1}{8}}}{\left(\log \frac{3}{2} - \frac{1}{3}\right)}$ is equal to

Sol. (3) We have, $y = \left(1 + \frac{1}{x}\right)^x$

On taking logarithm both sides, we get

$$\begin{aligned} \log y &= x \left[\log \left(1 + \frac{1}{x}\right) \right] \\ \Rightarrow \frac{1}{y} y_1(x) &= \frac{x^2}{x+1} \left(\frac{-1}{x^2} \right) + \log \left(1 + \frac{1}{x}\right) \\ &= -\frac{1}{x+1} + \log \left(1 + \frac{1}{x}\right) \quad \dots(i) \end{aligned}$$

Since, $y(2) = \left(1 + \frac{1}{2}\right)^2 = \frac{9}{4}$

So, $y_1(2) = \left(\frac{9}{4}\right) \left(-\frac{1}{3} + \log \frac{3}{2}\right)$

On multiplying Eq. (i) by y and then differentiating, we get

$$\begin{aligned} y_2(x) &= y_1(x) \left\{ -\frac{1}{x+1} + \log \left(1 + \frac{1}{x}\right) \right\} \\ &\quad + y(x) \left\{ \frac{1}{(x+1)^2} + \frac{x}{x+1} \left(-\frac{1}{x^2} \right) \right\} \end{aligned}$$

$$\begin{aligned} \text{So, } y_2(2) &= y_1(2) \left(-\frac{1}{3} + \log \frac{3}{2} \right) + y(2) \left(\frac{1}{9} - \frac{1}{6} \right) \\ &= \left(\frac{9}{4} \right) \left(-\frac{1}{3} + \log \frac{3}{2} \right)^2 - \frac{1}{8} \\ \therefore \frac{2\sqrt{y_2(2) + \frac{1}{8}}}{\left(\log \frac{3}{2} - \frac{1}{3}\right)} &= \frac{2 \cdot \frac{3}{2} \left(\log \frac{3}{2} - \frac{1}{3} \right)}{\left(\log \frac{3}{2} - \frac{1}{3}\right)} = 3 \end{aligned}$$

Ex 49. Let $F(x) = f(x) g(x) h(x)$ for all real x , where $f(x)$, $g(x)$ and $h(x)$ are differentiable functions. At some point x_0 ,

$$F'(x_0) = 21F(x_0), \quad f'(x_0) = 4f(x_0),$$

$$g'(x_0) = -7g(x_0) \quad \text{and} \quad h'(x_0) = kh(x_0).$$

Then, $\frac{k}{6}$ is equal to _____.

Sol. (4) $F'(x) = f'(x) g(x) h(x) + f(x) g'(x) h(x) + f(x) g(x) h'(x)$

$$\begin{aligned} \text{So, } 21F(x_0) &= F'(x_0) = f'(x_0) g(x_0) h(x_0) \\ &\quad + f(x_0) g'(x_0) h(x_0) + f(x_0) g(x_0) h'(x_0) \\ &= 4f(x_0) g(x_0) h(x_0) - 7f(x_0) g(x_0) h(x_0) \\ &\quad + kf(x_0) g(x_0) h(x_0) \\ &= (-3 + k)f(x_0) g(x_0) h(x_0) \\ &= (-3 + k)F(x_0) \end{aligned}$$

Hence, $21 = -3 + k$

$$\Rightarrow k = 24$$

$$\therefore \frac{k}{6} = 4$$

Target Exercises

Type 1. Only One Correct Option

- If $y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$, then $\frac{dy}{dx}$ is equal to
 (a) y (b) $y + \frac{x^n}{n!}$
 (c) $y - \frac{x^n}{n!}$ (d) $y - 1 - \frac{x^n}{n!}$
- If $y = \log_{\sin x}(\tan x)$, then $\frac{dy}{dx}$ at $x = \frac{1}{4}$ is equal to
 (a) $\frac{4}{\log 2}$ (b) $-4 \log 2$
 (c) $-\frac{4}{\log 2}$ (d) None of these
- If $y = \cos^{-1} \left(\frac{5 \cos x - 12 \sin x}{13} \right)$, where $x \in \left(0, \frac{\pi}{2} \right)$, then $\frac{dy}{dx}$ is equal to
 (a) 1 (b) -1
 (c) 0 (d) None of these
- If $\frac{d}{dx} \left(\frac{1 + x^4 + x^8}{1 + x^2 + x^4} \right) = ax^3 + bx$, then
 (a) $a = 4, b = 2$
 (b) $a = 4, b = -2$
 (c) $a = -2, b = 4$
 (d) None of the above
- If $y = \sum_{r=1}^x \tan^{-1} \left(\frac{1}{1+r+r^2} \right)$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{1}{1+x^2}$ (b) $\frac{1}{1+(1+x)^2}$
 (c) 0 (d) None of these
- If $y = \sin^{-1} \left(\frac{5x + 12\sqrt{1-x^2}}{13} \right)$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{1}{\sqrt{1-x^2}}$ (b) $-\frac{1}{\sqrt{1-x^2}}$
 (c) $\frac{3}{\sqrt{1-x^2}}$ (d) None of these
- If $y = \tan^{-1} \left(\frac{\sin x + \cos x}{\cos x - \sin x} \right)$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{1}{2}$ (b) 0
 (c) 1 (d) None of these
- $\frac{d}{dx} \left\{ \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right) \right\}$ is equal to
 (a) $\frac{3}{1+9x^2}, \forall |x| < \frac{1}{\sqrt{3}}$ (b) $\frac{1}{9+x^2}, \forall |x| < \frac{1}{\sqrt{3}}$
 (c) $\sec^2 3x, \forall |x| < \frac{1}{\sqrt{3}}$ (d) $\frac{3}{1+x^2}, \forall |x| < \frac{1}{\sqrt{3}}$
- If $y = \cos^{-1} \left(\frac{2 \cos x - 3 \sin x}{\sqrt{13}} \right)$, then $\frac{dy}{dx}$ is equal to
 (a) 1 (b) 0
 (c) constant ($\neq 1$) (d) None of these
- If $\sin^{-1} \left(\frac{x^2 - y^2}{x^2 + y^2} \right) = \log a$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{x^2 - y^2}{x^2 + y^2}$ (b) $\frac{y}{x}$
 (c) $\frac{y}{x^2}$ (d) $\frac{x}{y}$
- If $f(x) = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, then $(1-x^2) f'(x) - xf(x)$ is equal to
 (a) 1 (b) -1
 (c) 0 (d) None of these
- If $f(x) = \cos^{-1} \left[\frac{1 - (\log x)^2}{1 + (\log x)^2} \right]$, then $f'(e)$ is equal to
 (a) Does not exist (b) $\frac{2}{e}$
 (c) $\frac{1}{e}$ (d) 1
- If $y = \sin^{-1} [\sqrt{x-ax} - \sqrt{a-ax}]$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{1}{\sin \sqrt{a-ax}}$ (b) $\sin \sqrt{x} \sin \sqrt{a}$
 (c) $\frac{1}{2\sqrt{x}\sqrt{1-x}}$ (d) zero
- If $y = \sin^{-1} \left(\frac{\sin \alpha \sin x}{1 - \cos \alpha \sin x} \right)$, then $y'(0)$ is equal to
 (a) 1 (b) $2 \tan \alpha$ (c) $\left(\frac{1}{2} \right) \tan \alpha$ (d) $\sin \alpha$
- If $f(x) = \cot^{-1} \left(\frac{x^x - x^{-x}}{2} \right)$, then $f'(1)$ equals
 (a) -1 (b) 1 (c) $\log_e 2$ (d) $-\log_e 2$

- 16.** If $y = \sin^{-1}(\sin x)$, then $\frac{dy}{dx}$ at $x = \frac{\pi}{2}$ is equal to
 (a) 1 (b) -1
 (c) Does not exist (d) None of these
- 17.** If $y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \dots}}}}$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{y+x}{y^2-2x}$ (b) $\frac{y^3-x}{2y^2-2xy-1}$
 (c) $\frac{y^3+x}{2y^2-x}$ (d) None of these
- 18.** If $y = (1+x)(1+x^2)(1+x^4)\dots(1+x^{2^n})$, then $\frac{dy}{dx}$ at $x=0$ is equal to
 (a) -1 (b) 1
 (c) 0 (d) None of these
- 19.** If y is a function of x and $\log(x+y) - 2xy = 0$, then the value of $y'(0)$ is
 (a) 1 (b) -1
 (c) 2 (d) 0
- 20.** If $y = \log_{x^2+4}(7x^2 - 5x + 1)$, then $\frac{dy}{dx}$ is equal to
 (a) $\log_e(x^2+4) \left\{ \frac{14x-5}{7x^2-5x+1} - \frac{2xy}{x^2+4} \right\}$
 (b) $\frac{1}{\log_e(x^2+4)} \left\{ \frac{14x-5}{7x^2-5x+1} - \frac{2xy}{x^2+4} \right\}$
 (c) $\log_e(7x^2-5x+1) \left\{ \frac{2x}{x^2+4} - \frac{(14x-5)y}{7x^2-5x+1} \right\}$
 (d) $\frac{1}{\log_e(7x^2-6x+1)} \left\{ \frac{2x}{x^2+4} - \frac{(14x-5)y}{7x^2-5x+1} \right\}$
- 21.** If $f(x) = \log \left[\frac{m(x)}{n(x)} \right]$, $m(1) = n(1) = 1$ and $m'(1) = n'(1) = 2$, then $f'(1)$ is equal to
 (a) 0 (b) 1
 (c) -1 (d) None of these
- 22.** If $y = [(\tan x)^{\tan x}]^{\tan x}$, then $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$ is equal to
 (a) 1 (b) 2
 (c) 0 (d) None of the above
- 23.** If $f(x) = \log|x|$, $x \neq 0$, then $f'(x)$ equals
 (a) $\frac{1}{|x|}$ (b) $\frac{1}{x}$
 (c) $-\frac{1}{x}$ (d) None of these
- 24.** If $f(x) = |\log x|$, then for $x \neq 1$, $f'(x)$ equals
 (a) $\frac{1}{x}$ (b) $\frac{1}{|x|}$
 (c) $-\frac{1}{x}$ (d) None of these
- 25.** If $f(x) = |\log_e|x||$, then $f'(x)$ equals
 (a) $\frac{1}{|x|}$, $x \neq 0$
 (b) $\frac{1}{x}$, for $|x| > 1$ and $-\frac{1}{x}$, for $|x| < 1$
 (c) $-\frac{1}{x}$, for $|x| > 1$ and $\frac{1}{x}$, for $|x| < 1$
 (d) $\frac{1}{x}$, for $x > 0$ and $-\frac{1}{x}$, for $x < 0$
- 26.** If $f(x) = |x-2|$ and $g(x) = f\{f(x)\}$, then for $2 < x < 4$, $g'(x)$ equals
 (a) -1 (b) 1
 (c) 0 (d) None of these
- 27.** If $f(x) = |x|^{\sin x}$, then $f'\left(-\frac{\pi}{4}\right)$ equals
 (a) $\left(\frac{\pi}{4}\right)^{\frac{1}{\sqrt{2}}} \left(\frac{\sqrt{2}}{2} \log \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi} \right)$
 (b) $\left(\frac{\pi}{4}\right)^{\frac{1}{\sqrt{2}}} \left(\frac{\sqrt{2}}{2} \log \frac{4}{\pi} + \frac{2\sqrt{2}}{\pi} \right)$
 (c) $\left(\frac{\pi}{4}\right)^{\frac{1}{\sqrt{2}}} \left(\frac{\sqrt{2}}{2} \log \frac{\pi}{4} - \frac{2\sqrt{2}}{\pi} \right)$
 (d) $\left(\frac{\pi}{4}\right)^{\frac{1}{\sqrt{2}}} \left(\frac{\sqrt{2}}{2} \log \frac{\pi}{4} + \frac{2\sqrt{2}}{\pi} \right)$
- 28.** If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{4}\right)$ is equal to
 (a) $\sqrt{2}$
 (b) $-\sqrt{2}$
 (c) 0
 (d) None of the above
- 29.** If $y = |\cos x| + |\sin x|$, then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is equal to
 (a) $\frac{1-\sqrt{3}}{2}$ (b) 0
 (c) $\frac{\sqrt{3}-1}{2}$ (d) None of these
- 30.** If $f(x) = |\cos x|$, then $f'\left(\frac{3\pi}{4}\right)$ is equal to
 (a) $-\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{2}}$
 (c) 1 (d) None of these
- 31.** If $f(x) = |x|^{\tan x}$, then $f'\left(-\frac{\pi}{6}\right)$ is equal to
 (a) $\left(\frac{\pi}{6}\right)^{\frac{1}{\sqrt{3}}} \left(\frac{2\sqrt{3}}{\pi} - \frac{4}{3} \log \frac{6}{\pi} \right)$
 (b) $\left(\frac{\pi}{6}\right)^{\frac{1}{\sqrt{3}}} \left(\frac{-2\sqrt{3}}{\pi} + \frac{4}{3} \log \frac{6}{\pi} \right)$
 (c) $\left(\frac{\pi}{6}\right)^{\frac{1}{\sqrt{3}}} \left(\frac{2\sqrt{3}}{\pi} + \frac{4}{3} \log \frac{6}{\pi} \right)$
 (d) None of the above

32. If $f(x) = |x - 1|$ and $g(x) = f(f(f(x)))$, then $x > 2$, $g'(x)$ is equal to

- (a) 1, $\forall x > 2$
(b) 1, for $2 < x < 3$
(c) -1, for $2 < x < 3$
(d) None of the above

33. If $f(x) = 2 \tan^{-1} x + \cos^{-1} \left(\frac{1-x^2}{1+x^2} \right)$, then

- (a) $f'(-2) = \frac{4}{5}$ (b) $f'(-1) = -1$
(c) $f'(x) = 0, \forall x < 0$ (d) None of these

34. Let $f(x) = \sqrt{x-1} + \sqrt{x+24-10\sqrt{x-1}}$, $1 < x < 26$ be real valued function, then $f'(x)$, for $1 < x < 26$, is equal to

- (a) 0 (b) $\frac{1}{\sqrt{x-1}}$
(c) $2\sqrt{x-1}$ (d) None of these

35. If $f(x) = \frac{1}{1-x}$, then the derivative of the composite function $f(f(f(x)))$ is equal to

- (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

36. If $f(x) = |x - 1|$ and $g(x) = f(f(f(x)))$, then $g'(4)$ is equal to

- (a) -1 (b) -2
(c) 1 (d) None of these

37. If $u = f(x)^3$, $v = g(x^2)$, $f'(x) = \cos x$ and $g'(x) = \sin x$, then $\frac{du}{dv}$ is equal to

- (a) $\frac{3}{2} x \cos x^3 \operatorname{cosec} x^2$ (b) $\frac{2}{3} \sin x^3 \sec x^2$
(c) $\tan x$ (d) None of these

38. If $f(x) = e^{ax} + e^{bx}$, where $a \neq b$ and $f''(x) - 2f'(x) - 15f(x) = 0, \forall x$. Then, the product ab is

- (a) 25 (b) 9 (c) -15 (d) -9

39. The function $f(x) = e^x + x$, being differentiable and one-one, has a differentiable inverse $f^{-1}(x)$. The value of $\frac{d}{dx}(f^{-1})$ at the point $f(\log 2)$ is

- (a) $\frac{1}{\ln 2}$ (b) $\frac{1}{3}$
(c) $\frac{1}{4}$ (d) None of these

40. If $f''(x) = -f(x)$, $g(x) = f'(x)$,

$$F(x) = \left(f\left(\frac{x}{2}\right) \right)^2 + \left(g\left(\frac{x}{2}\right) \right)^2 \text{ and } F(5) = 5, \text{ then}$$

$F(10)$ is equal to

- (a) 5 (b) 10 (c) 0 (d) 15

41. Let $y = \ln(1 + \cos x)^2$, then the value of

$$\frac{d^2 y}{dx^2} + \frac{2}{e^{y/2}} \text{ equals}$$

- (a) 0 (b) $\frac{2}{1 + \cos x}$
(c) $\frac{4}{(1 + \cos x)}$ (d) $\frac{-4}{(1 + \cos x)^2}$

42. Let $u(x)$ and $v(x)$ be differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left[\frac{u(x)}{v(x)} \right]' = q$, then

$$\frac{p+q}{p-q} \text{ has the value equal to}$$

- (a) 1 (b) 0 (c) 7 (d) -7

43. If $x = \sec \theta - \cos \theta$ and $y = \sec^n \theta - \cos^n \theta$, then

$$(x^2 + 4) \left(\frac{dy}{dx} \right)^2 \text{ is equal to}$$

- (a) $n^2(y^2 - 4)$ (b) $n^2(4 - y^2)$
(c) $n^2(y^2 + 4)$ (d) None of these

44. If $x = f(t) \cos t - f'(t) \sin t$ and

$$y = f(t) \sin t + f'(t) \cos t, \text{ then } \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 \text{ is}$$

equal to

- (a) $f(t) - f''(t)$ (b) $[f(t) - f''(t)]^2$
(c) $[f(t) + f''(t)]^2$ (d) None of these

45. If $x = f(t)$ and $y = \phi(t)$, then $\frac{d^2 y}{dx^2}$ is equal to

- (a) $\frac{f_1 \phi_2 - \phi_1 f_2}{f_1^2}$ (b) $\frac{f_1 \phi_2 - \phi_1 f_2}{f_1^3}$
(c) $\frac{\phi_1 f_2 - f_1 \phi_2}{f_1^3}$ (d) None of these

46. If $P = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, then $P + \frac{d^2 P}{d\theta^2}$ is equal

to

- (a) $2a^2 + 2b^2 + 3P$ (b) $2a^2 + 2b^2 - 3P$
(c) $2a^2 - 2b^2 - 3P$ (d) None of these

47. If $y = \frac{ax+b}{x^2+c}$, where a, b, c are constants, then

$$(2xy' + y)y''' \text{ is equal to}$$

- (a) $3(xy'' + y')y''$
(b) $3(xy' + y'')y''$
(c) $3(xy'' + y')y'$
(d) None of the above

48. Let $f(x) = (x^3 + 2)^{30}$. If $f^n(x)$ is a polynomial of degree 20, where $f^n(x)$ denotes the n th derivative of $f(x)$ w.r.t. x , then the value of n is

- (a) 60 (b) 40
(c) 70 (d) None of these

49. Let $f(x) = 3x^2 + 4x g'(1) + g''(2)$ and $g(x) = 2x^2 + 3x f''(2) + f''(3)$, then
- $f'(1) = 22 + 12f''(2)$
 - $f''(2) = 44 + 12f'(1)$
 - $f''(3) + g''(2) = 2$
 - None of the above
50. If $y = x^{n-1} \log x$, then $x^2 y_2 + (3-2n)xy_1$ is equal to
- $-(n-1)^2 y$
 - $(n-1)^2 y$
 - $-n^2 y$
 - $n^2 y$
51. If $2x = y^{1/5} + y^{-1/5}$, then $(x^2 - 1)y_2 + xy_1 = ky$, where k is equal to
- 25
 - 25
 - 16
 - 16
52. If $f(x)$ is a polynomial of degree 3 such that $f(3) = 1$, $f'(3) = -1$, $f''(3) = 0$ and $f'''(3) = 12$. Then, the value of $f'(1)$ is
- 12
 - 23
 - 13
 - None of these
53. Let $f(x) = \sin x$, $g(x) = 2x$ and $h(x) = \cos x$. If $\phi(x) = g[(fh)x]$, then $\phi''\left(\frac{\pi}{4}\right)$ is equal to
- 4
 - 0
 - 4
 - None of these
54. If the parametric equation of a curve is given by $x = \cos \theta + \log \tan \frac{\theta}{2}$ and $y = \sin \theta$, then the points for which $\frac{d^2 y}{dx^2} = 0$, are given by
- $\theta = n\pi, n \in I$
 - $\theta = (2n+1)\frac{\pi}{2}, n \in I$
 - $\theta = (2n+1)\pi, n \in I$
 - None of these
55. If $x^2 + y^2 = a^2$, then a is equal to
- $\frac{\sqrt{(1+y'^2)^3}}{|y''|}$
 - $\frac{|y''|}{\sqrt{(1+y'^2)^3}}$
 - $\frac{\sqrt{(1+y'^2)^3}}{2|y''|}$
 - None of these
56. If $y = \tan^{-1} \left[\frac{\log(e/x^3)}{\log(ex^3)} \right] + \tan^{-1} \left[\frac{\log(e^4 x^3)}{\log(e/x^{12})} \right]$, then $\frac{d^2 y}{dx^2}$ is equal to
- 1
 - 0
 - 1
 - None of the above
57. Let $f(x)$ be a polynomial in x . Then, the second order derivative of $f(e^x)$ w.r.t. x , is
- $f''(e^x) \cdot e^x + f'(e^x)$
 - $f'''(e^x) \cdot e^{2x} + f''(e^x) \cdot e^{2x}$
 - $f''(e^x) \cdot e^{2x}$
 - $f'''(e^x) \cdot e^{2x} + f'(e^x) \cdot e^x$
58. Let $f(x) = \sin x$, $g(x) = x^2$ and $h(x) = \log_e x$. If $F(x) = (hogof) x$, then $F''(x)$ is equal to
- $2 \operatorname{cosec}^3 x$
 - $2 \cot x^2 - 4x^2 \operatorname{cosec}^2 x^2$
 - $2 \cot x^2$
 - $-2 \operatorname{cosec}^2 x$
59. The functions $u = e^x \sin x$, $v = e^x \cos x$ satisfy the equation
- $v \frac{du}{dx} - u \frac{dv}{dx} = u^2 + v^2$
 - $\frac{d^2 u}{dx^2} = 2v$
 - $\frac{d^2 v}{dx^2} = -2u$
 - All of these
60. The derivative of $\operatorname{cosec}^{-1} \left(\frac{1}{2x^2 - 1} \right)$ w.r.t. $\sqrt{1-x^2}$ at $x = \frac{1}{2}$ is equal to
- 4
 - 4
 - 1
 - None of the above
61. The derivative of $\sin^{-1} (3x - 4x^3)$ w.r.t. $\sin^{-1} x$ is equal to
- 3, for $|x| < 1$
 - 3, for $|x| < \frac{1}{2}$ and -3, for $\frac{1}{2} < |x| < 1$
 - 3, for $|x| < 1$
 - 3, for $|x| \leq \frac{1}{2}$ and 3, for $\frac{1}{2} < |x| < 1$
62. If a curve is given by $x = a \cos t + \frac{b}{2} \cos 2t$ and $y = a \sin t + \frac{b}{2} \sin 2t$, then the points for which $\frac{d^2 y}{dx^2} = 0$, are given by
- $\sin t = \frac{2a^2 + b^2}{3ab}$
 - $\cos t = -\left(\frac{a^2 + 2b^2}{3ab} \right)$
 - $\tan t = \frac{a}{b}$
 - None of these
63. Let f be a differentiable function satisfying $f(x) + f(y) + f(z) + f(x)f(y)f(z) = 14, \forall x, y, z \in R$. Then,
- $f'(x) < 0, \forall x \in R$
 - $f'(x) = 0, \forall x \in R$
 - $f'(x) < 0, \forall x \in R$
 - None of the above
64. Let f be a real function such that $f(x-y)$, $f(x)f(y)$ and $f(x+y)$ are in AP for all $x, y \in R$. If $f(0) \neq 0$, then
- $f(1) + f(-1) = 0$
 - $f(2) + f(-2) = 0$
 - $f'(3) + f'(-3) = 0$
 - $f'(2) = f'(-2)$

65. If $f(x)$ and $g(x)$ are two differentiable functions on $[0, 2]$ such that $f''(x) - g''(x) = 0$, $f'(1) = 2g'(1) = 4$ and $f(2) = 3g(2) = 9$, then $f(x) - g(x)$ at $x = \frac{3}{2}$ is equal to
- (a) 0
(b) 2
(c) 10
(d) 5
66. Let $h(x) = g'(x)$, where $g(x)$ is the inverse function of $f(x)$ and $f(x)$ is invertible and

- $f'(x) = \frac{f^2(x) - 1}{f(x) + (f^2(x) - 1)e^{af(x)} \sin(2f(x) + 3)}$, then $h'(x)$ is equal to
- (a) $-\frac{1}{(x+1)^2} - \frac{2x}{(x^2-1)^2} + 2e^{ax} \cos(2x+3) + ae^{ax} \sin(2x+3)$
(b) $\frac{1}{(x+1)^2} + \frac{x}{(x^2-1)^3} + e^{ax} \cos(2x-3) + ae^{ax} \sin(2x-3)$
(c) $\frac{2}{(2x+1)^2} + \frac{x}{(x^2-1)} + e^{ax} \cos(2x+3) - ae^{ax} \sin(2x-3)$
(d) None of the above

Type 2. More than One Correct Option

67. Which of the following functions are not derivable at $x = 0$?

- (a) $f(x) = \sin^{-1}(2x\sqrt{1-x^2})$
(b) $g(x) = \sin^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$
(c) $h(x) = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right)$
(d) $k(x) = \sin^{-1}(\cos x)$

68. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$, is equal to

- (a) $\frac{1}{\sqrt{2}}$
(b) $\sin^{-1}\left(\sin \frac{1}{\sqrt{2}}\right)$
(c) 1
(d) None of these

69. If $y = 10^{10x}$, then $\frac{1}{y} \frac{dy}{dx} = 10^x \cdot \lambda$, where λ is equal to

- (a) $\ln 10$
(b) $(\ln 10)^2$
(c) $e^{\ln(\ln 10)^2}$
(d) $(\log_{10} e)^2$

70. Let $\sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$, then $\frac{dy}{dx}$ is equal to

- (a) $\frac{1}{2y-1}$
(b) $\frac{x}{x+2y}$
(c) $\frac{1}{\sqrt{1-4x}}$
(d) $\frac{y}{2x+y}$

71. If $f(x) = \begin{vmatrix} x^n & \sin x & -\cos x \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$, then the value of $\frac{d^n}{dx^n}(f(x))$ at $x = 0$ for $n = 2m + 1$ is

- (a) -1
(b) 0
(c) a^6
(d) independent of a

72. Let $f(x) = \frac{\sqrt{x-2}\sqrt{x-1}}{\sqrt{x-1}-1}$, then

- (a) $f'(10) = 1$
(b) $f'\left(\frac{3}{2}\right) = -1$
(c) domain of $f(x)$ is $x \geq 1$
(d) range of $f(x)$ is $(-2, -1] \cup (2, \infty)$

73. If $f_n(x) = e^{f_{n-1}(x)}$, $\forall n \in \mathbb{N}$ and $f_0(x) = (x)$, then $\frac{d}{dx}\{f_n(x)\}$ is equal to

- (a) $f_n(x) \frac{d}{dx}\{f_{n-1}(x)\}$
(b) $f_n(x) f_{n-1}(x)$
(c) $f_n(x) f_{n-1}(x) \dots f_2(x) \cdot f_1(x)$
(d) None of the above

74. If $x^3 - 2x^2y^2 + 5x + y - 5 = 0$ and $y(1) = 1$, then

- (a) $y'(1) = 4/3$
(b) $y''(1) = -4/3$
(c) $y''(1) = -8 \frac{22}{27}$
(d) $y'(1) = \frac{2}{3}$

Type 3. Assertion and Reason

Directions (Q. Nos. 75-76) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation of Statement I
(b) Statement I is true, Statement II is true; Statement II is not the correct explanation of Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

75. **Statement I** Let $f(x) = x[x]$, where $[]$ denotes the greatest integral function. When x is not an integral, then rule for $f'(x)$ is given by $[x]$.

Statement II $f'(x)$ does not exist for any $x \in \text{integer}$.

76. **Statement I** Let f be a differentiable function satisfying $f(x+y) = f(x) + f(y) + 2xy - 1$, $\forall x, y \in \mathbb{R}$ and $f'(0) = \sin \phi$. Then, $f(x) > 0, \forall x$.

Statement II f in Statement I is of the form $x^2 + x \sin \phi + 1$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 77-79)

Let $f(x) = x^2 + xg'(1) + g''(2)$
and $g(x) = f(1)x^2 + xf'(x) + f''(x)$

77. The value of $f(3)$ is

- (a) 1 (b) 0 (c) -1 (d) -2

78. The value of $g(0)$ is

- (a) 0
(b) -3
(c) 2
(d) None of the above

79. The domain of the function $\sqrt{\frac{f(x)}{g(x)}}$ is

- (a) $(-\infty, 1] \cup (2, 3]$ (b) $(-2, 0] \cup (1, \infty)$
(c) $(-\infty, 0] \cup \left(\frac{2}{3}, 3\right]$ (d) None of these

Passage II (Q. Nos. 80-82) $f(x)$ is a polynomial function $f: R \rightarrow R$ such that $f(2x) = f'(x)f''(x)$.

80. The value of $f(3)$ is

- (a) 4
(b) 12
(c) 15
(d) None of the above

81. $f(x)$ is

- (a) one-one and onto
(b) one-one and into
(c) many-one and onto
(d) many-one and into

82. Equation $f(x) = x$ has

- (a) three real and positive roots
(b) three real and negative roots
(c) one real root
(d) three real roots such that sum of roots is zero

Passage III (Q. Nos. 83-85) Let $g(x+y) = g(x) + g(y) + 3xy(x+y)$, $\forall x, y \in R$ and $g'(0) = -4$.

83. Number of real roots of the equation $g(x) = 0$ is

- (a) 2 (b) 0 (c) 1 (d) 3

84. For which of the following values of x , $\sqrt{g(x)}$ is not defined?

- (a) $[-2, 0]$ (b) $[2, \infty)$
(c) $[-1, 1]$ (d) None of these

85. The value of $g'(1)$ is

- (a) 0 (b) 1
(c) -1 (d) None of these

Type 5. Match the Columns

86. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $f'(x) = \sqrt{3x^2 + 6}$ and $y = f(x^3)$, then at $x = 1$, $\frac{dy}{dx}$ is equal to	p. -2
B. If f is a difference function such that $f(xy) = f(x) + f(y)$; $x, y \in R$, then $f(e) + f(1/e)$ is equal to	q. -1
C. Let h be a twice function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = (f(x))^2 + (g(x))^2$ and $h(5) = 9$, then $h(10)$ is equal to	r. 0
D. If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, $\frac{\pi}{2} < x < \pi$, then $\frac{dy}{dx}$ is equal to	s. 9

87. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $y = \cos^{-1}(\cos x)$, then y' at $x = 5$ is equal to	p. -1
B. For the function $f(x) = \ln \tan x $, $f'\left(-\frac{\pi}{4}\right)$ is equal to	q. 0

Column I	Column II
C. The derivative of $\tan^{-1}\left(\frac{1+x}{1-x}\right)$ at $x = -1$ is	r. $\frac{1}{2}$
D. The derivative of $\frac{\ln x }{x}$ at $x = -1$ is	s. 1

88. Match the statements of Column I with values of Column II.

Column I	Column II
A. Let $y = f(x)$ be given by $x = t^5 - 5t^3 - 20t + 7$ and $y = 4t^3 - 3t^2 - 18t + 3$, then $-5\frac{dy}{dx}$ at $t = 1$ is equal to	p. 0
B. $P(x)$ be a polynomial of degree 4, with $P(2) = -1$, $P'(2) = 0$, $P''(2) = 2$, $P'''(2) = -12$ and $P^{iv}(2) = 24$, then $P''(3)$ equals	q. -2
C. If $y = \frac{1}{x}$, then $\frac{\sqrt{1+y^4}}{\sqrt{1+x^4}}$ is equal to	r. 2
D. If $f\left(\frac{2x+3y}{5}\right) = \frac{2f(x)+3f(y)}{5}$, $f'(0) = p$ and $f(0) = q$, then $f''(0)$ is equal to	s. -1

Type 6. Single Integer Answer Type Questions

89. If $f(x) = \sqrt{x+2} + \sqrt{2x-4}$, then the value of $10 f'(102^+)$ is _____.
90. Let $h(x)$ be differentiable for all x and $f(x) = (kx + e^x)h(x)$, where k is some constant. If $h(0) = 5$, $h'(0) = -2$ and $f'(0) = 18$, then the value of k is _____.
91. If the function $f(x) = x^3 + e^{x/2}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is _____.
92. If $2x = y^{1/5} + y^{-1/5}$, then $(x^2 - 1) \frac{d^2 y}{dx^2} + x \frac{dy}{dx} = ky$, where $\frac{k}{5}$ is equal to _____.
93. If $y = \frac{a + bx^{3/2}}{x^{5/4}}$ and $y' = 0$ at $x = 5$, then the value of $\frac{a^2}{b^2}$ is _____.
94. Let $y = \frac{2^{\log_2 1/4 x} - 3^{\log_{27} (x^2 + 1)^3} - 2x}{7^{4 \log_{49} x} - x - 1}$ and $\frac{dy}{dx} = ax + b$, then the value of $a + b$ is _____.
95. If graph of $y = f(x)$ is symmetrical about the point $(5, 0)$ and $f'(7) = 3$, then the value of $f'(3)$ is _____.
96. A function is represented parametrically by the equations $x = \frac{(1+t)}{t^3}$ and $y = \frac{3}{2t^2} + \frac{2}{t}$, then the value of $\left| \frac{dy}{dx} - x \left(\frac{dy}{dx} \right)^3 \right|$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Let $f(x) = \sin \pi x$, $x > 0$. Then, for all natural numbers n , $f'(x)$ vanishes at [2013]
 (a) a unique point in the interval $\left(n, n + \frac{1}{2}\right)$
 (b) a unique point in the interval $\left(n + \frac{1}{2}, n + 1\right)$
 (c) a unique point in the interval $(n, n + 1)$
 (d) two points in the interval $(n, n + 1)$
2. If $f(\theta) = \sin \left[\tan^{-1} \left(\frac{\sin \theta}{\sqrt{\cos 2\theta}} \right) \right]$, where $-\frac{\pi}{4} < \theta < \frac{\pi}{4}$, then the value of $\frac{d}{d(\tan \theta)} [f(\theta)]$ is [2012]
 (a) 1 (b) 0
 (c) 2 (d) None of these

JEE Main/AIEEE

5. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is equal to [2014]
 (a) $1+x^5$ (b) $5x^4$ (c) $\frac{1}{1+\{g(x)\}^5}$ (d) $1+\{g(x)\}^5$
6. If $f(x) = \frac{x^2 - x}{x^2 + 2x}$, where $x \neq 0, -2$, then $\frac{d}{dx} [f^{-1}(x)]$ (wherever it is defined) is equal to [2013]
 (a) $\frac{-1}{(1-x)^2}$ (b) $\frac{3}{(1-x)^2}$ (c) $\frac{1}{(1-x)^2}$ (d) $\frac{-3}{(1-x)^2}$

3. Let $p(x)$ be a polynomial of degree 4 having extremum at $x = 1, 2$ and $\lim_{x \rightarrow 0} \left[1 + \frac{p(x)}{x^2} \right] = 2$. Then, the value of $p(2)$ is [2010]
4. Let f be a real valued function defined on the interval $(0, \infty)$ by $f(x) = \ln x + \int_0^x \sqrt{1 + \sin t} dt$. Then, which of the following statement(s) is/are true? [2010]
 (a) $f''(x)$ exists for all $x \in (0, \infty)$
 (b) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$ but not differentiable on $(0, \infty)$
 (c) There exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$, $\forall x \in (\alpha, \infty)$
 (d) There exists $\beta > 0$ such that $|f(x)| + |f'(x)| \leq \beta$, $\forall x \in (0, \infty)$

7. For $a > 0$, $t \in \left(0, \frac{\pi}{2}\right)$, let $x = \sqrt{a^{\sin^{-1} t}}$ and $y = \sqrt{a^{\cos^{-1} t}}$. Then, $1 + \left(\frac{dy}{dx}\right)^2$ equals [2013]
 (a) $\frac{x^2}{y^2}$ (b) $\frac{y^2}{x^2}$ (c) $\frac{x^2 + y^2}{y^2}$ (d) $\frac{x^2 + y^2}{x^2}$
8. If $f(x) = \sin(\sin x)$ and $f'''(x) + \tan x f'(x) + g(x) = 0$, then $g(x)$ is equal to [2013]
 (a) $\cos^2 x \cos(\sin x)$ (b) $\sin^2 x \cos(\cos x)$
 (c) $\sin^2 x \sin(\cos x)$ (d) $\cos^2 x \sin(\sin x)$

9. If $y = \sec(\tan^{-1} x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to [2013]

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt{2}$

10. $\frac{d^2x}{dy^2}$ is equal to [2011]

- (a) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$
(c) $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2y}{dx^2}\right)^{-1}$

11. If y is an implicit function of x defined by $x^{2x} - 2x^x \cot y - 1 = 0$, then $y'(1)$ is equal to [2009]

- (a) -1 (b) 1
(c) $\log 2$ (d) $-\log 2$

12. Let $f(x) = x|x|$ and $g(x) = \sin x$.

Statement I gof is differentiable at $x = 0$ and its derivative is continuous at that point.

Statement II gof is twice differentiable at $x = 0$.

[2009]

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

13. If $x^m y^n = (x + y)^{m+n}$, then $\frac{dy}{dx}$ is equal to [2006]

- (a) $\frac{x+y}{xy}$ (b) xy (c) $\frac{x}{y}$ (d) $\frac{y}{x}$

14. If $f(x) = x^n$, then the value of

$$f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \dots + \frac{(-1)^n f^n(1)}{n!}$$

is

[2003]

- (a) 2^n (b) 2^{n-1}
(c) 0 (d) 1

15. Let $f(x)$ be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in AP, then $f'(a), f'(b)$ and $f'(c)$ are in [2003]

- (a) AP
(b) GP
(c) HP
(d) AGP

16. If $y = (x + \sqrt{1+x^2})^n$, then $(1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx}$ is

equal to

[2002]

- (a) n^2y (b) $-n^2y$
(c) $-y$ (d) $2x^2y$

17. If $\sin y = x \sin(a+y)$, then $\frac{dy}{dx}$ is equal to [2002]

- (a) $\frac{\sin a}{\sin^2(a+y)}$ (b) $\frac{\sin^2(a+y)}{\sin a}$
(c) $\sin a \sin^2(a+y)$ (d) $\frac{\sin^2(a-y)}{\sin a}$

18. If $x^y = e^{x-y}$, then $\frac{dy}{dx}$ is equal to [2002]

- (a) $\frac{1+x}{1+\log x}$ (b) $\frac{1-\log x}{1+\log x}$
(c) not defined (d) $\frac{\log x}{(1+\log x)^2}$

Other Engineering Entrances

19. If $x \sin(a+y) + \sin a \cos(a+y) = 0$, then $\frac{dy}{dx}$ is

equal to

[Manipal 2014]

- (a) $\frac{\sin^2(a+y)}{\sin a}$
(b) $\frac{\cos^2(a+y)}{\cos a}$
(c) $\frac{\sin^2(a+y)}{\cos a}$
(d) $\frac{\cos^2(a+y)}{\sin a}$

20. If $y = |\sin x|^{|x|}$, then the value of $\frac{dy}{dx}$ at $x = \frac{-\pi}{6}$ is

[Manipal 2014]

- (a) $\frac{2^{-\pi/6}}{6}$ [6 $\log 2 - \sqrt{3}\pi$] (b) $\frac{2^{\pi/6}}{6}$ [6 $\log 2 + \sqrt{3}\pi$]
(c) $\frac{2^{-\pi/6}}{6}$ [6 $\log 2 + \sqrt{3}\pi$] (d) None of these

21. If $y = (1+x)(1+x^2)(1+x^4)$, then $\frac{dy}{dx}$ at $x = 1$ is

[Karnataka CET 2014]

- (a) 20 (b) 28
(c) 1 (d) 0

22. If the function $f(x)$ defined by $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$, then $f'(0)$ is equal to

[Karnataka CET 2014]

- (a) $100f'(0)$
(b) 100
(c) 1
(d) -1

23. If $y = f(x^2 + 2)$ and $f'(3) = 5$, then $\frac{dy}{dx}$ at $x = 1$ is

[Kerala CEE 2014]

- (a) 5 (b) 25 (c) 15
(d) 20 (e) 10

24. If $f(x) = x^2 + bx + 7$ and $f'(5) = 2f'\left(\frac{7}{2}\right)$, then the value of b is [Kerala CEE 2014]
(a) 4 (b) 3 (c) -4
(d) -3 (e) 2

25. If $f(x) = \frac{x}{1+x}$ and $g(x) = f[f(x)]$, then $g'(x)$ is equal to [EAMCET 2014]
(a) $\frac{1}{(2x+3)^2}$ (b) $\frac{1}{(x+1)^2}$ (c) $\frac{1}{x^2}$ (d) $\frac{1}{(2x+1)^2}$

26. Find the derivative of $\sqrt{2}x + 2\sqrt{x} - \frac{1}{\sqrt{x}}$. [J&K CET 2014]
(a) $\sqrt{2} + \frac{1}{\sqrt{x}}\left(1 - \frac{1}{2x}\right)$ (b) $\sqrt{2} - \frac{1}{\sqrt{x}}\left(1 + \frac{1}{2x}\right)$
(c) $\sqrt{2} - \frac{1}{\sqrt{x}}\left(1 - \frac{1}{2x}\right)$ (d) $\sqrt{2} + \frac{1}{\sqrt{x}}\left(1 + \frac{1}{2x}\right)$

27. If $y = (\tan^{-1} x)^2$, then $(x^2 + 1)^2 y_2 + 2x(x^2 + 1)y_1$ is equal to [Karnataka CET 2014]
(a) 4 (b) 0 (c) 2 (d) 1

28. If $x = \sin t$ and $y = \tan t$, then $\frac{dy}{dx}$ is equal to [Kerala CEE 2014]
(a) $\cos^3 t$ (b) $\frac{1}{\cos^3 t}$ (c) $\frac{1}{\cos^2 t}$
(d) $\sin^2 t$ (e) $\frac{1}{\sin^2 t}$

29. If $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$, then $1 + \left(\frac{dy}{dx}\right)^2$ is equal to [Kerala CEE 2014]
(a) $\tan \theta$ (b) $\tan^2 \theta$ (c) 1
(d) $\sec^2 \theta$ (e) $\sec \theta$

30. If $y = \sin^{-1}(2x\sqrt{1-x^2})$, $-\frac{1}{\sqrt{2}} \leq x \leq \frac{1}{\sqrt{2}}$, then $\frac{dy}{dx}$ is equal to [Kerala CEE 2014]
(a) $\frac{x}{\sqrt{1-x^2}}$ (b) $\frac{1}{\sqrt{1-x^2}}$ (c) $\frac{2}{\sqrt{1-x^2}}$
(d) $\frac{2x}{\sqrt{1-x^2}}$ (e) $\frac{-2x}{\sqrt{1-x^2}}$

31. Suppose that $f(x)$ is a differentiable function such that $f'(x)$ is continuous, $f'(0) = 1$ and $f''(0)$ does not exist. If $g(x) = x f'(x)$, then [WB JEE 2014]
(a) $g'(0)$ does not exist (b) $g'(0) = 0$
(c) $g'(0) = 1$ (d) $g'(0) = 2$

32. If $y = \tan^{-1}\left(\frac{\sqrt{1+a^2x^2}-1}{ax}\right)$, then $(1+a^2x^2)y'' + 2a^2xy'$ is equal to [EAMCET 2014]
(a) $-2a^2$ (b) a^2
(c) $2a^2$ (d) 0

33. For $y = \frac{x}{x^2-1}$, $\frac{d^n y}{dx^n}$ is equal to [GGSIPU 2014]

(a) $\frac{n!}{2} \left[\frac{1}{(x+1)^n} + \frac{1}{(x-1)^n} \right]$
(b) $\frac{(-1)^n n!}{2} \left[\frac{1}{(x+1)^n} - \frac{1}{(x-1)^n} \right]$
(c) $\frac{n!}{2} \left[\frac{1}{(x+1)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right]$
(d) $\frac{(-1)^n n!}{2} \left[\frac{1}{(x+1)^{n+1}} + \frac{1}{(x-1)^{n+1}} \right]$

34. If $\sin(x+y) = \log(x+y)$, then $\frac{dy}{dx}$ is equal to [BITSAT 2013]
(a) 2 (b) -2 (c) 1 (d) -1

35. If $y = \tan^{-1}\left(\frac{a \cos x - b \sin x}{b \cos x + a \sin x}\right)$, then $\frac{dy}{dx}$ is equal to [Kerala CEE 2013]
(a) 2 (b) -1 (c) $\frac{a}{b}$
(d) 0 (e) None of these

36. If $x = e^{y+e^{y+e^{y+\dots}}}$, then $\frac{dy}{dx}$ is equal to UP SEE 2013]
(a) $\frac{1}{x}$ (b) $\frac{1-x}{x}$
(c) $\frac{x}{1+x}$ (d) None of these

37. The derivative of $y = (1-x)(2-x)\dots(n-x)$ at $x = 1$ is equal to [OJEE 2013]
(a) 0
(b) $(-1)(n-1)!$
(c) $n! - 1$
(d) $(-1)^{n-1}(n-1)!$

38. If $\Delta_1 = \begin{vmatrix} x & a & b \\ b & x & a \\ a & b & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$ are the given determinants, then [BITSAT 2012]
(a) $\Delta_1 = 3(\Delta_2)^2$ (b) $\left(\frac{d}{dx}\right)(\Delta_1) = 3\Delta_2$
(c) $\left(\frac{d}{dx}\right)(\Delta_1) = 3(\Delta_2)^2$ (d) $\Delta_1 = 3(\Delta_2)^{3/2}$

39. If f is twice differentiable function satisfying $f(1) = 1$, $f(2) = 4$ and $f(3) = 9$, then [Manipal 2012]
(a) $f''(x) = 2, \forall x \in \mathbb{R}$
(b) $f'(x) = 5 = f''(x)$, for some $x \in (1, 3)$
(c) there exists atleast one $x \in (1, 3)$ such that $f''(x) = 2$
(d) None of the above

40. If $y = \cos(3\cos^{-1} x)$, then $\frac{d^3 y}{dx^3}$ is equal to [MP PET 2012]
(a) 24 (b) 27
(c) $3 - 12x^2$ (d) -24

41. If $x = \sin^{-1}(3t - 4t^3)$ and $y = \cos^{-1}(\sqrt{1-t^2})$, then $\frac{dy}{dx}$ is equal to [Kerala CEE 2012]
 (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{3}$
 (d) $\frac{2}{5}$ (e) $\frac{1}{5}$
42. If $f(x) = \tan^{-1} x$, then $f'(x) + f''(x) = 0$, when x is equal to [WB JEE 2012]
 (a) 0 (b) 1 (c) i (d) $-i$
43. If $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right)$, then $x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y}$ is equal to [OJEE 2012]
 (a) $-\sin 2u$ (b) $\sin 2u$
 (c) $\cos 2u$ (d) $-\cos 2u$
44. If x is measured in degrees, then $\frac{d}{dx}(\cos x)$ is equal to [BITSAT 2012]
 (a) $-\sin x$ (b) $-\frac{180}{\pi} \sin x$
 (c) $-\frac{\pi}{180} \sin x$ (d) $\sin x$
45. $\frac{d}{dx} \left\{ \tan^{-1} \sqrt{\frac{1 + \sin \frac{x}{2}}{1 - \sin \frac{x}{2}}} \right\}$ is equal to [MP PET 2012]
 (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $-\frac{1}{4}$
46. If $y = e^{\log_e[1+x+x^2+\dots]}$, then $\frac{dy}{dx}$ is equal to [Karnataka CET 2012]
 (a) $\frac{1}{(1+x)^2}$ (b) $\frac{1}{(1-x)^2}$ (c) $\frac{-1}{(1+x)^2}$ (d) $\frac{-1}{(1-x)^2}$
47. If $-\frac{\pi}{2} < x < \frac{3\pi}{2}$, then the value of $\frac{d}{dx} \left\{ \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) \right\}$ is equal to [WB JEE 2012]
 (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
 (c) 1 (d) $\frac{\sin x}{(1 + \sin x)^2}$
48. If $y = \left(\frac{3^x - 1}{3^x + 1} \right) \sin x + \log_e(1+x)$, $x > -1$, then at $x = 0$, $\frac{dy}{dx}$ is equal to [WB JEE 2012]
 (a) 1 (b) 0 (c) -1 (d) -2
49. If $f(x) = \log_x \{ \ln(x) \}$, then $f'(e)$ is equal to [AMU 2012]
 (a) $\frac{1}{e}$ (b) e
 (c) $-e$ (d) e^2
50. If $e^x + e^y = e^{x+y}$, then $\frac{dy}{dx}$ at $(2, 2)$ is [OJEE 2012]
 (a) 1 (b) -1
 (c) 2 (d) None of these
51. If $f(x) = \log_e(\log_e x)$, then $f'(e)$ is equal to [MP PET 2011]
 (a) e^{-1} (b) e
 (c) 1 (d) 0
52. If $F(x) = \int_x^3 \log t \, dt$; $x > 0$, then $F'(x)$ is equal to [MP PET 2011]
 (a) $(9x^2 - 4x) \log x$ (b) $(4x - 9x^2) \log x$
 (c) $(9x^2 + 4x) \log x$ (d) $(3x^2 - 2x) \log x$
53. If $f(x) = \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x}$, then $f'\left(\frac{\pi}{4}\right)$ is equal to [Karnataka CET 2011]
 (a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$
 (c) 0 (d) $-\sqrt{3}$
54. If $a < 0$ and $\left| \frac{x}{a} \right| < 1$, then $\frac{d}{dx} \left[\sin^{-1} \frac{x}{a} \right]$ is equal to [Guj CET 2011]
 (a) $\frac{1}{\sqrt{a^2 - x^2}}$ (b) $\frac{1}{\sqrt{x^2 - a^2}}$ (c) $\frac{-1}{\sqrt{a^2 - x^2}}$ (d) $\frac{-1}{\sqrt{x^2 - a^2}}$
55. If $\pi < x < 2\pi$, then $\frac{d}{dx} \left[\tan^{-1} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{1/2} \right]$ is [Guj CET 2011]
 (a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$
56. If $y = \sin x \cdot \sin 2x \cdot \sin 3x \dots \sin nx$, then y' is [J&K CET 2011]
 (a) $\sum_{k=1}^n k \tan kx$ (b) $y \cdot \sum_{k=1}^n k \cot kx$
 (c) $y \cdot \sum_{k=1}^n k \tan kx$ (d) $\sum_{k=1}^n \cot kx$
57. If $x \neq 0$ and $y = \log_e |2x|$, then $\frac{dy}{dx}$ is equal to [J&K CET 2011]
 (a) $\frac{1}{x}$ (b) $-\frac{1}{x}$
 (c) $\pm \frac{1}{2x}$ (d) None of these
58. Let $g(x) = [f\{2f(x) + 2\}]^2$. If $f: (-1, 1) \rightarrow R$ is a differentiable function with $f(0) = -1$ and $f'(0) = 1$, then $g'(0)$ is equal to [J&K CET 2010]
 (a) 4 (b) -4
 (c) 0 (d) -2
59. If $y = \sin^{-1} \sqrt{1-x}$, then $\frac{dy}{dx}$ is equal to [BITSAT 2010]
 (a) $\frac{1}{\sqrt{1-x}}$ (b) $\frac{-1}{2\sqrt{1-x}}$ (c) $\frac{1}{\sqrt{x}}$ (d) $\frac{-1}{2\sqrt{x}\sqrt{1-x}}$

60. If $y = \tan^{-1} x + \cot^{-1} x + \sec^{-1} x + \operatorname{cosec}^{-1} x$, then $\frac{dy}{dx}$ is equal to [Kerala CEE 2010]
- (a) $\frac{x^2 - 1}{x^2 + 1}$ (b) π (c) 0
(d) 1 (e) $\frac{1}{x\sqrt{x^2 - 1}}$

61. If $f(x) = |x - 2| + |x + 1| - x$, then $f'(-10)$ is equal to [Kerala CEE 2010]
- (a) -3 (b) -2 (c) -1
(d) 0 (e) 1

62. If $x^y = e^{2(x-y)}$, then $\frac{dy}{dx}$ is equal to [Kerala CEE 2010]
- (a) $\frac{2(1 + \log x)}{(2 + \log x)^2}$ (b) $\frac{1 + \log x}{(2 + \log x)^2}$
(c) $\frac{2}{2 + \log x}$ (d) $\frac{2(1 - \log x)}{(2 + \log x)^2}$
(e) $\frac{2 + \log x}{(2 - \log x)^2}$

63. If $y = (1 + x)(1 + x^2)(1 + x^4) \dots (1 + x^{2^n})$, then the value of $\frac{dy}{dx}$ at $x = 0$ is [WB JEE 2010]

- (a) 0
(b) -1
(c) 1
(d) 2

64. If $x = a(1 + \cos \theta)$ and $y = a(\theta + \sin \theta)$, then $\frac{d^2 y}{dx^2}$ at $\theta = \frac{\pi}{2}$ is equal to [Kerala CEE 2010]

- (a) $-\frac{1}{a}$ (b) $\frac{1}{a}$ (c) -1
(d) -2 (e) $-\frac{2}{a}$

65. The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to $\sin^{-1}(3x - 4x^3)$ is [Kerala CEE 2010]

- (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$
(d) 1 (e) 0

Answers

Work Book Exercise 6.1

1. (a) 2. (b) 3. (a) 4. (b) 5. (a) 6. (c) 7. (a) 8. (c) 9. (a)

Work Book Exercise 6.2

1. (b) 2. (c) 3. (b) 4. (b) 5. (b) 6. (b) 7. (c) 8. (a)

Work Book Exercise 6.3

1. (c) 2. (d) 3. (c) 4. (a) 5. (b) 6. (a) 7. (b)

Target Exercises

1. (c)	2. (c)	3. (a)	4. (b)	5. (b)	6. (a)	7. (c)	8. (d)	9. (a)	10. (b)
11. (a)	12. (c)	13. (c)	14. (d)	15. (a)	16. (c)	17. (d)	18. (b)	19. (a)	20. (b)
21. (a)	22. (b)	23. (b)	24. (d)	25. (b)	26. (a)	27. (a)	28. (d)	29. (c)	30. (b)
31. (b)	32. (c)	33. (c)	34. (a)	35. (c)	36. (c)	37. (a)	38. (c)	39. (b)	40. (a)
41. (a)	42. (a)	43. (c)	44. (d)	45. (b)	46. (b)	47. (a)	48. (c)	49. (a)	50. (a)
51. (b)	52. (b)	53. (c)	54. (a)	55. (a)	56. (b)	57. (d)	58. (d)	59. (d)	60. (a)
61. (b)	62. (b)	63. (b)	64. (c)	65. (d)	66. (a)	67. (a,b)	68. (a,b)	69. (b,c)	70. (a,c,d)
71. (b,d)	72. (a,b,d)	73. (a,c)	74. (a,c)	75. (a)	76. (a)	77. (b)	78. (c)	79. (c)	80. (b)
81. (a)	82. (d)	83. (d)	84. (c)	85. (c)	86. (*)	87. (*)	88. (*)	89. (1)	90. (3)
91. (2)	92. (5)	93. (5)	94. (3)	95. (3)	96. (1)				

*86. $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow s$; $D \rightarrow p$

87. $A \rightarrow p$; $B \rightarrow s$; $C \rightarrow r$; $D \rightarrow s$

88. $A \rightarrow q$; $B \rightarrow r$; $C \rightarrow s$; $d \rightarrow p$

Entrances Gallery

1. (b,d)	2. (a)	3. (0)	4. (b,c)	5. (d)	6. (b)	7. (d)	8. (d)	9. (a)	10. (c)
11. (a)	12. (c)	13. (d)	14. (c)	15. (a)	16. (a)	17. (b)	18. (d)	19. (a)	20. (a)
21. (b)	22. (c)	23. (e)	24. (c)	25. (d)	26. (d)	27. (c)	28. (b)	29. (d)	30. (c)
31. (c)	32. (d)	33. (d)	34. (d)	35. (b)	36. (b)	37. (b)	38. (b)	39. (c)	40. (a)
41. (c)	42. (b)	43. (b)	44. (c)	45. (c)	46. (b)	47. (b)	48. (a)	49. (a)	50. (b)
51. (a)	52. (a)	53. (c)	54. (a)	55. (d)	56. (b)	57. (a)	58. (b)	59. (d)	60. (c)
61. (a)	62. (a)	63. (c)	64. (a)	65. (a)					

Explanations

Target Exercises

$$1. \text{ Given, } y = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!}$$

$$\Rightarrow \frac{dy}{dx} = 0 + 1 + x + \frac{x^2}{2!} + \dots + \frac{x^{n-1}}{(n-1)!}$$

$$\Rightarrow \frac{dy}{dx} + \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \dots + \frac{x^n}{n!}$$

$$\Rightarrow \frac{dy}{dx} = y - \frac{x^n}{n!}$$

$$2. \text{ We have, } y = \frac{\log \tan x}{\log \sin x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\log \sin x) \left(\frac{\sec^2 x}{\tan x} \right) - (\log \tan x)(\cot x)}{(\log \sin x)^2}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=\pi/4} = \frac{-4}{\log 2} \quad [\text{on simplification}]$$

$$3. \text{ Let } \cos \alpha = \frac{5}{13}, \text{ then } \sin \alpha = \frac{12}{13}$$

$$\text{So, } y = \cos^{-1}(\cos \alpha \cdot \cos x - \sin \alpha \cdot \sin x)$$

$$\Rightarrow y = \cos^{-1}\{\cos(x + \alpha)\} = x + \alpha$$

$$[\because (x + \alpha) \text{ is in the first or second quadrant}]$$

$$\therefore \frac{dy}{dx} = 1$$

$$4. \text{ We have,}$$

$$\frac{d}{dx} \left[\frac{(1 + x^2 + x^4)(1 - x^2 + x^4)}{(1 + x^2 + x^4)} \right] = ax^3 + bx$$

$$\Rightarrow \frac{d}{dx} (1 - x^2 + x^4) = ax^3 + bx$$

$$\Rightarrow -2x + 4x^3 = ax^3 + bx$$

$$\Rightarrow a = 4 \text{ and } b = -2$$

$$5. \text{ We have,}$$

$$y = \sum_{r=1}^x \tan^{-1} \frac{1}{1+r+r^2} = \sum_{r=1}^x \tan^{-1} \left[\frac{(r+1)-r}{1+(r+1)r} \right]$$

$$= \sum_{r=1}^x [\tan^{-1}(r+1) - \tan^{-1} r]$$

$$= [\tan^{-1} 2 - \tan^{-1} 1 + \tan^{-1} 3 - \tan^{-1} 2 + \dots +$$

$$\tan^{-1} x - \tan^{-1}(x-1) + \tan^{-1}(x+1) - \tan^{-1} x]$$

$$= [\tan^{-1}(x+1) - \tan^{-1} 1]$$

$$\therefore \frac{dy}{dx} = \frac{1}{1+(x+1)^2}$$

$$6. \text{ Given, } y = \sin^{-1} \left[\frac{5x + 12\sqrt{1-x^2}}{13} \right]$$

On putting $x = \sin \theta$, $5 = r \cos \alpha$ and $12 = r \sin \alpha$, we get

$$y = \sin^{-1} \left(\frac{r \cos \alpha \sin \theta + r \sin \alpha \cos \theta}{13} \right)$$

$$= \sin^{-1} \left(\frac{13 \sin(\theta + \alpha)}{13} \right) \quad [\because r = \sqrt{25 + 144} = 13]$$

$$= \theta + \alpha = \sin^{-1} x + \tan^{-1} \frac{12}{5} \quad \left[\because \tan \alpha = \frac{12}{5} \right]$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$7. \text{ We have, } y = \tan^{-1} \left(\frac{1 + \tan x}{1 - \tan x} \right) = \tan^{-1} \left[\tan \left(\frac{\pi}{4} + x \right) \right]$$

$$\Rightarrow y = \frac{\pi}{4} + x \therefore \frac{dy}{dx} = 1$$

$$8. \text{ Let } y = \tan^{-1} \left(\frac{3x - x^3}{1 - 3x^2} \right)$$

$$\text{Put } x = \tan \theta$$

$$\text{Then, } y = \tan^{-1}(\tan 3\theta) = 3\theta = 3 \tan^{-1} x, \text{ when}$$

$$|3\theta| < \frac{\pi}{2}$$

$$\Rightarrow \tan^{-1} x < \frac{\pi}{6} \Rightarrow |x| < \frac{1}{\sqrt{3}}$$

$$\therefore \frac{dy}{dx} = \frac{3}{1+x^2}$$

$$9. \text{ Given, } y = \cos^{-1} \left(\frac{2 \cos x - 3 \sin x}{\sqrt{13}} \right)$$

$$\text{Put } 2 = r \cos \theta \text{ and } 3 = r \sin \theta$$

$$\text{Then, } r = \sqrt{13} \text{ and } \tan \theta = \frac{3}{2}$$

$$\therefore y = \cos^{-1} \left[\frac{r(\cos \theta \cos x - \sin \theta \sin x)}{\sqrt{13}} \right]$$

$$= \cos^{-1}(\cos(\theta + x)) = \theta + x = x + \tan^{-1} \frac{3}{2}$$

$$\therefore \frac{dy}{dx} = 1$$

$$10. \text{ We have, } \frac{x^2 - y^2}{x^2 + y^2} = \sin(\log a) = \text{constant} = k \quad [\text{say}]$$

$$\Rightarrow x^2 - y^2 = kx^2 + ky^2$$

$$\Rightarrow (k+1)y^2 = (1-k)x^2$$

$$\Rightarrow y^2 = \frac{1-k}{1+k} x^2 \quad \dots (i)$$

On differentiating both sides w.r.t. x , we get

$$2y \frac{dy}{dx} = \frac{1-k}{1+k} \cdot 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{1-k}{1+k} \cdot \frac{x}{y} = \frac{y^2}{x^2} \cdot \frac{x}{y} \quad [\text{from Eq. (i)}]$$

$$\therefore \frac{dy}{dx} = \frac{y}{x}$$

11. We have,

$$f(x) = \frac{\sin^{-1} x}{\sqrt{1-x^2}} \Rightarrow [f(x)]^2 = \frac{(\sin^{-1} x)^2}{1-x^2}$$

$$\Rightarrow (1-x^2)[f(x)]^2 = (\sin^{-1} x)^2$$

On differentiating both sides w.r.t. x , we get

$$(1-x^2)2f(x) \cdot f'(x) + [f(x)]^2(-2x) = \frac{2\sin^{-1} x}{\sqrt{1-x^2}} = 2f(x)$$

$$\Rightarrow (1-x^2)f'(x) - xf(x) = 1 \quad [\text{dividing by } 2f(x)]$$

12. We have, $f(x) = 2 \tan^{-1}(\log x)$

$$\therefore f'(x) = \frac{2}{1+(\log x)^2} \cdot \frac{1}{x}$$

$$\therefore f'(e) = \frac{2}{1+(\log e)^2} \cdot \frac{1}{e} = \frac{2}{2} \cdot \frac{1}{e} = \frac{1}{e}$$

13. Given, $y = \sin^{-1}[\sqrt{x-ax} - \sqrt{a-ax}]$

Put $x = \sin^2 \theta$ and $a = \sin^2 \phi$, then

$$y = \sin^{-1}(\sin \theta \cos \phi - \cos \theta \sin \phi)$$

$$= \sin^{-1}[\sin(\theta - \phi)] = \theta - \phi$$

$$= \sin^{-1} \sqrt{x} - \sin^{-1} \sqrt{a}$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}}$$

14. We have,

$$y = \sin^{-1} \left(\frac{\sin \alpha \sin x}{1 - \cos \alpha \sin x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - \frac{\sin^2 \alpha \sin^2 x}{(1 - \cos \alpha \sin x)^2}}} \cdot \frac{d}{dx} \left(\frac{\sin \alpha \sin x}{1 - \cos \alpha \sin x} \right)$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1 - \cos \alpha \sin x)}{\sqrt{1 + \cos 2\alpha \sin^2 x - 2 \cos \alpha \sin x}} \cdot \frac{\sin \alpha}{\cos \alpha} \cdot \frac{d}{dx} \left(\frac{1}{1 - \cos \alpha \sin x} \right)$$

$$\Rightarrow \frac{dy}{dx} = - \frac{\tan \alpha (1 - \cos \alpha \sin x)}{\{\sqrt{1 + \cos 2\alpha \sin^2 x - 2 \sin x \cos \alpha}\} \left[\frac{-\cos \alpha \cos x}{(1 - \cos \alpha \sin x)^2} \right]}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin \alpha \cos x}{(1 - \cos \alpha \sin x) \sqrt{1 + \cos 2\alpha \sin^2 x - 2 \cos \alpha \sin x}}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=0} = \sin \alpha \Rightarrow y'(0) = \sin \alpha$$

15. We have,

$$f(x) = \cot^{-1} \left(\frac{x^x - x^{-x}}{2} \right)$$

$$\Rightarrow f'(x) = - \frac{1}{1 + \left(\frac{x^x - x^{-x}}{2} \right)^2} \cdot \frac{d}{dx} \left(\frac{x^x - x^{-x}}{2} \right)$$

$$\Rightarrow f'(x) = \frac{-2}{4 + (x^x - x^{-x})^2} \cdot \frac{d}{dx} (x^x - x^{-x})$$

$$\Rightarrow f'(x) = \frac{-2}{(x^x + x^{-x})^2} \cdot \frac{d}{dx} (e^{x \log x} - e^{-x \log x})$$

$$\Rightarrow f'(x) = \frac{-2}{(x^x + x^{-x})^2}$$

$$\left\{ e^{x \log x} \cdot \frac{d}{dx} (x \log x) - e^{-x \log x} \cdot \frac{d}{dx} (-x \log x) \right\}$$

$$\Rightarrow f'(x) = \frac{-2}{(x^x + x^{-x})^2} \{ x^x (1 + \log x) + x^{-x} (1 + \log x) \}$$

$$\Rightarrow f'(x) = \frac{-2(1 + \log x)}{(x^x + x^{-x})^2} \cdot (x^x + x^{-x}) = \frac{-2(1 + \log x)}{x^x + x^{-x}}$$

$$\therefore f'(1) = \frac{-2}{(1+1)} = -1$$

16. We have,

$$y = \sin^{-1}(\sin x) = \begin{cases} x, & \text{if } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2} \\ \pi - x, & \text{if } \frac{\pi}{2} \leq x \leq \frac{3\pi}{2} \\ x - 2\pi, & \text{if } \frac{3\pi}{2} \leq x \leq 2\pi \end{cases}$$

We observe that,

$$\left(\text{LHD at } x = \frac{\pi}{2} \right) = 1 \text{ and } \left(\text{RHD at } x = \frac{\pi}{2} \right) = -1$$

Hence, $\left[\frac{dy}{dx} \right]_{x=\pi/2}$ does not exist.

17. We have,

$$y = \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \dots}}}}$$

$$\Rightarrow y^2 = x + \sqrt{y + \sqrt{x + \sqrt{y + \dots}}}$$

$$\Rightarrow y^2 = x + \sqrt{y + y}$$

$$\Rightarrow (y^2 - x)^2 = 2y$$

On differentiating both sides w.r.t. x , we get

$$2(y^2 - x) \left(2y \frac{dy}{dx} - 1 \right) = 2 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x}{y^3 - xy - 1}$$

18. We have,

$$y = (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n})$$

$$= \frac{(1-x)(1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n})}{1-x}$$

$$= \frac{1-x^{2^{n+1}}}{1-x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x) \cdot (-2^{n+1}) \cdot x^{2^{n+1}-1} + (1-x^{2^{n+1}})}{(1-x)^2}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=0} = 1$$

19. We have, $\log(x+y) - 2xy = 0$

When $x = 0$, we have

$$\log(0+y) - 2 \times 0 \times y = 0$$

$$\Rightarrow y = 1$$

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{1}{x+y} \left(1 + \frac{dy}{dx} \right) - 2y - 2x \frac{dy}{dx} = 0$$

On putting $x = 0$ and $y = 1$, we get

$$\left(1 + \frac{dy}{dx} \right) - 2 - 2 \times 0 = 0$$

$$\Rightarrow \frac{dy}{dx} = 1$$

20. We have, $y = \log_{x^2+4}(7x^2 - 5x + 1)$

$$= \frac{\log_e(7x^2 - 5x + 1)}{\log_e(x^2 + 4)}$$

$$= \frac{\log_e f(x)}{\log_e g(x)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\log_e \{g(x)\} \cdot \frac{f'(x)}{f(x)} - \log_e f(x) \cdot \frac{g'(x)}{g(x)}}{\{\log_e g(x)\}^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log_e g(x)} \left\{ \frac{f'(x)}{f(x)} - y \frac{g'(x)}{g(x)} \right\}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\log_e(x^2 + 4)} \left\{ \frac{14x - 5}{7x^2 - 5x + 1} - \frac{2xy}{x^2 + 4} \right\}$$

21. We have, $f(x) = \log \left[\frac{m(x)}{n(x)} \right]$

$$\Rightarrow f'(x) = \frac{n(x)}{m(x)} \left[\frac{n(x)m'(x) - m(x)n'(x)}{[n(x)]^2} \right]$$

$$\therefore f'(1) = \frac{n(1)}{m(1)} \left[\frac{n(1)m'(1) - m(1)n'(1)}{[n(1)]^2} \right] = 0$$

22. We have, $y = [(\tan x)^{\tan x}]^{\tan x}$

Taking log on both sides, we get
 $\log y = \tan^2 x \cdot \log \tan x$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 2 \tan x \sec^2 x \cdot \log \tan x$$

$$+ \tan^2 x \cdot \frac{1}{\tan x} \cdot \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} = [(\tan x)^{\tan x}]^{\tan x} \cdot \tan x \sec^2 x (2 \log \tan x + 1)$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=\pi/4} = 1 \cdot 1 \cdot 2(2 \cdot 0 + 1) = 2$$

23. We have,

$$f(x) = \log |x| = \begin{cases} \log x, & \text{if } x > 0 \\ \log(-x), & \text{if } x < 0 \end{cases}$$

$$\therefore f'(x) = \begin{cases} \frac{1}{x}, & \text{if } x > 0 \\ -\frac{1}{x} \times (-1) = \frac{1}{x}, & \text{if } x < 0 \end{cases}$$

$$\Rightarrow f'(x) = \frac{1}{x}, \forall x \neq 0$$

...(i)

24. We have,

$$f(x) = |\log x| = \begin{cases} -\log x, & \text{if } 0 < x < 1 \\ \log x, & \text{if } x \geq 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -\frac{1}{x}, & \text{if } 0 < x < 1 \\ \frac{1}{x}, & \text{if } x > 1 \end{cases}$$

25. For $x > 1$, we have

$$f(x) = |\log |x|| = \log x$$

$$\Rightarrow f'(x) = \frac{1}{x}$$

For $x < -1$, we have

$$f(x) = |\log |x|| = \log(-x)$$

$$\Rightarrow f'(x) = \frac{1}{x}$$

For $0 < x < 1$, we have

$$f(x) = |\log |x|| = -\log x \Rightarrow f'(x) = -\frac{1}{x}$$

For $-1 < x < 0$, we have

$$f(x) = -\log(-x) \Rightarrow f'(x) = -\frac{1}{x}$$

$$\text{Hence, } f'(x) = \begin{cases} \frac{1}{x}, & \text{if } |x| > 1 \\ -\frac{1}{x}, & \text{if } |x| < 1 \end{cases}$$

26. We have,

$$f(x) = \begin{cases} x - 2, & \text{if } 2 \leq x < 4 \\ -(x - 2), & \text{if } x < 2 \end{cases}$$

$$\therefore g(x) = f\{f(x)\}$$

$$\Rightarrow g(x) = \begin{cases} f(x - 2), & \text{if } 2 \leq x < 4 \\ f\{-(x - 2)\}, & \text{if } 0 < x < 2 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} -(x - 2 - 2), & \text{if } 2 \leq x < 4 \\ -(-x + 2 - 2), & \text{if } 0 < x < 2 \end{cases}$$

$$\Rightarrow g'(x) = \begin{cases} -1, & \text{if } 2 < x < 4 \\ 1, & \text{if } 0 < x < 2 \end{cases}$$

27. In the neighbourhood of $-\frac{\pi}{4}$, we have

$$f(x) = (-x)^{-\sin x} = e^{-\sin x \log(-x)}$$

$$\Rightarrow f'(x) = e^{-\sin x \log(-x)} \left[-\cos x \cdot \log(-x) - \frac{\sin x}{x} \right]$$

$$\Rightarrow f'(x) = (-x)^{-\sin x} \left[-\cos x \cdot \log(-x) - \frac{\sin x}{x} \right]$$

$$\therefore f'\left(-\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^{1/\sqrt{2}} \left[\frac{-1}{\sqrt{2}} \log \frac{\pi}{4} + \frac{4}{\pi} \times \left(\frac{-1}{\sqrt{2}}\right) \right]$$

$$\Rightarrow f'\left(-\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^{1/\sqrt{2}} \left(\frac{\sqrt{2}}{2} \log \frac{4}{\pi} - \frac{2\sqrt{2}}{\pi} \right)$$

28. We have, $f(x) = |\cos x - \sin x|$

$$\Rightarrow f(x) = \begin{cases} \cos x - \sin x, & \text{if } 0 < x \leq \frac{\pi}{4} \\ \sin x - \cos x, & \text{if } \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

Clearly,

$$\begin{aligned} \text{(LHD at } x = \pi/4) &= \left[\frac{d}{dx} (\cos x - \sin x) \right]_{\text{at } x = \pi/4} \\ &= [-\sin x - \cos x]_{x = \pi/4} \\ &= -\sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{and (RHD at } x = \pi/4) &= \left[\frac{d}{dx} (\sin x - \cos x) \right]_{\text{at } x = \pi/4} \\ &= [\cos x + \sin x]_{x = \pi/4} \\ &= \sqrt{2} \end{aligned}$$

$\therefore$ (LHD at $x = \pi/4$) $\neq$ (RHD at $x = \pi/4$)

Thus, $f'\left(\frac{\pi}{4}\right)$ does not exist.

29. In the neighbourhood of $x = \frac{2\pi}{3}$, we have

$$\begin{aligned} \Rightarrow \quad \cos x < 0 \quad \text{and} \quad \sin x > 0 \\ \Rightarrow \quad |\cos x| = -\cos x \quad \text{and} \quad |\sin x| = \sin x \\ \therefore \quad y = |\cos x| + |\sin x| \\ \Rightarrow \quad y = -\cos x + \sin x \\ \Rightarrow \quad \frac{dy}{dx} = \sin x + \cos x \\ \therefore \left[\frac{dy}{dx} \right]_{x = 2\pi/3} = \sin \frac{2\pi}{3} + \cos \frac{2\pi}{3} = \frac{\sqrt{3}}{2} - 1 \end{aligned}$$

30. In the neighbourhood of $x = \frac{3\pi}{4}$, we have

$$\begin{aligned} \cos x < 0 &= |\cos x| = -\cos x \\ \therefore \quad f(x) &= -\cos x \\ \Rightarrow \quad f'(x) &= \sin x \\ \therefore \quad f'\left(\frac{3\pi}{4}\right) &= \sin \frac{3\pi}{4} = \frac{1}{\sqrt{2}} \end{aligned}$$

31. In the neighbourhood of $x = -\frac{\pi}{6}$, we have

$$\begin{aligned} |x| &= -x \quad \text{and} \quad |\tan x| = -\tan x \\ \therefore \quad f(x) &= (-x)^{-\tan x} \\ \Rightarrow \quad f(x) &= e^{-\tan x \cdot \log(-x)} \\ \Rightarrow \quad f'(x) &= (-x)^{-\tan x} \left\{ -\sec^2 x \cdot \log(-x) - \frac{\tan x}{x} \right\} \\ \therefore \quad f'\left(-\frac{\pi}{6}\right) &= \left(\frac{\pi}{6}\right)^{1/\sqrt{3}} \left\{ -\frac{4}{3} \log \frac{\pi}{6} - \frac{2\sqrt{3}}{\pi} \right\} \\ \Rightarrow \quad f'\left(-\frac{\pi}{6}\right) &= \left(\frac{\pi}{6}\right)^{1/\sqrt{3}} \left\{ \frac{4}{3} \log \frac{6}{\pi} - \frac{2\sqrt{3}}{\pi} \right\} \end{aligned}$$

32. We have, $f(x) = |x - 1|$

$$\begin{aligned} \therefore \quad g(x) &= f(f(f(x))) \Rightarrow g(x) = f(f(|x - 1|)) \\ \Rightarrow \quad g(x) &= f(|x - 1| - 1) \\ \Rightarrow \quad g(x) &= \begin{cases} |x - 2| - 1, & \text{if } x \geq 1 \\ |-x| - 1, & \text{if } x < 1 \end{cases} \\ \Rightarrow \quad g(x) &= \begin{cases} |x - 3|, & \text{if } x \geq 2 \\ |2 - x - 1|, & \text{if } 1 \leq x < 2 \\ |x - 1|, & \text{if } 0 \leq x < 1 \\ |-x - 1|, & \text{if } x < 0 \end{cases} \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad g(x) &= \begin{cases} x - 3, & \text{if } x \geq 3 \\ 3 - x, & \text{if } 2 \leq x < 3 \\ x - 1, & \text{if } 1 \leq x < 2 \\ 1 - x, & \text{if } 0 \leq x < 1 \\ x + 1, & \text{if } -1 \leq x < 0 \\ -x - 1, & \text{if } x < -1 \end{cases} \\ \Rightarrow \quad g'(x) &= \begin{cases} 1, & \text{if } x > 3 \\ -1, & \text{if } 2 \leq x < 3 \\ 1, & \text{if } 1 < x < 2 \\ -1, & \text{if } 0 < x < 1 \\ 1, & \text{if } -1 < x < 0 \\ -1, & \text{if } x < -1 \end{cases} \end{aligned}$$

33. We know that,

$$\begin{aligned} \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) &= \begin{cases} 2 \tan^{-1} x, & \text{if } x \geq 0 \\ -2 \tan^{-1} x, & \text{if } x \leq 0 \end{cases} \\ \therefore \quad f(x) &= \begin{cases} 4 \tan^{-1} x, & \text{if } x \geq 0 \\ 0, & \text{if } x \leq 0 \end{cases} \\ \Rightarrow \quad f'(x) &= \begin{cases} \frac{4}{1+x^2}, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases} \end{aligned}$$

34. We have,

$$\begin{aligned} f(x) &= \sqrt{x-1} + \sqrt{(x-1)+25-10\sqrt{x-1}} \\ \Rightarrow \quad f(x) &= \sqrt{x-1} + \sqrt{(\sqrt{x-1}-5)^2} \\ \Rightarrow \quad f(x) &= \sqrt{x-1} + |\sqrt{x-1}-5| \\ \Rightarrow \quad f(x) &= \sqrt{x-1} - (\sqrt{x-1}-5) \\ &\quad [\because \sqrt{x-1}-5 < 0 \text{ for } 1 < x < 26] \\ \therefore \quad f'(x) &= 0, \forall x \in (1, 26) \end{aligned}$$

35. Clearly, $f(x)$ is not defined at $x = 1$.

For $x \in R - \{1\}$, we have

$$\begin{aligned} g(x) &= f(f(f(x))) = f\left[f\left(\frac{1}{1-x}\right)\right] = f\left(\frac{1}{1-\frac{1}{1-x}}\right) \\ \Rightarrow \quad g(x) &= f\left(\frac{x-1}{x}\right) = \frac{1}{1-\frac{x-1}{x}} = x \\ \therefore \quad g'(x) &= 1, \forall x \in R - \{1\} \end{aligned}$$

36. We have, $f(x) = |x - 1|$

$$\begin{aligned} \text{and} \quad g(x) &= f(f(f(x))) \\ \therefore \quad g(x) &= f(f(|x - 1|)) \\ \Rightarrow \quad g(x) &= f(|x - 1| - 1) \\ \Rightarrow \quad g(x) &= \begin{cases} |x - 2| - 1, & \text{if } x \geq 1 \\ |-x| - 1, & \text{if } x < 1 \end{cases} \\ \Rightarrow \quad g(x) &= \begin{cases} |x - 3|, & \text{if } x \geq 2 \\ |2 - x - 1|, & \text{if } 1 \leq x < 2 \\ |x - 1|, & \text{if } 0 \leq x < 1 \\ |-x - 1|, & \text{if } x < 0 \end{cases} \end{aligned}$$

$$\Rightarrow g(x) = \begin{cases} x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } 2 \leq x < 3 \\ x-1, & \text{if } 1 \leq x < 2 \\ 1-x, & \text{if } 0 \leq x < 1 \\ x+1, & \text{if } -1 \leq x < 0 \\ -x-1, & \text{if } x < -1 \end{cases}$$

$$\Rightarrow g'(x) = \begin{cases} 1, & \text{if } x > 3 \\ -1, & \text{if } 2 < x < 3 \\ 1, & \text{if } 1 < x < 2 \\ -1, & \text{if } 0 < x < 1 \\ 1, & \text{if } -1 < x < 0 \\ -1, & \text{if } x < -1 \end{cases}$$

$$\therefore g'(4) = 1$$

Aliter

For $x > 2$, we have

$$f(x) = |x-1| = x-1$$

$$\therefore f(f(x)) = f(x-1) = |x-1-1| = x-2 \quad [\because x > 2]$$

$$\Rightarrow f(f(x)) = x-2$$

$$\Rightarrow f(f(f(x))) = f(x-2) = |x-2-1|$$

$$= \begin{cases} x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } 2 \leq x < 3 \\ x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } 2 \leq x < 3 \end{cases}$$

$$\Rightarrow g(x) = \begin{cases} x-3, & \text{if } x \geq 3 \\ 3-x, & \text{if } 2 \leq x < 3 \end{cases}$$

$$\Rightarrow g'(x) = \begin{cases} 1, & \text{if } x > 3 \\ -1, & \text{if } 2 < x < 3 \end{cases}$$

$$\therefore g'(4) = 1$$

37. We have,

$$u = f(x)^3 \text{ and } v = g(x)^2$$

$$\Rightarrow \frac{dy}{dx} = f'(x)^3 \cdot 3x^2 \text{ and } \frac{dv}{dx} = g'(x)^2 \cdot 2x$$

$$\therefore \frac{du}{dv} = \frac{\frac{dy}{dx}}{\frac{dv}{dx}} = \frac{f'(x)^3 \cdot 3x^2}{g'(x)^2 \cdot 2x} = \frac{\cos^3 x \cdot 3x^2}{\sin^2 x \cdot 2x}$$

$$[\because f'(x) = \cos x \text{ and } g'(x) = \sin x]$$

$$= \frac{3}{2} x \cos^3 x \operatorname{cosec}^2 x$$

38. We have, $f'(x) = ae^{ax} + be^{bx}$ and $f''(x) = a^2e^{ax} + b^2e^{bx}$

$$\therefore a^2e^{ax} + b^2e^{bx} - 2ae^{ax} - 2be^{bx} - 15e^{ax} - 15e^{bx} = 0$$

$$\Rightarrow (a^2 - 2a - 15)e^{ax} + (b^2 - 2b - 15)e^{bx} = 0$$

$$\Rightarrow (a^2 - 2a - 15) = 0 \text{ and } (b^2 - 2b - 15) = 0$$

$$\Rightarrow (a-5)(a+3) = 0 \text{ and } (b-5)(b+3) = 0$$

$$\Rightarrow a = 5 \text{ or } -3 \text{ and } b = 5 \text{ or } -3$$

$$\therefore a \neq b, \text{ hence } a = 5 \text{ and } b = -3$$

$$\text{or } a = -3 \text{ and } b = 5$$

$$\therefore ab = -15$$

39. $f(g(x)) = x \Rightarrow f'(g(x)) g'(x) = 1$

$$\Rightarrow (e^{g(x)} + 1) g'(x) = 1$$

$$\Rightarrow (e^{f(\log 2)} + 1) g'(f(\log 2)) = 1$$

$$\Rightarrow (e^{\log 2} + 1) g'(f(\log 2)) = 1$$

$$\Rightarrow g'(f(\log 2)) = \frac{1}{3}$$

$$40. F'(x) = \left[f\left(\frac{x}{2}\right) \cdot f'\left(\frac{x}{2}\right) + g\left(\frac{x}{2}\right) g'\left(\frac{x}{2}\right) \right]$$

Here, $g(x) = f'(x)$

and $g'(x) = f''(x) = -f(x)$

$$\text{So, } F'(x) = f\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) - f\left(\frac{x}{2}\right) g\left(\frac{x}{2}\right) = 0$$

$\therefore F(x)$ is a constant function.

Hence, $F(10) = 5$

41. We have, $y = 2 \ln(1 + \cos x)$

$$\Rightarrow \frac{dy}{dx} = \frac{-2 \sin x}{1 + \cos x}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -2 \left[\frac{(1 + \cos x) \cos x - \sin x(-\sin x)}{(1 + \cos x)^2} \right]$$

$$= -2 \left[\frac{\cos x + 1}{(1 + \cos x)^2} \right] = \frac{-2}{(1 + \cos x)}$$

$$\text{Now, } e^{\frac{2}{y/2}} = 2 \cdot e^{\frac{-2 \ln(1 + \cos x)}{2}} = \frac{2}{(1 + \cos x)}$$

$$\therefore \frac{d^2y}{dx^2} + \frac{2}{e^{y/2}} = 0$$

42. $u(x) = 7v(x) \Rightarrow u'(x) = 7v'(x)$

$$\Rightarrow p = 7$$

[given]

$$\text{Again, } \frac{u(x)}{v(x)} = 7 \Rightarrow \left[\frac{u(x)}{v(x)} \right]' = 0 \Rightarrow q = 0$$

$$\text{Now, } \frac{p+q}{p-q} = \frac{7+0}{7-0} = 1$$

43. We have, $x = \sec \theta - \cos \theta$ and $y = \sec^n \theta - \cos^n \theta$

$$\Rightarrow \frac{dx}{d\theta} = \sec \theta \tan \theta + \sin \theta = \tan \theta (\sec \theta + \cos \theta)$$

$$\text{and } \frac{dy}{d\theta} = n [\sec^n \theta \tan \theta + \cos^{n-1} \theta \sin \theta]$$

$$= n \tan \theta (\sec^n \theta + \cos^n \theta)$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n \tan \theta (\sec^n \theta + \cos^n \theta)}{\tan \theta (\sec \theta + \cos \theta)}$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = n^2 \cdot \frac{(\sec^n \theta - \cos^n \theta)^2 + 4}{(\sec \theta - \cos \theta)^2 + 4} = \frac{n^2(y^2 + 4)}{(x^2 + 4)}$$

$$\Rightarrow (x^2 + 4) \left(\frac{dy}{dx} \right)^2 = n^2(y^2 + 4)$$

44. We have, $x = f(t) \cos t - f'(t) \sin t$

and $y = f(t) \sin t + f'(t) \cos t$

$$\Rightarrow \frac{dx}{dt} = -f(t) \sin t + f'(t) \cos t - f''(t) \cos t - f'''(t) \sin t$$

$$= -[f(t) + f''(t)] \sin t$$

$$\text{and } \frac{dy}{dt} = f(t) \cos t + f'(t) \sin t - f''(t) \sin t + f'''(t) \cos t$$

$$= [f(t) + f''(t)] \cos t$$

$$\therefore \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 = [f(t) + f''(t)]^2$$

45. We have, $\frac{dx}{dt} = \frac{df}{dt} = f_1, \frac{dy}{dt} = \frac{d\phi}{dt} = \phi_1$

$$\therefore \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\phi_1}{f_1}$$

$$\begin{aligned} \text{Now, } \frac{d^2y}{dx^2} &= \frac{d}{dx} \left(\frac{\phi_1}{f_1} \right) = \frac{d}{dt} \left(\frac{\phi_1}{f_1} \right) \cdot \frac{dt}{dx} \\ &= \frac{f_1 \phi_2 - \phi_1 f_2}{f_1^2} \times \frac{1}{f_1} = \frac{f_1 \phi_2 - \phi_1 f_2}{f_1^3} \end{aligned}$$

46. We have, $P = a^2 \cos^2 \theta + b^2 \sin^2 \theta$

$$\Rightarrow \frac{dP}{d\theta} = -2a^2 \cos \theta \sin \theta + 2b^2 \sin \theta \cos \theta$$

$$= (b^2 - a^2) \sin 2\theta$$

$$\Rightarrow \frac{d^2P}{d\theta^2} = 2(b^2 - a^2) \cos 2\theta$$

$$\begin{aligned} \therefore 4P + \frac{d^2P}{d\theta^2} &= 4a^2 \cos^2 \theta + 4b^2 \sin^2 \theta \\ &\quad + 2(b^2 - a^2)(\cos^2 \theta - \sin^2 \theta) \\ &= \cos^2 \theta (4a^2 + 2b^2 - 2a^2) \\ &\quad + \sin^2 \theta (4b^2 - 2b^2 + 2a^2) \\ &= (2a^2 + 2b^2)(\sin^2 \theta + \cos^2 \theta) \\ &= 2a^2 + 2b^2 \end{aligned}$$

$$\therefore P + \frac{d^2P}{d\theta^2} = 2a^2 + 2b^2 - 3P$$

47. We have, $yx^2 + yc = ax + b$

On differentiating w.r.t. x , we get

$$y'x^2 + 2xy' + y'c = a$$

Again, differentiating w.r.t. x , we get

$$y''(x^2 + c) + 2xy' + 2xy' + 2y = 0$$

$$\Rightarrow y'' = -\frac{4xy' + 2y}{x^2 + c}$$

$$\Rightarrow x^2 + c = \frac{-2(2xy' + y)}{y''}$$

Again, differentiating w.r.t. x , we get

$$2x = -2 \left[\frac{y''(2xy'' + 3y') - y'''(2xy' + y)}{(y'')^2} \right]$$

$$\Rightarrow y'''(2xy' + y) = 3y''(xy'' + y')$$

48. $f(x)$ is a polynomial of degree 20. $f'(x)$ reduces the degree of $f(x)$ by one. Thus, in order to get a polynomial of degree 20, we must reduce the degree of $f(x)$ by 70. Hence, $f(x)$ should be differentiated 70 times to get a polynomial of degree 20.

$$\therefore n = 70$$

49. We have,

$$\begin{aligned} f'(x) &= 6x + 4g'(1) && \dots(i) \\ f''(x) &= 6 && \dots(ii) \\ g'(x) &= 4x + 3f'(2) && \dots(iii) \\ g''(x) &= 4 && \dots(iv) \end{aligned}$$

From Eq. (i), we have

$$\begin{aligned} f'(1) &= 6 + 4g'(1) = 6 + 4[4 + 3f'(2)] \\ &= 22 + 12f'(2) \end{aligned}$$

From Eq. (iii), we have

$$\begin{aligned} g'(2) &= 8 + 3f'(2) = 8 + 3[12 + 4g'(1)] \\ &= 44 + 12g'(1) \end{aligned}$$

Also, from Eqs. (ii) and (iv), we have

$$f''(3) + g''(2) = 6 + 4 = 10$$

50. We have,

$$y_1 = (n-1)x^{n-2} \log x + x^{n-1} \cdot \frac{1}{x}$$

$$\begin{aligned} \Rightarrow xy_1 &= (n-1)x^{n-1} \log x + x^{n-1} \\ &= (n-1)y + x^{n-1} \end{aligned} \quad \dots(i)$$

On differentiating w.r.t. x , we get

$$\begin{aligned} y_1 + xy_2 &= (n-1)y_1 + (n-1)x^{n-2} \\ \Rightarrow x^2y_2 + xy_1 &= (n-1)xy_1 + (n-1)x^{n-1} \\ &= (n-1)xy_1 + (n-1)(xy_1 - (n-1)y) \end{aligned}$$

[from Eq. (i)]

$$\Rightarrow x^2y_2 + (3-2n)xy_1 + (n-1)^2y = 0$$

51. We have, $4x^2 = (y^{1/5} + y^{-1/5})^2 = y^{2/5} + y^{-2/5} + 2$

$$= (y^{1/5} - y^{-1/5})^2 + 4$$

$$\Rightarrow y^{1/5} - y^{-1/5} = 2\sqrt{x^2 - 1}$$

On adding with the given relation, we get

$$\begin{aligned} 2y^{1/5} &= 2[x + \sqrt{x^2 - 1}] \\ \Rightarrow y &= [x + \sqrt{x^2 - 1}]^5 \\ \Rightarrow y_1 &= 5(x + \sqrt{x^2 - 1})^4 \left(1 + \frac{x}{\sqrt{x^2 - 1}} \right) \end{aligned}$$

$$\Rightarrow y_1 = \frac{5(x + \sqrt{x^2 - 1})^5}{\sqrt{x^2 - 1}} = \frac{5y}{\sqrt{x^2 - 1}}$$

$$\Rightarrow (x^2 - 1)y_1^2 = 25y^2$$

Again, differentiating w.r.t. x , we get

$$\begin{aligned} (x^2 - 1)2y_1y_2 + y_1^2 \cdot 2x &= 50yy_1 \\ \Rightarrow (x^2 - 1)y_2 + xy_1 &= 25y \quad [\text{dividing by } 2y_1] \\ \therefore k &= 25 \end{aligned}$$

52. Let $f(x) = a(x-3)^3 + b(x-3)^2 + c(x-3) + d$

$$\begin{aligned} \text{Then, } f(3) &= 1 = d \Rightarrow d = 1 \\ f'(3) &= -1 = c \Rightarrow c = -1 \\ f''(3) &= 0 = 2b \Rightarrow b = 0 \\ f'''(3) &= 12 = 6a \Rightarrow a = 2 \\ \Rightarrow f'(x) &= 6(x-3)^2 - 1 \\ \therefore f'(1) &= 6(4) - 1 = 23 \end{aligned}$$

53. We have, $(f \cdot h)(x) = f(x) \cdot h(x) = \sin x \cos x$

$$\begin{aligned} \therefore [g \circ f]h(x) &= g[f \cdot h(x)] = g[\sin x \cos x] \\ &= g(\sin x \cos x) = 2 \sin x \cos x = \sin 2x \end{aligned}$$

$$\begin{aligned} \text{i.e. } \phi(x) &= \sin 2x \\ \Rightarrow \phi'(x) &= 2 \cos 2x \text{ and } \phi''(x) = -4 \sin 2x \\ \therefore \phi''\left(\frac{\pi}{4}\right) &= -4 \sin 2 \cdot \frac{\pi}{4} = -4 \end{aligned}$$

54. We have,

$$\begin{aligned} \frac{dx}{d\theta} &= -\sin \theta + \frac{\sec^2 \frac{\theta}{2}}{2 \tan \frac{\theta}{2}} \\ &= -\sin \theta + \frac{1}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \\ &= -\sin \theta + \frac{1}{\sin \theta} = \frac{1 - \sin^2 \theta}{\sin \theta} = \frac{\cos^2 \theta}{\sin \theta} \end{aligned}$$

$$\text{and } \frac{dy}{d\theta} = \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos \theta \sin \theta}{\cos^2 \theta} = \tan \theta$$

$$\text{Also, } \frac{d^2y}{dx^2} = \sec^2 \theta \cdot \frac{d\theta}{dx} = \frac{\sin \theta}{\cos^4 \theta}$$

$$\therefore \frac{d^2y}{dx^2} = 0 \Rightarrow \sin \theta = 0$$

$$\Rightarrow \theta = n\pi, n \in \mathbb{I}$$

55. We have, $x^2 + y^2 = a^2$

On differentiating w.r.t. x , we get

$$2x + 2yy' = 0$$

$$\Rightarrow y' = -\frac{x}{y}$$

Again, differentiating w.r.t. x , we get

$$y'^2 + yy'' + 1 = 0$$

$$\Rightarrow y = -\frac{1 + y'^2}{y''}$$

$$\begin{aligned} \therefore a &= |\sqrt{x^2 + y^2}| = \left| y \sqrt{1 + \left(\frac{x}{y}\right)^2} \right| \\ &= \left| -\frac{(1 + y'^2)}{y''} \cdot \sqrt{1 + y'^2} \right| = \frac{\sqrt{(1 + y'^2)^3}}{|y''|} \end{aligned}$$

56. We have,

$$\begin{aligned} y &= \tan^{-1}\left(\frac{1 - 3\log x}{1 + 3\log x}\right) + \tan^{-1}\left(\frac{4 + 3\log x}{1 - 12\log x}\right) \\ &= \tan^{-1}(1) - \tan^{-1}(3\log x) + \tan^{-1}(4) + \tan^{-1}(3\log x) \\ &\quad \left[\begin{array}{l} \text{using } \tan^{-1} A - \tan^{-1} B = \tan^{-1}\left(\frac{A - B}{1 + AB}\right) \\ \text{and } \tan^{-1} A + \tan^{-1} B = \tan^{-1}\left(\frac{A + B}{1 - AB}\right) \end{array} \right] \\ &= \tan^{-1}(1) + \tan^{-1}(4) \end{aligned}$$

$$\therefore \frac{dy}{dx} = 0 \text{ and hence } \frac{d^2y}{dx^2} = 0$$

57. Let $\phi(x) = f(e^x)$. Then,

$$\phi'(x) = f'(e^x) \cdot e^x$$

$$\begin{aligned} \Rightarrow \phi''(x) &= f''(e^x) \cdot e^x \cdot e^x + f'(e^x) \cdot e^x \\ &= f''(e^x) \cdot e^{2x} + f'(e^x) e^x \end{aligned}$$

58. We have,

$$F(x) = (h \circ g \circ f)(x)$$

$$\Rightarrow F(x) = h\{g(f(x))\} = h\{g(\sin x)\} = h(\sin^2 x)$$

$$\Rightarrow F(x) = \log_e \sin^2 x = 2 \log_e \sin x$$

$$\therefore F'(x) = 2 \cot x \text{ and } F''(x) = -2 \operatorname{cosec}^2 x$$

59. We have, $u = e^x \sin x$ and $v = e^x \cos x$

$$\Rightarrow \frac{du}{dx} = e^x \sin x + e^x \cos x = u + v$$

$$\text{Also, } \frac{dv}{dx} = e^x \cos x - e^x \sin x = v - u$$

$$\therefore v \frac{du}{dx} - u \frac{dv}{dx} = v(u + v) - u(v - u) = u^2 + v^2$$

$$\frac{d^2u}{dx^2} = \frac{du}{dx} + \frac{dv}{dx} = u + v + v - u = 2v$$

$$\text{and } \frac{d^2v}{dx^2} = \frac{dv}{dx} - \frac{du}{dx} = (v - u) - (u + v) = -2u$$

60. Let $y = \operatorname{cosec}^{-1}\left(\frac{1}{2x^2 - 1}\right)$ and $z = \sqrt{1 - x^2}$

$$\text{We have, } y = \operatorname{cosec}^{-1}\left(\frac{1}{2x^2 - 1}\right)$$

$$\Rightarrow y = \sin^{-1}(2x^2 - 1) = \frac{\pi}{2} - \cos^{-1}(2x^2 - 1)$$

$$\Rightarrow y = \begin{cases} \frac{\pi}{2} - 2 \cos^{-1} x, & \text{if } 0 \leq x \leq 1 \\ -\frac{3\pi}{2} + 2 \cos^{-1} x, & \text{if } -1 \leq x \leq 0 \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} \frac{2}{\sqrt{1 - x^2}}, & \text{if } 0 < x < 1 \\ -\frac{2}{\sqrt{1 - x^2}}, & \text{if } -1 < x < 0 \end{cases}$$

$$\text{and } z = \sqrt{1 - x^2}$$

$$\Rightarrow \frac{dz}{dx} = \frac{-x}{\sqrt{1 - x^2}}, \forall x \in (-1, 1)$$

$$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \begin{cases} -\frac{2}{x}, & \text{if } 0 < x < 1 \\ \frac{2}{x}, & \text{if } -1 < x < 0 \end{cases}$$

$$\Rightarrow \left[\frac{dy}{dz}\right]_{x=1/2} = -4$$

61. We have,

$$y = \sin^{-1}(3x - 4x^3) = \begin{cases} 3 \sin^{-1} x, & \text{if } |x| \leq \frac{1}{2} \\ \pi - 3 \sin^{-1} x, & \text{if } \frac{1}{2} < x \leq 1 \\ -\pi - 3 \sin^{-1} x, & \text{if } -1 \leq x < -\frac{1}{2} \end{cases}$$

Let $z = \sin^{-1} x$. Then,

$$y = \begin{cases} 3z, & \text{if } |x| \leq \frac{1}{2} \\ \pi - 3z, & \text{if } \frac{1}{2} < x \leq 1 \\ -\pi - 3z, & \text{if } -1 \leq x < -\frac{1}{2} \end{cases}$$

$$\Rightarrow \frac{dy}{dz} = \begin{cases} 3, & \text{if } |x| < \frac{1}{2} \\ -3, & \text{if } \frac{1}{2} < |x| < 1 \end{cases}$$

62. We have, $x = a \cos t + \frac{b}{2} \cos 2t$

and $y = a \sin t + \frac{b}{2} \sin 2t$

$$\Rightarrow \frac{dx}{dt} = -a \sin t - b \sin 2t$$

and $\frac{dy}{dt} = a \cos t + b \cos 2t$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{a \cos t + b \cos 2t}{-a \sin t - b \sin 2t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx}$$

$$= -\frac{d}{dt} \left(\frac{a \cos t + b \cos 2t}{a \sin t + b \sin 2t} \right) \cdot \frac{dt}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\left[\frac{a^2 + 2b^2 + 3ab \cos t}{(a \sin t + b \sin 2t)^3} \right]$$

$$\therefore \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow a^2 + 2b^2 + 3ab \cos t = 0$$

$$\Rightarrow \cos t = -\left(\frac{a^2 + 2b^2}{3ab} \right)$$

63. We have, $f(x) + f(y) + f(z) + f(x)f(y)f(z) = 14$,

$$\forall x, y, z \in R \quad \dots(i)$$

On putting $x = y = z = 0$, we get

$$3f(0) + \{f(0)\}^3 = 14$$

$$\Rightarrow \{f(0)\}^3 + 3f(0) - 14 = 0 \Rightarrow f(0) = 2$$

Now, on putting $y = z = x$ in Eq. (i), we get

$$3f(x) + \{f(x)\}^3 = 14, \forall x \in R$$

On differentiating w.r.t x , we get

$$3f'(x) + 3\{f(x)\}^2 f'(x) = 0, \forall x \in R$$

$$\Rightarrow [\{f(x)\}^2 + 1]f'(x) = 0, \forall x \in R$$

$$\Rightarrow f'(x) = 0, \forall x \in R$$

64. It is given that, $f(x - y)$, $f(x)f(y)$ and $f(x + y)$ are in AP

for all $x, y \in R$. Therefore,

$$2f(x)f(y) = f(x - y) + f(x + y), \forall x, y \in R \quad \dots(ii)$$

On putting $x = y = 0$, we get $2\{f(0)\}^2 = 2f(0)$

$$\Rightarrow f(0)\{f(0) - 1\} = 0$$

$$\Rightarrow f(0) = 1 \quad [\because f(0) \neq 0]$$

On putting $x = 0$ and $y = x$ in Eq. (i), we get

$$2f(0)f(x) = f(-x) + f(x), \forall x \in R$$

$$\Rightarrow 2f(x) = f(-x) + f(x), \forall x \in R$$

$$\Rightarrow f(-x) = f(x), \forall x \in R$$

$\Rightarrow f(x)$ is an even function.

$\Rightarrow f'(x)$ is an odd function.

$$\therefore f(-1) = f(1) \text{ and } f'(-3) = -f'(3)$$

$$\Rightarrow f(1) = f(-1) \text{ and } f'(3) + f'(-3) = 0$$

65. We have, $f''(x) - g''(x) = 0, \forall x$

$$\Rightarrow f'(x) - g'(x) = C, \forall x$$

$$\Rightarrow f'(1) - g'(1) = C$$

$$\Rightarrow 4 - 2 = C \Rightarrow C = 2$$

$$\therefore f'(x) - g'(x) = 2, \forall x$$

$$\Rightarrow f(x) - g(x) = 2x + C_1, \forall x \quad \dots(i)$$

$$\Rightarrow f(2) - g(2) = 4 + C_1$$

$$\Rightarrow 9 - 3 = 4 + C_1$$

$$\therefore C_1 = 2$$

On substituting the value of C_1 in Eq. (i), we get

$$f(x) - g(x) = 2x + 2, \forall x$$

$$\therefore f\left(\frac{3}{2}\right) - g\left(\frac{3}{2}\right) = 3 + 2 = 5$$

66. Since, $g(x)$ is the inverse function of $f(x)$, therefore

$$f(g(x)) = x.$$

$$\therefore f'(g(x)) \cdot g'(x) = 1$$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))}$$

$$\text{Given, } f'(x) = \frac{f^2(x) - 1}{f(x) + (f^2(x) - 1)e^{af(x)} \sin(2f(x) + 3)}$$

$$\therefore f'(g(x)) = \frac{x^2 - 1}{x + (x^2 - 1)e^{ax} \sin(2x + 3)} \quad [\because f(g(x)) = x]$$

$$\Rightarrow g'(x) = \frac{1}{f'(g(x))} = \frac{x + (x^2 - 1)e^{ax} \sin(2x + 3)}{x^2 - 1}$$

$$\text{or } h(x) = \frac{x + (x^2 - 1)e^{ax} \sin(2x + 3)}{x^2 - 1}$$

$$\Rightarrow h(x) = \frac{x}{x^2 - 1} + e^{ax} \sin(2x + 3)$$

$$\therefore h'(x) = -\frac{1}{(x + 1)^2} - \frac{2x}{(x^2 - 1)^2} + 2e^{ax} \cos(2x + 3) + ae^{ax} \sin(2x + 3)$$

67. (a) Not derivable at $x = 0$ check with $x = \sin \theta$ or direct differentiation.

$$(b) g'(x) = \frac{1}{\sqrt{1 - \left(\frac{2 \cdot 2^x}{1 + 2^{2x}} \right)^2}}$$

$$\times \frac{(1 + 4^x)(2^{x+1} \ln 2 - 2^{x+1} \cdot 4^x \cdot \ln 4)}{(1 + 4^x)^2}$$

$$= \frac{(1 + 2^{2x})^2}{\sqrt{(1 + 2^{2x})^2 - (2 \cdot 2^x)^2}} = \frac{1}{|1 - 2^{2x}|}$$

$$\Rightarrow \text{Not derivable at } x = 0.$$

68. Given, $y = \sec(\tan^{-1} x)$

$$\therefore \frac{dy}{dx} = \frac{\sec(\tan^{-1} x) \cdot \tan(\tan^{-1} x)}{1+x^2}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=1} = \frac{1}{\sqrt{2}} = \sin^{-1} \left(\sin \frac{1}{\sqrt{2}} \right)$$

69. Given, $y = 10^{10^x}$

$$\Rightarrow \frac{dy}{dx} = 10^{10^x} \ln 10 \cdot 10^x \ln 10$$

$$= y 10^x (\ln 10)^2$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = 10^x (\ln 10)^2$$

$$\therefore \lambda = (\ln 10)^2 = e^{\ln(\ln 10)^2}$$

70. Let $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$

$$\Rightarrow y = \sqrt{x + y}$$

$$y^2 = x + y \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

$$\text{Also, } y = \frac{x}{y} + 1 \Rightarrow \frac{dy}{dx} = \frac{y}{2x+y}$$

Make a quadratic in y to get explicit function to find the option (c).

$$71. \text{ Given, } f(x) = \begin{vmatrix} x^n & \sin x & -\cos x \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$$

$$\Rightarrow \left[\frac{d^n}{dx^n} (f(x)) \right] = \begin{vmatrix} n! & \sin(x+n\pi/2) & -\cos(x+n\pi/2) \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$$

$$\therefore \left[\frac{d^n}{dx^n} (f(x)) \right]_{x=0} = \begin{vmatrix} n! & \sin(n\pi/2) & -\cos(n\pi/2) \\ n! & \sin(n\pi/2) & \cos(n\pi/2) \\ a & a^2 & a^3 \end{vmatrix}$$

$$= \begin{vmatrix} n! & \sin(n\pi/2) & 0 \\ n! & \sin(n\pi/2) & 0 \\ a & a^2 & a^3 \end{vmatrix} = 0 \quad [n \text{ is odd}]$$

$$72. f(x) = \frac{\sqrt{(\sqrt{x-1})^2 + 1} - 2\sqrt{x-1}}{\sqrt{x-1} - 1} x$$

$$= \frac{|\sqrt{x-1}-1|}{\sqrt{x-1}-1} x = \begin{cases} -x, & \text{if } x \in [1, 2) \\ x, & \text{if } x \in (2, \infty) \end{cases}$$

$$73. \frac{d}{dx} \{f_n(x)\} = \frac{d}{dx} \{e^{f_{n-1}(x)}\}$$

$$= e^{f_{n-1}(x)} \frac{d}{dx} \{f_{n-1}(x)\} = f_n(x) \frac{d}{dx} \{f_{n-1}(x)\}$$

$$= f_n(x) \cdot \frac{d}{dx} \{e^{f_{n-2}(x)}\} = f_n(x) \cdot e^{f_{n-2}(x)} \frac{d}{dx} \{f_{n-2}(x)\}$$

$$= f_n(x) \cdot f_{n-1}(x) \frac{d}{dx} \{f_{n-2}(x)\}$$

$$\vdots$$

$$= f_n(x) \cdot f_{n-1}(x) \dots f_2(x) \frac{d}{dx} \{f_1(x)\}$$

$$= f_n(x) \cdot f_{n-1}(x) \dots f_2(x) \frac{d}{dx} \{e^{f_0(x)}\}$$

$$= f_n(x) \cdot f_{n-1}(x) \dots f_2(x) e^{f_0(x)} \frac{d}{dx} \{f_0(x)\}$$

$$\text{Use } e^{f_0(x)} = f_1(x) \text{ and } f_0(x) = x$$

74. Given, $x^3 - 2x^2y^2 + 5x + y - 5 = 0$

On differentiating w.r.t. x , we get

$$3x^2 - 4xy^2 - 4x^2y \frac{dy}{dx} + 5 + \frac{dy}{dx} = 0$$

$$\Rightarrow y' = \frac{dy}{dx} = \frac{3x^2 - 4xy^2 + 5}{4x^2y - 1}$$

$$\therefore y'(1) = \frac{3-4+5}{4-1} = \frac{4}{3}$$

$$\text{Also, } y'' = \frac{[(6x - 4y^2 - 8xyy')(4x^2y - 1) - (8xy + 4x^2y')](3x^2 - 4xy^2 + 5)}{(4x^2y - 1)^2}$$

$$\Rightarrow y''(1) = \frac{(6-4-8 \cdot \frac{4}{3})(4-1) - (8+4 \cdot \frac{4}{3})(3-4+5)}{(4-1)^2}$$

$$= -8 \frac{22}{27}$$

$$75. f(x) = x[x] = \begin{cases} -x, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ x, & 1 \leq x < 2 \\ 2x, & 2 \leq x < 3 \\ \vdots & \vdots \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 1, & 1 < x < 2 \\ 2, & 2 < x < 3 \\ \vdots & \vdots \end{cases}$$

$$\Rightarrow f'(x) = [x]$$

76. By putting $y = 0$ in the given equation, we have

$$f(x) = f(x) + f(0) - 1 \Rightarrow f(0) = 1$$

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x) + f(h) + 2hx - 1 - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(h) - 1}{h} + 2x$$

$$= \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} + 2x$$

$$= f'(0) + 2x = \frac{d}{dx} (xf'(0) + x^2)$$

$$f(x) = xf'(0) + x^2 + C$$

$$\text{But } f(0) = 1, \text{ so } C = 1$$

$$\text{Thus, } f(x) = x \sin \phi + x^2 + 1$$

$$= \left(x + \frac{\sin \phi}{2} \right)^2 + \frac{\cos^2 \phi}{4} + \frac{3}{4} > 0$$

Solutions (Q. Nos. 77-79)

$$\text{Here, put } g'(1) = a \text{ and } g''(2) = b \quad \dots(i)$$

$$\text{Then, } f(x) = x^2 + ax + b, f(1) = 1 + a + b$$

$$\Rightarrow f'(x) = 2x + a \text{ and } f''(x) = 2$$

$$\therefore g(x) = (1+a+b)x^2 + (2x+a)x + 2 = x^2(3+a+b) + ax + 2$$

$$\begin{aligned} \Rightarrow g'(x) &= 2x(3+a+b) + a \\ \text{and } g''(x) &= 2(3+a+b) \\ \text{Hence, } g'(1) &= 2(3+a+b) + a \quad \dots (ii) \\ g''(2) &= 2(3+a+b) \quad \dots (iii) \end{aligned}$$

From Eqs. (i), (ii) and (iii), we have

$$\begin{aligned} a &= 2(3+a+b) + a \\ \text{and } b &= 2(3+a+b) \end{aligned}$$

$$\Rightarrow 3+a+b=0$$

$$\text{and } b+2a+6=0$$

Hence, $b=0$ and $a=-3$.

$$\text{So, } f(x) = x^2 - 3x \text{ and } g(x) = -3x + 2.$$

$$\therefore f(3) = 0 \text{ and } g(0) = 0$$

$$\text{and } \sqrt{\frac{f(x)}{g(x)}} = \sqrt{\frac{x^2-3x}{-3x+2}} \text{ is defined, if } \frac{x^2-3x}{-3x+2} \geq 0$$

$$\Rightarrow \frac{x(x-3)}{\left(x-\frac{2}{3}\right)} \leq 0 \Rightarrow x \in (-\infty, 0] \cap \left(\frac{2}{3}, 3\right]$$

Solutions (Q. Nos. 80-82)

Suppose degree of $f(x) = n$, then degree of $f' = n-1$ and $\deg f'' \geq n-2$.

$$\text{So, } n = n-1 + n-2$$

$$\text{Hence, } n = 3.$$

$$\text{So, put } f(x) = ax^3 + bx^2 + cx + d. \quad [\text{where, } a \neq 0]$$

From $f(2x) = f'(x) \cdot f''(x)$, we have

$$\begin{aligned} 8ax^3 + 4bx^2 + 2cx + d &= (3ax^2 + 2bx + c)(6ax + 2b) \\ &= 18a^2x^3 + 18abx^2 + (6ac + 4b^2)x + 2bc \end{aligned}$$

On comparing the coefficients of terms, we have

$$18a^2 = 8a \Rightarrow a = \frac{4}{9},$$

$$18ab = 4b \Rightarrow b = 0,$$

$$2c = 6ac + 4b^2 \Rightarrow c = 0$$

$$\text{and } d = 2bc \Rightarrow d = 0$$

$$\Rightarrow f(x) = \frac{4x^3}{9}, \text{ which is clearly one-one and onto.}$$

$$\Rightarrow f(3) = 12$$

$$\text{Also, } \frac{4x^3}{9} = x \Rightarrow x = 0, x = \pm \frac{3}{2}$$

Hence, sum of roots of equation is zero.

Solutions (Q. Nos. 83-85)

$$g(x+y) = g(x) + g(y) + 3x^2y + 3xy^2 \quad \dots (i)$$

$$\Rightarrow g'(x+y) = g'(x) + 6yx + 3y^2$$

[differentiating w.r.t. x keeping y as constant]

Put $x=0$,

$$\Rightarrow g'(y) = g'(0) + 3y^2$$

$$\Rightarrow g'(y) = -4 + 3y^2$$

$$\Rightarrow g'(x) = -4 + 3x^2$$

$$\Rightarrow g(x) = -4x + x^3 + C$$

Now put $x=y=0$ in Eq. (i), we get

$$g(0) = g(0) + g(0) + 0$$

$$\Rightarrow g(0) = 0$$

$$\Rightarrow g(x) = x^3 - 4x$$

$$g(x) = 0 \Rightarrow x^3 - 4x = 0 \Rightarrow x = 0, 2, -2.$$

Hence, three roots.

$$\sqrt{g(x)} = \sqrt{x^3 - 4x} \text{ is defined, if } x^3 - 4x \geq 0 \text{ or}$$

$$x \in [-2, 0] \cup [2, \infty)$$

$$\text{Also, } g'(x) = 3x^2 - 4 \Rightarrow g'(1) = -1$$

86. A. $y = f(x)$

$$\therefore \frac{dy}{dx} = f'(x^3) \cdot 3x^2$$

$$\text{So, } \left[\frac{dy}{dx} \right]_{x=1} = f'(1) \cdot 3 = 9$$

B. $f(xy) = f(x) + f(y)$

$$f(1) = f(1) + f(1)$$

$$\therefore f(1) = 0$$

$$\text{Also, } f(1) = f(e) + f\left(\frac{1}{e}\right)$$

$$\therefore f(e) + f\left(\frac{1}{e}\right) = 0$$

C. $f''(x) = -f(x)$, $f'(x) = g(x)$

$$\therefore g'(x) = f''(x) - f(x)$$

$$h(x) = (f(x))^2 + (g(x))^2$$

$$\begin{aligned} \text{Now, } h'(x) &= 2f(x) \cdot f'(x) + 2g(x) \cdot g'(x) \\ &= 2f(x) \cdot g(x) + 2g(x) \cdot (-f(x)) = 0 \end{aligned}$$

$$\therefore h(x) = c, x \in R$$

$$\text{Hence, } h(10) = h(5) = 9$$

D. $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, $\frac{\pi}{2} < x < \pi$

$$\begin{aligned} \frac{dy}{dx} &= \frac{-\operatorname{cosec}^2 x}{1 + \cot^2 x} + \frac{-1}{1 + \tan^2 x} \cdot \sec^2 x \\ &= -1 - 1 = -2 \end{aligned}$$

87. A. $y = \cos^{-1}(\cos x)$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 - \cos^2 x}} \cdot (-\sin x) = \frac{\sin x}{|\sin x|}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=5} = -1$$

B. $y = f(x) = \ln |\tan x|$

$$\Rightarrow f'(x) = \left(\frac{1}{\tan x} \right) (\sec^2 x) \cdot \left(\frac{|\tan x|}{\tan x} \right)$$

$$\therefore f'\left(-\frac{\pi}{4}\right) = 1$$

$$\begin{aligned} \text{C. } \frac{d}{dx} \tan^{-1}\left(\frac{1+x}{1-x}\right) &= \frac{1}{1 + \left(\frac{1+x}{1-x}\right)^2} \cdot \frac{d}{dx} \left(\frac{1+x}{1-x}\right) \\ &= \frac{(1-x)^2}{2(1+x^2)} \cdot \frac{2}{(1-x)^2} = \frac{1}{1+x^2} \end{aligned}$$

At $x = -1$,

$$\frac{d}{dx} \tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{1}{2}$$

$$D. \frac{d}{dx} \frac{\ln|x|}{x} = \frac{x \cdot \frac{1}{x} - \ln|x|}{x^2} = \frac{1 - \ln|x|}{x^2}$$

At $x = -1$,

$$\frac{d}{dx} \frac{\ln|x|}{x} = 1$$

$$88. A. \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{12t^2 - 6t - 18}{5t^4 - 15t^2 - 20}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{t=1} = \frac{12 - 6 - 18}{5 - 15 - 20} = \frac{2}{5}$$

$$\Rightarrow -5 \left[\frac{dy}{dx} \right]_{t=1} = -2$$

B. Let us take, $P(x) = a(x-2)^4 + b(x-2)^3 + c(x-2)^2 + d(x-2) + e$

$$-1 = P(2) = e$$

$$0 = P'(2) = d$$

$$2 = P''(2) = 2c \Rightarrow c = 1$$

$$-12 = P'''(2) = 6b \Rightarrow b = -2$$

$$24 = P''''(2) = 24a \Rightarrow a = 1$$

Thus, $P''(x) = 12(x-2)^2 - 12(x-2) + 2$

$$\Rightarrow P''(3) = 12 - 12(1) + 2 = 2$$

C. Here, $\sqrt{1+y^4} = \sqrt{\left(1 + \frac{1}{x^4}\right)} = \frac{\sqrt{1+x^4}}{x^2} \left[\because y = \frac{1}{x} \right]$

$$\Rightarrow \frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} = \frac{1}{x^2} \quad \dots(i)$$

But $y = \frac{1}{x}$

$$\therefore \frac{dy}{dx} = -\frac{1}{x^2} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\begin{aligned} \frac{\sqrt{1+y^4}}{\sqrt{1+x^4}} &= -\frac{dy}{dx} \\ \frac{dy}{\sqrt{1+y^4}} &= -\frac{dx}{\sqrt{1+x^4}} \\ \Rightarrow \frac{dy}{\sqrt{1+y^4}} &= -1 \end{aligned}$$

D. Obviously, $f(x)$ is a linear function.

Also, from $f'(0) = p$ and $f(0) = q$, $f(x) = px + q$.

$$\Rightarrow f''(0) = 0$$

$$89. f(x) = \sqrt{x+2} \sqrt{2x-4} + \sqrt{x-2} \sqrt{2x-4}$$

$$\begin{aligned} \therefore f(x) &= \sqrt{(\sqrt{x-2} + \sqrt{2})^2} + \sqrt{(\sqrt{x-2} - \sqrt{2})^2} \\ &= |\sqrt{x-2} + \sqrt{2}| + |\sqrt{x-2} - \sqrt{2}| \end{aligned}$$

For $\sqrt{x-2}$ to exist $x \geq 2$.

Also, $\sqrt{x-2} + \sqrt{2} > 0$ only, if $x \geq 4$ < 0 only, if $x < 4$.

$\therefore$ Now, $f(x)$ becomes

$$\begin{aligned} f(x) &= \sqrt{x-2} + \sqrt{2} - \sqrt{x-2} + \sqrt{2} \text{ for } 2 \leq x < 4 \\ &= \sqrt{x-2} + \sqrt{2} + \sqrt{x-2} - \sqrt{2} \text{ for } x \geq 4 \end{aligned}$$

$$\therefore f(x) = 2\sqrt{2}, \text{ for } 2 \leq x < 4$$

$$= 2\sqrt{x-2} \text{ for } 4 \leq x < \infty$$

$\therefore f$ is continuous $[2, 4) \cup [4, \infty)$.

[verify]

$$\therefore f'(x) = 0, 2 \leq x < 4$$

$$= \frac{1}{\sqrt{x-2}}, 4 \leq x < \infty$$

$$\Rightarrow f'(102^+) = \frac{1}{\sqrt{102-2}} = \frac{1}{10}$$

$$\therefore 10f'(102^+) = 1$$

$$90. f'(x) = (kx + e^x)h'(x) + h(x)(k + e^x)$$

$$f'(0) = h'(0) + h(0)(k + 1)$$

$$18 = -2 + 5(k + 1)$$

$$\Rightarrow k = 3$$

$$91. f(x) = x^3 + e^{x/2} \text{ at } x = 0, f(0) = 1$$

$$g(f(x)) = x$$

$$\Rightarrow g'(f(x)) \cdot f'(x) = 1$$

$$\Rightarrow g'(1) \cdot f'(0) = 1$$

Here, $f'(x) = e^{x/2} + \frac{1}{2}e^{x/2}$

$$\Rightarrow f'(0) = \frac{1}{2}$$

$$\therefore g'(1) = 2$$

$$92. \frac{1}{5} \cdot y^{\frac{1}{5}-1} \cdot y_1 - \frac{1}{5} \cdot y^{\frac{1}{5}-1} \cdot y_1 = 2$$

$$\Rightarrow \frac{1}{5} \cdot \frac{y^{\frac{1}{5}} \cdot y_1}{y} - \frac{1}{5} \cdot \frac{y^{\frac{1}{5}} \cdot y_1}{y} = 2$$

$$\Rightarrow \frac{y^{\frac{1}{5}} - y^{\frac{1}{5}}}{y} = \frac{10y}{y_1}$$

$$\Rightarrow \left(y^{\frac{1}{5}} - y^{\frac{1}{5}} \right)^2 = \frac{100y^2}{y_1^2}$$

$$\Rightarrow \left(y^{\frac{1}{5}} - y^{\frac{1}{5}} \right)^2 - 4 = \frac{100y^2}{y_1^2}$$

$$\Rightarrow 4(x^2 - 1) = \frac{100y^2}{y_1^2}$$

$$\Rightarrow y_1^2(x^2 - 1) = 25y^2$$

$$\Rightarrow (x^2 - 1)2y_1y_2 + y_1^2 2x = 25 \cdot 2 \cdot yy_1$$

$$(x^2 - 1)y_2 + xy_1 = 25y \quad [\text{dividing by } 2y_1]$$

$$\Rightarrow (x^2 - 1) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = 25y$$

$$\Rightarrow k = 25 \Rightarrow \frac{k}{5} = 5$$

$$93. y = \frac{a + bx^{3/2}}{x^{5/4}}$$

$$\Rightarrow y' = \frac{\frac{3}{2}bx^{1/2}x^{5/4} - \frac{5}{4}x^{1/4}(a + bx^{3/2})}{x^{5/2}}$$

According to the question,

$$0 = \frac{\frac{3}{2}b5^{1/2}5^{5/4} - \frac{5}{4}5^{1/4}(a + b5^{3/2})}{5^{5/2}}$$

$$\Rightarrow \frac{3b}{2}5^{7/4} - a \frac{5^{5/4}}{4} - 5b \frac{5^{7/4}}{4} = 0$$

$$\Rightarrow b5^{7/4} = a5^{5/4}$$

$$\Rightarrow b\sqrt{5} = a$$

$$\Rightarrow a : b = \sqrt{5} : 1$$

$$\therefore \frac{a^2}{b^2} = 5$$

$$94. y = \frac{x^4 - (x^2 + 2x + 1)}{x^2 - x - 1} = x^2 + x + 1$$

$$\therefore \frac{dy}{dx} = 2x + 1 = ax + b$$

Hence, $a = 2$ and $b = 1$.

$$\therefore a + b = 3$$

95. We have, $f(5 - x) = -f(5 + x)$

$$\Rightarrow -f'(5 - x) = -f'(5 + x)$$

$$\Rightarrow f'(5 - 2) = f'(5 + 2) \Rightarrow f'(3) = f'(7) = 3$$

$$96. \frac{dx}{dt} = -\frac{3}{t^4} - \frac{2}{t^3} = -\left(\frac{3+2t}{t^4}\right)$$

$$\frac{dy}{dt} = -\left(\frac{3}{t^3} + \frac{2}{t^2}\right) = -\left(\frac{3+2t}{t^3}\right) \Rightarrow \frac{dy}{dx} = t$$

$$\Rightarrow \frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^3 = t - \left(\frac{1+t}{t^3}\right) \cdot t^3 = -1$$

$$\therefore \left| \frac{dy}{dx} - x\left(\frac{dy}{dx}\right)^3 \right| = 1$$

Entrances Gallery

1. We have,

$$f'(x) = \sin \pi x + \pi x \cos \pi x = 0$$

$$\Rightarrow \tan \pi x = -\pi x$$

$$\Rightarrow \pi x \in \left(\frac{2n+1}{2}\pi, (n+1)\pi\right)$$

$$\Rightarrow x \in \left(n + \frac{1}{2}, n+1\right) \in (n, n+1)$$

$$2. \text{ Let } \tan^{-1}\left(\frac{\sin \theta}{\sqrt{\cos 2\theta}}\right) = \phi$$

$$\Rightarrow \tan \phi = \frac{\sin \theta}{\sqrt{\cos 2\theta}}$$

$$\therefore \sin \phi = \frac{\sin \theta}{\sqrt{\sin^2 \theta + \cos 2\theta}} = \tan \theta$$

$$\therefore f(\theta) = \sin \phi = \tan \theta$$

$$\Rightarrow \frac{df(\theta)}{d(\tan \theta)} = 1$$

$$3. \text{ Let } p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$\Rightarrow p'(x) = 4ax^3 + 3bx^2 + 2cx + d$$

$$\therefore p'(1) = 4a + 3b + 2c + d = 0 \quad \dots(i)$$

$$\text{and } p'(2) = 32a + 12b + 4c + d = 0 \quad \dots(ii)$$

$$\therefore \lim_{x \rightarrow 0} \left[1 + \frac{p(x)}{x^2}\right] = 2 \quad [\text{given}]$$

$$\therefore \lim_{x \rightarrow 0} \frac{ax^4 + bx^3 + (c+1)x^2 + dx + e}{x^2} = 2$$

$$\Rightarrow c + 1 = 2, d = 0, e = 0$$

$$\Rightarrow c = 1$$

From Eqs. (i) and (ii), we get

$$4a + 3b = -2 \text{ and } 32a + 12b = -4$$

$$\Rightarrow a = \frac{1}{4} \text{ and } b = -1$$

$$\therefore p(x) = \frac{x^4}{4} - x^3 + x^2 \Rightarrow p(2) = \frac{16}{4} - 8 + 4$$

$$\therefore p(2) = 0$$

4. Here, $f'(x) = \frac{1}{x} + \sqrt{1 + \sin x}$, $x > 0$ but $f(x)$ is not differentiable in $(0, \infty)$ as $\sin x$ may be -1 and then

$$f''(x) = -\frac{1}{x^2} + \frac{\cos x}{2\sqrt{1 + \sin x}} \text{ will not exist.}$$

So, $f'(x)$ is continuous for all $x \in (0, \infty)$ but $f'(x)$ is not differentiable on $(0, \infty)$.

$\therefore$ Option (b) is true.

Also, $f'(x) \leq 3$, if $x > 1$

and $f(x) > 3$, if $x > e^3$

$$\text{Let } \alpha = e^3$$

Hence, option (c) is true.

(d) is not possible as $f(x) \rightarrow \infty$ when $x \rightarrow \infty$.

5. Here, g is the inverse of $f(x)$.

$$\Rightarrow fog(x) = x$$

On differentiating w.r.t. x , we get

$$f'\{g(x)\} \times g'(x) = 1$$

$$\Rightarrow g'(x) = \frac{1}{f'\{g(x)\}} = \frac{1}{\frac{1}{1 + \{g(x)\}^5}} \left[\because f'(x) = \frac{1}{1 + x^5} \right]$$

$$\therefore g'(x) = 1 + \{g(x)\}^5$$

$$6. \text{ Let } y = \frac{x^2 - x}{x^2 + 2x}$$

$$\Rightarrow x = \frac{2y + 1}{-y + 1}; x \neq 0$$

$$\Rightarrow f^{-1}(x) = \frac{2x + 1}{-x + 1}$$

$$\therefore \frac{d}{dx} \{f^{-1}(x)\} = \frac{(-x + 1) \cdot 2 - (2x + 1)(-1)}{(-x + 1)^2} = \frac{3}{(-x + 1)^2}$$

$$7. \text{ Since, } \frac{dx}{dt} = \frac{1}{2\sqrt{a^{\sin^{-1}t}}} \left(a^{\sin^{-1}t} \times \frac{1}{\sqrt{1-t^2}} \right)$$

$$\text{and } \frac{dy}{dt} = \frac{1}{2\sqrt{a^{\cos^{-1}t}}} \left(a^{\cos^{-1}t} \times \frac{-1}{\sqrt{1-t^2}} \right)$$

$$\therefore \frac{dy}{dx} = -\frac{\sqrt{a^{\sin^{-1}t}}}{\sqrt{a^{\cos^{-1}t}}} \left(\frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} \times 1 \right) = -\frac{a^{\sqrt{\cos^{-1}t}}}{a^{\sqrt{\sin^{-1}t}}}$$

$$\therefore 1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{a^{\cos^{-1}t}}{a^{\sin^{-1}t}} = 1 + \frac{y^2}{x^2} = \frac{x^2 + y^2}{x^2}$$

8. We have, $f(x) = \sin(\sin x)$

$$\Rightarrow f'(x) = \cos x \cdot \cos(\sin x)$$

$$\Rightarrow f''(x) = -\sin x \cdot \cos(\sin x) - \cos^2 x \cdot \sin(\sin x)$$

$$\text{Now, } g(x) = -[f''(x) + f'(x) \cdot \tan x]$$

$$= \sin x \cdot \cos(\sin x) + \cos^2 x \cdot \sin(\sin x) \\ - \tan x \cdot \cos x \cdot \cos(\sin x)$$

$$= \sin x \cdot \cos(\sin x) + \cos^2 x \cdot \sin(\sin x)$$

$$- \sin x \cdot \cos(\sin x)$$

$$= \cos^2 x \cdot \sin(\sin x)$$

9. Given, $y = \sec(\tan^{-1} x)$

$$\text{Let } \tan^{-1} x = \theta \Rightarrow x = \tan \theta$$

$$\therefore y = \sec \theta = \sqrt{1+x^2}$$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=1} = \frac{1}{\sqrt{2}}$$

10. Since, $\frac{dx}{dy} = \left(\frac{dy}{dx} \right)^{-1}$

$$\therefore \frac{d^2x}{dy^2} = - \left(\frac{dy}{dx} \right)^{-2} \cdot \frac{d^2y}{dx^2} \cdot \frac{dx}{dy} = - \left(\frac{d^2y}{dx^2} \right) \left(\frac{dy}{dx} \right)^{-3}$$

11. Given $x^{2x} - 2x^x \cot y - 1 = 0$... (i)

Now, at $x = 1$,

$$1 - 2 \cot y - 1 = 0$$

$$\Rightarrow \cot y = 0$$

$$\Rightarrow y = \frac{\pi}{2}$$

On differentiating Eq. (i) w.r.t. x , we get

$$2x^{2x}(1 + \log x) - 2[x^x(-\operatorname{cosec}^2 y) \frac{dy}{dx} \\ + \cot y \cdot x^x(1 + \log x)] = 0$$

$$\text{At } x = \left(1, \frac{\pi}{2}\right),$$

$$2(1 + \log 1) - 2 \left\{ 1(-1) \left(\frac{dy}{dx} \right)_{\left(1, \frac{\pi}{2}\right)} + 0 \right\} = 0$$

$$\Rightarrow 2 + 2 \left[\frac{dy}{dx} \right]_{\left(1, \frac{\pi}{2}\right)} = 0$$

$$\therefore \left[\frac{dy}{dx} \right]_{\left(1, \frac{\pi}{2}\right)} = -1$$

12. We have, $f(x) = x|x|$ and $g(x) = \sin x$

$$gof(x) = \sin(x|x|) = \begin{cases} -\sin x^2, & x < 0 \\ \sin x^2, & x > 0 \end{cases}$$

$$\therefore (gof)'(x) = \begin{cases} -2x \cos x^2, & x < 0 \\ 2x \cos x^2, & x > 0 \end{cases}$$

Clearly, $L(gof)'(0) = 0 = R(gof)'(0)$

Hence, gof is differentiable at $x = 0$ and also its derivative is continuous at $x = 0$.

$$\text{Now, } (gof)''(x) = \begin{cases} -2 \cos x^2 + 4x^2 \sin x^2, & x < 0 \\ 2 \cos x^2 - 4x^2 \sin x^2, & x \geq 0 \end{cases}$$

$$\therefore L(gof)''(0) = -2 \text{ and } R(gof)''(0) = 2$$

$$\therefore L(gof)''(0) \neq R(gof)''(0)$$

Hence, $gof(x)$ is not twice differentiable at $x = 0$.

13. Given, $x^m y^n = (x+y)^{m+n}$... (i)

Taking log on both sides of Eq. (i), we get

$$m \log x + n \log y = (m+n) \log(x+y)$$

$$\Rightarrow \frac{m}{x} + \frac{n}{y} \frac{dy}{dx} = \frac{(m+n)}{(x+y)} \left(1 + \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx} \left\{ \frac{n}{y} - \frac{(m+n)}{(x+y)} \right\} = \frac{m+n}{x+y} - \frac{m}{x}$$

$$\Rightarrow \frac{dy}{dx} \left\{ \frac{nx + ny - my - ny}{y(x+y)} \right\} = \frac{mx + nx - mx - ny}{x(x+y)}$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{nx - my}{y} \right) = \frac{(nx - my)}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

14. We have, $f(x) = x^n \Rightarrow f(1) = 1$

$$f'(x) = nx^{n-1} \Rightarrow f'(1) = n$$

$$f''(x) = n(n-1)x^{n-2} \Rightarrow f''(1) = n(n-1)$$

$$\vdots \quad \vdots$$

$$f^n(x) = n(n-1)(n-2) \dots 2 \cdot 1$$

$$\Rightarrow f^n(1) = n(n-1)(n-2) \dots 2 \cdot 1$$

$$\text{Now, } f(1) - \frac{f'(1)}{1!} + \frac{f''(1)}{2!} - \frac{f'''(1)}{3!} + \dots + \frac{(-1)^n f^n(1)}{n!} \\ = 1 - \frac{n}{1!} + \frac{n(n-1)}{2!} - \frac{n(n-1)(n-2)}{3!} + \dots \\ + \frac{(-1)^n n(n-1)(n-2) \dots 2 \cdot 1}{n!} \\ = (1-1)^n = 0$$

15. Let $f(x) = Ax^2 + Bx + C$

$$\therefore f(1) = A + B + C$$

$$\text{and } f(-1) = A - B + C$$

$$\therefore f(1) = f(-1) \quad [\text{given}]$$

$$\Rightarrow A + B + C = A - B + C$$

$$\Rightarrow 2B = 0 \Rightarrow B = 0$$

$$\therefore f(x) = Ax^2 + C$$

$$\Rightarrow f'(x) = 2Ax$$

$$\therefore f'(a) = 2Aa$$

$$f'(b) = 2Ab$$

$$\text{and } f'(c) = 2Ac$$

Also, a, b, c are in AP.

$$\therefore 2Aa, 2Ab, 2Ac \text{ are in AP.}$$

$$\Rightarrow f'(a), f'(b), f'(c) \text{ are also in AP.}$$

16. Given, $y = (x + \sqrt{1+x^2})^n$... (i)

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = n(x + \sqrt{1+x^2})^{n-1} \cdot \left(1 + \frac{2x}{2\sqrt{1+x^2}}\right)$$

$$= \frac{n(x + \sqrt{1+x^2})^n}{\sqrt{1+x^2}}$$

$$\Rightarrow (1+x^2) \left(\frac{dy}{dx}\right)^2 = n^2 y^2 \quad [\text{from Eq. (i)}]$$

Again, differentiating w.r.t. x , we get

$$(1+x^2) \cdot 2 \frac{dy}{dx} \cdot \frac{d^2y}{dx^2} + 2x \left(\frac{dy}{dx}\right)^2 = n^2 2y \frac{dy}{dx}$$

$$\Rightarrow (1+x^2) \frac{d^2y}{dx^2} + x \frac{dy}{dx} = n^2 y$$

17. Given, $\sin y = x \sin(a+y)$

$$\Rightarrow x = \frac{\sin y}{\sin(a+y)}$$

On differentiating w.r.t. y , we get

$$\frac{dx}{dy} = \frac{\sin(a+y) \cos y - \sin y \cos(a+y)}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dx}{dy} = \frac{\sin a}{\sin^2(a+y)} \Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$$

18. Given, $x^y = e^{x-y}$

Taking log on both sides, we get

$$y \log x = (x-y) \log e$$

$$\Rightarrow y = \frac{x}{1 + \log x}$$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{(1 + \log x) - x \cdot \frac{1}{x}}{(1 + \log x)^2} = \frac{\log x}{(1 + \log x)^2}$$

19. Given, $x \sin(a+y) + \sin a \cos(a+y) = 0$... (i)

On differentiating w.r.t. x , we get

$$\frac{d}{dx} [x \sin(a+y)] + \frac{d}{dx} [\sin a \cos(a+y)] = 0$$

$$\Rightarrow \left[x \frac{d}{dx} \{\sin(a+y)\} + \sin(a+y) \frac{d}{dx} (x) \right]$$

$$+ \sin a \frac{d}{dx} \cos(a+y) = 0$$

[using product rule and chain rule]

$$\Rightarrow \left[x \cos(a+y) \frac{d}{dx} (a+y) + \sin(a+y) \right]$$

$$+ \sin a \left[-\sin(a+y) \frac{d}{dx} (a+y) \right] = 0$$

$$\Rightarrow \left[x \cos(a+y) \left(0 + \frac{dy}{dx}\right) + \sin(a+y) \right]$$

$$- \sin a \sin(a+y) \left(0 + \frac{dy}{dx}\right) = 0$$

$$\Rightarrow x \cos(a+y) \frac{dy}{dx} + \sin(a+y) - \sin a \sin(a+y) \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} [x \cos(a+y) - \sin a \sin(a+y)] = -\sin(a+y)$$

$$\left[\text{from Eq. (i), putting } x = -\sin a \frac{\cos(a+y)}{\sin(a+y)} \right]$$

$$\Rightarrow \frac{dy}{dx} \left[-\sin a \frac{\cos^2(a+y)}{\sin(a+y)} - \sin a \sin(a+y) \right]$$

$$= -\sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} \left[\frac{\sin a \cos^2(a+y) + \sin a \sin^2(a+y)}{\sin(a+y)} \right] = \sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \sin(a+y) \left[\frac{\sin(a+y)}{\sin a \{\cos^2(a+y) + \sin^2(a+y)\}} \right]$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a} \quad [\because \sin^2 \theta + \cos^2 \theta = 1]$$

20. Given, $y = |\sin x|^x$

In the neighbourhood of $-\frac{\pi}{6}$, $|x|$ and $|\sin x|$ both are negative.

i.e. $y = (-\sin x)^{-x}$

Taking log on both sides, we get

$$\log y = -x \log(-\sin x)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = (-x) \frac{1}{-\sin x} (-\cos x) + \log(-\sin x) (-1)$$

$$= -x \cot x - \log(-\sin x)$$

$$= -[x \cot x + \log(-\sin x)]$$

$$\Rightarrow \frac{dy}{dx} = -y [x \cot x + \log(-\sin x)]$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=-\pi/6} = (2)^{-\pi/6} \cdot \frac{[6 \log 2 - \sqrt{3}\pi]}{6}$$

21. Given, $y = (1+x)(1+x^2)(1+x^4)$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = (1+x)(1+x^2) \frac{d}{dx} (1+x^4) + (1+x)(1+x^4) \frac{d}{dx} (1+x^2)$$

$$+ (1+x)(1+x^2)(4x^3) + (1+x)(1+x^4)(2x)$$

$$+ (1+x^2)(1+x^4)(1)$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=1} = (1+1)(1+1^2)(4 \times 1^3)$$

$$+ (1+1)(1+1^4)(2 \times 1) + (1+1^2)(1+1^4)$$

$$= 2 \times 2 \times 4 + 2 \times 2 \times 2 + 2 \times 2$$

$$= 16 + 8 + 4 = 28$$

22. Given, $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + \frac{x}{1} + 1$

On differentiating both sides w.r.t. x , we get

$$f'(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0$$

$$\Rightarrow f'(x) = x^{99} + x^{98} + \dots + x + 1$$

On putting $x = 0$, we get

$$f'(0) = 0 + 0 + \dots + 0 + 1$$

$$\Rightarrow f'(0) = 1$$

23. Given, $y = f(x^2 + 2)$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = f'(x^2 + 2) \times 2x$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=1} = f'(1^2 + 2) \times 2 = f'(3) \times 2$$

$$= 5 \times 2 = 10 \quad [\because f'(3) = 5, \text{ given}]$$

24. Given, $f(x) = x^2 + bx + 7$

On differentiating both sides w.r.t. x , we get

$$f'(x) = 2x + b$$

Also, $f'(5) = 2f'\left(\frac{7}{2}\right)$

$$\therefore 2(5) + b = 2\left(2 \times \frac{7}{2} + b\right)$$

$$\Rightarrow 10 + b = 14 + 2b$$

$$\therefore b = -4$$

25. Given, $f(x) = \frac{x}{1+x}$ and $g(x) = f[f(x)]$

$$\therefore g(x) = f\left(\frac{x}{x+1}\right) + \frac{\frac{x}{x+1}}{1 + \frac{x}{x+1}}$$

$$\Rightarrow g(x) = \frac{x}{2x+1}$$

On differentiating both sides w.r.t. x , we get

$$g'(x) = \frac{(2x+1)1 - x(2)}{(2x+1)^2} = \frac{1}{(2x+1)^2}$$

26. The derivative of $\sqrt{2}x + 2\sqrt{x} - \frac{1}{\sqrt{x}}$

$$= \sqrt{2} + 2 \cdot \frac{1}{2} x^{\frac{1}{2}-1} + \frac{1}{2} x^{-\frac{1}{2}-1}$$

$$= \sqrt{2} + x^{-\frac{1}{2}} + \frac{1}{2} x^{-\frac{3}{2}}$$

$$= \sqrt{2} + \frac{1}{\sqrt{x}} + \frac{1}{2x\sqrt{x}} = \sqrt{2} + \frac{1}{\sqrt{x}} \left(1 + \frac{1}{2x}\right)$$

27. Given, $y = (\tan^{-1} x)^2$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 2(\tan^{-1} x) \times \frac{d}{dx}(\tan^{-1} x)$$

$$\Rightarrow \frac{dy}{dx} = \frac{2 \tan^{-1} x}{1+x^2}$$

$$\Rightarrow (1+x^2) \frac{dy}{dx} = 2 \tan^{-1} x$$

Again, differentiating both sides w.r.t. x , we get

$$(1+x^2) \frac{d}{dx} \left(\frac{dy}{dx} \right) + \frac{dy}{dx} \frac{d}{dx} (1+x^2) = 2 \frac{d}{dx} (\tan^{-1} x)$$

$$\Rightarrow y_2(1+x^2) + 2xy_1 = \frac{2}{1+x^2}$$

$$\Rightarrow y_2(1+x^2)^2 + 2x(1+x^2)y_1 = 2$$

28. Given, $x = \sin t$ and $y = \tan t$

On differentiating both sides w.r.t. t , we get

$$\frac{dx}{dt} = \cos t \text{ and } \frac{dy}{dt} = \sec^2 t$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\sec^2 t}{\cos t} = \frac{1}{\cos^3 t}$$

29. Given, $x = a \cos^3 \theta$ and $y = a \sin^3 \theta$

On differentiating both sides w.r.t. θ , we get

$$\frac{dx}{d\theta} = 3a \cos^2 \theta (-\sin \theta)$$

and $\frac{dy}{d\theta} = 3a \sin^2 \theta (\cos \theta)$

Now, $\frac{d\theta}{dx} = \frac{3a \sin^2 \theta (\cos \theta)}{3a \cos^2 \theta (-\sin \theta)}$

$$\Rightarrow \frac{dy}{dx} = -\tan \theta$$

$$\therefore 1 + \left(\frac{dy}{dx} \right)^2 = 1 + (-\tan \theta)^2$$

$$= 1 + \tan^2 \theta = \sec^2 \theta$$

30. Given, $y = \sin^{-1}(2x\sqrt{1-x^2})$

Put $x = \sin \theta$

$$\therefore y = \sin^{-1}(2 \sin \theta \sqrt{1-\sin^2 \theta})$$

$$= \sin^{-1}(2 \sin \theta \cos \theta) = \sin^{-1}(\sin 2\theta) = 2\theta$$

$$\Rightarrow y = 2 \sin^{-1} x$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{2}{\sqrt{1-x^2}}$$

31. Given, $f'(0) = 1$ and $f''(0)$ does not exist.

Also, given $g(x) = x f'(x)$

$$\therefore g'(x) = x f''(x) + f'(x)$$

On putting $x = 0$, we get

$$g'(0) = 0 f''(0) + f'(0) = 0 + 1 = 1$$

32. Given, $y = \tan^{-1} \left(\frac{\sqrt{1+a^2x^2}-1}{ax} \right)$

Put $ax = \tan \theta$

$$\therefore y = \tan^{-1} \left(\frac{\sqrt{1+\tan^2 \theta}-1}{\tan \theta} \right)$$

$$= \tan^{-1} \left(\frac{\sec \theta - 1}{\tan \theta} \right) = \tan^{-1} \left(\frac{1-\cos \theta}{\sin \theta} \right)$$

$$= \tan^{-1} \left(\frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} \right)$$

$$= \tan^{-1}\left(\tan \frac{\theta}{2}\right) = \frac{\theta}{2} = \frac{1}{2} \tan^{-1} ax$$

$$\Rightarrow y = \frac{1}{2} \tan^{-1} ax$$

On differentiating w.r.t. x , we get

$$y' = \frac{a}{2(1+a^2x^2)} \quad \dots(i)$$

Again, differentiating both sides, we get

$$y'' = -\frac{1}{2} \cdot \frac{(a^3 2x)}{(1+a^2x^2)^2}$$

$$\Rightarrow (1+a^2x^2)y'' = -\frac{a^3x}{1+a^2x^2} \\ = -a^2x(2y') \quad [\text{from Eq. (i)}]$$

$$\Rightarrow (1+a^2x^2)y'' + 2a^2xy' = 0$$

$$33. \text{ We have, } y = \frac{x}{x^2-1}$$

$$\Rightarrow y = \frac{2x}{2(x+1)(x-1)}$$

$$\Rightarrow y = \frac{1}{2} \left[\frac{x+1+x-1}{(x+1)(x-1)} \right]$$

$$\Rightarrow y = \frac{1}{2} \left[\frac{1}{x-1} + \frac{1}{x+1} \right]$$

On taking derivative both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{2} \left[\frac{-1}{(x-1)^2} - \frac{1}{(x+1)^2} \right]$$

Again, taking derivative both sides w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \frac{1}{2} \left[\frac{2}{(x-1)^3} + \frac{2}{(x+1)^3} \right]$$

Again, taking derivative both sides w.r.t. x , we get

$$\frac{d^3y}{dx^3} = \left[\frac{-6}{(x-1)^4} - \frac{6}{(x+1)^4} \right]$$

Hence, n th derivative of y w.r.t. x can be written as

$$\frac{d^ny}{dx^n} = \frac{(-1)^n \cdot n!}{2} \left[\frac{1}{(x-1)^{n+1}} + \frac{1}{(x+1)^{n+1}} \right]$$

$$34. \text{ We have, } \sin(x+y) = \log(x+y)$$

On differentiating w.r.t. x , we get

$$\cos(x+y) \left(1 + \frac{dy}{dx} \right) = \frac{1}{x+y} \left(1 + \frac{dy}{dx} \right)$$

$$\Rightarrow \left[\cos(x+y) - \frac{1}{x+y} \right] \left(1 + \frac{dy}{dx} \right) = 0$$

$$\therefore \frac{dy}{dx} = -1$$

$$35. \text{ Given, } y = \tan^{-1} \left(\frac{\frac{a}{b} - \tan x}{1 + \frac{a}{b} \tan x} \right)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{a}{b} \right) - \tan^{-1}(\tan x)$$

$$\Rightarrow y = \tan^{-1} \left(\frac{a}{b} \right) - x$$

$$\therefore \frac{dy}{dx} = 0 - 1 = -1$$

$$36. \text{ We have, } x = e^{y+e^y+e^{e^y}+\dots}$$

$$\Rightarrow x = e^{y+x}$$

Taking log on both sides, we get

$$\log x = (x+y) \log e \\ \Rightarrow \frac{1}{x} \cdot \frac{dy}{dx} + 1 = \frac{dy}{dx} = \frac{1-x}{x}$$

$$37. \text{ We have, } y = (1-x)(2-x) \dots (n-x)$$

Taking log on both sides, we get

$$\log y = \log(1-x) + \log(2-x) + \dots + \log(n-x) \\ \Rightarrow \frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{(1-x)}(-1) + \frac{1}{(2-x)}(-1) + \dots + \frac{1}{(n-x)}(-1) \\ \Rightarrow \frac{dy}{dx} = y \left[\frac{(2-x)(3-x) \dots (n-x)(-1)}{y} + \dots \right] \\ \therefore \left[\frac{dy}{dx} \right]_{x=1} = 1 \cdot 2 \dots (n-1)(-1) = (-1)(n-1)!$$

$$38. \text{ Given, } \Delta_1 = \begin{vmatrix} x & a & b \\ b & x & a \\ a & b & x \end{vmatrix} \text{ and } \Delta_2 = \begin{vmatrix} x & b \\ a & x \end{vmatrix}$$

$$\therefore \frac{d}{dx}(\Delta_1) = \begin{vmatrix} 1 & 0 & 0 \\ b & x & a \\ a & b & x \end{vmatrix} + \begin{vmatrix} x & a & b \\ 0 & 1 & 0 \\ a & b & x \end{vmatrix} + \begin{vmatrix} x & a & b \\ b & x & a \\ 0 & 0 & 1 \end{vmatrix} \\ = \begin{vmatrix} x & a \\ b & x \end{vmatrix} + \begin{vmatrix} x & b \\ a & x \end{vmatrix} + \begin{vmatrix} x & a \\ b & x \end{vmatrix} = 3\Delta_2$$

$$39. \text{ Let } g(x) = f(x) - x^2$$

$\Rightarrow g(x)$ has at least 3 real roots which are

$$x = 1, 2, 3 \quad [\text{by mean value theorem}]$$

$\Rightarrow g'(x)$ has at least 2 real roots in $x \in (1, 3)$

$\Rightarrow g''(x)$ has at least 1 real root in $x \in (1, 3)$

$\therefore f''(x) = 2$ for at least one $x \in (1, 3)$.

$$40. \text{ Given, } y = \cos(3\cos^{-1}x) \quad \dots(i)$$

Also, we know that,

$$3\cos^{-1}x = \cos^{-1}(4x^3 - 3x) \quad \dots(ii)$$

On substituting Eq. (ii) in Eq. (i), we get

$$y = \cos[\cos^{-1}(4x^3 - 3x)]$$

$$\Rightarrow y = 4x^3 - 3x$$

$$\therefore \frac{dy}{dx} = 12x^2 - 3 \Rightarrow \frac{d^2y}{dx^2} = 24x$$

$$\text{Hence, } \frac{d^3y}{dx^3} = 24$$

$$41. \text{ We have, } x = \sin^{-1}(3t - 4t^3)$$

$$\text{and } y = \cos^{-1}(\sqrt{1-t^2})$$

$$\text{Let } t = \sin \theta$$

$$\text{Then, } x = \sin^{-1}(3\sin \theta - 4\sin^3 \theta)$$

$$= \sin^{-1}(\sin 3\theta) = 3\theta$$

Also, $y = \cos^{-1}(\sqrt{1 - \sin^2 \theta}) = \cos^{-1}(\cos \theta) = \theta$

Hence, $x = 3y \Rightarrow y = \frac{x}{3}$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{3}$$

42. We have, $f(x) = \tan^{-1} x \Rightarrow f'(x) = \frac{1}{1+x^2}$

and $f''(x) = \frac{-2x}{(1+x^2)^2}$

$$\begin{aligned} \therefore f'(x) + f''(x) &= \frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2} \\ &= \frac{1+x^2-2x}{(1+x^2)^2} = \frac{(x-1)^2}{(1+x^2)^2} \end{aligned}$$

Given, $f'(x) + f''(x) = 0$

$$\Rightarrow \frac{(x-1)^2}{(1+x^2)^2} = 0$$

$\therefore x = 1$

43. Given, $u = \tan^{-1}\left(\frac{x^3 + y^3}{x - y}\right) \Rightarrow \tan u = \frac{x^3 + y^3}{x - y}$

Here, $\tan u$ is a homogeneous in x and y of degree 2.

$$\therefore x \frac{\delta}{\delta x}(\tan u) + y \frac{\delta}{\delta y}(\tan u) = 2 \tan u$$

$$\Rightarrow x \sec^2 u \frac{\delta u}{\delta x} + y \sec^2 u \frac{\delta u}{\delta y} = 2 \tan u$$

$$\Rightarrow x \frac{\delta u}{\delta x} + y \frac{\delta u}{\delta y} = \frac{2 \tan u}{\sec^2 u} = \sin 2u$$

44. $\frac{d}{dx}(\cos x) = -\frac{\pi}{180} \sin x$

45. We have, $\frac{d}{dx} \left(\tan^{-1} \sqrt{\frac{1 + \sin \frac{x}{2}}{1 - \sin \frac{x}{2}}} \right)$

$$= \frac{d}{dx} \left[\tan^{-1} \sqrt{\frac{\left(\sin \frac{x}{4} + \cos \frac{x}{4}\right)^2}{\left(\cos \frac{x}{4} - \sin \frac{x}{4}\right)^2}} \right]$$

$$= \frac{d}{dx} \left[\tan^{-1} \left(\frac{\sin \frac{x}{4} + \cos \frac{x}{4}}{\cos \frac{x}{4} - \sin \frac{x}{4}} \right) \right] = \frac{d}{dx} \left[\tan^{-1} \left(\frac{1 + \tan \frac{x}{4}}{1 - \tan \frac{x}{4}} \right) \right]$$

$$= \frac{d}{dx} \left[\tan^{-1} \left\{ \tan \left(\frac{\pi}{4} + \frac{x}{4} \right) \right\} \right] = \frac{d}{dx} \left(\frac{\pi}{4} + \frac{x}{4} \right) = \frac{1}{4}$$

46. Given, $y = e^{\log_e [1 + x + x^2 + \dots]}$

$$\Rightarrow y = 1 + x + x^2 + \dots$$

$$\Rightarrow y = \frac{1}{1-x} \quad [\text{sum of infinite GP}]$$

$$\therefore \frac{dy}{dx} = \frac{1}{(1-x)^2}$$

$$\begin{aligned} 47. \text{ We have, } \frac{d}{dx} \left\{ \tan^{-1} \left(\frac{\cos x}{1 + \sin x} \right) \right\} \\ &= \frac{1}{1 + \left(\frac{\cos x}{1 + \sin x} \right)^2} \cdot \frac{d}{dx} \left(\frac{\cos x}{1 + \sin x} \right) \\ &= \frac{(1 + \sin x)^2}{1 + \sin^2 x + 2 \sin x + \cos^2 x} \\ &\quad \times \frac{(1 + \sin x)(-\sin x) - \cos x(\cos x)}{(1 + \sin x)^2} \\ &= \frac{1}{2 + 2 \sin x} \times [-(\sin x + \sin^2 x + \cos^2 x)] \\ &= \frac{1}{2(1 + \sin x)} \times \{-(1 + \sin x)\} = -\frac{1}{2} \end{aligned}$$

48. Given, $y = \left(\frac{3^x - 1}{3^x + 1} \right) \sin x + \log_e(1 + x)$, $x > -1$

$$\begin{aligned} \Rightarrow \frac{dy}{dx} &= \cos x \left(\frac{3^x - 1}{3^x + 1} \right) + \sin x \\ &\quad \times \left[\frac{(3^x \log 3)(3^x + 1) - (3^x - 1)(3^x \log 3)}{(3^x + 1)^2} \right] + \frac{1}{1 + x} \end{aligned}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=0} = 0 + 0 + 1 = 1$$

49. Given, $f(x) = \log_x \{ \ln(x) \} = \frac{\ln \{ \ln(x) \}}{\ln(x)}$

$$\Rightarrow f'(x) = \frac{\frac{\ln(x)}{x \ln(x)} - \ln \{ \ln(x) \}}{\{\ln(x)\}^2} = \frac{[1 - \ln \{ \ln(x) \}]}{x \{\ln(x)\}^2}$$

$$\therefore f'(e) = \frac{1}{e}$$

50. Given, $e^x + e^y = e^{x+y}$

On differentiating w.r.t. x , we get

$$e^x + e^y \frac{dy}{dx} = e^{x+y} \left(1 + \frac{dy}{dx} \right) \Rightarrow \frac{dy}{dx} = \frac{e^x - e^{(x+y)}}{e^{(x+y)} - e^y}$$

$$\therefore \left[\frac{dy}{dx} \right]_{(2,2)} = \frac{e^2 - e^4}{e^4 - e^2} = -1$$

51. Given, $f(x) = \log_e(\log_e x)$

$$\Rightarrow f'(x) = \frac{1}{x \log_e(x)}$$

$$\therefore f'(e) = \frac{1}{e}$$

52. $F(x) = \frac{d}{dx} \left\{ \int_{x^2}^{x^3} \log t \, dt \right\}$, $x > 0$

$$= (\log x^3) \cdot 3x^2 - (\log x^2) \cdot 2x = (9x^2 - 4x) \log x$$

53. Given, $f(x) = \frac{\sin^2 x}{1 + \cot x} + \frac{\cos^2 x}{1 + \tan x}$

$$= \frac{\sin^2 x \tan x}{1 + \tan x} + \frac{\cos^2 x}{1 + \tan x}$$

$$= \frac{\sin^2 x \tan x + \cos^2 x}{1 + \tan x}$$

$$\Rightarrow f'(x) = \frac{\begin{bmatrix} (2 \sin x \cos x \tan x + \sin^2 x) \\ \sec^x - 2 \cos x \sin x \\ (1 + \tan x) - \sec^2 x \\ (\sin^2 x \tan x + \cos^2 x) \end{bmatrix}}{(1 + \tan x)^2}$$

$$\therefore f'\left(\frac{\pi}{4}\right) = \frac{(1 + 1 - 1)(2) - (2)\left(\frac{1}{2} + \frac{1}{2}\right)}{(2)^2} = 0$$

$$54. \frac{d}{dx} \left[\sin^{-1} \frac{x}{a} \right] = \frac{1}{a} \cdot \frac{1}{\sqrt{1 - \frac{x^2}{a^2}}} = \frac{1}{\sqrt{a^2 - x^2}}$$

$$55. \frac{d}{dx} \left[\tan^{-1} \left(\frac{1 - \cos x}{1 + \cos x} \right)^{1/2} \right] = \frac{d}{dx} \left[\tan^{-1} \left(\frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \right)^{1/2} \right]$$

$$= \frac{d}{dx} \tan^{-1} \left(-\tan \frac{x}{2} \right) = \frac{d}{dx} \left(-\frac{x}{2} \right) = -\frac{1}{2}$$

$$\left[\because \frac{\pi}{2} < \frac{x}{2} < \pi \Rightarrow \tan \frac{x}{2} < 0 \right]$$

56. Given, $y = \sin x \cdot \sin 2x \cdot \sin 3x \dots \sin nx$

Taking log on both sides, we get

$$\log y = \log \sin x + \log \sin 2x + \log \sin 3x + \dots + \log (\sin nx)$$

On differentiating w.r.t. x , we get

$$\frac{1}{y} \frac{dy}{dx} = \frac{\cos x}{\sin x} + \frac{2 \cos 2x}{\sin 2x} + \frac{3 \cos 3x}{\sin 3x} + \dots + \frac{n \cos nx}{\sin nx}$$

$$\Rightarrow \frac{dy}{dx} = y [\cot x + 2 \cot 2x + 3 \cot 3x + \dots + n \cot nx]$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \sum_{k=1}^n k \cot kx$$

57. Given, $y = \log_e |2x|$

For $x > 0$, $y = \log_e (2x)$, $\frac{dy}{dx} = \frac{2}{2x} = \frac{1}{x}$

For $x < 0$, $y = \log_e (-2x)$

$$\frac{dy}{dx} = \frac{-2}{-2x} = \frac{1}{x}$$

Hence, $\frac{dy}{dx} = \frac{1}{x}$ $[\because x \neq 0]$

58. Given, $g(x) = [f\{2f(x) + 2\}]^2$

$$\therefore g'(x) = 2 \cdot f\{2f(x) + 2\} \cdot f'\{2f(x) + 2\} \cdot 2f'(x)$$

$$= 4f\{2f(x) + 2\} f'\{2f(x) + 2\} f'(x)$$

$$\therefore g'(0) = 4f(0) f'(0) f'(0)$$

$$= -4$$

59. Given, $y = \sin^{-1} \sqrt{1-x}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-(1-x)}} \cdot \frac{1}{2\sqrt{1-x}} \cdot (-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1}{2\sqrt{x}\sqrt{1-x}}$$

60. Given, $y = \tan^{-1} x + \cot^{-1} x + \sec^{-1} x$

$$+ \operatorname{cosec}^{-1} x = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

$$\therefore \frac{dy}{dx} = 0$$

61. For $x < -1$, $f(x) = -(x-2) - (x+1) - x = -3x+1$

$$\therefore f'(x) = -3 \Rightarrow f'(-10) = -3$$

62. We have, $x^y = e^{2(x-y)} \Rightarrow y \log x = 2(x-y)$

$$\Rightarrow y(\log x + 2) = 2x \Rightarrow y = \frac{2x}{\log x + 2}$$

$$\therefore \frac{dy}{dx} = \frac{(\log x + 2)(2) - 2x \cdot \frac{1}{x}}{(\log x + 2)^2}$$

$$= \frac{2 \log x + 4 - 2}{(\log x + 2)^2} = \frac{2(\log x + 1)}{(\log x + 2)^2}$$

63. We have, $y = \frac{(1-x)(1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n})}{(1-x)}$

$$\Rightarrow y = \frac{1-x^{4n}}{1-x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(1-x)(-4nx^{4n-1}) - (1-x^{4n})(-1)}{(1-x)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-4n(1-x)x^{4n-1} + (1-x^{4n})}{(1-x)^2}$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=0} = 1$$

64. Given, $x = a(1 + \cos \theta)$, $y = a(\theta + \sin \theta)$

Then, $\frac{dx}{d\theta} = -a \sin \theta$ and $\frac{dy}{d\theta} = a(1 + \cos \theta)$

$$\therefore \frac{dy}{dx} = \frac{1 + \cos \theta}{-\sin \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}$$

$$\Rightarrow \frac{dy}{dx} = -\cot \frac{\theta}{2}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right) = \frac{d}{d\theta} \left(\frac{dy}{dx} \right) \cdot \frac{d\theta}{dx}$$

$$= \frac{d}{d\theta} \left(-\cot \frac{\theta}{2} \right) \cdot \frac{1}{-a \sin \theta}$$

$$= \frac{1}{2} \operatorname{cosec}^2 \frac{\theta}{2} \cdot \frac{1}{-a \sin \theta}$$

$$\therefore \left[\frac{d^2y}{dx^2} \right]_{\theta=\pi/2} = \frac{1}{2} \cdot 2 \cdot \frac{1}{-a} = -\frac{1}{a}$$

65. Let $y = \sin^{-1}(2x\sqrt{1-x^2})$ and $z = \sin^{-1}(3x-4x^3)$

Put $x = \sin \theta$,

$$\therefore y = 2\theta \text{ and } z = 3\theta$$

$$\Rightarrow y = 2 \sin^{-1} x \text{ and } z = 3 \sin^{-1} x$$

$$\Rightarrow y = \frac{2}{3} z$$

$$\therefore \frac{dy}{dz} = \frac{2}{3}$$

7

Application of Derivatives

Derivatives as the Rate of Change

If a variable quantity y is some function of time t , i.e. $y = f(t)$, then small change in time Δt have a corresponding change Δy in y .

Thus, the average rate of change = $\frac{\Delta y}{\Delta t}$

When limit $\Delta t \rightarrow 0$ is applied, the rate of change becomes instantaneous and we get the rate of change with respect to at the instant t .

i.e. $\lim_{\Delta t \rightarrow 0} \frac{\Delta y}{\Delta t} = \frac{dy}{dt}$

Hence, it is clear that the rate of change of any variable with respect to some other variable is derivative of first variable with respect to other variable.

✎ The differential coefficient of y with respect to x , i.e. $\frac{dy}{dx}$ is nothing but the rate of increase of y relative to x .

➡ **Example 1.** If $V = \frac{4}{3}\pi r^3$, then at what rate in cubic unit is V increasing, when

$r = 10$ and $\frac{dr}{dt} = 0.01$?

(a) π

(b) 4π

(c) 40π

(d) $\frac{4\pi}{3}$

Sol. (b) We have,

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ \Rightarrow \frac{dV}{dt} &= 4\pi r^2 \frac{dr}{dt} \\ \Rightarrow \frac{dV}{dt} &= 4\pi \times 100 \times 0.01 & \left[\because r = 10 \text{ and } \frac{dr}{dt} = 0.01 \right] \\ \Rightarrow \frac{dV}{dt} &= 4\pi \end{aligned}$$

Hence, V is increasing at the rate of 4π cubic units per unit of time.

Chapter Snapshot

- Derivatives as the Rate of Change
- Approximation and Differentials
- Errors
- Tangent and Normal
- Equations of Tangent and Normal
- Tangent and Normal in Parametric Form
- Angle of Intersection of Two Curves
- Length of Tangent, Sub-tangent, Normal and Sub-normal
- Rolle's Theorem
- Lagrange's Mean Value Theorem
- Increasing Function
- Decreasing Function
- Monotonic Function
- Operation of Monotonic Function
- Comparison of Functions using Calculus

➤ **Example 2.** The side of an equilateral triangle expands at the rate of 2 cm/s. The rate of increase of its area when each side is 10 cm, is

- (a) $10\sqrt{2}$ cm²/s (b) $10\sqrt{3}$ cm²/s
(c) 10 cm²/s (d) 5 cm²/s

Sol. (b) Let x be the side and A be the area of equilateral triangle at time t . Then,

$$A = \frac{\sqrt{3}}{4} x^2 \Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2} x \frac{dx}{dt}$$

$$\Rightarrow \frac{dA}{dt} = \frac{\sqrt{3}}{2} \times 10 \times 2 = 10\sqrt{3} \text{ cm}^2/\text{s}$$

$$\left[\because x = 10 \text{ and } \frac{dx}{dt} = 2 \right]$$

➤ **Example 3.** The radius of a sphere is changing at the rate of 0.1 cm/s. The rate of change of its surface area when the radius is 200 cm, is

- (a) 8π cm²/s (b) 12π cm²/s
(c) 160π cm²/s (d) 200π cm²/s

Sol. (c) Let r be the radius and S be the surface area of the sphere at any time t . Then,

$$S = 4\pi r^2$$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\Rightarrow \left[\frac{dS}{dt} \right]_{r=200} = 8\pi \times 200 \times 0.1$$

$$= 160\pi \text{ cm}^2/\text{s}$$

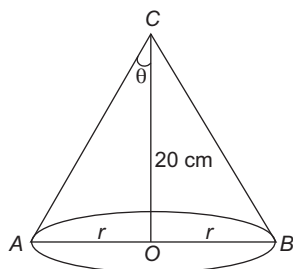
➤ **Example 4.** The altitude of a cone is 20 cm and its semi-vertical angle is 30°. If the semi-vertical angle is increasing at the rate of 2° per second, then the radius of the base is increasing at the rate of

- (a) 30 cm/s (b) $\frac{160}{3}$ cm/s (c) 10 cm/s (d) 160 cm/s

Sol. (b) Let θ be the semi-vertical angle and r be the radius of the cone at any time t . Then,

$$r = 20 \tan \theta$$

$$\Rightarrow \frac{dr}{dt} = 20 \sec^2 \theta \frac{d\theta}{dt}$$



$$\Rightarrow \frac{dr}{dt} = 20 \sec^2 30^\circ \times 2 \left[\because \theta = 30^\circ \text{ and } \frac{d\theta}{dt} = 2 \right]$$

$$\Rightarrow \frac{dr}{dt} = 20 \times \frac{4}{3} \times 2 \text{ cm/s}$$

$$= \frac{160}{3} \text{ cm/s}$$

➤ **Example 5.** For what values of x is the rate of increase of $x^3 - 5x^2 + 5x + 8$ is twice the rate of increase of x ?

- (a) $-3, -\frac{1}{3}$ (b) $-3, \frac{1}{3}$
(c) $3, -\frac{1}{3}$ (d) $3, \frac{1}{3}$

Sol. (d) Let $y = x^3 - 5x^2 + 5x + 8$. Then,

$$\frac{dy}{dt} = (3x^2 - 10x + 5) \frac{dx}{dt}$$

When $\frac{dy}{dt} = 2 \frac{dx}{dt}$, we have

$$(3x^2 - 10x + 5) \frac{dx}{dt} = 2 \frac{dx}{dt}$$

$$\Rightarrow 3x^2 - 10x + 3 = 0$$

$$\Rightarrow (3x - 1)(x - 3) = 0$$

$$\Rightarrow x = 3, \frac{1}{3}$$

➤ **Example 6.** The radius of the base of a cone is increasing at the rate of 3 cm/min and the altitude is decreasing at the rate of 4 cm/min. The rate of change of lateral surface when the radius is 7 cm and altitude is 24 cm, is

- (a) 54π cm²/min
(b) 7π cm²/min
(c) 27π cm²/min
(d) None of the above

Sol. (a) Let r , l and h denote respectively the radius, slant height and height of the cone at any time t . Then,

$$l^2 = r^2 + h^2$$

$$\Rightarrow 2l \frac{dl}{dt} = 2r \frac{dr}{dt} + 2h \frac{dh}{dt}$$

$$\Rightarrow l \frac{dl}{dt} = r \frac{dr}{dt} + h \frac{dh}{dt}$$

$$\Rightarrow l \frac{dl}{dt} = 7 \times 3 + 24 \times (-4) \left[\because \frac{dr}{dt} = 3 \text{ and } \frac{dh}{dt} = -4 \right]$$

$$\Rightarrow l \frac{dl}{dt} = -75$$

When $r = 7$ and $h = 24$, we have

$$l^2 = 7^2 + 24^2 \quad [\because l^2 = r^2 + h^2]$$

$$\Rightarrow l = 25$$

$$\therefore l \frac{dl}{dt} = -75$$

$$\Rightarrow \frac{dl}{dt} = -3$$

Let S denote the lateral surface area.

Then,

$$S = \pi r l$$

$$\Rightarrow \frac{dS}{dt} = \pi \frac{d}{dt}(rl)$$

$$= \pi \left\{ \frac{dr}{dt} \cdot l + r \cdot \frac{dl}{dt} \right\}$$

$$= \pi \{ 3 \times 25 + 7 \times (-3) \}$$

$$= \pi \{ 54 \}$$

$$= 54\pi \text{ cm}^2/\text{min}$$

Example 7. The surface area of a sphere when its volume is increasing at the same rate as its radius, is

- (a) 1 sq unit (b) $\frac{1}{2\sqrt{\pi}}$ sq unit
(c) 4π sq unit (d) $\frac{4\pi}{3}$ sq unit

Sol. (a) At any time t , let V , S and r denote respectively the volume, surface area and radius of the sphere. Then,

$$V = \frac{4}{3}\pi r^3 \quad \text{and} \quad S = 4\pi r^2$$

Now, $\frac{dV}{dt} = \frac{dr}{dt} \Rightarrow 4\pi r^2 \frac{dr}{dt} = \frac{dS}{dt}$

$\Rightarrow 4\pi r^2 = 1 \Rightarrow S = 1$ sq unit

Example 8. A variable triangle is inscribed in a circle of radius R . If the rate of change of a side is R times the rate of change of the opposite angle, then that angle is

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Sol. (c) Let the side BC be a and A be the opposite angle. Then,

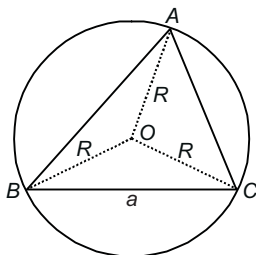
$$R = \frac{a}{2\sin A}$$

$$\Rightarrow a = 2R\sin A$$

$$\Rightarrow \frac{da}{dt} = 2R\cos A \frac{dA}{dt}$$

$$\Rightarrow R \frac{dA}{dt} = 2R\cos A \cdot \frac{dA}{dt} \quad \left[\because \frac{da}{dt} = R \frac{dA}{dt} \right]$$

$$\Rightarrow \cos A = \frac{1}{2} \Rightarrow A = \frac{\pi}{3}$$



Example 9. The surface area of a cube is increasing at the rate of $2 \text{ cm}^2/\text{s}$. When its edge is 90 cm, the volume is increasing at the rate of

- (a) $1620 \text{ cm}^3/\text{s}$ (b) $810 \text{ cm}^3/\text{s}$
(c) $405 \text{ cm}^3/\text{s}$ (d) $45 \text{ cm}^3/\text{s}$

Sol. (d) Let at any time t , the length of each edge of the cube be x cm. Let S denotes the surface area and V denotes the volume of the cube. Then,

$$S = 6x^2 \quad \text{and} \quad V = x^3$$

$$\Rightarrow \frac{dS}{dt} = 12x \frac{dx}{dt} \quad \text{and} \quad \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow 2 = 12 \times 90 \frac{dx}{dt}$$

$$\text{and} \quad \frac{dV}{dt} = 3 \times 90^2 \times \frac{dx}{dt} \quad \left[\because x = 90 \text{ cm and } \frac{dS}{dt} = 2 \right]$$

$$\Rightarrow 90 \times \frac{dx}{dt} = \frac{1}{6}$$

$$\text{and} \quad \frac{dV}{dt} = 3 \times 90 \times \left(90 \times \frac{dx}{dt} \right)$$

$$\Rightarrow \frac{dV}{dt} = 3 \times 90 \times \frac{1}{6} \text{ cm}^3/\text{s}$$

$$= 45 \text{ cm}^3/\text{s}$$

Example 10. A ladder 10 m long rests with one end against a vertical wall, the other end on the floor. The lower end moves away from the wall at the rate of 2 m/min. The rate at which the upper end falls when its base is 6 m away from the wall, is

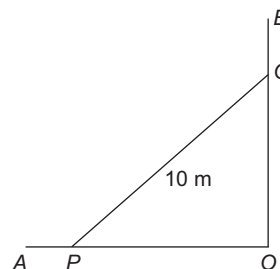
- (a) 3 m/min (b) $\frac{2}{3}$ m/min
(c) $-\frac{3}{2}$ m/min (d) None of these

Sol. (c) Let at time t , the lower end P of ladder PQ be at a distance of x metre from the wall and the upper end Q be at height of y metre from the ground. Then,

$$x^2 + y^2 = 10^2$$

It is given that, $\frac{dx}{dt} = 2$

We have to find $\frac{dy}{dt}$, when $x = 6$.



On putting $x = 6$ in $x^2 + y^2 = 10^2$, we get $y = 8$

Now, $x^2 + y^2 = 10^2$

$$\Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$$

$$\Rightarrow 6 \times 2 + 8 \times \frac{dy}{dt} = 0 \quad \left[\text{putting } x = 6, y = 8 \text{ and } \frac{dx}{dt} = 2 \right]$$

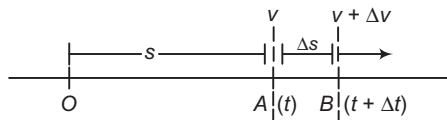
$$\Rightarrow \frac{dy}{dt} = -\frac{3}{2} \text{ m/min}$$

Velocity and Acceleration in Rectilinear Motion

Students who are studying Physics, are quite familiar with the definition of velocity and acceleration.

The velocity of a moving particle is defined as the rate of change of its displacement with respect to time and the acceleration is defined as the rate of change of its velocity with respect to time.

Let a particle A move rectilinearly as shown in figure.



Let s be the displacement from a fixed point O along the path at time t , s is considered to be positive on right of the point O and negative on the left of it.

Also, Δs is positive when s increases, i.e. when the particle moves towards right.

Thus, if Δs is the increment in s in time Δt , then the average velocity in this interval is

$$\frac{\Delta s}{\Delta t}$$

and the instantaneous velocity i.e., velocity at time t is

$$v = \lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

If the velocity varies, then there is change of velocity Δv in time Δt .

Hence, the acceleration at time t

$$= \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$

➤ **Example 11.** The distance moved by the particle in time t is given by $x = t^3 - 12t^2 + 6t + 8$. At the instant when its acceleration is zero, the velocity is

- (a) 42 m/s (b) -42 m/s (c) 48 m/s (d) -48 m/s

Sol. (b) We have,

$$x = t^3 - 12t^2 + 6t + 8$$

$$\Rightarrow \frac{dx}{dt} = 3t^2 - 24t + 6$$

$$\text{and } \frac{d^2x}{dt^2} = 6t - 24$$

Now, acceleration = 0

$$\Rightarrow \frac{d^2x}{dt^2} = 0$$

$$\Rightarrow 6t - 24 = 0 \Rightarrow t = 4$$

At $t = 4$, we have

$$\text{Velocity} = \left(\frac{dx}{dt} \right)_{t=4}$$

$$= 3 \times 4^2 - 24 \times 4 + 6 = -42 \text{ m/s}$$

➤ **Example 12.** A particle's velocity v at time t is given by $v = 2e^{2t} \cos \frac{\pi t}{3}$. The least value of t at which the acceleration becomes zero, is

- (a) 0 (b) $\frac{3}{2}$
(c) $\frac{3}{\pi} \tan^{-1} \left(\frac{6}{\pi} \right)$ (d) $\frac{3}{\pi} \cot^{-1} \left(\frac{6}{\pi} \right)$

Sol. (c) We have,

$$v = 2e^{2t} \cos \frac{\pi t}{3}$$

$$\Rightarrow \frac{dv}{dt} = 4e^{2t} \cos \frac{\pi t}{3} - \frac{2\pi}{3} e^{2t} \sin \frac{\pi t}{3}$$

Now, acceleration = 0

$$\Rightarrow 4e^{2t} \cos \frac{\pi t}{3} - \frac{2\pi}{3} e^{2t} \sin \frac{\pi t}{3} = 0$$

$$\Rightarrow \tan \frac{\pi t}{3} = \frac{6}{\pi}$$

$$\Rightarrow t = \frac{3}{\pi} \tan^{-1} \left(\frac{6}{\pi} \right)$$

➤ **Example 13.** If a particle is moving such that the velocity acquired is proportional to the square root of the distance covered, then its acceleration is

(a) a constant

(b) $\propto s^2$

(c) $\propto \frac{1}{s^2}$

(d) $\propto \frac{1}{s}$

Sol. (a) Let v be the velocity of the particle when the distance covered is s . Then,

$$v \propto \sqrt{s} \quad [\text{given}]$$

$$\Rightarrow v = \lambda \sqrt{s}$$

$$\Rightarrow \frac{dv}{ds} = \frac{\lambda}{2\sqrt{s}}$$

$$\Rightarrow v \frac{dv}{ds} = \frac{\lambda v}{2\sqrt{s}} = \frac{\lambda^2}{2} = \text{constant}$$

$\therefore$ The acceleration is constant.

➤ **Example 14.** If a particle moving along a line follows the law $s = \sqrt{1+t}$, then the acceleration is proportional to

(a) square of the velocity

(b) cube of the displacement

(c) cube of the velocity

(d) square of the displacement

Sol. (c) We have, $s = \sqrt{1+t}$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{2\sqrt{1+t}}$$

$$\Rightarrow \frac{ds}{dt} = \frac{1}{2s}$$

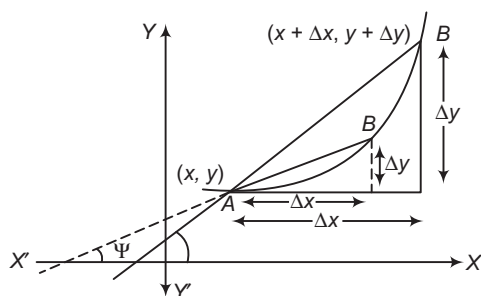
$$\Rightarrow \frac{d^2s}{dt^2} = -\frac{1}{2s^2} \cdot \frac{ds}{dt} = -\frac{1}{2s^2} \times \frac{1}{2s} = -\frac{1}{4s^3}$$

$$\Rightarrow \frac{d^2s}{dt^2} = -2 \times \left(\frac{1}{2s} \right)^3 = -2 \left(\frac{ds}{dt} \right)^3$$

7 Approximation and Differentials

From the figure, it is clear that, if Δx and Δy are sufficiently small quantities, then

$$\frac{\Delta y}{\Delta x} = \tan \psi \cong \frac{dy}{dx} = f'(x)$$



Hence, approximate change in the value of y , called its differential, is given by

$$\Delta y = f'(x) \cdot \Delta x$$

➤ **Example 15.** The value of $\sqrt{101}$ is

- (a) 10.04 (b) 10.05 (c) 10.1 (d) 10.02

Sol. (b) Let $f(x) = y = \sqrt{x}$, $y + \Delta y = \sqrt{x + \Delta x}$

$$\therefore \Delta y = \sqrt{x + \Delta x} - \sqrt{x} = f'(x) \cdot \Delta x$$

Put $x = 100$, $\Delta x = 1$

$$\sqrt{101} = 10 + \frac{1}{20} \times 1$$

$$= 10 + 0.05 = 10.05$$

➤ **Example 16.** The value of $(127)^{1/3}$ to four decimal places is

- (a) 5.0267 (b) 5.4267
(c) 5.5267 (d) 5.001

Sol. (a) Let $y = x^{1/3}$, $x = 125$ and $x + \Delta x = 127$. Then,

$$\frac{dy}{dx} = \frac{1}{3x^{2/3}} \text{ and } \Delta x = 2$$

When $x = 125$, we have

$$y = 5 \text{ and } \frac{dy}{dx} = \frac{1}{75}$$

$$\therefore \Delta y = \frac{dy}{dx} \Delta x$$

$$\Rightarrow \Delta y = \frac{1}{75} \times 2 = \frac{2}{75}$$

$$\therefore (127)^{1/3} = y + \Delta y = 5 + \frac{2}{75} = 5 + \frac{8}{3} \times \frac{1}{100}$$

$$\Rightarrow (127)^{1/3} = 5 + \frac{(2.6667)}{100} = 5.02667 = 5.0267$$

➤ **Example 17.** The approximate change in the volume of a cube of side x m caused by increasing the side by 3% is

- (a) $0.06x^3 m^3$ (b) $0.6x^3 m^3$
(c) $0.09x^3 m^3$ (d) $0.9x^3 m^3$

Sol. (c) We know that, the volume V of a cube of side x metre is given by

$$V = x^3 \Rightarrow \frac{dV}{dx} = 3x^2$$

Let Δx be change inside.

$$\Rightarrow 3\% \text{ of } x = 0.03x$$

Now, change in volume,

$$\Delta V = \left(\frac{dV}{dx} \right) \Delta x = (3x^2) \Delta x = (3x^2)(0.03x) \quad [\because \Delta x = 0.03x]$$

$$= 0.09x^3 m^3$$

Hence, the approximate change in the volume of the cube is $0.09x^3 m^3$.

➤ **Example 18.** If $1^\circ = \alpha$ radian, then the approximate value of $\cos 60^\circ 1'$ is

- (a) $\frac{1}{2} + \frac{\alpha\sqrt{3}}{120}$ (b) $\frac{1}{2} - \frac{\alpha}{120}$
(c) $\frac{1}{2} - \frac{\alpha\sqrt{3}}{120}$ (d) None of these

Sol. (c) Let $y = \cos x$, $x = 60^\circ$ and $x + \Delta x = 60^\circ 1'$

Then, $\frac{dy}{dx} = -\sin x$ and $\Delta x = 1'$

When $x = 60^\circ$, we have $y = \frac{1}{2}$ and $\frac{dy}{dx} = -\frac{\sqrt{3}}{2}$

$$\therefore \Delta y = \frac{dy}{dx} \cdot \Delta x$$

$$\Rightarrow \Delta y = -\frac{\sqrt{3}}{2} \cdot 1'$$

$$= -\frac{\sqrt{3}}{2} \times \frac{\alpha}{60}$$

$$\Rightarrow \cos 60^\circ 1' = y + \Delta y = \frac{1}{2} - \frac{\alpha\sqrt{3}}{120}$$

Errors

Let $y = f(x)$. Then,

$$\lim_{\delta x \rightarrow 0} \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x)$$

$$\therefore \frac{f(x + \delta x) - f(x)}{\delta x} = f'(x) + \epsilon$$

where, $\epsilon \rightarrow 0$ when $\delta x \rightarrow 0$

$$\Rightarrow f(x + \delta x) - f(x) = f'(x) \cdot \delta x + \epsilon \cdot \delta x$$

$$\Rightarrow f(x + \delta x) - f(x) = f'(x) \cdot \delta x \text{ (approximately)}$$

$$\Rightarrow \delta y = f'(x) \cdot \delta x \text{ (approximately)}$$

$$[\because f(x + \delta x) - f(x) = \delta y]$$

Thus, if δx is an error in x , then the corresponding error in y is δy . These small values δx and δy are called differentials.

(i) **Absolute error** δx is called an absolute error in x .

(ii) **Relative error** $\frac{\delta x}{x}$ is called the relative error.

(iii) **Percentage error** $\left(\frac{\delta x}{x} \times 100 \right) \%$ is called the percentage error.

➤ **Example 19.** In a ΔABC , if sides a and b remain constant such that α is the error in C , then relative error in its area is

- (a) $\alpha \cot C$ (b) $\alpha \sin C$
(c) $\alpha \tan C$ (d) $\alpha \cos C$

Sol. (a) We have, $S = \frac{1}{2}ab \sin C \Rightarrow \frac{dS}{dC} = \frac{1}{2}ab \cos C$

Let ΔS be the error in area S . Then,

$$\begin{aligned} \Delta S &= \frac{dS}{dC} \Delta C \\ \Rightarrow \Delta S &= \frac{1}{2}ab \cos C \alpha \quad [\because \Delta C = \alpha] \\ \Rightarrow \frac{\Delta S}{S} &= \frac{\frac{1}{2}ab \cos C}{\frac{1}{2}ab \sin C} \alpha = \alpha \cot C \end{aligned}$$

➤ **Example 20.** The focal length of a mirror is given by $\frac{1}{v} - \frac{1}{u} = \frac{2}{f}$. If errors made in measuring u and v are α , then the relative error in f is

- (a) $\frac{2}{\alpha}$ (b) $\alpha \left(\frac{1}{u} + \frac{1}{v} \right)$
(c) $\alpha \left(\frac{1}{u} - \frac{1}{v} \right)$ (d) None of these

Sol. (b) We have,

$$\frac{1}{v} - \frac{1}{u} = \frac{2}{f} \Rightarrow d \left(\frac{1}{v} - \frac{1}{u} \right) = d \left(\frac{2}{f} \right)$$

$$\begin{aligned} \Rightarrow -\frac{1}{v^2} dv + \frac{1}{u^2} du &= -\frac{2}{f^2} df \\ \Rightarrow \left(\frac{1}{v^2} - \frac{1}{u^2} \right) \alpha &= \frac{2}{f^2} df \quad [\because du = dv = \alpha] \\ \Rightarrow \alpha \left(\frac{1}{v} + \frac{1}{u} \right) \left(\frac{1}{v} - \frac{1}{u} \right) &= \frac{2}{f^2} df \\ \Rightarrow \alpha \left(\frac{1}{v} + \frac{1}{u} \right) \times \frac{2}{f} &= \frac{2}{f^2} df \quad \left[\because \frac{1}{v} - \frac{1}{u} = \frac{2}{f} \right] \\ \Rightarrow \frac{df}{f} &= \alpha \left(\frac{1}{u} + \frac{1}{v} \right) \\ \Rightarrow \text{Relative error in } f &= \alpha \left(\frac{1}{u} + \frac{1}{v} \right) \end{aligned}$$

➤ **Example 21.** In a ΔABC , the sides b and c are given. If there is an error ΔA in measuring $\angle A$, then the error Δa inside a is given by

- (a) $\frac{S}{2a} \Delta A$ (b) $\frac{2S}{a} \Delta A$
(c) $b \sin A \Delta A$ (d) None of these

Sol. (b) In ΔABC , we have

$$\begin{aligned} 2bc \cos A &= b^2 + c^2 - a^2 \\ \Rightarrow d(2bc \cos A) &= d(b^2 + c^2 - a^2) \\ \Rightarrow -2bc \sin A dA &= -2a da \\ \Rightarrow b \sin A dA &= a da \\ \Rightarrow \frac{2}{a} \left(\frac{1}{2} b \sin A \right) dA &= da \\ \Rightarrow da &= \frac{2S}{a} dA, \text{ where } S = \text{area of } \Delta ABC \\ \Rightarrow \Delta a &= \frac{2S}{a} \Delta A \quad [\because da \cong \Delta a \text{ and } dA = \Delta A] \end{aligned}$$

Work Book Exercise 7.1

- A balloon is pumped at the rate of a $\text{cm}^3/\text{minute}$. The rate of increase of its surface area when the radius is b cm, is
a $\frac{2a^2}{b^4} \text{ cm}^2/\text{min}$ b $\frac{a}{2b} \text{ cm}^2/\text{min}$
c $\frac{2a}{b} \text{ cm}^2/\text{min}$ d None of these
- x and y are the sides of two squares such that $y = x - x^2$. The rate of change of the area of the second square with respect to that of the first square is
a $2(1 - x^2)x$
b $2x^2 - 3x + 1$
c $2(2x^2 - 3x + 1)$
d None of the above
- On the curve $x^3 = 12y$, the abscissa changes at a faster rate than the ordinate. Then, x belongs to the interval
a $(-2, 2)$
b $(-1, 1)$
c $(0, 2)$
d None of the above

- Two cyclists start from the junction of two perpendicular roads, their velocities being $3v$ m/min and $4v$ m/min. The rate at which the two cyclists are separating is
a $\frac{7}{2}v$ m/min
b $5v$ m/min
c v m/min
d None of the above
- Moving along the X -axis, there are two points with $x = 10 + 6t$, $x = 3 + t^2$. The speed with which they are reaching from each other at the time of encounter is (x is in centimetre and t is in second)
a 16 cm/s
b 20 cm/s
c 8 cm/s
d 12 cm/s
- The position of a point in time t is given by $x = a + bt - ct^2$, $y = at + bt^2$. Its acceleration at time t is
a $b - c$ b $b + c$
c $2b - 2c$ d $2\sqrt{b^2 + c^2}$

- 7 Gas is being pumped into a spherical balloon at the rate of $30 \text{ ft}^3/\text{min}$. Then, the rate at which the radius increases when it reaches the value 15 ft, is
- a $\frac{1}{30\pi} \text{ ft/min}$ b $\frac{1}{15\pi} \text{ ft/min}$
 c $\frac{1}{20} \text{ ft/min}$ d $\frac{1}{15} \text{ ft/min}$
- 8 Water is dripping out from a conical funnel of semi-vertical angle $\pi/4$ at the uniform rate of $2 \text{ cm}^2/\text{s}$ in the surface area, through a tiny hole at the vertex of the bottom. When the slant height of cone is 4 cm, the rate of decrease of the slant height of water, is
- a $\frac{\sqrt{2}}{4\pi} \text{ cm/s}$ b $\frac{1}{4\pi} \text{ cm/s}$
 c $\frac{1}{\pi\sqrt{2}} \text{ cm/s}$ d None of these
- 9 A stick of length a cm rests against a vertical wall and the horizontal floor. If the foot of the stick slides with a constant velocity of b cm/s, then the magnitude of the velocity of the middle point of the stick when it is equally inclined with the floor and the wall, is
- a $\frac{b}{\sqrt{2}} \text{ cm/s}$ b $\frac{b}{2} \text{ cm/s}$
 c $\frac{ab}{2} \text{ cm/s}$ d None of these
- 10 If $y = \int_0^x \frac{t^2}{\sqrt{t^2 + 1}} dt$, then the rate of change of y with respect to x when $x = 1$, is
- a $\sqrt{2}$ b $\frac{1}{2}$
 c $\frac{1}{\sqrt{2}}$ d None of these
- 11 The rate of change of the volume of a sphere w.r.t its surface area, when the radius is 2 cm is
- a 1 b 2 c 3 d 4
- 12 The approximate value of $(25)^{1/3}$ is
- a 3 b 2.926
 c 2.629 d None of these
- 13 A cube of ice melts without changing its shape at the uniform rate of $4 \text{ cm}^3/\text{min}$. The rate of change of the surface area of the cube, in cm^2/min , when the volume of the cube is 125 cm^3 is
- a -4 b $-\frac{16}{5}$ c $-\frac{16}{6}$ d $-\frac{8}{15}$
- 14 If the radius of a sphere is measured as 7 m with an error of 0.02 m, then the approximate error in calculating its volume is
- a $3.12 \pi \text{ m}^3$ b $3.92 \pi \text{ m}^3$
 c $3.56 \pi \text{ m}^3$ d $4.01 \pi \text{ m}^3$

Tangent and Normal

Slope of Tangent

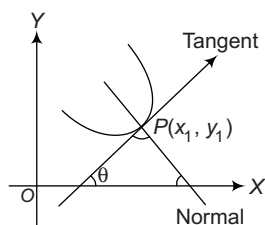
Let $y = f(x)$ be a continuous curve and $P(x_1, y_1)$ be the point on it.

Then, $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ is the slope of tangent to the

curve $y = f(x)$ at a point P .

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = \tan \theta = \text{Slope of tangent at } P$$

where, θ is the angle which the tangent at $P(x_1, y_1)$ makes with the positive direction of X -axis as shown in the figure.



• If tangent is parallel to X -axis, i.e. $\theta = 0^\circ \Rightarrow \tan \theta = 0$

$$\therefore \left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

• If tangent is perpendicular to X -axis (or parallel to Y -axis), then

$$\theta = 90^\circ$$

$$\Rightarrow \tan \theta = \infty$$

$$\text{or } \cot \theta = 0$$

$$\therefore \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

Slope of Normal

We know that, normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ passing through P .

$\therefore$ Slope of the normal at P

$$= -\frac{1}{\text{Slope of the tangent at } P}$$

$$\Rightarrow \text{Slope of normal at } P(x_1, y_1) = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

$$\text{or slope of normal at } P(x_1, y_1) = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

- If normal is parallel to X-axis, then

$$-\left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

$$\Rightarrow \left(\frac{dx}{dy}\right)_{(x_1, y_1)} = 0$$

- If normal is perpendicular to X-axis (or parallel to Y-axis), then

$$-\left(\frac{dy}{dx}\right)_{(x_1, y_1)} = 0$$

Equations of Tangent and Normal

We know that, the equation of line passing through (x_1, y_1) and having slope $\tan \theta$ (or m) is $(y - y_1) = \tan \theta (x - x_1)$ or $(y - y_1) = m(x - x_1)$.

Equation of Tangent

We know that, slope of tangent at

$$P(x_1, y_1) = m = \tan \theta = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Thus, we can write equation of tangent as,

$$\frac{y - y_1}{x - x_1} = \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

Equation of Normal

Since, the normal at $P(x_1, y_1)$ passes through $P(x_1, y_1)$ and has slope $-\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$.

$\therefore$ Equation of normal at $P(x_1, y_1)$ to the curve is,

$$\frac{y - y_1}{x - x_1} = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}}$$

$$\Rightarrow \frac{y - y_1}{x - x_1} = -\left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

- Students are advised to use direct method for finding equation of tangent on the curve of the form $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

i.e. we must replace equation in the above as

$$x^2 \text{ to } xx_1$$

$$y^2 \text{ to } yy_1$$

$$2x \text{ to } x + x_1$$

$$2y \text{ to } y + y_1$$

$$2xy \text{ to } xy_1 + yx_1$$

Therefore, equation of tangent is

$$axx_1 + h(xy_1 + yx_1) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

Tangent and Normal in Parametric Form

If the equation(s) of the curve $y = f(x)$ are represented in the form of the parameter t , such that $x = f_x(t)$ and $y = f_y(t)$

i. Slope of tangent

$$m = \left(\frac{dy}{dx}\right) \Rightarrow m = \frac{dy}{dt} \cdot \frac{dt}{dx} \Rightarrow m = \frac{f'_y}{f'_x}$$

ii. Slope of normal

$$M = -\frac{1}{m} = -\frac{f'_x}{f'_y}$$

iii. Equation of tangent

$$y - f_y(t) = \frac{f'_y}{f'_x} [x - f_x(t)]$$

iv. Equation of normal

$$y - f_y(t) = -\frac{f'_x}{f'_y} [x - f_x(t)]$$

➤ **Example 22.** The slope of the normal to the curve $x = 1 - a \sin \theta$, $y = b \cos^2 \theta$ at $\theta = \frac{\pi}{2}$, is

(a) $\frac{a}{2b}$

(b) $\frac{2a}{b}$

(c) $\frac{a}{b}$

(d) $-\frac{a}{2b}$

Sol. (d) It is given that, $x = 1 - a \sin \theta$ and $y = b \cos^2 \theta$.

On differentiating x and y both w.r.t. θ , we get

$$\frac{dx}{d\theta} = \frac{d}{d\theta} [1 - a \sin \theta] = -a \cos \theta$$

$$\text{and } \frac{dy}{d\theta} = \frac{d}{d\theta} [b \cos^2 \theta] = 2b \cos \theta (-\sin \theta) = -2b \cos \theta \sin \theta$$

$$\text{Now, } \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-2b \cos \theta \sin \theta}{-a \cos \theta} = \frac{2b}{a} \sin \theta$$

$$\therefore \text{ Slope of normal at the point } \theta = \frac{\pi}{2} = -\left(\frac{dx}{dy}\right)_{\theta = \frac{\pi}{2}}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)_{\theta = \pi/2}} = \frac{-1}{\frac{2b}{a} \sin\left(\frac{\pi}{2}\right)} = \frac{-a}{2b}$$

➤ **Example 23.** The slope of the tangent to the curve $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $t = 2$ is

(a) $\frac{7}{6}$

(b) $\frac{5}{6}$

(c) $\frac{6}{7}$

(d) 1

Sol. (c) We have, $\frac{dx}{dt} = 2t + 3$ and $\frac{dy}{dt} = 4t - 2$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3}$$

Thus, slope of the tangent to the curve at the point $t = 2$ is

$$\left[\frac{dy}{dx} \right]_{t=2} = \frac{4(2) - 2}{2(2) + 3} = \frac{6}{7}$$

➤ **Example 24.** The normals to the curve $x = a(\theta + \sin \theta)$, $y = a(1 - \cos \theta)$ at the points $\theta = (2n + 1)\pi$, $n \in I$ are all

(a) parallel to X-axis

(b) parallel to Y-axis

(c) parallel to the line $y = x$

(d) None of the above

Sol. (a) We have, $x = a(\theta + \sin \theta)$ and $y = a(1 - \cos \theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(1 + \cos \theta)$$

$$\text{and } \frac{dy}{d\theta} = a \sin \theta$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\sin \theta}{1 + \cos \theta}$$

$$= \frac{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2 \cos^2 \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

$$\text{Now, } \left[\frac{dy}{dx} \right]_{\theta = (2n+1)\pi} = \tan(2n+1)\frac{\pi}{2} = \infty$$

$\therefore$ Slope of the normal is 0.

Hence, the normal is parallel to X-axis at $\theta = (2n + 1)\pi$.

➤ **Example 25.** The tangent to the curve $x^2 + y^2 = 25$ is parallel to the line $3x - 4y = 7$ at the point

(a) $(-3, -4)$

(b) $(3, -4)$

(c) $(3, 4)$

(d) None of these

Sol. (b) We have, $x^2 + y^2 = 25$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y} \quad \dots(i)$$

Now, slope of the line $3x - 4y = 7$ is $m = \frac{3}{4}$.

Since, the tangent is parallel to the given line.

$$\Rightarrow \frac{dy}{dx} = \frac{3}{4} \Rightarrow -\frac{x}{y} = \frac{3}{4}$$

$$\Rightarrow y = -\frac{4}{3}x \quad \dots(ii)$$

$$\text{From Eq. (i), } x^2 + \frac{16}{9}x^2 = 25$$

$$\Rightarrow x = \pm 3$$

$$\text{If } x = 3, \text{ from Eq. (ii), } y = -\frac{4}{3}(3) = -4$$

$$\text{If } x = -3, \text{ from Eq. (ii), } y = -\frac{4}{3}(-3) = 4$$

$\therefore$ The points are $(3, -4)$ and $(-3, 4)$.

➤ **Example 26.** If m is the slope of a tangent to the curve $e^{2y} = 1 + 4x^2$, then

(a) $m < 1$

(b) $|m| \leq 1$

(c) $|m| > 1$

(d) None of these

Sol. (b) We have,

$$e^{2y} = 1 + 4x^2 \Rightarrow e^{2y} \cdot 2 \frac{dy}{dx} = 8x$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x}{e^{2y}} = \frac{4x}{1 + 4x^2}$$

$$\therefore \text{Slope of tangent} = m = \frac{4x}{1 + 4x^2}$$

$$\Rightarrow |m| = \frac{4|x|}{1 + 4x^2} \leq 1 = \frac{4|x|}{1 + 4|x|^2} \leq 1$$

$$\left[\because (1 - 2|x|)^2 \geq 0 \Rightarrow 1 + 4|x|^2 - 4|x| \geq 0 \right]$$

$$\Rightarrow \frac{4|x|}{1 + 4|x|^2}$$

➤ **Example 27.** If the slope of the curve

$y = \frac{ax}{b - x}$ at the point $(1, 1)$ is 2, then the values of

a and b are respectively

(a) 1, -2

(b) -1, 2

(c) 1, 2

(d) None of these

Sol. (c) We have, $y = \frac{ax}{b - x}$

$$\Rightarrow \frac{dy}{dx} = \frac{(b - x)a - ax \cdot (-1)}{(b - x)^2} = \frac{ab}{(b - x)^2}$$

$$\therefore \left[\frac{dy}{dx} \right]_{(1,1)} = \frac{ab}{(b - 1)^2} = 2 \quad [\text{given}] \dots(i)$$

Since, the curve passes through the point $(1, 1)$, therefore

$$1 = \frac{a}{b - 1} \Rightarrow a = b - 1 \quad \dots(ii)$$

On putting $a = b - 1$ in Eq. (i), we get

$$\frac{(b - 1)b}{(b - 1)^2} = 2 \Rightarrow b = 2 \text{ and } b \neq 1$$

$$\therefore a = 2 - 1 = 1$$

$$\text{Hence, } a = 1, b = 2$$

➤ **Example 28.** If the tangent at each point of the curve $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$ makes an acute angle with the positive direction of X-axis, then

(a) $a \geq 1$

(b) $-1 \leq a \leq 1$

(c) $a \leq -1$

(d) None of these

Sol. (b) We have, $y = \frac{2}{3}x^3 - 2ax^2 + 2x + 5$

$$\Rightarrow \frac{dy}{dx} = 2x^2 - 4ax + 2$$

Since, the tangent makes an acute angle with the positive direction of X-axis, therefore

$$\frac{dy}{dx} \geq 0 \Rightarrow 2x^2 - 4ax + 2 \geq 0, \forall x$$

$$\Rightarrow 16a^2 - 16 \leq 0 \quad [\because \text{discriminant} = (4a)^2 - 4(2)(2) \leq 0]$$

$$\Rightarrow a^2 - 1 \leq 0 \Rightarrow (a - 1)(a + 1) \leq 0 \Rightarrow -1 \leq a \leq 1$$

➤ **Example 29.** The equation of the tangent to the curve $y = \sqrt{9 - 2x^2}$ at the point where the ordinate and the abscissa are equal, is

- (a) $2x + y - 3\sqrt{3} = 0$
 (b) $2x + y + 3\sqrt{3} = 0$
 (c) $2x - y - 3\sqrt{3} = 0$
 (d) None of the above

Sol. (a) On putting $y = x$ in $y = \sqrt{9 - 2x^2}$, we get

$$x = \sqrt{9 - 2x^2} \Rightarrow x^2 = 9 - 2x^2$$

$$\Rightarrow x = \sqrt{3}, -\sqrt{3}$$

Since, $y > 0$, therefore the point is $(\sqrt{3}, \sqrt{3})$.

Now, we have, $y^2 = 9 - 2x^2$

On differentiating w.r.t. x , we get

$$2y \frac{dy}{dx} = -4x \Rightarrow \frac{dy}{dx} = -\frac{2x}{y}$$

$$\therefore \left[\frac{dy}{dx} \right]_{(\sqrt{3}, \sqrt{3})} = -\frac{2\sqrt{3}}{\sqrt{3}} = -2$$

So, the equation of tangent at $(\sqrt{3}, \sqrt{3})$ is

$$(y - \sqrt{3}) = -2(x - \sqrt{3}) \Rightarrow 2x + y - 3\sqrt{3} = 0$$

➤ **Example 30.** The tangent to the curve $5x^2 + y^2 = 1$ at $\left(\frac{1}{3}, -\frac{2}{3}\right)$ passes through the point

- (a) $(0, 0)$ (b) $(1, -1)$
 (c) $(-1, 1)$ (d) None of these

Sol. (d) We have, $5x^2 + y^2 = 1$

$$\Rightarrow 10x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{5x}{y}$$

$$\therefore \left[\frac{dy}{dx} \right]_{\left(\frac{1}{3}, -\frac{2}{3}\right)} = \frac{-5\left(\frac{1}{3}\right)}{\left(-\frac{2}{3}\right)} = \frac{5}{2}$$

Hence, the equation of the tangent at $\left(\frac{1}{3}, -\frac{2}{3}\right)$ is

$$y + \frac{2}{3} = \frac{5}{2}\left(x - \frac{1}{3}\right) \Rightarrow 5x - 2y - 3 = 0$$

which clearly passes through the point $(1, 1)$.

➤ **Example 31.** Tangents are drawn from the origin to the curve $y = \sin x$. Their points of contact lie on the curve

- (a) $x^2 y^2 = x^2 + y^2$
 (b) $x^2 y^2 = x^2 - y^2$
 (c) $x^2 y^2 = y^2 - x^2$
 (d) None of these

Sol. (b) The given curve is $y = \sin x$... (i)

$$\Rightarrow \frac{dy}{dx} = \cos x$$

So, the equation of tangent through the origin $(0, 0)$ is

$$y - 0 = \frac{dy}{dx}(x - 0) = x \cos x$$

$$\therefore \frac{y}{x} = \cos x \quad \dots (ii)$$

On squaring and adding Eqs. (i) and (ii), we get

$$y^2 + \frac{y^2}{x^2} = 1,$$

$$\therefore x^2 y^2 = x^2 - y^2$$

➤ **Example 32.** If the line $y = 2x$ touches the curve $y = ax^2 + bx + c$ at the point where $x = 1$ and the curve passes through the point $(-1, 0)$, then the values of a, b, c are

$$(a) a = \frac{1}{2}, b = 1, c = \frac{1}{2}$$

$$(b) a = 1, b = \frac{1}{2}, c = \frac{1}{2}$$

$$(c) a = \frac{1}{2}, b = \frac{1}{2}, c = 1$$

(d) None of these

Sol. (a) The given curve is $y = ax^2 + bx + c$... (i)

Since, the point $(-1, 0)$ lies on it.

$$\therefore a - b + c = 0 \quad \dots (ii)$$

Also, $y = 2x$ is a tangent to Eq. (i) at $x = 1$, so that $y = 2$.

Since, the point $(1, 2)$ lies on Eq. (i).

$$\therefore a + b + c = 2 \quad \dots (iii)$$

$$\text{Also, } \left[\frac{dy}{dx} \right]_{(1, 2)} = (2ax + b)_{(1, 2)} = 2$$

$$\therefore 2a + b = 2 \quad \dots (iv)$$

On solving Eqs. (ii), (iii) and (iv), we get

$$a = \frac{1}{2}, b = 1, c = \frac{1}{2}$$

➤ **Example 33.** The points where the normal to the curve $\sqrt{xy} = a + x$ makes equal intercepts on the axes, are

$$(a) \pm \left(\frac{a}{\sqrt{2}}, \sqrt{2}a - \frac{a}{\sqrt{2}} \pm 2a \right)$$

$$(b) \pm \left(\frac{a}{\sqrt{2}}, -\sqrt{2}a + \frac{a}{\sqrt{2}} \pm 2a \right)$$

$$(c) \pm \left(\frac{a}{\sqrt{2}}, \sqrt{2}a + \frac{a}{\sqrt{2}} \pm 2a \right)$$

(d) None of the above

Sol. (c) We have, $xy = (a + x)^2$

$$\Rightarrow y = \frac{a^2}{x} + 2a + x$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{a^2}{x^2}$$

If the normal makes equal intercepts, then its slope is ± 1 .

$$\Rightarrow -\frac{x^2}{x^2 - a^2} = \pm 1$$

$$\Rightarrow x^2 = x^2 - a^2 \text{ or } x^2 = -(x^2 - a^2)$$

$$\Rightarrow a^2 = 0 \text{ or } x^2 = \frac{a^2}{2}$$

$$\text{i.e. } x = \pm \frac{a}{\sqrt{2}}$$

Then,
$$y = \frac{a^2}{\pm \frac{a}{\sqrt{2}}} + 2a \pm \frac{a}{\sqrt{2}}$$
$$= \pm \sqrt{2}a \pm \frac{a}{\sqrt{2}} + 2a$$

Hence, the points are $\pm \left(\frac{a}{\sqrt{2}}, \sqrt{2}a + \frac{a}{\sqrt{2}} \pm 2a \right)$.

➤ **Example 34.** The equation of the tangent to the curve $y = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$ at the origin is

- (a) $x = 0$ (b) $x = y$
(c) $y = 0$ (d) None of these

Sol. (c) We have,

$$\left[\frac{dy}{dx} \right]_{(0,0)} = \lim_{h \rightarrow 0} \frac{h^2 \sin \frac{1}{h} - 0}{h} = \lim_{h \rightarrow 0} h \sin \frac{1}{h} = 0$$

∴ Slope of the tangent at origin is 0.

Hence, the equation of the tangent at $(0, 0)$ is
 $y - 0 = 0(x - 0) \Rightarrow y = 0$

➤ **Example 35.** Number of possible tangents to the curve $y = \cos(x + y)$, $-3\pi \leq x \leq 3\pi$, that are parallel to the line $x + 2y = 0$, is

- (a) 1 (b) 2 (c) 3 (d) 4

Sol. (c) We have, $y = \cos(x + y)$

$$\Rightarrow \frac{dy}{dx} = -\sin(x + y) \left(1 + \frac{dy}{dx} \right)$$

Since, the tangents are parallel to the line $x + 2y = 0$.

$$\therefore -\frac{1}{2} = -\sin(x + y) \left(1 - \frac{1}{2} \right)$$

$$\Rightarrow \sin(x + y) = 1$$

$$\Rightarrow x + y = \frac{\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}$$

$$\text{Since, } y = \cos(x + y)$$

$$\therefore y = 0$$

$$\text{For } x + y = \frac{\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{2}, -\frac{3\pi}{2}, \frac{5\pi}{2} \quad [\because y = 0, \text{ in each case}]$$

➤ **Example 36.** The condition that the line $lx + my = 1$ may be normal to the curve $y^2 = 4ax$, is

- (a) $al^3 - 2alm^2 = m^2$
(b) $al^2 + 2alm^3 = m^2$
(c) $al^3 + 2alm^2 = m^3$
(d) $al^3 + 2alm^2 = m^2$

Sol. (d) Let $P(x_1, y_1)$ be a point on the curve

$$y^2 = 4ax$$

On differentiating w.r.t. x , we get

$$2y \frac{dy}{dx} = 4a \Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = \frac{2a}{y_1}$$

Hence, the equation of normal at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

$$\Rightarrow xy_1 + 2ay = y_1(x_1 + 2a)$$

But $lx + my = 1$ is also normal, hence coefficients must be proportional.

$$\therefore \frac{y_1}{l} = \frac{2a}{m} = \frac{y_1(x_1 + 2a)}{1}$$

$$\Rightarrow y_2 = \frac{2al}{m}, x_1 = \frac{1}{l} - 2a$$

Hence, from Eq. (i),

$$\frac{4a^2 l^2}{m^2} = 4a \left(\frac{1 - 2al}{l} \right)$$

$$\Rightarrow al^3 + 2alm^2 = m^2$$

➤ **Example 37.** The equation of the normal to the curve $y = e^{-2|x|}$ at the point where the curve cuts the line $x = 1/2$, is

- (a) $2e(ex + 2y) = e^2 - 4$
(b) $2e(ex - 2y) = e^2 - 4$
(c) $2e(ey - 2x) = e^2 - 4$
(d) None of the above

Sol. (b) At the point $x = \frac{1}{2}$, $y = e^{-1}$

$$\text{Since, } x = \frac{1}{2} > 0$$

$$\therefore y = e^{-2x}$$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = -2e^{-2x}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{\left(\frac{1}{2}, \frac{1}{e} \right)} = -2e^{-1} = -\frac{2}{e}$$

Thus, the equation of normal is

$$\left(y - \frac{1}{e} \right) = \frac{e}{2} \left(x - \frac{1}{2} \right)$$

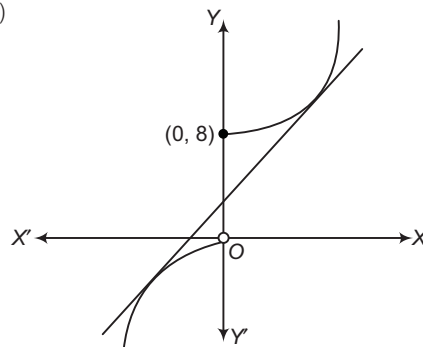
$$\Rightarrow 2e(ex - 2y) = e^2 - 4$$

➤ **Example 38.** Let $f(x) = \begin{cases} -x^2, & \text{for } x < 0 \\ x^2 + 8, & \text{for } x \geq 0 \end{cases}$.

Then, x -intercept of the line, i.e. the tangent to the graph of $f(x)$ is

- (a) zero (b) -1 (c) -2 (d) -4

Sol. (b)



Let $y = mx + c$ be a tangent to $f(x)$

$$y = x^2 + 8, \text{ for } x \geq 0$$

$$\Rightarrow mx + c = x^2 + 8$$

$$\Rightarrow x^2 - mx + 8 - c = 0 \text{ [for the line to be tangent, } D = 0]$$

$$\therefore m^2 = 4(8 - c)$$

Again $y = x^2$, for $x < 0$

$$\therefore mx + c = -x^2 \Rightarrow x^2 = mx + c = 0$$

$$\Rightarrow D = 0 \Rightarrow m^2 = 4c \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$c = 4, m = 4$$

$$\therefore y = 4x + 4$$

$$\text{Put } y = 0 \Rightarrow x = -1$$

➤ **Example 39.** The equation of the normal to the curve $y = (1+x)^y + \sin^{-1}(\sin^2 x)$ at $x=0$ is

$$(a) x + y + 1 = 0 \quad (b) x + 2y - 1 = 0$$

$$(c) x + y - 1 = 0 \quad (d) 2x + y + 1 = 0$$

Sol. (c) Given, $y = (1+x)^y + \sin^{-1}(\sin^2 x)$

Let $y = u + v$, where $u = (1+x)^y$, $v = \sin^{-1}(\sin^2 x)$,

On differentiating, we get

$$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \quad \dots(i)$$

Now,

$$u = (1+x)^y$$

On taking logarithm both sides, we get

$$\log_e u = y \log_e (1+x)$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = \frac{y}{1+x} + \frac{dy}{dx} \{ \log_e (1+x) \}$$

$$\Rightarrow \frac{du}{dx} = (1+x)^y \left[\frac{y}{1+x} + \frac{dy}{dx} \log_e (1+x) \right] \quad \dots(ii)$$

Again, $v = \sin^{-1}(\sin^2 x) \Rightarrow \sin v = \sin^2 x$

$$\Rightarrow \cos v \frac{dv}{dx} = 2 \sin x \cos x$$

$$\Rightarrow \frac{dv}{dx} = \frac{1}{\cos v} (2 \sin x \cos x)$$

$$\Rightarrow \frac{dv}{dx} = \frac{2 \sin x \cos x}{\sqrt{1 - \sin^2 v}} = \frac{2 \sin x \cos x}{\sqrt{1 - \sin^4 x}} \quad \dots(iii)$$

From Eq. (i),

$$\frac{dy}{dx} = (1+x)^y \left[\frac{y}{1+x} + \frac{dy}{dx} \log_e (1+x) \right] + \frac{2 \sin x \cos x}{\sqrt{1 - \sin^4 x}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(1+x)^{y-1} + \frac{2 \sin x \cos x}{\sqrt{1 - \sin^4 x}}}{1 - (1+x)^y \log_e (1+x)}$$

At $x=0$,

$$y = (1+0)^y + \sin^{-1} \sin^2(0) = 1$$

$$\therefore \frac{dy}{dx} = \frac{1(1+0)^{1-1} + \frac{2 \sin 0 \cdot \cos 0}{\sqrt{1 - \sin^4 0}}}{1 - (1+0)^1 \log_e (1+0)} \Rightarrow \frac{dy}{dx} = 1$$

Now, the slope of the normal is

$$m = -\frac{1}{\left(\frac{dy}{dx}\right)_{(0,1)}} = -1$$

Hence, the required equation of the normal is

$$y - 1 = (-1)(x - 0)$$

$$\text{i.e. } y + x - 1 = 0$$

➤ **Example 40.** If the curve $y = ax^3 + bx^2 + cx + 5$ touches the X-axis at $P(-2, 0)$ and cuts the Y-axis at a point $Q(0, 5)$ where its slope is 3, then the values of a , b and c are

$$(a) a = -\frac{1}{2}, b = -\frac{3}{4} \text{ and } c = 3$$

$$(b) a = -\frac{3}{4}, b = -\frac{1}{2} \text{ and } c = 3$$

$$(c) a = -\frac{1}{2}, b = 3 \text{ and } c = -\frac{3}{4}$$

$$(d) a = 3, b = -\frac{1}{2} \text{ and } c = -\frac{3}{4}$$

Sol. (a) Given, $y = ax^3 + bx^2 + cx + 5$ touches X-axis at $P(-2, 0)$ which implies that X-axis is tangent at $(-2, 0)$ and the curve is also passes through $(-2, 0)$.

The curve cuts Y-axis at $(0, 5)$ and gradient at this point is given 3, therefore at $(0, 5)$, slope of the tangent is 3.

$$\text{Now, } \frac{dy}{dx} = 3ax^2 + 2bx + c$$

Since, X-axis is tangent at $(-2, 0)$.

$$\therefore \left(\frac{dy}{dx}\right)_{x=-2} = 0$$

$$\Rightarrow 0 = 3a(-2)^2 + 2b(-2) + c$$

$$\Rightarrow 0 = 12a - 4b + c \quad \dots(i)$$

Again, slope of tangent at $(0, 5)$ is 3.

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,5)} = 3$$

$$\Rightarrow 3 = 3a(0)^2 + 2b(0) + c$$

$$\Rightarrow 3 = c \quad \dots(ii)$$

Since, the curve passes through $(-2, 0)$.

$$\therefore 0 = a(-2)^3 + b(-2)^2 + c(-2) + 5$$

$$\Rightarrow 0 = -8a + 4b - 2c + 5 \quad \dots(iii)$$

From Eqs. (i) and (ii), we get

$$12a - 4b = -3 \quad \dots(iv)$$

From Eqs. (iii) and (ii), we get

$$-8a + 4b = 1 \quad \dots(v)$$

On adding Eqs. (iv) and (v), we get

$$4a = -2$$

$$\Rightarrow a = -\frac{1}{2}$$

On putting $a = -\frac{1}{2}$ in Eq. (iv), we get

$$12\left(-\frac{1}{2}\right) - 4b = -3$$

$$\Rightarrow -6 - 4b = -3$$

$$\Rightarrow -3 = 4b$$

$$\Rightarrow b = -\frac{3}{4}$$

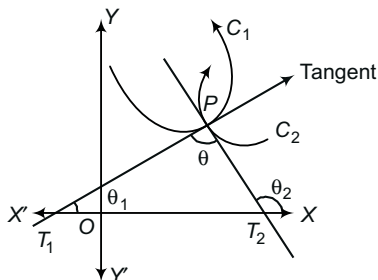
$$\text{Hence, } a = -\frac{1}{2}$$

$$b = -\frac{3}{4}$$

$$\text{and } c = 3$$

Angle of Intersection of Two Curves

The angle of intersection of two curves is defined as the angle between the tangents to the two curves at their point of intersection.



Let C_1 and C_2 be two curves having equations $y = f(x)$ and $y = g(x)$, respectively.

Let PT_1 and PT_2 be tangents to the curves C_1 and C_2 at their point of intersection.

Let θ be the angle between the two tangents PT_1 , PT_2 and θ_1, θ_2 are the angles made by tangents with the positive direction of X -axis in anti-clockwise sense.

$$\text{Then, } m_1 = \tan \theta_1 = \left(\frac{dy}{dx} \right)_{C_1}$$

$$m_2 = \tan \theta_2 = \left(\frac{dy}{dx} \right)_{C_2}$$

From the figure, it follows

$$\theta = \theta_2 - \theta_1$$

$$\Rightarrow \tan \theta = \tan(\theta_2 - \theta_1)$$

$$= \frac{\tan \theta_2 - \tan \theta_1}{1 + \tan \theta_2 \tan \theta_1}$$

$$\Rightarrow \tan \theta = \frac{\left(\frac{dy}{dx} \right)_{C_2} - \left(\frac{dy}{dx} \right)_{C_1}}{1 + \left(\frac{dy}{dx} \right)_{C_1} \left(\frac{dy}{dx} \right)_{C_2}}$$

Angle of intersection of these curves is defined as acute angle between the tangents.

Orthogonal Curves

If the angle of intersection of two curves is right angle, then the two curves are said to be orthogonal.

i.e. if the curves are orthogonal, then $\theta = \frac{\pi}{2}$.

$$\Rightarrow 1 + \left(\frac{dy}{dx} \right)_{C_1} \left(\frac{dy}{dx} \right)_{C_2} = 0$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{C_1} \left(\frac{dy}{dx} \right)_{C_2} = -1$$

Example 41. The angle between the tangents to the curve $y^2 = 2ax$ at the points where $x = \frac{a}{2}$, is

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Sol. (d) We have, $y^2 = 2ax$

$$\text{Put } x = \frac{a}{2}, y^2 = 2a \left(\frac{a}{2} \right)$$

$$\Rightarrow y = \pm a$$

$$\therefore \text{ The points are } \left(\frac{a}{2}, a \right) \text{ and } \left(\frac{a}{2}, -a \right).$$

On differentiating Eq. (i) w.r.t. x , we get

$$2y \frac{dy}{dx} = 2a \Rightarrow \frac{dy}{dx} = \frac{a}{y}$$

$$\text{At } \left(\frac{a}{2}, a \right), \quad \frac{dy}{dx} = \frac{a}{y} = \frac{a}{a} = 1 = m_1 \quad [\text{say}]$$

$$\text{At } \left(\frac{a}{2}, -a \right), \quad \frac{dy}{dx} = \frac{a}{y} = \frac{a}{-a} = -1 = m_2 \quad [\text{say}]$$

Since, $m_1 m_2 = -1$, the two tangents are at right angles.

Example 42. The angle between the tangents at those points on the curve $y = (x+1)(x-3)$, where it meets X -axis, is

- (a) $\pm \tan^{-1} \left(\frac{15}{8} \right)$ (b) $\pm \tan^{-1} \left(\frac{8}{15} \right)$
- (c) $\pm \frac{\pi}{4}$ (d) None of these

Sol. (b) We have,

$$y = (x+1)(x-3) = x^2 - 2x - 3 \quad \dots(i)$$

For points on X -axis, $y = 0$ which gives

$$(x+1)(x-3) = 0 \Rightarrow x = -1, 3$$

Therefore, the curve meets X -axis at $(-1, 0)$ and $(3, 0)$.

On differentiating the given equation w.r.t. x , we get

$$\frac{dy}{dx} = 2x - 2$$

$$\therefore \left[\frac{dy}{dx} \right]_{(-1, 0)} = -4 = \tan \theta_1$$

$$\Rightarrow \theta_1 = \tan^{-1}(-4)$$

$$\text{and } \left[\frac{dy}{dx} \right]_{(3, 0)} = 4 = \tan \theta_2$$

$$\Rightarrow \theta_2 = \tan^{-1} 4$$

Hence, the angle between these tangents is given by

$$\pm(\theta_1 - \theta_2) = \pm[\tan^{-1}(-4) - \tan^{-1}(4)]$$

$$= \pm \tan^{-1} \left(\frac{-4-4}{1+(-4)(4)} \right) = \pm \tan^{-1} \frac{8}{15}$$

Example 43. The curve $x^3 - 3xy^2 = a$ and $3x^2y - y^3 = b$, where a and b are constants, cut each other

- (a) at an angle $\frac{\pi}{3}$ (b) at an angle $\frac{\pi}{4}$
- (c) orthogonally (d) None of these

Sol. (c) The two curves are

$$x^3 - 3xy^2 = a \quad \dots(i)$$

and $3x^2y - y^3 = b \quad \dots(ii)$

On differentiating Eq. (i) w.r.t. x , we get

$$3x^2 - 3y^2 - 6xy \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$$

On differentiating Eq. (ii) w.r.t. x , we get

$$6xy + 3x^2 \frac{dy}{dx} - 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2xy}{x^2 - y^2}$$

The product of $\frac{dy}{dx}$ for the two curves

$$= \left(\frac{x^2 - y^2}{2xy} \right) \times \left(\frac{-2xy}{x^2 - y^2} \right) = -1$$

$\therefore$ The curves cut each other orthogonally.

➤ **Example 44.** The angle of intersection of the curves $y = 2\sin^2 x$ and $y = \cos 2x$ at $x = \frac{\pi}{6}$ is

(a) $\frac{\pi}{4}$

(b) $\frac{\pi}{3}$

(c) $\frac{\pi}{2}$

(d) None of these

Sol. (b) We have,

$$y = 2\sin^2 x \quad \dots(i)$$

and $y = \cos 2x \quad \dots(ii)$

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = 4 \sin x \cos x$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{x=\frac{\pi}{6}} = 4 \cdot \left(\frac{1}{2} \right) \cdot \frac{\sqrt{3}}{2} = \sqrt{3} = m_1 \quad [\text{say}]$$

On differentiating Eq. (ii) w.r.t. x , we get

$$\frac{dy}{dx} = -2 \sin 2x$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{x=\frac{\pi}{6}} = -2 \sin \frac{\pi}{3} = -\sqrt{3} = m_2 \quad [\text{say}]$$

Hence, angle between two curves is

$$\theta = \pm \tan^{-1} \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) = \pm \tan^{-1} \sqrt{3}$$

$$= \frac{\pi}{3} \text{ or } \frac{2\pi}{3}$$

➤ **Example 45.** The angle at which the curves $y = \sin x$ and $y = \cos x$ intersect in $[0, \pi]$, is

(a) $\pm \tan^{-1} 2\sqrt{2}$

(b) $\pm \tan^{-1} \sqrt{2}$

(c) $\pm \tan^{-1} \left(\frac{1}{\sqrt{2}} \right)$

(d) None of the above

Sol. (a) We have,

$$y = \sin x \quad \dots(i)$$

and $y = \cos x \quad \dots(ii)$

The two curves intersect at $x = \frac{\pi}{4}$ in $[0, \pi]$.

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = \cos x$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=\frac{\pi}{4}} = \frac{1}{\sqrt{2}} = m_1$$

[say]

On differentiating Eq. (ii) w.r.t. x , we get

$$\frac{dy}{dx} = -\sin x$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=\frac{\pi}{4}} = -\frac{1}{\sqrt{2}} = m_2 \quad (\text{say})$$

Hence, the angle between the two curves is

$$\theta = \pm \tan^{-1} \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right)$$

$$= \pm \tan^{-1} 2\sqrt{2}$$

➤ **Example 46.** The two tangents to the curve $ax^2 + 2hxy + by^2 = 1$, $a > 0$ at the points, where it crosses X -axis, are

(a) parallel

(b) perpendicular

(c) inclined at an angle $\frac{\pi}{4}$

(d) None of the above

Sol. (a) When the curve meets X -axis, then

$$\Rightarrow y = 0$$

$$ax^2 = 1$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{a}}$$

Hence, the points are $\left(\frac{1}{\sqrt{a}}, 0 \right)$ and $\left(-\frac{1}{\sqrt{a}}, 0 \right)$.

On differentiating the given equation w.r.t. x , we get

$$2ax + 2h \left(y + x \frac{dy}{dx} \right) + 2by \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{ax + hy}{hx + by}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{\left(\frac{1}{\sqrt{a}}, 0 \right)} = -\frac{a}{h}$$

and $\left[\frac{dy}{dx} \right]_{\left(-\frac{1}{\sqrt{a}}, 0 \right)} = -\frac{a}{h}$

Hence, the tangents at these points are parallel.

➤ **Example 47.** The curves $4x^2 + 9y^2 = 72$ and $x^2 - y^2 = 5$ at $(3, 2)$

(a) touch each other

(b) cut orthogonally

(c) intersect at 45°

(d) intersect at 60°

Sol. (b) We have, $4x^2 + 9y^2 = 72$

On differentiating w.r.t x , we get

$$8x + 18y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{4x}{9y}$$

$$\text{At } (3, 2), \frac{dy}{dx} = -\frac{4}{9} \times \frac{3}{2} = -\frac{2}{3}$$

$$\text{Also, } x^2 - y^2 = 5$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\text{At } (3, 2), \frac{dy}{dx} = \frac{3}{2}$$

Hence, the curves cut orthogonally.

Length of Tangent, Sub-tangent, Normal and Sub-normal

Length of Tangent

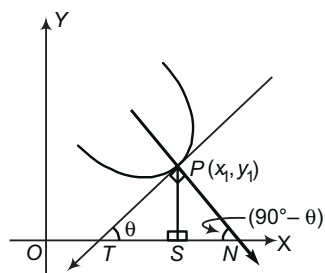
The length of the segment PT of the tangent between the point of contact and X -axis is called the length of tangent.

Sub-tangent

The projection of the segment PT along X -axis (ST) is called the sub-tangent.

Length of Normal

The length of segment PN , i.e. the portion of the normal intercepted between the point on the curve and X -axis is called the length of normal.



Sub-normal

The projection of the segment PN along X -axis (SN) is called the sub-normal. From the figure.

If PT makes an angle θ with X -axis, then

$$\tan \theta = \left(\frac{dy}{dx} \right)_{(x_1, y_1)}$$

$$\text{Thus, we can say, } \frac{ST}{PS} = \cot \theta$$

$$\Rightarrow \text{Sub-tangent} = ST = PS \cot \theta$$

$$\text{or length of sub-tangent} = ST = \left| y_1 \left(\frac{dx}{dy} \right)_{(x_1, y_1)} \right|$$

$$\Rightarrow \frac{SN}{PS} = \cot(90^\circ - \theta)$$

$$\Rightarrow \text{Sub-normal} = SN = PS \tan \theta$$

$$\text{or length of sub-normal} = SN = \left| y_1 \left(\frac{dy}{dx} \right)_{(x_1, y_1)} \right|$$

$$\text{Length of tangent} = PT = \sqrt{y_1^2 + y_1^2 \left(\frac{dx}{dy} \right)_{(x_1, y_1)}^2}$$

$$\Rightarrow PT = \left| y_1 \sqrt{1 + \left(\frac{dx}{dy} \right)_{(x_1, y_1)}^2} \right|$$

$$\text{Length of normal} = PN = \sqrt{y_1^2 + y_1^2 \left(\frac{dy}{dx} \right)_{(x_1, y_1)}^2}$$

$$\Rightarrow PN = \left| y_1 \sqrt{1 + \left(\frac{dy}{dx} \right)_{(x_1, y_1)}^2} \right|$$

➤ **Example 48.** The sub-normal at any point of the curve $x^2 y^2 = a^2 (x^2 - a^2)$ varies as

- (a) (abscissa)⁻³ (b) (abscissa)³
(c) (ordinate)⁻³ (d) None of these

Sol. (a) We have,

$$x^2 y^2 = a^2 (x^2 - a^2) \quad \dots(i)$$

$$\Rightarrow x^2 \cdot 2y \frac{dy}{dx} + y^2 \cdot 2x = a^2 \cdot 2x$$

$$\Rightarrow \frac{dy}{dx} = \frac{a^2 - y^2}{xy}$$

$$\begin{aligned} \therefore \text{Sub-normal} &= y \frac{dy}{dx} = \frac{a^2 - y^2}{x} \\ &= \frac{x^2 (a^2 - y^2)}{x^3} = \frac{a^4}{x^3} \\ &\quad [\text{from Eq. (i), } x^2 (a^2 - y^2) = a^4] \end{aligned}$$

Thus, the sub-normal varies inversely as the cube of its abscissa.

➤ **Example 49.** The sub-tangent at any point of the curve $x^m y^n = a^{m+n}$ varies as

- (a) (abscissa)² (b) (abscissa)³
(c) abscissa (d) ordinate

Sol. (c) We have, $x^m y^n = a^{m+n}$

$$\Rightarrow m \log x + n \log y = (m+n) \log a$$

$$\therefore \frac{m}{x} + \frac{n}{y} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dx}{dy} = -\frac{nx}{my}$$

$$\text{Hence, sub-tangent} = y \frac{dx}{dy} = \left(\frac{nx}{m} \right) \propto x$$

➤ **Example 50.** If at any point on a curve, the sub-tangent and sub-normal are equal, then the length of the normal is equal to

- (a) $\sqrt{2}$ ordinate (b) ordinate
(c) $\sqrt{2}$ ordinate (d) None of these

Sol. (a) We are given, sub-tangent = sub-normal

$$\Rightarrow \frac{y}{dy/dx} = y \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \pm 1$$

$$\text{So, length of the normal} = y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\ = y \cdot \sqrt{2} = \sqrt{2} \text{ (ordinate)}$$

➤ **Example 51.** For the parabola $y^2 = 4ax$, the ratio of the sub-tangent to the abscissa is

- (a) 1:1 (b) 2:1
(c) $x:y$ (d) $x^2:y$

Sol. (b) We have, $y^2 = 4ax$

$$\Rightarrow 2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\text{Sub-tangent} = \frac{y}{\frac{dy}{dx}} = \frac{y}{\frac{2a}{y}} = \frac{y^2}{2a} = \frac{4ax}{2a} = 2x$$

$$\therefore \text{Sub-tangent : Abscissa} = 2x : x = 2 : 1$$

Work Book Exercise 7.2

- 1 Equation of the line through the point $\left(\frac{1}{2}, 2\right)$

and tangent to the parabola $y = \frac{-x^2}{2} + 2$ and

secant to the curve $y = \sqrt{4 - x^2}$ is

- a $2x + 2y - 5 = 0$ b $2x + 2y - 3 = 0$
c $y - 2 = 0$ d None of these

- 2 If a variable tangent to the curve $x^2y = c^3$ makes intercepts a, b on X and Y -axes respectively, then the value of a^2b is

- a $27c^3$ b $\frac{4}{27}c^3$ c $\frac{27}{4}c^3$ d $\frac{4}{9}c^3$

- 3 The set of values of p for which the equation $|\ln x| - px = 0$ possess three distinct roots, is

- a $\left(0, \frac{1}{e}\right)$ b $(0, 1)$ c $(1, e)$ d $(0, e)$

- 4 The sum of the intercepts made on the axes of coordinates by any tangent to the curve $\sqrt{x} + \sqrt{y} = 2$ is equal to

- a 4 b 2
c 8 d None of these

- 5 The angle between the tangent lines to the graph of the function $f(x) = \int_2^x (2t - 5)dt$ at the points, where the graph cuts the X -axis, is

- a $\frac{\pi}{6}$ b $\frac{\pi}{4}$ c $\frac{\pi}{3}$ d $\frac{\pi}{2}$

- 6 Two curves $C_1 : y = x^2 - 3$ and $C_2 : y = kx^2$, $k \in R$ intersect each other at two different points. The tangent drawn to C_2 at one of the points of intersection $A \equiv (a, y_1)$, $a > 0$ meets C_1 again at $B(1, y_2)$ ($y_1 \neq y_2$). The value of a is

- a 4 b 3 c 2 d 1

- 7 A curve is represented by the equations $x = \sec^2 t$ and $y = \cot t$, where t is a parameter. If the tangent at the point P on the curve, where $t = \frac{\pi}{4}$ meet the curve again at the point Q , then $|PQ|$ is equal to

- a $\frac{5\sqrt{3}}{2}$ b $\frac{5\sqrt{5}}{2}$
c $\frac{2\sqrt{5}}{3}$ d $\frac{3\sqrt{5}}{2}$

- 8 The equation of common tangent to the curves $y^2 = 8x$ and $xy = -1$ is

- a $3y = 9x + 2$ b $y = 2x + 1$
c $2y = x + 8$ d $y = x + 2$

- 9 The slope of the tangent of the curve $y = \int_0^x \frac{dx}{1+x^3}$ at the point, where $x = 1$, is

- a $\frac{1}{2}$ b 1
c $\frac{1}{4}$ d None of these

- 10 The acute angle between the curves $y = |x^2 - 1|$ and $y = |x^2 - 3|$ at their points of intersection when $x > 0$, is

- a $\frac{\pi}{4}$ b $\tan^{-1}\left(\frac{4\sqrt{2}}{7}\right)$
c $\frac{\pi}{6}$ d $\tan^{-1}\left(\frac{4}{7}\right)$

- 11 In the curve $x^{m+n} = a^{m-n}y^{2n}$, if the m th power of the sub-tangent varies as the k th power of sub-normal, then k equals

- a m b n c $\frac{1}{n}$ d $\frac{1}{m}$

- 12 The equation of family of curves for which the length of normal at any point P is equal to the distance from origin, is

- a $x^2 = -y + C$ b $y^2 = \pm x^2 + C$
c $x = \pm y + C$ d $2x^2 = \pm y^2 + C$

- 13 The angle of intersection of curves $y = [|\sin x| + |\cos x|]$, where $[]$ denotes the greatest integer function and $x^2 + y^2 = 5$, is

- a $\tan^{-1} 2$ b $\tan^{-1} \frac{1}{2}$
c $\tan^{-1} \sqrt{2}$ d $\tan^{-1} \left(\frac{1}{\sqrt{2}}\right)$

Rolle's Theorem

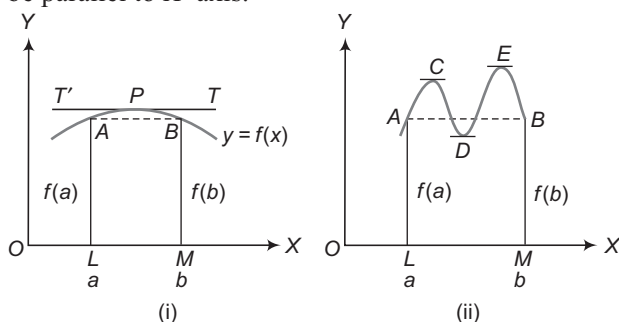
If a function $f(x)$ is

- continuous in the closed interval $[a, b]$, i.e. continuous at each point in the interval $[a, b]$
- differentiable in an open interval (a, b) , i.e. differentiable at each point in the open interval (a, b)
- $f(a) = f(b)$

Then, there will be atleast one point c in the interval (a, b) , such that $f'(c) = 0$.

Geometrical Meaning of Rolle's Theorem

If the graph of a function $y = f(x)$ is continuous at each point from the point $A\{a, f(a)\}$ to the point $B\{b, f(b)\}$ and tangent at each point between point A and B is unique, i.e. tangent at each point between A and B exists and ordinates, i.e. y -coordinates of points A and B are equal, then there will be atleast one point P on the curve between A and B at which tangent will be parallel to X -axis.



In Fig. (i), there is only one such point P , where tangent is parallel to X -axis but in Fig. (ii), there are more than one such point, where tangents are parallel to X -axis.

- Converse of Rolle's theorem may not be true, i.e. if a function $f(x)$ is such that $f'(c) = 0$ for atleast one c in the open interval (a, b) , then it is not necessary that
 - $f(x)$ is continuous in $[a, b]$
 - $f(x)$ is differentiable in (a, b)
 - $f(a) = f(b)$

Application of Rolle's Theorem for Isolating the Real Roots of an Equation $f(x) = 0$

Suppose a and b are two real numbers such that

- $f(x)$ and its derivative $f'(x)$ are continuous for $a \leq x \leq b$.
- $f(a)$ and $f(b)$ have opposite signs.
- $f'(x)$ is different from zero, for all values of x between a and b .

Then, there is one and only real root of the equation $f(x) = 0$ between a and b .

Example 52. The value of c in Rolle's theorem when $f(x) = 2x^3 - 5x^2 - 4x + 3$, $x \in \left[\frac{1}{2}, 3\right]$, is

- (a) 2 (b) $-\frac{1}{3}$ (c) -2 (d) $\frac{2}{3}$

Sol. (a) Clearly, $f(x)$ being a polynomial is continuous on $\left[\frac{1}{2}, 3\right]$ and differentiable on $\left(\frac{1}{2}, 3\right)$.

Also, $f\left(\frac{1}{2}\right) = f(3) = 0$

So, by Rolle's theorem, there exists $c \in \left(\frac{1}{2}, 3\right)$ such that

$$\begin{aligned} f'(c) &= 0 \\ \Rightarrow 6c^2 - 10c - 4 &= 0 \quad [\because f'(x) = 6x^2 - 10x - 4] \\ \Rightarrow 3c^2 - 5c - 2 &= 0 \\ \Rightarrow (c-2)(3c+1) &= 0 \\ \Rightarrow c &= 2 \in \left(\frac{1}{2}, 3\right) \end{aligned}$$

Example 53. In which of the following function, is Rolle's theorem applicable?

- $f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$ on $[0, 1]$
- $f(x) = \begin{cases} \frac{\sin x}{x}, & -\pi \leq x < 0 \\ 0, & x = 0 \end{cases}$ on $[-\pi, 0]$
- $f(x) = \frac{x^2 - x - 6}{x - 1}$ on $[-2, 3]$
- $f(x) = \begin{cases} \frac{x^3 - 2x - 5x + 6}{x - 1}, & \text{if } 0 \neq 1, \text{ on } [-2, 3] \\ -6, & \text{if } x = 1 \end{cases}$

Sol. (d)

- Discontinuous at $x = 1 \Rightarrow$ Not applicable.
- $f(x)$ is not continuous (jump discontinuity) at $x = 0$.
- Discontinuity (missing point) at $x = 1 \Rightarrow$ Not applicable.
- Notice that $x^3 - 2x^3 - 5x + 6 = (x-1)(x^2 - x - 6)$
Hence, $f(x) = x^2 - x - 6$, if $x \neq 1$ and $f(1) = -6$.
 $\Rightarrow f$ is continuous at $x = 1$. So, $f(x) = x^2 - x - 6$ is continuous in the interval $[-2, 3]$.
Also, note that $f(-2) = f(3) = 0$. Hence, Rolle's theorem applies $f'(x) = 2x - 1$.
For $f'(x) = 0$, we obtain $x = 1/2$ which lies between -2 and 3 .

Example 54. If $a + b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ in the interval $(0, 1)$, has

- (a) atleast one root (b) atmost one root
(c) no root (d) None of these

Sol. (a) Consider the polynomial $f(x)$ given by

$$f(x) = ax^3 + bx^2 + cx, x \in [0, 1]$$

$$\Rightarrow f'(x) = 3ax^2 + 2bx + c$$

Clearly, $f(x)$ being a polynomial, is continuous on $[0, 1]$ and differentiable on $(0, 1)$.

Also, $f(0) = 0$ and $f(1) = a + b + c = 0$ [given]

Thus, 0 and 1 are two roots of $f(x)$.

Therefore, by the algebraic interpretation of Rolle's theorem, $f'(x) = 0$ for atleast one $x \in (0, 1)$, i.e. $3ax^2 + 2bx + c = 0$ has atleast one root between 0 and 1.

➤ **Example 55.** If $f(x) = \begin{vmatrix} \sin x & \sin a & \sin b \\ \cos x & \cos a & \cos b \\ \tan x & \tan a & \tan b \end{vmatrix}$,

where $0 < a < b < \frac{\pi}{2}$, then the equation $f'(x) = 0$

in the interval (a, b) , has

- (a) atleast one root (b) atmost one root
(c) no root (d) None of these

Sol. (a) Here, $f(a) = \begin{vmatrix} \sin a & \sin a & \sin b \\ \cos a & \cos a & \cos b \\ \tan a & \tan a & \tan b \end{vmatrix} = 0$

Also, $f(b) = 0$.

Moreover, as $\sin x$, $\cos x$ and $\tan x$ are continuous and differentiable in $[a, b]$ for $0 < a < b < \frac{\pi}{2}$, therefore $f(x)$ is

also continuous and differentiable in $[a, b]$. Hence, by Rolle's theorem, there exists a real number c in (a, b) such that $f'(c) = 0$.

➤ **Example 56.** The equation $x \log x = 3 - x$ in the interval $(1, 3)$, has

- (a) exactly one root (b) atmost one root
(c) atleast one root (d) no root

Sol. (c) Let $f(x) = (x - 3) \log x$

Then, $f(1) = -2 \log 1 = 0$ and $f(3) = (3 - 3) \log 3 = 0$. As $(x - 3)$ and $\log x$ are continuous and differentiable in $[1, 3]$, therefore $(x - 3) \log x = f(x)$ is also continuous and differentiable in $[1, 3]$. Hence, by Rolle's theorem, there exists a value of x in $(1, 3)$ such that

$$f'(x) = 0$$

$$\Rightarrow \log x + (x - 3) \frac{1}{x} = 0$$

$$\Rightarrow x \log x = 3 - x$$

➤ **Example 57.** Between any two real roots of the equation $e^x \sin x = 1$, the equation $e^x \cos x = -1$, has

- (a) atleast one root (b) exactly one root
(c) atmost one root (d) no root

Sol. (a) Let $\alpha, \beta (\alpha < \beta)$ be any two real roots of

$$f(x) = e^{-x} - \sin x$$

$$\text{Then, } f(\alpha) = 0 = f(\beta)$$

Moreover, $f(x)$ is continuous and differentiable for all $x \in [\alpha, \beta]$.

Hence, from Rolle's theorem, there exists atleast one x in (α, β) such that

$$f'(x) = 0$$

$$\Rightarrow -e^{-x} - \cos x = 0$$

$$\Rightarrow -e^{-x}(1 + e^x \cos x) = 0$$

$$\Rightarrow e^x \cos x = -1$$

Lagrange's Mean Value Theorem

If a function $f(x)$ is

- continuous in the closed interval $[a, b]$, i.e. continuous at each point in the interval $[a, b]$.
- differentiable in an open interval (a, b) , i.e. differentiable at each point in the interval (a, b) .

Then, there will be atleast one point c , where $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Another Form of Lagrange's Mean Value Theorem

If a function $f(x)$ is

- continuous in the closed interval $[a, a + h]$
- differentiable in the open interval $(a, a + h)$

Then, there exists atleast one value $\theta, 0 < \theta < 1$ such that

$$f(a + h) = f(a) + hf'(a + \theta h)$$

Geometrical Meaning of Lagrange's Mean Value Theorem

Let $A\{a, f(a)\}$ and $B\{b, f(b)\}$ be two points on the curve $y = f(x)$.

Then, $OL = a, OM = b, AL = f(a), BM = f(b)$

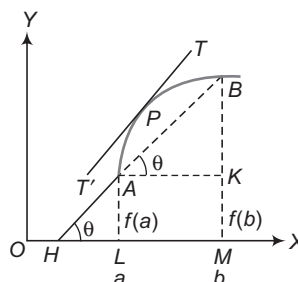
Now, slope of chord AB

$$\tan \theta = \frac{BK}{AK} = \frac{f(b) - f(a)}{b - a} \quad \dots(i)$$

By Lagrange's mean value theorem,

$$\frac{f(b) - f(a)}{b - a} = f'(c) = \text{Slope of tangent at point}$$

$$P\{c, f(c)\}$$



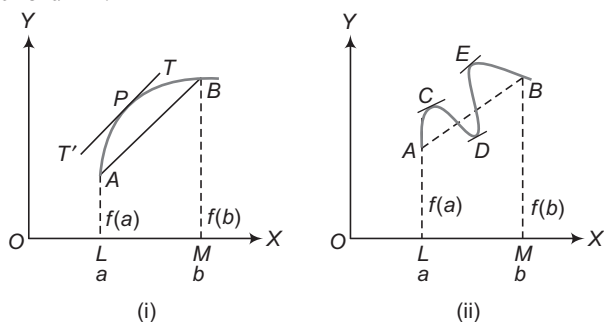
From Eq. (i), $\tan \theta = \text{Slope of tangent at } P$

$\Rightarrow \text{Slope of chord } AB = \text{Slope of tangent at } P$

Hence, chord $AB \parallel$ tangent PT

Thus, geometrical meaning of mean value theorem is as follows:

If the graph of a curve $y = f(x)$ is continuous at each point from the point $A\{a, f(a)\}$ to the point $B\{b, f(b)\}$ and tangent at each point between A and B exists, i.e. tangent is unique, then there will be atleast one point P on the curve between A and B , where tangent will be parallel to chord AB . In Fig. (i), there is only one such point P where tangent is parallel to chord AB but in Fig. (ii), there are more than one such points, where tangents are parallel to chord AB .



➤ **Example 58.** If $f(x)$ is differentiable in the interval $[2, 5]$ where $f(2) = \frac{1}{5}$ and $f(5) = \frac{1}{2}$, then there exists a number c , $2 < c < 5$ for which $f'(c)$ is equal to

- (a) $\frac{1}{2}$ (b) $\frac{1}{5}$
(c) $\frac{1}{10}$ (d) None of these

Sol. (c) As $f(x)$ is differentiable in $[2, 5]$, therefore it is also continuous in $[2, 5]$. Hence, by mean value theorem, there exists a real number c in $(2, 5)$ such that

$$f'(c) = \frac{f(5) - f(2)}{5 - 2} \Rightarrow f'(c) = \frac{\frac{1}{2} - \frac{1}{5}}{3} = \frac{1}{10}$$

➤ **Example 59.** In $[0, 1]$ Lagrange's mean value theorem is not applicable to

$$(a) f(x) = \begin{cases} \frac{1}{2} - x, & x < \frac{1}{2} \\ \left(\frac{1}{2} - x\right)^2, & x \geq \frac{1}{2} \end{cases}$$

$$(b) f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$(c) f(x) = x|x|$$

$$(d) f(x) = |x|$$

Sol. (a) For the function $f(x)$ given in option (a), we have

$$\left(\text{LHD at } x = \frac{1}{2}\right) = -1$$

and

$$\left(\text{RHD at } x = \frac{1}{2}\right) = 0$$

So, it is not differentiable at $x = \frac{1}{2} \in (0, 1)$.

Hence, Lagrange's mean value theorem is not applicable and options (b), (c) and (d) are continuous and differentiable in the interval $(0, 1)$.

➤ **Example 60.** Let $f(x)$ satisfy the requirement of Lagrange's mean value theorem in $[0, 2]$. If $f(0) = 0$ and $|f'(x)| \leq \frac{1}{2}$ for all $x \in [0, 2]$,

then

- (a) $f(x) \leq 2$
(b) $|f(x)| \leq 1$
(c) $f(x) = 2x$
(d) $f(x) = 3$ for atleast one x in $[0, 2]$

Sol. (b) Let $x \in (0, 2)$. Since, $f(x)$ satisfies the requirements of Lagrange's mean value theorem in $[0, 2]$. So, it also satisfies in $[0, x]$. Consequently, there exist $c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow f'(c) = \frac{f(x)}{x}$$

$$\Rightarrow \left|\frac{f(x)}{x}\right| = |f'(c)| \leq \frac{1}{2} \quad \left[\because |f'(x)| \leq \frac{1}{2}\right]$$

$$\Rightarrow |f(x)| \leq \frac{|x|}{2}$$

$$\Rightarrow |f(x)| \leq \frac{x}{2} \quad [\because x > 0]$$

$$\Rightarrow |f(x)| \leq 1 \quad [\because x \in (0, 2) \therefore x < 2]$$

➤ **Example 61.** Let $f'(x) = e^{x^2}$ and $f(0) = 10$. If $A < f(1) < B$ can be concluded from the mean value theorem, then the largest value of $(A - B)$ equals

- (a) e (b) $1 - e$
(c) $e - 1$ (d) $1 + e$

Sol. (b) Applying LMVT in $[0, 1]$ to the function $y = f(x)$, we get

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}, \text{ for some } c \in (0, 1)$$

$$\Rightarrow e^{c^2} = \frac{f(1) - f(0)}{1}$$

$$\Rightarrow f(1) - 10 = e^{c^2}, \text{ for some } c \in (0, 1)$$

$$\text{But } 1 < e^{c^2} < e \text{ in } (0, 1)$$

$$\therefore 1 < f(1) - 10 < e$$

$$\Rightarrow 11 < f(1) < 10 + e$$

$$\Rightarrow A = 11, B = 10 + e$$

$$\Rightarrow A - B = 1 - e$$

1 n is a positive integer. If the value of c prescribed in Rolle's theorem for the function $f(x) = 2x(x - 3)^n$ on the interval $[0, 3]$ is $\frac{3}{4}$, then

- the value of n is
- a 5 b 2
c 3 d 4
- 2 In the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies conditions of Rolle's theorem in $[1, 3]$ and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then values of a and b are respectively
- a 1, -6 b -1, 6
c -2, 1 d -1, $\frac{1}{2}$
- 3 For all $x \in (0, 1)$,
- a $e^x < 1 + x$ b $\log_e(1+x) < x$
c $\sin x > x$ d $\log_e x > x$
- 4 If $4a + 2b + c = 0$, then the equation $3ax^2 + 2bx + c = 0$ has atleast one real root lying in the interval
- a (0, 1) b (1, 2)
c (0, 2) d None of these
- 5 The length of longest interval in which Rolle's theorem can be applied for the function $f(x) = |x^2 - a^2|$, $a > 0$, is
- a $2a$ b $4a^2$ c $a\sqrt{2}$ d a

- 6** The value of c in Lagrange's theorem for the function $f(x) = \log_e \sin x$ in the interval $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$, is
- a** $\frac{\pi}{4}$ **b** $\frac{\pi}{2}$
c $\frac{2\pi}{3}$ **d** None of these
- 7** Let $f(x) = \sqrt{x-1} + \sqrt{x+24-10\sqrt{x-1}}$; $1 < x < 26$ be real valued function. Then, $f'(x)$ for $1 < x < 26$ is
- a** 0 **b** $\frac{1}{\sqrt{x-1}}$
c $2\sqrt{x-1} - 5$ **d** None of these
- 8** From mean value theorem,
 $f(b) - f(a) = (b-a)f'(a)$; $a \leq x_1 < b$, if $f(x) = \frac{1}{x}$, x_1 is equal to
- a** $\sqrt{ab}$ **b** $\frac{a+b}{2}$ **c** $\frac{2ab}{a+b}$ **d** $\frac{b-a}{b+a}$
- 9** If $f(x)$ satisfies the conditions of Rolle's theorem in $[1, 2]$ and $f(x)$ is continuous in $[1, 2]$, then $\int_1^2 f'(x) dx$ is equal to
- a** 3 **b** 0 **c** 1 **d** 2
- 10** If $f(x)$ satisfies the conditions for Rolle's theorem in $[3, 5]$, then $\int_3^5 f(x) dx$ is equal to
- a** 2 **b** -1 **c** 0 **d** $-\frac{4}{3}$

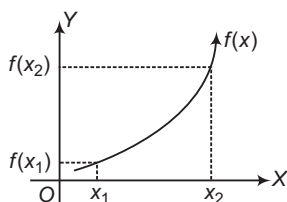
(i) Strictly Increasing Function

A function $f(x)$ is known as strictly increasing function in its domain, if $x_1 < x_2$

$$\Rightarrow f(x_1) < f(x_2)$$

i.e. for the smaller input, we have smaller output and for higher value of input, we have higher output.

Graphically, it can be expressed as shown in figure:



Here, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

Thus, the function is strictly increasing.

In the graph,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow \quad f'(x) = \lim_{h \rightarrow 0} \frac{+ve}{+ve} \quad \left[\begin{array}{l} \because \text{ as } x_1 < x_2 \\ \Rightarrow f(x_1) < f(x_2) \\ \Rightarrow f(x) < f(x+h) \end{array} \right.$$

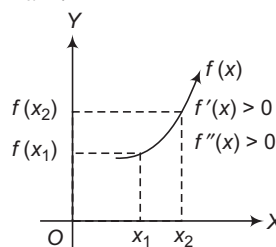
i.e. $f'(x) > 0$

➤ It means that the value of $f(x)$ will keep on increasing with an increase in the value of x or $f'(x) > 0, \forall x \in \text{domain}$.

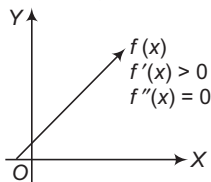
Classification of Strictly Increasing Function

Increasing function can be classified as

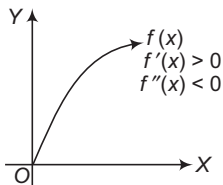
- (a) **Concave up** When $f'(x) > 0$ and $f''(x) > 0$,
 $\forall x \in \text{domain}$.



(b) When $f'(x) > 0$ and $f''(x) = 0, \forall x \in \text{domain}$



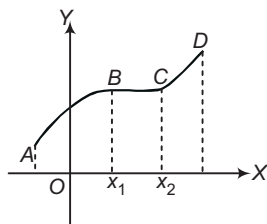
(c) **Concave down** When $f'(x) > 0$ and $f''(x) < 0, \forall x \in \text{domain}$.



(ii) Only Increasing or Non-decreasing Function

A function $f(x)$ is said to be non-decreasing, if for $x_1 < x_2$

$$\Rightarrow f(x_1) \leq f(x_2)$$



As shown in graph, for AB and CD portion, $x_1 < x_2$

$$\Rightarrow f(x_1) < f(x_2)$$

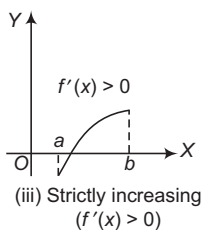
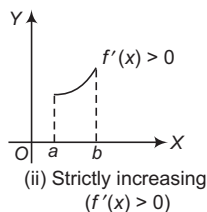
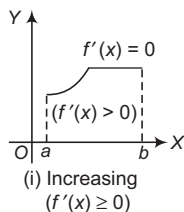
$$\text{and for BC, } x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$$

Hence, as a whole, we can say that for non-decreasing function,

$$x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

Obviously, for this, $f'(x) \geq 0$ where equality holds for horizontal path of the graph, i.e. in the interval of BC.

Fig. (i) shows only increasing or non-decreasing function and Fig. (ii), Fig. (iii) shows strictly increasing function on $[a, b]$.



Obviously in these type, question ask the interval in which a given function is increasing. While solving such types of problems, generally we follow the steps given below:

Step I First, we write the given function in standard form.

Step II Now, differentiate the given function with respect to given independent variable.

Step III Now, use the appropriate condition of increasing function and simplify for number line.

Step IV Get the range of given function from number line.

Step V Finally, we get the required interval in which the given function is increasing.

Example 62. The function $f(x) = 2 \log(x-2) - x^2 + 4x + 1$ increases on the interval

(a) (1, 2) (b) (2, 3)

(c) (1, 3) (d) (2, 4)

Sol. (b) Clearly, $f(x)$ is defined for all $x > 2$.

We have,

$$f(x) = 2 \log(x-2) - x^2 + 4x + 1$$

$$\Rightarrow f'(x) = \frac{2}{x-2} - 2x + 4$$

$$\Rightarrow f'(x) = 2 \left\{ \frac{1 - (x-2)^2}{x-2} \right\} = \frac{-2(x-1)(x-3)}{x-2}$$

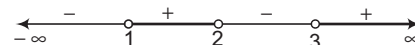
$$\Rightarrow f'(x) = -\frac{2(x-1)(x-3)(x-2)}{(x-2)^2}$$

$$\therefore f'(x) > 0$$

$$\Rightarrow -2(x-1)(x-3)(x-2) > 0$$

$$\Rightarrow (x-1)(x-2)(x-3) < 0$$

$$\Rightarrow x \in (-\infty, 1) \cup (2, 3)$$



Clearly, domain of $f(x)$ is $(2, \infty)$.

Thus, $f(x)$ is increasing on $(2, 3)$.

Example 63. The function $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$ is

an increasing on θ in the interval

(a) $\left[0, \frac{\pi}{3}\right]$ (b) $\left[\frac{\pi}{3}, \frac{\pi}{2}\right]$ (c) $\left[0, \frac{\pi}{2}\right]$ (d) $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$

Sol. (c) We have, $y = \frac{4 \sin \theta}{2 + \cos \theta} - \theta$

On differentiating w.r.t. θ , we get

$$\frac{dy}{d\theta} = \frac{d}{d\theta} \left[\frac{4 \sin \theta}{2 + \cos \theta} - \theta \right]$$

$$= \frac{(2 + \cos \theta) \frac{d}{d\theta} (4 \sin \theta) - 4 \sin \theta \frac{d}{d\theta} (2 + \cos \theta)}{(2 + \cos \theta)^2} - 1$$

$$= \frac{4 \cos \theta (2 + \cos \theta) - 4 \sin \theta (-\sin \theta)}{(2 + \cos \theta)^2} - 1$$

$$\begin{aligned}
 &= \frac{8 \cos \theta + 4 \cos^2 \theta + 4 \sin^2 \theta}{(2 + \cos \theta)^2} - 1 \\
 &= \frac{8 \cos \theta + 4(\cos^2 \theta + \sin^2 \theta)}{(2 + \cos \theta)^2} - 1 \\
 &= \frac{8 \cos \theta + 4}{(2 + \cos \theta)^2} - 1 = \frac{8 \cos \theta + 4 - (2 + \cos \theta)^2}{(2 + \cos \theta)^2} \\
 &= \frac{8 \cos \theta + 4 - 4 - \cos^2 \theta - 4 \cos \theta}{(2 + \cos \theta)^2} \\
 &= \frac{4 \cos \theta - \cos^2 \theta}{(2 + \cos \theta)^2} = \frac{\cos \theta (4 - \cos \theta)}{(2 + \cos \theta)^2}
 \end{aligned}$$

In the interval $\left[0, \frac{\pi}{2}\right]$, we have $\cos \theta \geq 0$.

Also, $4 > \cos \theta$
 $\Rightarrow (4 - \cos \theta) > 0$
 $\therefore \cos \theta (4 - \cos \theta) \geq 0$ and $(2 + \cos \theta)^2 > 0$
 $\Rightarrow \frac{\cos \theta (4 - \cos \theta)}{(2 + \cos \theta)^2} \geq 0 \Rightarrow \frac{dy}{d\theta} \geq 0$
Hence, given function is increasing in $\left[0, \frac{\pi}{2}\right]$.

Decreasing Function

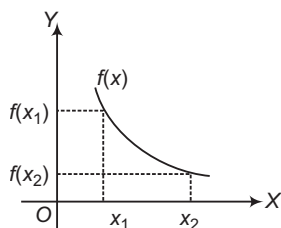
(i) Strictly Decreasing Function

A function $f(x)$ is known as strictly decreasing function in its domain, if

$$\begin{aligned}
 &x_1 < x_2 \\
 \Rightarrow &f(x_1) > f(x_2)
 \end{aligned}$$

i.e. for the smaller input, we have higher output and for higher value of input, we have smaller output.

Graphically, it can be expressed as shown in the figure:



Here, $x_1 < x_2$
and $f(x_1) > f(x_2)$, thus function is strictly decreasing.

In the above graph,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\text{as } x_1 < x_2$$

$$\Rightarrow f(x_1) > f(x_2)$$

$$\text{Thus, } f(x+h) < f(x)$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{-ve}{+ve}$$

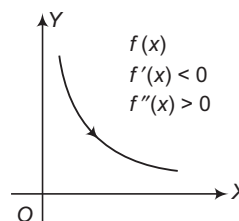
$$\Rightarrow f'(x) < 0$$

It means that the value of $f(x)$ will keep on decreasing with an increase in the value of x or $f'(x) < 0, \forall x \in \text{domain}$.

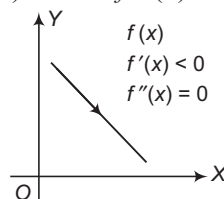
Classification of Strictly Decreasing Function

Decreasing function can be classified as

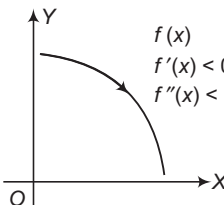
(a) **Concave up** When $f'(x) < 0$ and $f''(x) > 0, \forall x \in \text{domain}$.



(b) When $f'(x) < 0$ and $f''(x) = 0, \forall x \in \text{domain}$

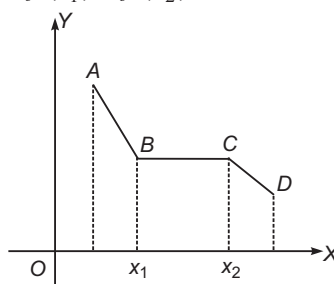


(c) **Concave down** When $f'(x) < 0$ and $f''(x) < 0, \forall x \in \text{domain}$.



(ii) Only Decreasing or Non-increasing Function

A function $f(x)$ is said to be non-increasing, if for $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$



As shown in graph,

For AB and CD portion, $x_1 < x_2$

$$\Rightarrow f(x_1) > f(x_2)$$

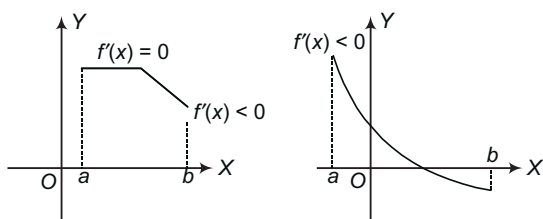
and for BC, $x_1 < x_2$

$$\Rightarrow f(x_1) = f(x_2)$$

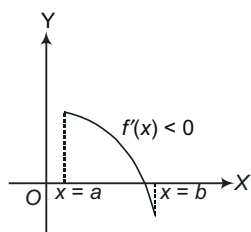
Hence, as a whole, we can say that, for non-increasing function, $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

Obviously, for this $f'(x) \leq 0$, where equality holds for horizontal path of the graph, i.e. in the interval of BC.

Fig. (i) shows only decreasing or non-increasing and Fig. (ii), Fig. (iii) show strictly decreasing on $[a, b]$.



(i) Only decreasing or non-increasing (ii) Strictly decreasing



(iii) Strictly decreasing

Example 64. On which of the following intervals is the function f given by $f(x) = x^{100} + \sin x - 1$ strictly decreasing?

- (a) $(0, 1)$ (b) $\left(\frac{\pi}{2}, \pi\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ (d) None of these

Sol. (d) Given, $f(x) = x^{100} + \sin x - 1$

$$\Rightarrow f'(x) = 100x^{99} + \cos x$$

(a) In the interval $(0, 1)$, $\cos x > 0$ and $100x^{99} > 0$.

$$\therefore f'(x) > 0$$

$$\left[\because \text{in } \left(0, \frac{\pi}{2}\right), \cos x > 0, \text{ i.e. } (0, 1.57), \cos x > 0 \right]$$

Thus, function f is strictly increasing in the interval $(0, 1)$.

(b) In the interval $\left(\frac{\pi}{2}, \pi\right)$ $\cos x < 0$ and $100x^{99} > 0$.

$$\text{Also, } 100x^{99} > 1$$

$$\therefore f'(x) > 0 \text{ in } \left(\frac{\pi}{2}, \pi\right)$$

Thus, function f is strictly increasing in the interval $\left(\frac{\pi}{2}, \pi\right)$.

(c) In the interval $\left(0, \frac{\pi}{2}\right)$, $\cos x > 0$ and $100x^{99} > 0$.

$$\therefore 100x^{99} + \cos x > 0$$

$$\Rightarrow f'(x) > 0 \text{ in } \left(0, \frac{\pi}{2}\right)$$

Thus, f is strictly increasing in the interval $\left(0, \frac{\pi}{2}\right)$.

Hence, function f is not strictly decreasing in the given intervals.

Example 65. Let $f(x) = -x^3 + ax^2 + bx + 5 \sin x \cos x$ be a decreasing function on the set R . Then, a and b satisfy

- (a) $a^2 + 3b + 15 > 0$ (b) $a^2 + 3b + 15 < 0$
(c) $a^2 - 3b - 15 > 0$ (d) $a^2 - 3b - 15 < 0$

Sol. (b) We have,

$$f(x) = -x^3 + ax^2 + bx + \frac{5}{2} \sin 2x$$

$$\Rightarrow f'(x) = -3x^2 + 2ax + b + 5 \cos 2x$$

For $f(x)$ to be decreasing on R , we must have $f'(x) < 0$, for all $x \in R$.

$$\Rightarrow -3x^2 + 2ax + b + 5 \cos 2x < 0, \forall x \in R$$

$$\Rightarrow -3x^2 + 2ax + b + 5 < 0, \forall x \in R$$

$$\Rightarrow 3x^2 - 2ax - b - 5 > 0, \forall x \in R$$

$$\Rightarrow 4a^2 - 4 \times 3(-b - 5) < 0 \quad [\because \text{discriminant} < 0]$$

$$\Rightarrow a^2 + 3b + 15 < 0$$

Monotonic Function

A function f is said to be monotonic or monotone in an interval I , if it is either increasing or decreasing in the interval I . $f(x) = \ln x$, $f(x) = 2^x$, $f(x) = -2x + 3$ are monotonic functions. $f(x) = x^2$ is monotonic in $(-\infty, 0)$ or $(0, \infty)$ but is not monotonic in R .

Properties of Monotonic Functions

- i. If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \leq 0$ ($f'(c) < 0$) for each $c \in (a, b)$, then $f(x)$ is monotonically (or strictly) decreasing function on $[a, b]$.
- ii. If $f(x)$ is continuous on $[a, b]$ such that $f'(c) \geq 0$ ($f'(c) > 0$) for each $c \in (a, b)$, then $f(x)$ is monotonically (strictly) increasing function on $[a, b]$.
- iii. If $f(x)$ and $g(x)$ are monotonically increasing (or decreasing) functions on $[a, b]$, then $gof(x)$ is a monotonically increasing (or decreasing) function on $[a, b]$.
- iv. If $f(x)$ and $g(x)$ are strictly increasing (or decreasing) on $[a, b]$, then gof is strictly increasing (or decreasing) on $[a, b]$.
- v. If one of the two functions $f(x)$ and $g(x)$ is strictly (or monotonically) increasing and other a strictly (monotonically) decreasing, then $gof(x)$ is strictly (monotonically) decreasing on $[a, b]$.
- vi. If $f(x)$ is strictly increasing function on an interval $[a, b]$ such that it is continuous, then f^{-1} is continuous on $[f(a), f(b)]$.

Operations of Monotonic Functions

I : Increasing, D : Decreasing, λ : Neither increasing nor decreasing				
$f(x)$	I	D	I	D
$g(x)$	I	I	D	D
$-f(x)$	D	I	D	I
$-g(x)$	D	D	I	I
$f(x) + g(x)$	I	I or D or λ	I or D or λ	D
$f(x) - g(x)$	I or D or λ	D	I	I or D or λ
$f(x) \cdot g(x)$	I	I or D or λ	I or D or λ	D
$\frac{f(x)}{g(x)}$	I or D or λ	D	I	I or D or λ
$\frac{1}{f(x)}$	D	I	D	I
$\frac{1}{g(x)}$	D	D	I	I
$(f \circ g)(x)$	I	D	D	I

➤ **Example 66.** The function $f(x) = x^3 - 3x$, is

- (a) increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $(-1, 1)$
 (b) decreasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $(-1, 1)$
 (c) increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$
 (d) decreasing on $(0, \infty)$ and increasing on $(-\infty, 0)$

Sol. (a) We have,

$$f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3$$

For $f(x)$ to be increasing, we must have

$$f'(x) \geq 0 \Rightarrow 3x^2 - 3 \geq 0 \Rightarrow x^2 - 1 \geq 0$$

$$\Rightarrow x \leq -1 \text{ or } x \geq 1$$

Hence, $f(x)$ is increasing on $(-\infty, -1] \cup [1, \infty)$ and decreasing on $(-1, 1)$.

➤ **Example 67.** Let the function

$f(x) = \tan^{-1}(\sin x + \cos x)$ be defined on $[0, 2\pi]$.

Then, $f(x)$ is

(a) strictly increasing on $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$

(b) strictly decreasing on $\left(\frac{\pi}{4}, 2\pi\right)$

(c) strictly increasing on $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, 2\pi\right)$

(d) strictly decreasing on $\left[\frac{\pi}{4}, \frac{7\pi}{4}\right]$

Sol. (a) We have, $f(x) = \tan^{-1}(\sin x + \cos x)$

$$\Rightarrow f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} \times (\cos x - \sin x)$$

$$\text{Now, } f'(x) > 0 \Rightarrow \cos x - \sin x > 0 \Rightarrow \sqrt{2} \cos\left(x + \frac{\pi}{4}\right) > 0$$

$$\Rightarrow x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$$

$$\text{and } f'(x) < 0 \Rightarrow \cos x - \sin x < 0 \Rightarrow x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$$

Hence, $f(x)$ is strictly increasing on $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$

and decreasing on $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$.

➤ **Example 68.** Function $f(x) = \frac{\lambda \sin x + 6 \cos x}{2 \sin x + 3 \cos x}$

is monotonic increasing, if

(a) $\lambda > 1$

(b) $\lambda < 1$

(c) $\lambda < 4$

(d) $\lambda > 4$

Sol. (d) We have, $f(x) = \frac{\lambda \sin x + 6 \cos x}{2 \sin x + 3 \cos x}$

$\therefore f$ is monotonic increasing, if $f'(x) > 0$

$$\text{i.e. } \frac{(2 \sin x + 3 \cos x)(\lambda \cos x - 6 \sin x)}{(2 \sin x + 3 \cos x)^2} - \frac{(\lambda \sin x + 6 \cos x)(2 \cos x - 3 \sin x)}{(2 \sin x + 3 \cos x)^2} > 0$$

$$\Rightarrow 3\lambda (\sin^2 x + 3 \cos^2 x) - 12 (\sin^2 x + \cos^2 x) > 0$$

$$\Rightarrow 3\lambda - 12 > 0$$

$$\Rightarrow \lambda > 4$$

➤ **Example 69.** If $a < 0$, $f(x) = e^{ax} + e^{-ax}$ and $S = \{x : f(x) \text{ is monotonically decreasing}\}$, then S is equal to

(a) $\{x : x > 0\}$

(b) $\{x : x < 0\}$

(c) $\{x : x > 1\}$

(d) $\{x : x < 1\}$

Sol. (b) Since, $f(x)$ is monotonically decreasing.

$$\therefore f'(x) = ae^{ax} - ae^{-ax} < 0$$

$$\Rightarrow e^{ax} > e^{-ax} \quad [\because a < 0]$$

$$\Rightarrow e^{2ax} > 1$$

$$\Rightarrow 2ax > 0$$

$$\Rightarrow 2x < 0$$

$$\therefore x < 0$$

Comparison of Functions using Calculus

If we want to compare $f(x)$ and $g(x)$, consider a function $\phi(x) = f(x) - g(x)$ or $\phi(x) = g(x) - f(x)$ and check whether $\phi(x)$ is increasing or decreasing in given domain of $f(x)$ and $g(x)$.

➔ **Example 70.** Which of the following is/are incorrect?

(a) $\ln(1+x) > x - \frac{x^2}{2}, \forall x(0, \infty)$

(b) $\tan x \geq x, \forall x \in \left[0, \frac{\pi}{2}\right)$

(c) $\sin x \leq x, \forall x \in \left[0, \frac{\pi}{2}\right)$

(d) $\left(\tan^{-1} \frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} > (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$

Sol. (d) (a) Consider the function

$$f(x) = \ln(1+x) - x + \frac{x^2}{2}, x \in (0, \infty)$$

$$\text{Then, } f'(x) = \frac{1}{1+x} - 1 + x$$

$$= \frac{x^2}{1+x} > 0, \forall x \in (0, \infty)$$

$$\Rightarrow f(x) \text{ is strictly increasing in } (0, \infty).$$

$$\Rightarrow f(x) > f(0^+) = 0$$

$$\text{i.e. } \ln(1+x) > x - \frac{x^2}{2}$$

(b) Consider the function

$$f(x) = \tan x - x, x \in \left[0, \frac{\pi}{2}\right)$$

$$f'(x) = \sec^2 x - 1 \geq 0, \forall x \in \left[0, \frac{\pi}{2}\right)$$

$$\text{Thus, } f(x) \text{ increases in } \left[0, \frac{\pi}{2}\right).$$

$$\Rightarrow f(x) \geq f(0) = 0$$

$$\text{i.e. } \tan x \geq x$$

(c) Now, consider the function, $g(x) = x - \sin x$, $x \in \left[0, \frac{\pi}{2}\right)$.

$$\text{Then, } g'(x) = 1 - \cos x = 2\sin^2\left(\frac{x}{2}\right) \geq 0,$$

$$\forall x \in \left[0, \frac{\pi}{2}\right)$$

$$\Rightarrow g(x) \text{ increases in } \left[0, \frac{\pi}{2}\right).$$

$$\Rightarrow g(x) \geq g(0) = 0$$

$$\text{i.e. } \sin x \leq x$$

(d) $\left(\tan^{-1} \frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$

Let us consider a function

$$f(x) = (\tan^{-1} x)^2 + \frac{2}{\sqrt{x^2+1}}, \forall x \in R$$

$$f'(x) = \frac{2 \tan^{-1} x}{1+x^2} - \frac{2x}{(x^2+1)^{3/2}}$$

$$= \frac{2}{1+x^2} \left[\tan^{-1} x - \frac{x}{\sqrt{x^2+1}} \right]$$

$$f'(x) = \frac{2}{1+x^2} g(x)$$

where $g(x) = \tan^{-1} x - \frac{x}{\sqrt{x^2+1}} \dots (i)$

$$\therefore g'(x) = \frac{1}{1+x^2} - \frac{1}{\sqrt{x^2+1}} + \frac{x^2}{(x^2+1)^{3/2}}$$

$$= \frac{1}{1+x^2} - \frac{1}{(x^2+1)^{3/2}}$$

$$= \frac{1}{1+x^2} \left(1 - \frac{1}{\sqrt{x^2+1}} \right) > 0,$$

$$\forall x \in R - \{0\}$$

$$\Rightarrow g(x) \text{ is strictly increasing for all } x \in R.$$

$$\Rightarrow g(x) > g(0), \forall x > 0$$

$$\Rightarrow \tan^{-1} x - \frac{x}{\sqrt{x^2+1}} > 0, \forall x > 0 \dots (ii)$$

From Eqs. (i) and (ii), we get

$$f'(x) > 0 \text{ for all } x > 0$$

$$\therefore f(x) \text{ is strictly increasing for all } x > 0.$$

$$\Rightarrow f\left(\frac{1}{e}\right) < f(e)$$

$$\Rightarrow \left(\tan^{-1} \frac{1}{e}\right)^2 + \frac{2e}{\sqrt{e^2+1}} < (\tan^{-1} e)^2 + \frac{2}{\sqrt{e^2+1}}$$

Work Book Exercise 7.4

- Let $f(x) = 2\sin^3 x - 3\sin^2 x + 12\sin x + 5$, $0 \leq x \leq \frac{\pi}{2}$. Then, $f(x)$ is
 - decreasing on $\left[0, \frac{\pi}{2}\right]$
 - increasing on $\left[0, \frac{\pi}{2}\right]$
 - increasing on $\left[0, \frac{\pi}{4}\right]$ and decreasing on $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
 - None of the above
- The function $f(x) = \log(1+x) - \frac{2x}{2+x}$ is increasing on
 - $(0, \infty)$
 - $(-\infty, 0)$
 - $(-\infty, \infty)$
 - None of these
- The function $f(x)$ given by

$$f(x) = \begin{vmatrix} x+1 & 1 & 1 \\ 1 & x+1 & 1 \\ 1 & 1 & x+1 \end{vmatrix}$$
 is increasing on
 - R
 - $(-2, 0)$
 - $R - [-2, 0]$
 - None of these
- If the function $f(x) = \frac{k \sin x + 2 \cos x}{\sin x + \cos x}$ is increasing for all values of x , then
 - $k < 1$
 - $k > 1$
 - $k < 2$
 - $k > 2$
- The function $y = x - \cot^{-1} x - \log(x + \sqrt{x^2 + 1})$ is increasing on
 - $(-\infty, 0)$
 - $(0, \infty)$
 - $(-\infty, \infty)$
 - None of the above
- The values of x for which the function $(a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically throughout for all real x , are
 - $a < -2$
 - $a > -2$
 - $-3 < a < 0$
 - $-\infty < a \leq -3$
- The set of all values of a for which the function

$$f(x) = \left(\frac{\sqrt{a+4}}{1-a} - 1 \right) x^5 - 3x + \log 5$$
 decreases for all real x , is
 - $(-\infty, \infty)$
 - $\left[-4, \frac{3 - \sqrt{21}}{2} \right] \cup [1, \infty)$
 - $\left(-3, 5 - \frac{\sqrt{27}}{2} \right) \cup (2, \infty)$
 - $[1, \infty)$
- The function $f(x) = \cos\left(\frac{\pi}{x}\right)$ is increasing in the interval
 - $(2n+1, 2n), n \in N$
 - $\left(\frac{1}{2n+1}, 2n\right), n \in N$
 - $\left(\frac{1}{2n+2}, \frac{1}{2n+1}\right), n \in N$
 - None of these
- Let $f(x) = x\sqrt{4ax - x^2}$ ($a > 0$) then $f(x)$ is
 - increasing in $(0, 3a)$, decreasing in $(3a, 4a)$
 - increasing in $(a, 4a)$, decreasing in $(5a, \infty)$
 - increasing in $(0, 4a)$
 - None of the above
- If $f(x) = \int \left(2 - \frac{1}{1+x^2} - \frac{1}{\sqrt{1+x^2}} \right) dx$, then f is
 - increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$
 - increasing in $(-\infty, 0)$ and decreasing in $(0, \infty)$
 - increasing in $(-\infty, \infty)$
 - decreasing in $(-\infty, \infty)$
- Let m and n be odd integers such that $0 < m < n$. If $f(x) = x^{m/n}$ for $x \in R$, then
 - $f(x)$ is differentiable everywhere
 - $f'(0)$ exists
 - f increases on $(0, \infty)$ and decreases on $(-\infty, 0)$
 - f increases on R
- If $f(x) = (ab - b^2 - 2)x + \int_0^x (\cos^4 \theta + \sin^4 \theta) d\theta$ is a decreasing function of x for all $x \in R$ and $b \in R$, b being independent of x , then
 - $a \in (0, \sqrt{6})$
 - $a \in (-\sqrt{6}, \sqrt{6})$
 - $a \in (-\sqrt{6}, 0)$
 - None of these
- Let $f(x) = \phi(2-x) + \phi(x)$ and $\phi''(x) < 0$ for $x \in [0, 2]$. Then,
 - $f(x)$ is m.i in $[0, 1]$
 - $f(x)$ is m.d. in $[0, 1]$
 - $f(x)$ is m.i in $[1, 2]$
 - None of these
- Let $f(x) = \tan^{-1}\{\phi(x)\}$, where $\phi(x)$ is m.i. for $0 < x < \frac{\pi}{2}$. Then, $f(x)$ is
 - increasing in $\left(0, \frac{\pi}{2}\right)$
 - decreasing in $\left(0, \frac{\pi}{2}\right)$
 - increasing in $\left(0, \frac{\pi}{4}\right)$ and decreasing in $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
 - None of the above
- If $f(x) = \frac{p^2 - 1}{p^2 + 1} x^3 - 3x + \log 2$ is a decreasing function of x in R , then the set of possible values of p (independent of x) is
 - $[-1, 1]$
 - $[1, \infty]$
 - $(-\infty, -1]$
 - None of these

WorkedOut Examples

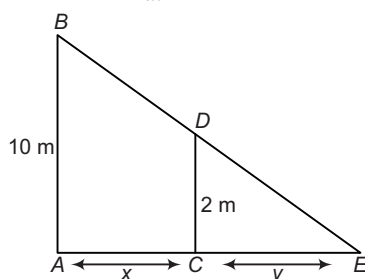
Type 1. Only One Correct Option

Ex 1. The rate of increase of length of the shadow of a man 2 m height, due to a lamp of 10 m height, when he is moving away from it at the rate of 2 m/s, is

- (a) $\frac{1}{2}$ m/s (b) $\frac{2}{5}$ m/s
(c) $\frac{1}{3}$ m/s (d) 5 m/s

Sol. Let $CE = y$ be the length of the shadow of the man at time t , when he is x m away from the lamp post.

It is given that, $\frac{dx}{dt} = 2$ m/s



Clearly, $\triangle ABE \sim \triangle CDE$

$$\Rightarrow \frac{AB}{CD} = \frac{AE}{CE} \Rightarrow \frac{10}{2} = \frac{x+y}{y}$$

$$\Rightarrow 4y = x \Rightarrow 4 \frac{dy}{dt} = \frac{dx}{dt}$$

$$\Rightarrow \frac{dy}{dt} = \frac{2}{4} = \frac{1}{2} \text{ m/s}$$

Hence, (a) is the correct answer

Ex 2. The surface area of a cube is increasing at the rate of $2\text{cm}^2/\text{s}$. When its edge is 90 cm, the volume is increasing at the rate of

- (a) $1620 \text{ cm}^3/\text{s}$ (b) $810 \text{ cm}^3/\text{s}$
(c) $405 \text{ cm}^3/\text{s}$ (d) $45 \text{ cm}^3/\text{s}$

Sol. Let at any time t , the length of each edge of the cube be x cm. Let S denotes the surface area and V denotes the volume of the cube. Then,

$$S = 6x^2 \text{ and } V = x^3$$

$$\Rightarrow \frac{dS}{dt} = 12x \frac{dx}{dt} \text{ and } \frac{dV}{dt} = 3x^2 \frac{dx}{dt}$$

$$\Rightarrow 2 = 12 \times 90 \times \frac{dx}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1}{540}$$

$$\therefore \frac{dV}{dt} = 3 \times (90)^2 \times \frac{1}{540} = 45 \text{ cm}^3/\text{s}$$

Hence, (d) is the correct answer.

Ex 3. Let $f(x)$ be a continuous, differentiable and bijective function. If the tangent to $y = f(x)$ at $x = a$, is also the normal to $y = f(x)$ at $x = b$, then there exists atleast one $c \in (a, b)$ such that

- (a) $f'(c) = 0$ (b) $f'(c) > 0$
(c) $f'(c) < 0$ (d) None of these

Sol. Since, same line is tangent at one point $x = a$ and normal at other point $x = b$.

$\Rightarrow$ Tangent at $x = b$ will be perpendicular to tangent at $x = a$.

$\Rightarrow$ Slope of tangent goes from positive to negative or negative to positive.

$\Rightarrow$ It takes the value zero somewhere.

$\therefore$ There will be a point $c \in (a, b)$, where $f'(c) = 0$.

Hence, (a) is the correct answer.

Ex 4. Let $f: [2, 7] \rightarrow [0, \infty)$ be a continuous and differentiable function. Then,

$$(f(7) - f(2)) \frac{(f(7))^2 + (f(2))^2 + f(2)f(7)}{3},$$

where $c \in (2, 7)$

- (a) $5f^2(c)f'(c)$ (b) $5f'(c)$
(c) $f(c)f'(c)$ (d) None of these

Sol. Let $g(x) = f^3(x)$

$$\Rightarrow g'(x) = 3f^2(x)f'(x)$$

$$\therefore f: [2, 7] \rightarrow [0, \infty)$$

$$\Rightarrow g: [2, 7] \rightarrow [0, \infty)$$

Using Lagrange's mean value theorem on $g(x)$, we get

$$g'(c) = \frac{g(7) - g(2)}{5}, c \in (2, 7)$$

$$\Rightarrow 5f^2(c)f'(c) = (f(7) - f(2)) \frac{(f(7))^2 + (f(2))^2 + f(2)f(7)}{3}$$

Hence, (a) is the correct answer.

Ex 5. If a, b, c are non-zero real numbers such that

$$\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c)dx$$

$$= \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c)dx = 0, \text{ then the}$$

equation $ax^2 + bx + c = 0$ will have

- (a) one root between 0 and 1 and other root between 1 and 2
(b) both the roots between 0 and 1
(c) both the roots between 1 and 2
(d) None of the above

Sol. Consider the function $\phi(x)$ given by

$$\phi(x) = \int_0^x (1 + \cos^8 t)(at^2 + bt + c)dt$$

$$\phi(1) = \int_0^1 (1 + \cos^8 t)(at^2 + bt + c)dt = 0$$

$$\text{and } \phi(2) = \int_0^2 (1 + \cos^8 t)(at^2 + bt + c)dt = 0$$

Therefore, 0, 1 and 2 are the roots of $\phi(x)$.

By Rolle's theorem $\phi'(x) = 0$ will have atleast one real root between 0 and 1 and atleast one real root between 1 and 2.

Hence, (a) is the correct answer.

Ex 6. Let $f : [0, 4] \rightarrow R$ be a continuous function such that $|f(x)| \leq 2$, for all $x \in [0, 4]$ and $\int_0^4 f(t)dt = 2$. Then, for all $x \in [0, 4]$, the value of $\int_0^x f(t)dt$ lies in the interval

- (a) $[-6 + 2x, 10 - 2x]$ (b) $[-12 + 2x, -7 + 2x]$
(c) $[11 - 2x, 17 + 2x]$ (d) $[-8 - 2x, 6 - 2x]$

Sol. Let $\phi(x) = \int_0^x f(t)dt$. Then, $\phi(x)$ is continuous on $[0, 4]$ and is differentiable on $(0, 4)$ as $\phi'(x) = f(x)$. Applying Lagrange's mean value theorem to $\phi(x)$ on $[x, 4]$ we get

$$\phi'(c) = \frac{\phi(4) - \phi(x)}{4 - x} \text{ for some } c \in (x, 4)$$

$$\Rightarrow f(c) = \frac{2 - \phi(x)}{4 - x}$$

$$\Rightarrow |f(c)| = \left| \frac{2 - \phi(x)}{4 - x} \right|$$

$$\Rightarrow |f(c)| = \left| \frac{\phi(x) - 2}{4 - x} \right| = \left| \frac{\phi(x) - 2}{4 - x} \right| \leq 2$$

$$\Rightarrow |\phi(x) - 2| \leq 8 - 2x$$

$$\Rightarrow -8 + 2x \leq \phi(x) - 2 \leq 8 - 2x$$

$$\Rightarrow 2x - 6 \leq \phi(x) \leq 10 - 2x$$

$$\Rightarrow \phi(x) \in [2x - 6, 10 - 2x]$$

Hence, (a) is the correct answer.

Ex 7. Let $f(x)$, $x \geq 0$ be a non-negative continuous function and $F(x) = \int_0^x f(t)dt$, $x \geq 0$. If for some $c > 0$, $f(x) \leq cF(x)$ for all $x \geq 0$, then

- (a) $f(x) = 0, \forall x \leq 1$
(b) $f(x) = 0, \forall x \geq 0$
(c) $f(x) = 1, \forall x \geq 0$
(d) $f(x) = -1, \forall x \geq 0$

Sol. $F'(x) = f(x)$

$$\Rightarrow F'(x) - cF(x) \leq 0$$

$$\Rightarrow (F(x))e^{-cx} \leq 0$$

$$\Rightarrow g(x) = F(x)e^{-cx} \text{ is decreasing}$$

$$\text{but } g(0) = 0$$

$$\Rightarrow F(x)e^{-cx} \leq 0$$

$$\Rightarrow f(x) \leq 0 \text{ but given that } f(x) \geq 0$$

$$\Rightarrow f(x) = 0, \forall x \geq 0$$

Hence, (b) is the correct answer.

Ex 8. The semi-vertical angle of a cone is 45° . If the height of the cone is 20.025, then its approximate lateral surface area is

- (a) $401\sqrt{2}\pi$ (b) $400\sqrt{2}\pi$
(c) $399\sqrt{2}\pi$ (d) None of these

Sol. Let r be the radius, h be the height and l be the slant height of a cone of semi-vertical angle 45° . Then, $r = h$ and $l = \sqrt{2}h$.

$$\text{Let } h = 20 \text{ and } h + \Delta h = 20.025$$

$$\Rightarrow \Delta h = 0.025$$

Let S be the lateral surface area. Then,

$$S = \sqrt{2}\pi h^2$$

$$[\because S = \pi r l \Rightarrow S = \pi h \times \sqrt{2}h]$$

$$\Rightarrow \frac{dS}{dh} = 2\sqrt{2}\pi h$$

$$\Rightarrow \left(\frac{dS}{dh} \right)_{h=20} = 40\sqrt{2}\pi$$

$$\therefore \Delta S = \frac{dS}{dh} \Delta h$$

$$\Rightarrow \Delta S = 40\sqrt{2}\pi \times 0.025 = \sqrt{2}\pi$$

$$\text{Also, at } h = 20, \text{ we have } S = 400\sqrt{2}\pi$$

$$\therefore S + \Delta S = 400\sqrt{2}\pi + \sqrt{2}\pi = 401\sqrt{2}\pi$$

Hence, (a) is the correct answer.

Ex 9. The area of a triangle is computed using the formula $S = \frac{1}{2}bc \sin A$. If the relative errors made in measuring b , c and calculating S are respectively 0.02, 0.01 and 0.13, then the approximate error in A when $A = \frac{\pi}{6}$ is

- (a) 0.05 radian (b) 0.01 radian
(c) 0.05 degree (d) 0.01 degree

Sol. We have,

$$S = \frac{1}{2}bc \sin A \Rightarrow dS = \frac{1}{2}d(bc \sin A)$$

$$\Rightarrow dS = \frac{1}{2}\{c \sin A db + b \sin A dc + bc \cos A dA\}$$

$$\Rightarrow \frac{dS}{S} = \frac{db}{b} + \frac{dc}{c} + \cot A dA$$

$$\Rightarrow 0.13 = 0.02 + 0.01 + \cot \frac{\pi}{6} dA$$

$$\left[\because \frac{dS}{S} = 0.13, \frac{db}{b} = 0.02 \text{ and } \frac{dc}{c} = 0.01 \right]$$

$$\Rightarrow 0.10 = \sqrt{3}dA$$

$$\Rightarrow dA = \frac{0.10}{\sqrt{3}}$$

$$= 0.05 \text{ radian}$$

Hence, (a) is the correct answer.

Ex 10. Line joining the points $(0, 3)$ and $(5, -2)$ is a tangent to the curve $y = \frac{ax}{1+x}$, then

- (a) $a = 1 \pm \sqrt{3}$ (b) $a \in \phi$
(c) $a = -1 \pm \sqrt{3}$ (d) $a = -2 \pm 2\sqrt{3}$

Sol. Equation of line joining the points $(0, 3)$ and $(5, -2)$ is $y = 3 - x$. If this line is tangent to $y = \frac{ax}{(x+1)}$, then

$(3-x)(x+1) = ax$ should have equal roots. Thus,

$$(a-2)^2 + 12 = 0$$

$\Rightarrow$ No value of $a \Rightarrow a \in \phi$

Hence, (b) is the correct answer.

Ex 11. If $f(x) = x^3 + bx^2 + cx + d$ and $0 < b^2 < c$, then in $(-\infty, \infty)$

(a) $f(x)$ is strictly increasing function

(b) $f(x)$ has a local maxima

(c) $f(x)$ is a strictly decreasing function

(d) $f(x)$ is unbounded

Sol. We have, $f(x) = x^3 + bx^2 + cx + d$

$$\Rightarrow f'(x) = 3x^2 + 2bx + c$$

Let D_1 be the discriminant of $f'(x) = 3x^2 + 2bx + c$.

$$\text{Then, } D_1 = 4b^2 - 12c = 4(b^2 - c) - 8c < 0$$

$$[\because b^2 < c \text{ and } c > 0]$$

$$\Rightarrow f'(x) > 0, \forall x \in (-\infty, \infty)$$

$\Rightarrow f(x)$ is strictly increasing function on $(-\infty, \infty)$.

Hence, (a) is the correct answer.

Ex 12. If $f(x)$ is increasing for all real values of x

$$\text{and } f(x) = \int_0^x |\log_2(\log_3(\log_4(\cos t + a)))| dt|,$$

then

(a) $a \in (-1, 1)$

(b) $a \in (1, 5)$

(c) $a \in (1, \infty)$

(d) $a \in (5, \infty)$

Sol. $f'(x) = |(\log_2(\log_3(\log_4(\cos x + a))))|$

Clearly, $f(x)$ is increasing for all values of x , if

$\log_2(\log_3(\log_4(\cos x + a)))$ is defined for all values of x .

$$\Rightarrow \log_3(\log_4(\cos x + a)) > 0, \forall x \in R$$

$$\Rightarrow \log_4(\cos x + a) > 1, \forall x \in R$$

$$\Rightarrow \cos x + a > 4, \forall x \in R$$

$$\Rightarrow a > 5$$

Hence, (d) is the correct answer.

Ex 13. Let $f''(x) > 0, \forall x \in R$ and $g(x) = f(2-x) + f(4+x)$. Then, $g(x)$ is increasing in

(a) $(-\infty, -1)$

(b) $(-\infty, 0)$

(c) $(-1, \infty)$

(d) None of the above

Sol. $f''(x) > 0, \forall x \in R$

$\Rightarrow f'(x)$ is increasing $\forall x \in R$.

$$g'(x) = -f'(2-x) + f'(4+x)$$

$$\text{If } g'(x) > 0$$

$$\Rightarrow f'(4+x) > f'(2-x)$$

$$\Rightarrow 4+x > 2-x \Rightarrow x > -1$$

Hence, (c) is the correct answer.

Ex 14. The function $f(x) = \frac{\ln(\pi+x)}{\ln(e+x)}$ is

(a) increasing on $[0, \infty)$

(b) decreasing on $[0, \infty)$

(c) decreasing on $\left[0, \frac{\pi}{e}\right)$ and increasing on

$$\left[\frac{\pi}{e}, \infty\right)$$

(d) increasing on $\left[0, \frac{\pi}{e}\right)$ and decreasing on

$$\left[\frac{\pi}{e}, \infty\right)$$

Sol. We have, $f(x) = \frac{\ln(\pi+x)}{(e+x)}$

$$\Rightarrow f'(x) = \frac{\ln(e+x) \times \frac{1}{(\pi+x)} - \ln(\pi+x) \times \frac{1}{e+x}}{\ln(e+x)^2}$$

$$= \frac{(e+x)\ln(e+x) - (\pi+x)\ln(\pi+x)}{\ln^2(e+x) \times (e+x) \times (\pi+x)}$$

$$\Rightarrow f'(x) < 0, \forall x \geq 0 \quad [\because \pi > e]$$

Thus, $f(x)$ is decreasing in $[0, \infty)$.

Hence, option (b) is correct.

Ex 15. Let $f(x)$ be a differentiable function such

that $f'(x) = \frac{1}{(\log_3(\log_{1/4}(\cos x + a)))}$. If $f(x)$ is

increasing for all values of x , then

(a) $a \in (5, \infty)$

(b) $a \in \left(1, \frac{5}{4}\right)$

(c) $a \in \left(\frac{5}{4}, 5\right)$

(d) None of the above

Sol. For $f(x)$ to be increasing $\forall x \in R$, we must have

$$\Rightarrow \log_3(\log_{1/4}(\cos x + a)) > 0, \forall x \in R$$

$$\Rightarrow \log_{1/4}(\cos x + a) > 1$$

$$\Rightarrow 0 < \cos x + a < \frac{1}{4}, \forall x \in R$$

$$\Rightarrow -\cos x < a < \frac{1}{4} - \cos x, \forall x \in R$$

$$\Rightarrow a > 1 \text{ and } a < -\frac{3}{4}, \text{ which is not possible.}$$

Hence, (d) is the correct answer.

Ex 16. The curve $C_1 : y = 1 - \cos x, x \in (0, \pi)$ and

$C_2 : y = \frac{\sqrt{3}}{2}|x| + a$ will touch each other, if

$$(a) a = \frac{3}{2} - \frac{\pi}{\sqrt{3}}$$

$$(b) a = \frac{3}{2} - \frac{\pi}{2\sqrt{3}}$$

$$(c) a = \frac{1}{2} - \frac{\pi}{\sqrt{3}}$$

$$(d) a = \frac{3}{4} - \frac{\pi}{\sqrt{3}}$$

Sol. Slope of C_1 is $\sin x$ for $x > 0$.

Slope of C_2 is $\frac{\sqrt{3}}{2}$ for $x > 0$.

Thus, for point of contact, $\sin x = \frac{\sqrt{3}}{2} \Rightarrow x = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

Hence, point of contact is $\left(\frac{\pi}{3}, \frac{1}{2}\right)$ or $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$.

For $\left(\frac{\pi}{3}, \frac{1}{2}\right)$, we get $a = \frac{1}{2} - \frac{\pi}{2\sqrt{3}}$

For $\left(\frac{2\pi}{3}, \frac{3}{2}\right)$, we get $a = \frac{3}{2} - \frac{\pi}{\sqrt{3}}$

Hence, (a) is the correct answer.

Ex 17. If the parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect orthogonally at point $p(x_1, y_1)$, where $x_1, y_1 \neq 0$, then

- (a) $b = a^2$
(b) $b = a^3$
(c) $b^3 = a^2$
(d) None of the above

Sol. On solving $y^2 = 4ax$ and $x^2 = 4by$, we get $x = 0$

or $x^3 = 64ab^2$. Slope of the curves at the common points are $\frac{2a}{y}$ and $\frac{x}{2b}$, respectively. If these

parabolas intersect orthogonally, then

$$\frac{2a}{y} \cdot \frac{x}{2b} = -1$$

$$\Rightarrow ax + by = 0$$

$$\Rightarrow ax + \frac{x^2}{4} = 0$$

$$\Rightarrow x = -4a \quad [\because x \neq 0]$$

$$\Rightarrow -x^3 = 64a^3$$

$$\Rightarrow 64ab^2 + 64a^3 = 0$$

$$\Rightarrow a^2 + b^2 = 0$$

which is not possible.

Hence, (d) is the correct answer.

Ex 18. Let $y = f(x)$ is a parabola, having its axis parallel to Y -axis. If the line $y = x$ touches this parabola at $x = 1$, then

- (a) $f''(1) - f'(0) = 1$ (b) $f''(0) - f'(1) = 1$
(c) $f''(1) + f'(0) = 1$ (d) $f''(0) + f'(1) = 1$

Sol. Let $y = f(x) = ax^2 + bx + c = x$.

We have, $f(1) = 1 \Rightarrow a + b + c = 1$.

Also, $ax^2 + bx + c = 0$ should have $x - 1$ as its repeated root.

$$\Rightarrow ax^2 + (b-1)x + c = a(x-1)^2$$

$$\Rightarrow 1 - b = 2a, a = c$$

We have, $f'(x) = 2ax + b$, $f''(x) = 2a$

$$\Rightarrow f''(1) = 2a, f'(0) = b$$

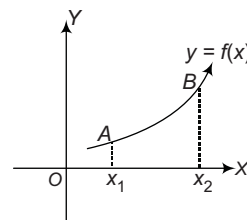
$$\Rightarrow f''(1) + f'(0) = 1$$

Hence, (c) is the correct answer.

Ex 19. If $f'(x) > 0$ and $f''(x) > 0, \forall x \in R$, then for any two real numbers x_1 and x_2 ,

- (a) $f\left(\frac{x_1 + x_2}{2}\right) > \frac{f(x_1) + f(x_2)}{2}$
(b) $f\left(\frac{x_1 + x_2}{2}\right) < \frac{f(x_1) + f(x_2)}{2}$
(c) $f'\left(\frac{x_1 + x_2}{2}\right) > \frac{f'(x_1) + f'(x_2)}{2}$
(d) $f'\left(\frac{x_1 + x_2}{2}\right) < \frac{f'(x_1) + f'(x_2)}{2}$

Sol. Let $A \equiv (x_1, f(x_1))$ and $B \equiv (x_2, f(x_2))$ be any two points on the graph of $y = f(x)$. Chord AB will lie completely above the graph of $y = f(x)$.



$$\therefore \frac{f(x_1) + f(x_2)}{2} > f\left(\frac{x_1 + x_2}{2}\right)$$

Hence, (b) is the correct answer.

Ex 20. Let $f'(\sin x) < 0$ and $f''(\sin x) > 0, \forall x \in \left(0, \frac{\pi}{2}\right)$ and $g(x) = f(\sin x) + f(\cos x)$, then $g(x)$ is decreasing in

- (a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(0, \frac{\pi}{4}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

Sol. $g'(x) = f'(\sin x) \cdot \cos x - f'(\cos x) \cdot \sin x$
 $\Rightarrow g''(x) = -f'(\sin x) \cdot \sin x + \cos^2 x \cdot f''(\sin x)$
 $+ f''(\cos x) \cdot \sin^2 x - f'(\cos x) \cdot \cos x > 0,$
 $\forall x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow g'(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$. Also, $g'\left(\frac{\pi}{4}\right) = 0$

$\Rightarrow g'(x) > 0, \forall x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ and $g'(x) < 0, \forall x \in \left(0, \frac{\pi}{4}\right)$

Thus, $g(x)$ is decreasing in $\left(0, \frac{\pi}{4}\right)$.

Hence, (b) is the correct answer.

Ex 21. The equation $x + \cos x = a$ has exactly one positive root. Complete set of values of a is

- (a) $(0, 1)$ (b) $(-\infty, 1)$
(c) $(-1, 1)$ (d) $(1, \infty)$

Sol. Let $f(x) = x + \cos x - a$

$$\Rightarrow f'(x) = 1 - \sin x \geq 0, \forall x \in R$$

Thus, $f(x)$ is increasing in $(-\infty, \infty)$, as zero of $f'(x)$ don't form an interval as $f'(0) = 1 - a$.

For positive root, $1 - a < 0$.

$$\Rightarrow a > 1$$

Hence, (d) is the correct answer.

Ex 22. If $f(x) = \cos x + a^2x + b$ is an increasing function for all values of x , then

- (a) $a \in [-1, 1]$
 (b) $a \in (-\infty, 1] \cup [1, \infty)$
 (c) $a \in [-1, \infty)$
 (d) $a \in (-\infty, 1]$

Sol. $f'(x) = -\sin x + a^2 \Rightarrow a^2 \geq \sin x, \forall x \in R$
 $\Rightarrow a^2 \geq 1 \Rightarrow a \in (-\infty, -1] \cup [1, \infty)$
 Hence, (b) is the correct answer.

Ex 23. Let $f(x) = 4x + 8\cos x - 4 \ln(\cos x(1 + \sin x)) + \tan x - 2\sec x - 6$. If $f(x) > 0, \forall x \in (0, a)$, then

- (a) $a = \frac{\pi}{6}$ (b) $a = \frac{\pi}{3}$
 (c) $a = \frac{\pi}{2}$ (d) None of these

Sol. $f'(x) = 4 - 8\sin x - \frac{4(-\sin x + \cos^2 x - \sin^2 x)}{\cos x(1 + \sin x)} + \sec^2 x - 2\sec x \cdot \tan x$
 $= 4(1 - 2\sin x) + \sec^2 x(1 - 2\sin x) - 4\sec x(1 - 2\sin x)$
 $= (\sec x - 2)^2(1 - 2\sin x)$
 $= f(0 + 0) = 0$
 If $f(x) > 0, \forall x \in (0, a)$, then $f(x)$ must be increasing in $(0, a)$.

$$\Rightarrow f'(x) > 0, \forall x \in (0, a) \Rightarrow a = \frac{\pi}{6}$$

Hence, (a) is the correct answer.

Ex 24. The equation of tangent drawn to the curve $y^2 - 2x^3 - 4y + 8 = 0$ from the point $(1, 2)$ is

- (a) $2x\sqrt{3} + 2y + (1 + \sqrt{3}) = 0$
 (b) $2\sqrt{3}x + y - 2(1 + \sqrt{3}) = 0$
 (c) $2x\sqrt{3} + y - (1 + \sqrt{2}) = 0$
 (d) $2x + y - 2(1 + \sqrt{3}) = 0$

Sol. Let the tangent at point (x_1, y_1) to the curve

$$y^2 - 2x^3 - 4y + 8 = 0 \quad \dots(i)$$

passes through point $(1, 2)$.

$$\frac{3x_1^2}{y_1 - 2} = \frac{y_1 - 2}{x_1 - 1}$$

$$\Rightarrow 3x_1^2(x_1 - 1) = (y_1 - 2)^2 \quad \dots(ii)$$

$$\text{Also, } y_1^2 - 2x_1^3 - 4y_1 + 8 = 0$$

$$\Rightarrow (y_1 - 2)^2 - 2x_1^3 + 4 = 0$$

From Eqs. (ii) and (iii), we get

$$3x_1^2(x_1 - 1) - 2x_1^3 + 4 = 0$$

$$\Rightarrow x_1^3 - 3x_1^2 + 4 = 0$$

$$\Rightarrow x_1 = -1, 2$$

Now, from Eq. (ii) corresponding to $x_1 = -1$, no real value of y_1 is possible and corresponding to $x_1 = 2$, $y_1 = 2 + 2\sqrt{3}$ or $2 - 2\sqrt{3}$.

$$\therefore \text{Equation of tangent is } (y - 2) = \frac{3 \times 4}{2\sqrt{3}}(x - 1)$$

$$\Rightarrow 2\sqrt{3}x - y + 2(1 - \sqrt{3}) = 0$$

$$\text{Also, } (y - 2) = \frac{3 \times 4}{-2\sqrt{3}}(x - 1)$$

$$\Rightarrow 2\sqrt{3}x + y - 2(1 + \sqrt{3}) = 0$$

Hence, (b) is the correct answer.

Ex 25. For $f(x) = x^2 - 2|x|$, test the continuity and differentiability of $g(x)$ in the interval $[-2, 3]$, where

$$g(x) = \begin{cases} \min \{f(t), -2 \leq t \leq x\} & -2 \leq x \leq 0 \\ \max \{f(t), 0 \leq t \leq x\} & 0 \leq x \leq 3 \end{cases}$$

are

- (a) $g(x)$ is continuous and differentiable everywhere except $x = 0$
 (b) $g(x)$ is non-differentiable at $x = 0$ and 2
 (c) $g(x)$ is non-differentiable at $x = 0, 1, 2$
 (d) None of the above

$$\text{Sol. Here, } f(x) = \begin{cases} x^2 + 2x, & x < 0 \\ 0, & x = 0 \\ x^2 - 2x, & x > 0 \end{cases}$$

Case I $-2 \leq x < -1$, $f(x)$ decreases $\Rightarrow g(x) = x^2 + 2x$

Case II $-1 \leq x < 0$, $f(x)$ increases $\Rightarrow g(x) = -1$

Case III $0 \leq x < 1$, $f(x)$ decreases $\Rightarrow g(x) = 0$

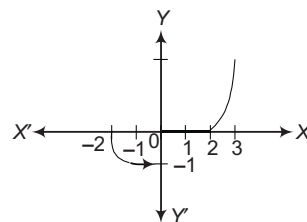
Case IV $1 \leq x < 2$, $f(x)$ increases but $f(x) < 0$

$$\Rightarrow g(x) = f(0) = 0$$

Case V $2 \leq x \leq 3$, $f(x)$ increases and $f(x) \geq 0$, $\forall x \in [2, 3]$

$$\Rightarrow g(x) = f(x) = x^2 - 2x, 2 \leq x \leq 3$$

Graph of $g(x)$



$$\text{Therefore, } g(x) = \begin{cases} x^2 + 2x, & -2 \leq x < -1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 2 \\ x^2 - 2x, & 2 \leq x \leq 3 \end{cases}$$

Clearly, $g(x)$ is continuous everywhere except at $x = 0$.

Also, $g(x)$ is not differentiable at $x = 0$ and 2 .

Hence, (b) is the correct answer.

Ex 26. Suppose $p(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$.

If $|p(x)| \leq |e^{x-1} - 1|$ for all $x \geq 0$, then

- (a) $|a_1 + 2a_2 + 3a_3 + \dots + na_n| \leq 1$
 (b) $|a_1 + 2a_2 + 3a_3 + \dots + na_n| \geq 1$
 (c) $|a_1 + 2a_2 + 3a_3 + \dots + na_n| \geq 0$
 (d) $|a_1 + 2a_2 + 3a_3 + \dots + na_n| \leq 0$

Sol. We know that, $p'(1) = \lim_{h \rightarrow 0} \frac{p(1+h) - p(1)}{h}$
 $\Rightarrow |p'(1)| = \left| \lim_{h \rightarrow 0} \frac{p(1+h) - p(1)}{h} \right|$
 $\lim_{h \rightarrow 0} \frac{|p(1+h) - p(1)|}{|h|} \leq \lim_{h \rightarrow 0} \frac{|p(1+h)| + |p(1)|}{|h|}$

Now, $|p(x)| \leq |e^{x-1} - 1|$
 $\Rightarrow |p(1+h)| \leq |e^h - 1|$ and $|p(1)| \leq 0$
 $\Rightarrow \frac{p(1)}{h} = 0$
 $|p'(1)| \leq \lim_{h \rightarrow 0} \frac{|e^h - 1|}{|h|} = 1$
 $\Rightarrow |p'(1)| \leq 1$
 $\Rightarrow |a_1 + 2a_2 + 3a_3 + \dots + na_n| \leq 1$
Hence, (a) is the correct answer.

Ex 27. Let $a + b = 4$, $a < 2$ and $g(x)$ be a monotonically increasing function of x .
Then, $f(x) = \int_0^a g(x) dx + \int_0^b g(x) dx$

- (a) increases with increase in $(b - a)$
(b) decreases with increase in $(b - a)$
(c) increases with decrease in $(b - a)$
(d) None of the above

Sol. $a + b = 4 \Rightarrow b = 4 - a$ and $b - a = 4 - 2a = t$ [say]

Now, $\int_0^a g(x) dx + \int_0^b g(x) dx = \int_0^a g(x) dx + \int_0^{4-a} g(x) dx$
 $= l(a)$

$\Rightarrow \frac{dl(a)}{da} = g(a) - g(4 - a)$

Now, as $a < 2$ and $g(x)$ is increasing,
 $4 - a > a$ and $g(a) - g(4 - a) < 0$

$\Rightarrow \frac{dl(a)}{da} < 0$

Now, $\frac{dl(a)}{da} = \frac{dl(a)}{dt} \cdot \frac{dt}{da} = -2 \cdot \frac{dl(a)}{dt} \Rightarrow \frac{dl(a)}{dt} > 0$

Thus, $l(a)$ is an increasing function of t . So, the given expression increases with increase in $(b - a)$.

Hence, (a) is correct answer.

Type 2. More than One Correct Option

Ex 28. If the tangent at any point $P(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is also a normal to the curve $x^3 - y^2 = 0$, then the value of m is

- (a) $m = \frac{\sqrt{2}}{3}$ (b) $m = -\frac{\sqrt{2}}{3}$
(c) $m = \frac{3}{\sqrt{2}}$ (d) $m = -\frac{3}{\sqrt{2}}$

Sol. $x^3 - y^2 = 0$... (i)

$\Rightarrow 2y \times \frac{dy}{dx} = 3x^2$

Slope of the tangent at P ,
 $= \left[\frac{dy}{dx} \right]_P = \left[\frac{3x^2}{2y} \right]_{(4m^2, 8m^3)} = 3m$

$\therefore$ Equation of the tangent at P is
 $y - 8m^3 = 3m(x - 4m^2)$ or $y = 3mx - 4m^3$... (ii)

It cuts the curve again at point Q .

On solving Eqs. (i) and (ii), we get $x = 4m^2, m^2$

Put $x = m^2$ in Eq. (ii)

$\Rightarrow y = 3m(m^2) - 4m^3 = -m^3$

$\therefore Q$ is $(m^2, -m^3)$

Slope of the tangent

$Q = \frac{dy}{dx} \Big|_{(m^2, -m^3)} = \frac{3(m^4)}{2 \times (-m^3)} = -\frac{3}{2}m$

Slope of the normal at $Q = \frac{-1}{\left(-\frac{3}{2}m\right)} = \frac{2}{3m}$

Since, tangent at P is normal at Q

$\Rightarrow \frac{2}{3m} = 3m$

$\Rightarrow 9m^2 = 2$

Hence, (a) and (b) are the correct answers.

Ex 29. A function $g(x)$ is defined as $g(x) = \frac{1}{4}f(2x^2 - 1) + \frac{1}{2}f(1 - x^2)$ and $f'(x)$ is an increasing function, then $g(x)$ is increasing in the interval

- (a) $(-1, 1)$ (b) $\left(\sqrt{\frac{2}{3}}, \infty\right)$
(c) $\left(-\sqrt{\frac{2}{3}}, 0\right)$ (d) None of these

Sol. $g'(x) = xf'(2x^2 - 1) - xf'(1 - x^2)$
 $= x(f'(2x^2 - 1) - f'(1 - x^2))$

If $x > 0$, then $2x^2 - 1 < 1 - x^2$
[as f' is an increasing function]

$\Rightarrow 3x^2 > 2 \Rightarrow x \in \left(-\infty, -\sqrt{\frac{2}{3}}\right) \cup \left(\sqrt{\frac{2}{3}}, \infty\right)$

$\Rightarrow x \in \left(\sqrt{\frac{2}{3}}, \infty\right)$

If $x < 0$, $2x^2 - 1 > 1 - x^2$

$\Rightarrow 3x^2 < 2 \Rightarrow x \in \left(-\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right) \in \left(-\sqrt{\frac{2}{3}}, 0\right)$

Hence, (b) and (c) are the correct answers.

Ex 30. Which of the following is/are correct?

- (a) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $\tan x = 1$
(b) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $\tan x = -1$
(c) Between any two roots of $e^x \cos x = 1$, there exists atleast one root of $e^x \sin x = 1$
(d) Between any two roots of $e^x \sin x = 1$, there exists atleast one root of $e^x \cos x = 1$

Sol. (a) Let $f(x) = e^x \cos x - 1$

$$\Rightarrow f'(x) = e^x (\cos x - \sin x) = 0$$

$\Rightarrow \tan x = 1$, which has a root between two roots of $f(x) = 0$.

(b) Let $f(x) = e^x \sin x - 1$,

$$\Rightarrow f'(x) = e^x (\sin x + \cos x) = 0$$

$\Rightarrow \tan x = -1$, which has a root between two roots of $f(x) = 0$.

(c) Let $f(x) = e^{-x} - \cos x$

$$\Rightarrow f'(x) = -e^{-x} + \sin x = 0$$

$\Rightarrow e^{-x} = \sin x$, which has a root between two roots of $f(x) = 0$.

Hence, (a), (b) and (c) are the correct answers.

Ex 31. If f is an odd continuous function in $[-1, 1]$ and differentiable in $(-1, 1)$, then

(a) $f'(A) = f(1)$ for some $A \in (-1, 0)$

(b) $f'(B) = f(1)$ for some $B \in (0, 1)$

(c) $n(f(A))^{n-1} f'(A) = (f(1))^n$ for some $A \in (-1, 0)$, $n \in \mathbb{N}$

(d) $n(f(B))^{n-1} f'(B) = (f(1))^n$ for some $B \in (1, 0)$, $n \in \mathbb{N}$

Sol. $f(-1) = -f(1)$, $f(0) = 0$. For options (a) and (b), apply LMVT for the function $f(x)$ in $(-1, 0)$ and $(0, 1)$, respectively. For option (d), apply LMVT for $(f(x))^n$ in $(0, 1)$.

Hence, (a), (b) and (d) are the correct answers.

Ex 32. The function $f(x) = \int_0^x \sqrt{1-t^4} dt$ is such that

(a) it is defined on the interval $[-1, 1]$

(b) it is an increasing function

(c) it is an odd function

(d) the point $(0, 0)$ is the point of inflection

Sol. $f'(x) = \sqrt{1-x^4} > 0$ in $(-1, 1) \Rightarrow f$ is increasing

$$\text{Now, } f(x) + f(-x) = \int_0^x \sqrt{1-t^4} dt + \int_0^{-x} \sqrt{1-t^4} dt$$

$$\Rightarrow \int_0^x \sqrt{1-t^4} dt + \left(-\int_0^x \sqrt{1-y^4} dy \right) = 0 \quad (t = -y)$$

$\Rightarrow f(x)$ is odd.

$$\text{Again, } f''(x) = \frac{-4x^3}{2\sqrt{1-x^4}}$$

which vanished at $x = 0$ and changes sign

$\Rightarrow (0, 0)$ is point of inflection, since f is well defined in $[-1, 1]$.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 33. Let $g(x) = 2f\left(\frac{x}{2}\right) + f(1-x)$ and $f''(x) < 0$

in $0 \leq x \leq 1$, then $g(x)$

(a) decreases in $\left[0, \frac{2}{3}\right]$ (b) decreases in $\left(\frac{2}{3}, 1\right]$

(c) increases in $\left[0, \frac{2}{3}\right]$ (d) increases in $\left(\frac{2}{3}, 1\right]$

Sol. $f''(x) < 0 \Rightarrow f'(x)$ is decreasing.

$$\text{Now, } g'(x) = f'\left(\frac{x}{2}\right) - f'(1-x)$$

For $g(x)$ to be increasing, $g'(x) > 0$.

$$\Rightarrow f'\left(\frac{x}{2}\right) > f'(1-x)$$

$$\Rightarrow \left(\frac{x}{2}\right) < (1-x)$$

$$\Rightarrow x < \frac{2}{3}$$

Hence, (b) and (c) are the correct answers.

Ex 34. Let $F(x) = 1 + f(x) + (f(x))^2 + (f(x))^3$, where $f(x)$ is an increasing differentiable function and $F(x) = 0$ has a positive root, then

(a) $F(x)$ is an increasing function

(b) $F(0) < 0$

(c) $F(0) < -1$

(d) $F'(0) > 0$

Sol. $F'(x) = 1 + 2f(x) + 3(f(x))^2 f'(x) > 0$.

So, $F(x)$ is an increasing function.

$$\Rightarrow F(0) < 0$$

$$\Rightarrow (1 + f(0))(1 + (f(0))^2) < 0$$

$$\Rightarrow f(0) < -1$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 35. The coordinates of the point(s) on the graph of the function, $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$,

where the tangent drawn cut-off intercepts from the coordinates axes which are equal in magnitude but opposite in sign, is

$$(a) \left(2, \frac{8}{3}\right)$$

$$(b) \left(3, \frac{7}{2}\right)$$

$$(c) \left(1, \frac{5}{6}\right)$$

(d) None of these

Sol. Since, intercepts are equal in magnitude but opposite in sign.

$$\therefore \left[\frac{dy}{dx}\right]_P = 1$$

$$\text{Now, } \frac{dy}{dx} = x^2 - 5x + 7 = 1$$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow x = 2 \text{ or } 3$$

Hence, (a) and (b) are the correct answers.

Ex 36. In the curve $y = ce^{x/a}$,

(a) sub-tangent is constant

(b) sub-normal varies as the square of the ordinate

(c) tangent at (x_1, y_1) on the curve intersects the X -axis at a distance of $(x_1 - a)$ from the origin

(d) equation of the normal at the point where the curve cuts Y -axis, is $cy + ax = c^2$

Sol. We have, $y = ce^{x/a}$

$$\Rightarrow \frac{dy}{dx} = \frac{c}{a} e^{x/a} \Rightarrow \frac{dy}{dx} = \frac{1}{a} y$$

$$\Rightarrow \frac{y}{\frac{dy}{dx}} = a = \text{Constant}$$

$\Rightarrow$ Sub-tangent = Constant

Now, sub-normal

$$= y \frac{dy}{dx} = y \frac{y}{a} = \frac{y^2}{a} \propto \text{Square of the ordinate}$$

Equation of the tangent at (x_1, y_1) is

$$y - y_1 = \frac{y_1}{a} (x - x_1)$$

This meets X -axis at a point given by

$$-y_1 = \frac{y_1}{a} (x - x_1) \Rightarrow x = x_1 - a$$

The curve meets Y -axis at $(0, c)$.

$$\therefore \left(\frac{dy}{dx} \right)_{(0, c)} = \frac{c}{a}$$

So, the equation of the normal at $(0, c)$ is

$$y - c = \frac{-1}{c/a} (x - 0) \Rightarrow ax + cy = c^2$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 37. Let $f(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x$, where a_i 's are real and $f(x) = 0$ has a positive root α_0 . Then,

(a) $f'(x) = 0$ has a root α_1 such that $0 < \alpha_1 < \alpha_0$

(b) $f''(x) = 0$ has atleast one real root

(c) $f'''(x) = 0$ has atleast one real root

(d) None of the above

Sol. Clearly, $f(0) = 0$. So, $f(0) = f(\infty)$, therefore $f'(x) = 0$ has a real root α_1 lying between 0 and α_0 . So, $0 < \alpha_1 < \alpha_0$

Again, $f'(x) = 0$ is a fourth degree equation. As imaginary roots occurs in conjugate pairs, $f'(x) = 0$ will have another real root α_2 . Therefore, $f''(x) = 0$ will have a real root lying between α_1 and α_2 . As $f(x) = 0$ is an equation of the fifth degree, it will have atleast three real roots and so $f'(x)$ will have atleast real roots.

Hence, (a), (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 38-45) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 38. Statement I The tangent at $x=1$ to the curve $y = x^3 - x^2 - x + 2$ again meets the curve at $x = -2$.

Statement II When an equation of a tangent solved with the curve, repeated roots are obtained at point of tangency.

Sol. When $x = 1$,

then $y = 1$.

Now, $y' = 3x^2 - 2x - 1$

$$\Rightarrow y'|_{x=1} = 0$$

Equation of tangent is $y = 1$.

Solving with the curve $x^3 - x^2 - x + 2 = 1$

$$\Rightarrow x^3 - x^2 - x + 1 = 0$$

$$\Rightarrow x = -1, 1$$

$\therefore$ The tangent meets the curve again at $x = -1$.

Thus, Statement I is false and Statement II is true.

Hence, (d) is the correct answer.

Ex 39. Statement I For any $\triangle ABC$, $\sin\left(\frac{A+B+C}{3}\right) \geq \frac{\sin A + \sin B + \sin C}{3}$.

Statement II $y = \sin x$ is concave downward for $x \in (0, \pi]$.

Sol. Statements I and II both are true but Statement II does not explain Statement I.

Hence, (b) is the correct answer.

Ex 40. Statement I The ratio of length of tangent to length of normal is inversely proportional to the ordinate of the point of tangency at the curve $y^2 = 4ax$.

Statement II Length of normal and tangent to a curve $y = f(x)$ are $|y\sqrt{1+m^2}|$ and

$$\left| \frac{y\sqrt{1+m^2}}{m} \right|, \text{ where } m = \frac{dy}{dx}.$$

Sol. Let the slope of the tangent be denoted by $\tan \psi$, then length of tangent = $y \operatorname{cosec} \psi$

Length of normal = $y \sec \psi$

$$\therefore \text{Ratio} = \frac{\text{Length of tangent}}{\text{Length of normal}} = \cot \psi \propto \frac{1}{y}$$

Thus, Statement I true.

$$\text{Length of normal} = y \sec \psi = |y\sqrt{1+m^2}|$$

$$\text{Length of tangent} = y \operatorname{cosec} \psi = \left| \frac{y\sqrt{1+m^2}}{m} \right|$$

Thus, Statement II is true and explains Statement I.

Hence, (a) is the correct answer.

Ex 41. Statement I If $g(x)$ is a differentiable function $g(1) \neq 0$, $g(-1) \neq 0$ and Rolle's theorem is not applicable to $f(x) = \frac{x^2 - 1}{g(x)}$ in $[-1, 1]$, then $g(x)$ has atleast one root in $(-1, 1)$.

Statement II If $f(a) = f(b)$, then Rolle's theorem is applicable for $x \in (a, b)$.

Sol. Statement I As $f(-1) = f(1)$ and Rolle's theorem is not applicable, then it implies $f(x)$ is either discontinuous or $f'(x)$ does not exist atleast one point in $(-1, 1)$.

$\Rightarrow g(x) = 0$ at atleast one value of x in $(-1, 1)$.

Statement II is false. Consider the example in Statement I.

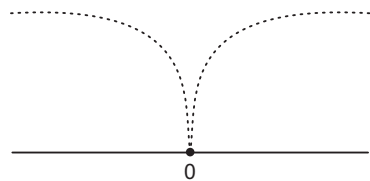
Hence, (c) is the correct answer.

Ex 42. Statement I If Rolle's theorem is applied in $f(x)$, then Lagrange's mean value theorem is also applied in $f(x)$.

Statement II Both Rolle's Theorem and LMVT cannot be applied in $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$.

Sol. For Rolle's theorem and LMVT, $f(x)$ must be continuous in $[a, b]$ and differentiable in (a, b) .

Hence, Statement I is true.



Since, $f(x) = |\sin|x||$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$ is not differentiable at $x = 0$.

Hence, Statement II is also true.

Hence, (b) is the correct answer.

Ex 43. Consider the ordinate of a point describing the circle $x^2 + y^2 = 25$ decreasing at the rate of 1.5 cm/s.

Statement I The rate of change of the abscissa of the point when ordinate equals to 4 cm is 2 cm/s.

Statement II $x dx + y dy = 0$

Sol. Given, $x^2 + y^2 = 25$

$$\Rightarrow 2x dx + 2y dy = 0 \Rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\Rightarrow \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\frac{x}{y} \Rightarrow -\frac{1.5}{\frac{dx}{dt}} = -\frac{3}{4}$$

$$\Rightarrow \frac{dx}{dt} = \frac{1.5 \times 4}{3} = 2 \text{ cm/s}$$

Hence, (b) is the correct answer.

Ex 44. Let $y = f(x)$ be polynomial of odd degree (≥ 3) with real coefficients and (a, b) is any point.

Statement I There always exists a line passing through (a, b) and touching the curve $y = f(x)$ at some point.

Statement II A polynomial of odd degree with real coefficients have atleast one real root.

Sol. Equation of a tangent at (h, k) on $y = f(x)$ is $y - k = f'(h)(x - h)$ must hold good for some (h, k) . Now, $hf'(h) - f(h) + b = 0$ represents an equation of odd degree in h .

there exist some h for which LHS vanishes.

Hence, (a) is the correct answer.

Ex 45. Consider the equation $x^3 - 3x + k = 0$, $k \in R$.

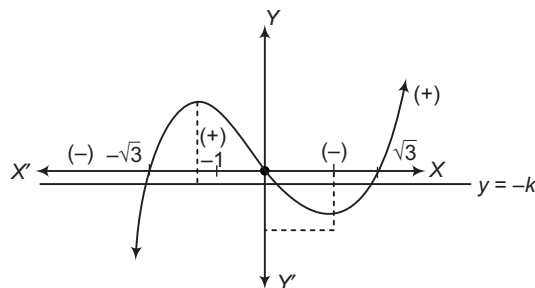
Statement I There is no value of k for which the given equation has two distinct roots in $(0, 1)$.

Statement II Between two consecutive roots of $f'(x) = 0$, $\{f(x) \text{ is a polynomial}\}$, $f(x) = 0$ must have one root.

Sol. $f(x) = x^3 - 3x = x(x - \sqrt{3})(x + \sqrt{3})$

Now, $f(x) = -k$ will have roots when $y = -k$ and $f(x) = y$ intersect.

Clearly, $f'(x) = 0$, at $x = \pm 1$.



Hence, (c) is the correct answer.

Type 4. Linked Comprehension Based Questions

7

Application of Derivatives

Passage I (Ex. Nos. 46-48) We say an equation $f(x) = g(x)$ to be consistent, if the curves $y = f(x)$ and $y = g(x)$ touch or intersect at atleast one point. If the curves $y = f(x)$ and $y = g(x)$ do not intersect or touch, then the equation $f(x) = g(x)$ is said to be inconsistent i.e. has no solution.

Ex 46. The equation $\cos x + \cos^{-1} x = \sin x + \sin^{-1} x$ is

- (a) consistent and has infinite number of solutions
- (b) consistent and has finite number of solutions
- (c) inconsistent
- (d) None of the above

Sol. Let $f(x) = \cos x - \sin x + \cos^{-1} x - \sin^{-1} x$, $x \in [-1, 1]$

$\therefore f(-1)f(1) < 0 \Rightarrow$ There exists atleast $c \in (-1, 1)$ such that $f(c) = 0$

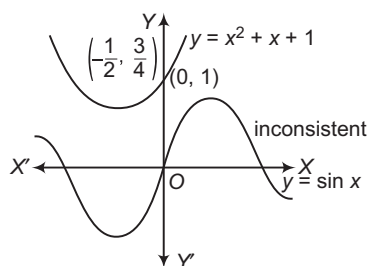
$\Rightarrow$ The curves $y = \cos x + \cos^{-1} x$ and $y = \sin x + \sin^{-1} x$ intersect atleast one point.

Hence, (b) is the correct answer.

Ex 47. The equation $\sin x = x^2 + x + 1$ is

- (a) consistent and has infinite number of solution
- (b) consistent and has finitely many number of solutions.
- (c) inconsistent
- (d) consistent and has unique solution

Sol. $\sin x = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$

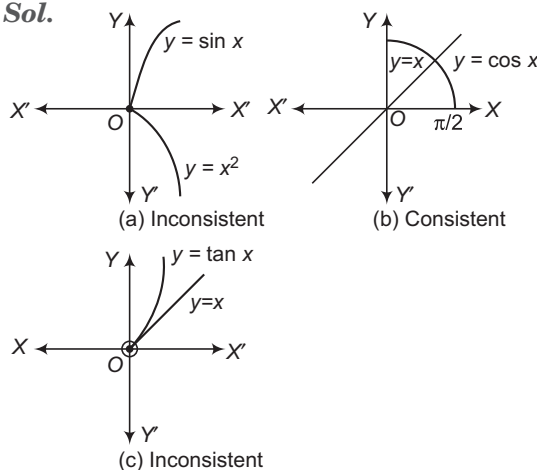


Hence, (c) is the correct answer.

Ex 48. Among the following equations, which is consistent in $\left(0, \frac{\pi}{2}\right)$?

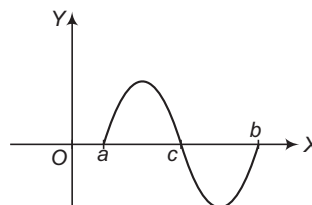
- (a) $\sin x + x^2 = 0$
- (b) $\cos x = x$
- (c) $\tan x = x$
- (d) All of the above

Sol.



Hence, (b) is the correct answer.

Passage II (Ex. Nos. 49-51) $f(x)$ is continuous and differentiable function. Given $f(x)$ takes values of the form $\pm \sqrt{l}$, where l belongs to the set of whole numbers whenever $x = a$ or b ; otherwise $f(x)$ takes real values. Also, $f(c) = -\frac{3}{2}$ and $|f(a)| \leq |f(b)|$



Ex 49. The number of rational values that $f(a) + f(b) + f(c)$ can take is/are

- (a) 4
- (b) 2
- (c) 3
- (d) 5

Ex 50. The number of values that $(f(a))^2 + (f(b))^2 + (f(c))^2$ can take is

- (a) 4
- (b) 2
- (c) 3
- (d) 7

Ex 51. The possible number of triplets $(f(a) \cdot f(b) \cdot f(c))$ is/are

- (a) 4
- (b) 5
- (c) 7
- (d) 6

Sol. (Q. Nos. 49-51) $f(x)f'(x) \geq 0$ in $[a, c]$ and ≤ 0 in $[c, b]$

$\therefore f(c)$ and $\frac{3}{2}f(x)f'(x) \neq 0$ in $(a, c) \cup (c, b)$

$\Rightarrow f(a), f(b) \leq 0$

$f(a)$	$f(b)$	$f(c)$
0	0	$-\frac{3}{2}$
0	-1	$-\frac{3}{2}$
0	$-\sqrt{2}$	$-\frac{3}{2}$
-1	-1	$-\frac{3}{2}$
-1	$-\sqrt{2}$	$-\frac{3}{2}$
$-\sqrt{2}$	$-\sqrt{2}$	$-\frac{3}{2}$

49. (c) 50. (a) 51. (d)

Passage III (Ex. Nos. 52-54) To find the point of contact $P \equiv (x_1, y_1)$ of a tangent to the graph of $y = f(x)$ passing through origin O , we equate the slope of tangent to $y = f(x)$ at P to the slope of OP .

Hence, we solve the equation $f'(x_1) = \frac{f(x_1)}{x_1}$ to get x_1 and y_1 .

Ex 52. The equation $|\ln mx| = px$, where m is a positive constant, has a single root for

- (a) $0 < p < \frac{m}{e}$ (b) $p < \frac{e}{m}$
 (c) $0 < p < \frac{e}{m}$ (d) $p > \frac{m}{e}$

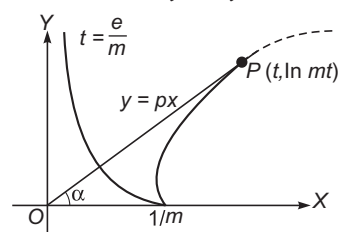
Ex 53. The equation $|\ln mx| = px$, where m is a positive constant has exactly two roots for

- (a) $p = \frac{m}{e}$ (b) $p = \frac{e}{m}$
 (c) $0 < p \leq \frac{e}{m}$ (d) $0 < p \leq \frac{m}{e}$

Ex 54. The equation $|\ln mx| = px$ where m is a positive constant has exactly three roots for

- (a) $p < \frac{m}{e}$ (b) $0 < p < \frac{m}{e}$
 (c) $0 < p < \frac{e}{m}$ (d) $p < \frac{e}{m}$

Sol. (Q. Nos. 52-54) Slope of tangent at P = Slope of OP
 $\Rightarrow \frac{1}{t} = \frac{\ln mt}{t}$



$\Rightarrow P \equiv \left(\frac{e}{m}, 1 \right)$

$\Rightarrow \tan \alpha = p = \frac{m}{e}$

52. (d) 53. (a) 54. (b)

Type 5. Match the Columns

Ex 55. Match the statements of Column I with values of Column II.

Column I	Column II
A. The equation $x \log x = 4 - x$ has atleast one root in	p. (0, 1)
B. If $27a + 9b + 3c + d = 0$, then the equation $4ax^3 + 3bx^2 + 2cx + d = 0$ has atleast one root in	q. (1, 4)
C. If $c = \sqrt{3}$ and $f(x) = x + \frac{1}{x}$, then interval of x in which LMVT is applicable for $f(x)$ is	r. (0, 3)
D. If $c = \frac{1}{2}$ and $f(x) = 2x - x^2$, then interval of x in which LMVT is applicable for $f(x)$ is	s. (-1, 1)

Sol. A. $f'(x) = \log x - \frac{4}{x} + 1$

$\Rightarrow f(x) = (x - 4) \log x + C$

$\therefore f(1) = f(4)$

$\therefore \exists x \in (1, 4)$ such that $f'(x) = 0$

B. $f'(x) = 4ax^3 + 3bx^2 + 2cx + d$

$\therefore f(x) = ax^4 + bx^3 + cx^2 + dx + e$

$\therefore f(0) = f(3) \quad [\because 27a + 9b + 3c + d = 0]$

$\therefore \exists x \in (0, 3)$ such that $f'(x) = 0$

C. $\frac{f(b) - f(a)}{b - a} = f'(\sqrt{3}) = \frac{2}{3}$

$\Rightarrow \frac{ab - 1}{ab} = \frac{2}{3}$

D. $\frac{f(b) - f(a)}{b - a} = f'\left(\frac{1}{2}\right) \Rightarrow a + b = 1$

A $\rightarrow$ q; B $\rightarrow$ r; C $\rightarrow$ q; D $\rightarrow$ p

Ex 56. Match the statements of Column I with the values of Column II.

Column I	Column II
A. The sides of a triangle vary slightly in such a way that its circumradius remains constant, if $\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 = m $, then the value of m is	p. 1
B. The length of sub-tangent to the curve $x^2 y^2 = 16$ at the point $(-2, 2)$ is $ k $, then the value of k is	q. -1
C. The curve $y = 2e^{2x}$ intersects the Y-axis at an angle $\cot^{-1}\left \frac{(8n-4)}{3}\right $, then the value of n is	r. 2
D. The area of a triangle formed by normal at the point $(1, 0)$, on the curve $x = e^{\sin y}$ with axes is $\frac{ 2t+1 }{6}$ sq units, then the value of t is	s. -2

Sol. A. Given, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ [say]

$$\begin{aligned} \therefore da &= 2R \cos A \, dA \\ db &= 2R \cos B \, dB \\ dc &= 2R \cos C \, dC \\ \therefore \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} &= 2R(dA + dB + dC) \quad \dots(i) \end{aligned}$$

Also, $A + B + C = \pi$
So, $dA + dB + dC = 0 \quad \dots(ii)$

From Eqs. (i) and (ii), we get

$$\begin{aligned} \frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} + 1 &= 1 \\ \Rightarrow m &= \pm 1 \end{aligned}$$

B. $x^2 y^2 = 16$
 $\Rightarrow xy = \pm 4 \quad \dots(1)$

$$L_{ST} = \left| \frac{y}{\frac{dy}{dx}} \right|$$

On differentiating Eq. (i) w.r.t. x , we get

$$y + xy' = 0$$

$$\Rightarrow y' = \frac{-y}{x} \Rightarrow L_{ST} = \left| \frac{y}{\frac{y}{x}} \right| = |x|$$

$$\begin{aligned} \Rightarrow L_{ST} &= 2 \\ \therefore k &= \pm 2 \end{aligned}$$

C. $y = 2e^{2x}$ intersects with Y-axis at $(0, 2)$.

$$\frac{dy}{dx} = 4e^{2x}$$

$$\therefore \left[\frac{dy}{dx} \right]_{\text{at } x=0} = 4$$

Angle of intersection with Y-axis

$$= \frac{\pi}{2} - \tan^{-1} 4 = \cot^{-1} 4$$

$$\Rightarrow n = 2 \text{ or } -1$$

$$D. \frac{dx}{dy} = e^{\sin y} \cos y$$

Slope of the normal at $(1, 0) = -1$

Equation of the normal is $x + y = 1$

$$\text{Area} = \frac{1}{2} \Rightarrow t = 1, -2$$

$A \rightarrow p, q; B \rightarrow r, s; C \rightarrow r, q; D \rightarrow p, s$

Ex 57. Match the statements of Column I with the values of Column II.

Column I (Curves)	Column II (Angle between curves)
A. $y^2 = 4x$ and $x^2 = 4y$	p. 90°
B. $2y^2 = x^2$ and $y^2 = 32x$	q. any one of $\tan^{-1} \frac{3}{4}$ or $\tan^{-1}(16^{1/3})$
C. $xy = a^2$ and $x^2 + y^2 = 2a^2$	r. 0°
D. $y^2 = x$ and $x^3 + y^3 = 3xy$ at other than origin	s. $\tan^{-1} \frac{1}{2}$

Sol. A. $y^2 = 4x$ and $x^2 = 4y$ intersect at point $(0, 0)$ and $(4, 4)$.

$$C_1 : y^2 = 4x \quad C_2 : x^2 = 4y$$

$$\frac{dy}{dx} = \frac{2}{y} \quad \frac{dy}{dx} = \frac{x}{2}$$

$$\left[\frac{dy}{dx} \right]_{0,0} = \infty \quad \left[\frac{dy}{dx} \right]_{0,0} = 0$$

Hence, $\tan \theta = 90^\circ$ at point $(0, 0)$

$$\frac{dy}{dx} \Big|_{(4,4)} = \frac{1}{2} \quad \frac{dy}{dx} \Big|_{(4,4)} = 2$$

$$\tan \theta = \left| \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}} \right| = \frac{3}{4}$$

B. Solving I: $2y^2 = x^3$ and II: $y^2 = 32x$, we get $(0, 0)$, $(8, 16)$ and $(8, -16)$.

$$\text{At } (0, 0), \left[\frac{dy}{dx} \right]_{(0,0)} = \infty \text{ for I}$$

$$\text{At } (0, 0), \left[\frac{dy}{dx} \right]_{(0,0)} = \infty \text{ for II}$$

Hence, angle $= 90^\circ$

$$\text{Now, } \left[\frac{dy}{dx} \right]_{(8,16)} = \frac{3}{4} \cdot \frac{64}{16} = 3 \text{ for I}$$

$$\left[\frac{dy}{dx} \right]_{(8,16)} = \frac{16}{16} = 1 \text{ for II}$$

$$\therefore \tan \theta = \left| \frac{3-1}{1+3} \right| = \frac{2}{4} = \frac{1}{2}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(8,-16)} = \frac{3(8)^2}{4(-16)} = -3 \text{ for I}$$

$$\left[\frac{dy}{dx} \right]_{(8,-16)} = \frac{32}{2(-16)} = -1 \text{ for II}$$

$$\therefore \tan \theta = \left| \frac{-3+1}{1+3} \right| = \left| \frac{-1}{2} \right| = \frac{1}{2}$$

C. The two curves are

$$xy = a^2 \quad \dots(i)$$

$$\text{and } x^2 + y^2 = 2a^2 \quad \dots(ii)$$

Solving Eqs. (i) and (ii), the points of intersection are (a, a) and $(-a, -a)$.

On differentiating (i), we get $\frac{dy}{dx} = \frac{-y}{x} = m_1$ [say]

On differentiating Eq. (ii), $\frac{dy}{dx} = \frac{-x}{y} = m_2$ [say]

At both points, $m_1 = -1 = m_2$

Hence, the two curves touch each other.

D. $y^2 = x, x^3 + y^3 = 3xy$

For the 1st curve, $2y \frac{dy}{dx} = 1$

$$\Rightarrow \left[\frac{dy}{dx} \right]_P = \frac{1}{2y_1}$$

Again, for the 2nd curve, $\left[\frac{dy}{dx} \right]_P = \frac{y_1 - x_1^2}{y_1^2 - x_1}$

Solving $y^2 = x$ and $x^3 + y^3 = 3xy$

$$y^6 + y^3 = 3y^3$$

$$\Rightarrow y^3 + 1 = 3$$

$$\Rightarrow y^3 = 2$$

$$\therefore y_1 = 2^{1/3} \text{ and } x_1 = 2^{2/3}$$

$$\text{Now, } m_1 = \frac{1}{2 \times 2^{1/3}} = \frac{1}{2^{4/3}}$$

$$\text{and } m_2 = \frac{2^{1/3} - 2^{4/3}}{2^{2/3} - 2^{2/3}} \rightarrow \infty$$

$$\text{Also, } \tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| = \left| \frac{1 - \frac{m_1}{m_2}}{\frac{1}{m_2} + m_1} \right|$$

$$= \left| \frac{1}{m_1} \right| = 2^{4/3} = 16^{1/3}$$

$A \rightarrow p, q; B \rightarrow p, s; C \rightarrow r; D \rightarrow q$

$$\therefore \theta = \tan^{-1}(16^{1/3})$$

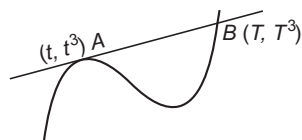
Type 6. Single Integer Answer Type Questions

Ex 58. Let C be the curve $y = x^3$ (where x takes all real values). The tangent at A meets the curve again at B . If the gradient at B is K times the gradient at A , then K is equal to _____.

Sol. (4) $\frac{dy}{dx} = 3x^2 = 3t^2$ at A

$$\therefore 3t^2 = \frac{T^3 - t^3}{T - t}$$

$$= T^2 + Tt + t^2$$



$$\text{Now, } T^2 + Tt - t^2 = 0$$

$$\Rightarrow (T - t)(T + 2t) = 0$$

$$\Rightarrow T = t$$

or $T = -2t$ [$\because T = t$ is not possible]

$$\text{Now, } m_A = 3t^2$$

$$m_B = 3T^2$$

$$\text{Also, } \frac{m_B}{m_A} = \frac{T^2}{t^2} = \frac{4t^2}{t^2} \quad [\because T = -2t]$$

$$\therefore m_B = 4m_A$$

$$\Rightarrow k = 4$$

Ex 59. Consider the two graphs $y = 2x$ and $x^2 - xy + 2y^2 = 28$. The absolute value of the tangent of the angle between the two curves at the points where they meet, is _____.

Sol. (2) $y = 2x, x^2 - xy + 2y^2 = 28$

On solving, the points of intersection are $(2, 4)$ and $(-2, -4)$.

$$\text{For 1st curve, } \frac{dy}{dx} = 2 = m_1 \quad \dots(i)$$

$$\text{For 2nd curve, } \frac{dy}{dx} = \frac{y - 2x}{4y - x} = 0 = m_2 \quad \dots(ii)$$

$$\therefore \tan \theta = 2$$

Ex 60. There is a point (q, p) on the graph of $f(x) = x^2$ and a point (r, s) on the graph of $g(x) = \frac{-8}{x}$, where $p > 0$ and $r > 0$. If the line

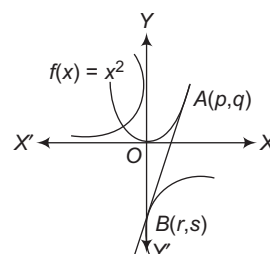
through (p, q) and (r, s) is also tangent to both the curves at these points respectively, then the value of $p + r$ is

Sol. (5) $y = x^2$ and $y = -\frac{8}{x}; q = p^2$ and $s = -\frac{8}{r} \quad \dots(i)$

Equating $\frac{dy}{dx}$ at A and B , we get

$$2p = \frac{8}{r^2} \quad \dots(ii)$$

$$\Rightarrow pr^2 = 4$$



$$\begin{aligned}\text{Now, } m_{AB} &= \frac{q-s}{p-r} \Rightarrow 2p = \frac{p^2 + \frac{8}{r}}{p-r} \\ \Rightarrow p^2 &= 2pr + \frac{8}{r} \\ \Rightarrow p^2 \frac{16}{r} & \\ \Rightarrow \frac{16}{r^4} &= \frac{16}{r} \Rightarrow r = 1 \ (r \neq 0) \\ \Rightarrow p &= 4 \\ \therefore r &= 1, p = 1 \\ \text{Hence, } p + r &= 5\end{aligned}$$

Ex 61. The radius of a right circular cylinder increase at a constant rate. Its altitude is a linear function of the radius and increase three times as fast as radius. When the radius is 1 cm, then the altitude is 6 cm. When the radius is 6 cm, the volume is increasing at the rate of $1 \text{ cm}^3/\text{s}$. When the radius is 36 cm, then the volume is increasing at the rate of $n \text{ cm}^3/\text{s}$. The value of $\frac{n}{11}$ is equal to _____.

Sol. (3) $\frac{dr}{dt} = C$ and $h = ar + b$

$$\text{Also, } \frac{dh}{dt} = 3 \frac{dr}{dt} \quad [\text{given}]$$

$$\therefore a \frac{dr}{dt} = 3 \frac{dr}{dt}$$

$$\Rightarrow a = 3$$

$$\text{Hence, } h = 3r + b$$

$$\text{When } r = 1, h = 6$$

$$\Rightarrow 6 = 3 + b$$

$$\Rightarrow b = 3$$

$$\therefore h = 3(r + 1)$$

$$V = \pi r^2 h$$

$$= 3\pi r^2 (r + 1)$$

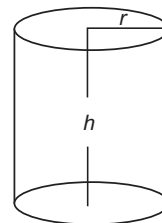
$$= 3\pi (r^3 + r^2)$$

$$\frac{dV}{dt} = 3\pi(3r^2 + 2r) \frac{dr}{dt}$$

$$\text{When } r = 6, \frac{dV}{dt} = 1 \text{ cu cm/s}$$

$$1 = 3\pi(108 + 12) \frac{dr}{dt}$$

$$\Rightarrow 360\pi \frac{dr}{dt} = 1$$



$$\text{Again, when } r = 36, \frac{dV}{dt} = n$$

$$\Rightarrow n = 3\pi[3(36)^2 + 2(36)] \frac{dr}{dt}$$

$$\Rightarrow n = 3\pi \cdot 36(110) \cdot \frac{1}{360\pi}$$

$$\Rightarrow n = 33$$

$$\Rightarrow \frac{n}{11} = \frac{33}{11} = 3$$

Ex 62. Let C be a curve defined by $y = e^{a+bx^2}$. The curve C passes through the point $P(1, 1)$ and the slope of the tangent at P is (-2) . Then, the value of $2a - 3b$ is

Sol. (5) $y = e^{a+bx^2}$, passes through $(1, 1)$.

$$\Rightarrow 1 = e^{a+b}$$

$$\text{Also, } \left(\frac{dy}{dx} \right)_{(1,1)} = -2$$

$$\Rightarrow [e^{a+bx^2} \cdot 2bx]_{(1,1)} = -2$$

$$\Rightarrow e^{a+b} \cdot 2b(1) = -2$$

$$\Rightarrow b = -1 \text{ and } a = 1$$

$$\Rightarrow 2a - 3b = 5$$

Target Exercises

Type 1. Only One Correct Option

- If the rate of change of area of a square plate is equal to that of the rate of change of its perimeter, then length of the side is
(a) 1 unit (b) 2 units (c) 3 units (d) 4 units
- The side of a square is equal to the diameter of a circle. If the side and radius change at the same rate, then the ratio of the change of their areas is
(a) $1 : \pi$ (b) $\pi : 1$ (c) $2 : \pi$ (d) $1 : 2$
- The radius and height of a cylinder are equal. If the radius of the sphere is equal to the height of the cylinder, then the ratio of the rates of increase of the volume of the sphere and the volume of the cylinder is
(a) 4 : 3 (b) 3 : 4
(c) $4 : 3\pi$ (d) $3\pi : 4$
- If the length of the diagonal of a square is increasing at the rate of 0.2 cm/s, then the rate of increase of its area when its side is $\frac{30}{\sqrt{2}}$ cm, is
(a) $3 \text{ cm}^2/\text{s}$ (b) $\frac{6}{\sqrt{2}} \text{ cm}^2/\text{s}$
(c) $3\sqrt{2} \text{ cm}^2/\text{s}$ (d) $6 \text{ cm}^2/\text{s}$
- The edge of a cube is equal to the radius of a sphere. If the edge and the radius increase at the same rate, then the ratio of the surface areas increases in cube and sphere is
(a) $2\pi : 3$ (b) $3 : 2\pi$
(c) $6 : \pi$ (d) None of these
- If the semi-vertical angle of a cone is 45° , then the rate of change of volume of the cone is
(a) curved surface area times the rate of change of r
(b) base area times the rate of change of l
(c) base area times the rate of change of r
(d) None of the above
- If errors of 1% each are made in the base radius and height of a cylinder, then the percentage error in its volume is
(a) 1% (b) 2%
(c) 3% (d) None of these
- If $T = 2\pi\sqrt{\frac{l}{g}}$, then relative errors in T and l are in the ratio
(a) $\frac{1}{2}$ (b) 2
(c) $\frac{1}{2\pi}$ (d) None of these
- If $\log_e 4 = 1.3868$, then $\log_e 4.01$ equals
(a) 1.3968 (b) 1.3898
(c) 1.3893 (d) None of these
- If a sphere of radius 100 mm shrinks to radius 98 mm, then the approximate decrease in its volume is
(a) $12000 \pi \text{ mm}^3$ (b) $800 \pi \text{ mm}^3$
(c) $80000 \pi \text{ mm}^3$ (d) $120 \pi \text{ mm}^3$
- If the ratio of base radius and height of a cone is 1 : 2 and percentage error in radius is $\lambda\%$, then the error in its volume is
(a) $\lambda\%$ (b) $2\lambda\%$
(c) $3\lambda\%$ (d) None of these
- The pressure P and volume V of a gas are connected by the relation $PV^{1/4} = \text{constant}$. The percentage increase in the pressure corresponding to a deminution of $\frac{1}{2}\%$ in the volume is
(a) $\frac{1}{2}\%$ (b) $\frac{1}{4}\%$
(c) $\frac{1}{8}\%$ (d) None of these
- The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point
(a) (1, 1)
(b) at no point
(c) (0, 1)
(d) (1, 0)
- The equation of normal at the point θ of the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$ is
(a) $x \cos \theta + y \sin \theta = a \cos 2\theta$
(b) $x \cos \theta - y \sin \theta = a \cos 2\theta$
(c) $x \sin \theta - y \cos \theta = a \cos 2\theta$
(d) None of the above
- The equation of the tangent to the curve $x = \frac{3at}{1+t^2}$,
 $y = \frac{3at^2}{1+t^2}$ at the point $t = 2$ is
(a) $4x + 3y + a = 0$ (b) $3x + 4y - a = 0$
(c) $4x + 3y - 12a = 0$ (d) None of these
- If the normal at the point t_1 on the curve $xy = c^2$ meets the curve again at t_2 , then
(a) $t_1^3 t_2 = 1$ (b) $t_1^3 t_2 = -1$
(c) $t_1 t_2^3 = -1$ (d) $t_1 t_2^3 = 1$

- 17.** The tangent to the curve $3xy^2 - 2x^2y = 1$ at $(1, 1)$ meets the curve again at
 (a) $\left(-\frac{16}{5}, -\frac{1}{20}\right)$ (b) $\left(\frac{16}{5}, -\frac{1}{20}\right)$
 (c) $\left(-\frac{16}{5}, \frac{1}{20}\right)$ (d) None of these
- 18.** The equation of the normal to the curve $y = 1 - 2^{x/2}$ at the point of intersection with the Y -axis is
 (a) $2y - x \log 2 = 0$ (b) $2y + x = 0$
 (c) $2y + x \log 2 = 0$ (d) None of these
- 19.** The area of the triangle formed by the positive X -axis, the tangent and the normal to the circle $x^2 + y^2 = 9$ at $(2, \sqrt{5})$ is
 (a) $9\sqrt{5}$ sq units (b) $\frac{9\sqrt{5}}{2}$ sq units
 (c) $\frac{9\sqrt{5}}{4}$ sq units (d) None of these
- 20.** If $y = f(x)$ is the equation of an ellipse to which the line $y = 2x + 3$ is a tangent at the point where $x = 2$, then
 (a) $f'(2) = 2$
 (b) $f(2) = 2f'(2)$
 (c) $f(2) + f'(2) + f''(2) = 2$
 (d) None of the above
- 21.** If the curve $y = px^2 + qx + r$ passes through the point $(1, 2)$ and the line $y = x$ touches it at the origin then the values of p, q and r are
 (a) $p = 1, q = -1, r = 0$ (b) $p = 1, q = 1, r = 0$
 (c) $p = -1, q = 1, r = 0$ (d) None of these
- 22.** If a tangent to the parabola $y^2 = 8x$ makes an angle $\frac{\pi}{4}$ with the straight line $y = 3x + 5$, then the point of contact is
 (a) $\left(\frac{1}{2}, 2\right)$ (b) $\left(-\frac{1}{2}, 2\right)$
 (c) $\left(\frac{1}{2}, -2\right)$ (d) None of these
- 23.** If the curve $y = x^2 + bx + c$ touches the line $y = x$ at the point $(1, 1)$, then the values of x for which the curve has a negative gradient, are
 (a) $x < \frac{1}{2}$ (b) $x > \frac{1}{2}$ (c) $\left(\frac{1}{2}, -2\right)$ (d) $x > -\frac{1}{2}$
- 24.** The intercepts made on the coordinate axes by the tangent at any point on the curve $y^m = ax^{m-1} + x^m$ are
 (a) $\frac{-ax}{(m-1)a + mx}, \frac{ay}{m(a+y)}$
 (b) $\frac{-ax}{(m-1)a + mx}, \frac{ay}{m(a+x)}$
 (c) $\frac{ax}{(m-1)a + mx}, \frac{ay}{m(a+x)}$
 (d) None of the above
- 25.** The angle made by the tangent at any point on the curve $x = a(t + \sin t \cos t), y = a(1 + \sin t)^2$, with X -axis is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2} + \frac{t}{2}$
 (c) $\frac{\pi}{4} + \frac{t}{2}$ (d) None of these
- 26.** The normal to the curve $5x^5 - 10x^3 + x + 2y + 6 = 0$ at $P(0, -3)$ meets the curve again at the point
 (a) $(-1, 1)$ (b) $(1, -1)$
 (c) $(-1, 5)$ (d) None of these
- 27.** Any tangent to the curve $y = 3x^7 + 5x + 3$
 (a) is parallel to X -axis
 (b) is parallel to Y -axis
 (c) makes an acute angle with the X -axis
 (d) makes an obtuse angle with the X -axis
- 28.** On the ellipse $4x^2 + 9y^2 = 1$, the points at which the tangents are parallel to the line $8x = 9y$, are
 (a) $\left(\frac{2}{5}, \frac{1}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{1}{5}\right)$
 (c) $\left(-\frac{2}{5}, -\frac{1}{5}\right)$ (d) None of these
- 29.** If $3(a + 2c) = 4(b + 3d)$, then the equation $ax^3 + bx^2 + cx + d = 0$ will have
 (a) no real solution
 (b) atleast one real root in $(-1, 0)$
 (c) atleast one real root in $(0, 1)$
 (d) None of the above
- 30.** A curve $y = f(x)$ passes through the point $P(1, 1)$. The normal to the curve at P is $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, then the equation of the curve is
 (a) $y = e^{a(x-1)}$ (b) $y = e^{a(1-x)}$
 (c) $y = e^{\frac{a}{2}(x-1)}$ (d) $y = e^{\frac{a}{2}(x+1)}$
- 31.** The point on the curve $9y^2 = x^3$, where the normal to the curve makes equal intercepts with the axes, is
 (a) $\left(4, \frac{8}{3}\right)$ (b) $\left(4, \frac{3}{8}\right)$
 (c) $\left(-4, -\frac{8}{3}\right)$ (d) None of these
- 32.** For the curve $x = t^2 - 1, y = t^2 - t$, the tangent is parallel to X -axis, where
 (a) $t = \frac{1}{\sqrt{3}}$ (b) $t = -\frac{1}{\sqrt{3}}$ (c) $t = 0$ (d) $t = \frac{1}{2}$
- 33.** For the curve $x = 3\cos \theta, y = 3\sin \theta, 0 \leq \theta \leq \pi$, the tangent is parallel to the X -axis when
 (a) $\theta = \pi$ (b) $\theta = 0$ (c) $\theta = \frac{\pi}{3}$ (d) $\theta = \frac{\pi}{2}$

- 34.** The equation of the normal to the curve $y = \sin x$ at $(0, 0)$ is
 (a) $x - y = 0$ (b) $x + y = 0$
 (c) $y = 0$ (d) $x = 0$
- 35.** The slope of tangent to the curves $x = t^2 + 3t - 8$, $y = 2t^2 - 2t - 5$ at the point $(2, -1)$, is
 (a) $\frac{22}{7}$ (b) $\frac{6}{7}$
 (c) -6 (d) None of these
- 36.** The curve $y = ax^3 + bx^2 + cx$ is inclined at 45° to X -axis at $(0, 0)$ but it touches X -axis at $(1, 0)$, then the values of a, b are given by
 (a) $a = 1, b = -2, c = 1$
 (b) $a = 1, b = 1, c = -2$
 (c) $a = -2, b = 1, c = 1$
 (d) $a = -1, b = 2, c = 1$
- 37.** The angle between the tangents to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points $(a, 0)$ and $(0, b)$ is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
 (c) $\frac{\pi}{3}$ (d) None of these
- 38.** The two curves $y^2 = 4x$ and $x^2 + y^2 - 6x + 1 = 0$ at the point $(1, 2)$
 (a) intersect orthogonally (b) intersect at an angle $\frac{\pi}{3}$
 (c) touch each other (d) None of these
- 39.** The length of the perpendicular from the origin to the normal to curve $x = a(\cos \theta + \theta \sin \theta)$, $y = a(\sin \theta - \theta \cos \theta)$ at any point θ is
 (a) a (b) $\frac{a}{2}$
 (c) $\frac{a}{3}$ (d) None of these
- 40.** If the tangent to the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ meets the coordinate axes at A and B , where $p = OA$ and $q = OB$, O being the origin, then the locus of (p, q) is
 (a) $\frac{p^2}{b^2} + \frac{q^2}{a^2} = 1$
 (b) $\frac{p^2}{a^2} + \frac{q^2}{b^2} = 2$
 (c) $\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1$
 (d) None of the above
- 41.** The part of the tangent to the curve $xy = c^2$ included between the coordinate axes is divided by the point of tangency in the ratio
 (a) $1 : 1$ (b) $1 : 2$
 (c) $1 : 3$ (d) None of these
- 42.** The portion of the tangent to the curve $x = \sqrt{a^2 - y^2} + \frac{a}{2} \log \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}}$ intercepted between the curve and X -axis, is of length
 (a) $\frac{a}{2}$ (b) a
 (c) $2a$ (d) None of these
- 43.** The length of the perpendicular from the origin to the tangent to the curve $y = e^{4x} + 2$ drawn at the point $x = 0$, is
 (a) $\frac{4}{\sqrt{17}}$ unit
 (b) $\frac{3}{\sqrt{17}}$ unit
 (c) $\frac{2}{\sqrt{17}}$ unit
 (d) None of the above
- 44.** The area of the triangle formed by a tangent to the curve $2xy = a^2$ and the coordinate axes is
 (a) $2a^2$ sq units (b) a^2 sq units
 (c) $3a^2$ sq units (d) None of these
- 45.** Tangents are drawn to the parabola $x^2 = 4y$ at its points of intersection with another parabola $y^2 = 4x$. The point of intersection of the tangents drawn, is given by
 (a) $(2, 0)$ (b) $(0, 2)$
 (c) $(-2, 0)$ (d) None of these
- 46.** The points where the curves $x^2 + y^2 = 2a^2$ and $2y^2 - x^2 = a^2$ cut orthogonally, are
 (a) $\left(\pm\sqrt{\frac{2}{3}}a, \pm\sqrt{\frac{4}{3}}a\right)$ (b) $\left(\pm\sqrt{\frac{4}{3}}a, \pm\sqrt{\frac{2}{3}}a\right)$
 (c) $\left(\pm\frac{a}{\sqrt{3}}, \pm\sqrt{\frac{2}{3}}a\right)$ (d) None of these
- 47.** The two curves $y = 3^x$ and $y = 5^x$ intersect at an angle
 (a) $\tan^{-1}\left(\frac{\log 3 - \log 5}{1 + \log 3 \cdot \log 5}\right)$
 (b) $\tan^{-1}\left(\frac{\log 3 + \log 5}{1 - \log 3 \cdot \log 5}\right)$
 (c) $\tan^{-1}\left(\frac{\log 3 + \log 5}{1 + \log 3 \cdot \log 5}\right)$
 (d) None of the above
- 48.** If $f(x)$ and $g(x)$ are differentiable functions for $0 \leq x \leq 1$ such that $f(0) = 10, g(0) = 2, f(1) = 2, g(1) = 4$, then in the interval $(0, 1)$,
 (a) $f'(x) = 0$ for all x
 (b) $f'(x) + 4g'(x) = 0$ for atleast one x
 (c) $f'(x) = 2g'(x)$ for atleast one x
 (d) None of the above

49. The tangent to the graph of the function $y = f(x)$ at the point with abscissa $x = 1$ forms an angle of $\frac{\pi}{6}$ and at the point $x = 2$ form an angle of $\frac{\pi}{3}$ and at the point $x = 3$ form an angle of $\frac{\pi}{4}$ with the positive direction of X -axis. The value of $\int_1^3 f'(x)f''(x)dx + \int_2^3 f''(x)dx$, $f''(x)$ being continuous, is
- (a) $\frac{4\sqrt{3}-1}{3\sqrt{3}}$ (b) $\frac{3\sqrt{3}-1}{2}$
 (c) $\frac{4-\sqrt{3}}{3}$ (d) None of these
50. The abscissa of the point on the curve $xy = (c+x)^2$, the normal at which cut-off numerically equal intercepts from the axes of coordinates, is
- (a) $\frac{c}{\sqrt{2}}$ (b) c
 (c) $\sqrt{2}c$ (d) None of these
51. If $g(x) = \frac{x+2}{x-1}$ and the line $3x + y - 10 = 0$, then the line is
- (a) tangent to $g(x)$ (b) normal to $g(x)$
 (c) chord of $g(x)$ (d) None of these
52. The sum of tangent and sub-tangent at any point of the curve $y = a \log(x^2 - a^2)$ varies as
- (a) abscissa
 (b) product of the coordinates
 (c) ordinate
 (d) None of the above
53. The length of the normal at any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ varies as
- (a) (abscissa)³
 (b) (ordinate)³
 (c) the perpendicular r from origin on tangent
 (d) None of the above
54. The sub-tangent, ordinate and sub-normal to the parabola $y^2 = 4ax$ at a point (different from the origin) are in
- (a) GP
 (b) AP
 (c) HP
 (d) None of the above
55. If α and β ($\alpha < \beta$) are two different real roots of the equation $ax^2 + bx + c = 0$, then
- (a) $\alpha > -\frac{b}{2a}$ (b) $\beta < -\frac{b}{2a}$
 (c) $\alpha < -\frac{b}{2a} < \beta$ (d) $\beta < -\frac{b}{2a} < \alpha$
56. If $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0$, then equation $a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n = 0$ has in the interval $(0, 1)$
- (a) exactly one root (b) atleast one root
 (c) atmost one root (d) no root
57. If the function $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) , then the equation $\begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix}$ has in the interval $[a, b]$
- (a) atleast one root
 (b) exactly one root
 (c) atmost one root
 (d) no root
58. If $f'(x) = \frac{1}{1+x^2}$ for all x and $f(0) = 0$, then
- (a) $f(2) < 0.4$ (b) $f(2) > 2$
 (c) $0.4 < f(2) < 2$ (d) $f(2) = 2$
59. If the equation $ax^2 + bx + c = 0$ has two positive and real roots, then the equation $ax^2 + (b+6a)x + (c+3b) = 0$ has
- (a) no solution
 (b) atleast one positive solution
 (c) atmost one positive solution
 (d) None of the above
60. The value of c in Lagrange's theorem for the function $f(x) = \begin{cases} x \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ in the interval $[-1, 1]$, is
- (a) 0
 (b) $\frac{1}{2}$
 (c) $-\frac{1}{2}$
 (d) not exist in the interval
61. Let f be a function which is continuous and differentiable for all real x . If $f(2) = -4$ and $f'(x) \geq 6$ for all $x \in [2, 4]$, then
- (a) $f(4) < 8$ (b) $f(4) \geq 8$
 (c) $f(4) \geq 12$ (d) None of these
62. The function $f(x) = 2x^2 - \log|x|$, $x \neq 0$ is strictly increasing in the interval
- (a) $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$
 (b) $\left(-\infty, -\frac{1}{2}\right) \cup \left(0, \frac{1}{2}\right)$
 (c) $\left(-\frac{1}{2}, \frac{1}{2}\right)$
 (d) None of the above

- 63.** If $0 < x < \frac{\pi}{2}$, then
 (a) $\cos x < 1 - \frac{x^2}{2}$ (b) $\cos x > 1 - \frac{x^2}{2}$
 (c) $\cos x = 1 - \frac{x^2}{2}$ (d) $\cos x \geq 1 - \frac{x^2}{2}$
- 64.** The range of value of x for which the function $f(x) = \frac{x}{\log x}$, $x > 0$ and $x \neq 1$, may be decreasing, is
 (a) $(0, e)$ (b) (e, ∞)
 (c) $(0, e) - \{1\}$ (d) None of these
- 65.** $\log x - \tan^{-1} x$ increases in the interval
 (a) $(-\infty, 0)$ (b) $(0, \infty)$
 (c) $(-\infty, \infty)$ (d) None of these
- 66.** If $0 < x < \frac{\pi}{2}$, then
 (a) $\cos(\sin x) < \sin(\cos x)$ (b) $\cos(\sin x) < \cos x$
 (c) $\cos(\sin x) > \sin(\cos x)$ (d) None of these
- 67.** The function $f(x) = \log x - \frac{2x}{2+x}$ is increasing interval
 (a) $(-\infty, 0)$ (b) $(0, \infty)$
 (c) $(-\infty, 1)$ (d) None of these
- 68.** If $y = x^3(x-2)^2$, then the values of x for which y , increases, are
 (a) $x < \frac{6}{5}$ or $x > 3$ (b) $\frac{6}{5} < x < 2$
 (c) $x < \frac{6}{5}$ or $x > 2$ (d) None of these
- 69.** The function $f(x) = -2x^3 + 21x^2 - 60x + 41$, in the interval $(-\infty, 1)$, is
 (a) < 0 (b) ≤ 0 (c) > 0 (d) ≥ 0
- 70.** If $0 < \alpha < \beta < \frac{\pi}{2}$, then
 (a) $\frac{\tan \beta}{\tan \alpha} < \frac{\alpha}{\beta}$ (b) $\frac{\tan \beta}{\tan \alpha} > \frac{\alpha}{\beta}$
 (c) $\frac{\tan \alpha}{\tan \beta} < \frac{\alpha}{\beta}$ (d) $\frac{\tan \alpha}{\tan \beta} > \frac{\alpha}{\beta}$
- 71.** The function $f(x) = \frac{1}{x}$ on its domain is
 (a) increasing (b) decreasing
 (c) constant (d) information insufficient
- 72.** Let $y = x^2 e^{-x}$, then the interval in which y increases w.r.t. x , is
 (a) $(-\infty, \infty)$ (b) $(-2, 0)$
 (c) $(2, \infty)$ (d) $(0, 2)$
- 73.** The function $f(x) = |x+2| + |x-1|$ is
 (a) increasing in $(1, \infty)$
 (b) increasing in $[1, \infty)$
 (c) decreasing in $(-\infty, -1]$
 (d) None of the above
- 74.** If $g(x) = f(x) + f(1-x)$ and $f'''(x) < 0$ for $0 \leq x \leq 1$, then
 (a) $g(x)$ increases in $(-\infty, \frac{1}{2})$
 (b) $g(x)$ increases in $(0, \frac{1}{2})$
 (c) $g(x)$ decreases in $(\frac{1}{2}, \infty)$
 (d) None of the above
- 75.** The function $f(x) = \frac{4 \sin x - 2x - x \cos x}{2 + \cos x}$, $0 < x < 2\pi$
 (a) increases in $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$
 (b) increases in $(\frac{\pi}{2}, \frac{3\pi}{2})$
 (c) decreases in $(0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$
 (d) None of the above
- 76.** The range of values of a for which the function $f(x) = x^3 + (a+2)x^2 + 3ax + 5$ may be monotonic in R , is
 (a) $a < 1$ (b) $1 < a < 4$
 (c) $a > 4$ (d) None of these
- 77.** The function $f(x) = \frac{|x-1|}{x^2}$
 (a) increases in $(-\infty, 0) \cup (1, 2)$
 (b) increases in $(0, 1) \cup (2, \infty)$
 (c) decreases in $(-\infty, 0) \cup (1, 2)$
 (d) None of the above
- 78.** The function $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$, $0 \leq x \leq \pi$ is
 (a) increasing in $(\frac{\pi}{2}, \frac{2\pi}{3})$
 (b) increasing in $(0, \frac{\pi}{2}) \cup (\frac{2\pi}{3}, \pi)$
 (c) decreasing in $(\frac{\pi}{2}, \frac{2\pi}{3})$
 (d) None of the above
- 79.** The values of k for which the function $f(x) = kx^3 - 9x^2 + 9x + 3$ may be increasing on R are
 (a) $k > 3$
 (b) $k < 3$
 (c) $k \leq 3$
 (d) None of the above
- 80.** The least possible value of k for which the function $f(x) = x^2 + kx + 1$ may be increasing on $[1, 2]$, is
 (a) 2
 (b) -2
 (c) 0
 (d) None of the above

81. If $f(x) = 2x^3 + 9x^2 + \lambda x + 20$ is a decreasing function of x in the largest possible interval $(-2, -1)$, then λ is equal to
(a) 12 (b) -12 (c) 6 (d) None of the above
82. The value of a in order that $f(x) = \sin x - \cos x - ax + b$ decreases for all real values, is given by
(a) $a \geq \sqrt{2}$ (b) $a < \sqrt{2}$ (c) $a \geq 1$ (d) $a < 1$
83. If $f(x) = \begin{cases} 3x^2 + 12x - 1, & -1 \leq x \leq 2 \\ 37 - x, & 2 < x \leq 3 \end{cases}$, then
(a) $f(x)$ is increasing on $[-1, 2]$
(b) $f(x)$ is continuous on $[-1, 3]$
(c) $f'(2)$ does not exist
(d) All of the above
84. $f(x) = (x-8)^4(x-9)^5, 0 \leq x \leq 10$, monotonically decreases in
(a) $\left(\frac{76}{9}, 10\right]$ (b) $\left(8, \frac{76}{9}\right)$ (c) $[0, 8]$ (d) $\left(\frac{76}{9}, 10\right]$
85. The function $y = x^3 + 5x^2 - 1$ is decreasing in the interval
(a) $-\infty < x < \frac{10}{3}$ (b) $0 < x < \infty$
(c) $-3 < x < 3$ (d) $-\frac{10}{3} < x < 0$
86. Given that $f'(x) > g'(x)$ for all real x and $f(0) = g(0)$, then
(a) $f(x) < g(x), \forall x \in (-\infty, 0)$
(b) $f(x) > g(x), \forall x \in (-\infty, 0)$
(c) $f(x) < g(x), \forall x \in (0, \infty)$
(d) None of the above
87. If $f''(x) < 0, \forall x \in (a, b)$, then $f'(x) = 0$
(a) exactly once in (a, b) (b) atmost once in (a, b)
(c) atleast once in (a, b) (d) None of these

Type 2. More than One Correct Option

88. Let the parabolas $y = x(c-x)$ and $y = x^2 + ax + b$ touch each other at the point $(1, 0)$, then
(a) $a + b + c = 0$ (b) $a + b = 2$
(c) $b - c = 1$ (d) $a + c = -2$
89. Points on the curve $f(x) = \frac{x}{1-x^2}$ where the tangent is inclined at an angle of $\frac{\pi}{4}$ to the X -axis are
(a) $(0, 0)$ (b) $\left(\sqrt{3}, -\frac{\sqrt{3}}{2}\right)$
(c) $\left(-2, \frac{2}{3}\right)$ (d) $\left(-\sqrt{3}, \frac{\sqrt{3}}{2}\right)$
90. The abscissa of the point on the curve $\sqrt{xy} = a + x$, the tangent at which cut-off equal intercepts from the coordinate axes is ($a > 0$)
(a) $\frac{a}{\sqrt{2}}$ (b) $-\frac{a}{\sqrt{2}}$ (c) $a\sqrt{2}$ (d) $-a\sqrt{2}$
91. Let $f(x) = \int_{x^2}^{\frac{1}{x}} \frac{dt}{\log t}, x > 0$, then
(a) $f'(x) = \frac{x^2 - x}{\log x}$
(b) f is an increasing function
(c) f has minimum at $x = 1$
(d) f is an increasing function on $[1, \infty)$
92. Let $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$ for every real number x , then
(a) h is increasing, whenever f is increasing
(b) h is decreasing, whenever f is decreasing
(c) h is decreasing, whenever f is decreasing
(d) nothing can be said in general
93. Given,
$$f(x) = 4 - \left(\frac{1}{2} - x\right)^{2/3}, g(x) = \begin{cases} \frac{\tan [x]}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

 $h(x) = \{x\}, k(x) = |x|$, where $[]$ and $\{ \}$ represent greatest integer function and fractional part function respectively, then in $[0, 1]$, Lagrange's mean value theorem is not applicable to
(a) f (b) g (c) k (d) h
94. On which of the following intervals, the function $x^{100} + \sin x - 1$ is strictly increasing?
(a) $(-1, 1)$ (b) $(0, 1)$
(c) $\left(\frac{\pi}{2}, \pi\right)$ (d) $\left(0, \frac{\pi}{2}\right)$
95. Let $f(x)$ satisfies the requirements of Lagrange's mean value theorem in $[0, 1]$, $f(0) = 0$ and $f'(x) \leq 1 - x, \forall x \in (0, 1)$, then
(a) $f(x) \geq x$
(b) $|f(x)| \geq x$
(c) $f(x) \leq x(1-x)$
(d) $f(x) \leq \frac{1}{4}$
96. For function $f(x) = \frac{\ln x}{x}$, which of the following statements are true?
(a) $f(x)$ has horizontal tangent at $x = e$
(b) $f(x)$ cuts the X -axis only at one point
(c) $f(x)$ is many-one function
(d) $f(x)$ has one vertical tangent

Type 3. Assertion and Reason

Directions (Q. Nos. 97-104) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

97. Statement I Let $f: [0, \infty) \rightarrow [0, \infty)$ and $g: [0, \infty) \rightarrow [0, \infty)$ be non-increasing and non-decreasing functions respectively and $h(x) = g(f(x))$. If f and g are differentiable for all points in their respective domains and $h(0) = 0$, then $h(x)$ is constant function.

Statement II $g(x) \in [0, \infty) \Rightarrow h(x) \geq 0$ and $h'(x) \leq 0$.

98. Statement I The equation $3x^2 + 4ax + b = 0$ has atleast one root in $(0, 1)$, if $3 + 4a = 0$.

Statement II $f(x) = 3x^2 + 4ax + b$ is continuous and differentiable in the interval $(0, 1)$.

99. Statement I If $f(x)$ is increasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

Statement II If $f(x)$ is decreasing function with concavity upwards, then concavity of $f^{-1}(x)$ is also upwards.

100. Let $f: R \rightarrow R$ be differentiable and strictly increasing function throughout its domain.

Statement I If $|f(x)|$ is also strictly increasing function, then $f(x) = 0$ has no real roots.

Statement II At ∞ or $-\infty$, $f(x)$ may approach to 0, but cannot be equal to zero.

101. Statement I If $f(x) = [x](\sin x + \cos x - 1)$, where $[]$ denotes the greatest integer function, then $f'(x) = [x](\cos x - \sin x)$ for any $x \in \text{integer}$.

Statement II $f'(x)$ does not exist for any $x \in \text{integer}$.

102. Statement I Both $\sin x$ and $\cos x$ are decreasing functions in $\left(\frac{\pi}{2}, \pi\right)$.

Statement II If a differentiable function decreases in (a, b) , then its derivative also decreases in (a, b) .

103. Statement I Lagrange's mean value theorem is not applicable to $f(x) = |x - 1|(x - 1)$.

Statement II $|x - 1|$ is not differentiable at $x = 1$.

104. Statement I The greatest of the numbers $1, 2^{1/2}, 3^{1/3}, 4^{1/4}, 5^{1/5}, 6^{1/6}, 7^{1/7}$ is $3^{1/3}$.

Statement II $x^{1/x}$ is increasing for $x < e$ and decreasing for $x > e$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 105-109) Let $f(x) = x^3 + ax^2 + bx + c$ be the given cubic polynomial and $f(x) = 0$ be the corresponding cubic equation, where $a, b, c \in R$. Now, $f'(x) = 3x^2 + 2ax + b$. Let $D = 4a^2 - 12b = 4(a^2 - 3b)$ be the discriminant of the equation $f'(x) = 0$.

105. If $D = 4a^2 - 12b = 4(a^2 - 3b)$ is the discriminant of the equation $f'(x) = 0$. Then,

- (a) $f(x)$ has all real roots
(b) $f(x)$ has one real and two imaginary roots
(c) $f(x)$ has repeated roots
(d) None of the above

106. If $D = 4(a^2 - 3b) > 0$ and $f(x_1) \cdot f(x_2) > 0$, where x_1, x_2 are the roots of $f(x)$, then

- (a) $f(x)$ has all real and distinct roots
(b) $f(x)$ would have three real roots but one of the root would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

107. If $D = 4(a^2 - 3b) > 0$ and $f(x_1) \cdot f(x_2) < 0$, where x_1, x_2 are the roots of $f(x)$, then

- (a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the root would be repeated
(c) $f(x)$ would have just one real root
(d) None of the above

108. If $D = 4(a^2 - 3b) > 0$ and $f(x_1) \cdot f(x_2) < 0$, where x_1, x_2 are the roots of $f(x)$, then

- (a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the root would be repeated
(c) $f(x)$ would have just one real root
(d) $f(x)$ has three real roots but all same

109. If $D = 4(a^2 - 3b) = 0$, then

- (a) $f(x)$ has all real and distinct roots
(b) $f(x)$ has three real roots but one of the root would be repeated
(c) $f(x)$ would have one real root
(d) None of the above

Passage II (Q. Nos. 110-112) If $y = f(x)$ is a curve and if there exists two points $A(x_1, f(x_1))$ and $B(x_2, f(x_2))$ on it such that $f'(x_1) = -\frac{1}{f'(x_2)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$, then the tangent at x_1 is normal at x_2 for that curve.

- 110.** Number of such lines on the curve $y = \sin x$ is
(a) 1 (b) 0 (c) 2 (d) infinite
- 111.** Number of such lines on the curve $y = |\ln x|$ is
(a) 1 (b) 2 (c) 0 (d) infinite
- 112.** Number of such lines on the curve $y^2 = x^3$ is
(a) 1 (b) 2 (c) 3 (d) 0

Passage III (Q. Nos. 113-115) Let f and g be two functions such that $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) . Then, atleast one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- (i) If $f(x) = f(b)$, then $f'(c) = 0$ (RMVT)
(ii) If $f(a) \neq f(b)$ and $a \neq b$, then $f'(c) = \frac{f(b) - f(a)}{b - a}$ (LMVT)
(iii) If $g'(x) \neq 0$, then $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$ then (Cauchy's theorem)

- 113.** The set of values of k , for which equation $x^3 - 3x + k = 0$ has two distinct roots in $(0, 1)$ is
(a) $(1, 4)$ (b) $(0, \infty)$ (c) $(0, 1)$ (d) ϕ
- 114.** Which of the following is true?
(a) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|, \forall x, y \in R$
(b) $|\tan^{-1} x - \tan^{-1} y| \geq |x - y|, \forall x, y \in R$
(c) $|\sin x - \sin y| \geq |x - y|, \forall x, y \in R$
(d) None of the above

- 115.** Let $0 < \alpha < \theta < \beta < \frac{\pi}{2}$, then $\frac{\sin \alpha - \sin \beta}{\cos \alpha - \cos \beta}$ is equal to
(a) $\tan \theta$ (b) $-\tan \theta$
(c) $\cot \theta$ (d) $-\cot \theta$

Type 5. Match the Columns

116. Match the statements of Column I with the values of Column II.

Column I	Column II
A. Circular plate is expanded by heat from radius 5 cm to 5.06 cm. Approximate increases in area is	p. 4
B. If an edge of a cube increases by 1%, then percentage increase in volume is	q. 0.6π
C. If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is twice the rate of decrease of x , then x is equal to (rate of decreases in non-zero)	r. 3
D. Rate of increase in area of equilateral triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s, is	s. $\frac{3\sqrt{3}}{4}$

117. Let $f(x) = (2^x - 1)(2^x - 2)$ and $g(x) = 2\sin x + \cos 2x$, then match the column.

Column I	Column II
A. f increases on	p. (π, ∞)
B. f decreases on	q. $(-\infty, -\pi)$

Column I	Column II
C. g decreases on	r. $\left(\frac{\pi}{2}, \frac{5\pi}{6}\right)$
D. g increases on	s. $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$

118. Let the function defined in Column I have domain $(0, \pi)$ then match the functions of Column I with the nature of $f(x)$ given in Column II.

Column I	Column II
A. $x^2 + 2\cos x$	p. increasing
B. $8\log x + x + \frac{16}{x}$	q. decreasing
C. $\log(\sqrt{1+x^2} - x)$	r. neither increasing nor decreasing
D. $\log(1+x) - \left(x - \frac{x^2}{2}\right)$	s. increases on $\left(0, \frac{\pi}{2}\right)$

Type 6. Single Integer Answer Type Questions

- 119.** Consider the function $f(x) = 8x^2 - 7x + 5$ on the interval $[-6, 6]$. The value of c that satisfies the conclusion of the mean value theorem, is _____.
- 120.** At the point $P(a, a^n)$ on the graph of $y = x^n$ ($n \in N$) in the first quadrant a normal is drawn. The normal intersects the Y -axis at the point $(0, b)$. If $\lim_{a \rightarrow 0} b = \frac{1}{2}$, then n equal _____.

- 121.** Suppose that f is differentiable for all x and that $f'(x) \leq 2$ for all x . If $f(1) = 2$ and $f(4) = 8$, then $f(2)$ has the value equal to _____.
- 122.** The set of all points where $f(x)$ is increasing is $(a, b) \cup (c, \infty)$, then find $[a+b+c]$, where $[]$ denotes greatest integer function if $\forall x \in R$ and $f''(x) > 0, \forall x \in R$.

Entrances Gallery

JEE Advanced/IITJEE

1. Let $f: (0, \infty) \rightarrow \mathbb{R}$ be given by $f(x) = \int_{1/x}^x e^{-\left(t+\frac{1}{t}\right)} \frac{dt}{t}$

Then,

[2014]

- (a) $f(x)$ is monotonically increasing on $[1, \infty)$
- (b) $f(x)$ is monotonically decreasing on $[0, 1)$
- (c) $f(x) + f\left(\frac{1}{x}\right) = 0$ for all $x \in (0, \infty)$
- (d) $f(2^x)$ is an odd function of x on $\mathbb{R}$

2. The slope of tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point $(1, 3)$ is _____.

[2014]

3. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is _____.

[2011]

Direction (Q. No. 4) Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$.

[2010]

JEE Main/AIEEE

6. The normal to the curve $x^2 + 2xy - 3y^2 = 0$ at $(1, 1)$

[2014]

- (a) does not meet the curve again
- (b) meets the curve again in the second quadrant
- (c) meets the curve again in the third quadrant
- (d) meets the curve again in the fourth quadrant

7. The intercepts on X -axis made by tangents to the curve, $y = \int_0^x |t| dt$, $x \in \mathbb{R}$, which are parallel to the line $y = 2x$, are equal to

[2013]

- (a) ± 1
- (b) ± 2
- (c) ± 3
- (d) ± 4

8. The real number k for which the equation, $2x^3 + 3x + k = 0$ has two distinct real roots in $[0, 1]$

[2013]

- (a) lies between 1 and 2
- (b) lies between 2 and 3
- (c) lies between -1 and 0
- (d) Does not exist

9. **Statement I** The equation $x \log x = 2 - x$ is satisfied by atleast one value of x lying between 1 and 2.

Statement II The function $f(x) = x \log x$ is an increasing function in $[1, 2]$ and $g(x) = 2 - x$ is a decreasing function in $[1, 2]$ and the graphs represented by these functions intersect at a point in $[1, 2]$.

[2013]

- (a) Statement I is true, Statement II is false
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c)

4. The function $f'(x)$ is

- (a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$
- (b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$
- (c) increasing in $(-t, t)$
- (d) decreasing in $(-t, t)$

5. Let f be a real-valued function defined on the interval $\left(-\frac{1}{4}, 1\right)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt$, for all $x \in \left(-\frac{1}{4}, 1\right)$ and let f^{-1} be the inverse function of f . Then, $(f^{-1})'(2)$ is equal to

[2010]

- (a) 1
- (b) $\frac{1}{3}$
- (c) $\frac{1}{2}$
- (d) $\frac{1}{e}$

Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(d) Statement I is false, Statement II is true

10. If the surface area of a sphere of radius r is increasing uniformly at the rate $8\text{ cm}^2/\text{s}$, then the rate of change of its volume is

[2013]

- (a) constant
- (b) proportional to $\sqrt{r}$
- (c) proportional to r^2
- (d) proportional to r

11. **Statement I** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$.

Statement II The function $x^2 e^x$ and $x^2 e^{-x}$ are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b) .

[2013]

- (a) Statement I is true, Statement II is false
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- (d) Statement I is false, Statement II is true

12. If the curves $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ and $y^3 = 16x$ intersect at right angled, then the value of α is

[2013]

- (a) 2
- (b) $\frac{4}{3}$
- (c) $\frac{1}{2}$
- (d) $\frac{3}{4}$

- 13.** A spherical balloon is being inflated at the rate of 35 cc/min. The rate of increase in the surface area (in cm^2/min) of the balloon when its diameter is 14 cm, is [2013]
(a) 10 (b) $\sqrt{10}$ (c) 100 (d) $10\sqrt{10}$
- 14.** If an equation of a tangent to the curve, $y = \cos(x + y)$, where $-1 \leq x \leq 1 + \pi$, is $x + 2y = k$, then k is equal to [2013]
(a) 1 (b) 2 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$
- 15.** A spherical balloon is filled with 4500π cu m of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cu m/min, then the rate (in m/min) at which the radius of the balloon decreases 49 min after the leakage began, is [2012]
(a) $\frac{9}{7}$ (b) $\frac{7}{9}$ (c) $\frac{2}{9}$ (d) $\frac{9}{2}$
- 16.** The equation of the tangent to the curve $y = x + \frac{4}{x^2}$, that is parallel to the X -axis, is [2010]
(a) $y = 0$ (b) $= 1$ (c) $y = 2$ (d) $y = 3$
- 17.** How many real solutions does the equation $x^7 + 14x^5 + 16x^3 + 30x - 560 = 0$ have? [2007]
(a) 5 (b) 7 (c) 1 (d) 3
- 18.** A value of c for which the conclusion of mean value theorem holds for the function $f(x) = \log_e x$ on the interval $[1, 3]$ is [2007]
(a) $2\log_3 e$ (b) $\frac{1}{2}\log_e 3$ (c) $\log_3 e$ (d) $\log_e 3$
- 19.** The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in [2007]
(a) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (b) $\left(-\frac{\pi}{2}, \frac{\pi}{4}\right)$
(c) $\left(0, \frac{\pi}{2}\right)$ (d) $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
- 20.** Angle between the tangents to the curve $y = x^2 - 5x + 6$ at the points (2, 0) and (3, 0) is [2006]
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$
- 21.** A spherical iron ball 10 cm in radius is coated with a layer of ice of uniform thickness that melts at a rate of $50 \text{ cm}^3/\text{min}$. When the thickness of ice is 15 cm, then the rate at which the thickness of ice decreases, is [2005]
(a) $\frac{5}{6\pi} \text{ cm/min}$ (b) $\frac{1}{54\pi} \text{ cm/min}$
(c) $\frac{1}{18\pi} \text{ cm/min}$ (d) $\frac{1}{39\pi} \text{ cm/min}$
- 22.** A lizard, at an initial distance of 21 cm behind an insect, moves from rest with an acceleration of 2 cm/s^2 and pursues the insect which is crawling uniformly along a straight line at a speed of 20 cm/s. Then, the lizard will catch the insect after [2005]
(a) 24 s (b) 21 s
(c) 1 s (d) 20 s
- 23.** A point on the parabola $y^2 = 18x$ at which the ordinate increases at twice the rate of the abscissa, is [2004]
(a) (2, 4) (b) (2, -4)
(c) $\left(-\frac{9}{8}, \frac{9}{2}\right)$ (d) $\left(\frac{9}{8}, \frac{9}{2}\right)$
- 24.** A function $y = f(x)$ has a second order derivative $f'' = 6(x - 1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is [2004]
(a) $(x - 1)^2$ (b) $(x - 1)^3$
(c) $(x + 1)^3$ (d) $(x + 1)^2$
- 25.** The normal to the curve $x = a(1 + \cos \theta)$, $y = a \sin \theta$ at θ always passes through the fixed point [2004]
(a) (a, 0) (b) (0, a)
(c) (0, 0) (d) (a, a)
- 26.** The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 - 2 = 0$ [2002]
(a) cut at right angle (b) touch each other
(c) cut at an angle $\frac{\pi}{3}$ (d) cut at an angle $\frac{\pi}{4}$
- 27.** The function $f(x) = \cot^{-1} x + x$ increases in the interval [2002]
(a) (1, ∞) (b) (-1, ∞)
(c) $(-\infty, \infty)$ (d) (0, ∞)

Other Engineering Entrances

- 28.** If there is an error of $k\%$ in measuring the edge of a cube, then the per cent error in estimating its volume is [VITEEE 2014]
(a) k
(b) $3k$
(c) $\frac{k}{3}$
(d) None of these
- 29.** The approximate value of $f(5.001)$, where $f(x) = x^3 - 7x^2 + 15$ is [Mainpal 2014]
(a) -34.995
(b) -33.995
(c) -33.335
(d) -35.995

- 30.** A stone is dropped into a quiet lake and waves move in circles at the speed of 5 cm/s. At that instant, when the radius of circular wave is 8 cm, how far is the enclosed area increasing?
[Karnataka CET 2014]
(a) $6\pi \text{ cm}^2/\text{s}$ (b) $8\pi \text{ cm}^2/\text{s}$
(c) $\frac{8}{3}\pi \text{ cm}^2/\text{s}$ (d) $80\pi \text{ cm}^2/\text{s}$
- 31.** The radius of a cylinder is increasing at the rate of 5 cm/min, so that its volume is constant. When its radius is 5 cm and height is 3 cm, then the rate of decreasing of its height is [Kerala CEE 2014]
(a) 6 cm/min (b) 3 cm/min
(c) 4 cm/min (d) 5 cm/min
(e) 2 cm/min
- 32.** If there is an error ± 0.04 cm in the measurement of the diameter of a sphere, then the approximate percentage error in its volume, when the radius is 10 cm, is [EAMCET 2014]
(a) ± 1.2 (b) ± 0.06
(c) ± 0.006 (d) ± 0.6
- 33.** Let $f'(x)$ be differentiable for all x . If $f(1) = -2$ and $f''(x) \geq 2, \forall x \in [1, 6]$, then [VITEEE 2014]
(a) $f(6) < 8$ (b) $f(6) \geq 8$
(c) $f(6) \geq 5$ (d) $f(6) \leq 5$
- 34.** The value of c from the Lagrange's mean value theorem for which $f(x) = \sqrt{25 - x^2}$ in $[1, 5]$ is [VITEEE 2014]
(a) 5 (b) 1
(c) $\sqrt{15}$ (d) None of these
- 35.** What are the values of c for which Rolle's theorem for the functions $f(x) = x^3 - 3x^2 + 2x$ in the interval $[0, 2]$ is verified? [Manipal 2014]
(a) $c = \pm 1$ (b) $c = 1 \pm \frac{1}{\sqrt{3}}$
(c) $c = \pm 2$ (d) None of these
- 36.** The length of the longest interval, in which $f(x) = 3\sin x - 4\sin^3 x$ is increasing, is [Manipal 2014]
(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π
- 37.** The function x^x is increasing, when [Manipal 2014]
(a) $x > \frac{1}{e}$ (b) $x < \frac{1}{e}$ (c) $x < 0$ (d) $\forall x$
- 38.** If $f(x) = x^3$ and $g(x) = x^3 - 4x$ in $-2 \leq x \leq 2$, then consider the statements [Karnataka CET 2014]
I. $f(x)$ and $g(x)$ satisfy mean value theorem.
II. $f(x)$ and $g(x)$ both satisfy Rolle's theorem.
III. Only $g(x)$ satisfied Rolle's theorem.
Of these statements
(a) I and II are correct (b) Only I is correct
(c) None is correct (d) I and III are correct
- 39.** The function $f(x) = 2x^3 - 15x^2 + 36x + 6$ is strictly decreasing in the interval [Kerala CEE 2014]
(a) (2, 3) (b) $(-\infty, 2)$
(c) (3, 4) (d) $(-\infty, 3) \cup (4, \infty)$
(e) $(-\infty, 2) \cup (3, \infty)$
- 40.** Applying Lagrange's mean value theorem for a suitable function $f(x)$ in $[0, h]$, we have $f(h) = f(0) + hf'(\theta h)$, $0 < \theta < 1$. Then, for $f(x) = \cos x$, the value of $\lim_{h \rightarrow 0^+} \theta$ is [WB JEE 2014]
(a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\frac{1}{3}$
- 41.** If $f(x)$ is a differentiable function in $[2, 7]$. If $f(2) = 3$ and $f'(x) \leq 5$ for all x in $(2, 7)$, then the maximum possible value of $f(x)$ at $x = 7$ is [WB JEE 2014]
(a) 7 (b) 15 (c) 28 (d) 14
- 42.** If R is the set of all real numbers and $f: [-1, 1] \rightarrow R$ is defined by $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then [WB JEE 2014]
(a) f satisfies the conditions of Rolle's theorem on $[-1, 1]$
(b) f satisfies the condition of Lagrange's mean value theorem on $[-1, 1]$
(c) f satisfies the conditions of Rolle's theorem on $[0, 1]$
(d) f satisfies the condition of Lagrange's mean value theorem on $[0, 1]$
- 43.** If $f: [0, 1] \rightarrow R$ (the set of all real numbers) is a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$ at $x = \frac{1}{4}$. Then, which of the following is correct? [AMU 2014]
(a) $f'(x) < f(x), \frac{1}{4} < x < \frac{3}{4}$
(b) $f'(x) > f(x), 0 < x < \frac{1}{4}$
(c) $f'(x) < f(x), 0 < x < \frac{1}{4}$
(d) $f'(x) < f(x), \frac{3}{4} < x < 1$
- 44.** The value of c in the Lagrange's mean value theorem for $f(x) = \sqrt{x-2}$ in the interval $[2, 6]$ is [EAMCET 2014]
(a) $\frac{9}{2}$ (b) $\frac{5}{2}$ (c) 3 (d) 4
- 45.** The triangle formed by the tangent to the curve $f(x) = x^2 + bx - b$ at the point (1, 1) and the coordinate axes lies in the first quadrant. If its area is 2, then the value of b is [Manipal 2014]
(a) -1 (b) 3
(c) -3 (d) 1

- 46.** The equation of the tangent of the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at (a, b) is [Manipal 2014]
 (a) $\frac{x}{a} + \frac{y}{b} = 2$ (b) $\frac{x}{a} + \frac{y}{b} = \frac{1}{2}$
 (c) $\frac{x}{b} - \frac{y}{b} = 2$ (d) $ax + by = 2$
- 47.** The slope of the tangent to the curve $y^2 e^{xy} = 9e^{-3} x^2$ at $(-1, 3)$ is [Kerala CEE 2014]
 (a) $-\frac{15}{2}$ (b) $-\frac{9}{2}$ (c) 15
 (d) $\frac{15}{2}$ (e) $\frac{9}{2}$
- 48.** The slope of the normal to the curve $y = x^2 - \frac{1}{x^2}$ at $(-1, 0)$ is [Kerala CEE 2014]
 (a) $\frac{1}{4}$ (b) $-\frac{1}{4}$ (c) 4
 (d) -4 (e) 0
- 49.** The angle of intersection between the curves $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 10$, where $[x]$ denotes the greatest integer $\leq x$, is [WB JEE 2014]
 (a) $\tan^{-1} 3$ (b) $\tan^{-1}(-3)$
 (c) $\tan^{-1} \sqrt{3}$ (d) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
- 50.** Find the slope of the normal to the curve $4x^3 - 3xy^2 + 6x^2 - 5xy + 9x + 14 = 0$ at the point $(-2, 3)$. [GGSIU 2014]
 (a) ∞ (b) 1 (c) $\frac{9}{2}$ (d) $\frac{2}{9}$
- 51.** The equation of the tangent to the curve $y = e^{-|x|}$ at the point, where the curve cuts the line $x = 1$ is [Karnataka CET 2013]
 (a) $e(x + y) = 1$ (b) $y = ex = 1$
 (c) $x + y = e$ (d) None of these
- 52.** If at each point of the curve $y = x^3 - ax^2 + x + 1$ tangent is inclined at an acute angle with the positive direction of the X -axis, then [UP SEE 2013]
 (a) $a \leq \sqrt{3}$ (b) $a > 0$
 (c) $-\sqrt{3} \leq a \leq \sqrt{3}$ (d) None of these
- 53.** If $f(x)$ is differentiable and strictly increasing function, then the value of $\lim_{x \rightarrow 0} \frac{f(x^2) - f(x)}{f(x) - f(0)}$ is [MP PET 2013]
 (a) -1 (b) 0
 (c) 1 (d) 2
- 54.** If displacement x of a moving particle in a plane at time t is given by $x = 2\cos(t\sqrt{3}) + 3\sin(t\sqrt{3})$, then its acceleration is proportional to [MP PET 2012]
 (a) $\frac{1}{x}$ (b) x (c) x^2 (d) $\frac{1}{x^2}$
- 55.** The function $x - \cot x$ [MP PET 2012]
 (a) always increases
 (b) always decreases
 (c) never decreases
 (d) sometime increases and sometimes decreases
- 56.** If f is a real valued differentiable function, such that $f(x)f'(x) < 0$ for all real x , then [WB JEE 2012]
 (a) $f(x)$ must be an increasing function
 (b) $f(x)$ must be a decreasing function
 (c) $|f(x)|$ must be an increasing function
 (d) $|f(x)|$ must be a decreasing function
- 57.** Rolle's theorem is applicable in the interval $[-2, 2]$ for the function [WB JEE 2012]
 (a) $f(x) = x^3$ (b) $f(x) = 4x^4$
 (c) $f(x) = 2x^3 + 3$ (d) $f(x) = \pi|x|$
- 58.** Which of the following function is decreasing on $\left(0, \frac{\pi}{2}\right)$? [AMU 2011]
 (a) $\sin 2x$ (b) $\cos 3x$
 (c) $\tan x$ (d) $\cos 2x$
- 59.** The value of c in the mean value theorem for the function $f(x) = 2x^2 + 3x + 4$ in the interval $[1, 2]$, is [MP PET 2011]
 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$
 (c) $\frac{3}{2}$ (d) $\frac{2}{3}$
- 60.** The function $f(x) = [x(x - 2)]^2$ is increasing in the set [Kerala CEE 2011]
 (a) $(-\infty, 0) \cup (2, \infty)$ (b) $(-\infty, 1)$
 (c) $(0, 1) \cup (2, \infty)$ (d) $(1, 2)$
 (e) $(0, 2)$
- 61.** A sphere increases its volume at the rate of π cc/s. The rate at which its surface area increases, when the radius is 1 cm is [Karnataka CET 2011]
 (a) 2π sq cm/s (b) π sq cm/s
 (c) $\frac{3\pi}{2}$ sq cm/s (d) $\frac{\pi}{2}$ sq cm/s
- 62.** The total revenue (in ₹) received from the sale of x units of a product is given by $R(x) = 13x^2 + 26x + 15$. Then, the marginal revenue (in ₹) when $x = 15$, is [Kerala CEE 2011]
 (a) 116 (b) 126 (c) 136
 (d) 416 (e) 146
- 63.** The approximate value of $\sqrt[3]{33}$ correct to 4 decimal places is [WB JEE 2011]
 (a) 2.0000 (b) 2.1001
 (c) 2.0125 (d) 2.0500
- 64.** If $y = 2x^3 - 2x^2 + 3x - 5$, then for $x = 2$ and $\Delta x = 0.1$, value of Δy is [WB JEE 2011]
 (a) 2.002 (b) 1.9
 (c) 0 (d) 0.9

- 65.** The distance (in metre) travelled by a vehicle in time t (in second) is given by the equation $s = 3t^3 + 2t^2 + t + 1$. The difference in the acceleration between $t = 2$ and $t = 4$ is [UP SEE 2011]
(a) 36 m/s^2 (b) 38 m/s^2 (c) 45 m/s^2 (d) 46 m/s^2
- 66.** The distance travelled by a bus in t second after the brakes are applied is $1 + 2t - 2t^2$ metre. The distance travelled by the bus before it stops is equal to [J&K CET 2011]
(a) 0.5 m (b) 1 m (c) 1.5 m (d) 2.5 m
- 67.** The point on the curve $y = x^3$ at which tangent is parallel to the point (1, 1), is [MP PET 2011]
(a) (0, 0) (b) (1, -1) (c) (-1, 1) (d) (-1, -1)
- 68.** The length of the normal to the curve $y = a \cosh\left(\frac{x}{a}\right)$ at any point varies as [GGSIPIU 2011]
(a) ordinate (b) abscissa
(c) square of the abscissa (d) square of the ordinate
- 69.** If $f(x) = \begin{cases} 0, & x = 0 \\ x - 3, & x > 0 \end{cases}$, then the function $f(x)$ is [WB JEE 2011]
(a) increasing when $x \geq 0$
(b) strictly increasing when $x > 0$
(c) strictly increasing at $x = 0$
(d) not continuous at $x = 0$ and so it is not increasing when $x > 0$
- 70.** The function $f(x) = ax + b$ is strictly increasing for all real x , if [WB JEE 2011]
(a) $a > 0$ (b) $a < 0$ (c) $a = 0$ (d) $a \leq 0$
- 71.** If $f(x) = xe^{x(1-x)}$, then $f(x)$ is
(a) increasing on R (b) decreasing on $\left[-\frac{1}{2}, 1\right]$
(c) increasing on $\left[-\frac{1}{2}, 1\right]$ (d) decreasing on R
- 72.** If $f(x) = kx - \cos x$ is monotonically increasing for all $x \in R$, then [J&K CET 2011]
(a) $k > -1$ (b) $k < 1$
(c) $x > 1$ (d) None of these
- 73.** If $f(x) = 2x^2 - |x| + 4$, $x \in [-1, 2]$, then for some $c \in (-1, 2)$, $f'(c)$ is equal to [J&K CET 2011]
(a) $\frac{f(2) - f(0)}{2 - 0}$ (b) $\frac{f(2) - f(-1)}{2 - (-1)}$
(c) $\frac{f(1) - f(-1)}{1 - (-1)}$ (d) None of these
- 74.** An edge of a variable cube is increasing at the rate of 10 cm/s. How fast the volume of the cube will increase, when the edge is 5 cm long? [Kerala CEE 2010]
(a) $750 \text{ cm}^3/\text{s}$ (b) $75 \text{ cm}^3/\text{s}$
(c) $300 \text{ cm}^3/\text{s}$ (d) $150 \text{ cm}^3/\text{s}$
(e) $25 \text{ cm}^3/\text{s}$
- 75.** If the error committed in measuring the radius of the circle is 0.05%, then the corresponding error in calculating the area is [Kerala CEE 2010]
(a) 0.05% (b) 0.0025% (c) 0.25%
(d) 0.1% (e) 0.2%
- 76.** If the normal to the curve $y = f(x)$ at the point (3, 4) make an angle $\frac{3\pi}{4}$ with the positive X -axis, then $f'(3)$ is [WB JEE 2010; BITSAT 2010]
(a) 1 (b) -1 (c) $-\frac{3}{4}$ (d) $\frac{3}{4}$
- 77.** The coordinates of the point on the curve $y = x^2 - 3x + 2$, where the tangent is perpendicular to the straight line $y = x$ are [WB JEE 2010; VITEEE 2010]
(a) (0, 2) (b) (1, 0) (c) (-1, 6) (d) (2, -2)
- 78.** The point on the curve $x^2 + y^2 = a^2$, $y \geq 0$ at which the tangent is parallel to X -axis is [Kerala CEE 2010]
(a) (a, 0) (b) (-a, 0) (c) $\left(\frac{a}{2}, \frac{\sqrt{3}}{2}a\right)$
(d) (0, a) (e) (0, a^2)
- 79.** The angle between the curves $y = x^2$ and $y^2 - x = 0$ at the point (1, 1) is [Kerala CEE 2010]
(a) $\frac{\pi}{2}$ (b) $\tan^{-1} \frac{4}{3}$ (c) $\frac{\pi}{3}$
(d) $\frac{\pi}{4}$ (e) $\tan^{-1} \frac{3}{4}$
- 80.** The distance between the origin and the normal to the curve $y = e^{2x} + x^2$ at $x = 0$, is [Kerala CEE 2010]
(a) 2 units (b) $\frac{2}{\sqrt{3}}$ units (c) $\frac{2}{\sqrt{5}}$ unit
(d) $\frac{1}{2}$ unit (e) $\frac{1}{\sqrt{5}}$ unit
- 81.** If the line $ax + by + c = 0$ is a tangent to the curve $xy = 4$, then [WB JEE 2010]
(a) $a < 0, b > 0$ (b) $a \leq 0, b > 0$
(c) $a < 0, b < 0$ (d) $a \leq 0, b < 0$
- 82.** The value of c in (0, 2) satisfying the mean value theorem for the function $f(x) = x(x-1)^2$, $x \in [0, 2]$ is equal to [Kerala CEE 2010]
(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{1}{3}$
(d) $\frac{2}{3}$ (e) $\frac{5}{3}$
- 83.** In which of the following functions, Rolle's theorem is applicable? [WB JEE 2010]
(a) $f(x) = |x|$ in $-2 \leq x \leq 2$
(b) $f(x) = \tan x$ in $0 \leq x \leq \pi$
(c) $f(x) = 1 + (x-2)^{2/3}$ in $1 \leq x \leq 3$
(d) $f(x) = x(x-2)^2$ in $0 \leq x \leq 2$

Answers

Work Book Exercise 7.1

1. (c)	2. (b)	3. (c)	4. (b)	5. (c)	6. (d)	7. (a)	8. (a)	9. (a)	10. (c)
11. (a)	12. (b)	13. (b)	14. (b)						

Work Book Exercise 7.2

1. (a)	2. (c)	3. (a)	4. (a)	5. (d)	6. (b)	7. (d)	8. (d)	9. (a)	10. (d)
11. (b)	12. (b)	13. (b)							

Work Book Exercise 7.3

1. (c)	2. (a)	3. (b)	4. (c)	5. (a)	6. (b)	7. (a)	8. (a)	9. (b)	10. (d)

Work Book Exercise 7.4

1. (b)	2. (a)	3. (c)	4. (d)	5. (c)	6. (d)	7. (b)	8. (d)	9. (a)	10. (c)
11. (d)	12. (b)	13. (a)	14. (a)	15. (a)					

Target Exercises

1. (b)	2. (c)	3. (a)	4. (d)	5. (b)	6. (c)	7. (c)	8. (a)	9. (c)	10. (c)
11. (c)	12. (c)	13. (d)	14. (b)	15. (c)	16. (b)	17. (a)	18. (d)	19. (c)	20. (a)
21. (b)	22. (c)	23. (a)	24. (b)	25. (c)	26. (b)	27. (c)	28. (b)	29. (b)	30. (a)
31. (a)	32. (d)	33. (d)	34. (b)	35. (b)	36. (a)	37. (b)	38. (c)	39. (a)	40. (c)
41. (a)	42. (b)	43. (b)	44. (b)	45. (a)	46. (b)	47. (a)	48. (b)	49. (d)	50. (a)
51. (a)	52. (b)	53. (d)	54. (a)	55. (c)	56. (b)	57. (a)	58. (c)	59. (b)	60. (d)
61. (b)	62. (a)	63. (b)	64. (c)	65. (b)	66. (c)	67. (b)	68. (c)	69. (c)	70. (b)
71. (b)	72. (d)	73. (a)	74. (b)	75. (a)	76. (b)	77. (a)	78. (a)	79. (a)	80. (b)
81. (a)	82. (a)	83. (d)	84. (b)	85. (d)	86. (a)	87. (b)	88. (a,b,d)	89. (a,b,d)	90. (a,b)
91. (a,d)	92. (a,b)	93. (a,b,d)	94. (b,c,d)	95. (c,d)	96. (a,b,c)	97. (a)	98. (d)	99. (d)	100. (b)
101. (b)	102. (c)	103. (d)	104. (a)	105. (b)	106. (c)	107. (a)	108. (b)	109. (d)	110. (b)
111. (c)	112. (b)	113. (d)	114. (a)	115. (d)	116. (*)	117. (*)	118. (*)	119. (0)	120. (2)
121. (4)	122. (0)								

*

116. $A \rightarrow q; B \rightarrow r; C \rightarrow p; D \rightarrow s$

117. $A \rightarrow p; B \rightarrow q; C \rightarrow s; D \rightarrow r$

118. $A \rightarrow p, s; B \rightarrow r; C \rightarrow q; D \rightarrow p, s$

Entrances Gallery

1. (a,c,d)	2. (8)	3. (2)	4. (b)	5. (b)	6. (d)	7. (a)	8. (d)	9. (c)	10. (d)
11. (a)	12. (b)	13. (a)	14. (d)	15. (c)	16. (d)	17. (c)	18. (a)	19. (b)	20. (a)
21. (c)	22. (b)	23. (d)	24. (b)	25. (a)	26. (a)	27. (c)	28. (b)	29. (a)	30. (d)
31. (a)	32. (d)	33. (b)	34. (c)	35. (b)	36. (a)	37. (a)	38. (d)	39. (a)	40. (c)
41. (c)	42. (d)	43. (c)	44. (c)	45. (c)	46. (a)	47. (c)	48. (a)	49. (a)	50. (d)
51. (d)	52. (c)	53. (a)	54. (b)	55. (a)	56. (d)	57. (b)	58. (d)	59. (c)	60. (c)
61. (a)	62. (d)	63. (c)	64. (b)	65. (a)	66. (c)	67. (d)	68. (d)	69. (b)	70. (a)
71. (c)	72. (a)	73. (a)	74. (a)	75. (d)	76. (a)	77. (b)	78. (d)	79. (e)	80. (c)
81. (c)	82. (b)	83. (d)							

Explanations

Target Exercises

1. Let at any time t , the length of a side of the square be x . Then, $A = \text{area} = x^2$ and $P = \text{perimeter} = 4x$

$$\Rightarrow \frac{dA}{dt} = 2x \frac{dx}{dt} \quad \text{and} \quad \frac{dP}{dt} = 4 \frac{dx}{dt}$$

It is given that

$$\frac{dA}{dt} = \frac{dP}{dt}$$

$$\Rightarrow 2x \frac{dx}{dt} = 4 \frac{dx}{dt} \Rightarrow x = 2 \text{ units}$$

2. Let x be the length of a side of the square and r be the radius of the circle. Then,

$$\frac{x}{r} = \frac{2r}{r} \quad [\text{given}]$$

$$\text{and} \quad \frac{dx}{dt} = \frac{dr}{dt}$$

Let A_1 and A_2 be the areas of the square and circle, respectively.

$$\text{Then,} \quad A_1 = x^2 \quad \text{and} \quad A_2 = \pi r^2$$

$$\Rightarrow \frac{dA_1}{dt} = 2x \frac{dx}{dt} \quad \text{and} \quad \frac{dA_2}{dt} = 2\pi r \frac{dr}{dt}$$

$$\Rightarrow \frac{\frac{dA_1}{dt}}{\frac{dA_2}{dt}} = \frac{2x \frac{dx}{dt}}{2\pi r \frac{dr}{dt}} = \frac{2}{\pi} \quad \left[\because x = 2r \text{ and } \frac{dx}{dt} = \frac{dr}{dt} \right]$$

3. Let r be the radius of the cylinder. Let V_1 and V_2 be the volumes of the sphere and cylinder, respectively. Then,

$$V_1 = \frac{4}{3}\pi r^3 \quad \text{and} \quad V_2 = \pi r^3 \quad [\because h = r]$$

$$\Rightarrow \frac{dV_1}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \text{and} \quad \frac{dV_2}{dt} = 3\pi r^2 \frac{dr}{dt}$$

$$\text{Now,} \quad \frac{\frac{dV_1}{dt}}{\frac{dV_2}{dt}} = 4:3$$

4. Let at any time t , the length of the diagonal be x cm.

$$\text{Then, each side} = \frac{x}{\sqrt{2}} \text{ cm}$$

$$\text{We have,} \quad \frac{dx}{dt} = 0.2 \text{ cm/s}$$

Let A be the area of the square. Then,

$$A = \left(\frac{x}{\sqrt{2}} \right)^2 = \frac{1}{2} x^2 \Rightarrow \frac{dA}{dt} = x \frac{dx}{dt}$$

$$\Rightarrow \left(\frac{dA}{dt} \right)_{\frac{x}{\sqrt{2}} = \frac{30}{\sqrt{2}}} = 30 \times 0.2 = 6 \text{ cm}^2/\text{s}$$

5. Let the length of an edge of the cube be x units. Let S_1 and S_2 be the surface areas of the cube and sphere respectively. Then,

$$S_1 = 6x^2 \quad \text{and} \quad S_2 = 4\pi x^2$$

$$\Rightarrow \frac{dS_1}{dt} = 12x \frac{dx}{dt} \quad \text{and} \quad S_2 = 8\pi x \frac{dx}{dt}$$

$$\text{Now,} \quad \frac{\frac{dS_1}{dt}}{\frac{dS_2}{dt}} = \frac{3}{2\pi}$$

6. Let r be the radius of the base, h be the height and l be slant height of the cone with semi-vertical angle 45° .

$$\text{Then,} \quad r = h \quad \text{and} \quad l^2 = r^2 + h^2 = 2r^2$$

Let V be the volume.

$$\text{Then,} \quad V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi r^3$$

$$\Rightarrow \frac{dV}{dt} = \pi r^2 \frac{dr}{dt}$$

$$= \text{Area of the base} \times \frac{dr}{dt}$$

7. Let V be the volume of the cone of base radius r and height h . Then,

$$V = \pi r^2 h$$

$$\Rightarrow dV = d(\pi r^2 h)$$

$$\Rightarrow dV = \pi(hdr^2 + r^2 dh)$$

$$\Rightarrow dV = \pi(2rhdr + r^2 dh)$$

$$\Rightarrow \frac{dV}{V} = \frac{\pi(2rhdr + r^2 dh)}{\pi r^2 h}$$

$$\frac{dV}{V} = \frac{2}{r} dr + \frac{dh}{h}$$

$$\Rightarrow \frac{dV}{V} \times 100 = 2 \frac{dr}{r} \times 100 + \frac{dh}{h} \times 100$$

$$\Rightarrow \frac{\Delta V}{V} \times 100 = 2 \left(\frac{\Delta r}{r} \times 100 \right) + \left(\frac{\Delta h}{h} \times 100 \right)$$

$$= 2 \times 1 + 1 = 3\%$$

8. We have,

$$T = 2\pi \sqrt{\frac{l}{g}}$$

$$\Rightarrow \log T = \log 2\pi + \frac{1}{2} \log l - \frac{1}{2} \log g$$

$$\Rightarrow \frac{1}{T} dT = 0 + \frac{1}{2l} dl - 0$$

$$\Rightarrow \frac{dT}{T} = \frac{1}{2} \frac{dl}{l}$$

$$\Rightarrow \frac{\Delta T}{T} = \frac{1}{2} \frac{\Delta l}{l} \quad [\because dT \equiv \Delta T \text{ and } dl \equiv \Delta l]$$

$$\Rightarrow \frac{\frac{\Delta T}{T}}{\frac{\Delta l}{l}} = \frac{1}{2}$$

9. Let $y = f(x) = \log_e x$, $x = 4$ and $x + \Delta x = 4.01$

Then,

$$\frac{dy}{dx} = \frac{1}{x} \quad \text{and} \quad \Delta x = 0.01$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{x=4} = \frac{1}{4}$$

$$\Delta y = \frac{dy}{dx} \Delta x$$

$$= \frac{1}{4} \times 0.01 = 0.0025$$

$$\therefore \log_e 4.01 = y + \Delta y = \log_e 4 + 0.0025$$

$$= 1.3868 + 0.0025 = 1.3893$$

10. Let x be the radius and V be the volume of the sphere.

$$\begin{aligned} \text{Then, } V &= \frac{4}{3} \pi r^3 \\ \therefore \frac{dV}{dx} &= 4\pi x^2 \\ \text{We have, } x &= 100 \text{ and } x + \Delta x = 98 \\ \therefore \Delta x &= -2 \\ \text{Now, } \Delta V &= \frac{dV}{dx} \Delta x \\ \Delta V &= 4\pi \times 100^2 \times -2 \\ \left[\because \left(\frac{dV}{dx} \right)_{x=100} = 4\pi \times 100^2 \right] \end{aligned}$$

$$\Rightarrow \Delta V = -80000\pi$$

Hence, decrease in volume is $80000\pi \text{ mm}^3$.

11. Let r be the base radius and h be the height of the cone. Then, $2r = h$. Let V be the volume of the cone, then

$$\begin{aligned} V &= \frac{1}{3} \pi r^2 h = \frac{2\pi r^3}{3} \\ \Rightarrow \frac{dV}{dr} &= 2\pi r^2 \\ \therefore \Delta V &= \frac{dV}{dr} \Delta r \\ \Rightarrow \Delta V &= 2\pi r^2 \Delta r \\ \Rightarrow \frac{\Delta V}{V} \times 100 &= \frac{2\pi r^2 \Delta r}{\frac{2}{3} \pi r^3} \times 100 \\ &= 3 \left(\frac{\Delta r}{r} \times 100 \right) = 3\lambda\% \end{aligned}$$

12. We have, $PV^{1/4} = \lambda$ (constant)

$$\begin{aligned} \Rightarrow \log P + \frac{1}{4} \log V &= \log \lambda \\ \Rightarrow \frac{1}{P} \cdot \frac{dP}{dV} + \frac{1}{4V} &= 0 \\ \Rightarrow \frac{dP}{dV} &= -\frac{1}{4} \frac{P}{V} \\ \therefore \Delta P &= \frac{dP}{dV} \Delta V \\ \Rightarrow \Delta P &= -\frac{1}{4} \frac{P}{V} \Delta V \\ \frac{\Delta P}{P} &= -\frac{1}{4} \frac{\Delta V}{V} \\ \Rightarrow \frac{\Delta P}{P} \times 100 &= -\frac{1}{4} \frac{\Delta V}{V} \times 100 \\ \Rightarrow \frac{\Delta P}{P} \times 100 &= -\frac{1}{4} \times -\frac{1}{2} = \frac{1}{8} \% \\ \left[\because \frac{\Delta V}{V} \times 100 = -\frac{1}{2}, \text{ given} \right] \end{aligned}$$

13. We have, $y - e^{xy} + x = 0$

$$\begin{aligned} \Rightarrow 1 - e^{xy} \left(x + y \frac{dx}{dy} \right) + \frac{dx}{dy} &= 0 \\ \text{Now, } \frac{dx}{dy} = \frac{xe^{xy} - 1}{1 - ye^{xy}} &= 0 \text{ for vertical tangents and } y = 0 \\ \therefore xe^{xy} - 1 = 0, y = 0 &\Rightarrow e^{xy} = \frac{1}{x}, y = 0 \\ \Rightarrow x &= 1 \end{aligned}$$

14. We have, $x = a \cos^3 \theta$, $y = a \sin^3 \theta$

$$\Rightarrow \frac{dx}{d\theta} = -3a \cos^2 \theta \sin \theta \text{ and } \frac{dy}{d\theta} = 3a \sin^2 \theta \cos \theta$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = -\frac{3a \sin^2 \theta \cos \theta}{3a \cos^2 \theta \sin \theta} = -\tan \theta$$

$$\text{Slope of tangent} = -\tan \theta$$

$$\therefore \text{Slope of normal} = \frac{-1}{-\tan \theta} = \cot \theta$$

So, the equation of normal at $(a \cos^3 \theta, a \sin^3 \theta)$ is

$$y - a \sin^3 \theta = \frac{\cos \theta}{\sin \theta} (x - a \cos^3 \theta)$$

$$\Rightarrow y \sin \theta - a \sin^4 \theta = x \cos \theta - a \cos^4 \theta$$

$$\Rightarrow x \cos \theta - y \sin \theta = a(\cos^4 \theta - \sin^4 \theta)$$

$$= a(\cos^2 \theta - \sin^2 \theta)(\cos^2 \theta + \sin^2 \theta) = a \cos 2\theta$$

$\therefore$ The equation of normal is $x \cos \theta - y \sin \theta = a \cos 2\theta$

15. We have, $\frac{dx}{dt} = \frac{3a(1+t^2) - 3at \cdot 2t}{(1+t^2)^2}$

$$= \frac{3a(1-t^2)}{(1+t^2)^2} \text{ and } \frac{dy}{dt} = \frac{6at}{(1+t^2)^2}$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{6at}{3a(1-t^2)} = \frac{2t}{1-t^2}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{t=2} = -\frac{4}{3}$$

Also, when $t = 2$, then

$$x = \frac{6a}{5}, y = \frac{12a}{5}$$

Hence, the equation of the tangent is

$$y - \frac{12a}{5} = -\frac{4}{3} \left(x - \frac{6a}{5} \right)$$

$$\Rightarrow 4x + 3y - 12a = 0$$

16. We have, $xy = c^2$

$$\Rightarrow x \frac{dy}{dx} + y \cdot 1 = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\therefore \left[\frac{dy}{dx} \right]_{(ct_1, c/t_1)} = -\frac{1}{t_1^2}$$

The equation of the normal at $\left(ct_1, \frac{c}{t_1} \right)$ is

$$y - \frac{c}{t_1} = t_1^2 (x - ct_1)$$

Since, this normal passes through $\left(ct_2, \frac{c}{t_2} \right)$.

$$\therefore \frac{c}{t_2} - \frac{c}{t_1} = t_1^2 (ct_2 - ct_1)$$

$$\Rightarrow t_1^3 t_2 = -1 \quad [\because t_1 - t_2 \neq 0]$$

17. We have, $3xy^2 - 2x^2y = 1$

On differentiating w.r.t. x , we get

$$3y^2 + 6xy \frac{dy}{dx} - 4xy - 2x^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(1,1)} = \frac{1}{4}$$

The equation of the tangent (1, 1) is

$$(y - 1) = \frac{1}{4}(x - 1) \Rightarrow x = 4y - 3$$

When this tangent meets the curve again, then

$$3 \cdot (4y - 3)y^2 - 2(4y - 3)^2 y = 1$$

$$\Rightarrow 20y^3 - 39y^2 + 18y + 1 = 0$$

$$\Rightarrow y = 1, 1, \frac{-1}{20}$$

18. We have, $y = 1 - 2^{x/2}$... (i)

For Y-axis, $x = 0$

$$\therefore y = 1 - 2^0 = 1 - 1 = 0$$

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = -2^{x/2} \cdot \frac{1}{2} \log 2$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(0,0)} = -\frac{1}{2} \log 2$$

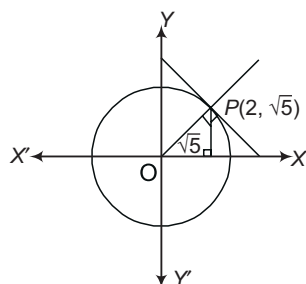
Therefore, equation of normal is

$$(y - 0) = \frac{2}{\log 2}(x - 0)$$

$$\Rightarrow 2x - y \log 2 = 0$$

19. We have, $x^2 + y^2 = 9$

$$\Rightarrow 2x + 2y \frac{dy}{dx} = 0$$



$$\therefore \left[\frac{dy}{dx} \right]_{(2, \sqrt{5})} = -\frac{2}{\sqrt{5}}$$

The equation of tangent is

$$(y - \sqrt{5}) = -\frac{2}{\sqrt{5}}(x - 2) \quad \dots (i)$$

and the equation of normal is

$$(y - \sqrt{5}) = \frac{\sqrt{5}}{2}(x - 2) \quad \dots (ii)$$

The tangent and the normal intersect X-axis at the points $\left(\frac{9}{2}, 0\right)$ and $O(0, 0)$, respectively.

Therefore, area of ΔOAP

$$= \frac{1}{2} \times \frac{9}{2} \times \sqrt{5} = \frac{9\sqrt{5}}{4} \text{ sq units}$$

20. We have, $y = f(x)$

$$\Rightarrow \frac{dy}{dx} = f'(x)$$

$\therefore$ Slope of the tangent to the ellipse at the point where

$$x = 2 \text{ is } f'(2) = 2$$

$$[\because y = 2x + 3 \text{ is a tangent to the ellipse at } x = 2]$$

Hence, $f'(2) = 2$

21. Since, the curve passes through the point (1, 2).

$$\therefore 2 = p + q + r \quad \dots (i)$$

Also, the curve passes through the origin.

$$\therefore r = 0$$

The equation of the tangent at (0, 0) is

$$y - 0 = \left[\frac{dy}{dx} \right]_{(0,0)} (x - 0)$$

$$\Rightarrow y = qx$$

But $y = x$ is the tangent at origin.

$$\therefore q = 1$$

$$\text{From Eq. (i), } 2 = p + 1 + 0 \Rightarrow p = 1$$

$$\text{Hence, } p = 1, q = 1, r = 0.$$

22. Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 8x$ (i)

On differentiating Eq. (i) w.r.t. x , we get

$$2y \frac{dy}{dx} = 8$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = \frac{4}{y_1} = m_1 \quad [\text{say}]$$

Also, the slope of the given line $= 3 = m_2$ [say]

$$\therefore \tan \frac{\pi}{4} = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow 1 = \left| \frac{\frac{4}{y_1} - 3}{1 + \frac{12}{y_1}} \right|$$

$$\Rightarrow y_1 = 8 \text{ or } y_1 = -2$$

The corresponding values of x_1 are $8, \frac{1}{2}$.

Hence, the points of contact are $(8, 8)$ and $\left(\frac{1}{2}, -2\right)$.

23. We have, $y = x^2 + bx + c$

$$\Rightarrow \frac{dy}{dx} = 2x + b$$

Since, the curve touches the line $y = x$ at the point (1, 1).

$$[2x + b]_{(1,1)} = 1 \Rightarrow 2 + b = 1$$

$$\Rightarrow b = -1$$

Also, the curve passes through the point (1, 1).

$$\therefore 1 = 1 + b + c$$

$$\Rightarrow c = -b = 1$$

$$\therefore y = x^2 - x + 1$$

$$\Rightarrow \frac{dy}{dx} = 2x - 1$$

$$\therefore \frac{dy}{dx} < 0$$

$$\Rightarrow 2x - 1 < 0 \Rightarrow x < \frac{1}{2}$$

24. We have, $y^m = ax^{m-1} + x^m$

$$\Rightarrow my^{m-1} \frac{dy}{dx} = a(m-1)x^{m-2} + mx^{m-1}$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^{m-2}(am - a + mx)}{my^{m-1}}$$

$\therefore$ Equation of tangent is

$$Y - y = \frac{x^{m-2}(am - a + mx)}{my^{m-1}}(X - x)$$

For x-intercept, put $Y = 0$

$$\therefore X = x - \frac{mx^{m-1}(a+x)}{x^{m-2}(am-a+mx)} = \frac{-ax}{(m-1)a+mx}$$

$$\text{Similarly, intercept on Y-axis} = \frac{ay}{m(a+x)}$$

25. We have,

$$\begin{aligned} x &= a(t + \sin t \cos t) \text{ and } y = a(1 + \sin t)^2 \\ \frac{dx}{dt} &= a(1 + \cos 2t) \\ \Rightarrow \frac{dy}{dt} &= 2a \cos t (1 + \sin t) \\ &= a(2 \cos t + \sin 2t) \\ \therefore \frac{dy}{dx} &= \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2 \cos t + \sin 2t}{1 + \cos 2t} = \frac{1 + \sin t}{\cos t} \\ &= \frac{\cos \frac{t}{2} + \sin \frac{t}{2}}{\cos \frac{t}{2} - \sin \frac{t}{2}} = \frac{1 + \tan \frac{t}{2}}{1 - \tan \frac{t}{2}} = \tan \left(\frac{\pi}{4} + \frac{t}{2} \right) \end{aligned}$$

Therefore, the tangent makes an angle $\frac{\pi}{4} + \frac{t}{2}$ with X-axis.

26. We have, $5x^5 - 10x^3 + x + 2y + 6 = 0$... (i)

$$\begin{aligned} \Rightarrow 25x^4 - 30x^2 + 1 + 2 \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= -\frac{1}{2}(25x^4 - 30x^2 + 1) \\ \therefore \left[\frac{dy}{dx} \right]_{(0, -3)} &= -\frac{1}{2} \end{aligned}$$

Equation of the normal to the curve at $P(0, -3)$ is

$$(y + 3) = 2(x - 0), \text{ i.e. } y = 2x - 3 \quad \dots (ii)$$

On substituting the value of y from Eq. (ii) in Eq. (i), we get

$$\begin{aligned} 5x^5 - 10x^3 + 5x &= 0 \\ \Rightarrow 5x(x^2 - 1)^2 &= 0 \Rightarrow x = 0, 1, -1 \end{aligned}$$

When $x = 0$, $y = -3$, when $x = 1$, $y = -1$ and when $x = -1$, $y = -5$.

27. We have, $\frac{dy}{dx} = 21x^6 + 5$, which is positive, $\forall x \in R$.

Therefore, the tangent makes an acute angle with X-axis.

28. We have, $4x^2 + 9y^2 = 1$... (i)

On differentiating w.r.t. x , we get

$$8x + 18y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{4x}{9y}$$

The tangent at point (x, y) will be parallel to the line

$$8x = 9y, \text{ if } -\frac{4x}{9y} = \frac{8}{9} \Rightarrow x = -2y$$

On substituting $x = -2y$ in Eq. (i), we get

$$\begin{aligned} 4(-2y)^2 + 9y^2 &= 1 \\ \Rightarrow 25y^2 &= 1 \\ \Rightarrow y &= \pm \frac{1}{5} \end{aligned}$$

Thus, the points where the tangents are parallel to

$$8x = 9y, \text{ are } \left(-\frac{2}{5}, \frac{1}{5} \right) \text{ and } \left(\frac{2}{5}, -\frac{1}{5} \right).$$

29. Let $f(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$, which is continuous and differentiable.

$$\begin{aligned} f(0) &= 0, f(-1) = \frac{a}{4} - \frac{b}{3} + \frac{c}{2} - d \\ &= \frac{1}{4}(a + 2c) - \frac{1}{3}(b + 3d) = 0 \end{aligned}$$

So, according to Rolle's theorem, there exists atleast one root of $f'(x) = 0$ in $(-1, 0)$.

30. Slope of the normal at $(1, 1)$ is $-\frac{1}{a}$.

$\Rightarrow$ Slope of the tangent at $(1, 1)$ is a .

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(1, 1)} = a \quad \dots (i)$$

We are given that, $\frac{dy}{dx} \propto y$

$$\frac{dy}{dx} = ky, \text{ where } k \text{ is any constant.}$$

$$\Rightarrow \frac{dy}{y} = k dx$$

$$\Rightarrow \log |y| = kx + c, \text{ where } c \text{ is a constant}$$

$$\Rightarrow |y| = e^{kx+c}$$

$$\Rightarrow y = \pm e^c e^{kx} = Ae^{kx}, \text{ where } A \text{ is a constant.}$$

Since, the curve passes through $(1, 1)$, therefore

$$1 = Ae^k \Rightarrow A = e^{-k}$$

$$\text{Thus, } y = e^{-k} \cdot e^{kx} = e^{k(x-1)}$$

$$\Rightarrow \frac{dy}{dx} = ke^{k(x-1)}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(1, 1)} = k$$

$$\Rightarrow a = k \quad [\text{from Eq. (i)}]$$

Thus, the required curve is $y = e^{a(x-1)}$.

31. We have, $9y^2 = x^3$... (i)

On differentiating w.r.t. x , we get

$$18y \frac{dy}{dx} = 3x^2 \Rightarrow \frac{dy}{dx} = \frac{x^2}{6y}$$

$$\text{Slope of the normal} = -\frac{6y}{x^2} = \pm 1 \quad \dots (ii)$$

[$\because$ any line making equal intercept on axes will have its slope as 1 or -1]

Now, from Eq. (ii), we have

$$y = -\frac{x^2}{6} \text{ or } y = \frac{x^2}{6}$$

On solving these with the Eq. (i), we get the points

$$\left(4, \frac{8}{3} \right), \left(4, -\frac{8}{3} \right)$$

32. We have, $x = t^2 - 1$, $y = t^2 - t$

$$\Rightarrow \frac{dx}{dt} = 2t \text{ and } \frac{dy}{dt} = 2t - 1$$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t - 1}{2t}$$

Since, the tangent is parallel to X-axis.

$$\therefore \frac{dy}{dx} = 0 \Rightarrow t = \frac{1}{2}$$

33. We have, $x = 3\cos \theta$ and $y = 3\sin \theta$
 $\Rightarrow \frac{dx}{d\theta} = -3\sin \theta$ and $\frac{dy}{d\theta} = 3\cos \theta$
 $\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3\cos \theta}{-3\sin \theta} = -\frac{\cos \theta}{\sin \theta} = -\cot \theta$

Since, the tangent is parallel to X-axis.

$\therefore \frac{dy}{dx} = 0 \Rightarrow \cot \theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

34. We have, $y = \sin x$
 $\Rightarrow \frac{dy}{dx} = \cos x$
 $\therefore \left[\frac{dy}{dx} \right]_{(0,0)} = \cos 0 = 1$

$\therefore$ Slope of normal $= -\frac{1}{1} = -1$

The equation of normal at (0,0) is

$y - 0 = -1(x - 0)$
 $\Rightarrow y + x = 0$

35. We have, $x = t^2 + 3t - 8$ and $y = 2t^2 - 2t - 5$... (i)

On differentiating Eq. (i) w.r.t. t , we get

$\frac{dx}{dt} = 2t + 3$ and $\frac{dy}{dt} = 4t - 2$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4t - 2}{2t + 3}$

Thus, slope of the tangent to the curve at (2, -1)

$= \left[\frac{dy}{dx} \right]_{t=2} = \frac{4 \times 2 - 2}{2 \times 2 + 3} = \frac{6}{7}$

36. We have, $y = ax^3 + bx^2 + cx$

$\Rightarrow \frac{dy}{dx} = 3ax^2 + 2bx + c$

$\therefore \left[\frac{dy}{dx} \right]_{(0,0)} = c = \tan 45^\circ = 1$ [given]

$\Rightarrow c = 1$

$\left[\frac{dy}{dx} \right]_{(1,0)} = 3a + 2b + c = 0$

[$\because$ X-axis is tangent at (1, 0)]

$\Rightarrow 3a + 2b + 1 = 0$

Also, (1, 0) satisfy equation of curve, therefore we have $a + b + c = 0 \Rightarrow a + b + 1 = 0$

Hence, $a = 1, b = -2, c = 1$.

37. We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-b^2 x}{a^2 y}$

$\therefore \tan \theta_1 = \left[\frac{dy}{dx} \right]_{(a,0)} = \infty$

$\Rightarrow \theta_1 = \frac{\pi}{2}$ and $\tan \theta_2 = \left[\frac{dy}{dx} \right]_{(0,b)} = 0$

$\Rightarrow \theta_2 = 0$

Hence, the angle between the two tangents is

$\theta = \theta_1 - \theta_2 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

38. We have, $y^2 = 4x$... (i)

and $x^2 + y^2 - 6x + 1 = 0$... (ii)

On differentiating Eq. (i) w.r.t. x , we get

$2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$
 $\left[\frac{dy}{dx} \right]_{(1,2)} = 2/2 = 1 = m_1$ [say]

On differentiating Eq. (ii), w.r.t. x , we get

$2x + 2y \frac{dy}{dx} - 6 = 0 \Rightarrow \frac{dy}{dx} = \frac{3-x}{y}$

$\therefore \left[\frac{dy}{dx} \right]_{(1,2)} = \frac{3-1}{2} = 1 = m_2$ [say]

Since, $m_1 = m_2$, therefore the two curves touch each other at (1, 2).

39. We have, $x = a(\cos \theta + \theta \sin \theta)$

and $y = a(\sin \theta - \theta \cos \theta)$

$\Rightarrow \frac{dx}{d\theta} = a(-\sin \theta + \theta \cos \theta + \sin \theta) = a\theta \cos \theta$

and $\frac{dy}{d\theta} = a(\cos \theta + \theta \sin \theta - \cos \theta) = a\theta \sin \theta$

$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\theta \sin \theta}{a\theta \cos \theta} = \tan \theta$

Hence, the equation of the normal at any point θ on the curve is given by

$[y - a(\sin \theta - \theta \cos \theta)] = -\frac{\cos \theta}{\sin \theta} [x - a(\cos \theta + \theta \sin \theta)]$

$\Rightarrow x \cos \theta + y \sin \theta = a(\cos^2 \theta + \sin^2 \theta) = a$

Therefore, the length of the perpendicular from the origin to the normal

$= \frac{|0 \cdot \cos \theta + 0 \cdot \sin \theta - a|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = a$

40. Let $P(x_1, y_1)$ be a point on the curve

$\left(\frac{x}{a} \right)^{2/3} + \left(\frac{y}{b} \right)^{2/3} = 1$... (i)

On differentiating w.r.t. x , we get

$\frac{2}{3} \left(\frac{x}{a} \right)^{\frac{2}{3}-1} \cdot \frac{1}{a} + \frac{2}{3} \left(\frac{y}{b} \right)^{\frac{2}{3}-1} \cdot \frac{1}{b} \frac{dy}{dx} = 0$

$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = -\left(\frac{b}{a} \right)^{2/3} \left(\frac{y_1}{x_1} \right)^{1/3}$

The equation of the tangent at (x_1, y_1) is

$y - y_1 = -\left(\frac{b}{a} \right)^{2/3} \left(\frac{y_1}{x_1} \right)^{1/3} (x - x_1)$

$\Rightarrow \frac{x}{x_1^{1/3} a^{2/3}} + \frac{y}{y_1^{1/3} b^{2/3}} = \left(\frac{x_1}{a} \right)^{2/3} + \left(\frac{y_1}{b} \right)^{2/3} = 1$

[from Eq. (i)]

Hence, $p = x_1^{1/3} a^{2/3}$ and $q = y_1^{1/3} b^{2/3}$

$\therefore \frac{p^2}{a^2} + \frac{q^2}{b^2} = \frac{x_1^{2/3} a^{4/3}}{a^2} + \frac{y_1^{2/3} b^{4/3}}{b^2}$

$= \left(\frac{x_1}{a} \right)^{2/3} + \left(\frac{y_1}{b} \right)^{2/3} = 1$ [from Eq. (i)]

41. Let $P(x_1, y_1)$ be a point on the curve $xy = c^2$ (i)

On differentiating w.r.t. x , we get

$$x \frac{dy}{dx} + y \cdot 1 = 0$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = -\frac{y_1}{x_1}$$

Equation of tangent at (x_1, y_1) is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow xy_1 + yx_1 = 2x_1y_1$$

$$\Rightarrow \frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

Therefore, the tangent at $P(x_1, y_1)$ meets X -axis at $A(2x_1, 0)$ and Y -axis at $B(0, 2y_1)$. Clearly, $P(x_1, y_1)$ is the mid-point of AB .

42. Let $P(x_1, y_1)$ be a point on the curves

$$x = \sqrt{a^2 - y^2} + \frac{a}{2} \log \frac{a - \sqrt{a^2 - y^2}}{a + \sqrt{a^2 - y^2}} \quad \dots (i)$$

On differentiating w.r.t. x , we get

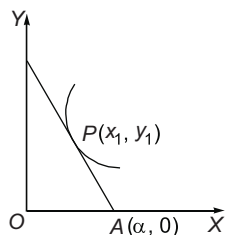
$$\frac{dx}{dy} = \frac{\sqrt{a^2 - y^2}}{y}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = \frac{y_1}{\sqrt{a^2 - y_1^2}}$$

Equation of the tangent at (x_1, y_1) is

$$y - y_1 = \frac{y_1}{\sqrt{a^2 - y_1^2}}(x - x_1)$$

which passes through point A .



$$\therefore \alpha = x_1 - \sqrt{a^2 - y_1^2}$$

$$\therefore AP^2 = (x_1 - \alpha)^2 + (y_1 - 0)^2$$

$$= a^2 - y_1^2 + y_1^2 = a^2$$

43. We have, $y = e^{4x} + 2$

On putting $x = 0$, we get $y = 3$. So, the given point is $(0, 3)$.

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = 4e^{4x} \Rightarrow \left[\frac{dy}{dx} \right]_{(0, 3)} = 4$$

The equation of the tangent at $(0, 3)$ is

$$(y - 3) = 4(x - 0)$$

$$\Rightarrow 4x - y + 3 = 0$$

$\therefore$ Length of the perpendicular r from origin to the tangent

$$= \frac{|4(0) - 0 + 3|}{\sqrt{16 + 1}}$$

$$= \frac{3}{\sqrt{17}} \text{ unit}$$

44. Let $P(x_1, y_1)$ be a point on the curve $2xy = a^2$ (i)

On differentiating Eq. (i) w.r.t. x , we get

$$2x \frac{dy}{dx} + 2y \cdot 1 = 0$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = -\frac{y_1}{x_1}$$

Equation of the tangent at $P(x_1, y_1)$ is

$$y - y_1 = -\frac{y_1}{x_1}(x - x_1)$$

$$\Rightarrow xy_1 + yx_1 = 2x_1y_1$$

$$\Rightarrow \frac{x}{2x_1} + \frac{y}{2y_1} = 1$$

Hence, intercepts on X -axis and Y -axis are $2x_1$ and $2y_1$ respectively.

$\therefore$ The area of the required triangle

$$= \frac{1}{2}(2x_1)(2y_1) = 2(x_1y_1) = a^2 \text{ sq unit}$$

45. We have, $x^2 = 4y$... (i)

$$\text{and } y^2 = 4x \quad \dots (ii)$$

On solving the Eqs. (i) and (ii), we get $x = 0$ and $x = 4$. Thus, the two parabolas intersect at the points $(0, 0)$ and $(4, 4)$.

Now, differentiating Eq. (i) w.r.t. x , we get

$$2x = 4 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(0, 0)} = 0 \quad \text{and} \quad \left[\frac{dy}{dx} \right]_{(4, 4)} = 2$$

Thus, the two tangents to the parabola $x^2 = 4y$ are

$$(y - 0) = 0(x - 0) \quad \text{and} \quad (y - 4) = 2(x - 4)$$

$$\Rightarrow y = 0 \quad \text{and} \quad 2x - y - 4 = 0$$

Clearly, the two tangents intersect at $(2, 0)$.

46. Let (x_1, y_1) be the point of intersection where the curves cut orthogonally.

$$\text{Now, } x^2 + y^2 = 2a^2$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = -\frac{x_1}{y_1} = m_1 \quad [\text{say}]$$

$$\text{and } 2y^2 - x^2 = a^2$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(x_1, y_1)} = \frac{x_1}{2y_1} = m_2 \quad [\text{say}]$$

Now, the product of the slopes at (x_1, y_1)

$$= m_1 \times m_2 = -\frac{x_1^2}{2y_1^2} = -1$$

$$\Rightarrow x_1^2 = 2y_1^2 \quad \dots (i)$$

Also, (x_1, y_1) lies on the curve, $x^2 + y^2 = 2a^2$

$$x_1^2 + y_1^2 = 2a^2 \quad \dots (ii)$$

From Eqs. (i) and (ii),

$$3y_1^2 = 2a^2 \Rightarrow y_1 = \pm \sqrt{\frac{2}{3}}a$$

$$\text{and hence } x_1 = \pm \sqrt{\frac{4}{3}}a$$

Thus, the points of intersection of the two curves are

$$\left(\pm \sqrt{\frac{4}{3}}a, \pm \sqrt{\frac{2}{3}}a \right).$$

47. We have,

$$y = 3^x \quad \dots (i)$$

and

$$y = 5^x$$

Clearly, the two curves intersect at the point (0, 1).

On differentiating Eq. (i) w.r.t. x, we get

$$\frac{dy}{dx} = 3^x \log 3$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(0,1)} = \log 3 = m_1 \quad [\text{say}]$$

On differentiating Eq. (ii) w.r.t. x, we get

$$\frac{dy}{dx} = 5^x \log 5$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(0,1)} = \log 5 = m_2 \quad [\text{say}]$$

∴ The angle between the two curves is given by

$$\theta = \tan^{-1} \left(\frac{m_1 - m_2}{1 + m_1 m_2} \right) = \tan^{-1} \left(\frac{\log 3 - \log 5}{1 + \log 3 \cdot \log 5} \right)$$

48. We know that, there exists atleast one x in (0, 1) for

$$\text{which } \frac{f(1) - f(0)}{g(1) - g(0)} = \frac{f'(x)}{g'(x)}$$

$$\text{or } \frac{2 - 10}{4 - 2} = \frac{f'(x)}{g'(x)} \text{ or } f'(x) + 4g'(x) = 0 \text{ for atleast one } x \text{ in } (0, 1).$$

49. We have, $f'(1) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

$$f'(2) = \tan \frac{\pi}{3} = \sqrt{3} \quad \text{and} \quad f'(3) = \tan \frac{\pi}{4} = 1$$

$$\begin{aligned} \therefore \int_1^3 f'(x) f''(x) dx + \int_2^3 f''(x) dx \\ = \left[\frac{(f'(x))^2}{2} \right]_1^3 + [f''(x)]_2^3 \\ = \frac{(f'(3))^2 - (f'(1))^2}{2} + f''(3) - f''(2) \\ = \left(\frac{1 - \frac{1}{3}}{2} \right) + (1 - \sqrt{3}) \\ = \frac{1}{3} + 1 - \sqrt{3} = \frac{4}{3} - \sqrt{3} = \frac{4 - 3\sqrt{3}}{3} \end{aligned}$$

50. We have, $xy = (c + x)^2$

$$\Rightarrow x \frac{dy}{dx} + y = 2(c + x) \Rightarrow \frac{dy}{dx} = \frac{2(c + x) - y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{2(c + x) - \frac{(c + x)^2}{x}}{x} = \frac{(x^2 - c^2)}{x^2}$$

At the point where the normal cut-off numerically equal intercepts on axes,

$$\text{Slope of normal} = - \left(\frac{dx}{dy} \right) = \pm 1$$

$$\Rightarrow \frac{-x^2}{x^2 - c^2} = \pm 1$$

$$\Rightarrow -x^2 = x^2 - c^2 \quad \text{or} \quad -x^2 = -(x^2 - c^2)$$

$$\Rightarrow x^2 = \frac{c^2}{2} \quad \text{or} \quad c^2 = 0$$

$$\Rightarrow x = \pm \frac{c}{\sqrt{2}} \quad [\because c^2 = 0 \text{ is not possible}]$$

51. We have, $g(x) = \frac{x + 2}{x - 1}$

$$\Rightarrow g'(x) = \frac{-3}{(x - 1)^2}$$

Slope of given line = -3

$$\Rightarrow \frac{-3}{(x - 1)^2} = -3$$

$$\Rightarrow x = 2$$

Also, $g(2) = 4$ and (2, 4) lie on the given line. Hence, the given line is tangent to the curve.

52. We have, $y = a \log(x^2 - a^2)$

$$\Rightarrow \frac{dy}{dx} = a \cdot \frac{2x}{x^2 - a^2}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x^2 - a^2}{2ax}$$

∴ Sum of tangent and sub-tangent

$$= y \sqrt{1 + \left(\frac{dx}{dy} \right)^2} + y \frac{dx}{dy} = y \frac{x^2 + a^2}{2ax} + y \frac{x^2 - a^2}{2ax}$$

$$= \frac{xy}{a} \propto \text{Product of the coordinates}$$

53. We have, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\Rightarrow \frac{2x}{a^2} + \frac{2y}{b^2} \frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

p = Length of the perpendicular from (0, 0) on the tangent

$$= \frac{x \frac{dy}{dx} - y}{\sqrt{1 + \left(\frac{dy}{dx} \right)^2}}$$

$$p' = \text{Length of normal} = y \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

$$\therefore p \times p' = y \left(x \cdot \frac{dy}{dx} - y \right) = y \left(-\frac{b^2 x^2}{a^2 y} - y \right)$$

$$= -b^2 = \text{constant}$$

$$\Rightarrow p' \propto \frac{1}{p}$$

54. We have, $y^2 = 4ax$

$$\Rightarrow 2y \frac{dy}{dx} = 4a$$

$$\text{i.e. } \frac{dy}{dx} = \frac{2a}{y}$$

$$\text{Sub-tangent} = \frac{y}{\frac{dy}{dx}} = \frac{y}{\frac{2a}{y}} = \frac{y^2}{2a}$$

$$\text{Sub-normal} = y \frac{dy}{dx} = y \times \frac{2a}{y} = 2a$$

Clearly, $\frac{y^2}{2a}$, y , $2a$ are in GP.

55. Let $f(x) = ax^2 + bx + c$

Then, $f(\alpha) = 0 = f(\beta)$

Also, $f(x)$ is continuous and differentiable in $[\alpha, \beta]$ as it is a polynomial function of x .

Hence, by Rolle's theorem, there exists a real number k in (α, β) such that

$$f'(k) = 0 \Rightarrow 2ak + b = 0 \Rightarrow k = -\frac{b}{2a}$$

$$\therefore \alpha < -\frac{b}{2a} < \beta$$

56. Let $f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + a_2 \frac{x^{n-1}}{n-1} + \dots + a_{n-1} \frac{x^2}{2} + a_n x$

Then, $f(x)$ is continuous and differentiable in $[0, 1]$ as it is a polynomial function of x .

Also, $f(0) = 0$

$$\text{and } f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \frac{a_{n-1}}{2} + a_n = 0 \text{ [given]}$$

Hence, by Rolle's theorem, there exists atleast one real number $c \in (0, 1)$ such that $f'(c) = 0$, i.e. c is a root of the equation $a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0$

57. Let $h(x) = \begin{vmatrix} f(a) & f(x) \\ g(a) & g(x) \end{vmatrix} = f(a)g(x) - g(a)f(x)$

$$\text{Then, } h'(x) = f(a)g'(x) - g(a)f'(x) = \begin{vmatrix} f(a) & f'(x) \\ g(a) & g'(x) \end{vmatrix}$$

Since, $f(x)$ and $g(x)$ are continuous in $[a, b]$ and differentiable in (a, b) , therefore $h(x)$ is also continuous in $[a, b]$ and differentiable in (a, b) . So, by mean value theorem, there exists atleast one real number c , $a < c < b$ for which

$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

$$\therefore h(b) - h(a) = (b - a)h'(c) \quad \dots(i)$$

$$\text{Here, } h(a) = \begin{vmatrix} f(a) & f(a) \\ g(a) & g(a) \end{vmatrix} = 0$$

$$h(b) = \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix}$$

$$\text{From Eq. (i), } \begin{vmatrix} f(a) & f(b) \\ g(a) & g(b) \end{vmatrix} = (b - a)h'(c)$$

$$= (b - a) \begin{vmatrix} f(a) & f'(c) \\ g(a) & g(b) \end{vmatrix}$$

58. We have, $f'(x) = \frac{1}{1+x^2}$, $\forall x$... (i)

$$\Rightarrow f'(x) > 0, \forall x \quad [\because 1+x^2 > 0]$$

From Eq. (i), it follows that $f(x)$ is differentiable at all x , therefore $f(x)$ is also continuous at all x .

By mean value theorem in $[0, 2]$,

$$\frac{f(2) - f(0)}{2 - 0} = f'(c) = \frac{1}{1+c^2}, \text{ where } 0 < c < 2$$

$$\Rightarrow \frac{f(2) - 0}{2} = \frac{1}{1+c^2} \text{ or } f(2) = \frac{2}{1+c^2} \quad \dots(ii)$$

$$\text{Now, } \frac{2}{1+c^2} < \frac{2}{1+0^2} \Rightarrow \frac{2}{1+c^2} < 2 \quad \dots(iii)$$

$$\text{and } \frac{2}{1+c^2} > \frac{2}{1+2^2} = \frac{2}{5} = 0.4 \quad \dots(iv)$$

From Eqs. (ii), (iii) and (iv), it follows that $0.4 < f(2) < 2$.

59. Consider, $f(x) = e^{3x} (ax^2 + bx + c)$

Now, $f(x) = 0$ has two positive real roots say α and β , then

$$f(\alpha) = f(\beta) = 0$$

So, using Rolle's theorem, we can say that $f'(x) = 0$ has atleast are real root between two roots of $f(x) = 0$.

$\Rightarrow e^{3x} (ax^2 + (b + 6a)x + c + 3b) = 0$ has atleast one positive real root.

$\Rightarrow ax^2 + (b + 6a)x + (c + 3b) = 0$ has atleast one positive real root.

60. $f(x)$ is continuous in the interval $[-1, 1]$ but $f(x)$ is not differentiable at $x = 0$. Hence, mean value theorem is not applicable. So, no c can be found.

61. By mean value theorem, there exists a real number $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2} \Rightarrow f'(c) = \frac{f(4) + 4}{2}$$

Since, $f'(x) \geq 6, \forall x \in [2, 4]$

$$\therefore f'(c) \geq 6 \Rightarrow f'(c) = \frac{f(4) + 4}{2}$$

$$\Rightarrow f(4) + 4 \geq 12 \Rightarrow f(4) \geq 8$$

62. We have, $f(x) = 2x^2 - \log|x|, x \neq 0$

Case I When $x < 0$, then

$$f(x) = 2x^2 - \log|x| = 2x^2 - \log(-x)$$

$$\therefore f'(x) = 4x - \frac{1}{(-x)} = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x}$$

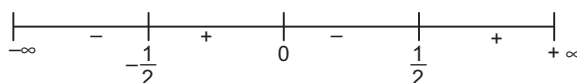
Case II When $x > 0$,

$$f(x) = 2x^2 - \log|x| = 2x^2 - \log x$$

$$\therefore f'(x) = 4x - \frac{1}{x}$$

$$\text{Thus, when } x < 0 \text{ or } x > 0, f'(x) = 4x - \frac{1}{x} = \frac{4x^2 - 1}{x}$$

$$\text{Sign scheme for } \frac{4x^2 - 1}{x}$$



Thus, $f(x)$ is an increasing function in the interval $\left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$.

63. Let $f(x) = \cos x - 1 + \frac{x^2}{2}$

$$\Rightarrow f'(x) = -\sin x + x$$

$$\text{Let } g(x) = x - \sin x$$

$$\text{Then, } g'(x) = 1 - \cos x$$

$g(x)$ is an increasing function, for $0 < x < \frac{\pi}{2}$

$$\Rightarrow g(x) > g(0), \text{ for } 0 < x < \frac{\pi}{2}$$

$$\Rightarrow f(x) > f(0), \text{ for } 0 < x < \frac{\pi}{2}$$

$$\Rightarrow \cos x - 1 + \frac{x^2}{2} > 0, \text{ for } 0 < x < \frac{\pi}{2}$$

$$\Rightarrow \cos x > 1 - \frac{x^2}{2}, \text{ for } 0 < x < \frac{\pi}{2}$$

64. We have, $f(x) = \frac{x}{\log x}$, $x > 0$, $x \neq 1$
 $\Rightarrow f'(x) = \frac{\log x - 1}{(\log x)^2}$
 $f'(x) < 0$, if $\log x - 1 < 0$, i.e. $\log x < 1$
 $\Rightarrow x < e$
 But $x > 0$ and $x \neq 1$
 $f'(x) < 0$, if $x > 0$, $x \neq 1$, $x < e$
 Thus, $f(x)$ is decreasing for $(0, e) - \{1\}$.

65. Let $f(x) = \log x - \tan^{-1} x$
 $\Rightarrow f'(x) = \frac{1}{x} - \frac{1}{1+x^2} = \frac{1+x^2-x}{x(1+x^2)}$
 $= \frac{\left(x - \frac{1}{2}\right)^2 + \frac{3}{4}}{x(1+x^2)} > 0$, $\forall x > 0$

Hence, $f(x)$ is an increasing function in the interval $(0, \infty)$.

66. We know that, $\sin x < x$, if $0 < x < \frac{\pi}{2}$... (i)

Since, $\cos x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$,

$\cos(\sin x) > \cos x$ and $0 < x < \frac{\pi}{2}$.

$\therefore 0 < \cos x < 1 < \frac{\pi}{2}$

$\Rightarrow \sin(\cos x) < \cos x$ [from Eq. (i)]

Hence, $\cos(\sin x) > \cos x > \sin(\cos x)$, if $0 < x < \frac{\pi}{2}$.

67. We have, $f(x) = \log x - \frac{2x}{2+x}$
 $\Rightarrow f'(x) = -\frac{1}{x} \left[\frac{2(2+x) - 2x}{(2+x)^2} \right] = \frac{(x+2)^2 - 4x}{x(2+x)^2}$
 $= \frac{x^2 + 4}{x(x+2)^2} = \frac{(x^2 + 4)x}{x^2(x+2)^2} > 0$, for $x > 0$

$\therefore f(x)$ is increasing for $x > 0$.

68. We have, $y = x^3(x-2)^2$
 $\Rightarrow \frac{dy}{dx} = 3x^2(x-2)^2 + 2x^3(x-2)$
 $= x^2(x-2)(5x-6)$

$\frac{dy}{dx} > 0$, if $(x-2)(5x-6) > 0$

$\Rightarrow x < \frac{6}{5}$ or $x > 2$

$\therefore y$ is increasing for $x < \frac{6}{5}$ or $x > 2$.

69. We have, $f(x) = -2x^3 + 21x^2 - 60x + 41$
 $\Rightarrow f'(x) = -6x^2 + 42x - 60$
 $= -6(x-5)(x-2)$

Clearly, $f'(x) < 0$, for $x \in (-\infty, 1)$

$\Rightarrow f(x)$ is decreasing for $x \in (-\infty, 1)$.

$\Rightarrow f(x) > f(1)$ as $x < 1$

Since, $f(1) = 0$

$\therefore f(x) > 0$, for $x \in (-\infty, 1)$

70. Let $f(x) = x \tan x$

$\Rightarrow f'(x) = \tan x \cdot 1 + \sec^2 x > 0$ for $x \in \left(0, \frac{\pi}{2}\right)$

So, $f(x)$ is increasing for $x \in \left(0, \frac{\pi}{2}\right)$.

Since, $0 < \alpha < \beta < \frac{\pi}{2}$

$\therefore f(\beta) > f(\alpha)$

$\Rightarrow \beta \tan \beta > \alpha \tan \alpha$

$\Rightarrow \frac{\tan \beta}{\tan \alpha} > \frac{\alpha}{\beta}$

71. Clearly, $f(x)$ is defined for all non-zero real values of x .

$f'(x) = -\frac{1}{x^2} < 0$, for all real $x \neq 0$

$\Rightarrow f(x)$ is decreasing function on its domain.

72. We have, $y = x^2 e^{-x}$

$\Rightarrow \frac{dy}{dx} = x^2 e^{-x}(-1) + 2xe^{-x}$
 $= e^{-x}x(2-x)$

y increasing in the interval, where

$\frac{dy}{dx} > 0$

$\Rightarrow e^{-x}x(2-x) > 0$

$\Rightarrow x(2-x) > 0$

$\Rightarrow x(x-2) < 0$

$\Rightarrow 0 < x < 2$

73. We have,

$f(x) = |x+2| + |x-1| = \begin{cases} -2x-1, & x < -2 \\ 3, & -2 \leq x \leq 1 \\ 2x+1, & x > 1 \end{cases}$

$\therefore f'(x) = \begin{cases} -2, & x < -2 \\ \text{Does not exist,} & x = -2 \\ 0, & -2 < x < 1 \\ \text{Does not exist,} & x = 1 \\ 2, & x \geq 1 \end{cases}$

Thus, $f(x)$ is increasing in $(1, \infty)$ and decreasing in $(-\infty, -2)$.

74. We have, $f''(x) < 0$, for $0 \leq x \leq 1$

$\Rightarrow f'(x)$ is a decreasing function in $[0, 1]$.

$g(x)$ is increasing, if $g'(x) > 0$

$\Rightarrow f'(x) > f'(1-x)$

$\Rightarrow x < 1-x$ [$\because f'(x)$ is decreasing]

i.e. $x < \frac{1}{2}$

$\therefore g(x)$ is increasing function for $0 < x < \frac{1}{2}$.

Also, $g(x)$ is decreasing, if $g'(x) < 0$.

$\Rightarrow f'(x) < f'(1-x)$

$\Rightarrow x > 1-x$ [$\because f'(x)$ is decreasing]

i.e. $x > \frac{1}{2}$

$\therefore g(x)$ is a decreasing function for $\frac{1}{2} < x < 1$.

$$75. f(x) = \frac{4\sin x - 2x - x \cos x}{2 + \cos x}, 0 < x < 2\pi$$

$$= \frac{4\sin x}{2 + \cos x} - x$$

$$\Rightarrow f'(x) = \frac{(2 + \cos x)4\cos x + 4\sin^2 x}{(2 + \cos x)^2} - 1$$

$$= \frac{\cos x(4 - \cos x)}{(2 + \cos x)^2}$$

Now, $f'(x) = 0$ at $\cos x = 0$

i.e. at $x = \frac{\pi}{2}, \frac{3\pi}{2}$

Also, $f'(x) > 0$, for $0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$

and $f'(x) < 0$, for $\frac{\pi}{2} < x < \frac{3\pi}{2}$

Hence, $f(x)$ is an increasing function in $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ and a decreasing function in $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

$$76. \text{ We have, } f(x) = x^3 + (a+2)x^2 + 3ax + 5$$

$$\Rightarrow f'(x) = 3x^2 + 2(a+2)x + 3a$$

$f(x)$ is monotonic in R , if $f'(x) < 0, \forall x \in R$
or $f'(x) > 0, \forall x \in R$

Since, the coefficient of $x^2 = 3 > 0$, therefore

$$f'(x) < 0, \forall x \in R$$

$$\therefore f'(x) > 0, \forall x \in R$$

Now, discriminant $= 4(a+2)^2 - 4 \cdot 3 \cdot 3a < 0$

$$\Rightarrow a^2 - 5a + 4 < 0$$

$$\Rightarrow (a-1)(a-4) < 0 \Rightarrow 1 < a < 4$$

$$77. f(x) = \frac{|x-1|}{x^2} = \begin{cases} \frac{x-1}{x^2}, & x \geq 1 \\ \frac{1-x}{x^2}, & x < 1, x \neq 0 \end{cases}$$

$$\therefore f'(x) = \begin{cases} \frac{x-2}{x^3}, & x < 1, x \neq 0 \\ \text{Does not exist,} & x = 1 \\ \frac{2-x}{x^3}, & x > 1 \end{cases}$$

Clearly, $f'(x) > 0$ for $x < 0$ or $1 < x < 2$

and $f'(x) < 0$ for $0 < x < 1$ or $x > 2$

Thus, $f(x)$ is increasing for $(-\infty, 0) \cup (1, 2)$ and decreasing for $(0, 1) \cup (2, \infty)$.

$$78. f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3, 0 \leq x \leq \pi$$

$$\Rightarrow f'(x) = -12\cos^3 x \sin x - 30\cos^2 x \sin x$$

$$= -12\cos x \sin x$$

$$= -6\sin x \cos x (\cos x + 2)(2\cos x + 1)$$

$$f'(x) = 0, \text{ for } x = 0, \frac{\pi}{2}, \frac{2\pi}{3}, \pi$$

Clearly, $f'(x) > 0$, for $\frac{\pi}{2} < x < \frac{2\pi}{3}$

and $f'(x) < 0$, for $0 < x < \frac{\pi}{2}$ or $\frac{2\pi}{3} < x < \pi$

Thus, $f(x)$ is increasing in the interval $\left(\frac{\pi}{2}, \frac{2\pi}{3}\right)$ and

decreasing in the interval $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{2\pi}{3}, \pi\right)$.

$$79. \text{ We have, } f(x) = kx^3 - 9x^2 + 9x + 3$$

$$f'(x) = 3kx^2 - 18x + 9$$

Since, $f(x)$ is increasing on R , therefore

$$\Rightarrow f'(x) > 0, \forall x \in R$$

$$\Rightarrow 3kx^2 - 18x + 9 > 0, \forall x \in R$$

$$\Rightarrow kx^2 - 6x + 3 > 0, \forall x \in R$$

$$\Rightarrow k > 0 \text{ and } 36 - 12k < 0$$

$$\left[\begin{array}{l} \because ax^2 + bx + c > 0, \forall x \in R \\ \Rightarrow a > 0 \text{ and discriminant} < 0 \end{array} \right]$$

$$\Rightarrow k > 0 \text{ and } k > 3$$

Hence, $f(x)$ is increasing on R , if $k > 3$.

$$80. \text{ We have, } f(x) = x^2 + kx + 1$$

$$\Rightarrow f'(x) = 2x + k. \text{ Also, } f''(x) = 2$$

$$\text{Now, } f''(x) = 2, \forall x \in [1, 2]$$

$$\Rightarrow f''(x) > 0, \forall x \in [1, 2]$$

$\Rightarrow f'(x)$ is an increasing function in the interval $[1, 2]$.

$\Rightarrow f'(1)$ is the least value of $f'(x)$ on $[1, 2]$.

$$\text{But } f'(x) > 0, \forall x \in [1, 2]$$

$[\because f(x) \text{ is increasing in } [1, 2]]$

$$\therefore f'(1) > 0, \forall x \in [1, 2] \Rightarrow k > -2$$

Thus, the least value of k is -2 .

$$81. \text{ Since, } f(x) \text{ is decreasing in the interval } (-2, -1), \text{ therefore}$$

$$f'(x) < 0$$

$$\Rightarrow 6x^2 + 18x + \lambda < 0$$

The value of λ should be such that the equation

$$6x^2 + 18x + \lambda = 0 \text{ has roots } -2 \text{ and } -1.$$

$$\text{Therefore, } (-2)(-1) = \frac{\lambda}{6}$$

$$\therefore \lambda = 12$$

$$82. \text{ We have, } f(x) = \sin x - \cos x - ax + b$$

$$\Rightarrow f'(x) = \cos x + \sin x - a$$

$f(x)$ is a decreasing function for all real values of x , if $f'(x) < 0, \forall x \in R$

$$\Rightarrow \cos x + \sin x < a, \forall x \in R$$

As the maximum value of $\cos x + \sin x$ is $\sqrt{2}$, the above is possible when $a \geq \sqrt{2}$.

$\therefore f(x)$ decreases for each $x \in R$, if $a \geq \sqrt{2}$.

$$83. \text{ In the interval } [-1, 2], f'(x) = 6x + 12 > 0$$

Hence, $f(x)$ is increasing in $[-1, 2]$.

Now, $f(x)$ being a polynomial in x_2 continuous in $-1 \leq x < 2$ and in $2 < x \leq 3$. We check at $x = 2$

$$\lim_{h \rightarrow 0} f(2-h) = \lim_{h \rightarrow 0} 3(2-h)^2 + 12(2-h) - 1$$

$$= 12 + 24 - 1 = 35$$

$$\lim_{h \rightarrow 0} f(2+h) = \lim_{h \rightarrow 0} 37 - (2+h) = 35$$

$$\text{Also, } f(2) = 3(2)^2 + 12(2) + 1 = 35$$

$\therefore f(x)$ is continuous at $x = 2$ and hence in the interval $[-1, 3]$.

$$\text{Now, } Lf'(2) = \lim_{h \rightarrow 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{3(2-h)^2 + 12(2-h) - 1 - 35}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{3h^2 - 24h}{-h} = 24$$

$$Rf'(2) = \lim_{h \rightarrow 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{37 - (2+h) - 35}{h} = -1$$

Since, $Lf'(2) \neq Rf'(2)$, therefore $f'(2)$ does not exist.

84. $f(x) = (x-8)^4(x-9)^5, 0 \leq x \leq 10$

$$\Rightarrow f'(x) = 4(x-8)^3(x-9)^5 + 5(x-9)^4(x-8)^4$$

$$= (x-8)^3(x-9)^4[4(x-9) + 5(x-8)]$$

$$= 9(x-8)^3(x-9)^4 \left(x - \frac{76}{9} \right)$$

Sign scheme of $f'(x)$

$$\begin{array}{ccccccc} & + & & - & & + & & - \\ & | & & | & & | & & | \\ & 8 & & 76/9 & & 9 & & \end{array}$$

$$f'(x) < 0, \text{ if } x \in \left(8, \frac{76}{9} \right) \Rightarrow f(x) \text{ decreases if } x \in \left(8, \frac{76}{9} \right)$$

85. We have, $y = x^3 + 5x^2 - 1$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + 10x = x(3x + 10)$$

The given function is decreasing, if $\frac{dy}{dx} < 0$

$$\Rightarrow x(3x + 10) < 0$$

$$\Rightarrow -\frac{10}{3} < x < 0$$

86. Let $h(x) = f(x) - g(x)$

Since, $f'(x) > g'(x), \forall x \in R$

$$h'(x) = f'(x) - g'(x) > 0, \forall x \in R$$

$\Rightarrow h(x)$ is an increasing function, $\forall x \in R$.

Also, $h(0) = f(0) - g(0) = 0$

So, for $x > 0$, we must have $h(x) > h(0) = 0$ and for $x < 0$, we have $h(x) < h(0) = 0$

$$\Rightarrow f(x) > g(x), \forall x \in (0, \infty)$$

$$\Rightarrow f(x) < g(x), \forall x \in (-\infty, 0)$$

87. Suppose, there are two points x_1 and x_2 in (a, b) such that $f'(x_1) = f'(x_2) = 0$. If Rolle's theorem applied to $f'(x)$ on $[x_1, x_2]$, there must be $\alpha \in (x_1, x_2)$ such that $f''(\alpha) = 0$. This contradicts the given condition $f''(x) < 0, \forall x \in (a, b)$. Hence, our assumption is wrong. Therefore, there can be atmost one point in (a, b) at which $f'(x)$ is zero.

88. $y = x(c - x)$... (i)

$$y = x^2 + ax + b$$
 ... (ii)

Slope of Eq. (i), $m_1 = c - 2x$

At $(1, 0)$,

$$c - 2 = m_1$$
 [say]

Slope of Eq. (ii), $m_2 = 2x + a$

At $(1, 0)$, $2 + a = m_2$ [say]

Since, curves touches at $(1, 0)$.

$$\therefore m_1 = m_2$$
 ... (iii)
$$\Rightarrow 2 + a = c - 2$$

Also, $(1, 0)$ lies on both the curves

$$\Rightarrow 0 = c - 1 \text{ and } 0 = 1 + a + b$$
 ... (iv)

Solving Eqs. (iii) and (iv), we get

$$a = -3, b = 2 \text{ and } c = 1$$

89. $f(x) = \frac{x}{1-x^2}$

$$\therefore f'(x) = \frac{1+x^2}{(1-x^2)^2} = 1, \text{ i.e. } x = 0, -\sqrt{3}, \sqrt{3}$$

Hence, the points are $(0, 0), \left(\pm\sqrt{3}, \mp\frac{\sqrt{3}}{2} \right)$.

90. $xy = a^2 + x^2 + 2ax$

$$\Rightarrow y = \frac{a^2}{x} + x + 2a \Rightarrow \frac{dy}{dx} = -\frac{a^2}{x^2} + 1 = -1$$

$$\Rightarrow 2x^2 = a^2 \Rightarrow x = \pm \frac{a}{\sqrt{2}}$$

91. $f'(x) = \frac{1}{\log x} [x^2 - x]$ and increasing in the interval $[1, \infty)$.

$$\therefore \log |x| > 0$$

$$\therefore f'(x) > 0, \text{ hence } f(x) \text{ is increasing.}$$

92. Here, $h(x) = f(x) - \{f(x)\}^2 + \{f(x)\}^3$

$$\Rightarrow h'(x) = f'(x) - 2f(x)f'(x) + 3\{f(x)\}^2 f'(x)$$

$$= f'(x)[1 - 2f(x) + 3\{f(x)\}^2]$$

$$f'(x)(3y^2 - 2y + 1), \text{ where } y = f(x)$$

The discriminant of $3y^2 - 2y + 1 = 4 - 12 = -8 < 0$, thus its sign is same as the coefficient of y^2 , i.e. > 0 .

$\therefore h'(x) = f'(x)$ (a positive quantity)

$\Rightarrow$ Sign of $h'(x)$ is the same as that of $f'(x)$.

Either $h'(x) > 0$ and $f'(x) > 0$ or $h'(x) < 0$ and $f'(x) < 0$

Hence, $h(x)$ and $f(x)$ increase and decrease together.

93. (a) f is not differentiable at $x = \frac{1}{2}$.

(b) g is not continuous in $[0, 1]$ at $x = 0$.

(c) h is not continuous in $[0, 1]$ at $x = 1$.

(d) $k(x) = |x|$ is continuous and differentiable in $[0, 1]$ and $(0, 1)$ respectively.

94. $f'(x) = 100x^{99} + \cos x$

For $x \in (0, 1)$ and $\left(0, \frac{\pi}{2} \right)$,

$\cos x$ and x are both positive $\Rightarrow$ increasing

For $x \in \left(\frac{\pi}{2}, \pi \right), \frac{\pi}{2} > 1$

Hence, $100x^{99} > \cos x \Rightarrow$ increasing

95. $\frac{f(x) - f(0)}{x - 0} = f'(c) \leq 1 - x$ for some $c \in (0, 1)$

$$\Rightarrow f(x) \leq x(1 - x) \leq \frac{1}{4}$$

96. $f(x) = \frac{\ln x}{x}$... (i)

Clearly, domain is R^+

$$\text{and } f'(x) = \frac{x \cdot \frac{1}{x} - \ln x}{x^2} = \frac{1 - \ln x}{x^2}$$

(a) For horizontal tangent, $f'(x) = 0$

$$\Rightarrow \ln x = 1 \Rightarrow x = e$$

(b) If Eq. (i) cuts the X-axis

$$\Rightarrow \frac{\ln x}{x} = 0 \Rightarrow x = 1$$

- (c) $f'(x)$ is positive, if $x \in (0, e)$ and $f'(x)$ negative, if $x \in (e, \infty)$
 $\therefore f(x)$ is not monotonic
Hence, $f(x)$ is many-one

- (d) For vertical tangent $f'(x) = \infty$
 $\Rightarrow \frac{1 - \ln x}{x^2} = \infty \Rightarrow \frac{x^2}{1 - \ln x} = 0$
 $\Rightarrow x = 0$, which is not in the domain of $f(x)$.

97. $h(x) = g(f(x))$ and $f(x) \in [0, \infty)$
 $\therefore h(x) \geq 0$... (i)
 $h(0) = 0$... (ii)
and $h'(x) = g'(f(x))f'(x) \leq 0$... (iii)
From Eqs. (i), (ii) and (iii), $h(x)$ is a constant function.

98. If $b < 0$, then $f(0) = b < 0$, $f(1) = b < 0$
 $\therefore 0, 1$ lie between the roots.
Hence, Statement I is false

99. Let $g(x)$ be the inverse function of $f(x)$. Then, $f(g(x)) = x$
 $f'(g(x)) \cdot g'(x) = 1$, i.e. $g'(x) = \frac{1}{f'(g(x))}$

$$\therefore g''(x) = -\frac{1}{(f'(g(x)))^2} (f''(g(x)) \cdot g'(x))$$

Statement I $f''(g(x)) > 0$ and $g'(x) < 0$
 $\Rightarrow g''(x) < 0$

$\Rightarrow$ Concavity of $f^{-1}(x)$ is upwards.

$\therefore$ Statement I is false.

Statement II $f''(g(x)) > 0$ and $g'(x) < 0$

$\Rightarrow g''(x) < 0$

$\Rightarrow$ Concavity of $f^{-1}(x)$ is upwards.

$\therefore$ Statement II is true.

100. Suppose $f(x) = 0$ has a real root say $x = a$, then $f(x) < 0$ for all $x < a$. Thus, $|f(x)|$ becomes strictly decreasing on $(-\infty, a)$ which is a contradiction.

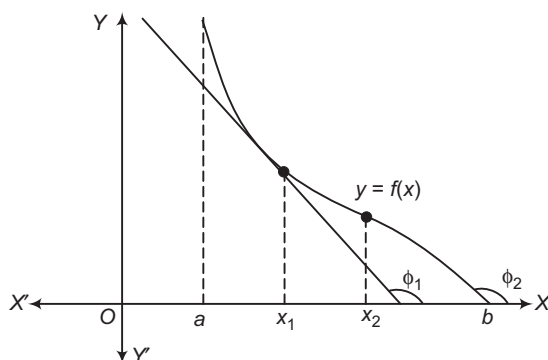
101. Statement II is true as $f'(a)^+ \neq f'(a)^-, \forall a \in I$

Statement I is true and is obtained from differentiable rule.

102. **Statement I** We know that, $\sin x$ and $\cos x$ decrease in $\frac{\pi}{2} < x < \pi$.

So, Statement I is true.

Statement II Consider the following graph



It is clear from the graph that, $f(x)$ is decreasing and differentiable function in (a, b) .

Also, $a < x_1 < x_2 < b$

But $f'(x_1) = \tan \phi_1 < \tan \phi_2 = f'(x_2)$

Thus, Statement II is false.

103. Clearly, $|x - 1|$ is not differentiable at $x = 1$, but $(x - 1)|x - 1|$ is differentiable at $x = 1$, for which Lagrange's mean value theorem is applicable.

104. Statement I and Statement II are correct.

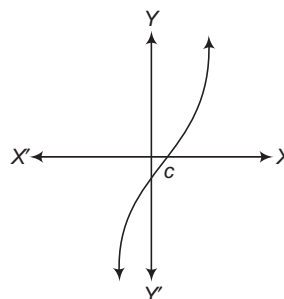
Solutions (Q. Nos. 105-109)

Case I If $D < 0 \Rightarrow f'(x) > 0, \forall x \in R$

That means $f(x)$ would be an increasing function of x .

Also, $\lim_{x \rightarrow -\infty} f(x) = -\infty$ and $\lim_{x \rightarrow \infty} f(x) = \infty$

Thus, the graph of $f(x)$ would be as follows:



It is clear that, the graph of $f(x)$ would cut the X-axis only at once.

That means we would have just one real root, (say x_0).

Case II Clearly, $x_0 > 0$, if $c < 0$, if $x_0 < 0$, $c > 0$.

If $D > 0$, $f'(x) = 0$ would have two real roots (say x_1 and x_2 , let $x_1 < x_2$).

$$\Rightarrow f'(x) = 3(x - x_1)(x - x_2)$$

Clearly, $f'(x) < 0, x \in (x_1, x_2)$

$$f'(x) > 0, x \in (-\infty, x_1) \cup (x_2, \infty)$$

That means $f(x)$ would increase in $(-\infty, x_1)$ and (x_2, ∞) and would decrease in (x_1, x_2) . Hence, $x = x_1$ would be a point of local maxima and $x = x_2$ would be a point of local minima. Thus, the graph of $y = f(x)$ could have these five possibilities.

Thus, the following results are obtained from above graphs.

- (a) $f(x_1) \cdot f(x_2) > 0$, $f(x) = 0$ would have just one real root.
(b) $f(x_1) \cdot f(x_2) < 0$, $f(x) = 0$ would have three real and distinct roots.
(c) $f(x_1) \cdot f(x_2) = 0$, $f(x) = 0$ would have three real roots about one of the root would be repeated.

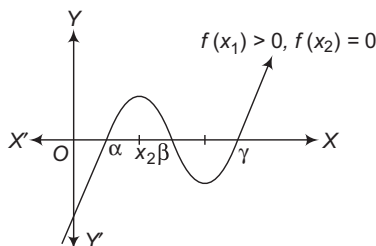
Case III If $D = 0$, $f'(x) = 3(x - x_1)^2$, where x_1 is root of $f'(x) = 0$.

$$\Rightarrow f(x) = (x - x_1)^3 + k$$

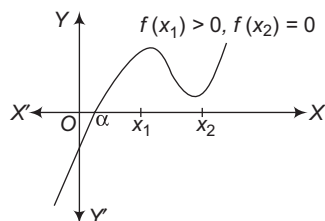
$\therefore f(x) = 0$ has three real roots, if $k = 0$

$f(x) = 0$ have one real root, if $k \neq 0$

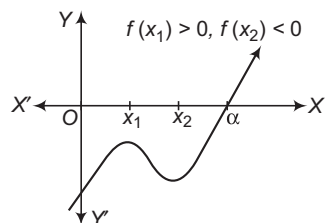
105. Here, $f(x) = 0$ with three distinct roots $x = \alpha, \beta, \gamma$.



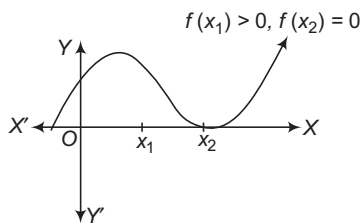
106. Here, $f(x) = 0$ with one real root $x = \alpha$ and other two imaginary roots.



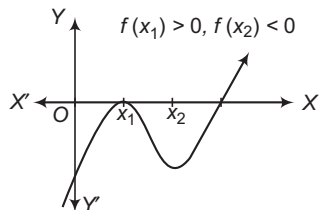
107. Here, $f(x) = 0$ with one real root $x = \alpha$ and other two imaginary roots.



108. Here, $f(x) = 0$ with three roots $x = \alpha, x_2$ (x_2 being repeated root).



109. Here, $f(x) = 0$ with three real roots $x = x_1, x_2, \alpha$ (x_1 being repeated root).



110. $f(x) = y = \sin x$

$$f'(x) = \frac{dy}{dx} = \cos x$$

$$\therefore \cos x_1 = \frac{-1}{\cos x_2} = \frac{\sin x_2 - \sin x_1}{x_2 - x_1}$$

$$\text{i.e. } \cos x_1 \cos x_2 = -1$$

$$\Rightarrow \sin x_1 = \sin x_2 = 0$$

$\therefore$ There is no solution.

111. $f(x) = y = |\ln x|$

$$\therefore f'(x) = -\frac{1}{f'(x_2)} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\Rightarrow \frac{\ln x_1}{x_1 |\ln x_1|} = -\frac{x_2 |\ln x_2|}{\ln x_2} = \frac{|\ln x_2| - |\ln x_1|}{x_2 - x_1} \quad \dots (i)$$

$$\Rightarrow \ln x_1 \cdot \ln x_2 < 0$$

Let $0 < x_1 < 1$, then $1 < x_2$ and $x_1 \cdot x_2 = 1$

From Eq. (i), we get

$$-\frac{1}{x_1} = -x_2 = \frac{\ln x_2 + \ln x_1}{x_2 - x_1} = \frac{\ln x_1 x_2}{x_2 - x_1} = 0$$

which is not possible.

$\therefore$ There is no solution.

112. $y^2 = x^3 \therefore y = \sqrt{x^3}$ or $-\sqrt{x^3}$

$$2y \frac{dy}{dx} = 3x^2$$

$$\frac{3x_1^2}{2y_1} = -\frac{2y_2}{3x_2^2} \Rightarrow \frac{9}{4} x_1^2 x_2^2 = -y_1 y_2$$

$$\therefore y_1 y_2 < 0$$

$$\text{Let } y_1 = \sqrt{x_1^3} \text{ and } y_2 = -\sqrt{x_2^3}$$

$$\text{Thus, } \frac{3x_1^2}{2\sqrt{x_1^3}} = \frac{2\sqrt{x_2^3}}{3x_2^2} = -\frac{-\sqrt{x_2^3}}{x} \cdot \frac{-\sqrt{x_1^2}}{x_1}$$

$$\frac{3\sqrt{x_1}}{2} = \frac{2}{3\sqrt{x_2}} = \frac{\sqrt{x_2^3} + \sqrt{x_1^2}}{x_1 - x_2}$$

$$\sqrt{x_1 x_2} = \frac{4}{9}$$

$$3x_1 \sqrt{x_1} - 3\sqrt{x_1} x_2 = 2\sqrt{x_2^3} + 2\sqrt{x_1^3}$$

$$3(\sqrt{x_1})^3 - \frac{3 \times 16}{31\sqrt{x_1}} = 2 \cdot \frac{64}{729\sqrt{x_1^3}} + 2\sqrt{x_1^3}$$

$$3x_1^3 - \frac{16}{27} x_1 = \frac{128}{729} + 2x_1^2$$

$$x_1^3 - \frac{16}{27} x_1 = \frac{128}{729}$$

$$739x_1^3 - 432x_1 - 128 = 0$$

$$\text{Consider } h(t) = 729x^3 - 432t - 128$$

$$h'(t) = 3 \times 729t^2 - 432 = 0 \text{ gives } t = \pm \frac{4}{9}, h\left(-\frac{4}{9}\right) = 0$$

$\therefore$ There are two distinct solutions of

$$729x_1^3 - 432x_1 - 128 = 0$$

113. Let $0 < \alpha < \beta < 1$ and α, β are the roots of

$$f'(x) = x^3 - 3x + k = 0 \Rightarrow f(\alpha) = f(\beta) = 0$$

$\Rightarrow f(x)$ satisfied RMVT.

$$\Rightarrow f'(c) = 0 \Rightarrow 3c^2 = 3 \Rightarrow c = \pm 1$$

But c must lie between α and β . Hence, $k \in \phi$.

114. Let $f(x) = \tan^{-1} x$, then for some $\alpha \in (x, y)$.

$$f'(\alpha) = \frac{\tan^{-1} y - \tan^{-1} x}{y - x} \quad (\text{LMVT})$$

$$\Rightarrow \left| \frac{1}{1 + \alpha^2} \right| = \left| \frac{\tan^{-1} x - \tan^{-1} y}{x - y} \right|$$

$$\Rightarrow |\tan^{-1} x - \tan^{-1} y| \leq |x - y|$$

$$\therefore \left| \frac{1}{1 + \alpha^2} \right| \leq 1$$

115. Let $f(x) = \sin x$ and $g(x) = \cos x$. Also, $\sin x \neq 0$ for $x \in \left(0, \frac{\pi}{2}\right)$, then by Cauchy's theorem,

$$\frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} = \frac{f'(\theta)}{g'(\theta)}$$

$$\Rightarrow \frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} = \frac{\cos \theta}{-\sin \theta} = -\cot \theta$$

116. A. $r = 5$ cm, $\delta r = 0.06$

$$A = \pi r^2$$

$$\delta A = 2\pi r \delta r = 10\pi \times 0.06$$

$$= 0.6\pi \text{ sq units}$$

- B. $V = x^3$, $\delta V = 3x^2 \delta x$

$$\frac{\delta V}{V} \times 100 = 3 \frac{\delta x}{x} \times 100$$

$$= 3 \times 1 = 3 \text{ cubic units}$$

- C. $(x-2) \frac{dx}{dt} = 2 \frac{dx}{dt}$

$$x = 4$$

- D. $A = \frac{\sqrt{3}}{4} x^2$

$$\frac{dA}{dt} = \frac{\sqrt{3}}{2} x \frac{dx}{dt} = \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}$$

117. $f'(x) = 2^x(2^{x+1} - 3) \log 2$.

Since, $2^x > 0$ and $\log 2 > 0$, so $y'(x) > 0$, if $2^{x+1} - 3 > 0$, i.e. $x + 1 > \frac{\log 3}{\log 2} = \log_2 3$

Thus, $f'(x) > 0$ for $x > \log_2 \frac{3}{2}$

Hence, f increases on $\left(\log_2 \frac{3}{2}, \infty\right)$ and decreases on

$$\left(-\infty, \log_2 \frac{3}{2}\right) \cap (-\infty, -\pi).$$

The period of g is 2π , so it is enough to consider g on $[0, 2\pi]$.

$$g'(x) = 2 \cos x - 2 \sin 2x = 2 \cos x(1 - 2 \sin x)$$

$$> 0 \text{ if } x \in \left(0, \frac{\pi}{6}\right) \cup \left(\frac{\pi}{2}, \frac{5\pi}{6}\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$$

$$< 0 \text{ if } x \in \left(\frac{\pi}{6}, \frac{\pi}{2}\right) \cup \left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$$

118. A. Let $f(x) = x^2 + 2 \cos x$, so $f'(x) = 2(x - \sin x)$, since $x \geq \sin x$ on $(0, \pi)$, so f is increasing.

- B. Given, $f(x) = 8 \log x + x + \frac{16}{x}$,

$$\Rightarrow f'(x) = \frac{8}{x} + 1 - \frac{16}{x^2} = \frac{x^2 + 8x - 16}{x^2}$$

$$= \frac{(x+4+4\sqrt{2})(x+4-4\sqrt{2})}{x^2}$$

Hence, f neither increases nor decreases.

- C. If $f(x) = \log(\sqrt{1+x^2} - x)$, then $f'(x) = -\frac{1}{\sqrt{1+x^2}} < 0$

for all x .

- D. $f'(x) = \frac{x^2}{1+x} > 0$ for $x > 0$

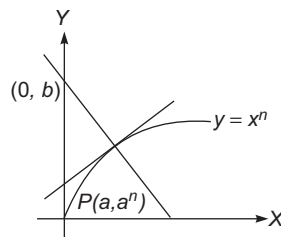
119. $f'(c) = 16c - 7 = \frac{f(6) - f(-6)}{12}$

$$= \frac{\{8 \cdot (36) - 7 \cdot (6) + 5\} - \{8 \cdot (36) + 7 \cdot (6) + 5\}}{12}$$

$$= -\frac{2 \cdot 7 \cdot 6}{12} = -7 \Rightarrow 16c = 0 \Rightarrow c = 0$$

120. $y = x^n \Rightarrow \frac{dy}{dx} = nx^{n-1}$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(a, a^n)} = na^{n-1}$$



$$\text{Slope of normal} = -\frac{1}{na^{n-1}}$$

$\therefore$ Equation of normal is

$$y - a^n = -\frac{1}{na^{n-1}}(x - a)$$

Put $x = 0$ to get y-intercept

$$\Rightarrow y = a^n + \frac{1}{na^{n-2}}$$

$$\text{Hence, } b = a^n + \frac{1}{na^{n-2}}, \lim_{a \rightarrow 0} b = \begin{cases} 0, & \text{if } n < 2 \\ \frac{1}{2}, & \text{if } n = 2 \\ \infty, & \text{if } n > 2 \end{cases}$$

121. Using LMVT for f in $[1, 2]$, we have for $c \in (1, 2)$,

$$\frac{f(2) - f(1)}{2 - 1} = f'(c) \leq 2$$

$$f(2) - f(1) \leq 2 \Rightarrow f(2) \leq 4 \quad \dots(i)$$

Again, using LMVT in $[2, 4]$, we have for $d \in (2, 4)$,

$$\frac{f(4) - f(2)}{4 - 2} = f'(d) \leq 2$$

$$\therefore f(4) - f(2) \leq 4 \Rightarrow 8 - f(2) \leq 4$$

$$\Rightarrow 4 \leq f(2) \Rightarrow f(2) \geq 4 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get $f(2) = 4$

122. $f(x) = 2f\left(\frac{x^2}{2}\right) + f(6 - x^2)$

$$\Rightarrow f'(x) = 2f'\left(\frac{x^2}{2}\right)x - 2xf'(6 - x^2)$$

$$\Rightarrow f'(x) = 2x \left[f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) \right]$$

$$\Rightarrow f'\left(\frac{x^2}{2}\right) > f'(6 - x^2), \text{ If } \frac{x^2}{2} > 6 - x^2$$

[$\because f'(x)$ is increasing]

$$\Rightarrow \frac{x^2}{2} > 6 - x^2 \Rightarrow x^2 > 4$$

$$\Rightarrow f'\left(\frac{x^2}{2}\right) - f'(6 - x^2) > 0, \text{ when } x < -2 \text{ or } x > 2$$

$$\Rightarrow f'(x) > 0, \text{ when } x \in (-2, 0) \cup (2, \infty)$$

$$\therefore a + b + c = 0 \Rightarrow [a + b + c] = 0$$

Entrances Gallery

1. Given, $f(x) = \int_{1/x}^x \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$$f'(x) = 1 \cdot \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} - \left(\frac{-1}{x^2}\right) \frac{e^{-\left(\frac{1}{x}+x\right)}}{\frac{1}{x}}$$

$$= \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} + \frac{e^{-\left(x+\frac{1}{x}\right)}}{x} = \frac{2e^{-\left(x+\frac{1}{x}\right)}}{x}$$

As $f'(x) > 0, \forall x \in (0, \infty)$

$\therefore f(x)$ is monotonically increasing on $(0, \infty)$.

$\Rightarrow$ Option (a) is correct and, option (b) is wrong.

Now, $f(x) + f\left(\frac{1}{x}\right) = \int_{1/x}^x \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt + \int_x^{1/x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$$= 0, \forall x \in (0, \infty)$$

Let $g(x) = f(2^x) = \int_{2^{-x}}^{2^x} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt$

$$g(-x) = f(2^{-x}) = \int_{2^x}^{2^{-x}} \frac{e^{-\left(t+\frac{1}{t}\right)}}{t} dt = -g(x)$$

$\therefore f(2^x)$ is an odd function.

2. Given curve is $(y - x^5)^2 = x(1 + x^2)^2$

$$\Rightarrow 2(y - x^5) \left(\frac{dy}{dx} - 5x^4 \right) = (1 + x^2)^2 + 2x(1 + x^2) \cdot 2x$$

Put $x = 1, y = 3 \therefore \frac{dy}{dx} = 8$

3. $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$$f'(x) = 4x^3 - 12x^2 + 24x + 1$$

$$f''(x) = 12x^2 - 24x + 24 = 12(x^2 - 2x + 2)$$

$$= 12\{(x-1)^2 + 1\} > 0, \forall x$$

$\Rightarrow f'(x)$ is increasing.

Since, $f'(x)$ is cubic and increasing.

Therefore, $f'(x)$ has only one real root and two imaginary roots i.e. atmost two real roots.

Now, $f(-1) = 15, f(0) = -1, f(1) = 9$

$\therefore f(x)$ must have one root in $(-1, 0)$ and other in $(0, 1)$.

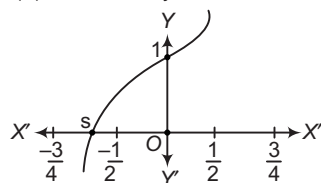
$\Rightarrow$ Two real roots.

4. Given, $f(x) = 4x^3 + 3x^2 + 2x + 1$

Now, $f'(x) = 12x^2 + 6x + 2$

Here, $D = 9 - 24 < 0$

Hence, $f(x) = 0$ has only one real root.



Here, $f\left(-\frac{1}{2}\right) > 0$ and $f\left(-\frac{3}{4}\right) < 0$

$\therefore f(x)$ changes its sign in $\left(-\frac{3}{4}, -\frac{1}{2}\right)$.

5. We have, $e^{-x}f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt, x \in (-1, 1)$

On differentiating w.r.t. x , we get

$$e^{-x}(f'(x) - f(x)) = \sqrt{x^4 + 1}$$

$$\Rightarrow f'(x) = f(x) + e^x \sqrt{x^4 + 1}$$

$\therefore f^{-1}$ is the inverse of f .

$$\therefore (f^{-1})(f(x)) = x$$

$$\Rightarrow f^{-1}(f(x))f'(x) = 1$$

$$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f'(x)}$$

$$\Rightarrow (f^{-1})'(f(x)) = \frac{1}{f(x) + e^x \sqrt{x^4 + 1}}$$

At $x = 0, f(x) = 2$

$$\Rightarrow (f^{-1})'(2) = \frac{1}{2 + 1} = \frac{1}{3}$$

6. Given equation of curve is

$$x^2 + 2xy - 3y^2 = 0 \quad \dots(i)$$

On differentiating w.r.t. x , we get

$$2x + 2xy' + 2y - 6yy' = 0$$

$$\Rightarrow y' = \frac{x + y}{3y - x}$$

At $x = 1, y = 1, y' = 1$

i.e. $\left(\frac{dy}{dx}\right)_{(1,1)} = 1$

Equation of normal at $(1, 1)$ is

$$y - 1 = -\frac{1}{1}(x - 1)$$

$$\Rightarrow y - 1 = -(x - 1)$$

$$\Rightarrow x + y = 2 \quad \dots(ii)$$

On solving Eqs. (i) and (ii) simultaneously, we get

$$x^2 + 2x(2 - x) - 3(2 - x)^2 = 0$$

$$\Rightarrow x^2 + 4x - 2x^2 - 3(4 + x^2 - 4x) = 0$$

$$\Rightarrow -x^2 + 4x - 12 - 3x^2 + 12x = 0$$

$$\Rightarrow -4x^2 + 16x - 12 = 0$$

$$\Rightarrow 4x^2 - 16x + 12 = 0$$

$$\Rightarrow x^2 - 4x + 3 = 0$$

$$\Rightarrow (x - 1)(x - 3) = 0$$

$$\Rightarrow x = 1, 3$$

Now, when $x = 1$, then $y = 1$

and when $x = 3$, then $y = -1$

$$\therefore P = (1, 1) \text{ and } Q = (3, -1)$$

Hence, normal meets the curve again at $(3, -1)$ in fourth quadrant.

Aliter

Given, $x^2 + 2xy - 3y^2 = 0$

$$\Rightarrow (x - y)(x + 3y) = 0$$

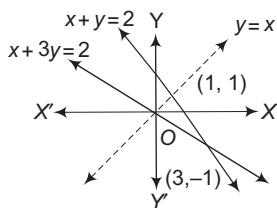
$$\Rightarrow x - y = 0$$

or $x + 3y = 0$

Equation of normal at (1, 1) is

$$y - 1 = -1(x - 1)$$

$$\Rightarrow x + y - 2 = 0$$



It intersects $x + 3y = 0$ at (3, -1) and hence meet the curve in fourth quadrant.

7. Given, $y = \int_0^x |t| dt$

$$\therefore \frac{dy}{dx} = |x| = 2$$

$$\Rightarrow x = \pm 2$$

$$\therefore \text{Points, } y = \int_0^{\pm 2} |t| dt = \pm 2$$

Equation of tangent is

$$y - 2 = 2(x - 2) \text{ or } y + 2 = 2(x + 2)$$

For x-intercept, put $y = 0$, we get

$$0 - 2 = 2(x - 2) \text{ or } 0 + 2 = 2(x + 2)$$

$$\therefore x = \pm 1$$

8. Let $f(x) = 2x^3 + 3x + k$

On differentiating w.r.t. x , we get

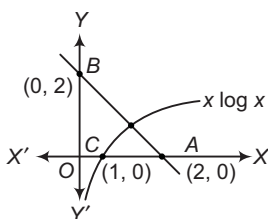
$$f'(x) = 6x^2 + 3 > 0, \forall x \in R$$

So, $f(x)$ is strictly increasing function.

Hence, $f(x) = 0$ has only one real root, so two roots are not possible.

9. Curve of $g(x) = 2 - x$

and $f(x) = x \log x$



From graph, we observe that $f(x)$ and $g(x)$ are increasing and decreasing in the interval $[1, 2]$ respectively and atleast one point in $[1, 2]$ will be exist, where both the curves intersect each other.

10. Surface area, $S = 4\pi r^2$

$$\Rightarrow \frac{dS}{dt} = 8\pi r \cdot \frac{dr}{dt}$$

$$\Rightarrow 8\pi r \cdot \frac{dr}{dt} = 8 \quad \left[\because \frac{dS}{dt} = 8\text{cm}^2/\text{s} \right]$$

$$\frac{dr}{dt} = \frac{1}{\pi r}$$

Now, $V = \frac{4\pi r^3}{3}$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt} = 4\pi r^2 \cdot \frac{1}{\pi r} = 4r$$

$$\therefore \frac{dV}{dt} \propto r$$

11. **Statement I** x^2 is an increasing function and $e^x + e^{-x}$ is also an increasing function for $x > 0$. Therefore, $x^2(e^x + e^{-x})$ is an increasing function.

Statement II Here, $x^2 e^x$ is an increasing function but $x^2 e^{-x}$ is not always increasing.

12. Slope of the curve $\frac{x^2}{\alpha} + \frac{y^2}{4} = 1$ is $m_1 = \frac{-4x}{\alpha y}$

Now, slope of the curve $y^3 = 16x$ is $m_2 = \frac{16}{3y^2}$

Apply the condition of perpendicularity of two curves,

i.e. $m_1 \cdot m_2 = -1$

and get $\alpha = \frac{4}{3}$ with the help of equation of curves.

13. $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = \frac{4}{3}\pi \cdot 3r^2 \cdot \frac{dr}{dt}$

$$\Rightarrow 35 = 4\pi(7)^2 \frac{dr}{dt} \Rightarrow \frac{dr}{dt} = \frac{5}{28\pi}$$

Surface area of balloon, $S = 4\pi r^2$

$$\therefore \frac{dS}{dt} = 8\pi r \frac{dr}{dt} = 8\pi \times 7 \times \frac{5}{28\pi} = 10\text{cm}^2/\text{min}$$

14. $\therefore y = \cos(x + y)$

$$\therefore \frac{dy}{dx} = -\sin(x + y) \left(1 + \frac{dy}{dx} \right)$$

$$\Rightarrow \frac{dy}{dx} [1 + \sin(x + y)] = -\sin(x + y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sin(x + y)}{1 + \sin(x + y)} = -\frac{1}{2}$$

$$\left[\because x + 2y = k \Rightarrow m = \frac{dy}{dx} = \frac{-1}{2} \right]$$

$$\Rightarrow \sin(x + y) = 1 \Rightarrow x + y = \frac{\pi}{2}$$

$$\therefore y = \cos\left(\frac{\pi}{2}\right) = 0 \Rightarrow x = \frac{\pi}{2}$$

$$\therefore x + 2y = k$$

$$\therefore \frac{\pi}{2} + 2(0) = k$$

$$\Rightarrow k = \frac{\pi}{2}$$

15. Since, the balloon is spherical in shape, hence the volume of the balloon is $V = \frac{4}{3}\pi r^3$.

On differentiating both sides w.r.t. t , we get

$$\frac{dV}{dt} = \frac{4}{3}\pi \left(3r^2 \times \frac{dr}{dt} \right)$$

$$\Rightarrow \frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2}$$

Now, to find $\frac{dr}{dt}$ after the time $t = 49$ min, we have the rate of change in volume at the stage is

$$\frac{dV}{dt} = -72\pi \text{ m}^3/\text{min}$$

Also, amount of volume lost in 49 min $= 72\pi \times 49 \text{ m}^3$
Final volume at the end of 49 min

$$= (4500\pi - 3528\pi) \text{ m}^3$$

$$= 972\pi \text{ m}^3$$

If r is the radius at the end of 49 min, then

$$\frac{4}{3}\pi r^3 = 972\pi$$

$$\Rightarrow r^3 = 729 \Rightarrow r = 9\text{m}$$

$\therefore$ Radius of the balloon at the end of 49 min = 9 m

From Eq. (i),

$$\frac{dr}{dt} = \frac{\frac{dV}{dt}}{4\pi r^2} \Rightarrow \left(\frac{dr}{dt}\right)_{t=49} = \frac{\left(\frac{dV}{dt}\right)_{t=49}}{4\pi(r^2)_{t=49}}$$

$$\Rightarrow \left(\frac{dr}{dt}\right)_{t=49} = \frac{-72\pi}{4\pi(9^2)} = \frac{-2}{9}\text{m/min}$$

16. We have, $y = x + \frac{4}{x^2}$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = 1 - \frac{8}{x^3}$$

Since, the tangent is parallel to X-axis, therefore

$$\frac{dy}{dx} = 0 \Rightarrow x^3 = 8$$

$$\therefore x = 2 \text{ and } y = 3$$

17. Let $f(x) = x^7 + 14x^5 + 16x^3 + 30x - 560$

$$f'(x) = 7x^6 + 70x^4 + 48x^2 + 30 > 0, \forall x \in R$$

Since, $f(x)$ is strictly increasing.

So, $f(x) = 0$ has only one solution.

18. Using mean value theorem,

$$f'(c) = \frac{f(3) - f(1)}{3 - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log_e 3 - \log_e 1}{3 - 1}$$

$$\therefore c = \frac{2}{\log_e 3} = 2\log_3 e$$

19. $f'(x) = \frac{1}{1 + (\sin x + \cos x)^2} (\cos x - \sin x)$

$$= \frac{\sqrt{2} \cos\left(x + \frac{\pi}{4}\right)}{1 + (\sin x + \cos x)^2}$$

For $f(x)$ to be increasing,

$$\sqrt{2} \cos\left(x + \frac{\pi}{4}\right) > 0$$

$$\Rightarrow -\frac{\pi}{2} < x + \frac{\pi}{4} < \frac{\pi}{2}$$

$$\therefore -\frac{3\pi}{4} < x < \frac{\pi}{4}$$

20. Since, $y = x^2 - 5x + 6$

$$\therefore \frac{dy}{dx} = 2x - 5$$

Now, $m_1 = \left(\frac{dy}{dx}\right)_{(2, 0)} = 4 - 5 = -1$

and $m_2 = \left(\frac{dy}{dx}\right)_{(3, 0)} = 6 - 5 = 1$

Since, $m_1 m_2 = -1$

$\therefore$ Angle between the tangents is $\frac{\pi}{2}$.

21. Given, $\frac{dV}{dt} = 50 \text{ cm}^3/\text{min}$

$$\therefore \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 50 \Rightarrow 3r^2 \frac{dr}{dt} = \frac{50 \times 3}{4\pi}$$

$$\Rightarrow \frac{dr}{dt} = \frac{50}{4\pi r^2} \Rightarrow \left(\frac{dr}{dt}\right)_{r=15} = \frac{50}{4\pi \times 225} = \frac{1}{18\pi} \text{ cm/min}$$

22. Let lizard catch the insect C and distance covered by insect = S time taken by insect,

$$t = \frac{S}{20} \quad \dots(i)$$

Distance covered by lizard = 21 + S

$$21 + S = \frac{1}{2}(20t)^2 \quad \dots(ii)$$

$$\Rightarrow 21 + 20t = t^2 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow t^2 - 20t - 21 = 0$$

$$\Rightarrow t^2 - 21t + t - 21 = 0$$

$$\Rightarrow t(t - 21) + 1(t - 21) = 0$$

$$\Rightarrow (t + 1)(t - 21) = 0$$

$$\Rightarrow t = 21\text{s}$$

23. Equation of parabola is $y^2 = 18x$

On differentiating w.r.t. t , we get

$$2y \frac{dy}{dt} = 18 \frac{dx}{dt}$$

$$\Rightarrow 2 \cdot 2y = 18 \quad \left[\because \frac{dy}{dt} = 2 \frac{dx}{dt}, \text{ given}\right]$$

$$\Rightarrow y = \frac{9}{2}$$

From equation of parabola,

$$\left(\frac{9}{2}\right)^2 = 18x$$

$$\Rightarrow \frac{81}{4} = 18x$$

$$\Rightarrow x = \frac{9}{8}$$

Hence, required point is $\left(\frac{9}{8}, \frac{9}{2}\right)$.

24. Since, $f''(x) = 6(x - 1)$

On integrating, we get

$$f'(x) = 3(x - 1)^2 + c \quad \dots(i)$$

Also, at the point (2, 1), the tangent to graph is

$$y = 3x - 5$$

Slope of tangent = 3 $\Rightarrow f'(2) = 3$

$$\therefore f'(2) = 3(2 - 1)^2 + c = 3 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 3 + c = 3 \Rightarrow c = 0$$

From Eq. (i),

$$f'(x) = 3(x - 1)^2$$

On integrating, we get

$$f(x) = (x - 1)^3 + k \quad \dots(ii)$$

Since, graph passes through (2, 1).

$$\therefore 1 = (2 - 1)^2 + k \Rightarrow k = 0$$

Hence, the function is

$$f(x) = (x - 1)^3$$

25. Given that,

$$x = a(1 + \cos \theta), y = a \sin \theta$$

On differentiating w.r.t. θ , we get

$$\frac{dx}{d\theta} = a(-\sin \theta), \frac{dy}{d\theta} = a \cos \theta$$

$$\therefore \frac{dy}{dx} = \frac{-\cos \theta}{\sin \theta}$$

Equation of normal at the given points is

$$y - a \sin \theta = \frac{\sin \theta}{\cos \theta} [x - a(1 + \cos \theta)]$$

It is clear that from the given options, normal passes through the point $(a, 0)$.

26. The equation of two curves are

$$x^3 - 3xy^2 + 2 = 0 \quad \dots(i)$$

$$\text{and} \quad 3x^2y - y^3 - 2 = 0 \quad \dots(ii)$$

On differentiating Eqs. (i) and (ii) w.r.t. x , we get

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{x^2 - y^2}{2xy}$$

$$\text{and} \quad \left(\frac{dy}{dx}\right)_{c_2} = \frac{-2xy}{x^2 - y^2}$$

$$\text{Now, } \left(\frac{dy}{dx}\right)_{c_1} \times \left(\frac{dy}{dx}\right)_{c_2} = \left(\frac{x^2 - y^2}{2xy}\right) \left(\frac{-2xy}{x^2 - y^2}\right) = -1$$

Hence, the two curves cut at right angle.

27. Since, $f(x) = \cot^{-1} x + x$

On differentiating w.r.t. x , we get

$$f'(x) = -\frac{1}{1+x^2} + 1 = \frac{x^2}{1+x^2} \geq 0$$

28. We know that, the volume V of a cube of side x is given by

$$V = x^3$$

On differentiating both sides w.r.t. x , we get

$$\frac{dV}{dx} = 3x^2$$

Let the change in x be $\Delta x = k\%$ of $x = \frac{kx}{100}$

Now, the change in volume

$$\Delta V = \left(\frac{dV}{dx}\right) \Delta x = 3x^2(\Delta x) = 3x^2\left(\frac{kx}{100}\right) = \frac{3x^3 \cdot k}{100}$$

$$\therefore \text{Approximate change in volume} = \frac{3kx^3}{100} = \frac{3k}{100} x^3$$

$= 3k\%$ of original volume,

29. Consider $f(x) = x^3 - 7x^2 + 15$

$$\Rightarrow f'(x) = 3x^2 - 14x$$

Let $x = 5$

and $\Delta x = 0.001$

Therefore, $f(x + \Delta x) \approx (x^3 - 7x^2 + 15) + \Delta x(3x^2 - 14x)$

$$\Rightarrow f(5.001) = (5^3 - 7 \times 5^2 + 15) + (3 \times 5^2 - 14 \times 5)(0.001)$$

$$[\text{as } x = 5, \Delta x = 0.001]$$

$$= (125 - 175 + 15) + (75 - 70)(0.001)$$

$$= -35 + (5)(0.001)$$

$$= -35 + (5)(0.001) = -35 + 0.005 = -34.995$$

30. Given, $\frac{dr}{dt} = 5 \text{ cm/s}$

$\therefore$ Area of circular wave, $A = \pi r^2$

$$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi(8) \cdot 5 = 80\pi \text{ cm}^2/\text{s} \quad [\because r = 8 \text{ cm}]$$

31. Given, $\frac{dr}{dt} = 5 \text{ cm/min}$, radius $(r) = 5 \text{ cm}$ and height

$(h) = 3 \text{ cm}$

$$\therefore \text{Volume of cylinder, } V = \pi r^2 h \quad \dots(i)$$

On differentiating both sides w.r.t. t , we get

$$\frac{dV}{dt} = 2\pi r \frac{dr}{dt} \cdot h \quad \dots(ii)$$

Again, differentiating Eq. (i) w.r.t. t , (keeping r constant) we get

$$\frac{dV}{dt} = \pi r^2 \frac{dh}{dt} \quad \dots(iii)$$

From Eqs. (ii) and (iii),

$$2\pi r \cdot \frac{dr}{dt} \cdot h = \pi r^2 \frac{dh}{dt}$$

$$\Rightarrow 2 \frac{dr}{dt} \cdot h = r \frac{dh}{dt}$$

$$\Rightarrow 2 \cdot 5 \cdot 3 = \frac{dh}{dt}$$

$$\therefore \frac{dh}{dt} = 6 \text{ cm/min}$$

32. Given, $\Delta r = \pm \frac{0.04}{2} = \pm 0.02$

$$\text{Volume of sphere, } V = \frac{4}{3} \pi r^3$$

On differentiating w.r.t. r , we get

$$\frac{dV}{dr} = \frac{4}{3} \pi \times 3r^2 = 4\pi r^2$$

$$\therefore \Delta V = \frac{dV}{dr} \Delta r = 4\pi r^2 \Delta r$$

$$\therefore \text{Relative per cent error, } \frac{dV}{V} \times 100 = \frac{4\pi r^2 \Delta r}{\frac{4}{3} \pi r^3} \times 100$$

$$= \frac{3\Delta r}{r} \times 100 = \frac{3 \times (\pm 0.02)}{10} \times 100$$

$$= \pm \frac{6}{10} = \pm 0.6$$

33. Since, $f'(x)$ is differentiable, $\forall x \in [1, 6]$.

By Lagrange's mean value theorem,

$$f'(x) = \frac{f(6) - f(1)}{6 - 1}$$

$$\therefore f'(x) \geq 2, \forall x \in [1, 6] \quad [\text{given}]$$

$$\therefore \frac{f(6) - 2}{5} \geq 2 \quad [\because f(1) = -2]$$

$$\Rightarrow f(6) \geq 10 - 2 \Rightarrow f(6) \geq 8$$

34. It is clear that, $f(x)$ has a definite and unique value for each $x \in [1, 5]$.

Thus, for every point in the interval $[1, 5]$, the value of $f(x)$ is equal to the limit of $f(x)$.

So, $f(x)$ is continuous in the interval $[1, 5]$. Also,

$$f'(x) = \frac{-x}{\sqrt{25 - x^2}}, \text{ which clearly exists for all } x \text{ in an}$$

open interval $(1, 5)$.

Hence, there must be a value $c \in (1, 5)$, such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1} = \frac{0 - \sqrt{24}}{4} \\ = \frac{0 - 2\sqrt{6}}{4} = \frac{-\sqrt{6}}{2}$$

But $f'(c) = \frac{-c}{\sqrt{25 - c^2}}$

$$\therefore \frac{-c}{\sqrt{25 - c^2}} = -\frac{\sqrt{6}}{2}$$

$$\Rightarrow 4c^2 = 6(25 - c^2)$$

$$\Rightarrow 4c^2 = 150 - 6c^2$$

$$\Rightarrow 10c^2 = 150$$

$$\Rightarrow c^2 = 15$$

$$\Rightarrow c = \pm \sqrt{15}$$

$$\therefore c = \sqrt{15} \in (1, 5)$$

35. Here, we observe that

(i) $f(x)$ is a polynomial, so it is continuous in the interval $[0, 2]$.

(ii) $f'(x) = 3x^2 - 6x + 2$ exists for all $x \in (0, 2)$.

So, $f(x)$ is differentiable for all $x \in (0, 2)$

(iii) $f(0) = 0$, $f(2) = 2^3 - 3(2)^2 + 2(2) = 0 \Rightarrow f(0) = f(2)$

Thus, all the three conditions of Rolle's theorem are satisfied.

So, there exists $c \in (0, 2)$ such that $f'(c) = 0$.

$$\Rightarrow f'(c) = 3c^2 - 6c + 2 = 0$$

$$\Rightarrow c = 1 \pm \frac{1}{\sqrt{3}} \in (0, 2)$$

36. Let $f(x) = 3\sin x - 4\sin^3 x = \sin 3x$

Since, $\sin x$ is increasing in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

$$\therefore -\frac{\pi}{2} \leq 3x \leq \frac{\pi}{2} \Rightarrow -\frac{\pi}{6} \leq x \leq \frac{\pi}{6}$$

$$\text{Thus, the length of interval} = \left[\frac{\pi}{6} - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3}$$

37. Let $y = x^x \Rightarrow \frac{dy}{dx} = x^x(1 + \log x)$

For increasing function, $\frac{dy}{dx} > 0$

$$\Rightarrow x^x(1 + \log x) > 0 \Rightarrow 1 + \log x > 0$$

$$\Rightarrow \log_e x > \log_e \frac{1}{e} \Rightarrow x > \frac{1}{e}$$

So, the function is increasing, when $x > \frac{1}{e}$.

38. Given, $f(x) = x^3$ and $g(x) = x^3 - 4x$, $-2 \leq x \leq 2$

$f(x)$ and $g(x)$ are polynomial functions, so they are continuous in $[-2, 2]$.

$f(x)$ and $g(x)$ are polynomial functions, so they are differentiable in $(-2, 2)$.

Thus, $f(x)$ and $g(x)$ satisfy mean value theorem.

$$\text{Also, } f(-2) = -8, f(2) = 8$$

$$\therefore f(-2) \neq f(2)$$

$$\text{and } g(-2) = -8 + 8 = 0, g(2) = 8 - 8 = 0$$

$$\therefore g(-2) = g(2)$$

Thus, $f(x)$ does not satisfy the Rolle's theorem but $g(x)$ satisfy the Rolle's theorem.

39. Given, $f(x) = 2x^3 - 15x^2 + 36x + 6$

On differentiating both sides w.r.t. x , we get

$$f'(x) = 6x^2 - 30x + 36$$

For strictly decreasing, $f'(x) < 0$

$$\Rightarrow 6x^2 - 30x + 36 < 0$$

$$\Rightarrow (x - 3)(x - 2) < 0$$

$$\Rightarrow 2 < x < 3$$

$$\therefore x \in (2, 3)$$

40. We know that, in Lagrange's mean value theorem, there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\therefore f'(\theta h) = \frac{f(h) - f(0)}{h - 0}$$

$$\Rightarrow \sin(\theta h) = \frac{\cos h - \cos 0}{h - 0} \quad [\because f(x) = \cos x]$$

$$= \frac{\cos h - 1}{h} = \frac{\left(1 - \frac{h^2}{2!}\right) - 1}{h}$$

[neglecting higher power of h]

$$\Rightarrow \sin(\theta h) = \frac{h^2}{2h} \Rightarrow \sin(\theta h) = \frac{h}{2}$$

$$\Rightarrow \theta h = \sin^{-1}\left(\frac{h}{2}\right) \Rightarrow \theta = \frac{\sin^{-1} \frac{h}{2} \times \frac{1}{2}}{h \times \frac{1}{2}}$$

$$\therefore \lim_{h \rightarrow 0^+} \theta = \frac{1}{2} \lim_{h \rightarrow 0^+} \frac{\sin^{-1} \frac{h}{2}}{\frac{h}{2}} = \frac{1}{2} \times 1 = \frac{1}{2}$$

41. Given, $f'(x) \leq 5$

$$\therefore \frac{f(7) - f(2)}{7 - 2} \leq 5$$

[by Lagrange's mean value theorem]

$$\Rightarrow \frac{f(7) - 3}{7 - 2} \leq 5 \Rightarrow f(7) \leq 5(7 - 2) + 3$$

$$\therefore f(7) \leq 28$$

42. Given, $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Continuity at $x = 0$

$$\text{LHL} = \lim_{x \rightarrow 0^-} x \sin \frac{1}{x} = 0$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} x \sin \frac{1}{x} = 0 \text{ and } f(0) = 0$$

$$\therefore \text{LHL} = \text{RHL} = f(0)$$

Hence, $f(x)$ is continuous for all values of x .

Differentiability at $x = 0$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h \sin \frac{1}{h}}{h} = -\lim_{h \rightarrow 0} \sin \frac{1}{h}$$

= An oscillating value between -1 and 1 .

Thus, $f(x)$ is not differentiable at $x = 0$.

Now, $f(0) = 0$ and $f(1) = 1 \sin 1$

So, $f(0) \neq f(1)$ and Rolle's theorem is not satisfy.

Now, $f'(c) = \frac{f(1) - f(0)}{1 - 0} = \frac{1 \sin(1) - 0}{1 - 0} = \sin 1 = \sin 1$

which lies between 0 to 1.

So, there exists a real number $c \in (0, 1)$.

Hence, Lagrange's mean value theorem is satisfy on $[0, 1]$.

43. Here, $e^{-x}f(x) < 0$, $x \in \left(0, \frac{1}{4}\right)$

and $[e^{-x}f(x)] > 0$, $x \in \left(\frac{1}{4}, 1\right)$

$\Rightarrow e^{-x}f'(x) - e^{-x}f(x) < 0$, $x \in \left(0, \frac{1}{4}\right)$

$\Rightarrow f'(x) < f(x)$, $0 < x < \frac{1}{4}$

44. Given, $f(x) = \sqrt{x-2}$, $x \in [2, 6]$

We know that, by Lagrange's mean value theorem, there exists $c \in (2, 6)$, such that

$$f'(c) = \frac{f(6) - f(2)}{6 - 2}$$

$\Rightarrow \frac{1}{2\sqrt{c-2}} = \frac{f(6) - f(2)}{6 - 2}$

$\Rightarrow \frac{1}{2\sqrt{c-2}} = \frac{\sqrt{6-2} - \sqrt{2-2}}{4}$

$\Rightarrow \frac{1}{\sqrt{c-2}} = \frac{\sqrt{4-0}}{2}$

$\Rightarrow \frac{1}{\sqrt{c-2}} = 1 \Rightarrow \sqrt{c-2} = 1$

$\Rightarrow c - 2 = 1 \Rightarrow c = 3$

45. Given curve is $y = f(x) = x^2 + bx - b$

On differentiating w.r.t. x , we get $\frac{dy}{dx} = 2x + b$

The equation of the tangent at $(1, 1)$ is

$$y - 1 = \left(\frac{dy}{dx}\right)_{(1,1)} (x - 1)$$

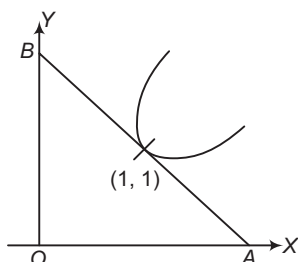
$\Rightarrow y - 1 = (b + 2)(x - 1)$

$\Rightarrow (2 + b)x - y = 1 + b$

$\Rightarrow \frac{x}{(1+b)} - \frac{y}{(1+b)} = 1$

So, $OA = \frac{1+b}{2+b}$

and $OB = -(1+b)$



Now, area of $\triangle AOB = \frac{1}{2} \left(\frac{1+b}{2+b}\right) [-(1+b)] = 2$ [given]

$\Rightarrow 4(2+b) + (1+b)^2 = 0$

$\Rightarrow 8 + 4b + 1 + b^2 + 2b = 0$

$\Rightarrow b^2 + 6b + 9 = 0$

$\Rightarrow (b+3)^2 = 0$

$\Rightarrow b = -3$

46. The given curve is $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$

$\Rightarrow n\left(\frac{x}{a}\right)^{n-1} \cdot \frac{1}{a} + n\left(\frac{y}{b}\right)^{n-1} \cdot \frac{1}{b} \frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = -\frac{b}{a} \cdot \left(\frac{x}{a}\right)^{n-1} \cdot \left(\frac{y}{b}\right)^{n-1}$

At (a, b) , $\frac{dy}{dx} = -\frac{b}{a} \cdot \frac{(1)^{n-1}}{(1)^{n-1}} = -\frac{b}{a}$

So, the tangent at (a, b) is

$\Rightarrow y - b = -\frac{b}{a}(x - a)$

$\Rightarrow ay - ab = -bx + ab$

$\Rightarrow bx + ay = 2ab$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2$

47. Given curve is $y^2 e^{xy} = 9e^{-3}x^2$

On differentiating both sides w.r.t. x , we get

$$e^{xy} \frac{d}{dx}(y^2) + y^2 \frac{d}{dx}(e^{xy}) = 9e^{-3} \frac{d}{dx}(x^2)$$

$\Rightarrow e^{xy} \left(2y \frac{dy}{dx}\right) + y^2 \left[e^{xy} \left(y + x \frac{dy}{dx}\right)\right] = 9e^{-3}(2x)$

Put $x = -1$ and $y = 3$, we get

$$e^{-1 \times 3} \left(2 \times 3 \frac{dy}{dx}\right) + 3^2 \left[e^{-1 \times 3} \left\{3 + (-1) \frac{dy}{dx}\right\}\right] = 9e^{-3}[2(-1)]$$

$\Rightarrow e^{-3} \left(6 \frac{dy}{dx}\right) + 9 \left[e^{-3} \left(3 - \frac{dy}{dx}\right)\right] = 9(-2e^{-3})$

On dividing both sides by e^{-3} , we get

$$6 \frac{dy}{dx} + 27 - 9 \frac{dy}{dx} = -18$$

$\Rightarrow -3 \frac{dy}{dx} = -45$

$\Rightarrow \frac{dy}{dx} = 15$

48. Given curve is $y = x^2 - \frac{1}{x^2}$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 2x + \frac{2}{x^3}$$

At point $(-1, 0)$, $\frac{dy}{dx} = 2(-1) + \frac{2}{(-1)^3} = -4$

$\therefore$ Slope of normal to the curve

$$= -\frac{1}{\frac{dy}{dx}} = -\frac{1}{-4} = \frac{1}{4}$$

49. Given $y = [|\sin x| + |\cos x|]$ and $x^2 + y^2 = 10$

We know that, $(|\sin x| + |\cos x|) \in [1, \sqrt{2}]$

$$\therefore y = 1$$

The point of intersection of given curve is

$$x^2 + 1^2 = 10$$

$$\Rightarrow x^2 = 9 \Rightarrow x = \pm 3$$

So, the point of intersection is $(\pm 3, 1)$.

$$\text{Now, } x^2 + y^2 = 10 \Rightarrow 2x + 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

$$\text{At point } (-3, 1), \frac{dy}{dx} = \frac{3}{1} = 3 \Rightarrow m_1 = 3$$

Hence, the slope of line $y = 1$ is $m_2 = 0$.

So, the angle between two curves is

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = 3$$

$$\Rightarrow \theta = \tan^{-1}(3)$$

50. The given curve is

$$4x^3 - 3xy^2 + 6x^2 - 5xy - 8y^2 + 9x + 14 = 0$$

On differentiating both sides w.r.t. x , we get

$$12x^2 - 3\left(y^2 + 2yx \frac{dy}{dx}\right) + 12x - 5\left(x \frac{dy}{dx} + y\right) - 16y \frac{dy}{dx} + 9 = 0$$

$$\Rightarrow \frac{dy}{dx} [-6xy - 5x - 16y] + 12x^2 - 3y^2 + 12x - 5y + 9 = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{12x^2 - 3y^2 + 12x - 5y + 9}{6xy + 5x + 16y}$$

$$\therefore \left(\frac{dy}{dx}\right)_{(-2, 3)} = \frac{48 - 27 - 24 - 15 + 9}{-36 - 10 + 48} = -\frac{9}{2}$$

Since, the tangent and normal are perpendicular to each other.

$$\therefore (\text{Slope of tangent}) \times (\text{Slope of normal}) = -1$$

$$\Rightarrow -\frac{9}{2} \times \text{Slope of normal} = -1$$

$$\Rightarrow \text{Slope of normal} = \frac{2}{9}$$

51. Since, the point of intersection is $\left(1, \frac{1}{e}\right)$.

Around $x = 1$, $|x| = x$

$$\text{So, } y = e^{-x} \Rightarrow \frac{dy}{dx} = -e^{-x}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{x=1} = -e^{-1}$$

So, the equation of the tangent at $\left(1, \frac{1}{e}\right)$ is

$$y - \frac{1}{e} = -\frac{1}{e}(x - 1)$$

$$\Rightarrow x + ey = 2$$

52. $\frac{dy}{dx} = 3x^2 - 2ax + 1$

$$\text{Also, } \frac{dy}{dx} \geq 0$$

$$\Rightarrow 3x^2 - 2ax + 1 \geq 0, \forall x$$

$$\therefore D \leq 0 \text{ or } 4a^2 - 12 \leq 0$$

$$\Rightarrow -\sqrt{3} \leq a \leq \sqrt{3}$$

53. $\lim_{x \rightarrow 0} \frac{f(x^2) - f(x)}{f(x) - f(0)} \quad \left[\left(\frac{0}{0} \right) \text{ form} \right]$

Using L' Hospital's rule,

$$\begin{aligned} &= \lim_{x \rightarrow 0} \frac{f'(x^2) \cdot 2x - f'(x)}{f'(x) - 0} \\ &= -1 + \lim_{x \rightarrow 0} \frac{2xf'(x^2)}{f'(x)} = -1, f'(0) \neq 0 \end{aligned}$$

So, f is strictly increasing.

54. Given, $x = 2 \cos(t\sqrt{3}) + 3 \sin(t\sqrt{3})$

$$\text{Now, } \frac{dx}{dt} = -2\sqrt{3} \sin(t\sqrt{3}) + 3\sqrt{3} \cos(t\sqrt{3})$$

$$\text{and } \frac{d^2x}{dt^2} = -6\cos(t\sqrt{3}) - 9\sin(t\sqrt{3})$$

$$\Rightarrow \frac{d^2x}{dt^2} = -3[2\cos(t\sqrt{3}) + 3\sin(t\sqrt{3})] = -3x$$

$$\therefore \frac{d^2x}{dt^2} \propto x$$

55. We have, $f(x) = x - \cot x$

$$\therefore f'(x) = 1 + \operatorname{cosec}^2 x$$

Evidently, $1 + \operatorname{cosec}^2 x > 0$

Thus, the function $x - \cot x$ always increases.

56. Given, $f(x)f'(x) < 0$

So, $f(x)$ and $f'(x) < 0$ must be of opposite sign.

(i) Let $f(x) = e^{-x}$

$$\therefore f'(x) = -e^{-x} \Rightarrow f(x) > 0$$

$$\text{and } f'(x) < 0, \forall x \in \mathbb{R}$$

(ii) Let $f(x) = -e^{-x}$

$$\therefore f'(x) = e^{-x} \Rightarrow f(x) < 0$$

$$\text{and } f'(x) > 0, \forall x \in \mathbb{R}$$

$$\text{But } |f(x)| = | \pm e^{-x} | = e^{-x} \text{ in both cases}$$

$$\therefore \frac{d}{dx} |f(x)| = -e^{-x} < 0 \text{ in both cases, } \forall x \in \mathbb{R}$$

Hence, $|f(x)|$ must be a decreasing function.

57. If we take $f(x) = 4x^4$, then

(i) $f(x)$ is continuous in $[-2, 2]$

(ii) $f(x)$ is differentiable in $(-2, 2)$

(iii) $f(-2) = f(2)$

So, $f(x) = 4x^4$ satisfied all the conditions of Rolle's theorem, therefore there exists a point $c \in (-2, 2)$ such that $f'(c) = 0$.

$$\Rightarrow 16c^3 = 0 \Rightarrow c = 0 \in (-2, 2)$$

58. $f(x) = \cos 2x, f'(x) = -2 \sin 2x < 0$ in $\left(0, \frac{\pi}{2}\right)$.

So, $\cos 2x$ is decreasing in $\left(0, \frac{\pi}{2}\right)$.

59. $\therefore f'(x) = 4x + 3$

$$\Rightarrow f'(c) = 4c + 3$$

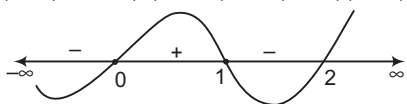
$$\text{Also, } f(1) = 2 \times 1^2 + 3 \times 1 + 4 = 9$$

$$\text{and } f(2) = 2 \times 2^2 + 3 \times 2 + 4 = 18$$

By mean value theorem,

$$f'(c) = \frac{f(2) - f(1)}{2 - 1} \Rightarrow 4c + 3 = \frac{18 - 9}{1} \Rightarrow c = \frac{3}{2}$$

60. $f'(x) = 2(x^2 - 2x) \cdot (2x - 2) = 4x(x - 2)(x - 1)$



Thus, $f(x)$ is increasing in $(0, 1) \cup (2, \infty)$.

61. Let volume of sphere, $V = \frac{4}{3}\pi r^3$

$$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \pi = 4\pi r^2 \frac{dr}{dt} \left[\because \frac{dV}{dt} = \pi \right]$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{4r^2} \quad \dots(i)$$

Now, $\frac{dS}{dt} = \frac{d}{dt}(4\pi r^2) = 4\pi \left(2r \frac{dr}{dt} \right)$

$$\therefore \left(\frac{dS}{dt} \right)_{r=1} = 4\pi \left(2 \cdot 1 \cdot \frac{1}{4} \right) = 2\pi \text{ cm}^2/\text{s} \quad [\text{from Eq. (i)}]$$

62. We have, $R(x) = 13x^2 + 26x + 15$

Marginal revenue

$$= \frac{d}{dx}[R(x)] = \frac{d}{dx}(13x^2 + 26x + 15) = 26x + 26$$

$$\therefore \text{Marginal revenue at } (x = 15) = 26 \times 15 + 26 = 416$$

63. Consider the function $y = x^{1/5}$

Let $x = 32$ and $\Delta x = 1$

Then, $\Delta y = (x + \Delta x)^{1/5} - x^{1/5}$

$$= (33)^{1/5} - (32)^{1/5} = (33)^{1/5} - 2$$

$$\Rightarrow (33)^{1/5} = 2 + \Delta y \quad \dots(i)$$

Now, dy is approximately equal to Δy and is given by

$$dy = \left(\frac{dy}{dx} \right) \Delta x = \frac{1}{5x^{4/5}} (1)$$

$$\text{At } x = 32, \quad dy = \frac{1}{5(32)^{4/5}} = \frac{1}{80} = 0.0125$$

Thus, the approximate value of $\sqrt[5]{33}$ is given by
 $2 + 0.0125 = 2.0125$

64. We have, $y = 2x^3 - 2x^2 + 3x - 5$

Now, $\frac{dy}{dx} = 6x^2 - 4x + 3$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{x=2} = 24 - 8 + 3 = 19$$

$$\therefore \Delta y = \left(\frac{dy}{dx} \right)_{x=2} \Delta x$$

$$\Rightarrow \Delta y = 19 \times (0.1) = 1.9$$

65. Given, $s = 3t^3 + 2t^2 + t + 1$

$$\Rightarrow \frac{ds}{dt} = 9t^2 + 4t + 1$$

$$\Rightarrow \frac{d^2s}{dt^2} = 18t + 4$$

At $t = 2$, acceleration, $\left(\frac{d^2s}{dt^2} \right)_{t=2} = 18 \times 2 + 4 = 40 \text{ m/s}^2$

At $t = 4$, acceleration, $\left(\frac{d^2s}{dt^2} \right)_{t=4} = 18 \times 4 + 4 = 76 \text{ m/s}^2$

$$\therefore \left(\frac{d^2s}{dt^2} \right)_{t=4} - \left(\frac{d^2s}{dt^2} \right)_{t=2} = 76 - 40 = 36 \text{ m/s}^2$$

66. Given, $s = 1 + 2t - 2t^2$

$$\therefore v = \frac{ds}{dt} = 2 - 4t$$

When the bus stops, then $v = 0 \Rightarrow 2 - 4t = 0 \Rightarrow t = \frac{1}{2}$

$$\therefore s = 1 + 2\left(\frac{1}{2}\right) - 2\left(\frac{1}{2}\right)^2 = \frac{3}{2} = 1.5 \text{ m}$$

67. Given curve is $y = x^3$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 3x^2$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{(1,1)} = 3 \Rightarrow 3x^2 = 3 \Rightarrow x^2 = 1 \Rightarrow x = \pm 1$$

$$\therefore y = \pm 1$$

Therefore, the point is $(-1, -1)$.

68. Given, $y = a \cosh\left(\frac{x}{a}\right) \Rightarrow \frac{dy}{dx} = \sinh\left(\frac{x}{a}\right)$

$$\therefore \text{Length of normal} = \cosh\left(\frac{x}{a}\right) \sqrt{1 + \sinh^2\left(\frac{x}{a}\right)}$$

$$= a \sinh\left(\frac{n}{a}\right) \cdot \sqrt{\cosh^2\left(\frac{x}{a}\right)} = a \cosh\left(\frac{x}{a}\right)$$

$$\text{Length of normal} = \frac{ay^2}{a^2} = \frac{y^2}{a}$$

$$\Rightarrow \text{Length of normal} \propto y^2$$

Length of normal varies as the square of ordinate.

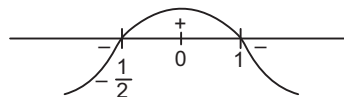
69. $f'(x) = 1 > 0$, when $x > 0$

So, $f(x)$ is strictly increasing when $x > 0$.

70. Given, $f(x) = ax + b \Rightarrow f'(x) = a$

For strictly increasing, $f'(x) > 0 \Rightarrow a > 0$

71. $f'(x) = x \cdot e^{x(1-x)}(1-2x) + e^{x(1-x)} = e^{x(1-x)}[x - 2x^2 + 1]$
 $= e^{x(1-x)}(-2x^2 + x + 1) \geq 0$ in $\left[-\frac{1}{2}, 1\right]$ [$\because e^{x(1-x)} > 0$]



So, $f(x)$ is increasing on $\left[-\frac{1}{2}, 1\right]$.

72. $f'(x) = k + \sin x > 0$

[since, $f(x)$ is monotonically increasing]

$$\text{or } k > -\sin x \text{ or } k > -1$$

73. Given, $f(x) = 2x^2 - |x| + 4$, $x \in [-1, 2]$

We can easily verify that $f(x)$ satisfies the mean value theorem in the interval $[0, 2]$, therefore

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}, c \in (0, 2)$$

74. Let l be the length of an edge and V be the volume of cube at any time t .

Then,

$$V = l^3$$

$$\therefore \frac{dV}{dt} = 3l^2 \frac{dl}{dt}$$

$$= 3 \times 5^2 \times 10$$

$$= 750 \text{ cm}^3/\text{s}$$

75. We know that, area of circle, $A = \pi r^2$
On taking log both sides, we get
 $\log A = \log \pi + 2 \log r$
 $\therefore \frac{\Delta A}{A} \times 100 = 2 \times \frac{\Delta r}{r} \times 100 = 2 \times 0.05 = 0.1\%$

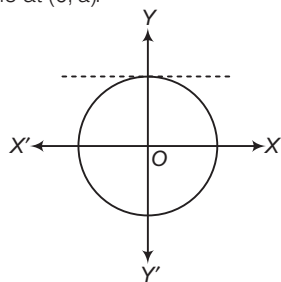
76. $\therefore \frac{dy}{dx} = f'(x)$
Slope of normal at (3, 4) = $-\frac{1}{f'(3)}$
But given slope is $\tan \frac{3\pi}{4} = -1$
 $\therefore -\frac{1}{f'(3)} = -1 \Rightarrow f'(3) = 1$

77. Given, curve is $y = x^2 - 3x + 2$
 $\Rightarrow \frac{dy}{dx} = 2x - 3$
 $\Rightarrow (2x - 3) \times 1 = -1$
 $\Rightarrow 2x - 3 = -1 \Rightarrow 2x = 2 \Rightarrow x = 1$
 $\therefore y = 0$
So, the required point is (1, 0).

78. Given curve is $x^2 + y^2 = a^2$
 $\therefore 2x + 2y \frac{dy}{dx} = 0$
 $\Rightarrow \frac{dy}{dx} = -\frac{x}{y}$
Given, $\frac{dy}{dx} = 0$
 $\Rightarrow -\frac{x}{y} = 0$
 $\Rightarrow x = 0$
 $\therefore y = \pm a$
But $y \geq 0$ [given]
 $\therefore y = a$
So, the required point is (0, a).

Aliter

It is clear from the graph, tangent is parallel to X-axis for $y \geq 0$ is at (0, a).



79. Given, $y = x^2$
 $\therefore \frac{dy}{dx} = 2x \Rightarrow m_1 = \left(\frac{dy}{dx}\right)_{(1,1)} = 2$
and $y^2 = x \Rightarrow 2y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$

$$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(1,1)} = \frac{1}{2}$$

If θ is the angle between the two curves, then

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{2 - \frac{1}{2}}{1 + 2 \times \frac{1}{2}} \right| = \frac{3}{4}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{3}{4}\right)$$

80. Given, curve is $y = e^{2x} + x^2$

At $x = 0$, $y = 1$

$$\text{Now, } \frac{dy}{dx} = 2e^{2x} + 2x$$

$$\text{Slope of normal at } (0, 1) = -\frac{1}{2+0} = -\frac{1}{2}$$

$$\therefore \text{Equation of normal is } y - 1 = -\frac{1}{2}(x - 0)$$

$$\Rightarrow 2y - 2 = -x$$

$$\Rightarrow x + 2y - 2 = 0$$

$$\therefore \text{Required distance} = \left| \frac{0+0-2}{\sqrt{1+4}} \right| = \frac{2}{\sqrt{5}} \text{ unit}$$

81. Given curve is $xy = 4 \Rightarrow y = \frac{4}{x}$

$$\therefore \frac{dy}{dx} = -\frac{4}{x^2}$$

$$\Rightarrow -\frac{4}{x^2} = -\frac{a}{b}$$

$$\Rightarrow x^2 = \frac{4b}{a} > 0$$

which is true, when $a > 0$, $b > 0$ or $a < 0$, $b < 0$.

82. Given, $f(x) = x(x-1)^2 \Rightarrow f'(x) = 2x(x-1) + (x-1)^2$
 $= (x-1)(2x+x-1)$
 $= (x-1)(3x-1)$

$$\therefore f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$\Rightarrow (c-1)(3c-1) = \frac{2-0}{2} = 1$$

$$\Rightarrow 3c^2 - 4c = 0$$

$$\Rightarrow c(3c-4) = 0$$

$$\Rightarrow c = 0 \text{ or } c = \frac{4}{3}$$

So, the value of c in $(0, 2)$ is $\frac{4}{3}$.

83. (a) $f(x) = |x|$ is not differentiable at $x = 0$.

(b) $f(x) = \tan x$ is discontinuous at $x = \frac{\pi}{2}$.

(c) $f(x) = 1 + (x-2)^{2/3}$ gives real and imaginary values at $x = 1$.

(d) Only function which satisfies Rolle's theorem, is $f(x) = x(x-2)^2$ in $0 \leq x \leq 2$

8

Maxima and Minima

Introduction

The notion of optimizing function is one of the most useful application of calculus used in almost every sphere of life including geometry, business, trade, industries, economics, medicines and even at home. In this section, we shall see how calculus defines the notion of maxima and minima and distinguishes it from the greatest and least value or global maxima and global minima of a function.

Maximum and Minimum Values of a Function

Maximum Let $f(x)$ be a function with domain $D \subset R$. Then, $f(x)$ is said to attain the maximum value at a point $a \in D$, if $f(x) \leq f(a)$ for all $x \in D$.

In such a case, a is called the point of maximum and $f(a)$ is known as the maximum value or the greatest value or the absolute maximum value of $f(x)$.

Minimum Let $f(x)$ be a function with domain $D \subset R$. Then, $f(x)$ is said to attain the minimum value at a point $a \in D$, if $f(x) \geq f(a)$ for all $x \in D$.

In such a case, the point a is called the point of minimum and $f(a)$ is known as the minimum value or the least value or the absolute minimum value of $f(x)$.

➤ **Example 1.** If m and M are respectively the minimum and maximum of $f(x) = (x-1)^2 + 3$ for $x \in [-3, 1]$, then the ordered pair (m, M) is equal to

- (a) $(-3, 19)$
- (b) $(3, 19)$
- (c) $(-19, 3)$
- (d) $(-19, -3)$

Sol. (b) We have, $f(x) = (x-1)^2 + 3$

$$\Rightarrow f'(x) = 2(x-1) \leq 0, \forall x \in [-3, 1]$$

So, $f(x)$ is decreasing in $[-3, 1]$.

$$\Rightarrow m = f(1) \quad \text{and} \quad M = f(-3)$$

$$\Rightarrow m = 3 \quad \text{and} \quad M = 19$$

$$\therefore (m, M) = (3, 19)$$

Chapter Snapshot

- Introduction
- Maximum and Minimum Values of a Function
- Critical Points
- Point of Inflection
- Local Maxima and Local Minima
- Test for Local Maximum/Minimum
- Concept of Global Maximum/Minimum
- Application of Maxima and Minima to Problems

➤ **Example 2.** Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

- (a) $[0, 1]$
(b) $(0, 1/2]$
(c) $[1/2, 1]$
(d) $(0, 1]$

Sol. (a) We have,

$$\begin{aligned} f(x) &= (1 + b^2)x^2 + 2bx + 1 \\ \Rightarrow f(x) &= (1 + b^2) \left\{ x^2 + \frac{2b}{1 + b^2}x + \frac{1}{1 + b^2} \right\} \\ \Rightarrow f(x) &= (1 + b^2) \left\{ \left(x + \frac{b}{1 + b^2} \right)^2 - \frac{b^2}{(1 + b^2)^2} + \frac{1}{(1 + b^2)} \right\} \\ \Rightarrow f(x) &= (1 + b^2) \left\{ x + \frac{b}{1 + b^2} \right\}^2 + \frac{1}{(1 + b^2)} \end{aligned}$$

It is evident from this that the minimum value of $f(x)$ is $\frac{1}{1 + b^2}$, which it attains at $x = -\frac{b}{1 + b^2}$.

$$\therefore m(b) = \frac{1}{1 + b^2}$$

Clearly, $0 < m(b) \leq 1$

Hence, range of $m(b)$ is $[0, 1]$.

Critical Points

Critical point of a function of a real value is any value in the domain, where either the function is not differentiable or its derivative is zero.

Or

It is a collection of points for which

- (i) $f(x)$ does not exist
(ii) $f'(x)$ does not exist or
(iii) $f'(x) = 0$

All the value of x obtained from above conditions are said to be critical points.

It should be noted that critical points are the interior points of an interval.

➤ **Example 3.** The number of the critical points for $f(x) = (x - 2)^{2/3}(2x + 1)$ is

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. (c) Given, $f(x) = (x - 2)^{2/3}(2x + 1)$

$$\begin{aligned} \text{Now, } f'(x) &= \frac{2}{3}(x - 2)^{-1/3}(2x + 1) + (x - 2)^{2/3} \cdot 2 \\ \Rightarrow f'(x) &= 2 \left[\frac{(2x + 1)}{3(x - 2)^{1/3}} + \frac{(x - 2)^{2/3}}{1} \right] \end{aligned}$$

Clearly, $f'(x)$ is not defined at $x = 2$, so $x = 2$ is a critical point. Another critical point is given by

$$f'(x) = 0 \Rightarrow 2 \left[\frac{(2x + 1) + 3(x - 2)}{3(x - 2)^{1/3}} \right] = 0$$

$$\Rightarrow 5x - 5 = 0 \Rightarrow x = 1$$

Hence, $x = 1$ and $x = 2$ are two critical points of $f(x)$.

➤ **Example 4.** The number of critical points of $f(x) = \frac{|x - 1|}{x^2}$ is

- (a) 1 (b) 2
(c) 3 (d) None of these

Sol. (c) We have, $f(x) = \frac{|x - 1|}{x^2} = \begin{cases} \frac{x - 1}{x^2}, & x \geq 1 \\ \frac{1 - x}{x^2}, & x < 1 \end{cases}$

Clearly, $f(x)$ is not differentiable at $x = 0$ and $x = 1$.

So, by definition, these are two of the critical points.

For points other than these two, we have

$$f'(x) = \begin{cases} \frac{-x + 2}{x^3}, & x \geq 1 \\ \frac{x - 2}{x^3}, & x < 1 \end{cases}$$

Clearly, $f'(x) = 0$ at $x = 2$.

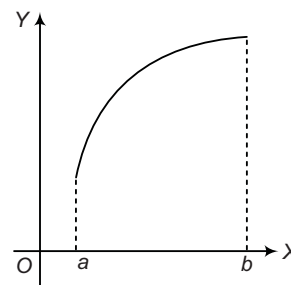
So, $x = 2$ is also a critical point.

Hence, $f(x)$ has three critical points, viz. 0, 1 and 2.

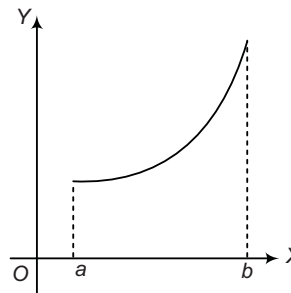
Point of Inflection

For continuous function $f(x)$, if $f''(x_0) = 0$ or $f''(x_0)$ does not exist at points, where $f'(x_0)$ exists and if $f''(x)$ changes sign when passing through $x = x_0$, then x_0 is called the point of inflection. At the point of inflection, the curve changes its concavity, i.e.

- (a) If $f''(x) < 0$, $x \in (a, b)$, then the curve $y = f(x)$ is concave downward in (a, b) .



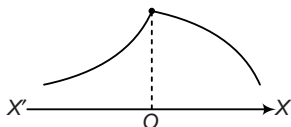
- (b) If $f''(x) > 0$, $x \in (a, b)$, then the curve $y = f(x)$ is concave upward in (a, b) .



Consider, function $f(x) = x^3$. At $x = 0$, $f'(x) = 0$.

Also, $f''(x) = 0$ at $x = 0$. Such a point is called the point of inflection. Here, the 2nd derivative is zero.

Consider, the function $f(x)$ whose graph is given below.



Here, $f(x)$ is non-differentiable at $x=c$, but curve changes its concavity.

Hence, $x=c$ is the point of inflection.

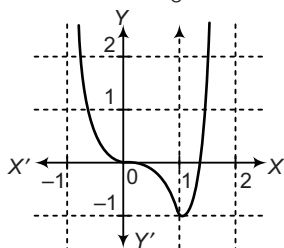
➤ **Example 5.** The points of inflection for $f(x) = 3x^4 - 4x^3$ are

- (a) 0 (b) $\frac{2}{3}$ (c) 1 (d) 2

Sol. (a,b) $f(x) = 3x^4 - 4x^3$

$$\Rightarrow f'(x) = 12x^3 - 12x^2 \Rightarrow f''(x) = 36x^2 - 24x$$

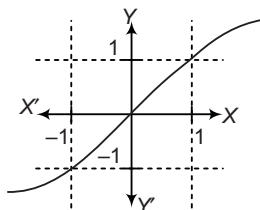
Now, $f''(x) = 0 \Rightarrow x=0$ and $\frac{2}{3}$ are the points of inflection.



➤ **Example 6.** The points of inflection for $f(x) = x^{1/3}$ is

- (a) 1 (b) 2
(c) 0 (d) None of these

Sol. (c) $f(x) = x^{1/3} \Rightarrow f'(x) = \frac{1}{3x^{2/3}}$



$\therefore f(x)$ is non-differentiable at $x=0$, but curve changes its concavity, hence $x=0$ is the point of inflection.

Local Maxima and Local Minima

There may be some points in the domain of a function, where the function does not attain the greatest or the least value but the values at these points are greater than or less than the values of the function at the neighbouring points. Such points are known as the points of local minimum or local maximum and we will be mainly discussing about the local maximum and local minimum values of a function.

Local Maximum

A function $f(x)$ is said to attain a local maximum at $x=a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that

$$f(x) < f(a), \forall x \in (a-\delta, a+\delta), x \neq a$$

$$\text{or } f(x) - f(a) < 0, \forall x \in (a-\delta, a+\delta), x \neq a$$

In such a case, $f(a)$ is called the local maximum value of $f(x)$ at $x=a$.

Local Minimum

A function $f(x)$ is said to attain a local minimum at $x=a$, if there exists a neighbourhood $(a-\delta, a+\delta)$ of a such that

$$f(x) > f(a), \forall x \in (a-\delta, a+\delta), x \neq a$$

$$\text{or } f(x) - f(a) > 0, \forall x \in (a-\delta, a+\delta), x \neq a$$

The value of the function at $x=a$, i.e. $f(a)$ is called the local minimum value of $f(x)$ at $x=a$.

The points at which a function attains either the local maximum values or local minimum values are known as the extreme points or turning points and both local maximum and local minimum values are called the extreme values of $f(x)$.

Thus, a function attains an extreme value at $x=a$, if $f(a)$ is either a local maximum value or a local minimum value. Consequently, at an extreme point a , $f(x) - f(a)$ keeps the same sign for all values of x in a deleted nbd of a .

- The maximum and minimum values of a function are also known as local/relative maxima or local/relative minima as these are the greatest and least values of the function relative to some neighbourhood of the point in question.
- The term extreme or external or turning value is used for both maximum and minimum values.
- A maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.
- A function can have several maximum and minimum values and a minimum value may even be greater than a maximum value.
- The maximum and minimum values of a continuous function occur alternately and between two consecutive maximum values, there is a minimum value and vice-versa.

➤ **Example 7.** The function

$f(x) = (x^2 - 4)^n (x^2 - x + 1)$, $n \in \mathbb{N}$ assumes a local minimum value at $x=2$, then find the possible values of n .

- (a) 2 (b) 3
(c) 5 (d) None of these

Sol. (a) $f(x) = (x^2 - 4)^n (x^2 - x + 1)$

$$\text{At } f(2) = 0,$$

$$\text{Now, } x^2 - x + 1 > 0, \forall x$$

$$\begin{aligned} \therefore f(2^+) &= \lim_{x \rightarrow 2^+} (x^2 - 4)^n (x^2 - x + 1) \\ &= 3 \lim_{h \rightarrow 0} \{((h+2)^2 - 4)^n\} \end{aligned}$$

$$\begin{aligned}
 &= 3 \lim_{h \rightarrow 0} (4h + h^2)^n > 0 \\
 f(2^-) &= \lim_{x \rightarrow 2^-} (x^2 - 4)^n (x^2 - x + 1) \\
 &= 3 \lim_{h \rightarrow 0} \{(h-2)^2 - 4\}^n \\
 &= 3 \lim_{h \rightarrow 0} (h^2 - 4h)^n \\
 &= 3 \times (\text{very small negative value})^n
 \end{aligned}$$

For $x = 0$ to be a point of minima, we must have $f(2^-) > 0$ for which n must be an even integer.

Test for Local Maximum/Minimum

We have two cases to consider:

- (a) **Test for local maximum/minimum at $x = a$, if $f(x)$ is differentiable at $x = a$.**

If $f(x)$ is differentiable at $x = a$ and it is a critical point of the function, i.e. $f'(a) = 0$, then we have the following three tests to decide whether $f(x)$ has a local maximum or local minimum or neither at $x = a$.

First Derivative Test

If $f'(a) = 0$ and $f'(x)$ changes its sign while passing through the point $x = a$, then

- $f(x)$ would have a local maximum at $x = a$, if $f'(a-0) > 0$ and $f'(a+0) < 0$. It means that $f'(x)$ should change its sign from positive to negative.
- $f(x)$ would have local minimum at $x = a$, if $f'(a-0) < 0$ and $f'(a+0) > 0$. It means that $f'(x)$ should change its sign from negative to positive.
- If $f'(x)$ does not change its sign while passing through $x = a$, then $f(x)$ would have neither a maximum nor minimum at $x = a$.

➤ **Example 8.** Let $f(x) = x + \frac{1}{x}$, $x \neq 0$, then $f(x)$

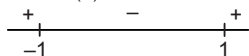
assumes maximum and minimum are respectively

- $-1, 1$
- $1, -1$
- $0, 1$
- None of the above

Sol. (a) Here, $f'(x) = 1 - \frac{1}{x^2}$

$$\Rightarrow f'(x) = \frac{x^2 - 1}{x^2} = \frac{(x-1)(x+1)}{x^2}$$

Sign scheme for $f'(x)$



Using number line rule, we have maximum at $x = -1$ and minimum at $x = 1$.

At $x = -1$, we have local maximum $\Rightarrow f_{\max}(x) = -2$
and at $x = 1$, we have local minimum $\Rightarrow f_{\min}(x) = 2$

➤ **Example 9.** The value of a , so that the sum of the cubes of the roots of the equation $x^2 - ax + (2a-3) = 0$ assumes the minimum value, is

- $a = 1$
- $a = 3$
- $a = 0$
- None of these

Sol. (b) Let α and β be the roots of the given equation.

Then, $\alpha + \beta = a$ and $\alpha\beta = 2a - 3$

Let $S = \alpha^3 + \beta^3$

Then, $S = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$\Rightarrow S = a^3 - 3a(2a - 3)$$

$$\Rightarrow S = a^3 - 6a^2 + 9a$$

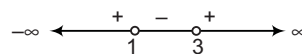
$$\Rightarrow \frac{dS}{da} = 3a^2 - 12a + 9$$

$$= 3(a^2 - 4a + 3) = 3(a-1)(a-3)$$

Clearly, $\frac{dS}{da} = 0$ when $a = 1, 3$.

The changes in the signs of $\frac{dS}{da}$ are shown in the figure.

Clearly, S is minimum at $a = 3$.



Second Derivative Test

First we find the roots of $f'(x) = 0$. Suppose, $x = a$ is one of the roots of $f'(x) = 0$. Now, find $f''(x)$ at $x = a$.

- If $f''(a)$ is negative, then $f(x)$ is maximum at $x = a$.
- If $f''(a)$ is positive, then $f(x)$ is minimum at $x = a$.
- If $f''(a)$ is zero, then we find $f'''(x)$ at $x = a$. If $f'''(a) \neq 0$, then $f(x)$ has neither maximum nor minimum (inflection point) at $x = a$. But, if $f'''(a) = 0$, then find $f^{iv}(a)$.

If $f^{iv}(a)$ is positive, then $f(x)$ is minimum at $x = a$.

If $f^{iv}(a)$ is negative, then $f(x)$ is maximum at $x = a$ and so on, process is repeated till point is discussed.

n th Derivative Test

It is nothing but the general version of the second derivative test, it says that, if $f'(a) = f''(a) = f'''(a) = \dots = f^{(n-1)}(a) = 0$ and is $f^{(n)}(a) \neq 0$ all derivatives of the function up to order n vanishes and $(n+1)$ th order derivative does not vanish at $x = a$, then $f(x)$ would have a local maximum or local minimum at $x = a$, if n is odd natural number and that $x = a$ would be a point of local maxima, if $f^{(n)}(a) < 0$ and would be a point of local minima, if $f^{(n)}(a) > 0$. However, if n is even, then f has neither a maxima nor a minima at

$x = a$. It is clear that the last two tests are basically the mathematical representation of the first derivative test. But that should not diminish the importance of these tests. Because at that time, it becomes very difficult to decide whether $f'(x)$ changes its sign or not while passing through point $x = a$ and the remaining tests may come handy in these kinds of situations.

It must be remembered that this method is not applicable to those critical points, where $f'(x)$ remains undefined.

Example 10. If $f(x) = 2x^3 - 21x^2 + 36x - 30$, then which one of the following is correct?

- (a) $f(x)$ has minimum at $x = 1$
 (b) $f(x)$ has maximum at $x = 6$
 (c) $f(x)$ has maximum at $x = 1$
 (d) $f(x)$ has no maximum or minimum

Sol. (c) We have, $f(x) = 2x^3 - 21x^2 + 36x - 30$

$$\Rightarrow f'(x) = 6x^2 - 42x + 36 \text{ and } f''(x) = 12x - 42$$

At points of local maximum or minimum, we must have

$$\Rightarrow f'(x) = 0$$

$$\Rightarrow 6(x^2 - 7x + 6) = 0$$

$$\Rightarrow x = 1, 6$$

$$\text{Clearly, } f''(1) = 12 - 42 = -30 < 0$$

$$\text{and } f''(6) = 72 - 42 = 30 > 0$$

So, $f(x)$ has local maximum at $x = 1$ and local minimum at $x = 6$.

Example 11. The minimum value of $\frac{x}{\log x}$ is

- (a) e (b) $1/e$
 (c) 1 (d) None of these

Sol. (a) Let $f(x) = \frac{x}{\log x}$

$$\Rightarrow f'(x) = \frac{\log x - 1}{(\log x)^2}$$

For maximum or minimum, $f'(x) = 0$

$$\Rightarrow \log x = 1 \Rightarrow x = e$$

$$\text{Now, } f''(x) = \frac{1 \cdot (\log x)^2 - (\log x - 1) \cdot \frac{2 \log x}{x}}{(\log x)^4}$$

$$\Rightarrow f''(x)_{x=e} = \frac{\frac{1}{e} - 0}{1} = \frac{1}{e} > 0$$

So, $f(x)$ is minimum at $x = e$.

$$\therefore \text{Minimum value of } f(x) = f(e) = \frac{e}{1} = e$$

Example 12. The maximum value of the function $y = x(x-1)^2$, is

- (a) 0
 (b) $\frac{4}{27}$
 (c) -4
 (d) None of the above

Sol. (b) We have, $y = x(x-1)^2$

$$\Rightarrow \frac{dy}{dx} = (x-1)^2 \cdot 1 + 2x(x-1) = 3x^2 - 4x + 1 = (x-1)(3x-1)$$

For maximum or minimum, put $\frac{dy}{dx} = 0$.

$$\Rightarrow (x-1)(3x-1) = 0$$

$$\Rightarrow x = \frac{1}{3}, 1$$

$$\text{Now, } \frac{d^2y}{dx^2} = 6x - 4$$

$$\Rightarrow \left[\frac{d^2y}{dx^2} \right]_{x=\frac{1}{3}} = -2 < 0$$

So, y is maximum when $x = \frac{1}{3}$ and maximum value is

$$[y]_{x=\frac{1}{3}} = \frac{4}{27}$$

$$\text{Also, } \left[\frac{d^2y}{dx^2} \right]_{x=1} = 2 > 0$$

So, y is minimum when $x = 1$.

Example 13. The value of a , so that the sum of the squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value, is

- (a) 2 (b) 0
 (c) 3 (d) 1

Sol. (d) Let α and β be the roots of the given equation, so that $\alpha + \beta = a - 2$ and $\alpha\beta = -(a - 1)$

$$\text{Let } S = \alpha^2 + \beta^2$$

$$\begin{aligned} \text{Then, } S &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a-2)^2 + 2(a-1) \\ &= a^2 - 2a + 2 \end{aligned}$$

$$\therefore \frac{dS}{da} = 2a - 2$$

$$\text{Now, } \frac{dS}{da} = 0$$

$$\Rightarrow 2a - 2 = 0$$

$$\Rightarrow a = 1$$

$$\text{Also, } \frac{d^2S}{da^2} = 2 > 0, \forall a$$

Hence, S is minimum when $a = 1$.

Example 14. The maximum ordinate of a point on the graph of the function $f(x) = \sin x(1 + \cos x)$, is

- (a) $\frac{2+\sqrt{3}}{4}$ (b) $\frac{3\sqrt{3}}{4}$
 (c) 1 (d) None of these

Sol. (b) We have, $f(x) = \sin x(1 + \cos x)$

$$\Rightarrow f(x) = \sin x + \frac{1}{2} \sin 2x$$

$$\Rightarrow f'(x) = \cos x + \cos 2x$$

$$\text{and } f''(x) = -\sin x - 2\sin 2x$$

For local maximum or minimum, we have

$$f'(x) = 0$$

$$\Rightarrow \cos x + \cos 2x = 0$$

$$\begin{aligned} \Rightarrow 2\cos^2 x + \cos x - 1 &= 0 \\ \Rightarrow (2\cos x - 1)(\cos x + 1) &= 0 \\ \Rightarrow \cos x &= \frac{1}{2}, -1 \\ \Rightarrow x &= \frac{\pi}{3}, \pi, \frac{5\pi}{3} \\ \text{Now, } f''\left(\frac{\pi}{3}\right) &= -\sin\frac{\pi}{3} - 2\sin\frac{2\pi}{3} < 0, f''(\pi) = 0 \\ \text{and } f''\left(\frac{5\pi}{3}\right) &= -\sin\frac{5\pi}{3} - 2\sin\frac{10\pi}{3} > 0 \\ \text{Thus, } f(x) &\text{ attains a local maximum at } x = \frac{\pi}{3}. \\ \therefore \text{Maximum ordinate} &= f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3}\left(1 + \cos\frac{\pi}{3}\right) \\ &= \frac{\sqrt{3}}{2} \times \frac{3}{2} \\ &= \frac{3\sqrt{3}}{4} \end{aligned}$$

- **Example 15.** If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals
- 3
 - 1
 - 2
 - $1/2$

Sol. (c) We have, $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$
 $\therefore f'(x) = 6x^2 - 18ax + 12a^2 = 0$
 $\Rightarrow 6[x^2 - 3ax + 2a^2] = 0$
 $\Rightarrow x^2 - 3ax + 2a^2 = 0$
 $\Rightarrow x^2 - 2ax - ax + 2a^2 = 0$
 $\Rightarrow x(x - 2a) - a(x - 2a) = 0$
 $\Rightarrow (x - a)(x - 2a) = 0$
 $\Rightarrow x = a \text{ and } x = 2a$
 Now, $f''(x) = 12x - 18a$
 $\therefore f''(a) = 12a - 18a = -6a < 0$
 $\therefore f(x)$ will be maximum at $x = a$
 i.e. $p = a$
 Also, $f''(2a) = 24a - 18a = 6a$
 $\therefore f(x)$ will be minimum at $x = 2a$
 i.e. $q = 2a$
 Given, $p^2 = q \Rightarrow a^2 = 2a \Rightarrow a = 2$

- (b) **Test for local maximum/minimum at $x = a$, if $f(x)$ is not differentiable at $x = a$.**

Case I When $f(x)$ is continuous at $x = a$ and $f'(a - h)$ and $f'(a + h)$ exist and are non-zero, then $f(x)$ has a local maximum or minimum at $x = a$, if $f'(a - h)$ and $f'(a + h)$ are of opposite signs.

If $f'(a - h) > 0$ and $f'(a + h) < 0$, then $x = a$ will be a point of local maximum.

If $f'(a - h) < 0$ and $f'(a + h) > 0$, then $x = a$ will be a point of local minimum.

Case II When $f(x)$ is continuous and $f'(a - h)$ and $f'(a + h)$ exist but one of them is zero, then we should infer the following about the existence of local maxima/minima from the basic definition of local maxima/minima.

Case III If $f(x)$ is not continuous at $x = a$ and $f'(a - h)$ and/or $f'(a + h)$ are not finite, then compare the values of $f(x)$ at the neighbouring points of $x = a$.

Remark It is advisable to draw the graph of the function in the vicinity of the point $x = a$ because the graph would give us the clear picture about the existence of local maxima/minima at $x = a$.

- **Example 16.** Let $f(x) = \begin{cases} x^3 + x^2 + 10x, & x < 0 \\ -3\sin x, & x \geq 0 \end{cases}$

then at $x = 0$, $f(x)$ is

- local minimum
- local maximum
- Neither maximum nor minimum
- None of the above

Sol. (b) Clearly, $f(x)$ is continuous at $x = 0$ but not differentiable at $x = 0$ as $f(0) = f(0 - 0) = f(0 + 0) = 0$

$$\begin{aligned} f'^-(0) &= \lim_{h \rightarrow 0} \frac{f(-h) - f(0)}{-h} \\ &= \lim_{h \rightarrow 0} \frac{-h^3 + h^2 - 10h - 0}{-h} = 10 \end{aligned}$$

$$\text{But } f'^+(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{-3\sin h}{h} = -3$$

Since, $f'^-(0) > 0$ and $f'^+(0) < 0$, $x = 0$ is the point of local maximum.

- **Example 17.** If $f(x) = |x| + |x - 1| + |x - 2|$, then

- $f(x)$ has minima at $x = 1$
- $f(x)$ has maxima at $x = 1$
- $f(x)$ has neither maxima nor minima at $x = 1$
- None of the above

Sol. (a) We have,

$$f(x) = |x| + |x - 1| + |x - 2| = \begin{cases} -3x + 3, & x < 0 \\ -x + 3, & 0 \leq x < 1 \\ x + 1, & 1 \leq x < 2 \\ 3x - 3, & x \geq 2 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} -3, & x < 0 \\ \text{does not exist,} & x = 0 \\ -1, & 0 < x < 1 \\ \text{does not exist,} & x = 1 \\ 1, & 1 < x < 2 \\ \text{does not exist,} & x = 2 \\ 3, & x > 2 \end{cases}$$

Clearly, $f(x)$ has minima at $x = 1$ and neither maxima nor minima at $x = 0$ and $x = 2$.

Concept of Global Maximum/Minimum

Let $y = f(x)$ be a given function with domain D . Let $[a, b] \subseteq D$. Global maximum/minimum of $f(x)$ in $[a, b]$ is basically the greatest/least value of $f(x)$ in $[a, b]$.

Global maximum and minimum in $[a, b]$ would occur at the critical point of $f(x)$ within $[a, b]$ or at the end points of the interval.

Global Maximum/Minimum in $[a, b]$

In order to find the global maximum and minimum of $f(x)$ in $[a, b]$, find all the critical points of $f(x)$ in (a, b) . Let $c_1, c_2, \dots, c_n$ be the different critical points. Then, find the value of the function at these critical points.

Let $f(c_1), f(c_2), \dots, f(c_n)$ be the values of the function at critical points.

Say, $M_1 = \max$

$$\{f(a), f(c_1), f(c_2), \dots, f(c_n), f(b)\}$$

and $M_2 = \min$

$$\{f(a), f(c_1), f(c_2), \dots, f(c_n), f(b)\}$$

Then, M_1 is the greatest value of $f(x)$ in $[a, b]$ and M_2 is the least value of $f(x)$ in $[a, b]$.

Global Maximum/Minimum in (a, b)

The method for obtaining the greatest and least values of $f(x)$ in (a, b) is almost same as the method used for obtaining the greatest and least values in $[a, b]$, however with a caution.

Let $y = f(x)$ be a function and $c_1, c_2, \dots, c_n$ be the different critical points of the function in (a, b) .

Let $M_1 = \max \{f(c_1), f(c_2), f(c_3), \dots, f(c_n)\}$

and $M_2 = \min \{f(c_1), f(c_2), f(c_3), \dots, f(c_n)\}$

Now, if $\lim_{x \rightarrow a+0} f(x) > M_1$

or $(x \rightarrow b-0)$

or $< M_2$, then $f(x)$ will not have global maximum or global minimum in (a, b) .

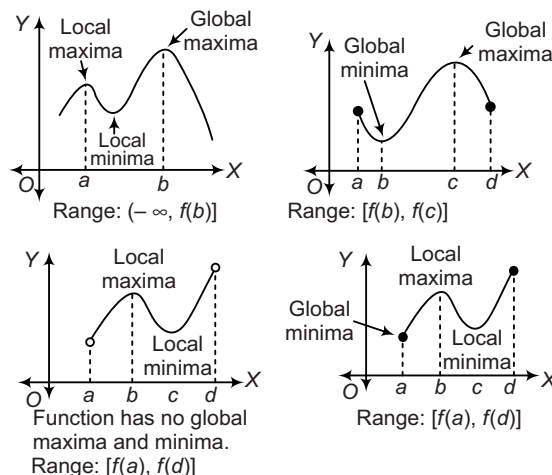
This means that if the limiting values at the end points are greater than M_1 or less than M_2 , then $f(x)$ will not have global maximum/minimum in (a, b) , then M_1 and M_2 will respectively be the global maximum or global minimum of $f(x)$ in (a, b) .

On the other hand, if $M_1 > \lim_{x \rightarrow a+0} f(x)$

and $M_2 < \lim_{x \rightarrow b-0} f(x)$,

then M_1 and M_2 will respectively be the global maximum global minimum of $f(x)$ in (a, b) .

Consider, the following cases:



➤ **Example 18.** The greatest value of

$f(x) = (x+1)^{1/3} - (x-1)^{1/3}$ on $[0, 1]$ is

- (a) 1 (b) 2
(c) 3 (d) $\frac{1}{3}$

Sol. (b) We have, $f(x) = (x+1)^{1/3} - (x-1)^{1/3}$

$$\therefore f'(x) = \frac{1}{3} \left[\frac{1}{(x+1)^{2/3}} - \frac{1}{(x-1)^{2/3}} \right]$$

Clearly, $f'(x)$ does not exist at $x = \pm 1$.

Now, $f'(x) = 0$

$$\Rightarrow (x-1)^{2/3} = (x+1)^{2/3} \Rightarrow x = 0$$

$f'(x) \neq 0$ for any other value of $x \in [0, 1]$. The value of $f(x)$ at $x = 0$ is 2. Hence, the greatest value of $f(x)$ is 2.

➤ **Example 19.** Let

$f(x) = \cos \pi x + 10x + 3x^2 + x^3$, $x \in [-2, 3]$. Then, the absolute minimum value of $f(x)$ is

- (a) 0
(b) -15
(c) $3 - 2\pi$
(d) None of the above

Sol. (b) We have, $f(x) = \cos \pi x + 10x + 3x^2 + x^3$

$$\Rightarrow f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$$

$$\Rightarrow f'(x) = 3(x+1)^2 + (7 - \pi \sin \pi x) > 0 \text{ for all } x \in [-2, 3]$$

$\Rightarrow f(x)$ is strictly increasing on $[-2, 3]$.

$\therefore$ Absolute minimum value of $f(x)$

$$= f(-2) = \cos 2\pi - 20 + 12 - 8 = -15$$

➤ **Example 20.** The difference between the greatest and least values of the function

$f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is

- (a) $\frac{\sqrt{3} + \sqrt{2}}{2}$ (b) $\frac{\sqrt{3} + \sqrt{2}}{2} + \frac{\pi}{6}$
(c) $\frac{\sqrt{3}}{2} + \frac{\pi}{3}$ (d) $\frac{\sqrt{3} + \sqrt{2}}{2} - \frac{\pi}{3}$

Sol. (c) We have, $f'(x) = 2\cos 2x - 1$

$$\begin{aligned}\therefore f'(x) &= 0 \\ \Rightarrow 2\cos 2x - 1 &= 0 \\ \Rightarrow \cos 2x &= \frac{1}{2} \\ \Rightarrow 2x &= -\frac{\pi}{3}, \frac{\pi}{3} \\ x &= -\frac{\pi}{6}, \frac{\pi}{6}\end{aligned}$$

$$\text{Now, } f\left(-\frac{\pi}{6}\right) = \frac{\pi}{2} \Rightarrow f\left(\frac{\pi}{6}\right) = -\frac{\pi}{2}$$

$$\begin{aligned}\text{Now, } f\left(-\frac{\pi}{6}\right) &= -\frac{\sqrt{3}}{2} + \frac{\pi}{6} \\ \text{and } f\left(\frac{\pi}{6}\right) &= \frac{\sqrt{3}}{2} - \frac{\pi}{6}\end{aligned}$$

Clearly, $\frac{\sqrt{3}}{2} - \frac{\pi}{6}$ is the greatest value of $f(x)$ and its least value is $-\frac{\pi}{2}$.

$$\text{Hence, the required difference is } \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \left(-\frac{\pi}{2}\right) = \frac{\sqrt{3}}{2} + \frac{\pi}{3}$$

Work Book Exercise 8.1

1 The minimum value of the polynomial $x(x+1)(x+2)(x+3)$ is

- a 0 b $\frac{9}{16}$ c -1 d $-\frac{3}{2}$

2 The minimum value of $\frac{\tan\left(x + \frac{\pi}{6}\right)}{\tan x}$ is

- a 0 b $\frac{1}{2}$ c 1 d 3

3 If $a > b > 0$, then the minimum value of $a \sec \theta - b \tan \theta$ is

- a $b - a$ b $\sqrt{a^2 + b^2}$
c $\sqrt{a^2 - b^2}$ d $2\sqrt{a^2 - b^2}$

4 For all $a, b \in R$, the function $f(x) = 3x^4 - 4x^3 + 6x^2 + ax + b$ has

- a no extremum
b exactly one extremum
c exactly two extremum
d three extremum

5 Difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

- a 2 b e c e^2 d 1

6 If the function $f(x) = \frac{t + 3x - x^2}{x - 4}$, where t is a

parameter, has a minimum and a maximum, then the range of t is

- a (0, 4) b (0, ∞) c $(-\infty, 4)$ d $(4, \infty)$

7 The set of value(s) of a for which the function $f(x) = \frac{ax^3}{3} + (a+2)x^2 + (a-1)x + 2$ possess a negative point of inflection, is

- a $(-\infty, -2) \cup (0, \infty)$ b $\left\{-\frac{4}{5}\right\}$
c $(-2, 0)$ d empty set

8 The least value of a for which the equation, $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has atleast one solution on the interval $\left(0, \frac{\pi}{2}\right)$, is

- a 3 b 5 c 7 d 9

9 If λ and μ are real numbers such that $x^3 - \lambda x^2 + \mu x - 6 = 0$ has its roots real and positive, then the minimum value of μ is

- a $3 \times \sqrt[3]{36}$ b 11
c 0 d None of these

10 Let $f(x) = \frac{\tan^n x}{\sum_{r=0}^{2n} \tan^r x}$; $n \in N$, where $x \in \left[0, \frac{\pi}{2}\right)$, then

- a $f(x)$ is bounded and it takes both of its bounds and the range of $f(x)$ contains exactly one integral point
b $f(x)$ is bounded and it takes both of its bounds and the range of $f(x)$ contains more than one integral point
c $f(x)$ is bounded but minimum and maximum does not exist
d $f(x)$ is not bounded as the upper bound does not exist

Application of Maxima and Minima to Problems

Optimization Problems

➤ **Example 21.** If the two positive numbers x and y such that $x + y = 60$ and $x^3 y$ is maximum, then

- (a) $x = 15, y = 45$ (b) $x = 45, y = 15$
(c) $x = 30, y = 30$ (d) $x = 20, y = 40$

Sol. (b) We have, $x + y = 60$

$$\Rightarrow y = 60 - x$$

$$\Rightarrow x^3 y = (60 - x)(x^3); x \in (0, 60)$$

$$\text{Let } f(x) = x^3(60 - x) \Rightarrow f'(x) = 3x^2(60 - x) - x^3$$

For maxima or minima, put $f'(x) = 0$

$$\therefore 3x^2(60 - x) - x^3 = 0$$

$$\Rightarrow x^2(180 - 3x - x) = 0$$

$$\Rightarrow x = 45$$

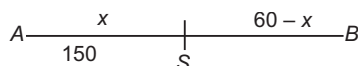
$$\therefore f'(45^+) < 0 \text{ and } f'(45^-) > 0 \quad [\because x \neq 0]$$

Hence, local maxima is at $x = 45$.

$$\text{So, } x = 45 \text{ and } y = 15$$

- **Example 22.** Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B . If the total distance to be travelled by 200 students is to be as small as possible, then the school be built at
- (a) town B (b) 45 km from town A
 (c) town A (d) 45 km from town B

Sol. (c) Given that, $AB = 60$



Let the school be at a distance x from A (with 150 students), then the distance travelled by 200 students is

$$D = 150x + 50(60 - x) = 100x + 3000$$

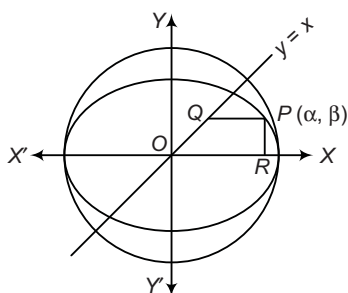
D will be least and equal to 3000, if $x = 0$, i.e. school is built at A .

Coordinate Geometry

- **Example 23.** Find the point (α, β) on the ellipse $4x^2 + 3y^2 = 12$, in the first quadrant, so that the area enclosed by the lines $y = x$, $y = \beta$, $x = \alpha$ and X -axis is maximum.

- (a) $\left(\frac{3}{2}, 1\right)$ (b) $\left(1, \frac{3}{2}\right)$
 (c) $\left(\sqrt{2}, \frac{2}{\sqrt{3}}\right)$ (d) $(0, 2)$

Sol. (a) Given, equation of the ellipse is $\frac{x^2}{3} + \frac{y^2}{4} = 1$. Let point P be $(\sqrt{3}\cos\theta, 2\sin\theta)$, $\theta \in \left(0, \frac{\pi}{2}\right)$.



Clearly, line PQ is $y = 2\sin\theta$, line PR is $x = \sqrt{3}\cos\theta$, OQ is $y = x$ and Q is $(2\sin\theta, 2\sin\theta)$.

Area of the trapezium $PQORP$,

$$\begin{aligned} A &= \frac{1}{2}(OR + PQ)PR \\ &= \frac{1}{2}[\sqrt{3}\cos\theta + (\sqrt{3}\cos\theta - 2\sin\theta)]2\sin\theta \\ &= (2\sqrt{3}\cos\theta\sin\theta - 2\sin^2\theta) = (\sqrt{3}\sin 2\theta + \cos 2\theta - 1) \\ &= 2\cos\left(2\theta - \frac{\pi}{3}\right) - \frac{1}{2} \end{aligned}$$

which is maximum when $\cos\left(2\theta - \frac{\pi}{3}\right) = 1$

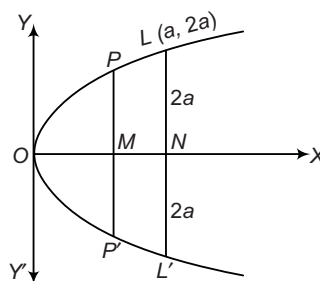
$$\text{or } 2\theta - \frac{\pi}{3} = 0 \text{ or } \theta = \frac{\pi}{6}$$

Hence, point P is $\left(\frac{3}{2}, 1\right)$.

- **Example 24.** If LL' is the latusrectum of the parabola $y^2 = 4ax$ and PP' is a double ordinate drawn between the vertex and the latusrectum, then the area of the trapezium $PP'LL'$ is maximum when the distance PP' from the vertex is

- (a) $\frac{a}{2}$ (b) $\frac{a}{3}$ (c) $\frac{a}{6}$ (d) $\frac{a}{9}$

Sol. (d) Let $LL' = 4a$ be the latusrectum of the parabola $y^2 = 4ax$ and $(at^2, 2at)$ be the coordinates of the point P .



Here, PP' is the double ordinate of the parabola.

$$\Rightarrow OM = at^2$$

$$\Rightarrow MN = ON - OM = a - at^2$$

$$\text{and } PP' = 2PM = 4at$$

Now, area of the trapezium $PP'LL'$

$$A = \frac{1}{2}(PP' + LL') \times MN$$

$$\Rightarrow A = \frac{1}{2}(4at + 4a)(a - at^2)$$

$$\Rightarrow A = 2a^2(-t^3 - t^2 + t + 1)$$

$$\text{Now, } \frac{dA}{dt} = 2a^2(-3t^2 - 2t + 1)$$

$$\text{and } \frac{d^2A}{dt^2} = 2a^2(-6t - 2)$$

For maximum or minimum of A , put $\frac{dA}{dt} = 0$.

$$\Rightarrow -2a^2(3t - 1)(t + 1) = 0$$

$$\Rightarrow t = -1, \frac{1}{3}$$

$$\text{When } t = -1, \frac{d^2A}{dt^2} = 8a^2(+ve)$$

$\Rightarrow A$ is minimum when $t = -1$

$$\text{and when } t = \frac{1}{3}, \frac{d^2A}{dt^2} = -8a^2(-ve)$$

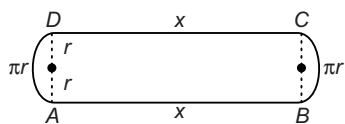
$$\Rightarrow A \text{ is maximum when } t = \frac{1}{3} \text{ (only point of maxima).}$$

For the area of the trapezium $PP'LL'$ to be maximum, distance of PP' from vertex $O = OM = at^2 = a\left(\frac{1}{3}\right)^2 = \frac{a}{9}$

Plane Geometry

- **Example 25.** A running track of 440 ft is to be laid out enclosing a football field, the shape of which is a rectangle with a semi-circle at each end. If the area of the rectangular portion is to be maximum, then find the lengths of its sides.
- (a) 35 ft (b) 45 ft (c) 100 ft (d) 110 ft

Sol. (d)

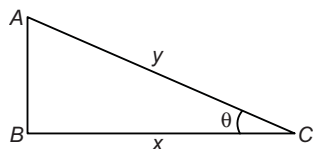


$$\begin{aligned} \therefore \text{Perimeter} &= 440 \text{ ft} \\ \Rightarrow 2x + \pi r + \pi r &= 440 \text{ or } 2x + 2\pi r = 440 \\ \therefore \text{Area of the rectangular portion, } A &= x \cdot 2r \\ \Rightarrow A &= x \cdot \frac{(440 - 2x)}{\pi} \\ \Rightarrow A &= \frac{1}{\pi} (440x - 2x^2) \\ \text{Let } \frac{dA}{dx} &= \frac{1}{\pi} (440 - 4x) = 0 \\ \Rightarrow x &= 110 \text{ for which } \frac{d^2A}{dx^2} < 0 \\ \Rightarrow A \text{ is maximum when } x &= 110. \\ \Rightarrow 2r &= \frac{440 - 2x}{\pi} = \frac{440 - 220}{\pi} = \frac{220}{\pi} = 70 \\ \therefore r &= 35 \text{ ft and } x = 110 \text{ ft} \end{aligned}$$

➤ **Example 26.** If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, the area of the triangle is maximum when the angle between these sides is

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Sol. (c) Let ABC be a right angled triangle in which side $BC = x$ (say) and hypotenuse $AC = y$ (say).
Given, $x + y = k$ (constant)
 $\Rightarrow y = k - x$



$$\begin{aligned} \text{Now, the area of } \triangle ABC \text{ is given by} \\ A &= \frac{1}{2} \cdot BC \cdot AB = \frac{1}{2} x \sqrt{y^2 - x^2} \\ &= \frac{1}{2} x \sqrt{(k-x)^2 - x^2} \end{aligned}$$

$$\begin{aligned} \text{Let } u &= A^2 = \frac{1}{4} x^2 (k^2 - 2kx) \\ \Rightarrow \frac{du}{dx} &= \frac{1}{2} k(kx - 3x^2) \text{ and } \frac{d^2u}{dx^2} = \frac{1}{2} k(k - 6x) \end{aligned}$$

For maximum or minimum of u , put $\frac{du}{dx} = 0$.

$$\Rightarrow x = \frac{k}{3} \quad [\because x \neq 0]$$

$$\text{When } x = \frac{k}{3}, \frac{d^2u}{dx^2} = \frac{1}{2} k \left(k - 6 \times \frac{1}{3} k \right) = -\frac{1}{2} k^2 (-ve)$$

$$\Rightarrow u, \text{ i.e. } A \text{ is maximum when } x = \frac{k}{3}$$

$$\text{and when } y = k - x = \frac{2k}{3}$$

$$\text{Now, } \cos \theta = \frac{BC}{AC} = \frac{x}{y} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\text{Hence, the required angle is } \frac{\pi}{3}$$

Solid Geometry

Some useful formulae of mensuration are given below:

- i. Volume of cuboid = lbh
- ii. Surface area of cuboid = $2(lb + bh + hl)$
- iii. Volume of cube = a^3
- iv. Surface area of cube = $6a^2$
- v. Volume of cone = $\frac{1}{3} \pi r^2 h$
- vi. Curved surface area of cone = $\pi r l$, where l is slant height.
- vii. Curved surface area of a cylinder = $2\pi r h$
- viii. Total surface area of a cylinder = $2\pi r h + 2\pi r^2$
- ix. Volume of a sphere = $\frac{4}{3} \pi r^3$
- x. Surface area of sphere = $4\pi r^2$
- xi. Area of a circular sector = $\frac{1}{2} r^2 \theta$, where, θ is in radians.
- xii. Volume of a prism = Area of the base $\times$ Height
- xiii. Lateral surface area of a prism = Perimeter of a base $\times$ Height
- xiv. Total surface area of a prism = Lateral surface area + 2 Area of the base

➤ The lateral surface area of a prism are all rectangle.

- xv. Volume of a pyramid = $\frac{1}{3}$ Area of the base $\times$ Height
- xvi. Curved surface area of a pyramid = $\frac{1}{2}$ Perimeter of the base $\times$ Slant height

➤ The slant surfaces of a pyramid are triangles.

➤ **Example 27.** An open box is to be made out-of a piece of paper of cardboard measuring $(24 \text{ cm} \times 24 \text{ cm})$ by cutting off equal squares from the corners and turning up the sides, then the height of the box when it has maximum volume, is

- (a) 2 cm (b) 4 cm (c) 6 cm (d) 8 cm

Sol. (b) Let the length of the side of each square cut-off from the corners be x cm.

Then, height of the box = x cm.

$$\therefore V = (24 - 2x)^2 \times x = 4x^3 - 96x^2 + 576x$$

$$\Rightarrow \frac{dV}{dx} = 12(x^2 - 16x + 48) \text{ and } \frac{d^2V}{dx^2} = 24(x - 8)$$

For maximum and minimum of V , put $\frac{dV}{dx} = 0$.

$$\begin{aligned} \Rightarrow x^2 - 16x + 48 &= 0 \\ \Rightarrow (x - 12)(x - 4) &= 0 \\ \Rightarrow x &= 4 \quad [\because x \neq 12] \\ \therefore \left[\frac{d^2V}{dx^2} \right]_{x=4} &= -96 < 0 \end{aligned}$$

So, V is maximum at $x = 4$.

Hence, the volume of the box is maximum when its height is 4 cm.

➤ **Example 28.** The height of the cylinder of maximum volume that can be inscribed in a sphere of radius R and the volume of the largest cylinder inscribed in a sphere of radius R , are

- (a) $\frac{2R}{\sqrt{3}}, \frac{4\pi R^3}{3\sqrt{3}}$ (b) $\frac{2R}{3}, \frac{4\pi R^3}{3}$
(c) $\frac{2R}{3\sqrt{3}}, \frac{4\pi R^3}{\sqrt{3}}$ (d) None of these

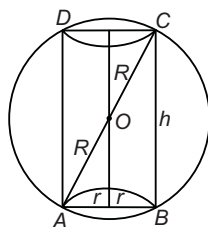
Sol. (a) Let r be the radius and h be the height of the inscribed cylinder $ABCD$. Let V be its volume.

Then, $V = \pi r^2 h$... (i)

Clearly, $AC = 2R$

Also, $AC^2 = AB^2 + BC^2$

$$\Rightarrow (2R)^2 = (2r)^2 + h^2$$



$$\Rightarrow r^2 = \frac{1}{4}(4R^2 - h^2) \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$V = \frac{\pi h}{4}(4R^2 - h^2)$$

$$\Rightarrow \frac{dV}{dh} = \left(\pi R^2 - \frac{3}{4} \pi h^2 \right)$$

$$\text{and } \frac{d^2V}{dh^2} = -\frac{3}{2} \pi h$$

For maxima or minima, put $\frac{dV}{dh} = 0$.

$$\Rightarrow \pi R^2 - \frac{3}{4} \pi h^2 = 0$$

$$\Rightarrow h = \frac{2R}{\sqrt{3}}$$

$$\therefore \left[\frac{d^2V}{dh^2} \right]_{h=\left(\frac{2R}{\sqrt{3}}\right)} = -\frac{3}{2} \pi \times \frac{2R}{\sqrt{3}} = -\pi R \sqrt{3} < 0$$

So, V is maximum when $h = \frac{2R}{\sqrt{3}}$.

Hence, the height of the cylinder of maximum volume is $\frac{2R}{\sqrt{3}}$.

Now, largest volume of the cylinder

$$\begin{aligned} &= \pi \times \frac{1}{4} \left[4R^2 - \frac{4R^2}{3} \right] \times \frac{2R}{\sqrt{3}} \\ &= \frac{4\pi R^3}{3\sqrt{3}} \end{aligned}$$

Work Book Exercise 8.2

- If the sum of two numbers is 3, then the maximum value of the product of the first and the square of second is
a 4 b 1 c 3 d 0
- If $xy = a^2$ and $S = b^2x + c^2y$, where a, b and c are constants, then the minimum value of S is
a abc b $bc\sqrt{a}$
c $2abc$ d None of these
- The sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle will be maximum, if the angle between them is
a $\frac{\pi}{6}$ b $\frac{\pi}{4}$ c $\frac{\pi}{3}$ d $\frac{5\pi}{12}$
- A rectangle with one side lying along X-axis is to be inscribed in the closed region of the XY-plane bounded by the lines $y = 0$, $y = 3x$ and $y = 30 - 2x$. The largest area of such a rectangle is
a $\frac{135}{8}$ b 45 c $\frac{135}{2}$ d 90
- Rectangles are inscribed in a circle of radius r . The dimensions of the rectangle which has the maximum area, are
a r, r b $2r, 2r$
c $\sqrt{2}r, \sqrt{2}r$ d None of these
- Of all the closed cylindrical cans (right circular), which enclosed a given volume 100 cm^3 , the dimensions of the minimum surface area is
a $\left(\frac{50}{\pi}\right)^{1/3} \text{ cm}$ b $2\left(\frac{50}{\pi}\right)^{1/3} \text{ cm}$
c $2\left(\frac{25}{\pi}\right)^{1/3} \text{ cm}$ d None of these
- A point on the hypotenuse of a triangle is at distances a and b from the sides of the triangle, then the minimum length of the hypotenuse is
a $\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{\frac{3}{2}}$ b $\left(\frac{1}{a^3} + \frac{1}{b^3}\right)^{\frac{3}{2}}$
c $\left(\frac{2}{a^3} + \frac{2}{b^3}\right)^{\frac{3}{2}}$ d $\left(\frac{3}{a^2} + \frac{3}{b^2}\right)^{\frac{2}{3}}$
- If the vertices of a triangle are $(0, 0)$, $(x, \cos x)$ and $(\sin^3 x, 0)$, where $0 < x < \frac{\pi}{2}$. Then, the maximum area for such a triangle (in sq units), is
a $\frac{3\sqrt{3}}{32}$ b $\frac{\sqrt{3}}{32}$
c $\frac{4}{32}$ d $\frac{6\sqrt{3}}{32}$

WorkedOut Examples

Type 1. Only One Correct Option

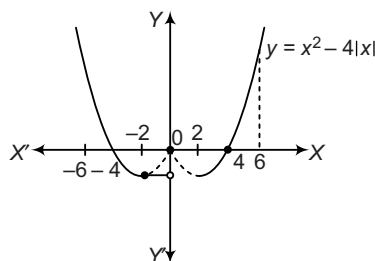
Ex 1. If $f(x) = x^2 - 4|x|$ and

$$g(x) = \begin{cases} \min\{f(t) : -6 \leq t \leq x\}, & x \in [-6, 0] \\ \max\{f(t) : 0 < t \leq x\}, & x \in (0, 6] \end{cases}$$

then $g(x)$ has

- (a) exactly one point of local minima
- (b) exactly one point of local maxima
- (c) no point of local maxima but exactly one point of local minima
- (d) neither a point of local maxima nor minima

Sol. Bold line represents the graph of $y = g(x)$, clearly $g(x)$ has neither a point of local maxima nor a point of local minima.



Hence, (d) is the correct answer.

Ex 2. The function $f(x) = \frac{x^2 - 2}{x^2 - 4}$ has

- (a) no point of local minima
- (b) no point of local maxima
- (c) exactly one point of local minima
- (d) exactly one point of local maxima

Sol. For $y = \frac{x^2 - 2}{x^2 - 4} \Rightarrow \frac{dy}{dx} = \frac{-4x}{(x^2 - 4)^2}$
 $\Rightarrow \frac{dy}{dx} > 0$ for $x < 0$ and $\frac{dy}{dx} < 0$ for $x > 0$. Thus, $x = 0$ is the point of local maxima for y .
 Now, $y|_{x=0} = \frac{1}{2}$ (positive). Thus, $x = 0$ is also the point of local maximum for $y = \left| \frac{x^2 - 2}{x^2 - 4} \right|$.

Hence, (d) is the correct answer.

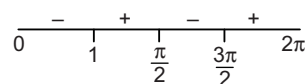
Ex 3. If $f(x) = \int_0^x (t^2 - 1) \cos t \, dt$, $x \in (0, 2\pi)$, then

$f(x)$ attains local maximum value at

- (a) $x = \frac{\pi}{2}$
- (b) $x = 1$
- (c) $x = \frac{3\pi}{2}$
- (d) None of the above

Sol. We have, $f'(x) = (x^2 - 1) \cos x$

Sign scheme of $f'(x)$, clearly $x = \frac{\pi}{2}$ is the point of local maxima, is given below:



Hence, (a) is the correct answer.

Ex 4. Let $y = f(x)$ be a parametrically defined expression such that $x = 3t^2 - 18t + 7$ and $y = 2t^3 - 15t^2 + 24t + 10$, $\forall t \in [0, 6]$.

Then, the minimum and maximum values of $y = f(x)$ are

- (a) 36, 3
- (b) 46, 6
- (c) 40, -6
- (d) 46, -6

Sol. We have, $\frac{dy}{dt} = 6t^2 - 30t + 24 = 6(t-1)(t-4)$ and

$$\frac{dx}{dt} = 6t - 18 = 6(t-3).$$

Thus, $\frac{dy}{dx} = \frac{(t-1)(t-4)}{(t-3)}$, which indicates that $t = 1, 3$

and 4 are the critical points of $y = f(x)$.

$$\text{Now, } \frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) \cdot \frac{dt}{dx} = \frac{t^2 - 6t + 11}{(t-3)^2} \times \frac{1}{6(t-3)}$$

At $t = 1$, $\frac{d^2y}{dx^2} < 0 \Rightarrow t = 1$ is a point of local maxima.

At $t = 4$, $\frac{d^2y}{dx^2} > 0 \Rightarrow t = 4$ is a point of local minima.

At $t = 3$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are not defined and change its sign.

$\frac{d^2y}{dx^2}$ is unknown in the vicinity of $t = 3$, thus $t = 3$ is a point of neither maxima nor minima.

Finally, for $t = 1$, $y = 21$; $t = 4$, $y = -6$; $t = 0$, $y = 10$; $t = 6$, $y = 46$

Thus, maximum and minimum values of expression $y = f(x)$ are 46 and -6, respectively.

Hence, (d) is the correct answer.

Ex 5. Let

$$f(x) = \begin{cases} 4x - x^3 + \ln(a^2 - 3a + 3), & 0 \leq x < 3 \\ x - 18, & x \geq 3 \end{cases}$$

Complete set of values of a such that $f(x)$ has a local maxima at $x = 3$, is

- (a) $[-1, 2]$
- (b) $(-\infty, 1) \cup (2, \infty)$
- (c) $[1, 2]$
- (d) $(-\infty, -1) \cup (2, \infty)$

Sol. Clearly, $f(x)$ is increasing just before $x = 3$ and decreasing after $x = 3$. For $x = 3$ to be the point of local maxima,

$$\begin{aligned} f(3) &\geq f(3-0) \\ \Rightarrow -15 &\geq 12 - 27 + \ln(a^2 - 3a + 3) \\ \Rightarrow 0 &< a^2 - 3a + 3 \leq 1 \\ \Rightarrow 1 &\leq a \leq 2 \end{aligned}$$

Hence, (c) is the correct answer.

Ex 6. If the function $f(x) = \frac{ax+b}{(x-1)(x-4)}$ has a local maxima at $(2, -1)$, then

- (a) $b = 1, a = 0$ (b) $a = 1, b = 0$
(c) $b = -1, a = 0$ (d) $a = -1, b = 0$

Sol. Clearly, $f(2) = -1$

$$\Rightarrow -1 = \frac{2a+b}{(2-1)(2-4)}$$

$$\Rightarrow 2a+b=2$$

$$\text{Now, } f'(x) = \frac{4a+5b-2bx-ax^2}{(x-1)^2(x-4)^2}, f'(2)=0$$

$$\Rightarrow b=0 \Rightarrow a=1$$

$$\Rightarrow f'(x) = -\frac{(x-2)(x+2)}{(x-1)^2(x-4)^2}$$

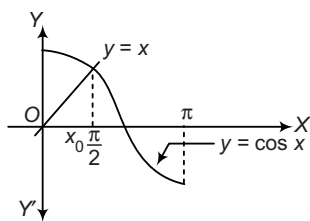
Clearly, for $x > 2$, $f'(x) < 0$ and for $x < 2$, $f'(x) > 0$. Thus, $x = 2$ is indeed the point of local maxima for $y = f(x)$.

Hence, (b) is the correct answer.

Ex 7. If $f(x) = \frac{x}{1+x \tan x}$, $x \in \left(0, \frac{\pi}{2}\right)$, then

- (a) $f(x)$ has exactly one point of minima
(b) $f(x)$ has exactly one point of maxima
(c) $f(x)$ is increasing in $\left(0, \frac{\pi}{2}\right)$
(d) $f(x)$ is decreasing in $\left(0, \frac{\pi}{2}\right)$

$$\begin{aligned} \text{Sol. } f'(x) &= \frac{1-x^2 \sec^2 x}{(1+x \tan x)^2} \\ &= \frac{\sec^2 x (\cos x + x)(\cos x - x)}{(1+x \tan x)^2} \end{aligned}$$



Clearly, $f'(x_0) = 0$

and $f'(x) > 0, \forall x \in (0, x_0)$

$$f'(x) < 0, \forall x \in \left(x_0, \frac{\pi}{2}\right)$$

Thus, $x = x_0$ is the only point of local maxima for $y = f(x)$.

Hence, (b) is the correct answer.

Ex 8. If $f(x) = a \sec x - b \tan x$, $a > b > 0$, then the minimum value of $f(x)$ is

- (a) $a^2 + b^2$ (b) $a^2 - b^2$
(c) $\sqrt{a^2 + b^2}$ (d) $\sqrt{a^2 - b^2}$

$$\text{Sol. } f(x) = \frac{a - b \sin x}{\cos x}$$

$$\Rightarrow f'(x) = \frac{a \sin x - b}{\cos^2 x}, f'(x) = 0$$

$$\Rightarrow x = \sin^{-1}\left(\frac{b}{a}\right) \text{ or } x = \pi - \sin^{-1}\left(\frac{b}{a}\right)$$

Sign scheme of $f'(x)$ is given below:

$$\begin{array}{c} - \quad + \quad - \\ \sin^{-1}\left(\frac{b}{a}\right) \quad \pi - \sin^{-1}\left(\frac{b}{a}\right) \end{array}$$

Clearly, $x = \sin^{-1}\left(\frac{b}{a}\right)$ is the point of minima for $y = f(x)$.

When $x = \sin^{-1}\left(\frac{b}{a}\right)$,

$$\sec x = \frac{a}{\sqrt{a^2 - b^2}}, \tan x = \frac{b}{\sqrt{a^2 - b^2}}$$

$$\Rightarrow f(x) = \sqrt{a^2 - b^2}$$

Hence, (d) is the correct answer.

Ex 9. A straight line is drawn through the point $P(3, 4)$ meeting the positive direction of coordinate axes at the points A and B . If O is the origin, then minimum area of ΔOAB is

- (a) 12 sq units (b) 6 sq units
(c) 24 sq units (d) 48 sq units

Sol. Let the equation of drawn line be $\frac{x}{a} + \frac{y}{b} = 1$, where

$a > 3, b > 4$, as the line passes through $(3, 4)$ and meets the positive direction of coordinate axes.

$$\text{We have, } \frac{3}{a} + \frac{4}{b} = 1 \Rightarrow b = \frac{4a}{(a-3)}$$

Now, area of ΔOAB ,

$$\Delta = \frac{1}{2} ab = \frac{2a^2}{(a-3)}$$

$$\Rightarrow \frac{d\Delta}{da} = \frac{2a(a-6)}{(a-3)^2}$$

Clearly, $a = 6$ is the point of minima for Δ .

$$\text{Thus, } \Delta_{\min} = \frac{2 \times 36}{3} = 24 \text{ sq units}$$

Hence, (c) is the correct answer.

Ex 10. The function $f(x) = |ax - b| + c|x|$, $\forall x \in (-\infty, \infty)$, where $a > 0, b > 0, c > 0$, assumes its minimum value only at one point, if

- (a) $a \neq b$
(b) $a \neq c$
(c) $b \neq c$
(d) $a = b = c$

Sol.
$$f(x) = \begin{cases} b - (a + c)x, & x < 0 \\ b + (c - a)x, & 0 \leq x < \frac{b}{a} \\ (a + c)x + b, & x \geq \frac{b}{a} \end{cases}$$

(i) $c < a$ (ii) $c > a$ (iii) $c = a$

These figures clearly indicate that for exactly one point of minima, $a \neq c$.

Hence, (b) is the correct answer.

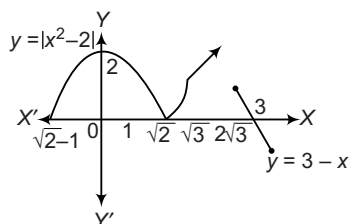
Ex 11. Let
$$f(x) = \begin{cases} |x^2 - 2|, & -1 \leq x < \sqrt{3} \\ \frac{x}{\sqrt{3}}, & \sqrt{3} \leq x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \leq x \leq 4 \end{cases}$$

Then, the points where $f(x)$ takes maximum and minimum values, are

- (a) 1, 4 (b) 0, 4
(c) 2, 4 (d) None of these

Sol.
$$f(x) = \begin{cases} |x^2 - 2|, & -1 \leq x < \sqrt{3} \\ \frac{x}{\sqrt{3}}, & \sqrt{3} \leq x < 2\sqrt{3} \\ 3 - x, & 2\sqrt{3} \leq x \leq 4 \end{cases}$$

From the graph,



Maximum occurs at $x = 0$ and minimum at $x = 4$.

Hence, (b) is the correct answer.

Ex 12. The maximum value of function

$f(x) = \sin x (1 + \cos x), x \in R$ is

- (a) $\frac{3^{3/2}}{4}$ (b) $\frac{3^{5/3}}{4}$
(c) $\frac{3}{2}$ (d) $\frac{3^{7/5}}{4}$

Sol. Given, $f(x) = \sin x (1 + \cos x)$

$$\Rightarrow f(x) = \sin x + \frac{1}{2} \sin 2x$$

On differentiating w.r.t. x , we get

$$f'(x) = \cos x + \cos 2x$$

On putting $f'(x) = 0$,

$$\cos x + \cos 2x = 0$$

$$\Rightarrow 2 \cos \left(\frac{3x}{2} \right) \cos \left(\frac{x}{2} \right) = 0$$

$$\Rightarrow \cos \frac{x}{2} = 0, \cos \frac{3x}{2} = 0$$

$$\Rightarrow x = \pi, x = \frac{\pi}{3}$$

Now, $f''(x) = -\sin x - 2 \sin 2x$

At $x = \frac{\pi}{3}$, $f''(x) = -\sin \frac{\pi}{3} - 2 \sin \frac{2\pi}{3}$

$$= -\frac{\sqrt{3}}{2} - \sqrt{3} < 0 \quad [\text{maxima}]$$

$$\therefore \text{Maximum value, } f\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right) \left(1 + \cos \frac{\pi}{3}\right)$$

$$= \frac{\sqrt{3}}{2} \left(1 + \frac{1}{2}\right)$$

$$= \frac{3\sqrt{3}}{4} = \frac{3^{3/2}}{4}$$

Hence, (a) is the correct answer.

Ex 13. The minimum value of $e^{(2x^2 - 2x + 1) \sin^2 x}$ is

- (a) 0 (b) 1
(c) 2 (d) 3

Sol. Given, $y = e^{(2x^2 - 2x + 1) \sin^2 x}$

For minimum or maximum, put $\frac{dy}{dx} = 0$.

$$\therefore e^{(2x^2 - 2x + 1) \sin^2 x} [(4x - 2) \sin^2 x$$

$$+ 2(2x^2 - 2x + 1) \sin x \cos x] = 0$$

$$\Rightarrow (4x - 2) \sin^2 x + 2(2x^2 - 2x + 1) \sin x \cos x = 0$$

$$\Rightarrow 2 \sin x [(2x - 1) \sin x + (2x^2 - 2x + 1) \cos x] = 0$$

$$\Rightarrow \sin x = 0$$

So, y is minimum for $\sin x = 0$.

$$\therefore \text{Minimum value of } y = e^{(2x^2 - 2x + 1) (0)} = e^0 = 1$$

Hence, (b) is the correct answer.

Aliter

Let $f(x) = e^{(2x^2 - 2x + 1) \sin^2 x}$

Since, $\sin^2 x \geq 0$

and $2x^2 - 2x + 1 \geq \frac{4(2)(1) - 2^2}{4(2)}$

$$\left[\because \text{minima} = \frac{4ac - b^2}{4a} \right]$$

$$\Rightarrow 2x^2 - 2x + 1 \geq \frac{1}{2}$$

$$\therefore (2x^2 - 2x + 1) \sin^2 x \geq 0$$

$$\therefore f(x)_{\min} = e^0 = 1$$

Ex 14. If the function $f(x) = x^2 e^{-2x}$, $x > 0$, then the maximum value of $f(x)$ is

- (a) $\frac{1}{e}$ (b) $\frac{1}{2e}$ (c) $\frac{1}{e^2}$ (d) $\frac{4}{e^4}$

Sol. Given, $f(x) = x^2 e^{-2x}$

$$\therefore f'(x) = 2x e^{-2x} - 2x^2 e^{-2x} = 2x(1 - x) e^{-2x}$$

For maxima or minima, put $f'(x) = 0$.

$$2x(1 - x) e^{-2x} = 0$$

$$\Rightarrow x = 0, 1$$

$$\text{Now, } f''(x) = 2x(-1)e^{-2x} + 2(1-x)e^{-2x} - 2 \cdot 2x(1-x)e^{-2x}$$

$$f''(0) = 0 + 2e^0 = 2 > 0 \quad [\text{minima}]$$

$$\text{and } f''(1) = -2e^{-2} + 0 - 0 = -\frac{2}{e^2} < 0 \quad [\text{maxima}]$$

$$\text{Thus, maximum value is } f(1) = 1 \cdot e^{-2} = \frac{1}{e^2}$$

Hence, (c) is the correct answer.

Ex 15. A missile is fired from the ground level rises x metre vertically upwards in t s, where $x = 100t - \frac{25}{2}t^2$. The maximum height reached is

- (a) 200 m
(b) 125 m
(c) 160 m
(d) 190 m

Sol. We have, $x = 100t - \frac{25}{2}t^2$

On differentiating both sides w.r.t. t , we get

$$\frac{dx}{dt} = 100 - 25t \quad (2t) = 100 - 25t$$

At maximum height, velocity $\frac{dx}{dt} = 0$

$$\Rightarrow 100 - 25t = 0$$

$$\Rightarrow t = 4$$

$$\therefore x = 100 \times 4 - \frac{25 \times 16}{2} = 200 \text{ m}$$

Hence, (a) is the correct answer.

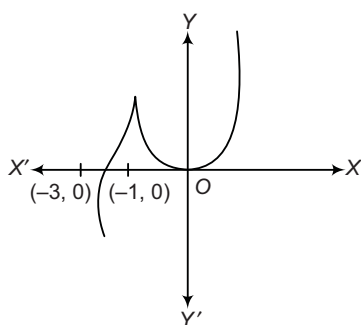
Ex 16. The total number of local maxima and local minima of the function

$$f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 < x < 2 \end{cases} \text{ is}$$

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. Given that, $f(x) = \begin{cases} (2+x)^3, & -3 < x \leq -1 \\ x^{2/3}, & -1 < x < 2 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} 3(2+x)^2, & -3 < x \leq -1 \\ \frac{2}{3}x^{-1/3}, & -1 < x < 2 \end{cases}$$



Clearly, $f'(x)$ changes its sign at $x = -1$ from +ve to -ve, so $f(x)$ has local maxima at $x = -1$.

Also, $f'(0)$ does not exist but $f'(0^-) < 0$ and $f'(0^+) > 0$. It can only be inferred that $f(x)$ has a possibility of a minima at $x = 0$. Therefore, the given function has one local maxima at $x = -1$ and one local minima at $x = 0$.

Hence, (c) is the correct answer.

Ex 17. If $f(x) = x^2 + 2bx + 2c^2$ and

$$g(x) = -x^2 - 2cx + b^2 \text{ such that}$$

$\min f(x) > \max g(x)$, then the relation between b and c is

- (a) no real value of b and c
(b) $0 < c < b\sqrt{2}$
(c) $|c| < |b|\sqrt{2}$
(d) $|c| > |b|\sqrt{2}$

Sol. Given, $f(x) = x^2 + 2bx + 2c^2$

$$\text{and } g(x) = -x^2 - 2cx + b^2$$

Then, $f(x)$ is minimum and $g(x)$ is maximum at

$$x = -\frac{b}{2a} \text{ and } x = -\frac{D}{2A}, \text{ respectively.}$$

$$\therefore \min f(x) = \frac{-(4b^2 - 8c^2)}{4} = (2c^2 - b^2)$$

$$\text{and } \max g(x) = -\frac{(4c^2 + 4b^2)}{4(-1)} = (b^2 + c^2)$$

Now, $\min f(x) > \max g(x)$

$$\Rightarrow 2c^2 - b^2 > b^2 + c^2$$

$$\Rightarrow c^2 > 2b^2$$

$$\Rightarrow |c| > \sqrt{2}|b|$$

Hence, (d) is the correct answer.

Ex 18. If θ and ϕ are acute angles related by

$$a \sec \theta + b \sec \phi = c; a, b, c > 0, \text{ then}$$

$$a \cos \theta + b \cos \phi \text{ is}$$

- (a) minimum when $\theta = \phi$
(b) maximum when $\theta = \phi$
(c) minimum when $\theta \neq \phi$
(d) maximum when $\theta \neq \phi$

Sol. We know that, for positive $a \cos \theta$ and $b \cos \phi$,

$$a \cos \theta + b \cos \phi \geq 2(a \cos \theta \cdot b \cos \phi)^{1/2} \dots (i)$$

$$\text{Also, } a \sec \theta + b \sec \phi \geq 2(a \sec \theta \cdot b \sec \phi)^{1/2} \dots (ii)$$

$$\text{i.e. } c \geq 2(a \sec \theta \cdot b \sec \phi)^{1/2} \dots (iii)$$

On multiplying Eqs. (i) and (iii), we get

$$a \cos \theta + b \cos \phi \geq \frac{4ab}{c}$$

Hence, the minimum value of

$$a \cos \theta + b \cos \phi = \frac{4ab}{c}$$

For $a \cos \theta = b \cos \phi$ and $a \sec \theta = b \sec \phi$,

$$a^2 = b^2$$

$$\Rightarrow a = b$$

$$\Rightarrow \cos \theta = \cos \phi$$

$$\Rightarrow \theta = \phi$$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 19. The function $\frac{\sin(x+a)}{\sin(x+b)}$ has no maxima or minima, if

- (a) $b - a = n\pi, n \in I$
 (b) $b - a = (2n+1)\pi, n \in I$
 (c) $b - a = 2n\pi, n \in I$
 (d) None of the above

Sol. $f(x) = \frac{\sin(x+a)}{\sin(x+b)}$

$$\Rightarrow f'(x) = \frac{[\sin(x+b) \times \cos(x+a)] - [\sin(x+a) \cos(x+b)]}{\sin^2(x+b)} = \frac{\sin(b-a)}{\sin^2(x+b)}$$

If $\sin(b-a) = 0$, then $f'(x) = 0$

$\Rightarrow f(x)$ will be constant.

i.e. $b - a = n\pi$
 or $b - a = (2n+1)\pi$ or $b - a = 2n\pi$

Then, $f(x)$ has no minima.

Hence, (a), (b) and (c) are the correct answers.

Ex 20. The parabola $y = x^2 + px + q$ cuts the straight line $y = 2x - 3$ at a point with abscissa 1. If the distance between the vertex of the parabola and X -axis is least, then

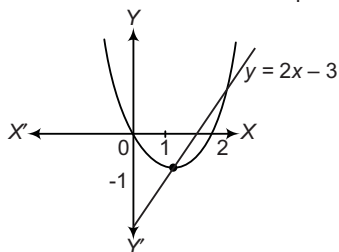
- (a) $p = 0$ and $q = -2$
 (b) $p = -2$ and $q = 0$
 (c) least distance between the parabola and X -axis is 2
 (d) least distance between the parabola and X -axis is 1

Sol. When $x = 1, y = -1$ [from the line]
 Thus, it must lie on the parabola $y^2 = x^2 + px + q$
 $\Rightarrow -1 = 1 + p + q \Rightarrow p + q = -2$
 Now, distance of the vertex of the parabola from X -axis is

$$d = f\left(-\frac{p}{2}\right) = \frac{p^2}{4} - \frac{p^2}{2} + q = q - \frac{p^2}{4} \quad \dots(i)$$

On substituting $q = -2 - p$ in Eq.(i), we get

$$f(p) = -2 - p - \frac{p^2}{4}$$



$$\therefore f'(p) = -1 - \frac{p}{2} = 0 \Rightarrow p = -2$$

So, $q = 0$

Note that least distance of the vertex from X -axis is 1.

Hence, (b) and (d) are the correct answers.

Ex 21. The extremum values of the function $f(x) = \frac{1}{\sin x + 4} - \frac{1}{\cos x - 4}$, where $x \in R$, are

- (a) $\frac{4}{8-\sqrt{2}}$ (b) $\frac{2\sqrt{2}}{8-\sqrt{2}}$ (c) $\frac{2\sqrt{2}}{4\sqrt{2}+1}$ (d) $\frac{4}{8+\sqrt{2}}$

Sol. $f'(x) = 0$
 $\Rightarrow (\sin x + \cos x) \text{ (non-zero quantity)} = 0$
 $\Rightarrow \tan x = -1$
 $\Rightarrow x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}$

Global minimum at $x = 2n\pi + \left(\frac{3\pi}{4}\right)$

Global maximum at $x = 2n\pi + \left(\frac{7\pi}{4}\right)$

$$M = \frac{4}{8-\sqrt{2}}, m = \frac{4}{8+\sqrt{2}}$$

Hence, (a) and (d) are the correct answers.

Ex 22. The function

$$f(x) = \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$$

has a local minimum at x , equals to

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. $f(x) = \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$



$$\Rightarrow f'(x) = \frac{d}{dx} \int_{-1}^x t(e^t - 1)(t-1)(t-2)^3(t-3)^5 dt$$

$$= x(e^x - 1)(x-1)(x-2)^3(x-3)^5 \times 1$$

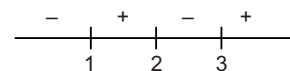
$$\left[\because \frac{d}{dx} \int_{\phi(x)}^{\psi(x)} f(t) dt = f\{\psi(x)\} \psi'(x) - f\{\phi(x)\} \phi'(x) \right]$$

For local minimum, $f'(x) = 0$

$$\Rightarrow x = 0, 1, 2, 3$$

$$\text{Let } f'(x) = g(x) = x(e^x - 1)(x-1)(x-2)^3(x-3)^5$$

Using sign scheme rule,



This shows that $f(x)$ has a local minimum at $x = 1$ and $x = 3$ and maximum at $x = 2$.

Hence, (b) and (d) are the correct answers.

Ex 23. Let $f(x)$ be cubic polynomial which has local maximum at $x = -1$. If $f(2) = 18$, $f(1) = -1$ and $f'(x)$ has local minimum at $x = 0$, then

- (a) the distance between $(-1, 2)$ and $(a, f(a))$, where $x = a$ is the point of local minima, is $2\sqrt{5}$
 (b) $f(x)$ is increasing for $x \in [1, 2\sqrt{5}]$
 (c) $f(x)$ has local minima at $x = 1$
 (d) the value of $f(0) = 5$

Sol. Since, $f(x)$ has local maxima at $x = -1$ and $f'(x)$ has local minima at $x = 0$.

$$\therefore \begin{aligned} f''(x) &= \lambda x \\ f'(x) &= \lambda \frac{x^2}{2} + c \quad [f'(-1) = 0] \end{aligned}$$

$$\Rightarrow \frac{\lambda}{2} + c = 0 \Rightarrow \lambda = -2c \quad \dots(i)$$

Again, integrating both sides, we get

$$f(x) = \lambda \frac{x^3}{6} + cx + d \quad \dots(ii)$$

$$\begin{aligned} f(2) &= \lambda \left(\frac{8}{6} \right) + 2c + d \\ &= 18 \end{aligned}$$

$$\text{and } f(1) = \frac{\lambda}{6} + c + d = -1 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\begin{aligned} f(x) &= \frac{1}{4} (19x^3 - 57x + 34) \\ \therefore f'(x) &= \frac{1}{4} (57x^2 - 57) \\ &= \frac{57}{4} (x-1)(x+1), \text{ using number line rule} \end{aligned}$$

So, $f(x)$ is increasing for $[1, 2\sqrt{5}]$ and $f(x)$ has local maximum at $x = -1$ and $f(x)$ has local minimum at $x = 1$.

$$\text{Also, } f(0) = \frac{34}{4}$$

Hence, (b) and (c) are the correct answers.

$$\text{Ex 24. If } f(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$$

and $g(x) = \int_0^x f(t)dt$, $x \in [1, 3]$, then

- $g(x)$ has local maxima at $x = 1 + \log_e 2$ and local minima at $x = e$
- $f(x)$ has local maxima at $x = 1$ and local minima at $x = 2$
- $g(x)$ has no local minima
- $f(x)$ has no local maxima

$$\text{Sol. Here, } f(x) = \begin{cases} e^x, & 0 \leq x \leq 1 \\ 2 - e^{x-1}, & 1 < x \leq 2 \\ x - e, & 2 < x \leq 3 \end{cases}$$

$$\text{and } g(x) = \int_0^x f(t)dt \Rightarrow g'(x) = f(x)$$

where, $g'(x) = 0 \Rightarrow x = 1 + \log_e 2$ and $x = e$

$$\text{Also, } g''(x) = \begin{cases} -e^{x-1}, & 1 < x \leq 2 \\ 1, & 2 < x \leq 3 \end{cases}$$

$g''(1 + \log_e 2) = -e^{\log_e 2} < 0$, hence at $x = 1 + \log_e 2$, $g(x)$ has a local maxima. Also, $g''(e) = 1 > 0$, hence at $x = e$, $g(x)$ has a local minima. Since, $f(x)$ is discontinuous at $x = 1$, so we get local maxima at $x = 1$ and local minima at $x = 2$.

Hence, (a) and (b) are the correct answers.

Type 3. Assertion and Reason

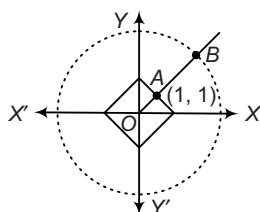
Directions (Ex. Nos. 25-28) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
- Statement I is true, Statement II is false
- Statement I is false, Statement II is true

Ex 25. Statement I Shortest distance between $|x| + |y| = 2$ and $x^2 + y^2 = 16$ is $4 - \sqrt{2}$.

Statement II Shortest distance between the two smooth curves lies along the common normal.

Sol. Common normal is $y = x$.



On solving $x + y = 2$, we get (1, 1) and on solving $x^2 + y^2 = 16$, we get $(2\sqrt{2}, 2\sqrt{2})$

The distance between them is $AB = 4 - \sqrt{2}$.

But as curves are not smooth, check at slope points. The coordinates in 1st quadrant are (2, 0) and (4, 0) and here distance = 2

$\therefore 4 - \sqrt{2}$ is not shortest.

Hence, (d) is the correct answer.

Ex 26. Statement I $f(x) = xe^{-x}$ has maximum at $x = 1$.

Statement II $f'(1) = 0$ and $f''(1) < 0$.

Sol. Given, $f(x) = xe^{-x}$

$$\begin{aligned} \therefore f'(x) &= e^{-x} - xe^{-x} \\ \text{and } f''(x) &= -e^{-x} - e^{-x} + xe^{-x} \\ &= -2e^{-x} + xe^{-x} \end{aligned}$$

For maximum, put $f'(x) = 0$

$$\Rightarrow x = 1$$

$$\text{and } f''(1) = -1 < 0$$

So, both the statements are true and Statement II is the correct explanation for Statement I.

Hence, (a) is the correct answer.

Ex 27. Statement I The absolute maximum value of $f(x) = \frac{1}{3x^4 + 8x^3 - 18x^2 + 60}$ is $\frac{1}{53}$.

Statement II The minimum value of $f_1(x) = 1/f(x)$ is at $x = 1$ and $x = -3$.

Sol. $f_1(x) = 3x^4 + 8x^3 - 18x^2 + 60$
 $\Rightarrow f'_1(x) = 12x^3 + 24x^2 - 36x$
 $= 12x(x^2 + 2x - 3)$
 $= 12x(x-1)(x+3)$
 and $f''_1(x) = 12(3x^2 + 4x - 3)$

The critical points of $f_1(x)$ are

$$x_1 = -3, x_2 = 0, x_3 = 1$$

$$\Rightarrow f''_1(-3) > 0, f''_1(0) = -36, f''_1(1) = 48 > 0$$

So, $f_1(x)$ has minimum at $x = -3$ and $x = 1$

$f(x)$ has maximum at $x = -3$.

Also, $f(x)$ has maximum at $x = 1$ and $f(1) = \frac{1}{53}$.

So, the absolute maximum of f is $\frac{1}{53}$.

Hence, (a) is the correct answer.

Ex 28. Let $f(x) = 2 \sin x + \tan x - 3x$

Statement I $f(x)$ has no maximum and minimum on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Statement II f increases on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Sol. The derivative of f is

$$\begin{aligned} f'(x) &= 2 \cos x + \frac{1}{\cos^2 x} - 3 \\ &= \frac{(1 - \cos x)(1 + \cos x - 2 \cos^2 x)}{\cos^2 x} \\ &= \frac{4 \sin^3\left(\frac{x}{2}\right) \sin\left(\frac{3x}{2}\right)}{\cos^2 x} > 0 \end{aligned}$$

in the interval $(-\pi/2, 0)$ and $(0, \pi/2)$ and vanishes only at $x = 0$.

Therefore, f increases on $(-\pi/2, \pi/2)$, so f has no maximum and minimum.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 29-33) The absolute maximum and minimum values of functions can be found by their monotonic and asymptotic behaviour provided they exist. We may agree that finite limiting values may be regarded as absolute maximum or minimum. For instance the absolute maximum value of $\frac{1}{1+x^2}$ is

unity. It is attained at $x = 0$ while absolute minimum value of the same function is zero which is a limiting value of $x \rightarrow \infty$ or $x \rightarrow -\infty$.

Ex 29. The function $x^4 - 4x + 1$ will have

- (a) absolute maximum value
- (b) absolute minimum value
- (c) both absolute maximum and minimum values
- (d) None of the above

Sol. Since, $f'(x) = 4x^3 - 4 = 0$

$$\Rightarrow x = 1$$

So, there is only extremum which is minima.

$\Rightarrow 1$ is a point of absolute minima.

Hence, (b) is the correct answer.

Ex 30. The absolute minimum value of the function

$$\frac{x-2}{\sqrt{x^2+1}} \text{ is}$$

- (a) -1
- (b) $\frac{1}{2}$
- (c) $-\sqrt{5}$
- (d) None of these

Sol. Since, $f'(x) = \frac{1+2x}{(x^2+1)^{3/2}}$

$\Rightarrow f(x)$ is increasing in $\left(-\frac{1}{2}, \infty\right)$ and decreasing in $\left(-\infty, -\frac{1}{2}\right)$.

$\Rightarrow$ Absolute minimum occurs at $x = -\frac{1}{2}$, we have the minimum value $= -\sqrt{5}$

Hence, (c) is the correct answer.

Ex 31. The absolute minimum and maximum values

of the function $\frac{x^2 - x + 1}{x^2 + x + 1}$ are

- (a) 1 and 3
- (b) $\frac{1}{2}$ and 3
- (c) $\frac{1}{3}$ and 3
- (d) None of the above

Sol. Let $y = \frac{x^2 - x + 1}{x^2 + x + 1}$

$$yx^2 + yx + y = x^2 - x + 1$$

$$\Rightarrow x^2(1-y) - x(1+y) + (1-y) = 0$$

$$D \geq 0$$

$$\Rightarrow (1+y)^2 - 4(1-y)^2 \geq 0$$

$$\Rightarrow -(3y-1)(y-3) \geq 0$$

$$\Rightarrow \frac{1}{3} \leq y \leq 3$$

Hence, (c) is the correct answer.

Ex 32. The function $f(x) = \frac{4}{x-1} - \frac{9}{x+1}$ will

- (a) have absolute maximum value $-\frac{1}{2}$
- (b) have absolute minimum value $-\frac{25}{2}$
- (c) not lie between $-\frac{25}{2}$ and $-\frac{1}{2}$
- (d) always be negative

Sol. Option (c) should be the correct choice because we can prove by using monotonically that f cannot lie between $-\frac{25}{2}$ and $-\frac{1}{2}$. This is an example of a function whose maximum (local) value is smaller than minimum value.
Hence, (c) is the correct answer.

Ex 33. Which of the following functions will have absolute minimum value?

- (a) $\cot(\sin x)$
- (b) $\tan(\log x)$
- (c) $x^{2005} - x^{1947} + 1$
- (d) $x^{2006} + x^{1947} + 1$

Sol. As even degree polynomial will have absolute minimum essentially.
Hence, (d) is the correct answer.

Passage II (Ex. Nos. 34-36) Consider, the function $f: (-\infty, \infty) \rightarrow (-\infty, \infty)$ defined by

$$f(x) = \frac{x^2 - ax + 1}{x^2 + ax + 1}, 0 < a < 2$$

Ex 34. Which one of the following is true?

- (a) $(2+a)^2 f''(1) + (2-a)^2 f''(-1) = 0$
- (b) $(2-a)^2 f''(1) - (2+a)^2 f''(-1) = 0$
- (c) $f'(1)f'(-1) = (2-a)^2$
- (d) $f'(1)f'(-1) = -(2+a)^2$

Sol. $\because f(x) = \frac{(x^2 + ax + 1) - 2ax}{x^2 + ax + 1}$
 $= 1 - \frac{2ax}{x^2 + ax + 1}$
 $\therefore f'(x) = -\left[\frac{(x^2 + ax + 1)2a - 2ax(2x + a)}{(x^2 + ax + 1)^2} \right]$
 $= -\left[\frac{-2ax^2 + 2a}{(x^2 + ax + 1)^2} \right] = 2a \left[\frac{(x^2 - 1)}{(x^2 + ax + 1)^2} \right]$... (i)

$$\text{and } f''(x) = 2a \left[\frac{(x^2 + ax + 1)^2 (2x) - 2(x^2 - 1)}{(x^2 + ax + 1)^4} \right] \dots (ii)$$

Now, $f''(1) = \frac{4a(a+2)}{(a+2)^3} = \frac{4a}{(a+2)^2}$
 and $f''(-1) = \frac{4a(a-2)}{(2-a)^3} = -\frac{4a}{(a-2)^2}$
 $\therefore (2+a)^2 f''(1) + (2-a)^2 f''(-1) = 4a - 4a = 0$
 Hence, (a) is the correct answer.

Ex 35. Which one of the following is true?

- (a) $f(x)$ is decreasing on $(-1, 1)$ and has a local minimum at $x = 1$
- (b) $f(x)$ is increasing on $(-1, 1)$ and has a local maximum at $x = 1$
- (c) $f(x)$ is increasing on $(-1, 1)$ but it has neither a local maximum nor a local minimum at $x = 1$
- (d) $f(x)$ is decreasing on $(-1, 1)$ but it has neither a local maximum nor a local minimum at $x = 1$

Sol. When $x \in (-1, 1)$, then

$$x^2 < 1 \Rightarrow x^2 - 1 < 0$$

$\therefore f'(x) < 0 \Rightarrow f(x)$ is decreasing.

$$\text{Also, at } x = 1, f''(1) = \frac{4a}{(a+2)^2} > 0 \quad [\because 0 < a < 2]$$

So, $f(x)$ has a local minimum at $x = 1$.

Hence, (a) is the correct answer.

Ex 36. Let $g(x) = \int_0^x \frac{f'(t)}{1+t^2} dt$, then which one of the following is true?

- (a) $g'(x)$ is positive on $(-\infty, 0)$ and negative on $(0, \infty)$
- (b) $g'(x)$ is negative on $(-\infty, 0)$ and positive on $(0, \infty)$
- (c) $g'(x)$ changes sign on both $(-\infty, 0)$ and $(0, \infty)$
- (d) $g'(x)$ does not change sign on $(-\infty, \infty)$

Sol. $\because g'(x) = \frac{f'(e^x)}{1+(e^x)^2} \cdot e^x$
 $= 2a \left[\frac{e^{2x} - 1}{(e^{2x} + ae^x + 1)^2} \right] \left(\frac{e^x}{1+e^{2x}} \right)$

$$g'(x) = 0, \text{ if } e^{2x} - 1 = 0, \text{ i.e. } x = 0$$

$$\text{If } x < 0, e^{2x} < 1 \Rightarrow g'(x) < 0$$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 37. Match the statements of Column I with values of Column II.

Column I	Column II
A. The least value of the function $f(x) = 2 \cdot 3^{3x} - 3^{2x} \cdot 4 + 2 \cdot 3^x$ in $[-1, 1]$ is	p. 5
B. The minimum value of the polynomial $f(x) = (x-1) \times (x+1)$ is	q. -1
C. The value of the polynomial $\int_{-1}^3 (x-2 - [x]) dx$, where $[]$ denotes the greatest integer function, is	r. 3
D. If period of the function $f(x) = \sin 36x \tan 42x$ is p , then $\frac{18p}{\pi}$ equals	s. 0

Sol. A. Let $3^x = t \Rightarrow Q(t) = 2t^3 - 4t^2 + 2t$ in $t \in \left[\frac{1}{3}, 3\right]$

$$\Rightarrow Q'(t) < 0 \text{ in } \left(\frac{1}{3}, 1\right) \text{ and } Q'(t) > 0 \text{ in } (1, 3)$$

$$\Rightarrow f(x)_{\min} = Q(1) = 0$$

B. Put $x+1 = t \Rightarrow x = t-1$

$$\therefore Q(t) = t(t-2) \Rightarrow Q(t)_{\min} = -1$$

$$C. \int_{-1}^3 |x-2| dx = 5$$

$$\text{and } \int_{-1}^3 [x] dx = 2 \Rightarrow I = 3$$

$$D. \text{ Period of } \sin 36x = \frac{\pi}{18}$$

$$\text{and period of } \tan 42x = \frac{\pi}{42} \Rightarrow p = \frac{\pi}{6}$$

$$\therefore \frac{18p}{\pi} = \frac{18 \times \pi/6}{\pi} = 3$$

$$A \rightarrow s; B \rightarrow q; C \rightarrow r; D \rightarrow r$$

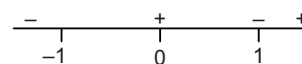
Ex 38. Match the statements of Column I with values of Column II.

Column I	Column II
A. $f(x) = \int_0^x 8t(t-1)(t+1)dt$ is/has in $(-1, 1)$	p. local maxima
B. $f(x) = \begin{cases} \sin\left(\frac{\pi x}{4}\right), & x \leq 2 \\ 9-4x, & x > 2 \end{cases}$ is/has in $(0, 2)$	q. local minima
C. $f(x) = \{2x\}$, where $\{ \}$ denotes fractional part of x , in $(0, 1)$, is/has	r. continuous
D. If $f(x) = \begin{cases} x-2 - 2, & x < 2 \\ [x], & x \geq 2 \end{cases}$ (where, $[]$ denotes the greatest integer function, then $f(x)$ is/has in $(-1, 4)$	s. non-differentiable

Sol. A. We have, $f(x) = \int_0^x 8t(t-1)(t+1)dt$

$$\Rightarrow f'(x) = 8x(x-1)(x+1)$$

$$f'(x) = 0 \text{ at } x=0, 1, -1$$



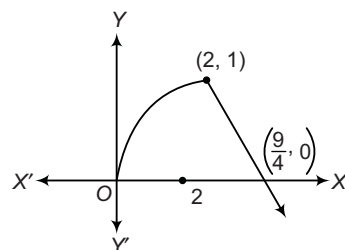
At $x=0$ sign change +ve to -ve

$\therefore$ Local maximum at $x=0$

and it is continuous function for $x \in (-1, 1)$.

$$B. \text{ We have, } f(x) = \begin{cases} \frac{\sin \pi x}{4}, & x < 2 \\ 9-4x, & x > 2 \end{cases}$$

Graph of $f(x)$ is in $(0, 2)$



Clearly from graph,

$f(x)$ is continuous in $(0, 2)$

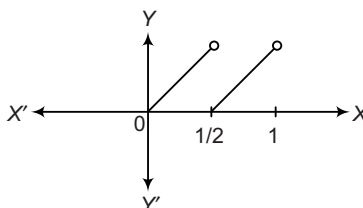
Local maximum at $x=2$

$\therefore$ Sharp edges at $x=2$

$\therefore$ It is non-differentiable.

C. We have, $f(x) = \{2x\}$, $\{x\}$ is fractional part in x

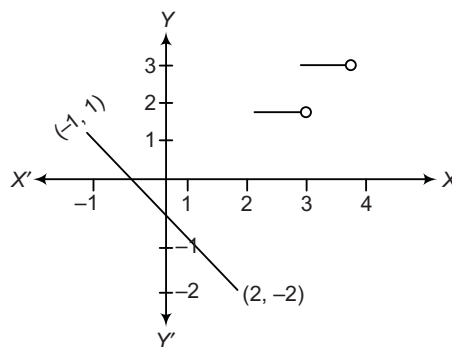
Graph of $f(x)$



In $(0,1)$ non-differentiable, non-continuous and local minimum at $x=\frac{1}{2}$.

$$D. \text{ We have, } f(x) = \begin{cases} [x-2] - 2, & x < 2 \\ [x], & x \geq 2 \end{cases}$$

Graph of $f(x)$ is



Clearly, from graph, local minimum at $x=2$ and non-differentiable in $(-1, 4)$.

$A \rightarrow p, r; B \rightarrow p, r, s; C \rightarrow q, s; D \rightarrow q, s$

Ex 39. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $\theta + \phi = \frac{\pi}{3}$, then $\sin \theta \cdot \sin \phi$ has a maximum value	p. 2
B. Let $f(x) = (x-1)^2 + (x-2)^2 + (x-3)^2$. Then, $f(x)$ has a minimum at $x = \lambda$, where λ is	q. $3\sqrt[3]{36}$
C. The sum of two non-zero numbers is 8. The minimum value of the sum of their reciprocals is	r. $\frac{1}{4}$
D. If λ and μ are real numbers such that $x^3 - \lambda x^2 + \mu x - 6 = 0$ has its roots real and positive, then the minimum value of μ is	s. $\frac{1}{2}$

Sol. A. Here, $y = \sin \theta \cdot \sin \phi = \sin \theta \cdot \sin \left(\frac{\pi}{3} - \theta \right)$

$$= \frac{1}{2} \left\{ \cos \left(2\theta - \frac{\pi}{3} \right) - \cos \frac{\pi}{3} \right\}$$

$$= \frac{1}{2} \cos \left(2\theta - \frac{\pi}{3} \right) - \frac{1}{4}$$

This has a maximum at $2\theta - \frac{\pi}{3} = 0$

$\therefore$ Maximum value of $\sin \theta \cdot \sin \phi = \frac{1}{4}$

B. $f(x) = 3x^2 - 12x + 14$

$\therefore f'(x) = 6x - 12$

$\Rightarrow f'(x) = 0$

$\therefore x = 2$

Also, $f''(x) = 6 > 0$

C. Here, $a + b = 8$

Let $y = \frac{1}{a} + \frac{1}{b} = \frac{1}{a} + \frac{1}{8-a} = \frac{8}{a(8-a)}$

$\therefore$

$$\frac{dy}{da} = \frac{-8}{a^2(8-a)^2} \cdot (8-2a)$$

$\therefore \frac{dy}{da} = 0$, if $8-2a = 0 \Rightarrow a = 4$

Also, $\left(\frac{dy}{dx} \right)_{a=4-\epsilon} = \frac{-8}{(4-\epsilon)^2 \cdot (4+\epsilon)^2} (2\epsilon) < 0$

$$\left(\frac{dy}{dx} \right)_{a=4+\epsilon} = \frac{-8}{(4+\epsilon)^2 \cdot (4-\epsilon)^2} (-2\epsilon) > 0$$

$\therefore$ At $a = 4$, y is minimum.

So, minimum value is

$$y = \frac{1}{4} + \frac{1}{8-4} = \frac{1}{2}$$

D. Let a, b and c be the roots. Then, $ab + bc + ca = \mu$ and $abc = 6$

On dividing by abc , $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{\mu}{6}$

$\therefore$ AM $\geq$ GM

$$\therefore \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} \geq \sqrt[3]{\frac{1}{abc}}$$

or $\mu \geq 18 \cdot \sqrt[3]{\frac{1}{abc}} = 3 \cdot 6^{2/3}$

$\therefore$ The minimum value of $\mu = 3 \cdot 6^{2/3}$

$A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$

Type 6. Single Integer Answer Type Questions

Ex 40. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x . Then, the largest value which $f(2)$ can attain, is _____.

Sol. (7) Using LMVT in $[0, 2]$,

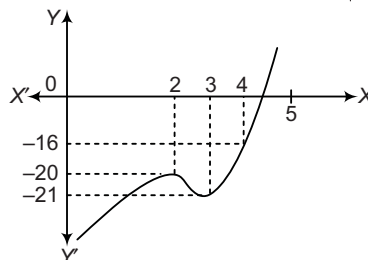
$$\frac{f(2) - f(0)}{2 - 0} = f'(c), \text{ where } c \in (0, 2)$$

$$\Rightarrow \frac{f(2) + 3}{2} \leq 5$$

$$\Rightarrow f(2) \leq 7$$

Ex 41. The maximum value of the function $f(x) = 2x^3 - 15x^2 + 36x - 48$ on the set $A = \{x | x^2 + 20 \leq 9x\}$ is _____.

Sol. (7) Given, $A = \{x | x^2 + 20 \leq 9x\} = \{x | x \in [4, 5]\}$



Now, $f'(x) = 6x^2 - 30x + 36$

Put $f'(x) = 0 \Rightarrow x = 2, 3$

$f(2) = -20, f(3) = -21, f(4) = -16, f(5) = 7$

From graph, maximum value of $f(x)$ on set A is $f(5) = 7$.

Target Exercises

Type 1. Only One Correct Option

- A function f such that $f'(a) = f''(a) = f'''(a) = \dots = f^{(2n)}(a) = 0$ and f has a local maximum value b at $x = a$, if $f(x)$ is
 - $(x - a)^{2n+2}$
 - $b - 1 - (x + 1 - a)^{2n-1}$
 - $b - (x - a)^{2n+2}$
 - $(x - a)^{2n+2} - b$
- If $f(x) = a - (x - 3)^{89}$, then the greatest value of $f(x)$ is
 - 3
 - a
 - no maximum value
 - None of these
- If $P = x^3 - \frac{1}{x^3}$ and $Q = x - \frac{1}{x}, x \in (0, x)$, then minimum value of $\frac{P}{Q^2}$ is
 - $2\sqrt{3}$
 - $-2\sqrt{3}$
 - Does not exist
 - None of these
- Let $f(x) = \begin{cases} x^3 - x^2 + 10x - 5, & x \leq 1 \\ -2x + \log_2(b^2 - 2), & x > 1 \end{cases}$. The set of values of b for which $f(x)$ has greatest value at $x = 1$, is given by
 - $1 \leq b \leq 2$
 - $b = \{1, 2\}$
 - $b \in (-\infty, -1)$
 - None of these
- The maximum value of $\sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$, $0 \leq x \leq \frac{\pi}{2}$, is
 - $1 + 2\sqrt{2}$
 - $2\sqrt{2} - 1$
 - $1 - 2\sqrt{2}$
 - None of these
- If $y = a \log_e x + bx^2 + x$ has its extreme values (i.e. maximum or minimum value) at $x = 1$ and $x = 2$, then the values of a and b are
 - $a = -\frac{1}{6}, b = \frac{4}{3}$
 - $a = -\frac{4}{3}, b = \frac{1}{6}$
 - $a = \frac{4}{3}, b = -\frac{1}{6}$
 - None of these
- If the function $y = \frac{ax + b}{(x - 4)(x - 1)}$ has an extremum at $P(2, -1)$, then the values of a and b are
 - $a = 0, b = 1$
 - $a = 0, b = -1$
 - $a = 1, b = 0$
 - $a = -1, b = 0$
- If $y = \frac{\sin(x + a)}{\sin(x + b)}$; $a \neq b$, then y has
 - maximum at $x = 0$
 - minimum at $x = 0$
 - neither maximum nor minimum
 - None of the above
- For the function $y = x + \frac{1}{x}$
 - $x = 1$ is a point of maximum
 - $x = -1$ is a point of minimum
 - maximum value $>$ minimum value
 - maximum value $<$ minimum value
- If $xz = 1$, where $x > 0$, then the least value of $x + z$ is
 - 1
 - 2
 - 2
 - None of these
- The minimum value of $\log_a x + \log_x a, 0 < x < a$, is
 - 1
 - 2
 - 2
 - None of these
- The minimum value of $2\log_{10} x - \log_x 0.01, x > 1$, is
 - 1
 - 1
 - 2
 - None of the above
- The function $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2] dt$ has
 - maximum at $x = 1$
 - minimum at $x = \frac{7}{5}$
 - neither maximum nor minimum at $x = 2$
 - All of the above
- For a function $y = f(x)$, if $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} = 0$ at a point $x = a$, then
 - y must be maximum at $x = a$
 - y must be minimum at $x = a$
 - y may not have a maximum or minimum at $x = a$
 - it is a constant function
- The fraction exceeding its p th power by the greatest number possible, where $p \geq 2$, is
 - $\left(\frac{1}{p}\right)^{\frac{1}{p-1}}$
 - $\left(\frac{1}{p}\right)^{p-1}$
 - $\frac{1}{p^{p-1}}$
 - None of these

16. If $y = x^2 + ax + b$ has a minimum at $x = 3$ and the minimum value is 5, then the values of a and b are
 (a) $a = 6, b = -14$
 (b) $a = -6, b = 14$
 (c) $a = 14, b = -14$
 (d) $a = -14, b = 6$
17. The greatest value of the function $f(x) = xe^{-x}$ in $[0, \infty)$ is
 (a) 0
 (b) $\frac{1}{e}$
 (c) $-e$
 (d) None of these
18. The least value of the function $f(x) = \cos^{-1} x^2$ in the interval $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ is
 (a) $\frac{\pi}{3}$
 (b) $\frac{\pi}{2}$
 (c) $-\frac{\pi}{3}$
 (d) None of these
19. The greatest value of the function $f(x) = \tan^{-1} x - \frac{1}{2} \log x$ in $\left[\frac{1}{\sqrt{3}}, \sqrt{3}\right]$ is
 (a) $\frac{\pi}{6} + \frac{1}{4} \log 3$
 (b) $\frac{\pi}{6} - \frac{1}{4} \log 3$
 (c) $\frac{\pi}{3} - \frac{1}{4} \log 3$
 (d) $\frac{\pi}{3} + \frac{1}{2} \log 3$
20. The maximum value of the function $y = x(x-1)^2$, is
 (a) 0
 (b) $\frac{4}{27}$
 (c) -4
 (d) None of these
21. The shortest distance of the point $(0, 0)$ from the curve $y = \frac{1}{2}(e^x + e^{-x})$ is
 (a) 2
 (b) 1
 (c) 3
 (d) None of these
22. The greatest height of the graph of the curve $y = 6 \cos x - 8 \sin x$ above the X -axis is
 (a) 4
 (b) 8
 (c) 10
 (d) None of these
23. The minimum value of $a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x, 0 < a < b$, is
 (a) $a + b$
 (b) $(a + b)^2$
 (c) $(a + b)^4$
 (d) None of these
24. The points on the curve $xy^2 = 1$ which are nearest to the origin, are
 (a) $\left[\left(\frac{1}{2}\right)^{1/3}, \pm \left(\frac{1}{2}\right)^{-1/6}\right]$
 (b) $\left[\left(\frac{1}{2}\right)^{1/3}, \pm 2^{-1/6}\right]$
 (c) $\left[2^{1/3}, \pm \left(\frac{1}{2}\right)^{-1/6}\right]$
 (d) None of these
25. N characters of information are held on magnetic tape in batches of x character each; the batch processing time is $\alpha + \beta x^2$ seconds, α and β are constants. The optimum value of x for fast processing is
 (a) $\frac{\alpha}{\beta}$
 (b) $\frac{\beta}{\alpha}$
 (c) $\sqrt{\frac{\alpha}{\beta}}$
 (d) $\sqrt{\frac{\beta}{\alpha}}$
26. AB is a diameter of a circle and C is any point on the circumference of the circle, then
 (a) area of $\triangle ABC$ is maximum when it is an isosceles
 (b) area of $\triangle ABC$ is minimum when it is isosceles
 (c) the perimeter of $\triangle ABC$ is minimum when it is isosceles
 (d) the perimeter of $\triangle ABC$ is maximum when it is isosceles
27. If $xy = k$ and $z = lx + my$, where k, l and m are positive constants, then the minimum value of z is
 (a) $\sqrt{mlk}$
 (b) $2\sqrt{mlk}$
 (c) $2mlk$
 (d) None of these
28. If the slope of the tangent to the curve $y = e^x \cos x$ is minimum at $x = 0, 0 \leq a \leq 2\pi$, then the value of a is
 (a) 0
 (b) π
 (c) 2π
 (d) None of these
29. If $\alpha + \beta = \frac{\pi}{2}$, then $\cos \alpha \cdot \cos \beta$ has a maximum value at β is equal to
 (a) $\frac{\pi}{4}$
 (b) $\frac{\pi}{2}$
 (c) $\frac{\pi}{6}$
 (d) None of these
30. Let $f(x) = 1 + 3x^2 + 3^2 \cdot x^4 + \dots + 3^{30} \cdot x^{60}$. Then, $f(x)$ has
 (a) atleast one maximum
 (b) exactly one maximum
 (c) atleast one minimum
 (d) exactly one minimum
31. Let the function $f(x)$ be defined as $f(x) = \begin{cases} \tan^{-1} \alpha - 3x^2, & 0 < x < 1 \\ -6x, & x \geq 1 \end{cases}$, $f(x)$ can have a maximum at $x = 1$, if the value of α is
 (a) 0
 (b) 2
 (c) 1
 (d) None of these
32. If $f(x) = x^5 - 5x^4 + 5x^3 - 10$ has maximum and minimum at $x = p$ and $x = q$ respectively, then (p, q) is equal to
 (a) 0, 1
 (b) 1, 3
 (c) 1, 0
 (d) None of the above
33. Let $f(x) = (x-2)^2 \cdot x^n, n \in N$. Then, $f(x)$ has a
 (a) minimum at $x = 2, \forall n \in N$
 (b) minimum at $x = 2$, if n is even
 (c) minimum at $x = 0$, if n is odd
 (d) None of the above

- 34.** For the equation $f'(x) = e^{-\frac{x^4}{4}} (4 - x^2)$, $f(x)$ has
- maximum occurs at $x = 2$
 - minimum occurs at $x = 2$
 - maximum occurs at $x = -2$
 - None of the above
- 35.** A function f is such that $f'(4) = f''(4) = 0$ and f has minimum value 10 at $x = 4$. Then, $f(x)$ is equal to
- $4 + (x - 4)^4$
 - $10 + (x - 4)^4$
 - $(x - 4)^4$
 - None of these
- 36.** The difference between the greatest and the least values of the function $f(x) = \int_0^x (6t^2 - 24)dt$ on $[1, 3]$, is
- 14
 - 10
 - 4
 - None of these
- 37.** The range of values of k for which the function $f(x) = (k^2 - 7k + 12)\cos x + 2(k - 4)x + \log 2$ does not possess critical points, is
- $(1, 5)$
 - $(1, 5) - \{4\}$
 - $(1, 4)$
 - None of these
- 38.** The range of values of k for which the function $f(x) = (2k - 3)(x + \tan 2) + (k - 1)(\sin^4 x + \cos^4 x)$ does not possess critical points, is
- $\left(-\infty, \frac{4}{3}\right) \cup (2, \infty)$
 - $\left(\frac{4}{3}, 2\right)$
 - $\left(\frac{4}{3}, \infty\right)$
 - $(2, \infty)$
- 39.** The minimum value of $e^{(x^4 - x^3 + x^2)}$ is
- e
 - e^2
 - 1
 - None of these
- 40.** The minimum value of the function $f(x) = 2|x - 2| + 5|x - 3|$, $\forall x \in R$ is
- 3
 - 2
 - 5
 - 7
- 41.** The maximum value of $x^{2/3} + (x - 2)^{2/3}$ is
- 0
 - 2
 - $2^{2/3}$
 - None of these
- 42.** The minimum value of the function $f(x) = \frac{x^p}{p} + \frac{x^{-q}}{q}$, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, is
- 1
 - 0
 - 2
 - None of the above
- 43.** On the interval $[0, 1]$, the function $x^{25} \cdot (1 - x)^{75}$ takes its maximum value at the point
- 0
 - $\frac{1}{4}$
 - $\frac{1}{2}$
 - $\frac{3}{4}$
- 44.** Let $f(x) = \begin{cases} |x|, & 0 < |x| \leq 2 \\ 1, & x = 0 \end{cases}$. Then, at $x = 0$, f has
- a local maximum
 - no local maximum
 - a local minimum
 - no extremum
- 45.** If $f(x) = \frac{x^2 - 1}{x^2 + 1}$, for every real number x , then the minimum value of $f(x)$ is
- does not exist because f is unbounded
 - is not attained even though f is bounded
 - is equal to 1
 - is equal to -1
- 46.** The number of values of x , where the function $f(x) = \cos x + \cos(\sqrt{2}x)$ attains its maximum, is
- 0
 - 1
 - 2
 - infinite
- 47.** The coordinates of the point on the parabola $y^2 = 8x$ which is at a minimum distance from the circle $x^2 + (y + 6)^2 = 1$ are
- $(2, 4)$
 - $(2, -4)$
 - $(-2, 4)$
 - None of these
- 48.** If $h(x) = f(x) + f(-x)$, then $h(x)$ has got an extreme value at a point, where $f'(x)$ is
- even function
 - odd function
 - zero
 - None of the above
- 49.** The least value of a for which the equation $\frac{4}{\sin x} + \frac{1}{1 - \sin x} = a$ has atleast one solution on the interval $\left(0, \frac{\pi}{2}\right)$, is
- 4
 - 1
 - 9
 - 8
- 50.** The minimum value of $2^{(x^2 - 3)^3 + 27}$ is
- 1
 - 2
 - 2^{27}
 - None of these
- 51.** If a differentiable function $f(x)$ has a relative minimum at $x = 0$, then the function $y = f(x) + ax + b$ has a relative minimum at $x = 0$ for
- all $a > 0$
 - all $b > 0$
 - all a and b
 - all b , if $a = 0$
- 52.** $f(x) = \int_{\cos x}^{\sin x} (1 - t + 2t^3)dt$ has in $[0, 2\pi]$,
- a maximum at $\frac{\pi}{4}$ and a minimum at $\frac{3\pi}{4}$
 - a maximum at $\frac{3\pi}{4}$ and a minimum at $\frac{7\pi}{4}$
 - a maximum at $\frac{5\pi}{4}$ and a minimum at $\frac{7\pi}{4}$
 - neither a maximum nor a minimum

53. The function $f(x) = \frac{x}{1+x \tan x}$ has

- (a) one point of minimum in the interval $\left(0, \frac{\pi}{2}\right)$
- (b) one point of maximum in the interval $\left(0, \frac{\pi}{2}\right)$
- (c) no point of maximum, no point of minimum in the interval $\left(0, \frac{\pi}{2}\right)$
- (d) two points of maxima in the interval $\left(0, \frac{\pi}{2}\right)$

54. The denominator of a fraction is greater than 16 of the square of numerator, then least value of fraction is

- (a) $\frac{1}{4}$ (b) $-\frac{1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{16}$

55. Let $f(x) = ax^3 + 5x^2 + cx + 1$ be a polynomial function. If $f(x)$ has extreme at $x = \alpha$ and β such that $\alpha\beta < 0$ and $f(\alpha)f(\beta) < 0$. Then, the equation $f(x) = 0$ has

- (a) three distinct real roots
- (b) one positive root, if $f(\alpha) < 0$ and $f(\beta) > 0$
- (c) one negative root, if $f(\alpha) > 0$ and $f(\beta) < 0$
- (d) All of the above

Type 2. More than One Correct Option

56. Let $f(x) = \begin{cases} x+2, & -1 \leq x < 0 \\ 1, & x = 0 \\ \frac{x}{2}, & 0 < x \leq 1 \end{cases}$. Then, on $[-1, 1]$, this

function has

- (a) a minimum
- (b) a maximum
- (c) neither a maximum nor a minimum
- (d) $f'(0)$ does not exist

57. If composite function $f_1(f_2(f_3(\dots(f_n(x))))n$ times is an increasing function and if r of f_i 's are decreasing function while rest are increasing, then maximum value of $r(n-r)$ is

- (a) $\frac{n^2-1}{4}$, where n is an even number
- (b) $\frac{n^2}{4}$, when n is an odd number
- (c) $\frac{n^2-1}{4}$, when n is an odd number
- (d) $\frac{n^2}{4}$, when n is an even number

58. Which of the following is true about point of extremum $x = a$ of function $y = f(x)$?

- (a) At $x = a$, function $y = f(x)$ may be discontinuous
- (b) At $x = a$, function $y = f(x)$ may be continuous but non-differentiable
- (c) At $x = a$, function $y = f(x)$ may have point of inflection
- (d) None of the above

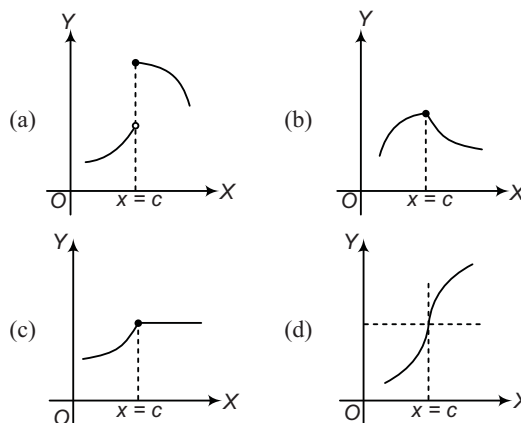
59. Let $f(x) = (x-1)^4(x-2)^n$, $n \in N$. Then, $f(x)$ has

- (a) a maximum at $x = 1$, if n is odd
- (b) a maximum at $x = 1$, if n is even
- (c) a minimum at $x = 1$, if n is even
- (d) a minimum at $x = 2$, if n is even

60. For the function $f(x) = x^4(12 \log_e x - 7)$,

- (a) the point $(1, -7)$ is the point of inflection
- (b) $x = e^{1/3}$ is the point of minima
- (c) the graph is concave downwards in $(0, 1)$
- (d) the graph is concave upwards in $(1, \infty)$

61. In which of the following graphs is $x = c$ the point of inflection?



Type 3. Assertion and Reason

Directions (Q. Nos. 62-67) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is correct explanation of Statement I

(b) Statement I is true, Statement II is true; Statement II is not the correct explanation of Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

62. **Statement I** Let $f(x) = 5 - 4(x-2)^{2/3}$, then at $x = 2$, the function $f(x)$ attains neither least value nor greatest value.

Statement II $x=2$ is the only critical point of $f(x)$.

- 63. Statement I** The minimum distance of the fixed point $(0, y_0)$, where $0 \leq y_0 \leq \frac{1}{2}$, from the curve $y = x^2$ is y_0 .

Statement II Maxima and minima of a function is always a root of the equation $f'(x) = 0$.

- 64.** Let $f(x) = (x^3 - 6x^2 + 12x - 8)e^x$.

Statement I $f(x)$ is neither maximum nor minimum at $x = 2$

Statement II If a function $x = 2$ is a point of inflection, then it is not a point of extremum.

- 65. Statement I** $e^\pi > \pi^e$.

Statement II The function $x^{1/x}$ ($x > 0$) has local maximum at $x = e$.

- 66.** Let $f(x) = \tan^{-1} \left(\frac{1-x}{1+x} \right)$.

Statement I The difference of the greatest and smallest value of $f(x)$ on $[0, 1]$ is $\frac{\pi}{4}$.

Statement II If a function g is decreases on $[a, b]$, then greatest value of $g = g(a)$ and least value of g is $g(b)$.

- 67.** Let $f(x) = \begin{cases} 2^x + 1 & , -1 \leq x < 0 \\ 2^x & , x = 0 \\ 2^x - 1 & , 0 < x \leq 1 \end{cases}$

Statement I f is bounded but never reaches its maximum and minimum.

Statement II f has a discontinuity at 0.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 68-72)

If $f: D \rightarrow R$, $f(x) = \frac{x^2 + bx + c}{x^2 + b_1x + c_1}$, where α and β are the

roots of the equation $x^2 + bx + c = 0$ and α_1, β_1 are the roots of $x^2 + b_1x + c_1 = 0$.

- 68.** If $\alpha_1 < \alpha < \beta_1 < \beta$, then

- (a) $f(x)$ is increasing in (α_1, β_1)
(b) $f(x)$ is decreasing in (α, β)
(c) $f(x)$ is decreasing in (β_1, β)
(d) $f(x)$ is decreasing in $(-\infty, \alpha)$

- 69.** If $\alpha_1 < \beta_1 < \alpha < \beta$, then

- (a) $f(x)$ has a maxima in (α_1, β_1) and a minima in (α, β)
(b) $f(x)$ has a minima in (α_1, β_1) and a maxima in (α, β)
(c) $f''(x) > 0$ wherever defined
(d) $f''(x) < 0$ wherever defined

- 70.** If the equations $x^2 + bx + c = 0$ and $x^2 + b_1x + c_1 = 0$ do not have real roots, then

- (a) $f'(x) = 0$ has real and distinct roots
(b) $f'(x) = 0$ has real and equal roots
(c) $f'(x) = 0$ has imaginary roots
(d) Nothing can be said

- 71.** In the above problem, $\lim_{x \rightarrow \infty} [f(x)] \cdot \lim_{x \rightarrow -\infty} [f(x)]$,

where $[]$ denotes the greatest integer function, is equal to

- (a) 1 (b) 0
(c) -1 (d) Does not exist

- 72.** In the last problem, if $b > b_1$, then

- (a) x-coordinate of point of minima is greater than the x-coordinate of point of maxima
(b)

x-coordinate of point of minima is less than x-coordinate of point of maxima

(c) it also depends upon c and c_1

(d) Nothing can be said

Passage II (Q. Nos. 73-75)

Let $f(x) = 4x^2 - 4ax + a^2 - 2a + 2$ and the global minimum value of $f(x)$ for $x \in [0, 2]$ is equal to 3.

- 73.** The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs at the end point of interval $[0, 2]$, is

- (a) 1 (b) 2
(c) 3 (d) 0

- 74.** The number of values of a for which the global minimum value equal to 3 for $x \in [0, 2]$ occurs for the value of x lying in $(0, 2)$, is

- (a) 1 (b) 2 (c) 3 (d) 0

- 75.** The values of a for which $f(x)$ is monotonic for $x \in [0, 2]$, are

- (a) $a \leq 0$ or $a \geq 4$ (b) $0 \leq a \leq 4$
(c) $a > 0$ (d) None of these

Passage III (Q. Nos. 76-78) Consider, a polynomial $y = P(x)$ of the least degree passing through $A(-1, 1)$ and whose graph has two points of inflection $B(1, 2)$ and C with abscissa 0 at which the curve is inclined to the positive axis of abscissa at an angle of $\sec^{-1} \sqrt{2}$.

- 76.** The value of $P(-1)$ is

- (a) -1 (b) 0
(c) 1 (d) 2

- 77.** The value of $P(0)$ is

- (a) 1 (b) 0 (c) $\frac{3}{4}$ (d) $\frac{1}{2}$

78. The equation $P'(x) = 0$ has

- (a) three distinct real roots
(b) one real root
(c) three real roots such that one root is repeated
(d) None of the above

Type 5. Match the Columns

79. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $f(x) = \cos \pi x + 10x + 3x^2 + x^3$, $-2 \leq x \leq 3$. Then, absolute minimum value of $f(x)$ is	p. $\frac{3}{4}$
B. If $x \in [-1, 1]$, then the minimum value of $f(x) = x^2 + x + 1$ is	q. 2
C. Let $f(x) = \frac{4}{3}x^3 - 4x$, $0 \leq x \leq 2$. Then, the global minimum value of the function is	r. -15
D. Let $f(x) = 6 - 12x + 9x^2 - 2x^3$, $1 \leq x \leq 4$. Then, the absolute maximum value of $f(x)$ in the interval is	s. $-\frac{8}{3}$

80. Match the statements of Column I with values of Column II.

Column I	Column II
A. Number of values of x lying in $\left(0, \frac{\pi}{2}\right)$ at which $f(x) = \ln(\sin x)$ is not monotonic, is	p. 0
B. If the greatest interval of decrease of the function $f(x) = x^3 - 3x + 2$ is $[a, b]$, then $(a + b)$ is equal to	q. 2

Column I	Column II
C. Let $f(x) = \frac{x^2 + 2}{[x]}$, $1 \leq x \leq 3$, where $[]$ denotes the greatest integer function, then least value of $f(x)$ is	r. -3
D. Set of all possible values of a such that $f(x) = e^{2x} - (a + 1)e^x + 2x$ is monotonically increasing for all $x \in \mathbb{R}$ is $(-\infty, a]$, then a is equal to	s. 3

81. Match the statements of Column I with values of Column II.

Column I	Column II
A. The dimensions of the rectangle of perimeter 36 cm, which sweeps out the largest volume when revolved about one of its sides, are	p. 6
B. Let $A(-1, 2)$ and $B(2, 3)$ be two fixed points, A point P lying on $y = x$ such that perimeter of $\triangle PAB$ is minimum, then sum of the abscissa and ordinate of point P , is	q. 12
C. If x_1 and x_2 are abscissae of two points on the curve $f(x) = x - x^2$ in the interval $[0, 1]$, then maximum value of expression $(x_1 + x_2) - (x_1^2 + x_2^2)$ is	r. 4
D. The number of non-zero integral values of a for which the function $f(x) = x^4 + ax^3 + \frac{3x^2}{2} + 1$ is concave upward along the entire real line is	s. $\frac{1}{2}$

Type 6. Single Integer Answer Type Questions

82. If m is the minimum value of $f(x, y) = x^2 - 4x + y^2 + 6y$, when x and y are subjected to the restrictions $0 \leq x \leq 1$ and $0 \leq y \leq 1$, then the value of $|m|$ is _____.

83. Let $f(x) = \begin{cases} x^2 - 3x + a, & 0 \leq x < \frac{3}{2} \\ -2x + 3, & x \geq \frac{3}{2} \end{cases}$. If $f(x)$ has a local maxima at $x = \frac{3}{2}$, then the greatest value of $|4a|$ is _____.

84. A rectangle with one side lying along the X -axis is to be inscribed in the closed region of the XY -plane

bounded by the lines $y = 0$, $y = 3x$ and $y = 30 - 2x$. If M is the largest area of such a rectangle, then the value of $\frac{2M}{27}$ is _____.

85. From a given solid cone of height H , another inverted cone is curved whose height is h such that its volume is maximum. Then, the ratio $\frac{H}{h}$ is _____.

86. Let $f(x) = \begin{cases} x^{3/5}, & \text{if } x \leq 1, \\ -(x - 2)^3, & \text{if } x > 1 \end{cases}$

then the number of critical points on the graph of the function is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Let $f(x)$ be a polynomial of degree four having extreme values at $x=1$ and $x=2$. If $\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2} \right] = 3$, then $f(2)$ is equal to [2014]
(a) -8 (b) -4 (c) 0 (d) 4

2. A vertical line passing through the point $(h, 0)$ intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P and Q . Let the tangents to the ellipse at P and Q meet at the point R . If $\Delta(h) = \text{area of } \Delta PQR$, $\Delta_1 = \max_{\frac{1}{2} \leq h \leq 1} \Delta(h)$ and $\Delta_2 = \min_{\frac{1}{2} \leq h \leq 1} \Delta(h)$, then $\frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2$ is equal to [2013]

Passage (Q. Nos. 3-4) Let $f: [0, 1] \rightarrow R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f'''(x) - 2f''(x) + f(x) \geq e^x$, $x \in [0, 1]$. [2013]

3. Which one of the following is true for $0 < x < 1$?
(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$
(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$
4. If the function $e^{-x} f(x)$ assumes its minimum in the interval $[0, 1]$ at $x = \frac{1}{4}$, then which one of the following is true?
(a) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$ (b) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$
(c) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$ (d) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$
5. A line $L: y = mx + 3$ meets Y -axis at $E(0, 3)$ the arc of the parabola $y^2 = 16x$, $0 \leq y \leq 6$ at the point $F(x_0, y_0)$. The tangent to the parabola at $F(x_0, y_0)$ intersects the Y -axis at $G(0, y_1)$. The slope m of the line L is chosen such that the area of ΔEFG has a local maximum.

Match List I with List II and select the correct answer using the codes given below the lists.

List I	List II
A. m is equal to	p. $1/2$
B. Maximum area of ΔEFG is	q. 4
C. y_0 is equal to	r. 2
D. y_1 is equal to	s. 1

[2013]

Codes

A	B	C	D	A	B	C	D
(a) s	p	q	r	(b) r	s	p	q
(c) p	r	q	s	(d) p	r	s	q

6. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are [2013]

- (a) 24 (b) 32
(c) 45 (d) 60

7. The function $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$ has a local minimum or a local maximum at [2013]

- (a) -2 (b) $-\frac{2}{3}$ (c) 2 (d) $\frac{2}{3}$

8. If $f(x) = \int_0^x e^{t^2} (t-2)(t-3) dt$, $\forall x \in (0, \infty)$, then [2013]

- (a) f has a local maximum at $x = 2$
(b) f is decreasing on $(2, 3)$
(c) there exists some $c \in (0, \infty)$ such that $f'''(c) = 0$
(d) f has a local minimum at $x = 3$

9. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is [2012]

10. Let $f: R \rightarrow R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is [2012]

11. Let f, g and h be real valued functions defined on the interval $[0, 1]$ by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2 e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on $[0, 1]$, then [2010]

- (a) $a = b$ and $c \neq b$ (b) $a = c$ and $a \neq b$
(c) $a \neq b$ and $c \neq b$ (d) $a = b = c$

12. Let f be a function defined on R (the set of all real numbers) such that $f'(x) = 2010(x - 2009)(x - 2010)^2(x - 2011)^3(x - 2012)^4$, $\forall x \in R$, if g is a function defined on R with values in the interval $(0, \infty)$ such that $f(x) = \ln(g(x))$, $\forall x \in R$, then the number of points in R at which g has a local maximum, is [2010]

13. The maximum value of the expression $\frac{1}{\sin^2 \theta + 3 \sin \theta \cos \theta + 5 \cos^2 \theta}$ is [2010]

JEE Main/AIEEE

14. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then [2014]

(a) $\alpha = -6, \beta = \frac{1}{2}$ (b) $\alpha = -6, \beta = -\frac{1}{2}$
(c) $\alpha = 2, \beta = -\frac{1}{2}$ (d) $\alpha = 2, \beta = \frac{1}{2}$

15. The maximum area of a right angled triangle with hypotenuse h is [2013]

(a) $\frac{h^3}{2\sqrt{2}}$ (b) $\frac{h^2}{2}$ (c) $\frac{h^2}{\sqrt{2}}$ (d) $\frac{h^2}{4}$

16. The cost of running a bus from A to B , is $\left(av + \frac{b}{v}\right)$,

where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be 75 while at 40 km/h, it is 65. Then, the most economical speed (in km/h) of the bus is [2013]

(a) 45 (b) 50 (c) 60 (d) 40

17. A line is drawn through the point $(1, 2)$ to meet the coordinate axes at P and Q such that it forms a $\triangle OPQ$, where O is the origin, if the area of the $\triangle OPQ$ is least, then the slope of the line PQ is [2012]

(a) $-\frac{1}{4}$ (b) -4 (c) -2 (d) $-\frac{1}{2}$

18. Let $a, b \in R$ be such that the function f given by $f(x) = \log|x| + bx^2 + ax, x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement I f has local maximum at $x = -1$ and at $x = 2$.

Statement II $a = \frac{1}{2}$ and $b = \frac{-1}{4}$. [2012]

- (a) Statement I is incorrect, Statement II is correct
(b) Statement I is correct, Statement II is correct;
Statement II is a correct explanation for Statement I
(c) Statement I is correct, Statement II is correct;
Statement II is not a correct explanation for Statement I
(d) Statement I is correct, Statement II is incorrect

19. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t} \sin t \, dt$. Then,

f has [2011]

- (a) local maximum at π and 2π
(b) local minimum at π and local maximum at 2π
(c) local maximum at π and local minimum at 2π
(d) local minimum at π and 2π

20. Let f be a function defined by

$$f(x) = \begin{cases} \tan x, & x \neq 0 \\ x, & x = 0 \end{cases}$$

Statement I $x = 0$ is point of minima of f .

Statement II $f'(0) = 0$ [2011]

- (a) Statement I is incorrect, Statement II is correct
(b) Statement I is correct, Statement II is correct;
Statement II is a correct explanation for Statement I
(c) Statement I is correct, Statement II is correct;
Statement II is not a correct explanation for Statement I
(d) Statement I is correct, Statement II is incorrect

21. Let $f: R \rightarrow R$ be a continuous function defined by

$$f(x) = \frac{1}{e^x + 2e^{-x}}.$$

Statement I $f(c) = \frac{1}{3}$, for some $c \in R$.

Statement II $0 < f(x) \leq \frac{1}{2\sqrt{2}}, \forall x \in R$ [2010]

- (a) Statement I is correct, Statement II is correct;
Statement II is the correct explanation for Statement I
(b) Statement I is correct, Statement II is correct;
Statement II is not the correct explanation for Statement I
(c) Statement I is correct, Statement II is incorrect
(d) Statement I is incorrect, Statement II is correct

22. The point in the interval $[0, 2\pi]$, where $f(x) = e^x \sin x$ has maximum slope, is [2010]

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) $\frac{3\pi}{2}$

23. Let $f: R \rightarrow R$ be defined by $f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$. If f has a local minimum at $x = -1$, then a possible value of k is [2010]

(a) 1 (b) 0
(c) $-\frac{1}{2}$ (d) -1

24. Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$ such that $x = 0$ is the only real root of $P'(x) = 0$. If $P(-1) < P(1)$, then in the interval $[-1, 1]$ [2009]

- (a) $P(-1)$ is the minimum and $P(1)$ is the maximum of P
(b) $P(-1)$ is not minimum but $P(1)$ is the maximum of P
(c) $P(-1)$ is the minimum and $P(1)$ is not the maximum of P
(d) Neither $P(-1)$ is the minimum nor $P(1)$ is the maximum of P

25. Suppose the cubic $x^3 - px + q$ has three distinct real roots, where $p > 0$ and $q > 0$. Then, which one of the following holds? [2008]

- (a) The cubic has maxima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$
(b) The cubic has minima at $\sqrt{\frac{p}{3}}$ and maxima at $-\sqrt{\frac{p}{3}}$
(c) The cubic has minima at $-\sqrt{\frac{p}{3}}$ and maxima at $\sqrt{\frac{p}{3}}$
(d) The cubic has minima at both $\sqrt{\frac{p}{3}}$ and $-\sqrt{\frac{p}{3}}$

- 26.** The function $f(x) = \frac{x}{2} + \frac{2}{x}$ has a local minimum at [2006]
(a) $x = -2$ (b) $x = 0$ (c) $x = 1$ (d) $x = 2$
- 27.** If x is real, then maximum value of $\frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$ is [2006]
(a) 41 (b) 1 (c) $\frac{17}{7}$ (d) $\frac{1}{4}$
- 28.** If $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta} + \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$, then the difference between the maximum and minimum values of u^2 is given by [2004]
(a) $2(a^2 + b^2)$ (b) $2\sqrt{a^2 + b^2}$ (c) $(a + b)^2$ (d) $(a - b)^2$
- 29.** The real numbers x when added to its inverse gives the minimum value of the sum at x equals [2003]
(a) 2 (b) 1 (c) -1 (d) -2
- 30.** If the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$, attains its maximum and minimum at p and q respectively such that $p^2 = q$, then a equals [2003]
(a) 3 (b) 1 (c) 2 (d) $\frac{1}{2}$
- 31.** The greatest value of $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$ on $[0, 1]$ is [2002]
(a) 1 (b) 2 (c) 3 (d) $\frac{1}{3}$

Other Engineering Entrances

- 32.** The minimum value of $\frac{x}{\log x}$ is [VITEEE 2014]
(a) e (b) $\frac{1}{e}$ (c) e^2 (d) e^3
- 33.** If $f(x) = \begin{cases} |x^3 + x^2 + 3x + \sin x| \left(3 + \sin \frac{1}{x} \right), & x \neq 0 \\ 0, & x = 0 \end{cases}$, then the number of points, where $f(x)$ attains its minimum value, is [Manipal 2014]
(a) 1 (b) 2 (c) 3 (d) infinitely many
- 34.** The minimum radius vector of the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ is of length [Manipal 2014]
(a) $a - b$ (b) $a + b$ (c) $2a + b$ (d) None of the above
- 35.** The local minimum value of the function f' given by $f(x) = 3 + |x|$, $x \in R$ is [Karnataka CET 2014]
(a) -1 (b) 3 (c) 1 (d) 0
- 36.** If x is real, then the minimum value of $y = \frac{x^2 - x + 1}{x^2 + x + 1}$ is [EAMCET 2014]
(a) 3 (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 2
- 37.** The condition that $f(x) = ax^3 + bx^2 + cx + d$ has no extreme value, is [EAMCET 2014]
(a) $b^2 > 3ac$ (b) $b^2 = 4ac$ (c) $b^2 = 3ac$ (d) $b^2 < 3ac$
- 38.** If $P(x) = a_0 + a_1x^2 + a_2x^4 + \dots + a_nx^{2n}$ is a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \dots < a_n$. Then, the function $P(x)$ has [BITSAT 2013]
(a) neither maximum nor minimum
(b) only one maximum
(c) only one minimum
(d) None of the above
- 39.** The difference between the greatest and the least values of the function $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ is [AMU 2013]
(a) π (b) $\sqrt{3} - \frac{\pi}{3}$ (c) $-\sqrt{3} + \frac{\pi}{3}$ (d) None of these
- 40.** The maximum value of xe^{-x} is [Karnataka CET 2012]
(a) e (b) $\frac{1}{e}$ (c) $-e$ (d) $\frac{-1}{e}$
- 41.** The maximum value of the function $f(x) = \frac{x}{8} + \frac{2}{x}$ on interval $[1, 6]$ is [WB JEE 2012]
(a) 1 (b) $\frac{9}{8}$ (c) $\frac{13}{12}$ (d) $\frac{17}{8}$
- 42.** The least value of the function $f(x) = ax + \frac{b}{x}$, $a > 0$, $b > 0$, $x > 0$ is [AMU 2012]
(a) $\sqrt{ab}$ (b) $2\sqrt{\frac{a}{b}}$ (c) $2\sqrt{\frac{b}{a}}$ (d) $2\sqrt{ab}$

43. If P is a point on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci S and S' , then the maximum value of $\Delta SPS'$ is
[Karnataka CET 2011]

- (a) ab
(b) abe^2
(c) abe
(d) $\frac{ab}{e}$

44. If $f(x) = x^3 e^{-3x}$, $x > 0$, then the maximum value of $f(x)$ is
[WB JEE 2011]

- (a) e^{-3} (b) $3e^{-3}$ (c) $27e^{-9}$ (d) ∞

45. The maximum value of $f(x) = 2\sin x + \cos 2x$, $0 \leq x \leq \frac{\pi}{2}$ occurs at x is
[UP SEE 2011]

- (a) 0 (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{2}$ (d) None of these

46. If $f(x) = \frac{x^2 - 1}{x^2 + 1}$ for every real number x , then the minimum value of f is
[AMU 2011]

- (a) -1 (b) Does not exist
(c) 0 (d) 1

Answers

Work Book Exercise 8.1

1. (c) 2. (d) 3. (c) 4. (b) 5. (b) 6. (c) 7. (a) 8. (d) 9. (a) 10. (a)

Work Book Exercise 8.2

1. (a) 2. (c) 3. (c) 4. (c) 5. (c) 6. (b) 7. (c) 8. (a)

Target Exercises

1. (c) 2. (c) 3. (c) 4. (d) 5. (d) 6. (d) 7. (c) 8. (c) 9. (d) 10. (b)
11. (b) 12. (d) 13. (d) 14. (c) 15. (a) 16. (b) 17. (b) 18. (a) 19. (c) 20. (b)
21. (b) 22. (c) 23. (b) 24. (a) 25. (c) 26. (a) 27. (b) 28. (b) 29. (a) 30. (d)
31. (d) 32. (b) 33. (a) 34. (a) 35. (b) 36. (a) 37. (b) 38. (a) 39. (c) 40. (b)
41. (b) 42. (a) 43. (b) 44. (a) 45. (d) 46. (b) 47. (b) 48. (a) 49. (c) 50. (a)
51. (d) 52. (b) 53. (b) 54. (b) 55. (d) 56. (c,d) 57. (c,d) 58. (a,b,c) 59. (a,c,d) 60. (all)
61. (a,b,d) 62. (d) 63. (c) 64. (c) 65. (b) 66. (a) 67. (a) 68. (a) 69. (a) 70. (a)
71. (b) 72. (b) 73. (b) 74. (d) 75. (a) 76. (c) 77. (d) 78. (c) 79. (*) 80. (*)
81. (*) 82. (3) 83. (9) 84. (5) 85. (3) 86. (3)

*

79. $A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$
80. $A \rightarrow p; B \rightarrow p; C \rightarrow s; D \rightarrow s$
81. $A \rightarrow p, q; B \rightarrow r; C \rightarrow s; D \rightarrow r$

Entrances Gallery

1. (c) 2. (9) 3. (d) 4. (c) 5. (a) 6. (a,c) 7. (a,b) 8. (all) 9. (9) 10. (5)
11. (d) 12. (1) 13. (2) 14. (c) 15. (d) 16. (c) 17. (c) 18. (c) 19. (c) 20. (c)
21. (b) 22. (b) 23. (d) 24. (b) 25. (b) 26. (d) 27. (a) 28. (d) 29. (b) 30. (c)
31. (b) 32. (a) 33. (a) 34. (b) 35. (a) 36. (b) 37. (d) 38. (c) 39. (b) 40. (b)
41. (d) 42. (d) 43. (c) 44. (a) 45. (b) 46. (a)

Explanations

Target Exercises

1. Since, $f^{2n+1}(a) = 0$, $f^{2n+2}(a) = -ve$ and $f(a) = b$
 $\therefore f(x) = -(x-a)^{2n+2} + b = b - (x-a)^{2n+2}$

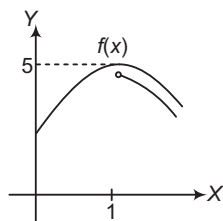
2. We have, $f(x) = a - (x-3)^{89}$
 $f'(x) = -89(x-3)^{88}$
 $\Rightarrow f'(x) = -89(88)(x-3)^{87}$
 For maximum or minimum, $f'(x) = 0$
 $\Rightarrow x = 3 \Rightarrow f'(3) = 0$
 $\therefore f(x)$ have neither maximum nor minimum.

3. $P = x^3 - \frac{1}{x^3}$, $Q = x - \frac{1}{x}$
 $\therefore \frac{P}{Q^2} = \frac{\left(x - \frac{1}{x}\right)\left(x^2 + 1 + \frac{1}{x^2}\right)}{\left(x - \frac{1}{x}\right)^2}$
 $= \frac{\left(x - \frac{1}{x}\right)^2 + 3}{\left(x - \frac{1}{x}\right)} = \left(x - \frac{1}{x}\right) + \frac{3}{\left(x - \frac{1}{x}\right)}$

Clearly, the minimum does not exist.

4. For $x \leq 1$, $f'(x) = 3x^2 - 2x + 10 = 3\left[\left(x - \frac{1}{3}\right)^2 + \frac{29}{9}\right] > 0$

So, $f(x)$ is an increasing function for $x \leq 1$
 For $x > 1$, $f'(x) = -2$.
 So, $f(x)$ is decreasing function for $x > 1$.
 Now, $f(x)$ will have greatest value at $x = 1$.



If $\lim_{x \rightarrow 1^+} f(x) \leq f(1)$, then $\lim_{h \rightarrow 0} f(1+h) \leq 5$

$$\begin{aligned} \Rightarrow \lim_{h \rightarrow 0} -2(1+h) + \log_2(b^2 - 2) &\leq 1 + 5 \\ \Rightarrow -2 + \log_2(b^2 - 2) &\leq 5 \\ \Rightarrow \log_2(b^2 - 2) &\leq 7 \\ \Rightarrow b^2 - 2 &\leq 128 \\ \Rightarrow b^2 &\leq 130 \text{ but } b^2 > 2 \\ \Rightarrow 2 < b^2 &\leq 130 \\ \therefore b &\in [-\sqrt{130}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{130}] \end{aligned}$$

5. Let $y = \sin x + \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x$
 $\Rightarrow \frac{dy}{dx} = \cos x + \cos 2x + \cos 3x$
 $= \cos 2x + 2 \cos 2x \cos x$
 $= \cos 2x (1 + 2 \cos x)$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$\begin{aligned} \Rightarrow \cos 2x (1 + 2 \cos x) &= 0 \\ \Rightarrow \cos 2x = 0 \text{ or } \cos x &= -\frac{1}{2} \\ \Rightarrow 2x = \frac{\pi}{2} \text{ or } x = \frac{2\pi}{3} \\ \therefore x = \frac{\pi}{4}, \text{ as } x = \frac{2\pi}{3} &\text{ is not required.} \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{d^2y}{dx^2} &= -\sin x - 2 \sin 2x - 3 \sin 3x \\ \Rightarrow \left[\frac{d^2y}{dx^2}\right]_{x=\frac{\pi}{4}} &= -\frac{1}{\sqrt{2}} - 2 - 3 \cdot \frac{1}{\sqrt{2}} < 0 \end{aligned}$$

$\therefore y$ is maximum, when $x = \frac{\pi}{4}$ and maximum value is

$$\begin{aligned} [y]_{x=\frac{\pi}{4}} &= \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{\sqrt{2}} = \frac{8 + 3\sqrt{2}}{6\sqrt{2}} \\ &= \left(\frac{1}{2} + \frac{4}{3\sqrt{2}}\right) \end{aligned}$$

6. We have, $y = a \log_e x + bx^2 + x$

$$\Rightarrow \frac{dy}{dx} = \frac{a}{x} + 2bx + 1$$

Since, y has its extreme values at $x = 1$ and $x = 2$

$$\therefore \left[\frac{dy}{dx}\right]_{x=1} = 0$$

$$\text{and } \left[\frac{dy}{dx}\right]_{x=2} = 0$$

$$\Rightarrow a + 2b + 1 = 0 \quad \dots(i)$$

$$\text{and } a + 8b + 2 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$a = -\frac{2}{3}$$

$$\text{and } b = -\frac{1}{6}$$

7. We have, $y = \frac{ax+b}{(x-4)(x-1)}$

$$\Rightarrow \frac{dy}{dx} = \frac{a(x-4)(x-1) - (ax+b)(2x-5)}{(x-4)^2(x-1)^2} \quad \dots(i)$$

For an extremum (maximum or minimum),

$$\frac{dy}{dx} = 0$$

$$\Rightarrow a(x-4)(x-1) - (ax+b)(2x-5) = 0 \quad \dots(ii)$$

Since, y has an extremum at $P(2, -1)$, therefore $x = 2$ satisfies the Eq. (ii) and $y = -1$ when $x = 2$.

$$\therefore a(-2)(1) - (2a+b)(4-5) = 0$$

$$\Rightarrow b = 0$$

$$\text{and } -1 = \frac{2a+b}{(2-4)(2-1)} \quad [\text{putting } x=2, y=-1 \text{ in Eq. (i)}]$$

$$\Rightarrow 2 = 2a + b$$

$$\Rightarrow 2 = 2a \Rightarrow a = 1$$

Hence, $a = 1$ and $b = 0$

8. We have, $y = \frac{\sin(x+a)}{\sin(x+b)}$

$$\Rightarrow \frac{dy}{dx} = \frac{[\sin(x+b) \cdot \cos(x+a) \cdot 1] - [\sin(x+a) \cdot \cos(x+b) \cdot 1]}{\sin^2(x+b)}$$

$$= \frac{\sin(b-a)}{\sin^2(x+b)} \neq 0 \text{ for any } x \text{ as } a \neq b$$

9. We have, $y = x + \frac{1}{x}$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$\Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x^2 = 1$$

i.e. $x = \pm 1$

Now, $\frac{d^2y}{dx^2} = \frac{2}{x^3}$

$$\therefore \left[\frac{d^2y}{dx^2} \right]_{x=1} = 2 > 0$$

$\Rightarrow y$ is maximum when $x = -1$.
Also, $\max(y) = -2$ and $\min(y) = 2$
 $\therefore \max(y) < \min(y)$

10. Let $y = x + z = x + \frac{1}{x}$ $\left[\because xz = 1 \therefore z = \frac{1}{x}, x > 0 \right]$

$$= 2 \left[\frac{x + \frac{1}{x}}{2} \right] \geq \left[\sqrt{x \cdot \frac{1}{x}} \right] \quad [\because \text{AM} \geq \text{GM}]$$

$\therefore y \geq 2$
 $\therefore \min(y) = 2$

11. Let $z = \log_a x$

$$\Rightarrow \frac{1}{z} = \frac{1}{\log_a x} = \log_x a$$

$$\therefore \log_a x + \log_x a = z + \frac{1}{z} = \left(\sqrt{z} - \frac{1}{\sqrt{z}} \right)^2 + 2$$

whose minimum value is 2.

12. Let $z = 2 \log_{10} x$

$$\therefore \log_x 0.01 = \log_x \frac{1}{100}$$

$$= \log_x 10^{-2} = -2 \log_x 10$$

$$= \frac{-2}{\log_{10} x} = -\frac{4}{z}$$

$$\therefore 2 \log_{10} x - \log_x 0.01 = z + \frac{4}{z} = \left(\sqrt{z} - \frac{2}{\sqrt{z}} \right)^2 + 4$$

whose minimum value is 4.

13. We have, $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2] dt$

$$\Rightarrow f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$$

$$= (x-1)(x-2)^2 [2(x-2) + 3(x-1)]$$

$$= (x-1)(x-2)^2 (5x-7)$$

For maximum or minimum, put $f'(x) = 0$.
 $\therefore (x-1)(x-2)^2 (5x-7) = 0$

$$\Rightarrow x = 1, 2, \frac{7}{5}$$

Sign scheme

$$- \quad +ve \quad 1 \quad -ve \quad \frac{7}{5} \quad +ve$$

Clearly, $f(x)$ has maximum value at $x = 1$, minimum value at $x = \frac{7}{5}$ and neither maximum nor minimum at $x = 2$.

14. When $\frac{d^2y}{dx^2} = 0$ at $x = a$, then we have to obtain $\frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}, \dots$ and so on to ascertain the existence of points of extremum.

15. Let $y = x - x^p$, where x is the fraction.

$$\Rightarrow \frac{dy}{dx} = 1 - px^{p-1}$$

For maximum or minimum, put $\frac{dy}{dx} = 0$.

$$\therefore 1 - px^{p-1} = 0$$

$$\Rightarrow x = \left(\frac{1}{p} \right)^{\frac{1}{(p-1)}}$$

Now, $\frac{d^2y}{dx^2} = -p(p-1)x^{p-2}$

$$\therefore \left[\frac{d^2y}{dx^2} \right]_{x=\left(\frac{1}{p}\right)^{\frac{1}{(p-1)}}} = -p(p-1) \left(\frac{1}{p} \right)^{\frac{p-2}{(p-1)}} < 0$$

$$\therefore y \text{ is maximum at } x = \left(\frac{1}{p} \right)^{\frac{1}{(p-1)}}$$

16. We have, $y = x^2 + ax + b$

$$\Rightarrow \frac{dy}{dx} = 2x + a \quad \text{and} \quad \frac{d^2y}{dx^2} = 2 > 0$$

Now, $\left[\frac{dy}{dx} \right]_{x=3} = 0$ [given]

$$\Rightarrow 6 + a = 0$$

$$\therefore a = -6$$

Also, $[y]_{x=3} = 5$

$$\Rightarrow 9 + (-6)(3) + b = 5$$

$$\Rightarrow b = 14$$

Thus, $a = -6$ and $b = 14$

17. We have, $f(x) = xe^{-x}$

$$\Rightarrow f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x)$$

$$f'(x) = 0$$

$$\Rightarrow e^{-x}(1-x) = 0$$

$$\Rightarrow x = 1$$

Now, $f(0) = 0$, $f(x) \rightarrow 0$ as $x \rightarrow \infty$ and $f(1) = \frac{1}{e}$.

Hence, the greatest value of the function is $\frac{1}{e}$.

18. We have, $f(x) = \cos^{-1} x^2 \Rightarrow f'(x) = \frac{-2x}{\sqrt{1-x^4}}$

$$\therefore f'(x) = 0 \Rightarrow x = 0$$

Now, $f(0) = \frac{\pi}{2}$ and $f\left(\pm \frac{1}{\sqrt{2}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$

Hence, the least value of the function is $\frac{\pi}{3}$.

19. We have, $f(x) = \tan^{-1} x - \frac{1}{2} \log x$

$$\Rightarrow f'(x) = \frac{1}{1+x^2} - \frac{1}{2x} = \frac{-(x-1)^2}{2x(1+x^2)}$$

$$\therefore f'(x) = 0 \Rightarrow x = 1$$

$$\text{Now, } f(1) = \tan^{-1} 1 - \frac{1}{2} \log 1 = \frac{\pi}{4}$$

$$f\left(\frac{1}{\sqrt{3}}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) + \frac{1}{4} \log 3 = \frac{\pi}{6} + \frac{1}{4} \log 3$$

$$\text{and } f(\sqrt{3}) = \tan^{-1}(\sqrt{3}) - \frac{1}{4} \log 3 = \frac{\pi}{3} - \frac{1}{4} \log 3$$

$$\text{Hence, the greatest value of } f(x) \text{ is } \frac{\pi}{3} - \frac{1}{4} \log 3.$$

20. We have, $y = x(x-1)^2$

$$\Rightarrow \frac{dy}{dx} = (x-1)^2 + 2x(x-1) = 3x^2 - 4x + 1 = (x-1)(3x-1)$$

For maximum or minimum, put $\frac{dy}{dx} = 0$.

$$\therefore (x-1)(3x-1) = 0 \Rightarrow x = \frac{1}{3}, 1$$

$$\text{Now, } \frac{d^2y}{dx^2} = 6x - 4$$

$$\therefore \left[\frac{d^2y}{dx^2} \right]_{x=\frac{1}{3}} = -2 < 0$$

$\therefore y$ is maximum when $x = \frac{1}{3}$ and maximum value is

$$[y]_{x=\frac{1}{3}} = \frac{4}{27}$$

$$\text{Also, } \left[\frac{d^2y}{dx^2} \right]_{x=1} = 2 > 0$$

$\therefore y$ is minimum when $x = 1$.

21. Let $P(x, y)$ be the point on the curve which is nearest to $O(0, 0)$.

$$\text{Let } z = OP^2 = x^2 + y^2 = x^2 + \frac{1}{4}(e^{2x} + e^{-2x} + 2)$$

$$\Rightarrow \frac{dz}{dx} = 2x + \frac{1}{2}(e^{2x} - e^{-2x})$$

For maximum or minimum, put $\frac{dz}{dx} = 0$

$$\therefore 2x + \frac{1}{2}(e^{2x} - e^{-2x}) = 0$$

$$\Rightarrow \frac{e^{-2x} - e^{2x}}{2} = 2x$$

$$\Rightarrow x = 0 \text{ is a solution and then } y = \frac{1}{2}(e^0 + e^0) = 1$$

$$\text{Also, } \frac{d^2z}{dx^2} = 2 + e^{2x} + e^{-2x} > 0, \text{ hence } z \text{ is minimum.}$$

$$\therefore \text{The shortest distance, } OP = \sqrt{0^2 + 1^2} = 1$$

22. We have, $y = 6 \cos x - 8 \sin x$

$$\Rightarrow \frac{dy}{dx} = -6 \sin x - 8 \cos x$$

$$\therefore \frac{dy}{dx} = 0 \Rightarrow -6 \sin x = 8 \cos x$$

$$\Rightarrow \tan x = -\frac{4}{3}$$

$$\Rightarrow \sin x = \pm \frac{4}{5}, \cos x = \mp \frac{3}{5}$$

$$\text{When } \sin x = \frac{4}{5}, \cos x = -\frac{3}{5}$$

$$\text{then } y = 6\left(-\frac{3}{5}\right) - 8\left(\frac{4}{5}\right) = -10$$

Hence, the greatest height above the X-axis is 10.

23. Let $f(x) = a^2 \sec^2 x + b^2 \operatorname{cosec}^2 x$

$$\Rightarrow f'(x) = 2a^2 \sec^2 x \tan x - 2b^2 \operatorname{cosec}^2 x \cot x$$

$$\therefore f'(x) = 0$$

$$\Rightarrow \frac{2a^2 \sin x}{\cos^3 x} - \frac{2b^2 \cos x}{\sin^3 x} = 0$$

$$\Rightarrow \tan^4 x = \frac{b^2}{a^2} \Rightarrow \tan^2 x = \frac{b}{a}$$

$$\text{i.e. } \cot^2 x = \frac{a}{b}$$

$$\text{Hence, } \sec^2 x = 1 + \frac{b}{a} \text{ and } \operatorname{cosec}^2 x = 1 + \frac{a}{b}$$

$$\text{Also, } f''(x) = 2a^2 (\sec^4 x + 2 \sec^2 x \tan^2 x) + 2b^2 (\operatorname{cosec}^4 x + 2 \operatorname{cosec}^2 x \cot^2 x) > 0, \forall x$$

$$\Rightarrow f(x) \text{ is minimum at } \tan^4 x = \frac{b^2}{a^2}$$

$$\text{Hence, the minimum value of } f(x) = a^2 \cdot \frac{a+b}{a} + b^2 \cdot \frac{b+a}{b} = (a+b)^2$$

24. Let $P(x, y)$ be such a point, then $OP^2 = x^2 + y^2$

$$\text{Let } S = OP^2 = x^2 + \frac{1}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{1}{x^2}$$

$$\therefore \frac{dS}{dx} = 0 \Rightarrow 2x - \frac{1}{x^2} = 0$$

$$\Rightarrow x^3 = \frac{1}{2} \Rightarrow x = \left(\frac{1}{2}\right)^{1/3}$$

$$\text{Also, } \frac{d^2S}{dx^2} = 2 + \frac{2}{x^3}$$

$$\Rightarrow \left[\frac{d^2S}{dx^2} \right]_{x=\left(\frac{1}{2}\right)^{1/3}} = 2 + \frac{2}{\frac{1}{2}} > 0$$

$$\text{So, } S \text{ is minimum at } x = \left(\frac{1}{2}\right)^{1/3}$$

$$\text{Thus, for nearest point, } x = \left(\frac{1}{2}\right)^{1/3}, y = \pm \left(\frac{1}{2}\right)^{-1/6}$$

$$25. T = \frac{N}{x}(\alpha + \beta x^2) = N\left(\frac{\alpha}{x} + \beta x\right)$$

$$\Rightarrow \frac{dT}{dx} = N\left(-\frac{\alpha}{x^2} + \beta\right)$$

$$\therefore \frac{dT}{dx} = 0$$

$$\Rightarrow x^2 - \frac{\alpha}{\beta} = 0$$

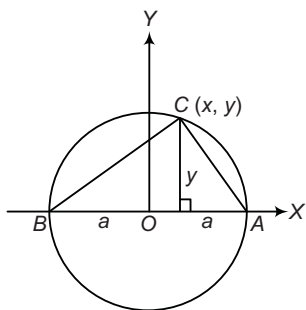
$$\Rightarrow x = \sqrt{\frac{\alpha}{\beta}}$$

$$\text{Also, } \frac{d^2T}{dx^2} = N\left(\frac{2\alpha}{x^3}\right) > 0 \text{ at } x = \sqrt{\frac{\alpha}{\beta}}$$

$$\therefore T \text{ is minimum for } x = \sqrt{\frac{\alpha}{\beta}}$$

26. Let the equation of the circle be $x^2 + y^2 = a^2$.

Let $A(a, 0)$, $B(-a, 0)$ be the ends of the diameter and $C(x, y)$ be any point on the circle.



Then, area of $\triangle ABC$

$$A = \frac{1}{2} \times AB \times y = ay = a\sqrt{a^2 - x^2}$$

$\therefore A$ is maximum, if $x = 0$, i.e. C lies on Y -axis and then $\triangle CAB$ is an isosceles triangle.

27. We have, $z = lx + my = lx + \frac{mk}{x}$ [$\because xy = k$]

$$\Rightarrow \frac{dz}{dx} = l - \frac{mk}{x^2}$$

$$\therefore \frac{dz}{dx} = 0 \Rightarrow l - \frac{mk}{x^2} = 0$$

$$\Rightarrow x = \pm \sqrt{\frac{mk}{l}}$$

$$\text{Now, } \frac{d^2z}{dx^2} = \frac{2mk}{x^3}$$

$$\Rightarrow \left[\frac{d^2z}{dx^2} \right]_{x=\sqrt{\frac{mk}{l}}} = \frac{2mk}{\left(\sqrt{\frac{mk}{l}}\right)^3} > 0$$

$$\therefore z \text{ is minimum at } x = \sqrt{\frac{mk}{l}}$$

$$\text{Also, min } z = l \cdot \sqrt{\frac{mk}{l}} + m \cdot \frac{k}{\sqrt{\frac{mk}{l}}} = \sqrt{lmk} + \sqrt{lmk} = 2\sqrt{lmk}$$

28. The slope of the tangent to the curve $y = e^x \cos x$ is

$$S = \frac{dy}{dx} = e^x (-\sin x + \cos x)$$

$$\text{Now, } \frac{dS}{dx} = e^x (-\sin x + \cos x - \cos x - \sin x) = -2e^x \sin x$$

$$\text{Put } \frac{dS}{dx} = 0 \Rightarrow -2e^x \sin x = 0$$

$$\Rightarrow x = 0, \pi, 2\pi \quad [\because 0 \leq x \leq 2\pi]$$

$\therefore S$ is minimum at $x = \pi$.

Thus, $a = \pi$

29. Let $y = \cos \alpha \cdot \cos \beta = \frac{1}{2} (2 \cos \alpha \cos \beta)$

$$= \frac{1}{2} [\cos(\alpha + \beta) + \cos(\alpha - \beta)]$$

$$= \frac{1}{2} \left[\cos \frac{\pi}{2} + \cos \left(\frac{\pi}{2} - 2\beta \right) \right] = \frac{1}{2} \sin 2\beta$$

Clearly, y is maximum when $2\beta = \frac{\pi}{2}$, i.e. $\beta = \frac{\pi}{4}$

30. We have,

$$f(x) = 1 + 3x^2 + 3^2 \cdot x^4 + \dots + 3^{30} \cdot x^{60}$$

$$\Rightarrow f'(x) = x(6 + 4 \cdot 3^2 \cdot x^2 + \dots + 60 \cdot 3^{30} \cdot x^{58})$$

$$f'(x) = 0$$

$$\Rightarrow x = 0$$

$$\text{Also, } f''(x) = 1 \cdot (6 + 4 \cdot 3^2 \cdot x^2 + \dots + 60 \cdot 3^{30} \cdot x^{58}) + x(4 \cdot 3^2 \cdot 2x + \dots + 60 \cdot 3^{30} \cdot 58x^{57})$$

$$\Rightarrow f''(0) = 6 > 0$$

$\therefore f(x)$ has minimum at $x = 0$ only.

31. We have, $f'(x) = \begin{cases} -6x, & 0 < x < 1 \\ -6, & x \geq 1 \end{cases}$

$$\therefore f'(1-h) = -6(1-h) < 0$$

$$\text{and } f'(1+h) = -6 < 0$$

Since, $f'(x)$ does not change sign as x passes through 1, therefore $f(x)$ does not have a maximum or minimum at $x = 1$, whatever be the value of α .

32. $f(x) = x^5 - 5x^4 + 5x^3 - 10$

$$\therefore f'(x) = 5x^4 - 20x^3 + 15x^2$$

$$\Rightarrow 5x^2(x^2 - 4x + 3) = 0$$

$$\Rightarrow x^2(x-3)(x-1) = 0$$

$$\Rightarrow x = 0, x = 3, x = 1$$

$$f''(x) = 20x^3 - 60x^2 + 30x$$

$$= 10x(2x^2 - 6x + 3)$$

$$f''(0) = 0$$

$$f''(3) = \text{Positive (Minima)}$$

$$f''(1) = \text{Negative (Maxima)}$$

$$\therefore (p, q) = (1, 3)$$

33. We have, $f(x) = (x-2)^2 \cdot x^n$

$$\Rightarrow f'(x) = 2(x-2) \cdot x^n + (x-2)^2 \cdot nx^{n-1}$$

$$= (x-2) \cdot x^{n-1} [2x + n(x-2)]$$

$$= x^{n-1} \cdot (x-2) [(n+2)x - 2n]$$

$$\therefore f'(x) = 0 \Rightarrow x = 0, 2, \frac{2n}{n+2}$$

$$\text{Now, } f'(0-h) = (-h)^{n-1} (-h-2) [(n+2)(-h) - 2n]$$

$$= (-ve)^{n-1} \cdot (-ve) \cdot (-ve) = (-ve)^{n-1}$$

$$\text{and } f'(0+h) = h^{n-1} (h-2) [(n+2)h - 2n]$$

$$= (+ve) \cdot (-ve) \cdot (-ve) = +ve$$

$\therefore f(x)$ has a minimum at $x = 0$, if n is even.

$$\text{Also, } f'(2-h) = (2-h)^{n-1} \cdot (-h) \cdot [2(2-h) - nh]$$

$$= (2-h)^{n-1} \cdot (-h) \cdot (4-2h-nh)$$

$$= (+ve)^{n-1} \cdot (-ve) \cdot (+ve) = -ve$$

$$\text{and } f'(2+h) = (2+h)^{n-1} \cdot h \cdot [2(2+h) - nh] = +ve$$

$\therefore f(x)$ has a minimum at $x = 2, \forall n \in \mathbb{N}$

34. We have, $f'(x) = e^{-x^4/4} (4 - x^2)$

For maximum or minimum, put $f'(x) = 0$.

$$\therefore e^{-x^4/4} (4 - x^2) = 0$$

$$\Rightarrow x = -2, 2$$

$$\text{Also, } f''(x) = e^{-x^4/4} (-2x) + e^{-x^4/4} \cdot (4 - x^2) \cdot (-x^3)$$

$$\Rightarrow f''(x) = \begin{cases} < 0, & \text{if } x = 2 \\ > 0, & \text{if } x = -2 \end{cases}$$

$$\Rightarrow f(x) \text{ has a maximum at } x = 2$$

$$\text{and minimum at } x = -2.$$

35. Since, $f'(4) = f''(4) = 0$, therefore

$$f(x) = (x - 4)^n + k, \text{ when } n \geq 3$$

But f has minimum at $x = 4$, so $n = 4$

$$\therefore f(x) = (x - 4)^4 + k$$

Since, $f(4) = 10$, therefore $k = 10$

$$\text{Thus, } f(x) = (x - 4)^4 + 10$$

36. We have, $f(x) = \int_0^x (6t^2 - 24) dt$

$$\Rightarrow f'(x) = (6x^2 - 24) \cdot 1$$

$$f'(x) = 0$$

$$\Rightarrow 6x^2 - 24 = 0 \Rightarrow x = \pm 2$$

But $x = -2 \notin [1, 3]$. So, $x = 2$ is the only critical point.

$$\text{Now, } f(1) = \int_0^1 (6t^2 - 24) dt = \left[\frac{6t^3}{3} - 24t \right]_0^1 = -22$$

$$f(2) = \int_0^2 (6t^2 - 24) dt = \left[\frac{6t^3}{3} - 24t \right]_0^2 = -32$$

$$\text{and } f(3) = \int_0^3 (6t^2 - 24) dt = \left[\frac{6t^3}{3} - 24t \right]_0^3 = -18$$

Hence, the greatest value of $f(x)$ is -18 which is attained at $x = 3$ and the least value is -32 which is attained at $x = 2$. Thus, the difference between the greatest and the least values of the function

$$= f(3) - f(2) = -18 + 32 = 14$$

37. We have,

$$f(x) = (k^2 - 7k + 12) \cos x + 2(k - 4)x + \log 2$$

$$= (k - 3)(k - 4) \cos x + 2(k - 4)x + \log 2$$

$$\Rightarrow f'(x) = -(k - 3)(k - 4) \sin x + 2(k - 4)$$

$$= (k - 4)[-(k - 3) \sin x + 2]$$

Since, $f(x)$ does not have critical points, therefore

$f'(x) = 0$ does not have any solution in R .

$\Rightarrow k \neq 4$ and $2 - (k - 3) \sin x = 0$ is not solvable in R .

$\Rightarrow k \neq 4$ and $\sin x = \frac{2}{k - 3}$ is not solvable in R .

$$\Rightarrow k \neq 4 \text{ and } \left| \frac{2}{k - 3} \right| > 1$$

$$\Rightarrow k \neq 4 \text{ and } |k - 3| < 2$$

$$\Rightarrow k \neq 4 \text{ and } -2 < (k - 3) < 2$$

$$\Rightarrow k \neq 4 \text{ and } 1 < k < 5$$

$$\therefore k \in (1, 5) - \{4\}$$

38. We have,

$$f(x) = (2k - 3)(x + \tan x) + (k - 1)(\sin^4 x + \cos^4 x)$$

$$\Rightarrow f'(x) = (2k - 3) + (k - 1)(4 \sin^3 x \cos x$$

$$- 4 \cos^3 x \sin x)$$

$$= (2k - 3) - 4(k - 1) \sin x \cos x (\cos^2 x - \sin^2 x)$$

$$= (2k - 3) - (k - 1) 2 \sin 2x \cos 2x$$

$$= (2k - 3) - (k - 1) \sin 4x$$

Since, $f(x)$ does not have critical points, therefore

$f'(x) = 0$ does not have any solution in R .

$\Rightarrow (2k - 3) - (k - 1) \sin 4x = 0$ as no solution in R .

$\Rightarrow \sin 4x = \frac{2k - 3}{k - 1}$ is not solvable in R .

$$\Rightarrow \left| \frac{2k - 3}{k - 1} \right| > 1$$

$$\Rightarrow \frac{2k - 3}{k - 1} < -1$$

$$\text{or } \frac{2k - 3}{k - 1} > 1$$

$$\Rightarrow 2k - 3 < -k + 1$$

$$2k - 3 > k - 1$$

$$\Rightarrow k < \frac{4}{3} \text{ or } k > 2$$

$$\Rightarrow k \in \left(-\infty, \frac{4}{3}\right) \cup (2, \infty)$$

39. $e^{(x^4 - x^3 + x^2)}$ is minimum when $(x^4 - x^3 + x^2)$ is minimum.

$$\text{Since, } x^4 - x^3 + x^2 = x^2(x^2 - x + 1)$$

$$= x^2 \left[\left(x - \frac{1}{2}\right)^2 + \frac{3}{4} \right] \geq 0, \forall x \in R$$

$\therefore$ The minimum value of $(x^4 - x^3 + x^2)$ is 0 for $x = 0$.

Hence, the minimum value of $e^{(x^4 - x^3 + x^2)}$ is $e^0 = 1$.

40. We have,

$$f(x) = \begin{cases} 2(2 - x) + 5(3 - x) = 19 - 7x, & \text{if } x < 2 \\ 2(2 - 2) + 5(3 - 2) = 5, & \text{if } x = 2 \\ 2(x - 2) + 5(3 - x) = 11 - 3x, & \text{if } 2 < x < 3 \\ 2(3 - 2) = 2, & \text{if } x = 3 \\ 2(x - 2) + 5(x - 3) = 7x - 19, & \text{if } x > 3 \end{cases}$$

Thus, we find that $f(x)$ has a minimum value 2 at $x = 3$.

41. Let $f(x) = x^{2/3} + (x - 2)^{2/3}$

$$\Rightarrow f'(x) = \frac{2}{3} x^{-1/3} + \frac{2}{3} (x - 2)^{-1/3}$$

$$\text{Now, } f'(x) = 0$$

$$\Rightarrow x^{-1/3} + (x - 2)^{-1/3} = 0$$

$$\Rightarrow (x - 2)^{-1/3} = -x^{-1/3}$$

$$\Rightarrow x - 2 = -x$$

$$\Rightarrow x = 1 \text{ is the critical point.}$$

$$\text{Also, } f''(x) = \frac{2}{3} \left[-\frac{1}{3} x^{-4/3} - \frac{1}{3} (x - 2)^{-4/3} \right]$$

$$= -\frac{2}{9} \left[x^{-4/3} + (x - 2)^{-4/3} \right]$$

$$\therefore f''(1) = -\frac{2}{9} [1^{-4/3} + (-1)^{-4/3}] = -\frac{4}{9} < 0$$

$\therefore f(x)$ has maximum at $x = 1$ and the maximum value is

$$f(1) = 1^{2/3} + (-1)^{2/3} = 2$$

42. We have, $f(x) = \frac{x^p}{p} + \frac{x^{-q}}{q}$

$$\Rightarrow f'(x) = x^{p-1} - x^{-q-1}$$

$$= x^{p-1} - \frac{1}{x^{q+1}} = (x^{p+q} - 1) / x^{q+1}$$

$$f'(x) = 0 \text{ gives } x^{p+q} = 1$$

$$\Rightarrow x = 1$$

When $x < 1$, $f'(x) < 0$ and when $x > 1$, $f'(x) > 0$.

Hence, $f'(x)$ changes sign from negative to positive as x passes through the value $x = 1$ and hence $f(x)$ has a minimum value at $x = 1$.

43. Let $f(x) = x^{25} \cdot (1 - x)^{75}$

$$\Rightarrow f'(x) = 25x^{24} (1 - x)^{75} - 75x^{25} (1 - x)^{74}$$

$$= 25x^{24} (1 - x)^{74} (1 - 4x)$$

$$\therefore f'(x) = 0$$

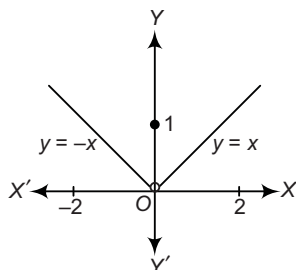
$\therefore x = 0, 1, \frac{1}{4}$ are the critical points.

Now, $f(0) = 0; f(1) = 0$ and $f\left(\frac{1}{4}\right) = \left(\frac{1}{4}\right)^{25} \cdot \left(\frac{3}{4}\right)^{75}$

$\therefore f(x)$ takes its maximum value at $x = \frac{1}{4}$.

44. We have, $f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ 1, & x = 0 \\ x, & 0 < x \leq 2 \end{cases}$

The graph of $f(x)$ is



Clearly, from the graph,

$$f(0) = 1, f(0 - \epsilon) < 1$$

$f(0 + \epsilon) > 1$, where ϵ is small and positive.

$\therefore f(x)$ has a local maximum at $x = 0$.

45. We have, $f(x) = \frac{x^2 - 1}{x^2 + 1} = 1 - \frac{2}{x^2 + 1}$

$f(x)$ will be minimum when $\frac{2}{x^2 + 1}$ is maximum,

i.e. when $x^2 + 1$ is minimum, i.e. at $x = 0$.

$\therefore$ Minimum value of $f(x) = f(0) = -1$

46. We have, $f(x) = \cos x + \cos(\sqrt{2}x)$

$$\Rightarrow |f(x)| = |\cos x + \cos \sqrt{2}x| \leq |\cos x| + |\cos \sqrt{2}x| = 1 + 1 = 2, \forall x \in \mathbb{R}$$

$\therefore$ Maximum value of $f(x) = 2$

This requires $\cos x = 1$ and $\cos \sqrt{2}x = 1$

$$\Rightarrow x = 2n\pi \text{ and } \sqrt{2}x = 2m\pi; n, m \in \mathbb{I}$$

$$\Rightarrow 2n\pi = \frac{2m\pi}{\sqrt{2}} \Rightarrow n = \frac{m}{\sqrt{2}}$$

This is possible only when $n = m = 0$.

$\therefore$ There is only one value ($x = 0$) at which $f(x)$ attains its maximum value.

47. Let $P(2t^2, 4t)$ be any point on the parabola.

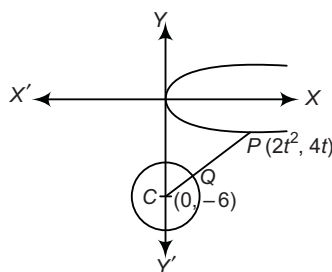
$$\text{Then, } CP^2 = (2t^2)^2 + (4t + 6)^2 = 4t^4 + 4(2t + 3)^2$$

$$\text{Let } f(t) = 4t^4 + 4(2t + 3)^2$$

$$\Rightarrow f'(t) = 16t^3 + 16(2t + 3) = 16(t^3 + 2t + 3)$$

$$= 16(t + 1)(t^2 - t + 3)$$

$$= 16(t + 1)\left[\left(t - \frac{1}{2}\right)^2 + \frac{11}{4}\right]$$



$$\therefore f'(t) = 0$$

$$\Rightarrow t = -1$$

$$\text{Also, } f''(t) = 16(3t^2 + 2)$$

$$\therefore [f''(t)]_{t=-1} = 16(3 + 2) = 80 > 0$$

Hence, $f(t)$ is least when $t = -1$.

Thus, the required coordinates are $(2, -4)$.

48. We have, $h'(x) = f'(x) - f'(-x)$

For extreme value of $h(x)$, $h'(x) = 0$

$$\Rightarrow f'(x) = f'(-x) \Rightarrow f'(x) \text{ is an even function.}$$

49. The required value of a is the minimum value of

$$f(x) = \frac{4}{\sin x} + \frac{1}{1 - \sin x} \text{ on } \left(0, \frac{\pi}{2}\right).$$

$$\text{We have, } f'(x) = -\frac{4\cos x}{\sin^2 x} + \frac{\cos x}{(1 - \sin x)^2}$$

$$\therefore f'(x) = 0$$

$$\Rightarrow \cos x \left[\frac{-4}{\sin^2 x} + \frac{1}{(1 - \sin x)^2} \right] = 0$$

$$\Rightarrow \cos x = 0 \text{ or } 4(1 - \sin x)^2 = \sin^2 x$$

$$\text{But } \cos x \neq 0 \quad \left[\because x \in \left(0, \frac{\pi}{2}\right) \right]$$

$$\therefore 2(1 - \sin x) = \pm \sin x$$

$$\Rightarrow \sin x = \frac{2}{3} \text{ or } \sin x = 2 \quad [\text{not possible}]$$

$$\therefore \sin x = \frac{2}{3}$$

$$\Rightarrow x = \sin^{-1}\left(\frac{2}{3}\right)$$

$$\therefore \text{Minimum value of } f(x) = \frac{4}{\frac{2}{3}} + \frac{1}{1 - \frac{2}{3}} = 6 + 3 = 9$$

Hence, the minimum value of $a = 9$.

50. $2^{(x^2 - 3)^3 + 27}$ is minimum when $(x^2 - 3)^3 + 27$ is minimum.

$$\begin{aligned} \text{Since, } (x^2 - 3)^3 + 27 &= x^6 - 9x^4 + 27x^2 \\ &= x^2(x^4 - 9x^2 + 27) \\ &= x^2(x^4 - 9x^2 + 27) \\ &= x^2 \left[\left(x^2 - \frac{9}{2}\right)^2 + \frac{27}{4} \right] \\ &\geq 0, \forall x \end{aligned}$$

$$\therefore \text{Minimum value of } (x^2 - 3)^3 + 27 = 0$$

Thus, minimum value of $2^{(x^2 - 3)^3 + 27}$ is $2^0 = 1$

51. Since, $f(x)$ has a relative minimum at $x = 0$

$$\therefore f'(0) = 0 \text{ and } f''(0) > 0$$

$$\text{Now, } y = f(x) + ax + b$$

$$\Rightarrow \frac{dy}{dx} = f'(x) + a$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=0} = f'(0) + a = a = 0, \text{ if } a = 0$$

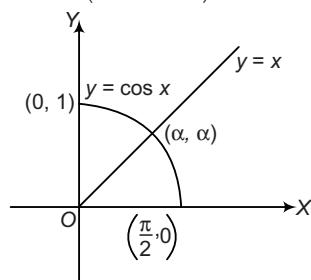
$$\text{Also, } \frac{d^2y}{dx^2} = f''(x)$$

$$\therefore \left[\frac{d^2y}{dx^2} \right]_{x=0} = f''(0) > 0$$

Hence, y has a relative minimum at $x = 0$, if $a = 0$ and for all b .

52. We have, $f(x) = \int_{\cos x}^{\sin x} (1-t+2t^3) dt$
 $\Rightarrow f'(x) = \cos x (1 - \sin x + 2 \sin^3 x)$
 $\quad + \sin x (1 - \cos x + 2 \cos^3 x)$
 $\quad \quad \quad \text{[using Leibnitz's rule]}$
 $\Rightarrow f'(x) = \cos x + \sin x - 2 \sin x \cos x$
 $\quad + 2 \sin x \cos x (\sin^2 x + \cos^2 x)$
 $\Rightarrow f'(x) = \cos x + \sin x$
 For maximum or minimum, put $f'(x) = 0$
 $\therefore \tan x = -1 \Rightarrow x = \frac{3\pi}{4}, \frac{7\pi}{4}$
 Now, $f''(x) = -\sin x + \cos x$
 $\therefore f''\left(\frac{3\pi}{4}\right) < 0$ and $f''\left(\frac{7\pi}{4}\right) > 0$
 Hence, $f(x)$ attains a local maximum at $x = \frac{3\pi}{4}$ and a minimum at $x = \frac{7\pi}{4}$.

53. We have, $f(x) = \frac{x}{1+x \tan x}$
 $\Rightarrow f'(x) = \frac{1+x \tan x - x(\tan x + x \sec^2 x)}{(1+x \tan x)^2}$
 $\Rightarrow f'(x) = \frac{1-x^2 \sec^2 x}{(1+x \tan x)^2}$



For local maximum or minimum, put $f'(x) = 0$.
 $\therefore \frac{1-x^2 \sec^2 x}{(1+x \tan x)^2} = 0 \Rightarrow \cos^2 x = x^2 \Rightarrow \cos x = \pm x$
 $\Rightarrow \cos x = x$
 $\left[\because x \text{ and } \cos x \text{ are positive for } x \in \left(0, \frac{\pi}{2}\right) \right]$

In order to find the number of solutions of this equation, let us draw the graphs of the curves $y = \cos x$ and $y = x$ in $\left(0, \frac{\pi}{2}\right)$. We find that the two

curves intersect at one point (α, α) only.

$\therefore f'(x) = 0 \Rightarrow \cos \alpha = \alpha$

Now, $f'(x) = \frac{1-x^2 \sec^2 x}{(1+x \tan x)^2}$

$\Rightarrow (1+x \tan x)^2 f'(x) = 1-x^2 \sec^2 x$

$\Rightarrow (1+x \tan x)^2 f''(x) + 2(1+x \tan x)(x \sec^2 x + \tan x) f'(x)$
 $= -2x \sec^2 x - 2x^2 \sec^2 x \tan x$

$\Rightarrow (1+\alpha \tan \alpha)^2 f''(\alpha) = -2\alpha \sec^2 \alpha - 2\alpha^2 \sec^2 \alpha \tan \alpha$

$\Rightarrow f''(\alpha) < 0$

Therefore, $x = \alpha$ is a point of local maximum.

Hence, $f(x)$ attains local maximum at one point in the interval $\left(0, \frac{\pi}{2}\right)$.

54. Let the number be x , then the function is

$$f(x) = \frac{x}{x^2 + 16}$$

On differentiating w.r.t. x , we get

$$f'(x) = \frac{(x^2 + 16)1 - x(2x)}{(x^2 + 16)^2}$$

$$= \frac{x^2 + 16 - 2x^2}{(x^2 + 16)^2}$$

$$= \frac{16 - x^2}{(x^2 + 16)^2}$$

For maxima or minima, put $f'(x) = 0$

$$\therefore 16 - x^2 = 0$$

$$\Rightarrow x = 4, -4$$

Again, differentiating

$$f''(x) = \frac{(x^2 + 16)^2 (-2x) - (16 - x^2) 2(x^2 + 16) 2x}{(x^2 + 16)^4}$$

At $x = 4$, $f''(x) < 0$

and $x = -4$, $f''(x) > 0$

$$\therefore \text{Least value of } f(x) = \frac{-4}{16 + 16}$$

$$= -\frac{1}{8}$$

55. We have, $\alpha\beta < 0$

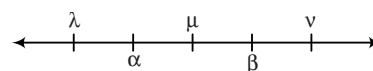
$\Rightarrow \alpha$ and β are of opposite signs.

Let $\alpha < 0$

and $\beta > 0$

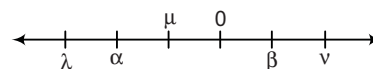
It is given that $f(x)$ has extremum at $x = \alpha, \beta$. Therefore, α and β are two distinct real roots of $f'(x) = 0$. But we know that between two distinct real roots of a polynomial, there is atleast one real root of its derivative. Therefore, $f(x)$ has three distinct real roots λ, μ and v (say) such that

$$\lambda < \alpha < \mu < \beta < v.$$



Thus, option (a) is correct.

If $f(x) = 0$ has exactly one positive root, then it is evident from given figure that $v > 0$ and $\lambda, \mu < 0$.



Therefore, $\alpha < \mu < 0$

$\Rightarrow f(\alpha) f(0) < 0$ [$\because \mu$ lies between α and 0]

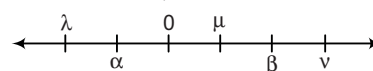
$\Rightarrow f(\alpha) < 0$ [$\because f(0) = 1 > 0$]

$\Rightarrow f(\beta) > 0$ [$\because f(\alpha) f(\beta) < 0$]

Thus, option (b) is correct.

If $f(x) = 0$ has exactly one negative real root, then from the given figure, we have $\lambda < 0$

and $\mu > 0$



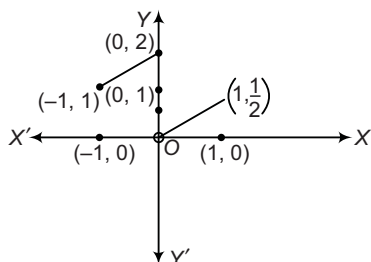
$\Rightarrow 0 < \mu < \beta$

$\Rightarrow f(0) f(\beta) < 0$ [$\because \mu$ lies between 0 and β]

$\Rightarrow f(\beta) < 0$ [$\because f(0) = 1 > 0$]

Thus, option (c) is correct.

56. From the graph, it is clear that $f(x)$ has neither a maximum nor a minimum at $x = 0$ and $f'(0)$ does not exist.

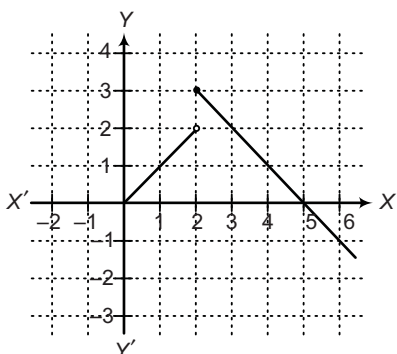


57. r must be an even integer because two decreasing functions are required to make it increasing function. Let $y = r(n - r)$

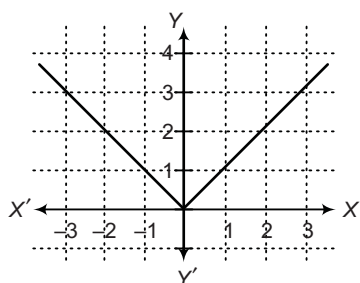
When n is odd, then $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ for maximum values of y and when n is even, then $r = \frac{n}{2}$ for maximum value of y .

Therefore, maximum $(y) = \frac{n^2 - 1}{4}$, when n is odd and $\frac{n^2}{4}$, when n is even.

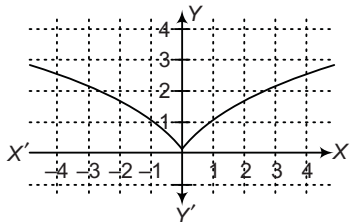
58. The following function is discontinuous at $x = 2$, but it has point of maxima.



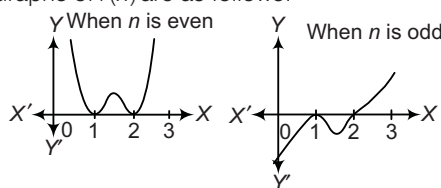
$f(x) = |x|$ has point of minima at $x = 0$, though it is non-differentiable at $x = 0$.



$f(x) = x^{2/3}$ has point of inflection at $x = 0$ as curve changes its concavity at $x = 0$, however $x = 0$ is point of minima for the function.



59. Graphs of $f(x)$ are as follows:



From graph of $f(x)$, $f(n)$ is minimum of $x = 1, 2$, if n is even and $f(x)$ is maximum at $x = 1$, if n is odd.

60. $f(x) = x^4 (12 \log_e x - 7)$; $x > 0$

$$\Rightarrow \frac{dy}{dx} = 16x^3 (3 \log_e x - 1) \text{ and } \frac{d^2y}{dx^2} = x^2 (9 \log_e x)$$

Put $\frac{dy}{dx} = 0 \Rightarrow x = e^{1/3}$

At $x = e^{1/3}$, $\frac{d^2y}{dx^2} > 0$, hence $x = e^{1/3}$ is point of minima.

Also, for $0 < x < 1$, $\frac{d^2y}{dx^2} < 0$ and for $x > 1$, $\frac{d^2y}{dx^2} > 0$

Hence, $x = 1$ is the point of inflection and for $0 < x < 1$, graph is concave downward and for $x > 1$, graph is concave upward.

61. At the point of inflection, concavity of the curve changes irrespective of any other factor.

62. **Statement I** $5 - 4(x - 2)^{2/3}$ attains greatest value at $x = 2$.

Statement II is obviously true.

63. Let the point on the parabola be (t, t^2) . Let d be the distance between (t, t^2) and $(0, y_0)$, then

$$d^2 = t^2 + (t^2 - y_0)^2 = t^4 + (1 - 2y_0)t^2 + y_0^2$$

$$= z^2 + (1 - 2y_0)z + y_0^2, z \geq 0$$

Its vertex is at $x = y_0 - \frac{1}{2} < 0$.

So, the minimum value of d^2 is at $z = 0$, i.e. $t^2 = 0$.

$$\therefore d = y_0$$

Hence, Statement I is true. Statement II is false because extremum can occur at a point, where $f'(x)$ does not exist.

64. $f(x) = (x^3 - 6x^2 + 12x - 8)e^x$

$$\Rightarrow f'(x) = e^x(x^3 - 6x^2 + 12x - 8) + e^x(3x^2 - 12x + 12)$$

$$= e^x(x^3 - 3x^2 + 4)$$

$$\Rightarrow f''(x) = e^x(x^3 - 3x^2 + 4) + e^x(3x^2 - 6x)$$

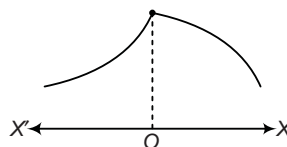
$$= e^x(x^3 - 6x + 4)$$

$$\Rightarrow f'''(x) = e^x(x^3 - 6x + 4) + e^x(3x^2 - 6)$$

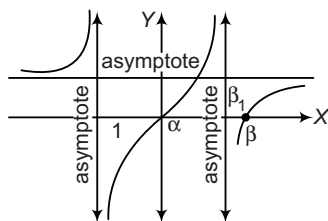
$$= e^x(x^3 + 3x^2 - 6x - 2)$$

Clearly, $f'(2) = f''(2) = 0$ and $f'''(2) \neq 0$, hence $x = 2$ is the point of inflection and hence not a point of extrema. Thus, Statement I is true.

But Statement II is false, as it is not necessary that at point of inflection, extrema does not occur. Consider, the following graph:

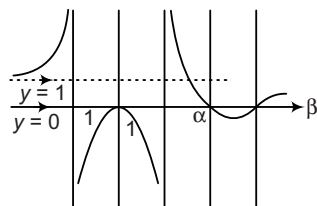


65. Let $f(x) = x^{1/x} \Rightarrow \log f(x) = \frac{\log x}{x}$
 $\Rightarrow \frac{f'(x)}{f(x)} = \frac{-\log x + 1}{x^2} \Rightarrow f'(x) = x^{1/x} \left(\frac{-\log x + 1}{x^2} \right)$
 For $x > e$, $f'(x) < 0$ and for $0 < x < e$, $f'(x) > 0$
 Thus, f has local maximum at $x = e$.
 Since, $\pi > e$ and f decreases on (e, ∞) , so $f(\pi) < f(e)$
 $\Rightarrow \pi^{1/\pi} < e^{1/e} \Rightarrow \pi^e < e^\pi$
66. $f(x) = \frac{\pi}{4} - \tan^{-1} x \Rightarrow f'(x) = \frac{-1}{1+x^2} \neq 0$ for all x and
 $f'(x) < 0$ for all x . Hence, the greatest value of
 $f = f(0) = \frac{\pi}{4}$ and the least value $= f(1) = 0$
67. In the interval $[-1, 0)$, the function increases from $3/2$ to 2 and in $(0, 1]$, it increases from 0 to 1 . It does not attain either the value 2 or 0 . Therefore, the function is bounded but never reaches its maximum or minimum. This is because there is a discontinuity at the point $x = 0$.
68. Graph of $f(x)$ is shown below:

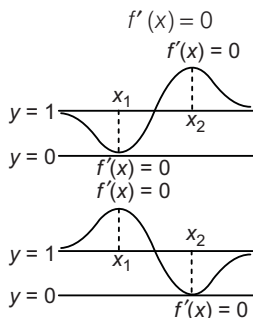


Clearly, $f(x)$ is increasing in (α, β) .

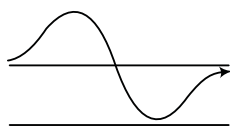
69. Clearly, $f(x)$ has a maximum in $[\alpha, \beta]$ and a minima in $[\alpha, \beta]$, shown as below:



70. $f(x)$ has one of the two graphs.
 $\Rightarrow f'(x) = 0$ has real and distinct roots.



71. Clearly, $\lim_{x \rightarrow \infty} [f(x)] = 0$ and $\lim_{x \rightarrow -\infty} [f(x)] = 1$
 $\therefore \lim_{x \rightarrow \infty} [f(x)] = \lim_{x \rightarrow -\infty} [f(x)] = 0$



$$72. f(x) = \frac{x^2 + bx + c}{x^2 + b_1x + c_1} = 1 + \frac{(b - b_1)x + (c - c_1)}{x^2 + b_1x + c_1}$$

$$= 1 + \frac{\frac{(b - b_1)}{x} + \frac{(c - c_1)}{x^2}}{1 + \frac{b_1}{x} + \frac{c_1}{x^2}}$$

$$\text{For } b > b_1, \lim_{x \rightarrow \infty} f(x) \rightarrow 1^+$$

$\therefore$ Point of maxima is greater than point of minima.

Solutions (Q. Nos. 73-75)

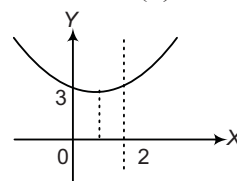
$$f(x) = 4x^2 - 4ax + a^2 - 2a + 2$$

Vertex of this parabola is $\left(\frac{a}{2}, 2 - 2a\right)$.

Case I When $0 < \frac{a}{2} < 2$

In this case, $f(x)$ will attain the minimum value at $x = \frac{a}{2}$.

$$\text{Thus, } f\left(\frac{a}{2}\right) = 3$$



$$\Rightarrow 3 = -2a + 2 \Rightarrow a = -\frac{1}{2} \quad [\text{rejected}]$$

Case II When $\frac{a}{2} \geq 2$

In this case, $f(x)$ attains the global minimum value at $x = 2$.

$$\text{Thus, } f(2) = 3$$

$$\Rightarrow 3 = 16 - 8a + a^2 - 2a + 2$$

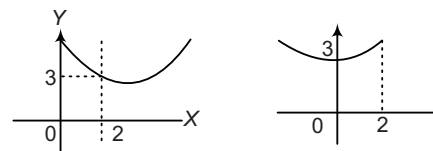
$$\Rightarrow a = 5 \pm \sqrt{10}$$

$$\text{Thus, } a = 5 + \sqrt{10}$$

Case III When $\frac{a}{2} \leq 0$

In this case, $f(x)$ attains the global minimum value at $x = 0$. Thus, $f(0) = 3$

Convert the following graph:



$$\Rightarrow 3 = a^2 - 2a + 2 \Rightarrow a = 1 \pm \sqrt{2}$$

$$\text{Thus, } a = 1 - \sqrt{2}$$

Hence, the permissible values of a are $1 - \sqrt{2}$ and $5 + \sqrt{10}$.

$f(x) = 4x^2 - 49x + a^2 - 2a + 2$ is monotonic in $[0, 2]$.

Hence, the point of minima of function should not lie in $[0, 2]$.

$$\text{Now, } f'(x) = 0 \Rightarrow 8x - 4a = 0 \Rightarrow x = \frac{a}{2}$$

$$\text{If } \frac{a}{2} \in [0, 2] \Rightarrow a \in [0, 4]$$

For $f(x)$ to be monotonic in $[0, 2]$, $a \notin [0, 4]$
 $\Rightarrow a \leq 0$ or $a \geq 4$

Solutions (Q. Nos. 76-78)

Since, two points of inflection occur at $x = 1$ and $x = 0$.

$$\Rightarrow P''(1) = P''(0) = 0$$

$$\therefore P''(x) = a(x^2 - x)$$

$$\Rightarrow P'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + b$$

$$\text{Also, given } \left(\frac{dy}{dx}\right)_{x=0} = \sec^{-1} \sqrt{2} = \tan^{-1} 1$$

Hence, $P'(0) = 1$, so $b = 1$

$$\Rightarrow P'(x) = a\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + 1$$

$$\therefore P(x) = a\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + c$$

$$\text{As, } P(-1) = 1$$

$$\Rightarrow a\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + c = 1 \Rightarrow \frac{a}{4} + c = 2 \quad \dots(i)$$

$$\Rightarrow a\left(\frac{1}{12} - \frac{1}{6}\right) + 1 + c = 1$$

$$\Rightarrow -\frac{a}{12} + c = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\begin{aligned} a &= 6 \\ \text{and } c &= \frac{1}{2} \end{aligned}$$

$$\Rightarrow P(x) = 6\left(\frac{x^4}{12} - \frac{x^3}{6}\right) + x + \frac{1}{2}$$

$$\Rightarrow P(-1) = 6\left(\frac{1}{12} + \frac{1}{6}\right) - 1 + \frac{1}{2} = 1$$

$$\text{and } P(0) = \frac{1}{2}$$

$$\begin{aligned} \text{Now, } P'(x) &= 6\left(\frac{x^3}{3} - \frac{x^2}{2}\right) + 1 \\ &= (x-1)^2(2x+1) \end{aligned}$$

79. A. $f'(x) = -\pi \sin \pi x + 10 + 6x + 3x^2$

$$= 3(x+1)^2 + 7 - \pi \sin \pi x > 0, \forall x$$

$\therefore f(x)$ is minimum in $-2 \leq x \leq 3$. So, the absolute minimum value $= f(-2) = -15$

B. $\therefore f'(x) = 2x + 1$,

therefore for $- \leq x < -\frac{1}{2}$, we get

$$f'(x) < 0 \text{ and for } -\frac{1}{2} < x \leq 1, \text{ we get } f'(x) > 0$$

$$\therefore f(x) \text{ is monotonically decreasing in } f\left[-1, -\frac{1}{2}\right]$$

$$\text{and monotonically increasing in } \left[-\frac{1}{2}, 1\right]$$

$$\therefore \min f(x) = f\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 1 = \frac{3}{4}$$

C. $f'(x) = 4(x^2 - 1)$

So, for $0 \leq x < 1$, we get $f'(x) < 0$, i.e. $f(x)$ is monotonically decreasing and for $1 < x \leq 2$, we get $f'(x) > 0$, $f(x)$ is monotonically increasing.

$$\therefore \min f(x) = f(1) = \frac{4}{3} - 4 = -\frac{8}{3}$$

D. $f'(x) = -12 + 18x - 6x^2$

$$= -6(x^2 - 3x + 2) = -6(x-1)(x-2)$$

$$\therefore f'(x) > 0, \text{ if } 1 < x < 2$$

$$\text{and } f'(x) < 0, \text{ if } 2 < x \leq 4$$

$\therefore f(x)$ is monotonically increasing in $1 < x < 2$ and monotonically decreasing in $2 < x \leq 4$.

$\therefore$ Absolute maximum

$$= \text{The greatest among } \{f(1), f(2)\}$$

$$= \text{The greatest among } \{1, 2\} = 2$$

80. A. $f(x) = \ln(\sin x)$

$$f'(x) = \frac{\cos x}{\sin x} > 0$$

$\therefore$ Required number of values of x is 0.

B. $f'(x) = 3x^2 - 3 \leq 0$, if $-1 \leq x \leq 1$

$$\therefore a = -1, b = 1$$

$$\text{Hence, } a + b = 0$$

$$C. f(x) = \begin{cases} x^2 + 2, & 1 \leq x < 2 \\ \frac{x^2 + 2}{2}, & 2 \leq x < 3 \\ \frac{x^2 + 2}{3}, & x = 3 \end{cases}$$

$$f'(x) = \begin{cases} 2x, & 1 < x < 2 \\ x, & 2 < x < 3 \end{cases}$$

$\therefore$ Least value of $f(x)$ in $[1, 2]$ is 3 and least value of $f(x)$ in $[2, 3]$ is 3.

$$\Rightarrow f(3) = \frac{11}{3}$$

Hence, least value of $f(x)$ is 3.

D. $f(x) = e^{2x} - (a+1)e^x + 2x$

$$f'(x) = 2e^{2x} - (a+1)e^x + 2$$

$$\text{Now, } 2e^{2x} - (a+1)e^x + 2 \geq 0, \forall x \in R$$

$$\text{i.e. } 2\left(e^x + \frac{1}{e^x}\right) - (a+1) \geq 0, \forall x \in R$$

$$\text{i.e. } 4 - (a+1) \geq 0$$

$$\text{i.e. } a \leq 3 \therefore a = 3$$

81. A. Perimeter of the rectangle = 36 cm

If one side is x , then the other side is $18 - x$.

If the rectangle is revolved around the side x , then volume swept out $V = \pi x(18 - x)^2$

$$\Rightarrow \frac{dV}{dx} = \pi [(18 - x)^2 - 2x(18 - x)]$$

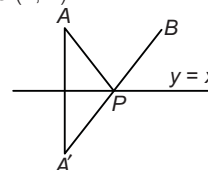
$$= \pi (18 - x)(18 - x - 2x)$$

$$\therefore x = 6 \text{ and } y = 12$$

B. $A(-1, 2)$, $B(2, 3)$ and P in a point on $y = x$ perimeter of $\triangle PAB$ is minimum when $PA + PB$ is minimum image of $A(-1, 2)$ in the line $y = x$ is $A'(2, -1)$.

Equation of $A'B$ is $x = 2$

Hence, P is $(2, 2)$.



C. Let (x_1, y_1) and (x_2, y_2) be two points.

$$\therefore y_1 + y_2 = (x_1 + x_2) - (x_1^2 + x_2^2)$$

$$\text{Now, } y = x - x^2 = -\left(x - \frac{1}{2}\right)^2 + \frac{1}{4}$$

$$\therefore (y_1 + y_2)_{\max} = 2 \times \frac{1}{4} = \frac{1}{2}$$

D. $f''(x) = 12x^2 + 6ax + 3 \geq 0, \forall x \in R$

$$\Rightarrow a \in [-2, 2]$$

$\therefore$ Number of non-zero integer values of a is 4.

82. We have, $f(x, y) = x^2 + y^2 - 4x + 6y$

$$\text{Let } (x, y) = (\cos \theta, \sin \theta), \text{ then } \theta \in \left[0, \frac{\pi}{2}\right]$$

$$\text{and } f(x, y) = f(\theta) = \cos^2 \theta + \sin^2 \theta - 4\cos \theta + 6\sin \theta$$

$$f'(\theta) = 6\cos \theta + 4\sin \theta > 0, \forall \theta \in \left[0, \frac{\pi}{2}\right]$$

So, $f'(\theta)$ is strictly increasing in $\left[0, \frac{\pi}{2}\right]$

$$\therefore f(\theta)_{\min} = f(0) = 1 - 4 + 0 = -3$$

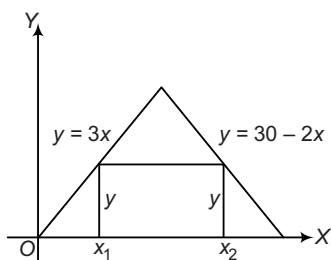
$$83. f\left(\frac{3}{2}\right) = 0$$

$$\Rightarrow \lim_{x \rightarrow \frac{3}{2}} |x^2 - 3x| + a \leq 0$$

$$\Rightarrow a \leq -\frac{9}{4}$$

Hence, the greatest value of $|4a|$ is 9.

84.



$$A = (x_2 - x_1)y$$

$$y = 3x_1 \text{ and } y = 30 - 2x_2$$

$$A(y) = \left(\frac{30 - y}{2} - \frac{y}{3}\right)y$$

$$6A(y) = (90 - 3y - 2y)y = 90y - 5y^2$$

$$\Rightarrow 6A'(y) = 90 - 10y = 0$$

$$\Rightarrow y = 9$$

$$\Rightarrow A''(y) = -10 < 0$$

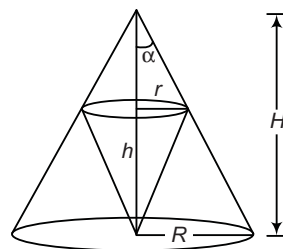
$$\therefore x_1 = 3; x_2 = \frac{21}{2}$$

$$\therefore A_{\max} = \left(\frac{21}{2} - 3\right)9$$

$$= \frac{15 \times 9}{2} = \frac{135}{2}$$

$$\therefore \text{Value of } \frac{2M}{27} = \frac{2 \times 135}{27 \times 2} = 5$$

85.



$$\frac{r}{R} = \frac{H-h}{H}$$

$\Rightarrow$

$$r = \frac{R(H-h)}{H}$$

$\therefore$

$$\text{Volume } (V) = \frac{1}{3} \pi \frac{R^2 (H-h)^2}{H^2} \cdot h$$

$\Rightarrow$

$$V = \frac{\pi R^2}{3H^2} (H-h)^2 h$$

$\therefore$

$$\frac{dV}{dh} = \frac{\pi R^2}{3H^2} [(H-h)^2 - 2h(H-h)]$$

$$= \frac{\pi R^2}{3H^2} (H-h)(H-h-2h)$$

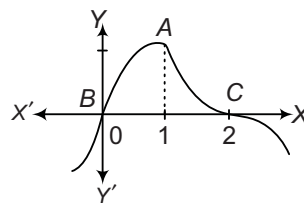
$$\therefore \frac{dV}{dh} = 0, \text{ if } h = \frac{H}{3}$$

and $h = \frac{H}{3}$ is a point of maximum

$\Rightarrow$

$$\frac{H}{h} = 3$$

86. A, B and C are the 3 critical points of $y = f(x)$



At B, it has vertical tangent, hence it is non-differentiable.

At A, it is non-differentiable.

$$\text{At C, } \frac{dy}{dx} = 0$$

Entrances Gallery

1. Since, the function have extreme values at $x = 1$ and $x = 2$.

$$\therefore f'(x) = 0 \text{ at } x = 1 \text{ and } x = 2$$

$$\Rightarrow f'(1) = 0 \text{ and } f'(2) = 0$$

Also, it is given that

$$\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2}\right] = 3 \Rightarrow 1 + \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2$$

$\Rightarrow f(x)$ will be of the form

$$ax^4 + bx^3 + 2x^2$$

[$\therefore f(x)$ is of four degree polynomial]

Let

$$f(x) = ax^4 + bx^3 + 2x^2$$

$\Rightarrow$

$$f'(x) = 4ax^3 + 3bx^2 + 4x$$

$\Rightarrow$

$$f'(1) = 4a + 3b + 4 = 0 \quad \dots (i)$$

and

$$f'(2) = 32a + 12b + 8 = 0$$

$\Rightarrow$

$$8a + 3b + 2 = 0 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

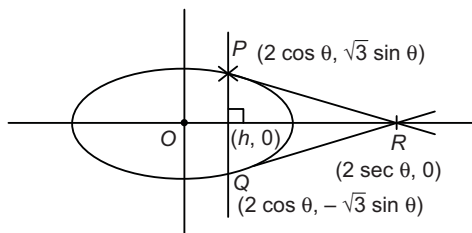
$$a = \frac{1}{2}, b = -2$$

$$\therefore f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$$

$$\Rightarrow f(2) = 8 - 16 + 8 = 0$$

2. Here, tangent at $P(2 \cos \theta, \sqrt{3} \sin \theta)$ is

$$\frac{x}{2} \cos \theta + \frac{y}{\sqrt{3}} \sin \theta = 1$$



$$\therefore R = (2 \sec \theta, 0)$$

$$\begin{aligned} \Rightarrow \Delta &= \text{Area of } \triangle PQR \\ &= \frac{1}{2} (2\sqrt{3} \sin \theta) (2 \sec \theta - 2 \cos \theta) \\ &= 2\sqrt{3} \cdot \frac{\sin^3 \theta}{\cos \theta} \end{aligned} \quad \dots(i)$$

$$\text{Since, } \frac{1}{2} \leq h \leq 1$$

$$\therefore \frac{1}{2} \leq 2 \cos \theta \leq 1$$

$$\Rightarrow \frac{1}{4} \leq \cos \theta \leq \frac{1}{2} \quad \dots(ii)$$

$$\begin{aligned} \therefore \frac{d\Delta}{d\theta} &= \frac{2\sqrt{3} \{ \cos \theta \cdot 3 \sin^2 \theta \cos \theta - \sin^3 \theta (-\sin \theta) \}}{\cos^2 \theta} \\ &= \frac{2\sqrt{3} \cdot \sin^2 \theta}{\cos^2 \theta} [3 \cos^2 \theta + \sin^2 \theta] \\ &= \frac{2\sqrt{3} \sin^2 \theta}{\cos^2 \theta} \cdot [2 \cos^2 \theta + 1] \\ &= 2\sqrt{3} \tan^2 \theta (2 \cos^2 \theta + 1) > 0 \end{aligned}$$

$$\text{When } \frac{1}{4} \leq \cos \theta \leq \frac{1}{2}$$

$$\therefore \Delta_1 = \Delta_{\max} \text{ occurs at } \cos \theta = \frac{1}{4} = \left(\frac{2\sqrt{3} \cdot \sin^3 \theta}{\cos \theta} \right)$$

$$\cos \theta = \frac{1}{2} = \frac{45\sqrt{5}}{8}$$

$$\begin{aligned} \Delta_2 = \Delta_{\min} \text{ occurs at } \cos \theta &= \frac{1}{2} = \left(\frac{2\sqrt{3} \cdot \sin^3 \theta}{\cos \theta} \right) \\ &= \frac{9}{2} \end{aligned}$$

$$\text{Hence, } \frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2 = 45 - 36 = 9$$

3. Here, $f''(x) - 2f'(x) + f(x) \geq e^x$

$$\Rightarrow f''(x)e^{-x} - f'(x)e^{-x} - f'(x)e^{-x} + f(x)e^{-x} \geq 1$$

$$\Rightarrow \frac{d}{dx} (f'(x)e^{-x}) - \frac{d}{dx} (f(x)e^{-x}) \geq 1$$

$$\Rightarrow \frac{d}{dx} (f'(x)e^{-x} - f(x)e^{-x}) \geq 1$$

$$\Rightarrow \frac{d^2}{dx^2} (e^{-x}f(x)) \geq 1 \text{ for all } x \in [0, 1]$$

$\therefore \phi(x) = e^{-x}f(x)$ is concave.

$$\Rightarrow f(0) = f(1) = 0 \Rightarrow \phi(0) = 0 = \phi(1)$$

$$\Rightarrow \phi(x) < 0 \Rightarrow e^{-x}f(x) < 0$$

$$\therefore f(x) < 0$$

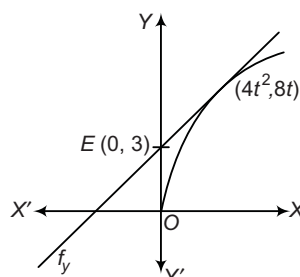
4. Here, $\phi'(x) < 0, x \in \left(0, \frac{1}{4}\right)$

$$\text{and } \phi'(x) > 0, x \in \left(\frac{1}{4}, 1\right)$$

$$\Rightarrow e^{-x} f'(x) - e^{-x} f(x) < 0, x \in \left(0, \frac{1}{4}\right)$$

$$\Rightarrow f'(x) < f(x), 0 < x < \frac{1}{4}$$

5. Here, $y^2 = 16x, 0 \leq y \leq 6$



Tangent at F, $yt = x + at^2$

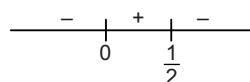
$$\text{At } x = 0, y = at = 4t$$

$$\text{Also, } (4t^2, 8t) \text{ satisfy } y = mx + c$$

$$\therefore 8t = 4mt^2 + 3 \Rightarrow 4mt^2 - 8t + 3 = 0$$

$$\therefore \text{Area of triangle} = \frac{1}{2} \begin{vmatrix} 0 & 3 & 1 \\ 0 & 4t & 1 \\ 4t^2 & 8t & 1 \end{vmatrix} = \frac{1}{2} \cdot 4t^2(3 - 4t)$$

$$A = 2(3t^2 - 4t^3) \Rightarrow \frac{dA}{dt} = 2(6t - 12t^2) = -12t(2t - 1)$$



$$\therefore \text{Maximum at } t = \frac{1}{2}$$

and

$$4mt^2 - 8t + 3 = 0 \Rightarrow m - 4 + 3 = 0$$

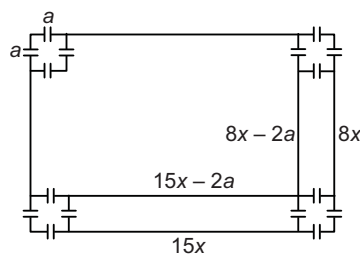
$$\therefore m = 1$$

$$\therefore G(0, 4t) \Rightarrow G(0, 2), y_1 = 2$$

$$\therefore (x_0, y_0) = (4t^2, 8t) = (1, 4) \Rightarrow y_0 = 4$$

$$\text{Now, maximum area at } t = \frac{1}{2} \text{ is } 2\left(\frac{3}{4} - \frac{1}{2}\right), \text{ i.e. } \frac{1}{2}$$

6. Here, $l = 15x - 2a, b = 8x - 2a, h = a$



$$\begin{aligned} \therefore \text{Volume} &= (8x - 2a)(15x - 2a) \cdot a \\ V &= 2a \cdot (4x - a)(15x - 2a) \end{aligned}$$

...

$$\frac{dV}{da} = 2 [6a^2 - 46ax + 60x^2]$$

$$\text{Again, } \frac{d^2V}{da^2} = 2 [12a - 46x]$$

$$\text{Also, give } 4a^2 = 100 \Rightarrow a = 5$$

$$\text{Here, } \frac{dV}{da} = 0$$

$$\Rightarrow 6x^2 - 23x + 15 = 0$$

$$\therefore x = 3, \frac{5}{6}$$

$$\text{Now, } \left(\frac{d^2V}{da^2} \right) = 4(30 - 23x) \quad [\because a = 5]$$

$$\text{At } x = 3, \left(\frac{d^2V}{da^2} \right) = 4(30 - 69) < 0$$

So, maximum, when $x = 3$

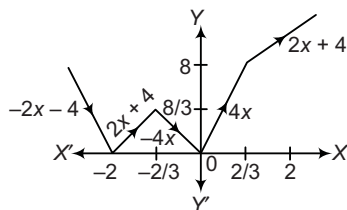
$$\text{Also, at } x = \frac{5}{6}, \left(\frac{d^2V}{da^2} \right) > 0$$

Hence, at $x = \frac{5}{6}$, volume is minimum.

Thus, the sides are $8x = 24$ and $15x = 45$.

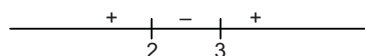
$$\begin{aligned} 7. \text{ Here, } f(x) &= 2|x| + |x+2| - ||x+2| - 2|x|| \\ &= \begin{cases} -2x - (x+2) + (x-2), & \text{when } x \leq -2 \\ -2x + x + 2 + 3x, & \text{when } -2 < x \leq -\frac{2}{3} \\ -4x, & \text{when } -\frac{2}{3} < x \leq 0 \\ 4x, & \text{when } 0 < x \leq 2 \\ 2x + 4, & \text{when } x > 2 \end{cases} \\ &= \begin{cases} -2x - 4, & x \leq -2 \\ 2x + 4, & -2 < x \leq -\frac{2}{3} \\ -4x, & -\frac{2}{3} < x \leq 0 \\ 4x, & 0 < x \leq 2 \\ 2x + 4, & x > 2 \end{cases} \end{aligned}$$

Graph for $y = f(x)$ is shown below:



$$8. \text{ Here, } f(x) = \int_0^x e^{t^2} (t-2)(t-3) dt$$

$$\Rightarrow f'(x) = e^{x^2} (x-2)(x-3)$$



$\therefore$ Maximum at $x = 2$

and minimum at $x = 3$

Decreasing on $(2, 3)$.

Also, $f'(x) = 0$ has two roots $x = 2$ and $x = 3$

i.e. $f'(2) = f'(3) = 0$

Thus, by Rolle's theorem,

$f''(c) = 0$ must have atleast one root $\in (2, 3)$.

$$9. \text{ Here, } p'(x) = \lambda (x-1)(x-3) = \lambda (x^2 - 4x + 3)$$

Integrating both sides between 1 to 3.

$$\int_1^3 p'(x) dx = \int_1^3 \lambda (x^2 - 4x + 3) dx$$

$$\Rightarrow [p(x)]_1^3 = \lambda \left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$\Rightarrow p(3) - p(1) = \lambda \left[(9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3 \right) \right]$$

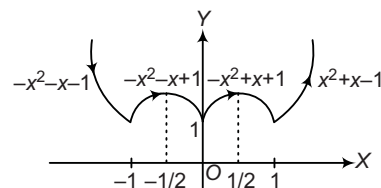
$$\Rightarrow 2 - 6 = \lambda \left(\frac{-4}{3} \right) \Rightarrow \lambda = 3$$

$$\therefore p'(x) = 3(x-1)(x-3) \Rightarrow p'(0) = 9$$

$$10. y = |x| + |x^2 - 1| = \begin{cases} -x + x^2 - 1, & x \leq -1 \\ -x + 1 - x^2, & -1 \leq x \leq 0 \\ x + 1 - x^2, & 0 \leq x \leq 1 \\ x + x^2 - 1, & x \geq 1 \end{cases}$$

$$= \begin{cases} x^2 - x - 1, & x \leq -1 \\ -x^2 - x + 1, & -1 \leq x \leq 0 \\ -x^2 + x + 1, & 0 \leq x \leq 1 \\ x^2 + x - 1, & x \geq 1 \end{cases}$$

which could be graphically shown as:



Thus, $f(x)$ attains maximum at $x = \frac{1}{2}, -\frac{1}{2}$ and $f(x)$ attains minimum at $x = -1, 0, 1$.

$\therefore$ Total number of points = 5

$$11. \text{ Given function, } f(x) = e^{x^2} + e^{-x^2}.$$

$$g(x) = xe^{x^2} + e^{-x^2} \text{ and}$$

$h(x) = x^2e^{x^2} + e^{-x^2}$ are strictly increasing on $[0, 1]$. Hence, at $x = 1$, the given function attains absolute maximum all equal to $e + \frac{1}{e}$.

$$\Rightarrow a = b = c$$

$$12. \text{ Let } g(x) = e^{f(x)}, \forall x \in R \Rightarrow g''(x) = e^{f(x)} \cdot f'(x)$$

$\Rightarrow f'(x)$ changes its sign from positive to negative in the neighbourhood of $x = 2009$.

$\Rightarrow f(x)$ has local maxima at $x = 2009$.

So, the number of local maximum is one.

$$13. \text{ Let } f(\theta) = \frac{1}{\sin^2 \theta + 3 \sin \theta \cos \theta + 5 \cos^2 \theta}$$

Again, let $g(\theta) = \sin^2 \theta + 3 \sin \theta \cos \theta + 5 \cos^2 \theta$

$$= \frac{1 - \cos 2\theta}{2} + 5 \left(\frac{1 + \cos 2\theta}{2} \right) + \frac{3}{2} \sin 2\theta$$

$$= 3 + 2 \cos 2\theta + \frac{3}{2} \sin 2\theta$$

$$\therefore g(\theta)_{\min} = 3 - \sqrt{4 + \frac{9}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$$

$$\therefore f(\theta) = \frac{1}{g(\theta)_{\min}} = 2$$

14. Here, $x = -1$ and $x = 2$ are extreme points of

$$f(x) = \alpha \log |x| + \beta x^2 + x, \text{ then } f'(x) = \frac{\alpha}{x} + 2\beta x + 1$$

$$\therefore f'(-1) = -\alpha - 2\beta + 1 = 0 \quad \dots(i)$$

[at extreme point, $f'(x) = 0$]

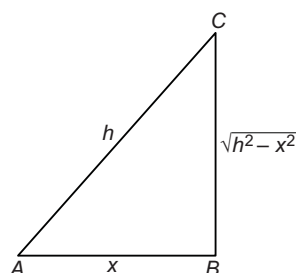
$$\text{and } f'(2) = \frac{\alpha}{2} + 4\beta + 1 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\alpha = 2 \text{ and } \beta = -\frac{1}{2}$$

15. Area of triangle, $\Delta = \frac{1}{2} x \sqrt{h^2 - x^2}$

$$\Rightarrow \frac{d\Delta}{dx} = \frac{1}{2} \left[\sqrt{h^2 - x^2} + \frac{x(-2x)}{2\sqrt{h^2 - x^2}} \right] = 0$$



$$x = \frac{h}{\sqrt{2}}$$

$$\text{Also, } \frac{d^2\Delta}{dx^2} < 0 \text{ at } x = \frac{h}{\sqrt{2}}$$

$$\therefore \Delta = \frac{1}{2} x \frac{h}{\sqrt{2}} \sqrt{h^2 - \frac{h^2}{2}} = \frac{h^2}{4}$$

16. Let $c = av + \frac{b}{v}$... (i)

When $v = 30$ km/h, then $c = ₹ 75$

$$\therefore 75 = 30a + \frac{b}{30} \quad \dots(ii)$$

When $v = 40$ km/h, then $c = ₹ 65$

$$\therefore 65 = 40a + \frac{b}{40} \quad \dots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$a = \frac{1}{2} \text{ and } b = 1800$$

On differentiating w.r.t. v in Eq. (i), we get

$$\frac{dc}{dv} = a - \frac{b}{v^2}$$

For maximum or minimum of c , $\frac{dc}{dv} = 0$.

$$\Rightarrow v = \pm \sqrt{\frac{b}{a}} \Rightarrow \frac{d^2c}{dv^2} = \frac{2b}{v^3}$$

$$\text{At } v = \sqrt{\frac{b}{a}}, \frac{d^2c}{dv^2} > 0$$

So, at $v = \sqrt{\frac{b}{a}}$, the speed is most economical.

$\therefore$ Most economical speed is

$$c = a \sqrt{\frac{b}{a}} + b \sqrt{\frac{a}{b}} = 2\sqrt{ab}$$

$$\Rightarrow c = 2\sqrt{\frac{1}{2} \times 1800} = 2 \times 30$$

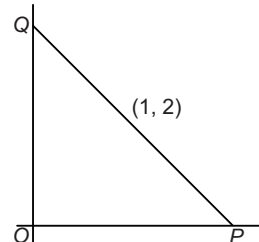
$$\therefore c = 60$$

17. Let m be the slope of the line PQ , then the equation of line PQ be

$$y - 2 = m(x - 1)$$

Now, PQ meets X -axis at $P\left(1 - \frac{2}{m}, 0\right)$ and Y -axis at

$Q(0, 2 - m)$.



$$\therefore OP = 1 - \frac{2}{m} \text{ and } OQ = 2 - m$$

Also, area of $\Delta OPQ = \frac{1}{2} (OP)(OQ)$

$$\begin{aligned} &= \frac{1}{2} \left(1 - \frac{2}{m}\right)(2 - m) \\ &= \frac{1}{2} \left(2 - m - \frac{4}{m} + 2\right) \\ &= \frac{1}{2} \left(4 - \left(m + \frac{4}{m}\right)\right) \end{aligned}$$

$$\text{Let } f(m) = 4 - \left(m + \frac{4}{m}\right)$$

$$\Rightarrow f'(m) = -1 + \frac{4}{m^2}$$

$$\text{Now, } f'(m) = 0 \Rightarrow m^2 = 4$$

$$\Rightarrow m = \pm 2$$

$$\Rightarrow f(2) = 0 \text{ and } f(-2) = 8$$

Since, the area cannot be zero, hence the required value of m is -2 .

18. **Statement I** A function f , such that $f(x) = \log |x| + bx^2 + ax, x \neq 0$

Statement II The function f has extreme at $x = -1$ and $x = 2$, i.e. $f'(-1) = f'(2) = 0$ and $f''(-1) \neq 0 \neq f''(2)$

Now, given function f is given by

$$f(x) = \log |x| + bx^2 + ax$$

$$\Rightarrow f'(x) = \frac{1}{x} + 2bx + a \Rightarrow f''(x) = \frac{-1}{x^2} + 2b$$

Since, f has extreme at $x = -1$ and $x = 2$.

$$\therefore f'(-1) = 0 = f'(2)$$

$$\Rightarrow f'(-1) = 0 \Rightarrow a - 2b = 1 \quad \dots(i)$$

$$\text{and } f'(2) = 0 \Rightarrow a + 4b = \frac{-1}{2} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$a = \frac{1}{2} \text{ and } b = \frac{-1}{4}$$

$$\Rightarrow f''(x) = \frac{-1}{x^2} + \frac{-1}{2} = -\left(\frac{x^2 + 2}{2x^2}\right)$$

$$\Rightarrow f''(-1) < 0 \text{ and } f''(2) < 0$$

So, f has local maxima at both $x = -1$ and $x = 2$.

Hence, obviously Statement I is correct.

Also, while solving for Statement I, we found the values of a and b , which justify that Statement II is also correct. However, Statement II does not explain Statement I in anyway.

19. Here, $f(x) = \int_0^x \sqrt{t} \sin t \, dt$, where $x \in \left(0, \frac{5}{2}\right)$

$$f'(x) = \{\sqrt{x} \sin x - 0\} \quad \dots(i)$$

[using Newton Leibnitz's formula]

$$\therefore f'(x) = \sqrt{x} \sin x = 0 \Rightarrow \sin x = 0$$

$$\therefore x = \pi, 2\pi$$

$$f''(x) = \sqrt{x} \cos x + \frac{1}{2\sqrt{x}} \sin x$$

$$\Rightarrow f''(\pi) = -\sqrt{\pi} < 0$$

So, local maximum at $x = \pi$

$$\Rightarrow f''(2\pi) = -\sqrt{2\pi} > 0$$

So, local minimum at $x = 2\pi$

20. Statement I As, $\frac{\tan x}{x} < 1, \forall x \neq 0$

$$\therefore f(0+h) > f(0) \text{ and } f(0-h) > f(0)$$

At $x = 0$, $f(x)$ attains minima.

$$\text{Statement II } f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h} = \lim_{h \rightarrow 0} \frac{\tan h - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\tan h - h}{h^2} \quad [\text{using L' Hospital's rule}]$$

$$= \lim_{h \rightarrow 0} \frac{\sec^2 h - 1}{2h} = \lim_{h \rightarrow 0} \frac{\tan^2 h}{2h^2} \cdot h = \frac{1}{2} \cdot 0 = 0$$

So, Statement II is correct.

Hence, both statements are correct but Statement II is not the correct explanation for Statement I.

21. We have, $f(x) = \frac{1}{e^x + \frac{2}{e^x}}$

$$\text{Using AM} \geq \text{GM}, \frac{e^x + \frac{2}{e^x}}{2} \geq \left(e^x \cdot \frac{2}{e^x}\right)^{\frac{1}{2}} \text{ as } e^x > 0$$

$$\Rightarrow e^x + \frac{2}{e^x} \geq 2\sqrt{2}$$

$$\Rightarrow 0 < \frac{1}{e^x + \frac{2}{e^x}} \leq \frac{1}{2\sqrt{2}}$$

$$\therefore 0 < f(x) \leq \frac{1}{2\sqrt{2}}, \forall x \in \mathbb{R}$$

Hence, Statement II is correct and Statement I is also correct as $\frac{1}{3} \in \left(0, \frac{1}{2\sqrt{2}}\right)$.

Aliter

$$\therefore f(x) = \frac{1}{e^x + 2e^{-x}} = \frac{e^x}{e^{2x} + 2}$$

$$\Rightarrow f'(x) = \frac{(e^{2x} + 2)e^x - 2e^{2x} \cdot e^x}{(e^{2x} + 2)^2}$$

$$\Rightarrow f'(x) = 0 \Rightarrow e^{2x} + 2 = 2e^{2x}$$

$$\Rightarrow e^{2x} = 2 \Rightarrow e^x = \sqrt{2}$$

$$\therefore \text{Maximum value of } f(x) = \frac{\sqrt{2}}{4} = \frac{1}{2\sqrt{2}}$$

22. We have, $f(x) = e^x \sin x$

$$\therefore f'(x) = e^x \cos x + \sin x e^x$$

$$\text{and } f''(x) = -e^x \sin x + e^x \cos x + e^x \cos x + e^x \sin x$$

$$= 2e^x \cos x$$

For maximum slope, $f''(x) = 0$

$$\Rightarrow 2e^x \cos x = 0 \Rightarrow \cos x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \forall x \in [0, 2\pi]$$

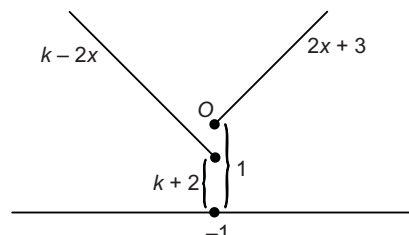
$$f'''(x) = 2[-e^x \sin x + e^x \cos x]$$

$$\text{At } x = \frac{\pi}{2}, f'''(x) < 0$$

$$\text{and at } x = \frac{3\pi}{2}, f'''(x) > 0$$

So, slope is maximum at $x = \frac{\pi}{2}$.

23.



$$\therefore k+2 \leq 1$$

$$\therefore k \leq -1$$

24. Given, $P(x) = x^4 + ax^3 + bx^2 + cx + d$

$$\Rightarrow P'(x) = 4x^3 + 3ax^2 + 2bx + c$$

Since, $x = 0$ is a solution for

$$P'(x) = 0 \Rightarrow c = 0$$

$$\therefore P(x) = x^4 + ax^3 + bx^2 + d \quad \dots(i)$$

Also, we have $P(-1) < P(1)$

$$\Rightarrow 1 - a + b + d < 1 + a + b + d$$

$$\Rightarrow a > 0$$

Since, $P'(x) = 0$ only when $x = 0$ and $P(x)$ is differentiable in $(-1, 1)$, we should have the maximum and minimum at the points $x = -1, 0$ and 1 only.

Also, we have $P(-1) < P(1)$

$$\therefore \text{Maximum of } P(x) = \max\{P(0), P(1)\}$$

$$\text{and minimum of } P(x) = \min\{P(-1), P(0)\}$$

In the interval $[0, 1]$,

$$P'(x) = 4x^3 + 3ax^2 + 2bx$$

$$= x(4x^2 + 3ax + 2b)$$

Since, $P'(x)$ has only root $x = 0$, then

$$4x^2 + 3ax + 2b = 0 \text{ has no real roots.}$$

$$\therefore (3a)^2 - 32b < 0$$

$$\Rightarrow \frac{9a^2}{32} < b$$

$$b > 0$$

Thus, we have $a > 0$ and $b > 0$.

$$\therefore P'(x) = 4x^3 + 4ax^2 + 2bx > 0, \forall x \in (0, 1)$$

Hence, $P(x)$ is increasing in $[0, 1]$.

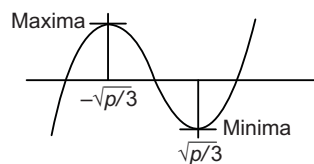
$$\therefore \text{Maximum of } P(x) = P(1).$$

Similarly, $P(x)$ is decreasing in $[-1, 0]$.

Therefore, minimum $P(x)$ does not occur at $x = -1$.

25. Let $f(x) = x^3 - px + q$

$$\text{Then, } f'(x) = 3x^2 - p$$



Put $f'(x) = 0$
 $\Rightarrow x = \sqrt{\frac{p}{3}}, -\sqrt{\frac{p}{3}}$

Now, $f''(x) = 6x$
 At $x = \sqrt{\frac{p}{3}}, f''(x) = 6\sqrt{\frac{p}{3}} > 0$ [minima]
 and at $x = -\sqrt{\frac{p}{3}}, f''(x) < 0$ [maxima]

26. Since, $f(x) = \frac{x}{2} + \frac{2}{x}$
 $\therefore f'(x) = \frac{1}{2} - \frac{2}{x^2}$

For maxima or minima, put $f'(x) = 0$.

$\therefore \frac{1}{2} - \frac{2}{x^2} = 0$
 $\Rightarrow x^2 = 4$
 $\Rightarrow x = \pm 2$
 Now, $f''(x) = \frac{4}{x^3}$
 $\Rightarrow f''(2) = \frac{4}{8} = \frac{1}{2} > 0$ [minima]
 and $f''(-2) = -\frac{4}{8} = -\frac{1}{2} < 0$ [maxima]

Hence, $f(x)$ is minimum at $x = 2$.

27. Let $f(x) = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$
 $= 1 + \frac{10}{3\left(x^2 + 3x + \frac{7}{3}\right)}$
 $= 1 + \frac{10}{3\left[\left(x + \frac{3}{2}\right)^2 + \frac{1}{12}\right]}$
 $\therefore f(x)$ will be maximum at $x = -\frac{3}{2}$.
 So, the maximum value of $f(x) = 1 + \frac{10}{3\left(\frac{1}{12}\right)}$
 $= 1 + 40 = 41$

28. Given that, $u = \sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}$
 $+ \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$
 $\therefore u^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta + a^2 \sin^2 \theta$
 $+ b^2 \cos^2 \theta + 2\sqrt{(a^2 \cos^2 \theta + b^2 \sin^2 \theta)(a^2 \sin^2 \theta + b^2 \cos^2 \theta)}$
 $\Rightarrow u^2 = a^2 + b^2 + 2\sqrt{x(a^2 + b^2 - x)}$
 where, $x = a^2 \cos^2 \theta + b^2 \sin^2 \theta$
 $\Rightarrow u^2 = (a^2 + b^2) + 2\sqrt{(a^2 + b^2)x - x^2}$
 On differentiating w.r.t. θ , we get
 $\frac{du^2}{d\theta} = \frac{2}{2\sqrt{(a^2 + b^2)x - x^2}} (a^2 + b^2 - 2x) \times \frac{dx}{d\theta}$
 and $\frac{dx}{d\theta} = (b^2 - a^2) \sin 2\theta$
 $\frac{du^2}{d\theta} = \frac{(a^2 + b^2 - 2x)}{\sqrt{(a^2 + b^2)x - x^2}} \times (b^2 - a^2) \sin 2\theta$

For maxima and minima, put $\frac{du^2}{d\theta} = 0$.
 $a^2 + b^2 = 2(a^2 \cos^2 \theta + b^2 \sin^2 \theta)$

and $\sin 2\theta = 0$
 $\Rightarrow \cos 2\theta (b^2 - a^2) = 0$
 and $\sin 2\theta = 0$
 $\Rightarrow \cos 2\theta = 0$ and $\theta = 0$
 $\Rightarrow 2\theta = \frac{\pi}{2}$ and $\theta = 0$
 $\Rightarrow \theta = \frac{\pi}{4}$ and $\theta = 0$

u^2 will be minimum at $\theta = 0$ and maximum at $\theta = \frac{\pi}{4}$.

$\therefore u_{\min}^2 = (a + b)^2$
 and $u_{\max}^2 = 2(a^2 + b^2)$
 Hence, $u_{\max}^2 - u_{\min}^2 = 2(a^2 + b^2) - (a + b)^2$
 $= (a - b)^2$

29. Let $f(x) = x + \frac{1}{x}$
 $\Rightarrow f'(x) = 1 - \frac{1}{x^2}$
 For maxima and minima, put $f'(x) = 0$.
 $\Rightarrow 1 - \frac{1}{x^2} = 0 \Rightarrow x = \pm 1$
 Now, $f''(x) = \frac{2}{x^3}$
 At $x = 1, f''(x) = +ve$ [minima]
 and at $x = -1, f''(x) = -ve$ [maxima]
 Thus, $f(x)$ attains minimum value at $x = 1$.

30. Since, $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$
 $f'(x) = 6x^2 - 18ax + 12a^2$
 For maxima or minima, put $f'(x) = 0$
 $6(x^2 - 3ax + 2a^2) = 0$
 $\Rightarrow x^2 - 3ax + 2a^2 = 0$
 $\Rightarrow x^2 - 2ax - ax + 2a^2 = 0$
 $\Rightarrow x(x - 2a) - a(x - 2a) = 0$
 $\Rightarrow (x - a)(x - 2a) = 0$
 $\Rightarrow x = a$ and $x = 2a$
 Now, $f''(x) = 12x - 18a$
 At $x = a$, $f''(x) = 12a - 18a = -6a$
 $\therefore f(x)$ will be maximum at $x = a$
 i.e. $p = a$
 Again at $x = 2a$, $f''(x) = 24a - 18a = 6a$
 $\therefore f(x)$ will be maximum at $x = 2a$.
 i.e. $q = 2a$
 Given, $p^2 = q$
 $\Rightarrow a^2 = 2a$
 $\Rightarrow a = 2$

31. Given that, $f(x) = (x + 1)^{1/3} - (x - 1)^{1/3}$
 On differentiating w.r.t. x , we get
 $f'(x) = \frac{1}{3} \left[\frac{1}{(x + 1)^{2/3}} - \frac{1}{(x - 1)^{2/3}} \right]$
 $= \frac{(x - 1)^{2/3} - (x + 1)^{2/3}}{3(x^2 - 1)^{2/3}}$

Clearly, $f'(x)$ does not exist at $x = \pm 1$.

Now, put $f'(x) = 0$, then

$$(x-1)^{2/3} = (x+1)^{2/3}$$

$$\Rightarrow x = 0$$

At $x = 0$,

$$f(x) = (0+1)^{1/3} - (0-1)^{1/3} = 2$$

Hence, the greatest value of $f(x)$ is 2.

32. Let $f(x) = \frac{x}{\log x}$

On differentiating both sides w.r.t. x , we get

$$f'(x) = \frac{\log x - 1}{(\log x)^2}$$

For maxima and minima, put $f'(x) = 0$.

$$\Rightarrow \log x - 1 = 0$$

$$\Rightarrow x = e$$

$$\text{Now, } f''(x) = \frac{(\log x)^2 \cdot \frac{1}{x} - (\log x - 1) \cdot \frac{2 \log x}{x}}{(\log x)^4}$$

$$\Rightarrow f''(e) = \frac{\frac{1}{e} - 0}{1} = \frac{1}{e} > 0$$

So, $f(x)$ is minimum at $x = e$.

Hence, the minimum value of $f(x)$ at $x = e$ is

$$f(e) = \frac{e}{\log e} = e$$

33. We have,

$$f(x) = \begin{cases} x^3 + x^2 + 3x + \sin x & \left(3 + \sin \frac{1}{x}\right), x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{Let } g(x) = x^3 + x^2 + 3x + \sin x$$

$$\therefore g'(x) = 3x^2 + 2x + 3 + \cos x$$

$$= 3 \left(x^2 + \frac{2x}{3} + 1 \right) + \cos x$$

$$\Rightarrow g'(x) = 3 \left\{ \left(x + \frac{1}{3} \right)^2 + \frac{8}{9} \right\} + \cos x > 0$$

$$\text{and } 2 < 3 + \sin \left(\frac{1}{x} \right) < 4$$

So, the minimum value of $f(x)$ is 0 at $x = 0$.

Hence, the number of points = 1.

34. Given curve is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$.

Let radius vector be r .

$$\therefore r^2 = x^2 + y^2$$

$$\Rightarrow r^2 = \frac{a^2 y^2}{y^2 - b^2} + y^2 \left[\because \frac{a^2}{x^2} + \frac{b^2}{y^2} = 1 \right]$$

For minimum value of r ,

$$\frac{d(r^2)}{dy} = 0$$

$$\Rightarrow \frac{-2y b^2 a^2}{(y^2 - b^2)^2} + 2y = 0$$

$$\Rightarrow y^2 = b(a+b)$$

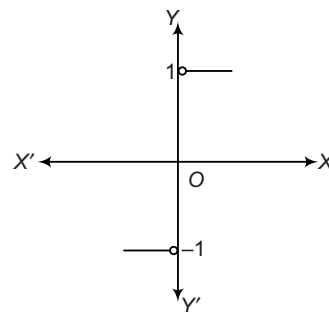
$$\therefore x^2 = a(a+b)$$

$$\Rightarrow r^2 = (a+b)^2$$

$$\Rightarrow r = a+b$$

35. Given, $f(x) = 3 + |x| = \begin{cases} 3 + x, & x \geq 0 \\ 3 - x, & x < 0 \end{cases}$

$$\therefore f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



Hence, the local minimum value of $f'(x)$ is -1 .

36. Given, $y = \frac{1-x+x^2}{1+x+x^2}$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{[(1+x+x^2)(-1+2x) - (1-x+x^2)(1+2x)]}{(1+x+x^2)^2}$$

$$= \frac{[-1+2x-x+2x^2-x^2+2x^3-1-2x+x+2x^2-x^2-2x^3]}{(1+x+x^2)^2}$$

$$= \frac{-2+2x^2-2x^2+2x^2}{(1+x+x^2)^2} = \frac{2x^2-2}{(1+x+x^2)^2} = \frac{2(x^2-1)}{(1+x+x^2)^2}$$

Put $\frac{dy}{dx} = 0$, then $x^2 = 1 \Rightarrow x = \pm 1$

$$\text{Now, } \frac{d^2y}{dx^2} = \frac{2[(1+x+x^2)^2(2x) - (x^2-1)(2)]}{(1+x+x^2)^4}$$

$$= \frac{4(1+x+x^2)[(1+x+x^2)x - (x^2-1)(1+2x)]}{(1+x+x^2)^4}$$

$$= \frac{4[x+x^2+x^3-x^2-2x^3+1+2x]}{(1+x+x^2)^3} = \frac{4(1+3x-x^3)}{(1+x+x^2)^3}$$

$$\text{At } x = 1, \left(\frac{d^2y}{dx^2} \right)_{x=1} = \frac{4[1+3(1)-1^3]}{(1+1+1^2)^3} = \frac{4(3)}{3^3} = \frac{4}{9} > 0$$

$$\text{At } x = -1, \left(\frac{d^2y}{dx^2} \right)_{x=-1} = \frac{4[1+3(-1)-(-1)^3]}{[1+(-1)+(-1)^2]^3}$$

$$= \frac{4(1-3+1)}{(1-1+1)^3} = 4(-1) = -4 < 0$$

By second derivative test, f is minimum at $x = 1$ and the minimum value is given by

$$y = \frac{1-1+1}{1+1+1} = \frac{1}{3}$$

37. Given curve is $df(x) = ax^3 + bx^2 + cx + d$

On differentiating w.r.t. x , we get

$$f'(x) = 3ax^2 + 2bx + c$$

$$\text{For extremum, } f'(x) = 0 \Rightarrow 3ax^2 + 2bx + c = 0$$

Since, it has no extremum value.

$$\therefore b^2 - 4ac < 0 \Rightarrow (2b)^2 - 4 \times 3a \times c < 0$$

$$\Rightarrow 4b^2 - 12ac < 0 \Rightarrow b^2 - 3ac < 0 \Rightarrow b^2 < 3ac$$

38. $P'(x) = 2x(a_1 + 2a_2x^2 + \dots + na_nx^{2n-2})$

For maximum or minimum, put $P'(x) = 0$.

This gives $x = 0$, since other factor cannot be zero because of a condition $0 < a_1 < a_2 < \dots < a_n$.

Now, $P''(x) = 2[a_1 + 6a_2x^2 + \dots + n(2n-1)a_nx^{2n-2}]$

So, $P''(x) = 2a_1 > 0$ at $x = 0$.

Hence, $P(x)$ has only one minimum at $x = 0$.

39. $f(x) = \sin 2x - x$ on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow f'(x) = 2 \cos 2x - 1$

Put $f'(x) = 0 \Rightarrow 2 \cos 2x - 1 = 0$

$\Rightarrow \cos 2x = \frac{1}{2} \Rightarrow 2x = 2n\pi \pm \frac{\pi}{3}$

$\Rightarrow x = n\pi \pm \frac{\pi}{6}$, where $n = 0, 1, 2, \dots$

For $n = 0$, $x = \pm \frac{\pi}{6}$

Now, $f''(x) = -4 \sin 2x$

At $x = \frac{\pi}{6}$, $f''\left(\frac{\pi}{6}\right) = -4 \sin 2 \cdot \frac{\pi}{6} = -4 \sin \frac{\pi}{3}$

$= -4 \cdot \frac{\sqrt{3}}{2} = -2\sqrt{3} < 0$ [maximum]

At $x = \frac{\pi}{6}$, $f''\left(-\frac{\pi}{6}\right) = -4 \sin 2 \cdot \left(-\frac{\pi}{6}\right) = 4 \sin \frac{\pi}{3}$

$= 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3} > 0$ [minimum]

Now, $f\left(\frac{\pi}{6}\right) = \sin\left(2 \times \frac{\pi}{6}\right) - \frac{\pi}{6} = \left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$

and $f\left(-\frac{\pi}{6}\right) = \sin\left(2 \times \frac{-\pi}{6}\right) + \frac{\pi}{6} = \left(-\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$

$\therefore$ Required difference $= f\left(\frac{\pi}{6}\right) - f\left(-\frac{\pi}{6}\right)$

$= \frac{\sqrt{3}}{2} - \frac{\pi}{6} - \left(-\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

40. Let $f(x) = xe^{-x}$

$\therefore f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$

For maxima or minima, put $f'(x) = 0$.

$\Rightarrow e^{-x}(1-x) = 0 \Rightarrow x = 1$ [$\because e^{-x} > 0, \forall x \in R$]

Further, $f''(x) < 0$ at $x = 1$.

Therefore, $f(x)$ attains its maxima at $x = 1$ and the maximum value is $\frac{1}{e}$.

41. $f'(x) = \frac{1}{8} - \frac{2}{x^2} = \frac{x^2 - 16}{8x^2}$

For maximum or minimum, $f'(x)$ must be vanish.

$\therefore f'(x) = 0 \Rightarrow \frac{x^2 - 16}{8x^2} = 0 \Rightarrow x = 4, -4$

$\therefore x \in [1, 6] \therefore x \neq -4$

Also, in $[1, 4]$, $f'(x) < 0$, then $f(x)$ is decreasing.

In $[4, 6]$, $f'(x) > 0$, then $f(x)$ is increasing.

$\therefore f(1) = \frac{1}{8} + \frac{2}{1} = \frac{17}{8}$

$f(4) = \frac{4}{8} + \frac{2}{4} = \frac{1}{2} + \frac{1}{2} = 1$

and $f(6) = \frac{6}{8} + \frac{2}{6} = \frac{3}{4} + \frac{1}{3} = \frac{13}{12}$

Hence, the maximum value of $f(x)$ in $[1, 6]$ is $\frac{17}{8}$.

42. Given, $f(x) = ax + \frac{b}{x}$; $a, b, x > 0$

$\Rightarrow f'(x) = a - \frac{b}{x^2}$

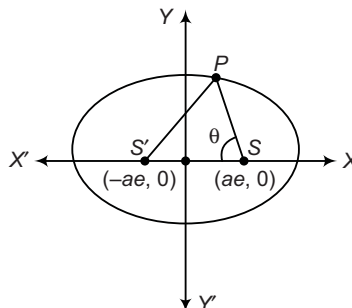
For maxima or minima, put $f'(x) = 0$.

$\Rightarrow x^2 = \frac{b}{a} \Rightarrow x = \pm \sqrt{\frac{b}{a}}$

At $x = \sqrt{\frac{b}{a}}$, $\frac{d^2y}{dx^2} > 0$

So, $f(x)$ attains its minima at $x = \sqrt{b/a}$ and the minimum value is $2\sqrt{ab}$.

43. Let $P(x, y)$ be a point on the ellipse.



$\therefore$ Maximum area of $\Delta SS'P = \frac{1}{2} \times 2ae \times b$ [$\because h_{\max} = b$]
 $= aeb$

44. Given, $f(x) = x^3 e^{-3x}$, $x > 0$

$\Rightarrow f'(x) = 3x^2 e^{-3x}(1-x)$

For maxima or minima, put $f'(x) = 0$.

$\Rightarrow 3x^2 e^{-3x}(1-x) = 0 \Rightarrow x = 1$
[$\because 3x^2 e^{-3x} > 0, \forall x \in R$]

Further at $x = 1$, $f''(x) < 0$.

Hence, at $x = 1$, $f(x)$ attains its maximum value e^{-3} .

45. Given, $f(x) = 2 \sin x + \cos 2x$, $0 \leq x \leq \frac{\pi}{2}$

$\Rightarrow f'(x) = 2 \cos x - 2 \sin 2x$

For maxima or minima, put $f'(x) = 0$.

$\Rightarrow 2 \cos x = 2 \sin 2x \Rightarrow \cos x = 2 \sin x \cos x$

$\Rightarrow \sin x = \frac{1}{2}$

$\therefore x = \frac{\pi}{6}$

At $x = \frac{\pi}{6}$, $f''(x) < 0$.

Therefore, $f(x)$ attains its maximum at $x = \frac{\pi}{6}$.

46. Given, $f(x) = \frac{x^2 - 1}{x^2 + 1}$

$\Rightarrow f'(x) = \frac{4x}{(x+1)^2}$

For maxima or minima, put $f'(x) = 0$.

$\Rightarrow \frac{4x}{(x+1)^2} = 0$

$\therefore x = 0$

At $x = 0$, $f''(x) > 0$.

Therefore, $f(x)$ has a minima at $x = 0$ and the minimum value is -1 .

9

Indefinite Integrals

Integration as Reverse Process of Differentiation

The concept of integration originated during the course of finding the area of a plane figure. It is based on the limit of the sum of the series whose each term tends to zero and the number of terms that tends to infinity. Infact, it is called integration because of the process of summation, as integration means summation. But later it was observed that integration is just the inverse process of differentiation. This helps to find the function whose differential coefficient is known.

Thus, a function $F(x)$ is called a primitive or an anti-derivative of a function $f(x)$, if

$$F'(x) = f(x)$$

Since, the differential coefficient of a constant is zero.

$$\therefore \frac{d}{dx}[F(x) + C] = \frac{d}{dx}F(x) + 0 = f(x)$$

Hence,

$$\int f(x)dx = F(x) + C$$

where, $\int$ is the notation of integration, $f(x)$ is called the integrand and x is called the variable of integration. C is constant of integration and can take any constant value. This shows that $F(x)$ and $F(x) + C$ are both integrals of the same function $f(x)$. Thus, for different value of C , we obtain different integral of $f(x)$. This implies that the integral of $f(x)$ is not definite. Due to this, $F(x)$ is called the indefinite integral of $f(x)$.

Fundamental Formulae on Integration

$$\text{Since, } \frac{d}{dx}\{\phi(x)\} = f(x) \Rightarrow \int f(x)dx = \phi(x) + C$$

Therefore, based upon this definition and various standard differentiation formulae, we obtain the following integration formulae.

$$(i) \int x^n dx = \frac{x^{n+1}}{n+1} + C, n \neq -1$$

$$(ii) \int \frac{1}{x} dx = \log |x| + C$$

Chapter Snapshot

- Integration as Reverse Process of Differentiation
- Fundamental Formulae on Integration
- Some Standard Results on Integration
- Method of Integration
- Integration by Substitution
- Integration using Trigonometric Identities
- Some Standard Integrals
- Evaluation of the Various Forms of Integrals by use of Standard Results
- Integration by Parts
- Integration by Partial Fraction
- Integration of Some Special Irrational Algebraic Functions
- Reduction Formulae

$$(iii) \int e^x dx = e^x + C$$

$$(iv) \int a^x dx = \frac{a^x}{\log a} + C, a > 0, a \neq 1$$

$$(v) \int \sin x dx = -\cos x + C$$

$$(vi) \int \cos x dx = \sin x + C$$

$$(vii) \int \sec^2 x dx = \tan x + C$$

$$(viii) \int \operatorname{cosec}^2 x dx = -\cot x + C$$

$$(ix) \int \sec x \tan x dx = \sec x + C$$

$$(x) \int \operatorname{cosec} x \cot x dx = -\operatorname{cosec} x + C$$

$$(xi) \int \cot x dx = \log |\sin x| + C$$

$$(xii) \int \tan x dx = -\log |\cos x| + C$$

$$(xiii) \int \sec x dx = \log |\sec x + \tan x| + C$$

$$(xiv) \int \operatorname{cosec} x dx = \log |\operatorname{cosec} x - \cot x| + C$$

$$(xv) \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$(xvi) \int -\frac{1}{\sqrt{a^2 - x^2}} dx = \cos^{-1} \left(\frac{x}{a} \right) + C$$

$$(xvii) \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$(xviii) \int -\frac{1}{a^2 + x^2} dx = \frac{1}{a} \cot^{-1} \left(\frac{x}{a} \right) + C$$

$$(xix) \int \frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \sec^{-1} \left(\frac{x}{a} \right) + C$$

$$(xx) \int -\frac{1}{x\sqrt{x^2 - a^2}} dx = \frac{1}{a} \operatorname{cosec}^{-1} \left(\frac{x}{a} \right) + C$$

Some Standard Results on Integration

$$(i) \int k f(x) dx = k \int f(x) dx, \text{ where } k \text{ is constant}$$

i.e. the integral of the product of a constant and a function = constant $\times$ integral of the function

$$(ii) \int \{f_1(x) \pm f_2(x) \pm \dots \pm f_n(x)\} dx$$

$$= \int f_1(x) dx \pm \int f_2(x) dx \pm \dots \pm \int f_n(x) dx$$

i.e. the integral of the sum or difference of a finite number of functions is equal to the sum or difference of the integrals of the various functions.

➤ **Example 1.** $\int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx$ is equal to

$$(a) \frac{\left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3}{3} + C$$

$$(b) 2 \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right) \left(x^{\frac{3}{2}} + \frac{1}{x^{\frac{3}{2}}} \right) + C$$

$$(c) \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^3 \left(\frac{2}{3} x^{3/2} + \frac{3}{2} \frac{1}{x^{3/2}} \right) + C$$

(d) None of the above

$$\begin{aligned} \text{Sol. (d)} \int \left(\sqrt{x} + \frac{1}{\sqrt{x}} \right)^2 dx &= \int \left(x + \frac{1}{x} + 2 \right) dx \\ &= \int x dx + \int \frac{1}{x} dx + \int 2 dx \\ &= \frac{x^2}{2} + \log |x| + 2x + C \end{aligned}$$

➤ **Example 2.** $\int \frac{x^3 + 5x^2 + 4x + 1}{x^2} dx$ is equal to

$$(a) \frac{\frac{x^4}{4} + \frac{5x^3}{3} + \frac{4x^2}{2} + x}{\frac{x^3}{3}} + C$$

$$(b) \frac{x^2}{2} + 5x + 4 \log |x| - \frac{1}{x} + C$$

$$(c) \frac{3x^2 + 10x + 4}{2x} + C$$

(d) None of the above

$$\begin{aligned} \text{Sol. (b)} \int \frac{x^3 + 5x^2 + 4x + 1}{x^2} dx &= \int \left(\frac{x^3}{x^2} + \frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2} \right) dx \\ &= \int x dx + \int 5 dx + \int \frac{4}{x} dx + \int \frac{1}{x^2} dx \\ &= \frac{x^2}{2} + 5x + 4 \log |x| - \frac{1}{x} + C \end{aligned}$$

➤ **Example 3.** $\int \frac{dx}{\sin^2 x \cos^2 x}$ is equal to

$$(a) \tan x + \cot x + C$$

$$(b) \tan x - \cot x + C$$

$$(c) \tan x \cot x + C$$

$$(d) \tan x - \cot 2x + C$$

$$\begin{aligned} \text{Sol. (b)} \int \frac{dx}{\sin^2 x \cos^2 x} &= \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x} dx \\ &= \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} \\ &= \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx \\ &= \tan x - \cot x + C \end{aligned}$$

➤ **Example 4.** $\int \frac{2^x + 3^x}{5^x} dx$ is equal to

$$\begin{aligned} (a) & \frac{2^x}{\log 2} + \frac{3^x}{\log 3} - \frac{5^x}{\log 5} + C \\ (b) & \frac{2/5}{x \log_e (2/5)} + \frac{3/5}{x \log_e (3/5)} + C \\ (c) & \frac{\left(\frac{2}{5}\right)^x}{\log_e \left(\frac{2}{5}\right)} + \frac{\left(\frac{3}{5}\right)^x}{\log_e \left(\frac{3}{5}\right)} + C \\ (d) & \text{None of the above} \end{aligned}$$

$$\begin{aligned} \text{Sol. (c)} \int \frac{2^x + 3^x}{5^x} dx &= \int \left(\frac{2^x}{5^x} + \frac{3^x}{5^x} \right) dx \\ &= \int \left[\left(\frac{2}{5}\right)^x + \left(\frac{3}{5}\right)^x \right] dx \\ &= \frac{\left(\frac{2}{5}\right)^x}{\log_e \frac{2}{5}} + \frac{\left(\frac{3}{5}\right)^x}{\log_e \frac{3}{5}} + C \end{aligned}$$

➤ **Example 5.** $\int \frac{(a^x + b^x)^2}{a^x b^x} dx$ is equal to

$$\begin{aligned} (a) & \frac{(b/a)^x}{\log_e (a/b)} + \frac{(a/b)^x}{\log_e (b/a)} + x + C \\ (b) & \frac{(a/b)^x}{\log_e (b/a)} + \frac{(b/a)^x}{\log_e (a/b)} + x + C \\ (c) & \frac{(a/b)^x}{\log_e (b/a)} + \frac{(b/a)^x}{\log_e (a/b)} + 2x + C \\ (d) & \frac{(a/b)^x}{\log_e (a/b)} + \frac{(b/a)^x}{\log_e (b/a)} + 2x + C \end{aligned}$$

$$\begin{aligned} \text{Sol. (d)} \int \frac{(a^x + b^x)^2}{a^x b^x} dx &= \int \frac{a^{2x} + b^{2x} + 2a^x b^x}{a^x b^x} dx \\ &= \int \left(\frac{a^x}{b^x} + \frac{b^x}{a^x} + 2 \right) dx \\ &= \int \left\{ \left(\frac{a}{b}\right)^x + \left(\frac{b}{a}\right)^x + 2 \right\} dx \\ &= \frac{(a/b)^x}{\log_e (a/b)} + \frac{(b/a)^x}{\log_e (b/a)} + 2x + C \end{aligned}$$

➤ **Example 6.** $\int \frac{e^{5 \log_e x} - e^{4 \log_e x}}{e^{3 \log_e x} - e^{2 \log_e x}} dx$ is equal to

$$(a) \frac{x^3}{3} + C \quad (b) \frac{x^2}{3} + C \quad (c) \frac{x^4}{3} + C \quad (d) \frac{x^3}{2} + C$$

$$\begin{aligned} \text{Sol. (a)} \int \frac{e^{5 \log_e x} - e^{4 \log_e x}}{e^{3 \log_e x} - e^{2 \log_e x}} dx &= \int \frac{x^5 - x^4}{x^3 - x^2} dx \quad [\because e^{a \log_e x} = x^a] \\ &= \int \frac{x^4(x-1)}{x^2(x-1)} dx \\ &= \int x^2 dx = \frac{x^3}{3} + C \end{aligned}$$

Methods of Integration

We have the following methods of integration:

Method I Integration by substitution or change of independent variable.

Method II Integration by parts

Method III Integration of rational algebraic function by using partial fractions.

In the following sections, we shall learn about these methods.

Integration by Substitution

Integrations of certain functions cannot be obtained directly, if they are not in one of the standard forms discussed in previous sections, but they may be reduced to standard forms by proper substitution. The process of evaluating an integral by reducing it to standard form by a proper substitution is called integration by substitution.

I. If $\phi(x)$ is a continuous differentiable function, then to evaluate integrals of the form

$$\int f(\phi(x)) \phi'(x) dx,$$

we substitute $\phi(x) = t$ and $\phi'(x) dx = dt$

This substitution reduces the integral $\int f(\phi(x)) \phi'(x) dx$ to the form $\int f(t) dt$

This integral can be evaluated by using the standard results.

After evaluating this integral, we substitute back the value of t as $\phi(x)$.

The above substitution can also be performed in the following manner

$$\begin{aligned} \int f(\phi(x)) \phi'(x) dx &= \int f(\phi(x)) d(\phi(x)) \\ &[\because d\phi(x) = \phi'(x) dx] \end{aligned}$$

$$\Rightarrow \int f(\phi(x)) \phi'(x) dx = \int f(t) dt, \text{ where } t = \phi(x)$$

II. (i) If $\int f(x)dx = g(x) + C$, then

$$\int f(ax+b)dx = \frac{1}{a} g(ax+b) + C$$

(ii) If $\int \frac{1}{x} dx = \log |x| + C$, then

$$\int \frac{1}{ax+b} dx = \frac{1}{a} \log |ax+b| + C$$

Thus, in any fundamental integral formulae given in first article, if in place of x , we have $(ax+b)$, then same formula is applicable but we must divide by coefficient of x or derivative of $(ax+b)$, i.e. a .

We give the list of some frequently used formulae

$$(i) \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C, n \neq -1$$

$$(ii) \int \frac{1}{ax+b} dx = \frac{1}{a} \log |ax+b| + C$$

$$(iii) \int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$$

$$(iv) \int a^{bx+d} dx = \frac{1}{b} \cdot \frac{a^{bx+d}}{\log a} + C$$

$$(v) \int \sin(ax+b) dx = -\frac{1}{a} \cos(ax+b) + C$$

$$(vi) \int \cos(ax+b) dx = \frac{1}{a} \sin(ax+b) + C$$

$$(vii) \int \sec^2(ax+b) dx = \frac{1}{a} \tan(ax+b) + C$$

$$(viii) \int \operatorname{cosec}^2(ax+b) dx = -\frac{1}{a} \cot(ax+b) + C$$

$$(ix) \int \sec(ax+b) \tan(ax+b) dx = \frac{1}{a} \sec(ax+b) + C$$

$$(x) \int \operatorname{cosec}(ax+b) \cot(ax+b) dx = -\frac{1}{a} \operatorname{cosec}(ax+b) + C$$

$$(xi) \int \tan(ax+b) dx = -\frac{1}{a} \log |\cos(ax+b)| + C$$

$$(xii) \int \cot(ax+b) dx = \frac{1}{a} \log |\sin(ax+b)| + C$$

$$(xiii) \int \sec(ax+b) dx$$

$$= \frac{1}{a} \log |\sec(ax+b) + \tan(ax+b)| + C$$

$$(xiv) \int \operatorname{cosec}(ax+b) dx$$

$$= \frac{1}{a} \log |\operatorname{cosec}(ax+b) - \cot(ax+b)| + C$$

➤ **Example 7.** $\int \frac{dx}{x(\log x)^m}$ is equal to

$$(a) (\log x)^{1-m} + C$$

$$(b) \frac{1}{m} (\log x)^m + C$$

$$(c) \frac{(\log x)^{1-m}}{1-m} + C$$

$$(d) \text{None of the above}$$

Sol. (c) Let $\log x = t$

$$\Rightarrow \frac{1}{x} = \frac{dt}{dx}$$

$$\Rightarrow dx = x dt$$

$$\therefore \int \frac{dx}{x(\log x)^m} = \int \frac{x}{x(t)^m} dt = \int t^{-m} dt$$

$$= \frac{t^{-m+1}}{-m+1} + C = \frac{(\log x)^{1-m}}{1-m} + C$$

➤ **Example 8.** $\int \frac{e^{2x}-1}{e^{2x}+1} dx$ is equal to

$$(a) \frac{e^x + e^{-x}}{e^x - e^{-x}} + C$$

$$(b) \log \frac{e^x + e^{-x}}{e^x - e^{-x}} + C$$

$$(c) \log |e^x + e^{-x}| + C$$

$$(d) \log |e^x - e^{-x}| + C$$

Sol. (c) $\int \frac{e^{2x}-1}{e^{2x}+1} dx = \int \frac{(e^x \cdot e^x - 1)}{(e^x \cdot e^x + 1)} dx$

$$= \int \frac{e^x \left(e^x - \frac{1}{e^x} \right)}{e^x \left(e^x + \frac{1}{e^x} \right)} dx = \int \frac{(e^x - e^{-x})}{(e^x + e^{-x})} dx$$

$$\text{Let } (e^x + e^{-x}) = t \Rightarrow e^x - e^{-x} = \frac{dt}{dx}$$

$$\Rightarrow dx = \frac{dt}{e^x - e^{-x}}$$

$$\therefore \int \frac{e^{2x}-1}{e^{2x}+1} dx = \int \frac{(e^x - e^{-x})}{t} \cdot \frac{dt}{(e^x - e^{-x})}$$

$$= \int \frac{dt}{t} = \log |t| + C$$

$$= \log |e^x + e^{-x}| + C$$

➤ **Example 9.** $\int x^2 e^{x^3} \cos(e^{x^3}) dx$ is equal to

$$(a) \sin(e^{x^3}) + C$$

$$(b) 3 \sin(e^{x^3}) + C$$

$$(c) \frac{1}{3} \sin(e^{x^3}) + C$$

$$(d) e^x \sin(e^{x^3}) + C$$

Sol. (c) Let $e^{x^3} = t$

$$\Rightarrow 3x^2 e^{x^3} dx = dt$$

$$\therefore \int x^2 e^{x^3} \cos(e^{x^3}) dx = \frac{1}{3} \int \cos t dt$$

$$= \frac{1}{3} \sin t + C = \frac{1}{3} \sin(e^{x^3}) + C$$

➤ **Example 10.** The value of

$$\int \left(x + \frac{1}{x} \right)^{\frac{3}{2}} \left(\frac{x^2 - 1}{x^2} \right) dx \text{ is}$$

(a) $\frac{2}{3} \left(x + \frac{1}{x} \right)^{\frac{3}{2}} + C$

(b) $\frac{2}{5} \left(x + \frac{1}{x} \right)^{\frac{5}{2}} + C$

(c) $2 \left(x + \frac{1}{x} \right)^{\frac{1}{2}} + C$

(d) None of the above

Sol. (b) Let $x + \frac{1}{x} = t$

$$\Rightarrow \left(1 - \frac{1}{x^2} \right) dx = dt$$

$$\begin{aligned} \therefore \int \left(x + \frac{1}{x} \right)^{\frac{3}{2}} \left(\frac{x^2 - 1}{x^2} \right) dx &= \int t^{\frac{3}{2}} dt \\ &= \frac{2}{5} t^{\frac{5}{2}} + C \\ &= \frac{2}{5} \left(x + \frac{1}{x} \right)^{\frac{5}{2}} + C \end{aligned}$$

➤ **Example 11.** If $\int \frac{2^{1/x}}{x^2} dx = a 2^{1/x} + C$, then a is

equal to

(a) $-\log_2 e$

(b) $-\log_e 2$

(c) -1

(d) $\frac{1}{2}$

Sol. (a) Let $\frac{1}{x} = t \Rightarrow -\frac{1}{x^2} dx = dt$

$$\begin{aligned} \therefore \int \frac{2^{1/x}}{x^2} dx &= -\int 2^t dt = -\frac{2^t}{\log_e 2} + C \\ &= \frac{-2^{\frac{1}{x}}}{\log_e 2} + C \\ &= -\log_2 e \cdot 2^{\frac{1}{x}} + C \\ \therefore a &= -\log_2 e \end{aligned}$$

➤ **Example 12.** $\int \frac{x}{x - \sqrt{x^2 - 1}} dx$ is equal to

(a) $\frac{x^3}{3} + \frac{1}{2} (x^2 - x - 1)^{3/2} + C$

(b) $\frac{x^3}{3} + \frac{1}{3} (x^2 - 1)^{3/2} + C$

(c) $\frac{x^3}{3} + \frac{2}{3} (x^2 - 1)^{3/2} + C$

(d) $\frac{x^3}{3} + \frac{2}{3} (x^2 - x - 1)^{3/2} + C$

Sol. (b) We have, $\int \frac{x}{x - \sqrt{x^2 - 1}} dx$

$$\begin{aligned} &= \int x (x + \sqrt{x^2 - 1}) dx \\ &= \int x^2 dx + \int x \sqrt{x^2 - 1} dx \\ &= \frac{x^3}{3} + \frac{1}{2} \int 2x \cdot \sqrt{x^2 - 1} dx \\ &= \frac{x^3}{3} + \frac{1}{2} \times \frac{2}{3} (x^2 - 1)^{3/2} + C \\ &= \frac{x^3}{3} + \frac{1}{3} (x^2 - 1)^{3/2} + C \end{aligned}$$

Integration using Trigonometric Identities

1. Integrals of the form $\int \sin mx \cos nx dx$,

$$\int \sin mx \sin nx dx, \int \cos mx \sin nx dx \text{ and } \int \cos mx \cos nx dx$$

To evaluate integrals of the form

$$\int \sin mx \cos nx dx, \int \sin mx \sin nx dx,$$

$$\int \cos mx \cos nx dx \text{ and } \int \cos mx \sin nx dx, \text{ we use}$$

the following trigonometrical identities:

(i) $2 \sin A \cdot \cos B = \sin (A + B) + \sin (A - B)$

(ii) $2 \cos A \cdot \sin B = \sin (A + B) - \sin (A - B)$

(iii) $2 \cos A \cdot \cos B = \cos (A + B) + \cos (A - B)$

(iv) $2 \sin A \cdot \sin B = \cos (A - B) - \cos (A + B)$

(v) $2 \sin A \cdot \cos A = \sin 2A$

(vi) $\cos^2 A = \left(\frac{1 + \cos 2A}{2} \right)$

(vii) $\sin^2 A = \left(\frac{1 - \cos 2A}{2} \right)$

(viii) $\cos^2 A - \sin^2 A = \cos 2A$

(ix) $\sin^2 A + \cos^2 A = 1$

➤ **Example 13.** $\int \cos 4x \cos 7x dx$ is equal to

(a) $\sin 3x + \sin 11x + C$ (b) $\frac{\sin 3x}{3} + \frac{\sin 11x}{11} + C$

(c) $\frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$ (d) None of these

Sol. (c) Let $I = \int \cos 4x \cos 7x dx$

$$\text{Here, } \cos 4x \cos 7x = \frac{1}{2} (\cos 3x + \cos 11x)$$

$$I = \int \cos 4x \cos 7x dx = \frac{1}{2} \int (\cos 3x + \cos 11x) dx$$

$$= \frac{1}{2} \int \cos 3x dx + \frac{1}{2} \int \cos 11x dx$$

$$= \frac{\sin 3x}{6} + \frac{\sin 11x}{22} + C$$

➤ **Example 14.** The value of $\int \frac{1}{1 + \cos 16x} dx$ is

- (a) $\frac{\tan 4x}{16} + C$ (b) $\frac{\tan 16x}{16} + C$
 (c) $\frac{\tan 8x}{4} + C$ (d) $\frac{\tan 8x}{16} + C$

Sol. (d) Let $I = \int \frac{dx}{1 + \cos 16x} = \int \frac{dx}{2 \cos^2 8x}$
 $= \frac{1}{2} \int \sec^2 8x dx = \frac{1}{2} \cdot \frac{\tan 8x}{8} + C$
 $= \frac{\tan 8x}{16} + C$

2. Integrals of the form $\int \sin^m x \cos^n x dx$,

$\int \sin^m x dx$, $\int \cos^n x dx$

If $m, n \in \mathbb{N}$, then to evaluate integrals of the forms, $\int \sin^m x \cos^n x dx$, $\int \sin^m x dx$ and

$\int \cos^n x dx$, we use the following algorithm:

Algorithm

Step I Find m and n .

Step II If m is odd, i.e. power or index of $\sin x$ is odd, then put $\cos x = t$ and reduce the integrand in terms of t .

If n is odd, i.e. power of $\cos x$ is odd, then put $\sin x = t$ and reduce the integrand in terms of t .

If m and n both are odd, then either of the above substitutions can be used.

Step III Evaluate the integral obtained in Step II and replace t by its value.

➤ **Example 15.** $\int \sin^3 x \cos^3 x dx$ is equal to

- (a) $\frac{\sin^6 x}{6} - \frac{\cos^4 x}{4} + C$
 (b) $\frac{\sin^6 x}{3} - \frac{\cos^4 x}{2} + C$
 (c) $\frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C$
 (d) None of the above

Sol. (c) Put $\cos x = t$

$$\Rightarrow -\sin x = \frac{dt}{dx}$$

$$\Rightarrow dx = -\frac{dt}{\sin x}$$

$$\begin{aligned} \therefore \int \sin^3 x \cos^3 x dx &= \int \sin^2 x \cdot t^3 \cdot \frac{dt}{-\sin x} \\ &= -\int \sin x \cdot t^3 dt = -\int t^3 (1 - t^2) dt \\ &= -\int (t^3 - t^5) dt = -\left(\frac{t^4}{4} - \frac{t^6}{6}\right) + C \\ &= -\left[\frac{t^4}{4} - \frac{t^6}{6}\right] + C = \frac{\cos^6 x}{6} - \frac{\cos^4 x}{4} + C \end{aligned}$$

➤ **Example 16.** If $\int \cos^7 x dx = A \sin^7 x + B \sin^5 x + C \sin^3 x + \sin x + k$, then

- (a) $A = \frac{1}{7}, B = \frac{3}{5}, C = -1$
 (b) $A = -\frac{1}{7}, B = \frac{3}{5}, C = -1$
 (c) $A = -\frac{1}{7}, B = \frac{1}{5}, C = -1$
 (d) $A = \frac{1}{7}, B = \frac{3}{5}, C = 1$

Sol. (b) We have, $I = \int \cos^7 x dx = \int (1 - \sin^2 x)^3 d(\sin x)$

$$\Rightarrow I = \int (1 - 3\sin^2 x + 3\sin^4 x - \sin^6 x) d(\sin x)$$

$$\Rightarrow I = \sin x - \sin^3 x + \frac{3}{5} \sin^5 x - \frac{1}{7} \sin^7 x + k$$

$$\therefore A = -\frac{1}{7}, B = \frac{3}{5} \text{ and } C = -1$$

3. Integrals of the form $\int \sin^m x \cos^n x dx$,

where $m, n \in \mathbb{Q}$ and $m + n$ is a negative even integer

In order to evaluate this form of integrals, we use the following algorithm:

Algorithm

Step I Change the integrand in terms of $\tan x$ and $\sec^2 x$ by dividing numerator and denominator by $\cos^k x$, where $k = -(m + n)$.

Step II Substitute $\tan x = t$

Step III Evaluate the integral obtained in step II and replace t by its value.

➤ **Example 17.** $\int \sin^{-7/5} x \cos^{-3/5} x dx$ is equal to

- (a) $\frac{-2}{5} (\tan x)^{2/5} + C$ (b) $-\frac{5}{2} (\cot x)^{2/5} + C$
 (c) $\frac{-5}{2} (\tan x)^{2/5} + C$ (d) None of these

Sol. (b) Here, $p = -\frac{7}{5}, q = -\frac{3}{5}$

$$p + q = -2$$

$$\text{Let } I = \int \sin^{-7/5} x \cos^{-3/5} x dx = \int \frac{\cos^{-3/5} x \cdot \cos^2 x}{\sin^{7/5} x \cos^2 x} dx$$

$$= \int \frac{\cos^{7/5} x}{\sin^{7/5} x \cos^2 x} dx$$

$$= \int \cot^{7/5} x \sec^2 x dx$$

$$= \int \frac{\sec^2 x}{\tan^{7/5} x} dx$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int \frac{dt}{t^{7/5}} = \int t^{-7/5} dt = -\frac{5}{2} t^{-2/5} + C$$

$$= -\frac{5}{2} (\tan x)^{-2/5} + C = -\frac{5}{2} (\cot x)^{2/5} + C$$

Work Book Exercise 9.1

1 $\int (3 \operatorname{cosec}^2 x + 2 \sin 3x) dx$ is equal to

- a $3 \cot x + \frac{2}{3} \cos 3x + C$
 b $-\left(3 \cot x + \frac{2}{3} \cos 3x\right) + C$
 c $3 \cot x - \frac{2}{3} \cos 3x + C$
 d None of the above

2 $\int \left(2 \sin x + \frac{1}{x}\right) dx$ is equal to

- a $-2 \cos x + \log x + C$
 b $2 \cos x + \log x + C$
 c $-2 \sin x - \frac{1}{x^2} + C$
 d $-2 \cos x + \frac{1}{x^2} + C$

3 $\int \frac{\tan(\log x)}{x} dx$ is equal to

- a $\log \cos(\log x) + C$
 b $\log \sin(\log x) + C$
 c $\log \sec(\log x) + C$
 d $\log \operatorname{cosec}(\log x) + C$

4 If $\int e^x(1+x) \cdot \sec^2(xe^x) dx = f(x) + C$, then $f(x)$ is equal to

- a $\cos(xe^x)$ b $\sin(xe^x)$
 c $2 \tan^{-1}(x)$ d $\tan(xe^x)$

5 $\int \tan(3x-5) \sec(3x-5) dx$ is equal to

- a $\sec(3x-5) + C$ b $\frac{1}{3} \sec(3x-5) + C$
 c $\tan(3x-5) + C$ d None of these

6 $\int \frac{dx}{x\sqrt{1-(\log x)^2}}$ is equal to

- a $\cos^{-1}(\log x) + C$ b $x \log(1-x^2) + C$
 c $\sin^{-1}(\log x) + C$ d $\frac{1}{2} \cos^{-1}(\log x) + C$

7 $\int \sin^3 x \cdot \cos x dx$ is equal to

- a $\frac{\sin^4 x \cos^2 x}{8} + C$ b $\frac{\sin^4 x}{4} + C$
 c $\frac{\sin^2 x}{2} + C$ d $4 \sin^4 x + C$

8 $\int 5^{\log_e x} dx$ is

- a $\frac{x^{\log_e 5+1}}{\log_e 5+1}$ b $5^{\log_e x} + C$
 c $\frac{5^{\log_e x} + 1}{\log_e x + 1}$ d None of these

9 $\int \frac{d\theta}{\sin \theta \cos^3 \theta}$ is equal to

- a $\log \tan \theta + \tan^2 \theta + C$
 b $\log \tan \theta - \frac{1}{2} \tan^2 \theta + C$
 c $\log \tan \theta + \frac{1}{2} \tan^2 \theta + C$
 d None of the above

10 $\int \cos^5 x dx$ is equal to

- a $\sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$
 b $\sin x + \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x + C$
 c $\sin x - \frac{2}{3} \sin^3 x - \frac{1}{5} \sin^5 x + C$
 d None of the above

11 $\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$ is equal to

- a $\log \sin 3x - \log \sin 5x + C$
 b $\frac{1}{3} \log \sin 3x + \frac{1}{5} \log \sin 5x + C$
 c $\frac{1}{3} \log \sin 3x - \frac{1}{5} \log \sin 5x + C$
 d $3 \log \sin 3x - 5 \log \sin 5x + C$

12 $\int \tan^4 x dx$ is equal to

- a $\tan^3 x - \tan x + x + C$
 b $\frac{1}{3} \tan^3 x - \tan x + x + C$
 c $\frac{1}{3} \tan^3 x + \tan x + x + C$
 d $\frac{1}{3} \tan^3 x + \tan x + 2x + C$

13 $\int \sec^{2/3} x \operatorname{cosec}^{4/3} x dx$ is equal to

- a $-3(\tan x)^{1/3} + C$ b $-3(\tan x)^{-1/3} + C$
 c $3(\tan x)^{-1/3} + C$ d $(\tan x)^{-1/3} + C$

Some Standard Integrals

In this section, we shall first discuss some standard integrals. The following substitutions are very useful to evaluate these integrals:

Standard Substitutions

	Integrand form	Substitution
(i)	$\sqrt{a^2 - x^2}, \frac{1}{\sqrt{a^2 - x^2}}, a^2 - x^2$	$x = a \sin \theta$ or $x = a \cos \theta$
(ii)	$\sqrt{x^2 + a^2}, \frac{1}{\sqrt{x^2 + a^2}}, x^2 + a^2$	$x = a \tan \theta, a \cot \theta$ or $x = a \sinh \theta$

	Integrand form	Substitution
(iii)	$\sqrt{x^2 - a^2}, \frac{1}{x^2 - a^2}, x^2 - a^2$	$x = a \sec \theta, a \operatorname{cosec} \theta$ or $x = a \cosh \theta$
(iv)	$\sqrt{\frac{x}{a+x}}, \sqrt{\frac{a+x}{x}}, \sqrt{x(a+x)}, \frac{1}{\sqrt{x \cdot (a+x)}}$	$x = a \tan^2 \theta, a \cot^2 \theta$
(v)	$\sqrt{\frac{x}{a-x}}, \sqrt{\frac{a-x}{x}}, \sqrt{x(a-x)}, \frac{1}{\sqrt{x(a-x)}}$	$x = a \sin^2 \theta, a \cos^2 \theta$
(vi)	$\sqrt{\frac{x}{x-a}}, \sqrt{\frac{x-a}{x}}, \sqrt{x(x-a)}, \frac{1}{\sqrt{x(x-a)}}$	$x = a \sec^2 \theta,$ $a \operatorname{cosec}^2 \theta$
(vii)	$\sqrt{\frac{a-x}{a+x}}, \sqrt{\frac{a+x}{a-x}}$	$x = a \cos 2\theta$
(viii)	$\sqrt{\frac{x-\alpha}{\beta-x}}, \sqrt{(x-\alpha)(\beta-x)}, (\beta > \alpha)$	$x = \alpha \cos^2 \theta + \beta \sin^2 \theta$

Some More Results

$$1. \int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\text{or } \int \frac{1}{a^2 + x^2} dx = -\frac{1}{a} \cot^{-1} \left(\frac{x}{a} \right) + C$$

$$2. \int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| + C$$

$$3. \int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \left| \frac{a+x}{a-x} \right| + C$$

$$4. \int \frac{1}{\sqrt{a^2 + x^2}} dx = \log |x + \sqrt{a^2 + x^2}| + C$$

$$5. \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$\text{or } \int \frac{1}{\sqrt{a^2 - x^2}} dx = -\cos^{-1} \left(\frac{x}{a} \right) + C$$

$$6. \int \frac{1}{\sqrt{x^2 - a^2}} dx = \log |x + \sqrt{x^2 - a^2}| + C$$

$$7. \int \sqrt{a^2 - x^2} dx = \frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \left(\frac{x}{a} \right) + C$$

$$8. \int \sqrt{a^2 + x^2} dx = \frac{1}{2} x \sqrt{a^2 + x^2} + \frac{1}{2} a^2 \log |x + \sqrt{a^2 + x^2}| + C$$

$$9. \int \sqrt{x^2 - a^2} dx = \frac{1}{2} x \sqrt{x^2 - a^2} - \frac{1}{2} a^2 \log |x + \sqrt{x^2 - a^2}| + C$$

➤ **Example 18.** $\int \frac{dx}{\sqrt{9 - 25x^2}}$ is equal to

$$(a) \sin^{-1} \left(\frac{5x}{3} \right) + C \quad (b) \frac{3}{2} \sin^{-1} \left(\frac{5x}{3} \right) + C$$

$$(c) \frac{1}{5} \sin^{-1} \left(\frac{5x}{3} \right) + C \quad (d) \text{None of these}$$

Sol. (c) $\int \frac{dx}{\sqrt{9 - 25x^2}} = \int \frac{dx}{\sqrt{25 \left[\left(\frac{3}{5} \right)^2 - x^2 \right]}}$

$$= \frac{1}{5} \int \frac{dx}{\sqrt{\left(\frac{3}{5} \right)^2 - x^2}} = \frac{1}{5} \sin^{-1} \left(\frac{x}{3/5} \right) + C$$

$$\left[\because \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \left(\frac{x}{a} \right) \right]$$

$$= \frac{1}{5} \sin^{-1} \left(\frac{5x}{3} \right) + C$$

➤ **Example 19.** $\int \frac{dx}{\sqrt{1 + 4x^2}}$ is equal to

$$(a) \frac{1}{2} \log |x + \sqrt{2x^2 + 1}| + C$$

$$(b) \frac{1}{2} \log |2x + \sqrt{4x^2 + 1}| + C$$

$$(c) \log |2x + \sqrt{4x^2 + 1}| + C$$

$$(d) \text{None of the above}$$

Sol. (b) $\int \frac{dx}{\sqrt{1 + 4x^2}} = \int \frac{dx}{\sqrt{4 \left\{ \left(\frac{1}{2} \right)^2 + x^2 \right\}}}$

$$= \frac{1}{2} \int \frac{dx}{\sqrt{x^2 + \left(\frac{1}{2} \right)^2}} = \frac{1}{2} \log \left| x + \sqrt{x^2 + \left(\frac{1}{2} \right)^2} \right| + C$$

$$\left[\because \int \frac{1}{\sqrt{x^2 + a^2}} dx = \log |x + \sqrt{x^2 + a^2}| \right]$$

$$= \frac{1}{2} \log \left| x + \frac{\sqrt{4x^2 + 1}}{2} \right| + C_1$$

$$= \frac{1}{2} \log |2x + \sqrt{4x^2 + 1}| - \frac{1}{2} \log 2 + C_1$$

$$\left[\because \log \left(\frac{m}{n} \right) = \log m - \log n \right]$$

$$= \frac{1}{2} \log |2x + \sqrt{4x^2 + 1}| + C$$

$$\left[\because \frac{1}{2} \log 2 \text{ is constant and constant + constant} = C \right]$$

Evaluation of the Various Forms of Integrals by use of Standard Results

Working Steps

Step I Make the coefficient of x^2 unity by taking the coefficient of x^2 outside the quadratic equation.

Step II Complete the square in the terms involving x by making it perfect square, i.e. $a[(x \pm \alpha)^2 \pm \beta^2]$.

Step III The integrand is converted to one of the nine special integrals, as given below.

Step IV Integrate the function, as usually using the standard formulae.

Some Various Forms

(i) If the integrals are of the form $\int \frac{dx}{ax^2 + bx + c}$,

$$\int \frac{dx}{\sqrt{ax^2 + bx + c}} \text{ and } \int \sqrt{ax^2 + bx + c} dx, \text{ then}$$

express $ax^2 + bx + c$ as the sum or difference of two squares, i.e. in the form of perfect square and then apply the standard results.

Working Rule

We write $\int \frac{dx}{ax^2 + bx + c} = \frac{1}{a} \int \frac{dx}{x^2 + \frac{b}{a}x + \frac{c}{a}}$

$$= \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \frac{c}{a} - \frac{b^2}{4a^2}}, \text{ which is of the form}$$

$$\int \frac{dx}{X^2 - A^2}, \int \frac{dx}{X^2 + A^2}$$

or $\int \frac{dx}{A^2 - X^2}.$

Similarly, $\int \frac{dx}{\sqrt{ax^2 + bx + c}}$ can be reduced to one

of the form $\int \frac{dx}{\sqrt{X^2 - A^2}}, \int \frac{dx}{\sqrt{A^2 - X^2}}$ or

$\int \frac{dx}{\sqrt{A^2 + X^2}}$ and also $\int \sqrt{ax^2 + bx + c} dx$ can be

reduced to one of the form $\int \sqrt{A^2 - X^2} dx,$

$\int \sqrt{X^2 - A^2} dx$ or $\int \sqrt{A^2 + X^2} dx$ and then apply the standard result.

➤ **Example 20.** $\int \frac{dx}{9x^2 + 6x + 5}$ is equal to

(a) $\tan^{-1}\left(\frac{3x+1}{2}\right) + C$ (b) $6 \tan^{-1}\left(\frac{3x+1}{2}\right) + C$

(c) $\cot^{-1}\left(\frac{3x+1}{2}\right) + C$ (d) $\frac{1}{6} \tan^{-1}\left(\frac{3x+1}{2}\right) + C$

Sol. (d) $\int \frac{dx}{9x^2 + 6x + 5} = \frac{1}{9} \int \frac{dx}{x^2 + \frac{2}{3}x + \frac{5}{9}}$

$$= \frac{1}{9} \int \frac{dx}{x^2 + 2 \cdot \frac{1}{3} \cdot x + \left(\frac{1}{3}\right)^2 + \frac{5}{9} - \left(\frac{1}{3}\right)^2}$$

$$= \frac{1}{9} \int \frac{dx}{x^2 + \frac{2}{3}x + \left(\frac{1}{3}\right)^2 + \frac{5}{9} - \frac{1}{9}}$$

$$= \frac{1}{9} \int \frac{dx}{\left(x + \frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2} = \frac{1}{9} \left(\frac{1}{2/3}\right) \tan^{-1} \left(\frac{x + \frac{1}{3}}{2/3}\right) + C$$

$$\left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a}\right) \right]$$

$$= \frac{1}{6} \tan^{-1} \left(\frac{3x+1}{2}\right) + C$$

➤ **Example 21.** $\int \frac{dx}{\sqrt{x^2 + 2x + 2}}$ is equal to

(a) $\log |(x+1) + \sqrt{x^2 + 2x + 2}| + C$

(b) $\log |x + \sqrt{x^2 + 2x + 2}| + C$

(c) $\log |(x+1) - \sqrt{x^2 + 2x + 2}| + C$

(d) None of the above

Sol. (a) Let $I = \int \frac{dx}{\sqrt{x^2 + 2x + 2}} = \int \frac{dx}{\sqrt{(x^2 + 2x + 1) + 1}}$

$$= \int \frac{1}{\sqrt{(x+1)^2 + (1)^2}} dx \quad [\text{let } x+1 = t \Rightarrow dx = dt]$$

$$\therefore I = \int \frac{dt}{\sqrt{t^2 + 1}} = \log |t + \sqrt{t^2 + 1}| + C$$

$$\left[\because \int \frac{dx}{\sqrt{x^2 + a^2}} = \log |x + \sqrt{x^2 + a^2}| \right]$$

$$= \log |(x+1) + \sqrt{(x+1)^2 + 1}| + C$$

$$= \log |(x+1) + \sqrt{x^2 + 2x + 2}| + C$$

➤ **Example 22.** $\int \sqrt{x^2 + 4x + 1} dx$ is equal to

(a) $(x+2)\sqrt{x^2 + 4x + 1} - \frac{3}{2} \log |(x+2) + \sqrt{x^2 + 4x + 1}| + C$

(b) $\frac{(x+2)}{2} \sqrt{x^2 + 4x + 1} + \frac{3}{2} \log |(x+2) + \sqrt{x^2 + 4x + 1}| + C$

$$(c) \frac{(x+2)}{2} \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log |(x+2) + \sqrt{x^2 + 4x + 1}| + C$$

(d) None of the above

Sol. (c) Let

$$I = \int \sqrt{x^2 + 4x + 1} dx = \int \sqrt{x^2 + 4x + 1 - 2^2 + 2^2} dx = \int \sqrt{(x+2)^2 - (\sqrt{3})^2} dx$$

$$\left[\because \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log |x + \sqrt{x^2 - a^2}| \right]$$

$$\therefore I = \frac{x+2}{2} \sqrt{x^2 + 4x + 1} - \frac{3}{2} \log |(x+2) + \sqrt{x^2 + 4x + 1}| + C$$

(ii) If the integrals are of the form

$$\int \frac{px + q}{ax^2 + bx + c} dx, \int \frac{px + q}{\sqrt{ax^2 + bx + c}} dx \text{ and } \int (px + q) \sqrt{ax^2 + bx + c} dx$$

Then, put $px + q = A$ (differential coefficient of $ax^2 + bx + c$) + B

Find A and B by comparing the coefficient of x and constant term on both the sides of

$$\text{equation. Then, } \int \frac{px + q}{ax^2 + bx + c} dx = A \int \frac{2ax + b}{ax^2 + bx + c} dx + B \int \frac{1}{ax^2 + bx + c} dx$$

Now, we can integrate it easily. Similarly, other two cases.

➤ **Example 23.** $\int \frac{5x-2}{1+2x+3x^2} dx$ is equal to

$$(a) \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C$$

$$(b) \frac{5}{3} \log |1+2x+3x^2| - \frac{11}{3} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C$$

$$(c) \frac{1}{3} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C$$

(d) None of the above

Sol. (a) $\int \frac{5x-2}{1+2x+3x^2} dx$

$$\text{Let } 5x-2 = A \frac{d}{dx} (1+2x+3x^2) + B$$

$$\Rightarrow 5x-2 = A(2+6x) + B$$

$$\Rightarrow 5x-2 = 6Ax + (2A+B)$$

On equating the coefficient of x and constant on both sides, we get

$$5 = 6A \Rightarrow A = \frac{5}{6}$$

$$\text{and } 2A + B = -2$$

$$\Rightarrow B = \frac{-11}{3}$$

$$\therefore I = \int \frac{5x-2}{1+2x+3x^2} dx$$

$$= \int \frac{\frac{5}{6}(2+6x) - \frac{11}{3}}{1+2x+3x^2} dx$$

$$= \frac{5}{6} \int \frac{2+6x}{1+2x+3x^2} dx - \frac{11}{3} \int \frac{dx}{1+2x+3x^2}$$

$$\text{Let } I_1 = \int \frac{2+6x}{1+2x+3x^2} dx$$

$$\text{and } I_2 = \int \frac{dx}{1+2x+3x^2}$$

$$\therefore I = \frac{5}{6} I_1 - \frac{11}{3} I_2 \quad \dots(i)$$

$$\text{Now, } I_1 = \int \frac{2+6x}{1+2x+3x^2} dx$$

$$\text{Let } 1+2x+3x^2 = t$$

$$\Rightarrow (2+6x) dx = dt$$

$$\Rightarrow I_1 = \int \frac{dt}{t} = \log |t| + C_1$$

$$\therefore I_1 = \log |1+2x+3x^2| + C_1 \quad \dots(ii)$$

$$\text{Also, } I_2 = \int \frac{dx}{1+2x+3x^2}$$

$$1+2x+3x^2 \text{ can be written as } 1+3\left(x^2 + \frac{2}{3}x\right)$$

$$\text{Therefore, } 1+3\left(x^2 + \frac{2}{3}x\right) = 1+3\left(x^2 + \frac{2}{3}x + \frac{1}{9} - \frac{1}{9}\right)$$

$$= 1+3\left(x + \frac{1}{3}\right)^2 - \frac{1}{3} = \frac{2}{3} + 3\left(x + \frac{1}{3}\right)^2$$

$$= 3\left[\left(x + \frac{1}{3}\right)^2 + \frac{2}{9}\right] = 3\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]$$

$$\therefore I_2 = \frac{1}{3} \int \frac{dx}{\left[\left(x + \frac{1}{3}\right)^2 + \left(\frac{\sqrt{2}}{3}\right)^2\right]}$$

$$= \frac{1}{3} \left[\frac{1}{\sqrt{2}/3} \tan^{-1} \left(\frac{x + \frac{1}{3}}{\sqrt{2}/3} \right) \right] + C_2$$

$$\left[\because \int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) \right]$$

$$= \frac{1}{3} \left[\frac{3}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \right] + C_2$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C_2 \quad \dots(iii)$$

On substituting the values of I_1 and I_2 in Eq. (i), we get

$$I = \frac{5}{6} [\log |1+2x+3x^2|] - \frac{11}{3} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) \right] + C$$

$$\left[\because \frac{5}{6} C_1 - \frac{11}{3} C_2 = C \right]$$

$$= \frac{5}{6} \log |1+2x+3x^2| - \frac{11}{3\sqrt{2}} \tan^{-1} \left(\frac{3x+1}{\sqrt{2}} \right) + C$$

➤ **Example 24.** $\int \frac{x+2}{\sqrt{x^2+2x+3}} dx$ is equal to

- (a) $\sqrt{x^2+2x+3} + \log|x+1+\sqrt{x^2+2x+3}| + C$
 (b) $\sqrt{x^2+2x+3} - \log|x+1+\sqrt{x^2+2x+3}| + C$
 (c) $\sqrt{x^2+2x+3} + \log|x+1-\sqrt{x^2+2x+3}| + C$
 (d) None of the above

Sol. (a) Let $x+2 = A \frac{d}{dx}(x^2+2x+3) + B$

$$\Rightarrow x+2 = A(2x+2) + B$$

$$\Rightarrow x+2 = 2Ax + (2A+B)$$

On equating the coefficient of x and constant term on both sides, we get

$$2A = 1 \Rightarrow A = \frac{1}{2}$$

$$\text{and } 2A + B = 2$$

$$\Rightarrow 2 \times \frac{1}{2} + B = 2$$

$$\Rightarrow B = 2 - 1 = 1$$

$$\Rightarrow x+2 = \frac{1}{2}(2x+2) + 1$$

$$\begin{aligned} \therefore \int \frac{x+2}{\sqrt{x^2+2x+3}} dx &= \int \frac{\frac{1}{2}(2x+2) + 1}{\sqrt{x^2+2x+3}} dx \\ &= \frac{1}{2} \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx + \int \frac{dx}{\sqrt{x^2+2x+3}} \end{aligned}$$

$$\text{Let } I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$$

$$\text{and } I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}}$$

$$\text{Then, } \int \frac{x+2}{\sqrt{x^2+2x+3}} dx = \frac{1}{2} I_1 + I_2 \quad \dots(i)$$

$$\text{Now, } I_1 = \int \frac{2x+2}{\sqrt{x^2+2x+3}} dx$$

$$\text{Let } x^2+2x+3 = t$$

$$\Rightarrow (2x+2) dx = dt$$

$$I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-1/2} dt = \frac{t^{(-1/2)+1}}{\left(\frac{-1}{2}\right)+1} + C$$

$$= 2\sqrt{t} + C_1 = 2\sqrt{x^2+2x+3} + C_1$$

$$\text{and } I_2 = \int \frac{dx}{\sqrt{x^2+2x+3}} = \int \frac{dx}{\sqrt{x^2+2x+(1)^2+3-(1)^2}}$$

$$= \int \frac{dx}{\sqrt{(x+1)^2 + (\sqrt{2})^2}}$$

$$\text{Let } x+1 = t \Rightarrow dx = dt$$

$$I_2 = \int \frac{dt}{\sqrt{t^2 + (\sqrt{2})^2}} = \log|t + \sqrt{t^2+2}| + C_2$$

$$\left[\because \int \frac{dx}{\sqrt{x^2+a^2}} = \log|x + \sqrt{x^2+a^2}| \right]$$

$$= \log|x+1 + \sqrt{(x+1)^2+2}| + C_2$$

$$= \log|x+1 + \sqrt{x^2+2x+3}| + C_2$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$\begin{aligned} \int \frac{x+2}{\sqrt{x^2+2x+3}} dx &= \frac{1}{2} [2\sqrt{x^2+2x+3}] \\ &\quad + \log|x+1 + \sqrt{x^2+2x+3}| + C \\ &\quad \left[\because \frac{1}{2} C_1 + C_2 = C \right] \\ &= \sqrt{x^2+2x+3} + \log|x+1 + \sqrt{x^2+2x+3}| + C \end{aligned}$$

➤ **Example 25.** If $\int (x+1)\sqrt{1-x-x^2} dx$

$$= -\frac{1}{3}(1-x-x^2)^{3/2} + A + C, \text{ then the value of } A \text{ is}$$

$$(a) \sqrt{1-x-x^2} + \frac{5}{16} \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$$

$$(b) \frac{1}{8} \sqrt{1-x-x^2} + \frac{\sqrt{5}}{2} \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$$

$$(c) \frac{1}{8} (2x+1) \sqrt{1-x-x^2} + \frac{5}{16} \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right)$$

(d) None of the above

Sol. (c) Let $x+1 = \lambda \cdot \frac{d}{dx}(1-x-x^2) + \mu$

$$\text{Then, } x+1 = \lambda(-1-2x) + \mu$$

On comparing the coefficients of like powers of x , we get $-2\lambda = 1$ and $\mu - \lambda = 1$

$$\Rightarrow \lambda = -\frac{1}{2} \quad \text{and} \quad \mu = \frac{1}{2}$$

$$x+1 = -\frac{1}{2}(-1-2x) + \frac{1}{2}$$

$$\begin{aligned} \text{So, } \int (x+1)\sqrt{1-x-x^2} dx &= \int \left\{ -\frac{1}{2}(-1-2x) + \frac{1}{2} \right\} \sqrt{1-x-x^2} dx \\ &= -\frac{1}{2} \int (-1-2x)\sqrt{1-x-x^2} dx + \frac{1}{2} \int \sqrt{1-x-x^2} dx \\ &= -\frac{1}{2} \int \sqrt{1-x-x^2} d(1-x-x^2) \\ &\quad + \frac{1}{2} \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - \left(x + \frac{1}{2}\right)^2} dx \\ &= -\frac{1}{3}(1-x-x^2)^{3/2} + \frac{1}{8}(2x+1)\sqrt{1-x-x^2} \\ &\quad + \frac{5}{16} \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + C \end{aligned}$$

(iii) If the integrals are of the form

$$\int \frac{(ax^2+bx+c)}{(px^2+qx+r)} dx, \int \frac{(ax^2+bx+c)}{\sqrt{(px^2+qx+r)}} dx,$$

$$\int (ax^2+bx+c) \sqrt{(px^2+qx+r)} dx$$

Substitute, $ax^2+bx+c = \lambda(px^2+qx+r)$

$$+ \mu \left\{ \frac{d}{dx}(px^2+qx+r) \right\} + \gamma$$

Find λ , μ and γ . This substitution reduce integration to integration of three independent functions.

➤ **Example 26.** $\int \frac{(x^2 + 4x + 7)}{\sqrt{x^2 + x + 1}} dx$ is equal to

- (a) $\left(\frac{x+\frac{1}{2}}{2}\right) + \sqrt{x^2 + x + 1} + \frac{3}{8} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + 3\sqrt{x^2 + x + 1} + \frac{9}{2} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$
 (b) $\left(x + \frac{1}{2}\right) + \sqrt{x^2 + x + 1} + \frac{1}{8} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + \sqrt{x^2 + x + 1} + \frac{1}{2} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$
 (c) $\left(x - \frac{1}{2}\right) + \sqrt{x^2 + x + 1} + \frac{1}{8} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) - \sqrt{x^2 + x + 1} + \frac{1}{2} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$
 (d) None of the above

Sol. (a) Let $x^2 + 4x + 7 = A(x^2 + x + 1) + B(2x + 1) + C$

Comparing the coefficient of x^2 , x and constant term, we get

$$A = 1, A + 2B = 4, A + B + C = 7$$

$$\Rightarrow A = 1, B = \frac{3}{2}, C = \frac{9}{2}$$

$$\text{So, } I = \int \sqrt{x^2 + x + 1} dx + \frac{3}{2} \int \frac{(2x + 1)}{\sqrt{x^2 + x + 1}} dx + \frac{9}{2} \int \frac{dx}{\sqrt{x^2 + x + 1}}$$

$$\text{Now, } x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$$

$$I = \left(\frac{x+\frac{1}{2}}{2}\right) + \sqrt{x^2 + x + 1} + \frac{3}{8} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + 3\sqrt{x^2 + x + 1} + \frac{9}{2} \log \left(x + \frac{1}{2} + \sqrt{x^2 + x + 1}\right) + C$$

(iv) Integrals of the form

$$\int \frac{dx}{a + b \cos^2 x}, \int \frac{dx}{a + b \sin^2 x}, \int \frac{dx}{a \sin^2 x + b \cos^2 x},$$

$$\int \frac{dx}{(a \sin x + b \cos x)^2}, \int \frac{dx}{a + b \sin^2 x + c \cos^2 x}$$

To evaluate the above forms of integrals proceed as follows:

- (a) Divide both the numerator and denominator by $\cos^2 x$.
 (b) Replace $\sec^2 x$ in the denominator, if any by $(1 + \tan^2 x)$.
 (c) Put $\tan x = t \Rightarrow \sec^2 x dx = dt$
 (d) Now, evaluate the integral thus obtained, by the method discussed earlier.

➤ **Example 27.** $\int \frac{1}{1 + \cos^2 x} dx$ is equal to

- (a) $\frac{1}{\sqrt{2}} \tan^{-1}(\tan x) + C$
 (b) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{2} \tan x\right) + C$
 (c) $\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}} \tan x\right) + C$
 (d) None of the above

Sol. (c) $\int \frac{dx}{1 + \cos^2 x} = \int \frac{\sec^2 x}{\sec^2 x + 1} dx = \int \frac{\sec^2 x}{\tan^2 x + 2} dx$
 $= \int \frac{dt}{t^2 + 2} = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C$ [putting $\tan x = t$]
 $= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{1}{\sqrt{2}} \tan x\right) + C$

➤ **Example 28.** $\int \frac{dx}{1 + 3 \sin^2 x}$ is equal to

- (a) $\frac{1}{3} \tan^{-1}(3 \tan^2 x) + C$ (b) $\frac{1}{2} \tan^{-1}(2 \tan x) + C$
 (c) $\tan^{-1}(\tan x) + C$ (d) None of these

Sol. (b) $\int \frac{dx}{1 + 3 \sin^2 x} = \int \frac{dx}{\sin^2 x + \cos^2 x + 3 \sin^2 x}$
 $= \int \frac{dx}{4 \sin^2 x + \cos^2 x} = \int \frac{\sec^2 x}{4 \tan^2 x + 1} dx = \frac{1}{4} \int \frac{\sec^2 x}{\tan^2 x + \frac{1}{4}} dx$
 Put $t = \tan x \Rightarrow dt = \sec^2 x dx$, then it reduces to
 $\frac{1}{4} \int \frac{dt}{t^2 + \left(\frac{1}{2}\right)^2} = \frac{1}{4} \cdot 2 \tan^{-1}(2t) + C = \frac{1}{2} \tan^{-1}(2t) + C$
 $= \frac{1}{2} \tan^{-1}(2 \tan x) + C$

(v) Integrals of the form $\int \frac{dx}{a + b \cos x}$ and $\int \frac{dx}{a + b \sin x}$

To evaluate such form of integrals, proceed as follows:

- (a) Put $\cos x = \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$ and $\sin x = \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}$
 (b) Replace $1 + \tan^2 \frac{x}{2}$ in the numerator by $\sec^2 \frac{x}{2}$
 (c) Put $\tan \frac{x}{2} = t$, so that $\frac{1}{2} \sec^2 \frac{x}{2} dx = dt$
 (d) Now, evaluate the integral obtained which will be of the form $\int \frac{dt}{at^2 + bt + c}$ by the method discussed earlier.

$$\int \frac{dx}{a + b \cos x}$$

Case I When $a^2 > b^2$, then

$$\int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left(\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + C$$

Case II When $a^2 < b^2$, then

$$\int \frac{dx}{a + b \cos x} = \frac{1}{\sqrt{b^2 - a^2}} \log \left| \frac{\sqrt{b-a} \tan \frac{x}{2} + \sqrt{b+a}}{\sqrt{b-a} \tan \frac{x}{2} - \sqrt{b+a}} \right| + C$$

Case III When $a^2 = b^2$

In this case either $a = b$ or $b = -a$

▪ When $b = a$, then $\int \frac{dx}{a + b \cos x} = \frac{1}{a} \tan \frac{x}{2} + C$

▪ When $b = -a$, then $\int \frac{dx}{a + b \cos x} = \frac{-1}{a} \cot \frac{x}{2} + C$

Case I When $a^2 > b^2$ or $a > 0$ and $a > b$, then

$$\int \frac{dx}{a + b \sin x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{a \tan \frac{x}{2} + b}{\sqrt{a^2 - b^2}} \right] + C$$

Case II When $a^2 < b^2$, then $\int \frac{dx}{a + b \sin x}$

$$= \frac{1}{\sqrt{b^2 - a^2}} \log \left| \frac{\left(a \tan \frac{x}{2} + b \right) - (\sqrt{b^2 - a^2})}{\left(a \tan \frac{x}{2} + b \right) + \sqrt{b^2 - a^2}} \right| + C$$

Case III When $a^2 = b^2$

In this case, either $b = a$ or $b = -a$

When $b = a$, then

$$\int \frac{dx}{a + b \cos x} = \frac{1}{a} \tan \frac{x}{2} + C$$

When $b = -a$, then

$$\int \frac{dx}{a + b \cos x} = \frac{-1}{a} + C$$

➤ **Example 29.** $\int \frac{dx}{5 + 4 \cos x}$ is equal to

(a) $\frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan x \right) + C$

(b) $\frac{1}{3} \tan^{-1} \left(\frac{1}{3} \tan x \right) + C$

(c) $\frac{2}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2} \right) + C$

(d) $\frac{1}{3} \tan^{-1} \left(\frac{1}{3} \tan \frac{x}{2} \right) + C$

Sol. (c) $\int \frac{dx}{5 + 4 \cos x} = \int \frac{dx}{5 + 4 \left[\frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \right]} = \int \frac{\sec^2 \frac{x}{2} dx}{9 + \tan^2 \frac{x}{2}}$

Put $\tan \frac{x}{2} = t$, then it reduces to

$$2 \int \frac{dt}{3^2 + t^2} = \frac{2}{3} \tan^{-1} \left[\frac{1}{3} \tan \frac{x}{2} \right] + C$$

Aliter

Apply direct formula

i.e. $\int \frac{1}{a + b \cos x} dx, \{a^2 > b^2\}$

$$= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right] + C$$

We get, $\int \frac{dx}{5 + 4 \cos x} = \frac{2}{3} \tan^{-1} \left\{ \frac{1}{3} \tan \frac{x}{2} \right\} + C$

(vi) Integrals of the form

$$\int \frac{dx}{a + b \cos x + c \sin x}, \int \frac{dx}{a \sin x + b \cos x}$$

(a) Integral of the form $\int \frac{dx}{a + b \cos x + c \sin x}$

▪ To evaluate such integrals, we put $b = r \cos \alpha$ and $c = r \sin \alpha$.

So that, $r^2 = b^2 + c^2$ and $\alpha = \tan^{-1} \left(\frac{c}{b} \right)$

$$\begin{aligned} \therefore \int \frac{dx}{a + b \cos x + c \sin x} &= \int \frac{dx}{a + r(\cos \alpha \cos x + \sin \alpha \sin x)} \\ &= \int \frac{dx}{a + r \cos(x - \alpha)} \end{aligned}$$

Again, put $x - \alpha = t \Rightarrow dx = dt$, we have

$$I = \int \frac{dt}{a + r \cos t}$$

Which can be evaluated by the method discussed earlier.

(b) Integral of the form $\int \frac{dx}{a \sin x + b \cos x}$

▪ To evaluate this type of integrals, we substitute $a = r \cos \theta$, $b = r \sin \theta$ and so that $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1} \left(\frac{b}{a} \right)$.

$$\begin{aligned} \therefore \int \frac{dx}{a \sin x + b \cos x} &= \frac{1}{r} \int \frac{dx}{\sin(x + \theta)} \\ &= \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx = \frac{1}{r} \log \left| \tan \left(\frac{x}{2} + \frac{\theta}{2} \right) \right| \\ &= \frac{1}{\sqrt{a^2 + b^2}} \log \left| \tan \left(\frac{x}{2} + \frac{1}{2} \tan^{-1} \frac{b}{a} \right) \right| + C \end{aligned}$$

- The integral of the above form can be evaluated by using $\cos x = \frac{1 - \tan^2\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$ and

$$\sin x = \frac{2 \tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}$$

➤ **Example 30.** $\int \frac{dx}{2 + \sin x + \cos x}$ is equal to

(a) $\sqrt{2} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{\sqrt{2}} \right) + C$

(b) $\tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{\sqrt{2}} \right) + C$

(c) $\sqrt{2} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{2}} \right) + C$

(d) None of the above

Sol. (a) Let $I = \int \frac{dx}{2 + \sin x + \cos x}$

$$\Rightarrow I = \int \frac{dx}{2 + \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}}$$

$$= \int \frac{\sec^2 \frac{x}{2} dx}{2 + 2 \tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 1 - \tan^2 \frac{x}{2}}$$

$$I = \int \frac{\sec^2 \frac{x}{2} dx}{\tan^2 \frac{x}{2} + 2 \tan \frac{x}{2} + 3}$$

Put $\tan \frac{x}{2} = t \Rightarrow \frac{1}{2} \sec^2 \frac{x}{2} dx = dt$

$$\therefore I = \int \frac{2dt}{t^2 + 2t + 3} = 2 \int \frac{dt}{t^2 + 2t + 1 + 2}$$

$$= 2 \int \frac{dt}{(t+1)^2 + (\sqrt{2})^2} = 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t+1}{\sqrt{2}} \right) + C$$

$$\Rightarrow I = \sqrt{2} \tan^{-1} \left(\frac{\tan \frac{x}{2} + 1}{\sqrt{2}} \right) + C$$

(vii) Integrals of the form

$$\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx \text{ and } \int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx$$

(a) Integral of the form $\int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$

Such rational functions of $\sin x$ and $\cos x$ may

be integrated by expressing the numerator of the integrand as follows:

Numerator = M [Differentiation of denominator] + N [Denominator]

i.e. $a \sin x + b \cos x = M \frac{d}{dx} (c \sin x + d \cos x) + N(c \sin x + d \cos x)$

The arbitrary constants M and N are determined by comparing the coefficients of $\sin x$ and $\cos x$ from two sides of the above identity. Then, the given integral is

$$I = \int \frac{a \sin x + b \cos x}{c \sin x + d \cos x} dx$$

$$= \int \frac{M(c \cos x - d \sin x) + N(c \sin x + d \cos x)}{c \sin x + d \cos x} dx$$

$$= M \int \frac{c \cos x - d \sin x}{c \sin x + d \cos x} dx + N \int \frac{1}{c \sin x + d \cos x} dx$$

$$= M \log |c \sin x + d \cos x| + Nx + C$$

(b) Integral of the form $\int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx$

To evaluate this type of integrals, we express the numerator as follows:

Numerator = M [Denominator] + N (Differentiation of denominator) + P

$$(a \sin x + b \cos x + q) = M(c \sin x + d \cos x + r) + N(c \cos x - d \sin x) + P$$

where, M , N and P are constants to be determined by comparing the coefficients of $\sin x$, $\cos x$ and constant term on both sides.

$$\therefore \int \frac{a \sin x + b \cos x + q}{c \sin x + d \cos x + r} dx$$

$$= \int M dx + N \int \frac{\text{Differentiation of denominator}}{\text{Denominator}} dx + \int \frac{dx}{c \sin x + d \cos x + r}$$

$$= Mx + N \log |\text{Denominator}| + P \int \frac{dx}{c \sin x + d \cos x + r}$$

➤ **Example 31.** $\int \frac{3 \sin x + 2 \cos x}{3 \cos x + 2 \sin x} dx$ is equal to

(a) $\frac{12}{13} x - \frac{5}{13} \log (3 \cos x + 2 \sin x) + C$

(b) $\frac{12}{13} x + \frac{5}{13} \log (3 \cos x + 2 \sin x) + C$

(c) $\frac{13}{12} x \times \frac{5}{13} \log (3 \cos x + 2 \sin x) + C$

(d) None of the above

Sol. (a) Write $N' = I(D') + m$ (differential coefficient of D')

$$\text{Let } 3\sin x + 2\cos x = I(3\cos x + 2\sin x) + m(-3\sin x + 2\cos x)$$

Comparing coefficients of $\sin x$ and $\cos x$ on both sides,

$$3 = 2I - 3m \text{ and } 2 = 3I + 2m$$

$$\text{Solving, we get } I = \frac{12}{13}, m = -\frac{5}{13}$$

$$\begin{aligned} \therefore I &= I \int dx + m \int \frac{-3\sin x + 2\cos x}{3\cos x + 2\sin x} dx \\ &= Ix + m \log(3\cos x + 2\sin x) + C \\ &= \frac{12}{13}x - \frac{5}{13} \log(3\cos x + 2\sin x) + C \end{aligned}$$

(viii) Integrals of the form $\int \frac{x^2 + 1}{x^4 + \lambda x^2 + 1} dx$,

$$\int \frac{x^2 - 1}{x^4 + \lambda x^2 + 1} dx, \int \frac{1}{x^4 + \lambda x^2 + 1} dx, \text{ where } \lambda$$

is a constant.

To evaluate this type of integrals, we divide the numerator and denominator by x^2 and put

$$x + \frac{1}{x} = t \text{ or } x - \frac{1}{x} = t, \text{ whichever on}$$

differentiation gives the numerator of the resulting integrand.

➤ **Example 32.** $\int \frac{x^2 - 1}{x^4 + x^2 + 1} dx$ is equal to

$$(a) \frac{1}{2} \log \left(\frac{x^2 + x + 1}{x^2 - x + 1} \right) + C$$

$$(b) \frac{1}{2} \log \left(\frac{x^2 - x - 1}{x^2 + x + 1} \right) + C$$

$$(c) \log \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) + C$$

$$(d) \frac{1}{2} \log \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) + C$$

Sol. (d) The given integral can be written as

$$\int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right)^2 - 1} dx$$

$$\text{Put } x + \frac{1}{x} = t$$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt, \text{ then it reduces to}$$

$$\begin{aligned} \int \frac{dt}{t^2 - 1} &= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C \\ &= \frac{1}{2} \log \left(\frac{x + \frac{1}{x} - 1}{x + \frac{1}{x} + 1} \right) + C \\ &= \frac{1}{2} \log \left(\frac{x^2 - x + 1}{x^2 + x + 1} \right) + C \end{aligned}$$

➤ **Example 33.** If

$$\int \frac{1}{x^4 + 1} dx = \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + A + C. \text{ Then, } A$$

is equal to

$$(a) -\frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right|$$

$$(b) -\frac{1}{4\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right|$$

$$(c) -\frac{1}{2} \log \left| \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right|$$

$$(d) -\frac{1}{4\sqrt{2}} \log \left| \frac{x^2 + \sqrt{2}x + 1}{x^2 - \sqrt{2}x + 1} \right|$$

$$\text{Sol. (b)} \int \frac{1}{x^4 + 1} dx = \int \frac{1/x^2}{x^2 + 1/x^2} dx = \frac{1}{2} \int \frac{2/x^2}{x^2 + \frac{1}{x^2}} dx$$

$$\begin{aligned} &= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right) - \left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} dx \\ &= \frac{1}{2} \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx - \frac{1}{2} \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx \\ &= \frac{1}{2} \int \frac{1}{\left(x - \frac{1}{x}\right)^2 + 2} d\left(x - \frac{1}{x}\right) \\ &\quad - \frac{1}{2} \int \frac{1}{\left(x + \frac{1}{x}\right)^2 - 2} d\left(x + \frac{1}{x}\right) \\ &= \frac{1}{2} \int \frac{du}{u^2 + (\sqrt{2})^2} - \frac{1}{2} \int \frac{dv}{v^2 - (\sqrt{2})^2} \end{aligned}$$

$$\text{where, } u = x - \frac{1}{x} \text{ and } v = x + \frac{1}{x}$$

$$\begin{aligned} &= \frac{1}{2} \times \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) - \frac{1}{2} \times \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + C \\ &= \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) - \frac{1}{4\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + C \end{aligned}$$

➤ **Example 34.** $\int \sqrt{\tan \theta} d\theta$ is equal to

$$(a) \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + C$$

$$(b) \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + C$$

$$(c) \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{x^2 + 1}{\sqrt{2}x} \right) + \frac{1}{\sqrt{2}} \log \left| \frac{x^2 + \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + C$$

(d) None of the above

Sol. (a) On putting $\tan \theta = x^2 \Rightarrow d(\tan \theta) = dx^2$

$$\Rightarrow \sec^2 \theta d\theta = 2x dx$$

$$\Rightarrow d\theta = \frac{2x}{1+x^4} dx, \text{ we get } \int \sqrt{\tan \theta} d\theta$$

$$= \int \sqrt{x^2} \frac{2x}{1+x^4} dx = \int \frac{2x^2}{x^4+1} dx = \int \frac{2}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{\left(1 + \frac{1}{x^2}\right) + \left(1 - \frac{1}{x^2}\right)}{x^2 + \frac{1}{x^2}} dx = \int \frac{1 + \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx + \int \frac{1 - \frac{1}{x^2}}{x^2 + \frac{1}{x^2}} dx$$

$$= \int \frac{1}{\left(x - \frac{1}{x}\right)^2 + 2} d\left(x - \frac{1}{x}\right) + \int \frac{1}{\left(x + \frac{1}{x}\right)^2 - 2} d\left(x + \frac{1}{x}\right)$$

$$= \int \frac{du}{u^2 + (\sqrt{2})^2} + \int \frac{1}{v^2 - (\sqrt{2})^2} dv$$

$$\text{where, } u = x - \frac{1}{x} \text{ and } v = x + \frac{1}{x}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{u}{\sqrt{2}} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{v - \sqrt{2}}{v + \sqrt{2}} \right| + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 - 1}{\sqrt{2}x} \right) + \frac{1}{2\sqrt{2}} \log \left| \frac{x^2 - \sqrt{2}x + 1}{x^2 + \sqrt{2}x + 1} \right| + C$$

(ix) Integrals of the form

$$\int x^m (a + bx^n)^p dx \quad (p \neq 0).$$

Here, we have the following cases:

(a) If p is a natural number, then expand $(a + bx^n)^p$ by binomial theorem and integrate.

(b) If p is a negative integer and m and n are rational number, put $x = t^k$, when k is the LCM of denominator of m and n .

(c) If $\frac{m+1}{n}$ is an integer and p is rational number, put $(a + bx^n) = t^k$, when k is the denominator of p .

(d) If $\left(\frac{m+p}{n}\right)$ is an integer and p is a rational number, put $\frac{a + bx^n}{x^n} = t^k$, where k is the denominator of p .

➤ **Example 35.** Evaluate $\int x^{-1/3} (1 + x^{1/3})^{1/4} dx$.

$$(a) -\frac{4}{15} (1 + x^{1/3})^{5/4} (4 + 9x^{1/3}) + C$$

$$(b) \frac{1}{15} (1 + x^{1/3})^{5/4} (4 + 9x^{1/3}) + C$$

$$(c) \frac{2}{15} (1 + x^{1/3})^{1/4} (4 + 9x^{1/3}) + C$$

(d) None of the above

Sol. (d) Here, $m = -\frac{1}{3}$, $n = \frac{1}{3}$, $p = \frac{1}{4}$, $\frac{m+1}{n} = 2$, which is an integer.

$$\text{Put } (1 + x^{1/3}) = t^4 \Rightarrow \frac{dx}{3x^{2/3}} = 4t^3 dt$$

$$\therefore I = 12 \int (t^4 - 1)t^4 dt = -\frac{4}{15} (1 + x^{1/3})^{5/4} (4 - 5x^{1/3}) + C$$

➤ **Example 36.** $\int \frac{1}{x^{1/2} + x^{1/3}} dx$ is equal to

$$(a) x^{1/2} - 3x^{1/3} - 6x^{1/6} + 6 \log |x^{1/6} + 1| + C$$

$$(b) 2x^{1/2} + 3x^{1/3} - x^{1/6} - 6 \log |x^{1/6} + 1| + C$$

$$(c) 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6 \log |x^{1/6} + 1| + C$$

(d) None of the above

Sol. (c) Let $I = \int \frac{1}{x^{1/2} + x^{1/3}} dx = \int x^{-1/2} (1 + x^{-1/6})^{-1} dx$

$$\text{Put } x^{1/6} = t \Rightarrow x = t^6 \Rightarrow dx = 6t^5 dt$$

$$\therefore I = \int \frac{6t^5}{(t^6)^{1/2} + (t^6)^{1/3}} dt = 6 \int \frac{t^5}{t^3 + t^2} dt$$

$$= 6 \int \frac{t^5}{t^2(t+1)} dt = 6 \int \frac{t^3}{(t+1)} dt = 6 \int \left(\frac{t^3 + 1 - 1}{t+1} \right) dt$$

$$= 6 \int \left[\frac{(t+1)(t^2 - t + 1) - 1}{t+1} \right] dt$$

$$= 6 \int \left\{ t^2 - t + 1 - \frac{1}{t+1} \right\} dt$$

$$= 6 \left\{ \frac{t^3}{3} - \frac{t^2}{2} + t - \log |t+1| \right\} + C$$

$$= 6 \left\{ \frac{(x^{1/6})^3}{3} - \frac{(x^{1/6})^2}{2} + (x^{1/6}) - \log |x^{1/6} + 1| \right\} + C$$

$$= 2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6 \log |x^{1/6} + 1| + C$$

$$[\because t = x^{1/6}]$$

- 1 If $\int \frac{a^x}{\sqrt{1-a^{2x}}} dx = \frac{1}{m} \sin^{-1}(a^x) + C$, then
 - a $m = \log a$
 - b $m = \log e$
 - c $2 \log a$
 - d None of these
- 2 $\int \frac{2x}{\sqrt{1-x^2-x^4}} dx$ will be
 - a $\frac{1}{\sqrt{5}} \sin^{-1} \left(\frac{2x^2+1}{\sqrt{5}} \right) + C$
 - b $\sin^{-1} \left(\frac{2x^2+1}{\sqrt{5}} \right) + C$
 - c $\frac{1}{\sqrt{5}} \tan^{-1} \left(\frac{2x^2+1}{\sqrt{5}} \right) + C$
 - d None of these
- 3 If $\int \frac{1}{x^2+2x+2} dx = f(x) + C$, then $f(x)$ is equal to
 - a $\tan^{-1}(x+1)$
 - b $2 \tan^{-1}(x+1)$
 - c $-\tan^{-1}(x+1)$
 - d $3 \tan^{-1}(x+1)$
- 4 $\int \frac{1}{x(x^7+1)} dx$ is equal to
 - a $\log \left(\frac{x^7}{x^7+1} \right) + C$
 - b $\frac{1}{7} \log \left(\frac{x^7}{x^7+1} \right) + C$
 - c $\log \left(\frac{x^7+1}{x^7} \right) + C$
 - d $\frac{1}{7} \log \left(\frac{x^7+1}{x^7} \right) + C$
- 5 If $\int \frac{1}{\sqrt{2ax-x^2}} dx = f \circ g(x) + C$, then
 - a $f(x) = \sin^{-1} x$ and $g(x) = \frac{x+a}{a}$
 - b $f(x) = \sin^{-1} x$ and $g(x) = \frac{x-a}{a}$
 - c $f(x) = \cos^{-1} x$ and $g(x) = \frac{x-a}{a}$
 - d $f(x) = \tan^{-1} x$ and $g(x) = \frac{x-a}{a}$
- 6 $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}} dx$ is equal to
 - a $\frac{1}{2} \log \left| \frac{\sqrt{1-e^x}-1}{\sqrt{1-e^x}+1} \right| + C$
 - b $\sin^{-1} \left(\frac{e^x+2}{3} \right) + C$
 - c $\frac{1}{3} \tan^{-1} \left(\frac{e^x+2}{3} \right) + C$
 - d None of these
- 7 The value of $\int \frac{dx}{3-2x-x^2}$ will be
 - a $\frac{1}{4} \log \left(\frac{3+x}{1-x} \right)$
 - b $\frac{1}{3} \log \left(\frac{3+x}{1-x} \right)$
 - c $\frac{1}{2} \log \left(\frac{3+x}{1-x} \right)$
 - d $\log \left(\frac{1-x}{3+x} \right)$
- 8 If $\int x \sqrt{1+x-x^2} dx = -\frac{1}{3}(1+x-x^2)^{3/2} + A + C$, then A is equal to
 - a $\frac{1}{2} \left[\frac{1}{2} \left(x - \frac{1}{2} \right) \sqrt{1+x-x^2} + \frac{5}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) \right]$
 - b $\frac{1}{2} \left[\left(x - \frac{1}{2} \right) \sqrt{1+x-x^2} + \sin^{-1} \left(\frac{2x-1}{\sqrt{5}} \right) \right]$
 - c $\frac{1}{2} \left[\left(x - \frac{1}{2} \right) \sqrt{1+x-x^2} + \frac{1}{8} \sin^{-1} \left(\frac{2x-1}{\sqrt{5/2}} \right) \right]$
 - d None of the above
- 9 $\int \frac{1}{7+5\cos x} dx$ is equal to
 - a $\frac{1}{\sqrt{6}} \tan^{-1} \left(\frac{1}{\sqrt{6}} \tan \frac{x}{2} \right) + C$
 - b $\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{1}{\sqrt{3}} \tan \frac{x}{2} \right) + C$
 - c $\frac{1}{4} \tan^{-1} \left(\frac{x}{2} \right) + C$
 - d $\frac{1}{7} \tan^{-1} \left(\tan \frac{x}{2} \right) + C$
- 10 If $\int \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx = \frac{1}{12} \tan^{-1}(3 \tan x) + C$, then the maximum value of $a \sin x + b \cos x$ is
 - a $\sqrt{41}$
 - b $\sqrt{40}$
 - c $\sqrt{39}$
 - d $\sqrt{38}$
- 11 $\int \frac{dx}{x^4+5x^2+1}$ will be
 - a $\frac{1}{2} \left[\frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{x-1}{\sqrt{7}x} \right) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+\frac{1}{x}}{\sqrt{3}} \right) \right] + C$
 - b $\frac{1}{3} \left[\frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{x-1}{\sqrt{7}x} \right) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+\frac{1}{x}}{\sqrt{3}} \right) \right] + C$
 - c $\frac{1}{6} \left[\frac{1}{\sqrt{7}} \tan^{-1} \left(\frac{x-1}{\sqrt{7}x} \right) - \frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{x+\frac{1}{x}}{\sqrt{3}} \right) \right] + C$
 - d None of the above
- 12 $\int \frac{\sin x + \cos x}{9+16\sin 2x} dx$ is equal to
 - a $\frac{1}{40} \log \left| \frac{5+4(\sin x - \cos x)}{5-4(\sin x - \cos x)} \right| + C$
 - b $\frac{1}{4} \log \left| \frac{5+4(\sin x + \cos x)}{5-4(\sin x + \cos x)} \right| + C$
 - c $\frac{1}{40} \log \left| \frac{5-4(\sin x - \cos x)}{5+4(\sin x - \cos x)} \right| + C$
 - d None of the above

13 $\int x^{-11} (1+x^4)^{-1/2} dx$ is equal to

- a $-\frac{1}{2} \left[\frac{1}{5} \left(\frac{x^4+1}{x^4} \right)^{5/2} - \frac{2}{3} \left(\frac{x^4+1}{x^4} \right)^{3/2} + \left(\frac{x^4+1}{x^4} \right)^{1/2} \right] + C$
 b $-\frac{1}{10} \left(\frac{x^4+1}{x^4} \right)^{5/2} + \frac{1}{3} \left(\frac{x^4+1}{x^4} \right)^{3/2} + C$
 c $-\frac{1}{8} \left(\frac{x^4+1}{x^4} \right)^{5/2} + \frac{1}{3} \left(\frac{x^4+1}{x^4} \right)^{3/2} + \left(\frac{x^4+1}{x^4} \right)^{1/2} + C$
 d None of the above

14 Evaluate $\int \frac{5\sin x + 6}{\sin x + 2\cos x + 3} dx$.

- a $\tan^{-1} \left(\frac{1+\tan \frac{x}{2}}{2} \right) + C$
 b $\cot^{-1} \left(\frac{1+\tan \frac{x}{2}}{2} \right) + C$
 c $\frac{1}{2} \tan^{-1} \left(\frac{1+\tan \frac{x}{2}}{2} \right) + C$
 d None of the above

Integration by Parts

1. When integrand involves more than one type of functions. We may solve such integrals by a rule which is known as integration by parts.

We know that, $\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$

$$\Rightarrow d(uv) = u dv + v du$$

$$\Rightarrow \int d(uv) = \int u dv + \int v du$$

If u and v are two functions of x , then

$$\int_1^u v \cdot \frac{du}{dx} dx = u \int v dx - \int \left\{ \frac{du}{dx} \cdot \int v dx \right\} dx, \text{ i.e. the}$$

integral of the product of two functions = (First function) $\times$ (Integral of second function) – Integral of {(Differentiation of first function) $\times$ (Integral of second function)}

Before applying this rule proper choice of first and second function is necessary. Normally, we use the following methods:

Method I In the product of two functions, one of the function is not directly integrable (i.e. $\log|x|$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, etc.), then we take it as the first function and the remaining is taken as the second function.

Method II If there is no other function, then unity is taken as the second function e.g. In the integration of $\int \sin^{-1} x dx$, $\int \log x dx$, 1 is taken as the second function.

Method III If both of the function are directly integrable, then the first function is chosen in such a way that the derivative of the function, thus obtained under integral sign is easily integrable. Usually, we use the following preference order for the first function.

(Inverse Trigonometric, Logarithmic, Algebraic, Exponential). This rule is simply called as 'ILATE'.

➤ **Example 37.** $\int x^2 \log x dx$ is equal to

- (a) $\frac{x^3}{3} \log x - \frac{x^3}{9} + C$ (b) $\frac{x^2}{2} \log x - \frac{x^3}{9} + C$
 (c) $\frac{x}{3} \log x - \frac{x^3}{3} + C$ (d) None of these

Sol. (a) On taking $\log x$ as first function and x^2 as second function ($\because$ log function comes before algebraic function in ILATE) and integrating by parts, we get

$$I = \int x^2 \log x dx = \log x \int x^2 dx - \int \left[\frac{d}{dx}(\log x) \int x^2 dx \right] dx$$

$$= \frac{x^3}{3} \log x - \int \left[\frac{1}{x} \cdot \frac{x^3}{3} \right] dx = \frac{x^3}{3} \log x - \frac{1}{3} \int x^2 dx$$

$$= \frac{x^3}{3} \log x - \frac{x^3}{9} + C$$

➤ **Example 38.** $\int x \sin^{-1} x dx$ is equal to

- (a) $\frac{\sin^{-1} x}{2} (2x^2 - 1) + \frac{1}{4} x \sqrt{1-x^2} + C$
 (b) $\frac{\sin^{-1} x}{4} (2x^2 - 1) + \frac{1}{4} x \sqrt{1-x^2} + C$
 (c) $\frac{\sin^{-1} x}{4} (2x^2 - 1) + \frac{1}{2} x \sqrt{1-x^2} + C$
 (d) None of the above

Sol. (b) Let $I = \int x \sin^{-1} x dx$

On taking $\sin^{-1} x$ as first function and x as second function ($\because$ inverse function comes before algebraic function in ILATE) and integrating by parts, we get

$$I = \sin^{-1} x \int x dx - \int \left[\frac{d}{dx}(\sin^{-1} x) \int x dx \right] dx$$

$$= \sin^{-1} x \cdot \frac{x^2}{2} - \int \left[\frac{1}{\sqrt{1-x^2}} \cdot \frac{x^2}{2} \right] dx$$

$$= \frac{x^2}{2} \cdot \sin^{-1} x + \int \left[\frac{1-1-x^2}{\sqrt{1-x^2}} \cdot \frac{1}{2} \right] dx$$

[add and subtract 1 in numerator of second term]

$$= \frac{x^2}{2} \cdot \sin^{-1} x - \frac{1}{2} \int \frac{1}{\sqrt{1-x^2}} dx + \frac{1}{2} \int \frac{1-x^2}{\sqrt{1-x^2}} dx$$

$$\begin{aligned}
 &= \frac{x^2}{2} \cdot \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \int \sqrt{1-x^2} dx \\
 \Rightarrow I &= \frac{x^2}{2} \cdot \sin^{-1} x - \frac{1}{2} \sin^{-1} x + \frac{1}{2} \left[\frac{1}{2} x \sqrt{1-x^2} + \frac{1}{2} \sin^{-1} x \right] + C \\
 \Rightarrow I &= \sin^{-1} x \left[\frac{x^2}{2} - \frac{1}{4} \right] + \frac{1}{4} x \sqrt{1-x^2} + C \\
 \therefore I &= \frac{\sin^{-1} x}{4} (2x^2 - 1) + \frac{1}{4} x \sqrt{1-x^2} + C
 \end{aligned}$$

➤ **Example 39.** $\int 1 \cdot \tan^{-1} x dx$ is equal to

- (a) $x \tan^{-1} x + \log |1+x^2| + C$
 (b) $x \tan^{-1} x - \log |1+x^2| + C$
 (c) $\frac{x}{2} \tan^{-1} x - \log |1+x^2| + C$
 (d) $x \tan^{-1} x - \frac{1}{2} \log |1+x^2| + C$

Sol. (d) There is no standard integration for $\tan^{-1} x$.

So, take $\tan^{-1} x$ as first function and 1 as second function and integrating by parts, we get

$$I = \tan^{-1} x \int 1 dx - \int \left[\frac{d}{dx} (\tan^{-1} x) \int 1 dx \right] dx$$

$$\text{Let } 1+x^2 = t \Rightarrow 2x = \frac{dt}{dx} \Rightarrow dx = \frac{dt}{2x}$$

$$\begin{aligned}
 \therefore I &= x \tan^{-1} x - \int \frac{x}{t} \cdot \frac{dt}{2x} = x \tan^{-1} x - \frac{1}{2} \int \frac{1}{t} dt \\
 &= x \tan^{-1} x - \frac{1}{2} \log |t| + C \\
 &= x \tan^{-1} x - \frac{1}{2} \log |1+x^2| + C
 \end{aligned}$$

2. Integral is of the form $\int e^x \{f(x) + f'(x)\} dx$:

If the integral is of the form

$\int e^x \{f(x) + f'(x)\} dx$, then by breaking this integral into two integrals integrate one integral by parts and keeping other integral as it is, by doing so, we get

$$(i) \int e^x [f(x) + f'(x)] dx = e^x f(x) + C$$

$$(ii) \int e^{mx} [mf(x) + f'(x)] dx = e^{mx} f(x) + C$$

$$(iii) \int e^{mx} \left[f(x) + \frac{f'(x)}{m} \right] dx = \frac{e^{mx} f(x)}{m} + C$$

➤ **Example 40.** $\int e^x (\log \sin x + \cot x) dx$ is equal to

- (a) $e^x \cot x + C$ (b) $e^x \log \sin x + C$
 (c) $e^x \log \sin x + \tan x + C$ (d) $e^x + \sin x + C$

Sol. (b) $\int e^x (\log \sin x + \cot x) dx$

$$f(x) = \log \sin x \text{ and } f'(x) = \cot x$$

Using integrating by parts,

$$\int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C$$

$$\therefore \int e^x (\log \sin x + \cot x) dx = e^x \log \sin x + C$$

3. Integrals of the form $\int e^{ax} \sin bx dx$, $\int e^{ax} \cos bx dx$

$$\int e^{ax} \cdot \sin bx dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left[bx - \tan^{-1} \left(\frac{b}{a} \right) \right] + C$$

$$\int e^{ax} \cdot \cos bx dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left[bx - \tan^{-1} \left(\frac{b}{a} \right) \right] + C$$

$$\int e^{ax} \cdot \sin (bx + c) dx$$

$$= \frac{e^{ax}}{a^2 + b^2} [a \sin (bx + c) - b \cos (bx + c)] + K$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin \left[(bx + c) - \tan^{-1} \left(\frac{b}{a} \right) \right] + K$$

$$\int e^{ax} \cdot \cos (bx + c) dx$$

$$= \frac{e^{ax}}{a^2 + b^2} [a \cos (bx + c) + b \sin (bx + c)] + K$$

$$= \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos \left[(bx + c) - \tan^{-1} \left(\frac{b}{a} \right) \right] + K$$

➤ **Example 41.** $\int e^{2x} \sin x dx$ is equal to

- (a) $\frac{e^x}{5} (\sin x - \cos x) + C$ (b) $\frac{e^{2x}}{5} (2 \sin x - \cos x) + C$
 (c) $\frac{e^{2x}}{5} (2 \cos x - \sin x) + C$ (d) $\frac{e^x}{5} (2 \cos x - \sin x) + C$

Sol. (b) Let $I = \int e^{2x} \sin x dx$

On taking $\sin x$ as first function and e^{2x} as second function and integrating by parts, we get

$$\begin{aligned}
 \int e^{2x} \sin x dx &= \sin x \cdot \int e^{2x} dx - \int \left[\frac{d}{dx} (\sin x) \cdot \int e^{2x} dx \right] dx \\
 &= \sin x \cdot \frac{e^{2x}}{2} - \int \cos x \cdot \frac{e^{2x}}{2} dx = \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2} \int e^{2x} \cos x dx \\
 \Rightarrow \int e^{2x} \sin x dx &= \sin x \cdot \frac{e^{2x}}{2} - \frac{1}{2} I_1 \quad \dots (i)
 \end{aligned}$$

$$\text{where, } I_1 = \int e^{2x} \cos x dx$$

On taking $\cos x$ as first function and e^{2x} as second function and integrating by parts, we get

$$\begin{aligned}
 I_1 &= \cos x \cdot \int e^{2x} dx - \int \left[\frac{d}{dx} (\cos x) \cdot \int e^{2x} dx \right] dx \\
 &= \cos x \cdot \frac{e^{2x}}{2} - \int (-\sin x) \cdot \frac{e^{2x}}{2} dx + C_1 \\
 &= \frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx + C_1
 \end{aligned}$$

On putting the value of I_1 in Eq. (i), we get

$$\begin{aligned}
 \int e^{2x} \sin x dx &= \frac{e^{2x} \sin x}{2} \\
 &\quad - \frac{1}{2} \left[\frac{e^{2x} \cos x}{2} + \frac{1}{2} \int e^{2x} \sin x dx + C_1 \right] \\
 \Rightarrow \int e^{2x} \sin x dx &= \frac{e^{2x} \sin x}{2} - \frac{e^{2x} \cos x}{4} - \frac{1}{4} \int e^{2x} \sin x dx - \frac{1}{2} C_1 \\
 \Rightarrow I + \frac{1}{4} I &= \frac{e^{2x}}{4} (2 \sin x - \cos x) - \frac{1}{2} C_1
 \end{aligned}$$

$$\Rightarrow \frac{5}{4} I = \frac{e^{2x}}{4} (2 \sin x - \cos x) - \frac{1}{2} C_1$$

$$\Rightarrow I = \frac{e^{2x}}{5} (2 \sin x - \cos x) + C \text{ where, } C = -\frac{2}{5} C_1$$

➤ **Example 42.** $\int e^{-2x} \sin 3x \, dx$ is equal to

- (a) $\frac{1}{13} e^{-2x} [\sin 3x + \cos 3x] + C$
 (b) $-\frac{1}{13} e^{-2x} [\sin 3x + \cos 3x] + C$
 (c) $\frac{1}{13} e^{-2x} [2 \sin 3x + 3 \cos 3x] + C$
 (d) $-\frac{1}{13} e^{-2x} [2 \sin 3x + 3 \cos 3x] + C$

Sol. (d) Let $I = \int e^{-2x} \sin 3x \, dx$

$$= -\frac{e^{-2x} \cos 3x}{3} - \int \frac{2e^{-2x} \cos 3x}{3} \, dx$$

$$= -\frac{e^{-2x} \cos 3x}{3} - \frac{2}{3} \left[\frac{e^{-2x} \sin 3x}{3} + \int \frac{2e^{-2x} \sin 3x}{3} \, dx \right]$$

$$\Rightarrow I = -\frac{e^{-2x} \cos 3x}{3} - \frac{2e^{-2x} \sin 3x}{9} - \frac{4}{9} I + C$$

$$\Rightarrow \frac{13}{9} I = -e^{-2x} \left[\frac{3 \cos 3x + 2 \sin 3x}{9} \right] + C$$

Hence, $I = -\frac{1}{13} e^{-2x} [3 \cos 3x + 2 \sin 3x] + C$

Work Book Exercise 9.3

1 If $\int x \sin x \, dx = -x \cos x + A$, then A is equal to

- a $\sin x + C$
 b $\cos x + C$
 c C
 d None of these

2 $\int \sin(\log x) \, dx$ is equal to

- a $\frac{1}{2} x [\cos(\log x) - \sin(\log x)] + C$
 b $\cos(\log x) - x + C$
 c $\frac{1}{2} x [\sin(\log x) - \cos(\log x)]$
 d $-\cos \log x$

3 $\int x^3 \log x \, dx$ is

- a $\frac{x^4}{4} \log x - \frac{x^4}{16} + C$
 b $\frac{x^4}{4} \log x + C$
 c $\frac{x^3}{3} \log x - \frac{x^3}{8} + C$
 d None of the above

4 $\int x \sin 2x \cdot \cos 3x \, dx$ is

- a $\frac{-x \cos 5x}{10} + \frac{\sin 5x}{50} + C$
 b $-\frac{x \cos 5x}{10} + \frac{\sin 5x}{50} + \frac{x \cos x}{2} + \frac{\sin x}{2} + C$
 c $-\frac{x \cos 5x}{10} + \frac{\sin 5x}{50} - \frac{x \cos x}{2} + \frac{\sin x}{2} + C$
 d None of the above

5 $\int \log(x+1) \, dx$ is equal to

- a $(x+1) \log(x+1) - x + C$
 b $(x+1) \log(x+1) + x + C$
 c $(x-1) \log(x+1) - x + C$
 d $(x-1) \log(x+1) + x + C$

6 $\int x \sin x \sec^3 x \, dx$ is equal to

- a $\frac{1}{2} [\sec^2 x - \tan x] + C$
 b $\frac{1}{2} [x \sec^2 x - \tan x] + C$
 c $\frac{1}{2} [x \sec^2 x + \tan x] + C$
 d $\frac{1}{2} [\sec^2 x + \tan x] + C$

7 $\int \frac{\log x}{(1 + \log x)^2} \, dx$ is equal to

- a $\frac{1}{1 + \log x} + C$
 b $\frac{x}{(1 + \log x)^2} + C$
 c $\frac{x}{1 + \log x} + C$
 d $\frac{1}{(1 + \log x)^2} + C$

8 The value of $\int e^x \frac{2 - \sin 2x}{1 - \cos 2x} \, dx$ is

- a $e^x \tan x + C$
 b $-e^x \tan x + C$
 c $e^x \cot x + C$
 d $-e^x \cot x + C$

9 $\int \left(\frac{x+2}{x+4} \right)^2 e^x \, dx$ is equal to

- a $e^x \left(\frac{x}{x+4} \right) + C$ b $e^x \left(\frac{x+2}{x+4} \right) + C$
 c $e^x \left(\frac{x-2}{x+4} \right) + C$ d $\left(\frac{2xe^x}{x+4} \right) + C$

10 If $\int x f(x) \, dx = \frac{f(x)}{2}$, then $f(x)$ is equal to

- a e^x b e^{-x} c $\log x$ d e^{x^2}

Integration by Partial Fractions

If a function is of the form $\frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are two polynomials of degree m and n , respectively.

If $m < n$, this fraction is proper fraction and if $m \geq n$ this fraction is improper fraction to convert it into a proper fraction divide $P(x)$ by $Q(x)$, so that $\frac{P(x)}{Q(x)}$

is expressed in the form $\frac{P(x)}{Q(x)} = M(x) + \frac{P_1(x)}{Q(x)}$

where, $M(x)$ and $P_1(x)$ are polynomials such that the degree of $P_1(x)$ is less than that of $Q(x)$.
(where, $\frac{P_1(x)}{Q(x)}$ is proper fraction)

Any proper rational function $\frac{P(x)}{Q(x)}$ can be expressed as the sum of rational functions, each having a simple factor of $Q(x)$. Each such fraction is called a partial fraction and the process of obtaining them is called the resolution or decomposition of $\frac{P(x)}{Q(x)}$ into

partial fractions and it depends upon the nature of the factors of $Q(x)$.

Case I If $Q(x)$ has non-repeating linear factors like $Q(x) = (a_1x + b_1)(a_2x + b_2) \dots (a_nx + b_n)$, then

$$\frac{P_1(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n}$$

$A_1, A_2, \dots, A_n$ are independent of x , i.e. constants. To find A_1 put $x = -\frac{b_1}{a_1}$ in the

LHS except in the factor $(a_1x + b_1)$, we get A_1 .

Similarly, find $A_2, A_3, \dots, A_n$. Now,

$$\begin{aligned} & \text{integrate } \int \frac{P_1(x)}{Q(x)} dx \\ &= \int \left(\frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{a_nx + b_n} \right) dx \end{aligned}$$

Case II If $Q(x)$ has repeating linear factors like

$Q(x) = (a_1x + b_1)(a_2x + b_2)^n$, then

$$\begin{aligned} \frac{P_1(x)}{Q(x)} &= \frac{P_1(x)}{(a_1x + b_1)(a_2x + b_2)^n} \\ &= \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \dots + \frac{A_n}{(a_nx + b_n)^n} \end{aligned}$$

Now, find the constants $A_1, A_2, \dots, A_n$ and then integrate.

Case III When some of the factors of denominator $Q(x)$ are quadratic but non-repeating. Corresponding to each quadratic factor $ax^2 + bx + c$, assume partial fraction of the type $\frac{Ax + B}{ax^2 + bx + c}$, where A and B are constants.

Now, if $Q(x) = (a_1x + b_1)(ax^2 + bx + c)(ex^2 + fx + g)$, then

$$\begin{aligned} \frac{P_1(x)}{Q(x)} &= \frac{A_1}{a_1x + b_1} + \frac{Ax + B}{ax^2 + bx + c} \\ &\quad + \frac{Cx + D}{ex^2 + fx + g} \end{aligned}$$

Now, find the constants A_1, A, B, C and D and then integrate.

Case IV When some of the factors of $Q(x)$ are quadratic and repeating like $Q(x) = (a_1x + b_1)(a_2x + b_2)(ax^2 + bx + c)(ax^2 + bx + c)^2$, then

$$\begin{aligned} \frac{P_1(x)}{Q(x)} &= \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \frac{A_3x + B_1}{ax^2 + bx + c} \\ &\quad + \frac{A_4x + B_2}{(ax^2 + bx + c)^2} \end{aligned}$$

Now, find the constants A_1, A_2, A_3, B_1, A_4 and B_2 and then integrate.

➤ **Example 43.** $\int \frac{dx}{(x+2)(x^2+1)}$ is equal to

(a) $\frac{1}{5} \log|x+2| - \frac{1}{10} \log(x^2+1) + \frac{2}{5} \tan^{-1}x + C$

(b) $\log|x+2| + \frac{1}{10} \log(x^2+1) + \frac{1}{5} \tan^{-1}x + C$

(c) $5 \log|x+2| + 10 \log(x^2+1) + \tan^{-1}x + C$

(d) None of the above

Sol. (a) Let $\frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$

$$\Rightarrow 1 = A(x^2+1) + (Bx+C)(x+2)$$

Put $x = -2$, we get $A = \frac{1}{5}$

Now, comparing the coefficients of x^2 and constant term, we get

$$0 = A + B \text{ and } 1 = A + 2C$$

$$\Rightarrow B = -\frac{1}{5}, C = \frac{2}{5}$$

$$\text{So, given integral} = \frac{1}{5} \int \frac{dx}{x+2} - \frac{1}{5} \int \frac{x}{x^2+1} dx$$

$$= \frac{1}{5} \log|x+2| - \frac{1}{10} \log(x^2+1) + \frac{2}{5} \tan^{-1}x + C$$

➤ **Example 44.** $\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx$ is equal to

- (a) $\log|x-1| - 5\log|x-2| + 4\log|x-3| + C$
 (b) $\log|x-1| - \log|x-2| + 4\log|x-3| + C$
 (c) $5\log|x-1| - \log|x-2| + 4\log|x-3| + C$
 (d) None of the above

Sol. (a) Let $\frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)}$

$$\Rightarrow \frac{3x-1}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$\Rightarrow 3x-1 = A(x^2-5x+6) + B(x^2-4x+3) + C(x^2-3x+2)$$

$$\Rightarrow 3x-1 = x^2(A+B+C) + x(-5A-4B-3C) + (6A+3B+2C)$$

On equating the coefficients of x^2 , x and constant term on both sides, we get

$$\begin{aligned} A+B+C &= 0 & \dots(i) \\ -5A-4B-3C &= 3 & \dots(ii) \\ 6A+3B+2C &= -1 & \dots(iii) \end{aligned}$$

From Eq. (i), we get $A = -(B+C)$

On putting the value of A in Eqs. (ii) and (iii), we get

$$\begin{aligned} -5\{-B-C\} - 4B - 3C &= 3 \\ \Rightarrow 5B + 5C - 4B - 3C &= 3 \\ \Rightarrow B + 2C &= 3 & \dots(iv) \end{aligned}$$

$$\begin{aligned} \text{and } 6\{-(B+C)\} + 3B + 2C &= -1 \\ \Rightarrow -6B - 6C + 3B + 2C &= -1 \\ \Rightarrow -3B - 4C &= -1 & \dots(v) \end{aligned}$$

On solving Eqs. (iv) and (v), we get $C = 4$

On putting the value of C in Eq. (iv), we get

$$B + 2 \times 4 = 3 \Rightarrow B = -5$$

On putting the value of B and C in Eq. (i), we get

$$A + (-5) + 4 = 0 \Rightarrow A = 1$$

$$\therefore A = 1, B = -5, C = 4$$

Now, $\int \frac{3x-1}{(x-1)(x-2)(x-3)} dx$

$$= \int \left[\frac{A}{(x-1)} + \frac{B}{(x-2)} + \frac{C}{(x-3)} \right] dx$$

$$= \int \frac{1}{(x-1)} dx + \int \frac{(-5)}{(x-2)} dx + \int \frac{4}{(x-3)} dx$$

$$= \log|x-1| - 5\log|x-2| + 4\log|x-3| + C$$

$\left[\because \int \frac{1}{x} dx = \log|x| \right]$

➤ **Example 45.** If $\int \sqrt[3]{\tan x} dx = -\frac{1}{2} \log|\tan^{2/3} x + 1| + \frac{1}{4} \log|\tan^{4/3} x - \tan^{2/3} x + 1| + A$. Then, A is equal to

- (a) $\frac{\sqrt{3}}{2} \tan^{-1} \left\{ \frac{2 \tan^{2/3} x - \tan^{4/3} x}{\sqrt{3}} \right\} + C$
 (b) $\frac{\sqrt{3}}{2} \tan^{-1} \left\{ \frac{2 \tan^{2/3} x - 1}{\sqrt{3}} \right\} + C$

(c) $\frac{\sqrt{3}}{4} \tan^{-1} \left\{ \frac{2 \tan^{2/3} x - \tan^{4/3} x}{3} \right\} + C$

(d) None of the above

Sol. (b) On putting $\tan x = t^{3/2} \Rightarrow \sec^2 x dx = \frac{3}{2} t^{1/2} dt$

or $dx = \frac{3t^{1/2}}{2(1+t^3)} dt$, we get

$$I = \int \frac{t^{1/2} \times 3t^{1/2}}{2(1+t^3)} dt = \frac{3}{2} \int \frac{t}{1+t^3} dt$$

$$= \frac{3}{2} \int \frac{t}{(1+t)(t^2-t+1)} dt$$

Now, let $\frac{t}{(1+t)(t^2-t+1)} = \frac{A}{t+1} + \frac{Bt+C}{t^2-t+1}$

$$\Rightarrow t = A(t^2-t+1) + (Bt+C)(t+1)$$

On equating the coefficient of like power of t , we get

$$A+B=0, -A+B+C=1 \text{ and } A+C=0$$

$$\Rightarrow A = \frac{1}{3}, B = \frac{1}{3}, C = 0$$

$$\therefore \frac{t}{(1+t)(t^2-t+1)} = \frac{-1}{3(t+1)} + \frac{t+1}{3(t^2-t+1)}$$

$$\Rightarrow I = \frac{3}{2} \int \left\{ -\frac{1}{3(t+1)} + \frac{t+1}{3(t^2-t+1)} \right\} dt$$

$$= -\frac{1}{2} \int \frac{1}{t+1} dt + \frac{1}{2} \int \frac{t+1}{t^2-t+1} dt$$

$$= -\frac{1}{2} \log|t+1| + \frac{1}{4} \int \frac{2t-1+3}{t^2-t+1} dt$$

$$= -\frac{1}{2} \log|t+1| + \frac{1}{4} \int \frac{2t-1}{t^2-t+1} dt + \frac{3}{4} \int \frac{1}{t^2-t+1} dt$$

$$= -\frac{1}{2} \log|t+1| + \frac{1}{4} \log|t^2-t+1| + \frac{3}{4} \int \frac{1}{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dt$$

$$= -\frac{1}{2} \log|t+1| + \frac{1}{4} \log|t^2-t+1| + \frac{\sqrt{3}}{2} \tan^{-1} \left(\frac{2t-1}{\sqrt{3}} \right) + C$$

$$= -\frac{1}{2} \log|\tan^{2/3} x + 1| + \frac{1}{4} \log|\tan^{4/3} x - \tan^{2/3} x + 1| + \frac{\sqrt{3}}{2} \tan^{-1} \left\{ \frac{2 \tan^{2/3} x - 1}{\sqrt{3}} \right\} + C$$

➤ **Example 46.** $\int \frac{1-x^2}{x(1-2x)} dx$ is equal to

- (a) $x + \log|x| - \frac{3}{4} \log|x-1| + C$
 (b) $\frac{x}{2} + \log|x| + \frac{1}{4} \log|x-1| + C$
 (c) $\frac{x}{2} + \log|x| - \frac{3}{4} \log|2x-1| + C$
 (d) None of the above

Sol. (c) In rational algebraic functions, if the degree of numerator is greater than or equal to the degree of denominator, then always divide the numerator by denominator and use the result.

$$\frac{\text{Numerator}}{\text{Denominator}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Denominator}}$$

Thus, we can convert the given integrand into simple form and then integrate easily.

Let $I = \int \frac{1-x^2}{x(1-2x)} dx$. Here, degree of numerator is equal to degree of denominator, so divide by denominator.

$$\text{Thus, } \frac{1-x^2}{x(1-2x)} = \frac{x^2-1}{2x^2-x} = \frac{1}{2} + \frac{\frac{1}{2}x-1}{2x^2-x}$$

$$\therefore I = \int \frac{1}{2} dx + \int \frac{\frac{1}{2}x-1}{(2x^2-x)} dx$$

$$\Rightarrow I = I_1 + I_2 \quad \dots(i)$$

$$\text{where, } I_1 = \int \frac{1}{2} dx$$

$$\text{and } I_2 = \int \frac{\frac{1}{2}x-1}{(2x^2-x)} dx$$

$$\text{Now, } I_1 = \int \frac{1}{2} dx = \frac{1}{2}x + C_1 \text{ and } I_2 = \int \frac{\frac{1}{2}x-1}{2x^2-x} dx$$

$$\text{Let } \frac{\frac{1}{2}x-1}{(2x^2-x)} = \frac{A}{x} + \frac{B}{(2x-1)}$$

$$\Rightarrow \frac{\frac{1}{2}x-1}{(2x^2-x)} = \frac{A(2x-1) + Bx}{x(2x-1)}$$

$$\Rightarrow \frac{1}{2}x-1 = 2Ax - A + Bx$$

$$\Rightarrow \frac{1}{2}x-1 = x(2A+B) - A$$

On comparing the coefficient of x and constant term on both sides, we get

$$2A+B = \frac{1}{2} \quad \dots(ii)$$

$$\text{and } -A = -1 \Rightarrow A = 1 \quad \dots(iii)$$

From Eq. (ii),

$$2 \times 1 + B = \frac{1}{2}$$

$$\Rightarrow B = \frac{1}{2} - 2 = -\frac{3}{2}$$

$$\therefore I_2 = \int \left[\frac{1}{x} - \frac{3}{2(2x-1)} \right] dx$$

$$= \int \frac{1}{x} dx - \frac{3}{2} \int \frac{1}{2x-1} dx$$

$$\Rightarrow I_2 = \log x - \frac{3}{2} \cdot \frac{\log|2x-1|}{2} + C_2$$

On putting the values of I_1 and I_2 in Eq. (i), we get

$$I = \frac{1}{2}x + \log x - \frac{3}{4} \log|2x-1| + C$$

$$[\because C_1 + C_2 = C]$$

➤ **Example 47.** The value of $\int \frac{3x+2}{(x-2)^2(x-3)} dx$

is

(a) $11 \log \left| \frac{x-3}{x-2} \right| - \frac{8}{x-2} + K$

(b) $11 \log \left| \frac{x+3}{x+2} \right| - \frac{8}{x-2} + K$

(c) $11 \log \left| \frac{x-3}{x-2} \right| + \frac{8}{x-2} + K$

(d) $11 \log \left| \frac{x+3}{x+2} \right| + \frac{8}{x-2} + K$

Sol. (c) Let $\frac{3x+2}{(x-2)^2(x-3)} = \frac{A}{(x-2)} + \frac{B}{(x-2)^2} + \frac{C}{(x-3)}$

$$\therefore 3x+2 = A(x-2)(x-3) + B(x-3) + C(x-2)^2 \quad \dots(i)$$

After solving Eq. (i), we get $A = -11, B = -8, C = 11$

$$\therefore \int \frac{3x+2}{(x-2)^2(x-3)} dx = \int \frac{-11dx}{(x-2)} + \int \frac{-8dx}{(x-2)^2} + \int \frac{11dx}{(x-3)}$$

$$= -11 \log|x-2| + \frac{8}{(x-2)} + 11 \log|x-3| + K$$

$$= 11 \log \left| \frac{(x-3)}{(x-2)} \right| + \frac{8}{(x-2)} + K$$

Work Book Exercise 9.4

1 $\int \frac{x-1}{(x-3)(x-2)} dx$ is equal to

- a $\log(x-3) - \log(x-2) + C$
- b $\log(x-3)^2 - \log(x-2) + C$
- c $\log(x-3) + \log(x-2) + C$
- d $\log(x-3)^2 + \log(x-2) + C$

2 Correct evaluation of $\int \frac{x}{(x-2)(x-1)} dx$ is

(where, P is an arbitrary constant)

- a $\log_e \frac{(x-2)^2}{(x-1)} + P$
- b $\log_e \frac{(x-1)}{(x-2)} + P$
- c $\frac{x-1}{x-2} + P$
- d $2 \log_e \left(\frac{x-2}{x-1} \right) + P$

3 $\int \frac{x^2+x-1}{x^2+x-6} dx$ is equal to

- a $x + \log(x+3) + \log(x-2) + C$
- b $x - \log(x+3) + \log(x-2) + C$
- c $x - \log(x+3) - \log(x-2) + C$
- d None of the above

4 $\int \frac{dx}{1+x+x^2+x^3}$ is equal to

- a $\log \sqrt{1+x} - \frac{1}{2} \log \sqrt{1+x^2} + \frac{1}{2} \tan^{-1} x + C$
- b $\log \sqrt{1+x} - \log \sqrt{1+x^2} + \tan^{-1} x + C$
- c $\log \sqrt{1+x^2} - \log \sqrt{1+x} + \frac{1}{2} \tan^{-1} x + C$
- d $\log \sqrt{1+x} + \tan^{-1} x + \log \sqrt{1+x^2} + C$

5 $\int \frac{e^x}{(1+e^x)(2+e^x)} dx$ is equal to

a $\log[(1+e^x)(2+e^x)] + C$ b $\log\left[\frac{1+e^x}{2+e^x}\right] + C$

c $\log[(1+e^x)\sqrt{2+e^x}] + C$ d None of these

6 $\int \frac{x}{x^4-1} dx$ is equal to

a $\frac{1}{4} \log\left[\frac{x^2-1}{x^2+1}\right] + C$ b $\frac{1}{4} \log\left[\frac{x^2+1}{x^2-1}\right] + C$

c $\frac{1}{2} \log\left[\frac{x^2-1}{x^2+1}\right] + C$

d $\frac{1}{2} \log\left[\frac{x^2+1}{x^2-1}\right] + C$

7 $\int \frac{1}{\cos x (1+\cos x)} dx$ is equal to

a $\log(\sec x + \tan x) + 2 \tan \frac{x}{2} + C$

b $\log(\sec x + \tan x) - 2 \tan \frac{x}{2} + C$

c $\log(\sec x + \tan x) + \tan \frac{x}{2} + C$

d $\log(\sec x + \tan x) - \tan \frac{x}{2} + C$

Integration of Some Special Irrational Algebraic Functions

In this section we shall discuss six integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$, where P and Q are polynomial functions of x .

1. Integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$, where P and

Q are both linear expression of x .

To evaluate this type of integrals, we put $Q = t^2$, i.e. to evaluate integrals of the form

$$\int \frac{1}{(ax+b)\sqrt{cx+d}} dx, \text{ put } cx+d = t^2$$

2. Integrals of the form $\int \frac{\phi(x)}{P\sqrt{Q}} dx$, where P is

quadratic expression and Q is linear expression. To evaluate this type of integrals, we put $Q = t^2$, i.e. to evaluate integrals of the form

$$\int \frac{1}{(ax^2+bx+c)\sqrt{px+q}} dx, \text{ put } px+q = t^2$$

3. Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, where P is linear

expression and Q is a quadratic expression. To evaluate this type of integral we put, $P = \frac{1}{t}$, i.e.

to evaluate integrals of the form $\int \frac{1}{(ax+b)\sqrt{px^2+qx+r}} dx$, put $ax+b = \frac{1}{t}$

4. Integrals of the form $\int \frac{dx}{P\sqrt{Q}}$, where P and Q

both are pure quadratic expression in x .

i.e. $P = ax^2 + b$ and $Q = cx^2 + d$

i.e. $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$

To evaluate this type of integrals, we put $x = \frac{1}{t}$.

5. Integrals of the form $\int \frac{dx}{(x-k)^r \sqrt{ax^2+bx+c}}$,

where $r \geq 2$ and $r \in I$.

Here, we substitute, $x-k = \frac{1}{t}$.

6. Integrals of the form

$$\int \frac{ax^2+bx+c}{(dx+e)\sqrt{fx^2+gx+h}} dx$$

Here, we write,

$$ax^2+bx+c = A_1(dx+e)(2fx+g) + B_1(dx+e) + C_1$$

where A_1 , B_1 and C_1 are constants which can be obtained by comparing the coefficients of like terms on both sides.

➤ **Example 48.** $\int \frac{1}{(x+1)\sqrt{x^2-1}} dx$ is equal to

(a) $\frac{1}{2} \sqrt{\frac{x-1}{x+1}} + C$

(b) $\sqrt{\frac{x+1}{x-1}} + C$

(c) $\frac{1}{2} \sqrt{\frac{x+1}{x-1}} + C$

(d) $\sqrt{\frac{x-1}{x+1}} + C$

Sol. (d) Let $I = \int \frac{1}{(x+1)\sqrt{x^2-1}} dx$

Let $x+1 = \frac{1}{t}$, then $dx = -\frac{1}{t^2} dt$

$$I = \int \frac{1}{\frac{1}{t} \sqrt{\left(\frac{1}{t}-1\right)^2-1}} \cdot \left(-\frac{1}{t^2}\right) dt$$

$$\begin{aligned}
 &= -\int \frac{dt}{\sqrt{1-2t}} = -\int (1-2t)^{-1/2} dt \\
 &= -\frac{(1-2t)^{1/2}}{(-2)\left(\frac{1}{2}\right)} + C = \sqrt{1-2t} + C \\
 &= \sqrt{1-\frac{2}{x+1}} + C = \sqrt{\frac{x-1}{x+1}} + C
 \end{aligned}$$

➤ **Example 49.** $\int \frac{1}{x^2 \sqrt{1+x^2}} dx$ is equal to

- (a) $-\sqrt{\frac{1}{1+x^2}} + C$ (b) $-\frac{\sqrt{1+x^2}}{x} + C$
 (c) $\frac{\sqrt{1+x^2}}{x} + C$ (d) None of these

Sol. (b) Let $I = \int \frac{1}{x^2 \sqrt{1+x^2}} dx$

Let $x = \frac{1}{t}$, then $-\frac{1}{x^2} dx = dt \Rightarrow dx = -x^2 dt$

$$\begin{aligned}
 \therefore I &= \int \frac{-dt}{\sqrt{1+\frac{1}{t^2}}} = -\int \frac{t dt}{\sqrt{t^2+1}} = -\int \frac{u du}{\sqrt{u^2}} \\
 &\quad \text{[where, } t^2 + 1 = u^2]
 \end{aligned}$$

$$\begin{aligned}
 &= \int (-1) du = -u + C \\
 &= -\sqrt{t^2+1} + C = -\sqrt{\frac{1}{x^2}+1} + C \\
 &= -\frac{\sqrt{1+x^2}}{x} + C
 \end{aligned}$$

➤ **Example 50.** If $\int \frac{\sqrt{1+x^2}}{1-x^2} dx$

$$= -\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{1+x^2} - \sqrt{2}x}{\sqrt{1+x^2} + \sqrt{2}x} \right| - A + C. \text{ Then, } A \text{ is}$$

equal to

- (a) $\log |x + \sqrt{1+x^2}|$
 (b) $\log |\sqrt{1+x^2} - x|$
 (c) $\log |\sqrt{1-x^2} + x|$
 (d) None of these

Sol. (a) We have, $I = \int \frac{\sqrt{1+x^2}}{1-x^2} dx$

$$\begin{aligned}
 &= \int \frac{\sqrt{1+x^2}}{1-x^2} \times \frac{\sqrt{1+x^2}}{\sqrt{1+x^2}} dx \\
 &= \int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2}} dx \\
 &= \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx + \int \frac{x^2}{(1-x^2)\sqrt{1+x^2}} dx \\
 &= \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx - \int \frac{-x^2}{(1-x^2)\sqrt{1+x^2}} dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx - \int \frac{(1-x^2)-1}{(1-x^2)\sqrt{1+x^2}} dx \\
 &= \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx \\
 &\quad - \int \left\{ \frac{1-x^2}{(1-x^2)\sqrt{1+x^2}} - \frac{1}{(1-x^2)\sqrt{1+x^2}} \right\} dx \\
 &= \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx - \int \frac{1}{\sqrt{1+x^2}} dx \\
 &\quad + \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx \\
 &= 2 \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx - \int \frac{1}{\sqrt{1+x^2}} dx \\
 &= 2I_1 - \log |x + \sqrt{1+x^2}| + C
 \end{aligned}$$

where, $I_1 = \int \frac{1}{(1-x^2)\sqrt{1+x^2}} dx$

Let $x = \frac{1}{t}$, then $dx = -\frac{1}{t^2} dt$

$$\begin{aligned}
 \therefore I_1 &= \int \frac{\left(-\frac{1}{t^2}\right)}{\left(1-\frac{1}{t^2}\right)\sqrt{1+\frac{1}{t^2}}} dt \\
 &= -\int \frac{t}{(t^2-1)\sqrt{t^2+1}} dt \\
 &= -\int \frac{u du}{(u^2-2)\sqrt{u^2}} \text{ where } t^2+1=u^2 \\
 &= -\int \frac{du}{u^2-(\sqrt{2})^2} = -\frac{1}{2\sqrt{2}} \log \left| \frac{u-\sqrt{2}}{u+\sqrt{2}} \right| \\
 &= -\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{t^2+1}-\sqrt{2}}{\sqrt{t^2+1}+\sqrt{2}} \right| \\
 &= -\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{\frac{1}{x^2}+1}-\sqrt{2}}{\sqrt{\frac{1}{x^2}+1}+\sqrt{2}} \right| \\
 &= -\frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x} \right| \\
 \therefore I &= -\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{1+x^2}-\sqrt{2}x}{\sqrt{1+x^2}+\sqrt{2}x} \right| \\
 &\quad - \log |x + \sqrt{1+x^2}| + C
 \end{aligned}$$

Reduction Formulae

$$\begin{aligned}
 1. I_n &= \int \tan^n x dx = \int \tan^2 x \cdot \tan^{n-2} x dx \\
 &= \int (\sec^2 x - 1) \tan^{n-2} x dx \\
 &= \int \sec^2 x \cdot \tan^{n-2} x dx - I_{n-2} \\
 \Rightarrow I_n &= \frac{\tan^{n-1} x}{n-1} - I_{n-2}, n \geq 2
 \end{aligned}$$

$$\begin{aligned} 2. I_n &= \int \cot^n x \, dx = \int \cot^2 x \cdot \cot^{n-2} x \, dx \\ &= \int (\operatorname{cosec}^2 x - 1) \cot^{n-2} x \, dx \\ &= \int \operatorname{cosec}^2 x \cdot \cot^{n-2} x \, dx - I_{n-2} \end{aligned}$$

$$\Rightarrow I_n = \frac{-\cot^{n-1} x}{n-1} - I_{n-2}, n \geq 2$$

$$3. I_n = \int \sec^n x \, dx = \int \sec^2 x \cdot \sec^{n-2} x \, dx$$

$$\Rightarrow I_n = \tan x \cdot \sec^{n-2} x$$

$$- \int (\tan x) (n-2) \sec^{n-3} x \sec x \cdot \tan x \, dx$$

$$\Rightarrow I_n = \tan x \cdot \sec^{n-2} x$$

$$- (n-2) \int (\sec^2 x - 1) \cdot \sec^{n-2} x \, dx$$

$$\Rightarrow (n-1) I_n = \tan x \cdot \sec^{n-2} x + (n-2) I_{n-2}$$

$$I_n = \frac{\tan x \cdot \sec^{n-2} x}{n-1} + \frac{(n-2)}{(n-1)} \cdot I_{n-2}$$

$$4. I_n = \int \operatorname{cosec}^n x \, dx = \int \operatorname{cosec}^2 x \cdot \operatorname{cosec}^{n-2} x \, dx$$

$$\Rightarrow I_n = -\cot x \cdot \operatorname{cosec}^{(n-2)} x$$

$$+ \int (\cot x)(n-2)(-\operatorname{cosec}^{n-3} x \cdot \operatorname{cosec} x \cdot \cot x) \, dx$$

$$\Rightarrow I_n = \cot x \cdot \operatorname{cosec}^{n-2} x - (n-2)$$

$$\int \cot^2 x \operatorname{cosec}^{n-2} x \, dx$$

$$\Rightarrow I_n = -\cot x \cdot \operatorname{cosec}^{n-2} x - (n-2)$$

$$\int (\operatorname{cosec}^2 x - 1) \cdot \operatorname{cosec}^{n-2} x \, dx$$

$$\Rightarrow (n-1) I_n = -\cot x \operatorname{cosec}^{n-2} x + (n-2) \cdot I_{n-2}$$

$$\Rightarrow I_n = \frac{\cot x \cdot \operatorname{cosec}^{n-2} x}{-(n-1)} + \frac{(n-2)}{(n-1)} \cdot I_{n-2}$$

$$5. \int \sin^n x \, dx = \frac{-\sin^{n-1} x \cdot \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$$

$$6. \int x^n \sin x \, dx = -x^n \cos x + n \int x^{n-1} \cos x \, dx$$

$$7. \int x^n \cos x \, dx = x^n \sin x - n \int x^{n-1} \sin x \, dx$$

$$8. \int x^n e^{ax} \, dx = \frac{1}{a} x^n \cdot e^{ax} - \frac{n}{a} \int x^{n-1} \cdot e^{ax} \, dx$$

$$9. \int (\log x)^n \, dx = x (\log x)^n - n \int (\log x)^{n-1} \, dx$$

➤ **Example 51.** $\int \tan^8 x \, dx$ is equal to

$$(a) \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} - \frac{\tan x}{1} + x + C$$

$$(b) \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \frac{\tan x}{1} + x + C$$

$$(c) \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \tan x + x + C$$

(d) None of the above

$$\text{Sol. (c)} I_8 = \int \tan^8 x \, dx = \frac{\tan^7 x}{7} - I_6 = \frac{\tan^7 x}{7} - \left[\frac{\tan^5 x}{5} - I_4 \right]$$

$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - I_2$$

$$= \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} + \frac{\tan x}{1} + I_0 \quad \dots(i)$$

$$\text{Now, } I_0 = \int dx = x + C \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$I_8 = \frac{\tan^7 x}{7} - \frac{\tan^5 x}{5} + \frac{\tan^3 x}{3} - \frac{\tan x}{1} + x + C$$

Work Book Exercise 9.5

1 $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ is equal to

a $-\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{1-x^2}}{\sqrt{2}x} \right) + C$

b $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{1-x^2}}{\sqrt{2}x} \right) + C$

c $-\tan^{-1} \left(\frac{\sqrt{1-x^2}}{\sqrt{2}x} \right) + C$

d None of the above

2 $\int \frac{dx}{x\sqrt{ax-x^2}}$ is equal to

a $-\frac{1}{a} \sqrt{\frac{x-a}{x}} + C$

b $\frac{-2}{a} \sqrt{\frac{a-x}{x}} + C$

c $\frac{2}{a} \sqrt{\frac{a-x}{x}} + C$

d None of these

3 $\int \frac{1}{(x-1)\sqrt{x^2+4}}$ dx is equal to

a $-\frac{1}{\sqrt{5}} \log \left| \frac{1}{x-1} + \frac{1}{5} + \sqrt{\frac{x^2+4}{(x-1)^2}} \right| + C$

b $-\frac{1}{\sqrt{5}} \log \left| \frac{1}{x-1} + \frac{1}{5} + \sqrt{\frac{x^2+4}{5(x-1)^2}} \right| + C$

c $-\frac{1}{\sqrt{5}} \log \left| \frac{1}{x-1} + \frac{1}{5} + \sqrt{\frac{x^2+4}{(x-1)^2}} \right| + C$

d None of the above

4 $\int \frac{2x^2+7x+11}{(x+2)\sqrt{x^2+4x+8}}$ is equal to

a $2\sqrt{x^2+4x+8} - \log |(x+2) + \sqrt{x^2+4x+8}|$
 $-\frac{5}{2} \log \left| \frac{1}{(x+2)} + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}} \right| + C$

- b $\sqrt{x^2 + 4x + 8} + \log |(x+2) + \sqrt{x^2 + 4x + 8}|$
 $+ 5 \log \left| (x+2) + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}} \right| + C$
- c $\sqrt{x^2 + 4x + 8} + \log |(x+2) + \sqrt{x^2 + 4x + 8}|$
 $+ 5 \log \left| (x+2) + \sqrt{(x+2)^2 + 4} \right| + C$
- d None of the above

5 $\int \frac{x dx}{(2x^2 + 3)\sqrt{x^2 - 1}}$ is equal to

- a $\tan^{-1} \left(\sqrt{\frac{2}{5}} \cdot \sqrt{x^2 - 1} \right) + C$
- b $10 \tan^{-1} \left(\sqrt{\frac{2}{5}} \cdot \sqrt{x^2 - 1} \right) + C$
- c $\frac{1}{10} \tan^{-1} \left(\sqrt{\frac{2}{5}} \cdot \sqrt{x^2 - 1} \right) + C$
- d None of the above

6 $\int \frac{x+2}{(x^2 + 3x + 3)\sqrt{x+1}} dx$ is equal to

- a $\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{x+1} \right) + C$
- b $\frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{x}{\sqrt{3(x+1)}} \right) + C$
- c $\frac{2}{\sqrt{3}} \tan^{-1} \left[\frac{x}{(x+1)^2} \right] + C$
- d None of the above

7 $\int \frac{(4x+7)}{(x+2)\sqrt{x^2 + 4x + 8}}$ is equal to

- a $\log(x+2 + \sqrt{x^2 + 4x + 8}) +$
 $\frac{1}{2} \log \left| \frac{1}{(x+2)} + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}} \right| + C$
- b $4 \log(x+2 + \sqrt{x^2 + 4x + 8})$
 $+ \frac{1}{2} \log \left| \frac{1}{(x+2)} + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}} \right| + C$
- c $\log(x+2 + \sqrt{x^2 + 4x + 8})$
 $+ \log \left| (x+2) + \sqrt{\frac{1}{(x+2)^2} + \frac{1}{4}} \right| + C$
- d None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $I = \int \frac{\tan x}{\sqrt{a + b \tan^2 x}} dx$, $a < b$, then I equals

- (a) $\frac{1}{\sqrt{b-a}} \sin^{-1} \left(\sqrt{\frac{a + b \tan^2 x}{b-a}} \right) + C$
 (b) $\frac{1}{\sqrt{b-a}} \cos^{-1} \left(\sqrt{\frac{a + b \tan^2 x}{b-a}} \right) + C$
 (c) $\frac{1}{\sqrt{b-a}} \tan^{-1} \left(\sqrt{\frac{a + b \tan^2 x}{b-a}} \right) + C$
 (d) $\frac{1}{\sqrt{b-a}} \tan^{-1} \left(\sqrt{\frac{b-a}{a + b \tan^2 x}} \right) + C$

Sol. Put $a + b \tan^2 x = t^2$

$$\Rightarrow 2b \tan x \sec^2 x dx = 2t dt$$

$$\Rightarrow \tan x dx = \frac{t}{b} \cdot \frac{b}{b + (t^2 - a)} dt$$

$$\begin{aligned} \therefore I &= \int \frac{t}{t} \cdot \frac{dt}{t^2 + (b-a)} \\ &= \frac{1}{\sqrt{b-a}} \tan^{-1} \left(\frac{t}{\sqrt{b-a}} \right) + C \\ &= \frac{1}{\sqrt{b-a}} \tan^{-1} \left(\sqrt{\frac{a + b \tan^2 x}{b-a}} \right) + C \end{aligned}$$

Hence, (c) is the correct answer.

Aliter

$$I = \int \frac{\sin x}{\sqrt{a \cos^2 x + b \sin^2 x}} dx$$

Put $\cos x = t$, so that

$$\begin{aligned} I &= \int \frac{-dt}{\sqrt{b - (b-a)t^2}} \\ &= \frac{1}{\sqrt{b-a}} \cos^{-1} \left(\sqrt{\frac{b-a}{b}} \cos x \right) + C \end{aligned}$$

$$\text{But } \cos^{-1} x = \tan^{-1} \left(\frac{\sqrt{1-x^2}}{x} \right), \text{ therefore}$$

$$\begin{aligned} I &= \frac{1}{\sqrt{b-a}} \tan^{-1} \left(\sqrt{\frac{b}{(b-a)\cos^2 x} - 1} \right) + C \\ &= \frac{1}{\sqrt{b-a}} \tan^{-1} \left(\sqrt{\frac{b(1 + \tan^2 x) - b + a}{b-a}} \right) + C \\ &= \frac{1}{\sqrt{b-a}} \tan^{-1} \left(\sqrt{\frac{a + b \tan^2 x}{b-a}} \right) + C \end{aligned}$$

Ex 2. If $I = \int \sqrt{e^{2x} + ae^x} dx$, then I equals

- (a) $\sqrt{e^{2x} + ae^x} + a \log |e^x + ae^{2x}| + C$
 (b) $\sqrt{e^{2x} + ae^x} + a \log |e^{x/2} + ae^x| + C$
 (c) $\sqrt{e^{2x} + ae^x} + a \log |e^{\frac{x}{2}} + \sqrt{e^x + a}| + C$
 (d) None of the above

Sol. Write $I = \int e^x \sqrt{1 + ae^{-x}} dx$, put $e^x = t^2$, so that

$$\begin{aligned} I &= \int \sqrt{\left(1 + \frac{a}{t^2}\right)} (2t) dt = 2 \int \sqrt{t^2 + a} dt \\ &= t\sqrt{t^2 + a} + a \log |t + \sqrt{t^2 + a}| + C \\ &= e^{\frac{x}{2}} \sqrt{e^x + a} + a \log |e^{x/2} + \sqrt{e^x + a}| + C \\ &= \sqrt{e^{2x} + ae^x} + a \log |e^{x/2} + \sqrt{e^x + a}| + C \end{aligned}$$

Hence, (c) is the correct answer.

Ex 3. If $I = \int \frac{(3 \sin \theta - 2) \cos \theta}{5 - \cos^2 \theta - 4 \sin \theta} d\theta$, then I equals

- (a) $3 \log (2 - \sin \theta) + \frac{4}{2 - \sin \theta} + C$
 (b) $3 \log |2 - \sin \theta| + \frac{4}{2 + \sin \theta} + C$
 (c) $3 \log (2 + \sin \theta) - \frac{2}{2 + \cos \theta} + C$
 (d) $3 \log (2 + \sin \theta) - \frac{2}{2 - \cos \theta} + C$

Sol. Put $\sin \theta = x$, so that

$$\begin{aligned} I &= \int \frac{(3x - 2)}{5 - (1 - x^2) - 4x} dx \\ &= \int \frac{3x - 2}{(x - 2)^2} dx \end{aligned}$$

Put $x - 2 = t$, so that

$$\begin{aligned} I &= \int \frac{3(t + 2) - 2}{t^2} dt \\ &= \int \left(\frac{3}{t} + \frac{4}{t^2} \right) dt \\ &= 3 \log |t| - \frac{4}{t} + C \\ &= 3 \log |\sin \theta - 2| - \frac{4}{\sin \theta - 2} + C \\ &= 3 \log (2 - \sin \theta) + \frac{4}{2 - \sin \theta} + C \end{aligned}$$

Hence, (a) is the correct answer.

Ex 4. If $I = \int e^x (x \cos x + \sin x) dx$, then I equals

- (a) $\frac{1}{2} e^x (x \sin x - \cos x) + C$
 (b) $\frac{1}{2} e^x (x \sin x + \cos x) + C$
 (c) $\frac{1}{2} e^x (x \cos x - \sin x) + C$
 (d) None of the above

Sol. We have, $\int e^x \cos x dx = \frac{1}{2} e^x (\sin x + \cos x)$, therefore

$$\begin{aligned} I &= \frac{x}{2} e^x (\sin x + \cos x) \\ &\quad - \frac{1}{2} \int e^x (\sin x + \cos x) dx + \int e^x \sin x dx \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} \int e^x (\cos x - \sin x) dx \\ &= \frac{1}{2} x e^x (\sin x + \cos x) - \frac{1}{2} e^x \cos x + C \\ &= \frac{1}{2} e^x (x \sin x + x \cos x - \cos x) + C \end{aligned}$$

Hence, (d) is the correct answer.

Ex 5. If $I = \int \sec^{-1} \sqrt{x} dx$, then I equals

- (a) $x \sec^{-1} \sqrt{x} - \log (1+x) + C$
 (b) $\sec^{-1} \sqrt{x} - \tan^{-1} \sqrt{x} + C$
 (c) $x \sec^{-1} \sqrt{x} - \sqrt{x-1} + C$
 (d) None of the above

Sol. Putting $\sqrt{x} = \sec \theta$ or $x = \sec^2 \theta$

$$\begin{aligned} \therefore I &= \int \theta (2 \sec^2 \theta \tan \theta) d\theta \\ &= \int \theta \frac{d(\sec^2 \theta)}{d\theta} d\theta = \theta \sec^2 \theta - \int \sec^2 \theta d\theta \\ &= \theta \sec^2 \theta - \tan \theta + C \\ &= \theta \sec^2 \theta - \sqrt{\sec^2 \theta - 1} + C \\ &= x \sec^{-1} \sqrt{x} - \sqrt{x-1} + C \end{aligned}$$

Hence, (c) is the correct answer.

Ex 6. If $I = \int \frac{dx}{\sin x (3 + 2 \cos x)}$, then I equals

- (a) $\left(\frac{2}{5}\right) \log (3 + 2 \cos x) - \left(\frac{1}{2}\right) \log (1 - \cos x) + \left(\frac{1}{10}\right) \log (1 + \cos x) + C$
 (b) $\left(\frac{2}{5}\right) \log (3 + 2 \cos x) + \left(\frac{1}{2}\right) \log (1 - \cos x) - \left(\frac{1}{10}\right) \log (1 + \cos x) + C$
 (c) $\left(\frac{2}{5}\right) \log (3 + 2 \cos x) - \left(\frac{1}{2}\right) \log (1 + \cos x) + \left(\frac{1}{10}\right) \log (1 - \cos x) + C$
 (d) $\left(\frac{2}{5}\right) \log (3 + 2 \cos x) + \left(\frac{1}{10}\right) \log \sin x + C$

Sol. Write $I = \int \frac{\sin x dx}{\sin^2 x (3 + 2 \cos x)}$ and put $3 + 2 \cos x = t$,

$$\begin{aligned} \text{so that } -\sin x dx &= \frac{dt}{2} \text{ and } \sin^2 x = 1 - \left(\frac{t-3}{2}\right)^2 \\ &= \frac{1}{4} [2^2 - (t-3)^2] \\ &= \frac{1}{4} (t-1)(5-t) \\ \therefore I &= 2 \int \frac{dt}{(t-1)(5-t)} \\ &= 2 \int \left[\frac{1}{5} \cdot \frac{1}{t} - \frac{1}{4} \cdot \frac{1}{t-1} + \frac{1}{20} \cdot \frac{1}{t-5} \right] dt \\ &= 2 \left[\frac{1}{5} \log |t| - \frac{1}{4} \log |t-1| + \frac{1}{20} \log |t-5| \right] + C \\ &= \frac{2}{5} \log (3 + 2 \cos x) - \frac{1}{2} \log (1 + \cos x) \\ &\quad + \frac{1}{10} \log (1 - \cos x) + C' \end{aligned}$$

where, $C' = C - \frac{1}{2} \log 2 + \frac{1}{10} \log 2$

Hence, (c) is the correct answer.

Ex 7. $\int \frac{x^2 - 2}{(x^4 + 5x^2 + 4) \tan^{-1} \left(\frac{x^2 + 2}{x} \right)} dx$ is equal to

- (a) $\log \left| \tan^{-1} \left(2 + \frac{x}{2} \right) \right| + C$
 (b) $\log \left| \tan^{-1} \left(2 + \frac{2}{x} \right) \right| + C$
 (c) $\log \left| \tan^{-1} \left(x + \frac{2}{x} \right) \right| + C$
 (d) $\log \left| \tan^{-1} \left(x + \frac{4}{x} \right) \right| + C$

Sol. We have, $\int \frac{x^2 - 2}{(x^4 + 5x^2 + 4) \tan^{-1} \left(\frac{x^2 + 2}{x} \right)} dx$

$$\begin{aligned} &= \int \frac{1 - \frac{2}{x^2}}{\left(x^2 + 5 + \frac{4}{x^2} \right) \tan^{-1} \left(x + \frac{2}{x} \right)} dx \\ &= \int \frac{1}{\left\{ \left(x + \frac{2}{x} \right)^2 + 1 \right\} \tan^{-1} \left(x + \frac{2}{x} \right)} d \left(x + \frac{2}{x} \right) \\ &= \int \frac{1}{(t^2 + 1) \tan^{-1} t} dt, \text{ where } t = x + \frac{2}{x} \\ &= \int \frac{1}{\tan^{-1} t} d(\tan^{-1} t) \\ &= \log |\tan^{-1} t| + C \\ &= \log \left| \tan^{-1} \left(x + \frac{2}{x} \right) \right| + C \end{aligned}$$

Hence, (c) is the correct answer.

Ex 8. The value of the integral

$\int \{(x+a)^{-3} (x+b)^{-5}\}^{1/4} dx$ is equal to

(a) $\frac{1}{b-a} \left(\frac{x+a}{x+b} \right)^{1/4} + C$

(b) $\frac{1}{b-a} \left(\frac{x-a}{x+b} \right)^{1/4} + C$

(c) $\frac{4}{b-a} \left(\frac{x+a}{x+b} \right)^{1/4} + C$

(d) $\frac{1}{b-a} \left(\frac{x+a}{x-b} \right)^{1/4} + C$

Sol. Let $I = \int \{(x+a)^{-3} (x+b)^{-5}\}^{1/4} dx$

$$= \int \left\{ \left(\frac{x+a}{x+b} \right)^{-3} (x+b)^{-8} \right\}^{1/4} dx$$

$$= \int \left(\frac{x+a}{x+b} \right)^{-3/4} \frac{1}{(x+b)^2} dx$$

Let $\frac{x+a}{x+b} = t$, then $d\left(\frac{x+a}{x+b}\right) = dt$

$$\Rightarrow \frac{b-a}{(x+b)^2} dx = dt$$

$$\Rightarrow \frac{1}{(x+b)^2} dx = \frac{1}{b-a} dt$$

$$\begin{aligned} \therefore I &= \frac{1}{b-a} \int t^{-3/4} dt \\ &= \frac{1}{b-a} \times 4t^{1/4} + C \\ &= \frac{4}{b-a} \left(\frac{x+a}{x+b} \right)^{1/4} + C \end{aligned}$$

Hence, (c) is the correct answer.

Ex 9. $\int \sec^{4/3} x \operatorname{cosec}^{8/3} x dx$ is equal to

(a) $-\frac{3}{5} \tan^{-8/3} x + 3 \tan^{-2/3} x + C$

(b) $-\frac{3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C$

(c) $\frac{3}{5} \tan^{-8/3} x + 3 \tan^{-2/3} x + C$

(d) $-\frac{3}{5} \tan^{-8/3} x - 3 \tan^{-2/3} x + C$

Sol. Let $I = \int \sec^{4/3} x \operatorname{cosec}^{8/3} x dx$

$$= \int \frac{1}{\cos^{4/3} x \sin^{8/3} x} dx = \int \cos^{-4/3} x \sin^{-8/3} x dx$$

Since, $-\left(\frac{4}{3} + \frac{8}{3}\right) = -4$, which is an even integer.

So, we divide both numerator and denominator by $\cos^4 x$.

$$\begin{aligned} \therefore I &= \int \frac{\sec^4 x}{\tan^{8/3} x} dx = \int \frac{(1 + \tan^2 x)}{\tan^{8/3} x} \cdot \sec^2 x dx \\ &= \int \frac{1+t^2}{t^{8/3}} dt, \text{ where } t = \tan x \end{aligned}$$

$$\begin{aligned} &= \int (t^{-8/3} + t^{-2/3}) dt = -\frac{3}{5} t^{-5/3} + 3t^{1/3} + C \\ &= -\frac{3}{5} \tan^{-5/3} x + 3 \tan^{1/3} x + C \end{aligned}$$

Hence, (b) is the correct answer.

Ex 10. $\int \frac{\cos^4 x}{\sin^3 x (\sin^5 x + \cos^5 x)^{3/5}} dx$ is equal to

(a) $-\frac{1}{5} \left(\frac{\tan^5 x + 1}{\tan^5 x} \right)^{2/5} + C$

(b) $-\frac{1}{2} \left(\frac{\tan^5 x + 1}{\tan^5 x} \right)^{2/5} + C$

(c) $-\frac{1}{5} \left(\frac{\tan^{3/5} x + 1}{\tan^5 x} \right)^{5/3} + C$

(d) $-\frac{1}{2} \left(\frac{\tan^{3/5} x + 1}{\tan^5 x} \right)^{5/3} + C$

Sol. Let $I = \int \frac{\cos^4 x}{\sin^3 x (\sin^5 x + \cos^5 x)^{3/5}} dx$

$$= \int \frac{\cos^4 x}{\sin^6 x (1 + \cot^5 x)^{3/5}} dx$$

$$= \int \frac{\sec^2 x}{\tan^6 x \left(1 + \frac{1}{\tan^5 x} \right)^{3/5}} dx$$

$$= \int \frac{1}{t^6 \left(1 + \frac{1}{t^5} \right)^{3/5}} dt, \text{ where } t = \tan x$$

$$= -\frac{1}{5} \int \left(1 + \frac{1}{t^5} \right)^{-3/5} \left(\frac{-5}{t^6} \right) dt$$

$$= -\frac{1}{5} \int \left(1 + \frac{1}{t^5} \right)^{-3/5} d \left(1 + \frac{1}{t^5} \right)$$

$$= -\frac{1}{5} \left[\frac{\left(1 + \frac{1}{t^5} \right)^{2/5}}{\frac{2}{5}} \right] + C$$

$$= -\frac{1}{2} \left(\frac{t^5 + 1}{t^5} \right)^{2/5} + C$$

$$= -\frac{1}{2} \left(\frac{\tan^5 x + 1}{\tan^5 x} \right)^{2/5} + C$$

Hence, (b) is the correct answer.

Ex 11. If $\int \frac{\sin(x-\theta)}{\sin(x+\theta)} dx = -A - \sin \theta \log |\sin x + \sqrt{\sin^2 x - \sin^2 \theta}| + C$. Then, the value of A is equal to

(a) $\cos \theta \cos^{-1} \left(\frac{\sin x}{\sin \theta} \right)$ (b) $\cos \theta \left(\frac{\cos x}{\cos \theta} \right)$

(c) $\cos \theta \cos^{-1} \left(\frac{\sin x}{\cos \theta} \right)$ (d) $\cos \theta \sin^{-1} \left(\frac{\cos x}{\cos \theta} \right)$

Sol. We have, $\int \frac{\sin(x-\theta)}{\sin(x+\theta)} dx = \int \frac{\sin(x-\theta)}{\sqrt{\sin^2 x - \sin^2 \theta}} dx$

$$= \cos \theta \int \frac{\sin x}{\sqrt{\sin^2 x - \sin^2 \theta}} dx - \sin \theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2 \theta}} dx$$

$$= \cos \theta \int \frac{\sin x}{\sqrt{\cos^2 \theta - \cos^2 x}} dx - \sin \theta \int \frac{\cos x}{\sqrt{\sin^2 x - \sin^2 \theta}} dx$$

$$= -\cos \theta \int \frac{1}{\sqrt{\cos^2 \theta - \cos^2 x}} d(\cos x) - \sin \theta \int \frac{1}{\sqrt{\sin^2 x - \sin^2 \theta}} d(\sin x)$$

$$= -\cos \theta \int \frac{1}{\sqrt{\cos^2 \theta - u^2}} du - \sin \theta \int \frac{dv}{\sqrt{v^2 - \sin^2 \theta}}$$

where, $u = \cos x$ and $v = \sin x$

$$= -\cos \theta \sin^{-1} \left(\frac{u}{\cos \theta} \right) - \sin \theta \log |v + \sqrt{v^2 - \sin^2 \theta}| + C$$

$$= -\cos \theta \sin^{-1} \left(\frac{\cos x}{\cos \theta} \right) - \sin \theta \log |\sin x + \sqrt{\sin^2 x - \sin^2 \theta}| + C$$

Hence, (d) is the correct answer.

Ex 12. If $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}} dx$

$$= \frac{1}{2} \log \left| \frac{\sqrt{x + \frac{1}{x} + 1} - 1}{\sqrt{x + \frac{1}{x} + 1} + 1} \right| - A + C$$

Then, the value of A is equal to

- (a) $\cos^{-1} \sqrt{1 + \frac{1}{x}}$
 (b) $\tan^{-1} \sqrt{x + \frac{1}{x} + 1}$
 (c) $\cot^{-1} \sqrt{x + \frac{1}{x}}$
 (d) $\sin^{-1} \sqrt{1 + \frac{1}{x}}$

Sol. We have, $\int \frac{x(x-1)}{(x^2+1)(x+1)\sqrt{x^3+x^2+x}} dx$

$$= \int \frac{x(x^2-1)}{(x^2+1)(x+1)^2\sqrt{x^3+x^2+x}} dx$$

$$= \int \frac{x^3 \left(1 - \frac{1}{x^2}\right)}{x^3 \left(x + \frac{1}{x}\right) \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 \sqrt{x + \frac{1}{x} + 1}} dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right)}{\left(x + \frac{1}{x}\right) \left(x + \frac{1}{x} + 2\right) \sqrt{x + \frac{1}{x} + 1}} dx$$

$$I = \int \frac{2t}{(t^2-1)(t^2+1)\sqrt{t^2}} dt$$

where, $x + \frac{1}{x} + 1 = t^2$

$$= \int \frac{2}{(t^2-1)(t^2+1)} dt$$

$$= \int \left(\frac{1}{t^2-1} - \frac{1}{t^2+1} \right) dt$$

$$= \int \frac{1}{t^2-1} dt - \int \frac{1}{t^2+1} dt$$

$$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| - \tan^{-1} t + C$$

$$= \frac{1}{2} \log \left| \frac{\sqrt{x + \frac{1}{x} + 1} - 1}{\sqrt{x + \frac{1}{x} + 1} + 1} \right| - \tan^{-1} \sqrt{x + \frac{1}{x} + 1} + C$$

Hence, (b) is the correct answer.

Ex 13. If $y(x-y)^2 = x$, then $\int \frac{1}{x-3y} dx$ is equal to

- (a) $\frac{1}{3} \log \{(x-y)^2 + 1\}$
 (b) $\frac{1}{4} \log \{(x-y)^2 - 1\}$
 (c) $\frac{1}{2} \log \{(x-y)^2 - 1\}$
 (d) $\frac{1}{6} \log \{(x^2 - y^2) - 1\}$

Sol. We have, $y(x-y)^2 = x$

$$\Rightarrow d\{y(x-y)^2\} = dx$$

[taking differential on both sides]

$$\Rightarrow (x-y)^2 dy + 2y(x-y)(dx-dy) = dx$$

$$\Rightarrow (x-y)^2 dy - (x-y)^2 dx + 2y(x-y)(dx-dy) = [1 - (x-y)^2] dx$$

$$\Rightarrow [(x-y)^2 - 2y(x-y)](dy-dx) = [1 - (x-y)^2] dx$$

$$\Rightarrow (x-y)(x-3y)(dx-dy) = [(x-y)^2 - 1] dx$$

$$\Rightarrow (x-y) \frac{(dx-dy)}{(x-y)^2 - 1} = \frac{dx}{x-3y}$$

$$\Rightarrow \frac{dx}{x-3y} = \frac{(x-y) d(x-y)}{(x-y)^2 - 1}$$

$$\Rightarrow \int \frac{1}{x-3y} dx = \frac{1}{2} \int \frac{2(x-y) d(x-y)}{(x-y)^2 - 1}$$

$$\Rightarrow \int \frac{1}{x-3y} dx = \frac{1}{2} \log \{(x-y)^2 - 1\}$$

Hence, (c) is the correct answer.

Ex 14. $\int \frac{1}{(1-x^8)} \left\{ \cos^{-1} \left(\frac{2x}{1+x^2} \right) + \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right\} dx$ is equal to

(a) $\frac{\pi}{16} \log \left| \frac{1+x}{1-x} \right| + \frac{\pi}{8} \tan^{-1} x + \frac{\pi}{8\sqrt{2}} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) - \frac{\pi}{16\sqrt{2}} \log \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + C$

(b) $\frac{\pi}{16} \log \left| \frac{1-x}{1+x} \right| + \frac{\pi}{8} \tan^{-1} x + \frac{\pi}{8\sqrt{2}} \tan^{-1} \left(\frac{x^2+1}{\sqrt{2}x} \right) - \frac{\pi}{16\sqrt{2}} \log \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + C$

(c) $\frac{\pi}{16} \log \left| \frac{1-x}{1+x} \right| + \frac{\pi}{8} \tan^{-1} x + \frac{\pi}{8\sqrt{2}} \tan^{-1} \left(\frac{x^2+1}{\sqrt{2}x} \right) - \frac{\pi}{16\sqrt{2}} \log \left| \frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1} \right| + C$

(d) None of the above

Sol. We have,

$$\begin{aligned} & \int \frac{1}{(1-x^8)} \left\{ \cos^{-1} \left(\frac{2x}{1+x^2} \right) + \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right\} dx \\ &= \int \frac{1}{(1-x^8)} \left\{ \frac{\pi}{2} - \sin^{-1} \left(\frac{2x}{1+x^2} \right) + \tan^{-1} \left(\frac{2x}{1-x^2} \right) \right\} dx \\ &= \int \frac{1}{(1-x^8)} \left\{ \frac{\pi}{2} - 2 \tan^{-1} x + 2 \tan^{-1} x \right\} dx \\ &= \frac{\pi}{2} \int \frac{1}{1-x^8} dx = \frac{\pi}{4} \int \left(\frac{1}{1-x^4} + \frac{1}{1+x^4} \right) dx \\ &= \frac{\pi}{4} \int \frac{1}{1-x^4} dx + \frac{\pi}{4} \int \frac{1}{1+x^4} dx \\ &= \frac{\pi}{8} \int \left(\frac{1}{1-x^2} + \frac{1}{1+x^2} \right) dx + \frac{\pi}{8} \int \left(\frac{x^2+1}{x^4+1} - \frac{x^2-1}{x^4+1} \right) dx \\ &= \frac{\pi}{8} \int \frac{1}{1-x^2} dx + \frac{\pi}{8} \int \frac{1}{1+x^2} dx \\ &\quad + \frac{\pi}{8} \int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx - \frac{\pi}{8} \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}} dx \\ &= \frac{\pi}{8} \int \frac{1}{1-x^2} dx + \frac{\pi}{8} \int \frac{1}{1+x^2} dx \\ &\quad + \frac{\pi}{8} \int \frac{1}{\left(x-\frac{1}{x}\right)^2 + (\sqrt{2})^2} d\left(x-\frac{1}{x}\right) \\ &\quad - \frac{\pi}{8} \int \frac{d\left(x+\frac{1}{x}\right)}{\left(x+\frac{1}{x}\right)^2 - (\sqrt{2})^2} \end{aligned}$$

$$\begin{aligned} &= \frac{\pi}{16} \log \left| \frac{1+x}{1-x} \right| + \frac{\pi}{8} \tan^{-1} x + \frac{\pi}{8\sqrt{2}} \tan^{-1} \left(\frac{x-\frac{1}{x}}{\sqrt{2}} \right) \\ &\quad - \frac{\pi}{16\sqrt{2}} \log \left| \frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}} \right| + C \\ &= \frac{\pi}{16} \log \left| \frac{1+x}{1-x} \right| + \frac{\pi}{8} \tan^{-1} x + \frac{\pi}{8\sqrt{2}} \tan^{-1} \left(\frac{x^2-1}{\sqrt{2}x} \right) \\ &\quad - \frac{\pi}{16\sqrt{2}} \log \left| \frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1} \right| + C \end{aligned}$$

Hence, (a) is the correct answer.

Ex 15. $\int \frac{\cos^2 x + \sin 2x}{(2 \cos x - \sin x)^2} dx$ is equal to

(a) $\frac{\cos x}{2 \cos x - \sin x} - \frac{1}{5} x - \frac{2}{5} \log |2 \cos x - \sin x| + C$

(b) $\frac{\sin x}{2 \sin x - \cos x} - \frac{1}{5} x - \frac{2}{5} \log |2 \cos x - \sin x| + C$

(c) $\frac{\cos x + \sin x}{2 \sin x - \cos x} - \frac{1}{5} x - \frac{2}{5} \log |2 \sin x - \cos x| + C$

(d) None of the above

Sol. Let $I = \int \frac{\cos^2 x + \sin 2x}{(2 \cos x - \sin x)^2} dx$

$$\begin{aligned} &= \int \frac{\cos x (2 \sin x + \cos x)}{(2 \cos x - \sin x)^2} dx \\ &= - \int \cos x \frac{d(2 \cos x - \sin x)}{(2 \cos x - \sin x)^2} \\ &= - \left[\cos x \times \left(\frac{-1}{(2 \cos x - \sin x)} \right) - \int (-\sin x) \left(\frac{-1}{(2 \cos x - \sin x)} \right) dx \right] \\ &= \frac{\cos x}{2 \cos x - \sin x} + \int \frac{\sin x}{2 \cos x - \sin x} dx \\ &= \frac{\cos x}{2 \cos x - \sin x} + I_1 \end{aligned}$$

where, $I_1 = \int \frac{\sin x}{2 \cos x - \sin x} dx$

To evaluate the integral I_1 , let us express $\sin x$ in terms of the denominator $2 \cos x - \sin x$ and its derivative.

So, let $\sin x = K(2 \cos x - \sin x) + L(-2 \sin x - \cos x)$
On equating the coefficient of $\sin x$ and $\cos x$, we get

$$\begin{aligned} 1 &= -K - 2L \text{ and } 0 = 2K - L \\ \Rightarrow K &= -\frac{1}{5} \text{ and } L = -\frac{2}{5} \\ \therefore I_1 &= \int \frac{K(2 \cos x - \sin x) + L(-2 \sin x - \cos x)}{2 \cos x - \sin x} dx \end{aligned}$$

$$\Rightarrow I_1 = \int K dx + L \int \frac{-2 \sin x - \cos x}{2 \cos x - \sin x} dx$$

$$\Rightarrow I_1 = Kx + L \log |2 \cos x - \sin x| + C$$

$$\Rightarrow I_1 = -\frac{1}{5} x - \frac{2}{5} \log |2 \cos x - \sin x| + C$$

On substituting the value of I_1 , we get

$$I = \frac{\cos x}{2 \cos x - \sin x} - \frac{1}{5}x - \frac{2}{5} \log |2 \cos x - \sin x| + C$$

Hence, (a) is the correct answer.

Ex 16. $\int e^{2ax} \frac{1 - \cos 2ax}{1 + \sin 2ax} dx$ is equal to

- (a) $-\frac{1}{a} e^{2ax} \cos\left(\frac{\pi}{4} + ax\right) + C$
 (b) $-\frac{1}{2a} e^{2ax} \cot\left(\frac{\pi}{4} + ax\right) + C$
 (c) $-\frac{1}{2a} e^{2ax} \cos\left(\frac{\pi}{4} + ax\right) + C$
 (d) $-\frac{1}{a} e^{2ax} \operatorname{cosec}\left(\frac{\pi}{4} + ax\right) + C$

Sol. Let $I = \int e^{2ax} \frac{1 - \cos 2ax}{1 + \sin 2ax} dx = \frac{1}{a} \int e^{2t} \frac{1 - \cos 2t}{1 + \sin 2t} dt$,

where $ax = t$

$$\begin{aligned} &= \frac{1}{a} \int e^{2t} \frac{1 - \sin\left(\frac{\pi}{2} + 2t\right)}{1 - \cos\left(\frac{\pi}{2} + 2t\right)} dt \\ &= \frac{1}{a} \int e^{2t} \frac{1 - 2\sin\left(\frac{\pi}{4} + t\right)\cos\left(\frac{\pi}{4} + t\right)}{2\sin^2\left(\frac{\pi}{4} + t\right)} dt \\ &= \frac{1}{a} \int e^{2t} \left(\frac{1}{4} \operatorname{cosec}^2\left(\frac{\pi}{4} + t\right) - \cot\left(\frac{\pi}{4} + t\right) \right) dt \\ &= \frac{1}{2a} \int e^{2t} \operatorname{cosec}^2\left(\frac{\pi}{4} + t\right) dt - \frac{1}{a} \int e^{2t} \cot\left(\frac{\pi}{4} + t\right) dt \\ &= -\frac{1}{2a} e^{2t} \cot\left(\frac{\pi}{4} + t\right) + \frac{1}{a} \int e^{2t} \cot\left(\frac{\pi}{4} + t\right) dt \\ &\quad - \frac{1}{a} \int e^{2t} \cot\left(\frac{\pi}{4} + t\right) dt + C \\ &= -\frac{1}{2a} e^{2t} \cot\left(\frac{\pi}{4} + t\right) + C \\ &= -\frac{1}{2a} e^{2ax} \cot\left(\frac{\pi}{4} + ax\right) + C \end{aligned}$$

Hence, (b) is the correct answer.

Ex 17. $\int \frac{1}{x} \{\log_{ex} e \cdot \log_{e^2x} e \cdot \log_{e^3x} e\} dx$ is equal to

- (a) $\frac{1}{2} \log\{\log_e e^2x\} - \log\{\log_e e^3x\}$
 $\quad + \log\{\log_e e^4x\} + C$
 (b) $\frac{1}{2} \log\{\log_e x\} - \log\{\log_e x\} + \log\{\log_e x\} + C$
 (c) $\frac{1}{2} \log\{\log_e ex\} - \log\{\log_e e^2x^2\}$
 $\quad + \log\{\log_e e^3x^3\} + C$
 (d) $\frac{1}{2} \log\{\log_e ex\} - \log\{\log_e e^2x\}$
 $\quad + \frac{1}{2} \log\{\log_e e^3x\} + C$

Sol. We have, $\int \frac{1}{x} [\log_{ex} e \cdot \log_{e^2x} e \cdot \log_{e^3x} e] dx$

$$\begin{aligned} &= \int \frac{1}{x \log_e ex \cdot \log_e e^2x \cdot \log_e e^3x} dx \\ &= \int \frac{1}{x(\log_e e + \log_e x)(\log_e e^2 + \log_e x)(\log_e e^3 + \log_e x)} dx \\ &= \int \frac{1}{(1 + \log_e x)(2 + \log_e x)(3 + \log_e x)} d(\log_e x) \\ &= \int \frac{1}{(1+t)(2+t)(3+t)} dt, \text{ where } t = \log_e x \\ &= \int \left(\frac{1}{2} \cdot \frac{1}{1+t} - \frac{1}{2+t} + \frac{1}{2} \cdot \frac{1}{3+t} \right) dt \\ &\quad \text{[using partial fractions]} \\ &= \frac{1}{2} \log|1+t| - \log|(2+t)| + \frac{1}{2} \log|(3+t)| + C \\ &= \frac{1}{2} \log|1 + \log_e x| - \log|2 + \log_e x| \\ &\quad + \frac{1}{2} \log|3 + \log_e x| + C \\ &= \frac{1}{2} \log\{\log_e ex\} - \log\{\log_e e^2x\} \\ &\quad + \frac{1}{2} \log\{\log_e e^3x\} + C \end{aligned}$$

Hence, (d) is the correct answer.

Ex 18. $\int 7^{7^{7^x}} \cdot 7^{7^x} \cdot 7^x dx$ is equal to

- (a) $\frac{7^{7^{7^x}}}{(\log_e 7)^3} + C$
 (b) $\frac{7^{7^{7^x}}}{(\log_e 7)^2} + C$
 (c) $7^{7^{7^x}} \cdot (\log 7)^3 + C$
 (d) None of the above

Sol. We have, $\int 7^{7^{7^x}} \cdot 7^{7^x} \cdot 7^x dx$

$$\begin{aligned} &= \frac{1}{(\log_e 7)^3} \int (7^{7^{7^x}} \log_e 7) \cdot (7^{7^x} \log_e 7) \cdot (7^x \log_e 7) dx \\ &= \frac{1}{(\log_e 7)^3} \int 1 \cdot d(7^{7^{7^x}}) = \frac{7^{7^{7^x}}}{(\log_e 7)^3} + C \end{aligned}$$

Hence, (a) is the correct answer.

Ex 19. Let $x^2 \neq n\pi - 1, n \in \mathbb{N}$. Then, the value of

$\int x \sqrt{\frac{2 \sin(x^2 + 1) - \sin 2(x^2 + 1)}{2 \sin(x^2 + 1) + \sin 2(x^2 + 1)}} dx$ is equal to

- (a) $\log \left| \frac{1}{2} \sec(x^2 + 1) \right| + C$
 (b) $\log \left| \sec \left(\frac{x^2 + 1}{2} \right) \right| + C$
 (c) $\frac{1}{2} \log |\sec(x^2 + 1)| + C$
 (d) None of the above

Sol. We have, $\int x \sqrt{\frac{2 \sin(x^2 + 1) - \sin 2(x^2 + 1)}{2 \sin(x^2 + 1) + \sin 2(x^2 + 1)}} dx$

$$= \int x \sqrt{\frac{2 \sin(x^2 + 1) - 2 \sin(x^2 + 1) \cos(x^2 + 1)}{2 \sin(x^2 + 1) + 2 \sin(x^2 + 1) \cos(x^2 + 1)}} dx$$

$$= \int x \sqrt{\frac{1 - \cos(x^2 + 1)}{1 + \cos(x^2 + 1)}} dx = \int x \tan\left(\frac{x^2 + 1}{2}\right) dx$$

$$= \int \tan\left(\frac{x^2 + 1}{2}\right) d\left(\frac{x^2 + 1}{2}\right) = \log \left| \sec\left(\frac{x^2 + 1}{2}\right) \right| + C$$

Hence, (b) is the correct answer.

Ex 20. If $f(x) = \begin{vmatrix} 0 & x^2 - \sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1 - 2x \\ 2 - \cos x & 2x - 1 & 0 \end{vmatrix}$,

then $\int f(x) dx$ is equal to

- (a) $\frac{x^3}{3} - x^2 \sin x + \sin 2x + C$
 (b) $\frac{x^3}{3} - x^2 \sin x - \cos 2x + C$
 (c) $\frac{x^3}{3} - x^2 \cos x - \cos 2x + C$
 (d) None of the above

Sol. We have, $f(x) = \begin{vmatrix} 0 & x^2 - \sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1 - 2x \\ 2 - \cos x & 2x - 1 & 0 \end{vmatrix}$

$$\Rightarrow f(x) = \begin{vmatrix} 0 & \sin x - x^2 & 2 - \cos x \\ x^2 - \sin x & 0 & 2x - 1 \\ \cos x - 2 & 1 - 2x & 0 \end{vmatrix}$$

[interchanging rows and columns]

$$\Rightarrow f(x) = (-1)^3 \begin{vmatrix} 0 & x^2 - \sin x & \cos x - 2 \\ \sin x - x^2 & 0 & 1 - 2x \\ 2 - \cos x & 2x - 1 & 0 \end{vmatrix}$$

[taking (-1) common from each column]

$$\Rightarrow f(x) = -f(x)$$

$$\Rightarrow f(x) = 0$$

$$\Rightarrow \int f(x) dx = 0$$

Hence, (d) is the correct answer.

Ex 21. If $\int f(x) dx = F(x)$, then $\int x^3 f(x^2) dx$ is equal to

- (a) $\frac{1}{2} [x^2 \{F(x)\}^2 - \int \{F(x)\}^2 dx]$
 (b) $\frac{1}{2} [x^2 F(x^2) - \int F(x^2) d(x^2)]$
 (c) $\frac{1}{2} \left[x^2 F(x) - \frac{1}{2} \int \{F(x)\}^2 dx \right]$
 (d) None of the above

Sol. We have, $\int f(x) dx = F(x)$

$$\therefore \int x^3 f(x^2) dx = \frac{1}{2} \int_1^{x^2} f(t) d(t)$$

$$= \frac{1}{2} [x^2 F(x^2) - \int F(x^2) d(x^2)]$$

Hence, (b) is the correct answer.

Ex 22. Let $g(x)$ be a differentiable function satisfying $\frac{d}{dx} \{g(x)\} = g(x)$ and $g(0) = 1$, then

$\int g(x) \left(\frac{2 - \sin 2x}{1 - \cos 2x} \right) dx$ is equal to

- (a) $g(x) \cot x + C$ (b) $-g(x) \cot x + C$
 (c) $\frac{g(x)}{1 - \cos 2x} + C$ (d) None of these

Sol. We have, $\frac{d}{dx} \{g(x)\} = g(x) \Rightarrow g'(x) = g(x)$

$$\Rightarrow \int \frac{g'(x)}{g(x)} dx = \int 1 dx$$

$$\Rightarrow \log_e \{g(x)\} = x + \log C_1$$

$$\Rightarrow g(x) = C_1 e^x$$

Now, $g(0) = 1$

$$\Rightarrow C_1 = 1$$

$$\therefore g(x) = e^x$$

So, $\int g(x) \frac{2 - \sin 2x}{1 - \cos 2x} dx = \int e^x (\operatorname{cosec}^2 x - \cot x) dx$

$$= -e^x \cot x + C = -g(x) \cot x + C$$

Hence, (b) is the correct answer.

Ex 23. Let $f(x)$ be a polynomial of degree three satisfying $f(0) = -1$ and $f(1) = 0$. Also, 0 is a stationary point of $f(x)$ does not have an extremum at $x = 0$, then the value of the integral $\int \frac{f(x)}{x^3 - 1} dx$ is

- (a) $\frac{x^2}{2} + C$ (b) $x + C$
 (c) $\frac{x^3}{b} + C$ (d) None of these

Sol. Let $f(x) = ax^3 + bx^2 + cx + d$

Then, $f(0) = -1$ and $f(1) = 0$

$$\Rightarrow d = -1 \text{ and } a + b + c + d = 0$$

$$\Rightarrow d = -1 \text{ and } a + b + c = 1 \quad \dots(i)$$

It is given that $x = 0$ is a stationary point of $f(x)$ but it is not a point of extremum. Therefore,

$$f'(0) = 0, f''(0) = 0$$

and $f'''(0) \neq 0$

Now, $f(x) = ax^3 + bx^2 + cx + d$

$$\Rightarrow f'(x) = 3ax^2 + 2bx + c$$

$$\Rightarrow f''(x) = 6ax + b \text{ and } f'''(x) = 6a$$

$$\therefore f'(0) = 0, f''(0) = 0 \text{ and } f'''(0) \neq 0$$

$$\Rightarrow c = 0, b = 0 \text{ and } a \neq 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$a = 1, b = c = 0 \text{ and } d = -1$$

$$\therefore f(x) = x^3 - 1$$

$$\text{Thus, } \int \frac{f(x)}{x^3 - 1} dx = \int 1 dx = x + C$$

Hence, (b) is the correct answer.

Ex 24. Let $f(x)$ be a polynomial satisfying $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$. Then, $f(4)$ is equal to

(a) $\frac{5(e^8 + 1)}{2e^4}$

(b) $\frac{5e^8 - 1}{2e^4}$

(c) $\frac{2e^4}{5(e^8 - 1)}$

(d) $\frac{2e^4}{5(e^8 + 1)}$

Sol. We have, $f''(x) = f(x)$

$$\Rightarrow 2f'(x)f''(x) = 2f(x)f'(x)$$

$$\Rightarrow \frac{d}{dx}\{f'(x)\}^2 = \frac{d}{dx}\{f(x)\}^2$$

$$\Rightarrow \{f'(x)\}^2 = \{f(x)\}^2 + C$$

...(i)

$$\text{Now, } f(0) = 2 \text{ and } f'(0) = 3$$

From Eq. (i), we get

$$\{f'(0)\}^2 = \{f(0)\}^2 + C$$

$$\Rightarrow 9 = 4 + C$$

$$\Rightarrow C = 5$$

$$\Rightarrow \{f'(x)\}^2 = \{f(x)\}^2 + 5$$

$$\Rightarrow f'(x) = \sqrt{5 + \{f(x)\}^2}$$

$$\Rightarrow \int \frac{1}{\sqrt{(\sqrt{5})^2 + \{f(x)\}^2}} d\{f(x)\} = \int 1 dx$$

$$\Rightarrow \log\{f(x) + \sqrt{5 + \{f(x)\}^2}\} = x + C_1$$

$$\therefore f(0) = 2$$

$$\Rightarrow \log\{2 + 3\} = C_1 \Rightarrow C_1 = \log 5$$

$$\therefore \log\{f(x) + \sqrt{5 + \{f(x)\}^2}\} = x + \log 5$$

$$\Rightarrow \log\left\{\frac{f(x) + \sqrt{5 + \{f(x)\}^2}}{5}\right\} = x$$

$$\Rightarrow f(x) + \sqrt{5 + \{f(x)\}^2} = 5e^x$$

$$\Rightarrow \sqrt{5 + \{f(x)\}^2} + f(x) = 5e^x$$

$$\text{and } \sqrt{5 + \{f(x)\}^2} - f(x) = e^{-x}$$

$$\Rightarrow 2f(x) = 5e^x - e^{-x}$$

$$\Rightarrow f(x) = \frac{1}{2}(5e^x - e^{-x})$$

$$\Rightarrow f(4) = \frac{1}{2}(5e^4 - e^{-4})$$

$$\Rightarrow f(4) = \frac{5e^8 - 1}{2e^4}$$

Hence, (b) is the correct answer.

Ex 25. $\int \frac{x^2 - 1}{x\sqrt{(x^2 + \alpha x + 1)(x^2 + \beta x + 1)}} dx$ is equal to

(a) $2 \log \left\{ \frac{\sqrt{x^2 + \alpha x + 1} + \sqrt{x^2 + \beta x + 1}}{\sqrt{x}} \right\} + C$

(b) $2 \log \left\{ \frac{\sqrt{x^2 + \alpha x + 1} - \sqrt{x^2 + \beta x + 1}}{\sqrt{x}} \right\} + C$

(c) $\log\{\sqrt{x^2 + \alpha x + 1} - \sqrt{x^2 + \beta x + 1}\} + C$

(d) None of the above

Sol. We have, $\int \frac{x^2 - 1}{x\sqrt{(x^2 + \alpha x + 1)(x^2 + \beta x + 1)}} dx$

$$= \int \frac{1 - \frac{1}{x^2}}{\sqrt{\left(x + \frac{1}{x} + \alpha\right)\left(x + \frac{1}{x} + \beta\right)}} dx$$

$$= \int \frac{1}{\sqrt{\left(x + \frac{1}{x}\right)^2 + (\alpha + \beta)\left(x + \frac{1}{x}\right) + \alpha\beta}} d\left(x + \frac{1}{x}\right)$$

$$= \int \frac{1}{\sqrt{t^2 + (\alpha + \beta)t + \alpha\beta}} dt, \text{ where } t = x + \frac{1}{x}$$

$$= \int \frac{1}{\sqrt{\left(t + \frac{\alpha + \beta}{2}\right)^2 - \left(\frac{\alpha - \beta}{2}\right)^2}} dt$$

$$= \log \left| \left(t + \frac{\alpha + \beta}{2}\right) + \sqrt{\left(t + \frac{\alpha + \beta}{2}\right)^2 - \left(\frac{\alpha - \beta}{2}\right)^2} \right| + C_1$$

$$= \log \left| t + \frac{\alpha + \beta}{2} + \sqrt{t^2 + (\alpha + \beta)t + \alpha\beta} \right| + C_1$$

$$= \log |(t + \alpha) + (t + \beta) + 2\sqrt{(t + \alpha)(t + \beta)}| + C_1 - \log 2$$

$$= \log \{\sqrt{t + \alpha} + \sqrt{t + \beta}\}^2 + C_1 - \log 2$$

$$= 2 \log \{\sqrt{t + \alpha} + \sqrt{t + \beta}\} + C_1 - \log 2$$

$$= 2 \log \left\{ \frac{\sqrt{x^2 + \alpha x + 1} + \sqrt{x^2 + \beta x + 1}}{\sqrt{x}} \right\} + C$$

Hence, (a) is the correct answer.

Ex 26. If $\int f(x) dx = f(x)$, then $\int \{f(x)\}^2 dx$ equals

(a) $\frac{1}{2}\{f(x)\}^2$

(b) $\{f(x)\}^3$

(c) $\frac{\{f(x)\}^3}{3}$

(d) $\{f(x)\}^2$

Sol. We have, $\int f(x) dx = f(x)$

$$\Rightarrow \frac{d}{dx}(f(x)) = f(x)$$

$$\Rightarrow \frac{1}{f(x)} d(f(x)) = dx$$

$$\begin{aligned} \Rightarrow \log(f(x)) &= x + \log C \\ \Rightarrow f(x) &= Ce^x \Rightarrow \{f(x)\}^2 = C^2 e^{2x} \\ \Rightarrow \int \{f(x)\}^2 dx &= \int C^2 e^{2x} dx \\ &= \frac{C^2 e^{2x}}{2} = \frac{1}{2} \{f(x)\}^2 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 27. If $I = \int \frac{x^2 + 20}{(x \sin x + 5 \cos x)^2} dx$, then I equals

- (a) $-\frac{x}{\cos x (x \sin x + 5 \cos x)} + \tan x + C$
 (b) $\frac{x}{\sin x (x \sin x + 5 \cos x)} + \cot x + C$
 (c) $(x \sin x - 5 \cos x)^{-1} \sin x + 7x + C$
 (d) None of the above

Type 2. More than One Correct Option

Ex 28. If $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = f(x) + C$, then $f(x)$ is equal to

- (a) $\sqrt{2} \sin^{-1}(\sin x - \cos x)$
 (b) $\frac{\pi}{\sqrt{2}} - \sqrt{2} \cos^{-1}(\sin x - \cos x)$
 (c) $\sqrt{2} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2} \sqrt{\tan x}} \right)$
 (d) None of the above

Sol. $I = \int (\sqrt{\tan x} + \sqrt{\cot x}) dx = \int \sqrt{2} \frac{\sin x + \cos x}{\sqrt{2 \sin x \cos x}} dx$

If $\sin x - \cos x = p$, then $(\cos x + \sin x) dx = dp$

$$\begin{aligned} I &= \sqrt{2} \int \frac{dp}{\sqrt{1-p^2}} = \sqrt{2} \sin^{-1} p + C \\ &= \sqrt{2} \sin^{-1}(\sin x - \cos x) + C \\ &= \frac{\pi}{\sqrt{2}} - \sqrt{2} \cos^{-1}(\sin x - \cos x) \\ &= \sqrt{2} \tan^{-1} \frac{\sin x - \cos x}{\sqrt{1 - (\sin x - \cos x)^2}} \\ &= \sqrt{2} \tan^{-1} \frac{\sin x - \cos x}{\sqrt{2 \sin x \cos x}} \\ &= \sqrt{2} \tan^{-1} \left(\frac{\tan x - 1}{\sqrt{2 \tan x}} \right) \end{aligned}$$

Hence, (a), (b) and (c) are the correct answers.

Ex 29. $\int \frac{\sqrt{4+x^2}}{x^6} dx = \frac{A(4+x^2)^{\frac{3}{2}}(Bx^2-6)}{x^5} + C$

- (a) $A = \frac{1}{120}$
 (b) $B = 1$
 (c) $A = -\frac{1}{120}$
 (d) $B = -1$

Sol. We have, $I = \int \frac{x^8 (x^2 + 20)}{(x^5 \sin x + 5x^4 \cos x)^2} dx$
 $= \int \frac{(x^5 + 20x^3) \cos x}{(x^5 \sin x + 5x^4 \cos x)^2} \left(\frac{x^5}{\cos x} \right) dx$

Note that

$$\frac{d}{dx} [x^5 \sin x + 5x^4 \cos x] = (x^5 + 20x^3) \cos x$$

Integrating by parts, we get

$$\begin{aligned} I &= -\frac{x^5}{\cos x (x^5 \sin x + 5x^4 \cos x)} \\ &\quad + \int \frac{5x^4 \cos x + x^5 \sin x}{(x^5 \sin x + 5x^4 \cos x)^2} \cdot \sec^2 x dx \\ &= -\frac{x}{\cos x (x \sin x + 5 \cos x)} + \tan x + C \end{aligned}$$

Hence, (a) is the correct answer.

Sol. Let $I = \int \frac{\sqrt{4+x^2}}{x^6} dx = \int \frac{\sqrt{1+\frac{4}{x^2}}}{x^5} dx = \int \frac{\sqrt{1+\frac{4}{x^2}}}{x^2 \cdot x^3} dx$

Put $t = \sqrt{1 + \frac{4}{x^2}}$

$$\begin{aligned} \therefore I &= \frac{1}{16} \int (t^2 - t^4) dt \\ &= \frac{1}{120} \cdot \frac{(4+x^2)^{3/2}}{x^5} (x^2 - 6) + C \end{aligned}$$

Thus, $A = \frac{1}{120}, B = 1$

Hence, (a) and (b) are the correct answers.

Ex 30. If $\int x \log(1+x^2) dx = \phi(x) \log(1+x^2) + \psi(x) + C$, then

- (a) $\phi(x) = \frac{1+x^2}{2}$ (b) $\psi(x) = \frac{1+x^2}{2}$
 (c) $\psi(x) = -\frac{1+x^2}{2}$ (d) $\phi(x) = -\frac{1+x^2}{2}$

Sol. $\int x \log(1+x^2) dx$
 $= \log(1+x^2) \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot 2x \cdot \frac{x^2}{2} dx$
 $= \frac{x^2}{2} \log(1+x^2) - \int \frac{x^3}{x^2+1} dx$
 $\therefore \phi(x) = \frac{x^2+1}{2}, \psi(x) = -\left(\frac{1+x^2}{2}\right)$

Hence, (a) and (c) are the correct answers.

Ex 31. If $\int \frac{dx}{(x+1)(x-2)} = A \log(x+1)$

+ $B \log(x-2) + C$, then

- (a) $A+B=0$ (b) $AB=0$
 (c) $\frac{A}{B} = -1$ (d) None of these

$$\text{Sol. } \int \frac{dx}{(x+1)(x-2)} = \int \left(-\frac{1}{3(x+1)} + \frac{1}{3(x-2)} \right) dx$$

$$= -\frac{1}{3} \log(x+1) + \frac{1}{3} \log(x-2) + C$$

$$\therefore A = -\frac{1}{3}, B = \frac{1}{3}$$

$$\therefore A + B = 0 \text{ and } \frac{A}{B} = \frac{-1/3}{1/3} = -1$$

Hence, (a) and (c) are the correct answers.

Ex 32. If $\int \frac{dx}{(x^2+1)(x^2+4)} = k \tan^{-1} x + I$

$$\tan^{-1} \frac{x}{2} + C, \text{ then}$$

$$(a) k = \frac{1}{3} \quad (b) I = \frac{3}{2} \quad (c) k = -\frac{1}{3} \quad (d) I = -\frac{1}{6}$$

$$\text{Sol. } \frac{1}{3} \int \left(\frac{1}{(x^2+1)} - \frac{1}{x^2+4} \right) dx = \frac{1}{3} \tan^{-1} x - \frac{1}{6} \tan^{-1} \frac{x}{2} + C$$

$$= k \tan^{-1} x + I \tan^{-1} \frac{x}{2} + C$$

$$\therefore k = \frac{1}{3} \text{ and } I = -\frac{1}{6}$$

Hence, (a) and (d) are the correct answers.

Ex 33. If $I_n = \int \cot^n x \, dx$ and $I_0 + I_1 + 2(I_2$

$$+ \dots + I_8) + I_9 + I_{10} = A \left(u + \frac{u^2}{2} + \dots + \frac{u^9}{9} \right)$$

+ constant, where $u = \cot x$, then

$$(a) A \text{ is constant} \quad (b) A = -1$$

$$(c) A = 1 \quad (d) A \text{ is dependent on } x$$

$$\text{Sol. } I_n = \int \cot^n x \, dx = \int \cot^{n-2} x \cot^2 x \, dx$$

$$= \int \cot^{n-2} x (\operatorname{cosec}^2 x - 1) \, dx$$

$$= \int \cot^{n-2} x \operatorname{cosec}^2 x \, dx - I_{n-2} \quad \dots(i)$$

$$\text{Thus, } I_n + I_{n-2} = -\frac{\cot^{n-1} x}{n-1} + C$$

$$\therefore I_0 + I_1 + 2(I_2 + \dots + I_8) + I_9 + I_{10}$$

$$= (I_2 + I_0) + (I_3 + I_1) + (I_4 + I_2) + (I_5 + I_3)$$

$$+ (I_6 + I_4) + (I_7 + I_5) + (I_8 + I_6)$$

$$+ (I_9 + I_7) + (I_{10} + I_8)$$

$$= -\left(\frac{\cot x}{1} + \frac{\cot^2 x}{2} + \dots + \frac{\cot^9 x}{9} \right) + C$$

$$\therefore A = -1 \quad [\text{from Eq. (i)}]$$

Hence, (a) and (b) are the correct answers.

Ex 34. If $I = \int \frac{dx}{\sqrt{2x+3} + \sqrt{x+2}}$, then I is

$$(a) 2(u-v) + \log \left| \frac{u-1}{u+1} \right| + \log \left| \frac{v-1}{v+1} \right| + C$$

$$u = \sqrt{2x+3}, v = \sqrt{x+2}$$

$$(b) \log \left| \frac{\sqrt{x+2} + \sqrt{2x+3}}{\sqrt{x+2} - \sqrt{2x+3}} \right| + C$$

$$(c) \log(\sqrt{x+2} + \sqrt{2x+3}) + C$$

(d) transcendental function in u and v , where

$$u = \sqrt{2x+3}, v = \sqrt{x+2}$$

$$\text{Sol. } I = \int \frac{\sqrt{2x+3} - \sqrt{x+2}}{x+1} dx = I_1 - I_2$$

$$\text{where, } I_1 = \int \frac{\sqrt{2x+3}}{x+1} dx$$

$$\text{and } I_2 = \int \frac{\sqrt{x+2}}{x+1} dx$$

Put $2x+3 = t^2$ in I_1 , so that

$$I_1 = \int \frac{2t \cdot t}{t^2-1} dt = 2 \int \left[1 + \frac{1}{t^2-1} \right] dt$$

$$= 2 \left[t + \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| \right] + C_1$$

In I_2 , put $x+2 = y^2$, so that

$$I_2 = \int \frac{2y^2}{y^2-1} dy = 2y + \log \left| \frac{y-1}{y+1} \right| + C_2$$

$$\text{Thus, } I = 2(\sqrt{2x+3} - \sqrt{x+2}) + \log \left| \frac{\sqrt{2x+3}-1}{\sqrt{2x+3}+1} \right|$$

$$+ \log \left| \frac{\sqrt{x+2}-1}{\sqrt{x+2}+1} \right| + C$$

Hence, (a) and (d) are the correct answers.

Ex 35. If $\int \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} dx$ is expressed as

$$K \sin 3x + M \sin 2x + C, \text{ then}$$

$$(a) K = -\frac{1}{3} \quad (b) K = \frac{1}{3}$$

$$(c) M = -\frac{1}{2} \quad (d) M = \frac{1}{2}$$

$$\text{Sol. } \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} = \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} \cdot \frac{2 \sin 5x}{2 \sin 5x}$$

$$= \frac{\sin 13x - \sin 3x - \sin 12x + \sin 2x}{2(\sin 5x + \sin 10x)}$$

$$= \frac{1}{2} \cdot \frac{\sin 13x + \sin 2x - (\sin 3x + \sin 12x)}{\sin 5x + \sin 10x}$$

$$= \frac{1}{2} \cdot \frac{2 \sin \left(\frac{15x}{2} \right) \cos \left(\frac{11x}{2} \right) - 2 \sin \left(\frac{15x}{2} \right) \cos \left(\frac{9x}{2} \right)}{2 \sin \left(\frac{15x}{2} \right) \cos \left(\frac{5x}{2} \right)}$$

$$= \frac{1}{2} \cdot \frac{\cos \left(\frac{11x}{2} \right) - \cos \left(\frac{9x}{2} \right)}{\cos \left(\frac{5x}{2} \right)} = \frac{-2 \sin 5x \sin \left(\frac{x}{2} \right)}{2 \cos \left(\frac{5x}{2} \right)}$$

$$= -2 \sin \left(\frac{5x}{2} \right) \sin \left(\frac{x}{2} \right) = \cos 3x - \cos 2x$$

So, the given integral is equal to

$$\int (\cos 3x - \cos 2x) dx = \frac{\sin 3x}{3} - \frac{\sin 2x}{2} + C$$

Hence, (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 36-42) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

Ex 36. Statement I $\int \frac{d(x^2 + 1)}{\sqrt{x^2 + 2}} = 2\sqrt{x^2 + 2} + C$

Statement II

$$\int \frac{x^{9/2}}{\sqrt{1+x^{11}}} dx = \frac{2}{11} \ln |x + \sqrt{1+x^{11}}| + C$$

Sol. $I = \int \frac{dt}{\sqrt{1+t}}$, where $x^2 + 1 = t$
 $= 2\sqrt{1+t} + C$
 $= 2\sqrt{x^2 + 2} + C$

and for $I = \int \frac{x^{9/2}}{\sqrt{1+x^{11}}} dx$

Put $x^{11/2} = t$, then

$$I = \frac{2}{11} \ln |(x^{11/2} + \sqrt{1+x^{11}})| + C$$

Hence, (c) is the correct answer.

Ex 37. Statement I If $a > 0$ and $b^2 - 4ac < 0$, then the value of the integral $\int \frac{dx}{ax^2 + bx + c}$ will be of the type of $\mu \tan^{-1} \left(\frac{x+A}{B} \right) + C$, where A, B, C and μ are constants.

Statement II If $a > 0, b^2 - 4ac < 0$, then $ax^2 + bx + c$ can be written as sum of two squares.

Sol. If $a > 0$ and $b^2 - 4ac < 0$, then

$$ax^2 + bx + c = a \left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a}$$

$$\Rightarrow \int \frac{dx}{ax^2 + bx + c} = \int \frac{dx}{a \left(x + \frac{b}{2a} \right)^2 + k^2}$$

where, $k^2 = \frac{4ac - b^2}{4a} > 0$

which will have an answer of the type

$$\frac{1}{a} \cdot \frac{1}{k/\sqrt{a}} \tan^{-1} \left(\frac{x + \frac{b}{2a}}{k/\sqrt{a}} \right) + C$$

or $\mu \tan^{-1} \left(\frac{x+A}{B} \right) + C$

Hence, (a) is the correct answer.

Ex 38. Statement I If y is a function of x such that $y(x-y)^2 = x$, then

$$\int \frac{dx}{x-3y} = \frac{1}{2} [\log(x+y)^2 - 1]$$

Statement II $\int \frac{dx}{x-3y} = \log(x-3y) + C$

Sol. Statement II is false, since in writing

$$\int \frac{dx}{x-3y} = \log(x-3y) + C,$$

We are assuming that y is a constant.

We will now prove the Statement I.

From the given relation, $(x-y)^2 = \frac{x}{y}$

and $2 \log(x-y) = \log x - \log y$

Also, $\frac{dy}{dx} = \left(-\frac{y}{x} \right) \cdot \frac{x+y}{x-3y}$

To prove the integral relation, it is sufficient to show

that $\frac{d}{dx} \text{RHS} = \frac{1}{x-3y}$

Now, $\text{RHS} = \frac{1}{2} \log \left[\frac{x}{y} - 1 \right] \quad \left[\because (x-y)^2 = \frac{x}{y} \right]$

$$= \frac{1}{2} [\log(x-y) - \log y]$$

$$= \frac{1}{2} \left[\frac{\log x - \log y}{2} - \log y \right]$$

$$= \frac{1}{4} [\log x - 3 \log y]$$

$$\Rightarrow \frac{d}{dx} \text{RHS} = \frac{1}{4} \left[\frac{1}{x} - \frac{3}{y} \cdot \frac{dy}{dx} \right]$$

$$= \frac{1}{4} \left[\frac{1}{x} - \frac{3}{y} \left(-\frac{y}{x} \right) \frac{x+y}{x-3y} \right] = \frac{1}{x-3y}$$

Thus, Statement I is true.

Hence, (c) is the correct answer.

Ex 39. Statement I $\int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}} dx$

$$= \sec^{-1} \left| \frac{x^2 + 1}{x\sqrt{2}} \right| + C$$

Statement II $\int \frac{dt}{t\sqrt{t^2 - a}} = \frac{1}{\sqrt{a}} \sec^{-1} \left| \frac{t}{\sqrt{a}} \right| + C$

$$\begin{aligned} \text{Sol. } \int \frac{x^2 - 1}{(x^2 + 1)\sqrt{x^4 + 1}} dx &= \int \frac{1 - \frac{1}{x^2}}{\left(x + \frac{1}{x}\right)\sqrt{x^2 + \frac{1}{x^2}}} dx \\ &= \int \frac{dt}{t\sqrt{t^2 - 2}} \quad \left[\text{put } t = x + \frac{1}{x} \right] \\ &= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{t}{\sqrt{2}} \right| + C \\ &= \frac{1}{\sqrt{2}} \sec^{-1} \left| \frac{x^2 + 1}{x\sqrt{2}} \right| + C \end{aligned}$$

Hence, (d) is the correct answer.

$$\begin{aligned} \text{Ex 40. Statement I } \int \sqrt{x - \sqrt{x^2 - 4}} dx \\ = \left[\frac{1}{3} (\sqrt{x+2} - \sqrt{x-2})^3 \right. \\ \left. + 2(\sqrt{x+2} + \sqrt{x-2}) \right] + C \end{aligned}$$

Statement II The integral in Statement I can be computed by substituting $(\sqrt{x+2} - \sqrt{x-2})^2 = 2t$.

$$\begin{aligned} \text{Sol. Putting } (\sqrt{x+2} - \sqrt{x-2})^2 &= 2t \\ \Rightarrow x - \sqrt{x^2 - 4} &= t \Rightarrow (x - t)^2 = x^2 - 4 \\ \Rightarrow t^2 + 4 &= 2tx \\ \Rightarrow x &= \frac{1}{2} \left(t + \frac{4}{t} \right) \\ \text{Thus, } dx &= \frac{1}{2} \left(1 - \frac{4}{t^2} \right) dt \\ \text{Therefore, the given integral} \\ &= \frac{1}{2} \int \sqrt{t} \left(1 - \frac{4}{t^2} \right) dt = \frac{1}{2} \left(\frac{2}{3} t^{3/2} + 8t^{-1/2} \right) + C \\ &= \frac{1}{2} \left[\frac{1}{3\sqrt{2}} (\sqrt{x+2} - \sqrt{x-2})^3 \right. \\ &\quad \left. + 8\sqrt{2} \frac{1}{\sqrt{x+2} - \sqrt{x-2}} \right] + C \end{aligned}$$

$$= \frac{1}{2\sqrt{2}} \left[\frac{1}{3} (\sqrt{x+2} - \sqrt{x-2})^3 \right. \\ \left. + 4(\sqrt{x+2} + \sqrt{x-2}) \right] + C$$

Hence, (d) is the correct answer.

$$\text{Ex 41. Statement I } \int \left(\frac{1}{1+x^4} \right) dx = \tan^{-1}(x^2) + C$$

$$\text{Statement II } \int \frac{1}{1+x^2} dx = \tan^{-1} x + C$$

$$\begin{aligned} \text{Sol. } \frac{1}{2} \int \frac{\frac{2}{x^2} dx}{x^2 + \frac{1}{x^2}} &= \frac{1}{2} \int \frac{\left(1 + \frac{1}{x^2}\right) dx}{\left(x - \frac{1}{x}\right)^2 + 2} - \frac{1}{2} \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x}\right)^2 - 2} \\ &\neq \tan^{-1}(x^2) + C \end{aligned}$$

Hence, (d) is the correct answer.

$$\begin{aligned} \text{Ex 42. Let } \int \sin^2 x \cos^4 x dx \\ = Ax + B \sin 2x + C \sin 4x + D \sin 6x + E \end{aligned}$$

$$\text{Statement I } A = \frac{1}{16}, B = \frac{1}{64}, C = -\frac{1}{64},$$

$$D = -\frac{1}{192}$$

$$\begin{aligned} \text{Statement II } \sin^2 x &= \frac{1 - \cos 2x}{2} \\ \text{and } \cos^2 x &= \frac{1 + \cos 2x}{2} \end{aligned}$$

$$\begin{aligned} \text{Sol. } \int \sin^2 x \cdot \cos^4 x dx \\ = \frac{1}{8} \int (4 \cdot \sin^2 x \cdot \cos^2 x) \cdot (1 + \cos 2x) dx \\ = \frac{1}{8} \int (\sin^2 2x + \sin^2 2x \cdot \cos 2x) dx \\ = \int \left(\frac{1}{16} + \frac{1}{32} \cos 2x - \frac{1}{16} \cos 4x - \frac{1}{32} \cos 6x \right) dx \end{aligned}$$

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 43-45) Integrals of the form $\int \frac{p_m(x)}{\sqrt{ax^2 + bx + c}} dx$, where $p_m(x)$ is a polynomial of degree m , are calculated by the reduction formula.

$\int \frac{p_m(x)}{\sqrt{ax^2 + bx + c}} dx = p_{m-1}(x) \sqrt{ax^2 + bx + c}$ $+ \lambda \int \frac{dx}{\sqrt{ax^2 + bx + c}}$, where $p_{m-1}(x)$ is a polynomial of degree $m-1$ and λ is some constant number.

e.g. $I = \int \frac{x^3 - x - 1}{\sqrt{x^2 + 2x + 2}} dx$, then applying the above formula, we can write

$$\int \frac{x^3 - x - 1}{\sqrt{x^2 + 2x + 2}} dx = (Ax^2 + Bx + C) \sqrt{x^2 + 2x + 2} + \lambda \int \frac{dx}{\sqrt{x^2 + 2x + 2}}$$

On differentiating both sides, we get

$$\begin{aligned} \frac{x^3 - x - 1}{\sqrt{x^2 + 2x + 2}} &= (Ax^2 + Bx + C) \cdot \frac{2(x+1)}{2\sqrt{x^2 + 2x + 2}} \\ &\quad + (2Ax + B) \sqrt{x^2 + 2x + 2} + \frac{\lambda}{\sqrt{x^2 + 2x + 2}} \end{aligned}$$

$$x^3 - x - 1 = (Ax^2 + Bx + C)(x+1) + (2Ax + B)(x^2 + 2x + 2) + \lambda$$

On comparing coefficient of like powers of x , we obtain the values of A, B, C and λ .

Ex 43. If $\int \frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} dx$
 $= (Ax^2 + Bx + C)\sqrt{x^2 + 4x + 3}$
 $+ \lambda \int \frac{dx}{\sqrt{x^2 + 4x + 3}}$, then the value of A is
 (a) $\frac{1}{3}$ (b) -1
 (c) 3 (d) $-1/3$

Ex 44. In Q. No. 43, the value of C is
 (a) -37 (b) $-\frac{14}{3}$ (c) $\frac{14}{3}$ (d) 37

Ex 45. In Q. No. 43, the value of λ is
 (a) 66 (b) -66
 (c) $\frac{37}{3}$ (d) $-\frac{37}{3}$

Sol. (Q. Nos. 43-45)

Differentiating both sides, we get

$$\frac{x^3 - 6x^2 + 11x - 6}{\sqrt{x^2 + 4x + 3}} = (Ax^2 + Bx + C) \frac{(x+2)}{\sqrt{x^2 + 4x + 3}} + (2Ax + B) \sqrt{x^2 + 4x + 3} + \frac{\lambda}{\sqrt{x^2 + 4x + 3}}$$

$$\Rightarrow x^3 - 6x^2 + 11x - 6 = (Ax^2 + Bx + C)(x+2) + (2Ax + B)(x^2 + 4x + 3) + \lambda$$

Comparing coefficient of like powers of x , we get

$$x^3 : 1 = A + 2A \Rightarrow A = \frac{1}{3}$$

$$x^2 : -6 = 2A + B + 8A + B$$

$$\Rightarrow 2B = -6 - 10 \cdot \frac{1}{3}$$

$$\Rightarrow B = -\frac{14}{3}$$

$$x : 11 = 2B + C + 6A + 4B$$

$$C = 11 - 6\left(-\frac{14}{3}\right) - 6 \cdot \frac{1}{3}$$

$$= 11 + 28 - 2 = 37$$

and constant terms : $-6 = 2C + 3B + \lambda$

$$\lambda = -6 - 2 \cdot 37 - 3 \cdot \left(-\frac{14}{3}\right)$$

$$= -6 - 74 + 14$$

$$= -66$$

43. (a) 44. (d) 45. (b)

Passage II (Ex. Nos. 46-48)

Let $I_{(n,m)} = \int \sin^n x \cos^m x dx$.

Then, we can relate $I_{(n,m)}$ with each of the following:

$$(i) I_{(n-2,m)} \quad (ii) I_{(n+2,m)} \quad (iii) I_{(n,m-2)}$$

$$(iv) I_{(n,m+2)} \quad (v) I_{(n-2,m+2)} \quad (vi) I_{(n+2,m-2)}$$

Suppose we want to establish a relation between $I_{(n,m)}$ and $I_{(n,m-2)}$, then let

$$P(x) = \sin^{(n+1)} x \cos^{(m-1)} x \quad \dots(i)$$

In $I_{(n,m)}$ and $I_{(n,m-2)}$, the exponent of $\cos x$ is m and $m-2$ respectively, the minimum of the two is $m-2$, adding 1 to the minimum, we get $m-2+1=m-1$. Now, choose the exponent $m-1$ of $\cos x$ in $P(x)$. Similarly, choose the exponent of $\sin x$ for $P(x)$. Now, differentiating both sides of Eq. (i), we get

$$\frac{d}{dx} P(x) = (n+1) \sin^n x \cos^m x - (m-1) \sin^{n+1} x \cos^{m-2} x$$

$$= (n+1) \sin^n x \cos^m x - (m-1) \sin^n x \cos^{m-2} x (1 - \cos^2 x)$$

$$= (n+1) \sin^n x \cos^m x - (m-1) \sin^n x \cos^{m-2} x + (m-1) \sin^{n+1} x \cos^{m-2} x$$

$$= (n+m) \sin^n x \cos^m x - (m-1) \sin^n x \cos^{m-2} x$$

Now, integrating both sides, we get

$$\sin^{(n+1)} x \cos^{(m-1)} x = (n+m) I_{(n,m)} - (m-1) I_{(n,m-2)}$$

Similarly, we can establish the other relations.

Ex 46. The relation between $I_{(4,2)}$ and $I_{(2,2)}$ is

$$(a) I_{(4,2)} = \frac{1}{6} [-\sin^3 x \cos^3 x + 3I_{(2,2)}]$$

$$(b) I_{(4,2)} = \frac{1}{6} [\sin^3 x \cos^3 x + 3I_{(2,2)}]$$

$$(c) I_{(4,2)} = \frac{1}{6} [\sin^3 x \cos^3 x - 3I_{(2,2)}]$$

$$(d) I_{(4,2)} = \frac{1}{4} [-\sin^3 x \cos^3 x + 2I_{(2,2)}]$$

Sol. Let $P = \sin^3 x \cos^3 x$

$$\therefore \frac{dP}{dx} = 3 \sin^2 x \cos^4 x - 3 \sin^4 x \cos^2 x$$

$$= 3 \sin^2 x (1 - \sin^2 x) \cos^2 x - 3 \sin^4 x \cos^2 x$$

$$= 3 \sin^2 x \cos^2 x - 6 \sin^4 x \cos^2 x$$

$$\therefore P = 3I_{(2,2)} - 6I_{(4,2)}$$

$$\therefore I_{(4,2)} = \frac{1}{6} [-P + 3I_{(2,2)}]$$

Hence, (a) is the correct answer.

Ex 47. The relation between $I_{(4,2)}$ and $I_{(6,2)}$ is

$$(a) I_{(4,2)} = \frac{1}{5} [\sin^5 x \cos^3 x + 8I_{(6,2)}]$$

$$(b) I_{(4,2)} = \frac{1}{5} [-\sin^5 x \cos^3 x + 8I_{(6,2)}]$$

$$(c) I_{(4,2)} = \frac{1}{5} [\sin^5 x \cos^3 x - 8I_{(6,2)}]$$

$$(d) I_{(4,2)} = \frac{1}{6} [\sin^5 x \cos^3 x + 8I_{(6,2)}]$$

Sol. Let $P = \sin^5 x \cos^3 x$

$$\therefore \frac{dP}{dx} = 5 \sin^4 x \cos^4 x - 3 \sin^6 x \cos^2 x$$

$$= 5 \sin^4 x (1 - \sin^2 x) \cos^2 x - 3 \sin^6 x \cos^2 x$$

$$= 5 \sin^4 x \cos^2 x - 8 \sin^6 x \cos^2 x$$

$$\therefore P = 5I_{(4,2)} - 8I_{(6,2)}$$

$$\therefore I_{(4,2)} = \frac{1}{5} [P + 8I_{(6,2)}]$$

Hence, (a) is the correct answer.

Ex 48. The relation between $I_{(4,2)}$ and $I_{(4,4)}$ is

- (a) $I_{(4,2)} = \frac{1}{3} [\sin^5 x \cos^3 x + 8I_{(4,4)}]$
 (b) $I_{(4,2)} = \frac{1}{3} [-\sin^5 x \cos^3 x + 8I_{(4,4)}]$
 (c) $I_{(4,2)} = \frac{1}{3} [\sin^3 x \cos^5 x + 8I_{(4,4)}]$
 (d) $I_{(4,2)} = \frac{1}{3} [\sin^5 x \cos^3 x + 6I_{(4,4)}]$

Sol. Let $P = \sin^5 x \cos^3 x$

$$\begin{aligned} \therefore \frac{dP}{dx} &= 5 \sin^4 x \cos^4 x - 3 \sin^6 x \cos^2 x \\ &= 5 \sin^4 x \cos^4 x - 3 \sin^4 x (1 - \cos^2 x) \cos^2 x \\ &= 8 \sin^4 x \cos^4 x - 3 \sin^4 x \cos^2 x \\ \therefore P &= 8I_{(4,4)} - 3I_{(4,2)} \\ \therefore I_{(4,2)} &= \frac{1}{3} [-P + 8I_{(4,4)}] \end{aligned}$$

Hence, (b) is the correct answer.

Passage III (Q. Nos. 49-52) Integrals of class of functions following a definite pattern can be found by the method of reduction and recursion. Reduction formula make it possible to reduce an integral dependent on the index $n > 0$, called the order of the integral, to an integral of the same type with a smaller index. Integration by parts helps us to derive reduction formulae.

Ex 49. If $I_n = \int \frac{dx}{(x^2 + a^2)^n}$, then $I_{n+1} + \frac{1-2n}{2n} \cdot \frac{1}{a^2} I_n$

is equal to

- (a) $\frac{x}{(x^2 + a^2)^n}$
 (b) $\frac{1}{2na^2} \cdot \frac{1}{(x^2 + a^2)^{n-1}}$
 (c) $\frac{1}{2na^2} \cdot \frac{x}{(x^2 + a^2)^n}$
 (d) $\frac{1}{2na^2} \cdot \frac{x}{(x^2 + a^2)^{n-1}}$

Sol. Using integration by parts

$$\begin{aligned} I_n &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{x^2}{(x^2 + a^2)^{n+1}} dx \\ \Rightarrow I_n &= \frac{x}{(x^2 + a^2)^n} + 2n \int \frac{1}{(x^2 + a^2)^n} dx - 2na^2 \int \frac{1}{(x^2 + a^2)^{n+1}} dx \\ \Rightarrow I_n &= \frac{x}{(x^2 + a^2)^n} + 2n I_n - 2na^2 I_{n+1} \\ \Rightarrow I_{n+1} + \frac{1-2n}{2n} \cdot \frac{1}{a^2} I_n &= \frac{1}{2na^2} \cdot \frac{x}{(x^2 + a^2)^n} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 50. If $I_{(n,-m)} = \int \frac{\sin^n x}{\cos^m x} dx$, then

$I_{(n,-m)} + \frac{n-1}{m-1} I_{(n-2,2-m)}$ is equal to

- (a) $\frac{\sin^{n-1} x}{\cos^{m-1} x}$ (b) $\frac{1}{(m-1)} \cdot \frac{\sin^{n-1} x}{\cos^{m-1} x}$
 (c) $\frac{1}{(n-1)} \cdot \frac{\sin^{n-1} x}{\cos^{m-1} x}$ (d) $\frac{n-1}{m-1} \cdot \frac{\sin^{n-1} x}{\cos^{m-1} x}$

Sol. Using integration by parts

$$\begin{aligned} I_{(n,-m)} &= \frac{\sin^{n-1} x}{(m-1) \cos^{m-1} x} - \frac{n-1}{m-1} \int \frac{\sin^{n-2} x}{\cos^{m-2} x} dx \\ &= \frac{\sin^{n-1} x}{(m-1) \cos^{m-1} x} - \frac{n-1}{m-1} I_{(n-2,2-m)} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 51. If $u_n = \int \frac{x^n}{\sqrt{ax^2 + 2bx + c}} dx$, then

$(n+1) au_{n+1} + (2n+1) bu_n + nc u_{n-1}$ is equal to

- (a) $x^{n-1} \sqrt{ax^2 + 2bx + c}$
 (b) $\frac{x^{n-2}}{\sqrt{ax^2 + 2bx + c}}$
 (c) $\frac{x^n}{\sqrt{ax^2 + 2bx + c}}$
 (d) $x^n \sqrt{ax^2 + 2bx + c}$

Sol. Consider,

$$\begin{aligned} u_{n+1} &= \int \frac{x^{n+1}}{\sqrt{ax^2 + 2bx + c}} dx \\ &= \frac{1}{2a} \int \frac{x^n (2ax + 2b) - 2bx^n}{\sqrt{ax^2 + 2bx + c}} dx \\ &= \frac{1}{2a} \int \frac{x^n (2ax + 2b)}{\sqrt{ax^2 + 2bx + c}} dx - \frac{b}{a} u_n \\ &= I_n - \frac{b}{a} u_n \quad \dots(i) \end{aligned}$$

where $I_n = \frac{1}{2a} \int \frac{x^n (2ax + 2b)}{\sqrt{ax^2 + 2bx + c}} dx$

$$\begin{aligned} &= \frac{1}{2a} \left[\int \frac{x^n 2\sqrt{ax^2 + 2bx + c}}{\sqrt{ax^2 + 2bx + c}} dx - \int \frac{n x^{n-1} 2\sqrt{ax^2 + 2bx + c}}{\sqrt{ax^2 + 2bx + c}} dx \right] \\ &= \frac{x^n}{a} \sqrt{ax^2 + 2bx + c} - \frac{n}{a} \int \frac{x^{n-1} (ax^2 + 2bx + c)}{\sqrt{ax^2 + 2bx + c}} dx \\ \Rightarrow aI_n &= x^n \sqrt{ax^2 + 2bx + c} - na u_{n+1} - 2bn u_n - nc u_{n-1} \end{aligned}$$

Putting this value in Eq. (i), we get

$$\begin{aligned} au_{n+1} &= x^n \sqrt{ax^2 + 2bx + c} - na u_n - 2bn u_n - nc u_{n-1} - bu_n \\ \Rightarrow (n+1) au_{n+1} + (2n+1) bu_n + nc u_{n-1} &= x^n \sqrt{ax^2 + 2bx + c} \end{aligned}$$

Hence, (d) is the correct answer.

Ex 52. The value of $I_m = \int (x^{3m} + x^{2m} + x^m)(2x^{2m} + 3x^m + 6)^{1/m} dx$ is

- (a) $\frac{1}{6(m+1)} (2x^{2m} + 3x^m + 6)^{(m+1)/m}$
 (b) $\frac{1}{(m+1)} (2x^{2m} + 3x^m + 6)^{(m+1)/m}$
 (c) $\frac{1}{6(m+1)} (2x^{3m} + 3x^{2m} + 6x^m)^{(m+1)/m}$
 (d) $\frac{1}{6(2m+1)} (x^{3m} + x^{2m} + x^m)^{(m+1)/m}$

Sol. $I_m = \int x^m (x^{2m} + x^m + 1)(2x^{2m} + 3x^m + 6)^{1/m} dx$

$$\begin{aligned} &= \int x^{m-1} (x^{2m} + x^m + 1)(x^m (2x^{2m} + 3x^m + 6))^{1/m} dx \\ &= \int x^{m-1} (x^{2m} + x^m + 1)(2x^{3m} + 3x^{2m} + 6x^m)^{1/m} dx \end{aligned}$$

Put $x^m = t \Rightarrow mx^{m-1} dx = dt$

$$\therefore I_m = \frac{1}{m} \int (t^2 + t + 1)(2t^3 + 3t^2 + 6t)^{1/m} dt$$

Put $2t^3 + 3t^2 + 6t = u^m$

$$\Rightarrow (6t^2 + 6t + 6) dt = mu^{m-1} du$$

$$\Rightarrow \frac{1}{m} (t^2 + t + 1) dt = \frac{1}{6} u^{m-1} du$$

$$\begin{aligned} \therefore I_m &= \frac{1}{6} \int u^{m-1} u du = \frac{1}{6} \int u^m du = \frac{1}{6(m+1)} u^{m+1} \\ &= \frac{1}{6(m+1)} (2t^3 + 3t^2 + 6t)^{\frac{m+1}{m}} \\ &= \frac{1}{6(m+1)} (2x^{3m} + 3x^{2m} + 6x^m)^{\frac{m+1}{m}} \end{aligned}$$

Hence, (c) is the correct answer.

Type 5. Match the Columns

Ex 53. Match the following:

	Column I		Column II
A.	If $I = \int \frac{\sin x - \cos x}{ \sin x - \cos x } dx$, where $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then I is equal to	p.	$\sin x$
B.	If $\int \frac{x^2}{(x^3 + 1)(x^3 + 2)} dx = \frac{1}{3} f\left(\frac{x^3 + 1}{x^3 + 2}\right) + C$, then $f(x)$ is equal to	q.	$x + C$
C.	If $\int \sin^{-1} x \cdot \cos^{-1} x dx = f^{-1}(x) \left[\frac{\pi}{2} x - xf^{-1}(x) - 2\sqrt{1-x^2} \right] + \frac{\pi}{2} \sqrt{1-x^2} + 2x + C$, then $f(x)$ equals	r.	$\ln x $
D.	If $\int \frac{dx}{xf(x)} = f(f(x)) + C$, then $f(x)$ is equal to	s.	$\sin^{-1} x$

Sol. A. If $\frac{\pi}{4} < x < \frac{3\pi}{8}$, then $\sin x > \cos x$

$$\therefore \int \frac{\sin x - \cos x}{|\sin x - \cos x|} dx = \int 1 dx = x + C$$

B. $\int \frac{x^2 dx}{(x^3 + 1)(x^3 + 2)}$
 $= \frac{1}{3} \int 3x^2 \left(\frac{1}{x^3 + 1} - \frac{1}{x^3 + 2} \right) dx = \frac{1}{3} \ln \left| \frac{x^3 + 1}{x^3 + 2} \right| + C$

C. $\int \sin^{-1} x \cos^{-1} x dx$
 $= \int \left[\frac{\pi}{2} \sin^{-1} x - (\sin^{-1} x)^2 \right] dx$

$$\begin{aligned} &= \frac{\pi}{2} (x \sin^{-1} x + \sqrt{1-x^2}) - \{x(\sin^{-1} x)^2 \\ &\quad + 2 \sin^{-1} x \sqrt{1-x^2} - 2x\} + C \\ &\quad \text{[using integration by parts]} \\ &= \sin^{-1} x \left[\frac{\pi}{2} x - x \sin^{-1} x - 2\sqrt{1-x^2} \right] \\ &\quad + \frac{\pi}{2} \sqrt{1-x^2} + 2x + C \end{aligned}$$

$$\therefore f^{-1}(x) = \sin^{-1} x, f(x) = \sin x$$

D. $\int \frac{dx}{x \ln|x|} = \ln|\ln|x|| + C \therefore f(x) = \ln|x|$

A $\rightarrow$ q; B $\rightarrow$ r; C $\rightarrow$ p; D $\rightarrow$ r

Ex 54. Match the following:

	Column I		Column II
A.	If $F(x) = \int \frac{x + \sin x}{1 + \cos x} dx$ and $F(0) = 0$, then $F\left(\frac{\pi}{2}\right)$ is equal to	p.	$\frac{\pi}{2}$
B.	Let $F(x) = \int e^{\sin^{-1} x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx$ and $F(0) = 1$, if $F\left(\frac{1}{2}\right) = \frac{k\sqrt{3}e^{\pi/6}}{\pi}$, then k equals	q.	$\frac{\pi}{3}$
C.	Let $F(x) = \int \frac{dx}{(x^2 + 1)(x^2 + 9)}$ and $F(0) = 0$, if $F(\sqrt{3}) = \frac{5}{36}k$, then k equals	r.	$\frac{\pi}{4}$
D.	Let $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$ and $F(0) = 0$, if $F\left(\frac{\pi}{4}\right) = \frac{2k}{\pi}$, then k equals	s.	π

Sol. A. $F(x) = \int \frac{x + \sin x}{1 + \cos x} dx$
 $= \int \left(x \frac{1}{2} \sec^2 \frac{x}{2} + \tan \frac{x}{2} \right) dx = x \tan \frac{x}{2} + C$

Since, $0 = F(0)$
 $\therefore C = 0$

and $F\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$

B. $F(x) = \int e^{\sin^{-1} x} \left(1 - \frac{x}{\sqrt{1-x^2}} \right) dx$
 $= \int e^{\sin^{-1} x} \left(\frac{1}{\sqrt{1-x^2}} \sqrt{1-x^2} + \frac{-x}{\sqrt{1-x^2}} \right) dx$

$F(x) = e^{\sin^{-1} x} \sqrt{1-x^2} + C$

$F(0) = 1 + C$
 $\Rightarrow C = 0$

$F\left(\frac{1}{2}\right) = e^{\frac{\pi}{6}} \cdot \frac{\sqrt{3}}{2}$
 $= \frac{k\sqrt{3}}{\pi} e^{\frac{\pi}{6}}$

$\therefore k = \frac{\pi}{2}$

C. $F(x) = \int \frac{dx}{(x^2 + 1)(x^2 + 9)}$
 $= \frac{1}{8} \int \left(\frac{1}{x^2 + 1} - \frac{1}{x^2 + 9} \right) dx$
 $= \frac{1}{8} \left[\tan^{-1} x - \frac{1}{3} \tan^{-1} \frac{x}{3} \right] + C$

$\Rightarrow F(0) = C$
 $C = 0$

$\therefore F(\sqrt{3}) = \frac{1}{8} \left(\frac{\pi}{3} - \frac{1}{3} \cdot \frac{\pi}{6} \right)$
 $= \frac{5\pi}{144} = \frac{5k}{36}$

$\therefore k = \frac{\pi}{4}$

D. $F(x) = \int \frac{\sqrt{\tan x}}{\sin x \cos x} dx$
 $= \int (\tan x)^{-1/2} \sec^2 x dx$
 $= 2\sqrt{\tan x} + C$

$F(0) = C$
 $C = 0$

$\therefore F\left(\frac{\pi}{4}\right) = 2 = \frac{2k}{\pi}$

$\therefore k = \pi$

$A \rightarrow p; B \rightarrow p; C \rightarrow r; D \rightarrow s$

Ex 55. Let $I = \int \frac{dx}{a + b \cos x}$, where $a, b > 0$ and

$a + b = u, a - b = v$, then match the following.

Column I	Column II
A. $v = 0$	p. $\frac{1}{\sqrt{u}} \log \left(\frac{\sqrt{u} + \tan \frac{x}{2}}{\sqrt{u} - \tan \frac{x}{2}} \right) + C$
B. $v > 0$	q. $\frac{2}{\sqrt{uv}} \tan^{-1} \left(\tan \frac{x}{2} \sqrt{\frac{v}{u}} \right) + C$
C. $v < 0$	r. $\frac{1}{\sqrt{-uv}} \log \left \frac{\sqrt{u} + \sqrt{-v} \tan \frac{x}{2}}{\sqrt{u} - \sqrt{-v} \tan \frac{x}{2}} \right + C$
D. $v = -1$	s. $\frac{2}{u} \tan \frac{x}{2} + C$

Sol. Putting $\tan \frac{x}{2} = t$ in the given integral, we have

$I = 2 \int \frac{dt}{u + vt^2}$

A. If $v = 0$, then $I = \frac{2}{u} t + C$
 $= \frac{2}{u} \tan \frac{x}{2} + C$

B. If $v > 0$, then $I = \frac{2}{v} \int \frac{dt}{t^2 + \frac{u}{v}}$
 $= \frac{2}{v} \sqrt{\frac{v}{u}} \tan^{-1} \left(t \sqrt{\frac{v}{u}} \right) + C$
 $= \frac{2}{\sqrt{vu}} \tan^{-1} \left(\left(\tan \frac{x}{2} \right) \sqrt{\frac{v}{u}} \right) + C$

C. If $v < 0$, then $I = \frac{2}{-v} \int \frac{dt}{\frac{u}{-v} - t^2}$
 $= \frac{2\sqrt{-v}}{-v \cdot 2\sqrt{u}} \log \left| \frac{\sqrt{\frac{u}{-v}} + t}{\sqrt{\frac{u}{-v}} - t} \right| + C$
 $= \frac{1}{\sqrt{-uv}} \log \left| \frac{\sqrt{u} + \sqrt{-v} \tan \frac{x}{2}}{\sqrt{u} - \sqrt{-v} \tan \frac{x}{2}} \right| + C$

D. If $v = -1$, then $I = \frac{1}{\sqrt{u}} \log \left| \frac{\sqrt{u} + \tan \frac{x}{2}}{\sqrt{u} - \tan \frac{x}{2}} \right| + C$

$A \rightarrow s; B \rightarrow q; C \rightarrow r; D \rightarrow p$

Type 6. Single Integer Answer Type Questions

Ex 56. If $\int \frac{\cos 6x + \cos 9x}{1 - 2\cos 5x} dx = -\frac{\sin 4x}{k} - \sin x + C$,

then the value of k is _____.

$$\text{Sol. (4)} \quad \frac{\cos 6x + \cos 9x}{1 - 2\cos 5x} dx = \frac{2\cos \frac{15}{2}x \cos \frac{3}{2}x}{1 - 2\left(2\cos^2 \frac{5x}{2} - 1\right)}$$

$$= \frac{2\left(4\cos^3 \frac{5x}{2} - 3\cos \frac{5x}{2}\right)\cos \frac{3x}{2}}{3 - 4\cos^2 \frac{5x}{2}}$$

$$= -2\cos \frac{5x}{2} \cos \frac{3x}{2}$$

$$= -(\cos 4x + \cos x)$$

$$\text{Now, } \int \frac{\cos 6x + \cos 9x}{1 - 2\cos 5x} dx = -\frac{\sin 4x}{4} - \sin x + C$$

$$\therefore k = 4$$

Ex 57. The value of $\int \frac{\tan x}{1 + \tan x + \tan^2 x} dx$
 $= x - \frac{2}{\sqrt{A}} \tan^{-1} \left(\frac{2 \tan x + 1}{\sqrt{A}} \right) + C$, then the
 value of A is _____.

$$\text{Sol. (3)} \quad \text{Let } I = \int \frac{\tan x}{1 + \tan x + \tan^2 x} dx$$

$$= \int \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos^2 x} + \frac{\sin x}{\cos x}} dx = \int \frac{\sin 2x}{2 + \sin 2x} dx$$

$$= \int dx - 2 \int \frac{dx}{2 + \sin 2x}$$

$$= x - 2 \int \frac{\sec^2 x}{2\sec^2 x + 2\tan x} dx$$

$$\text{Let } \tan x = t$$

$$\Rightarrow dt = \sec^2 x dx$$

$$\therefore I = x - \frac{2}{2} \int \frac{dt}{t^2 + t + 1} = x - \int \frac{dt}{\left(t + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= x - \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{2 \tan x + 1}{\sqrt{3}} \right) + C$$

$$\therefore A = 3$$

Ex 58. If $I = \int \frac{e^x (2 - x^2)}{(1 - x)\sqrt{1 - x^2}} dx = \mu e^x \left(\frac{1 + x}{1 - x} \right)^\lambda + C$,

then $2(\lambda + \mu)$ is equal to _____.

$$\text{Sol. (3)} \quad I = \int e^x \left(\frac{1}{(1 - x)\sqrt{1 - x^2}} + \frac{\sqrt{1 + x}}{\sqrt{1 - x}} \right) dx$$

$$\text{As, } \frac{d}{dx} \left(\sqrt{\frac{1 + x}{1 - x}} \right) = \frac{1}{(1 - x)\sqrt{1 - x^2}}$$

$$\therefore I = e^x \sqrt{\frac{1 + x}{1 - x}} + C$$

$$\Rightarrow I = e^x \left(\frac{1 + x}{1 - x} \right)^{\frac{1}{2}} + C$$

$$\Rightarrow \mu = 1, \lambda = \frac{1}{2}$$

$$\therefore 2(\mu + \lambda) = 2 \left(1 + \frac{1}{2} \right) = 2 \times \frac{3}{2} = 3$$

Ex 59. If $\int \sin^{5/2} x \cos^3 x dx$

$= 2 \sin^{\frac{A}{2}} x \left[\frac{1}{B} - \frac{1}{C} \sin^2 x \right] + C$, then the value
 of $(A + B) - C$ is equal to _____.

$$\text{Sol. (3)} \quad \int \sin^{5/2} x \cos^3 x dx = \int \sin^2 x \sin^{1/2} x \cos^2 x \cos x dx$$

$$= \int \sin^2 x \sin^{1/2} x (1 - \sin^2 x) \cos x dx$$

$$= \int t^2 t^{1/2} (1 - t^2) dt = \int [t^{5/2} - t^{9/2}] dt$$

$$= \int t^{5/2} dt - \int t^{9/2} dt = \frac{t^{7/2}}{\frac{7}{2}} - \frac{t^{11/2}}{\frac{11}{2}} + C$$

$$= \frac{2}{7} t^{7/2} - \frac{2}{11} t^{11/2} + C$$

$$= \frac{2}{7} \sin^{7/2} x - \frac{2}{11} \sin^{11/2} x + C$$

$$= 2 \sin^{7/2} x \left[\frac{1}{7} - \frac{1}{11} \sin^2 x \right] + C$$

$$\Rightarrow A = 7, B = 7, C = 11$$

$$\therefore (A + B) - C = (7 + 7) - 11 = 3$$

Ex 60. If $\int \frac{1}{(x^2 - 1)} \ln \left(\frac{x - 1}{x + 1} \right) dx$

$$= 6A \left[\ln \left(\frac{x - 1}{x + 1} \right) \right]^2 + C$$
, then find $24A$.

$$\text{Sol. (1)} \quad \text{Let } t = \ln \left(\frac{x - 1}{x + 1} \right) \Rightarrow \frac{dt}{dx} = \frac{2}{x^2 - 1}$$

$$I = \int \frac{1}{2} t dt = \frac{1}{4} t^2 + C = \frac{1}{4} \left[\ln \left(\frac{x - 1}{x + 1} \right) \right]^2 + C$$

$$\Rightarrow 6A = \frac{1}{4}$$

$$\therefore 24A = 6A \times A = \frac{1}{4} \times 4 = 1$$

Ex 61. If $I = \int \frac{dx}{1 + \sqrt{x^2 + 2x + 2}} = \frac{A}{7}$

$$\log |x + 1 + \sqrt{x^2 + 2x + 2}| - \frac{\sqrt{x^2 + 2x + 2} - 1}{x + 1} + C,$$

then A is equal to _____.

$$\text{Sol. (7)} I = \int \frac{dx}{\sqrt{(x+1)^2 + 1} + 1}$$

$$I = \int \frac{\sqrt{(x+1)^2 + 1} - 1}{(x+1)^2} dx$$

Put $(x+1)^2 + 1 = t^2$, so that

$$I = \int \frac{t-1}{t^2-1} \cdot \frac{t}{\sqrt{t^2-1}} dt$$

$$= \int \frac{t}{t+1} \cdot \frac{dt}{\sqrt{t^2-1}}$$

$$= \int \frac{dt}{\sqrt{t^2-1}} - \int \frac{dt}{(t+1)\sqrt{t^2-1}}$$

$$= \log \left| t + \sqrt{t^2-1} \right| - I_1$$

In I_1 , put $t+1 = \frac{1}{u}$, so that

$$I_1 = \int \frac{\left(-\frac{1}{u^2}\right) du}{\left(\frac{1}{u}\right) \sqrt{\left(\frac{1}{u}-1\right)^2 - 1}} = \int \frac{du}{\sqrt{1-2u}}$$

$$= \sqrt{1-2u} = \sqrt{\frac{t-1}{t+1}}$$

$$\text{Thus, } I = \log \left| t + \sqrt{t^2-1} \right| - \frac{t-1}{\sqrt{t^2-1}} + C$$

$$= \log \left| x+1 + \sqrt{x^2+2x+2} \right| - \frac{\sqrt{x^2+2x+2}-1}{x+1} + C$$

$$\therefore A = 7$$

Ex 62. If $\int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}} dx$ is equal to $\frac{\sqrt{2}}{4} A \tan^{-1} \left(\frac{1}{\sqrt{2}} \sqrt{x^2 + \frac{1}{x^2}} \right) + C$, then A is equal to _____.

$$\text{Sol. (2)} \int \frac{x^2-1}{(x^2+1)\sqrt{x^4+1}} dx = \int \frac{1-\frac{1}{x^2}}{\left(x+\frac{1}{x}\right) \left(\sqrt{x^2+\frac{1}{x^2}}\right)} dx$$

$$= \int \frac{dt}{t\sqrt{t^2-2}} = \int \frac{tdt}{t^2\sqrt{t^2-2}} \quad \left[\because t = x + \frac{1}{x} \right]$$

$$= \int \frac{ydy}{(y^2+2)y} = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{y}{\sqrt{2}} \right) + C \quad [\because y^2 = t^2 - 2]$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{1}{\sqrt{2}} \left(\sqrt{\left(x+\frac{1}{x}\right)^2 - 2} \right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{1}{\sqrt{2}} \sqrt{x^2 + \frac{1}{x^2}} \right) + C$$

$$\therefore A = 2$$

Ex 63. If $\int \frac{x^2-3}{x^3-2x^2-x+2} dx = \frac{A}{3} \log \left(C \left| \frac{(x-2)(x-1)^3}{x+1} \right| \right)$, then A is equal to _____.

$$\text{Sol. (1)} \int \frac{x^2-3}{x^3-2x^2-x+2} dx = \int \frac{x^2-3}{(x+1)(x-1)(x-2)} dx$$

$$= \int \left(-\frac{1}{3(x+1)} + \frac{1}{x-1} + \frac{1}{3(x-2)} \right) dx$$

$$= -\left(\frac{1}{3}\right) \log |x+1| + \log |x-1| + \left(\frac{1}{3}\right) \log |x-2| + \log C_1$$

$$= \frac{1}{3} \log (|x+1|)^{-1} + \frac{1}{3} \log |x-1|^3 + \frac{1}{3} \log |x-2| + \frac{1}{3} \log C, \text{ where } C = C_1^3$$

$$= \left(\frac{1}{3}\right) \log \left(C \left| \frac{(x-2)(x-1)^3}{x+1} \right| \right)$$

$$\therefore A = 1$$

Target Exercises

Type 1. Only One Correct Option

1. If $f'(x) = \frac{1}{-x + \sqrt{x^2 + 1}}$ and $f(0) = -\frac{1 + \sqrt{2}}{2}$, then

$f(1)$ is equal to

- (a) $-\log(\sqrt{2} + 1)$ (b) 1
(c) $1 + \sqrt{2}$ (d) None of these

2. $\int \frac{3 + 2\cos x}{(2 + 3\cos x)^2} dx$ is equal to

- (a) $\left(\frac{\sin x}{3\cos x + 2}\right) + C$ (b) $\left(\frac{2\cos x}{3\sin x + 2}\right) + C$
(c) $\left(\frac{2\cos x}{3\cos x + 2}\right) + C$ (d) $\left(\frac{2\sin x}{3\sin x + 2}\right) + C$

3. $\int \sec^4 x \operatorname{cosec}^2 x dx$ is equal to

- (a) $\tan^2 x + 2\tan x - \cot x + C$
(b) $\frac{\tan^3 x}{3} + 2\tan x - \cot x + C$
(c) $\frac{\tan^2 x}{2} + \tan x - \cot x + C$
(d) $\frac{\tan^2 x}{2} + 2\tan x - 2\cot x + C$

4. $\int \frac{\operatorname{cosec} \theta - \cot \theta}{\operatorname{cosec} \theta + \cot \theta} d\theta$ is equal to

- (a) $2 \operatorname{cosec} \theta - 2 \cot \theta - \theta + C$
(b) $2 \operatorname{cosec} \theta - 2 \cot \theta + \theta + C$
(c) $2 \operatorname{cosec} \theta + 2 \cot \theta - \theta + C$
(d) None of the above

5. $\int \frac{\sin(x+a)}{\sin(x+b)} dx$ is equal to

- (a) $(x+b) \cos(x+b) + \sin(a+b) \log |\sin(x-b)| + C$
(b) $(x+b) \cos(a-b) + \sin(a-b) \log |\sin(x+b)| + C$
(c) $(x+b) \cos(x-b) + \sin(a+b) \log |\sin(x-b)| + C$
(d) None of the above

6. $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$ is equal to

- (a) $2\sin x + x + C$
(b) $2\cos x + x + C$
(c) $\frac{2\sin x + \cos x}{\cos x} + C$
(d) $2(\sin x + \cos x) + C$

7. $\int (1 + 2x + 3x^2 + 4x^3 + \dots) dx$ is equal to

- (a) $(1+x)^{-1} + C$ (b) $(1-x)^{-1} + C$
(c) $(1-x)^{-1} - 1 + C$ (d) None of these

8. $\int \frac{\sin^{3/2} x + \cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}} dx$
 $= \frac{2}{\cos \theta} \sqrt{\tan x \cos \theta + \sin \theta} - A + C$

Then, A is equal to

- (a) $\frac{2}{\sin \theta} \sqrt{\cos \theta + \cot x \sin \theta}$ (b) $\frac{2}{\sin \theta} \cos \theta + \cot x \sin \theta$
(c) $\frac{2}{\sin \theta} \sqrt{\cot x \cos \theta + \sin \theta}$ (d) None of these

9. $\int \sqrt{\frac{x}{a^3 - x^3}} dx$ is equal to

- (a) $\frac{2}{3} \cos^{-1} \left(\frac{x^{2/3}}{a^{2/3}} \right) + C$ (b) $\frac{2}{3} \sin^{-1} \left(\frac{x^{2/3}}{a^{2/3}} \right) + C$
(c) $\frac{2}{3} \tan^{-1} \left(\frac{x^{2/3}}{a^{2/3}} \right) + C$ (d) $\frac{2}{3} \sin^{-1} \left(\frac{x^{3/2}}{a^{3/2}} \right) + C$

10. $\int \sqrt{\frac{x^4}{a^6 + x^6}} dx$ is equal to

- (a) $\frac{1}{3} \log \left| \frac{1}{\sqrt{(a^3)^2 + x^2}} \right| + C$
(b) $\frac{1}{3} \log |x^3 + \sqrt{a^6 + x^6}| + C$
(c) $\frac{1}{3} \log |x^2 + \sqrt{a^6 + x^6}| + C$
(d) None of the above

11. $\int \sec^{25/13} x \operatorname{cosec}^{27/13} x dx$ is equal to

- (a) $-\frac{15}{13} (\tan x)^{-15/13} + \frac{3}{2} (\tan x)^{3/2} + C$
(b) $-\frac{13}{14} (\tan x)^{-14/13} + \frac{13}{12} (\tan x)^{12/13} + C$
(c) $-\frac{14}{13} (\tan x)^{-13/14} + \frac{2}{3} (\tan x)^{3/2} + C$
(d) None of the above

12. $\int \frac{1}{\sin x + \sec x} dx = \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x} \right| + A + C$

Then, A is equal to

- (a) $\frac{1}{3} \tan^{-1}(\sin x + \cos x)$
(b) $\frac{1}{3} \tan^{-1}(\sin x - \cos x)$
(c) $\frac{2}{3} \tan^{-1}(\sin x - \cos x)$
(d) $\tan^{-1}(\sin x + \cos x)$

13. $\int \frac{1}{\cos^6 x + \sin^6 x} dx$ is equal to
 (a) $\tan^{-1}(\tan x - \cot x) + C$ (b) $\tan^{-1}(\tan x - \sec x) + C$
 (c) $\sec^{-1}(\sec x - \tan x) + C$ (d) $\sec^{-1}(\tan x - \cot x) + C$

14. $\int \frac{2\sin 2\phi - \cos \phi}{6 - \cos^2 \phi - 4\sin \phi} d\phi$ is equal to
 (a) $2\log\left(\frac{2\sin \phi - 4}{\cos^2 \phi - 4\cos \phi + 5}\right) + \frac{7}{\tan^{-1}(\sin \phi - 2)} + C$
 (b) $2\log\left(\frac{2\sin \phi - 4}{\sin^2 \phi - 4\sin \phi + 5}\right) + \frac{7}{\tan^{-1}(\sin \phi - 2)} + C$
 (c) $2\log|\sin^2 \phi - 4\sin \phi + 5| + 7\tan^{-1}(\sin \phi - 2) + C$
 (d) None of the above

15. $\int \frac{1}{2 + \cos x} dx$ is equal to
 (a) $\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$ (b) $\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$
 (c) $\sqrt{3} \tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$ (d) $2\tan^{-1}\left(\frac{\tan x/2}{\sqrt{3}}\right) + C$

16. $\int \frac{1}{1 - 2\sin x} dx$ is equal to
 (a) $\frac{1}{2\sqrt{3}} \log\left|\frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}}\right| + C$
 (b) $\frac{\sqrt{3}}{2} \log\left|\frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}}\right| + C$
 (c) $\frac{1}{\sqrt{3}} \log\left|\frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}}\right| + C$
 (d) None of the above

17. $\int \frac{1}{\sin x + \sqrt{3}\cos x} dx$ is equal to
 (a) $\frac{1}{\sqrt{3}} \log|\operatorname{cosec}(x + \pi)| + C$
 (b) $\frac{1}{2} \log\left|\tan\left(\frac{x}{2} + \frac{\pi}{6}\right)\right| + C$
 (c) $\frac{1}{2} \log\left|\frac{1}{\sin(x + \pi)}\right| + C$
 (d) $\frac{1}{2} \log\left|\sin\left(\frac{x}{2} + \frac{\pi}{\sqrt{3}}\right)\right| + C$

18. $\int \frac{5(x^6 + 1)}{x^2 + 1} dx$ is equal to
 (a) $5(x^7 + x)\tan^{-1} x + C$
 (b) $x^5 - \frac{5}{3}x^3 + 5x + C$
 (c) $3x^4 - 5x^2 + 15x + C$
 (d) $5\tan^{-1}(x^2 + 1) + \log(x^2 + 1) + C$

19. $\int \frac{1}{1 - \sin^4 x} dx$ is equal to
 (a) $\frac{1}{2}\tan x + \frac{1}{2}\tan^{-1}(\sqrt{2}\tan x) + C$
 (b) $\frac{1}{2\sqrt{2}}\tan x + \frac{1}{2}\tan^{-1}(\sqrt{2}\tan x) + C$
 (c) $\frac{1}{2}\tan x + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$
 (d) $\frac{1}{2}\tan^{-1} x + \frac{1}{2\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) + C$

20. If $\int \frac{1}{(\sin x + 4)(\sin x - 1)} dx$
 $= A \frac{1}{\tan \frac{x}{2} - 1} + B \tan^{-1}\{f(x)\} + C$, then
 (a) $A = \frac{1}{5}, B = \frac{-2}{5\sqrt{15}}, f(x) = \frac{4\tan x + 3}{\sqrt{15}}$
 (b) $A = -\frac{1}{5}, B = \frac{1}{\sqrt{15}}, f(x) = \frac{4\tan\left(\frac{x}{2}\right) + 1}{\sqrt{15}}$
 (c) $A = \frac{2}{5}, B = \frac{-2}{5}, f(x) = \frac{4\tan x + 1}{5}$
 (d) $A = \frac{2}{5}, B = \frac{-2}{5\sqrt{15}}, f(x) = \frac{4\tan \frac{x}{2} + 1}{\sqrt{15}}$

21. If $\int f(x) \sin x \cos x dx = \frac{1}{2(b^2 - a^2)} \log\{f(x)\} + C$,
 then $f(x)$ is equal to
 (a) $\frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$
 (b) $\frac{1}{a^2 \sin^2 x - b^2 \cos^2 x}$
 (c) $\frac{1}{a^2 \cos^2 x + b^2 \sin^2 x}$
 (d) $\frac{1}{a^2 \cos^2 x - b^2 \sin^2 x}$

22. If $\int \frac{x^2}{\sqrt{1-x}} dx = p \sqrt{(1-x)}(3x^2 + 4x + 8)$, then the
 value of p is
 (a) $\frac{-2}{15}$ (b) $\frac{2}{15}$
 (c) $\frac{4}{15}$ (d) None of these

23. If $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$ and $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx$,
 then for any arbitrary constant C , the value of $J - I$
 is
 (a) $\frac{1}{2} \log\left|\frac{e^{4x} - e^{2x} + 1}{e^{4x} + e^{2x} + 1}\right| + C$ (b) $\frac{1}{2} \log\left|\frac{e^{2x} + e^x + 1}{e^{2x} - e^x + 1}\right| + C$
 (c) $\frac{1}{2} \log\left|\frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1}\right| + C$ (d) $\frac{1}{2} \log\left|\frac{e^{4x} + e^{2x} + 1}{e^{4x} - e^{2x} + 1}\right| + C$

24. The value of $\int \frac{\log(x+1) - \log x}{x(x+1)} dx$ is
- (a) $-\frac{1}{2} [\log(x+1)]^2 - \frac{1}{2} (\log x)^2 + \log(x+1) \cdot \log x + C$
 (b) $-\frac{1}{2} [\log(x+1)]^2 - (\log x)^2 + \log(x+1) \cdot \log x + C$
 (c) $\left[\log \left(1 + \frac{1}{x} \right) \right]^2 + C$
 (d) None of the above
25. $\int \frac{\sin x + 8 \cos x}{4 \sin x + 6 \cos x} dx$ is equal to
- (a) $x + \frac{1}{2} \log(4 \sin x + 6 \cos x) + C$
 (b) $2x + \log(2 \sin x + 3 \cos x) + C$
 (c) $x + 2 \log(2 \sin x + 3 \cos x) + C$
 (d) $\frac{1}{2} \log(4 \sin x + 6 \cos x) + C$
26. $\int \frac{1}{x^2} (2x+1)^3 dx$ is equal to
- (a) $4x^2 + 12x + 6 \log x - \frac{1}{x} + C$
 (b) $4x^2 + 12x - 6 \log x - \frac{2}{x} + C$
 (c) $2x^2 + 8x + 3 \log x - \frac{2}{x} + C$
 (d) $8x^2 + 6x + 6 \log x + \frac{2}{x} + C$
27. $\int \sqrt{1 + \sin \left(\frac{x}{4} \right)} dx$ is equal to
- (a) $8 \left(\sin \frac{x}{8} + \cos \frac{x}{8} \right) + C$
 (b) $8 \left(\sin \frac{x}{8} - \cos \frac{x}{8} \right) + C$
 (c) $8 \left(\cos \frac{x}{8} - \sin \frac{x}{8} \right) + C$
 (d) $\frac{1}{8} \left(\sin \frac{x}{8} - \cos \frac{x}{8} \right) + C$
28. $\int \frac{2 \sin x}{(3 + \sin 2x)} dx$ is equal to
- (a) $\frac{1}{2} \ln \left| \frac{2 + \sin x - \cos x}{2 - \sin x + \cos x} \right| - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right) + C$
 (b) $\frac{1}{2} \ln \left| \frac{2 + \sin x - \cos x}{2 - \sin x + \cos x} \right| - \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right) + C$
 (c) $\frac{1}{4} \ln \left| \frac{2 + \sin x - \cos x}{2 - \sin x + \cos x} \right| - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right) + C$
 (d) None of the above
29. If $\int x \log \left(1 + \frac{1}{x} \right) dx = f(x) \log(x+1) + g(x)x^2 + Ax + C$, then
- (a) $f(x) = \frac{1}{2}x^2$
 (b) $g(x) = \log x$
 (c) $A = 1$
 (d) None of the above
30. $\int \frac{dx}{4 \cos^3 2x - 3 \cos 2x}$ is equal to
- (a) $\frac{1}{3} \log(\sec 6x + \tan 6x) + C$
 (b) $\frac{1}{6} \log(\sec 6x + \tan 6x) + C$
 (c) $\log(\sec 6x + \tan 6x) + C$
 (d) None of the above
31. If $f(x) = \int \frac{x^2 dx}{(1+x^2)(1+\sqrt{1+x^2})}$ and $f(0) = 0$, then the value of $f(1)$ be
- (a) $\log(1 + \sqrt{2})$ (b) $\log(1 + \sqrt{2}) - \frac{\pi}{4}$
 (c) $\log(1 + \sqrt{2}) + \frac{\pi}{2}$ (d) None of these
32. $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx$ is equal to
- (a) $-\frac{1}{2} \cdot \frac{1}{(10^x + x^{10})^2} + C$ (b) $\log(10^x + x^{10}) + C$
 (c) $\frac{1}{2} \cdot \frac{1}{(10^x + x^{10})^2} + C$ (d) None of these
33. If $\int \frac{dx}{x^2 (x^n + 1)^{\frac{n}{n-1}}} = -[f(x)]^{1/n} + C$, then $f(x)$ is
- (a) $(1 + x^n)$
 (b) $1 + x^{-n}$
 (c) $x^n + x^{-n}$
 (d) None of the above
34. $\int \frac{1}{\sqrt{e^{5x}} \sqrt[4]{(e^{2x} + e^{-2x})^3}} dx$ is equal to
- (a) $-\frac{1}{4} (1 + e^{-4x})^{1/4} + C$
 (b) $-\frac{(1 + e^{-4x})^{1/4}}{1/4} + C$
 (c) $-(1 + e^{-4x})^{1/4} + C$
 (d) None of the above
35. $\int \frac{(x - \sin x)^{3/2}}{\sqrt{x}} \left\{ \frac{6x^2 \sin^2 \frac{x}{2}}{x - \sin x} + 3x \right\} dx$ is equal to
- (a) $\frac{2}{3} (2x^2 - x \sin x)^{3/2} + C$
 (b) $2 (x^2 - x \sin x)^{3/2} + C$
 (c) $3 (x^2 - x \sin x)^{3/2} + C$
 (d) $3 (2x^2 - x \sin x)^{3/2} + C$
36. $\int \frac{(1 + \log x)^2}{1 + \log x^{x+1} + (\log x^{\sqrt{x}})^2} dx$ is equal to
- (a) $(x \log x) + C$
 (b) $\log |1 + x \log x| + C$
 (c) $\log |\log x + 1| + C$
 (d) None of the above

37. $\int x(x^x)^x (2 \log x + 1) dx$ is equal to

- (a) $(x^x)^x + C$
 (b) $\log(x^x)^x + C$
 (c) $x^x + C$
 (d) None of the above

38. $\int \frac{1}{x^4 \sqrt{x^2 - 3}} dx$ is equal to

- (a) $\frac{\sqrt{x^2 - 3}}{27x^3} (2x^2 + 3) + C$
 (b) $\frac{1}{108} \sqrt{x^2 - 3} (2x^2 + 3) + C$
 (c) $\frac{1}{27} \cdot \frac{\sqrt{x^2 - 3}}{9x^3} (2x^2 + 3) + C$
 (d) None of the above

39. $\int \frac{1}{(x^2 + 2x + 2)^2} dx$ is equal to

- (a) $\left\{ \frac{\tan^{-1}(\sqrt{x+1})}{2} + \frac{x+1}{x^2 + 2x + 2} \right\} + C$
 (b) $\frac{1}{2} \left\{ \sec^{-1}(\sqrt{x+1}) + \frac{x+1}{x^2 + 2x + 2} \right\} + C$
 (c) $\frac{1}{2} \left\{ \tan^{-1}(x+1) + \frac{x+1}{x^2 + 2x + 2} \right\} + C$
 (d) $\left\{ \frac{\sec^{-1} \sqrt{x+1}}{2} + \frac{x+1}{x^2 + 2x + 2} \right\} + C$

40. $\frac{1}{xI(x)I^2(x) \dots I^{n-1}(x)I^n(x)}$ dx is equal to, where

$$I^n(x) = \log_e \underbrace{(\log_e (\log_e \dots (\log_e x)))}_{n \text{ times}}$$

- (a) $I^n(x) + C$ (b) $I^{n+1}(x) + C$
 (c) $I^{n-1}(x) + C$ (d) None of these

41. $\int (x + \sqrt{x^2 + 1})^n dx$ is equal to

- (a) $\frac{1}{2} \left[\frac{(x + \sqrt{x^2 + 1})^{n+1}}{n+1} + \frac{(x + \sqrt{x^2 + 1})^{n-1}}{n-1} \right] + C$
 (b) $\frac{1}{2} \left[\frac{(x + \sqrt{x^2 + 1})^{n+1}}{n+1} + \frac{(x + \sqrt{x^2 - 1})^{n-1}}{n-1} \right] + C$
 (c) $\frac{1}{2} \left[\frac{(x - \sqrt{x^2 + 1})^{n+1}}{n+1} + \frac{(x + \sqrt{x^2 + 1})^{n-1}}{n-1} \right] + C$
 (d) None of the above

42. If $\int \frac{x^4 + 4x^3 + 11x^2 + 12x + 8}{(x^2 + 2x + 3)^2 (x+1)} dx$

$$= \log |x+1| - \frac{1}{2(x^2 + 2x + 3)} - A + C.$$

Then, A is equal to

- (a) $\frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{x+1}{2\sqrt{2}} \right) + \frac{\sqrt{2}(x+1)}{x^2 + 2x + 3} \right]$
 (b) $\frac{1}{2\sqrt{2}} \left[\tan^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + \frac{\sqrt{2}(x+1)}{x^2 + 2x + 3} \right]$
 (c) $\frac{1}{\sqrt{2}} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + \frac{\sqrt{2}(x+1)}{x^2 + 2x + 3} \right]$
 (d) $\frac{1}{2\sqrt{2}} \left[\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x+1}{\sqrt{2}} \right) + \frac{\sqrt{2}(x+1)}{x^2 + 2x + 3} \right]$

43. If $f(x) = \frac{\sin^2 \pi x}{1 + \pi^x}$, then $\int \{f(x) + f(-x)\} dx$ is

equal to

- (a) 0 (b) $x + C$
 (c) $\frac{x}{2} - \frac{\cos \pi x}{2\pi} + C$ (d) $\frac{x}{2} - \frac{\sin 2\pi x}{4\pi} + C$

44. $\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^2+x^4}} dx$ is equal to

- (a) $-\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{x^2 + \frac{1}{x^2} + 1} - \sqrt{3}}{\sqrt{x^2 + x + 1} + \sqrt{3}} \right| + C$
 (b) $\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{x^2 + \frac{1}{x^2} + 1} - \sqrt{3}}{\sqrt{x^2 + \frac{1}{x^2} + 1} + \sqrt{3}} \right| + C$
 (c) $-\frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{x^2 + x + 1} - \sqrt{3}}{\sqrt{x^2 + x + 1} + \sqrt{3}} \right| + C$
 (d) None of the above

45. If $\int \frac{\sqrt{\cos 2x}}{\sin x} dx = -\log |\cot x + \sqrt{\cot^2 x - 1}| + A + C$, then A is equal to

- (a) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \sec^2 x}}{\sqrt{2} - \sqrt{1 - \sec^2 x}} \right|$
 (b) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right|$
 (c) $\frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \sin^2 x}}{\sqrt{2} - \sqrt{1 - \sin^2 x}} \right|$
 (d) None of the above

46. $\int x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} dx$ is equal to

- (a) $\frac{1}{2} a^2 \sin^{-1} \left(\frac{x^2}{a^2} \right) + \frac{1}{2} \sqrt{a^4 - x^4} + C$
 (b) $\frac{1}{2} a^2 \sin^{-1} \left(\frac{x^2}{a^2} \right) + \frac{1}{4} \sqrt{a^4 - x^4} + C$
 (c) $\frac{1}{2} a^2 \sin^{-1} \left(\frac{x^2}{a^2} \right) - \frac{1}{4} \sqrt{a^4 - x^4} + C$
 (d) None of the above

47. If $\int \frac{6x^3 + 15x^2 - 7x + 6}{\sqrt{2x^2 - 2x + 1}} dx$
 $= (x^2 + 5x + 3)\sqrt{2x^2 - 2x + 1} + A + C$, then the
 value of A is

- (a) $2\sqrt{2} \log \left| \left(x - \frac{1}{2\sqrt{2}} \right) + \sqrt{x^2 - x + \frac{1}{2}} \right|$
 (b) $4 \log \left| \left(x - \frac{1}{2} \right) + \sqrt{x^2 - x + \frac{1}{2}} \right|$
 (c) $2\sqrt{2} \left| \left(x - \frac{1}{2} \right) + \sqrt{x^2 - x + \frac{1}{2}} \right|$
 (d) $2\sqrt{2} \log \left| \left(x - \frac{1}{2} \right) + \sqrt{x^2 - x + \frac{1}{2}} \right|$

48. If $\int x \tan^{-1} x dx = \frac{x^2}{2} \tan^{-1} x - A + C$, then the
 value of A is

- (a) $\frac{1}{2} [1 - \tan^{-1} x]$
 (b) $\frac{1}{2} \left[1 - \frac{x^2 - 1}{x^2 + 1} \right]$
 (c) $\frac{1}{2} \left[1 - \frac{1}{x^2 + 1} \right]$
 (d) $\frac{1}{2} [x - \tan^{-1} x]$

49. If $\int \log(1 + x^2) dx = x \log(1 + x^2) - A + C$, then the
 value of A is

- (a) $2x - 2 \tan^{-1} x$
 (b) $2x + \tan^{-1} x$
 (c) $2 - 2x \tan^{-1} x$
 (d) None of the above

50. If $\int \tan^{-1} \left(\frac{2 \cos^2 \theta}{2 - \sin 2\theta} \right) \sec^2 \theta d\theta = \theta \tan \theta + \log \cos \theta$
 $- A + \frac{1}{2} \log(2 - 2 \tan \theta + \tan^2 \theta) + C$, then A is equal to

- (a) $(1 - \tan^{-1} \theta)(1 - \tan \theta)$ (b) $(1 - \tan \theta) \tan^{-1}(1 - \tan \theta)$
 (c) $\tan^{-1}(1 - \tan \theta)$ (d) $\frac{1}{2} \tan^{-1}(1 - \tan \theta)$

51. If $\int \frac{x^2(x \sec^2 x + \tan x)}{(x \tan x + 1)^2} dx = \frac{-x^2}{x \tan x + 1} + A + C$,

then A is equal to

- (a) $2 \log |x \sin x + \sin x|$ (b) $2 \log |x \cos x + \cos x|$
 (c) $2 \log |x \cos x + \sin x|$ (d) $2 \log |x \sin x + \cos x|$

52. If $\int \left\{ \log(1 + \cos x) - x \tan \frac{x}{2} \right\} dx$ is equal to

- (a) $x \log \tan \frac{x}{2} + C$
 (b) $x \log(1 + \cos x) + C$
 (c) $x \tan \frac{x}{2} + C$
 (d) None of the above

53. $\int e^{x \sin x + \cos x} \left\{ \frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x} \right\} dx$ is

equal to

- (a) $x^2 e^{x \sin x + \cos x} + \frac{e^{x \sin x + \cos x}}{x \cos x} + C$
 (b) $e^{x \sin x + \cos x} + \frac{e^{x \sin x + \cos x}}{x \cos x} + C$
 (c) $x e^{x \sin x + \cos x} - \frac{e^{x \sin x + \cos x}}{x \cos x} + C$
 (d) $x^2 e^{x \sin x + \cos x} + \frac{1}{x \cos x} + C$

54. $\int \sin^{-1} \left\{ \frac{2x + 2}{\sqrt{4x^2 + 8x + 13}} \right\} dx$ is equal to

- (a) $(x + 1) \tan^{-1} \left(\frac{2x + 2}{3} \right) - \frac{3}{4} \log(4x^2 + 8x + 13) + C$
 (b) $\frac{3}{2} \tan^{-1} \left(\frac{2x + 2}{3} \right) - \frac{3}{4} \log(4x^2 + 8x + 13) + C$
 (c) $(x + 1) \tan^{-1} \left(\frac{2x + 2}{3} \right) - \frac{3}{2} \log(4x^2 + 8x + 13) + C$
 (d) $\frac{3}{2} (x + 1) \tan^{-1} \left(\frac{2x + 2}{3} \right) - \frac{3}{4} \log(4x^2 + 8x + 13) + C$

55. $\int e^x \frac{2 - x^2}{(1 - x)\sqrt{1 - x^2}} dx$ is equal to

- (a) $e^x \sqrt{\frac{1 - x}{1 + x}} \cdot \frac{1}{(1 - x)^2} + C$
 (b) $e^x \sqrt{\frac{1 + x}{1 - x}} + C$
 (c) $e^x \sqrt{\frac{1 - x}{1 + x}} + C$
 (d) None of the above

56. $\int x^3 \sqrt{\frac{1 + x^2}{1 - x^2}} dx$ is equal to

- (a) $-\frac{1}{2} \sqrt{1 - x^4} - \frac{1}{2} x^2 \sqrt{1 - x^4} + \frac{1}{4} \cos^{-1} x^2 + C$
 (b) $-\frac{1}{2} \sqrt{1 - x^4} - \frac{1}{2} x^2 \sqrt{1 - x^4} + \frac{1}{4} \sin^{-1} x^2 + C$
 (c) $-\frac{1}{2} \sqrt{1 - x^4} - \frac{1}{4} x^2 \sqrt{1 - x^4} + \frac{1}{4} \sin^{-1} x^2 + C$
 (d) $-\frac{1}{2} \sqrt{1 - x^4} - \frac{1}{2} x^2 \sqrt{1 - x^4} + \frac{1}{4} \tan^{-1} x^2 + C$

57. $\int \frac{1}{x^6 - a^6 + a^2 x^2 (x^2 - a^2)} dx$ is equal to

- (a) $\frac{1}{8a^5} \log \left| \frac{x - a}{x + a} \right| - \frac{1}{2a^5} \tan^{-1} \left(\frac{x}{a} \right) - \frac{x}{4a^4(x^2 + a^2)} + C$
 (b) $\frac{1}{4a^4} \log \left| \frac{x - a}{x + a} \right| - \frac{1}{8a^5} \tan^{-1} \left(\frac{x}{2a^2} \right) - \frac{x}{4a^4(x^2 + a^2)} + C$
 (c) $\frac{1}{2a^5} \log \left| \frac{x - a}{x + a} \right| - \frac{1}{4a^4} \tan^{-1} \left(\frac{x}{2a^2} \right) - \frac{x}{4a^4(x^2 + a^2)} + C$
 (d) None of the above

58. If $\int \frac{1}{x + \sqrt{x^2 - x + 1}} dx = P \log(x + \sqrt{x^2 - x + 1}) + Q \log\{(2x - 1) + 2\sqrt{x^2 - x + 1}\} - R \left\{ \frac{3}{(2x - 1) + 2\sqrt{x^2 - x + 1}} \right\}$. Then,

- (a) $P = -2$
 (b) $Q = -\frac{3}{2}$
 (c) $R = \frac{3}{4}$
 (d) None of the above

59. $\int \frac{1 + x \cos x}{x(1 - x^2 e^{2 \sin x})} dx$ is equal to

- (a) $\frac{1}{2} \log \left| \frac{x^2 e^{2 \sin x}}{1 - x^2 e^{2 \sin x}} \right| + C$
 (b) $\log \left| \frac{(x e^{\sin x})^2}{1 - x^2 e^{2 \sin x}} \right| + C$
 (c) $\frac{1}{2} \log \left| \frac{x e^{\sin x}}{1 - x^2 e^{2 \sin x}} \right| + C$
 (d) $\frac{1}{2} \log \left| \frac{x e^{\sin x}}{1 - x e^{\sin x}} \right| + C$

60. $\int \frac{2 - x^2}{(1 + x)\sqrt{1 - x^2}} dx$ is equal to

- (a) $\tan^{-1} x + \sqrt{1 - x^2} + \sqrt{\frac{1 - x}{1 + x}} + C$
 (b) $\sin^{-1} x + \sqrt{1 - x^2} - \sqrt{\frac{1 - x}{1 + x}} + C$
 (c) $\sin^{-1} x + \sqrt{1 + x^2} + \sqrt{\frac{1 - x}{1 + x}} + C$
 (d) $\tan^{-1} x + \sqrt{1 + x^2} + \sqrt{\frac{1 - x}{1 + x}} + C$

61. $\int \sqrt{\tan^2 x + 2} dx$ is equal to

- (a) $\log \left| \sqrt{\tan^2 x + 2} + \tan x \right| + \tan^{-1} \left(\frac{\tan x}{\sqrt{\tan^2 x + 2}} \right) + C$
 (b) $\log \left| \sqrt{\tan^2 x + 2} + \tan x \right| + \tan^{-1} \left(\frac{\sqrt{\tan^2 x + 2}}{\tan x} \right) + C$
 (c) $\log \left| \sqrt{\tan^2 x - 2} + \tan x \right| + \tan^{-1} \left(\frac{\sqrt{\tan^2 x - 2}}{\tan x} \right) + C$
 (d) $\log \left| \sqrt{\tan^2 x - 2} + \tan x \right| + \tan^{-1} \left(\frac{\sqrt{\tan^2 x + 2}}{\tan x} \right) + C$

62. $\int \frac{\sqrt{x}}{\sqrt{x} - \sqrt[3]{x}} dx$ is equal to

- (a) $6 \left\{ \frac{x}{6} + \frac{x^{6/5}}{5} + \frac{x^{1/2}}{2} + \frac{x^{1/3}}{3} + \log(x^{1/6} - 1) \right\} + C$
 (b) $6 \left\{ \frac{x}{6} + \frac{x^{6/5}}{5} + \frac{x^{1/2}}{3} + \frac{x^{1/3}}{2} + \log(x^{1/6} - 1) \right\} + C$
 (c) $6 \left\{ \frac{x}{6} + \frac{x^{5/6}}{5} + \frac{x^{2/3}}{4} + \frac{x^{1/2}}{3} + \frac{x^{1/3}}{2} + x^{1/6} + \log(x^{1/6} - 1) \right\} + C$
 (d) None of the above

63. $\int \frac{\sec x}{\sqrt{\sin(2x + \alpha)} + \sin \alpha} dx$ is equal to

- (a) $\sqrt{2} \sec \alpha (\tan x + \tan \alpha) + C$
 (b) $\sqrt{2} \sec \alpha (\tan x - \tan \alpha) + C$
 (c) $\sqrt{2} \sec \alpha (\tan \alpha - \tan x) + C$
 (d) None of the above

64. $\int \sqrt{x}(1 + \sqrt[3]{x})^4 dx$ is equal to

- (a) $\frac{24}{11} x^{3/2} + \frac{2}{3} x^{11/6} + \frac{8}{5} x^{13/6} + \frac{36}{13} x^{5/2} + \frac{6}{17} x^{17/6} + C$
 (b) $\frac{2}{3} x^{3/2} + \frac{24}{11} x^{11/6} + \frac{36}{13} x^{13/6} + \frac{8}{5} x^{5/2} + \frac{6}{17} x^{17/6} + C$
 (c) $\frac{2}{3} x^{3/2} + \frac{24}{11} x^{11/6} + \frac{8}{5} x^{13/6} + \frac{36}{13} x^{5/2} + \frac{6}{17} x^{17/6} + C$
 (d) $\frac{2}{3} x^{3/2} + \frac{24}{11} x^{11/6} + \frac{8}{5} x^{13/6} + \frac{36}{13} x^{5/2} + \frac{13}{15} x^{17/6} + C$

65. $\int x^5 \sqrt[3]{(1 + x^3)^2} dx$ is equal to

- (a) $\frac{1}{8} (1 + x^3)^{3/8} - \frac{1}{5} (1 + x^3)^{5/3} + C$
 (b) $\frac{1}{8} (1 + x^3)^{8/3} - \frac{1}{5} (1 + x^3)^{5/3} + C$
 (c) $\frac{1}{5} (1 + x^3)^{3/8} - \frac{1}{8} (1 + x^3)^{5/3} + C$
 (d) $\frac{1}{5} (1 + x^3)^{3/8} - \frac{1}{8} (1 + x^3)^{2/3} + C$

66. If $\int x^{13/2} (1 + x^{5/2})^{1/2} dx = \frac{4}{35} (1 + x^{5/2})^{7/2} - \frac{8}{25} (1 + x^{5/2})^{5/2} + A + C$.

Then, the value of A is equal to

- (a) $\frac{2}{9} (1 + x^{5/2})^{5/2}$ (b) $\frac{4}{15} (1 + x^{5/2})^{3/2}$
 (c) $\frac{2}{9} (1 + x^{7/2})^{3/2}$ (d) $\frac{4}{15} (1 + x^{7/2})^{3/2}$

Type 2. More than One Correct Option

67. If $\int \tan^5 x \, dx = A \tan^4 x + B \tan^2 x + g(x) + C$, then

- (a) $A = \frac{1}{4}, B = -\frac{1}{2}$
 (b) $g(x) = \ln |\sec x|$
 (c) $g(x) = \ln |\cos x|$
 (d) $A = -\frac{1}{4}, B = \frac{1}{3}$

68. The value of the integral

$$\int e^{\sin^2 x} (\cos x + \cos^3 x) \sin x \, dx \text{ is}$$

- (a) $\frac{1}{2} e^{\sin^2 x} (3 - \sin^2 x) + C$
 (b) $e^{\sin^2 x} \left(1 + \frac{1}{2} \cos^2 x\right) + C$
 (c) $e^{\sin^2 x} (3 \cos^2 x + 2 \sin^2 x) + C$
 (d) $e^{\sin^2 x} (2 \cos^2 x + 3 \sin^2 x) + C$

69. If $\int \operatorname{cosec} 2x \, dx = f\{g(x)\} + C$, then

- (a) range $g(x) = (-\infty, \infty)$
 (b) domain $f(x) = (-\infty, \infty) - \{0\}$
 (c) $g'(x) = \sec^2 x$
 (d) $f'(x) = \frac{1}{x}, \forall x \in (0, \infty)$

70. If $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} \, dx = Ax + B \log_e (9e^{2x} - 4) + C$,

then

- (a) $A = \frac{3}{2}$
 (b) $B = \frac{35}{36}$
 (c) C is indefinite
 (d) $A + B = -\frac{19}{36}$

71. If $\int \sqrt{\operatorname{cosec} x + 1} \, dx = k \log (x) + C$, where k is real constant, then

- (a) $k = -2, f(x) = \cot^{-1} x, g(x) = \sqrt{\operatorname{cosec} x - 1}$
 (b) $k = -2, f(x) = \tan^{-1} x, g(x) = \sqrt{\operatorname{cosec} x - 1}$
 (c) $k = 2, f(x) = \tan^{-1} x, g(x) = \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}}$
 (d) $k = 2, f(x) = \cot^{-1} x, g(x) = \frac{\cot x}{\sqrt{\operatorname{cosec} x + 1}}$

72. If $\int \sin x \, d(\sec x) = f(x) - g(x) + C$, then

- (a) $f(x) = \sec x$
 (b) $f(x) = \tan x$
 (c) $g(x) = 2x$
 (d) $g(x) = x$

73. If a curve $g(x) = \int x^{27} (1+x+x^2)^6 (6x^2 + 5x + 4) \, dx$ is passing through origin, then

- (a) $g(1) = \frac{3^7}{7}$ (b) $g(1) = \frac{2^7}{7}$
 (c) $g(-1) = \frac{1}{7}$ (d) $g(-1) = \frac{3^7}{14}$

74. If $I = \int \frac{\sin x + \sin^3 x}{\cos 2x} \, dx = P \cos x + Q \log |f(x)| + R$, then

- (a) $P = \frac{1}{2}, Q = -\frac{3}{4\sqrt{2}}$
 (b) $P = \frac{1}{4}, Q = -\frac{1}{\sqrt{2}}$
 (c) $f(x) = \frac{\sqrt{2} \cos x + 2}{\sqrt{2} \cos x - 1}$
 (d) $f(x) = \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1}$

Type 3. Assertion and Reason

Directions (Q. Nos. 75-82) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b) (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

75. **Statement I**

$$\int \frac{x^{9/2}}{\sqrt{1+x^{11}}} \, dx = \frac{2}{11} \ln |x^{11/2} + \sqrt{1-x^{11}}| + C$$

Statement II $\int \frac{dx}{\sqrt{1+x^2}} = \ln |x + \sqrt{1+x^2}| + C$

76. **Statement I** If $x > 0, x \neq 1$, then $\int \{\log_x e - (\log_x e)^2\} \, dx = x \log_x e + C$.

Statement II $\int e^x \{f(x) + f'(x)\} \, dx = e^x f(x) + C$ and $e^y = x$ iff $y = \ln x$.

77. **Statement I** If $y = \sin^{-1} x$, then

$$\int \sin^{-1} x \, dx = \int y \cos y \, dy + C$$

Statement II If $y = f^{-1}(x)$, then

$$\int f^{-1}(x) \, dx = \int y f'(y) \, dy + C$$

78. **Statement I** $\int 2^{\tan^{-1} x} d(\cot^{-1} x) = \frac{2^{\tan^{-1} x}}{\ln 2} + C$

Statement II $\frac{d}{dx} (a^x + C) = a^x \ln a$

79. Statement I Integral of an even function is not always an odd function.

Statement II Integral of an odd function is an even function.

80. Statement I For $-1 < a < 4$, $\int \frac{dx}{x^2 + 2(a-1)x + a + 5} = \lambda \log |g(x)| + C$, where λ and C are constants.

Statements II For $-1 < a < 4$, $\frac{1}{x^2 + 2(a-1)x + a + 5}$ is continuous function

81. Statement I $\int \frac{\sin x \, dx}{x} (x > 0)$ cannot be evaluated.

Statement II Only differentiable functions can be integrated.

82. Statement I If the primitive of $f(x) = \pi \sin \pi x + 2x - 4$ has the value -2 for $x = 1$, then there are exactly two values of x for which primitive of $f(x)$ vanishes.

Statement II $\cos \pi x$ has period 2.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 83-85) Let us consider the integral of the following forms

$$\int f(x, \sqrt{mx^2 + nx + p}) \, dx$$

Case I If $m > 0$, then put

$$\sqrt{mx^2 + nx + p} = u \pm x\sqrt{m}$$

Case II If $p > 0$, then put $\sqrt{mx^2 + nx + p} = ux \pm \sqrt{p}$

Case III If quadratic equation $mx^2 + nx + p = 0$ has real roots α and β , then put

$$\sqrt{mx^2 + nx + p} = (x - \alpha)u \text{ or } (x - \beta)u$$

83. If $I = \int \frac{dx}{x - \sqrt{9x^2 + 4x + 6}}$, then to evaluate I , one of

the most proper substitution could be

- (a) $\sqrt{9x^2 + 4x + 6} = u \pm 3x$ (b) $\sqrt{9x^2 + 4x + 6} = 3u \pm x$
(c) $x = \frac{1}{t}$ (d) $9x^2 + 4x + 6 = \frac{1}{t}$

84. $\int \frac{(x + \sqrt{1+x^2})^{15}}{\sqrt{1+x^2}} \, dx$ is equal to

- (a) $\frac{(x + \sqrt{1+x^2})^{16}}{10} + C$ (b) $\frac{1}{15(\sqrt{1+x^2} + x)} + C$
(c) $\frac{15}{(\sqrt{1+x^2} - x)} + C$ (d) $\frac{(x + \sqrt{1+x^2})^{15}}{15} + C$

85. To evaluate $\int \frac{dx}{(x-1)\sqrt{-x^2 + 3x - 2}}$, one of the

most suitable substitution could be

- (a) $\sqrt{-x^2 + 3x - 2} = u$ (b) $\sqrt{-x^2 + 3x - 2} = (ux - \sqrt{2})$
(c) $\sqrt{-x^2 + 3x - 2} = u(1-x)$ (d) $\sqrt{-x^2 + 3x - 2} = u(x-2)$

Passage II (Q. Nos. 86-88) For integral $\int f\left(x - \frac{a}{x}\right) \cdot \left(1 + \frac{a}{x^2}\right) dx$, put $x - \frac{a}{x} = t$.

For integral $\int f\left(x + \frac{a}{x}\right) \cdot \left(1 - \frac{a}{x^2}\right) dx$, put $x + \frac{a}{x} = t$

For integral $\int f\left(x^2 - \frac{a}{x^2}\right) \cdot \left(x + \frac{a}{x^3}\right) dx$, put $x^2 - \frac{a}{x^2} = t$

For integral $\int f\left(x^2 + \frac{a}{x^2}\right) \cdot \left(x - \frac{a}{x^3}\right) dx$, put $x^2 + \frac{a}{x^2} = t$

Many integrands can be brought into above forms by suitable reductions or transformations.

86. $\int \frac{x^4 - 2}{x^2 \sqrt{x^4 + x^2 + 2}} \, dx$ equals

- (a) $\sqrt{x^2 + 1 + \frac{1}{x^2}} + C$ (b) $\sqrt{x^2 + 1 + \frac{2}{x^2}} + C$
(c) $\sqrt{x^2 + \frac{1}{x^2}} + C$ (d) $\sqrt{x^2 + \frac{2}{x^2}} + C$

87. $\int \frac{(x-1)}{(x+1)\sqrt{x^3 + x^2 + x}} \, dx$ equals

- (a) $\tan^{-1}\left(x + \frac{1}{x} + 1\right) + C$ (b) $\tan^{-1}\left(\sqrt{x + \frac{1}{x} + 1}\right) + C$
(c) $2\tan^{-1}\left(\sqrt{x + \frac{1}{x} + 1}\right) + C$ (d) None of these

88. $\int \frac{5x^4 + 4x^5}{(x^5 + x + 1)^2} \, dx$ equals

- (a) $x^5 + x + 1 + C$
(b) $\frac{1}{x^5 + x + 1} + C$
(c) $x^{-4} + x^{-5} + C$
(d) $\frac{x^5}{x^5 + x + 1} + C$

Passage III (Q. Nos. 89-91) In some of the cases, we can split the integrand into the sum of the two functions such that the integration of one of them by parts produces an integral which cancels the other integral. Suppose we have an integral of the type $\int [f(x)h(x) + g(x)] \, dx$.

Let $\int f(x)h(x) dx = I_1$ and $\int g(x) dx = I_2$

Integrating I_1 by parts, we get

$$I_1 = f(x) \int h(x) dx - \int \{f'(x) \int h(x) dx\} dx$$

Suppose $\int \{f'(x) \int h(x) dx\} dx$ converts to I_2 , then we

get $I_1 + I_2 = f(x) \int h(x) dx + C$, which is the desired

integral. In particular, consider the integral of the kind

$$I = \int e^x \{f(x) + f'(x)\} dx = \int e^x f(x) dx + \int e^x f'(x) dx$$

Integrating first integral by parts, we get

$$I = e^x f(x) - \int e^x f'(x) dx + \int e^x f'(x) dx = e^x f(x) + C$$

89. $\int \left(\frac{1}{\log x} - \frac{1}{(\log x)^2} \right) dx$ ($x > 0$)

- (a) $\log(\log x) + C$ (b) $x \log x + C$
(c) $\frac{x}{\log x} + C$ (d) $x + \log x + C$

90. $\int \frac{xe^x}{(1+x)^2} dx$ is equal to

- (a) $xe^x + C$ (b) $\frac{e^x}{(x+1)^2} + C$
(c) $e^x - \frac{1}{x+1} + C$ (d) $\frac{e^x}{x+1} + C$

91. The integral of $\int e^{\sin x} (x \cos x - \sec x \tan x) dx$ is

- (a) $xe^{\sin x} - e^{\sin x} \cdot \sec x + C$ (b) $(x + \sec x) e^{\sin x} + C$
(c) $e^{\sin x} \cos x + C$ (d) $e^{\sin x} (\cos x - \sec x) + C$

Passage IV (Q. Nos. 92-94) Integrals of the form

$\int R(x, \sqrt{ax^2 + bx + c}) dx$ are calculated with the aid of

one of the three Euler substitutions

1. $\sqrt{ax^2 + bx + c} = t \pm x\sqrt{a}$, if $a > 0$

2. $\sqrt{ax^2 + bx + c} = tx \pm \sqrt{c}$, if $c > 0$

3. $\sqrt{ax^2 + bx + c} = (x - \alpha)t$, if

$$ax^2 + bx + c = a(x - \alpha)(x - \beta)$$

i.e. if α and β are real roots of $ax^2 + bx + c = 0$.

Type 5. Match the Columns

98. Match the statements of Column I with the values of Column II.

Column I	Column II
A. If $\int \frac{e^{2x} - 2e^x}{e^{2x} + 1} dx = A \ln(e^{2x} + 1) + B \tan^{-1}(e^x) + C$, then	p. $A = -\frac{1}{2}, B = -\frac{1}{4}$
B. If $\int \sqrt{x + \sqrt{x^2 + 2}} dx = A \{x + \sqrt{x^2 + 2}\}^{3/2} + \frac{B}{\sqrt{x + \sqrt{x^2 + 2}}} + C$, then	q. $A = \frac{1}{2}, B = -2$
C. If $\int \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} dx = A \sin 3x + B \sin 2x + C$, then	r. $A = \frac{1}{3}, B = -2$

92. Which of the following functions does not appear in the primitive of $\frac{1}{1 + \sqrt{x^2 + 2x + 2}}$, if t is a function

of x ?

- (a) $\log_e |t + 1|$ (b) $\log_e |t + 2|$
(c) $\frac{1}{t + 2}$ (d) None of these

93. Which of the following functions does not appear in the primitive of $\frac{dx}{x + \sqrt{x^2 - x + 1}}$, if t is a function

of x ?

- (a) $\log_e |t|$ (b) $\log_e |t - 2|$
(c) $\log_e |t - 1|$ (d) $\log_e |t + 1|$

94. $\int \frac{xdx}{(\sqrt{7x - 10 - x^2})^3}$ can be evaluated by

substituting for x as

- (a) $x = \frac{5 + 2t^2}{t^2 + 1}$ (b) $x = \frac{5 - t^2}{t^2 + 2}$
(c) $x = \frac{2t^2 - 5}{3t^2 - 1}$ (d) None of these

Passage V (Q. Nos. 95-97) $y = f(x)$ is a polynomial function passing through point $(0, 1)$ and which increases in the intervals $(1, 2)$ and $(3, \infty)$ and decreases in the intervals $(-\infty, 1)$ and $(2, 3)$.

95. If $f(1) = -8$, then the value of $f(2)$ is

- (a) -3 (b) -6 (c) -20 (d) -7

96. If $f(3) = -8$, then the range of $f(x)$ is

- (a) $[3, \infty)$ (b) $[-8, \infty)$ (c) $[-7, \infty)$ (d) $(-\infty, 6]$

97. If $f(x) = 0$ has four real roots, then the range of values of leading coefficient of polynomial is

- (a) $[4/9, 1/2]$ (b) $[4/9, 1]$
(c) $[1/3, 1/2]$ (d) None of these

Column I	Column II
D. If $\int \frac{\ln x}{x^3} dx = A \frac{\ln x}{x^2} + \frac{B}{x^2} + C$, then	s. $A = \frac{1}{3}, B = -\frac{1}{2}$
99. Let a, b, c be positive integers, match the following.	
Column I	Column II
A. The value of $\int \frac{c}{(x-a)(x-b)} dx$ is/are	p. $\frac{c}{a-b} \log \left \frac{x-a}{x-b} \right + K$
B. The value of $\int \frac{x-c}{(x-a)(x-b)} dx$ is/are	q. $\frac{c}{a} \log ax - b + \frac{1}{b} \log \left a - \frac{b}{x} \right + K$

Column I	Column II
C. The value of $\int \frac{cx+1}{x(ax-b)} dx$ is/are	r. $xc + (a+bc)\log x-b + K$
D. The value of $\int \frac{cx+a}{x-b} dx$ is/are	s. $\frac{1}{a-b}\{(a-c)\log x-a - (b-c)\log x-b \} + K$

100. Match the following:

Column I	Column II
A. If $\int \left(\frac{x^2 + \cos^2 x}{1+x^2} \right) \operatorname{cosec}^2 x dx$ = $A \cot^{-1} x + B \frac{\operatorname{cosec} x}{\sec x}$, then	p. $A = 1$
B. If $\int \frac{\cos 5x + \cos 4x}{1-2\cos 3x} dx$ = $\frac{A}{2} \sin 2x + B \sin x + C$, then	q. $B = -1$
C. If $\int \frac{\sqrt{2-x-x^2}}{x^2} dx = A \frac{\sqrt{2-x-x^2}}{x} + \frac{B}{4\sqrt{2}}$ $\log \left \frac{(4-x) + 4\sqrt{2-x-x^2}}{x} \right $ $-\sin^{-1} \left(\frac{2x+1}{3} \right)$, then	r. $B = 2$
D. If $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = B \cot^{-1}(\tan^2 x)$, then	s. $A = -1$

101. Which of the following functions given in Column II appear in integration of function in Column I.

Column I	Column II
A. $\int \frac{x^2 - x + 1}{x^3 - 4x^2 + 4x} dx$	p. $\log x $
B. $\int \frac{x^2 - 1}{x(x-2)^3} dx$	q. $\log x-2 $
C. $\int \frac{x^3 + 1}{x(x-2)^2} dx$	r. $\frac{1}{(x-2)}$
D. $\int \frac{x^5 + 1}{x(x-2)^3} dx$	s. x

102. Match the following:

Column I	Column II
A. If $\int \frac{2^x}{\sqrt{1-4^x}} dx = k \sin^{-1}(f(x)) + C$, then k is greater than	p. 0
B. If $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6} dx = a \ln \frac{x^k}{x^k + 1} + C$, then ak is less than	q. 1
C. $\int \frac{x^4 + 1}{x(x^2 + 1)^2} dx = k \ln x + \frac{m}{1+x^2} + n$, where n is the constant of integration, then mk is greater than	r. 3
D. $\int \frac{dx}{5 + 4\cos x} = k \tan^{-1} \left(m \tan \frac{x}{2} \right) + C$, then k/m is greater than	s. 4

Type 6. Single Integer Answer Type Questions

103. Let $f(x) = \int x^{\sin x} (1 + x \cos x \cdot \ln x + \sin x) dx$ and $f\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$, then the value of $|\cos\{f(\pi)\}|$ is ____.

104. Suppose $\begin{vmatrix} f'(x) & f(x) \\ f''(x) & f'(x) \end{vmatrix} = 0$, where $f(x)$ is continuously differentiable function with $f'(x) \neq 0$ and satisfies $f(0) = 1$ and $f'(0) = 2$, where $f(x) = e^{\lambda x} + k$, then $(\lambda + k)$ is equal to ____.

105. Let $F(x)$ be the primitive of $\frac{3x+2}{\sqrt{x-9}}$ w.r.t. x . If

$F(10) = 60$, then the sum of digits of the value of $F(13)$, is ____.

106. If $\int \frac{\cos x - \sin x + 1 - x}{e^x + \sin x + x} dx = \ln\{f(x)\} + g(x) + C$, where C is the constant of integration and $f(x)$ is positive, then $\frac{f(x) + g(x)}{e^x + \sin x}$ is equal to ____.

107. $\int \sin(101x) \cdot \sin^{99} x dx = \frac{\sin(100x)(\sin x)^\lambda}{\mu}$, then $\frac{\lambda}{\mu}$ is equal to ____.

108. If $\int \frac{2\cos x - \sin x + \lambda}{\cos x + \sin x - 2} dx = A \ln|\cos x + \sin x - 2| + Bx + C$. Then the value of $A + B + |\lambda|$ is ____.

109. Let $k(x) = \int \frac{(x^2 + 1)dx}{\sqrt[3]{x^3 + 3x + 6}}$ and $k(-1) = \frac{1}{\sqrt[3]{2}}$, then the value of $k(-2)$ is ____.

110. If $\int \left[\left(\frac{x}{e} \right)^x + \left(\frac{e}{x} \right)^x \right] \ln x dx = A \left(\frac{x}{e} \right)^x + B \left(\frac{e}{x} \right)^x + C$, then the value of $(A + B)$ is ____.

111. If $f(x) = \sqrt{x}$, $g(x) = e^x - 1$ and $\int fog(x) dx = A fog(x) + B \tan^{-1}\{fog(x)\} + C$, then $(A + B)$ is equal to ____.

Entrances Gallery

JEE Advanced/IIT JEE

1. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K) [2012]

(a) $\frac{-1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$
 (b) $\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} - \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$
 (c) $\frac{-1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$
 (d) $\frac{1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

2. If $f\left(\frac{3x-4}{3x+4}\right) = x+2$, then $\int f(x) dx$ is equal to [2010]

(a) $e^{x+2} \log \left| \frac{3x-4}{3x+4} \right| + C$
 (b) $-\frac{8}{3} \log |1-x| + \frac{2}{3}x + C$
 (c) $\frac{8}{3} \log |1-x| + \frac{x}{3} + C$
 (d) $e^{[(3x-4)/(3x+4)]} - \frac{x^2}{2} - 2x + C$

JEE Main/AIEEE

3. The integral $\int \frac{dx}{x^2 (x^4 + 1)^{3/4}}$ equals [2014]

(a) $\left(\frac{x^4 + 1}{x^4} \right)^{1/4} + C$ (b) $(x^4 + 1)^{1/4} + C$
 (c) $-(x^4 + 1)^{1/4} + C$ (d) $-\left(\frac{x^4 + 1}{x^4} \right)^{1/4} + C$

4. The integral $\int \left(1+x - \frac{1}{x} \right) e^{x + \frac{1}{x}} dx$ is equal to [2014]

(a) $(x-1)e^{x + \frac{1}{x}} + C$ (b) $x e^{x + \frac{1}{x}} + C$
 (c) $(x+1)e^{x + \frac{1}{x}} + C$ (d) $-x e^{x + \frac{1}{x}} + C$

5. If $\int \frac{dx}{x+x^7} = p(x)$, then $\int \frac{x^6}{x+x^7} dx$ is equal to [2013]

(a) $\log |x| - p(x) + C$
 (b) $\log |x| + p(x) + C$
 (c) $x - p(x) + C$
 (d) $x + p(x) + C$

6. If $\int \frac{x^2 - x + 1}{x^2 + 1} e^{\cot^{-1} x} dx = A e^{\cot^{-1} x} + C$, then A is equal to [2013]

(a) $\frac{-x}{\sqrt{1-x}}$ (b) $\frac{x}{\sqrt{1+x}}$
 (c) $\sqrt{1-x}$ (d) $\sqrt{1+x}$

7. The integral $\int \frac{x dx}{2-x^2 + \sqrt{2-x^2}}$ is equal to [2013]

(a) $\log |1 + \sqrt{2+x^2}| + C$
 (b) $-\log |1 + \sqrt{2-x^2}| + C$
 (c) $-x \log |1 - \sqrt{2-x^2}| + C$
 (d) $x \log |1 - \sqrt{2+x^2}| + C$

8. If the integral $\int \frac{\cos 8x + 1}{\cot 2x - \tan 2x} dx = A \cos 8x + k$, where k is an arbitrary constant, then A is equal to [2013]

(a) $-\frac{1}{16}$ (b) $\frac{1}{16}$ (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

9. If $\int f(x) dx = \psi(x)$, then $\int x^5 f(x^3) dx$ is equal to [2013]

(a) $\frac{1}{3} [x^3 \psi(x^3) - \int x^2 \psi(x^3) dx] + C$
 (b) $\frac{1}{3} [x^3 \psi(x^3)] - 3 \int x^3 \psi(x^3) dx + C$
 (c) $\frac{1}{3} [x^3 \psi(x^3)] - \int x^2 \psi(x^3) dx + C$
 (d) $\frac{1}{3} [x^3 \psi(x^3) - \int x^3 \psi(x^3) dx] + C$

10. If the integral $\int \frac{5 \tan x}{\tan x - 2} dx = x + a \ln |\sin x - 2 \cos x| + K$, then a is equal to [2012]

(a) -1 (b) -2 (c) 1 (d) 2

11. The value of $\sqrt{2} \int \frac{\sin x dx}{\sin \left(x - \frac{\pi}{4} \right)}$ is [2008]

(a) $x - \log \left| \cos \left(x - \frac{\pi}{4} \right) \right| + C$
 (b) $x + \log \left| \cos \left(x - \frac{\pi}{4} \right) \right| + C$
 (c) $x - \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + C$
 (d) $x + \log \left| \sin \left(x - \frac{\pi}{4} \right) \right| + C$

12. $\int \frac{dx}{\cos x + \sqrt{3} \sin x}$ equals [2007]

- (a) $\frac{1}{2} \log \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) + C$ (b) $\frac{1}{2} \log \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) + C$
(c) $\log \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) + C$ (d) $\log \tan \left(\frac{x}{2} - \frac{\pi}{12} \right) + C$

13. $\int \left\{ \frac{(\log x - 1)}{1 + (\log x)^2} \right\}^2 dx$ is equal to [2005]

- (a) $\frac{x}{(\log x)^2 + 1} + C$ (b) $\frac{xe^x}{1 + x^2} + C$
(c) $\frac{x}{x^2 + 1} + C$ (d) $\frac{\log x}{(\log x)^2 + 1} + C$

14. If $\int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B \log \sin(x - \alpha) + C$, then

- the value of (A, B) is [2004]
(a) $(\sin \alpha, \cos \alpha)$
(b) $(\cos \alpha, \sin \alpha)$
(c) $(-\sin \alpha, \cos \alpha)$
(d) $(-\cos \alpha, \sin \alpha)$

15. $\int \frac{dx}{\cos x - \sin x}$ is equal to [2004]

- (a) $\frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{\pi}{8} \right) \right| + C$
(b) $\frac{1}{\sqrt{2}} \log \left| \cot \left(\frac{x}{2} \right) \right| + C$
(c) $\frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} - \frac{3\pi}{8} \right) \right| + C$
(d) $\frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + C$

16. $\int \frac{dx}{x(x^n + 1)}$ is equal to [2002]

- (a) $\frac{1}{n} \log \left(\frac{x^n}{x^n + 1} \right) + C$
(b) $\frac{1}{n} \log \left(\frac{x^n + 1}{x^n} \right) + C$
(c) $\log \left(\frac{x^n}{x^n + 1} \right) + C$
(d) None of the above

Other Engineering Entrances

17. $\int \frac{dx}{\sin x - \cos x + \sqrt{2}}$ is equal to [Manipal 2014]

- (a) $-\frac{1}{\sqrt{2}} \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) + C$ (b) $\frac{1}{\sqrt{2}} \tan \left(\frac{x}{2} + \frac{\pi}{8} \right) + C$
(c) $\frac{1}{\sqrt{2}} \cot \left(\frac{x}{2} + \frac{\pi}{8} \right) + C$ (d) $-\frac{1}{\sqrt{2}} \cot \left(\frac{x}{2} + \frac{\pi}{8} \right) + C$

18. $\int \sqrt{1 + \cos x} dx$ is equal to [Kerala CEE 2014]

- (a) $2 \sin \left(\frac{x}{2} \right) + C$ (b) $\sqrt{2} \sin \left(\frac{x}{2} \right) + C$
(c) $\frac{1}{2} \sin \left(\frac{x}{2} \right) + C$ (d) $\frac{\sqrt{2}}{2} \sin \left(\frac{x}{2} \right) + C$
(e) $2\sqrt{2} \sin \left(\frac{x}{2} \right) + C$

19. If $\int \frac{dx}{(1 + \sqrt{x})(\sqrt{x} - x^2)} = \frac{A\sqrt{x}}{\sqrt{1-x}} + \frac{B}{\sqrt{1-x}} + C$,

where C is real constant, then $A + B$ equals

- [EAMCET 2014]
(a) 3 (b) 0 (c) 1 (d) 2

20. If $\int \frac{(x^2 - 1)}{(x+1)^2 \sqrt{x(x^2 + x + 1)}} dx$

$= A \tan^{-1} \left(\sqrt{\frac{x^2 + x + 1}{x}} \right) + C$, where C is a constant,

- then A equals [EAMCET 2014]
(a) $1/2$ (b) 3
(c) 2 (d) 1

21. $\int a^x dx$ is equal to [GGSIU 2014]

- (a) $\frac{a^x}{x \log_e a} + C$
(b) $a^x \log_e a + C$
(c) $\frac{a^x}{\log_e a} + C$
(d) $\frac{xa^x}{\log_e a} + C$

22. The value of $\int \frac{\cos 2x - 1}{\cos 2x + 1} dx$ is

- [RPET 2014]
(a) $\tan x - x + C$
(b) $\tan x + x + C$
(c) $x - \tan x + C$
(d) $-x - \cot x + C$

23. $\int \frac{1}{\sqrt{7-x^2}} dx$ is equal to [J&K CET 2014]

- (a) $\frac{1}{2\sqrt{7}} \log \left| \frac{\sqrt{7+x}}{\sqrt{7-x}} \right| + C$ (b) $\sin^{-1} \left(\frac{x}{\sqrt{7}} \right) + C$
(c) $\log |x + \sqrt{x^2 - 7}| + C$ (d) $\frac{1}{2\sqrt{7}} \log \left| \frac{x - \sqrt{7}}{x + \sqrt{7}} \right| + C$

24. $\int \sin^{-1} \cos 3x dx$ is equal to [J&K CET 2014]

- (a) $\frac{3\pi}{2} x - \frac{x^2}{2} + C$ (b) $\frac{\pi}{2} x - \frac{3x^2}{2} + C$
(c) $\frac{3\pi}{2} x + \frac{x^2}{2} + C$ (d) $\frac{\pi}{2} x - \frac{x^3}{3} + C$

25. $\int \frac{x^2 - 1}{(x^4 + 3x^2 + 1) \tan^{-1}\left(x + \frac{1}{x}\right)} dx$ is equal to [Manipal 2014]

- (a) $\tan^{-1}\left(x + \frac{1}{x}\right) + C$
 (b) $\cot^{-1}\left(x + \frac{1}{x}\right) + C$
 (c) $\log\left(x + \frac{1}{x}\right) + C$
 (d) $\log\left[\tan^{-1}\left(x + \frac{1}{x}\right)\right] + C$

26. $\int \frac{\cos x}{\sin^2 x (\sin x + \cos x)} dx$ is equal to [Manipal 2014]

- (a) $\log\left|\frac{1 + \tan x}{\tan x}\right| - \cot x + C$
 (b) $\log\left|\frac{\tan x}{1 + \tan x}\right| + C$
 (c) $\log\left|\frac{\tan x}{1 + \tan x}\right| - \tan x + C$
 (d) $\log\left|\frac{\tan x}{1 + \tan x}\right| + \cot x + C$

27. $\int \frac{\sin 2x}{\sin^2 x + 2\cos^2 x} dx$ is equal to [Karnataka CET 2014]

- (a) $-\log(1 + \sin^2 x) + C$ (b) $\log(1 + \cos^2 x) + C$
 (c) $-\log(1 + \cos^2 x) + C$ (d) $\log(1 + \tan^2 x) + C$

28. If linear function $f(x)$ and $g(x)$ satisfy $\int [(3x - 1)\cos x + (1 - 2x)\sin x] dx = f(x)\cos x + g(x)\sin x + C$, then [Karnataka CET 2014]

- (a) $f(x) = 3(x - 1)$ (b) $f(x) = 3x - 5$
 (c) $g(x) = 3(x - 1)$ (d) $g(x) = 3 + x$

29. $\int \frac{(1+x)e^x}{\sin^2(xe^x)} dx$ is equal to [Kerala CEE 2014]

- (a) $-\cot(e^x) + C$ (b) $\tan(xe^x) + C$
 (c) $\tan(e^x) + C$ (d) $\cot(xe^x) + C$
 (e) $-\cot(xe^x) + C$

30. $\int \frac{xe^x}{(1+x)^2} dx$ is equal to [Kerala CEE 2014]

- (a) $\frac{-e^x}{x+1} + C$ (b) $\frac{e^x}{x+1} + C$ (c) $\frac{xe^x}{x+1} + C$
 (d) $\frac{-xe^x}{x+1} + C$ (e) $\frac{e^x}{(x+1)^2} + C$

31. $\int e^x (\sin x + 2\cos x) \sin x dx$ is equal to [Kerala CEE 2014]

- (a) $e^x \cos x + C$
 (b) $e^x \sin x + C$
 (c) $e^x \sin^2 x + C$
 (d) $e^x \sin 2x + C$
 (e) $e^x (\cos x + \sin x) + C$

32. $\int \frac{\sqrt{x^2 - 1}}{x} dx$ is equal to [Kerala CEE 2014]

- (a) $\sqrt{x^2 - 1} - \sec^{-1} x + C$ (b) $\sqrt{x^2 - 1} + \tan^{-1} x + C$
 (c) $\sqrt{x^2 - 1} + \sec^{-1} x + C$ (d) $\sqrt{x^2 - 1} - \tan x + C$
 (e) $\sqrt{x^2 - 1} + \sec x + C$

33. $\int \frac{\sqrt{5+x^2}}{x^4} dx$ is equal to [Kerala CEE 2014]

- (a) $\frac{1}{15} \left(1 + \frac{5}{x^2}\right)^{3/2} + C$ (b) $\frac{-1}{15} \left(1 + \frac{1}{x^2}\right)^{3/2} + C$
 (c) $\frac{-1}{15} \left(1 + \frac{5}{x^2}\right)^{3/2} + C$ (d) $\frac{1}{15} \left(1 + \frac{1}{x^2}\right)^{3/2} + C$
 (e) $\frac{-1}{10} \left(1 + \frac{1}{x^2}\right)^{3/2} + C$

34. If $\int \frac{dx}{\sqrt{\sin^3 x \cos x}} = g(x) + C$, then $g(x)$ is equal to [EAMCET 2014]

- (a) $\frac{-2}{\sqrt{\cot x}}$ (b) $\frac{-2}{\sqrt{\tan x}}$
 (c) $\frac{2}{\sqrt{\cot x}}$ (d) $\frac{2}{\sqrt{\tan x}}$

35. For any integer $n \geq 2$, let $I_n = \int \tan^n x dx$. If $I_n = \frac{1}{a} \tan^{n-1} x - b I_{n-2}$ for $n \geq 2$, then the ordered pair (a, b) equals [EAMCET 2014]

- (a) $\left(n-1, \frac{n-1}{n-2}\right)$ (b) $\left(n-1, \frac{n-2}{n-1}\right)$
 (c) $(n, 1)$ (d) $(n-1, 1)$

36. $\int \frac{xdx}{(1+x)^{3/2}}$ is equal to [GGSIU 2014]

- (a) $2 \left(\frac{2+x}{\sqrt{1+x}} \right) + C$ (b) $\frac{(2+x)}{\sqrt{1+x}} + C$
 (c) $\frac{3}{2} \cdot \frac{x}{\sqrt{1-x}} + C$ (d) $\frac{3}{2} \cdot \frac{(2+x)}{\sqrt{1+x}} + C$

37. $\int \frac{\sec^2(\sin^{-1} x)}{\sqrt{1-x^2}} dx$ is equal to [J&K CET 2014]

- (a) $\sin(\tan^{-1} x) + C$
 (b) $\tan(\sec^{-1} x) + C$
 (c) $\tan(\sin^{-1} x) + C$
 (d) $-\tan(\cos^{-1} x) + C$

38. $\int \sec^3 x dx$ is equal to [J&K CET 2014]

- (a) $\frac{1}{2} [\sec x \tan x - \log |\sec x + \tan x|] + C$
 (b) $\frac{1}{2} [\sec x \tan x + \log |\sec x - \tan x|] + C$
 (c) $\frac{1}{2} [\sec x \tan x - \log |\sec x - \tan x|] + C$
 (d) $\frac{1}{2} [\sec x \tan x + \log |\sec x + \tan x|] + C$

39. $\int \frac{1}{x^3(x^3-1)} dx$ is equal to [J&K CET 2014]

- (a) $\frac{1}{3} \log \left| \frac{x^3}{x^3-1} \right| + C$
 (b) $\frac{1}{3} \log \left| \frac{1-x^3}{x^3} \right| + C$
 (c) $\log \left| \frac{x^3}{x^3-1} \right| + C$
 (d) $\frac{1}{3} \log |x-1| + \frac{1}{2x^2} - \frac{1}{6} \log |x^2+x+1| - \frac{1}{\sqrt{6}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$

40. If $I_1 = \int \sin^{-1} x dx$ and $I_2 = \int \sin^{-1} \sqrt{1-x^2} dx$, then [Kerala CEE 2013]

- (a) $I_1 = I_2$ (b) $I_2 = \frac{\pi}{2} I_1$
 (c) $I_1 + I_2 = \frac{\pi}{2} x + C$ (d) $I_1 + I_2 = \frac{\pi}{2}$
 (e) None of these

41. $\int \frac{(x^2-1)}{x^3 \sqrt{2x^4-2x^2+1}} dx$ is equal to [AMU 2013]

- (a) $-\frac{1}{2} \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + C$ (b) $\frac{1}{2} \log (2x^4 - 2x^2 + 1) + C$
 (c) $\frac{1}{2} \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + C$ (d) None of these

42. If $f(x)$ and $g(x)$ are twice differentiable function on $(0, 3)$ satisfying $f''(x) = g''(x)$, $f'(1) = 4$, $g'(1) = 6$, $f(2) = 3$, $g(2) = 9$, then $f(1) - g(1)$ is equal to [WB JEE 2012]

- (a) 4 (b) -4
 (c) 0 (d) -2

43. $\int \frac{2dx}{(e^x + e^{-x})^2}$ is equal to [BITSAT 2012]

- (a) $-\frac{e^{-x}}{(e^x + e^{-x})} + C$ (b) $-\frac{1}{(e^x + e^{-x})} + C$
 (c) $\frac{1}{(e^x + 1)^2} + C$ (d) $\frac{1}{(e^x - e^{-x})^2} + C$

44. $\int e^x \cdot \frac{x^2+1}{(x+1)^2} dx$ is equal to [MP PET 2012]

- (a) $-\frac{e^x}{x+1} + C$ (b) $\frac{e^x}{(x+1)} + C$
 (c) $\frac{xe^x}{x+1} + C$ (d) $e^x \left(\frac{x-1}{x+1} \right) + C$

45. If $n \in N$ and $I_n = \int (\log x)^n dx$, then $I_n + nI_{n-1}$ is equal to [OJEE 2012]

- (a) $\frac{(\log x)^{n+1}}{n+1}$ (b) $x(\log x)^n + C$
 (c) $(\log x)^{n-1}$ (d) $\frac{(\log x)^n}{n}$

46. If $f(x) = \int \frac{dx}{(1+x^2)^{3/2}}$ and $f(0) = 0$, then the value of $f(1)$ is [OJEE 2012]

- (a) $\frac{1}{\sqrt{2}}$ (b) $-\frac{1}{\sqrt{2}}$
 (c) $\sqrt{2}$ (d) $-\sqrt{2}$

47. If $\int \frac{\sqrt{\cot x}}{\sin x \cos x} dx = P\sqrt{\cot x} + Q$, then the value of P is [OJEE 2012]

- (a) -2 (b) 2
 (c) 3 (d) -3

48. $\int \frac{2^x}{\sqrt{1-4^x}} dx$ is equal to [AMU 2012]

- (a) $(\log 2) \sin^{-1}(2^x) + C$
 (b) $\frac{1}{2} \sin^{-1}(2^x) + C$
 (c) $\frac{1}{\log 2} \sin^{-1}(2^x) + C$
 (d) $2 \log 2 \sin^{-1}(2^x) + C$

49. $\int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx$ is equal to [WB JEE 2011]

- (a) $-\frac{1}{2} \sin 2x + C$ (b) $\frac{1}{2} \sin 2x + C$
 (c) $\frac{1}{2} \sin x + C$ (d) $-\frac{1}{2} \sin x + C$

50. The value of $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx$ is [UP SEE 2011]

- (a) $\sqrt{1 + \cos 2x} + C$ (b) $\sqrt{x} + C$
 (c) $x + C$ (d) $\sqrt{1 + 2x} + C$

51. $\int \frac{1}{\sin(x-a) \sin(x-b)} dx$ is equal to [GGSI PU 2011]

- (a) $\frac{1}{\sin(b-a)} \log \left| \frac{\sin(x+b)}{\sin(x+a)} \right| + C$
 (b) $\frac{1}{\sin(b+a)} \log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| - C$
 (c) $\frac{1}{\sin(b-a)} \log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| + C$
 (d) None of the above

52. $\int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$ is equal to [BITSAT 2011]

- (a) $2 \tan^{-1}(\tan^2 x) + C$
 (b) $\tan^{-1}(x \tan^2 x) + C$
 (c) $\tan^{-1}(\tan^2 x) + C$
 (d) None of the above

53. $\int \frac{1 + \cot^2 x}{1 + \cot x} dx$ is equal to [MP PET 2011]

- (a) $-\log(1 + \cot x) + C$
 (b) $\log(1 + \tan x) + C$
 (c) $\log(1 + \cot x) + C$
 (d) $\log \cot x + C$

54. $\int \sqrt{\frac{e^x + a}{e^x - a}} dx$ is equal to [MP PET 2011]

- (a) $\cos h^{-1}\left(\frac{e^x}{a}\right) + \sec^{-1}\left(\frac{e^x}{a}\right) + C$
 (b) $\cos h^{-1}\left(\frac{e^x}{a}\right) - \sec^{-1}\left(\frac{e^x}{a}\right) + C$
 (c) $\frac{1}{a} \left[\cos h^{-1}\left(\frac{e^x}{a}\right) + \sec^{-1}\left(\frac{e^x}{a}\right) \right] + C$
 (d) $\frac{1}{a} \left[\cos h^{-1}\left(\frac{e^x}{a}\right) - \sec^{-1}\left(\frac{e^x}{a}\right) \right] + C$

55. $\int e^x \left[\frac{\sin x + \cos x}{1 - \sin^2 x} \right] dx$ is equal to [Karnataka CET 2011]

- (a) $e^x \cdot \operatorname{cosec} x + C$ (b) $e^x \cdot \cot x + C$
 (c) $e^x \cdot \sec x + C$ (d) $e^x \cdot \tan x + C$

56. $\int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx$ is equal to [Kerala CEE 2011]

- (a) $\frac{3}{2}x + \frac{35}{35} \log |9e^{2x} - 4| + C$
 (b) $\frac{3}{2}x + \frac{36}{36} \log |9e^{2x} - 4| + C$
 (c) $-\frac{3}{2}x + \frac{35}{36} \log |9e^{2x} - 4| + C$
 (d) $-\frac{5}{2}x + \frac{35}{36} \log |9e^{2x} - 4| + C$
 (e) $\frac{5}{2}x + \frac{35}{66} \log |9e^{2x} - 4| + C$

57. $\int \frac{dx}{x^2 \sqrt{4 - x^2}}$ is equal to [GGSIPU 2011]

- (a) $\frac{1}{4} \left(\frac{\sqrt{4 - x^2}}{x} \right) + C$ (b) $\frac{1}{2} \left(\frac{\sqrt{4 - x^2}}{x} \right) + C$
 (c) $-\frac{1}{4} \left(\frac{\sqrt{4 - x^2}}{x} \right) + C$ (d) $-\frac{1}{2} \left(\frac{\sqrt{4 - x^2}}{x} \right) + C$

58. $\int e^{e^x} \cdot e^{e^x} \cdot e^x dx = \dots + C$ [Guj CET 2011]

- (a) e^{e^x} (b) $\frac{1}{2} e^2 \cdot e^x$
 (c) $e^{e^{e^x}}$ (d) $\frac{1}{2} e^{e^x}$

59. $\int \frac{d(\cos \theta)}{\sqrt{1 - \cos^2 \theta}}$ is equal to [BITSAT 2010]

- (a) $\cos^{-1} \theta + C$
 (b) $\theta + C$
 (c) $\sin^{-1} \theta + C$
 (d) $\sin^{-1}(\cos \theta) + C$

60. $\int [\sin(\log x) + \cos(\log x)] dx$ is equal to [MHT CET 2010]

- (a) $x \cos(\log x) + C$ (b) $\cos(\log x) + C$
 (c) $x \sin(\log x) + C$ (d) $\sin(\log x) + C$

61. $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1} dx$ is equal to [MP PET 2010]

- (a) $\frac{x^3}{3} - \frac{x^2}{2} + x + C$ (b) $\frac{x^3}{3} + \frac{x^2}{2} + x + C$
 (c) $\frac{x^3}{3} - \frac{x^2}{2} - x + C$ (d) $\frac{x^3}{3} + \frac{x^2}{2} - x + C$

62. $\int \operatorname{cosec}(x - a) \operatorname{cosec} x dx$ is equal to [Karnataka CET 2010]

- (a) $\frac{-1}{\sin a} \log |\sin x \operatorname{cosec}(x - a)| + C$
 (b) $\frac{-1}{\sin \alpha} \log [\sin(x - a) \sin x] + C$
 (c) $\frac{-1}{\sin a} \log [\sin(x - a) \operatorname{cosec} x] + C$
 (d) $\frac{1}{\sin a} \log [\sin(x - a) \sin x] + C$

63. $\int (x + 1)(x + 2)^7 (x + 3) dx$ is equal to [Kerala CEE 2010]

- (a) $\frac{(x + 2)^{10}}{10} - \frac{(x + 2)^8}{8} + C$
 (b) $\frac{(x + 1)^2}{2} - \frac{(x + 2)^2}{8} + \frac{(x + 3)^2}{2} + C$
 (c) $\frac{(x + 2)^{10}}{10} + C$
 (d) $\frac{(x + 1)^2}{2} + \frac{(x + 2)^8}{8} + \frac{(x + 3)^2}{2} + C$
 (e) $\frac{(x + 2)^9}{9} - \frac{(x + 2)^7}{7} + C$

64. If $\frac{d}{dx} \{f(x)\} = \frac{1}{1 + x^2}$, then $\frac{d}{dx} \{f(x^3)\}$ is equal to [Kerala CEE 2010]

- (a) $\frac{3x}{1 + x^3}$ (b) $\frac{3x^2}{1 + x^6}$ (c) $\frac{-6x^5}{(1 + x^6)^2}$
 (d) $\frac{-6x^5}{1 + x^6}$ (e) None of these

65. If $u = -f''(\theta) \sin \theta + f'(\theta) \cos \theta$ and $v = f''(\theta) \cos \theta + f'(\theta) \sin \theta$, then

$\int \left[\left(\frac{du}{d\theta} \right)^2 + \left(\frac{dv}{d\theta} \right)^2 \right]^{1/2} d\theta$ is equal to [Kerala CEE 2010]

- (a) $f(\theta) - f'(\theta) + C$
 (b) $f(\theta) + f''(\theta) + C$
 (c) $f'(\theta) + f''(\theta) + C$
 (d) $f'(\theta) - f''(\theta) + C$
 (e) $f(\theta) + f'(\theta) + C$

66. $\int \frac{x^4 - 1}{x^2 (x^4 + x^2 + 1)^{1/2}} dx$ is equal to [Kerala CEE 2010]

- (a) $\sqrt{\frac{x^4 + x^2 + 1}{x}} + C$ (b) $\frac{x^2}{\sqrt{x^4 + x^2 + 1}} + C$
 (c) $x(x^4 + x^2 + 1)^{3/2} + C$ (d) $\frac{\sqrt{x^4 + x^2 + 1}}{x} + C$
 (e) $\sqrt{x^4 + x^2 + 1} + C$

67. If $\frac{d}{dx} \left[a \tan^{-1} x + b \log \left(\frac{x-1}{x+1} \right) \right] = \frac{1}{x^4 - 1}$, then

- $a - 2b$ is equal to [EAMCET 2010]
(a) 1 (b) -1 (c) 0 (d) 2

68. $\int [e^{a \log x} + e^{x \log a}] dx$; $a, x > 1$, equals [Guj CET 2010]

- (a) $\frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + C$ (b) $\frac{e^{a \log x}}{a \log x} + \frac{e^{x \log a}}{x \log a} + C$
(c) $\frac{x^{a-1}}{a-1} + a^x \cdot \log a + C$ (d) $\frac{e^{a \log x}}{a/x} + \frac{e^{x \log a}}{x} + C$

69. $\int \frac{dx}{\sqrt{(1-x)(x-2)}}$ is equal to [BCECE 2010]

- (a) $\sin^{-1}(2x-3) + C$ (b) $\sin^{-1}(2x+5) + C$
(c) $\sin^{-1}(3-2x) + C$ (d) $\sin^{-1}(5x-2x) + C$

70. $\int \frac{(\sec^2 x - 7)}{\sin^7 x} dx$ is equal to [BCECE 2010]

- (a) $\frac{\tan x}{\sin^7 x} + C$ (b) $\frac{\cos x}{\sin^7 x} + C$
(c) $\frac{\sin x}{\cos^7 x} + C$ (d) $\frac{\sin x}{\tan^7 x} + C$

71. If $f(x) = \lim_{y \rightarrow x} \frac{\sin^2 y - \sin^2 x}{y^2 - x^2}$, then $\int 4x f(x) dx$ equals [J&K CET 2010]

- (a) $\cos 2x + C$ (b) $2 \cos 2x + C$
(c) $-\cos 2x + C$ (d) $-2 \cos 2x + C$

72. $\int e^x \frac{(x-1)}{x^2} dx$ is equal to [MHT CET 2010]

- (a) $\frac{e^x}{x^2} + C$ (b) $\frac{-e^x}{x^2} + C$ (c) $\frac{e^x}{x} + C$ (d) $\frac{-e^x}{x} + C$

73. $\int \frac{\log \sqrt{x}}{3x} dx$ is equal to [WB JEE 2010]

- (a) $\frac{1}{3} (\log \sqrt{x})^2 + C$ (b) $\frac{2}{3} (\log \sqrt{x})^2 + C$
(c) $\frac{2}{3} (\log x)^2 + C$ (d) $\frac{1}{3} (\log x)^2 + C$

74. $\int e^x \left(\frac{2}{x} - \frac{2}{x^2} \right) dx$ is equal to

- [WB JEE 2010; VITEEE 2010]
(a) $\frac{e^x}{x} + C$ (b) $\frac{e^x}{2x^2} + C$ (c) $\frac{2e^x}{x} + C$ (d) $\frac{2e^x}{x^2} + C$

75. $\int (x^2 + 1) \sqrt{x+1} dx$ is equal to [Kerala CEE 2010]

- (a) $\frac{(x+1)^{7/2}}{7} - 2 \frac{(x+1)^{5/2}}{5} + 2 \frac{(x+1)^{3/2}}{3} + C$
(b) $2 \left[\frac{(x+1)^{7/2}}{7} - 2 \frac{(x+1)^{5/2}}{5} + 2 \frac{(x+1)^{3/2}}{3} \right] + C$
(c) $\frac{(x+1)^{7/2}}{7} - 2 \frac{(x+1)^{5/2}}{5} + 5 + C$
(d) $\frac{(x+1)^{7/2}}{7} - 3 \frac{(x+1)^{5/2}}{5} + 11(x+1)^{1/2} + C$
(e) $(x+1)^{7/2} + (x+1)^{5/2} + (x+1)^{3/2} + C$

76. $\int \frac{1+x}{x+e^{-x}} dx$ is equal to [Kerala CEE 2010]

- (a) $\log|(x-e^{-x})| + C$ (b) $\log|(x+e^{-x})| + C$
(c) $\log|(1+xe^x)| + C$ (d) $(1+xe^x)^2 + C$
(e) $\log|(1-xe^x)| + C$

77. $\int \frac{\cos x + x \sin x}{x^2 + x \cos x} dx$ is equal to [Kerala CEE 2010]

- (a) $\log \left| \frac{\sin x}{1 + \cos x} \right| + C$
(b) $\log \left| \frac{\sin x}{x + \cos x} \right| + C$
(c) $\log \left| \frac{2 \sin x}{x + \cos x} \right| + C$
(d) $\log \left| \frac{x \sin x}{x + \cos x} \right| + C$
(e) $\log \left| \frac{x}{x + \cos x} \right| + C$

78. $\int \frac{dx}{\sqrt{1-e^{2x}}}$ is equal to [Kerala CEE 2010]

- (a) $\log|e^{-x} + \sqrt{e^{-2x} - 1}| + C$
(b) $\log|e^x + \sqrt{e^{2x} - 1}| + C$
(c) $-\log|e^{-x} + \sqrt{e^{-2x} - 1}| + C$
(d) $-\log|e^{-2x} + \sqrt{e^{-2x} - 1}| + C$
(e) $\log|e^{-2x} + \sqrt{e^{-2x} - 1}| + C$

79. If $\int \frac{x+2}{2x^2+6x+5} dx = P \int \frac{4x+6}{2x^2+6x+5} dx$,
+ $\frac{1}{2} \int \frac{dx}{2x^2+6x+5}$, then the value of P is [Kerala CEE 2010]

- (a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$
(d) 2 (e) 1

80. $\int \frac{\log(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$ is equal to [Kerala CEE 2010]

- (a) $[\log(x + \sqrt{1+x^2})]^2 + C$
(b) $x \log(x + \sqrt{1+x^2}) + C$
(c) $\frac{1}{2} \log(x + \sqrt{1+x^2}) + C$
(d) $\frac{1}{2} [\log(x + \sqrt{1+x^2})]^2 + C$
(e) $\frac{x}{2} \log(x + \sqrt{1+x^2}) + C$

81. If $I_n = \int \sin^n x dx$, then $nI_n - (n-1)I_{n-2}$ is equal to [EAMCET 2010]

- (a) $\sin^{n-1} x \cos x$
(b) $\cos^{n-1} x \sin x$
(c) $-\sin^{n-1} x \cos x$
(d) $-\cos^{n-1} x \sin x$

Answers

Work Book Exercise 9.1

1. (b) 2. (a) 3. (c) 4. (d) 5. (b) 6. (c) 7. (b) 8. (a) 9. (c) 10. (a)
11. (c) 12. (b) 13. (b)

Work Book Exercise 9.2

1. (a) 2. (b) 3. (a) 4. (b) 5. (b) 6. (b) 7. (a) 8. (a) 9. (a) 10. (b)
11. (a) 12. (a) 13. (a) 14. (a)

Work Book Exercise 9.3

1. (a) 2. (c) 3. (a) 4. (b) 5. (a) 6. (b) 7. (c) 8. (d) 9. (a) 10. (d)

Work Book Exercise 9.4

1. (b) 2. (a) 3. (b) 4. (a) 5. (b) 6. (a) 7. (d)

Work Book Exercise 9.5

1. (a) 2. (b) 3. (b) 4. (a) 5. (c) 6. (b) 7. (b)

Target Exercises

1. (d) 2. (a) 3. (b) 4. (a) 5. (b) 6. (a) 7. (b) 8. (a) 9. (d) 10. (b)
11. (b) 12. (d) 13. (a) 14. (c) 15. (a) 16. (c) 17. (b) 18. (b) 19. (c) 20. (d)
21. (a) 22. (a) 23. (c) 24. (a) 25. (a) 26. (a) 27. (b) 28. (c) 29. (d) 30. (b)
31. (b) 32. (b) 33. (b) 34. (c) 35. (b) 36. (b) 37. (a) 38. (a) 39. (c) 40. (b)
41. (a) 42. (b) 43. (d) 44. (d) 45. (b) 46. (a) 47. (d) 48. (d) 49. (a) 50. (b)
51. (d) 52. (b) 53. (c) 54. (a) 55. (b) 56. (c) 57. (a) 58. (b) 59. (a) 60. (b)
61. (a) 62. (c) 63. (a) 64. (b) 65. (b) 66. (b) 67. (a,b) 68. (a,b) 69. (a,b,c) 70. (b,c,d)
71. (b,d) 72. (b,d) 73. (a,c) 74. (a,d) 75. (d) 76. (a) 77. (a) 78. (d) 79. (c) 80. (d)
81. (b) 82. (b) 83. (a) 84. (d) 85. (d) 86. (b) 87. (c) 88. (d) 89. (c) 90. (d)
91. (a) 92. (d) 93. (b) 94. (a) 95. (d) 96. (b) 97. (a) 98. (*) 99. (*) 100. (*)
101. (*) 102. (*) 103. (1) 104. (2) 105. (6) 106. (1) 107. (1) 108. (3) 109. (2) 110. (0)
111. (0)

*

98. $A \rightarrow q; B \rightarrow r; C \rightarrow s; D \rightarrow p$

99. $A \rightarrow p; B \rightarrow s; C \rightarrow q; D \rightarrow r$

100. $A \rightarrow p, q; B \rightarrow s, q; C \rightarrow r, s; D \rightarrow q$

101. $A \rightarrow p, q, r; B \rightarrow p, q, r; C \rightarrow p, q, r, s; D \rightarrow p, q, r, s$

102. $A \rightarrow p, q, r; B \rightarrow r, s; C \rightarrow p; D \rightarrow p, q;$

Entrances Gallery

1. (c) 2. (b) 3. (d) 4. (b) 5. (a) 6. (b) 7. (b) 8. (a) 9. (c) 10. (d)
11. (d) 12. (a) 13. (a) 14. (b) 15. (d) 16. (a) 17. (d) 18. (e) 19. (b) 20. (c)
21. (c) 22. (c) 23. (b) 24. (b) 25. (d) 26. (a) 27. (c) 28. (c) 29. (e) 30. (b)
31. (c) 32. (a) 33. (c) 34. (b) 35. (d) 36. (a) 37. (c) 38. (d) 39. (d) 40. (c)
41. (c) 42. (b) 43. (a) 44. (d) 45. (b) 46. (a) 47. (a) 48. (c) 49. (a) 50. (c)
51. (c) 52. (d) 53. (a) 54. (a) 55. (c) 56. (c) 57. (c) 58. (c) 59. (d) 60. (c)
61. (b) 62. (a) 63. (a) 64. (b) 65. (b) 66. (d) 67. (b) 68. (a) 69. (a) 70. (a)
71. (c) 72. (c) 73. (a) 74. (c) 75. (b) 76. (c) 77. (e) 78. (c) 79. (c) 80. (d)
81. (c)

Explanations

Target Exercises

1. We have, $f'(x) = \frac{1}{-x + \sqrt{x^2 + 1}}$

$$\Rightarrow \int f'(x) dx = \int \frac{dx}{-x + \sqrt{x^2 + 1}}$$

By rationalising the integrand, the above integral can be written as

$$f(x) = \int (x + \sqrt{x^2 + 1}) dx$$

$$= \frac{x^2}{2} + \frac{x}{2} \sqrt{x^2 + 1} + \frac{1}{2} \log |x + \sqrt{x^2 + 1}| + C$$

Putting $x = 0$, we have $f(0) = C$, so $C = -\frac{1}{2} - \frac{1}{\sqrt{2}}$

and $f(1) = \frac{1}{2} + \frac{1}{2} \cdot \sqrt{2} + \frac{1}{2} \log |1 + \sqrt{2}| + \left(-\frac{1}{2} - \frac{1}{\sqrt{2}}\right)$

$$= \frac{1}{2} \log (1 + \sqrt{2})$$

$$= \frac{-1}{2} \log (\sqrt{2} - 1)$$

2. Let $I = \int \frac{3 + 2 \cos x}{(2 + 3 \cos x)^2} dx$

On multiplying numerator and denominator by $\operatorname{cosec}^2 x$, we get

$$I = \int \frac{(3 \operatorname{cosec}^2 x + 2 \cot x \operatorname{cosec} x)}{(2 \operatorname{cosec} x + 3 \cot x)^2} dx$$

$$= - \int \frac{-3 \operatorname{cosec}^2 x - 2 \cot x \operatorname{cosec} x}{(2 \operatorname{cosec} x + 3 \cot x)^2} dx$$

$$= \frac{1}{2 \operatorname{cosec} x + 3 \cot x} + C = \left(\frac{\sin x}{2 + 3 \cos x} \right) + C$$

3. We have, $\int \sec^4 x \operatorname{cosec}^2 x dx$

$$= \int \frac{1}{\cos^4 x \sin^2 x} = \int \frac{(\sin^2 x + \cos^2 x)^2}{\cos^4 x \sin^2 x} dx$$

$$= \int \frac{\sin^4 x + \cos^4 x + 2 \sin^2 x \cos^2 x}{\cos^4 x \sin^2 x} dx$$

$$= \int \tan^2 x \sec^2 x dx + 2 \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$

$$= \int \tan^2 x d(\tan x) + 2 \int \sec^2 x dx + \int \operatorname{cosec}^2 x dx$$

$$= \frac{\tan^3 x}{3} + 2 \tan x - \cot x + C$$

4. $\int \frac{\operatorname{cosec} \theta - \cot \theta}{\operatorname{cosec} \theta + \cot \theta} d\theta = \int (\operatorname{cosec} \theta - \cot \theta)^2 d\theta$

$$= \int \operatorname{cosec}^2 \theta d\theta + \int \cot^2 \theta d\theta - 2 \int \operatorname{cosec} \theta \cot \theta d\theta$$

$$= \int (2 \operatorname{cosec}^2 \theta - 1) d\theta - 2 \int \operatorname{cosec} \theta \cot \theta d\theta$$

$$= 2 \operatorname{cosec} \theta - 2 \cot \theta - \theta + C$$

5. Let $I = \int \frac{\sin(x+a)}{\sin(x+b)} dx$ and $x+b=t$. Then, $dx=dt$

Also, $\sin(x+a) = \sin(t-b+a)$

$$= \sin t \cos(a-b) + \cos t \sin(a-b)$$

$$\therefore I = \int \frac{\sin(x+a)}{\sin(x+b)} dx$$

$$= \int \frac{\sin t \cos(a-b) + \cos t \sin(a-b)}{\sin t} dt$$

$$= \int [\cos(a-b) + \cot t \sin(a-b)] dt$$

$$= \cos(a-b) \int 1 dt + \sin(a-b) \int \cot t dt$$

$$= t \cos(a-b) + \sin(a-b) \log |\sin t| + C$$

$$= (x+b) \cos(a-b) + \sin(a-b) \log |\sin(x+b)| + C$$

6. We have, $\int \frac{\cos x - \cos 2x}{1 - \cos x} dx$

$$= \int \frac{\cos x - (2 \cos^2 x - 1)}{1 - \cos x} dx$$

$$= \int \frac{-(2 \cos^2 x - \cos x - 1)}{-(\cos x - 1)} dx$$

$$= \int \frac{(2 \cos x + 1)(\cos x - 1)}{(\cos x - 1)} dx$$

$$= \int (2 \cos x + 1) dx$$

$$= 2 \sin x + x + C$$

7. $\int (1 + 2x + 3x^2 + 4x^3 + \dots) dx$

$$= \int (1 - x)^{-2} dx = (1 - x)^{-1} + C$$

8. $\int \frac{\sin^{3/2} x + \cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}} dx$

$$= \int \frac{\sin^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}} dx$$

$$+ \int \frac{\cos^{3/2} x}{\sqrt{\sin^3 x \cos^3 x \sin(x+\theta)}} dx$$

$$= \int \frac{1}{\sqrt{\cos^3 x \sin(x+\theta)}} dx + \int \frac{1}{\sqrt{\sin^3 x \sin(x+\theta)}} dx$$

$$= \int \frac{1}{\sqrt{\cos^4 x (\tan x \cdot \cos \theta + \sin \theta)}} dx$$

$$+ \int \frac{1}{\sqrt{\sin^4 x (\cos \theta + \cot x \sin \theta)}} dx$$

$$= \int \frac{\sec^2 x}{\sqrt{\tan x \cos \theta + \sin \theta}} dx + \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cos \theta + \cot x \sin \theta}} dx$$

$$= \frac{1}{\cos \theta} \int \frac{\cos \theta \sec^2 x}{\sqrt{\tan x \cos \theta + \sin \theta}} dx$$

$$- \frac{1}{\sin \theta} \int \left(\frac{-\operatorname{cosec}^2 x \sin \theta}{\sqrt{\cos \theta + \cot x \sin \theta}} \right) dx$$

$$= \frac{1}{\cos \theta} \int \frac{1}{\sqrt{\tan x \cos \theta + \sin \theta}} d(\tan x \cos \theta + \sin \theta)$$

$$- \frac{1}{\sin \theta} \int \frac{1}{\sqrt{\cos \theta + \cot x \sin \theta}} d(\cos \theta + \cot x \sin \theta)$$

$$= \frac{1}{\cos \theta} \int \frac{du}{\sqrt{u}} - \frac{1}{\sin \theta} \int \frac{dv}{\sqrt{v}}$$

where $u = \tan x \cos \theta + \sin \theta$, $v = \cos \theta + \cot x \sin \theta$

$$= \frac{2}{\cos \theta} \sqrt{u} - \frac{2}{\sin \theta} \sqrt{v} + C$$

$$= \frac{2}{\cos \theta} \sqrt{\tan x \cos \theta + \sin \theta} - \frac{2}{\sin \theta} \sqrt{\cos \theta + \cot x \sin \theta} + C$$

9. We have, $\int \sqrt{\frac{x}{a^3 - x^3}} dx$

$$= \int \frac{\sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} dx$$

$$= \frac{2}{3} \int \frac{\frac{3}{2} \sqrt{x}}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} dx$$

$$= \frac{2}{3} \int \frac{1}{\sqrt{(a^{3/2})^2 - (x^{3/2})^2}} d(x^{3/2})$$

$$= \frac{2}{3} \int \frac{dt}{\sqrt{(a^{3/2})^2 - t^2}}, \text{ where } t = x^{3/2}$$

$$= \frac{2}{3} \sin^{-1} \left(\frac{t}{a^{3/2}} \right) + C = \frac{2}{3} \sin^{-1} \left(\frac{x^{3/2}}{a^{3/2}} \right) + C$$

10. $\int \sqrt{\frac{x^4}{a^6 + x^6}} dx = \int \frac{x^2}{\sqrt{a^6 + x^6}} dx$

$$= \int \frac{x^2}{\sqrt{(a^3)^2 + (x^3)^2}} dx = \frac{1}{3} \int \frac{3x^2}{\sqrt{(a^3)^2 + (x^3)^2}} dx$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{(a^3)^2 + (x^3)^2}} d(x^3)$$

$$= \frac{1}{3} \int \frac{1}{\sqrt{(a^3)^2 + t^2}} dt, \text{ where } t = x^3$$

$$= \frac{1}{3} \log \left| t + \sqrt{a^6 + t^2} \right| + C$$

$$= \frac{1}{3} \log \left| x^3 + \sqrt{a^6 + x^6} \right| + C$$

11. We have, $I = \int \sec^{25/13} x \csc^{27/13} x dx$

$$\Rightarrow I = \int \cos^{-25/13} x \sin^{-27/13} x dx$$

Here, $\frac{-25}{13} + \frac{-27}{13} = -4$. So, we multiply and divide by $\cos^4 x$.

$$\therefore I = \int \cos^{27/13} x \sin^{-27/13} x \sec^4 x dx$$

$$= \int \frac{\sin^{-27/13} x}{\cos^{-27/13} x} \sec^4 x dx$$

$$= \int \tan^{-27/13} x (1 + \tan^2 x) \sec^2 x dx$$

$$= \int t^{-27/13} (1 + t^2) dt, \text{ where } t = \tan x$$

$$= \int (t^{-27/13} + t^{-1/13}) dt$$

$$= \frac{-13}{14} t^{-14/13} + \frac{13}{12} t^{12/13} + C$$

$$= \frac{-13}{14} (\tan x)^{-14/13} + \frac{13}{12} (\tan x)^{12/13} + C$$

12. We have, $\int \frac{1}{\sin x + \sec x} dx$

$$= \int \frac{\cos x}{1 + \sin x \cos x} dx$$

$$= \int \frac{2 \cos x}{2 + 2 \sin x \cos x} dx$$

$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + \sin 2x} dx$$

$$= \int \frac{\cos x + \sin x}{2 + \sin 2x} dx + \int \frac{\cos x - \sin x}{2 + \sin 2x} dx$$

$$= \int \frac{1}{2 + \{1 - (\sin x - \cos x)^2\}} d(\sin x - \cos x)$$

$$+ \int \frac{1}{2 + \{(\sin x + \cos x)^2 - 1\}} d(\sin x + \cos x)$$

$$= \int \frac{1}{(\sqrt{3})^2 - (\sin x - \cos x)^2} d(\sin x - \cos x)$$

$$+ \int \frac{1}{1^2 + (\sin x + \cos x)^2} d(\sin x + \cos x)$$

$$= \int \frac{1}{(\sqrt{3})^2 - u^2} du + \int \frac{1}{1^2 + v^2} dv,$$

where $u = \sin x - \cos x$
and $v = \sin x + \cos x$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + u}{\sqrt{3} - u} \right| + \tan^{-1}(v) + C$$

$$= \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x} \right| + \tan^{-1}(\sin x + \cos x) + C$$

13. $\int \frac{1}{\cos^6 x + \sin^6 x} dx$

$$= \int \frac{1}{(\cos^2 x + \sin^2 x)(\cos^4 x + \sin^4 x - \cos^2 x \sin^2 x)} dx$$

$$= \int \frac{1}{(\cos^2 x + \sin^2 x)^2 - 3 \sin^2 x \cos^2 x} dx$$

$$= \int \frac{1}{1 - 3 \sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sec^4 x}{\sec^4 x - 3 \tan^2 x} dx$$

[on dividing numerator and denominator by $\cos^4 x$]

$$= \int \frac{(1 + \tan^2 x) \sec^2 x}{(1 + \tan^2 x)^2 - 3 \tan^2 x} dx$$

$$= \int \frac{1 + \tan^2 x}{1 + \tan^4 x - \tan^2 x} d(\tan x)$$

$$= \int \frac{1 + \frac{1}{t^2}}{t^2 + \frac{1}{t^2} - 1} dt = \int \frac{d\left(t - \frac{1}{t}\right)}{\left(t - \frac{1}{t}\right)^2 + 1}$$

$$= \int \frac{du}{u^2 + 1}, \text{ where } u = t - \frac{1}{t}$$

$$= \tan^{-1} u + C$$

$$= \tan^{-1}(\tan x - \cot x) + C$$

$$\begin{aligned}
 14. \text{ Let } I &= \int \frac{2 \sin 2\phi - \cos \phi}{6 - \cos^2 \phi - 4 \sin \phi} d\phi \\
 &= \int \frac{(4 \sin \phi - 1) \cos \phi}{6 - (1 - \sin^2 \phi) - 4 \sin \phi} d\phi \\
 &= \int \frac{(4 \sin \phi - 1) \cos \phi}{\sin^2 \phi - 4 \sin \phi + 5} d\phi \\
 I &= \int \frac{4t - 1}{t^2 - 4t + 5} dt, \text{ where } t = \sin \phi \\
 \therefore I &= \int \frac{2(2t - 4) + 7}{t^2 - 4t + 5} dt \\
 &= 2 \int \frac{2t - 4}{t^2 - 4t + 5} dt + 7 \int \frac{1}{t^2 - 4t + 5} dt \\
 &= 2 \log |t^2 - 4t + 5| + 7 \int \frac{1}{(t - 2)^2 + 1^2} dt \\
 &= 2 \log |t^2 - 4t + 5| + 7 \tan^{-1}(t - 2) + C \\
 &= 2 \log |\sin^2 \phi - 4 \sin \phi + 5| + 7 \tan^{-1}(\sin \phi - 2) + C
 \end{aligned}$$

$$\begin{aligned}
 15. \text{ Let } I &= \int \frac{1}{2 + \cos x} dx \\
 \text{On putting } \cos x &= \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \text{ we get} \\
 I &= \int \frac{1}{2 + \frac{1 - \tan^2 x/2}{1 + \tan^2 x/2}} dx \\
 &= \int \frac{1 + \tan^2 \frac{x}{2}}{2 \left(1 + \tan^2 \frac{x}{2} \right) + \left(1 - \tan^2 \frac{x}{2} \right)} dx \\
 &= \int \frac{\sec^2 \frac{x}{2}}{\tan^2 \frac{x}{2} + 3} dx \\
 &= 2 \int \frac{1}{\left(\tan \frac{x}{2} \right)^2 + (\sqrt{3})^2} d \left(\tan \frac{x}{2} \right) \\
 &= \frac{2}{\sqrt{3}} \tan^{-1} \left(\frac{\tan \frac{x}{2}}{\sqrt{3}} \right) + C
 \end{aligned}$$

$$\begin{aligned}
 16. \text{ Let } I &= \int \frac{1}{1 - 2 \sin x} dx \\
 \text{On putting } \sin x &= \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}, \text{ we get} \\
 I &= \int \frac{1}{1 - \frac{4 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}}} dx \\
 &= \int \frac{1 + \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx
 \end{aligned}$$

$$\begin{aligned}
 &= \int \frac{\sec^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2}} dx \\
 &= 2 \int \frac{1}{\left(\tan \frac{x}{2} - 2 \right)^2 - (\sqrt{3})^2} d \left(\tan \frac{x}{2} \right) \\
 &= 2 \int \frac{dt}{(t - 2)^2 - (\sqrt{3})^2}, \text{ where } t = \tan \frac{x}{2} \\
 &= 2 \cdot \frac{1}{2\sqrt{3}} \log \left| \frac{t - 2 - \sqrt{3}}{t - 2 + \sqrt{3}} \right| + C \\
 &= \frac{1}{\sqrt{3}} \log \left| \frac{\tan \frac{x}{2} - 2 - \sqrt{3}}{\tan \frac{x}{2} - 2 + \sqrt{3}} \right| + C
 \end{aligned}$$

$$\begin{aligned}
 17. \text{ Let } 1 &= r \cos \theta \text{ and } \sqrt{3} = r \sin \theta. \text{ Then,} \\
 r &= \sqrt{1^2 + (\sqrt{3})^2} = 2 \text{ and } \tan \theta = \frac{\sqrt{3}}{1} \Rightarrow \theta = \frac{\pi}{3}
 \end{aligned}$$

$$\begin{aligned}
 \int \frac{1}{\sin x + \sqrt{3} \cos x} dx &= \frac{1}{r} \int \frac{1}{\sin x \cos \theta + \cos x \sin \theta} dx \\
 &= \frac{1}{r} \int \frac{1}{\sin(x + \theta)} dx \\
 &= \frac{1}{r} \int \operatorname{cosec}(x + \theta) dx \\
 &= \frac{1}{r} \log \left| \tan \left(\frac{x}{2} + \frac{\theta}{2} \right) \right| + C \\
 &= \frac{1}{2} \log \left| \tan \left(\frac{x}{2} + \frac{\pi}{6} \right) \right| + C
 \end{aligned}$$

$$\begin{aligned}
 18. \int \frac{5(x^6 + 1)}{x^2 + 1} dx &= \int \frac{5(x^2 + 1)(x^4 - x^2 + 1)}{(x^2 + 1)} dx \\
 &= \int 5(x^4 - x^2 + 1) dx \\
 &= x^5 - \frac{5}{3} x^3 + 5x + C
 \end{aligned}$$

$$\begin{aligned}
 19. \text{ We have, } I &= \int \frac{1}{1 - \sin^4 x} dx \\
 &= \int \frac{1}{\cos^2 x (1 + \sin^2 x)} dx = \int \frac{\sec^4 x}{\sec^2 x + \tan^2 x} dx \\
 &= \int \frac{1 + \tan^2 x}{1 + 2 \tan^2 x} d(\tan x) \\
 &= \int \frac{1 + t^2}{1 + 2t^2} dt, \text{ where } t = \tan x \\
 &= \frac{1}{2} \int 1 dt + \frac{1}{2} \int \frac{1}{1 + 2t^2} dt \\
 &= \frac{1}{2} t + \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2} t) + C \\
 &= \frac{1}{2} \tan x + \frac{1}{2\sqrt{2}} \tan^{-1}(\sqrt{2} \tan x) + C
 \end{aligned}$$

$$\begin{aligned}
 20. \text{ Let } I &= \int \frac{1}{(\sin x + 4)(\sin x - 1)} dx \\
 &= \frac{1}{5} \int \frac{(\sin x + 4) - (\sin x - 1)}{(\sin x + 4)(\sin x - 1)} dx
 \end{aligned}$$

$$= \frac{1}{5} \int \frac{1}{\sin x - 1} dx - \frac{1}{5} \int \frac{1}{\sin x + 4} dx$$

$$= \frac{1}{5} \int \frac{\sec^2 \frac{x}{2}}{2 \tan \frac{x}{2} - 1 - \tan^2 \frac{x}{2}} dx$$

$$- \frac{1}{5} \int \frac{\sec^2 \frac{x}{2} dx}{2 \tan \frac{x}{2} + 4 + 4 \tan^2 \frac{x}{2}}$$

Putting $\tan \frac{x}{2} = t \Rightarrow \sec^2 \frac{x}{2} dx = 2 dt$

$$\therefore I = \frac{1}{5} \int \frac{2 dt}{2t - 1 - t^2} - \frac{1}{5} \int \frac{2 dt}{[2t + 4(1+t^2)]}$$

$$= -\frac{2}{5} \int \frac{dt}{t^2 - 2t + 1} - \frac{1}{10} \int \frac{dt}{t^2 + \frac{1}{2}t + 1}$$

$$= -\frac{2}{5} \int \frac{1}{(t-1)^2} dt - \frac{1}{10} \int \frac{dt}{\left(t + \frac{1}{4}\right)^2 + \left(\frac{\sqrt{15}}{4}\right)^2}$$

$$= \frac{2}{5} \cdot \frac{1}{(t-1)} - \frac{2}{5\sqrt{15}} \tan^{-1} \left(\frac{4t+1}{\sqrt{15}} \right) + C$$

$$= \frac{2}{5} \cdot \frac{1}{\tan \frac{x}{2} - 1} - \frac{2}{5\sqrt{15}} \tan^{-1} \left(\frac{4 \tan \frac{x}{2} + 1}{\sqrt{15}} \right) + C$$

$$\therefore A = \frac{2}{5}, B = \frac{-2}{5\sqrt{15}}, f(x) = \frac{4 \tan \frac{x}{2} + 1}{\sqrt{15}}$$

21. Given,

$$\int f(x) \sin x \cos x dx = \frac{1}{2(b^2 - a^2)} \log \{f(x)\} + C$$

On differentiating both sides, we get

$$f(x) \sin x \cos x = \frac{1}{2(b^2 - a^2)} \cdot \frac{1}{f(x)} f'(x)$$

$$\Rightarrow 2(b^2 - a^2) \sin x \cos x = \frac{f'(x)}{[f(x)]^2}$$

$\Rightarrow$

$$\int (2b^2 \sin x \cos x - 2a^2 \sin x \cos x) dx = \int \frac{f'(x)}{[f(x)]^2} dx$$

$$\Rightarrow I_1 - I_2 = \int \frac{f'(x) dx}{[f(x)]^2} \quad \dots (i)$$

$$\text{where, } I_1 = \int 2b^2 \sin x \cos x dx$$

$$\text{and } I_2 = \int 2a^2 \sin x \cos x dx$$

$$\text{Now, } I_1 = \int 2b^2 \sin x \cos x dx$$

$$\text{Putting } \cos x = t \Rightarrow -\sin x dx = dt$$

$$\therefore I_1 = -\int 2b^2 t dt = -b^2 t^2 = -b^2 \cos^2 x$$

$$\text{and } I_2 = \int 2a^2 \sin x \cos x dx$$

$$\text{Putting } \sin x = t \Rightarrow \cos x dx = dt$$

$$\therefore I_2 = \int 2a^2 t dt = a^2 t^2 = a^2 \sin^2 x$$

$$\text{From Eq. (i), } -b^2 \cos^2 x - a^2 \sin^2 x = -\frac{1}{f(x)}$$

$$\therefore f(x) = \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x}$$

$$22. \int \frac{x^2}{\sqrt{1-x}} dx = p \sqrt{1-x} (3x^2 + 4x + 8)$$

Putting $1-x = t^2$ in LHS, we get

$$I = \int \frac{x^2}{\sqrt{1-x}} dx$$

$$= -\int \frac{2t}{t} (1-t^2)^2 dt = -2 \int (1+t^4 - 2t^2) dt$$

$$= -2 \left[t + \frac{t^5}{5} - \frac{2t^3}{3} \right] + C = -2t \left[1 + \frac{t^4}{5} - \frac{2}{3}t^2 \right] + C$$

$$= -2\sqrt{1-x} \left[1 + \frac{(1-x)^2}{5} - \frac{2}{3}(1-x) \right] + C$$

$$= -2\sqrt{1-x} \left[\frac{15 + 3(1+x^2 - 2x) - 10(1-x)}{15} \right] + C$$

$$= \frac{-2}{15} \sqrt{1-x} (3x^2 + 4x + 8) + C$$

$$23. \text{ Given, } I = \int \frac{e^x}{e^{4x} + e^{2x} + 1} dx$$

$$\text{and } J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1} dx = \int \frac{e^{3x}}{1 + e^{2x} + e^{4x}} dx$$

$$\therefore J - I = \int \frac{(e^{3x} - e^x)}{1 + e^{2x} + e^{4x}} dx = \int \frac{(u^2 - 1)}{1 + u^2 + u^4} du$$

[let $u = e^x$]

$$= \int \frac{\left(1 - \frac{1}{u^2}\right)}{1 + \frac{1}{u^2} + u^2} du = \int \frac{\left(1 - \frac{1}{u^2}\right)}{\left(u + \frac{1}{u}\right)^2 - 1} du$$

$$= \int \frac{dt}{t^2 - 1} \quad \left[\text{put } u + \frac{1}{u} = t \Rightarrow \left(1 - \frac{1}{u^2}\right) du = dt \right]$$

$$= \frac{1}{2} \log \left| \frac{t-1}{t+1} \right| + C = \frac{1}{2} \log \left| \frac{u^2 - u + 1}{u^2 + u + 1} \right| + C$$

$$= \frac{1}{2} \log \left| \frac{e^{2x} - e^x + 1}{e^{2x} + e^x + 1} \right| + C$$

$$24. \text{ Let } I = \int \frac{\log(x+1) - \log x}{x(x+1)} dx$$

$$= \int \frac{\log \left(1 + \frac{1}{x}\right)}{x(x+1)} dx$$

$$= \int \frac{\log \left(1 + \frac{1}{x}\right)}{x^2 \left(1 + \frac{1}{x}\right)} dx$$

$$\text{Putting } 1 + \frac{1}{x} = t \Rightarrow -\frac{1}{x^2} dx = dt$$

$$\therefore I = -\int \frac{\log t}{t} dt = -\frac{1}{2} (\log t)^2 + C$$

$$= -\frac{1}{2} \left[\log \left(1 + \frac{1}{x}\right) \right]^2 + C$$

$$= -\frac{1}{2} [\log(x+1) - \log x]^2 + C$$

$$= -\frac{1}{2} \{ \log(x+1) \}^2 - \frac{1}{2} (\log x)^2$$

$$+ \log(x+1) \cdot \log x + C$$

25. Let $I = \int \frac{\sin x + 8 \cos x}{4 \sin x + 6 \cos x} dx$

We can write,

$$\sin x + 8 \cos x = A(4 \sin x + 6 \cos x) + B \frac{d}{dx}(4 \sin x + 6 \cos x)$$

$$\Rightarrow \sin x + 8 \cos x = A(4 \sin x + 6 \cos x) + B(4 \cos x - 6 \sin x)$$

On equating the coefficients of $\sin x$ and $\cos x$, we get

$$1 = 4A - 6B, 8 = 6A + 4B \Rightarrow A = 1 \text{ and } B = \frac{1}{2}$$

$$\begin{aligned} \therefore I &= \int \frac{(4 \sin x + 6 \cos x) + \frac{1}{2}(4 \cos x - 6 \sin x)}{4 \sin x + 6 \cos x} dx \\ &= \int \left(1 + \frac{1}{2} \cdot \frac{4 \cos x - 6 \sin x}{4 \sin x + 6 \cos x} \right) dx \\ &= x + \frac{1}{2} \log(4 \sin x + 6 \cos x) + C \end{aligned}$$

26. $\int \frac{1}{x^2} (2x + 1)^3 dx = \int \frac{(8x^3 + 1 + 12x^2 + 6x)}{x^2} dx$

$$= \int \left(8x + 12 + \frac{6}{x} + \frac{1}{x^2} \right) dx$$

$$= 4x^2 + 12x + 6 \log x - \frac{1}{x} + C$$

27. $\int \sqrt{1 + \sin\left(\frac{x}{4}\right)} dx = \int \sqrt{1 + 2 \sin \frac{x}{8} \cos \frac{x}{8}} dx$

$$= \int \sqrt{\sin^2 \frac{x}{8} + \cos^2 \frac{x}{8} + 2 \sin \frac{x}{8} \cos \frac{x}{8}} dx$$

$$= \int \sqrt{\left(\sin \frac{x}{8} + \cos \frac{x}{8}\right)^2} dx = \int \left(\sin \frac{x}{8} + \cos \frac{x}{8}\right) dx$$

$$= \frac{-\cos \frac{x}{8}}{\frac{1}{8}} + \frac{\sin \frac{x}{8}}{\frac{1}{8}} + C = 8 \left(\sin \frac{x}{8} - \cos \frac{x}{8}\right) + C$$

28. $I = \int \frac{2 \sin x}{(3 + \sin 2x)} dx$

$$= \int \frac{\sin x + \cos x + \sin x - \cos x}{(3 + \sin 2x)} dx$$

$$= \int \frac{\sin x + \cos x}{3 + \sin 2x} dx - \int \frac{\sin x - \cos x}{(3 + \sin 2x)} dx$$

I_1 I_2

Putting $t_1 = \sin x - \cos x$ in I_1 and $t_2 = \sin x + \cos x$ in I_2 , we get

$$\begin{aligned} I &= \int \frac{dt_1}{[3 + (1 - t_1^2)]} - \int \frac{dt_2}{[3 + (t_2^2 - 1)]} \\ &= \int \frac{dt_1}{4 - t_1^2} - \int \frac{dt_2}{2 + t_2^2} \\ &= \frac{1}{4} \ln \left| \frac{2 + t_1}{2 - t_1} \right| - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{t_2}{\sqrt{2}} \right) + C \\ &= \frac{1}{4} \ln \left| \frac{2 + \sin x - \cos x}{2 - \sin x + \cos x} \right| \\ &\quad - \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sin x + \cos x}{\sqrt{2}} \right) + C \end{aligned}$$

29. $\int x \log \left(1 + \frac{1}{x} \right) dx$

$$= \int x \log(x + 1) dx - \int x \log x dx$$

$$= \frac{x^2}{2} \log(x + 1) - \frac{1}{2} \int \frac{x^2}{x + 1} dx - \frac{x^2}{2} \log x + \frac{1}{2} \int \frac{x^2}{x} dx$$

$$= \frac{x^2}{2} \log(x + 1) - \frac{1}{2} \int \left(x - 1 + \frac{1}{x + 1} \right) dx - \frac{x^2}{2} \log x + \frac{1}{4} x^2$$

$$= \frac{x^2}{2} \log(x + 1) - \frac{x^2}{2} \log x - \frac{1}{2} \left(\frac{x^2}{2} - x \right)$$

$$- \frac{1}{2} \log(x + 1) + \frac{1}{4} x^2 + C$$

$$= \frac{x^2}{2} \log(x + 1) - \frac{x^2}{2} \log x - \frac{1}{2} \log(x + 1) + \frac{1}{2} x + C$$

Hence, $f(x) = \frac{x^2}{2} - \frac{1}{2} \cdot g(x)$

$$= -\frac{1}{2} \log x \text{ and } A = \frac{1}{2}$$

30. $\int \frac{dx}{4 \cos^3 2x - 3 \cos 2x} = \int \frac{dx}{\cos 6x} = \int \sec 6x dx$

$$= \frac{1}{6} \log(\sec 6x + \tan 6x) + C$$

31. $f(x) = \int \frac{x^2 dx}{(1 + x^2)(1 + \sqrt{1 + x^2})}$

Let $x = \tan \theta \Rightarrow dx = \sec^2 \theta d\theta$

$$\therefore f(x) = \int \frac{x^2}{(1 + x^2)(1 + \sqrt{1 + x^2})} dx$$

$$= \int \frac{\tan^2 \theta \sec^2 \theta}{\sec^2 \theta (1 + \sec \theta)} d\theta$$

$$= \int \frac{\tan^2 \theta}{1 + \sec \theta} d\theta$$

$$= \int \frac{\sin^2 \theta}{\cos \theta (1 + \cos \theta)} d\theta$$

$$= \int \frac{(1 - \cos^2 \theta)}{\cos \theta (1 + \cos \theta)} d\theta$$

$$= \int \frac{(1 - \cos \theta)}{\cos \theta} d\theta$$

$$= \int \sec \theta d\theta - \int d\theta$$

$$= \log(x + \sqrt{1 + x^2}) - \tan^{-1} x + C$$

Given, $f(0) = 0$

$$\Rightarrow 0 = \log 1 - 0 + C$$

$$\Rightarrow C = 0$$

$$\therefore f(1) = \log(1 + \sqrt{1 + 1}) - \tan^{-1}(1)$$

$$= \log(1 + \sqrt{2}) - \frac{\pi}{4}$$

32. Put $x^{10} + 10^x = t$

$$\Rightarrow (10x^9 + 10^x \log_e 10) dx = dt$$

Then, $\int \frac{10x^9 + 10^x \log_e 10}{10^x + x^{10}} dx = \int \frac{1}{t} dt = \log t + C$

$$= \log(x^{10} + 10^x) + C$$

33. We have, $\int \frac{dx}{x^2(x^n+1)^{(n-1)/n}}$

$$= \int \frac{dx}{x^2 x^{n-1} \left(1 + \frac{1}{x^n}\right)^{(n-1)/n}}$$

$$= \int \frac{dx}{x^{n+1} (1+x^{-n})^{(n-1)/n}}$$

Put $1+x^{-n}=t$

$$\therefore -nx^{-n-1} dx = dt \Rightarrow \frac{dx}{x^{n+1}} = -\frac{dt}{n}$$

$$\Rightarrow \int \frac{dx}{x^2 (x^n+1)^{(n-1)/n}} = -\frac{1}{n} \int \frac{dt}{t^{(n-1)/n}}$$

$$= -\frac{1}{n} \int t^{-1+1/n} dt = -\frac{1}{n} \cdot \frac{t^{1/n}}{1/n} + C$$

$$= -t^{1/n} + C = -[1+x^{-n}]^{1/n} + C$$

34. $\int \frac{1}{\sqrt{e^{5x}} \sqrt[4]{(e^{2x}+e^{-2x})^3}} dx$

$$= \int \frac{1}{\sqrt{e^{5x}} e^{3x/2} (1+e^{-4x})^{3/4}} dx$$

$$= \int \frac{1}{e^{4x} (1+e^{-4x})^{3/4}} dx = \int (1+e^{-4x})^{-3/4} e^{-4x} dx$$

$$= -\frac{1}{4} \int (1+e^{-4x})^{-3/4} d(1+e^{-4x})$$

$$= -\frac{1}{4} \left\{ \frac{(1+e^{-4x})^{1/4}}{1/4} \right\} + C$$

$$= -(1+e^{-4x})^{1/4} + C$$

35. We have, $\int \frac{(x-\sin x)^{3/2}}{\sqrt{x}} \left\{ \frac{6x^2 \sin^2 \frac{x}{2}}{x-\sin x} + 3x \right\} dx$

$$= \int \frac{(x-\sin x)^{3/2}}{\sqrt{x}} \left\{ \frac{3x^2(1-\cos x) + 3x(x-\sin x)}{x-\sin x} \right\} dx$$

$$= \int \frac{\sqrt{x-\sin x}}{\sqrt{x}} \{3x^2(1-\cos x) + 3x(x-\sin x)\} dx$$

$$= 3 \int \sqrt{x^2-x\sin x} (2x-x\cos x-\sin x) dx$$

$$= 3 \int \sqrt{x^2-x\sin x} d(x^2-x\sin x)$$

$$= 3 \times \frac{2}{3} (x^2-x\sin x)^{3/2} + C = 2(x^2-x\sin x)^{3/2} + C$$

36. We have, $\int \frac{(1+\log x)^2}{1+\log x^{x+1} + (\log x^{\sqrt{x}})^2} dx$

$$= \int \frac{(1+\log x)^2}{1+(x+1)\log x + (\sqrt{x}\log x)^2} dx$$

$$= \int \frac{(1+\log x)^2}{1+(x+1)\log x + x(\log x)^2} dx$$

$$= \int \frac{(1+\log x)^2}{(1+\log x) + x\log x(1+\log x)} dx$$

$$= \int \frac{(1+\log x)^2}{(1+x\log x)(1+\log x)} dx$$

$$= \int \frac{1+\log x}{(1+x\log x)} dx = \int \frac{1}{(1+x\log x)} d(1+x\log x)$$

$$= \log |1+x\log x| + C$$

37. $\int x (x^x)^x (2\log x + 1) dx$

Let $(x^x)^x = t$

Taking log on both sides, we get

$$\log (x^x)^x = \log t$$

$$\Rightarrow x \log x^x = \log t \Rightarrow x \cdot x \log x = \log t$$

$$\Rightarrow x^2 \log x = \log t$$

On differentiating w.r.t. x , we get

$$2x \log x + x^2 \cdot \frac{1}{x} = \frac{1}{t} \cdot \frac{dt}{dx}$$

$$\Rightarrow (2\log x + 1) x t dx = dt$$

$$\Rightarrow (2\log x + 1) x (x^x)^x dx = dt$$

$$\therefore \int 1 dt = t + C = (x^x)^x + C$$

38. Let $I = \int \frac{1}{x^4 \sqrt{x^2-3}} dx$

On putting $x = \sqrt{3} \sec \theta$ and $dx = \sqrt{3} \sec \theta \tan \theta d\theta$, we get

$$I = \int \frac{1}{9 \sec^4 \theta \sqrt{3 \sec^2 \theta - 3}} \sqrt{3} \sec \theta \tan \theta d\theta$$

$$= \frac{1}{9} \int \cos^3 \theta d\theta = \frac{1}{9} \int \frac{\cos 3\theta + 3 \cos \theta}{4} d\theta$$

$$= \frac{1}{36} \left(\frac{\sin 3\theta}{3} + 3 \sin \theta \right) + C$$

$$= \frac{1}{108} (12 \sin \theta - 4 \sin^3 \theta) + C$$

$$= \frac{1}{27} (3 \sin \theta - \sin^3 \theta) + C$$

$$= \frac{\sqrt{x^2-3}}{27x^3} (2x^2+3) + C$$

39. Let $I = \int \frac{1}{(x^2+2x+2)^2} dx = \int \frac{1}{[(x+1)^2+1]^2} dx$

Put $x+1 = \tan \theta$

$$\Rightarrow dx = \sec^2 \theta d\theta$$

$$\therefore I = \int \frac{1}{(\tan^2 \theta + 1)^2} \sec^2 \theta d\theta$$

$$= \int \cos^2 \theta d\theta = \frac{1}{2} \int (1 + \cos 2\theta) d\theta$$

$$= \frac{1}{2} \left(\theta + \frac{\sin 2\theta}{2} \right) + C$$

$$= \frac{1}{2} (\theta + \sin \theta \cos \theta) + C$$

$$= \frac{1}{2} \left\{ \tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2} \right\} + C$$

40. We have, $d\{I(x)\} = d(\log_e x) = \frac{1}{x} dx$

$$d\{I^2(x)\} = d\{\log_e (\log_e x)\} = \frac{1}{x \log_e x} dx = \frac{1}{x I(x)} dx$$

$$d\{I^3(x)\} = d\{\log_e (\log_e (\log_e x))\}$$

$$= \frac{1}{x \log_e x \log_e (\log_e x)}$$

$$= \frac{1}{x I(x) I^2(x)} dx \text{ and so on.}$$

$$\therefore d\{I^n(x)\} = \frac{1}{x I(x) I^2(x) I^3(x) \dots I^{n-1}(x)} dx$$

$$\begin{aligned}\text{Thus, } \int \frac{1}{x I(x) I^n(x) \dots I^{n-1}(x) I^n(x)} dx \\&= \int \frac{1}{t} dt, \text{ where } t = I^n(x) \\&= \log_e(t) + C \\&= \log_e\{I^n(x)\} + C = I^{n+1}(x) + C\end{aligned}$$

41. On putting $x + \sqrt{x^2 + 1} = t$ and $\left(1 + \frac{x}{\sqrt{x^2 + 1}}\right) dx = dt$

$$\Rightarrow \frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}} dx = dt$$

$$\Rightarrow \frac{t}{\sqrt{x^2 + 1}} dx = dt, \text{ we get}$$

$$I = \int (x + \sqrt{x^2 + 1})^n dx = \int t^n \frac{\sqrt{x^2 + 1}}{t} dt$$

$$= \int t^{n-1} \sqrt{1 + \left(\frac{t^2 - 1}{2t}\right)^2} dt \left[\begin{array}{l} \because x + \sqrt{x^2 + 1} = t \\ \Rightarrow t - x = \sqrt{x^2 + 1} \\ \Rightarrow (t - x)^2 = x^2 + 1 \\ \Rightarrow t^2 - 2tx + x^2 = x^2 + 1 \\ \Rightarrow t^2 - 2tx = 1 \\ \Rightarrow x = \frac{t^2 - 1}{2t} \end{array} \right]$$

$$I = \frac{1}{2} \int t^{n-2} (t^2 + 1) dt = \frac{1}{2} \int (t^n + t^{n-2}) dt$$

$$= \frac{1}{2} \left(\frac{t^{n+1}}{n+1} + \frac{t^{n-1}}{n-1} \right) + C$$

$$= \frac{1}{2} \left[\frac{(x + \sqrt{x^2 + 1})^{n+1}}{n+1} + \frac{(x + \sqrt{x^2 + 1})^{n-1}}{n-1} \right] + C$$

42. We have, $(x^2 + 2x + 3)^2 = x^4 + 4x^3 + 10x^2 + 12x + 9$

$$\begin{aligned}\therefore x^4 + 4x^3 + 11x^2 + 12x + 8 \\&= (x^4 + 4x^3 + 10x^2 + 12x + 9) + (x^2 - 1) \\&= (x^2 + 2x + 3)^2 + (x^2 - 1)\end{aligned}$$

$$\text{Hence, } I = \int \frac{x^4 + 4x^3 + 11x^2 + 12x + 8}{(x^2 + 2x + 3)^2 (x + 1)} dx$$

$$= \int \frac{(x^2 + 2x + 3)^2 + (x^2 - 1)}{(x^2 + 2x + 3)^2 (x + 1)} dx$$

$$= \int \frac{1}{x + 1} dx + \int \frac{x - 1}{(x^2 + 2x + 3)^2} dx$$

$$= \int \frac{1}{x + 1} dx + \frac{1}{2} \int \frac{2x - 2}{(x^2 + 2x + 3)^2} dx$$

$$= \int \frac{1}{x + 1} dx + \frac{1}{2} \int \frac{2x + 2 - 4}{(x^2 + 2x + 3)^2} dx$$

$$= \int \frac{1}{x + 1} dx + \frac{1}{2} \int \frac{2x + 2}{(x^2 + 2x + 3)^2} dx - 2 \int \frac{1}{(x^2 + 2x + 3)^2} dx$$

$$= \log |x + 1| - \frac{1}{2(x^2 + 2x + 3)} - 2 \int \frac{1}{[(x + 1)^2 + (\sqrt{2})^2]^2} dx$$

On putting $x + 1 = \sqrt{2} \tan \theta$ and $dx = \sqrt{2} \sec^2 \theta d\theta$, we get

$$I = \log |x + 1| - \frac{1}{2(x^2 + 2x + 3)} - 2 \int \frac{\sqrt{2} \sec^2 \theta d\theta}{(2 \tan^2 \theta + 2)^2}$$

$$\begin{aligned}&= \log |x + 1| - \frac{1}{2(x^2 + 2x + 3)} - \frac{1}{\sqrt{2}} \int \cos^2 \theta d\theta \\&= \log |x + 1| - \frac{1}{2(x^2 + 2x + 3)} - \frac{1}{2\sqrt{2}} \int (1 + \cos 2\theta) d\theta \\&= \log |x + 1| - \frac{1}{2(x^2 + 2x + 3)} \\&\quad - \frac{1}{2\sqrt{2}} (\theta + \sin \theta \cos \theta) + C\end{aligned}$$

$$\begin{aligned}&= \log |x + 1| - \frac{1}{2(x^2 + 2x + 3)} \\&\quad - \frac{1}{2\sqrt{2}} \left[\tan^{-1} \left(\frac{x + 1}{\sqrt{2}} \right) + \frac{\sqrt{2}(x + 1)}{x^2 + 2x + 3} \right] + C\end{aligned}$$

43. $\int [f(x) + f(-x)] dx = \int \left[\frac{\sin^2 \pi x}{1 + \pi^x} + \frac{\sin^2 (-\pi x)}{1 + \pi^{-x}} \right] dx$

$$\begin{aligned}&= \int \left(\frac{\sin^2 \pi x}{1 + \pi^x} + \frac{\pi^x \sin^2 \pi x}{1 + \pi^x} \right) dx \\&= \int \sin^2(\pi x) \left(\frac{1 + \pi^x}{1 + \pi^x} \right) dx \\&= \int \sin^2 \pi x dx = \frac{1}{2} \int (1 - \cos 2\pi x) dx \\&= \frac{1}{2} \left[x - \frac{\sin 2\pi x}{2\pi} \right] + C = \frac{x}{2} - \frac{\sin 2\pi x}{4\pi} + C\end{aligned}$$

44. Let $I = \int \frac{1 + x^2}{(1 - x^2) \sqrt{1 + x^2 + x^4}} dx$

$$\text{Then, } I = \int \frac{\frac{1 + x^2}{x^2}}{\left(\frac{1}{x} - x\right) \sqrt{\frac{1 + x^2 + x^4}{x^2}}} dx$$

[on dividing numerator and denominator by x^2]

$$\Rightarrow I = - \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right) \sqrt{x^2 + \frac{1}{x^2} + 1}} dx$$

$$\Rightarrow I = - \int \frac{1}{\left(x - \frac{1}{x}\right) \sqrt{\left(x - \frac{1}{x}\right)^2 + 3}} d\left(x - \frac{1}{x}\right)$$

$$\Rightarrow I = - \int \frac{dt}{t \sqrt{t^2 + 3}}, \text{ where } x - \frac{1}{x} = t$$

$$\Rightarrow I = - \int \frac{udu}{(u^2 - 3) \sqrt{u^2}}$$

where $t^2 + 3 = u^2$ and $2t dt = 2u du$

$$\Rightarrow I = - \int \frac{1}{u^2 - 3} du$$

$$\Rightarrow I = - \frac{1}{2\sqrt{3}} \log \left| \frac{u - \sqrt{3}}{u + \sqrt{3}} \right| + C$$

$$\Rightarrow I = - \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{t^2 + 3} - \sqrt{3}}{\sqrt{t^2 + 3} + \sqrt{3}} \right| + C$$

$$\Rightarrow I = - \frac{1}{2\sqrt{3}} \log \left| \frac{\sqrt{x^2 + \frac{1}{x^2} + 1} - \sqrt{3}}{\sqrt{x^2 + \frac{1}{x^2} + 1} + \sqrt{3}} \right| + C$$

45. We have, $I = \int \frac{\sqrt{\cos 2x}}{\sin x} dx$

$$= \int \sqrt{\frac{\cos^2 x - \sin^2 x}{\sin^2 x}} dx$$

$$= \int \sqrt{\cot^2 x - 1} dx$$

On putting $\cot x = \sec \theta$

$$\Rightarrow -\operatorname{cosec}^2 x dx = \sec \theta \tan \theta d\theta$$

$$\therefore I = \int \sqrt{\sec^2 \theta - 1} \times \frac{\sec \theta \tan \theta}{-\operatorname{cosec}^2 x} d\theta$$

$$= - \int \frac{\sec \theta \tan^2 \theta}{1 + \sec^2 \theta} d\theta$$

$$= - \int \frac{\sin^2 \theta}{\cos \theta + \cos^3 \theta} d\theta$$

$$= - \int \frac{1 - \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= - \int \frac{(1 + \cos^2 \theta) - 2 \cos^2 \theta}{\cos \theta (1 + \cos^2 \theta)} d\theta$$

$$= - \int \sec \theta d\theta + 2 \int \frac{\cos \theta}{1 + \cos^2 \theta} d\theta$$

$$= - \int \sec \theta d\theta + 2 \int \frac{d(\sin \theta)}{1 + \cos^2 \theta}$$

$$= - \int \sec \theta d\theta + 2 \int \frac{d(\sin \theta)}{2 - \sin^2 \theta}$$

$$= - \log |\sec \theta + \tan \theta|$$

$$+ 2 \times \frac{1}{2\sqrt{2}} \log \left| \frac{\sqrt{2} + \sin \theta}{\sqrt{2} - \sin \theta} \right| + C$$

$$= - \log \left| \sec \theta + \sqrt{\sec^2 \theta - 1} \right|$$

$$+ \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \cos^2 \theta}}{\sqrt{2} - \sqrt{1 - \cos^2 \theta}} \right| + C$$

$$= - \log \left| \cot x + \sqrt{\cot^2 x - 1} \right|$$

$$+ \frac{1}{\sqrt{2}} \log \left| \frac{\sqrt{2} + \sqrt{1 - \tan^2 x}}{\sqrt{2} - \sqrt{1 - \tan^2 x}} \right| + C$$

46. We have, $I = \int x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} dx$

$$I = \int x \sqrt{\frac{a^2 - t}{a^2 + t}} \cdot \frac{dt}{2x}, \text{ where } t = x^2$$

$$= \frac{1}{2} \int \sqrt{\frac{a^2 - t}{a^2 + t}} dt = \frac{1}{2} \int \sqrt{\frac{a^2 - t}{a^2 + t}} \times \frac{a^2 - t}{a^2 - t} dt$$

$$= \frac{1}{2} \int \frac{a^2 - t}{\sqrt{a^4 - t^2}} dt = \frac{1}{2} \int \frac{a^2}{\sqrt{a^4 - t^2}} dt - \frac{1}{2} \int \frac{t dt}{\sqrt{a^4 - t^2}}$$

$$= \frac{1}{2} a^2 \int \frac{1}{\sqrt{(a^2)^2 - t^2}} dt + \frac{1}{4} \int \frac{d(a^4 - t^2)}{\sqrt{a^4 - t^2}}$$

$$= \frac{1}{2} a^2 \sin^{-1} \left(\frac{t}{a^2} \right) + \frac{1}{2} \sqrt{a^4 - t^2} + C$$

$$= \frac{1}{2} a^2 \sin^{-1} \left(\frac{x^2}{a^2} \right) + \frac{1}{2} \sqrt{a^4 - x^4} + C$$

47. Let $\int \frac{6x^3 + 15x^2 - 7x + 6}{\sqrt{2x^2 - 2x + 1}} dx$

$$= (b_0 x^2 + b_1 x + b_2) \sqrt{2x^2 - 2x + 1} + b_3 \int \frac{1}{\sqrt{2x^2 - 2x + 1}} dx \quad \dots (i)$$

On differentiating both sides w.r.t. x , we get

$$\frac{6x^3 + 15x^2 - 7x + 6}{\sqrt{2x^2 - 2x + 1}}$$

$$= (2b_0 x + b_1) \sqrt{2x^2 - 2x + 1} + \frac{(b_0 x^2 + b_1 x + b_2)(2x - 1)}{\sqrt{2x^2 - 2x + 1}} + \frac{b_3}{\sqrt{2x^2 - 2x + 1}}$$

$$\Rightarrow 6x^3 + 15x^2 - 7x + 6 = (2b_0 x + b_1)(2x^2 - 2x + 1) + (b_0 x^2 + b_1 x + b_2)(2x - 1) + b_3$$

On comparing the coefficients of x^3, x^2, x and constant terms on both sides, we get

$$6b_0 = 6, -5b_0 + 4b_1 = 15, 2b_0 - 3b_1 + 2b_2 = -7$$

$$\text{and } b_1 - b_2 + b_3 = 6$$

$$\Rightarrow b_0 = 1, b_1 = 5, b_2 = 3 \text{ and } b_3 = 4$$

On substituting these values in Eq. (i), we get

$$\int \frac{6x^3 + 15x^2 - 7x + 6}{\sqrt{2x^2 - 2x + 1}} dx$$

$$= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 4 \int \frac{1}{\sqrt{2x^2 - 2x + 1}} dx$$

$$= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 2\sqrt{2} \int \frac{1}{\sqrt{x^2 - x + \frac{1}{2}}} dx$$

$$= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 2\sqrt{2} \int \frac{1}{\sqrt{\left(x - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}} dx$$

$$= (x^2 + 5x + 3) \sqrt{2x^2 - 2x + 1} + 2\sqrt{2} \log \left| \left(x - \frac{1}{2}\right) + \sqrt{x^2 - x + \frac{1}{2}} \right| + C$$

48. $\int \frac{x \tan^{-1} x}{1 + x^2} dx = (\tan^{-1} x) \frac{x^2}{2} - \int \frac{1}{1 + x^2} \cdot \frac{x^2}{2} dx$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2 + 1 - 1}{x^2 + 1} dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \left(1 - \frac{1}{x^2 + 1}\right) dx$$

$$= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] + C$$

49. $\int \log(1 + x^2) dx = \int \log(1 + x^2) \cdot 1 dx$

$$= x \log(1 + x^2) - \int \left(\frac{1}{1 + x^2} \cdot 2x \right) x dx$$

$$= x \log(1 + x^2) - 2 \int \frac{x^2}{x^2 + 1} dx$$

$$= x \log(x^2 + 1) - 2 \int 1 - \frac{1}{x^2 + 1} dx$$

$$= x \log(x^2 + 1) - 2(x - \tan^{-1} x) + C$$

$$= x \log(x^2 + 1) - 2x + 2 \tan^{-1} x + C$$

50. Let $I = \int \tan^{-1} \left(\frac{2 \cos^2 \theta}{2 - \sin 2\theta} \right) \cdot \sec^2 \theta d\theta$

Then, $I = \int \tan^{-1} \left(\frac{1 + \cos 2\theta}{2 - \sin 2\theta} \right) \sec^2 \theta d\theta$

$= \int \tan^{-1} \left(\frac{1}{1 - \tan \theta + \tan^2 \theta} \right) \sec^2 \theta d\theta$

$= \int \tan^{-1} \left(\frac{1}{1 - t + t^2} \right) dt$, where $t = \tan \theta$

$= \int \tan^{-1} \left\{ \frac{t + (1-t)}{1 - t(1-t)} \right\} dt$

$= \int \{ \tan^{-1} t + \tan^{-1}(1-t) \} dt$

$= t \sin^{-1} t - \frac{1}{2} \log(1+t^2) - (1-t) \tan^{-1}(1-t)$

$+ \frac{1}{2} \log \{1 + (1-t)^2\} + C$

$= \theta \tan \theta + \log \cos \theta - (1 - \tan \theta) \tan^{-1}(1 - \tan \theta)$

$+ \frac{1}{2} \log(2 - 2 \tan \theta + \tan^2 \theta) + C$

51. We have, $I = \int \frac{x^2 (x \sec^2 x + \tan x)}{(x \tan x + 1)^2} dx$

$= \int x^2 \cdot \frac{d(x \tan x + 1)}{(x \tan x + 1)^2} dx$

$= x^2 \times \left(\frac{-1}{x \tan x + 1} \right) - 2 \int x \times \left(\frac{-1}{x \tan x + 1} \right) dx$

$= \frac{-x^2}{x \tan x + 1} + 2 \int \frac{x \cos x}{x \sin x + \cos x} dx$

$= \frac{-x^2}{x \tan x + 1} + 2 \int \frac{d(x \sin x + \cos x)}{x \sin x + \cos x}$

$= \frac{-x^2}{x \tan x + 1} + 2 \log |x \sin x + \cos x| + C$

52. We have, $\int \left\{ \log(1 + \cos x) - x \tan \frac{x}{2} \right\} dx$

$= \int \log(1 + \cos x) dx - \int x \tan \frac{x}{2} dx$

$= x \log(1 + \cos x) - \int \frac{x(-\sin x)}{1 + \cos x} dx - \int x \tan \frac{x}{2} dx$

$= x \log(1 + \cos x) + \int x \tan \frac{x}{2} dx$

$- \int x \tan \frac{x}{2} dx + C$

$= x \log(1 + \cos x) + C$

53. We have,

$\int e^{x \sin x + \cos x} \cdot \left\{ \frac{x^4 \cos^3 x - x \sin x + \cos x}{x^2 \cos^2 x} \right\} dx$

$= \int e^{x \sin x + \cos x} \cdot x^2 \cos x dx$

$+ \int e^{x \sin x + \cos x} \cdot \frac{\cos x - x \sin x}{(x \cos x)^2} dx$

$= \int x e^{x \sin x + \cos x} d(x \sin x + \cos x)$

$+ \int e^{x \sin x + \cos x} \frac{d(x \cos x)}{(x \cos x)^2}$

$= x e^{x \sin x + \cos x} - \int e^{x \sin x + \cos x} dx$

$+ e^{x \sin x + \cos x} \times \left(\frac{-1}{x \cos x} \right)$

$- \int e^{x \sin x + \cos x} \cdot x \cos x \times \left(\frac{-1}{x \cos x} \right) dx + C$

$= x e^{x \sin x + \cos x} - \frac{e^{x \sin x + \cos x}}{x \cos x} + C$

54. We have, $I = \int \sin^{-1} \left\{ \frac{2x+2}{\sqrt{4x^2+8x+13}} \right\} dx$

$= \int \sin^{-1} \left\{ \frac{2x+2}{\sqrt{(2x+2)^2+3^2}} \right\} dx$

On substituting $2x+2 = 3 \tan \theta$

and $dx = \frac{3}{2} \sec^2 \theta d\theta$, we get

$I = \int \sin^{-1} \left(\frac{3 \tan \theta}{3 \sec \theta} \right) \times \frac{3}{2} \sec^2 \theta d\theta$

$= \frac{3}{2} \int \theta \sec^2 \theta d\theta = \frac{3}{2} [\theta \tan \theta - \int \tan \theta d\theta]$

$= \frac{3}{2} [\theta \tan \theta - \log |\sec \theta|] + C$

$= \frac{3}{2} \left[\left(\frac{2x+2}{3} \right) \tan^{-1} \left(\frac{2x+2}{3} \right) - \log \sqrt{1 + \left(\frac{2x+2}{3} \right)^2} \right] + C$

$= (x+1) \tan^{-1} \left(\frac{2x+2}{3} \right) - \frac{3}{4} \log(4x^2+8x+13) + C$

55. We have, $\int e^x \frac{2-x^2}{(1-x)\sqrt{1-x^2}} dx$

$= \int e^x \frac{1 + (1-x^2)}{(1-x)\sqrt{1-x^2}} dx$

$= \int e^x \left(\frac{1}{(1-x)\sqrt{1-x^2}} + \frac{1+x}{\sqrt{1-x^2}} \right) dx$

$= \int e^x \left(\frac{1}{(1-x)^{3/2} \sqrt{1+x}} + \sqrt{\frac{1+x}{1-x}} \right) dx$

$= \int e^x \left(\sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} + \sqrt{\frac{1+x}{1-x}} \right) dx$

$= \int e^x \sqrt{\frac{1+x}{1-x}} dx + \int e^x \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} dx$

$= \sqrt{\frac{1+x}{1-x}} \cdot e^x - \int e^x \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} dx$

$+ \int e^x \sqrt{\frac{1-x}{1+x}} \cdot \frac{1}{(1-x)^2} dx + C$

$= e^x \sqrt{\frac{1+x}{1-x}} + C$

56. We have, $I = \int x^3 \sqrt{\frac{1+x^2}{1-x^2}} dx$

$= \int \sqrt{\frac{1+x^2}{1-x^2}} \cdot x^2 \cdot x dx$

$$\begin{aligned}
 &= \frac{1}{2} \int \sqrt{\frac{1+x^2}{1-x^2}} \cdot x^2 \cdot 2x \, dx \\
 &= \frac{1}{2} \int \sqrt{\frac{1+x^2}{1-x^2}} \cdot x^2 \, d(x^2) \\
 &= \frac{1}{2} \int \sqrt{\frac{1+t}{1-t}} t \, dt, \text{ where } t = x^2 \\
 &= \frac{1}{2} \int \frac{t+t^2}{\sqrt{1-t^2}} dt \\
 &= -\frac{1}{2} \sqrt{1-t^2} - \frac{1}{4} t \sqrt{1-t^2} + \frac{1}{4} \sin^{-1} t + C \\
 &= -\frac{1}{2} \sqrt{1-x^4} - \frac{1}{4} x^2 \sqrt{1-x^4} + \frac{1}{4} \sin^{-1} x^2 + C
 \end{aligned}$$

57. We have, $\int \frac{1}{x^6 - a^6 + a^2 x^2 (x^2 - a^2)} dx$

$$\begin{aligned}
 &= \int \frac{1}{(x^2 - a^2)(x^4 + a^4 + 2a^2 x^2)} dx \\
 &= \int \frac{1}{(x^2 - a^2)(x^2 + a^2)^2} dx
 \end{aligned}$$

Let $x^2 = y$. Then,

$$\frac{1}{(x^2 - a^2)(x^2 + a^2)^2} = \frac{1}{(y - a^2)(y + a^2)^2}$$

Let $\frac{1}{(y - a^2)(y + a^2)^2} = \frac{A}{y - a^2} + \frac{B}{y + a^2} + \frac{C}{(y + a^2)^2}$

$$\Rightarrow 1 = A(y + a^2)^2 + B(y - a^2)(y + a^2) + C(y - a^2)$$

On putting $y = a^2$ and $y = -a^2$ respectively, we get

$$A = \frac{1}{4a^4}, C = -\frac{1}{2a^2}$$

Now, putting $y = 0$, we get

$$1 = Aa^4 - Ba^4 - Ca^2$$

$$\Rightarrow 1 = \frac{1}{4} - Ba^4 + \frac{1}{2}$$

$$\Rightarrow B = -\frac{1}{4a^4}$$

$$\therefore \frac{1}{(y - a^2)(y + a^2)^2} = \frac{1}{4a^4} \cdot \frac{1}{(y - a^2)} - \frac{1}{4a^4} \cdot \frac{1}{(y + a^2)} - \frac{1}{2a^2} \cdot \frac{1}{(y + a^2)^2}$$

$$\Rightarrow \frac{1}{(x^2 - a^2)(x^2 + a^2)^2} = \frac{1}{4a^4} \cdot \frac{1}{x^2 - a^2} - \frac{1}{4a^4} \cdot \frac{1}{x^2 + a^2} - \frac{1}{2a^2} \cdot \frac{1}{(x^2 + a^2)^2}$$

$$\Rightarrow \int \frac{1}{(x^2 - a^2)(x^2 + a^2)^2} dx = \frac{1}{4a^4} \int \frac{1}{x^2 - a^2} dx - \frac{1}{4a^4} \int \frac{1}{x^2 + a^2} dx - \frac{1}{2a^2} \int \frac{1}{(x^2 + a^2)^2} dx$$

$$= \frac{1}{4a^4} \times \frac{1}{2a} \log \left| \frac{x-a}{x+a} \right| - \frac{1}{4a^5} \tan^{-1} \frac{x}{a} - \frac{1}{2a^2} \left[\frac{x}{2a^2(x^2 + a^2)} + \frac{1}{2a^3} \tan^{-1} \left(\frac{x}{a} \right) \right] + C$$

$$= \frac{1}{8a^5} \log \left| \frac{x-a}{x+a} \right| - \frac{1}{2a^5} \tan^{-1} \left(\frac{x}{a} \right) - \frac{x}{4a^4(x^2 + a^2)} + C$$

58. Let $I = \int \frac{1}{x + \sqrt{x^2 - x + 1}} dx$

Let $x + \sqrt{x^2 - x + 1} = t$

Then, $\sqrt{x^2 - x + 1} = t - x$

$$\Rightarrow x^2 - x + 1 = (t - x)^2$$

$$\Rightarrow -x + 1 = t^2 - 2tx \Rightarrow x = \frac{t^2 - 1}{2t - 1}$$

$$\therefore dx = \frac{(2t - 1)2t - 2(t^2 - 1)}{(2t - 1)^2} dt = \frac{2t^2 - 2t + 2}{(2t - 1)^2} dt$$

On substituting these values, we get

$$I = \int \frac{1}{t} \times \frac{2t^2 - 2t + 2}{(2t - 1)^2} dt = 2 \int \frac{t^2 - t + 1}{t(2t - 1)^2} dt$$

Let $\frac{t^2 - t + 1}{t(2t - 1)^2} = \frac{A}{t} + \frac{B}{2t - 1} + \frac{C}{(2t - 1)^2}$... (i)

On multiplying both sides by t and putting $t = 0$, we get $A = 1$.

Similarly, multiplying both sides of Eq. (i) by $(2t - 1)^2$ and putting $t = \frac{1}{2}$, we get

$$C = \frac{\frac{1}{4} - \frac{1}{2} + 1}{\frac{1}{2}} = \frac{3}{2}$$

From Eq. (i), we have

$$t^2 - t + 1 = A(2t - 1)^2 + B(2t - 1)t + Ct$$

On equating the coefficient of t^2 on both sides, we get

$$1 = 4A + 2B$$

$$\Rightarrow B = -\frac{3}{2}$$

On substituting the values of A , B and C in Eq. (i), we get

$$\frac{t^2 - t + 1}{t(2t - 1)^2} = \frac{1}{t} - \frac{3}{2(2t - 1)} + \frac{3}{2} \cdot \frac{1}{(2t - 1)^2}$$

$$\therefore I = 2 \int \frac{1}{t} dt - 3 \int \frac{1}{2t - 1} dt + 3 \int \frac{1}{(2t - 1)^2} dt$$

$$\Rightarrow I = 2 \log t - \frac{3}{2} \log (2t - 1) - \frac{3}{2} \cdot \frac{1}{(2t - 1)} + C$$

$$\begin{aligned}
 \Rightarrow I &= 2 \log (x + \sqrt{x^2 - x + 1}) \\
 &\quad - \frac{3}{2} \log \{ (2x - 1) + 2\sqrt{x^2 - x + 1} \} \\
 &\quad - \frac{3}{2 \{ (2x - 1) + 2\sqrt{x^2 - x + 1} \}} + C
 \end{aligned}$$

59. We have, $I = \int \frac{1 + x \cos x}{x \{1 - (x e^{\sin x})^2\}} dx$

$$\begin{aligned}
 &= \int \frac{(1 + x \cos x) e^{\sin x}}{x e^{\sin x} \{1 - (x e^{\sin x})^2\}} dx \\
 &= \int \frac{1}{x e^{\sin x} \{1 - (x e^{\sin x})^2\}} d(x e^{\sin x}) \\
 &= \int \frac{dt}{t(1 - t^2)}, \text{ where } t = x e^{\sin x}
 \end{aligned}$$

$$= \int \left(\frac{1}{t} + \frac{1}{2(1-t)} - \frac{1}{2(1+t)} \right) dt$$

$$= \log |t| - \frac{1}{2} \log |1-t| - \frac{1}{2} \log |1+t| + C$$

$$= \frac{1}{2} \log \left| \frac{t^2}{1-t^2} \right| + C = \frac{1}{2} \log \left| \frac{x^2 e^{2 \sin x}}{1-x^2 e^{2 \sin x}} \right| + C$$

60. We have, $\int \frac{2-x^2}{(1+x)\sqrt{1-x^2}} dx = \int \frac{(1-x^2)+1}{(1+x)\sqrt{1-x^2}} dx$

$$= \int \frac{1-x^2}{(1+x)\sqrt{1-x^2}} dx + \int \frac{1}{(1+x)\sqrt{1-x^2}} dx$$

$$= \int \frac{1-x}{\sqrt{1-x^2}} dx + \int \frac{1}{(1+x)\sqrt{1-x^2}} dx$$

$$= \int \frac{1}{\sqrt{1-x^2}} dx + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} dx + \int \frac{1}{(1+x)\sqrt{1-x^2}} dx$$

$$= \sin^{-1} x + \sqrt{1-x^2} + I_1, \text{ where } I_1 = \int \frac{1}{(1+x)\sqrt{1-x^2}} dx$$

On putting $1+x = \frac{1}{t}$, $dx = -\frac{1}{t^2} dt$ and $x = \frac{1-t}{t}$ in I_1 ,

we get

$$I_1 = \int \frac{1}{\frac{1}{t} \sqrt{1-\left(\frac{1-t}{t}\right)^2}} \times \left(\frac{-1}{t^2} dt \right)$$

$$= - \int \frac{1}{\sqrt{2t-1}} dt = -\sqrt{2t-1}$$

$$= -\sqrt{\frac{2}{1+x}} - 1 = -\sqrt{\frac{1-x}{1+x}}$$

$$\therefore \int \frac{2-x^2}{(1+x)\sqrt{1-x^2}} dx = \sin^{-1} x + \sqrt{1-x^2} - \sqrt{\frac{1-x}{1+x}} + C$$

61. Let $I = \int \sqrt{\tan^2 x + 2} dx$

On putting $\tan^2 x + 2 = t^2$

$$\Rightarrow 2 \tan x \sec^2 x dx = 2t dt$$

$$\Rightarrow dx = \frac{2t dt}{2 \tan x \sec^2 x} = \frac{t dt}{\sqrt{t^2-2} (t^2-1)}$$

$$\therefore I = \int \sqrt{t^2} \cdot \frac{t dt}{\sqrt{t^2-2} (t^2-1)}$$

$$= \int \frac{t^2}{(t^2-1)\sqrt{t^2-2}} dt = \int \frac{(t^2-1)+1}{(t^2-1)\sqrt{t^2-2}} dt$$

$$= \int \frac{1}{\sqrt{t^2-2}} dt + \int \frac{1}{(t^2-1)\sqrt{t^2-2}} dt$$

$$= \log \left| t + \sqrt{t^2-2} \right| + I_1,$$

where, $I_1 = \int \frac{1}{(t^2-1)\sqrt{t^2-2}} dt$

On putting $t = \frac{1}{u}$ and $dt = -\frac{1}{u^2} du$ in I_1 , we get

$$I_1 = - \int \frac{u du}{(1-u^2)\sqrt{1-2u^2}}$$

Now, on putting $1-2u^2 = v^2$ and $-2u du = v dv$, we get

$$I_1 = \frac{1}{2} \int \frac{v dv}{\left(1-\frac{1-v^2}{2}\right)\sqrt{v^2}} = \int \frac{dv}{(1+v^2)} = \tan^{-1} v$$

$$= \tan^{-1} \sqrt{1-2u^2} = \tan^{-1} \left(\frac{\sqrt{t^2-2}}{t} \right)$$

$$\therefore I = \log \left| t + \sqrt{t^2-2} \right| + \tan^{-1} \left(\frac{\sqrt{t^2-2}}{t} \right) + C$$

$$= \log \left| \sqrt{\tan^2 x + 2} + \tan x \right| + \tan^{-1} \left(\frac{\tan x}{\sqrt{\tan^2 x + 2}} \right) + C$$

62. We have, $I = \int \frac{x^{1/2}}{x^{1/2} - x^{1/3}} dx$

Clearly, LCM of 2 and 3 is 6. So, putting $x = t^6$ and $dx = 6t^5 dt$, we get

$$I = \int \frac{t^3}{t^3 - t^2} \cdot 6t^5 dt$$

$$= 6 \int \frac{t^6}{t-1} dt = 6 \int \frac{t^6-1+1}{t-1} dt$$

$$= 6 \int \left(t^5 + t^4 + t^3 + t^2 + t + 1 + \frac{1}{t-1} \right) dt$$

$$= 6 \left\{ \frac{t^6}{6} + \frac{t^5}{5} + \frac{t^4}{4} + \frac{t^3}{3} + \frac{t^2}{2} + t + \log(t-1) \right\} + C$$

$$= 6 \left\{ \frac{x}{6} + \frac{x^{5/6}}{5} + \frac{x^{2/3}}{4} + \frac{x^{1/2}}{3} + \frac{x^{1/3}}{2} + x^{1/6} + \log(x^{1/6}-1) \right\} + C$$

63. $\int \frac{\sec x}{\sqrt{\sin(2x+\alpha) + \sin \alpha}} dx$

$$= \int \frac{\sec x dx}{\sqrt{2 \sin(x+\alpha) \cos x}} = \int \frac{\sec^2 x dx}{\sqrt{2 \sin(x+\alpha) \cos x}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{\sec^2 x}{\sqrt{\tan x \cos \alpha + \sin \alpha}} dx$$

$$= \frac{1}{\sqrt{2} \cos \alpha} \int \frac{dt}{\sqrt{t}} \quad [\text{put } \tan x \cos \alpha + \sin \alpha = t]$$

$$= \frac{1}{\sqrt{2} \cos \alpha} 2\sqrt{t} + C$$

$$= \sqrt{2} \sec \alpha \sqrt{\tan x \cos \alpha + \sin \alpha} + C$$

$$= \sqrt{2} \sec \alpha (\tan x + \tan \alpha) + C$$

64. Here, $m = \frac{1}{2}$ and $n = \frac{1}{3}$

Clearly, LCM of 2 and 3 is 6. So, we substitute $x = t^6$ and $dx = 6t^5 dt$.

$$\therefore \int \sqrt{x}(1+\sqrt[3]{x})^4 dx = \int t^3(1+t^2)^4 \cdot 6t^5 dt$$

$$= 6 \int t^8(1+4t^2+6t^4+4t^6+t^8) dt$$

$$= 6 \int (t^8+4t^{10}+6t^{12}+4t^{14}+t^{16}) dt$$

$$= 6 \left\{ \frac{t^9}{9} + \frac{4t^{11}}{11} + \frac{6t^{13}}{13} + \frac{4t^{15}}{15} + \frac{t^{17}}{17} \right\} + C$$

$$= \frac{2}{3} x^{3/2} + \frac{24}{11} x^{11/6} + \frac{36}{13} x^{13/6} + \frac{8}{5} x^{5/2} + \frac{6}{17} x^{17/6} + C$$

65. We have, $\int x^5 \sqrt[3]{(1+x^3)^2} dx = \int x^5 (1+x^3)^{2/3} dx$
 So, we substituting $1+x^3=t^2$ and $3x^2 dx = 2t dt$ to obtain

$$\begin{aligned} \int x^5 (1+x^3)^{2/3} dx &= \int x^5 (t^2)^{2/3} \cdot \frac{2t}{3x^2} dt \\ &= \frac{2}{3} \int x^3 t^{7/3} dt = \frac{2}{3} \int (t^2-1)t^{7/3} dt \\ &= \frac{2}{3} \int (t^{13/3} - t^{7/3}) dt \\ &= \frac{2}{3} \left[\frac{3}{16} t^{16/3} - \frac{3}{10} t^{10/3} \right] + C \\ &= \frac{1}{8} (1+x^3)^{8/3} - \frac{1}{5} (1+x^3)^{5/3} + C \end{aligned}$$

66. We have, $\int x^{13/2} (1+x^{5/2})^{1/2} dx$
 We substitute $1+x^{5/2}=t^3$ and $\frac{5}{2} x^{3/2} dx = 3t^2 dt$

$$\begin{aligned} \therefore \int x^{13/2} (1+x^{5/2})^{1/2} dx &= \int x^{13/2} t^{3/2} \cdot \frac{6t^2}{5x^{3/2}} dt = \frac{6}{5} \int x^5 \cdot t^{7/2} dt \\ &= \frac{6}{5} \int (t^3-1)^2 t^{7/2} dt = \frac{6}{5} \int (t^6-2t^3+1)t^{7/2} dt \\ &= \frac{6}{5} \int (t^{19/2} - 2t^{13/2} + t^{7/2}) dt \\ &= \frac{6}{5} \left\{ \frac{2}{21} t^{21/2} - \frac{4}{15} t^{15/2} + \frac{2}{9} t^{9/2} \right\} + C \\ &= \frac{4}{35} (1+x^{5/2})^{7/2} - \frac{8}{25} (1+x^{5/2})^{5/2} + \frac{4}{15} (1+x^{5/2})^{3/2} + C \end{aligned}$$

67. $\int \tan^3 x (\sec^2 x - 1) dx = \int \tan^3 x \sec^2 x dx - \int \tan^3 x dx$

$$I_1 = \int \tan^3 x \sec^2 x dx = \frac{\tan^4 x}{4} + C_1$$

$$I_2 = \int \tan x (\sec^2 x - 1) dx = \frac{\tan^2 x}{2} - \ln |\sec x| + C_2$$

68. Put $t = \sin^2 x$
 The integral reduces to

$$I = \frac{1}{2} \int e^t (2-t) dt = \frac{3}{2} e^t - \frac{te^t}{2} + C$$

$$\Rightarrow I = \frac{1}{2} e^{\sin^2 x} (3 - \sin^2 x) + C$$

 or
$$I = e^{\sin^2 x} \left(1 + \frac{1}{2} \cos^2 x \right) + C$$

69. $\int \operatorname{cosec} 2x dx = \frac{1}{2} \int \frac{2 \operatorname{cosec} 2x (\operatorname{cosec} 2x - \cot 2x)}{\operatorname{cosec} 2x - \cot 2x} dx$

$$= \frac{1}{2} \log |\operatorname{cosec} 2x - \cot 2x| + C$$

$$= \frac{1}{2} \log \left| \frac{1 - \cos 2x}{\sin 2x} \right| + C$$

Thus, $f(x) = \frac{1}{2} \log |x|$ and $g(x) = \tan x$

Now, range $g(x) = (-\infty, \infty)$

and domain $f(x) = (-\infty, \infty) - \{0\}$

Moreover, $g'(x) = \sec^2 x$

and $f'(x) = \frac{1}{2x}, \forall x \in (0, \infty)$

70. $I = \int \frac{4e^{2x} + 6}{9e^{2x} - 4} dx$, put $9e^{2x} - 4 = z$
 Then, $I = \int \frac{1}{z} \left\{ 4 \cdot \frac{z+4}{9} + 6 \right\} \frac{1}{18} \cdot \frac{dz}{z+4}$

$$= \int \frac{1}{z(z+4)} \left\{ \frac{2z+8}{9} + 3 \right\} dz$$

$$= \frac{1}{9} \int \frac{2z+35}{z(z+4)} dz = \frac{1}{9} \int \frac{2(z+4)+27}{z(z+4)} dz$$

$$= \frac{2}{9} \int \frac{dz}{z} + \frac{3}{4} \int \left(\frac{1}{z} - \frac{1}{z+4} \right) dz$$

$$= \left(\frac{2}{9} + \frac{3}{4} \right) \log z - \frac{3}{4} \log (z+4) + C$$

$$= \frac{35}{36} \log (9e^{2x} - 4) - \frac{3}{4} \log (9e^{2x}) + C$$

$$= -\frac{3}{2} x + \frac{35}{36} \log (9e^{2x} - 4) - \frac{3}{2} \log 3 + C$$

71. Let $I = \int \sqrt{\operatorname{cosec} x + 1} dx = \int \frac{\cot x}{\sqrt{\operatorname{cosec} x - 1}} dx$
 Put $\operatorname{cosec} x - 1 = t^2 \Rightarrow -\operatorname{cosec} x \cot x dx = 2t dt$

$$\therefore I = - \int \frac{-\cot x \operatorname{cosec} x}{\operatorname{cosec} x \sqrt{\operatorname{cosec} x - 1}} dx = - \int \frac{2t dt}{1+t^2}$$

$$= -2 \tan^{-1} t + C = -2 \tan^{-1} \sqrt{\operatorname{cosec} x - 1} + C$$

$$= -2 \left[\frac{\pi}{2} - \cot^{-1} \sqrt{\operatorname{cosec} x - 1} \right] + C$$

$$= 2 \cot^{-1} \sqrt{\operatorname{cosec} x - 1} + C = 2 \cot^{-1} \frac{\cot x}{\sqrt{\operatorname{cosec} x + 1}} + C$$

72. $\int \sin x d(\sec x)$

$$= \int \sin x \frac{d(\sec x)}{dx} dx = \int \sin x \sec x \tan x dx$$

$$= \int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$$

$$\Rightarrow f(x) = \tan x, g(x) = x$$

73. $g(x) = \int x^{27} (1+x+x^2)^6 (6x^2+5x+4) dx$

$$= \int (x^4 + x^5 + x^6)^6 (6x^5 + 5x^4 + 4x^3) dx$$

 Let $x^6 + x^5 + x^4 = t$

$$\Rightarrow (6x^5 + 5x^4 + 4x^3) dx = dt$$

$$\therefore g(x) = \int t^6 dt = \frac{t^7}{7} + C = \frac{1}{7} (x^4 + x^5 + x^6)^7 + C$$

$$\therefore g(0) = 0 \Rightarrow C = 0 \Rightarrow g(1) = \frac{3^7}{7}, \text{ also } g(-1) = \frac{1}{7}$$

74. We have, $I = \int \frac{\sin x + \sin^3 x}{\cos 2x} dx$
 Put $\cos x = t \Rightarrow \cos 2x = 2t^2 - 1$
 and $dt = -\sin x dx$

$$\therefore I = \int \frac{t^2 - 2}{2t^2 - 1} dt = \frac{1}{2} \int \frac{2t^2 - 4}{2t^2 - 1} dt$$

$$= \frac{1}{2} \int dt - \frac{3}{2} \int \frac{dt}{2t^2 - 1}$$

$$= \frac{1}{2} t - \frac{3}{2\sqrt{2}} \times \frac{1}{2} \log \left| \frac{\sqrt{2}t - 1}{\sqrt{2}t + 1} \right| + R$$

$$= \frac{1}{2} \cos x - \frac{3}{4\sqrt{2}} \log \left| \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1} \right| + R$$

$$\text{So, } P = 1/2, Q = -\frac{3}{4\sqrt{2}}, f(x) = \frac{\sqrt{2} \cos x - 1}{\sqrt{2} \cos x + 1}$$

$$\text{or } P = 1/2, Q = \frac{3}{4\sqrt{2}}, f(x) = \frac{\sqrt{2} \cos x + 1}{\sqrt{2} \cos x - 1}$$

75. Put $x^{11/2} = t \Rightarrow \frac{11}{2} x^{\frac{11}{2}-1} dx = dt$

$$\Rightarrow x^{9/2} dx = \frac{2}{11} dt$$

$$\therefore \text{Given integral} = \frac{2}{11} \int \frac{dt}{\sqrt{1+t^2}}$$

$$= \frac{2}{11} \ln \left| t + \sqrt{1+t^2} \right| + C$$

$$= \frac{2}{11} \ln \left| x^{11/2} + \sqrt{1+x^{11}} \right| + C$$

76. $\int \{(\log_x e) - (\log_x e)^2\} dx$

$$= \int \left(\frac{1}{\ln x} - \frac{1}{(\ln x)^2} \right) dx = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt$$

(where $t = \ln x$)

$$= \frac{e^t}{t} + C = \frac{x}{\ln x} + C = x \log_x e + C$$

77. Since, $y = f^{-1}(x)$, therefore $x = f(y)$

$$\therefore dx = f'(y) dy$$

Thus, $\int f^{-1}(x) dx = \int y f'(y) dy + C$

$\therefore$ Statement II is true.

Statement I is true for $f^{-1}(x) = \sin^{-1}(x)$ in Statement II.

78. Since, $\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$,

$$\therefore d(\cot^{-1} x) = -d(\tan^{-1} x)$$

Thus, $\int 2^{\tan^{-1} x} d(\cot^{-1} x) = -\int 2^{\tan^{-1} x} d(\tan^{-1} x)$

$$= -\frac{2^{\tan^{-1} x}}{\ln 2} + C$$

$\therefore$ Statement I is false and Statement II is true.

79. Let $g(x) = f(x) + f(-x)$

Assuming $\int f(x) dx = F(x) + C$

$$\therefore \int g(x) dx = \int \{f(x) + f(-x)\} dx$$

$$= \int f(x) dx + \int f(-x) dx$$

$$= F(x) + C + [-F(-x) + C']$$

$$= F(x) - F(-x) + C + C'$$

which may be an odd function, if $C + C' = 0$.

Similarly, integral of an odd function is not always an even function.

Hence, Statement I is true and Statement II is false.

80. For $x^2 + 2(a-1)x + a + 5 = 0$

if $D < 0 \Rightarrow 4(a-1)^2 - 4(a+5) < 0$

$$\Rightarrow a^2 - 3a - 4 < 0 \text{ or } (a-4)(a+1) < 0 \text{ or } -1 < a < 4$$

Thus, for these values of a , $x^2 + 2(a-1)x + a + 5$ cannot factorised, so

$$\int \frac{dx}{x^2 + 2(a-1)x + a + 5} = \lambda \tan^{-1} |g(x)| + C$$

Hence, Statement I is false and Statement II is true.

81. $\int \frac{\sin x dx}{x}$ cannot be evaluated as there does not exist any method for evaluating this (integration by parts also does not work); however $\frac{\sin x}{x}$ ($x > 0$) is a differentiable function. Hence, both the statements are true but Statement II is not a correct explanation of Statement I.

82. $f(x) = \pi \sin \pi x + 2x - 4$

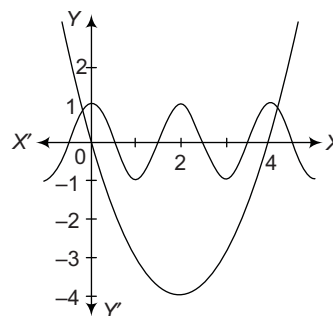
$$\Rightarrow g(x) = \int (\pi \sin \pi x + 2x - 4) dx$$

$$= -\cos \pi x + x^2 - 4x + C$$

Also, $g(1) = -2$

$$\Rightarrow 1 + 1 - 4 + C = -2 \Rightarrow C = 0$$

$$\Rightarrow g(x) = -\cos \pi x + x^2 - 4x$$



Hence, both statements are true but Statement II is not a correct explanation for Statement I.

83. As $m = 9 > 0$, hence we can substitute $\sqrt{9x^2 + 4x + 6} = u \pm 3x$.

84. Here, as per notations given, we can substitute $\sqrt{1+x^2} = (u-x)$.

As, $m = 1 > 0$ and $p = 1 > 0$

$$\Rightarrow I = \int \frac{u^{15}}{u} du = \int u^{14} du = \frac{1}{15} u^{15} + C$$

$$= \frac{1}{15} (x + \sqrt{1+x^2})^{15} + C$$

85. Here, $m = -1 < 0 \Rightarrow p = -2 < 0$

Also, $-x^2 + 3x - 2 = -(x-1)(x-2)$

We can use, case III

Putting $\sqrt{-x^2 + 3x - 2} = u(x-2)$

or $(x-1)u$

86. Let $I = \int \frac{x - \frac{2}{x^3}}{\sqrt{x^2 + 1 + \frac{2}{x^2}}} dx$

Put $x^2 + \frac{2}{x^2} + 1 = t$

$$\therefore I = \frac{1}{2} \int \frac{dt}{\sqrt{t}} = \frac{1}{2} \cdot \frac{t^{1/2}}{1/2} + C = \sqrt{t} + C$$

$$= \sqrt{x^2 + \frac{2}{x^2} + 1} + C$$

$$\Rightarrow 2 \left(x - \frac{2}{x^3} \right) dx = dt$$

$$87. \int \frac{x^2 - 1}{(x+1)^2 \sqrt{x^3 + x^2 + x}} dx$$

$$= \int \frac{\left(1 - \frac{1}{x^2}\right) dx}{\left(x + \frac{1}{x} + 2\right) \sqrt{x + \frac{1}{x} + 1}}$$

Put $x + \frac{1}{x} + 1 = t^2$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = 2t dt$$

$$= \int \frac{2t}{(t^2 + 1)t} dt$$

$$= 2 \int \frac{1}{(t^2 + 1)} dt$$

$$= 2 \tan^{-1}(t) + C$$

$$= 2 \tan^{-1} \left(\sqrt{x + \frac{1}{x} + 1} \right) + C$$

88. Divide numerator and denominator by x^{10} , we get

$$I = \int \frac{5x^{-6} + 4x^{-5}}{(1 + x^{-4} + x^{-5})^2} dx$$

Put $1 + x^{-4} + x^{-5} = t$

$$\Rightarrow (-4x^{-5} - 5x^{-6}) dx = dt$$

$$\therefore I = - \int \frac{dt}{t^2} = \frac{1}{t} + C$$

$$= \frac{1}{1 + x^{-4} + x^{-5}} + C$$

$$= \frac{x^5}{x^5 + x + 1} + C$$

$$89. I = \int \left(\frac{1}{\log_e x} - \frac{1}{(\log_e x)^2} \right) dx$$

Put $\log_e x = t$

$$\Rightarrow x = e^t$$

$$\Rightarrow dx = e^t dt$$

$$\therefore I = \int \left(\frac{1}{t} - \frac{1}{t^2} \right) e^t dt$$

$$= e^t \cdot \frac{1}{t} + C$$

$$= \frac{x}{\log x} + C$$

$$90. \int e^x \frac{x}{(1+x)^2} dx = \int e^x \left(\frac{1}{1+x} - \frac{1}{(1+x)^2} \right) dx$$

$$= \frac{e^x}{x+1} + C$$

$$91. I = \int e^{\sin x} \cos x \cdot x dx - \int e^{\sin x} (\sec x \tan x) dx$$

$$= x \cdot e^{\sin x} - \int 1 \cdot e^{\sin x} dx - \int e^{\sin x} \cdot (\sec x \tan x) dx$$

$$= x \cdot e^{\sin x} - \int e^{\sin x} dx - e^{\sin x} \cdot \sec x + \int e^{\sin x} dx$$

$$= x \cdot e^{\sin x} - e^{\sin x} \cdot \sec x + C$$

92. Here, $a = 1 > 0$; therefore we make the substitution

$\sqrt{x^2 + 2x + 2} = t - x$. Squaring both sides of this equality and reducing the similar terms, we get

$$2x + 2tx = t^2 - 2 \Rightarrow x = \frac{t^2 - 2}{2(1+t)} \Rightarrow dx = \frac{t^2 + 2t + 2}{2(1+t)^2} dt,$$

$$1 + \sqrt{x^2 + 2x + 2} = 1 + t - \frac{t^2 - 2}{2(1+t)} = \frac{t^2 + 4t + 4}{2(1+t)}.$$

Substituting into the integral, we get

$$I = \int \frac{2(1+t)(t^2 + 2t + 2)}{(t^2 + 4t + 4)2(1+t)^2} dt = \int \frac{(t^2 + 2t + 2)}{(1+t)(t+2)^2} dx$$

Now, let us expand the obtained proper rational fraction into partial fractions.

$$\frac{t^2 + 2t + 2}{(t+1)(t+2)^2} = \frac{A}{t+1} + \frac{B}{t+2} + \frac{D}{(t+2)^2}$$

$$93. I = \int \frac{dx}{x + \sqrt{x^2 - x + 1}}$$

Since, $c = 1 > 0$, we can apply the second Euler substitution.

$$\therefore \sqrt{x^2 - x + 1} = tx - 1 \Rightarrow (2t - 1)x = (t^2 - 1)x^2$$

$$\Rightarrow x = \frac{2t - 1}{t^2 - 1}$$

Substituting into I , we get an integral of a rational fraction

$$\int \frac{dx}{x + \sqrt{x^2 - x + 1}} = \int \frac{-2t^2 + 2t - 2}{t(t-1)(t+1)^2} dt$$

$$\text{Now, } \frac{-2t^2 + 2t - 2}{t(t-1)(t+1)^2} = \frac{A}{t} + \frac{B}{t-1} + \frac{D}{t+1} + \frac{E}{(t+1)^2}$$

94. In this case, $a < 0$ and $c < 0$; therefore neither first nor second Euler's substitution is applicable. But the quadratic $7x - 10 - x^2$ has real roots $\alpha = 2$, $\beta = 5$; therefore we use the third Euler's substitution.

$$\therefore \sqrt{7x - 10 - x^2} = \sqrt{(x-2)(5-x)} = (x-2)t$$

$$\Rightarrow 5 - x = (x-2)t^2 \Rightarrow x = \frac{5 + 2t^2}{t^2 + 1}$$

Solutions (Q. Nos. 95-97)

From the given data, we can conclude that $\frac{dy}{dx} = 0$,

at $x = 1, 2, 3$.

Hence, $f'(x) = a(x-1)(x-2)(x-3)$, $a > 0$

$$\Rightarrow f(x) = \int a(x^3 - 6x^2 + 11x - 6) dx$$

$$= a \int (x^3 - 6x^2 + 11x - 6) dx$$

$$= a \left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x \right) + C$$

Also, $f(0) = 1 \Rightarrow C = 1$

$$\Rightarrow f(x) = a \left(\frac{x^4}{4} - 2x^3 + \frac{11x^2}{2} - 6x \right) + 1 \dots (i)$$

$$f(1) = a \left(-\frac{9}{4} \right) + 1, f(2) = -2a + 1,$$

$$f(3) = a \left(-\frac{9}{4} \right) + 1 \dots (ii)$$

So, the graph is symmetrical about line $x = 2$ and the range is $[f(1), \infty)$ or $[f(3), \infty)$.

95. $f(1) = -8 \Rightarrow a = 4$

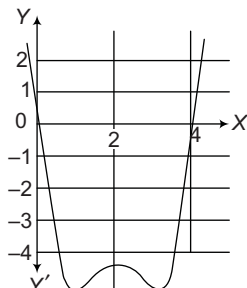
$\Rightarrow f(2) = -7$

96. $f(3) = -8$

Hence, the range is $[-8, \infty)$.

97. If $f(2) = 0$, then $a = 1/2$

If $f(1) = 0$, then $a = 4/9$



For four roots of $f(x) = 0$, $a \in [4/9, 1/2]$

98. A. $\int \frac{e^{2x}}{e^{2x} + 1} dx - 2 \int \frac{e^x}{e^{2x} + 1} dx$

$= \frac{1}{2} \ln(1 + e^{2x}) - 2 \tan^{-1}(e^x) + C$

B. $x + \sqrt{x^2 + 2} = t \Rightarrow x^2 + 2 = t^2 + x^2 - 2tx$

$\Rightarrow x = \frac{1}{2} \left(t - \frac{2}{t} \right)$

So, $I = \frac{1}{2} \int t^{1/2} \left(1 + \frac{2}{t^2} \right) dt$

$I = \frac{1}{2} \int t^{1/2} dt + \int t^{-3/2} dt$
 $= \frac{1}{3} t^{3/2} - \frac{2}{\sqrt{t}} + C$

C. $\int \frac{\cos 8x - \cos 7x}{1 + 2 \cos 5x} dx$

$= \int \frac{-2 \sin \frac{5x}{2} \sin \frac{x}{2} (3 - 2 + 2 \cos 5x)}{1 + 2 \cos 5x} dx$

$= \int (\cos 3x - \cos 2x) dx = \frac{\sin 3x}{3} - \frac{\sin 2x}{2} + C$
 $\left[\because \sin 3\left(\frac{5x}{2}\right) = 3 \sin \frac{5x}{2} - 4 \sin^3 \frac{5x}{2} \right]$

D. $\int \frac{\ln x}{x^3} dx = (\ln x) \left(-\frac{1}{2x^2} \right) + \int \frac{1}{x} \cdot \frac{1}{2x^2} dx$
 $= \left(-\frac{1}{2} \right) \frac{\ln x}{x^2} + \left(-\frac{1}{4} \right) \frac{1}{x^2} + C$

99. A. $\int \frac{c}{(x-a)(x-b)} dx$

$= \frac{c}{b-a} \int \left\{ \frac{(x-a)}{(x-a)(x-b)} - \frac{(x-b)}{(x-a)(x-b)} \right\} dx$

$= \frac{c}{b-a} \int \left(\frac{1}{x-b} - \frac{1}{x-a} \right) dx$

$= \frac{c}{b-a} \log \left| \frac{x-b}{x-a} \right| + K$

or $\frac{c}{a-b} \log \left| \frac{x-a}{x-b} \right| + K$

[from Eq. (ii)]

B. $I = \int \frac{x-c}{(x-a)(x-b)} dx$

Using partial fraction,

$I = \int \left\{ \left(\frac{a-c}{a-b} \right) \cdot \frac{1}{x-a} + \left(\frac{b-c}{b-a} \right) \cdot \frac{1}{x-b} \right\} dx$

$= \frac{1}{a-b} \{ (a-c) \log |x-a| - (b-c) \log |x-b| \} + K$

C. $I = \int \frac{cx+1}{x(ax-b)} dx$

Using partial fractions,

$I = \int \left\{ -\frac{1}{b} \cdot \frac{1}{x} + \frac{(cb+a)}{b} \cdot \frac{1}{(ax-b)} \right\} dx$

$= \frac{1}{b} \left\{ -\log x + \frac{(cb+a)}{a} \log |ax-b| \right\} + K$

$= \frac{c}{a} \log |ax-b| + \frac{1}{b} \log \left| \frac{ax-b}{x} \right| + K$

D. $I = \int \frac{cx+a}{(x-b)} dx$

$I = \int \left(\frac{c(x-b)}{x-b} + \frac{bc+a}{x-b} \right) dx$

$= cx + (bc+a) \log |x-b| + K$

100. A. $I = \int \left(\frac{x^2 + \cos^2 x}{1 + x^2} \right) \cdot \operatorname{cosec}^2 x dx$

$= \int \left(\frac{x^2 + 1 - \sin^2 x}{1 + x^2} \right) \cdot \operatorname{cosec}^2 x dx$

$= \int \left(\operatorname{cosec}^2 x - \frac{1}{1+x^2} \right) dx$

$= -\cot x + \cot^{-1} x + K \Rightarrow A = 1, B = -1$

B. Here, $I = \int \frac{\cos 5x + \cos 4x}{1 - 2 \cos 3x} dx$

$= \int \frac{2 \cos \frac{9x}{2} \cdot \cos \frac{x}{2}}{1 - 2 \left(2 \cos^2 \frac{3x}{2} - 1 \right)} dx$

$= \int \frac{2 \cos \frac{x}{2} \cos \frac{9x}{2}}{3 - 4 \cos^2 \frac{3x}{2}} dx$

Multiplying and dividing it by $\cos \frac{3x}{2}$, we get

$I = \int \frac{2 \cos \frac{x}{2} \cdot \cos \frac{3x}{2} \cdot \cos \frac{9x}{2}}{3 \cos \frac{3x}{2} - 4 \cos^3 \frac{3x}{2}} dx$

$= \int \frac{2 \cos x/2 \cdot \cos 3x/2 \cdot \cos 9x/2}{-\cos 9x/2} dx$

$[\because \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta]$

$= - \int 2 \cos \frac{x}{2} \cos \frac{3x}{2} dx$

$= - \int (\cos 2x + \cos x) dx$

$= - \left(\frac{\sin 2x}{2} + \sin x \right) + C$

$\Rightarrow A = -1 \text{ and } B = -1$

$$\begin{aligned}
 \text{C. } I &= \int \frac{2-x-x^2}{x^2 \sqrt{2-x-x^2}} dx \\
 &= -\int \frac{dx}{\sqrt{2-x-x^2}} + 2 \int \frac{dx}{x^2 \sqrt{2-x-x^2}} \\
 &\quad - \int \frac{dx}{x \sqrt{2-x-x^2}} \\
 &= -\int \frac{dx}{\sqrt{\frac{9}{4} - \left(x + \frac{1}{2}\right)^2}} + 2 \int \frac{dt}{\sqrt{2 - \frac{1}{t} - \frac{1}{t^2}}} \\
 &\quad + \int \frac{\frac{1}{t} dt}{\sqrt{2 - \frac{1}{t} - \frac{1}{t^2}}} \quad \left[\text{put } x = \frac{1}{t} \right] \\
 &= -\sin^{-1} \left(\frac{2x+1}{3} \right) - \frac{1}{2} \int \frac{(4t-1)dt}{\sqrt{2t^2-t-1}} \\
 &\quad + \frac{1}{2} \int \frac{dt}{\sqrt{2t^2-t-1}} \\
 &= -\sin^{-1} \left(\frac{2x+1}{3} \right) - \frac{1}{2} \cdot \frac{\sqrt{2t^2-t-1}}{1/2} \\
 &\quad + \frac{1}{2\sqrt{2}} \int \frac{dt}{t^2 - \frac{t}{2} - \frac{1}{2} + \frac{1}{16} - \frac{1}{16}} \\
 &= -\sin^{-1} \left(\frac{2x+1}{3} \right) - \frac{\sqrt{2-x-x^2}}{x} \\
 &\quad + \frac{1}{2\sqrt{2}} \log \left| \left(t - \frac{1}{4} \right) + \sqrt{2t^2-t-1} \right| + K \\
 &= -\frac{\sqrt{2-x-x^2}}{x} + \frac{1}{2\sqrt{2}} \log \left| \frac{(4-x)+4\sqrt{2-x-x^2}}{4x} \right| \\
 &\quad - \sin^{-1} \left(\frac{2x+1}{3} \right) + K
 \end{aligned}$$

$$\Rightarrow A = -1, B = 2$$

$$\begin{aligned}
 \text{D. } \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx \\
 &= \int \frac{2 \tan x \cdot \sec^2 x}{\tan^4 x + 1} dx \\
 &\quad [\text{put } \tan^2 x = t \Rightarrow 2 \tan x \cdot \sec^2 x dx = dt] \\
 &= \int \frac{dt}{t^2 + 1} = -\cot^{-1}(t) + C = -\cot^{-1}(\tan^2 x) + C
 \end{aligned}$$

$$\therefore B = -1$$

$$101. \text{ A. } \int \frac{x^2 - x + 1}{x^3 - 4x^2 + 4x} dx = \int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \right] dx$$

$$\text{B. } \int \frac{x^2 - 1}{x(x-2)^3} dx = \int \left[\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} + \frac{D}{(x-2)^3} \right] dx$$

$$\begin{aligned}
 \text{C. } \int \frac{x^3 + 1}{x(x-2)^2} dx &= \int \left[\left(\frac{x^3 + 1}{x(x-2)^2} - 1 \right) + 1 \right] dx \\
 &= \int \left[\left(\frac{x^3 + 1 - x(x-2)^2}{x(x-2)^2} \right) + 1 \right] dx \\
 &= \int \left[\left(\frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2} \right) + 1 \right] dx
 \end{aligned}$$

$$\text{D. } \int \frac{x^5 + 1}{x(x-2)^3} dx = \int \left[x + k + \frac{g(x)}{x(x-2)^3} \right] dx$$

where, k is constant $\neq 0$ and $g(x)$ is a polynomial of degree less than 4.

$$102. \text{ A. Let } I = \int \frac{2^x}{\sqrt{1-4^x}} dx = \frac{1}{\log 2} \int \frac{1}{\sqrt{1-t^2}} dt$$

$$[\text{putting } 2^x = t \Rightarrow 2^x \log 2 dx = dt]$$

$$I = \frac{1}{\log 2} \sin^{-1} \left(\frac{t}{1} \right) + C = \frac{1}{\log 2} \sin^{-1}(2^x) + C$$

$$\therefore k = \frac{1}{\log 2}$$

$$\text{B. } \int \frac{dx}{(\sqrt{x})^2 + (\sqrt{x})^7} = \int \frac{dx}{(\sqrt{x})^7 \left(1 + \frac{1}{(\sqrt{x})^5} \right)}$$

$$\text{Put } \frac{1}{(\sqrt{x})^5} = y \Rightarrow \frac{dy}{dx} = -\frac{5}{2(\sqrt{x})^7}$$

$$\therefore I = \int \frac{-2 dy}{5(1+y)} = -\frac{2}{5} \ln |1+y| + C$$

$$= \frac{2}{5} \ln \left(\frac{1}{1 + \frac{1}{(\sqrt{x})^5}} \right) + C$$

$$\Rightarrow a = \frac{2}{5}, k = \frac{5}{2}$$

C. Add and subtract $2x^2$ in the numerator, then on simplifying, we get $k = 1$ and $m = 1$.

$$\begin{aligned}
 \text{D. } I &= \int \frac{dx}{5 + 4 \cos x} \\
 &= \int \frac{dx}{5 \left(\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2} \right) + 4 \left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)}
 \end{aligned}$$

$$= \int \frac{dx}{9 \cos^2 \frac{x}{2} + \sin^2 \frac{x}{2}} = \int \frac{\sec^2 \frac{x}{2}}{9 + \tan^2 \frac{x}{2}} dx$$

$$\text{Put } t = \tan \frac{x}{2} \Rightarrow 2 dt = \sec^2 \frac{x}{2} dx$$

$$\therefore I = \int \frac{2 dt}{9 + t^2} = \frac{2}{3} \tan^{-1} \left(\frac{t}{3} \right) + C$$

$$= \frac{2}{3} \tan^{-1} \left(\frac{\tan \left(\frac{x}{2} \right)}{3} \right) + C$$

$$\text{Hence, } k = \frac{2}{3}, m = \frac{1}{3}$$

$$103. f(x) = \int x^{\sin x} (1 + x \cos x \cdot \ln x + \sin x) dx$$

If $F(x) = x^{\sin x} = e^{\sin x \ln x}$, then

$$f(x) = \int [F(x) + x F'(x)] dx = x F(x) + C$$

$$\Rightarrow f(x) = x \cdot x^{\sin x} + C$$

$$f\left(\frac{\pi}{2}\right) = \frac{\pi}{2} \cdot \frac{\pi}{2} + C \Rightarrow C = 0$$

$$\therefore f(x) = x(x)^{\sin x} \cdot f(\pi) = \pi(\pi)^0 = \pi$$

104. $f'(x) \cdot f'(x) - f(x) \cdot f''(x) = 0$

or $\frac{\{f'(x)\}^2 - f(x)f''(x)}{\{f'(x)\}^2} = 0$

$\Rightarrow \frac{d}{dx} \left[\frac{f(x)}{f'(x)} \right] = 0$

On integrating, we get $\frac{f(x)}{f'(x)} = C$

Put $x = 0$, $\frac{f(0)}{f'(0)} = C \Rightarrow C = \frac{1}{2}$

Hence, $\frac{f(x)}{f'(x)} = \frac{1}{2}$... (i)

From Eq. (i), $2f(x) = f'(x)$

$\therefore \frac{f'(x)}{f(x)} = 2$

Again integrating, we get $\ln[f(x)] = 2x + k$

Put $x = 0$, we get $k = 0$

$f(x) = e^{2x}$

$\Rightarrow \lambda + k = 2 + 0 = 2$

105. $F(x) = \int \frac{3x+2}{\sqrt{x-9}} dx$,

Put $x - 9 = t^2 \Rightarrow dx = 2t dt$

$\therefore F(x) = \int \left(\frac{3(t^2+9)+2}{t} \cdot 2t \right) dt$
 $= 2 \int (29 + 3t^2) dt = 2[29t + t^3] + C$
 $F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}] + C$

Given, $F(10) = 60 = 2[29 + 1] + C \Rightarrow C = 0$

$\therefore F(x) = 2[29\sqrt{x-9} + (x-9)^{3/2}]$

$F(13) = 2[29 \times 2 + 4 \times 2] = 4 \times 33 = 132$

Hence, sum of digits = $1 + 3 + 2 = 6$

106. $I = \int \frac{(e^x + \cos x + 1) - (e^x + \sin x + x)}{e^x + \sin x + x} dx$

$= \ln(e^x + \sin x + x) - x + C$

$\therefore f(x) = e^x + \sin x + x$ and $g(x) = -x$

$f(x) + g(x) = e^x + \sin x$

$\Rightarrow \frac{f(x) + g(x)}{e^x + \sin x} = 1$

107. $I = \int \sin(100x + x) \cdot (\sin x)^{99} dx$

$= \int \{\sin(100x)\cos x + \cos(100x) \cdot \sin x\} \cdot (\sin x)^{99} dx$

$= \int \sin(100x) (\cos x \cdot (\sin x)^{99}) dx$
 $+ \int \cos(100x) \cdot (\sin x)^{100} dx$

$= \frac{\sin(100x)(\sin x)^{100}}{100} - \frac{100}{100} \int \cos(100x)(\sin x)^{100} dx$

$+ \int \cos(100x) \cdot (\sin x)^{100} dx$

$= \frac{\sin(100x)(\sin x)^{100}}{100} + C$

$\Rightarrow \lambda = 100, \mu = 100$

$\therefore \frac{\lambda}{\mu} = \frac{100}{100} = 1$

108. $\frac{d}{dx} (A \ln |\cos x + \sin x - 2| + Bx + C)$

$= A \frac{\cos x - \sin x}{\cos x + \sin x - 2} + B$

$= \frac{A \cos x - A \sin x + B \cos x + B \sin x - 2B}{\cos x + \sin x - 2}$

$\therefore 2 = A + B, -1 = -A + B, \lambda = -2B$

$\therefore A = 3/2, B = 1/2, \lambda = -1$

109. $k(x) = \int \frac{(x^2 + 1) dx}{(x^3 + 3x + 6)^{1/3}}$

Put $x^3 + 3x + 6 = t^3$

$\Rightarrow 3(x^2 + 1) dx = 3t^2 dt$

$\therefore k(x) = \int \frac{t^2 dt}{t} = \frac{t^2}{2} + C$

$k(x) = \frac{1}{2} (x^3 + 3x + 6)^{2/3} + C$

$k(-1) = \frac{1}{2} (2)^{2/3} + C \Rightarrow C = 0$

$\therefore k(x) = \frac{1}{2} (x^3 + 3x + 6)^{2/3}$

$\Rightarrow k(-2) = \frac{1}{2} (-8)^{2/3}$
 $= \frac{1}{2} [(-2)^3]^{2/3} = 2$

110. $\int \left[\left(\frac{x}{e} \right)^x + \left(\frac{e}{x} \right)^x \right] \ln x dx$

Put $\left(\frac{x}{e} \right)^x = t$

or $x \ln \left(\frac{x}{e} \right) = \ln t$

$\therefore \left(x \cdot \frac{1}{x/e} \cdot \frac{1}{e} + \ln \left(\frac{x}{e} \right) \right) dx = \frac{1}{t} dt$

$\therefore (1 + \ln x - \ln e) dx = \frac{1}{t} dt$

$\therefore (\ln e + \ln x - \ln e) dx = \frac{1}{t} dt$

$\therefore (\ln x) dx = \frac{1}{t} dt$

$\therefore I = \int \left(t + \frac{1}{t} \right) \frac{1}{t} dt = \int 1 \cdot dt + \int \frac{1}{t^2} dt$

$= t - \frac{1}{t} + C$

or $I = \left(\frac{x}{e} \right)^x - \left(\frac{e}{x} \right)^x + C$

$\therefore A + B = 1 - 1 = 0$

111. $f \circ g(x) = \sqrt{e^x - 1}$

$\therefore I = \int \sqrt{e^x - 1} dx = \int \frac{2t^2}{t^2 + 1} dt$ [where, $\sqrt{e^x - 1} = t$]

$= 2t - 2 \tan^{-1} t + C$

$= 2\sqrt{e^x - 1} - 2 \tan^{-1} (\sqrt{e^x - 1}) + C$

$= 2f \circ g(x) - 2 \tan^{-1} \{f \circ g(x)\} + C$

$\therefore A + B = 2 + (-2) = 0$

Entrances Gallery

9

Indefinite Integrals

Target Exercises

1. Let $I = \int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$

Put $\sec x + \tan x = t$
 $\Rightarrow (\sec x \tan x + \sec^2 x) dx = dt$
 $\Rightarrow \sec x \cdot t dx = dt \Rightarrow \sec x dx = \frac{dt}{t}$
 $\therefore \sec x - \tan x = \frac{1}{t} \Rightarrow \sec x = \frac{1}{2} \left(t + \frac{1}{t} \right)$
 $\therefore I = \int \frac{\sec x \cdot \sec x}{(\sec x + \tan x)^{9/2}} dx$
 $\Rightarrow I = \int \frac{\frac{1}{2} \left(t + \frac{1}{t} \right) \cdot \frac{dt}{t}}{\frac{9}{t^2}} = \frac{1}{2} \int \left(\frac{1}{t^{9/2}} + \frac{1}{t^{13/2}} \right) dt$
 $= -\frac{1}{2} \left\{ \frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}} \right\} + K$
 $= -\left[\frac{1}{7(\sec x + \tan x)^{7/2}} + \frac{1}{11(\sec x + \tan x)^{11/2}} \right] + K$
 $= \frac{-1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7} (\sec x + \tan x)^2 \right\} + K$

2. Given, $f\left(\frac{3x-4}{3x+4}\right) = x+2$

Let $\frac{3x-4}{3x+4} = X$
 $\Rightarrow \frac{3x-4+3x+4}{3x-4-(3x+4)} = \frac{X+1}{X-1}$
 [applying componendo and dividendo rule]
 $\Rightarrow \frac{6x}{-8} = \frac{X+1}{X-1}$
 $\Rightarrow x = \frac{4(X+1)}{3(1-X)}$
 $\therefore f(X) = \frac{4(X+1)}{3(1-X)} + 2 = \frac{10-2X}{3(1-X)}$
 Now, $\int f(x) dx = \int \frac{10-2x}{3(1-x)} dx = \frac{2}{3} \int \frac{5-x}{1-x} dx$
 $= \frac{2}{3} \left[\int dx + \int \frac{4}{1-x} dx \right]$
 $= \frac{2}{3} [x - 4 \log |1-x|] + C$
 $= \frac{2}{3} x - \frac{8}{3} \log |1-x| + C$

3. Let $I = \int \frac{dx}{x^2 (x^4 + 1)^{3/4}} = \int \frac{dx}{x^5 \left(1 + \frac{1}{x^4}\right)^{3/4}}$

Put $1 + \frac{1}{x^4} = t^4$
 $\Rightarrow -\frac{4}{x^5} dx = 4t^3 dt$
 $\Rightarrow \frac{dx}{x^5} = -t^3 dt$
 $\therefore I = -\int dt = -t + C = -\left(1 + \frac{1}{x^4}\right)^{1/4} + C$

4. $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}} dx = \int e^{x+\frac{1}{x}} dx + \int x \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx$
 $= \int e^{x+\frac{1}{x}} dx + x e^{x+\frac{1}{x}} - \int \frac{d}{dx} (x) e^{x+\frac{1}{x}} dx + C$
 $\left[\because \int \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}} dx = e^{x+\frac{1}{x}} \right]$
 $= \int e^{x+\frac{1}{x}} dx + x \cdot e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}} dx + C$
 $= x \cdot e^{x+\frac{1}{x}} + C$

5. Let $I = \int \frac{x^6}{x + x^7} dx = \int \frac{x^6}{x(1 + x^6)} dx$
 $\Rightarrow I = \int \frac{(1 + x^6) - 1}{x(1 + x^6)} dx$
 $\Rightarrow I = \int \frac{dx}{x} - \int \frac{dx}{x + x^7} = \log |x| - p(x) + C$

6. LHS $= \int \left[\frac{x^2 + 1}{x^2 + 1} - \frac{x}{x^2 + 1} \right] e^{\cot^{-1} x} dx$
 $= \int 1 \cdot e^{\cot^{-1} x} dx - \int \frac{x}{x^2 + 1} e^{\cot^{-1} x} dx$
 $= x e^{\cot^{-1} x} - \int x \cdot e^{\cot^{-1} x} \left(-\frac{1}{1 + x^2} \right) dx$
 $- \int \frac{x}{1 + x^2} e^{\cot^{-1} x} dx + C$
 $= x e^{\cot^{-1} x} + C$

7. Let $I = \int \frac{x}{2 - x^2 + \sqrt{2 - x^2}} dx$
 Put $t^2 = 2 - x^2 \Rightarrow t dt = -x dx$
 $\therefore I = \int \frac{dt}{t+1} = -\log |1+t| + C = -\log |1 + \sqrt{2 - x^2}| + C$

8. LHS $= \int \frac{2 \cos^2 4x}{\cos^2 2x - \sin^2 2x} dx$
 $\frac{\cos 2x \sin 2x}{\cos 4x}$
 $= \int \frac{2 \cos^2 4x \times \cos 2x \sin 2x}{\cos 4x} dx$
 $= \int \cos 4x \times \sin 4x dx = \frac{1}{2} \int \sin 8x dx$
 $= \frac{1}{2} \left(-\frac{\cos 8x}{8} \right) + k$

9. Given, $\int f(x) dx = \psi(x)$
 Let $I = \int x^5 f(x^3) dx$
 Put $x^3 = t \Rightarrow x^2 dx = \frac{dt}{3}$... (i)
 $\therefore I = \frac{1}{3} \int t f(t) dt = \frac{1}{3} [t \psi(t) - \int \psi(t) dt]$
 $= \frac{1}{3} [x^3 \psi(x^3) - 3 \int x^2 \psi(x^3) dx] + C$
 [from Eq. (i)]
 $= \frac{1}{3} x^3 \psi(x^3) - \int x^2 \psi(x^3) dx + C$

10. Given, $\int \frac{5 \tan x}{\tan x - 2} dx = x + a \ln |\sin x - 2 \cos x| + K$... (i)

Now, let us assume that

$$I = \int \frac{5 \tan x}{\tan x - 2} dx$$

On multiplying by $\cos x$ in numerator and denominator, we get

$$I = \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

$$\text{Let } 5 \sin x = A(\sin x - 2 \cos x) + B(\cos x + 2 \sin x)$$

$$\Rightarrow 0 \cos x + 5 \sin x = (A + 2B) \sin x + (B - 2A) \cos x$$

On comparing the coefficients of $\sin x$ and $\cos x$, we get

$$A + 2B = 5 \text{ and } B - 2A = 0$$

$$\Rightarrow A = 1 \text{ and } B = 2$$

$$\Rightarrow 5 \sin x = (\sin x - 2 \cos x) + 2(\cos x + 2 \sin x)$$

$$\therefore I = \int \frac{5 \sin x}{\sin x - 2 \cos x} dx$$

$$\Rightarrow I = \int \frac{(\sin x - 2 \cos x) + 2(\cos x + 2 \sin x)}{(\sin x - 2 \cos x)} dx$$

$$\Rightarrow I = \int \frac{\sin x - 2 \cos x}{\sin x - 2 \cos x} dx + 2 \int \frac{(\cos x + 2 \sin x)}{(\sin x - 2 \cos x)} dx$$

$$\Rightarrow I = \int 1 dx + 2 \int \frac{d(\sin x - 2 \cos x)}{(\sin x - 2 \cos x)}$$

$$I = x + 2 \log |(\sin x - 2 \cos x)| + K \quad \dots (ii)$$

where, K is the constant of integration.

By comparing the value of I in Eqs. (i) and (ii), we get

$$a = 2$$

11. Let $I = \sqrt{2} \int \frac{\sin x}{\sin \left(x - \frac{\pi}{4}\right)} dx$

Put $x - \frac{\pi}{4} = t \Rightarrow dx = dt$

$$\begin{aligned} \therefore I &= \sqrt{2} \int \frac{\sin \left(\frac{\pi}{4} + t\right)}{\sin t} \\ &= \sqrt{2} \int \left[\frac{1}{\sqrt{2}} \cot t + \frac{1}{\sqrt{2}} \right] dt \\ &= \log |\sin t| + t + c \\ &= x + \log \left| \sin \left(x - \frac{\pi}{4}\right) \right| + C \end{aligned}$$

where, $C = c - \frac{\pi}{4}$

12. Now, $\int \frac{dx}{\cos x + \sqrt{3} \sin x}$

$$\begin{aligned} &= \int \frac{dx}{2 \left(\frac{1}{2} \cos x + \frac{\sqrt{3}}{2} \sin x \right)} \\ &= \frac{1}{2} \int \sec \left(x - \frac{\pi}{3} \right) dx \\ &= \frac{1}{2} \log \tan \left(\frac{x}{2} - \frac{\pi}{6} + \frac{\pi}{4} \right) + C \\ &= \frac{1}{2} \log \tan \left(\frac{x}{2} + \frac{\pi}{12} \right) + C \end{aligned}$$

13. $\int \left\{ \frac{(\log x - 1)}{1 + (\log x)^2} \right\}^2 dx = \int \frac{(\log x)^2 + 1 - 2 \log x}{[(\log x)^2 + 1]^2} dx$

$$= \int \frac{(\log x)^2 + 1 - 2x \left(\log x \cdot \frac{1}{x} \right)}{[(\log x)^2 + 1]^2} dx$$

$$= \int \frac{d}{dx} \left(\frac{x}{(\log x)^2 + 1} \right) dx = \frac{x}{(\log x)^2 + 1} + C$$

14. Let $I = \int \frac{\sin x}{\sin(x - \alpha)} dx$

Put $x - \alpha = t \Rightarrow dx = dt$

$$\therefore I = \int \frac{\sin(t + \alpha)}{\sin t} dt$$

$$\Rightarrow I = \int \frac{\sin t \cos \alpha + \cos t \sin \alpha}{\sin t} dt$$

$$\Rightarrow I = \cos \alpha \int 1 dt + \sin \alpha \int \frac{\cos t}{\sin t} dt$$

$$\Rightarrow I = \cos \alpha(t) + \sin \alpha \log \sin t + C_1$$

$$\Rightarrow I = \cos \alpha(x - \alpha) + \sin \alpha \log \sin(x - \alpha) + C_1$$

$$\Rightarrow I = x \cos \alpha + \sin \alpha \log \sin(x - \alpha) + C$$

[let $C = -\alpha \cos \alpha + C_1$]

But $I = Ax + B \log \sin(x - \alpha) + C$

$$\therefore x \cos \alpha + \sin \alpha \log \sin(x - \alpha) + C = Ax + B \log \sin(x - \alpha) + C$$

On comparing both sides, we get

$$A = \cos \alpha \text{ and } B = \sin \alpha$$

Aliter

Given that,

$$\int \frac{\sin x}{\sin(x - \alpha)} dx = Ax + B \log \sin(x - \alpha) + C$$

On differentiating both sides w.r.t. x , we get

$$\frac{\sin x}{\sin(x - \alpha)} = A + B \frac{\cos(x - \alpha)}{\sin(x - \alpha)}$$

$$\Rightarrow \sin x = A \sin(x - \alpha) + B \cos(x - \alpha)$$

$$\Rightarrow \sin x = A(\sin x \cos \alpha - \cos x \sin \alpha) + B(\cos x \cos \alpha + \sin x \sin \alpha)$$

$$\Rightarrow \sin x = \sin x(A \cos \alpha + B \sin \alpha) + \cos x(B \cos \alpha - A \sin \alpha)$$

On comparing the coefficients of $\sin x$ and $\cos x$ both sides, we get

$$A \cos \alpha + B \sin \alpha = 1 \quad \dots (i)$$

$$\text{and } B \cos \alpha - A \sin \alpha = 0 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$A = \cos \alpha \text{ and } B = \sin \alpha$$

15. Let $I = \int \frac{dx}{\cos x - \sin x}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\left(\frac{1}{\sqrt{2}} \cos x - \frac{1}{\sqrt{2}} \sin x \right)}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\cos \left(x + \frac{\pi}{4} \right)} = \frac{1}{\sqrt{2}} \int \sec \left(x + \frac{\pi}{4} \right) dx$$

$$= \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{\pi}{4} + \frac{x}{2} + \frac{\pi}{8} \right) \right| + C$$

$$= \frac{1}{\sqrt{2}} \log \left| \tan \left(\frac{x}{2} + \frac{3\pi}{8} \right) \right| + C$$

16. Let $I = \int \frac{dx}{x(x^n + 1)} = \int \frac{x^{n-1}}{x^n(x^n + 1)} dx$
 Put $x^n + 1 = t \Rightarrow nx^{n-1} dx = dt$
 $\therefore I = \frac{1}{n} \int \frac{dt}{t(t-1)} = \frac{1}{n} \int \left(\frac{1}{t-1} - \frac{1}{t} \right) dt$
 $= \frac{1}{n} \log \left(\frac{t-1}{t} \right) + C$
 $= \frac{1}{n} \log \left(\frac{x^n}{x^n + 1} \right) + C$

17. $\int \frac{dx}{\sin x - \cos x + \sqrt{2}}$
 $= \int \frac{dx}{\sqrt{2} \left(\frac{1}{\sqrt{2}} \sin x - \frac{1}{\sqrt{2}} \cos x \right) + \sqrt{2}}$
 $= \frac{1}{\sqrt{2}} \int \frac{dx}{1 - \cos \left(x + \frac{\pi}{4} \right)} = \frac{1}{\sqrt{2}} \int \frac{dx}{2 \sin^2 \left(\frac{x}{2} + \frac{\pi}{8} \right)}$
 $= \frac{1}{2\sqrt{2}} \int \frac{dx}{\sin^2 \left(\frac{x}{2} + \frac{\pi}{8} \right)}$
 $= \frac{1}{2\sqrt{2}} \int \operatorname{cosec}^2 \left(\frac{x}{2} + \frac{\pi}{8} \right) dx$
 $= \frac{1}{2\sqrt{2}} \left[-\cot \left(\frac{x}{2} + \frac{\pi}{8} \right) \right] + C$
 $= -\frac{1}{\sqrt{2}} \cot \left(\frac{x}{2} + \frac{\pi}{8} \right) + C$

18. $\int \sqrt{1 + \cos x} dx = \int \sqrt{2 \cos^2 \left(\frac{x}{2} \right)} dx$
 $= \sqrt{2} \int \cos \left(\frac{x}{2} \right) dx$
 $= \sqrt{2} \cdot \frac{\sin \left(\frac{x}{2} \right)}{\frac{1}{2}} + C = 2\sqrt{2} \sin \left(\frac{x}{2} \right) + C$

19. Given, $\int \frac{dx}{(1 + \sqrt{x})(\sqrt{x} - x^2)} = \frac{A\sqrt{x}}{\sqrt{1-x}} + \frac{B}{\sqrt{1-x}} + C$
 On differentiating both sides, we get
 $\frac{1}{(1 + \sqrt{x})(\sqrt{x} - x^2)} = \frac{A\sqrt{x}}{\sqrt{1-x}} + \frac{B}{\sqrt{1-x}} + C$
 $= \frac{\sqrt{1-x} \left(A \frac{1}{2\sqrt{x}} \right) - (A\sqrt{x} + B) \frac{1}{2\sqrt{1-x}} (-1)}{(\sqrt{1-x})^2}$
 $\Rightarrow \frac{1}{(1 + \sqrt{x})(\sqrt{x} - x^2)} = \frac{(1-x)A + Ax + B\sqrt{x}}{(1-x)2\sqrt{x}\sqrt{1-x}}$
 $\Rightarrow \frac{2(1-x)}{(1 + \sqrt{x})} = A + B\sqrt{x}$
 $\Rightarrow 2 - 2\sqrt{x} = A + B\sqrt{x}$
 $\Rightarrow A = 2, B = -2$
 $\therefore A + B = 0$

20. Given, $\int \frac{x^2 - 1}{(x+1)^2 \sqrt{x(x^2 + x + 1)}} dx$
 $= A \tan^{-1} \left(\sqrt{\frac{x^2 + x + 1}{x}} \right) + C$
 Now, $\frac{d}{dx} \left(\tan^{-1} \sqrt{\frac{x^2 + x + 1}{x}} \right)$
 $= \frac{1}{1 + \frac{x^2 + x + 1}{x}} \times \frac{d}{dx} \left(\sqrt{\frac{x^2 + x + 1}{x}} \right)$
 $= \frac{x}{x^2 + 2x + 1} \left[\frac{\sqrt{x} \times \frac{2x+1}{2\sqrt{x^2+x+1}} - \sqrt{x^2+x+1} \times \frac{1}{2\sqrt{x}}}{(\sqrt{x})^2} \right]$
 $= \frac{1}{x^2 + 2x + 1} \left[\frac{\sqrt{x}(2x+1)}{2\sqrt{x(x^2+x+1)}} - \frac{\sqrt{x^2+x+1}}{2\sqrt{x}} \right]$
 $= \frac{1}{x^2 + 2x + 1} \left[\frac{x(2x+1) - (x^2 + x + 1)}{2\sqrt{x(x^2+x+1)}} \right]$
 $= \frac{1}{(x+1)^2} \left[\frac{x^2 - 1}{2\sqrt{x(x^2+x+1)}} \right]$
 $= \frac{x^2 - 1}{2(x+1)^2 \sqrt{x(x^2+x+1)}}$
 $\therefore I = \int \frac{d}{dx} \left(2 \tan^{-1} \sqrt{\frac{x^2 + x + 1}{x}} \right) dx$
 $= 2 \tan^{-1} \left(\sqrt{\frac{x^2 + x + 1}{x}} \right) + C$

$\therefore A = 2$

21. $\int a^x dx = \frac{a^x}{\log_e a} + C$ [by formula]

22. $\int \frac{\cos 2x - 1}{\cos 2x + 1} dx$
 $= - \int \frac{1 - \cos 2x}{1 + \cos 2x} dx$
 $= - \int \frac{2 \sin^2 x}{2 \cos^2 x} dx = - \int \tan^2 x dx$
 $= - \int (\sec^2 x - 1) dx$
 $= - \tan x + x + C = x - \tan x + C$

23. $\int \frac{dx}{\sqrt{7-x^2}} = \int \frac{dx}{\sqrt{(\sqrt{7})^2 - x^2}} = \sin^{-1} \left(\frac{x}{\sqrt{7}} \right) + C$
 $\left[\because \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \left(\frac{x}{a} \right) \right]$

24. $\int \sin^{-1} \cos 3x dx$
 $= \int \sin^{-1} \left\{ \sin \left(\frac{\pi}{2} - 3x \right) \right\} dx$ [$\because \cos \theta = \sin (90^\circ - \theta)$]
 $= \int \left(\frac{\pi}{2} - 3x \right) dx = \frac{\pi}{2} x - \frac{3x^2}{2} + C$

25. Let $I = \int \frac{x^2 - 1}{(x^4 + 3x^2 + 1) \tan^{-1}\left(x + \frac{1}{x}\right)} dx$

On dividing numerator and denominator by x^2 , we get

$$I = \int \frac{1 - \frac{1}{x^2}}{\left(x^2 + \frac{1}{x^2} + 3\right) \tan^{-1}\left(x + \frac{1}{x}\right)} dx$$

Putting $x + \frac{1}{x} = t$

$$\Rightarrow \left(1 - \frac{1}{x^2}\right) dx = dt \text{ and } x^2 + \frac{1}{x^2} = t^2 - 2$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{(t^2 - 2 + 3) \tan^{-1} t} \\ &= \int \frac{dt}{(1 + t^2) \tan^{-1} t} = \log(\tan^{-1} t) + C \\ &= \log \left[\tan^{-1} \left(x + \frac{1}{x} \right) \right] + C \end{aligned}$$

26. Let $I = \int \frac{\cos x}{\sin^2 x (\sin x + \cos x)} dx$

On dividing numerator and denominator by $\cos^3 x$, we get

$$I = \int \frac{\sec^2 x}{\tan^2 x (1 + \tan x)} dx$$

Putting $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I = \int \frac{dt}{t^2 (1+t)} = \int \left[-\frac{1}{t} + \frac{1}{t^2} + \frac{1}{1+t} \right] dt$$

[using partial fraction]

$$\begin{aligned} \therefore I &= -\int \frac{1}{t} dt + \int t^{-2} dt + \int \frac{1}{1+t} dt \\ &= -\log |t| - \frac{1}{t} + \log |1+t| + C \\ &= -\frac{1}{\tan x} + \log \left| \frac{1 + \tan x}{\tan x} \right| + C \\ &= -\cot x + \log \left| \frac{1 + \tan x}{\tan x} \right| + C \end{aligned}$$

27. Let $I = \int \frac{\sin 2x}{\sin^2 x + 2 \cos^2 x} dx$

$$= \int \frac{\sin 2x}{1 - \cos^2 x + 2 \cos^2 x} dx$$

$$= \int \frac{\sin 2x}{1 + \cos^2 x} dx$$

Putting $1 + \cos^2 x = t$

$$\Rightarrow -2 \cos x \sin x dx = dt$$

$$\Rightarrow -\sin 2x dx = dt$$

$$\therefore I = -\int \frac{dt}{t} = -\log |t| + C = -\log (1 + \cos^2 x) + C$$

28. Given, $f(x) \cos x + g(x) \sin x + C$

$$= \int [(3x - 1) \cos x + (1 - 2x) \sin x] dx$$

$$= \int (3x - 1) \cos x dx + \int (1 - 2x) \sin x dx$$

$$= (3x - 1) \sin x - \int 3 \sin x dx - (1 - 2x) \cos x$$

$$+ \int (-2) \cos x dx \text{ [using integration by parts]}$$

$$\begin{aligned} &= (3x - 1) \sin x + 3 \cos x - (1 - 2x) \cos x - 2 \sin x + C \\ &= (3x - 1 - 2) \sin x + (3 - 1 + 2x) \cos x + C \\ &\Rightarrow f(x) \cos x + g(x) \sin x + C = 3(x - 1) \sin x \\ &\quad + 2(x + 1) \cos x + C \end{aligned}$$

On comparing the coefficients of $\cos x$ and $\sin x$, we get

$$f(x) = 2(x + 1) \text{ and } g(x) = 3(x - 1)$$

29. Let $I = \int \frac{(1+x)e^x}{\sin^2(xe^x)} dx$

Putting $xe^x = t \Rightarrow (1 \cdot e^x + x \cdot e^x) dx = dt$

$$\Rightarrow (1+x)e^x dx = dt$$

$$\begin{aligned} \therefore I &= \int \frac{dt}{\sin^2 t} = \int \operatorname{cosec}^2 t dt = -\cot t + C \\ &= -\cot(xe^x) + C \end{aligned}$$

30. Let $I = \int \frac{xe^x}{(1+x)^2} dx = \int \frac{(x+1-1)e^x}{(1+x)^2} dx$

$$= \int \frac{e^x}{(1+x)} dx - \int \frac{e^x}{(1+x)^2} dx$$

Applying integration by parts in first integral, we get

$$I = \frac{1}{(1+x)} e^x - \int \left(\frac{1}{(1+x)^2} \right) e^x dx - \int \frac{e^x}{(1+x)^2} dx + C$$

$$\Rightarrow I = \frac{e^x}{(1+x)} + C$$

31. Let $I = \int e^x (\sin x + 2 \cos x) \sin x dx$

$$= \int e^x \sin^2 x dx + \int 2e^x \sin x \cos x dx$$

Applying integration by parts in first integral, we get

$$\begin{aligned} I &= e^x \sin^2 x - \int 2 \sin x \cos x \cdot e^x dx \\ &\quad + \int 2e^x \sin x \cos x dx + C \\ &= e^x \cdot \sin^2 x + C \end{aligned}$$

32. Let $I = \int \frac{\sqrt{x^2 - 1}}{x} dx$

Putting $x = \sec \theta \Rightarrow dx = \sec \theta \tan \theta d\theta$

$$\therefore I = \int \frac{\sqrt{\sec^2 \theta - 1} \sec \theta \tan \theta d\theta}{\sec \theta}$$

$$= \int \tan \theta \cdot \tan \theta d\theta$$

$$= \int \tan^2 \theta d\theta = \int (\sec^2 \theta - 1) d\theta$$

$$= \tan \theta - \theta + C = \sqrt{\sec^2 \theta - 1} - \theta + C$$

$$= \sqrt{x^2 - 1} - \sec^{-1} x + C$$

33. Let $I = \int \frac{\sqrt{5+x^2}}{x^4} dx = \int \frac{x \sqrt{\frac{5}{x^2} + 1}}{x^4} dx = \int \frac{\sqrt{\frac{5}{x^2} + 1}}{x^3} dx$

Putting $\frac{5}{x^2} + 1 = t \Rightarrow -\frac{2 \times 5}{x^3} dx = dt$

$$\Rightarrow \frac{1}{x^3} dx = -\frac{1}{10} dt$$

$$\therefore I = -\frac{1}{10} \int \sqrt{t} dt = -\frac{1}{10} \cdot \frac{t^{3/2}}{3/2} + C$$

$$= -\frac{1}{15} t^{3/2} + C = -\frac{1}{15} \left(\frac{5}{x^2} + 1 \right)^{3/2} + C$$

34. Given, $\int \frac{dx}{\sqrt{\sin^3 x \cos x}} = g(x) + C$

Now, $\int \frac{dx}{\sqrt{\sin^4 x \cot x}} = \int \frac{dx}{\sin^2 x \sqrt{\cot x}}$
 $= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot x}} dx$

Putting $\cot x = t \Rightarrow -\operatorname{cosec}^2 x dx = dt$

$\therefore \int -\frac{1}{\sqrt{t}} dt = -\frac{t^{1/2}}{1/2} + C = -2\sqrt{\cot x} + C$
 $= -\frac{2}{\sqrt{\tan x}} + C$

$\therefore g(x) = -\frac{2}{\sqrt{\tan x}}$

35. Given, $I_n = \int \tan^n x dx$

$= \int \tan^{n-2} x \cdot \tan^2 x dx = \int \tan^{n-2} x (\sec^2 x - 1) dx$

$= \int \tan^{n-2} x \sec^2 x dx - \int \tan^{n-2} x dx$

$= \frac{\tan^{n-1} x}{n-1} - I_{n-2} \left[\text{put } \tan x = t \Rightarrow \sec^2 x dx = dt \right]$
 $\therefore \int t^{n-2} dt = \frac{t^{n-1}}{n-1} = \frac{\tan^{n-1} x}{n-1}$

But it given that, $I_n = \frac{1}{a} \tan^{n-1} x - b I_{n-2}$

$\therefore \frac{1}{a} = \frac{1}{n-1}$ and $b = 1$

$\Rightarrow a = n-1, b = 1$

$\therefore (a, b) = (n-1, 1)$

36. Let $I = \int \frac{x dx}{(1+x)^{3/2}}$

Putting $1+x = t \Rightarrow dx = dt$

$\therefore I = \int \frac{(t-1) dt}{t^{3/2}}$

$= \int (t^{-1/2} - t^{-3/2}) dt = (2t^{1/2} + 2t^{-1/2}) + C$

$= 2\left(\frac{t+1}{\sqrt{t}}\right) + C = 2\left(\frac{x+2}{\sqrt{1+x}}\right) + C$

37. Let $I = \int \frac{\sec^2(\sin^{-1} x)}{\sqrt{1-x^2}} dx$

Putting $\sin^{-1} x = t$

$\Rightarrow \frac{dt}{dx} = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow dt = \frac{1}{\sqrt{1-x^2}} dx$

$\therefore I = \int \sec^2 t dt = \tan t + C$

$= \tan(\sin^{-1} x) + C$

38. Let $I = \int \sec^3 x dx = \int \sec x \cdot \sec^2 x dx$

$\Rightarrow I = \sec x \tan x - \int \sec x \tan^2 x dx$

[using integration by parts]

$\Rightarrow I = \sec x \tan x - \int \sec x (\sec^2 x - 1) dx$

$\Rightarrow I = \sec x \tan x - \int \sec^3 x dx + \int \sec x dx$

$\Rightarrow I = \sec x \tan x - I + \log |\sec x + \tan x| + 2C$

$\Rightarrow 2I = \sec x \tan x + \log |\sec x + \tan x| + 2C$

$\therefore I = \frac{1}{2} [\sec x \tan x + \log |\sec x + \tan x|] + C$

39. $\int \frac{1}{x^3(x^3-1)} dx = \int \frac{1}{x^3-1} dx - \int \frac{1}{x^3} dx$

$= \int \frac{1}{(x-1)(x^2+x+1)} dx + \frac{1}{2x^2}$

$= \int \left[\frac{1}{3(x-1)} + \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1} \right] dx + \frac{1}{2x^2}$

$= \int \frac{1}{3(x-1)} dx - \frac{1}{3} \int \frac{x+2}{x^2+x+1} dx + \frac{1}{2x^2}$

$= \frac{1}{3} \log |x-1| + \frac{1}{2x^2} - \frac{1}{3} \int \frac{(2x+1) + \frac{5}{2}}{x^2+x+1} dx$

$= \frac{1}{3} \log |x-1| + \frac{1}{2x^2} - \frac{1}{6} \int \frac{2x+1}{x^2+x+1} dx$

$- \frac{1}{2} \int \frac{dx}{x^2+x+1}$

$= \frac{1}{3} \log |x-1| + \frac{1}{2x^2} - \frac{1}{6} \log |x^2+x+1|$

$- \frac{1}{2} \int \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= \frac{1}{3} \log |x-1| + \frac{1}{2x^2} - \frac{1}{6} \log |x^2+x+1|$

$- \frac{1}{2} \cdot \frac{1}{\frac{\sqrt{3}}{2}} \tan^{-1} \left(\frac{x + \frac{1}{2}}{\frac{\sqrt{3}}{2}} \right) + C$

$= \frac{1}{3} \log |x-1| + \frac{1}{2x^2} - \frac{1}{6} \log |x^2+x+1|$

$- \frac{1}{\sqrt{6}} \tan^{-1} \left(\frac{2x+1}{\sqrt{3}} \right) + C$

40. We have, $I_1 = \int \sin^{-1} x dx$

$I_2 = \int \sin^{-1} \sqrt{1-x^2} dx$

$= \int \cos^{-1} x dx$

[$\because \sin^{-1} \sqrt{1-x^2} = \cos^{-1} x$]

$\therefore I_1 + I_2 = \int (\sin^{-1} x + \cos^{-1} x) dx$

$= \int \frac{\pi}{2} dx = \frac{\pi}{2} x + C$

41. Let $I = \int \frac{(x^2-1)}{x^3 \sqrt{2x^4-2x^2+1}} dx$

$= \int \frac{(x^2-1)}{x^5 \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$

$= \int \frac{\left(\frac{1}{x^3} - \frac{1}{x^5}\right)}{\sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}}} dx$

$$\begin{aligned} \text{Putting } 2 - \frac{2}{x^2} + \frac{1}{x^4} &= t^2 \\ \Rightarrow \left(\frac{4}{x^3} - \frac{4}{x^5} \right) dx &= 2t dt \\ \Rightarrow \left(\frac{1}{x^3} - \frac{1}{x^5} \right) dx &= \frac{t}{2} dt \\ \therefore I &= \frac{1}{2} \int \frac{t dt}{t} = \frac{1}{2} t + C \\ &= \frac{1}{2} \sqrt{2 - \frac{2}{x^2} + \frac{1}{x^4}} + C \end{aligned}$$

42. We have, $f''(x) = g''(x)$, $f'(1) = 4$, $g'(1) = 6$
 $f(2) = 3$, $g(2) = 9$

Consider, $f''(x) = g''(x)$

On integrating both sides w.r.t. x , we get

$$\begin{aligned} f'(x) &= g'(x) + C_1 \\ \Rightarrow f'(1) &= g'(1) + C_1 \\ \Rightarrow C_1 &= 4 - 6 = -2 \\ \Rightarrow f'(x) &= g'(x) - 2 \end{aligned}$$

Again, integrating both sides w.r.t. x , we get

$$\begin{aligned} f(x) &= g(x) - 2x + C_2 \\ \Rightarrow f(2) &= g(2) - 4 + C_2 \\ \Rightarrow C_2 &= 3 - 9 + 4 = -2 \\ \Rightarrow f(x) &= g(x) - 2x - 2 \\ \Rightarrow f(x) - g(x) &= -2x - 2 \\ \therefore f(1) - g(1) &= -4 \end{aligned}$$

43. Let $I = \int \frac{2 dx}{(e^x + e^{-x})^2} = \int \frac{2 dx}{e^{2x} + e^{-2x} + 2}$
 $= \int \frac{2e^{2x} dx}{(e^{2x})^2 + 2e^{2x} + 1}$

Putting $e^{2x} = t$

$\Rightarrow 2e^{2x} dx = dt$

$$\begin{aligned} \therefore I &= \int \frac{dt}{t^2 + 2t + 1} = \int \frac{dt}{(t+1)^2} \\ &= \frac{-1}{1+t} + C = \frac{-1}{1+e^{2x}} + C = \frac{-e^{-x}}{e^{-x} + e^x} + C \end{aligned}$$

44. $\int e^x \cdot \left(\frac{x^2 + 1}{(x+1)^2} \right) dx = \int e^x \left[\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right] dx$
 $= e^x \left(\frac{x-1}{x+1} \right) + C$
 $[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + C]$

45. We have, $I_n = \int (\log x)^n dx = \int \frac{1}{x} (\log x)^n dx$
 $= \left[(\log x)^n \cdot x - \int n (\log x)^{n-1} \cdot \frac{1}{x} \cdot x dx \right] + C$
 $= x \cdot (\log x)^n - n \int (\log x)^{n-1} dx + C$
 $= x \cdot (\log x)^n - n I_{n-1} + C$
 $\Rightarrow I_n + n I_{n-1} = x (\log x)^n + C$

46. We have, $f(x) = \int \frac{dx}{(1+x^2)^{3/2}}$
 Let $I = \int \frac{dx}{(1+x^2)^{3/2}} = \int \frac{dx}{x^3 \left(1 + \frac{1}{x^2} \right)^{3/2}}$

Putting $1 + \frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3} dx = dt \Rightarrow \frac{dx}{x^3} = \frac{-1}{2} dt$
 $\therefore I = \frac{-1}{2} \int \frac{dt}{t^{3/2}} = \frac{-1}{2} \cdot \frac{t^{-1/2}}{\left(-\frac{1}{2}\right)} + C$
 $= \frac{1}{\sqrt{t}} + C = \frac{1}{\sqrt{1 + \frac{1}{x^2}}} + C$

$\Rightarrow f(x) = \frac{x}{\sqrt{1+x^2}} + C$

$\therefore f(0) = 0$

$\therefore C = 0$

$\therefore f(x) = \frac{x}{\sqrt{1+x^2}} \Rightarrow f(1) = \frac{1}{\sqrt{2}}$

47. Let $I = \int \frac{\sqrt{\cot x}}{\sin x \cos x} dx = \int \frac{\operatorname{cosec}^2 x \sqrt{\cot x}}{\cot x} dx$
 [dividing numerator and denominator by $\sin^2 x$]

$= \int \frac{\operatorname{cosec}^2 x}{\sqrt{\cot x}} dx$

Putting $\cot x = t \Rightarrow -\operatorname{cosec}^2 x dx = dt$

$\Rightarrow I = - \int \frac{dt}{\sqrt{t}} = -2\sqrt{t} + Q = -2\sqrt{\cot x} + Q$

$\therefore P = -2$

48. Let $I = \int \frac{2^x}{\sqrt{1-4^x}} dx = \int \frac{2^x}{\sqrt{1-(2^x)^2}} dx$

Putting $2^x = t$

$\Rightarrow 2^x \cdot \log 2 dx = dt$

$\Rightarrow 2^x dx = \frac{dt}{\log 2}$

$\therefore I = \frac{1}{\log 2} \int \frac{dt}{\sqrt{1-t^2}} = \frac{1}{\log 2} \sin^{-1}(t) + C$
 $= \frac{1}{\log 2} \sin^{-1}(2^x) + C$

49. $\int \frac{\sin^8 x - \cos^8 x}{1 - 2 \sin^2 x \cos^2 x} dx$
 $= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx$
 $= \int \frac{(\sin^4 x - \cos^4 x)(\sin^4 x + \cos^4 x)}{(\sin^4 x + \cos^4 x)} dx$
 $= \int (\sin^4 x - \cos^4 x) dx$
 $= \int (\sin^2 x - \cos^2 x)(\sin^2 x + \cos^2 x) dx$
 $= - \int \cos 2x dx = -\frac{\sin 2x}{2} + C$

50. $\int \frac{\sin x + \cos x}{\sqrt{1 + \sin 2x}} dx = \int \frac{\sin x + \cos x}{\sqrt{(\sin x + \cos x)^2}} dx$
 $= \int dx = x + C$

51. $\int \frac{dx}{\sin(x-a) \sin(x-b)}$
 $= \frac{1}{\sin(a-b)} \int \frac{\sin\{(x-b)-(x-a)\}}{\sin(x-a) \sin(x-b)} dx$

$$= \frac{1}{\sin(a-b)} \left[\int \cot(x-a) dx - \int \cot(x-b) dx \right]$$

$$= \frac{1}{\sin(a-b)} [\log |\sin(x-a)| - \log |\sin(x-b)|] + C$$

$$= \frac{1}{\sin(b-a)} \log \left| \frac{\sin(x-b)}{\sin(x-a)} \right| + C$$

52. Let $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx$

$$= \int \frac{\sin 2x}{(\sin^2 x + \cos^2 x)^2 - 2 \sin^2 x \cos^2 x} dx$$

$$= \int \frac{\sin 2x}{1 - \frac{1}{2}(\sin 2x)^2} dx = \int \frac{\sin 2x}{\frac{1}{2}(1 + \cos^2 2x)} dx$$

Putting $\cos 2x = t \Rightarrow -2 \sin 2x dx = dt$

$$\therefore I = - \int \frac{dt}{1+t^2} = -\tan^{-1}(\cos 2x) + C$$

53. Let $I = \int \frac{1 + \cot^2 x}{1 + \cot x} dx = \int \frac{\operatorname{cosec}^2 x}{1 + \cot x} dx$

Putting $1 + \cot x = t \Rightarrow -\operatorname{cosec}^2 x dx = dt$

$$\therefore I = - \int \frac{dt}{t} = -\log t + C = -\log(1 + \cot x) + C$$

54. $\int \frac{\sqrt{e^x + a}}{\sqrt{e^x - a}} dx = \int \frac{e^x}{\sqrt{e^{2x} - a^2}} dx + \int \frac{ae^x}{e^x \sqrt{e^{2x} - a^2}} dx$

Putting $e^x = t \Rightarrow e^x dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{t^2 - a^2}} + a \int \frac{dt}{t \sqrt{t^2 - a^2}}$$

$$= \cosh^{-1} \left(\frac{t}{a} \right) + \sec^{-1} \left(\frac{t}{a} \right) + C$$

$$= \cosh^{-1} \left(\frac{e^x}{a} \right) + \sec^{-1} \left(\frac{e^x}{a} \right) + C$$

55. $\int e^x \left[\frac{\sin x + \cos x}{1 - \sin^2 x} \right] dx = \int e^x \left(\frac{\sin x + \cos x}{\cos^2 x} \right) dx$

$$\int e^x (\sec x \tan x + \sec x) dx = e^x \sec x + C$$

$[\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C]$

56. Let $I = \int \frac{4e^x + 6e^{-x}}{9e^x - 4e^{-x}} dx = \int \frac{4e^{2x} + 6}{9e^{2x} - 4} dx$

$$= 4 \int \frac{e^{2x}}{9e^{2x} - 4} dx + 6 \int \frac{e^{-2x}}{9 - 4e^{-2x}} dx$$

Putting $t_1 = 9e^{2x} - 4$ and $t_2 = 9 - 4e^{-2x}$

$$\Rightarrow dt_1 = 18e^{2x} dx \text{ and } dt_2 = 8e^{-2x} dx$$

$$\therefore I = 4 \int \frac{1}{t_1} \cdot \frac{dt_1}{18} + 6 \int \frac{1}{t_2} \cdot \frac{dt_2}{8}$$

$$= \frac{2}{9} \int \frac{dt_1}{t_1} + \frac{3}{4} \int \frac{dt_2}{t_2} = \frac{2}{9} \log |t_1| + \frac{3}{4} \log |t_2| + C$$

$$= \frac{2}{9} \log |9e^{2x} - 4| + \frac{3}{4} \log |9 - 4e^{-2x}| + C$$

$$= \frac{2}{9} \log |9e^{2x} - 4| + \frac{3}{4} \log |9e^{2x} - 4| - \frac{3}{4} \log |e^{2x}| + C$$

$$= \left(\frac{2}{9} + \frac{3}{4} \right) \log |9e^{2x} - 4| - \frac{3}{4} (2x) + C \quad [\because \log e = 1]$$

$$= -\frac{3}{2} x + \frac{35}{36} \log |9e^{2x} - 4| + C$$

57. Let $I = \int \frac{dx}{x^2 \sqrt{4-x^2}}$

Putting $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2} dt$

$$\therefore I = \int \frac{\left(\frac{-1}{t^2} \right) dt}{\left(\frac{1}{t^2} \right) \sqrt{4 - \frac{1}{t^2}}} = \int \frac{-t dt}{\sqrt{4t^2 - 1}}$$

$$= \frac{-1}{8} \int \frac{8t dt}{\sqrt{4t^2 - 1}}$$

Now, again put $4t^2 - 1 = u^2$

$$\Rightarrow 8t dt = 2u du$$

$$\therefore I = \frac{-1}{8} \int \frac{2u}{u} du$$

$$= \frac{-1}{4} u + C = \frac{-1}{4} \sqrt{4t^2 - 1} + C$$

$$= \frac{-1}{4} \left(\frac{\sqrt{4-x^2}}{x} \right) + C$$

58. Let $I = \int e^{e^x} \cdot e^{e^x} \cdot e^x dx$

Putting $e^{e^x} = t$

$$\Rightarrow e^{e^x} \cdot e^{e^x} \cdot e^x dx = dt$$

$$\therefore I = \int dt = t + C = e^{e^x} + C$$

59. Given integral $= \int \frac{d(\cos \theta)}{\sqrt{1 - \cos^2 \theta}}$

We know that, $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C$

$$\therefore \int \frac{d(\cos \theta)}{\sqrt{1 - \cos^2 \theta}} = \sin^{-1}(\cos \theta) + C$$

60. $\int [\sin(\log x) + \cos(\log x)] dx$

$$= \int \frac{d}{dx} \{x \sin(\log x)\} dx$$

$$= x \sin(\log x) + C$$

61. $\int \frac{x^4 + x^2 + 1}{x^2 - x + 1} dx = \int (x^2 + x + 1) dx$

$$= \frac{x^3}{3} + \frac{x^2}{2} + x + C$$

62. $\int \operatorname{cosec}(x-a) \operatorname{cosec} x dx = \int \frac{\sin a}{\sin a \sin(x-a) \sin x} dx$

$$= -\frac{1}{\sin a} \int \frac{\sin[(x-a)-x]}{\sin(x-a) \sin x} dx$$

$$= -\frac{1}{\sin a} \int \left[\frac{\sin(x-a) \cos x - \cos(x-a) \sin x}{\sin(x-a) \sin x} \right] dx$$

$$= -\frac{1}{\sin a} \int [\cot x - \cot(x-a)] dx$$

$$= -\frac{1}{\sin a} [\log |\sin x| - \log |\sin (x-a)|] + C$$

$$= -\frac{1}{\sin a} \log \left| \frac{\sin x}{\sin (x-a)} \right| + C$$

$$= -\frac{1}{\sin a} [\log |\sin x \operatorname{cosec} (x-a)|] + C$$

63. Let $I = \int (x+1)(x+2)^7(x+3)dx$

Put $(x+1)(x+3) = (x+2-1)(x+2+1) = (x+2)^2 - 1$

$$\therefore I = \int \{(x+2)^9 - (x+2)^7\} dx$$

$$= \frac{(x+2)^{10}}{10} - \frac{(x+2)^8}{8} + C$$

64. Given, $\frac{d}{dx} \{f(x)\} = \frac{1}{1+x^2}$

On integrating both sides, we get

$$f(x) = \tan^{-1} x + C$$

$$\therefore \frac{d}{dx} f(x^3) = \frac{d}{dx} (\tan^{-1} x^3 + C)$$

$$= \frac{1}{1+(x^3)^2} \cdot 3x^2 = \frac{3x^2}{1+x^6}$$

65. Given, $u = -f'''(\theta) \sin \theta + f''(\theta) \cos \theta$

and $v = f'''(\theta) \cos \theta + f''(\theta) \sin \theta$

On differentiating w.r.t. θ respectively, we get

$$\frac{du}{d\theta} = -f'''(\theta) \sin \theta - f''(\theta) \cos \theta$$

$$+ f'''(\theta) \cos \theta - f''(\theta) \sin \theta$$

$$= -f'''(\theta) \sin \theta - f''(\theta) \sin \theta$$

and $\frac{dv}{d\theta} = f'''(\theta) \cos \theta - f''(\theta) \sin \theta$

$$+ f'''(\theta) \sin \theta + f''(\theta) \cos \theta$$

$$= f'''(\theta) \cos \theta + f''(\theta) \cos \theta$$

$$\therefore \left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2 = [f'''(\theta)]^2 + [f''(\theta)]^2 + 2f''(\theta)f'''(\theta)$$

$$= [f'''(\theta) + f''(\theta)]^2$$

$$\text{Now, } \int \left[\left(\frac{du}{d\theta}\right)^2 + \left(\frac{dv}{d\theta}\right)^2 \right]^{1/2} d\theta = \int [f'''(\theta) + f''(\theta)] d\theta$$

$$= f''(\theta) + f'(\theta) + C$$

66. $\int \frac{x^4 - 1}{x^2(x^4 + x^2 + 1)^{1/2}} dx = \int \frac{2x^4 + x^2 - (x^4 + x^2 + 1)}{x^2 \sqrt{x^4 + x^2 + 1}} dx$

$$= \int \frac{x(4x^3 + 2x)}{2\sqrt{x^4 + x^2 + 1}} - \frac{\sqrt{x^4 + x^2 + 1}}{x^2} dx$$

$$= \int \frac{d}{dx} \left(\frac{\sqrt{x^4 + x^2 + 1}}{x} \right) dx$$

$$= \frac{\sqrt{x^4 + x^2 + 1}}{x} + C$$

67. Given, $\frac{d}{dx} \left[a \tan^{-1} x + b \log \left(\frac{x-1}{x+1} \right) \right] = \frac{1}{x^4 - 1}$

On integrating both sides, we get

$$a \tan^{-1} x + b \log \left(\frac{x-1}{x+1} \right) = \frac{1}{2} \int \left[\frac{1}{x^2 - 1} - \frac{1}{x^2 + 1} \right] dx$$

$$\Rightarrow a \tan^{-1} x + b \log \left(\frac{x-1}{x+1} \right) = \frac{1}{4} \log \left(\frac{x-1}{x+1} \right) - \frac{1}{2} \tan^{-1} x + C$$

$$\Rightarrow a = -\frac{1}{2}, b = \frac{1}{4}$$

$$\therefore a - 2b = -\frac{1}{2} - 2 \left(\frac{1}{4} \right) = -1$$

68. $\int [e^{a \log x} + e^{x \log a}] dx$

$$= \int [e^{\log_e x^a} + e^{\log_e a^x}] dx$$

$$= \int [x^a + a^x] dx = \frac{x^{a+1}}{a+1} + \frac{a^x}{\log a} + C$$

69. $\int \frac{dx}{\sqrt{(1-x)(x-2)}} = \int \frac{dx}{\sqrt{-x^2 + 3x - 2}} = \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x - \frac{3}{2}\right)^2}}$

$$= \sin^{-1} \left[\frac{\left(x - \frac{3}{2}\right)}{\frac{1}{2}} \right] + C = \sin^{-1} (2x - 3) + C$$

70. $\int \frac{\sec^2 x - 7}{\sin^7 x} dx = \int \frac{\sin^7 x \sec^2 x - 7 \sin^7 x}{(\sin^7 x)^2} dx$

$$= \int \frac{d}{dx} \left(\frac{\tan x}{\sin^7 x} \right) dx = \frac{\tan x}{\sin^7 x} + C$$

71. Given, $f(x) = \lim_{y \rightarrow x} \frac{\sin^2 y - \sin^2 x}{y^2 - x^2} \left[\frac{0}{0} \text{ form} \right]$

$$= \lim_{y \rightarrow x} \frac{2 \sin y \cos y - 0}{2y - 0} \quad [\text{using L' Hospital's rule}]$$

$$= \frac{\sin 2x}{2x}$$

$$\therefore \int 4x f(x) dx = \int 4x \left(\frac{\sin 2x}{2x} \right) dx$$

$$= 2 \int \sin 2x dx = -\cos 2x + C$$

72. $\int e^x \left(\frac{x-1}{x^2} \right) dx = \int e^x \left(\frac{1}{x} - \frac{1}{x^2} \right) dx = \frac{e^x}{x} + C$

$$[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + C]$$

73. Let $I = \int \frac{\log \sqrt{x}}{3x} dx$

Putting $\sqrt{x} = t \Rightarrow \frac{1}{\sqrt{x}} dx = 2 dt$

$$\therefore I = \int \frac{2 \log t}{3t} dt = \frac{2}{3} \int \frac{\log t}{t} dt$$

$$= \frac{2}{3} \cdot \frac{(\log t)^2}{2} + C = \frac{1}{3} (\log \sqrt{x})^2 + C$$

74. $\int e^x \left(\frac{2}{x} - \frac{2}{x^2} \right) dx = \frac{2e^x}{x} + C$

$$\left[\because f(x) = \frac{2}{x} \text{ and } f'(x) = -\frac{2}{x^2} \right]$$

75. Let $I = \int (x^2 + 1) \sqrt{x+1} dx$

Putting $x+1 = t^2 \Rightarrow dx = 2t dt$

$$\Rightarrow x^2 + 1 = (t^2 - 1)^2 + 1 = t^4 - 2t^2 + 2$$

$$\therefore I = \int (t^4 - 2t^2 + 2) t \cdot 2t dt$$

$$= 2 \int (t^6 - 2t^4 + 2t^2) dt$$

$$= 2 \left[\frac{t^7}{7} - 2 \frac{t^5}{5} + 2 \frac{t^3}{3} \right] + C$$

$$= 2 \left[\frac{(x+1)^{7/2}}{7} - 2 \frac{(x+1)^{5/2}}{5} + 2 \frac{(x+1)^{3/2}}{3} \right] + C$$

76. Let $I = \int \frac{1+x}{x+e^{-x}} dx = \int \frac{e^x(x+1)}{xe^x+1} dx$

Putting $xe^x + 1 = t$

$$\Rightarrow (xe^x + e^x) dx = dt$$

$$\Rightarrow (x+1)e^x dx = dt$$

$$\therefore I = \int \frac{dt}{t} = \log |t| + C = \log |xe^x + 1| + C$$

77. $\int \frac{\cos x + x \sin x}{x^2 + x \cos x} dx = \int \frac{x + \cos x - x + x \sin x}{x^2 + x \cos x} dx$

$$= \int \frac{dx}{x} - \int \frac{x(1 - \sin x)}{x(x + \cos x)} dx$$

$$= \log |x| - \log |x + \cos x| + C$$

$$= \log \left| \frac{x}{x + \cos x} \right| + C$$

78. Let $I = \int \frac{dx}{\sqrt{1-e^{2x}}} = \int \frac{e^{-x}}{\sqrt{e^{-2x}-1}} dx$

Putting $e^{-x} = t \Rightarrow e^{-x} dx = -dt$

$$\therefore I = - \int \frac{dt}{\sqrt{t^2-1}} = -\log \left| t + \sqrt{t^2-1} \right| + C$$

$$= -\log \left| e^{-x} + \sqrt{e^{-2x}-1} \right| + C$$

79. $\int \frac{x+2}{2x^2+6x+5} dx = \frac{1}{4} \int \frac{4x+6+2}{2x^2+6x+5} dx$

$$= \frac{1}{4} \int \frac{4x+6}{2x^2+6x+5} dx + \frac{1}{2} \int \frac{dx}{2x^2+6x+5}$$

$$\therefore P = \frac{1}{4}$$

80. Let $I = \int \frac{\log(x + \sqrt{1+x^2})}{\sqrt{1+x^2}} dx$

Putting $x + \sqrt{1+x^2} = t$

$$\Rightarrow \left(1 + \frac{1}{2\sqrt{1+x^2}} \cdot 2x \right) dx = dt$$

$$\Rightarrow \frac{x + \sqrt{1+x^2}}{\sqrt{1+x^2}} dx = dt$$

$$\Rightarrow \frac{dx}{\sqrt{1+x^2}} = \frac{dt}{t}$$

$$\therefore I = \int \frac{\log t}{t} dt = \frac{(\log t)^2}{2} + C$$

$$= \frac{1}{2} [\log(x + \sqrt{1+x^2})]^2 + C$$

81. We know that, if $I_n = \int \sin^n x dx$, then

$$I_n = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} I_{n-2}$$

where, n is a positive integer.

$$\Rightarrow nI_n - (n-1)I_{n-2} = -\sin^{n-1} x \cos x$$

10

Definite Integrals

Introduction

Every definite integral has a unique value.

Let $\phi(x)$ be the anti-derivative of a function $f(x)$ defined on $[a, b]$, i.e. $\frac{d}{dx} [\phi(x)] = f(x)$.

Definite integral of $f(x)$ over $[a, b]$ is denoted by $\int_a^b f(x) dx$ and is defined as $[\phi(b) - \phi(a)]$,

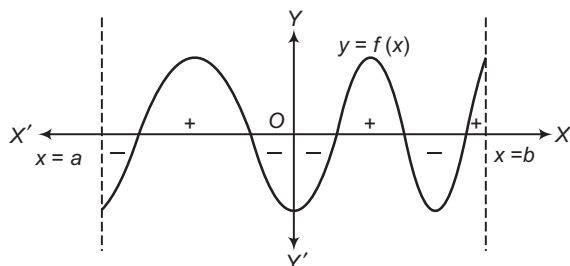
i.e. $\int_a^b f(x) dx = \phi(b) - \phi(a)$.

This is also called Newton-Leibnitz's formula.

The numbers a and b are called the limits of integration, where a is called lower limit and b is upper limit. The interval $[a, b]$ is called the interval of integration. The interval $[a, b]$ is also known as range of integration.

Geometrical Interpretation of Definite Integral

Let $f(x)$ be a function defined on a closed interval $[a, b]$. Then, $\int_a^b f(x) dx$ represents the algebraic sum of the areas of the region bounded by the curve $y = f(x)$, X -axis and the lines $x = a$, $x = b$. Here, algebraic sum means that area which is above the X -axis will be added in this sum with '+' sign and area which is below the X -axis will be added in this sum with '-' sign. So, value of the definite integral may be positive, zero or negative.



Chapter Snapshot

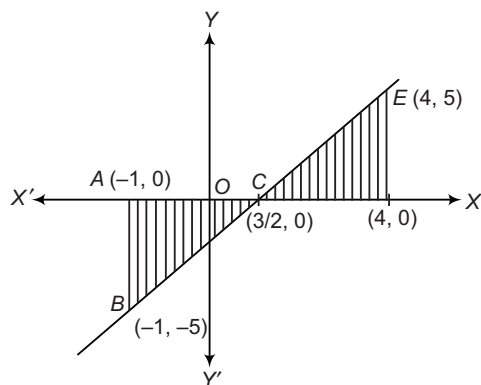
- Introduction
- Geometrical Interpretation of Definite Integral
- Fundamental Theorems of Calculus
- Evaluation of Definite Integrals by Substitution
- Properties of Definite Integrals
- Definite Integral as a Limit of Sum
- Gamma Function
- Reduction Formulae in Definite Integrals

- In a definite integral, there is no need to keep the constant of integration.
- The indefinite integral $\int f(x) dx$ is a function of x , whereas definite integral $\int_a^b f(x) dx$ is a number.
- In $\int f(x) dx$, we can find $\int_a^b f(x) dx$ but if $\int_a^b f(x) dx$ we cannot find $\int f(x) dx$.

➤ **Example 1.** $\int_{-1}^4 (2x - 3) dx$ is equal to

- (a) $\frac{3}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 0

Sol. (d) $y = 2x - 3$ is a straight line, which lie below the X-axis in $\left[-1, \frac{3}{2}\right]$ and above in $\left(\frac{3}{2}, 4\right]$.



Now, area of $\triangle ABC = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$
 Area of $\triangle CDE = \frac{1}{2} \times \frac{5}{2} \times 5 = \frac{25}{4}$
 $\therefore \int_{-1}^4 (2x - 3) dx = -\frac{25}{4} + \frac{25}{4} = 0$

Fundamental Theorems of Calculus

The fundamental theorems of calculus are theorems that links the concept of the derivative of a function with the concept of the integral.

First Fundamental Theorem of Calculus

Let f be a continuous real valued function defined on a closed interval $[a, b]$ and F be the function defined for all x in $[a, b]$ by

$$F(x) = \int_a^x f(t) dt$$

Then, F is continuous on $[a, b]$, differentiable on the open interval (a, b) and $F'(x) = f(x)$ for all x in (a, b) .

Second Fundamental Theorem of Calculus/Newton-Leibnitz's Axiom

Let f and g be real valued functions defined on a closed interval $[a, b]$ such that the derivative of g is f , i.e. f and g are functions such that for all x in $[a, b]$,

$$f(x) = g'(x)$$

If f is Riemann integrable on $[a, b]$, then

$$\int_a^b f(x) dx = g(b) - g(a)$$

➤ **Example 2.** The value of $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$ is

- (a) $\frac{\pi}{2} + 1$ (b) $\frac{\pi}{2} - 1$
 (c) -1 (d) 1

Sol. (b) We have,

$$\begin{aligned} \int_0^1 \sqrt{\frac{1-x}{1+x}} dx &= \int_0^1 \frac{1-x}{\sqrt{1-x^2}} dx \\ &= \int_0^1 \frac{1}{\sqrt{1-x^2}} dx - \int_0^1 \frac{x}{\sqrt{1-x^2}} dx \\ &= [\sin^{-1} x]_0^1 + [\sqrt{1-x^2}]_0^1 = \frac{\pi}{2} - 1 \end{aligned}$$

➤ **Example 3.** $\int_{\pi/4}^{3\pi/4} \frac{1}{1+\cos x} dx$ is equal to

- (a) 2 (b) -2 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Sol. (a) Let $I = \int_{\pi/4}^{3\pi/4} \frac{1}{1+\cos x} dx$
 $= \int_{\pi/4}^{3\pi/4} \frac{1-\cos x}{1-\cos^2 x} dx$
 $= \int_{\pi/4}^{3\pi/4} (\operatorname{cosec}^2 x - \cot x \operatorname{cosec} x) dx$
 $= [-\cot x + \operatorname{cosec} x]_{\pi/4}^{3\pi/4}$
 $= [(1 + \sqrt{2}) - (-1 + \sqrt{2})] = 2$

➤ **Example 4.** If $af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5, x \neq 0$,

$a \neq b$, then $\int_1^2 f(x) dx$ equals

- (a) $\frac{(\ln 2 - 5)a + \frac{13}{2}b}{a^2 - b^2}$ (b) $\frac{(\ln 2 - 5)a + \frac{7b}{2}}{a^2 - b^2}$
 (c) $\frac{(5 - \ln 2)a + \frac{7b}{2}}{a^2 - b^2}$ (d) None of these

Sol. (b) We have,

$$af(x) + bf\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad \dots(i)$$

$$\Rightarrow bf(x) + af\left(\frac{1}{x}\right) = x - 5 \quad \left[\text{replacing } x \text{ by } \frac{1}{x}\right] \dots(ii)$$

On solving Eqs. (i) and (ii) for $f(x)$, we get

$$f(x) = \frac{1}{a^2 - b^2} \left\{ \frac{a}{x} - bx - 5a + 5b \right\}$$

$$\therefore \int_1^2 f(x) dx = \frac{1}{a^2 - b^2} \left[a \ln x - b \frac{x^2}{2} - 5(a - b)x \right]_1^2$$

$$\Rightarrow \int_1^2 f(x) dx = \frac{1}{a^2 - b^2} \left[a \ln 2 - \frac{3b}{2} - 5(a - b) \right]$$

$$\Rightarrow \int_1^2 f(x) dx = \frac{1}{a^2 - b^2} \left[(\ln 2 - 5)a + \frac{7b}{2} \right]$$

Evaluation of Definite Integrals by Substitution

When the variable in a definite integral is changed, then the substitution in terms of new variable should be effected at three places.

- In the integrand
- In the differential (say dx)
- In the limits

The limits of the new variable (say t) are simply the values of t corresponding to the values of the original variable (say x) and so they can be easily obtained by putting the values of x in the substitutional relation between x and t .

➤ **Example 5.** $\int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$ is equal to

- (a) $\frac{1}{6}$ (b) $\frac{1}{12}$
(c) $\frac{1}{4}$ (d) None of these

Sol. (a) Let $I = \int_0^{\pi/4} \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$
 $= \int_0^{\pi/4} \frac{\tan^2 x \sec^2 x}{(1 + \tan^3 x)^2} dx$
 $= \int_1^2 \frac{dz}{3z^2} \quad \left[\begin{array}{l} \text{putting } 1 + \tan^3 x = z \\ \Rightarrow \tan^2 x \sec^2 x dx = \frac{1}{3} dz \end{array} \right]$
 $= \left[\frac{-1}{3z} \right]_1^2 = -\frac{1}{3} \left(\frac{1}{2} - 1 \right) = \frac{1}{6}$

➤ **Example 6.** $\int_0^a \frac{dx}{a + \sqrt{a^2 - x^2}}$ is equal to

- (a) $\frac{\pi}{2} + 1$ (b) $\frac{\pi}{2} - 1$
(c) $1 - \frac{\pi}{2}$ (d) None of these

Sol. (b) Let $I = \int_0^a \frac{dx}{a + \sqrt{a^2 - x^2}} = \int_0^{\pi/2} \frac{a \cos \theta}{a + a \cos \theta} d\theta$
 $\left[\text{putting } x = a \sin \theta \Rightarrow dx = a \cos \theta d\theta \right]$
 $= \int_0^{\pi/2} \frac{\cos \theta}{1 + \cos \theta} d\theta = \int_0^{\pi/2} \left(1 - \frac{1}{1 + \cos \theta} \right) d\theta$
 $= \int_0^{\pi/2} 1 d\theta - \frac{1}{2} \int_0^{\pi/2} \sec^2 \frac{\theta}{2} d\theta$
 $= \left[\theta - \tan \frac{\theta}{2} \right]_0^{\pi/2} = \frac{\pi}{2} - \tan \frac{\pi}{4} = \left(\frac{\pi}{2} - 1 \right)$

➤ **Example 7.** The value of $\int_0^r xy dx$, where

$y = \sqrt{r^2 - x^2}$, is

- (a) $\frac{r^3}{6}$ (b) $\frac{r^3}{3}$
(c) $\frac{r^3}{2}$ (d) None of these

Sol. (b) Let $I = \int_0^r xy dx = \int_0^r x \sqrt{r^2 - x^2} dx = \int_{r^2}^0 \left(-\frac{1}{2} \sqrt{z} \right) dz$
 $\left[\text{put } r^2 - x^2 = z \Rightarrow x dx = \left(-\frac{1}{2} \right) dz \right]$
 $= \frac{1}{2} \int_0^{r^2} \sqrt{z} dz = \left[\frac{z^{3/2}}{3} \right]_0^{r^2} = \frac{1}{3} (r^3)$

➤ **Example 8.** If $I_n = \int_0^{\pi/4} \tan^n \theta d\theta$, then $I_8 + I_6$ equals

- (a) $\frac{1}{4}$ (b) $\frac{1}{5}$ (c) $\frac{1}{6}$ (d) $\frac{1}{7}$

Sol. (d) We have, $I_n = \int_0^{\pi/4} \tan^n \theta d\theta$
 $\therefore I_8 + I_6 = \int_0^{\pi/4} (\tan^8 \theta + \tan^6 \theta) d\theta$
 $= \int_0^{\pi/4} \tan^6 \theta \sec^2 \theta d\theta = \int_0^1 t^6 dt$
 $\left[\text{put } \tan \theta = t \Rightarrow \sec^2 \theta d\theta = dt \right]$
 $= \left[\frac{t^7}{7} \right]_0^1 = \frac{1}{7}$

➤ **Example 9.** $\int_0^{2\pi} e^{x/2} \sin \left(\frac{x}{2} + \frac{\pi}{4} \right) dx$ equals

- (a) 2π (b) e^π
(c) 0 (d) $2\sqrt{2}$

Sol. (c) Let $I = \int_0^{2\pi} e^{x/2} \sin \left(\frac{x}{2} + \frac{\pi}{4} \right) dx$
 $\Rightarrow I = 2 \int_0^\pi e^t \sin \left(\frac{\pi}{4} + t \right) dt \left[\text{put } x = 2t \Rightarrow dx = 2dt \right]$
 $\Rightarrow I = \frac{2}{\sqrt{2}} \int_0^\pi e^t (\sin t + \cos t) dt$
 $\Rightarrow I = \sqrt{2} [e^t \sin t]_0^\pi = \sqrt{2} \times 0 = 0$

➤ **Example 10.** The value of $\int_0^{2a} \sqrt{2ax - x^2} dx$ is

- (a) πa^2
(b) $\frac{\pi a^2}{4}$
(c) $\frac{\pi a^2}{2}$
(d) None of the above

Sol. (c) Let $I = \int_0^{2a} \sqrt{2ax - x^2} dx$
 $= \int_0^{2a} \sqrt{a^2 - (a - x)^2} dx$
 $= \int_0^{2a} \sqrt{a^2 - (a - x)^2} dx$
 $= \left[\frac{-(a-x)}{2} \sqrt{a^2 - (a-x)^2} - \frac{a^2}{2} \sin^{-1} \left(\frac{a-x}{a} \right) \right]_0^{2a}$
 $= \frac{a^2}{2} \sin^{-1} 1 + \frac{a^2}{2} \cdot \sin^{-1} 1 = 2 \cdot \frac{a^2}{2} \cdot \sin^{-1} 1$
 $= 2 \cdot \frac{a^2}{2} \cdot \frac{\pi}{2} = \frac{a^2 \pi}{2}$

Work Book Exercise 10.1

10

Definite Integrals

1 If $\int_0^a \frac{1}{1+4x^2} dx = \frac{\pi}{8}$, then a equals

$\frac{\pi}{2}$ $\frac{\pi}{4}$ 1 $\frac{1}{2}$

2 $\int_0^{\pi/2} \cos^5\left(\frac{x}{2}\right) \cdot \sin x dx$ is equal to

$\frac{2}{7}\left(1 - \frac{1}{8\sqrt{2}}\right)$ $-\frac{4}{7}\left(1 - \frac{1}{8\sqrt{2}}\right)$
 $\frac{4}{7}\left(1 - \frac{1}{8\sqrt{2}}\right)$ None of these

3 $\int_0^{2/3} \frac{1}{4+9x^2} dx$ is equal to

$\frac{\pi}{6}$ $\frac{\pi}{12}$ $\frac{\pi}{24}$ $\frac{\pi}{4}$

4 The value of $\int_{1/3}^1 \frac{(x-x^3)^{1/3}}{x^4} dx$ is

6 0 3 4

5 $\int_2^3 \frac{2x^2}{x^4+3x^2+1} dx$ is equal to

$\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{7}{54}\right) + \tan^{-1}\left(\frac{5}{26}\right)$

$\frac{2}{\sqrt{5}} \tan^{-1}\left(\frac{\sqrt{5}}{54}\right) + \tan^{-1}\left(\frac{5}{26}\right)$

$\frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{7\sqrt{5}}{54}\right) + \tan^{-1}\left(\frac{5}{56}\right)$

None of the above

6 The value of $\int_0^{\pi/4} \frac{\sin x + \cos x}{3 + \sin 2x} dx$ is

$\log 2$ $\log 3$
 $\frac{1}{4} \log 3$ $\frac{1}{8} \log 3$

7 The value of $\int_0^a \frac{x^4}{(a^2+x^2)^4} dx$ is

$\frac{1}{16a^3}\left(\frac{\pi}{4} - \frac{1}{3}\right)$ $\frac{1}{16a^3}\left(\frac{\pi}{4} + \frac{1}{3}\right)$
 $\frac{a^3}{16}\left(\frac{\pi}{4} - \frac{1}{3}\right)$ $\frac{a^3}{16}\left(\frac{\pi}{4} + \frac{1}{3}\right)$

8 If $\int_1^2 e^{x^2} dx = a$, then the value of $\int_e^{e^4} \sqrt{\log_e x} dx$ is

$e^4 - e$
 $e^4 - a$
 $2e^4 - a$
 $2e^4 - e - a$

9 The value of $\int_0^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 3} dx$ is

$3 + 2\pi$
 $4 - \pi$
 $2 + \pi$
 None of the above

10 If $\int_0^1 \frac{\sin t}{1+t} dt = \alpha$, then the value of the integral

$\int_{4\pi-2}^{4\pi} \frac{\sin t/2}{4\pi+2-t} dt$ in terms of α is given by
 2α -2α α $-\alpha$

Properties of Definite Integrals

i. $\int_a^b f(x) dx = \int_a^b f(t) dt$

i.e. the integration is independent of the change of variable.

ii. $\int_a^b f(x) dx = -\int_b^a f(x) dx$

i.e. if the limits of definite integral are interchanged, then its value changes by minus sign only.

iii. $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$, where $c \in R$

Generalisation The above property can be generalised into the following form:

$$\int_a^b f(x) dx = \int_a^{c_1} f(x) dx + \int_{c_1}^{c_2} f(x) dx + \dots + \int_{c_n}^b f(x) dx$$

where, $a < c_1 < c_2 < c_3 \dots < c_n < b$

Example 11. $\int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) π

Sol. (b) Let $I = \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x} dx$

Put $\cos x = t \Rightarrow -\sin x dx = dt$
 $\Rightarrow dx = -\frac{dt}{\sin x}$

For limit, when $x = 0 \Rightarrow t = \cos 0 = 1$ [$\because t = \cos x$]

and when $x = \frac{\pi}{2} \Rightarrow t = \cos \frac{\pi}{2} = 0$

$$\begin{aligned} \therefore I &= \int_1^0 \frac{\sin x}{1+t^2} \cdot \left(-\frac{dt}{\sin x}\right) \\ &= -\int_1^0 \frac{1}{1+t^2} dt = \int_0^1 \frac{1}{1+t^2} dt \\ &= \left[\frac{1}{2} \tan^{-1}\left(\frac{t}{1}\right) \right]_0^1 \left[\because \int \frac{1}{a^2+x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} \right] \\ &= -\left(0 - \frac{\pi}{4}\right) = \frac{\pi}{4} \end{aligned}$$

➤ **Example 12.** $\int_{1/2}^2 \frac{1}{x} \operatorname{cosec}^{101} \left(x - \frac{1}{x} \right) dx$ is

equal to

- (a) $\frac{1}{4}$ (b) 1 (c) 0 (d) $\frac{101}{2}$

Sol. (c) Let $I = \int_{1/2}^2 \frac{1}{x} \operatorname{cosec}^{101} \left(x - \frac{1}{x} \right) dx$

$$\text{Put } \frac{1}{x} = t \Rightarrow -\frac{1}{x^2} dx = dt$$

$$\therefore I = - \int_{1/2}^0 \frac{1}{t} \operatorname{cosec}^{101} \left(t - \frac{1}{t} \right) dt$$

$$\Rightarrow I = -I$$

$$\Rightarrow 2I = 0$$

$$\Rightarrow I = 0$$

➤ **Example 13.** $\int_2^8 |x-5| dx$ is equal to

- (a) 9 (b) 3 (c) 6 (d) 15

Sol. (a) Let $I = \int_2^8 |x-5| dx$

It can be seen that $(x-5) \leq 0$ on $[2, 5]$ and $(x-5) \geq 0$ on $[5, 8]$.

$$\therefore \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

$$\begin{aligned} \therefore I &= \int_2^5 -(x-5) dx + \int_5^8 (x-5) dx \\ &= \left[5x - \frac{x^2}{2} \right]_2^5 + \left[\frac{x^2}{2} - 5x \right]_5^8 \\ &= \left(25 - \frac{25}{2} \right) - \left(10 - \frac{4}{2} \right) + \left(\frac{64}{2} - 40 \right) - \left(\frac{25}{2} - 25 \right) \\ &= \frac{25}{2} - 8 - 8 + \frac{25}{2} = 25 - 16 = 9 \end{aligned}$$

➤ **Example 14.** The value of

$$\int_1^3 |(x-1)(x-2)(x-3)| dx \text{ is}$$

- (a) $\frac{1}{3}$ (b) $\frac{1}{2}$
(c) $\frac{9}{4}$ (d) None of these

Sol. (b) Let $I = \int_1^3 |(x-1)(x-2)(x-3)| dx$

$$\begin{aligned} &= \int_1^2 |(x-1)(x-2)(x-3)| dx + \int_2^3 |(x-1)(x-2)(x-3)| dx \\ &= \int_1^2 (x^3 - 6x^2 + 11x - 6) dx - \int_2^3 (x^3 - 6x^2 + 11x - 6) dx \\ &= \left[\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x \right]_1^2 - \left[\frac{x^4}{4} - 2x^3 + \frac{11}{2}x^2 - 6x \right]_2^3 \\ &= \left[(4 - 16 + 22 - 12) - \left(\frac{1}{4} - 2 + \frac{11}{2} - 6 \right) \right] \\ &\quad - \left[\left(\frac{81}{4} - 54 + \frac{99}{2} - 18 \right) - (4 - 16 + 22 - 12) \right] \\ &= \frac{1}{2} \end{aligned}$$

➤ **Example 15.** $\int_{-1}^{3/2} |x \sin \pi x| dx$ is equal to

- (a) $\frac{1}{\pi} \left(1 + \frac{1}{\pi} \right)$ (b) $\frac{1}{\pi} \left(1 - \frac{1}{\pi} \right)$
(c) $\frac{1}{\pi} \left(\frac{1}{\pi} - 1 \right)$ (d) None of these

Sol. (d) $x \sin \pi x = 0$ gives $x = 0$ or $\pi x = n\pi, n \in \text{integer}$.
 $x = 0; x = n, n = 0, \pm 1, \pm 2, \dots$ out of which $0, 1 \in \left(-1, \frac{3}{2} \right)$.

$$\begin{aligned} \therefore \int_{-1}^{3/2} |x \sin \pi x| dx &= \int_{-1}^0 x \sin \pi x dx + \int_0^1 |x \sin \pi x| dx + \int_1^{3/2} |x \sin \pi x| dx \\ &= \int_{-1}^0 x \sin \pi x dx + \int_0^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx \\ &= \int_{-1}^1 x \sin \pi x dx - \int_1^{3/2} x \sin \pi x dx \\ &= \left[\frac{-x}{\pi} \cos \pi x + \frac{1}{\pi^2} \sin \pi x \right]_{-1}^1 - \left[\frac{-x}{\pi} \cos \pi x + \frac{1}{\pi^2} \sin \pi x \right]_1^{3/2} \\ &= \left(\frac{1}{\pi} + \frac{1}{\pi} \right) - \left(\frac{-1}{\pi^2} - \frac{1}{\pi} \right) = \frac{1}{\pi} \left(3 + \frac{1}{\pi} \right) \end{aligned}$$

➤ **Example 16.** $\int_{-1}^1 [x] dx$, where $[]$ denotes the greatest integer function, is equal to

- (a) 0 (b) 1
(c) -1 (d) None of these

Sol. (c) $\int_{-1}^1 [x] dx = \int_{-1}^0 (-1) dx + \int_0^1 0 dx$

$$\begin{aligned} \because [x] &= \begin{cases} -1, & \text{if } -1 \leq x < 0 \\ 0, & \text{if } 0 \leq x < 1 \end{cases} \\ &= -[x]_{-1}^0 = -1 \end{aligned}$$

➤ **Example 17.** $\int_0^2 [x^2] dx$ is equal to

- (a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $\sqrt{2} - 1$ (d) $-\sqrt{2} - \sqrt{3} + 5$

Sol. (d) $\int_0^2 [x^2] dx$

$$\begin{aligned} &= \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx + \int_{\sqrt{3}}^2 [x^2] dx \\ &= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx \\ &= 0 + 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + 3(2 - \sqrt{3}) = 5 - \sqrt{3} - \sqrt{2} \end{aligned}$$

➤ **Example 18.** $\int_0^3 [\sqrt{x}] dx$ is equal to

- (a) 1 (b) 2
(c) -1 (d) -2

Sol. (b) $\int_0^3 [\sqrt{x}] dx = \int_0^1 [\sqrt{x}] dx + \int_1^3 [\sqrt{x}] dx$

$$\begin{aligned} \because [\sqrt{x}] &= \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x < 3 \end{cases} \\ &= \int_0^1 0 dx + \int_1^3 1 dx = [x]_1^3 = 2 \end{aligned}$$

➤ **Example 19.** $\int_0^{\sqrt{n}} [t^2] dt$, where n is any positive integer, is equal to

- (a) $n\sqrt{n} + (1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n})$
 (b) $n\sqrt{n} - (1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n})$
 (c) $-n\sqrt{n} + (1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n})$
 (d) None of the above

Sol. (b) $I = \int_0^{\sqrt{n}} [t^2] dt$
 $= \int_0^1 [t^2] dt + \int_1^{\sqrt{2}} [t^2] dt + \int_{\sqrt{2}}^{\sqrt{3}} [t^2] dt + \dots + \int_{\sqrt{n-1}}^{\sqrt{n}} [t^2] dt$

$$\therefore [t^2] = \begin{cases} 0, & \text{if } 0 \leq t < 1 \\ 1, & \text{if } 1 \leq t < 2 \\ 2, & \text{if } \sqrt{2} \leq t < \sqrt{3} \\ 3, & \text{if } \sqrt{3} \leq t < 2 \\ \vdots & \\ (n-1), & \text{if } \sqrt{n-1} \leq t < \sqrt{n} \end{cases}$$

 $= \int_0^1 0 dt + \int_1^{\sqrt{2}} 1 dt + \dots + \int_{\sqrt{n-1}}^{\sqrt{n}} (n-1) dt$
 $= 1(\sqrt{2} - 1) + 2(\sqrt{3} - \sqrt{2}) + \dots + (n-1)(\sqrt{n} - \sqrt{n-1})$
 $= n\sqrt{n} - (1 + \sqrt{2} + \sqrt{3} + \dots + \sqrt{n})$

iv. $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

➤ **Example 20.** $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ is equal to

- (a) 0
 (b) $\frac{\pi}{4}$
 (c) $\frac{\pi}{2}$
 (d) None of these

Sol. (b) Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$... (i)

Then, $I = \int_0^{\pi/2} \frac{\sqrt{\sin(\frac{\pi}{2} - x)}}{\sqrt{\sin(\frac{\pi}{2} - x)} + \sqrt{\cos(\frac{\pi}{2} - x)}} dx$

$$[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx]$$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx$... (ii)

$$[\because \sin(\frac{\pi}{2} - x) = \cos x \text{ and } \cos(\frac{\pi}{2} - x) = \sin x]$$

On adding Eqs. (i) and (ii), we get

$\Rightarrow 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$

$\Rightarrow 2I = \int_0^{\pi/2} 1 dx$

$\Rightarrow 2I = [x]_0^{\pi/2} = \frac{\pi}{2} - 0$

$\Rightarrow I = \frac{\pi}{4}$

➤ **Example 21.** If $m, n \in \mathbb{N}$, then

$\int_0^{\pi/2} \frac{\sqrt[n]{\sin^m x}}{\sqrt[n]{\sin^m x} + \sqrt[n]{\cos^m x}} dx$ is equal to

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2n}$ (d) $\frac{\pi}{4n}$

Sol. (b) Let $I = \int_0^{\pi/2} \frac{\sqrt[n]{\sin^m x}}{\sqrt[n]{\sin^m x} + \sqrt[n]{\cos^m x}} dx$... (i)

Then, $I = \int_0^{\pi/2} \frac{\sqrt[n]{\cos^m x}}{\sqrt[n]{\cos^m x} + \sqrt[n]{\sin^m x}} dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}$

$\Rightarrow I = \frac{\pi}{4}$

➤ **Example 22.** The value of $\int_0^{\pi/2} \log \tan x dx$ is

- (a) π (b) $\frac{\pi}{2}$
 (c) 0 (d) None of these

Sol. (c) Let $I = \int_0^{\pi/2} \log \tan x dx$... (i)

Then, $I = \int_0^{\pi/2} \log \tan\left(\frac{\pi}{2} - x\right) dx$

$\Rightarrow I = \int_0^{\pi/2} \log \cot x dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/2} (\log \tan x + \log \cot x) dx$

$\Rightarrow 2I = \int_0^{\pi/2} \log 1 dx = \int_0^{\pi/2} 0 dx = 0$

$\Rightarrow I = 0$

➤ **Example 23.** $\int_0^{\pi/4} \log(1 + \tan x) dx$ equals

- (a) $\frac{\pi}{8} \log_e 2$ (b) $\frac{\pi}{4} \log_e 2$
 (c) $-\frac{\pi}{8} \log_e 2$ (d) $-\frac{\pi}{4} \log_e 2$

Sol. (a) Let $I = \int_0^{\pi/4} \log(1 + \tan x) dx$... (i)

$\Rightarrow I = \int_0^{\pi/4} \log \left\{ 1 + \tan\left(\frac{\pi}{4} - x\right) \right\} dx$

$\Rightarrow I = \int_0^{\pi/4} \log \left(1 + \frac{1 - \tan x}{1 + \tan x} \right) dx$

$\Rightarrow I = \int_0^{\pi/4} \log \left(\frac{2}{1 + \tan x} \right) dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/4} \log 2 dx$

$\Rightarrow 2I = \frac{\pi}{4} \log_e 2$

$\Rightarrow I = \frac{\pi}{8} \log_e 2$

➤ **Example 24.** The value of $\int_0^\pi e^{\cos^2 x} \cdot \cos^3(2n+1)x \, dx$, $n \in \text{integer}$, is

- (a) 0 (b) π
(c) 2π (d) None of these

Sol. (a) Let $I = \int_0^\pi e^{\cos^2 x} \cdot \cos^3(2n+1)x \, dx$

$$= \int_0^\pi e^{\cos^2(\pi-x)} \cdot \cos^3(2n+1)(\pi-x) \, dx$$

$$= \int_0^\pi e^{\cos^2 x} \cos^3[(2n+1)\pi - (2n+1)x] \, dx$$

$$= - \int_0^\pi e^{\cos^2 x} \cos^3(2n+1)x \, dx = -I$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

➤ **Example 25.** $\int_0^{\pi/2} \sin 2x \log \tan x \, dx$ is equal to

- (a) π (b) $\frac{\pi}{2}$ (c) 0 (d) 1

Sol. (c) Let $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$... (i)

Then, $I = \int_0^{\pi/2} \sin 2\left(\frac{\pi}{2} - x\right) \log \tan\left(\frac{\pi}{2} - x\right) \, dx$

$$\Rightarrow I = \int_0^{\pi/2} \sin 2x \log \cot x \, dx$$
 ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \sin 2x (\log \tan x + \log \cot x) \, dx$$

$$2I = \int_0^{\pi/2} \sin 2x \cdot \log 1 \, dx$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

➤ **Example 26.** The value of the integral $\int_0^\pi \frac{1}{e^{\cos x} + 1} \, dx$ is

- (a) π (b) 0 (c) 2π (d) $\frac{\pi}{2}$

Sol. (d) Let $I = \int_0^\pi \frac{1}{e^{\cos x} + 1} \, dx$... (i)

$$\Rightarrow I = \int_0^\pi \frac{1}{e^{\cos(\pi-x)} + 1} \, dx$$

$$\Rightarrow I = \int_0^\pi \frac{1}{e^{-\cos x} + 1} \, dx$$

$$\Rightarrow I = \int_0^\pi \frac{e^{\cos x}}{e^{\cos x} + 1} \, dx$$
 ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^\pi 1 \, dx = \pi \Rightarrow I = \frac{\pi}{2}$$

➤ **Example 27.** The value of $\int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx$ is

- (a) $\frac{\pi^2}{2}$ (b) $\frac{\pi^2}{4}$
(c) $\frac{\pi^2}{8}$ (d) $\frac{\pi^2}{16}$

Sol. (b) Let $I = \int_0^\pi \frac{x \sin x}{1 + \cos^2 x} \, dx$... (i)

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x) \sin(\pi-x)}{1 + \cos^2(\pi-x)} \, dx$$

$$\Rightarrow I = \int_0^\pi \frac{(\pi-x) \sin x}{1 + \cos^2 x} \, dx$$
 ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^\pi \frac{\sin x}{1 + \cos^2 x} \, dx$$

$$\Rightarrow 2I = -\pi \int_1^{-1} \frac{1}{1+t^2} \, dt$$

[put $t = \cos x \Rightarrow dt = -\sin x \, dx$]

$$\Rightarrow 2I = -\pi [\tan^{-1} t]_1^{-1}$$

$$\Rightarrow 2I = -\pi \left(-\frac{\pi}{4} - \frac{\pi}{4}\right) = \frac{\pi^2}{2}$$

$$\Rightarrow I = \frac{\pi^2}{4}$$

➤ **Example 28.** $\int_0^\pi x f(\sin x) \, dx$ is equal to

- (a) $\frac{\pi}{2} \int_0^\pi f(\sin x) \, dx$ (b) $\frac{\pi}{4} \int_0^\pi f(\sin x) \, dx$
(c) $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x) \, dx$ (d) $\frac{\pi}{2} \int_0^{\pi/2} f(\cos x) \, dx$

Sol. (a) Let $I = \int_0^\pi x f(\sin x) \, dx$... (i)

$$\Rightarrow I = \int_0^\pi (\pi-x) f(\sin(\pi-x)) \, dx$$

$$\Rightarrow I = \int_0^\pi (\pi-x) f(\sin x) \, dx$$
 ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^\pi f(\sin x) \, dx$$

$$\Rightarrow I = \frac{\pi}{2} \int_0^\pi f(\sin x) \, dx$$

v.
$$\int_0^{2a} f(x) \, dx = \begin{cases} 2 \int_0^a f(x) \, dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

➤ **Example 29.** The value of $\int_0^{2\pi} \cos^5 x \, dx$ is

- (a) 0 (b) π
(c) 2π (d) $\frac{\pi}{2}$

Sol. (a) Let $f(x) = \cos^5 x$

Then, $f(2\pi-x) = |\cos(2\pi-x)|^5 = \cos^5 x$

$$\therefore \int_0^{2\pi} \cos^5 x \, dx = 2 \int_0^\pi \cos^5 x \, dx$$

Now, $f(\pi-x) = |\cos(\pi-x)|^5$

$$= -\cos^5 x = -f(x)$$

$$\therefore \int_0^\pi \cos^5 x \, dx = 0$$

Hence, $\int_0^{2\pi} \cos^5 x \, dx = 2 \int_0^\pi \cos^5 x \, dx$

$$= 2 \times 0 = 0$$

➤ **Example 30.** $\int_0^{\pi/2} \log \sin x \, dx$ is equal to

- (a) $\frac{\pi}{2} \log 2$ (b) $-\frac{\pi}{2} \log 2$
(c) $\pi \log 2$ (d) None of these

Sol. (b) Let $I = \int_0^{\pi/2} \log \sin x \, dx = \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - x \right) dx$

$$\Rightarrow I = \int_0^{\pi/2} \log \cos x \, dx$$

$$\therefore 2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) dx$$

$$= \int_0^{\pi/2} \log (\sin x \cos x) dx$$

$$= \int_0^{\pi/2} \log \left(\frac{\sin 2x}{2} \right) dx = \int_0^{\pi/2} (\log \sin 2x - \log 2) dx$$

$$= \int_0^{\pi/2} \log \sin 2x \, dx - \log 2 \int_0^{\pi/2} dx$$

$$= \frac{1}{2} \int_0^{\pi} \log \sin z \, dz - \log 2 \left[x \right]_0^{\pi/2}$$

$$\left[\because \text{putting } 2x = z \Rightarrow 2dx = dz \right]$$

$$= \frac{1}{2} \cdot 2 \int_0^{\pi/2} \log \sin z \, dz - \frac{\pi}{2} \log 2$$

$$\left[\because \log \sin z = \log \sin (\pi - z) \right. \\ \left. \therefore \int_0^{\pi} \log \sin z \, dz = 2 \int_0^{\pi/2} \log \sin z \, dz \right]$$

$$= \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2 = I - \frac{\pi}{2} \log 2$$

$$\therefore I = -\frac{\pi}{2} \log 2$$

➤ **Example 31.** $\int_0^{\pi} \frac{dx}{1 + 2 \sin^2 x}$ is equal to

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{3\sqrt{3}}$ (c) $\frac{\pi}{\sqrt{3}}$ (d) 0

Sol. (c) Let $I = \int_0^{\pi} \frac{dx}{1 + 2 \sin^2 x} = 2 \int_0^{\pi/2} \frac{dx}{1 + 2 \sin^2 x}$

$$\left[\because \int_0^{2a} f(x) dx = 2 \int_0^a f(x) dx, \text{ if } f(2a - x) = f(x) \right]$$

$$= 2 \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 2 \tan^2 x} dx = 2 \int_0^{\pi/2} \frac{\sec^2 x}{1 + 3 \tan^2 x} dx$$

$$= 2 \int_0^{\infty} \frac{dt}{1 + 3t^2} \quad [\text{put } \tan x = t \Rightarrow \sec^2 x dx = dt]$$

$$= 2 \cdot \frac{1}{\sqrt{3}} [\tan^{-1} t \sqrt{3}]_0^{\infty} = \frac{2}{\sqrt{3}} \times \frac{\pi}{2} = \frac{\pi}{\sqrt{3}}$$

➤ **Example 32.** $\int_0^{2\pi} \frac{1}{1 + e^{\sin x}} dx$ is equal to

- (a) π (b) 2π (c) $\frac{\pi}{2}$ (d) None of these

Sol. (a) Let $I = \int_0^{2\pi} \left(\frac{1}{1 + e^{\sin x}} \right) dx$

$$\Rightarrow 2I = \int_0^{2\pi} \left\{ \frac{1}{1 + e^{\sin x}} + \frac{1}{e^{\sin(2\pi - x)}} \right\} dx$$

$$\Rightarrow 2I = \int_0^{2\pi} \left\{ \frac{1}{1 + e^{\sin x}} + \frac{e^{\sin x}}{1 + e^{\sin x}} \right\} dx = \int_0^{2\pi} 1 \cdot dx$$

$$2I = 2\pi \Rightarrow I = \pi$$

➤ **Example 33.** $\int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx, n > 0$, is

equal to

- (a) π (b) 2π
(c) π^2 (d) $\frac{1}{2} \pi^2$

Sol. (c) Let $I = \int_0^{2\pi} \frac{x \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \quad \dots(i)$

$$\Rightarrow I = \int_0^{2\pi} \frac{(2\pi - x) \sin^{2n} (2\pi - x)}{\sin^{2n} (2\pi - x) + \cos^{2n} (2\pi - x)} dx$$

$$\Rightarrow I = \int_0^{2\pi} \frac{(2\pi - x) \sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = 2\pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow I = \pi \int_0^{2\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow I = 2\pi \int_0^{\pi} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow I = 4\pi \int_0^{\pi/2} \frac{\sin^{2n} x}{\sin^{2n} x + \cos^{2n} x} dx$$

$$\Rightarrow I = 4\pi \times \frac{\pi}{4} = \pi^2$$

vi.

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

➤ **Example 34.** The value of

$$\int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x + 1) dx \text{ is}$$

- (a) 0 (b) 2
(c) π (d) 1

Sol. (c) Let $I = \int_{-\pi/2}^{\pi/2} (x^3 + x \cos x + \tan^5 x + 1) dx$

$$I = \int_{-\pi/2}^{\pi/2} x^3 dx + \int_{-\pi/2}^{\pi/2} x \cos x dx \\ + \int_{-\pi/2}^{\pi/2} \tan^5 x dx + \int_{-\pi/2}^{\pi/2} 1 dx$$

$$\text{We know that, } \int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

$$\therefore I = 0 + 0 + 0 + 2 \int_0^{\pi/2} 1 dx$$

$$[\because x^3, x \cos x \text{ and } \tan^5 x \text{ are odd functions}]$$

$$\therefore I = 2 \left[x \right]_0^{\pi/2} = \frac{2\pi}{2} = \pi$$

➤ **Example 35.** The value of $\int_{-2}^2 (ax^3 + bx + c) dx$

depends on

- (a) b
(b) c
(c) a
(d) None of the above

Sol. (b) Let $I = \int_{-2}^2 (ax^3 + bx + c) dx$

$$\Rightarrow I = a \int_{-2}^2 x^3 dx + b \int_{-2}^2 x dx + \int_{-2}^2 c dx$$

$$\Rightarrow I = a \times 0 + b \times 0 + 4c = 4c$$

Hence, I depends on c only.

➤ **Example 36.** The value of $\int_{-2}^0 [x^3 + 3x^2 + 3x$

$+ 3 + (x+1) \cos(x+1)] dx$, is

- (a) 0 (b) 3 (c) 4 (d) 1

Sol. (c) Let $I = \int_{-2}^0 [x^3 + 3x^2 + 3x + 3 + (x+1) \cos(x+1)] dx$

$$\Rightarrow I = \int_{-2}^0 \{(x+1)^3 + 2 + (x+1) \cos(x+1)\} dx$$

$$\Rightarrow I = \int_{-1}^1 (t^3 + 2 + t \cos t) dt \quad [\text{put } x+1=t \Rightarrow dx=dt]$$

$$\Rightarrow I = \int_{-1}^1 t^3 dt + \int_{-1}^1 2 dt + \int_{-1}^1 t \cos t dt$$

$$\Rightarrow I = 0 + 2 \times 2 + 0 = 4$$

vii. $\boxed{\int_{-a}^a f(x) dx = \int_0^a \{f(x) + f(-x)\} dx}$

➤ **Example 37.** $\int_{-\pi/2}^{\pi/2} \frac{1}{e^{\sin x} + 1} dx$ is equal to

- (a) 0 (b) 1 (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

Sol. (d) Let $I = \int_{-\pi/2}^{\pi/2} \frac{1}{e^{\sin x} + 1} dx$

$$= \int_0^{\pi/2} \left\{ \frac{1}{e^{\sin x} + 1} + \frac{1}{e^{\sin(-x)} + 1} \right\} dx$$

$$I = \int_0^{\pi/2} \left\{ \frac{1}{e^{\sin x} + 1} + \frac{e^{\sin x}}{e^{\sin x} + 1} \right\} dx$$

$$= \int_0^{\pi/2} 1 \cdot dx = \frac{\pi}{2}$$

➤ **Example 38.** $\int_{-\pi}^{\pi} \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} dx$ equals

- (a) $\frac{\pi}{2}$ (b) 2π (c) π (d) $\frac{\pi}{4}$

Sol. (c) We have,

$$I = \int_0^{\pi} \left\{ \frac{e^{\sin x}}{e^{\sin x} + e^{-\sin x}} + \frac{e^{-\sin x}}{e^{-\sin x} + e^{\sin x}} \right\} dx$$

$$= \int_1^{\pi} 1 \cdot dx = [x]_0^{\pi} = \pi$$

➤ **Example 39.** The value of $\int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx$ is

- (a) 0 (b) -1
(c) 1 (d) None of these

Sol. (c) Let $I = \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx = \int_0^{\pi/2} \left\{ \frac{\cos x}{1+e^x} + \frac{\cos(-x)}{1+e^{-x}} \right\} dx$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} \frac{\cos x}{1+e^x} dx = \int_0^{\pi/2} \cos x dx$$

$$= [\sin x]_0^{\pi/2} = 1$$

➤ **Example 40.** $\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$ is equal to

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{4}$ (d) π

Sol. (a) Let $I = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$

$$\text{Here, } f(x) = \frac{1}{(1+e^x)(1+x^2)}$$

$$\Rightarrow f(-x) = \frac{1}{(1+e^{-x})(1+(-x)^2)} = \frac{e^x}{(1+e^x)(1+x^2)}$$

$$\left[\because \int_{-a}^a f(x) dx = \int_0^a \{f(x) + f(-x)\} dx \right]$$

$$\Rightarrow \int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)} = \int_0^{\sqrt{3}} \left(\frac{1}{(1+e^x)(1+x^2)} + \frac{e^x}{(1+e^x)(1+x^2)} \right) dx$$

$$\text{So, } I = \int_0^{\sqrt{3}} \frac{dx}{1+x^2} = [\tan^{-1} x]_0^{\sqrt{3}} = \frac{\pi}{3}$$

viii. $\boxed{\int_a^b f(x) dx = \int_a^b f(a+b-x) dx}$

Important Result

$$\boxed{\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx = \frac{b-a}{2}}$$

➤ **Example 41.** $\int_2^7 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} dx$ is equal to

- (a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{2}$ (d) 0

Sol. (c) Let $I = \int_2^7 \frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} dx$... (i)

$$\Rightarrow I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}} dx$$

$$\Rightarrow I = \int_2^7 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{x}} dx \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_2^7 \left(\frac{\sqrt{x}}{\sqrt{x} + \sqrt{9-x}} + \frac{\sqrt{9-x}}{\sqrt{x} + \sqrt{9-x}} \right) dx$$

$$\Rightarrow 2I = \int_2^7 dx = [x]_2^7 = 5$$

$$\text{So, } I = \frac{5}{2}$$

➤ **Example 42.** The value of $\int_3^4 \frac{[x^2]}{[x^2 - 14x + 49] + [x^2]} dx$, where $[]$ denotes the greatest integer function, is

- (a) 1 (b) $\frac{3}{2}$
(c) $\frac{1}{2}$ (d) None of these

Sol. (c) Let $I = \int_3^4 \frac{[x^2]}{[x^2 - 14x + 49] + [x^2]} dx$... (i)

$$= \int_3^4 \frac{[x^2]}{[(7-x)^2] + [x^2]} dx$$

$$= \int_3^4 \frac{[(7-x)^2]}{[(7-(7-x))^2] + [(7-x)^2]} dx$$

$$[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx]$$

$$\Rightarrow I = \int_3^4 \frac{[(7-x)^2]}{[x^2] + [(7-x)^2]} dx \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_3^4 1 dx = 1$$

$$\therefore I = \frac{1}{2}$$

➤ **Example 43.** The value of $\int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}} dx$ is

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{12}$ (d) 0

Sol. (c) Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \sqrt{\tan x}} dx$... (i)

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots (ii)$$

This is of the form $\int_a^b \frac{f(x)}{f(x) + f(a+b-x)} dx$.

$$\therefore I = \frac{\pi/3 - \pi/6}{2} = \frac{\pi}{12}$$

➤ **Example 44.** $\int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x + \cos x} dx$ is equal to

- (a) π
(b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$
(d) None of the above

Sol. (d) Let $I = \int_{\pi/5}^{3\pi/10} \frac{\sin x}{\sin x + \cos x} dx$... (i)

$$\text{Then, } I = \int_{\pi/5}^{3\pi/10} \frac{\sin\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)} dx$$

$$\Rightarrow I = \int_{\pi/5}^{3\pi/10} \frac{\cos x}{\cos x + \sin x} dx \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{\pi/5}^{3\pi/10} 1 dx = \frac{3\pi}{10} - \frac{\pi}{5} = \frac{\pi}{10}$$

$$\Rightarrow I = \frac{\pi}{20}$$

ix. $\int_a^b f(x) dx = (b-a) \int_0^1 f|(b-a)x + a| dx$

➤ **Example 45.** The value of $\int_{-4}^{-5} e^{(x+5)^2} dx + 3 \int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$ is

- (a) 1 (b) 0
(c) -1 (d) None of these

Sol. (b) Let $I_1 = \int_{-4}^{-5} e^{(x+5)^2} dx$ and $I_2 = \int_{1/3}^{2/3} e^{9(x-2/3)^2} dx$

$$\Rightarrow I_1 = (-5 + 4) \int_0^1 e^{[(-5+4)x - 4 + 5]^2} dx$$

$$\Rightarrow I_1 = - \int_0^1 e^{(-x+1)^2} dx = - \int_0^1 e^{(x-1)^2} dx$$

$$\text{and } I_2 = \left(\frac{2}{3} - \frac{1}{3}\right) \int_0^1 e^{9[(2/3 - 1/3)x + 1/3 - 2/3]^2} dx$$

$$\Rightarrow I_2 = \frac{1}{3} \int_0^1 e^{9(x/3 - 1/3)^2} dx = \frac{1}{3} \int_0^1 e^{(x-1)^2} dx$$

$$\therefore I = I_1 + 3I_2 = - \int_0^1 e^{(x-1)^2} dx + \int_0^1 e^{(x-1)^2} dx = 0$$

x. If $f(x)$ is a periodic function with period T , then

(a) $\int_0^{nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}$

(b) $\int_a^{a+nT} f(x) dx = n \int_0^T f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$

(c) $\int_m^{nT} f(x) dx = (n-m) \int_0^T f(x) dx, m, n \in \mathbb{Z}$

(d) $\int_{nT}^{a+nT} f(x) dx = \int_0^a f(x) dx, n \in \mathbb{Z}, a \in \mathbb{R}$

(e) $\int_{a+nT}^{b+nt} f(x) dx = \int_a^b f(x) dx, n \in \mathbb{Z}, a, b \in \mathbb{R}$

where, $\int_a^{a+T} f(x) dx$ is independent of a

➤ **Example 46.** The value of $\int_0^{400\pi} \sqrt{1 - \cos 2x} dx$ is

- (a) $200\sqrt{2}$ (b) $400\sqrt{2}$
(c) $800\sqrt{2}$ (d) None of these

Sol. (c) Let $I = \int_0^{400\pi} \sqrt{1 - \cos 2x} dx = \int_0^{400\pi} \sqrt{2} |\sin x| dx$

$$\Rightarrow I = \sqrt{2} \times 400 \int_0^{\pi} |\sin x| dx$$

[$\because |\sin x|$ is periodic with period π]

$$\Rightarrow I = 400\sqrt{2} \int_0^{\pi} \sin x dx$$

$$= 400\sqrt{2} [-\cos x]_0^{\pi}$$

$$= 800\sqrt{2}$$

➤ **Example 47.** $\int_{-3/2}^{10} \{2x\} dx$, where $\{\}$ denotes the fractional part of x , is equal to

- (a) $\frac{1}{2}$ (b) $\frac{13}{2}$
(c) $\frac{23}{2}$ (d) $\frac{23}{4}$

Sol. (d) $f(x) = \{2x\}$ is a periodic function with period $\frac{1}{2}$.

$$\begin{aligned} \text{Let } I &= \int_{-3/2}^{10} \{2x\} dx \\ &= \int_{-3/2}^{20(1/2)} \{2x\} dx = 23 \int_0^{1/2} 2x dx \\ &\quad \left[\text{as } \{2x\} = 2x - [2x] \in \left(0, \frac{1}{2}\right) \Rightarrow [2x] = 0 \right] \\ &= 23 [x^2]_0^{1/2} = \frac{23}{4} \end{aligned}$$

➤ **Example 48.** $\int_{10\pi + \pi/6}^{10\pi + \pi/3} (\sin x + \cos x) dx$ is

equal to

(a) $\sqrt{3}$ (b) $1 + \sqrt{3}$ (c) $\sqrt{3} - 1$ (d) 1

Sol. (c) $f(x) = \sin x + \cos x$ is periodic with period 2π .

$$\begin{aligned} \text{Let } I &= \int_{10\pi + \pi/6}^{10\pi + \pi/3} (\sin x + \cos x) dx \\ &= \int_{\pi/6}^{\pi/3} (\sin x + \cos x) dx \\ &= [\sin x - \cos x]_{\pi/6}^{\pi/3} \\ &= \left[\frac{\sqrt{3}}{2} - \frac{1}{2} \right] - \left[\frac{1}{2} - \frac{\sqrt{3}}{2} \right] = (\sqrt{3} - 1) \end{aligned}$$

➤ **Example 49.** The value of $\int_0^{100} e^{x-[x]} dx$ is

(a) $100e$ (b) $100(e-1)$
(c) $100(e+1)$ (d) None of these

Sol. (b) Since, $x - [x]$ is a periodic function with period one unit. Therefore, it is $e^{x-[x]}$ and hence

$$\begin{aligned} I &= \int_0^{100} e^{x-[x]} dx \\ &= 100 \int_0^1 e^{x-[x]} dx = 100 \int_0^1 e^{x-0} dx \\ \Rightarrow I &= 100 \int_0^1 e^x dx = 100(e-1) \end{aligned}$$

xi. Leibnitz Theorem

Let $\phi(x)$ and $\Psi(x)$ be two functions defined on $[\alpha, \beta]$ and differentiable at a point $x \in [\alpha, \beta]$ and $f(x, t)$ is continuous function.

(a) **For implicit function**

$$\begin{aligned} \frac{d}{dx} \left[\int_{\phi(x)}^{\Psi(x)} f(x, t) dt \right] &= \left[\int_{\phi(x)}^{\Psi(x)} \frac{\partial}{\partial x} f(x, t) dt + \frac{d}{dx} \right. \\ &\quad \left. + \left\{ [\Psi(x)] f[x, \Psi(x)] - \frac{d}{dx} [\phi(x)] f[x, \phi(x)] \right\} \right] \end{aligned}$$

where, $f(x, t)$ is an implicit function.

(b) **For explicit function**

If functions $\phi(x)$ and $\psi(x)$ are defined on $[\alpha, \beta]$ and differentiable at a point $x \in [\alpha, \beta]$ and $f(t)$ is continuous on $[\phi(\alpha), \phi(\beta)]$, then

$$\begin{aligned} \frac{d}{dx} \left[\int_{\phi(x)}^{\Psi(x)} f(t) dt \right] &= \left\{ \frac{d}{dx} \Psi(x) \right\} f[\Psi(x)] \\ &\quad - \left\{ \frac{d}{dx} [\phi(x)] \right\} f[\phi(x)] \end{aligned}$$

(c) Let a function $f(x, \alpha)$ be continuous for $a \leq x \leq b$ and $c \leq \alpha \leq d$, then for any $\alpha \in [c, d]$, if $I(\alpha) = \int_a^b f(x, \alpha) dx$, then

$$\frac{dI(\alpha)}{d\alpha} = \int_a^b \frac{\partial f(x, \alpha)}{\partial \alpha} dx$$

➤ **Example 50.** $\frac{d}{dx} \left(\int_{x^2}^{x^3} \frac{1}{\log t} dt \right)$ is equal to

(a) $\frac{1}{\log x}$ (b) $\frac{x^2}{\log x}$
(c) $\frac{x^2 - x}{\log x}$ (d) None of these

$$\begin{aligned} \text{Sol. (c)} \quad \frac{d}{dx} \left(\int_{x^2}^{x^3} \frac{1}{\log t} dt \right) &= \frac{1}{\log x^3} \cdot \frac{d}{dx} (x^3) - \frac{1}{\log x^2} \cdot \frac{d}{dx} (x^2) \\ &= \frac{3x^2}{3 \log x} - \frac{2x}{2 \log x} \end{aligned}$$

$$\therefore \frac{d}{dx} \left(\int_{x^2}^{x^3} \frac{1}{\log t} dt \right) = \frac{1}{\log x} \cdot (x^2 - x)$$

➤ **Example 51.** If $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$ for all $x \in \left[0, \frac{\pi}{2}\right]$, then $f\left(\frac{1}{\sqrt{3}}\right)$ equals

(a) 3
(b) $\sqrt{3}$
(c) $\frac{1}{3}$
(d) None of the above

Sol. (a) We have, $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$

$$\begin{aligned} \Rightarrow \frac{d}{dx} \left(\int_{\sin x}^1 t^2 f(t) dt \right) &= -\cos x \\ &= 0 - \cos x \sin^2 x f(\sin x) \\ &= -\cos x \quad [\text{using Leibnitz's rule}] \\ \Rightarrow f(\sin x) &= \frac{1}{\sin^2 x} \\ \Rightarrow f\left(\frac{1}{\sqrt{3}}\right) &= 3 \end{aligned}$$

➤ **Example 52.** $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^{x^2} \sin \sqrt{t} dt$ is equal to

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $-\frac{1}{3}$ (d) $-\frac{2}{3}$

Sol. (b) $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^{x^2} \sin \sqrt{t} dt = \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sin \sqrt{t} dt}{x^3} \quad \left[\frac{0}{0} \text{ form} \right]$

$$= \lim_{x \rightarrow 0} \frac{[\sin \sqrt{t}]_{t=0}^{t=x^2} \cdot \frac{d}{dx}(x^2)}{3x^2}$$

$$= \lim_{x \rightarrow 0} \frac{(\sin x)(2x)}{3x^2} = \frac{2}{3} \quad [\text{using L' Hospital's rule}]$$

xii. Approximation in Definite Integration

- (a) If $f(x) \geq g(x)$, then $\int_a^b f(x) dx \geq \int_a^b g(x) dx$, where $b > a$.
- (b) If $f(x) \leq g(x)$, then $\int_a^b f(x) dx \leq \int_a^b g(x) dx$, where $b < a$.
- (c) If $f(x) \geq 0$ for all $x \in [a, b]$, then $\int_a^b f(x) dx \geq 0$.
- (d) If f is continuous on $[a, b]$, then $\left| \int_a^b f(x) dx \right| \leq \int_a^b |f(x)| dx$
- (e) If f is continuous on $[a, b]$ and $|f(x)| \leq k$, $\forall x \in [a, b]$, then $\left| \int_a^b f(x) dx \right| \leq k(b-a)$.
- (f) If $f_1(x)$, $f(x)$ and $f_2(x)$ are continuous functions in $[a, b]$ such that $f_1(x) \leq f(x) \leq f_2(x)$, $\forall x \in [a, b]$, then $\int_a^b f_1(x) dx \leq \int_a^b f(x) dx \leq \int_a^b f_2(x) dx$
- (g) $\left| \int_a^b f(x)g(x) dx \right| \leq \sqrt{\left\{ \int_a^b f^2(x) dx \right\} \left\{ \int_a^b g^2(x) dx \right\}}$
where, $f^2(x)$ and $g^2(x)$ are integrable on $[a, b]$.
- (h) If m and M are global minima and global maxima of $f(x)$ in $[a, b]$, i.e. $m \leq f(x) \leq M$ for $a \leq x \leq b$, then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$

➤ **Example 53.** The value of $\int_0^1 e^{x^2} dx$ is

- (a) greater than e (b) less than e
(c) less than 1 (d) None of these

Sol. (b) We have, $0 < x^2 < 1$

$$\Rightarrow e^0 < e^{x^2} < e^1$$

$$\Rightarrow \int_0^1 1 dx < \int_0^1 e^{x^2} dx < \int_0^1 e dx$$

$$\Rightarrow 1 < \int_0^1 e^{x^2} dx < e$$

➤ **Example 54.** Which one of the following is correct?

- (a) $\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{\pi}{4\sqrt{2}}$
(b) $\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$
(c) $\int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} = 0$

(d) None of the above

Sol. (a) $\because 0 \leq x \leq 1 \Rightarrow 0 \leq x^3 \leq x^2 \leq 1$

$$\Rightarrow -x^2 \leq -x^3 \leq 0$$

$$\Rightarrow 4-x^2-x^2 \leq 4-x^2-x^3 \leq 4-x^2$$

$$\Rightarrow \frac{1}{\sqrt{4-x^2}} \leq \frac{1}{\sqrt{4-x^2-x^3}} \leq \frac{1}{\sqrt{4-2x^2}}$$

$$\therefore \int_0^1 \frac{dx}{\sqrt{4-x^2}} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\Rightarrow \frac{\pi}{6} \leq \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} \leq \frac{\pi}{4\sqrt{2}}$$

➤ **Example 55.** If $I_1 = \int_0^{\pi/2} \cos(\sin x) dx$,

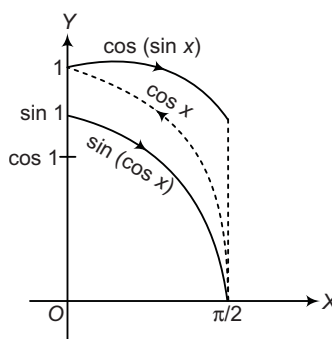
$I_2 = \int_0^{\pi/2} \sin(\cos x) dx$ and $I_3 = \int_0^{\pi/2} \cos x dx$, then

- (a) $I_1 > I_3 > I_2$
(b) $I_3 > I_1 > I_2$
(c) $I_1 > I_2 > I_3$
(d) $I_3 > I_2 > I_1$

Sol. (a) $I_1 = \int_0^{\pi/2} \cos(\sin x) dx = \int_0^{\pi/2} \cos(\cos x) dx$

$$I_2 = \int_0^{\pi/2} \sin(\cos x) dx$$

$$I_3 = \int_0^{\pi/2} \cos x dx$$



$$f_1(x) = \cos(\sin x)$$

Area under curve f_1 is more than that of f_2 , where

$$f_2(x) = \sin(\cos x), x \in [0, \pi/2]$$

$$\therefore I_1 > I_2$$

Since, area under f_3 is more than that of others.

$$\therefore I_1 > I_3 > I_2$$

- xiii.** If $f(t)$ is an odd function, then $\phi(x) = \int_a^x f(t)dt$ is an even function.
- xiv.** If $f(t)$ is an even function, then $\phi(x) = \int_0^x f(t)dt$ is an odd function.
- xv.** If $f(x)$ is continuous on $[a, \infty]$, then $\int_a^\infty f(x)dx$ is called an improper integral and is defined as
- $$\int_a^\infty f(x)dx = \lim_{b \rightarrow \infty} \int_a^b f(x)dx \quad \dots(i)$$
- If there exists a finite limit on the right hand side of Eq. (i), then we can say that the improper integral is convergent; otherwise it is divergent.

➤ **Example 56.** $\int_0^\infty \frac{dx}{(x + \sqrt{1+x^2})^n}$, where n is an integer > 1 , is equal to

- (a) $\frac{n}{1-n^2}$ (b) $\frac{n}{n^2-1}$ (c) $\frac{n}{n^2+1}$ (d) $\frac{-n}{n^2+1}$

Sol. (b) Put $x + \sqrt{1+x^2} = z$

$$\Rightarrow \left(1 + \frac{x}{\sqrt{1+x^2}}\right) dx = dz$$

$$\Rightarrow z dx = \sqrt{1+x^2} dz$$

$$\therefore \int_0^\infty \frac{dx}{(x + \sqrt{1+x^2})^n} = \int_0^\infty \frac{\sqrt{1+x^2} dz}{z^n \cdot z} = \int_0^\infty \frac{1}{z^{n+1}} \left(z + \frac{1}{z}\right) dz$$

$$\left[\because (\sqrt{1+x^2} - x)(\sqrt{1+x^2} + x) = 1 \right]$$

$$\Rightarrow \sqrt{1+x^2} - x = \frac{1}{z} \Rightarrow 2\sqrt{1+x^2} = z + \frac{1}{z}$$

$$= \frac{1}{2} \int_1^\infty \frac{z^2 + 1}{z^{n+2}} dz = \frac{1}{2} \int_0^\infty (z^{-n} + z^{-n-2}) dz$$

$$= \frac{1}{2} \left[\frac{z^{1-n}}{1-n} - \frac{z^{-n-1}}{(n+1)} \right]_1^\infty$$

$$= \frac{1}{2} \left[\lim_{z \rightarrow \infty} \left(\frac{z^{1-n}}{1-n} - \frac{z^{-n-1}}{n+1} \right) - \left(\frac{1}{1-n} - \frac{1}{1+n} \right) \right]$$

$$= \frac{1}{2} \left[(0 - 0) - \left(\frac{2n}{1-n^2} \right) \right] = \frac{n}{n^2-1}$$

➤ **Example 57.** If $f(x)$ is an odd function, then $\int_0^x f(t)dt$ is

- (a) odd function (b) even function
(c) neither even nor odd (d) None of these

Sol. (b) Let $\phi(x) = \int_a^x f(t)dt$

Then, $\phi(-x) = \int_a^{-x} f(t)dt = - \int_a^x f(-z)dz$

[putting $t = -z \Rightarrow dt = -dz$]

$$= \int_a^x f(z)dz \quad [\because f \text{ is an odd function}]$$

$$\int_{-a}^a f(z)dz + \int_a^x f(z)dz = 0 + \int_a^x f(t)dt$$

$$\left[\because f(x) \text{ is odd } \therefore f(-x) = -f(x) \right]$$

$$\Rightarrow \int_{-a}^a f(x)dx = 0$$

$$= \phi(x)$$

$\therefore \phi(x)$ is even.

➤ **Example 58.** If $f(x)$ is an even function, then $\int_0^x f(t)dt$ is

- (a) odd function (b) even function
(c) neither even nor odd (d) None of these

Sol. (a) Let $\phi(x) = \int_0^x f(t)dt$

Then, $\phi(-x) = \int_0^{-x} f(t)dt = - \int_0^x f(-z)dz$

[putting $t = -z \Rightarrow dt = -dz$]

$$= - \int_0^x f(z)dz \quad [\because f \text{ is an even function}]$$

$$= - \int_0^x f(t)dt = -\phi(x)$$

$\therefore \phi(x)$ is an odd function.

Work Book Exercise 10.2

1 The value of $\int_0^\pi |\cos x| dx$ is

- 1 0 2 4

2 $\int_0^3 [x] dx$ is equal to

- 2 3 4 1

3 $\int_0^{1.5} [x^2] dx$ is equal to

- 2 2 + $\sqrt{2}$ 2 - $\sqrt{2}$ $\sqrt{2}$

4 $\int_0^2 |x^2 + 2x - 3| dx$ is equal to

- 4 6 3 2

5 For any real number x , the value of $\int_0^x [x] dx$ is

- $x[x]$ $x[x] - [x]([x] + 1)$
 $x[x] - \frac{1}{2}[x]([x] + 1)$ None of these

6 The value of $\int_{\pi/6}^{5\pi/6} \sqrt{4 - 4\sin^2 t} dt$ is

- 0 2
1 None of these

- 7 If for a real number y , $[y]$ is the greatest integer less than or equal to y , then the value of the integral $\int_{\pi/2}^{3\pi/2} [2 \sin x] dx$, is
 a $-\pi$ b 0 c $-\frac{\pi}{2}$ d $\frac{\pi}{2}$
- 8 The value of $\int_0^{\pi/2} (2 \log \sin x - \log \sin 2x) dx$ is
 a $\frac{\pi}{2} \log 2$ b $-\frac{\pi}{2} \log 2$ c $\pi \log 2$ d $-\pi \log 2$
- 9 The value of $\int_0^{\pi/2n} \frac{1}{1 + \cot nx} dx$ is
 a 0 b $\frac{\pi}{4n}$ c $\frac{\pi}{2n}$ d $\frac{\pi}{2}$
- 10 The value of $\int_0^{32\pi/3} \sqrt{1 + \cos 2x} dx$ is
 a $\frac{44 + \sqrt{3}}{\sqrt{2}}$ b $\frac{44 - \sqrt{3}}{\sqrt{2}}$
 c $\frac{22 - \sqrt{3}}{\sqrt{2}}$ d $\frac{22 + \sqrt{3}}{\sqrt{2}}$
- 11 The value of $\int_0^{\sin^2 x} \sin^{-1} \sqrt{t} dt + \int_0^{\cos^2 x} \cos^{-1} \sqrt{t} dt$ is
 a π b $\frac{\pi}{2}$
 c $\frac{\pi}{4}$ d None of these
- 12 The value of $\int_0^1 e^{x^2} dx$ lies in the interval
 a $(0, 1)$ b $(-1, 0)$
 c $(1, e)$ d None of these

- 13 If $\phi(x) = \int_{1/x}^{\sqrt{x}} \sin(t^2) dt$, then $\phi'(1)$ is equal to
 a $\sin 1$ b $2 \sin 1$
 c $\left(\frac{3}{2}\right) \sin 1$ d None of these

- 14 Let $f(a) > 0$ and $f(x)$ be a non-decreasing continuous function in $[a, b]$. Then,

$$\frac{1}{b-a} \int_a^b f(x) dx \text{ has}$$

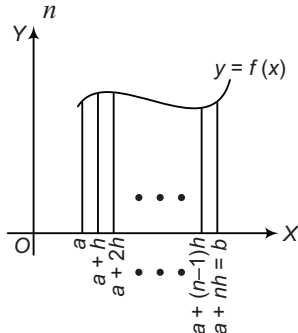
- a maximum value $f(b)$ and minimum value $f(a)$
 b maximum value $bf(b)$ and minimum value $af(a)$
 c maximum value $f(a)$ and minimum value $f(b)$
 d None of the above
- 15 The smallest interval $[a, b]$ such that $\int_0^1 \frac{1}{\sqrt{1+x^4}} dx \in [a, b]$, is given by
 a $\left[\frac{1}{\sqrt{2}}, 1\right]$ b $[0, 1]$ c $\left[\frac{1}{2}, 1\right]$ d $\left[\frac{3}{4}, 1\right]$
- 16 If $f(x) = \int_0^x f(t) dt$, then the value of $f(\ln 5)$ is
 a 5 b 0 c 1 d -5
- 17 The least value of the function $\phi(x) = \int_{5\pi/4}^x (3 \sin t + 4 \cos t) dt$ on the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$ is
 a $\sqrt{3} + \frac{3}{2}$ b $-2\sqrt{3} + \frac{3}{2} + \frac{1}{\sqrt{2}}$
 c $\frac{3}{2} + \frac{1}{\sqrt{2}}$ d None of these

Definite Integral as a Limit of Sum

Let $f(x)$ be a continuous real valued function defined on the closed interval $[a, b]$ which is divided into n parts. The points of division on X -axis are

$$a, a+h, a+2h, \dots, a+(n-1)h, a+nh$$

$$\text{where, } h = \frac{b-a}{n}$$



Let S_n denotes the area of these n rectangles. Then,

$$S_n = hf(a) + hf(a+h) + hf(a+2h) + \dots + hf\{a+(n-1)h\}$$

Clearly, S_n is the area very close to the area of the region bounded by curve $y = f(x)$, X -axis and the ordinates $x = a, x = b$.

$$\begin{aligned} \text{Hence, } \int_a^b f(x) dx &= \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} hf(a+rh) \\ &= \lim_{n \rightarrow \infty} \sum_{r=0}^{n-1} \left(\frac{b-a}{n}\right) f\left\{a + \frac{(b-a)r}{n}\right\} \end{aligned}$$

$$\begin{aligned} &\bullet \sin \alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin[\alpha + (n-1)\beta] \\ &= \frac{\sin\left(\alpha + \frac{(n-1)\beta}{2}\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}} \\ &\bullet \cos \alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (n-1)\beta] \\ &= \cos \frac{\left(\frac{\text{1st angle} + \text{Last angle}}{2}\right)}{\sin \left(\frac{\beta}{2}\right)} \cdot \sin \frac{n\beta}{2} \end{aligned}$$

Methods to Express the Infinite Series as Definite Integral

- Express the given series in the form $\sum \frac{1}{n} f\left(\frac{r}{n}\right)$.
- Then, the limit is its sum when $n \rightarrow \infty$, i.e.

$$\lim_{n \rightarrow \infty} \sum \frac{1}{n} f\left(\frac{r}{n}\right)$$
- Replace $\frac{r}{n}$ by x , $\frac{1}{n}$ by dx and $\lim_{n \rightarrow \infty} \sum$ by the sign of $\int$.
- The lower and upper limits of integration are limiting values of $\frac{r}{n}$ for the first and last terms of r , respectively.

Some Particular Cases

- $$\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} f\left(\frac{r}{n}\right) \text{ or } \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} f\left(\frac{r}{n}\right)$$

$$= \int_0^1 f(x) dx$$
- $$\lim_{n \rightarrow \infty} \sum_{r=1}^{pn} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_\alpha^\beta f(x) dx$$

where, $\alpha = \lim_{n \rightarrow \infty} \frac{r}{n} = 0$ [$\because r=1$]

and $\beta = \lim_{n \rightarrow \infty} \frac{r}{n} = p$ [$\because r=pn$]

The method to evaluate the integral as limit of the sum of an infinite series is known as integration by first principle.

Example 59. $\lim_{n \rightarrow \infty} \frac{1^4 + 2^4 + 3^4 + \dots + n^4}{n^5}$

– $\lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{n^5}$ is equal to

- (a) $\frac{1}{30}$ (b) zero (c) $\frac{1}{4}$ (d) $\frac{1}{5}$

Sol. (d) $\lim_{n \rightarrow \infty} \frac{1^4 + 2^4 + 3^4 + \dots + n^4}{n^5} = \lim_{n \rightarrow \infty} \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{n^5}$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^4 = \lim_{n \rightarrow \infty} \frac{1}{n} \times \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n}\right)^3$$

$$= \int_0^1 x^4 dx = \lim_{n \rightarrow \infty} \frac{1}{n} \times \int_0^1 x^3 dx = \left[\frac{x^5}{5}\right]_0^1 = \frac{1}{5}$$

Example 60.

$\lim_{n \rightarrow \infty} \left\{ \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right\}$ is equal to

- (a) 1 (b) 0 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

Sol. (c) Let $S = \lim_{n \rightarrow \infty} \left\{ \frac{n}{n^2 + 1^2} + \frac{n}{n^2 + 2^2} + \dots + \frac{n}{n^2 + n^2} \right\}$

$$\Rightarrow S = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2 + r^2}$$

$$\Rightarrow S = \frac{1}{n} \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{1 + \left(\frac{r}{n}\right)^2}$$

$$\Rightarrow S = \int_0^1 \frac{1}{1 + x^2} dx$$

$$= [\tan^{-1} x]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \frac{\pi}{4}$$

Example 61. $\lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n}$ equals

- (a) e
 (b) e^{-1}
 (c) 1
 (d) None of the above

Sol. (b) Let $A = \lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n}$

Then, $\log A = \lim_{n \rightarrow \infty} \log \left(\frac{1 \cdot 2 \cdot 3 \dots n}{n^n} \right)^{1/n}$

$\Rightarrow \log A = \lim_{n \rightarrow \infty} \log \left(\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \dots \frac{n}{n} \right)^{1/n}$

$\Rightarrow \log A = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left(\frac{r}{n} \right)$

$\Rightarrow \log A = \int_0^1 \log x dx = [x \log x - x]_0^1 = -1$

$\therefore A = e^{-1}$

Gamma Function

If n is a positive number, then the improper integral $\int_0^\infty e^{-x} x^{n-1} dx$ is defined as gamma function and is denoted by $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$, where $x \in \mathbb{Q}^+$.

Properties of Gamma Function

- $\Gamma 1 = 1$, $\Gamma 0 = \infty$ and $\Gamma(n+1) = n\Gamma n$
 e.g. $\Gamma 5 = 4\Gamma 4 = 4 \times 3\Gamma 3 = 4 \times 3 \times 2\Gamma 2$
 $= 4 \times 3 \times 2 \times 1\Gamma 1 = 4 \times 3 \times 2 \times 1 = 24$
- If $n \in \mathbb{N}$, then $\Gamma(n+1) = (n)!$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$
- $\int_0^{\pi/2} \sin^m x \cdot \cos^n x dx = \frac{\Gamma\left(\frac{m+1}{2}\right) \Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}$

➤ **Example 62.** The value of $\int_0^\pi x(\sin^4 x \cos^4 x) dx$ is

- (a) $\frac{3\pi^2}{64}$ (b) $\frac{3\pi^2}{128}$
(c) $\frac{3\pi^2}{256}$ (d) None of these

Sol. (c) Let $I = \int_\pi^0 x(\sin^4 x \cos^4 x) dx$
 $= \int_0^\pi (\pi - x) \sin^4 (\pi - x) \cos^4 (\pi - x) dx$
 $= \pi \int_0^\pi \sin^4 x \cos^4 x dx - I$
 $= \frac{\pi}{2} \int_0^\pi \sin^4 x \cos^4 x dx = 2 \cdot \frac{\pi}{2} \int_0^{\pi/2} \sin^4 x \cos^4 x dx$
 $= \pi \cdot \frac{3 \cdot 1 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{3\pi^2}{256}$

➤ **Example 63.** $\int_{-100\pi}^{100\pi} (\sin^4 x + \cos^4 x) dx$ is equal to

- (a) 100π (b) 150π
(c) 200π (d) None of these

Sol. (b) $\int_{-100\pi}^{100\pi} (\sin^4 x + \cos^4 x) dx$
 $= 2 \int_0^{100\pi} (\sin^4 x + \cos^4 x) dx$ [$\because$ integrand is even]
 $= 2 \int_0^{\pi/2(200)} (\sin^4 x + \cos^4 x) dx$
 $= 2 \cdot 200 \int_0^{\pi/2} (\sin^4 x + \cos^4 x) dx$
 $\left[\because \sin^4 x + \cos^4 x \text{ is a periodic function of period } \frac{\pi}{2} \right]$
 $= 400 \left[\int_0^{\pi/2} \sin^4 x + \int_0^{\pi/2} \cos^4 \left(\frac{\pi}{2} - x \right) dx \right]$
 $= 800 \int_0^{\pi/2} \sin^4 x dx = 800 \cdot \frac{3 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2} = 150\pi$

➤ **Example 64.** $\int_0^\pi \frac{x^3 \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2} dx$ is equal to

- (a) $\frac{\pi^2}{16}$ (b) $\frac{\pi^2}{8}$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{32}$

Sol. (d) Let $I = \int_0^\pi \frac{x^3 \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2} dx$
 $= \int_0^\pi \frac{(\pi - x)^3 \cos^4 (\pi - x) \sin^2 (\pi - x)}{\pi^2 - 3\pi (\pi - x) + 3(\pi - x)^2} dx$
 $= \int_0^\pi \frac{(\pi^3 - 3\pi^2 x + 3\pi x^2 - x^3) \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2} dx$
 $= \int_0^\pi \frac{(\pi^3 - 3\pi^2 x + 3\pi x^2) \cos^4 x \sin^2 x}{\pi^2 - 3\pi x + 3x^2} dx - I$
 $\Rightarrow 2I = \pi \cdot \int_0^\pi \frac{(\pi^2 - 3\pi x + 3x^2) \cos^4 x \sin^2 x}{(\pi^2 - 3\pi x + 3x^2)} dx$
 $\therefore I = \pi \int_0^{\pi/2} \cos^4 x \sin^2 x dx = \pi \cdot \frac{3 \cdot 1 \cdot 1}{6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$
 $\Rightarrow I = \frac{\pi^2}{32}$

➤ **Example 65.** The value of $\int_{-1}^1 \sin^{11} x dx$ is

- (a) $\frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}$ (b) $\frac{10}{11} \cdot \frac{8}{9} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{2}$
(c) 1 (d) 0

Sol. (d) Let $f(x) = \sin^{11} x$
 $\Rightarrow f(-x) = \sin^{11}(-x) = -\sin^{11} x = -f(x)$
 So, $f(x)$ is an odd function.
 $\therefore \int_{-1}^1 \sin^{11} x dx = 0$

➤ **Example 66.** If $I_n = \int_0^{\pi/4} \tan^n \theta d\theta$, where n is a positive integer, then $n(I_{n-1} + I_{n+1})$ is equal to

- (a) 1
(b) $n-1$
(c) $\frac{1}{n-1}$
(d) None of the above

Sol. (a) We have, $I_n = \int_0^{\pi/4} \tan^n \theta d\theta$
 Now, $I_{n+1} = \int_0^{\pi/4} \tan^{n+1} \theta d\theta = \int_0^{\pi/4} \tan^{n-1} \theta \cdot \tan^2 \theta d\theta$
 $= \int_0^{\pi/4} \tan^{n-1} \theta \cdot (\sec^2 \theta - 1) d\theta$
 $= \int_0^{\pi/4} \tan^{n-1} \theta \sec^2 \theta d\theta - \int_0^{\pi/4} \tan^{n-1} \theta d\theta$
 $= \left[\frac{\tan^n \theta}{n} \right]_0^{\pi/4} - I_{n-1}$
 $\Rightarrow I_{n-1} + I_{n+1} = \frac{1}{n} \Rightarrow n(I_{n-1} + I_{n+1}) = 1$

➤ **Example 67.** If $f(p, q) = \int_0^{\pi/2} \cos^p x \cos qx dx$, then

- (a) $f(p, q) = \frac{q}{p+q} f(p-1, q-1)$
(b) $f(p, q) = \frac{p}{p+q} f(p-1, q-1)$
(c) $f(p, q) = -\frac{p}{p+q} f(p-1, q-1)$
(d) $f(p, q) = -\frac{q}{p+q} f(p-1, q-1)$

Sol. (b) $f(p, q) = \int_0^{\pi/2} \cos^p x \cos qx dx$
 $= \left[\cos^p x \cdot \frac{\sin qx}{q} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{p}{q} \cos^{p-1} x \sin x \sin qx dx$
 $= 0 + \frac{p}{q} \int_0^{\pi/2} \cos^{p-1} x [\cos(q-1)x - \cos qx \cos x] dx$
 $\left[\because \cos(q-1)x = \cos qx \cos x + \sin qx \sin x \right]$
 $\left[\because \cos(q-1)x - \cos qx \cos x = \sin qx \sin x \right]$
 $= \frac{p}{q} f(p-1, q-1) - \frac{p}{q} f(p, q)$

$$\Rightarrow \left(1 + \frac{p}{q}\right) f(p, q) = \frac{p}{q} f(p-1, q-1)$$

$$\Rightarrow f(p, q) = \frac{p}{p+q} f(p-1, q-1)$$

➤ **Example 68.** If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, then

$\lim_{n \rightarrow \infty} n[I_n + I_{n+2}]$ is equal to

- (a) $\frac{1}{2}$ (b) 1
(c) ∞ (d) zero

Sol. (b) $I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x (1 + \tan^2 x) \, dx$

$$= \int_0^{\pi/4} \tan^n x \sec^2 x \, dx = \int_0^1 t^n \, dt$$

[put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$]

$$\therefore I_n + I_{n+2} = \frac{1}{n+1}$$

$$\text{Now, } \lim_{n \rightarrow \infty} n[I_n + I_{n+2}] = \lim_{n \rightarrow \infty} n \cdot \frac{1}{n+1}$$

$$= \lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{n}{n\left(1 + \frac{1}{n}\right)} = 1$$

➤ **Example 69.** If m and n are positive integers and $f(m, n) = \int_0^1 x^{n-1} (\log x)^m \, dx$, then $f(m, n)$ is equal to

- (a) $\frac{m}{n} f(m-1, n)$ (b) $-\frac{m}{n} f(m-1, n)$
(c) $\frac{n}{m} f(m-1, n)$ (d) None of these

Sol. (b) $f(m, n) = \int_0^1 (\log x)^m \cdot x^{n-1} \, dx$

$$= \left[(\log x)^m \cdot \frac{x^n}{n} \right]_0^1 - \int_0^1 m (\log x)^{m-1} \cdot \frac{1}{x} \cdot \frac{x^n}{n} \, dx$$

$$= -\frac{m}{n} \int_0^1 x^{n-1} (\log x)^{m-1} \, dx = -\frac{m}{n} f(m-1, n)$$

Reduction Formulae in Definite Integrals

$$i. I_n = \int_0^{\pi/2} \sin^n x \, dx = [-\sin^{n-1} x \cdot \cos x]_0^{\pi/2}$$

$$+ \int_0^{\pi/2} (n-1) \sin^{(n-2)} x \cdot \cos^2 x \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot (1 - \sin^2 x) \, dx$$

$$= (n-1) \int_0^{\pi/2} \sin^{n-2} x \cdot dx - (n-1) \int_0^{\pi/2} \sin^n x \, dx$$

$$\Rightarrow I_n + (n-1)I_n = (n-1)I_{n-2}$$

$$\Rightarrow I_n = \left(\frac{n-1}{n}\right) I_{n-2}$$

According to n is even or odd, $I_0 = \frac{\pi}{2}$, $I_1 = 1$

So,

$$I_n = \begin{cases} \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \left(\frac{n-5}{n-4}\right) \dots \left(\frac{1}{2}\right) \cdot \frac{\pi}{2}, & \text{if } n \text{ is even} \\ \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \left(\frac{n-5}{n-4}\right) \dots \left(\frac{2}{3}\right) \cdot 1, & \text{if } n \text{ is odd} \end{cases}$$

$$\bullet \int_0^{\pi/2} \sin^n x \, dx = \int_0^{\pi/2} \cos^n x \, dx$$

$$\bullet I_n = \left(\frac{n-1}{n}\right) \left(\frac{n-3}{n-2}\right) \left(\frac{n-5}{n-4}\right) \dots I_0 \text{ or } I_1$$

$$ii. I_n = \int_0^{\pi/4} \tan^n x \, dx$$

$$I_n = \int_0^{\pi/4} (\tan x)^{(n-2)} \cdot \tan^2 x \, dx$$

$$= \int_0^{\pi/4} (\tan x)^{(n-2)} (\sec^2 x - 1) \, dx$$

$$= \int_0^{\pi/4} (\tan x)^{(n-2)} \cdot \sec^2 x \, dx - \int_0^{\pi/4} (\tan x)^{(n-2)} \, dx$$

$$= \left[\frac{(\tan x)^{n-1}}{n-1} \right]_0^{\pi/4} - I_{n-2} \Rightarrow I_n = \frac{1}{n-1} - I_{n-2}$$

$$\therefore I_n + I_{n-2} = \frac{1}{n-1}$$

$$iii. I_{mn} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \int_0^{\pi/2} \sin^{m-1} x (\sin x \cos^n x) \, dx$$

$$= \left[-\frac{\sin^{m-1} x \cdot \cos^{n+1} x}{n+1} \right]_0^{\pi/2}$$

$$+ \int_0^{\pi/2} \frac{\cos^{n+1} x}{n+1} \cdot (m-1) \cdot \sin^{m-2} x \cos x \, dx$$

$$= \frac{(m-1)}{(n+1)} \int_0^{\pi/2} \sin^{m-2} x \cdot \cos^n x \cdot \cos^2 x \, dx$$

$$= \frac{(m-1)}{(n+1)} \int_0^{\pi/2} (\sin^{m-2} x \cdot \cos^n x - \sin^m x \cdot \cos^n x) \, dx$$

$$= \frac{(m-1)}{(n+1)} I_{(m-2, n)} - \left(\frac{m-1}{n+1}\right) I_{(m, n)}$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right) I_{(m, n)} = \left(\frac{m-1}{m+1}\right) I_{(m-2, n)}$$

$$\Rightarrow I_{(m, n)} = \left(\frac{m-1}{m+n}\right) I_{(m-2, n)}$$

➤ According to m is even or odd,

$$I_{m, n} = \left(\frac{m-1}{m+n}\right) \left(\frac{m-3}{m+n-2}\right) \left(\frac{m-5}{m+n-4}\right) \dots I_{(0, n)} \text{ or } I_{(1, n)}$$

$$I_{(0, n)} = \int_0^{\pi/2} \cos^n x \, dx$$

$$\text{and } I_{(1, n)} = \int_0^{\pi/2} \sin x \cdot \cos^n x \, dx = \frac{1}{n+1}$$

Work Book Exercise 10.3

10

Definite Integrals

1 $\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right) \dots \left(1 + \frac{n^2}{n^2}\right) \right]^{1/n}$ is equal to

- a $\frac{e^{\pi/2}}{2e^2}$ b $2e^2 e^{\pi/2}$
c $\frac{2}{e^2} \cdot e^{\pi/2}$ d None of these

2 $\lim_{n \rightarrow \infty} \frac{\pi}{n} \left\{ \sin \frac{\pi}{n} + \sin \frac{2\pi}{n} + \dots + \sin \frac{(n-1)\pi}{n} \right\}$ is equal to

- a 0 b π
c 2 d None of these

3 $\lim_{n \rightarrow \infty} \left\{ \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{n+n} \right\}$ is equal to

- a $\log_e 3$ b 0 c $\log_e 2$ d 1

4 $\lim_{n \rightarrow \infty} \left\{ \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n} \right\}$ is equal to

- a $\log 2$ b $\log 3$ c $\log 5$ d 0

5 $\lim_{n \rightarrow \infty} \left\{ \frac{1^m + 2^m + 3^m + \dots + n^m}{n^{m+1}} \right\}$ equals

- a $\frac{1}{m+1}$ b $\frac{1}{m+2}$
c $\frac{1}{m}$ d None of these

6 The value of $\int_0^{\pi/2} \sin^4 x \cos^6 x \, dx$ is

- a $\frac{3\pi}{216}$ b $\frac{3\pi}{512}$
c $\frac{\pi}{512}$ d None of these

7 $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$ equals

- a $1 + \sqrt{5}$
b $-1 + \sqrt{5}$
c $-1 + \sqrt{2}$
d $1 + \sqrt{2}$

8 If $I_n = \int_0^{\pi/4} \tan^n x \, dx$, then $\frac{1}{I_2 + I_4}, \frac{1}{I_3 + I_5}, \frac{1}{I_4 + I_6}, \dots$ form

- a an AP b a GP
c a HP d None of these

9 If $S_n = \left[\frac{1}{2n} + \frac{1}{\sqrt{4n^2 - 1}} + \frac{1}{\sqrt{4n^2 - 4}} + \dots + \frac{1}{\sqrt{3n^2 + 2n - 1}} \right]$, then

$\lim_{n \rightarrow \infty} S_n$ is equal to

- a $\frac{\pi}{4}$ b $\frac{\pi}{6}$
c $\frac{\pi}{3}$ d $\frac{\pi}{2}$

10 The value of

$$\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2 - 1}} + \frac{1}{\sqrt{n^2 - 2^2}} + \dots + \frac{1}{\sqrt{n^2 - (n-1)^2}} \right] \text{ is}$$

- a $\frac{\pi}{4}$ b $\frac{\pi}{3}$
c $\frac{\pi}{2}$ d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $I = \int_{-\pi/6}^{\pi/6} \frac{\pi + 4x^5}{1 - \sin\left(|x| + \frac{\pi}{6}\right)} dx$, then I equals

- (a) 4π (b) $2\pi + \frac{1}{\sqrt{3}}$
(c) $2\pi - \sqrt{3}$ (d) $4\pi + \sqrt{3} - \frac{1}{\sqrt{3}}$

Sol. Since, $\frac{4x^5}{1 - \sin\left(|x| + \frac{\pi}{6}\right)}$ is an odd function and

$\frac{\pi}{1 - \sin\left(|x| + \frac{\pi}{6}\right)}$ is an even function.

$$\therefore I = 2\pi \int_0^{\pi/6} \frac{dx}{1 - \sin\left(x + \frac{\pi}{6}\right)}$$

Put $x + \frac{\pi}{6} = t \Rightarrow dx = dt$

$$\begin{aligned} \therefore I &= 2\pi \int_{\pi/6}^{\pi/3} \frac{dt}{1 - \sin t} = 2\pi \int_{\pi/6}^{\pi/3} \frac{1 + \sin t}{\cos^2 t} dt \\ &= 2\pi [\tan t + \sec t]_{\pi/6}^{\pi/3} \\ &= 2\pi \left\{ (\sqrt{3} + 2) - \left(\frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} \right) \right\} = 4\pi \end{aligned}$$

Hence, (a) is the correct answer.

Ex 2. If $I = \int_0^{\pi} e^{\left(\frac{1}{2}\right)^{\cos x}} \left\{ 2\sin\left(\frac{1}{2}\cos x\right) + 3\cos\left(\frac{1}{2}\cos x\right) \right\} \sin x dx$, then I equals

- (a) $7\sqrt{e} \cos\left(\frac{1}{2}\right)$
(b) $7\sqrt{e} \left[\cos\left(\frac{1}{2}\right) - \sin\left(\frac{1}{2}\right) \right]$
(c) 0
(d) None of the above

Sol. Put $\frac{1}{2}\cos x = t$, so that $-\sin x dx = 2dt$

and $I = \int_{1/2}^{-1/2} e^{|t|} (2\sin t + 3\cos t) (-2) dt$

As $e^{|t|} \sin t$ is an odd function and $e^{|t|} \cos t$ is an even function.

$$\therefore I = 12 \int_0^{1/2} e^t \cos t dt = 12[e^t \cos t]_0^{1/2} + 12 \int_0^{1/2} e^t \sin t dt$$

$$\begin{aligned} \Rightarrow I &= 12 \left[\sqrt{e} \cos\left(\frac{1}{2}\right) - 1 \right] + 12[e^t \sin t]_0^{1/2} \\ &\quad - 12 \int_0^{1/2} e^t \cos t dt \end{aligned}$$

$$\Rightarrow 2I = 12\sqrt{e} \left[\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) - 1 \right]$$

$$\Rightarrow I = 6\sqrt{e} \left[\cos\left(\frac{1}{2}\right) + \sin\left(\frac{1}{2}\right) - 1 \right]$$

Hence, (d) is the correct answer.

Ex 3. If $I = \int_{\alpha}^{\beta} \left[\log \log x + \frac{1}{(\log x)^2} \right] dx$, then I

equals

- (a) $\alpha \log \log \alpha - \beta \log \log \beta$
(b) $\frac{1}{\alpha} - \frac{1}{\beta} + \log \log \alpha - \log \log \beta$
(c) $\frac{\beta - \alpha}{\alpha\beta} + \alpha \log \log \alpha - \beta \log \log \beta$
(d) None of the above

Sol. We have, $I = \int_{\alpha}^{\beta} \left[\log \log x + \frac{1}{(\log x)^2} \right] dx$

Put $\log x = t$ or $x = e^t$

$$\Rightarrow I = \int_a^b \left[\log t + \frac{1}{t^2} \right] e^t dt$$

where, $a = \log \alpha$, $b = \log \beta$

$$\Rightarrow I = \int_a^b \left(\log t + \frac{1}{t} + \left(-\frac{1}{t} \right) + \frac{1}{t^2} \right) e^t dt$$

$$= \left(\log t - \frac{1}{t} e^t \right) \Big|_a^b \quad [\because \int e^x (f(x) + f'(x)) = e^x f(x)]$$

$$= \left(\log b - \frac{1}{b} \right) e^b - \left(\log a - \frac{1}{a} \right) e^a$$

$$= \left(\log \log \beta - \frac{1}{\log \beta} \right) \beta - \left(\log \log \alpha - \frac{1}{\log \alpha} \right) \alpha$$

Hence, (d) is the correct answer.

Ex 4. If $I = \int_0^1 (1-x)^{\frac{2007}{x}} x dx$, then I equals

- (a) $\frac{2007}{2}$ (b) $\frac{2011}{2}$
(c) $\frac{2009}{2}$ (d) None of these

Sol. Using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we get

$$I = \int_0^1 [1 - (1-x)]^{\frac{2007}{2}} \cdot (1-x) dx$$

$$I = \int_0^1 (x)^{\frac{2007}{2}} \cdot (1-x) dx$$

$$= \int_0^1 \left[(x)^{\frac{2007}{2}} - x^{\frac{2007}{2} + 1} \right] dx$$

$$= \left(\frac{2}{2009} x^{\frac{2009}{2}} - \frac{2}{2011} x^{\frac{2011}{2}} \right) \Bigg|_0^1$$

$$= \frac{4}{(2009)(2011)}$$

Hence, (d) is the correct answer.

Ex 5. If $b > a$, then $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$ is equal to

- (a) $\frac{\pi}{2}$ (b) π
 (c) $\frac{\pi}{2}(b-a)$ (d) $\frac{\pi}{4}(b-a)$

Sol. We have, $\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}}$

$$= \int_a^b \frac{1}{\sqrt{-x^2 + (a+b)x - ab}} dx$$

$$= \int_a^b \frac{1}{\sqrt{\left(\frac{b-a}{2}\right)^2 - \left(x - \frac{a+b}{2}\right)^2}} dx$$

$$= \left[\sin^{-1} \left(\frac{x - \frac{a+b}{2}}{\frac{b-a}{2}} \right) \right]_a^b$$

$$= \sin^{-1} 1 - \sin^{-1}(-1) = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$

Hence, (b) is the correct answer.

Ex 6. The value of $\int_0^{\pi/2} \frac{\cos 3x + 1}{2 \cos x - 1} dx$ is

- (a) 2 (b) 1 (c) $\frac{1}{2}$ (d) 0

Sol. We have, $\int_0^{\pi/2} \frac{\cos 3x + 1}{2 \cos x - 1} dx = \int_0^{\pi/2} \frac{\cos 3x - \cos \frac{\pi}{3}}{2 \left(\cos x - \cos \frac{\pi}{3} \right)} dx$

$$= \int_0^{\pi/2} \frac{(4 \cos^3 x - 3 \cos x) - \left(4 \cos^3 \frac{\pi}{3} - 3 \cos \frac{\pi}{3} \right)}{2 \left(\cos x - \cos \frac{\pi}{3} \right)} dx$$

$$= 2 \int_0^{\pi/2} \left(\cos^2 x + \cos^2 \frac{\pi}{3} + \cos x \cos \frac{\pi}{3} \right) dx - \frac{3}{2} \int_0^{\pi/2} 1 dx$$

$$= \int_0^{\pi/2} \left(\frac{3}{2} + \cos 2x + \cos x \right) dx - \frac{3\pi}{4}$$

$$= \frac{3\pi}{4} + 1 - \frac{3\pi}{4} = 1$$

Hence, (b) is the correct answer.

Ex 7. $\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx$

+ $\int_{2\pi}^{\pi/4} (\cos x - \sin x) dx$ is equal to

- (a) $\sqrt{2} - 2$ (b) $2\sqrt{2} - 2$
 (c) $3\sqrt{2} - 2$ (d) $4\sqrt{2} - 2$

Sol. $\int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx$

$$+ \int_{2\pi}^{\pi/4} (\cos x - \sin x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4}$$

$$+ [\sin x + \cos x]_{2\pi}^{\pi/4}$$

$$= \left[\frac{1}{\sqrt{2}} (1+1) - 1 \right] - \frac{1}{\sqrt{2}} (-1-1-1-1)$$

$$+ \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - 1 \right)$$

$$= \left[\frac{2}{\sqrt{2}} - 1 \right] - \left[-\frac{4}{\sqrt{2}} \right] + \left[\frac{2}{\sqrt{2}} - 1 \right]$$

$$= (\sqrt{2} - 1) + 2\sqrt{2} + (\sqrt{2} - 1) = 4\sqrt{2} - 2$$

Hence, (d) is the correct answer.

Ex 8. The least value of the function $\phi(x) = \int_{5\pi/4}^x (3 \sin t + 4 \cos t) dt$ on the interval

$\left[\frac{5\pi}{4}, \frac{4\pi}{3} \right]$ is

- (a) $\sqrt{3} + \frac{3}{2}$ (b) $-2\sqrt{3} + \frac{3}{2} + \frac{1}{\sqrt{2}}$
 (c) $\frac{3}{2} + \frac{1}{\sqrt{2}}$ (d) None of these

Sol. We have,

$$\phi(x) = \int_{5\pi/4}^x (3 \sin t + 4 \cos t) dt$$

$$\Rightarrow \phi'(x) = (3 \sin x + 4 \cos x)$$

Clearly, $\phi'(x) < 0, \forall x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3} \right]$

$$\Rightarrow \phi(x) \text{ is decreasing function on } \left[\frac{5\pi}{4}, \frac{4\pi}{3} \right]$$

$$\therefore \phi\left(\frac{4\pi}{3}\right) \text{ is the least value of } \phi(x) \text{ on } \left[\frac{5\pi}{4}, \frac{4\pi}{3} \right]$$

Now, $\phi\left(\frac{4\pi}{3}\right) = \int_{5\pi/4}^{4\pi/3} (3 \sin t + 4 \cos t) dt$

$$\Rightarrow \phi\left(\frac{4\pi}{3}\right) = [-3 \cos t + 4 \sin t]_{5\pi/4}^{4\pi/3} = \frac{3}{2} - 2\sqrt{3} + \frac{1}{\sqrt{2}}$$

Hence, (b) is the correct answer.

Ex 9. If $f(x)$ is an odd periodic function defined on the interval $\left[-\frac{T}{2}, \frac{T}{2} \right]$, where T is the

period of $f(x)$. Then, $\phi(x) = \int_a^x f(t) dt$ is

- (a) periodic with period T
 (b) non-periodic
 (c) periodic with period $2T$
 (d) periodic with period aT

Sol. We have,

$$\phi(x+T) = \int_a^{x+T} f(t) dt = \int_a^x f(t) dt + \int_x^{x+T} f(t) dt$$

$$\Rightarrow \phi(x+T) = \phi(x) + \int_x^{x+T} f(t) dt$$

$$\Rightarrow \phi(x+T) = \phi(x) + \psi(x)$$

$$\text{where, } \psi(x) = \int_x^{x+T} f(t) dt$$

$$\text{Now, } \psi(x) = \int_x^T f(y+T) dy + \int_T^{x+T} f(y) dy$$

$$\therefore \psi(x) = \int_x^0 f(t) dt + \int_0^T f(t) dt + \int_0^x f(t) dt$$

$$= \int_0^T f(t) dt$$

$$\Rightarrow \psi(x) \text{ is independent of } x.$$

$$\Rightarrow \psi(x) = \psi\left(-\frac{T}{2}\right)$$

$$\Rightarrow \psi(x) = \int_{-T/2}^{T/2} f(t) dt$$

$$\left[\text{putting } x = \frac{T}{2} \text{ in } \psi(x) = \int_x^{x+T} f(t) dt \right]$$

$$\Rightarrow \psi(x) = 0 \quad [\because f(t) \text{ is an odd function}]$$

$$\therefore \phi(x+T) = \phi(x), \forall x,$$

So, $\phi(x)$ is periodic with period T .

Hence, (a) is the correct answer.

Ex 10. If $\phi(x) = \int_{1/x}^{\sqrt{x}} \sin(t^2) dt$, then $\phi'(1)$ is equal to

- (a) $\sin 1$ (b) $2\sin 1$
(c) $\left(\frac{3}{2}\right)\sin 1$ (d) None of these

Sol. By Leibnitz's rule, we have

$$\phi'(x) = \frac{1}{2\sqrt{x}} \sin x - \left(-\frac{1}{x^2}\right) \sin \frac{1}{x^2}$$

$$\therefore \phi'(1) = \frac{1}{2} \sin 1 + \sin 1 = \frac{3}{2} \sin 1$$

Hence, (c) is the correct answer.

Ex 11. If $f(x)$ is a periodic function with period T , then

$$(a) \int_a^b f(x) dx = \int_a^{b+T} f(x) dx$$

$$(b) \int_a^b f(x) dx = \int_{a+T}^b f(x) dx$$

$$(c) \int_a^b f(x) dx = \int_{a+T}^{b+T} f(x) dx$$

$$(d) \int_a^b f(x) dx = \int_{a+T}^{b+2T} f(x) dx$$

Sol. We have,

$$\int_{a+T}^{b+T} f(x) dx = \int_a^b f(y+T) dy, \text{ where } x = y+T$$

$$= \int_a^b f(y) dy \quad \left[\because f \text{ is periodic with period } T \right]$$

$$\quad \left[\because f(y+t) = f(y) \right]$$

$$= \int_a^b f(x) dx$$

Hence, (c) is the correct answer.

Ex 12. If $\frac{1}{\sqrt{a}} \int_1^a \left(\frac{3}{2} \sqrt{x} + 1 - \frac{1}{\sqrt{x}} \right) dx < 4$, then a may

- take values
(a) 6 (b) 4
(c) 9 (d) None of these

Sol. We have, $\frac{1}{\sqrt{a}} \int_1^a \left(\frac{3}{2} \sqrt{x} + 1 - \frac{1}{\sqrt{x}} \right) dx < 4$

$$\Rightarrow \frac{1}{\sqrt{a}} [x\sqrt{x} + x - 2\sqrt{x}]_1^a < 4$$

$$\Rightarrow \frac{1}{\sqrt{a}} (a\sqrt{a} - 1 + a - 1 - 2\sqrt{a} + 2) < 4$$

$$\Rightarrow a + \sqrt{a} - 2 < 4$$

$$\Rightarrow t^2 + t - 6 < 0, \text{ where } t = \sqrt{a}$$

$$\Rightarrow t \in (-3, 2)$$

$$\Rightarrow \sqrt{a} \in (-3, 2)$$

$$\Rightarrow \sqrt{a} \in [0, 2) \Rightarrow a \in [0, 4)$$

Clearly, the given inequality is defined for $a \neq 0$.

Hence, (d) is the correct answer.

Ex 13. If $\phi'(x) = \frac{\log_e |\sin x|}{x}$, $x \neq n\pi$, $n \in I$ and

$\int_1^3 \frac{3 \log_e |\sin x^3|}{x} dx = \phi(k) - \phi(1)$, then the possible value of k is

- (a) 27 (b) 18
(c) 9 (d) None of these

Sol. We have, $\int_1^3 \frac{3 \log_e |\sin x^3|}{x} dx$

$$= \int_1^3 \frac{3x^2 \log_e |\sin x^3|}{x^3} dx$$

$$= \int_1^3 \frac{3 \log_e |\sin x^3|}{x^3} d(x^3)$$

$$= \int_1^{27} \phi'(t) [\phi(t)]_1^{27}, \text{ where } t = x^3$$

$$= \phi(27) - \phi(1)$$

$$\therefore k = 27$$

Hence, (a) is the correct answer.

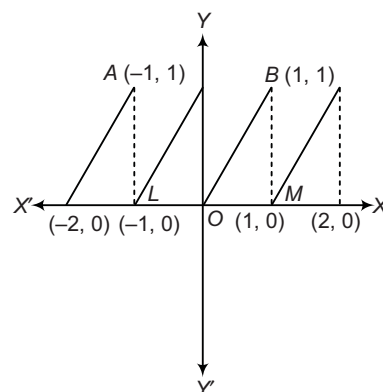
Ex 14. Let $f(x) = x - [x]$, for every real number x , where $[x]$ is integral part of x . Then,

$$\int_{-1}^1 f(x) dx \text{ is}$$

- (a) 1 (b) 2 (c) 0 (d) $\frac{1}{2}$

Sol. We have, $\int_{-1}^1 (x - [x]) dx$

$$= \int_{-1}^0 (x - [x]) dx + \int_0^1 (x - [x]) dx$$



$$= \int_{-1}^0 (x+1)dx + \int_0^1 (x-0)dx$$

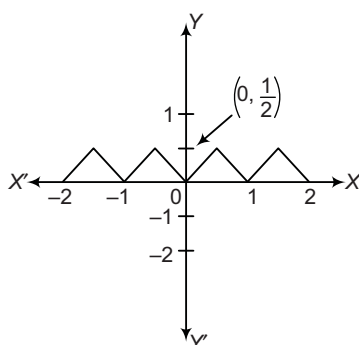
$$= \left[\frac{(x+1)^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

Hence, (a) is the correct answer.

- Ex 15.** $\int_{-2}^2 \min(x - [x], -x - [-x]) dx$, where $[x]$ represents greatest integer less than or equal to x , equals
- (a) 2 (b) 1
(c) 4 (d) 0

Sol. The graph of $f(x) = \min(x - [x], -x - [-x])$ is as shown in figure.



$$\therefore \int_{-2}^2 f(x)dx = 4 \int_0^1 f(x)dx$$

$$= 4 \times \frac{1}{2} \times 1 \times \frac{1}{2} = 1$$

Hence, (b) is the correct answer.

- Ex 16.** The expression $\frac{\int_0^n [x]dx}{\int_0^n \{x\}dx}$, where $[x]$ and $\{x\}$ represent integral and fractional part of x and $n \in N$, is equal to

- (a) $\frac{1}{n-1}$ (b) $\frac{1}{n}$
(c) n (d) $n-1$

Sol. We have, $\int_0^n [x]dx = \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx$

$$+ \dots + \int_{n-1}^n (n-1)dx$$

$$= 1(2-1) + 2(3-2) + 3(4-3) + \dots + (n-1)(n-(n-1))$$

$$= 1 + 2 + 3 + \dots + (n-1) = \frac{n(n-1)}{2}$$

and $\int_0^n \{x\}dx = \int_0^n (x - [x])dx = \frac{n^2}{2}$

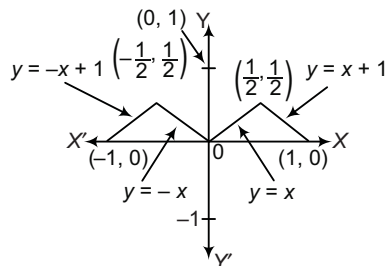
$$\therefore \frac{\int_0^n [x]dx}{\int_0^n \{x\}dx} = n-1$$

Hence, (d) is the correct answer.

- Ex 17.** If $f(x) = \min\{|x-1|, |x|, |x+1|\}$, then $\int_{-1}^1 f(x)dx$ equals

- (a) 1 (b) 0
(c) 2 (d) None of these

Sol. The graph of $f(x)$ is as shown in figure.



$$\therefore \int_{-1}^1 f(x)dx = 2 \int_0^1 f(x)dx$$

$$= 2 \left(\frac{1}{2} \times 1 \times \frac{1}{2} \right) = \frac{1}{2}$$

Hence, (d) is the correct answer.

- Ex 18.** The value of $\int_0^{100} [\tan^{-1} x]dx$ is

- (a) 100 (b) $100 - \tan^{-1} 1$
(c) $100 - \tan 1$ (d) None of these

Sol. We have, $\int_0^{100} [\tan^{-1} x]dx$

$$= \int_0^{\tan^{-1} 1} [\tan^{-1} x]dx + \int_{\tan^{-1} 1}^{100} [\tan^{-1} x]dx$$

$$= \int_0^{\tan^{-1} 1} 0dx + \int_{\tan^{-1} 1}^{100} 1dx = 100 - \tan 1$$

Hence, (c) is the correct answer.

- Ex 19.** If $\int_0^x \{t\}dt = \int_0^{\{x\}} t dt$, where $x > 0$, $x \notin I$ and $\{ \}$ represents fractional part function, then

- (a) $x \in (0, 1)$ (b) $[x] = 1$
(c) $x \in (1, 6) - I$ (d) None of these

Sol. Let $x \in (k, k+1)$, where $k \in I$

Then, $\int_0^x \{t\}dt = \int_0^{\{x\}} t dt$

$$\Rightarrow \int_0^k (t - [t])dt + \int_k^x (t - [t])dt = \int_0^{\{x\}} t dt$$

$$\Rightarrow \sum_{r=1}^k \int_{(r-1)}^r (t - [t])dt + \int_k^x (t - [t])dt = \int_0^{\{x\}} t dt$$

$$\Rightarrow \sum_{r=1}^k \int_{(r-1)}^r \{t - (r-1)\}dt + \int_k^x (t - k)dt = \frac{\{x\}^2}{2}$$

$$\Rightarrow \frac{1}{2} \sum_{r=1}^k [\{t - (r-1)\}^2]_{r-1}^r + \left[\frac{1}{2} (t - k)^2 \right]_k^x = \frac{1}{2} \{x\}^2$$

$$\Rightarrow \sum_{r=1}^k 1 + (x - k)^2 = \{x\}^2$$

$$\Rightarrow k + (x - k)^2 = (x - k)^2$$

$$\Rightarrow [\because x \in (k, k+1) \therefore \{x\} = x - k]$$

$$\Rightarrow k = 0$$

$$\therefore x \in (0, 1)$$

Hence, (a) is the correct answer.

Ex 20. Let $f(x) = \max \{x + |x|, x - [x]\}$, where $[x]$ denotes the greatest integer less than or equal to x , then $\int_{-2}^2 f(x) dx$ is equal to

- (a) 3 (b) 5
(c) 1 (d) None of these

Sol. Since, $x + |x| \geq x - [x]$ for all $x \geq 0$
and $x + |x| = 0 \leq x - [x]$ for all $x < 0$
 $\therefore \int_{-2}^2 f(x) dx = \int_{-2}^0 (x - [x]) dx + \int_0^2 (x + |x|) dx$
 $\Rightarrow \int_{-2}^2 f(x) dx = 2 \times \frac{1}{2} + \int_0^2 2x dx = 1 + 4 = 5$
Hence, (b) is the correct answer.

Ex 21. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by
$$f(x) = \begin{cases} |x - [x]|, & \text{when } [x] \text{ is odd} \\ |x - [x] - 1|, & \text{when } [x] \text{ is even} \end{cases}$$
where $[]$ denotes the greatest integer function, then $\int_{-2}^4 f(x) dx$ is equal to

- (a) $\frac{5}{2}$ (b) $\frac{3}{2}$
(c) 5 (d) 3

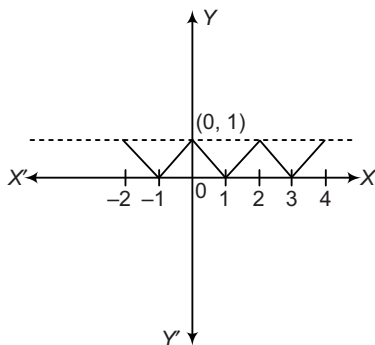
Sol. We have,

$$f(x) = \begin{cases} |x - [x]|, & \text{when } [x] \text{ is odd} \\ |x - [x] - 1|, & \text{when } [x] \text{ is even} \end{cases}$$

$$[\cdot: [x + 1] = [x] + 1]$$

$$\Rightarrow f(x) = \begin{cases} \{x\}, & \text{when } [x] \text{ is odd} \\ 1 - \{x\}, & \text{when } [x] \text{ is even} \end{cases}$$

The graph of $f(x)$ is shown in figure.



Clearly, $\int_{-2}^4 f(x) dx = 6 \int_{-2}^{-1} f(x) dx = 6 \times \frac{1}{2} = 3$

Hence, (d) is the correct answer.

Ex 22. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all non-zero x , then

$\int_{\sin \theta}^{\operatorname{cosec} \theta} f(x) dx$ equals

- (a) $\sin \theta + \operatorname{cosec} \theta$
(b) $\sin^2 \theta$
(c) $\operatorname{cosec}^2 \theta$
(d) None of the above

Sol. We have, $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$

$$\Rightarrow f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right)$$

$$\text{Let } I = \int_{\sin \theta}^{\operatorname{cosec} \theta} f(x) dx$$

$$= \int_{\sin \theta}^{\operatorname{cosec} \theta} \left\{ -\frac{1}{x^2} f\left(\frac{1}{x}\right) \right\} dx = \int_{\operatorname{cosec} \theta}^{\sin \theta} f(t) dt$$

where, $t = \frac{1}{x}$

$$\Rightarrow I = - \int_{\sin \theta}^{\operatorname{cosec} \theta} f(t) dt = -I$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

Hence, (d) is the correct answer.

Ex 23. The value of $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx$ is

- (a) $\frac{3}{2}$ (b) $\frac{5}{2}$
(c) 3 (d) 5

Sol. We have, $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx$

$$= \int_{1/e}^{e^2} \frac{|\log_e x|}{x} dx$$

$$= \int_{1/e}^1 \frac{|\log_e x|}{x} dx + \int_1^{e^2} \frac{|\log_e x|}{x} dx$$

$$= - \int_{1/e}^1 \frac{\log_e x}{x} dx + \int_1^{e^2} \frac{\log_e x}{x} dx$$

$$\left[\begin{array}{l} \because \log_e x < 0, \text{ for } \frac{1}{e} < x < 1 \\ \text{and } \log_e x > 0, \text{ for } x > 1 \end{array} \right]$$

$$= - \int_{1/e}^1 \log_e x d(\log_e x) + \int_1^{e^2} \log_e x d(\log_e x)$$

$$= - \left[\frac{1}{2} (\log_e x)^2 \right]_{1/e}^1 + \left[\frac{1}{2} (\log_e x)^2 \right]_1^{e^2}$$

$$= - \left[\frac{1}{2} \times 0 - \frac{1}{2} \right] + \left[\frac{1}{2} \times 4 - 0 \right] = \frac{1}{2} + 2 = \frac{5}{2}$$

Hence, (b) is the correct answer.

Ex 24. Let $g(x) = \int_0^x f(t) dt$, where f is such that

$\frac{1}{2} \leq f(t) \leq 1$ for $t \in [0, 1]$ and $0 \leq f(t) \leq \frac{1}{2}$, for

$t \in [1, 2]$. Then, $g(2)$ satisfies the inequality

- (a) $-\frac{3}{2} \leq g(2) < \frac{1}{2}$ (b) $\frac{1}{2} \leq g(2) \leq \frac{3}{2}$
(c) $\frac{3}{2} < g(2) \leq \frac{5}{2}$ (d) $2 < g(2) < 4$

Sol. We have, $g(x) = \int_0^x f(t) dt$

$$\Rightarrow g(2) = \int_0^2 f(t) dt$$

$$\Rightarrow g(2) = \int_0^1 f(t) dt + \int_1^2 f(t) dt$$

We know that,

$$m \leq f(x) \leq M, \text{ for } x \in [a, b]$$

$$\Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$$

$$\therefore \frac{1}{2} \leq f(t) \leq 1, \text{ for } t \in [0, 1]$$

$$\text{and } 0 \leq f(t) \leq \frac{1}{2}, \text{ for } t \in [1, 2]$$

$$\Rightarrow \frac{1}{2}(1-0) \leq \int_0^1 f(t)dt \leq 1(1-0)$$

$$\Rightarrow 0(2-1) \leq \int_1^2 f(t)dt \leq \frac{1}{2}(2-1)$$

$$\text{and } 0 \leq \int_1^2 f(t)dt \leq \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \leq \int_0^1 f(t)dt \leq \frac{1}{2}(2-1)$$

$$\text{and } 0 \leq \int_1^2 f(t)dt \leq \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} \leq \int_0^1 f(t)dt + \int_1^2 f(t)dt \leq \frac{3}{2}$$

$$\Rightarrow \frac{1}{2} \leq g(2) \leq \frac{3}{2}$$

Hence, (b) is the correct answer.

Ex 25. If $\left| \int_a^b f(x)dx \right| = \int_a^b |f(x)|dx$, $a < b$, then

$f(x) = 0$ has

(a) exactly one root in (a, b)

(b) atleast one root in (a, b)

(c) no root in (a, b)

(d) None of the above

Sol. We have, $\left| \int_a^b f(x)dx \right| = \int_a^b |f(x)|dx$

and the equality occurs only, if the graph of $y = f(x)$ has entirely above or below the X -axis for all $x \in (a, b)$. Thus, $f(x) = 0$ has no real root in (a, b) .

Hence, (c) is the correct answer.

Ex 26. Let $f(x)$ be an odd continuous function which is periodic with period 2. If $g(x) = \int_0^x f(t)dt$, then

(a) $g(x)$ is an odd function

(b) $g(n) = 0$ for all $n \in \mathbb{N}$

(c) $g(2n) = 0$ for all $n \in \mathbb{N}$

(d) $g(x)$ is non-periodic

Sol. It is given that $f(x)$ is an odd function.

Therefore, $g(x) = \int_0^x f(t)dt$ is an even function.

$$\Rightarrow g(x+2) = \int_0^{x+2} f(t)dt$$

$$\Rightarrow g(x+2) = \int_0^2 f(t)dt + \int_2^{x+2} f(t)dt$$

$$\Rightarrow g(x+2) = g(2) + \int_0^x f(t)dt$$

[$\because f(x)$ is periodic with period 2]

$$\Rightarrow g(x+2) = g(2) + g(x), \text{ for all } x \quad \dots(i)$$

Now, $g(x+2) = g(2) + g(x)$

$$\Rightarrow g(1) = g(2) + g(-1) \quad [\text{replacing } x \text{ by } -1]$$

$$\Rightarrow g(2) = \int_0^1 f(t)dt - \int_0^{-1} f(t)dt$$

$$\Rightarrow g(2) = \int_0^1 f(t)dt + \int_{-1}^0 f(t)dt$$

$$\Rightarrow g(2) = \int_{-1}^1 f(t)dt$$

$$\Rightarrow g(2) = 0 \quad [\because f \text{ is an odd function}]$$

On putting $g(2) = 0$ in Eq. (i), we get

$$g(x+2) = g(x), \text{ for all } x$$

$\Rightarrow g(x)$ is periodic with period 2 units.

Also, $g(x+2) = g(x)$, for all x

$$\Rightarrow g(2) = g(4) = g(6) = \dots = g(2n)$$

$$\Rightarrow g(2n) = 0, \forall n \in \mathbb{N}$$

Hence, (c) is the correct answer.

Ex 27. Let $f(x)$ be a function defined by $f(x) = \int_1^x (t^2 - 3t + 2)dt$, $1 \leq x \leq 3$. Then, the range of $f(x)$ is

$$(a) [0, 2] \quad (b) \left[-\frac{1}{4}, 4\right]$$

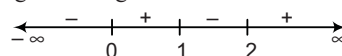
$$(c) \left[-\frac{1}{4}, 2\right] \quad (d) \text{None of these}$$

Sol. We have, $f(x) = \int_1^x (t^2 - 3t + 2)dt$

$$\Rightarrow f'(x) = x^2 - 3x + 2$$

$$\Rightarrow f'(x) = x(x-2)(x-1)$$

The changes in sign of $f'(x)$ for different values of x are given in figure.



$$\therefore \min\{f(x)\} = f(2) = \int_1^2 (t^2 - 3t + 2)dt$$

$$= \left[\frac{t^4}{4} - t^3 + t^2 \right]_1^2 = -\frac{1}{4}$$

and $\max\{f(x)\} = f(3)$

$$= \int_1^3 (t^2 - 3t + 2)dt = \left[\frac{t^4}{4} - t^3 + t^2 \right]_1^3 = 2$$

Since, $f(x)$ is continuous on $[1, 3]$. Therefore, range is

$$f(x) = [f(2), f(3)] = \left[-\frac{1}{4}, 2\right]$$

Hence, (c) is the correct answer.

Ex 28. For any $n \in \mathbb{N}$, $\int_0^\pi \frac{\sin^2 nx}{\sin^2 x} dx$ is equal to

(a) π

(b) $n\pi$

(c) 0

(d) None of these

Sol. Let $I_n = \int_0^\pi \frac{\sin^2 nx}{\sin^2 x} dx$

$$\text{Then, } I_{n+1} = \int_0^\pi \frac{\sin^2(n+1)x}{\sin^2 x} dx$$

$$\therefore I_{n+1} - I_n = \int_0^\pi \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx$$

$$\Rightarrow I_{n+1} - I_n = \int_0^\pi \frac{\sin(2n+1)x}{\sin^2 x} dx$$

$$\Rightarrow I_{n+1} - I_n = \pi$$

Thus, $I_1, I_2, \dots, I_n, \dots$, form an AP with common difference π .

$$\therefore I_n = I_1 + (n-1)\pi$$

$$\Rightarrow I_n = \pi + (n-1)\pi = n\pi \left[\because I_1 = \int_0^\pi \frac{\sin^2 x}{\sin^2 x} dx = \pi \right]$$

Hence, (b) is the correct answer.

Ex 29. The value of $\int_0^{\pi/2} \sin 2nx \cot x dx$, when n is positive, is

- (a) $\frac{\pi}{2}$
(b) $-\pi$
(c) π
(d) None of the above

Sol. We have, $\cos x + \cos 3x + \cos 5x + \dots + \cos(2n-1)x$

$$= \frac{\cos [x + (n-1)x] \sin nx}{\sin x}$$

$$\Rightarrow 2[\cos x + \cos 3x + \cos 5x + \dots + \cos(2n-1)x] = \frac{\sin 2nx}{\sin x}$$

$$\Rightarrow \sin 2nx \cot x = 2 \cos x \{ \cos x + \cos 3x + \cos 5x + \dots + \cos(2n-1)x \}$$

$$\therefore \int_0^{\pi/2} \sin 2nx \cot x dx = \frac{\pi}{2} + 2 \int_0^{\pi/2} \{ \cos 2x + \cos 4x + \dots + \cos(2n-2)x \} dx + \int_0^{\pi/2} \cos 2nx dx$$

$$= \frac{\pi}{2} + 2 \times 0 + 0 = \frac{\pi}{2}$$

Hence, (a) is the correct answer.

Ex 30. If $f(x) = \sin x + \cos x$ and $g(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases}$,

then the value of $\int_{-\pi/4}^{2\pi} g \circ f(x) dx$ is equal to

- (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{4}$
(c) π (d) None of these

Sol. We have, $f(x) = \sin x + \cos x = \sqrt{2} \sin \left(x + \frac{\pi}{4} \right)$

and $g(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 2, & x = 0 \end{cases} \Rightarrow g(x) = \begin{cases} 1, & x > 0 \\ 2, & x = 0 \\ -1, & x < 0 \end{cases}$

$$\therefore g \circ f(x) = \begin{cases} 1, & \text{if } x \in \left(-\frac{\pi}{4}, \frac{3\pi}{4} \right) \cup \left(\frac{7\pi}{4}, 2\pi \right) \\ 2, & \text{if } x = -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{7\pi}{4} \\ -1, & \text{if } x \in \left(\frac{3\pi}{4}, \frac{7\pi}{4} \right) \end{cases}$$

$$\begin{aligned} \Rightarrow \int_{-\pi/4}^{2\pi} g \circ f(x) dx &= \int_{-\pi/4}^{3\pi/4} 1 dx + \int_{3\pi/4}^{7\pi/4} -1 dx + \int_{7\pi/4}^{2\pi} 1 dx \\ &= 1 \left(\frac{3\pi}{4} + \frac{\pi}{4} \right) - \left(\frac{7\pi}{4} - \frac{3\pi}{4} \right) + \left(2\pi - \frac{7\pi}{4} \right) \\ &= \pi - \pi + \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 31. If $f: R \rightarrow R$ is continuous and differentiable function such that $\int_{-1}^x f(t) dt + f'''(3) \int_x^0 dt = \int_1^x t^3 dt - f'(1) \int_0^x t^2 dt + f''(2) \int_x^3 t dt$, then the value $f'(4)$ is
 (a) $48 - 8f'(1) + f''(2)$ (b) $48 - 8f'(1) - f''(2)$
 (c) $48 + 8f'(1) - f''(2)$ (d) None of these

Sol. We have, $\int_{-1}^x f(t) dt + f'''(3) \int_x^0 dt = \int_1^x t^3 dt - f'(1) \int_0^x t^2 dt + f''(2) \int_x^3 t dt$

$$\Rightarrow \int_{-1}^x f(t) dt - x f'''(3) = \left(\frac{x^4}{4} - \frac{1}{4} \right) - f'(1) \frac{x^3}{3} + f''(2) \left(\frac{9}{2} - \frac{x^2}{2} \right)$$

On differentiating w.r.t. x , we get

$$f(x) - f'''(3) = x^3 - x^2 f'(1) - x f''(2)$$

Again differentiating w.r.t. x , we get

$$f'(x) = 3x^2 - 2x f'(1) - f''(2)$$

$$\Rightarrow f'(4) = 48 - 8f'(1) - f''(2)$$

Hence, (b) is the correct answer.

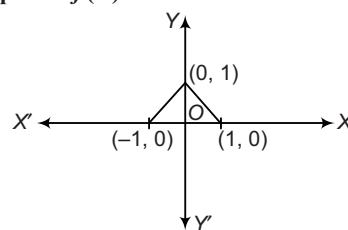
Ex 32. Let $f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$ and

$g(x) = f(x-1) + f(x+1)$, $\forall x \in R$. Then, the value of $\int_{-3}^3 g(x) dx$ is

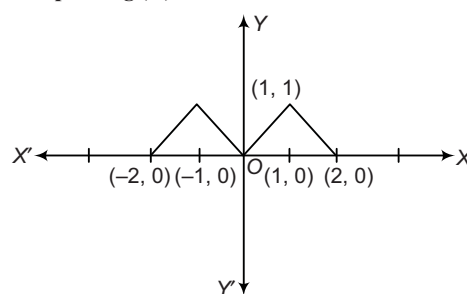
- (a) 2 (b) 3 (c) 4 (d) 5

Sol. The graphs of $f(x)$ and $g(x)$ are shown in figures, respectively.

Graph for $f(x)$



Graph for $g(x)$



Clearly, $\int_{-3}^3 g(x) dx = \int_{-3}^{-2} 0 dx + \int_{-2}^0 f(x) dx + \int_0^2 f(x) dx + \int_2^3 0 dx$

$$\therefore \int_{-3}^3 g(x) dx = 2 \int_0^2 f(x) dx = 2 \left(\frac{1}{2} \times 2 \times 1 \right) = 2$$

Hence, (a) is the correct answer.

Ex 33. If $f(x) = \frac{x-1}{x+1}$, $f^2(x) = f(f(x))$, ..., $f^{k+1}(x) = f(f^k(x))$; $k=1,2,3,\dots$ and $g(x) = f^{1998}(x)$, then $\int_{1/e}^1 g(x)dx$ is equal to
(a) 0 (b) 1 (c) -1 (d) e

Sol. We have, $f(x) = \frac{x-1}{x+1}$

$$\Rightarrow f^2(x) = f(f(x)) = f\left(\frac{x-1}{x+1}\right) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = -\frac{1}{x}$$

$$\Rightarrow f^4(x) = f^2(f^2(x)) = f^2\left(-\frac{1}{x}\right) = \frac{-\frac{1}{x}}{-\frac{1}{x}+1} = x$$

$$\therefore g(x) = f^{1998}(x) = f^2 \circ f^{1996}(x)$$

$$\Rightarrow g(x) = f^2(f^{1996}(x))$$

$$\Rightarrow g(x) = f^2(x)$$

$$\left[\because f^{1996}(x) = \left(\underbrace{f^4 \circ f^4 \circ \dots \circ f^4}_{499 \text{ times}} \right)(x) = x \right]$$

$$\Rightarrow g(x) = -\frac{1}{x}$$

$$\therefore \int_{1/e}^1 g(x)dx = \int_{1/e}^1 \left(-\frac{1}{x}\right)dx = -[\log_e x]_{1/e}^1$$

$$\Rightarrow \int_{1/e}^1 g(x)dx = -[\log_e 1 - \log_e 1/e] = -[0 + 1] = -1$$

Hence, (c) is the correct answer.

Ex 34. If $f : D \rightarrow R$ is such that

$$f(x) = \sqrt{\sin(\cos x)} + \ln(-2\cos^2 x + 3\cos x - 1),$$

$$\text{then } \int_{x_1}^{x_2} \left[\cos x - \frac{1}{2} \right] dx, \text{ where } x_1, x_2 \in D \text{ and}$$

$[]$ denotes the greatest integer function, is equal to

$$(a) 0 \quad (b) \frac{1}{2}(x_2 - x_1)$$

$$(c) x_1 - x_2 \quad (d) \frac{1}{2}(x_1 - x_2)$$

Sol. Clearly, $\sqrt{\sin(\cos x)}$ is defined for all $x \in R$.

So, $\ln(-2\cos^2 x + 3\cos x - 1)$ is defined, if

$$-2\cos^2 x + 3\cos x - 1 > 0$$

$$\Rightarrow 2\cos^2 x - 3\cos x + 1 < 0$$

$$\Rightarrow (2\cos x - 1)(\cos x - 1) < 0$$

$$\Rightarrow \frac{1}{2} < \cos x < 1$$

$$\Rightarrow 0 < \cos x - \frac{1}{2} < \frac{1}{2}$$

$$\Rightarrow \left[\cos x - \frac{1}{2} \right] = 0$$

$$\therefore \int_{x_1}^{x_2} \left[\cos x - \frac{1}{2} \right] dx = \int_{x_1}^{x_2} 0 dx = 0$$

Hence, (a) is the correct answer.

Ex 35. If $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$, then x can be equal to

$$(a) \frac{2}{\sqrt{3}}$$

$$(b) \sqrt{3}$$

$$(c) 2$$

$$(d) \text{None of these}$$

Sol. $\int_1^x \frac{dt}{|t|\sqrt{t^2-1}} = \frac{\pi}{6}$

$$\Rightarrow [\sec^{-1}(t)]_1^x = \frac{\pi}{6}$$

$$\Rightarrow \sec^{-1} x - 0 = \frac{\pi}{6}$$

$$\Rightarrow x = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$$

Hence, (a) is the correct answer.

Ex 36. If $\int_0^1 \cot^{-1}(1-x+x^2)dx = \lambda \int_0^1 \tan^{-1} x dx$,

then λ is equal to

$$(a) 1$$

$$(b) 2$$

$$(c) 3$$

$$(d) 4$$

Sol. $\int_0^1 \cot^{-1}(1-x+x^2)dx$

$$= \int_0^1 \tan^{-1} \left(\frac{1}{1-x+x^2} \right) dx = \int_0^1 \tan^{-1} \left[\frac{x-(x-1)}{1+x(x-1)} \right] dx$$

$$= \int_0^1 [\tan^{-1} x - \tan^{-1}(x-1)] dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}(1-x) dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1}[1-(1-x)] dx$$

$$= 2 \int_0^1 \tan^{-1} x dx$$

$$\Rightarrow \lambda = 2$$

Hence, (b) is the correct answer.

Ex 37. $\int_0^\pi |1+2\cos x| dx$ is equal to

$$(a) \frac{\pi}{2} + 2\sqrt{3}$$

$$(b) \frac{\pi}{3} + \sqrt{3}$$

$$(c) \frac{\pi}{3} + 2\sqrt{3}$$

$$(d) \text{None}$$

of these

Sol. Let $I = \int_0^\pi |1+2\cos x| dx$

$$= \int_0^{2\pi/3} |1+2\cos x| dx + \int_{2\pi/3}^\pi |1+2\cos x| dx$$

$$= \int_0^{2\pi/3} (1+2\cos x) dx - \int_{2\pi/3}^\pi (1+2\cos x) dx$$

$$= [x+2\sin x]_0^{2\pi/3} - [x+2\sin x]_{2\pi/3}^\pi = \frac{\pi}{3} + 2\sqrt{3}$$

Hence, (c) is the correct answer.

Ex 38. $\int_0^x [\cos t] dt$, where $x \in \left(2n\pi, (4n+1)\frac{\pi}{2} \right)$,

$n \in N$ and $[]$ denotes the greatest integer function, is equal to

$$(a) -n\pi$$

$$(b) -(n+1)\pi$$

$$(c) -2n\pi$$

$$(d) -(2n+1)\pi$$

Sol. $\int_0^x [\cos t] dt = \int_0^{2n\pi} [\cos t] dt + \int_{2n\pi}^x [\cos t] dt$
 $= n \int_0^{2\pi} [\cos t] dt + \int_{2n\pi}^x [\cos t] dt$
 $= n \left(\int_0^{\pi/2} [\cos t] dt + \int_{\pi/2}^{3\pi/2} [\cos t] dt + \int_{3\pi/2}^{2\pi} [\cos t] dt \right)$
 $+ \int_{2n\pi}^x [\cos t] dt$
 $= n \left[0 - \left(\frac{3\pi}{2} - \frac{\pi}{2} \right) + 0 \right] + 0 = -n\pi$
Hence, (a) is the correct answer.

Ex 39. $\int_0^x [\sin t] dt$, where $x \in \{(2n\pi, (2n+1)\pi), n \in \mathbb{N}\}$ and $[]$ denotes the greatest integer function, is equal to

- (a) $-n\pi$ (b) $-(n+1)\pi$
(c) $-2n\pi$ (d) $-(2n+1)\pi$

Sol. Let $I = \int_0^x [\sin t] dt = \int_0^{2n\pi} [\sin t] dt + \int_{2n\pi}^x [\sin t] dt$
 $= n \int_0^{2\pi} [\sin t] dt + \int_{2n\pi}^x [\sin t] dt$
 $= n \left(\int_0^{\pi/2} [\sin t] dt + \int_{\pi/2}^{\pi} [\sin t] dt + \int_{\pi}^{3\pi/2} [\sin t] dt + \int_{3\pi/2}^{2\pi} [\sin t] dt \right)$
 $+ \int_{2n\pi}^x [\sin t] dt$
 $= n \left(0 + 0 - \int_{\pi}^{2\pi} dt \right) + 0 = -n\pi$
Hence, (a) is the correct answer.

Ex 40. If $f(x)$ is continuous function for all real values of x and satisfies

$\int_0^x f(t) dt = \int_x^1 t^2 f(t) dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$, then the value of a is

- (a) $-\frac{1}{24}$ (b) $\frac{18}{168}$
(c) $\frac{1}{7}$ (d) None of these

Sol. $\int_0^x f(t) dt = \int_x^1 t^2 f(t) dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$... (i)

For $x = 1$, $\int_0^1 f(t) dt = 0 + \frac{1}{8} + \frac{1}{3} + a = \frac{11}{24} + a$

On differentiating both sides of Eq. (i) w.r.t. x , we get

$f(x) = 0 - x^2 f(x) + 2x^{15} + 2x^5$

$\Rightarrow f(x) = \frac{2(x^{15} + x^5)}{1 + x^2}$

$\therefore \int_0^1 f(t) dt = 2 \int_0^1 \frac{t^{15} + t^5}{1 + t^2} dt = \frac{11}{24} + a$

$\Rightarrow 2 \int_0^1 (x^{13} - x^{11} + x^9 - x^7 + x^5) dx = \frac{11}{24} + a$

$\Rightarrow 2 \left(\frac{1}{14} - \frac{1}{12} + \frac{1}{10} - \frac{1}{8} + \frac{1}{6} \right) = \frac{11}{24} + a$

$\Rightarrow a = -\frac{167}{840}$

Hence, (d) is the correct answer.

Ex 41. $\int_{-1}^1 \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$ is equal to

- (a) $\ln 2$ (b) $2 \ln 2$
(c) $\frac{1}{2} \ln 2$ (d) None of these

Sol. Let $I = \int_{-1}^1 \frac{x^3 + |x| + 1}{x^2 + 2|x| + 1} dx$
 $= \int_{-1}^1 \frac{x^3}{x^2 + 2|x| + 1} dx + \int_{-1}^1 \frac{|x| + 1}{x^2 + 2|x| + 1} dx$
 $= 0 + 2 \int_0^1 \frac{(|x| + 1)}{(|x| + 1)^2} dx = 2 \int_0^1 \frac{dx}{1 + x}$
 $= 2 [\ln(1 + x)]_0^1 = 2 \ln 2$

Hence, (b) is the correct answer.

Ex 42. $\lim_{x \rightarrow 0} \int_0^x \frac{(\tan^{-1} t)^2}{\sqrt{1 + x^2}} dt$ is equal to

- (a) π^2 (b) $\frac{\pi^2}{2}$
(c) $\frac{\pi^2}{4}$ (d) None of these

Sol. $\lim_{x \rightarrow 0} \int_0^x \frac{(\tan^{-1} t)^2}{\sqrt{1 + x^2}} dt = \lim_{x \rightarrow 0} \int_0^x \frac{(\tan^{-1} t)^2}{\sqrt{1 + x^2}} dt$
 $= \lim_{x \rightarrow 0} \frac{(\tan^{-1} x)^2 \sqrt{1 + x^2}}{x}$
 $= \lim_{x \rightarrow 0} \left(\frac{\tan^{-1} x}{x} \right)^2 \cdot x \sqrt{1 + x^2} = 0$

Hence, (d) is the correct answer.

Ex 43. Equation of tangent to $y = \int_{x^2}^{x^3} \frac{dt}{\sqrt{1 + t^2}}$ at $x = 1$

- is
(a) $y = x\sqrt{2} - 1$ (b) $y\sqrt{2} = x - 1$
(c) $y = x\sqrt{2} + 1$ (d) None of these

Sol. $[y]_{x=1} = 0 \Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1 + x^6}} \cdot 3x^2 - \frac{1}{\sqrt{1 + x^4}} \cdot 2x$
 $\Rightarrow \left[\frac{dy}{dx} \right]_{x=1} = \frac{1}{\sqrt{2}}$

$\therefore$ Required equation is $y\sqrt{2} = x - 1$.

Hence, (b) is the correct answer.

Ex 44. If $f(x)$ is a continuous for all real values of x and satisfies $\int_n^{n+1} f(x) dx = \frac{n^2}{2}, \forall n \in \mathbb{I}$,

then $\int_{-3}^5 f(|x|) dx$ is equal to

- (a) $\frac{19}{2}$ (b) $\frac{35}{2}$
(c) $\frac{17}{2}$ (d) None of these

$$\begin{aligned}
 \text{Sol. } \int_{-3}^5 f(|x|) dx &= \int_{-3}^0 f(|x|) dx + \int_0^5 f(|x|) dx \\
 &= 2 \int_0^3 f(x) dx + \int_3^5 f(x) dx \\
 &= 2 \left(\int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx \right) \\
 &\quad + \left(\int_3^4 f(x) dx + \int_4^5 f(x) dx \right) \\
 &= 2 \left(0 + \frac{1}{2} + \frac{2^2}{2} \right) + \left(\frac{9}{2} + \frac{16}{2} \right) = \frac{35}{2}
 \end{aligned}$$

Hence, (b) is the correct answer.

Ex 45. If $f(x) = \min(\{x+1\}, \{x-1\})$, $\forall x \in R$, where $\{ \}$ denotes the fractional part, then $\int_{-5}^4 f(x) dx$ is equal to

- (a) $\frac{9}{2}$
 (b) $\frac{7}{2}$
 (c) 4
 (d) None of the above

$$\begin{aligned}
 \text{Sol. } \{x+1\} &= \{x-1\} = \{x\} \\
 \Rightarrow f(x) &= \{x\} \\
 \Rightarrow \int_{-5}^4 f(x) dx &= \int_0^9 \{x\} dx \\
 &= 9 \int_0^1 \{x\} dx = 9 \int_0^1 x dx = \frac{9}{2}
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 46. If $I_{1(n)} = \int_0^{\pi/2} \frac{\sin(2n-1)}{\sin x} dx$,

$$I_{2(n)} = \int_0^{\pi/2} \frac{\sin^2 nx}{\sin^2 x} dx, n \in N, \text{ then}$$

- (a) $I_{2(n+1)} - I_{2(n)} = I_{1(n)}$
 (b) $I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$
 (c) $I_{2(n+1)} + I_{1(n)} = I_{2(n)}$
 (d) $I_{2(n+1)} + I_{1(n+1)} = I_{2(n)}$

$$\begin{aligned}
 \text{Sol. } I_{2(n)} - I_{2(n-1)} &= \int_0^{\pi/2} \frac{(\sin^2 nx - \sin^2 (n-1)x)}{\sin^2 x} dx \\
 &= \int_0^{\pi/2} \frac{\sin(2n-1)x \cdot \sin x}{\sin^2 x} dx \\
 &= \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx = I_{1(n)}
 \end{aligned}$$

$$\Rightarrow I_{2(n+1)} - I_{2(n)} = I_{1(n+1)}$$

Hence, (b) is the correct answer.

Ex 47. If $f(x)$ is continuous for all real values of x ,

then $\sum_{r=1}^n \int_0^1 f(r-1+x) dx$ is equal to

- (a) $\int_0^n f(x) dx$ (b) $\int_0^1 f(x) dx$
 (c) $n \int_0^1 f(x) dx$ (d) $(n-1) \int_0^1 f(x) dx$

$$\begin{aligned}
 \text{Sol. } \sum_{r=1}^n \int_0^1 f(r-1+x) dx &= \int_0^1 f(x) dx + \int_0^1 f(1+x) dx + \int_0^1 f(2+x) dx \\
 &\quad + \dots + \int_0^1 f(n-1+x) dx \\
 &= \int_0^1 f(x) dx + \int_1^2 f(x) dx + \int_2^3 f(x) dx \\
 &\quad + \dots + \int_{n-1}^n f(x) dx + \dots + \int_{n-1}^n f(x) dx \\
 &= \int_0^n f(x) dx
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 48. $\int_0^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] dx$, where $[]$ denotes the greatest integer function, is equal to

- (a) $\frac{\pi}{2} (\sin 1 + \cos 1)$ (b) $\frac{\pi}{2} (\sin 1 - \sin 2)$
 (c) $\frac{\pi}{2} (\sin 1 - \cos 1)$ (d) $\frac{\pi}{2} (\sin 1 + \sin 2)$

$$\begin{aligned}
 \text{Sol. } \int_0^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] dx &= \int_0^{\pi/2} \sin \left[\frac{2x}{\pi} \right] dx \\
 &\quad + \int_{\pi/2}^{\pi} \sin \left[\frac{2x}{\pi} \right] dx + \int_{\pi}^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] dx \\
 &= 0 + \sin 1 \int_{\pi/2}^{\pi} dx + \sin 2 \int_{\pi}^{3\pi/2} dx \\
 &= \frac{\pi}{2} (\sin 1 + \sin 2)
 \end{aligned}$$

Hence, (d) is the correct answer.

Ex 49. $\int_{-1}^1 [x^2 - x + 1] dx$, where $[]$ denotes the greatest integer function, is equal to

- (a) $\frac{5+\sqrt{5}}{2}$ (b) $\frac{5-\sqrt{5}}{2}$
 (c) $5+\sqrt{5}$ (d) $5-\sqrt{5}$

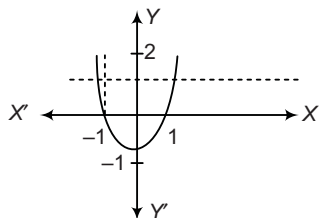
$$\begin{aligned}
 \text{Sol. } \int_{-1}^1 [x^2 - x + 1] dx &= \int_{-1}^1 dx + \int_{-1}^1 [x^2 - x] dx \\
 &= 2 + \int_{-1}^1 [x^2 - x] dx
 \end{aligned}$$

$$\text{If } x^2 - x = 1 \Rightarrow x^2 - x - 1 = 0$$

$$\Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

$$\begin{aligned}
 \Rightarrow \int_{-1}^1 [x^2 - x] dx &= \int_{-1}^{\frac{1-\sqrt{5}}{2}} [x^2 - x] dx + \int_{\frac{1-\sqrt{5}}{2}}^0 [x^2 - x] dx \\
 &\quad + \int_0^{\frac{1+\sqrt{5}}{2}} [x^2 - x] dx + \int_{\frac{1+\sqrt{5}}{2}}^1 [x^2 - x] dx \\
 &= \int_{-1}^{\frac{1-\sqrt{5}}{2}} dx + 0 - \int_0^{\frac{1+\sqrt{5}}{2}} dx \\
 &= \frac{1-\sqrt{5}}{2} + 1 - 1 \\
 &= \frac{1-\sqrt{5}}{2}
 \end{aligned}$$

Graph for $y = x^2 - x$ is given below:



$$\therefore \int_{-1}^1 [x^2 - x + 1] dx = 2 + \frac{1 - \sqrt{5}}{2} = \frac{5 - \sqrt{5}}{2}$$

Hence, (b) is the correct answer.

Ex 50. $f(x) = \begin{cases} x|x|, & x \leq -1 \\ [1+x] + [1-x], & -1 < x < 1, \text{ where } [] \\ -x|x|, & x \geq 1 \end{cases}$

denotes the greatest integer function, is equal to

(a) $-\frac{8}{3}$ (b) $-\frac{4}{3}$ (c) $\frac{8}{3}$ (d) $\frac{4}{3}$

Sol. $f(x) = \begin{cases} -x^2, & x \leq -1 \\ 2 + [x] + [-x], & -1 < x < 1 \\ -x^2, & x \geq 1 \end{cases}$

$$\Rightarrow f(x) = \begin{cases} -x^2, & x < -1 \\ 1, & -1 < x < 0 \\ 2, & x = 0 \\ 1, & 0 < x < 1 \\ -x^2, & x \geq 1 \end{cases}$$

Thus, $f(x)$ is an even function.

$$\begin{aligned} \Rightarrow \int_{-2}^2 f(x) dx &= 2 \int_0^2 f(x) dx \\ &= 2 \left(\int_0^1 f(x) dx + \int_1^2 f(x) dx \right) \\ &= 2 \left[1 - \frac{x^3}{3} \right]_0^1 = 2 \left(1 - \frac{7}{3} \right) = -\frac{8}{3} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 51. $\int_1^{10\pi} ([\sec^{-1} x] + [\cot^{-1} x]) dx$, where $[]$ denotes the greatest integer function, is equal to
(a) $10\pi - \sec 1$ (b) $10\pi + \sec 1$
(c) 10π (d) $\sec 1$

Sol. $\int_1^{10\pi} ([\sec^{-1} x] + [\cot^{-1} x]) dx$
 $= \int_1^{\sec 1} ([\sec^{-1} x] + [\cot^{-1} x]) dx$
 $+ \int_{\sec 1}^{10\pi} ([\sec^{-1} x] + [\cot^{-1} x]) dx$
 $= \int_1^{\sec 1} (0 + 0) dx + \int_{\sec 1}^{10\pi} (1 + 0) dx = (10\pi - \sec 1)$

Hence, (a) is the correct answer.

Ex 52. $\int_{-1}^1 \{x^2 + x - 3\} dx$, where $\{x\}$ denotes the fractional part of x , is equal to
(a) $\frac{1}{3}(1 + 3\sqrt{5})$ (b) $\frac{1}{6}(1 + 3\sqrt{5})$
(c) $\frac{1}{3}(3\sqrt{5} - 1)$ (d) $\frac{1}{6}(3\sqrt{5} - 1)$

Sol. $\int_{-1}^1 \{x^2 + x - 3\} dx = \int_{-1}^1 \{x^2 + x\} dx$
 $= \int_{-1}^1 (x^2 + x - [x^2 + x]) dx$
 $= \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_{-1}^1 - \int_{-1}^0 [x^2 + x] dx$
 $- \int_0^{\frac{\sqrt{5}-1}{2}} [x^2 + x] dx - \int_{\frac{\sqrt{5}-1}{2}}^1 [x^2 + x] dx$
 $= \frac{2}{3} + 1 - 0 - 1 \left(1 - \frac{\sqrt{5}-1}{2} \right)$
 $= \frac{2}{3} + \frac{\sqrt{5}-1}{2} = \frac{1}{6}(1 + 3\sqrt{5})$

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 53. Let $I = \int_0^{\pi/2} \frac{\sin x}{x} dx$, then

(a) $I > 1$ (b) $I > \frac{\pi}{2}$ (c) $I < \frac{\pi}{2}$ (d) $I < 1$

Sol. Let $f(x) = \frac{\sin x}{x}$, then

$$f'(x) = \frac{x \cos x - \sin x}{x^2} = \frac{\cos x (x - \tan x)}{x^2} < 0, \text{ for}$$

$$x \in \left(0, \frac{\pi}{2} \right)$$

Thus, f is a decreasing function.

Hence, $M = \text{maximum value of } f = f(0^+) = 1$,

$m = \text{minimum value of } f = f(\pi/2) = 2/\pi$

So, $m \left(\frac{\pi}{2} - 0 \right) < I < M \left(\frac{\pi}{2} - 0 \right)$

$$\Rightarrow 1 < I < \frac{\pi}{2}$$

Hence, (a) and (c) are the correct answers.

Ex 54. Let $f(x)$ be a function defined by $f(x) = \int_1^x x(x^2 - 3x + 2) dx$, $1 \leq x \leq 4$. Then,

(a) maximum value of $f(x)$ is $\frac{53}{4}$
 (b) maximum value of $f(x)$ is $\frac{63}{4}$
 (c) minimum value of $f(x)$ is $-\frac{1}{2}$
 (d) minimum value of $f(x)$ is $-\frac{1}{4}$

Sol. We have, $f(x) = \int_1^x (x^2 - 3x + 2) dx$... (i)

On differentiating Eq. (i), we get $f'(x) = x(x-1)(x-2) = 0$

Clearly, the function has its minimum value at $x = 2$.

i.e. $f_{\min} = \int_1^2 (x^2 - 3x + 2) dx = -\frac{1}{4}$

When $x = 1$, then $f(1) = 0$

When $x = 4$, $f(4) = \int_1^4 (x^2 - 3x + 2) dx = \frac{63}{4}$

Hence, range of $f(x)$ is $\left[-\frac{1}{4}, \frac{63}{4}\right]$

Hence, (b) and (d) are the correct answers.

Ex 55. If $I = \int_0^{\pi/2} \frac{dx}{\sqrt{1 + \sin^3 x}}$, then

- (a) $0 < I < 1$ (b) $I > \frac{\pi}{2\sqrt{2}}$
(c) $I < \pi/2$ (d) $I > 2\pi$

Sol. Since, $x \in \left[0, \frac{\pi}{2}\right]$

$\therefore 1 \leq 1 + \sin^3 x \leq 2$

$\Rightarrow \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1 + \sin^3 x}} \leq 1$

$\Rightarrow \int_0^{\pi/2} \frac{1}{\sqrt{2}} dx \leq \int_0^{\pi/2} \frac{dx}{\sqrt{1 + \sin^3 x}} \leq \int_0^{\pi/2} dx$

$\Rightarrow \frac{\pi}{2\sqrt{2}} \leq I \leq \frac{\pi}{2}$

Hence, (b) and (c) are the correct answers.

Ex 56. If $I(a) = \int_0^1 \frac{x^{\cos a} - 1}{\ln x} dx$, a being a real number other than an odd multiple of π , then

(a) $I(a) = \ln(\cos a)$ (b)

$I(a) = \ln(1 + \cos a)$

(c) $I'(a) = \tan \frac{a}{2}$ (d) $I'(a) = -\tan \frac{a}{2}$

Sol. We have, $I(a) = \int_0^1 \frac{x^{\cos a} - 1}{\ln x} dx$

$\Rightarrow I'(a) = \int_0^1 \frac{x^{\cos a} \ln x (-\sin a)}{\ln x} dx$

$= -\int_0^1 x^{\cos a} (\sin a) dx = -\left[\frac{\sin a \cdot x^{\cos a + 1}}{\cos a + 1}\right]_0^1$

$= -\frac{\sin a}{1 + \cos a} = -\tan \frac{a}{2}$

$\Rightarrow I(a) = -\int \frac{\sin a}{1 + \cos a} da = \ln |1 + \cos a| + C$

As $I\left(\frac{\pi}{2}\right) = 0 \Rightarrow C = 0$

$\therefore I(a) = \ln(1 + \cos a)$

Hence, (b) and (d) are the correct answers.

Ex 57. Let $F(x) = \int_0^x (t-1)(t-2)^2 dt$, then

(a) $\left(1, -\frac{17}{12}\right)$ is a point of minimum

(b) $\left(2, -\frac{4}{3}\right)$ is a point of inflection

(c) $\left(\frac{4}{3}, -\frac{112}{81}\right)$ is a point of inflection

(d) $(1, -3)$ is a point of minimum

Sol. $F'(x) = (x-1)(x-2)^2$, so $F'(x) = 0$ implies that $x = 1$ and $x = 2$. Around $x = 1$, $F'(x)$ changes sign from negative to positive. Hence, $F(x)$ is minimum at $x = 1$

and $F(1) = \int_0^1 (t^3 - 5t^2 + 8t - 4) dt$

$= \frac{1}{4} - \frac{5}{3} + 4 - 4 = -\frac{17}{12}$

Hence, $\left(1, -\frac{17}{12}\right)$ is a point of minimum.

Moreover, $F''(x) = 3x^2 - 10x + 8 = (3x-4)(x-2)$

and $F''(x) = 0$ implies that $x = 2, \frac{4}{3}$. As $F''(x)$

changes sign around $x = 2$ and $\frac{4}{3}$. So, $x = 2$ and $x = \frac{4}{3}$

are points of inflection. Also, $F(2) = -\frac{4}{3}$ and

$F\left(\frac{4}{3}\right) = -\frac{112}{81}$.

So, $\left(2, -\frac{4}{3}\right)$ and $\left(\frac{4}{3}, -\frac{112}{81}\right)$ are points of inflection.

Hence, (a), (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos 58-66) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 58. Statement I If $f(x) = \int_1^x \frac{\log t}{1+t+t^2} dt$ ($x > 0$),

then $f(x) = -f\left(\frac{1}{x}\right)$.

Statement II If $f(x) = \int_1^x \frac{\log t}{t+1} dt$, then

$f(x) + f\left(\frac{1}{x}\right) = \frac{1}{2} (\log x)^2$.

Sol. $f\left(\frac{1}{x}\right) = \int_1^{1/x} \frac{\log t}{1+t+t^2} dt$

On putting $t = \frac{1}{z} \Rightarrow dt = -\frac{1}{z^2} dz$

$$\begin{aligned} \therefore f\left(\frac{1}{x}\right) &= \int_1^x \frac{\log 1/z}{1+\frac{1}{z}+\frac{1}{z^2}} \left(-\frac{dz}{z^2}\right) \\ &= \int_1^x \frac{\log z}{1z^2+z+1} dz \\ &= \int_1^x \frac{\log t}{1+t+t^2} dt = f(x) \end{aligned}$$

$\Rightarrow$ Statement I is false.

Statement II is true which can be proved in the same manner in which Statement I has been disproved.

Hence, (d) is the correct answer.

Ex 59. Statement I If $I = \int_0^1 \frac{\cos x}{1+x^2} dx$, then

$I < (\pi/4) \cos 1$.

Statement II If f and ϕ are continuous functions on $[a, b]$, then for some $c \in (a, b)$, $\int_a^b f(x)\phi(x) dx = f(c) \int_a^b \phi(x) dx$.

Sol. $\int_0^1 \frac{\cos x}{1+x^2} dx = \cos c \int_0^1 \frac{dx}{1+x^2} = \cos c [\tan^{-1} x]_0^1$
 $= \cos c \frac{\pi}{4}$ (for some $c, 0 < c < 1$)

But the function $\cos x$ decreases on $(0, 1)$, so $\cos c > \cos 1$. Thus, $I > (\cos 1) \frac{\pi}{4}$. Statement II is

known as generalised mean value theorem.

Hence, (a) is the correct answer.

Ex 60. Statement I $\int_0^{\pi/4} \sec x \sqrt{\frac{1-\sin x}{1+\sin x}} dx = 2 - \sqrt{2}$

Statement II $\int_0^{\pi/4} \frac{\sec x}{1+2\sin^2 x} dx$
 $= \frac{1}{2} \ln(\sqrt{2}-1) + \frac{\pi}{6\sqrt{2}}$

Sol. Statement I Let $I = \int_0^{\pi/4} \frac{1}{\cos x} \sqrt{\frac{1-\sin x}{1+\sin x}} dx$
 $= \int_0^{\pi/4} \frac{1-\sin x}{\cos x \cdot \cos x} dx$
 $= \int_0^{\pi/4} (\sec^2 x - \sec x \tan x) dx$
 $= [\tan x - \sec x]_0^{\pi/4} = 2 - \sqrt{2}$

Statement II Let $I = \int_0^{\pi/4} \frac{\sec x}{1+2\sin^2 x} dx$
 $= \int_0^{\pi/4} \frac{\cos x}{(1-\sin^2 x)(1+2\sin^2 x)} dx$
 $= \int_0^{1/\sqrt{2}} \frac{dt}{(1-t^2)(1+2t^2)}$
 $= \frac{1}{3} \int_0^{1/\sqrt{2}} \left(\frac{1}{1-t^2} + \frac{2}{1+2t^2} \right) dt$

$$\begin{aligned} &= \left[\frac{1}{3} \cdot \frac{1}{2} \cdot \ln \frac{1+t}{1-t} + \frac{2}{3\sqrt{2}} \tan^{-1} \sqrt{2}t \right]_0^{1/\sqrt{2}} = \frac{1}{6} \\ \Rightarrow \ln \frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{\sqrt{2}}{3} \left(\frac{\pi}{4} \right) &= \frac{1}{6} \ln(3+2\sqrt{2}) + \frac{\pi}{6\sqrt{2}} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 61. Let $I_n = \int_1^e (\ln x)^n dx, n \in N$

Statement I $I_1, I_2, I_3, \dots$ is an increasing sequence.

Statement II $\ln x$ is an increasing function.

Sol. Statement II is true, as if $f(x) = \ln x$, then $f'(x) = \frac{1}{x} > 0$ as $x > 0$, so that $f(x)$ is defined.

Statement I is not true as $0 < \ln x < 1, \forall x \in (1, e)$ and hence $(\ln x)^n$ decreases as n is increasing, so that I_n is a decreasing sequence.

Hence, (d) is the correct answer.

Ex 62. Statement I $\int_0^1 2^x x^2 dx$
 $= \frac{2}{\ln 2} - \frac{4}{(\ln 2)^2} + \frac{2}{(\ln 2)^3}$

Statement II If $\int_0^1 e^x (x-1)^n dx = 16 - 6e$, then the value of n is 3.

Sol. Statement I Let $I = \int_0^1 2^x \cdot x^2 dx$

$$\begin{aligned} &= \left[x^2 \cdot \frac{2^x}{\ln 2} \right]_0^1 - \int_0^1 \frac{2x \cdot 2^x}{\ln 2} dx \\ &= \frac{2}{\ln 2} - \frac{2}{\ln 2} \left\{ \left[x \cdot \frac{2^x}{\ln 2} \right]_0^1 - \int_0^1 \frac{2x}{\ln 2} dx \right\} \\ &= \frac{2}{\ln 2} - \frac{4}{(\ln 2)^2} + \frac{2}{(\ln 2)^3} \end{aligned}$$

Statement II $I_n = \int_0^1 e^x (x-1)^n dx$
 $= [e^x (x-1)^n]_0^1 - n \int_0^1 (x-1)^{n-1} e^x dx$
 $= -(-1)^n - nI_{n-1}$

$\therefore I_n + nI_{n-1} = -(-1)^n$

$\Rightarrow I_1 + (e-1) = 1 \Rightarrow I_1 = 2 - e$

$\Rightarrow I_2 + 2(2-e) = -1$

$\therefore I_2 = 2e - 5$

and $I_3 + 3(2e-5) = 1 \Rightarrow I_3 = 16 - 6e$

$\therefore n = 3$

Hence, (b) is the correct answer.

Ex 63. Let f be a periodic function of period 2. If $\int_0^x f(t) dt$ and $h(x) = g(x+2) - g(x)$, then

Statement I h is a periodic function.

Statement II $g(x+2) - g(x) = g(2)$

Sol. Statement II is true because

$$g(x+2) - g(x) = \int_0^{x+2} f(t) dt - \int_0^x f(t) dt$$

$$= \int_x^{x+2} f(t) dt$$

$$\Rightarrow \frac{d}{dx} \{g(x+2) - g(x)\} = f(x+2) - f(x) = 0$$

Since, f is periodic function of period 2.

$\therefore g(x+2) - g(x)$ is constant.

So, $g(x+2) - g(x) = g(0+2) - g(0) = g(2) - g(0)$

$\Rightarrow h$ is a constant function and hence periodic.

Also, Statement II is the correct reason for Statement I.

Hence, (a) is the correct answer.

Ex 64. Statement I $\int_{\pi/6}^{\pi/3} \frac{1}{1 + \tan^3 x} dx = \frac{\pi}{12}$

Statement II $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

Sol. Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1 + \tan^3 x} dx$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(i)$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \frac{\pi}{3} - \frac{\pi}{6} \Rightarrow I = \frac{\pi}{12}$$

Hence, (a) is the correct answer.

Ex 65. Statement I If f satisfies

$$f(x+y) = f(x) + f(y), \forall x, y \in R, \text{ then } \int_{-5}^5 f(x) dx = 0.$$

Statement II If f is an odd function, then

$$\int_{-a}^a f(x) dx = 0.$$

Sol. $f(x+y) = f(x) + f(y)$

Put $x, y = 0 \Rightarrow f(0) = 0$

Put $y = -x$

$$\Rightarrow f(-x) + f(x) = 0 \Rightarrow f(-x) = -f(x)$$

So, $f(x)$ is odd.

$$\therefore \int_{-5}^5 f(x) dx = 0$$

Hence, (a) is the correct answer.

Ex 66. Statement I $\int_0^{n\pi+t} |\sin x| dx = (2n+1) - \cos t, 0 \leq t \leq \pi$

Statement II $\int_a^b f(x) dx$

$$= \int_a^e f(x) dx + \int_e^b f(x) dx$$

and $\int_0^{na} f(x) dx = n \int_0^a f(x) dx,$

if $f(a+x) = f(x)$

Sol. $\int_0^{n\pi+t} |\sin x| dx = \int_0^t |\sin x| dx + \int_t^{n\pi+t} |\sin x| dx$

$$= (2n+1) - \cos t$$

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 67-69) *Definite integral of any discontinuous or non-differentiable function is normally solved by the property*

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

where, $c \in (a, b)$ is the point of discontinuity or non-differentiability.

Ex 67. If $A = \int_1^\infty [\operatorname{cosec}^{-1} x] dx$, where $[]$ denotes

greatest integer function, then A is equal to

- (a) $\operatorname{cosec} 1 - 1$ (b) 1
(c) $1 - \sin 1$ (d) None of these

Sol. $A = \int_1^\infty [\operatorname{cosec}^{-1} x] dx$

$$= \int_1^{\operatorname{cosec} 1} 1 dx + \int_{\operatorname{cosec} 1}^\infty 0 dx = \operatorname{cosec} 1 - 1$$

Hence, (a) is the correct answer.

Ex 68. If $B = \int_1^{100} [\sec^{-1} x] dx$, where $[]$ denotes

greatest integer function, then B is equal to

- (a) $\sec 1$ (b) $100 - \sec 1$
(c) $99 - \sec 1$ (d) None of these

Ex 70. Evaluate $\int_{-\pi/4}^{\pi/4} \ln \left(\frac{\sin x + \cos x}{\cos x - \sin x} \right) dx.$

- (a) $\pi \ln 2$ (b) $\frac{\pi \ln 2}{2}$
(c) 0 (d) $-\pi \ln 2$

Sol. $I = \int_{-\pi/4}^{\pi/4} \ln \left(\frac{\sin x + \cos x}{\cos x - \sin x} \right) dx$
 $\Rightarrow I = 0$ [$\because f(x) = -f(-x)$]
Hence, (c) is the correct answer.

Ex 71. Evaluate $\int_{-\pi/4}^{\pi/4} \ln (\sin x + \cos x) dx$.

- (a) $\frac{\pi \ln 2}{2}$ (b) $-\frac{\pi \ln 2}{4}$
(c) $\pi \ln 2$ (d) 0

Sol. Let $I = \int_{-\pi/4}^{\pi/4} \ln (\sin x + \cos x) dx$
 $\Rightarrow I = \int_{-\pi/4}^{\pi/4} \ln (\cos x + \sin x) dx$
 $\therefore 2I = \int_{-\pi/4}^{\pi/4} \ln (\cos^2 x - \sin^2 x) dx$
 $\Rightarrow 2I = \int_{-\pi/4}^{\pi/4} \ln (\cos 2x) dx = 2 \int_0^{\pi/4} \ln (\cos 2x) dx$
 $\therefore I = \int_0^{\pi/4} \ln (\cos 2x) dx = \frac{1}{2} \int_0^{\pi/2} \ln (\cos t) dt$
 $= \frac{1}{2} \left(-\frac{\pi}{2} \ln 2 \right) = -\frac{\pi}{4} \ln 2$

Hence, (b) is the correct answer.

Ex 72. Evaluate $\int_0^{\pi/4} \ln (\sin 2x) dx$

- (a) $-\frac{\pi \ln 2}{2}$ (b) $\pi \ln 2$
(c) $\frac{\pi \ln 2}{4}$ (d) $-\frac{\pi \ln 2}{4}$

Sol. Let $I = \int_0^{\pi/4} \ln (\sin 2x) dx$
Put $2x = t \Rightarrow 2 dx = dt$
 $\therefore I = \frac{1}{2} \int_0^{\pi/2} \ln \sin t dt$
 $= \frac{1}{2} \left(-\frac{\pi}{2} \ln 2 \right) = -\frac{\pi \ln 2}{4}$

Hence, (d) is the correct answer.

Passage IV (Q. Nos. 73-75) Integral $\int_a^b f(x) dx$ can

be represented as a limit of a sum of infinite series

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{r=na+c}^{nb+c} \frac{1}{n} f\left(\frac{r}{n}\right)$$

where, $na+c \leq r \leq nb+c$, $r, n \in N$, $c \in R$ and any limit of sum of series of same form can be changed to definite integral by replacing

- $\lim_{n \rightarrow \infty} \Sigma \rightarrow \int$
- $\frac{1}{n} \rightarrow dx$
- $\frac{r}{n} \rightarrow x$
- Lower limit $= \lim_{n \rightarrow \infty} \left(\frac{r}{n} \right)_{min} = \lim_{n \rightarrow \infty} \left(\frac{na+c}{n} \right) = a$
- Upper limit $= \lim_{n \rightarrow \infty} \left(\frac{r}{n} \right)_{max} = \lim_{n \rightarrow \infty} \left(\frac{nb+c}{n} \right) = b$

Ex 73. Find the value of

$$\lim_{n \rightarrow \infty} \left(\frac{n}{(n+1)\sqrt{2n+1}} + \frac{n}{(n+2)\sqrt{2(2n+2)}} + \frac{n}{(n+3)\sqrt{3(2n+3)}} + \dots + \frac{1}{2n\sqrt{3}} \right)$$

- (a) $\pi/3$
(b) $\pi/2$
(c) $\pi/4$
(d) None of the above

Sol. $S = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{(n+r)\sqrt{r(2n+r)}}$
 $= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n \left(1 + \frac{r}{n} \right) \sqrt{\frac{r}{n} \left(2 + \frac{r}{n} \right)}}$
 $= \int_0^1 \frac{dx}{(1+x)\sqrt{2x+x^2}}$
 $= \int_0^1 \frac{dx}{(1+x)\sqrt{(1+x)^2 - 1}}$
 $= [\sec^{-1}(1+x)]_0^1$
 $= \sec^{-1} 2 - \sec^{-1} 1 = \frac{\pi}{3}$

Hence, (a) is the correct answer.

Ex 74. The n th term of the corresponding series of $\int_0^1 \tan^{-1} x dx$ is

- (a) $\frac{\pi}{4n}$
(b) $\frac{1}{n} \tan^{-1}(n-1)$
(c) $\frac{\pi}{2n}$
(d) $\tan^{-1} n$

Sol. $f(x) = \tan^{-1} x$

$\therefore f(n) = \frac{1}{n} f\left(\frac{n}{n}\right) = \frac{1}{n} f(1) = \frac{\pi}{4n}$

Hence, (a) is the correct answer.

Ex 75. $\lim_{n \rightarrow \infty} \sum_{r=0}^{2n-1} \frac{1}{n} \sec^2 \left(\frac{r}{n} \right)$ is

- (a) $\sec 2$
(b) $\tan 2$
(c) $\sec^2 2$
(d) Not defined

Sol. $S = \lim_{n \rightarrow \infty} \sum_{r=0}^{2n-1} \frac{1}{n} \sec^2 \left(\frac{r}{n} \right) = \int_0^2 \sec^2 x dx$
 $= [\tan x]_0^2 = \tan 2$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 76. Match the statements of Column I with values of Column II.

Column I	Column II
A. $\int_{-1}^1 \frac{dx}{(1+x^2)(e^x+1)}$ equals	p. 4
B. $\int_{-1}^2 \frac{[x]}{1+x^2} dx$, [] denotes the greatest integer function equals	q. $\tan^{-1} 2 - \frac{\pi}{2}$
C. $\int_0^{2\pi} \sin 2x dx$ equals	r. $\ln 6$
D. $\lim_{n \rightarrow \infty} \left(\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{6n} \right)$ equals	s. $\frac{\pi}{4}$

Sol. A. $\int_0^1 \frac{dx}{(1+x^2)(e^x+1)} + \int_0^1 \frac{dx}{(1+x^2)(e^{-x}+1)}$
 $= \int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$

B. $\int_{-1}^2 \frac{[x]}{1+x^2} dx = \int_{-1}^0 \frac{-1}{1+x^2} dx + \int_0^1 0 dx + \int_1^2 \frac{1}{1+x^2} dx$
 $= -[\tan^{-1} x]_{-1}^0 + \tan^{-1} 2 - \tan^{-1} 1$
 $= \tan^{-1} 2 - \frac{\pi}{2}$

C. $\int_0^{2\pi} |\sin 2x| dx = 4 \int_0^{\pi/2} |\sin 2x| dx$
 $= 4 \int_0^{\pi/2} \sin 2x dx$
 $= -\frac{4}{2} [\cos 2x]_0^{\pi/2}$
 $= -2[-1-1] = 4$

D. $\lim_{n \rightarrow \infty} \sum_{r=0}^{5n} \frac{1}{n+r} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{5n} \frac{1}{1+(r/n)} = \int_0^5 \frac{dx}{1+x}$
 $= [\ln(1+x)]_0^5 = \ln 6$

A $\rightarrow$ s; B $\rightarrow$ q; C $\rightarrow$ p; D $\rightarrow$ r

Ex 77. Match the statements of Column I with values of Column I.

Column I	Column II
A. If $\int_0^1 \frac{\cos x}{1+x} dx = k$ and $\int_0^{6\pi-3} \frac{\cos(x/3)}{6\pi+3-x} dx = mk$, then m is	p. 4
B. $\int_{-1}^1 \left(\sin^{-1} \left[x + \frac{3}{4} \right] \right) dx = \frac{\pi}{k}$, then k is {where [] denotes the greatest integer function}	q. 1
C. If $f(x) = \max(x - [x], x + [x])$ and $\int_{-3}^3 f(x) dx = -\frac{3}{k}$, then k is {where [] denotes greatest integer function}	r. 3
D. If $\int_0^{20} \sqrt{1 - \cos \pi x} dx = \frac{10k\sqrt{2}}{\pi}$, then k is	s. 2

Sol. A. Let $I = \int_0^{6\pi-3} \frac{\cos(x/3)}{6\pi+3-x} dx$

Put $6\pi - 3t = x$

$\Rightarrow -3dt = dx$

$\therefore I = \int_0^1 \frac{3 \cos(2\pi - t)}{3(1+t)} dt$

$= \int_0^1 \frac{\cos t}{1+t} dt = \int_0^1 \frac{\cos x}{1+x} dx = k$

B. $\int_{-1}^1 \left(\sin^{-1} \left[x + \frac{3}{4} \right] \right) dx$

$= \int_{-1}^{3/4} -\frac{\pi}{2} dx + \int_{3/4}^1 0 dx + \int_{1/4}^1 \frac{\pi}{2} dx$

$= -\frac{\pi}{8} + \frac{\pi}{2} \times \frac{3}{4} = \frac{2\pi}{8} = \frac{\pi}{4}$

C. $\therefore -|x| \leq [x]$ for all positive x

$-|x| \geq [x]$ for all negative x

$\therefore \int_{-3}^3 f(x) dx = \int_{-3}^0 (x - |x|) dx + \int_0^3 (x + [x]) dx$
 $= \int_{-3}^0 2x dx + \int_0^3 x dx + \int_0^3 [x] dx$
 $= -\frac{3}{2}$

D. $\int_0^{20} \sqrt{1 - \cos \pi x} dx = \int_0^{20} \sqrt{2} \left| \sin \frac{\pi x}{2} \right| dx$

$= 10\sqrt{2} \int_0^2 \left| \sin \frac{\pi x}{2} \right| dx$

$\left[\because \left| \sin \frac{\pi x}{2} \right| \text{ is periodic with period } 2 \right]$

$= 10\sqrt{2} \int_0^2 \sin \left(\frac{\pi x}{2} \right) dx$

$= 10\sqrt{2} \left[-\frac{\cos \left(\frac{\pi x}{2} \right)}{\frac{\pi}{2}} \right]_0^2 = \frac{20\sqrt{2}}{\pi}$

A $\rightarrow$ q; B $\rightarrow$ p; C $\rightarrow$ s; D $\rightarrow$ r

Ex 78. Match the statements of Column I with values of Column II.

Column I	Column II
A. $\int_{-1}^1 \frac{3x^2}{1+4^{\tan x}} dx$ equals	p. 7
B. $\int_6^8 \frac{\sin x^2}{\sin x^2 + \sin(x-14)^2} dx$ equals	q. $\frac{1}{2}$
C. $\frac{1}{156} \int_1^{13} [x] dx$ equals	r. 1
D. $\frac{1}{\pi \ln 2} \int_{\pi/2}^0 \ln \sin 2x dx$ equals	s. 2

Sol. A. Let $I = \int_{-1}^1 \frac{3x^2}{1+4^{\tan x}} dx$... (i)

$\Rightarrow I = \int_{-1}^1 \frac{3x^2}{1+4^{-\tan x}} dx$

$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$

$$\Rightarrow I = \int_{-1}^1 \frac{3x^2 4^{\tan x}}{1 + 4^{\tan x}} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-1}^1 \frac{3x^2 (1 + 4^{\tan x})}{1 + 4^{\tan x}} dx$$

$$\Rightarrow 2I = [x^3]_{-1}^1 \Rightarrow I = 1$$

$$\text{B. Let } I = \int_6^8 \frac{\sin x^2}{\sin x^2 + \sin(x-14)^2} dx \quad \dots(i)$$

$$\Rightarrow I = \int_6^8 \frac{\sin(14-x)^2}{\sin x^2 + \sin(x-14)^2} dx \quad \dots(ii)$$

$$[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_6^8 dx \Rightarrow I = 1$$

C. Let

$$I = \frac{1}{156} \left[\int_1^2 1 dx + \int_2^3 2 dx + \int_3^4 3 dx + \dots + \int_{12}^{13} 12 dx \right]$$

$$\Rightarrow I = \frac{1}{156} [(2-1) + 2(3-2) + 3(4-3) + \dots + 12(13-12)]$$

$$\Rightarrow I = \frac{1}{156} [1 + 2 + 3 + \dots + 12] = \frac{1}{2}$$

$$\text{D. Let } I = \int_0^{\pi/2} \ln \sin 2x dx = -\frac{\pi}{2} \ln 2$$

$$\therefore \frac{1}{\pi \ln 2} \int_{\pi/2}^0 \ln \sin 2x dx = \frac{1}{2}$$

$$A \rightarrow r; B \rightarrow r; C \rightarrow q; D \rightarrow q$$

Ex 79. Let $I = \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}}$

and $I_1 = \int_0^{1/2} \frac{dx}{\sqrt{1-x^4}}$

Then, match the statements of Column I with values of Column II.

A.	The value of I is less than	p.	1
B.	The value of I is more than	q.	1/2
C.	The value of I_1 is less than	r.	$\pi/4$
D.	The value of I_1 is more than	s.	1/4

Sol. We know that for $0 \leq x \leq 1$,

$$4 - x^2 \geq 4 - x^2 - x^3 \geq 4 - 2x^2$$

$$\text{So, } \int_0^1 \frac{dx}{\sqrt{4-x^2}} \leq I \leq \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-x^2}} = \left[\sin^{-1} \frac{1}{2} \right]_0^1 = \frac{\pi}{6}$$

$$\Rightarrow \int_0^1 \frac{dx}{\sqrt{4-2x^2}} = \frac{1}{\sqrt{2}} \int_0^1 \frac{dx}{\sqrt{2-x^2}}$$

$$= \left[\frac{1}{\sqrt{2}} \sin^{-1} \frac{x}{\sqrt{2}} \right]_0^1$$

$$= \frac{\pi}{4\sqrt{2}}$$

$$\text{Hence, } \frac{\pi}{6} < I < \frac{\pi}{4\sqrt{2}}.$$

$$\text{It is clear that } \frac{\pi}{4\sqrt{2}} < 1 \text{ and } \frac{\pi}{4\sqrt{2}} < \frac{\pi}{4}. \text{ Also,}$$

$$\frac{\pi}{6} > \frac{1}{2} > \frac{1}{4}.$$

$$\text{For } 0 \leq x \leq \frac{1}{2}, \sqrt{1-x^2} \leq \sqrt{1-x^4} \leq 1$$

$$\therefore \frac{1}{2} = \int_0^{1/2} 1 dx \leq I_1 \leq \int_0^{1/2} \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{6}$$

$$A \rightarrow p, r; B \rightarrow q, s; C \rightarrow p, r; D \rightarrow q, s;$$

Type 6. Single Integer Answer Type Questions

Ex 80. A positive integer $n \leq 5$ such that

$$\int_0^1 e^{2x-1} (x-1)^n dx = \frac{1}{4} \left(\frac{7}{e} - e \right) \text{ is } \underline{\hspace{2cm}}.$$

Sol. (3) For $n \geq 0$, let $I_n = \int_0^1 e^{2x-1} (x-1)^n dx$

$$\text{We have, } I_0 = \int_0^1 e^{2x-1} dx = \frac{1}{2} [e^{2x-1}]_0^1 = \frac{1}{2} \left(e - \frac{1}{e} \right)$$

For $n \geq 1$,

$$I_n = \frac{1}{2} [e^{2x-1} (x-1)^n]_0^1 - \frac{n}{2} \int_0^1 e^{2x-1} (x-1)^{n-1} dx$$

$$= -\frac{1}{2} e^{-1} (-1)^n - \frac{n}{2} I_{n-1}$$

$$= \frac{1}{2} (-1)^{n+1} e^{-1} - \frac{n}{2} I_{n-1}$$

$$\therefore I_1 = \frac{1}{2} e^{-1} - \frac{1}{2} \cdot \frac{1}{2} \left(e - \frac{1}{e} \right)$$

$$= -\frac{1}{4} \left(\frac{3}{e} - e \right)$$

$$I_2 = -\frac{1}{2e} - I_1$$

$$= -\frac{1}{2e} - \frac{1}{4} \left(\frac{3}{e} - e \right) = \frac{1}{4} \left(e - \frac{5}{e} \right)$$

and

$$I_3 = \frac{1}{2e} - I_2$$

$$= \frac{1}{2e} - \frac{1}{4} \left(e - \frac{5}{e} \right) = \frac{1}{4} \left(\frac{7}{e} - e \right)$$

Hence,

$$n = 3$$

Ex 81. The value $\int_{-2}^2 |1 - x^2| dx$ is _____.

Sol. (4) $\int_{-2}^2 |1 - x^2| dx = 2 \int_0^2 |1 - x^2| dx$
 $= 2 \left(\int_0^1 |1 - x^2| dx + \int_1^2 |1 - x^2| dx \right)$
 $= 2 \left(\int_0^1 (1 - x^2) dx + \int_1^2 (x^2 - 1) dx \right)$
 $= 2 \left\{ \left[x - \frac{x^3}{3} \right]_0^1 + \left[\frac{x^3}{3} - x \right]_1^2 \right\}$
 $= 2 \left[1 - \frac{1}{3} + \frac{8}{3} - 2 - \left(\frac{1}{3} - 1 \right) \right]$
 $= 2(2) = 4$

Ex 82. If $\int_a^b f(x) dx = \frac{(b-a)\lambda}{8} \int_0^1 f((b-a)x + a) dx$, then the value of λ is _____.

Sol. (8) Let $x = \alpha t + \beta$, where α and β are to be chosen such that

$$a = \alpha \cdot 0 + \beta \quad \text{and} \quad b = \alpha \cdot 1 + \beta$$

$$\Rightarrow \beta = a$$

$$\alpha = b - a$$

$$\text{Now, } x = (b - a)t + a$$

$$\Rightarrow dx = (b - a) dt$$

$$\text{Hence, } \int_a^b f(x) dx = (b - a) \int_0^1 f((b - a)t + a) dt$$

$$= (b - a) \int_0^1 f((b - a)x + a) dx$$

$$\therefore \lambda = 8$$

Ex 83. If $\int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx = \pi^\alpha$, then α is equal to _____.

Sol. (2) Let $I = \int_0^{2\pi} \frac{x \sin^8 x}{\sin^8 x + \cos^8 x} dx$
 $= \int_0^{2\pi} \frac{(2\pi - x) \sin^8 x}{\sin^8 x + \cos^8 x} dx$
 $= \int_0^{2\pi} \frac{\pi \sin^8 x}{\sin^8 x + \cos^8 x} dx$
 $= 4 \int_0^{\pi/2} \frac{\pi \sin^8 x}{\sin^8 x + \cos^8 x} dx \quad \dots(i)$
 $= 4 \int_0^{\pi/2} \frac{\pi \cos^8 x}{\cos^8 x + \sin^8 x} dx \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

$$2I = 4\pi \int_0^{\pi/2} \frac{\sin^8 x + \cos^8 x}{\cos^8 x + \sin^8 x} dx$$

$$= 4\pi \times \frac{\pi}{2} = 2\pi^2 \Rightarrow I = \pi^2$$

$$\therefore \alpha = 2$$

Ex 84. The value of the integral $\int_{-3\pi/4}^{5\pi/4} \frac{\cos x + \sin x}{1 + e^{x - \frac{\pi}{4}}} dx$ is _____.

Sol. (0) Let $I = \int_{-3\pi/4}^{5\pi/4} \frac{\sin x + \cos x}{e^{x - \frac{\pi}{4}} + 1} dx$
 $= \int_{-3\pi/4}^{5\pi/4} \frac{\sqrt{2} \cos \left(x - \frac{\pi}{4} \right)}{e^{x - \frac{\pi}{4}} + 1} dx$

$$\text{Put } x - \frac{\pi}{4} = t \text{ and get } I = 0.$$

Ex 85. Evaluate $\int_{-1}^1 [x[1 + \sin \pi x] + 1] dx$, where $[]$ denotes the greatest integer function.

Sol. (2) Let $I = \int_{-1}^0 [x[1 + \sin \pi x] + 1] dx$
 $+ \int_0^1 [x[1 + \sin \pi x] + 1] dx$

$$\text{Now, } [1 + \sin \pi x] = \begin{cases} 0, & \text{if } -1 < x < 0 \\ 1, & \text{if } 0 < x < 1 \end{cases}$$

$$\therefore I = \int_{-1}^0 1 dx + \int_0^1 [x + 1] dx$$

$$= 1 + 1 \int_0^1 dx$$

$$= 1 + 1 = 2$$

Ex 86. Find the value of the integral $\int_{-4}^{-5} \sin(x^2 + 3) dx + \int_{-2}^1 \sin(x^2 + 12x + 33) dx$.

Sol. (0) Transforming each of the integrals into integrable with limits 0 and 1, with the help of transformation $x = (b - a)t + a$.

$$\text{Now, } I_1 = \int_{-4}^{-5} \sin(x^2 - 3) dx$$

$$\text{Put } x = (-5 + 4)t - 4 = -t - 4$$

$$\therefore I_1 = \int_0^1 \sin(t^2 + 8t + 16 - 3) (-dt)$$

$$= - \int_0^1 \sin(t^2 + 8t + 13) dt$$

In the second integral,

$$I_2 = \int_0^1 \sin(x^2 + 12x + 33) dx$$

Now, on putting $x = (-1 + 2)t - 2 = t - 2$

$$\therefore I_1 = \int_0^1 \sin(t^2 - 4t + 4 + 12t - 24 + 33) dt$$

$$= \int_0^1 \sin(t^2 + 8t + 13) dt$$

$$\Rightarrow I_1 + I_2 = 0$$

Target Exercises

Type 1. Only One Correct Option

- $\int_0^{\pi/2} \frac{dx}{\sin\left(x - \frac{\pi}{3}\right) \cdot \sin\left(x - \frac{\pi}{6}\right)}$ is equal to
(a) $4 \log \sqrt{3}$ (b) $-4 \log \sqrt{3}$
(c) $2 \log \sqrt{3}$ (d) None of these
- If $\int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx = a + \frac{b}{\log 2}$, then
(a) $a = e, b = -2$ (b) $a = e, b = 2$
(c) $a = -e, b = 2$ (d) None of these
- $\int_0^{\pi/2} \sin \theta \cos \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta$ is equal to
(a) $\frac{a^2 + ab + b^2}{2(a+b)}$ (b) $\frac{a^2 + ab + b^2}{(a+b)}$
(c) $\frac{a^2 + ab + b^2}{3(a+b)}$ (d) None of these
- The value of $\int_1^{3^{1/n}} \frac{dx}{x(x^n + 1)}$ is
(a) $\frac{1}{n} \log\left(\frac{2}{3}\right)$ (b) $n \log\left(\frac{2}{3}\right)$
(c) $\frac{1}{n} \log\left(\frac{3}{2}\right)$ (d) $n \log\left(\frac{3}{2}\right)$
- The value of $\int_0^1 \frac{4^{2x+1} - 3^{2x-1}}{12^x} dx$ is
(a) $\frac{4}{3 \log\left(\frac{4}{3}\right)} + \frac{1}{12 \log\left(\frac{3}{4}\right)}$ (b) $\frac{4}{3 \log\left(\frac{4}{3}\right)} + \frac{1}{4 \log\left(\frac{3}{4}\right)}$
(c) $\frac{3}{3 \log\left(\frac{4}{3}\right)} + \frac{1}{3 \log\left(\frac{3}{4}\right)}$ (d) None of these
- If $I = \int_0^{\pi/2} \cos^n x \sin^n x dx = \lambda \int_0^{\pi/2} \sin^n x dx$, then λ equals
(a) 2^{-n+1} (b) 2^{-n-1} (c) 2^{-n} (d) 2^{-1}
- $\int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx$ is equal to
(a) $\frac{\pi}{2ab}$ (b) $\frac{\pi}{ab}$ (c) $\frac{\pi^2}{2ab}$ (d) $\frac{\pi^2}{ab}$
- If $I = \int_{1/e}^{\pi} |\log x| \frac{dx}{x^2}$, then I equals
(a) 2 (b) $\frac{2}{e}$ (c) $2\left(1 - \frac{1}{e}\right)$ (d) 0
- $\int_0^{\pi/2} x \sin^2 x \cos^2 x dx$ is equal to
(a) $\frac{\pi^2}{32}$ (b) $\frac{\pi^2}{16}$
(c) $\frac{\pi}{32}$ (d) None of these
- If $\int_0^{\pi/2} \frac{dx}{5 + 3 \cos x} = \lambda \tan^{-1}\left(\frac{1}{2}\right)$, then the value of λ is
(a) 1 (b) $\frac{1}{2}$
(c) $\frac{1}{3}$ (d) $\frac{1}{4}$
- $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$ is equal to
(a) $\frac{\pi^2}{4}$ (b) π^2 (c) zero (d) $\frac{\pi}{2}$
- The value of $\int_0^{\pi} \log(1 + \cos x) dx$, is
(a) $\frac{\pi}{2} \log 2$ (b) $\pi \log 2$
(c) $-\pi \log 2$ (d) None of these
- The value of $\int_0^{\pi} \frac{x^2 \sin x}{(2x - \pi)(1 + \cos^2 x)} dx$ is
(a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{2}$
(c) $\frac{\pi^2}{6}$ (d) None of these
- The value of $\int_a^b \frac{|x|}{x} dx, a < b$, is
(a) $b - a$ (b) $a - b$
(c) $b + a$ (d) $|b| - |a|$
- $\int_0^1 \frac{\tan^{-1} x}{x} dx$ equals
(a) $\int_0^{\pi/2} \frac{x}{\sin x} dx$ (b) $\frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx$
(c) $\int_0^{\pi/2} \frac{\sin x}{x} dx$ (d) None of these
- $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx$ is equal to
(a) $\frac{\pi}{2}$ (b) 0
(c) $\frac{\pi}{4}$ (d) 1

17. If $\frac{d}{dx} g(x) = f(x)$, where $f(x)$ is continuous in $[a, b]$, then $\int_a^b f(x) \cdot g(x) dx$ equals
- (a) $\frac{1}{2}[\{f(b)\}^2 - \{f(a)\}^2]$
 (b) $f(b) - f(a)$
 (c) $\frac{1}{2}[\{g(b)\}^2 - \{g(a)\}^2]$
 (d) $g(b) - g(a)$
18. $\int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ is equal to
- (a) 1 (b) -1
 (c) 0 (d) None of these
19. $\int_{-2}^2 |1-x^2| dx$ is equal to
- (a) 4 (b) 2
 (c) 1 (d) None of these
20. $\int_0^{2a} x^{9/2} (2a-x)^{-1/2} dx$ is equal to
- (a) $\frac{63\pi}{4} a^5$ (b) $\frac{63\pi}{2} a^5$
 (c) $63\pi a^5$ (d) $\frac{63\pi}{8} a^5$
21. The integral $\int_{-1}^3 \left[\tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x} \right] dx$ is equal to
- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) π (d) 2π
22. If $P = \int_0^{3\pi} f(\cos^2 x) dx$ and $Q = \int_0^{\pi} f(\cos^2 x) dx$, then
- (a) $P - Q = 0$ (b) $P - 2Q = 0$
 (c) $P - 3Q = 0$ (d) $P - 5Q = 0$
23. $\int_0^2 \sin \frac{\pi[x]}{2} dx$ is equal to
- (a) 1 (b) -1
 (c) 0 (d) None of these
24. $\int_0^{\pi} |\cos \theta - \sin \theta| d\theta$ is equal to
- (a) $\sqrt{2}$ (b) $2\sqrt{2}$
 (c) $3\sqrt{3}$ (d) None of these
25. $\int_{-2}^2 [x^2] dx$ is equal to
- (a) $10 - 2\sqrt{3} - 2\sqrt{2}$
 (b) $10 + 2\sqrt{3} - 2\sqrt{2}$
 (c) $10 - 2\sqrt{3} + 2\sqrt{2}$
 (d) None of the above
26. $\int_0^{\pi} |\sin x + \cos x| dx$ is equal to
- (a) $\sqrt{2}$
 (b) $2\sqrt{2}$
 (c) $3\sqrt{3}$
 (d) None of the above
27. $\int_{-2}^1 [x+1] dx$ is equal to
- (a) 0 (b) 1
 (c) -1 (d) None of these
28. $\int_0^{n^2} [\sqrt{x}] dx$ is equal to
- (a) $\frac{n(n+1)(4n+1)}{6}$ (b) $\frac{n(n-1)(4n+1)}{6}$
 (c) $\frac{n(n-1)(4n-1)}{6}$ (d) None of these
29. If $f(x)$ and $\phi(x)$ are continuous functions on the interval $[0, 4]$ satisfying $f(x) = f(4-x)$, $\phi(x) + \phi(4-x) = 3$ and $\int_0^4 f(x) dx = 2$, then $\int_0^4 f(x) \phi(x) dx$ is equal to
- (a) 3 (b) 6
 (c) 2 (d) None of these
30. If $2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1}{x} - 2$, $x \neq 0$, then $\int_1^2 f(x) dx$ is equal to
- (a) $-\frac{2}{5} \log 2 + \frac{1}{2}$ (b) $-\frac{2}{5} \log 2 - \frac{1}{2}$
 (c) $\frac{2}{5} \log 2 + \frac{1}{2}$ (d) None of these
31. Let $f(x)$ be a polynomial of degree 2 satisfying $f(0) = 1$, $f'(0) = -2$ and $f''(0) = 6$, then $\int_{-1}^2 f(x) dx$ is equal to
- (a) 6 (b) 0
 (c) 9 (d) None of these
32. If $(-1, 2)$ and $(2, 4)$ are two points on the curve $y = f(x)$ and if $g(x)$ is the gradient of the curve at the point (x, y) , then $\int_{-1}^2 g(x) dx$ is equal to
- (a) 2 (b) -2
 (c) 0 (d) 1
33. If $f(x)$ is a function satisfying $f\left(\frac{1}{x}\right) + x^2 f(x) = 0$ for all $x(x \neq 0)$, then the value of $\int_{\tan \theta}^{\cot \theta} f(x) dx$ is
- (a) $\tan^2 \theta$ (b) $2 \tan \theta$
 (c) 0 (d) None of these
34. $\int_0^1 \frac{\log(1-x)}{x} dx$ is equal to
- (a) $\frac{\pi^2}{8}$ (b) $-\frac{\pi^2}{8}$ (c) $\frac{\pi^2}{6}$ (d) $-\frac{\pi^2}{6}$
35. $\int_0^1 \frac{\log x}{1+x} dx$ is equal to
- (a) $\frac{\pi^2}{12}$ (b) $-\frac{\pi^2}{12}$
 (c) $\frac{\pi^2}{6}$ (d) $-\frac{\pi^2}{6}$

36. The value of $\int_0^1 |x \cos \pi x| dx$ is
 (a) $\frac{2}{\pi}$ (b) $\frac{1}{\pi}$
 (c) $-\frac{2}{\pi}$ (d) $-\frac{1}{\pi}$
37. If $A = \int_1^{\sin \theta} \frac{t}{1+t^2} dt$ and $B = \int_1^{\csc \theta} \frac{dt}{t(1+t^2)}$, then
 the value of $\begin{vmatrix} A & A^2 & B \\ e^A e^B & B^2 & -1 \\ 1 & A^2 + B^2 & -1 \end{vmatrix}$ is
 (a) $\sin \theta$ (b) $\csc \theta$
 (c) 0 (d) 1
38. If $\int_a^b (x-a)^3 (b-x)^4 dx = \frac{(b-a)^m}{n}$, then (m, n) is
 (a) (6, 260) (b) (8, 280)
 (c) (4, 240) (d) None of these
39. If $f(x)$ is a periodic function of period, then
 $\int_0^{na} f(x) dx$ is equal to
 (a) $\int_0^a f(x) dx$ (b) $n \int_0^a f(x) dx$
 (c) $(n-1) \int_0^a f(x) dx$ (d) None of these
40. $\int_0^3 \{\sqrt{x}\} dx$, where $\{x\}$ denotes the fractional part of x ,
 is equal to
 (a) $2\sqrt{3} - 2$ (b) $\frac{7}{2}$
 (c) $\frac{3}{2}$ (d) None of these
41. If $I_1 = \int_0^a [x] dx$ and $I_2 = \int_0^a \{x\} dx$, where $[x]$ and $\{x\}$
 denote respectively the integral and fractional parts
 of x and a is a positive integer, then
 (a) $I_2 = (a-1)I_1$ (b) $I_1 = (a-1)I_2$
 (c) $I_1 = aI_2$ (d) $I_2 = aI_1$
42. The value of $\int_a^{a+\frac{\pi}{2}} (|\sin x| + |\cos x|) dx$ is
 (a) $a\pi$
 (b) $2a\pi$
 (c) $\frac{a\pi}{2}$
 (d) independent of a
43. If $f(x)$ is a periodic function with period T , then
 $\int_{a+2T}^{b+2T} f(x) dx$ is equal to
 (a) $\int_{a+T}^{b+T} f(x) dx$
 (b) $\int_{a+T}^b f(x) dx$
 (c) $\int_a^{b+T} f(x) dx$
 (d) None of the above
44. $\int_0^\infty \frac{x}{1+e^x} dx$ is equal to
 (a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{8}$
 (c) $\frac{\pi^2}{12}$ (d) None of these
45. The value of $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx$ is
 (a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) 3 (d) 5
46. If $\int_1^2 e^{x^2} dx = a$, then the value of $\int_e^{e^4} \sqrt{\ln(x)} dx$ is
 (a) $e^4 - e$ (b) $e^4 - a$
 (c) $2e^4 - a$ (d) $2e^4 - e - a$
47. For $x \in R$ and a continuous function f , let
 $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f\{x(2-x)\} dx$
 and $I_2 = \int_{\sin^2 t}^{1+\cos^2 t} f\{x(2-x)\} dx$, then I_1/I_2 is
 (a) 0 (b) 1 (c) 2 (d) 3
48. If $m = \int_2^0 \frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$, $n = \int_0^2 \frac{|\sin x|}{\left[\frac{x}{\pi}\right] + \frac{1}{2}} dx$, where $[]$
 denotes greatest integer function, then
 (a) $m = n$ (b) $m = -n$
 (c) $m = 2n$ (d) None of these
49. The value of $\int_{-1/2}^{1/2} \sqrt{\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2} - 2 dx$ is
 (a) $2 \log\left(\frac{4}{3}\right)$ (b) $4 \log\left(\frac{4}{3}\right)$
 (c) $\log\left(\frac{4}{3}\right)$ (d) None of these
50. Suppose for every integer n , $\int_n^{n+1} f(x) dx = n^2$.
 The value of $\int_{-2}^4 f(x) dx$ is
 (a) 16
 (b) 14
 (c) 19
 (d) None of the above
51. $\left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right|$ is less than
 (a) 10^{-10} (b) 10^{-11}
 (c) 10^{-7} (d) 10^{-9}
52. The smallest interval $[a, b]$ such that
 $\int_0^1 \frac{dx}{\sqrt{1+x^4}} \in [a, b]$, is given by
 (a) $\left[\frac{1}{\sqrt{2}}, 1\right]$ (b) $[0, 1]$ (c) $\left[\frac{1}{2}, 1\right]$ (d) $\left[\frac{3}{4}, 1\right]$

53. Let a, b, c be non-zero real numbers such that $\int_0^1 (1 + \cos^8 x)(ax^2 + bx + c) dx = \int_0^2 (1 + \cos^8 x)(ax^2 + bx + c) dx$, then the quadratic equation $ax^2 + bx + c = 0$ has

- (a) no root in $(0, 2)$
(b) atleast one root in $(1, 2)$
(c) atleast one root in $(0, 1)$
(d) two imaginary roots

54. If $I = \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}}$, then

- (a) $\frac{1}{2} < I < \frac{1}{3}$ (b) $\frac{1}{4} < I < \frac{1}{3}$
(c) $\frac{1}{3} < I < 1$ (d) None of these

55. The value of $\int_0^{[x]} (x - [x]) dx$ is

- (a) $[x]$ (b) $\frac{1}{2}[x]$ (c) $3[x]$ (d) $2[x]$

56. The value of $\int_0^a \frac{e^x}{e^{[x]}} dx, a \in I^+,$ is

- (a) ae (b) $a(e+1)$ (c) $a(1-e)$ (d) $a(e-1)$

57. $\int_0^{100\pi} \sum_{r=1}^{10} \tan rx dx$ is equal to

- (a) 100π (b) -100π
(c) 1 (d) None of these

58. If $x = \int_0^{\sin t} \sin^{-1} z dz$ and $y = \int_n^{\sqrt{t}} \frac{\sin z^2}{z} dz$, then

$\frac{dy}{dx}$ is equal to

- (a) $\frac{\tan t}{2t^2}$ (b) $\frac{2t^2}{\tan t}$
(c) $\frac{\tan t}{t^2}$ (d) None of these

59. If $f(x)$ is differentiable and $\int_0^{t^2} x f(x) dx = \frac{2}{5} t^5$,

then $f\left(\frac{4}{25}\right)$ equals

- (a) $2/5$ (b) $-5/2$ (c) 1 (d) $\frac{5}{2}$

60. If $\int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x, x \in [0, \pi/2]$, then

$f\left(\frac{1}{\sqrt{3}}\right)$ is

- (a) 3 (b) $\sqrt{3}$
(c) $\frac{1}{3}$ (d) None of these

61. The value of $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sec^2 t dt}{x \sin x}$ is

- (a) 3 (b) 2 (c) 1 (d) -1

62. If $f(x) = \int_{1/x}^{\sqrt{x}} \cos t^2 dt, x > 0$, then $f'(x)$ is equal to

- (a) $\frac{1}{x^2} \cos \frac{1}{x^2} + \frac{1}{2\sqrt{x}} \sin x$
(b) $\frac{1}{x^2} \cos \frac{1}{x^2} - \frac{1}{2\sqrt{2}} \cos x$
(c) $\frac{1}{x^2} \cos \frac{1}{x^2} + \frac{1}{2\sqrt{x}} \cos x$
(d) None of the above

63. The derivative of $f(x) = \int_{x^2}^{x^3} \frac{dt}{\log t}, x > 0$, is

- (a) $\frac{1}{3 \log x}$ (b) $\frac{1}{3 \log x} - \frac{1}{2 \log x}$
(c) $(\log x)^{-1} x(x-1)$ (d) $\frac{3x^2}{\log x}$

64. If $a_n = \int_0^{\pi/2} \frac{\cos^2 nx}{\sin x} dx$, then $a_2 - a_1, a_3 - a_2,$

$a_4 - a_3$ are in

- (a) GP (b) AP
(c) HP (d) None of these

65. If $a_n = \int_0^{\pi/4} \cot^n x dx$, then $a_2 + a_4, a_3 + a_5,$

$a_4 + a_6$ are in

- (a) GP (b) AP
(c) HP (d) None of the above

66. Let $I_n = \int_0^{\pi/2} \sin^n x dx$, where n is a positive integer. Then,

- (a) $I_n : I_{n-2} = n : (n-1)$ (b) $I_n > I_{n-2}$
(c) $n(I_{n-2} - I_n) = I_{n-2}$ (d) None of these

67. $\lim_{n \rightarrow \infty} \left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \frac{n+3}{n^2+3^2} + \dots + \frac{1}{n} \right]$ is equal to

- (a) $\frac{\pi}{4} + \frac{1}{2} \log 2$ (b) $\frac{\pi}{4} - \frac{1}{2} \log 2$
(c) $-\left(\frac{\pi}{4} + \frac{1}{2} \log 2\right)$ (d) None of these

68. $\lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{2n-1^2}} + \frac{1}{\sqrt{4n-2^2}} + \frac{1}{\sqrt{6n-3^2}} + \dots + \frac{1}{n} \right]$

is equal to

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$

69. The value of $\lim_{n \rightarrow \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{1}{2n} \right]$

is

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) π (d) None of these

70. $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r}{n^2} \cdot \sec^2 \frac{r^2}{n^2}$ is equal to
 (a) $\tan 1$ (b) $\frac{1}{3} \tan 1$
 (c) $\frac{1}{2} \tan 1$ (d) None of these
71. The value of $\int_0^\pi \left(\sum_{r=0}^3 a_r \cos^{3-r} x \sin^r x \right) dx$ depends upon
 (a) a_1 and a_2 (b) a_0 and a_3
 (c) a_2 and a_3 (d) a_1 and a_3
72. $\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right) \dots \left(1 + \frac{n}{n} \right) \right]^{1/n}$ is equal to
 (a) $\frac{2}{e}$ (b) $\frac{e}{2}$ (c) $\frac{e}{4}$ (d) $\frac{4}{e}$
73. $\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1^2}{n^2} \right) \left(1 + \frac{2^2}{n^2} \right) \dots \left(1 + \frac{n^2}{n^2} \right) \right]^{1/n}$ is equal to
 (a) $2e^{\frac{\pi+4}{2}}$ (b) $e^{\frac{\pi+4}{2}}$
 (c) $2e^{\frac{\pi-4}{2}}$ (d) $e^{\frac{\pi-4}{2}}$
74. $\lim_{n \rightarrow \infty} \frac{1}{n^{p+1}} [1^p + 2^p + \dots + n^p]$, $p > -1$, is equal to
 (a) $\frac{1}{p+1}$ (b) $\frac{1}{p}$
 (c) $\frac{1}{p+2}$ (d) None of these
75. $\lim_{n \rightarrow \infty} \left[\frac{1^2}{1^3 + n^3} + \frac{2^2}{2^3 + n^3} + \dots + \frac{1}{2n} \right]$ is equal to
 (a) $\frac{1}{3} \log 2$ (b) $\frac{1}{2} \log 2$
 (c) $\log 2$ (d) None of these
76. The value of $\int_0^\pi \cos^n x dx$, where n is an even integer, is
 (a) $\frac{\pi(2n)!}{2^{2n} \cdot (n!)^2}$ (b) $\frac{\pi \cdot n!}{2^n \left[\left(\frac{n}{2} \right)! \right]^2}$
 (c) $\frac{\pi \cdot n!}{2^{n+1} \left[\left(\frac{n}{2} \right)! \right]^2}$ (d) None of these
77. If $I_n = \int_0^{\pi/2} x^n \sin x dx$, where $n > 1$, then
 (a) $I_n + n(n-1)I_{n-2} = n \left(\frac{\pi}{2} \right)^n$
 (b) $I_n + n(n-1)I_{n-2} = n \left(\frac{\pi}{2} \right)^{n-1}$
 (c) $I_n - n(n-1)I_{n-2} = n \left(\frac{\pi}{2} \right)^n$
 (d) None of the above
78. $\int_0^a \frac{x^n}{\sqrt{ax-x^2}} dx$ is equal to
 (a) $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} (-\pi a^n)$ (b) $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} (\pi a^n)$
 (c) $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} (\pi a^{n-1})$ (d) None of these
79. $\int_0^\pi x \sin^4 x dx$ is equal to
 (a) $\frac{3\pi}{16}$ (b) $\frac{3\pi^2}{16}$ (c) $\frac{16\pi}{3}$ (d) $\frac{16\pi^2}{3}$
80. If $\int_0^{\pi/2} \sin^6 x dx = \frac{5\pi}{32}$, then the value of $\int_{-\pi}^\pi (\sin x^6 + \cos x^6) dx$ is
 (a) $\frac{5\pi}{8}$ (b) $\frac{5\pi}{6}$ (c) $\frac{5\pi}{2}$ (d) $\frac{5\pi}{4}$

Type 2. More than One Correct Option

81. Let $f(x) = \int_0^{\sin^2 x} \sin^{-1}(\sqrt{t}) dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) dt$, then
 (a) $f(x)$ is a constant function
 (b) $f\left(\frac{\pi}{4}\right) = 0$
 (c) $f\left(\frac{\pi}{3}\right) = \frac{\pi}{4}$
 (d) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$
82. Let $f(x) = \int_2^x f(t^2 - 3t + 4) dt$, then
 (a) $f'(2) = 0$ (b) $f'(-2) = 0$
 (c) $f(2) = 0$ (d) $f(2) = 2$
83. Let f be an odd continuous function which is periodic with period 2. If $g(x) = \int_0^x f(t) dt$, then
 (a) $g(x)$ is even
 (b) $g(n) = 0, n \in \mathbb{N}$
 (c) $g(2n) = 0, n \in \mathbb{N}$
 (d) $g(x)$ is periodic
84. If $I_n = \int_0^\pi \frac{\sin\left(n + \frac{1}{2}\right)x}{\sin \frac{x}{2}} dx$, where n is non-negative integer, then
 (a) I_n is dependent on n
 (b) I_n is independent on n
 (c) $I_{10} + I_5 = I_{20}$
 (d) $I_n = \pi$

85. The values of α which satisfy

$$\int_{\pi/2}^{\alpha} \sin x \, dx = \sin 2\alpha \quad (\alpha \in [0, 2\pi]) \text{ are}$$

- (a) $\frac{\pi}{2}$ (b) $\frac{3\pi}{2}$ (c) $\frac{7\pi}{6}$ (d) $\frac{11\pi}{6}$

86. If $A_n = \int_0^{\pi/2} \frac{\sin(2n-1)x}{\sin x} dx$

$$\text{and } B_n = \int_0^{\pi/2} \left(\frac{\sin nx}{\sin x} \right)^2 dx \text{ for } n \in N, \text{ then}$$

- (a) $A_{n+1} = A_n$
(b) $B_{n+1} = B_n$
(c) $A_{n+1} - A_n = B_{n+1}$
(d) $B_{n+1} - B_n = A_{n+1}$

87. The values of a for which $\int_0^2 |x-a| dx \geq 1$ is satisfied, are

- (a) $[2, \infty)$ (b) $(-\infty, 0]$
(c) $(0, 2)$ (d) None of these

88. If $f(x) = \int_0^x |t-1| dt$, where $0 \leq x \leq 2$, then

- (a) range of $f(x)$ is $[0, 1]$
(b) $f(x)$ is differentiable at $x = 1$
(c) $f(x) = \cos^{-1} x$ has two real roots
(d) $f'\left(\frac{1}{2}\right) = \frac{1}{2}$

89. Which of the following statement(s) is/are true?

- (a) If function $y = f(x)$ is continuous at $x = c$ such that $f(c) \neq 0$, then $f(x)f(c) > 0, \forall x \in (c-h, c+h)$, where h is sufficiently small positive quantity
(b) $\lim_{n \rightarrow \infty} \frac{1}{n} \ln \left(\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right) = 1 + 2 \ln 2$
(c) Let f be a continuous and non-negative function defined on $[a, b]$. If $\int_a^b f(x) dx = 0$, then $f(x) = 0, \forall x \in [a, b]$
(d) Let f be a continuous function defined on $[a, b]$ such that $\int_a^b f(x) dx = 0$, then there exists atleast one $c \in (a, b)$ for which $f(c) = 0$

90. If $f(x) = \int_0^x (\cos(\sin t) + \cos(\cos t)) dt$, then

- $f(x + \pi)$ is
(a) $f(x) + f(\pi)$
(b) $f(x) + 2f(\pi)$
(c) $f(x) + f\left(\frac{\pi}{2}\right)$
(d) $f(x) + 2f\left(\frac{\pi}{2}\right)$

Type 3. Assertion and Reason

Directions (Q. Nos. 91-99) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

91. **Statement I** $\int_0^{\pi} \sqrt{1 - \sin^2 x} \, dx = 0$

Statement II $\int_0^{\pi} \cos x \, dx = 0$

92. **Statement I** $\int_0^{\pi} |\cos x| \, dx = 2$

Statement II $\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$,
 $a < c < b$

93. **Statement I** If $\frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}$, $x > 0$ and

$\int_1^{43} \frac{e^{\sin x^3}}{x} dx = f(k) - f(1)$, then one possible value of k is 64.

Statement II If $f(x)$ is a function satisfying

$$f\left(\frac{1}{x}\right) + x^2 f(x) = 0, \forall x \in R_0, \text{ then}$$

$$\int_{\sin \theta}^{\operatorname{cosec} \theta} f(x) dx = \sin \theta - \operatorname{cosec} \theta$$

94. **Statement I** If $n > 1$, then $\int_0^{\infty} \frac{dx}{1+x^n} = \int_0^1 \frac{dx}{(1-x^n)^{1/n}}$.

Statement II $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$.

95. **Statement I** $\int_{-\pi}^{\pi} \frac{2x(1 + \sin^2 x)}{1 + \cos^2 x} dx = 0$

Statement II $\int_0^{\pi/2} \sin^2 x \cos^2 x (\sin x + \cos x) dx = \frac{4}{15}$

96. **Statement I** $\int_a^x f(t) dt$ is an even function, if $f(x)$ is an odd function.

Statement II $\int_a^x f(t) dt$ is an odd function, if $f(x)$ is an even function.

97. Let f be a polynomial function of degree n .

Statement I There exists a number $x \in [a, b]$ such that $\int_a^x f(t) dt = \int_x^b f(t) dt$.

Statement II $f(x)$ is a continuous function.

98. Statement I In the interval $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$, the least value of the function $f(x) = \int_{5\pi/4}^x (3\sin t + 4\cos t) dt$ is 0.

Statement II If $f(x)$ is a decreasing function on the interval $[a, b]$, then the least value of $f(x)$ is $f(b)$.

99. Statement I If $f(x)$ is continuous on $[a, b]$, then there exists a point $c \in (a, b)$ such that $\int_a^b f(x) dx = f(c)(b-a)$.

Statement II For $a < b$, if m and M are respectively the smallest and greatest values of $f(x)$ on $[a, b]$, then $m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 100-103) Let $f(x)$ and $\phi(x)$ are two continuous functions on R satisfying $\phi(x) = \int_a^x f(t) dt$, $a \neq 0$ and another continuous function $g(x)$ satisfying $g(x + \alpha) + g(x) = 0$, $\forall x \in R$, $\alpha > 0$ and $\int_b^{2k} g(t) dt$ is independent of b .

100. If $f(x)$ is an odd function, then

- (a) $\phi(x)$ is also an odd function
- (b) $\phi(x)$ is also an even function
- (c) $\phi(x)$ is neither as even nor an odd function
- (d) for $\phi(x)$ to be an even function, it must satisfy

$$\int_0^a f(x) dx = 0$$

101. If $f(x)$ is an even function, then

- (a) $\phi(x)$ is also an even function
- (b) $\phi(x)$ is an odd function
- (c) If $f(a-x) = -f(x)$, then $\phi(x)$ is an even function
- (d) If $f(a-x) = -f(x)$, then $\phi(x)$ is an odd function

102. Least positive value of c , if c, k, b are in AP, is

- (a) 0
- (b) 1
- (c) α
- (d) 2α

103. If m, n are even integers and $p, q \in R$, then

$$\int_{p+m\alpha}^{q+n\alpha} g(t) dt \text{ is equal to}$$

- (a) $\int_p^q g(x) dx$
- (b) $(n-m) \int_0^\alpha g(x) dx$
- (c) $\int_p^q g(x) dx + (n-m) \int_0^\alpha g(2x) dx$
- (d) $\int_p^q g(x) dx + (n-m) \int_0^\alpha g(x) dx$

Passage II (Q. Nos. 104-106) By using properties of definite integral, $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$.

104. If $I_1 = \int_{1-k}^k x \sin[x(1-x)] dx$ and

$$I_2 = \int_{1-k}^k \sin(x(1-x)) dx, \text{ then}$$

- (a) $I_1 = 2I_2$
- (b) $2I_1 = I_2$
- (c) $I_1 = I_2$
- (d) None of these

105. Let $I_1 = \int_{\sec^2 z}^{2 - \tan^2 z} x f[(x)(3-x)] dx$ and

$I_2 = \int_{\sec^2 z}^{2 - \tan^2 z} f[(x)(3-x)] dx$, where f is a continuous function and z is any real number, then $\frac{I_1}{I_2}$ is equal to

- (a) $\frac{3}{2}$
- (b) $\frac{1}{2}$
- (c) 1
- (d) None of these

106. If f and g are continuous functions on $[0, \pi]$ satisfying $f(x) + f(\pi-x) = g(x) + g(\pi-x) = 1$, then $\int_0^\pi [f(x) + g(x)] dx$ is equal to

- (a) π
- (b) 2π
- (c) $\frac{\pi}{2}$
- (d) $\frac{3\pi}{2}$

Passage III (Q. Nos. 107-108) If m and M are the smallest and greatest values of a function $f(x)$ defined on an interval $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

(i) $f(x) \geq 0$ on $[a, b] \Rightarrow \int_a^b f(x) dx \geq 0$

(ii) $f(x) \leq g(x)$ on $[a, b] \Rightarrow \int_a^b f(x) dx \leq \int_a^b g(x) dx$

107. If $a \leq \int_0^1 e^{x^2} dx \leq b$, then (a, b) is

- (a) $(1, e^2)$
- (b) $(1, e)$
- (c) (e^{-1}, e)
- (d) (e, e^2)

108. Consider $I_1 = \int_0^1 e^{-x} \cos^2 x dx$, $I_2 = \int_0^1 e^{-x^2} \cos x dx$,

$$I_3 = \int_0^1 e^{-x^2} dx \text{ and } I_4 = \int_0^1 e^{-x^2/2} dx. \text{ Let } I \text{ be the}$$

greatest integral among I_1, I_2, I_3, I_4 , then

- (a) $I = I_1$
- (b) $I = I_2$
- (c) $I = I_3$
- (d) $I = I_4$

Passage IV (Q. Nos. 109-111) Let $f(x)$ be a differentiable function such that

$$f'(x) = f(x) + \int_0^2 f(x) dx. \text{ If } f(0) = \frac{4-e^2}{3}, \text{ then}$$

$$f'(x) = f(x) + \int_0^2 f(x) dx$$

$$\text{Let } \int_0^2 f(x) dx = k \Rightarrow f'(x) - f(x) = k$$

$$\Rightarrow e^{-x} f(x) = k \int e^{-x} dx \Rightarrow f(x) = Ce^x - k$$

Since, $\int_0^2 f(x) dx = k \Rightarrow k = \frac{\alpha(e^2 - 1)}{3}$

and $f(0) = \frac{4 - e^2}{3} = c = 1$

$\therefore k = \frac{e^2 - 1}{3}$

109. $f(x)$ is equal to

- (a) $e^x - \left(\frac{e^2 - 1}{3}\right)$ (b) $e^x + \left(\frac{e^2 - 1}{3}\right)$
(c) $e^x - \left(\frac{e^2 + 1}{3}\right)$ (d) None of these

110. The number of solutions of $x + f(x) = 0$ is

- (a) 0 (b) 1
(c) 2 (d) None of these

111. $\int_0^1 f(x) dx$ is equal to

- (a) $\frac{3e - e^2 - 2}{3}$
(b) $\frac{3e + e^2 - 4}{3}$
(c) $\frac{3e - e^2 - 4}{3}$
(d) None of the above

Type 5. Match the Columns

112. Match the statements of Column I with the values of Column II.

Column I	Column II
A. $\lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{\sqrt{n^2 - r^2}}{n^2}$ is equal to	p. $\frac{\pi}{4}$
B. $\int_0^{\pi} (\sqrt{\tan x} + \sqrt{\cot x}) dx$ is equal to	q. $\frac{\pi}{\sqrt{2}}$
C. $\int_{-1}^1 \sin^3 x \cos^2 x dx$ is equal to	r. 0
D. $\int_0^{\pi/2} \frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x} + \sqrt{\cos^3 x}} dx$ is equal to	s. $\frac{\pi}{2}$

113. Match the statements of Column I with the values of Column II.

Column I	Column II
A. Let $f(x)$ be a continuous function such that $f(2 - 4) + f(x) = 0$ for $x \in [0, 2]$, then $\int_0^2 \frac{dx}{1 + 2^{f(x)}}$ is equal to	p. 0
B. $\int_{1/2}^2 \frac{1}{x} \operatorname{cosec}^{101} \left(x - \frac{1}{x}\right) dx$ is equal to	q. 1
C. $\int_{-\pi/2}^{\pi/2} \frac{2 \cos x}{1 + 2^{x^2}} dx$ is equal to	r. 3
D. $\int_0^{\pi} \frac{\sin 6x}{\sin x} dx$ is equal to	s. 2

114. Match the statement of Column I with the values of Column II.

Column I	Column II
A. $\int_{\pi}^{2\pi} [2 \sin x] dx$, where $[]$ denotes the greatest integer function, is equal to	p. $\frac{\pi}{2} - 1$
B. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r}{\sqrt{n^2 + r^2}}$ is equal to	q. 4
C. $\int_0^1 \sqrt{\frac{1-x}{1+x}} dx$ is equal to	r. $-\frac{5\pi}{3}$
D. $\int_2^3 1 - x^2 dx$ is equal to	s. $\sqrt{5} - 1$

115. Match the statements of Column I with the values of Column II.

Column I	Column II
A. $\int_4^{10} \frac{[x^2]}{[x^2 - 28x + 196] + [x^2]} dx$ is equal to {where, $[]$ denotes greatest integer function}	p. $\frac{1}{100}$
B. $\int_{-1}^2 \frac{[x]}{x} dx$ is equal to	q. 3
C. $\lim_{n \rightarrow \infty} \frac{1^{99} + 2^{99} + \dots + n^{99}}{n^{100}}$ is equal to	r. $\frac{1}{3}$
D. If $5050 \int_{-1}^1 \sqrt{x^{200}} dx = \frac{1}{\alpha}$, then α is equal to	s. 1

116. Match the statements of Column I with the values of Column II.

Column I	Column II
A. If $I = \int_{-2}^2 (\alpha x^3 + \beta x + \gamma) dx$, then I is	p. independent of α
B. Let α, β be the distinct positive roots of the equation $\tan x = 2x$, then $\gamma \int_0^1 (\sin \alpha x \cdot \sin \beta x) dx$, where $\gamma \neq 0$, is	q. independent of β
C. If $f(x + \alpha) + f(x) = 0$, where $\alpha > 0$, then $\int_{\beta}^{\beta + 2\gamma\alpha} f(x) dx$, where $\gamma \in N$, is	r. independent of γ
D. $\gamma \int_0^{\alpha} [\sin x] dx$, where $\gamma \neq 0$, $\alpha \in [(2\beta + 1)\pi, (2\beta + 2)\pi]$, $n \in N$ and $[]$ denotes the greatest integer function, is	s. depends on α

117. If $[]$ denotes the greatest integer function, then match the statement of Column I with the values of Column II.

Column I	Column II
A. $\int_{-1}^1 [x + [x + [x]]] dx$ is equal to	p. 3
B. $\int_2^5 ([x] + [-x]) dx$ is equal to	q. 5
C. $\int_{-1}^3 \operatorname{sgn}(x - [x]) dx$ is equal to	r. 4
D. $25 \int_0^{\pi/4} \{(\tan^6(x - [x])) + \tan^4(x - [x])\} dx$ is equal to	s. -3

Type 6. Single Integer Answer Type Questions

118. If the value of the definite integral $\int_0^1 \frac{\sin^{-1} \sqrt{x}}{x^2 - x + 1} dx = \frac{\pi^2}{\sqrt{n}}$, where $n \in N$, then the value of $\frac{n}{27}$ is _____.

119. If the value of the definite integral $\int_0^1 {}^{207}C_7 x^{200} (1-x)^7 dx$ is equal to $\frac{1}{k}$, where $k \in N$, then the value of $\frac{k}{26}$ is _____.

120. Let $J = \int_{-5}^{-4} (3-x^2) \tan(3-x^2) dx$ and $K = \int_{-2}^{-1} (6-6x+x^2) \tan(6x-x^2-6) dx$, then $(J+K)$ equals _____.

121. If $\int_1^4 (\{x\})^{\{x\}} dx = \frac{a}{b}$, where, $\{ \}$ and $[]$ denote the fractional part and the greatest integer function respectively, then $a-b$ is equal to _____.

122. Let $f(x) = x - [x]$, for every real number x , where $[x]$ denotes integral part of x , then $\int_{-1}^1 f(x) dx$ is _____.

123. If f, g and h are continuous functions on $[0, a]$ such that $f(a-x) = f(x)$, $g(a-x) = -g(x)$ and $3h(x) - 4h(a-x) = 5$, then $\int_0^a f(x) \cdot g(x) \cdot h(x) dx$ is equal to _____.

124. If $\int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx = \frac{a}{b}$, then $a+b$ is _____.

125. If $f(x) = \begin{cases} 0, & \text{where } x = \frac{n}{n+1}, n = 1, 2, 3, \dots \\ 1, & \text{elsewhere} \end{cases}$, then the value of $\int_0^2 f(x) dx$ is _____.

126. The value of $\int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

Directions (Q. Nos. 1-2) Given that for each $a \in (0, 1)$, $\lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a} (1-t)^{a-1} dt$ exists. Let this limit be $g(a)$. In addition, it is given that the function $g(a)$ is differentiable on $(0, 1)$. [2014]

1. The value $g\left(\frac{1}{2}\right)$ is

- (a) π (b) 2π
(c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

2. The value $g'\left(\frac{1}{2}\right)$ is

- (a) $\frac{\pi}{2}$
(b) π
(c) $-\frac{\pi}{2}$
(d) 0

3. The value of $\int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} (1-x^2)^5 \right\} dx$ is _____. [2014]

4. Match the statements of Column I with values of Column II. [2014]

Column I	Column II
P. The number of polynomials $f(x)$ with 1. non-negative integer coefficients of degree ≤ 2 , satisfying $f(0) = 0$ and $\int_0^1 f(x) dx = 1$, is	8
Q. The number of points in the interval $[-\sqrt{13}, \sqrt{13}]$ at which $f(x) = \sin(x^2) + \cos(x^2)$ attains its maximum value, is	2
R. $\int_{-2}^2 \frac{3x^2}{(1+e^x)} dx$ equals	3. 4
S. $\left(\int_{-1/2}^{1/2} \cos 2x \log \left(\frac{1+x}{1-x} \right) dx \right) \left(\int_0^{1/2} \cos 2x \log \left(\frac{1+x}{1-x} \right) dx \right)$ equals	4. 0

Codes

	P	Q	R	S		P	Q	R	S
(a)	3	2	4	1	(b)	2	3	4	1
(c)	3	2	1	4	(d)	2	3	1	4

5. $\int_{\pi/4}^{\pi/2} (2\operatorname{cosec} x)^{17} dx$ is equal to [2014]
- (a) $\int_0^{\log(1+\sqrt{2})} 2(e^u + e^{-u})^{16} du$
 (b) $\int_0^{\log(1+\sqrt{2})} (e^u + e^{-u})^{17} du$
 (c) $\int_0^{\log(1+\sqrt{2})} (e^u - e^{-u})^{17} du$
 (d) $\int_0^{\log(1+\sqrt{2})} 2(e^u - e^{-u})^{16} du$

6. If $f: \left[\frac{1}{2}, 1\right] \rightarrow R$ (the set of all real numbers) is a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$, then the value of $\int_{1/2}^1 f(x) dx$ lies in the interval [2013]
- (a) $(2e-1, 2e)$ (b) $(e-1, 2e-1)$
 (c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

7. For $a \in R$ (the set of all real numbers), $a \neq -1$,
 $\lim_{n \rightarrow \infty} \frac{(1^a + 2^a + \dots + n^a)}{(n+1)^{a-1}[(na+1) + (na+2) + \dots + (na+n)]}$
 $= \frac{1}{60}$. Then, a is equal to [2013]
- (a) 5 (b) 7 (c) $-\frac{15}{2}$ (d) $-\frac{17}{2}$

8. The value of $\int_{-\pi/2}^{\pi/2} \left(x^2 + \log \frac{\pi-x}{\pi+x}\right) \cos x dx$ is [2012]
- (a) 0 (b) $\frac{\pi^2}{2} - 4$
 (c) $\frac{\pi^2}{2} + 4$ (d) $\frac{\pi^2}{2}$

9. The value of $\int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx$ is [2011]
- (a) $\frac{1}{4} \log\left(\frac{3}{2}\right)$ (b) $\frac{1}{2} \log\left(\frac{3}{2}\right)$
 (c) $\log\left(\frac{3}{2}\right)$ (d) $\frac{1}{6} \log\left(\frac{3}{2}\right)$

10. Let $f: (1, \infty) \rightarrow (2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6 \int_1^x f(t) dt = 3xf(x) - x^3$, $\forall x \geq 1$, then the value of $f(2)$ is [2011]
- (a) $-\frac{8}{3}$ (b) $\frac{8}{3}$ (c) 0 (d) $\frac{3}{8}$

11. For any real number x , let $[x]$ denotes the largest integer less than or equal to x . If f is a real valued function defined on the interval $[-10, 10]$ by
 $f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$
 Then, the value of $\frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x dx$ is _____. [2010]

12. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2} dx$ is/are [2010]
- (a) $\frac{22}{7} - \pi$ (b) $\frac{2}{105}$ (c) 0 (d) $\frac{71}{15} - \frac{3\pi}{2}$

13. If f is a real valued function defined on the interval $(-1, 1)$ such that $e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt$, $\forall x \in (-1, 1)$ and f^{-1} is the inverse function of f . Then, $(f^{-1})'(2)$ is equal to [2010]
- (a) 1 (b) $\frac{1}{3}$
 (c) $\frac{1}{2}$ (d) $\frac{1}{e}$

JEE Main/AIEEE

14. $\int_0^{\pi} \sqrt{1 + 4 \sin^2 \frac{x}{2}} - 4 \sin \frac{x}{2} dx$ is equal to [2014]
- (a) $\pi - 4$
 (b) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$
 (c) $4\sqrt{3} - 4$
 (d) $4\sqrt{3} - 4 - \frac{\pi}{3}$

15. The value of $\int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx$ is [2013]
- (a) $\log 2\sqrt{2}$ (b) $\log 2$ (c) $2 \log 2$ (d) $\log \sqrt{2}$

16. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$ is [2013]
- (a) π (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

17. For $0 \leq x \leq \frac{\pi}{2}$, the value of
 $\int_0^{\sin^2 x} \sin^{-1}(\sqrt{t}) dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) dt$ is [2013]
- (a) $\frac{\pi}{4}$ (b) 0
 (c) 1 (d) $-\frac{\pi}{4}$

18. Let $f: [-2, 3] \rightarrow [0, \infty)$ be a continuous function such that $f(1-x) = f(x)$ for all $x \in [-2, 3]$. If R_1 is the numerical value of the area of the region bounded by $y = f(x)$, $x = -2$, $x = 3$, X -axis and $R_2 = \int_{-2}^3 x f(x) dx$, then [2013]
- (a) $3R_1 = 2R_2$ (b) $2R_1 = 3R_2$
 (c) $R_1 = R_2$ (d) $R_1 = 2R_2$

19. If $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$, then $\frac{d^2y}{dx^2}$ is equal to [2013]

- (a) y (b) $\sqrt{1+y^2}$
(c) $\frac{y}{\sqrt{1+y^2}}$ (d) y^2

20. Statement I The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is $\frac{\pi}{6}$.

Statement II $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$ [2013]

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

21. If $g(x) = \int_0^x \cos 4t dt$, then $g(x+\pi)$ is equal to [2012]

- (a) $g(x)$ (b) $g(x) + g(\pi)$
(c) $g(x) - g(\pi)$ (d) $g(x) \cdot g(\pi)$

22. The value of $\int_0^1 \frac{8 \log(1+x)}{1+x^2} dx$ is [2011]

- (a) $\frac{\pi}{8} \log 2$ (b) $\frac{\pi}{2} \log 2$
(c) $\log 2$ (d) $\pi \log 2$

23. If $[]$ denotes the greatest integer function, then the value of $\int_0^{1.5} x[x^2] dx$ is [2011]

- (a) $\frac{5}{4}$ (b) 0 (c) $\frac{3}{2}$ (d) $\frac{3}{4}$

24. Let $p(x)$ be a function defined on R such that $\lim_{x \rightarrow \infty} \frac{f(3x)}{f(x)} = 1$, $p'(x) = p'(1-x)$, for all $x \in [0, 1]$, $p(0) = 1$ and $p(1) = 41$. Then, $\int_0^1 p(x)dx$ equals [2010]

- (a) $\sqrt{41}$ (b) 21 (c) 41 (d) 42

25. $\int_0^\pi [\cot x] dx$, where $[]$ denotes the greatest integer function, is equal to [2009]

- (a) $\frac{\pi}{2}$ (b) 1
(c) -1 (d) $-\frac{\pi}{2}$

26. Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$. Then, which one of the following is true? [2008]

- (a) $I > \frac{2}{3}$ and $J < 2$ (b) $I > \frac{2}{3}$ and $J > 2$
(c) $I < \frac{2}{3}$ and $J < 2$ (d) $I < \frac{2}{3}$ and $J > 2$

27. Let $F(x) = f(x) + f\left(\frac{1}{x}\right)$, where $f(x) = \int_1^x \frac{\log t}{1+t} dt$.

Then, $F(e)$ equals [2007]

- (a) $1/2$ (b) 0 (c) 1 (d) 2

28. The value of the integral $\int_3^6 \frac{\sqrt{x}}{\sqrt{9-x} + \sqrt{x}} dx$ is [2006]

- (a) $\frac{3}{2}$ (b) 2 (c) 1 (d) $\frac{1}{2}$

29. $\int_{-\pi/2}^{-\pi/2} [(x+\pi)^3 + \cos^2(x+3\pi)] dx$ is equal to [2006]

- (a) $\left(\frac{\pi}{32}\right)^4 + \left(\frac{\pi}{2}\right)$ (b) $\frac{\pi}{2}$
(c) $\left(\frac{\pi}{4}\right) - 1$ (d) $\frac{\pi^4}{32}$

30. $\int_0^\pi x f(\sin x) dx$ is equal to [2006]

- (a) $\pi \int_0^\pi f(\sin x) dx$ (b) $\frac{\pi}{2} \int_0^{\pi/2} f(\sin x) dx$
(c) $\pi \int_0^{\pi/2} f(\cos x) dx$ (d) $\pi \int_0^\pi f(\cos x) dx$

31. The value of $\int_1^a [x] f'(x) dx$, $a > 1$, where $[x]$ denotes the greatest integer not exceeding x , is [2006]

- (a) $[a]f(a) - \{f(1) + f(2) + \dots + f([a])\}$
(b) $[a]f([a]) - \{f(1) + f(2) + \dots + f(a)\}$
(c) $af([a]) - \{f(1) + f(2) + \dots + f(a)\}$
(d) $af(a) - \{f(1) + f(2) + \dots + f([a])\}$

32. $\lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \dots + \frac{n}{n^2} \sec^2 1 \right]$ equals [2005]

- (a) $\frac{1}{2} \tan 1$ (b) $\tan 1$
(c) $\frac{1}{2} \operatorname{cosec} 1$ (d) $\frac{1}{2} \sec 1$

33. Let $f: R \rightarrow R$ be a differentiable function having $f(2) = 6$, $f'(2) = \left(\frac{1}{48}\right)$. Then, $\lim_{x \rightarrow 2} \int_0^{f(x)} \frac{4t^3}{x-2} dt$ equals [2005]

- (a) 18 (b) 12 (c) 36 (d) 24

34. Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f'(x)$, X -axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$, is

$\left(\beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2} \beta\right)$. Then, $f\left(\frac{\pi}{2}\right)$ is [2005]

- (a) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$ (b) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$
(c) $\left(\frac{\pi}{4} - \sqrt{2} + 1\right)$ (d) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$

35. If $I_1 = \int_0^1 2^{x^2} dx$, $I_2 = \int_0^1 2^{x^3} dx$, $I_3 = \int_1^2 2^{x^2} dx$ and $I_4 = \int_1^2 2^{x^3} dx$, then [2005]

- (a) $I_3 > I_4$ (b) $I_3 = I_4$
(c) $I_1 > I_2$ (d) $I_2 > I_1$

36. The value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+a^x} dx$, $a > 0$, is [2005]

- (a) 2π (b) $\frac{\pi}{a}$ (c) $\frac{\pi}{2}$ (d) $a\pi$

37. The value of $\int_{-2}^3 |1-x^2| dx$ is [2004]

- (a) $\frac{28}{3}$ (b) $\frac{14}{3}$
(c) $\frac{7}{3}$ (d) $\frac{1}{3}$

38. The value of $\int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1+\sin 2x}} dx$ is [2004]

- (a) 0 (b) 1 (c) 2 (d) 3

39. If $\int_0^{\pi} xf(\sin x) dx = A \int_0^{\pi/2} f(\sin x) dx$, then A is equal to [2004]

- (a) 0 (b) π
(c) $\frac{\pi}{4}$ (d) 2π

40. If $f(x) = \frac{e^x}{1+e^x}$, $I_1 = \int_{f(-a)}^{f(a)} xg[x(1-x)] dx$ and $I_2 = \int_{f(-a)}^{f(a)} g[x(1-x)] dx$, then the value of $\frac{I_2}{I_1}$ is [2004]

- (a) 2 (b) -3
(c) -1 (d) 1

41. $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} e^{r/n}$ is [2004]

- (a) e (b) $e-1$
(c) $1-e$ (d) $e+1$

42. If $f(y) = e^y$, $g(y) = y$, $y > 0$ and $F(t) = \int_0^t f(t-y)g(y) dy$, then [2003]

- (a) $F(t) = 1 - e^{-t}(1+t)$ (b) $F(t) = e^t - (1+t)$
(c) $F(t) = te^t$ (d) $F(t) = te^{-t}$

43. If $f(a+b-x) = f(x)$, then $\int_a^b x f(x) dx$ is equal to [2003]

- (a) $\frac{a+b}{2} \int_a^b f(b-x) dx$ (b) $\frac{a+b}{2} \int_a^b f(x) dx$
(c) $\frac{b-a}{2} \int_a^b f(x) dx$ (d) $\frac{a+b}{2} \int_a^b f(a+b+x) dx$

44. The value of $\lim_{x \rightarrow 0} \frac{\int_0^x \sec^2 t dt}{x \sin x}$ is [2003]

- (a) 3 (b) 2
(c) 1 (d) -1

45. The value of $\int_0^1 x(1-x)^n dx$ is [2003]

- (a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$
(c) $\frac{1}{n+1} - \frac{1}{n+2}$ (d) $\frac{1}{n+1} + \frac{1}{n+2}$

46. $\lim_{n \rightarrow \infty} \frac{1+2^4+3^4+\dots+n^4}{n^5} - \lim_{n \rightarrow \infty} \frac{1+2^3+3^3+\dots+n^3}{n^5}$ is equal to [2003]

- (a) $\frac{1}{30}$ (b) 0
(c) $\frac{1}{4}$ (d) $\frac{1}{5}$

47. Let $\frac{d}{dx} F(x) = \left(\frac{e^{\sin x}}{x} \right)$, $x > 0$.

If $\int_1^{43} \frac{e^{\sin x^3}}{x} dx = F(k) - F(1)$, then one of the possible values of k , is [2003]

- (a) 15 (b) 16
(c) 63 (d) 64

48. Let $f(x)$ be a function satisfying $f'(x) = f(x)$ with $f(0) = 1$ and $g(x)$ be a function that satisfies $f(x) + g(x) = x^2$. Then, the value of $\int_0^1 f(x)g(x) dx$ is [2003]

- (a) $e - \frac{e^2}{2} - \frac{5}{2}$ (b) $e + \frac{e^2}{2} - \frac{3}{2}$
(c) $e - \frac{e^2}{2} - \frac{3}{2}$ (d) $e + \frac{e^2}{2} + \frac{5}{2}$

49. $\int_0^{10\pi} |\sin x| dx$ is equal to [2002]

- (a) 20 (b) 8
(c) 10 (d) 18

50. If $I_n = \int_0^{\pi/4} \tan^n x dx$, then $\lim_{n \rightarrow \infty} n[I_n + I_{n+2}]$ equals [2002]

- (a) $\frac{1}{2}$ (b) 1 (c) ∞ (d) 0

51. $\int_0^2 [x^2] dx$ is equal to [2002]

- (a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $\sqrt{2} - 1$ (d) $-\sqrt{2} - \sqrt{3} + 5$

52. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x} dx$ is equal to [2002]

- (a) $\frac{\pi^2}{4}$ (b) π^2 (c) 0 (d) $\frac{\pi}{2}$

53. Evaluate $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$. [2002]

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) 0 (d) 1

Other Engineering Entrances

54. If $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$,

then $\int_0^{\pi/4} \Delta(x) dx$ is equal to [VITEEE 2014]

- (a) $\frac{1}{4}$ (b) $\frac{1}{2}$
(c) 0 (d) $-\frac{1}{4}$

55. $\int_0^1 \frac{dx}{1+x+x^2}$ is equal to [GGSIPU 2014]

- (a) $\frac{\pi}{\sqrt{3}}$ (b) $\frac{\pi}{2\sqrt{3}}$
(c) $\frac{2\pi}{3\sqrt{3}}$ (d) $\frac{\pi}{3\sqrt{3}}$

56. $\int_0^{\pi/2} \left| \cos\left(\frac{x}{2}\right) \right| dx$ is equal to [RPET 2014]

- (a) 1 (b) -2 (c) $\sqrt{2}$ (d) 0

57. If $l(m, n) = \int_0^1 t^m (1+t)^n dt$, then the expression for $l(m, n)$ in terms of $l(m+1, n-1)$, is [VITEEE 2014]

- (a) $\frac{2^n}{m+1} - \frac{n}{m+1} \cdot l(m+1, n-1)$
(b) $\frac{n}{m+1} \cdot l(m+1, n-1)$
(c) $\frac{2n}{m+1} + \frac{n}{m+1} \cdot l(m+1, n-1)$
(d) $\frac{m}{n+1} \cdot l(m+1, n-1)$

58. The value of $\int_0^1 \frac{dx}{e^x + e}$ is [Kerala CEE 2014]

- (a) $\frac{1}{e} \log\left(\frac{1+e}{2}\right)$ (b) $\log\left(\frac{1+e}{2}\right)$
(c) $\frac{1}{e} \log(1+e)$ (d) $\log\left(\frac{2}{1+e}\right)$
(e) $\frac{1}{e} \log\left(\frac{2}{1+e}\right)$

59. The value of $\int_1^e \frac{1 + \log x}{3x} dx$ is [Kerala CEE 2014]

- (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$
(d) e (e) $\frac{1}{e}$

60. The value of $\int_0^1 \frac{x^3}{1+x^8} dx$ is [Kerala CEE 2014]

- (a) $\frac{\pi}{8}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{16}$
(d) $\frac{\pi}{6}$ (e) $\frac{\pi}{12}$

61. The value of $\int_2^4 \left(\frac{\log t}{t} \right) dt$ is [Kerala CEE 2014]

- (a) $\frac{1}{2}(\log 2)^2$ (b) $\frac{5}{2}(\log 2)^2$ (c) $\frac{3}{2}(\log 2)^2$
(d) $(\log 2)^2$ (e) $\frac{3}{2}(\log 2)$

62. If $\int_0^{\pi/3} \frac{\cos x}{3+4\sin x} dx = k \log\left(\frac{3+2\sqrt{3}}{3}\right)$, then k is equal to [AMU 2014]

- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{8}$

63. $\int_0^1 x \tan^{-1} x dx$ is equal to [GGSIPU 2014]

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{4} + \frac{1}{2}$ (c) $\frac{\pi}{4} - \frac{1}{2}$ (d) $\frac{1}{2}$

64. Find the value of $\int_0^2 f'(x^2) 2x dx$. [RPET 2014]

- (a) $f(4) - f(0)$ (b) $f(4) + f(0)$
(c) $f(0) - f(4)$ (d) None of these

65. If $\int_0^{25} e^{x-[x]} dx = k(e-1)$, then the value of k is [BITSAT 2014]

- (a) 12 (b) 25 (c) 23 (d) 24

66. $\int_0^{2\pi} (\sin x + |\sin x|) dx$ is equal to [VITEEE 2014]

- (a) 0 (b) 4
(c) 8 (d) 1

67. The value of $\int_0^{\sqrt{2}} [x^2] dx$, where $[]$ is the greatest integer function, is [VITEEE 2014]

- (a) $2 - \sqrt{2}$ (b) $2 + \sqrt{2}$
(c) $\sqrt{2} - 1$ (d) $\sqrt{2} - 2$

68. $\int_0^{3\pi/2} \sin\left[\frac{2x}{\pi}\right] dx$, where $[]$ denotes the greatest integer function, is equal to [Manipal 2014]

- (a) $\frac{\pi}{2}(\sin 1 + \cos 1)$ (b) $\frac{\pi}{2}(\sin 1 + \sin 2)$
(c) $\frac{\pi}{2}(\sin 1 - \cos 1)$ (d) $\frac{\pi}{2}(\sin \pi + \sin 2)$

69. The value of $\int_{1/n}^{(an-1)/n} \frac{\sqrt{x}}{\sqrt{a-x} + \sqrt{x}} dx$ is [Manipal 2014]

- (a) $\frac{a}{2}$ (b) $\frac{n \cdot a + 2}{2n}$
(c) $\frac{n \cdot a - 2}{2n}$ (d) $\frac{n \cdot a}{2}$

70. $\int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} dx$ is equal to [Manipal 2014]

- (a) 0 (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{2}$ (d) π

71. If $f(x) = f(\pi + e - x)$ and $\int_e^\pi f(x) dx = \frac{2}{e + \pi}$, then

$\int_e^\pi xf(x) dx$ is equal to [Karnataka CET 2014]

- (a) $\pi - e$ (b) $\frac{\pi + e}{2}$
(c) 1 (d) $\frac{\pi - e}{2}$

72. The value of $\int_{-\pi/4}^{\pi/4} \log(\sec \theta - \tan \theta) d\theta$ is

[Karnataka CET 2014]

- (a) 0 (b) $\frac{\pi}{4}$
(c) π (d) $\frac{\pi}{2}$

73. If $M = \int_0^{\pi/2} \frac{\cos x}{x+2} dx$ and $N = \int_0^{\pi/4} \frac{\sin x \cos x}{(x+1)^2} dx$.

Then, the value of $(M - N)$ is [WB JEE 2014]

- (a) $\frac{3}{\pi + 2}$ (b) $\frac{2}{\pi - 4}$
(c) $\frac{4}{\pi - 2}$ (d) $\frac{2}{\pi + 4}$

74. If $f(x) = \begin{cases} 2x^2 + 1, & x \leq 1 \\ 4x^3 - 1, & x > 1 \end{cases}$, then $\int_0^2 f(x) dx$ is equal to [WB JEE 2014]

- (a) $\frac{47}{3}$ (b) $\frac{50}{3}$ (c) $\frac{1}{3}$ (d) $\frac{47}{2}$

75. If f is the function on $[-\pi, \pi]$ given by $f(0) = 9$

and $f(x) = \frac{\sin(\frac{9x}{2})}{\sin(\frac{x}{2})}$ for $x \neq 0$. Then, the value of

$\frac{2}{\pi} \int_{-\pi}^{\pi} f(x) dx$ is [AMU 2014]

- (a) 0 (b) 4
(c) 8 (d) None of these

76. For any real number x , let $[x]$ denotes the largest integer less than or equal to x . If f is a real valued function defined on the interval $[-10, 10]$ by

$$f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$$

Then, the value of $\int_{-10}^{10} f(x) \cos \pi x dx$ is

[AMU 2014]

- (a) 2 (b) 4
(c) 0 (d) None of these

77. $\int_{-\pi/2}^{\pi/2} \cos x \ln \left(\frac{1+x}{1-x} \right) dx$ is equal to [GGSIPU 2014]

- (a) 0 (b) $\frac{\pi^2}{4} \left(-1 + \frac{\pi}{2} \right)$
(c) 1 (d) $\frac{\pi^2}{2}$

78. $\int_{-\pi}^{\pi} (\cos px - \sin qx)^2 dx$, where p and q are integers, is equal to [GGSIPU 2014]

- (a) $-\pi$ (b) 0 (c) π (d) 2π

79. $\int_0^\pi \sqrt{\frac{1 + \cos 2x}{2}} dx$ is equal to [GGSIPU 2014]

- (a) 0 (b) 2 (c) 4 (d) -2

80. The value of $\int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$ is [RPET 2014]

- (a) π (b) 0
(c) 2π (d) $\frac{\pi}{4}$

81. $\int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta d\theta$ equals [EAMCET 2014]

- (a) $\frac{\pi}{96}$ (b) $\frac{5}{192}$
(c) $\frac{5\pi}{256}$ (d) $\frac{5\pi}{192}$

82. $\int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx$ is equal to [MP PET 2013]

- (a) $\frac{\pi^2}{4}$ (b) $\frac{\pi^2}{8}$
(c) $\frac{\pi^2}{16}$ (d) $\frac{\pi^2}{32}$

83. $\int_{-\pi}^{\pi} \sin x [f(\cos x)] dx$ is equal to [AMU 2013]

- (a) $2f(\pi)$ (b) $2f(2)$
(c) $2f(1)$ (d) None of these

84. The value of $\int_0^{2\pi} \frac{dx}{e^{\sin x} + 1}$ is [MP PET 2012]

- (a) π (b) 0
(c) 2π (d) $\frac{\pi}{2}$

85. $\int_0^{2\pi} |\sin x| dx$ is equal to [MP PET 2012]

- (a) 2 (b) $\sqrt{3}$
(c) 4 (d) 0

86. If $[x]$ is the greatest integer function not greater than x , then $\int_0^{11} [x] dx$ is equal to [Karnataka CET 2012]

- (a) 45 (b) 66
(c) 35 (d) 55

87. The value of $\int_{-2}^2 (1 + 2 \sin x) e^{|x|} dx$ is [WB JEE 2012]

- (a) 0 (b) $e^2 - 1$
(c) $2(e^2 - 1)$ (d) 1

88. If $[x]$ denotes the greatest integer less than or equal to x , then the value of $\int_{-1}^1 (|x| - 2[x]) dx$ is [WB JEE 2012]

- (a) 3 (b) 2
(c) -2 (d) -3

- 89.** The value of $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^3}{r^4 + n^4}$ is [WB JEE 2012]
 (a) $\frac{1}{2} \log_e \left(\frac{1}{2} \right)$ (b) $\frac{1}{4} \log_e \left(\frac{1}{2} \right)$
 (c) $\frac{1}{4} \log_e 2$ (d) $\frac{1}{2} \log_e 2$
- 90.** The value of $\int_{\pi/6}^{\pi/2} \left(\frac{1 + \sin 2x + \cos 2x}{\sin x + \cos x} \right) dx$ is equal to [WB JEE 2012]
 (a) 16 (b) 8 (c) 4 (d) 1
- 91.** If $I_n = \int_0^{\pi/4} \tan^n x dx$, where n is a positive integer, then $I_{10} + I_8$ is equal to [Karnataka CET 2011]
 (a) $\frac{1}{9}$ (b) $\frac{1}{8}$ (c) $\frac{1}{7}$ (d) 9
- 92.** The value of $\int_{-2}^2 (x \cos x + \sin x + 1) dx$ is [WB JEE 2011]
 (a) 2 (b) 0 (c) -2 (d) 4
- 93.** The value of $\int_0^4 |x - 1| dx$ is [Karnataka CET 2011]
 (a) $\frac{5}{2}$ (b) 5 (c) 4 (d) 1
- 94.** If $[x]$ denotes the greatest integer less than or equal to x , then the value of $\int_0^2 (|x - 2| + [x]) dx$ is [Kerala CEE 2011]
 (a) 2 (b) 3 (c) 1
 (d) 4 (e) $\frac{3}{2}$
- 95.** $\int_{\pi}^{16\pi} |\sin x| dx$ is equal to [WB JEE 2011]
 (a) 0 (b) 32 (c) 30 (d) 28
- 96.** The value of $\int_0^{\pi} \sin^{50} x \cos^{49} x dx$ is [WB JEE 2011]
 (a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) 1
- 97.** If $I_1 = \int_0^{\pi/2} f(\sin 2x) \sin x dx$ and $I_2 = \int_0^{\pi/4} f(\cos 2x) \cos x dx$, then $\frac{I_1}{I_2}$ is equal to [AMU 2011]
 (a) 1 (b) $\sqrt{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) 2
- 98.** If $\frac{d}{dx} [\phi(x)] = f(x)$, then $\int_1^2 f(x) dx$ is equal to [BITSAT 2011]
 (a) $f(1) - f(2)$ (b) $\phi(1) - \phi(2)$
 (c) $f(2) - f(1)$ (d) $\phi(2) - \phi(1)$
- 99.** If a function $f(x)$ satisfied $f'(x) = g(x)$. Then, the value of $\int_a^b f(x)g(x) dx$ is [WB JEE 2010]
 (a) $\frac{1}{2} [\{f(b)\}^2 - \{f(a)\}^2]$ (b) $\frac{1}{2} [\{f(b)\}^2 + \{f(a)\}^2]$
 (c) $\frac{1}{2} [f(b) - f(a)]^2$ (d) None of these
- 100.** $\int_a^b \sqrt{(x-a)(b-x)} dx, b > a$, is equal to [DCE 2010]
 (a) $\frac{\pi(b-a)^2}{8}$ (b) $\frac{\pi(b+a)^2}{8}$
 (c) $(b-a)^2$ (d) $(b+a)^2$
- 101.** The value of $\int_0^{\pi/2} \sin^5 x dx$ is [WB JEE 2010]
 (a) $\frac{4}{15}$ (b) $\frac{8}{5}$ (c) $\frac{8}{15}$ (d) $\frac{4}{5}$
- 102.** $\int_{-1}^1 (e^{x^3} + e^{-x^3})(e^x - e^{-x}) dx$ is equal to [Kerala CEE 2010]
 (a) $\frac{e^2}{2} - 2e$ (b) $e^2 - 2e$
 (c) $2(e^2 - e)$ (d) $(2e^2 - 2e)$
 (e) 0
- 103.** If $\int_0^a f(2a-x) dx = m$ and $\int_0^a f(x) dx = n$, then $\int_0^{2a} f(x) dx$ is equal to [Kerala CEE 2010]
 (a) $2m + n$ (b) $m + 2n$ (c) $m - n$
 (d) $n - m$ (e) $m + n$
- 104.** $\int_0^1 \frac{2 \sin^{-1} \frac{x}{2}}{x} dx$ is equal to [Kerala CEE 2010]
 (a) $\int_0^{\pi/6} \frac{x}{\tan x} dx$ (b) $\int_0^{\pi/6} \frac{2x}{\tan x} dx$
 (c) $\int_0^{\pi/2} \frac{2x}{\tan x} dx$ (d) $\int_0^{\pi/6} \frac{x}{\sin x} dx$
 (e) $\int_0^{\pi/6} \frac{2x}{\sin x} dx$
- 105.** $\int_{-100}^{100} f(x) dx$ is equal to [Kerala CEE 2010]
 (a) $\int_{-100}^{100} f(x^2) dx$ (b) $\int_{-100}^{100} f(-x^2) dx$
 (c) $\int_{-100}^{100} f\left(\frac{1}{x}\right) dx$ (d) $\int_{-100}^{100} f(-x) dx$
 (e) $\int_{-100}^{100} [f(x) + f(-x)] dx$
- 106.** If $I_1 = \int_0^{3\pi} f(\cos^2 x) dx$ and $I_2 = \int_0^{\pi} f(\cos^2 x) dx$, then [WB JEE 2010; BITSAT 2010]
 (a) $I_1 = I_2$ (b) $3I_1 = I_2$
 (c) $I_1 = 3I_2$ (d) $I_1 = 5I_2$
- 107.** If $I = \int_0^1 \frac{dx}{1+x^{\pi/2}}$, then [WB JEE 2010; VITEEE 2010]
 (a) $\log_e 2 < I < \frac{\pi}{4}$ (b) $\log_e 2 > I$
 (c) $I = \frac{\pi}{4}$ (d) $I = \log_e 2$
- 108.** The value of $\int_{-\pi/2}^{\pi/2} |\sin x| dx$ is [WB JEE 2010]
 (a) 0 (b) 2
 (c) -2 (d) $-2 < I < 2$

Answers

Work Book Exercise 10.1

1. (b) 2. (c) 3. (c) 4. (a) 5. (c) 6. (c) 7. (a) 8. (d) 9. (b) 10. (d)

Work Book Exercise 10.2

1. (b) 2. (c) 3. (b) 4. (a) 5. (c) 6. (b) 7. (c) 8. (b) 9. (b) 10. (b)
11. (c) 12. (c) 13. (c) 14. (a) 15. (a) 16. (b) 17. (b)

Work Book Exercise 10.3

1. (c) 2. (c) 3. (c) 4. (b) 5. (a) 6. (b) 7. (b) 8. (a) 9. (b) 10. (c)

Target Exercises

1. (b) 2. (a) 3. (c) 4. (c) 5. (a) 6. (c) 7. (c) 8. (c) 9. (d) 10. (b)
11. (b) 12. (c) 13. (a) 14. (d) 15. (b) 16. (c) 17. (c) 18. (a) 19. (a) 20. (d)
21. (d) 22. (c) 23. (a) 24. (b) 25. (a) 26. (b) 27. (a) 28. (b) 29. (a) 30. (a)
31. (c) 32. (a) 33. (c) 34. (d) 35. (b) 36. (b) 37. (c) 38. (b) 39. (c) 40. (a)
41. (b) 42. (d) 43. (a) 44. (c) 45. (b) 46. (d) 47. (b) 48. (b) 49. (b) 50. (c)
51. (c) 52. (a) 53. (b) 54. (c) 55. (a) 56. (d) 57. (d) 58. (a) 59. (a) 60. (a)
61. (c) 62. (c) 63. (c) 64. (c) 65. (c) 66. (a) 67. (a) 68. (b) 69. (a) 70. (c)
71. (d) 72. (d) 73. (c) 74. (a) 75. (a) 76. (b) 77. (b) 78. (b) 79. (b) 80. (d)
81. (a,c,d) 82. (a,c) 83. (a,c,d) 84. (b,d) 85. (all) 86. (a,d) 87. (a,b) 88. (a,b) 89. (a,c) 90. (a,d)
91. (d) 92. (a) 93. (c) 94. (b) 95. (b) 96. (c) 97. (a) 98. (d) 99. (a) 100. (b)
101. (d) 102. (d) 103. (d) 104. (b) 105. (a) 106. (a) 107. (b) 108. (d) 109. (a) 110. (b)
111. (a) 112. (*) 113. (*) 114. (*) 115. (*) 116. (*) 117. (*) 118. (4) 119. (8) 120. (0)
121. (1) 122. (1) 123. (0) 124. (3) 125. (2) 126. (1)

*

112. $A \rightarrow p; B \rightarrow q; C \rightarrow r; D \rightarrow p$
113. $A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow p$
114. $A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$
115. $A \rightarrow q; B \rightarrow s; C \rightarrow p; D \rightarrow p$
116. $A \rightarrow p, q; B \rightarrow p, q, r; C \rightarrow q, s; D \rightarrow s$
117. $A \rightarrow s; B \rightarrow s; C \rightarrow r; D \rightarrow q$

Entrances Gallery

1. (a) 2. (d) 3. (2) 4. (d) 5. (a) 6. (d) 7. (b, d) 8. (b) 9. (a) 10. (b)
11. (4) 12. (a) 13. (b) 14. (d) 15. (a) 16. (d) 17. (a) 18. (d) 19. (a) 20. (d)
21. (c) 22. (d) 23. (d) 24. (b) 25. (d) 26. (c) 27. (a) 28. (a) 29. (b) 30. (c)
31. (a) 32. (a) 33. (a) 34. (a) 35. (c) 36. (c) 37. (a) 38. (c) 39. (b) 40. (a)
41. (b) 42. (b) 43. (b) 44. (c) 45. (c) 46. (d) 47. (d) 48. (c) 49. (a) 50. (b)
51. (d) 52. (b) 53. (a) 54. (d) 55. (d) 56. (c) 57. (a) 58. (a) 59. (b) 60. (c)
61. (c) 62. (c) 63. (c) 64. (a) 65. (b) 66. (b) 67. (c) 68. (b) 69. (c) 70. (a)
71. (c) 72. (a) 73. (d) 74. (a) 75. (b) 76. (b) 77. (a) 78. (d) 79. (b) 80. (d)
81. (d) 82. (c) 83. (d) 84. (a) 85. (c) 86. (d) 87. (c) 88. (a) 89. (c) 90. (d)
91. (a) 92. (d) 93. (b) 94. (b) 95. (c) 96. (a) 97. (b) 98. (d) 99. (a) 100. (a)
101. (c) 102. (e) 103. (e) 104. (b) 105. (d) 106. (c) 107. (a) 108. (b)

Explanations

Target Exercises

$$\begin{aligned}
 1. \text{ Let } I &= \int_0^{\pi/2} \frac{dx}{\sin\left(x - \frac{\pi}{3}\right) \sin\left(x - \frac{\pi}{6}\right)} \\
 &= 2 \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)}{\sin\left(x - \frac{\pi}{3}\right) \sin\left(x - \frac{\pi}{6}\right)} dx \\
 &= 2 \int_0^{\pi/2} \frac{\sin\left[\left(x - \frac{\pi}{6}\right) - \left(x - \frac{\pi}{3}\right)\right]}{\sin\left(x - \frac{\pi}{3}\right) \sin\left(x - \frac{\pi}{6}\right)} dx \\
 &= 2 \int_0^{\pi/2} \left[\cot\left(x - \frac{\pi}{3}\right) - \cot\left(x - \frac{\pi}{6}\right) \right] dx \\
 &= 2 \left[\log \frac{\sin\left(x - \frac{\pi}{3}\right)}{\sin\left(x - \frac{\pi}{6}\right)} \right]_0^{\pi/2} = -4 \log \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 2. \text{ Put } \log x = z \Rightarrow x = e^z \text{ and } dx = e^z dz \\
 \therefore \int_2^e \left[\frac{1}{\log x} - \frac{1}{(\log x)^2} \right] dx = \int_{\log 2}^1 \left(\frac{1}{z} - \frac{1}{z^2} \right) e^z dz \\
 = \int_{\log 2}^1 e^z \left(\frac{1}{z} + d \left(\frac{1}{z} \right) \right) dz \\
 = \left[e^z \cdot \frac{1}{z} \right]_{\log 2}^1 = e - \frac{2}{\log 2}
 \end{aligned}$$

$$\therefore a = e \text{ and } b = -2$$

$$\begin{aligned}
 3. \text{ Let } I &= \int_0^{\pi/2} \sin \theta \cos \theta \sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta} d\theta \\
 \text{Put } a^2 \sin^2 \theta + b^2 \cos^2 \theta &= z^2 \\
 \Rightarrow (a^2 - b^2) 2 \sin \theta \cos \theta d\theta &= 2z dz \\
 \therefore I &= \frac{1}{a^2 - b^2} \int_b^a z \cdot z dz = \frac{1}{3(a^2 - b^2)} [z^3]_b^a \\
 &= \frac{a^3 - b^3}{3(a^2 - b^2)} = \frac{a^2 + ab + b^2}{3(a + b)}
 \end{aligned}$$

$$\begin{aligned}
 4. \text{ Let } I &= \int_1^{3^{1/n}} \frac{dx}{x(x^n + 1)} \\
 \text{Put } x^n &= t \Rightarrow nx^{n-1} dx = dt \\
 \Rightarrow \frac{nx^n}{x} dx &= dt \\
 \Rightarrow \frac{1}{x} dx &= \frac{dt}{nt} \\
 \therefore I &= \int_1^3 \frac{1}{nt(t+1)} dt = \frac{1}{n} \int_1^3 \frac{dt}{t(t+1)} \\
 &= \frac{1}{n} \int_1^3 \left(\frac{1}{t} - \frac{1}{t+1} \right) dt \\
 &= \frac{1}{n} [\log t - \log(t+1)]_1^3 = \frac{1}{n} \left[\log \left(\frac{t}{t+1} \right) \right]_1^3 \\
 &= \frac{1}{n} \left[\log \left(\frac{3}{4} \right) - \log \left(\frac{1}{2} \right) \right] = \frac{1}{n} \log \left(\frac{3}{2} \right)
 \end{aligned}$$

$$\begin{aligned}
 5. \text{ Let } I &= \int_0^1 \frac{4^{2x+1} - 3^{2x-1}}{12^x} dx \\
 &= \int_0^1 \frac{4^{2x} \cdot 4 - 3^{2x} \cdot 3^{-1}}{4^x \cdot 3^x} dx = 4 \int_0^1 \left(\frac{4}{3} \right)^x dx - \frac{1}{3} \int_0^1 \left(\frac{3}{4} \right)^x dx \\
 &= \left[4 \cdot \frac{\left(\frac{4}{3} \right)^x}{\log \frac{4}{3}} - \frac{1}{3} \cdot \frac{\left(\frac{3}{4} \right)^x}{\log \left(\frac{3}{4} \right)} \right]_0^1 = \left[\frac{4 \left(\frac{4}{3} - 1 \right)}{\log \left(\frac{4}{3} \right)} - \frac{1}{3} \cdot \frac{\left(\frac{3}{4} - 1 \right)}{\log \left(\frac{3}{4} \right)} \right] \\
 &= \frac{4}{3 \log \left(\frac{4}{3} \right)} + \frac{1}{12 \log \left(\frac{3}{4} \right)}
 \end{aligned}$$

$$\begin{aligned}
 6. I &= \frac{1}{2^n} \int_0^{\pi/2} (2 \sin x \cos x)^n dx = \frac{1}{2^n} \int_0^{\pi/2} (\sin 2x)^n dx \\
 \text{Put } 2x &= \theta \Rightarrow 2dx = d\theta \\
 I &= \frac{1}{2^n} \int_0^{\pi} (\sin^n \theta) \frac{1}{2} d\theta \\
 &= \frac{1}{2^{n+1}} \left[\int_0^{\pi/2} [(\sin^n \theta) + (\sin(\pi - \theta))^n] d\theta \right] \\
 &\quad \left[\because \int_0^{2a} f(x) dx = \int_0^a [f(x) + f(2a - x)] dx \right]
 \end{aligned}$$

$$\text{Thus, } I = \frac{1}{2^n} \int_0^{\pi/2} \sin^n \theta d\theta$$

$$\therefore \lambda = 2^{-n}$$

$$7. \text{ Let } I = \int_0^{\pi} \frac{x}{a^2 \cos^2 x + b^2 \sin^2 x} dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)}{a^2 \cos^2 x + b^2 \sin^2 x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = 2\pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$\Rightarrow 2I = 2\pi \int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x} dx$$

$$\text{Now, put } \tan x = t \Rightarrow dx = \frac{dt}{\sec^2 x}$$

$$\begin{aligned}
 \therefore 2I &= 2\pi \int_0^{\infty} \frac{dt}{a^2 + b^2 t^2} \\
 &= \frac{1}{b^2} \times 2\pi \int_0^{\infty} \frac{dt}{\frac{a^2}{b^2} + t^2} = \left[\frac{2\pi}{b^2} \cdot \frac{b}{a} \tan^{-1} \frac{bt}{a} \right]_0^{\infty} \\
 &= \frac{2\pi}{ab} [\tan^{-1} \infty - \tan^{-1} 0] = \frac{2\pi}{ab} \times \frac{\pi}{2} \Rightarrow I = \frac{\pi^2}{2ab}
 \end{aligned}$$

$$8. \text{ Put } \frac{1}{x} = t$$

$$\begin{aligned}
 I &= \int_e^{1/e} \left| \log \frac{1}{t} \right| (-1) dt = \int_{1/e}^e |-\log t| dt \\
 &= \int_{1/e}^1 (-\log t) dt + \int_1^e (\log t) dt \\
 &= [-t \log t + t]_{1/e}^1 + [t \log t - t]_1^e \\
 &= 1 - \frac{1}{e} - \frac{1}{e} + e - e + 1 = 2 \left(1 - \frac{1}{e} \right)
 \end{aligned}$$

9. Let $I = \int_0^{\pi/2} x \sin^2 x \cos^2 x \, dx$... (i)

$\Rightarrow I = \int_0^{\pi/2} \left(\frac{\pi}{2} - x\right) \sin^2 x \cos^2 x \, dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \frac{\pi}{2} \int_0^{\pi/2} \sin^2 x \cos^2 x \, dx = \frac{\pi}{8} \int_0^{\pi/2} \sin^2 2x \, dx$

$\Rightarrow 2I = \frac{\pi}{8} \int_0^{\pi/2} \left(\frac{1 - \cos 4x}{2}\right) dx$

$\Rightarrow 2I = \frac{\pi}{16} \left[x - \frac{\sin 4x}{4} \right]_0^{\pi/2}$

$\Rightarrow 2I = \frac{\pi}{16} \left[\frac{\pi}{2} \right] \Rightarrow I = \frac{\pi^2}{64}$

10. Let $I = \int_0^{\pi/2} \frac{dx}{5 + 3\cos x}$

Put $\tan\left(\frac{x}{2}\right) = t \Rightarrow dx = \frac{2}{1+t^2} dt$

and $5 + 3\cos x = \frac{8+2t^2}{1+t^2}$

$\therefore I = \int_0^1 \frac{dt}{4+t^2} = \frac{1}{2} \left[\tan^{-1}\left(\frac{t}{2}\right) \right]_0^1 = \frac{1}{2} \tan^{-1}\left(\frac{1}{2}\right)$

Thus, $\lambda = \frac{1}{2}$

11. Let $I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx + 2 \int_{-\pi}^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$

$\Rightarrow I = 0 + 4 \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x} dx$... (i)

$\left[\text{since, } \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x} dx \text{ is an odd function} \right]$

$\Rightarrow I = 4 \int_0^{\pi} \frac{\pi(\pi-x) \sin x}{1+\cos^2 x} dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

Put $\cos x = t \Rightarrow \sin x \, dx = -dt$

$\therefore I = 2\pi \int_1^{-1} \frac{-dt}{1+t^2} = 2\pi [\tan^{-1} t]_{-1}^1$

$= 2\pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right] = \pi^2$

12. Let $I = \int_0^{\pi} \log(1 + \cos x) dx$... (i)

$\Rightarrow I = \int_0^{\pi} \log[1 + \cos(\pi - x)] dx$

$\Rightarrow I = \int_0^{\pi} \log(1 - \cos x) dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi} \log(1 + \cos x)(1 - \cos x) dx$

$\Rightarrow 2I = \int_0^{\pi} \log(1 - \cos^2 x) dx = \int_0^{\pi} \log \sin^2 x \, dx$

$\Rightarrow 2I = 2 \int_0^{\pi} \log \sin x \, dx = 2 \cdot 2 \int_0^{\pi/2} \log \sin x \, dx$

$\Rightarrow 2I = 4 \cdot \left(-\frac{\pi}{2} \log 2 \right) \left[\because \int_0^{\pi/2} \log \sin x \, dx = -\frac{\pi}{2} \log 2 \right]$

$\therefore I = -\pi \log 2$

13. Let $I = \int_0^{\pi} \frac{x^2 \sin x}{(2x - \pi)(1 + \cos^2 x)} dx$

$= \int_0^{\pi} \frac{(\pi - x)^2 \sin(\pi - x)}{(2\pi - 2x - \pi)(1 + \cos^2(\pi - x))} dx$

$= \int_0^{\pi} \frac{(\pi^2 - 2\pi x + x^2) \sin x}{(\pi - 2x)(1 + \cos^2 x)} dx$

$= \int_0^{\pi} \frac{(\pi^2 - 2\pi x) \sin x}{(\pi - 2x)(1 + \cos^2 x)} dx - I$

$\Rightarrow 2I = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx = -\pi \int_1^{-1} \frac{dt}{1+t^2}$

$[\text{let } \cos x = t \Rightarrow \sin x \, dx = -dt]$

$\therefore 2I = \pi [\tan^{-1} t]_{-1}^1 = \pi \left(\frac{\pi}{4} + \frac{\pi}{4} \right) = \frac{\pi^2}{2} \Rightarrow I = \frac{\pi^2}{4}$

14. Let $I = \int_a^b \frac{|x|}{x} dx$

(i) If $0 \leq a < b$, then $|x| = x$

$\therefore I = \int_a^b \frac{x}{x} dx = \int_a^b 1 dx = b - a = |b| - |a|$

(ii) If $a < b \leq 0$, then $|x| = -x$

$\therefore I = \int_a^b \frac{(-x)}{x} dx = a - b = -b - (-a) = |b| - |a|$

(iii) If $a < 0 < b$, then

$I = \int_a^0 \frac{|x|}{x} dx + \int_0^b \frac{|x|}{x} dx = \int_a^0 \left(\frac{-x}{x} \right) dx + \int_0^b \frac{x}{x} dx$
 $= - \int_a^0 dx + \int_0^b dx = a + b = b - (-a) = |b| - |a|$

15. Let $I = \int_0^1 \frac{\tan^{-1} x}{x} dx = \int_2^{\pi/2} \frac{\frac{z}{2} \left(\frac{1}{2} \sec^2 \frac{z}{2} \right)}{\tan \frac{z}{2}} dz$

$\left[\text{putting } \tan^{-1} x = \frac{z}{2} \right]$
 $\left[\text{i.e. } x = \tan \frac{z}{2} \Rightarrow dx = \frac{1}{2} \sec^2 \frac{z}{2} dz \right]$

$= \frac{1}{2} \int_0^{\pi/2} \frac{z}{\sin z} dz = \frac{1}{2} \int_0^{\pi/2} \frac{x}{\sin x} dx$

16. Let $I = \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx$

Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta \, d\theta$

$\therefore I = \int_0^{\pi/2} \frac{\tan \theta \sec^2 \theta}{(1 + \tan \theta)(1 + \tan^2 \theta)} d\theta$

$\Rightarrow I = \int_0^{\pi/2} \frac{\tan \theta}{1 + \tan \theta} d\theta$

$\Rightarrow I = \int_0^{\pi/2} \frac{\sin \theta}{\cos \theta + \sin \theta} d\theta$... (i)

$\Rightarrow I = \int_0^{\pi/2} \frac{\sin \left(\frac{\pi}{2} - \theta \right)}{\cos \left(\frac{\pi}{2} - \theta \right) + \sin \left(\frac{\pi}{2} - \theta \right)} d\theta$

$\Rightarrow I = \int_0^{\pi/2} \frac{\cos \theta}{\sin \theta + \cos \theta} d\theta$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi/2} 1 d\theta = [\theta]_0^{\pi/2} = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$

$$17. \int_a^b f(x) \cdot g(x) dx = \int_a^b \frac{d}{dx} g(x) \cdot g(x) dx = \frac{1}{2} \int_a^b \frac{d}{dx} [g(x)]^2 dx$$

$$= \frac{1}{2} [(g(x))^2]_a^b = \frac{1}{2} [\{g(b)\}^2 - \{g(a)\}^2]$$

$$18. \int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)} dt$$

$$= \int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \left(\frac{1}{t} - \frac{t}{1+t^2} \right) dt$$

$$= \int_{\cot x}^{\tan x} \frac{t}{1+t^2} dt + [\log t]_{1/e}^{\cot x}$$

$$= \frac{1}{2} [\log (1+t^2)]_{\cot x}^{\tan x} + [\log t]_{1/e}^{\cot x}$$

$$= \frac{1}{2} \log \left(\frac{\sec^2 x}{\operatorname{cosec}^2 x} \right) + \log \cot x - \log \frac{1}{e}$$

$$= \log \tan x + \log \cot x + 1$$

$$= \log (\tan x \cdot \cot x) + 1$$

$$= 0 + 1 = 1$$

$$19. \int_{-2}^2 |1-x^2| dx$$

$$= \int_{-2}^{-1} |1-x^2| dx + \int_{-1}^0 |1-x^2| dx + \int_0^1 |1-x^2| dx + \int_1^2 |1-x^2| dx$$

$$= -\int_{-2}^{-1} (1-x^2) dx + \int_{-1}^0 (1-x^2) dx + \int_0^1 (1-x^2) dx - \int_1^2 (1-x^2) dx$$

$$= -\left[x - \frac{x^3}{3} \right]_{-2}^{-1} + \left[x - \frac{x^3}{3} \right]_{-1}^0 + \left[x - \frac{x^3}{3} \right]_0^1 - \left[x - \frac{x^3}{3} \right]_1^2$$

$$= -1 + \frac{7}{3} + 1 - \frac{1}{3} + 1 - \frac{1}{3} - 1 + \frac{7}{3} = 4$$

$$20. \text{ Let } I = \int_0^{2a} x^{9/2} (2a-x)^{-1/2} dx$$

$$\therefore I = \int_0^{\pi/2} \frac{(2a)^{9/2} \sin^9 \theta \cdot 4a \sin \theta \cos \theta d\theta}{\sqrt{2a-2a\sin^2 \theta}}$$

$$\text{Put } x = 2a \sin^2 \theta \Rightarrow dx = 4a \sin \theta \cos \theta d\theta$$

$$= 64a^5 \int_0^{\pi/2} \sin^{10} \theta d\theta$$

$$= 64a^5 \cdot \frac{9 \cdot 7 \cdot 5 \cdot 3 \cdot 1}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2} = \frac{63\pi}{8} a^5$$

$$21. \text{ Let } I = \int_{-1}^3 \left[\tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x} \right] dx$$

$$\text{Now, } \tan^{-1} \frac{x}{x^2+1} + \tan^{-1} \frac{x^2+1}{x}$$

$$= \tan^{-1} \left[\frac{\frac{x}{x^2+1} + \frac{x^2+1}{x}}{1 - \frac{x}{x^2+1} \cdot \frac{x^2+1}{x}} \right]$$

$$= \tan^{-1} \left[\frac{x^2 + (x^2+1)t^2}{x(x^2+1) \cdot 0} \right] = \tan^{-1}(\infty) = \frac{\pi}{2}$$

$$\text{Hence, } I = \int_{-1}^3 \frac{\pi}{2} dx = 2\pi$$

$$22. \text{ Given, } P = \int_0^{3\pi} f(\cos^2 x) dx \quad \dots(i)$$

$$Q = \int_0^{\pi} f(\cos^2 x) dx \quad \dots(ii)$$

From Eq. (i),

$$P = 3 \int_0^{\pi} f(\cos^2 x) dx = 3Q$$

$$\therefore P - 3Q = 0$$

$$23. \int_0^2 \sin \frac{\pi [x]}{2} dx = \int_0^1 \sin \frac{\pi [x]}{2} dx + \int_1^2 \sin \frac{\pi [x]}{2} dx$$

$$= \int_0^1 \sin \frac{\pi \cdot 0}{2} dx + \int_1^2 \sin \frac{\pi \cdot 1}{2} dx$$

$$\left[\because [x] = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x < 2 \end{cases} \right]$$

$$= 0 + [x]_1^2 = 2 - 1 = 1$$

$$24. \int_0^{\pi} |\cos \theta - \sin \theta| d\theta$$

$$= \int_0^{\pi/4} |\cos \theta - \sin \theta| d\theta + \int_{\pi/4}^{\pi} |\cos \theta - \sin \theta| d\theta$$

$$\left[\because \cos \theta - \sin \theta = 0, \text{ gives } \cos \theta = \sin \theta \right]$$

$$\therefore \tan \theta = 1 = \tan \frac{\pi}{4}$$

$$\therefore \theta = n\pi + \frac{\pi}{4}, n = 0, \pm 1, \pm 2, \dots,$$

$$\text{out of which only } \theta = \frac{\pi}{4} \in (0, \pi)$$

$$= \int_0^{\pi/4} (\cos \theta - \sin \theta) d\theta - \int_{\pi/4}^{\pi} (\cos \theta - \sin \theta) d\theta$$

$$= [\sin \theta + \cos \theta]_0^{\pi/4} - [\sin \theta + \cos \theta]_{\pi/4}^{\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) - 1 \right] - \left[(-1) - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right] = 2\sqrt{2}$$

$$25. \int_{-2}^2 [x^2] dx = 2 \int_0^2 [x^2] dx$$

$$[\because \text{integrand is an even function}]$$

$$= 2 \left[\int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^2 [x^2] dx + \int_{\sqrt{3}}^2 [x^2] dx \right]$$

$$\left[\because [x^2] = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < \sqrt{2} \\ 2, & \sqrt{2} \leq x < \sqrt{3} \\ 3, & \sqrt{3} \leq x < 2 \end{cases} \right]$$

$$= 2 \left[\int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^2 2 dx + \int_{\sqrt{3}}^2 3 dx \right]$$

$$= 2 [x]_1^{\sqrt{2}} + 4 [x]_{\sqrt{2}}^{\sqrt{3}} + 6 [x]_2^{\sqrt{3}} = (10 - 2\sqrt{3} - 2\sqrt{2})$$

$$26. \int_0^{\pi} |\sin x + \cos x| dx$$

$$= \int_0^{3\pi/4} |\sin x + \cos x| dx + \int_{3\pi/4}^{\pi} |\sin x + \cos x| dx$$

$$\left[\because \sin x + \cos x = 0 \Rightarrow \cos x = -\sin x \right]$$

$$\Rightarrow \tan x = -1 \Rightarrow x = \frac{3\pi}{4} \in (0, \pi)$$

$$= \int_0^{3\pi/4} (\sin x + \cos x) dx - \int_{3\pi/4}^{\pi} (\sin x + \cos x) dx$$

$$= [-\cos x + \sin x]_0^{3\pi/4} - [-\cos x + \sin x]_{3\pi/4}^{\pi}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) + 1 \right] - \left[1 - \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right) \right] = 2\sqrt{2}$$

$$27. \int_{-2}^1 [x+1] dx = \int_{-2}^{-1} [x+1] dx + \int_{-1}^0 [x+1] dx + \int_0^1 [x+1] dx$$

$$\therefore [x+1] = \begin{cases} -1, & \text{if } -2 \leq x < -1 \\ 0, & \text{if } -1 \leq x < 0 \\ 1, & \text{if } 0 \leq x < 1 \end{cases}$$

$$= \int_{-2}^{-1} (-1) dx + \int_{-1}^0 0 dx + \int_0^1 1 dx = -[x]_{-2}^{-1} + [x]_0^1 = 0$$

$$28. \int_0^{n^2} [\sqrt{x}] dx = \int_0^1 [\sqrt{x}] dx + \int_1^4 [\sqrt{x}] dx + \int_4^9 [\sqrt{x}] dx$$

$$+ \dots + \int_{(n-1)^2}^{n^2} [\sqrt{x}] dx$$

$$\therefore [\sqrt{x}] = \begin{cases} 0, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x < 4 \\ 2, & \text{if } 4 \leq x < 9 \\ \vdots & \vdots \\ (n-1), & \text{if } (n-1)^2 \leq x < n^2 \end{cases}$$

$$= \int_0^1 0 dx + \int_1^4 1 dx + \int_4^9 2 dx + \dots + \int_{(n-1)^2}^{n^2} (n-1) dx$$

$$= 1(4-1) + 2(9-4) + \dots + (n-1)[n^2 - (n-1)^2]$$

$$= -(1^2 + 2^2 + 3^2 + \dots + n^2) + n^3$$

$$= n^3 - \frac{n(n+1)(2n+1)}{6} = \frac{n(n-1)(4n+1)}{6}$$

$$29. \text{ Let } I = \int_0^4 f(x) \phi(x) dx = \int_0^4 f(4-x) \phi(4-x) dx$$

$$= \int_0^4 f(x) \cdot (3 - \phi(x)) dx \quad \left[\begin{array}{l} \because f(x) = f(4-x) \\ \text{and } \phi(x) + \phi(4-x) = 3 \end{array} \right]$$

$$= 3 \int_0^4 f(x) dx - I$$

$$\Rightarrow 2I = 3 \cdot 2$$

$$\therefore I = 3$$

$$30. \text{ We have, } 2f(x) + 3f\left(\frac{1}{x}\right) = \frac{1}{x} - 2 \quad \dots (i)$$

$$\Rightarrow 2f\left(\frac{1}{x}\right) + 3f(x) = x - 2 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$f(x) = -\frac{2}{5x} + \frac{3x}{5} - \frac{2}{5}$$

$$\therefore \int_1^2 f(x) dx = \int_1^2 \left(-\frac{2}{5x} + \frac{3x}{5} - \frac{2}{5} \right) dx$$

$$= \left[-\frac{2}{5} \log x + \frac{3x^2}{10} - \frac{2}{5} x \right]_1^2$$

$$= \left(-\frac{2}{5} \log 2 + \frac{6}{5} - \frac{4}{5} \right) - \left(\frac{3}{10} - \frac{2}{5} \right)$$

$$= \left(-\frac{2}{5} \log 2 + \frac{1}{2} \right)$$

$$31. \text{ Let } f(x) = ax^2 + bx + c$$

$$\Rightarrow f'(x) = 2ax + b \quad \text{and} \quad f''(x) = 2a$$

We are given $f(0) = c = 1, f'(0) = b = -2$

and $f''(0) = 2a = 6$

$\Rightarrow a = 3, b = -2$ and $c = 1$

Now, $f(x) = 3x^2 - 2x + 1$

$$\therefore \int_{-1}^2 f(x) dx = \int_{-1}^2 (3x^2 - 2x + 1) dx = [x^3 - x^2 + x]_{-1}^2$$

$$= (8 - 4 + 2) - (-1 - 1 - 1) = 6 + 3 = 9$$

$$32. \text{ Since, } g(x) = f'(x)$$

$$\therefore \int_{-1}^2 g(x) dx = \int_{-1}^2 f'(x) dx = [f(x)]_{-1}^2 = f(2) - f(-1)$$

$$= 4 - 2 = 2 \quad [\because f(-1) = 2 \text{ and } f(2) = 4]$$

$$33. \text{ Let } I = \int_{\tan \theta}^{\cot \theta} f(x) dx = \int_{\tan \theta}^{\cot \theta} \left(-\frac{1}{x^2} f\left(\frac{1}{x}\right) \right) dx$$

$$\left[\because f(x) = -\frac{1}{x^2} f\left(\frac{1}{x}\right) \right]$$

$$= \int_{\cot \theta}^{\tan \theta} f(z) dz \quad \left[\text{put } \frac{1}{x} = z \Rightarrow -\frac{1}{x^2} dx = dz \right]$$

$$= - \int_{\tan \theta}^{\cot \theta} f(x) dx = -I \Rightarrow 2I = 0$$

$$\therefore I = 0$$

$$34. \int_0^1 \frac{\log(1-x)}{x} dx = \int_0^1 \frac{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots}{x} dx$$

$$= \int_0^1 \left(-1 - \frac{x}{2} - \frac{x^2}{3} - \frac{x^3}{4} - \dots \right) dx$$

$$= - \left[x + \frac{x^2}{2^2} + \frac{x^3}{3^2} + \frac{x^4}{4^2} + \dots \right]_0^1$$

$$= - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \right) = -\frac{\pi^2}{6}$$

$$35. \int_0^1 \frac{\log x}{1+x} dx = \int_0^1 \log x \cdot \frac{1}{1+x} dx$$

$$= [\log x \cdot \log(1+x)]_0^1 - \int_0^1 \frac{1}{x} \cdot \log(1+x) dx$$

$$= - \int_0^1 \frac{\log(1+x)}{x} dx = -\frac{\pi^2}{12}$$

$$36. x \cos \pi x = 0 \text{ gives } x = 0, \pi x = \frac{\pi}{2}, \frac{3\pi}{2}, \text{ etc}$$

$$\Rightarrow x = 0, \frac{1}{2} \in [0, 1]$$

$$\therefore \int_0^1 x \cos \pi x dx$$

$$= \int_0^{1/2} x \cos \pi x dx + \int_{1/2}^1 x \cos \pi x dx$$

$$= \int_0^{1/2} x \cos \pi x dx - \int_{1/2}^1 x \cos \pi x dx$$

$$= \left[\frac{x}{\pi} \sin \pi x + \frac{1}{\pi^2} \cos \pi x \right]_0^{1/2} - \left[\frac{x}{\pi} \sin \pi x + \frac{1}{\pi^2} \cos \pi x \right]_{1/2}^1$$

$$= \left(\frac{1}{2\pi} - \frac{1}{\pi^2} \right) - \left(\frac{-1}{\pi^2} - \frac{1}{2\pi} \right) = \frac{1}{\pi}$$

$$37. A = \int_1^{\sin \theta} \frac{t}{1+t^2} dt$$

$$\text{Put } t = \frac{1}{x} \Rightarrow dt = -\frac{1}{x^2} dx$$

$$\therefore A = \int_1^{\csc \theta} \frac{1}{x} \cdot \left(-\frac{1}{x^2} \right) \cdot \frac{dx}{x^2 + 1}$$

$$= - \int_1^{\csc \theta} \frac{dx}{x(1+x^2)} = -B$$

$$\therefore A + B = 0$$

$$\text{Now, } \Delta = \begin{vmatrix} A & A^2 & B \\ 1 & A^2 + B^2 & -1 \end{vmatrix} = \begin{vmatrix} A & A^2 & -A \\ 1 & A^2 & -1 \end{vmatrix} = 0$$

38. Let $x = a \cos^2 \theta + b \sin^2 \theta \Rightarrow dx = 2(b-a) \sin \theta \cos \theta d\theta$

$$\begin{aligned} \therefore I &= \int_a^b (x-a)^3 (b-x)^4 dx \\ &= 2(b-a) \int_0^{\pi/2} (a \cos^2 \theta + b \sin^2 \theta - a)^3 \\ &\quad (b - a \cos^2 \theta - b \sin^2 \theta)^4 \sin \theta \cos \theta d\theta \\ &= 2(b-a)^8 \int_0^{\pi/2} \sin^7 \theta \cos^9 \theta d\theta \\ &= 2(b-a)^8 \int_0^{\pi/2} \sin^7 \theta (1 - \sin^2 \theta)^4 \cos \theta d\theta \end{aligned}$$

Put $\sin \theta = t \Rightarrow \cos \theta d\theta = dt$

$$\begin{aligned} \therefore I &= 2(b-a)^8 \int_0^1 t^7 (1-t^2)^4 dt \\ &= 2(b-a)^8 \int_0^1 t^7 (1 - 4t^2 + 6t^4 - 4t^6 + t^8) dt \\ &= \frac{(b-a)^8}{280} \end{aligned}$$

39. $\int_0^{na} f(x) dx = (n-1) \int_0^a f(x) dx$

40. $\int_0^3 \{\sqrt{x}\} dx = \int_0^3 (\sqrt{x} - [\sqrt{x}]) dx = \int_0^3 \sqrt{x} dx - \int_0^3 [\sqrt{x}] dx$

$$\begin{aligned} &= \left[\frac{x^{3/2}}{3/2} \right]_0^3 - \left(\int_0^1 [\sqrt{x}] dx + \int_1^3 [\sqrt{x}] dx \right) \\ &= 2\sqrt{3} - \left(\int_0^1 0 dx + \int_1^3 1 dx \right) = 2\sqrt{3} - 2 \end{aligned}$$

41. We have, $I_1 = \int_0^a [x] dx$

$$\begin{aligned} &= \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx + \dots + \int_{a-1}^a (a-1) dx \\ &= 1 + 2 + \dots + (a-1) = \frac{a(a-1)}{2} \quad \dots (i) \end{aligned}$$

and $I_2 = \int_0^a \{x\} dx = \int_0^a (x - [x]) dx = \int_0^a x dx - \int_0^a [x] dx$

$$= \left[\frac{x^2}{2} \right]_0^a - \frac{a(a-1)}{2} = \frac{a^2}{2} - \frac{a(a-1)}{2} = \frac{a}{2} \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\frac{I_1}{I_2} = (a-1) \Rightarrow I_1 = (a-1) I_2$$

42. Since, $(|\sin x| + |\cos x|)$ is a periodic function with period $\frac{\pi}{2}$, therefore $\int_a^{a+\frac{\pi}{2}} (|\sin x| + |\cos x|) dx$ is independent of a .

43. $\int_{a+2T}^{b+2T} f(x) dx = \int_{a+T}^{b+T} f(y+T) dy$ [put $x = y+T \Rightarrow dx = dy$]

$$\begin{aligned} &= \int_a^b f(z) dz \quad [\text{put } z = y+T \Rightarrow dz = dy] \\ &= \int_a^b f(x) dx \end{aligned}$$

44. $\int_0^\infty \frac{x}{1+e^x} dx = \int_0^\infty \frac{x \cdot e^{-x}}{e^{-x} + 1} dx$

$$\begin{aligned} &= \int_1^0 \frac{-\log t (-dt)}{t+1} \quad [\text{put } e^{-x} = t \Rightarrow -e^{-x} dx = dt] \\ &= \int_1^0 \frac{\log t}{t+1} dt = [\log t \cdot \log(t+1)]_1^0 - \int_1^0 \frac{1}{t} \log(t+1) dt \\ &= \int_0^1 \frac{\log(1+t)}{t} dt = \frac{\pi^2}{12} \end{aligned}$$

45. $\int_{e^{-1}}^{e^2} \left| \frac{\log_e x}{x} \right| dx = \int_{e^{-1}}^1 \left| \frac{\log_e x}{x} \right| dx + \int_1^{e^2} \left| \frac{\log_e x}{x} \right| dx$

$$\begin{aligned} &= - \int_{e^{-1}}^1 \frac{\log x}{x} dx + \int_1^{e^2} \frac{\log x}{x} dx = - \int_1^0 z dz + \int_0^2 z dz \\ &\quad \left[\text{put } \log x = z \Rightarrow \frac{1}{x} dx = dz \right] \\ &= \left[-\frac{z^2}{2} \right]_{-1}^0 + \left[\frac{z^2}{2} \right]_0^2 = \frac{1}{2} + 2 = \frac{5}{2} \end{aligned}$$

46. Given, $\int_1^2 e^{x^2} dx = a$

Let $I = \int_e^{e^4} \sqrt{\ln(x)} dx$

Put $\ln(x) = t^2 \Rightarrow \frac{1}{x} dx = 2t dt$

$$\begin{aligned} \therefore I &= \int_1^2 e^{t^2} \cdot 2t^2 dt \\ &= \left[t e^{t^2} \right]_1^2 - \int_1^2 e^{t^2} dt = 2e^4 - e - a \end{aligned}$$

47. $I_1 = \int_{\sin^2 t}^{1+\cos^2 t} x f\{x(2-x)\} dx$

$$\begin{aligned} &= \int_{\sin^2 t}^{1+\cos^2 t} (2-x) f(x(2-x)) dx = 2 \cdot I_2 - I_1 \\ \Rightarrow 2I_1 &= 2I_2 \Rightarrow \frac{I_1}{I_2} = 1 \end{aligned}$$

48. Put $x = -t$ in m

$$\begin{aligned} \therefore m &= \int_2^0 \left| \frac{\sin(-t)}{\left[-\frac{t}{\pi}\right] + \frac{1}{2}} \right| (-dt) \\ &= \int_0^2 \frac{|\sin t|}{\left[-\frac{t}{\pi}\right] + \frac{1}{2}} dt \\ &= \int_0^2 \frac{|\sin t|}{-\left[\frac{t}{\pi}\right] - 1 + \frac{1}{2}} dt \\ &= - \int_0^2 \frac{|\sin t|}{\left[\frac{t}{\pi}\right] + \frac{1}{2}} dt = -n \end{aligned}$$

49. $\int_{-1/2}^{1/2} \sqrt{\left(\frac{x+1}{x-1}\right)^2 + \left(\frac{x-1}{x+1}\right)^2} - 2 dx$

$$\begin{aligned} &= \int_{-1/2}^{1/2} \sqrt{\left(\frac{x+1}{x-1} - \frac{x-1}{x+1}\right)^2} dx = \int_{-1/2}^{1/2} \left| \frac{x+1}{x-1} - \frac{x-1}{x+1} \right| dx \\ &= \int_{-1/2}^{1/2} \left| \frac{4x}{x^2-1} \right| dx = 2 \int_0^{1/2} \left| \frac{4x}{x^2-1} \right| dx \quad [\because \text{integrand is an even function}] \\ &= -2 \int_0^{1/2} \left(\frac{4x}{x^2-1} \right) dx \\ &\quad \left[\because \frac{4x}{x^2-1} < 0 \text{ in the interval } \left(0, \frac{1}{2}\right) \right] \\ &= -4 [\log(1-x^2)]_0^{1/2} \\ &= -4 \left(\log \frac{3}{4} \right) = 4 \log \left(\frac{4}{3} \right) \end{aligned}$$

$$\begin{aligned} 50. \int_{-2}^4 f(x) dx &= \int_{-2}^{-1} f(x) dx + \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx \\ &\quad + \int_1^2 f(x) dx + \int_2^3 f(x) dx + \int_3^4 f(x) dx \\ &= 4 + 1 + 0 + 1 + 4 + 9 = 19 \end{aligned}$$

$$\begin{aligned} 51. \left| \int_{10}^{19} \frac{\sin x}{1+x^8} dx \right| &\leq \int_{10}^{19} \left| \frac{\sin x}{1+x^8} \right| dx \leq \int_{10}^{19} \frac{1}{1+x^8} dx < \int_{10}^{19} \frac{dx}{x^8} \\ &= -\frac{1}{7} [x^{-7}]_{10}^{19} \\ &= \frac{1}{7} 10^{-7} - \frac{1}{7} 19^{-7} < \frac{1}{7} 10^{-7} < 10^{-7} \end{aligned}$$

$$52. \text{ Since, } 0 \leq x \leq 1 \Rightarrow 1 \leq 1+x^4 \leq 2$$

$$\Rightarrow 1 \leq \sqrt{1+x^4} \leq \sqrt{2}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{1+x^4}} \leq 1$$

$$\Rightarrow \frac{1}{\sqrt{2}} \leq \int_0^1 \frac{dx}{\sqrt{1+x^4}} \leq 1$$

Hence, $\left[\frac{1}{\sqrt{2}}, 1 \right]$ is the smallest interval such that

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} \in \left[\frac{1}{\sqrt{2}}, 1 \right]$$

$$53. \text{ Let } f(x) = \int_0^x (1 + \cos^8 t)(at^2 + bt + c) dt$$

$$\Rightarrow f'(x) = (1 + \cos^8 x)(ax^2 + bx + c)$$

Clearly, $f(x)$ is continuous on $[1, 2]$ and derivable on $(1, 2)$.

$$\text{Also, } f(1) = f(2) \quad [\text{given}]$$

By Rolle's theorem, there exists a point $t \in (1, 2)$ such that $f'(t) = 0$.

$$\Rightarrow (1 + \cos^8 t)(at^2 + bt + c) = 0$$

$$\Rightarrow at^2 + bt + c = 0 \quad [\because 1 + \cos^8 t \neq 0]$$

$\therefore$ The quadratic equation $ax^2 + bx + c = 0$ has at least one root in $(1, 2)$.

$$54. \text{ Let } f(x) = 2x^3 - 9x^2 + 12x + 4, \text{ then } f(x) \text{ is a decreasing function on the interval } [1, 2].$$

$$\therefore f(2) < f(x) < f(1) = 9$$

$$\text{Now, } \frac{1}{3} < \frac{1}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}}$$

$$\Rightarrow \frac{1}{3} \int_1^2 dx < \int_1^2 \frac{dx}{\sqrt{2x^3 - 9x^2 + 12x + 4}} < \frac{1}{\sqrt{8}} \int_1^2 dx$$

$$\Rightarrow \frac{1}{3} < I < \frac{1}{\sqrt{8}} < 1$$

$$\text{Hence, } \frac{1}{3} < I < 1$$

$$55. \int_0^{2[x]} (x - [x]) dx = \int_0^{2[x]-1} (x - [x]) dx$$

$$= 2[x] \int_0^1 (x - [x]) dx$$

$[\because x - [x] \text{ is a periodic function of period } 1]$

$$= 2[x] \left(\left(\frac{x^2}{2} \right)_0^1 - \int_0^1 [x] dx \right) = 2[x] \left(\frac{1}{2} - 0 \right) = [x]$$

$$\begin{aligned} 56. \int_0^a \frac{e^x}{e^{[x]}} dx &= \int_0^{a-1} e^{x-[x]} dx + a \int_0^1 e^{x-[x]} dx \\ &\quad [\because x - [x] \text{ is a periodic function of period } 1] \\ &= a \int_0^1 e^x dx = a(e - 1) \end{aligned}$$

$$57. I = \int_0^{100\pi} (\tan x + \tan 2x + \tan 3x + \dots + \tan 10x) dx$$

Period of $f(x) = \tan x + \tan 2x + \dots + \tan 10x$ is π

$$\therefore I = 100 \int_0^\pi (\tan x + \tan 2x + \tan 3x + \dots + \tan 10x) dx$$

$$\text{Now, } f(x) = -f(\pi - x)$$

$$\therefore I = 0 \quad [\because \int_0^{2a} f(x) dx = 0, \text{ if } f(x) = -f(2a - x)]$$

$$\begin{aligned} 58. \text{ We have, } \frac{dx}{dt} &= [\sin^{-1} z]_{z=\sin t} \cdot \frac{d}{dt} (\sin t) \\ &= \sin^{-1}(\sin t) \cdot \cos t = t \cos t \end{aligned}$$

$$\text{and } \frac{dy}{dt} = \left[\frac{\sin z^2}{z} \right]_{z=\sqrt{t}} \cdot \frac{d}{dt} (\sqrt{t})$$

$$= \frac{\sin t}{\sqrt{t}} \cdot \frac{1}{2\sqrt{t}} = \frac{\sin t}{2t}$$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\sin t}{2t} \times \frac{1}{t \cos t} = \frac{\tan t}{2t^2}$$

$$59. \text{ Here } \int_0^{t^2} \{x f(x)\} dx = \frac{2}{5} t^5$$

On differentiating both sides, we get

$$t^2 \{f(t^2)\} \cdot \left\{ \frac{d}{dt} (t^2) \right\} - 0 \cdot f(0) \left\{ \frac{d}{dt} (0) \right\} = 2t^4$$

$$\Rightarrow t^2 f(t^2) \cdot 2t = 2t^4$$

$$\Rightarrow f(t^2) = t$$

$$\therefore f\left(\frac{4}{25}\right) = \pm \frac{2}{5} \quad \left[\text{putting } t = \pm \frac{2}{5} \right]$$

$$\Rightarrow f\left(\frac{4}{25}\right) = \frac{2}{5} \quad [\text{neglecting negative sign}]$$

$$60. \text{ Given, } \int_{\sin x}^1 t^2 f(t) dt = 1 - \sin x$$

$$\text{Now, } \frac{d}{dx} \int_{\sin x}^1 t^2 f(t) dt = \frac{d}{dx} (1 - \sin x)$$

$$\Rightarrow [t^2 f(t)] \cdot (0) - (\sin^2 x) \cdot f(\sin x) \cdot \cos x = -\cos x \quad [\text{by Leibnitz's formula}]$$

$$\Rightarrow f(\sin x) = \frac{1}{\sin^2 x}$$

$$\Rightarrow \sin x = \frac{1}{\sqrt{3}}$$

$$\therefore f\left(\frac{1}{3}\right) = (\sqrt{3})^2 = 3$$

$$61. \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sec^2 t}{x \sin x} dt = \lim_{x \rightarrow 0} \frac{\sec^2 x^2 \cdot 2x}{x \cos x + \sin x}$$

[using L'Hospital's rule]

$$= \lim_{x \rightarrow 0} \frac{2x \cdot 2 \sec^2 x^2 \cdot \tan x^2 \cdot 2x + 2 \sec^2 x^2}{-x \sin x + \cos x + \cos x}$$

[using L'Hospital's rule]

$$= \frac{0 + 2 \sec^2 0}{0 + 2 \cos 0} = 1$$

$$\begin{aligned}
 62. f(x) &= \int_{1/x}^{\sqrt{x}} \cos t^2 dt = \int_{1/x}^0 \cos t^2 dt + \int_0^{\sqrt{x}} \cos t^2 dt \\
 &= -\int_0^{1/x} \cos t^2 dt + \int_0^{\sqrt{x}} \cos t^2 dt \\
 \therefore f'(x) &= -[\cos t^2]_{t=1/x}^0 \cdot \frac{d}{dx} \left(\frac{1}{x} \right) + [\cos t^2]_{t=0}^{\sqrt{x}} \cdot \frac{d}{dx} (\sqrt{x}) \\
 &= \frac{1}{x^2} \cos \frac{1}{x^2} + \frac{1}{2\sqrt{x}} \cos x
 \end{aligned}$$

$$\begin{aligned}
 63. \text{ We have, } f(x) &= \int_{x^2}^{x^3} \frac{dt}{\log t} \\
 \Rightarrow f'(x) &= \frac{d}{dx} (x^3) \cdot \frac{1}{\log x^3} - \frac{d}{dx} (x^2) \cdot \frac{1}{\log x^2} \\
 &= \frac{3x^2}{3 \log x} - \frac{2x}{\log x} \\
 &= \frac{x^2 - x}{\log x} = x(x-1)(\log x)^{-1}
 \end{aligned}$$

$$\begin{aligned}
 64. a_n - a_{n-1} &= \int_0^{\pi/2} \frac{\cos^2 nx - \cos^2(n-1)x}{\sin x} dx, n \geq 2 \\
 &= -\frac{1}{2} \int_0^{\pi/2} \frac{\cos 2(n-1)x - \cos 2nx}{\sin x} dx \\
 &= -\int_0^{\pi/2} \frac{\sin(2n-1)x \sin x}{\sin x} dx \\
 &= \int_0^{\pi/2} \sin(2n-1)x dx \\
 &= \left[\frac{\cos(2n-1)x}{(2n-1)} \right]_0^{\pi/2} \\
 &= -\frac{1}{(2n-1)}
 \end{aligned}$$

$\therefore a_2 - a_1, a_3 - a_2, \dots$ are in HP.

$$\begin{aligned}
 65. a_n + a_{n+2} &= \int_0^{\pi/4} (\cot^n x + \cot^{n+2} x) dx, n \geq 2 \\
 &= \int_0^{\pi/4} \cot^n x \operatorname{cosec}^2 x dx \\
 &= \left[\frac{-\cot^{n+1} x}{n+1} \right]_0^{\pi/4} = \frac{-1}{n+1}
 \end{aligned}$$

$\therefore a_2 + a_4, a_3 + a_5, a_4 + a_6$ are in HP.

$$\begin{aligned}
 66. I_n &= \int_0^{\pi/2} \sin^n x dx = \int_0^{\pi/2} \sin^{n-1} x \sin x dx \\
 &= [-\sin^{n-1} x \cos x]_0^{\pi/2} + \int_0^{\pi/2} (n-1) \sin^{n-2} x \cos^2 x dx \\
 &= (n-1) \int_0^{\pi/2} \sin^{n-2} x (1 - \sin^2 x) dx \\
 &= (n-1) \int_0^{\pi/2} \sin^{n-2} x dx - (n-1) \int_0^{\pi/2} \sin^n x dx \\
 \therefore I_n &= (n-1) I_{n-2} - (n-1) I_n \\
 \Rightarrow n I_n &= (n-1) I_{n-2} \\
 \Rightarrow I_n &= \frac{n-1}{n} I_{n-2}
 \end{aligned}$$

$$\Rightarrow n(I_{n-2} - I_n) = I_{n-2}$$

$$\text{Also, } I_n : I_{n-2} = (n-1) : n$$

$$\text{and } I_{n-2} > I_n$$

$$\begin{aligned}
 67. \lim_{n \rightarrow \infty} &\left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \frac{n+3}{n^2+3^2} + \dots + \frac{1}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \frac{n+3}{n^2+3^2} + \dots + \frac{n+n}{n^2+n^2} \right]
 \end{aligned}$$

$$\begin{aligned}
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n+r}{n^2+r^2} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1+\frac{r}{n}}{\frac{r^2}{n^2}} \cdot \frac{1}{n} \\
 &= \int_0^1 \frac{1+x}{1+x^2} dx = \int_0^1 \frac{dx}{1+x^2} + \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx \\
 &= [\tan^{-1} x]_0^1 + \frac{1}{2} [\log(1+x^2)]_0^1 = \frac{\pi}{4} + \frac{1}{2} \log 2
 \end{aligned}$$

$$\begin{aligned}
 68. \lim_{n \rightarrow \infty} &\left[\frac{1}{\sqrt{2n-1^2}} + \frac{1}{\sqrt{4n-2^2}} + \frac{1}{\sqrt{6n-3^2}} + \dots + \frac{1}{n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{1}{\sqrt{2n-1^2}} + \frac{1}{\sqrt{4n-2^2}} + \frac{1}{\sqrt{6n-3^2}} + \dots + \frac{1}{\sqrt{2n \cdot n - n^2}} \right] \\
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\sqrt{2m-r^2}} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \cdot \frac{1}{\sqrt{\frac{2r}{n} - \frac{r^2}{n^2}}} \\
 &= \int_0^1 \frac{1}{\sqrt{2x-x^2}} dx = \int_0^1 \frac{dx}{\sqrt{1-(x^2-2x+1)}} \\
 &= \int_0^1 \frac{dx}{\sqrt{1-(x-1)^2}} = [\sin^{-1}(x-1)]_0^1 = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}
 \end{aligned}$$

$$\begin{aligned}
 69. \lim_{n \rightarrow \infty} &\left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{1}{2n} \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \dots + \frac{n}{n^2+n^2} \right] \\
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{n}{n^2+r^2} = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{\frac{r^2}{n^2}} \cdot \frac{1}{n} = \int_0^1 \frac{1}{1+x^2} dx \\
 &= [\tan^{-1} x]_0^1 = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 70. \lim_{n \rightarrow \infty} &\sum_{r=1}^n \frac{r}{n^2} \cdot \sec^2 \frac{r^2}{n^2} \\
 &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{r}{n} \cdot \sec^2 \frac{r^2}{n^2} \right) \frac{1}{n} = \int_0^1 x \sec^2 x^2 dx \\
 &= \frac{1}{2} \int_0^1 \sec^2 t dt \quad [\text{put } x^2 = t \Rightarrow 2x dx = dt] \\
 &= \frac{1}{2} [\tan t]_0^1 = \frac{1}{2} \tan 1
 \end{aligned}$$

$$\begin{aligned}
 71. \text{ Let } I &= \int_0^\pi \sum_{r=0}^3 a_r \cos^{3-r} x \sin^r x dx \\
 &= \int_0^\pi a_0 \cos^3 x dx + \int_0^\pi a_1 \cos^2 x \sin x dx \\
 &\quad + \int_0^\pi a_2 \cos x \sin^2 x dx + \int_0^\pi a_3 \sin^3 x dx
 \end{aligned}$$

$$\text{Since, } \int_0^{2a} f(x) dx$$

$$= \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(2a-x) = f(x) \\ 0, & \text{if } f(2a-x) = -f(x) \end{cases}$$

$\therefore$ Integral Ist and Illrd becomes zero.

Hence, the given integral depends upon a_1 and a_3 .

72. Let $y = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{1/n}$

$$\Rightarrow \log y = \lim_{n \rightarrow \infty} \frac{1}{n} \times \log \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left(1 + \frac{r}{n}\right) = \int_0^1 \log(1+x) dx$$

$$= [x \log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x} dx$$

$$= \log 2 - \int_0^1 \frac{1+x-1}{1+x} dx = \log 2 - \int_0^1 \left(1 - \frac{1}{1+x}\right) dx$$

$$= \log 2 - [x - \log(1+x)]_0^1 = \log 2 - [(1 - \log 2) - 0]$$

$$= 2 \log 2 - \log e = \log \frac{4}{e}$$

$$\therefore y = \frac{4}{e}$$

73. Let $y = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1^2}{n^2}\right) \left(1 + \frac{2^2}{n^2}\right) \dots \left(1 + \frac{n^2}{n^2}\right) \right]^{1/n}$

$$\Rightarrow \log y = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \left(1 + \frac{1^2}{n^2}\right) + \log \left(1 + \frac{2^2}{n^2}\right) + \dots + \log \left(1 + \frac{n^2}{n^2}\right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left(1 + \frac{r^2}{n^2}\right) = \int_0^1 \log(1+x^2) dx$$

$$= [x \log(1+x^2)]_0^1 - \int_0^1 x \cdot \frac{2x}{1+x^2} dx$$

$$= \log 2 - 2 \int_0^1 \left(1 - \frac{1}{1+x^2}\right) dx$$

$$= \log 2 - 2 [x - \tan^{-1} x]_0^1 = \log 2 + \frac{\pi}{2} - 2$$

$$\therefore y = e^{\log 2 + \frac{\pi}{2} - 2} = e^{\log 2} \cdot e^{\frac{\pi}{2}} \cdot e^{-2} = 2e^{\frac{\pi}{2} - 2}$$

74. $\lim_{n \rightarrow \infty} \frac{1}{n^{p+1}} [1^p + 2^p + \dots + n^p]$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^{p+1}} \sum_{r=1}^n r^p = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} \cdot \left(\frac{r}{n}\right)^p$$

$$= \int_0^1 x^p dx = \left[\frac{x^{p+1}}{p+1} \right]_0^1 = \frac{1}{p+1}$$

75. $\lim_{n \rightarrow \infty} \left[\frac{1^2}{1^3 + n^3} + \frac{2^2}{2^3 + n^3} + \dots + \frac{1}{2n} \right]$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{r^2}{r^3 + n^3} \right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{\left(\frac{r}{n}\right)^2}{\left(\frac{r}{n}\right)^3 + 1}$$

$$= \int_0^1 \frac{x^2}{x^3 + 1} dx$$

$$= \frac{1}{3} [\log(x^3 + 1)]_0^1$$

$$= \frac{1}{3} (\log 2 - \log 1)$$

$$= \frac{1}{3} \log 2$$

76. $\int_0^\pi \cos^n x dx = 2 \int_0^{\pi/2} \cos^n x dx$ $[\because f(x) = f(\pi - x)]$

$$= 2 \cdot \frac{(n-1)(n-3) \dots 3 \cdot 1}{n(n-2) \dots 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \pi \cdot \frac{(2m-1)(2m-3) \dots 3 \cdot 1}{2m(2m-2) \dots 4 \cdot 2}$$

where, $2m = n$

$$= \pi \cdot \frac{2m(2m-1)(2m-2)(2m-3) \dots 4 \cdot 3 \cdot 2 \cdot 1}{[2m(2m-2) \dots 4 \cdot 2]^2}$$

$$= \pi \cdot \frac{(2m)!}{[2^m m(m-1)(m-2) \dots 2 \cdot 1]^2}$$

$$= \pi \cdot \frac{(2m)!}{(2^m \cdot m!)^2} = \pi \cdot \frac{(2m)!}{2^{2m} \cdot (m!)^2}$$

$$= \frac{\pi \cdot n!}{2^n \cdot \left(\left(\frac{n}{2}\right)!\right)^2} \quad [\because 2m = n]$$

77. $I_n = \int_0^{\pi/2} x^n \sin x dx$

$$= [-x^n \cos x]_0^{\pi/2} + \int_0^{\pi/2} nx^{n-1} \cos x dx$$

$$= n \int_0^{\pi/2} x^{n-1} \cos x dx$$

$$= n \left[[x^{n-1} \sin x]_0^{\pi/2} - \int_0^{\pi/2} (n-1)x^{n-2} \sin x dx \right]$$

$$= n \left[\left(\frac{\pi}{2}\right)^{n-1} - (n-1)I_{n-2} \right]$$

Hence, $I_n + n(n-1)I_{n-2} = n \left(\frac{\pi}{2}\right)^{n-1}$

78. Put $x = a \sin \theta$

$$\Rightarrow dx = 2a \sin \theta \cos \theta d\theta$$

$$\therefore \int_0^a \frac{x^n}{\sqrt{a^2 - x^2}} dx = \int_0^{\pi/2} \frac{a^n \sin^{2n} \theta \cdot 2a \sin \theta \cos \theta d\theta}{a \sin \theta \sqrt{1 - \sin^2 \theta}}$$

$$= 2a^n \int_0^{\pi/2} \sin^{2n} \theta d\theta$$

$$= 2a^n \cdot \frac{(2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1}{2n(2n-2) \dots 6 \cdot 4 \cdot 2} \cdot \frac{\pi}{2}$$

$$= \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots 2n} \cdot \pi a^n$$

79. Let $I = \int_0^\pi x \sin^4 x dx$... (i)

$$\Rightarrow I = \int_0^\pi (\pi - x) \sin^4 x dx$$
 ... (ii)

On adding Eqs. (i) and (ii), we get

$$2I = \pi \int_0^\pi \sin^4 x dx$$

$$= 2\pi \int_0^{\pi/2} \sin^4 x dx$$

$$= 2\pi \cdot \frac{4-1}{4} \cdot \frac{4-3}{4-2} \cdot \frac{\pi}{2}$$

$$= 2\pi \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{3\pi^2}{8}$$

$$\Rightarrow I = \frac{3\pi^2}{16}$$

$$\begin{aligned}
 80. \text{ Let } I &= \int_{-\pi}^{\pi} (\sin^6 x + \cos^6 x) dx \\
 &= 2 \int_0^{\pi} (\sin^6 x + \cos^6 x) dx \\
 &= 4 \int_0^{\pi/2} \sin^6 x dx + 4 \int_0^{\pi/2} \cos^6 x \left(\frac{\pi}{2} - x \right) dx \\
 &= 8 \int_0^{\pi/2} \sin^6 x dx = 8 \times \frac{5 \cdot 3 \cdot 1}{6 \cdot 4 \cdot 2} \times \frac{\pi}{2} = \frac{5\pi}{4}
 \end{aligned}$$

$$81. \text{ Given, } f(x) = \int_0^{\sin^2 x} \sin^{-1}(\sqrt{t}) dt + \int_0^{\cos^2 x} \cos^{-1}(\sqrt{t}) dt$$

$$\text{where, } f'(x) = 0$$

$$\Rightarrow f(x) = C \text{ (constant)}$$

$$\text{But } f\left(\frac{\pi}{4}\right) = \int_0^{1/2} (\sin^{-1} \sqrt{t} + \cos^{-1} \sqrt{t}) dt$$

$$\therefore \int_0^{1/2} \left(\frac{\pi}{2}\right) dt = \frac{\pi}{4}$$

$$\Rightarrow f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$\text{and } f\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$$

$$82. f(2) = \int_2^2 (t^2 - 3t + 4) dt = 0$$

$$\text{Also, } f'(x) = f(x^2 - 3x + 4)$$

$$f'(2) = f(2) = 0$$

$$83. g(-x) = \int_0^{-x} f(t) dt$$

$$\text{Put } t = -z$$

$$\therefore g(-x) = - \int_0^x f(-z) dz \quad [\because f \text{ is odd, so } f(-z) = f(z)]$$

$$= \int_0^x f(z) dz = g(x)$$

$$\therefore g(x) \text{ is even.}$$

$$\text{Now, } g(x+2) = \int_0^{x+2} f(t) dt$$

$$= \int_0^2 f(t) dt + \int_2^{x+2} f(t) dt = g(2) + \int_0^x f(t) dt$$

$$= g(2) + g(x) = g(2) \quad [\because f \text{ is odd}]$$

$$\therefore g(x+2) = g(x)$$

$$\Rightarrow g(x) \text{ is periodic.}$$

$$\Rightarrow g(2) = g(4) = g(6) = \dots = g(2n) = 0$$

$$84. I_{n+1} - I_n = \int_0^{\pi} \frac{\sin\left(n + \frac{3}{2}\right)x - \sin\left(n + \frac{1}{2}\right)x}{\sin \frac{x}{2}} dx$$

$$= \int_0^{\pi} \frac{2 \cos(n+1)x \cdot \sin \frac{x}{2}}{\sin \frac{x}{2}} dx = 0$$

$$\text{Hence, } I_0 = I_1 = I_2 = I_3$$

$$\text{Also, } I_n = I_0 = \pi$$

$$85. \sin 2\alpha = \int_{\pi/2}^{\alpha} \sin x dx$$

$$\Rightarrow \sin 2\alpha = -\cos \alpha$$

$$\Rightarrow \cos \alpha (2 \sin \alpha + 1) = 0$$

$$\text{Hence, } \cos \alpha = 0 \text{ or } \sin \alpha = -\frac{1}{2}$$

$$\text{Therefore, } \frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$

$$86. A_{n+1} - A_n = \int_0^{\pi/2} \frac{\sin(2n+1)x - \sin(2n-1)x}{\sin x} dx$$

$$= \int_0^{\pi/2} 2 \cos 2nx dx = 0$$

$$\Rightarrow A_{n+1} = A_n$$

$$\text{Now, } B_{n+1} - B_n = \int_0^{\pi/2} \frac{\sin^2(n+1)x - \sin^2 nx}{\sin^2 x} dx$$

$$= \int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x} dx = A_{n+1}$$

$$87. \text{ For } a \leq 0,$$

$$\text{Given equation becomes}$$

$$\int_0^2 (x-a) dx \geq 1 \Rightarrow a \leq \frac{1}{2}$$

$$\Rightarrow a \leq 0$$

$$\text{For } 0 < a < 2, \int_0^2 |x-a| dx \geq 1$$

$$\Rightarrow \int_0^a (a-x) dx + \int_a^2 (x-a) dx \geq 1$$

$$\Rightarrow \frac{a^2}{2} + 2 - 2a + \frac{a^2}{2} \geq 1$$

$$\Rightarrow a^2 - 2a + 1 \geq 0$$

$$\Rightarrow (a-1)^2 \geq 0$$

$$\text{For } a \geq 2,$$

$$\int_0^2 |x-a| dx \geq 1$$

$$\Rightarrow \int_0^2 (a-x) dx \geq 1$$

$$\Rightarrow 2a - 2 \geq 1$$

$$\Rightarrow a \geq \frac{3}{2}$$

$$\Rightarrow a \geq 2$$

$$88. \text{ Given, } f(x) = \int_0^x |t-1| dt$$

$$\Rightarrow f(x) = \int_0^x (1-t) dt, 0 \leq x \leq 1$$

$$\Rightarrow f(x) = x - \frac{x^2}{2}$$

$$\text{Also, } f(x) = \int_0^1 (1-t) dt + \int_1^x (t-1) dt, \text{ where } 1 \leq x \leq 2$$

$$\Rightarrow f(x) = \frac{1}{2} + \frac{x^2}{2} - x + \frac{1}{2} = \frac{x^2}{2} - x + 1$$

$$\text{Thus, } f(x) = \begin{cases} x - \frac{x^2}{2}, & 0 \leq x \leq 1 \\ \frac{x^2}{2} - x + 1, & 1 \leq x \leq 2 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1-x, & 0 \leq x \leq 1 \\ x-1, & 1 < x < 2 \end{cases}$$

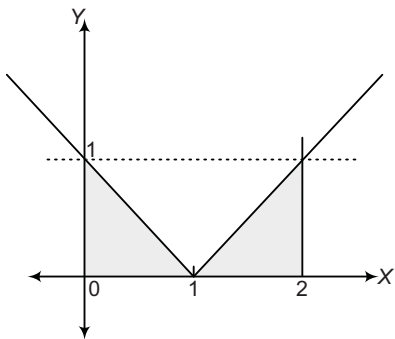
$$\text{Hence, } f(x) \text{ is continuous as well as differentiable at } x=1.$$

$$\text{Also, } f(x) = \cos^{-1} x \text{ has one real root, draw the graph and verify.}$$

$$\text{For range of } f(x), f(x) = \int_0^x |t-1| dt \text{ is the value of area bounded by the curve } y=|t-1| \text{ and X-axis between the limits } t=0 \text{ and } t=x.$$

$$\text{Obviously, minimum area is obtained when } t=0 \text{ and } t=x \text{ coincide or } x=0.$$

Maximum value of area occurs when $t = 2$.
Hence, $f(2) = \text{Area of shaded region} = 1$



89. (a) The expression $f(x)f(c)$, $\forall x \in (c-h, c+h)$, where $h \rightarrow 0^+$ is equivalent to $\lim_{x \rightarrow 0} f(x)f(c)$ which equals

to $(f(c))^2$ because $f(x)$ is continuous.

Therefore, $f(x)f(c) > 0$, $\forall x \in (c-h, c+h)$, where $h \rightarrow 0^+$.

$$\begin{aligned} \text{(b) We have, } I &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \ln \prod_{k=1}^n \left(1 + \frac{k}{n}\right) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \ln \left(1 + \frac{k}{n}\right) \\ &= \int_1^2 \ln x \, dx = [x(\ln x - 1)]_1^2 = -1 + 2 \ln 2 \end{aligned}$$

$$\text{(c) Given, } f(x) \geq 0 \Rightarrow \int_a^b f(x) \, dx \geq 0$$

But $\int_a^b f(x) \, dx = 0$, so this can be true only when $f(x) = 0$.

$$\text{(d) } \int_a^b f(x) \, dx = 0 \Rightarrow y = f(x) \text{ cuts } X\text{-axis at least once.}$$

So, there exists at least one $c \in (a, b)$ for which $f(c) = 0$.

$$\begin{aligned} 90. f(x + \pi) &= \int_0^{x+\pi} (\cos(\sin t) + \cos(\cos t)) \, dt \\ &= \int_0^\pi (\cos(\sin t) + \cos(\cos t)) \, dt \\ &\quad + \int_\pi^{x+\pi} (\cos(\sin t) + \cos(\cos t)) \, dt \\ &= f(\pi) + \int_0^x (\cos(\sin t) + \cos(\cos t)) \, dt \\ [\because \text{for } g(x) &= \cos(\sin x) + \cos(\cos x), f(x + \pi) = f(x)] \\ &= f(\pi) + f(x) \\ &= f(x) + 2f\left(\frac{\pi}{2}\right) \quad [\because g(x) \text{ has period } \pi/2] \end{aligned}$$

$$\begin{aligned} 91. \text{(d) We have, } \int_0^\pi \sqrt{1 - \sin^2 x} \, dx &= \int_0^\pi \sqrt{\cos^2 x} \, dx \\ &= \int_0^\pi |\cos x| \, dx \\ &= \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^\pi \cos x \, dx \\ &= [\sin]_0^{\pi/2} - [\sin x]_{\pi/2}^\pi \\ &= [1 - 0] - [0 - 1] \\ &= 1 + 1 = 2 \end{aligned}$$

Hence, Statement I is false.

However, Statement II is true.

$$\begin{aligned} 92. \int_0^\pi |\cos x| \, dx &= \int_0^{\pi/2} |\cos x| \, dx + \int_{\pi/2}^\pi |\cos x| \, dx \\ &= \int_0^{\pi/2} \cos x \, dx - \int_{\pi/2}^\pi \cos x \, dx \\ &= (1 - 0) - (0 - 1) = 2 \end{aligned}$$

$$93. \text{Statement I } \frac{d}{dx} f(x) = \frac{e^{\sin x}}{x}, x > 0$$

$$\Rightarrow f(x) = \int \frac{e^{\sin x}}{x} \, dx$$

$$\begin{aligned} \therefore I &= \int_1^4 \frac{3e^{\sin x^3}}{x} \, dx \\ &= \int_1^4 \frac{3x^2 e^{\sin x^3}}{x^3} \, dx = \int_1^{64} \frac{e^{\sin t}}{t} \, dt \\ &= [f(x)]_1^{64} = f(64) - f(1) \end{aligned}$$

$$\therefore f(64) - f(1) = f(k) - f(1) \therefore k = 64$$

$$\text{Statement II } I = \int_{\sin \theta}^{\operatorname{cosec} \theta} f(x) \, dx$$

$$= \int_{\operatorname{cosec} \theta}^{\sin \theta} f\left(\frac{1}{t}\right) \left(-\frac{1}{t^2}\right) \, dt$$

$$\Rightarrow I = \int_{\operatorname{cosec} \theta}^{\sin \theta} f(t) \, dt = -1 \therefore I = 0$$

94. Statement I can be proved by showing that both integrals are equal to a third integral. Put $x^n = \tan^2 \theta$ in the integral on LHS and $x^n = \sin^2 \theta$ in the integral on RHS, then both integrals will be equal to $\frac{2}{n} \int_0^{\pi/2} \tan^{n-1} \phi \, d\phi$ and $\frac{2}{n} \int_0^{\pi/2} \tan^{n-1} \theta \, d\theta$, respectively.

Since, the last two integrals are equal, so Statement I is proved.

Statement II has no role to play here.

$$\begin{aligned} 95. \text{Statement I Let } I &= \int_{-\pi}^\pi \frac{2x(1 + \sin^2 x)}{1 + \cos^2 x} \, dx \\ I &= 0 \quad [\because f(x) = -f(-x)] \end{aligned}$$

$$\text{Statement II Let } I = \int_0^{\pi/2} \sin^2 x \cos^2 x (\sin x + \cos x) \, dx$$

Put $\sin x - \cos x = t$

$$\Rightarrow 1 - 2 \sin x \cos x = t^2$$

$$\Rightarrow \frac{1 - t^2}{2} = \sin x \cos x$$

$$\Rightarrow (\sin x + \cos x) \, dx = dt$$

$$\begin{aligned} \therefore I &= \int_{-1}^1 \left(\frac{1 - t^2}{2}\right)^2 \, dt \\ &= 2 \int_0^1 \frac{(1 - t^2)^2}{4} \, dt \\ &= \frac{1}{2} \int_0^1 (1 - 2t^2 + t^4) \, dt \\ &= \frac{1}{2} \left[t - \frac{2t^3}{3} + \frac{t^5}{5} \right]_0^1 = \frac{4}{15} \end{aligned}$$

96. Statement I is true as it is a fundamental property.

$$\text{Let } g(x) = \int_a^x f(t) \, dt$$

If $f(x)$ is an even function.

$$\begin{aligned}\text{Then, } g(-x) &= \int_a^{-x} f(t) dt \\ &= -\int_{-a}^x f(-y) dy \\ &= -\int_{-a}^x f(y) dy \\ &= -\int_{-a}^a f(y) dy - \int_a^x f(y) dy \\ &\neq -g(x)\end{aligned}$$

Hence, Statement II is false.

97. Let $g(x) = \int_a^x f(t) dt - \int_x^b f(t) dt$, where $x \in [a, b]$

We have, $g(a) = -\int_a^b f(t) dt$

and $g(b) = \int_a^b f(t) dt$

$$\Rightarrow g(a)g(b) = -\left(\int_a^b f(t) dt\right)^2 \leq 0$$

Clearly, $g(x)$ is continuous in $[a, b]$ and $g(a)g(b) \leq 0$. It implies that $g(x)$ will become zero at least once in $[a, b]$. Thus, $\int_a^x f(t) dt = \int_x^b f(t) dt$ for at least one value of $x \in [a, b]$.

Hence, both statements are true and Statement II is the correct explanation for Statement I.

98. $f(x) = \int_{5\pi/4}^x (3\sin t + 4\cos t) dt$

$$\Rightarrow f'(x) = 3\sin x + 4\cos x, x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$$

These values of x are in third quadrant, where both $\sin x$ and $\cos x$ are negative.

Then, $f'(x) < 0$ for $x \in \left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$

Hence, $f(x)$ is decreasing for these values of x .

Thus, the least value of function occurs at $x = \frac{4\pi}{3}$.

$$\begin{aligned}\therefore f_{\min} &= \int_{5\pi/4}^{4\pi/3} (3\sin t + 4\cos t) dt \\ &= \frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}\end{aligned}$$

99. For $a < b$, if m and M are the smallest and greatest values of $f(x)$ on $[a, b]$, then

$$m(b-a) \leq \int_a^b f(x) dx \leq M(b-a)$$

or $m \leq \frac{1}{(b-a)} \int_a^b f(x) dx \leq M$

Since, $f(x)$ is continuous on $[a, b]$, it takes on all intermediate values between m and M .

Therefore, for some values $f(c)$, $a \leq f(c) \leq b$, we will have $\frac{1}{(b-a)} \int_a^b f(x) dx = f(c)$

or $\int_a^b f(x) dx = f(c)(b-a)$

Hence, both statements are true and Statement II is the correct explanation of Statement I.

100. Since, $f(x)$ is an odd function.

$$\begin{aligned}\therefore f(x) &= -f(-x) \\ \phi(-x) &= \int_a^{-x} f(t) dt\end{aligned}$$

$$\begin{aligned}\text{Put } t &= -y \Rightarrow dt = -dy \\ \Rightarrow \phi(-x) &= \int_a^x f(-t)(-dt) = \int_{-a}^x f(t) dt = \int_{-a}^a f(t) dt \\ &\quad + \int_a^x f(t) dt = 0 + \int_a^x f(t) dt = \phi(x)\end{aligned}$$

101. If $f(x)$ is an even function, then

$$\begin{aligned}\phi(-x) &= -\int_{-a}^x f(t) dt = -\int_{-a}^a f(t) dt - \int_a^x f(t) dt \\ &= -2\int_0^a f(t) dt - \int_a^x f(t) dt\end{aligned}$$

[since, $f(x)$ is an even function]

$$\begin{aligned}\text{Now, } \int_0^a f(t) dt &= \int_0^a f(a-t) dt \\ &= \int_0^a -f(t) dt \quad [\because f(a-x) = -f(x)]\end{aligned}$$

$$\Rightarrow \int_0^a f(t) dt = 0$$

$$\Rightarrow \phi(-x) = -\int_a^x f(t) dt = -f(x)$$

$\therefore \phi(x)$ is an odd function.

102. $g(x + \alpha) + g(x) = 0$

$$\Rightarrow g(x + 2\alpha) + g(x + \alpha) = 0$$

$$\Rightarrow g(x + 2\alpha) = -g(x + \alpha)$$

$\Rightarrow g(x)$ is periodic with period 2α .

$$\Rightarrow \int_b^{2k} g(t) dt = \int_b^{b+c} g(x) dx \quad [\because b, k, c \text{ are in AP}]$$

This is independent of b , then c has least value 2α .

$$\begin{aligned}103. \int_{p+m\alpha}^{q+n\alpha} g(t) dt &= \int_{p+m\alpha}^p g(x) dx + \int_p^q g(x) dx + \\ &\quad \int_q^{q+n\alpha} g(x) dx\end{aligned}$$

$$= -m \int_0^\alpha g(x) dx + \int_p^q g(x) dx + n \int_0^\alpha g(x) dx$$

$$= \int_p^q g(x) dx + (n-m) \int_0^\alpha g(x) dx$$

104. $I_1 = \int_{1-k}^k (1-x) \sin[x(1-x)] dx$

$$\begin{aligned}&= \int_{1-k}^k \sin x(1-x) dx - \int_{1-k}^k x \sin x(1-x) dx \\ &= I_2 - I_1\end{aligned}$$

$$\Rightarrow 2I_1 = I_2$$

105. We have, $I_1 = \int_{\sec^2 z}^{2-\tan^2 z} x f[(x)(3-x)] dx$

$$\Rightarrow I_1 = \int_{\sec^2 z}^{2-\tan^2 z} (2 - \tan^2 z + \sec^2 z - x)$$

$$\begin{aligned}&\quad f[(2 - \tan^2 z + \sec^2 z - x) \\ &\quad (3 - (2 - \tan^2 z + \sec^2 z - x))] dx\end{aligned}$$

$$\Rightarrow I_1 = \int_{\sec^2 z}^{2-\tan^2 z} (3-x) f[(3-x)(x)] dx$$

$$\Rightarrow I_1 = \int_{\sec^2 z}^{2-\tan^2 z} 3f[(3-x)(x)] dx$$

$$\int_{\sec^2 z}^{2-\tan^2 z} x f[(3-x)(x)] dx$$

$$\Rightarrow I_1 = 3I_2 - I_1$$

$$\Rightarrow 2I_1 = 3I_2$$

$$\Rightarrow \frac{I_1}{I_2} = \frac{3}{2}$$

106. Let $I = \int_0^\pi [f(x) + g(x)] dx$

$$\Rightarrow I = \int_0^\pi [f(\pi - x) + g(\pi - x)] dx$$

$$\Rightarrow I = \int_0^\pi [1 - f(x) + 1 - g(x)] dx$$

$$\Rightarrow I = 2 \int_0^\pi dx - \int_0^\pi [f(x) + g(x)] dx$$

$$\Rightarrow 2I = 2\pi \Rightarrow I = \pi$$

107. $a \leq \int_0^1 e^{x^2} dx \leq b \Rightarrow 1 \leq \int_0^1 e^{x^2} dx < e$

$$\therefore (a, b) = (1, e)$$

108. $x \in (0, 1)$

$$x > x^2 \Rightarrow -x < -x^2 \Rightarrow e^{-x} < e^{-x^2}$$

$$\Rightarrow e^{-x} \cos^2 x < e^{-x^2} \cos^2 x$$

$$I_1 < I_2 \text{ and } -x^2 < -\frac{x^2}{2}$$

$$\Rightarrow e^{-x^{1/2}} < e^{-x^{1/2}} \Rightarrow I_4 > I_3$$

Obviously, $I_4 > I_2$

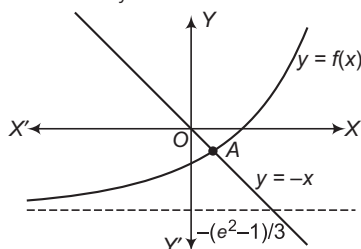
Hence, $I = I_4$

109. $f(x) = e^x - \left(\frac{e^2 - 1}{3}\right)$

110. Number of $f(x) + x = 0$

$$\Rightarrow f(x) = -x$$

Point A is the only solution.



111. $\int_0^1 f(x) dx = \int_0^1 \left(e^x - \left(\frac{e^2 - 1}{3}\right) \right) dx$

$$= \left[e^x - \left(\frac{e^2 - 1}{3}\right)x \right]_0^1 = \frac{3e - e^2 + 1}{3} - 1$$

$$= \frac{3e - e^2 - 2}{3}$$

112. A. $\lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{\sqrt{n^2 - r^2}}{n^2} = \lim_{n \rightarrow \infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{1 - \left(\frac{r}{n}\right)^2}$

$$= \int_0^1 \sqrt{1 - x^2} dx = \frac{\pi}{4}$$

B. $\int_0^{\pi/4} (\sqrt{\tan x} + \sqrt{\cot x}) dx = 2$

$$\Rightarrow \int_0^{\pi/4} \frac{1}{2} \cdot \frac{\sqrt{\tan x} + \sqrt{\cot x}}{\tan x + \cot x} (\tan x + \cot x) dx = 2$$

$$\Rightarrow \int_0^{\pi/4} \frac{d(\sqrt{\tan x} - \sqrt{\cot x})}{(\sqrt{\tan x} - \sqrt{\cot x})^2 + 2} dx$$

$$= 2 \cdot \frac{1}{\sqrt{2}} \tan^{-1} \left[\frac{\sqrt{\tan x} - \sqrt{\cot x}}{\sqrt{2}} \right]_0^{\pi/4} = \frac{\pi}{\sqrt{2}}$$

C. $\int_{-1}^1 \sin^3 x \cos^2 x dx = 0$ [$\because$ integrand is odd]

D. $\int_0^{\pi/2} \frac{\sqrt{\sin^3 x}}{\sqrt{\sin^3 x} + \sqrt{\cos^3 x}} dx$

$$= \int_0^{\pi/2} \frac{\sqrt{\cos^3 x}}{\sqrt{\sin^3 x} + \sqrt{\cos^3 x}} dx$$

$$\therefore 2I = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

113. A. $I = \int_0^2 \frac{dx}{1 + 2^{f(x)}} \Rightarrow I = \int_0^2 \frac{dx}{1 + 2^{f(2-x)}} = \int_0^2 \frac{dx}{1 + 2^{-f(x)}}$

$$I = \int_0^2 \frac{2^{f(x)}}{1 + 2^{f(x)}} dx$$

$$\Rightarrow 2I = \int_0^2 \frac{dx}{1 + 2^{f(x)}} + \int_0^2 \frac{2^{f(x)}}{1 + 2^{f(x)}} dx$$

$$= \int_0^2 dx = 2$$

$$\Rightarrow I = 1$$

B. Let $I = \int_{1/2}^2 \frac{1}{x} \operatorname{cosec}^{101} \left(x - \frac{1}{x} \right) dx$

Put $\frac{1}{x} = t$

$$\Rightarrow dt = -\frac{1}{x^2} dx \Rightarrow dx = -\frac{1}{t^2} dt$$

$$\therefore I = - \int_2^{1/2} \operatorname{cosec}^{101} \left(\frac{1}{t} - t \right) \frac{dt}{t}$$

$$= - \int_2^{1/2} -\frac{1}{t} \operatorname{cosec}^{101} \left(t - \frac{1}{t} \right) dt$$

$$= - \int_{1/2}^2 \frac{1}{t} \operatorname{cosec}^{101} \left(t - \frac{1}{t} \right) dt = -I$$

$$\Rightarrow 2I = 0 \Rightarrow I = 0$$

C. Let $I = \int_{-\pi/2}^{\pi/2} \frac{2 \cos x}{1 + 2^{x^2}} dx$

$$= \int_{-\pi/2}^0 \frac{2 \cos x}{1 + 2^{x^2}} dx + \int_0^{\pi/2} \frac{2 \cos x}{1 + 2^{x^2}} dx = I_1 + I_2$$

$$I_1 = \int_{-\pi/2}^0 \frac{2 \cos x}{1 + 2^{x^2}} dx$$

Put $x = -t$

$$\Rightarrow dx = -dt$$

$$\therefore I_1 = - \int_{\pi/2}^0 \frac{2 \cos(-t)}{1 + 2^{(-t)^2}} dt = \int_0^{\pi/2} \frac{2 \cos t \cdot 2^{t^3}}{1 + 2^{t^2}} dt$$

$$\Rightarrow I_1 + I_2 = 2 \int_0^{\pi/2} \cos x dx = 2 [\sin x]_0^{\pi/2} = 2$$

D. Let $I = \int_0^{\pi/2} \left[\frac{\sin 6x}{\sin x} + \frac{\sin 6(\pi - x)}{\sin(\pi - x)} \right] dx = 0$

114. A. $\int_\pi^{2\pi} [2 \sin x] dx = \int_\pi^{\pi + \pi/6} -1 dx +$

$$\int_{\pi + \pi/6}^{2\pi - \pi/6} -2 dx + \int_{2\pi - \pi/6}^{2\pi} -1 dx$$

$$= 1 \cdot \frac{\pi}{6} - 2 \cdot \frac{2\pi}{3} - 1 \cdot \frac{\pi}{6} = -\frac{5\pi}{3}$$

B. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \frac{r/n}{\sqrt{1 + \left(\frac{r}{n}\right)^2}} = \int_0^2 \frac{x}{\sqrt{1 + x^2}} dx$

$$= [\sqrt{1 + x^2}]_0^2$$

$$= \sqrt{5} - 1$$

C. Put $x = \cos 2\theta \Rightarrow dx = -2 \sin 2\theta d\theta$

$$\begin{aligned} \therefore \int_0^1 \frac{\sqrt{1-x}}{1+x} dx &= -2 \int_{\pi/4}^0 \tan \theta \cdot \sin 2\theta d\theta \\ &= 4 \int_0^{\pi/4} \sin^2 \theta d\theta = 2 \int_0^{\pi/4} (1 - \cos 2\theta) d\theta \\ &= 2 \left[\theta - \frac{\sin 2\theta}{2} \right]_0^{\pi/4} = \frac{\pi}{2} - 1 \end{aligned}$$

D. $\int_{-2}^2 |1-x^2| dx = 2 \int_0^2 |1-x^2| dx$

$$\begin{aligned} &= 2 \int_0^1 (1-x^2) dx + 2 \int_1^2 (x^2-1) dx \\ &= 2 \left[x - \frac{x^3}{3} \right]_0^1 + 2 \left[\frac{x^3}{3} - x \right]_1^2 = 4 \end{aligned}$$

115. A. Let $I = \int_4^{10} \frac{[x^2]}{[x^2-28x+196]+[x^2]} dx$

$$\Rightarrow I = \int_4^{10} \frac{[x^2]}{[(x-14)^2]+[x^2]} dx \quad \dots(i)$$

$$\Rightarrow I = \int_4^{10} \frac{[(14-x)^2] dx}{[x^2]+[(14-x)^2]} \quad \dots(ii)$$

$$[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx]$$

On adding Eqs. (i) and (ii), we get

$$I = \int_4^{10} dx = \frac{10-4}{2} = 3$$

B. $\int_{-1}^2 \frac{|x|}{x} dx = \int_{-1}^0 -1 dx + \int_0^2 1 dx = -[x]_{-1}^0 + [x]_0^2$

$$= -[0 - (-1)] + [2 - 0] = 1$$

C. $\lim_{n \rightarrow \infty} \frac{1^{99} + 2^{99} + \dots + n^{99}}{n^{100}}$

$$= \lim_{n \rightarrow \infty} \sum_{r=0}^n \frac{1}{n} \left(\frac{r}{n} \right)^{99} = \int_0^1 x^{99} dx = \left[\frac{x^{100}}{100} \right]_0^1 = \frac{1}{100}$$

D. $\frac{1}{\alpha} = 5050 \int_{-1}^1 |x|^{100} dx = 5050 \times 2 \int_0^1 x^{100} dx$

$$= 2 \times 5050 \left[\frac{x^{101}}{101} \right]_0^1 = 2 \times 5050 \left[\frac{1}{101} - 0 \right]$$

$$\Rightarrow \frac{1}{\alpha} = 100 \Rightarrow \alpha = \frac{1}{100}$$

116. A. $I = \int_{-2}^2 (\alpha x^3 + \beta x + \gamma) dx$

Since, $\alpha x^3 + \beta x$ is an odd function.

$$\therefore I = 0 + 2 \int_0^2 \gamma dx = 2 \cdot 2\gamma = 4\gamma$$

B. $I = \frac{1}{2} \int_0^1 2 \sin \alpha x \sin \beta x dx$

$$= \frac{1}{2} \int_0^1 \{ \cos(\alpha - \beta)x - \cos(\alpha + \beta)x \} dx$$

$$= \frac{1}{2} \left[\frac{\sin(\alpha - \beta)x}{\alpha - \beta} - \frac{\sin(\alpha + \beta)x}{\alpha + \beta} \right]_0^1$$

$$= \frac{1}{2} \left[\frac{\sin(\alpha - \beta)}{\alpha - \beta} - \frac{\sin(\alpha + \beta)}{\alpha + \beta} \right] \quad \dots(i)$$

Also, $2\alpha = \tan \alpha$ and $2\beta = \tan \beta$

$$\Rightarrow 2(\alpha - \beta) = \tan \alpha - \tan \beta$$

and $2(\alpha + \beta) = \tan \alpha + \tan \beta$

$$\Rightarrow 2(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos \alpha \cos \beta}$$

and $2(\alpha + \beta) = \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}$

On substituting these values, we get

$$I = (\cos \alpha \cos \beta) - (\cos \alpha \cos \beta) = 0$$

C. $f(x + \alpha) + f(x) = 0$

$$\Rightarrow f(x + 2\alpha) + f(x + \alpha) = 0$$

$$\Rightarrow f(x + 2\alpha) = f(x)$$

$\Rightarrow f(x)$ is periodic with period 2α .

$$\therefore \int_{\beta}^{\beta+2\gamma\alpha} (ax^3 + \beta x + \gamma) dx = \gamma \int_0^{2\alpha} f(x) dx$$

D. Let $I = \int_0^a [\sin x] dx$, $a \in [2\beta + 1]\pi$, $[2\beta + 2]\pi$, $\beta \in N$,

where $[]$ denotes the greatest integer function.

$$\begin{aligned} \text{Then, } I &= \int_0^{2\beta\pi} [\sin x] dx + \int_{2\beta\pi}^{(2\beta+1)\pi} [\sin x] dx \\ &\quad + \int_{(2\beta+1)\pi}^a [\sin x] dx \end{aligned}$$

$$= \beta \int_0^{2\pi} [\sin x] dx + 0 + \int_{(2\beta+1)\pi}^a (-1) dx$$

$$= -\beta\pi + (2\beta + 1)\pi - \alpha = (\beta + 1)\pi - \alpha$$

So, $\gamma \int_0^a [\sin x] dx$ depends on α , β and γ .

117. A. $\int_{-1}^1 [x + [x + [x]]] dx$

$$= \int_{-1}^1 3[x] dx = 3 \int_{-1}^1 [x] dx$$

$$[\because [x + n] = [x] + n, \text{ if } n \text{ is an integer}]$$

$$= 3 \int_0^1 ([x] + [-x]) dx$$

$$= -3 \quad [\because [x] + [-x] = -1]$$

B. $\int_2^5 ([x] + [-x]) dx = \int_2^5 -dx = -3$

C. $\text{sgn}(x - [x]) = \begin{cases} 1, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer} \end{cases}$

$$\text{Hence, } \int_{-1}^3 \text{sgn}(x - [x]) dx = 4(1 - 0) = 4$$

D. Let $I = 25 \int_0^{\pi/4} \{ \tan^6(x - [x]) + \tan^4(x - [x]) \} dx$

$$[\because 0 < x < \frac{\pi}{4} \Rightarrow [x] = 0]$$

$$\begin{aligned} \therefore I &= 25 \int_0^{\pi/4} (\tan^6 x + \tan^4 x) dx \\ &= 25 \int_0^{\pi/4} \tan^4 x (\tan^2 x + 1) dx \\ &= 25 \int_0^{\pi/4} \tan^4 x \sec^2 x dx \\ &= 25 \left[\frac{\tan^5 x}{5} \right]_0^{\pi/4} = 25 \times \frac{1}{5} = 5 \end{aligned}$$

118. Let $I = \int_0^1 \frac{\sin^{-1} \sqrt{x}}{x^2 - x + 1} dx \quad \dots(i)$

$$\Rightarrow I = \int_0^1 \frac{\sin^{-1} \sqrt{1-x}}{x^2 - x + 1} dx = \int_0^1 \frac{\cos^{-1} \sqrt{x}}{x^2 - x + 1} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^1 \frac{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}{x^2 - x + 1} dx$$

$$= \frac{\pi}{2} \int_0^1 \frac{dx}{x^2 - x + 1} = \frac{\pi}{2} \int_0^1 \frac{dx}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\pi}{2} \left[\frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \left[\tan^{-1} \left(\frac{2x-1}{\sqrt{3}} \right) \right]_0^1 \right] = \frac{\pi^2}{3\sqrt{3}}$$

$$\text{Hence, } I = \frac{\pi^2}{6\sqrt{3}} = \frac{\pi^2}{\sqrt{108}} \Rightarrow \frac{n}{27} = \frac{108}{27} = 4$$

119. Let $I = \int_0^{207} C_7 \cdot \underbrace{x^{200}}_{\text{zero}} \cdot \underbrace{(1-x)^7}_{\text{zero}} dx$

$$\Rightarrow I = {}^{207}C_7 \left\{ \left[\underbrace{(1-x)^7 \cdot \frac{x^{201}}{201}}_{\text{zero}} \right]_0^{207} + \frac{7}{201} \int_0^{207} (1-x)^6 \cdot x^{201} dx \right\}$$

$$\Rightarrow I = - {}^{207}C_7 \cdot \frac{7}{201} \int_0^1 (1-x)^6 \cdot x^{201} dx$$

Integrating by parts again 6 more times, we get

$$I = {}^{207}C_7 \cdot \frac{7!}{201 \cdot 202 \cdot 203 \cdot 204 \cdot 205 \cdot 206 \cdot 207} \int_0^1 x^{207} dx$$

$$= \frac{207!}{7! 200!} \cdot \frac{7!}{201 \cdot 202 \cdots 207} \cdot \frac{1}{208}$$

$$= \frac{207!}{207! 7!} \cdot \frac{7!}{208} = \frac{1}{208} = \frac{1}{k}$$

$$\Rightarrow k = 208 \therefore \frac{k}{26} = \frac{208}{26} = 8$$

120. We have, $J = \int_{-5}^{-4} (3-x^2) \tan(3-x^2) dx$

Put $(x+5)=t \Rightarrow dx=dt$

$$J = \int_0^1 (3-(t-5)^2) \tan(3-(t-5)^2) dt$$

$$= \int_0^1 (-22+10t-t^2) \tan(-22+10t-t^2) dt$$

Now, $K = \int_{-2}^{-1} (6-6x+x^2) \tan(6x-x^2-6) dx$

Put $(x+2)=z \Rightarrow dx=dz$

$$\therefore K = \int_0^1 (6-6(z-2)+(z-2)^2) \tan(6(z-2)-(z-2)^2-6) dz$$

$$= \int_0^1 (22-10z+z^2) \tan(-22+10z-z^2) dz$$

Hence, $(J+K)=0$

121. Let $I = \int_1^4 (\{x\})^{[x]} dx = \int_1^4 (x-[x])^{[x]} dx$

$$= \int_1^2 (x-[x])^{[x]} dx + \int_2^3 (x-[x])^{[x]} dx + \int_3^4 (x-[x])^{[x]} dx$$

$$= \int_1^2 (x-1)^1 dx + \int_2^3 (x-2)^2 dx + \int_3^4 (x-3)^3 dx$$

$$= \left[\frac{(x-1)^2}{2} \right]_1^2 + \left[\frac{(x-2)^3}{3} \right]_2^3 + \left[\frac{(x-3)^4}{4} \right]_3^4$$

$$= \left[\frac{1}{2} - 0 \right] + \left[\frac{1}{3} - 0 \right] + \left[\frac{1}{4} - 0 \right] = \frac{13}{12}$$

$$\Rightarrow \frac{a}{b} = \frac{13}{12} \therefore a-b = 13-12 = 1$$

122. Let $f(x) = x - [x]$

$$\therefore I = \int_{-1}^1 (x - [x]) dx$$

$$= \int_{-1}^0 (x - [x]) dx + \int_0^1 (x - [x]) dx$$

$$= \int_{-1}^0 (x+1) dx + \int_0^1 x dx = \left[\frac{x^2}{2} + x \right]_{-1}^0 + \left[\frac{x^2}{2} \right]_0^1 = 1$$

123. Let $I = \int_0^a f(a-x) g(a-x) \cdot h(a-x) dx$

Let $7I = 3I + 4I$

$$= 3 \int_0^a f(x) g(x) h(x) dx + 4 \int_0^a f(x) g(x) h(x) dx$$

$$= 3 \int_0^a f(x) g(x) h(x) dx + 4 \int_0^a f(a-x) g(a-x) h(a-x) dx$$

$$= 3 \int_0^a f(x) g(x) h(x) dx - 4 \int_0^a f(x) g(x) h(a-x) dx$$

$$= \int_0^a f(x) g(x) \{3h(x) - 4h(a-x)\} dx$$

$$= 5 \int_0^a f(x) g(x) dx = 0$$

$\therefore f(a-x) g(a-x) = -f(x) g(x)$

$\therefore I = 0$

124. Using property, $\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$

$$I = \int_2^3 \frac{\sqrt{2+3-x}}{\sqrt{5-(2+3-x)} + \sqrt{5-x}} dx$$

$$\Rightarrow I = \int_2^3 \frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} dx \quad \dots(i)$$

$$\Rightarrow I = \int_2^3 \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_2^3 \left(\frac{\sqrt{5-x}}{\sqrt{x} + \sqrt{5-x}} + \frac{\sqrt{x}}{\sqrt{5-x} + \sqrt{x}} \right) dx$$

$$= \int_2^3 1 dx = 3-2=1 \Rightarrow I = \frac{1}{2} \Rightarrow \frac{a}{b} = \frac{1}{2}$$

$\therefore a+b=1+2=3$

125. $\int_0^2 f(x) dx = \int_0^{1/2} 1 dx + \int_{1/2}^{2/3} 1 dx + \int_{2/3}^{3/4} 1 dx$

$$+ \dots + \int_{(n-1)/n}^{n/(n+1)} 1 dx + \dots + \int_1^2 1 dx$$

$$= \frac{1}{2} + \left[\frac{2}{3} - \frac{1}{2} \right] + \left[\frac{3}{4} - \frac{2}{3} \right] + \dots + \left[\frac{n}{n+1} - \frac{n-1}{n} \right] + \dots + 1$$

$$= \frac{n}{n+1} + \dots + 1$$

We take $n \rightarrow \infty \Rightarrow \int_0^2 f(x) dx = 1+1=2$

126. $I(x) = \int_{1/e}^{\tan x} \frac{t}{1+t^2} dt + \int_{1/e}^{\cot x} \frac{dt}{t(1+t^2)}$

On differentiating w.r.t. x , we get

$$\frac{dI(x)}{dx} = \frac{\tan x}{(1+\tan^2 x)} \sec^2 x$$

$$+ \frac{1}{\cot x (1+\cot^2 x)} (-\operatorname{cosec}^2 x) = 0$$

$I(x) = \text{constant}$

Let $x = \frac{\pi}{4}$

Thus, $I\left(\frac{\pi}{4}\right) = \int_{1/e}^1 \frac{t}{1+t^2} dt + \int_{1/e}^1 \frac{dt}{t(1+t^2)}$

$$= \int_{1/e}^1 \frac{t^2+1}{t(t^2+1)} dt = \int_{1/e}^1 \frac{dt}{t} = [\ln t]_{1/e}^1 = 1$$

Entrances Gallery

$$1. g'(a) = \int_h^{1-h} \frac{\partial}{\partial a} t^{-a}(1-t)^{a-1} dt$$

$$= - \int_h^{1-h} t^{-a}(1-t)^{a-1} dt + t^{-a}(1-t)^{a-1} dt = 0$$

$$g(a) = \text{constant} \Rightarrow g(a) = \lambda$$

$$\therefore g(a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} \frac{1}{\sqrt{t(1-t)}} dt$$

$$= \int_h^{1-h} \frac{dt}{\sqrt{-\left(t - \frac{1}{2}\right)^2 + \frac{1}{4}}} = \left[\sin^{-1} \frac{t - \frac{1}{2}}{\frac{1}{2}} \right]_h^{1-h}$$

$$= [\sin^{-1}(2t-1)]_h^{1-h}$$

$$= \sin^{-1}(1-2h) - \sin^{-1}(2h-1) = \pi$$

$$2. g(a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1} dt$$

$$g(1-a) = \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{-(1-a)}(1-t)^{(1-a)-1} dt$$

$$= \lim_{h \rightarrow 0^+} \int_h^{1-h} t^{a-1}(1-t)^{-a} dt$$

$$= \lim_{h \rightarrow 0^+} \int_h^{1-h} (1-t)^{a-1} (1-(1-t))^{-a} dt$$

$$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$$

$$= \lim_{h \rightarrow 0^+} \int_h^{1-h} (1-t)^{a-1} t^{-a} dt$$

$$\therefore g(1-a) = g(a) \Rightarrow -g'(1-a) = g'(a)$$

$$\text{At } a = \frac{1}{2}, -g'\left(\frac{1}{2}\right) = g'\left(\frac{1}{2}\right) \Rightarrow g'\left(\frac{1}{2}\right) = 0$$

$$3. \text{ Let } I = \int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} (1-x^2)^5 \right\} dx$$

$$= [4x^3 \cdot \frac{d}{dx} (1-x^2)^5]_0^1 - 12 \int_0^1 x^2 \cdot \frac{d}{dx} (1-x^2)^5 dx$$

$$= -12 \left[[x^2 \cdot (1-x^2)^5]_0^1 - 2 \int_0^1 x \cdot (1-x^2)^5 dx \right]$$

$$= 12 \int_0^1 2x(1-x^2)^5 dx = -12 \int_0^1 t^5 dt = \frac{12}{6} [t^6]_0^1 = 2$$

Aliter

$$\text{Let } I = \int_0^1 4x^3 \left\{ \frac{d^2}{dx^2} (1-x^2)^5 \right\} dx$$

$$= \frac{d}{dx} \left(\frac{d(1-x^2)^5}{dx} \right) = \frac{d}{dx} [5(1-x^2)^4 (-2x)]$$

$$= -10 \frac{d}{dx} [x(1-x^2)^4]$$

$$= -10 [(1-x^2)^4 + x^4(1-x^2)^3 (-2x)]$$

$$= -10 (1-x^2)^3 [1-x^2-8x^2]$$

Hence, given integral is

$$I = -40 \int_0^1 x^3 (1-x^2)^3 (1-9x^2) dx$$

$$= -40 \int_0^{\pi/2} \sin^3 \theta \cos^7 \theta d\theta + 360 \int_0^1 \sin^5 \theta \cos^7 \theta d\theta$$

$$= -40 \cdot 1 \cdot \frac{2 \cdot 6 \cdot 4 \cdot 2}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} + 360 \cdot \frac{4 \cdot 2 \cdot 6 \cdot 4 \cdot 2}{12 \cdot 10 \cdot 8 \cdot 6 \cdot 4 \cdot 2}$$

$$= -1 + 3 = 2$$

$$4. \text{ P. Let } f(x) = ax^2 + bx, a, b \in W \quad [\because f(0) = 0]$$

$$\therefore \int_0^1 ax^2 + bx = \frac{a}{3} + \frac{b}{2} = 1$$

$$\Rightarrow 2a + 3b = 6$$

$$\Rightarrow (a, b) \equiv (3, 0), (0, 2)$$

$$\therefore \text{Number of such polynomials} = 2$$

$$Q. f(x) = \sqrt{2} \sin \left(x^2 + \frac{\pi}{4} \right)$$

$$x^2 + \frac{\pi}{4} = 2n\pi + \frac{\pi}{2}, \text{ if } f(x) \text{ is maximum.}$$

$$\Rightarrow x^2 = 2n\pi + \frac{\pi}{4} \text{ for } n = 0, 1$$

$$\text{when } x^2 \in [0, 13]$$

$$R. \int_{-2}^2 \frac{3x^2}{1+e^x} dx = \int_0^2 3x^2 \left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}} \right) dx$$

$$\left[\because \int_{-a}^a f(x) dx = \int_0^a \{f(x) + f(-x)\} dx \right]$$

$$= \int_0^2 3x^2 \left(\frac{1}{1+e^x} + \frac{e^x}{1+e^x} \right) dx$$

$$= \int_0^2 3x^2 dx = [x^3]_0^2 = 8$$

$$S. \int_{-1/2}^{1/2} \cos 2x \log \left(\frac{1+x}{1-x} \right) dx = 0$$

[since, it is an odd function]

$$5. \text{ Let } I = \int_{\pi/4}^{\pi/2} (2 \operatorname{cosec} x)^{17} dx$$

$$\text{Put } \log \tan \frac{x}{2} = u$$

$$\Rightarrow \tan \frac{x}{2} = e^u$$

$$\Rightarrow \sin x = \frac{2e^u}{1+e^{2u}}$$

$$\operatorname{cosec} x = \frac{e^u + e^{-u}}{2}$$

$$\therefore I = 2 \int_{\log(\sqrt{2}-1)}^0 (e^u + e^{-u})^{16} du$$

$$= 2 \int_0^{\log(\sqrt{2}+1)} (e^u + e^{-u})^{16} du$$

Since, $(e^u + e^{-u})^{16}$ is an even function.

$$\text{Hence, } I = \int_0^{\log(\sqrt{2}+1)} 2(e^u + e^{-u})^{16} du$$

$$6. \text{ Here, } f'(x) < 2f(x)$$

On multiplying by e^{-2x} , we get

$$f'(x) \cdot e^{-2x} - 2e^{-2x} f(x) < 0$$

$$\Rightarrow \frac{d}{dx} \{f(x) \cdot e^{-2x}\} < 0$$

$$\text{So, } \phi(x) = f(x)e^{-2x} \text{ is decreasing for } x \in \left[\frac{1}{2}, 1 \right]$$

$$\text{Thus, when } x > \frac{1}{2}$$

$$\phi(x) < \phi\left(\frac{1}{2}\right)$$

$$\Rightarrow e^{-2x} f(x) < e^{-1} \cdot f\left(\frac{1}{2}\right)$$

$$\begin{aligned} \Rightarrow f(x) &< e^{2x-1} \cdot 1 \quad \left[\because f\left(\frac{1}{2}\right) = 1 \right] \\ \Rightarrow 0 &< \int_{1/2}^1 f(x) dx < \int_{1/2}^1 e^{2x-1} dx \\ \Rightarrow 0 &< \int_{1/2}^1 f(x) dx < \left[\frac{e^{2x-1}}{2} \right]_{1/2}^1 \\ \therefore 0 &< \int_{1/2}^1 f(x) dx < \frac{e-1}{2} \end{aligned}$$

$$7. \text{ Required limit } = \frac{\int_0^1 x^a dx}{\int_0^1 (a+x) dx} = \frac{2}{(2a+1)(a+1)} = \frac{2}{120}$$

$$\Rightarrow a = 7 \text{ or } -\frac{17}{2}$$

$$8. \text{ Let } I = \int_{-\pi/2}^{\pi/2} \left(x^2 + \log \left(\frac{\pi-x}{\pi+x} \right) \right) \cos x dx$$

$$\text{As } \int_{-a}^a f(x) dx = 0, \text{ when } f(-x) = -f(x)$$

$$\begin{aligned} \therefore I &= \int_{-\pi/2}^{\pi/2} x^2 \cos x dx + 0 \\ &= 2 \int_0^{\pi/2} x^2 \cos x dx \\ &= 2 \left\{ [x^2 \sin x]_0^{\pi/2} - \int_0^{\pi/2} 2x \cdot \sin x dx \right\} \\ &= 2 \left\{ \frac{\pi^2}{4} - 2 \left\{ [-x \cos x]_0^{\pi/2} - \int_0^{\pi/2} 1(-\cos x) dx \right\} \right\} \\ &= 2 \left\{ \frac{\pi^2}{4} - 2 [\sin x]_0^{\pi/2} \right\} = 2 \left(\frac{\pi^2}{4} - 2 \right) = \left(\frac{\pi^2}{2} - 4 \right) \end{aligned}$$

$$9. \text{ Let } I = \int_{\sqrt{\log 2}}^{\sqrt{\log 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\log 6 - x^2)} dx$$

$$\text{Put } x^2 = t \Rightarrow x dx = dt/2$$

$$\therefore I = \int_{\log 2}^{\log 3} \frac{\sin t \frac{dt}{2}}{\sin t + \sin(\log 6 - t)} \quad \dots (i)$$

$$\Rightarrow I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin(\log 2 + \log 3 - t)}{\sin(\log 2 + \log 3 - t) + \sin t} dt$$

$$\Rightarrow I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \frac{1}{2} \int_{\log 2}^{\log 3} \frac{\sin t + \sin(\log 6 - t)}{\sin(\log 6 - t) + \sin t} dt$$

$$\Rightarrow 2I = \frac{1}{2} [t]_{\log 2}^{\log 3} = \frac{1}{2} (\log 3 - \log 2)$$

$$\therefore I = \frac{1}{4} \log \left(\frac{3}{2} \right)$$

10. Using Newton-Leibnitz's formula,

$$6f(x) \cdot 1 - 0 = 3f(x) + 3xf'(x) - 3x^2$$

$$\Rightarrow f'(x) - \frac{1}{x} f(x) = x \Rightarrow \frac{xf'(x) - f(x)}{x^2} = 1$$

$$\Rightarrow \frac{d}{dx} \left\{ \frac{f(x)}{x} \right\} = 1 \Rightarrow \frac{f(x)}{x} = x + C \quad \left[\because f(1) = \frac{1}{3} \right]$$

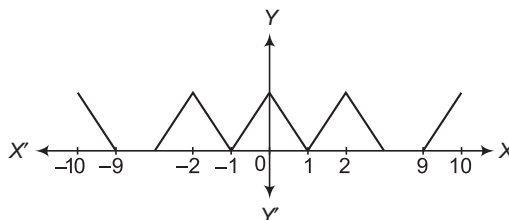
$$\Rightarrow \frac{1}{3} = 1 + C \Rightarrow C = -\frac{2}{3}$$

$$\therefore f(x) = x^2 - \frac{2}{3}x \Rightarrow f(2) = 4 - \frac{4}{3} = \frac{8}{3}$$

$$11. \text{ Given, } f(x) = \begin{cases} x - [x], & \text{if } [x] \text{ is odd} \\ 1 + [x] - x, & \text{if } [x] \text{ is even} \end{cases}$$

Since, $f(x)$ and $\cos \pi x$ both are periodic with period 2 and both are even.

$$\begin{aligned} \therefore \int_{-10}^{10} f(x) \cos \pi x dx &= 2 \int_0^{10} f(x) \cos \pi x dx \\ &= 10 \int_0^2 f(x) \cos \pi x dx \end{aligned}$$



$$\begin{aligned} \text{Now, } \int_0^1 f(x) \cos \pi x dx &= \int_0^1 (1-x) \cos \pi x dx \\ &= - \int_0^1 u \cos \pi u du \end{aligned}$$

$$\begin{aligned} \text{and } \int_1^2 f(x) \cos \pi x dx &= \int_1^2 (x-1) \cos \pi x dx \\ &= - \int_0^1 u \cos \pi u du \end{aligned}$$

$$\therefore \int_{-10}^{10} f(x) \cos \pi x dx = -20 \int_0^1 u \cos \pi u du = \frac{40}{\pi^2}$$

$$\text{Hence, } \frac{\pi^2}{10} \int_{-10}^{10} f(x) \cos \pi x dx = 4$$

$$\begin{aligned} 12. \int_0^1 \frac{x^4 (1-x)^4}{1+x^2} dx &= \int_0^1 \frac{(x^4-1)(1-x)^4 + (1-x)^4}{(1+x^2)} dx \\ &= \int_0^1 (x^2-1)(1-x)^4 dx + \int_0^1 \frac{(1+x^2-2x)^2}{(1+x^2)} dx \\ &= \int_0^1 \left\{ (x^2-1)(1-x)^4 + (1+x^2) - 4x + \frac{4x^2}{(1+x^2)} \right\} dx \\ &= \int_0^1 \left\{ (x^2-1)(1-x)^4 + (1+x^2) - 4x + 4 - \frac{4}{1+x^2} \right\} dx \\ &= \int_0^1 \left\{ x^6 - 4x^5 + 5x^4 - 4x^2 + 4 - \frac{4}{1+x^2} \right\} dx \\ &= \left[\frac{x^7}{7} - \frac{4x^6}{6} + \frac{5x^5}{5} - \frac{4x^3}{3} + 4x - 4 \tan^{-1} x \right]_0^1 = \frac{22}{7} - \pi \end{aligned}$$

$$13. \text{ We have, } e^{-x} f(x) = 2 + \int_0^x \sqrt{t^4 + 1} dt, x \in (-1, 1)$$

On differentiating w.r.t. x , we get

$$e^{-x} [f'(x) - f(x)] = \sqrt{x^4 + 1}$$

$$\Rightarrow f'(x) = f(x) + e^x \sqrt{x^4 + 1}$$

Since, f^{-1} is the inverse of f .

$$\therefore f^{-1}[f(x)] = x$$

$$\Rightarrow f^{-1'}[f(x)] f'(x) = 1$$

$$\Rightarrow f^{-1'}[f(x)] = \frac{1}{f'(x)}$$

$$\Rightarrow f^{-1'}[f(x)] = \frac{1}{f(x) + e^x \sqrt{x^4 + 1}}$$

$$\text{At } x = 0, f(x) = 2 \text{ and } f^{-1}(2) = \frac{1}{2+1} = \frac{1}{3}$$

14. Use the formula $|x - a| = \begin{cases} x - a, & x \geq a \\ -(x - a), & x < a \end{cases}$ to break given integral in two parts and then integrate separately.

$$\begin{aligned} \therefore \int_0^{\pi} \sqrt{1 - 2 \sin \frac{x}{2}} dx &= \int_0^{\pi} \left| 1 - 2 \sin \frac{x}{2} \right| dx \\ &= \int_0^{\pi/3} \left(1 - 2 \sin \frac{x}{2} \right) dx - \int_{\pi/3}^{\pi} \left(1 - 2 \sin \frac{x}{2} \right) dx \\ &= \left[x + 4 \cos \frac{x}{2} \right]_0^{\pi/3} - \left[x + 4 \cos \frac{x}{2} \right]_{\pi/3}^{\pi} \\ &= 4\sqrt{3} - 4 - \frac{\pi}{3} \end{aligned}$$

$$\begin{aligned} 15. I &= \int_{7\pi/4}^{7\pi/3} \sqrt{\tan^2 x} dx = \int_{7\pi/4}^{2\pi} |\tan x| dx + \int_{2\pi}^{7\pi/3} |\tan x| dx \\ &= -\int_{7\pi/4}^{2\pi} \tan x dx + \int_{2\pi}^{7\pi/3} \tan x dx \\ &= [\log \cos x]_{7\pi/4}^{2\pi} - [\log \cos x]_{2\pi}^{7\pi/3} \\ &= \left[\log \sin \frac{\pi}{2} - \log \sin \frac{\pi}{4} \right] - \left[\log \cos \frac{\pi}{3} - \log \sin \frac{\pi}{2} \right] \\ &= \left[0 - \log \frac{1}{\sqrt{2}} \right] - [-\log 2 - 0] \\ &= \log \sqrt{2} + \log 2 \\ &= \frac{3}{2} \log 2 = \log 2\sqrt{2} \end{aligned}$$

$$16. \text{ Let } I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx \quad \dots(i)$$

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \frac{2^x x \sin^2 x}{1 + 2^x} dx \quad \dots(ii) \\ &\quad \left[\because \int_a^b f(x) dx = \int_a^b f(a + b - x) dx \right] \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} \Rightarrow 2I &= \int_{-\pi/2}^{\pi/2} \sin^2 x dx \\ &= \int_0^{\pi/2} 2 \sin^2 x dx \\ &= \int_0^{\pi/2} (1 - \cos 2x) dx \\ \Rightarrow 2I &= \left[x - \frac{\sin 2x}{2} \right]_0^{\pi/2} = \frac{\pi}{2} \\ \therefore I &= \frac{\pi}{4} \end{aligned}$$

17. Put $t = \sin^2 z$ in 1st integral and $t = \cos^2 u$ in 2nd integral, we get

$$\begin{aligned} dt &= 2 \sin z \cos z dz \text{ and } dt = -2 \cos u \sin u du \\ \therefore I &= \int_0^x z^2 \sin z \cos z dz + \int_{\pi/2}^x (-u^2 \cos u \sin u) du \\ &= \int_0^x \frac{z}{2} \sin 2z dz - \int_{\pi/2}^x \frac{u}{2} \sin 2u du \\ &= \left[-z \cdot \frac{\cos 2z}{2} + \frac{\sin 2z}{4} \right]_0^x - \left[\frac{-u \cos 2u}{2} + \frac{\sin 2u}{4} \right]_{\pi/2}^x \\ &= \left[-x \cdot \frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \{0 + 0\} \right] \\ &\quad - \left[-x \cdot \frac{\cos 2x}{2} + \frac{\sin 2x}{4} - \left(\frac{\pi}{4} + 0 \right) \right] \\ &= \frac{\pi}{4} \end{aligned}$$

$$18. \text{ Given, } R_1 = \int_{-2}^3 f(x) dx$$

$$\begin{aligned} \therefore R_2 &= \int_{-2}^3 xf(x) dx = \int_{-2}^3 (1 - x)f(1 - x) dx \\ \Rightarrow R_2 &= \int_{-2}^3 (1 - x)f(x) dx = \int_{-2}^3 f(x) dx - \int_{-2}^3 xf(x) dx \\ \Rightarrow R_2 &= R_1 - R_2 \\ \Rightarrow 2R_2 &= R_1 \end{aligned}$$

$$19. \text{ We have, } x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$$

By Leibnitz's rule, we get

$$\begin{aligned} 1 &= \frac{1}{\sqrt{1+y^2}} \cdot \frac{dy}{dx} \\ \Rightarrow \frac{dy}{dx} &= \sqrt{1+y^2} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{y}{\sqrt{1+y^2}} \cdot \frac{dy}{dx} \\ \therefore \frac{d^2y}{dx^2} &= \frac{y}{\sqrt{1+y^2}} \cdot \sqrt{1+y^2} = y \end{aligned}$$

$$20. \text{ Let } I = \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan x}} \quad \dots(i)$$

$$\begin{aligned} \therefore I &= \int_{\pi/6}^{\pi/3} \frac{dx}{1 + \sqrt{\tan\left(\frac{\pi}{2} - x\right)}} \\ \Rightarrow I &= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}}{1 + \sqrt{\tan x}} dx \quad \dots(ii) \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2I &= \int_{\pi/6}^{\pi/3} dx \\ \Rightarrow 2I &= [x]_{\pi/6}^{\pi/3} \\ \Rightarrow I &= \frac{1}{2} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{12} \end{aligned}$$

Hence, Statement I is false but Statement II $\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$ is a true statement by property of definite integrals.

$$21. \text{ Given integral is } g(x) = \int_0^x \cos 4t dt$$

$$\begin{aligned} \Rightarrow g(x + \pi) &= \int_0^{x+\pi} \cos 4t dt \\ &= \int_0^x \cos 4t dt + \int_x^{x+\pi} \cos 4t dt \\ &= g(x) + I_1 \quad [\text{say}] \end{aligned}$$

Now,

$$\begin{aligned} I_1 &= \int_x^{x+\pi} \cos 4t dt \\ &= \int_0^{\pi} \cos 4t dt \\ &\quad [\text{by definite integrals property}] \\ &= g(\pi) \end{aligned}$$

$$\Rightarrow g(x + \pi) = g(x) + g(\pi) \text{ but the value of } I_1 \text{ is zero.}$$

$$\Rightarrow I_1 = \left[\frac{\sin 4t}{4} \right]_0^{\pi} = \left(\frac{\sin 4\pi}{4} - \frac{\sin 0}{4} \right) = 0$$

$$\therefore g(x + \pi) = g(x) - g(\pi) \text{ or } g(x) + g(\pi)$$

22. Let $I = \int_0^1 \frac{8 \log(1+x)}{(1+x^2)} dx$

Put $x = \tan \theta$
 $\Rightarrow dx = \sec^2 \theta d\theta$

When $x = 0$

$\Rightarrow \tan \theta = 0$

$\therefore \theta = 0$

When $x = 1 = \tan \theta \Rightarrow \theta = \frac{\pi}{4}$

$\therefore I = \int_0^{\pi/4} \frac{8 \log(1+\tan \theta)}{(1+\tan^2 \theta)} \cdot \sec^2 \theta d\theta$

$\Rightarrow I = 8 \int_0^{\pi/4} \log(1+\tan \theta) d\theta$

$\Rightarrow I = 8 \int_0^{\pi/4} \log(1+\tan \theta) d\theta \quad \dots(i)$

$\Rightarrow I = 8 \int_0^{\pi/4} \log \left\{ 1 + \tan \left(\frac{\pi}{4} - \theta \right) \right\} d\theta$

$\therefore \int_0^a f(x) dx = \int_0^a f(a-x) dx$

$\Rightarrow I = 8 \int_0^{\pi/4} \log \left\{ 1 + \frac{1-\tan \theta}{1+\tan \theta} \right\} d\theta$

$\Rightarrow I = 8 \int_0^{\pi/4} \log \left\{ \frac{2}{1+\tan \theta} \right\} d\theta \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

$$2I = 8 \int_0^{\pi/4} \left[\log \{1 + \tan \theta\} + \log \left\{ \frac{2}{1 + \tan \theta} \right\} \right] d\theta$$

$\Rightarrow I = 4 \int_0^{\pi/4} \log 2 d\theta$

$= 4 \cdot \log 2 [\theta]_0^{\pi/4}$

$= 4 \log 2 \cdot \left(\frac{\pi}{4} - 0 \right) = \pi \log 2$

23. Here, $\int_0^{1.5} x [x^2] dx$

$= \int_0^1 x \cdot 0 dx + \int_1^{\sqrt{2}} x \cdot 1 dx + \int_{\sqrt{2}}^{1.5} x \cdot 2 dx$

$= 0 + \left[\frac{x^2}{2} \right]_1^{\sqrt{2}} + [x^2]_{\sqrt{2}}^{1.5}$

$= \frac{1}{2} \{2 - 1\} + \{(1.5)^2 - 2\}$

$= \frac{1}{2} + 2.25 - 2 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

24. We have, $p'(x) = p'(1-x), \forall x \in [0, 1]$

$p(0) = 1, p(1) = 41$

$\Rightarrow p(x) = -p(1-x) + C$

$\Rightarrow 1 = -41 + C \Rightarrow C = 42$

$\therefore p(x) + p(1-x) = 42$

Now, $I = \int_0^1 p(x) dx \quad \dots(i)$

$\Rightarrow I = \int_0^1 p(1-x) dx \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

$2I = \int_0^1 \{p(x) + p(1-x)\} dx$

$\Rightarrow 2I = \int_0^1 42 dx = 42$

$\therefore I = 21$

25. Let $I = \int_0^{\pi} [\cot x] dx \quad \dots(i)$

$\Rightarrow I = \int_0^{\pi} [\cot(\pi - x)] dx$

$\Rightarrow I = \int_0^{\pi} [-\cot x] dx \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{\pi} [\cot x] dx + \int_0^{\pi} [-\cot x] dx = \int_0^{\pi} [-1] dx$

$[\because [x] + [-x] = -1, \text{ if } x \notin \mathbb{Z} = 0, \text{ if } x \in \mathbb{Z}]$
 $= [-x]_0^{\pi} = -\pi$

$\therefore I = -\frac{\pi}{2}$

26. Since, $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx$, because in

$x \in (0, 1), x > \sin x$

$\therefore I < \int_0^1 \sqrt{x} dx = \frac{2}{3} [x^{3/2}]_0^1 \Rightarrow I < \frac{2}{3}$

and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 x^{-1/2} dx = 2$

$\therefore J < 2$

27. Since, $f(x) = \int_1^x \frac{\log t}{1+t} dt$

and $F(e) = f(e) + f\left(\frac{1}{e}\right)$

$\Rightarrow F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{1+t} dt$

Put $t = \frac{1}{t}$ in second integration, we get

$F(e) = \int_1^e \frac{\log t}{1+t} dt + \int_1^{1/e} \frac{\log t}{t(1+t)} dt$

$= \int_1^e \frac{\log t}{t} dt = \left[\frac{(\log t)^2}{2} \right]_1^e$

$= \frac{1}{2} [(\log e)^2 - (\log 1)^2] = \frac{1}{2}$

28. Let $I = \int_3^6 \frac{\sqrt{x}}{\sqrt{9-x} + \sqrt{x}} dx \quad \dots(i)$

$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{9-x} + \sqrt{9-x}} dx$

$\Rightarrow I = \int_3^6 \frac{\sqrt{9-x}}{\sqrt{x} + \sqrt{9-x}} dx \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

$2I = \int_3^6 \frac{\sqrt{x} + \sqrt{9-x}}{\sqrt{x} + \sqrt{9-x}} dx$

$\Rightarrow 2I = [x]_3^6 = 6 - 3 \Rightarrow I = \frac{3}{2}$

29. Let $I = \int_{-\pi/2}^{-\pi/2} [(x + \pi)^3 + \cos^2(x + 3\pi)] dx$

or $I = \int_{-\pi/2}^{-\pi/2} [(x + \pi)^3 + \cos^2 x] dx \quad \dots(i)$

$\Rightarrow I = \int_{-\pi/2}^{-\pi/2} \left[\left(-\frac{\pi}{2} - \frac{3\pi}{2} - x + \pi \right)^3 + \cos^2 \left(-\frac{\pi}{2} - \frac{3\pi}{2} - x \right) \right] dx$

$\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right]$

$$\Rightarrow I = \int_{-3\pi/2}^{-\pi/2} [-(x + \pi)^3 + \cos^2 x] dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2I &= \int_{-3\pi/2}^{-\pi/2} 2 \cos^2 x dx = \int_{-3\pi/2}^{-\pi/2} (1 + \cos 2x) dx \\ &= \left[x + \frac{\sin 2x}{2} \right]_{-3\pi/2}^{-\pi/2} \\ &= \left[-\frac{\pi}{2} + \frac{\sin(-\pi)}{2} - \left(-\frac{3\pi}{2} + \frac{\sin(-3\pi)}{2} \right) \right] \\ &= -\frac{\pi}{2} + \frac{3\pi}{2} = \pi \Rightarrow I = \frac{\pi}{2} \end{aligned}$$

$$30. \text{ Let } I = \int_0^\pi x f(\sin x) dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^\pi (\pi - x) f(\sin(\pi - x)) dx$$

$$\Rightarrow I = \int_0^\pi (\pi - x) f(\sin x) dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2I &= \int_0^\pi \pi f(\sin x) dx \\ \Rightarrow I &= \frac{\pi}{2} \int_0^\pi f(\sin x) dx \quad \dots(iii) \\ \Rightarrow I &= \pi \int_0^{\pi/2} f(\sin x) dx \end{aligned}$$

$$\text{Put } \frac{\pi}{2} - x = t$$

$$\Rightarrow x = \frac{\pi}{2} - t$$

On putting $dx = -dt$ in Eq. (iii), we get

$$\begin{aligned} I &= \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} f(\cos t) dt \\ &= \frac{\pi}{2} \int_{-\pi/2}^{\pi/2} f(\cos x) dx \\ &= \pi \int_0^{\pi/2} f(\cos x) dx \\ &\quad [\because f(\cos x) \text{ is an even function}] \end{aligned}$$

$$31. \text{ Since, } \int_1^a [x] f'(x) dx = \int_1^2 f'(x) dx + \int_2^3 2f'(x) dx + \dots + \int_{[a]}^a [a] f'(x) dx$$

$$\begin{aligned} &= [f(x)]_1^2 + 2[f(x)]_2^3 + \dots + [a][f(x)]_{[a]}^a \\ &= f(2) - f(1) + 2f(3) - 2f(2) + \dots + [a]f(a) - [a]f([a]) \\ &= [a]f(a) - \{f(1) + f(2) + \dots + f([a])\} \end{aligned}$$

32. Let

$$\begin{aligned} A &= \lim_{n \rightarrow \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \dots + \frac{n}{n^2} \sec^2 1 \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{1}{n} \sec^2 \left(\frac{1}{n} \right)^2 + \frac{2}{n} \sec^2 \left(\frac{2}{n} \right)^2 + \dots + \frac{n}{n} \sec^2 \left(\frac{n}{n} \right)^2 \right] \\ &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right) \sec^2 \left(\frac{r}{n} \right)^2 \\ \Rightarrow A &= \int_0^1 x \sec^2(x^2) dx \\ \text{Put } x^2 &= t \\ \Rightarrow x dx &= \frac{dt}{2} \\ \therefore A &= \frac{1}{2} \int_0^1 \sec^2 t dt \\ &= \frac{1}{2} [\tan t]_0^1 = \frac{1}{2} \tan 1 \end{aligned}$$

$$\begin{aligned} 33. \lim_{x \rightarrow 2} \int_6^{f(x)} \frac{4t^3}{x-2} dt \\ &= \lim_{x \rightarrow 2} \frac{4\{f(x)\}^3}{1} f'(x) \quad [\text{using L' Hospital's rule}] \\ &= 4\{f(2)\}^3 f'(2) = 4 \times (6)^3 \times \frac{1}{48} \\ &\quad \left[\because f(2) = 6, f'(2) = \frac{1}{48}, \text{ given} \right] \\ &= 18 \end{aligned}$$

34. According to the given condition,

$$\int_{\pi/4}^\beta f(x) dx = \beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2} \beta$$

On differentiating both sides w.r.t. β , we get

$$f(\beta) = \sin \beta + \beta \cos \beta - \frac{\pi}{4} \sin \beta + \sqrt{2}$$

$$\begin{aligned} \therefore f\left(\frac{\pi}{2}\right) &= 1 + 0 - \frac{\pi}{4} + \sqrt{2} \\ &= 1 - \frac{\pi}{4} + \sqrt{2} \end{aligned}$$

35. Given,

$$I_1 = \int_0^1 2^{x^2} dx$$

$$I_2 = \int_0^1 2^{x^3} dx$$

$$I_3 = \int_1^2 2^{x^2} dx$$

$$I_4 = \int_1^2 2^{x^3} dx$$

and

Since, $2^{x^3} < 2^{x^2}$, for $0 < x < 1$

and $2^{x^3} > 2^{x^2}$, for $x > 1$

$$\therefore \int_0^1 2^{x^3} dx < \int_0^1 2^{x^2} dx$$

$$\text{and } \int_1^2 2^{x^3} dx > \int_1^2 2^{x^2} dx$$

$$\Rightarrow \begin{aligned} I_2 &< I_1 \\ \text{and } I_4 &> I_3 \end{aligned}$$

$$36. \text{ Let } I = \int_{-\pi}^\pi \frac{\cos^2 x}{1+a^x} dx, a > 0 \quad \dots(i)$$

Put $x = -x$, we get

$$I = \int_{-\pi}^\pi \frac{\cos^2 x}{1+a^{-x}} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_{-\pi}^\pi \frac{(1+a^x)\cos^2 x}{(1+a^x)} dx$$

$$\Rightarrow 2I = \int_{-\pi}^\pi \cos^2 x dx$$

$$\Rightarrow 2I = \int_{-\pi}^\pi \left(\frac{\cos 2x + 1}{2} \right) dx$$

$$\Rightarrow 2I = \frac{1}{2} \left[\left(\frac{\sin 2x}{2} + x \right) \right]_{-\pi}^\pi$$

$$\Rightarrow 2I = \frac{1}{2} \left[\frac{\sin 2\pi}{2} + \pi - \left(\frac{\sin(-2\pi)}{2} - \pi \right) \right]$$

$$\Rightarrow 2I = \frac{1}{2} (\pi + \pi)$$

$$\Rightarrow I = \frac{\pi}{2}$$

$$\begin{aligned}
 37. \int_{-2}^3 |1-x^2| dx &= \int_{-2}^{-1} (x^2-1) dx + \int_{-1}^1 (1-x^2) dx \\
 &\quad + \int_1^3 (x^2-1) dx \\
 &= \left[\frac{x^3}{3} - x \right]_{-2}^{-1} + \left[x - \frac{x^3}{3} \right]_{-1}^1 + \left[\frac{x^3}{3} - x \right]_1^3 \\
 &= \left[-\frac{1}{3} + 1 + \frac{8}{3} - 2 \right] + \left[1 - \frac{1}{3} + 1 - \frac{1}{3} \right] + \left[9 - 3 - \frac{1}{3} + 1 \right] \\
 &= \frac{4}{3} + \frac{4}{3} + \frac{20}{3} = \frac{28}{3}
 \end{aligned}$$

$$\begin{aligned}
 38. \text{ Now, } \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{1+\sin 2x}} dx \\
 &= \int_0^{\pi/2} \frac{(\sin x + \cos x)^2}{\sqrt{(\sin x + \cos x)^2}} dx \\
 &= \int_0^{\pi/2} (\sin x + \cos x) dx \\
 &= [-\cos x + \sin x]_0^{\pi/2} \\
 &= -\cos \frac{\pi}{2} + \sin \frac{\pi}{2} + \cos 0 - \sin 0 \\
 &= -0 + 1 + 1 - 0 = 2
 \end{aligned}$$

$$39. \text{ Let } I = \int_0^{\pi} x f(\sin x) dx \quad \dots(i)$$

$$\therefore I = \int_0^{\pi} (\pi - x) f[\sin(\pi - x)] dx$$

$$\Rightarrow I = \int_0^{\pi} (\pi - x) f(\sin x) dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} (x + \pi - x) f(\sin x) dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} f(\sin x) dx \Rightarrow 2I = 2\pi \int_0^{\pi/2} f(\sin x) dx$$

$$\Rightarrow I = \pi \int_0^{\pi/2} f(\sin x) dx$$

$$\Rightarrow A \int_0^{\pi/2} f(\sin x) dx = \pi \int_0^{\pi/2} f(\sin x) dx$$

$$\left[\because I = A \int_0^{\pi/2} f(\sin x) dx, \text{ given} \right]$$

$$\Rightarrow A = \pi$$

$$40. \text{ Given, } f(x) = \frac{e^x}{1+e^x}$$

$$\therefore f(a) = \frac{e^a}{1+e^a} \quad \dots(i)$$

$$\text{and } f(-a) = \frac{e^{-a}}{1+e^{-a}} = \frac{1}{1+e^a} \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$f(a) + f(-a) = 1$$

$$\Rightarrow f(a) = 1 - f(-a)$$

$$\text{Let } f(-a) = t \Rightarrow f(a) = 1 - t$$

$$\text{Now } I_1 = \int_t^{1-t} xg[x(1-x)] dx \quad \dots(iii)$$

$$\Rightarrow I_1 = \int_t^{1-t} (1-x)g[x(1-x)] dx \quad \dots(iv)$$

On adding Eqs. (iii) and (iv), we get

$$2I_1 = \int_t^{1-t} g[x(1-x)] dx = I_2$$

$$\Rightarrow \frac{I_2}{I_1} = 2$$

$$41. \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{n} e^{r/n} = \int_0^1 e^x dx = [e^x]_0^1 = e - 1$$

$$42. \text{ Given, } f(y) = e^y, g(y) = y$$

$$\text{and } F(t) = \int_0^t f(t-y)g(y) dy$$

$$= \int_0^t e^{t-y} \cdot y dy = e^t \int_0^t e^{-y} y dy$$

$$= e^t \left\{ [-ye^{-y}]_0^t - \int_0^t 1(-e^{-y}) dy \right\}$$

$$= e^t \{-te^{-t} - 0\} - [e^{-y}]_0^t$$

$$= e^t \{-te^{-t} - e^{-t} + 1\}$$

$$\therefore F(t) = e^t - (1+t)$$

$$43. \text{ Let } I = \int_a^b x f(x) dx \quad \dots(i)$$

$$\Rightarrow I = \int_a^b (a+b-x)f(a+b-x) dx$$

$$\Rightarrow I = \int_a^b (a+b-x)f(x) dx$$

$$\Rightarrow I = (a+b) \int_a^b f(x) dx - \int_a^b x f(x) dx \quad [\because f(a+b-x) = f(x), \text{ given}]$$

$$\Rightarrow I = (a+b) \int_a^b f(x) dx - I \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 2I = (a+b) \int_a^b f(x) dx$$

$$\Rightarrow I = \left(\frac{a+b}{2} \right) \int_a^b f(x) dx$$

$$\begin{aligned}
 44. \lim_{x \rightarrow 0} \frac{\int_0^{x^2} \sec^2 t dt}{x \sin x} \\
 &= \lim_{x \rightarrow 0} \frac{\sec^2 x^2 \cdot 2x}{x \cos x + \sin x} \quad [\text{using L'Hospital's rule}] \\
 &= \lim_{x \rightarrow 0} \frac{2x \cdot 2 \sec^2 x^2 \cdot \tan x^2 \cdot 2x + 2 \sec^2 x^2}{-x \sin x + \cos x + \cos x} \\
 &\quad [\text{using L'Hospital's rule}] \\
 &= \frac{0 + 2 \sec^2 0}{0 + 2 \cos 0} = 1
 \end{aligned}$$

$$45. \text{ Given, } I = \int_0^1 x(1-x)^n dx$$

$$\text{Put } 1-x = z \Rightarrow -dx = dz$$

$$\therefore I = \int_1^0 (1-z)z^n(-dz)$$

$$= \int_0^1 (1-z)z^n dz = \int_0^1 (z^n - z^{n+1}) dz$$

$$= \left[\frac{z^{n+1}}{n+1} - \frac{z^{n+2}}{n+2} \right]_0^1 = \frac{1}{n+1} - \frac{1}{n+2}$$

$$\begin{aligned}
 46. \lim_{n \rightarrow \infty} \frac{1+2^4+3^4+\dots+n^4}{n^5} \\
 - \lim_{n \rightarrow \infty} \frac{1+2^3+3^3+\dots+n^3}{n^5} \\
 &= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^4 - \lim_{n \rightarrow \infty} \frac{1}{n} \times \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(\frac{r}{n} \right)^3 \\
 &= \int_0^1 x^4 dx - \lim_{n \rightarrow \infty} \frac{1}{n} \times \int_0^1 x^3 dx \\
 &= \left[\frac{x^5}{5} \right]_0^1 - 0 = \frac{1}{5}
 \end{aligned}$$

47. Given, $\frac{d}{dx} F(x) = \frac{e^{\sin x}}{x}$, $x > 0$

On integrating both sides, we get

$$f(x) = \int \frac{e^{\sin x}}{x} dx \quad \dots(i)$$

Also, $\int_1^4 \frac{3}{x} e^{\sin x^3} dx = \int_1^4 \frac{3x^2}{x^3} \cdot e^{\sin x^3} dx$

$$= F(k) - F(1)$$

Let $x^3 = z \Rightarrow 3x^2 dx = dz$

$$\int_1^{64} \frac{e^{\sin z}}{z} dz = F(k) - F(1)$$

$$\Rightarrow [F(z)]_1^{64} = F(k) - F(1) \quad [\text{from Eq. (i)}]$$

$$\Rightarrow F(64) - F(1) = F(k) - F(1)$$

$$\therefore k = 64$$

48. Given, $f'(x) = f(x)$ and $f(0) = 1$

Let $f(x) = e^x \quad \dots(i)$

Also, $f(x) + g(x) = x^2$

$$\Rightarrow g(x) = x^2 - e^x \quad \dots(ii)$$

Now, $\int_0^1 f(x)g(x) dx = \int_0^1 e^x (x^2 - e^x) dx$

[from Eqs. (i) and (ii)]

$$= \int_0^1 x^2 e^x dx - \int_0^1 e^{2x} dx$$

$$= [x^2 e^x - \int 2x e^x dx]_0^1 - \frac{1}{2} [e^{2x}]_0^1$$

$$= [x^2 e^x - 2x e^x + 2e^x]_0^1 - \frac{1}{2} (e^2 - 1)$$

$$= [(x^2 - 2x + 2)e^x]_0^1 - \frac{1}{2} (e^2 + 1)$$

$$= [(1 - 2 + 2)e^1 - (0 - 0 + 2)e^0] - \frac{1}{2} (e^2 + 1)$$

$$= e - 2 - \frac{e^2}{2} + \frac{1}{2}$$

$$= e - \frac{e^2}{2} - \frac{3}{2}$$

49. Since, $|\sin x|$ is a periodic function with period π .

$$\therefore \int_0^{10\pi} |\sin x| dx = 10 \int_0^\pi |\sin x| dx$$

$$= 10 \int_0^\pi \sin x dx$$

$$= 10 [-\cos x]_0^\pi$$

$$= 10 [-\cos \pi + \cos 0]$$

$$= 10 [1 + 1]$$

$$= 20$$

50. Since, $I_n = \int_0^{\pi/4} \tan^n x dx$

$$\therefore I_{n+2} = \int_0^{\pi/4} \tan^{n+2} x dx$$

Now,

$$I_n + I_{n+2} = \int_0^{\pi/4} \tan^n x dx + \int_0^{\pi/4} \tan^{n+2} x dx$$

$$= \int_0^{\pi/4} \tan^n x (1 + \tan^2 x) dx$$

$$= \int_0^{\pi/4} \sec^2 x \tan^n x dx$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$\therefore I_n + I_{n+2} = \int_0^1 t^n dt$$

$$= \left[\frac{t^{n+1}}{n+1} \right]_0^1 = \frac{1}{n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} n [I_n + I_{n+2}] = \lim_{n \rightarrow \infty} \frac{n}{n+1}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}} = 1$$

51. $\int_0^2 [x^2] dx = \int_0^1 [x^2] dx + \int_1^{\sqrt{2}} [x^2] dx + \int_{\sqrt{2}}^{\sqrt{3}} [x^2] dx + \int_{\sqrt{3}}^2 [x^2] dx$

$$= \int_0^1 0 dx + \int_1^{\sqrt{2}} 1 dx + \int_{\sqrt{2}}^{\sqrt{3}} 2 dx + \int_{\sqrt{3}}^2 3 dx$$

$$= [x]_1^{\sqrt{2}} + [2x]_{\sqrt{2}}^{\sqrt{3}} + [3x^2]_{\sqrt{3}}^2$$

$$= \sqrt{2} - 1 + 2\sqrt{3} - 2\sqrt{2} + 6 - 3\sqrt{3}$$

$$= 5 - \sqrt{3} - \sqrt{2}$$

52. Let $I = \int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$

$$= \int_{-\pi}^{\pi} \frac{2x}{1 + \cos^2 x} dx + \int_{-\pi}^{\pi} \frac{2x \sin x}{1 + \cos^2 x} dx$$

$$\Rightarrow I = 0 + 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \quad \dots(i)$$

$$\left[\because \frac{2x}{1 + \cos^2 x} \text{ is an odd function} \right]$$

$$\Rightarrow I = 4 \int_0^{\pi} \frac{\pi - x}{1 + \cos^2(\pi - x)} dx$$

$$\Rightarrow I = 4 \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - 4 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$$

$$\Rightarrow I = 4 \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx - I \quad [\text{from Eq. (i)}]$$

$$\Rightarrow I = 2 \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx$$

Put $\cos x = t$

$$\Rightarrow -\sin x dx = dt$$

$$\therefore I = -2\pi \int_1^{-1} \frac{1}{1 + t^2} dt$$

$$= 2\pi [\tan^{-1} t]_{-1}^1$$

$$= 2\pi \left[\frac{\pi}{4} + \frac{\pi}{4} \right] = \pi^2$$

53. Let

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin\left(\frac{\pi}{2} - x\right)}}{\cos\left(\frac{\pi}{2} - x\right) + \sqrt{\sin\left(\frac{\pi}{2} - x\right)}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} 1 dx = [x]_0^{\pi/2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

54. Given, $\Delta(x) = \begin{vmatrix} 1 & \cos x & 1 - \cos x \\ 1 + \sin x & \cos x & 1 + \sin x - \cos x \\ \sin x & \sin x & 1 \end{vmatrix}$

Applying $C_3 \rightarrow C_3 + C_2 - C_1$, we get

$$\Delta(x) = \begin{vmatrix} 1 & \cos x & 0 \\ 1 + \sin x & \cos x & 0 \\ \sin x & \sin x & 1 \end{vmatrix}$$

$$= \cos x - \cos x (1 + \sin x)$$

[expanding along C_3]

$$= -\cos x \sin x = -\frac{1}{2} \sin 2x$$

$$\therefore \int_0^{\pi/4} \Delta(x) dx = -\frac{1}{2} \int_0^{\pi/4} \sin 2x dx = -\frac{1}{2} \left[-\frac{\cos 2x}{2} \right]_0^{\pi/4}$$

$$= \frac{1}{2 \times 2} \left[\cos \frac{\pi}{2} - \cos 0 \right] = \frac{1}{4} (0 - 1) = -\frac{1}{4}$$

55. $\int_0^1 \frac{dx}{1+x+x^2} = \int_0^1 \frac{dx}{x^2+x+\frac{1}{4}-\frac{1}{4}+1}$

$$= \int_0^1 \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \left[\tan^{-1} \frac{\left(x+\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}} \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \left(\frac{3}{\sqrt{3}} \right) - \tan^{-1} \left(\frac{1}{\sqrt{3}} \right) \right]$$

$$= \frac{2}{\sqrt{3}} \left[\tan^{-1} \sqrt{3} - \tan^{-1} \frac{1}{\sqrt{3}} \right]$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{2\pi}{6\sqrt{3}} = \frac{\pi}{3\sqrt{3}}$$

56. $\int_0^{\pi/2} \left| \cos \left(\frac{x}{2} \right) \right| dx = \int_0^{\pi/2} \cos \left(\frac{x}{2} \right) dx$

$$\left[\because 0 < x < \frac{\pi}{2}, \left| \cos \left(\frac{x}{2} \right) \right| = \cos \left(\frac{x}{2} \right) \right]$$

$$= 2 \left[\sin \left(\frac{x}{2} \right) \right]_0^{\pi/2} = 2 \left[\sin \frac{\pi}{4} - \sin 0 \right]$$

$$= 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$$

57. We have, $I(m, n) = \int_0^1 t^m (1+t)^n dt$

$$\Rightarrow I(m, n) = \left[(1+t)^n \cdot \frac{t^{m+1}}{m+1} \right]_0^1 - \frac{n}{m+1} \int_0^1 (1+t)^{n-1} \cdot t^{m+1} dt$$

$$= \frac{2^n}{m+1} - \frac{n}{m+1} \cdot I(m+1, n-1)$$

58. Let $I = \int_0^1 \frac{dx}{e^x + e} = \int_0^1 \frac{dx}{e^x \left(1 + \frac{e}{e^x} \right)}$

Put $1 + \frac{e}{e^x} = t$

$$\Rightarrow 0 - \frac{e}{e^x} dx = dt$$

$$\Rightarrow \frac{1}{e^x} dx = -\frac{1}{e} dt$$

$$\therefore I = -\frac{1}{e} \int_{1+e}^2 \frac{1}{t} dt = -\frac{1}{e} [\log t]_{1+e}^2$$

$$= -\frac{1}{e} [\log 2 - \log (1+e)]$$

$$= -\frac{1}{e} \log \left(\frac{2}{1+e} \right) = \frac{1}{e} \log \left(\frac{1+e}{2} \right)$$

59. Let $I = \int_1^e \frac{1 + \log x}{3x} dx = \int_1^e \frac{1}{3x} dx + \frac{1}{3} \int_1^e \frac{\log x}{x} dx$

Put $\log x = t \Rightarrow \frac{1}{x} dx = dt$

$$\therefore I = \int_1^e \frac{1}{3x} dx + \frac{1}{3} \int_0^1 t dt = \left[\frac{1}{3} \log x \right]_1^e + \frac{1}{3} \left[\frac{t^2}{2} \right]_0^1$$

$$= \frac{1}{3} [\log e - \log 1] + \frac{1}{6} [1^2 - 0]$$

$$= \frac{1}{3} [1 - 0] + \frac{1}{6} = \frac{3}{6} = \frac{1}{2}$$

60. Let $I = \int_0^1 \frac{x^3}{1+x^8} dx$

Put $x^4 = t \Rightarrow 4x^3 dx = dt$

$$\therefore I = \int_0^1 \frac{1}{1+t^2} \cdot \frac{1}{4} dt = \frac{1}{4} [\tan^{-1} t]_0^1$$

$$= \frac{1}{4} [\tan^{-1} 1 - \tan^{-1} 0] = \frac{1}{4} \left[\frac{\pi}{4} - 0 \right] = \frac{\pi}{16}$$

61. Let $I = \int_2^4 \left(\frac{\log t}{t} \right) dt$

Put $\log t = z \Rightarrow \frac{1}{t} dt = dz$

$$\therefore I = \int_{\log 2}^{\log 4} z dz = \left[\frac{z^2}{2} \right]_{\log 2}^{\log 4} = \frac{1}{2} [(\log 4)^2 - (\log 2)^2]$$

$$= \frac{1}{2} [2 \log 2)^2 - (\log 2)^2] = \frac{3}{2} (\log 2)^2$$

62. We have,

$$\int_0^{\pi/3} \frac{\cos x}{3+4 \sin x} dx = k \log \left(\frac{3+2\sqrt{3}}{3} \right) \quad \dots(i)$$

Let $I = \int_0^{\pi/3} \frac{\cos x}{3+4 \sin x} dx$

Put $3+4 \sin x = t$

$$\Rightarrow 4 \cos x dx = dt$$

Upper limit,

$$x = \frac{\pi}{3}, t = 3 + 4 \sin \frac{\pi}{3}$$

$$= 3 + 4 \times \frac{\sqrt{3}}{2} = 3 + 2\sqrt{3}$$

and lower limit, $x = 0, t = 3 + 4 \sin 0 = 3$

$$I = \int_3^{3+2\sqrt{3}} \frac{dt}{4t} = \frac{1}{4} [\log t]_3^{3+2\sqrt{3}}$$

$$= \frac{1}{4} [\log (3+2\sqrt{3}) - \log 3]$$

$$= \frac{1}{4} \log \left(\frac{3+2\sqrt{3}}{3} \right) \quad \dots(ii)$$

From Eqs. (i) and (ii), $k = \frac{1}{4}$

$$\begin{aligned}
 63. \int x \tan^{-1} x \, dx &= \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{1+x^2} \times \frac{x^2}{2} \, dx \\
 &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{1+x^2-1}{1+x^2} \, dx \\
 &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \left[\int dx - \int \frac{dx}{1+x^2} \right] \\
 &= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} [x - \tan^{-1} x] \\
 &= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2}
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \int_0^1 x \tan^{-1} x \, dx &= \left[\frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{\tan^{-1} x}{2} \right]_0^1 \\
 &= \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8} = \frac{2\pi}{8} - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2}
 \end{aligned}$$

$$64. \text{ Let } I = \int_0^2 f'(x^2) 2x \, dx$$

$$\text{Put } x^2 = t \Rightarrow 2x \, dx = dt$$

$$\therefore I = \int_0^4 f'(t) \, dt = [f(t)]_0^4 = f(4) - f(0)$$

65. We know that, $x - [x]$ is periodic function with period one.

So, $e^{x-[x]}$ has period one.

Since, $f(x)$ is periodic with period T , then

$$\int_0^{nT} f(x) \, dx = n \int_0^T f(x) \, dx$$

$$\begin{aligned}
 \therefore \int_0^{25} e^{x-[x]} \, dx &= 25 \int_0^1 e^{x-[x]} \, dx = 25 \int_0^1 e^{x-0} \, dx \\
 &= 25 [e^x]_0^1 = 25 [e - 1]
 \end{aligned}$$

$$\Rightarrow 25(e - 1) = k(e - 1) \quad [\text{given}]$$

$$\therefore k = 25$$

$$\begin{aligned}
 66. \int_0^{2\pi} (\sin x + |\sin x|) \, dx &= \int_0^\pi (\sin x + \sin x) \, dx + \int_\pi^{2\pi} (\sin x - \sin x) \, dx \\
 &= 2 \int_0^\pi \sin x \, dx + 0 = -2 [\cos x]_0^\pi \\
 &= -2 (\cos \pi - \cos 0) \\
 &= -2 (-1 - 1) = 4
 \end{aligned}$$

$$\begin{aligned}
 67. \int_0^{\sqrt{2}} [x^2] \, dx &= \int_0^1 [x^2] \, dx + \int_1^{\sqrt{2}} [x^2] \, dx \\
 &= \int_0^1 0 \, dx + \int_1^{\sqrt{2}} 1 \, dx = [x]_1^{\sqrt{2}} = \sqrt{2} - 1
 \end{aligned}$$

$$\begin{aligned}
 68. \int_0^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] \, dx &= \int_0^{\pi/2} \sin \left[\frac{2x}{\pi} \right] \, dx + \int_{\pi/2}^\pi \sin \left[\frac{2x}{\pi} \right] \, dx + \int_\pi^{3\pi/2} \sin \left[\frac{2x}{\pi} \right] \, dx \\
 &= 0 + \sin 1 \int_{\pi/2}^\pi dx + \sin 2 \int_\pi^{3\pi/2} dx \\
 &= \frac{\pi}{2} (\sin 1 + \sin 2)
 \end{aligned}$$

$$\begin{aligned}
 69. \text{ Let } I &= \int_{1/n}^{(an-1)/n} \frac{\sqrt{x}}{\sqrt{a-x} + \sqrt{x}} \, dx \\
 \Rightarrow I &= \int_{1/n}^{a-1/n} \frac{\sqrt{x}}{\sqrt{a-x} + \sqrt{x}} \, dx \quad \dots(i)
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow I &= \int_{1/n}^{a-1/n} \frac{\left(\sqrt{\frac{1}{n} + a - \frac{1}{n} - x} \right)}{\sqrt{a - \left(\frac{1}{n} + a - \frac{1}{n} - x \right)} + \sqrt{\frac{1}{n} + a - \frac{1}{n} - x}} \, dx \\
 \Rightarrow I &= \int_{1/n}^{a-1/n} \frac{\sqrt{a-x}}{\sqrt{x} + \sqrt{a-x}} \, dx \quad \dots(ii)
 \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned}
 2I &= \int_{1/n}^{a-1/n} 1 \, dx = [x]_{1/n}^{a-1/n} \\
 &= a - \frac{1}{n} - \frac{1}{n} = \frac{na-2}{n} \\
 \Rightarrow I &= \frac{na-2}{2n}
 \end{aligned}$$

$$70. \text{ Let } I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \sin x \cos x} \, dx \quad \dots(i)$$

$$\begin{aligned}
 \Rightarrow I &= \int_0^{\pi/2} \frac{\sin \left(\frac{\pi}{2} - x \right) - \cos \left(\frac{\pi}{2} - x \right)}{1 + \sin \left(\frac{\pi}{2} - x \right) \cos \left(\frac{\pi}{2} - x \right)} \, dx \\
 &[\because \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx]
 \end{aligned}$$

$$= \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \cos x \sin x} \, dx \quad \dots(ii)$$

$$[\because \sin \left(\frac{\pi}{2} - x \right) = \cos x \text{ and } \cos \left(\frac{\pi}{2} - x \right) = \sin x]$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{0}{1 + \sin x \cos x} \, dx = 0 \Rightarrow I = 0$$

$$71. \text{ Given, } f(x) = f(\pi + e - x) \quad \dots(i)$$

$$\text{and } \int_e^\pi f(x) \, dx = \frac{2}{e + \pi} \quad \dots(ii)$$

$$\text{Let } I = \int_e^\pi x f(x) \, dx = \int_e^\pi (e + \pi - x) f(e + \pi - x) \, dx$$

$$\Rightarrow I = \int_e^\pi (e + \pi) f(e + \pi - x) \, dx - \int_e^\pi x f(e + \pi - x) \, dx$$

$$\Rightarrow I = \int_e^\pi (e + \pi) f(x) \, dx - \int_e^\pi x f(x) \, dx \quad [\text{from Eq. (i)}]$$

$$\Rightarrow I = (e + \pi) \int_e^\pi f(x) \, dx - I$$

$$\Rightarrow 2I = (e + \pi) \times \frac{2}{e + \pi}$$

$$\therefore I = 1 \quad [\text{from Eq. (ii)}]$$

$$72. \text{ Let } I = \int_{-\pi/4}^{\pi/4} \log (\sec \theta - \tan \theta) \, d\theta$$

Again, let $f(\theta) = \log (\sec \theta - \tan \theta)$

$$\therefore f(-\theta) = \log [\sec (-\theta) - \tan (-\theta)]$$

$$= \log \left[(\sec \theta + \tan \theta) \times \frac{\sec \theta - \tan \theta}{\sec \theta - \tan \theta} \right]$$

$$= \log \left[\frac{\sec^2 \theta - \tan^2 \theta}{\sec \theta - \tan \theta} \right] = \log \left[\frac{1}{\sec \theta - \tan \theta} \right]$$

$$= \log 1 - \log (\sec \theta - \tan \theta)$$

$$= 0 - \log (\sec \theta - \tan \theta)$$

$$\Rightarrow f(-\theta) = -f(\theta)$$

Hence, $f(\theta)$ is an odd function.

$$\therefore I = 0$$

73. Given, $M = \int_0^{\pi/2} \frac{\cos x}{(x+2)} dx$
 and $N = \int_0^{\pi/4} \frac{\sin x \cos x}{(x+1)^2} dx$
 $\Rightarrow N = \int_0^{\pi/4} \frac{1}{2} \cdot \frac{\sin 2x}{(x+1)^2} dx$
 Put $2x = t \Rightarrow dx = \frac{dt}{2}$
 $\therefore N = \int_0^{\pi/2} \frac{\sin t}{4 \left(\frac{t}{2} + 1\right)^2} dt = \int_0^{\pi/2} \frac{\sin t}{(t+2)^2} dt$
 $= \int_0^{\pi/2} \frac{\sin x}{(x+2)^2} dx$
 $\therefore M - N = \int_0^{\pi/2} \cos x \times \frac{1}{(x+2)} dx - \int_0^{\pi/2} \frac{\sin x}{(x+2)^2} dx$
 $= \left[\frac{\sin x}{x+2} \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin x}{(x+2)^2} dx - \int_0^{\pi/2} \frac{\sin x}{(x+2)^2} dx$
 $= \frac{\sin \frac{\pi}{2}}{\frac{\pi}{2} + 2} = \frac{1}{\frac{\pi}{2} + 2} = \frac{2}{\pi + 4}$

74. Given, $f(x) = \begin{cases} 2x^2 + 1, & x \leq 1 \\ 4x^3 - 1, & x > 1 \end{cases}$
 $\therefore \int_0^2 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx$
 $= \int_0^1 (2x^2 + 1) dx + \int_1^2 (4x^3 - 1) dx$
 $= \left[\frac{2x^3}{3} + x \right]_0^1 + \left[\frac{4x^4}{4} - x \right]_1^2$
 $= \frac{2}{3}(1)^3 + 1 - (0 + 0) + [(2)^4 - 2 - \{(1)^4 - 1\}]$
 $= \frac{2}{3} + 1 + [16 - 2 - 0] = \frac{2}{3} + 15 = \frac{2 + 45}{3} = \frac{47}{3}$

75. Given, $f(0) = 9$ and $f(x) = \frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)}$
 Now, $f(-x) = \frac{\sin\left(-\frac{9x}{2}\right)}{\sin\left(-\frac{x}{2}\right)} = \frac{-\sin\left(\frac{9x}{2}\right)}{-\sin\left(\frac{x}{2}\right)} = f(x)$
 So, $f(x)$ is an even function.
 $\therefore I = \frac{2}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{2}{\pi} \times 2 \int_0^{\pi} f(x) dx$
 $= \frac{4}{\pi} \int_0^{\pi} \frac{\sin\left(\frac{9x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx = \frac{4}{\pi} \int_0^{\pi} \frac{\sin\left(4x + \frac{x}{2}\right)}{\sin\left(\frac{x}{2}\right)} dx$
 $= \frac{4}{\pi} \int_0^{\pi} \left(\frac{\sin 4x \cos \frac{x}{2} + \sin \frac{x}{2} \cos 4x}{\sin \frac{x}{2}} \right) dx$
 $= \frac{4}{\pi} \left[\int_0^{\pi} \left(\sin 4x \cot \frac{x}{2} \right) dx + \int_0^{\pi} \cos 4x dx \right]$

$$= \frac{4}{\pi} \left[\int_0^{\pi} \sin 4x \cot \frac{x}{2} dx \right] + 0$$

$$\left[\begin{aligned} &\because \int_0^{\pi} \cos 4x dx = \left[\frac{\sin 4x}{4} \right]_0^{\pi} \\ &= \frac{1}{4} (\sin 4\pi - \sin 0) = \frac{1}{4} (0 - 0) = 0 \end{aligned} \right]$$

$$\therefore I = \frac{4}{\pi} \int_0^{\pi} \sin 4x \cot \frac{x}{2} dx \quad \dots(i)$$

$$\Rightarrow I = \frac{4}{\pi} \int_0^{\pi} \sin (4\pi - 4x) \cot \left(\frac{\pi - x}{2} \right) dx$$

$$\Rightarrow I = \frac{4}{\pi} \int_0^{\pi} -\sin 4x \tan \frac{x}{2} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \frac{4}{\pi} \left[\int_0^{\pi} \sin 4x \left(\cot \frac{x}{2} - \tan \frac{x}{2} \right) dx \right]$$

$$= \frac{4}{\pi} \left[\int_0^{\pi} \sin 4x \frac{\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2} \right)}{\sin \frac{x}{2} \cos \frac{x}{2}} dx \right]$$

$$= \frac{4}{\pi} \left[\int_0^{\pi} \frac{\sin 4x \cos x}{\frac{1}{2} \sin x} dx \right]$$

$$= \frac{8}{\pi} \left[\int_0^{\pi} \frac{2 \sin 2x \cos 2x \cos x}{\sin x} dx \right]$$

$$= \frac{16}{\pi} \left[\int_0^{\pi} \frac{2 \sin x \cos x \cos 2x \cos x}{\sin x} dx \right]$$

$$= \frac{32}{\pi} \int_0^{\pi} \cos^2 x \cos 2x dx$$

$$= \frac{32}{\pi} \int_0^{\pi} \left(\frac{\cos 2x + 1}{2} \right) \cos 2x dx$$

$$= \frac{16}{\pi} \left[\int_0^{\pi} \cos^2 2x dx + \int_0^{\pi} \cos 2x dx \right]$$

$$= \frac{16}{\pi} \left[\int_0^{\pi} \frac{\cos 4x + 1}{2} dx + \int_0^{\pi} \cos 2x dx \right]$$

$$= \frac{16}{\pi} \left\{ \frac{1}{2} \left[\frac{\sin 4x}{4} + x \right]_0^{\pi} + \left[\frac{\sin 2x}{2} \right]_0^{\pi} \right\}$$

$$= \frac{16}{\pi} \left\{ \frac{1}{2} [0 + \pi - 0 - 0] + \left[\frac{0 - 0}{2} \right] \right\} = \frac{16}{\pi} \left[\frac{\pi}{2} \right]$$

$$\therefore 2I = 8 \Rightarrow I = 4$$

76. Let $I = \frac{\pi^2}{10} \left[\int_{-10}^{10} f(x) \cos \pi x dx \right]$
 $= \frac{\pi^2}{10} \left[\int_{-10}^{-9} f(x) \cos \pi x dx + \int_{-9}^{-8} f(x) \cos \pi x dx \right.$
 $\left. + \dots + \int_9^{10} f(x) \cos \pi x dx \right]$

Again, let $I_1 = \int_{-10}^{-9} f(x) \cos \pi x dx$

For $-10 < x < -9$, $f(x)$ is an even function.

Put $[x] = -10$

$$\therefore I_1 = \int_{-10}^{-9} (1 - 10 - x) \cos \pi x dx$$

$$= - \int_{-10}^{-9} (9 + x) \cos \pi x dx$$

$$\begin{aligned}
 &= - \left[(9+x) \frac{\sin \pi x}{\pi} - \int \frac{1 \times \sin \pi x}{\pi} dx \right]_{-10}^9 \\
 &= - \left[(9+x) \frac{\sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_{-10}^9 \\
 &= - \left[0 + \frac{\cos(-9\pi)}{\pi^2} - \left\{ 0 + \frac{\cos(-10\pi)}{\pi^2} \right\} \right] \\
 &= - \left[-\frac{1}{\pi^2} - \frac{1}{\pi^2} \right] = \frac{2}{\pi^2}
 \end{aligned}$$

Similarly, $I_2 = \int_{-9}^{-8} \{x - (-9)\} \cos \pi x dx = \frac{2}{\pi^2}$.

$\therefore I = \frac{\pi^2}{10} \left[\left(\frac{2}{\pi^2} \right) \times 20 \right] = 4$

Since, in $\int_{-10}^{10}$, there are such 20 intervals

In all intervals, we get equal values, i.e. $\frac{2}{\pi^2}$

77. Let $I = \int_{-\pi/2}^{\pi/2} \cos x \ln \left(\frac{1+x}{1-x} \right) dx$

and $f(x) = \cos x \ln \left(\frac{1+x}{1-x} \right)$

Now, $f(-x) = \cos(-x) \ln \left(\frac{1-x}{1+x} \right)$

$$= \cos x \ln \left[\left(\frac{1+x}{1-x} \right)^{-1} \right] = -\cos x \ln \left(\frac{1+x}{1-x} \right) = -f(x)$$

$\Rightarrow f(x)$ is an odd function.

$$\therefore \int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \\ 0, & \text{if } f(x) \text{ is odd} \end{cases}$$

$\therefore I = 0$

78. We have, $\int_{-\pi}^{\pi} (\cos px - \sin qx)^2 dx$

$$= \int_{-\pi}^{\pi} (\cos^2 px + \sin^2 qx - 2 \cos px \sin qx) dx$$

$$[\because (a-b)^2 = a^2 + b^2 - 2ab]$$

$$= \int_{-\pi}^{\pi} (\cos^2 px + \sin^2 qx) dx - 2 \int_{-\pi}^{\pi} \cos px \sin qx dx$$

$$= 2 \int_0^{\pi} (\cos^2 px + \sin^2 qx) dx - 0$$

$$\left[\because \int_{-a}^a f(x) dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd} \\ 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even} \end{cases} \right]$$

$$= 2 \int_0^{\pi} \left[\left(\frac{1 + \cos 2px}{2} \right) + \left(\frac{1 - \cos 2qx}{2} \right) \right] dx$$

$$= \int_0^{\pi} (2 + \cos 2px - \cos 2qx) dx$$

$$= \left[2x + \frac{\sin 2px}{2} - \frac{\sin 2qx}{2} \right]_0^{\pi} = 2\pi - 0 = 2\pi$$

79. $\int_0^{\pi} \sqrt{\frac{1 + \cos 2x}{2}} dx = \int_0^{\pi} \sqrt{\frac{2 \cos^2 x}{2}} dx = \int_0^{\pi} \sqrt{\cos^2 x} dx$

$$= \int_0^{\pi} |\cos x| dx \quad [\because \sqrt{x^2} = |x|]$$

$$= \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{\pi} \cos x dx$$

$$\left[\because \cos x > 0 \text{ for } 0 < x < \frac{\pi}{2} \text{ and } \cos x < 0 \text{ for } \frac{\pi}{2} < x < \pi \right]$$

$$= [\sin x]_0^{\pi/2} - [\sin x]_{\pi/2}^{\pi} = [1 - 0] - [0 - 1] = 1 + 1 = 2$$

80. Let $I = \int_0^{\pi/2} \frac{dx}{1 + \tan^3 x}$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 x}{\cos^3 x + \sin^3 x} dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^3 \left(\frac{\pi}{2} - x \right)}{\cos^3 \left(\frac{\pi}{2} - x \right) + \sin^3 \left(\frac{\pi}{2} - x \right)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^3 x}{\sin^3 x + \cos^3 x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\cos^3 x + \sin^3 x}{\sin^3 x + \cos^3 x} dx = \int_0^{\pi/2} dx = [x]_0^{\pi/2} = \frac{\pi}{2}$$

$$\Rightarrow I = \frac{\pi}{4}$$

81. Let $I = \int_0^{\pi/6} \cos^4 3\theta \sin^2 6\theta d\theta$

Put $3\theta = t \Rightarrow d\theta = \frac{dt}{3}$

$$\begin{aligned}
 \therefore I &= \frac{1}{3} \int_0^{\pi/2} \cos^4 t \sin^2 2t dt \\
 &= \frac{1}{3} \int_0^{\pi/2} \cos^4 t \times (2 \sin t \cos t)^2 dt \\
 &= \frac{4}{3} \int_0^{\pi/2} \cos^6 t \sin^2 t dt = \frac{4}{3} \left[\frac{5 \cdot 3 \cdot 1}{8 \cdot 6 \cdot 4 \cdot 2} \times \frac{\pi}{2} \right] \\
 &= \frac{5\pi}{192}
 \end{aligned}$$

82. Let $I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x} dx \quad \dots(i)$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x \right) \sin \left(\frac{\pi}{2} - x \right) \cos \left(\frac{\pi}{2} - x \right)}{\cos^4 \left(\frac{\pi}{2} - x \right) + \sin^4 \left(\frac{\pi}{2} - x \right)} dx$$

$$\therefore I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x \right) \sin x \cos x}{\sin^4 x + \cos^4 x} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} dx$$

Put $\sin^2 x = t \Rightarrow 2 \sin x \cos x dx = dt$

$$\Rightarrow t = 0, \text{ when } x = 0 \text{ and } t = 1, \text{ when } x = \frac{\pi}{2}$$

$$\text{Now, } I = \frac{\pi}{4} \int_0^1 \frac{1}{2t^2 - 2t + 1} dt$$

$$= \frac{\pi}{8} \times \frac{1}{2} \int_0^1 \frac{dt}{\left(t - \frac{1}{2} \right)^2 + \left(\frac{1}{2} \right)^2} = \frac{\pi}{8} \left[\tan^{-1} \left(\frac{t - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1$$

$$= \frac{\pi}{8} [\tan^{-1}(1) - \tan^{-1}(-1)]$$

$$= \frac{\pi}{8} \cdot \left(\frac{\pi}{4} + \frac{\pi}{4} \right)$$

$$= \frac{\pi^2}{16}$$

83. $\int_{-\pi}^{\pi} \sin x [f(\cos x)] dx$

Let $F(x) = \sin x [f(\cos x)]$

Then, $F(-x) = \sin(-x) [f\{\cos(-x)\}] = -\sin x [f(\cos x)]$

$\therefore F(-x) = -F(x)$

$F(x) = \sin x [f(\cos x)]$ is an odd function.

$\Rightarrow I = \int_{-\pi}^{\pi} \sin x [f(\cos x)] dx = 0$

84. Let $I = \int_0^{2\pi} \frac{dx}{e^{\sin x} + 1}$... (i)

$\Rightarrow I = \int_0^{2\pi} \frac{dx}{e^{\sin(2\pi-x)} + 1} = \int_0^{2\pi} \frac{dx}{e^{-\sin x} + 1}$

$\Rightarrow I = \int_0^{2\pi} \frac{e^{\sin x}}{1 + e^{\sin x}} dx$... (ii)

On adding Eqs. (i) and (ii), we get

$2I = \int_0^{2\pi} \frac{(1 + e^{\sin x})}{(1 + e^{\sin x})} dx = \int_0^{2\pi} 1 dx = [x]_0^{2\pi} = 2\pi$

$\therefore I = \pi$

85. $\int_0^{2\pi} |\sin x| dx = \int_0^{\pi} \sin x dx + \int_{\pi}^{2\pi} (-\sin x) dx$

$= -[\cos x]_0^{\pi} + [\cos x]_{\pi}^{2\pi}$

$= -[\cos \pi - \cos 0] + [\cos 2\pi - \cos \pi]$

$= -[-1 - 1] + [1 + 1] = 2 + 2 = 4$

86. $\int_0^{11} [x] dx = \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx + \int_3^4 [x] dx$
 $+ \int_4^5 [x] dx + \int_5^6 [x] dx + \int_6^7 [x] dx + \int_7^8 [x] dx$
 $+ \int_8^9 [x] dx + \int_9^{10} [x] dx + \int_{10}^{11} [x] dx$
 $= \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx + \int_3^4 3 dx + \int_4^5 4 dx$
 $+ \int_5^6 5 dx + \int_6^7 6 dx + \int_7^8 7 dx + \int_8^9 8 dx$
 $+ \int_9^{10} 9 dx + \int_{10}^{11} 10 dx$
 $= [x]_0^1 + 2[x]_1^2 + 3[x]_2^3 + 4[x]_3^4 + 5[x]_4^5 + 6[x]_5^6$
 $+ 7[x]_6^7 + 8[x]_7^8 + 9[x]_8^9 + 10[x]_{10}^{11}$
 $= 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10 = 55$

87. $\int_{-2}^2 (1 + 2 \sin x) e^{|x|} dx = \int_{-2}^2 e^{|x|} dx + 2 \int_{-2}^2 \sin x e^{|x|} dx$
 $= 2 \int_0^2 e^x dx + 2 \cdot 0 = 2 [e^x]_0^2 = 2(e^2 - 1)$

[since, first function is even and second function is odd]

88. $\int_{-1}^1 (|x| - 2[x]) dx$

$= \int_{-1}^0 (|x| - 2[x]) dx + \int_0^1 (|x| - 2[x]) dx$

$= \int_{-1}^0 \{-x - 2(-1)\} dx + \int_0^1 \{x - 2(0)\} dx$

$= \int_{-1}^0 (-x + 2) dx + \int_0^1 x dx$

$= \left[-\frac{x^2}{2} + 2x \right]_{-1}^0 + \left[\frac{x^2}{2} \right]_0^1$

$= -\left(-\frac{1}{2} + 2(-1)\right) + \frac{1}{2}$

$= \frac{1}{2} + 2 + \frac{1}{2} = 1 + 2 = 3$

89. Let $I = \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{r^3}{r^4 + n^4} = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \frac{(r/n)^3}{(r/n)^4 + 1}$

Put $\frac{r}{n} = x \Rightarrow \frac{1}{n} = dx$

$\therefore I = \int_0^1 \frac{x^3}{1 + x^4} dx$
 $= \frac{1}{4} [\log(1 + x^4)]_0^1 = \frac{1}{4} \log_e 2$

90. $\int_{\pi/6}^{\pi/2} \left(\frac{1 + \sin 2x + \cos 2x}{\sin x + \cos x} \right) dx$
 $= \int_{\pi/6}^{\pi/2} \left(\frac{1 + 2 \sin x \cos x + 2 \cos^2 x - 1}{(\sin x + \cos x)} \right) dx$
 $= \int_{\pi/6}^{\pi/2} \frac{2 \cos x (\sin x + \cos x)}{(\sin x + \cos x)} dx$
 $= \int_{\pi/6}^{\pi/2} 2 \cos x dx = 2 [\sin x]_{\pi/6}^{\pi/2}$
 $= 2 \left(\sin \frac{\pi}{2} - \sin \frac{\pi}{6} \right) = 2 \left(1 - \frac{1}{2} \right) = 2 \times \frac{1}{2} = 1$

91. Given, $I_n = \int_0^{\pi/4} \tan^n x dx$, then $I_{10} = \int_0^{\pi/4} \tan^{10} x dx$

and $I_8 = \int_0^{\pi/4} \tan^8 x dx$

$\therefore I_{10} + I_8 = \int_0^{\pi/4} (\tan^{10} x + \tan^8 x) dx$

$= \int_0^{\pi/4} \tan^8 x (1 + \tan^2 x) dx$

$= \int_0^{\pi/4} \tan^8 x \sec^2 x dx$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$\Rightarrow t = 0$, when $x = 0$ and $t = 1$, when $x = \frac{\pi}{4}$

Now, $I_{10} + I_8 = \int_0^1 t^8 dt = \left[\frac{t^9}{9} \right]_0^1 = \frac{1}{9}$

92. $\int_{-2}^2 (x \cos x + \sin x + 1) dx$
 $= \int_{-2}^2 x \cos x dx + \int_{-2}^2 \sin x dx + \int_{-2}^2 1 dx$
 $= [x \sin x]_{-2}^2 - \int_{-2}^2 \sin x dx + \int_{-2}^2 \sin x dx + [x]_{-2}^2$
 $= 2 \sin 2 - 2 \sin 2 - \int_{-2}^2 \sin x dx + \int_{-2}^2 \sin x dx + (2 + 2)$
 $= 4$

93. $\int_0^4 |(x-1)| dx = \int_0^1 (1-x) dx + \int_1^4 (x-1) dx$
 $= \left[x - \frac{x^2}{2} \right]_0^1 + \left[\frac{x^2}{2} - x \right]_1^4$
 $= \left(1 - \frac{1}{2} \right) - 0 + \left(8 - 4 - \frac{1}{2} + 1 \right) = \frac{1}{2} + 5 - \frac{1}{2} = 5$

94. $I = \int_0^2 |x-2| dx + \int_0^2 [x] dx$
 $= -\int_0^2 (x-2) dx + \int_0^1 0 dx + \int_1^2 1 dx$
 $= -[2 - 4] + [2 - 1] = 3$

95. Since, period of $|\sin x|$ is π .

$\therefore I = \int_{\pi}^{16\pi} |\sin x| dx = 15 \int_0^{\pi} \sin x dx = 15 [-\cos x]_0^{\pi}$
 $= 15 (-\cos \pi + \cos 0) = 30$

96. Let $I = \int_0^{\pi} \sin^{50} x \cos^{49} x \, dx$... (i)

$$\Rightarrow I = \int_0^{\pi} \sin^{50} (\pi - x) \cos^{49} (\pi - x) \, dx$$

$$\Rightarrow I = \int_0^{\pi} -\sin^{50} x \cos^{49} x \, dx \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\pi} 0 \, dx$$

$$\therefore I = 0$$

97. Given, $I_1 = \int_0^{\pi/2} f(\sin 2x) \sin x \, dx$

$$\text{and } I_2 = \int_0^{\pi/4} f(\cos 2x) \cos x \, dx$$

$$\text{Put } x = \frac{\pi}{4} - t \Rightarrow dx = -dt$$

$$\Rightarrow t = \frac{\pi}{4}, \text{ when } x = 0 \text{ and } t = -\frac{\pi}{4}, \text{ when } x = \frac{\pi}{2}$$

$$\begin{aligned} \therefore I_1 &= \int_{\pi/4}^{-\pi/4} -f\left[\sin\left(\frac{\pi}{2} - 2t\right)\right] \cdot \sin\left(\frac{\pi}{4} - t\right) dt \\ &= \int_{-\pi/4}^{\pi/4} f(\cos 2t) \cdot \left(\frac{1}{\sqrt{2}} \cos t - \frac{1}{\sqrt{2}} \sin t\right) dt \\ &= \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f(\cos 2t) \cos t \, dt \\ &\quad - \frac{1}{\sqrt{2}} \int_{-\pi/4}^{\pi/4} f(\cos 2t) \sin t \, dt \end{aligned}$$

$$= 2 \times \frac{1}{\sqrt{2}} \int_0^{\pi/4} f(\cos 2t) \cos t \, dt$$

[since, $f(\cos 2t) \cos t$ is an even function and $f(\cos 2t) \sin t$ is an odd function]

$$\therefore I_1 = \sqrt{2} \int_0^{\pi/4} f(\cos 2x) \cos x \, dx = \sqrt{2} \cdot I_2$$

$$\Rightarrow I_1 / I_2 = \sqrt{2}$$

98. If $\frac{d}{dx} \phi(x) = f(x)$, then

$$\int f(x) \, dx = \phi(x) + C \text{ and } \int_1^2 f(x) \, dx = \phi(2) - \phi(1)$$

[by definition of definite integrals]

99. $\therefore f'(x) = g(x)$

$$\begin{aligned} \therefore \int_a^b f(x) g(x) \, dx &= \int_a^b f(x) f'(x) \, dx = \left[\frac{\{f(x)\}^2}{2} \right]_a^b \\ &= \frac{1}{2} [\{f(b)\}^2 - \{f(a)\}^2] \end{aligned}$$

100. Let $I = \int_a^b \sqrt{(x-a)(b-x)} \, dx$

$$\text{Put } x = a \cos^2 \theta + b \sin^2 \theta$$

$$dx = -2a \cos \theta \sin \theta + 2b \sin \theta \cos \theta \, d\theta$$

$$\Rightarrow dx = \sin 2\theta (b-a) \, d\theta$$

$$\begin{aligned} \therefore I &= \int_0^{\pi/2} \sqrt{(b-a) \sin^2 \theta (b-a) \cos^2 \theta} \\ &\quad (b-a) \sin 2\theta \, d\theta \end{aligned}$$

$$= \int_0^{\pi/2} \frac{(b-a)^2}{2} \sin^2 2\theta \, d\theta$$

$$= \frac{(b-a)^2}{2} \int_0^{\pi/2} \left[\frac{1 - \cos 4\theta}{2} \right] d\theta$$

$$= \frac{(b-a)^2}{4} \left[\theta - \frac{\sin 4\theta}{4} \right]_0^{\pi/2} = \frac{(b-a)^2}{4} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{\pi (b-a)^2}{8}$$

101. Using Walli's formula, we get

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \dots \frac{2}{3}, \text{ if } n \text{ is odd.}$$

$$\Rightarrow \int_0^{\pi/2} \sin^5 x \, dx = \frac{5-1}{5} \cdot \frac{5-3}{5-2} = \frac{4}{5} \cdot \frac{2}{3} = \frac{8}{15}$$

102. Let $f(x) = (e^{x^3} + e^{-x^3})(e^x - e^{-x})$

$$\Rightarrow f(-x) = (e^{-x^3} + e^{x^3})(e^{-x} - e^x)$$

$$= -(e^{x^3} + e^{-x^3})(e^x - e^{-x}) = -f(x)$$

So, $f(x)$ is an odd function.

$$\therefore \int_{-1}^1 f(x) \, dx = 0$$

103. $\int_0^{2a} f(x) \, dx = \int_0^a \{f(2a-x) + f(x)\} \, dx$

$$= \int_0^a f(2a-x) \, dx + \int_0^a f(x) \, dx = m + n$$

104. Let $I = \int_0^1 \frac{2 \sin^{-1} \frac{x}{2}}{x} \, dx$

$$\text{Put } \sin^{-1} \frac{x}{2} = t \Rightarrow \frac{x}{2} = \sin t$$

$$\Rightarrow x = 2 \sin t$$

$$\Rightarrow dx = 2 \cos t \, dt$$

$$\therefore I = \int_0^{\pi/6} \frac{2t}{2 \sin t} \cdot 2 \cos t \, dt$$

$$= \int_0^{\pi/6} \frac{2t}{\tan t} \, dt$$

$$= \int_0^{\pi/6} \frac{2x}{\tan x} \, dx$$

105. Let $I = \int_{-100}^{100} f(x) \, dx$

$$\text{Put } x = -t \Rightarrow dx = -dt$$

$$\therefore I = - \int_{100}^{-100} f(-t) \, dt$$

$$= \int_{-100}^{100} f(-t) \, dt$$

$$= \int_{-100}^{100} f(-x) \, dx$$

106. $\therefore \cos^2(\pi + x) = \cos^2 x$

$$\therefore I_1 = \int_0^{3\pi} f(\cos^2 x) \, dx$$

$$= 3 \int_0^{\pi} f(\cos^2 x) \, dx = 3 I_2$$

107. $\therefore x^2 < x^{\pi/2} < x$; for $0 < x < 1$

$$\therefore 1 + x^2 < 1 + x^{\pi/2} < 1 + x$$

$$\Rightarrow \frac{1}{1+x} < \frac{1}{1+x^{\pi/2}} < \frac{1}{1+x^2}$$

$$\Rightarrow \int_0^1 \frac{dx}{1+x} < \int_0^1 \frac{1}{1+x^{\pi/2}} \, dx < \int_0^1 \frac{dx}{1+x^2}$$

$$\Rightarrow [\log_e(1+x)]_0^1 < I < [\tan^{-1} x]_0^1$$

$$\Rightarrow \log_e 2 < I < \frac{\pi}{4}$$

108. $\int_{-\pi/2}^{\pi/2} |\sin x| \, dx = 2 \int_0^{\pi/2} \sin x \, dx = -2 [\cos x]_0^{\pi/2}$

$$= -2 \left(\cos \frac{\pi}{2} - \cos 0 \right) = 2$$

11

Area Bounded by Curves

Introduction

Continuous curve which is bounded by under the certain condition, the space which is occupied, is called area bounded by curves.

Curve Sketching

For the evaluation of area of bounded regions, it is very essential to know the rough sketch of the curves. The following points are very useful to draw a rough sketch of a curve.

(i) Symmetry

- (a) **Symmetric about X-axis** If all powers of y in the equation of the given curve are even, then it is symmetric about X -axis i.e. the shape of the curve above X -axis is exactly identical to its shape below X -axis.
e.g. $y^2 = 4ax$ is symmetric about X -axis.
- (b) **Symmetric about Y-axis** If all powers of x in the equation of the given curve are even, then it is symmetric about Y -axis.
e.g. $x^2 = 4ay$ is symmetric about Y -axis.
- (c) **Symmetric in opposite quadrants** By putting $-x$ for x and $-y$ for y , if the equation of curve remains same, then it is symmetric in opposite quadrants.
e.g. $xy = c^2$, $x^2 + y^2 = a^2$ are symmetric in opposite quadrants.
- (d) **Symmetric about the line $y = x$** If the equation of a given curve remains unaltered by interchanging x and y , then it is symmetric about the line $y = x$ which passes through the origin and makes an angle of 45° with positive direction of X -axis.

(ii) Origin and Tangents at the Origin

See whether the curve passes through the origin or not. If the point $(0, 0)$ satisfies the equation of the curve, then it passes through the origin and in such a case to find the equations of the tangents at the origin, equate the lowest degree term to zero.

Chapter Snapshot

- Introduction
- Curve Sketching
- Area Bounded by Curves
- Formulae for Finding the Area under the Curves
- Area of Some Important Curves
- Variable Area, Greatest and Least Values

e.g. $y^2 = 4ax$ passes through the origin. The lowest degree term in this equation is $4ax$. Equating $4ax$ to zero, we get $x = 0$. So, $x = 0$ i.e. Y -axis is tangent at the origin to $y^2 = 4ax$.

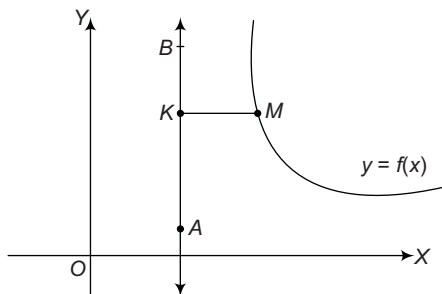
(iii) Points of Intersection of the Curve with the Coordinate Axes

By putting $y = 0$ in the equation of the given curve, find points where the curve crosses the X -axis. Similarly, by putting $x = 0$ in the equation of the given curve, we can find points where the curve crosses the Y -axis.

e.g. To find the points where the curve $xy^2 = 4a^2(2a - x)$ meets X -axis, we put $y = 0$ in the equation which gives $4a^2(2a - x) = 0$ or $x = 2a$. So, the curve $xy^2 = 4a^2(2a - x)$ meets X -axis at $(2a, 0)$. This curve does not intersect Y -axis because by putting $x = 0$ in the equation of the given curve, we get an absurd result.

(iv) Asymptotes

The straight line AB is called the asymptote of curve $y = f(x)$, if the distance MK from M , a point on the curve $y = f(x)$ to the straight line AB tends to zero as M recedes infinity. In other words, the straight line AB meets the curve $y = f(x)$ at infinity (where, K is a point on AB).



- (a) If $f(x) \rightarrow \pm \infty$ for $x \rightarrow a$, then the straight line $x = a$ is the asymptote of the curve $y = f(x)$.
- (b) If in the right hand member of the equation of the curve $y = f(x)$, it is possible to single out a linear part, so that the remaining part tends to zero as $x \rightarrow \pm \infty$, i.e. if $y = f(x) = Kx + b + g(x)$ and $g(x) \rightarrow 0$ for $x \rightarrow \pm \infty$, then the straight line $y = Kx + b$ is the asymptote of the curve.

- (c) If there exist finite limits, $\lim_{x \rightarrow \pm \infty} \frac{f(x)}{x} = K$ and $\lim_{x \rightarrow \pm \infty} [f(x) - Kx] = b$, then the straight line $y = Kx + b$ is the asymptote of the curve.

(v) Regions where the Curve does not Exist

Determine the regions in which the curve does not exist. For this, find the value of y in terms of x from the equation of the curve and find the value of x for which y is imaginary. Similarly, find the value of x in terms of y and determine the values of y for which x is imaginary. The curve does not exist for these values of x and y .

e.g. The values of y obtained from $y^2 = 4ax$ are imaginary for negative values of x . So, the curve does not exist on the left side of Y -axis. Similarly, the curve $a^2 y^2 = x^2(a - x)$ does not exist for $x > a$ as the values of y are imaginary for $x > a$.

(vi) Special Points

Find the points at which $\frac{dy}{dx} = 0$. At these points, the tangent to the curve is parallel to X -axis.

Find the points at which $\frac{dx}{dy} = 0$. At these points, the tangent to the curve is parallel to Y -axis.

(vii) Sign of $\frac{dy}{dx}$ and Points of Maxima and Minima

Find the interval in which $\frac{dy}{dx} > 0$. In this interval, the function is monotonically increasing, find the interval in which $\frac{dy}{dx} < 0$. In this interval, the function is monotonically decreasing. Put $\frac{dy}{dx} = 0$ and check the sign of $\frac{d^2y}{dx^2}$ at the points so obtained, to find the points of maxima and minima.

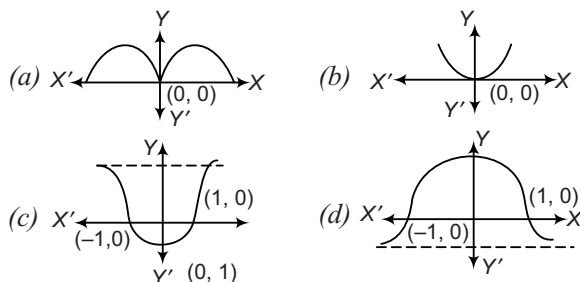
Keeping the above facts in mind and plotting some points on the curve, one can easily have a rough sketch of the curve.

- While constructing the graphs of functions, it is expedient to follow the procedure given below:
 - > Find the domain of definition of the function.
 - > Determine the odd-even nature of the function.
 - > Find the period of the function, if it is periodic.
 - > Find the asymptotes of the function.
 - > Check the behaviour of the function for $x \rightarrow 0$.
 - > Find the value of x , if possible for which $f(x) \rightarrow 0$.

- The interval of the function increases and decreases in its range. Hence, determine the greatest and the least values of the function, if any.
- If function is periodic and we can find its period, then plot curve for the interval equal to one period and repeat it.
- If $f(a)f(b) < 0$, curve intersects X-axis atleast once. Similarly, if $f(a)f(b) > 0$, curve intersects X-axis even number of times.

➤ **Example 1.** The graph of the function

$$f(x) = \frac{x^2 - 1}{x^2 + 1} \text{ is}$$



Sol. (c) Here, $f(x) = \frac{x^2 - 1}{x^2 + 1}$ [say]

Symmetry On putting $x = -x$, the graph is unaltered.
 $\therefore$ Graph is symmetrical about Y-axis.

Passes through the origin Since, $(0, 0)$ does not satisfy the given curve.

$\therefore$ Graph is not passing through origin.

Intersection with coordinate axes Put $x = 0$, then $y = -1$ and put $y = 0$, then $x = \pm 1$.

$\therefore$ Graph cuts Y-axis at $(0, -1)$ and X-axis at $(1, 0)$ and $(-1, 0)$.

Asymptotes $y = \frac{x^2 - 1}{x^2 + 1}$

Here, no asymptotes is parallel to Y-axis.

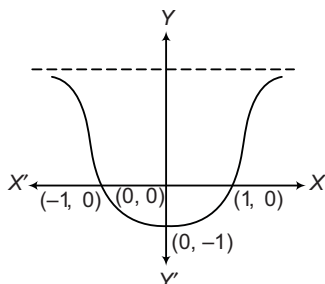
[since, $x^2 + 1 \neq 0$ for any value of x]

Now, $x^2 = \frac{y+1}{1-y}$

$\Rightarrow 1 - y = 0$ at $y = 1$

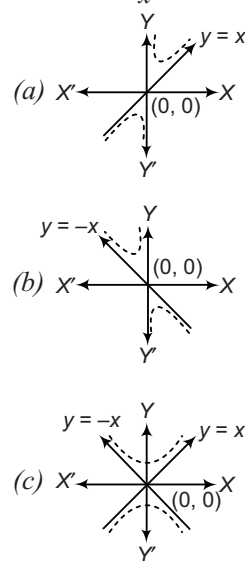
So, $y = 1$ is an asymptote parallel to X-axis.

Hence, required graph is given below :



➤ **Example 2.** The graph of the function

$$f(x) = x + \frac{1}{x} \text{ is}$$



(d) None of the above

Sol. (a) The function is defined for all x except for $x = 0$.

It is an odd function for $x \neq 0$.

It is not a periodic function.

For $x \rightarrow 0^+$, $f(x) \rightarrow \infty$; for $x \rightarrow 0^-$, $f(x) \rightarrow -\infty$

For $x \rightarrow -\infty$, $f(x) \rightarrow -\infty$; for $x \rightarrow \infty$, $f(x) \rightarrow \infty$

$\therefore \lim_{x \rightarrow \pm \infty} \{f(x) - x\} = 0$

So, the straight lines $x = 0$ and $y = x$ are the asymptotes of the graph of the given function.

Now, consider $f(x_2) - f(x_1)$

$$= (x_2 - x_1) + \frac{1}{x_2} - \frac{1}{x_1} \text{ for } x_2 > x_1$$

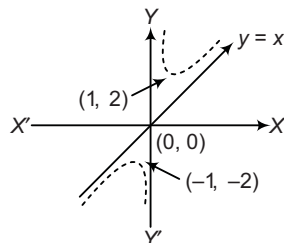
$$= (x_2 - x_1) \left[1 - \frac{1}{x_1 x_2} \right] < 0, \text{ for } x_1 x_2 \in (0, 1]$$

$$\text{or } (x_2 - x_1) \left[1 - \frac{1}{x_1 x_2} \right] > 0 \text{ for } x_1 x_2 \in [1, \infty)$$

Thus, $f(x)$ increases for $x \in [1, \infty)$ and decreases for $x \in (0, 1]$.

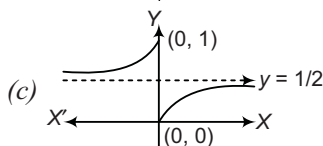
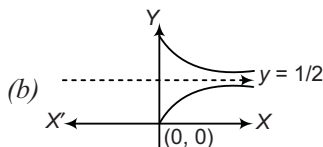
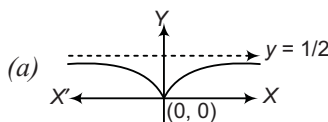
Therefore, the least value of the function is at $x = 1$ which is $f(1) = 2$.

Thus, its graph can be drawn as shown in figure :



➤ **Example 3.** The graph of the function

$$f(x) = \frac{1}{1 + e^{1/x}} \text{ is}$$



(d) None of the above

Sol. (c) The function is defined for all x except for $x = 0$. It is neither even nor an odd function. It is not a periodic function.

For $x \rightarrow 0^+$, $f(x) \rightarrow 0$; for $x \rightarrow 0^-$, $f(x) \rightarrow 1$

For $x \rightarrow \infty$, $f(x) \rightarrow \frac{1}{2}$; for $x \rightarrow -\infty$, $f(x) \rightarrow \frac{1}{2}$

$$\therefore \lim_{x \rightarrow \pm \infty} f(x) = \frac{1}{2}$$

So, the straight line $y = \frac{1}{2}$ is asymptote of the graph of the given function.

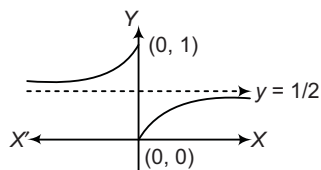
As x increases from $(0, \infty)$, $\frac{1}{x}$ decreases from $(0, \infty)$ and $e^{1/x}$ decreases from $(0, \infty)$. Thus, $(1 + e^{1/x})$ decreases from $(2, \infty)$.

$\therefore f(x)$ increases from $\left(0, \frac{1}{2}\right)$, for $x \in (0, \infty)$.

Similarly, $f(x)$ increases from $\left(\frac{1}{2}, 1\right)$, for $x \rightarrow (-\infty, 0)$.

i.e. $f(x)$ is an increasing function except for $x = 0$.

Thus, its graph can be drawn as shown in figure :



➤ **Example 4.** The asymptote of $y = e^{-x}$ is

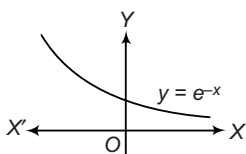
(a) $y = 0$

(b) $x = 0$

(c) $y = x$

(d) None of these

Sol. (a) $\lim_{x \rightarrow \infty} y = \lim_{x \rightarrow \infty} e^{-x} = 0$



Hence, $y = 0$ is an asymptote.

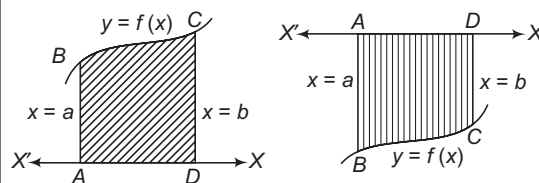
Area Bounded by Curves

Let $f(x)$ be a continuous non-negative function in the interval $[a, b]$. The area of the region bounded by the curve $y = f(x)$, X -axis and the lines $x = a$, $x = b$, is given by $\int_a^b f(x) dx$.

Formulae for Finding the Area under the Curves

i. Area of $ABCD$ bounded by the curve $y = f(x)$, X -axis and two ordinates $x = a$, $x = b$, is given by

$$\int_a^b |y| dx = \begin{cases} \int_a^b y dx, & \text{if } y \geq 0 \text{ for } x \in [a, b] \\ -\int_a^b y dx, & \text{if } y \leq 0 \text{ for } x \in [a, b] \end{cases}$$

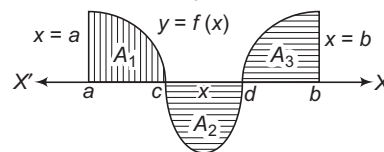


If however y i.e. $f(x)$ changes sign in the interval $[a, b]$, say $y \geq 0$ in $[a, c]$, $y \leq 0$ in $[c, d]$ and $y \geq 0$ in $[d, b]$, where $a < c < d < b$, then area bounded by the curve $y = f(x)$, X -axis and the lines $x = a$, $x = b$ is given by

$$\int_a^b |y| dx = \int_a^c y dx - \int_c^d y dx + \int_d^b y dx$$

$$= A_1 - A_2 + A_3,$$

where A_1 , A_2 and A_3 are algebraic areas.

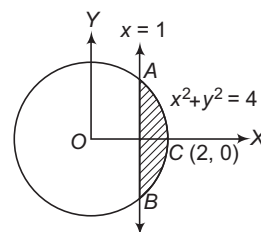


➤ **Example 5.** The area of smaller portion of the circle $x^2 + y^2 = 4$ cut-off by the line $x = 1$, is

(a) $\frac{4\pi + 3\sqrt{3}}{3}$ sq units (b) $\frac{4\pi}{3}$ sq units

(c) $\frac{4\pi - 3\sqrt{3}}{3}$ sq units (d) None of these

Sol. (c) Given equation of the circle is $x^2 + y^2 = 4$ and equation of the line is $x = 1$.



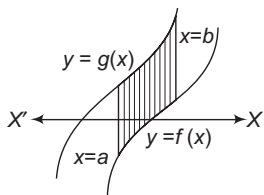
∴ Required area = Area of region ABCA

$$\begin{aligned}
 &= 2 \int_1^2 y \, dx = 2 \int_1^2 \sqrt{4 - x^2} \, dx \\
 &= 2 \left[\frac{x\sqrt{4 - x^2}}{2} + \frac{2^2 \sin^{-1} \frac{x}{2}}{2} \right]_1^2 \\
 &= 0 + 2 \sin^{-1} 1 - 2 \left[\frac{\sqrt{3}}{2} + 2 \sin^{-1} \left(\frac{1}{2} \right) \right] \\
 &= \frac{4\pi - 3\sqrt{3}}{3} \text{ sq units}
 \end{aligned}$$

ii. Area of ABCDA bounded by two curves $y = f(x)$, $y = g(x)$ and two ordinates $x = a$, $x = b$, is given by

$$\begin{aligned}
 &\int_a^b |f(x) - g(x)| \, dx \\
 &= \begin{cases} \int_a^b \{f(x) - g(x)\} \, dx, & \text{if } f(x) \geq g(x) \\ -\int_a^b \{f(x) - g(x)\} \, dx, & \text{if } f(x) \leq g(x) \end{cases} \quad \text{for } a \leq x \leq b
 \end{aligned}$$

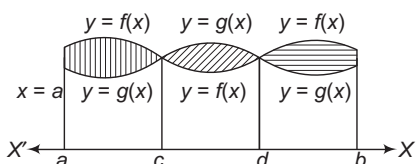
While using this formula, $f(x)$ is taken from the curve which lies above and $g(x)$ is taken from the curve which lies below



If $a < c < d < b$ and $f(x) \geq g(x)$ for $a \leq x \leq c$

$f(x) \leq g(x)$ for $c \leq x \leq d$

$f(x) \geq g(x)$ for $d \leq x \leq b$



Then, shaded area

$$\begin{aligned}
 &= \int_a^c \{f(x) - g(x)\} \, dx + \int_c^d \{g(x) - f(x)\} \, dx \\
 &\quad + \int_d^b \{f(x) - g(x)\} \, dx \\
 &= \int_a^c \{f(x) - g(x)\} \, dx - \int_c^d \{f(x) - g(x)\} \, dx \\
 &\quad + \int_d^b \{f(x) - g(x)\} \, dx
 \end{aligned}$$

Example 6. The area included between the line $y = x$ and the parabola $x^2 = 4y$ is

- (a) $8/3$ sq units (b) $16/3$ sq units
(c) $4/3$ sq units (d) None of these

Sol. (a) Given equation of parabola is $x^2 = 4y$ and equation of line is $y = x$.

On solving, we get $x^2 = 4$

$$\text{or } x(x - 4) = 0$$

$$\therefore x = 0, 4$$

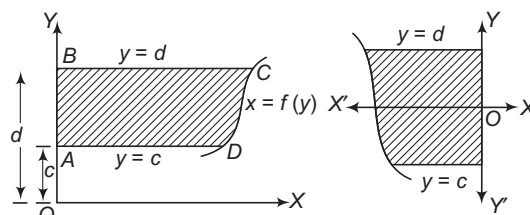
So, line $y = x$ cuts parabola at two points O and B, x-coordinate of O is 0 and x-coordinate of B is 4.

∴ Required area = Area of region OCBAB

$$\begin{aligned}
 &= \int_0^4 (y_1 - y_2) \, dx = \int_0^4 \left(x - \frac{x^2}{4} \right) \, dx \\
 &= \left[\frac{x^2}{2} - \frac{x^3}{12} \right]_0^4 = \left[\frac{16}{2} - \frac{64}{12} \right] = \frac{8}{3} \text{ sq units}
 \end{aligned}$$

iii. Area of ABCDA enclosed by the curve $x = f(y)$, Y-axis and two abscissae $y = c$, $y = d$, is given by

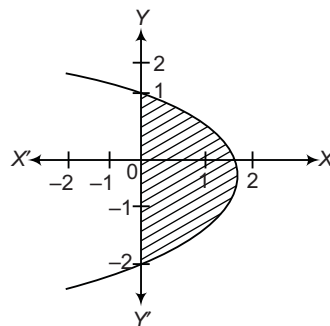
$$\int_c^d |x| \, dy = \begin{cases} \int_c^d x \, dy, & \text{if } x \geq 0 \text{ for } c \leq y \leq d \\ -\int_c^d x \, dy, & \text{if } x \leq 0 \text{ for } c \leq y \leq d \end{cases}$$



Example 7. The area bounded by the curve $x = 2 - y - y^2$ and Y-axis, is

- (a) $\frac{9}{7}$ sq units (b) $\frac{9}{2}$ sq units
(c) 9 sq units (d) 3 sq units

Sol. (b) Required area = $\int_{-2}^1 x \, dy$

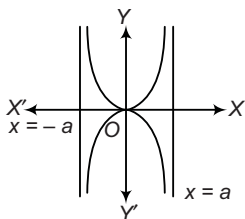


$$= \int_{-2}^1 (2 - y - y^2) \, dy = \left[2y - \frac{y^2}{2} - \frac{y^3}{3} \right]_{-2}^1 = \frac{9}{2} \text{ sq units}$$

Example 8. The whole area included between the curve $x^2 y^2 = a^2 (y^2 - x^2)$ and its asymptotes, is

- (a) $4\pi a^2$ sq units (b) $4a^2$ sq units
(c) $2a^2$ sq units (d) None of these

Sol. (b) Given curve is $x^2y^2 = a^2(y^2 - x^2)$ which is symmetrical about both the axes. Its asymptotes are $x = \pm a$.



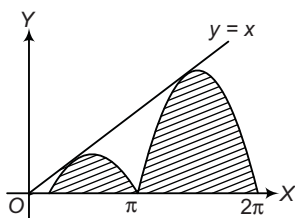
$$\therefore \text{Required area, } A = 4 \int_0^a y dx = 4 \int_0^a \frac{ax}{\sqrt{a^2 - x^2}} dx$$

$$\begin{aligned} \text{Put } x &= a \sin \theta \Rightarrow dx = a \cos \theta d\theta \\ \therefore A &= 4a^2 \int_0^{\pi/2} \frac{\sin \theta}{a \sqrt{1 - \sin^2 \theta}} a \cos \theta d\theta \\ &= 4a^2 \int_0^{\pi/2} \sin \theta d\theta = 4a^2 [-\cos \theta]_0^{\pi/2} = 4a^2 \text{ sq units} \end{aligned}$$

➤ **Example 9.** The area bounded by $y = x |\sin x|$ and X -axis between $x = 0, x = 2\pi$, is

- (a) 2π sq units (b) 3π sq units
(c) π sq units (d) 4π sq units

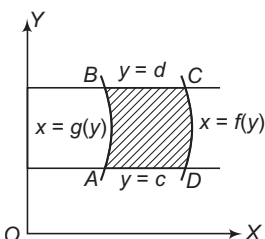
Sol. (d) $y = \begin{cases} x \sin x, & \text{if } \sin x \geq 0 \text{ i.e. } 0 \leq x \leq \pi \\ -x \sin x, & \text{if } \sin x < 0 \text{ i.e. } \pi < x \leq 2\pi \end{cases}$



$$\begin{aligned} \therefore \text{Required area} &= \int_0^\pi x \sin x dx + \int_\pi^{2\pi} (-x \sin x) dx \\ &= [x(-\cos x)]_0^\pi - \int_0^\pi (-\cos x) dx - [x(-\cos x)]_\pi^{2\pi} \\ &\quad + \int_\pi^{2\pi} (-\cos x) dx \\ &= [\pi + \sin x]_0^\pi + [(2\pi + \pi) - \sin x]_\pi^{2\pi} = 4\pi \text{ sq units} \end{aligned}$$

iv. Area of ABCDA bounded by the two curves $x = f(y), x = g(y)$ and two abscissae $y = c, y = a$, is given by

$$\begin{aligned} &\int_c^d |x_1 - x_2| dy \\ &= \begin{cases} \int_c^d (x_1 - x_2) dy, & \text{if } x_1 \geq x_2 \text{ for } c \leq y \leq d \\ -\int_c^d (x_1 - x_2) dy, & \text{if } x_1 \leq x_2 \text{ for } c \leq y \leq d \end{cases} \end{aligned}$$



➤ **Example 10.** The area enclosed by the two curves given by $y^2 = x + 1$ and $y^2 = -x + 1$, is

- (a) $\frac{8}{3}$ sq units (b) 4 sq units
(c) 3 sq units (d) None of these

Sol. (a) Given curves are

$$y^2 = x + 1 \quad \dots (i)$$

$$\text{and } y^2 = -x + 1 \quad \dots (ii)$$

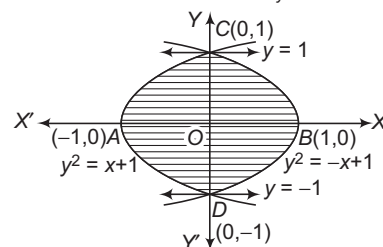
Curve (i) is the parabola having axis $y = 0$ and vertex $(-1, 0)$.

Curve (ii) is the parabola having axis $y = 0$ and vertex $(1, 0)$.

On subtracting Eq. (i) from Eq. (ii), we get
 $2x = 0 \Rightarrow x = 0$

From Eq. (i),

$$x = 0 \Rightarrow y = \pm 1$$



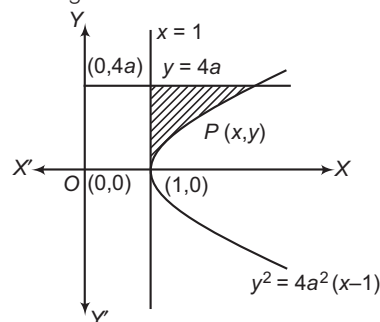
$$\begin{aligned} \therefore \text{Required area} &= \int_{-1}^1 (x_1 - x_2) dy \\ &= \int_{-1}^1 [(1 - y^2) - (y^2 - 1)] dy \\ &= 2 \int_{-1}^1 (1 - y^2) dy = 2 \left[y - \frac{y^3}{3} \right]_{-1}^1 \\ &= 2 \left[\left(1 - \frac{1}{3}\right) - \left(-1 + \frac{1}{3}\right) \right] = \frac{8}{3} \text{ sq units} \end{aligned}$$

➤ **Example 11.** The area of the region bounded by the curves $y^2 = 4a^2(x - 1)$ and the lines $x = 1, y = 4a$, is

- (a) $\frac{21a}{2}$ sq units (b) $\frac{16}{3}$ sq units
(c) $\frac{17a}{3}$ sq units (d) $\frac{16a}{3}$ sq units

Sol. (d) We have, $y^2 = 4a^2(x - 1) \Rightarrow (y - 0)^2 = 4a^2(x - 1)$

Clearly, this equation represents a parabola with vertex at $(1, 0)$ as shown in figure. The area enclosed by $y^2 = 4a^2(x - 1), x = 1$ and $y = 4a$ is the area of shaded portion in figure.



When we slice the area of the shaded portion in horizontal strips, we observe that each strip has its one end on the line $x = 1$ and the other end on the parabola $y^2 = 4a^2(x - 1)$.

So, the approximating rectangle shown in figure has length $x - 1$, width Δy and area $(x - 1) \Delta y$. Since, the approximating rectangle can move from $y = 0$ to $y = 4a$.

$$\begin{aligned} \therefore \text{Required area} &= \int_0^{4a} (x - 1) dy = \int_0^{4a} \frac{y^2}{4a^2} dy \\ &= \frac{1}{4a^2} \left[\frac{y^3}{3} \right]_0^{4a} = \frac{1}{4a^2} \left(\frac{64a^3}{3} \right) \\ &= \frac{16a}{3} \text{ sq units} \end{aligned}$$

➤ **Example 12.** The area of the region bounded by the curves $y = x^2 + 2$, $y = x$, $x = 0$ and $x = 3$, is

- (a) $\frac{21}{2}$ sq units (b) $\frac{17}{2}$ sq units
(c) $\frac{27}{2}$ sq units (d) $\frac{9}{2}$ sq units

Sol. (a) The equations of the given curves are

$$y = x^2 + 2 \quad \dots(i)$$

$$y = x \quad \dots(ii)$$

$$x = 0 \quad \dots(iii)$$

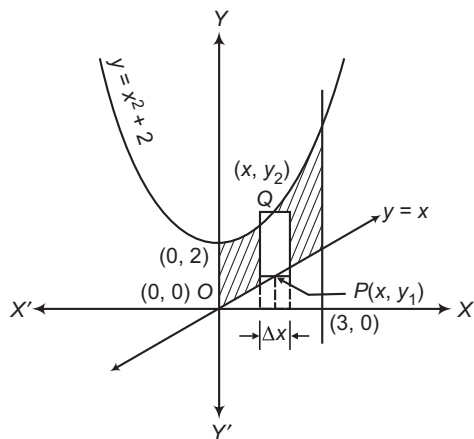
$$\text{and } x = 3 \quad \dots(iv)$$

$$\text{Now, } y = x^2 + 2 \quad \dots(v)$$

$$\Rightarrow (x - 0)^2 = (y - 2) \quad \dots(v)$$

So, Eq. (i) represents a parabola with vertex at $(0, 2)$ and axis along positive direction of Y -axis. This parabola opens upward as shown in figure. Clearly, $y = x$ is a line passing through the origin and makes 45° angle with X -axis, $x = 0$ is Y -axis and $x = 3$ is a line parallel to X -axis at a distance of 3 units from it.

A rough sketch of these curves is shown in figure and the region bounded by these curves is shaded.



Here, we slice this region into vertical strips. We observe that each vertical strip runs from the line $y = x$ to the parabola $y = x^2 + 2$. So, the approximating rectangle shown in figure has width Δx , length $(y_2 - y_1)$ and area $(y_2 - y_1) \Delta x$. Since, the approximating rectangle can move from $x = 0$ to $x = 3$.

$$\therefore \text{Required area} = \int_0^3 (y_2 - y_1) dx = \int_0^3 [(x^2 + 2) - x] dx$$

[$\because P(x, y_1)$ and $Q(x, y_2)$ lie on respectively curves

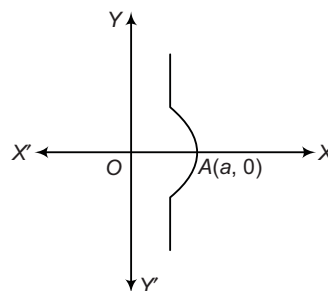
(ii) and (i) $\therefore y_1 = x$ and $y_2 = x^2 + 2$]

$$= \left[\frac{x^3}{3} + 2x - \frac{x^2}{2} \right]_0^3 = 9 + 6 - \frac{9}{2} = \frac{21}{2} \text{ sq units}$$

➤ **Example 13.** The area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptotes, is

- (a) a^2 sq units (b) πa^2 sq units
(c) $4\pi a$ sq units (d) None of these

Sol. (d) The given curve is symmetrical about X -axis and meet it at $(2a, 0)$. The line $x = 0$ i.e. Y -axis is an asymptote.



$$\therefore \text{Required area, } A = 2 \int_0^{2a} y dx = 2 \int_0^{2a} 2a \sqrt{\frac{2a - x}{x}} dx$$

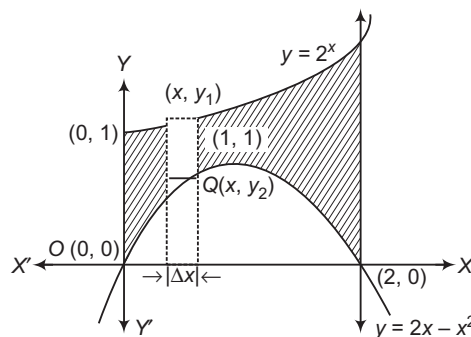
$$\text{Put } x = 2a \sin^2 \theta \Rightarrow dx = 4a \sin \theta \cos \theta d\theta$$

$$\therefore A = 16a^2 \int_0^{\pi/2} \cos^2 \theta d\theta = 4\pi a^2 \text{ sq units}$$

➤ **Example 14.** The area of the region bounded by the straight lines $x = 0$, $x = 2$ and the curves $y = 2^x$, $y = 2x - x^2$ is

- (a) $\left(\frac{4}{\log 2} - \frac{8}{3} \right)$ sq unit (b) $\left(\frac{4}{\log 2} + \frac{8}{3} \right)$ sq unit
(c) $\left(\frac{8}{\log 3} - \frac{4}{3} \right)$ sq unit (d) $\left(\frac{3}{\log 2} - \frac{4}{3} \right)$ sq unit

Sol. (d) Given equation $y = 2x - x^2$ represents a parabola opening downwards having vertex at $(1, 1)$ and crossing X -axis at $(0, 0)$ and $(2, 0)$.



The equation $y = 2^x$ represents the exponential curve as shown in figure. Lines $x = 0$ and $x = 2$ are shown in figure. The area bounded by these curves is shaded in figure. We slice the shaded region into vertical strips. For the approximating rectangle shown in figure, we have length $(y_1 - y_2)$, width Δx and area $(y_1 - y_2) \Delta x$.

The approximating rectangle can move horizontally between $x = 0$ and $x = 2$.

$$\begin{aligned} \therefore \text{Required area} &= \int_0^2 (y_1 - y_2) dx = \int_0^2 (2^x - 2x + x^2) dx \\ &\left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } y = 2^x \right. \\ &\quad \left. \text{and } y = 2x - x^2 \text{ respectively} \right] \\ \therefore y_1 &= 2^x \text{ and } y_2 = 2x - x^2 \\ &= \left[\frac{2^x}{\log 2} - x^2 + \frac{x^3}{3} \right]_0^2 \\ &= \frac{4}{\log 2} - 4 + \frac{8}{3} - \frac{1}{\log 2} \\ &= \left(\frac{3}{\log 2} - \frac{4}{3} \right) \text{ sq unit} \end{aligned}$$

➤ **Example 15.** The area bounded by the curves $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$, where $a, b > 0$, is

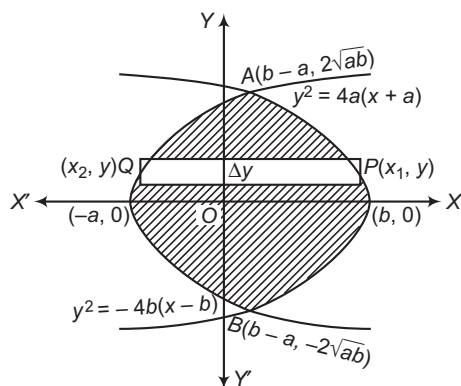
- (a) $\frac{8}{3}(a+b)\sqrt{ab}$ sq units (b) $\frac{2}{3}(a+b)\sqrt{ab}$ sq unit
(c) $\frac{4}{3}(a+b)\sqrt{ab}$ sq units (d) None of these

Sol. (a) Given curves are

$$y^2 = 4a(x + a) \quad \dots(i)$$

$$\text{and } y^2 = -4b(x - b) \quad \dots(ii)$$

Clearly, these two curves represent two parabolas having their vertices at $(-a, 0)$ and $(b, 0)$ respectively as shown in figure.



To find the coordinates of their points of intersection, we solve Eqs. (i) and (ii) together. Solving these equations, we find that the two curves intersect at $A(b - a, 2\sqrt{ab})$ and $B(b - a, -2\sqrt{ab})$. We have to find the area of the shaded region in figure. Let us slice this region into horizontal strips. For the approximating rectangle shown in figure, we have length $(x_1 - x_2)$, width Δy and area $(x_1 - x_2) \Delta y$.

The approximating rectangle can move vertically between B and A .

$$\begin{aligned} \therefore \text{Required area} &= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} (x_1 - x_2) dy \\ &= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} \left\{ \left(b - \frac{y^2}{4b} \right) - \left(\frac{y^2}{4a} - a \right) \right\} dy \\ &\left[\because P(x_1, y) \text{ and } Q(x_2, y) \text{ lie on Eqs. (i) and (ii) respectively} \right] \\ \therefore y^2 &= -4b(x_1 - b) \text{ and } y^2 = 4a(x_2 + a) \end{aligned}$$

$$\begin{aligned} &= \int_{-2\sqrt{ab}}^{2\sqrt{ab}} \left\{ a + b - \left(\frac{a+b}{4ab} \right) y^2 \right\} dy \\ &= 2 \int_0^{2\sqrt{ab}} \left\{ a + b - \left(\frac{a+b}{4ab} \right) y^2 \right\} dy \\ &= 2 \left[(a+b)y - \left(\frac{a+b}{4ab} \right) \frac{y^3}{3} \right]_0^{2\sqrt{ab}} \\ &= 2 \left[(a+b)2\sqrt{ab} - \left(\frac{a+b}{4ab} \right) \frac{8ab\sqrt{ab}}{3} \right] \\ &= 2 \left[(a+b)2\sqrt{ab} - \frac{2}{3}(a+b)\sqrt{ab} \right] \\ &= \frac{8}{3}(a+b)\sqrt{ab} \text{ sq units} \end{aligned}$$

➤ **Example 16.** The parabola $y^2 = ax$ cuts $x^2 - y^2 = 2a^2$ at P and Q . If tangent at P to the hyperbola cuts the parabola again in R , then the area of the curvilinear ΔPQR is

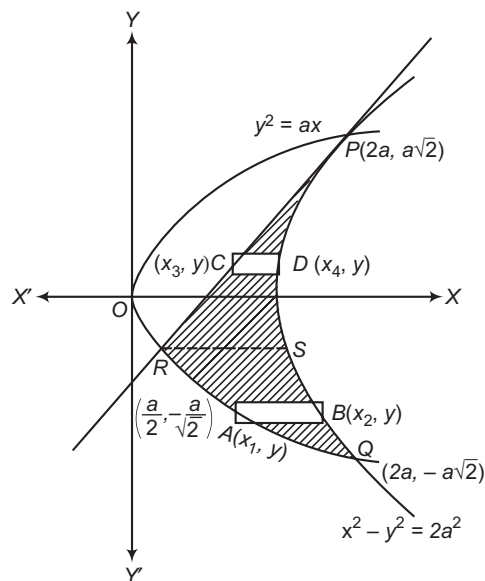
- (a) $\left[2\sqrt{2}a^2 \log(1 - \sqrt{2}) + \frac{11}{24}a^2 \right]$ sq units
(b) $\left[2\sqrt{2}a^2 \log(1 + \sqrt{2}) + \frac{11}{24}a^2 \right]$ sq units
(c) $\left[2\sqrt{2}a^2 \log(1 + \sqrt{2}) - \frac{11\sqrt{2}}{24}a^2 \right]$ sq units
(d) $\left[2a^2 \log(1 + \sqrt{2}) - \frac{11\sqrt{2}}{24}a^2 \right]$ sq units

Sol. (d) On solving $y^2 = ax$ and $x^2 - y^2 = 2a^2$ together, we obtain the coordinates of P and Q as $(2a, a\sqrt{2})$ and $(2a, -a\sqrt{2})$ respectively. The equation of the tangent to $x^2 - y^2 = 2a^2$ at $P(2a, a\sqrt{2})$ is

$$2ax - a\sqrt{2}y = 2a^2$$

$$\Rightarrow 2x - \sqrt{2}y = 2a$$

On solving this equation with $y^2 = ax$, we obtain the coordinates of R as $\left(\frac{a}{2}, -\frac{a}{\sqrt{2}} \right)$.



Now,

Area of curvilinear PQR = Area of curvilinear QRS

+ Area of curvilinear PRS

$$\begin{aligned}
 &= \int_{-a/\sqrt{2}}^{a/\sqrt{2}} (x_2 - x_1) dy + \int_{-a/\sqrt{2}}^{a/\sqrt{2}} (x_4 - x_3) dy \\
 &= \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left(\sqrt{2a^2 + y^2} - \frac{y^2}{a} \right) dy \\
 &\quad + \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left\{ \sqrt{2a^2 + y^2} - \left(a + \frac{1}{\sqrt{2}} y \right) \right\} dy \\
 &= \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \sqrt{2a^2 + y^2} dy + \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \sqrt{2a^2 + y^2} dy \\
 &\quad - \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \frac{y^2}{a} dy - \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left(a + \frac{y}{\sqrt{2}} \right) dy \\
 &= \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \sqrt{2a^2 + y^2} dy - \frac{1}{a} \int_{-a/\sqrt{2}}^{a/\sqrt{2}} y^2 dy - \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left(a + \frac{y}{\sqrt{2}} \right) dy \\
 &= 2 \int_0^{a/\sqrt{2}} \sqrt{2a^2 + y^2} dy - \frac{1}{a} \int_{-a/\sqrt{2}}^{a/\sqrt{2}} y^2 dy - \int_{-a/\sqrt{2}}^{a/\sqrt{2}} \left(a + \frac{y}{\sqrt{2}} \right) dy \\
 &= [y\sqrt{2a^2 + y^2} + 2a^2 \log \{y + \sqrt{2a^2 + y^2}\}]_0^{a/\sqrt{2}} \\
 &\quad - \frac{1}{3a} [y^3]_{-a/\sqrt{2}}^{a/\sqrt{2}} - \left[ay + \frac{y^2}{2\sqrt{2}} \right]_{-a/\sqrt{2}}^{a/\sqrt{2}} \\
 &= [2\sqrt{2}a^2 + 2a^2 \log(a\sqrt{2} + 2a) - 2a^2 \log(\sqrt{2}a)] \\
 &\quad - \frac{1}{3a} \left[-\frac{a^3}{2\sqrt{2}} + 2\sqrt{2}a^3 \right] - \left[\left(\sqrt{2}a^2 + \frac{a^2}{\sqrt{2}} \right) \right. \\
 &\quad \left. - \left(-\frac{a^2}{\sqrt{2}} + \frac{a^2}{4\sqrt{2}} \right) \right] \\
 &= \left[2\sqrt{2}a^2 + 2a^2 \log(1 + \sqrt{2}) - \frac{7a^2}{6\sqrt{2}} - \frac{3a^2}{\sqrt{2}} - \frac{3a^2}{4\sqrt{2}} \right] \\
 &= \left[2a^2 \log(1 + \sqrt{2}) - \frac{11\sqrt{2}}{24} a^2 \right] \text{ sq units}
 \end{aligned}$$

Example 17. The area of the region bounded by the curves $(x-1)^2 + 4(y-1)^2 = 4$ and $4(y-1)^2 = 3(x-1)$, is

(a) $\left(\frac{2\pi}{3} + \frac{1}{2\sqrt{3}} \right)$ sq units

(b) $\left(\frac{3\pi}{2} + \frac{2}{2\sqrt{3}} \right)$ sq units

(c) $\left(\frac{2\pi}{5} + \frac{1}{2\sqrt{5}} \right)$ sq units

(d) None of the above

Sol. (a) The equation $(x-1)^2 + 4(y-1)^2 = 4$ represents an ellipse with centre (1, 1) and the equation $4(y-1)^2 = 3(x-1)$ represents a parabola with vertex (1, 1). If we shift the origin at (1, 1), then the equation of the two curves become

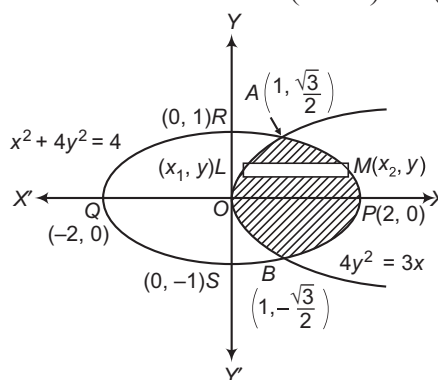
$$x^2 + 4y^2 = 4 \quad \dots(i)$$

and $4y^2 = 3x \quad \dots(ii)$

Since, the area of the region bounded by the given curves does not change by the shifting of origin. So, required area is the area enclosed by curves (i) and (ii) in the new system.

On solving Eqs. (i) and (ii), we obtain the coordinates of their points of intersection as marked in figure. The region enclosed by the curves (i) and (ii) is shaded in figure. For the approximating rectangle LM shown in figure, we have length $(x_2 - x_1)$ and width dy .

As it can move vertically from $B\left(1, -\frac{\sqrt{3}}{2}\right)$ to $A\left(1, \frac{\sqrt{3}}{2}\right)$.



$$\begin{aligned}
 \therefore \text{Required area} &= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} (x_2 - x_1) dy \\
 &= 2 \int_0^{\sqrt{3}/2} (x_2 - x_1) dy
 \end{aligned}$$

[since, both curves are symmetric about X-axis]

$$\begin{aligned}
 &= 2 \int_0^{\sqrt{3}/2} \left\{ \sqrt{4 - 4y^2} - \frac{4y^2}{3} \right\} dy \\
 &= 4 \int_0^{\sqrt{3}/2} \sqrt{1 - y^2} dy - \frac{8}{3} \int_0^{\sqrt{3}/2} y^2 dy \\
 &= 2 [y\sqrt{1 - y^2} + \sin^{-1} y]_0^{\sqrt{3}/2} - \frac{8}{9} [y^3]_0^{\sqrt{3}/2} \\
 &= 2 \left[\frac{\sqrt{3}}{4} + \frac{\pi}{3} \right] - \frac{8}{9} \times \frac{3\sqrt{3}}{8} \\
 &= \left(\frac{2\pi}{3} + \frac{1}{2\sqrt{3}} \right) \text{ sq units}
 \end{aligned}$$

Example 18. The area of the region bounded by the parabola $(y-2)^2 = (x-1)$, the tangent to it at the point with ordinate 3 and X-axis, is

(a) 7 sq units

(b) 8 sq units

(c) 9 sq units

(d) 10 sq units

Sol. (c) On putting $y = 3$ in $(y-2)^2 = (x-1)$, we obtain $x = 2$. The equation of the parabola is $(y-2)^2 = (x-1)$.

On differentiating w.r.t. x , we get

$$2(y-2) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y-2)}$$

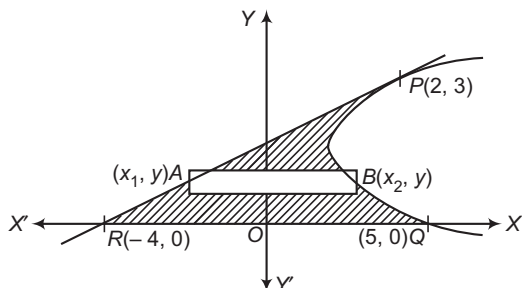
$$\therefore \left[\frac{dy}{dx} \right]_{(2,3)} = \frac{1}{2}$$

The equation of the tangent at (2, 3) to the parabola $(y-2)^2 = (x-1)$ is

$$y - 3 = \frac{1}{2}(x - 2)$$

$$\Rightarrow x - 2y + 4 = 0$$

The parabola $(y-2)^2 = (x-1)$ cuts X-axis at Q (5, 0).
The tangent P (2, 3) cuts X-axis at R (-4, 0).



The shaded region in figure is the region enclosed by the given curves. We slice this region into horizontal strips. The approximating rectangle shown in figure has length $x_2 - x_1$, width Δy and it can move between $y = 0$ and $y = 3$.

$$\begin{aligned}\therefore \text{Required area} &= \int_0^3 (x_2 - x_1) dy \\ &= \int_0^3 \{(y^2 - 4y + 5) - (2y - 4)\} dy \\ &= \int_0^3 (y^2 - 6y + 9) dy \\ &= \left[\frac{y^3}{3} - 3y^2 + 9y \right]_0^3 = 9 - 27 + 27 \\ &= 9 \text{ sq units}\end{aligned}$$

➤ **Example 19.** The area contained between the two arms of curve $(y-x)^2 = x^3$ between $x=0$ and $x=1$, is

- (a) $\frac{3}{5}$ sq unit (b) $\frac{4}{5}$ sq unit
(c) $\frac{2}{3}$ sq unit (d) 2 sq units

Sol. (b) We have, $(y-x)^2 = x^3$

$$\Rightarrow y = x \pm x^{3/2}$$

For arm $y = x + x^{3/2}$,

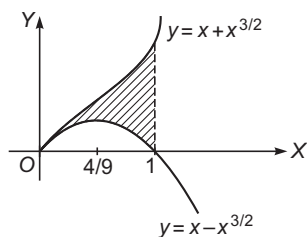
$$\frac{dy}{dx} = 1 + \frac{3}{2}x^{1/2} > 0, x \geq 0$$

So, y is an increasing function.

For arm $y = x - x^{3/2}$,

$$\frac{dy}{dx} = 1 - \frac{3}{2}x^{1/2} \Rightarrow \frac{dy}{dx} = 0 \Rightarrow x = \frac{4}{9}$$

$$\text{Now, } \frac{d^2y}{dx^2} = -\frac{3}{4}x^{-1/2} < 0 \text{ at } x = \frac{4}{9}$$



At $x = \frac{4}{9}$, $y = x - x^{3/2}$ has maxima.

$$\therefore \text{Required area} = \int_0^1 (x + x^{3/2} - x + x^{3/2}) dx$$

$$= 2 \int_0^1 x^{3/2} dx = \left[\frac{2x^{5/2}}{5/2} \right]_0^1 = \frac{4}{5} \text{ sq unit}$$

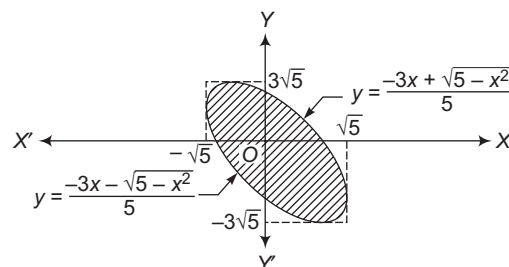
➤ **Example 20.** The area contained by ellipse $2x^2 + 6xy + 5y^2 = 1$ is

- (a) π sq units
(b) $\frac{\pi}{2}$ sq units
(c) 2π sq units
(d) None of the above

Sol. (a) We have, $5y^2 + 6xy + 2x^2 - 1 = 0$

$$\Rightarrow y = \frac{-6x \pm \sqrt{36x^2 - 20(2x^2 - 1)}}{10}$$

$$\Rightarrow y = \frac{-3x \pm \sqrt{5 - x^2}}{5}$$



Since, y is real. So, RHS is also real.

$$\Rightarrow -\sqrt{5} \leq x \leq \sqrt{5}$$

If $x = -\sqrt{5}$, then $y = 3\sqrt{5}$

If $x = \sqrt{5}$, then $y = -3\sqrt{5}$

If $x = 0$, then $y = \pm \frac{1}{\sqrt{5}}$

If $y = 0$, then $x = \pm \frac{1}{\sqrt{2}}$

$\therefore$ Required area,

$$\begin{aligned}A &= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x + \sqrt{5 - x^2}}{5} - \frac{-3x - \sqrt{5 - x^2}}{5} \right) dx \\ &= \int_{-\sqrt{5}}^{\sqrt{5}} \left(\frac{-3x + \sqrt{5 - x^2} + 3x + \sqrt{5 - x^2}}{5} \right) dx \\ &= \frac{2}{5} \int_{-\sqrt{5}}^{\sqrt{5}} \sqrt{5 - x^2} dx \\ &= \frac{4}{5} \int_0^{\sqrt{5}} \sqrt{5 - x^2} dx\end{aligned}$$

Put $x = \sqrt{5} \sin \theta \Rightarrow dx = \sqrt{5} \cos \theta d\theta$

Lower limit, when $x = 0$, then $\theta = 0$

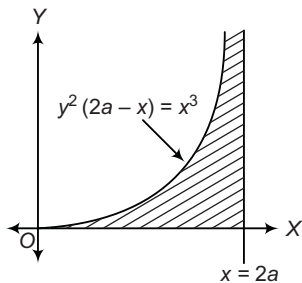
Upper limit, when $x = \sqrt{5}$, then $\theta = \frac{\pi}{2}$

$$\begin{aligned}\therefore A &= \frac{4}{5} \int_0^{\pi/2} \sqrt{5 - 5\sin^2 \theta} \sqrt{5} \cos \theta d\theta \\ &= 4 \int_0^{\pi/2} \cos^2 \theta d\theta \\ &= 4 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \pi \text{ sq units}\end{aligned}$$

➤ **Example 21.** The area enclosed between the curve $y^2(2a-x) = x^3$ and the line $x=2a$ above X-axis, is

- (a) πa^2 sq units (b) $\frac{3\pi a^2}{2}$ sq units
(c) $2\pi a^2$ sq units (d) $3\pi a^2$ sq units

Sol. (b) Required area, $A = \int_0^{2a} \sqrt{\frac{x^3}{2a-x}} dx$



Put $x = 2a \sin^2 \theta$

$$\Rightarrow dx = 2a \cdot 2 \sin \theta \cos \theta d\theta$$

$$\begin{aligned} \therefore A &= 8a^2 \int_0^{\pi} \left(\frac{1 - \cos 2\theta}{2} \right)^2 d\theta \\ &= 2a^2 \int_0^{\pi} (1 - 2\cos 2\theta + \cos^2 2\theta) d\theta \\ &= 2a^2 \int_0^{\pi} \left(1 - 2\cos 2\theta + \frac{1 + \cos 4\theta}{2} \right) d\theta \\ &= \frac{3\pi a^2}{2} \text{ sq units} \end{aligned}$$

Area of Some Important Curves

Equation of curves	Point of intersection	Area of shaded region	Shaded region
(i) $f(x, y) : ax^2 \leq y \leq mx$ $\therefore y = ax^2, y = mx$	$B(0, 0), A\left(\frac{m}{a}, \frac{m^2}{a}\right)$	$\text{Area} = \frac{1}{6} \frac{m^3}{a^2} \text{ sq unit}$	
(ii) $f(x, y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1$ $\frac{x}{a} + \frac{y}{b} \geq 1$ $\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \leq \frac{x}{a} + \frac{y}{b}$	$A(a, 0), B(0, b)$	$\text{Area} = ab \frac{(\pi - 2)}{4} \text{ sq unit}$	
(iii) Area of parabola $y^2 = 4ax$ and its latusrectum $x = a$	$A(a, 2a), B(a, -2a)$	$\text{Area} = \frac{8}{3} a^2 \text{ sq units}$	
(iv) $f(x, y) : y^2 = 4ax$ and $y = mx $	$O(0, 0), A\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$	$\text{Area} = \frac{8a^2}{3m^3} \text{ sq units}$	
(v) $f(x, y) : x^2 + y^2 \leq 2ax$ and $y^2 \geq ax$	$O(0, 0), A(a, a), B(a, -a)$	(i) For $x \geq 0, y \geq 0$ (area of 1st quadrant) $\text{Area} = a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right) \text{ sq unit}$ (ii) For $x \geq 0$ (for 1st and IVth quadrants) $\text{Area} = 2a^2 \left(\frac{\pi}{4} - \frac{2}{3} \right) = a^2 \left(\frac{\pi}{2} - \frac{4}{3} \right)$	

Equation of curves	Point of intersection	Area of shaded region	Shaded region
(vi) $f(x, y) : x^2 = 4ay$ and $y^2 = 4bx$	$O(0, 0)$, $A(4a^{2/3}b^{1/3}, 4a^{1/3}b^{2/3})$	$\text{Area} = \frac{16}{3}ab$ sq units	
(vii) $y = k \cos 3x, \forall 0 \leq x \leq \frac{\pi}{6}$	As $0 \leq x \leq \frac{\pi}{6}$ $\therefore 0 \leq 3x \leq \frac{\pi}{2}$	$\text{Area} = \frac{k}{3}$ sq unit	
(viii) Area bounded by $y^2 = 4a(x + a)$, $y^2 = 4b(b - x)$	$A(b - a, 2\sqrt{ab})$, $B(b - a, -2\sqrt{ab})$	$\text{Area} = \frac{8}{3}\sqrt{ab}(a + b)$ sq units	
(ix) Common area bounded by the ellipses $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{1}{a^2b^2}$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2b^2}$, $0 < a < b$	$x = y = \frac{1}{\sqrt{a^2 + b^2}}$, $x = y = -\frac{1}{\sqrt{a^2 + b^2}}$	$\text{Area} = \text{Area of } PQRS = 4 \times \text{Area of } OLQM = \frac{4}{ab} \tan^{-1}\left(\frac{a}{b}\right)$ sq units	
(x) If $\alpha, \beta > 0, \alpha > \beta$, the area between the hyperbola $xy = p^2$, X-axis and the ordinates $x = \alpha, x = \beta$	No point of intersection	$\text{Area} = p^2 \log\left(\frac{\alpha}{\beta}\right)$	

Work Book Exercise 11.1

- 1 If the area bounded by the curve $y = f(x)$, X-axis and the lines $x = 1, x = b$ is equal to $\sqrt{b^2 + 1} - \sqrt{2}$ for all $b > 1$, then the value of $f(x)$ is

$$\frac{(\sqrt{x-1})}{(\sqrt{x^2+1})}$$

$$\frac{(\sqrt{x+1})}{(x/\sqrt{x^2+1})}$$

- 2 If a curve $y = a\sqrt{x} + bx$ passes through the point $(1, 2)$ and the area bounded by the curve, line $x = 4$ and X-axis is 8 sq units, then
- $a = 3, b = -1$
 $a = 3, b = 1$
 $a = -3, b = 1$
 $a = -3, b = -1$



- 3 If the area enclosed between the curves $y = ax^2$ and $x = ay^2$ ($a > 0$) is 1 sq unit, then the value of a is

$\frac{1}{\sqrt{3}}$	$\frac{1}{2}$
1	$\frac{1}{3}$

- 4 The area of the region bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and X-axis, is

4 sq units	2 sq units
3 sq units	1 sq unit

- 5 Area common to the curve $y = \sqrt{9 - x^2}$ and $x^2 + y^2 = 6x$ is

$\left(\frac{\pi + \sqrt{3}}{4}\right)$ sq unit
$\left(\frac{\pi - \sqrt{3}}{4}\right)$ sq unit
$3\left(\pi + \frac{3\sqrt{3}}{4}\right)$ sq unit
$3\left(\pi - \frac{3\sqrt{3}}{4}\right)$ sq unit

- 6 If $y = 2 \sin x + \sin 2x$ for $0 \leq x < 2\pi$, then the area enclosed by the curve and the X-axis is

$\frac{9}{2}$ sq units	8 sq units
9 sq units	4 sq units

- 7 The area bounded by the curve $y = x^2 + 4x + 5$, the axes of coordinates and the minimum ordinate, is

$3\frac{2}{3}$ sq units
$4\frac{2}{3}$ sq units
$5\frac{2}{3}$ sq units
None of the above

- 8 The area enclosed by the curve $y^2 + x^4 = x^2$ is

$\frac{2}{3}$ sq unit
$\frac{4}{3}$ sq units
$\frac{8}{3}$ sq units
$\frac{10}{3}$ sq units

- 9 The area bounded by the curve $y = x^2 + 1$ and the tangents to it drawn from the origin, is

$\frac{2}{3}$ sq unit
$\frac{4}{3}$ sq units
$\frac{1}{3}$ sq unit
1 sq unit

- 10 The area enclosed by the curves $y = \sqrt{x}$ and $x = -\sqrt{y}$, the circle $x^2 + y^2 = 2$ above the X-axis, is

$\frac{\pi}{4}$ sq unit	$\frac{3\pi}{2}$ sq units
π sq units	$\frac{\pi}{2}$ sq units

- 11 The area bounded by the curve $y = f(x)$, the coordinate axes and the line $x = x_1$ is given by $x_1 \cdot c^{x_1}$. Then, $f(x)$ equals

e^x sq unit	xe^x sq unit
$(xe^x - e^x)$ sq unit	$(xe^x + e^x)$ sq unit

- 12 The limit of the area under the curve $y = e^{-x}$ from $x = 0$ to $x = h$ at $h \rightarrow \infty$, is

$\frac{2}{e}$	e
$\frac{1}{e}$	1

- 13 The slope of the tangent to a curve $y = f(x)$ at $(x, f(x))$ is $(2x + 1)$. If the curve passes through the point $(1, 2)$, then the area of the region bounded by the curve, X-axis and line $x = 1$, is

$\frac{5}{6}$ sq unit	$\frac{6}{5}$ sq units
$\frac{1}{6}$ sq unit	1 sq unit

- 14 The graphs of $f(x) = x^2$ and $g(x) = cx^3$ ($c > 0$) intersect at the points $(0, 0)$ and $\left(\frac{1}{c}, \frac{1}{c^2}\right)$. If the region which lies between these graphs and over the interval $\left[0, \frac{1}{c}\right]$ has the area equal to $\frac{2}{3}$, then the value of c is

1	$\frac{1}{3}$
$\frac{1}{2}$	2

- 15 Area bounded by the curve $xy^2 = a^2(a - x)$ and the Y-axis, is

$\frac{\pi a^2}{2}$ sq units	πa^2 sq units
$3\pi a^2$ sq units	None of these

- 16 The area of the region bounded by $x^2 + y^2 - 6x - 4y + 12 \leq 0$, $y \leq x$ and $x \leq \frac{5}{2}$, is

$\left(\frac{\pi}{6} - \frac{\sqrt{3} + 1}{8}\right)$ sq unit
$\left(\frac{\pi}{6} + \frac{\sqrt{3} - 1}{8}\right)$ sq unit
$\left(\frac{\pi}{6} - \frac{\sqrt{3} - 1}{8}\right)$ sq unit

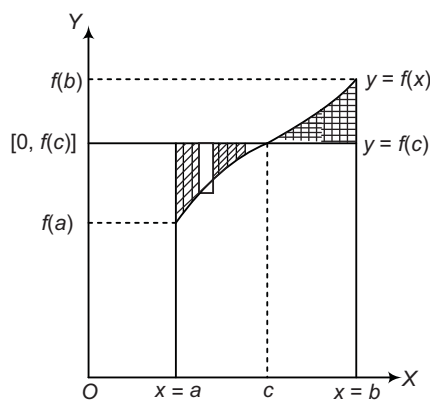
- d None of the above

Variable Area, Greatest and Least Values

An Important Concept

If $y = f(x)$ is a monotonic function in (a, b) , then the area bounded by the ordinates at $x = a, x = b, y = f(x)$ and $y = f(c)$, where $c \in (a, b)$, is minimum, when $c = \frac{a+b}{2}$.

$$\begin{aligned} \text{Proof } A &= \int_a^c [f(c) - f(x)] dx + \int_c^b [f(x) - f(c)] dx \\ &= f(c)(c-a) - \int_a^c [f(x)] dx \\ &\quad + \int_c^b [f(x)] dx - f(c)(b-c) \\ &= [(c-a) - (b-c)] f(c) \\ &\quad + \int_c^b [f(x)] dx - \int_a^c [f(x)] dx \end{aligned}$$



$$A = [2c - (a+b)] f(c) + \int_c^b [f(x)] dx - \int_a^c [f(x)] dx$$

On differentiating both sides w.r.t. c , we get

$$\begin{aligned} \frac{dA}{dc} &= [2c - (a+b)] f'(c) + 2f(c) + 0 - f(c) \\ &\quad - [f(c) - 0] \end{aligned}$$

For maximum and minimum, put

$$\begin{aligned} \frac{dA}{dc} &= 0 \\ \Rightarrow f'(c) [2c - (a+b)] &= 0 \quad [\text{as } f'(c) \neq 0] \end{aligned}$$

$$\therefore c = \frac{a+b}{2}$$

$$\text{Also, } c < \frac{a+b}{2} \Rightarrow \frac{dA}{dc} < 0$$

$$\text{and } c > \frac{a+b}{2} \Rightarrow \frac{dA}{dc} > 0$$

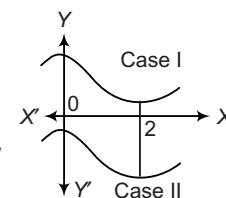
Hence, A is minimum, when

$$c = \frac{a+b}{2}$$

Example 22. If the area bounded by

$f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x=0, x=2$ and the X -axis, is minimum, then the value of a is

- (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{5}{3}$ (d) $-\frac{2}{3}$



Sol. (b) We have,

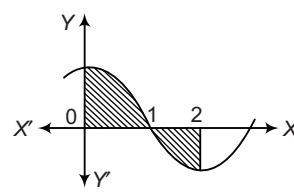
$$\begin{aligned} f'(x) &= x^2 - 2x \\ &= x(x-2) = 0 \end{aligned}$$

[since, $f(x)$ is monotonic function in $(0, 2)$]

Thus, for the minimum

and $f(x)$ must cross the X -axis at $\frac{0+2}{2} = 1$.

$$\text{Hence, } f(1) = \frac{1}{3} - 1 + a = 0 \Rightarrow a = \frac{2}{3}$$



Example 23. The value of the parameter a for which the area of the figure bounded by the X -axis, the graph of the function $y = x^3 + 3x^2 + x + a$ and the straight lines, which are parallel to X -axis and cut the X -axis at the point of extremum of the function, is the least, is

- (a) 1 (b) -1 (c) 0 (d) 2

Sol. (b) We have, $f(x) = x^3 + 3x^2 + x + a$

$$\Rightarrow f'(x) = 3x^2 + 6x + 1 = 0 \Rightarrow x = -1 \pm \frac{\sqrt{6}}{3}$$

Hence, $f(x)$ cuts X -axis at

$$\frac{1}{2} \left[\left(-1 + \frac{\sqrt{6}}{3} \right) + \left(-1 - \frac{\sqrt{6}}{3} \right) \right] = -1$$

$$\Rightarrow f(-1) = -1 + 3 - 1 + a = 0 \Rightarrow a = -1$$

Example 24. Let $f(x) = \max \{x^2, (1-x)^2, 2x(1-x)\}$. Then, the area of the region bounded by the curve $y = f(x)$, X -axis, $x=0$ and $x=1$, is

- (a) $\frac{2}{81}$ sq unit (b) $\frac{17}{27}$ sq unit
(c) $\frac{16}{81}$ sq unit (d) $\frac{17}{29}$ sq unit

Sol. (b) Let us first draw the graphs of the following curves

$$y = x^2 \quad \dots(i)$$

$$y = (1-x)^2 \quad \dots(ii)$$

$$\text{and } y = 2x(1-x) \quad \dots(iii)$$

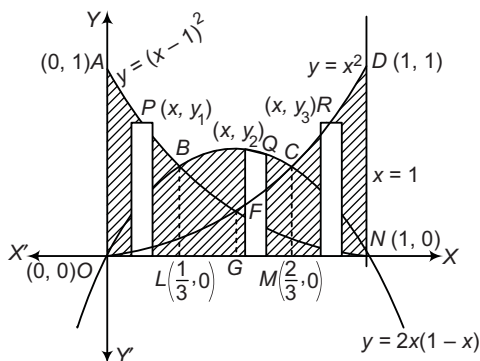
in the interval $[0, 1]$.

All these curves are parabolas. So, their points of intersection are marked in figure.

It is evident from the graph that,

$$f(x) = \begin{cases} (1-x)^2, & 0 \leq x \leq \frac{1}{3} \\ 2x(1-x), & \frac{1}{3} \leq x \leq \frac{2}{3} \\ x^2, & \frac{2}{3} \leq x \leq 1 \end{cases}$$

We have to find the area of the shaded region in figure.



∴ Required area

$$\begin{aligned}
 &= \text{Area of } OABLO + \text{Area of } LBCML + \text{Area of } CMNDC \\
 &= \int_0^{1/3} y_1 dx + \int_{1/3}^{2/3} y_2 dx + \int_{2/3}^1 y_3 dx \\
 &= \int_0^{1/3} (x-1)^2 dx + \int_{1/3}^{2/3} 2x(1-x) dx + \int_{2/3}^1 x^2 dx \\
 &= \left[\frac{(x-1)^3}{3} \right]_0^{1/3} + \left[x^2 - \frac{2x^3}{3} \right]_{1/3}^{2/3} + \left[\frac{x^3}{3} \right]_{2/3}^1 \\
 &= \frac{1}{3} \left[-\frac{8}{27} + 1 \right] + \left[\left(\frac{4}{9} - \frac{16}{81} \right) - \left(\frac{1}{9} - \frac{2}{81} \right) \right] + \frac{1}{3} \left[1 - \frac{8}{27} \right] \\
 &= \frac{38}{81} + \frac{20}{81} - \frac{7}{81} = \frac{51}{81} = \frac{17}{27} \text{ sq unit}
 \end{aligned}$$

➤ **Example 25.** Let $f(x) = \min \left\{ e^x, \frac{3}{2}, 1 + e^{-x} \right\}$,

$0 \leq x \leq 1$. Then, the area bounded by $y = f(x)$, X-axis, Y-axis and the line $x = 1$, is

(a) $\left\{ 2 - \frac{1}{e} + \frac{3}{2} \log \frac{4}{3} - \log 2 \right\}$ sq unit

(b) $\left\{ \frac{1}{e} + \frac{3}{2} \log \frac{4}{3} - \log 2 \right\}$ sq unit

(c) $\left\{ 3 - \frac{1}{e} + \frac{3}{2} \log \frac{4}{3} + \log 2 \right\}$ sq unit

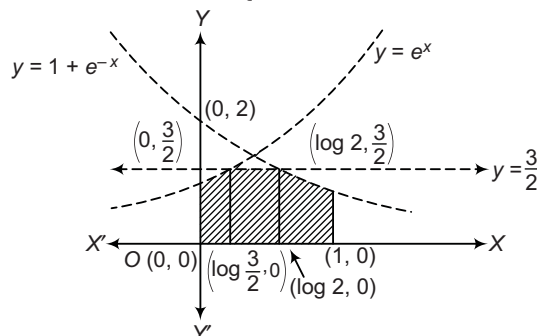
(d) $\left\{ 3 - \frac{2}{e} + \frac{3}{2} \log \frac{3}{4} + \log 2 \right\}$ sq unit

Sol. (a) It is evident from the graph of

$$y = e^x, y = \frac{3}{2}$$

and

$$y = 1 + e^{-x}, \text{ then } f(x) = \begin{cases} e^x, & 0 \leq x \leq \log(3/2) \\ 3/2, & \log(3/2) \leq x \leq \log 2 \\ 1 + e^{-x}, & \log 2 \leq x \leq 1 \end{cases}$$



∴ Required area

$$\begin{aligned}
 &= \int_0^{\log(3/2)} e^x dx + \int_{\log(3/2)}^{\log 2} \frac{3}{2} dx + \int_{\log 2}^1 (1 + e^{-x}) dx \\
 &= [e^x]_0^{\log(3/2)} + \frac{3}{2} [x]_{\log(3/2)}^{\log 2} + [x - e^{-x}]_{\log 2}^1 \\
 &= \left\{ \frac{3}{2} - 1 \right\} + \frac{3}{2} \left\{ \log 2 - \log \frac{3}{2} \right\} + \left\{ 1 - \frac{1}{e} \right\} - \left\{ \log 2 - \frac{1}{2} \right\} \\
 &= \left\{ 2 - \frac{1}{e} + \frac{3}{2} \log \frac{4}{3} - \log 2 \right\} \text{ sq units}
 \end{aligned}$$

Work Book Exercise 11.2

- 1 If $f(x) = \min\{x + 1, \sqrt{1-x}\}$ for all $x \leq 1$, then the area bounded by $y = f(x)$ and X-axis, is

$\frac{7}{3}$ sq units $\frac{1}{6}$ sq unit
 $\frac{11}{6}$ sq units $\frac{7}{6}$ sq units

- 2 The value of the parameter a such that the area bounded by $y = a^2 x^2 + ax + 1$, coordinate axes and the line $x = 1$ attains its least value, is equal to

$-\frac{1}{4}$ $-\frac{1}{2}$ $-\frac{3}{4}$ -1

- 3 The area of the region of the plane bounded by $\max(|x|, |y|) \leq 1$ and $xy \leq \frac{1}{2}$, is

$\left(\frac{1}{2} + \ln 2 \right)$ sq unit $(3 + \ln 2)$ sq units
 $\frac{31}{4}$ sq units $(1 + 2 \ln 2)$ sq units

- 4 Let $f(x) = \begin{cases} \min\{(x-1)^2, 2x(1-x)\}, & \text{if } 0 \leq x < 1/2 \\ \min\{2x(1-x), x^2\}, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$

Then, the area bounded by the curve and X-axis, is

$\frac{40}{118}$ sq unit
 $\frac{21}{108}$ sq unit
 $\frac{40}{128}$ sq unit
 $\frac{31}{108}$ sq unit

- 5 If $f(x) = \max\left\{ \sin x, \cos x, \frac{1}{2} \right\}$, then the area of

the region bounded by the curves $y = f(x)$, X-axis, Y-axis and $x = \frac{5\pi}{3}$, is

$\left(\sqrt{2} - \sqrt{3} + \frac{5\pi}{12} \right)$ sq units
 $\left(\sqrt{2} + \frac{\sqrt{3}}{2} + \frac{5\pi}{12} \right)$ sq units
 $\left(\sqrt{2} + \sqrt{3} + \frac{5\pi}{12} \right)$ sq units

None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The area bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is

- (a) $\frac{9}{4}$ sq units (b) $\frac{9}{8}$ sq units
(c) $\frac{3}{4}$ sq unit (d) None of these

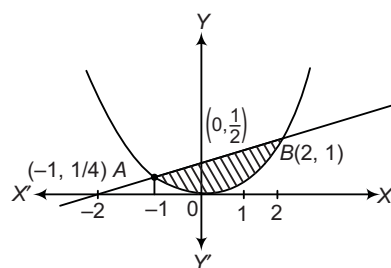
Sol. Given curves $x^2 = 4y$ and $x = 4y - 2$ intersect, when $x^2 = x + 2$.

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow x = 2, -1$$

$$\therefore y = 1, \frac{1}{4}$$

So, points of intersection are $A(-1, \frac{1}{4})$, $B(2, 1)$.



$\therefore$ Required area

$$= \int_{-1}^2 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx = \frac{1}{4} \left[\frac{x^2}{2} + 2x - \frac{x^3}{3} \right]_{-1}^2$$

$$= \frac{1}{4} \left[\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right]$$

$$= \frac{1}{4} \left[\frac{10}{3} - \left(-\frac{7}{6} \right) \right]$$

$$= \frac{1}{4} \left[\frac{27}{6} \right] = \frac{9}{8} \text{ sq units}$$

Hence, (b) is the correct answer.

Ex 2. Area enclosed by the curve $y = f(x)$ defined parametrically as $x = \frac{1-t^2}{1+t^2}$ and $y = \frac{2t}{1+t^2}$, is equal to

- (a) π sq units (b) $\frac{\pi}{2}$ sq units
(c) $\frac{3\pi}{4}$ sq units (d) $\frac{3\pi}{2}$ sq units

Sol. Clearly, t can be any real number.

Let $t = \tan \theta \Rightarrow x = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$

$$\Rightarrow x = \cos 2\theta$$

and $y = \frac{2 \tan \theta}{1 + \tan^2 \theta} = \sin 2\theta$

$\therefore x^2 + y^2 = 1$

Thus, required area = π sq units

Hence, (a) is the correct answer.

Ex 3. The area bounded by $y = 3 - |3 - x|$ and

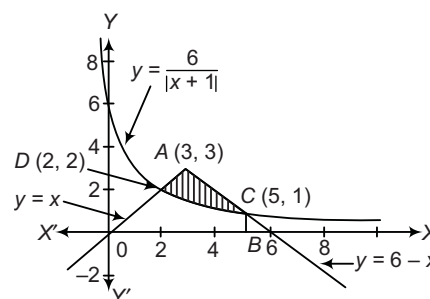
$y = \frac{6}{|x+1|}$ is

- (a) $\left(\frac{15}{2} - 6 \ln 2 \right)$ sq units
(b) $\left(\frac{13}{2} - 3 \ln 2 \right)$ sq units
(c) $\left(\frac{13}{2} - 6 \ln 2 \right)$ sq units
(d) None of the above

Sol. Consider, $y = 3 - |3 - x|$

For $x < 3$, $y = 3 - (3 - x) = x$

For $x \geq 3$, $y = 3 - (x - 3) = 6 - x$



Also, consider $y = \frac{6}{|x+1|}$

For $x < -1$, $y = \frac{6}{-1-x}$

$\Rightarrow (1+x)y = -6$

For $x > -1$, $y = \frac{6}{x+1}$

$\therefore$ Required area

$$= \left[\int_2^3 \left(x - \frac{6}{x+1} \right) dx + \int_3^5 \left((6-x) - \frac{6}{x+1} \right) dx \right]$$

$$= \left\{ \left[\frac{x^2}{2} \right]_2^3 + \left[6x - \frac{x^2}{2} \right]_3^5 - [6 \log(x+1)]_2^5 \right\}$$

$$= \left(\frac{5}{2} + 4 - 6 \log 2 \right)$$

$$= \left(\frac{13}{2} - 6 \ln 2 \right) \text{ sq units}$$

Hence, (c) is the correct answer.

Ex 4. The area bounded by the two branches of curve $(y-x)^2 = x^3$ and the straight line $x=1$, is

- (a) $\frac{1}{5}$ sq unit (b) $\frac{3}{5}$ sq unit
(c) $\frac{4}{5}$ sq unit (d) $\frac{8}{4}$ sq units

Sol. We have, $(y-x)^2 = x^3$, where $x \geq 0 \Rightarrow y-x = \pm x^{3/2}$

$$\Rightarrow y = x + x^{3/2} \quad \dots(i)$$

$$\text{and } y = x - x^{3/2} \quad \dots(ii)$$

Eq. (i) is an increasing function.

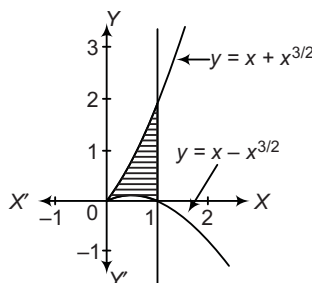
Eq. (ii) meets X -axis, when $x - x^{3/2} = 0$ or $x = 0, 1$.

Also, for $0 < x < 1$, $x - x^{3/2} > 0$

and for $x > 1$, $x - x^{3/2} < 0$

When $x \rightarrow \infty$, $x - x^{3/2} \rightarrow -\infty$

From these information, we can plot the graph as below:



$\therefore$ Required area

$$= \int_0^1 [(x + x^{3/2}) - (x - x^{3/2})] dx = 2 \int_0^1 x^{3/2} dx$$

$$= 2 \left[\frac{x^{5/2}}{5/2} \right]_0^1 = \frac{4}{5} \text{ sq unit}$$

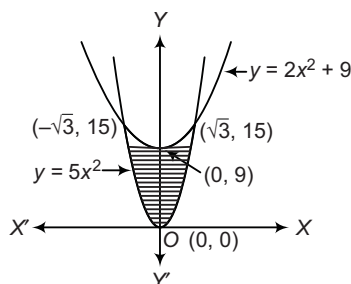
Hence, (c) is the correct answer.

Ex 5. The area inside the parabola $5x^2 - y = 0$ but outside the parabola $2x^2 - y + 9 = 0$, is

- (a) $12\sqrt{3}$ sq units (b) $6\sqrt{3}$ sq units
(c) $8\sqrt{3}$ sq units (d) $4\sqrt{3}$ sq units

Sol. Given, $5x^2 - y = 0$

and $2x^2 - y + 9 = 0$



On eliminating y , we get

$$5x^2 - (2x^2 + 9) = 0$$

$$\Rightarrow 3x^2 = 9$$

$$\Rightarrow x = -\sqrt{3}, \sqrt{3}$$

$$\therefore \text{ Required area} = 2 \int_0^{\sqrt{3}} \{(2x^2 + 9) - 5x^2\} dx$$

$$= 2 \int_0^{\sqrt{3}} (9 - 3x^2) dx$$

$$= 2 [9x - x^3]_0^{\sqrt{3}} = 2 [9\sqrt{3} - 3\sqrt{3}]$$

$$= 12\sqrt{3} \text{ sq units}$$

Hence, (a) is the correct answer.

Ex 6. The area bounded by $y = \sec^{-1} x$, $y = \operatorname{cosec}^{-1} x$ and line $x-1=0$, is

(a) $\left(\log(3 + 2\sqrt{2}) - \frac{\pi}{2} \right)$ sq unit

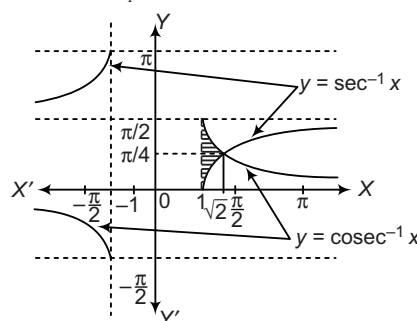
(b) $\left(\frac{\pi}{2} - \log(3 + 2\sqrt{2}) \right)$ sq unit

(c) $(\pi - \log_e 3)$ sq unit

(d) None of the above

Sol. Integrating along X -axis, we get

$$A = \int_1^{\sqrt{2}} (\operatorname{cosec}^{-1} x - \sec^{-1} x) dx$$



Integrating along Y -axis, we get

$$A = 2 \int_0^{\pi/4} (\sec y - 1) dy$$

$$= 2 [\log |\sec y + \tan y| - y]_0^{\pi/4}$$

$$= 2 \left[\log |\sqrt{2} + 1| - \frac{\pi}{4} \right]$$

$$= \left(\log(3 + 2\sqrt{2}) - \frac{\pi}{2} \right) \text{ sq unit}$$

Hence, (a) is the correct answer.

Ex 7. The area of the part of the square $\{(x, y): 1 \leq x \leq 4, 1 \leq y \leq 4\}$ that is enclosed by the two curves $x^2 = 4[\sqrt{x}]y$ and $y^2 = 4[\sqrt{y}]x$ and the line $x+y=3$, is

(a) $\frac{17}{6}$ sq units (b) $\frac{19}{6}$ sq units

(c) $\frac{17}{3}$ sq units (d) $\frac{22}{3}$ sq units

Sol. We have, $1 \leq x \leq 4$ and $1 \leq y \leq 4$

$$\Rightarrow 1 \leq \sqrt{x} \leq 2 \text{ and } 1 \leq \sqrt{y} \leq 2$$

$$\Rightarrow [\sqrt{x}] = 1 \text{ and } [\sqrt{y}] = 1$$

for all (x, y) lying within the square.

Thus, the curves $x^2 = 4[\sqrt{x}]y$ and $y^2 = 4[\sqrt{y}]x$ are the parts of the two parabolas $x^2 = 4y$ and

$y^2 = 4x$, respectively lying within the given squares as shown in figure. We have to find the area of the shaded region. Let us slice this region into vertical strips.

We observe that the vertical strips change their character at point E . So, we draw a line EG parallel to Y -axis which divides the area into two regions EGF and ECF .

In the region EGF , the approximating rectangle has length $(y_2 - y_1)$ and width Δx and it can move from $x = 1$ to $x = 2$.

Area of region EGF ,

$$A_1 = \int_1^2 (y_2 - y_1) dx = \int_1^2 [\sqrt{4x} - (3 - x)] dx$$

$$\left[\because P(x, y_1) \text{ and } Q(x, y_2) \text{ lie on } \right.$$

$$\left. y^2 = 4x \text{ and } x + y = 3 \text{ respectively} \right]$$

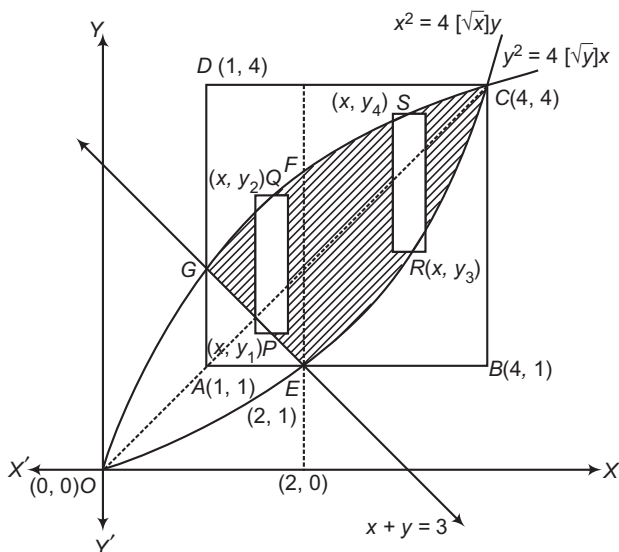
Area of region ECF ,

$$A_2 = \int_2^4 (y_4 - y_3) dx$$

$$= \int_2^4 \left(\sqrt{4x} - \frac{x^2}{4} \right) dx$$

$$\left[\because R(x, y_3) \text{ and } S(x, y_4) \text{ lie on } \right.$$

$$\left. x^2 = 4y \text{ and } y^2 = 4x, \text{ respectively.} \right]$$



$\therefore$ Required area $= A_1 + A_2$

$$= \int_1^2 (2\sqrt{x} - 3 + x) dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx$$

$$= \left[\frac{4}{3} x^{3/2} - 3x + \frac{x^2}{2} \right]_1^2 + \left[\frac{4}{3} x^{3/2} - \frac{x^3}{12} \right]_2^4$$

$$= \left(\frac{8\sqrt{2}}{3} - 6 + 2 \right) - \left(\frac{4}{3} - 3 + \frac{1}{2} \right)$$

$$+ \left(\frac{32}{3} - \frac{64}{12} \right) - \left(\frac{8\sqrt{2}}{3} - \frac{8}{12} \right)$$

$$= \frac{8\sqrt{2}}{3} - 4 + \frac{7}{6} + \frac{16}{3} - \frac{8\sqrt{2}}{3} + \frac{2}{3} = \frac{19}{6} \text{ sq units}$$

Hence, (b) is the correct answer.

Ex 8. Let $f(x)$ be defined by

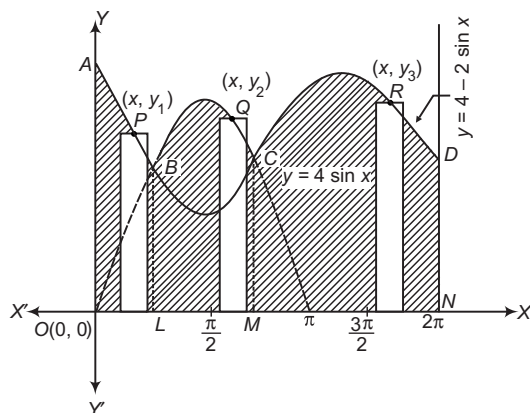
$$f(x) = \max(4 \sin x, 4 - 2 \sin x), 0 \leq x \leq 2\pi.$$

Then, the area bounded by the curve $y = f(x)$, Y -axis, X -axis and the ordinate at $x = 2\pi$, is

- (a) $\left(\pi + 4 \sin^{-1} \frac{2}{3} + \frac{\sqrt{5}}{3} \right)$ sq units
- (b) $\left(\pi + 4 \sin^{-1} \frac{4}{3} + \frac{\sqrt{5}}{3} \right)$ sq units
- (c) $\left(8\pi + 4 \sin^{-1} \frac{4}{3} + 4\sqrt{5} \right)$ sq units
- (d) $\left(4\pi + 8 \sin^{-1} \frac{2}{3} + 4\sqrt{5} \right)$ sq units

Sol. The graph of $f(x)$ is shown by the continuous curve shown in figure. At the points of intersection of $y = 4 \sin x$ and $y = 4 - 2 \sin x$, we have

$$4 \sin x = 4 - 2 \sin x \Rightarrow \sin x = \frac{2}{3}$$



Thus, the coordinates of B and C are $\left(\sin^{-1} \frac{2}{3}, \frac{8}{3} \right)$ and $\left(\pi - \sin^{-1} \frac{2}{3}, \frac{8}{3} \right)$, respectively.

$\therefore$ Required area

$$= \text{Area of region } OABL + \text{Area of region } LBCM$$

$$+ \text{Area of region } MCDN$$

$$= \int_0^{\sin^{-1} 2/3} (4 - 2 \sin x) dx + \int_{\sin^{-1} 1/3}^{\pi - \sin^{-1} 2/3} 4 \sin x dx$$

$$+ \int_{\pi - \sin^{-1} 2/3}^{2\pi} (4 - 2 \sin x) dx$$

$$= [4x + 2 \cos x]_0^{\sin^{-1} 2/3} - 4 [\cos x]_{\sin^{-1} 2/3}^{\pi - \sin^{-1} 2/3}$$

$$+ [4x + 2 \cos x]_{\pi - \sin^{-1} 2/3}^{2\pi}$$

$$= \left[4 \sin^{-1} \frac{2}{3} + \frac{2\sqrt{5}}{3} - 2 \right] - 4 \left[-\frac{\sqrt{5}}{3} - \frac{\sqrt{5}}{3} \right]$$

$$+ \left[(8\pi + 2) - \left(4\pi - 4 \sin^{-1} \frac{2}{3} \right) - 2 \left(1 - \frac{\sqrt{5}}{3} \right) \right]$$

$$= \left(4\pi + 8 \sin^{-1} \frac{2}{3} + 4\sqrt{5} \right) \text{ sq units}$$

Hence, (d) is the correct answer.

707

$$\begin{aligned}\text{Now, } A &= \frac{1}{6} \cdot \frac{b^2}{(1+b^2)^2} \\ \Rightarrow \frac{dA}{db} &= \frac{1}{6} \left[\frac{(1+b^2)^2 \cdot 2b - b^2 \cdot 2(1+b^2) \cdot 2b}{(1+b^2)^4} \right] \\ \Rightarrow \frac{dA}{db} &= \frac{b}{3(1+b^2)^3} [1+b^2-2b^2] \\ \Rightarrow \frac{dA}{db} &= \frac{b(1-b)(1+b)}{3(1+b^2)^3}\end{aligned}$$

For maximum value of A , put $\frac{dA}{db} = 0$.

$$\begin{aligned}\Rightarrow b(1-b)(1+b) &= 0 \\ \Rightarrow 1-b &= 0 \\ \Rightarrow b &= 1 \quad [\because b > 0]\end{aligned}$$

Since, $b > 0$. Therefore, $1+b > 0$. Thus, $\frac{dA}{db}$ changes

its sign from positive to negative in the neighbourhood of $b = 1$.

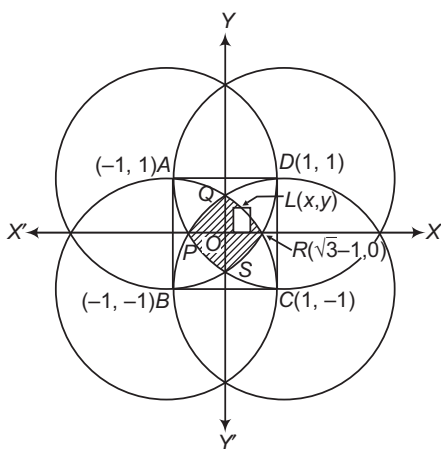
Therefore, A is maximum, when $b = 1$.

Hence, (a) is the correct answer.

Ex 11. A square is constructed by joining the points $A(-1, 1)$, $B(-1, -1)$, $C(1, -1)$ and $D(1, 1)$. Considering A, B, C and D as centres, four circles each of radius 2 units are described on the square. Then, the area common to these circles is

- (a) $3\left(\frac{\pi}{4} - \sqrt{3}\right)$ sq units
(b) $4\left(\frac{\pi}{3} - \sqrt{3}\right)$ sq units
(c) $\left(\frac{\pi}{3} - \sqrt{3}\right)$ sq unit
(d) $\left(\frac{\pi}{4} - \sqrt{3}\right)$ sq unit

Sol. We have to find the area of the shaded region in figure.



From the symmetry,

Required area = 4 (Area of the region $ORQO$) ... (i)
Clearly, R is the point of intersection of the circle having centre at $(-1, -1)$ and radius 2 i.e. $(x+1)^2 + (y+1)^2 = 2^2$ with X -axis.

Therefore, abscissa of R is given by

$$\begin{aligned}(x+1)^2 + (0+1)^2 &= 2^2 \\ \Rightarrow (x+1)^2 &= 3 \\ \Rightarrow x &= \sqrt{3} - 1\end{aligned}$$

$\therefore$ Area of the region $ORQO$

$$\begin{aligned}&= \int_0^{\sqrt{3}-1} y \, dx = \int_0^{\sqrt{3}-1} \{-1 + \sqrt{2^2 - (x+1)^2}\} \, dx \\ &\quad \left[\because L(x, y) \text{ lies on } (x+1)^2 + (y+1)^2 = 2^2 \right. \\ &\quad \left. \therefore y = -1 + \sqrt{2^2 - (x+1)^2} \right] \\ &= \left[-x + \frac{1}{2}(x+1)\sqrt{2^2 - (x+1)^2} + \frac{4}{2} \sin^{-1}\left(\frac{x+1}{2}\right) \right]_0^{\sqrt{3}-1} \\ &= \left[-(\sqrt{3}-1) + \frac{\sqrt{3}}{2} + \frac{4}{2} \sin^{-1}\frac{\sqrt{3}}{2} \right] \\ &\quad - \left[\frac{\sqrt{3}}{2} + \frac{4}{2} \sin^{-1}\left(\frac{1}{2}\right) \right]\end{aligned}$$

$$\begin{aligned}&= -\sqrt{3} + 1 + \frac{\sqrt{3}}{2} + 2 \times \frac{\pi}{3} - \frac{\sqrt{3}}{2} - 2 \times \frac{\pi}{6} \\ &= \left(\frac{\pi}{3} - \sqrt{3}\right)\end{aligned}$$

Thus, required area = $4\left(\frac{\pi}{3} - \sqrt{3}\right)$ sq units

Hence, (b) is the correct answer.

Ex 12. The area bounded by the curve $y = f(x)$, $y = x$ and the lines $x = 1$, $x = t$ is $[(t + \sqrt{1+t^2}) - \sqrt{2} - 1]$ sq unit for all $t > 1$. If $f(x)$ satisfying $f(x) > x$, $\forall x > 1$, then $f(x)$ is equal to

- (a) $x + 1 + \frac{x}{\sqrt{1+x^2}}$ (b) $x + \frac{x}{\sqrt{1+x^2}}$
(c) $1 + \frac{x}{\sqrt{1+x^2}}$ (d) $\frac{x}{\sqrt{1+x^2}}$

Sol. It is given that, $f(x) > x$, $\forall x > 1$

So, area bounded by $y = f(x)$, $y = x$ and the lines $x = 1$, $x = t$, is given by $\int_1^t \{f(x) - x\} \, dx$.

But this area is equal to $(t + \sqrt{1+t^2} - \sqrt{2} - 1)$ sq unit.

Therefore,

$$\int_1^t \{f(x) - x\} \, dx = t + \sqrt{1+t^2} - \sqrt{2} - 1, \forall t > 1$$

On differentiating both sides w.r.t. t , we get

$$f(t) - t = 1 + \frac{t}{\sqrt{1+t^2}}, \forall t > 1$$

$$\Rightarrow f(t) = t + 1 + \frac{t}{\sqrt{1+t^2}}, \forall t > 1$$

$$\therefore f(x) = x + 1 + \frac{x}{\sqrt{1+x^2}}, \forall x > 1$$

Hence, (a) is the correct answer.

Ex 13. Consider a square with vertices at $(1, 1)$, $(-1, 1)$, $(-1, -1)$ and $(1, -1)$. Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Then, the area of region S is equal to

- (a) $\frac{4}{3}(4\sqrt{2} - 5)$ sq units
 (b) $\frac{3}{4}(\sqrt{2} - 5)$ sq unit
 (c) $\frac{3}{4}(3\sqrt{2} - 5)$ sq unit
 (d) $\frac{3}{4}(3\sqrt{2} - 4)$ sq unit

Sol. Recall that a parabola is the locus of a point which is equidistant from a fixed point and a fixed line. The fixed point is the focus and the fixed line is the directrix. The points which are nearer to the fixed point (focus), then the fixed line (directrix) lie inside the parabola.

Curve C_1 represents a parabola having focus at $(0, 0)$ and directrix AD . So, its equation is

$$(y - 0)^2 = -2 \left(x - \frac{1}{2} \right) \quad \dots(i)$$

Similarly, equations of parabolas C_2 , C_3 and C_4 are respectively

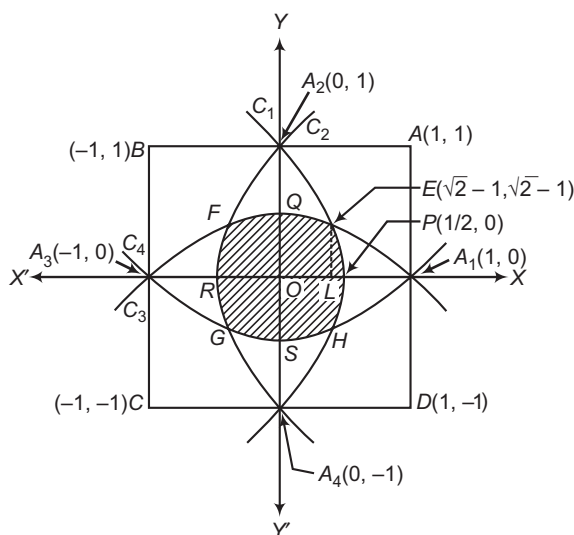
$$(y - 0)^2 = 2 \left(x + \frac{1}{2} \right) \quad \dots(ii)$$

$$(x - 0)^2 = -2 \left(y - \frac{1}{2} \right) \quad \dots(iii)$$

and $(x - 0)^2 = 2 \left(y + \frac{1}{2} \right) \quad \dots(iv)$

The region S is the shaded region enclosed by these four parabolas.

Parabolas C_1 and C_3 intersect at point E whose coordinates are obtained by solving Eqs. (i) and (iii) simultaneously.



We have, $y^2 + 2x - 1 = 0 \quad \dots(v)$
 and $x^2 + 2y - 1 = 0 \quad \dots(vi)$

On subtracting Eq. (vi) from Eq. (v), we get

$$y^2 - x^2 + 2(x - y) = 0$$

$$\Rightarrow (y - x)(y + x - 2) = 0$$

$$\Rightarrow y = x \text{ or } y = 2 - x$$

On putting $y = 1$ in Eq. (v), we get

$$y^2 + 2y - 1 = 0$$

$$\Rightarrow y = \sqrt{2} - 1$$

Thus, the coordinates of E are $(\sqrt{2} - 1, \sqrt{2} - 1)$.

From the symmetry, we observe that

Area of region $S = 4$ (Area of region $OPEQO$)

$$= 4 (\text{Area of region } OLEQO + \text{Area of region } LPEL)$$

$$= 4 \left[\int_0^{\sqrt{2}-1} \left(\frac{1-x^2}{2} \right) dx + \int_{\sqrt{2}-1}^{1/2} (\sqrt{1-2x}) dx \right]$$

$$= 4 \left\{ \left[\frac{1}{2} \left(x - \frac{x^3}{3} \right) \right]_0^{\sqrt{2}-1} - \frac{1}{3} [(1-2x)^{3/2}]_{\sqrt{2}-1}^{1/2} \right\}$$

$$= 4 \left[(\sqrt{2}-1) \frac{\sqrt{2}}{3} + \frac{1}{3} (\sqrt{2}-1)^3 \right]$$

$$= \frac{4}{3} (\sqrt{2}-1) [3 - \sqrt{2}] = \frac{4}{3} (4\sqrt{2} - 5) \text{ sq units}$$

Hence, (a) is the correct answer.

Ex 14. If $y = f(x)$ makes positive intercepts of 2 units and 1 unit on x and y -coordinate axes and encloses an area of $\frac{3}{4}$ sq units with the

axes, then $\int_0^2 x f'(x) dx$ is equal to

- (a) $\frac{3}{2}$ (b) 1
 (c) $\frac{5}{4}$ (d) $-\frac{3}{4}$

Sol. Clearly, $y = f(x)$ passes through $(2, 0)$ and $(0, 1)$.

$$\therefore 0 = f(2) \text{ and } 1 = f(0)$$

Also, $\int_0^2 f(x) dx = \frac{3}{4}$ [given]

Now, $\int_0^2 x f'(x) dx = [x f(x)]_0^2 - \int_0^2 f(x) dx$

$$\Rightarrow \int_0^2 x f'(x) dx = [2f(2) - 0f(0)] - \frac{3}{4}$$

$$\Rightarrow \int_0^2 x f'(x) dx = 2 \times 0 - 0 \times 1 - \frac{3}{4} = -\frac{3}{4}$$

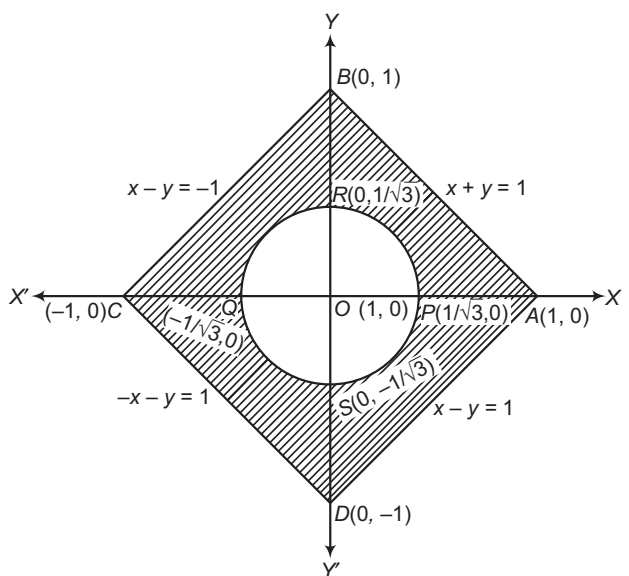
Hence, (d) is the correct answer.

Ex 15. The area of the bounded region by the curves $|x + y| \leq 1$, $|x - y| \leq 1$ and $3x^2 + 3y^2 = 1$, is

- (a) $\left(2 - \frac{2\pi}{3} \right)$ sq unit
 (b) $\left(2 - \frac{\pi}{3} \right)$ sq units
 (c) $\left(3 - \frac{\pi}{2} \right)$ sq units
 (d) None of the above

Sol. We have, $|x + y| \leq 1 \Leftrightarrow -1 \leq x + y \leq 1$
and $|x - y| \leq 1 \Leftrightarrow -1 \leq x - y \leq 1$
Clearly, $|x - y| \leq 1$ represents that part of XY -plane which lies between the lines $x + y = 1$ and $x + y = -1$.
Also, $|x - y| \leq 1$ represents the region lying between the lines $x - y = 1$ and $x - y = -1$.
The equation $3x^2 + 3y^2 = 1$ represents a circle having centre at the origin and radius $\frac{1}{\sqrt{3}}$.

The required area is shaded in figure.



$$\begin{aligned} \therefore \text{Required area} &= 4 (\text{Area of } \triangle OAB - \text{Area of quadrant } OPRO) \\ &= 4 \left\{ \frac{1}{2} \times 1 \times 1 - \frac{\pi}{4} \left(\frac{1}{\sqrt{3}} \right)^2 \right\} \\ &= 4 \left(\frac{1}{2} - \frac{\pi}{12} \right) = \left(2 - \frac{\pi}{3} \right) \text{ sq units} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 16. The area of the region containing the points satisfying $|y| + \frac{1}{2} \leq e^{-|x|}$, $\max(|x|, |y|) \leq 2$, is

- (a) $4 \log \left(\frac{e}{3} \right)$ sq units
- (b) $2 \log \left(\frac{e}{2} \right)$ sq units
- (c) $3 \log \left(\frac{e}{4} \right)$ sq units
- (d) None of the above

Sol. We observe that,

$$\max(|x|, |y|) \leq 2$$

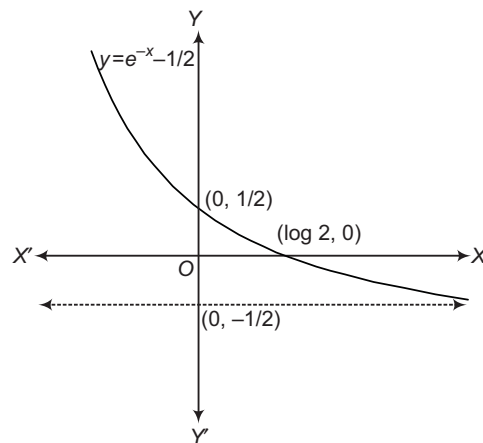
represents the region enclosed by the square whose sides are $x = 2, x = -2, y = 2$ and $y = -2$.

In order to draw the graph of $|y| + \frac{1}{2} \leq e^{-|x|}$, let us consider the curve given by

$$y + \frac{1}{2} = e^{-x}$$

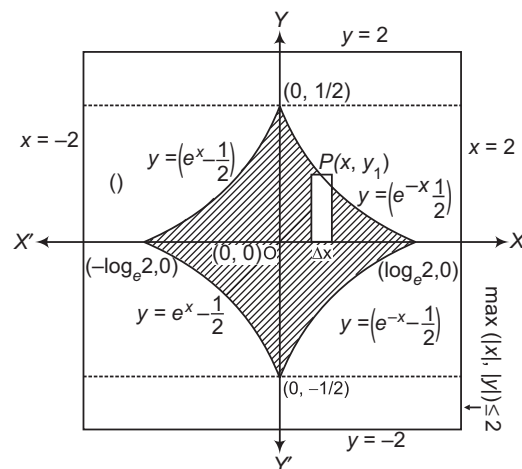
$$\text{or } y = e^{-x} - \frac{1}{2}$$

The graph of this curve is shown in figure.



By using transformations of graphs, the graph of $|y| = e^{-|x|} - \frac{1}{2}$ is as shown in figure.

The region containing the points satisfying the two inequalities is shaded in figure.



$\therefore$ Required area

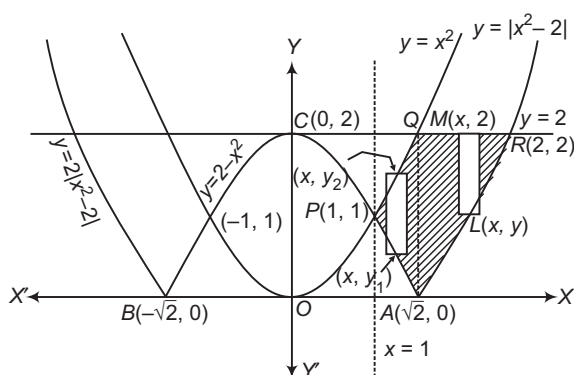
$$\begin{aligned} &= 4 \int_0^{\log 2} \left(e^{-x} - \frac{1}{2} \right) dx = 4 \left[-e^{-x} - \frac{x}{2} \right]_0^{\log 2} \\ &= 4 \left\{ -\frac{1}{2} - \frac{1}{2} \log 2 + 1 \right\} \\ &= 2 (1 - \log 2) = 2 \log \left(\frac{e}{2} \right) \text{ sq units} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 17. The area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$ which lies to the right of the line $x = 1$, is

- (a) $\left(\frac{12 - 20\sqrt{3}}{2}\right)$ sq unit
 (b) $\left(\frac{20 - 12\sqrt{2}}{3}\right)$ sq units
 (c) $\left(\frac{20 - \sqrt{2}}{3}\right)$ sq units
 (d) $\left(\frac{12 - 20\sqrt{2}}{3}\right)$ sq unit

Sol. The region bounded by the given curves on the right side of $x = 1$ is shaded in figure. Let us slice the shaded region into vertical strips.



$$\begin{aligned} \therefore \text{Required area} &= \text{Area of the region } APQA \\ &\quad + \text{Area of the region } AQR A \\ &= \int_1^{\sqrt{2}} (y_2 - y_1) dx + \int_{\sqrt{2}}^2 (2 - y) dx \\ &= \int_1^{\sqrt{2}} \{x^2 - (2 - x^2)\} dx + \int_{\sqrt{2}}^2 (4 - x^2) dx \\ &= \int_1^{\sqrt{2}} (2x^2 - 2) dx + \int_{\sqrt{2}}^2 (4 - x^2) dx \\ &= \left[\frac{2}{3}x^3 - 2x \right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3} \right]_{\sqrt{2}}^2 \\ &= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} \right) - \left(\frac{2}{3} - 2 \right) + \left(8 - \frac{8}{3} \right) \\ &\quad - \left(4\sqrt{2} - \frac{2\sqrt{2}}{3} \right) \\ &= \left(-4\sqrt{2} + \frac{20}{3} \right) = \left(\frac{20 - 12\sqrt{2}}{3} \right) \text{ sq units} \end{aligned}$$

Hence, (b) is the correct answer.

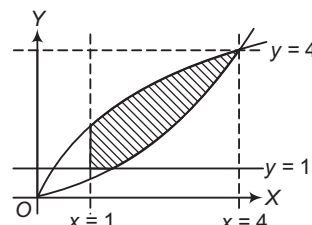
Ex 18. Consider two curves $C_1 : y^2 = 4[\sqrt{y}]x$ and $C_2 : x^2 = 4[\sqrt{x}]y$, where $[\]$ denotes the greatest integer function. Then, the area of region enclosed by these two curves within the square formed by the lines $x = 1$, $y = 1$, $x = 4$, $y = 4$, is

- (a) $\frac{8}{3}$ sq units
 (b) $\frac{10}{3}$ sq units
 (c) $\frac{11}{3}$ sq units
 (d) $\frac{11}{4}$ sq units

Sol. We have, $y^2 = 4[\sqrt{y}]x$

For $y \in [1, 4)$, $[\sqrt{y}] = 1 \Rightarrow y^2 = 4x$

Similarly, for $x \in [1, 4)$, $[\sqrt{x}] = 1$ and $x^2 = 4[\sqrt{x}]y$ would transform into $x^2 = 4y$.



The required area is being shaded.

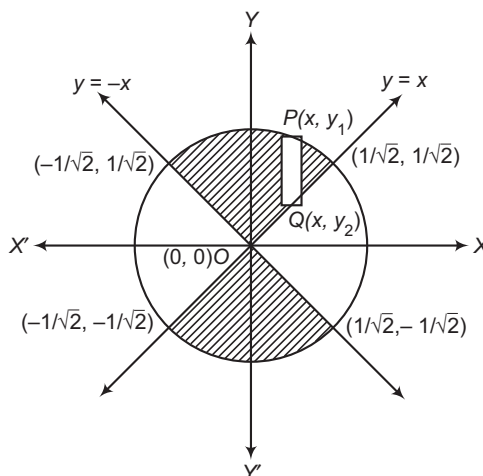
$$\begin{aligned} \therefore A &= \int_1^2 (2\sqrt{x} - 1) dx + \int_2^4 \left(2\sqrt{x} - \frac{x^2}{4} \right) dx \\ &= \left[\frac{4}{3}x^{3/2} - x \right]_1^2 + \left[\frac{4}{3}x^{3/2} - \frac{x^3}{12} \right]_2^4 = \frac{11}{3} \text{ sq units} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 19. The area of the region $R = \{(x, y) : |x| \leq |y| \text{ and } x^2 + y^2 \leq 1\}$, is

- (a) $\frac{3\pi}{8}$ sq unit
 (b) $\frac{5\pi}{8}$ sq units
 (c) $\frac{\pi}{2}$ sq units
 (d) $\frac{\pi}{8}$ sq unit

Sol. Required area = Area of the shaded region
 = 4 (Area of the shaded region in first quadrant)
 = $4 \int_0^{1/\sqrt{2}} (y_1 - y_2) dx = 4 \int_0^{1/\sqrt{2}} (\sqrt{1 - x^2} - x) dx$



$$\begin{aligned} &= 4 \left[\frac{1}{2}x\sqrt{1 - x^2} + \frac{1}{2}\sin^{-1}x - \frac{x^2}{2} \right]_0^{1/\sqrt{2}} \\ &= 4 \left[\frac{1}{2\sqrt{2}} \times \frac{1}{\sqrt{2}} + \frac{1}{2} \times \frac{\pi}{4} - \frac{1}{4} \right] = \frac{\pi}{2} \text{ sq units} \end{aligned}$$

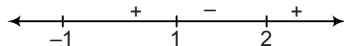
Hence, (c) is the correct answer.

Ex 20. Area bounded by $f(x) = \frac{(x-1)(x+1)}{x-2}$,

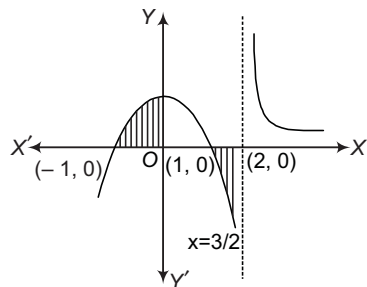
X -axis and ordinates $x=0, x=\frac{3}{2}$, is

- (a) $\frac{4}{5}$ sq unit (b) $\frac{7}{8}$ sq unit
(c) 1 sq unit (d) None of these

Sol. Clearly, $f(x)$ is continuous everywhere and differentiable except at $x=2$. The signs of $f(x)$ for different values of x are shown in figure.



A rough sketch of the curve $y = f(x)$ is as shown in figure.



$$\begin{aligned} \therefore \text{Required area} &= \int_0^1 \frac{(x-1)(x+1)}{x-2} dx - \int_1^{3/2} \frac{(x-1)(x+1)}{x-2} dx \\ &= \int_0^1 \left(x+2 + \frac{3}{x-2} \right) dx - \int_1^{3/2} \left(x+2 + \frac{3}{x-2} \right) dx \\ &= \left[\frac{(x+2)^2}{2} + 3 \log|x-2| \right]_0^1 \\ &\quad - \left[\frac{(x+2)^2}{2} + 3 \log|x-2| \right]_1^{3/2} \\ &= \left(\frac{9}{2} - 2 - 3 \log 2 \right) - \left[\left(\frac{49}{8} + 3 \log \frac{1}{2} \right) - \frac{9}{2} \right] \\ &= \frac{7}{8} \text{ sq unit} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 21. The area of the loop of the curve $ay^2 = x^2(a-x)$ is

- (a) $4a^2$ sq units (b) $\frac{8a^2}{15}$ sq unit
(c) $\frac{16a^2}{9}$ sq units (d) None of these

Sol. We have, $ay^2 = x^2(a-x) \Rightarrow y = \pm x\sqrt{\frac{a-x}{a}}$

For curve tracing, $y = x\sqrt{\frac{a-x}{a}}$

We must have, $x \leq a$

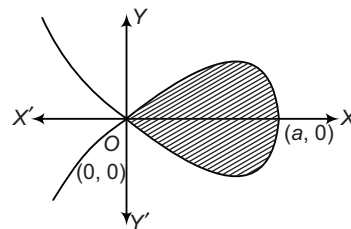
For $0 < x \leq a$, $y > 0$ and for $x < 0$, $y < 0$

Also, $y = 0 \Rightarrow x = 0, a$

So, curve is symmetrical about X -axis.

When $x \rightarrow -\infty$, $y \rightarrow -\infty$

Also, it can be verified that y has only one point of maxima for $0 < x < a$.



$\therefore$ Required area, $A = 2 \int_0^a x \sqrt{\frac{a-x}{a}} dx$

Put $\sqrt{\frac{a-x}{a}} = t \Rightarrow 1 - \frac{x}{a} = t^2$

$\Rightarrow x = a(1-t^2) \Rightarrow dx = -2at dt$

$\therefore A = 2 \int_1^0 a(1-t^2)t(-2at) dt$

$$= 4a^2 \int_0^1 (t^2 - t^4) dt = 4a^2 \left[\frac{t^3}{3} - \frac{t^5}{5} \right]_0^1$$

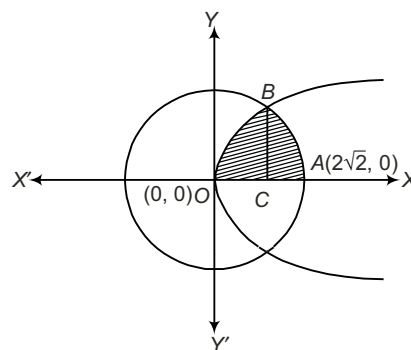
$$= 4a^2 \left[\frac{1}{3} - \frac{1}{5} \right] = \frac{8a^2}{15} \text{ sq unit}$$

Hence, (b) is the correct answer.

Ex 22. If the parabola $y^2 = 2x$ divides the circle $x^2 + y^2 = 8$ in two parts, then the ratio of the areas of these parts is

- (a) $\frac{3\pi-2}{10\pi+2}$ (b) $\frac{3\pi+2}{9\pi-2}$ (c) $\frac{6\pi-3}{11\pi-5}$ (d) $\frac{2\pi-9}{9\pi+2}$

Sol. The graph of $y^2 = 2x$ (a parabola having axis as the positive X -axis and vertex at the origin) and $x^2 + y^2 = 8$ (a circle having radius $2\sqrt{2}$ and centre at origin) are given in the following figure:



Let the area of the smaller part of circle be A_1 and that of the bigger part be A_2 . We have to find $\frac{A_1}{A_2}$.

The point B is a point of intersection (lying in the first quadrant) of the given parabola and the circle, whose coordinates can be obtained by solving the two equations $y^2 = 2x$ and $x^2 + y^2 = 8$.

$$\Rightarrow x^2 + 2x = 8 \Rightarrow (x-2)(x+4) = 0$$

$$\Rightarrow x = 2, -4$$

But $x = -4$ is not possible as both the points of intersection have the same positive x -coordinate.

Thus, $C \equiv (2, 0)$



$$\begin{aligned} \text{Now, } A_1 &= 2 [\text{Area of region } OBCO \\ &\quad + \text{Area of region } CBAC] \\ &= 2 \left[\int_0^2 y_1 dx + \int_2^{2\sqrt{2}} y_2 dx \right] \end{aligned}$$

where, y_1 and y_2 are respectively the values of y from the equations of the parabola and that of the circle.

$$\text{or } A_1 = 2 \left[\int_0^2 \sqrt{2x} dx + \int_2^{2\sqrt{2}} \sqrt{8-x^2} dx \right]$$

[since, $y^2 = 2x \Rightarrow y = \pm \sqrt{2x}$, but we choose $y = \sqrt{2x}$, as for the arc OB , y is positive and same is the reasoning for circle]

$$\begin{aligned} \Rightarrow A_1 &= 2 \left[\sqrt{2} \cdot \frac{2}{3} \cdot x^{3/2} \right]_0^2 + \left[\frac{x}{2} \sqrt{8-x^2} \right. \\ &\quad \left. + \frac{8}{2} \sin^{-1} \frac{x}{2\sqrt{2}} \right]_2^{2\sqrt{2}} \\ &= \frac{16}{3} + 2 \left[2\pi - \left(2 + 4 \times \frac{\pi}{4} \right) \right] = \left(\frac{4}{3} + 2\pi \right) \text{ sq units} \end{aligned}$$

Area of the circle = $\pi (2\sqrt{2})^2 = 8\pi$ sq units

Thus, $A_2 = 8\pi - A_1 = 6\pi - 4/3$.

Then, required ratio = $\frac{4/3 + 2\pi}{6\pi - 4/3} = \frac{2 + 3\pi}{9\pi - 2}$

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

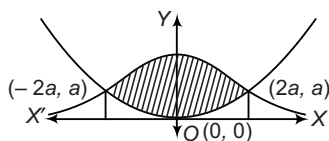
Ex 23. Area bounded by curves $y = \frac{x^2}{4a}$ and

$$y = \frac{8ab}{x^2 + 4a^2}, \text{ is}$$

$$(a) \frac{1}{3} (6\pi - 4), \text{ if } a = 1 \quad (b) \frac{1}{3} (4\pi + 3), \text{ if } a = 1$$

$$(c) \frac{4}{3} (6\pi - 4), \text{ if } a = 2 \quad (d) \frac{1}{3} (2\pi + 3), \text{ if } a = 1$$

Sol. The curve $y(x^2 + 4a^2) = 8a^3$ is symmetrical about Y -axis and cut it at $A(0, 2a)$.



Tangent at A is parallel to X -axis and X -axis is an asymptote.

This curve meets $x^2 = 4ay$, where $\frac{x^2}{4a} = \frac{8a^3}{x^2 + 4a^2}$

$$\Rightarrow x^4 + 4a^2x^2 - 32a^4 = 0$$

$$\Rightarrow (x^2 - 4a^2)(x^2 + 8a^2) = 0$$

$$\Rightarrow x = \pm 2a$$

$$\therefore \text{Required area} = 2 \left[\int_0^{2a} \frac{8a^3}{x^2 + 4a^2} dx - \int_0^{2a} \frac{x^2}{4a} dx \right]$$

$$= 16a^3 \cdot \frac{1}{2a} \left[\tan^{-1} \frac{x}{2a} \right]_0^{2a} - \frac{1}{2a} \left[\frac{x^3}{3} \right]_0^{2a}$$

$$= 8a^2 \cdot \frac{\pi}{4} - \frac{1}{6a} \cdot 8a^3 = 2\pi a^2 - \frac{4}{3}a^2 = \frac{a^2}{3} (6\pi - 4)$$

Hence, (a) and (c) are the correct answers.

Ex 24. If A_i is the area bounded by $|x - a_i| + |y| = b_i$;

$i \in N$, where $a_{i+1} = a_i + \frac{3}{2}b_i$ and $b_{i+1} = \frac{b_i}{2}$;

$a_1 = 0, b_1 = 32$, then

$$(a) A_3 = 128$$

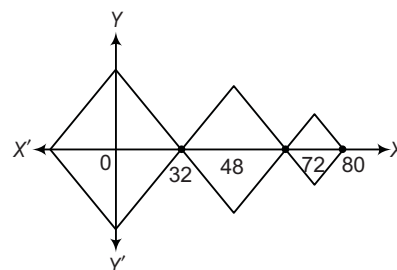
$$(b) A_3 = 256$$

$$(c) \lim_{n \rightarrow \infty} \sum_{i=1}^n A_i = \frac{8}{3} (32)^2$$

$$(d) \lim_{n \rightarrow \infty} \sum_{i=1}^n A_i = \frac{4}{3} (16)^2$$

Sol. $a_1 = 0, b_1 = 32, a_2 = a_1 + \frac{3}{2}b_1 = 48, b_2 = \frac{b_1}{2} = 16$

$$a_3 = 48 + \frac{3}{2} \times 16 = 72, b_3 = \frac{16}{2} = 8$$



So, the three loops from $i = 1$ to $i = 3$ are alike.

Now, area of i th loop (square) = $\frac{1}{2} (\text{diagonal})^2$

$$A_i = \frac{1}{2} (2b_i)^2 = 2(b_i)^2$$

$$\Rightarrow A_3 = 2(8)^2 = 128$$

$$\therefore \frac{A_{i+1}}{A_i} = \frac{2(b_{i+1})^2}{2(b_i)^2} = \frac{1}{4}$$

So, the areas form a GP series.

Thus, sum of GP upto infinite terms

$$= A_1 \frac{1}{1-r} = 2(32)^2 \times \frac{1}{1-\frac{1}{4}}$$

$$= 2 \times (32)^2 \times \frac{4}{3} = \frac{8}{3} (32)^2 \text{ sq units}$$

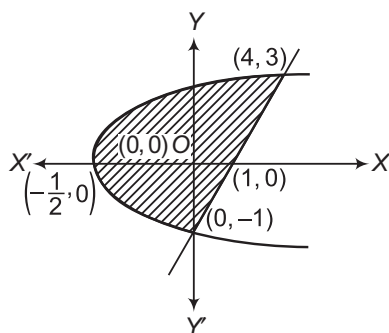
Hence, (a) and (c) are the correct answers.

Ex 25. Let $L: x - y - 1 = 0$ be a line and $C: y^2 = 2x + 1$ be a parabola, then

- (a) area bounded by L and C lying in the upper half plane, is $\frac{32}{3}$ sq units
 (b) area bounded by L and C lying in the plane, is $\frac{16}{3}$ sq units
 (c) area bounded by L and C in the upper half plane, is $\frac{9}{2}$ sq units
 (d) area bounded by L and C in the lower half plane, is $\frac{5}{6}$ sq unit

Sol. Area above the upper half plane

$$= \int_{-1/2}^4 (\sqrt{2x+1}) dx - \int_1^4 (x-1) dx = \frac{9}{2} \text{ sq units}$$



Area below the lower half plane

$$= \left| \int_{-1/2}^0 -\sqrt{2x+1} dx \right| + \left| \int_0^1 (x-1) dx \right| = \frac{5}{6} \text{ sq unit}$$

$$\therefore \text{Required area} = \frac{9}{2} + \frac{5}{6} = \frac{16}{3} \text{ sq units}$$

Hence, (b), (c) and (d) are the correct answers.

Ex 26. The area bounded by the parabola $y = x^2 - 7x + 10$ and X -axis, equals

Type 3. Assertion and Reason

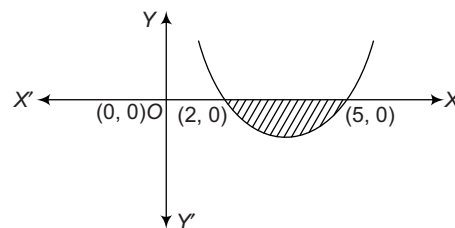
Directions (Ex. Nos. 28-32) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

Ex 28. Statement I The area of an ellipse $2x^2 + 3y^2 = 6$ will be more than the area of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

- (a) area bounded by $y = -x^2 + 7x - 10$ and X -axis
 (b) $\frac{1}{6}$ sq unit
 (c) $\frac{5}{6}$ sq unit
 (d) $\frac{9}{2}$ sq units

Sol. Area bounded by parabola $x^2 - 7x + 10$ and X -axis is given by



$$\int_2^5 |x^2 - 7x + 10| dx = \int_2^5 -x^2 + 7x - 10 dx = \frac{9}{2} \text{ sq units}$$

Hence, (a) and (d) are the correct answers.

Ex 27. Let f and g be continuous functions on $a \leq x \leq b$ and set $p(x) = \max\{f(x), g(x)\}$ and $q(x) = \min\{f(x), g(x)\}$, then the area bounded by the curves $y = p(x)$, $y = q(x)$ and the ordinates $x = a$, $x = b$, is given by

- (a) $\int_a^b |f(x) - g(x)| dx$ (b) $\int_a^b |p(x) - q(x)| dx$
 (c) $\int_a^b \{f(x) - g(x)\} dx$ (d) $\int_a^b \{p(x) - q(x)\} dx$

$$\text{Sol. } \max\{f(x), g(x)\} = \frac{1}{2} [|f(x) + g(x)| + |f(x) - g(x)|]$$

$$\min\{f(x), g(x)\} = \frac{1}{2} [|f(x) + g(x)| - |f(x) - g(x)|]$$

[by definitions]

Hence, (a), (b) and (d) are the correct answers.

Statement II The length of the semi-major axis of an ellipse $2x^2 + 3y^2 = 6$ is more than the radius of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

$$\text{Sol. The ellipse is } \frac{x^2}{3} + \frac{y^2}{2} = 1 \text{ and the circle is}$$

$$(x-1)^2 + (y+2)^2 = 1.$$

$$\therefore \text{Area of an ellipse} = \pi \sqrt{3} \sqrt{2} = \sqrt{6} \pi$$

$$\text{and area of a circle} = \pi \cdot 1^2 = \pi$$

So, Statement I is true.

Statement II is true in this particular example, in general may not be true.

Hence, (b) is the correct answer.



Ex 29. Statement I The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is least, if $k = 0$.

Statement II The area bounded by the curves $y = x^2 - 3$ and $y = kx + 2$, is $\sqrt{k^2 + 20}$.

Sol. Let the line $y = kx + 2$ cuts $y = x^2 - 3$ at $x = \alpha$ and $x = \beta$, where α and β are the roots of the equation $x^2 - kx - 5 = 0$, $\alpha < \beta$.

Then, area bounded by these two curves

$$\begin{aligned} f(x) &= \int_{\alpha}^{\beta} (y_1 - y_2) dx \\ &= \int_{\alpha}^{\beta} [(kx + 2) - (x^2 - 3)] dx = \left[\frac{kx^2}{2} + 2x - \frac{x^3}{3} + 3x \right]_{\alpha}^{\beta} \\ &= \left(\frac{k\beta^2}{2} + 2\beta - \frac{\beta^3}{3} + 3\beta \right) - \left(\frac{k\alpha^2}{2} + 2\alpha - \frac{\alpha^3}{3} + 3\alpha \right) \\ &= \frac{k}{2} (\beta^2 - \alpha^2) + 2(\beta - \alpha) - \frac{1}{3} (\beta^3 - \alpha^3) + 3(\beta - \alpha) \\ &= (\beta - \alpha) \left[\frac{k}{2} (\beta + \alpha) + 2 - \frac{1}{3} (\beta^2 + \alpha^2 + \alpha\beta) + 3 \right] \end{aligned}$$

Now, α, β are the roots of the equation

$$x^2 - kx - 5 = 0, \alpha + \beta = k; \alpha\beta = -5$$

where, $|\beta - \alpha| = \sqrt{k^2 + 20}$

Thus, $f(k)$ can be easily computed, we will get

$$f(k) = \frac{(k^2 + 20)^{3/2}}{6}$$

This shows that Statement II is false but $f(k)$ is least, when $dk = 0$.

Thus, Statement I is true.

Hence, (c) is the correct answer.

Ex 30. Consider the curve $f(x) = y = e^{x^3}$.

Statement I Area enclosed by the curve $f(x)$ between the lines $x = a, x = b$ and X -axis, is $\int_a^b e^{x^3} dx$.

Statement II $f(x)$ is an increasing function.

Sol. Since, $y = e^{x^3} \Rightarrow \frac{dy}{dx} = e^{x^3} \cdot 3x^2 > 0$

Hence, y is an increasing function and area enclosed by the curve $y = e^{x^3}$ between the lines $x = a, x = b$ and X -axis, is $\int_a^b e^{x^3} dx$.

Hence, (b) is the correct answer.

Ex 31. Statement I The area enclosed by the curve $|x| + |y| = 2$ is 8 units.

Statement II $|x| + |y| = 2$ represents a square of side length $\sqrt{8}$ units.

Sol. Clearly, $|x| + |y| = 2$ represents a square of $\sqrt{8}$ units and area of square is equal to square of the side length. Hence, (a) is the correct answer.

Ex 32. Statement I The area of the curve $y = \sin^2 x$ from 0 to π will be more than that of the curve $y = \sin x$ from 0 to π .

Statement II $x^2 > x$, if $x > 1$.

Sol. $\because \sin^2 x \leq \sin x, \forall x \in (0, \pi)$

Therefore, area of $y = \sin^2 x$ will be lesser from area of $y = \sin x$.

Statement II is obviously true.

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 33-35) Two curves $C_1 \equiv [f(y)]^{2/3} + [f(x)]^{1/3} = 0$ and $C_2 \equiv [f(y)]^{2/3} + [f(x)]^{2/3} = 12$, satisfying the relation $(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2)$.

Ex 33. The area bounded by C_1 and C_2 is

- (a) $(2\pi - \sqrt{3})$ sq units (b) $(2\pi + \sqrt{3})$ sq units
(c) $(\pi + \sqrt{6})$ sq units (d) $(2\sqrt{3} - \pi)$ sq unit

Sol. Given,

$$\begin{aligned} (x - y)f(x + y) - (x + y)f(x - y) &= 4xy(x^2 - y^2) \\ &= (x^2 - y^2)[(x + y)^2 - (x - y)^2] \\ &= (x - y)(x + y)^3 - (x + y)(x - y)^3 \end{aligned}$$

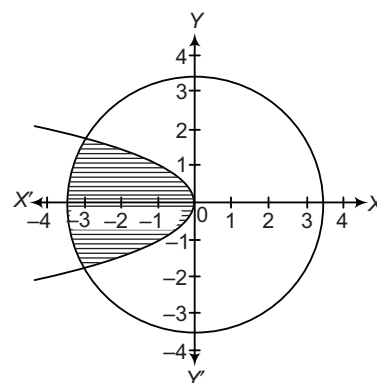
$$\Rightarrow f(x + y) = (x + y)^3$$

$$\Rightarrow f(x) = x^3, f(y) = y^3$$

Now, equations of given curves are

$$y^2 + x = 0 \quad \dots(i)$$

$$\text{and} \quad x^2 + y^2 = 12 \quad \dots(ii)$$



On solving Eqs. (i) and (ii), we get

$$x = -3, y = \pm \sqrt{3}$$

$\therefore$ Area bounded by curves,

$$A = 2 \left[\int_{-2\sqrt{3}}^{-3} \sqrt{12 - x^2} dx + \left| \int_{-3}^0 \sqrt{-x} dx \right| \right]$$

$$\text{Let } I_1 = 2 \int_{-2\sqrt{3}}^{-3} \sqrt{12 - x^2} dx = 2 \int_{-\pi/2}^{-\pi/3} 12 \cos^2 \theta d\theta$$

$$\begin{aligned}
 &= 12 \left[\int_{-\pi/2}^{-\pi/3} (1 + \cos 2\theta) d\theta \right] \\
 &= 12 \left[\theta + \frac{\sin \theta}{2} \right]_{-\pi/2}^{-\pi/3} = 12 \left[-\frac{\pi}{3} - \frac{\sqrt{3}}{4} + \frac{\pi}{2} \right] \\
 &= 12 \left[\frac{\pi}{6} - \frac{\sqrt{3}}{4} \right] = 2\pi - 3\sqrt{3} \\
 \text{and } I_2 &= 2 \int_{-3}^0 \sqrt{-x} dx = \frac{2 [(-x)^{3/2}]_{-3}^0}{-3/2} \\
 &= -\frac{4}{3} [0 - 3^{3/2}] = 4\sqrt{3} \\
 \therefore A &= 2\pi - 3\sqrt{3} + 4\sqrt{3} = (2\pi + \sqrt{3}) \text{ sq units}
 \end{aligned}$$

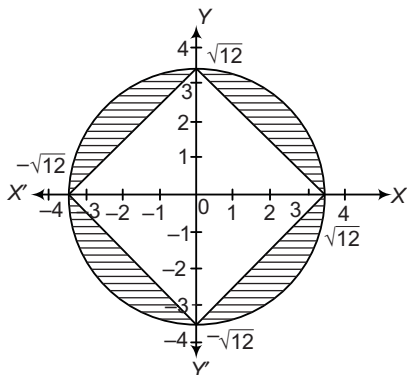
Hence, (b) is the correct answer.

Ex 34. The area bounded by the curve C_2 and

$$|x| + |y| = \sqrt{12} \text{ is}$$

- (a) $(12\pi - 24)$ sq units
 (b) $(6 - \sqrt{12})$ sq units
 (c) $(2\sqrt{12} - 6)$ sq units
 (d) None of the above

Sol. Required area = Area of circle - Area of square
 $= \pi(r)^2 - (\text{side})^2 = \pi(\sqrt{12})^2 - (\sqrt{24})^2$
 $= (12\pi - 24)$ sq units

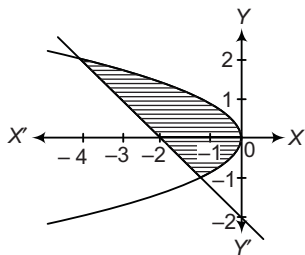


Hence, (a) is the correct answer.

Ex 35. The area bounded by C_1 and $x + y + 2 = 0$ is

- (a) $\frac{5}{2}$ sq units
 (b) $\frac{7}{2}$ sq units
 (c) $\frac{9}{2}$ sq units
 (d) None of the above

Sol. Required area = $\int_{-1}^2 [-y^2 - (-y - 2)] dy$



$$\begin{aligned}
 &= \left[\frac{y^2}{2} + 2y - \frac{y^3}{3} \right]_{-1}^2 \\
 &= \left[\frac{4}{2} + 4 - \frac{8}{3} - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right] = \frac{9}{2} \text{ sq units}
 \end{aligned}$$

Hence, (c) is the correct answer.

Passage II (Ex. Nos. 36-40) $f(x)$ is a continuous function satisfying $x^4 - 4x^2 \leq f(x) \leq 2x^2 - x^3$, $\forall x \in [0, 2]$, such that the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, Y -axis and the line $x = t$ ($0 \leq t \leq 2$) is k times the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, Y -axis and the line $x = t$ ($0 \leq t \leq 2$).

Ex 36. The function $f(x)$ is given by

- (a) $\frac{1}{k+1} [x^4 - kx^3 + 2(k-2)x^2]$
 (b) $\frac{1}{k+1} [x^4 - (k-2)x^3 + 2(k-2)x^2]$
 (c) $\frac{1}{k+1} [x^4 - (k-2)x^3 + 2(k-2)x]$
 (d) $\frac{1}{k+1} [x^4 - kx^3 + 2(k-2)x]$

Sol. Let A_1 be the area bounded by $y = f(x)$, $y = x^4 - 4x^2$, Y -axis and $x = t$.

$$\text{Since, } f(x) \geq x^4 - 2x^2, \forall x \in [0, 2]$$

$$\therefore A_1 = \int_0^t [f(x) - (x^4 - 2x^2)] dx \quad \dots(i)$$

Let A_2 be the area bounded by $y = f(x)$, $y = 2x^2 - x^3$, Y -axis and $x = t$, then

$$A_2 = \int_0^t [(2x^2 - x^3) - f(x)] dx$$

$$[\because f(x) \leq (2x^2 - x^3), \forall x \in [0, 2]]$$

It is given that, $A_1 = kA_2$

$$\therefore \int_0^t [f(x) - (x^4 - 4x^2)] dx = k \int_0^t [(2x^2 - x^3) - f(x)] dx$$

On differentiating both sides w.r.t. t , we get

$$f(t) - (t^4 - 4t^2) = k[(2t^2 - t^3) - f(t)]$$

$$\Rightarrow (1+k)f(t) = k(2t^2 - t^3) + (t^4 - 4t^2)$$

$$\Rightarrow f(t) = \frac{k}{k+1} (2t^2 - t^3) + \frac{t^4 - 4t^2}{k+1}, \quad \forall t \in [0, 2]$$

$$\text{Thus, } f(x) = \frac{1}{k+1} [x^4 - kx^3 + 2(k-2)x^2]$$

Hence, (a) is the correct answer.

Ex 37. If $k = 0$, then $f(x)$ is symmetric about the

- (a) origin
 (b) Y -axis
 (c) X -axis
 (d) curve $y = x$

Sol. If $k = 0$, $f(x) = x^4 - 4x^2$

which is symmetric about Y -axis.

Hence, (b) is the correct answer.

Ex 38. If $k = 0$, then $f(x)$ attains local maxima at

- (a) 0 (b) $\sqrt{2}$
(c) $-\sqrt{2}$ (d) None of these

Sol. We have, $f'(x) = 4x^3 - 8x = 4x(x^2 - 2)$

So, the function is local maximum is at $x = 0$.

Hence, (a) is the correct answer.

Ex 39. If $k = 2$, then $f(x)$ attains point of inflection at

- (a) 0 (b) $\sqrt{2}$
(c) $-\sqrt{2}$ (d) None of these

Sol. If $k = 2$, then $f(x) = \frac{1}{3}(x^4 - 2x^3)$

$$\therefore f''(x) = \frac{1}{3}(4x^3 - 6x^2) = \frac{2}{3}x^2(2x - 3)$$

which shows that at $x = 0$, $f(x)$ attains point of inflection, when $k = 2$.

Hence, (a) is the correct answer.

Ex 40. The value of $\int_{-1}^1 f(x) dx$ is

- (a) $\frac{2}{15(k+1)}(23 - 10k)$
(b) $\frac{2}{15(k+1)}(23 + 10k)$
(c) $\frac{2}{15(k+1)}(10k - 17)$
(d) $\frac{2}{15(k+1)}(10k + 17)$

Sol. We have, $\int_{-1}^1 f(x) dx$

$$\begin{aligned} &= \int_{-1}^1 \frac{1}{k+1} \{x^4 - kx^3 + 2(k-2)x^2\} dx \\ &= \int_{-1}^1 \frac{1}{k+1} \{x^4 + 2(k-2)x^2\} dx \quad \left[\because \int_{-1}^1 x^3 dx = 0 \right] \\ &= \frac{2}{k+1} \left\{ \frac{1}{5} + \frac{2}{3}(k-2) \right\} \\ &= \frac{2}{15(k+1)}(3 + 10k - 20) \\ &= \frac{2}{15(k+1)}(10k - 17) \end{aligned}$$

Hence, (c) is the correct answer.

Passage III (Ex. Nos. 41-44) If

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}, \text{ then } f(x) \text{ is a}$$

quadratic function and its maximum value occurs at a point V . A is a point of intersection of $y = f(x)$ with X -axis and point B is such that chord AB subtends a right angle at V .

Ex 41. The function $f(x)$ is given by

- (a) $\frac{x^2 - 4}{4}$ (b) $\frac{4 - x^2}{4}$
(c) $\frac{7 - x^2}{4}$ (d) None of these

Sol. Given, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$

$$\Rightarrow 4a^2 f(-1) + 4a f(1) + f(2) = 3a^2 + 3a \quad \dots(i)$$

$$4b^2 f(-1) + 4b f(1) + f(2) = 3b^2 + 3b \quad \dots(ii)$$

$$\text{and } 4c^2 f(-1) + 4c f(1) + f(2) = 3c^2 + 3c \quad \dots(iii)$$

where, $f(x)$ is a quadratic expression given by $f(x) = ax^2 + bx + c$ and Eqs. (i), (ii), (iii).

$$\Rightarrow 4x^2 f(-1) + 4x f(1) + f(2) = 3x^2 + 3x$$

$$\text{or } [4f(-1) - 3]x^2 + [4f(1) - 3]x + f(2) = 0 \quad \dots(iv)$$

Eq. (iv) has three roots a, b, c . But it is quadratic in x . Therefore, it is an identity i.e. coefficients must be zero.

$$\Rightarrow f(-1) = \frac{3}{4}, f(1) = \frac{3}{4}, f(2) = 0 \quad \dots(v)$$

$$\therefore f(x) = ax^2 + bx + c$$

$$\Rightarrow a = -\frac{1}{4}, b = 0, c = 1 \quad [\text{from Eq. (v)}]$$

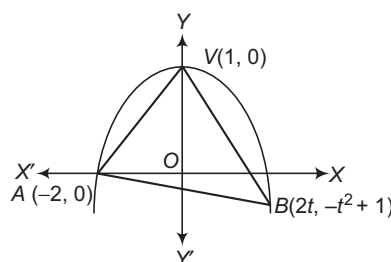
$$\text{Thus, } f(x) = \frac{4 - x^2}{4}$$

Hence, (b) is the correct answer.

Ex 42. The point V at which function attains maximum, is

- (a) (0, 1) (b) (1, 0) (c) (2, -2) (d) (2, 1)

Sol. From the figure given below, it is clear that $f(x)$ attains maximum at $V(1, 0)$.



Hence, (a) is the correct answer.

Ex 43. The point B at which chord AB subtends right angle at V , is

- (a) (3, 5) (b) (8, -15) (c) (9, 5) (d) (4, 3)

Sol. Let $A(-2, 0)$, $B(2t, -t^2 + 1)$.

Since, AB subtends right angle at vertex $V(0, 1)$.

$$\therefore \frac{1}{2} \left(\frac{-t^2}{2t} \right) = -1 \Rightarrow t = 4$$

$$\therefore B \equiv (8, -15)$$

Hence, (b) is the correct answer.

Ex 44. The area enclosed by $f(x)$ and the chord AB , is

- (a) $\frac{25}{3}$ sq units (b) $\frac{65}{3}$ sq units
(c) $\frac{25}{6}$ sq units (d) $\frac{125}{3}$ sq units

Sol. Required area = $\int_{-2}^8 \left(\frac{4-x^2}{4} + \frac{3x+6}{2} \right) dx$

$$= \left[x - \frac{x^3}{12} + \frac{3x^2}{4} + 3x \right]_{-2}^8 = \frac{125}{3} \text{ sq units}$$

Hence, (d) is the correct answer.

Passage IV (Ex. Nos. 45-47) Let

$f(x) = \frac{ax^2 + bx + c}{x^2 + 1}$ such that $y = -2$ is an asymptote of the curve $y = f(x)$. The curve $y = f(x)$ is symmetric about Y-axis and its maximum value is 4.

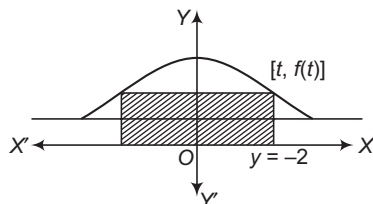
Ex 45. Maximum area of rectangle with two vertices on asymptote and two vertices on curve $y = f(x)$, is

- (a) 12 (b) 8 (c) 6 (d) 4

Sol. Applying the given conditions, we get

$$f(x) = \frac{-2x^2 + 4}{x^2 + 1}$$

$$\therefore \text{Required area} = 2t \left(\frac{-2t^2 + 4}{t^2 + 1} + 2 \right) = \frac{12t}{t^2 + 1}$$



$$\therefore t > 0 \Rightarrow t + \frac{1}{t} \geq 2$$

$$\Rightarrow \frac{1}{t + \frac{1}{t}} \leq \frac{1}{2} \Rightarrow \frac{12t}{t^2 + 1} \leq 6$$

$\therefore$ Maximum area = 6

Hence, (c) is the correct answer.

Ex 46. If $\lambda_1, \lambda_2 > 0$ such that $\lambda_1 + \lambda_2 = 1$, then which of the following holds?

- (a) $\lambda_1 f(x_1) + \lambda_2 f(x_2) > f(\lambda_1 x_1 + \lambda_2 x_2)$
for $x_1, x_2 \in \left(\infty, -\frac{1}{\sqrt{3}} \right)$ or $x_1, x_2 \in \left(\frac{1}{\sqrt{3}}, \infty \right)$
(b) $\lambda_1 f(x_1) + \lambda_2 f(x_2) > f(\lambda_1 x_1 + \lambda_2 x_2)$
for $x_1, x_2 \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$
(c) $\lambda_1 f(x_1) + \lambda_2 f(x_2) < f(\lambda_1 x_1 + \lambda_2 x_2)$
for $x_1, x_2 \in \left(-\infty, -\frac{1}{\sqrt{3}} \right)$ or $x_1, x_2 \in \left(\frac{1}{\sqrt{3}}, \infty \right)$
(d) $\lambda_1 f(x_1) + \lambda_2 f(x_2) < f(\lambda_1 x_1 + \lambda_2 x_2)$
for $x_1, x_2 \in R$

Sol. We have, $f''(x) = \frac{12(3x^2 - 1)}{(x^2 + 1)^3}$

$$f''(x) < 0 \text{ for } x \in \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$$

$\therefore y = f(x)$ is concave downward.

$$f''(x) > 0 \text{ for } x \in \left(-\infty, -\frac{1}{\sqrt{3}} \right) \cup \left(\frac{1}{\sqrt{3}}, \infty \right)$$

$\therefore y = f(x)$ is concave upward.

The result follows by considering points $[x_1, f(x_1)]$, $[x_2, f(x_2)]$ and point dividing these in the ratio $\lambda_2 : \lambda_1$.

Hence, (a) is the correct answer.

Ex 47. Let $g(x) = \begin{cases} ke^{-kx}, & x < 0 \\ f(x), & x \geq 0 \end{cases}$, then the values of k

such that $y = g(x)$ has a maxima at $x = 0$, is

- (a) $(-\infty, 4]$ (b) $(-\infty, 4)$
(c) $(0, 4)$ (d) $[4, \infty)$

Sol. Since, $y = g(x)$ is maximum at $x = 0$.

$$\therefore \lim_{x \rightarrow 0^-} g(x) \leq \lim_{x \rightarrow 0^+} g(x) = g(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} (ke^{-kx}) \leq f(0)$$

$$\Rightarrow k \leq 4.$$

Hence, (a) is the correct answer.

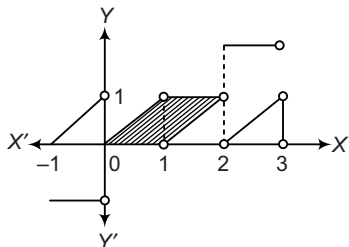
Type 5. Match the Columns

Ex 48. Match the statements of Column I with values of Column II.

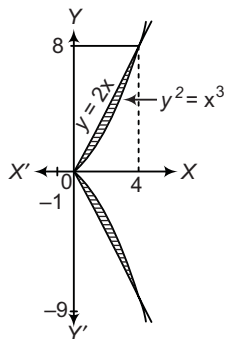
Column I	Column II
A. Area enclosed by $y = [x]$ and $y = \{x\}$, where $[]$ and $\{ \}$ represent the greatest integer and fractional part functions respectively, is	p. $\frac{32}{5}$
B. The area bounded by the curves $y^2 = x^3$ and $ y = 2x$, is	q. 1

Column I	Column II
C. The smaller area included between the curves $\sqrt{x} + \sqrt{ y } = 1$ and $ x + y = 1$, is	r. 4
D. Area bounded by the curves $y = \left[\frac{x^2}{64} + 2 \right]$, where $[]$ denotes the greatest integer function, $y = x - 1$ and $x = 0$, above X-axis, is	s. $\frac{2}{3}$

Sol. A. Required area = $2 \left(\frac{1}{2} \cdot 1 \cdot 1 \right) = 1$ sq unit



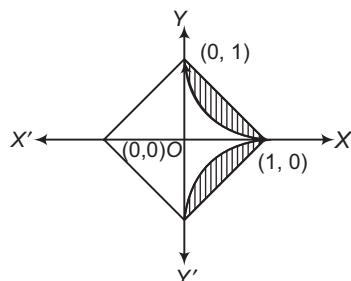
B. $y^2 = x^3$ and $|y| = 2x$, both curves are symmetric about X -axis.



$$\text{So, } 4x^2 = x^3 \Rightarrow x = 0, 4$$

$$\therefore \text{ Required area} = 2 \int_0^4 (2x - x^{3/2}) dx \\ = \frac{32}{5} \text{ sq units}$$

C. $\sqrt{x} + \sqrt{|y|} = 1$



The curve is symmetrical about X -axis.

$$\text{So, } \sqrt{|y|} = 1 - \sqrt{x} \text{ and } \sqrt{x} = 1 - \sqrt{|y|}$$

$$\text{For } x > 0, y > 0, \sqrt{y} = 1 - \sqrt{x}$$

$$\Rightarrow \frac{1}{2\sqrt{y}} \cdot \frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{y}{x}}$$

$$\therefore \frac{dy}{dx} < 0, \text{ i.e. function is decreasing.}$$

$\therefore$ Required area

$$= 2 \int_0^1 [(1-x) - (1-2\sqrt{x}+x)] dx$$

$$= 4 \int_0^1 (\sqrt{x} - x) dx$$

$$= 4 \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1 = 4 \left[\frac{2}{3} - \frac{1}{2} \right] = \frac{2}{3} \text{ sq unit}$$

D. If $-8 < x < 8$, then $y = 2$.

$$\text{If } x \in (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2}), \text{ then } y = 3$$

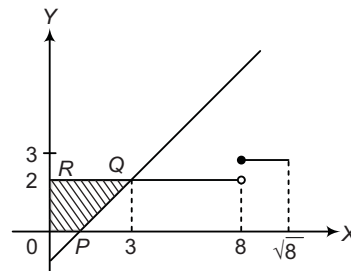
and so on.

Intersection of $y = x - 1$ and $y = 2$ is

$$x = 3 \in (-8, 8)$$

Intersection of $y = x - 1$ and $y = 3$ is $x = 4 \notin (-8\sqrt{2}, -8] \cup [8, 8\sqrt{2})$.

Similarly, $y = x - 1$ will not intersect $y = \left[\frac{x^2}{64} + 2 \right]$ at any other integral, except in the interval $x \in (-8, 8)$.



$$\therefore \text{ Required area} = 2 \times 3 - \frac{1}{2} \times 2 \times 2 \\ = 4 \text{ sq units}$$

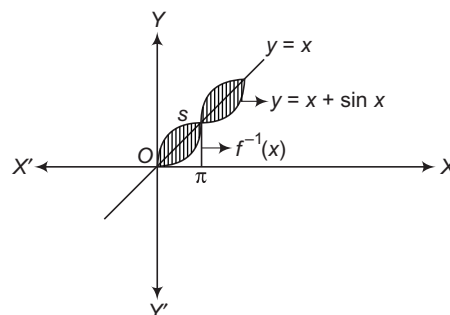
$$A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow r$$

Ex 49. Match the statements of Column I with values of Column II.

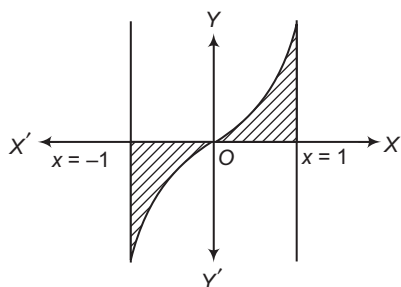
Column I	Column II
A. If the area bounded by the curve $y = x + \sin x$ and its inverse function between the ordinates $x = 0$ to $x = 2\pi$ is $4s$. Then, the value of s is	p. 2
B. The area bounded by the curve $y = xe^{ x }$ and lies between $ x = 1$, $y = 0$, is	q. 1

Sol. A. Required area = $4s$

$$s = \int_0^\pi (x + \sin x) dx - \int_0^\pi x dx \\ = \frac{\pi^2}{2} - \cos \pi + \cos 0 - \frac{\pi^2}{2} \\ = 2 \text{ sq units}$$



$$\text{B. Required area} = 2 \int_0^1 xe^x dx = 2 [xe^x - e^x]_0^1 \\ = 2 \text{ sq units}$$

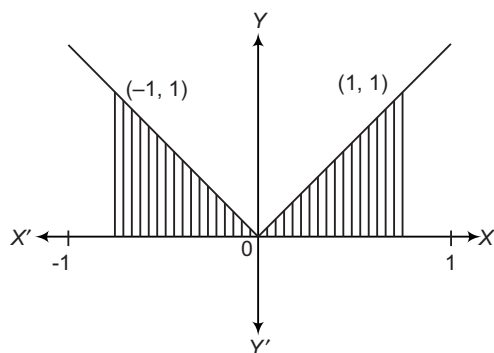


A $\rightarrow$ p; B $\rightarrow$ p

Ex 50. Match the statements of Column I with values of Column II.

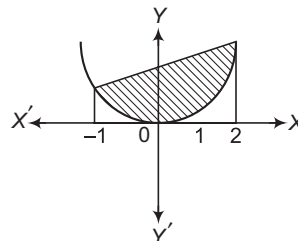
Column I	Column II
A. Area enclosed by $y = x $, $ x = 1$ and $y = 0$, is	p. 2
B. Area enclosed by the curves $y = \sin x$, $x = 0$, $x = \pi$ and $y = 0$, is	q. 1
C. If the area of the region bounded by $x^2 \leq y$ and $y \leq x + 2$, is $\frac{k}{4}$, then k is equal to	r. 27
D. Area of the quadrilateral formed by tangents at the ends of latusrectum of ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is	s. $\frac{9}{2}$

Sol. A. Required area $= 2 \int_0^1 x \cdot dx = 2 \left(\frac{x^2}{2} \right)_0^1 = 1$.



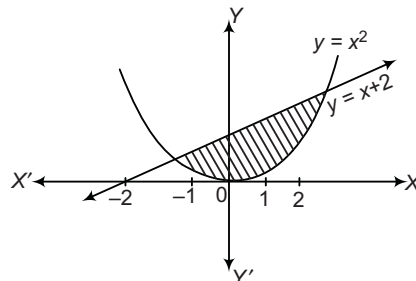
A $\rightarrow$ p; B $\rightarrow$ p

B. Area enclosed $= \int_0^\pi \sin x \, dx = 2$ sq units



C. The line $y = x + 2$ intersects $y = x^2$ at $x = -1$ and $x = 2$.

The given region is shaded region area

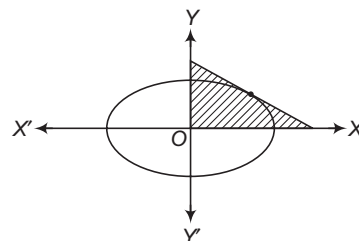


$\therefore$ Required area $= \frac{15}{2} - \int_{-1}^2 x^2 dx = \frac{9}{2}$ sq units

D. Here, $a^2 = 9$, $b^2 = 5$, $b^2 = a^2(1 - e^2)$

$$\Rightarrow e^2 = \frac{4}{9} \Rightarrow e = \frac{2}{3}$$

Equation of tangent at $\left(2, \frac{5}{3}\right)$ is $\frac{2x}{9} + \frac{y}{3} = 1$.



x-intercept $= \frac{9}{2}$, y-intercept $= 3$

$\therefore$ Required area $= 4 \times \frac{9}{2} \times 3 \times \frac{1}{2}$

$= 27$ sq units

A $\rightarrow$ q; B $\rightarrow$ p; C $\rightarrow$ s; D $\rightarrow$ r

Type 6. Single Integer Answer Type Questions

Ex 51. If a ($a > 0$) is the value of parameter for each of which the area of the figure bounded by the straight line, $y = \frac{a^2 - ax}{1 + a^4}$ and the parabola $y = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$, is the greatest, then the value of a^4 is _____.

Sol. (3) We have, $y = \frac{a^2 - ax}{1 + a^4}$... (i)

and $y = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$... (ii)

Point of intersection of Eqs. (i) and (ii) is

$$\frac{a^2 - ax}{1 + a^4} = \frac{x^2 + 2ax + 3a^2}{1 + a^4}$$

$$\Rightarrow (x + a)(x + 2a) = 0$$

$$\Rightarrow x = -a, -2a$$

Required area

$$= \int_{-2a}^{-a} \left[\left(\frac{a^2 - ax}{1 + a^4} \right) - \left(\frac{x^2 + 2ax + 3a^2}{1 + a^4} \right) \right] dx$$

$$\therefore f(a) = \frac{a^3}{6(1 + a^4)}$$

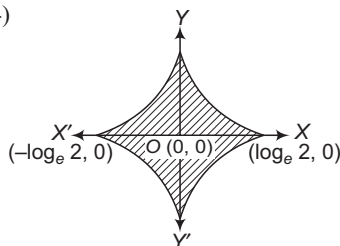
Now, $f(a)$ is maximum when $f'(a) = 0$

$$\Rightarrow 3 + 3a^4 - 4a^4 = 0$$

$$\therefore a^4 = 3$$

Ex 52. If the area bounded by region $|y| + 1 \leq 2e^{-|x|}$ is $A(1 - \log_e 2)$, then the value of A is _____.

Sol. (4)

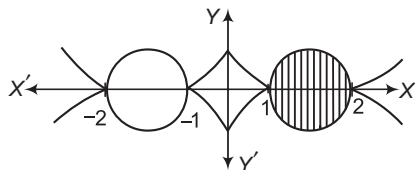


$$\text{Required area} = 4 \int_0^{\log_e 2} (2e^{-x} - 1) dx = 4(1 - \log_e 2)$$

$$\therefore A = 4$$

Ex 53. Consider $f(x) = x^2 - 3x + 2$. If the area bounded by $|y| = |f(|x|)|$, $x \geq 1$ is A , then the value of $3A + 2$ is _____.

Sol. (3)



Area is given by

$$A = 2 \left| \int_1^2 (x^2 - 3x + 2) dx \right| = \frac{1}{3}$$

$$\therefore 3A + 2 = 3 \cdot \frac{1}{3} + 2 = 3$$

Ex 54. The value of $c + 2$ for which the area of the figure bounded by the curve $y = 8x^2 - x^5$, the straight lines $x = 1$, $x = c$ and X -axis is equal to $16/3$, is _____.

Sol. (1) For $c < 1$, $\int_c^1 (8x^2 - x^5) dx = \frac{16}{3}$

$$\Rightarrow \frac{8}{3} \left[\frac{1}{3} - \frac{c^3}{3} \right] - \frac{1}{6} \left[\frac{1}{6} - \frac{c^6}{6} \right] = \frac{16}{3}$$

$$\Rightarrow c^3 \left[-\frac{8}{3} + \frac{c^3}{6} \right] = \frac{16}{3} - \frac{8}{3} + \frac{1}{6} = \frac{17}{6} \Rightarrow c = -1$$

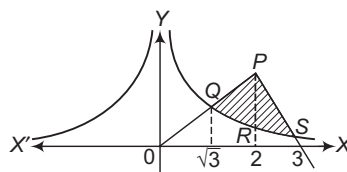
Again, for $c \geq 1$, none of the values of c satisfy the required condition that

$$\int_1^c (8x^2 - x^5) dx = \frac{16}{3}$$

$$\therefore c + 2 = -1 + 2 = 1$$

Ex 55. If the area bounded by $y = 2 - |2 - x|$, $y = \frac{3}{|x|}$ is $\frac{k - 3 \ln 3}{2}$, then k is equal to _____.

Sol. (4) $y = \begin{cases} x; & x < 2 \\ 4 - x; & x \geq 2 \end{cases}$ and $y = \begin{cases} 3/x; & x > 0 \\ -3/x; & x < 0 \end{cases}$



$\therefore$ Required area = Area of region $PQRP$ + Area of region $PRSP$

$$= \left| \int_{\sqrt{3}}^2 \left(x - \frac{3}{x} \right) dx \right| + \left| \int_2^3 \left(4 - x - \frac{3}{x} \right) dx \right|$$

$$= \left(\frac{4 - 3 \ln 3}{2} \right) \text{ sq unit}$$

$$\therefore k = 4$$

Ex 56. The area of $\triangle ABC$, coordinates of whose vertices are $A(2, 0)$, $B(4, 5)$ and $C(6, 3)$, is _____.

Sol. (7) Equation of AB , $y = \frac{5}{2}(x - 2)$

Equation of BC ,

$$y - 5 = \frac{3 - 5}{6 - 4}(x - 4) \Rightarrow y = -x + 9$$

Equation of CA ,

$$y - 3 = \frac{0 - 3}{2 - 6}(x - 6) \Rightarrow y = \frac{3}{4}(x - 2)$$

$\therefore$ Required area

$$= \frac{5}{2} \int_2^4 (x - 2) dx + \int_4^6 (-x + 9) dx - \frac{3}{4} \int_2^6 (x - 2) dx$$

$$= \frac{5}{2} \left[\frac{(x - 2)^2}{2} \right]_2^4 - \left[\frac{(x - 9)^2}{2} \right]_4^6 - \frac{3}{4} \left[\frac{(x - 2)^2}{2} \right]_2^6$$

$$= \frac{5}{4} [2^2 - 0] - \frac{1}{2} [(-3)^2 - (-5)^2] - \frac{3}{8} [4^2 - 0]$$

$$= \frac{5}{4} \times 4 - \frac{1}{2} [9 - 25] - \frac{3}{8} [16 - 0]$$

$$= 5 - \frac{1}{2} [-16] - \frac{3}{8} \times 16 = 5 + 8 - 6 = 7 \text{ sq units}$$

Target Exercises

Type 1. Only One Correct Option

- The area between the curve $y = 2x^4 - x^2$, X -axis and ordinates of the two minima of the curve, is
 (a) $\frac{11}{60}$ sq unit (b) $\frac{7}{120}$ sq unit
 (c) $\frac{1}{30}$ sq unit (d) $\frac{7}{90}$ sq unit
- The area bounded by the curve $a^2y = x^2(x + a)$ and X -axis, is
 (a) $\frac{a^2}{3}$ sq unit (b) $\frac{a^2}{4}$ sq unit
 (c) $\frac{3a^2}{4}$ sq unit (d) $\frac{a^2}{12}$ sq unit
- The area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1 - y^2}$, is
 (a) 1 sq unit (b) $\frac{4}{3}$ sq units
 (c) $\frac{2}{3}$ sq unit (d) 2 sq units
- The area bounded by the loop of the curve $4y^2 = x^2(4 - x^2)$ is
 (a) $\frac{7}{3}$ sq units (b) $\frac{8}{3}$ sq units
 (c) $\frac{11}{3}$ sq units (d) $\frac{16}{3}$ sq units
- The area enclosed by the curves $xy^2 = a^2(a - x)$ and $(a - x)y^2 = a^2x$, is
 (a) $(\pi - 2)a^2$ sq units
 (b) $(4 - \pi)a^2$ sq units
 (c) $\frac{\pi a^2}{3}$ sq units
 (d) None of the above
- The area bounded by the curves $y = xe^x$, $y = xe^{-x}$ and the line $x = 1$, is
 (a) $\left(\frac{2}{e}\right)$ sq unit (b) $\left(1 - \frac{2}{e}\right)$ sq unit
 (c) $\left(\frac{1}{e}\right)$ sq unit (d) $\left(1 - \frac{1}{e}\right)$ sq unit
- The area bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to it at the point with the ordinate $x = 3$ and X -axis, is
 (a) 7 sq units
 (b) 6 sq units
 (c) 9 sq units
 (d) None of the above
- The area bounded by $y = xe^{|x|}$ and lines $|x| = 1$, $y = 0$, is
 (a) 4 sq units (b) 6 sq units
 (c) 1 sq unit (d) 2 sq units
- The area bounded by the curves $y = \ln x$, $y = \ln |x|$, $y = |\ln x|$ and $y = |\ln |x||$, along X -axis and $y > 0$, is
 (a) 5 sq units (b) 2 sq units
 (c) 4 sq units (d) None of these
- Let $f(x)$ be a continuous function such that the area bounded by the curve $y = f(x)$, X -axis and the lines $x = 0$ and $x = a$, is $1 + \frac{a^2}{2} \sin a$. Then,
 (a) $f\left(\frac{\pi}{2}\right) = 1 + \frac{\pi^2}{8}$
 (b) $f(a) = 1 + \frac{a^2}{2} \sin a$
 (c) $f(a) = a \sin a + \frac{a^2}{2} \cos a$
 (d) None of the above
- The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is
 (a) 1 sq unit (b) 2 sq units
 (c) $2\sqrt{2}$ sq units (d) 4 sq units
- The area bounded by $y = \log_e x$, X -axis and the ordinate $x = e$, is given by
 (a) 4 sq units (b) $\frac{1}{2}$ sq unit
 (c) 1 sq unit (d) None of these
- The area bounded by the curve $|x| + y = 1$ and X -axis, is given by
 (a) 2 sq units (b) 1 sq unit
 (c) 4 sq units (d) None of these
- The area (in sq units) bounded by $y = \cos x$, $y = 0$ and $|x| = 1$, is given by
 (a) $\sin 1$
 (b) $2 \sin 1$
 (c) $4 \sin 1$
 (d) None of the above
- The area bounded by $y = |\sin x|$, X -axis and the line $|x| = \pi$, is
 (a) 2 sq units
 (b) 1 sq unit
 (c) 4 sq units
 (d) None of the above

- 16.** If the area bounded by the curve $y = f(x)$, X -axis and the ordinates $x = 1$ and $x = b$ is $(b - 1)\sin(3b + 4)$, then
 (a) $f(x) = \cos(3x + 4) + 3(x - 1)\sin(3x + 4)$
 (b) $f(x) = \sin(3x + 4) + 3(x - 1)\cos(3x + 4)$
 (c) $f(x) = \sin(3x + 4) - 3(x - 1)\cos(3x + 4)$
 (d) None of the above
- 17.** The area bounded by the curve $y = \sin^{-1} x$ and the lines $x = 0$, $|y| = \frac{\pi}{2}$ is
 (a) 2 sq units (b) 4 sq units
 (c) 8 sq units (d) 16 sq units
- 18.** The total area enclosed by the lines $y = |x|$, $y = 0$ and $|x| = 1$ is
 (a) 2 sq units (b) 4 sq units
 (c) 1 sq unit (d) None of these
- 19.** The area of the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the ordinates $x = \pm ae$, where $b^2 = a^2(1 - e^2)$ and $e < 1$, is given by
 (a) $2ab(e\sqrt{1 - e^2} + \sin^{-1} e)$ sq units
 (b) $2ab(e\sqrt{1 - e^2} - \sin^{-1} e)$ sq units
 (c) $ab(e\sqrt{1 - e^2} - \sin^{-1} e)$ sq unit
 (d) None of the above
- 20.** The area of the closed figure bounded by $y = \frac{x^2}{2} - 2x + 2$ and the tangents to it at $(1, 1/2)$ and $(4, 2)$, is
 (a) $9/8$ sq units
 (b) $3/8$ sq unit
 (c) $3/2$ sq units
 (d) $9/4$ sq units
- 21.** The area of the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$, is
 (a) $\left(\frac{5\pi}{4} + \frac{1}{2}\right)$ sq units (b) $\left(\frac{3\pi}{4} + \frac{1}{2}\right)$ sq units
 (c) $\left(\frac{5\pi}{4} - \frac{1}{2}\right)$ sq units (d) $\left(\frac{3\pi}{4} - \frac{1}{2}\right)$ sq units
- 22.** The area bounded by the curves $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$, above X -axis, is
 (a) $\left(2 + \frac{25}{2}\sin^{-1}\frac{4}{5}\right)$ sq units
 (b) $\left(2 + \frac{25}{4}\sin^{-1}\frac{4}{5}\right)$ sq units
 (c) $\left(2 + \frac{25}{2}\sin^{-1}\frac{1}{5}\right)$ sq units
 (d) None of the above
- 23.** The area bounded by $y = x^3 - 4x$ and X -axis is
 (a) 4 sq units (b) 8 sq units
 (c) 16 sq units (d) None of these
- 24.** The area bounded by the curve $y = x(3 - x)^2$, X -axis and the ordinates of the maximum and minimum points of the curve, is given by
 (a) 1 sq unit
 (b) 2 sq units
 (c) 4 sq units
 (d) None of the above
- 25.** If the area bounded by the lines $y = 2$, $x = 1$, $x = a$ and the curve $y = f(x)$, which cuts the last two lines above the first line for all $a \geq 1$, is equal to $\frac{2}{3}[(2a^{3/2} - 3a + 3 - 2\sqrt{2})]$. Then, $f(x)$ is equal to
 (a) $2\sqrt{2x}$, $x \geq 1$ (b) $\sqrt{2x}$, $x \geq 1$
 (c) $2\sqrt{x}$, $x \geq 1$ (d) None of these
- 26.** The area of the loop of the curve $y^2 = x(1 - x)^2$, is
 (a) $\frac{8}{15}$ sq unit (b) $\frac{7}{15}$ sq unit
 (c) $\frac{4}{15}$ sq unit (d) None of these
- 27.** For any real t , $x = \frac{e^t + e^{-t}}{2}$ and $y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. The area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$, is
 (a) $\frac{1}{2}t_1$ sq unit
 (b) t_1 sq unit
 (c) $2t_1$ sq units
 (d) None of the above
- 28.** The area of the region bounded by the curves $y = ex \log_e x$ and $y = \frac{\log_e x}{ex}$ is
 (a) $\left(\frac{e^2 - 5}{2e}\right)$ sq unit (b) $\left(\frac{e^2 + 5}{2e}\right)$ sq unit
 (c) $\left(\frac{e^2 + 5}{4e}\right)$ sq unit (d) $\left(\frac{e^2 - 5}{4e}\right)$ sq unit
- 29.** The area enclosed between the curves $y = \log_e(x + e)$, $x = \log_e \frac{1}{y}$ and X -axis, is
 (a) 2 sq units
 (b) 1 sq unit
 (c) 4 sq units
 (d) None of the above
- 30.** The area bounded by the curves $\sqrt{x} + \sqrt{y} = 1$ and $x + y = 1$, is
 (a) $\frac{1}{3}$ sq unit
 (b) $\frac{1}{6}$ sq unit
 (c) $\frac{1}{2}$ sq unit
 (d) None of the above

- 31.** The area above X -axis, bounded by the line $x = 4$ and the curve $y = f(x)$, where $f(x) = x^2$, $0 \leq x \leq 1$ and $f(x) = \sqrt{x}$, $x \geq 1$, is
 (a) 1 sq unit (b) 2 sq units
 (c) 4 sq units (d) 5 sq units
- 32.** The area bounded by the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ is
 (a) $a^2(\pi - 2)$ sq units (b) $a^2(\pi + 2)$ sq units
 (c) $a^2(\pi - 1)$ sq units (d) $a^2(\pi + 1)$ sq units
- 33.** The area bounded by the parabolas $y^2 = 4a(x + a)$ and $y^2 = -4a(x - a)$ is
 (a) $\frac{16}{3}a^2$ sq units
 (b) $\frac{8}{3}a^2$ sq units
 (c) $\frac{4}{3}a^2$ sq units
 (d) None of the above
- 34.** If A_n is the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$, then for $n > 2$,
 (a) $A_n + A_{n-2} = \frac{2}{2n-1}$
 (b) $A_n + A_{n-2} < \frac{1}{n-1}$
 (c) $A_n > \frac{1}{2n-2}$
 (d) $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$
- 35.** The slope of the tangent to a curve $y = f(x)$ at $[x, f(x)]$ is $2x + 1$. If the curve passes through the point $(1, 2)$, then the area of the region bounded by the curve, X -axis and the line $x = 1$, is
 (a) $\frac{5}{6}$ sq unit
 (b) $\frac{6}{5}$ sq units
 (c) $\frac{1}{6}$ sq unit
 (d) 6 sq units
- 36.** The area of the region bounded by $y = [x - 2]$, X -axis and the lines $x = -1$ and $x = 2$, where $[]$ denotes greatest integer function, is
 (a) 6 sq units
 (b) 8 sq units
 (c) 4 sq units
 (d) None of the above
- 37.** The area of the triangle formed by the lines $y = 2x$, $x = 0$ and $y = 2$, is
 (a) 1 sq unit
 (b) $\frac{1}{2}$ sq unit
 (c) 2 sq units
 (d) 3 sq units
- 38.** The area of the smaller portion of the circle $x^2 + y^2 = a^2$, $a > 0$ cut-off by the line $x = \frac{a}{2}$, is
 (a) $\frac{\pi a^2}{3}$ sq unit
 (b) $a^2 \left[\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right]$ sq unit
 (c) $a^2 \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right]$ sq unit
 (d) $a^2 \left[\frac{\pi}{2} - \sqrt{3} \right]$ sq unit
- 39.** If the curve $y = x^2 - 7x + 10$ intersect X -axis at the points A and B , then the area bounded by the curve and the line AB is
 (a) $4\frac{1}{2}$ sq units (b) 4 sq units
 (c) 6 sq units (d) 2 sq units
- 40.** If $f(x) = a + bx + cx^2$, where $c > 0$ and $b^2 - 4ac < 0$, then the area enclosed by the coordinate axes, the line $x = 2$ and the curve $y = f(x)$, is given by
 (a) $\frac{1}{3} \{4f(1) + f(2)\}$ sq unit
 (b) $\frac{1}{2} \{f(0) + 4f(1) + f(2)\}$ sq unit
 (c) $\frac{1}{2} \{f(0) + 4f(1)\}$ sq unit
 (d) $\frac{1}{3} \{f(0) + 4f(1) + f(2)\}$ sq unit
- 41.** The area bounded by the curve $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ and $x^2 + y^2 = a^2$, $a > 0$, is
 (a) $\left\{ \pi a^2 - \frac{2}{3} a^2 \right\}$ sq unit
 (b) $\left\{ 2\pi a^2 - \frac{a^2}{3} \right\}$ sq unit
 (c) $\left\{ \pi a^2 - \frac{3a^2}{2} \right\}$ sq unit
 (d) None of the above
- 42.** A curve $y = f(x)$ passes through the point $P(1, 1)$, the normal to the curve at P is $a(y - 1) + (x - 1) = 0$. If the slope of the tangent at any point on the curve $y = e^{a(x-1)}$ is proportional to the ordinate of that point, then the area bounded by Y -axis, the curve and the normal to the curve at P , is
 (a) $\left(1 + \frac{e^{-a}}{a} + \frac{1}{2a} \right)$ sq unit
 (b) $\left(1 + \frac{e^{-a}}{a} - \frac{1}{2a} \right)$ sq unit
 (c) $\left(1 + \frac{e^{-a}}{2a} + \frac{1}{a} \right)$ sq unit
 (d) $\left(1 + \frac{2e^{-a}}{a} + \frac{1}{a} \right)$ sq unit



Type 2. More than One Correct Option

- 43.** Which of the following have the same bounded area?
- (a) $f(x) = \sin x$, $g(x) = \sin^2 x$, where $0 \leq x \leq 10\pi$
 (b) $f(x) = \sin x$, $g(x) = |\sin x|$, where $0 \leq x \leq 20\pi$
 (c) $f(x) = |\sin x|$, $g(x) = \sin^3 x$, where $0 \leq x \leq 10\pi$
 (d) $f(x) = \sin x$, $g(x) = \sin^4 x$, where $0 \leq x \leq 10\pi$
- 44.** If the curve $y = ax^{1/2} + bx$ passes through the point (1, 2) and lies above X -axis for $0 \leq x \leq 9$ and the area enclosed by the curve, X -axis and the line $x = 4$, is 8 sq units, then
- (a) $a = 1$ (b) $b = 1$ (c) $a = 3$ (d) $b = -1$
- 45.** The area enclosed by the curves $x = a \sin^3 t$ and $y = a \cos^3 t$ is equal to
- (a) $12a^2 \int_0^{\pi/2} \cos^4 t \sin^2 t \, dt$
 (b) $12a \int_0^{\pi/2} \cos^2 t \sin^4 t \, dt$
 (c) $2 \int_{-a}^a (a^{2/3} - x^{2/3})^{3/2} dx$
 (d) $4 \int_0^a (a^{2/3} - x^{2/3})^{3/2} dx$
- 46.** There is a curve in which the length of the perpendicular from the origin to tangent at any point is equal to abscissa of that point. Then,
- (a) $x^2 + y^2 = 2x$ is one such curve
 (b) $y^2 = 4x$ is one such curve
 (c) $x^2 + y^2 = 2cx$ (c being a parameter) is one such curve
 (d) there is no such curves
- 47.** If $C_1 : x^2 + y^2 - 2x - 4y + 1 = 0$, $C_2 : x^2 + y^2 + 8x = 0$ are two circles, then
- (a) the length of their common chord is $\sqrt{\frac{175}{29}}$
 (b) the length of their common tangent is 5
 (c) the centre of C_2 is an interior point of C_1
 (d) the area of C_2 is more than C_1
- 48.** If the area enclosed by $y^2 = 4ax$ is $\frac{1}{3}$ sq unit, then the roots of the equation $x^2 + 2x = a$, are
- (a) -4
 (b) 2
 (c) -2
 (d) 8

Type 3. Assertion and Reason

Directions (Q. Nos. 49-54) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement II
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

- 49. Statement I** The area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$ is same as that of bounded by curves $y^2 = -4x$ and $x^2 = y$.

Statement II Shifting of origin to point (h, k) does not change the bounded area.

- 50. Statement I** The area of the region bounded by the curves $2y = \log_e x$, $y = e^{2x}$ and the pair of lines $(x + y - 1) \times (x + y - 3) = 0$ is $2k$ sq units.

Statement II The area of the region bounded by the curves $y = e^{2x}$, $y = x$ and the pair of lines $x^2 + y^2 + 2xy - 4x - 4y + 3 = 0$ is k sq unit.

- 51.** Consider two regions

R_1 : Point P is nearer to (1, 0) than to $x = -1$.
 R_2 : Point P is nearer to (0, 0) than to (8, 0).

Statement I Area of the region common to R_1 and R_2 is $128/3$ sq units.

Statement II Area bounded by $x = 4\sqrt{y}$ and $y = 4$ is $32/3$ sq units.

- 52. Statement I** Area bounded by $\max\{|x - y|, |x + y|\} \leq 2$ is 8 sq units.

Statement II Area of the square of side length 4 is 16 sq units.

- 53. Statement I** The area of the ellipse $2x^2 + 3y^2 = 6$ will be more than the area of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

Statement II The length of the semi-major axis of ellipse $2x^2 + 3y^2 = 6$ is more than the radius of the circle $x^2 + y^2 - 2x + 4y + 4 = 0$.

- 54. Statement I** Area enclosed by curve $(y - \sin^{-1} x)^2 = x - x^2$ is equal to $\int_0^1 (\sin^{-1} x + \sqrt{x - x^2}) dx$.

Statement II If $y = f(x)$ and $y = g(x)$ are two curves such that $f(x) \geq g(x)$, $\forall x \in [a, b]$, where $f(x) = g(x) \Rightarrow x = a, b$, then area enclosed by these two curves is given by $\int_a^b \{f(x) - g(x)\} dx$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 55-57) Consider the function defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

55. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2})$ is equal to

- (a) $\frac{4\sqrt{2}}{7^3 3^2}$ (b) $-\frac{4\sqrt{2}}{7^3 3^2}$
(c) $\frac{4\sqrt{2}}{7^3 3}$ (d) $-\frac{4\sqrt{2}}{7^3 3}$

56. The area of the region bounded by the curve $y = f(x)$, X -axis and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

- (a) $\int_a^b \frac{x}{3\{f(x)\}^2 - 1} dx + bf(b) - af(a)$
(b) $-\int_a^b \frac{x}{3\{f(x)\}^2 - 1} dx - bf(b) + af(a)$
(c) $\int_a^b \frac{x}{3\{f(x)\}^2 - 1} dx - bf(b) + af(a)$
(d) $-\int_a^b \frac{x}{3\{f(x)\}^2 - 1} dx - bf(b) + af(a)$

57. $\int_{-1}^1 g'(x) dx$ is equal to

- (a) $2g(-1)$
(b) 0
(c) $-2g(1)$
(d) $2g(1)$

Passage II (Q. Nos. 58-60) Consider the function defined implicitly by the equation

$y^2 - 2ye^{\sin^{-1} x} + x^2 - 1 + [x] + e^{2\sin^{-1} x} = 0$, where $[x]$ denotes the greatest integer function.

58. The area of the region bounded by the curve and the line $x = -1$ is

- (a) $(\pi + 1)$ sq units
(b) $(\pi - 1)$ sq units
(c) $\left(\frac{\pi}{2} + 1\right)$ sq units
(d) $\left(\frac{\pi}{2} - 1\right)$ sq unit

59. Line $x = 0$ divides the region mentioned above in two parts. The ratio of area of left hand side of line to that of right hand side of line is

- (a) $1 + \pi : \pi$
(b) $2 - \pi : \pi$
(c) 1:1
(d) $\pi + 2 : \pi$

60. The area of the region of curve and lines $x = 0$ and $x = \frac{1}{2}$ is

- (a) $\left(\frac{\sqrt{3}}{4} + \frac{\pi}{6}\right)$ sq units
(b) $\left(\frac{\sqrt{3}}{2} + \frac{\pi}{6}\right)$ sq units
(c) $\left(\frac{\sqrt{3}}{4} - \frac{\pi}{6}\right)$ sq units
(d) $\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$ sq units

Passage III (Q. Nos. 61-63) If the boundary of a figure is represented by parametric equations i.e. $x = x(t)$, $y = y(t)$, then the area of the figure is evaluated by one of the three formulae $S = -\int_{\alpha}^{\beta} y(t) \cdot x'(t) dt$, $S = \int_{\alpha}^{\beta} x(t) \cdot y'(t) dt$ and $S = \frac{1}{2} \int_{\alpha}^{\beta} (xy' - yx') dt$, where α and β are the values of the parameter t corresponding, respectively to the beginning and the end of the traversal of the curve corresponding to increasing t .

61. The area enclosed by the asteroid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{a}\right)^{2/3} = 1, \text{ is}$$

- (a) $\frac{3}{4} a^2 \pi$ sq units
(b) $\frac{3}{18} \pi a^2$ sq unit
(c) $\frac{3}{8} \pi a^2$ sq units
(d) $\frac{3}{4} \pi a$ sq units

62. The area of the region bounded by an arc of the cycloid $x = a(1 - \sin t)$, $y = a(1 - \cos t)$ and X -axis, is

- (a) $6\pi a^2$ sq units
(b) $3\pi a^2$ sq units
(c) $4\pi a^2$ sq units
(d) None of the above

63. Area of the loop described as $x = \frac{t}{3}(6 - t)$ and

$$y = \frac{t^2}{8}(6 - t), \text{ is}$$

- (a) $\frac{27}{5}$ sq units
(b) $\frac{24}{5}$ sq units
(c) $\frac{27}{6}$ sq units
(d) $\frac{21}{5}$ sq units



Type 5. Match the Columns

64. Match the statements of Column I with values of Column II.

Column I	Column II
A. The area bounded by $y = 3x$ and $y = x^2$ is	p. $\frac{2}{3}$
B. The area bounded by $x = 4 - y^2$ and Y -axis is	q. $\frac{17}{12}$
C. The area bounded by the curve $x^2 = 4y$, the line $x = 2$ and X -axis is	r. $\frac{32}{3}$
D. The area bounded by the curves $y = x^3$, $y = x^2$ and the lines $x = 1$, $x = 2$ is	s. $\frac{9}{2}$

65. Match the statements of Column I with values of Column II.

Column I	Column II
A. The area bounded by the curves $y^2 = 16x$ and $x = 16y$ is	p. $\frac{8}{3}$
B. The area bounded by $y^2 = 4x$ and $y = 4x$ is	q. $\frac{8}{3} \sqrt{ab}(a + b)$

Column I	Column II
C. The area bounded by $y^2 = 4x$ and $x = 1$ is	r. $\frac{1}{24}$
D. The area bounded by $y^2 = 4a(x + a)$ and $y^2 = 4b(b - x)$ is	s. $\frac{256}{3}$

66. If $[]$ represents the greatest integer function, then match the statements of Column I with values of Column II.

Column I	Column II
A. The area enclosed by $[x]^2 = [y]^2$ for $1 \leq x \leq 4$ is	p. 8
B. The area enclosed by $[x] + [y] = 2$ is	q. 6
C. The area enclosed by $[x][y] = 2$ is	r. 4
D. The area enclosed by $\frac{[x]}{[y]} = 2$ for $-5 \leq x \leq 5$ is	s. 12

Type 6. Single Integer Answer Type Questions

67. The area enclosed by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at $(2, 3)$ and X -axis is _____.

68. A point P moves in XY -plane in such a way that $[|x|] + [|y|] = 1$, where $[]$ denotes the greatest integer function. Area of the region representing all possible positions of the point P , is equal to _____.

69. If the area enclosed by the lines $x = 0$, $y = 0$, $x = \frac{3}{2}$ and the curve $y = 1 + 4x - x^2$, is A , where $A = \lambda/\mu$, then $\lambda - 4\mu$ is equal to _____.

70. If the area of the region $\{(x, y) : 0 \leq y \leq x^2 + 1, 0 \leq y \leq x + 1, 0 \leq x \leq 2\}$ is A , then the value of $6A - 17$ is _____.

71. If S is the sum of possible values of c for which the area of the figure bounded by the curve $y = \sin 2x$, the straight lines $x = \frac{\pi}{6}$, $x = c$ and X -axis, is equal to $\frac{1}{2}$, then the value of $\frac{\pi}{S}$ is _____.

72. If A is the area bounded by the curves $y = \sqrt{1 - x^2}$ and $y = x^3 - x$, then the value of $\frac{\pi}{A}$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

- If the common tangent to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P, Q and the parabola at the points R, S . Then, the area of the quadrilateral $PQRS$ is [2014]
 (a) 3 sq units (b) 6 sq units
 (c) 9 sq units (d) 15 sq units
- The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$, is [2013]
 (a) $4(\sqrt{2} - 1)$ sq units (b) $2\sqrt{2}(\sqrt{2} - 1)$ sq units
 (c) $2(\sqrt{2} + 1)$ sq units (d) $2\sqrt{2}(\sqrt{2} + 1)$ sq units
- If S is the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then, [2012]
 (a) $S \geq \frac{1}{e}$ (b) $S \geq 1 - \frac{1}{e}$
 (c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$
- If the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts R_1 ($0 \leq x \leq b$) and R_2 ($b \leq x \leq 1$) such that $R_1 - R_2 = \frac{1}{4}$. Then, b equals [2011]
 (a) $\frac{3}{4}$ (b) $\frac{1}{2}$
 (c) $\frac{1}{3}$ (d) $\frac{1}{4}$
- Let $f: [-1, 2] \rightarrow [0, \infty)$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. If $R_1 = \int_{-1}^2 xf(x) dx$ and R_2 are the areas of the region bounded by $y = f(x)$, $x = -1$, $x = 2$ and X -axis. Then, [2011]
 (a) $R_1 = 2R_2$
 (b) $R_1 = 3R_2$
 (c) $2R_1 = R_2$
 (d) $3R_1 = R_2$

JEE Main/AIEEE

- The area of the region described by $\{(x, y): y^2 \leq 2x \text{ and } y \geq 4x - 1\}$ is [2015]
 (a) $\frac{7}{32}$ sq unit
 (b) $\frac{5}{64}$ sq unit
 (c) $\frac{15}{64}$ sq unit
 (d) $\frac{9}{32}$ sq unit
- The area of the quadrilateral formed by the tangents at the end points of the latusrectum to the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$, is [2015]
 (a) $\frac{27}{4}$ sq units
 (b) 18 sq units
 (c) $\frac{27}{2}$ sq units
 (d) 27 sq units
- The area of the region described by $A = \{(x, y): x^2 + y^2 \leq 1 \text{ and } y^2 \leq 1 - x\}$ is [2014]
 (a) $\left(\frac{\pi}{2} + \frac{4}{3}\right)$ sq units (b) $\left(\frac{\pi}{2} - \frac{4}{3}\right)$ sq units
 (c) $\left(\frac{\pi}{2} - \frac{2}{3}\right)$ sq units (d) $\left(\frac{\pi}{2} + \frac{2}{3}\right)$ sq units
- The area bounded by the curve $y = \ln(x)$ and the lines $y = 0$, $y = \ln(3)$ and $x = 0$, is equal to [2013]
 (a) 3 sq units (b) $(3 \ln 3 - 2)$ sq units
 (c) $(3 \ln 3 + 2)$ sq units (d) 2 sq units
- The area of the region in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$, is [2013]
 (a) $\frac{7}{9}$ sq unit (b) $\frac{14}{3}$ sq units
 (c) $\frac{7}{3}$ sq units (d) $\frac{14}{9}$ sq units
- The area under the curve $y = |\cos x - \sin x|$, $0 \leq x \leq \frac{\pi}{2}$ and above X -axis, is [2013]
 (a) $2\sqrt{2}$ sq units
 (b) $(2\sqrt{2} - 2)$ sq units
 (c) $(2\sqrt{2} + 2)$ sq units
 (d) 0
- The area bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, X -axis and lying in the first quadrant, is [2013]
 (a) 9 sq units
 (b) 36 sq units
 (c) 18 sq units
 (d) $\frac{27}{4}$ sq units

- 13.** The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$ and the straight line $y = 2$, is [2012]
 (a) $20\sqrt{2}$ sq units (b) $\frac{10\sqrt{2}}{3}$ sq units
 (c) $\frac{20\sqrt{2}}{3}$ sq units (d) $10\sqrt{2}$ sq units
- 14.** The area bounded by the curves $y^2 = 4x$ and $x^2 = 4y$, is [2011]
 (a) 0 (b) $\frac{32}{3}$ sq units
 (c) $\frac{16}{3}$ sq units (d) $\frac{8}{3}$ sq units
- 15.** The area of the region enclosed by the curves $y = x$, $x = e$, $y = \frac{1}{x}$ and the positive X -axis, is [2011]
 (a) 1 sq unit (b) $\frac{3}{2}$ sq units
 (c) $\frac{5}{2}$ sq units (d) $\frac{1}{2}$ sq unit
- 16.** The area bounded by the curves $y = \cos x$ and $y = \sin x$ between the ordinates $x = 0$ and $x = \frac{3\pi}{2}$, is [2010]
 (a) $(4\sqrt{2} - 2)$ sq units (b) $(4\sqrt{2} + 2)$ sq units
 (c) $(4\sqrt{2} - 1)$ sq units (d) $(4\sqrt{2} + 1)$ sq units
- 17.** The area of the region bounded by the parabola $(y - 2)^2 = x - 1$, the tangent to the parabola at the point $(2, 3)$ and X -axis, is [2009]
 (a) 6 sq units (b) 9 sq units
 (c) 12 sq units (d) 3 sq units
- 18.** The area of the plane region bounded by the curves $x + 2y^2 = 0$ and $x + 3y^2 = 1$, is equal to [2008]
 (a) $\frac{4}{3}$ sq units (b) $\frac{5}{3}$ sq units
 (c) $\frac{1}{3}$ sq unit (d) $\frac{2}{3}$ sq unit
- 19.** The area enclosed between the curves $y^2 = x$ and $y = |x|$ is [2008]
 (a) $\frac{2}{3}$ sq unit (b) 1 sq unit
 (c) $\frac{1}{6}$ sq unit (d) $\frac{1}{3}$ sq unit
- 20.** The parabolas $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the lines $x = 4$, $y = 4$ and the coordinate axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom, then $S_1 : S_2 : S_3$ is [2005]
 (a) 2 : 1 : 2 (b) 1 : 1 : 1 (c) 1 : 2 : 1 (d) 1 : 2 : 3
- 21.** Let $f(x)$ be a non-negative continuous function such that the area bounded by the curve $y = f(x)$, X -axis and the ordinates $x = \frac{\pi}{4}$, $x = \beta$ ($\beta > \frac{\pi}{4}$), is $\left(\beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2}\beta \right)$, then $f\left(\frac{\pi}{2}\right)$ is equal to [2005]
 (a) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$ (b) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$
 (c) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$ (d) $\left(\frac{\pi}{4} - \sqrt{2} + 1\right)$
- 22.** The area of the region bounded by the curves $y = |x - 2|$, $x = 1$, $x = 3$ and X -axis, is [2004]
 (a) 1 sq unit (b) 2 sq units
 (c) 3 sq units (d) 4 sq units
- 23.** The area of the region bounded by the curves $y = |x - 1|$ and $y = 3 - |x|$, is [2003]
 (a) 2 sq units (b) 3 sq units
 (c) 4 sq units (d) 6 sq units
- 24.** The area bounded by the curve $y = 2x - x^2$ and the straight line $y = -x$, is given by [2002]
 (a) $\frac{9}{2}$ sq units (b) $\frac{43}{6}$ sq units
 (c) $\frac{35}{6}$ sq units (d) None of these

Other Engineering Entrances

- 25.** The area bounded by the curves $y = -\sqrt{-x}$ and $x = -\sqrt{-y}$, where $x, y \leq 0$, is [BITSAT 2014]
 (a) $\frac{1}{3}$ sq unit (b) $\frac{1}{4}$ sq unit
 (c) $\frac{1}{5}$ sq unit (d) $\frac{1}{2}$ sq unit
- 26.** The area in the first quadrant between $x^2 + y^2 = \pi^2$ and $y = \sin x$, is [VITEEE 2014]
 (a) $\left(\frac{\pi^3 - 8}{4}\right)$ sq units (b) $\frac{\pi^3}{4}$ sq units
 (c) $\left(\frac{\pi^3 - 16}{4}\right)$ sq units (d) $\left(\frac{\pi^3 - 8}{2}\right)$ sq units
- 27.** The area bounded by $y = x|\sin x|$ and X -axis between $x = 0$, $x = 2\pi$, is [Manipal 2014]
 (a) 2π sq units
 (b) 3π sq units
 (c) π sq units
 (d) 4π sq units
- 28.** The area enclosed by the curves $|y + x| \leq 1$, $|y - x| \leq 1$ and $2x^2 + 2y^2 = 1$, is [Manipal 2014]
 (a) $\left(2 + \frac{\pi}{2}\right)$ sq units (b) $\left(2 - \frac{\pi}{2}\right)$ sq unit
 (c) $\left(3 + \frac{\pi}{2}\right)$ sq units (d) $\left(3 - \frac{\pi}{4}\right)$ sq units

- 29.** The area of the region bounded by $1 - y^2 = |x|$ and $|x| + |y| = 1$, is [Manipal 2014]
 (a) $\frac{1}{3}$ sq unit (b) $\frac{2}{3}$ sq unit (c) $\frac{4}{3}$ sq units (d) 1 sq unit
- 30.** If the area of the region bounded by the lines $y = mx$, $x = 1$, $x = 2$ and X -axis, is 6 sq units, then m is equal to [Karnataka CET 2014]
 (a) 3 (b) 1 (c) 2 (d) 4
- 31.** The area of the region bounded by two parabolas $y = x^2$ and $x = y^2$ is [Karnataka CET 2014; WB JEE 2014]
 (a) $\frac{1}{4}$ sq unit (b) $\frac{1}{3}$ sq unit (c) 4 sq units (d) 3 sq units
- 32.** The area bounded by the curves $y = e^x$, $y = e^{-x}$ and the straight line $x = 1$, is [Kerala CEE 2014]
 (a) $\left(e + \frac{1}{e}\right)$ sq units (b) $\left(e + \frac{1}{e} + 2\right)$ sq units
 (c) $\left(e + \frac{1}{e} - 2\right)$ sq units (d) $\left(e - \frac{1}{e} + 2\right)$ sq units
 (e) $\left(e - \frac{1}{e}\right)$ sq units
- 33.** The area bounded by the curve $x = 0$ and $x + 2|y| = 1$ is [AMU 2014]
 (a) $\frac{1}{4}$ sq unit (b) $\frac{1}{2}$ sq unit (c) 1 sq unit (d) 2 sq units
- 34.** The area of the region bounded by $x = -1$, $x = 2$, $y = x^2 + 1$ and $y = 2x - 2$, is [EAMCET 2014]
 (a) 10 sq units (b) 7 sq units
 (c) 8 sq units (d) 9 sq units
- 35.** The area enclosed between the curves $y = x^3$ and $y = \sqrt{x}$ is [RPET 2014]
 (a) $\frac{5}{3}$ sq units (b) $\frac{5}{4}$ sq units
 (c) $\frac{5}{12}$ sq unit (d) $\frac{12}{5}$ sq units
- 36.** The area of the region bounded by the curves $y^2 = 4ax$ and $x^2 = 4ay$, $a > 0$, is [Kerala CEE 2013]
 (a) $\frac{16a^2}{3}$ sq units (b) $\frac{14a^2}{3}$ sq units
 (c) $\frac{13a^2}{3}$ sq units (d) $16a^2$ sq units
 (e) None of these
- 37.** The area included between the parabola $y = \frac{x^2}{4a}$ and the curve $y = \frac{8a^3}{x^2 + 4a^2}$ is [UP SEE 2013]
 (a) $\frac{a^2}{3}(6\pi - 4)$ sq unit (b) $\frac{a^2}{3}(4\pi + 3)$ sq unit
 (c) $\frac{a^2}{3}(8\pi + 3)$ sq unit (d) None of these
- 38.** The area bounded by the curves $y = x$ and $y = x^3$, is equal to [AMU 2013]
 (a) 0 (b) $\frac{1}{4}$ sq unit (c) $\frac{1}{2}$ sq unit (d) 1 sq unit
- 39.** The area of the region bounded by the curves $y = x^3$, $y = \frac{1}{x}$ and $x = 2$, is [WB JEE 2012]
 (a) $(4 - \log_e 2)$ sq units (b) $\left(\frac{1}{4} + \log_e 2\right)$ sq unit
 (c) $(3 - \log_e 2)$ sq units (d) $\left(\frac{15}{3} - \log_e 2\right)$ sq units
- 40.** The area in the positive quadrant enclosed by the circle $x^2 + y^2 = 4$, the line $x = y\sqrt{3}$ and X -axis, is [MP PET 2011]
 (a) $\frac{\pi}{2}$ sq units (b) $\frac{\pi}{4}$ sq unit
 (c) $\frac{\pi}{3}$ sq units (d) π sq units
- 41.** If the area between $y = mx^2$ and $x = my^2$ ($m > 0$) is $\frac{1}{4}$ sq unit, then the value of m is [Karnataka CET 2011]
 (a) $\pm 3\sqrt{2}$ (b) $\pm \frac{2}{\sqrt{3}}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$
- 42.** The area enclosed between $y^2 = x$ and $y = x$ is [WB JEE 2011]
 (a) $\frac{2}{3}$ sq unit (b) $\frac{1}{2}$ sq unit
 (c) $\frac{1}{3}$ sq unit (d) $\frac{1}{6}$ sq unit
- 43.** Find the area of a curve $xy = 4$, bounded by the lines $x = 1$ and $x = 3$ and X -axis. [Guj CET 2011]
 (a) $\log 12$ (b) $\log 64$ (c) $\log 81$ (d) $\log 27$
- 44.** The area bounded by the curve $y = 1 + \log_e x$, X -axis and the straight line $x = e$, is equal to [J&K CET 2011]
 (a) $(3e - 2)$ sq unit (b) e sq unit
 (c) $\left(e - \frac{1}{e}\right)$ sq unit (d) $\left(e + \frac{1}{e}\right)$ sq unit
- 45.** The area enclosed by $y = 3x - 5$, $y = 0$, $x = 3$ and $x = 5$, is [VITEEE 2010]
 (a) 12 sq units (b) 13 sq units
 (c) $13\frac{1}{2}$ sq units (d) 14 sq units
- 46.** The area of the plane region bounded by the curve $x = y^2 - 2$ and the line $y = -x$, is [Kerala CEE 2010]
 (a) $\frac{13}{3}$ sq units (b) $\frac{2}{5}$ sq unit
 (c) $\frac{9}{2}$ sq units (d) $\frac{5}{2}$ sq units
 (e) $\frac{13}{2}$ sq units

Answers

Work Book Exercise 11.1

1. (d) 2. (a) 3. (a) 4. (d) 5. (d) 6. (b) 7. (b) 8. (b) 9. (a) 10. (d)
11. (d) 12. (d) 13. (a) 14. (c) 15. (b) 16. (c)

Work Book Exercise 11.2

1. (d) 2. (c) 3. (b) 4. (d) 5. (b)

Target Exercises

1. (b) 2. (d) 3. (d) 4. (d) 5. (a) 6. (a) 7. (c) 8. (c) 9. (d) 10. (c)
11. (b) 12. (c) 13. (b) 14. (b) 15. (c) 16. (b) 17. (a) 18. (c) 19. (a) 20. (a)
21. (c) 22. (a) 23. (b) 24. (c) 25. (a) 26. (a) 27. (b) 28. (d) 29. (a) 30. (b)
31. (d) 32. (a) 33. (a) 34. (d) 35. (a) 36. (a) 37. (a) 38. (c) 39. (a) 40. (d)
41. (b) 42. (b) 43. (b,c,d) 44. (c,d) 45. (a,c,d) 46. (b,c) 47. (a, b,d) 48. (a,b,) 49. (a) 50. (a)
51. (d) 52. (b) 53. (b) 54. (d) 55. (b) 56. (d) 57. (d) 58. (a) 59. (d) 60. (a)
61. (c) 62. (b) 63. (a) 64. (*) 65. (*) 66. (*) 67. (9) 68. (8) 69. (7) 70. (6)
71. (6) 72. (6)

*

64. $A \rightarrow s; B \rightarrow r; C \rightarrow p; D \rightarrow q$

65. $A \rightarrow s; B \rightarrow r; C \rightarrow p; D \rightarrow q$

66. $A \rightarrow q; B \rightarrow s; C \rightarrow p; D \rightarrow p$

Entrances Gallery

1. (d) 2. (b) 3. (b,d) 4. (b) 5. (c) 6. (d) 7. (d) 8. (a) 9. (d) 10. (d)
11. (b) 12. (a) 13. (c) 14. (c) 15. (b) 16. (a) 17. (b) 18. (a) 19. (c) 20. (b)
21. (b) 22. (a) 23. (c) 24. (a) 25. (a) 26. (a) 27. (d) 28. (b) 29. (b) 30. (d)
31. (b) 32. (b) 33. (b) 34. (d) 35. (c) 36. (a) 37. (a) 38. (c) 39. (b) 40. (c)
41. (b) 42. (d) 43. (c) 44. (d) 45. (d) 46. (c)

Explanations

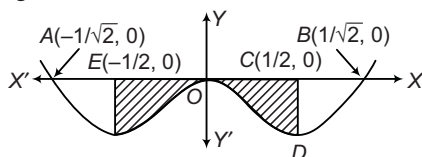
Target Exercises

1. The curve is $y = 2x^4 - x^2 = x^2(2x^2 - 1)$.

The curve is symmetrical about Y-axis. Also, it is a polynomial of 4 degree having roots $0, 0, \pm \frac{1}{\sqrt{2}}$. But

$x = 0$ is repeated root. Hence, graph touches at $(0, 0)$.

The curve intersects the axes at $O(0, 0)$, $A(-1/\sqrt{2}, 0)$ and $B(1/\sqrt{2}, 0)$. Thus, the graph of the curve is shown in figure.

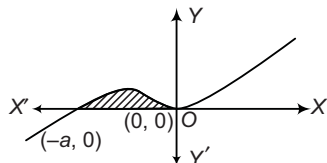


Here, $y \leq 0$, as x varies from $x = -1/2$ to $x = 1/2$.

∴ Required area = $2 \times$ Area of region OCDO

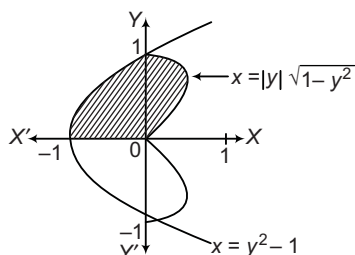
$$\begin{aligned} &= 2 \left| \int_0^{1/2} y dx \right| \\ &= 2 \left| \int_0^{1/2} (2x^4 - x^2) dx \right| \\ &= 7/120 \text{ sq unit} \end{aligned}$$

2. The curve is $y = \frac{x^2(x+a)}{a^2}$, which is a cubic polynomial. Since, $\frac{x^2(x+a)}{a^2} = 0$ has repeated root $x = 0$, so it touches X-axis at $(0, 0)$ and intersects at $(-a, 0)$.



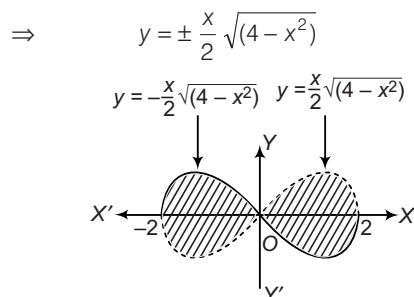
$$\begin{aligned} \therefore \text{Required area} &= \int_{-a}^0 y dx = \int_{-a}^0 \left[\frac{x^2(x+a)}{a^2} \right] dx \\ &= a^2/12 \text{ sq unit} \end{aligned}$$

3. Required area = $2 \int_0^1 [y\sqrt{1-y^2} - (y^2 - 1)] dy$
 $= 2 \text{ sq units}$



4. We have, $4y^2 = x^2(4 - x^2)$

$$\Rightarrow y = \pm \frac{1}{2} \sqrt{x^2(4 - x^2)}$$



$$\begin{aligned} \therefore \text{Required area (A)} &= 4 \int_0^2 \left[\frac{x}{2} \sqrt{4 - x^2} \right] dx \\ &= 2 \int_0^2 [x \sqrt{4 - x^2}] dx \end{aligned}$$

$$\text{Put } 4 - x^2 = t \Rightarrow -2x dx = dt$$

$$\begin{aligned} \therefore A &= \int_0^4 \sqrt{t} dt = \left[\frac{t^{3/2}}{3/2} \right]_0^4 = \frac{2}{3} [\sqrt{64} - 0] \\ &= \frac{16}{3} \text{ sq units} \end{aligned}$$

5. Given curves are

$$xy^2 = a^2(a - x) \Rightarrow x = \frac{a^3}{a^2 + y^2} \quad \dots(i)$$

and

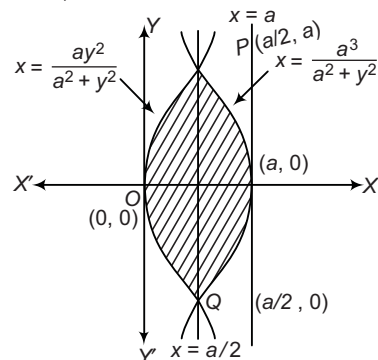
$$(a - x)y^2 = a^2x$$

$$\Rightarrow x = \frac{ay^2}{a^2 + y^2} = \frac{ay^2 + a^3 - a^3}{a^2 + y^2} = a - \frac{a^3}{a^2 + y^2} \quad \dots(ii)$$

Curve (i) is symmetrical about X-axis and have Y-axis as the asymptote.

Curve (ii) is symmetrical about X-axis, tangent at origin as Y-axis and has the asymptote $x = a$.

The two curves intersect at the points $P(a/2, a)$ and $Q(a/2, -a)$.



∴ Required area

$$\begin{aligned} &= 2 \int_0^a \left[-a + \frac{a^3}{a^2 + y^2} + \frac{a^3}{a^2 + y^2} \right] dy \\ &= 2 \left[-ay + 2a^2 \tan^{-1} \frac{y}{a} \right]_0^a \\ &= 2 \left[-a^2 + 2a^2 \cdot \frac{\pi}{4} \right] = (\pi - 2)a^2 \text{ sq units} \end{aligned}$$



6. Curve tracing is $y = xe^x$

$$\text{Let } \frac{dy}{dx} = 0 \Rightarrow e^x + xe^x = 0 \Rightarrow x = -1$$

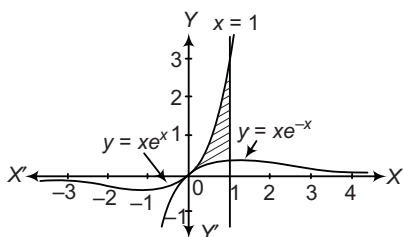
Also, at $x = -1$, $\frac{dy}{dx}$ changes sign from -ve to +ve.

So, $x = -1$ is a point of minima.

When $x \rightarrow \infty$, then $y \rightarrow \infty$

$$\text{Also, } \lim_{x \rightarrow -\infty} xe^x = \lim_{x \rightarrow -\infty} \frac{x}{e^{-x}} = \lim_{x \rightarrow -\infty} \frac{1}{-e^{-x}} = 0$$

With similar types of arguments, we can draw the graph of $y = xe^{-x}$.



$\therefore$ Required area

$$\begin{aligned} &= \int_0^1 xe^x dx - \int_0^1 xe^{-x} dx \\ &= [xe^x]_0^1 - \int_0^1 e^x dx - \{[-xe^{-x}]_0^1 + \int_0^1 e^{-x} dx\} \\ &= e - (e - 1) - \{-e^{-1} - (e^{-1} - 1)\} \\ &= \frac{2}{e} \text{ sq units} \end{aligned}$$

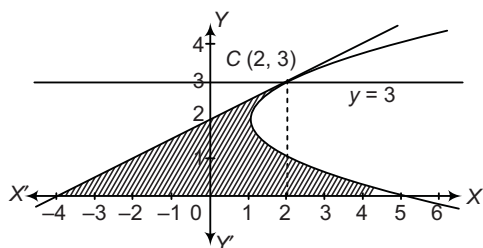
7. Given parabola is $(y - 2)^2 = x - 1$.

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2(y - 2)}$$

When $y = 3$, then $x = 2$

$$\therefore \frac{dy}{dx} = \frac{1}{2(3 - 2)} = \frac{1}{2}$$

Tangent at $(2, 3)$ is $y - 3 = \frac{1}{2}(x - 2) \Rightarrow x - 2y + 4 = 0$

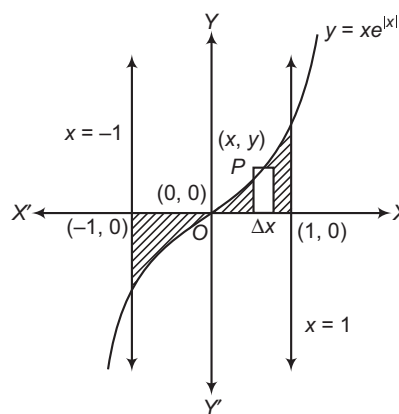


$\therefore$ Required area

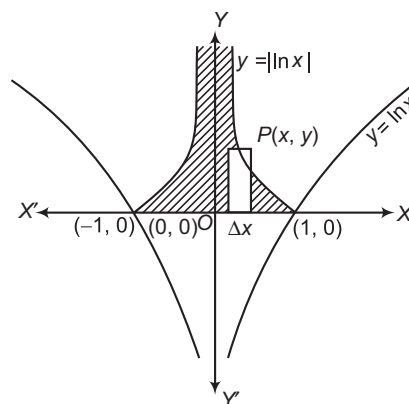
$$\begin{aligned} &= \int_0^3 [(y - 2)^2 + 1] dy - \int_0^3 (2y - 4) dy \\ &= \left[\frac{(y - 2)^3}{3} + y \right]_0^3 - [y^2 - 4y]_0^3 \\ &= \frac{1}{3} + 3 + \frac{8}{3} - (9 - 12) = 9 \text{ sq units} \end{aligned}$$

8. Required area = 2 $\times$ Area in the first quadrant

$$\begin{aligned} &= 2 \int_0^1 y dx = 2 \int_0^1 xe^{x^2} dx \\ &= 2 \int_0^1 xe^x dx = 2 [(x - 1)e^x]_0^1 \\ &= 2 [0 - (-1)] = 2 \text{ sq units} \end{aligned}$$



9. Required area = 2 $\times$ Area in the first quadrant



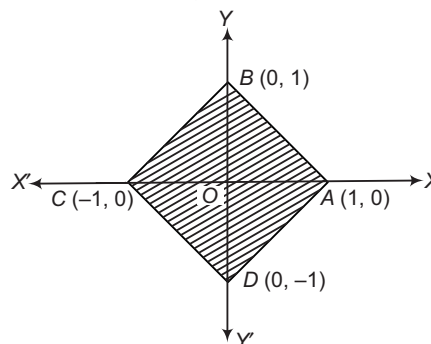
$$\begin{aligned} &= 2 \int_0^1 |\ln x| dx = -2 \int_0^1 \ln x dx \\ &= -2 [x(\log x - 1)]_0^1 = -2 [1(0 - 1) - 0] = 2 \text{ sq units} \end{aligned}$$

10. It is given that, $\int_0^a f(x) dx = 1 + \frac{a^2}{2} \sin a$

On differentiating both sides w.r.t. a , we get

$$f(a) = a \sin a + \frac{a^2}{2} \cos a$$

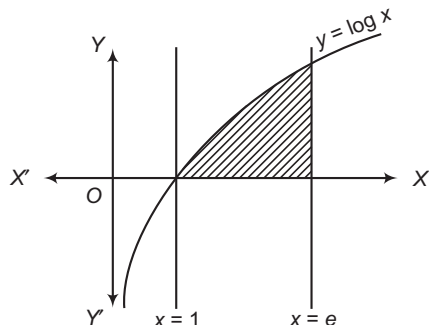
11. Required area = Area enclosed by square ABCD
= 2 sq units



Aliter

$$\begin{aligned} \text{Required area} &= 4 \times (\text{Area of } \triangle COB) \\ &= 4 \int_0^1 y dx = 4 \int_0^1 (-x + 1) dx \\ &= 4 \left[-\frac{x^2}{2} + x \right]_0^1 \\ &= 2 \text{ sq units} \end{aligned}$$

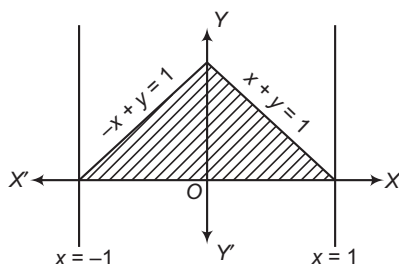
12. Required area = $\int_1^e y \, dx = \int_1^e \log x \, dx$



$$= [x \log x - x]_1^e = (e \log e - e) - (0 - 1)$$

$$= e - e + 1 = 1 \text{ sq unit}$$

13. Required area = $\int_{-1}^0 y \, dx + \int_0^1 y \, dx$



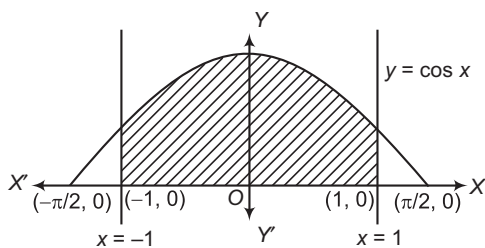
$$= \int_{-1}^0 (x+1) \, dx + \int_0^1 (1-x) \, dx$$

$$= \left[\frac{x^2}{2} + x \right]_{-1}^0 + \left[x - \frac{x^2}{2} \right]_0^1$$

$$= \left[0 - \left(\frac{1}{2} - 1 \right) \right] + \left[\left(1 - \frac{1}{2} \right) - 0 \right]$$

$$= \frac{1}{2} + \frac{1}{2} = 1 \text{ sq unit}$$

14. Required area = $\int_{-1}^1 y \, dx = \int_{-1}^1 \cos x \, dx$

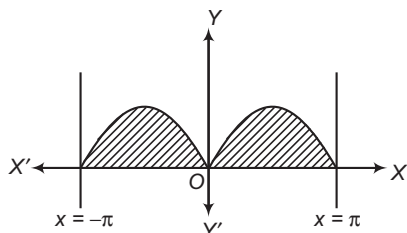


$$= [\sin x]_{-1}^1 = \sin 1 - \sin(-1)$$

$$= 2 \sin 1 \text{ sq units}$$

15. Required area = $\int_{-\pi}^{\pi} |\sin x| \, dx = 2 \int_0^{\pi} \sin x \, dx$

$$= -2 [\cos x]_0^{\pi}$$



$$= -2(\cos \pi - \cos 0) = 4 \text{ sq units}$$

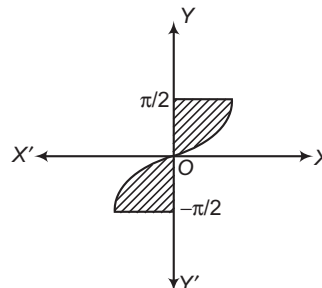
16. Given that, $\int_1^b f(x) \, dx = (b-1) \sin(3b+4)$

On differentiating both sides w.r.t. b , we get

$$[f(x)]_{x=b} = \sin(3b+4) + (b-1) \cdot 3 \cos(3b+4)$$

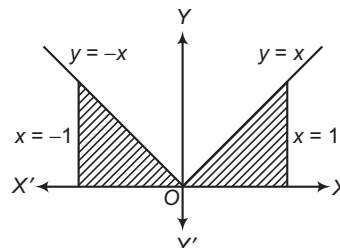
$$\therefore f(x) = \sin(3x+4) + 3(x-1) \cos(3x+4)$$

17. Required area = $2 \int_0^{\pi/2} x \, dy$, where $y = \sin^{-1} x$
i.e. $x = \sin y$



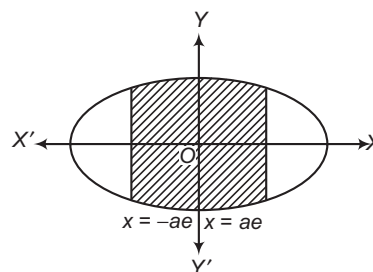
$$= 2 \int_0^{\pi/2} \sin y \, dy = -2 [\cos y]_0^{\pi/2} = 2 \text{ sq units}$$

18. Required area = $2 \int_0^1 y \, dx$, where $y = x$



$$= 2 \int_0^1 x \, dx = 2 \left[\frac{x^2}{2} \right]_0^1 = 1 \text{ sq unit}$$

19. Required area = $4 \int_0^{ae} y \, dx = \frac{4b}{a} \int_0^{ae} \sqrt{a^2 - x^2} \, dx$



$$= \frac{4b}{a} \left[\frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_0^{ae}$$

$$= \frac{4b}{2a} [ae \sqrt{a^2 - a^2 e^2} + a^2 \sin^{-1} e]$$

$$= 2ab (e \sqrt{1 - e^2} + \sin^{-1} e) \text{ sq units}$$

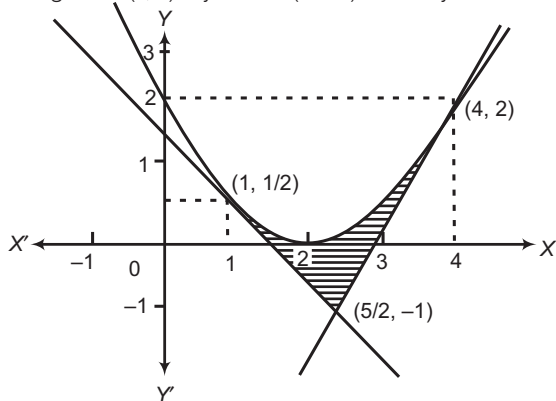
20. We have, $y = \frac{x^2}{2} - 2x + 2 = \frac{(x-2)^2}{2}$

Now, $\frac{dy}{dx} = x - 2$

$$\therefore \left[\frac{dy}{dx} \right]_{x=1} = -1 \text{ and } \left[\frac{dy}{dx} \right]_{x=4} = 2$$

Tangent at $(1, 1/2)$ is $y - 1/2 = -1(x - 1)$
or $2x + 2y - 3 = 0$

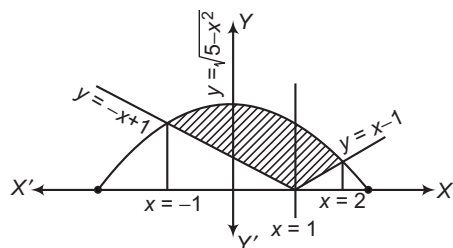
Tangent at (4, 2) is $y - 2 = 2(x - 4)$ or $2x - y - 6 = 0$



$$\begin{aligned} \therefore \text{Area, } A &= \int_1^{5/2} \left(\frac{x^2}{2} - 2x + 2 - \frac{3-2x}{2} \right) dx \\ &\quad + \int_{5/2}^4 \left(\frac{x^2}{2} - 2x + 2 - (2x - 6) \right) dx \\ &= \int_1^4 \left(\frac{x^2}{2} - 2x + 2 \right) dx - \int_1^{5/2} \left(\frac{3-2x}{2} \right) dx \\ &\quad - \int_{5/2}^4 (2x - 6) dx \\ &= \left[\frac{x^3}{6} - x^2 + 2x \right]_1^4 - \frac{1}{2} [3x - x^2]_1^{5/2} - [x^2 - 6x]_{5/2}^4 \\ &= \left(\frac{63}{6} - 15 + 6 \right) - \frac{1}{2} \left[3 \times \frac{3}{2} - \left(\frac{25}{4} - 1 \right) \right] \\ &\quad - \left[\left(16 - \frac{25}{4} \right) - 6 \left(4 - \frac{5}{2} \right) \right] \\ &= \frac{3}{2} - \frac{1}{2} \left(\frac{9}{2} - \frac{21}{4} \right) - \left[\frac{39}{4} - 6 \left(\frac{3}{2} \right) \right] = \frac{9}{8} \text{ sq units} \end{aligned}$$

21. Required area

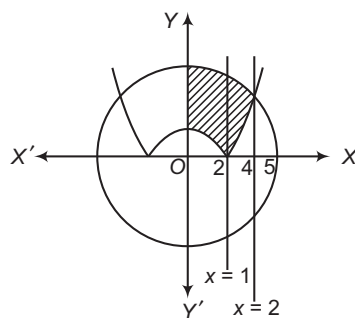
$$= \int_{-1}^2 \sqrt{5-x^2} dx - \int_{-1}^1 (-x+1) dx - \int_1^2 (x-1) dx$$



$$\begin{aligned} &= \left[\frac{x}{2} \sqrt{5-x^2} + \frac{5}{2} \sin^{-1} \frac{x}{\sqrt{5}} \right]_{-1}^2 - \left[\frac{-x^2}{2} + x \right]_{-1}^1 - \left[\frac{x^2}{2} - x \right]_1^2 \\ &= \left(1 + \frac{5}{2} \sin^{-1} \frac{2}{\sqrt{5}} \right) - \left(-1 + \frac{5}{2} \sin^{-1} \left[-\frac{1}{\sqrt{5}} \right] \right) \\ &\quad - \left(-\frac{1}{2} + 1 + \frac{1}{2} + 1 \right) - \left(2 - 2 - \frac{1}{2} + 1 \right) \\ &= \frac{5}{2} \left(\sin^{-1} \frac{2}{\sqrt{5}} + \sin^{-1} \frac{1}{\sqrt{5}} \right) - \frac{1}{2} \\ &= \frac{5}{2} \sin^{-1} \left(\frac{2}{\sqrt{5}} \sqrt{1 - \frac{1}{5}} + \frac{1}{\sqrt{5}} \sqrt{1 - \frac{4}{5}} \right) - \frac{1}{2} \\ &= \frac{5}{2} \sin^{-1} 1 - \frac{1}{2} = \left(\frac{5\pi}{4} - \frac{1}{2} \right) \text{ sq units} \end{aligned}$$

22. Required area

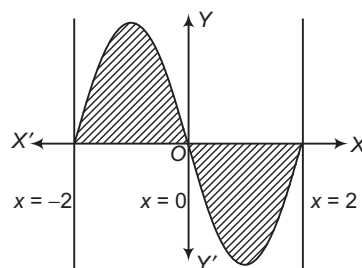
$$= \int_0^4 \sqrt{25-x^2} dx - \int_0^2 \frac{24-x^2}{4} dx - \int_2^4 \frac{x^2-4}{4} dx$$



$$\begin{aligned} &= \left[\frac{x}{2} \sqrt{25-x^2} + \frac{25}{2} \sin^{-1} \frac{x}{5} \right]_0^4 \\ &\quad - \frac{1}{4} \left[4x - \frac{x^3}{3} \right]_0^2 - \frac{1}{4} \left[\frac{x^3}{3} - 4x \right]_2^4 \\ &= \left(2 \times 3 + \frac{25}{2} \sin^{-1} \frac{4}{5} \right) - \frac{1}{4} \left(8 - \frac{8}{3} \right) \\ &\quad - \frac{1}{4} \left(\frac{64}{3} - 16 \right) + \frac{1}{4} \left(\frac{8}{3} - 8 \right) \\ &= \left(2 + \frac{25}{2} \sin^{-1} \frac{4}{5} \right) \text{ sq units} \end{aligned}$$

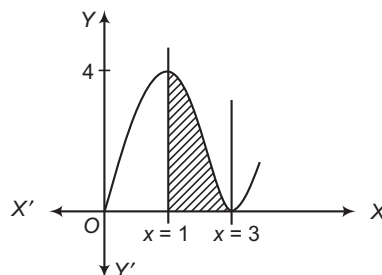
23. Required area

$$= \int_{-2}^2 y dx = \int_{-2}^0 (x^3 - 4x) dx - \int_0^2 (x^3 - 4x) dx$$



$$\begin{aligned} &= \left[\frac{x^4}{4} - 2x^2 \right]_{-2}^0 - \left[\frac{x^4}{4} - 2x^2 \right]_0^2 \\ &= (-4 - 8) - (4 - 8) = 8 \text{ sq units} \end{aligned}$$

24. Clearly, the curve $y = x(3-x)^2$ has maximum at $x = 1$ and minimum at $x = 3$.



$$\begin{aligned} \therefore \text{Required area} &= \int_1^3 x(3-x)^2 dx \\ &= \int_1^3 (x^3 - 6x^2 + 9x) dx = \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} \right]_1^3 \\ &= \left(\frac{81}{4} - 54 + \frac{81}{2} \right) - \left(\frac{1}{4} - 2 + \frac{9}{2} \right) = 4 \text{ sq units} \end{aligned}$$

25. Given that,

$$\int_1^a [f(x) - 2] dx = \frac{2}{3} [(2a)^{3/2} - 3a + 3 - 2\sqrt{2}]$$

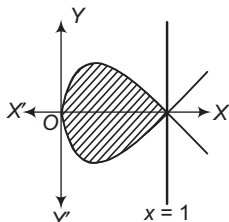
On differentiating both sides w.r.t. a , we get

$$f(a) - 2 = \frac{2}{3} \left[\frac{3}{2} \sqrt{2a} \cdot 2 - 3 \right]$$

$$\Rightarrow f(a) = 2\sqrt{2a}, a \geq 1$$

$$\therefore f(x) = 2\sqrt{2x}, x \geq 1$$

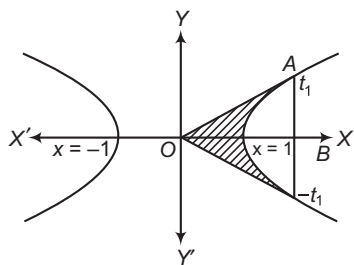
26. Since, the curve is symmetrical about X-axis.



$$\begin{aligned} \therefore \text{Required area} &= 2 \int_0^1 y dx = 2 \int_0^1 \sqrt{x} (1-x) dx \\ &= 2 \int_0^1 (x^{1/2} - x^{3/2}) dx \\ &= 2 \left[\frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{5} \right]_0^1 = \frac{8}{15} \text{ sq unit} \end{aligned}$$

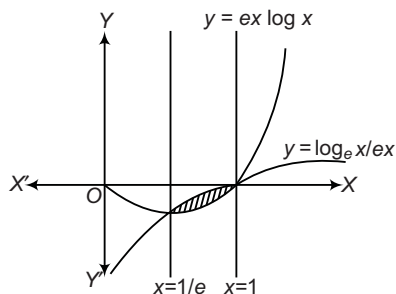
27. Required area

$$\begin{aligned} &= 2 \left[\text{Area of } \triangle OAB - \int_0^{\frac{1}{2}(e^{t_1} + e^{-t_1})} \frac{1}{2} (e^t - e^{-t}) dt \right] \\ &\quad d \left[\frac{1}{2} (e^t + e^{-t}) \right] \end{aligned}$$



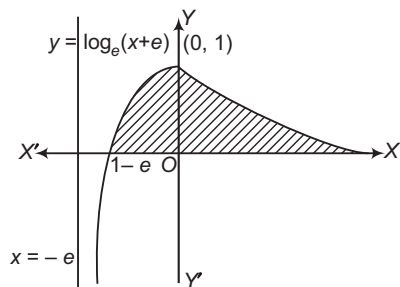
$$\begin{aligned} &= 2 \left[\frac{1}{2} \cdot \frac{1}{2} (e^{t_1} + e^{-t_1}) \cdot \frac{1}{2} (e^{t_1} - e^{-t_1}) \right. \\ &\quad \left. - \int_0^{t_1} \frac{1}{4} (e^t - e^{-t})^2 dt \right] \\ &= \frac{1}{4} (e^{2t_1} - e^{-2t_1}) - \frac{1}{2} \int_0^{t_1} (e^{2t} + e^{-2t} - 2) dt \\ &= \frac{1}{4} (e^{2t_1} - e^{-2t_1}) - \frac{1}{4} (e^{2t_1} - e^{-2t_1} - 4t_1) = t_1 \text{ sq unit} \end{aligned}$$

28. The two curves meet at the points $x = 1$ and $x = \frac{1}{e}$.



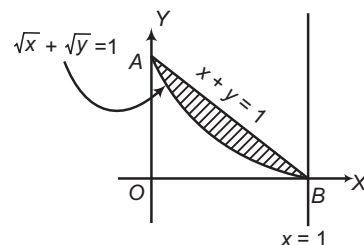
$$\begin{aligned} \therefore \text{Required area} &= \left| \int_{1/e}^1 ex \log_e x dx - \int_{1/e}^1 \frac{\log_e x}{ex} dx \right| \\ &= \left| \left[e \log x \cdot \frac{x^2}{2} - \frac{x^2 e}{4} \right]_{1/e}^1 - \frac{1}{2e} [(\log x)^2]_{1/e}^1 \right| \\ &= \left| -\frac{e}{4} + \frac{1}{2e} + \frac{1}{4e} + \frac{1}{2e} \right| \\ &= \left(\frac{e^2 - 5}{4e} \right) \text{ sq unit} \end{aligned}$$

$$29. \text{ Required area} = \int_{1-e}^0 \log_e (x+e) + \int_0^\infty e^{-x} dx$$



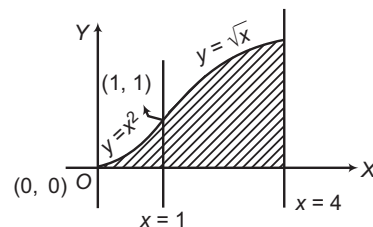
$$\begin{aligned} &= \left[x \log (x+e) - \int \frac{x}{x+e} dx \right]_{1-e}^0 - [e^{-x}]_0^\infty \\ &= [x \log (x+e) - x + e \log (x+e)]_{1-e}^0 + 1 \\ &= e + (1-e) + 1 = 2 \text{ sq units} \end{aligned}$$

$$30. \text{ Required area} = \text{Area of } \triangle OAB - \int_0^1 (1 - \sqrt{x})^2 dx$$



$$\begin{aligned} &= \frac{1}{2} \times 1 \times 1 - \left[x - \frac{2x^{3/2}}{3} \right]_0^1 \\ &= \frac{1}{6} \text{ sq unit} \end{aligned}$$

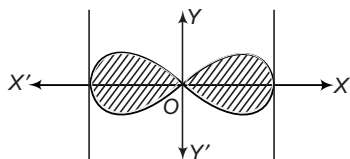
$$31. \text{ Required area} = \int_0^1 x^2 dx + \int_1^4 \sqrt{x} dx$$



$$\begin{aligned} &= \left[\frac{x^3}{3} \right]_0^1 + \left[\frac{2x^{3/2}}{3} \right]_1^4 \\ &= \frac{1}{3} + \frac{2}{3} (8 - 1) \\ &= 5 \text{ sq units} \end{aligned}$$



32. Since, the curve is symmetrical about both the axes.



$$\therefore \text{Required area, } A = 4 \int_0^a y \, dx$$

$$= 4 \int_0^a x \sqrt{\frac{a^2 - x^2}{a^2 + x^2}} \, dx$$

$$\text{Put } x^2 + a^2 = z^2 \Rightarrow x \, dx = z \, dz$$

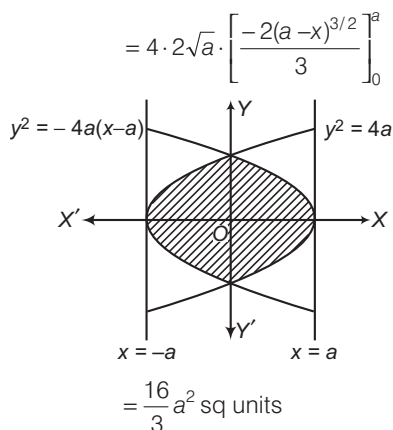
$$\text{Also, } a^2 - x^2 = 2a^2 - z^2$$

$$\therefore A = 4 \int_0^{\sqrt{2}a} \frac{\sqrt{(2a^2 - z^2)} \cdot z \, dz}{z^2}$$

$$= 4 \left[\frac{z}{2} \sqrt{(2a^2 - z^2)} + \frac{(\sqrt{2}a)^2}{2} \sin^{-1} \frac{z}{\sqrt{2}a} \right]_0^{\sqrt{2}a}$$

$$= a^2(\pi - 2) \text{ sq units}$$

33. Required area = $4 \int_0^a \sqrt{4a(a-x)} \, dx$



34. We have, $A_n = \int_0^{\pi/4} y \, dx = \int_0^{\pi/4} \tan^n x \, dx$

$$= \int_0^{\pi/4} \tan^{n-2} x (\sec^2 x - 1) \, dx$$

$$= \left[\frac{\tan^{n-1} x}{n-1} \right]_0^{\pi/4} - \int_0^{\pi/4} \tan^{n-2} x \, dx = \frac{1}{n-1} - A_{n-2}$$

$$\therefore A_n + A_{n-2} = \frac{1}{n-1} \quad \dots(i)$$

Now, in the interval $\left(0, \frac{\pi}{4}\right)$, $0 < \tan x < 1$

$$\Rightarrow \tan^n x < \tan^{n-2} x$$

$$\Rightarrow A_n < A_{n-2}$$

$$\therefore 2A_n = A_n + A_n < A_n < A_n + A_{n-2} = \frac{1}{n-1}$$

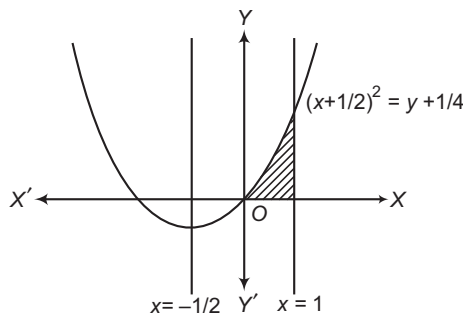
$$\Rightarrow A_n < \frac{1}{2n-2}$$

$$\text{Also, } 2A_n = A_n + A_n > A_{n+2} + A_n = \frac{1}{n+1}$$

[replacing n by $n+2$ in Eq. (i)]

$$\Rightarrow A_n > \frac{1}{2n+2}$$

35. Given, $\frac{dy}{dx} = 2x + 1 \Rightarrow y = x^2 + x + k$



Since, the curve passes through the point (1, 2).

$$\therefore 2 = 1 + 1 + k$$

$$\Rightarrow k = 0$$

So, the curve is $y = x^2 + x$.

$$\therefore \text{Required area} = \int_0^1 (x^2 + x) \, dx = \left[\frac{x^3}{3} + \frac{x^2}{2} \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} = \frac{5}{6} \text{ sq unit}$$

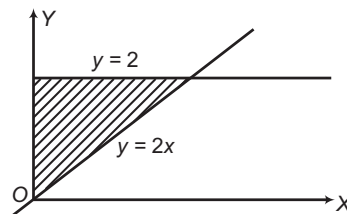
36. Required area = $\int_{-1}^2 |x-2| \, dx$

$$= \int_{-1}^0 |x-2| \, dx + \int_0^1 |x-2| \, dx + \int_1^2 |x-2| \, dx$$

$$= \int_{-1}^0 3 \, dx + \int_0^1 2 \, dx + \int_1^2 1 \, dx$$

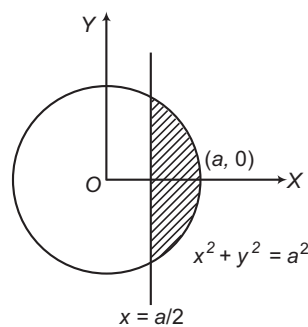
$$= [3x]_{-1}^0 + [2x]_0^1 + [x]_1^2 = 3 + 2 + 1 = 6 \text{ sq units}$$

37. Required area = $\int_0^2 x \, dy$



$$= \int_0^2 \frac{1}{2} y \, dy = \frac{1}{2} \left[\frac{y^2}{2} \right]_0^2 = \frac{1}{4} [4 - 0] = 1 \text{ sq unit}$$

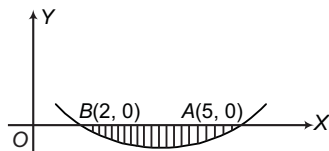
38. Required area = $2 \int_{a/2}^a y \, dx$



$$= 2 \int_{a/2}^a \sqrt{a^2 - x^2} \, dx$$

$$= a^2 \left[\frac{\pi}{3} - \frac{\sqrt{3}}{4} \right] \text{ sq unit}$$

39. The curve $y = x^2 - 7x + 10$ intersects X-axis at the points (2, 0) and (5, 0).



∴ Required area

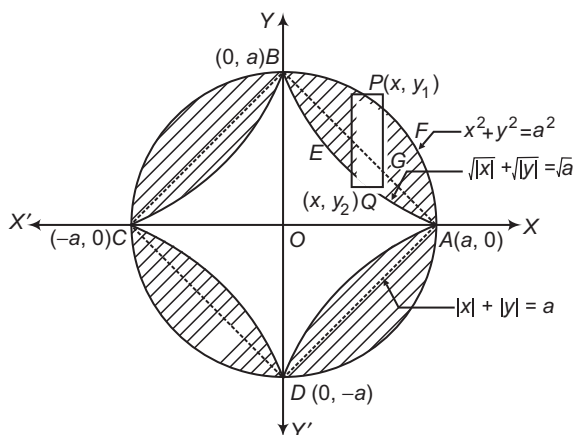
$$\begin{aligned} &= -\int_2^5 y \, dx = -\int_2^5 (x^2 - 7x + 10) \, dx \\ &= -\left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x \right]_2^5 = 4\frac{1}{2} \text{ sq units} \end{aligned}$$

40. Area of OABL = $\int_0^2 y \, dx$

$$\begin{aligned} &= \int_0^2 (a + bx + cx^2) \, dx \\ &= 2a + 2b + \frac{8}{3}c \\ &= \frac{1}{3} [6a + 6b + 8c] \end{aligned} \quad \dots(i)$$

But $f(x) = a + bx + cx^2$
 $\Rightarrow f(0) = a, f(1) = a + b + c$
 and $f(2) = a + 2b + 4c$
 $\therefore \frac{1}{3} \{f(0) + 4f(1) + f(2)\}$
 $= \frac{1}{3} \{a + 4(a + b + c) + (a + 2b + 4c)\}$
 $= \frac{1}{3} (6a + 6b + 8c)$

41. The shaded region in figure represents the region enclosed by $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$.

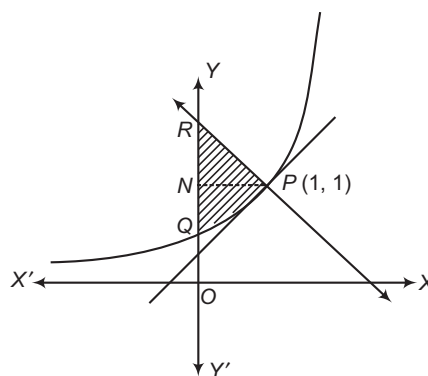


From the symmetry, it is evident that
 Required area = 4 × Area of region bounded by the two curves in first quadrant only
 $= 4 \int_0^a (y_1 - y_2) \, dx$

$$\begin{aligned} &= 4 \int_0^a \{ \sqrt{a^2 - x^2} - (\sqrt{a} - \sqrt{x})^2 \} \, dx \\ &= 4 \int_0^a \{ \sqrt{a^2 - x^2} - a - x + 2\sqrt{a}\sqrt{x} \} \, dx \\ &= 4 \left[\frac{1}{2} x \sqrt{a^2 - x^2} + \frac{1}{2} a^2 \sin^{-1} \frac{x}{a} - ax \right. \\ &\quad \left. - \frac{x^2}{2} + \frac{4\sqrt{a}}{3} x^{3/2} \right]_0^a \\ &= 4 \left[\left\{ \frac{1}{2} a^2 \times \frac{\pi}{2} - a^2 - \frac{a^2}{2} + \frac{4}{3} a^2 \right\} - 0 \right] \\ &= \left\{ \pi a^2 - \frac{2}{3} a^2 \right\} \text{ sq unit} \end{aligned}$$

42. The graph of $y = e^{a(x-1)}$ is shown in figure.

It cuts Y-axis at $Q(0, e^{-a})$. The normal to $y = e^{a(x-1)}$ at $P(1, 1)$ meets Y-axis at $R(0, 1 + \frac{1}{a})$.

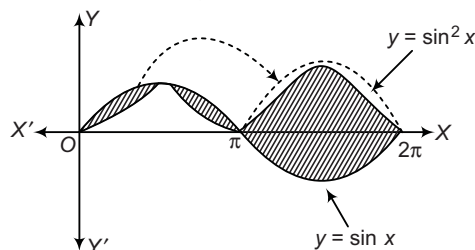


Let PN be perpendicular from P on Y-axis.

∴ Required area = Area of the region PQN
 + Area of the region PNR

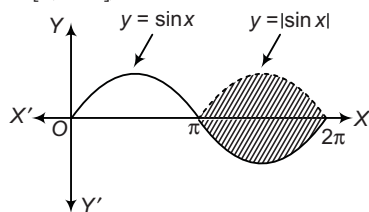
$$\begin{aligned} &= \int_{e^{-a}}^1 x_1 \, dy + \int_1^{1+1/a} x_2 \, dy \\ &= \frac{1}{a} \int_{e^{-a}}^1 (\log y + a) \, dy + \int_1^{1+1/a} \{ (1+a) - ay \} \, dy \\ &= \frac{1}{a} [y(\log y - 1) + ay]_{e^{-a}}^1 + \left[(1+a)y - \frac{a}{2} y^2 \right]_1^{1+1/a} \\ &= \frac{1}{a} [-1 + a + ae^{-a} + e^{-a} - ae^{-a}] \\ &= \frac{1}{a} (-1 + a + e^{-a}) + \left(\frac{1}{a} + 1 - 1 - \frac{1}{2a} \right) \\ &\quad + \left[\frac{1+a}{a} - \frac{a}{2} \left(2 + \frac{1}{a} \right) \frac{1}{a} \right] \\ &= -\frac{1}{a} + 1 + \frac{e^{-a}}{a} + \frac{1}{2a} = \left(1 + \frac{e^{-a}}{a} - \frac{1}{2a} \right) \text{ sq unit} \end{aligned}$$

43. We know that area bounded by $y = \sin x$ and X-axis for $x \in [0, \pi]$ is 2 sq units.



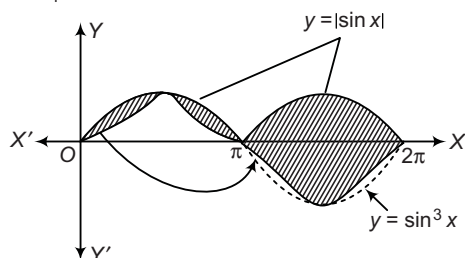
For $x \in [0, 2\pi]$, the area bounded by $y = \sin x$ and $y = \sin^2 x$ is 4 sq units.

For $x \in [0, 10\pi]$, the area bounded is 20 sq units.



The area bounded by $y = \sin x$ and $y = |\sin x|$ for $x \in [0, 2\pi]$ is 4 sq units.

Then, for $x \in [0, 20\pi]$, the area bounded is 40 sq units.



The area bounded by $y = \sin x$ and $y = \sin^3 x$ for $x \in [0, 2\pi]$ is 4 sq units.

Then, for $x \in [0, 10\pi]$, the area bounded is 20 sq units. Similarly, the area bounded by $y = \sin x$ and $y = \sin^4 x$ for $x \in [0, 10\pi]$ is 20 sq units.

44. Since, the curve $y = ax^{1/2} + bx$ passes through the point (1, 2).

$$\therefore 2 = a + b \quad \dots (i)$$

By observation, the curve also passes through (0, 0). Therefore, the area enclosed by the curve, X-axis and $x = 4$, is given by

$$A = \int_0^4 (ax^{1/2} + bx) dx = 8$$

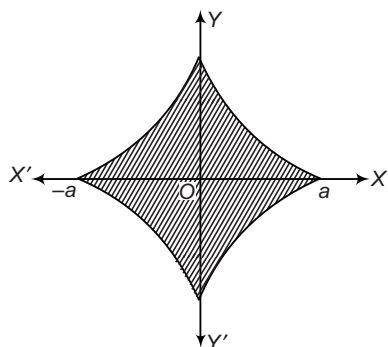
$$\Rightarrow \frac{2a}{3} \times 8 + \frac{b}{2} \times 16 = 8$$

$$\Rightarrow \frac{2a}{3} + b = 1 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get $a = 3$ and $b = -1$.

45. Eliminating t , we have $x^{2/3} + y^{2/3} = a^{2/3}$

$$\Rightarrow y = (a^{2/3} - x^{2/3})^{3/2}$$

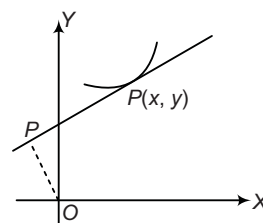


From, the above figure,

$$A = 2 \int_{-a}^a (a^{2/3} - x^{2/3})^{3/2} dx = 4 \int_0^a (a^{2/3} - x^{2/3})^{3/2} dx$$

$$\therefore A = 4 \int_0^a y dx = 4a^2 \int_0^{\pi/2} 3 \cos^3 t \sin^2 t \cos t dt$$

46. $OP = x$



If slope is $\frac{dy}{dx}$, then equation of tangent is

$$Y - y = \frac{dy}{dx} (X - x)$$

Length of perpendicular from origin to this tangent is

$$x = \frac{y - x \frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}$$

$$\Rightarrow x^2 \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} = y^2 + x^2 \left(\frac{dy}{dx}\right)^2 - 2xy \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^2 - y^2}{2xy}$$

which is a homogeneous form.

$$(a) \frac{dy}{dx} = \frac{1-x}{y} = \frac{x-x^2}{xy}$$

$$= \frac{\left[\frac{x^2 + y^2}{2} \right] - x^2}{xy} = \frac{y^2 - x^2}{2xy}$$

Hence, option (a) is true (equation of a circle).

If option (a) is true, then option (b) cannot be true (it is parabola).

If option (a) is true, then option (c) is also true, when $c = 1$.

47. The centres are (1, 2) and (-4, 0).

$$\text{So, } r_1 = 2, r_2 = 4$$

$$\text{The equation of common chord AB is}$$

$$10x + 4y - 1 = 0$$

- (a) Length of the perpendicular $C_1 M$ from centre C_1 to common chord AB is

$$C_1 M = \frac{10 \times 1 + 4 \times 2 - 1}{\sqrt{100 + 16}}$$

$$= \frac{17}{\sqrt{116}}$$

$$\Rightarrow AM = \sqrt{4 - \frac{289}{116}}$$

$$= \frac{\sqrt{175}}{2\sqrt{29}}$$

$$\Rightarrow AB = \frac{\sqrt{175}}{\sqrt{29}}$$

- (b) Length of the common tangent
- $$= \sqrt{d^2 - (r_1 - r_2)^2} = 5$$

- (c) Option (c) is not true, as centre of C_2 is an exterior point of C_1 .

- (d) Option (d) is obviously correct, as radius of C_2 is more than C_1 .

48. $\therefore y = \int_0^{4/a} (ax - \sqrt{4ax}) dx$

$$\therefore \frac{1}{3} = \int_0^{4/a} ax dx - \int_0^{4/a} \sqrt{4ax} dx$$

$$\therefore \frac{1}{3} = \left[\frac{ax^2}{2} \right]_0^{4/a} - 2 \left[\frac{(4ax)^{3/2}}{3} \right]_0^{4/a}$$

$$\Rightarrow \frac{1}{3} = \frac{16a}{2} - \frac{2}{3} \left[4a \left(\frac{4}{a} \right)^{3/2} \right]$$

$$\Rightarrow a = 8$$

On putting the value of a in $x^2 + 2x - a = 0$, we get its roots as -4 and 2 .

49. Given curves are $y^2 - 2y + 4x + 5 = 0$

and $x^2 + 2x - y + 2 = 0$

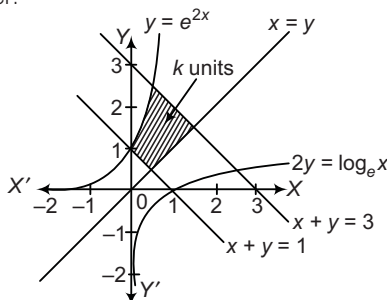
or $(y - 1)^2 = -4(x + 1)$

and $(x + 1)^2 = y - 1$

Shifting origin to $(-1, 1)$, equation of given curves changes to $Y^2 = -4X$ and $X^2 = Y$.

Hence, Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I.

50. Since, $y = e^{2x}$ and $2y = \log_e x$ are inverse of each other.



So, the shaded area is given as k sq units.

Hence, the required area is $2k$ sq units.

51. R_1 : Point $P(x, y)$ is nearer to $(1, 0)$ than to $x = -1$.

$$\Rightarrow \sqrt{(x - 1)^2 + y^2} < |x + 1|$$

$$\Rightarrow y^2 < 4x$$

So, point P lies inside parabola $y^2 = 4x$.

R_2 : Point $P(x, y)$ is nearer to $(0, 0)$ than to $(8, 0)$.

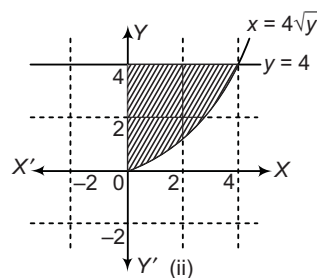
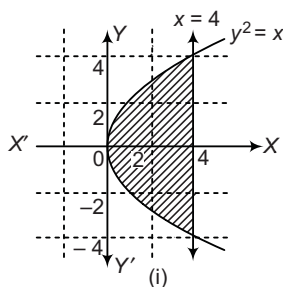
$$\Rightarrow |x| < |x - 8|$$

$$\Rightarrow x^2 < x^2 - 16x + 64$$

$$\Rightarrow x < 4$$

So, point P is towards left side of line $x = 4$.

The area of common region of R_1 and R_2 is the area bounded by $x = 4$ and $y^2 = 4x$.



Thus, area is twice the area bounded by $x = 4\sqrt{y}$ and $y = 4$.

Now, area bounded by $x = 4\sqrt{y}$ and $y = 4$ is

$$A = \int_0^4 \left(4 - \frac{x^2}{4} \right) dx = \left[4x - \frac{x^3}{12} \right]_0^4$$

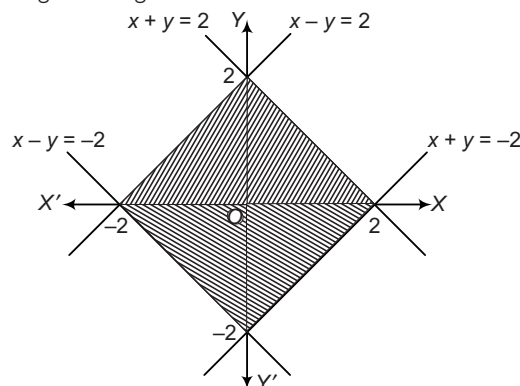
$$= \left[16 - \frac{64}{12} \right] = \frac{32}{3} \text{ sq units}$$

So, the area bounded by R_1 and R_2 is $\frac{64}{3}$ sq units.

Hence, Statement I is false but Statement II is true.

52. We have, $\max \{ |x - y|, |x + y| \} \leq 2$

$\Rightarrow |x - y| \leq 2$ and $|x + y| \leq 2$, which form a square of diagonal length 4 units.



$\therefore$ Area of the region $= \frac{1}{2} \times 4 \times 4 = 8$ sq units, which is equal to the area of square of side length $2\sqrt{2}$.

53. The ellipse $\frac{x^2}{3} + \frac{y^2}{2} = 1$ and the circle is

$$(x - 1)^2 + (y + 2)^2 = 1$$

Area of ellipse $= \pi \sqrt{3} \sqrt{2} = \sqrt{6} \pi$

and area of circle $= \pi \cdot (1)^2 = \pi$

Thus, Statement II is true in this particular example. In general, this may not be true.

54. We have, $(y - \sin^{-1} x) = \pm \sqrt{x - x^2}$, whose domain $\in (0, 1)$

$\therefore$ Area enclosed by curve

$$= \int_0^1 \{ (\sin^{-1} x + \sqrt{x - x^2}) - (\sin^{-1} x - \sqrt{x - x^2}) \} dx$$

Thus, Statement I is false and Statement II is true.

55. We have, $y^3 - 3y + x = 0$

On differentiating it w.r.t. y , we get

$$3y^2 y' - 3y' + 1 = 0 \Rightarrow y' = \frac{1}{3(1 - y^2)} \quad \dots (i)$$

$$\Rightarrow y'(-10\sqrt{2}) = \frac{1}{3\{1 - (2\sqrt{2})^2\}} = \frac{1}{3(1 - 8)} = -\frac{1}{21}$$

Again, differentiating Eq. (i) w.r.t. y , we get

$$y'' = \frac{6yy'^2}{3(1-y^2)}$$

$$\Rightarrow y''(-10\sqrt{2}) = \frac{6 \cdot 2\sqrt{2} \left(\frac{1}{21}\right)^2}{3(1-8)} = -\frac{4\sqrt{2}}{7^3 \cdot 3^2}$$

56. Required area $= \int_a^b f(x) dx = [x f(x)]_a^b - \int_a^b x f'(x) dx$

$$= bf(b) - af(a) + \int_a^b \frac{x}{3\{f(x)\}^2 - 1} dx$$

57. $y' = \frac{1}{3[1 - \{f(x)\}^2]}$, which is an even function.

$$\therefore \int_{-1}^1 g'(x) dx = g(1) - g(-1) = 2g(1)$$

58. For $-1 \leq x < 0$, $(y - e^{\sin^{-1} x})^2 = 2 - x^2$

$$\Rightarrow y = e^{\sin^{-1} x} \pm \sqrt{2 - x^2}$$

$$\therefore A = \int_{-1}^0 (e^{\sin^{-1} x} + \sqrt{2 - x^2}) - (e^{\sin^{-1} x} - \sqrt{2 - x^2}) dx$$

$$= 2 \int_{-1}^0 \sqrt{2 - x^2} dx$$

$$= 2 \left\{ \left[\frac{1}{2} x \sqrt{2 - x^2} \right]_{-1}^0 + \left[\frac{2}{2} \sin^{-1} \frac{x}{\sqrt{2}} \right]_{-1}^0 \right\}$$

$$= \left[1 + 2 \left\{ 0 - \left(-\frac{\pi}{4} \right) \right\} \right] = \left(\frac{\pi}{2} + 1 \right) \text{ sq units}$$

For $0 \leq x < 1$, $y = \sin^{-1} x \pm \sqrt{1 - x^2}$

$$\therefore A = 2 \int_0^1 \sqrt{1 - x^2} dx$$

$$= 2 \left\{ \left[\frac{x}{2} \sqrt{1 - x^2} \right]_0^1 + \left[\frac{1}{2} \sin^{-1} x \right]_0^1 \right\}$$

$$= 0 + \sin^{-1}(1) = \frac{\pi}{2} \text{ sq units}$$

$$\therefore \text{Total area} = \left(\frac{\pi}{2} + 1 \right) + \frac{\pi}{2} = (\pi + 1) \text{ sq units}$$

59. Required ratio $= \frac{\pi/2 + 1}{\pi/2} = \frac{\pi + 2}{\pi}$

60. $A = 2 \int_0^{1/2} \sqrt{1 - x^2} dx$

$$= 2 \left\{ \left[\frac{x}{2} \sqrt{1 - x^2} \right]_0^{1/2} + \left[\frac{1}{2} \sin^{-1} x \right]_0^{1/2} \right\}$$

$$= \left(\frac{\sqrt{3}}{4} + \frac{\pi}{6} \right) \text{ sq units}$$

61. Clearly, $x = a \sin^3 t$, $y = a \cos^3 t$; $0 \leq t \leq 2\pi$

$$\therefore S = \int_0^{2\pi} a \sin^3 t \cdot a \cdot 3 \cos^2 t (-\sin t) dt$$

$$= -3a^2 \int_0^{2\pi} \sin^4 t \cos^2 t dt$$

$$= -3a^2 \cdot \frac{\sqrt{\frac{5}{2}} \sqrt{\frac{3}{2}}}{2\sqrt{\frac{4+2+2}{2}}} = -3a^2 \cdot \frac{\frac{3}{2} \times \frac{1}{2} \sqrt{\pi} \times \frac{1}{2} \sqrt{\pi}}{2 \times 3 \times 2 \times 1}$$

$$= -\frac{3}{8} \pi a^2 = \frac{3}{8} \pi a^2 \text{ sq units} \quad [\text{absolute value}]$$

62. $S = - \int_0^{2\pi} a(1 - \cos t) a(1 - \cos t) dt$

$$= -a^2 \int_0^{2\pi} (1 - 2 \cos t + \cos^2 t) dt$$

$$= -a^2 \int_0^{2\pi} \left[1 - 2 \cos t + \left(\frac{1 + \cos 2t}{2} \right) \right] dt$$

$$= -\frac{a^2}{2} \int_0^{2\pi} [3 - 4 \cos t + \cos 2t] dt$$

$$= -3\pi a^2 = 3\pi a^2 \quad [\text{absolute value}]$$

63. We have, $x = \frac{t}{3} (6 - t) = \frac{1}{3} (6t - t^2)$

and $y = \frac{t^2}{8} (6 - t) = \frac{1}{8} (6t^2 - t^3)$

$$\therefore \text{Area} = \int_0^6 \frac{t}{3} (6 - t) \cdot \frac{1}{8} (12t - 3t^2) dt$$

$$= \frac{1}{8} \int_0^6 (t^4 + 24t^2 - 10t^3) dt$$

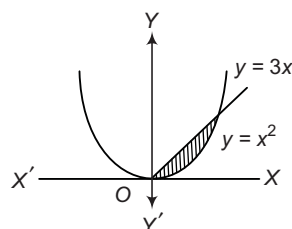
$$= \frac{27}{5} \text{ sq units}$$

64. A. Required area $= \int_0^3 (3x - x^2) dx$

$$= \left[\frac{3}{2} x^2 - \frac{x^3}{3} \right]_0^3$$

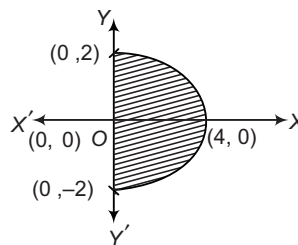
$$= \left(\frac{27}{2} - 9 \right) = \frac{9}{2}$$

$$= 4.5 \text{ sq units}$$

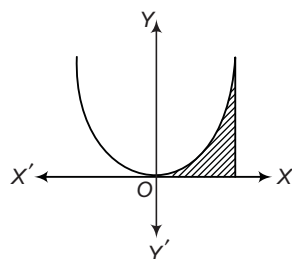


B. Required area $= 2 \int_0^2 (4 - y^2) dy = 2 \left[4y - \frac{y^3}{3} \right]_0^2$

$$= 2 \left[8 - \frac{8}{3} \right] = \frac{32}{3} \text{ sq units}$$



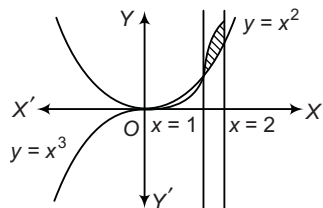
C. Required area $= \int_0^2 \frac{x^2}{4} dx = \left[\frac{x^3}{12} \right]_0^2 = \frac{8}{12} = \frac{2}{3} \text{ sq unit}$



D. Required area = $\int_1^2 (x^3 - x^2) dx$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} \right]_1^2 = \left\{ \left(4 - \frac{8}{3} \right) - \left(\frac{1}{4} - \frac{1}{3} \right) \right\}$$

$$= \frac{4}{3} + \frac{1}{12} = \frac{17}{12} \text{ sq units}$$



65. A. Area bounded by $y^2 = 4ax$ and $x^2 = 4ay$

$$= \frac{16a^2}{3} \text{ sq units.}$$

B. Area bounded by $y^2 = 4ax$ and $y = mx$

$$= \frac{8a^2}{3m^3} \text{ sq units.}$$

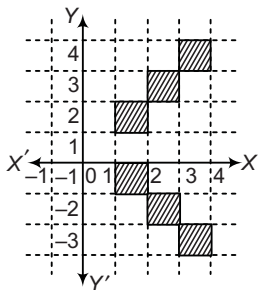
C. Area bounded by $y^2 = 4ax$ and $x = a$

$$= \frac{8}{3} a^2 \text{ sq units.}$$

$$D. A = 2 \left[\int_{-a}^{b-a} \sqrt{4a(x+a)} dx + \int_{b-a}^b \sqrt{4b(x+b)} dx \right]$$

$$= \frac{8}{3} \sqrt{ab}(a+b) \text{ sq units}$$

66. A. We have, $[x]^2 = [y]^2$, where $1 \leq x \leq 4 \Rightarrow [x] = \pm [y]$



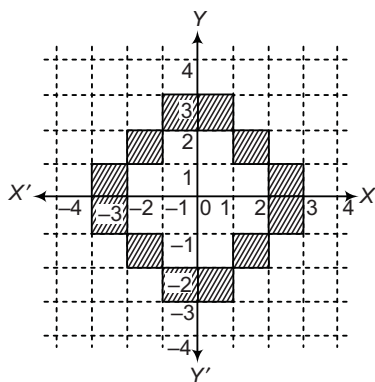
B. We have, $[|x|] + [|y|] = 2$

The graph is symmetrical about both X and Y-axes.

For $x, y > 0$, $[x] + [y] = 2$

$\Rightarrow [x] = 0$ and $[y] = 2$, $[x] = 1$ and $[y] = 1$

or $[x] = 2$ and $[y] = 0$

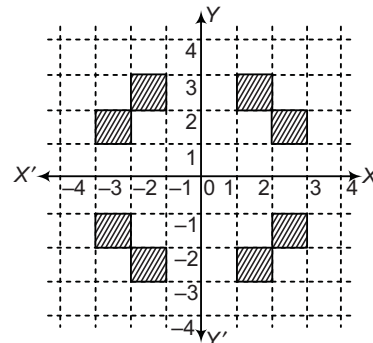


C. We have, $[|x|][|y|] = 2$

The graph is symmetrical about both X and Y-axes. For $x, y > 0$, $[x][y] = 2$

$\Rightarrow [x] = 1$ and $[y] = 2$

or $[x] = 2$ and $[y] = 1$

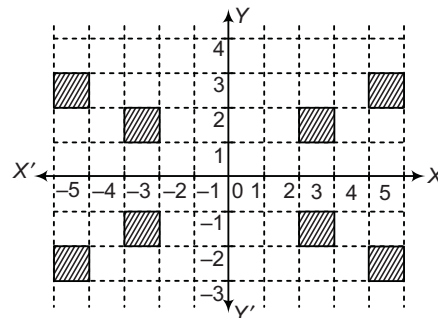


D. We have, $\frac{[|x|]}{[|y|]} = 2$, where $-5 \leq x \leq 5$

The graph is symmetrical about both the axes.

For $x, y > 0$, $[x] = 2[y]$, $[y] \neq 0 \Rightarrow [x] = 2$ and $[y] = 1$

or $[x] = 4$ and $[y] = 2$

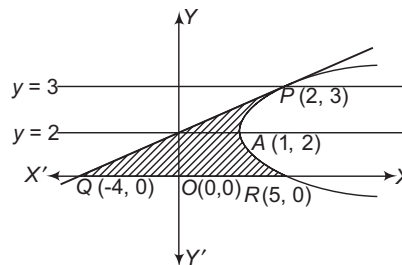


67. Given parabola is $(y - 2)^2 = x - 1$... (i)

Its axis is $y = 2$ and vertex is $(1, 2)$. Let $P \equiv (2, 3)$

From Eq. (i), $2(y - 2) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2(y - 2)}$

At $P(2, 3)$, $\frac{dy}{dx} = \frac{1}{2}$



$\therefore$ Equation of tangent at $P(2, 3)$ is

$$y - 3 = \frac{1}{2}(x - 2) \text{ or } x - 2y + 4 = 0 \quad \dots (ii)$$

Line (ii) cuts X-axis at $(-4, 0)$ and Y-axis at $(0, 2)$.

$\therefore$ Required area = $\int_0^3 (x_1 - x_2) dy$

$$= \int_0^3 [(y - 2)^2 + 1 - (2y - 4)] dy$$

$$= \int_0^3 (y^2 - 6y + 9) dy = \left[\frac{y^3}{3} - 3y^2 + 9y \right]_0^3$$

$$= (9 - 27 + 27) - 0 = 9 \text{ sq units}$$



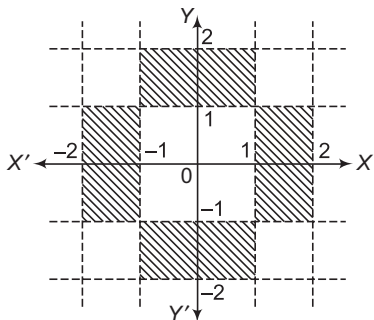
68. If $[|x|] = 1$ and $[|y|] = 0$, then

$$1 \leq |x| < 2, 0 \leq |y| < 1$$

$$\Rightarrow x \in (-2, -1] \cup [1, 2), y \in (-1, 1)$$

If $[|x|] = 0, [y] = 1$, then

$$x \in (-1, 1), y \in (-2, -1] \cup [1, 2)$$

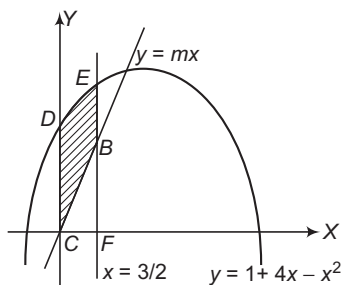


$\therefore$ Area of required region

$$= 4(2-1)\{1-(-1)\} = 8 \text{ sq units}$$

69. The given curve is $y - 5 = -(x - 2)^2$.

Thus, given curve is a parabola with vertex at (2, 5) and $x = 2$.



$$\therefore \text{Area, } A = \int_0^{3/2} (1 + 4x - x^2) dx$$

$$= \left[x + 2x^2 - \frac{x^3}{3} \right]_0^{3/2}$$

$$= \frac{3}{2} + 2\left(\frac{9}{4}\right) - \frac{9}{8} = \frac{39}{8} \text{ sq units}$$

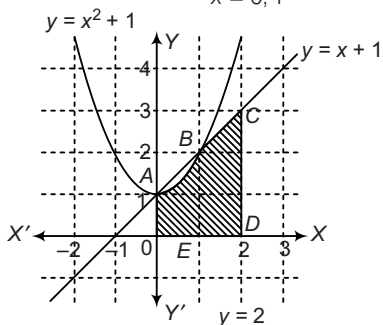
Here, $\lambda = 39, \mu = 8$

$$\therefore \lambda - 4\mu = 39 - 4 \times 8 = 39 - 32 = 7$$

70. Draw the given region of point of intersection of $y = x^2 + 1$ and $y = x + 1$

$$\Rightarrow x + 1 = x^2 + 1$$

$$\therefore x = 0, 1$$



$\therefore$ Required area = Area of OABCDEO

$$= \int_0^1 (x^2 + 1) dx + \int_1^2 (x + 1) dx$$

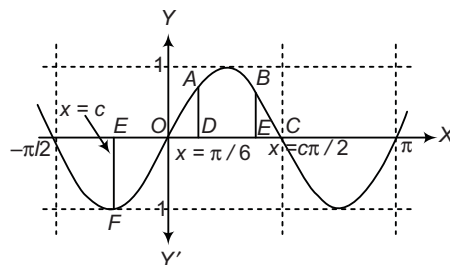
$$= \left[\frac{x^3}{3} + x \right]_0^1 + \left[\frac{x^2}{2} + x \right]_1^2$$

$$= \frac{23}{6} \text{ sq units}$$

$$\therefore 6A - 17 = 6 \times \frac{23}{6} - 17$$

$$= 6$$

71. Area of OABCO = $\int_0^{\pi/2} \sin 2x dx = 1$



$$\text{Area of OAD} = \int_0^{\pi/6} \sin 2x dx = \frac{1}{4}$$

Since, $\sin 2x$ is symmetric about origin.

So, $c = -\frac{\pi}{6}$, because area of OAD = area of OEF.

$$\therefore \int_{\pi/6}^c \sin 2x dx = \frac{1}{2}$$

$$\text{Now, } \cos 2c = -\frac{1}{2} \cos 2c = \frac{3}{2} \quad [\text{not possible}]$$

$$\text{Hence, } c = -\frac{\pi}{6}, \frac{\pi}{3}$$

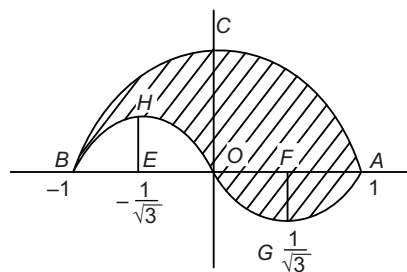
$$\Rightarrow S = \frac{\pi}{6}$$

$$\therefore \pi/S = 6$$

72. We have, $y = \sqrt{1 - x^2}$... (i)

and $y = x^3 - x$... (ii)

Put $y = 0$ in Eq. (ii), we get $x = 0, 1, -1$



$\therefore$ Required area = Area of region BCAGOH

$$= \text{Area of semi-circle } BCAOB = \frac{\pi}{2}$$

$$[\because \text{area of } BHOEB = \text{area of } OFAGO]$$

1. Let equation of tangent to parabola be $y = mx + 2/m$.

It also touches the circle $x^2 + y^2 = 2$.

$$\therefore \left| \frac{2}{m\sqrt{1+m^2}} \right| = \sqrt{2}$$

$$\Rightarrow m^4 + m^2 = 2$$

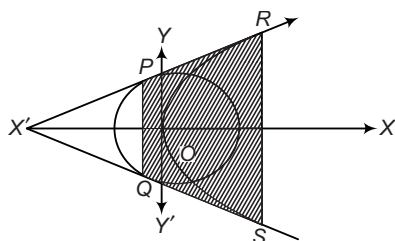
$$\Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2 - 1)(m^2 + 2) = 0$$

$$\Rightarrow m = \pm 1, m^2 = -2 \quad [\because m^2 = -2, \text{ rejected}]$$

So, tangents are $y = x + 2$, $y = -x - 2$

They intersect at $(-2, 0)$.



Equation of chord PQ is

$$-2x = 2 \Rightarrow x = -1$$

Equation of chord RS is

$$0 = 4(x - 2) \Rightarrow x = 2$$

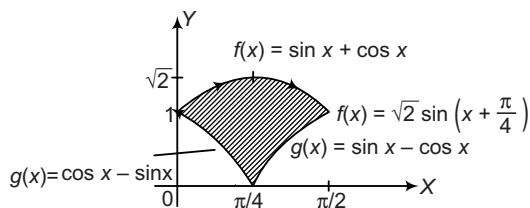
$\therefore$ Coordinates of P, Q, R, S are $P(-1, 1)$, $Q(-1, -1)$, $R(2, 4)$, $S(2, -4)$.

$$\text{Hence, area of quadrilateral PQRS} = \frac{(2 + 8) \times 3}{2}$$

$$= 15 \text{ sq units}$$

2. Here, $f(x) = y = \sin x + \cos x$, when $0 \leq x \leq \frac{\pi}{2}$ and

$$g(x) = y = |\cos x - \sin x| = \begin{cases} \cos x - \sin x, & 0 \leq x \leq \frac{\pi}{4} \\ \sin x - \cos x, & \frac{\pi}{4} \leq x \leq \frac{\pi}{2} \end{cases}$$



$\therefore$ Area of bounded region

$$= \int_0^{\pi/4} \{(\sin x + \cos x) - (\cos x - \sin x)\} dx$$

$$+ \int_{\pi/4}^{\pi/2} \{(\sin x + \cos x) - (\sin x - \cos x)\} dx$$

$$= \int_0^{\pi/4} 2 \sin x dx + \int_{\pi/4}^{\pi/2} 2 \cos x dx$$

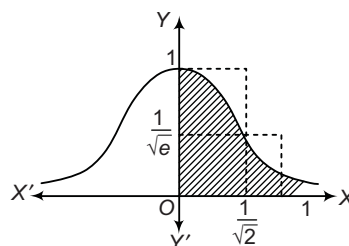
$$= 2 [-\cos x]_0^{\pi/4} + 2 [\sin x]_{\pi/4}^{\pi/2}$$

$$= 2 \left[-\frac{1}{\sqrt{2}} + 1 \right] + 2 \left[1 - \frac{1}{\sqrt{2}} \right]$$

$$= 4 - 2\sqrt{2}$$

$$= 2\sqrt{2} (\sqrt{2} - 1) \text{ sq units}$$

3. Given curve is $y = e^{-x^2}$.



Since, $x^2 \leq x$ when $x \in [0, 1]$

$$\therefore -x^2 \geq -x \Rightarrow e^{-x^2} \geq e^{-x}$$

$$\therefore \int_0^1 e^{-x^2} dx \geq \int_0^1 e^{-x} dx$$

$$\Rightarrow S \geq -[e^{-x}]_0^1 = 1 - \frac{1}{e} \quad \dots (i)$$

Also, $\int_0^1 e^{-x^2} dx \leq \text{Area of two rectangles}$

$$\leq \left(1 \times \frac{1}{\sqrt{2}}\right) + \left(1 - \frac{1}{\sqrt{2}}\right) \times \frac{1}{\sqrt{e}}$$

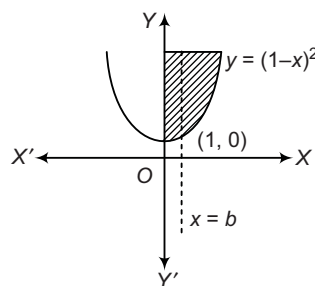
$$\leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}} \left(1 - \frac{1}{\sqrt{2}}\right) \quad \dots (ii)$$

$$\therefore \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}} \left(1 - \frac{1}{\sqrt{2}}\right) \geq S \geq 1 - \frac{1}{e}$$

[from Eqs. (i) and (ii)]

4. Given area between 0 to b is R_1 and b to 1 is R_2 .

$$\text{Also, } R_1 - R_2 = \frac{1}{4} \quad [\text{given}]$$



$$\therefore \int_0^b (1-x)^2 dx - \int_b^1 (1-x)^2 dx = \frac{1}{4}$$

$$\Rightarrow \left[\frac{(1-x)^3}{-3} \right]_0^b - \left[\frac{(1-x)^3}{-3} \right]_b^1 = \frac{1}{4}$$

$$\Rightarrow -\frac{1}{3} \{(1-b)^3 - 1\} + \frac{1}{3} \{0 - (1-b)^3\} = \frac{1}{4}$$

$$\Rightarrow (1-b)^3 = \frac{1}{8}$$

$$\therefore b = \frac{1}{2}$$

5. Given, $R_1 = \int_{-1}^2 x f(x) dx \quad \dots (i)$

$$\text{Using } \int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

$$R_1 = \int_{-1}^2 (1-x) f(1-x) dx,$$

[$\because f(x) = f(1-x)$, given]



$$\therefore R_1 = \int_{-1}^2 (1-x) f(x) dx \quad \dots(ii)$$

Since, R_2 is area bounded by $f(x)$, $x = -1$ and $x = 2$.

$$\therefore R_2 = \int_{-1}^2 f(x) dx \quad \dots(iii)$$

On adding Eqs. (i) and (ii), we get

$$2R_1 = \int_{-1}^2 f(x) dx \quad \dots(iv)$$

From Eqs. (iii) and (iv), we get $2R_1 = R_2$

6. Given region is $\{(x, y) : y^2 \leq 2x \text{ and } y \geq 4x - 1\}$

$y^2 \leq 2x$ represents a region inside the parabola

$$y^2 = 2x \quad \dots(i)$$

and $y \geq 4x - 1$ represents a region to the left of the line

$$y = 4x - 1 \quad \dots(ii)$$

The point of intersection of curves (i) and (ii) is

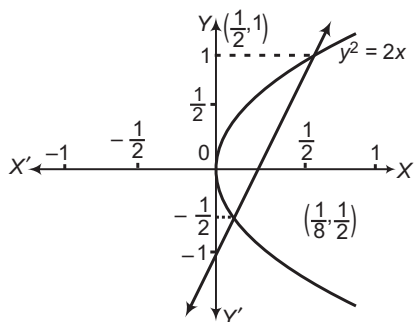
$$(4x - 1)^2 = 2x$$

$$\Rightarrow 16x^2 + 1 - 8x = 2x$$

$$\Rightarrow 16x^2 - 10x + 1 = 0$$

$$\Rightarrow x = \frac{1}{2}, \frac{1}{8}$$

So, the points where these curves intersect, are $\left(\frac{1}{2}, 1\right)$ and $\left(\frac{1}{8}, -\frac{1}{2}\right)$.



$$\therefore \text{Required area} = \int_{-1/2}^1 \left(\frac{y+1}{4} - \frac{y^2}{2} \right) dy$$

$$= \frac{1}{4} \left[\frac{y^2}{2} + y \right]_{-1/2}^1 - \frac{1}{6} [y^3]_{-1/2}^1$$

$$= \frac{1}{4} \left\{ \left(\frac{1}{2} + 1 \right) - \left(\frac{1}{8} - \frac{1}{2} \right) \right\} - \frac{1}{6} \left\{ 1 + \frac{1}{8} \right\}$$

$$= \frac{1}{4} \left(\frac{3}{2} + \frac{3}{8} \right) - \frac{1}{6} \left(\frac{9}{8} \right)$$

$$= \frac{1}{4} \times \frac{15}{8} - \frac{3}{16}$$

$$= \frac{9}{32} \text{ sq units}$$

7. Given equation of ellipse is

$$\frac{x^2}{9} + \frac{y^2}{5} = 1 \quad \dots(i)$$

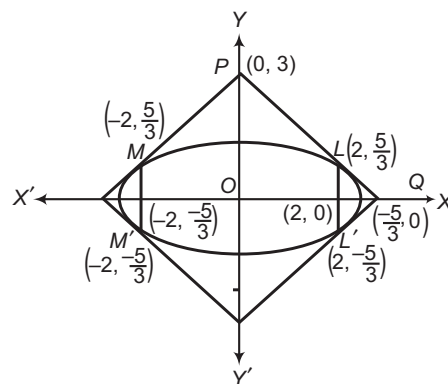
$$\therefore a^2 = 9, b^2 = 5$$

$$\Rightarrow a = 3, b = \sqrt{5}$$

$$\begin{aligned} \text{Now, } e &= \sqrt{1 - \frac{b^2}{a^2}} \\ &= \sqrt{1 - \frac{5}{9}} = \frac{2}{3} \end{aligned}$$

$$\therefore \text{Foci} = (\pm ae, 0) = (\pm 2, 0)$$

$$\text{and } \frac{b^2}{a} = \frac{5}{3}$$



$\therefore$ Extremities of one of latusrectum are

$$\left(2, \frac{5}{3} \right) \text{ and } \left(2, -\frac{5}{3} \right)$$

$\therefore$ Equation of tangent at $\left(2, \frac{5}{3} \right)$ is

$$\frac{x(2)}{9} + \frac{y(5/3)}{5} = 1$$

or

$$2x + 3y = 9 \quad \dots(ii)$$

Eq. (ii) intersects X and Y-axes at $\left(\frac{9}{2}, 0 \right)$ and $(0, 3)$, respectively.

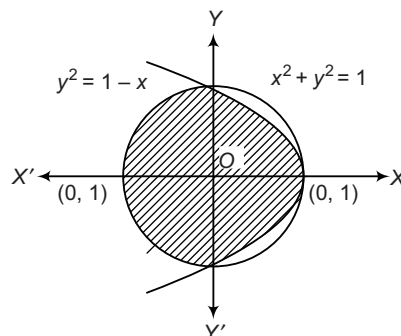
$\therefore$ Area of quadrilateral $= 4 \times \text{Area of } \triangle POQ$

$$= 4 \times \left(\frac{1}{2} \times \frac{9}{2} \times 3 \right)$$

$$= 27 \text{ sq units}$$

8. Given, $A = \{(x, y) : x^2 + y^2 \leq 1$

and $y^2 \leq 1 - x\}$



$$\therefore \text{Required area} = \frac{1}{2} \pi r^2 + 2 \int_0^1 (1 - y^2) dy$$

$$= \frac{1}{2} \pi (1)^2 + 2 \left[y + \frac{y^3}{3} \right]_0^1$$

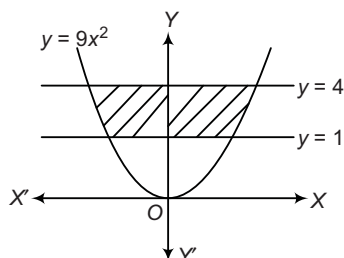
$$= \left(\frac{\pi}{2} + \frac{4}{3} \right) \text{ sq units}$$

$$9. \text{ Required area} = \int_0^{\log 3} x dy = \int_0^{\log 3} e^y dy = [e^y]_0^{\log 3}$$

$$= [e^{\log 3} - e^0]$$

$$= 3 - 1 = 2 \text{ sq units}$$

10. Required area = $\int_1^4 \frac{\sqrt{y}}{3} dy = \frac{1}{3} \left[\frac{y^{3/2}}{3/2} \right]_1^4$



$$= \frac{2}{9} (4^{3/2} - 1^{3/2}) = \frac{2}{9} \times 7 = \frac{14}{9} \text{ sq units}$$

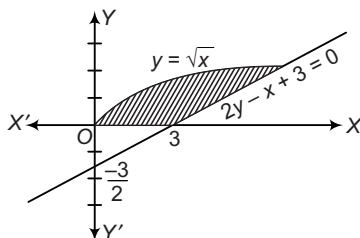
11. Required area = $\int_0^{\pi/2} |\cos x - \sin x| dx$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) - (1 - \sqrt{2}) = (2\sqrt{2} - 2) \text{ sq units}$$

12. Given curves are $y = \sqrt{x}$... (i)
and $2y - x + 3 = 0$... (ii)



On solving Eqs. (i) and (ii), we get

$$2\sqrt{x} - (\sqrt{x})^2 + 3 = 0$$

$$\Rightarrow (\sqrt{x})^2 - 2\sqrt{x} - 3 = 0$$

$$\Rightarrow (\sqrt{x} - 3)(\sqrt{x} + 1) = 0$$

$$\Rightarrow \sqrt{x} = 3 \quad [\because \sqrt{x} = -1 \text{ is not possible}]$$

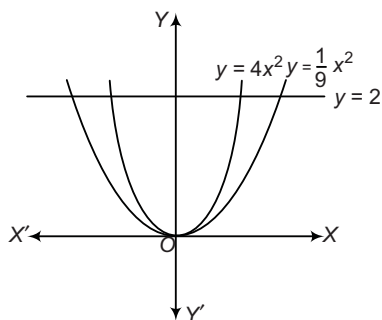
$$\therefore y = 3$$

$\therefore$ Required area = $\int_0^3 (x_2 - x_1) dy$

$$= \int_0^3 \{ (2y + 3) - y^2 \} dy = \left[y^2 + 3y - \frac{y^3}{3} \right]_0^3$$

$$= 9 + 9 - 9 = 9 \text{ sq units}$$

13. Required area = $2 \int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2} \right) dy = 2 \int_0^2 \left(\frac{5\sqrt{y}}{2} \right) dy$



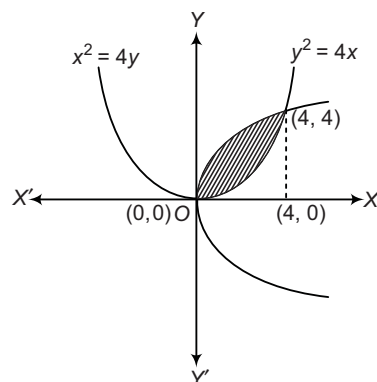
$$= 5 \left[\frac{y^{3/2}}{3/2} \right]_0^2 = \frac{10}{3} (2^{3/2} - 0) = \frac{20\sqrt{2}}{3} \text{ sq units}$$

14. For the point of intersection, for

$$\left(\frac{x^2}{4} \right)^2 = 4x \quad [\text{for } y^2 = 4x \text{ and } x^2 = 4y]$$

$$\Rightarrow x^4 = 4^3 x$$

$$\Rightarrow x = 0, 4$$



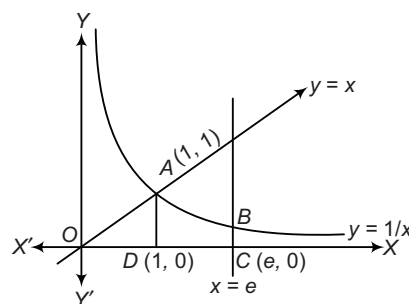
$\therefore$ Area bounded between the curves

$$= \int_0^4 \left(\sqrt{4x} - \frac{x^2}{4} \right) dx = \left[2 \cdot \frac{x^{3/2}}{3/2} - \frac{x^3}{12} \right]_0^4$$

$$= \frac{4}{3} \cdot (4)^{3/2} - \frac{(4)^3}{12} = \frac{32}{3} - \frac{16}{3} = \frac{16}{3} \text{ sq units}$$

15. Given, $y = x$, $x = e$ and $y = \frac{1}{x}$, $x \geq 0$

Since, $y = x$ and $x \geq 0 \Rightarrow y \geq 0$



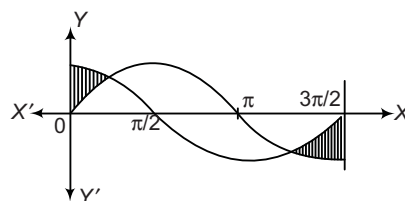
Therefore, area to be calculated in 1st quadrant shown in figure = Area of $\triangle ODA$ + Area of $DABCD$

$$= \frac{1}{2} (1 \times 1) + \int_1^e \frac{1}{x} dx$$

$$= \frac{1}{2} + [\log |x|]_1^e = \frac{1}{2} + 1 = \frac{3}{2} \text{ sq units}$$

16. Required area = $\int_0^{\pi/4} (\cos x - \sin x) dx$

$$+ \int_{\pi/4}^{5\pi/4} (\sin x - \cos x) dx + \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x) dx$$

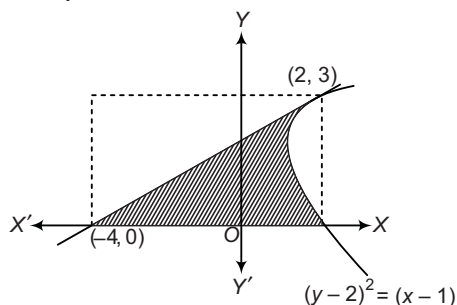


$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{5\pi/4} + [\sin x + \cos x]_{5\pi/4}^{3\pi/2}$$

$$= (4\sqrt{2} - 2) \text{ sq units}$$



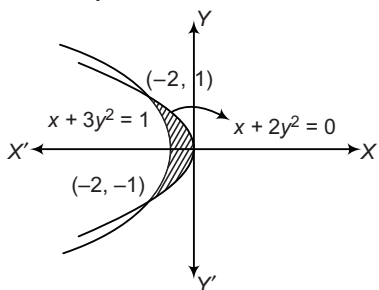
17. The equation of tangent at (2, 3) to the given parabola is $x = 2y - 4$.



$$\begin{aligned} \therefore \text{Required area} &= \int_0^3 \{(y-2)^2 + 1 - 2y + 4\} dy \\ &= \left[\frac{(y-2)^3}{3} - y^2 + 5y \right]_0^3 \\ &= \frac{1}{3} - 9 + 15 + \frac{8}{3} = 9 \text{ sq units} \end{aligned}$$

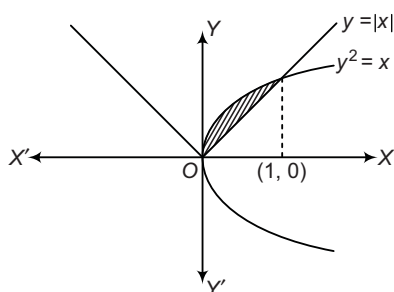
18. Given curves are $x + 3y^2 = 1$... (i)
and $x + 2y^2 = 0$... (ii)

On solving Eqs. (i) and (ii), we get
 $y = \pm 1$ and $x = -2$



$$\begin{aligned} \therefore \text{Required area} &= \left| \int_{-1}^1 (x_1 - x_2) dy \right| \\ &= \left| \int_{-1}^1 (1 - 3y^2 + 2y^2) dy \right| \\ &= \left| \int_{-1}^1 (1 - y^2) dy \right| = \left| 2 \int_0^1 (1 - y^2) dy \right| \\ &= \left| 2 \left[y - \frac{y^3}{3} \right]_0^1 \right| = \left| 2 \left(1 - \frac{1}{3} \right) \right| \\ &= \frac{4}{3} \text{ sq units} \end{aligned}$$

19. Required area $= \int_0^1 (\sqrt{x} - x) dx$

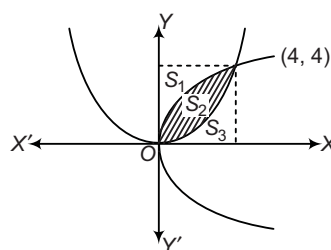


$$= \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} \right]_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6} \text{ sq unit}$$

20. Given, $y^2 = 4x$, $x^2 = 4y$ are symmetric about $y = x$.

$$\therefore \text{Required area} = \int_0^4 (2\sqrt{x} - x) dx = \frac{16}{3} \text{ sq units}$$

$$\Rightarrow \text{Area of } S_2 = \frac{16}{3} \text{ sq units}$$



$$\therefore \text{Area of } S_1 = \text{Area of } S_3 = \frac{16}{3} \text{ sq units}$$

$$\text{Now, } S_1 : S_2 : S_3 = 1 : 1 : 1$$

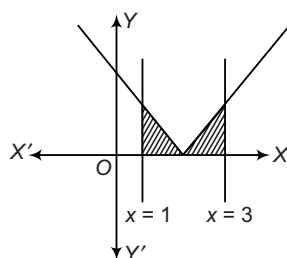
21. $\int_{\pi/4}^{\beta} f(x) dx = \beta \sin \beta + \frac{\pi}{4} \cos \beta + \sqrt{2} \beta$

On differentiating w.r.t. β , we get

$$f(\beta) = \sin \beta + \beta \cos \beta - \frac{\pi}{4} \sin \beta + \sqrt{2}$$

$$\therefore f\left(\frac{\pi}{2}\right) = \left(1 - \frac{\pi}{4} + \sqrt{2}\right)$$

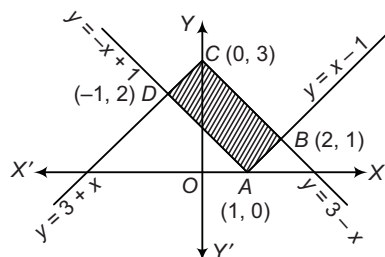
22. Required area $= \int_1^3 y dx = \int_1^3 |x - 2| dx$



$$\begin{aligned} &= \int_1^2 -(x-2) dx + \int_2^3 (x-2) dx \\ &= \int_1^2 (2-x) dx + \int_2^3 (x-2) dx \\ &= \left[2x - \frac{x^2}{2} \right]_1^2 + \left[\frac{x^2}{2} - 2x \right]_2^3 \\ &= \left[4 - 2 - \left(2 - \frac{1}{2} \right) \right] + \left[\frac{9}{2} - 6 - (2 - 4) \right] \\ &= 2 - \frac{3}{2} - \frac{3}{2} + 2 \text{ sq unit} \end{aligned}$$

23. Since, $y = |x - 1| = \begin{cases} x - 1, & x > 1 \\ -x + 1, & x \leq 1 \end{cases}$

and $y = 3 - |x| = \begin{cases} 3 + x, & x \leq 0 \\ 3 - x, & x > 0 \end{cases}$



On solving $y = x - 1$ and $y = 3 - x$, we get

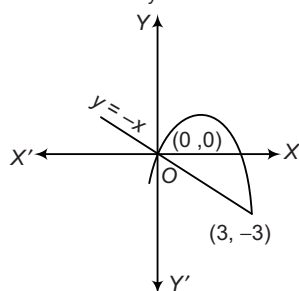
$$\begin{aligned} x - 1 &= 3 - x \\ \Rightarrow x &= 2 \\ \text{and } y &= 3 - 2 \\ \Rightarrow y &= 1 \\ \text{Now, } AB^2 &= (2 - 1)^2 + (1 - 0)^2 \\ &= 1 + 1 = 2 \\ \Rightarrow AB &= \sqrt{2} \\ \text{and } BC^2 &= (0 - 2)^2 + (3 - 1)^2 \\ &= 4 + 4 = 8 \\ \Rightarrow BC &= 2\sqrt{2} \end{aligned}$$

$$\begin{aligned} \therefore \text{Area of rectangle } ABCD &= AB \times BC \\ &= \sqrt{2} \times 2\sqrt{2} = 4 \text{ sq units} \end{aligned}$$

24. The equations of given curve and a line are

$$y = 2x - x^2 \quad \dots(i)$$

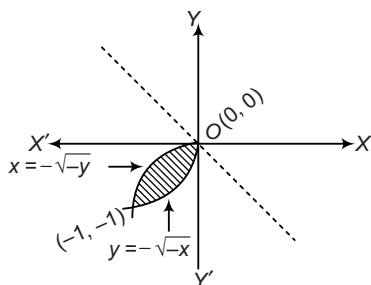
$$\text{and } y = -x \quad \dots(ii)$$



On solving Eqs. (i) and (ii), we get the points of intersection of curves as $(0, 0)$ and $(3, -3)$.

$$\begin{aligned} \therefore \text{Required area} &= \int_0^3 [(2x - x^2) - (-x)] dx \\ &= \int_0^3 (3x - x^2) dx \\ &= \left[\frac{3x^2}{2} - \frac{x^3}{3} \right]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{9}{2} \text{ sq units} \end{aligned}$$

25. Given, $y = -\sqrt{-x}$



$\Rightarrow y^2 = -x$, where x and y both are negative.

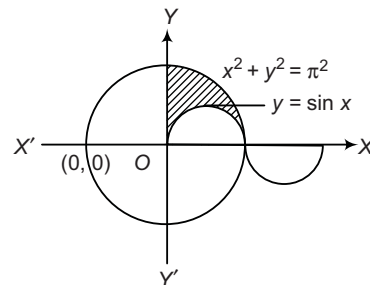
Also, $x = -\sqrt{-y}$

$\Rightarrow x^2 = -y$, where x and y both are negative.

So, the point of intersection of two curves are $(0, 0)$ and $(-1, -1)$.

$$\begin{aligned} \therefore \text{Required area} &= \left| \int_{-1}^0 -\sqrt{-x} dx - \int_{-1}^0 -x^2 dx \right| \\ &= \left| \left[-\frac{(-x)^{3/2}}{3/2} (-1) + \frac{x^3}{3} \right]_{-1}^0 \right| \\ &= \left| 0 + 0 - \left(\frac{2}{3} - \frac{1}{3} \right) \right| = \frac{1}{3} \text{ sq unit} \end{aligned}$$

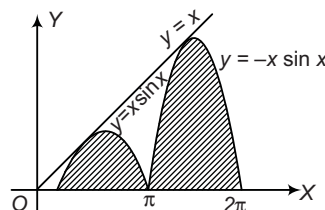
26. Given curve $x^2 + y^2 = \pi^2$ is a circle of radius π and centre at origin.



$$\begin{aligned} \therefore \text{Required area} &= \text{Area of circle (1st quadrant)} \\ &\quad - \int_0^\pi \sin x dx \end{aligned}$$

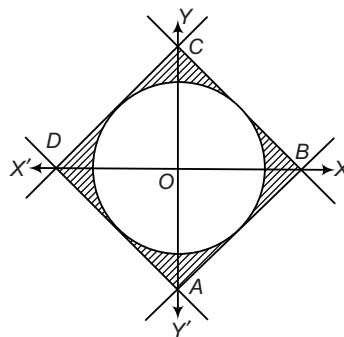
$$\begin{aligned} &= \frac{\pi \cdot \pi^2}{4} - [-\cos x]_0^\pi = \frac{\pi^3}{4} + (\cos \pi - \cos 0) \\ &= \frac{\pi^3}{4} + (-1 - 1) = \frac{\pi^3}{4} - 2 = \left(\frac{\pi^3 - 8}{4} \right) \text{ sq units} \end{aligned}$$

27. Given, $y = x |\sin x| \Rightarrow y = \begin{cases} x \sin x, & 0 \leq x \leq \pi \\ -x \sin x, & \pi < x \leq 2\pi \end{cases}$



$$\begin{aligned} \therefore \text{Required area} &= \int_0^\pi x \sin x dx + \int_\pi^{2\pi} (-x \sin x) dx \\ &= [x(-\cos x)]_0^\pi - \int_0^\pi (-\cos x) dx \\ &\quad - [x(-\cos x)]_\pi^{2\pi} + \int_\pi^{2\pi} (-\cos x) dx \\ &= \pi + [\sin x]_0^\pi + (2\pi + \pi) - [\sin x]_\pi^{2\pi} \\ &= 4\pi \text{ sq units} \end{aligned}$$

28. Given curves $|y + x| \leq 1$ and $|y - x| \leq 1$ form a square $ABCD$ and $2x^2 + 2y^2 = 1$ is a circle of radius $\frac{1}{\sqrt{2}}$ and centre $(0, 0)$.



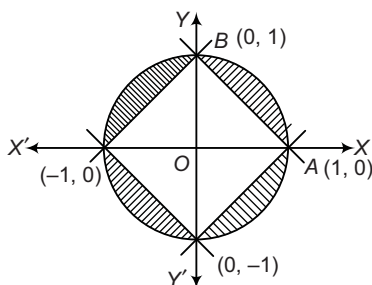
$$\begin{aligned} \therefore \text{Required area} &= \text{Area of square} - \text{Area of circle} \\ &= \frac{1}{2} d_1 d_2 - \pi r^2 = \frac{1}{2} \times 2 \times 2 - \pi \left(\frac{1}{\sqrt{2}} \right)^2 \\ &= \left(2 - \frac{\pi}{2} \right) \text{ sq units} \end{aligned}$$

29. Given curves are $|x| + |y| = 1$

$$\Rightarrow \begin{cases} x + y = 1; & x > 0, y > 0 \\ x - y = 1; & x > 0, y < 0 \\ -x + y = 1; & x < 0, y > 0 \\ -x - y = 1; & x < 0, y < 0 \end{cases}$$

and $1 - y^2 = |x|$

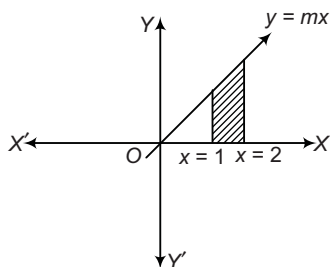
$$\Rightarrow \begin{cases} 1 - y^2 = x, & x \geq 0 \\ 1 - y^2 = -x, & x < 0 \end{cases}$$



$\therefore$ Required area

$$\begin{aligned} &= \left| 2 \int_0^1 \sqrt{1-x} dx \right| + \left| 2 \int_{-1}^0 \sqrt{x+1} dx \right| - 4 \left(\frac{1}{2} \cdot 1 \cdot 1 \right) \\ &= \left[-2 \frac{(1-x)^{3/2}}{\frac{3}{2}} \right]_0^1 + 2 \left[\frac{(x+1)^{3/2}}{\frac{3}{2}} \right]_{-1}^0 - 2 \\ &= -\frac{4}{3} (0-1) + \frac{4}{3} (1) - 2 = \frac{2}{3} \text{ sq unit} \end{aligned}$$

30. Given equation of line is $y = mx$ and bounded by $x = 1$, $x = 2$ and X -axis.



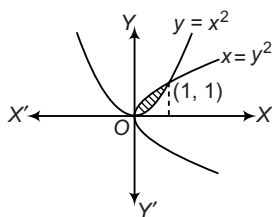
$\therefore$ Required area $= \int_1^2 mx dx$

$$\Rightarrow 6 = m \left[\frac{x^2}{2} \right]_1^2 \Rightarrow 6 = m \left(\frac{4}{2} - \frac{1}{2} \right)$$

$$\Rightarrow 6 = m \times \frac{3}{2} \Rightarrow m = 4$$

31. The intersection point of parabolas $y = x^2$ and $x = y^2$ is

$$\begin{aligned} y &= (y^2)^2 \Rightarrow y = y^4 \\ \Rightarrow y &= 0, y = 1 \Rightarrow x = 0, x = 1 \end{aligned}$$



So, the intersection points are $O(0, 0)$ and $(1, 1)$.

$$\begin{aligned} \therefore \text{Required area} &= \int_0^1 (y_2 - y_1) dx = \int_0^1 (\sqrt{x} - x^2) dx \\ &= \left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3} \right]_0^1 = \left[\frac{2}{3} x^{3/2} - \frac{x^3}{3} \right]_0^1 \\ &= \left[\frac{2}{3} (1) - \frac{1}{3} + 0 - 0 \right] \\ &= \left[\frac{2}{3} - \frac{1}{3} \right] = \frac{1}{3} \text{ sq unit} \end{aligned}$$

Aliter

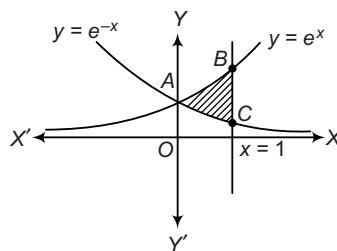
We know that, if parabolas are $y^2 = 4ax$ and $x^2 = 4by$, then area of bounded region is $\frac{4a \cdot 4b}{3}$.

$$\begin{aligned} \therefore \text{Required area} &= \frac{1 \times 1}{3} \\ &= \frac{1}{3} \text{ sq unit} \end{aligned}$$

32. Given curves are $y = e^x$ and $y = e^{-x}$.

The point of intersection is $e^{-x} = e^x$

$$\Rightarrow x = 0 \Rightarrow y = 1$$



So, the point of intersection is $(0, 1)$.

$\therefore$ Area of bounded region ABC

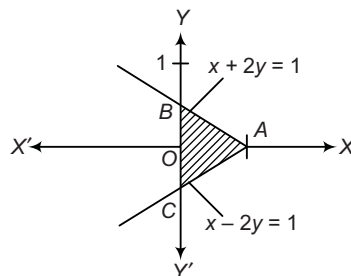
$$\begin{aligned} &= \int_0^1 (y_2 - y_1) dx = \int_0^1 (e^x - e^{-x}) dx = [e^x + e^{-x}]_0^1 \\ &= [e^1 + e^{-1} - (e^0 + e^{-0})] = \left[e + \frac{1}{e} - (1 + 1) \right] \\ &= \left(e + \frac{1}{e} - 2 \right) \text{ sq units} \end{aligned}$$

33. Given curves are $x = 0$ and $x + 2|y| = 1$.

Now, $x + 2|y| = 0$

When $y > 0$, then $x + 2y = 1$

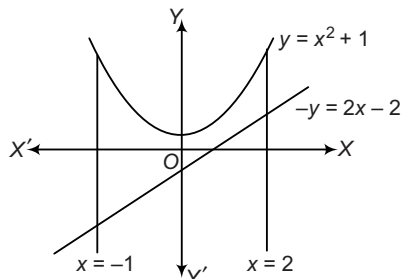
When $y < 0$, then $x - 2y = 1$



$\therefore$ Area of bounded region ABC

$$\begin{aligned} &= 2AOB = 2 \int_0^1 \left(\frac{1-x}{2} \right) dx = \left[x - \frac{x^2}{2} \right]_0^1 \\ &= \left[1 - \frac{1}{2} - (0 - 0) \right] = \frac{1}{2} \text{ sq unit} \end{aligned}$$

34. Given curve is $y = x^2 + 1$ i.e. $x^2 = y - 1$ and line is $y = 2x - 2$.



The intersection point of curve and line is
 $x^2 = 2x - 2 - 1 \Rightarrow x^2 - 2x + 3 = 0$

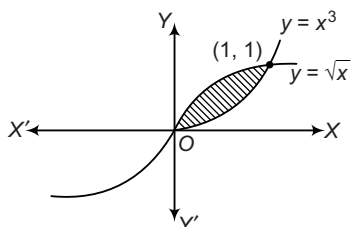
Now, $b^2 - 4ac = 4 - 12 = -8 < 0$

So, there is no point of intersection.

$$\begin{aligned} \therefore \text{Required area} &= \int_{-1}^2 (y_2 - y_1) dx \\ &= \int_{-1}^2 [(x^2 + 1) - (2x - 2)] dx \\ &= \left[\frac{x^3}{3} + x \right]_{-1}^2 - \left[x^2 - 2x \right]_{-1}^2 \\ &= \left[\frac{8}{3} + 2 - \left(-\frac{1}{3} - 1 \right) \right] - [4 - 4 - (1 + 2)] \\ &= \frac{14}{3} + \frac{4}{3} - [-3] \\ &= 6 + 3 = 9 \text{ sq units} \end{aligned}$$

35. The points of intersection of $y = \sqrt{x}$ and $y = x^3$ are

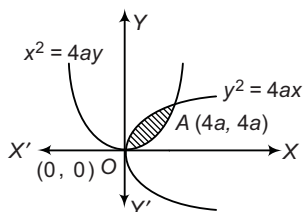
$$\begin{aligned} (\sqrt{x})^2 &= (x^3)^2 \\ \Rightarrow x &= x^6 \\ \Rightarrow x &= 0, 1 \Rightarrow y = 0, 1 \end{aligned}$$



So, intersection points are (0, 0) and (1, 1).

$$\begin{aligned} \therefore \text{Required area} &= \int_0^1 (\sqrt{x} - x^3) dx \\ &= \left[\frac{x^{3/2}}{3/2} - \frac{x^4}{4} \right]_0^1 = \frac{2}{3} - \frac{1}{4} = \frac{5}{12} \text{ sq unit} \end{aligned}$$

36. The equations of given curves are $y^2 = 4ax$ and $x^2 = 4ay$



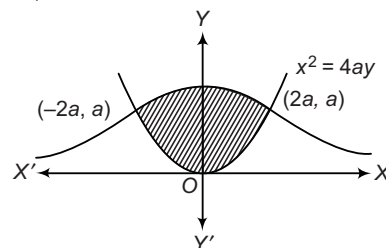
On solving these equations, we get the intersection points as (0, 0) and (4a, 4a).

$$\begin{aligned} \therefore \text{Required area} &= \int_0^{4a} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4a} \right) dx \\ &= 2\sqrt{a} \left[\frac{x^{3/2}}{3/2} \right]_0^{4a} - \left[\frac{x^3}{12a} \right]_0^{4a} \\ &= \frac{32a^2}{3} - \frac{16a^2}{3} \\ &= \frac{16a^2}{3} \text{ sq units} \end{aligned}$$

37. Given two curves intersect at $\frac{x^2}{4a} = \frac{8a^3}{x^2 + 4a^2}$.

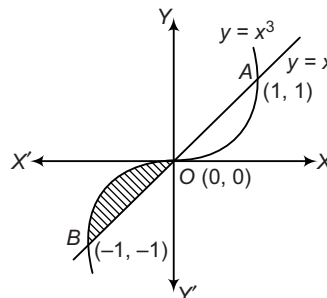
$$\begin{aligned} \Rightarrow x^4 + 4a^2x^2 - 32a^4 &= 0 \\ \Rightarrow (x^2 - 4a^2)(x^2 + 8a^2) &= 0 \\ \Rightarrow x &= \pm 2a \text{ and } y = a \end{aligned}$$

Thus, the points of intersection are (2a, a) and (-2a, a).



$$\begin{aligned} \therefore \text{Required area} &= 2 \left[\int_0^{2a} \frac{8a^3}{x^2 + 4a^2} dx - \int_a^{2a} \frac{x^2}{4a} dx \right] \\ &= 2 \left\{ \left[8a^3 \times \frac{1}{2a} \tan^{-1} \frac{x}{2a} \right]_0^{2a} - \left[\frac{1}{4a} \times \frac{x^3}{3} \right]_0^{2a} \right\} \\ &= 2 \left\{ 4a^2 [\tan^{-1} 1 - \tan^{-1} 0] - \frac{1}{12a} \times 8a^3 \right\} \\ &= 2 \left\{ 4a^2 \cdot \frac{\pi}{4} - \frac{2}{3} a^2 \right\} \\ &= 2a^2 \pi - \frac{4}{3} a^2 \\ &= \frac{a^2}{3} (6\pi - 4) \text{ sq units} \end{aligned}$$

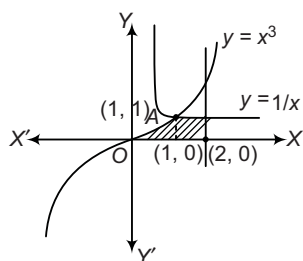
38. The intersection points of both curves $y = x$ and $y = x^3$ are (-1, -1), (0, 0) and (1, 1).



$$\begin{aligned} \therefore \text{Required area} &= 2 \int_0^1 (x - x^3) dx = 2 \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 \\ &= 2 \left(\frac{1}{2} - \frac{1}{4} \right) = 2 \times \frac{1}{4} \\ &= \frac{1}{2} \text{ sq unit} \end{aligned}$$



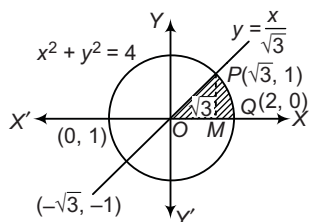
39. The point of intersection of the curves $y = x^3$ and $y = \frac{1}{x}$ is $(1, 1)$.



∴ Required area

$$\begin{aligned} &= \int_0^1 x^3 dx + \int_1^2 \frac{1}{x} dx = \left[\frac{x^4}{4} \right]_0^1 + [\log_e x]_1^2 \\ &= \left(\frac{1}{4} - 0 \right) + \log_e 2 - \log_e 1 \\ &= \left(\frac{1}{4} + \log_e 2 \right) \text{ sq units} \end{aligned}$$

40. The intersection points of curves $x^2 + y^2 = 4$ and $y = \sqrt{3}x$ are $(0, 0)$ and $P(\sqrt{3}, 1)$.



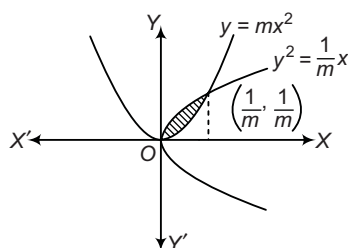
∴ Area of $\triangle OPM = \frac{1}{2} \times \sqrt{3} \times 1 = \frac{\sqrt{3}}{2}$

and area of curve $MPQ = \int_{\sqrt{3}}^2 \sqrt{4 - x^2} dx$

$$\begin{aligned} &= \left[\frac{x}{2} \sqrt{4 - x^2} + \frac{4}{2} \sin^{-1} \left(\frac{x}{2} \right) \right]_{\sqrt{3}}^2 \\ &= \left[0 + 2 \left(\frac{\pi}{2} \right) - \left(\frac{\sqrt{3}}{2} + 2 \times \frac{\pi}{3} \right) \right] = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2} \right) \end{aligned}$$

∴ Required area = $\frac{\sqrt{3}}{2} + \frac{\pi}{3} - \frac{\sqrt{3}}{2} = \frac{\pi}{3}$ sq units

41. The points of intersection of the curves $my^2 = x$ and $y = mx^2$ are $(0, 0)$ and $\left(\frac{1}{m}, \frac{1}{m} \right)$.

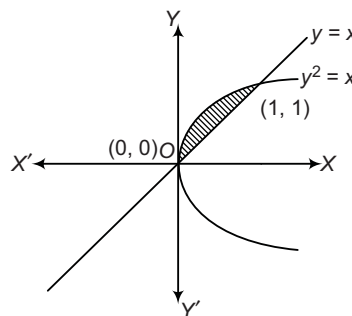


$$\begin{aligned} \therefore \text{Required area} &= \int_0^{1/m} \left(\frac{1}{\sqrt{m}} \sqrt{x} - mx^2 \right) dx \\ &= \left[\frac{1}{\sqrt{m}} \times \frac{x^{3/2}}{3/2} \right]_0^{1/m} - \left[m \cdot \frac{x^3}{3} \right]_0^{1/m} \\ &= \frac{2}{3m^2} - \frac{1}{3m^2} = \frac{1}{3m^2} \end{aligned}$$

Also given that, $A = \frac{1}{4}$ sq unit

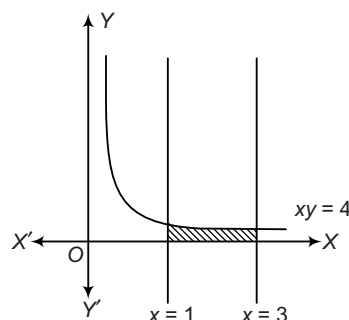
$$\begin{aligned} \therefore \quad &\frac{1}{3m^2} = \frac{1}{4} \\ \Rightarrow \quad &m^2 = \frac{4}{3} \\ \Rightarrow \quad &m = \pm \frac{2}{\sqrt{3}} \end{aligned}$$

42. The points of intersection of the curves $y = x$ and $y^2 = x$ are $(0, 0)$ and $(1, 1)$.



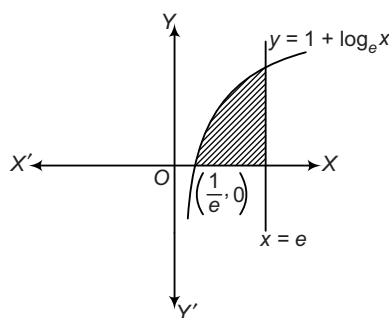
$$\begin{aligned} \therefore \text{Required area} &= \int_0^1 (\sqrt{x} - x) dx = \left[\frac{2}{3} x^{3/2} - \frac{x^2}{2} \right]_0^1 \\ &= \frac{2}{3} - \frac{1}{2} \\ &= \frac{1}{6} \text{ sq unit} \end{aligned}$$

43. Given curve is $xy = 4$.



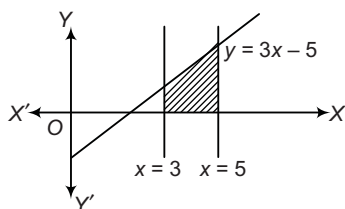
$$\begin{aligned} \therefore \text{Required area} &= \int_1^3 \frac{4}{x} dx = 4 \cdot [\log x]_1^3 \\ &= 4 (\log 3 - \log 1) \\ &= 4 \log 3 = \log 81 \end{aligned}$$

44. Given curve $y = 1 + \log_e x$ cuts X-axis at the point $\left(\frac{1}{e}, 0 \right)$.



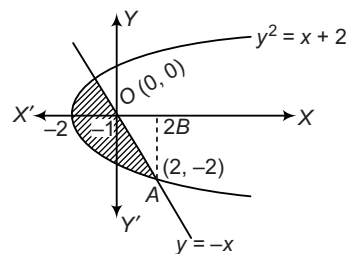
$$\begin{aligned}
 \therefore \text{Required area} &= \int_{1/e}^e (1 + \log_e x) dx \\
 &= \int_{1/e}^e dx + \int_{1/e}^e \log_e x dx \\
 &= [x]_{1/e}^e + [x \log_e x]_{1/e}^e - \int_{1/e}^e 1 dx \\
 &= [x]_{1/e}^e + \left[e \cdot 1 - \frac{1}{e} \cdot \log \frac{1}{e} \right] - [x]_{1/e}^e \\
 &= \left(e + \frac{1}{e} \right) \text{sq unit}
 \end{aligned}$$

45. The region is bounded by the curves $y = 3x - 5$, $y = 0$, $x = 3$ and $x = 5$.



$$\begin{aligned}
 \therefore \text{Required area} &= \int_3^5 (3x - 5) dx \\
 &= \left[\frac{3x^2}{2} - 5x \right]_3^5 = \left(\frac{75}{2} - 25 \right) - \left(\frac{27}{2} - 15 \right) \\
 &= \frac{75}{2} - 25 - \frac{27}{2} + 15 = \frac{48}{2} - 10 = 14 \text{ sq units}
 \end{aligned}$$

46. Since, the plane region is bounded by the curve $x = y^2 - 2$ and the line $y = -x$.



$$\begin{aligned}
 \therefore \text{Required area} &= 2 \int_{-2}^{-1} \sqrt{x+2} dx + \int_{-1}^0 -x dx \\
 &\quad + \left| \left[\int_{-1}^2 \sqrt{x+2} dx - \text{Area of } \triangle ABO \right] \right| \\
 &= \frac{4}{3} (x+2)^{3/2} - \left[\frac{x^2}{2} \right]_{-1}^0 + \left| \left[\frac{2}{3} (x+2)^{3/2} \right]_{-1}^2 - \frac{1}{2} \times 2 \times 2 \right| \\
 &= \frac{4}{3} (1-0) - \left(0 - \frac{1}{2} \right) + \left| \frac{16}{3} - \frac{2}{3} - 2 \right| \\
 &= \frac{4}{3} + \frac{1}{2} + \frac{8}{3} = \frac{12}{3} + \frac{1}{2} \\
 &= 4 + \frac{1}{2} = \frac{9}{2} \text{ sq units}
 \end{aligned}$$

12

Differential Equations

Introduction

An equation involving independent variable, dependent variable, derivatives of dependent variable with respect to independent variable and constant is called a differential equation.

e.g. $\frac{dy}{dx} = \log x$, $\frac{dy}{dx} + y = xe^x$, $\frac{d^2 y}{dx^2} + y = 0$

$$\left(\frac{dy}{dx}\right)^2 = px + q, \frac{d^2 y}{dx^2} + 2\left(\frac{dy}{dx}\right) + 3y = 0, \text{ etc.}$$

Here, y is some function of only one variable x .

Ordinary and Partial Differential Equation

(i) Ordinary

If the unknown function depends on only one argument (independent variable), then the differential equation is called an ordinary differential equation.

e.g.

$$(i) \quad \frac{dy}{dx} = x \ln x$$

$$(ii) \quad dy = \cos x \, dx$$

$$(iii) \quad y = x \frac{dy}{dx} + a$$

$$(iv) \quad \left(1 + \frac{dy}{dx}\right)^4 = b \left(\frac{d^2 y}{dx^2}\right)^2$$

(ii) Partial

A differential equation is said to be partial differential equation, if there are two or more independent variables occurring in it.

e.g. (i) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz$

$$(ii) \quad \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

Chapter Snapshot

- Introduction
- Ordinary and Partial Differential Equation
- Order and Degree of a Differential Equation
- Linear and Non-linear Differential Equation
- Solution of Differential Equation
- Formation of Differential Equation
- Differential Equation of First Order and First Degree
- Methods of Solving First Order and First Degree Differential Equations
- Orthogonal Trajectory
- Applications of First Order Differential Equation

Order and Degree of a Differential Equation

The order of highest differential coefficient (or highest order derivative) appearing in a differential equation is the order of differential equation.

e.g. The differential equation

$\frac{d^3 y}{dx^3} + x \frac{dy}{dx} + xy \frac{d^2 y}{dx^2} + 4 = 0$ is of order 3 and the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$ is of order 2.

The highest exponent of the highest derivative is called degree of a differential equation provided exponent of each derivative and the unknown variable appearing in the differential equation is a non-negative integer.

e.g.

$$(i) \frac{d^2 y}{dx^2} + \sqrt{\frac{dy}{dx}} + x = 0 \Rightarrow \frac{dy}{dx} = \left(\frac{d^2 y}{dx^2} + x\right)^2$$

$$\Rightarrow \left(\frac{d^2 y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right) + 2x \frac{d^2 y}{dx^2} + x^2 = 0$$

Here, degree is 2.

$$(ii) \left(\frac{d^3 y}{dx^3}\right)^{2/3} + x + y = 0 \Rightarrow \left(\frac{d^3 y}{dx^3}\right)^2 = (-x - y)^3$$

Here, degree is 2.

➤ **Example 1.** The degree of the differential

equation $\frac{d^2 y}{dx^2} + x \left(\frac{dy}{dx}\right)^4 = 4 \log \left(\frac{d^4 y}{dx^4}\right)$ is

- (a) 1
(b) 3
(c) 4
(d) None of the above

Sol. (d) Since, the given differential equation is not a polynomial in differential coefficients, so its degree is not defined.

➤ **Example 2.** The degree of the differential

equation $\left(\frac{d^2 y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^2 + \sin \left(\frac{dy}{dx}\right) + 1 = 0$ is

- (a) 3
(b) 2
(c) 1
(d) None of the above

Sol. (d) Since, the given differential equation is not a polynomial equation in $\frac{dy}{dx}$. Therefore, its degree is not defined.

➤ **Example 3.** The order of the differential

equation $2x^2 \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} + y = 0$ is

- (a) 2 (b) 1 (c) 0 (d) 3

Sol. (a) The highest order derivative present in the given differential equation as $\frac{d^2 y}{dx^2}$. Therefore, its order is two.

➤ **Example 4.** The order and degree of the

differential equation $\rho = \frac{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{3/2}}{d^2 y/dx^2}$ are

respectively

- (a) 2, 2
(b) 2, 3
(c) 2, 1
(d) None of the above

Sol. (a) The given equation when expressed as a polynomial in derivatives, is $\rho^2 \left(\frac{d^2 y}{dx^2}\right)^2 = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^3$.

Clearly, it is a second order differential equation of degree 2.

➤ **Example 5.** If m and n are the order and degree of the differential equation

$$\left(\frac{d^2 y}{dx^2}\right)^5 + 4 \frac{\left(\frac{d^2 y}{dx^2}\right)^3}{d^3 y/dx^3} + \frac{d^3 y}{dx^3} = x^2 - 1, \text{ then}$$

- (a) $m = 3, n = 3$ (b) $m = 3, n = 2$
(c) $m = 3, n = 5$ (d) $m = 3, n = 1$

Sol. (b) The given differential equation can be written as

$$\left(\frac{d^3 y}{dx^3}\right)^2 - (x^2 - 1) \frac{d^3 y}{dx^3} + \left(\frac{d^3 y}{dx^3}\right) \left(\frac{d^2 y}{dx^2}\right)^5 + 4 \left(\frac{d^2 y}{dx^2}\right)^3 = 0$$

Clearly, its order is 3 and degree is 2.

$$\therefore m = 3 \text{ and } n = 2$$

Linear and Non-linear Differential Equation

A differential equation is said to be linear, if the dependent variable (say y) and all its derivatives occur in the first degree, otherwise it is non-linear.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = 0$$

where, $a_0, a_1, a_2, \dots, a_{n-1}, a_n$ are either constants or functions of independent variable x .

If a differential equation when expressed in the form of a polynomial involves the derivatives and dependent variable in the first power and there are no product of these and also the coefficient of the various terms are either constants or functions of the independent variable. Then, it is said to be linear differential equation, otherwise it is non-linear.

e.g. The differential equation $\left(\frac{d^3 y}{dx^3}\right)^3 - 6\left(\frac{d^2 y}{dx^2}\right)^2 - 4y = 0$, is a non-linear differential equation because its degree is 3, more than one. The differential equation $\frac{d^2 y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 + 9y = x$, is also a non-linear differential equation because differential coefficient $\frac{dy}{dx}$ has exponent 2.

- A differential equation will be non-linear differential equation, if
 - > its degree is more than one.
 - > any of the differential coefficient has exponent more than one.
 - > exponent of dependent variable is more than unity.
 - > products containing dependent variable and its differential coefficients are present.
- Every differential equation of degree more than 1, is a non-linear differential equation.

Solution of Differential Equation

The solution of the differential equation is a relation between the variables of the equation not containing the derivatives but satisfying the given differential equation (i.e. from which the given differential equation can be derived).

Thus, the solution of $\frac{dy}{dx} = e^x$ could be obtained by simply integrating both sides, i.e. $y = e^x + C$ and that of, $\frac{dy}{dx} = px + q$ is $y = p\frac{x^2}{2} + qx + C$, where C is an arbitrary constant.

- (i) A general solution or an integral of a differential equation is a relation between the variables (not involving the derivatives) which contains the same number of the arbitrary constants as the order of the differential equation.

e.g. A general solution of the differential equation $\frac{d^2 y}{dt^2} = -4x$ is $x = A \cos 2t + b \sin 2t$, where A and B are the arbitrary constants.

- (ii) Particular solution or particular integral is that solution of the differential equation obtained

from the general solution by assigning particular value to the arbitrary constant in the general solution.

e.g. $x = 10 \cot 2t + 5 \sin 2t$ is a particular solution of differential equation $\frac{d^2 y}{dt^2} = -4x$.

➤ **Example 6.** The solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$, is

- (a) $y = 2$ (b) $y = 2x$
(c) $y = 2x - 4$ (d) $y = 2x^2 - 4$

Sol. (c) If $y = 2x - 4$, then $\frac{dy}{dx} = 2$.

Clearly, this value satisfy the differential equation.

➤ **Example 7.** The solution of the differential equation $(1 + x^2)\frac{dy}{dx} + 1 + y^2 = 0$, is

- (a) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} C$
(b) $\tan^{-1} y - \tan^{-1} x = \tan^{-1} C$
(c) $\tan^{-1} y \pm \tan^{-1} x = \tan C$
(d) $\tan^{-1} y + \tan^{-1} x = \tan^{-1} C$

Sol. (d) If $\tan^{-1} y + \tan^{-1} x = \tan^{-1} C$, then

$$\begin{aligned} \frac{d}{dx}(\tan^{-1} y) + \frac{d}{dx}(\tan^{-1} x) &= 0 \\ \Rightarrow \frac{1}{1+y^2} \cdot \frac{dy}{dx} + \frac{1}{1+x^2} &= 0 \\ \Rightarrow (1+x^2)\frac{dy}{dx} + 1 + y^2 &= 0 \end{aligned}$$

Hence, $\tan^{-1} x + \tan^{-1} y = \tan^{-1} C$ is the solution of the given differential equation.

Formation of Differential Equation

If an equation in independent and dependent variables involving some arbitrary constants is given, then a differential equation is obtained as follows:

- Differentiate the given equation with respect to the independent variable (say x) as many times as the number of arbitrary constants in it.
- Eliminate the arbitrary constants.
- The eliminant is the required differential equation, i.e. if we have an equation $f(x, y, c_1, c_2, \dots, c_n) = 0$ containing n arbitrary constants $c_1, c_2, c_3, \dots, c_n$, then by differentiating this n times, we shall get n equations.

Now, among these n equations and the given equation, in all $(n+1)$ equations, if n arbitrary

constants $c_1, c_2, c_3, \dots, c_n$ are eliminated, we shall evidently get a differential equation of the n th order. For there being n differentiation, the resulting equation must contain a derivative of the n th order.

➤ **Example 8.** The differential equation of the family of curves $y = ae^{3x} + be^{-2x}$, where a and b are arbitrary constants, is

(a) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 6y = 0$ (b) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 - 6y = 0$
 (c) $\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$ (d) None of these

Sol. (c) Given family of curves, $y = ae^{3x} + be^{-2x}$... (i)

On differentiating w.r.t. x , we get

$$\begin{aligned} \frac{dy}{dx} &= a \frac{d}{dx} (e^{3x}) + b \frac{d}{dx} (e^{-2x}) \\ \Rightarrow \frac{dy}{dx} &= ae^{3x} \cdot \frac{d}{dx} (3x) + be^{-2x} \cdot \frac{d}{dx} (-2x) \\ \Rightarrow \frac{dy}{dx} &= 3ae^{3x} - 2be^{-2x} \end{aligned} \quad \dots (ii)$$

Again, differentiating w.r.t. x , we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= 3a e^{3x} \cdot \frac{d}{dx} (3x) - 2b e^{-2x} \cdot \frac{d}{dx} (-2x) \\ &= 9ae^{3x} + 4be^{-2x} \\ \Rightarrow \frac{d^2y}{dx^2} &= 3(3ae^{3x}) + 4be^{-2x} \\ &= 3 \left(\frac{dy}{dx} + 2be^{-2x} \right) + 4be^{-2x} \\ &\quad \left[\because \frac{dy}{dx} + 2be^{-2x} = 3ae^{3x} \right] \\ &= 3 \frac{dy}{dx} + 6be^{-2x} + 4be^{-2x} \\ &= 3 \frac{dy}{dx} + 10be^{-2x} \end{aligned} \quad \dots (iii)$$

On multiplying Eq. (i) by 3 and then subtracting from Eq. (iii), we get

$$\begin{aligned} \frac{dy}{dx} - 3y &= -2be^{-2x} - 3be^{-2x} \\ \Rightarrow 3y - \frac{dy}{dx} &= 5be^{-2x} \Rightarrow b = \frac{3y - \frac{dy}{dx}}{5e^{-2x}} \end{aligned} \quad \dots (iv)$$

On putting the value of b in Eq. (iii), we get

$$\begin{aligned} \frac{d^2y}{dx^2} &= 3 \frac{dy}{dx} + 10 \frac{3y - \frac{dy}{dx}}{5e^{-2x}} \cdot e^{-2x} \\ \Rightarrow \frac{d^2y}{dx^2} &= 3 \frac{dy}{dx} + 2 \left(3y - \frac{dy}{dx} \right) \\ \Rightarrow \frac{d^2y}{dx^2} &= 3 \frac{dy}{dx} + 6y - 2 \frac{dy}{dx} \\ \Rightarrow \frac{d^2y}{dx^2} &= \frac{dy}{dx} + 6y \Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0 \end{aligned}$$

which is the required differential equation.

➤ **Example 9.** The differential equation of the family of parabolas with foci at the origin and axis along the X -axis, is

(a) $y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} - y = 0$
 (b) $x \left(\frac{dy}{dx} \right)^2 + 2y \frac{dy}{dx} - y = 0$
 (c) $y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} + y = 0$
 (d) None of the above

Sol. (a) Let the directrix be $x = -2a$ and latusrectum be $4a$. Then, the equation of the parabola is (distance from focus = distance from directrix)
 i.e. $x^2 + y^2 = (2a + x)^2 \Rightarrow y^2 = 4a(a + x)$... (i)

On differentiating w.r.t. x , we get

$$y \frac{dy}{dx} = 2a \Rightarrow a = \frac{1}{2} y \frac{dy}{dx}$$

On putting this value of a in Eq. (i), we get the differential equation as

$$y^2 = 2y \frac{dy}{dx} \left(\frac{y}{2} \cdot \frac{dy}{dx} + x \right) \Rightarrow y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} - y = 0$$

➤ **Example 10.** The differential equation of all non-horizontal lines in a plane, is

(a) $\frac{d^2y}{dx^2} = 0$ (b) $\frac{d^2x}{dy^2} = 0$ (c) $\frac{dy}{dx} = 0$ (d) $\frac{dx}{dy} = 0$

Sol. (b) The general equation of all non-horizontal lines in XY -plane is $ax + by = 1$, where $a \neq 0$.

Now, $ax + by = 1 \Rightarrow a \frac{dx}{dy} + b = 0$ [differentiate w.r.t. y]

$$\Rightarrow a \frac{d^2x}{dy^2} = 0 \quad [\text{differentiate w.r.t. } y]$$

$$\therefore \frac{d^2x}{dy^2} = 0 \quad [\because a \neq 0]$$

➤ **Example 11.** The differential equation representing the family of curves $y^2 = 4a(x + b)$, where a, b are arbitrary constants, is

(a) $y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$ (b) $y \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$
 (c) $x \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$ (d) $x \frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Sol. (a) We have, $y^2 = 4a(x + b)$

On differentiating Eq. (i) w.r.t. x , we get

$$2y \cdot \frac{dy}{dx} = 4a \Rightarrow y \frac{dy}{dx} = 2a$$

$$\text{Again, differentiating w.r.t. } x, \text{ we get } y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$$

which is the required differential equation.

Thus, the elimination of arbitrary constants lead to the formation of a differential equation.

Work Book Exercise 12.1

12

Differential Equations

- The order and degree of the differential equation $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{5/2} = \frac{d^3y}{dx^3}$, are respectively
 - 3, 2
 - 2, 3
 - 3, 10
 - 3, 5
- The differential equation of all circles which passes through the origin and whose centre lies on Y-axis, is
 - $(x^2 - y^2) \frac{dy}{dx} - 2xy = 0$
 - $(x^2 - y^2) \frac{dy}{dx} + 2xy = 0$
 - $(x^2 - y^2) \frac{dy}{dx} - xy = 0$
 - $(x^2 - y^2) \frac{dy}{dx} + xy = 0$
- The order and degree of the differential equation $5\left(\frac{d^2y}{dx^2}\right)^5 + 4\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{dy}{dx}\right)^3 + 2y + x^3 = 0$ are respectively
 - 2, 5
 - 3, 2
 - 1, 3
 - 2, 3
- The order of the differential equation whose solution is $y = a \cos x + b \sin x + ce^{-x}$, is
 - 3
 - 1
 - 2
 - 4
- The differential equation of all non-vertical lines in a plane is
 - $\frac{d^2y}{dx^2} = 0$
 - $\frac{d^2x}{dy^2} = 0$
 - $\frac{dy}{dx} = 0$
 - $\frac{dx}{dy} = 0$
- The differential equation of the family of circles passing through the fixed points $(a, 0)$ and $(-a, 0)$ is
 - $y_1(y^2 - x^2) + 2xy + a^2 = 0$
 - $y_1y^2 + xy + a^2x^2 = 0$
 - $y_1(y^2 - x^2 + a^2) + 2xy = 0$
 - $y_1(y^2 + x^2) - 2xy + a^2 = 0$
- The differential equation of the family of curves $y = e^{2x}(a \cos x + b \sin x)$, where a and b are arbitrary constants, is given by
 - $y_2 - 4y_1 + 5y = 0$
 - $2y_2 - y_1 + 5y = 0$
 - $y_2 + 4y_1 - 5y = 0$
 - $y_2 - 2y_1 + 5y = 0$
- The differential equation of all straight lines passing through origin is
 - $y = \sqrt{x} \frac{dy}{dx}$
 - $\frac{dy}{dx} = y + x$
 - $\frac{dy}{dx} = y - x$
 - None of the above
- The order and degree of the differential equation $\sqrt{\sin x} (dx + dy) = \sqrt{\cos x} (dx - dy)$ are
 - 1, 2
 - 2, 2
 - 1, 1
 - 2, 1
- If C_1, C_2, C_3, C_4, C_5 and C_6 are constants, then the order of the differential equation whose general solution is given by $y = C_1 \cos(x + C_2) + C_3 \sin(x + C_4) + C_5 e^x + C_6$, is
 - 6
 - 5
 - 4
 - 3

Differential Equation of First Order and First Degree

A differential equation of first order and first degree involves x , y and $\frac{dy}{dx}$. So, it can be put in anyone

of the forms $\frac{dy}{dx} = f(x, y)$ or $f\left(x, y, \frac{dy}{dx}\right) = 0$ or $f(x, y) dx + g(x, y) dy = 0$, where $f(x, y)$ and $g(x, y)$ are obviously the functions of x and y .

Methods of Solving First Order and First Degree Differential Equations

(i) Variable Separable Method

Equations in which the variables are separable, are those equations which can be expressed that the coefficient of dx is only a function of x and that of dy is only a function of y .

Thus, the general form of such an equation is $f(x) dx + g(y) dy = 0$. The solution of this equation is obtained by integrating $f(x)$ and $g(y)$ with respect to x and y respectively, i.e. solution is given by

$$\int f(x) dx = \int g(y) dy + C$$

➤ **Example 12.** The solution of the differential

equation $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$, is

(a) $\tan^{-1} x + \cot^{-1} x = C$ (b) $\sin^{-1} x + \sin^{-1} y = C$

(c) $\sec^{-1} x + \operatorname{cosec}^{-1} x = C$ (d) None of these

Sol. (b) We have, $\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$

$$\Rightarrow \frac{1}{\sqrt{1-y^2}} dy + \frac{1}{\sqrt{1-x^2}} dx = 0$$

On integrating, we get $\sin^{-1} y + \sin^{-1} x = C$

➤ **Example 13.** The solution of the differential

equation $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$, is

(a) $y = \tan^{-1} x$ (b) $y - x = C(1 + xy)$

(c) $x = \tan^{-1} y$ (d) $\tan(xy) = C$

Sol. (b) We have, $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$$\Rightarrow \frac{1}{1+y^2} dy = \frac{1}{1+x^2} dx$$

On integrating both sides, we get

$$\tan^{-1} y = \tan^{-1} x + \tan^{-1} C$$

$$\Rightarrow \tan^{-1} y - \tan^{-1} x = \tan^{-1} C$$

$$\Rightarrow \tan^{-1} \left(\frac{y-x}{1+xy} \right) = \tan^{-1} C$$

$$\Rightarrow \frac{y-x}{1+xy} = C$$

$$\Rightarrow y - x = C(1 + xy)$$

➤ **Example 14.** The differential equation

$y \frac{dy}{dx} + x = C$ represents

(a) a family of hyperbolas

(b) a family of circles whose centres are on Y-axis

(c) a family of parabolas

(d) a family of circles whose centres are on X-axis

Sol. (d) We have, $y \frac{dy}{dx} + x = C$

$$\Rightarrow y dy + (x - C) dx = 0$$

$$\Rightarrow \frac{y^2}{2} + \frac{(x - C)^2}{2} = C_1$$

$$\Rightarrow (x - C)^2 + y^2 = 2C_1$$

Clearly, it represents a family of circles having their centres on X-axis.

➤ **Example 15.** The solution of the differential equation $\frac{dx}{x} + \frac{dy}{y} = 0$, is

(a) $\frac{1}{x} + \frac{1}{y} = C$

(b) $\log x \log y = C$

(c) $xy = C$

(d) $x + y = C$

Sol. (c) We have, $\frac{1}{x} dx + \frac{1}{y} dy = 0$

On integrating both sides, we get

$$\log x + \log y = \log C$$

$$\Rightarrow xy = C$$

(ii) Differential Equation Reducible to Variable Separable Method

Sometimes, differential equation of the first order cannot be solved directly by variable separable method. But by some substitution, we can reduce it to a differential equation with separable variables. Let the differential equation is of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

which can be reduced to variable separable form by the substitution $ax + by + c = z$.

$$\therefore (a + b) \frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \left(\frac{dz}{dx} - a \right) \frac{1}{b} = f(z) \Rightarrow \frac{dz}{dx} = a + bf(z)$$

Now, apply variable separable method.

➤ **Example 16.** The solution of the differential

equation $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$ is

(a) $\log \left| \tan \left(\frac{x+y}{2} \right) + 1 \right| = x + C$

(b) $\log \left| \cot \left(\frac{x+y}{2} \right) + 1 \right| = x + C$

(c) $\log \left| \tan \left(\frac{x-y}{2} \right) - 1 \right| = C$

(d) None of the above

Sol. (a) We have, $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$... (i)

$$\text{Let } z = x + y \Rightarrow \frac{dz}{dx} = 1 + \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 1$$

From Eq. (i),

$$\frac{dz}{dx} - 1 = \sin z + \cos z$$

$$\Rightarrow \frac{dz}{\sin z + \cos z + 1} = dx \quad [\text{variables are separable}]$$

On integrating both sides, we get

$$\int \frac{dz}{\sin z + \cos z + 1} = \int 1 \cdot dx$$

$$\Rightarrow \int \frac{dz}{\frac{2 \tan \frac{z}{2}}{1 + \tan^2 \frac{z}{2}} + \frac{1 - \tan^2 \frac{z}{2}}{1 + \tan^2 \frac{z}{2}}} = x + C$$

$$\Rightarrow \int \frac{\sec^2 \frac{z}{2} dz}{2 \tan \frac{z}{2} + 2} = x + C$$

$$\Rightarrow \int \frac{dt}{t+1} = x + C, \quad \text{where } t = \tan \frac{z}{2}$$

$$\Rightarrow \log |t+1| = x + C$$

$$\Rightarrow \log \left| \tan \left(\frac{x+y}{2} \right) + 1 \right| = x + C$$

which is the required general solution.

➤ **Example 17.** The solution of the differential

equation $\frac{dy}{dx} + 1 = e^{x+y}$, is

- (a) $(x+y)e^{x+y} = 0$ (b) $(x+C)e^{x+y} = 0$
 (c) $(x-C)e^{x+y} = 1$ (d) $(x+C)e^{x+y} + 1 = 0$

Sol. (d) We have, $\frac{dy}{dx} + 1 = e^{x+y}$

Let $x+y=v$. Then, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

$$\therefore \frac{dy}{dx} + 1 = e^{x+y}$$

$$\therefore \frac{dv}{dx} = e^v$$

$$\Rightarrow e^{-v} dv = dx$$

$$\Rightarrow \int e^{-v} dv = \int dx$$

$$\Rightarrow -e^{-v} = x + C$$

$$\Rightarrow -e^{-(x+y)} = x + C$$

$$\Rightarrow (x+C)e^{x+y} + 1 = 0$$

➤ **Example 18.** The equation of a curve passing through the origin and satisfying the

differential equation $\frac{dy}{dx} = (x-y)^2$, is

- (a) $e^{2x}(1-x+y) = 1+x-y$
 (b) $e^{2x}(1+x-y) = 1-x+y$
 (c) $e^{2x}(1-x+y) + (1+x-y) = 0$
 (d) $e^{2x}(1+x+y) = 1-x+y$

Sol. (a) We have, $\frac{dy}{dx} = (x-y)^2$

Let $x-y=v$. Then, $1 - \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

$$\therefore \frac{dy}{dx} = (x-y)^2$$

$$\Rightarrow 1 - \frac{dv}{dx} = v^2$$

$$\Rightarrow 1 - v^2 = \frac{dv}{dx}$$

$$\Rightarrow dx = \frac{1}{1-v^2} dv$$

$$\Rightarrow 2 \int dx = 2 \int \frac{1}{1-v^2} dv$$

$$\Rightarrow 2x = \log \left(\frac{1+v}{1-v} \right) + \log C$$

$$\Rightarrow C \left(\frac{1+v}{1-v} \right) = e^{2x}$$

$$\Rightarrow C \left(\frac{x-y+1}{y-x+1} \right) = e^{2x}$$

$$\Rightarrow C(x-y+1) = e^{2x}(y-x+1)$$

Taking $C=1$, we find that option (a) is correct.

➤ **Example 19.** The solution of the differential

equation $\left(\frac{x+y-1}{x+y-2} \right) \frac{dy}{dx} = \left(\frac{x+y+1}{x+y+2} \right)$, given that

$y=1$ when $x=1$, is

(a) $\log \left| \frac{(x-y)^2 - 2}{2} \right| = 2(x+y)$

(b) $\log \left| \frac{(x-y)^2 + 2}{2} \right| = 2(x-y)$

(c) $\log \left| \frac{(x+y)^2 + 2}{2} \right| = 2(x-y)$

(d) None of the above

Sol. (d) Let $x+y=u$

Then, $1 + \frac{dy}{dx} = \frac{du}{dx}$

So, the given differential equation becomes

$$\left(\frac{u-1}{u-2} \right) \left(\frac{du}{dx} - 1 \right) = \frac{u+1}{u+2}$$

$$\Rightarrow \frac{du}{dx} - 1 = \left(\frac{u+1}{u-2} \right) \left(\frac{u-2}{u-1} \right)$$

$$\Rightarrow \frac{du}{dx} = \frac{u^2 - u - 2}{u^2 + u - 2} + 1$$

$$\Rightarrow \frac{du}{dx} = \frac{2u^2 - 4}{u^2 + u - 2}$$

$$\Rightarrow \frac{u^2 + u - 2}{u^2 - 2} du = 2 dx$$

$$\Rightarrow \left(1 + \frac{u}{u^2 - 2} \right) du = 2 dx$$

On integrating both sides, we get

$$u + \frac{1}{2} \log |u^2 - 2| = 2x + C$$

$$\Rightarrow (x+y) + \frac{1}{2} \log |(x+y)^2 - 2| = 2x + C$$

$$\Rightarrow 2(y-x) + \log |(x+y)^2 - 2| = 2C \quad \dots(i)$$

When $x=1$, then we have $y=1$

$$\therefore \log 2 = 2C$$

Substituting the value of C in Eq. (i), we get

$$2(y-x) + \log |(x+y)^2 - 2| = \log 2$$

$$\Rightarrow 2(y-x) + \log \left| \frac{(x+y)^2 - 2}{2} \right| = 0$$

- 1 The solution of the differential equation

$$\frac{dy}{dx} = \frac{xy + y}{xy + x} \text{ is}$$

a $x + y = \log \left(\frac{Cy}{x} \right)$

b $x + y = \log (Cxy)$

c $x - y = \log \left(\frac{Cx}{y} \right)$

d $y - x = \log \left(\frac{Cx}{y} \right)$

- 2 The differential equation $(e^x + 1)y dy = (y + 1)e^x dx$, has the solution

a $(y - 1)(e^x - 1) = Ce^y$

b $(y - 1)(e^x + 1) = Ce^y$

c $(y + 1)(e^x - 1) = Ce^y$

d $(y + 1)(e^x + 1) = Ce^y$

- 3 The solution of the differential equation

$$\frac{dy}{dx} = \frac{x - y + 3}{2(x - y) + 5} \text{ is}$$

a $2(x - y) + \log(x - y) = x + C$

b $2(x - y) - \log(x - y + 2) = x + C$

c $2(x - y) + \log(x - y + 2) = x + C$

d None of the above

- 4 A particular solution of $\log \left(\frac{dy}{dx} \right) = 3x + 4y$,

$y(0) = 0$, is

a $e^{3x} + 3e^{-4y} = 4$

b $4e^{3x} - 3e^{-4y} = 3$

c $3e^{3x} + 4e^{-4y} = 7$

d $4e^{3x} + 3e^{-4y} = 7$

- 5 The solution of the equation

$$y - x \frac{dy}{dx} = a \left(y^2 + \frac{dy}{dx} \right) \text{ is}$$

a $y = C(x + a)(1 - ay)$

b $y = C(x + a)(1 + ay)$

c $y = C(x - a)(1 + ay)$

d None of these

- 6 The solution of the differential equation

$$\frac{dy}{dx} = e^{3x-2y} + x^2e^{-2y} \text{ is}$$

a $e^{2y} = e^{3x} + x^3 + C$

b $\frac{1}{2}e^{2y} = \frac{1}{3}(e^{3x} + x^3) + C$

c $\frac{1}{2}e^{2y} = (e^{3x} + x^3) + C$

d $e^{2y} = e^{3x} + x^2 + C$

- 7 The solution of the differential equation

$$\frac{dy}{dx} = \frac{x - 2y + 1}{2x - 4y} \text{ is}$$

a $(x - 2y)^2 + 2x = C$

b $(x - 2y)^2 + x = C$

c $(x - 2y) + 2x^2 = C$

d $(x - 2y) + x^2 = C$

(iii) Homogeneous Differential Equation

A differential equation of the form $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$,

where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of x and y of the same degree, is called a homogeneous differential equation.

For solving such equations, put $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$.

These substitutions transform the given equation into an equation of the form

$$v + x \frac{dv}{dx} = f(v)$$

i.e. $x \frac{dv}{dx} = f(v) - v$

The variables are now separable. Separating the variables and integrating, we get

$$\int \frac{dv}{f(v) - v} = \log x + C, \text{ where } C \text{ is any arbitrary}$$

constant. Now, replacing v by $\left(\frac{y}{x} \right)$ a factor integration,

we get the required solution.

- The function $P(x, y)$ is homogeneous function of degree n , if for any real number $t (\neq 0)$, we have $P(tx, ty) = t^n P(x, y)$

For instance, $P(x, y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$ is a homogeneous function of degree $2/3$.

$$\therefore P(tx, ty) = a(tx)^{2/3} + h(tx)^{1/3} \cdot (ty)^{1/3} + b(ty)^{2/3} \\ = t^{2/3} (ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}) = t^{2/3} P(x, y)$$

- A differential equation of the form $\frac{dy}{dx} = f(x, y)$ is homogeneous, if $f(x, y)$ is a homogeneous function of degree zero, i.e. $f(tx, ty) = t^0 f(x, y) = f(x, y)$.

➤ **Example 20.** Which one of the following functions is not homogeneous?

(a) $f(x, y) = \frac{x - y}{x^2 + y^2}$

(b) $f(x, y) = x^{1/3} \cdot y^{-2/3} \tan^{-1} \frac{x}{y}$

(c) $f(x, y) = x(\ln \sqrt{x^2 + y^2} - \ln y) + ye^{x/y}$

(d) $f(x, y) = x \left[\ln \left(\frac{2x^2 + y^2}{x} \right) - \ln(x + y) \right] \\ + y^2 \tan \left(\frac{x + 2y}{3x - y} \right)$

Sol. (d) (a) $f(\lambda x, \lambda y) = \frac{\lambda(x-y)}{\lambda^2(x^2+y^2)} = \lambda^{-1} f(x, y)$

$\therefore$ Homogeneous of degree (-1) .

(b) $f(\lambda x, \lambda y) = (\lambda x)^{1/3} (\lambda y)^{-2/3} \tan^{-1} \left(\frac{\lambda x}{\lambda y} \right)$
 $= \lambda^{-1/3} x^{1/3} y^{-2/3} \tan^{-1} \frac{x}{y} = \lambda^{-1/3} f(x, y)$

$\therefore$ Homogeneous

(c) $f(\lambda x, \lambda y) = \lambda x (\ln \sqrt{\lambda^2(x^2+y^2)} - \ln \lambda y) + \lambda y e^{x/y}$
 $= \lambda x \left[\ln \left(\frac{\lambda \sqrt{x^2+y^2}}{\lambda y} \right) \right] + \lambda y e^{x/y}$
 $= \lambda [x (\ln \sqrt{x^2+y^2} - \ln y) + y e^{x/y}]$
 $= \lambda f(x, y)$

$\therefore$ Homogeneous

(d) $f(\lambda x, \lambda y) = \lambda x \left[\ln \left(\frac{2\lambda^2 x^2 + \lambda^2 y^2}{\lambda x \cdot \lambda(x+y)} \right) \right]$
 $+ \lambda^2 y^2 \tan \left(\frac{x+2y}{3x-y} \right)$
 $= \lambda x \left[\ln \left(\frac{2x^2+y^2}{x(x+y)} \right) \right] + \lambda^2 y^2 \tan \left(\frac{x+2y}{3x-y} \right)$

$\therefore$ Non-homogeneous

➤ **Example 21.** The solution of the differential equation $(x-y) dy - (x+y) dx = 0$ is

(a) $\tan^{-1} \frac{x}{y} + \frac{1}{2} \log(x^2 + y^2) = C$

(b) $\tan^{-1} \frac{y}{x} - \frac{1}{2} \log(x^2 + y^2) = C$

(c) $\cot^{-1} \left(\frac{y}{x} \right) + \frac{1}{2} \log(x^2 + y^2) = C$

(d) None of the above

Sol. (b) Given, $(x-y) dy - (x+y) dx = 0$

$\Rightarrow (x-y) \frac{dy}{dx} = (x+y)$... (i)

Thus, the given differential equation is homogeneous.

So, put $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

On putting values of dy/dx and y in Eq. (i), we get

$v + x \frac{dv}{dx} = \frac{x+vx}{x-vx} \Rightarrow v + x \frac{dv}{dx} = \frac{1+v}{1-v}$
 $\Rightarrow x \frac{dv}{dx} = \frac{1+v}{1-v} - v \Rightarrow x \frac{dv}{dx} = \frac{1+v-v+v^2}{1-v}$
 $\Rightarrow \frac{1-v}{1+v^2} dv = \frac{dx}{x}$

On integrating both sides, we get

$\int \frac{1-v}{1+v^2} dv = \int \frac{dx}{x}$
 $\Rightarrow \int \frac{1}{1+v^2} dv - \int \frac{v}{1+v^2} dv = \log |x| + C$

Put $1+v^2 = t \Rightarrow 2v = \frac{dt}{dv} \Rightarrow dv = \frac{dt}{2v}$

$\therefore \tan^{-1} v - \int \frac{v}{t} \times \frac{dt}{2v} = \log |x| + C$

$\Rightarrow \tan^{-1} v - \frac{1}{2} \log |t| = \log |x| + C$

$\Rightarrow 2 \tan^{-1} v - [\log(1+v^2) + 2 \log(x)] = 2C$

$\Rightarrow 2 \tan^{-1} v - \log[(1+v^2)x^2] = 2C$ [put $t = 1+v^2$]

$\Rightarrow 2 \tan^{-1} \frac{y}{x} - \log \left[\left(\frac{x^2+y^2}{x^2} \right) x^2 \right] = 2C$ [put $v = \frac{y}{x}$]

$\Rightarrow 2 \tan^{-1} \frac{y}{x} - \log(x^2 + y^2) = 2C$

$\Rightarrow \tan^{-1} \frac{y}{x} - \frac{1}{2} \log(x^2 + y^2) = C$

which is the required solution of the given differential equation.

➤ **Example 22.** The solution of the differential

equation $x \sin \frac{y}{x} dy = \left(y \sin \frac{y}{x} - x \right) dx$ is

(a) $\log x = \cos \frac{y}{x} + C$ (b) $\log x = \cos \frac{x}{y} + C$

(c) $\log y = \cos \frac{x}{y} + C$ (d) None of these

Sol. (a) We have, $\frac{dy}{dx} = \frac{y \sin \frac{y}{x} - x}{x \sin \frac{y}{x}}$

Put $y = vx$

$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

Hence, $v + x \frac{dv}{dx} = \frac{v \sin v - 1}{\sin v} = v - \frac{1}{\sin v}$

$\Rightarrow x \frac{dv}{dx} = -\frac{1}{\sin v}$

$\Rightarrow \int \frac{dx}{x} + \int \sin v dv = C \Rightarrow \log x - \cos v = C$

$\Rightarrow \log x = \cos \frac{y}{x} + C$ [put $v = \frac{y}{x}$]

➤ **Example 23.** The solution of the equation

$x dy = \left(y + x \frac{f(y/x)}{f'(y/x)} \right) dx$ is

(a) $f\left(\frac{x}{y}\right) = Cy$ (b) $f\left(\frac{y}{x}\right) = Cx$

(c) $f\left(\frac{y}{x}\right) = Cxy$ (d) None of these

Sol. (b) We have, $x dy = \left(y + x \frac{f(y/x)}{f'(y/x)} \right) dx$

$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \frac{f(y/x)}{f'(y/x)}$ which is homogeneous.

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

We obtain, $v + x \frac{dv}{dx} = v + \frac{f(v)}{f'(v)} \Rightarrow \frac{f'(v)}{f(v)} dv = \frac{dx}{x}$

On integrating both sides, we get

$\log f(v) = \log x + \log C \Rightarrow \log f(v) = \log Cx$

$\Rightarrow f\left(\frac{y}{x}\right) = Cx$ [put $v = \frac{y}{x}$]

➤ **Example 24.** If a curve passes through the point $\left(1, \frac{\pi}{4}\right)$ and its slope at any point is given by $\frac{y}{x} - \cos^2\left(\frac{y}{x}\right)$. Then, the curve has the equation

- (a) $y = x \tan^{-1}\left(\ln \frac{e}{x}\right)$ (b) $y = x \tan^{-1}(\ln 2)$
 (c) $y = \frac{1}{x} \tan^{-1}\left(\ln \frac{e}{x}\right)$ (d) None of these

Sol. (a) We have, $\frac{dy}{dx} = \frac{y}{x} - \cos^2 \frac{y}{x}$
 Put $y = vx$
 $\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$
 $\therefore v + x \frac{dv}{dx} = v - \cos^2 v$
 $\Rightarrow \int \frac{dv}{\cos^2 v} + \int \frac{dx}{x} = C$
 $\Rightarrow \tan v + \ln x = C$
 $\Rightarrow \tan \frac{y}{x} + \ln x = C$... (i)
 If $x = 1, y = \frac{\pi}{4} \Rightarrow C = 1$
 From Eq. (i), $\tan \frac{y}{x} = 1 - \ln x = \ln \frac{e}{x}$
 $\therefore y = x \tan^{-1}\left(\ln \frac{e}{x}\right)$

(iv) Differential Equation Reducible to Homogeneous Form

A differential equation of the form

$$\frac{dy}{dx} = \frac{ax + by + c}{a_1x + b_1y + c_1}, \text{ where } \frac{a}{a_1} \neq \frac{b}{b_1}$$

can be reduced to homogeneous form by taking new variables x and y such that $x = X + h$ and $y = Y + k$, where h and k are constants to be so chosen as to make the given equation homogeneous.

With the above substitutions, we get $dx = dX$ and $dy = dY$, so that $\frac{dy}{dx} = \frac{dY}{dX}$.

Hence, the given equation becomes

$$\begin{aligned} \frac{dY}{dX} &= \frac{a(X+h) + b(Y+k) + c}{a_1(X+h) + b_1(Y+k) + c_1} \\ &= \frac{aX + bY + (ah + bk + c)}{a_1X + b_1Y + (a_1h + b_1k + c_1)} \end{aligned}$$

Now, choose h and k such that

$$ah + bk + c = 0 \quad \text{and} \quad a_1h + b_1k + c_1 = 0$$

Then, the differential equation becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a_1X + b_1Y}, \text{ which is homogeneous.}$$

Now, this equation can be solved as in homogeneous equations by substituting $Y = vX$. Finally, by replacing X by $(x - h)$ and Y by $(Y - k)$, we shall get the solution in original variables x and y .

If however $\frac{a}{a_1} = \frac{b}{b_1} = m$ (say), then the differential

equation becomes of the form $\frac{dy}{dx} = \frac{m(a_1x + b_1y) + c}{a_1x + b_1y + c_1}$

To solve such a differential equation, put $v = a_1x + b_1y$, get rid of y and then the transformed equation will be such that the variables are separable.

➤ **Example 25.** The solution of the differential equation $\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}$ is

- (a) $\log[(y+3)^2 + (x+2)^2] + 2 \tan^{-1}\{(y+3)/(x+2)\} = C$
 (b) $\log[(y+3)^2 + (x+2)^2] + \tan^{-1}\{(y+3)/(x+2)\} = C$
 (c) $\log[(y+3)^2 - (x+2)^2] + 2 \cot^{-1}\{(y+3)/(x+2)\} = C$
 (d) None of the above

Sol. (a) Here, $\frac{a}{a_1} \neq \frac{b}{b_1}$, i.e. the coefficients of x and y in the numerator and denominator of the expression for $\frac{dy}{dx}$ are

not proportional. Such equations can be reduced to homogeneous form by taking new variables X and Y such that $x = X + h$ and $y = Y + k$, where h and k are constants to be taken at our choice. With these substitutions, the given equation reduces to

$$\frac{dY}{dX} = \frac{Y + k - X - h + 1}{Y + k + X + h + 5} = \frac{Y - X + (k - h + 1)}{Y + X + (k + h + 5)} \dots (i)$$

Choose h and k such that $k - h + 1 = 0$ and $k + h + 5 = 0$. Solving these, we get $k = -3, h = -2$, with these values of h and k , Eq. (i) becomes

$$\frac{dY}{dX} = \frac{Y - X}{Y + X} \dots (ii)$$

Now, Eq. (ii) is a homogeneous equation, so putting

$Y = vX$ and $\frac{dY}{dX} = v + X \frac{dv}{dX}$, we have

$$\begin{aligned} v + X \frac{dv}{dX} &= \frac{v - 1}{v + 1} \\ \Rightarrow X \frac{dv}{dX} &= \frac{v - 1}{v + 1} - v = \frac{-1 - v^2}{v + 1} \\ \Rightarrow \frac{v + 1}{v^2 + 1} dv &= \frac{-dX}{X} \\ \Rightarrow \frac{v}{v^2 + 1} dv + \frac{dv}{v^2 + 1} &= \frac{-dX}{X} \\ \Rightarrow \frac{2v}{v^2 + 1} dv + \frac{2}{v^2 + 1} dv &= \frac{-2}{X} dX \end{aligned}$$

On integrating both sides, we get

$$\log(v^2 + 1) + 2 \tan^{-1} v = -2 \log X + C$$

$$\Rightarrow \log\{(v^2 + 1)X^2\} = -2 \tan^{-1} v + C$$

$$\Rightarrow \log\{Y^2 + X^2\} = -2 \tan^{-1} \left(\frac{Y}{X}\right) + C \quad \left[\because v = \frac{Y}{X}\right]$$

$$\Rightarrow \log[(x+2)^2 + (y+3)^2] + 2 \tan^{-1}\{(y+3)/(x+2)\} = C$$

$$[\because Y = y - k = y + 3, X = x - h = x + 2]$$

which is the required solution.

- **Example 26.** The solution of the equation $(x - y)(2dy - dx) = 3dx - 5dy$ is
- (a) $2y - x = \log(x - y + 2) + C$
 (b) $2x - y = \log(y - x + 2) + C$
 (c) $2y + x = \log(x - y + 2) + C$
 (d) None of the above

Sol. (a) We have,

$$(2x - 2y + 5) \frac{dy}{dx} = (x - y + 3) \frac{dx}{dx} \quad \dots(i)$$

Put $x - y = v$, so that $1 - \frac{dy}{dx} = \frac{dv}{dx}$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dv}{dx}$$

$\therefore$ Eq. (i) becomes

$$1 - \frac{dv}{dx} = \frac{v + 3}{2v + 5}$$

$$\Rightarrow \frac{dv}{dx} = 1 - \frac{v + 3}{2v + 5} = \frac{v + 2}{2v + 5}$$

$$\Rightarrow dx = \frac{2v + 5}{v + 2} dv = \left(2 + \frac{1}{v + 2}\right) dv$$

On integrating both sides, we get

$$x = 2v + \log(v + 2) + C$$

$$\Rightarrow x = 2(x - y) + \log(x - y + 2) + C \quad [\because v = x - y]$$

$$\Rightarrow 2y - x = \log(x - y + 2) + C$$

which is the required solution.

- **Example 27.** The solution of the differential equation $\frac{dy}{dx} = \frac{x + y + 3}{2x + 2y + 1}$ is

- (a) $x + C = \frac{1}{3}(x + y) - \frac{5}{4} \log(3x + 3y + 4)$
 (b) $C = \frac{1}{3}(x + y) - \frac{5}{9} \log(3x + 3y + 4)$
 (c) $x + C = \frac{2}{3}(x + y) + \frac{4}{9} \log(3x + 3y + 4)$
 (d) $x + C = \frac{2}{3}(x + y) - \frac{5}{9} \log(3x + 3y + 4)$

Sol. (d) Here, $\frac{a}{a_1} = \frac{b}{b_1} = \frac{1}{2}$, i.e. the coefficients of x and y

in the numerator and denominator of the expression for $\frac{dy}{dx}$ are proportional. Proper substitution in this case, will be to put v for $x + y$.

Let $x + y = v$. Then, $1 + \frac{dy}{dx} = \frac{dv}{dx}$

With these substitutions, the given equation reduces to

$$\frac{dv}{dx} - 1 = \frac{v + 3}{2v + 1} \Rightarrow \frac{dv}{dx} = \frac{v + 3}{2v + 1} + 1 = \frac{3v + 4}{2v + 1}$$

$$\Rightarrow dx = \frac{2v + 1}{3v + 4} dv = \left[\frac{2}{3} - \frac{5/3}{3v + 4}\right] dv$$

On integrating both sides, we get

$$x + C = \frac{2}{3}v - \frac{5}{9} \cdot \frac{1}{3} \log(3v + 4)$$

$$\Rightarrow x + C = \frac{2}{3}v - \frac{5}{9} \log(3v + 4)$$

$$\Rightarrow x + C = \frac{2}{3}(x + y) - \frac{5}{9} \log(3x + 3y + 4) \quad [\because v = x + y]$$

which is the required solution.

Work Book Exercise 12.3

- 1 The solution of the differential equation

$$x dy - y dx - \sqrt{x^2 + y^2} dx = 0 \text{ is}$$

- a $y - \sqrt{x^2 + y^2} = Cx^2$ b $y + \sqrt{x^2 + y^2} = Cx^2$
 c $y + \sqrt{x^2 + y^2} = Cy^2$ d $x - \sqrt{x^2 + y^2} = Cy^2$

- 2 The solution of the differential equation

$$(3xy + y^2) dx + (x^2 + xy) dy = 0 \text{ is}$$

- a $x^2(2xy + y^2) = C^2$ b $x^2(2xy - y^2) = C^2$
 c $x^2(y^2 - 2xy^2) = C^2$ d None of these

- 3 The solution of the differential equation

$$x(x - y) \frac{dy}{dx} = y(x + y), \text{ is}$$

- a $\frac{x}{y} + \log xy = C$
 b $\frac{y}{x} + \log xy = C$
 c $\frac{x}{y} + y \log x = C$
 d $\frac{x}{y} + x \log y = C$

- 4 The solution of the differential equation

$$xy^2 dy - (x^3 + y^3) dx = 0 \text{ is}$$

- a $y^3 = 3x^3 + C$
 b $y^3 = 3x^3 \log(Cx)$
 c $y^3 = 3x^3 + \log(Cx)$
 d $y^3 + 3x^3 = \log(Cx)$

- 5 The solution of $\frac{dy}{dx} = \left(\frac{x + 2y - 3}{2x + y + 3}\right)^2$ is

- a $(x + 3)^3 - (y - 3)^3 = C(x - y + 6)^4$
 b $(x + 3)^3 - (y - 3)^3 = C$
 c $(x + 3)^4 + (y - 3)^4 = C$
 d None of the above

- 6 If $x \frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is

- a $\log\left(\frac{x}{y}\right) = Cy$ b $\log\left(\frac{y}{x}\right) = Cx$
 c $x \log\left(\frac{y}{x}\right) = Cy$ d $y \log\left(\frac{x}{y}\right) = Cx$

7 The solution of $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ is

- a $e^{y/x} = kx$
b $e^{y/x} = ky$
c $e^{x/y} = kx$
d $e^{-y/x} = ky$

(v) Linear Differential Equation

A differential equation is said to be linear, if the dependent variable and its differential coefficients occur in the first degree only and are not multiplied together. The most general form of a linear differential equation of the first order is

$$\frac{dy}{dx} + Py = Q \quad \dots(i)$$

where, P and Q are any functions of x .

To solve such equations, first find integrating factor,

i.e. $IF = e^{\int P dx}$

On multiplying Eq.(i) by $e^{\int P dx}$ on both sides, we get

$$\begin{aligned} e^{\int P dx} \left(\frac{dy}{dx} + Py \right) &= Q \cdot e^{\int P dx} \\ \Rightarrow \frac{dy}{dx} \cdot e^{\int P dx} + y \cdot \frac{d}{dx} e^{\int P dx} &= Q \cdot e^{\int P dx} \\ \left[\because \frac{d}{dx} (e^{\int P dx}) &= P \cdot e^{\int P dx} \right] \\ \Rightarrow \frac{d}{dx} (y \cdot e^{\int P dx}) &= Q \cdot e^{\int P dx} \\ \Rightarrow \int \frac{d}{dx} (y \cdot e^{\int P dx}) dx &= \int Q \cdot e^{\int P dx} dx \\ \Rightarrow y \cdot e^{\int P dx} &= \int Q \cdot e^{\int P dx} dx + C \end{aligned}$$

which is the required solution of the given differential equation.

➤ If linear equation is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are the functions of y , then $IF = e^{\int P dy}$ and solution is given by

$$x \cdot IF = \int Q \cdot IF dy + C$$

➤ **Example 28.** The integrating factor of the differential equation $(1-x^2) \frac{dy}{dx} - xy = 1$, is

- (a) $-x$ (b) $\frac{x}{1+x^2}$
(c) $\sqrt{1-x^2}$ (d) $\frac{1}{2} \log(1-x^2)$

8 If $dx + dy = (x+y)(dx-dy)$, then $\log(x+y)$ is equal to

- a $x+y+C$ b $x+2y+C$
c $x-y+C$ d $2x+y+C$

9 If $\frac{dy}{dx} = \frac{y+x \tan y/x}{x}$, then $\sin \frac{y}{x}$ is equal to

- a Cx^2 b Cx c Cx^3 d Cx^4

Sol. (c) We have, $(1-x^2) \frac{dy}{dx} - xy = 1$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{x}{1-x^2} \right) y = \frac{1}{1-x^2}$$

It is a linear differential equation with integrating factor

$$IF = e^{-\int \frac{x}{1-x^2} dx} = e^{\frac{1}{2} \log(1-x^2)} = \sqrt{1-x^2}$$

➤ **Example 29.** The integrating factor of the differential equation $\cos x \frac{dy}{dx} + y \sin x = 1$, is

- (a) $\sin x$ (b) $\sec x$ (c) $\tan x$ (d) $\cos x$

Sol. (b) We have,

$$\frac{dy}{dx} + (\tan x)y = \sec x$$

$$\therefore IF = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

➤ **Example 30.** The integrating factor of the differential equation $(x \log x) \frac{dy}{dx} + y = 2 \log x$, is

given by

- (a) $\log(\log x)$ (b) e^x
(c) $\log x$ (d) x

Sol. (c) We have,

$$\frac{dy}{dx} + \frac{1}{x \log x} y = \frac{2}{x}$$

$$\therefore IF = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)} = \log x$$

➤ **Example 31.** The differential equation $x \frac{dy}{dx} - y = x^2$, has the general solution

- (a) $y - x^3 = 2Cx$ (b) $2y - x^3 = 2Cx$
(c) $2y + x^2 = 2Cx$ (d) $y + x^2 = 2Cx$

Sol. (b) We have, $\frac{dy}{dx} + \left(-\frac{1}{x} \right) y = x^2 \quad \dots(i)$

It is a linear differential equation with integrating factor

$$IF = e^{\int -\frac{1}{x} dx} = e^{-\log x} = \frac{1}{x}$$

On multiplying Eq. (i) by $\frac{1}{x}$ and then integrating, we get

$$\frac{y}{x} = \frac{x^2}{2} + C$$

or $2y - x^3 = 2Cx$

➤ **Example 32.** The solution of

$$x \frac{dy}{dx} + y - x + xy \cot x = 0, \quad x \neq 0, \text{ is}$$

$$(a) y = x - \cot x + \frac{C}{\sin x} \quad (b) y = \frac{1}{x} - \cot x - \frac{x C}{x \sin x}$$

$$(c) y = \frac{1}{x} - \cot x + \frac{C}{x \sin x} \quad (d) y = \frac{1}{x} + \cot x - \frac{C}{\sin x}$$

Sol. (c) Given, $x \frac{dy}{dx} + y - x + xy \cot x = 0$

$$\Rightarrow x \frac{dy}{dx} + y(1 + x \cot x) = x$$

On dividing by x both sides, we get

$$\frac{dy}{dx} + \frac{y(1 + x \cot x)}{x} = 1$$

On comparing with the form $\frac{dy}{dx} + Py = Q$, we get

$$P = \frac{1 + x \cot x}{x} \text{ and } Q = 1$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \left(\frac{1}{x} + \cot x\right) dx}$$

$$= e^{\log x + \log |\sin x|} = e^{\log(x \sin x)}$$

$$\Rightarrow \text{IF} = x \sin x \quad [\because e^{\log ax} = ax]$$

The general solution the differential equation is given by

$$y \cdot \text{IF} = \int Q \times \text{IF} dx + C$$

$$\Rightarrow y(x \sin x) = \int x \sin x dx + C$$

$$\Rightarrow y(x \sin x) = -x \cos x + \int \cos x dx + C$$

[using integration by parts]

$$\Rightarrow y(x \sin x) = -x \cos x + \sin x + C$$

$$\Rightarrow y = \frac{-x \cos x}{x \sin x} + \frac{\sin x}{x \sin x} + \frac{C}{x \sin x}$$

$$\Rightarrow y = \frac{1}{x} - \cot x + \frac{C}{x \sin x}$$

➤ **Example 33.** A function $y = f(x)$ satisfies

$$(x+1) \cdot f'(x) - 2(x^2 + x) f(x) = \frac{e^{x^2}}{(x+1)}, \quad \forall x > -1.$$

If $f(0) = 5$, then $f(x)$ is

$$(a) \left(\frac{3x+5}{x+1}\right) \cdot e^{x^2} \quad (b) \left(\frac{6x+5}{x+1}\right) \cdot e^{x^2}$$

$$(c) \left(\frac{6x+5}{(x+1)^2}\right) \cdot e^{x^2} \quad (d) \left(\frac{5-6x}{x+1}\right) \cdot e^{x^2}$$

Sol. (b) $f'(x) - \frac{2x(x+1)}{x+1} f(x) = \frac{e^{x^2}}{(x+1)^2}$

$$\therefore \text{IF} = e^{\int -2x dx} = e^{-x^2}$$

$\therefore$ General solution is given by

$$f(x) \cdot e^{-x^2} = \int \frac{dx}{(x+1)^2} + C \Rightarrow f(x) \cdot e^{-x^2} = -\frac{1}{x+1} + C$$

$$\text{At } x = 0, f(0) = 5 \Rightarrow C = 6$$

$$\therefore f(x) = \left(\frac{6x+5}{x+1}\right) \cdot e^{x^2}$$

(vi) Differential Equation Reducible to Linear Form

Bernoulli's Equation

Sometimes, a differential equation is not linear but it can be converted into a linear differential equation by a suitable substitution.

An equation of the form

$$\frac{dy}{dx} + Py = Qy^n; \quad n \neq 0, 1$$

where, P and Q are functions of x only, is known as Bernoulli's equation. For $n = 0$, the equation is linear.

It is easy to reduce the above equation into linear form as given below :

On dividing both sides by y^n , we get

$$\frac{1}{y^n} \cdot \frac{dy}{dx} + \frac{1}{y^{n-1}} \cdot P = Q$$

On putting $y^{1-n} = z$ and hence

$$(1-n) y^{-n} \frac{dy}{dx} = \frac{dz}{dx}, \quad \text{the equation becomes}$$

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q, \text{ which is linear in } z.$$

$$\text{Here, IF} = e^{\int (1-n)P dx}$$

$\therefore$ The solution is given by

$$z \cdot e^{\int (1-n)P dx} = \int \{(1-n) \cdot Q \cdot e^{\int (1-n)P dx}\} dx + C$$

➤ **Example 34.** The solution of

$$x \left(\frac{dy}{dx}\right) + y = y^2 \log x \text{ is}$$

$$(a) C = y(1 + \log x) + xy \quad (b) 1 = y(1 + \log x) + Cxy$$

$$(c) 1 = y(1 + \log x) - Cxy \quad (d) C = y(1 + \log x) - xy$$

Sol. (b) Given equation can be written as

$$\frac{dy}{dx} + \frac{1}{x} \cdot y = \frac{y^2}{x} \log x \quad [\text{dividing throughout by } x]$$

$$\text{or} \quad \frac{1}{y^2} \cdot \frac{dy}{dx} + \frac{1}{x} \cdot \frac{1}{y} = \frac{\log x}{x} \quad \dots(i)$$

[dividing throughout by y^2]

$$\text{Now, put } \frac{1}{y} = v, \text{ so that } \left(-\frac{1}{y^2}\right) \frac{dy}{dx} = \frac{dv}{dx}$$

With these substitutions, Eq. (i) becomes

$$-\frac{dv}{dx} + \frac{1}{x} \cdot v = \frac{\log x}{x}$$

$$\text{or} \quad \frac{dv}{dx} - \frac{1}{x} \cdot v = -\frac{\log x}{x}$$

which is linear with v as the dependent variable.

$$\text{Here, } P = -\frac{1}{x} \text{ and } Q = -(\log x)/x$$

$$\therefore \text{IF} = e^{\int P dx} = e^{\int \left(-\frac{1}{x}\right) dx} = e^{-\log x} = e^{\log \left(\frac{1}{x}\right)} = \frac{1}{x}$$

Hence, the solution is

$$\begin{aligned} v \left(\frac{1}{x} \right) &= \int \{ -(\log x) / x \} \cdot \left(\frac{1}{x} \right) dx + C \\ \Rightarrow \frac{v}{x} &= - \int \frac{1}{x^2} \log x dx + C \\ &= - \left[\left(-\frac{1}{x} \right) \log x - \int \left(\frac{1}{x} \right) \left(-\frac{1}{x} \right) dx \right] + C \\ &\quad \text{[using integration by parts]} \\ &= \left(\frac{1}{x} \right) \log x - \int \left(\frac{1}{x^2} \right) dx + C \\ &= \frac{1}{x} \log x + \frac{1}{x} + C \\ \text{or } \frac{1}{xy} &= \frac{1}{x} (1 + \log x) + C \text{ or } 1 = y(1 + \log x) + Cxy \end{aligned}$$

➤ **Example 35.** The equation of the curve satisfying the equation $(xy - x^2) \frac{dy}{dx} = y^2$ and passing through the point $(-1, 1)$, is

- (a) $y = (\log y - 1)x$ (b) $y = (\log y + 1)x$
(c) $x = (\log x - 1)y$ (d) $x = (\log x + 1)y$

Sol. (a) We have, $(xy - x^2) \frac{dy}{dx} = y^2$

$$\Rightarrow y^2 \frac{dx}{dy} = xy - x^2$$

$$\Rightarrow \frac{1}{x^2} \cdot \frac{dx}{dy} - \frac{1}{x} \cdot \frac{1}{y} = \frac{-1}{y^2}$$

Let $\frac{1}{x} = v \Rightarrow -\frac{1}{x^2} \cdot \frac{dx}{dy} = \frac{dv}{dy}$, we get

$$\frac{dv}{dy} + \frac{v}{y} = \frac{1}{y^2}, \text{ which is linear.}$$

$$\therefore \text{IF} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

Hence, the solution is $vy = \int \frac{1}{y^2} \cdot y dy + C$

$$\Rightarrow \frac{y}{x} = \log y + C$$

$$\Rightarrow y = x(\log y + C) \quad \dots(i)$$

which passes through the point $(-1, 1)$.

$$\therefore 1 = -1(\log 1 + C) \Rightarrow C = -1$$

So, solution is $y = x(\log y - 1)$.

➤ **Example 36.** The general solution of the differential equation $\frac{dy}{dx} = y \tan x - y^2 \sec x$ is

- (a) $\tan x = (C + \sec x)y$ (b) $\sec y = (C + \tan y)x$
(c) $\sec x = (C + \tan x)y$ (d) None of these

Sol. (c) We have, $\frac{dy}{dx} = y \tan x - y^2 \sec x$

$$\Rightarrow \frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{1}{y} \tan x = -\sec x$$

Putting $\frac{1}{y} = v \Rightarrow -\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$, we get

$$\frac{dv}{dx} + \tan x \cdot v = \sec x, \text{ which is linear.}$$

$$\therefore \text{IF} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$$

The solution is

$$v \sec x = \int \sec^2 x dx + C$$

$$\Rightarrow \frac{1}{y} \sec x = \tan x + C$$

$$\Rightarrow \sec x = y(C + \tan x)$$

(vii) Exact Differential Equation

A differential equation is said to be exact, if it can be obtained from its primitive (solution) directly by differentiation, without involving any subsequent process of multiplication, elimination, etc. Thus, the differential equation of the form $M dx + N dy = 0$, where M and N are the some functions of x and y is exact, if it can be obtained directly by differentiating an equation of the form $u = C$, where u is some functions of x and y and C is an arbitrary constant.

➤ The necessary and sufficient condition for the ordinary differential equation $Mdx + Ndy = 0$ to be exact is that

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Working Rule for Solving an Exact Differential Equation

- Integrate M with respect to x treating y as constant.
- Integrate with respect to y only those terms of N which do not contain x .
- Equate the sum of these two integrals [found in points (i) and (ii)] to an arbitrary constant. Thus, we get the required solution.

Equation Reducible to the Exact Form

Sometimes, a differential equation of the form $Mdx + Ndy = 0$, which is not exact, can be reduced to an exact form by multiplying by a suitable function $f(x, y)$ which is not identically zero. This function $f(x, y)$, which then multiplied to a non-exact differential equation, makes it exact, is known as integrating factor. One can find integrating factors by inspection but for that some experience and practice is required.

For finding the integrating factors by inspection, the following identities must be remembered.

- $x dy + y dx = d(xy)$
- $x dx + y dy = \frac{1}{2} d(x^2 + y^2)$
- $\frac{xdy - ydx}{x^2} = d\left(\frac{y}{x}\right)$
- $\frac{ydx - xdy}{y^2} = d\left(\frac{x}{y}\right)$
- $\frac{xdy - ydx}{xy} = \frac{dy}{y} - \frac{dx}{x} = d\left[\log\left(\frac{y}{x}\right)\right]$

$$(vi) \frac{y dx - x dy}{xy} = d \left[\log \left(\frac{x}{y} \right) \right]$$

$$(vii) \frac{xdy - ydx}{x^2 + y^2} = \frac{\frac{xdy - ydx}{x^2}}{1 + y^2/x^2} = \frac{d \left(\frac{y}{x} \right)}{1 + \left(\frac{y}{x} \right)^2} = d \left[\tan^{-1} \left(\frac{y}{x} \right) \right]$$

$$(viii) \frac{dx - dy}{x + y} = d [\ln (x + y)]$$

$$(ix) d (\ln (xy)) = \frac{x dy + y dx}{xy}$$

$$(x) d \left(\frac{1}{2} \ln (x^2 + y^2) \right) = \frac{x dx + y dy}{x^2 + y^2}$$

$$(xi) d \left(-\frac{1}{xy} \right) = \frac{x dy + y dx}{x^2 y^2}$$

$$(xii) d \left(\frac{e^y}{x} \right) = \frac{x e^y dy - e^y dx}{x^2}$$

$$(xiii) d \left(\frac{e^x}{y} \right) = \frac{y e^x dx - e^x dy}{y^2}$$

$$(xiv) d (x^m y^n) = x^{m-1} y^{n-1} (m y dx + n x dy)$$

$$(xv) \frac{d[f(x, y)]^{1-n}}{1-n} = \frac{f'(x, y)}{(f(x, y))^n}$$

- Sometimes, transformation to the polar coordinates facilitates separation of variables. It is convenient to remember the following differentials.
- If $x = r \cos \theta, y = r \sin \theta$, then
 - > $x dx + y dy = r dr$
 - > $x dy - y dx = r^2 d\theta$
 - > $dx^2 + dy^2 = dr^2 + r^2 d\theta^2$

➤ **Example 37.** The solution of the differential equation $(x^2 - ay) dx - (ax - y^2) dy = 0$ is

$$(a) x^3 + y^3 - 3axy = 3C$$

$$(b) x^3 + y^3 + 3axy = C$$

$$(c) x^3 + y^3 - axy = C$$

$$(d) x^3 + y^3 - 3xy = 3C$$

Sol. (a) Here, $M = x^2 - ay$ and $N = -(ax - y^2)$, then $\frac{\partial M}{\partial y} = -a$ and $\frac{\partial N}{\partial x} = -a$, i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and hence the given equation is exact.

Now, $\int M dx$ (regarding y as a constant)

$$= \int (x^2 - ay) dx = \frac{1}{3} x^3 - axy \quad \dots(i)$$

and $\int N dy$ (taking in N , only those terms which do not contain x)

$$= \int y^2 dy = \frac{1}{3} y^3 \quad \dots(ii)$$

On adding Eqs. (i) and (ii) equal to C (an arbitrary constant), we get

$$\frac{1}{3} x^3 - axy + \frac{1}{3} y^3 = C$$

$$\Rightarrow x^3 + y^3 - 3axy = 3C$$

➤ **Example 38.** The solution of the equation

$$x dx + y dy + \frac{x dy - y dx}{x^2 + y^2} = 0$$

$$(a) y = x \tan \left(\frac{C + x^2 + y^2}{2} \right)$$

$$(b) x = y \tan \left(\frac{C + x^2 + y^2}{2} \right)$$

$$(c) y = x \tan \left(\frac{C - x^2 - y^2}{2} \right)$$

(d) None of the above

Sol. (c) We have, $x dx + y dy + \frac{x dy - y dx}{x^2 + y^2} = 0$

$$\Rightarrow \frac{1}{2} d(x^2 + y^2) + d \tan^{-1} \left(\frac{y}{x} \right) = 0$$

On integrating both sides, we get

$$\frac{1}{2} (x^2 + y^2) + \tan^{-1} \frac{y}{x} = \frac{C}{2}$$

$$\Rightarrow x^2 + y^2 + 2 \tan^{-1} \frac{y}{x} = C$$

$$\therefore y = x \tan \left(\frac{C - x^2 - y^2}{2} \right)$$

which is the required solution.

➤ **Example 39.** The solution of

$$\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}, \text{ satisfying } y(1) = 0, \text{ is given by}$$

(a) a hyperbola

(b) a circle

$$(c) y^2 = x(1+x) - 10$$

$$(d) (x-2)^2 + (y-3)^2 = 5$$

Sol. (a) $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$

$$\Rightarrow 2xy dy = (x^2 + 1) dx + y^2 dx$$

$$\Rightarrow \frac{xd(y^2) - y^2 dx}{x^2} = \left(\frac{x^2 + 1}{x^2} \right) dx$$

$$\Rightarrow \int d(y^2/x) = \int \left(1 + \frac{1}{x^2} \right) dx$$

$$\Rightarrow \frac{y^2}{x} = x - \frac{1}{x} + C$$

$$\Rightarrow y^2 = (x^2 - 1 + Cx)$$

$$\therefore x = 1, y = 0$$

$$\therefore 0 = 1 - 1 + C \Rightarrow C = 0$$

$$\text{So, } y^2 = x^2 - 1$$

$$\Rightarrow x^2 - y^2 = 1$$

Orthogonal Trajectory

Any curve, which cuts every member of a given family of curves at right angle, is called an orthogonal trajectory of the family.

Procedure for Finding the Orthogonal Trajectory

- i. Let $f(x, y, c) = 0$ be the equation of the given family of curves, where c is an arbitrary parameter.
- ii. Differentiate $f = 0$; with respect to x and eliminate c , i.e. form a differential equation.
- iii. Substitute $\left(-\frac{dx}{dy}\right)$ for $\frac{dy}{dx}$ in the above differential equation. This will give the differential equation of the orthogonal trajectories.
- iv. By solving this differential equation, we get the required orthogonal trajectories.

➤ **Example 40.** The equation of the family of curves which intersect the hyperbola $xy = 2$ orthogonally, is

- (a) $y = \frac{x^3}{6} + C$
- (b) $y = \frac{x^2}{4} + C$
- (c) $y = -\frac{x^3}{6} + C$
- (d) $y = -\frac{x^2}{4} + C$

Sol. (a) Let $m_1 = \frac{dy}{dx}$ for required family of curves at (x, y)

and $m_2 = \frac{dy}{dx}$ for the hyperbola $xy = 2$.

Then, $m_2 = \frac{dy}{dx} = -\frac{2}{x^2}$

Since, the required family of curves is orthogonal to the hyperbola.

$$\therefore m_1 \times m_2 = -1$$

$$\Rightarrow \frac{dy}{dx} \times \left(-\frac{2}{x^2}\right) = -1$$

$$\Rightarrow \frac{dy}{dx} = \frac{x^2}{2}$$

$$\Rightarrow dy = \frac{x^2}{2} dx$$

On integrating both sides, we get $y = \frac{x^3}{6} + C$

which is the required family of curves.

➤ **Example 41.** The orthogonal trajectories of the circles $x^2 + y^2 - ay = 0$, where a is a parameter, is

- (a) $x^2 + y^2 = Cy$
- (b) $x^2 + y^2 = Cx$
- (c) $x^2 + y^2 = C$
- (d) $x^2 + y^2 = C(x + y)$

Sol. (b) We have,

$$x^2 + y^2 - ay = 0 \quad \dots(i)$$

$$\Rightarrow 2x + 2y \frac{dy}{dx} - a \frac{dy}{dx} = 0$$

$$\Rightarrow a = \frac{2 \left(x + y \frac{dy}{dx} \right)}{\frac{dy}{dx}}$$

On substituting the value of a in Eq. (i), we get

$$x^2 + y^2 - 2y \left(\frac{x + y \frac{dy}{dx}}{\frac{dy}{dx}} \right) = 0$$

$$\Rightarrow (x^2 - y^2) \frac{dy}{dx} - 2xy = 0$$

which is the differential equation of the family of circles given in Eq. (i).

The differential equation representing the orthogonal trajectories is

$$-(x^2 - y^2) \frac{dx}{dy} - 2xy = 0$$

$$\Rightarrow \frac{dy}{dx} = - \left(\frac{x^2 - y^2}{2xy} \right)$$

$$\Rightarrow 2xy dy - y^2 dx = -x^2 dx$$

$$\Rightarrow \frac{x d(y^2) - y^2 dx}{x^2} = -dx$$

$$\Rightarrow d \left(\frac{y^2}{x} \right) = -dx$$

On integrating both sides, we get

$$\frac{y^2}{x} = -x + C$$

$$\Rightarrow y^2 + x^2 = Cx$$

➤ If the orthogonal trajectories of a family of curves is same as the given family of curves, then the given family of curves is called self-orthogonal.

Work Book Exercise 12.4

12

Differential Equations

- The integrating factor of the differential equation $x \log x \frac{dy}{dx} + y = 2 \log x$ is given by
a e^x b $\log x$ c $\log (\log x)$ d x
- An integrating factor of the differential equation $(1 + y + x^2 y) dx + (x + x^3) dy = 0$ is
a $\log x$ b x c e^x d $1/x$
- The solution of the differential equation $x \frac{dy}{dx} = 2y + x^3 e^x$, where $y = 0$, when $x = 1$, is
a $y = x^3 (e^x - e)$ b $y = x^3 (e - e^x)$
c $y = x^2 (e^x - e)$ d $y = x^2 (e - e^x)$
- The solution of the differential equation $\frac{dy}{dx} = \frac{x+y}{x}$ satisfying the condition $y(1) = 1$ is
a $y = x \log x + x$ b $y = \log x + x$
c $y = x \log x + x^2$ d $y = x e^{(x-1)}$
- The solution of the differential equation $\left(e^{-2\sqrt{x}} - \frac{y}{\sqrt{x}} \right) \frac{dx}{dy} = 1$ is given by
a $y e^{2\sqrt{x}} = x + C$ b $y e^{-2\sqrt{x}} = \sqrt{x} + C$
c $y = \sqrt{x}$ d $y = 3\sqrt{x}$
- If $\cos^2 x \frac{dy}{dx} - y \tan 2x = \cos^4 x$, where $|x| < \frac{\pi}{4}$ and $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$, then the constant term in the solution of the differential equation is
a 1 b 0
c -1 d None of these
- The solution of the equation $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$, when $y(0) = 1$ is
a $y^3 = 3x^3 \sin x$ b $x^3 = 3y^3 \sin x$
c $x^3 = y^3 \sin x$ d $x^3 = y^3 \cos x$
- The solution of the differential equation $\frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{1}{(1+x^2)^2}$ is
a $y(1-x^2) = \tan^{-1} x + C$
b $y(1+x^2) = \tan^{-1} x + C$
c $y(1+x^2)^2 = \tan^{-1} x + C$
d $y(1-x^2)^2 = \tan^{-1} x + C$
- The solution of $(x+y+1) \frac{dy}{dx} = 1$ is
a $y = (x+2) + C e^x$ b $y = -(x+2) + C e^x$
c $x = -(y+2) + C e^y$ d $x = (y+2)^2 + C e^y$

Applications of First Order Differential Equation

- i. **Growth and Decay Problems** Let $N(t)$ denotes the amount of substance (or population), i.e. either growing or decaying. If we assume that $\frac{dN}{dt}$, the time rate of change of this amount of substance is proportional to the amount of substance present, then $\frac{dN}{dt} = kN$ or $\frac{dN}{dt} - kN = 0$... (i) where, k is the constant of proportionality. We are assuming that $N(t)$ is a differentiable, hence it is continuous function of time.

- **Example 42.** If the population of a country doubles in 50 yr, then the number of years in which population will be triple under the assumption that the rate of increase is proportional to the number of inhabitants, is
(a) 79 yr (b) 78 yr
(c) 77 yr (d) 76 yr

Sol. (a) Let x denotes the population at a time t (in years).

Then, $\frac{dx}{dt} \propto x \Rightarrow \frac{dx}{dt} = kx$

where, k is a constant of proportionality.

Solving $\frac{dx}{dt} = kx$, we get

$$\int \frac{dx}{x} = \int k dt \Rightarrow \log x = kt + C$$

$$\Rightarrow x = e^{kt+C} \Rightarrow x = x_0 e^{kt}$$

where, x_0 is the population at time $t = 0$.

Since, it doubles in 50 yr at $t = 50$, so we must have $x = 2x_0$.

Hence, $2x_0 = x_0 e^{50k} \Rightarrow 50k = \log 2$

$\Rightarrow k = \frac{\log 2}{50}$, so that $x = x_0 e^{\frac{\log 2}{50} t}$

When $x = 3x_0 \Rightarrow t = \frac{50 \log 3}{\log 2} = 79 \text{ yr}$

- **Example 43.** Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops is proportional to the square root of water depth y , where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = \frac{1}{5}$, then the time to drain the tank, if the water is 4 m deep to start with, is
(a) 30 min (b) 45 min (c) 60 min (d) 80 min

Sol. (c) $\frac{dy}{dt} = -k\sqrt{y} \Rightarrow \int_4^0 \frac{dy}{\sqrt{y}} = -k \int_0^t dt$
 $\Rightarrow [2\sqrt{y}]_4^0 = -kt$
 $\Rightarrow 0 - 4 = -\frac{t}{15} \quad [\because k = 1/15]$
 $\Rightarrow t = 60 \text{ min}$

ii. Temperature Problems Newton's law of cooling which is equally applicable to heating, states that the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium. Let T denotes the temperature of the body and T_m denotes the temperature of the surrounding medium. Then, the time rate of change temperature of the body is $\frac{dT}{dt}$ and Newton's law of cooling can be formulated as

$$\frac{dT}{dt} = -k(T - T_m)$$

or $\frac{dT}{dt} + KT = KT_m \quad \dots(ii)$

where, k is a positive constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make $\frac{dT}{dt}$ negative in a cooling process, when T is greater than T_m and positive in a heating process, when T is less than T_m .

Example 44. A metal bar at a temperature of 100°F is placed in a room at a constant temperature of 0°F . If after 20 min, the temperature of the bar is 50°F , then the time it will take the bar to reach a temperature of 25°F and the temperature of the bar after 10 min are respectively

(a) 39.6 min and $100 e^{\frac{1}{20} \log \frac{1}{2} 10}^\circ \text{F}$

(b) 36.9 min and $100 e^{\frac{1}{2} \log \frac{1}{2} 10}^\circ \text{F}$

(c) 38.9 min and $100 e^{-\frac{1}{2} \log \frac{1}{2} 10}^\circ \text{F}$

(d) None of the above

Sol. (a) Using Eq. (ii) with $T_m = 0$; the medium, here is the room which is being held at a constant temperature of 0°F . Thus, we get

$$\frac{dT}{dt} + kT = 0$$

whose solution is $T = Ce^{-kt} \quad \dots(i)$

Since, $T = 100$ at $t = 0$ (the temperature of the bar is initially 100°F), it follows Eq. (i) and $100 = ce^{-k(0)}$ or $100 = c$. Substituting this value into Eq. (i), we get

$$T = 100e^{-kt} \quad \dots(ii)$$

At $t = 20$, we are given that $T = 50$, hence from Eq. (ii),

$$50 = 100e^{-20k} \text{ from which}$$

$$k = -\frac{1}{20} \ln \frac{50}{100}$$

On substituting this value into Eq. (ii), we get the temperature of the bar at any time t as

$$T = 100e^{\frac{1}{20} \ln \frac{1}{2} t} \quad \dots(iii)$$

We require t , when $T = 25$.

Substituting $T = 25$ into Eq. (iii), we get

$$25 = 100e^{\left(\frac{1}{20} \ln \frac{1}{2}\right)t}$$

On solving, we find that $t = 39.6 \text{ min}$.

We require T , when $t = 10$.

Substituting $t = 10$ into Eq. (iii) and then solving for T , we find that

$$T = 100e^{\frac{1}{20} \ln \frac{1}{2} 10}^\circ \text{F}$$

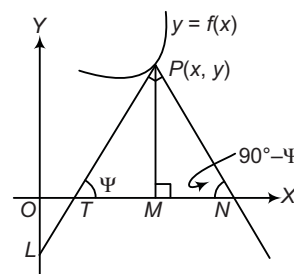
It should be noted that, Newton's law is valid only for small temperature differences, hence the above calculations represent only a first approximation to the physical situation.

iii. Geometrical Applications Let $P(x, y)$ be any point on the curve $y = f(x)$. Let the tangent and normal at $P(x, y)$ to the curve meets X -axis and Y -axis at T and N , respectively.

Now, draw perpendicular from P on X -axis.

$$\therefore PM = y$$

If tangent at P makes angle ψ with positive direction of X -axis, then $\frac{dy}{dx} = \tan \psi$.



(a) **Length of sub-tangent** TM is defined as sub-tangent. In $\triangle PTM$,

$$TM = |y \cot \psi| = \left| \frac{y}{\tan \psi} \right| = \left| y \frac{dx}{dy} \right|$$

$$\therefore \text{Length of sub-tangent} = \left| y \frac{dx}{dy} \right|$$

(b) **Length of sub-normal** MN is defined as sub-normal. In $\triangle PMN$,

$$MN = |y \cot (90^\circ - \psi)| = |y \tan \psi| = \left| y \frac{dy}{dx} \right|$$

$$\therefore \text{Length of sub-normal} = \left| y \frac{dy}{dx} \right|$$

(c) **Length of tangent** PT is defined as length of tangent. In ΔPMT ,

$$PT = |y \operatorname{cosec} \psi| = \left| y \sqrt{1 + \cot^2 \psi} \right|$$

$$\therefore \text{Length of tangent} = \left| y \sqrt{1 + \left(\frac{dx}{dy} \right)^2} \right|$$

(d) **Length of normal** PN is defined as length of normal. In ΔPMN ,

$$PN = |y \operatorname{cosec} (90^\circ - \psi)| = |y \sec \psi|$$

$$= \left| y \sqrt{1 + \tan^2 \psi} \right| = \left| y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} \right|$$

$$\therefore \text{Length of normal} = \left| y \sqrt{1 + (dy/dx)^2} \right|$$

(e) **Intercepts made by the tangent on the coordinate axes** The equation of tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx} (X - x) \quad \dots(iii)$$

On putting $Y = 0$ in Eq. (iii), we get

$$X = x - y \frac{dx}{dy}$$

Hence, the length of intercept OT that the tangent cuts-off from the X -axis, is

$$x - y \frac{dx}{dy}$$

Again, if the tangent meets Y -axis, then putting $X = 0$ in Eq. (i), we get

$$Y = y - x \frac{dy}{dx}$$

Hence, the length of intercept OL that the tangent cuts-off from the Y -axis, is

$$y - x \frac{dy}{dx}$$

➤ **Example 45.** The equation of the curve which passes through the point $(2a, a)$ and for which the sum of the cartesian sub-tangent and the abscissa is equal to the constant a , is

$$(a) y(x - a) = a^2 \quad (b) y(x + a) = a^2$$

$$(c) x(y - a) = a^2 \quad (d) x(y + a) = a^2$$

Sol. (a) We have,

Cartesian sub-tangent + Abscissa = Constant

$$\Rightarrow \frac{y}{dy/dx} + x = a \Rightarrow y \frac{dx}{dy} + x = a$$

$$\Rightarrow \frac{dy}{y} = \frac{dx}{a - x}$$

On integrating both sides, we get

$$\log y + \log(x - a) = \log C$$

$$\therefore y(x - a) = C$$

As the curve passes through the point $(2a, a)$, so we have $C = a^2$.

Hence, the required curve is $y(x - a) = a^2$.

➤ **Example 46.** A curve is such that the portion of X -axis cut-off between the origin and the tangent at a point is twice the abscissa and which passes through the point $(1, 2)$. The equation of the curve is

$$(a) xy = 1$$

$$(b) xy = 2$$

$$(c) xy = 3$$

$$(d) \text{None of these}$$

Sol. (b) Let $P(x, y)$ be any point on the curve, PM be the perpendicular to X -axis, PT be the tangent at P meeting the X -axis at T . As given $OT = 2$, $OM = 2x$. Equation of the tangent at $P(x, y)$ is

$$Y - y = \frac{dy}{dx} (X - x)$$

It intersects X -axis, where $Y = 0$.

$$\Rightarrow -y = \frac{dy}{dx} (X - x) \Rightarrow X = x - y \frac{dx}{dy} = OT$$

$$\text{Hence, } x - y \frac{dx}{dy} = 2x \Rightarrow \frac{dx}{x} + \frac{dy}{y} = 0$$

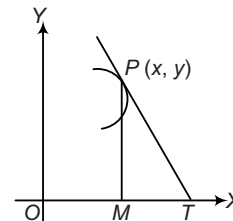
On integrating both sides, we get $\log x + \log y = \log C$

$$\Rightarrow xy = C$$

which this passes through $(1, 2)$.

$$\therefore C = 2$$

Hence, the required curve is $xy = 2$.



➤ **Example 47.** The x -intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point $(1, 1)$ is

$$(a) ye^{x/y} = e$$

$$(b) xe^{x/y} = e$$

$$(c) xe^{y/x} = e$$

$$(d) ye^{y/x} = e$$

Sol. (a) $Y - y = m \cdot (X - x)$

For x -intercept, $Y = 0$

$$\therefore X = x - \frac{y}{m}$$

$$\text{Hence, } x - \frac{y}{m} = y$$

$$\text{or } \frac{\frac{y}{m}}{\frac{dy}{dx}} = \frac{y}{x - y}$$

Put $y = vx$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v}{1 - v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v}{1 - v} - v = \frac{v - v + v^2}{1 - v}$$

$$\Rightarrow \int \frac{1 - v}{v^3} dv = \int \frac{dx}{x}$$

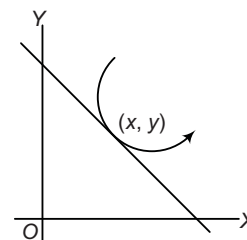
$$\Rightarrow -\frac{1}{v} - \ln v = \ln x + C$$

$$\Rightarrow -\frac{x}{y} - \ln \frac{y}{x} = \ln x + C \Rightarrow -\frac{x}{y} = \ln y + C$$

$$\therefore x = 1, y = 1 \Rightarrow C = -1$$

$$\therefore 1 - \frac{x}{y} = \ln y \Rightarrow y = e \cdot e^{-x/y}$$

$$\Rightarrow e^{x/y} = \frac{e}{y} \Rightarrow ye^{x/y} = e$$



Work Book Exercise 12.5

- The equation of the curve whose tangent at any point (x, y) makes an angle $\tan^{-1}(2x + 3y)$ with X-axis and which passes through $(1, 2)$, is
 - $6x + 9y + 2 = 26e^{3(x-1)}$
 - $6x - 9y + 2 = 26e^{3(x-1)}$
 - $6x + 9y - 2 = 26e^{3(x-1)}$
 - $6x - 9y - 2 = 26e^{3(x-1)}$
- A curve having the condition that the slope of tangent at some point is two times the slope of the straight line joining the same point to the origin of coordinates, is a/an
 - circle
 - ellipse
 - parabola
 - hyperbola
- The equation of the curve satisfying the differential equation $y_2(x^2 + 1) = 2xy_1$ passing through the point $(0, 1)$ and having slope of tangent at $x = 0$ as 3, is
 - $y = x^3 + 3x + 1$
 - $y = x^3 - 3x + 1$
 - $y = x^2 + 3x + 1$
 - $y = x^2 - 3x + 1$
- The equation of the curve whose sub-normal is equal to a constant a , is
 - $y = ax + b$
 - $y^2 = 2ax + 2b$
 - $ay^2 - x^3 = a$
 - None of the above
- The slope at any point of a curve $y = f(x)$ is given by $\frac{dy}{dx} = 3x^2$ and it passes through $(-1, 1)$.
The equation of the curve is
 - $y = x^3 + 2$
 - $y = -x^3 - 2$
 - $y = 3x^3 + 4$
 - $y = -x^3 + 2$
- The curve from which the length of normal is equal to the radius vector, is
 - $y^2 \left(\frac{dy}{dx} \right)^2 = x^2$
 - $y \left(\frac{dy}{dx} \right)^2 = x$
 - $x^2 \left(\frac{dy}{dx} \right) = y^2$
 - None of the above
- The curve for which the intercept cut-off by any tangent on Y-axis is proportional to the square of the ordinate of the point of tendency, is
 - $\frac{C_1}{x} - \frac{C_2}{y} = 1$
 - $\frac{C_1}{x} + \frac{C_2}{y} = 1$
 - $\frac{C_1}{x} + \frac{C_2}{y} = \frac{1}{xy}$
 - None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The solution of the differential equation $e^{-x} (y+1) dy + (\cos^2 x - \sin 2x) y dx = 0$ subjected to the condition that $y=1$, when $x=0$, is

- (a) $y + \log y + e^x \cos^2 x = 2$
- (b) $\log (y+1) + e^x \cos^2 x = 1$
- (c) $y + \log y = e^x \cos^2 x$
- (d) $(y+1) + e^x \cos^2 x = 2$

Sol. Given equation can be rewritten as $\left(1 + \frac{1}{y}\right) dy = -e^x (\cos^2 x - \sin 2x) dx$

On integrating both sides, we get

$$y + \log y = -e^x \cos^2 x + \int e^x \sin 2x dx + C$$

$$\Rightarrow y + \log y = -e^x \cos^2 x + C$$

At $x=0$ and $y=1$, [given]

$$1 + 0 = -e^0 \cos^2 0 + C \Rightarrow C = 2$$

$\therefore$ Required solution is

$$y + \log y = -e^x \cos^2 x + 2$$

$$\Rightarrow y + \log y + e^x \cos^2 x = 2$$

Hence, (a) is the correct answer.

Ex 2. The solution of the differential equation

$$\frac{dy}{dx} = \frac{1}{x+y^2} \text{ is}$$

- (a) $y = -x^2 - 2x - 2 + Ce^x$
- (b) $y = x^2 + 2x + 2 - Ce^x$
- (c) $x = -y^2 - 2y + 2 - Ce^y$
- (d) $x = -y^2 - 2y - 2 + Ce^y$

Sol. Given, differential equation is

$$\frac{dy}{dx} = \frac{1}{x+y^2}$$

$$\Rightarrow \frac{dx}{dy} - x = y^2$$

Here, $P = -1$, $Q = y^2$

$$\Rightarrow \text{IF} = e^{\int -1 dy} = e^{-y}$$

$\therefore$ Required solution is

$$xe^{-y} = \int e^{-y} y^2 dy = -e^{-y} y^2 + \int 2e^{-y} y dy$$

$$\Rightarrow xe^{-y} = -e^{-y} y^2 + 2[-e^{-y} y + \int e^{-y} dy] + C$$

$$\Rightarrow xe^{-y} = -e^{-y} y^2 + 2[-e^{-y} y - e^{-y}] + C$$

$$\Rightarrow xe^{-y} = e^{-y}(-y^2 - 2y - 2) + C$$

$$\Rightarrow x = -y^2 - 2y - 2 + Ce^y$$

Hence, (d) is the correct answer.

Ex 3. The differential equation of the straight lines touching the circle $x^2 + y^2 = a^2$ is

- (a) $\left(y - \frac{dy}{dx}\right)^2 = a^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]$
- (b) $\left(y - x \frac{dy}{dx}\right)^2 = a^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]$
- (c) $\left(y - x \frac{dy}{dx}\right) = a^2 \left[1 + \frac{dy}{dx}\right]$
- (d) $\left(y - \frac{dy}{dx}\right) = a^2 \left[1 - \frac{dy}{dx}\right]$

Sol. The equation of straight line touching the given circle is

$$x \cos \theta + y \sin \theta = a \quad \dots(i)$$

On differentiating both sides w.r.t. x , regarding θ as a constant, we get

$$\cos \theta + \frac{dy}{dx} \sin \theta = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), $\cos \theta = \frac{a \frac{dy}{dx}}{x \frac{dy}{dx} - y}$

and $\sin \theta = -\frac{a}{x \frac{dy}{dx} - y}$

$$\therefore \cos^2 \theta + \sin^2 \theta = 1$$

$$\therefore \frac{a^2 \left(\frac{dy}{dx}\right)^2 + a^2}{\left(x \frac{dy}{dx} - y\right)^2} = 1$$

$$\Rightarrow \left(y - x \frac{dy}{dx}\right)^2 = a^2 \left[1 + \left(\frac{dy}{dx}\right)^2\right]$$

Hence, (b) is the correct answer.

Ex 4. General solution of the differential equation

$$\frac{dy}{dx} = \frac{x+y+1}{x+y-1} \text{ is given by}$$

- (a) $x + y = \log |x + y| + C$
- (b) $x - y = \log |x + y| + C$
- (c) $y = x + \log |x + y| + C$
- (d) $y = x \log |x + y| + C$

Sol. Given, $\frac{dy}{dx} = \frac{x+y+1}{x+y-1}$

Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx} \Rightarrow \frac{dy}{dx} = \frac{dt}{dx} - 1$

$$\therefore \frac{dt}{dx} - 1 = \frac{t+1}{t-1} \Rightarrow \frac{dt}{dx} = \frac{t+1+t-1}{t-1}$$

$$\Rightarrow \frac{dt}{dx} = \frac{2t}{t-1}$$

$$\Rightarrow \left(\frac{t-1}{2t} \right) dt = dx$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{2t} \right) dt = dx$$

On integrating both sides, we get

$$\frac{1}{2}t - \frac{1}{2} \log |t| = x + C$$

$$\left[\because \int \frac{1}{x} dx = \log |x| + C \right]$$

$$\Rightarrow t - \log |t| = 2x + 2C$$

$$\Rightarrow x + y - \log |x + y| = 2x + 2C$$

$$\Rightarrow y = x + \log |x + y| + C$$

Hence, (c) is the correct answer.

Ex 5. The solution of $dy = \cos x (2 - y \operatorname{cosec} x) dx$, where $y = \sqrt{2}$, when $x = \pi/4$, is

(a) $y = \sin x + \frac{1}{2} \operatorname{cosec} x$

(b) $y = \tan (x/2) + \cot (x/2)$

(c) $y = (1/\sqrt{2}) \sec (x/2) + \sqrt{2} \cos (x/2)$

(d) None of the above

Sol. Given, $\frac{dy}{dx} = 2 \cos x - y \cos x \operatorname{cosec} x$

$$\Rightarrow \frac{dy}{dx} + y \cot x = 2 \cos x$$

Here, $P = \cot x$ and $Q = 2 \cos x$

$$\therefore \text{IF} = e^{\int \cot x dx} = e^{\log (\sin x)} = \sin x$$

Hence, required solution is

$$y \sin x = \int 2 \cos x \sin x dx + C$$

$$\Rightarrow y \sin x = \int \sin 2x dx + C$$

$$\Rightarrow y \sin x = \frac{-\cos 2x}{2} + C$$

At $x = \frac{\pi}{4}$ and $y = \sqrt{2}$,

$$\therefore \sqrt{2} \sin \frac{\pi}{4} = \frac{-\cos 2(\pi/4)}{2} + C \Rightarrow C = 1$$

Now, $y \sin x = -\frac{1}{2} \cos 2x + 1$

$$\Rightarrow y = -\frac{1}{2} \cdot \frac{\cos 2x}{\sin x} + \operatorname{cosec} x$$

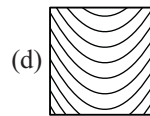
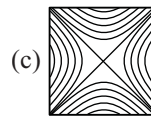
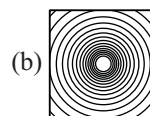
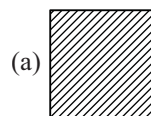
$$\Rightarrow y = -\frac{1}{2 \sin x} (1 - 2 \sin^2 x) + \operatorname{cosec} x$$

$$\therefore y = \frac{1}{2} \operatorname{cosec} x + \sin x$$

Hence, (a) is the correct answer.

Ex 6. The general solution of the differential equation $\frac{dy}{dx} = \frac{1-x}{y}$ is a family of curves

which looks most like which of the following?



Sol. $\int y dy = \int (1-x) dx$

$$\Rightarrow \frac{y^2}{2} = x - \frac{x^2}{2} + C$$

$$\Rightarrow x^2 + y^2 - 2x = C$$

$$\Rightarrow (x-1)^2 + y^2 = C + 1 = C$$

Hence, (b) is the correct answer.

Ex 7. The degree of the differential equation

$$\left(\frac{d^3 y}{dx^3} \right)^{2/3} + 4 - 3 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} = 0 \text{ is}$$

(a) 1

(b) 2

(c) 3

(d) None of these

Sol. We have, $\left(\frac{d^3 y}{dx^3} \right)^{2/3} + 4 - 3 \frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} = 0$

$$\Rightarrow \left(\frac{d^3 y}{dx^3} \right)^2 = \left(3 \frac{d^2 y}{dx^2} - 5 \frac{dy}{dx} - 4 \right)^3$$

Clearly, it is a differential equation of degree 2.

Hence, (b) is the correct answer.

Ex 8. The degree of the differential equation satisfying the relation

$$\sqrt{1+x^2} + \sqrt{1+y^2} = \lambda (x\sqrt{1+y^2} - y\sqrt{1+x^2})$$

is

(a) 1

(b) 2

(c) 3

(d) None of the above

Sol. On putting $x = \tan A$ and $y = \tan B$ in the given relation, we get

$$\cos A + \cos B = \lambda (\sin A - \sin B)$$

$$\Rightarrow \tan \left(\frac{A+B}{2} \right) = \frac{1}{\lambda}$$

$$\tan^{-1} x - \tan^{-1} y = 2 \tan^{-1} \left(\frac{1}{\lambda} \right)$$

On differentiating w.r.t. x , we get

$$\frac{1}{1+x^2} - \frac{1}{1+y^2} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{1+y^2}{1+x^2}$$

Clearly, it is a differential equation of degree 1.

Hence, (a) is the correct answer.

Ex 9. The degree of the differential equation corresponding to the family of curves $y = a(x+a)^2$, where a is an arbitrary constant, is

- (a) 1 (b) 2
(c) 3 (d) None of these

Sol. We have, $y = a(x+a)^2$... (i)

$$\Rightarrow \frac{dy}{dx} = 2a(x+a) \quad \dots (ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\Rightarrow x + a = \frac{2y}{y_1}$$

where $y_1 = \frac{dy}{dx}$

On substituting $a = \frac{2y}{y_1} - x$ in Eq. (i), we get

$$y = \left(\frac{2y}{y_1} - x \right) \left(\frac{2y}{y_1} \right)^2$$

$$\Rightarrow y_1^3 y = 4(2y - xy_1) y^2$$

Clearly, it is a differential equation of degree 3.

Hence, (c) is the correct answer.

Ex 10. The degree of the differential equation of all curves having normal of constant length C , is

- (a) 1 (b) 3
(c) 4 (d) 2

Sol. We have, $y \sqrt{1 + \left(\frac{dy}{dx} \right)^2} = C$

$$\Rightarrow y^2 \left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\} = C^2$$

$$\Rightarrow y^2 \left(\frac{dy}{dx} \right)^2 + y^2 = C^2$$

Clearly, it is a differential equation of degree 2.

Hence, (d) is the correct answer.

Ex 11. The equation of the curve in which sub-normal varies as the square of the ordinate, is (λ is constant of proportionality)

- (a) $y = Ce^{2\lambda x}$
(b) $y = Ce^{\lambda x}$
(c) $\frac{y^2}{2} + \lambda x = C$
(d) $y^2 + \lambda x^2 = C$

Sol. We have, $y \frac{dy}{dx} = \lambda y^2$

$$\Rightarrow \frac{dy}{dx} = \lambda y$$

$$\Rightarrow \frac{1}{y} dy = \lambda dx$$

$$\Rightarrow \log y = \lambda x + \log C$$

$$\Rightarrow y = Ce^{\lambda x}$$

Hence, (b) is the correct answer.

Ex 12. The orthogonal trajectories of the family of curves $a^{n-1}y = x^n$ are given by (a is the arbitrary constant)

- (a) $x^n + n^2 y = \text{constant}$
(b) $ny^2 + x^2 = \text{constant}$
(c) $n^2 x + y^n = \text{constant}$
(d) $n^2 x - y^n = \text{constant}$

Sol. The equation of the given family of curves is

$$a^{n-1}y = x^n \quad \dots (i)$$

$$\Rightarrow (n-1) \log a + \log y = n \log x$$

On differentiating w.r.t. x , we get

$$\frac{1}{y} \frac{dy}{dx} = \frac{n}{x} \quad \dots (ii)$$

This is the differential equation of the family of curves given in Eq. (i).

The differential equation of the orthogonal trajectories of Eq. (i) is obtained by replacing $\frac{dy}{dx}$ by

$$\left(-\frac{dx}{dy} \right) \text{ in Eq. (ii).}$$

$$\therefore \frac{1}{y} \times \left(-\frac{dx}{dy} \right) = \frac{n}{x}$$

$$\Rightarrow x dx + ny dy = 0$$

On integrating, we get

$$\frac{x^2}{2} + n \frac{y^2}{2} = C$$

$$\Rightarrow x^2 + ny^2 = 2C$$

Hence, (b) is the correct answer.

Ex 13. The orthogonal trajectories of the family of circles given by $x^2 + y^2 - 2ay = 0$ (a is parameter), is

- (a) $x^2 + y^2 - 2kx = 0$
(b) $x^2 + y^2 - 2ky = 0$
(c) $x^2 + y^2 - 2k_1 x - 2k_2 y = 0$
(d) None of the above

Sol. The equation of the family of circles is

$$x^2 + y^2 - 2ay = 0 \quad \dots (i)$$

On differentiating w.r.t. x , we get

$$2x + 2y \frac{dy}{dx} - 2a \frac{dy}{dx} = 0 \Rightarrow a = \frac{x + y \frac{dy}{dx}}{\frac{dy}{dx}}$$

On putting the value of a in Eq. (i), we get

$$x^2 + y^2 - 2y \left\{ \frac{x + y \frac{dy}{dx}}{\frac{dy}{dx}} \right\} = 0$$

$$\Rightarrow x^2 - y^2 - 2xy \frac{dx}{dy} = 0 \quad \dots (ii)$$

This is the differential equation of the family of circles given by Eq. (i).

The differential equation representing the family of orthogonal trajectories of Eq. (i) is obtained by replacing $\left(\frac{-dx}{dy}\right)$ by $\left(\frac{dy}{dx}\right)$ in Eq. (ii). So, the differential equation of the orthogonal trajectories is

$$\begin{aligned} x^2 - y^2 + 2xy \frac{dy}{dx} &= 0 \\ \Rightarrow (x^2 - y^2) dx + 2xy dy &= 0 \\ \Rightarrow 2xy dy - y^2 dx &= -x^2 dx \\ \Rightarrow \frac{xd(y^2) - y^2 dx}{x^2} &= -dx \\ \Rightarrow d\left(\frac{y^2}{x}\right) &= -dx \\ \Rightarrow \frac{y^2}{x} &= -x + 2k \\ \Rightarrow x^2 + y^2 - 2kx &= 0 \end{aligned}$$

This is the required family of orthogonal trajectories.

Hence, (a) is the correct answer.

Ex 14. The equation of the curve in which the portion of the tangent included between the coordinate axes is bisected at the point of contact, is

- (a) a parabola
- (b) an ellipse
- (c) a circle
- (d) a hyperbola

Sol. The equation of the tangent at any point $P(x, y)$ is $Y - y = \frac{dy}{dx}(X - x)$.

This cuts the coordinate axes at $A\left(x - y \frac{dx}{dy}, 0\right)$ and $B\left(0, y - x \frac{dy}{dx}\right)$.

It is given that $P(x, y)$ is the mid-point of AB .

$$\therefore x - y \frac{dx}{dy} = 2x \text{ and } y - x \frac{dy}{dx} = 2y$$

$$\Rightarrow x + y \frac{dx}{dy} = 0 \text{ and } y + x \frac{dy}{dx} = 0$$

$$\Rightarrow x dy + y dx = 0 \text{ and } y dx + x dy = 0$$

$$\Rightarrow d(xy) = 0 \Rightarrow xy = C$$

Clearly, it represents a rectangular hyperbola.

Hence, (d) is the correct answer.

Ex 15. The solution of the differential equation

$$\frac{x}{x^2 + y^2} dy = \left(\frac{y}{x^2 + y^2} - 1 \right) dx \text{ is}$$

- (a) $y = x \cot(C - x)$
- (b) $\cos^{-1} \frac{y}{x} = (-x + C)$
- (c) $y = x \tan(C - x)$
- (d) $\frac{y^2}{x^2} = x \tan(C - x)$

$$\text{Sol. We have, } \frac{x}{x^2 + y^2} dy = \left(\frac{y}{x^2 + y^2} - 1 \right) dx$$

$$\Rightarrow \frac{xdy - ydx}{x^2 + y^2} = -dx$$

$$\Rightarrow d \left\{ \tan^{-1} \left(\frac{y}{x} \right) \right\} = -dx$$

$$\Rightarrow \tan^{-1} \frac{y}{x} = -x + C$$

$$\Rightarrow y = x \tan(C - x)$$

Hence, (c) is the correct answer.

Ex 16. If $\phi(x)$ is a differentiable function, then the solution of the differential equation $dy + \{y\phi'(x) - \phi(x)\phi'(x)\} dx = 0$ is

- (a) $y = \{\phi(x) - 1\} + Ce^{-\phi(x)}$
- (b) $y\phi(x) = \{\phi(x)\}^2 + C$
- (c) $y e^{\phi(x)} = \phi(x) e^{\phi(x)} + C$
- (d) None of the above

Sol. We have, $dy + \{y\phi'(x) - \phi(x)\phi'(x)\} dx = 0$

$$\Rightarrow \frac{dy}{dx} + \phi'(x) \cdot y = \phi(x) \phi'(x) \quad \dots(i)$$

This is a linear differential equation with

$$IF = e^{\int \phi'(x) dx} = e^{\phi(x)}$$

On multiplying Eq. (i) by $\phi(x)$ and integrating, we get

$$ye^{\phi(x)} = \int \phi(x) \phi'(x) e^{\phi(x)} dx$$

$$\Rightarrow ye^{\phi(x)} = \int e^{\phi(x)} \phi(x) \phi'(x) dx$$

$$\Rightarrow ye^{\phi(x)} = \int \phi(x) e^{\phi(x)} \phi'(x) dx$$

$$\Rightarrow ye^{\phi(x)} = \phi(x) e^{\phi(x)} - \int \phi'(x) e^{\phi(x)} dx$$

$$\Rightarrow ye^{\phi(x)} = \phi(x) e^{\phi(x)} - e^{\phi(x)} + C$$

$$\Rightarrow y = \{\phi(x) - 1\} + Ce^{-\phi(x)}$$

Hence, (a) is the correct answer.

Ex 17. The solution of the differential equation

$$x^3 \frac{dy}{dx} + 4x^2 \tan y = e^x \sec y \text{ satisfying } y(1) = 0,$$

$$(a) \tan y = (x - 2) e^x \log 3$$

$$(b) \sin y = e^x (x - 1) x^{-4}$$

$$(c) \tan y = (x - 1) e^x x^{-3}$$

$$(d) \sin y = e^x (x - 1) x^{-3}$$

Sol. We have, $x^3 \frac{dy}{dx} + 4x^2 \tan y = e^x \sec y$

$$\Rightarrow x^3 \cos y \frac{dy}{dx} + 4x^2 \sin y = e^x$$

$$\Rightarrow x^4 \cos y dy + 4x^3 \sin y dx = xe^x dx$$

$$\Rightarrow d(x^4 \sin y) = xe^x dx$$

On integrating, we get

$$x^4 \sin y = (x - 1) e^x + C \quad \dots(i)$$

It is given that, $y = 0$, when $x = 1$.

On putting $x = 1$ and $y = 0$ in Eq. (i), we get

$$C = 0$$

On putting $C = 0$ in Eq. (i), we get

$$x^4 \sin y = (x-1)e^x$$

$$\sin y = e^x (x-1)x^{-4}$$

Hence, (b) is the correct answer.

Ex 18. Solution of the differential equation $y(xy + 2x^2 y^2) dx + x(xy + x^2 y^2) dy = 0$ is by

(a) $2 \log |x| - \log |y| - \frac{1}{xy} = C$

(b) $2 \log |y| - \log |x| - \frac{1}{xy} = C$

(c) $2 \log |x| + \log |y| + \frac{1}{xy} = C$

(d) $2 \log |y| + \log |x| + 1/xy = C$

Sol. We have, $(xy^2 + 2x^2 y^3) dx + (x^2 y - x^3 y^2) dy = 0$

$$\Rightarrow xy(y dx + x dy) + x^2 y^2 (2y dx - x dy) = 0$$

$$\Rightarrow \frac{d(xy)}{x^2 y^2} + \left(\frac{2}{x} dx - \frac{1}{y} dy \right) = 0 \quad [\text{dividing by } x^3 y^3]$$

On integrating, we get

$$-\frac{1}{xy} + 2 \log |x| - \log |y| = C$$

Hence, (a) is the correct answer.

Ex 19. Solution of the differential equation

$$\frac{\sqrt{x} dx + \sqrt{y} dy}{\sqrt{x} dx - \sqrt{y} dy} = \sqrt{\frac{y^3}{x^3}} \text{ is given by}$$

(a) $\frac{3}{2} \log \left(\frac{y}{x} \right) + \log \left| \frac{x^{3/2} + y^{3/2}}{x^{3/2}} \right| + \tan^{-1} \left(\frac{y}{x} \right)^{3/2} + C = 0$

(b) $\frac{2}{3} \log \left(\frac{y}{x} \right) + \log \left| \frac{x^{3/2} + y^{3/2}}{x^{3/2}} \right| + \tan^{-1} \frac{y}{x} + C = 0$

(c) $\frac{2}{3} \log \left(\frac{y}{x} \right) + \log \left(\frac{x+y}{x} \right) + \tan^{-1} \left(\frac{y^{3/2}}{x^{3/2}} \right) + C = 0$

(d) None of the above

Sol. We have, $\frac{\sqrt{x} dx + \sqrt{y} dy}{\sqrt{x} dx - \sqrt{y} dy} = \sqrt{\frac{y^3}{x^3}}$

$$\Rightarrow \frac{d(x^{3/2}) + d(y^{3/2})}{d(x^{3/2}) - d(y^{3/2})} = \frac{y^{3/2}}{x^{3/2}} \Rightarrow \frac{du + dv}{du - dv} = \frac{v}{u}$$

where, $u = x^{3/2}$ and $v = y^{3/2}$

$$\Rightarrow u du + u dv = v du - v dv$$

$$\Rightarrow u du + v dv = v du - u dv$$

$$\Rightarrow \frac{u du + v dv}{u^2 + v^2} = \frac{v du - u dv}{u^2 + v^2}$$

$$\Rightarrow \frac{d(u^2 + v^2)}{u^2 + v^2} = -2d \tan^{-1} \left(\frac{v}{u} \right) + C$$

On integrating, we get

$$\log(u^2 + v^2) = -2 \tan^{-1} \left(\frac{v}{u} \right) + C$$

$$\Rightarrow \frac{1}{2} \log(x^3 + y^3) + \tan^{-1} \left(\frac{y}{x} \right)^{3/2} = \frac{C}{2}$$

Hence, (d) is the correct answer.

Ex 20. Solution of the differential equation

$$x = 1 + xy \frac{dy}{dx} + \frac{(xy)^2}{2!} \left(\frac{dy}{dx} \right)^2 + \frac{(xy)^3}{3!} \left(\frac{dy}{dx} \right)^3 + \dots \text{is}$$

(a) $y = \log_e(x) + C$

(b) $y = (\log_e x)^2 + C$

(c) $y = \pm \sqrt{(\log_e x)^2 + 2C}$

(d) $xy = x^y + k$

Sol. We have, $x = e^{xy \frac{dy}{dx}}$

$$\Rightarrow \log x = xy \frac{dy}{dx}$$

$$\Rightarrow y dy = \frac{\log x}{x} dx$$

$$\Rightarrow y dy = \log x d(\log x)$$

$$\text{On integrating, we get } \frac{y^2}{2} = \frac{(\log x)^2}{2} + C$$

$$\Rightarrow y^2 = (\log_e x)^2 + 2C$$

$$\Rightarrow y = \pm \sqrt{(\log_e x)^2 + 2C}$$

Hence, (c) is the correct answer.

Ex 21. The order of the differential equation satisfying $\sqrt{1-x^4} + \sqrt{1-y^4} = a(x^2 - y^2)$ is

(a) 1

(b) 2

(c) 3

(d) None of these

Sol. Put $x^2 = \sin \alpha$, $y^2 = \sin \beta$

$\therefore$ Given equation reduces to

$$\cos \alpha + \cos \beta = a(\sin \alpha - \sin \beta)$$

$$\Rightarrow 2 \cos \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)$$

$$= 2a \cos \left(\frac{\alpha + \beta}{2} \right) \cdot \sin \left(\frac{\alpha - \beta}{2} \right)$$

$$\Rightarrow \cot \left(\frac{\alpha - \beta}{2} \right) = a$$

$$\Rightarrow \alpha - \beta = 2 \cot^{-1} a$$

$$\Rightarrow \sin^{-1} x^2 - \sin^{-1} y^2 = 2 \cot^{-1} a$$

On differentiating with respect to x , we get

$$\frac{1}{\sqrt{1-x^4}} \cdot 2x - \frac{1}{\sqrt{1-y^4}} \cdot 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x}{y} \sqrt{\frac{1-y^4}{1-x^4}},$$

which is a differential equation of first order and first degree.

Hence, (a) is the correct answer.

Ex 22. If the solution of the differential equation $\frac{dy}{dx} = \frac{ax+3}{2y+f}$ represents a circle, then the value of a is
(a) 2 (b) -2 (c) 3 (d) -4

Sol. We have, $\frac{dy}{dx} = \frac{ax+3}{2y+f}$
 $\Rightarrow (ax+3) dx = (2y+f) dy$
On integrating, we get
 $a \frac{x^2}{2} + 3x = y^2 + f y + C$

$$\Rightarrow -\frac{a}{2}x^2 + y^2 - 3x + fy + C = 0$$

This will represent a circle, if

$$-\frac{a}{2} = [\because \text{Coeff. of } x^2 = \text{Coeff. of } y^2]$$

$$\text{and } \frac{9}{4} + f^2 - C > 0 \quad [\text{using } g^2 + f^2 - C > 0]$$

$$\Rightarrow a = -2$$

$$\text{and } 9 + 4f^2 - 4C > 0$$

Hence, (b) is the correct answer.

Ex 23. If $\int_a^x ty(t)dt = x^2 + y(x)$, then y as a function of x is

$$(a) y = 2 - (2 + a^2)e^{\frac{x^2 - a^2}{2}}$$

$$(b) y = 1 - (2 + a^2)e^{\frac{x^2 - a^2}{2}}$$

$$(c) y = 2 - (1 + a^2)e^{\frac{x^2 - a^2}{2}}$$

(d) None of the above

Sol. On differentiating both the sides, we get

$$xy(x) = 2x - y'(x)$$

$$\text{So, } \frac{dy}{dx} - xy = -2x \quad \left[\because y'(x) = \frac{dy}{dx}; y(x) = y \right]$$

$$\therefore \text{IF} = e^{\int -x dx} = e^{-\frac{x^2}{2}}$$

$$\Rightarrow ye^{\frac{-x^2}{2}} = \int -2xe^{\frac{-x^2}{2}} dx$$

$$\text{Let } \frac{x^2}{e^2} = t \Rightarrow -xe^{\frac{x^2}{2}} dx = dt$$

$$I = \int 2dt$$

$$\frac{x^2}{ye^2} = 2e^{\frac{x^2}{2}} + C$$

$$y = 2 + Ce^{\frac{x^2}{2}}$$

$$\text{If } x = a \Rightarrow a^2 + y = 0 \Rightarrow y = -a^2$$

[from the given equation]

$$\text{Thus, } -a^2 = 2 + Ce^{\frac{a^2}{2}}; Ce^{\frac{a^2}{2}} = -(2 + a^2)$$

$$C = -(2 + a^2)e^{-\frac{a^2}{2}}; y = 2 - (2 + a^2)e^{\frac{x^2 - a^2}{2}}$$

Hence, (a) is the correct answer.

Ex 24. A curve passes through the point (0, 1) and the gradient at (x, y) on it is $y(xy-1)$. The equation of the curve is

- (a) $y(x-1) = 1$ (b) $y(x+1) = 1$
(c) $x(y+1) = 1$ (d) $x(y-1) = 1$

Sol. We have, $\frac{dy}{dx} = y(xy-1)$

$$\Rightarrow dy = xy^2 dx - y dx$$

$$\Rightarrow y dx + dy = xy^2 dx$$

$$\Rightarrow \frac{y dx + dy}{y^2} = x dx$$

$$\Rightarrow \frac{ye^{-x} dx + e^{-x} dy}{y^2} = xe^{-x} dx$$

$$\Rightarrow -d\left(\frac{e^{-x}}{y}\right) = xe^{-x} dx$$

On integrating, we get

$$-\frac{e^{-x}}{y} = -xe^{-x} - e^{-x} - C$$

$$\Rightarrow \frac{1}{y} = x + 1 + Ce^x \quad \dots(i)$$

This passes through (0, 1). Therefore,

$$1 = 1 + C \Rightarrow C = 0$$

On putting $C = 0$ in Eq. (i), we get

$$\frac{1}{y} = x + 1 \Rightarrow y(x+1) = 1$$

Hence, (b) is the correct answer.

Ex 25. The equation of the curve passing through the origin and satisfying the differential equation $\left(\frac{dy}{dx}\right)^2 = (x-y)^2$, is

$$(a) e^{2x} (1-x+y) = 1+x-y$$

$$(b) e^{2x} (1+x-y) = 1-x+y$$

$$(c) e^{2x} (1-x+y) = 1+x+y$$

$$(d) e^{2x} (1+x+y) = 1-x+y$$

Sol. We have, $\frac{dy}{dx} = \pm (x-y)$

Case I When $\frac{dy}{dx} = (x-y)$

In this case, we have

$$1 - \frac{dv}{dx} = v, \text{ where } x - y = v$$

$$\Rightarrow \frac{dv}{dx} = 1 - v \Rightarrow \frac{1}{1-v} dv = dx$$

$$\Rightarrow -\log(1-v) = x + \log C$$

$$\Rightarrow (1-v)^{-1} = Ce^{-x}$$

$$\Rightarrow \frac{1}{1-x+y} = Ce^x$$

It passes through the origin.

$$\therefore C = 1$$

$$\text{Hence, } \frac{1}{1-x+y} = e^x \quad \dots(i)$$

Case II When $\frac{dy}{dx} = -(x - y) = y - x$

In this case, we have

$$\frac{du}{dx} + 1 = u, \text{ where } u = y - x$$

$$\Rightarrow \frac{du}{u-1} = dx$$

$$\Rightarrow \log(u-1) = x + \log C$$

$$\Rightarrow u-1 = Ce^x \Rightarrow y-x-1 = Ce^x$$

It passes through the origin.

$$\therefore C = -1$$

$$\text{Hence, } y-x-1 = -e^x$$

$$\Rightarrow x-y+1 = e^x \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{x-y+1}{1-x+y} = e^{2x}$$

$$\Rightarrow (x-y+1) = (1-x+y)e^{2x}$$

Hence, (a) is the correct answer.

Ex 26. A spherical rain drop evaporates at a rate proportional to its surface area at any instant t . The differential equation giving the rate of change of the radius of the rain drop, is

$$(a) \frac{d^2r}{dt^2} + 2r = 0 \quad (b) \frac{d^2r}{dt^2} - 3r = 0$$

$$(c) \frac{d^2r}{dt^2} = 0 \quad (d) \text{ None of these}$$

Sol. Volume of rain drop (if radius is r) is

$$V = \frac{4}{3}\pi r^3$$

$$\therefore \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \quad \dots(i)$$

$$\text{Given, } \frac{dV}{dt} \propto 4\pi r^2$$

$$\Rightarrow \frac{dV}{dt} = 4\pi k r^2 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$4\pi r^2 \frac{dr}{dt} = 4\pi k r^2 \Rightarrow \frac{dr}{dt} = k$$

On differentiating both sides w.r.t. t , we get

$$\frac{d^2r}{dt^2} = 0$$

Hence, (c) is the correct answer.

Ex 27. The differential equation of rectangular hyperbolas whose axes are asymptotes of the hyperbola, $x^2 - y^2 = a^2$, is

$$(a) y \frac{dy}{dx} = x \quad (b) x \frac{dy}{dx} = -y$$

$$(c) x \frac{dy}{dx} = y \quad (d) x dy + y dx = C$$

Sol. The equation of the rectangular hyperbola whose axes are asymptotes of the hyperbola $x^2 - y^2 = a^2$ is $xy = C^2$, where $C^2 = a^2/2$.

This is one parameter of family of curves.

On differentiating with respect to x , we get

$$x \frac{dy}{dx} + y = 0 \Rightarrow x \frac{dy}{dx} = -y$$

Hence, (b) is the correct answer.

Ex 28. If $\sin x$ is an integrating factor of the differential equation $\frac{dy}{dx} + Py = Q$, then P can

be

$$(a) \log \sin x \quad (b) \cot x$$

$$(c) \sin x \quad (d) \log \cos x$$

Sol. We have, IF = $\sin x$

$$\Rightarrow e^{\int P dx} = \sin x$$

$$\Rightarrow \int P dx = \log \sin x$$

$$\Rightarrow P = \frac{d}{dx} (\log \sin x) = \cot x$$

Hence, (b) is the correct answer.

Ex 29. If $y = y(x)$ and $\frac{2 + \sin x}{y+1} \left(\frac{dy}{dx} \right) = -\cos x$,

$y(0) = 1$, then $y(\pi/2)$ is equal to

$$(a) \frac{1}{3} \quad (b) \frac{2}{3}$$

$$(c) -\frac{1}{3} \quad (d) 1$$

Sol. We have, $\frac{2 + \sin x}{y+1} \left(\frac{dy}{dx} \right) = -\cos x$

$$\Rightarrow \frac{1}{y+1} dy = -\frac{\cos x}{2 + \sin x} dx$$

$$\Rightarrow \int \frac{1}{y+1} dy = -\int \frac{\cos x}{2 + \sin x} dx$$

$$\Rightarrow \log(y+1) = -\log(2 + \sin x) + \log C$$

$$\Rightarrow y+1 = \frac{C}{2 + \sin x} \quad \dots(i)$$

On putting $x = 0$ and $y = 1$ in Eq. (i), we get

$$y+1 = \frac{4}{2 + \sin x}$$

$$C = 4$$

On putting $x = \frac{\pi}{2}$ in Eq. (i), we get

$$y+1 = \frac{4}{3}$$

$$\Rightarrow y = \frac{1}{3}$$

$$\Rightarrow y\left(\frac{\pi}{2}\right) = \frac{1}{3}$$

Hence, (a) is the correct answer.

Ex 30. A function $y = f(x)$ has a second order derivative $f''(x) = 6(x-1)$. If its graph passes through the point (2, 1) and at that point the tangent to the graph is $y = 3x - 5$, then the function is

$$(a) (x+1)^2 \quad (b) (x-1)^3$$

$$(c) (x+1)^3 \quad (d) (x-1)^2$$

Sol. We have, $f''(x) = 6(x-1)$
 $\Rightarrow f'(x) = 3(x-1)^2 + C$ [on integrating] ... (i)
 It is given that, $y = 3x - 5$ is tangent to the curve $y = f(x)$ at the point $(2, 1)$.
 $\Rightarrow \left(\frac{dy}{dx}\right)_{(2, 1)} = (\text{Slope of the line } y = 3x - 5)$
 $\Rightarrow \left(\frac{dy}{dx}\right)_{(2, 1)} = 3$
 $\Rightarrow \{f'(x)\}_{(2, 1)} = 3 \Rightarrow f'(2) = 3$
 On putting $x = 2$, $f'(2) = 3$ in Eq. (i), we get $C = 0$
 $\therefore f'(x) = 3(x-1)^2$ [putting $C = 0$ in Eq. (i)]
 $\Rightarrow f(x) = (x-1)^3 + C_1$ [on integrating] ... (ii)
 The curve $y = f(x)$ passes through $(2, 1)$.
 $\therefore f(2) = 1$
 On putting $x = 2$, $f(2) = 1$, in Eq. (ii), we get
 $C_1 = 0$
 On putting $C_1 = 0$ in Eq. (ii), we get $f(x) = (x-1)^3$
 Hence, (b) is the correct answer.

Ex 31. The equation of the curve for which the tangent at $P(x, y)$ cuts the Y -axis at $(0, y^3)$ is
 (a) $xy^2 = x + y$ (b) $x^2(y^2 - 1) = Cy^2$
 (c) $y(x^2 - 1) = Cx^2$ (d) $yx + x^2 = C$

Sol. The equation of the tangent at $P(x, y)$ to the curve is
 $Y - y = \frac{dy}{dx}(X - x)$
 Since, it passes through $(0, y^3)$.
 $\Rightarrow y^3 - y = \frac{dy}{dx}(0 - x) \Rightarrow \frac{dy}{y^3 - y} = -\frac{dx}{x}$
 $\Rightarrow \frac{y^2 - 1}{y^2} = \frac{C}{x^2} \Rightarrow x^2(y^2 - 1) = Cy^2$
 Hence, (b) is the correct answer.

Ex 32. A tangent to the curve $y = f(x)$ cuts the line $y = x$ at a point which is at a distance of 1 unit from Y -axis. The equation of the curve is
 (a) $\frac{x-1}{y-1} = C$ (b) $\frac{x}{y} = C$
 (c) $xy = C$ (d) None of these

Sol. The equation of the tangent at $P(x, y)$ is
 $Y - y = \frac{dy}{dx}(X - x)$
 It cuts the line $Y = x$ at the point $\left(\frac{y-x\frac{dy}{dx}}{1-\frac{dy}{dx}}, \frac{y-x\frac{dy}{dx}}{1-\frac{dy}{dx}}\right)$
 $\Rightarrow \frac{y-x\frac{dy}{dx}}{1-\frac{dy}{dx}} = 1 \Rightarrow y - x\frac{dy}{dx} = 1 - \frac{dy}{dx}$
 $\Rightarrow (1-x)dy = (1-y)dx$
 $\Rightarrow \frac{dx}{x-1} = \frac{dy}{y-1} \Rightarrow (x-1) = C(y-1)$
 Hence, (a) is the correct answer.

Ex 33. If the slope of tangent to the curve is maximum at $x = 1$ and curve has a minimum value 1 at $x = 0$, then the curve which also satisfies the equation $y''' = 4x - 3$, is
 (a) $y + 2x + \frac{dy}{dx} = 0$
 (b) $y = 1 + \frac{x^2}{2} - \frac{x^3}{2} + \frac{x^4}{6}$
 (c) $y = 1 + x + x^2 + x^3$
 (d) None of the above

Sol. We have, $\frac{d^3y}{dx^3} = 4x - 3$
 On integrating both sides w.r.t. x , we get
 $\frac{d^2y}{dx^2} = 2x^2 - 3x + C$
 Given slope of tangent to the curve is maximum at $x = 1$, then $\left(\frac{d^2y}{dx^2}\right)_{x=1} = 0$
 $\therefore 0 = 2 - 3 + C$
 $\Rightarrow C = 1$
 $\Rightarrow \left(\frac{d^2y}{dx^2}\right) = 2x^2 - 3x + 1$
 On integrating both sides, we get
 $\frac{dy}{dx} = \frac{2x^3}{3} - \frac{3x^2}{2} + x + D$
 $\therefore$ Curve has a minimum value = 1 at $x = 0$.
 $\therefore \left(\frac{dy}{dx}\right)_{x=0} = 0$
 $\Rightarrow 0 = 0 + D \Rightarrow D = 0$
 Then, $\frac{dy}{dx} = \frac{2x^3}{3} - \frac{3x^2}{2} + x$
 Again integrating both sides, we get
 $y = \frac{x^4}{6} - \frac{x^3}{2} + \frac{x^2}{2} + E$
 Given minimum value of curve is 1 at $x = 0$, then
 $1 = 0 - 0 + 0 + E$
 $\therefore E = 1$
 Hence, curve is $y = \frac{x^4}{6} - \frac{x^3}{2} + \frac{x^2}{2} + 1$
 Hence, (b) is the correct answer.

Ex 34. The differential equation of the family of hyperbolas with asymptotes as the lines $x + y = 1$ and $x - y = 1$, is
 (a) $yy' + x = 0$ (b) $yy' = (x-1)$
 (c) $yy'' + y' = 0$ (d) $y' + xy = 0$

Sol. Solving the equations of the asymptotes, the centre is $x = 1$ and $y = 0$. Since, $e = \sqrt{2}$, then the equation of the family of the hyperbola is
 $\frac{(x-1)^2}{a^2} - \frac{(y-0)^2}{a^2} = 1 \Rightarrow 2(x-1) - 2y\frac{dy}{dx} = 0$
 $\Rightarrow (x-1) - y\frac{dy}{dx} = 0$ is the differential equation.
 Hence, (b) is the correct answer.

Ex 35. Solution of $\frac{x dx + y dy}{x dy - y dx} = \frac{\sqrt{1 - (x^2 + y^2)}}{\sqrt{x^2 + y^2}}$ is

- (a) $\sin^{-1} \sqrt{x^2 + y^2} = C$
 (b) $\tan^{-1} \frac{y}{x} = C$
 (c) $\sin^{-1} \sqrt{x^2 + y^2} = \tan^{-1} \frac{y}{x} = C$
 (d) None of the above

Sol. Put $x = r \cos \theta$, $y = r \sin \theta$

$$\begin{aligned} \therefore dx &= -r \sin \theta d\theta + dr \cos \theta \\ dy &= r \cos \theta d\theta + dr \sin \theta \\ \therefore \frac{r \cos \theta (-r \sin \theta d\theta + dr \cos \theta) + \sin \theta (r \cos \theta d\theta + dr \sin \theta)}{r \cos \theta (r \sin \theta d\theta + dr \sin \theta) - r \sin \theta (-r \sin \theta d\theta + dr \cos \theta)} &= \frac{\sqrt{1 - r^2}}{\sqrt{r^2}} \\ \Rightarrow \frac{\begin{bmatrix} -r^2 \sin \theta \cos \theta d\theta + r \cos^2 \theta dr \\ + r^2 \sin \theta \cos \theta d\theta + r \sin^2 \theta dr \end{bmatrix}}{\begin{bmatrix} r^2 \cos^2 \theta d\theta + r \cos \theta \sin \theta dr \\ + r^2 \sin^2 \theta d\theta - r \sin \theta \cos \theta dr \end{bmatrix}} &= \frac{\sqrt{1 - r^2}}{r} \\ \Rightarrow \frac{r dr}{r^2 d\theta} = \frac{\sqrt{1 - r^2}}{r} &\Rightarrow \frac{dr}{\sqrt{1 - r^2}} = d\theta \\ \Rightarrow \sin^{-1} r &= \theta + C \\ \Rightarrow \sin^{-1} \sqrt{x^2 + y^2} &= \tan^{-1} \frac{y}{x} + C \end{aligned}$$

Hence, (c) is the correct answer.

Ex 36. The solution of the differential equation

$$(e^{x^2} + e^{y^2})y \frac{dy}{dx} + e^{x^2} (xy^2 - x) = 0 \text{ is}$$

- (a) $e^{x^2} (y^2 - 1) + e^{y^2} = C$
 (b) $e^{y^2} (x^2 - 1) + e^{x^2} = C$
 (c) $e^{y^2} (y^2 - 1) + e^{x^2} = C$
 (d) $e^{x^2} (y - 1) + e^{y^2} = C$

Sol. Let $y^2 = t$; $2y \frac{dy}{dx} = \frac{dt}{dx}$

Hence, the differential equation becomes

$$\begin{aligned} (e^{x^2} + e^t) \frac{dt}{dx} + 2e^{x^2} (xt - x) &= 0 \\ e^{x^2} + e^t + 2e^{x^2} x(t - 1) \frac{dx}{dt} &= 0 \\ \text{Put } e^{x^2} &= z; e^{x^2} 2x \frac{dx}{dt} = \frac{dz}{dt} \\ \Rightarrow z + e^t + \frac{dz}{dt} (t - 1) &= 0 \\ \frac{dz}{dt} + \frac{z}{t - 1} &= -\frac{e^t}{(t - 1)} \\ \Rightarrow \text{IF} &= e^{\int \frac{dt}{t-1}} \\ &= e^{\ln(t-1)} = t - 1 \\ \Rightarrow z(t - 1) &= -\int (e^t) dt \end{aligned}$$

$$\begin{aligned} \Rightarrow e^{x^2} (y^2 - 1) &= -e^{y^2} + C \\ \Rightarrow e^{x^2} (y^2 - 1) + e^{y^2} &= C \end{aligned}$$

Hence, (a) is the correct answer.

Ex 37. The eccentricity of the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is

changed at the rate of 0.1/s. The time at which it will touch the auxiliary circle is

- (a) 2 s (b) 3 s
 (c) 5 s (d) 6 s

Sol. $\frac{de}{dt} = -0.1 \Rightarrow e = -0.1t + e_0$

Here, $e_0 = 1/2$

$$\Rightarrow e_t = -0.1t + \frac{1}{2}, \text{ given that } e_t = 0$$

$$\Rightarrow 0.1t = \frac{1}{2} \Rightarrow t = 5 \text{ s}$$

Hence, (c) is the correct answer.

Ex 38. An inverted conical tank of 2 m radius and 4m height is initially full of water, has an outlet at bottom. The outlet is opened at some instant. The rate of flow through the outlet at any time t is $6h^{3/2}$, when h is height of water level above the outlet at time t . Then, the time it takes to empty the tank, is

- (a) $\frac{2\pi}{9}$ unit (b) $\frac{\pi}{9}$ unit
 (c) $\frac{2\pi}{8}$ unit (d) None of these

Sol. Given that, $\frac{dv}{dt} = -6h^{3/2}$

At time t , the volume of water in the tank, $V = \frac{1}{3} \pi r^2 h$.

Here, $\triangle OAB$ and $\triangle OA'B'$ are similar.

$$\Rightarrow \frac{OA'}{A'B'} = \frac{OA}{AB}$$

$$\Rightarrow \frac{h}{r} = \frac{4}{2} \Rightarrow r = \frac{h}{2}$$

$$\Rightarrow V = \frac{1}{3} \pi \frac{h^3}{4}$$

$$\Rightarrow h^3 = \frac{12V}{\pi}$$

$$\Rightarrow \frac{dV}{dt} = \frac{-6\sqrt{12}}{\sqrt{\pi}} \sqrt{V}$$

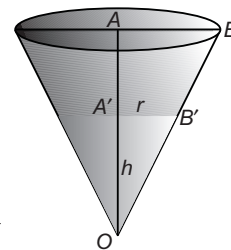
$$\Rightarrow \int_{V_0}^0 \frac{dV}{\sqrt{V}} = \frac{-6\sqrt{12}}{\sqrt{\pi}} \int_0^t dt$$

$$\Rightarrow (2\sqrt{V})_{V_0}^0 = \frac{-6\sqrt{12}}{\sqrt{\pi}} t$$

$$\Rightarrow \frac{2\sqrt{V_0} \sqrt{\pi}}{6\sqrt{12}} = t = \frac{2\sqrt{\pi}}{6\sqrt{12}} \sqrt{\frac{16}{3} \pi}$$

$$\therefore t = \frac{2\pi}{9} \text{ unit}$$

Hence, (a) is the correct answer.



Ex 39. The sum of squares of the perpendiculars drawn from the points $(0, 1)$ and $(0, -1)$ to any tangent to a curve is 2. Then, the equation of the curve is

- (a) $2y' = c(x+2)$ (b) $y = c'(x+1)$
(c) $2y = c'(x+2)$ (d) $yy' = c(x+1)$

Sol. Equation of tangent to the curve at (x, y) is

$$Y - y = dy/dx (X - x) \quad \dots(i)$$

The length of perpendicular from $(0, 1)$ to the curve is

$$D_1 = \left| \frac{1 + xy' - y}{\sqrt{1 + (y')^2}} \right| \quad \dots(ii)$$

where, $y' = \frac{dy}{dx}$

Then, length of perpendicular from $(0, -1)$ to the curve is

$$D_2 = \left| \frac{-1 + xy' - y}{\sqrt{1 + (y')^2}} \right| \quad \dots(iii)$$

$$\begin{aligned} \Rightarrow D_1^2 + D_2^2 &= 2 \\ \Rightarrow 2 + 2x^2 y'^2 + 2y^2 - 4xyy' &= 2(1 + y'^2) \\ \Rightarrow x^2 y'^2 + y^2 - 2xyy' &= y'^2 \\ \Rightarrow \left(\frac{dy}{dx} \right)^2 (x^2 - 1) - 2xy \left(\frac{dy}{dx} \right) + y^2 &= 0 \\ \Rightarrow \frac{dy}{dx} &= \frac{2xy \pm \sqrt{4y^2}}{2(x^2 - 1)} = \frac{xy \pm y}{(x^2 - 1)} \\ \Rightarrow \frac{dy}{y} &= \frac{(x \pm 1) dx}{x^2 - 1} \end{aligned}$$

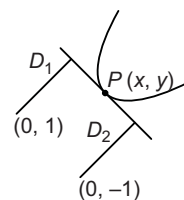
Case I Taking positive sign,

$$\frac{dy}{y} = \frac{dx}{x-1} \Rightarrow y = c(x-1)$$

Case II Taking negative sign,

$$\frac{dy}{y} = \frac{dx}{x+1} \Rightarrow y = c'(x+1)$$

Hence, (b) is the correct answer.



Type 2. More than One Correct Option

Ex 40. Family of curves whose tangent at a point with its intersection with the curve $xy = c^2$ form an angle of $\frac{\pi}{4}$, is

- (a) $y^2 - 2xy - x^2 = k$
(b) $y^2 + 2xy - x^2 = k$
(c) $y = x - 2c \tan^{-1} \left(\frac{x}{c} \right) + k$
(d) $y = c \ln \left| \frac{c+x}{c-x} \right| - x + k$

Sol. Let $m = \frac{dy}{dx}$ be the slope of tangent at (x, y) to the required curve.

$m_1 =$ Slope of the tangent at the curve $xy = c^2$

$$= -\frac{c^2}{x^2} \text{ or } -\frac{y}{x}$$

$$\text{Hence, } \frac{m + \frac{c^2}{x^2}}{1 - \frac{c^2}{x^2} m} = \pm 1 \text{ or } \frac{m + \frac{y}{x}}{1 - \frac{y}{x} m} = \pm 1$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 41. The general solution of the differential equation, $x \left(\frac{dy}{dx} \right) = y \cdot \ln \left(\frac{y}{x} \right)$ is

- (a) $y = xe^{1-Cx}$
(b) $y = xe^{1+Cx}$
(c) $y = ex \cdot e^{Cx}$
(d) $y = xe^{Cx}$

where, C is an arbitrary constant.

Sol. $x \frac{dy}{dx} = y \log \left(\frac{y}{x} \right)$

$$\frac{dy}{dx} = \frac{y}{x} \log \left(\frac{y}{x} \right), \text{ put } y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore v + \frac{xdv}{dx} = v \log v$$

$$\Rightarrow x \frac{dv}{dx} = v(\log v - 1)$$

$$\Rightarrow \int \frac{dv}{v(\log v - 1)} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dt}{t-1} = \int \frac{dx}{x}, \text{ let } \log v = t$$

$$\Rightarrow \log(t-1) = \log(x) + \log C$$

$$\Rightarrow \log(t-1) = \log(xC)$$

$$\Rightarrow t-1 = xC$$

$$\Rightarrow \log \frac{y}{x} = 1 + xC$$

$$\Rightarrow y = x \cdot e^{1+Cx}$$

$$\text{or } y = x \cdot e^{1-Cx}$$

$$\text{and } \log \frac{y}{x} = \log e + Cx$$

$$\Rightarrow y = ex \cdot e^{Cx}$$

Hence, (a), (b) and (c) are the correct answers.

Ex 42. Which of the following equation(s) is/are linear?

- (a) $\frac{dy}{dx} + \frac{y}{x} = \ln x$ (b) $y \left(\frac{dy}{dx} \right) + 4x = 0$

(c) $dx + dy = 0$ (d) $\frac{d^2 y}{dx^2} = \cos x$

Sol. Clearly, options (a) and (c) are of the form

$\frac{dy}{dx} + Py = Q$, which is linear in y .

Also, option (d) is $\frac{d^2 y}{dx^2} = \cos x$

On integrating Eq. (i), we get

$\frac{dy}{dx} = \sin x + C$

which is also linear in y .

Hence, (a), (c) and (d) are the correct answers.

Ex 43. The function $f(x)$ satisfying the equation.

$f^2(x) + 4f'(x) \cdot f(x) + [f'(x)]^2 = 0$

(a) $f(x) = C \cdot e^{(2-\sqrt{3})x}$ (b) $f(x) = C \cdot e^{(2+\sqrt{3})x}$
(c) $f(x) = C \cdot e^{(\sqrt{3}-2)x}$ (d) $f(x) = C \cdot e^{-(2+\sqrt{3})x}$

Sol. $(f'(x))^2 + 4f(x) \cdot (f'(x)) + f^2(x) = 0$

$f'(x) = (-2 \pm \sqrt{3}) f(x)$

or $\frac{dy}{dx} = (-2 \pm \sqrt{3}) y$

On integrating both sides, we get

$\log y = (-2 \pm \sqrt{3})x + C$

$\Rightarrow y = e^{(-2 \pm \sqrt{3})x} \cdot C$

Hence, (c) and (d) are the correct answers.

Ex 44. The equation of the curve passing through (3, 4) and satisfying the differential equation

$y \left(\frac{dy}{dx} \right)^2 + (x - y) \frac{dy}{dx} - x = 0$ can be

(a) $x - y + 1 = 0$
(b) $x^2 + y^2 = 25$
(c) $x^2 + y^2 - 5x - 10 = 0$
(d) $x + y - 7 = 0$

Sol. $\frac{dy}{dx} = \frac{(y-x) \pm \sqrt{(x-y)^2 + 4xy}}{2y}$

$\Rightarrow \frac{dy}{dx} = 1$

or $\frac{dy}{dx} = -\frac{x}{y}$

$\Rightarrow x - y + 1 = 0$

and $x^2 + y^2 = 25$

Hence, (a) and (b) are the correct answers.

Ex 45. The orthogonal trajectories of the system of

curves $\left(\frac{dy}{dx} \right)^2 = a/x$ are

(a) $9a(y + C)^2 = 4x^3$
(b) $y + C = \frac{-2}{3\sqrt{a}} x^{3/2}$

(c) $y + C = \frac{2}{3\sqrt{a}} x^{3/2}$

(d) None of the above

Sol. Replacing $\left(\frac{dy}{dx} \right)$ by $(-dx/dy)$, we get

$\left(\frac{dy}{dx} \right)^2 = x/a$

$\Rightarrow dy = \pm (x^{1/2}/a^{1/2}) dx$

On integrating both sides, we get

$y + C = \pm \frac{2}{3a^{1/2}} x^{3/2}$

$\Rightarrow 9a(y + C)^2 = 4x^3$

Hence, (a), (b) and (c) are the correct answers.

Ex 46. The solution of $\frac{xdx + ydy}{xdy - ydx} = \frac{a^2 - x^2 - y^2}{x^2 + y^2}$ is

(a) $\sqrt{x^2 + y^2} = a(\sin(\tan^{-1} y/x + C))$

(b) $\sqrt{x^2 + y^2} = a(\cos(\tan^{-1} y/x + C))$

(c) $\sqrt{x^2 + y^2} = a(\tan(\sin^{-1} y/x + C))$

(d) $y = x \tan \left(C + \sin^{-1} \frac{1}{a} \sqrt{x^2 + y^2} \right)$

Sol. Taking, $x = r \cos \theta$ and $y = r \sin \theta$, so that $x^2 + y^2 = r^2$ and $y/x = \tan \theta$, we have $xdx + ydy = r dr$ and $x dy - y dx = x^2 \sec^2 \theta d\theta = r^2 d\theta$. The given equation can be transformed into

$\frac{rdr}{r^2 d\theta} = \frac{a^2 - r^2}{r^2} \Rightarrow \frac{dr}{d\theta} = \sqrt{a^2 - r^2}$

$\Rightarrow C + \sin^{-1} r/a = \theta = \tan^{-1} y/x$

$\Rightarrow y = x \tan \left(C + \sin^{-1} \frac{1}{a} \sqrt{x^2 + y^2} \right)$

or $\sqrt{x^2 + y^2} = a \sin(C + \tan^{-1} y/x)$

Hence, (a) and (d) are the correct answers.

Ex 47. The solution of $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$ satisfying

$y(1) = 1$ is given by

(a) a system of hyperbola

(b) a system of circles

(c) $y^2 = x(1+x) - 1$

(d) $(x-2)^2 + (y-3)^2 = 5$

Sol. Rewriting, the given equation as

$2xy \frac{dy}{dx} - y^2 = 1 + x^2$

$\Rightarrow 2y \frac{dy}{dx} - \frac{1}{x} y^2 = \frac{1}{x} + x$

Putting $y^2 = u$,

we have $\frac{du}{dx} - \frac{1}{x} u = \frac{1}{x} + x$

The IF of this equation is $1/x$, so

$u \cdot \frac{1}{x} = \int \left(\frac{1}{x^2} + 1 \right) dx = -\frac{1}{x} + x + C$

$\Rightarrow y^2 = (x^2 - 1) + Cx$
 Since, $y(1) = 1$, so $C = 1$, hence $y^2 = x(1 + x) - 1$
 which represents a system of hyperbola.

Type 3. Assertion and Reason

Directions (Ex. Nos. 48-52) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation of Statement I
- (b) Statement I is true, Statement II is true; Statement II is not the correct explanation of Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Ex 48. Statement I The order of the differential equation whose general solution is

$$y = c_1 \cos 2x + c_2 \sin^2 x + c_3 \cos^2 x + c_4 e^{2x} + c_5 e^{2x+c_6} + c_6 e^{2x+c_6}, \text{ is } 3.$$

Statement II Total number of arbitrary parameters in the given general solution in the Statement I is 6.

Sol. $y = c_1 \cos 2x + c_2 \sin^2 x + c_3 \cos^2 x + c_4 e^{2x} + c_5 e^{2x+c_6} + c_6 e^{2x+c_6}$
 $= c_1 \cos 2x + c_2 \left[\frac{1 - \cos 2x}{2} \right] + c_3 \left[\frac{\cos 2x + 1}{2} \right] + c_4 e^{2x} + c_5 e^{2x} \cdot e^{c_6}$
 $= \left(c_1 - \frac{c_2}{2} + \frac{c_3}{2} \right) \cos 2x + \left(\frac{c_2}{2} + \frac{c_3}{2} \right) + (c_4 + c_5 e^{c_6}) e^{2x}$
 $= \lambda_1 \cos 2x + \lambda_2 e^{2x} + \lambda_3$

So, total number of independent parameters in the given general solution is 3.

Hence, (c) is the correct answer.

Ex 49. Consider the differential equation

$$(x^2 + 1) \cdot \frac{d^2 y}{dx^2} = 2x \cdot \frac{dy}{dx}$$

Statement I For any member of this family $y \rightarrow \infty$ as $x \rightarrow \infty$.

Statement II Any solution of this differential equation is a polynomial of odd degree with positive coefficient of maximum power.

Sol. The given differential equation is

$$\frac{d \left(\frac{dy}{dx} \right)}{\frac{dy}{dx}} = \frac{2x}{x^2 + 1} dx$$

$$\Rightarrow \ln \left(\frac{dy}{dx} \right) = \ln (x^2 + 1) + \ln C, C > 0$$

$$\Rightarrow \frac{dy}{dx} = C(x^2 + 1)$$

Hence, (a) and (c) are the correct answers.

$$\Rightarrow y = C \left(\frac{x^3}{3} + x \right) + C', C' \in R$$

Obviously $y \rightarrow \infty$, as $x \rightarrow \infty$, as $C > 0$.

Hence, (a) is the correct answer.

Ex 50. Statement I Order of differential equation of family of parabolas whose axis is parallel to Y-axis and latusrectum is fixed, is 2.

Statement II Order of differential equation is same as actual number of arbitrary constant present in given curve.

Sol. $(x - h)^2 = 4b(y - k)$, here $b = 1/2$ is constant and h, k are parameters.

Hence, (a) is the correct answer.

Ex 51. Statement I The differential equation of all non-vertical lines in a plane is $\frac{d^2 x}{dy^2} = 0$.

Statement II The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

Sol. The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

Now, $ax + by = 1$

$$\Rightarrow a + b \frac{dy}{dx} = 0 \quad [\text{differentiating w.r.t. } x]$$

$$\Rightarrow b \frac{d^2 y}{dx^2} = 0 \quad [\text{differentiating w.r.t. } x]$$

$$\Rightarrow \frac{d^2 y}{dx^2} = 0 \quad [\because b \neq 0]$$

So, the differential equation is $\frac{d^2 y}{dx^2} = 0$.

Hence, (d) is the correct answer.

Ex 52. Statement I The order of differential equation of all conics whose centre lies at origin, is 2.

Statement II The order of differential equation is same as number of arbitrary unknowns in the given curve.

Sol. The equation $ax^2 + 2hxy + by^2 = 1$ represents the family of all conics whose centre lies at the origin for different values of a, h, b .

$\therefore$ Order is 3.

So, Statement I is false and Statement II is true.

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 53-55) Let $y = f(x)$ and $y = g(x)$ be the pair of curves such that the tangents at point with equal abscissae intersect on Y -axis, the normals drawn at points with equal abscissae intersect on X -axis. One curve passes through $(1, 1)$ and the other passes through $(2, 3)$.

Ex 53. The curve $f(x)$ is given by

- (a) $\frac{2}{x} - x$ (b) $2x^2 - \frac{1}{x}$
(c) $\frac{2}{x^2} - x$ (d) None of these

Ex 54. The curve $g(x)$ is given by

- (a) $x - \frac{1}{x}$ (b) $x + \frac{2}{x}$
(c) $x^2 - \frac{1}{x^2}$ (d) None of these

Ex 55. The number of positive integer solutions for $f(x) = g(x)$ are

- (a) 4 (b) 5
(c) 6 (d) None of these

Sol. (Ex. Nos. 53-55) Let $y = f(x)$ and $y = g(x)$ be the required curves. The equations of the tangents to these two curves at points with equal abscissae x , are

$$Y - f(x) = f'(x)(X - x)$$

and

$$Y - g(x) = g'(x)(X - x)$$

These two lines intersect on Y -axis.

$$\therefore Y - f(x) = -xf'(x), Y - g(x) = -xg'(x)$$

$$\Rightarrow Y = f(x) - xf'(x) = g(x) - xg'(x)$$

$$\Rightarrow f(x) - g(x) = x\{f'(x) - g'(x)\}$$

$$\Rightarrow f(x) - g(x) = x \frac{d}{dx} \{f(x) - g(x)\}$$

$$\Rightarrow \frac{d\{f(x) - g(x)\}}{f(x) - g(x)} = \frac{dx}{x}$$

On integrating both sides, we get

$$\log \{f(x) - g(x)\} = \log x + \log C$$

$$f(x) - g(x) = Cx \quad \dots(i)$$

The equation of the normals at points with equal abscissae x to the two curves are

$$Y - f(x) = -\frac{1}{f'(x)}(X - x)$$

$$\text{and } Y - g(x) = -\frac{1}{g'(x)}(X - x)$$

These intersect on X -axis.

$$\therefore 0 - f(x) = -\frac{1}{f'(x)}(X - x)$$

$$\text{and } 0 - g(x) = -\frac{1}{g'(x)}(X - x)$$

$$\Rightarrow X = x + f(x) f'(x)$$

$$\text{and } X = x + g(x) g'(x)$$

$$\Rightarrow x + f(x) f'(x) = x + g(x) g'(x)$$

$$\Rightarrow f(x) f'(x) = g(x) g'(x)$$

On integrating both sides, we get

$$\{f(x)\}^2 - \{g(x)\}^2 = C_1$$

$$\Rightarrow \{f(x) + g(x)\} Cx = C_1$$

$$\Rightarrow f(x) + g(x) = \frac{C_1}{Cx} \quad \dots(ii)$$

Solving Eqs. (i) and (ii), we get

$$f(x) = \frac{1}{2} \left(Cx + \frac{C_1}{Cx} \right) \quad \dots(iii)$$

$$\text{and } g(x) = \frac{1}{2} \left(\frac{C_1}{Cx} - Cx \right) \quad \dots(iv)$$

Since, Eq. (iii) passes through $(1, 1)$ and Eq. (iv) passes through $(2, 3)$.

$$\therefore 2 = C + \frac{C_1}{C} \text{ and } 6 = \frac{C_1}{2C} - 2C$$

$$\Rightarrow C = -2 \text{ and } C_1 = -8$$

$$\therefore f(x) = -x + \frac{2}{x} \text{ and } g(x) = \frac{2}{x} + x$$

For number of positive integral solutions for $f(x) = g(x)$,

$$-x + \frac{2}{x} = \frac{2}{x} + x$$

$\therefore x = 0$, i.e. no solution

53. (a) 54. (b) 55. (d)

Passage II (Ex. Nos. 56-58) Let $f(x)$ be non-positive continuous function and $F(x) = \int_0^x f(t) dt$, $\forall x \geq 0$ and $f(x) \geq cF(x)$, where $c > 0$ and let $g: [0, \infty) \rightarrow R$ be a function such that

$$\frac{dg(x)}{dx} < g(x), \forall x > 0 \text{ and } g(0) = 0.$$

Ex 56. The total number of roots of the equation $f(x) = g(x)$ is/are

- (a) ∞ (b) 1
(c) 2 (d) 0

Sol. $f(x) \leq 0$ and $F'(x) = f(x)$

$$f(x) \geq cF(x)$$

$$\Rightarrow F'(x) - cF(x) \geq 0$$

$$\Rightarrow e^{-cx} F'(x) - ce^{-cx} F(x) \geq 0$$

$$\Rightarrow \frac{d}{dx} (e^{-cx} F(x)) \geq 0$$

$\Rightarrow e^{-cx} F(x)$ is an increasing function.

$$\Rightarrow e^{-cx} F(x) \geq e^{-c(0)} F(0) \Rightarrow e^{-cx} F(x) \geq 0$$

$$\Rightarrow F(x) \geq 0 \Rightarrow f(x) \geq 0$$

$$[\because f(x) \geq cF(x) \text{ and } c \text{ is positive}]$$

$$\therefore f(x) = 0$$

$$\text{Also, } \left(\frac{dg(x)}{dx} \right) < g(x), \forall x > 0$$

$$\Rightarrow e^{-x} \frac{d[g(x)]}{dx} - e^{-x} g(x) < 0 \Rightarrow \frac{d}{dx} [e^{-x} g(x)] < 0$$

$\Rightarrow e^{-x} g(x)$ is decreasing function.

$$\Rightarrow e^{-x} g(x) < e^{-(0)} g(0)$$

$$\Rightarrow g(x) < 0 \quad [\because g(0) = 0]$$

Thus, $f(x) = g(x)$ has one solution $x = 0$.

Hence, (b) is the correct answer.

Ex 57. The number of solutions of the equation $|x^2 + x - 6| = f(x) + g(x)$ is/are

- (a) 2 (b) 1
(c) 0 (d) 3

Sol. We have, $|x^2 + x - 6| = f(x) + g(x)$
 $\Rightarrow |x^2 + x - 6| = g(x)$

$\therefore$ No solution

Hence, (c) is the correct answer.

Ex 58. The solution set of inequation $g(x) (\cos^{-1} x - \sin^{-1} x) \leq 0$ is

- (a) $\left[-1, \frac{1}{\sqrt{2}}\right]$ (b) $\left[\frac{1}{\sqrt{2}}, 1\right]$
(c) $\left[0, \frac{1}{\sqrt{2}}\right]$ (d) $\left(0, \frac{1}{\sqrt{2}}\right)$

Sol. We have, $g(x) (\cos^{-1} x - \sin^{-1} x) \leq 0$
 $\Rightarrow (\cos^{-1} x - \sin^{-1} x) \geq 0 \Rightarrow x \in \left[-1, \frac{1}{\sqrt{2}}\right]$

Hence, (a) is the correct answer.

Passage III (Ex. Nos. 59-61) For certain curves $y = f(x)$ satisfying $\frac{d^2 y}{dx^2} = 6x - 4$, $f(x)$ has local minimum value 5 when $x = 1$.

Ex 59. Number of critical points for $y = f(x)$ for $x \in [0, 2]$ is

- (a) 0 (b) 1
(c) 2 (d) 3

Ex 60. Global minimum value of $y = f(x)$ for $x \in [0, 2]$ is

- (a) 5 (b) 7
(c) 8 (d) 9

Ex 61. Global maximum value of $y = f(x)$ for $x \in [0, 2]$ is

- (a) 5 (b) 7
(c) 8 (d) 9

Sol. (Ex. Nos. 59-61)

On integrating $\frac{d^2 y}{dx^2} = 6x - 4$, we get

$$\frac{dy}{dx} = 3x^2 - 4x + A$$

When $x = 1$, $\frac{dy}{dx} = 0$, so that $A = 1$

Hence, $\frac{dy}{dx} = 3x^2 - 4x + 1$... (i)

On integrating, we get $y = x^3 - 2x^2 + x + B$

When $x = 1$, $y = 5$, so that $B = 5$

Thus, we have $y = x^3 - 2x^2 + x + 5$

From Eq. (i), we get the critical points $x = \frac{1}{3}$, $x = 1$.

At the critical point $x = \frac{1}{3}$, $\frac{d^2 y}{dx^2}$ is negative.

Therefore, at $x = \frac{1}{3}$, y has a local maximum.

At $x = 1$, $\frac{d^2 y}{dx^2}$ is positive.

Therefore, at $x = 1$, y has a local minimum.

Also, $f(1) = 5$, $f\left(\frac{1}{3}\right) = \frac{139}{27}$, $f(0) = 5$, $f(2) = 7$

Hence, the global maximum value is 7 and the global minimum value is 5.

59. (c) 60. (a) 61. (b)

Passage IV (Ex. Nos. 62-64) A chemical manufacturing company has 1000 kL holding tank which it uses to control the release of pollutants into a sewage system. Initially, the tank has 360 kL of water containing 2 kg of pollutant per kL. Water containing 3 kg of pollutant per kL enters the tank at the rate 80 kL/h and is uniformly mixed with the water already in the tank. Simultaneously water is released from the tank at the rate of 40 kL/h. If $P(t)$ is the amount of pollutant (in kg) in the tank at time t .

Ex 62. The value of $P(10)$ is approximately

- (a) 2000
(b) 3100
(c) 2109
(d) 4321

Ex 63. The value of $\int_0^1 P(t) dt$ is

- (a) $1140 - \log 5/3$
(b) $1140 - 3240 \log \frac{10}{9}$
(c) $2140 - 1140 \log \frac{8}{9}$
(d) $2140 - 3240 \log \frac{10}{9}$

Ex 64. $\frac{dP}{dt}$ at $t = 10$ is approximately equal to

- (a) 158 (b) 149 (c) 129 (d) 212

Sol. (Ex. Nos. 62-64) The rate of change of pollutant in the tank $\frac{dP}{dt}$ is given by

$$\frac{dP}{dt} = (\text{rate in}) - (\text{rate out})$$

The pollutant is entering the tank at the rate of 3×80 kg/h (= rate in). Water is entering the tank at the rate of 80 kL/h and is leaving at the rate of 40 kL/h. Therefore, amount of water at time t is $(360 + 40t)$ kL. Hence, amount of pollutant at time t is $\frac{P(t)}{360 + 40t}$ and the rate at which pollutant is

leaving the tank is $40 \frac{P(t)}{360 + 40t} = \frac{P(t)}{9 + t}$.

Thus, we have $\frac{dP}{dt} = 240 - \frac{P}{9+t}$
 $\Rightarrow \frac{dP}{dt} + \frac{P}{9+t} = 240$, which is a linear differential

equation whose IF = $e^{\int \frac{dt}{9+t}} = e^{\log(9+t)} = 9+t$

Thus, $\frac{d}{dt} \{P(9+t)\} = 240(9+t)$

$$\Rightarrow P(9+t) = \frac{240}{2}(9+t)^2 + C$$

$$\Rightarrow P = 120(9+t) + \frac{C}{9+t}$$

Since, the amount of pollutant at $t = 0$ is 2×360 kg.

$$\therefore P(0) = 120 \times 9 + \frac{C}{9} = 720$$

$$\Rightarrow C = -3240$$

$$\text{So, } P(t) = 120(9+t) - \frac{3240}{9+t}$$

$$\text{Now, } P(10) = 120 \times 19 - \frac{3240}{19}$$

$$= \frac{40080}{19} \text{ kg} \approx 2109 \text{ kg}$$

$$\therefore \int_0^1 P(t) dt = 120 \left(9 + \frac{1}{2} \right) - 3240 (\log 10 / 9)$$

62. (c) 63. (b) 64. (c)

Passage V (Ex. Nos. 65-67) A rumour spreads through a population of 5000 people at a rate proportional to the product of the number of people who have heard it and the number who have not. Suppose that 100 people initiate the rumour and that a total of 500 people know the rumour after two days.

Let $y(t)$ denotes the number of people who know the rumour at time t . Take $\log 9 / \log 49 = 129 / 229$.

Ex 65. The time required for half the population to hear rumour is

- (a) 4 days
- (b) 6.2 days
- (c) 4.58 days
- (d) 3.32 days

Ex 66. $\int_0^1 y(t) dt$ is equal to

- (a) $5000 \log \left(\frac{49 + e^{5000k}}{49} \right)$
- (b) $5000 \log (1 + e^{5000k})$
- (c) $5000 \log (1 + e^{500k})$
- (d) $5000 \log (1 - e^{-5000k})$

Ex 67. The function $y(t)$ is given by

- (a) $\frac{5000}{1 + 50e^{-5000kt}}$
- (b) $\frac{5000}{1 + 49e^{-5000kt}}$
- (c) $\frac{5000}{1 - 50e^{-5000kt}}$
- (d) $\frac{500}{1 + 49e^{-5000kt}}$

Sol. (Ex. Nos. 65-67) Let $y(t)$ denotes the number of people who know the rumour at time t . Maximum value of $y(t)$ is 5000, $y(0) = 100$ and $y(2) = 500$. Also,

$$\frac{dy}{dt} \propto y(5000 - y)$$

$$\Rightarrow \frac{dy}{dt} = ky(5000 - y)$$

Separating the variables and integrating, we get

$$\int \frac{dy}{y(5000 - y)} = kt + C$$

$$\Rightarrow \frac{1}{5000} \left[\int \frac{1}{y} dy + \int \frac{dy}{5000 - y} \right] = kt + C$$

$$\Rightarrow \frac{1}{5000} \log \frac{y}{5000 - y} = kt + C$$

$$\Rightarrow \frac{y}{5000 - y} = C e^{5000kt} = k e^{5000kt}$$

$$\Rightarrow y(1 + k e^{5000kt}) = 5000 k e^{5000kt} \quad \dots(i)$$

Put $y(0) = 100$, we have $100(1 + k) = 5000k$

$$\Rightarrow k = 1/49$$

Using $y(2) = 500$, we have

$$500(1 + k e^{10000kt}) = 5000 k e^{10000kt}$$

$$\Rightarrow 1 = 9 k e^{10000kt}$$

$$\Rightarrow e^{10000kt} = \frac{49}{9}$$

$$\Rightarrow k = \frac{1}{10000} \log \frac{49}{9}$$

Therefore, from Eq. (i),

$$y = \frac{5000/49}{e^{-5000kt} + 1/49}$$

$$= \frac{5000}{49 e^{-5000kt} + 1}$$

To determine how long it will take for half the population to hear rumour, we solve

$$2500 = \frac{5000}{1 + 49e^{-5000kt}}$$

$$\Rightarrow 1 + 49 e^{-5000kt} = 2$$

$$\Rightarrow e^{-5000kt} = \frac{1}{49}$$

$$\Rightarrow 5000kt = \log 49$$

$$\Rightarrow t = \frac{\log 49}{5000k}$$

$$\Rightarrow t = \frac{2 \log 49}{\log 49 - \log 9}$$

$$= \frac{2}{1 - \frac{\log 9}{\log 49}} = \frac{2}{1 - \frac{129}{229}}$$

$$= 4.58 \text{ days}$$

$$\therefore \int_0^1 y(t) dt = 5000 \int_0^1 \frac{e^{5000kt}}{49 + e^{5000kt}} dt$$

$$= 5000 \log \left(\frac{49 + e^{5000k}}{49} \right)$$

65. (c) 66. (a) 67. (b)

Type 5. Match the Columns

Ex 68. Match the statements of Column I with values of Column II.

Column I	Column II
A. The solution of $(\sin x + \cos x) dy + (\cos x - \sin x) dx = 0$ is	p. $\sec y + \tan y = C$ ($\operatorname{cosec} x + \cot x$)
B. The solution of $\sin x dy + \cos y dx = 0$ is	q. $2(x^2 + y^2) + 2(x \sin 2x + y \sin 2y) + (\cos 2x + \cos 2y) = C$
C. The solution of $x^{-1} \cos^2 y dy + y^{-1} \cos^2 x dx = 0$ is	r. $y = C - \log \sin x + \cos x $
D. The solution of $\sin y \cos y dx + \sin x \cos x dy = 0$ is	s. $\tan x \tan y = C$

Sol. A. The equation $(\sin x + \cos x) dy + (\cos x - \sin x) dx = 0$ can be written as $dy + \frac{\cos x - \sin x}{\sin x + \cos x} dx = 0$
 $\Rightarrow y + \log |\sin x + \cos x| = C$
 B. The equation $\sin x dy + \cos y dx = 0$ can be written as $\sec y dy + \operatorname{cosec} x dx = 0$
 $\Rightarrow \log |\sec y + \tan y| - \log |\operatorname{cosec} x + \cot x| = C$
 $\Rightarrow \frac{\sec y + \tan y}{\operatorname{cosec} x + \cot x} = C$
 C. The equation $x^{-1} \cos^2 y dy + y^{-1} \cos^2 x dx = 0$ can be written as
 $y \left(\frac{1 + \cos 2y}{2} \right) dy + x \left(\frac{1 + \cos 2x}{2} \right) dx = 0$
 On integrating by parts, we get
 $2(y^2 + x^2) + 2y \sin 2y + \cos 2y + 2x \sin 2x + \cos 2x = C$
 D. The equation $\sin y \cos y dx + \sin x \cos x dy = 0$ can be written as
 $2 \operatorname{cosec} 2x dx + 2 \operatorname{cosec} 2y dy = 0$
 $\Rightarrow \frac{2 \log \tan x}{2} + \frac{2 \log \tan y}{2} = \log C$
 $\Rightarrow \log \tan x \tan y = \log C$
 $\Rightarrow \tan x \tan y = C$
 $A \rightarrow r; B \rightarrow p; C \rightarrow q; D \rightarrow s$

Ex 69. Match the statements of Column I with values of Column II.

Column I	Column II
A. $y = e^x (A \cos x + B \sin x)$ is of	p. order 1
B. $xy = ae^x + be^{-x} + x^2$ is of	q. degree 2
C. All circles of radius a is of	r. degree 1
D. All circles of fixed radius r with centre on Y-axis is of	s. order 2

Sol. A. For y in $y = e^x (A \cos x + B \sin x)$,
 $y_1 = e^x (-A \sin x + B \cos x) + y$

$$\Rightarrow y_2 = -e^x (A \cos x + B \sin x) + e^x (-A \sin x + B \cos x) + y_1$$

So, $y_2 = -y + (y_1 - y) + y_1$
 $\Rightarrow y_2 - 2y_1 + 2y = 0$, which is of degree 1 and order 2. Note that it is not necessary to find the differential equation for determining the order.

B. We can show in case of y in $xy = ae^x + be^{-x} + x^2$, the differential equation is $xy_2 + 2y_1 - xy + x^2 - 2 = 0$ which is of order 2 and degree 1.

C. The required differential equation in all circles of radius a is $(y_1^2 + 1)^3 = a^2 y_2^2$ which is of order 2 and degree 2.

D. The differential equation in all circles of fixed radius r with centre on Y-axis is $y_1^2(r^2 - x^2) = 0$, which is of order 1 and degree 2.

$A \rightarrow r, s; B \rightarrow r, s; C \rightarrow q, s; D \rightarrow p, q$

Ex 70. Match the statements of Column I with values of Column II.

Column I	Column II
A. The solution of $\frac{xdx + ydy}{xdy - ydx} = \frac{\sqrt{a^2 - x^2 - y^2}}{x^2 + y^2}$	p. $y = \frac{C}{x^3} e^{-1/x}$
B. Solution of $\cos^2 x \frac{dy}{dx} - \tan 2x \cdot y = \cos^4 x$, where $ x < \pi/4$ and $y\left(\frac{\pi}{6}\right) = \frac{3\sqrt{3}}{8}$, is	q. $\sqrt{x^2 + y^2} = a \sin \left\{ C + \tan^{-1} \left(\frac{y}{x} \right) \right\}$
C. The equation of all possible curves that will cut each member of the family of circles $x^2 + y^2 - 2cx = 0$ at right angle, is	r. $x^2 + y^2 + Cy = 0$
D. Solution of the equation $x \int_0^x y(t) dt = (x+1) \int_0^x t y(t) dt, x > 0$, is	s. $y = \frac{\sin 2x}{2(1 - \tan^2 x)}$

Sol. A. Let $x = r \cos \theta, y = r \sin \theta$
 $\therefore x^2 + y^2 = r^2 (\sin^2 \theta + \cos^2 \theta) = r^2 \dots(i)$

and $\tan \theta = \frac{y}{x} \dots(ii)$

$\therefore d(x^2 + y^2) = d(r^2)$

From Eq. (i),
 $x dx + y dy = r dr \dots(iii)$

From Eq. (ii),
 $d\left(\frac{y}{x}\right) = d(\tan \theta)$

$\Rightarrow \frac{xdy - ydx}{x^2} = \sec^2 \theta d\theta$

$\therefore xdy - ydx = x^2 \sec^2 \theta d\theta$
 $= r^2 \cos^2 \theta \sec^2 \theta d\theta = r^2 d\theta \dots(iv)$

From Eqs. (iii) and (iv),

$$\frac{rdr}{r^2 d\theta} = \frac{\sqrt{a^2 - r^2}}{r^2} \quad \text{or} \quad \frac{dr}{\sqrt{a^2 - r^2}} = d\theta$$

On integrating both sides, we get

$$\begin{aligned}\sin^{-1}\left(\frac{r}{a}\right) &= \theta + C \\ \Rightarrow r &= a \sin(\theta + C) \\ \Rightarrow \sqrt{x^2 + y^2} &= a \sin\left\{C + \tan^{-1}\left(\frac{y}{x}\right)\right\}\end{aligned}$$

B. We have, $\frac{dy}{dx} - \sec^2 x \tan 2x = \cos^2 x$

$$\begin{aligned}\text{IF} = e^{\int \tan 2x \sec^2 x dx} &= e^{\int \frac{2 \tan x}{\tan^2 x - 1} \sec^2 x dx} \\ &= e^{\int \frac{dt}{t}}, \text{ where } t = \tan^2 x - 1 \\ &= e^{\ln |t|} = |t| = |\tan^2 x - 1|\end{aligned}$$

Given that, $|x| < \frac{\pi}{4}$

$$\therefore \tan^2 x < 1$$

$$\text{IF} = 1 - \tan^2 x$$

Hence, solution is

$$\begin{aligned}y(1 - \tan^2 x) &= \int \cos^2 x (1 - \tan^2 x) dx \\ &= \int (\cos^2 x - \sin^2 x) dx \\ &= \int \cos 2x dx = \frac{\sin 2x}{2} + C\end{aligned}$$

When $x = \frac{\pi}{6}$, $y = \frac{3\sqrt{3}}{8}$, then

$$\begin{aligned}\frac{3\sqrt{3}}{8} \left(1 - \frac{1}{3}\right) &= \frac{1}{2} \times \frac{\sqrt{3}}{2} + C \\ \Rightarrow C &= 0 \\ \therefore y &= \frac{\sin 2x}{2(1 - \tan^2 x)}\end{aligned}$$

C. We have, $x^2 + y^2 - 2cx = 0$... (i)

$$\therefore 2x + 2y \frac{dy}{dx} - 2c = 0$$

$$\text{or } c = x + \frac{y dy}{dx} \quad \dots (ii)$$

From Eqs. (i) and (ii),

$$x^2 + y^2 - 2\left(x + \frac{y dy}{dx}\right)x = 0$$

$$\text{or } -x^2 + y^2 - 2xy \frac{dy}{dx} = 0$$

which is the differential equation representing the given family of circles. To find the orthogonal trajectories,

$$y^2 - x^2 = -2xy \frac{dx}{dy}$$

$$\text{or } y^2 dy = x^2 dy - 2xy dx$$

$$\text{or } -dy = \frac{y d(x^2) - x^2 dy}{y^2}$$

$$\text{or } -dy = d\left(\frac{x^2}{y}\right)$$

$$\text{or } -y = \frac{x^2}{y} + C$$

$$\Rightarrow x^2 + y^2 + Cy = 0$$

D. On differentiating the equation with respect to x , we get

$$xy(x) + 1 \int_0^x y(t) dt = (x+1)xy(x) + 1 \int_0^x ty(t) dt$$

$$\text{or } \int_0^x y(t) dt = x^2 y(x) + \int_0^x ty(t) dt$$

Again, differentiating w.r.t. x , we get

$$y(x) = x^2 y'(x) + 2xy(x) + xy(x)$$

$$\text{or } (1 - 3x)y(x) = \frac{x^2 dy(x)}{dx}$$

$$\text{or } \frac{(1 - 3x) dx}{x^2} = \frac{dy(x)}{y(x)}$$

$$\text{or } y = \frac{C}{x^3} e^{-1/x}$$

$$A \rightarrow q; B \rightarrow s; C \rightarrow r; D \rightarrow p$$

Ex 71. Match the statements of Column I with values of Column II.

Column I	Column II
A. Circular plate is expanded by heat from p. radius 5 cm to 5.06 cm. Approximate increase in area is	4
B. If side of cube is increased by 1%, then q. percentage increase in volume is	0.6π
C. If the rate of decrease of $\frac{x^2}{2} - 2x + 5$ is r. twice the rate of decrease of x , then x is equal to	3
D. Rate of increase in area of equilateral s. triangle of side 15 cm, when each side is increasing at the rate of 0.1 cm/s, is	$\frac{3\sqrt{3}}{4}$

Sol. A. $r = 5$ cm, $\delta r = 0.06$

$$\begin{aligned}\therefore \delta A &= 2\pi r \delta r \\ &= 10\pi \times 0.06 = 0.6\pi\end{aligned}$$

$$\text{B. } \therefore V = x^3$$

$$\therefore \delta V = 3x^2 \delta x$$

$$\Rightarrow \frac{\delta V}{V} \times 100 = 3 \frac{\delta x}{x} \times 100 = 3 \times 1 = 3\%$$

$$\text{C. } (x-2) \frac{dx}{dt} = 2 \frac{dx}{dt}$$

$$\therefore x = 4$$

$$\text{D. } \therefore A = \frac{\sqrt{3}}{4} x^2$$

$$\begin{aligned}\therefore \frac{dA}{dt} &= \frac{\sqrt{3}}{2} \times x \frac{dx}{dt} \\ &= \frac{\sqrt{3}}{2} \cdot 15 \cdot \frac{1}{10} = \frac{3\sqrt{3}}{4}\end{aligned}$$

$$A \rightarrow q; B \rightarrow r; C \rightarrow p; D \rightarrow s$$

Type 6. Single Integer Answer Type Questions

Ex 72. Find the constant of integration of the general solution of the differential equation $(2x^2y - 2y^4)dx + (2x^3 + 3xy^3)dy = 0$, if curve passes through $(1, 1)$.

Sol. (1) $2x^2ydx - 2y^4dx + 2x^3dy + 3xy^3dy = 0$

On dividing throughout by x^3y , we get

$$2\frac{dx}{x} - \frac{2y^3}{x^3}dx + 2\frac{dy}{y} + \frac{3y^2}{x^2}dy = 0$$

$$\Rightarrow 2\frac{dx}{x} + 2\frac{dy}{y} + \frac{3y^2x^4dy - 2y^3x^3dx}{x^6} = 0$$

$$\Rightarrow 2\frac{dx}{x} + 2\frac{dy}{y} + \frac{3y^2x^2dy - 2y^3xdx}{x^4} = 0$$

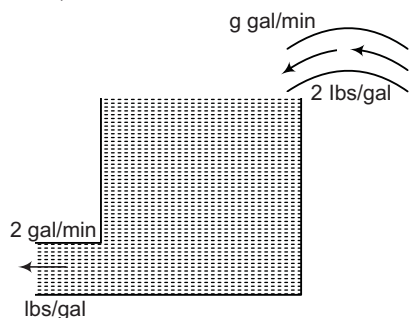
$$\Rightarrow 2d(\ln x) + 2d(\ln y) + d\left(\frac{y^3}{x^2}\right) = 0$$

$$\Rightarrow 2\ln|x| + 2\ln|y| + \frac{y^3}{x^2} = C$$

When $x = 1, y = 1$

So, $C = 1$

Ex 73. A tank initially contains 50 gallons of fresh water. Brine contains 2 pounds per gallon of salt, flows into the tank at the rate of 2 gallons per minute and the mixture kept uniform by stirring, runs out at the same rate. If it will take for the quantity of salt in the tank to increase from 40 to 80 pounds (in seconds) is 206, then find λ .



Sol. (8) Let the salt content at time t be u lbs, so that its rate of change is $du/dt = 2 \text{ gal} \times 2 \text{ lbs} = 4 \text{ lbs/min}$

If c is the concentration of the brine at time t , then rate at which the salt content decreases due to the out flow, is

$$2 \text{ gal} \times c \text{ lbs/min} = 2c \text{ lbs/min}$$

$$\therefore \frac{du}{dt} = 4 - 2c \quad \dots(i)$$

Since, there is no increase in the volume of the liquid, so the concentration is

$$c = \frac{u}{50}$$

$$\therefore \text{Eq. (i) becomes } \frac{du}{dt} = 4 - \frac{2u}{50}$$

On separating the variables and then integrating, we get

$$\int dt = 25 \int \frac{du}{100 - u}$$

$$\text{or } t = -25 \ln(100 - u) + k \quad \dots(ii)$$

Initially, when

$$t = 0, u = 0$$

$$\Rightarrow 0 = -25 \ln 100 + k \quad \dots(iii)$$

On eliminating k from Eqs. (ii) and (iii), we get

$$t = 25 \ln \left(\frac{100}{100 - u} \right)$$

On taking $t = t_1$ when $u = 40$ and $t = t_2$ when $u = 80$, we get

$$t_1 = 25 \ln \left(\frac{100}{60} \right)$$

$$\text{and } t_2 = 25 \ln \left(\frac{100}{20} \right)$$

$\therefore$ The required time $(t_2 - t_1)$

$$= 25 \left(\ln 5 - \ln \frac{5}{3} \right) = 25 \ln 3$$

$$= 25 \times 1.0986$$

$$= 1648 \text{ s} = 206 \times 8 = 206 \times \lambda$$

$$\therefore \lambda = 8$$

Ex 74. If $f: R - \{1\} \rightarrow R$ and f is differentiable function satisfying

$$f(x + f(y) + xf(y)) = y + f(x) + yf(x),$$

$$\forall x, y \in R - \{1\},$$

then find the value of $2010[1 + f(2009)]$.

Sol. (1) We have, $f(x + f(y) + xf(y)) = y + f(x) + yf(x) \quad \dots(i)$

On differentiating w.r.t. x keeping y as constant, we get

$$f'(x + f(y) + xf(y))(1 + f(y)) = f'(x) + yf'(x) \quad \dots(ii)$$

Again differentiating w.r.t. y keeping x as constant, $f'(x + f(y) + xf(y))(1 + x)f'(y) = 1 + f(x) \quad \dots(iii)$

From Eqs. (ii) and (iii), we get

$$\frac{1 + f(y)}{(1 + x)f'(y)} = \frac{(1 + y)f'(x)}{1 + f(x)}$$

$$\Rightarrow \frac{(1 + y)f'(y)}{1 + f(y)} = \frac{1 + f(x)}{(1 + x)f'(x)} = \lambda \quad [\text{say}]$$

$$\therefore f'(y) = \frac{1 + f(y)}{1 + y} \lambda$$

$$\text{and } f'(x) = \frac{1 + f(x)}{\lambda(1 + x)}$$

$$\therefore \lambda = \frac{1}{\lambda}$$

$$\Rightarrow \lambda = \pm 1$$

$$\therefore \frac{f'(x)}{1 + f(x)} = \pm \frac{1}{1 + x}$$

On integrating both sides, we get

$$f(x) = C(1 + x)^{\pm 1} - 1$$

From Eq. (i), put $x = y = 0$
 $\therefore f(f(0)) = f(0)$
 From Eq. (ii), $f(0) = C - 1$
 Thus, $f(C - 1) = C - 1$
 $\therefore C = 0, 1$ [taking +ve sign]
 Hence, $f(x) = -1$
 $f(x) = (1 + x) - 1 = x$
 and $C = 1$
 $\therefore f(x) = (1 + x)^{-1} - 1 = \frac{-x}{1 + x}$
 $\Rightarrow 1 + f(x) = \frac{1}{1 + x}$
 $\therefore 1 + f(2009) = \frac{1}{2010}$
 $\Rightarrow (2010)[1 + f(2009)] = 1$

Ex 75. If $\phi(x)$ is a differentiable real valued function satisfying $\phi'(x) + 2\phi(x) \leq 1$, then the value of $2\phi(x)$ is always less than or equal to _____.

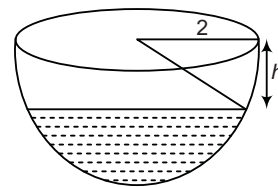
Sol. (1) $\phi'(x) + 2\phi(x) \leq 1 \Rightarrow e^{2x}\phi'(x) + 2\phi(x)e^{2x} \leq e^{2x}$
 $\Rightarrow \frac{d}{dx} \left(e^{2x}\phi(x) - \frac{1}{2}e^{2x} \right) \leq 0$
 $\therefore \left(e^{2x}\phi(x) - \frac{1}{2} \right)$ is a non-increasing function of x
 and $\phi(x) - \frac{1}{2}$ is also a non-increasing function of x .
 Hence, $2\phi(x)$ is always less than or equal to 1.

Ex 76. The degree of the differential equation satisfied by the curves $\sqrt{1+x} - a\sqrt{1+y} = 1$ is _____.

Sol. (1) We have, $\frac{1}{2}(1+x)^{-1/2} - \frac{a}{2}(1+y)^{-1/2} \frac{dy}{dx} = 0$
 $\Rightarrow \frac{1}{\sqrt{1+x}} = \frac{a}{\sqrt{1+y}} \cdot \frac{dy}{dx}$
 $\Rightarrow a = \sqrt{\frac{1+y}{1+x}} \cdot \frac{1}{dy/dx}$
 On putting this value in the given equation, we get
 $\frac{dy}{dx} \sqrt{1+x} - \frac{1+y}{\sqrt{1+x}} = \frac{dy}{dx}$
 $\Rightarrow (1+x) \frac{dy}{dx} = 1+y + \sqrt{1+x} \frac{dy}{dx}$
 $\Rightarrow (1+x - \sqrt{1+x}) \frac{dy}{dx} = 1+y$
 $\therefore$ Degree of the given equation is 1.

Ex 77. A hemispherical tank of radius 2m is initially full of water and has an outlet of 12 cm^2 cross-sectional area at the bottom. The outlet is opened at some instant. The flow through the outlet is according to the law $V(t) = 0.6\sqrt{2gh(t)}$, where $V(t)$ and $h(t)$ are respectively the velocity of the flow through

the outlet and the height of water level above the outlet at time t and g is the acceleration due to gravity. If t is the time taken to empty the tank, then the value of $\frac{27T\sqrt{g}}{2\pi} 10^{-5}$ is equal to _____.



Sol. (7) Let the height of the water level above the base at time t be $h(t)$ cm and it fall $dh(t)$ in time dt . Let the decrease in volume be dV . Let r be the radius of the surface of water at time t , then

$$r^2 = (200)^2 - (200 - h)^2$$

$$= (400h - h^2) \text{ cm}$$

and $dV = -\pi r^2 dh$

$$\Rightarrow \frac{dV}{dt} = -\pi r^2 \frac{dh}{dt}$$

We are given velocity of water flow is, $V = \frac{3}{5}\sqrt{2gh}$.

Rate of flow of water = $12V$ _____.

$$\text{Thus, } \frac{dV}{dt} = 12V = \frac{36}{5}\sqrt{2gh}$$

$$\therefore -\pi r^2 \frac{dh}{dt} = \frac{36}{5}\sqrt{2g} \sqrt{h}$$

$$\Rightarrow \frac{dh}{dt} = -\frac{36}{5\pi} \sqrt{2g} \frac{\sqrt{h}}{400h - h^2}$$

$$= -\frac{36}{5\pi} \sqrt{2g} \frac{1}{400\sqrt{h} - h^{3/2}}$$

On separating variables, we get

$$(400\sqrt{h} - h^{3/2}) dh = -\frac{36}{5\pi} \sqrt{2g} dt$$

$$\Rightarrow \frac{800}{3} h^{3/2} - \frac{2}{5} h^{5/2} = -\frac{36}{5\pi} \sqrt{2g} t + C$$

At $t = 0$, $h = 200$ cm

$$C = \frac{800}{3} (200)^{3/2} - \frac{2}{5} (200)^{5/2}$$

$$= \sqrt{2} \frac{56}{15} \times 10^5 \text{ cm}$$

We have to find t , so that $h = 0$.

Let $t = T$

When $h = 0$, then

$$0 = -\frac{36}{5\pi} \sqrt{2g} T + \sqrt{2} \frac{56}{15} \times 10^5$$

$$\Rightarrow T = \frac{\sqrt{2} \times 56 \times 5\pi}{15 \times 36 \times \sqrt{2g}} 10^5$$

$$= \frac{14}{27} \cdot \frac{\pi}{\sqrt{g}} \cdot 10^5 \text{ units}$$

$$\text{Hence, } \frac{27T\sqrt{g}}{2\pi} 10^{-5} = 7$$

Target Exercises

Type 1. Only One Correct Option

- The differential equation representing the family of curves $y^2 = 2c(x + c^{2/3})$, where c is a positive parameter, is of
(a) order 3, degree 3 (b) order 2, degree 4
(c) order 1, degree 5 (d) order 5, degree 1
- The degree of the differential equation $\left(\frac{d^4 y}{dx^4}\right)^{3/5} - 5\frac{d^3 y}{dx^3} + 6\frac{d^2 y}{dx^2} - 8\frac{dy}{dx} + 5 = 0$ is
(a) 2 (b) 3 (c) 4 (d) 5
- The differential equation of the family $y = ae^x + bx e^x + cx^2 e^x$ of curves, where a , b and c are arbitrary constants, is
(a) $y''' + 3y'' + 3y' + y = 0$ (b) $y''' + 3y'' - 3y' - y = 0$
(c) $y''' - 3y'' - 3y' + y = 0$ (d) $y''' - 3y'' + 3y' - y = 0$
- The differential equation of the system of all circles of radius r in the XY -plane is
(a) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = r^2 \left(\frac{d^2 y}{dx^2}\right)^2$
(b) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2} = r^2 \left(\frac{d^2 y}{dx^2}\right)^3$
(c) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{2/3} = r^2 \left(\frac{d^2 y}{dx^2}\right)^2$
(d) $\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{2/3} = r^2 \left(\frac{d^2 y}{dx^2}\right)^3$
- The differential equation which is satisfied by all the curves, $y = Ae^{2x} + Be^{-x/2}$, where A and B are non-zero constants, is
(a) $2\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} + 2y = 0$
(b) $2\frac{d^2 y}{dx^2} - 3\frac{dy}{dx} - 2y = 0$
(c) $2\frac{d^2 y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$
(d) None of the above
- The differential equation satisfied by $ax^2 + by^2 = 1$ is
(a) $xyy'' + xy'^2 + yy' = 0$
(b) $xyy'' + xy'^2 - yy' = 0$
(c) $xyy'' - xy'^2 + yy' = 0$
(d) None of the above
- The differential equation of the family of general circles is
(a) $y'''(1 + y'^2) - 3y'y''^2 = 0$
(b) $y'''(1 + y'^2) + 3y'y''^2 = 0$
(c) $y'''(1 + y'^2) - 3y''y'^2 = 0$
(d) None of the above
- The differential equation of all ellipses centred at the origin is
(a) $xyy_2 - xy_1^2 + yy_1 = 0$ (b) $xyy_2 + xy_1^2 - yy_1 = 0$
(c) $xyy_2 + xy_1^2 + yy_1 = 0$ (d) None of these
- The differential equation for $y = A \cos \alpha x + B \sin \alpha x$, where A and B are arbitrary constants, is
(a) $\frac{d^2 y}{dx^2} + \alpha y = 0$ (b) $\frac{d^2 y}{dx^2} - \alpha y = 0$
(c) $\frac{d^2 y}{dx^2} - \alpha^2 y = 0$ (d) $\frac{d^2 y}{dx^2} + \alpha^2 y = 0$
- The particular solution of $\cos y dx + (1 + 2e^{-x}) \sin y dy = 0$, when $x = 0$, $y = \frac{\pi}{4}$, is
(a) $e^x - 2 = 3\sqrt{2} \cos y$ (b) $e^x + 2 = \sqrt{2} \cos y$
(c) $e^x + 2 = 3\sqrt{2} \cos y$ (d) None of these
- The equation of the curve for which the cartesian sub-tangent varies as the reciprocal of the square of the abscissa, is
(a) $x = Ce^{y^{3/3k}}$ (b) $x = Ce^{y^{2/3k}}$
(c) $x^2 = 2Cy + k$ (d) None of these
- The solution of the equation $\log \frac{dy}{dx} = 9x - 6y + 6$, given that $y = 1$ when $x = 0$, is
(a) $3e^{6y} = 2e^{9x-6} + 6e^6$
(b) $3e^{6y} = 2e^{9x+6} - 6e^6$
(c) $3e^{6y} = 2e^{9x+6} + e^6$
(d) None of the above
- The solution of $\frac{dy}{dx} = \frac{ax+g}{by+f}$ represents a circle, when
(a) $a = b$ (b) $a = -b$
(c) $a = -2b$ (d) $a = 2b$
- If $2f(x) = f'(x)$ and $f(0) = 3$, then $f(2)$ equals
(a) $4e^3$ (b) $3e^4$
(c) $2e^3$ (d) $3e^2$

15. The solution of the differential equation $x dy - y dx = 0$ represents
 (a) parabola whose vertex is at origin
 (b) circle whose centre is at origin
 (c) a rectangular hyperbola
 (d) straight line passing through origin
16. The general solution of the differential equation $\frac{dy}{dx} = \frac{y}{x}$ is
 (a) $\log y = kx$ (b) $y = kx$ (c) $y = \frac{k}{x}$ (d) $y = k \log x$
17. The number of straight lines which satisfy the differential equation $\frac{dy}{dx} + x \left(\frac{dy}{dx} \right)^2 - y = 0$, is
 (a) 1 (b) 3
 (c) 2 (d) 4
18. The solution of the differential equation $2x \frac{dy}{dx} - y = 3$ represents
 (a) a circle
 (b) a pair of straight lines
 (c) an ellipse
 (d) a parabola
19. The solution of the differential equation $\frac{dy}{dx} = \frac{y(2y-x)}{x(2y+x)}$, which satisfies $y = 1$ at $x = 1$, is
 (a) $x \log(xy) = 2(x-y)$
 (b) $y \log(xy) = 2(y-x)$
 (c) $x \log(xy) = 2(x+y)$
 (d) None of the above
20. The differential equation of parabola having their axis of symmetry coinciding with X -axis, is
 (a) $y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$ (b) $y \frac{d^2 x}{dy^2} + \left(\frac{dx}{dy} \right)^2 = 0$
 (c) $y \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$ (d) None of these
21. The solution of the differential equation $x^2 \frac{dy}{dx} - xy = 1 + \cos \frac{y}{x}$ is
 (a) $\cos \frac{y}{x} = 1 + \frac{C}{x}$
 (b) $x^2 = (C + x^2) \tan \frac{y}{x}$
 (c) $\tan \frac{y}{2x} = C - \frac{1}{2x^2}$
 (d) $\tan \frac{y}{x} = C + \frac{1}{x}$
22. The solution of the equation $e^{\frac{dy}{dx}} = x + 3$, given that $x = -2$, $y = 3$, is
 (a) $y = (x+3) \log(x+3) - x + 1$
 (b) $y = (x+3) \log(x+3) + x + 1$
 (c) $y = (x+3) \log(x+3) - x - 1$
 (d) None of the above
23. The general solution of the differential equation $(1 + \tan y)(dx - dy) + 2x dy = 0$ is
 (a) $x(\sin y + \cos y) = \sin y + Ce^y$
 (b) $x(\sin y + \cos y) = \sin y + Ce^{-y}$
 (c) $x(\sin x + \cos x) = \sin x + Ce^x$
 (d) None of the above
24. The solution of the equation $(1 + e^{x/y}) dx + e^{x/y} \left(1 - \frac{x}{y} \right) dy = 0$ is
 (a) $x + ye^{x/y} = C$
 (b) $x - ye^{x/y} = C$
 (c) $y + xe^{x/y} = C$
 (d) None of the above
25. If the solution of the differential equation $\frac{dy}{dx} = \frac{1}{x \cos y + \sin 2y}$ is $x = Ce^{\sin y} - k(1 + \sin y)$, then k equals
 (a) 1 (b) 2
 (c) 3 (d) 4
26. The solution of the differential equation $\sqrt{a+x} \frac{dy}{dx} + xy = 0$ is
 (a) $y = Ae^{\frac{2}{3}(2a-x)\sqrt{x+a}}$
 (b) $y = Ae^{\frac{-2}{3}(a-x)\sqrt{x+a}}$
 (c) $y = Ae^{\frac{2}{3}(2a+x)\sqrt{x+a}}$
 (d) $y = Ae^{\frac{-2}{3}(2a-x)\sqrt{x+a}}$
27. To reduce the differential equation $\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n$ to the linear form, the substitution is
 (a) $v = \frac{1}{y^n}$ (b) $v = \frac{1}{y^{n-1}}$
 (c) $v = y^n$ (d) $v = y^{n-1}$
28. The general solution of the differential equation $\frac{dy}{dx} + yg'(x) = g(x)g'(x)$, where $g(x)$ is a given function of x , is
 (a) $g(x) + \log[1 + y + g(x)] = C$
 (b) $g(x) + \log[1 + y - g(x)] = C$
 (c) $g(x) - \log[1 + y - g(x)] = C$
 (d) None of the above
29. The equation of the curve satisfying the differential equation $y(x + y^3) dx = x(y^3 - x) dy$ and passing through the point $(1, 1)$, is
 (a) $y^3 - 2x + 3x^2 y = 0$
 (b) $y^3 + 2x + 3x^2 y = 0$
 (c) $y^3 + 2x - 3x^2 y = 0$
 (d) None of the above

30. The solution of the equation $\left(xy^2 - e \cdot \frac{1}{x^3}\right)dx - x^2 y dy = 0$, given $y = 0$ when $x = 1$, is

(a) $y^2 = \frac{e}{2} \left(\frac{1}{x^4} - x^2\right)$ (b) $y^2 = \frac{e}{3} \left(\frac{1}{x^4} - x^2\right)$
(c) $y = \frac{e}{3} \left(\frac{1}{x^4} + x^2\right)$ (d) None of these

31. The solution of the equation $y \sin x \frac{dy}{dx} = \cos x \left(\sin x - \frac{y^2}{2}\right)$, given $y = 1$ when $x = \frac{\pi}{2}$, is

(a) $y^2 \sin x = 2 \sin^2 x + 1$ (b) $y^2 = 2 \sin x$
(c) $x^2 = \sin y$ (d) $x^2 = 2 \sin y$

32. The solution of the equation

$x \left(\frac{dy}{dx}\right)^2 + (y - x) \frac{dy}{dx} - y = 0$ is

(a) $(x - y + C)(xy - C) = 0$
(b) $(x + y + C)(xy - C) = 0$
(c) $(x - y + C)(2xy - C) = 0$
(d) $(y - x + C)(xy - C) = 0$

33. The solution of the equation $(xy^4 + y)dx - x dy = 0$ is

(a) $4x^4 y^3 + 3x^3 = Cy^3$ (b) $3x^3 y^4 + 4x^3 = Cx^3$
(c) $3x^4 y^3 + 4x^3 = Cy^3$ (d) None of these

34. The solution of the equation $xdy - [y + xy^3(1 + \log x)]dx = 0$ is

(a) $-\frac{x^2}{y^2} = \frac{2x^3}{3} \left(\frac{2}{3} + \log x\right) + C$
(b) $\frac{x^2}{y^2} = \frac{2x^3}{3} \left(\frac{2}{3} + \log x\right) + C$
(c) $-\frac{x^2}{y^2} = \frac{x^3}{3} \left(\frac{2}{3} + \log x\right) + C$
(d) None of the above

35. The equation of a curve passing through $(0, 1)$ and having gradient $\frac{-(y + y^3)}{1 + x + xy^2}$ at (x, y) , is

(a) $xy + \tan^{-1} y = \frac{\pi}{2}$ (b) $xy + \tan^{-1} y = \frac{\pi}{4}$
(c) $xy - \tan^{-1} y = \frac{\pi}{2}$ (d) $xy - \tan^{-1} y = \frac{\pi}{4}$

36. Water is drained from a vertical cylindrical tank by opening a valve at the base of the tank. It is known that the rate at which the water level drops, is proportional to the square root of water depth y , where the constant of proportionality $k > 0$ depends on the acceleration due to gravity and the geometry of the hole. If t is measured in minutes and $k = 1/15$, then the time to drain the tank, if the water is 4 m deep to start with, is

(a) 30 min (b) 45 min (c) 60 min (d) 80 min

37. The general solution of the differential equation $y(x^2 y + e^x)dx - e^x dy = 0$ is

(a) $x^3 y - 3e^x = Cy$ (b) $x^3 y + 3e^x = Cy$
(c) $y^3 x - 3e^y = Cx$ (d) $y^3 x + 3e^y = Cx$

38. The solution of the equation $x^2 y - x^3 \frac{dy}{dx} = y^4 \cos x$, when $y(0) = 1$, is

(a) $y^3 = 3x^3 \sin x$ (b) $x^3 = 3y^3 \sin x$
(c) $x^3 = y^3 \sin x$ (d) None of these

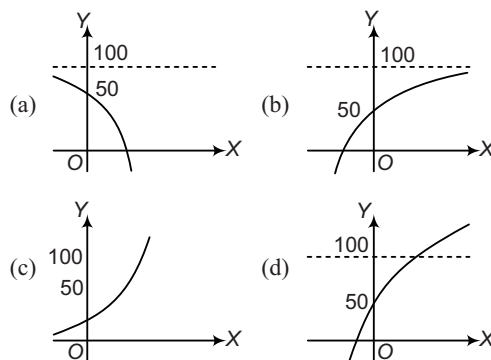
39. The equation of the curve in which the perpendicular from the origin upon the tangent is equal to the abscissa of the point of contact, is

(a) $x^2 + y^2 = Cx$ (b) $x^2 + y^2 = Cy$
(c) $x^2 + y^2 = C$ (d) None of these

40. A normal is drawn at a point $P(x, y)$ of a curve. It meets the X -axis at Q . If PQ is of constant length k , then the differential equation describing such a curve is

(a) $y \frac{dy}{dx} = \pm \sqrt{k^2 - y^2}$ (b) $x \frac{dy}{dx} = \pm \sqrt{k^2 - x^2}$
(c) $y \frac{dy}{dx} = \pm \sqrt{y^2 - k^2}$ (d) $x \frac{dy}{dx} = \pm \sqrt{x^2 - k^2}$

41. Which one of the following curves represents the solution of the initial value problem $Dy = 100 - y$, where $y(0) = 50$?



42. If a population grows at the rate of 5% per year. Then, the population will be doubled in

(a) $10 \log 2$ yr
(b) $20 \log 2$ yr
(c) $30 \log 2$ yr
(d) None of the above

43. The line normal to a given curve at each point (x, y) on the curve passes through the point $(3, 0)$. If the curve contains the point $(3, 4)$, then its equation is

(a) $x^2 + y^2 + 6x - 7 = 0$
(b) $x^2 + y^2 - 6x + 7 = 0$
(c) $x^2 + y^2 - 6x - 7 = 0$
(d) None of the above

Type 2. More than One Correct Option

44. Identify the statement(s) which is/are true.

- (a) $f(x, y) = e^{y/x} + \tan \frac{y}{x}$ is homogeneous of degree zero
 (b) $x \log \frac{y}{x} dx + \frac{y^2}{x} \sin^{-1} \frac{y}{x} dy = 0$ is homogeneous differential equation
 (c) $f(x, y) = x^2 + \sin x \cdot \cos y$ is not homogeneous
 (d) $(x^2 + y^2)dx - (xy^2 - y^3)dy = 0$ is a homogeneous differential equation

45. The graph of the function $y = f(x)$ passing through the point $(0, -1)$ and satisfying the differential equation $\frac{dy}{dx} + y \cos x = \cos x$ is such

that

- (a) it is a constant function
 (b) it is periodic
 (c) it is neither an even nor an odd function
 (d) it is continuous and differentiable for all x

46. If a function $y = f(x)$ satisfying the differential equation $\frac{dy}{dx} \sin x - y \cos x + \frac{\sin^2 x}{x^2} = 0$ is such that $y \rightarrow 0$ as $x \rightarrow \infty$, then which of the statement(s) is/are correct?

- (a) $\lim_{x \rightarrow 0} f(x) = 1$
 (b) $\int_0^{\pi/2} f(x) dx$ is less than $\frac{\pi}{2}$
 (c) $\int_0^{\pi/2} f(x) dx$ is greater than unity
 (d) $f(x)$ is an odd function

47. Which one of the following function(s) is/are homogeneous?

- (a) $f(x, y) = \frac{x - y}{x^2 + y^2}$
 (b) $f(x, y) = x^{1/3} y^{-2/3} \tan^{-1} \frac{x}{y}$
 (c) $f(x, y) = x (\ln \sqrt{x^2 + y^2} - \ln y) + y e^{x/y}$
 (d) $f(x, y) = x \left[\ln \frac{2x^2 + y^2}{x} - \ln(x + y) \right] + y^2 \tan \left(\frac{x + 2y}{3x - y} \right)$

48. The differential equation $\frac{d^2 y}{dx^2} + y + \cot^2 x = 0$ must be satisfied by

- (a) $y = 2 + C_1 \cos x + \sqrt{C_2} \sin x$
 (b) $y = \cos x \cdot \ln \left(\tan \frac{x}{2} \right) + 2$
 (c) $y = 2 + C_1 \cos x + C_2 \sin x + \cos x \ln \left(\tan \frac{x}{2} \right)$
 (d) All of the above

49. The equation of the curve satisfying the differential equation $y_2(x^2 + 1) = 2xy_1$ passing through the point $(0, 1)$ and having slope of tangent at $x = 0$ as 3, where y_2 and y_1 represent 2nd and 1st order derivatives, then

- (a) $y = f(x)$ is a strictly increasing function
 (b) $y = f(x)$ is a non-monotonic function
 (c) $y = f(x)$ has three distinct real roots
 (d) $y = f(x)$ has only one negative root

50. If a differentiable function satisfies

$$f(x) = \int_0^x \{f(t) \cos t - \cos(t - x)\} dt. \text{ Then which}$$

of the following hold good?

- (a) $f(x)$ has a minimum value $1 - e$
 (b) $f(x)$ has a maximum value $1 - e^{-1}$
 (c) $f''\left(\frac{\pi}{2}\right) = e$
 (d) $f'(0) = 1$

51. Identify the statement (s) which is/are true?

- (a) The order of differential equation $\sqrt{1 + \frac{d^2 y}{dx^2}} = x$ is 1
 (b) The solution of the differential equation $x dy - y dx = \sqrt{x^2 + y^2} dx$ is $y + \sqrt{x^2 + y^2} = Cx^2$
 (c) $\frac{d^2 y}{dx^2} = 2 \left(\frac{dy}{dx} - y \right)$ is differential equation of family of curves $y = e^x (A \cos x + B \sin x)$
 (d) The solution of differential equation $(1 + y^2) + (x - 2e^{\tan^{-1} y}) \frac{dy}{dx} = 0$ is $xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + C$

52. Let $y = (A + Bx) e^{3x} B e^{3x}$ be a solution of the differential equation $\frac{d^2 y}{dx^2} + m \frac{dy}{dx} + ny = 0; m, n \in I$, then

- (a) $m = -6$ (b) $n = -6$ (c) $m = 9$ (d) $n = 9$

Type 3. Assertion and Reason

Directions (Q. Nos. 53-61) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

53. Statement I $y = a \sin x + b \cos x$ is general solution of $y'' + y = 0$.

Statement II $y = a \sin x + b \cos x$ is a trigonometric function.

54. Statement I The equation $y = cx^2$, where c is any arbitrary constant, intersect the curves $x^2 + 2y^2 = 2c$ at right angle.

Statement II As above curves traces orthogonal trajectory.

55. Statement I The differential equation of the family of hyperbolas with asymptotes as the line $x + y = 1$ and $x - y = 1$, is $(x - 1) = y \frac{dy}{dx}$.

Statement II The eccentricity of the rectangular hyperbola is $\sqrt{2}$.

56. Statement I The solution of the differential equation $(xy - 1) \frac{dy}{dx} + y^2 = 0$, is $xy - \log y = C$.

Statement II The integrating factor of above differential equation is $\left(\frac{1}{y}\right)$.

57. Statement I The differential equation of all circles in a plane must be of order 3.

Statement II There is only one circle passing through three non-collinear points.

58. Statement I A ray of light from origin after reflection at the point $P(x, y)$ of any curve becomes parallel to X -axis, the equation of curve may be $y^2 = 2x + 1$.

Statement II A ray of light parallel to axis after reflection from parabolic mirror always passes through focus.

59. Statement I The eccentricity of ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ is changed at the rate of 0.1 per second. The line at which it will touch the auxiliary circle is, 5 s.

Statement II Eccentricity of auxiliary circle is always zero.

60. Statement I The solution of $(y dx - x dy) \cot\left(\frac{x}{y}\right) = ny^2 dx$ is $\sin\left(\frac{x}{y}\right) = Ce^{mx}$.

Statement II Such type of differential equations can only be solved by the substitution $x = vy$.

61. Statement I Degree of the differential equation $2x - 3y + 2 = \log\left(\frac{dy}{dx}\right)$ is not defined.

Statement II In the given differential equation, the power of highest order derivative when expressed as the polynomials of derivatives, is called degree.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 62-64) If any differential equation in the form

$$f(f_1(x, y)) d(f_1(x, y)) + \phi(f_2(x, y)) d(f_2(x, y)) + \dots = 0.$$

Then, each term can be integrated separately.

$$\text{e.g. } \int \sin xy d(xy) + \int \left(\frac{x}{y}\right) d\left(\frac{x}{y}\right) = -\cos xy + \frac{1}{2} \left(\frac{x}{y}\right)^2 + C$$

62. The solution of the differential equation $xy dy - y dx = \sqrt{x^2 - y^2} dx$ is

- (a) $Cx = e^{\sin^{-1} y/x}$ (b) $x e^{\sin^{-1} y/x} = C$
(c) $x + e^{\sin^{-1} y/x} = C$ (d) None of these

63. The solution of the differential equation $(xy^4 + y) dx - x dy = 0$ is

- (a) $\frac{x^3}{4} + \frac{1}{2} \left(\frac{x}{y}\right)^2 = C$
(b) $\frac{x^4}{4} + \frac{1}{3} \left(\frac{x}{y}\right)^3 = C$
(c) $\frac{x^4}{4} - \frac{1}{2} \left(\frac{x}{y}\right)^2 = C$
(d) $\frac{x^3}{4} - \frac{1}{2} \left(\frac{x}{y}\right)^2 = C$

64. The solution of the differential equation $(2x \cos y + y^2 \cos x) dx + (2y \sin x - x^2 \sin y) dy = 0$ is

- (a) $x^2 \cos y + y^2 \sin x = C$
(b) $x \cos y - y \sin x = C$
(c) $x^2 \cos^2 y + y^2 \sin^2 x = C$
(d) None of the above

Passage II (Q. Nos. 65-67) The differential equation $\frac{dy}{dx} = f(x)g(x)$ can be solved by separating variable $\frac{dy}{g(y)} = f(x) dx$.

65. The equation of the curve to the point $(1, 0)$ which satisfies the differential equation

$$(1 + y^2) dx - xy dy = 0 \text{ is}$$

- (a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$
(c) $x^2 + y^2 = 2$ (d) $x^2 - y^2 = 2$

66. The solution of the differential equation

$$\frac{dy}{dx} + \frac{1 + y^2}{\sqrt{1 - x^2}} = 0 \text{ is}$$

- (a) $\tan^{-1} y + \sin^{-1} x = C$ (b) $\tan^{-1} x + \sin^{-1} y = C$
(c) $\tan^{-1} y \cdot \sin^{-1} x = C$ (d) $\tan^{-1} y - \sin^{-1} x = C$

67. If $\frac{dy}{dx} = 1 + x + y + xy$ and $y(-1) = 0$, then y is equal to

- (a) $e^{\frac{(1-x)^2}{2}}$
 (b) $e^{\frac{(1+x)^2}{2}} - 1$
 (c) $\ln(1+x) - 1$
 (d) $1+x$

Passage III (Q. Nos. 68-70) A curve passing through the point (1, 1) has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of P from X-axis.

68. The differential equation formed is

- (a) $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ (b) $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$
 (c) $\frac{d^2y}{dx^2} = \frac{2xy}{y^2 - x^2}$ (d) $\frac{dy}{dx} = \frac{2xy}{y^2 - x^2}$

69. The order and degree of the differential equation will be

- (a) 1, 2 (b) 2, 1
 (c) 1, 1 (d) None of these

Type 5. Match the Columns

74. Match the statements of Column I with values of Column II.

Column I	Column II
A. The differential equation of the family of curves $y = e^x (A \cos x + B \sin x)$, where A, B are arbitrary constants, has the degree n and order m . Then, the values of n and m are, respectively	p. 2, 1
B. The degree and order of the differential equation of the family of all parabolas whose axis is X-axis, are respectively	q. 1, 1
C. The order and degree of the differential equations of the family of circles touching X-axis at the origin, are respectively	r. 2, 2
D. The degree and order of the differential equation of the family of ellipses having the same foci, are respectively	s. 1, 2

75. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $x dy = y(dx + y dy)$, $y > 0$, $y(1) = 1$ and $y(x_0) = -3$, then x_0 is equal to	p. $\frac{1}{4}$
B. If $y(t)$ is the solution of $(t+1)\frac{dy}{dt} - ty = 1$, then y is equal to	q. -15

70. The solution of the differential equation will be

- (a) $x^2 + 2y^2 = 3x$ (b) $x^2 + y^2 = 2x$
 (c) $x^2 + y^2 = 4x$ (d) None of these

Passage IV (Q. Nos. 71-73) A certain radioactive material is known to decay at a rate proportional to the amount present. Initially, there is 50 kg of the material present and after two hour, it is observed that the material has lost 10% of its original mass.

71. The expression for the mass of the material remaining at any time t is

- (a) $N = 50e^{-(1/2)(\ln 0.9)t}$ (b) $50e^{-(1/4)(\ln 0.9)t}$
 (c) $N = 50e^{-(\ln 0.9)t}$ (d) None of these

72. The mass of the material after four hours is

- (a) $50^{-0.5 \ln 9}$
 (b) $50e^{-2 \ln 9}$
 (c) $50e^{-2 \ln 0.9}$
 (d) None of the above

73. The time at which the material has decayed to one half of its initial mass, is

- (a) $(\ln 1/2) / (1/2 \ln 9)$ hr
 (b) $(\ln 2) / (-1/2 \ln 0.9)$ hr
 (c) $(\ln 1/2) / (-1/2 \ln 0.9)$ hr
 (d) None of the above

76. Match the statements of Column I with values of Column II.

Column I	Column II
C. If $(x^2 + y^2) dy = xy dx$, $y(1) = 1$ and $y(x_0) = e$, then x_0 is equal to	r. $-\frac{1}{2}$
D. If $\frac{dy}{dx} + 2\frac{y}{x} = 0$, $y(1) = 1$, then $y(2)$ is equal to	s. $\sqrt{3}e$
A. $(x+y)^2 \frac{dy}{dx} = a^2$	p. $\frac{1}{x^5 y^5} = \frac{5}{2x^2} + C$
B. $\sec^2 x \tan y dx + \sec^2 y \tan x dy = 0$	q. $7 = 3e^{-4y} + 4e^{3x}$
C. $\log\left(\frac{dy}{dx}\right) = 3x + 4y$, $(y(0) = 0)$	r. $\tan x \tan y = C$
D. $x \frac{dy}{dx} + y = x^3 y^6$	s. $y = a \tan^{-1}\left(\frac{x+y}{a}\right) + C$

Type 6. Single Integer Answer Type Questions

77. A curve passing through (2, 3) and satisfying differential equation $\int_0^x ty(t)dt = x^2 y(x)$, where $x > 0$, then C is _____.
78. If the substitution of $y = z^\alpha$ transforms the differential equation $(x^2 y^2 - 1)dy + 2xy^3 dx = 0$ into a homogeneous differential equation, then $1 + \alpha$ is _____.
79. If a function $f(x)$ satisfying $\int_0^1 f(tx)dt = nf(x)$ tends to $f(x) = \frac{c}{x^\alpha}$, as $n \rightarrow \infty$, then α is _____.
80. Consider the two statements.
Statement I $y = \sin kt$ satisfies the differential equation $y'' + 9y = 0$.
Statement II $y = e^{kt}$ satisfy the differential equation $y'' + y' - 6y = 0$.
 The value of $k + 4$ for which both statements are correct, is _____.
81. A curve C passes through origin and has the property that at each point (x, y) on it, the normal line at the point passes through (1, 0). If the equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is $x = \alpha$, then α is equal to _____.
82. Let $y = y(t)$ be a solution of the differential equation $y' + 2ty = t^2$, then $16 \lim_{t \rightarrow \infty} \frac{y}{t}$ equals _____.
83. Tangent is drawn at the point (x_i, y_i) on the curve $y = f(x)$, which intersects X -axis at $(x_{i+1}, 0)$. Now, again a tangent is drawn at (x_{i+1}, y_{i+1}) on the curve which intersects X -axis at $(x_{i+2}, 0)$ and the process is repeated n times, i.e. $i = 1, 2, 3, \dots, n$. If $x_1, x_2, x_3, \dots, x_n$ form an arithmetic progression with common difference equal to $\log_2 e$ and curve passes through (0, 2). If curve passes through the point $(-2, k)$, then the value of k is _____.
84. If the solution of the differential equation $dy/dx - y = 1 - e^{-x}$ and $y(0) = y_0$ has a finite value, when $x \rightarrow \infty$, then the value of $|2/y_0|$ is _____.

Entrances Gallery

JEE Advanced/IITJEE

1. If the function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2 - 1} = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then, $\int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x)dx$ is [2014]
 (a) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (b) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (c) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (d) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$
2. A curve passes through the point $(1, \frac{\pi}{6})$. If the slope of the curve at each point (x, y) is $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$. Then, the equation of the curve is [2013]
 (a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$
 (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$
 (c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$
 (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$
3. Let $f: \left[\frac{1}{2}, 1\right] \rightarrow \mathbb{R}$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then, the value of $\int_{1/2}^1 f(x)dx$ lies in the interval [2013]
 (a) $(2e - 1, 2e)$ (b) $(e - 1, 2e - 1)$
 (c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$
4. If $y(x)$ satisfies the differential equation $y' - y \tan x = 2 \sec x$ and $y(0) = 0$, then [2012]
 (a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$
 (c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$
5. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in \mathbb{R}$, where $f'(x)$ denotes $\frac{df(x)}{dx}$ and $g(x)$ is a given non-constant, differentiable function on $\mathbb{R}$ with $g(0) = g(2) = 0$. Then, the value of $y(2)$ is [2011]

JEE Main/AIEEE

12

Differential Equations

Target Exercises

6. Let $y(x)$ be the solution of the differential equation $(x \log x) \frac{dy}{dx} + y = 2x \log x, x \geq 1$. Then, $y(e)$ is equal to [2015]
(a) e (b) 0 (c) 2 (d) $2e$
7. Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2} p(t) - 200$. If $p(0) = 100$, then $p(t)$ is equal to [2014]
(a) $400 - 300e^{t/2}$
(b) $300 - 200e^{t/2}$
(c) $600 - 500e^{t/2}$
(d) $400 - 300e^{-t/2}$
8. **Statement I** The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P .
Statement II The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1. [2013]
(a) Statement I is true, Statement II is false
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(d) Statement I is false, Statement II is true
9. Consider the differential equation $\frac{dy}{dx} = \frac{y^3}{2(xy^2 - x^2)}$
Statement I The substitution $z = y^2$ transforms the above equation into a first order homogeneous differential equation.
Statement II The solution of this differential equation is $y^2 e^{-y^2/x} = C$. [2013]
(a) Both statements are false
(b) Statement I is true and Statement II is false
(c) Statement I is false and Statement II is true
(d) Both statements are true
10. If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1 - \frac{1}{x^2}\right)$ at any point (x, y) on it, then the ordinate of the point on the curve, whose abscissa is -2 , is [2013]
(a) $-\frac{3}{2}$
(b) $\frac{3}{2}$
(c) $\frac{5}{2}$
(d) $-\frac{5}{2}$
11. The equation of the curve passing through the origin and satisfying the differential equation $(1 + x^2) \frac{dy}{dx} + 2xy = 4x^2$, is [2013]
(a) $(1 + x^2)y = x^3$ (b) $3(1 + x^2)y = 2x^3$
(c) $(1 + x^2)y = 3x^3$ (d) $3(1 + x^2)y = 4x^3$
12. At present, a firm is manufacturing 2000 items. It is estimated that the rate of change of production P with respect to additional number of workers x is given by $\frac{dp}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is [2013]
(a) 2500 (b) 3000 (c) 3500 (d) 4500
13. The population $p(t)$ at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero, is [2012]
(a) $2 \log 18$ (b) $\log 9$ (c) $\frac{1}{2} \log 18$ (d) $\log 18$
14. If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\log 2)$ is equal to [2011]
(a) 5 (b) 13 (c) -2 (d) 7
15. Let I be the purchase value of an equipment and $V(t)$ be the value after it has been used for t yr. The value $V(t)$ depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T - t)$, where $k > 0$, is a constant and T is the total life (in years) of the equipment. Then, the scrap value $V(T)$ of the equipment is [2011]
(a) $I - \frac{kT^2}{2}$ (b) $I - \frac{k(T - t)^2}{2}$ (c) e^{-kT} (d) $T^2 - \frac{1}{k}$
16. Consider the differential equation $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. If $y(1) = 1$, then x is given by [2011]
(a) $1 - \frac{1}{y} + \frac{e^{1/y}}{e}$ (b) $4 - \frac{2}{y} - \frac{e^{1/y}}{e}$
(c) $3 - \frac{1}{y} + \frac{e^{1/y}}{e}$ (d) $1 + \frac{1}{y} - \frac{e^{1/y}}{e}$
17. $\frac{d^2 x}{dy^2}$ equals [2011]
(a) $-\left(\frac{d^2 y}{dx^2}\right)^{-1} \left(\frac{dy}{dx}\right)^{-3}$ (b) $\left(\frac{d^2 y}{dx^2}\right) \left(\frac{dy}{dx}\right)^{-2}$
(c) $-\left(\frac{d^2 y}{dx^2}\right) \left(\frac{dy}{dx}\right)^{-3}$ (d) $\left(\frac{d^2 y}{dx^2}\right)^{-1}$

- 18.** The solution of differential equation $\cos x dy = y(\sin x - y)dx$, where $0 < x < \frac{\pi}{2}$, is [2010]
 (a) $\sec x = (\tan x + C)y$
 (b) $y \sec x = \tan x + C$
 (c) $y \tan x = \sec x + C$
 (d) $\tan x = (\sec x + C)y$
- 19.** The differential equation which represents the family of curves $y = C_1 e^{C_2 x}$, where C_1 and C_2 are arbitrary constants, is [2009]
 (a) $y' = y^2$ (b) $y'' = y' y$
 (c) $yy'' = y'$ (d) $yy'' = (y')^2$
- 20.** The differential equation of the family of circles with fixed radius 5 units and centre on the line $y = 2$, is [2008]
 (a) $(x - 2)^2 y'^2 = 25 - (y - 2)^2$
 (b) $(x - 2)y'^2 = 25 - (y - 2)^2$
 (c) $(y - 2)y'^2 = 25 - (y - 2)^2$
 (d) $(y - 2)^2 y'^2 = 25 - (y - 2)^2$
- 21.** The solution of the differential equation $\frac{dy}{dx} = \frac{x + y}{x}$ satisfying the condition $y(1) = 1$, is [2008]
 (a) $y = x \log x + x$
 (b) $y = \log x + x$
 (c) $y = x \log x + x^2$
 (d) $y = x e^{(x-1)}$
- 22.** The differential equation of all circles passing through the origin and having their centres on the X -axis, is [2007]
 (a) $x^2 = y^2 + xy \frac{dy}{dx}$
 (b) $x^2 = y^2 + 3xy \frac{dy}{dx}$
 (c) $y^2 = x^2 + 2xy \frac{dy}{dx}$
 (d) $y^2 = x^2 - 2xy \frac{dy}{dx}$
- 23.** The differential equation whose solution is $Ax^2 + By^2 = 1$, where A and B are arbitrary constants, is of [2006]
 (a) first order and second degree
 (b) first order and first degree
 (c) second order and first degree
 (d) second order and second degree
- 24.** If $x \frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is [2005]
 (a) $\log \left(\frac{x}{y} \right) = Cy$ (b) $\log \left(\frac{y}{x} \right) = Cx$
 (c) $x \log \left(\frac{y}{x} \right) = Cy$ (d) $y \log \left(\frac{x}{y} \right) = Cx$
- 25.** The differential equation representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where $c > 0$ is a parameter, is of [2005]
 (a) order 2, degree 2 (b) order 1, degree 3
 (c) order 1, degree 1 (d) order 1, degree 2
- 26.** The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant, is [2004]
 (a) $2(x^2 - y^2)y' = xy$
 (b) $2(x^2 + y^2)y' = xy$
 (c) $(x^2 - y^2)y' = 2xy$
 (d) $(x^2 + y^2)y' = 2xy$
- 27.** The solution of the differential equation $y dx + (x + x^2 y)dy = 0$ is [2004]
 (a) $-\frac{1}{xy} = C$ (b) $-\frac{1}{xy} + \log y = C$
 (c) $\frac{1}{xy} + \log y = C$ (d) $\log y = Cx$
- 28.** The degree and order of the differential equation of the family of all parabolas whose axis is X -axis, are respectively [2003]
 (a) 2, 1 (b) 1, 2 (c) 3, 2 (d) 2, 3
- 29.** The solution of the differential equation $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$ is [2003]
 (a) $(x - 2) = Ce^{-2 \tan^{-1} y}$
 (b) $2xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + C$
 (c) $xe^{\tan^{-1} y} = \tan^{-1} y + C$
 (d) $xe^{2 \tan^{-1} y} = e^{\tan^{-1} y} + C$
- 30.** The order and degree of the differential equation $\left(1 + 3 \frac{dy}{dx}\right)^{2/3} = 4 \frac{d^3 y}{dx^3}$ are [2002]
 (a) 1, $\frac{2}{3}$ (b) 3, 1 (c) 3, 3 (d) 1, 0
- 31.** The solution of the equation $\frac{d^2 y}{dx^2} = e^{-2x}$ is [2002]
 (a) $\frac{e^{-2x}}{4}$ (b) $\frac{e^{-2x}}{4} + Cx + d$
 (c) $\frac{1}{4} e^{-2x} + Cx^2 + d$ (d) $\frac{1}{4} e^{-2x} + C + d$
- 32.** The differential equation of all non-vertical lines in a plane is [2002]
 (a) $\frac{d^2 y}{dx^2} = 0$
 (b) $\frac{d^2 x}{dy^2} = 0$
 (c) $\frac{dy}{dx} = 0$
 (d) $\frac{dx}{dy} = 0$

Other Engineering Entrances

12

Differential Equations

33. The form of the differential equation of the central conics $ax^2 + by^2 = 1$, is [BITSAT 2014]

- (a) $x = y \frac{dy}{dx}$
 (b) $x \left(\frac{dy}{dx} \right)^2 + xy \frac{d^2y}{dx^2} - y \frac{dy}{dx} = 0$
 (c) $x + y \frac{d^2y}{dx^2} = 0$
 (d) None of the above

34. The differential equation of the rectangular hyperbola, where axes are the asymptotes of the hyperbola, is [VITEEE 2014]

- (a) $y \frac{dy}{dx} = x$
 (b) $x \frac{dy}{dx} = -y$
 (c) $x \frac{dy}{dx} = y$
 (d) $xdy + ydx = C$

35. The order and degree of the differential equation $y = x \frac{dy}{dx} + \frac{2}{\frac{dy}{dx}}$, are [Karnataka 2014]

- (a) 1, 2 (b) 1, 3 (c) 2, 1 (d) 1, 1

36. The curve $y = (\cos x + y)^{1/2}$ satisfies the differential equation [WB JEE 2014]

- (a) $(2y-1) \frac{d^2y}{dx^2} + 2 \left(\frac{dy}{dx} \right)^2 + \cos x = 0$
 (b) $\frac{d^2y}{dx^2} - 2y \left(\frac{dy}{dx} \right)^2 + \cos x = 0$
 (c) $(2y-1) \frac{d^2y}{dx^2} - 2 \left(\frac{dy}{dx} \right)^2 + \cos x = 0$
 (d) $(2y-1) \frac{d^2y}{dx^2} - \left(\frac{dy}{dx} \right)^2 + \cos x = 0$

37. The differential equation of the family of parabolas with vertex at $(0, -1)$ and having axis along Y -axis, is [EAMCET 2014]

- (a) $yy' + 2xy + 1 = 0$
 (b) $xy' + y + 1 = 0$
 (c) $xy' - 2y - 2 = 0$
 (d) $xy' - y - 1 = 0$

38. The solution of the differential equation

$$\frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} = \frac{dx - dy}{dx + dy} \text{ is [BITSAT 2014]}$$

- (a) $2ye^{2x} = Ce^{2x} + 1$
 (b) $2ye^{2x} = Ce^{2x} - 1$
 (c) $ye^{2x} = Ce^{2x} + 2$
 (d) None of the above

39. The solution of the differential equation

$$x = 1 + xy \frac{dy}{dx} + \frac{x^2 y^2}{2!} \left(\frac{dy}{dx} \right)^2 + \frac{x^3 y^3}{3!} \left(\frac{dy}{dx} \right)^3 + \dots \text{ is [BITSAT 2014]}$$

- (a) $y = \log x + C$ (b) $y^2 = (\log x)^2 + C$
 (c) $y = \log x + xy$ (d) $xy = x^y + C$

40. If $x \cdot \frac{dy}{dx} + y = x \cdot \frac{f(xy)}{f'(xy)}$, then $f(xy)$ is equal to [VITEEE 2014]

- (a) $k \cdot e^{x^{1/2}}$ (b) $k \cdot e^{y^{1/2}}$
 (c) $k \cdot e^{x^2}$ (d) $k \cdot e^{xy/2}$

41. The solution of $\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$, satisfying

- $y(1) = 0$, is given by [VITEEE 2014]
 (a) hyperbola (b) circle
 (c) ellipse (d) parabola

42. Which one of the following is correct solution of

$$\left(\frac{dy}{dx} \right) \tan y = \sin(x+y) + \sin(x-y)? \text{ [AMU 2014]}$$

- (a) $\sec x = C - 2 \sec y$
 (b) $\sec y = C + 2 \cos y$
 (c) $\sec y = C - 2 \cos x$
 (d) $\sec x = C - 2 \cos y$

43. The real value of m for which the substitution $y = u^m$ will transform the differential equation $2x^4 \cdot y \frac{dy}{dx} + y^4 = 4x^6$ into a homogeneous equation, is [Manipal 2014]

- (a) $m = 0$ (b) $m = 1$ (c) $m = \frac{3}{2}$ (d) $m = \frac{2}{3}$

44. The solution of the differential equation

$$y \frac{dy}{dx} = x \left[\frac{y^2}{x^2} + \frac{\phi \left(\frac{y^2}{x^2} \right)}{\phi' \left(\frac{y^2}{x^2} \right)} \right] \text{ is [WB JEE 2014]}$$

- (a) $\phi \left(\frac{y^2}{x^2} \right) = Cx$ (b) $x\phi \left(\frac{y^2}{x^2} \right) = C$
 (c) $\phi \left(\frac{y^2}{x^2} \right) = Cx^2$ (d) $x^2\phi \left(\frac{y^2}{x^2} \right) = C$

45. Which of the following is correct solution of

$$(1 + e^{x/y}) dx + e^{x/y} \left[1 - \left(\frac{x}{y} \right) \right] dy = 0? \text{ [AMU 2014]}$$

- (a) $x + ye^{x/y} = C$
 (b) $y + ye^{x/y} = C$
 (c) $x + y = Ce^{-x/y}$
 (d) $y = x + Ce^{x/y}$

Target Exercises

46. The solution of $x \frac{dy}{dx} = y + xe^{y/x}$ with $y(1) = 0$ is [EAMCET 2014]
 (a) $e^{y/x} + \log x = 1$ (b) $e^{-y/x} = \log x$
 (c) $e^{-y/x} + 2\log x = 1$ (d) $e^{-y/x} + \log x = 1$
47. The solution of the differential equation $(x^2 - y^2)dx + 2xy dy = 0$ is [GGSIPU 2014]
 (a) $x^2 - y^2 = Cx$ (b) $x^2 - y^2 = Cy$
 (c) $x^2 + y^2 = Cx$ (d) $x^2 + y^2 = Cy$
48. The general solution of the differential equation $\frac{dy}{dx} + \frac{y}{x} = 3x$ is [Karnataka CET 2014]
 (a) $y = x - \frac{C}{x}$ (b) $y = x + \frac{C}{x}$
 (c) $y = x^2 - \frac{C}{x}$ (d) $y = x^2 + \frac{C}{x}$
49. The integrating factor of the differential equation $(1+x^2)\frac{dy}{dx} + y = e^{\tan^{-1}x}$ is [WB JEE 2014]
 (a) $\tan^{-1}x$ (b) $1+x^2$
 (c) $e^{\tan^{-1}x}$ (d) $\log_e(1+x^2)$
50. The solution of the differential equation $\frac{dy}{dx} + \frac{y}{x \log_e x} = \frac{1}{x}$ under the condition $y = 1$, when $x = e$, is [WB JEE 2014]
 (a) $2y = \log_e x + \frac{1}{\log_e x}$ (b) $y = \log_e x + \frac{1}{\log_e x}$
 (c) $y \log_e x = \log_e x + 1$ (d) $y = \log_e x + e$
51. Which of the following is a correct solution of $x \cos x \left(\frac{dy}{dx} \right) + y(x \sin x + \cos x) = 1$? [AMU 2014]
 (a) $yx \sec y = C + \tan x$
 (b) $yx \cos y = C + \tan x$
 (c) $yx \sec x = C + \tan x$
 (d) None of the above
52. If $f(x)$ is differentiable on the interval $(0, \infty)$ such that $f(1) = 1$ and $\lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$ for each $x > 0$. Then, $f(x)$ is [AMU 2014]
 (a) $\frac{1}{3x} + \frac{2x^2}{3}$ (b) $-\frac{1}{3x} + \frac{4x^2}{3}$
 (c) $-\frac{1}{x} + \frac{2}{x^2}$ (d) $\frac{1}{x}$
53. Which of the following is a correct solution of $y^2 + \left(x - \frac{1}{y}\right) \frac{dy}{dx} = 0$? [AMU 2014]
 (a) $y = Ce^{1/x} + \frac{1}{y} + 1$ (b) $x = Ce^{1/y} + \frac{1}{y} + 1$
 (c) $x = Ce^{1/x} + \frac{1}{y} + 1$ (d) $y = Ce^{1/y} + \frac{1}{x} + 1$
54. The solution of $\cos y + (x \sin y - 1) \frac{dy}{dx} = 0$, is [EAMCET 2014]
 (a) $x \sec y = \tan y + C$ (b) $\tan y - \sec y = Cx$
 (c) $\tan y + \sec y = Cx$ (d) $x \sec y + \tan y = C$
55. Solve $\frac{dy}{dx} + \frac{y}{x} = x^3$. [J&K CET 2014]
 (a) $y = \frac{x^4}{5} + \frac{C}{x}$
 (b) $y = \frac{x^3}{3} + \frac{C}{x}$
 (c) $y = \frac{x^3}{3} + C$
 (d) $y = \frac{x^4}{4} + x$
56. The equation of the curve for which the square of the ordinate is twice the rectangle contained by the abscissa and the intercept of the normal on the X -axis and passing through $(2, 1)$, is [Manipal 2014]
 (a) $x^2 + y^2 - x = 0$
 (b) $4x^2 + 2y^2 - 9y = 0$
 (c) $2x^2 + 4y^2 - 9x = 0$
 (d) $4x^2 + 2y^2 - 9x = 0$
57. A gardener is digging a plot of land. As he gets tired, he works more slowly. After t min, he is digging at a rate of $\frac{2}{\sqrt{t}}$ m²/min. How long will it take him to digging an area of 40 sq m? [Karnataka CET 2014]
 (a) 100 min (b) 10 min (c) 30 min (d) 40 min
58. If $\sqrt{y} = \cos^{-1} x$, then it satisfies the differential equation $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = C$, where C is equal to [WB JEE 2014]
 (a) 0 (b) 3 (c) 1 (d) 2
59. The solution of the differential equation $\frac{d^2 y}{dx^2} + 3y = -2x$ is [GGSIPU 2014]
 (a) $C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x - \frac{2}{3}x$
 (b) $C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x - \frac{4}{5}$
 (c) $C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x - 2x^2 + \frac{4}{9}$
 (d) $C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x - \frac{2}{3}x^2 + \frac{4}{9}$
60. The solution of the differential equation $\left(\frac{dy}{dx}\right)^2 - x \frac{dy}{dx} + y = 0$ is [AMU 2013]
 (a) $y = 2x$ (b) $y = -2x$
 (c) $y = 2x - 4$ (d) $y = 2x + 4$

61. The solution of the equation $(x^2 + xy)dy = (x^2 + y^2)dx$ is [Karnataka CET 2013]
 (a) $\log x = \log(x - y) + \frac{y}{x} + C$
 (b) $\log x = 2\log(x - y) + \frac{y}{x} + C$
 (c) $\log y = \log(x - y) + \frac{y}{x} + C$
 (d) None of the above
62. An integrating factor of the differential equation $x \frac{dy}{dx} + y \log x = xe^x x^{-\frac{1}{2} \log x}$ ($x > 0$) is [Manipal 2013]
 (a) $x^{\log x}$ (b) $(\sqrt{x})^{\log x}$
 (c) $(\sqrt{e})^{(\log x)^2}$ (d) e^{x^2}
63. The order and degree of the differential equation $\left(\frac{d^3 y}{dx^3}\right)^{4/5} - 2 \frac{dy}{dx} \left(\frac{d^2 y}{dx^2}\right)^2 = 0$, are [MP PET 2013]
 (a) 2, 10 (b) 3, 10 (c) 3, 5 (d) 3, 4
64. The differential equation of system of circles touching Y-axis at origin, is [MP PET 2013]
 (a) $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$ (b) $\frac{dy}{dx} = \frac{y^2 - x^2}{xy}$
 (c) $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ (d) $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$
65. Order of the equation $\left(1 + 5 \frac{dy}{dx}\right)^{3/2} = 10 \frac{d^3 y}{dx^3}$ is [OJEE 2012]
 (a) 2 (b) 3 (c) 1 (d) 0
66. The degree of the differential equation satisfying $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ is [AMU 2012]
 (a) 1 (b) 3
 (c) 2 (d) None of these
67. The solution of $e^{dy/dx} = x$, when $x = 1$ and $y = 0$, is [Karnataka CET 2012]
 (a) $y = x(\log x - 1) + 4$
 (b) $y = x(\log x - 1) + 3$
 (c) $y = x(\log x + 1) + 1$
 (d) $y = x(\log x - 1) + 1$
68. The general solution of the differential equation $\frac{dy}{dx} = \frac{x + y + 1}{2x + 2y + 1}$ is [WB JEE 2012]
 (a) $\log_e |3x + 3y + 2| + 3x + 6y = C$
 (b) $\log_e |3x + 3y + 2| - 3x + 6y = C$
 (c) $\log_e |3x + 3y + 2| - 3x - 6y = C$
 (d) $\log_e |3x + 3y + 2| + 3x - 6y = C$
69. If $y(t)$ is solution of $(1+t) \frac{dy}{dt} - ty = 1$ and $y(0) = -1$, then $y(1)$ is equal to [Manipal 2012]
 (a) $-\frac{1}{2}$ (b) $e + \frac{1}{2}$ (c) $e - \frac{1}{2}$ (d) $\frac{1}{2}$
70. The solution of the differential equation $(x+a) \frac{dy}{dx} - 3y = (x+a)^5$, when $y = 16a^5$ at $x = a$, is [MP PET 2012]
 (a) $y = (x+a)^5$
 (b) $2y = (x+a)^5$
 (c) $2y = (x+a)^3 + (x+a)^5$
 (d) $y = (x+a)^3 - (x+a)^5$
71. The solution of $(x + 2y^3) \frac{dy}{dx} = y$ is [OJEE 2012]
 (a) $x = y^3 + Cy$ (b) $x = y^2 + Cy$
 (c) $x^3 = y^3 + Cy$ (d) None of these
72. The order of the differential equation $\frac{d^2 y}{dx^2} + 5 \frac{dy}{dx} + \int y dx = x^3$ is [MP PET 2011]
 (a) 2 (b) 3 (c) 1 (d) 4
73. If $\sqrt{r} = ae^{\theta \cot \alpha}$, where a and α are real numbers, then $\frac{d^2 r}{d\theta^2} - 4r \cot^2 \alpha$ is equal to [Karnataka CET 2011]
 (a) r (b) $\frac{1}{r}$
 (c) 1 (d) 0
74. If $y = (\sin^{-1} x)^2$, then $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx}$ is equal to [Kerala CEE 2011]
 (a) 0 (b) -1 (c) -2
 (d) 1 (e) 2
75. If $y = \frac{A}{x} + Bx^2$, then $x^2 \frac{d^2 y}{dx^2}$ is equal to [WB JEE 2011]
 (a) $2y$ (b) y^2
 (c) y^3 (d) y^4
76. The degree of the differential equation $\frac{d^2 y}{dx^2} = \frac{5y + \frac{dy}{dx}}{\sqrt{\frac{d^2 y}{dx^2}}}$ is [J&K CET 2011]
 (a) 2 (b) 3
 (c) 4 (d) $\frac{5}{2}$
77. The general solution of $\left(\frac{dy}{dx}\right)^2 = 1 - x^2 - y^2 + x^2 y^2$ is [Karnataka CET 2011]
 (a) $2 \sin^{-1} y = x \sqrt{1-x^2} + \sin^{-1} x + C$
 (b) $\cos^{-1} y = \cos^{-1} x + C$
 (c) $\sin^{-1} y = \frac{1}{2} \sin^{-1} x + C$
 (d) $2 \sin^{-1} y = x \sqrt{1-y^2} + C$

- 78.** The general solution of the differential equation $\frac{dy}{dx} = e^y (e^x + e^{-x} + 2x)$ is [Kerala CEE 2011]
 (a) $e^{-y} = e^x - e^{-x} + x^2 + C$
 (b) $e^{-y} = e^{-x} - e^x - x^2 + C$
 (c) $e^{-y} = -e^{-x} - e^x - x^2 + C$
 (d) $e^y = e^{-x} + e^x + x^2 + C$
 (e) $e^y = e^{-x} + e^x + C$
- 79.** The solution of differential equation [UPSEE 2011]
 $4xy \frac{dy}{dx} = \frac{3(1+x)^2(1+y^2)}{(1+x^2)}$ is
 (a) $\log(1+y) = \log x + 2 \tan x + C$
 (b) $\log(1+y^2) = 3 \log \frac{1}{x} + 6 \tan^{-1} x + C$
 (c) $2 \log(1+y^2) = 3 \log x + 6 \tan^{-1} x + C$
 (d) None of the above
- 80.** An integrating factor of the differential equation $(1+x^2) \frac{dy}{dx} + xy = x$ is [Kerala CEE 2011]
 (a) $\frac{x}{1+x^2}$ (b) $\frac{1}{2} \log(1+x^2)$
 (c) $\sqrt{1+x^2}$ (d) x
 (e) $\frac{1}{1+x^2}$
- 81.** An integrating factor of the differential equation $\frac{dy}{dx} - \frac{3x^2 y}{1+x^3} = \frac{\sin^2 x}{1+x}$ is [WB JEE 2011]
 (a) e^{1+x^3} (b) $\log(1+x^3)$
 (c) $1+x^3$ (d) $\frac{1}{1+x^3}$
- 82.** The solution of $y dx - x dy + \log x dx = 0$ is [GGSIU 2011]
 (a) $y - \log x - 1 = Cx$
 (b) $x + \log y + 1 = Cx$
 (c) $y + \log x + 1 = Cx$
 (d) $y + \log x - 1 = Cx$
- 83.** The general solution of the differential equation $\frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 16y = 0$ is [WB JEE 2011]
 (a) $(A + Bx)e^{5x}$
 (b) $(A + Bx)e^{-4x}$
 (c) $(A + Bx^2)e^{4x}$
 (d) $(A + Bx^4)e^{4x}$
- 84.** The degree and order of the differential equation $y = px + \sqrt{a^2 p^2 + b^2}$, where $p = \frac{dy}{dx}$, are respectively [BITSAT 2010]
 (a) 3, 1 (b) 1, 3
 (c) 1, 1 (d) 3, 3
- 85.** The family of curves $y = e^{a \sin x}$, where a is an arbitrary constant, is represented by the differential equation [Kerala CEE 2010]
 (a) $\log y = \tan x \frac{dy}{dx}$
 (b) $y \log y = \tan x \frac{dy}{dx}$
 (c) $y \log y = \sin x \frac{dy}{dx}$
 (d) $\log y = \cos x \frac{dy}{dx}$
 (e) $y \log y = \cos x \frac{dy}{dx}$
- 86.** The degree of the differential equation $x = 1 + \left(\frac{dy}{dx}\right) + \frac{1}{2!} \left(\frac{dy}{dx}\right)^2 + \frac{1}{3!} \left(\frac{dy}{dx}\right)^3 + \dots$, is
 (a) 3 (b) 2
 (c) 1 (d) Not defined
- 87.** The solution of the differential equation $(x^2 - yx^2) \frac{dy}{dx} + y^2 + xy^2 = 0$ is [UPSEE 2010]
 (a) $\log \left(\frac{x}{y}\right) = \frac{1}{x} + \frac{1}{y} + C$ (b) $\log \left(\frac{y}{x}\right) = \frac{1}{x} + \frac{1}{y} + C$
 (c) $\log(xy) = \frac{1}{x} + \frac{1}{y} + C$ (d) $\log(xy) + \frac{1}{x} + \frac{1}{y} = C$
- 88.** The solution of the differential equation $xdy - ydx = 0$ represents a [DCE 2010]
 (a) parabola (b) circle
 (c) hyperbola (d) straight line
- 89.** The integrating factor of $x \frac{dy}{dx} + (1+x)y = x$ is [VITEEE 2010]
 (a) x (b) $2x$
 (c) $e^{x \log x}$ (d) xe^x
- 90.** The equation of one of the curves whose slope at any point is equal to $y + 2x$, is [WB JEE 2010]
 (a) $y = 2(e^x + x - 1)$
 (b) $y = 2(e^x - x - 1)$
 (c) $y = 2(e^x - x + 1)$
 (d) $y = 2(e^x + x + 1)$
- 91.** The general solution of the differential equation $100 \frac{d^2 y}{dx^2} - 20 \frac{dy}{dx} + y = 0$ is [WB JEE 2010]
 (a) $y = (C_1 + C_2 x)e^x$
 (b) $y = (C_1 + C_2 x)e^{-x}$
 (c) $y = (C_1 + C_2 x)e^{x/10}$
 (d) $y = C_1 e^x + C_2 e^{-x}$
- 92.** If $y'' - 3y' + 2y = 0$, where $y(0) = 1$, $y'(0) = 0$, then the value of y at $x = \log 2$ is [WB JEE 2010]
 (a) 1 (b) -1
 (c) 2 (d) 0

Answers

Work Book Exercise 12.1

1. (a) 2. (a) 3. (b) 4. (a) 5. (a) 6. (c) 7. (a) 8. (c) 9. (c) 10. (c)

Work Book Exercise 12.2

1. (d) 2. (d) 3. (c) 4. (d) 5. (a) 6. (b) 7. (a)

Work Book Exercise 12.3

1. (b) 2. (a) 3. (a) 4. (b) 5. (a) 6. (b) 7. (b) 8. (c) 9. (b)

Work Book Exercise 12.4

1. (b) 2. (b) 3. (b) 4. (c) 5. (a) 6. (b) 7. (b) 8. (b) 9. (c)

Work Book Exercise 12.5

1. (a) 2. (c) 3. (a) 4. (b) 5. (a) 6. (a) 7. (b)

Target Exercises

1. (c) 2. (b) 3. (d) 4. (c) 5. (b) 6. (b) 7. (d) 8. (b) 9. (d) 10. (c)
 11. (d) 12. (c) 13. (b) 14. (b) 15. (d) 16. (b) 17. (c) 18. (d) 19. (a) 20. (a)
 21. (c) 22. (a) 23. (b) 24. (a) 25. (b) 26. (a) 27. (b) 28. (b) 29. (c) 30. (b)
 31. (a) 32. (a) 33. (c) 34. (a) 35. (b) 36. (c) 37. (b) 38. (b) 39. (a) 40. (a)
 41. (b) 42. (b) 43. (c) 44. (a,b,c) 45. (a,b,d) 46. (a,b,c) 47. (a,b,c) 48. (b,c) 49. (a,d) 50. (a,b,c)
 51. (b,c,d) 52. (a,d) 53. (b) 54. (a) 55. (a) 56. (a) 57. (b) 58. (b) 59. (a) 60. (c)
 61. (d) 62. (a) 63. (b) 64. (a) 65. (b) 66. (a) 67. (b) 68. (a) 69. (c) 70. (b)
 71. (a) 72. (c) 73. (c) 74. (*) 75. (*) 76. (*) 77. (6) 78. (0) 79. (1) 80. (1)
 81. (0) 82. (8) 83. (2) 84. (4)

*

74. $A \rightarrow s; B \rightarrow s; C \rightarrow q; D \rightarrow p$

75. $A \rightarrow q; B \rightarrow r; C \rightarrow s; D \rightarrow p$

76. $A \rightarrow s; B \rightarrow r; C \rightarrow q; D \rightarrow p$

Entrances Gallery

1. (b) 2. (a) 3. (d) 4. (a,d) 5. (0) 6. (c) 7. (a) 8. (b) 9. (d) 10. (a)
 11. (d) 12. (c) 13. (a) 14. (d) 15. (a) 16. (d) 17. (c) 18. (a) 19. (d) 20. (d)
 21. (a) 22. (c) 23. (c) 24. (b) 25. (b) 26. (c) 27. (b) 28. (b) 29. (b) 30. (c)
 31. (b) 32. (a) 33. (b) 34. (b) 35. (a) 36. (a) 37. (c) 38. (b) 39. (b) 40. (a)
 41. (a) 42. (c) 43. (c) 44. (c) 45. (a) 46. (d) 47. (c) 48. (c,d) 49. (c) 50. (a)
 51. (c) 52. (a) 53. (b) 54. (a) 55. (a) 56. (d) 57. (a) 58. (d) 59. (a) 60. (c)
 61. (b) 62. (c) 63. (d) 64. (c) 65. (b) 66. (a) 67. (d) 68. (d) 69. (a) 70. (b)
 71. (a) 72. (b) 73. (d) 74. (e) 75. (a) 76. (b) 77. (a) 78. (b) 79. (c) 80. (c)
 81. (d) 82. (c) 83. (b) 84. (a) 85. (b) 86. (c) 87. (a) 88. (d) 89. (d) 90. (b)
 91. (c) 92. (d)

Explanations

Target Exercises

1. Given differential equation is $y^2 = 2c(x + c^{2/3})$

$$\Rightarrow 2y \frac{dy}{dx} = 2c$$

$$\Rightarrow c = y \frac{dy}{dx}$$

$$\therefore y^2 = 2y \frac{dy}{dx} \left[x + \left(y \frac{dy}{dx} \right)^{2/3} \right]$$

$$\Rightarrow \left(\frac{y}{2 \frac{dy}{dx}} - x \right) = \left(y \frac{dy}{dx} \right)^{2/3}$$

$$\Rightarrow \left(y - 2x \frac{dy}{dx} \right)^3 = \left(2 \frac{dy}{dx} \right)^3 \left(y \frac{dy}{dx} \right)^2$$

$$\Rightarrow \left(y - 2x \frac{dy}{dx} \right)^3 = 8y^2 \left(\frac{dy}{dx} \right)^5$$

Hence, order = 1, degree = 5

2. We have,

$$\left(\frac{d^4 y}{dx^4} \right)^{3/5} = 5 \frac{d^3 y}{dx^3} - 6 \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} - 5$$

$$\Rightarrow \left(\frac{d^4 y}{dx^4} \right)^3 = \left(5 \frac{d^3 y}{dx^3} - 6 \frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} - 5 \right)^5$$

which is a differential equation of order 4 and degree 3.

3. Given, $y = ae^x + bxe^x + cx^2e^x$... (i)

On differentiating both sides w.r.t. x, we get

$$y' = ae^x + b(xe^x + e^x) + c(x^2e^x + 2xe^x)$$

$$\Rightarrow y' = ae^x + bxe^x + cx^2e^x + be^x + 2cxe^x$$

$$\Rightarrow y' = y + be^x + 2cxe^x$$

$$[\because y = ae^x + bxe^x + cx^2e^x] \dots (ii)$$

Again, differentiating both sides w.r.t. x, we get

$$y'' = y' + be^x + 2c(xe^x + e^x)$$

$$\Rightarrow y'' = y' + be^x + 2cxe^x + 2ce^x$$

$$\Rightarrow y'' = 2y' - y + 2ce^x \quad [\text{from Eq. (ii)}] \dots (iii)$$

Again, differentiating both sides w.r.t. x, we get

$$y''' = 2y'' - y' + 2ce^x$$

$$\Rightarrow y''' = 2y'' - y' + (y'' - 2y' + y) \quad [\text{from Eq. (iii)}]$$

$$\Rightarrow y''' - 3y'' + 3y' - y = 0$$

4. The equation of the family of circles of radius r is

$$(x-a)^2 + (y-b)^2 = r^2 \quad \dots (i)$$

where, a and b are arbitrary constants.

On differentiating w.r.t. x, we get

$$2(x-a) + 2(y-b) \frac{dy}{dx} = 0$$

$$\Rightarrow (x-a) + (y-b) \frac{dy}{dx} = 0 \quad \dots (ii)$$

$$\Rightarrow 1 + (y-b) \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx} \right)^2 = 0$$

[differentiating w.r.t. x]

$$\Rightarrow (y-b) = - \frac{1 + \left(\frac{dy}{dx} \right)^2}{\left(\frac{d^2 y}{dx^2} \right)} \quad \dots (iii)$$

$$\text{From Eq. (ii), } (x-a) = \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right] \frac{dy}{dx}}{\frac{d^2 y}{dx^2}} \quad \dots (iv)$$

On putting the values of (y-b) and (x-a) in Eq. (i), we get

$$\frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^2 \left(\frac{dy}{dx} \right)^2}{\left(\frac{d^2 y}{dx^2} \right)^2} + \frac{\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^2}{\left(\frac{d^2 y}{dx^2} \right)^2} = r^2$$

$$\Rightarrow \left[1 + \left(\frac{dy}{dx} \right)^2 \right]^3 = r^2 \left(\frac{d^2 y}{dx^2} \right)^2$$

5. We have, $y = Ae^{2x} + Be^{-x/2}$... (i)

$$\Rightarrow \frac{dy}{dx} = 2Ae^{2x} - \frac{1}{2}Be^{-x/2} \quad \dots (ii)$$

$$\text{and } \frac{d^2 y}{dx^2} = 4Ae^{2x} + \frac{1}{4}Be^{-x/2} \quad \dots (iii)$$

On eliminating A from Eqs. (i) and (ii), we get

$$Be^{-x/2} = \frac{2}{5} \left(2y - \frac{dy}{dx} \right) \quad \dots (iv)$$

$$\text{From Eq. (i), } Ae^{2x} = y - Be^{-x/2} = y - \frac{2}{5} \left(2y - \frac{dy}{dx} \right)$$

$$\Rightarrow Ae^{2x} = \frac{1}{5} \left(y + 2 \frac{dy}{dx} \right) \quad \dots (v)$$

So, from Eqs. (iii), (iv) and (v), we get

$$\frac{d^2 y}{dx^2} = \frac{4}{5} \left(y + 2 \frac{dy}{dx} \right) + \frac{1}{4} \cdot \frac{2}{5} \left(2y - \frac{dy}{dx} \right)$$

$$\Rightarrow 2 \frac{d^2 y}{dx^2} - 3 \frac{dy}{dx} - 2y = 0$$

6. Given solution is $ax^2 + by^2 = 1$... (i)

Again, differentiating w.r.t. x, we get

$$2ax + 2byy' = 0 \quad \dots (ii)$$

Again, differentiating, we get

$$2a + 2b(y'^2 + yy'') = 0 \quad \dots (iii)$$

$$\text{From Eq. (ii), we have } \frac{a}{b} = \frac{-yy'}{x}$$

$$\text{Also, from Eq. (iii), we have } \frac{a}{b} = -(y'^2 + yy'')$$

$$\Rightarrow \frac{-yy'}{x} = -(y'^2 + yy'')$$

$$\Rightarrow yy' = xy'^2 + xyy''$$

$$\Rightarrow xyy'' + xy'^2 - yy' = 0$$

which is the required differential equation.

7. The equation of the general circle is given by
 $x^2 + y^2 + 2gx + 2fy + c = 0$... (i)

On differentiating w.r.t. x , we get
 $2x + 2y' + 2g + 2fy' = 0$... (ii)

Again, differentiating, we get
 $1 + y'^2 + yy'' + fy'' = 0$... (iii)

Again, differentiating, we get
 $2y'y'' + yy''' + y'y'' + fy''' = 0$... (iv)

On eliminating f from Eqs. (iii) and (iv), we get
 $y'''(1 + yy'' + y'^2) - y''(yy''' + 3y'y'') = 0$

$\Rightarrow y'''(1 + y'^2) - 3y'y''^2 = 0$
 which is the required differential equation.

8. The general equation of all ellipses centred at the origin is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

On differentiating w.r.t. x , we get
 $\frac{2x}{a^2} + \frac{2y}{b^2} \cdot y_1 = 0$
 $\Rightarrow \frac{x}{a^2} + \frac{y}{b^2} \cdot y_1 = 0$... (i)

Again, differentiating, we get
 $\frac{1}{a^2} + \frac{1}{b^2}(y_1^2 + yy_2) = 0$... (ii)

From Eqs. (i) and (ii), we get
 $\frac{1}{b^2}(yy_1 - xy_1^2 - xy_2) = 0$
 $\Rightarrow xyy_2 + xy_1^2 - yy_1 = 0$

which is the required differential equation.

9. We have, $\frac{dy}{dx} = -A\alpha \sin \alpha x + B\alpha \cos \alpha x$
 $\Rightarrow \frac{d^2x}{dx^2} = -A\alpha^2 \cos \alpha x - B\alpha^2 \sin \alpha x = -\alpha^2 y$
 $\therefore \frac{d^2y}{dx^2} + \alpha^2 y = 0$

10. We have, $\cos y \, dx + (1 + 2e^{-x}) \sin y \, dy = 0$

$\Rightarrow \frac{dx}{1 + 2e^{-x}} + \frac{\sin y}{\cos y} dy = 0$

$\Rightarrow \int \frac{e^x}{e^x + 2} dx - \int \left(\frac{-\sin y}{\cos y} \right) dy = 0$

$\Rightarrow \log(e^x + 2) - \log \cos y = \log C$

$\Rightarrow \log \left(\frac{e^x + 2}{\cos y} \right) = \log C$

$\Rightarrow e^x + 2 = C \cos y$

where, $1 + 2 = C \cdot \frac{1}{\sqrt{2}} \Rightarrow C = 3\sqrt{2}$

$\therefore$ The solution is $e^x + 2 = 3\sqrt{2} \cos y$.

11. We have,

Cartesian sub-tangent $\propto \frac{1}{\text{Square of abscissa}}$

$\Rightarrow \frac{y}{dy/dx} = \frac{k}{x^2} \Rightarrow \frac{dy}{y} = \frac{x^2}{k} dx$

On integrating, we $\log y = \frac{x^3}{3k} + \log C$

$\Rightarrow y = Ce^{x^3/3k}$

12. We have, $\log \frac{dy}{dx} = 9x - 6y + 6$

$\Rightarrow \frac{dy}{dx} = e^{9x-6y+6} = e^{9x+6} \cdot e^{-6y}$

$\Rightarrow e^{6y} dy = e^{9x+6} dx$

On integrating, we get $\frac{e^{6y}}{6} = \frac{e^{9x+6}}{9} + C$

On putting $x = 0, y = 1$, we get

$\frac{e^6}{6} = \frac{e^6}{9} + C \Rightarrow C = \frac{e^6}{18}$

$\therefore$ The solution is $\frac{e^{6y}}{6} = \frac{e^{9x+6}}{9} + \frac{e^6}{18}$

$\Rightarrow 3e^{6y} = 2e^{9x+6} + e^6$

13. We have, $\frac{dy}{dx} = \frac{ax+g}{by+f} \Rightarrow (by+f)dy = (ax+g)dx$

On integrating, we get

$\frac{by^2}{2} + fy = \frac{ax^2}{2} + gx + C$

$\Rightarrow ax^2 - by^2 + 2gx - 2fy + c = 0$

which represents a circle, if $a = -b$.

14. We have, $2f(x) = f'(x) \Rightarrow \frac{f'(x)}{f(x)} = 2$

On integrating, we get $\log f(x) = 2x + C_1$

$\Rightarrow f(x) = e^{2x+C_1} = e^{C_1} \cdot e^{2x} = Ce^{2x}$

where, $C = e^{C_1}$

On putting $x = 0, f(0) = 3$, we get $C = 3$

$\therefore f(x) = 3e^{2x} \Rightarrow f(2) = 3e^4$

15. We have, $x dy - y dx = 0 \Rightarrow \frac{dy}{y} - \frac{dx}{x} = 0$

On integrating, we get $\log y - \log x = \log C$

$\Rightarrow \log \frac{y}{x} = \log C \Rightarrow \frac{y}{x} = C$

$\Rightarrow y = Cx$

which is a straight line through the origin.

16. We have, $\frac{dy}{dx} = \frac{y}{x}$

$\Rightarrow \frac{dy}{y} = \frac{dx}{x}$

On integrating, we get

$\log y = \log x + \log k = \log kx$

$\Rightarrow y = kx$

17. We have, $y = ky + b$

$\Rightarrow \frac{dy}{dx} = k$

$\Rightarrow kx + b = k + xk^2$

$\Rightarrow k = k^2$ and $b = k$

$\therefore k = 0$

or $k = 1$

18. We have, $2x \frac{dy}{dx} = y + 3 \Rightarrow \frac{2}{y+3} dy = \frac{dx}{x}$

On integrating, we get

$2 \log(y+3) = \log x + \log C = \log Cx$

$\Rightarrow \log(y+3)^2 = \log Cx$

$\Rightarrow (y+3)^2 = Cx$

which represents a parabola.

19. Put $y = vx$, so that $\frac{dy}{dx} = v + x \frac{dv}{dx}$, we get

$$v + x \frac{dv}{dx} = \frac{v(2v-1)}{(2v+1)}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-2v}{2v+1}$$

$$\Rightarrow 2 \frac{dx}{x} + \left(2 + \frac{1}{v}\right) dv = 0$$

On integrating, we get

$$2 \log x + 2v + \log v = C$$

$$\Rightarrow \log(x^2 v) = -2v + C$$

$$\Rightarrow \log(xy) = C - \frac{2y}{x}$$

When $x = 1, y = 1$

$$\therefore C = \log 1 + 2 = 2$$

$$\text{So, the solution is } \log(xy) = 2 - \frac{2y}{x}$$

$$\Rightarrow x \log(xy) = 2(x - y)$$

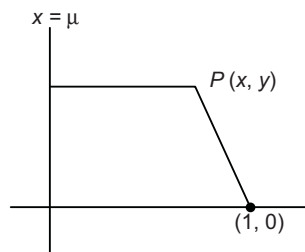
20. Directrix $\perp$ to X-axis. Let $x = \alpha$ and focus is on X-axis.

Let $(\beta, 0)$.

$$\text{Now, } (x - \beta)^2 + y^2 = (x - \alpha)^2$$

$$\Rightarrow \beta^2 - 2\beta x + y^2 = \alpha^2 - 2\alpha x$$

$$\Rightarrow y^2 = 2(\beta - \alpha)x + \alpha^2 - \beta^2$$



In general, $y^2 = mx + c$

$$2y \frac{dy}{dx} = m \Rightarrow 2y \frac{d^2 y}{dx^2} + 2 \left(\frac{dy}{dx}\right)^2 = 0$$

$$\Rightarrow y \frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$$

21. We have, $x^2 \frac{dy}{dx} - xy = 1 + \cos \frac{y}{x} = 2 \cos^2 \frac{y}{2x}$

$$\Rightarrow \sec^2 \left(\frac{y}{2x}\right) \cdot \left[x^2 \frac{dy}{dx} - xy\right] = 2$$

$$\Rightarrow \frac{1}{2} \sec^2 \left(\frac{y}{2x}\right) \cdot \frac{x \frac{dy}{dx} - y}{x^2} = \frac{1}{x^3}$$

$$\Rightarrow \frac{d}{dx} \left(\tan \frac{y}{2x} \right) = \frac{1}{x^3}$$

$$\text{On integrating, we get } \tan \frac{y}{2x} = C - \frac{1}{2x^2}$$

22. We have, $e^{dy/dx} = x + 3$

$$\Rightarrow \frac{dy}{dx} = \log(x + 3)$$

$$\Rightarrow dy = \log(x + 3) dx$$

On integrating, we get

$$y = x \log(x + 3) - \int \frac{x}{x+3} dx + C$$

$$\Rightarrow y = x \log(x + 3) - \int \frac{x+3-3}{x+3} dx + C$$

$$\Rightarrow y = x \log(x + 3) - x + 3 \log(x + 3) + C$$

$$\Rightarrow y = (x + 3) \log(x + 3) - x + C$$

On putting $x = -2, y = 3$, we get

$$3 = \log 1 + 2 + C \Rightarrow C = 1$$

$$\therefore \text{The solution is } y = (x + 3) \log(x + 3) - x + 1$$

23. We have, $(1 + \tan y)(dx - dy) + 2x dy = 0$

$$\Rightarrow (1 + \tan y) dx = (1 + \tan y - 2x) dy$$

$$\Rightarrow \frac{dx}{dy} + \frac{2}{1 + \tan y} x = 1, \text{ which is linear in } x.$$

$$\therefore \text{IF} = e^{\int \frac{2}{1 + \tan y} dy} = e^{\int \frac{2 \cos y}{\sin y + \cos y} dy}$$

$$= e^{\int \left(1 + \frac{\cos y - \sin y}{\cos y + \sin y}\right) dy}$$

$$= e^{y + \log(\cos y + \sin y)}$$

$$= (\cos y + \sin y) e^y$$

So, the solution is

$$x e^y (\sin y + \cos y) = \int e^y (\sin y + \cos y) dy + C$$

$$\Rightarrow x e^y (\sin y + \cos y) = e^y \sin y + C$$

$$\Rightarrow x (\sin y + \cos y) = \sin y + C e^{-y}$$

$$24. \text{ We have, } \frac{dx}{dy} = \frac{e^{x/y} \left(\frac{x}{y} - 1\right)}{1 + e^{x/y}}$$

On putting $x = vy$, so that $\frac{dx}{dy} = v + y \frac{dv}{dy}$, we get

$$v + y \frac{dv}{dy} = \frac{e^v (v - 1)}{1 + e^v}$$

$$\Rightarrow y \frac{dv}{dy} = -\frac{(v + e^v)}{e^v + 1}$$

$$\Rightarrow -\frac{e^v + 1}{v + e^v} dv = \frac{dy}{y}$$

On integrating, we get

$$\log y = -\log(v + e^v) + \log C$$

$$\Rightarrow y = \frac{C}{v + e^v} = \frac{C}{\frac{x}{y} + e^{x/y}}$$

$$\Rightarrow x + y e^{x/y} = C$$

which is the required solution.

$$25. \frac{dy}{dx} = \frac{1}{x \cos y + 2 \sin y \cos y}$$

$$\therefore \frac{dx}{dy} = x \cos y + 2 \sin y \cos y$$

$$\Rightarrow \frac{dx}{dy} + (-\cos y) x = 2 \sin y \cos y$$

$$\therefore \text{IF} = e^{\int \cos y dy} = e^{-\sin y}$$

Hence, the solution is

$$x \cdot e^{-\sin y} = 2 \int e^{-\sin y} \cdot \sin y \cos y dy + C$$

$$\Rightarrow x \cdot e^{-\sin y} = -2 \sin y e^{-\sin y} - 2 \int (-e^{-\sin y}) \cos y dy + C$$

$$\Rightarrow x \cdot e^{-\sin y} = -2 \sin y e^{-\sin y} + 2 \int e^{-\sin y} \cos y dy + C$$

$$= -2 \sin y e^{-\sin y} - 2 e^{-\sin y} + C$$

$$\Rightarrow x = -2 \sin y - 2 + C e^{\sin y} = C e^{\sin y} - 2(1 + \sin y)$$

$$\therefore k = 2$$

$$\begin{aligned}
 26. \int \frac{dy}{y} &= - \int \frac{x}{\sqrt{a+x}} dx \\
 \Rightarrow \ln y &= - \left\{ \frac{2}{3} (a+x)^{3/2} - 2a(a+x)^{1/2} \right\} - \ln A \\
 \Rightarrow \ln y &= \frac{2}{3} (2a-x) \sqrt{a+x} - \ln A \\
 \Rightarrow y &= Ae^{\frac{2}{3}(2a-x)\sqrt{a+x}}
 \end{aligned}$$

27. Given differential equation is

$$\frac{dy}{dx} + P(x) \cdot y = Q(x) \cdot y^n$$

$$\Rightarrow \frac{1}{y^n} \cdot \frac{dy}{dx} + y^{-n+1} P(x) = Q(x)$$

Put $\frac{1}{y^{n-1}} = v$

$$\Rightarrow (-n+1)y^{-n} \frac{dy}{dx} = \frac{dv}{dx}$$

$$\therefore \frac{1}{(-n+1)} \cdot \frac{dv}{dx} + P(x) \cdot v = Q(x)$$

$$\Rightarrow \frac{dv}{dx} + (1-n)P(x) \cdot v = (1-n)Q(x)$$

Hence, required substitution is $v = \frac{1}{y^{n-1}}$.

28. We have, $\frac{dy}{dx} = [g(x) - y] \cdot g'(x)$

Put $g(x) - y = v \Rightarrow g'(x) - \frac{dy}{dx} = \frac{dv}{dx}$

Hence, $g'(x) - \frac{dv}{dx} = v \cdot g'(x)$

$$\Rightarrow \frac{dv}{dx} = (1-v)g'(x) \Rightarrow \frac{dv}{1-v} = g'(x)dx$$

$$\Rightarrow \int \frac{dv}{1-v} = \int g'(x)dx$$

$$\Rightarrow -\log(1-v) = g(x) - C$$

$$\Rightarrow g(x) + \log(1-v) = C$$

$$\therefore g(x) + \log[1+y-g(x)] = C$$

29. We have, $y(x+y^3)dx = x(y^3-x)dy$

$$\Rightarrow y^3(ydx - xdy) + x(ydx + xdy) = 0$$

$$\Rightarrow x^2y^3 \cdot \left(\frac{ydx - xdy}{x^2} \right) + xd(xy) = 0$$

$$\Rightarrow -\frac{y}{x}d\left(\frac{y}{x}\right) + \frac{d(xy)}{x^2y^2} = 0$$

On integrating, we get

$$\frac{-(y/x)^2}{2} - \frac{1}{xy} = C$$

$$\Rightarrow \frac{y^2}{2x^2} + \frac{1}{xy} + C = 0$$

$$\Rightarrow y^3 + 2x + 2Cxy = 0$$

Since, it passes through the point (1, 1).

Therefore, $1 + 2 + 2C = 0$ or $C = -\frac{3}{2}$

$\therefore$ The curve is $y^3 + 2x - 3xy = 0$.

30. We have, $xy^2dx - x^2ydy = \frac{e}{x^3}dx$

$$\Rightarrow -\frac{1}{2}(x^2 \cdot 2ydy - y^2 \cdot 2xdx) = \frac{e}{x^3}dx$$

$$\Rightarrow \frac{x^2d(y^2) - y^2d(x^2)}{x^4} = -\frac{2e}{x^3} \cdot \frac{1}{x^4}dx$$

$$\Rightarrow d\left(\frac{y^2}{x^2}\right) = -\left(\frac{2e}{x^7}\right)dx$$

$$\Rightarrow \int d\left(\frac{y^2}{x^2}\right) = \int \left(-\frac{2e}{x^7}\right)dx$$

$$\Rightarrow \frac{y^2}{x^2} = -2e \int x^{-7}dx = (-2e) \cdot \left(\frac{x^{-6}}{-6}\right) + C$$

On putting $y = 0, x = 1$, we get

$$0 = \frac{e}{3} + C$$

$$\Rightarrow C = -\frac{e}{3}$$

$\therefore$ The solution is $y^2 = \frac{e}{3} \left(\frac{1}{x^4} - x^2 \right)$.

31. Put $y^2 \sin x = v$

$$\Rightarrow 2y \frac{dy}{dx} \sin x + y^2 \cos x = \frac{dv}{dx}$$

So, the given equation becomes $\frac{1}{2} \cdot \frac{dv}{dx} = \sin x \cos x$

$$\Rightarrow dv = 2 \sin x \cos x dx$$

On integrating, we get

$$v = 2 \sin^2 x + C$$

$$\Rightarrow y^2 \sin x = 2 \sin^2 x + C$$

On putting $y = 1, x = \frac{\pi}{2}$, we get $1 = 2 + C \Rightarrow C = -1$

So, the solution is $y^2 \sin x = 2 \sin^2 x - 1$.

32. We have, $x\left(\frac{dy}{dx}\right)^2 + (y-x)\frac{dy}{dx} - y = 0$

$$\Rightarrow \left(\frac{dy}{dx} - 1\right)\left(x\frac{dy}{dx} + y\right) = 0$$

$$\Rightarrow \frac{dy}{dx} = 1$$

or $x\frac{dy}{dx} = -y$

$\therefore$ The solution of $\frac{dy}{dx} = 1$ is $y = x + C$

and solution of $x\frac{dy}{dx} = -y$, i.e. $\frac{dy}{y} + \frac{dx}{x} = 0$ is

$$\log(xy) = \log C$$

$$\Rightarrow xy = C$$

Hence, general solution is $(x-y+C)(xy-C) = 0$.

33. We have, $xy^4dx + ydx - xdy = 0$.

$$\Rightarrow xdx + \frac{ydx - xdy}{y^4} = 0$$

$$\Rightarrow x^3dx + \left(\frac{x}{y}\right)^2 \cdot \frac{ydx - xdy}{y^2} = 0$$

$$\Rightarrow x^3dx + \left(\frac{x}{y}\right)^2 \cdot d\left(\frac{x}{y}\right) = 0$$

On integrating, we get $\frac{x^4}{4} + \frac{1}{3}\left(\frac{x}{y}\right)^3 = C$

$$\Rightarrow 3x^4y^3 + 4x^3 = Cy^3$$

which is the required solution.

34. We have, $xdy - ydx = xy^3(1 + \log x)dx$

$$\Rightarrow -\left(\frac{ydx - xdy}{y^2}\right) = xy(1 + \log x)dx$$

$$\Rightarrow -d\left(\frac{x}{y}\right) = xy(1 + \log x)dx$$

$$\Rightarrow -\frac{x}{y}d\left(\frac{x}{y}\right) = x^2(1 + \log x)dx$$

On integrating, we get

$$-\frac{\left(\frac{x}{y}\right)^2}{2} = (1 + \log x) \frac{x^3}{3} - \int \frac{x^3}{3} \cdot \frac{1}{x} dx$$

$$\Rightarrow -\frac{x^2}{2y^2} = \frac{x^3}{3}(1 + \log x) - \frac{x^3}{9} + \frac{C}{2}$$

$$\Rightarrow -\frac{x^2}{y^2} = \frac{2x^3}{3} \left(\frac{2}{3} + \log x\right) + C$$

35. We have, $\frac{dy}{dx} = \frac{-(y + y^3)}{1 + x(1 + y^2)} \Rightarrow \frac{dx}{dy} = -\frac{1 + x(1 + y^2)}{y + y^3}$

$$= -\left[\frac{1}{y(y^2 + 1)} + \frac{x(1 + y^2)}{y(1 + y^2)}\right]$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y} = \frac{-1}{y(1 + y^2)}$$

$$\therefore \text{IF} = e^{\int \frac{dy}{y}} = e^{\log y} = y$$

Hence, the solution is

$$x \cdot y = \int \frac{-1}{y(1 + y^2)} \cdot y dy + C$$

$$= -\int \frac{dy}{1 + y^2} + C = -\tan^{-1} y + C$$

$$\Rightarrow xy + \tan^{-1} y = C$$

which passes through (0, 1).

Therefore, $\tan^{-1} 1 = C \Rightarrow C = \pi/4$

Thus, the equation of the curve is

$$xy + \tan^{-1} y = \frac{\pi}{4}$$

36. $\frac{dy}{dx} = -k\sqrt{y}$

When $t = 0$; $y = 4$

$$\therefore \int_4^0 \frac{dy}{\sqrt{y}} = -k \int_0^t dt$$

$$\Rightarrow [2\sqrt{y}]_4^0 = -kt = -\frac{t}{15}$$

$$\Rightarrow 0 - 4 = -\frac{t}{15}$$

$$\Rightarrow t = 60 \text{ min}$$

37. We have, $y(x^2y + e^x)dx - e^x dy = 0$

$$\Rightarrow e^x \frac{dy}{dx} = x^2y^2 + ye^x$$

On dividing by y^2e^x , we get $\frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{1}{y} = x^2e^{-x}$

Put $\frac{1}{y} = v$, so that $-\frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dv}{dx}$

Thus, $\frac{dv}{dx} + v = -x^2e^{-x}$, which is linear.

$$\therefore \text{IF} = e^{\int 1 dx} = e^x$$

Hence, the solution is

$$v \cdot e^x = -\int x^2 e^{-x} \cdot e^x dx + \frac{C}{3}$$

$$\Rightarrow \frac{1}{y} e^x = -\frac{x^3}{3} + \frac{C}{3}$$

$$\Rightarrow x^3y + 3e^x = Cy$$

38. We have, $x^2y - x^3 \frac{dy}{dx} = y^4 \cos x$

$$\Rightarrow x^3 \frac{dy}{dx} - x^2y = -y^4 \cos x$$

On dividing by $-y^4x^3$, we get

$$-\frac{1}{y^4} \cdot \frac{dy}{dx} + \frac{1}{y^3} \cdot \frac{1}{x} = \frac{1}{x^3} \cos x$$

On putting $\frac{1}{y^3} = v$, so that $-\frac{1}{y^4} \cdot \frac{dy}{dx} = \frac{1}{3} \cdot \frac{dv}{dx}$, we get

$$\frac{1}{3} \cdot \frac{dv}{dx} + \frac{1}{x} v = \frac{1}{x^3} \cos x$$

$$\Rightarrow \frac{dv}{dx} + \frac{3}{x} v = \frac{3}{x^3} \cos x$$

which is linear in v .

$$\therefore \text{IF} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = x^3$$

So, the solution is

$$x^3v = \int x^3 \cdot \frac{3}{x^3} \cos x dx + C = 3 \sin x + C$$

$$\Rightarrow \frac{x^3}{y^3} = 3 \sin x + C$$

On putting $x = 0$, $y = 1$, we get $C = 0$

Hence, the solution is $x^3 = 3y^3 \sin x$.

39. Let $P(x, y)$ be any point on the curve.

Equation of the tangent at P is

$$Y - y = \frac{dy}{dx} (X - x)$$

$$\Rightarrow X \frac{dy}{dx} - Y + y - x \frac{dy}{dx} = 0 \quad \dots(i)$$

Now, length of perpendicular from origin (0, 0) on Eq. (i) is x .

$$\therefore \frac{0 - 0 + y - x \frac{dy}{dx}}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}} = x$$

$$\Rightarrow \left(y - x \frac{dy}{dx}\right)^2 = x^2 \left(1 + \left(\frac{dy}{dx}\right)^2\right)$$

$$\Rightarrow y^2 - 2xy \frac{dy}{dx} = x^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \dots(ii)$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v^2x^2 - x^2}{2vx^2} = \frac{v^2 - 1}{2v}$$

[putting $y = vx$ in Eq. (ii)]

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{-1 - v^2}{2v}$$

$$\Rightarrow \frac{2v}{1 + v^2} dv + \frac{dx}{x} = 0$$

On integrating, $(1 + v^2)x = C$

$$\Rightarrow x^2 + y^2 = Cx$$

40. We have, $k = PQ = \text{Length of normal}$

$$\Rightarrow k = y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \Rightarrow \frac{k^2}{y^2} = 1 + \left(\frac{dy}{dx}\right)^2$$

$$\therefore y \frac{dy}{dx} = \pm \sqrt{k^2 - y^2}$$

41. $\int \frac{dy}{100 - y} = \int dx$

$$-\ln(100 - y) = x + C$$

$$\ln(100 - y) = -x + C$$

$$\Rightarrow x = 0, y = 50$$

$$\text{Since, } C = \ln 50$$

$$\therefore x = \ln 50 - \ln(100 - y)$$

$$\Rightarrow \ln \frac{50}{100 - y} = x \Rightarrow \frac{50}{100 - y} = e^x$$

$$\Rightarrow 100 - y = 50e^{-x}$$

$$\Rightarrow y = 100 - 50e^{-x}$$

42. Let the population be p at any time t .

$$\text{Then, } \frac{dp}{dt} = 5\% \text{ at } p$$

$$\Rightarrow \frac{dp}{dt} = \frac{p}{20}$$

$$\Rightarrow \frac{dp}{p} = \frac{dt}{20}$$

$$\text{On integrating, we get } \log p = \frac{t}{20} + C \quad \dots (i)$$

Let the population be p_0 at $t = 0$.

On putting $p = p_0$ and $t = 0$ in Eq. (i), we get

$$C = \log p_0$$

$$\therefore \log p = \frac{t}{20} + \log p_0 \quad \dots (ii)$$

Let $p = 2p_0$ at $t = t_1$

$$\text{Then, } \log(2p_0) = \frac{t_1}{20} + \log p_0$$

$$\Rightarrow \log 2 + \log p_0 = \frac{t_1}{20} + \log p_0$$

$$\Rightarrow \log 2 = \frac{t_1}{20} \Rightarrow t_1 = 20(\log 2)$$

which is the required time.

43. The equation of normal to a curve at point (x, y) is

$$(Y - y) \frac{dy}{dx} + (X - x) = 0$$

Since, it passes through the point $(3, 0)$.

$$\therefore (0 - y) \frac{dy}{dx} + (3 - x) = 0$$

$$\Rightarrow y \frac{dy}{dx} = (3 - x) \Rightarrow y dy = (3 - x) dx$$

$$\text{On integrating, we get } \frac{y^2}{2} = 3x - \frac{x^2}{2} + C$$

$$\Rightarrow x^2 + y^2 - 6x - 2C = 0$$

Since, the curve passes through $(3, 4)$.

$$\therefore 9 + 16 - 18 - 2C = 0$$

$$\Rightarrow C = \frac{7}{2}$$

$$\therefore x^2 + y^2 - 6x - 7 = 0$$

which is the required equation of the curve.

44. (a) $f(x, y) = e^{y/x} + \tan \frac{y}{x} \Rightarrow f(x, t x) = e^t + \tan^{-1}(t)$,

independent of x .

$\therefore$ Homogeneous differential equation.

(b) $\log \left(\frac{y}{x}\right) dx + \frac{y^2}{x^2} \sin^{-1} \frac{y}{x} dy = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{\log \left(\frac{y}{x}\right)}{\frac{y^2}{x^2} \sin^{-1} \left(\frac{y}{x}\right)}$$

$$\therefore f(x, y) = \frac{\log \left(\frac{y}{x}\right)}{\frac{y^2}{x^2} \sin^{-1} \left(\frac{y}{x}\right)}$$

$$\text{So, } f(x, t x) = \frac{\log(t)}{t^2 \sin^{-1}(t)}, \text{ independent of } x.$$

$\therefore$ Homogeneous differential equation.

(c) $f(x, y) = x^2 + \sin x \cdot \cos y$

$$\Rightarrow f(x, t x) = x^2 + \sin x \cdot \cos(t x), \text{ not independent of } x.$$

$\therefore$ Not homogeneous differential equation.

(d) $f(x, y) = \frac{x^2 + y^2}{xy^2 - y^3}$

$$\Rightarrow f(x, t x) \text{ is not independent of } x.$$

$\therefore$ Not homogeneous differential equation.

45. $\frac{dy}{dx} + y \cos x = \cos x$

$$\therefore \text{IF} = e^{\int \cos x dx} = e^{-\sin x}$$

Hence, required solution is

$$y \cdot e^{-\sin x} = \int e^{-\sin x} \cos x dx = -e^{-\sin x} + C$$

$$\therefore y = 1$$

46. $\frac{dy}{dx} - y \cdot \cot x = -\frac{\sin x}{x^2}$

$$\therefore \text{IF} = e^{\int -\cot x dx} = e^{-\log \sin x} = \frac{1}{\sin x}$$

Hence, required solution is

$$y \cdot \frac{1}{\sin x} = \int -\frac{\sin x}{x^2} \cdot \frac{1}{\sin x} dx + C$$

$$\Rightarrow y \cdot \frac{1}{\sin x} = \frac{1}{x} + C$$

$$\text{As } x \rightarrow \infty, y \rightarrow 0 \Rightarrow C = 0$$

$$\therefore y = \frac{\sin x}{x} \Rightarrow \lim_{x \rightarrow 0} f(x) = 1 \quad \dots (i)$$

$$\text{Now, } I = \int_0^{\pi/2} \frac{\sin x}{x} dx$$

Since, $\frac{\sin x}{x}$ is decreasing.

$$\therefore x > 0$$

$$\Rightarrow f(x) < f(0)$$

$$\Rightarrow \int_0^{\pi/2} f(x) dx < \frac{\pi}{2} \text{ and } x < \frac{\pi}{2}$$

$$\Rightarrow f(x) > f\left(\frac{\pi}{2}\right)$$

$$\therefore \int_0^{\pi/2} f(x) dx > 1$$

47. (a) $f(\lambda x, \lambda y) = \frac{\lambda(x-y)}{\lambda^2(x^2+y^2)} = \lambda^{-1}f(x, y)$

$\therefore$ Homogeneous of degree (-1) .

(b) $f(\lambda x, \lambda y) = (\lambda x)^{1/3}(\lambda y)^{-2/3} \tan^{-1} \frac{x}{y}$
 $= \lambda^{-1/3} x^{1/3} y^{-2/3} \tan^{-1} \frac{x}{y}$
 $= \lambda^{-1/3} f(x, y)$

$\therefore$ Homogeneous.

(c) $f(\lambda x, \lambda y) + \lambda x (\ln \sqrt{\lambda^2(x^2+y^2)} - \ln \lambda y) + \lambda y e^{x/y}$
 $= \lambda x \left[\ln \left(\frac{\lambda \sqrt{x^2+y^2}}{\lambda y} \right) \right] + \lambda y e^{x/y}$
 $= \lambda [x (\ln \sqrt{x^2+y^2} - \ln y) + y e^{x/y}] = \lambda f(x, y)$

$\therefore$ Homogeneous

(d) $f(\lambda x, \lambda y) = \lambda x \left[\ln \frac{2\lambda^2 x^2 + \lambda^2 y^2}{\lambda x \lambda (x+y)} \right] + \lambda^2 x^2 \tan \left(\frac{x+2y}{3x-y} \right)$
 $= \lambda x \left[\ln \frac{2x^2 + y^2}{x(x+y)} \right] + \lambda^2 x^2 \tan \left(\frac{x+2y}{3x-y} \right)$

$\therefore$ Not homogeneous.

48. (a) $\frac{dy}{dx} = -C_1 \sin x + \sqrt{C_2} \cos x$

$\Rightarrow \frac{d^2 y}{dx^2} = -C_1 \cos x - \sqrt{C_2} \sin x = 2 - y$

$\Rightarrow \frac{d^2 y}{dx^2} + y - 2 = 0$

(b) $\frac{dy}{dx} = \cos x \cdot \frac{\sec^2 x/2}{2 \tan \frac{x}{2}} - \sin x \ln \left(\tan \frac{x}{2} \right)$

$\Rightarrow \frac{dy}{dx} = \cot x - \sin x \ln \left(\tan \frac{x}{2} \right)$

$\Rightarrow \frac{d^2 y}{dx^2} = \operatorname{cosec}^2 x \left[\sin x \cdot \frac{1}{\sin x} + \cos x \ln \left(\tan \frac{x}{2} \right) \right]$

$\Rightarrow \frac{d^2 y}{dx^2} = -\cot^2 x - 2 - \cos x \ln \left(\tan \frac{x}{2} \right)$

$\Rightarrow \frac{d^2 y}{dx^2} + y + \cot^2 x = 0$

(c) $\frac{dy}{dx} = -C_1 \sin x + C_2 \cos x + \frac{d}{dx} \left(\cos x \ln \tan \frac{x}{2} \right)$

$\Rightarrow \frac{d^2 y}{dx^2} = -(C_1 \cos x + C_2 \sin x)$

$+ \frac{d^2}{dx^2} \left(\cos x \ln \left(\tan \frac{x}{2} \right) \right)$

$\Rightarrow \frac{d^2 y}{dx^2} = -C_1 \cos x - C_2 \sin x - \cot^2 x$
 $- 2 - \cos x \ln \left(\tan \frac{x}{2} \right)$

$\Rightarrow \frac{d^2 y}{dx^2} + y + \cot^2 x = 0$

49. Given differential equation is

$y_2(x^2 + 1) = 2xy_1 \Rightarrow \frac{y_2}{y_1} = \frac{2x}{x^2 + 1}$

On integrating both sides, we get

$\log y_1 = \log(x^2 + 1) + \log C$

$\Rightarrow y_1 = C(x^2 + 1) \quad \dots (i)$

It is given that, $y_1 = 3$ at $x = 0$.

On putting $x = 0$, $y_1 = 3$ in Eq. (i), we get $3 = C$

On substituting the value of C in Eq. (i), we obtain

$y_1 = 3(x^2 + 1) \quad \dots (ii)$

On integrating both sides w.r.t. x , we get

$y = x^3 + 3x + C_2$

which passes through the point $(0, 1)$.

Therefore, $1 = C_2$. Hence, the required equation of the curve is $y = x^3 + 3x + 1$.

Obviously, it is strictly increasing from Eq. (ii).

Also, $f(0) = 1 > 0$, then the only root is negative.

50. $f(x) = \int_0^x \{f(t) \cos t - \cos(t-x)\} dx$
 $= \int_0^x f(t) \cos t dt - \int_0^x \cos(-t) dt$
 $\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$

$= \int_0^x f(t) \cos t dt - \sin x$

On differentiating both sides, we get

$f'(x) = f(x) \cos x - \cos x$

Let $f(x) = y$, $f'(x) = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} - y \cos x = -\cos x$

$\therefore IF = e^{-\int \cos x dx} = e^{-\sin x}$

Hence, the solution is $y \cdot e^{-\sin x} = \int e^{-\sin x} \cos x dx$

$\Rightarrow y \cdot e^{-\sin x} = C + e^{-\sin x}$

$\Rightarrow y = C e^{\sin x} + 1$

If $x = 0$, $y = 0$

$\Rightarrow C = -1$

Therefore, $f(x) = 1 - e^{\sin x}$

Now, minimum value $= 1 - e$, when $x = \pi/2$

and maximum value $= 1 - e^{-1}$, when $x = -\pi/2$

$\therefore f'(x) = -e^{\sin x} \cos x$

Therefore, $f'(0) = -1$

$\Rightarrow f''(x) = -[\cos^2 x \cdot e^{\sin x} - e^{\sin x} \sin x]$

$\therefore f''\left(\frac{\pi}{2}\right) = e$

51. (a) The order of the differential equation is 2.

(b) $\frac{xdy - ydx}{\sqrt{x^2 + y^2}} = dx$

$\Rightarrow \frac{xdy - ydx}{x^2} = \frac{dx}{\sqrt{1 + \frac{y^2}{x^2}}}$

$\therefore \ln \left| \frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} \right| = \ln |Cx|$

$\therefore \frac{y}{x} + \frac{\sqrt{x^2 + y^2}}{x} = Cx$

i.e. $y + \sqrt{x^2 + y^2} = Cx^2$

$$\begin{aligned}
 \text{(c) } y &= e^x (A \cos x + B \sin x) \\
 \Rightarrow \frac{dy}{dx} &= e^x (A \cos x + B \sin x) \\
 &\quad + e^x (-A \sin x + B \cos x) \\
 &= y + e^x (-A \sin x + B \cos x) \\
 \Rightarrow \frac{d^2 y}{dx^2} &= \frac{dy}{dx} + e^x (-A \sin x + B \cos x) \\
 &\quad + e^x (-A \cos x - B \sin x) \\
 \therefore \frac{d^2 y}{dx^2} &= \frac{dy}{dx} + e^x (-A \sin x + B \cos x) - y \\
 &= \frac{dy}{dx} + \frac{dy}{dx} - y - y = 2 \left(\frac{dy}{dx} - y \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{(d) } (1 + y^2) \frac{dx}{dy} + x &= 2e^{\tan^{-1} y} \\
 \Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2} x &= \frac{e^{\tan^{-1} y}}{1 + y^2} \\
 \therefore \text{IF} &= e^{\int \frac{1}{1 + y^2} dy} = e^{\tan^{-1} y}
 \end{aligned}$$

Hence, the solution is

$$xe^{\tan^{-1} y} = 2 \int e^{\tan^{-1} y} \cdot \frac{e^{\tan^{-1} y}}{1 + y^2} dy + C$$

$$\Rightarrow xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + C$$

$$\begin{aligned}
 52. \frac{dy}{dx} &= (A + Bx)e^{3x} + Be^{3x} \\
 \Rightarrow \frac{dy}{dx} + my &= (3 + m)(A + Bx)e^{3x} + Be^{3x} \\
 \Rightarrow \frac{d^2 y}{dx^2} + m \frac{dy}{dx} + ny &= (9 + 3m + n)(A + Bx)e^{3x} \\
 &\quad + B(6 + m)e^{3x} = 0 \\
 \Rightarrow 3m + n + 9 &= 0 \\
 \text{and } m + 6 &= 0 \\
 \Rightarrow m = -6 \text{ and } n &= 9
 \end{aligned}$$

53. We have, $y = a \sin x + b \cos x$

$$\begin{aligned}
 \Rightarrow \frac{dy}{dx} &= a \cos x - b \sin x \\
 \Rightarrow \frac{d^2 y}{dx^2} &= -a \sin x - b \cos x = -y \\
 \Rightarrow \frac{d^2 y}{dx^2} + y &= 0
 \end{aligned}$$

But Statement II is not the correct explanation of the Statement I.

54. The equation of the given family of curves is $y = cx^2$... (i)

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = 2cx \quad \dots (ii)$$

On eliminating c from Eqs. (i) and (ii), we get

$$y = \left(\frac{dy}{2x dx} \right) x^2 \Rightarrow 2y = x \frac{dy}{dx} \quad \dots (iii)$$

This is the differential equation of the family of curves given in Eq. (i). The differential equation of the orthogonal trajectory in Eq. (i) is obtained by replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (iii).

On replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in Eq. (iii), we get

$$2y = -x \frac{dx}{dy}$$

$$\Rightarrow 2y dy = -x dx$$

On integrating, we get

$$y^2 = -\frac{x^2}{2} + C$$

$$\Rightarrow x^2 + 2y^2 = 2C$$

Which is the required family of orthogonal trajectory.

55. Solving the equation of the asymptotes, the centre is $x = 1$ and $y = 0$, since $e = \sqrt{2}$. The equation of the family of the rectangular hyperbola, is

$$\frac{(x-1)^2}{a^2} - \frac{(y-0)^2}{a^2} = 1$$

$$\Rightarrow 2(x-1) - 2y \frac{dy}{dx} = 0$$

$$\Rightarrow (x-1) - y \frac{dy}{dx} = 0$$

56. Given equation can be written as

$$xy dy + y^2 dx - dy = 0$$

On multiplying it by $\left(\frac{1}{y}\right)$, we get

$$x dy + y dx - \frac{1}{y} dy = 0$$

$$\Rightarrow \int d(xy) - \int \frac{dy}{y} = 0$$

$$\Rightarrow xy - \log y = C$$

$$\Rightarrow \frac{1}{y} \text{ is an integrating factor.}$$

57. **Statement I** Equation of all circles can be given by $x^2 + y^2 + 2gx + 2fy + c = 0$, will be of order 3. Statement II is obviously true but it does not explain Statement I.

58. The slope of the ray = $\frac{y}{x}$

Slope of the normal at $P(x, y) = -\frac{dx}{dy}$

$$\frac{\frac{y}{x} + \frac{dx}{dy}}{1 - \frac{y}{x} \cdot \frac{dx}{dy}} = -\frac{dx}{dy}$$

$$\Rightarrow \frac{y}{x} \cdot \frac{dy}{dx} + 1 = -1 + \frac{y}{x} \cdot \frac{dx}{dy}$$

$$\Rightarrow y \left(\frac{dy}{dx} \right)^2 + 2x \cdot \frac{dy}{dx} = y$$

which is the differential equation of the curve satisfied by $y^2 = 2x + 1$.

$$59. \frac{de}{dt} = -0.1 \Rightarrow e = -0.1t + e_0$$

$$\text{Here, } e_0 = \frac{1}{2}$$

$$\Rightarrow e_t = -0.1t + \frac{1}{2}, \text{ given that } e_t = 0$$

$$\Rightarrow (0.1)t = \frac{1}{2} \Rightarrow t = 5s$$

60. $\frac{ydx - xdy}{y^2} \cdot \cot\left(\frac{x}{y}\right) = ndx$

$$\Rightarrow \int \cot\left(\frac{x}{y}\right) \cdot d\left(\frac{x}{y}\right) = \int ndx$$

$$\Rightarrow \int \cot t \, dt = nx + C_1$$

$$\Rightarrow \ln(\sin t) = nx + C_1$$

$$\Rightarrow \sin\left(\frac{y}{x}\right) = Ce^{nx}$$

61. Statement II is obviously true. But Statement I is false as

$$2x - 3y + 2 = \log\left(\frac{dy}{dx}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = e^{2x - 3y + 2}, \text{ which has degree 1.}$$

62. We have, $x dy - y dx = \sqrt{x^2 - y^2} \, dx$

$$\Rightarrow d\left(\frac{y}{x}\right) = \frac{\sqrt{1 - \left(\frac{y}{x}\right)^2}}{x} dx$$

$$\Rightarrow \int \frac{d\left(\frac{y}{x}\right)}{\sqrt{1 - \left(\frac{y}{x}\right)^2}} = \int \frac{dx}{x}$$

$$\Rightarrow \sin^{-1} \frac{y}{x} = \ln x + \ln C$$

$$\text{or } Cx = e^{\sin^{-1}\left(\frac{y}{x}\right)}$$

63. We have, $(xy^4 + y) dx - x dy = 0$

$$\Rightarrow xy^4 dx + y dx - x dy = 0$$

$$\Rightarrow \int x^3 dx - \int \left(\frac{x}{y}\right)^2 d\left(\frac{x}{y}\right) = 0$$

$$\Rightarrow \frac{x^4}{4} + \frac{1}{3} \left(\frac{x}{y}\right)^3 = C$$

64. We have, $2x \cos y \, dx - x^2 \sin y \, dy + y^2 \cos x \, dx + 2y \sin x \, dy = 0$

$$\Rightarrow \int d(x^2 \cos y) + \int d(y^2 \sin x) = 0$$

$$\Rightarrow x^2 \cos y + y^2 \sin x = C$$

65. We have, $\frac{dx}{dy} = \frac{ydy}{1+y^2}$

$$\Rightarrow \ln x = \frac{1}{2} \cdot \ln(1+y^2) + C$$

From the given condition, $C = 0$

$$\therefore x^2 - y^2 = 1$$

66. We have, $\frac{dy}{dx} + \frac{1+y^2}{\sqrt{1-x^2}} = 0$

$$\Rightarrow \frac{dy}{1+y^2} + \frac{dx}{\sqrt{1-x^2}} = 0$$

$$\Rightarrow \tan^{-1} y + \sin^{-1} x = C$$

67. $\frac{dy}{dx} = (1+x) \cdot (1+y)$ gives $y = e^{\frac{(1+x)^2}{2}} - 1$

68. Equation of normal is $Y - y = -\frac{dy}{dx}(X - x)$

$$\Rightarrow \frac{\left|x \frac{dx}{dy} + y\right|}{\sqrt{1 + \left(\frac{dx}{dy}\right)^2}} = |y|$$

$$\Rightarrow \frac{dx}{dy} = \frac{2xy}{y^2 - x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

69. From above, degree = 1 and order = 1

70. If $\frac{dx}{dy} = 0$, then $x = C$, when $x = 1$, $y = 1$, $C = 1$

$$\therefore \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \dots(i)$$

$$\text{Put } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{From Eq. (i), } \frac{(1-v^2)dv}{v(1+v^2)} = \frac{dy}{y}$$

$$\Rightarrow \int \left(\frac{1}{v} - \frac{2v}{1+v^2}\right) dv = \int \frac{dy}{y}$$

$$\Rightarrow \frac{v}{1+v^2} = Cy$$

$$\Rightarrow \frac{x}{x^2 + y^2} = C$$

$$\text{Put } x = 1, y = 1, \text{ then } C = \frac{1}{2}$$

$$\therefore x^2 + y^2 - 2x = 0$$

71. Let N denotes the amount of material present at time t . Then,

$$\frac{dN}{dt} - kN = 0$$

This differential equation is separable and linear.

Its solution is $N = Ce^{kt}$ $\dots(i)$

At $t = 0$, we are given that, $N = 50$

Therefore, from Eq. (i), $50 = Ce^{k(0)}$

or $C = 50$
Thus, $N = 50e^{kt}$ $\dots(ii)$

At $t = 2$, 10% of the original mass of 50 mg or 5 mg has decayed.

Hence, at $t = 2$, $N = 50 - 5 = 45$.

On substituting these values in Eq. (ii) and solving for k , we have $45 = 50e^{2k}$

$$\text{or } k = \frac{1}{2} \log \frac{45}{50}$$

On substituting this value in Eq. (ii), we get the amount of mass present at any time t as

$$50e^{-(1/2)(\ln 0.9)t} \quad \dots(iii)$$

72. We require N at $t = 4$.

On substituting $t = 4$ in Eq. (iii) and then solving for N , we get $N = 50e^{-2 \ln 0.9}$

73. We require t , when $N = \frac{50}{2} = 25$

On substituting $N = 25$ in Eq. (iii) and solving for t , we get

$$25 = 50e^{-\left(\frac{1}{2}\right)(\ln 0.9)t} \text{ or } t = \ln \frac{1}{2}(-1/2 \ln 0.9) \text{ h}$$

74. A. $\frac{dy}{dx} = e^x(A \cos x + B \sin x) + e^x(-A \sin x + B \cos x)$
 $= y + e^x(-A \sin x + B \cos x)$
 $\therefore \frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-A \sin x + B \cos x)$
 $+ e^x(-A \cos x - B \sin x)$
 $= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - y = 2\left(\frac{dy}{dx} - y\right)$

So, degree = 1 and order = 2.

B. The equation of the family is $y^2 = 4a(x - b)$, where a and b are arbitrary constants.

$$\therefore 2y \frac{dy}{dx} = 4a \text{ or } \left(\frac{dy}{dx}\right)^3 + y \frac{d^2y}{dx^2} = 0$$

So, degree = 1 and order = 2.

C. The equation of the family is $(x - c)^2 + y^2 = c^2$ or $x^2 + y^2 - 2cx = 0$

$$\text{or } \frac{x^2 + y^2}{x} = 2c$$

$$\Rightarrow \frac{(2x + 2y \frac{dy}{dx})x - (x^2 + y^2)1}{x^2} = 0$$

So, degree = 1 and order = 1.

D. The equation of the family is

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1,$$

because they have the same foci $(\pm \sqrt{a^2 - b^2}, 0)$.

On differentiating, we get

$$\frac{2x}{a^2 + \lambda} + \frac{2y}{b^2 + \lambda} \cdot \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2 + \lambda} + \frac{yp}{b^2 + \lambda} = 0 \quad \left[\text{let } p = \frac{dy}{dx} \right]$$

$$\Rightarrow x(b^2 + \lambda) + yp(a^2 + \lambda) = 0$$

$$\Rightarrow \lambda = \frac{-b^2x - a^2yp}{x + yp}$$

$\therefore$ The differential equation is

$$\frac{x^2}{a^2 + \frac{-b^2x - a^2yp}{x + yp}} + \frac{y^2}{b^2 + \frac{-b^2x - a^2yp}{x + yp}} = 1$$

$$\Rightarrow \frac{x(x + yp)}{a^2 - b^2} + \frac{y(x + yp)}{(b^2 - a^2)p} = 1$$

Therefore, order = 1 and degree = 2.

75. A. $x dy = y dx + y^2 dy$

$$\Rightarrow \frac{x dy - y dx}{y^2} = dy$$

$$\Rightarrow -d\left(\frac{x}{y}\right) = dy \Rightarrow -\frac{x}{y} = y + C$$

Put $x = 1, y = 1, C = -2$

$$-\frac{x}{y} = y - 2$$

$$\Rightarrow \frac{x}{3} = -5 \quad [\because y(x_0) = -3]$$

$$\Rightarrow x_0 = -15$$

B. $\frac{dy}{dt} - \frac{t}{t+1}y = \frac{1}{t+1}$

$$\therefore \text{IF} = e^{-\int \frac{t}{t+1} dt} = e^{-t + \ln(t+1)} = (t+1)e^{-t}$$

Hence, solution is $(t+1)e^{-t}y = -e^{-t} + C$

Put $t = 0$ and $y = -1 \Rightarrow C = 0$

$$\therefore 2e^{-1}y = -e^{-1} \Rightarrow y = -\frac{1}{2}$$

C. $(x^2 + y^2)dy = xy dx \Rightarrow \frac{dy}{dx} = \frac{xy}{x^2 + y^2}$

Put $y = vx \Rightarrow \frac{dv}{dx} = v + x \frac{dv}{dx}$

$$\Rightarrow \ln v - \frac{1}{2v^2} = -\ln x + C$$

$$\therefore C = -\frac{1}{2}$$

$$\ln \frac{y}{x} - \frac{x^2}{2y^2} = -\ln x - \frac{1}{2}$$

Put $y C x_0 = e$
 $\therefore C x_0 = \sqrt{3}e$

D. $\frac{dy}{dx} + 2\frac{y}{x} = 0 \Rightarrow x^2 y = C$

Put $x = 1, y = 1$ and we get $C = 1$

Put $x = 2 \Rightarrow y = \frac{1}{4}$

76. A. Given differential equation is $(x + y)^2 \frac{dy}{dx} = a^2 \dots (i)$

Put $x + y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$$\therefore t^2 \left(\frac{dt}{dx} - 1 \right) = a^2 \Rightarrow \frac{t^2 dt}{dx} = a^2 + t^2$$

$$\Rightarrow \frac{t^2 dt}{a^2 + t^2} = dx$$

On integrating, $\int \frac{t^2}{a^2 + t^2} dt = \int dx$

$$\Rightarrow \int \left(1 - \frac{a^2}{a^2 + t^2} \right) dt = \int dx$$

$$\Rightarrow t - a \tan^{-1} \frac{t}{a} = x + C$$

$$\Rightarrow x + y - a \tan^{-1} \frac{x + y}{a} + C = x$$

$$\therefore y = a \tan^{-1} \left(\frac{x + y}{a} \right) + C \text{ is the required solution.}$$

B. We have, $\sec^2 y \tan x dy = -\sec^2 x \tan y dx$

$$\Rightarrow \int \frac{\sec^2 y}{\tan y} dy + \int \frac{\sec^2 x}{\tan x} dx = 0$$

$$\Rightarrow \log \tan y + \log \tan x = \log C$$

$$\Rightarrow \log (\tan x + \tan y) = \log C$$

Hence, $\tan x \tan y = C$ is the required solution.

C. $\frac{dy}{dx} = e^{3x}e^{4y}$

$$\Rightarrow e^{-4y}dy = e^{3x}dx \Rightarrow \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + C$$

On putting $x = 0$, we have

$$-\frac{1}{4} - \frac{1}{3} = C \Rightarrow C = -\frac{7}{12}$$

$$\text{Hence, } \frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} - \frac{7}{12} \Rightarrow 7 = 3e^{-4y} + 4e^{3x}$$

D. $\int \frac{d(xy)}{x^6y^6} = \int \frac{dx}{x^3} \Rightarrow \frac{1}{(xy)^5} = \frac{5}{2x^2} + C$

77. Differentiating, $xy(x) = x^2y'(x) + 2xy(x)$

or $xy(x) + x^2y'(x) = 0$

$$x \frac{dy}{dx} + y = 0$$

$$\ln y + \ln x = \ln C$$

$xy = C$ passes through $(2, 3)$.

$$\Rightarrow C = 6$$

78. $(x^2z^{2\alpha} - 1)\alpha z^{\alpha-1}dz + 2xz^{3\alpha}dx = 0$

or $\alpha(x^2z^{3\alpha-1} - z^{\alpha-1})dz + 2xz^{3\alpha}dx = 0$

For homogenous, every term must be of the same degree.

$$3\alpha + 1 = \alpha - 1$$

$$\Rightarrow 1 + \alpha = 0$$

79. $\int_0^1 f(tx) dt = n \cdot f(x)$

Put $tx = y \Rightarrow dt = \frac{1}{x} dy$

$$\therefore \frac{1}{x} \int_0^x f(y) dy = nf(x)$$

$$\Rightarrow \int_0^x f(y) dy = x \cdot n \cdot f(x)$$

On differentiating, we get

$$f(x) = n[f(x) + x f'(x)]$$

$$\Rightarrow f(x)(1 - n) = nx f'(x)$$

$$\therefore \frac{f'(x)}{f(x)} = \frac{1-n}{nx}$$

On integrating, we get

$$\ln f(x) = \left(\frac{1-n}{n}\right) \ln Cx = \ln(Cx)^{\frac{1-n}{n}}$$

$$\therefore f(x) = C \cdot x^{\frac{1-n}{n}} \text{ as } n \rightarrow \infty, \frac{1-n}{n} \rightarrow -1$$

$$\Rightarrow f(x) = C \cdot x^{-1} = \frac{C}{x}$$

80. Statement I $y = \sin kt, y' = k \cos kt; y'' = -k^2 \sin kt$

$$\therefore -k^2 \sin kt + 9 \sin kt = 0$$

$$\Rightarrow \sin kt[9 - k^2] = 0$$

$$\Rightarrow k = 0, k = 3, k = -3$$

Statement II $y = e^{kt}, y' = ke^{kt}, y'' = k^2e^{kt}$

$$\therefore k^2e^{kt} + ke^{kt} - 6e^{kt} = 0$$

$$\Rightarrow e^{kt}[k^2 + k - 6] = 0$$

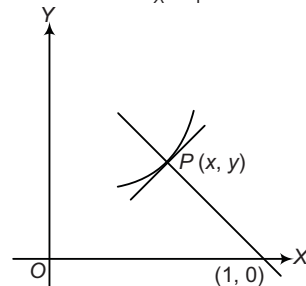
$$\Rightarrow (k + 3)(k - 2) = 0$$

$$\Rightarrow k = -3 \text{ or } 2$$

Hence, common value is $k = -3$

$$\therefore k + 4 = -3 + 4 = 1$$

81. Slope of the normal $= \frac{y}{x-1}$



$\therefore$

$$\frac{dy}{dx} = \frac{1-x}{y}$$

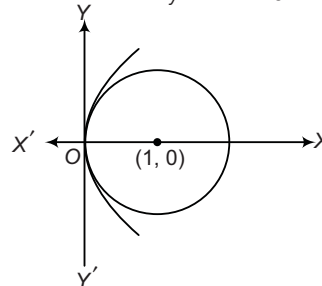
$$\frac{y^2}{2} = x - \frac{x^2}{2} + C$$

...(i)

$\therefore$ Eq. (i) passes through $(0, 0)$.

Hence, $C = 0$

$$x^2 + y^2 - 2x = 0$$



Now, tangent to $y^2 = 4x$ is

$$y = mx + \frac{1}{m}$$

...(ii)

If it touches the circle

$$x^2 + y^2 - 2x = 0$$

$$\text{Then, } \left| \frac{m + (1/m)}{\sqrt{1 + m^2}} \right| = 1$$

$$\Rightarrow 1 + m^2 = m^2 \Rightarrow m \rightarrow \infty$$

Hence, tangent is Y-axis, i.e. $x = 0$.

82. $\frac{dy}{dt} + 2ty = t^2$

$$\text{IF} = e^{t^2}$$

$$\therefore \text{Solution is } y \cdot e^{t^2} = \int t^2 e^{t^2} dt = \frac{1}{2} \int t \cdot (2t \cdot e^{t^2}) dt$$

$$y \cdot e^{t^2} = t \cdot \frac{e^{t^2}}{2} - \frac{1}{2} e^{t^2} dt$$

$$\lim_{t \rightarrow \infty} \frac{y}{t} = \frac{1}{2} - \lim_{t \rightarrow \infty} \frac{\int \frac{e^{t^2}}{2} dt}{te^{t^2}} + \frac{C}{t \cdot e^{t^2}} = \frac{1}{2}$$

$$\text{Hence, } 16 \lim_{t \rightarrow \infty} \frac{y}{t} = 16 \times \frac{1}{2} = 8$$

83. Equation of tangent at $P(x_1, y_1)$ of $y = f(x)$

$$y - y_1 = \frac{dy}{dx} (x - x_1)$$

...(i)

This tangent cuts the X-axis.

So, $x_2 = x_1 - \left(\frac{y_1}{\left(\frac{dy}{dx} \right)} \right)$

$\therefore x_1, x_2, x_3, \dots, x_n$ are in an AP.

$$\begin{aligned} \therefore x_2 - x_1 &= -\frac{y_1}{\frac{dy}{dx}} = \log_2 e \quad [\text{given}] \\ -y &= \log_2 e \frac{dy}{dx} \\ \frac{dy}{y} \log_2 e &= -dx \end{aligned}$$

On integrating both sides, we get

$$\log_e y = -x \log_2 e + C \Rightarrow y = ke^{-x \log_2 e}$$

$\therefore y = f(x)$ passes through $(0, 2)$.

$$\Rightarrow k = 2$$

$$\therefore y = 2e^{-x \log_2 e} \Rightarrow y = 2^{1-x}$$

84. $\frac{dy}{dx} - y = 1 - e^{-x}$

$$\begin{aligned} P &= -1, Q = 1 - e^{-x} \\ \text{IF} &= e^{\int P dx} = e^{\int (-1) dx} = e^{-x} \end{aligned}$$

Hence, solution is

$$y \cdot e^{-x} = \int e^{-x} (1 - e^{-x}) dx + C$$

$$\Rightarrow ye^{-x} = -e^{-x} + \frac{1}{2} e^{-2x} + C$$

$$\Rightarrow y = -1 + \frac{1}{2} e^{-x} + Ce^x$$

$$\therefore x = 0, y = y_0$$

$$\text{So, } C = y_0 + \frac{1}{2}$$

$$\Rightarrow y = -1 + \frac{1}{2} e^{-x} + \left(y_0 + \frac{1}{2}\right) e^x$$

$x \rightarrow \infty, y \rightarrow \text{finite value.}$

$$\text{So, } y_0 + \frac{1}{2} = 0$$

$$\Rightarrow y_0 = -\frac{1}{2}$$

Entrances Gallery

1. Given differential equation is

$$\frac{dy}{dx} + \frac{x}{x^2 - 1} y = \frac{x^4 + 2x}{\sqrt{1 - x^2}}$$

which is a linear differential equation

$$\therefore \text{IF} = e^{\int \frac{x}{x^2 - 1} dx} = e^{\frac{1}{2} \ln |x^2 - 1|} = \sqrt{1 - x^2}$$

Hence, solution is

$$y\sqrt{1 - x^2} = \int \frac{x(x^3 + 2)}{\sqrt{1 - x^2}} \cdot \sqrt{1 - x^2} dx$$

$$\text{or } y\sqrt{1 - x^2} = \int (x^4 + 2x) dx = \frac{x^5}{5} + x^2 + C$$

$$\Rightarrow f(0) = 0$$

$$\Rightarrow C = 0$$

$$\Rightarrow f(x)\sqrt{1 - x^2} = \frac{x^5}{5} + x^2$$

$$\text{Now, } \int_{-\sqrt{3}/2}^{\sqrt{3}/2} f(x) dx = \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}} dx$$

$$= 2 \int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1 - x^2}} dx$$

$$= 2 \int_0^{\pi/3} \frac{\sin^2 \theta}{\cos \theta} \cos \theta d\theta \quad [\text{taking } x = \sin \theta]$$

$$= 2 \int_0^{\pi/3} \sin^2 \theta d\theta = \int_0^{\pi/3} (1 - \cos 2\theta) d\theta$$

$$= \left(\theta - \frac{\sin 2\theta}{2} \right)_0^{\pi/3}$$

$$= \frac{\pi}{3} - \frac{\sin 2\pi/3}{2} = \frac{\pi}{3} - \frac{\sqrt{3}}{4}$$

2. Here, $\frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$

$$\text{Put } \frac{y}{x} = v$$

$$\therefore v + x \frac{dv}{dx} = v + \sec v$$

$$\Rightarrow x \frac{dv}{dx} = \sec v$$

$$\Rightarrow \frac{dv}{\sec v} = \frac{dx}{x}$$

$$\Rightarrow dv \cos v = \frac{dx}{x}$$

On integrating both sides, we get

$$\sin v = \log x + C$$

$$\Rightarrow \sin \frac{y}{x} = \log x + C$$

$$\text{At } x = 1, y = \frac{\pi}{6}$$

$$\therefore \sin \frac{\pi}{6} = \log 1 + C$$

$$\Rightarrow C = \frac{1}{2}$$

Thus, the required equation is $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$.

3. Here, $f'(x) < 2f(x)$, multiplying by $e^{-\int 2 dx}$

$$f'(x) \cdot e^{-2x} - 2e^{-2x} f(x) < 0$$

$$\Rightarrow \frac{d}{dx} (f(x) \cdot e^{-2x}) < 0$$

$$\therefore \phi(x) = f(x)e^{-2x} \text{ is decreasing for } x \in \left[\frac{1}{2}, 1\right]$$

Thus, when $x > \frac{1}{2}$

$$\text{Then, } \phi(x) < \phi\left(\frac{1}{2}\right) \Rightarrow e^{-2x} f(x) < e^{-1} \cdot f\left(\frac{1}{2}\right)$$

$$\Rightarrow f(x) < e^{2x-1} \cdot 1$$

$$\therefore f\left(\frac{1}{2}\right) = 1 \quad [\text{given}]$$

$$\Rightarrow 0 < \int_{1/2}^1 f(x) dx < \int_{1/2}^1 e^{2x-1} dx$$

$$\Rightarrow 0 < \int_{1/2}^1 f(x) dx < \left(\frac{e^{2x-1}}{2}\right)_{1/2}^1$$

$$\Rightarrow 0 < \int_{1/2}^1 f(x) dx < \frac{e-1}{2}$$

4. $y' - y \tan x = 2x \sec x$ and $y(0) = 0$

$$\Rightarrow \frac{dy}{dx} - y \tan x = 2x \sec x$$

$$\therefore \text{IF} = e^{\int -\tan x \, dx} = e^{\log |\cos x|} = \cos x$$

Hence, solution is $y \cdot \cos x = \int 2x \sec x \cdot \cos x \, dx + C$

$$\Rightarrow y \cdot \cos x = x^2 + C$$

$$\text{As } y(0) = 0$$

$$\Rightarrow C = 0$$

$$\therefore y = x^2 \sec x$$

$$\text{Now, } y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}} \Rightarrow y'\left(\frac{\pi}{4}\right) = \frac{\pi}{\sqrt{2}} + \frac{\pi^2}{8\sqrt{2}}$$

$$\text{and } y\left(\frac{\pi}{3}\right) = \frac{2\pi^2}{9}$$

$$\Rightarrow y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$$

5. We have, $\frac{dy}{dx} + y \cdot g'(x) = g(x) g'(x)$

$$\therefore \text{IF} = e^{\int g'(x) \, dx} = e^{g(x)}$$

So, solution is $y(e^{g(x)}) = \int g(x) \cdot g'(x) \cdot e^{g(x)} \, dx + C$

$$\text{Put } g(x) = t \Rightarrow g'(x) \, dx = dt$$

$$\therefore y(e^{g(x)}) = \int t \cdot e^t \, dt + C$$

$$= t \cdot e^t - \int 1 \cdot e^t \, dt + C$$

$$= t \cdot e^t - e^t + C$$

$$\Rightarrow y e^{g(x)} = (g(x) - 1) e^{g(x)} + C \quad \dots(i)$$

$$\text{Given, } y(0) = 0, g(0) = g(2) = 0$$

$\therefore$ Eq. (i) becomes

$$y(0) \cdot e^{g(0)} = \{g(0) - 1\} \cdot e^{g(0)} + C$$

$$\Rightarrow 0 = (-1) \cdot 1 + C$$

$$\Rightarrow C = 1$$

$$\therefore y(x) \cdot e^{g(x)} = \{g(x) - 1\} e^{g(x)} + 1$$

$$\Rightarrow y(2) \cdot e^{g(2)} = \{g(2) - 1\} e^{g(2)} + 1$$

$$\Rightarrow y(2) \cdot 1 = (-1) \cdot 1 + 1 \quad [\because g(2) = 0]$$

$$\Rightarrow y(2) = 0$$

6. Given differential equation is

$$(x \log x) \frac{dy}{dx} + y = 2x \log x, x \geq 1$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = 2$$

which is a linear differential equation.

$$\therefore \text{IF} = e^{\int \frac{1}{x \log x} \, dx} = e^{\log(\log x)} = \log x$$

Now, the solution of given differential equation is given by

$$y \cdot \log x = \int \log x \cdot 2 \, dx$$

$$\Rightarrow y \cdot \log x = 2 \int \log x \, dx$$

$$\Rightarrow y \cdot \log x = 2[x \log x - x] + C$$

$$\text{At } x = 1, C = 2$$

$$\therefore y \cdot \log x = 2[x \log x - x] + 2$$

$$\text{At } x = e, y = 2(e - e) + 2$$

$$\Rightarrow y = 2$$

7. Given differential equation $\frac{dp(t)}{dt} - \frac{1}{2} p(t) = -200$ is a linear differential equation.

$$\text{Here, } p(t) = \frac{-1}{2}, Q(t) = -200$$

$$\therefore \text{IF} = e^{\int \left(-\frac{1}{2}\right) dt} = e^{-t/2}$$

Hence, solution is $p(t) \cdot \text{IF} = \int Q(t) \cdot \text{IF} \, dt + C$

$$\Rightarrow p(t) \cdot e^{-t/2} = \int -200 \cdot e^{-t/2} \, dt + C$$

$$\Rightarrow p(t) \cdot e^{-t/2} = 400 e^{-t/2} + C$$

$$\Rightarrow p(t) = 400 + C e^{t/2}$$

$$\text{If } p(0) = 100, \text{ then } C = -300$$

$$\therefore p(t) = 400 - 300 e^{t/2}$$

8. **Statement I** Let the equation of parabola whose axis is X-axis and vertex at the origin, is

$$y^2 = 4ax$$

$$\Rightarrow 2y \frac{dy}{dx} = 4a \Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

$$\Rightarrow \frac{dy}{dx} \propto \frac{1}{y}$$

Where, a is parameter.

$$\text{Statement II } y^2 = 4ax \quad \dots(i)$$

$$\Rightarrow 2y \frac{dy}{dx} = 4a \Rightarrow a = \frac{y}{2} \cdot \frac{dy}{dx} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow y^2 = 4x \cdot \frac{y}{2} \cdot \frac{dy}{dx}$$

$$\Rightarrow y^2 = 2xy \frac{dy}{dx} \Rightarrow y = 2x \cdot \frac{dy}{dx}$$

$$\therefore \text{Order} = 1 \text{ and degree} = 1$$

9. **Statement I** Put $z = y^2 \Rightarrow \frac{dz}{dx} = 2y \frac{dy}{dx}$

$$\therefore \frac{dz}{dx} = \frac{2y^4}{2(xy^2 - x^2)} = \frac{z^2}{(xz - x^2)}$$

$$\text{Put } z = vx \Rightarrow \frac{dz}{dx} = v + x \frac{dv}{dx}$$

$$\therefore v + x \frac{dv}{dx} = \frac{v^2 x^2}{xvx - x^2} = \frac{v^2}{v - 1}$$

$$\Rightarrow \frac{x dv}{dx} = \frac{v^2 - v^2 + v}{v - 1}$$

$$\Rightarrow \frac{x dv}{dx} = \frac{v}{(v - 1)}$$

$$\Rightarrow \left(\frac{v-1}{v}\right) dv = \frac{dx}{x}$$

$$\Rightarrow \int \left(1 - \frac{1}{v}\right) dv = \int \frac{dx}{x}$$

$$\Rightarrow v - \log v = \log x - \log C$$

$$\Rightarrow \frac{z}{x} - \log \frac{z}{x} = \log x - \log C$$

$$\Rightarrow \frac{y^2}{x} - \log \frac{y^2}{x} = \log x - \log C$$

$$\Rightarrow \frac{y^2}{x} = \log \frac{y^2}{x} \cdot x - \log C$$

$$\Rightarrow e^{y^2/x} = y^2/C \Rightarrow y^2 = C \cdot e^{y^2/x}$$

$$\Rightarrow y^2 \cdot e^{-y^2/x} = C$$

10. Given, $\frac{dy}{dx} = 1 - \frac{1}{x^2} \Rightarrow dy = \left(1 - \frac{1}{x^2}\right) dx$

$$\Rightarrow y = x + \frac{1}{x} + C$$

Since, the curve passing through the point $\left(2, \frac{7}{2}\right)$.

$$\therefore \frac{7}{2} = 2 + \frac{1}{2} + C$$

$$\Rightarrow C = 1$$

$$\therefore y = x + \frac{1}{x} + 1 \quad \dots(i)$$

Given, $x = -2$

$$\therefore \text{Ordinate, } y = -2 - \frac{1}{2} + 1 = -\frac{3}{2}$$

11. We have, $\frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{4x^2}{1+x^2}$

$$\therefore \text{IF} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = (1+x^2)$$

Hence, solution is

$$y \cdot (1+x^2) = \int \frac{4x^2}{1+x^2} \cdot (1+x^2) dx + C$$

$$\Rightarrow y(1+x^2) = \int 4x^2 dx + C$$

$$\Rightarrow 3y(1+x^2) = 4x^3 + C$$

At origin, $C = 0$

$$\therefore 3y(1+x^2) = 4x^3$$

12. $\therefore dP = (100 - 12\sqrt{x}) dx$

$$\Rightarrow \int dP = \int (100 - 12\sqrt{x}) dx$$

$$\Rightarrow P = 100x - 8x^{3/2} + C$$

When $x = 0$, then $P = 2000$, so $C = 2000$

When $x = 25$, then

$$P = 100 \times 25 - 8 \times (25)^{3/2} + 2000$$

$$= 2500 - 8 \times 125 + 2000 = 3500$$

13. Given, $p'(t) = \frac{dp(t)}{dt} = 0.5 p(t) - 450$

$$\Rightarrow \frac{2dp(t)}{p(t) - 900} = dt$$

$$\Rightarrow \int \frac{2dp(t)}{p(t) - 900} = \int dt$$

$$\Rightarrow 2 \log |p(t) - 900| = t + C$$

To find the value of C , substitute $t = 0$.

$$\therefore 2 \log |p(0) - 900| = 0 + C$$

$$\Rightarrow C = 2 \log |850 - 900|$$

$$\Rightarrow C = 2 \log 50$$

Now,

$$2 \log |p(t) - 900| = t + 2 \log 50$$

Now, put $p(t) = 0$, then

$$2 \log |0 - 900| = t + 2 \log 50$$

$$\Rightarrow t = 2 \log \left| \frac{900}{50} \right| = 2 \log 18$$

14. Here, $dy/dx = y + 3 > 0$ and $y(0) = 2$

$$\Rightarrow \int \frac{dy}{y+3} = \int dx$$

$$\Rightarrow \log |y+3| = x + C$$

But $y(0) = 2$

$$\Rightarrow \log |2+3| = 0 + C$$

$$\Rightarrow C = \log_e 5$$

$$\therefore \log |y+3| = x + \log_e 5$$

When $x = \log_e 2$

$$\Rightarrow \log |y+3| = \log_e 2 + \log_e 5 = \log_e 10$$

$$\therefore y+3 = 10$$

$$\Rightarrow y = 7$$

15. Given, $\frac{d\{V(t)\}}{dt} = -k(T-t)$

$$\therefore d\{V(t)\} = -k(T-t) dt \quad \dots(i)$$

When $t = 0$, then $V(t) = I$

$$\Rightarrow \int_0^T d\{V(t)\} = \int_0^T -k(T-t) dt$$

$$\Rightarrow V(T) - V(0) = k \left[\frac{(t-T)^2}{2} \right]_0^T$$

$$\Rightarrow V(T) - I = \frac{k}{2} [(T-T)^2 - (0-T)^2]$$

$$\therefore V(T) = I - \frac{k}{2} T^2$$

16. Here, $\frac{dx}{dy} + \frac{1}{y^2} \cdot x = \frac{1}{y^3}$

which is linear differential equation in x .

$$\therefore \text{IF} = e^{\int \frac{1}{y^2} dy} = e^{-1/y}$$

$\therefore$ Complete solution is

$$x \cdot e^{-1/y} = \int \frac{1}{y^3} \cdot e^{-1/y} dy$$

$$\Rightarrow x \cdot e^{-1/y} = \int \frac{1}{y} \cdot \frac{1}{y^2} \cdot e^{1/y} dy + C$$

Put $-\frac{1}{y} = t$

$$\Rightarrow \frac{1}{y^2} dy = dt$$

$$\Rightarrow xe^{-1/y} = \int -t \cdot e^t dt + C$$

$$\Rightarrow xe^{-1/y} = -\{t e^t - \int 1 \cdot e^t dt\} + C$$

$$\Rightarrow xe^{-1/y} = -te^t + e^t + C$$

$$\Rightarrow xe^{-1/y} = \frac{1}{y} \cdot e^{-1/y} + e^{-1/y} + C \quad [\because y(1) = 1]$$

$$\therefore e^{-1} = e^{-1} + e^{-1} + C$$

$$\Rightarrow C = -\frac{1}{e}$$

$$\therefore xe^{-1/y} = \frac{1}{y} \cdot e^{-1/y} + e^{-1/y} - \frac{1}{e}$$

$$\Rightarrow x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$$

17. $\frac{d^2x}{dy^2} = \frac{d}{dy} \left(\frac{dx}{dy} \right) = \frac{d}{dy} \left(\frac{dy}{dx} \right)^{-1} = \frac{d}{dx} \left(\frac{dy}{dx} \right)^{-1} \times \frac{dx}{dy}$

$$= -1 \left(\frac{dy}{dx} \right)^{-2} \left(\frac{d^2y}{dx^2} \right) \times \left(\frac{dy}{dx} \right)^{-1}$$

$$= - \left(\frac{dy}{dx} \right)^{-3} \left(\frac{d^2y}{dx^2} \right)$$

$$\Rightarrow \frac{d^2x}{dy^2} = - \left(\frac{d^2y}{dx^2} \right) \left(\frac{dy}{dx} \right)^{-3}$$

18. Since, $\cos x \, dy = y \sin x \, dx - y^2 \, dx$

$$\Rightarrow \frac{1}{y^2} \cdot \frac{dy}{dx} - \frac{1}{y} \tan x = -\sec x$$

Put $-\frac{1}{y} = z \Rightarrow \frac{1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore \frac{dz}{dx} + (\tan x) z = -\sec x$$

which is a linear differential equation.

$$\therefore IF = e^{\int \tan x \, dx} = e^{\log \sec x} = \sec x$$

Hence, the solution is

$$z \cdot (\sec x) = \int -\sec x \cdot \sec x \, dx + C_1$$

$$\Rightarrow -\frac{1}{y} \sec x = -\tan x + C_1$$

$$\therefore \sec x = y (\tan x + C)$$

19. Given, $y = C_1 e^{C_2 x}$

$$\Rightarrow y' = C_2 C_1 e^{C_2 x}$$

$$\Rightarrow y' = C_2 y \quad \dots(i)$$

and $y'' = C_2 y' \quad \dots(ii)$

$$\Rightarrow y'' = \frac{(y')^2}{y} \quad [\text{from Eq. (i)}]$$

$$\therefore yy'' = (y')^2$$

20. The equation of family of circles with centre on $y = 2$ and of radius 5, is

$$(x - \alpha)^2 + (y - 2)^2 = 5^2 \quad \dots(i)$$

$$\Rightarrow x^2 + \alpha^2 - 2\alpha x + y^2 + 4 - 4y = 25$$

On differentiating w.r.t. x , we get

$$2x - 2\alpha + 2y \frac{dy}{dx} - 4 \frac{dy}{dx} = 0$$

$$\Rightarrow \alpha = x + \frac{dy}{dx} (y - 2)$$

On putting the value of α in Eq. (i), we get

$$\left(x - x - \frac{dy}{dx} (y - 2) \right)^2 + (y - 2)^2 = 5^2$$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 (y - 2)^2 = 25 - (y - 2)^2$$

$$\therefore y'^2 (y - 2)^2 = 25 - (y - 2)^2$$

21. Given equation can be rewritten as

$$\frac{dy}{dx} - \frac{1}{x} \cdot y = 1$$

Now, $IF = e^{-\int \frac{1}{x} \, dx} = e^{-\log x} = \frac{1}{x}$

$\therefore$ Required solution is

$$y \left(\frac{1}{x} \right) = \int \frac{1}{x} \, dx = \log x + C$$

Since, $y(1) = 1 \Rightarrow 1 = \log 1 + C \Rightarrow C = 1$

$$\therefore y = x \log x + x$$

22. Let equation of circle be

$$x^2 + y^2 + 2gx = 0$$

$$\Rightarrow 2g = - \left(2x + 2y \frac{dy}{dx} \right)$$

$$\therefore x^2 + y^2 + \left(-2x - 2y \frac{dy}{dx} \right) x = 0$$

$$\Rightarrow y^2 = x^2 + 2xy \frac{dy}{dx}$$

23. Given, $Ax^2 + By^2 = 1$

$$\Rightarrow 2Ax + 2By \frac{dy}{dx} = 0 \quad \dots(i)$$

$$\Rightarrow 2A + 2B \left\{ \left(\frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} \right\} = 0 \quad \dots(ii)$$

$$\therefore y \frac{d^2y}{dx^2} + \left(\frac{dy}{dx} \right)^2 - \frac{y}{x} \cdot \frac{dy}{dx} = 0$$

[from Eqs. (i) and (ii)]

24. Given, $\frac{dy}{dx} = \left(\frac{y}{x} \right) \left[\log \left(\frac{y}{x} \right) + 1 \right]$

Put $y = tx$,

$$\Rightarrow \frac{dy}{dx} = t + x \frac{dt}{dx}$$

$$\therefore t + x \frac{dt}{dx} = t \log t + t$$

$$\Rightarrow \frac{dt}{t \log t} = \frac{dx}{x}$$

$$\Rightarrow \int \frac{dt}{t \log t} = \int \frac{dx}{x}$$

$$\Rightarrow \log \log t = \log x + \log C$$

$$\therefore \log \left(\frac{y}{x} \right) = Cx$$

25. Given equation of family of curves is

$$y^2 = 2c(x + \sqrt{c}) \quad \dots(i)$$

On differentiating Eq. (i) w.r.t. x , we get

$$2yy_1 + 2c \Rightarrow c = yy_1$$

On putting the value of c in Eq. (i), we get

$$y^2 = 2yy_1(x + \sqrt{yy_1})$$

$$\Rightarrow (y^2 - 2yy_1x)^2 = 4(yy_1)^3$$

Thus, the degree and order of above equation are three and one, respectively.

26. The equation of the family of curves is

$$x^2 + y^2 - 2ay = 0 \quad \dots(i)$$

On differentiating w.r.t. x , we get

$$2x + 2yy' - 2ay' = 0$$

$$\Rightarrow \frac{2x + 2yy'}{y'} = 2a \quad \dots(ii)$$

From Eq. (i), $2a = \frac{x^2 + y^2}{y}$

On putting the value of $2a$ in Eq. (ii), we get

$$\frac{2x + 2yy'}{y'} = \frac{x^2 + y^2}{y}$$

$$\Rightarrow 2xy + 2y^2y' = x^2y' + y^2y'$$

$$\Rightarrow (x^2 - y^2)y' = 2xy$$

27. Given that, $y \, dx + (x + x^2y) \, dy = 0$

$$\therefore \frac{y \, dx + x \, dy}{x^2y^2} = -\frac{1}{y} \, dy$$

$$\Rightarrow d \left(-\frac{1}{xy} \right) = -\frac{1}{y} \, dy$$

On integrating both sides, we get

$$-\frac{1}{xy} = -\log y + C$$

$$\Rightarrow -\frac{1}{xy} + \log y = C$$

28. General equation of parabola whose axis is X-axis, is

$$y^2 = 4a(x + h)$$

On differentiating w.r.t. x, we get

$$2y \frac{dy}{dx} = 4a \Rightarrow y \frac{dy}{dx} = 2a$$

Again, differentiating w.r.t. x, we get

$$y \left(\frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} = 0$$

which is a differential equation whose degree and order are one and two, respectively.

29. Given differential equation can be rewritten as

$$(1 + y^2) \frac{dx}{dy} + x = e^{\tan^{-1} y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1 + y^2} x = \frac{e^{\tan^{-1} y}}{1 + y^2}$$

$$\therefore \text{IF} = e^{\int P dy} = e^{\int \frac{1}{1 + y^2} dy} = e^{\tan^{-1} y}$$

Therefore, required solution is

$$xe^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y}}{1 + y^2} dy + C_1$$

$$\Rightarrow xe^{\tan^{-1} y} = \frac{1}{2} e^{2 \tan^{-1} y} + C_1$$

$$\Rightarrow 2xe^{\tan^{-1} y} = e^{2 \tan^{-1} y} + C$$

30. Given differential equation can be rewritten as

$$\left(1 + 3 \frac{dy}{dx} \right)^2 = 4 \left(\frac{d^3y}{dx^3} \right)^3$$

which shows that the order and degree of given equation are 3 and 3, respectively.

31. Given, $\frac{d^2y}{dx^2} = e^{-2x}$

On integrating both sides, we get

$$\frac{dy}{dx} = \frac{e^{-2x}}{-2} + C$$

Again integrating on both sides, we get

$$y = \frac{e^{-2x}}{4} + Cx + d$$

32. The general equation of all non-vertical lines in a plane is $ax + by = 1$, where $b \neq 0$.

On differentiating both sides, we get

$$a + b \frac{dy}{dx} = 0$$

Again, differentiating, we get

$$b \frac{d^2y}{dx^2} = 0 \Rightarrow \frac{d^2y}{dx^2} = 0$$

33. We have, $ax^2 + by^2 = 1$

On differentiating both sides w.r.t. x, we get

$$2ax + 2by \frac{dy}{dx} = 0 \Rightarrow ax + by \frac{dy}{dx} = 0 \quad \dots (i)$$

$$\Rightarrow \frac{-a}{b} = \frac{y}{x} \cdot \frac{dy}{dx} \quad \dots (ii)$$

Again, differentiating Eq. (i) w.r.t. x, we get

$$a + b \left(\frac{dy}{dx} \right)^2 + by \frac{d^2y}{dx^2} = 0$$

$$\Rightarrow -\frac{a}{b} = \left(\frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2} \quad \dots (iii)$$

From Eqs. (ii) and (iii),

$$\frac{y}{x} \cdot \frac{dy}{dx} = \left(\frac{dy}{dx} \right)^2 + y \frac{d^2y}{dx^2}$$

$$\Rightarrow y \left(\frac{dy}{dx} \right) = x \left(\frac{dy}{dx} \right)^2 + xy \frac{d^2y}{dx^2}$$

34. The differential equation of the rectangular hyperbola $xy = c^2$ is

$$y + x \frac{dy}{dx} = 0 \Rightarrow x \frac{dy}{dx} = -y$$

35. Given differential equation is

$$y = x \frac{dy}{dx} + \frac{2}{\frac{dy}{dx}} \Rightarrow y \frac{dy}{dx} = x \left(\frac{dy}{dx} \right)^2 + 2$$

Here, order = 1 and degree = 2

36. Given curve is $y = (\cos x + y)^{1/2}$.

On differentiating both sides w.r.t. x, we get

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{2(\cos x + y)^{1/2}} \times \left(-\sin x + \frac{dy}{dx} \right) \\ &= \frac{\left(-\sin x + \frac{dy}{dx} \right)}{2y} \end{aligned}$$

$$\Rightarrow 2y \frac{dy}{dx} = -\sin x + \frac{dy}{dx} \Rightarrow (2y - 1) \frac{dy}{dx} = -\sin x$$

Again, differentiating both sides w.r.t. x, we get

$$(2y - 1) \frac{d^2y}{dx^2} + \frac{dy}{dx} \left(2 \frac{dy}{dx} \right) = -\cos x$$

$$\Rightarrow (2y - 1) \frac{d^2y}{dx^2} + \frac{dy}{dx} \left(2 \frac{dy}{dx} \right) + \cos x = 0$$

37. Equation of family of parabolas having vertex $(0, -1)$ and axis along Y-axis, is

$$x^2 = 4a(y + 1) \quad \dots (i)$$

On differentiating both sides w.r.t. x, we get

$$2x = 4ay' \Rightarrow a = \frac{x}{2y'}$$

On putting $a = \frac{x}{2y'}$ in Eq. (i), we get

$$x^2 = 4 \times \frac{x}{2y'} (y + 1)$$

$$\Rightarrow x = \frac{2}{y'} (y + 1) \Rightarrow xy' - 2y - 2 = 0$$

$$38. \text{ We have, } \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots} = \frac{dx - dy}{dx + dy}$$

On applying componendo and dividendo, we get

$$\begin{aligned} &\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) + \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \\ &\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots \right) - \left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots \right) \\ &= \frac{(dx - dy) + (dx + dy)}{(dx - dy) - (dx + dy)} \end{aligned}$$

$$\Rightarrow \frac{\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right)}{-\left(1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)} = \frac{2dx}{-2dy}$$

$$\Rightarrow \frac{e^x}{-e^{-x}} = -\frac{dx}{dy} \Rightarrow dy = \frac{dx}{e^{2x}}$$

$$\Rightarrow dy = e^{-2x} dx$$

On integrating both sides, we get

$$y = -\frac{e^{-2x}}{2} + C_1$$

$$\Rightarrow 2ye^{2x} = -1 + 2C_1 e^{2x}$$

$$\Rightarrow 2ye^{2x} = Ce^{2x} - 1 \quad [\text{where, } C = 2C_1]$$

39. The given equation is reduced to $x = e^{xy \left(\frac{dy}{dx}\right)}$.

On taking log both sides, we get

$$\log x = xy \frac{dy}{dx} \Rightarrow y dy = \frac{\log x}{x} dx$$

$$\Rightarrow \int y dy = \int \frac{1}{x} \log x dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{(\log x)^2}{2} + C'$$

$$\Rightarrow y^2 = (\log x)^2 + 2C'$$

$$\therefore y^2 = (\log x)^2 + C \quad [\text{where, } C = 2C']$$

40. Given, $x \cdot \frac{dy}{dx} + y = x \cdot \frac{f(xy)}{f'(xy)}$

$$\Rightarrow \frac{d}{dx}(xy) = x \cdot \frac{f(xy)}{f'(xy)}$$

$$\Rightarrow \frac{f'(xy)}{f(xy)} d(xy) = x dx$$

$$\Rightarrow \int \frac{f'(xy)}{f(xy)} d(xy) = \int x dx$$

$$\Rightarrow \log[f(xy)] = \frac{x^2}{2} + C$$

$$\therefore f(xy) = e^{(x^2/2 + C)} = e^{x^2/2} \cdot e^C = k \cdot e^{x^2/2}$$

41. Given differential equation is

$$\frac{dy}{dx} = \frac{x^2 + y^2 + 1}{2xy}$$

$$\Rightarrow 2xy dy = (x^2 + 1) dx + y^2 dx$$

$$\Rightarrow \frac{xd(y^2) - y^2 dx}{x^2} = \left(\frac{x^2 + 1}{x^2}\right) dx$$

$$\Rightarrow \int d\left(\frac{y^2}{x}\right) = \int \left(1 + \frac{1}{x^2}\right) dx$$

$$\Rightarrow \frac{y^2}{x} = x - \frac{1}{x} + C \Rightarrow y^2 = x^2 - 1 + Cx$$

When, $x = 1$ and $y = 0$, then $0 = 1 - 1 + C \Rightarrow C = 0$

So, the solution is $x^2 - y^2 = 1$, i.e. hyperbola.

42. Given differential equation is

$$\left(\frac{dy}{dx}\right) \tan y = \sin(x + y) + \sin(x - y)$$

$$\Rightarrow \left(\frac{dy}{dx}\right) \tan y = 2 \sin\left(\frac{x + y + x - y}{2}\right) \cos\left(\frac{x + y - x + y}{2}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right) \tan y = 2 \sin x \cos y$$

$$\Rightarrow \frac{\tan y}{\cos y} dy = 2 \sin x dx$$

$$\Rightarrow \frac{\sin y}{\cos^2 y} dy = 2 \sin x dx$$

On integrating both sides, we get

$$\int \frac{\sin y}{\cos^2 y} dy = 2 \int \sin x dx$$

$$\Rightarrow \frac{-(\cos y)^{-2+1}}{(-2+1)} = -2 \cos x + C$$

$$\Rightarrow \frac{1}{\cos y} = -2 \cos x + C$$

$$\Rightarrow \sec y = C - 2 \cos x$$

43. Given, $y = u^m \Rightarrow \frac{dy}{dx} = m \cdot u^{m-1} \cdot \frac{du}{dx}$

$$\text{Hence, } 2x^4 \cdot m u^{m-1} \cdot \frac{du}{dx} + u^{4m} = 4x^6$$

$$\Rightarrow \frac{du}{dx} = \frac{4x^6 - u^{4m}}{2mx^4 \cdot u^{2m-1}}$$

$$\therefore 4m = 6 \Rightarrow m = \frac{3}{2}$$

44. Given differential equation can be rewritten as

$$\frac{dy}{dx} = \frac{y}{x} + \frac{x \phi\left(\frac{y^2}{x^2}\right)}{y \phi'\left(\frac{y^2}{x^2}\right)}$$

$$\text{Put } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore v + x \frac{dv}{dx} = \frac{vx}{x} + \frac{x \phi\left(\frac{v^2 x^2}{x^2}\right)}{vx \phi'\left(\frac{v^2 x^2}{x^2}\right)}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{\phi(v^2)}{v \phi'(v^2)} \Rightarrow \frac{v \phi'(v^2)}{\phi(v^2)} dv = \frac{dx}{x}$$

On integrating both sides, we get

$$\frac{1}{2} \log \phi(v^2) = \log x + \log C_1$$

$$\Rightarrow \log \phi(v^2) = 2 \log x C_1$$

$$\Rightarrow \phi(v^2) = (xC_1)^2 \Rightarrow \phi\left(\frac{y^2}{x^2}\right) = x^2 C_1^2$$

$$\Rightarrow \phi\left(\frac{y^2}{x^2}\right) = x^2 C \quad [\text{where, } C_1^2 = C]$$

45. Given, $(1 + e^{x/y}) \frac{dx}{dy} + e^{x/y} \left(1 - \frac{x}{y}\right) = 0$

$$\text{Put } x = vy \Rightarrow \frac{dx}{dy} = v + y \frac{dv}{dy}$$

$$\therefore (1 + e^{vy/y}) \left(v + y \frac{dv}{dy}\right) + e^{vy/y} (1 - v) = 0$$

$$\Rightarrow (1 + e^v) \left(v + y \frac{dv}{dy}\right) + e^v (1 - v) = 0$$

$$\Rightarrow v + y \frac{dv}{dy} + ve^v + e^v y \frac{dv}{dy} + e^v - ve^v = 0$$

$$\Rightarrow y \frac{dv}{dy} (1 + e^v) + v + e^v = 0$$

$$\Rightarrow y \frac{dv}{dy} (1 + e^v) = -(v + e^v)$$

$$\Rightarrow \frac{(1 + e^v) dv}{(v + e^v)} = -\frac{1}{y} dy$$

On integrating both sides, we get

$$\int \frac{1 + e^v}{v + e^v} dv = -\int \frac{1}{y} dy$$

Put $v + e^v = t$

$$\Rightarrow (1 + e^v) dv = dt$$

$$\therefore \int \frac{1}{t} dt = -\int \frac{1}{y} dy$$

$$\Rightarrow \log t = -\log y + \log C$$

$$\Rightarrow \log (v + e^v) = \log \frac{C}{y}$$

$$\Rightarrow \log \left(\frac{x}{y} + e^{x/y} \right) = \log \frac{C}{y}$$

$$\Rightarrow x + ye^{x/y} = C$$

46. Given differential equation is

$$x \frac{dy}{dx} = y + xe^{y/x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

which is a homogeneous differential equation.

Put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{Now, } v + x \frac{dv}{dx} = \frac{vx}{x} + e^{vx/x}$$

$$\Rightarrow v + x \frac{dv}{dx} = v + e^v$$

$$\Rightarrow x \frac{dv}{dx} = e^v$$

$$\Rightarrow e^{-v} dv = \frac{1}{x} dx$$

On integrating both sides, we get

$$-e^{-v} = \log x + C \Rightarrow -e^{-y/x} = \log x + C$$

Given, $y(1) = 0$

$$\Rightarrow -e^{-0/1} = \log 1 + C - 1 = 0 + C \Rightarrow C = -1$$

$$\therefore -e^{-y/x} = \log x - 1 \Rightarrow 1 = \log x + e^{-y/x}$$

47. Given differential equation is

$$(x^2 - y^2) dx + 2xy dy = 0$$

$$\Rightarrow 2xy dy = (y^2 - x^2) dx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \dots (i)$$

which is a homogeneous differential equation.

$$\text{Put } y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

From Eq. (i),

$$v + x \frac{dv}{dx} = \frac{v^2 x^2 - x^2}{2vx^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1}{2v} - v$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2 - 1 - 2v^2}{2v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-v^2 - 1}{2v}$$

$$\Rightarrow \frac{2v}{v^2 + 1} dv = -\frac{dx}{x}$$

On integrating both sides, we get

$$\log |v^2 + 1| = -\log |x| + \log C$$

$$\Rightarrow \log |x(v^2 + 1)| = \log C$$

$$\Rightarrow x(v^2 + 1) = C \Rightarrow x \left(\frac{y^2}{x^2} + 1 \right) = C \quad \left[\because v = \frac{y}{x} \right]$$

$$\Rightarrow y^2 + x^2 = Cx$$

48. Given differential equation is

$$\frac{dy}{dx} + \frac{y}{x} = 3x$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

$$\text{where, } P = \frac{1}{x} \text{ and } Q = 3x$$

$$\Rightarrow IF = e^{\int P dx} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$\therefore$ Required solution is

$$yx = \int 3x \times x dx + C \quad \dots (i)$$

$$yx = 3 \left[\frac{x^3}{3} \right] + C \Rightarrow y = x^2 + \frac{C}{x}$$

Eq. (i) can also be written as $yx = \int 3x \times x dx - C$

$$\Rightarrow yx = x^3 - C \Rightarrow y = x^2 - \frac{C}{x}$$

49. Given equation is $(1 + x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{(1 + x^2)} y = \frac{e^{\tan^{-1} x}}{1 + x^2}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

$$\text{where, } P = \frac{1}{1 + x^2} \text{ and } Q = \frac{e^{\tan^{-1} x}}{1 + x^2}$$

$$\therefore IF = e^{\int P dx} = e^{\int \frac{1}{1 + x^2} dx} = e^{\tan^{-1} x}$$

50. Given differential equation is

$$\frac{dy}{dx} + \frac{y}{x \log_e x} = \frac{1}{x}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + Py = Q$$

$$\text{where, } P = \frac{1}{x \log_e x} \text{ and } Q = \frac{1}{x}$$

$$\therefore IF = e^{\int P dx} = e^{\int \frac{1}{x \log_e x} dx} = e^{\log (\log_e x)} = \log_e x$$

Hence, required solution of differential equation is

$$y \times \log_e x = \int \frac{1}{x} \log_e x dx + C$$

$$\Rightarrow y \times \log_e x = \frac{(\log_e x)^2}{2} + C \quad \dots (i)$$

When $y = 1$ and $x = e$, then

$$1 \times \log_e e = \frac{(\log_e e)^2}{2} + C$$

$$\Rightarrow 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

On putting $C = \frac{1}{2}$ in Eq.(i), we get

$$y \times \log_e x = \frac{(\log_e x)^2}{2} + \frac{1}{2}$$

$$\Rightarrow 2y = \log_e x + \frac{1}{\log_e x}$$

51. Given, $x \cos x \left(\frac{dy}{dx} \right) + y (x \sin x + \cos x) = 1$

$$\Rightarrow \frac{dy}{dx} + \frac{y (x \sin x + \cos x)}{x \cos x} = \frac{1}{x \cos x}$$

which is a linear differential equation of the form $\frac{dy}{dx} + py = Q$

where, $P = \frac{x \sin x + \cos x}{x \cos x} = \tan x + \frac{1}{x}$, $Q = \frac{1}{x \cos x}$

$$\therefore IF = e^{\int P dx} = e^{\int \left(\tan x + \frac{1}{x} \right) dx} = e^{(\log \sec x + \log x)}$$

$$= e^{\log (x \sec x)} = x \sec x$$

Hence, required solution is

$$y \times x \sec x = \int \frac{x \sec x}{x \cos x} dx = \int \sec^2 x dx$$

$$\Rightarrow xy \sec x = \tan x + C$$

52. Given, $\lim_{t \rightarrow x} \frac{t^2 f(x) - x^2 f(t)}{t - x} = 1$

On applying L' Hospital's rule, we get

$$\lim_{t \rightarrow x} \frac{2t f(x) - x^2 f'(t)}{1} = 1$$

$$\Rightarrow 2x f(x) - x^2 f'(x) = 1$$

$$\Rightarrow x^2 f'(x) - 2x f(x) = -1$$

$$\Rightarrow f'(x) - \frac{2}{x} f(x) = -\frac{1}{x^2}$$

which is a linear differential equation of the form

$$\frac{dy}{dx} + py = Q, \text{ where } P = \frac{-2}{x} \text{ and } Q = \frac{-1}{x^2}$$

$$\therefore IF = e^{\int \frac{-2}{x} dx} = e^{-2 \log x} = e^{\log \left(\frac{1}{x^2} \right)} = \frac{1}{x^2}$$

Hence, required solution is

$$f(x) \times \frac{1}{x^2} = \int \left(-\frac{1}{x^2} \times \frac{1}{x^2} \right) dx + C$$

$$\Rightarrow f(x) \times \frac{1}{x^2} = \frac{1}{3x^3} + C \Rightarrow f(x) = \frac{1}{3x} + Cx^2$$

At $f(1) = 1, 1 = \frac{1}{3} + C \Rightarrow C = \frac{2}{3}$

$$\therefore f(x) = \frac{1}{3x} + \frac{2}{3} x^2$$

53. Given differential equation is

$$y^2 + \left(x - \frac{1}{y} \right) \frac{dy}{dx} = 0$$

or $y^2 \frac{dx}{dy} + x = \frac{1}{y}$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3}$$

which is a linear differential equation of the form, $\frac{dx}{dy} + Px = Q$, where $P = \frac{1}{y^2}$ and $Q = \frac{1}{y^3}$

$$\therefore IF = e^{\int \frac{1}{y^2} dy} = e^{-y^{-1}} = e^{-1/y}$$

Hence, required solution is

$$xe^{-1/y} = \int e^{-1/y} \times \frac{1}{y^3} dy + C$$

Put $\frac{1}{y} = t \Rightarrow -\frac{1}{y^2} dy = dt$

$$\Rightarrow -\int e^{-t} t dt = -[-t e^{-t} - \int -1 e^{-t} dt] + C$$

$$= -[-t e^{-t} - e^{-t}] = e^{-t} (t + 1) + C$$

$$\Rightarrow xe^{-1/y} = e^{-1/y} \left(\frac{1}{y} + 1 \right) + C$$

$$\therefore x = \frac{1}{y} + 1 + C e^{1/y}$$

54. Given differential equation can be rewritten as

$$\cos y \frac{dx}{dy} + x \sin y - 1 = 0$$

$$\Rightarrow \frac{dx}{dy} + (\tan y) x = \sec y$$

which is a linear differential equation of the form

$$\frac{dx}{dy} + Px = Q$$

where, $P = \tan y$ and $Q = \sec y$

$$\therefore IF = e^{\int P dy} = e^{\int \tan y dy} = e^{\log \sec y} = \sec y$$

Hence, required solution is $x \sec y = \int \sec^2 y dy + C$

$$\Rightarrow x \sec y = \tan y + C$$

55. We have, $\frac{dy}{dx} + \frac{y}{x} = x^3$

which is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$, where $P = \frac{1}{x}$, $Q = x^3$

$$\therefore IF = e^{\int \frac{1}{x} dx} = e^{\log_e x} = x$$

Hence, required solution of given differential equation is $yx = \int x^3 \cdot x dx + C \Rightarrow yx = \int x^4 dx + C$

$$\Rightarrow yx = \frac{x^5}{5} + C$$

$$\Rightarrow y = \frac{x^4}{5} + \frac{C}{x}$$

56. Equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{dx}{dy} (x - x_1)$$

Put $y = 0$, then

$$x = x_1 + y_1 \frac{dy}{dx}$$

Also, $y_1^2 = 2x_1 x = 2x_1 \left(x_1 + y_1 \frac{dy}{dx} \right)$

$$\Rightarrow y_1^2 = 2x_1^2 + 2x_1 y_1 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y_1^2 - 2x_1^2}{2x_1 y_1} = \frac{\left(\frac{y_1}{x_1} \right)^2 - 2}{2 \left(\frac{y_1}{x_1} \right)}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\left(\frac{y}{x}\right)^2 - 2}{2y/x}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = \frac{v^2 - 2}{2v}$$

$$\Rightarrow x \cdot \frac{dv}{dx} = -\frac{(2 + v^2)}{2v}$$

$$\Rightarrow \frac{2v}{2 + v^2} dv + \frac{dx}{x} = 0$$

On integrating both sides, we get

$$\log(2 + v^2) + \log x = \log C$$

$$\Rightarrow \log(2 + v^2)x = \log C$$

$$\Rightarrow x \left(2 + \frac{y^2}{x^2}\right) = C$$

Since, it passes through (2, 1).

Therefore, $2 \left(2 + \frac{1}{4}\right) = C \Rightarrow C = \frac{9}{2}$

$$\therefore \frac{x(2x^2 + y^2)}{x^2} = \frac{9}{2}$$

$$\Rightarrow 2x^2 + y^2 = \frac{9}{2}x$$

$$\Rightarrow 4x^2 + 2y^2 - 9x = 0$$

57. Given, rate of digging a plot is

$$\frac{dA}{dt} = \frac{2}{\sqrt{t}} \Rightarrow dA = \frac{2}{\sqrt{t}} dt$$

On integrating both sides, we get

$$\int dA = \int \frac{2}{\sqrt{t}} dt$$

$$\Rightarrow A = \frac{2t^{1/2}}{1/2} + C$$

$$\Rightarrow A = 4\sqrt{t} + C$$

When $t = 0$, $A = 0$, then $C = 0$

$$\therefore A = 4\sqrt{t} \text{ but } A = 40$$

$$\therefore 40 = 4\sqrt{t} \Rightarrow \sqrt{t} = 10 \Rightarrow t = 100 \text{ min}$$

58. Given, $\sqrt{y} = \cos^{-1} x \Rightarrow y = (\cos^{-1} x)^2$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = 2(\cos^{-1} x) \times \frac{-1}{\sqrt{1-x^2}}$$

Again, differentiating both sides w.r.t. x , we get

$$-2 \left[\frac{\sqrt{1-x^2} \times \frac{-1}{\sqrt{1-x^2}} - \cos^{-1} x \times \left(\frac{1}{2}\right) \frac{(-2x)}{(1-x^2)^{1/2}}}{(\sqrt{1-x^2})^2} \right]$$

$$= -2 \left[\frac{-1 + \frac{x \cos^{-1} x}{(1-x^2)^{1/2}}}{(1-x^2)} \right] \Rightarrow \frac{d^2 y}{dx^2} = \left[\frac{2 - \frac{2x \cos^{-1} x}{(1-x^2)^{1/2}}}{(1-x^2)} \right]$$

$$\Rightarrow (1-x^2) \frac{d^2 y}{dx^2} = 2 + x \frac{dy}{dx} \Rightarrow (1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = 2$$

But it is given that, $(1-x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} = C$

$$\therefore C = 2$$

59. Given, $\frac{d^2 y}{dx^2} + 3y = -2x$

It can be rewritten as $(D^2 + 3)y = -2x$

$\therefore$ Auxiliary equation is

$$m^2 + 3 = 0 \Rightarrow m = \pm \sqrt{3}$$

$\therefore$ Complementary function $= C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x$

Now, $PI = \frac{-2x}{(D^2 + 3)} = \frac{1}{3} \cdot \frac{(-2x)}{\left(1 + \frac{D^2}{3}\right)}$

$$= -\frac{1}{3} \left(1 + \frac{D^2}{3}\right)^{-1} 2x = -\frac{1}{3} \left[1 - \frac{D^2}{3} + \dots\right] 2x = -\frac{2x}{3}$$

$\therefore$ Required solution is

$$C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x - \frac{2x}{3}$$

60. Let $y = 2x - 4 \Rightarrow \frac{dy}{dx} = 2$

$$\therefore \left(\frac{dy}{dx}\right)^2 - x \left(\frac{dy}{dx}\right) + y = (2)^2 - 2x + 2x - 4$$

$$= 4 - 4 = 0$$

So, $y = 2x - 4$ is a solution of given equation.

61. Given, $(x^2 + xy)dy = (x^2 + y^2)dx \Rightarrow \frac{dy}{dx} = \frac{x^2 + y^2}{x^2 + xy}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$$\therefore v + x \frac{dv}{dx} = \frac{x^2 + v^2 x^2}{x^2 + x^2 v}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{1 + v^2}{1 + v} \Rightarrow \frac{xdv}{dx} = \frac{1 + v^2}{1 + v} - v$$

$$= \frac{1 + v^2 - v - v^2}{1 + v} = \frac{1 - v}{1 + v}$$

$$\Rightarrow dv \left(\frac{1+v}{1-v}\right) = \frac{dx}{x} \Rightarrow dv \left(-1 + \frac{2}{1-v}\right) = \frac{dx}{x}$$

On integrating both sides, we get

$$-v - 2 \log(1-v) = \log x + C$$

$$\Rightarrow -\frac{y}{x} - 2 \log \left(1 - \frac{y}{x}\right) = \log x + C$$

$$\Rightarrow -\frac{y}{x} - 2 \log(x-y) + 2 \log x = \log x + C$$

$$\Rightarrow \frac{y}{x} + 2 \log(x-y) + C = \log x$$

62. Here, $\frac{dy}{dx} + \frac{y}{x} \log x = e^x x^{-(1/2) \log x}$

$$\therefore IF = e^{\int \frac{1}{x} \log x \, dx} = e^{\frac{(\log x)^2}{2}} = (\sqrt{e})^{(\log x)^2}$$

63. Given, $\left(\frac{d^3 y}{dx^3}\right)^{4/5} - 2 \frac{dy}{dx} \left(\frac{d^2 y}{dx^2}\right)^2 = 0$

$$\Rightarrow \left(\frac{d^3 y}{dx^3}\right)^{4/5} = 2 \frac{dy}{dx} \left(\frac{d^2 y}{dx^2}\right)^2$$

Both sides raise to the power 5.

$$\therefore \left(\frac{d^3 y}{dx^3}\right)^4 = 32 \left(\frac{dy}{dx}\right)^5 \left(\frac{d^2 y}{dx^2}\right)^{10}$$

Hence, the order of the given differential equation is 3 and degree is 4.

64. Let centre of circle on X-axis be (a, 0).

∴ Equation of family of circles is

$$(x-a)^2 + y^2 = a^2 \Rightarrow x^2 + y^2 - 2ax = 0 \quad \dots(i)$$

On differentiating both sides w.r.t. x, we get

$$2x + 2y \frac{dy}{dx} - 2a = 0 \Rightarrow a = x + y \frac{dy}{dx}$$

From Eq. (i), $x^2 + y^2 - 2x \left(x + y \frac{dy}{dx} \right) = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

65. Given, $\left(1 + 5 \frac{dy}{dx}\right)^{3/2} = 10 \frac{d^3y}{dx^3}$

On squaring both sides, we get

$$\left(1 + 5 \frac{dy}{dx}\right)^3 = 100 \left(\frac{d^3y}{dx^3}\right)^2$$

$$\Rightarrow 1 + 125 \left(\frac{dy}{dx}\right)^3 + 15 \frac{dy}{dx} \left(1 + 5 \frac{dy}{dx}\right) = 100 \left(\frac{d^3y}{dx^3}\right)^2$$

Clearly, the order of highest derivative occurring in the differential equation is 3. Hence, the order of given differential equation is 3.

66. Given, $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

On differentiating both sides w.r.t. x, we get

$$\frac{-2x}{2\sqrt{1-x^2}} + \frac{-2y}{2\sqrt{1-y^2}} \frac{dy}{dx} = a \left(1 - \frac{dy}{dx}\right)$$

$$\Rightarrow \frac{dy}{dx} \left(a - \frac{y}{\sqrt{1-y^2}}\right) = a + \frac{x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} \left(\frac{a\sqrt{1-y^2} - y}{\sqrt{1-y^2}}\right) = \frac{a\sqrt{1-x^2} + x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(\sqrt{1-y^2})(a\sqrt{1-x^2} + x)}{(\sqrt{1-x^2})(a\sqrt{1-y^2} - y)}$$

Hence, the degree of given differential equation is 1.

67. Given, $e^{dy/dx} = x$

On taking log both sides, we get

$$\frac{dy}{dx} = \log x \Rightarrow dy = \log x \, dx$$

On integrating both sides, we get

$$\int dy = \int \log x \, dx$$

$$\Rightarrow y = x \log x - \int \frac{x}{x} + C = x \log x - x + C \quad \dots(i)$$

Also, it is given that at $x=1$, $y=0$, then

$$0 = (1) \log 1 - 1 + C \Rightarrow C = 1$$

On substituting it in Eq. (i), we get

$$y = x \log x - x + 1$$

$$\Rightarrow y = x (\log x - 1) + 1$$

68. Given differential equation is

$$\frac{dy}{dx} = \frac{x+y+1}{2x+2y+1} \quad \dots(i)$$

$$\text{Put } x+y=v \Rightarrow 1 + \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dy}{dx} = \frac{dv}{dx} - 1$$

Eq. (i) becomes

$$\frac{dy}{dx} = \frac{v+1}{2v+1} \Rightarrow \frac{dv}{dx} - 1 = \frac{v+1}{2v+1}$$

$$\Rightarrow \frac{dv}{dx} = \frac{v+1}{2v+1} + 1 \Rightarrow \frac{dv}{dx} = \frac{v+1+2v+1}{2v+1} = \frac{3v+2}{2v+1}$$

$$\Rightarrow dv \left(\frac{2v+1}{3v+2} \right) = dx \Rightarrow dv \left(\frac{2}{3} - \frac{1}{3(3v+2)} \right) = dx$$

On integrating both sides, we get

$$\frac{2}{3} v - \frac{1}{9} \log |3v+2| = x + C_1$$

$$\Rightarrow \frac{2}{3} (x+y) - \frac{1}{9} \log |3x+3y+2| = x + C_1$$

$$\Rightarrow \frac{6x+6y - \log |3x+3y+2|}{9} = x + C_1$$

$$\Rightarrow \log |3x+3y+2| + 3x - 6y = C \text{ [where, } C = -9C_1]$$

69. Given, $(1+t) \frac{dy}{dt} - ty = 1$

$$\Rightarrow \frac{dy}{dt} - \frac{ty}{1+t} = \frac{1}{1+t}$$

$$\therefore \text{IF} = e^{\int \left(\frac{-t}{1+t} \right) dt} = e^{-t + \log(1+t)} = e^{-t} (1+t)$$

Hence, required solution is

$$ye^{-t}(1+t) = \int e^{-t} \frac{(1+t)}{1+t} + C$$

$$\Rightarrow ye^{-t}(1+t) = \int e^{-t} + C$$

$$\Rightarrow ye^{-t}(1+t) = -e^{-t} + C$$

Also, given that $y(0) = -1$.

$$\therefore (-1)e^{-0}(1+0) = -e^{-0} + C - 1 = -1 + C$$

$$\Rightarrow C = 0$$

$$\therefore y(1) = -\frac{1}{1+1} = -\frac{1}{2}$$

70. Given, $(x+a) \frac{dy}{dx} - 3y = (x+a)^5$

$$\Rightarrow \frac{dy}{dx} - \frac{3y}{(x+a)} = (x+a)^4$$

$$\therefore \text{IF} = e^{\int \left(\frac{-3}{(x+a)} \right) dx} = e^{-3 \log(x+a)} = \frac{1}{(x+a)^3}$$

Hence, required solution is

$$y \times \frac{1}{(x+a)^3} = \int (x+a) dx + C$$

$$\Rightarrow y \times \frac{1}{(x+a)^3} = \frac{x^2}{2} + ax + C$$

At $x=a$, $y=16a^5$

$$\frac{16a^5}{8a^3} = \frac{a^2}{2} + a^2 + C \Rightarrow C = \frac{a^2}{2}$$

∴ Required solution is

$$y \times \frac{1}{(x+a)^3} = \frac{x^2}{2} + ax + \frac{a^2}{2} \Rightarrow 2y = (x+a)^5$$

71. Given, $(x+2y^3) \frac{dy}{dx} = y$

$$\Rightarrow y \frac{dx}{dy} = x + 2y^3 \Rightarrow \frac{dx}{dy} - \frac{x}{y} = 2y^2$$

which is a linear differential equation of the form

$$\frac{dx}{dy} + Px = Q, \text{ where } P = -\frac{1}{y}, Q = 2y^2$$

$$\therefore \text{IF} = e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = y^{-1} = \frac{1}{y}$$

Hence, required solution is

$$\frac{x}{y} = \int \frac{2y^2}{y} dy + C$$

$$\Rightarrow \frac{x}{y} = y^2 + C \Rightarrow x = y^3 + Cy$$

72. Given, $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + \int y dx = x^3$

On differentiating both sides w.r.t. x , we get

$$\frac{d^3y}{dx^3} + 5\frac{d^2y}{dx^2} + y = 3x^2$$

Therefore, the order of given differential equation is 3.

73. Given, $\sqrt{r} = ae^{\theta \cot \alpha}$

On taking log both sides, we get

$$\frac{1}{2} \log r = \log a + \theta \cot \alpha$$

On differentiating both sides w.r.t. θ , we get

$$\frac{1}{2r} \cdot \frac{dr}{d\theta} = \cot \alpha \Rightarrow \frac{dr}{d\theta} = 2r \cot \alpha \quad \dots (i)$$

Again, differentiating both sides w.r.t. θ , we get

$$\frac{d^2r}{d\theta^2} = \frac{2dr}{d\theta} \cot \alpha \quad \dots (ii)$$

On substituting the value of $\frac{dr}{d\theta}$ from Eq. (i) in Eq. (ii), we get

$$\frac{d^2r}{d\theta^2} = 4r \cot^2 \alpha$$

$$\Rightarrow \frac{d^2r}{d\theta^2} - 4r \cot^2 \alpha = 0$$

74. Given, $y = (\sin^{-1} x)^2$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{2 \sin^{-1} x}{\sqrt{1-x^2}} \quad \dots (i)$$

Again, differentiating both sides w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \frac{\left(\frac{2\sqrt{1-x^2}}{\sqrt{1-x^2}} - \frac{2(-2x)\sin^{-1} x}{2\sqrt{1-x^2}} \right)}{(1-x^2)} \quad \dots (ii)$$

On substituting Eq. (i) in Eq. (ii), we get

$$(1-x^2) \frac{d^2y}{dx^2} = 2 + x \frac{dy}{dx}$$

$$\Rightarrow (1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} = 2$$

75. Given, $y = A/x + Bx^2$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = -\frac{A}{x^2} + 2Bx$$

Again, differentiating both sides w.r.t. x , we get

$$\frac{d^2y}{dx^2} = \frac{2A}{x^3} + 2B$$

On multiplying both sides by x^2 , we get

$$x^2 \frac{d^2y}{dx^2} = \frac{2A}{x} + 2Bx^2 = 2 \left(\frac{A}{x} + Bx^2 \right)$$

$$\Rightarrow x^2 \frac{d^2y}{dx^2} = 2y$$

76. Given, $\frac{d^2y}{dx^2} = \frac{5y + \frac{dy}{dx}}{\sqrt{\frac{d^2y}{dx^2}}}$

$$\Rightarrow \frac{d^2y}{dx^2} \sqrt{\frac{d^2y}{dx^2}} = 5y + \frac{dy}{dx}$$

$$\Rightarrow \left(\frac{d^2y}{dx^2} \right)^{3/2} = 5y + \frac{dy}{dx}$$

On squaring both sides, we get

$$\left(\frac{d^2y}{dx^2} \right)^3 = 25y^2 + \left(\frac{dy}{dx} \right)^2 + 10y \frac{dy}{dx}$$

Hence, the degree of given differential equation is 3.

77. Given, $\left(\frac{dy}{dx} \right)^2 = 1 - x^2 - y^2 + x^2 y^2$

$$\Rightarrow \left(\frac{dy}{dx} \right)^2 = 1 - y^2 - x^2 (1 - y^2) = (1 - y^2)(1 - x^2)$$

$$\therefore \frac{dy}{dx} = \sqrt{(1 - y^2)(1 - x^2)}$$

$$\Rightarrow \frac{dy}{\sqrt{1 - y^2}} = dx \sqrt{1 - x^2}$$

On integrating both sides, we get

$$\int \frac{dy}{\sqrt{1 - y^2}} = \int \sqrt{1 - x^2} dx$$

$$\Rightarrow \sin^{-1} y = \frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x + C$$

$$\Rightarrow 2 \sin^{-1} y = x \sqrt{1 - x^2} + \sin^{-1} x + C$$

78. Given, $\frac{dy}{dx} = e^y (e^x + e^{-x} + 2x)$

$$\Rightarrow \frac{dy}{e^y} = dx (e^x + e^{-x} + 2x)$$

On integrating both sides, we get

$$\int \frac{dy}{e^y} = \int (e^x + e^{-x} + 2x) dx$$

$$\Rightarrow e^{-y} = e^x - e^{-x} + x^2 + C$$

$$\Rightarrow e^{-y} = e^{-x} - e^x - x^2 + C$$

79. Given equation can be rewritten as

$$\frac{4y dy}{1 + y^2} = \frac{3(1 + x^2)}{x(1 + x^2)} dx$$

$$\Rightarrow \int \frac{4y}{1 + y^2} dy = \int \left(\frac{3}{x} + \frac{6}{1 + x^2} \right) dx$$

$$\therefore 2 \log (1 + y^2) = 3 \log x + 6 \tan^{-1} x + C$$

$$\left[\because \int \frac{1}{1 + x^2} dx = \tan^{-1} x \right]$$

80. Given, $(1 + x^2) \frac{dy}{dx} + xy = x \Rightarrow \frac{dy}{dx} + \frac{xy}{(1 + x^2)} = \frac{x}{1 + x^2}$

From the given equation, $P = \frac{x}{1 + x^2}$, $Q = \frac{x}{1 + x^2}$

Hence, IF = $e^{\int P dx} = e^{\int \frac{x}{1 + x^2} dx}$

$$= e^{\frac{1}{2} \log (1 + x^2)} = e^{\log (1 + x^2)^{1/2}} = \sqrt{1 + x^2}$$

81. Given, $\frac{dy}{dx} - \frac{3x^2 y}{1+x^3} = \frac{\sin^2 x}{1+x}$

From the given equation, $P = -\frac{3x^2}{1+x^3}$, $Q = \frac{\sin^2 x}{1+x}$

$\therefore IF = e^{\int P dx} = e^{\int \frac{-3x^2}{1+x^3} dx}$

Put $1+x^3 = t \Rightarrow 3x^2 dx = dt$

$\therefore IF = e^{\int \frac{-1}{t} dt} = e^{-\log t} = e^{-\log(1+x^3)} = e^{\log(1+x^3)^{-1}}$

Hence, $IF = (1+x^3)^{-1} = \frac{1}{1+x^3}$

82. Given, $ydx - xdy + \log x dx = 0$

$\Rightarrow dx(y + \log x) = x dy \Rightarrow \frac{y + \log x}{x} = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \frac{\log x}{x}$

$\therefore IF = e^{\int -\frac{1}{x} dx} = e^{-\log x} = x^{-1} = \frac{1}{x}$

Hence, required solution is

$\frac{y}{x} = \int \frac{\log x}{x^2} dx + C \quad \dots(i)$

Let $I = \int \frac{\log x}{x^2} dx$

Put $\frac{1}{x} = t \quad \dots(ii)$

$\Rightarrow -\frac{1}{x^2} dx = dt$

On taking log both sides of Eq. (ii), we get

$\log\left(\frac{1}{x}\right) = \log t \Rightarrow \log x = -\log t$

$\Rightarrow \log x = -\log t$

Hence, $I = \int \log t dt = t \log t - \int \frac{t}{t} dt = t \log t - 1 + C$

$\Rightarrow I = \frac{1}{x} \log\left(\frac{1}{x}\right) - \frac{1}{x} + C = -\frac{1}{x} \log x - \frac{1}{x} + C$

On substituting I in Eq. (i), we get

$\frac{y}{x} = -\frac{\log x}{x} - \frac{1}{x} + C$

$\Rightarrow y = -\log x - 1 + Cx$

$\Rightarrow y + \log x + 1 = Cx$

83. Given differential equation is $\frac{d^2 y}{dx^2} + 8 \frac{dy}{dx} + 16y = 0$

$\therefore$ Auxiliary equation is $m^2 + 8m + 16 = 0$

$\Rightarrow (m+4)^2 = 0$

$\Rightarrow m = -4$

Hence, the solution is $y = (A + Bx)e^{-4x}$

84. Given differential equation is

$y = x \frac{dy}{dx} + \left[a^2 \left(\frac{dy}{dx} \right)^2 + b^2 \right]^{1/3}$

$\Rightarrow \left(y - x \frac{dy}{dx} \right)^3 = a^2 \left(\frac{dy}{dx} \right)^2 + b^2$

So, order and degree of the above differential equation are 1 and 3, respectively.

85. Given curve is $y = e^{a \sin x}$

On taking log both sides, we get

$\sin x = \frac{\log y}{a}$

$\therefore \frac{dy}{dx} = e^{a \sin x} \cdot a \cos x$

$\Rightarrow \frac{dy}{dx} = y \cos x \cdot \frac{\log y}{\sin x}$

$\Rightarrow y \log y = \tan x \frac{dy}{dx}$

86. Given differential equation is

$x = 1 + \frac{dy}{dx} + \frac{1}{2!} \left(\frac{dy}{dx} \right)^2 + \frac{1}{3!} \left(\frac{dy}{dx} \right)^3 + \dots$

$\Rightarrow x = e^{\frac{dy}{dx}} \Rightarrow \frac{dy}{dx} = \log_e x$

Hence, degree of differential equation is 1.

87. Given equation can be rewritten as

$\frac{1-y}{y^2} dy + \frac{1+x}{x^2} dx = 0$

$\Rightarrow \int \left(\frac{1}{y^2} - \frac{1}{y} \right) dy + \int \left(\frac{1}{x^2} + \frac{1}{x} \right) dx = 0$

$\therefore \log\left(\frac{x}{y}\right) = \frac{1}{x} + \frac{1}{y} + C$

88. Given differential equation is

$xdy = ydx \Rightarrow \frac{dy}{y} = \frac{dx}{x} \Rightarrow \int \frac{dy}{y} = \int \frac{dx}{x}$

$\Rightarrow \log_e y = \log_e x + \log_e C \Rightarrow y = Cx$

which is a straight line.

89. Given, $x \frac{dy}{dx} + (1+x)y = x$

$\Rightarrow \frac{dy}{dx} + \left(\frac{1+x}{x} \right) y = 1$

$\therefore IF = e^{\int \left(\frac{1+x}{x} \right) dx} = e^{\int \left(\frac{1}{x} + 1 \right) dx} = e^{(\log x + x)} = xe^x$

90. We have, $\frac{dy}{dx} = y + 2x$

$\Rightarrow \frac{dy}{dx} - y = 2x$

$\therefore IF = e^{\int -1 dx} = e^{-x}$

Hence, solution of the differential equation is

$y \cdot e^{-x} = 2 \int xe^{-x} dx = 2(-xe^{-x} - e^{-x}) + C$

$\Rightarrow y = 2e^x(-xe^{-x} - e^{-x}) + Ce^x$

$\Rightarrow y = -2x - 2 + Ce^x$

On putting $C = 2$, we get

$y = 2(e^x - x - 1)$

91. Given differential equation is $100 \frac{d^2 y}{dx^2} - 20 \frac{dy}{dx} + y = 0$.

Here, auxiliary equation is

$(100m^2 - 20m + 1) = 0$

$\Rightarrow (10m - 1)^2 = 0$

$\Rightarrow m = \frac{1}{10}, \frac{1}{10}$

Hence, the required solution is

$y = (C_1 + C_2 x)e^{x/10}$

13

Vector Algebra

Scalar and Vector Quantities

Physical quantities are divided into two categories, scalar quantities and vector quantities. Those quantities which have only magnitude and which are not related to any fixed direction in space, also does not follow the rules of vector algebra, are called scalar quantities or briefly scalars. Examples of scalar quantities are mass, volume, density, work, temperature, etc. Vectors are those which have both magnitude and direction and follow the rules of vector algebra, are called vector quantities. Displacement, velocity, acceleration, momentum, weight, force, etc., are the examples of vector quantities.

➡ **Example 1.** Which among the following is scalar quantity?

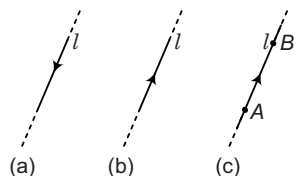
- (a) Force (b) Resistance
(c) Momentum (d) Weight

Sol. (b) Resistance is a scalar quantity.

Representation of a Vector

A vector is generally represented by a direction line segment, say $\overrightarrow{AB}$. A is called the initial point and B is called the terminal point. The positive number representing the measure of length of the line segment denoting the vector is called the modulus or magnitude of the vector and the arrow head indicates its direction. The modulus of vector $\overrightarrow{AB}$ is denoted by $|\overrightarrow{AB}|$.

➡ **Example 2.** Consider the figures (a), (b) and (c) given below.



Here, l is representing any straight line in plane or three dimensional space.

Based on above figures, which of the following is correct?

- (a) Figures (a) and (b) are representing a directed line
(b) Figure (c) is representing a directed line segment
(c) A directed line segment has magnitude as well as direction
(d) All of the above

Chapter Snapshot

- Scalar and Vector Quantities
- Types of Vectors
- Addition of Vectors
- Subtraction of Vectors
- Multiplication of a Vector by a Scalar
- Linear Combination of Vectors
- Linearly Independent and Dependent Vectors
- Component of a Vector
- Position Vector
- Section Formulae
- Scalar (Dot) Product of Two Vectors
- Vector (Cross) Product of Two Vectors
- Scalar Triple Product
- Vector Triple Product
- Reciprocal System of Vectors
- Tetrahedron

Sol. (d) Given that l is any straight line in plane or three dimensional space. This line can be given two directions by means of arrow heads. A line with one of these directions prescribed is called a directed line.

Now, observe that on restricting the line l to the line segment AB , a magnitude is prescribed on the line l with one of the two directions. So, we obtain a directed line segment. Thus, a directed line segment has magnitude as well as direction. So, all the given options are correct.

Types of Vectors

(i) Zero or Null Vector

A vector is zero magnitude i.e. which has the same initial and terminal points, is called a zero or null vector and it is denoted by $\mathbf{0}$.

(ii) Unit Vector

A vector is unit magnitude in the direction of a vector $\mathbf{a}$, is called a unit vector along $\mathbf{a}$ and it is denoted by $\hat{\mathbf{a}}$.

$$\text{Symbolically, } \hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|}$$

(iii) Equal Vectors

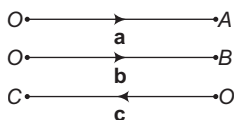
If magnitude and direction of two vectors are same, then those are called equal vectors.

(iv) Parallel Vectors

Two vectors $\mathbf{a}$ and $\mathbf{b}$ are said to be parallel, if either both have same line of action or support or there exists a scalar λ such that if $\lambda > 0$, then $\mathbf{a}$ and $\mathbf{b}$ have same direction or sense. If $\lambda < 0$, then $\mathbf{a}$ and $\mathbf{b}$ have opposite direction or sense.

(v) Like and Unlike Vectors

Two parallel vectors are said to be like when they have same sense of direction i.e. angle between them is zero, otherwise vectors are said to be unlike vectors and angle between them is π .



Vectors $\mathbf{a}$, $\mathbf{b}$ are like vectors and vectors $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{b}$, $\mathbf{a}$ are unlike vectors.

(vi) Negative Vector

A vector having the same magnitude as that of a given vector and direction opposite to that of given vector is called negative vector.

(vii) Coinitial and Coterminal Vectors

The vectors which have the same initial point, are coinitial vectors. Similarly, the vectors which have the same terminal point, are called coterminal vectors.

(viii) Free Vector

The vector whose initial point or tail is not fixed, is called free or non-localised vector.

(ix) Collinear Vectors

Two or more vectors are known as collinear vectors, if they are parallel to a given straight line. The magnitude of collinear vectors can be different.

(x) Coplanar Vectors

Vectors are said to be coplanar, if they occur in same or common plane.

(xi) Reciprocal of a Vector

A vector having the same direction as that of a given vector $\mathbf{a}$ but magnitude equal to the reciprocal of the given vector, is known as the reciprocal of $\mathbf{a}$ and it is denoted by $\mathbf{a}^{-1}$.

(xii) Localised Vector

A vector which is drawn parallel to a given vector through a specified point in space, is called a localised vector.

Force acting on a rigid body is a localised vector as its effect depends on the line of action of the force.

(xiii) Position Vector

Let O be a fixed origin, then the position vector of a point P is the vector $\mathbf{OP}$. e.g. If $\mathbf{a}$ and $\mathbf{b}$ are position vectors of two points A and B , then

$$\mathbf{AB} = \mathbf{b} - \mathbf{a} = \text{Position vector of } \mathbf{b} - \text{Position vector of } \mathbf{a}$$

➤ **Example 3.** A vector whose magnitude is the same as that of a given vector but direction is opposite to that of it, is called

- (a) negative of the given vector
- (b) equal vector
- (c) null vector
- (d) collinear vector

Sol. (a) A vector whose magnitude is the same as that of a given vector (say, $\mathbf{AB}$) but direction is opposite to that of it, is called negative of the given vector.

e.g. Vector $\mathbf{BA}$ is negative of the vector $\mathbf{AB}$ and written as $\mathbf{BA} = -\mathbf{AB}$.

➤ **Example 4.** Consider the statements given below and choose T for true statement and F for false statement.

- I. $\mathbf{a}$ and $-\mathbf{a}$ are collinear.
- II. Two collinear vectors are always equal in magnitude.
- III. Two vectors having same magnitude are collinear.
- IV. Two collinear vectors having the same magnitude are equal.

In their respective orders as given above, choose correct combination.

- (a) T, T, T, F (b) T, F, T, F
(c) T, F, F, F (d) T, F, F, T

Sol. (c) (i) **True** Vectors $\mathbf{a}$ and $-\mathbf{a}$ are parallel to the same line and hence collinear.

(ii) **False** Collinear vectors are those vectors which are parallel to the same line but they need not to be equal in magnitude.

(iii) **False** It is not necessary for two vectors having the same magnitude to be parallel to the same line.

(iv) **False** Two collinear vectors having the same magnitude may be in opposite direction. Thus, they may not be equal.

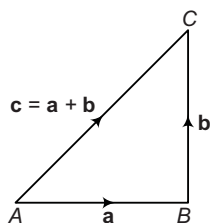
Addition of Vectors

The addition of two vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by $\mathbf{a} + \mathbf{b}$ and it is known as resultant of $\mathbf{a}$ and $\mathbf{b}$.

There are some methods of addition of vectors :

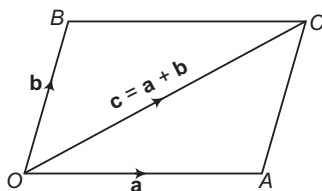
(i) Triangle Law of Vector Addition

If in a $\triangle ABC$, $\overrightarrow{AB} = \mathbf{a}$, $\overrightarrow{BC} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$, then $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$.



(ii) Parallelogram Law of Vector Addition

If in a parallelogram $OACB$, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$.



Then, $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OC}$ i.e. $\mathbf{a} + \mathbf{b} = \mathbf{c}$, where OC is a diagonal of the parallelogram $OACB$.

(iii) Vector Addition in Component Form

If the vectors are defined in terms of $\hat{i}$, $\hat{j}$ and $\hat{k}$,

i.e. $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$,

then their sum is defined as

$$\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$$

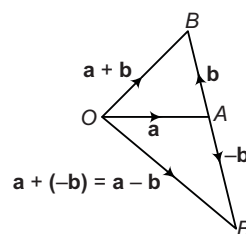
Properties of Vector Addition

Vector addition has the following properties:

- i. **Binary operation** The sum of two vectors is always a vector.
- ii. **Commutativity** For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.
- iii. **Associativity** For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.
- iv. **Identity** Zero vector is the identity for addition. For any vector $\mathbf{a}$, $\mathbf{0} + \mathbf{a} = \mathbf{a} = \mathbf{a} + \mathbf{0}$.
- v. **Additive inverse** For every vector $\mathbf{a}$, its negative vector $-\mathbf{a}$ exist such that, $\mathbf{a} + (-\mathbf{a}) = (-\mathbf{a}) + \mathbf{a} = \mathbf{0}$ i.e. $(-\mathbf{a})$ is the additive inverse of the vector $\mathbf{a}$.

Subtraction of Vectors

If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then their subtraction $\mathbf{a} - \mathbf{b}$ is defined as $\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b})$, where $(-\mathbf{b})$ is the negative of $\mathbf{b}$ having magnitude equal to that of $\mathbf{b}$ and direction opposite to $\mathbf{b}$.



If $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

Properties of Vector Subtraction

- i. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, $\mathbf{a} - \mathbf{b} \neq \mathbf{b} - \mathbf{a}$.
- ii. For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $(\mathbf{a} - \mathbf{b}) - \mathbf{c} \neq \mathbf{a} - (\mathbf{b} - \mathbf{c})$
- iii. Since, anyone side of a triangle is less than the sum and greater than the difference of the other two sides, so for any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have

(a) $ \mathbf{a} + \mathbf{b} \leq \mathbf{a} + \mathbf{b} $	(b) $ \mathbf{a} + \mathbf{b} \geq \mathbf{a} - \mathbf{b} $
(c) $ \mathbf{a} - \mathbf{b} \leq \mathbf{a} + \mathbf{b} $	(c) $ \mathbf{a} - \mathbf{b} \geq \mathbf{a} - \mathbf{b} $

Multiplication of a Vector by a Scalar

If $\mathbf{a}$ is a vector and m is a scalar i.e. a real number, then $m\mathbf{a}$ is a vector whose magnitude is m times that of $\mathbf{a}$ and whose direction is the same as that of $\mathbf{a}$, if m is positive and opposite to that of $\mathbf{a}$, if m is negative.

Properties of the Multiplication of Vectors by Scalars

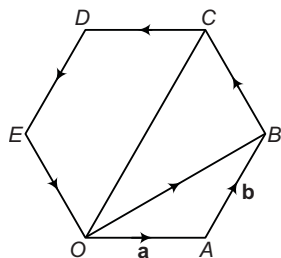
There are following properties of multiplication of vectors by scalars, for vectors $\mathbf{a}$, $\mathbf{b}$ and scalars m , n .

- i. $m(-\mathbf{a}) = (-m)\mathbf{a} = -(m\mathbf{a})$
- ii. $(-m)(-\mathbf{a}) = m\mathbf{a}$
- iii. $m(n\mathbf{a}) = (mn)\mathbf{a} = n(m\mathbf{a})$
- iv. $(m+n)\mathbf{a} = m\mathbf{a} + n\mathbf{a}$
- v. $m(\mathbf{a} + \mathbf{b}) = m\mathbf{a} + m\mathbf{b}$

➤ **Example 5.** If $\mathbf{a}$ and $\mathbf{b}$ are the vectors determined by two adjacent sides of a regular hexagon, then vector $\mathbf{EO}$ is

- (a) $(\mathbf{a} + \mathbf{b})$ (b) $(\mathbf{a} - \mathbf{b})$ (c) $2\mathbf{a}$ (d) $2\mathbf{b}$

Sol. (b) Here, $OABCDE$ is a regular hexagon.



Let $OA = \mathbf{a}$ and $AB = \mathbf{b}$. Join OB and OC , then

$$OB = OA + AB = \mathbf{a} + \mathbf{b}$$

Since, OC is parallel to AB and double of AB .

$$\therefore OC = 2AB = 2\mathbf{b}$$

$$\text{Now, } BC = OC - OB = 2\mathbf{b} - (\mathbf{a} + \mathbf{b}) = \mathbf{b} - \mathbf{a}$$

$$CD = -OA = -\mathbf{a}$$

$$\text{and } DE = -AB = -\mathbf{b}$$

$$\text{Also, } EO = -BC = -(\mathbf{b} - \mathbf{a}) = \mathbf{a} - \mathbf{b}$$

➤ **Example 6.** If $ABCD$ is a rhombus whose diagonals cut at the origin O , then $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}$ equals

- (a) $\mathbf{AB} + \mathbf{AC}$ (b) 0
(c) $2(\mathbf{AB} + \mathbf{BC})$ (d) $\mathbf{AC} + \mathbf{BD}$

Sol. (b) Since, the diagonals of a rhombus bisect each other.

$$\therefore OA = -OC \text{ and } OB = -OD$$

$$\Rightarrow OA + OB + OC + OD = 0$$

➤ **Example 7.** If $ABCDE$ is a pentagon, then the resultant of forces $\mathbf{AB}$, $\mathbf{AE}$, $\mathbf{BC}$, $\mathbf{DC}$, $\mathbf{ED}$ and $\mathbf{AC}$ is

- (a) $2\mathbf{AC}$ (b) $3\mathbf{AC}$ (c) $2\mathbf{AB}$ (d) $4\mathbf{AC}$

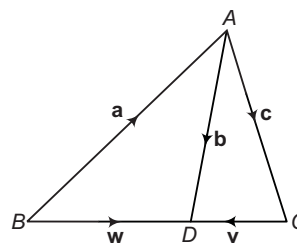
Sol. (b) Resultant of forces

$$= \mathbf{AB} + \mathbf{AE} + \mathbf{BC} + \mathbf{DC} + \mathbf{ED} + \mathbf{AC}$$

$$= (\mathbf{AB} + \mathbf{BC}) + (\mathbf{AE} + \mathbf{ED} + \mathbf{DC}) + \mathbf{AC}$$

$$= \mathbf{AC} + \mathbf{AC} + \mathbf{AC} = 3\mathbf{AC}$$

➤ **Example 8.** In the given figure, a vector $\mathbf{x}$ satisfies the equation $\mathbf{x} - \mathbf{w} = \mathbf{v}$. Then, $\mathbf{x}$ is equal to



- (a) $2\mathbf{a} + \mathbf{b} + \mathbf{c}$
(b) $\mathbf{a} + 2\mathbf{b} + \mathbf{c}$
(c) $\mathbf{a} + \mathbf{b} + 2\mathbf{c}$
(d) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

Sol. (b) We have, $\mathbf{v} = \mathbf{b} + \mathbf{c}$... (i)

$$\text{and } \mathbf{w} = \mathbf{b} + \mathbf{a} \quad \dots (ii)$$

$$\therefore \mathbf{x} = \mathbf{v} + \mathbf{w} = \mathbf{a} + 2\mathbf{b} + \mathbf{c}$$

Linear Combination of Vectors

A vector $\mathbf{r}$ is said to be a linear combination of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ..., etc., if there exist scalars x , y , z , ..., etc., such that

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + \dots$$

Note that a linear combination of vectors involves two linear compositions of the addition of vectors and the multiplication of vectors with scalars.

e.g. Vectors $\mathbf{r}_1 = 2\mathbf{a} + \mathbf{b} + 3\mathbf{c}$ and $\mathbf{r}_2 = \mathbf{a} + 3\mathbf{b} + \sqrt{2}\mathbf{c}$ are linear combinations of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Collinear and Non-collinear Vectors

Let $\mathbf{a}$, $\mathbf{b}$ be two collinear vectors and $\hat{\mathbf{x}}$ be the unit vector in the direction of $\mathbf{a}$. Then, the unit vector in the direction of $\mathbf{b}$ is $\hat{\mathbf{x}}$ or $-\hat{\mathbf{x}}$ according as $\mathbf{a}$ and $\mathbf{b}$ are like or unlike parallel vectors. Now, $\mathbf{a} = |\mathbf{a}| \hat{\mathbf{x}}$ and $\mathbf{b} = \pm |\mathbf{b}| \hat{\mathbf{x}}$.

$$\text{Then, } \mathbf{a} = \left(\frac{|\mathbf{a}|}{|\mathbf{b}|} \right) |\mathbf{b}| \hat{\mathbf{x}} \Rightarrow \mathbf{a} = \left(\pm \frac{|\mathbf{a}|}{|\mathbf{b}|} \right) \mathbf{b}$$

$$\Rightarrow \mathbf{a} = \lambda \mathbf{b}, \text{ where } \lambda = \pm \frac{|\mathbf{a}|}{|\mathbf{b}|}$$

Thus, if $\mathbf{a}$ and $\mathbf{b}$ are collinear vectors, then $\mathbf{a} = \lambda \mathbf{b}$ or $\mathbf{b} = \lambda \mathbf{a}$ for some scalar λ .

Relation between Two Parallel Vectors

- (i) If $\mathbf{a}$ and $\mathbf{b}$ are two parallel vectors, then there exists a scalar k such that $\mathbf{a} = k\mathbf{b}$ i.e. there exist two non-zero scalar quantities x and y , so that $x\mathbf{a} + y\mathbf{b} = \mathbf{0}$.

If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero, non-parallel vectors, then $x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow x = 0$ and $y = 0$.

$$\text{Obviously, } x\mathbf{a} + y\mathbf{b} = \mathbf{0} \Rightarrow \begin{cases} \mathbf{a} = \mathbf{0}, \mathbf{b} = \mathbf{0} \\ \text{or} \\ x = 0, y = 0 \\ \text{or} \\ \mathbf{a} \parallel \mathbf{b} \end{cases}$$

- (ii) If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then from the property of parallel vectors, we have

$$\mathbf{a} \parallel \mathbf{b} \Rightarrow \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

Test for Collinearity of Three Points

Three points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are collinear iff there exist scalars x , y and z not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} = \mathbf{0}$, where $x + y + z = 0$. If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}}$, then the points with position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ will be collinear iff

$$\begin{vmatrix} a_1 & a_2 & 1 \\ b_1 & b_2 & 1 \\ c_1 & c_2 & 1 \end{vmatrix} = 0$$

Test for Coplanarity of Three Vectors

Let $\mathbf{a}$ and $\mathbf{b}$ be two given non-zero and non-collinear vectors. Then, any vectors $\mathbf{r}$ coplanar with $\mathbf{a}$ and $\mathbf{b}$ can be uniquely expressed as $\mathbf{r} = x\mathbf{a} + y\mathbf{b}$ for some scalars x and y .

Test for Coplanarity of Four Points

Four points with position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are coplanar iff there exist scalars x , y , z and u not all zero such that $x\mathbf{a} + y\mathbf{b} + z\mathbf{c} + u\mathbf{d} = \mathbf{0}$, where $x + y + z + u = 0$.

Four points with position vectors

$\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ and $\mathbf{d} = d_1\hat{\mathbf{i}} + d_2\hat{\mathbf{j}} + d_3\hat{\mathbf{k}}$ will be

$$\text{coplanar iff } \begin{vmatrix} a_1 & a_2 & a_3 & 1 \\ b_1 & b_2 & b_3 & 1 \\ c_1 & c_2 & c_3 & 1 \\ d_1 & d_2 & d_3 & 1 \end{vmatrix} = 0.$$

Linearly Independent and Dependent Vectors

A set of non-zero vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ is said to be linearly independent, if $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$

$$\text{i.e. } x_1 = x_2 = \dots = x_n = 0.$$

A set of vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ is said to be linearly dependent, if there exist scalars $x_1, x_2, \dots, x_n$ not all zero such that $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \dots + x_n\mathbf{a}_n = \mathbf{0}$.

Three vectors $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$, $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ and $\mathbf{c} = c_1\hat{\mathbf{i}} + c_2\hat{\mathbf{j}} + c_3\hat{\mathbf{k}}$ will be

$$\text{linearly dependent vectors iff } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 0.$$

Properties of Linearly Independent and Dependent Vectors

- i. Two non-zero, non-collinear vectors are linearly independent.
- ii. Any two collinear vectors are linearly dependent.
- iii. Any three non-coplanar vectors are linearly independent.
- iv. Any three coplanar vectors are linearly dependent.
- v. Any four vectors in three dimensional space are linearly dependent.

Example 9. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors, no two of which are collinear. If the vector $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$ is equal to

- (a) $\lambda\mathbf{a}$ (b) $\lambda\mathbf{b}$
(c) $\lambda\mathbf{c}$ (d) $\mathbf{0}$

Sol. (d) Since, $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$.

$$\therefore \mathbf{a} + 2\mathbf{b} = x\mathbf{c}, x \in \mathbb{R}$$

and $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$.

$$\therefore \mathbf{b} + 3\mathbf{c} = y\mathbf{a}, y \in \mathbb{R}$$

$$\text{Now, } \mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (x + 6)\mathbf{c}$$

$$\text{Also, } \mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = (1 + 2y)\mathbf{a}$$

$$\Rightarrow (x + 6)\mathbf{c} = (1 + 2y)\mathbf{a}$$

$$\Rightarrow x + 6 = 0 \text{ and } 1 + 2y = 0$$

$$\Rightarrow x = -6 \text{ and } y = -\frac{1}{2}$$

$$\therefore \mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = \mathbf{0}$$

Example 10. If P, Q, R are three points with respective position vectors $\hat{i} + \hat{j}$, $\hat{i} - \hat{j}$ and $a\hat{i} + b\hat{j} + c\hat{k}$. The points P, Q, R are collinear, if

- (a) $a = b = c = 1$
 (b) $a = b = c = 0$
 (c) $a = 1, b, c \in R$
 (d) $a = 1, c = 0, b \in R$

Sol. (d) Since, P, Q and R are collinear.

$$\begin{aligned} \therefore PQ &= mQR \\ \Rightarrow -2\hat{j} &= m[(a-1)\hat{i} + (b+1)\hat{j} + c\hat{k}] \text{ for some non-zero scalar } m. \\ \Rightarrow (a-1)m &= 0, (b+1)m = -2, cm = 0 \\ \Rightarrow a &= 1, c = 0, b \in R \end{aligned}$$

Example 11. The vectors $2\hat{i} + 3\hat{j}$, $5\hat{i} + 6\hat{j}$ and $8\hat{i} + \lambda\hat{j}$ have their initial points at $(1, 1)$. The value of λ , so that the vectors terminate on one straight line, is

- (a) 0
 (b) 3
 (c) 6
 (d) 9

Sol. (d) Since, the vectors $2\hat{i} + 3\hat{j}$ and $5\hat{i} + 6\hat{j}$ have $(1, 1)$ as initial point, therefore their terminal points are $(3, 4)$ and $(6, 9)$, respectively. The equation of the line joining these two points is $x - y + 1 = 0$. The terminal point of $8\hat{i} + \lambda\hat{j}$ is $(9, \lambda + 1)$. Since, the vectors terminate on the same straight line, therefore $[9, (\lambda + 1)]$ lies on $x - y + 1 = 0$.

$$\begin{aligned} \Rightarrow 9 - (\lambda + 1) + 1 &= 0 \\ \Rightarrow \lambda &= 9 \end{aligned}$$

Example 12. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three non-coplanar vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$ and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$, then $\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}$ is equal to

- (a) 0
 (b) $\alpha\mathbf{a}$
 (c) $\beta\mathbf{b}$
 (d) $(\alpha + \beta)\mathbf{c}$

Sol. (a) We have, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \alpha\mathbf{d}$ and $\mathbf{b} + \mathbf{c} + \mathbf{d} = \beta\mathbf{a}$

$$\begin{aligned} \Rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} &= (\alpha + 1)\mathbf{d} \\ \text{and } \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} &= (\beta + 1)\mathbf{a} \\ \Rightarrow (\alpha + 1)\mathbf{d} &= (\beta + 1)\mathbf{a} \end{aligned}$$

$$\text{If } \alpha \neq -1, \text{ then } (\alpha + 1)\mathbf{d} = (\beta + 1)\mathbf{a} \Rightarrow \mathbf{d} = \left(\frac{\beta + 1}{\alpha + 1}\right)\mathbf{a}$$

$$\begin{aligned} \therefore \mathbf{a} + \mathbf{b} + \mathbf{c} &= \alpha\mathbf{d} \\ \therefore \mathbf{a} + \mathbf{b} + \mathbf{c} &= \alpha \left(\frac{\beta + 1}{\alpha + 1}\right)\mathbf{a} \end{aligned}$$

$$\Rightarrow \left(1 - \frac{\alpha(\beta + 1)}{\alpha + 1}\right)\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

So, $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar, which is a contradiction to the given condition.

$$\begin{aligned} \therefore \alpha &= -1 \\ \Rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} &= 0 \end{aligned}$$

Component of a Vector

The process of splitting a vector is called resolution of a vector. In simpler language, it would mean, determining the effect of a vector in a particular direction. The parts of the vector obtained after splitting the vectors are known as the components of the vector.

Components of a Vector in Two Dimensional Space

If a point P in a plane has coordinates (x, y) , then

- (i) $\mathbf{OP} = x\hat{i} + y\hat{j}$
 (ii) $|\mathbf{OP}| = \sqrt{x^2 + y^2}$
 (iii) The component of $\mathbf{OP}$ along X -axis is a vector $x\hat{i}$, whose magnitude is $|x|$ and direction is along OX or OX' according as x is positive or negative.
 (iv) The component of $\mathbf{OP}$ along Y -axis is a vector $y\hat{j}$, whose magnitude is $|y|$ and direction is along OY or OY' according as y is positive or negative.

Let $A(x_1, y_1)$ and $B(x_2, y_2)$ be any two points in XOY -plane. Let $\hat{i}$ and $\hat{j}$ be unit vectors along OX and OY , respectively.

$$\text{Then, } \mathbf{AB} = (x_2\hat{i} + y_2\hat{j}) - (x_1\hat{i} + y_1\hat{j})$$

$$\Rightarrow \mathbf{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j}$$

$\therefore$ Component of $\mathbf{AB}$ along X -axis $= (x_2 - x_1)\hat{i}$ and component of $\mathbf{AB}$ along Y -axis $= (y_2 - y_1)\hat{j}$

For any two vectors $\mathbf{a} = a_1\hat{i} + a_2\hat{j}$ and

$\mathbf{b} = b_1\hat{i} + b_2\hat{j}$, we define

- (a) $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j}$
 (b) $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j}$
 (c) $m\mathbf{a} = (ma_1)\hat{i} + (ma_2)\hat{j}$, where m is a scalar.
 (d) $\mathbf{a} = \mathbf{b} \Leftrightarrow a_1 = b_1 \text{ and } a_2 = b_2$

Components of a Vector in Three Dimensional Space

If a point P in space has coordinates (x, y, z) , then its position vector $\mathbf{r}$ is $x\hat{i} + y\hat{j} + z\hat{k}$ and $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$. The vectors $x\hat{i}$, $y\hat{j}$ and $z\hat{k}$ are known as the component vectors of $\mathbf{r}$ along x , y and z axes respectively. For any two vectors, $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\mathbf{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, we define

- (i) $\mathbf{a} + \mathbf{b} = (a_1 + b_1)\hat{i} + (a_2 + b_2)\hat{j} + (a_3 + b_3)\hat{k}$
 (ii) $\mathbf{a} - \mathbf{b} = (a_1 - b_1)\hat{i} + (a_2 - b_2)\hat{j} + (a_3 - b_3)\hat{k}$

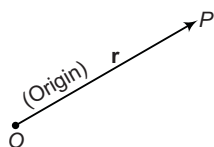
$$(iii) \quad m\mathbf{a} = (ma_1)\hat{\mathbf{i}} + (ma_2)\hat{\mathbf{j}} + (ma_3)\hat{\mathbf{k}},$$

where m is a scalar.

$$(iv) \quad \mathbf{a} = \mathbf{b} \Leftrightarrow a_1 = b_1, a_2 = b_2 \text{ and } a_3 = b_3$$

Position Vector

If a point O is fixed as the origin in space (or plane) and P is any point, then $\mathbf{OP}$ is called the position vector of P with respect to O . If we say that P is the point $\mathbf{r}$, then we mean that the position vector of P is $\mathbf{r}$ with respect to some origin O .



Let $\mathbf{a}$ and $\mathbf{b}$ be position vectors of points A and B respectively. Then,

$\mathbf{AB}$ = Position vector of B – Position vector of A

Section Formulae

Theorem 1 (Internal division) Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively and C be a point dividing AB internally in the ratio $m:n$. Then, the position vector of C is given by

$$\mathbf{OC} = \frac{m\mathbf{b} + n\mathbf{a}}{m + n}$$

Remark If C is the mid-point of AB , then it divides AB in the ratio 1:1.

$$\therefore \text{Position vector of } C = \frac{1 \cdot \mathbf{a} + 1 \cdot \mathbf{b}}{1 + 1} = \frac{\mathbf{a} + \mathbf{b}}{2}$$

Thus, the position vector of the mid-point of AB is $\frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Theorem 2 (External division) Let A and B be two points with position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively and C be a point dividing AB externally in the ratio $m:n$. Then, the position vector of C is given by

$$\mathbf{OC} = \frac{m\mathbf{b} - n\mathbf{a}}{m - n}$$

☞ **Example 13.** A vector $\mathbf{a}$ has components $2p$ and 1 with respect to a rectangular cartesian system. This system is rotated through a certain angle about the origin in the counter clockwise sense. With respect to new system, if $\mathbf{a}$ has components $p+1$ and 1 , then

$$(a) \quad p = 0 \quad (b) \quad p = 1 \text{ or } p = -\frac{1}{3}$$

$$(c) \quad p = -1 \text{ or } p = \frac{1}{3} \quad (d) \quad p = 1 \text{ or } p = -1$$

Sol. (b) We have, $\mathbf{a} = 2p\hat{\mathbf{i}} + \hat{\mathbf{j}}$

Let $\mathbf{b}$ be the vector obtained from $\mathbf{a}$ by rotating the axes. Then, the components of $\mathbf{b}$ are $p+1$ and 1 , therefore $\mathbf{b} = (p+1)\hat{\mathbf{i}} + \hat{\mathbf{j}}$.

$$\begin{aligned} \text{But} \quad & |\mathbf{b}| = |\mathbf{a}| \\ \Rightarrow & (2p)^2 + 1 = (p+1)^2 + 1 \\ \Rightarrow & 3p^2 - 2p - 1 = 0 \\ \therefore & p = 1, -\frac{1}{3} \end{aligned}$$

☞ **Example 14.** The position vectors of the points A , B and C are $(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$, $(3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$ and $(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ respectively. These points

- form an isosceles triangle
- form a right angled triangle
- are collinear
- form a scalene triangle

$$\begin{aligned} \text{Sol.} \quad (c) \quad \mathbf{AB} &= (3-2)\hat{\mathbf{i}} + (-2-1)\hat{\mathbf{j}} + (1-1)\hat{\mathbf{k}} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}} \\ \mathbf{BC} &= (1-3)\hat{\mathbf{i}} + (4+2)\hat{\mathbf{j}} + (-3-1)\hat{\mathbf{k}} = -2\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 4\hat{\mathbf{k}} \\ \mathbf{CA} &= (2-1)\hat{\mathbf{i}} + (1-4)\hat{\mathbf{j}} + (-1+3)\hat{\mathbf{k}} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}} \\ |\mathbf{AB}| &= \sqrt{1+9+4} = \sqrt{14} \\ |\mathbf{BC}| &= \sqrt{4+36+16} = \sqrt{56} = 2\sqrt{14} \\ |\mathbf{CA}| &= \sqrt{1+9+4} = \sqrt{14} \end{aligned}$$

So, $|\mathbf{AB}| + |\mathbf{AC}| = |\mathbf{BC}|$ and angle between $\mathbf{AB}$ and $\mathbf{BC}$ is 180° .

So, points A , B , C cannot form an isosceles triangle.

Hence, A , B , C are collinear.

☞ **Example 15.** If $\mathbf{a}$, $\mathbf{b}$ are the position vectors of A , B respectively and C is a point on AB produced such that $AC = 3AB$, then the position vector of C is

- $3\mathbf{a} - 2\mathbf{b}$
- $3\mathbf{b} - 2\mathbf{a}$
- $3\mathbf{b} + 2\mathbf{a}$
- $2\mathbf{a} - 3\mathbf{b}$

Sol. (b) Let the position vector for C be $\mathbf{c}$.

Clearly, B divides AC internally in the ratio 1:2.

$$\therefore \quad \mathbf{b} = \frac{2 \cdot \mathbf{a} + 1 \cdot \mathbf{c}}{2 + 1}$$

$$\Rightarrow \quad \mathbf{c} = 3\mathbf{b} - 2\mathbf{a}$$

☞ **Example 16.** If the sides of a parallelogram are $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then the unit vector parallel to one of the diagonals is

- $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
- $\frac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
- $\frac{1}{7}(-3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
- $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Sol. (a) Let $\mathbf{a} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Then, the diagonals of the parallelogram are

$$\begin{aligned} \mathbf{p} &= \mathbf{a} + \mathbf{b} \text{ and } \mathbf{q} = \mathbf{b} - \mathbf{a} \\ \Rightarrow \quad \mathbf{p} &= 3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}} \text{ and } \mathbf{q} = -\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}} \end{aligned}$$

So, unit vectors along the diagonals are

$$\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \text{ and } \frac{1}{\sqrt{69}}(-\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 8\hat{\mathbf{k}})$$

Example 17. The position vectors of the vertices A , B and C of ΔABC are $\hat{i} - \hat{j} - 3\hat{k}$, $2\hat{i} + \hat{j} - 2\hat{k}$ and $-5\hat{i} + 2\hat{j} - 6\hat{k}$, respectively. The length of the bisector AD of $\angle BAC$, where D is on the line segment BC , is

- (a) $\frac{15}{2}$ (b) $\frac{11}{2}$
(c) $\frac{1}{4}$ (d) None of these

Sol. (d) We have,

$$AB = \hat{i} + 2\hat{j} + \hat{k}, AC = -6\hat{i} + 3\hat{j} - 3\hat{k}$$

$$\Rightarrow |AB| = \sqrt{6} \text{ and } |AC| = 3\sqrt{6}$$

Clearly, point D divides BC in the ratio $AB : AC$ i.e. $1 : 3$.

$$\begin{aligned} \therefore \text{Position vector of } D &= \frac{(-5\hat{i} + 2\hat{j} - 6\hat{k}) + 3(2\hat{i} + \hat{j} - 2\hat{k})}{1 + 3} \\ &= \frac{1}{4}(\hat{i} + 5\hat{j} - 12\hat{k}) \end{aligned}$$

$$\therefore AD = \frac{1}{4}(\hat{i} + 5\hat{j} - 12\hat{k}) - (\hat{i} - \hat{j} - 3\hat{k}) = \frac{3}{4}(-\hat{i} + 3\hat{j})$$

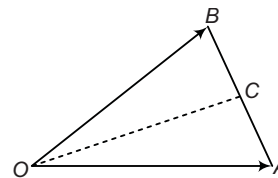
$$\Rightarrow AD = |AD| = \frac{3}{4}\sqrt{10}$$

Example 18. Let $OA = \hat{i} + 3\hat{j} - 2\hat{k}$ and $OB = 3\hat{i} + \hat{j} - 2\hat{k}$. The vector OC bisecting $\angle AOB$ and C being a point on the line AB , is

- (a) $4(\hat{i} + \hat{j} - \hat{k})$
(b) $2(\hat{i} + \hat{j} - \hat{k})$
(c) $\hat{i} + \hat{j} - \hat{k}$
(d) None of the above

Sol. (b) Taking O as the origin, the position vectors of A and B are $a = \hat{i} + 3\hat{j} - 2\hat{k}$ and $b = 3\hat{i} + \hat{j} - 2\hat{k}$, respectively.

$$\text{We have, } |a| = |b| = \sqrt{14}$$



So, the bisector OC of $\angle AOB$ meets AB at its mid-point C .

$$\therefore OC = \frac{1}{2}(OA + OB) = 2(\hat{i} + \hat{j} - \hat{k})$$

Aliter

The vector equation of the bisector of $\angle AOB$ is given by

$$\begin{aligned} r &= \lambda(\hat{a} + \hat{b}) \\ \Rightarrow r &= \frac{4\lambda}{\sqrt{14}}(\hat{i} + \hat{j} + \hat{k}) \quad \dots (i) \end{aligned}$$

The vector equation of line AB is

$$r = \hat{i} + 3\hat{j} - 2\hat{k} + \mu(2\hat{i} - 2\hat{j})$$

Since, point C lies on AB . Therefore,

$$\begin{aligned} \frac{4\lambda}{\sqrt{14}}(\hat{i} + \hat{j} - \hat{k}) &= (\hat{i} + 3\hat{j} - 2\hat{k}) + \mu(2\hat{i} - 2\hat{j}) \\ \Rightarrow \frac{-4\lambda}{\sqrt{14}} &= -2 \quad [\text{on equating the coefficients of } \hat{k}] \end{aligned}$$

$$\Rightarrow \lambda = \frac{\sqrt{14}}{2}$$

On putting $\lambda = \frac{\sqrt{14}}{2}$ in Eq. (i), we get

$$\begin{aligned} r &= 2(\hat{i} + \hat{j} - \hat{k}) \\ \Rightarrow OC &= 2(\hat{i} + \hat{j} - \hat{k}) \end{aligned}$$

Work Book Exercise 13.1

1 If m_1 , m_2 , m_3 and m_4 are respectively the magnitudes of the vectors $a_1 = 2\hat{i} - \hat{j} + \hat{k}$, $a_2 = 3\hat{i} - 4\hat{j} - 4\hat{k}$, $a_3 = \hat{i} + \hat{j} - \hat{k}$ and $a_4 = -\hat{i} + 3\hat{j} + \hat{k}$, then the correct order of m_1 , m_2 , m_3 and m_4 is

- a $m_3 < m_1 < m_4 < m_2$
b $m_3 < m_1 < m_2 < m_4$
c $m_3 < m_4 < m_1 < m_2$
d $m_3 < m_4 < m_2 < m_1$

2 A , B , C are three non-zero vectors, no two of them are parallel. If $A + B$ is collinear to C and $B + C$ is collinear to A , then $A + B + C$ is equal to

- a A b B
c C d 0

3 If the points with position vectors $60\hat{i} + 3\hat{j}$, $40\hat{i} - 8\hat{j}$ and $a\hat{i} - 52\hat{j}$ are collinear, then a is equal to

- a -40 b -20
c 20 d 40

4 If A , B and C are the vertices of a triangle whose position vectors are a , b and c respectively and G is the centroid of ΔABC , then $GA + GB + GC$ is equal to

- a 0 b $a + b + c$
c $\frac{a + b + c}{3}$ d $\frac{a - b - c}{3}$

5 The position vectors of P and Q are respectively a and b . If R is a point on PQ such that $PR = 5PQ$, then the position vector of R is

- a $5b - 4a$
b $5b + 4a$
c $4b - 5a$
d $4b + 5a$

6 If P is any point within a ΔABC , then $PA + PB + PC$ is equal to

- a $AC + CB$
b $BC + BA$
c $CB + AB$
d $CB + BA$

- 7 If the position vector of A with respect to O is $3\hat{i} - 2\hat{j} + 4\hat{k}$ and $AB = 3\hat{i} - \hat{j} + \hat{k}$, then the position vector of B with respect to O is
- a $-\hat{j} + 3\hat{k}$ b $6\hat{i} - 3\hat{j} + 5\hat{k}$
 c $\hat{j} - 3\hat{k}$ d $\hat{i} - 3\hat{j} + 5\hat{k}$

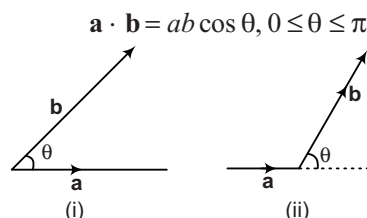
- 8 Let a and b be non-zero and non-collinear vectors, there exist scalars α, β such that $\alpha a + \beta b = 0$, then
- a $\alpha = \beta \neq 0$ b $\alpha + \beta = 0$
 c $\alpha = \beta = 0$ d $\alpha \neq \beta$

- 9 If $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices A, B and C respectively of $\triangle ABC$. The position vector of the point, where the bisector of $\angle A$ meets BC , is

- a $\frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$
 b $\frac{2}{3}(6\hat{i} + 12\hat{j} - 8\hat{k})$
 c $\frac{1}{3}(-6\hat{i} - 8\hat{j} - 9\hat{k})$
 d $\frac{2}{3}(-6\hat{i} - 12\hat{j} + 8\hat{k})$

Scalar (Dot) Product of Two Vectors

Let a and b be two non-zero vectors inclined at an angle θ . Then, the scalar product of a and b is denoted by $a \cdot b$ and defined as

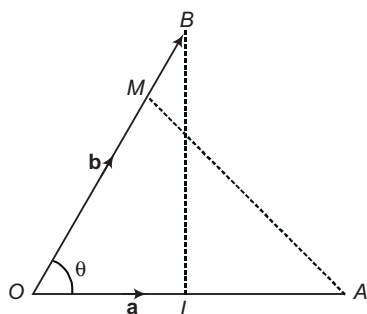


Since, scalar product of two vectors is a scalar quantity, so it is called scalar product.

✎ If a or b or both are equal to the null vector, then θ is not defined as 0 has no direction and in such a case, their dot product is defined as the scalar zero.

Geometrical Interpretation of Scalar Product

Let a and b be two vectors represented by OA and OB respectively. Let θ be the angle between OA and OB . Draw $BL \perp OA$ and $AM \perp OB$.



From $\triangle OBL$ and $\triangle OAM$, we have

$$OL = OB \cos \theta \text{ and } OM = OA \cos \theta$$

Here, OL and OM are known as projections of b on a and a on b , respectively.

Now, $a \cdot b = |a| |b| \cos \theta$

$$= |a| (OB \cos \theta)$$

$$= |a| (OL)$$

$$= (\text{Magnitude of } b) \cdot (\text{Projection of } a \text{ on } b) \quad \dots(i)$$

Again,

$$b \cdot a = |b| |a| \cos \theta$$

$$= |b| \{ |a| \cos \theta \}$$

$$= |b| (OA \cos \theta)$$

$$= |b| (OM)$$

$$= (\text{Magnitude of } b) \cdot (\text{Projection of } a \text{ on } b) \quad \dots(ii)$$

Thus, geometrically interpreted, the scalar product of two vectors is the product of modulus of either vector and the projection of the other in its direction.

From Eqs. (i) and (ii), we have

$$\text{Projection of } b \text{ on } a = \frac{b \cdot a}{|a|} = b \cdot \frac{a}{|a|} = b \cdot \hat{a}$$

$$\text{and projection of } a \text{ on } b = \frac{a \cdot b}{|b|} = a \cdot \frac{b}{|b|} = a \cdot \hat{b}$$

Thus, projection of a on b is the dot product of a with the unit vector along $\hat{b}$ and the projection of b on a is the dot product of b with the unit vector along $\hat{a}$.

Properties of Scalar Product

- i. **Commutativity** The scalar product of two vectors is commutative.

$$\text{i.e.} \quad a \cdot b = b \cdot a$$

- ii. **Distributivity** The scalar product of vectors is distributive over vector addition, i.e.

$$(a) \quad a \cdot (b + c) = a \cdot b + a \cdot c \quad [\text{left distributivity}]$$

$$(b) \quad (b + c) \cdot a = b \cdot a + c \cdot a$$

[right distributivity]

- iii. The dot product of two non-zero vectors is zero iff the vectors are perpendicular.

$$\text{i.e.} \quad a \cdot b = 0 \Leftrightarrow a \perp b$$

$$\therefore \hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{i} = 0, \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{j} = 0$$

$$\text{and } \hat{i} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0$$

- iv. For any vector a , we have $a \cdot a = |a|^2$

$$\therefore \hat{i} \cdot \hat{i} = |\hat{i}|^2 = 1, \hat{j} \cdot \hat{j} = |\hat{j}|^2 = 1$$

$$\text{and } \hat{k} \cdot \hat{k} = |\hat{k}|^2 = 1$$

- v. If m is a scalar and $\mathbf{a}, \mathbf{b}$ are two vectors, then

$$m\mathbf{a} \cdot \mathbf{b} = m(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot m\mathbf{b}$$
- vi. If m, n are scalars and $\mathbf{a}, \mathbf{b}$ are two vectors, then

$$m\mathbf{a} \cdot n\mathbf{b} = mn(\mathbf{a} \cdot \mathbf{b}) = (mna) \cdot \mathbf{b} = \mathbf{a} \cdot (mn\mathbf{b})$$
- vii. For any two vectors $\mathbf{a}$ and $\mathbf{b}$, we have
 (a) $|\mathbf{a} + \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a} \cdot \mathbf{b})$
 (b) $|\mathbf{a} - \mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 - 2(\mathbf{a} \cdot \mathbf{b})$
 (c) $(\mathbf{a} + \mathbf{b})^2 \cdot (\mathbf{a} - \mathbf{b})^2 = |\mathbf{a}|^2 - |\mathbf{b}|^2$
 (d) $\mathbf{a} \cdot \mathbf{b} < 0$ iff $\mathbf{a}$ and $\mathbf{b}$ are inclined at an obtuse angle.
 (e) $\mathbf{a} \cdot \mathbf{b} > 0$ iff $\mathbf{a}$ and $\mathbf{b}$ are inclined at an acute angle.
 (f) $\mathbf{a} \cdot \mathbf{b} = 0$ iff $\mathbf{a}$ and $\mathbf{b}$ are at right angle.
- viii. If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$, then

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$
- ix. If $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$ inclined at an angle θ , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}}$$
- x. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three vectors, then

$$|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$$

$$= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$$
- xi. If $\hat{\mathbf{r}}$ is a unit vector making angles α, β and γ with OX, OY and OZ respectively, then

$$\cos \alpha = \hat{\mathbf{r}} \cdot \hat{\mathbf{i}}, \cos \beta = \hat{\mathbf{r}} \cdot \hat{\mathbf{j}}, \cos \gamma = \hat{\mathbf{r}} \cdot \hat{\mathbf{k}}$$

$$\hat{\mathbf{r}} = (\cos \alpha)\hat{\mathbf{i}} + (\cos \beta)\hat{\mathbf{j}} + (\cos \gamma)\hat{\mathbf{k}}$$
- xii. A vector $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ makes an acute or obtuse angle with OX, OY, OZ respectively according as x, y, z are respectively positive or negative.
- xiii. If a vector $\mathbf{r}$ makes angles α, β and γ with OX, OY and OZ respectively, then

$$\mathbf{r} = (|\mathbf{r}|\cos \alpha)\hat{\mathbf{i}} + (|\mathbf{r}|\cos \beta)\hat{\mathbf{j}} + (|\mathbf{r}|\cos \gamma)\hat{\mathbf{k}}$$
- xiv. If $\mathbf{r}$ is a non-zero vector coplanar to two given vectors $\mathbf{a}$ and $\mathbf{b}$, then $\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \mathbf{b} = 0$.
 $\Rightarrow \mathbf{a}$ and $\mathbf{b}$ are collinear.

xv. If $\mathbf{r}$ is a non-zero vector in space and $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are three vectors, then $\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{c} = 0$.
 $\Rightarrow \mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar.

xvi. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors in space, then any vector $\mathbf{r}$ in space can be written as

$$\mathbf{r} = (\mathbf{r} \cdot \hat{\mathbf{a}})\hat{\mathbf{a}} + (\mathbf{r} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} + (\mathbf{r} \cdot \hat{\mathbf{c}})\hat{\mathbf{c}}$$
 where, $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors along $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, respectively.

Also,
$$\mathbf{r} = \left\{ \frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{a}|^2} \right\} \mathbf{a} + \left\{ \frac{\mathbf{r} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\} \mathbf{b} + \left\{ \frac{\mathbf{r} \cdot \mathbf{c}}{|\mathbf{c}|^2} \right\} \mathbf{c}$$

➤ **Example 19.** The angle between the vectors $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ is

- (a) 0 (b) $\frac{\pi}{2}$
 (c) $\cos^{-1}\left(\frac{5}{7}\right)$ (d) $\cos^{-1}\left(\frac{1}{7}\right)$

Sol. (c) Let $\mathbf{a} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$

Magnitude of $\mathbf{a}$,

$$|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 3^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Magnitude of $\mathbf{b}$,

$$|\mathbf{b}| = \sqrt{3^2 + (-2)^2 + 1^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

Now, $\mathbf{a} \cdot \mathbf{b} = (\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

$$= 1 \cdot 3 + (-2) \cdot (-2) + 3 \cdot 1$$

$$= 3 + 4 + 3 = 10$$

[since, dot product of two vectors is equal to the sum of the products of their corresponding components]

Let θ be the required angle between $\mathbf{a}$ and $\mathbf{b}$, then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

$$= \frac{10}{\sqrt{14}\sqrt{14}} = \frac{10}{14} = \frac{5}{7}$$

$$\therefore \theta = \cos^{-1}\left(\frac{5}{7}\right)$$

➤ **Example 20.** If $\mathbf{a}$ is a non-zero vector of magnitude a and λ is a non-zero scalar, then $\lambda\hat{\mathbf{a}}$ is a unit vector, if

- (a) $\lambda = 1$
 (b) $\lambda = -1$
 (c) $a = |\lambda|$
 (d) $a = \frac{1}{|\lambda|}$

Sol. (d) Vector $\lambda\hat{\mathbf{a}}$ is a unit vector, if $|\lambda\hat{\mathbf{a}}| = 1$

Now, $|\lambda\hat{\mathbf{a}}| = 1 \Rightarrow |\lambda||\hat{\mathbf{a}}| = 1$

$\Rightarrow |\hat{\mathbf{a}}| = \frac{1}{|\lambda|}$ [$\because \lambda \neq 0$]

$\Rightarrow a = \frac{1}{|\lambda|}$ [given, $|\hat{\mathbf{a}}| = a$]

➤ **Example 21.** The value of x for which the angle between $\mathbf{a} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\mathbf{b} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse and the angle between $\mathbf{b}$ and the Z-axis is acute and less than $\frac{\pi}{6}$, is

- (a) $0 < x < \frac{1}{2}$ (b) $\frac{1}{2} < x < 15$
(c) $x > \frac{1}{2}$ or $x < 0$ (d) None of these

Sol. (d) Since, the angle between $\mathbf{a}$ and $\mathbf{b}$ is obtuse.

$$\begin{aligned} \therefore \quad & \mathbf{a} \cdot \mathbf{b} < 0 \\ \Rightarrow & 14x^2 - 8x + x < 0 \\ \Rightarrow & 7x(2x - 1) < 0 \\ \Rightarrow & 0 < x < \frac{1}{2} \quad \dots(i) \end{aligned}$$

and the angle between $\mathbf{b}$ and Z-axis is acute and less than $\frac{\pi}{6}$.

$$\text{Therefore, } \frac{\mathbf{b} \cdot \mathbf{k}}{|\mathbf{b}||\mathbf{k}|} > \cos \frac{\pi}{6} \quad \left[\begin{array}{l} \because \theta < \frac{\pi}{6} \Rightarrow \cos \theta > \cos \frac{\pi}{6} \\ \Rightarrow \frac{\mathbf{b} \cdot \mathbf{k}}{|\mathbf{b}||\mathbf{k}|} > \cos \frac{\pi}{6} \end{array} \right]$$

$$\begin{aligned} \Rightarrow \quad & \frac{x}{\sqrt{x^2 + 53}} > \frac{\sqrt{3}}{2} \Rightarrow 4x^2 > 3x^2 + 159 \\ \Rightarrow \quad & x^2 > 159 \\ \Rightarrow \quad & x > \sqrt{159} \text{ or } x < -\sqrt{159} \quad \dots(ii) \end{aligned}$$

Clearly, Eqs. (i) and (ii) cannot hold together.

➤ **Example 22.** The values of x for which the angle between the vectors $\mathbf{a} = x\hat{i} - 3\hat{j} - \hat{k}$ and $\mathbf{b} = 2x\hat{i} + x\hat{j} - \hat{k}$ is acute and the angle between the vector $\mathbf{b}$ and Y-axis is obtuse, are

- (a) 1, 2 (b) -2, -3
(c) all $x < 0$ (d) all $x > 0$

Sol. (c) Since, the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is acute and the angle between the vector $\mathbf{b}$ and Y-axis is obtuse, therefore $\mathbf{a} \cdot \mathbf{b} > 0$ and $\mathbf{b} \cdot \hat{j} < 0$.

$$\begin{aligned} \therefore \quad & 2x^2 - 3x + 1 > 0 \text{ and } x < 0 \\ \Rightarrow \quad & (2x - 1)(x - 1) > 0 \text{ and } x < 0 \\ \Rightarrow \quad & x < \frac{1}{2} \text{ or } x > 1 \text{ and } x < 0 \\ \therefore \quad & x < 0 \end{aligned}$$

➤ **Example 23.** Vector $\mathbf{a}$ is equal to

- (a) $(\mathbf{a} \cdot \hat{i})\hat{i} + (\mathbf{a} \cdot \hat{j})\hat{j} + (\mathbf{a} \cdot \hat{k})\hat{k}$
(b) $(\mathbf{a} \cdot \hat{j})\hat{i} + (\mathbf{a} \cdot \hat{k})\hat{j} + (\mathbf{a} \cdot \hat{i})\hat{k}$
(c) $(\mathbf{a} \cdot \hat{k})\hat{i} + (\mathbf{a} \cdot \hat{i})\hat{j} + (\mathbf{a} \cdot \hat{j})\hat{k}$
(d) $(\mathbf{a} \cdot \mathbf{a})(\hat{i} + \hat{j} + \hat{k})$

Sol. (a) Let $\mathbf{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\text{Then, } \mathbf{a} \cdot \hat{i} = a_1, \mathbf{a} \cdot \hat{j} = a_2, \mathbf{a} \cdot \hat{k} = a_3$$

$$\therefore \quad \mathbf{a} = (\mathbf{a} \cdot \hat{i})\hat{i} + (\mathbf{a} \cdot \hat{j})\hat{j} + (\mathbf{a} \cdot \hat{k})\hat{k}$$

➤ **Example 24.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors each of magnitude unity, then $|\mathbf{a} + \mathbf{b} + \mathbf{c}|$ is equal to

- (a) 3 (b) 1
(c) $\sqrt{3}$ (d) None of these

Sol. (c) We have, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$

$$\text{and } \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$$

$$\begin{aligned} \therefore \quad |\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 &= (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) \\ &= |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2 \\ &\quad + 2\mathbf{a} \cdot \mathbf{b} + 2\mathbf{b} \cdot \mathbf{c} + 2\mathbf{c} \cdot \mathbf{a} = 3 \\ \Rightarrow \quad |\mathbf{a} + \mathbf{b} + \mathbf{c}| &= \sqrt{3} \end{aligned}$$

➤ **Example 25.** The length of the longer diagonal of the parallelogram constructed on $5\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$, if it is given that $|\mathbf{a}| = 2\sqrt{2}$,

$|\mathbf{b}| = 3$ and angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$, is

- (a) 15 (b) $\sqrt{113}$
(c) $\sqrt{593}$ (d) $\sqrt{369}$

Sol. (c) The diagonals of the parallelogram are

$$\alpha = 5\mathbf{a} + 2\mathbf{b} + \mathbf{a} - 3\mathbf{b} = 6\mathbf{a} - \mathbf{b}$$

$$\text{and } \beta = \pm(4\mathbf{a} + 5\mathbf{b})$$

$$\begin{aligned} \text{Now, } |\alpha| &= |6\mathbf{a} - \mathbf{b}| = \sqrt{36|\mathbf{a}|^2 + |\mathbf{b}|^2 - 12(\mathbf{a} \cdot \mathbf{b})} \\ &= \sqrt{36 \times 8 + 9 - 12 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}} = 15 \end{aligned}$$

$$\begin{aligned} \text{and } |\beta| &= |4\mathbf{a} + 5\mathbf{b}| \\ &= \sqrt{16|\mathbf{a}|^2 + 25|\mathbf{b}|^2 + 40(\mathbf{a} \cdot \mathbf{b})} \\ &= \sqrt{16 \times 8 + 25 \times 9 + 40 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}} \\ &= \sqrt{593} \end{aligned}$$

➤ **Example 26.** If $|\mathbf{a}| = |\mathbf{b}|$, then

- (a) $(\mathbf{a} + \mathbf{b})$ is parallel to $(\mathbf{a} - \mathbf{b})$
(b) $(\mathbf{a} + \mathbf{b})$ is perpendicular to $(\mathbf{a} - \mathbf{b})$
(c) $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 2|\mathbf{a}|^2$
(d) None of the above

Sol. (b) We have, $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = |\mathbf{a}|^2 - |\mathbf{b}|^2$

$$\begin{aligned} \Rightarrow \quad & (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = 0 \quad [\because |\mathbf{a}| = |\mathbf{b}|] \\ \Rightarrow \quad & (\mathbf{a} + \mathbf{b}) \perp (\mathbf{a} - \mathbf{b}) \end{aligned}$$

➤ **Example 27.** The projection of the vector $\hat{i} - 2\hat{j} + \hat{k}$ on the vector $4\hat{i} - 4\hat{j} + 7\hat{k}$ is

- (a) $\frac{5\sqrt{6}}{10}$ (b) $\frac{19}{9}$
(c) $\frac{9}{19}$ (d) $\frac{\sqrt{6}}{19}$

Sol. (b) Let $\mathbf{a} = \hat{i} - 2\hat{j} + \hat{k}$ and $\mathbf{b} = 4\hat{i} - 4\hat{j} + 7\hat{k}$.

Then, projection of $\mathbf{a}$ on $\mathbf{b}$

$$= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{4 + 8 + 7}{\sqrt{16 + 16 + 49}} = \frac{19}{9}$$

➤ **Example 28.** If $\hat{a}$ and $\hat{b}$ are two unit vectors and θ is angle between them, then $\left| \frac{\hat{a} - \hat{b}}{2} \right|$ is equal

to

- (a) $\sin \frac{\theta}{2}$ (b) $\sin \theta$
(c) $2 \sin \theta$ (d) $\sin 2\theta$

Sol. (a) We have, $|\hat{a} - \hat{b}|^2 = |\hat{a}|^2 + |\hat{b}|^2 - 2\hat{a} \cdot \hat{b}$

$$\Rightarrow |\hat{a} - \hat{b}|^2 = 1 + 1 - 2|\hat{a}||\hat{b}|\cos\theta$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 = 2 - 2\cos\theta$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 = 4\sin^2 \frac{\theta}{2}$$

$$\Rightarrow \left| \frac{\hat{a} - \hat{b}}{2} \right|^2 = \sin^2 \frac{\theta}{2}$$

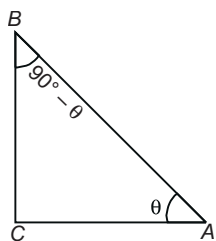
$$\Rightarrow \left| \frac{\hat{a} - \hat{b}}{2} \right| = \sin \frac{\theta}{2}$$

➤ **Example 29.** In a right angled $\triangle ABC$, if the hypotenuse $AB = p$, then

$\mathbf{AB} \cdot \mathbf{AC} + \mathbf{BC} \cdot \mathbf{BA} + \mathbf{CA} \cdot \mathbf{CB}$ is equal to

- (a) $2p^2$ (b) $\frac{p^2}{2}$
(c) p^2 (d) None of these

Sol. (c) We have, $\mathbf{AB} \cdot \mathbf{AC} + \mathbf{BC} \cdot \mathbf{BA} + \mathbf{CA} \cdot \mathbf{CB}$
 $= (AB)(AC)\cos\theta + (BC)(BA)\sin\theta + 0$



$$= AB(AC\cos\theta + BC\sin\theta)$$

$$= AB \left(\frac{(AC)^2}{AB} + \frac{(BC)^2}{AB} \right)$$

$$= AC^2 + BC^2 = AB^2 = p^2$$

➤ **Example 30.** Let $\mathbf{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\mathbf{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of $\mathbf{b}$ and $\mathbf{c}$ whose projection on $\mathbf{a}$ is a magnitude $\sqrt{\frac{2}{3}}$, is

- (a) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (b) $2\hat{i} + 3\hat{j} + 3\hat{k}$
(c) $-2\hat{i} - \hat{j} + 5\hat{k}$ (d) $\hat{i} - 5\hat{j} + 3\hat{k}$

Sol. (a, c) Any vector in the plane of $\mathbf{b}$ and $\mathbf{c}$ is
 $\mathbf{r} = \mathbf{b} + \lambda\mathbf{c} = (1 + \lambda)\hat{i} + (2 + \lambda)\hat{j} + (-1 - 2\lambda)\hat{k}$

$$\text{Projection of } \mathbf{r} \text{ on } \mathbf{a} = \frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{a}|} = \mathbf{r} \cdot \frac{\mathbf{a}}{|\mathbf{a}|} = \sqrt{\frac{2}{3}}$$

$$\Rightarrow \frac{2(1 + \lambda) - (2 + \lambda) - (1 + 2\lambda)}{\sqrt{4 + 1 + 1}} = \pm \sqrt{\frac{2}{3}}$$

$$\Rightarrow -\lambda - 1 = \pm 2 \Rightarrow \lambda = -3 \text{ or } 1$$

$$\therefore \mathbf{r} = -2\hat{i} - \hat{j} + 5\hat{k}$$

$$\text{or } \mathbf{r} = 2\hat{i} + 3\hat{j} - 3\hat{k}$$

➤ **Example 31.** If three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, $|\mathbf{c}| = 7$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

- (a) 30° (b) 45° (c) 60° (d) 90°

Sol. (c) Given, $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \Rightarrow \mathbf{a} + \mathbf{b} = -\mathbf{c}$

Squaring on both sides, we get

$$|\mathbf{a}|^2 + |\mathbf{b}|^2 + 2|\mathbf{a}||\mathbf{b}|\cos\theta = |\mathbf{c}|^2$$

$$\Rightarrow 9 + 25 + 30\cos\theta = 49 \Rightarrow \cos\theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ$$

Components of a Vector $\mathbf{b}$ along $\mathbf{a}$ and Perpendicular to Vector $\mathbf{a}$

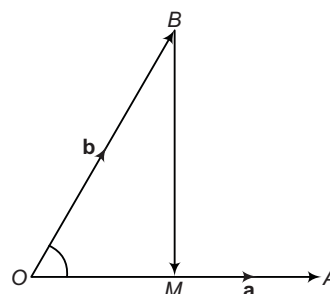
Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors represented by $\mathbf{OA}$, $\mathbf{OB}$ and θ be the angle between $\mathbf{a}$ and $\mathbf{b}$.

Draw $\mathbf{BM} \perp \mathbf{OA}$

In $\triangle OBM$, we have

$$\mathbf{OB} = \mathbf{OM} + \mathbf{MB}$$

$$\Rightarrow \mathbf{b} = \mathbf{OM} + \mathbf{MB}$$



Thus, $\mathbf{OM}$ and $\mathbf{MB}$ are components of $\mathbf{b}$ along $\mathbf{a}$ and perpendicular to $\mathbf{a}$ respectively.

Now, $\mathbf{OM} = (OM)\hat{\mathbf{a}} = (OB \cos\theta)\hat{\mathbf{a}}$

$$= (|\mathbf{b}|\cos\theta)\hat{\mathbf{a}}$$

$$= \left\{ |\mathbf{b}| \frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}||\mathbf{b}|} \right\} \hat{\mathbf{a}}$$

$$= \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right\} \hat{\mathbf{a}} = \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} \right\} \frac{\mathbf{a}}{|\mathbf{a}|} = \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right\} \mathbf{a}$$

$$\therefore \mathbf{b} = \mathbf{OM} + \mathbf{MB}$$

$$\Rightarrow \mathbf{MB} = \mathbf{b} - \mathbf{OM} = \mathbf{b} - \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right\} \mathbf{a}$$

Thus, the components of $\mathbf{b}$ along $\mathbf{a}$ and perpendicular to $\mathbf{a}$ are $\left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right\} \mathbf{a}$ and $\mathbf{b} - \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right\} \mathbf{a}$, respectively.

➤ **Example 32.** If $\mathbf{a} = 4\hat{i} + 6\hat{j}$ and $\mathbf{b} = 3\hat{j} + 4\hat{k}$, then the component of $\mathbf{a}$ along $\mathbf{b}$ is

- (a) $\frac{18}{10\sqrt{3}}(3\hat{j} + 4\hat{k})$ (b) $\frac{18}{25}(3\hat{j} + 4\hat{k})$
(c) $\frac{18}{\sqrt{3}}(3\hat{j} + 4\hat{k})$ (d) $(3\hat{j} + 4\hat{k})$

Sol. (b) Component of vector $\mathbf{a}$ along $\mathbf{b}$

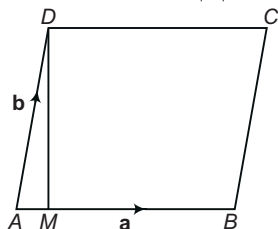
$$= \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} = \frac{18}{25}(3\hat{j} + 4\hat{k})$$

➤ **Example 33.** If $\mathbf{a}$ and $\mathbf{b}$ are two given vectors. On these vectors as adjacent sides, a parallelogram is constructed. The vector which is the altitude of the parallelogram and which is perpendicular to $\mathbf{a}$, is

- (a) $\frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{a}|^2} \mathbf{a} - \mathbf{b}$
(b) $\frac{1}{|\mathbf{a}|^2} [|\mathbf{a}|^2 \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}]$
(c) $\frac{(\mathbf{a} \cdot \mathbf{b})}{|\mathbf{b}|^2} \mathbf{a} - \mathbf{b}$
(d) None of the above

Sol. (a, b) We have,

$$\mathbf{AM} = \text{Projection of } \mathbf{b} \text{ on } \mathbf{a} = \frac{|\mathbf{a} \cdot \mathbf{b}|}{|\mathbf{a}|}$$



$$\therefore \mathbf{AM} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2} \right) \mathbf{a}$$

Now, in $\triangle ADM$, $\mathbf{AD} = \mathbf{AM} + \mathbf{MD}$
 $\Rightarrow \mathbf{DM} = \mathbf{AM} - \mathbf{AD} = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2} - \mathbf{b}$

Also, $\mathbf{DM} = \frac{1}{|\mathbf{a}|^2} [(\mathbf{a} \cdot \mathbf{b})\mathbf{a} - |\mathbf{a}|^2 \mathbf{b}]$
 $\Rightarrow \mathbf{MD} = \frac{1}{|\mathbf{a}|^2} [|\mathbf{a}|^2 \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}]$

Applications of Scalar Product in Mechanics to Find the Work Done

A force acting on a particle is said to do work, if the particle is displaced in a direction which is not perpendicular to the force.

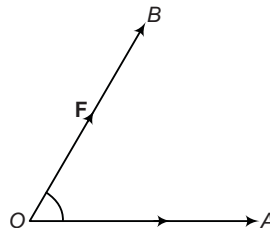
The work done by a force is a scalar quantity and its measure is equal to the product of the magnitude of the force and the resolved part to the displacement in the direction of the force.

Let a particle be placed at O and a force $\mathbf{F}$ represented by $\mathbf{OB}$ be acting on the particle at O . Due to the application of force $\mathbf{F}$, the particle is displaced in the direction of $\mathbf{OA}$. Let $\mathbf{OA}$ be the displacement. Then, the component of $\mathbf{OA}$ in the direction of the force $\mathbf{F}$ is $|\mathbf{OA}|\cos\theta$.

$$\therefore \text{Work done} = |\mathbf{F}||\mathbf{OA}|\cos\theta$$

$$= \mathbf{F} \cdot \mathbf{OA} = \mathbf{F} \cdot \mathbf{d}, \text{ where } \mathbf{d} = \mathbf{OA}$$

If a number of forces are acting on a particle, then the sum of the work done by the separate forces is equal to the work done by the resultant force.



➤ **Example 34.** If the constant forces $2\hat{i} - 5\hat{j} + 6\hat{k}$ and $-\hat{i} + 2\hat{j} - \hat{k}$ act on a particle due to which it is displaced from a point $A(4, -3, -2)$ to a point $B(6, 1, -3)$, then the work done by the forces is

- (a) 25 units (b) -15 units (c) 9 units (d) -9 units

Sol. (b) Resultant force is given by

$$\mathbf{F} = (2\hat{i} - 5\hat{j} + 6\hat{k}) + (-\hat{i} + 2\hat{j} - \hat{k}) = \hat{i} - 3\hat{j} + 5\hat{k}$$

and $\mathbf{d} = \text{Displacement}$

$$= \mathbf{AB} = (6\hat{i} + \hat{j} - 3\hat{k}) - (4\hat{i} - 3\hat{j} - 2\hat{k}) = 2\hat{i} + 4\hat{j} - \hat{k}$$

$$\therefore \text{Work done} = \mathbf{F} \cdot \mathbf{d} = (\hat{i} - 3\hat{j} + 5\hat{k}) \cdot (2\hat{i} + 4\hat{j} - \hat{k})$$

$$= 2 - 12 - 5 = -15 \text{ units}$$

➤ **Example 35.** The work done by the force $\mathbf{F} = 2\hat{i} - \hat{j} - \hat{k}$ in moving an object along the vector $3\hat{i} + 2\hat{j} - 5\hat{k}$ is

- (a) -9 units (b) 15 units (c) 9 units (d) 13 units

Sol. (c) Work done $= \mathbf{F} \cdot \mathbf{d}$

$$= (2\hat{i} - \hat{j} - \hat{k}) \cdot (3\hat{i} + 2\hat{j} - 5\hat{k}) = 9 \text{ units}$$

➤ **Example 36.** Two forces $\mathbf{F}_1 = 3\hat{i} - 2\hat{j} + \hat{k}$ and $\mathbf{F}_2 = \hat{i} + 3\hat{j} - 5\hat{k}$ acting on a particle at A move it to B . The work done, if the position vector of A and B are $-2\hat{i} + 5\hat{k}$ and $3\hat{i} - 7\hat{j} + 2\hat{k}$, is

- (a) 20 units (b) 7 units
(c) 25 units (d) None of these

Sol. (c) Let $\mathbf{R}$ be the resultant of two forces $\mathbf{F}_1, \mathbf{F}_2$ and $\mathbf{d}$ be the displacement.

$$\text{Then, } \mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 = (3\hat{i} - 2\hat{j} + \hat{k}) + (\hat{i} + 3\hat{j} - 5\hat{k})$$

$$= 4\hat{i} + \hat{j} - 4\hat{k}$$

$$\text{and } \mathbf{d} = (3\hat{i} - 7\hat{j} + 2\hat{k}) - (-2\hat{i} + 5\hat{k}) = 5\hat{i} - 7\hat{j} - 3\hat{k}$$

$\therefore$ Total work done

$$= \mathbf{R} \cdot \mathbf{d} = (4\hat{i} + \hat{j} - 4\hat{k}) \cdot (5\hat{i} - 7\hat{j} - 3\hat{k})$$

$$= 20 - 7 + 12 = 25 \text{ units}$$

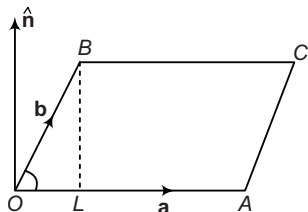
Objective Mathematics Vol. 2

- Objective Mathematics Vol. 2

842

Geometrical Interpretation of Vector Product

Let $\mathbf{a}$, $\mathbf{b}$ be two non-zero and non-parallel vectors represented by $\mathbf{OA}$ and $\mathbf{OB}$ respectively and θ be the angle between them. Complete the parallelogram $OACB$. Draw $BL \perp OA$.



In $\triangle OBL$, we have $\sin \theta = \frac{BL}{OB}$

$$\Rightarrow BL = OB \sin \theta = |\mathbf{b}| \sin \theta \quad \dots(i)$$

$$\begin{aligned} \text{Now, } \mathbf{a} \times \mathbf{b} &= |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}} \\ &= (OB)(BL) \hat{\mathbf{n}} \quad [\text{using Eq. (i)}] \\ &= (\text{Base} \times \text{Height}) \\ &= (\text{Area of parallelogram } OACB) \hat{\mathbf{n}} \\ &= \text{Vector area of the parallelogram } OACB \end{aligned}$$

Thus, $\mathbf{a} \times \mathbf{b}$ is a vector whose magnitude is equal to the area of the parallelogram having $\mathbf{a}$ and $\mathbf{b}$ as its adjacent sides and whose direction $\hat{\mathbf{n}}$ is perpendicular to the plane of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}$, $\mathbf{b}$, $\hat{\mathbf{n}}$ form a right handed system. Hence, $\mathbf{a} \times \mathbf{b}$ represents the vector area of the parallelogram having adjacent sides along $\mathbf{a}$ and $\mathbf{b}$.

Thus, area of parallelogram $OACB = |\mathbf{a} \times \mathbf{b}|$

$$\begin{aligned} \therefore \text{Area of } \triangle OAB &= \frac{1}{2} \times \text{Area of parallelogram } OACB \\ &= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{1}{2} |\mathbf{OA} \times \mathbf{OB}| \end{aligned}$$

Properties of Vector Product

- i. Vector product is not commutative i.e. if $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $\mathbf{a} \times \mathbf{b} \neq \mathbf{b} \times \mathbf{a}$, however $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$.
- ii. If $\mathbf{a}$, $\mathbf{b}$ are two vectors and m is a scalar, then $m\mathbf{a} \times \mathbf{b} = m(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times m\mathbf{b}$
- iii. If $\mathbf{a}$, $\mathbf{b}$ are two vectors and m , n are scalars, then $m\mathbf{a} \times n\mathbf{b} = mn(\mathbf{a} \times \mathbf{b}) = m(\mathbf{a} \times n\mathbf{b}) = n(m\mathbf{a} \times \mathbf{b})$
- iv. Vector product is distributive over vector addition.
i.e. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
and $(\mathbf{b} + \mathbf{c}) \times \mathbf{a} = \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a}$ for all $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

v. For any three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we have $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
[distributive law]

vi. The vector product of two non-zero vectors is the null vector iff they are collinear or parallel i.e. $\mathbf{a} \times \mathbf{b} = 0 \Leftrightarrow \mathbf{a} \parallel \mathbf{b}$, where $\mathbf{a}$, $\mathbf{b}$ are not null vectors.

$$\begin{aligned} vii. \quad \hat{\mathbf{i}} \times \hat{\mathbf{i}} &= \hat{\mathbf{j}} \times \hat{\mathbf{j}} = \hat{\mathbf{k}} \times \hat{\mathbf{k}} = 0 \\ \hat{\mathbf{i}} \times \hat{\mathbf{j}} &= \hat{\mathbf{k}}, \hat{\mathbf{j}} \times \hat{\mathbf{i}} = -\hat{\mathbf{k}}, \hat{\mathbf{j}} \times \hat{\mathbf{k}} = \hat{\mathbf{i}}, \\ \hat{\mathbf{k}} \times \hat{\mathbf{j}} &= -\hat{\mathbf{i}}, \hat{\mathbf{k}} \times \hat{\mathbf{i}} = \hat{\mathbf{j}}, \hat{\mathbf{i}} \times \hat{\mathbf{k}} = -\hat{\mathbf{j}} \end{aligned}$$

viii. If $\mathbf{a} = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}$
and $\mathbf{b} = b_1 \hat{\mathbf{i}} + b_2 \hat{\mathbf{j}} + b_3 \hat{\mathbf{k}}$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Some Useful Results

(i) If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-parallel vectors, then unit vectors normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are

$$\pm \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$$

(ii) Vectors of magnitude λ normal to the plane of $\mathbf{a}$ and $\mathbf{b}$ are

$$\pm \lambda \left(\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} \right)$$

(iii) The area of the parallelogram with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$.

(iv) The area of a triangle with adjacent sides $\mathbf{a}$ and $\mathbf{b}$ is $\frac{1}{2} |\mathbf{a} \times \mathbf{b}|$.

$$\begin{aligned} (v) \quad \text{Area of } \triangle ABC &= \frac{1}{2} |\mathbf{AB} \times \mathbf{AC}| \\ &= \frac{1}{2} |\mathbf{BC} \times \mathbf{BA}| = \frac{1}{2} |\mathbf{CA} \times \mathbf{CB}| \end{aligned}$$

(vi) The area of a plane quadrilateral $ABCD$ is $\frac{1}{2} |\mathbf{AC} \times \mathbf{BD}|$, where AC and BD are its diagonals.

Example 37. Let the vectors $\mathbf{a}$ and $\mathbf{b}$ be such that $|\mathbf{a}| = 3$ and $|\mathbf{b}| = \frac{\sqrt{2}}{3}$, then $\mathbf{a} \times \mathbf{b}$ is a unit vector, if the angle between $\mathbf{a}$ and $\mathbf{b}$ is

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

Sol. (b) It is given that, $|a| = 3$ and $|b| = \frac{\sqrt{2}}{3}$

Let θ be the angle between a and b , then $|a \times b| = 1$

$$\Rightarrow |a||b|\sin\theta = 1$$

$$\Rightarrow 3 \times \frac{\sqrt{2}}{3} \times \sin\theta = 1$$

$$\Rightarrow \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

Hence, $a \times b$ is a unit vector, if the angle between a and b is $\frac{\pi}{4}$.

➤ **Example 38.** A unit vector perpendicular to each of the vectors $a + b$ and $a - b$, where $a = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $b = \hat{i} + 2\hat{j} - 2\hat{k}$, is

(a) $\frac{1}{3}\hat{i} - \frac{1}{3}\hat{j} - \frac{1}{3}\hat{k}$ or $-\frac{1}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{1}{3}\hat{k}$

(b) $\frac{4}{3}\hat{i} - \frac{4}{3}\hat{j} - \frac{4}{3}\hat{k}$ or $-\frac{4}{3}\hat{i} + \frac{4}{3}\hat{j} + \frac{4}{3}\hat{k}$

(c) $\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k}$ or $-\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$

(d) None of the above

Sol. (c) Given that, $a = 3\hat{i} + 2\hat{j} + 2\hat{k}$

and $b = \hat{i} + 2\hat{j} - 2\hat{k}$

$$\therefore a + b = (3\hat{i} + 2\hat{j} + 2\hat{k}) + (\hat{i} + 2\hat{j} - 2\hat{k}) = 4\hat{i} + 4\hat{j} + 0\hat{k}$$

and $a - b = (3\hat{i} + 2\hat{j} + 2\hat{k}) - (\hat{i} + 2\hat{j} - 2\hat{k}) = 2\hat{i} + 4\hat{k}$

$$\text{Now, } (a + b) \times (a - b) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & 4 & 0 \\ 2 & 0 & 4 \end{vmatrix}$$

$$= (16 - 0)\hat{i} - (16 - 0)\hat{j} + (0 - 8)\hat{k}$$

$$= 16\hat{i} - 16\hat{j} - 8\hat{k}$$

$$\Rightarrow |(a + b) \times (a - b)| = \sqrt{(16)^2 + (-16)^2 + (-8)^2} = \sqrt{256 + 256 + 64} = \sqrt{576} = 24$$

$\therefore$ A unit vector perpendicular to $(a + b)$ and $(a - b)$

$$\pm \frac{(a + b) \times (a - b)}{|(a + b) \times (a - b)|} = \pm \frac{16\hat{i} - 16\hat{j} - 8\hat{k}}{24}$$

$$= \pm \frac{8(2\hat{i} - 2\hat{j} - \hat{k})}{24} = \pm \frac{1}{3}(2\hat{i} - 2\hat{j} - \hat{k})$$

$\therefore$ Required vector is either

$$\frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} - \frac{1}{3}\hat{k} \text{ or } -\frac{2}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

➤ **Example 39.** If $\hat{i} + 2\hat{j} + 3\hat{k}$ and $3\hat{i} - 2\hat{j} + \hat{k}$ represent the adjacent sides of a parallelogram, then the area of this parallelogram is

(a) $4\sqrt{3}$ (b) $6\sqrt{3}$ (c) $8\sqrt{3}$ (d) $16\sqrt{3}$

Sol. (c) Area of parallelogram

$$= |a \times b| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = |8\hat{i} + 8\hat{j} - 8\hat{k}| = 8\sqrt{3}$$

➤ **Example 40.** If $|a| = 2$, $|b| = 5$ and $|a \times b| = 8$, then $a \cdot b$ is equal to

(a) 0

(b) 2

(c) 4

(d) 6

Sol. (d) $a \cdot b = |a||b|\cos\theta$; but $|a \times b| = |a||b|\sin\theta$

$$\therefore \sin\theta = \frac{|a \times b|}{|a||b|} = \frac{4}{5}$$

$$\Rightarrow \cos\theta = \frac{3}{5}$$

Therefore, $a \cdot b = 2 \times 5 \times \frac{3}{5} = 6$

➤ **Example 41.** The vector r satisfying the conditions that it is perpendicular to $3\hat{i} + 2\hat{j} + 2\hat{k}$ and $18\hat{i} - 22\hat{j} - 5\hat{k}$, it makes an obtuse angle with Y-axis and $|r| = 14$, is

(a) $2(-2\hat{i} - 3\hat{j} + 6\hat{k})$

(b) $2(2\hat{i} - 3\hat{j} + 6\hat{k})$

(c) $4\hat{i} + 6\hat{j} - 12\hat{k}$

(d) None of the above

Sol. (a) Let $a = 3\hat{i} + 2\hat{j} + 2\hat{k}$ and $b = 18\hat{i} - 22\hat{j} - 5\hat{k}$.

Then, the required vector r is given by

$$\begin{aligned} r &= \lambda(a \times b) \\ &= \lambda(34\hat{i} + 51\hat{j} - 102\hat{k}) \\ &= 17\lambda(2\hat{i} + 3\hat{j} - 6\hat{k}) \end{aligned}$$

Now, $|r| = 14$

$$\Rightarrow 119|\lambda| = 14$$

$$\Rightarrow |\lambda| = \frac{2}{17}$$

Since, r makes an obtuse angle with Y-axis.

Therefore, $\lambda = -\frac{2}{17}$

$$\therefore r = -2(2\hat{i} + 3\hat{j} - 6\hat{k})$$

or $r = 2(-2\hat{i} - 3\hat{j} + 6\hat{k})$

➤ **Example 42.** If a , b and c are non-collinear vectors such that $a + b$ is parallel to c and $c + a$ is parallel to b , then

(a) $a + b = c$

(b) a , b , c taken order, form the sides of a triangle

(c) $b + c = a$

(d) None of the above

Sol. (b) It is given that, $(a + b) \parallel c$ and $(c + a) \parallel b$.

$$\therefore (a + b) \times c = 0$$

and $(c + a) \times b = 0$

$$\Rightarrow a \times c + b \times c = 0$$

and $c \times b + a \times b = 0$

$$\Rightarrow a \times b = b \times c = c \times a$$

Hence a , b , c are the sides of a triangle.

➤ **Example 43.** If $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = 0$, then

$\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is equal to

- (a) 0
(b) $6(\mathbf{b} \times \mathbf{c})$
(c) $2(\mathbf{b} \times \mathbf{c})$
(d) $3(\mathbf{c} \times \mathbf{a})$

Sol. (b, d) Given, $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c} = 0 \Rightarrow \mathbf{a} = -2\mathbf{b} - 3\mathbf{c}$
 $\therefore \mathbf{a} \times \mathbf{b} = -2\mathbf{b} \times \mathbf{b} - 3\mathbf{c} \times \mathbf{b} \Rightarrow \mathbf{a} \times \mathbf{b} = 3\mathbf{b} \times \mathbf{c}$
 and $\mathbf{c} \times \mathbf{a} = -2\mathbf{c} \times \mathbf{b} - 3\mathbf{c} \times \mathbf{c} \Rightarrow \mathbf{c} \times \mathbf{a} = 2\mathbf{b} \times \mathbf{c}$
 Now, $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = 3\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{c} + 2\mathbf{b} \times \mathbf{c}$
 $= 6(\mathbf{b} \times \mathbf{c})$
 or $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = 3(\mathbf{c} \times \mathbf{a})$

➤ **Example 44.** If $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}$ and

$\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$, then

- (a) $(\mathbf{a} - \mathbf{d}) = \lambda(\mathbf{b} - \mathbf{c})$
(b) $\mathbf{a} + \mathbf{d} = \lambda(\mathbf{b} + \mathbf{c})$
(c) $(\mathbf{a} - \mathbf{b}) = \lambda(\mathbf{c} + \mathbf{d})$
(d) None of the above

Sol. (a) We have,
 $(\mathbf{a} - \mathbf{d}) \times (\mathbf{b} - \mathbf{c}) = \mathbf{a} \times \mathbf{b} - \mathbf{a} \times \mathbf{c} - \mathbf{d} \times \mathbf{b} + \mathbf{d} \times \mathbf{c}$
 $= \mathbf{c} \times \mathbf{d} - \mathbf{b} \times \mathbf{d} - \mathbf{d} \times \mathbf{b} + \mathbf{d} \times \mathbf{c}$
 $= 0$ [$\because \mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{d}, \mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$]
 $\Rightarrow (\mathbf{a} - \mathbf{d}) \parallel (\mathbf{b} - \mathbf{c})$
 $\Rightarrow (\mathbf{a} - \mathbf{d}) = \lambda(\mathbf{b} - \mathbf{c})$

➤ **Example 45.** If $\mathbf{u} = \mathbf{a} - \mathbf{b}$, $\mathbf{v} = \mathbf{a} + \mathbf{b}$ and $|\mathbf{a}| = |\mathbf{b}| = 2$, then $|\mathbf{u} \times \mathbf{v}|$ is equal to

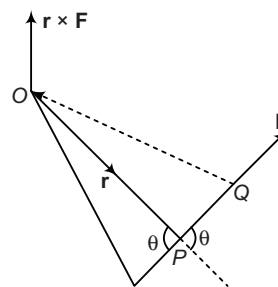
- (a) $2\sqrt{16 - (\mathbf{a} \cdot \mathbf{b})^2}$
(b) $2\sqrt{4 - (\mathbf{a} \cdot \mathbf{b})^2}$
(c) $\sqrt{16 - (\mathbf{a} \cdot \mathbf{b})^2}$
(d) $\sqrt{4 - (\mathbf{a} \cdot \mathbf{b})^2}$

Sol. (a) We have, $\mathbf{u} = \mathbf{a} - \mathbf{b}, \mathbf{v} = \mathbf{a} + \mathbf{b}$
 $\Rightarrow \mathbf{u} \times \mathbf{v} = (\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2(\mathbf{a} \times \mathbf{b})$
 $\therefore |\mathbf{u} \times \mathbf{v}| = 2|\mathbf{a} \times \mathbf{b}|$
 $= 2\sqrt{|\mathbf{a} \times \mathbf{b}|^2}$
 $= 2\sqrt{|\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2}$
 $= 2\sqrt{16 - (\mathbf{a} \cdot \mathbf{b})^2}$

Applications of Vector Product in Mechanics to Find the Moment of a Force

- (i) **About a point** Let a force $\mathbf{F}$ be applied at point P of a rigid body. Then, the moment of $\mathbf{F}$ about a point O measures the tendency (amount) of $\mathbf{F}$ to turn the body about point O . If this tendency of rotation about O is in anti-clockwise direction, the moment is positive, otherwise it is negative.

Let $\mathbf{r}$ be the position vector of P relative to O . Then, the moment or torque of $\mathbf{F}$ about the point O is defined as $\mathbf{M} = \mathbf{r} \times \mathbf{F}$.



This vector is perpendicular to the plane containing O and the line of action of the force. The vector $\mathbf{M}$ about O is independent of the choice of the point P on the line of action of the force. To prove this, let Q be any other point on the line of action of force $\mathbf{F}$. Then,

$$\begin{aligned} \mathbf{OQ} \times \mathbf{F} &= (\mathbf{OP} + \mathbf{PQ}) \times \mathbf{F} \quad [\because \mathbf{OQ} = \mathbf{OP} + \mathbf{PQ}] \\ &= \mathbf{OP} \times \mathbf{F} + \mathbf{PQ} \times \mathbf{F} = \mathbf{OP} \times \mathbf{F} + 0 \\ &\quad [\because \mathbf{PQ} \text{ and } \mathbf{F} \text{ are collinear}] \\ &= \mathbf{r} \times \mathbf{F} \end{aligned}$$

➤ If several forces are acting through the same point P , then the vector sum of the moments of the separate forces about O is equal to the moment of their resultant force about O .

- (ii) **About a line** The moment of a force $\mathbf{F}$ about a line is the resolve part (component) along this line of the moment of $\mathbf{F}$ about any point on the line.

➤ The moment of a force about a point is a vector quantity, while the moment about a straight line is a scalar quantity.

➤ **Example 46.** If the force represented by $3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ is acting through the point $5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, then its moment about the point $(1, 3, 1)$ is

- (a) $14\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$ (b) $-14\hat{\mathbf{i}} + 8\hat{\mathbf{j}} - 12\hat{\mathbf{k}}$
(c) $-6\hat{\mathbf{i}} - \hat{\mathbf{j}} + 9\hat{\mathbf{k}}$ (d) $6\hat{\mathbf{i}} + \hat{\mathbf{j}} - 9\hat{\mathbf{k}}$

Sol. (a) We have, $\mathbf{F} = 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

Let $P(5, 4, -3)$ and $O(1, 3, 1)$ be the given points.

Then, $\mathbf{r} = \mathbf{OP} = 4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

So, moment of $\mathbf{F}$ about O

$$\mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 4 & 1 & -4 \\ 0 & 3 & 2 \end{vmatrix} = 14\hat{\mathbf{i}} - 8\hat{\mathbf{j}} + 12\hat{\mathbf{k}}$$

➤ **Example 47.** A force of 39 kg-wt is acting at a point $P(-4, 2, 5)$ in the direction $12\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$. The moment of this force about a line through the origin having the direction of $2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is

- (a) 76 units (b) -76 units
(c) 72 units (d) None of these

Sol. (b) We have,

$$\text{Force (F)} = 39 \cdot \frac{(12\hat{i} - 4\hat{j} - 3\hat{k})}{\sqrt{144 + 16 + 9}} = 36\hat{i} - 12\hat{j} - 9\hat{k}$$

Let M be the moment of F about a line having the direction $\mathbf{a} = 2\hat{i} - 2\hat{j} + \hat{k}$, then

$$\mathbf{M} \cdot \hat{\mathbf{a}} = (42\hat{i} + 144\hat{j} - 24\hat{k}) \cdot \frac{(2\hat{i} - 2\hat{j} + \hat{k})}{\sqrt{4 + 4 + 1}} = -76 \text{ units}$$

➤ **Example 48.** The moment of the couple formed by the forces $5\hat{i} + \hat{k}$ and $-5\hat{i} - \hat{k}$ acting at the point $(9, -1, 2)$ and $(3, -2, 1)$ respectively, is

(a) $-\hat{i} + \hat{j} + 5\hat{k}$

(b) $\hat{i} - 11\hat{j} - 5\hat{k}$

(c) $-\hat{i} + 11\hat{j} + 5\hat{k}$

(d) $\hat{i} - \hat{j} - 5\hat{k}$

Sol. (d) Here,

$$\mathbf{r}_1 - \mathbf{r}_2 = (9\hat{i} - \hat{j} + 2\hat{k}) - (3\hat{i} - 2\hat{k} + \hat{k}) = 6\hat{i} + \hat{j} + \hat{k}$$

$$\text{and } \mathbf{F} = 5\hat{i} + \hat{k}$$

So, moment of the couple

$$= (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F} = (6\hat{i} + \hat{j} + \hat{k}) \times (5\hat{i} + \hat{k}) = \hat{i} - \hat{j} - 5\hat{k}$$

Work Book Exercise 13.3

- 1 If $\mathbf{a} = -\hat{i} + \hat{j} + 2\hat{k}$, $\mathbf{b} = 2\hat{i} - \hat{j} - \hat{k}$ and $\mathbf{c} = -2\hat{i} + \hat{j} + 3\hat{k}$, then the angle between $2\mathbf{a} - \mathbf{c}$ and $\mathbf{a} + \mathbf{b}$ is

a $\frac{\pi}{4}$

b $\frac{\pi}{3}$

c $\frac{\pi}{2}$

d $\frac{3\pi}{2}$

- 2 $(\mathbf{x} - \mathbf{y}) \times (\mathbf{x} + \mathbf{y})$, where $\mathbf{x}, \mathbf{y} \in R^3$, is equal to

a $2(\mathbf{x} \times \mathbf{y})$

b $|\mathbf{x}|^2 - |\mathbf{y}|^2$

c $\frac{1}{2}(\mathbf{x} \times \mathbf{y})$

d None of these

- 3 The value of $\frac{(\mathbf{a} \times \mathbf{b})^2 + (\mathbf{a} \cdot \mathbf{b})^2}{2a^2 b^2}$ is

a $\mathbf{a} \cdot \mathbf{b}$

b 1

c 0

d $\frac{1}{2}$

- 4 If $\mathbf{a}$ and $\mathbf{b}$ are any two vectors, then $(2\mathbf{a} + 3\mathbf{b}) \times (5\mathbf{a} + 7\mathbf{b}) + \mathbf{a} \times \mathbf{b}$ is equal to

a 0

b 0

c $\mathbf{a} \times \mathbf{b}$

d $\mathbf{b} \times \mathbf{a}$

- 5 If a force $\mathbf{F} = 3\hat{i} + 2\hat{j} - 4\hat{k}$ is acting at the point $P(1, -1, 2)$, then the magnitude of moment of F about the point $Q(2, -1, 3)$ is

a $\sqrt{57}$

b $\sqrt{39}$

c 12

d 17

- 6 If $\mathbf{a} + 2\mathbf{b} + 4\mathbf{c} = 0$ and

$(\mathbf{a} \times \mathbf{b}) + (\mathbf{b} \times \mathbf{c}) + (\mathbf{c} \times \mathbf{a}) = \lambda(\mathbf{b} \times \mathbf{c})$, then λ is equal to

a 4

b 7

c 8

d 9

- 7 If the projection of the vector $\mathbf{a}$ on $\mathbf{b}$ is $|\mathbf{a} \times \mathbf{b}|$ and $3\mathbf{b} = \hat{i} + \hat{j} + \hat{k}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is

a $\frac{\pi}{3}$

b $\frac{\pi}{2}$

c $\frac{\pi}{4}$

d $\frac{\pi}{6}$

- 8 If $2\mathbf{a} + 3\mathbf{b} + \mathbf{c} = 0$, then $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is equal to

a $6(\mathbf{b} \times \mathbf{c})$

b $3(\mathbf{b} \times \mathbf{c})$

c $2(\mathbf{b} \times \mathbf{c})$

d 0

- 9 Let $\mathbf{a} = 2\hat{i} + \hat{k}$, $\mathbf{b} = \hat{i} + \hat{j} + \hat{k}$ and $\mathbf{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$. If $\mathbf{r}$ is a vector such that $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$ and $\mathbf{r} \cdot \mathbf{a} = 0$, then the value of $\mathbf{r} \cdot \mathbf{b}$ is

a 7

b -7

c -5

d 5

- 10 If the area of the parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as two adjacent sides, is 15 sq units, then the area of the parallelogram having $3\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} + 3\mathbf{b}$ as two adjacent sides (in sq units), is

a 120

b 105

c 75

d 45

Scalar Triple Product

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors. Then, the scalar $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is called the scalar triple product of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and is denoted by $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$.

Geometrical Interpretation

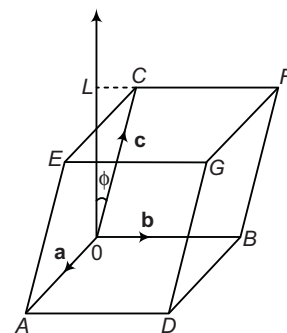
Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors. Consider a parallelepiped having coterminus edges OA , OB and OC such that $\mathbf{OA} = \mathbf{a}$, $\mathbf{OB} = \mathbf{b}$ and $\mathbf{OC} = \mathbf{c}$. Then, $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane $\mathbf{a}$ and $\mathbf{b}$. Let ϕ be the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$. If $\hat{\mathbf{n}}$ is a unit vector along $\mathbf{a} \times \mathbf{b}$ then ϕ is also the angle between $\hat{\mathbf{n}}$ and $\mathbf{c}$.

$$\text{Now, } [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = [\mathbf{a} \times \mathbf{b}] \cdot \mathbf{c} = (\text{Area of the parallelogram } OADB) \hat{\mathbf{n}} \cdot \mathbf{c}$$

$$= (\text{Area of the parallelogram } OADB) |\hat{\mathbf{n}}| |\mathbf{c}| \cos \phi$$

$$= (\text{Area of the parallelogram } OADB) (|\mathbf{c}| \cos \phi)$$

$$[\because |\hat{\mathbf{n}}| = 1]$$



= (Area of the parallelogram $OADB$) (OL)
 $[\because OC \cos \phi = OL]$
 = Area of the base of the parallelopiped $\times$ Height
 = Volume of the parallelopiped with coterminus edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$

Thus, the scalar triple product $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ represents the volume of the parallelopiped whose coterminus edges $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form a right handed system of vectors.

Properties of Scalar Triple Product

- i. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are cyclically permuted, then the value of scalar triple product remains same.
 i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$
 or $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = [\mathbf{b} \ \mathbf{c} \ \mathbf{a}] = [\mathbf{c} \ \mathbf{a} \ \mathbf{b}]$
- ii. The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not the magnitude.
 i.e. $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = -[\mathbf{b} \ \mathbf{a} \ \mathbf{c}]$
 $= -[\mathbf{c} \ \mathbf{b} \ \mathbf{a}] = -[\mathbf{a} \ \mathbf{c} \ \mathbf{b}]$
- iii. In scalar triple product, the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same
 i.e. $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
- iv. The scalar triple product of three vectors is zero, if any two of them are equal.
- v. For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and scalar λ ,
 $[\lambda \mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \lambda [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
- vi. The scalar triple product of three vectors is zero, if any two of them are parallel or collinear.
- vii. If $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ are four vectors, then
 $[\mathbf{a} + \mathbf{b} \ \mathbf{c} \ \mathbf{d}] = [\mathbf{a} \ \mathbf{c} \ \mathbf{d}] + [\mathbf{b} \ \mathbf{c} \ \mathbf{d}]$
- viii. The necessary and sufficient condition for three non-zero and non-collinear vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ to be coplanar is that $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$.
 i.e. $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are coplanar.
 $\Rightarrow [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$
- ix. Four points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$, will be coplanar, if
 $[\mathbf{d} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{d} \ \mathbf{c} \ \mathbf{a}] + [\mathbf{d} \ \mathbf{a} \ \mathbf{b}] = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
- x. If $\mathbf{a} = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}, \mathbf{b} = b_1 \hat{\mathbf{i}} + b_2 \hat{\mathbf{j}} + b_3 \hat{\mathbf{k}}$
 and $\mathbf{c} = c_1 \hat{\mathbf{i}} + c_2 \hat{\mathbf{j}} + c_3 \hat{\mathbf{k}}$ are three vectors, then

$$[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Example 49. If $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}, \mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, then $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is equal to
 (a) -15 (b) 15 (c) -10 (d) -5

Sol. (c) $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & 2 \end{vmatrix}$
 $= 2(4 - 1) + 1(3 + 2) + 3(-1 - 6)$
 $= 6 + 5 - 21 = 11 - 21 = -10$

Example 50. If four points $A(3, 2, 1), B(4, x, 5), C(4, 2, -2)$ and $D(6, 5, -1)$ are coplanar, then the value of x is
 (a) 2 (b) 3 (c) 5 (d) 0

Sol. (c) We know that four points A, B, C and D are coplanar, if the vectors $\overrightarrow{AB}, \overrightarrow{AC}$ and $\overrightarrow{AD}$ are coplanar.
 Then, $\overrightarrow{AB} = \hat{\mathbf{i}} + (x - 2)\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
 $\overrightarrow{AC} = \hat{\mathbf{i}} + 0\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$
 and $\overrightarrow{AD} = 3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
 $\therefore [\overrightarrow{AB} \ \overrightarrow{AC} \ \overrightarrow{AD}] = \begin{vmatrix} 1 & x-2 & 4 \\ 1 & 0 & -3 \\ 3 & 3 & -2 \end{vmatrix}$
 $= 1(9) - (x - 2)(7) + 4(3)$
 $= 9 - 7x + 14 + 12$
 $= 35 - 7x$
 Now, $[\overrightarrow{AB} \ \overrightarrow{AC} \ \overrightarrow{AD}] = 0$
 $\Rightarrow 35 - 7x = 0$
 $\Rightarrow x = 5$

Example 51. $[\mathbf{a} + \mathbf{b} \ \mathbf{b} + \mathbf{c} \ \mathbf{c} + \mathbf{a}]$ is equal to
 (a) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (b) $3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
 (c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (d) None of these

Sol. (a) $[\mathbf{a} + \mathbf{b} \ \mathbf{b} + \mathbf{c} \ \mathbf{c} + \mathbf{a}] = (\mathbf{a} + \mathbf{b}) \cdot ((\mathbf{b} + \mathbf{c}) \times (\mathbf{c} + \mathbf{a}))$
 $= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{c} + \mathbf{c} \times \mathbf{a})$
 $= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} \times \mathbf{c} + \mathbf{b} \times \mathbf{a} + \mathbf{c} \times \mathbf{a})$ $[\because \mathbf{c} \times \mathbf{c} = 0]$
 $= \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{a} \cdot (\mathbf{c} \times \mathbf{a}) + \mathbf{b} \cdot (\mathbf{b} \times \mathbf{c})$
 $\quad + \mathbf{b} \cdot (\mathbf{b} \times \mathbf{a}) + \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$
 $= [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{a}] + [\mathbf{a} \ \mathbf{c} \ \mathbf{a}] + [\mathbf{b} \ \mathbf{b} \ \mathbf{c}]$
 $\quad + [\mathbf{b} \ \mathbf{b} \ \mathbf{a}] + [\mathbf{b} \ \mathbf{c} \ \mathbf{a}]$
 $= 2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{a}] + [\mathbf{a} \ \mathbf{c} \ \mathbf{a}] + [\mathbf{b} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{b} \ \mathbf{b} \ \mathbf{a}]$
 $= 2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$

Example 52. If $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}, \mathbf{b} = \hat{\mathbf{j}} - \hat{\mathbf{k}}, \mathbf{c} = \hat{\mathbf{k}} - \hat{\mathbf{i}}$ and $\mathbf{d}$ is a unit vector such that $\mathbf{a} \cdot \mathbf{d} = 0 = [\mathbf{b} \ \mathbf{c} \ \mathbf{d}]$, then $\mathbf{d}$ is/are
 (a) $\pm \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}}{\sqrt{3}}$ (b) $\pm \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}}{\sqrt{6}}$
 (c) $\pm \frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$ (d) $\pm \hat{\mathbf{k}}$

Sol. (b) Let $\mathbf{d} = d_1 \hat{\mathbf{i}} + d_2 \hat{\mathbf{j}} + d_3 \hat{\mathbf{k}}$
 $\mathbf{a} \cdot \mathbf{d} = (\hat{\mathbf{i}} - \hat{\mathbf{j}}) \cdot (d_1 \hat{\mathbf{i}} + d_2 \hat{\mathbf{j}} + d_3 \hat{\mathbf{k}})$

$$\begin{aligned} \Rightarrow d_1 - d_2 &= 0 & [\because a \cdot d = 0] \\ \Rightarrow d_1 &= d_2 & \dots(i) \\ \text{Also, } d \text{ is a unit vector.} \\ \Rightarrow d_1^2 + d_2^2 + d_3^2 &= 1 & \dots(ii) \\ \text{Now, } [b \ c \ d] &= 0 \\ \Rightarrow \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ d_1 & d_2 & d_3 \end{vmatrix} &= 0 \\ \Rightarrow -1(-d_3 - d_1) - 1(-d_2) &= 0 \\ \Rightarrow d_1 + d_2 + d_3 &= 0 \\ \Rightarrow 2d_1 + d_3 &= 0 & [\text{from Eq. (i)}] \\ \Rightarrow d_3 &= -2d_1 & \dots(iii) \end{aligned}$$

Using Eqs. (iii) and (i) is Eq. (ii), we get

$$d_1^2 + d_1^2 + 4d_1^2 = 1 \Rightarrow d_1 = \pm \frac{1}{\sqrt{6}}$$

$$\therefore d_2 = \pm \frac{1}{\sqrt{6}} \text{ and } d_3 = \pm \frac{2}{\sqrt{6}}$$

Hence, the required vector is $\pm \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} - 2\hat{k})$.

- **Example 53.** If vectors $\mathbf{x} = 3\hat{i} - 6\hat{j} - \hat{k}$, $\mathbf{y} = \hat{i} + 4\hat{j} - 3\hat{k}$ and $\mathbf{z} = 3\hat{i} - 4\hat{j} - 12\hat{k}$, then the projection of $\mathbf{x} \times \mathbf{y}$ on vector $\mathbf{z}$ is
- (a) 14 (b) -14 (c) 12 (d) 15

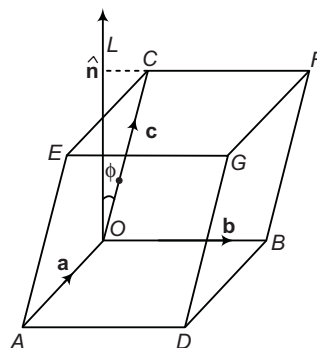
Sol. (b) The projection of $\mathbf{x} \times \mathbf{y}$ on $\mathbf{z}$ is given by

$$\begin{aligned} \frac{(\mathbf{x} \times \mathbf{y}) \cdot \mathbf{z}}{|\mathbf{z}|} &= \frac{1}{|\mathbf{z}|} [\mathbf{x} \ \mathbf{y} \ \mathbf{z}] \\ &= \frac{1}{13} \begin{vmatrix} 3 & -6 & -1 \\ 1 & 4 & -3 \\ 3 & -4 & -12 \end{vmatrix} = -14 \end{aligned}$$

➤ **Example 54.** For non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| = |\mathbf{a}| |\mathbf{b}| |\mathbf{c}|$ holds, if

- (a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
 (b) $\mathbf{a} \cdot \mathbf{b} = 0 = \mathbf{b} \cdot \mathbf{c}$
 (c) $\mathbf{b} \cdot \mathbf{c} = 0 = \mathbf{c} \cdot \mathbf{a}$
 (d) $\mathbf{c} \cdot \mathbf{a} = 0 = \mathbf{a} \cdot \mathbf{b}$

Sol. (a) Since, $|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|$ is the volume (V) of the parallelepiped whose adjacent edges are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.
 i.e. $V = |\mathbf{a} \times \mathbf{b}| (OL)$



$$\begin{aligned} \Rightarrow V &= |\mathbf{a}| |\mathbf{b}| (\sin \theta) (\cos \phi) |\mathbf{c}| \\ \text{where, } 0 \leq \theta \leq \pi &\text{ is the angle between } \mathbf{a} \text{ and } \mathbf{b}. \\ \text{Now, } |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}| &= |\mathbf{a}| |\mathbf{b}| |\mathbf{c}| \\ \Rightarrow |\sin \theta \cos \phi| &= 1 \\ \Rightarrow \sin \theta &= 1, \cos \phi = 1 \\ \therefore \theta &= \frac{\pi}{2}, \phi = 0 \\ \text{Now, } \mathbf{a} \cdot \mathbf{b} &= 0 = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} \end{aligned}$$

Work Book Exercise 13.4

- If $\mathbf{r} \cdot \mathbf{a} = 0$, $\mathbf{r} \cdot \mathbf{b} = 0$ and $\mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then the value of $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ is
 a 0 b $\frac{1}{2}$
 c 1 d 2
- If the vectors $2\hat{i} - 3\hat{j} + 4\hat{k}$, $\hat{i} + 2\hat{j} - \hat{k}$ and $m\hat{i} - \hat{j} + 2\hat{k}$ are coplanar, then the value of m is
 a $\frac{5}{8}$ b $\frac{8}{5}$
 c $-\frac{7}{4}$ d $\frac{2}{3}$
- If $\mathbf{a} \cdot \hat{i} = 4$, then $(\mathbf{a} \times \hat{j}) \cdot (2\hat{j} - 3\hat{k})$ is equal to
 a 12 b 2
 c 0 d -12
- If the vectors $\hat{i} - 2\hat{j} + 3\hat{k}$, $-2\hat{i} + 3\hat{j} - 4\hat{k}$ and $\lambda\hat{i} - \hat{j} + 2\hat{k}$ are linearly dependent, then the value of λ is
 a 0
 b 1
 c 2
 d 3
- If the volume of the parallelepiped with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as coterminal edges is 40 cu units, then the volume of the parallelepiped having $\mathbf{b} + \mathbf{c}$, $\mathbf{c} + \mathbf{a}$ and $\mathbf{a} + \mathbf{b}$ as coterminal edges (in cu units), is
 a 80 b 120 c 160 d 40
- Volume of the parallelepiped having vertices at $O \equiv (0, 0, 0)$, $A \equiv (2, -2, 1)$ and $C \equiv (1, -2, 4)$ is
 a 5 cu units b 10 cu units
 c 15 cu units d 20 cu units
- If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors such that $[\mathbf{a} \ \mathbf{b} \ \mathbf{a} \times \mathbf{b}] = \frac{1}{4}$, then the angle between $\mathbf{a}$ and $\mathbf{b}$ is
 a $\frac{\pi}{3}$ b $\frac{\pi}{4}$ c $\frac{\pi}{6}$ d $\frac{\pi}{2}$
- If the vectors $\alpha\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + \beta\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + \gamma\hat{k}$ (where, $\alpha, \beta, \gamma \neq 1$) are coplanar, then the value of $\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma}$ is
 a -1 b 0
 c 1 d $\frac{1}{2}$

Vector Triple Product

Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be any three vectors, then the vectors $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ and $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$ are called vector triple products of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Theorem For any three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, we have

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

Proof Let $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$. Since, cross product of two vectors is a vector perpendicular to both the vectors.

Therefore, $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$
 $\Rightarrow \mathbf{r} \perp \mathbf{a} \text{ and } \mathbf{r} \perp \mathbf{b} \times \mathbf{c} \quad \dots(i)$

But $(\mathbf{b} \times \mathbf{c})$ is a vector perpendicular to the plane of $\mathbf{b}$ and $\mathbf{c}$. Therefore, $\mathbf{r} \perp \mathbf{b} \times \mathbf{c}$.

$\Rightarrow \mathbf{r}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

$\Rightarrow \mathbf{r}$ is expressible as a linear combination of $\mathbf{b}$ and $\mathbf{c}$.

$\Rightarrow$ There exist scalars x, y such that

$$\mathbf{r} = x\mathbf{b} + y\mathbf{c} \quad \dots(ii)$$

Now, $\mathbf{r} \perp \mathbf{a}$

$$\Rightarrow \mathbf{r} \cdot \mathbf{a} = 0$$

$$\Rightarrow (x\mathbf{b} + y\mathbf{c}) \cdot \mathbf{a} = 0 \quad [\because \mathbf{r} = x\mathbf{b} + y\mathbf{c}]$$

$$\Rightarrow x(\mathbf{b} \cdot \mathbf{a}) + y(\mathbf{c} \cdot \mathbf{a}) = 0$$

$$\Rightarrow x(\mathbf{a} \cdot \mathbf{b}) = -y(\mathbf{a} \cdot \mathbf{c})$$

$$\Rightarrow \frac{x}{(\mathbf{a} \cdot \mathbf{c})} = \frac{-y}{(\mathbf{a} \cdot \mathbf{b})} = \lambda \quad [\text{say}]$$

$$\Rightarrow x = \lambda(\mathbf{a} \cdot \mathbf{c}) \text{ and } y = -\lambda(\mathbf{a} \cdot \mathbf{b})$$

On substituting the values of x, y in Eq. (ii), we get

$$\mathbf{r} = \lambda(\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - \lambda(\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$\Rightarrow \mathbf{r} = \lambda[(\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}]$$

$$\Rightarrow \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \lambda[(\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}] \quad \dots(iii)$$

This is a vector identity, so it is true for all $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

Taking $\mathbf{a} = \hat{\mathbf{i}}, \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, we have

$$\begin{aligned} \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \hat{\mathbf{i}} \times \{(\hat{\mathbf{i}} + \hat{\mathbf{j}}) \times (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\} \\ &= \hat{\mathbf{i}} \times \{\hat{\mathbf{i}} \times \hat{\mathbf{i}} + \hat{\mathbf{i}} \times \hat{\mathbf{j}} + \hat{\mathbf{i}} \times \hat{\mathbf{k}} + \hat{\mathbf{j}} \times \hat{\mathbf{i}} + \hat{\mathbf{j}} \times \hat{\mathbf{j}} + \hat{\mathbf{j}} \times \hat{\mathbf{k}}\} \\ &= \hat{\mathbf{i}} \times \{\hat{\mathbf{k}} - \hat{\mathbf{j}} - \hat{\mathbf{k}} + \hat{\mathbf{i}}\} = \hat{\mathbf{i}} \times (\hat{\mathbf{i}} - \hat{\mathbf{j}}) = -\hat{\mathbf{i}} + \hat{\mathbf{j}} = -\hat{\mathbf{k}} \end{aligned}$$

and $(\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$

$$\begin{aligned} &= \{\hat{\mathbf{i}} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})\}(\hat{\mathbf{i}} + \hat{\mathbf{j}}) - \{\hat{\mathbf{i}} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}})\}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \\ &= (\hat{\mathbf{i}} + \hat{\mathbf{j}}) - (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = -\hat{\mathbf{k}} \end{aligned}$$

On substituting these values in Eq. (iii), we get

$$-\hat{\mathbf{k}} = \lambda[-\hat{\mathbf{k}}]$$

$$\Rightarrow \lambda = 1$$

On putting $\lambda = 1$ in Eq. (iii), we get

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

Add to Memory

$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$ can also be written as $\mathbf{I} \times (\mathbf{II} \times \mathbf{III}) = (\mathbf{I} \cdot \mathbf{III}) \mathbf{II} - (\mathbf{I} \cdot \mathbf{II}) \mathbf{III}$

- The vector triple product $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a linear combination of those two vectors which are within brackets.
- The vectors $\mathbf{r} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is perpendicular to $\mathbf{a}$ and lies in the plane of $\mathbf{b}$ and $\mathbf{c}$.

Example 55. Given two vectors $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, the unit vector coplanar with the two vectors and perpendicular to first, is

- (a) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (b) $\frac{1}{\sqrt{5}}(2\hat{\mathbf{i}} + \hat{\mathbf{j}})$
 (c) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{k}})$ (d) None of these

Sol. (a) Let $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}}$. Since, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c})$ is a vector coplanar with $\mathbf{b}$ and $\mathbf{c}$ and perpendicular to $\mathbf{a}$. So, the required vector is along the vector

$$\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{b}) \mathbf{a} - (\mathbf{a} \cdot \mathbf{a}) \mathbf{b}$$

$$= -(\hat{\mathbf{i}} - \hat{\mathbf{j}}) - 2(\hat{\mathbf{i}} + 2\hat{\mathbf{j}}) = -3\hat{\mathbf{i}} - 3\hat{\mathbf{j}}$$

Hence, the required vectors are given by

$$\pm \frac{(-3\hat{\mathbf{i}} - 3\hat{\mathbf{j}})}{\sqrt{(-3)^2 + (-3)^2}} = \pm \frac{1}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$$

Example 56. The unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are perpendicular and unit vector $\hat{\mathbf{c}}$ is inclined at an angle θ to both $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$. If $\hat{\mathbf{c}} = \alpha\hat{\mathbf{a}} + \beta\hat{\mathbf{b}} + \gamma(\hat{\mathbf{a}} \times \hat{\mathbf{b}})$, then

- (a) $\alpha = \beta$ (b) $\gamma^2 = 1 - 2\alpha^2$
 (c) $\gamma^2 = -\cos 2\theta$ (d) $\beta^2 = \frac{1 + \cos 2\theta}{2}$

Sol. (a,b,c,d) Since, $\hat{\mathbf{a}}, \hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors inclined at an angle θ .

$$\text{Therefore, } |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = |\hat{\mathbf{c}}| = 1$$

$$\text{and } \cos \theta = \hat{\mathbf{a}} \cdot \hat{\mathbf{c}} = \hat{\mathbf{b}} \cdot \hat{\mathbf{c}}$$

$$\text{Now, } \hat{\mathbf{c}} = \alpha\hat{\mathbf{a}} + \beta\hat{\mathbf{b}} + \gamma(\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \quad \dots(i)$$

$$\Rightarrow \hat{\mathbf{a}} \cdot \hat{\mathbf{c}} = \alpha(\hat{\mathbf{a}} \cdot \hat{\mathbf{a}}) + \beta(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) + \gamma\{\hat{\mathbf{a}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}})\}$$

$$\Rightarrow \cos \theta = \alpha |\hat{\mathbf{a}}|^2 \quad [\because \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = 0, \hat{\mathbf{a}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) = 0]$$

$$\Rightarrow \cos \theta = \alpha$$

Similarly, by taking dot product on both sides of Eq. (i) by $\hat{\mathbf{b}}$ we get $\beta = \cos \theta$.

$$\therefore \alpha = \beta$$

$$\text{Again, } \hat{\mathbf{c}} = \alpha\hat{\mathbf{a}} + \beta\hat{\mathbf{b}} + \gamma(\hat{\mathbf{a}} \times \hat{\mathbf{b}})$$

$$|\hat{\mathbf{c}}|^2 = \alpha^2 |\hat{\mathbf{a}}|^2 + \beta^2 |\hat{\mathbf{b}}|^2 + \gamma^2 |\hat{\mathbf{a}} \times \hat{\mathbf{b}}|^2 + 2\alpha\beta(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) + 2\alpha\gamma\{\hat{\mathbf{a}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}})\} + 2\beta\gamma\{\hat{\mathbf{b}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}})\}$$

$$\Rightarrow 1 = \alpha^2 + \beta^2 + \gamma^2 |\hat{\mathbf{a}} \times \hat{\mathbf{b}}|^2$$

$$\Rightarrow 1 = 2\alpha^2 + \gamma^2 \left\{ |\hat{\mathbf{a}}|^2 |\hat{\mathbf{b}}|^2 \sin^2 \frac{\pi}{2} \right\}$$

$$\Rightarrow 1 = 2\alpha^2 + \gamma^2$$

$$\Rightarrow \alpha^2 = \frac{1 - \gamma^2}{2}$$

But $\alpha = \beta = \cos \theta$. Therefore,

$$\begin{aligned} 1 &= 2\alpha^2 + \gamma^2 \\ \Rightarrow \gamma^2 &= 1 - 2\cos^2 \theta = -2\cos \theta \\ \therefore \alpha^2 &= \beta^2 = \frac{1 - \gamma^2}{2} = \frac{1 + \cos 2\theta}{2} \end{aligned}$$

➤ **Example 57.** Let $\hat{a}$ be a unit vector and $\mathbf{b}$ a non-zero vector not parallel to $\hat{a}$. The angles of the triangle, two of whose sides are represented by $\sqrt{3}(\hat{a} \times \mathbf{b})$ and $\mathbf{b} - (\hat{a} \cdot \mathbf{b})\hat{a}$, are

- (a) $\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}$ (b) $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$
(c) $\frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}$ (d) None of these

Sol. (c) We have, $\hat{a} \times (\mathbf{b} \times \hat{a}) = (\hat{a} \cdot \hat{a})\mathbf{b} - (\hat{a} \cdot \mathbf{b})\hat{a}$
 $= \mathbf{b} - (\hat{a} \cdot \mathbf{b})\hat{a}$ [$\because \hat{a} \cdot \hat{a} = |\hat{a}|^2 = 1$]

Thus, two sides of the triangle are represented by the vectors $\sqrt{3}(\hat{a} \times \mathbf{b})$ and $\hat{a} \times (\mathbf{b} \times \hat{a})$.

Since, $\hat{a} \times (\mathbf{b} \times \hat{a})$ is perpendicular to $\hat{a}$ and $\mathbf{b} \times \hat{a}$.

Therefore, angle between the sides represented by the given vectors is a right angle.

Now, $|\mathbf{b} - (\hat{a} \cdot \mathbf{b})\hat{a}| = |\hat{a} \times (\mathbf{b} \times \hat{a})|$
 $= |\hat{a}| |\mathbf{b} \times \hat{a}| \sin \frac{\pi}{2}$ [$\because \hat{a} \perp (\mathbf{b} \times \hat{a})$]
 $= |\mathbf{b} \times \hat{a}| = |\hat{a} \times \mathbf{b}|$

Thus, the lengths of the sides containing the right angle of the triangle are in ratio $1 : \sqrt{3}$. So, the other two angles are $\frac{\pi}{6}$ and $\frac{\pi}{3}$.

➤ **Example 58.** Let $\mathbf{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\mathbf{b} = \hat{i} + \hat{j}$. If $\mathbf{c}$ is a vector such that $\mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|$, $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ is 30° , then $|\mathbf{a} \times \mathbf{b} \times \mathbf{c}|$ is equal to

- (a) $\frac{2}{3}$ (b) $\frac{3}{2}$
(c) 2 (d) 3

Sol. (b) We have, $\mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|$ and $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$

$$\Rightarrow \mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|$$

$$\text{and } |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2(\mathbf{a} \cdot \mathbf{c}) = 8$$

$$\Rightarrow |\mathbf{c}|^2 + 9 - 2|\mathbf{c}| = 8$$

$$\Rightarrow (|\mathbf{c}| - 1)^2 = 0$$

$$\Rightarrow |\mathbf{c}| = 1$$

$$\begin{aligned} \therefore |\mathbf{a} \times \mathbf{b} \times \mathbf{c}| &= |\mathbf{a} \times \mathbf{b}| |\mathbf{c}| \sin 30^\circ \\ &= \frac{1}{2} |\mathbf{a} \times \mathbf{b}| = \frac{3}{2} \end{aligned}$$

[$\because \mathbf{a} \times \mathbf{b} = 2\hat{j} - 2\hat{k} + \hat{k}$]

➤ **Example 59.** Let $\hat{a}$ and $\hat{b}$ be two non-collinear unit vectors. If $\mathbf{u} = \hat{a} - (\hat{a} \cdot \hat{b})\hat{b}$ and $\mathbf{v} = \hat{a} \times \hat{b}$, then $|\mathbf{v}|$ is equal to

- (a) $|\mathbf{u}|$ (b) $|\mathbf{u}| + |\mathbf{u} \cdot \hat{a}|$
(c) $|\mathbf{u}| + |\mathbf{u} \cdot \hat{b}|$ (d) $|\mathbf{u}| + \mathbf{u} \cdot (\hat{a} + \hat{b})$

Sol. (a,c) We have, $|\hat{a}| = |\hat{b}| = 1$

$$\text{Now, } \mathbf{u} = \hat{a} - (\hat{a} \cdot \hat{b})\hat{b}$$

$$\Rightarrow \mathbf{u} = (\hat{b} \cdot \hat{b})\hat{a} - (\hat{b} \cdot \hat{a})\hat{b}$$

$$\Rightarrow \mathbf{u} = \hat{b} \times (\hat{a} \times \hat{b}) \quad \dots (i)$$

$$\Rightarrow \mathbf{u} = \hat{b} \times \mathbf{v}$$

$$\Rightarrow |\mathbf{u}| = |\hat{b}| |\mathbf{v}| \sin \frac{\pi}{2} \quad [\because \mathbf{v} = \hat{a} \times \hat{b} \Rightarrow \mathbf{v} \perp \hat{b}]$$

$$\Rightarrow |\mathbf{u}| = |\mathbf{v}| \quad [\because |\hat{b}| = 1] \quad \dots (ii)$$

$$\text{Again, } \mathbf{u} = \hat{b} \times (\hat{a} \times \hat{b}) \quad [\text{using Eq. (i)}]$$

$\therefore \mathbf{u}$ is perpendicular to $\hat{b}$.

$$\Rightarrow \mathbf{u} \cdot \hat{b} = 0$$

$$\therefore |\mathbf{u}| + |\mathbf{u} \cdot \hat{b}| = |\mathbf{u}|$$

$$\Rightarrow |\mathbf{u}| + |\mathbf{u} \cdot \hat{b}| = |\mathbf{v}| \quad [\text{using Eq. (ii)}]$$

➤ **Example 60.** Let $\mathbf{a} = 2\hat{i} + \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} + 2\hat{j} - \hat{k}$ and a unit vector $\hat{c}$ be coplanar. If $\hat{c}$ is perpendicular to $\mathbf{a}$, then $\hat{c}$ is equal to

- (a) $\frac{1}{\sqrt{2}}(-\hat{j} + \hat{k})$ (b) $\frac{1}{\sqrt{3}}(-\hat{i} - \hat{j} - \hat{k})$
(c) $\frac{1}{\sqrt{5}}(\hat{i} - 2\hat{j})$ (d) $\frac{1}{\sqrt{3}}(\hat{i} - \hat{j} - \hat{k})$

Sol. (a) Required unit vector $= \frac{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})|}$

$$\begin{aligned} \text{Now, } \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) &= (\mathbf{a} \cdot \mathbf{b})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{b} \\ &= 3(2\hat{i} + \hat{j} + \hat{k}) - 6(\hat{i} + 2\hat{j} - \hat{k}) \\ &= -9\hat{j} + 9\hat{k} \end{aligned}$$

$$\text{So, the required unit vector } \hat{c} \text{ is } \pm \frac{1}{\sqrt{2}}(-\hat{j} + \hat{k}).$$

➤ **Example 61.** The vector $\mathbf{a}$ coplanar with the vectors $\hat{i}$ and $\hat{j}$, perpendicular to the vector $\mathbf{b} = 4\hat{i} - 3\hat{j} + 5\hat{k}$ such that $|\mathbf{a}| = |\mathbf{b}|$, is

- (a) $\sqrt{2}(3\hat{i} + 4\hat{j})$ or $-\sqrt{2}(3\hat{i} + 4\hat{j})$
(b) $\sqrt{2}(4\hat{i} + 3\hat{j})$ or $-\sqrt{2}(4\hat{i} + 3\hat{j})$
(c) $\sqrt{3}(4\hat{i} + 5\hat{j})$ or $-\sqrt{3}(4\hat{i} + 5\hat{j})$
(d) $\sqrt{3}(5\hat{i} + 4\hat{j})$ or $-\sqrt{3}(5\hat{i} + 4\hat{j})$

Sol. (a) We have, $\mathbf{a} = \lambda[\mathbf{b} \times (\hat{i} \times \hat{j})]$

$$= \lambda[(\mathbf{b} \cdot \hat{j})\hat{i} - (\mathbf{b} \cdot \hat{i})\hat{j}] = \lambda[-3\hat{i} - 4\hat{j}]$$

$$\text{Now, } |\mathbf{a}| = |\mathbf{b}| \Rightarrow 25\lambda^2 = 16 + 9 + 25$$

$$\text{Hence, } \mathbf{a} = \pm \sqrt{2}(3\hat{i} + 4\hat{j})$$

Reciprocal System of Vectors

Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be three non-coplanar vectors, so that $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \neq 0$.

We define another set of three vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ as given below

$$\mathbf{a} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}, \mathbf{b} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}, \mathbf{c} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

These three vectors are said to form a reciprocal system of vectors for the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

Properties of Reciprocal System of Vectors

- i. Scalar product of any vector of one system with the vector of other system is zero.
i.e. $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$
- ii. $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}][\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}'] = 1$
- iii. $\hat{\mathbf{i}} = \hat{\mathbf{i}}', \hat{\mathbf{j}} = \hat{\mathbf{j}}', \hat{\mathbf{k}} = \hat{\mathbf{k}}'$
- iv. Let $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ be the reciprocal system of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{r}$ be any vector, then

$$\mathbf{r} = (\mathbf{r} \cdot \mathbf{a}')\mathbf{a} + (\mathbf{r} \cdot \mathbf{b}')\mathbf{b} + (\mathbf{r} \cdot \mathbf{c}')\mathbf{c}'$$

$$= (\mathbf{r} \cdot \mathbf{a})\mathbf{a}' + (\mathbf{r} \cdot \mathbf{b})\mathbf{b}' + (\mathbf{r} \cdot \mathbf{c})\mathbf{c}'$$
- v. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar, then $\mathbf{a}', \mathbf{b}', \mathbf{c}'$ are the reciprocal system.
- vi. $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{a}' + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{b}' + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{c}' = 3$

Example 62. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be the vectors defined by the relations $\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$, $\mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$,

$\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$. Then, the value of $[\mathbf{p} \ \mathbf{q} \ \mathbf{r}]$ is

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
- (b) $\frac{1}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$
- (c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2$
- (d) None of the above

Sol. (b) $[\mathbf{p} \ \mathbf{q} \ \mathbf{r}] = \frac{[\mathbf{a} \times \mathbf{b} \ \mathbf{b} \times \mathbf{c} \ \mathbf{c} \times \mathbf{a}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^3}$

$$= \frac{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^3} = \frac{1}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

Example 63. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-coplanar vectors and $\mathbf{p}, \mathbf{q}, \mathbf{r}$ be the vectors defined by the relations $\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$, $\mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$, $\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$.

Then, $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$ is equal to

- (a) 0
- (b) 1
- (c) 2
- (d) 3

Sol. (d) We have, $\mathbf{a} \cdot \mathbf{p} = \mathbf{a} \cdot \frac{(\mathbf{b} \times \mathbf{c})}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} = \frac{1}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 1$

and $\mathbf{a} \cdot \mathbf{q} = \mathbf{a} \cdot \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} = \frac{1}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} \mathbf{a} \cdot (\mathbf{c} \times \mathbf{a}) = 0$

Similarly, $\mathbf{b} \cdot \mathbf{q} = \mathbf{c} \cdot \mathbf{r} = 1$

Also, $\mathbf{a} \cdot \mathbf{r} = \mathbf{b} \cdot \mathbf{p} = \mathbf{c} \cdot \mathbf{q} = 0$

$\therefore (\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$

$$= \mathbf{a} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{p} + \mathbf{b} \cdot \mathbf{q} + \mathbf{c} \cdot \mathbf{q} + \mathbf{c} \cdot \mathbf{r} + \mathbf{a} \cdot \mathbf{r}$$

$$= 1 + 1 + 1 = 3$$

Solutions of Vector Equations

Solving a vector equation means determining an unknown vector or a number of vectors satisfying the given conditions.

Generally, to solve a vector equation, we express the unknown vector as a linear combination of three known non-coplanar vectors and then we determine the coefficients from the given conditions.

Example 64. If $\mathbf{r}$ satisfies the equation $\mathbf{r} \times (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) = \hat{\mathbf{i}} - \hat{\mathbf{k}}$, then for any scalar m , $\mathbf{r}$ is equal to

- (a) $\hat{\mathbf{i}} + m(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$
- (b) $\hat{\mathbf{j}} + m(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$
- (c) $\hat{\mathbf{k}} + m(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$
- (d) $\hat{\mathbf{i}} - \hat{\mathbf{k}} + m(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}})$

Sol. (b) Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, then

$$\begin{aligned} \mathbf{r} \times (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) &= \hat{\mathbf{i}} - \hat{\mathbf{k}} \\ \Rightarrow (y - 2z)\hat{\mathbf{i}} + (z - x)\hat{\mathbf{j}} + (2x - y)\hat{\mathbf{k}} &= \hat{\mathbf{i}} - \hat{\mathbf{k}} \\ \Rightarrow y - 2z &= 1, z - x = 0 \\ \text{and } 2x - y &= -1 \\ \Rightarrow \frac{x}{1} = \frac{z}{1} = \frac{y-1}{2} &= m \quad [\text{say}] \\ \Rightarrow x = m, z = m \text{ and } y &= 2m + 1 \\ \therefore \mathbf{r} &= \hat{\mathbf{j}} + m(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}) \end{aligned}$$

Example 65. Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$. A vector $\mathbf{r}$ satisfying $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$ and $\mathbf{r} \cdot \mathbf{a} = 0$ is

- (a) $-2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
- (b) $-2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$
- (c) $-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$
- (d) $\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

Sol. (d) We have, $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$

$$\begin{aligned} \Rightarrow (\mathbf{r} - \mathbf{c}) \times \mathbf{b} &= 0 \\ \Rightarrow \mathbf{r} - \mathbf{c} &= \lambda \mathbf{b} \\ \Rightarrow \mathbf{r} &= \mathbf{c} + \lambda \mathbf{b} \\ \text{Now, } \mathbf{r} \cdot \mathbf{a} &= 0 \\ \Rightarrow (\mathbf{c} + \lambda \mathbf{b}) \cdot \mathbf{a} &= 0 \\ \Rightarrow \lambda = \frac{\mathbf{c} \cdot \mathbf{a}}{\mathbf{b} \cdot \mathbf{a}} &= -1 \\ \therefore \mathbf{r} &= \mathbf{c} - \mathbf{b} = \hat{\mathbf{i}} - 5\hat{\mathbf{j}} + 3\hat{\mathbf{k}} \end{aligned}$$

Example 66. If $\mathbf{a}$ and $\mathbf{b}$ are not perpendicular to each other and $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$, $\mathbf{r} \cdot \mathbf{c} = 0$, then $\mathbf{r}$ is equal to

- (a) $\mathbf{a} - \mathbf{c}$
- (b) $\mathbf{b} + x\mathbf{a}$ for all scalars x
- (c) $\mathbf{b} - \frac{(\mathbf{b} \cdot \mathbf{c})}{(\mathbf{a} \cdot \mathbf{c})} \mathbf{a}$
- (d) None of the above

Sol. (c) We have, $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$
 $\Rightarrow (\mathbf{r} - \mathbf{b}) \times \mathbf{a} = \mathbf{0}$
 $\Rightarrow \mathbf{r} - \mathbf{b}$ is parallel to $\mathbf{a}$.
 $\Rightarrow \mathbf{r} - \mathbf{b} = \lambda \mathbf{a}$... (i)
 Now, $\mathbf{r} \cdot \mathbf{c} = 0$
 $\Rightarrow (\mathbf{b} + \lambda \mathbf{a}) \cdot \mathbf{c} = 0$
 $\Rightarrow \mathbf{b} \cdot \mathbf{c} + \lambda (\mathbf{a} \cdot \mathbf{c}) = 0$
 $\Rightarrow \lambda = -\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{c}}$
 On putting the value of λ in Eq. (i), we get
 $\mathbf{r} = \mathbf{b} - \frac{(\mathbf{b} \cdot \mathbf{c})}{(\mathbf{a} \cdot \mathbf{c})} \mathbf{a}$

➤ **Example 67.** Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} - \hat{\mathbf{k}}$, then point of intersection of the lines $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$ and $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$

- (a) $-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$
 (b) $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$
 (c) $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$
 (d) $\hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$

Sol. (c) We have, $\mathbf{r} \times \mathbf{a} = \mathbf{b} \times \mathbf{a}$
 $\Rightarrow (\mathbf{r} - \mathbf{b}) \times \mathbf{a} = \mathbf{0}$
 $\Rightarrow \mathbf{r} - \mathbf{b} \parallel \mathbf{a}$
 $\Rightarrow \mathbf{r} - \mathbf{b} = \lambda \mathbf{a}$
 $\Rightarrow \mathbf{r} = \mathbf{b} + \lambda \mathbf{a}$
 Similarly, the equation of the line $\mathbf{r} \times \mathbf{b} = \mathbf{a} \times \mathbf{b}$ can be written as $\mathbf{r} = \mathbf{a} + \mu \mathbf{b}$.
 For the point of intersection of above two lines, we have
 $\mathbf{a} + \mu \mathbf{b} = \mathbf{b} + \lambda \mathbf{a} \Rightarrow \lambda = \mu = 1$
 Hence, the position vector of the point of intersection of the given lines are $\mathbf{r} = \mathbf{a} + \mathbf{b} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$.

Tetrahedron

A tetrahedron is a three dimensional figure formed by four triangles. In the following figure,

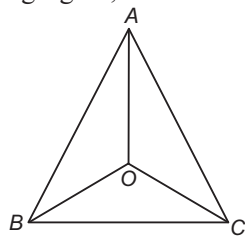
$OABC \rightarrow$ tetrahedron

$\triangle ABC \rightarrow$ base

$OAB, OBC, OCA \rightarrow$ faces

OA, OB, OC, AB, BC and $CA \rightarrow$ edges

OA, BC, OB, CA, OC and $AB \rightarrow$ pair of opposite edges.



Properties of Tetrahedron

- i. A tetrahedron in which all edges are equal, is called a regular tetrahedron.
- ii. If two pairs of opposite edges of a tetrahedron are perpendicular, then the opposite edges of the third pair are also perpendicular to each other.

- iii. The sum of the squares of two opposite edges is the same for each pair of opposite edges.
- iv. Any two opposite edges in a regular tetrahedron are perpendicular.
- v. Volume of a tetrahedron $ABCD$ is $\frac{1}{6} |[\mathbf{a} - \mathbf{d} \quad \mathbf{b} - \mathbf{d} \quad \mathbf{c} - \mathbf{d}]|$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are position vectors.
- vi. Volume of a tetrahedron whose three coterminal edges in the right handed system are $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, is given by $\frac{1}{6} [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}]$.
- vii. Centroid of tetrahedron is $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$ are position vectors.

➤ **Example 68.** The volume of the tetrahedron whose vertices are the points with position vectors $\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 10\hat{\mathbf{k}}, -\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 7\hat{\mathbf{k}}, 5\hat{\mathbf{i}} - \hat{\mathbf{j}} + \lambda\hat{\mathbf{k}}$ and $7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, is 11 cu units, if the value of λ is

- (a) -1 (b) 1 (c) -7 (d) 7

Sol. (b, d) Let A, B, C and D be the points with the given position vectors. Then,

$$AB = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}, AC = 4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + (\lambda - 10)\hat{\mathbf{k}}$$

$$\text{and } AD = 6\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

$\therefore$ Volume of tetrahedron = 11 cu units

$$\Rightarrow \frac{1}{6} [AB \quad AC \quad AD] = \pm 11 \text{ cu units}$$

$$\Rightarrow \frac{1}{6} \begin{vmatrix} -2 & 3 & -3 \\ 4 & 5 & \lambda - 10 \\ 6 & 2 & -3 \end{vmatrix} = \pm 11$$

$$\Rightarrow -88 + 22\lambda = \pm 66$$

$$\Rightarrow \lambda = 7, 1$$

➤ **Example 69.** Consider a tetrahedron with faces F_1, F_2, F_3 and F_4 . Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ be the vectors whose magnitudes are respectively equal to areas of F_1, F_2, F_3, F_4 and whose directions are perpendicular to these faces in outward direction. Then, $|\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4|$ equals

- (a) 1 (b) 4
 (c) 0 (d) None of these

Sol. (c) We have,

$$\mathbf{v}_1 = \frac{1}{2} (\mathbf{a} \times \mathbf{b}), \mathbf{v}_2 = \frac{1}{2} (\mathbf{b} \times \mathbf{c}), \mathbf{v}_3 = \frac{1}{2} (\mathbf{c} \times \mathbf{a})$$

$$\text{and } \mathbf{v}_4 = \frac{1}{2} \{(\mathbf{c} - \mathbf{a}) \times (\mathbf{b} - \mathbf{a})\}$$

$$\therefore \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4$$

$$= \frac{1}{2} (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b} - \mathbf{c} \times \mathbf{a} - \mathbf{a} \times \mathbf{b}) = 0$$

$$\Rightarrow |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4| = 0$$

Work Book Exercise 13.5

13

Vector Algebra

- [illegible]

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Let two non-collinear unit vectors $\hat{a}$ and $\hat{b}$ form an acute angle. A point P moves so that at any time t the position vector $\mathbf{OP}$ (where, O is the origin) is given by $\hat{a} \cos t + \hat{b} \sin t$. When P is farthest from origin O , let M be the length of $\mathbf{OP}$ and $\hat{u}$ be the unit vector along $\mathbf{OP}$. Then,

- (a) $\hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}$ and $M = (1 + \hat{a} \cdot \hat{b})^{1/2}$
 (b) $\hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|}$ and $M = (1 + \hat{a} \cdot \hat{b})^{1/2}$
 (c) $\hat{u} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}$ and $M = (1 + 2\hat{a} \cdot \hat{b})^{1/2}$
 (d) $\hat{u} = \frac{\hat{a} - \hat{b}}{|\hat{a} - \hat{b}|}$ and $M = (1 + 2\hat{a} \cdot \hat{b})^{1/2}$

Sol. $\mathbf{OP} = \hat{a} \cos t + \hat{b} \sin t$

$$\begin{aligned} M^2 = |\mathbf{OP}|^2 &= (\hat{a} \cos t + \hat{b} \sin t) \cdot (\hat{a} \cos t + \hat{b} \sin t) \\ &= (\cos^2 t + \sin^2 t + 2 \cos t \sin t \hat{a} \cdot \hat{b}) \\ &= (1 + \sin 2t \hat{a} \cdot \hat{b}) \end{aligned}$$

$$|\mathbf{OP}|_{\max} = (1 + \hat{a} \cdot \hat{b})^{1/2} \quad \left[\because t = \frac{\pi}{4} \right]$$

$$\text{Now, } \hat{u} = \frac{\mathbf{OP}}{|\mathbf{OP}|} = \frac{\frac{\hat{a}}{\sqrt{2}} + \frac{\hat{b}}{\sqrt{2}}}{\frac{|\hat{a} + \hat{b}|}{\sqrt{2}}} = \frac{\hat{a} + \hat{b}}{|\hat{a} + \hat{b}|}$$

Hence, (a) is the correct answer.

Ex 2. If $\mathbf{r} \cdot \mathbf{a} = 0$, $\mathbf{r} \cdot \mathbf{b} = -1$ and $[\mathbf{r} \ \mathbf{a} \ \mathbf{b}] = 1$, $\mathbf{a} \cdot \mathbf{b} \neq 0$, $|\mathbf{a} \times \mathbf{b}| = 1$, then the value of $\mathbf{r}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ is

- (a) $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times \mathbf{b}$
 (b) $\mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}$
 (c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{a}) + \frac{\mathbf{b} \times \mathbf{a}}{|\mathbf{b} \times \mathbf{a}|^2}$
 (d) $\frac{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times (\mathbf{a} \times \mathbf{b})|} + \mathbf{a} \times \mathbf{b}$

Sol. Writing $\mathbf{r}$ as linear combination of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, we have

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$$

For scalars x , y and z ,

$$\begin{aligned} 0 &= \mathbf{r} \cdot \mathbf{a} = x|\mathbf{a}|^2 + y\mathbf{a} \cdot \mathbf{b} - 1 \\ &= \mathbf{r} \cdot \mathbf{b} = x\mathbf{a} \cdot \mathbf{b} + y|\mathbf{b}|^2 \end{aligned}$$

On solving, we get $x = \mathbf{a} \cdot \mathbf{b}$, $y = -|\mathbf{a}|^2$

$$\text{Since, } |\mathbf{a} \times \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2$$

$$\text{Also, } 1 = [\mathbf{r} \ \mathbf{a} \ \mathbf{b}] = z|\mathbf{a} \times \mathbf{b}|^2 \Rightarrow z = \frac{1}{|\mathbf{a} \times \mathbf{b}|^2}$$

$$\begin{aligned} \text{Thus, } \mathbf{r} &= ((\mathbf{a} \cdot \mathbf{b})\mathbf{a} - |\mathbf{a}|^2 \mathbf{b}) + \mathbf{a} \times \mathbf{b} \\ &= \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{a} \times \mathbf{b} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 3. If $\mathbf{a} \cdot \hat{\mathbf{i}} = \mathbf{a} \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}}) = \mathbf{a} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 1$, then $\mathbf{a}$ is equal to

- (a) $\hat{\mathbf{i}} - \hat{\mathbf{k}}$ (b) $\frac{(3\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})}{3}$
 (c) $\frac{(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})}{3}$ (d) $\frac{(3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})}{3}$

Sol. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$

$$\therefore \mathbf{a} \cdot \hat{\mathbf{i}} = 1 \Rightarrow a_1 = 1$$

$$\therefore \mathbf{a} \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}}) = 1 \Rightarrow 2a_1 + a_2 = 1$$

$$\Rightarrow a_2 = 1 - 2$$

$$\Rightarrow a_2 = -1$$

$$\text{and } \mathbf{a} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 1$$

$$\Rightarrow a_1 + a_2 + 3a_3 = 1$$

$$\Rightarrow 1 - 1 + 3a_3 = 1$$

$$\Rightarrow a_3 = \frac{1}{3}$$

$$\therefore \mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \frac{1}{3}\hat{\mathbf{k}} = \frac{1}{3}(3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})$$

Hence, (d) is the correct answer.

Ex 4. If $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are any two non-collinear unit vectors and $\mathbf{a}$ is any vector, then

$$(\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} + (\mathbf{a} \cdot \hat{\mathbf{c}})\hat{\mathbf{c}} + \frac{\mathbf{a} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{c}})}{|\hat{\mathbf{b}} \times \hat{\mathbf{c}}|} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) \quad \text{is}$$

equal to

- (a) 0 (b) $\mathbf{a}$ (c) $\hat{\mathbf{b}}$ (d) $\hat{\mathbf{c}}$

Sol. Let $\hat{\mathbf{b}} = \hat{\mathbf{i}}$ and $\hat{\mathbf{c}} = \hat{\mathbf{j}}$

$$\therefore |\hat{\mathbf{b}} \times \hat{\mathbf{c}}| = |\hat{\mathbf{k}}| = 1$$

$$\text{Let } \mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$$

$$\text{Now, } \mathbf{a} \cdot \hat{\mathbf{b}} = \mathbf{a} \cdot \hat{\mathbf{i}} = a_1,$$

$$\mathbf{a} \cdot \hat{\mathbf{c}} = \mathbf{a} \cdot \hat{\mathbf{j}} = a_2$$

$$\text{and } \mathbf{a} \cdot \frac{\hat{\mathbf{b}} \times \hat{\mathbf{c}}}{|\hat{\mathbf{b}} \times \hat{\mathbf{c}}|} = \mathbf{a} \cdot \hat{\mathbf{k}} = a_3$$

$$\therefore (\mathbf{a} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}} + (\mathbf{a} \cdot \hat{\mathbf{c}})\hat{\mathbf{c}} + \frac{\mathbf{a} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{c}})}{|\hat{\mathbf{b}} \times \hat{\mathbf{c}}|} \cdot (\hat{\mathbf{b}} \times \hat{\mathbf{c}})$$

$$= a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3(\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}} = \mathbf{a}$$

Hence, (b) is the correct answer.

Ex 5. The equation of the plane which passes through the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_1 = q_1$, $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ and is parallel to the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_3 = q_3$ and $\mathbf{r} \cdot \mathbf{n}_4 = q_4$ given that

$$[\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4] = [\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4], \text{ is}$$

- (a) $\mathbf{r} \cdot \mathbf{n}_2 - q_2 = \mathbf{r} \cdot \mathbf{n}_3 - q_3$
 (b) $\mathbf{r} \cdot \mathbf{n}_1 - q_1 = [\mathbf{n}_1 \ \mathbf{n}_2 \ \mathbf{n}_3](\mathbf{r} \cdot \mathbf{n}_2 - q_2)$
 (c) $\mathbf{r} \cdot \mathbf{n}_1 - q_1 = \mathbf{r} \cdot \mathbf{n}_2 - q_2$
 (d) $\mathbf{r} \cdot \mathbf{n}_3 - q_3 = \mathbf{r} \cdot \mathbf{n}_1 - q_1$

Sol. Equation of any plane through the intersection of $\mathbf{r} \cdot \mathbf{n}_1 = q_1$ $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ is of the form

$$\mathbf{r} \cdot \mathbf{n}_1 + \lambda \mathbf{r} \cdot \mathbf{n}_2 = q_1 + \lambda q_2 \quad \dots(i)$$

where, λ is a parameter.

So, $\mathbf{n}_1 + \lambda \mathbf{n}_2$ is normal to the plane (i). Any plane containing planes $\mathbf{r} \cdot \mathbf{n}_3 = q_3$, $\mathbf{r} \cdot \mathbf{n}_4 = q_4$ is of the form $\mathbf{r} \cdot (\mathbf{n}_3 + \mu \mathbf{n}_4) = q_3 + \mu q_4$. Hence, we must have $\mathbf{n}_1 + \lambda \mathbf{n}_2 = k(\mathbf{n}_3 + \mu \mathbf{n}_4)$ for some k .

$$\Rightarrow [\mathbf{n}_1 + \lambda \mathbf{n}_2] \cdot [\mathbf{n}_3 \times \mathbf{n}_4] = 0$$

$$\Rightarrow [\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4] + \lambda [\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4] = 0$$

$$\Rightarrow \lambda = -1$$

Putting this value in Eq. (i), we get equation of required plane as

$$(\mathbf{r} \cdot \mathbf{n}_1 - q_1) = (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$$

$$\text{or } [\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4](\mathbf{r} \cdot \mathbf{n}_1 - q_1) = [\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4](\mathbf{r} \cdot \mathbf{n}_2 - q_2)$$

Hence, (c) is the correct answer.

Ex 6. For any four points P , Q , R and S , $|\mathbf{PQ} \times \mathbf{RS} - \mathbf{QR} \times \mathbf{PS} + \mathbf{RP} \times \mathbf{QS}|$ is equal to 4 times the area of

- (a) ΔPQR (b) ΔQSR (c) ΔPRS (d) ΔPQS

Sol. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the position vectors of points P , Q , R and S , respectively. Then,

$$\mathbf{PQ} \times \mathbf{RS} = (\mathbf{b} - \mathbf{a}) \times (\mathbf{d} - \mathbf{c})$$

$$= \mathbf{b} \times \mathbf{d} - \mathbf{b} \times \mathbf{c} + \mathbf{d} \times \mathbf{a} - \mathbf{c} \times \mathbf{a}$$

$$\mathbf{QR} \times \mathbf{PS} = (\mathbf{c} - \mathbf{b}) \times (\mathbf{d} - \mathbf{a})$$

$$= \mathbf{c} \times \mathbf{d} - \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{d} - \mathbf{a} \times \mathbf{b}$$

$$\mathbf{RP} \times \mathbf{QS} = (\mathbf{a} - \mathbf{c}) \times (\mathbf{d} - \mathbf{b})$$

$$= \mathbf{a} \times \mathbf{d} - \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{d} - \mathbf{b} \times \mathbf{c}$$

$$\therefore \mathbf{PQ} \times \mathbf{RS} - \mathbf{QR} \times \mathbf{PS} + \mathbf{RP} \times \mathbf{QS}$$

$$= \mathbf{b} \times \mathbf{d} - \mathbf{b} \times \mathbf{c} + \mathbf{d} \times \mathbf{a} - \mathbf{c} \times \mathbf{a} - \mathbf{c} \times \mathbf{d} + \mathbf{c} \times \mathbf{a}$$

$$+ \mathbf{b} \times \mathbf{d} + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{d} - \mathbf{a} \times \mathbf{b} - \mathbf{c} \times \mathbf{d} - \mathbf{b} \times \mathbf{c}$$

$$= 2(\mathbf{b} \times \mathbf{d}) - 2(\mathbf{b} \times \mathbf{c}) - 2(\mathbf{c} \times \mathbf{d})$$

$$= \{ \mathbf{b} \times \mathbf{d} + \mathbf{c} \times \mathbf{b} + \mathbf{d} \times \mathbf{c} \}$$

$$= 2(\mathbf{QS} \times \mathbf{QR}) = 4 \times \text{Area of } \Delta QSR$$

Hence, (b) is the correct answer.

Ex 7. Let G_1 , G_2 and G_3 be the centroids of the triangular faces OBC , OCA and OAB of a tetrahedron $OABC$. If V_1 denotes the volume of the tetrahedron $OABC$ and V_2 that of the parallelopiped with OG_1 , OG_2 and OG_3 as three concurrent edges, then

- (a) $4V_1 = 9V_2$ (b) $9V_1 = 4V_2$
 (c) $3V_1 = 2V_2$ (d) $3V_2 = 2V_1$

Sol. Taking O as the origin, let the position vectors of A , B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Then, the position

vectors G_1 , G_2 and G_3 are $\frac{\mathbf{b} + \mathbf{c}}{3}$, $\frac{\mathbf{c} + \mathbf{a}}{3}$ and $\frac{\mathbf{a} + \mathbf{b}}{3}$, respectively.

$$\therefore V_1 = \frac{1}{6}[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] \text{ and } V_2 = [\mathbf{OG}_1 \ \mathbf{OG}_2 \ \mathbf{OG}_3]$$

$$\text{Now, } V_2 = [\mathbf{OG}_1 \ \mathbf{OG}_2 \ \mathbf{OG}_3]$$

$$\Rightarrow V_2 = \left[\frac{\mathbf{b} + \mathbf{c}}{3} \ \frac{\mathbf{c} + \mathbf{a}}{3} \ \frac{\mathbf{a} + \mathbf{b}}{3} \right]$$

$$\Rightarrow V_2 = \frac{1}{27}[\mathbf{b} + \mathbf{c} \ \mathbf{c} + \mathbf{a} \ \mathbf{a} + \mathbf{b}] = \frac{2}{27}[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$$

$$\Rightarrow V_2 = \frac{2}{27} \times 6V_1$$

$$\Rightarrow 9V_2 = 4V_1$$

Hence, (a) is the correct answer.

Ex 8. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors of three non-collinear points A , B and C respectively, then the shortest distance of A from BC is

- (a) $\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})$

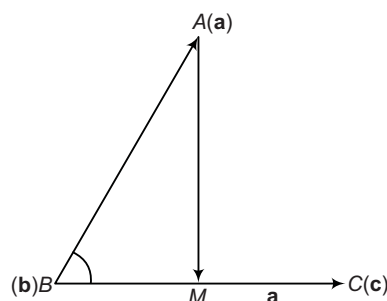
(b) $\sqrt{|\mathbf{b} - \mathbf{a}|^2 - \left\{ \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{c}|} \right\}^2}$

- (c) $|\mathbf{b} - \mathbf{a}|$

- (d) None of the above

Sol. The vector equation of the line BC is

$$\mathbf{r} = \mathbf{b} + \lambda(\mathbf{c} - \mathbf{b})$$



Clearly, $BM = \text{Projection of } \mathbf{BA} \text{ on } \mathbf{BC}$

$$= \frac{\mathbf{BA} \cdot \mathbf{BC}}{|\mathbf{BC}|}$$

$$= \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})}{|(\mathbf{c} - \mathbf{b})|}$$

$$\therefore \text{Required distance} = AM = \sqrt{AB^2 - BM^2}$$

$$= \sqrt{|\mathbf{b} - \mathbf{a}|^2 - \left\{ \frac{(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{b})}{|(\mathbf{c} - \mathbf{b})|} \right\}^2}$$

Hence, (d) is the correct answer.

Ex 9. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit coplanar vectors, then the scalar triple product $[2\hat{\mathbf{a}} - \hat{\mathbf{b}} \ 2\hat{\mathbf{b}} - \hat{\mathbf{c}} \ 2\hat{\mathbf{c}} - \hat{\mathbf{a}}]$ is equal to

- (a) 0
 (b) 1
 (c) $-\sqrt{3}$
 (d) $\sqrt{3}$

Sol. Let $\alpha = 2\hat{a} - \hat{b}$, $\beta = 2\hat{b} - \hat{c}$ and $\gamma = 2\hat{c} - \hat{a}$, then

$$[\alpha \ \beta \ \gamma] = \begin{vmatrix} 2 & -1 & 0 \\ 0 & 2 & -1 \\ -1 & 0 & 2 \end{vmatrix} [\hat{a} \ \hat{b} \ \hat{c}]$$

$$= 7[\hat{a} \ \hat{b} \ \hat{c}] = 7 \times 0 = 0$$

Hence, (a) is the correct answer.

Ex 10. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are unit vectors, then $|\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2$ does not exceed
(a) 4 (b) 9 (c) 8 (d) 6

Sol. We have, $|\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2$
 $= |\hat{a}|^2 + |\hat{b}|^2 - 2(\hat{a} \cdot \hat{b}) + |\hat{b}|^2 + |\hat{c}|^2 - 2(\hat{b} \cdot \hat{c})$
 $+ |\hat{c}|^2 + |\hat{a}|^2 - 2(\hat{c} \cdot \hat{a})$
 $= 2[|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 - (\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a})]$
 $= 2[3 - (\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a})]$
 $= 6 - 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \quad \dots(i)$

Now, $|\hat{a} \ \hat{b} \ \hat{c}|^2 \geq 0$
 $\Rightarrow |\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \geq 0$
 $\Rightarrow 3 + 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \geq 0$
 $\Rightarrow \hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a} \geq -\frac{3}{2}$
 $\Rightarrow -2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \leq 3 \dots(ii)$

From Eqs. (i) and (ii), we get

$$|\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \leq 6 + 3$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \leq 9$$

Hence, (b) is the correct answer.

Ex 11. Let $\mathbf{a} = \hat{i} - \hat{k}$, $\mathbf{b} = x\hat{i} + \hat{j} + (1-x)\hat{k}$ and $\mathbf{c} = y\hat{i} + x\hat{j} + (1+x-y)\hat{k}$. Then, $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ depends on
(a) only x (b) only y
(c) neither x nor y (d) None of these

Sol. We have, $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \begin{vmatrix} 1 & 0 & -1 \\ x & 1 & 1-x \\ y & x & 1+x-y \end{vmatrix}$
 $= \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ y & x & 1+x \end{vmatrix}$
 $[\text{applying } C_3 \rightarrow C_3 + C_1]$
 $= 1 + x - x = 1$

Clearly, it does not depend on x and y .

Hence, (c) is the correct answer.

Ex 12. Let $\mathbf{v} = 2\hat{i} + \hat{j} - \hat{k}$ and $\mathbf{w} = \hat{i} + 3\hat{k}$. If $\hat{u}$ is a unit vector, then maximum value of the scalar triple product $[\hat{u} \ \mathbf{v} \ \mathbf{w}]$ is
(a) -1 (b) $\sqrt{10} + \sqrt{6}$
(c) $\sqrt{59}$ (d) $\sqrt{60}$

Sol. We have, $[\hat{u} \ \mathbf{v} \ \mathbf{w}] = \hat{u} \cdot (\mathbf{v} \times \mathbf{w})$
 $[\hat{u} \ \mathbf{v} \ \mathbf{w}] \leq |\hat{u}| |\mathbf{v} \times \mathbf{w}| \quad [\because \mathbf{a} \cdot \mathbf{b} \leq |\mathbf{a}| |\mathbf{b}|]$
 $[\hat{u} \ \mathbf{v} \ \mathbf{w}] \leq |\mathbf{v} \times \mathbf{w}| \quad [\because |\hat{u}| = 1] \dots(i)$

Now, $\mathbf{v} \times \mathbf{w} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 0 & 3 \end{vmatrix} = 3\hat{i} - 7\hat{j} - \hat{k}$

$$\therefore |\mathbf{v} \times \mathbf{w}| = \sqrt{9 + 49 + 1} = \sqrt{59}$$

$$\Rightarrow [\hat{u} \ \mathbf{v} \ \mathbf{w}] \leq \sqrt{59} \quad [\text{from Eq. (i)}]$$

Hence, (c) is the correct answer.

Ex 13. A non-zero vector $\mathbf{a}$ is parallel to the line of intersection of the plane determined by the vectors $\hat{i}$, $\hat{i} + \hat{j}$ and the plane determined by the vectors $\hat{i} - \hat{j}$, $\hat{i} + \hat{k}$. The angle between $\mathbf{a}$ and $\hat{i} - 2\hat{j} + 2\hat{k}$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$

Sol. Let $\mathbf{n}_1$ and $\mathbf{n}_2$ be the vectors normal to the planes determined by $\hat{i}$, $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$, $\hat{i} + \hat{k}$ respectively.

Then, $\mathbf{n}_1 = \hat{i} \times (\hat{i} + \hat{j}) = \hat{k}$

and $\mathbf{n}_2 = (\hat{i} - \hat{j}) \times (\hat{i} + \hat{k}) = -\hat{i} - \hat{j} + \hat{k}$

Since, $\mathbf{a}$ is parallel to the line of intersection of the planes determined by $\hat{i}$, $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$, $\hat{i} + \hat{k}$.

Therefore,

$$\mathbf{a} \parallel (\mathbf{n}_1 \times \mathbf{n}_2) \Rightarrow \mathbf{a} = \lambda (\mathbf{n}_1 \times \mathbf{n}_2)$$

$$\Rightarrow \mathbf{a} = \lambda \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ -1 & -1 & 1 \end{vmatrix} \Rightarrow \mathbf{a} = \lambda(\hat{i} - \hat{j})$$

Let θ be the angle between $\mathbf{a}$ and $\hat{i} - 2\hat{j} + 2\hat{k}$.

$$\text{Then, } \cos \theta = \frac{\lambda(\hat{i} - \hat{j}) \cdot (\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{\lambda^2 + \lambda^2} \sqrt{1 + 4 + 4}}$$

$$= \frac{\lambda(1 + 2)}{\sqrt{2}\lambda \times 3} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$$

Hence, (a) is the correct answer.

Ex 14. If $\hat{a}$ and $\hat{b}$ are mutually perpendicular unit vectors. $\mathbf{r}$ is a vector satisfying $\mathbf{r} \cdot \hat{a} = 0$, $\mathbf{r} \cdot \hat{b} = 1$ and $[\mathbf{r} \ \hat{a} \ \hat{b}] = 1$, then $\mathbf{r}$ is equal to

(a) $\hat{a} + (\hat{a} \times \hat{b})$ (b) $\hat{b} + (\hat{a} \times \hat{b})$
(c) $\hat{a} + \hat{b}(\hat{a} \times \hat{b})$ (d) $\hat{a} - \hat{b} + (\hat{a} \times \hat{b})$

Sol. Let $\mathbf{r} = x_1\hat{a} + x_2\hat{b} + x_3(\hat{a} \times \hat{b})$

$$\Rightarrow \mathbf{r} \cdot \hat{a} = x_1 |\hat{a}|^2, \mathbf{r} \cdot \hat{b} = x_2 |\hat{b}|^2$$

and $\mathbf{r} \cdot (\hat{a} \times \hat{b}) = x_3 |\hat{a} \times \hat{b}|^2$

$$\Rightarrow x_1 = 0, x_2 = 1 \text{ and } x_3 = 1$$

$$\therefore \mathbf{r} = (\hat{a} \times \hat{b}) + \hat{b}$$

Hence, (b) is the correct answer.

Ex 15. If $\mathbf{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\mathbf{b} = 2\hat{i} + 3\hat{j} + 2\hat{k}$ and $\mathbf{c}$ is any vector not collinear with $\mathbf{b}$, then magnitude of

$$\frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{|\mathbf{b} \times \mathbf{c}|^2} (\mathbf{b} \times \mathbf{c}) + \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} + \frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{c}|^2} \mathbf{c} \text{ is equal to}$$

(a) $\sqrt{17}$ (b) 14
(c) 13 (d) None of these

Sol. Clearly, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{b} \times \mathbf{c}$ are non-coplanar vectors.

$$\therefore \mathbf{a} = \left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b} + \left(\frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{c}|^2} \right) \mathbf{c} + \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{|\mathbf{b} \times \mathbf{c}|^2} (\mathbf{b} \times \mathbf{c})$$

$$\therefore \text{Magnitude of the given vector} = |\mathbf{a}| = \sqrt{14}$$

Hence, (d) is the correct answer.

Ex 16. Given $|\mathbf{a}| = |\mathbf{b}| = 1$ and $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$. If $\mathbf{c}$ is a vector such that $\mathbf{c} - \mathbf{a} - 2\mathbf{b} = 3(\mathbf{a} \times \mathbf{b})$, then $\mathbf{c} \cdot \mathbf{b}$ is equal to

- (a) $-\frac{1}{2}$
(b) $\frac{1}{2}$
(c) $\frac{3}{2}$
(d) $\frac{5}{2}$

Sol. We have, $|\mathbf{a} + \mathbf{b}| = \sqrt{3}$

$$\Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 3$$

$$\Rightarrow |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2(\mathbf{a} \cdot \mathbf{b}) = 3$$

$$\Rightarrow 1 + 1 + 2(\mathbf{a} \cdot \mathbf{b}) = 3$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = \frac{1}{2} \quad \dots (i)$$

Now, $\mathbf{c} - \mathbf{a} - 2\mathbf{b} = 3(\mathbf{a} \times \mathbf{b})$

$$\Rightarrow (\mathbf{c} - \mathbf{a} - 2\mathbf{b}) \cdot \mathbf{b} = 3(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$$

$$\Rightarrow \mathbf{c} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{b} - 2(\mathbf{b} \cdot \mathbf{b}) = 0 \quad [\because [\mathbf{a} \ \mathbf{b} \ \mathbf{b}] = 0]$$

$$\Rightarrow \mathbf{c} \cdot \mathbf{b} - \frac{1}{2} - 2 \times 1 = 0 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \mathbf{c} \cdot \mathbf{b} = \frac{5}{2}$$

Hence, (d) is the correct answer.

Ex 17. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ are unit collinear vectors and $|\mathbf{b}| = 6$, then $\mathbf{b} - 3\hat{\mathbf{c}} = \lambda\hat{\mathbf{a}}$, where λ is equal to

- (a) $-9, 3$
(b) $9, 3$
(c) $3, -3$
(d) None of the above

Sol. We have, $\mathbf{b} - 3\hat{\mathbf{c}} = \lambda\hat{\mathbf{a}}$

Taking scalar product with $\hat{\mathbf{c}}$, we have

$$(\mathbf{b} - 3\hat{\mathbf{c}}) \cdot \hat{\mathbf{c}} = \lambda(\hat{\mathbf{a}} \cdot \hat{\mathbf{c}})$$

$$\Rightarrow \mathbf{b} \cdot \hat{\mathbf{c}} - 3(\hat{\mathbf{c}} \cdot \hat{\mathbf{c}}) = \lambda(\hat{\mathbf{a}} \cdot \hat{\mathbf{c}})$$

$$\Rightarrow \mathbf{b} \cdot \hat{\mathbf{c}} - 3 = \lambda$$

$$[\because |\hat{\mathbf{a}}| = |\hat{\mathbf{c}}| = 1 \text{ and } \hat{\mathbf{a}}, \hat{\mathbf{c}} \text{ are unit collinear vectors}]$$

$$\Rightarrow \mathbf{b} \cdot \hat{\mathbf{c}} = 3 + \lambda \quad \dots (i)$$

Again, $\mathbf{b} - 3\hat{\mathbf{c}} = \lambda\hat{\mathbf{a}}$

$$\Rightarrow |\mathbf{b} - 3\hat{\mathbf{c}}| = |\lambda\hat{\mathbf{a}}|$$

$$\Rightarrow |\mathbf{b} - 3\hat{\mathbf{c}}|^2 = \lambda^2 |\hat{\mathbf{a}}|^2$$

$$\Rightarrow |\mathbf{b}|^2 + 9|\hat{\mathbf{c}}|^2 - 6(\mathbf{b} \cdot \hat{\mathbf{c}}) = \lambda^2 |\hat{\mathbf{a}}|^2$$

$$\Rightarrow 36 + 9 - 6(3 + \lambda) = \lambda^2 \quad [\text{using Eq. (i)}]$$

$$\Rightarrow 27 - 6\lambda = \lambda^2$$

$$\Rightarrow \lambda^2 + 6\lambda - 27 = 0$$

$$\Rightarrow \lambda = -9, 3$$

Hence, (a) is the correct answer.

Ex 18. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors, then the projection of the vector $l\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{b}|} + n\frac{(\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|}$ along the angle bisector of the vectors $\mathbf{a}$ and $\mathbf{b}$ is

- (a) $\frac{l^2 + m^2}{\sqrt{l^2 + m^2 + n^2}}$ (b) $\sqrt{l^2 + m^2 + n^2}$
(c) $\frac{\sqrt{l^2 + m^2}}{\sqrt{l^2 + m^2 + n^2}}$ (d) $\frac{l + m}{\sqrt{2}}$

Sol. A vector parallel to the bisector of the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|} = \mathbf{a} + \mathbf{b}$

$\therefore$ Unit vector along the bisector

$$= \frac{\hat{\mathbf{a}} + \hat{\mathbf{b}}}{|\hat{\mathbf{a}} + \hat{\mathbf{b}}|} = \frac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{b}})$$

$$\left[\begin{aligned} \because |\hat{\mathbf{a}} + \hat{\mathbf{b}}|^2 &= |\hat{\mathbf{a}}|^2 + |\hat{\mathbf{b}}|^2 + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}} \\ \Rightarrow |\hat{\mathbf{a}} + \hat{\mathbf{b}}|^2 &= 1 + 1 + 0 = 2 \end{aligned} \right]$$

$\therefore$ Required projection

$$\begin{aligned} &= \left\{ l\frac{\mathbf{a}}{|\mathbf{a}|} + m\frac{\mathbf{b}}{|\mathbf{b}|} + n\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} \right\} \cdot \frac{1}{\sqrt{2}}(\hat{\mathbf{a}} + \hat{\mathbf{b}}) \\ &= \frac{1}{\sqrt{2}}(l + m) \end{aligned}$$

$$\left[\begin{aligned} \because |\hat{\mathbf{a}}| = |\hat{\mathbf{b}}| = 1 \text{ and } \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} &= \hat{\mathbf{a}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) \\ &= \hat{\mathbf{b}} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{b}}) = 0 \end{aligned} \right]$$

Hence, (d) is the correct answer.

Ex 19. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{r}$ be a variable vector such that $\mathbf{r} \cdot \hat{\mathbf{i}}$, $\mathbf{r} \cdot \hat{\mathbf{j}}$ and $\mathbf{r} \cdot \hat{\mathbf{k}}$ are positive integers. If $\mathbf{r} \cdot \mathbf{a} \leq 12$, then the total number of such vectors is

- (a) ${}^{12}C_9 - 1$ (b) ${}^{12}C_3$
(c) ${}^{12}C_8$ (d) None of these

Sol. Let $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, where x , y and z are integers.

We have, $\mathbf{r} \cdot \hat{\mathbf{i}} > 0$, $\mathbf{r} \cdot \hat{\mathbf{j}} > 0$

and $\mathbf{r} \cdot \hat{\mathbf{k}} > 0$

$$\Rightarrow x > 0, y > 0 \text{ and } z > 0$$

$\Rightarrow x, y$ and z are positive integers.

Also, $\mathbf{r} \cdot \mathbf{a} \leq 12$

$$\Rightarrow x + y + z \leq 12$$

Let $x + y + z + t = 12$, where $t \geq 0$.

Clearly, total number of vectors satisfying the given conditions is same as the total number of integral solutions of $x + y + z + t = 12$, where $x > 0$, $y > 0$, $z > 0$ and $t \geq 0$.

Let $x' = x - 1$, $y' = y - 1$ and $z' = z - 1$. Then, we have $x' + y' + z' + t = 9$, where $x' \geq 0$, $y' \geq 0$, $z' \geq 0$ and $t \geq 0$.

The total number of solutions of this equation

$$= {}^{9+4-1}C_{4-1} = {}^{12}C_3$$

$\therefore$ Required number of vectors = ${}^{12}C_3$

Hence, (b) is the correct answer.

Ex 20. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors of magnitude $\sqrt{3}$, 1 and 2 such that $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + 3\mathbf{b} = \mathbf{0}$. If θ is the angle between $\mathbf{a}$ and $\mathbf{c}$, then $\cos^2 \theta$ is equal to

- (a) $\frac{3}{4}$ (b) $\frac{1}{2}$
(c) $\frac{1}{4}$ (d) None of these

Sol. We have, $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + 3\mathbf{b} = \mathbf{0}$
 $\Rightarrow (\mathbf{a} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{c} + 3\mathbf{b} = \mathbf{0}$
 $\Rightarrow (2\sqrt{3} \cos \theta)\mathbf{a} - 3\mathbf{c} + 3\mathbf{b} = \mathbf{0}$
 $\Rightarrow (2 \cos \theta)\mathbf{a} - \sqrt{3}\mathbf{c} + \sqrt{3}\mathbf{b} = \mathbf{0}$
 $\Rightarrow |(2 \cos \theta)\mathbf{a} - \sqrt{3}\mathbf{c}| = |-\sqrt{3}\mathbf{b}|$
 $\Rightarrow |(2 \cos \theta)\mathbf{a} - \sqrt{3}\mathbf{c}|^2 = |-\sqrt{3}\mathbf{b}|^2$
 $\Rightarrow 4 \cos^2 \theta |\mathbf{a}|^2 + 3|\mathbf{c}|^2 - 4\sqrt{3} \cos \theta (\mathbf{a} \cdot \mathbf{c}) = 3|\mathbf{b}|^2$
 $\Rightarrow 12 \cos^2 \theta + 12 - 4\sqrt{3} \cos \theta \times \sqrt{3} \times 2 \cos \theta = 3$
 $\Rightarrow 12 \cos^2 \theta + 12 - 24 \cos^2 \theta = 0$
 $\Rightarrow 12 \cos^2 \theta = 9$
 $\Rightarrow \cos^2 \theta = \frac{9}{12} = \frac{3}{4}$

Hence, (a) is the correct answer.

Ex 21. Let us define the length of a vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ as $|a| + |b| + |c|$. This definition coincides with the usual definition of length of a vector $a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$ iff

- (a) $a = b = c = 0$
(b) any two of a , b and c are zero
(c) any one of a , b and c is zero
(d) $a + b + c = 0$

Sol. We have, $|a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}| = |a| + |b| + |c|$
 $\Rightarrow \sqrt{a^2 + b^2 + c^2} = |a| + |b| + |c|$
 $\Rightarrow a^2 + b^2 + c^2 = a^2 + b^2 + c^2 + 2(|a||b| + |b||c| + |c||a|)$
 $\Rightarrow |a||b| + |b||c| + |c||a| = 0$
 $\therefore$ Any two of a , b , and c are zero.
Hence, (b) is the correct answer.

Ex 22. If the position vector of a point P is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, where x , y and $z \in N$ and $\mathbf{a}$ is a vector given by $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, then the total number of possible positions of point P for which $\mathbf{r} \cdot \mathbf{a} = 10$, is

- (a) 36 (b) 72
(c) 66 (d) None of these

Sol. We have, $\mathbf{r} \cdot \mathbf{a} = 10$
 $\Rightarrow x + y + z = 10$
The total number of possible solutions of P is same as the number of solutions of the above equation in the set of natural numbers.
 $\therefore$ Total number of solutions
 $= {}^{10-1}C_{3-1} = {}^9C_2 = 36$

Hence, (a) is the correct answer.

Ex 23. If the vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other, then a vector $\mathbf{v}$ in terms of $\mathbf{a}$ and $\mathbf{b}$ satisfying the equations $\mathbf{v} \cdot \mathbf{a} = 0$, $\mathbf{v} \cdot \mathbf{b} = 1$ and $[\mathbf{v} \ \mathbf{a} \ \mathbf{b}] = 1$, is

- (a) $\frac{\mathbf{b}}{|\mathbf{b}|^2} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}$ (b) $\frac{\mathbf{b}}{|\mathbf{b}|} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|^2}$
(c) $\frac{\mathbf{b}}{|\mathbf{b}|^2} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|}$ (d) None of these

Sol. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors, then any vector $\mathbf{v}$ can be expressed as

$$\mathbf{r} = \left\{ \frac{\mathbf{r} \cdot \mathbf{a}}{|\mathbf{a}|^2} \right\} \mathbf{a} + \left\{ \frac{\mathbf{r} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\} \mathbf{b} + \left\{ \frac{\mathbf{r} \cdot \mathbf{c}}{|\mathbf{c}|^2} \right\} \mathbf{c}$$

Clearly, $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$ are three non-coplanar vectors.

$$\therefore \mathbf{v} = \left\{ \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{a}|^2} \right\} \mathbf{a} + \left\{ \frac{\mathbf{v} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\} \mathbf{b} + \left\{ \frac{\mathbf{v} \cdot (\mathbf{a} \times \mathbf{b})}{|\mathbf{a} \times \mathbf{b}|^2} \right\} (\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow \mathbf{v} = 0 \cdot \mathbf{a} + \frac{1}{|\mathbf{b}|^2} \cdot \mathbf{b} + \frac{1}{|\mathbf{a} \times \mathbf{b}|^2} \cdot (\mathbf{a} \times \mathbf{b})$$

Hence, (a) is the correct answer.

Ex 24. Let $\mathbf{u} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$, $\mathbf{v} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and $\mathbf{w} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$. If $\hat{\mathbf{n}}$ is a unit vector such that $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ and $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$, then $|\mathbf{w} \cdot \hat{\mathbf{n}}|$ is equal to
(a) 3 (b) 0 (c) 1 (d) 2

Sol. We have, $\mathbf{u} \cdot \hat{\mathbf{n}} = 0$ and $\mathbf{v} \cdot \hat{\mathbf{n}} = 0$
 $\Rightarrow \hat{\mathbf{n}} \perp \mathbf{u}$ and $\hat{\mathbf{n}} \perp \mathbf{v}$
 $\Rightarrow \hat{\mathbf{n}} = \pm \frac{\mathbf{u} \times \mathbf{v}}{|\mathbf{u} \times \mathbf{v}|}$

$$\text{Now, } \mathbf{u} \times \mathbf{v} = (\hat{\mathbf{i}} + \hat{\mathbf{j}}) \times (\hat{\mathbf{i}} - \hat{\mathbf{j}}) = -2\hat{\mathbf{k}}$$

$$\therefore \hat{\mathbf{n}} = \pm \hat{\mathbf{k}}$$

$$\text{Now, } |\mathbf{w} \cdot \hat{\mathbf{n}}| = |(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) \cdot (\pm \hat{\mathbf{k}})| = 3$$

Hence, (a) is the correct answer.

Ex 25. The value of a , so that the volume of the parallelopiped formed by $\hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ and $a\hat{\mathbf{j}} + \hat{\mathbf{k}}$ becomes minimum, is

- (a) $\frac{1}{3}$ (b) 3 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

Sol. Let V be the volume of the parallelopiped formed by the given vectors.

$$\text{Let } \alpha = \hat{\mathbf{i}} + a\hat{\mathbf{j}} + \hat{\mathbf{k}}, \beta = \hat{\mathbf{j}} + a\hat{\mathbf{k}} \text{ and } \gamma = a\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\text{Then, } [\alpha \ \beta \ \gamma] = \begin{vmatrix} 1 & a & 1 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 1 + a^3 - a$$

$$V = |[\alpha \ \beta \ \gamma]|$$

Clearly, V will be minimum when $\lambda = [\alpha \ \beta \ \gamma] = 1 + a^3 - a$ is minimum.

$$\text{Now, } \lambda = 1 + a^3 - a$$

$$\Rightarrow \frac{d\lambda}{da} = 3a^2 - 1$$

$$\text{and } \frac{d^2\lambda}{da^2} = 6a$$

For maximum or minimum value of λ , we must have

$$\frac{d\lambda}{da} = 0 \Rightarrow a = \pm \frac{1}{\sqrt{3}}$$

$$\text{Clearly, } \left(\frac{d^2\lambda}{da^2} \right)_{a=\frac{1}{\sqrt{3}}} = \frac{6}{\sqrt{3}} > 0$$

$$\text{So, } \lambda \text{ is minimum for } a = \frac{1}{\sqrt{3}}.$$

$$\therefore V \text{ is minimum for } a = \frac{1}{\sqrt{3}}.$$

Hence, (c) is the correct answer.

Ex 26. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three vectors having magnitudes 1, 1 and 2 respectively. If $\mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + \mathbf{b} = \mathbf{0}$, then the acute angle between $\mathbf{a}$ and $\mathbf{c}$ is

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{3}$ (d) None of these

Sol. We have, $|\mathbf{a}| = 1$, $|\mathbf{b}| = 1$ and $|\mathbf{c}| = 2$

$$\text{Also, } \mathbf{a} \times (\mathbf{a} \times \mathbf{c}) + \mathbf{b} = \mathbf{0}$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{a})\mathbf{c} + \mathbf{b} = \mathbf{0}$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \mathbf{c} + \mathbf{b} = \mathbf{0} [\because \mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2 = 1]$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \mathbf{c} = -\mathbf{b}$$

$$\Rightarrow |(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \mathbf{c}| = |-\mathbf{b}|$$

$$\Rightarrow |(\mathbf{a} \cdot \mathbf{c})\mathbf{a} - \mathbf{c}|^2 = |-\mathbf{b}|^2$$

$$\Rightarrow |(\mathbf{a} \cdot \mathbf{c})\mathbf{a}|^2 + |\mathbf{c}|^2 - 2\{(\mathbf{a} \cdot \mathbf{c})\mathbf{a} \cdot \mathbf{c}\} = |\mathbf{b}|^2$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})^2 |\mathbf{a}|^2 + |\mathbf{c}|^2 - 2(\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{c}) = |\mathbf{b}|^2$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})^2 \{|\mathbf{a}|^2 - 2\} + |\mathbf{c}|^2 = |\mathbf{b}|^2$$

$$\Rightarrow -(\mathbf{a} \cdot \mathbf{c})^2 + 4 = 1$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})^2 = 3$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{c} = \pm \sqrt{3}$$

$$\Rightarrow |\mathbf{a}||\mathbf{c}|\cos\theta = \sqrt{3}$$

$$\text{where, } \theta \text{ is an acute angle between } \mathbf{a} \text{ and } \mathbf{c}.$$

$$\therefore \cos\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{6}$$

$$\text{Hence, (b) is the correct answer.}$$

Ex 27. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors, then the greatest value of $|\hat{\mathbf{a}} + \hat{\mathbf{b}}| + |\hat{\mathbf{a}} - \hat{\mathbf{b}}|$ is

- (a) 2 (b) 4
(c) $2\sqrt{2}$ (d) $\sqrt{2}$

Sol. Let θ be an angle between unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$.

$$\text{Then, } \hat{\mathbf{a}} \cdot \hat{\mathbf{b}} = \cos\theta$$

$$\text{Now, } |\hat{\mathbf{a}} + \hat{\mathbf{b}}|^2 = |\hat{\mathbf{a}}|^2 + |\hat{\mathbf{b}}|^2 + 2\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}$$

$$= 2 + 2\cos\theta = 4\cos^2 \frac{\theta}{2}$$

$$\Rightarrow |\hat{\mathbf{a}} + \hat{\mathbf{b}}| = 2\cos \frac{\theta}{2}, |\hat{\mathbf{a}} - \hat{\mathbf{b}}| = 2\sin \frac{\theta}{2}$$

$$\Rightarrow |\hat{\mathbf{a}} + \hat{\mathbf{b}}| + |\hat{\mathbf{a}} - \hat{\mathbf{b}}| = 2 \left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right) \leq 2\sqrt{2}$$

Hence, (c) is the correct answer.

Ex 28. If the magnitude of the moment about the point $\hat{\mathbf{j}} + \hat{\mathbf{k}}$ of a force $\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} - \hat{\mathbf{k}}$ acting through the point $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ is $\sqrt{8}$, then the value of α is

- (a) 1 (b) 2
(c) 3 (d) -3

Sol. Let $\mathbf{M}$ be the moment of the force $\mathbf{F} = \hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} - \hat{\mathbf{k}}$ acting through the point $A(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ about the point $O(\hat{\mathbf{j}} + \hat{\mathbf{k}})$.

Then,

$$\mathbf{M} = \mathbf{OA} \times \mathbf{F}$$

$$= (\hat{\mathbf{i}} - \hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + \alpha\hat{\mathbf{j}} - \hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 0 & -1 \\ 1 & \alpha & -1 \end{vmatrix}$$

$$\Rightarrow = \alpha\hat{\mathbf{i}} + \alpha\hat{\mathbf{k}}$$

$$\therefore |\mathbf{M}| = \sqrt{\alpha^2 + \alpha^2} = \sqrt{2}|\alpha|$$

$$\Rightarrow \sqrt{2}|\alpha| = \sqrt{8} \quad [\because |\mathbf{M}| = \sqrt{8}]$$

$$\Rightarrow |\alpha| = 2$$

$$\Rightarrow \alpha = \pm 2$$

Hence, (b) is the correct answer.

Ex 29. If the volume of the parallelopiped formed by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as three coterminal edges is 27 cu units, then volume of the parallelopiped having $\alpha = \mathbf{a} + 2\mathbf{b} - \mathbf{c}$, $\beta = \mathbf{a} - \mathbf{b}$ and $\gamma = \mathbf{a} - \mathbf{b} - \mathbf{c}$ as three coterminal edges, is

- (a) 27 cu units
(b) 9 cu units
(c) 81 cu units
(d) None of the above

Sol. We have, $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 27$ cu units

$$\text{Now, } [[\alpha \ \beta \ \gamma]] = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & -1 \end{vmatrix} [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$$

$$[\alpha \ \beta \ \gamma] = 3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$$

$$\text{Required volume} = [[\alpha \ \beta \ \gamma]]$$

$$= 3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$$

$$= 3 \times 27 = 81 \text{ cu units}$$

Hence, (c) is the correct answer.

Ex 30. If V is the volume of the parallelopiped having three coterminal edges as $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then the volume of the parallelopiped having three coterminal edges as

$$\alpha = (\mathbf{a} \cdot \mathbf{a})\mathbf{a} + (\mathbf{a} \cdot \mathbf{b})\mathbf{b} + (\mathbf{a} \cdot \mathbf{c})\mathbf{c};$$

$$\beta = (\mathbf{a} \cdot \mathbf{b})\mathbf{a} + (\mathbf{b} \cdot \mathbf{b})\mathbf{b} + (\mathbf{b} \cdot \mathbf{c})\mathbf{c}$$

$$\text{and } \gamma = (\mathbf{a} \cdot \mathbf{c})\mathbf{a} + (\mathbf{b} \cdot \mathbf{c})\mathbf{b} + (\mathbf{c} \cdot \mathbf{c})\mathbf{c} \text{ is}$$

- (a) V^3 (b) $3V$
(c) V^2 (d) $2V$

Sol. We have, $[[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]] = V$

Let V_1 be the volume of the parallelopiped formed by the vectors α, β and γ .

Then, $V_1 = [[\alpha \ \beta \ \gamma]]$

$$\text{Now, } [[\alpha \ \beta \ \gamma]] = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} [[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]]$$

$$= [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2 [[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]] = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^3$$

$$\therefore V_1 = [[\alpha \ \beta \ \gamma]] = [[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]]^3 = V^3$$

Hence, (a) is the correct answer.

Ex 31. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be non-coplanar vectors and λ be a real number, then the vectors $\mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \lambda\mathbf{b} + 4\mathbf{c}$ and $(2\lambda - 1)\mathbf{c}$ are non-coplanar for

- (a) no value of λ
- (b) all except one value of λ
- (c) all except two values of λ
- (d) all values of λ

Sol. Let $\alpha = \mathbf{a} + 2\mathbf{b} + 3\mathbf{c}, \beta = \lambda\mathbf{b} + 4\mathbf{c}$ and $\gamma = (2\lambda - 1)\mathbf{c}$

$$\text{Then, } [[\alpha \ \beta \ \gamma]] = \begin{vmatrix} 1 & 2 & 3 \\ 0 & \lambda & 4 \\ 0 & 0 & (2\lambda - 1) \end{vmatrix} [[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]]$$

$$\Rightarrow [[\alpha \ \beta \ \gamma]] = \lambda(2\lambda - 1)[[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]]$$

$$\Rightarrow [[\alpha \ \beta \ \gamma]] = 0, \text{ if } \lambda = 0, \frac{1}{2} \quad [\because [[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]] \neq 0]$$

So, α, β and γ are non-coplanar for all values of λ except two values 0 and $\frac{1}{2}$.

Hence, (c) is the correct answer.

Ex 32. Let $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ be three vectors such that $|\mathbf{u}| = 1, |\mathbf{v}| = 2$ and $|\mathbf{w}| = 3$. If the projection of $\mathbf{v}$ along $\mathbf{u}$ is equal to that of $\mathbf{w}$ along $\mathbf{u}$ and $\mathbf{v}, \mathbf{w}$ are perpendicular to each other, then $|\mathbf{u} - \mathbf{v} + \mathbf{w}|$ equals

- (a) 14
- (b) $\sqrt{7}$
- (c) $\sqrt{14}$
- (d) 2

Sol. $\therefore$ Projection of $\mathbf{v}$ along $\mathbf{u}$

= Projection of $\mathbf{w}$ along $\mathbf{u}$

$$\therefore \frac{\mathbf{v} \cdot \mathbf{u}}{|\mathbf{u}|} = \frac{\mathbf{w} \cdot \mathbf{u}}{|\mathbf{u}|}$$

$$\Rightarrow \mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u} \quad \dots(i)$$

Also, $\mathbf{v}$ and $\mathbf{w}$ are perpendicular to each other.

$$\therefore \mathbf{v} \cdot \mathbf{w} = 0 \quad \dots(ii)$$

$$\text{Now, } |\mathbf{u} - \mathbf{v} + \mathbf{w}|^2 = |\mathbf{u}|^2 + |\mathbf{v}|^2 + |\mathbf{w}|^2 - 2(\mathbf{u} \cdot \mathbf{v}) - 2(\mathbf{v} \cdot \mathbf{w}) + 2(\mathbf{u} \cdot \mathbf{w})$$

$$\Rightarrow |\mathbf{u} - \mathbf{v} + \mathbf{w}|^2 = 1 + 4 + 9$$

$$\quad \quad \quad [\text{using Eqs. (i) and (ii)}]$$

$$\Rightarrow |\mathbf{u} - \mathbf{v} + \mathbf{w}| = \sqrt{14}$$

Hence, (c) is the correct answer.

Ex 33. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that no two are collinear and

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3} |\mathbf{b}| |\mathbf{c}| \mathbf{a}. \text{ If } \theta \text{ is the acute angle}$$

between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin \theta$ is equal to

- (a) $\frac{2\sqrt{2}}{3}$
- (b) $\frac{\sqrt{2}}{3}$
- (c) $\frac{2}{3}$
- (d) $\frac{1}{3}$

Sol. We have, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3} |\mathbf{b}| |\mathbf{c}| \mathbf{a}$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a} = \frac{1}{3} |\mathbf{b}| |\mathbf{c}| \mathbf{a}$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - \left\{ (\mathbf{b} \cdot \mathbf{c}) + \frac{1}{3} |\mathbf{b}| |\mathbf{c}| \right\} \mathbf{a} = \mathbf{0}$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{c} = 0 \text{ and } \mathbf{b} \cdot \mathbf{c} + \frac{1}{3} |\mathbf{b}| |\mathbf{c}| = 0$$

[since, $\mathbf{a}$ and $\mathbf{b}$ are non-collinear]

$$\Rightarrow |\mathbf{b}| |\mathbf{c}| \cos \theta + \frac{1}{3} |\mathbf{b}| |\mathbf{c}| = 0$$

$$\Rightarrow \cos \theta = -\frac{1}{3} \Rightarrow \sin \theta = \sqrt{\frac{8}{9}} = \frac{2\sqrt{2}}{3}$$

Hence, (a) is the correct answer.

Ex 34. If the resolved part of vector $\mathbf{a}$ along the vector $\mathbf{b}$ is $\mathbf{a}_1$ and that perpendicular to $\mathbf{b}$ is $\mathbf{a}_2$, then $\mathbf{a}_1 \times \mathbf{a}_2$ is equal to

- (a) $\frac{(\mathbf{a} \times \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$
- (b) $\frac{(\mathbf{a} \times \mathbf{b})\mathbf{a}}{|\mathbf{a}|^2}$
- (c) $\frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2}$
- (d) $\frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{a})}{|\mathbf{b} \times \mathbf{a}|}$

Sol. We have, $\mathbf{a}_1 = (\mathbf{a} \cdot \mathbf{b})\mathbf{b} = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$

$$\text{and } \mathbf{a}_2 = \mathbf{a} - \mathbf{a}_1 = \mathbf{a} - \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2}$$

$$\text{Thus, } \mathbf{a}_1 \times \mathbf{a}_2 = \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} \times \left(\mathbf{a} - \frac{(\mathbf{a} \cdot \mathbf{b})\mathbf{b}}{|\mathbf{b}|^2} \right) = \frac{(\mathbf{a} \cdot \mathbf{b})(\mathbf{b} \times \mathbf{a})}{|\mathbf{b}|^2}$$

Hence, (c) is the correct answer.

Ex 35. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{r}$ is any arbitrary vector, then $[\mathbf{b} \ \mathbf{c} \ \mathbf{r}]\mathbf{a} + [\mathbf{c} \ \mathbf{a} \ \mathbf{r}]\mathbf{b} + [\mathbf{a} \ \mathbf{b} \ \mathbf{r}]\mathbf{c}$ is always equal to

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
- (b) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
- (c) $3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
- (d) None of these

Sol. Let $\mathbf{r} = x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c}$

$$\Rightarrow \mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = x_1 \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$$

$$\Rightarrow x_1 = \frac{[\mathbf{r} \ \mathbf{b} \ \mathbf{c}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

$$\text{Also, } \mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) = x_2 \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$$

$$\Rightarrow x_2 = \frac{[\mathbf{r} \ \mathbf{c} \ \mathbf{a}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

$$\text{and } \mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = x_3 \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow x_3 = \frac{[\mathbf{r} \ \mathbf{a} \ \mathbf{b}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

$$\therefore \mathbf{r} = \frac{[\mathbf{r} \ \mathbf{b} \ \mathbf{c}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} \mathbf{a} + \frac{[\mathbf{r} \ \mathbf{c} \ \mathbf{a}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} \mathbf{b} + \frac{[\mathbf{r} \ \mathbf{a} \ \mathbf{b}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} \mathbf{c}$$

$$\Rightarrow [\mathbf{b} \ \mathbf{c} \ \mathbf{r}]\mathbf{a} + [\mathbf{c} \ \mathbf{a} \ \mathbf{r}]\mathbf{b} + [\mathbf{a} \ \mathbf{b} \ \mathbf{r}]\mathbf{c} = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$$

Hence, (a) is the correct answer.

- Ex 36.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{r}$ is any arbitrary vector, then $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{r} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) \times (\mathbf{r} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) \times (\mathbf{r} \times \mathbf{b})$ is equal to
- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
 (b) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
 (c) $3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
 (d) None of the above

Sol. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{r} \times \mathbf{c}) = \{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\}\mathbf{r} - \{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{r}\}\mathbf{c}$
 $= [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r} - [\mathbf{a} \ \mathbf{b} \ \mathbf{r}]\mathbf{c}$
 Similarly, $(\mathbf{b} \times \mathbf{c}) \times (\mathbf{r} \times \mathbf{a}) = [\mathbf{b} \ \mathbf{c} \ \mathbf{a}]\mathbf{r} - [\mathbf{b} \ \mathbf{c} \ \mathbf{r}]\mathbf{a}$
 and $(\mathbf{c} \times \mathbf{a}) \times (\mathbf{r} \times \mathbf{b}) = [\mathbf{c} \ \mathbf{a} \ \mathbf{b}]\mathbf{r} - [\mathbf{c} \ \mathbf{a} \ \mathbf{r}]\mathbf{b}$
 $\therefore (\mathbf{a} \times \mathbf{b}) \times (\mathbf{r} \times \mathbf{c}) + (\mathbf{b} \times \mathbf{c}) \times (\mathbf{r} \times \mathbf{a}) + (\mathbf{c} \times \mathbf{a}) \times (\mathbf{r} \times \mathbf{b})$
 $= 3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r} - \{[\mathbf{b} \ \mathbf{c} \ \mathbf{r}]\mathbf{a} + [\mathbf{c} \ \mathbf{a} \ \mathbf{r}]\mathbf{b} + [\mathbf{a} \ \mathbf{b} \ \mathbf{r}]\mathbf{c}\}$
 $= 3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r} - [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r} = 2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]\mathbf{r}$
 Hence, (b) is the correct answer.

Type 2. More than One Correct Option

- Ex 37.** Let $\alpha = a\hat{\mathbf{i}} + b\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, $\beta = b\hat{\mathbf{i}} + c\hat{\mathbf{j}} + a\hat{\mathbf{k}}$ and $\gamma = c\hat{\mathbf{i}} + a\hat{\mathbf{j}} + b\hat{\mathbf{k}}$ be three coplanar vectors with $a \neq b$, and $\mathbf{v} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$. Then, $\mathbf{v}$ is perpendicular to
- (a) α (b) β
 (c) γ (d) None of these

Sol. It is given that α , β and γ are coplanar vectors

$$\therefore \begin{vmatrix} \alpha & \beta & \gamma \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$\Rightarrow 3abc - a^3 - b^3 - c^3 = 0$$

$$\Rightarrow a^3 + b^3 + c^3 - 3abc = 0$$

$$\Rightarrow (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$$

$$\Rightarrow a + b + c = 0$$

$$\Rightarrow \mathbf{v} \cdot \alpha = \mathbf{v} \cdot \beta = \mathbf{v} \cdot \gamma = 0$$

$\therefore \mathbf{v}$ is perpendicular to α , β and γ .
 Hence, (a), (b) and (c) are the correct answers.

- Ex 38.** Let $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ be unit vectors. If $\mathbf{w}$ is a vector such that $\mathbf{w} + (\mathbf{w} \times \hat{\mathbf{u}}) = \hat{\mathbf{v}}$, then $\hat{\mathbf{u}} \cdot (\hat{\mathbf{v}} \times \mathbf{w})$ will be equal to
- (a) $1 - \hat{\mathbf{v}} \cdot \mathbf{w}$
 (b) $1 - |\mathbf{w}|^2$
 (c) $|\mathbf{w}|^2 - (\hat{\mathbf{u}} \cdot \mathbf{w})^2$
 (d) None of the above

Sol. We have, $\mathbf{w} + (\mathbf{w} \times \hat{\mathbf{u}}) = \hat{\mathbf{v}}$... (i)

Taking scalar product with $\hat{\mathbf{v}}$, we get

$$\{\mathbf{w} + (\mathbf{w} \times \hat{\mathbf{u}})\} \cdot \hat{\mathbf{v}} = \hat{\mathbf{v}} \cdot \hat{\mathbf{v}}$$

$$\Rightarrow \mathbf{w} \cdot \hat{\mathbf{v}} + [\hat{\mathbf{u}} \ \hat{\mathbf{v}} \ \mathbf{w}] = 1 \quad [\because |\hat{\mathbf{v}}| = 1]$$

$$\Rightarrow [\hat{\mathbf{u}} \ \hat{\mathbf{v}} \ \mathbf{w}] = 1 - \hat{\mathbf{v}} \cdot \mathbf{w} \quad \dots (ii)$$

So, option (a) is correct.

Taking cross-product of Eq. (i) with $\hat{\mathbf{u}}$, we get

$$\hat{\mathbf{u}} \times \mathbf{w} + \hat{\mathbf{u}} \times (\mathbf{w} \times \hat{\mathbf{u}}) = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$$

$$\Rightarrow \hat{\mathbf{u}} \times \mathbf{w} + (\hat{\mathbf{u}} \cdot \hat{\mathbf{u}})\mathbf{w} - (\hat{\mathbf{u}} \cdot \mathbf{w})\hat{\mathbf{u}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$$

$$\Rightarrow \hat{\mathbf{u}} \times \mathbf{w} + \mathbf{w} - (\hat{\mathbf{u}} \cdot \mathbf{w})\hat{\mathbf{u}} = \hat{\mathbf{u}} \times \hat{\mathbf{v}}$$

Taking dot product, we get

$$\mathbf{w} \cdot (\hat{\mathbf{u}} \times \mathbf{w}) + \mathbf{w} \cdot \mathbf{w} - (\hat{\mathbf{u}} \cdot \mathbf{w})(\mathbf{w} \cdot \hat{\mathbf{u}}) = \mathbf{w} \cdot (\hat{\mathbf{u}} \times \hat{\mathbf{v}})$$

$$\Rightarrow |\mathbf{w}|^2 - (\hat{\mathbf{u}} \cdot \mathbf{w})^2 = [\hat{\mathbf{u}} \ \hat{\mathbf{v}} \ \mathbf{w}]$$

$$\Rightarrow |\mathbf{w}|^2 - (\hat{\mathbf{u}} \cdot \mathbf{w})^2 = [\hat{\mathbf{u}} \ \hat{\mathbf{v}} \ \mathbf{w}] \quad \dots (iii)$$

So, option (c) is correct.

Taking scalar product of Eq. (i) with $\mathbf{w}$, we get

$$\mathbf{w} \cdot \mathbf{w} + \mathbf{w} \cdot (\mathbf{w} \times \hat{\mathbf{u}}) = \mathbf{w} \cdot \hat{\mathbf{v}}$$

$$\Rightarrow |\mathbf{w}|^2 + 0 = \hat{\mathbf{v}} \cdot \mathbf{w} \Rightarrow \hat{\mathbf{v}} \cdot \mathbf{w} = |\mathbf{w}|^2$$

Substituting the value of $\hat{\mathbf{v}} \cdot \mathbf{w}$ in Eq. (ii), we get

$$[\hat{\mathbf{u}} \ \hat{\mathbf{v}} \ \mathbf{w}] = 1 - |\mathbf{w}|^2$$

So, option (b) is correct.

Hence, (a), (b) and (c) are the correct answers.

- Ex 39.** The vector $\mathbf{c}$, directed along the internal bisector of the angle between the vectors $\mathbf{a} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{b} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ with $|\mathbf{c}| = 5\sqrt{6}$, is

- (a) $\frac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ (b) $\frac{5}{3}(\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
 (c) $\frac{5}{3}(-\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (d) $\frac{5}{3}(-\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

Sol. $\frac{7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}}{\sqrt{49 + 16 + 16}} + \frac{-2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}}{\sqrt{4 + 1 + 4}}$
 $= \frac{1}{9}(7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}} - 6\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$
 $= \frac{1}{9}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$

$\therefore$ Required vector

$$= \pm 5\sqrt{6} \frac{(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})}{\sqrt{1 + 49 + 4}}$$

$$= \pm \frac{5}{3}(\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

Hence, (a) and (c) are the correct answers.

- Ex 40.** In a four dimensional space, where unit vectors along axes are $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$ and $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ are four non-zero vectors such that no vector can be expressed as linear combination of others and

$$(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = 0, \text{ then}$$

(a) $\lambda = 1$ (b) $\mu = -\frac{2}{3}$
 (c) $\gamma = \frac{2}{3}$ (d) $\delta = \frac{1}{3}$

Sol. $(\lambda - 1)(\mathbf{a}_1 - \mathbf{a}_2) + \mu(\mathbf{a}_2 + \mathbf{a}_3) + \gamma(\mathbf{a}_3 + \mathbf{a}_4 - 2\mathbf{a}_2) + \mathbf{a}_3 + \delta\mathbf{a}_4 = 0$
 i.e. $(\lambda - 1)\mathbf{a}_1 + (1 - \lambda + \mu - 2\gamma)\mathbf{a}_2 + (\mu + \gamma + 1)\mathbf{a}_3 + (\gamma + \delta)\mathbf{a}_4 = 0$

Since, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ and $\mathbf{a}_4$ are linearly independent.

$$\begin{aligned}\therefore \lambda - 1 &= 0, 1 - \lambda + \mu - 2\gamma = 0, \mu + \gamma + 1 = 0 \\ \text{and } \gamma + \delta &= 0 \\ \text{i.e. } \lambda &= 1, \mu = 2\gamma, \mu + \gamma + 1 = 0, \gamma + \delta = 0 \\ \text{i.e. } \lambda &= 1, \mu = -\frac{2}{3}, \gamma = -\frac{1}{3} \text{ and } \delta = \frac{1}{3}\end{aligned}$$

Hence, (a), (b) and (d) are the correct answers.

Ex 41. The vectors $\mathbf{a} = x\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + y\hat{\mathbf{j}} - z\hat{\mathbf{k}}$ are collinear, if

- (a) $x = 1, y = -2$ and $z = -5$
 (b) $x = \frac{1}{2}, y = -4$ and $z = -10$
 (c) $x = -\frac{1}{2}, y = 4$ and $z = 10$
 (d) $x = -1, y = 2$ and $z = 5$

Sol. The given vectors are collinear, if $\mathbf{a} = k\mathbf{b}$ for some k

$$\begin{aligned}\text{i.e. } x\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 5\hat{\mathbf{k}} &= k(\hat{\mathbf{i}} + y\hat{\mathbf{j}} - z\hat{\mathbf{k}}) \\ \Rightarrow (x - k)\hat{\mathbf{i}} + (-2 - ky)\hat{\mathbf{j}} + (5 + kz)\hat{\mathbf{k}} &= 0 \\ \therefore x &= k, y = -\frac{2}{k} \text{ and } z = -\frac{5}{k}\end{aligned}$$

Taking $k = 1, \frac{1}{2}, -\frac{1}{2}$ and -1 , we get the values given

by (a), (b), (c) and (d), respectively.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 42. If the vectors

$(x^2 - 7)\hat{\mathbf{i}} + (x - 2y + 1)\hat{\mathbf{j}} + (y^2 - 3)\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} + \hat{\mathbf{k}}$ are parallel, then ordered pair (x, y) can be

- (a) (3, 2) (b) (-3, -1) (c) (-1, 0) (d) (1, 0)

Sol. $(x^2 - 7)\hat{\mathbf{i}} + (x - 2y + 1)\hat{\mathbf{j}} + (y^2 - 3)\hat{\mathbf{k}} = \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{k}})$

$$\therefore x^2 - 7 = 2\lambda, x - 2y + 1 = 0, y^2 - 3 = \lambda$$

$$\text{and } x^2 - 7 = 2y^2 - 6$$

$$\text{i.e. } x^2 = 2y^2 + 1$$

$$\text{and } x = 2y - 1$$

$$\therefore (2y - 1)^2 = 2y^2 + 1$$

$$\text{i.e. } 4y^2 - 4y + 1 = 2y^2 + 1$$

$$\text{i.e. } 2y^2 - 4y = 0$$

$$\Rightarrow y = 0$$

$$\text{or } y = 2$$

If $y = 0$, then $x = -1$ and if $y = 2$, then $x = 3$

$$\therefore (-1, 0) \text{ or } (3, 2)$$

Hence, (a) and (c) are the correct answers.

Ex 43. The vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are of the same length and taken pairwise, they form equal angles.

If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{b} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$, then the coordinates of $\mathbf{c}$ are

- (a) (1, 0, 1) (b) (1, 2, 3)
 (c) (-1, 1, 2) (d) $\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}\right)$

Sol. Let $\mathbf{c} = (c_1, c_2, c_3)$

$$\text{Then, } |\mathbf{c}| = |\mathbf{a}| = |\mathbf{b}| = \sqrt{2} = \sqrt{c_1^2 + c_2^2 + c_3^2}.$$

It is given that the angles between the vectors are identical and equal to ϕ (say).

$$\text{Then, } \cos \phi = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{0 + 1 + 0}{\sqrt{2}\sqrt{2}} = \frac{1}{2},$$

$$\frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}||\mathbf{c}|} = \frac{c_1 + c_2}{\sqrt{2}\sqrt{2}} = \frac{1}{2}$$

$$\text{and } \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} = \frac{c_2 + c_3}{2} = \frac{1}{2}$$

$$\therefore c_1 + c_2 = 1 \text{ and } c_2 + c_3 = 1$$

$$\begin{aligned}\text{i.e. } c_1 &= 1 - c_2 \text{ and } c_3 = 1 - c_2 \\ \Rightarrow 2 &= c_1^2 + c_2^2 + c_3^2 = (1 - c_2)^2 + c_2^2 + (1 - c_2)^2 \\ &= 3c_2^2 + 2 - 4c_2\end{aligned}$$

Therefore, $c_2 = 0$ or $c_2 = \frac{4}{3}$. If $c_2 = 0$, then $c_1 = 1$,

$$c_3 = 1 \text{ and if } c_2 = \frac{4}{3}, \text{ then } c_1 = -\frac{1}{3}, c_3 = -\frac{1}{3}.$$

So, the coordinates of $\mathbf{c}$ are (1, 0, 1) or $\left(-\frac{1}{3}, \frac{4}{3}, -\frac{1}{3}\right)$.

Hence, (a) and (d) are the correct answers.

Ex 44. If $\mathbf{a} \times \mathbf{b} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{a} + \mathbf{b} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, then

$$(a) \mathbf{a} = \frac{1}{4}\hat{\mathbf{i}} + \frac{1}{4}\hat{\mathbf{j}} + \hat{\mathbf{k}}, \mathbf{b} = \frac{3}{4}\hat{\mathbf{i}} - \frac{13}{4}\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$$

$$(b) \mathbf{a} = \hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \mathbf{b} = \hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$$

$$(c) \mathbf{a} = -\frac{1}{4}\hat{\mathbf{i}} + \frac{7}{4}\hat{\mathbf{j}} + 2\hat{\mathbf{k}}, \mathbf{b} = \frac{5}{4}\hat{\mathbf{i}} - \frac{19}{4}\hat{\mathbf{j}} - 7\hat{\mathbf{k}}$$

$$(d) \mathbf{a} = \frac{1}{2}\hat{\mathbf{i}} - \frac{1}{2}\hat{\mathbf{j}}, \mathbf{b} = \frac{3}{4}\hat{\mathbf{i}} - \frac{13}{4}\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$$

Sol. Let $\mathbf{a} = a_1\hat{\mathbf{i}} + a_2\hat{\mathbf{j}} + a_3\hat{\mathbf{k}}$ and $\mathbf{b} = b_1\hat{\mathbf{i}} + b_2\hat{\mathbf{j}} + b_3\hat{\mathbf{k}}$

Taking cross-product with $\mathbf{a}$ on both sides of $\mathbf{a} + \mathbf{b} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, we get

$$\mathbf{a} \times (\mathbf{a} + \mathbf{b}) = \mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ a_1 & a_2 & a_3 \\ 1 & -3 & -4 \end{vmatrix}$$

$$= (-4a_2 + 3a_3)\hat{\mathbf{i}} + (a_3 + 4a_1)\hat{\mathbf{j}} + (-3a_1 - a_2)\hat{\mathbf{k}}$$

$$\therefore 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}} = (-4a_2 + 3a_3)\hat{\mathbf{i}} + (a_3 + 4a_1)\hat{\mathbf{j}} + (-3a_1 - a_2)\hat{\mathbf{k}}$$

$$\text{So, } 2 = -4a_2 + 3a_3, 2 = a_3 + 4a_1$$

$$\text{and } -1 = -3a_1 - a_2$$

On solving these equations, we have

$$a_3 = k, a_2 = \frac{3k - 2}{4}, a_1 = \frac{2 - k}{4}$$

where, k is any real number.

$$\text{Also, } b_1 = 1 - a_1 = 1 - \frac{2 - k}{4} = \frac{2 + k}{4},$$

$$b_2 = -3 - a_2 = -3 - \frac{3k - 2}{4} = -\frac{10 + 3k}{4}$$

$$b_3 = -4a_3 = -4 - k$$

$$\therefore \mathbf{a} = \frac{2 - k}{4}\hat{\mathbf{i}} + \frac{3k - 2}{4}\hat{\mathbf{j}} + k\hat{\mathbf{k}}$$

$$\text{and } \mathbf{b} = \frac{2 + k}{4}\hat{\mathbf{i}} - \frac{(10 + 3k)}{4}\hat{\mathbf{j}} - (4 + k)\hat{\mathbf{k}}$$

Hence, (a), (b) and (c) are the correct answers.

- Ex 45.** For non-coplanar vectors $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, $[\mathbf{A} \ \mathbf{B} \ \mathbf{C}] = |\mathbf{A}| |\mathbf{B}| |\mathbf{C}|$ holds if and only if
- $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{C} = \mathbf{C} \cdot \mathbf{A} = 0$
 - $\mathbf{A} \cdot \mathbf{B} = 0 = \mathbf{B} \cdot \mathbf{C}$
 - $\mathbf{A} \cdot \mathbf{B} = 0 = \mathbf{C} \cdot \mathbf{A}$
 - $\mathbf{B} \cdot \mathbf{C} = 0 = \mathbf{C} \cdot \mathbf{A}$

Sol. Let θ be the angle between $\mathbf{A}$ and $\mathbf{B}$ and ϕ be the angle between $\mathbf{C}$ and $\mathbf{A} \times \mathbf{B}$ i.e. the angle between $\mathbf{C}$ and the perpendicular to the plane containing $\mathbf{A}$ and $\mathbf{B}$.

Then,

$|(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C}| = |\mathbf{A} \times \mathbf{B}| |\mathbf{C}| \cos \phi = |\mathbf{A}| |\mathbf{B}| |\mathbf{C}| \sin \theta \cos \phi$
So, if the given relation holds, we have $\sin \theta \cos \phi = 1$, since $|\sin \theta| \leq 1$, $|\cos \phi| \leq 1$, so we must have $\sin \theta = 1$, $\cos \phi = 1$ i.e. $\theta = \frac{\pi}{2}$, $\phi = 0$. The former

implies that $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, so that $\mathbf{A} \cdot \mathbf{B} = 0$. On the other hand, if ϕ is zero, then $\mathbf{C}$ must be perpendicular to both $\mathbf{A}$ and $\mathbf{B}$, so that $\mathbf{B} \cdot \mathbf{C} = 0 = \mathbf{A} \cdot \mathbf{C}$.

Hence, (a), (b), (c) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Q. Nos. 46-49) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- Statement I is true, Statement II is false
- Statement I is false, Statement II is true

- Ex 46. Statement I** If $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors inclined at an angle α and $\hat{\mathbf{x}}$ is a unit vector bisecting the angle between them, then
- $$\hat{\mathbf{x}} = \frac{\hat{\mathbf{u}} + \hat{\mathbf{v}}}{2 \cos \frac{\alpha}{2}}$$

Statement II If $\triangle ABC$ is an isosceles triangle with $AB = AC = 1$, then vector representing bisector of angle A is given by

$$\mathbf{AD} = \frac{\mathbf{AB} + \mathbf{AC}}{2}.$$

Sol. In an isosceles $\triangle ABC$ in which $AB = AC$, the median and bisector from A must be same line.
 $\therefore$ Statement II is true.

Now,
$$\mathbf{AD} = \frac{\hat{\mathbf{u}} + \hat{\mathbf{v}}}{2}$$

and
$$|\mathbf{AD}|^2 = \frac{1}{2} 2 \cos^2 \frac{\alpha}{2}$$

So,
$$|\mathbf{AD}| = \cos \frac{\alpha}{2}$$

Unit vector along $\mathbf{AD}$ i.e. $\hat{\mathbf{x}}$ is given by

$$\begin{aligned} \hat{\mathbf{x}} &= \frac{\mathbf{AD}}{|\mathbf{AD}|} \\ &= \frac{\hat{\mathbf{u}} + \hat{\mathbf{v}}}{2 \cos \frac{\alpha}{2}} \end{aligned}$$

Hence, (a) is the correct answer.

- Ex 47. Statement I** A relation between the vectors $\mathbf{r}$, $\mathbf{a}$ and $\mathbf{b}$ is $\mathbf{r} \times \mathbf{a} = \mathbf{b} \Rightarrow \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$.

Statement II $\mathbf{r} \cdot \mathbf{a} = 0$

Sol. Since, $\mathbf{b} = \mathbf{r} \times \mathbf{a}$
We have, $\mathbf{a} \times \mathbf{b} = \mathbf{a} \times (\mathbf{r} \times \mathbf{a})$
$$= (\mathbf{a} \cdot \mathbf{a}) \mathbf{r} - (\mathbf{a} \cdot \mathbf{r}) \mathbf{a} = (\mathbf{a} \cdot \mathbf{a}) \mathbf{r}$$

 $\therefore \mathbf{a} \cdot \mathbf{r} = 0$
 $\therefore \mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}$

Hence, (b) is the correct answer.

- Ex 48. Statement I** If $\mathbf{a} = 3\hat{\mathbf{i}} + \hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{d} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}}$, then there exist real numbers α , β and γ such that $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c} + \gamma\mathbf{d}$.

Statement II Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ be $\mathbf{d}$ are four vectors in a three dimensional space. If $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ are non-coplanar, then there exist real numbers α , β and γ such that $\mathbf{a} = \alpha\mathbf{b} + \beta\mathbf{c} + \gamma\mathbf{d}$.

Sol. Both statements are correct but Statement II is not correct explanation of Statement I, because vectors $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ in Statement I are coplanar.
Hence, (b) is the correct answer.

- Ex 49. Statement I** Let $A(\mathbf{a})$, $B(\mathbf{b})$ and $C(\mathbf{c})$ be three points such that $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{c} = -\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$, then $OABC$ is a tetrahedron.

Statement II Let $A(\mathbf{a})$, $B(\mathbf{b})$ and $C(\mathbf{c})$ be three points such that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar, then $OABC$ is a tetrahedron, where O is the origin.

Sol. Statement I is false and Statement II is true, since
$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

 $\therefore \mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar.
Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 50-53) Let two unit vectors along two lines OA and OB be $\hat{a}$ and $\hat{b}$ respectively. Take their point of intersection as the origin and let P be any point on the bisector of angle between the lines OA and OB . Draw PM parallel to AO cutting OB at M . $\angle AOP = \angle POM = \angle OPM$ and hence, $OM = PM$.

But $OM = t\hat{b}$ and $MP = t\hat{a}$ (since, $OM \parallel \hat{b}$, $MP \parallel \hat{a}$ and their magnitudes are same), then

$$\mathbf{OP} = \mathbf{r} = \mathbf{OM} + \mathbf{MP} = t(\hat{b} + \hat{a}) \quad \dots(i)$$

For external bisector OP' , the angle between OB and OA is the same as the internal bisector of the angle between the unit vectors along them being $-\hat{b}$ and $\hat{a}$ and hence, the equation of OP' is

$$\mathbf{OP}' = \mathbf{r}' = t(\hat{a} - \hat{b}) \quad \dots(ii)$$

For any two vectors $\hat{a}$ and $\hat{b}$, Eqs. (i) and (ii) reduce to

$$\mathbf{r} = t \left(\frac{\hat{a}}{|\hat{a}|} \pm \frac{\hat{b}}{|\hat{b}|} \right)$$

Ex 50. Let ABC be a triangle and $\hat{a}, \hat{b}, \hat{c}$ be the position vectors of the points A, B, C respectively. External bisectors of $\angle B$ and $\angle C$ meet at P with the sides of the triangle as a, b and c , then the position vector of P becomes

- (a) $\frac{(-b)\hat{b} + (-c)\hat{c}}{(b+c)}$
 (b) $\frac{a\hat{a} + (-b)\hat{b} + (-c)\hat{c}}{(a-b-c)}$
 (c) $\left(\frac{\hat{a} + \hat{b} + \hat{c}}{3} \right) (abc)$
 (d) $\frac{a\hat{a} + b\hat{b} + c\hat{c}}{(\hat{a} + \hat{b} + \hat{c})}$

Sol. Vectors along AB, BC and CA are $\hat{b} - \hat{a}, \hat{c} - \hat{b}$ and $\hat{a} - \hat{c}$ respectively. Therefore, the bisectors of the angle B and C respectively, are

$$\mathbf{r} = \hat{b} + \mu \left(\frac{\hat{b} - \hat{a}}{c} + \frac{\hat{c} - \hat{b}}{a} \right)$$

$$\text{and } \mathbf{r} = \hat{c} + t \left(\frac{\hat{b} - \hat{c}}{a} + \frac{\hat{c} - \hat{a}}{b} \right)$$

For the point P ,

$$\hat{b} + \mu \left(\frac{\hat{b} - \hat{a}}{c} + \frac{\hat{c} - \hat{b}}{a} \right) = \hat{c} + t \left(\frac{\hat{b} - \hat{c}}{a} + \frac{\hat{c} - \hat{a}}{b} \right)$$

$$\Rightarrow t = \frac{b\mu}{c} \text{ and } \mu = \frac{ac}{b+c-a}$$

$$\text{The position vector of } P \text{ is } \frac{b\hat{b} + c\hat{c} - a\hat{a}}{b+c-a}.$$

Hence, (b) is the correct answer.

Ex 51. If the interior and exterior bisectors of $\angle A$ of a ΔABC meet the base BC at D and E , then

- (a) $2BC = BD + BE$
 (b) $BC^2 = BD \times BE$
 (c) $\frac{2}{BC} = \frac{1}{BD} + \frac{1}{BE}$
 (d) None of the above

Sol. Let A be the initial point and the position vectors of B and C be $\hat{b}$ and $\hat{c}$ respectively.

Therefore, let $AB = c$ and $AC = b$. The internal bisector is $\mathbf{r} = t \left(\frac{\hat{b}}{c} + \frac{\hat{c}}{b} \right)$ and the equation of the line

BC is $\mathbf{r} = \hat{b} + k(\hat{c} - \hat{b})$ and thus, the position vector of D is $\frac{b\hat{b} + c\hat{c}}{b+c}$.

$$\Rightarrow \mathbf{BD} = \frac{b\hat{b} + c\hat{c}}{b+c} - \hat{b} = \frac{c(\hat{c} - \hat{b})}{b+c}$$

$$BD = \frac{c|\hat{c} - \hat{b}|}{b+c} = \frac{ca}{b+c}$$

$$\text{Similarly, } BE = \frac{c|\hat{c} - \hat{b}|}{c-b} = \frac{ca}{c-b}$$

$$\Rightarrow \frac{1}{BE} + \frac{1}{BD} = \frac{c-b}{ac} + \frac{c+b}{ac} = \frac{2c}{ac} = \frac{2}{a} = \frac{2}{BC}$$

Hence, (c) is the correct answer.

Ex 52. If ABC is a triangle of sides a, b and c with position vectors of A, B and C being $\hat{a}, \hat{b}$ and $\hat{c}$ respectively, then the position vectors of its incentre is

- (a) $\frac{(\hat{a} + \hat{b} + \hat{c})}{3}$
 (b) $\left(\frac{\hat{a} \times \hat{b} + \hat{b} \times \hat{c} + \hat{c} \times \hat{a}}{a^2 + b^2 + c^2} \right)$
 (c) $\left(\frac{a\hat{a} + b\hat{b} + c\hat{c}}{a+b+c} \right)$
 (d) None of the above

Sol. Incentre is the point of intersection of the internal bisectors of angles A, B and C .

Equation of the bisectors of angles A and B are

$$\mathbf{r} = \hat{a} + t \left(\frac{\hat{c} - \hat{a}}{b} + \frac{\hat{b} - \hat{a}}{c} \right)$$

$$\text{and } \mathbf{r} = \hat{b} + \mu \left(\frac{\hat{b} - \hat{a}}{c} + \frac{\hat{c} - \hat{b}}{a} \right)$$

For the coefficients of $\hat{a}, \hat{b}$ and $\hat{c}$, we find that the position vector of the incentre is

$$\frac{a\hat{a} + b\hat{b} + c\hat{c}}{a+b+c}$$

Hence, (c) is the correct answer.

Ex 53. ABC is a triangle, AD and AD' are internal and external bisectors of angle A , meeting BC at D and D' respectively, A' is the mid-point of DD' and B', C' are similar points on CA and AB . Then, A', B' and C'

- (a) lie on a plane
- (b) form an equilateral triangle
- (c) form an isosceles triangle
- (d) None of the above

Sol. Since, the three vertices A, B and C of the triangle are coplanar, so all the points lying on the linear AB, BC and CD are naturally coplanar.

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 54-57) If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three given non-coplanar vectors and any arbitrary vector $\mathbf{r}$ in space, where

$$\Delta_1 = \begin{vmatrix} \mathbf{r} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{a} \\ \mathbf{r} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{r} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}, \Delta_2 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{r} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{a} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{r} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{r} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix},$$

$$\Delta_3 = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{a} & \mathbf{r} \cdot \mathbf{a} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{r} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{r} \cdot \mathbf{c} \end{vmatrix} \text{ and } \Delta = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{a} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}$$

Ex 54. The vector $\mathbf{r}$ is expressible in the form

- (a) $\mathbf{r} = \frac{\Delta_1}{2\Delta} \mathbf{a} + \frac{\Delta_2}{2\Delta} \mathbf{b} + \frac{\Delta_3}{2\Delta} \mathbf{c}$
- (b) $\mathbf{r} = \frac{2\Delta_1}{\Delta} \mathbf{a} + \frac{2\Delta_2}{\Delta} \mathbf{b} + \frac{2\Delta_3}{\Delta} \mathbf{c}$
- (c) $\mathbf{r} = \frac{\Delta}{\Delta_1} \mathbf{a} + \frac{\Delta}{\Delta_2} \mathbf{b} + \frac{\Delta}{\Delta_3} \mathbf{c}$
- (d) $\mathbf{r} = \frac{\Delta_1}{\Delta} \mathbf{a} + \frac{\Delta_2}{\Delta} \mathbf{b} + \frac{\Delta_3}{\Delta} \mathbf{c}$

Ex 55. The vector $\mathbf{r}$ is expressible as

- (a) $\mathbf{r} = \frac{[\mathbf{r} \mathbf{a} \mathbf{b}]}{2[\mathbf{a} \mathbf{b} \mathbf{c}]} \mathbf{a} + \frac{[\mathbf{r} \mathbf{c} \mathbf{a}]}{2[\mathbf{a} \mathbf{b} \mathbf{c}]} \mathbf{b} + \frac{[\mathbf{r} \mathbf{a} \mathbf{b}]}{2[\mathbf{a} \mathbf{b} \mathbf{c}]} \mathbf{c}$
- (b) $\mathbf{r} = \frac{2[\mathbf{r} \mathbf{b} \mathbf{c}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]} \mathbf{a} + \frac{2[\mathbf{r} \mathbf{c} \mathbf{a}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]} \mathbf{b} + \frac{2[\mathbf{r} \mathbf{a} \mathbf{b}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]} \mathbf{c}$
- (c) $\mathbf{r} = [\mathbf{a} \mathbf{b} \mathbf{c}] \left(\frac{\mathbf{a}}{[\mathbf{r} \mathbf{b} \mathbf{c}]} + \frac{\mathbf{b}}{[\mathbf{r} \mathbf{c} \mathbf{a}]} + \frac{\mathbf{c}}{[\mathbf{r} \mathbf{a} \mathbf{b}]} \right)$
- (d) None of the above

Ex 56. If vector $\mathbf{r}$ is expressible as $\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$, then

- (a) $\mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{a}}{[\mathbf{a} \mathbf{b} \mathbf{c}]} (\mathbf{b} \times \mathbf{c}) + \frac{\mathbf{a} \cdot \mathbf{b}}{[\mathbf{a} \mathbf{b} \mathbf{c}]} (\mathbf{c} \times \mathbf{a}) + \frac{\mathbf{a} \cdot \mathbf{c}}{[\mathbf{a} \mathbf{b} \mathbf{c}]} (\mathbf{a} \times \mathbf{b})$
- (b) $\mathbf{a} = \mathbf{a} \cdot \mathbf{a} (\mathbf{b} \times \mathbf{c}) + \mathbf{a} \cdot \mathbf{b} (\mathbf{c} \times \mathbf{a}) + \mathbf{a} \cdot \mathbf{c} (\mathbf{a} \times \mathbf{b})$
- (c) $\mathbf{a} = [\mathbf{a} \mathbf{b} \mathbf{c}] (\mathbf{b} \times \mathbf{c}) + [\mathbf{a} \mathbf{b} \mathbf{c}] (\mathbf{c} \times \mathbf{a}) + [\mathbf{a} \mathbf{b} \mathbf{c}] (\mathbf{a} \times \mathbf{b})$
- (d) None of the above

Ex 57. The value of $\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{p} & \mathbf{b} \cdot \mathbf{p} & \mathbf{c} \cdot \mathbf{p} \\ \mathbf{a} \cdot \mathbf{q} & \mathbf{b} \cdot \mathbf{p} & \mathbf{c} \cdot \mathbf{q} \end{vmatrix}$ is

- (a) $(\mathbf{p} \times \mathbf{q}) [\mathbf{a} \times \mathbf{b} \mathbf{b} \times \mathbf{c} \mathbf{c} \times \mathbf{a}]$
- (b) $2 (\mathbf{p} \times \mathbf{q}) [\mathbf{a} \times \mathbf{b} \mathbf{b} \times \mathbf{c} \mathbf{c} \times \mathbf{a}]$
- (c) $4 (\mathbf{p} \times \mathbf{q}) [\mathbf{a} \times \mathbf{b} \mathbf{b} \times \mathbf{c} \mathbf{c} \times \mathbf{a}]$
- (d) $(\mathbf{p} \times \mathbf{q}) \sqrt{[\mathbf{a} \times \mathbf{b} \mathbf{b} \times \mathbf{c} \mathbf{c} \times \mathbf{a}]}$

Sol. (Ex. Nos. 54-57)

Since, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors. Therefore, there exist scalars x, y and z such that

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z\mathbf{c} \quad \dots(i)$$

Taking dot product with $\mathbf{b} \times \mathbf{c}, \mathbf{c} \times \mathbf{a}$ and $\mathbf{a} \times \mathbf{b}$ successively, we get

$$\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = (x\mathbf{a} + y\mathbf{b} + z\mathbf{c}) \cdot (\mathbf{b} \times \mathbf{c})$$

$$\mathbf{r} \cdot (\mathbf{c} \times \mathbf{a}) = (x\mathbf{a} + y\mathbf{b} + z\mathbf{c}) \cdot (\mathbf{c} \times \mathbf{a})$$

$$\mathbf{r} \cdot (\mathbf{a} \times \mathbf{b}) = (x\mathbf{a} + y\mathbf{b} + z\mathbf{c}) \cdot (\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow [\mathbf{r} \mathbf{b} \mathbf{c}] = x [\mathbf{a} \mathbf{b} \mathbf{c}]$$

$$[\mathbf{r} \mathbf{c} \mathbf{a}] = y [\mathbf{a} \mathbf{b} \mathbf{c}]$$

$$\text{and } [\mathbf{r} \mathbf{a} \mathbf{b}] = z [\mathbf{a} \mathbf{b} \mathbf{c}]$$

$$x = \frac{[\mathbf{r} \mathbf{b} \mathbf{c}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]}, y = \frac{[\mathbf{r} \mathbf{c} \mathbf{a}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$$

$$\text{and } z = \frac{[\mathbf{r} \mathbf{a} \mathbf{b}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$$

Substituting the values of x, y and z in Eq. (i), we get

$$\mathbf{r} = \left\{ \frac{[\mathbf{r} \mathbf{b} \mathbf{c}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]} \right\} \mathbf{a} + \left\{ \frac{[\mathbf{r} \mathbf{c} \mathbf{a}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]} \right\} \mathbf{b} + \left\{ \frac{[\mathbf{r} \mathbf{a} \mathbf{b}]}{[\mathbf{a} \mathbf{b} \mathbf{c}]} \right\} \mathbf{c}$$

Again, $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors.

$$\therefore [\mathbf{a} \mathbf{b} \mathbf{c}] \neq 0$$

$$\Rightarrow [\mathbf{a} \mathbf{b} \mathbf{c}]^2 \neq 0$$

$$\Rightarrow \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{a} \\ \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} = [\mathbf{a} \mathbf{b} \mathbf{c}]^2 \neq 0$$

Since, any vector $\mathbf{r}$ in space can be expressed as a linear combination of three non-coplanar vectors.

So, let $\mathbf{r} = l\mathbf{a} + m\mathbf{b} + n\mathbf{c} \quad \dots(ii)$

Taking dot products on both sides successively by $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, we get

$$\mathbf{r} \cdot \mathbf{a} = l\mathbf{a} \cdot \mathbf{a} + m\mathbf{b} \cdot \mathbf{a} + n\mathbf{c} \cdot \mathbf{a} \quad \dots(iii)$$

$$\mathbf{r} \cdot \mathbf{b} = l\mathbf{b} \cdot \mathbf{a} + m\mathbf{b} \cdot \mathbf{b} + n\mathbf{c} \cdot \mathbf{b} \quad \dots(iv)$$

$$\mathbf{r} \cdot \mathbf{c} = l\mathbf{c} \cdot \mathbf{a} + m\mathbf{b} \cdot \mathbf{c} + n\mathbf{c} \cdot \mathbf{c} \quad \dots(v)$$

On eliminating l, m and n from above four relations, we get

$$\begin{vmatrix} \mathbf{r} & \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{r} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{a} \\ \mathbf{r} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{b} \\ \mathbf{r} \cdot \mathbf{c} & \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} = 0$$

On expanding this determinant along first row, we obtain

$$\mathbf{r} = \left(\frac{\Delta_1}{\Delta} \right) \mathbf{a} + \left(\frac{\Delta_2}{\Delta} \right) \mathbf{b} + \left(\frac{\Delta_3}{\Delta} \right) \mathbf{c}$$

We know that,

$$[\mathbf{a} \times \mathbf{b} \mathbf{b} \times \mathbf{c} \mathbf{c} \times \mathbf{a}] = [\mathbf{a} \mathbf{b} \mathbf{c}]^2$$

$$\text{and } [\mathbf{a} \times \mathbf{b} \mathbf{b} \times \mathbf{c} \mathbf{c} \times \mathbf{a}] \neq 0 \quad [\because [\mathbf{a} \mathbf{b} \mathbf{c}] \neq 0]$$

$\Rightarrow \mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$ are non-coplanar.

We also know that, any vector in space can be expressed as a linear combination of any three non-coplanar vectors.

So, let $\mathbf{a} = l(\mathbf{b} \times \mathbf{c}) + m(\mathbf{c} \times \mathbf{a}) + n(\mathbf{a} \times \mathbf{b})$

Taking dot product on both sides successively by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we get

$$\mathbf{a} \cdot \mathbf{a} = l \{ \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) \}$$

$$\mathbf{a} \cdot \mathbf{b} = m \{ (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} \}$$

$$\text{and } \mathbf{c} \cdot \mathbf{a} = n \{ (\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})) \}$$

$$l = \frac{\mathbf{a} \cdot \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}, m = \frac{\mathbf{a} \cdot \mathbf{b}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} \text{ and } n = \frac{\mathbf{c} \cdot \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

$$\therefore \mathbf{a} = \frac{\mathbf{a} \cdot \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} (\mathbf{b} \times \mathbf{c}) + \frac{\mathbf{a} \cdot \mathbf{b}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} (\mathbf{c} \times \mathbf{a}) + \frac{\mathbf{a} \cdot \mathbf{c}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} (\mathbf{a} \times \mathbf{b})$$

Again, let $\mathbf{a} = a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}}$, $\mathbf{b} = b_1 \hat{\mathbf{i}} + b_2 \hat{\mathbf{j}} + b_3 \hat{\mathbf{k}}$

$$\mathbf{c} = c_1 \hat{\mathbf{i}} + c_2 \hat{\mathbf{j}} + c_3 \hat{\mathbf{k}}, \mathbf{p} = p_1 \hat{\mathbf{i}} + p_2 \hat{\mathbf{j}} + p_3 \hat{\mathbf{k}}$$

and $\mathbf{q} = q_1 \hat{\mathbf{i}} + q_2 \hat{\mathbf{j}} + q_3 \hat{\mathbf{k}}$

$$\text{Then, } \begin{vmatrix} a_1 \hat{\mathbf{i}} + a_2 \hat{\mathbf{j}} + a_3 \hat{\mathbf{k}} & b_1 \hat{\mathbf{i}} + b_2 \hat{\mathbf{j}} + b_3 \hat{\mathbf{k}} \\ a_1 p_1 + a_2 p_2 + a_3 p_3 & b_1 p_1 + b_2 p_2 + b_3 p_3 \\ a_1 q_1 + a_2 q_2 + a_3 q_3 & b_1 q_1 + b_2 q_2 + b_3 q_3 \end{vmatrix}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= (\mathbf{p} \times \mathbf{q}) [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \sqrt{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2} (\mathbf{p} \times \mathbf{q})$$

$$= \sqrt{[\mathbf{a} \times \mathbf{b} \ \mathbf{b} \times \mathbf{c} \ \mathbf{c} \times \mathbf{a}]} (\mathbf{p} \times \mathbf{q})$$

54. (d) 55. (d) 56. (a) 57. (d)

Passage III (Ex. Nos. 58-60) Let $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and $\mathbf{c} = -2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$ and let $\mathbf{a}_1$ be projection of $\mathbf{a}$ and $\mathbf{b}$, the projection of $\mathbf{a}_1$ on $\mathbf{c}$.

Ex 58. $\mathbf{a}_2$ is equal to

- (a) $\frac{943}{49} (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$
 (b) $\frac{943}{49^2} (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$
 (c) $\frac{943}{49} (-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$
 (d) $\frac{943}{49^2} (-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$

$$\text{Sol. } \mathbf{a}_1 = \left[2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}} \cdot \frac{(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7} \right] \frac{2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}}{7}$$

$$= \frac{-41}{49} (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

$$\mathbf{a}_2 = \frac{-41}{49} \left[(2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \cdot \frac{(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7} \right] \frac{(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{7}$$

$$= \frac{-41}{(49)^2} (-4 - 9 + 36) (-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

$$= \frac{943}{49^2} (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$$

Hence, (b) is the correct answer.

Ex 59. $\mathbf{a}_1 \cdot \mathbf{b}$ is equal to

- (a) -41
 (b) $-\frac{41}{7}$
 (c) 41
 (d) 287

$$\text{Sol. } \mathbf{a}_1 \cdot \mathbf{b} = \frac{41}{49} (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = -41$$

Hence, (a) is the correct answer.

Ex 60. Which of the following is true?

- (a) $\mathbf{a}$ and $\mathbf{a}_2$ are collinear
 (b) $\mathbf{a}_1$ and $\mathbf{c}$ are collinear
 (c) $\mathbf{a}$, $\mathbf{a}_1$ and $\mathbf{b}$ are coplanar
 (d) $\mathbf{a}$, $\mathbf{a}_1$ and $\mathbf{a}_2$ are coplanar

Sol. $\mathbf{a}$, $\mathbf{a}_1$ and $\mathbf{b}$ are coplanar because $\mathbf{a}_1$ and $\mathbf{b}$ are collinear.

Hence, (c) is the correct answer.

Passage IV (Ex. Nos. 61-63) Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be a system of three non-coplanar vectors. Then, the system $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ which satisfies

$$\mathbf{a} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{b}' = \mathbf{c} \cdot \mathbf{c}' = 1$$

and $\mathbf{a} \cdot \mathbf{b}' = \mathbf{a} \cdot \mathbf{c}' = \mathbf{b} \cdot \mathbf{a}' = \mathbf{b} \cdot \mathbf{c}' = \mathbf{c} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{b}' = 0$

is called the reciprocal system to the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Ex 61. If $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$ is a reciprocal system of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}][\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}'] = |\mathbf{a} \times \mathbf{a}'| |\mathbf{b} \times \mathbf{b}'| |\mathbf{c} \times \mathbf{c}'|$
 (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}][\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}'] = 1$
 (c) $\mathbf{a} \times \mathbf{a}' = \mathbf{b} \times \mathbf{b}' = \mathbf{c} \times \mathbf{c}'$
 (d) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}][\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}'] < 1$

Ex 62. $\mathbf{a} \times \mathbf{a}' + \mathbf{b} \times \mathbf{b}' + \mathbf{c} \times \mathbf{c}'$ is a

- (a) zero vector
 (b) non-zero vector
 (c) $\frac{1}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^3} [(\mathbf{a} \times \mathbf{a}') \times \mathbf{b}]$
 (d) scalar multiple of $\mathbf{a}' + \mathbf{b}' + \mathbf{c}'$

Ex 63. The value of $\mathbf{a}$ is

- (a) $\frac{\mathbf{a}' \times \mathbf{b}'}{[\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}']}$
 (b) $\frac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}']}$
 (c) $\frac{\mathbf{a}' \times \mathbf{c}'}{[\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}']}$
 (d) $\frac{1}{[\mathbf{a}' \ \mathbf{b}' \ \mathbf{c}']} [\mathbf{a}' \times \mathbf{b}' + \mathbf{b}' \times \mathbf{c}']$

Sol. (Ex. Nos. 61-63) Since, by definition of reciprocal systems, $\mathbf{b} \cdot \mathbf{a}' = \mathbf{c} \cdot \mathbf{a}' = 0$, $\mathbf{a}'$ is perpendicular to both $\mathbf{b}$ and $\mathbf{c}$ and hence parallel to $\mathbf{b} \times \mathbf{c}$, so $\mathbf{a}' = t(\mathbf{b} \times \mathbf{c})$.

Also, $\mathbf{a} \cdot \mathbf{a}' = 1$,
 $1 = \mathbf{a} \cdot \mathbf{a}' = t\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$
 $\Rightarrow t = \frac{1}{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})} = \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$

Thus, $\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$

Similarly, $\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$

and $\mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$

$\Rightarrow \mathbf{a} \times \mathbf{a}' + \mathbf{b} \times \mathbf{b}' + \mathbf{c} \times \mathbf{c}'$
 $= \frac{\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$

Now, $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$,

$\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = (\mathbf{b} \cdot \mathbf{a})\mathbf{c} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$

and $\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{c} \cdot \mathbf{b})\mathbf{a} - (\mathbf{c} \cdot \mathbf{a})\mathbf{b}$

$\therefore \mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{0}$

So, $\mathbf{a} \times \mathbf{a}' + \mathbf{b} \times \mathbf{b}' + \mathbf{c} \times \mathbf{c}' = \mathbf{0}$

Now, $[\mathbf{a}' \mathbf{b}' \mathbf{c}'] = (\mathbf{a}' \times \mathbf{b}') \cdot \mathbf{c}'$

$= \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]^3} [(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})] \cdot (\mathbf{a} \times \mathbf{b})$

$= \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]^3} [(\mathbf{e} \cdot \mathbf{a})\mathbf{c} - (\mathbf{e} \cdot \mathbf{c})\mathbf{a}] \cdot (\mathbf{a} \times \mathbf{b})$

$[\because \mathbf{e} = \mathbf{b} \times \mathbf{c}]$

$= \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]^3} [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}] [\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})]$

$= \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]^3} [(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}] [\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})]$

$= \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]^3} [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] [(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}]$

$= \frac{1}{[\mathbf{a} \mathbf{b} \mathbf{c}]}$

We can also show that,

$\mathbf{a} = \frac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}' \mathbf{b}' \mathbf{c}']}, \mathbf{b} = \frac{\mathbf{c}' \times \mathbf{a}'}{[\mathbf{a}' \mathbf{b}' \mathbf{c}']}$ and $\mathbf{c} = \frac{\mathbf{a}' \times \mathbf{b}'}{[\mathbf{a}' \mathbf{b}' \mathbf{c}']}$

61. (b)

62. (a)

63. (d)

Type 5. Match the Columns

Ex 64. Match the statements of Column I with values of Column II.

Column I	Column II
A. If $r = r_1 + \lambda a$ and $r = r_2 + \mu b$ are two skew-lines, then the vector along the line of shortest distance is parallel to	p. $(r_1 - r_2) \cdot (\mathbf{a} \times \mathbf{b}) = 0$
B. If the straight lines $r = r_1 + \lambda a$ and $r = r_2 + \mu b$ are coplanar and $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = 0$. Then, vector $\mathbf{c}$ in terms of the vectors r_1, r_2, a and b is	q. $(r_1 \times r_2) \cdot (\mathbf{a} \times \mathbf{b}) = 0$
C. If the straight lines $r = r_1 + \lambda a$ and $r = r_2 + \mu b$ are intersecting at a point, then	r. $(r_1 - r_2)$
D. If the point whose position vector is $\mathbf{a} \times \mathbf{b}$, lies on the straight line segment joining the points whose position vectors are r_1 and r_2 , then	s. $\mathbf{a} \times \mathbf{b}$

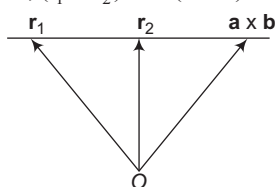
Sol. A. Line of shortest distance is perpendicular to both the lines. Hence, the vector is along the line of shortest distance $\mathbf{a} \times \mathbf{b}$.

B. The vector $\mathbf{r}_1 - \mathbf{r}_2$ is along the plane in which both the straight lines are lying and $\mathbf{a} \times \mathbf{b}$ is the vector perpendicular to the plane of line.

$\therefore (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{a} \times \mathbf{b}) = 0 \Rightarrow \mathbf{c} = \mathbf{r}_1 - \mathbf{r}_2$

C. If lines are intersecting, then
 $(\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{a} \times \mathbf{b}) = 0$

D. Since, $(\mathbf{r}_1 - \mathbf{r}_2)$ and $(\mathbf{a} \times \mathbf{b})$ are coplanar.



$\therefore (\mathbf{r}_1 \times \mathbf{r}_2) \cdot (\mathbf{a} \times \mathbf{b}) = 0$

$A \rightarrow s; B \rightarrow r; C \rightarrow q; D \rightarrow p$

Ex 65. Match the statements of Column I with values of Column II.

Column I	Column II
A. If the vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ form sides BA, CA and AB of $\triangle ABC$, then	p. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$
B. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ form three adjacent sides of regular tetrahedron, then	q. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
C. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}, \mathbf{b} \times \mathbf{c} = \mathbf{a}$, then	r. $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$
D. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors and $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, then	s. $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = -\frac{3}{2}$

Sol. A. $|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}| = 2\Delta ABC$

Also, directions of $\mathbf{a} \times \mathbf{b}, \mathbf{b} \times \mathbf{c}$ and $\mathbf{c} \times \mathbf{a}$ are same.

Hence, $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$.

B. For regular tetrahedron, all sides are of equal length. So, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}|$. Also, all the faces are equilateral triangle.

Thus, angle between $\mathbf{a}$ and $\mathbf{b}$ is 60° , $\mathbf{b}$ and $\mathbf{c}$ is 60° and $\mathbf{a}$ and $\mathbf{c}$ is 60° .

Hence, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a}$

C. $\mathbf{a} \times \mathbf{b} = \mathbf{c} \Rightarrow \mathbf{a} \perp \mathbf{c}$ and $\mathbf{b} \perp \mathbf{c}$

$\mathbf{b} \times \mathbf{c} = \mathbf{a} \Rightarrow \mathbf{b} \perp \mathbf{a}$ and $\mathbf{c} \perp \mathbf{a}$

$\therefore \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular.

Also, $|\mathbf{a} \times \mathbf{b}| = |\mathbf{c}| \Rightarrow |\mathbf{a}| |\mathbf{b}| = |\mathbf{c}|$

and $|\mathbf{b} \times \mathbf{c}| = |\mathbf{a}| \Rightarrow |\mathbf{b}| |\mathbf{c}| = |\mathbf{a}|$

$\Rightarrow |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$

D. Given, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$

$\Rightarrow a^2 + b^2 + c^2 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = 0$

$\Rightarrow \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = -\frac{3}{2}$

$A \rightarrow r; B \rightarrow p; C \rightarrow q; D \rightarrow s$

Ex 66. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be the vectors. Match the statements of Column I with values of Column II.

Column I	Column II
A. $\mathbf{r} \times \mathbf{a} + (\mathbf{r} \cdot \mathbf{b})\mathbf{c} = \mathbf{d}$	p. $\mathbf{r} = \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{b}}\mathbf{b}$
B. $\mathbf{r} = \mathbf{r} \times \mathbf{a} = \mathbf{b}$	q. $\mathbf{r} = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a} + \mathbf{a} \times \frac{\mathbf{a} \times (\mathbf{d} \times \mathbf{c})}{(\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{a})}$
C. $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$, $\mathbf{r} \cdot \mathbf{a} = 0$	r. $\mathbf{r} = -\frac{1}{ \mathbf{b} ^2}(\mathbf{a} \times \mathbf{b}) + y\mathbf{b}$, where y is a parameter
D. $\mathbf{r} \times \mathbf{b} = \mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are such that $\mathbf{a}$ is perpendicular to $\mathbf{b}$	s. $\mathbf{r} = \frac{1}{1 + \mathbf{a} ^2} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\lambda}\mathbf{a} + \mathbf{b} + \mathbf{a} \times \mathbf{b} \right)$

Sol. A. Taking cross-product with $\mathbf{c}$ on both sides of the given equation, we get

$$(\mathbf{r} \times \mathbf{a}) \times \mathbf{c} = \mathbf{d} \times \mathbf{c} \quad [\because \mathbf{c} \times \mathbf{c} = \mathbf{0}]$$

$$\Rightarrow (\mathbf{r} \cdot \mathbf{c})\mathbf{a} - (\mathbf{a} \cdot \mathbf{c})\mathbf{r} = \mathbf{d} \times \mathbf{c}$$

Taking cross-product with $\mathbf{a}$, we get

$$-(\mathbf{a} \cdot \mathbf{c})\mathbf{a} \times \mathbf{r} = \mathbf{a} \times (\mathbf{d} \times \mathbf{c})$$

$$\Rightarrow \mathbf{r} \times \mathbf{a} = \frac{\mathbf{a} \times (\mathbf{d} \times \mathbf{c})}{\mathbf{a} \cdot \mathbf{c}}$$

Again, taking cross-product with $\mathbf{a}$, we get

$$\mathbf{a} \times (\mathbf{r} \times \mathbf{a}) = \mathbf{a} \times \frac{\mathbf{a} \times (\mathbf{d} \times \mathbf{c})}{\mathbf{a} \cdot \mathbf{c}}$$

$$\Rightarrow (\mathbf{a} \cdot \mathbf{a})\mathbf{r} - (\mathbf{a} \cdot \mathbf{r})\mathbf{a} = \mathbf{a} \times \frac{\mathbf{a} \times (\mathbf{d} \times \mathbf{c})}{\mathbf{a} \cdot \mathbf{c}}$$

$$\Rightarrow \mathbf{r} = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a} \cdot \mathbf{a}}\mathbf{a} + \mathbf{a} \times \frac{\mathbf{a} \times (\mathbf{d} \times \mathbf{c})}{(\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{a})}$$

$$= \lambda\mathbf{a} + \mathbf{a} \times \frac{\mathbf{a} \times (\mathbf{d} \times \mathbf{c})}{(\mathbf{a} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{a})}$$

$$\text{where, } \lambda = \frac{\mathbf{a} \cdot \mathbf{r}}{\mathbf{a} \cdot \mathbf{a}}$$

B. Suppose that $\mathbf{r}$ is a solution. Expressing $\mathbf{r}$ as a linear combination of non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{a} \times \mathbf{b}$, we have

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$$

Substituting in the given equation, we get

$$\lambda[x\mathbf{a} + y\mathbf{b} + z\mathbf{a} \times \mathbf{b}] + y\mathbf{b} \times \mathbf{a} + z(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} = \mathbf{b}$$

$$\Rightarrow \lambda(x\mathbf{a} + y\mathbf{b} + z\mathbf{a} \times \mathbf{b}) - y\mathbf{a} \times \mathbf{b} + z[(\mathbf{a} \cdot \mathbf{a})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{a}] - \mathbf{b} = \mathbf{0}$$

$$\Rightarrow [\lambda x - z(\mathbf{a} \cdot \mathbf{b})]\mathbf{a} + (\lambda y + z(\mathbf{a} \cdot \mathbf{a}) - 1)\mathbf{b} + (\lambda z - y)(\mathbf{a} \times \mathbf{b}) = \mathbf{0}$$

$$[\because \mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b} \text{ are non-coplanar}]$$

$$\Rightarrow \lambda x - z(\mathbf{a} \cdot \mathbf{b}) = 0, \lambda y + z(\mathbf{a} \cdot \mathbf{a}) - 1 = 0,$$

$$\lambda z - y = 0.$$

$$\Rightarrow x = \frac{\mathbf{a} \cdot \mathbf{b}}{\lambda(\lambda^2 + |\mathbf{a}|^2)}, y = \frac{\lambda}{\lambda^2 + |\mathbf{a}|^2},$$

$$\text{and } z = \frac{1}{\lambda^2 + |\mathbf{a}|^2}$$

$$\text{Thus, } \mathbf{r} = \frac{1}{\lambda^2 + |\mathbf{a}|^2} \left(\frac{\mathbf{a} \cdot \mathbf{b}}{\lambda}\mathbf{a} + \lambda\mathbf{b} + \mathbf{a} \times \mathbf{b} \right)$$

C. The given equation can be written

$$(\mathbf{r} - \mathbf{c}) \times \mathbf{b} = \mathbf{0}$$

$\Rightarrow \mathbf{r} - \mathbf{c}$ and $\mathbf{b}$ are collinear, so that

$$\mathbf{r} - \mathbf{c} = \alpha\mathbf{b} \Rightarrow \mathbf{r} = \mathbf{c} + \alpha\mathbf{b}$$

where, α is a scalar.

Putting $\mathbf{r} \cdot \mathbf{a} = 0$ we get

$$\mathbf{c} \cdot \mathbf{a} + \alpha\mathbf{b} \cdot \mathbf{a} = 0 \Rightarrow \alpha = -\frac{\mathbf{c} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{b}}$$

$$\text{Thus, } \mathbf{r} = \mathbf{c} - \frac{\mathbf{c} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{b}}\mathbf{b}$$

D. As in part (B), we can write

$$\mathbf{r} = x\mathbf{a} + y\mathbf{b} + z(\mathbf{a} \times \mathbf{b})$$

Substituting this value in $\mathbf{r} \times \mathbf{b} = \mathbf{a}$, we get

$$x(\mathbf{a} \times \mathbf{b}) + y\mathbf{0} + z(\mathbf{a} \times \mathbf{b}) \times \mathbf{b} = \mathbf{a}$$

$$\Rightarrow x(\mathbf{a} \times \mathbf{b}) + z[(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{a}] = \mathbf{a}$$

$$\Rightarrow x(\mathbf{a} \times \mathbf{b}) + z[(\mathbf{a} \cdot \mathbf{b})\mathbf{b} - z|\mathbf{b}|^2\mathbf{a} - \mathbf{a}] = \mathbf{a}$$

$$\Rightarrow x = 0, (1 + z|\mathbf{b}|^2) = 0 \quad [\because \mathbf{a} \cdot \mathbf{b} = 0]$$

$$\Rightarrow \mathbf{r} = y\mathbf{b} - \frac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b})$$

So, $\mathbf{r}$ satisfy the given equation for any scalar y .

$$\text{Thus, } \mathbf{r} = -\frac{1}{|\mathbf{b}|^2}(\mathbf{a} \times \mathbf{b}) + y\mathbf{b}$$

$$A \rightarrow q; B \rightarrow s; C \rightarrow p; D \rightarrow r$$

Type 6. Single Integer Answer Type Questions

Ex 67. Find the value of the expression

$$\frac{\{(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} - \mathbf{b} - \mathbf{c})\} \cdot (\mathbf{a} + 2\mathbf{b} - \mathbf{c})}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

Sol. (3) The given expression reduces to

$$\frac{(-\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} - \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c}) \cdot (\mathbf{a} + 2\mathbf{b} - \mathbf{c})}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

$$= \frac{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] + 2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] - [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} = 3$$

Ex 68. Find the value of $x \in R$ for which the vectors $\mathbf{a} = (1, -2, 3)$, $\mathbf{b} = (-2, 3, -4)$ and $\mathbf{c} = (1, -1, x)$ form a linearly dependent system.

Sol. (1) As the vectors form a linearly dependent system, so there must be some scalars α , β and γ not all zero simultaneously

$$\text{and } \alpha(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \beta(-2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) + \gamma(\hat{\mathbf{i}} - \hat{\mathbf{j}} + x\hat{\mathbf{k}}) = \mathbf{0}$$

$$\text{We have, } \alpha - 2\beta + \gamma = 0 \quad \dots(i)$$

$$-2\alpha + 3\beta - \gamma = 0 \quad \dots(ii)$$

$$\text{and } 3\alpha - 4\beta + x\gamma = 0 \quad \dots(iii)$$

On eliminating α , β and γ from Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} 1 & -2 & 1 \\ -2 & 3 & -1 \\ 3 & -4 & x \end{vmatrix} = 0$$

$$\Rightarrow (3x - 4) + 2(-2x + 3) - (-1) = 0 \Rightarrow x = 1$$

Ex 69. Let $\mathbf{v} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{w} = \hat{\mathbf{i}} + 3\hat{\mathbf{k}}$. If $\hat{\mathbf{u}}$ is a unit vector and p is the maximum value of the scalar triple product $[\mathbf{u} \ \mathbf{v} \ \mathbf{w}]$, then the value of $p^2 - 52$ is _____.

Sol. (7) $[\mathbf{u} \ \mathbf{v} \ \mathbf{w}] = \mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = |\mathbf{u}| |\mathbf{v} \times \mathbf{w}| \cos \theta$
where, θ is the angle between $\mathbf{u}$ and $\mathbf{v} \times \mathbf{w}$.
For the maximum value, $\theta = 0^\circ$
 $\therefore p = |\mathbf{v} \times \mathbf{w}| = |(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) \times (\hat{\mathbf{i}} + 3\hat{\mathbf{k}})|$
 $= |3\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - \hat{\mathbf{k}}| = \sqrt{59}$
 $\therefore p^2 - 52 = 7$

Ex 70. If the area of the parallelogram whose diagonals are $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, is $\lambda\sqrt{3}$ sq units, then the value of λ is _____.

Sol. (5) Area of parallelogram $= \frac{1}{2} |\mathbf{d}_1 \times \mathbf{d}_2|$
 $= \frac{1}{2} |(3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}) \times (\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}})|$
 $= \frac{1}{2} |-2\hat{\mathbf{i}} - 14\hat{\mathbf{j}} - 10\hat{\mathbf{k}}| = 5\sqrt{3}$

Ex 71. The vector $\mathbf{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of $\triangle ABC$. If the length of the median through A is $\sqrt{7\lambda + 5}$ units, then the value of λ is _____.

Sol. (4) Let us take A as origin and the position vectors of points B and C as $3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ respectively.
Let D be the mid-point of BC .
Then, the position vector of point D is
 $\frac{1}{2} (3\hat{\mathbf{i}} + 4\hat{\mathbf{k}} + 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}})$
 $\Rightarrow \mathbf{AD} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}} \Rightarrow |\mathbf{AD}| = \sqrt{33}$ units
 $\Rightarrow \sqrt{7\lambda + 5} = \sqrt{33}$ units
Hence, $\lambda = 4$

Ex 72. Let $\mathbf{b} = 4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $\mathbf{c}$ be two vectors perpendicular to each other in the XY -plane. If $\mathbf{r}_i$, $i = 1, 2, \dots, n$ are the vectors in the same plane having projections 1 and 2 along $\mathbf{b}$ and $\mathbf{c}$ respectively, then $\sum_{i=1}^n |\mathbf{r}_i|^2 - 17$ is equal to _____.

Sol. (3) Let $\mathbf{r} = \lambda\mathbf{b} + \mu\mathbf{c}$ and $\mathbf{c} = \pm(\hat{\mathbf{i}} + \hat{\mathbf{j}})$.
Since, $\mathbf{c}$ and $\mathbf{b}$ are perpendicular, we have
 $4x + 3y = 0$
 $\Rightarrow \mathbf{c} = \pm x \left(\hat{\mathbf{i}} - \frac{4}{3}\hat{\mathbf{j}} \right)$
 $\therefore \pm 1 = \text{Projection of } \mathbf{r} \text{ on } \mathbf{b} = \frac{\mathbf{r} \cdot \mathbf{b}}{|\mathbf{b}|} = \frac{(\lambda\mathbf{b} + \mu\mathbf{c}) \cdot \mathbf{b}}{|\mathbf{b}|}$
 $= \frac{\lambda \mathbf{b} \cdot \mathbf{b}}{|\mathbf{b}|} = \lambda |\mathbf{b}| = 5\lambda \quad [\because \mathbf{b} \cdot \mathbf{c} = 0]$
 $\Rightarrow \lambda = \pm \frac{1}{5}$

Also, $\pm 2 = \text{Projection of } \mathbf{r} \text{ on } \mathbf{c} = \frac{\mathbf{r} \cdot \mathbf{c}}{|\mathbf{c}|}$
 $= \frac{(\lambda\mathbf{b} + \mu\mathbf{c}) \cdot \mathbf{c}}{|\mathbf{c}|} = \mu |\mathbf{c}| = \frac{5}{3} \mu x$

Thus, $\mu x = \pm \frac{6}{5}$

Therefore,

$$\mathbf{r} = \frac{1}{5} (4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) + \frac{6}{5} \left(\hat{\mathbf{i}} - \frac{4}{3}\hat{\mathbf{j}} \right) = \pm (2\hat{\mathbf{i}} - \hat{\mathbf{j}})$$

$$\text{and } \mathbf{r} = \frac{1}{5} (4\hat{\mathbf{i}} + 3\hat{\mathbf{j}}) - \frac{6}{5} \left(\hat{\mathbf{i}} - \frac{4}{3}\hat{\mathbf{j}} \right) = \pm \left(-\frac{2}{5}\hat{\mathbf{i}} + \frac{11}{5}\hat{\mathbf{j}} \right)$$

Thus, there are four such vectors such that

$$\sum_{i=1}^4 |\mathbf{r}_i|^2 = 2|2\hat{\mathbf{i}} - \hat{\mathbf{j}}|^2 + 2 \left| -\frac{2}{5}\hat{\mathbf{i}} + \frac{11}{5}\hat{\mathbf{j}} \right|^2 = 20$$

Hence, $\sum_{i=1}^4 |\mathbf{r}_i|^2 - 17 = 20 - 17 = 3$

Ex 73. If θ is the angle between the lines AB and AC , where A , B and C are the three points with coordinates $(1, 2, -1)$ and $(2, 0, 3)$ and $(3, -1, 2)$ respectively, then $\frac{\sqrt{462} \cos \theta}{10}$ is equal to _____.

Sol. (2) It is easy to see that

$$\mathbf{AB} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$$

$$\mathbf{AC} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

$$\mathbf{AB} \cdot \mathbf{AC} = 20, \mathbf{AB} \cdot \mathbf{AB} = 21$$

and $\mathbf{AC} \cdot \mathbf{AC} = 22$

Thus, $\cos \theta = \frac{\mathbf{AB} \cdot \mathbf{AC}}{|\mathbf{AB}| |\mathbf{AC}|} = \frac{20}{\sqrt{462}}$

Hence, $\frac{\sqrt{462} \cos \theta}{10} = 2$

Ex 74. If $[\mathbf{b} \ \mathbf{c} \ \mathbf{d}] = 24$ and $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) + k\mathbf{a} = 0$, then k is equal to _____.

Sol. (2) We know that, $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$
Putting $\mathbf{c} \times \mathbf{d} = \mathbf{e}$, we get
 $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{e} = (\mathbf{a} \cdot \mathbf{e})\mathbf{b} - (\mathbf{b} \cdot \mathbf{e})\mathbf{a}$
 $= \{\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})\} \mathbf{b} - \{\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})\} \mathbf{a}$
 $= [\mathbf{a} \ \mathbf{c} \ \mathbf{d}]\mathbf{b} - [\mathbf{b} \ \mathbf{c} \ \mathbf{d}]\mathbf{a} \quad \dots(i)$
Similarly,
 $(\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) = [\mathbf{a} \ \mathbf{d} \ \mathbf{b}]\mathbf{c} - [\mathbf{c} \ \mathbf{d} \ \mathbf{b}]\mathbf{a}$
 $= [\mathbf{a} \ \mathbf{d} \ \mathbf{b}]\mathbf{c} - [\mathbf{b} \ \mathbf{c} \ \mathbf{d}]\mathbf{a} \quad \dots(ii)$
Also, $(\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times (\mathbf{a} \times \mathbf{d})$
 $= (\mathbf{b} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{a})$
 $= [\mathbf{b} \ \mathbf{d} \ \mathbf{a}]\mathbf{c} - [\mathbf{c} \ \mathbf{d} \ \mathbf{a}]\mathbf{b}$
 $= -[\mathbf{a} \ \mathbf{d} \ \mathbf{b}]\mathbf{c} - [\mathbf{a} \ \mathbf{c} \ \mathbf{d}]\mathbf{b} \quad \dots(iii)$

On adding Eqs. (i), (ii) and (iii), we get
 $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$
 $= [\mathbf{a} \ \mathbf{c} \ \mathbf{d}]\mathbf{b} - [\mathbf{b} \ \mathbf{c} \ \mathbf{d}]\mathbf{a} + [\mathbf{a} \ \mathbf{d} \ \mathbf{b}]\mathbf{c} - [\mathbf{b} \ \mathbf{c} \ \mathbf{d}]\mathbf{a}$
 $- [\mathbf{a} \ \mathbf{d} \ \mathbf{b}]\mathbf{c} - [\mathbf{a} \ \mathbf{c} \ \mathbf{d}]\mathbf{b}$
 $= -2[\mathbf{b} \ \mathbf{c} \ \mathbf{d}]\mathbf{a} = -48\mathbf{a} \quad [\because [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 34]$
 $\therefore k\mathbf{a} = -48\mathbf{a} \Rightarrow k = -48$
Hence, $50 + k = 50 - 48 = 2$

Target Exercises

Type 1. Only One Correct Option

- If $ABCDEF$ is a regular hexagon with $\mathbf{AB} = \mathbf{a}$ and $\mathbf{BC} = \mathbf{b}$, then $\mathbf{CE}$ equals
 - $\mathbf{b} - \mathbf{a}$
 - $-\mathbf{b}$
 - $\mathbf{b} - 2\mathbf{a}$
 - None of these
- Let $ABCD$ be a parallelogram whose diagonals intersect at P and let O be the origin, then $\mathbf{OA} + \mathbf{OB} + \mathbf{OC} + \mathbf{OD}$ equals
 - $\mathbf{OA}$
 - $2\mathbf{OP}$
 - $3\mathbf{OP}$
 - $4\mathbf{OP}$
- Let D, E and F be the mid-points of the sides BC, CA and AB respectively of a $\triangle ABC$. Then, $\mathbf{AD} + \mathbf{BE} + \mathbf{CF}$ equals
 - $\mathbf{0}$
 - $2\mathbf{AB}$
 - $3\mathbf{AB}$
 - None of these
- If the vectors $\mathbf{a} + \lambda\mathbf{b} + 3\mathbf{c}, -2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$ and $\mathbf{a} - 3\mathbf{b} + 5\mathbf{c}$ are coplanar, then the value of λ is
 - 2
 - 1
 - 1
 - 2
- If $\mathbf{OA} + \mathbf{OB} = \mathbf{BO} + \mathbf{OC}$, then A, B and C are
 - coplanar
 - collinear
 - non-collinear
 - None of these
- If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are the position vectors of the vertices of an equilateral triangle whose orthocentre is at the origin, then
 - $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$
 - $\mathbf{a}^2 = \mathbf{b}^2 + \mathbf{c}^2$
 - $\mathbf{a} + \mathbf{b} = \mathbf{c}$
 - None of these
- If the position vectors of three points are $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}, 2\mathbf{a} + 3\mathbf{b} - 4\mathbf{c}$ and $-7\mathbf{b} + 10\mathbf{c}$, then the three points are
 - collinear
 - non-coplanar
 - non-collinear
 - None of these
- If $\mathbf{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\mathbf{b} = 3\hat{i} + 6\hat{j} + 2\hat{k}$, then the vector in the direction of $\mathbf{a}$ and having magnitude as $|\mathbf{b}|$, is
 - $7(\hat{i} + 2\hat{j} + 2\hat{k})$
 - $\frac{7}{9}(\hat{i} + 2\hat{j} + 2\hat{k})$
 - $\frac{7}{3}(\hat{i} + 2\hat{j} + 2\hat{k})$
 - None of the above
- If A, B have position vectors $\mathbf{a}, \mathbf{b}$ relative to the origin O and X, Y divide $\mathbf{AB}$ internally and externally respectively in the ratio 2:1. Then, $\mathbf{XY}$ is equal to
 - $\frac{3}{2}(\mathbf{b} - \mathbf{a})$
 - $\frac{4}{3}(\mathbf{a} - \mathbf{b})$
 - $\frac{5}{6}(\mathbf{b} - \mathbf{a})$
 - $\frac{4}{3}(\mathbf{b} - \mathbf{a})$
- If C is the mid-point of AB and P is any point outside AB , then
 - $\mathbf{PA} + \mathbf{PB} = \mathbf{PC}$
 - $\mathbf{PA} + \mathbf{PB} = 2\mathbf{PC}$
 - $\mathbf{PA} + \mathbf{PB} + \mathbf{PC} = \mathbf{0}$
 - $\mathbf{PA} + \mathbf{PB} + 2\mathbf{PC} = \mathbf{0}$
- The value of b such that the scalar product of the vector $\hat{i} + \hat{j} + \hat{k}$ with the unit vector parallel to the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $b\hat{i} + 2\hat{j} + 3\hat{k}$ is one, is
 - 2
 - 1
 - 0
 - 1
- If the median AD of $\triangle ABC$ is bisected at E , BE meets AC in F , then $AF : AC$ is equal to
 - $\frac{3}{4}$
 - $\frac{1}{3}$
 - $\frac{1}{2}$
 - $\frac{1}{4}$
- The vector $\mathbf{c}$, directed along the internal bisector of the angle between the vectors $\mathbf{a} = 7\hat{i} - 4\hat{j} - 4\hat{k}$ and $\mathbf{b} = -2\hat{i} - \hat{j} + 2\hat{k}$ with $|\mathbf{c}| = 5\sqrt{6}$, is
 - $\frac{5}{3}(\hat{i} - 7\hat{j} + 2\hat{k})$
 - $\frac{5}{3}(5\hat{i} + 5\hat{j} + 2\hat{k})$
 - $\frac{5}{3}(\hat{i} + 7\hat{j} + 2\hat{k})$
 - $\frac{5}{3}(-5\hat{i} + 5\hat{j} + 2\hat{k})$
- A unit vector making an obtuse angle with X -axis and perpendicular to the plane containing the points $\hat{i} + 2\hat{j} + 3\hat{k}, 2\hat{i} + 3\hat{j} + 4\hat{k}$ and $\hat{i} + 5\hat{j} + 7\hat{k}$
 - also makes an obtuse angle with Y -axis
 - also makes an obtuse angle with Z -axis
 - also makes an obtuse angle with Y and Z -axes
 - None of the above
- If the unit vectors $\mathbf{a}$ and $\mathbf{b}$ are inclined at an angle 2θ such that $|\mathbf{a} - \mathbf{b}| < 1$ and $0 \leq \theta \leq \pi$, then θ lies in the interval
 - $\left[0, \frac{\pi}{6}\right]$
 - $\left[\frac{5\pi}{6}, \pi\right]$
 - $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$
 - $\left[\frac{\pi}{2}, \frac{5\pi}{6}\right]$
- If $\mathbf{e}_1 = (1, 1, 1), \mathbf{e}_2 = (1, 1, -1)$ and $\mathbf{a}, \mathbf{b}$ are two vectors such that $\mathbf{e}_1 = 2\mathbf{a} + \mathbf{b}$ and $\mathbf{e}_2 = \mathbf{a} + 2\mathbf{b}$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is
 - $\cos^{-1}\left(\frac{7}{9}\right)$
 - $\cos^{-1}\left(\frac{7}{11}\right)$
 - $\cos^{-1}\left(-\frac{7}{11}\right)$
 - $\cos^{-1}\left(\frac{6\sqrt{2}}{11}\right)$

17. If the vector $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between the vector $\mathbf{c}$ and the vector $3\hat{i} + 4\hat{j}$, then the unit vector in the direction of $\mathbf{c}$ is
 (a) $\frac{1}{15}(1\hat{i} + 10\hat{j} + 2\hat{k})$ (b) $-\frac{1}{15}(1\hat{i} - 10\hat{j} + 2\hat{k})$
 (c) $-\frac{1}{15}(1\hat{i} + 10\hat{j} - 2\hat{k})$ (d) $-\frac{1}{15}(1\hat{i} + 10\hat{j} + 2\hat{k})$
18. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are unit vectors such that the vector $\hat{\mathbf{a}} + 3\hat{\mathbf{b}}$ is perpendicular to $7\hat{\mathbf{a}} - 5\hat{\mathbf{b}}$ and $\hat{\mathbf{a}} - 4\hat{\mathbf{b}}$ is perpendicular to $7\hat{\mathbf{a}} - 2\hat{\mathbf{b}}$, then the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
19. If $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$ and $2\hat{i} + 5\hat{j} + 7\hat{k}$ are the position vectors of the vertices A , B and C respectively of $\triangle ABC$. The position vector of the point where the bisector of angle A meets BC is
 (a) $\frac{2}{3}(-6\hat{i} - 8\hat{j} - 6\hat{k})$ (b) $\frac{2}{3}(6\hat{i} + 8\hat{j} + 6\hat{k})$
 (c) $\frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$ (d) $\frac{1}{3}(5\hat{j} + 12\hat{k})$
20. If θ is the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \cdot \mathbf{b} \geq 0$, then
 (a) $0 \leq \theta \leq \pi$ (b) $\frac{\pi}{2} \leq \theta \leq \pi$
 (c) $0 \leq \theta \leq \frac{\pi}{2}$ (d) $0 < \theta < \frac{\pi}{2}$
21. If $|\mathbf{a} + \mathbf{b}| = |\mathbf{a} - \mathbf{b}|$, then
 (a) $\mathbf{a}$ is parallel to $\mathbf{b}$ (b) $\mathbf{a} \perp \mathbf{b}$
 (c) $|\mathbf{a}| = |\mathbf{b}|$ (d) None of these
22. If $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{a} + \mathbf{b}| = 1$, then $|\mathbf{a} - \mathbf{b}|$ is equal to
 (a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2
23. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three mutually perpendicular vectors of equal magnitude, then the angle θ which $\mathbf{a} + \mathbf{b} + \mathbf{c}$ makes with any one of the three given vectors is given by
 (a) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{1}{3}\right)$
 (c) $\cos^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (d) None of these
24. A vector which makes equal angles with the vectors $\frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$, $\frac{1}{5}(-4\hat{i} - 3\hat{k})$ and $\hat{j}$, is
 (a) $5\hat{i} + \hat{j} + 5\hat{k}$ (b) $-5\hat{i} + \hat{j} + 5\hat{k}$
 (c) $5\hat{i} - \hat{j} - \hat{k}$ (d) $5\hat{i} + \hat{j} - 5\hat{k}$
25. The unit vector in ZOX -plane and making angle 45° and 60° respectively with $\mathbf{a} = 2\hat{i} + 2\hat{j} - \hat{k}$ and $\mathbf{b} = 0\hat{i} + \hat{j} - \hat{k}$, is
 (a) $-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{k}$ (b) $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}$
 (c) $\frac{1}{3\sqrt{2}}\hat{i} + \frac{1}{3\sqrt{2}}\hat{j} + \frac{1}{3\sqrt{2}}\hat{k}$ (d) None of these
26. If $\mathbf{a} = 4\hat{i} + 6\hat{j}$ and $\mathbf{b} = 3\hat{j} + 4\hat{k}$, then the vector form of component of $\mathbf{a}$ along $\mathbf{b}$, is
 (a) $\frac{18}{10\sqrt{3}}(3\hat{j} + 4\hat{k})$ (b) $\frac{18}{25}(3\hat{j} + 4\hat{k})$
 (c) $\frac{18}{10\sqrt{6}}(3\hat{j} + 4\hat{k})$ (d) $\frac{18}{35}(3\hat{j} + 4\hat{k})$
27. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors such that $\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot \mathbf{c}$, then
 (a) $\mathbf{b} = \mathbf{c}$ (b) $\mathbf{a} \perp \mathbf{b}, \mathbf{c}$
 (c) $\mathbf{a} \perp \mathbf{b} - \mathbf{c}$ (d) either $\mathbf{a} \perp \mathbf{b} - \mathbf{c}$ or $\mathbf{b} = \mathbf{c}$
28. Let $\mathbf{b} = 4\hat{i} + 3\hat{j}$ and $\mathbf{c}$ be two vectors perpendicular to each other in the XY -plane. Then, a vector in the same plane having projections 1 and 2 along $\mathbf{b}$ and $\mathbf{c}$ respectively, is
 (a) $\hat{i} + 2\hat{j}$ (b) $2\hat{i} - \hat{j}$
 (c) $2\hat{i} + \hat{j}$ (d) None of these
29. If $|\mathbf{a}| = 7$, $|\mathbf{b}| = 11$, $|\mathbf{a} + \mathbf{b}| = 10\sqrt{3}$, then $|\mathbf{a} - \mathbf{b}|$ equals
 (a) 10 (b) $\sqrt{10}$ (c) $2\sqrt{10}$ (d) 20
30. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are vectors such that $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} + \mathbf{b} = \mathbf{c}$, then
 (a) $|\mathbf{a}|^2 + |\mathbf{b}|^2 = |\mathbf{c}|^2$ (b) $|\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2$
 (c) $|\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{c}|^2$ (d) None of these
31. The projection of the vector $\mathbf{a} = 4\hat{i} - 3\hat{j} + 2\hat{k}$ on the axes, making equal acute angles with the coordinate axes, is
 (a) 3 (b) $\sqrt{3}$
 (c) $\frac{3}{\sqrt{3}}$ (d) None of these
32. If $\mathbf{a}$ is a vector of magnitude 50 collinear with the vector $\mathbf{b} = 6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k}$ and makes an acute angle with the positive direction of Z -axis, then $\mathbf{a}$ is equal to
 (a) $24\hat{i} - 32\hat{j} - 30\hat{k}$
 (b) $24\hat{i} + 32\hat{j} + 30\hat{k}$
 (c) $12\hat{i} - 16\hat{j} - 15\hat{k}$
 (d) None of the above
33. If the vectors $\mathbf{a} = (2, \log_3 x, a)$ and $\mathbf{b} = (-3, a \log_3 x, \log_3 x)$ are inclined at an acute angle, then
 (a) $a = 0$ (b) $a < 0$
 (c) $a > 0$ (d) None of these
34. Let $\mathbf{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} + 2\hat{j} - \hat{k}$ and $\mathbf{c} = \hat{i} + \hat{j} - 2\hat{k}$ be three vectors. A vector in the plane of $\mathbf{b}$ and $\mathbf{c}$ whose projection on $\mathbf{a}$ is of magnitude $\frac{\sqrt{2}}{3}$, is
 (a) $2\hat{i} + 3\hat{j} - 3\hat{k}$ (b) $2\hat{i} + 3\hat{j} + 3\hat{k}$
 (c) $-2\hat{i} - 5\hat{j} + 5\hat{k}$ (d) $2\hat{i} + \hat{j} + 5\hat{k}$

35. Let $\mathbf{p}$ and $\mathbf{q}$ be the position vectors of P and Q respectively with respect to O and $|\mathbf{p}| = p$, $|\mathbf{q}| = q$. The point R and S divide PQ internally and externally in the ratio 2:3 respectively. If $\mathbf{OR}$ and $\mathbf{OS}$ are perpendicular, then
(a) $9p^2 = 4q^2$ (b) $4p^2 = 9q^2$ (c) $9p = 4q$ (d) $4p = 9q$
36. A unit vector in XY -plane makes an angle of 45° with the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and an angle of 60° with the vector $3\hat{\mathbf{i}} - 4\hat{\mathbf{j}}$, is
(a) $\hat{\mathbf{i}}$ (b) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}}}{\sqrt{2}}$
(c) $\frac{\hat{\mathbf{i}} - \hat{\mathbf{j}}}{\sqrt{2}}$ (d) None of these
37. For any vector $\mathbf{a}$, $|\mathbf{a} \times \hat{\mathbf{i}}|^2 + |\mathbf{a} \times \hat{\mathbf{j}}|^2 + |\mathbf{a} \times \hat{\mathbf{k}}|^2$ is equal to
(a) $|\mathbf{a}|^2$ (b) $2|\mathbf{a}|^2$
(c) $3|\mathbf{a}|^2$ (d) None of these
38. If a vector $\mathbf{a}$ is expressed as the sum of two vectors α and β along and perpendicular to a given vector $\mathbf{b}$, then β is equal to
(a) $\frac{(\mathbf{a} \times \mathbf{b}) \times \mathbf{b}}{|\mathbf{b}|^2}$ (b) $\frac{\mathbf{b} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{b}|^2}$
(c) $\frac{\mathbf{b} \times (\mathbf{a} \times \mathbf{b})}{|\mathbf{b}|}$ (d) $\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b}$
39. The vectors $\mathbf{a} = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{b} = -\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$ are the adjacent sides of a parallelogram. Then, angle between its diagonals is
(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$
40. If the vectors $\mathbf{a} = (c \log_2 x)\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{b} = (\log_2 x)\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + (2c \log_2 x)\hat{\mathbf{k}}$ make an obtuse angle for any $x \in (0, \infty)$, then c belongs to
(a) $(-\infty, 0)$ (b) $\left(-\infty, -\frac{4}{3}\right)$
(c) $\left(-\frac{4}{3}, 0\right)$ (d) $\left(-\frac{4}{3}, \infty\right)$
41. The vectors $\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{\mathbf{i}} + (4x - 2)\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The value of x is
(a) $-\frac{2}{3}$ (b) $\frac{1}{3}$
(c) $\frac{2}{3}$ (d) None of these
42. If three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} \neq 0$ and $\mathbf{a} \times \mathbf{b} = 2(\mathbf{a} \times \mathbf{c})$, $|\mathbf{a}| = |\mathbf{c}| = 1$, $|\mathbf{b}| = 4$ and the angle between $\mathbf{b}$ and $\mathbf{c}$ is $\cos^{-1}\left(\frac{1}{4}\right)$, then $\mathbf{b} - 2\mathbf{c} = \lambda\mathbf{a}$, where λ is equal to
(a) 4 (b) 3 (c) 2 (d) -2
43. If the sides of a parallelogram are $2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then the unit vector parallel to one of the diagonals is
(a) $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (b) $\frac{1}{7}(3\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
(c) $\frac{1}{7}(-3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$ (d) $\frac{1}{7}(3\hat{\mathbf{i}} + 6\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$
44. If the vectors $\mathbf{c}$, $\mathbf{a} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{j}}$ such that $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{b}$ form a right handed system, then $\mathbf{c}$ is
(a) $z\hat{\mathbf{i}} - x\hat{\mathbf{k}}$ (b) $\mathbf{0}$
(c) $y\hat{\mathbf{j}}$ (d) $-z\hat{\mathbf{i}} + x\hat{\mathbf{k}}$
45. If $\mathbf{a}$ and $\mathbf{b}$ are two vectors such that $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} \times \mathbf{b} = \mathbf{0}$, then
(a) $\mathbf{a} \parallel \mathbf{b}$
(b) $\mathbf{a} \perp \mathbf{b}$
(c) either $\mathbf{a}$ or $\mathbf{b}$ is a null vector
(d) None of the above
46. If forces of magnitudes 6 and 7 units acting in the directions $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}}$ respectively act on a particle which is displaced from the point $P(2, -1, -3)$ to $Q(5, -1, 1)$, then the work done by the forces is
(a) 4 units (b) -4 unit (c) 7 units (d) -7 unit
47. If a vector $\mathbf{r}$ of magnitude $3\sqrt{6}$ is directed along the bisector of the angle between the vectors $\mathbf{a} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{b} = -2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, then $\mathbf{r}$ is equal to
(a) $\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ (b) $\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
(c) $-\hat{\mathbf{i}} + 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
48. If $\alpha = x(\mathbf{a} \times \mathbf{b}) + y(\mathbf{b} \times \mathbf{c}) + z(\mathbf{c} \times \mathbf{a})$ and $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = \frac{1}{8}$, then $x + y + z$ is equal to
(a) $8\alpha \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$ (b) $\alpha \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$
(c) $8(\mathbf{a} + \mathbf{b} + \mathbf{c})$ (d) None of these
49. The vectors $\mathbf{a} = x\hat{\mathbf{i}} + (x+1)\hat{\mathbf{j}} + (x+2)\hat{\mathbf{k}}$, $\mathbf{b} = (x+3)\hat{\mathbf{i}} + (x+4)\hat{\mathbf{j}} + (x+5)\hat{\mathbf{k}}$ and $\mathbf{c} = (x+6)\hat{\mathbf{i}} + (x+7)\hat{\mathbf{j}} + (x+8)\hat{\mathbf{k}}$ are coplanar for
(a) all values of x (b) $x < 0$
(c) $x > 0$ (d) None of these
50. If $(x, y, z) \neq (0, 0, 0)$ and $(\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})x + (3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}})y + (-4\hat{\mathbf{i}} + 5\hat{\mathbf{j}})z = a(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})$, then the value of a is
(a) 0, -2 (b) 2, 0 (c) 0, -1 (d) 1, 0
51. If $\mathbf{a} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{b} = 2\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + \hat{\mathbf{k}}$, then the vectors $\mathbf{c}$ satisfying the conditions, that it is coplanar with $\mathbf{a}$ and $\mathbf{b}$, that it is perpendicular to $\mathbf{b}$ and that $\mathbf{a} \cdot \mathbf{c} = 7$, is
(a) $-\frac{3}{2}\hat{\mathbf{i}} + \frac{5}{2}\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (b) $-3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$
(c) $-6\hat{\mathbf{i}} + 0\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ (d) $-\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

52. Let the unit vectors $\hat{a}$ and $\hat{b}$ be perpendicular to each other and the unit vector $\hat{c}$ be inclined at an angle θ to both $\hat{a}$ and $\hat{b}$. If $\hat{c} = x\hat{a} + y\hat{b} + z(\hat{a} \times \hat{b})$, then
- (a) $x = \cos \theta, y = \sin \theta, z = \cos 2\theta$
 (b) $x = \sin \theta, y = \cos \theta, z = -\cos 2\theta$
 (c) $x = y = \cos \theta, z^2 = \cos 2\theta$
 (d) $x = y = \cos \theta, z^2 = -\cos 2\theta$
53. If $\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{c} = 0$, where $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar, then
- (a) $\mathbf{r} \perp (\mathbf{c} \times \mathbf{a})$ (b) $\mathbf{r} \perp (\mathbf{a} \times \mathbf{b})$
 (c) $\mathbf{r} \perp (\mathbf{b} \times \mathbf{c})$ (d) $\mathbf{r} = \mathbf{0}$
54. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are linearly independent vectors and $\Delta = \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{a} \cdot \mathbf{c} & \mathbf{b} \cdot \mathbf{c} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix}$, then
- (a) $\Delta = 0$ (b) $\Delta = 1$
 (c) $\Delta = \text{any non-zero value}$ (d) None of these
55. If vector $\mathbf{a}$ lies in the plane of $\mathbf{b}$ and $\mathbf{c}$, then which of the following is correct?
- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$ (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 1$
 (c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 3$ (d) $[\mathbf{b} \ \mathbf{c} \ \mathbf{a}] = 1$
56. The value of $[\mathbf{a} - \mathbf{b} \ \mathbf{b} - \mathbf{c} \ \mathbf{c} - \mathbf{a}]$, where $|\mathbf{a}| = 1, |\mathbf{b}| = 5$ and $|\mathbf{c}| = 3$, is
- (a) 0 (b) 1
 (c) 6 (d) None of these
57. If $\mathbf{a} = 2\hat{i} + 3\hat{j} + \hat{k}$, $\mathbf{b} = 2\hat{i} + p\hat{j} + 3\hat{k}$ and $\mathbf{c} = 2\hat{i} + 17\hat{j} - 3\hat{k}$ are coplanar vectors, then the value of p is
- (a) -4 (b) -1 (c) 4 (d) -2
58. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are three non-coplanar mutually perpendicular unit vectors, then $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ is
- (a) 2 (b) 0 (c) 1 (d) 3
59. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, $(\mathbf{a} - \mathbf{b}) \cdot [(\mathbf{b} - \mathbf{c}) \times (\mathbf{c} - \mathbf{a})]$ equals
- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
 (b) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
 (c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2$
 (d) None of the above
60. If $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are the unit vectors such that $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = 1$ and $\mathbf{a} \cdot \mathbf{c} = \frac{1}{2}$, then
- (a) $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar
 (b) $\mathbf{a}, \mathbf{b}$ and $\mathbf{d}$ are non-coplanar
 (c) $\mathbf{b}$ and $\mathbf{d}$ are non-parallel
 (d) $\mathbf{a}$ and $\mathbf{d}$ are parallel and $\mathbf{b}, \mathbf{c}$ are parallel
61. If $\mathbf{a}$ and $\mathbf{b}$ are two non-zero and non-collinear vectors, then $2[\mathbf{a} \ \mathbf{b} \ \hat{i}] + 2[\mathbf{a} \ \mathbf{b} \ \hat{j}] + 2[\mathbf{a} \ \mathbf{b} \ \hat{k}] + [\mathbf{a} \ \mathbf{b} \ \mathbf{a}]$ is equal to
- (a) $2(\mathbf{a} \times \mathbf{b})$ (b) $\mathbf{a} \times \mathbf{b}$
 (c) $\mathbf{a} + \mathbf{b}$ (d) None of these
62. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and λ is a real number, then $[\lambda(\mathbf{a} + \mathbf{b}) \ \lambda^2 \mathbf{b} \ \lambda \mathbf{c}] = [\mathbf{a} \ \mathbf{b} + \mathbf{c} \ \mathbf{b}]$ for
- (a) exactly two values of λ
 (b) exactly three values of λ
 (c) no real value of λ
 (d) exactly one value of λ
63. Let $\hat{u}$ and $\hat{v}$ be unit vectors such that $\hat{u} \cdot \hat{v} = 0$. If $\hat{r}$ is any vector coplanar with $\hat{u}$ and $\hat{v}$, then the magnitude of the vector $\mathbf{r} \times (\hat{u} \times \hat{v})$ is
- (a) 0 (b) 1 (c) $|\mathbf{r}|$ (d) $2|\mathbf{r}|$
64. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \frac{1}{2} \mathbf{b}$, $\mathbf{b}$ and $\mathbf{c}$ being non-parallel. If θ_1 is the angle between $\mathbf{a}$ and $\mathbf{b}$ and θ_2 is the angle between $\mathbf{a}$ and $\mathbf{c}$, then
- (a) $\theta_1 = \frac{\pi}{6}, \theta_2 = \frac{\pi}{3}$ (b) $\theta_1 = \frac{\pi}{3}, \theta_2 = \frac{\pi}{6}$
 (c) $\theta_1 = \frac{\pi}{2}, \theta_2 = \frac{\pi}{3}$ (d) $\theta_1 = \frac{\pi}{3}, \theta_2 = \frac{\pi}{2}$
65. For any three vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, the value of $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b})$ is
- (a) 0 (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (c) $3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (d) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
66. If the vectors $\mathbf{a} = \hat{i} + \hat{j} + a^2 \hat{k}$, $\mathbf{b} = \hat{i} + \hat{j} + b^2 \hat{k}$ and $\mathbf{c} = \hat{i} + \hat{j} + c^2 \hat{k}$ are three non-coplanar vectors and $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$, then the value of abc is
- (a) 0 (b) 1
 (c) 2 (d) -1
67. If the points with position vectors $-(\hat{j} + \hat{k})$, $4\hat{i} + 5\hat{j} + \lambda \hat{k}$, $3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-4\hat{i} + 4\hat{j} + 4\hat{k}$ are coplanar, then the value of λ is
- (a) -1 (b) 0
 (c) 1 (d) None of these
68. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors, then $\frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})}{(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}} + \frac{\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})}{\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}$ is equal to
- (a) 0 (b) 2
 (c) 1 (d) None of these
69. Let $\mathbf{a} = a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k}$, $\mathbf{b} = b_1 \hat{i} + b_2 \hat{j} + b_3 \hat{k}$ and $\mathbf{c} = c_1 \hat{i} + c_2 \hat{j} + c_3 \hat{k}$ be three non-zero vectors such that $\mathbf{c}$ is a vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. If the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$, then $\left| \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \right|^2$ is equal to
- (a) 0 (b) 1
 (c) $\frac{1}{4} |\mathbf{a}|^2 |\mathbf{b}|^2$ (d) $\frac{3}{4} |\mathbf{a}|^2 |\mathbf{b}|^2$

70. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors, then

$$\begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \mathbf{b} & \mathbf{a} \cdot \mathbf{c} \\ \mathbf{b} \cdot \mathbf{a} & \mathbf{b} \cdot \mathbf{b} & \mathbf{b} \cdot \mathbf{c} \\ \mathbf{c} \cdot \mathbf{a} & \mathbf{c} \cdot \mathbf{b} & \mathbf{c} \cdot \mathbf{c} \end{vmatrix} \text{ equals}$$

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2$ (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
(c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^3$ (d) None of these

71. $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} \times \mathbf{c}) \cdot \mathbf{d}$ equals

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}](\mathbf{b} \cdot \mathbf{d})$ (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}](\mathbf{a} \cdot \mathbf{d})$
(c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}](\mathbf{c} \cdot \mathbf{d})$ (d) None of these

72. If $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$, then

- (a) $\mathbf{b} \times (\mathbf{c} \times \mathbf{a}) = 0$ (b) $(\mathbf{c} \times \mathbf{a}) \times \mathbf{b} = 0$
(c) $\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = 0$ (d) None of these

73. If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are reciprocal system of vectors, then $\mathbf{a} \times \mathbf{p} + \mathbf{b} \times \mathbf{q} + \mathbf{c} \times \mathbf{r}$ equals

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (b) $(\mathbf{p} + \mathbf{q} + \mathbf{r})$
(c) 0 (d) $\mathbf{a} + \mathbf{b} + \mathbf{c}$

74. Let a , b and c be distinct non-negative numbers. If the vectors $a\hat{\mathbf{i}} + a\hat{\mathbf{j}} + c\hat{\mathbf{k}}$, $\hat{\mathbf{i}} + \hat{\mathbf{k}}$ and $c\hat{\mathbf{i}} + c\hat{\mathbf{j}} + b\hat{\mathbf{k}}$ lie in a plane, then c is the

- (a) AM of a and b (b) GM of a and b
(c) HM of a and b (d) None of these

75. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{c}}$ is unit vector perpendicular to the vector $\mathbf{a}$ and coplanar with $\mathbf{a}$ and $\mathbf{b}$, then a unit vector $\hat{\mathbf{d}}$ perpendicular to both $\mathbf{a}$ and $\mathbf{c}$, is

- (a) $\frac{1}{\sqrt{6}}(2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$ (b) $\frac{1}{\sqrt{2}}(\hat{\mathbf{j}} + \hat{\mathbf{k}})$
(c) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{j}})$ (d) $\frac{1}{\sqrt{2}}(\hat{\mathbf{i}} + \hat{\mathbf{k}})$

76. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors, then $(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot [(\mathbf{a} + \mathbf{b}) \times (\mathbf{a} + \mathbf{c})]$ equals

- (a) 0 (b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$
(c) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (d) $-[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$

77. If $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ and $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, then

- (a) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are orthogonal in pairs and $|\mathbf{a}| = |\mathbf{c}|$, $|\mathbf{b}| = 1$
(b) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are not orthogonal to each other
(c) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are orthogonal in pairs but $|\mathbf{a}| \neq |\mathbf{c}|$
(d) $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are orthogonal but $|\mathbf{b}| \neq 1$

78. $[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c}) (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) (\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})]$ is equal to

- (a) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2$
(b) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^3$
(c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^4$
(d) None of the above

79. If the non-zero vectors $\mathbf{a}$ and $\mathbf{b}$ are perpendicular to each other, then the solution of the equation, $\mathbf{r} \times \mathbf{a} = \mathbf{b}$ is given by

- (a) $\mathbf{r} = x\mathbf{a} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a}|^2}$ (b) $\mathbf{r} = x\mathbf{b} + \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{b}|^2}$
(c) $\mathbf{r} = x(\mathbf{a} \times \mathbf{b})$ (d) None of these

80. A tetrahedron has vertices at $O(0, 0, 0)$, $A(1, 2, 1)$, $B(2, 1, 3)$ and $C(-1, 1, 2)$. Then, the angle between the faces OAB and ABC will be

- (a) $\cos^{-1}\left(\frac{19}{35}\right)$ (b) $\cos^{-1}\left(\frac{9}{35}\right)$
(c) 30° (d) 90°

81. If $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$ and $\mathbf{r} \cdot \mathbf{a} = 0$, where $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{b} = 3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, then $\mathbf{r}$ is equal to

- (a) $\frac{1}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
(b) $2(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
(c) $2(-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
(d) $\frac{1}{2}(\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}})$

82. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three non-zero and non-coplanar vectors, then any vector $\mathbf{r}$ is equal to

- (a) $z\mathbf{a} + x\mathbf{b} + y\mathbf{c}$
(b) $x\mathbf{a} + y\mathbf{b} + z\mathbf{c}$
(c) $y\mathbf{a} + z\mathbf{b} + x\mathbf{c}$
(d) None of the above

83. If $\mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{r}$ is a non-zero vector such that $p\mathbf{r} + (\mathbf{r} \cdot \mathbf{b})\mathbf{a} = \mathbf{c}$, then $\mathbf{r}$ is equal to

- (a) $\frac{\mathbf{c}}{p} - \frac{(\mathbf{b} \cdot \mathbf{c})\mathbf{a}}{p^2}$
(b) $\frac{\mathbf{a}}{p} - \frac{(\mathbf{c} \cdot \mathbf{a})\mathbf{b}}{p^2}$
(c) $\frac{\mathbf{b}}{p} - \frac{(\mathbf{a} \cdot \mathbf{c})\mathbf{c}}{p^2}$
(d) $\frac{\mathbf{c}}{p^2} - \frac{(\mathbf{b} \cdot \mathbf{c})\mathbf{a}}{p}$

84. If $\mathbf{p} = \lambda(\mathbf{u} \times \mathbf{v}) + \mu(\mathbf{v} \times \mathbf{w}) + \nu(\mathbf{w} \times \mathbf{u})$

and $[\mathbf{u} \ \mathbf{v} \ \mathbf{w}] = \frac{1}{5}$, then $\lambda + \mu + \nu$ is equal to

- (a) 5
(b) 10
(c) 15
(d) None of the above

85. If $\mathbf{a} = 2\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}}$, then $\mathbf{a} + t\mathbf{b}$ is perpendicular to $\mathbf{c}$, if t is equal to

- (a) 8 (b) 4 (c) 6 (d) 2

86. If the vectors $\mathbf{b} = \left(\tan \alpha, -1, 2\sqrt{\sin \frac{\alpha}{2}} \right)$ and

$\mathbf{c} = \left(\tan \alpha, \tan \alpha, -\frac{3}{\sqrt{\sin \frac{\alpha}{2}}} \right)$ are orthogonal and a

vector $\mathbf{a} = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the Z-axis, then the value of α is

- (a) $\alpha = (4n + 1)\pi - \tan^{-1} 2$
(b) $\alpha = (4n - 2)\pi + \tan^{-1} 2$
(c) $\alpha = (4n + 3)\pi$
(d) None of the above

Type 2. More than One Correct Option

87. If in $\triangle ABC$, $\mathbf{AB} = \frac{\mathbf{u}}{|\mathbf{u}|} - \frac{\mathbf{v}}{|\mathbf{v}|}$ and $\mathbf{AC} = \frac{2\mathbf{u}}{|\mathbf{u}|}$, where $|\mathbf{u}| \neq |\mathbf{v}|$, then
 (a) $1 + \cos 2A + \cos 2B + \cos 2C = 0$
 (b) $\sin A = \cos C$
 (c) projection of AC on BC is equal to BC
 (d) projection of AB on BC is equal to AB
88. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are non-coplanar unit vectors, also $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are non-collinear such that $2\hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \hat{\mathbf{b}} + \hat{\mathbf{c}}$, then
 (a) angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ is 60°
 (b) angle between $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ is 30°
 (c) angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is 120°
 (d) $\hat{\mathbf{b}}$ is perpendicular to $\hat{\mathbf{c}}$
89. A vector $\mathbf{d}$ is equally inclined to three vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = 3\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, let x , y and z be three vectors in the plane of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$; $\mathbf{c}$, $\mathbf{a}$ respectively, then
 (a) $\mathbf{x} \cdot \mathbf{d} = 14$
 (b) $\mathbf{v} \cdot \mathbf{d} = 3$
 (c) $\mathbf{z} \cdot \mathbf{d} = 0$
 (d) $\mathbf{r} \cdot \mathbf{d} = 0$
90. $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two non-collinear unit vectors and $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$, $x\hat{\mathbf{a}} - y\hat{\mathbf{b}}$ form a triangle, if
 (a) $x = -1$; $y = 1$ and $|\hat{\mathbf{a}} + \hat{\mathbf{b}}| = 2\cos\left(\frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}}{2}\right)$
 (b) $x = -1$; $y = 1$ and $\cos(\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}) + |\hat{\mathbf{a}} + \hat{\mathbf{b}}| \cos[\hat{\mathbf{a}} - (\hat{\mathbf{a}} + \hat{\mathbf{b}})] = -1$
 (c) $|\hat{\mathbf{a}} + \hat{\mathbf{b}}| = -2\cot\left(\frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}}{2}\right)\cos\left(\frac{\hat{\mathbf{a}} \cdot \hat{\mathbf{b}}}{2}\right)$ and $x = -1$, $y = 1$
 (d) None of the above
91. Let $\mathbf{a} = x\hat{\mathbf{i}} + x^2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = -3\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = (3x + 11)\hat{\mathbf{i}} + (x - 9)\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ be three vectors. Then, angle between $\mathbf{a}$ and $\mathbf{b}$ is acute angle and between $\mathbf{c}$ and $\mathbf{a}$ is obtuse, if x lies in
 (a) $(-\infty, 1) \cup (2, 3)$
 (b) $(-\infty, 1)$
 (c) $(2, 3)$
 (d) None of the above
92. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} + x\hat{\mathbf{j}} + y\hat{\mathbf{k}}$ are linearly dependent and $|\mathbf{c}| = \sqrt{3}$, then (x, y) is
 (a) $(1, 1)$
 (b) $(-2, 0)$
 (c) $\left(\frac{1}{5}, -\frac{7}{5}\right)$
 (d) $\left(-\frac{7}{5}, \frac{1}{5}\right)$
93. If $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = \mathbf{0}$ for non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then
 (a) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar
 (b) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$
 (c) $\mathbf{a} \times \mathbf{b}$ is coplanar with $\mathbf{c}$
 (d) $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{0}$
94. The angle of a triangle, two of whose sides are represented by vectors $\sqrt{3}(\hat{\mathbf{a}} \times \mathbf{b})$ and $\mathbf{b} - (\hat{\mathbf{a}} \cdot \mathbf{b})\hat{\mathbf{a}}$, where $\mathbf{b}$ is a non-zero vector and $\hat{\mathbf{a}}$ is a unit vector in the direction of $\mathbf{a}$, are
 (a) $\tan^{-1}(\sqrt{3})$
 (b) $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
 (c) $\cot^{-1}(0)$
 (d) $\tan^{-1}(1)$
95. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two non-collinear unit vectors and $\mathbf{u} = \hat{\mathbf{a}} - (\hat{\mathbf{a}} \cdot \hat{\mathbf{b}})\hat{\mathbf{b}}$ and $\mathbf{v} = \hat{\mathbf{a}} \times \hat{\mathbf{b}}$. Then, $|\mathbf{v}|$ is
 (a) $|\mathbf{u}| = |\mathbf{v}|$
 (b) $|\mathbf{u}| + |\mathbf{u} \cdot \hat{\mathbf{b}}|$
 (c) $|\mathbf{u}| + |\mathbf{u} \cdot \hat{\mathbf{a}}|$
 (d) None of the above
96. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-coplanar vectors and $\mathbf{d}$ be a non-zero vector, which is perpendicular to $(\mathbf{a} + \mathbf{b} + \mathbf{c})$. If $\mathbf{d} = (\mathbf{a} \times \mathbf{b})\sin x + (\mathbf{b} \times \mathbf{c})\cos y + 2(\mathbf{c} \times \mathbf{a})$. Then,
 (a) $\frac{\mathbf{d} \cdot (\mathbf{a} + \mathbf{c})}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} = 2$
 (b) $\frac{\mathbf{d} \cdot (\mathbf{a} + \mathbf{c})}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} = -2$
 (c) minimum value of $x^2 + y^2$ is $\frac{\pi^2}{4}$
 (d) minimum value of $x^2 + y^2$ is $\frac{5\pi^2}{4}$

Type 3. Assertion and Reason

Directions (Q. Nos. 97-105) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

97. **Statement I** Let $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$, $\mathbf{b} = 4\hat{\mathbf{i}} + 6\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$.

Statement II If $\mathbf{a} \neq \mathbf{0}$, $\mathbf{b} \neq \mathbf{0}$ and $\mathbf{a}$, $\mathbf{b}$ are non-collinear vectors, then $\mathbf{a} \times \mathbf{b} = ab \sin \theta \hat{\mathbf{n}}$, where θ is the smaller angle between the vectors $\mathbf{a}$ and $\mathbf{b}$; and $\hat{\mathbf{n}}$ is unit vector such that $\mathbf{a}$, $\mathbf{b}$ and $\hat{\mathbf{n}}$ taken in this order form right handed orientation.

98. **Statement I** Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ be position vectors of four points A , B , C and D and $3\mathbf{a} - 2\mathbf{b} + 5\mathbf{c} - 6\mathbf{d} = \mathbf{0}$, then points A , B , C and D are coplanar.

Statement II Three non-zero, linearly dependent coinitial vectors $\mathbf{PQ}$, $\mathbf{PR}$ and $\mathbf{PS}$ are coplanar.

99. Statement I Let $\mathbf{a} = 3\hat{i} - \hat{j}$, $\mathbf{b} = 2\hat{i} + \hat{j} - 3\hat{k}$. If $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ such that $\mathbf{b}_1$ is collinear with $\mathbf{a}$ and $\mathbf{b}_2$ is perpendicular to $\mathbf{a}$ is possible, then $\mathbf{b}_2 = \hat{i} + 3\hat{j} - 3\hat{k}$.

Statement II If $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-collinear vectors, then $\mathbf{b}$ can be expressed as $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$, where $\mathbf{b}_1$ is collinear with $\mathbf{a}$ and $\mathbf{b}_2$ is perpendicular to $\mathbf{a}$.

100. Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ be unit vectors makes an angle $\pi/3$ with each other. If $\{\hat{\mathbf{a}} \times (\mathbf{b} \times \hat{\mathbf{c}})\} \cdot (\hat{\mathbf{a}} \times \hat{\mathbf{c}}) = 5$, then

Statement I $[\hat{\mathbf{a}} \ \mathbf{b} \ \hat{\mathbf{c}}] = 0$

Statement II $[\mathbf{x} \ \mathbf{y} \ \mathbf{z}] = 0$, if $\mathbf{x} = \mathbf{y}$ or $\mathbf{y} = \mathbf{z}$ or $\mathbf{x} = \mathbf{z}$.

101. Statement I For all non-zero real and distinct values of $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2$, and c_3 , the vectors $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ are non-coplanar, if $|c_1| > |a_1| + |b_1|$, $|c_2| > |a_2| + |b_2|$, $|c_3| > |a_3| + |b_3|$.

Statement II Three vectors $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ will be

non-coplanar, if $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$.

102. Consider three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

Statement I $\mathbf{a} \times \mathbf{b} = \{(\hat{i} \times \mathbf{a}) \cdot \mathbf{b}\}\hat{i} + \{(\hat{j} \times \mathbf{a}) \cdot \mathbf{b}\}\hat{j} + \{(\hat{k} \times \mathbf{a}) \cdot \mathbf{b}\}\hat{k}$.

Statement II $\mathbf{c} = (\mathbf{i} \cdot \mathbf{c})\hat{i} + (\mathbf{j} \cdot \mathbf{c})\hat{j} + (\mathbf{k} \cdot \mathbf{c})\hat{k}$.

103. Statement I Distance of point $D(1, 0, -1)$ from the plane of points $A(1, -2, 0)$, $B(3, 1, 2)$ and $C(-1, 1, -1)$ is $8/\sqrt{229}$.

Statement II Volume of tetrahedron formed by the points A, B, C and D is $\sqrt{229}/2$.

104. Statement I If $a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ are three mutually perpendicular unit vectors, then $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ may be mutually perpendicular unit vectors.

Statement II Value of determinant and its transpose are the same.

105. Statement I A component of vector $\mathbf{b} = 4\hat{i} + 2\hat{j} + 3\hat{k}$ in the direction perpendicular to the direction of vector $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ is $\hat{i} - \hat{j}$.

Statement II A component of vector in the direction of $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$ is $2\hat{i} + 2\hat{j} + 2\hat{k}$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 106-108)

P is a point of intersection of $x^2 + y^2 = 20$... (i) and $y = 2$... (ii) in the 1st quadrant Q is a point on Eq. (ii), in the 1st quadrant such that it is an interior point of Eq. (i) and whose abscissae is an integer.

106. The sum of the projection of $\mathbf{OQ}$ on $\mathbf{OP}$ (O is the origin) for possible positions of Q is

- (a) $\frac{18\sqrt{5}}{5}$ (b) $3\sqrt{5}$ (c) $\frac{12\sqrt{5}}{5}$ (d) $\sqrt{5}$

107. The sum of $|\mathbf{OP} \times \mathbf{OQ}|$ for possible position of Q is

- (a) 6 (b) 8 (c) 12 (d) 10

108. Let R be $(-5, 2)$, then ratio of the maximum of $|\mathbf{QR}|$ and the sum of the projections of $\mathbf{OQ}$ on Y -axis for possible positions of Q , is

- (a) 1:2 (b) 1:1 (c) 4:3 (d) 5:4

Passage II (Q. Nos. 109-111) Consider a triangular pyramid $ABCD$, the position vectors of whose angular points are $A(3, 0, 1)$, $B(-1, 4, 1)$, $C(5, 2, 3)$ and $D(0, -5, 4)$. Let G be the point of intersection of the medians of $\triangle BCD$.

109. The length of vector $\mathbf{AG}$ is

- (a) $\sqrt{17}$ (b) $\frac{\sqrt{51}}{3}$ (c) $\frac{3}{\sqrt{6}}$ (d) $\frac{\sqrt{59}}{4}$

110. Area of $\triangle ABC$ (in sq units) is

- (a) 24 (b) $8\sqrt{6}$ (c) $4\sqrt{6}$ (d) 30

111. The length of the perpendicular from vertex D on the opposite face is

- (a) $\frac{14}{\sqrt{6}}$ (b) $\frac{2}{\sqrt{6}}$
(c) $\frac{3}{\sqrt{6}}$ (d) None of these

Passage III (Q. Nos. 112-114) Vectors $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, each of magnitude $\sqrt{2}$, makes an angle of 60° with each other and $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{a}$, $\mathbf{y} \times (\mathbf{z} \times \mathbf{x}) = \mathbf{b}$, $\mathbf{x} \times \mathbf{y} = \mathbf{c}$.

112. Vector $\mathbf{x}$ is

- (a) $\frac{1}{2}[(\mathbf{a} - \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})]$
(b) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} - \mathbf{b})]$
(c) $\frac{1}{2}[-(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + (\mathbf{a} + \mathbf{b})]$
(d) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} - (\mathbf{a} + \mathbf{b})]$

113. Vector $\mathbf{y}$ is

- (a) $\frac{1}{2}[(\mathbf{a} + \mathbf{c}) \times \mathbf{b} - \mathbf{b} - \mathbf{a}]$ (b) $\frac{1}{2}[(\mathbf{a} - \mathbf{c}) \times \mathbf{c} + \mathbf{b} + \mathbf{a}]$
(c) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + \mathbf{b} + \mathbf{a}]$ (d) $\frac{1}{2}[(\mathbf{a} - \mathbf{c}) \times \mathbf{a} + \mathbf{b} - \mathbf{a}]$

114. Vector $\mathbf{z}$ is

- (a) $\frac{1}{2}[(\mathbf{a} - \mathbf{c}) \times \mathbf{c} - \mathbf{b} + \mathbf{a}]$ (b) $\frac{1}{2}[(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + \mathbf{b} - \mathbf{a}]$
(c) $\frac{1}{2}[\mathbf{c} \times (\mathbf{a} - \mathbf{b}) + \mathbf{b} + \mathbf{a}]$ (d) None of these

Type 5. Match the Columns

115. Match the statements of Column I with the values of Column II.

Column I	Column II
A. Area of the triangle whose vertices are the points with rectangular cartesian coordinates $(1, 2, 3)$, $(-2, 1, -4)$ and $(3, 4, -2)$, is	p. 0
B. The value of $(a \times b) \cdot (c \times d) + (b \times c) \cdot (a \times d) + (c \times a) \cdot (b \times d)$ is	q. 2
C. A square PQRS of side length P is folded along the diagonal PR so that the point Q reaches at Q' and planes PRQ' and PRS are perpendicular to one another. If the shortest distance between PQ' and RS is $\frac{kP}{\sqrt{6}}$, then k is equal to	r. $\frac{\sqrt{1218}}{2}$
D. If $a = 2\hat{i} + 3\hat{j} - \hat{k}$, $b = -\hat{i} + 2\hat{j} - 4\hat{k}$, $c = \hat{i} + \hat{j} + \hat{k}$, and $d = 3\hat{i} + 2\hat{j} + \hat{k}$, then $(a \times b) \cdot (c \times d)$ is equal to	s. 21

116. Match the statements of Column I with the values of Column II.

Column I	Column II
A. Let $a = 2\hat{i} + \hat{k}$, $b = \hat{i} + \hat{j} + \hat{k}$ and $c = 4\hat{i} + 3\hat{j} + 7\hat{k}$, then the vector r which satisfies $r \times b = c \times b$ and $r \cdot a = 0$, is	p. $2\hat{i} - \hat{j}$
B. A vector r coplanar with the vector $q = 24\hat{i} - 32\hat{j} - 30\hat{k}$ $6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k}$ of magnitude 50, making an obtuse angle with Z-axis, is	q. $24\hat{i} - 32\hat{j} - 30\hat{k}$

Column I	Column II
C. A vector of magnitude 2 along the bisector of the angle between two vectors $2\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} - 2\hat{k}$, is	r. $-\hat{i} - 8\hat{j} + 2\hat{k}$
D. Let $a = 4\hat{i} + 3\hat{j}$ and b be two vectors perpendicular to each other in XY-plane. The vector in the same plane having projections 1 and 2 along a and b, is	s. $\frac{2}{\sqrt{10}}(3\hat{i} - \hat{k})$

117. Match the statements of Column I with the values of Column II.

Column I	Column II
A. If $ a = b = c $, angle between each pair of vectors is $\frac{\pi}{3}$ and $ a + b + c = \sqrt{6}$, then $2 a $ is equal to	p. 3
B. If a is perpendicular to $b + c$, b is perpendicular to $c + a$, c is perpendicular to $a + b$ and $ a = 2$, $ b = 3$, $ c = 6$, then $ a + b + c - 2$ is equal to	q. 2
C. If $a = 2\hat{i} + 3\hat{j} - \hat{k}$, $b = -\hat{i} + 2\hat{j} - 4\hat{k}$, $c = \hat{i} + \hat{j} + \hat{k}$ and $d = 3\hat{i} + 2\hat{j} + \hat{k}$, then $\frac{1}{7}(a \times b) \cdot (c \times d)$ is equal to	r. 4
D. If $ a = b = c = 2$ and $a \cdot b = b \cdot c = c \cdot a = 2$, then $[a \ b \ c] \cos 45^\circ$ is equal to	s. 5

Type 6. Single Integer Answer Type Questions

118. Given, $A = 2\hat{i} + 3\hat{j} + 6\hat{k}$, $B = \hat{i} + \hat{j} - 2\hat{k}$ and $C = \hat{i} + 2\hat{j} + \hat{k}$. If $|A \times [A \times (A \times B)] \cdot C| = abc$, then $a + b - c$ is equal to _____.

119. If a and b are non-zero, non-collinear vectors and the linear combination $(2x - y)a + 4b = 5a(x - 2y)b$ holds for real x and y, then $x + y$ is equal to _____.

120. Given an equilateral $\triangle ABC$ with side length equal to a. Let M and N be two points respectively on the side AB and AC such that $AN = kAC$ and $AM = \frac{AB}{3}$. If BN and CM are orthogonal, then the value of $k = \frac{\lambda}{\mu}$. The value of $\lambda + \mu$ is _____.

121. Let $a = \hat{i} + \hat{j}$, $b = \hat{j} + \hat{k}$ and $c = \alpha a + \beta b$. If the vectors $\hat{i} - 2\hat{j} + \hat{k}$, $3\hat{i} + 2\hat{j} - \hat{k}$ and c are coplanar, then $\frac{\alpha}{\beta} = -\lambda$. The value of λ is _____.

122. Let u be a vector on rectangular coordinate system with sloping angle 60° . Suppose that $|u - \hat{i}|$ is geometric mean of $|u|$ and $|u - 2\hat{i}|$, where $\hat{i}$ is the unit vector along X-axis. Then, the value of $(\sqrt{2} + 1)|u|$ is _____.

123. The absolute value of parameter t for which the area of the triangle whose vertices are $A(-1, 1, 2)$, $B(1, 2, 3)$ and $C(t, 1, 1)$ is minimum, is _____.

124. If x and y are two non-zero and non-collinear vectors satisfying $[(a-2)\alpha^2 + (b-3)\alpha + c]x + [(a-2)\beta^2 + (b-3)\beta + c]y + [(a-2)\gamma^2 + (b-3)\gamma + c](x \times y) = 0$, where α , β and γ are three distinct real numbers, then the value of $(a^2 + b^2 + c^2 - 4)$ is _____.

125. If $\hat{a}$, $\hat{b}$ and $\hat{c}$ are unit vectors such that $\hat{a} \cdot \hat{b} = 0 = \hat{a} \cdot \hat{c}$ and the angle between $\hat{b}$ and $\hat{c}$ is $\pi/3$, then the value of $|\hat{a} \times \hat{b} - \hat{a} \times \hat{c}|$ is _____.

Entrances Gallery

JEE Advanced/IITJEE

1. Let $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ be three vectors each of magnitude $\sqrt{2}$ and the angle between each pair of them is $\frac{\pi}{3}$. If

$\mathbf{a}$ is a non-zero vector perpendicular to $\mathbf{x}$ and $\mathbf{y} \times \mathbf{z}$ and $\mathbf{b}$ is a non-zero vector perpendicular to $\mathbf{y}$ and $\mathbf{z} \times \mathbf{x}$, then [2014]

- (a) $\mathbf{b} = (\mathbf{b} \cdot \mathbf{z})(\mathbf{z} - \mathbf{x})$ (b) $\mathbf{a} = (\mathbf{a} \cdot \mathbf{y})(\mathbf{y} - \mathbf{z})$
(c) $\mathbf{a} \cdot \mathbf{b} = -(\mathbf{a} \cdot \mathbf{y})(\mathbf{b} \cdot \mathbf{z})$ (d) $\mathbf{a} = (\mathbf{a} \cdot \mathbf{y})(\mathbf{z} - \mathbf{y})$

2. Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three non-coplanar unit vectors such that the angle between every pair of them is $\frac{\pi}{3}$.

If $\hat{\mathbf{a}} \times \hat{\mathbf{b}} + \hat{\mathbf{b}} \times \hat{\mathbf{c}} = p\hat{\mathbf{a}} + q\hat{\mathbf{b}} + r\hat{\mathbf{c}}$, where p , q and r are scalars, then the value of $\frac{p^2 + 2q^2 + r^2}{q^2}$ is [2014]

3. Match the Column I with Column II and select the correct answer using the code given below the Columns. [2013]

Column I	Column II
A. If the volume of parallelopiped determined by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 2. Then, the volume of the parallelopiped determined by vectors $2(\mathbf{a} \times \mathbf{b})$, $3(\mathbf{b} \times \mathbf{c})$ and $(\mathbf{c} \times \mathbf{a})$ is	p. 100
B. If the volume of parallelopiped determined by vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 5. Then, the volume of the parallelopiped determined by vectors $3(\mathbf{a} + \mathbf{b})$, $(\mathbf{b} + \mathbf{c})$ and $2(\mathbf{c} + \mathbf{a})$ is	q. 30
C. If the area of a triangle with adjacent sides determined by vectors $\mathbf{a}$ and $\mathbf{b}$ is 20. Then, the area of the triangle with adjacent sides determined by vectors $(2\mathbf{a} + 3\mathbf{b})$ and $(\mathbf{a} - \mathbf{b})$ is	r. 24
D. If the area of a parallelogram with adjacent sides determined by vectors $\mathbf{a}$ and $\mathbf{b}$ is 30. Then, the area of the parallelogram with adjacent sides determined by vectors $(\mathbf{a} + \mathbf{b})$ and $\mathbf{a}$ is	s. 60

Codes

A B C D	A B C D
(a) s q r p	(b) q r p s
(c) r s p q	(d) p s r q

4. Consider the set of eight vectors $V = \{\mathbf{a}\hat{\mathbf{i}} + \mathbf{b}\hat{\mathbf{j}} + \mathbf{c}\hat{\mathbf{k}}; \mathbf{a}, \mathbf{b}, \mathbf{c} \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then, p is [2013]
- (a) 5 (b) 6 (c) 4 (d) 3

5. Let $\mathbf{PR} = 3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{SQ} = \hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ determine diagonals of a parallelogram $PQRS$ and $\mathbf{PT} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ be another vector. Then, the volume of the parallelopiped determined by the vectors $\mathbf{PT}$, $\mathbf{PQ}$ and $\mathbf{PS}$ is [2013]
- (a) 5 (b) 20 (c) 10 (d) 30

6. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors satisfying $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2 = 9$, then $|2\hat{\mathbf{a}} + 5\hat{\mathbf{b}} + 5\hat{\mathbf{c}}|$ is [2012]

7. If $\mathbf{a}$ and $\mathbf{b}$ are vectors such that $|\mathbf{a} + \mathbf{b}| = \sqrt{29}$ and $\mathbf{a} \times (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) \times \mathbf{b}$, then a possible value of $(\mathbf{a} + \mathbf{b}) \cdot (-7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ is [2012]
- (a) 0 (b) 3 (c) 4 (d) 8

8. Let $\mathbf{a} = -\hat{\mathbf{i}} - \hat{\mathbf{k}}$, $\mathbf{b} = -\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ be three given vectors. If $\mathbf{r}$ is a vector such that $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$ and $\mathbf{r} \cdot \mathbf{a} = 0$, then the value of $\mathbf{r} \cdot \mathbf{b}$ is [2011]

9. Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$ be three vectors. A vector $\mathbf{v}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$, whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is given by [2011]
- (a) $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (b) $-3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - \hat{\mathbf{k}}$
(c) $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

10. The vectors which is/are coplanar with vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, are perpendicular to the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, is/are [2011]
- (a) $\hat{\mathbf{j}} - \hat{\mathbf{k}}$ (b) $-\hat{\mathbf{i}} + \hat{\mathbf{j}}$
(c) $\hat{\mathbf{i}} - \hat{\mathbf{j}}$ (d) $-\hat{\mathbf{j}} + \hat{\mathbf{k}}$

11. Let P, Q, R and S be the points on the plane with position vectors $-2\hat{\mathbf{i}} - \hat{\mathbf{j}}$, $4\hat{\mathbf{i}}$, $3\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$ and $-3\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, respectively. The quadrilateral $PQRS$ must be a [2010]
- (a) parallelogram, which is neither a rhombus nor a rectangle
(b) square
(c) rectangle, but not a square
(d) rhombus, but not a square

12. Two adjacent sides of a parallelogram $ABCD$ are given by $\mathbf{AB} = 2\hat{\mathbf{i}} + 10\hat{\mathbf{j}} + 11\hat{\mathbf{k}}$ and $\mathbf{AD} = -\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The side AD is rotated by an acute angle α in the plane of the parallelogram so that AD becomes AD' . If AD' makes a right angle with the side AB , then the cosine of the angle α is given by [2010]
- (a) $\frac{8}{9}$ (b) $\frac{\sqrt{17}}{9}$ (c) $\frac{1}{9}$ (d) $\frac{4\sqrt{5}}{9}$

13. If $\mathbf{a}$ and $\mathbf{b}$ are vectors in space given by $\mathbf{a} = \frac{\hat{\mathbf{i}} - 2\hat{\mathbf{j}}}{\sqrt{5}}$ and $\mathbf{b} = \frac{2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}}{\sqrt{14}}$, then the value of $(2\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})]$ is [2010]

JEE Main/AIEEE

13

Vector Algebra

- 14.** If $[\mathbf{a} \times \mathbf{b} \ \mathbf{b} \times \mathbf{c} \ \mathbf{c} \times \mathbf{a}] = \lambda[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]^2$, then λ is equal to [2014]
(a) 1 (b) 2 (c) 3 (d) 0
- 15.** If the vectors $\mathbf{AB} = 3\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$ and $\mathbf{AC} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ are the sides of a ΔABC , then the length of the median through A is [2013]
(a) $\sqrt{18}$ (b) $\sqrt{72}$
(c) $\sqrt{33}$ (d) $\sqrt{45}$
- 16.** Let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be two unit vectors. If the vectors $\mathbf{c} = \hat{\mathbf{a}} + 2\hat{\mathbf{b}}$ and $\mathbf{d} = 5\hat{\mathbf{a}} - 4\hat{\mathbf{b}}$ are perpendicular to each other, then the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is [2012]
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$
- 17.** Let $ABCD$ be a parallelogram such that $\mathbf{AB} = \mathbf{q}$, $\mathbf{AD} = \mathbf{p}$ and $\angle BAD$ be an acute angle. If $\mathbf{r}$ is the vector that coincides with the altitude directed from the vertex B to the side AD , then $\mathbf{r}$ is given by [2012]
(a) $\mathbf{r} = 3\mathbf{q} - \frac{3(\mathbf{p} \cdot \mathbf{q})}{(\mathbf{p} \cdot \mathbf{p})}\mathbf{p}$ (b) $\mathbf{r} = -\mathbf{q} + \left(\frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{p} \cdot \mathbf{p}}\right)\mathbf{p}$
(c) $\mathbf{r} = \mathbf{q} - \left(\frac{\mathbf{p} \cdot \mathbf{q}}{\mathbf{p} \cdot \mathbf{p}}\right)\mathbf{p}$ (d) $\mathbf{r} = -3\mathbf{q} + \frac{3(\mathbf{p} \cdot \mathbf{q})}{(\mathbf{p} \cdot \mathbf{p})}\mathbf{p}$
- 18.** If $\mathbf{a} = \frac{1}{\sqrt{10}}(3\hat{\mathbf{i}} + \hat{\mathbf{k}})$ and $\mathbf{b} = \frac{1}{7}(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$, then the value of $(2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})\}$ is [2011]
(a) -3 (b) 5 (c) 3 (d) -5
- 19.** If the vectors $\mathbf{a}, \mathbf{b}$ are not perpendicular and $\mathbf{c}, \mathbf{d}$ are two vectors satisfying $\mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$ and $\mathbf{a} \cdot \mathbf{d} = 0$. Then, the vector $\mathbf{d}$ is equal to [2011]
(a) $\mathbf{c} + \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (b) $\mathbf{b} + \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$
(c) $\mathbf{c} - \left(\frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{b}$ (d) $\mathbf{b} - \left(\frac{\mathbf{b} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}}\right)\mathbf{c}$
- 20.** If the vectors $p\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} + q\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + \hat{\mathbf{j}} + r\hat{\mathbf{k}}$ ($p \neq q \neq r \neq 1$) are coplanar, then the value of $pqr - (p + q + r)$ is [2011]
(a) -2 (b) 2 (c) 0 (d) -1
- 21.** Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three non-zero vectors which are pairwise non-collinear. If $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$, then $\mathbf{a} + 3\mathbf{b} + 6\mathbf{c}$ is [2011]
(a) $\mathbf{a} + \mathbf{c}$ (b) $\mathbf{a}$ (c) $\mathbf{c}$ (d) 0
- 22.** Let $\mathbf{a} = \hat{\mathbf{j}} - \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$. Then, the vector $\mathbf{b}$ satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{b} = 3$, is [2010]
(a) $-\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (b) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
(c) $\hat{\mathbf{i}} - \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (d) $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
- 23.** If the vectors $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, $\mathbf{b} = 2\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \lambda\hat{\mathbf{i}} + \hat{\mathbf{j}} + \mu\hat{\mathbf{k}}$ are mutually orthogonal, then (λ, μ) is equal to [2010]
(a) $(-3, 2)$ (b) $(2, -3)$ (c) $(-2, 3)$ (d) $(3, -2)$
- 24.** If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are non-coplanar vectors and p, q are real numbers, then the equality $[3\mathbf{u} \ p\mathbf{v} \ p\mathbf{w}] - [p\mathbf{v} \ w\mathbf{q} \ \mathbf{u}] - [2\mathbf{w} \ q\mathbf{v} \ q\mathbf{u}] = 0$ holds for [2009]
(a) exactly two values of (p, q)
(b) more than two but not all values of (p, q)
(c) all values of (p, q)
(d) exactly one value of (p, q)
- 25.** The vector $\mathbf{a} = \alpha\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \beta\hat{\mathbf{k}}$ lies in the plane of the vectors $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c} = \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and bisects the angle between $\mathbf{b}$ and $\mathbf{c}$. Then, which one of the following gives possible values of α and β ? [2008]
(a) $\alpha = 1, \beta = 1$ (b) $\alpha = 2, \beta = 2$
(c) $\alpha = 1, \beta = 2$ (d) $\alpha = 2, \beta = 1$
- 26.** If the non-zero vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ related by $\mathbf{a} = 8\mathbf{b}$ and $\mathbf{c} = -7\mathbf{b}$. Then, the angle between $\mathbf{a}$ and $\mathbf{c}$ is [2012]
(a) π (b) 0
(c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$
- 27.** If $\hat{\mathbf{u}}$ and $\hat{\mathbf{v}}$ are unit vectors and θ is the acute angle between them, then $2\hat{\mathbf{u}} \times 3\hat{\mathbf{v}}$ is a unit vector for [2007]
(a) exactly two values of θ
(b) more than two values of θ
(c) no value of θ
(d) exactly one value of θ
- 28.** Let $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and $\mathbf{c} = x\hat{\mathbf{i}} + (x - 2)\hat{\mathbf{j}} - \hat{\mathbf{k}}$. If the vector $\mathbf{c}$ lies in the plane of $\mathbf{a}$ and $\mathbf{b}$, then x equals [2007]
(a) 0 (b) 1 (c) -4 (d) -2
- 29.** If three vectors such that $\mathbf{a} \cdot \mathbf{b} \neq 0$, $\mathbf{b} \cdot \mathbf{c} \neq 0$, then $\mathbf{a}$ and $\mathbf{c}$ are [2006]
(a) inclined at an angle of $\frac{\pi}{6}$ between them
(b) perpendicular
(c) parallel
(d) inclined at an angle of $\frac{\pi}{3}$ between them
- 30.** The values of a , for which the points A, B, C with position vectors $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ and $a\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, respectively are the vertices of a right angled triangle with $C = \frac{\pi}{2}$, are [2006]
(a) -2 and -1
(b) -2 and 1
(c) 2 and -1
(d) 2 and 1

Target Exercises

- 31.** For any vector $\mathbf{a}$, the value of $(\mathbf{a} \times \hat{\mathbf{i}})^2 + (\mathbf{a} \times \hat{\mathbf{j}})^2 + (\mathbf{a} \times \hat{\mathbf{k}})^2$ is equal to [2005]
(a) $4a^2$ (b) $2a^2$ (c) a^2 (d) $3a^2$
- 32.** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be three non-zero vectors such that no two of these are collinear. If the vector $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$ and $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$ (λ being some non-zero scalar), then $\mathbf{a} + 2\mathbf{b} + 6\mathbf{c}$ equals [2004]
(a) $\lambda\mathbf{a}$ (b) $\lambda\mathbf{b}$ (c) $\lambda\mathbf{c}$ (d) $\mathbf{0}$
- 33.** A particle is acted upon by constant forces $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ and $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ which displace it from a point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ to the point $5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}$. The work done in standard units by the forces is given by [2004]
(a) 40 units (b) 30 units (c) 25 units (d) 15 units
- 34.** Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors such that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a}$. If θ is the acute angle between the vectors $\mathbf{b}$ and $\mathbf{c}$, then $\sin \theta$ equals [2004]
(a) $\frac{1}{3}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{3}$ (d) $\frac{2\sqrt{2}}{3}$
- 35.** If $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are three vectors, such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$, $|\mathbf{a}| = 1$, $|\mathbf{b}| = 2$, $|\mathbf{c}| = 3$, then $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$ is equal to [2003]
(a) 0 (b) -7 (c) 7 (d) 1
- 36.** Consider points A, B, C and D with position vectors $7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$, $\hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$, $-\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ and $5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$, respectively. Then, $ABCD$ is a [2003]
(a) square (b) rhombus (c) rectangle (d) None of the above
- 37.** If the vector $\hat{\mathbf{i}} + x\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is rotated through an angle θ and doubled in magnitude, then it becomes $4\hat{\mathbf{i}} + (4x - 2)\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$. The values of x are [2002]
(a) $-\frac{2}{3}, 2$ (b) $\frac{1}{3}, 2$ (c) $\frac{2}{3}, 0$ (d) 2, 7
- 38.** If the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ form the sides BC, CA and AB , respectively of a $\triangle ABC$, then [2002]
(a) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
(b) $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a} = 0$
(c) $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = 0$
(d) $\mathbf{a} \times \mathbf{a} + \mathbf{a} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = 0$

Other Engineering Entrances

- 39.** If $\lambda(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$ is a unit vector, then the values of λ are [Kerala CEE 2014]
(a) $\pm \frac{1}{7}$ (b) ± 7 (c) $\pm \sqrt{43}$ (d) $\pm \frac{1}{\sqrt{43}}$ (e) $\pm \frac{1}{\sqrt{7}}$
- 40.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and $\mathbf{d}$ is such that $\mathbf{d} = \frac{1}{x}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ and $\mathbf{d} = \frac{1}{y}(\mathbf{b} + \mathbf{c} + \mathbf{d})$, where x and y are non-zero real numbers, then $\frac{1}{xy}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$ equals [EAMCET 2014]
(a) $3\mathbf{c}$ (b) $-\mathbf{a}$ (c) $\mathbf{0}$ (d) $2\mathbf{a}$
- 41.** If three non-zero and non-collinear vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are such that $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$, $3\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$. Then, $\mathbf{a} + 3\mathbf{b} + 2\mathbf{c}$ is equal to [EAMCET 2014]
(a) $\mathbf{0}$ (b) $2\mathbf{a}$ (c) $3\mathbf{b}$ (d) $4\mathbf{c}$
- 42.** The length of longer diagonal of the parallelogram constructed on $5\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} - 3\mathbf{b}$, if it is given that $|\mathbf{a}| = 2\sqrt{2}$, $|\mathbf{b}| = 3$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$, is [VITEEE 2014]
(a) 15 (b) $\sqrt{113}$ (c) $\sqrt{593}$ (d) $\sqrt{369}$
- 43.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are reciprocal vectors, then $(\mathbf{a} + m\mathbf{b} + n\mathbf{c}) \cdot (\mathbf{p} + m\mathbf{q} + n\mathbf{r})$ is equal to [VITEEE 2014]
(a) $l + m + n$ (b) $l^3 + m^3 + n^3$ (c) $l^2 + m^2 + n^2$ (d) None of the above
- 44.** If a, b and c are p th, q th and r th terms of a GP, then the vectors $\log a\hat{\mathbf{i}} + \log b\hat{\mathbf{j}} + \log c\hat{\mathbf{k}}$ and $(q - r)\hat{\mathbf{i}} + (r - p)\hat{\mathbf{j}} + (p - q)\hat{\mathbf{k}}$ are [Manipal 2014]
(a) equal (b) parallel (c) perpendicular (d) None of the above
- 45.** If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two unit vectors inclined at an angle $\frac{\pi}{3}$, then the value of $|\hat{\mathbf{a}} + \hat{\mathbf{b}}|$ is [Karnataka CET 2014]
(a) equal to 1 (b) greater than 1 (c) equal to 0 (d) less than 1
- 46.** If $\mathbf{a} \cdot \mathbf{b} = 0$ and $\mathbf{a} + \mathbf{b}$ makes an angle of 60° with $\mathbf{a}$, then [Kerala CEE 2014]
(a) $|\mathbf{a}| = 2|\mathbf{b}|$ (b) $2|\mathbf{a}| = |\mathbf{b}|$ (c) $|\mathbf{a}| = \sqrt{3}|\mathbf{b}|$ (d) $|\mathbf{a}| = |\mathbf{b}|$ (e) $\sqrt{3}|\mathbf{a}| = |\mathbf{b}|$

47. If $\hat{i} + \hat{j}$, $\hat{j} + \hat{k}$ and $\hat{i} + \hat{k}$ are the position vectors of the vertices of $\triangle ABC$ taken in order, then $\angle A$ is equal to [Kerala CEE 2014]

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{5}$ (c) $\frac{\pi}{6}$
(d) $\frac{\pi}{4}$ (e) $\frac{\pi}{3}$

48. If $\mathbf{a} = \hat{i} - 2\hat{j} + 3\hat{k}$ and $\mathbf{b}$ is a vector such that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{b}|^2$ and $|\mathbf{a} - \mathbf{b}| = \sqrt{7}$, then $|\mathbf{b}|$ is equal to [Kerala CEE 2014]

(a) $\sqrt{7}$ (b) $\sqrt{3}$ (c) 7
(d) 3 (e) $7\sqrt{3}$

49. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-zero vectors such that each one of them being perpendicular to the sum of the other two vectors, then the value of $|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2$ is [Kerala CEE 2014]

(a) $|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2$
(b) $|\mathbf{a}| + |\mathbf{b}| + |\mathbf{c}|$
(c) $2(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2)$
(d) $\frac{1}{2}(|\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2)$
(e) 0

50. Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors such that $\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$. If $|\mathbf{u}| = 3$, $|\mathbf{v}| = 4$ and $|\mathbf{w}| = 5$, then $\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{u}$ is equal to [Kerala CEE 2014]

(a) 0 (b) -25 (c) 25
(d) 50 (e) 47

51. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors with magnitudes 2, 3 and 4 respectively, then the best upper bound of $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 + |\hat{\mathbf{b}} - \hat{\mathbf{c}}|^2 + |\hat{\mathbf{c}} - \hat{\mathbf{a}}|^2$ among the given values is [EAMCET 2014]

(a) 93 (b) 97
(c) 87 (d) 90

52. The angle between the vectors $\mathbf{a} = \hat{i} + 2\hat{j} + 2\hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + 2\hat{k}$ is [GGSIU 2014]

(a) $\sin^{-1}\left(\frac{1}{9}\right)$ (b) $\cos^{-1}\left(\frac{8}{9}\right)$
(c) $\sin^{-1}\left(\frac{8}{9}\right)$ (d) $\cos^{-1}\left(\frac{1}{9}\right)$

53. If $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are two unit vectors inclined at an angle 60° to each other, then which one of the following is correct? [RPET 2014]

(a) $|\hat{\mathbf{a}} + \hat{\mathbf{b}}| < 1$ (b) $|\hat{\mathbf{a}} + \hat{\mathbf{b}}| > 1$
(c) $|\hat{\mathbf{a}} - \hat{\mathbf{b}}| < 1$ (d) $|\hat{\mathbf{a}} - \hat{\mathbf{b}}| > 1$

54. Given two vectors are $\hat{i} - \hat{j}$ and $\hat{i} + 2\hat{j}$, the unit vector is coplanar with the two vectors and perpendicular to the first. Find the vector. [J&K CET 2014]

(a) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$ (b) $\frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$
(c) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (d) $\pm \frac{1}{\sqrt{2}}(2\hat{i} + 3\hat{j})$

55. The unit vector perpendicular to the vectors $\hat{i} - \hat{j}$ and $\hat{i} + \hat{j}$ forming a right handed system, is [BITSAT 2014]

(a) $\hat{k}$ (b) $-\hat{k}$ (c) $\frac{\hat{i} - \hat{j}}{\sqrt{2}}$ (d) $\frac{\hat{i} + \hat{j}}{\sqrt{2}}$

56. If $|\mathbf{a}| = 1$, $|\mathbf{b}| = 4$, $\mathbf{a} \cdot \mathbf{b} = 2$ and $\mathbf{c} = 2\mathbf{a} \times \mathbf{b} - 3\mathbf{b}$, then the angle between $\mathbf{b}$ and $\mathbf{c}$ is [Manipal 2014]

(a) $\frac{\pi}{6}$ (b) $\frac{5\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

57. The area of the parallelogram whose adjacent sides are $\hat{i} + \hat{k}$ and $2\hat{i} + \hat{j} + \hat{k}$, is [Karnataka CET 2014]

(a) 3 (b) $\sqrt{2}$
(c) 4 (d) $\sqrt{3}$

58. If $\mathbf{a} = \hat{i} + 2\hat{j} + 2\hat{k}$, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{6}$, then area of the triangle formed by these two vectors as two sides is [Kerala CEE 2014]

(a) $\frac{15}{4}$ (b) $\frac{15}{2}$ (c) 15
(d) $\frac{15\sqrt{3}}{2}$ (e) $15\sqrt{3}$

59. If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors such that $\hat{\mathbf{a}} + \hat{\mathbf{b}} + \hat{\mathbf{c}} = \mathbf{0}$. Then, which one of the following is correct? [AMU 2014]

(a) $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{b}} \times \hat{\mathbf{c}} = \hat{\mathbf{c}} \times \hat{\mathbf{a}} = \mathbf{0}$
(b) $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{b}} \times \hat{\mathbf{c}} = \hat{\mathbf{c}} \times \hat{\mathbf{a}} \neq \mathbf{0}$
(c) $\hat{\mathbf{a}} \times \hat{\mathbf{b}} = \hat{\mathbf{b}} \times \hat{\mathbf{c}} = \hat{\mathbf{a}} \times \hat{\mathbf{c}} = \mathbf{0}$
(d) $\hat{\mathbf{a}} \times \hat{\mathbf{b}}$, $\hat{\mathbf{b}} \times \hat{\mathbf{c}}$, $\hat{\mathbf{c}} \times \hat{\mathbf{a}}$ are mutually perpendicular

60. Let $\mathbf{a} = -\hat{i} - \hat{k}$, $\mathbf{b} = -\hat{i} + \hat{j}$ and $\mathbf{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\mathbf{r}$ is a vector such that $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$ and $\mathbf{r} \cdot \mathbf{a} = 0$, then the value of $\mathbf{r} \cdot \mathbf{b}$ is [AMU 2014]

(a) 3 (b) 6 (c) 9 (d) 12

61. If $|\mathbf{a}| = |\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$, then the area of the triangle constructed on the vectors $\mathbf{a} - 2\mathbf{b}$ and $3\mathbf{a} + 2\mathbf{b}$ is [GGSIU 2014]

(a) 50 (b) $50\sqrt{2}$ (c) $\frac{50}{\sqrt{2}}$ (d) 100

62. In a triangle with vertices $A(1, -1, 2)$, $B(5, -6, 2)$ and $C(1, 3, -1)$, find the altitude $n = |\mathbf{BD}|$. [GGSIU 2014]

(a) 5 (b) 10 (c) $5\sqrt{2}$ (d) $\frac{10}{\sqrt{2}}$

63. For any vector $\mathbf{a}$, what is the value of $|\mathbf{a} \times \hat{\mathbf{i}}|^2 + |\mathbf{a} \times \hat{\mathbf{j}}|^2 + |\mathbf{a} \times \hat{\mathbf{k}}|^2$? [RPET 2014]

(a) $|\mathbf{a}|^2$ (b) $2|\mathbf{a}|^2$
(c) $3|\mathbf{a}|^2$ (d) $4|\mathbf{a}|^2$

- 64.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three vectors such that $\mathbf{a} \times \mathbf{b} = \mathbf{c}$ and $\mathbf{b} \times \mathbf{c} = \mathbf{a}$, then [J&K CET 2014]
 (a) $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c}$
 (b) $\mathbf{a} = \mathbf{b} = \mathbf{c}$
 (c) $\mathbf{a} \neq \mathbf{b} \neq \mathbf{c} \neq 1$
 (d) $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are orthogonal in pairs
- 65.** If $\mathbf{r} = \alpha \mathbf{b} \times \mathbf{c} + \beta \mathbf{c} \times \mathbf{a} + \gamma \mathbf{a} \times \mathbf{b}$ and $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 2$, then $\alpha + \beta + \gamma$ is equal to [VITEEE 2014]
 (a) $\mathbf{r} \cdot [\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}]$
 (b) $\frac{1}{2} \mathbf{r} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$
 (c) $2\mathbf{r} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$
 (d) 4
- 66.** The value of $[(\mathbf{a} - \mathbf{b})(\mathbf{b} - \mathbf{c})(\mathbf{c} - \mathbf{a})]$ is [Karnataka CET 2014]
 (a) 0 (b) 1 (c) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (d) 2
- 67.** If $\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{c} = 0$ for some non-zero vector $\mathbf{r}$, then the value of $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ is [AMU 2014]
 (a) 2 (b) 3
 (c) 0 (d) None of these
- 68.** If x , y and z are non-zero real numbers and $\mathbf{a} = x\hat{\mathbf{i}} + 2\hat{\mathbf{j}}$, $\mathbf{b} = y\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ and $\mathbf{c} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$ are such that $\mathbf{a} \times \mathbf{b} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$, then $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ is equal to [EAMCET 2014]
 (a) 3 (b) 10 (c) 9 (d) 6
- 69.** If $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are the three-non-coplanar vectors, then $(\mathbf{u} + \mathbf{v} - \mathbf{w})[(\mathbf{u} - \mathbf{v}) \times (\mathbf{v} - \mathbf{w})]$ is equal to [J&K CET 2014]
 (a) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{u})$ (b) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
 (c) 0 (d) $3\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$
- 70.** Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$. If $\mathbf{c}$ is a vector such that $\mathbf{a} \cdot \mathbf{c} = |\mathbf{c}|$, $|\mathbf{c} - \mathbf{a}| = 2\sqrt{2}$ and the angle between $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ is 30° , then $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}|$ is equal to [Karnataka CET 2013]
 (a) $\frac{2}{3}$ (b) $\frac{3}{2}$
 (c) 2 (d) 3
- 71.** If $\mathbf{r} \cdot \mathbf{a} = \mathbf{r} \cdot \mathbf{b} = \mathbf{r} \cdot \mathbf{c} = 1$, where $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are any three non-coplanar vectors, then $\mathbf{r}$ is [UP SEE 2013]
 (a) coplanar with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$
 (b) parallel to $\mathbf{a} + \mathbf{b} + \mathbf{c}$
 (c) parallel to $\mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} + \mathbf{a} \times \mathbf{b}$
 (d) parallel to $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$
- 72.** The figure formed by four points $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}}$, $3\hat{\mathbf{i}} + 5\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and $\hat{\mathbf{k}} - \hat{\mathbf{j}}$ is a [MP PET 2012]
 (a) parallelogram (b) rectangle
 (c) trapezium (d) square
- 73.** If $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$, $|\mathbf{c}| = 7$, then angle between $\mathbf{a}$ and $\mathbf{b}$ is [OJEE 2012]
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
 (c) $\frac{\pi}{4}$ (d) π
- 74.** If $|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = 144$ and $|\mathbf{a}| = 4$, then $|\mathbf{b}|$ is equal to [Karnataka CET 2012]
 (a) 16 (b) 8 (c) 3 (d) 12
- 75.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are three non-coplanar vectors and $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are vectors defined by $\mathbf{p} = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$, $\mathbf{q} = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$ and $\mathbf{r} = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$, then the value of $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{p} + (\mathbf{b} + \mathbf{c}) \cdot \mathbf{q} + (\mathbf{c} + \mathbf{a}) \cdot \mathbf{r}$ is [Karnataka CET 2012]
 (a) 0 (b) 1 (c) 2 (d) 3
- 76.** If the volume of the parallelopiped formed by three non-coplanar vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ is 4 cu units, then $[\mathbf{a} \times \mathbf{b} \ \mathbf{b} \times \mathbf{c} \ \mathbf{c} \times \mathbf{a}]$ is equal to [Karnataka CET 2012]
 (a) 64 (b) 16 (c) 4 (d) 8
- 77.** If $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ are unit vectors such that $\hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{1}{2} \hat{\mathbf{b}}$, then angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{c}}$ is [MP PET 2012]
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
 (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$
- 78.** A unit vector in the XOY -plane that makes an angle 30° with the vector $\hat{\mathbf{i}} + \hat{\mathbf{j}}$ and makes an angle 60° with $\hat{\mathbf{i}} - \hat{\mathbf{j}}$, is [Kerala CEE 2011]
 (a) $\frac{1}{4}[(\sqrt{6} + \sqrt{2})\hat{\mathbf{i}} - (\sqrt{6} - \sqrt{2})\hat{\mathbf{j}}]$
 (b) $\frac{1}{2}[(\sqrt{6} - \sqrt{2})\hat{\mathbf{i}} + (\sqrt{6} + \sqrt{2})\hat{\mathbf{j}}]$
 (c) $\frac{1}{4}[(\sqrt{6} - \sqrt{2})\hat{\mathbf{i}} + (\sqrt{6} + \sqrt{2})\hat{\mathbf{j}}]$
 (d) $\frac{1}{3}[(\sqrt{6} + \sqrt{2})\hat{\mathbf{i}} + (\sqrt{6} - \sqrt{2})\hat{\mathbf{j}}]$
 (e) $\frac{1}{4}[(\sqrt{6} + \sqrt{2})\hat{\mathbf{i}} + (\sqrt{6} - \sqrt{2})\hat{\mathbf{j}}]$
- 79.** If S is the circumcentre, G is the centroid, O is the orthocentre of $\triangle ABC$, then $\mathbf{SA} + \mathbf{SB} + \mathbf{SC}$ is equal to [GGSIU 2011]
 (a) $\mathbf{SG}$ (b) $\mathbf{OS}$ (c) $\mathbf{SO}$ (d) $\mathbf{OG}$
- 80.** In R^2 , find the unit vector orthogonal to unit vector $\mathbf{x} = (\cos \alpha, \sin \alpha)$. [Guj CET 2011]
 (a) $\left(\cos \frac{\alpha}{2}, \sin \frac{\alpha}{2}\right)$ (b) $(-\cos \alpha, -\sin \alpha)$
 (c) $(-\sin \alpha, \cos \alpha)$ (d) $(\cos \alpha, \sin \alpha)$
- 81.** If a and b represent the adjacent sides of a parallelogram whose area is 15 sq units, then the area of the parallelogram whose adjacent sides are $3\mathbf{a} + 2\mathbf{b}$ and $\mathbf{a} + 3\mathbf{b}$, is [MP PET 2011]
 (a) 45 units
 (b) 75 units
 (c) 105 units
 (d) 165 units

- 82.** A unit vector perpendicular to both $\hat{i} + \hat{j} + \hat{k}$ and $2\hat{i} + \hat{j} + 3\hat{k}$ is [Karnataka CET 2011]
 (a) $(2\hat{i} - \hat{j} - \hat{k})\sqrt{6}$ (b) $\frac{(2\hat{i} - \hat{j} - \hat{k})}{\sqrt{6}}$
 (c) $2\hat{i} + \hat{j} + \hat{k}$ (d) $\frac{3\hat{i} + \hat{j} - 2\hat{k}}{\sqrt{6}}$
- 83.** If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are non-coplanar, then the value of $\mathbf{a} \cdot \left\{ \frac{\mathbf{b} \times \mathbf{c}}{3\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})} \right\} - \mathbf{b} \cdot \left\{ \frac{\mathbf{c} \times \mathbf{a}}{2\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})} \right\}$ is [Karnataka CET 2011]
 (a) $-\frac{1}{2}$ (b) $-\frac{1}{3}$
 (c) $-\frac{1}{6}$ (d) $\frac{1}{6}$
- 84.** For the non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, the relation $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$ is true, if [J&K CET 2011]
 (a) $\mathbf{b} \perp \mathbf{c}$ (b) $\mathbf{a} \perp \mathbf{b}$
 (c) $\mathbf{a} \parallel \mathbf{c}$ (d) $\mathbf{a} \perp \mathbf{c}$
- 85.** If $\mathbf{a}$ is perpendicular to $\mathbf{b}$, then the vector $\mathbf{a} \times [\mathbf{a} \times \{\mathbf{a} \times (\mathbf{a} \times \mathbf{b})\}]$ is equal to [Kerala CEE 2011]
 (a) $|\mathbf{a}|^2 \mathbf{b}$ (b) $|\mathbf{a}| \mathbf{b}$ (c) $|\mathbf{a}|^3 \mathbf{b}$
 (d) $|\mathbf{a}|^4 \mathbf{b}$ (e) 0
- 86.** If $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ are perpendicular to $\mathbf{q} + \mathbf{r}$, $\mathbf{r} + \mathbf{p}$ and $\mathbf{p} + \mathbf{q}$, respectively and if $|\mathbf{p} + \mathbf{q}| = 6$, $|\mathbf{q} + \mathbf{r}| = 4\sqrt{3}$ and $|\mathbf{r} + \mathbf{p}| = 4$, then $|\mathbf{p} + \mathbf{q} + \mathbf{r}|$ is [Kerala CEE 2010]
 (a) $5\sqrt{2}$ (b) 10
 (c) 15 (d) 5
 (e) 25
- 87.** If the scalar product of the vector $\hat{i} + \hat{j} + 2\hat{k}$ with the unit vector along $m\hat{i} + 2\hat{j} + 3\hat{k}$ is equal to 2, then one of the values of m is [Kerala CEE 2010]
 (a) 3 (b) 4
 (c) 5 (d) 6
 (e) 7
- 88.** Which one of the following vectors is of magnitude 6 and perpendicular to both $\mathbf{a} = 2\hat{i} + 2\hat{j} + \hat{k}$ and $\mathbf{b} = \hat{i} - 2\hat{j} + 2\hat{k}$? [Kerala CEE 2010]
 (a) $2\hat{i} - \hat{j} - 2\hat{k}$ (b) $2(2\hat{i} - \hat{j} + 2\hat{k})$
 (c) $3(2\hat{i} - \hat{j} - 2\hat{j})$ (d) $2(2\hat{i} + \hat{j} - 2\hat{k})$
 (e) $2(2\hat{i} - \hat{j} - 2\hat{k})$
- 89.** If $|\mathbf{a}| = 5$, $|\mathbf{b}| = 6$ and $\mathbf{a} \cdot \mathbf{b} = -25$, then $|\mathbf{a} \times \mathbf{b}|$ is equal to [Kerala CEE 2010]
 (a) 25 (b) $6\sqrt{11}$ (c) $11\sqrt{5}$
 (d) $11\sqrt{6}$ (e) $5\sqrt{11}$
- 90.** If the vectors $\mathbf{a} = 2\hat{i} + \hat{j} + 4\hat{k}$, $\mathbf{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$ and $\mathbf{c} = 2\hat{i} - 3\hat{j} - \lambda\hat{k}$ are coplanar, then the value of λ is [Kerala CEE 2010]
 (a) 2 (b) 1 (c) 3
 (d) -1 (e) 0
- 91.** If $\mathbf{a} = \hat{j} - \hat{k}$ and $\mathbf{c} = \hat{i} - \hat{j} - \hat{k}$. Then, the vector $\mathbf{b}$ satisfying $\mathbf{a} \times \mathbf{b} + \mathbf{c} = 0$ and $\mathbf{a} \cdot \mathbf{b} = 3$, is [BITSAT 2010]
 (a) $-\hat{i} + \hat{j} - 2\hat{k}$
 (b) $2\hat{i} - \hat{j} + 2\hat{k}$
 (c) $\hat{i} - \hat{j} - 2\hat{k}$
 (d) $\hat{i} + \hat{j} - 2\hat{k}$

Answers

Work Book Exercise 13.1

1. (a) 2. (d) 3. (a) 4. (a) 5. (a) 6. (d) 7. (b) 8. (c) 9. (a)

Work Book Exercise 13.2

1. (b) 2. (a) 3. (a) 4. (b) 5. (d) 6. (d) 7. (b) 8. (a) 9. (c)

Work Book Exercise 13.3

1. (b) 2. (a) 3. (d) 4. (a) 5. (a) 6. (b) 7. (a) 8. (b) 9. (b) 10. (b)

Work Book Exercise 13.4

1. (a) 2. (b) 3. (d) 4. (a) 5. (a) 6. (b) 7. (c) 8. (c)

Work Book Exercise 13.5

1. (a) 2. (a) 3. (a) 4. (d) 5. (a) 6. (d) 7. (b) 8. (a)

Target Exercises

1. (c) 2. (d) 3. (a) 4. (d) 5. (b) 6. (a) 7. (a) 8. (c) 9. (d) 10. (b)
 11. (d) 12. (b) 13. (a) 14. (b) 15. (a) 16. (c) 17. (d) 18. (c) 19. (c) 20. (c)
 21. (b) 22. (c) 23. (a) 24. (b) 25. (b) 26. (b) 27. (d) 28. (b) 29. (c) 30. (a)
 31. (b) 32. (a) 33. (d) 34. (a) 35. (a) 36. (d) 37. (b) 38. (b) 39. (a) 40. (c)
 41. (a) 42. (a) 43. (a) 44. (a) 45. (c) 46. (a) 47. (c) 48. (a) 49. (a) 50. (c)
 51. (a) 52. (d) 53. (d) 54. (c) 55. (a) 56. (a) 57. (a) 58. (c) 59. (d) 60. (c)
 61. (a) 62. (c) 63. (c) 64. (c) 65. (a) 66. (d) 67. (c) 68. (a) 69. (c) 70. (a)
 71. (b) 72. (b) 73. (c) 74. (b) 75. (b) 76. (d) 77. (a) 78. (c) 79. (a) 80. (a)
 81. (c) 82. (b) 83. (a) 84. (d) 85. (a) 86. (a) 87. (a,b,c) 88. (a,c) 89. (c,d) 90. (a,b)
 91. (a,b,c) 92. (a,d) 93. (a,b) 94. (a,b,c) 95. (a,b) 96. (b,d) 97. (b) 98. (a) 99. (d) 100. (d)
 101. (a) 102. (a) 103. (d) 104. (a) 105. (c) 106. (a) 107. (c) 108. (c) 109. (b) 110. (c)
 111. (a) 112. (d) 113. (c) 114. (b) 115. (*) 116. (*) 117. (*) 118. (4) 119. (1) 120. (6)
 121. (3) 122. (1) 123. (2) 124. (9) 125. (1)

*

115. $A \rightarrow r; B \rightarrow p; C \rightarrow q; D \rightarrow s$

116. $A \rightarrow r; B \rightarrow q; C \rightarrow s; D \rightarrow p$

117. $A \rightarrow q; B \rightarrow s; C \rightarrow p; D \rightarrow r$

Entrances Gallery

1. (a,b,c) 2. (4) 3. (c) 4. (a) 5. (c) 6. (3) 7. (c) 8. (9) 9. (c) 10. (a,d)
 11. (a) 12. (b) 13. (5) 14. (a) 15. (c) 16. (c) 17. (b) 18. (d) 19. (c) 20. (a)
 21. (d) 22. (a) 23. (a) 24. (d) 25. (a) 26. (a) 27. (d) 28. (d) 29. (c) 30. (d)
 31. (b) 32. (d) 33. (a) 34. (d) 35. (b) 36. (d) 37. (a) 38. (b) 39. (a) 40. (c)
 41. (a) 42. (c) 43. (c) 44. (c) 45. (b) 46. (e) 47. (e) 48. (a) 49. (a) 50. (b)
 51. (c) 52. (d) 53. (b) 54. (c) 55. (a) 56. (b) 57. (d) 58. (a) 59. (b) 60. (c)
 61. (b) 62. (a) 63. (b) 64. (d) 65. (b) 66. (a) 67. (c) 68. (c) 69. (b) 70. (b)
 71. (c) 72. (c) 73. (a) 74. (c) 75. (d) 76. (b) 77. (d) 78. (e) 79. (c) 80. (c)
 81. (c) 82. (b) 83. (c) 84. (c) 85. (d) 86. (a) 87. (d) 88. (e) 89. (e) 90. (b)
 91. (a)

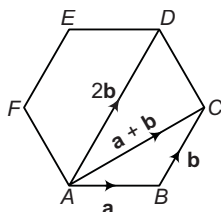
Explanations

Target Exercises

1. In $\triangle ABC$, $AB + BC = AC$

$$\Rightarrow AC = a + b \quad \dots (i)$$

AD is parallel to BC and $AD = 2BC$, therefore $AD = 2b$.



In $\triangle ACD$, $AC + CD = AD$

$$\Rightarrow CD = 2a - (a + b) = b - a$$

In $\triangle CDE$, $CE = CD + DE = b - a + (-a) = b - 2a$

2. Since, the diagonals of a parallelogram bisect each other. Therefore, P is the mid-point of AC and BD both.

$$\therefore OA + OC = 2OP$$

$$\text{and } OB + OD = 2OP$$

$$\Rightarrow OA + OB + OC + OD = 4OP$$

3. Let the position vectors of A, B and C be a, b and c respectively. Then, the position vectors of D, F and E are $\frac{b+c}{2}, \frac{c+a}{2}$ and $\frac{a+b}{2}$ respectively.

Therefore, $AD + BE + CF$

$$= \left(\frac{b+c}{2} - a \right) + \left(\frac{c+a}{2} - b \right) + \left(\frac{a+b}{2} - c \right) = 0$$

4. Since, three vectors are coplanar, therefore one of them should be expressible as a linear combination of the remaining two i.e. there should exist two scalars x and y such that

$$a + \lambda b + 3c = x(-2a + 3b - 4c) + y(a - 3b + 5c)$$

$$\Rightarrow -2x + y = 1, 3x - 3y = \lambda \text{ and } 3 = -4x + 5y$$

Solving first and third equations, we get

$$x = -\frac{1}{3}, y = \frac{1}{3}$$

Since, the vectors are coplanar, therefore these values of x and y also satisfy the third equation.

$$\text{i.e. } -1 - 1 = \lambda = -2$$

5. We have, $AO + OB = BO + OC \Rightarrow AB = BC$

Since, the initial point of BC is the terminal point of AB . Hence, A, B and C are collinear.

6. The position vector of the centroid of the triangle is $\frac{a+b+c}{3}$. Since, the triangle is an equilateral, therefore the orthocentre coincides with the centroid.

$$\therefore \frac{a+b+c}{3} = 0 \Rightarrow a + b + c = 0$$

7. If P, Q and R are the points, then $QR = -2PQ$.

Hence, P, Q and R are collinear.

$$8. \text{ Required vector } = |b| \hat{a} = \frac{|b|}{|a|} a = \frac{7}{3} (\hat{i} + 2\hat{j} + 2\hat{k})$$

9. Since, X and Y divide AB internally and externally in the ratio $2 : 1$, therefore the position vectors of X and Y are given by $\frac{2b+a}{3}$ and $2b-a$, respectively.

$$\text{Thus, } XY = (2b-a) - \frac{1}{3}(2b+a) = \frac{4}{3}(b-a)$$

10. We know that, if A and B are two points and P is any point, then $mPA + nPB = (m+n)PC$, where C divides AB in the ratio $n : m$.

Here, $m = n = 1$

$$\therefore PA + PB = 2PC$$

11. The unit vector parallel to the sum of the vectors $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $b\hat{i} + 2\hat{j} + 3\hat{k}$ is

$$\hat{n} = \frac{(2+b)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{(2+b)^2 + 6^2 + (-2)^2}} = \frac{(2+b)\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{b^2 + 4b + 44}}$$

$$\text{Now, } (\hat{i} + \hat{j} + \hat{k}) \cdot \hat{n} = 1$$

$$\Rightarrow 2 + b + 6 - 2 = \sqrt{b^2 + 4b + 44}$$

$$\Rightarrow b = 1$$

12. Taking A as the origin, let the position vectors of B and C be b and c respectively.

Equations of lines BF and AC are $r = b + \lambda \left(\frac{b+c}{4} - b \right)$ and $r = 0 + \mu c$ respectively.

For the point of intersection F , we have

$$b + \lambda \left(\frac{c+3b}{4} \right) = \mu c$$

$$\Rightarrow 1 - \frac{3\lambda}{4} = 0 \text{ and } \frac{\lambda}{4} = \mu$$

$$\Rightarrow \lambda = \frac{4}{3} \text{ and } \mu = \frac{1}{3}$$

Therefore, the position vector of F is $r = \frac{1}{3}c$

$$\text{Now, } AF = \frac{c}{3} \Rightarrow AF = \frac{1}{3}AC$$

$$\text{Hence, } AF : AC = \frac{1}{3} : 1 = \frac{1}{3}$$

13. The required vector c is given by

$$c = \lambda (\hat{a} + \hat{b}) = \lambda \left(\frac{a}{|a|} + \frac{b}{|b|} \right)$$

$$= \lambda \left\{ \frac{1}{9} (7\hat{i} - 4\hat{j} - 4\hat{k}) + \frac{1}{3} (-2\hat{i} - \hat{j} + 2\hat{k}) \right\}$$

$$\Rightarrow c = \frac{\lambda}{9} (\hat{i} - 7\hat{j} + 2\hat{k})$$

$$\Rightarrow |c| = \pm \frac{\lambda}{9} \sqrt{1 + 49 + 4} = \pm \frac{\lambda}{9} \sqrt{54}$$

$$\text{But } |c| = 5\sqrt{6} \quad [\text{given}]$$

$$\therefore \pm \frac{\lambda}{9} \sqrt{54} = 5\sqrt{6} \Rightarrow \lambda = \pm 15$$

$$\text{Hence, } c = \pm \frac{15}{9} (\hat{i} - 7\hat{j} + 2\hat{k}) = \pm \frac{5}{3} (\hat{i} - 7\hat{j} + 2\hat{k})$$

14. Let the given points be A, B and C respectively.

The unit vectors perpendicular to the plane containing A, B and C are given by

$$\pm \frac{\mathbf{AB} \times \mathbf{AC}}{|\mathbf{AB} \times \mathbf{AC}|} = \pm \frac{1}{\sqrt{26}} (-\hat{i} + 4\hat{j} - 3\hat{k})$$

Since, the required vector makes an obtuse angle with X -axis. So, it is equal to $\frac{1}{\sqrt{26}} (-\hat{i} + 4\hat{j} - 3\hat{k})$.

Clearly, it also makes an obtuse angle with Z -axis.

15. We have, $|a - b|^2 = |a|^2 + |b|^2 - 2(a \cdot b)$
 $\Rightarrow |a - b|^2 = |a|^2 + |b|^2 - 2|a||b|\cos\theta$
 $\Rightarrow |a - b|^2 = 2 - 2\cos\theta$ [$\because |a| = |b| = 1$]
 $\Rightarrow |a - b| = 2|\sin\theta|$
 Now, $|a - b| < 1$
 $\Rightarrow 2|\sin\theta| < 1$
 $\Rightarrow |\sin\theta| < \frac{1}{2}$
 $\Rightarrow \theta \in \left[0, \frac{\pi}{6}\right)$ or $\theta \in \left(\frac{5\pi}{6}, \pi\right]$

16. We have, $2a + b = \hat{i} + \hat{j} + \hat{k}$ and $a + 2b = \hat{i} + \hat{j} - \hat{k}$
 Solving these two equations, we get
 $a = \frac{1}{3}(\hat{i} + \hat{j} + 3\hat{k})$ and $b = \frac{1}{3}(\hat{i} + \hat{j} - 3\hat{k})$

$$\therefore \cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\Rightarrow \cos\theta = \frac{7}{11}$$

$$\Rightarrow \theta = \cos^{-1}\left(-\frac{7}{11}\right)$$

17. Let $x\hat{i} + y\hat{j} + z\hat{k}$ be the unit vector along c .

Since, $-\hat{i} + \hat{j} - \hat{k}$ bisects the angle between c and $3\hat{i} + \hat{j}$.

$$\text{Therefore, } \lambda(-\hat{i} + \hat{j} - \hat{k}) = (x\hat{i} + y\hat{j} + z\hat{k}) + \frac{3\hat{i} + 4\hat{j}}{5}$$

$$\Rightarrow x + \frac{3}{5} = -\lambda, y + \frac{4}{5} = \lambda \text{ and } z = -\lambda$$

$$\text{Now, } x^2 + y^2 + z^2 = 1$$

$$[\because x\hat{i} + y\hat{j} + z\hat{k} \text{ is a unit vector}]$$

$$\Rightarrow \left(-\lambda - \frac{3}{5}\right)^2 + \left(\lambda - \frac{4}{5}\right)^2 + \lambda^2 = 1$$

$$\Rightarrow \lambda = 0 \text{ or } \lambda = \frac{2}{15}$$

But $\lambda \neq 0$, because $\lambda = 0$ implies that the given vectors are parallel.

$$\therefore \lambda = \frac{2}{15}$$

$$\Rightarrow x = -\frac{11}{15}, y = -\frac{10}{15} \text{ and } z = -\frac{2}{15}$$

$$\text{Hence, } x\hat{i} + y\hat{j} + z\hat{k} = -\frac{1}{15}(11\hat{i} + 10\hat{j} + 2\hat{k})$$

18. $\therefore (\hat{a} + 3\hat{b}) \perp (7\hat{a} - 5\hat{b})$
 $\therefore (\hat{a} + 3\hat{b}) \cdot (7\hat{a} - 5\hat{b}) = 0$
 $\Rightarrow 7|\hat{a}|^2 + 16(\hat{a} \cdot \hat{b}) - 15|\hat{b}|^2 = 0$
 $\Rightarrow 7 + 16\cos\theta - 15 = 0$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

$$\text{and } (\hat{a} - 4\hat{b}) \perp (7\hat{a} - 2\hat{b})$$

$$\Rightarrow (\hat{a} - 4\hat{b}) \cdot (7\hat{a} - 2\hat{b}) = 0$$

$$\Rightarrow 7|\hat{a}|^2 + 8|\hat{b}|^2 - 30(\hat{a} \cdot \hat{b}) = 0$$

$$\Rightarrow 15 - 30\cos\theta = 0$$

$$\Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

19. Suppose the bisector of angle A meets BC at D . Then, AD divides BC in the ratio $AB : AC$.

$$\therefore \text{Position vector of } D$$

$$= \frac{|AB|(2\hat{i} + 5\hat{j} + 7\hat{k}) + |AC|(2\hat{i} + 3\hat{j} + 4\hat{k})}{|AB| + |AC|}$$

$$\text{But } AB = -2\hat{i} - 4\hat{j} - 4\hat{k}$$

$$\text{and } AC = -2\hat{i} - 2\hat{j} - \hat{k}$$

$$\text{Now, } |AB| = 6 \text{ and } |AC| = 3$$

$$\therefore \text{Position vector of } D$$

$$= \frac{6(2\hat{i} + 5\hat{j} + 7\hat{k}) + 3(2\hat{i} + 3\hat{j} + 4\hat{k})}{6 + 3}$$

$$= \frac{1}{3}(6\hat{i} + 13\hat{j} + 18\hat{k})$$

20. It is given that

$$a \cdot b \geq 0 \Rightarrow |a| + |b| \cos\theta \geq 0$$

$$\Rightarrow 0 \leq \theta \leq \frac{\pi}{2}$$

21. We have, $|a + b| = |a - b|$
 $\Rightarrow |a + b|^2 = |a - b|^2$
 $\Rightarrow |a|^2 + |b|^2 + 2a \cdot b = |a|^2 + |b|^2 - 2a \cdot b$
 $\Rightarrow 4a \cdot b = 0$
 $\Rightarrow a \cdot b = 0$
 $\Rightarrow a \perp b$

22. We have,
 $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$
 $\Rightarrow 1 + |a - b|^2 = 4$
 $\Rightarrow |a - b| = \sqrt{3}$

23. It is given that, $|a| = |b| = |c| = \lambda$ (say) and a, b, c are mutually perpendicular vectors.

$$\text{Therefore, } |a + b + c| = \sqrt{3}\lambda$$

Let θ be the angle which $a + b + c$ makes with a . Then,

$$\cos\theta = \frac{a \cdot (a + b + c)}{|a||a + b + c|} = \frac{|a|^2}{|a||a + b + c|}$$

$$= \frac{\lambda^2}{\lambda(\sqrt{3}\lambda)} = \frac{1}{\sqrt{3}}$$

$$\therefore \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

24. Let the required vector be $a = x\hat{i} + y\hat{j} + z\hat{k}$.

It makes equal angle with the vectors

$$\hat{b} = \frac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k}), \hat{c} = \frac{1}{5}(-4\hat{i} - 3\hat{k}), \hat{d} = \hat{j}$$

$$\therefore \hat{a} \cdot \hat{b} = \hat{a} \cdot \hat{c} = \hat{a} \cdot \hat{d}$$

[$\because \hat{b}, \hat{c}$ and $\hat{d}$ are unit vectors]

$$\therefore \frac{1}{3}(x - 2y + 2z) = \frac{1}{5}(-4x - 3z) = y$$

$$\Rightarrow x - 2y + 2z = 3y$$

$$\text{and } -4x - 5y - 3z = 0$$

$$\Rightarrow x - 5y + 2z = 0$$

$$\text{and } 4x + 5y + 3z = 0$$

Solving these equations, we get $x = -z$ and $y = -5y$.

$$\therefore \frac{x}{-5} = \frac{y}{1} = \frac{z}{5} \text{ or } \frac{x}{5} = \frac{y}{-1} = \frac{z}{-5}$$

$$\Rightarrow a = -5\hat{i} + \hat{j} + 5\hat{k} \text{ or } a = 5\hat{i} - \hat{j} - 5\hat{k}$$

- 25.** Let the required vector be $\hat{c} = x\hat{i} + z\hat{k}$. As, it is a unit vector.

Therefore,

$$\Rightarrow \frac{|\hat{c}|}{x^2 + y^2} = 1 \quad \dots (i)$$

Since, $\hat{a}$ and $\hat{c}$ are inclined at the angle 45° .

$$\therefore \cos 45^\circ = \frac{2x - y}{\sqrt{4 + 4 + 1}}$$

$$\Rightarrow 2x - z = \frac{3}{\sqrt{2}} \quad \dots (ii)$$

Since, $\hat{b}$ and $\hat{c}$ are inclined at an angle of 60° .

$$\therefore -\frac{y}{\sqrt{2}} = \cos 60^\circ$$

$$\Rightarrow y = -\frac{1}{\sqrt{2}} \quad \dots (iii)$$

From Eqs. (ii) and (iii), we get $x = \frac{1}{\sqrt{2}}$

Hence, the required vector is $\frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{k}$.

- 26.** The component of a along b is given by

$$\left(\frac{a \cdot b}{|b|^2} \right) b = \frac{18}{25}(3\hat{j} + 4\hat{k})$$

- 27.** We have, $a \cdot b = a \cdot c$

$$\Rightarrow a \cdot (b - c) = 0$$

$$\Rightarrow a \perp (b - c) \text{ or } b - c = 0$$

$$\Rightarrow a \perp (b - c) \text{ or } b = c$$

- 28.** Let $c = x\hat{i} + y\hat{j}$. Then,

$$b \perp c = b \cdot c = 0$$

$$\Rightarrow 4x + 3y = 0 \Rightarrow \frac{x}{3} = \frac{y}{-4} = \lambda$$

$$\Rightarrow x = 3\lambda \text{ and } y = -4\lambda$$

$$\therefore c = \lambda(3\hat{i} - 4\hat{j})$$

Let the required vectors be $\alpha = p\hat{i} + q\hat{j}$. Then, the

projections of α on b are c are $\frac{\alpha \cdot b}{|b|}$ and $\frac{\alpha \cdot c}{|c|}$,

respectively.

$$\therefore \alpha \cdot \frac{b}{|b|} = 1 \text{ and } \frac{\alpha \cdot c}{|c|} = 2 \quad [\text{given}]$$

$$\Rightarrow 4p + 3q = 5 \text{ and } 3p - 4q = 10$$

$$\Rightarrow p = 2 \text{ and } q = -1$$

$$\therefore \text{Required vector} = 2\hat{i} - \hat{j}$$

- 29.** We have, $|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2)$

$$\Rightarrow 300 + |a - b|^2 = 2(49 + 121)$$

$$\Rightarrow |a - b| = 2\sqrt{10}$$

- 30.** We have, $a + b = c$

$$\Rightarrow |c|^2 = |a|^2 + |b|^2 + 2a \cdot b$$

$$\Rightarrow |c|^2 = |a|^2 + |b|^2 + 2 \times 0$$

$$[\because a \cdot b = 0]$$

$$\Rightarrow |c|^2 = |a|^2 + |b|^2$$

- 31.** The direction cosines of vector making equal angles with the coordinate axes are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Therefore, the unit vector along the vector making equal angles with the coordinate axes is

$$\hat{b} = \frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}$$

$\therefore$ Projection of a on $b = a \cdot \hat{b}$

$$= (4\hat{i} - 3\hat{j} + 2\hat{k}) \cdot \left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k} \right) \\ = \frac{4 - 3 + 2}{\sqrt{3}} = \sqrt{3}$$

- 32.** Since, a is collinear to vector b , therefore $a = mb$ for some scalar m .

$$\text{i.e. } a = m \left(6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k} \right)$$

$$\Rightarrow |a| = |m| \sqrt{36 + 64 + \frac{225}{4}} = \frac{25}{2} |m|$$

$$\Rightarrow 50 = \frac{25}{2} |m|$$

$$\Rightarrow |m| = 4$$

$$\Rightarrow m = \pm 4 \quad [\because |a| = 50]$$

$$\therefore a = (24\hat{i} - 32\hat{j} - 30\hat{k})$$

- 33.** Since, the vectors $a = 2\hat{i} + \log_3 x\hat{j} + a\hat{k}$ and $b = -3\hat{i} + a\log_3 x\hat{j} + \log_3 x\hat{k}$ are inclined at acute angle.

Therefore, $a \cdot b > 0$

$$\Rightarrow -6 + a(\log_3 x)^2 + a\log_3 x > 0 \text{ for all } x > 0$$

$$\Rightarrow -6 + ay^2 + ay > 0, \text{ where } y = \log_3 x$$

$$\Rightarrow ay^2 + ay - 6 > 0 \text{ for all } y$$

$$\Rightarrow a > 0 \text{ and } a^2 + 24a < 0$$

$$\Rightarrow a > 0 \text{ and } a \in (-24, 0)$$

But this is not possible. Hence, the vectors a and b are not inclined at acute angle for any real value of a .

- 34.** Any vector in the plane of b and c is

$$r = mb + c = (m + 1)\hat{i} + (2m + 1)\hat{j} + (-m - 2)\hat{k}$$

$$\text{Projection of } r \text{ on } a = \frac{r \cdot a}{|a|} = \left| \frac{\sqrt{2}}{\sqrt{3}} \right|$$

$$\therefore \frac{2(m + 1) - (2m + 1) + (-m - 2)}{\sqrt{6}} = \pm \frac{\sqrt{2}}{3}$$

$$\Rightarrow -m - 1 = \pm 2$$

$$\Rightarrow m = -3 \text{ or } 1$$

$$\text{Hence, } r = -2\hat{i} - 5\hat{j} + \hat{k} \text{ for } m = -3$$

$$\text{and } r = 2\hat{i} + 3\hat{j} - 3\hat{k} \text{ for } m = 1$$

- 35.** The position vectors of R and S are $\frac{3p + 2q}{5}$ and

$$3p - 2q, \text{ respectively. Therefore, } OR = \frac{3p + 2q}{5} \text{ and}$$

$$OS = 3p - 2q. \text{ Since, } OR \perp OS.$$

Therefore, $OR \cdot OS = 0$

$$\Rightarrow \left(\frac{3p+2q}{5} \right) \cdot (3p-2q) = 0$$

$$\Rightarrow 9|p|^2 - 4|q|^2 = 0$$

$$\Rightarrow 9|p|^2 = 4|q|^2$$

$$\Rightarrow 9p^2 = 4q^2$$

36. Let the required unit vector be $\hat{r} = a\hat{i} + b\hat{j}$.

Then, $|\hat{r}| = 1$

$$\Rightarrow \sqrt{a^2 + b^2} = 1$$

$$\Rightarrow a^2 + b^2 = 1 \quad \dots(i)$$

Since, $\hat{r}$ makes 45° with $\hat{i} + \hat{j}$ and an angle of 60° with $3\hat{i} - 4\hat{j}$.

$$\text{Therefore, } \cos \frac{\pi}{4} = \frac{\hat{r} \cdot (\hat{i} + \hat{j})}{|\hat{r}| |\hat{i} + \hat{j}|}$$

$$\text{and } \cos \frac{\pi}{3} = \frac{\hat{r} \cdot (3\hat{i} - 4\hat{j})}{|\hat{r}| |3\hat{i} - 4\hat{j}|}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{a+b}{\sqrt{2}}$$

$$\text{and } \frac{1}{2} = \frac{3a-4b}{5}$$

$$\Rightarrow a+b=1 \quad \dots(ii)$$

$$\text{and } 3a-4b = \frac{5}{2} \quad \dots(iii)$$

There exists no real values of a and b satisfying Eqs. (i), (ii) and (iii). Hence, no such unit vector exists.

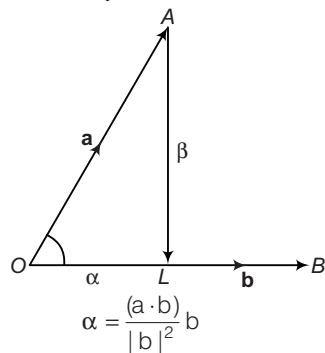
37. Let $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\text{Then, } a \times \hat{i} = -a_2\hat{k} + a_3\hat{j}, a \times \hat{j} = a_1\hat{k} - a_3\hat{i}$$

$$\text{and } a \times \hat{k} = -a_1\hat{j} + a_2\hat{i}$$

$$\begin{aligned} \therefore |a \times \hat{i}|^2 + |a \times \hat{j}|^2 + |a \times \hat{k}|^2 \\ = (a_2^2 + a_3^2) + (a_1^2 + a_3^2) + (a_1^2 + a_2^2) \\ = 2(a_1^2 + a_2^2 + a_3^2) = 2|a|^2 \end{aligned}$$

38. We have, α = Projection vector of a on b



$$\text{Now, } \alpha + \beta = a$$

$$\begin{aligned} \therefore \beta &= a - \frac{(a \cdot b)}{|b|^2} b \\ &= \frac{|b|^2 a - (a \cdot b)b}{|b|^2} = \frac{b \times (a \times b)}{|b|^2} \end{aligned}$$

39. The diagonals of the parallelogram are given by

$$\alpha = a + b \quad \text{and} \quad \beta = \pm(a - b)$$

$$\Rightarrow \alpha = 2\hat{i} - 2\hat{j} \quad \text{and} \quad \beta = \pm(4\hat{i} - 2\hat{j} + 4\hat{k})$$

Let θ be the angle between the diagonals.

$$\text{Then, } \cos \theta = \frac{\alpha \cdot \beta}{|\alpha| |\beta|}$$

$$\Rightarrow \cos \theta = \frac{1}{\sqrt{2}} \quad \text{or} \quad \cos \theta = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \theta = \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{3\pi}{4}$$

40. For the vectors a and b to be inclined at an obtuse angle, we must have

$$a \cdot b < 0 \quad \text{for all } x \in (0, \infty)$$

$$\Rightarrow c(\log_2 x)^2 - 12 + 6c(\log_2 x) < 0 \quad \text{for all } x \in (0, \infty)$$

$$\Rightarrow cy^2 + 6cy - 12 < 0 \quad \text{for all } y \in R, \text{ where } y = \log_2 x$$

$$\Rightarrow c < 0 \quad \text{and} \quad 36c^2 + 48c < 0$$

$$\Rightarrow c < 0 \quad \text{and} \quad c(3c + 4) < 0$$

$$\Rightarrow c < 0 \quad \text{and} \quad -\frac{4}{3} < c < 0$$

$$\Rightarrow c \in \left(-\frac{4}{3}, 0 \right)$$

41. We have, $2|\hat{i} + \hat{j} + 3\hat{k}| = |4\hat{i} + (4x-2)\hat{j} + 2\hat{k}|$

$$\Rightarrow 2\sqrt{x^2 + 10} = \sqrt{20 + (4x-2)^2}$$

$$\Rightarrow 4(x^2 + 10) = 20 + (4x-2)^2$$

$$\Rightarrow 3x^2 - 4x - 4 = 0$$

$$\therefore x = 2, -\frac{2}{3}$$

42. We have, $a \times b = 2(a \times c)$

$$\Rightarrow a \times (b - 2c) = 0$$

$\Rightarrow a$ is parallel to $b - 2c$.

$$\text{Now, } (b - 2c) = \lambda a$$

$$\Rightarrow |b - 2c|^2 = \lambda^2 |a|^2$$

$$\Rightarrow |b|^2 + 4|c|^2 - 4(b \cdot c) = \lambda^2 |a|^2$$

$$\Rightarrow 16 + 4 - 4 \times |b| \times |c| \times \frac{1}{4} = \lambda^2$$

$$\Rightarrow 20 - 4 = \lambda^2$$

$$\Rightarrow \lambda = \pm 4$$

43. We have, $R_1 = 2\hat{i} + 4\hat{j} - 5\hat{k}$ and $R_2 = \hat{i} + 2\hat{j} + 3\hat{k}$

$$R(\text{along AC}) = R_1 + R_2 = 3\hat{i} + 6\hat{j} - 2\hat{k}$$

$\therefore$ Required unit vector along AC ,

$$\therefore \hat{a} = \frac{R}{|R|} = \frac{3\hat{i} + 6\hat{j} - 2\hat{k}}{\sqrt{9 + 36 + 4}} = \frac{1}{7}(3\hat{i} + 6\hat{j} - 2\hat{k})$$

44. Since, a, b and c form a right handed system.

$$\therefore c = b \times a = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ x & y & z \end{vmatrix} = z\hat{i} - x\hat{k}$$

45. We have, $a \cdot b = 0$ and $a \times b = 0$

$$\Rightarrow |a||b| = \cos \theta = 0 \quad \text{and} \quad |a||b| \sin \theta = 0$$

$$\Rightarrow |a| = 0 \quad \text{or} \quad |b| = 0 \quad \text{or} \quad \cos \theta = 0$$

$$\text{and } |a| = 0 \quad \text{or} \quad |b| = 0 \quad \text{or} \quad \sin \theta = 0$$

$$\Rightarrow |a| = 0 \quad \text{or} \quad |b| = 0$$

46. The resultant force is given by

$$F = 6 \frac{(\hat{i} - 2\hat{j} + 2\hat{k})}{\sqrt{1+4+4}} + 7 \frac{(2\hat{i} - 3\hat{j} - 6\hat{k})}{\sqrt{4+9+36}} = 4\hat{i} - 7\hat{j} - 2\hat{k}$$

d = Displacement

$$= PQ = (5\hat{i} - \hat{j} + \hat{k}) - (2\hat{i} - \hat{j} - 3\hat{k}) = 3\hat{i} + 4\hat{k}$$

$$\therefore \text{Work done} = F \cdot d = 12 + 0 - 8$$

$$= 4 \text{ units}$$

47. Required vector $(r) = \lambda(\hat{a} + \hat{b})$, where λ is a scalar.

$$\Rightarrow r = \lambda \left[\frac{1}{9}(\hat{i} - 4\hat{j} - 4\hat{k}) + \frac{1}{3}(-2\hat{i} - \hat{j} + 2\hat{k}) \right]$$

$$= \frac{\lambda}{9}(\hat{i} - 7\hat{j} + 2\hat{k})$$

Now, $|r| = 3\sqrt{6}$

$$\Rightarrow |r|^2 = 54$$

$$\Rightarrow \frac{\lambda^2}{81}(1 + 49 + 4) = 54$$

$$\Rightarrow \lambda = \pm 9$$

Thus, the required vector r is $\pm(\hat{i} - 7\hat{j} + 2\hat{k})$.

48. We have, $\alpha = x(a \times b) + y(b \times c) + z(c \times a)$

Taking dot products with a, b and c , we get

$$\alpha \cdot a = y[(b \times c) \cdot a]$$

$$\Rightarrow \alpha \cdot a = y[a \cdot b \cdot c] \Rightarrow y = 8(\alpha \cdot a)$$

$$\alpha \cdot b = z[(c \times a) \cdot b]$$

$$\Rightarrow \alpha \cdot b = z[a \cdot b \cdot c] \Rightarrow z = 8(\alpha \cdot b)$$

and $\alpha \cdot c = x[(a \times b) \cdot c]$

$$\Rightarrow \alpha \cdot c = x[a \cdot b \cdot c] \Rightarrow x = 8(\alpha \cdot c)$$

$$\therefore x + y + z = 8\alpha \cdot (a + b + c)$$

49. a, b and c are coplanar iff $[a \ b \ c] = 0$

We have, $[a \ b \ c] = \begin{vmatrix} x & x+1 & x+2 \\ x+3 & x+4 & x+5 \\ x+6 & x+7 & x+8 \end{vmatrix}$

$$= \begin{vmatrix} x & x+1 & x+2 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} \quad \left[\begin{array}{l} \because R_1 \rightarrow R_2 - R_1 \\ \text{and } R_3 \rightarrow R_3 - R_1 \end{array} \right]$$

$$= 0 \text{ for all } x \quad [\because R_1 \text{ and } R_2 \text{ are proportional}]$$

50. The given vector equation can be written as

$$[(1-a)x + 3y - 4z]\hat{i} + [x - (3+a)y + 5z]\hat{j} + (3x + y - az)\hat{k} = 0$$

$$\Rightarrow (1-a)x + 3y - 4z = 0, \quad x - (3+a)y + 5z = 0$$

$[\because \hat{i}, \hat{j} \text{ and } \hat{k} \text{ are linearly independent}]$

and $3x + y - az = 0$

Eliminating x, y and z , we get

$$\begin{vmatrix} 1-a & 3 & -4 \\ 1 & -(3+a) & 5 \\ 3 & 1 & -a \end{vmatrix} = 0$$

$$\Rightarrow a = 0, -1$$

51. Required vector c is parallel to the vector

$$\alpha = b \times (a \times b) = |b|^2 a - (b \cdot a)b$$

$$= 5(-\hat{i} + \hat{j} + \hat{k}) + 2\hat{i} + 0\hat{j} + \hat{k}$$

$$= -3\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\therefore c = \lambda\alpha, \text{ for some scalar } \lambda$$

$$\Rightarrow c = \lambda(-3\hat{i} + 5\hat{j} + 6\hat{k})$$

Now, $a \cdot c = 7$

$$\Rightarrow (-\hat{i} + \hat{j} + \hat{k}) \cdot \lambda(-3\hat{i} + 5\hat{j} + 6\hat{k}) = 7$$

$$\Rightarrow \lambda(3 + 5 + 6) = 7$$

$$\Rightarrow \lambda = \frac{1}{2}$$

Hence, $c = \frac{1}{2}(-3\hat{i} + 5\hat{j} + 6\hat{k})$

52. We have, $c = xa + yb + z(a \times b)$

$$\Rightarrow c \cdot a = x = \cos \theta$$

and $c \cdot b = y = \cos \theta$

$$\Rightarrow c \cdot c = |c|^2$$

$$\Rightarrow \{xa + yb + z(a \times b)\} \cdot \{xa + yb + z(a \times b)\} = |c|^2$$

$$\Rightarrow 2x^2 + 2y^2 + z^2\{ |a|^2 |b|^2 - (a \cdot b)^2 \} = 1$$

$$\Rightarrow 2x^2 + 2y^2 = 1$$

$$\therefore z^2 = 1 - 2\cos^2 \theta$$

$$= -\cos 2\theta$$

53. Let $r \neq 0$. Then, $r \cdot a = r \cdot b = r \cdot c = 0 \Rightarrow a, b$ and c are coplanar, which is a contradiction. Therefore, $r = 0$.

54. If $\Delta = 0$, then

$$\begin{vmatrix} a & b & c \\ a \cdot a & a \cdot b & a \cdot c \\ a \cdot c & b \cdot c & c \cdot c \end{vmatrix} = 0$$

$$\Rightarrow \lambda a + \mu b + \nu c = 0$$

$\Rightarrow a, b$ and c are linearly dependent, which is a contradiction.

Hence, Δ can take any non-zero real value.

55. Since, a, b and c are coplanar.

$$a \perp (b \times c)$$

$$\Rightarrow a \cdot (b \times c) = 0$$

$$\Rightarrow [a \ b \ c] = 0$$

56. Since, $a - b + b - c + c - a = 0$, therefore the vectors $a - b, b - c$ and $c - a$ are coplanar.

$$\text{Hence, } [a - b \ b - c \ c - a] = 0$$

57. Since, $a = 2\hat{i} + 3\hat{j} + \hat{k}, \quad b = 2\hat{i} + \rho\hat{j} + 3\hat{k}$ and $c = 2\hat{i} + 17\hat{j} - 3\hat{k}$ are coplanar.

$$\therefore [a \ b \ c] = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 3 & 1 \\ 2 & \rho & 3 \\ 2 & 17 & -3 \end{vmatrix} = 0$$

$$\Rightarrow \rho = -4$$

58. Since, a, b and c are non-coplanar unit vectors, therefore $[a \ b \ c] = \text{Volume of a parallelepiped whose each edge is of one unit length} \Rightarrow [a \ b \ c] = \pm 1$

59. We have, $(a - b) \cdot [(b - c) \times (c - a)]$

$$\Rightarrow (a - b) \cdot (b \times c - b \times a + c \times a)$$

$$= a \cdot (b \times c) - a \cdot (b \times a) + a \cdot (c \times a) - b \cdot (b \times c) + b \cdot (b \times a) - b \cdot (c \times a) = 0$$

60. Let angle between a and b be θ_1 , c and d be θ_2 and $a \times b$ and $c \times d$ be θ .

Then, $(a \times b) = \sin \theta_1 \hat{n}_1$

and $(c \times d) = \sin \theta_2 \hat{n}_2$

where, $\hat{n}_1$ and $\hat{n}_2$ are perpendicular to a and $b \notin c, d$ respectively.

Since, $(a \times b) \cdot (c \times d) = 1$

$$\Rightarrow \sin \theta_1 \cdot \sin \theta_2 \cdot \cos \theta = 1$$

$$[\because |a| = |b| = |c| = |d| = 1]$$

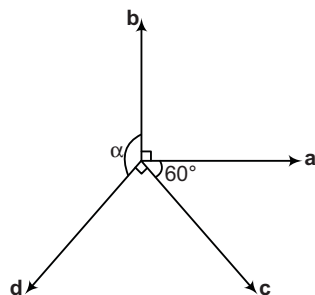
$$\Rightarrow \theta_1 = 90^\circ, \theta_2 = 90^\circ, \theta = 0^\circ$$

$$\Rightarrow a \perp b, c \perp d, (a \times b) \parallel (c \times d)$$

So, $a \times b = k(c \times d)$
 $\Rightarrow (a \times b) \cdot c = k(c \times d) \cdot c$
 and $(a \times b) \cdot d = k(c \times d) \cdot d$
 $\Rightarrow [a \ b \ c] = 0$ and $[a \ b \ d] = 0$
 Hence a, b, c and a, b, d are coplanar vectors, so options, (a) and (b) are incorrect.
 Let $b \parallel d \Rightarrow b = \pm d$
 As, $(a \times b) \cdot (c \times d) = 1$
 $\Rightarrow (a \times b) \cdot (c \times b) = \pm 1$
 $\Rightarrow [a \times b \ c \ b] = \pm 1$
 $\Rightarrow [c \ b \ a \times b] = \pm 1$
 $\Rightarrow c \cdot [b \times (a \times b)] = \pm 1$
 $\Rightarrow c \cdot [a - (b \cdot a)b] = \pm 1$
 $\Rightarrow c \cdot a = \pm 1$ $[\because a \cdot b = 0]$

which is a contradiction, so option (c) is correct.

Let option (d) be correct.



$\Rightarrow d = \pm a$ and $c = \pm b$
 As, $(a \times b) \cdot (c \times d) = 1$
 $\Rightarrow (a \times b) \cdot (b \times a) = \pm 1$
 which is a contradiction, so option (d) is incorrect.

61. Let $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$$\text{Now, } [a \ b \ \hat{i}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= a_1(0-0) - a_2(0-b_3) + a_3(0-b_2)$$

$$= a_2b_3 - a_3b_2$$

$$\therefore 2[a \ b \ \hat{i}]\hat{i} = 2(a_2b_3 - a_3b_2)\hat{i}$$

$$\text{Similarly, } 2[a \ b \ \hat{j}]\hat{j} = 2(a_3b_1 - a_1b_3)\hat{j}$$

$$\text{and } 2[a \ b \ \hat{k}]\hat{k} = 2(a_1b_2 - a_2b_1)\hat{k}$$

$$\therefore 2[a \ b \ \hat{i}]\hat{i} + 2[a \ b \ \hat{j}]\hat{j} + 2[a \ b \ \hat{k}]\hat{k} + 2[a \ b \ a]$$

$$= 2[(a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}]$$

$$= 2(a \times b) \quad [\because [a \ b \ c] = 0]$$

62. Given, $[\lambda(a+b)\lambda^2b\lambda c] = [a \ b \ c]$

$$\Rightarrow \begin{vmatrix} \lambda(a_1+b_1) & \lambda(a_2+b_2) & \lambda(a_3+b_3) \\ \lambda^2b_1 & \lambda^2b_2 & \lambda^2b_3 \\ \lambda c_1 & \lambda c_2 & \lambda c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1+c_1 & b_2+c_2 & b_3+c_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$= \lambda^4 \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 - R_2$ in LHS]
 [and $R_2 \rightarrow R_2 - R_2$ in RHS]

$$\Rightarrow \lambda^4 \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = - \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\therefore \lambda^4 = -1$$

Hence, no real value of λ exists.

63. $\hat{u} \cdot \hat{v} = 0 \Rightarrow |\hat{u}| |\hat{v}| \cos \theta = 0$

$$\Rightarrow 1 \times 1 \times \cos \theta = 0 \quad [\because |\hat{u}| = |\hat{v}| = 1]$$

$$\Rightarrow \cos \theta = 0 \Rightarrow \theta = 90^\circ$$

Let $\hat{n}$ be a unit vector perpendicular to the plane of vectors $\hat{u}$ and $\hat{v}$.

$$\Rightarrow \hat{u} \times \hat{v} = |\hat{u}| |\hat{v}| \sin 90^\circ \cdot \hat{n} = \hat{n}$$

Since, r is coplanar with $\hat{u}$ and $\hat{v}$.

$\therefore \hat{n}$ is perpendicular to r .

Let ϕ be the angle between $\hat{n}$ and $\hat{r}$.

Then, $\phi = 90^\circ$

$$\therefore |r \times (\hat{u} \times \hat{v})| = |r \times \hat{n}| = |r| |\hat{n}| \sin \phi$$

$$= |r| \times 1 \times \sin 90^\circ = |r|$$

64. Since, $a \times (b \times c) = \frac{1}{2}b$

$$\therefore (a \cdot c)b - (a \cdot b)c = \frac{1}{2}b$$

On comparing both sides, we get

$$a \cdot c = \frac{1}{2} \text{ and } a \cdot b = 0$$

$$\text{Now, } a \cdot c = \frac{1}{2}$$

$$\Rightarrow |a| |c| \cos \theta_2 = \frac{1}{2}$$

$$\Rightarrow \cos \theta_2 = \frac{1}{2} \Rightarrow \theta_2 = \frac{\pi}{3}$$

$$\text{and } a \cdot b = 0$$

$$\Rightarrow |a| |b| \cos \theta_1 = 0$$

$$\Rightarrow \cos \theta_1 = \cos \frac{\pi}{2}$$

$$\Rightarrow \theta_1 = \frac{\pi}{2}$$

65. We have, $a \times (b \times c) + b \times (c \times a) + c \times (a \times b)$

$$= (a \cdot c)b - (a \cdot b)c + (b \cdot a)c - (b \cdot c)a$$

$$+ (c \cdot b)a - (c \cdot a)b = 0$$

66. Since, a, b and c are non-coplanar vectors.

$$[a \ b \ c] \neq 0 \Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0 \Rightarrow \Delta \neq 0$$

$$\text{where, } \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$\text{Now, } \begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} + \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = 0$$

$$\Rightarrow \Delta(1+abc) = 0 \Rightarrow abc = -1 \quad [\because \Delta \neq 0]$$

67. Let the four points be P, Q, R and S , respectively.

$$\text{Then, } PQ = 4\hat{i} + 6\hat{j} + (\lambda + 1)\hat{k},$$

$$QR = -\hat{i} + 4\hat{j} + (4 - \lambda)\hat{k}$$

$$\text{and } RS = -7\hat{i} - 5\hat{j} + 0\hat{k}$$

The given points are coplanar, therefore PQ, QR and RS are coplanar.

$$\therefore [PQ \ QR \ RS] = 0$$

$$\Rightarrow \begin{vmatrix} 4 & 6 & \lambda + 1 \\ -1 & 4 & 4 - \lambda \\ -7 & -5 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1$$

$$\begin{aligned} 68. \text{ We have, } \frac{a \cdot (b \times c)}{(c \times a) \cdot b} + \frac{b \cdot (a \times c)}{c \cdot (a \times b)} &= \frac{[a \ b \ c]}{[b \ c \ a]} + \frac{[b \ a \ c]}{[c \ a \ b]} \\ &= \frac{[a \ b \ c]}{[a \ b \ c]} - \frac{[a \ b \ c]}{[a \ b \ c]} = 1 - 1 = 0 \end{aligned}$$

$$\begin{aligned} 69. \text{ We have, } \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 &= [a \ b \ c]^2 \\ &= \{(a \times b) \cdot c\}^2 = |a \times b|^2 |c|^2 \quad [\because c \parallel a \times b] \\ &= |a|^2 |b|^2 \sin^2 \frac{\pi}{6} |c|^2 \\ &= \frac{1}{4} |a|^2 |b|^2 \end{aligned}$$

70. Let $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $b = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $c = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$$\text{Then, } [a \ b \ c]^2 = [a \ b \ c][a \ b \ c]$$

$$\begin{aligned} &= \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\ &= \begin{vmatrix} a \cdot a & a \cdot b & a \cdot c \\ b \cdot a & b \cdot b & b \cdot c \\ c \cdot a & c \cdot b & c \cdot c \end{vmatrix} \end{aligned}$$

71. We have, $(a \times b) \times (a \times c) \cdot d$

$$\begin{aligned} &= \{(a \times b) \cdot c\} a - \{(a \times b) \cdot a\} c \cdot d \\ &= \{[a \ b \ c] a - 0\} \cdot d = [a \ b \ c](a \cdot d) \end{aligned}$$

72. We have, $(a \times b) \times c = a \times (b \times c)$

$$\begin{aligned} \Rightarrow -c \times (a \times b) &= a \times (b \times c) \\ \Rightarrow -\{(c \cdot b)a - (c \cdot a)b\} &= \{(a \cdot c)b - (a \cdot b)c\} \\ \Rightarrow (a \cdot b)c - (c \cdot b)a &= 0 \\ \Rightarrow (b \cdot a)c - (b \cdot c)a &= 0 \\ \Rightarrow b \times (c \times a) &= 0 \\ \Rightarrow (c \times a) \times b &= 0 \end{aligned}$$

73. We have, $p = \frac{b \times c}{[a \ b \ c]}$, $q = \frac{c \times a}{[a \ b \ c]}$, $r = \frac{a \times b}{[a \ b \ c]}$

$$\therefore a \times p = \frac{1}{[a \ b \ c]} a \times (b \times c),$$

$$b \times q = \frac{1}{[a \ b \ c]} b \times (c \times a)$$

$$\text{and } c \times r = \frac{1}{[a \ b \ c]} c \times (a \times b)$$

$$\begin{aligned} \therefore a \times p + b \times q + c \times r &= \frac{1}{[a \ b \ c]} \{a \times (b \times c) + b \times (c \times a) + c \times (a \times b)\} = 0 \end{aligned}$$

74. Since, the vectors $a\hat{i} + a\hat{j} + c\hat{k}$, $\hat{i} + \hat{k}$ and $c\hat{i} + c\hat{j} + b\hat{k}$ are coplanar, therefore

$$\begin{vmatrix} a & a & c \\ 1 & 0 & 1 \\ c & c & b \end{vmatrix} = 0$$

$$\Rightarrow -ac - ab + ac + c^2 = 0$$

$$\Rightarrow c^2 = ab$$

Hence, c is the GM of a and b .

75. We have, $\hat{c} = \pm \frac{a \times (a \times b)}{|a \times (a \times b)|}$

$$\begin{aligned} \text{Now, } a \times (a \times b) &= (a \cdot b)a - (a \cdot a)b \\ &= -\hat{i} - \hat{j} + \hat{k} - 3(\hat{i} - \hat{j} + \hat{k}) \\ &= -4\hat{i} + 2\hat{j} - 2\hat{k} \end{aligned}$$

$$\therefore \hat{c} = \pm \frac{1}{\sqrt{6}} (2\hat{i} - \hat{j} + \hat{k})$$

Since, $\hat{d}$ is a unit vector perpendicular to both a and $\hat{c}$.

$$\text{Therefore, } \hat{d} = \pm \frac{a \times \hat{c}}{|a \times \hat{c}|}$$

$$\begin{aligned} \text{Now, } a \times \hat{c} &= \pm \frac{1}{\sqrt{16}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & -1 \\ 2 & -1 & 1 \end{vmatrix} \\ &= \pm \frac{1}{\sqrt{6}} (-3\hat{j} - 3\hat{k}) \end{aligned}$$

$$\therefore \hat{d} = \pm \frac{1}{\sqrt{2}} (-\hat{j} - \hat{k}) = \pm \frac{1}{\sqrt{2}} (\hat{j} + \hat{k})$$

76. We have, $(a + b) \times (a + c) = a \times c + b \times a + b \times c$

$$= -a \times b + b \times c - c \times a$$

$$\begin{aligned} \therefore (a + b + c) \cdot \{(a + b) \times (a + c)\} &= (a + b + c) \cdot (-a \times b + b \times c - c \times a) \\ &= -[a \ a \ b] + [a \ b \ c] - [a \ c \ a] - [b \ a \ b] + [b \ b \ c] \\ &\quad - [b \ c \ a] - [c \ a \ b] + [c \ b \ c] - [c \ c \ a] \\ &= [a \ b \ c] - [b \ c \ a] - [c \ a \ b] = -[a \ b \ c] \end{aligned}$$

77. We Impw that, $c = a \times b \Rightarrow c \perp a$ and $c \perp b$

$$a = b \times c \Rightarrow a \perp b \text{ and } a \perp c$$

$$\Rightarrow a \perp b \perp c$$

$$\text{Now, } a \times b = c$$

$$\Rightarrow (b \times c) \times b = c \quad [\because a = b \times c]$$

$$\Rightarrow (b \cdot b)c - (b \cdot c)b = c$$

$$\Rightarrow |b|^2 c = c \quad [\because b \perp c, b \cdot c = 0]$$

$$\Rightarrow |b| = 1$$

$$\text{Also, } c = a \times b \Rightarrow |c| = |a \times b|$$

$$\Rightarrow |c| = |a| |b| \sin \frac{\pi}{2}$$

$$\Rightarrow |c| = |a| \quad [\because |b| = 1]$$

78. We have,

$$(a \times b) \times (b \times c) = \{(a \times b) \cdot c\} b - \{(a \times b) \cdot b\} c = [a \ b \ c] b$$

$$(b \times c) \times (c \times a) = \{(b \times c) \cdot a\} c - \{(b \times c) \cdot c\} a = [b \ c \ a] c$$

$$\text{and } (c \times a) \times (a \times b) = \{(c \times a) \cdot b\} a - \{(c \times a) \cdot a\} b = [c \ a \ b] a$$

$$\begin{aligned} \therefore [(a \times b) \times (b \times c) + (b \times c) \times (c \times a) + (c \times a) \times (a \times b)] &= [a \ b \ c] a + [b \ c \ a] b + [c \ a \ b] c \\ &= [a \ b \ c]^3 [a \ b \ c] = [a \ b \ c]^4 \end{aligned}$$

79. Since, a , b and $a \times b$ are non-coplanar.

$$\therefore r = xa + yb + z(a \times b) \text{ for some scalars } x, y, z \dots (i)$$

$$\begin{aligned} \text{Now, } b = r \times a &\Rightarrow b = \{xa + yb + z(a \times b)\} \times a \\ &= y(b \times c) + z\{(a \times b) \times a\} \\ &= y(b \times c) - z\{a \times (a \times b)\} \\ &= y(b \times c) - z\{(a \cdot b)a - (a \cdot a)b\} \\ &= y(b \times c) + z\{(a \cdot a)b\} \quad [\because a \cdot b = 0] \end{aligned}$$

Comparing the coefficients, we get

$$y = 0, z = \frac{1}{a \cdot a} = \frac{1}{|a|^2}$$

On putting the values of y and z in Eq. (i), we get

$$r = xa + \frac{1}{|a|^2}(a \times b)$$

80. Angle between the faces OAB and ABC is same as the angle between their normals.

We have, n_1 = Vector normal to face $OAB = OA \times OB$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 5\hat{i} - \hat{j} - 3\hat{k} \quad \dots (i)$$

and n_2 = Vector normal to face $ABC = AB \times AC$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ -2 & -1 & 1 \end{vmatrix} = \hat{i} - 5\hat{j} - 3\hat{k} \quad \dots (ii)$$

Let θ be the angle between the faces.

$$\begin{aligned} \text{Then, } \cos \theta &= \frac{n_1 \cdot n_2}{|n_1| |n_2|} \\ &= \frac{5 + 5 + 9}{\sqrt{35} \sqrt{35}} \\ &= \frac{19}{35} \end{aligned}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{19}{35}\right)$$

81. We have, $r \times b = c \times b \Rightarrow (r - c) \times b = 0$

$\Rightarrow r - c$ is parallel to b .

$$\Rightarrow r = c + \lambda b, \text{ for some } \lambda \in R \quad \dots (i)$$

$$\text{Now, } r \cdot a = 0$$

$$\Rightarrow (c + \lambda b) \cdot a = 0$$

$$\Rightarrow (c \cdot a) + \lambda(b \cdot a) = 0$$

$$\Rightarrow (2 + 3 - 3) + \lambda(6 - 3 - 1) = 0$$

$$\Rightarrow \lambda = -1$$

From Eq. (i),

$$\begin{aligned} r &= (\hat{i} + \hat{j} + 3\hat{k}) + (-1)(3\hat{i} - \hat{j} + \hat{k}) \\ &= 2(-\hat{i} + \hat{j} + \hat{k}) \end{aligned}$$

82. Let $r = ua + vb + wc$

$$\text{Then, } r \cdot (b \times c) = (ua + vb + wc) \cdot (b \times c)$$

$$\Rightarrow [r \ b \ c] = u[a \ b \ c] + v[b \ b \ c] + w[c \ b \ c]$$

$$\Rightarrow u = \frac{[r \ b \ c]}{[a \ b \ c]} = x$$

$$\text{Similarly, } v = \frac{[r \ c \ a]}{[a \ b \ c]} = y$$

$$\text{and } w = \frac{[r \ a \ b]}{[a \ b \ c]} = z$$

$$\text{Hence, } r = xa + yb + zc$$

83. We have, $pr + (r \cdot b)a = c \quad \dots (i)$

$$\Rightarrow p(r \cdot b) + (r \cdot b)(a \cdot b) = c \cdot b$$

$$\Rightarrow p(r \cdot b) = c \cdot b \quad [\because a \perp b, a \cdot b = 0]$$

$$\Rightarrow r \cdot b = \frac{c \cdot b}{p}$$

Substituting the value of $r \cdot b$ in Eq. (i), we get

$$pr + \left(\frac{c \cdot b}{p}\right)a = c$$

$$\Rightarrow r = \frac{c}{p} - \frac{c \cdot b}{p^2} a$$

84. We have, $p = \lambda(u \times v) + \mu(v \times w) + v(w \times u)$

$$\begin{aligned} \Rightarrow p \cdot w &= \lambda(u \times v) \cdot w + \mu(v \times w) \cdot w + v(w \times u) \cdot w \\ &= \lambda[u \ v \ w] + 0 + 0 = \frac{\lambda}{5} \quad \left[\because [u \ v \ w] = \frac{1}{5} \right] \end{aligned}$$

$$\Rightarrow \lambda = 5(p \cdot w)$$

$$\text{Similarly, } \mu = 5(p \cdot u)$$

$$\text{and } v = 5(p \cdot v)$$

$$\therefore \lambda + \mu + v = 5(p \cdot w) + 5(p \cdot u) + 5(p \cdot v) = 5P \cdot (u + v + w)$$

85. We have, $a = 2\hat{i} + 2\hat{j} + 3\hat{k}, b = -\hat{i} + 2\hat{j} + \hat{k}$

$$\text{and } c = 3\hat{i} + \hat{j}$$

$$\text{Then, } a + tb = (2 - t)\hat{i} + (2 + 2t)\hat{j} + (3 + t)\hat{k}$$

Now, $a + tb \perp c$

$$\Rightarrow (a + tb) \cdot c = 0$$

$$\Rightarrow 3(2 - t) + 1(2 + 2t) = 0$$

$$\Rightarrow 6 - 3t + 2 + 2t = 0$$

$$\Rightarrow 8 - t = 0$$

$$\Rightarrow t = 8$$

86. Since, the vector $a = (1, 3, \sin 2\alpha)$ makes an obtuse angle with the Z -axis, therefore its Z -component is negative.

$$\therefore \sin 2\alpha < 0$$

$$\Rightarrow -1 \leq \sin 2\alpha < 0 \quad \dots (i)$$

Since, $b \cdot c = 0$, we have

$$\tan^2 \alpha - \tan \alpha - 6 = 0$$

$$\Rightarrow (\tan \alpha - 3)(\tan \alpha + 2) = 0$$

$$\Rightarrow \tan \alpha = 3, -2$$

$$\text{Now, } \tan \alpha = 3$$

$$\Rightarrow \sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha} = \frac{6}{10} > 0$$

which is contradiction to Eq. (i).

$\therefore \tan \alpha = 3$ is not possible.

Thus, $\tan \alpha = -2$ and for this value of $\tan \alpha$, we have

$$\begin{aligned} \tan 2\alpha &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \\ &= \frac{4}{3} \end{aligned}$$

Since, $\sin 2\alpha < 0$ and $\tan 2\alpha > 0$, therefore 2α is in second quadrant.

Also, $\sqrt{\sin \frac{\alpha}{2}}$ is meaningful, if $0 < \sin \frac{\alpha}{2} < 1$.

when these conditions are satisfied, α is given by

$$(i) \alpha = (4n + 1)\pi - \tan^{-1} 2$$

$$(ii) \alpha = (4n + 2)\pi - \tan^{-1} 2$$

87. $AB + BC = AC$

$$BC = \frac{2u}{|u|} - \frac{u}{|u|} + \frac{v}{|v|} = \frac{u}{|u|} + \frac{v}{|v|}$$

$$AB \cdot BC = \left(\frac{u}{|u|} - \frac{v}{|v|} \right) \left(\frac{u}{|u|} + \frac{v}{|v|} \right) = (u - v)(u + v) = 1 - 1 = 0$$

$$\Rightarrow \angle B = 90^\circ$$

$$\Rightarrow 1 + \cos 2A + \cos 2B + \cos 2C = 0$$

88. $(\hat{a} \cdot \hat{c})\hat{b} - (\hat{a} \cdot \hat{b})\hat{c} = \frac{\hat{b} + \hat{c}}{2}$

$$\Rightarrow \hat{b} \left(\hat{a} \cdot \hat{c} - \frac{1}{2} \right) - \hat{c} \left(\hat{a} \cdot \hat{b} + \frac{1}{2} \right) = 0$$

$$\Rightarrow \hat{a} \cdot \hat{c} = \frac{1}{2} \text{ and } \hat{a} \cdot \hat{b} = -\frac{1}{2}$$

$$\therefore \theta_1 = \frac{\pi}{3} \text{ and } \theta_2 = \frac{2\pi}{3}$$

89. Since, $[a \ b \ c] = 0$

$\therefore a, b$ and c are coplanar vectors.

Since, d is equally inclined to a, b and c .

$$\therefore d \cdot a = d \cdot b = d \cdot c = 0$$

$$\Rightarrow d \cdot x = d \cdot y = d \cdot z = 0$$

$$\Rightarrow d \cdot r = 0$$

90. (a) If $x = -1$ and $y = 1$. Then, vertices are $\hat{a}, \hat{b}, -\hat{a} - \hat{b}$.

$$\therefore \hat{a} + \hat{b} + (-\hat{a} - \hat{b}) = 0$$

So, these form a triangle.

(b) If $x = -1, y = 1$, then

$$\begin{aligned} \cos(\hat{a} \cdot \hat{b}) + |\hat{a} + \hat{b}| \cos[\hat{a} - (\hat{a} + \hat{b})] \\ = \cos \alpha - \hat{a} \cdot (\hat{a} + \hat{b}) \\ = \cos \alpha - (1 + \hat{a} \cdot \hat{b}) \\ = \cos \alpha - (1 + \cos \alpha) = -1 \end{aligned}$$

$\therefore \hat{a}, \hat{b}$ and $\hat{x} - \hat{y}$ form a triangle.

(c) $|\hat{a} + \hat{b}| = \sqrt{2 + 2\cos \alpha} = 2 \cos \frac{\alpha}{2}$

$$\therefore |\hat{a} + \hat{b}| = -2 \cos \frac{\alpha}{2} \cdot \cos \frac{\alpha}{2} \quad [\text{given}]$$

$$\therefore 2 \cos \frac{\alpha}{2} = -2 \cos \frac{\alpha}{2} - \cos \frac{\alpha}{2}$$

$$\therefore \cot \frac{\alpha}{2} = -\frac{1}{2} \text{ or } \cos \frac{\alpha}{2} = 0$$

i.e. $\alpha = \pi$, which is not possible.

$$\therefore \frac{\alpha}{2} > \frac{\pi}{2} \text{ i.e. } \alpha > \pi \quad [\text{not possible}]$$

So, there is no such $\hat{a}$ and $\hat{b}$ exist.

91. Since, the angle between a and b is acute.

$$\therefore -3x + x^2 + 2 > 0$$

$$\text{i.e. } x \in (-\infty, 1) \cup (2, \infty)$$

Since, the angle between a and c is obtuse.

$$\therefore 3x^2 + 11x + x^3 - 9x^2 - 6x < 0$$

$$\text{i.e. } x^3 - 6x^2 + 11x - 6 < 0$$

$$\text{i.e. } (x-1)(x-2)(x-3) < 0$$

$$\therefore x \in (-\infty, 1) \cup (2, 3)$$

92. $c = \sqrt{la} + mb \Rightarrow l + 2m = 1$

$$\Rightarrow l - m = x \Rightarrow x = 3y - 2$$

$$\Rightarrow l + m = y$$

Also, $x^2 + y^2 + 1 = 3$

$$\Rightarrow 10y^2 - 12y + 2 = 0$$

$$\therefore y = 1, \frac{1}{5} \text{ and } x = 1, -\frac{7}{5}$$

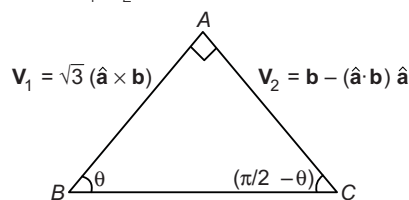
93. $c \cdot (a \times b) + c \cdot (b \times c) + c \cdot (c \times a) = 0$

$$\Rightarrow c \cdot (a \times b) = 0$$

$$\Rightarrow (a \times b) \cdot c = 0$$

$\Rightarrow a, b$ and c are coplanar.

94. Consider, $V_1 \cdot V_2 = 0 \Rightarrow A = 90^\circ$



Using the sine law, $\frac{|b - (\hat{a} \cdot \hat{b})\hat{a}|}{\sin \theta} = \frac{\sqrt{3}|\hat{a} \times \hat{b}|}{\cos \theta}$

$$\begin{aligned} \Rightarrow \tan \theta &= \frac{1}{\sqrt{3}} \frac{|b - (\hat{a} \cdot \hat{b})\hat{a}|}{|\hat{a} \times \hat{b}|} = \frac{1}{\sqrt{3}} \frac{|(\hat{a} \times \hat{b}) \times \hat{a}|}{|\hat{a} \times \hat{b}|} \\ &= \frac{1}{\sqrt{3}} \frac{|\hat{a} \times \hat{b}| |\hat{a}| \sin 90^\circ}{|\hat{a} \times \hat{b}|} = \frac{1}{\sqrt{3}} \end{aligned}$$

$$\Rightarrow \theta = \frac{\pi}{6}$$

95. Given,

$$u = \hat{a} - (\hat{a} \cdot \hat{b})\hat{b}$$

$$= \hat{a}(\hat{b} \cdot \hat{b}) - (\hat{a} \cdot \hat{b})\hat{b} = \hat{b} \times (\hat{a} \times \hat{b})$$

$$\Rightarrow |u| = |\hat{b} \times (\hat{a} \times \hat{b})|$$

$$= |\hat{b}| |\hat{a} \times \hat{b}| \sin 90^\circ$$

$$= |\hat{b}| |\hat{a} \times \hat{b}| = |v|$$

Also, $u \cdot \hat{b} = \hat{b} \cdot \hat{b} \times (\hat{a} \times \hat{b}) = [\hat{b} \ \hat{b} \ \hat{a} \times \hat{b}] = 0$

$$\Rightarrow |v| = |u| + |u \cdot \hat{b}|$$

96. $d \cdot a = [a \ b \ c] \cos y = -d \cdot (b + c)$

$$\Rightarrow \cos y = -\frac{d \cdot (b + c)}{[a \ b \ c]}$$

Similarly, $\sin x = -\frac{d \cdot (a + b)}{[a \ b \ c]}$

and $\frac{d \cdot (a + c)}{[a \ b \ c]} = -2$

$$\therefore \sin x + \cos y + 2 = 0$$

$$\Rightarrow \sin x + \cos y = -2$$

$$\Rightarrow \sin x = -1 \text{ and } \cos y = -1$$

Since, we want the minimum value of $x^2 + y^2, x = -\frac{\pi}{2}$

and $y = \pi$.

$$\therefore \text{The minimum value of } x^2 + y^2 \text{ is } \frac{5\pi^2}{4}.$$

97. Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.

98. $3a - 2b + 5c - 6d$

$$= (2a - 2b) + (-5a + 5c) + (6a - 6d)$$

$$= -2AB + 5AC - 6AD = 0$$

$\therefore AB, AC$ and AD are linearly dependent, hence by Statement II, Statement I is true.

99. Statement I $b_1 = \frac{(2\hat{i} + \hat{j} - 3\hat{k}) \cdot (3\hat{i} - \hat{j})}{|3\hat{i} - \hat{j}|}$

$$= \frac{3\hat{i} - \hat{j}}{|3\hat{i} - \hat{j}|} = \frac{3\hat{i}}{2} - \frac{\hat{j}}{2}$$

$$\therefore b_2 = 2\hat{i} + \hat{j} - 3\hat{k} - \frac{3\hat{i}}{2} + \frac{\hat{j}}{2} = \frac{\hat{i}}{2} + \frac{3\hat{j}}{2} - 3\hat{k}$$

$\therefore$ Statement I is false and Statement II is true.

100. $\{\hat{a} \times (\hat{b} \times \hat{c})\} \cdot (\hat{a} \times \hat{c}) = (\hat{a} \cdot \hat{c})[\hat{b} \cdot \hat{a} \times \hat{c}] = 5$

$$\Rightarrow [\hat{b} \hat{a} \hat{c}] = 10 \Rightarrow [\hat{a} \hat{b} \hat{c}] = -10$$

101. We have, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Consider the system of equations is

$$a_1x + b_1y + c_1z = 0$$

$$a_2x + b_2y + c_2z = 0$$

$$\text{and } a_3x + b_3y + c_3z = 0$$

Suppose system has a non-zero solution, let $x \leq y \leq z$, we have $a_1x + b_1y + c_1z = 0$

$$c_1z = -a_1x - b_1y$$

$$|c_1z| = |a_1x + b_1y|$$

$$|c_1z| \leq |a_1x| + |b_1y|$$

$$|c_1z| \leq |a_1z| + |b_1z|$$

$\Rightarrow$ which is a contradiction, as $|c_1| < |a_1| + |b_1|$.

102. Statement II is true. [see properties of dot product]

$$\text{Also, } (\hat{i} \times \hat{a}) \cdot \hat{b} = \hat{i} \cdot (\hat{a} \times \hat{b})$$

$$\Rightarrow \hat{a} \times \hat{b} = \{\hat{i} \cdot (\hat{a} \times \hat{b})\}\hat{i} + \{\hat{j} \cdot (\hat{a} \times \hat{b})\}\hat{j} + \{\hat{k} \cdot (\hat{a} \times \hat{b})\}\hat{k}$$

103. $AD = 2\hat{j} - \hat{k}$, $BD = -2\hat{i} - \hat{j} - 3\hat{k}$ and $CD = 2\hat{i} - \hat{j}$

$$\text{Volume of tetrahedron} = \frac{1}{6} [AD \ BD \ CD]$$

$$= \frac{1}{6} \begin{vmatrix} 0 & 2 & -1 \\ -2 & -1 & -3 \\ 2 & -1 & 0 \end{vmatrix} = \frac{8}{3}$$

$$\text{Also, area of the } \Delta ABC \text{ is } = \frac{1}{2} |AB \times AC|$$

$$= \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 2 \\ -2 & 3 & -1 \end{vmatrix} = \frac{1}{2} |-9\hat{i} - 2\hat{j} + 12\hat{k}| = \frac{\sqrt{229}}{2}$$

$$\text{Then, } \frac{8}{3} = \frac{1}{3} \times \text{Distance of } D \text{ from base } ABC \times \text{Area of } \Delta ABC$$

$$\therefore \text{Distance of } D \text{ from base } ABC = \frac{16}{\sqrt{229}}$$

104. Let the three given unit vectors be $\hat{a}$, $\hat{b}$ and $\hat{c}$. Since, they are mutually perpendicular, so $\hat{a} \cdot (\hat{b} \times \hat{c}) = 1$.

Therefore,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = 1 \Rightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 1$$

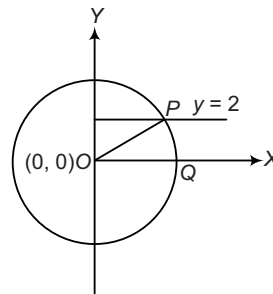
Hence, $a_1\hat{i} + b_1\hat{j} + c_1\hat{k}$, $a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$ and $a_3\hat{i} + b_3\hat{j} + c_3\hat{k}$ may be mutually perpendicular unit vectors.

105. Component of vector $b = 4\hat{i} + 2\hat{j} + 3\hat{k}$ in the direction of $a = \hat{i} + \hat{j} + \hat{k}$ is $\frac{a \cdot b}{|a|} \frac{a}{|a|}$ or $3\hat{i} + 3\hat{j} + 3\hat{k}$. Then,

component in the direction perpendicular to the direction of $a = \hat{i} + \hat{j} + \hat{k}$ is $b - (3\hat{i} + 3\hat{j} + 3\hat{k}) = \hat{i} - \hat{j}$.

106. Solving P is $(4, 2)$.

Q can be $(1, 2)$, $(2, 2)$ and $(3, 2)$.



$$OP = 4\hat{i} + 2\hat{j}$$

$$OQ = \hat{i} + 2\hat{j} \text{ or } 2\hat{i} + 2\hat{j} \text{ or } 3\hat{i} + 2\hat{j}$$

Sum of OQ for three positions is $6\hat{i} + 6\hat{j}$.

$$\begin{aligned} \text{Required projection} &= \frac{(4\hat{i} + 2\hat{j}) \cdot (6\hat{i} + 6\hat{j})}{\sqrt{16 + 4}} \\ &= \frac{36}{2\sqrt{5}} = \frac{18\sqrt{5}}{5} \end{aligned}$$

107. $|OP \times OQ|$

$$\begin{aligned} &= |(4\hat{i} + 2\hat{j}) \times (\hat{i} + 2\hat{j})| + |(4\hat{i} + 2\hat{j}) \times (2\hat{i} + 2\hat{j})| \\ &\quad + |(4\hat{i} + 2\hat{j}) \times (3\hat{i} + 2\hat{j})| \\ &= 6 + 4 + 2 = 12 \end{aligned}$$

108. $OR = -5\hat{i} + 2\hat{j}$ and $|QR| = 6, 7, 8$

Maximum of $|QR|$ is 8.

Sum of projections on Y-axis $= 2 + 2 + 2 = 6$

$\therefore$ Required ratio $= 4 : 3$

Solutions (Q. Nos. 109-111)

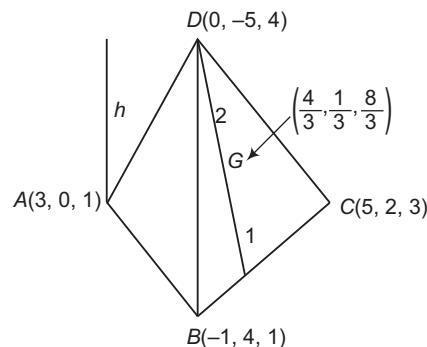
Point G is $(\frac{4}{3}, \frac{1}{3}, \frac{8}{3})$. Therefore,

$$|AG|^2 = \left(\frac{5}{3}\right)^2 + \frac{1}{9} + \left(\frac{5}{3}\right)^2 = \frac{51}{9}$$

$$\Rightarrow |AG| = \frac{\sqrt{51}}{3}$$

$$AB = -4\hat{i} + 4\hat{j} + 0\hat{k}$$

$$AC = 2\hat{i} + 2\hat{j} + 2\hat{k}$$



$$\therefore \quad \mathbf{AB} \times \mathbf{AC} = -8 \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 0 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 8(\hat{i} + \hat{j} - 2\hat{k})$$

$$\text{Area of } \triangle ABC = \frac{1}{2} |\mathbf{AB} \times \mathbf{AC}| = 4\sqrt{6} \text{ sq units}$$

$$\mathbf{AD} = -3\hat{i} - 5\hat{j} + 3\hat{k}$$

$\therefore$ Length of the perpendicular from the vertex D on the opposite face =

$$= \frac{|\text{Projection of } \mathbf{AD} \text{ on } \mathbf{AB} \times \mathbf{AC}|}{|\mathbf{AB} \times \mathbf{AC}|}$$

$$= \frac{|(-3\hat{i} - 5\hat{j} + 3\hat{k}) \cdot (\hat{i} + \hat{j} - 2\hat{k})|}{\sqrt{6}}$$

$$= \frac{|-3 - 5 - 6|}{\sqrt{6}} = \frac{14}{\sqrt{6}}$$

Solutions (Q. Nos. 112-114) Given that, $|\mathbf{x}| = |\mathbf{y}| = |\mathbf{z}| = \sqrt{2}$ and they are inclined at an angle of 60° with each other.

$$\therefore \quad \mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{z} = \mathbf{z} \cdot \mathbf{x} = \sqrt{2} \cdot \sqrt{2} \cos 60^\circ = 1$$

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = \mathbf{a} \Rightarrow (\mathbf{x} \cdot \mathbf{z})\mathbf{y} - (\mathbf{x} \cdot \mathbf{y})\mathbf{z} = \mathbf{a}$$

$$\Rightarrow \quad \mathbf{y} - \mathbf{z} = \mathbf{a} \quad \dots(i)$$

$$\text{Similarly, } \mathbf{y} \times (\mathbf{z} \times \mathbf{x}) = \mathbf{b} \Rightarrow \mathbf{z} - \mathbf{x} = \mathbf{b} \quad \dots(ii)$$

$$\mathbf{y} = \mathbf{a} + \mathbf{z}, \mathbf{x} = \mathbf{z} - \mathbf{b} \quad \dots(iii)$$

[from Eqs. (i) and (ii)]

$$\text{Now, } \mathbf{x} \times \mathbf{y} = \mathbf{c}$$

$$\Rightarrow (\mathbf{z} - \mathbf{b}) \times (\mathbf{z} + \mathbf{a}) = \mathbf{c}$$

$$\Rightarrow \mathbf{z} \times \mathbf{a} - \mathbf{b} \times \mathbf{z} - \mathbf{b} \times \mathbf{a} = \mathbf{c}$$

$$\Rightarrow \mathbf{z} \times (\mathbf{a} + \mathbf{b}) = \mathbf{c} + (\mathbf{b} \times \mathbf{a}) \quad \dots(iv)$$

$$\Rightarrow (\mathbf{a} + \mathbf{b}) \times \{\mathbf{z} \times (\mathbf{a} + \mathbf{b})\} = (\mathbf{a} + \mathbf{b}) \times \mathbf{c}$$

$$+ (\mathbf{a} + \mathbf{b}) \times (\mathbf{b} \times \mathbf{a})$$

$$\Rightarrow (\mathbf{a} + \mathbf{b})^2 \mathbf{z} - \{(\mathbf{a} + \mathbf{b}) \cdot \mathbf{z}\}(\mathbf{a} + \mathbf{b}) = (\mathbf{a} + \mathbf{b}) \times \mathbf{c}$$

$$+ |\mathbf{a}|^2 \mathbf{b} - |\mathbf{b}|^2 \mathbf{a} + (\mathbf{a} \cdot \mathbf{b})(\mathbf{b} - \mathbf{a}) \dots(v)$$

$$\text{From Eq. (i), we get } |\mathbf{a}|^2 = |\mathbf{y} - \mathbf{z}|^2 = 2 + 2 - 2 = 2$$

$$\text{Similarly, from Eq. (ii), we get } |\mathbf{b}|^2 = 2$$

$$\text{From Eqs. (i) and (ii), we get}$$

$$\mathbf{a} + \mathbf{b} = \mathbf{y} - \mathbf{x} \Rightarrow |\mathbf{a} + \mathbf{b}|^2 = 2 \quad \dots(vi)$$

$$\text{Also, } (\mathbf{a} + \mathbf{b}) \cdot \mathbf{z} = (\mathbf{y} - \mathbf{x}) \cdot \mathbf{z} = \mathbf{y} \cdot \mathbf{z} - \mathbf{x} \cdot \mathbf{z} = 1 - 1 = 0$$

$$\text{and } \mathbf{a} \cdot \mathbf{b} = (\mathbf{y} - \mathbf{z}) \cdot (\mathbf{z} - \mathbf{x})$$

$$= \mathbf{y} \cdot \mathbf{z} - \mathbf{x} \cdot \mathbf{y} - |\mathbf{z}|^2 + \mathbf{x} \cdot \mathbf{z} = -1$$

$$\text{From Eq. (v), we have}$$

$$2\mathbf{z} = (\mathbf{a} + \mathbf{b}) \times \mathbf{c} + 2(\mathbf{b} - \mathbf{a}) - (\mathbf{b} - \mathbf{a})$$

$$\text{or } \mathbf{z} = \left(\frac{1}{2}\right) [(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + \mathbf{b} - \mathbf{a}]$$

$$\therefore \quad \mathbf{y} = \mathbf{a} + \mathbf{z} = \left(\frac{1}{2}\right) [(\mathbf{a} + \mathbf{b}) \times \mathbf{c} + \mathbf{b} + \mathbf{a}]$$

$$\text{and } \mathbf{x} = \mathbf{z} - \mathbf{b} = \left(\frac{1}{2}\right) [(\mathbf{a} + \mathbf{b}) \times \mathbf{c} - (\mathbf{a} + \mathbf{b})]$$

$$115. \text{ A. } \mathbf{OA} = \hat{i} + 2\hat{j} + 3\hat{k}, \mathbf{OB} = -2\hat{i} + \hat{j} - 4\hat{k}$$

$$\text{and } \mathbf{OC} = 3\hat{i} + 4\hat{j} - 2\hat{k}$$

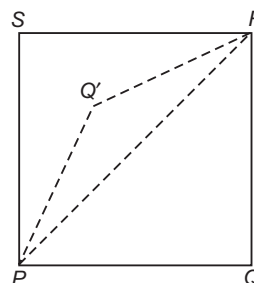
$$\text{Area of triangle} = \frac{1}{2} |\mathbf{AB} \times \mathbf{AC}| = \frac{\sqrt{1218}}{2}$$

$$\text{B. } \{(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}\} \cdot \mathbf{d} + \{(\mathbf{b} \times \mathbf{c}) \times \mathbf{a}\} \cdot \mathbf{d} + \{(\mathbf{c} \times \mathbf{a}) \times \mathbf{b}\} \cdot \mathbf{d} = 0$$

C. Taking P as origin, position vectors of Q, R and S are $P\hat{i}, P\hat{i} + P\hat{j}, P\hat{j}$

Equations of PQ' and RS are

$$\mathbf{r} = t(\hat{i} + \hat{j} + \sqrt{2}\hat{k}), \mathbf{r} = P\hat{i} + P\hat{j} + \lambda\hat{i}$$



$$\therefore \text{ Shortest distance} = \frac{2P}{\sqrt{6}}$$

$$\therefore \quad k = 2$$

$$\text{D. } (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{a} \cdot \mathbf{d}) = 21$$

$$116. \text{ A. If } \mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}, \text{ then } (\mathbf{r} - \mathbf{c}) \times \mathbf{b} = 0$$

$$\therefore \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-4 & y+3 & z-7 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow y - z = -10, z - x = 3, x - y = 7$$

$$\text{and } \mathbf{r} \cdot \mathbf{a} = 0 \Rightarrow 2x + z = 0$$

$$\therefore \quad x = -1, y = -8 \text{ and } z = 2$$

$$\text{So, the required vector is } -\hat{i} - 8\hat{j} + 2\hat{k}.$$

$$\text{B. A unit vector along } \mathbf{b} = 6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k}$$

$$= \pm \frac{2}{25} \left(6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k} \right)$$

$$\text{So, } \mathbf{r} = \pm 4 \left(6\hat{i} - 8\hat{j} - \frac{15}{2}\hat{k} \right), \text{ also } \mathbf{r} \cdot \mathbf{k} < 0$$

$$\text{C. We have, } |\mathbf{a}| = 3 \text{ and } |\mathbf{b}| = 3$$

$$\text{A vector along bisectors } \frac{\mathbf{a}}{|\mathbf{a}|} + \frac{\mathbf{b}}{|\mathbf{b}|} = \hat{i} - \frac{1}{3}\hat{k},$$

$$\frac{1}{3}\hat{i} - \frac{4}{5}\hat{j} + \hat{k}$$

$$\text{Required vector} = 2 \left(\frac{\hat{i} - \frac{1}{3}\hat{k}}{\sqrt{1^2 + \left(\frac{1}{3}\right)^2}} \right)$$

$$2 \left(\frac{\frac{1}{3}\hat{i} - \frac{4}{3}\hat{j} + \hat{k}}{\sqrt{\left(\frac{1}{3}\right)^2 + \left(-\frac{4}{3}\right)^2 + 1^2}} \right)$$

$$= \frac{2}{\sqrt{10}} (3\hat{i} - \hat{k}), \frac{2}{\sqrt{26}} (\hat{i} - 4\hat{j} + 3\hat{k})$$

$$\text{D. Let } \mathbf{b} = x\hat{i} + y\hat{j}, \text{ since } \mathbf{a} \perp \mathbf{b}, \text{ so } 4x + 3y = 0$$

$$\text{Thus, } \mathbf{b} = x \left(\hat{i} - \frac{4}{3}\hat{j} \right)$$

$$\text{Let } \mathbf{c} = u\hat{i} + v\hat{j}$$

$$\text{Then, } 1 = \frac{\mathbf{c} \cdot \mathbf{a}}{|\mathbf{a}|}$$

$$\Rightarrow 4u + 3v = 5$$

and $2 = \frac{c \cdot b}{|b|}$
 $\Rightarrow 3u - 4v = 10$
 On solving, we get $u = 2$ and $v = -1$.
 So, the required vector is $2\hat{i} - \hat{j}$.

117. A. We have, $|a + b + c| = \sqrt{6}$
 $\Rightarrow a^2 + b^2 + c^2 + 2a \cdot b + 2b \cdot c + 2c \cdot a = 6$
 $\Rightarrow |a| = 1$
 $\therefore 2|a| = 2$
 B. a is perpendicular to $b + c \Rightarrow a \cdot b + a \cdot c = 0 \dots (i)$
 b is perpendicular to $a + c \Rightarrow b \cdot a + b \cdot c = 0 \dots (ii)$
 c is perpendicular to $a + b$
 $\Rightarrow c \cdot a + c \cdot b = 0 \dots (iii)$
 From Eqs. (i), (ii) and (iii), we get
 $a \cdot b = b \cdot c = c \cdot a = 0$
 $\Rightarrow |a + b + c| = 7$
 $\therefore |a + b + c| - 2 = 7 - 2 = 5$
 C. $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (b \cdot c)(a \cdot d) = 21$
 $\therefore \frac{1}{7}(a \times b) \cdot (c \times d) = \frac{1}{7} \times 21 = 3$

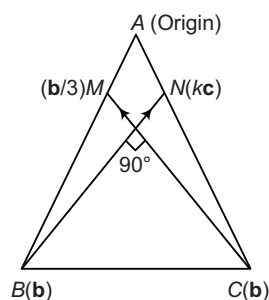
D. We know that, $[a \times b \ b \times c \ c \times a] = [a \ b \ c]^2$
 and $[a \ b \ c]^2 = \begin{vmatrix} a \cdot a & a \cdot b & a \cdot c \\ b \cdot a & b \cdot b & b \cdot c \\ c \cdot a & c \cdot b & c \cdot c \end{vmatrix} = \begin{vmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{vmatrix} = 32$
 $\Rightarrow [a \ b \ c] = 4\sqrt{2}$
 $\therefore [a \ b \ c] \cos 45^\circ = 4\sqrt{2} \times \frac{1}{\sqrt{2}} = 4$

118. $|A \times [A \times (A \times B) \cdot C]| = A \times [(A \cdot B)A - (A \cdot A)B] \cdot C$
 $= \left(\underbrace{A \times (A \cdot B)A}_{\text{zero}} - (A \cdot A)A \times B \right) \cdot C$
 $= -|A|^2 [A \ B \ C]$
 Now, $-|A|^2 = 4 + 9 + 36 = 49$
 Also, $[A \ B \ C] = \begin{vmatrix} 2 & 3 & 6 \\ 1 & 1 & -2 \\ 1 & 2 & 1 \end{vmatrix}$
 $= 2(1 + 4) - 1(3 - 12) + 1(-6 - 6)$
 $= 10 + 9 - 12 = 7$
 $\therefore -|A|^2 [A \ B \ C] = 49 \times 7 = 343 = abc$
 Hence, $a = 3, b = 4$ and $c = 3$
 $\Rightarrow a + b - c = 3 + 4 - 3 = 4$

119. $(2x - y - 5)a = (x - 2y - 4)b$
 $\therefore 2x - y = 5 \dots (i)$
 and $x - 2y = 4 \dots (ii)$
 From Eqs. (i) and (ii), we get
 $2(2y + 4) - y = 5$
 $\Rightarrow 3y = -3$
 $\Rightarrow y = -1$
 and $x = 2$
 Hence, $x + y = 1$

120. $\therefore BN \cdot CM = 0$

$\Rightarrow (kc - b) \cdot \left(\frac{b}{3} - c \right) = 0$



$\Rightarrow \frac{k}{3} \cdot \frac{a^2}{2} - ka^2 - \frac{a^2}{3} + \frac{a^2 2}{2} = 0$
 $\Rightarrow \frac{k}{6} - k + \frac{1}{6} = 0$
 $\Rightarrow \frac{5k}{6} = \frac{1}{6}$
 $\Rightarrow k = \frac{1}{5} = \frac{\lambda}{\mu}$
 $\Rightarrow \lambda + \mu = 1 + 5 = 6$

121. We have, $V_1 = \hat{i} - 2\hat{j} + \hat{k}$
 $V_2 = 3\hat{i} + 2\hat{j} - \hat{k}$
 $V_3 = c = \alpha a + \beta b = \alpha(\hat{i} + \hat{j}) + \beta(\hat{j} + \hat{k})$
 $= \alpha\hat{i} + (\alpha + \beta)\hat{j} + \beta\hat{k} = c$

Since, V_1, V_2 and V_3 are coplanar.

Now, $\begin{vmatrix} 1 & -2 & 1 \\ 3 & 2 & -1 \\ \alpha & \alpha + \beta & \beta \end{vmatrix} = 0$

On applying $C_2 \rightarrow C_2 - (C_1 + C_3)$, we get

$\begin{vmatrix} 1 & -4 & 1 \\ 3 & 0 & -1 \\ \alpha & 0 & \beta \end{vmatrix} = 0$

Hence, $4(3\beta + \alpha) = 0$
 $\Rightarrow 3\beta + \alpha = 0$
 $\Rightarrow \frac{\alpha}{\beta} = -3$

Hence, λ is equal to 3.

122. Since, angle between u and $\hat{i}$ is 60° .

$\therefore u \cdot \hat{i} = |u| |\hat{i}| \cos 60^\circ = \frac{|u|}{2}$

Given that, $|u - \hat{i}|, |u|, |u - 2\hat{i}|$ are in GP.

So, $|u - \hat{i}|^2 = |u| |u - 2\hat{i}|$

On squaring both sides, we get
 $[|u|^2 + |\hat{i}|^2 - 2u \cdot \hat{i}]^2 = |u|^2 [|u|^2 + 4|\hat{i}|^2 - 4u \cdot \hat{i}]$
 $\Rightarrow \left[|u|^2 + 1 - \frac{2|u|}{2} \right]^2 = |u|^2 \left[|u|^2 + 4 - \frac{4|u|}{2} \right]$
 $\Rightarrow |u|^2 + 2|u| - 1 = 0$
 $\Rightarrow |u| = \frac{2 \pm 2\sqrt{2}}{2}$
 $\Rightarrow |u| = \sqrt{2} - 1$
 $\therefore (\sqrt{2} + 1)|u| = (\sqrt{2} + 1)(\sqrt{2} - 1)$
 $= 2 - 1 = 1$

123. $AB = 2\hat{i} + \hat{j} + \hat{k}$, $AC = (t+1)\hat{i} + 0\hat{j} - \hat{k}$

$$AB \times AC = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & 1 \\ t+1 & 0 & -1 \end{vmatrix} = -\hat{i} + (t+3)\hat{j} - (t+1)\hat{k}$$

$$|AB \times AC| = \sqrt{1 + (t+3)^2 + (t+1)^2} = \sqrt{2t^2 + 8t + 11}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} |AB \times AC|$$

$$\Rightarrow \Delta = \frac{1}{2} \sqrt{2t^2 + 8t + 11}$$

$$\text{Let } f(t) = \Delta^2 = \frac{1}{4} (2t^2 + 8t + 11)$$

$$f'(t) = 0 \Rightarrow t = -2$$

$$\text{At } t = -2, f''(t) > 0$$

So, area of triangle is minimum at $t = -2$.

124. Since, x and y are non-collinear vectors, therefore x , y and $x \times y$ are non-coplanar vectors.

$$[(a-2)\alpha^2 + (b-3)\alpha + c]x + [(a+2)\beta^2 + (b-3)\beta + c]y + [(\alpha-2)\gamma^2 + (b-3)\gamma + c](x \times y) = 0$$

Coefficients of each vector x , y and $x \times y$ is zero.

$$\text{i.e. } (a-2)\alpha^2 + (b-3)\alpha + c = 0,$$

$$(a-2)\beta^2 + (b-3)\beta + c = 0$$

$$\text{and } (a-2)\gamma^2 + (b-3)\gamma + c = 0$$

The above three equations will satisfy, if the coefficients of α , β and γ are zero, because α , β and γ are three distinct real numbers.

$$\Rightarrow a - 2 = 0$$

$$\Rightarrow a = 2$$

$$\Rightarrow b - 3 = 0$$

$$\Rightarrow b = 3$$

$$\text{and } c = 0$$

$$\therefore a^2 + b^2 + c^2 = 2^2 + 3^2 + 0^2 = 4 + 9 = 13$$

$$\text{Now, } a^2 + b^2 + c^2 - 4 = 13 - 4 = 9$$

125. $\hat{a} \cdot \hat{b} = 0 \Rightarrow \hat{a} \perp \hat{b}$

$$\therefore \hat{a} \cdot \hat{c} = 0$$

$$\Rightarrow \hat{a} \perp \hat{c}$$

$$\Rightarrow \hat{a} \perp \hat{b} - \hat{c}$$

$$|\hat{a} \times \hat{b} - \hat{a} \times \hat{c}| = |\hat{a} \times (\hat{b} - \hat{c})|$$

$$= |\hat{a}| |\hat{b} - \hat{c}| = |\hat{b} - \hat{c}| \quad \dots (i)$$

$$\text{Now, } |\hat{b} - \hat{c}|^2 = |\hat{b}|^2 + |\hat{c}|^2 - 2|\hat{b}||\hat{c}|\cos \frac{\pi}{3}$$

$$= 2 - 2 \times \frac{1}{2} = 1$$

$$\Rightarrow |\hat{b} - \hat{c}| = 1$$

From Eq. (i), we get

$$(\hat{a} \times \hat{b} - \hat{a} \times \hat{c}) = 1$$

Entrance Gallery

1. $|x| = |y| = |z| = \sqrt{2}$ and $\theta = \frac{\pi}{3}$

$$a = \lambda x \times (y \times z)$$

$$b = \mu y \times (z \times x)$$

$$a = [(x \cdot z)y - (x \cdot y)z]$$

$$a = \lambda \left(2 \times \frac{1}{2} y - 2 \times \frac{1}{2} z \right)$$

$$a = \lambda (y - z) \quad \text{and} \quad b = \mu (z - x)$$

$$\text{Similarly, } a \cdot y = \lambda \left(2 - 2 \times \frac{1}{2} \right) = \lambda$$

$$a = (a \cdot y)(y - z)$$

$$b \cdot z = \mu \left(2 - 2 \times \frac{1}{2} \right)$$

$$\mu = b \cdot z$$

$$\therefore b = (b \cdot z)(z - x)$$

$$a \cdot b = (a \cdot y)(y - z) \cdot (b \cdot y)(z - x)$$

$$= (a \cdot y)(b \cdot z)(y \cdot z - y \cdot x - z \cdot x + z \cdot y)$$

$$= (a \cdot y)(b \cdot z)(-2 + 1) = -(a \cdot y)(b \cdot z)$$

2. We have, $p\hat{a} + q\hat{b} + r\hat{c} = \hat{a} \times \hat{b} + \hat{b} \times \hat{c}$

Taking dot product with $\hat{a}$, $\hat{b}$ and $\hat{c}$, we get

$$p + \frac{q}{2} + \frac{r}{2} = [\hat{a} \hat{b} \hat{c}] \quad \dots (i)$$

$$\frac{p}{2} + q + \frac{r}{2} = 0 \quad \dots (ii)$$

$$\text{and } \frac{p}{2} + \frac{q}{2} + r = [\hat{a} \hat{b} \hat{c}] \quad \dots (iii)$$

From Eqs. (i) and (iii), we get

$$\therefore \frac{p^2 + 2q^2 + r^2}{q^2} = \frac{p^2 + 2p^2 + p^2}{p^2} = 4$$

3. A. We have, $[a \ b \ c] = 2$

$$\therefore [2(a \times b) \ 3(b \times c) \ c \times a] = 6[a \ b \ c]^2 = 6 \times 4 = 24$$

B. We have, $[a \ b \ c] = 5$

$$\therefore [3(a + b) \ b + c \ 2(c + a)] =$$

$$6[a + b \ b + c \ c + a] = 12[a \ b \ c] = 60$$

C. We have, $\frac{1}{2}(a \times b) = 20$

$$\therefore = \frac{1}{2} |(2a + 3b) \times (a - b)|$$

$$= \frac{1}{2} |-2(a \times b) - 3(a \times b)|$$

$$= \frac{5}{2} \times 40 = 100$$

D. We have, $|a \times b| = 30$

$$\therefore |(a + b) \times a| = |b \times a| = 30$$

4. Let $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$ and $(-1, -1, 1)$ be the vectors a , b , c and d .

Rest of the vectors are $-a$, $-b$, $-c$ and $-d$.

Let us find the number of ways of selecting coplanar vectors.

Observe that, out of any 3 coplanar vectors, two will be collinear (anti-parallel).

Number of ways of selecting the anti-parallel pair = 4

Number of ways of selecting the third vector = 6

Total number of ways = 24

Number of non-coplanar selections

$$= {}^6C_3 - 24 = 32 - 24 = 8$$

$$\therefore 2^p = 2^5$$

$$\text{Hence, } p = 5$$

Aliter For non-coplanar vectors,

Selection of anyone out of 8 = 8

Selection of anyone from remaining 6 (excluding selected vector and its anti-parallel) = 6

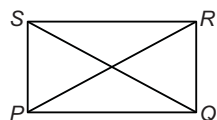
Selection of anyone from remaining 4 vectors = 4

Since, volume does not effected by change of order (permutation).

$$\begin{aligned} \therefore \text{Number of ways} &= {}^8C_3 - 24 = \frac{8 \times 7 \times 6 \times 5!}{5! \times 3 \times 2} - 24 \\ &= 32 = 2^5 \\ \Rightarrow p &= 5 \end{aligned}$$

5. We have, $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$, $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$

$\therefore \overrightarrow{PQ} + \overrightarrow{PS} = \overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$



$\overrightarrow{PQ} - \overrightarrow{PS} = \overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$

$\therefore 2\overrightarrow{PQ} = 4\hat{i} - 2\hat{j} - 6\hat{k}$

$2\overrightarrow{PS} = 2\hat{i} + 4\hat{j} + 2\hat{k}$

$\therefore \overrightarrow{PQ} = 2\hat{i} - \hat{j} + 3\hat{k}$, $\overrightarrow{PS} = \hat{i} + 2\hat{j} + \hat{k}$

Also, $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$

V = Volume of parallelopiped formed by $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$

$$\begin{aligned} &= [\overrightarrow{PT} \overrightarrow{PQ} \overrightarrow{PS}] \\ &= \begin{vmatrix} 2 & -1 & 3 \\ 1 & 2 & 1 \\ 1 & 2 & 3 \end{vmatrix} = 10 \text{ cu units} \end{aligned}$$

6. We know that,

$$|\hat{a} + \hat{b} + \hat{c}|^2 = |\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2 + 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a})$$

$$\begin{aligned} \text{and } |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \\ = \{2|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2\} - (\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \end{aligned}$$

$$\begin{aligned} \therefore |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \\ = 3\{|\hat{a}|^2 + |\hat{b}|^2 + |\hat{c}|^2\} - |\hat{a} + \hat{b} + \hat{c}|^2 \end{aligned}$$

$$\Rightarrow 9 = 3 \times 3 - |\hat{a} + \hat{b} + \hat{c}|^2$$

$$\Rightarrow |\hat{a} + \hat{b} + \hat{c}| = 0$$

$$\Rightarrow \hat{a} + \hat{b} + \hat{c} = 0$$

$$\Rightarrow \hat{b} + \hat{c} = -\hat{a}$$

$$\begin{aligned} |2\hat{a} + 5\hat{b} + 5\hat{c}| &= |2\hat{a} + 5(\hat{b} + \hat{c})| \\ &= |2\hat{a} + 5(-\hat{a})| \\ &= |-3\hat{a}| = 3 \end{aligned}$$

7. Here, $\mathbf{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \mathbf{b}$

$$\Rightarrow \mathbf{a} \times (2\hat{i} + 3\hat{j} + \hat{k}) - (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \mathbf{b} = 0$$

$$\Rightarrow (\mathbf{a} + \mathbf{b}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = 0$$

$$\Rightarrow \mathbf{a} + \mathbf{b} = \lambda(2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \dots(i)$$

Since, $|\mathbf{a} + \mathbf{b}| = \sqrt{29}$

$$\Rightarrow \pm \lambda \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$\Rightarrow \lambda = \pm 1$$

$$\therefore \mathbf{a} + \mathbf{b} = \pm (2\hat{i} + 3\hat{j} + 4\hat{k})$$

Now, $(\mathbf{a} + \mathbf{b}) \cdot (-7\hat{i} + 2\hat{j} + 3\hat{k})$

$$= \pm (-14 + 6 + 12) = \pm 4$$

8. $\mathbf{r} \times \mathbf{b} = \mathbf{c} \times \mathbf{b}$

$$\Rightarrow (\mathbf{r} - \mathbf{c}) \times \mathbf{b} = 0$$

$$\Rightarrow \mathbf{r} - \mathbf{c} = \lambda \mathbf{b}$$

$$\text{or } \mathbf{r} = \mathbf{c} + \lambda \mathbf{b} \quad \dots(i)$$

Given, $\mathbf{r} \cdot \mathbf{a} = 0$

Taking dot product with $\mathbf{a}$ for Eq. (i), we get

$$\mathbf{r} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{c} + \lambda \mathbf{a} \cdot \mathbf{b}$$

$$\therefore \lambda = \frac{-\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}} \quad [\because \mathbf{r} \cdot \mathbf{a} = 0] \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\mathbf{r} = \mathbf{c} - \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}} \mathbf{b}$$

Taking dot with $\mathbf{b}$, we get

$$\mathbf{r} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{b} - \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}} (\mathbf{b} \cdot \mathbf{b}) = (-1 + 2) - \frac{(-1 - 3)}{(1)} (1 + 1)$$

$$= 1 + 8 = 9 \quad \left[\begin{array}{l} \text{where, } \mathbf{a} = -\hat{i} - \hat{k}, \mathbf{b} = -\hat{i} + \hat{j} \\ \text{and } \mathbf{c} = \hat{i} + 2\hat{j} + 3\hat{k} \end{array} \right]$$

9. Let $\mathbf{v} = \mathbf{a} + \lambda \mathbf{b}$

$$\mathbf{v} = (1 + \lambda)\hat{i} + (1 - \lambda)\hat{j} - (1 + \lambda)\hat{k}$$

Now, projection of $\mathbf{v}$ on $\mathbf{c} = \frac{1}{\sqrt{3}}$

$$\Rightarrow \frac{\mathbf{v} \cdot \mathbf{c}}{|\mathbf{c}|} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{(1 + \lambda) - (1 - \lambda) - (1 + \lambda)}{\sqrt{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow 1 + \lambda - 1 + \lambda - 1 - \lambda = 1$$

$$\Rightarrow \lambda - 1 = 1$$

$$\Rightarrow \lambda = 2$$

$$\therefore \mathbf{v} = 3\hat{i} - \hat{j} + 3\hat{k}$$

10. Let $\mathbf{a} = \hat{i} + \hat{j} + 2\hat{k}$, $\mathbf{b} = \hat{i} + 2\hat{j} + \hat{k}$

and $\mathbf{c} = \hat{i} + \hat{j} + \hat{k}$

$\therefore$ A vector coplanar to $\mathbf{a}$ and $\mathbf{b}$ and perpendicular to $\mathbf{c}$.

i.e. $\lambda (\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \lambda \{(\mathbf{a} \cdot \mathbf{c}) \mathbf{v} - (\mathbf{b} \cdot \mathbf{c}) \mathbf{a}\}$

$$= \lambda \{(1 + 1 + 4)(\hat{i} + 2\hat{j} + \hat{k}) - (1 + 2 + 1)(\hat{i} + \hat{j} + 2\hat{k})\}$$

$$= \lambda \{6\hat{i} + 12\hat{j} + 6\hat{k} - 6\hat{i} - 6\hat{j} - 12\hat{k}\}$$

$$= \lambda \{6\hat{j} - 6\hat{k}\}$$

$$= 6\lambda \{\hat{j} - \hat{k}\}$$

For $\lambda = \frac{1}{6}$, option (a) is correct.

and for $\lambda = -\frac{1}{6}$, option (d) is correct.

11. We have, $\overrightarrow{PQ} = 6\hat{i} + \hat{j}$

$$\overrightarrow{SR} = 6\hat{i} + \hat{j}$$

$$\overrightarrow{PS} = -\hat{i} + 3\hat{j}$$

$$\overrightarrow{QR} = -\hat{i} + 3\hat{j}$$

$$\overrightarrow{PQ} = \overrightarrow{SR}$$

and $\overrightarrow{PS} = \overrightarrow{QR}$

Clearly, $\overrightarrow{PQ} \cdot \overrightarrow{PS} \neq 0$

Also, $\overrightarrow{PR} \cdot \overrightarrow{QS} \neq 0$

Hence, $PQRS$ is a parallelogram but neither a rectangle nor a rhombus.

12. We have, $\overrightarrow{AB} = 2\hat{i} + 10\hat{j} + 11\hat{k}$

and $\overrightarrow{AD} = -\hat{i} + 2\hat{j} + 2\hat{k}$

Angle between AB and AD is given by

$$\cos \theta = \frac{|\mathbf{AB} \cdot \mathbf{AD}|}{|\mathbf{AB}| \cdot |\mathbf{AD}|}$$

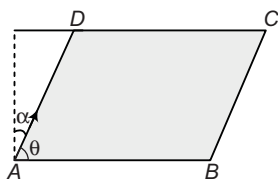
$$= \frac{|-2 + 20 + 22|}{(15)(3)} = \frac{8}{9}$$

$$\Rightarrow \sin \theta = \frac{\sqrt{17}}{9}$$

Since, $\alpha + \theta = 90^\circ$

$$\therefore \cos \alpha = \cos (90^\circ - \theta)$$

$$= \sin \theta = \frac{\sqrt{17}}{9}$$



13. From the given information, it is clear that

$$\mathbf{a} = \frac{\hat{i} - 2\hat{j}}{\sqrt{5}} \text{ and } \mathbf{b} = \frac{2\hat{i} - \hat{j} + 3\hat{k}}{\sqrt{14}}$$

$$\Rightarrow |\mathbf{a}| = 1, |\mathbf{b}| = 1 \text{ and } \mathbf{a} \cdot \mathbf{b} = 0$$

$$\text{Now, } (2\mathbf{a} + \mathbf{b}) \cdot [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} - 2\mathbf{b})]$$

$$= (2\mathbf{a} + \mathbf{b}) \cdot [\mathbf{a}^2 \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \cdot \mathbf{a} + 2\mathbf{b}^2 \cdot \mathbf{a} - 2(\mathbf{b} \cdot \mathbf{a}) \cdot \mathbf{a}]$$

$$= [2\mathbf{a} + \mathbf{b}] \cdot [\mathbf{b} + 2\mathbf{a}] = 4\mathbf{a}^2 + \mathbf{b}^2$$

$$= 4 \cdot 1 + 1 = 5 \quad [\because \mathbf{a} \cdot \mathbf{b} = 0]$$

14. $[\mathbf{a} \times \mathbf{b} \quad \mathbf{b} \times \mathbf{c} \quad \mathbf{c} \times \mathbf{a}] = (\mathbf{a} \times \mathbf{b}) \cdot \{(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})\}$

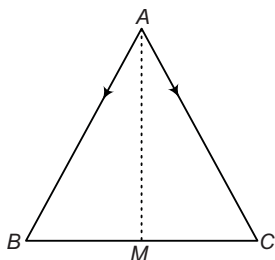
$$= (\mathbf{a} \times \mathbf{b}) \cdot \{[\mathbf{b} \mathbf{c} \mathbf{a}] \mathbf{c} - [\mathbf{b} \mathbf{c} \mathbf{c}] \mathbf{a}\}$$

$$= [\mathbf{b} \mathbf{c} \mathbf{a}] \{(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}\} - [\mathbf{b} \mathbf{c} \mathbf{c}] \cdot \mathbf{a}$$

$$= [\mathbf{a} \mathbf{b} \mathbf{c}]^2$$

$$\therefore \lambda = 1$$

15. We know that, the sum of three vectors of a triangle is zero.



$$\therefore \mathbf{AB} + \mathbf{BC} + \mathbf{CA} = \mathbf{0}$$

$$\Rightarrow \mathbf{BC} = \mathbf{AC} - \mathbf{AB}$$

$$\Rightarrow \mathbf{BM} = \frac{\mathbf{AC} - \mathbf{AB}}{2}$$

[since, M is the mid-point of BC]

$$\text{Also, } \mathbf{AB} + \mathbf{BM} + \mathbf{MA} = \mathbf{0}$$

[by properties of a triangle]

$$\Rightarrow \mathbf{AB} + \frac{\mathbf{AC} - \mathbf{AB}}{2} = \mathbf{AM}$$

$$\Rightarrow \mathbf{AM} = \frac{\mathbf{AB} + \mathbf{AC}}{2}$$

$$= \frac{3\hat{i} + 4\hat{k} + 2\hat{j} + 4\hat{k}}{2}$$

$$= 4\hat{i} - \hat{j} + 4\hat{k}$$

$$\therefore |\mathbf{AM}| = \sqrt{4^2 + 1^2 + 4^2}$$

$$= \sqrt{33}$$

16. Given that, $\hat{a}, \hat{b}$ are unit vectors i.e. $|\hat{a}| = |\hat{b}| = 1$
 $\mathbf{c} = \hat{a} + 2\hat{b}, \hat{b} = 5\hat{a} - 4\hat{b}$ and $\mathbf{c}, \mathbf{d}$ are perpendicular to each other i.e. $\mathbf{c} \cdot \mathbf{d} = 0$.

$$\text{Now, } \mathbf{c} \cdot \mathbf{d} = 0 \Rightarrow (\hat{a} + 2\hat{b}) \cdot (5\hat{a} - 4\hat{b}) = 0$$

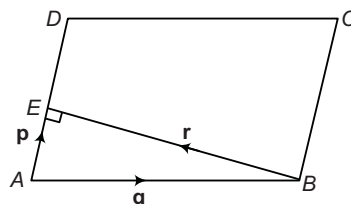
$$\Rightarrow 5\hat{a} \cdot \hat{a} - 4\hat{a} \cdot \hat{b} + 10\hat{b} \cdot \hat{a} - 8\hat{b} \cdot \hat{b} = 0$$

$$\Rightarrow 6\hat{a} \cdot \hat{b} = 3$$

$$\Rightarrow \hat{a} \cdot \hat{b} = \frac{1}{2}$$

So, the angle between $\hat{a}$ and $\hat{b}$ is $\frac{\pi}{3}$.

17. Given, parallelogram ABCD such that $\mathbf{AB} = \mathbf{q}, \mathbf{AD} = \mathbf{p}$ and the altitude from vertex B to side AD coincides with a vector $\mathbf{r}$.



To find the vector $\mathbf{r}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

Let E be the foot of perpendicular from B to side AD.

Then, $\mathbf{AE}$ = Projection of vector $\mathbf{q}$

$\therefore \mathbf{AE}$ = Vector along $\mathbf{AE}$ of length $\mathbf{AE}$

$$= |\mathbf{AE}| \frac{\mathbf{p}}{|\mathbf{p}|} = \frac{(\mathbf{q} \cdot \mathbf{p}) \mathbf{p}}{|\mathbf{p}|^2}$$

Now, applying triangle law in $\triangle ABE$, we get

$$\mathbf{AB} + \mathbf{BE} = \mathbf{AE}$$

$$\Rightarrow \mathbf{q} + \mathbf{r} = \frac{(\mathbf{q} \cdot \mathbf{p}) \mathbf{p}}{|\mathbf{p}|^2}$$

$$\Rightarrow \mathbf{r} = \frac{(\mathbf{q} \cdot \mathbf{p}) \mathbf{p}}{|\mathbf{p}|^2} - \mathbf{q}$$

$$\therefore \mathbf{r} = -\mathbf{q} + \left(\frac{\mathbf{q} \cdot \mathbf{p}}{\mathbf{p} \cdot \mathbf{p}} \right) \mathbf{p}$$

18. We have, $\mathbf{a} = \frac{1}{\sqrt{10}}(3\hat{i} + \hat{k})$ and $\mathbf{b} = \frac{1}{7}(2\hat{i} + 3\hat{j} - 6\hat{k})$

$$\therefore (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times (\mathbf{a} + 2\mathbf{b})\}$$

$$= (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \times \mathbf{b}) \times \mathbf{a} + (\mathbf{a} \times \mathbf{b}) \times 2\mathbf{b}\}$$

$$= (2\mathbf{a} - \mathbf{b}) \cdot \{(\mathbf{a} \cdot \mathbf{a}) \mathbf{b} + (\mathbf{b} \cdot \mathbf{a}) \mathbf{a} + 2(\mathbf{a} \cdot \mathbf{b}) \mathbf{b} - 2(\mathbf{b} \cdot \mathbf{b}) \mathbf{a}\}$$

$$= (2\mathbf{a} - \mathbf{b}) \cdot \{1(\mathbf{b}) - (0)\mathbf{a} + 2(0)\mathbf{b} - 2(1)\mathbf{a}\}$$

$$[\because \mathbf{a} \cdot \mathbf{b} = 0 \text{ and } \mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = 1]$$

$$= (2\mathbf{a} - \mathbf{b}) \cdot (\mathbf{b} - 2\mathbf{a})$$

$$= -(4|\mathbf{a}|^2 + 4\mathbf{a} \cdot \mathbf{b} + |\mathbf{b}|^2)$$

$$= -(4 + 0 + 1)$$

$$= -5$$

19. Given, $\mathbf{a} \cdot \mathbf{b} \neq 0, \mathbf{a} \cdot \mathbf{d} = 0$... (i)

$$\text{and } \mathbf{b} \times \mathbf{c} = \mathbf{b} \times \mathbf{d}$$

$$\Rightarrow \mathbf{b} \times (\mathbf{c} - \mathbf{d}) = \mathbf{0}$$

$$\therefore \mathbf{b} \parallel (\mathbf{c} - \mathbf{d})$$

$$\Rightarrow \mathbf{c} - \mathbf{d} = \lambda \mathbf{b}$$

$$\Rightarrow \mathbf{d} = \mathbf{c} - \lambda \mathbf{b} \quad \dots (ii)$$

Taking dot product with $\mathbf{a}$, we get

$$\mathbf{a} \cdot \mathbf{d} = \mathbf{a} \cdot \mathbf{c} - \lambda \mathbf{a} \cdot \mathbf{b}$$

$$\Rightarrow 0 = \mathbf{a} \cdot \mathbf{c} - \lambda (\mathbf{a} \cdot \mathbf{b})$$

$$\therefore \lambda = \frac{\mathbf{a} \cdot \mathbf{c}}{\mathbf{a} \cdot \mathbf{b}} \quad \dots (iii)$$

$$\therefore \mathbf{d} = \mathbf{c} - \frac{(\mathbf{a} \cdot \mathbf{c})}{(\mathbf{a} \cdot \mathbf{b})} \mathbf{b}$$

20. Given, $a = p\hat{i} + \hat{j} + \hat{k}$, $b = \hat{i} + q\hat{j} + \hat{k}$, $c = \hat{i} + \hat{j} + r\hat{k}$ are coplanar and $p \neq q \neq r \neq 1$.

Since, a , b and c are coplanar.

$$\therefore [a \ b \ c] = 0$$

$$\Rightarrow \begin{vmatrix} p & 1 & 1 \\ 1 & q & 1 \\ 1 & 1 & r \end{vmatrix} = 0$$

$$\Rightarrow p(qr - 1) - 1(r - 1) + 1(1 - q) = 0$$

$$\Rightarrow pqr - p - r + 1 + 1 - q = 0$$

$$\Rightarrow pqr - (p + q + r) = -2$$

21. As $a + 3b$ is collinear with c .

$$\Rightarrow a + 3b = \lambda c \quad \dots(i)$$

Also, $b + 2c$ is collinear with a .

$$\Rightarrow b + 2c = \mu a \quad \dots(ii)$$

From Eq. (i), we get

$$a + 3b + 6c = (\lambda + 6)c \quad \dots(iii)$$

From Eq. (ii), we get

$$a + 3b + 6c = (1 + 3\mu)a \quad \dots(iv)$$

From Eqs. (iii) and (iv), we get

$$(\lambda + 6)c = (1 + 3\mu)a$$

Since, a is not collinear with c .

$$\therefore \lambda + 6 = 1 + 3\mu = 0$$

From Eq. (iv), we get

$$a + 3b + 6c = 0$$

22. We have, $a \times b + c = 0$

$$\Rightarrow a \times (a \times b) + a \times 6c = 0$$

$$\Rightarrow (a \cdot b)a - (a \cdot a)b + a \times c = 0$$

$$\Rightarrow 3a - 2b + a \times c = 0$$

$$\Rightarrow 2b = 3a + a \times c$$

$$\Rightarrow 2b = 3\hat{j} - 3\hat{k} - 2\hat{i} - \hat{j} - \hat{k}$$

$$\Rightarrow 2b = -2\hat{i} + 2\hat{j} - 4\hat{k}$$

$$\therefore b = -\hat{i} + \hat{j} - 2\hat{k}$$

23. Since, the given vectors are mutually orthogonal.

$$\text{Therefore, } a \cdot b = 2 - 4 + 2 = 0$$

$$a \cdot c = \lambda - 1 + 2\mu = 0 \quad \dots(i)$$

$$\text{and } b \cdot c = 2\lambda + 4 + \mu = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\mu = 2$$

$$\text{and } \lambda = -3$$

$$\text{Hence, } (\lambda, \mu) = (-3, 2)$$

24. Since, $[3u \ p v \ p w] - [p v \ w q u] - [2w \ q v \ q u] = 0$

$$\therefore 3p^2[u \cdot (v \times w)] - pq[v \cdot (w \times u)] - 2q^2[w \cdot (v \times u)] = 0$$

$$\Rightarrow (3p^2 - pq + 2q^2)[u \cdot (v \times w)] = 0$$

$$\text{But } [u \ v \ w] \neq 0$$

$$\Rightarrow 3p^2 - pq + 2q^2 = 0$$

$$\Rightarrow p = q = 0$$

25. Given that, $b = \hat{i} + \hat{j}$ and $c = \hat{j} + \hat{k}$

The equation of bisector of b and c is

$$r = \lambda(b + c)$$

$$= \lambda \left(\frac{\hat{i} + \hat{j}}{\sqrt{2}} + \frac{\hat{j} + \hat{k}}{\sqrt{2}} \right)$$

$$= \frac{\lambda}{\sqrt{2}} (\hat{i} + 2\hat{j} + \hat{k}) \quad \dots(i)$$

Since, point C lies on AB .

$$\therefore \frac{\lambda}{\sqrt{2}} (\hat{i} + 2\hat{j} + \hat{k}) = (\hat{i} + \hat{j}) + \mu(\hat{j} + \hat{k})$$

On equating the coefficient of $\hat{i}$ both sides, we get

$$\frac{\lambda}{\sqrt{2}} = 1$$

$$\Rightarrow \lambda = \sqrt{2}$$

On putting $\lambda = \sqrt{2}$ in Eq. (i), we get

$$r = \hat{i} + 2\hat{j} + \hat{k}$$

Since, vector a represents the same bisector equation of r .

$$\therefore \alpha = 1 \text{ and } \beta = 1$$

Aliter

Since, a , b and c are coplanar.

$$\therefore \begin{vmatrix} \alpha & 2 & \beta \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \alpha(1 - 0) - 2(1 - 0) + \beta(1 - 0) = 0$$

$$\Rightarrow \alpha + \beta = 2, \text{ which is possible for } \alpha = 1 \text{ and } \beta = 1.$$

26. Since, $a = 8b$ and $c = -7b$

$\therefore a$ is parallel to b and c is anti-parallel to b .

$\Rightarrow a$ and c are anti-parallel.

Hence, the angle between a and c is π .

27. Since, $(2\hat{u} \times 3\hat{v})$ is a unit vector.

$$\therefore |2\hat{u} \times 3\hat{v}| = 1$$

$$\Rightarrow 6|\hat{u}| |\hat{v}| |\sin \theta| = 1$$

$$\Rightarrow \sin \theta = \frac{1}{6} \quad [\because |\hat{u}| = |\hat{v}| = 1]$$

Since, θ is an acute angle, so there is exactly one value of θ for which $(2\hat{u} \times 3\hat{v})$ is a unit vector.

28. Since, vectors a , b and c are coplanar.

$$\therefore \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ x & x-2 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 1\{1 - 2(x - 2)\} - 1(-1 - 2x) + 1(x - 2 + x) = 0$$

$$\Rightarrow 1 - 2x + 4 + 1 + 2x + 2x - 2 = 0$$

$$\Rightarrow 2x = -4$$

$$\Rightarrow x = -2$$

29. Since, $(a \times b) \times c = a \times (b \times c)$

$$\therefore (a \cdot c)b - (b \cdot c)a = (a \cdot c)b - (a \cdot b)c$$

$$\Rightarrow (b \cdot c)a = (a \cdot b)c$$

Hence, a is parallel to c .

30. Since, position vectors of A , B and C are $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ and $a\hat{i} - 3\hat{j} + \hat{k}$ respectively.

$$\text{Now, } AC = (a\hat{j} - 3\hat{j} + \hat{k}) - (2\hat{i} - \hat{j} + \hat{k})$$

$$= (a - 2)\hat{i} - 2\hat{j}$$

$$\text{and } BC = (a\hat{i} - 3\hat{j} + \hat{k}) - (\hat{i} - 3\hat{j} - 5\hat{k})$$

$$= (a - 1)\hat{i} + 6\hat{k}$$

Since, ΔABC is right angled at C .

$$\therefore AC \cdot BC = 0$$

$$\Rightarrow \{(a - 2)\hat{i} - 2\hat{j}\} \cdot \{(a - 1)\hat{i} + 6\hat{k}\} = 0$$

$$\Rightarrow (a - 2)(a - 1) = 0$$

$$\therefore a = 1 \text{ and } 2$$

31. Since, $(\mathbf{a} \times \hat{\mathbf{i}}) \cdot (\mathbf{a} \times \hat{\mathbf{i}}) = \begin{vmatrix} \mathbf{a} \cdot \mathbf{a} & \mathbf{a} \cdot \hat{\mathbf{i}} \\ \hat{\mathbf{i}} \cdot \mathbf{a} & 1 \end{vmatrix} = |\mathbf{a}|^2 - a_1^2$
 Similarly, $(\mathbf{a} \times \hat{\mathbf{j}})^2 = |\mathbf{a}|^2 - a_2^2$
 and $(\mathbf{a} \times \hat{\mathbf{k}})^2 = |\mathbf{a}|^2 - a_3^2$
 $\therefore (\mathbf{a} \times \hat{\mathbf{i}})^2 + (\mathbf{a} \times \hat{\mathbf{j}})^2 + (\mathbf{a} \times \hat{\mathbf{k}})^2$
 $= 3|\mathbf{a}|^2 - (a_1^2 + a_2^2 + a_3^2)$
 $= 3|\mathbf{a}|^2 - |\mathbf{a}|^2 = 2|\mathbf{a}|^2 = 2a^2$

32. If $\mathbf{a} + 2\mathbf{b}$ is collinear with $\mathbf{c}$, then
 $\mathbf{a} + 2\mathbf{b} = t\mathbf{c} \quad \dots(i)$

Also, $\mathbf{b} + 3\mathbf{c}$ is collinear with $\mathbf{a}$, then
 $\mathbf{b} + 3\mathbf{c} = \lambda\mathbf{a}$
 $\Rightarrow \mathbf{b} = \lambda\mathbf{a} - 3\mathbf{c} \quad \dots(ii)$

From Eqs. (i) and (ii), we get
 $\mathbf{a} + 2(\lambda\mathbf{a} - 3\mathbf{c}) = t\mathbf{c}$
 $\Rightarrow (\mathbf{a} - 6\mathbf{c}) = t\mathbf{c} - 2\lambda\mathbf{a}$
 On comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$, we get
 $1 = -2\lambda \Rightarrow \lambda = -\frac{1}{2}$

and $-6 = t \Rightarrow t = -6$

From Eq. (i), we get
 $\mathbf{a} + 2\mathbf{b} = -6\mathbf{c}$
 $\Rightarrow \mathbf{a} + 2\mathbf{b} + 6\mathbf{c} = 0$

33. Total force, $\mathbf{F} = (4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}) + (3\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$
 $= 7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$

The particle is displaced from $A(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ to $B(5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}})$. Now, displacement,

$$\mathbf{AB} = (5\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + \hat{\mathbf{k}}) - (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$$

$$= 4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$\therefore$ Work done $= \mathbf{F} \cdot \mathbf{AB}$
 $= (7\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \cdot (4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}})$
 $= 28 + 4 + 8$
 $= 40$ units

34. Given that, $\frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a} = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$

We know that,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

$$\therefore \frac{1}{3}|\mathbf{b}||\mathbf{c}|\mathbf{a} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

On comparing the coefficients of $\mathbf{a}$ and $\mathbf{b}$, we get

$$\frac{1}{3}|\mathbf{b}||\mathbf{c}| = -\mathbf{b} \cdot \mathbf{c} \text{ and } \mathbf{a} \cdot \mathbf{b} = 0$$

$$\Rightarrow \frac{1}{3}|\mathbf{b}||\mathbf{c}| = -|\mathbf{b}||\mathbf{c}|\cos\theta$$

$$\Rightarrow \cos\theta = -\frac{1}{3} \Rightarrow 1 - \sin^2\theta = \frac{1}{9}$$

$$\Rightarrow \sin^2\theta = \frac{8}{9} \Rightarrow \sin\theta = \frac{2\sqrt{2}}{3} \left[\because 0 \leq \theta \leq \frac{\pi}{2} \right]$$

35. Given that, $|\mathbf{a}| = 1, |\mathbf{b}| = 2, |\mathbf{c}| = 3$
 and $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$
 Now, $(\mathbf{a} + \mathbf{b} + \mathbf{c})^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + |\mathbf{c}|^2$
 $+ 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$
 $\Rightarrow 0 = 1^2 + 2^2 + 3^2 + 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a})$
 $\Rightarrow 2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = -14$
 $\Rightarrow \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = -7$

36. Given that, $\mathbf{OA} = 7\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 7\hat{\mathbf{k}}$,
 $\mathbf{OB} = \hat{\mathbf{i}} - 6\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$,
 $\mathbf{OC} = -\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 4\hat{\mathbf{k}}$
 and $\mathbf{OD} = 5\hat{\mathbf{i}} - \hat{\mathbf{j}} + 5\hat{\mathbf{k}}$

Now, $AB = \sqrt{(7-1)^2 + (-4-6)^2 + (7-10)^2}$
 $= \sqrt{36 + 4 + 9} = \sqrt{49} = 7$
 $BC = \sqrt{(1+1)^2 + (-6+3)^2 + (10-4)^2}$
 $= \sqrt{4 + 9 + 36} = \sqrt{49} = 7$
 $CD = \sqrt{(-1-5)^2 + (-3+1)^2 + (4-5)^2}$
 $= \sqrt{36 + 4 + 1} = \sqrt{41}$
 and $DA = \sqrt{(5-7)^2 + (-1+4)^2 + (5-7)^2}$
 $= \sqrt{4 + 9 + 4} = \sqrt{17}$

37. Since, the vector $\hat{\mathbf{i}} + \mathbf{x}\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is doubled in magnitude, so it becomes

$$4\hat{\mathbf{i}} + (4x-2)\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

$$\therefore 2|\hat{\mathbf{i}} + \mathbf{x}\hat{\mathbf{j}} + 3\hat{\mathbf{k}}| = |4\hat{\mathbf{i}} + (4x-2)\hat{\mathbf{j}} + 2\hat{\mathbf{k}}|$$

$$\Rightarrow 2\sqrt{1+x^2+9} = \sqrt{16+(4x-2)^2+4}$$

$$\Rightarrow 40+4x^2=20+(4x-2)^2$$

$$\Rightarrow 3x^2-4x-4=0$$

$$\Rightarrow (x-2)(3x+2)=0$$

$$\Rightarrow x=2, -\frac{2}{3}$$

38. Since, $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \Rightarrow \mathbf{a} + \mathbf{b} = -\mathbf{c}$

$$\Rightarrow (\mathbf{a} + \mathbf{b}) \times \mathbf{c} = -\mathbf{c} \times \mathbf{c}$$

$$\Rightarrow \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$$

Similarly, $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c}$
 $\therefore \mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{c} = \mathbf{c} \times \mathbf{a}$

39. Now, $|(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})| = \sqrt{3^2 + 2^2 + (-6)^2}$
 $= \sqrt{9 + 4 + 36} = \sqrt{49} = 7$

Since, $\lambda(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}})$ is a unit vector.

$$\therefore \lambda = \pm \frac{1}{|3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 6\hat{\mathbf{k}}|} = \pm \frac{1}{7}$$

40. Given, $d = \frac{1}{x}(\mathbf{a} + \mathbf{b} + \mathbf{c})$ and $d = \frac{1}{y}(\mathbf{b} + \mathbf{c} + \mathbf{d})$

$$\therefore \mathbf{a} + \mathbf{b} + \mathbf{c} - x\mathbf{d} = 0$$

and $\mathbf{b} + \mathbf{c} + \mathbf{d} - y\mathbf{d} = 0$
 $\Rightarrow \mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d} = 0$
 $\therefore \frac{1}{xy}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}) = \frac{1}{xy} \times 0 = 0$

41. Given, $\mathbf{a} + 3\mathbf{b}$ is collinear with $\mathbf{c}$.

$$\therefore \mathbf{a} + 3\mathbf{b} = \lambda\mathbf{c}$$

$$\Rightarrow \mathbf{a} + 3\mathbf{b} - \lambda\mathbf{c} = 0 \quad \dots(i)$$

and $3\mathbf{b} + 2\mathbf{c}$ is collinear with $\mathbf{a}$.

$$\therefore 3\mathbf{b} + 2\mathbf{c} = \mu\mathbf{a}$$

$$\Rightarrow 3\mathbf{b} + 2\mathbf{c} - \mu\mathbf{a} = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\mathbf{a} + 3\mathbf{b} - \lambda\mathbf{c} = 3\mathbf{b} + 2\mathbf{c} - \mu\mathbf{a}$$

On equating $\mathbf{c}$, we get

$$\lambda = -2$$

On putting $\lambda = -2$ in Eq. (i), we get

$$\mathbf{a} + 3\mathbf{b} + 2\mathbf{c} = 0$$

42. Given that, $|a| = 2\sqrt{2}$ and $|b| = 3$

The longer vector is

$$5a + 2b + a - 3b \text{ i.e. } 6a - b.$$

Length of one diagonal = $|6a - b|$

$$= \sqrt{36a^2 + b^2 - 2 \times 6|a||b| \cos 45^\circ}$$

$$= \sqrt{36 \times 8 + 9 - 12 \times 2\sqrt{2} \times 3 \times \frac{1}{\sqrt{2}}}$$

$$= \sqrt{288 + 9 - 12 \times 6} = \sqrt{225} = 15$$

Other diagonal is $4a + 5b$.

$$\therefore \text{Length} = \sqrt{(4a)^2 + (5b)^2 + 2 \times |4a||5b| \cos 45^\circ}$$

$$= \sqrt{16 \times 8 + 25 \times 9 + 40 \times 6}$$

$$= \sqrt{593}$$

43. Since, p, q and r are reciprocal vectors of a, b and c , respectively.

$$\therefore p \cdot a = 1, p \cdot b = 0 = p \cdot c$$

$$q \cdot a = 0, q \cdot b = 1, q \cdot c = 0$$

$$\text{and } r \cdot a = 0, r \cdot b = 0, r \cdot c = 1$$

$$\therefore (la + mb + nc) \cdot (lp + mq + nr) = l^2 + m^2 + n^2$$

44. Let first term and common ratio of a GP be α and β , then

$$a = \alpha \cdot \beta^{p-1}, b = \alpha \cdot \beta^{q-1} \text{ and } c = \alpha \cdot \beta^{r-1}$$

$$\therefore \log a = \log \alpha + (p-1) \log \beta$$

$$\log b = \log \alpha + (q-1) \log \beta$$

$$\text{and } \log c = \log \alpha + (r-1) \log \beta$$

Now, dot product of the given two vectors is

$$(q-r) \log a + (r-p) \log b + (p-q) \log c$$

$$= (q-r)[\log \alpha + (p-1) \log \beta] + (r-p)$$

$$[\log \alpha + (q-1) \log \beta] + (p-q)[\log \alpha + (r-1) \log \beta]$$

$$= \log \alpha [q-r+r-p+p-q] + \log \beta [(p-q)(q-r) + (r-p)(q-1) + (p-q)(r-1)]$$

$$= 0 + 0 = 0$$

So, two vectors are perpendicular.

45. Given, $|\hat{a}| = |\hat{b}| = 1$ and $\theta = \frac{\pi}{3}$

$$\text{Now, } |\hat{a} + \hat{b}|^2 = |\hat{a}|^2 + |\hat{b}|^2 + 2|\hat{a}||\hat{b}| \cos \theta$$

$$= 1^2 + 1^2 + 2 \times 1 \times 1 \times \cos \frac{\pi}{3}$$

$$= 1 + 1 + 2 \times \frac{1}{2} = 1 + 1 + 1 = 3$$

$$\Rightarrow |\hat{a} + \hat{b}| = \sqrt{3}$$

$$\therefore |\hat{a} + \hat{b}| > 1$$

46. Given, $a \cdot b = 0$

$$\text{Now, } (a+b) \cdot a = |a+b||a| \cos 60^\circ$$

$$a \cdot b + b \cdot a = |a+b||a| \frac{1}{2}$$

$$\Rightarrow |a|^2 + 0 = \frac{|a+b||a|}{2}$$

$$\Rightarrow 2|a| = |a+b|$$

On squaring both sides, we get

$$4|a|^2 = |a|^2 + |b|^2 + 2a \cdot b$$

$$\Rightarrow 3|a|^2 = |b|^2 + 0 [\because |a||b| \cos \theta = a \cdot b = 0]$$

$$\therefore \sqrt{3}|a| = |b|$$

47. Let position vector of the vertices be $OA = \hat{i} + \hat{j}$,

$$OB = \hat{j} + \hat{k} \text{ and } OC = \hat{i} + \hat{k}.$$

$$\text{Now, } AB = -\hat{i} + \hat{k} \text{ and } AC = \hat{k} - \hat{j}$$

$$\therefore \cos \theta = \frac{(AB) \cdot (AC)}{|AB||AC|} = \frac{(-\hat{i} + \hat{k}) \cdot (\hat{k} - \hat{j})}{\sqrt{1^2 + 1^2} \sqrt{1^2 + 1^2}}$$

$$= \frac{(\hat{k})^2}{\sqrt{2}\sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

48. Given, $a = \hat{i} - 2\hat{j} + 3\hat{k}$, $a \cdot b = |b|^2$ and $|a - b| = \sqrt{7}$

$$\Rightarrow |a - b|^2 = 7$$

$$\Rightarrow |a|^2 + |b|^2 - 2a \cdot b = 7$$

$$\Rightarrow (\sqrt{1+4+9})^2 + |b|^2 - 2|b|^2 = 7$$

$$\Rightarrow 14 - |b|^2 = 7$$

$$\Rightarrow |b|^2 = 7$$

$$\therefore |b| = \sqrt{7}$$

49. According to the given condition, each vector is perpendicular to the sum of two vectors.

$$\therefore a \cdot (b+c) = 0$$

$$b \cdot (a+c) = 0$$

$$\text{and } c \cdot (a+b) = 0$$

$$\Rightarrow a \cdot b + a \cdot c = 0, b \cdot a + b \cdot c = 0$$

$$\text{and } c \cdot a + c \cdot b = 0$$

$$\Rightarrow 2(a \cdot b + b \cdot c + c \cdot a) = 0$$

$$\Rightarrow a \cdot b + b \cdot c + c \cdot a = 0 \quad \dots (i)$$

$$\text{Now, } |a+b+c|^2 = |a|^2 + |b|^2 + |c|^2 + 2(a \cdot b + b \cdot c + c \cdot a)$$

$$= |a|^2 + |b|^2 + |c|^2 + 2(0) \quad [\text{from Eq. (i)}]$$

$$= |a|^2 + |b|^2 + |c|^2$$

50. Given, $|u| = 3, |v| = 4$ and $|w| = 5$

$$\text{Also, } u + v + w = 0$$

On squaring both sides, we get

$$|u|^2 + |v|^2 + |w|^2 + 2(u \cdot v + v \cdot w + w \cdot u) = 0$$

$$\Rightarrow 3^2 + 4^2 + 5^2 + 2(u \cdot v + v \cdot w + w \cdot u) = 0$$

$$\Rightarrow 9 + 16 + 25 + 2(u \cdot v + v \cdot w + w \cdot u) = 0$$

$$\therefore u \cdot v + v \cdot w + w \cdot u = -\frac{50}{2} = -25$$

51. Given, $|\hat{a}| = 2, |\hat{b}| = 3$ and $|\hat{c}| = 4$

$$\therefore |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2$$

$$= \hat{a}^2 + \hat{b}^2 - 2\hat{a} \cdot \hat{b} + \hat{b}^2 + \hat{c}^2 - 2\hat{b} \cdot \hat{c}$$

$$+ \hat{c}^2 + \hat{a}^2 - 2\hat{a} \cdot \hat{c}$$

$$= 2(\hat{a}^2 + \hat{b}^2 + \hat{c}^2) - 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a})$$

$$\Rightarrow 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) = 2(\hat{a}^2 + \hat{b}^2 + \hat{c}^2)$$

$$- (|\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2)$$

$$\text{We know that, } (\hat{a} + \hat{b} + \hat{c})^2 \geq 0$$

$$\Rightarrow \hat{a}^2 + \hat{b}^2 + \hat{c}^2 + 2(\hat{a} \cdot \hat{b} + \hat{b} \cdot \hat{c} + \hat{c} \cdot \hat{a}) \geq 0$$

$$\Rightarrow \hat{a}^2 + \hat{b}^2 + \hat{c}^2 + 2(\hat{a}^2 + \hat{b}^2 + \hat{c}^2)$$

$$- (|\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2) \geq 0$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \leq 3(\hat{a}^2 + \hat{b}^2 + \hat{c}^2)$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \leq 3(2^2 + 3^2 + 4^2)$$

$$\Rightarrow |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \leq 3(4 + 9 + 16)$$

$$\therefore |\hat{a} - \hat{b}|^2 + |\hat{b} - \hat{c}|^2 + |\hat{c} - \hat{a}|^2 \leq 87$$

52. We have, $a = \hat{i} + 2\hat{j} + 2\hat{k}$, $b = \hat{i} - 2\hat{j} + 2\hat{k}$

Clearly, $|a| = \sqrt{1+4+4} = 3$

and $|b| = \sqrt{1+4+4} = 3$

$\therefore a \cdot b = |a||b|\cos\theta$

$\Rightarrow \cos\theta = \frac{1 \cdot 1 + 2 \cdot (-2) + 2 \cdot 2}{(3)^2}$

$\Rightarrow \cos\theta = \frac{1-4+4}{9} = \frac{1}{9}$

$\therefore \theta = \cos^{-1}\left(\frac{1}{9}\right)$

53. $|\hat{a} + \hat{b}| = \sqrt{|\hat{a} + \hat{b}|^2}$
 $= \sqrt{|\hat{a}|^2 + |\hat{b}|^2 + 2|\hat{a}||\hat{b}|\cos 60^\circ}$
 $= \sqrt{1^2 + 1^2 + 2 \cdot 1 \cdot 1 \cdot \frac{1}{2}} = \sqrt{1+1+1} = \sqrt{3}$

$\therefore |\hat{a} + \hat{b}| > 1$

54. Let two vectors be $a = \hat{i} - \hat{j}$ and $b = \hat{i} + 2\hat{j}$.

Again, let third unit vector be c .

Since, c is coplanar with a and b .

$\therefore c = xa + yb = x(\hat{i} - \hat{j}) + y(\hat{i} + 2\hat{j})$

$\Rightarrow c = (x+y)\hat{i} + (-x+2y)\hat{j} \quad \dots(i)$

Also, c is perpendicular to a .

$\Rightarrow a \cdot c = 0$

$\Rightarrow (\hat{i} - \hat{j}) \cdot \{(x+y)\hat{i} + (-x+2y)\hat{j}\} = 0$

$\Rightarrow (x+y) - (-x+2y) = 0$

$\Rightarrow x+y+x-2y = 0$

$\Rightarrow 2x-y = 0 \Rightarrow y = 2x$

On putting this value of y in Eq. (i), we get

$c = (x+2x)\hat{i} + (-x+4x)\hat{j}$

$\Rightarrow c = 3x\hat{i} + 3x\hat{j} \quad \dots(ii)$

But c is a unit vector.

So, $|c| = 1$

$\Rightarrow (3x)^2 + (3x)^2 = 1 \Rightarrow 9x^2 + 9x^2 = 1$

$\Rightarrow x^2 = \frac{1}{18} \Rightarrow x = \pm \frac{1}{3\sqrt{2}}$

On putting this value of x in Eq. (ii), we get

$c = 3 \cdot \frac{1}{3\sqrt{2}}\hat{i} + 3 \cdot \frac{1}{3\sqrt{2}}\hat{j}$

$\therefore c = \frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$

55. Required unit vector = $\frac{(\hat{i} - \hat{j}) \times (\hat{i} + \hat{j})}{|(\hat{i} - \hat{j}) \times (\hat{i} + \hat{j})|}$
 $= \frac{\hat{k} + \hat{k}}{2} = \frac{2\hat{k}}{2} = \hat{k}$

56. $|a \times b|^2 = |a|^2 |b|^2 - (a \cdot b)^2 = 16 - 4 = 12$

and $|c|^2 = (2a \times b - 3b)^2$

$= 4|a \times b|^2 + 9|b|^2$

$= 4 \cdot 12 + 9 \cdot 16 = 192$

$\Rightarrow |c| = 8\sqrt{3}$

Now, $b \cdot c = b \cdot (2a \times b - 3b) = -3|b|^2 = -48$

$\therefore \cos\theta = \frac{b \cdot c}{|b||c|} = \frac{-48}{4 \cdot 8\sqrt{3}} = -\frac{\sqrt{3}}{2}$

$\Rightarrow \theta = \frac{5\pi}{6}$

57. Let adjacent sides of a parallelogram be

$a = \hat{i} + \hat{k}$

and

$b = 2\hat{i} + \hat{j} + \hat{k}$

$\therefore$ Area of parallelogram

$= |a \times b| = \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix} \right|$

$= |\hat{i}(0-1) - \hat{j}(1-2) + \hat{k}(1-0)|$

$= |-\hat{i} + \hat{j} + \hat{k}|$

$= \sqrt{(-1)^2 + (1)^2 + (1)^2} = \sqrt{3}$ sq units

58. Area of the triangle = $\frac{1}{2}|a \times b| = \frac{1}{2}|a||b|\sin\theta$

$= \frac{1}{2} \left[3 \times 5 \times \sin \frac{\pi}{6} \right] \quad [\because |a| = \sqrt{1+2^2+2^2} = 3]$

$= \frac{1}{2} \left[15 \times \frac{1}{2} \right] = \frac{15}{4}$ sq units $\left[\because \sin \frac{\pi}{6} = \frac{1}{2} \right]$

59. Given, $\hat{a} + \hat{b} + \hat{c} = 0$

Taking cross-product of both sides, we get

$(\hat{a} + \hat{b} + \hat{c}) \times \hat{a} = 0 \times \hat{a}$

$\Rightarrow \hat{a} \times \hat{a} + \hat{b} \times \hat{a} + \hat{c} \times \hat{a} = 0$

$\Rightarrow 0 + \hat{b} \times \hat{a} + \hat{c} \times \hat{a} = 0$

$\Rightarrow -\hat{a} \times \hat{b} + \hat{c} \times \hat{a} = 0$

$\hat{a} \times \hat{b} = \hat{c} \times \hat{a}$

Similarly, $\hat{b} \times \hat{c} = \hat{c} \times \hat{a}$

$\therefore \hat{a} \times \hat{b} = \hat{b} \times \hat{c} = \hat{c} \times \hat{a} \neq 0$

60. Let $r = x_1\hat{i} + x_2\hat{j} + x_3\hat{k}$

Now,

$c \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -1 & 1 & 0 \end{vmatrix} = (0-3)\hat{i} - (0+3)\hat{j} + (1+2)\hat{k}$

$= -3\hat{i} - 3\hat{j} + 3\hat{k}$

and $r \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & x_2 & x_3 \\ -1 & 1 & 0 \end{vmatrix}$

$= (0-x_3)\hat{i} - (0+x_3)\hat{j} + (x_1+x_2)\hat{k}$

$= -x_3\hat{i} - x_3\hat{j} + (x_1+x_2)\hat{k}$

Also, $r \cdot a = 0$

$\Rightarrow (x_1\hat{i} + x_2\hat{j} + x_3\hat{k}) \cdot (-\hat{i} - \hat{k}) = 0$

$\Rightarrow -x_1 - x_3 = 0 \quad \dots(i)$

But $r \times b = c \times b$

$\therefore -x_3\hat{i} - x_3\hat{j} + \hat{k}(x_1+x_2) = -3\hat{i} - 3\hat{j} + 3\hat{k}$

On comparing both sides, we get

$x_3 = 3$ and $x_1 + x_2 = 3 \quad \dots(ii)$

On solving Eqs. (i) and (ii), we get

$x_1 = -x_3 = -3$

and $x_2 = 6$

Now, $r \cdot b = (x_1\hat{i} + x_2\hat{j} + x_3\hat{k}) \cdot (-\hat{i} + \hat{j})$

$= -x_1 + x_2$

$= -(-3) + 6 = 9$

61. We have, $|a| = |b| = 5$ and angle between a and b is $\frac{\pi}{4}$.

Now, area of the triangle constructed on the vectors

$$= \frac{1}{2} |(a - 2b) \times (3a + 2b)|$$

$$= \frac{1}{2} |3(a \times a) + 2(a \times b) - 6(b \times a) - 4(b \times b)|$$

$$= \frac{1}{2} |2(a \times b) + 6(a \times b)|$$

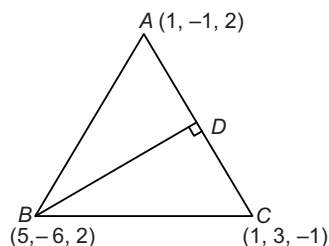
$$[\because a \times a = b \times b = 0 \text{ and } a \times b = -b \times a]$$

$$= \frac{1}{2} |8(a \times b)| = \frac{8}{2} |a \times b|$$

$$= 4 |a| |b| \left| \sin \frac{\pi}{4} \right| = 4 \cdot 5 \cdot 5 \cdot \frac{1}{\sqrt{2}}$$

$$= 2\sqrt{2} \times 5 \times 5 = 50\sqrt{2} \text{ sq units}$$

62. We have vertices $A(1, -1, 2)$, $B(5, -6, 2)$ and $C(1, 3, -1)$ of $\triangle ABC$.



To find, $n = |BD|$

Clearly, $AB = 4\hat{i} - 5\hat{j} + 0\hat{k}$

and $AC = 0\hat{i} + 4\hat{j} - 3\hat{k}$

Now, area of $\triangle ABC = \frac{1}{2} |AB \times AC|$

$$= \frac{1}{2} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -5 & 0 \\ 0 & 4 & -3 \end{vmatrix}$$

$$= \frac{1}{2} |(15)\hat{i} - (-12)\hat{j} + (16)\hat{k}|$$

$$= \frac{1}{2} |15\hat{i} + 12\hat{j} + 16\hat{k}|$$

$$= \frac{1}{2} \sqrt{(15)^2 + (12)^2 + (16)^2} = \frac{1}{2} \times 25 \quad \dots (i)$$

Also, area of $\triangle ABC = \frac{1}{2} |AC| \times |BD|$

$$= \frac{1}{2} \times \sqrt{0^2 + 4^2 + 3^2} \times |BD|$$

$$= \frac{5}{2} \times |BD| \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\frac{25}{2} = \frac{5}{2} \times |BD|$$

$\therefore |BD| = 5$

63. Let $a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$$\therefore a \times \hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 1 & 0 & 0 \end{vmatrix}$$

$$= (0)\hat{i} - (-a_3)\hat{j} + (-a_2)\hat{k} = a_3\hat{j} - a_2\hat{k}$$

$$a \times \hat{j} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 0 & 1 & 0 \end{vmatrix} = -a_3\hat{i} + a_1\hat{k}$$

$$\text{and } a \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 1 \end{vmatrix} = -a_2\hat{i} - a_1\hat{j}$$

$$\therefore |a \times \hat{i}|^2 + |a \times \hat{j}|^2 + |a \times \hat{k}|^2$$

$$= |a_3\hat{j} - a_2\hat{k}|^2 + |-a_3\hat{i} + a_1\hat{k}|^2 + |-a_2\hat{i} - a_1\hat{j}|^2$$

$$= a_3^2 + a_2^2 + a_3^2 + a_1^2 + a_2^2 + a_1^2$$

$$= 2(a_1^2 + a_2^2 + a_3^2) = 2|a|^2$$

64. Since, $a \times b = c$ and $b \times c = a$

So, a , b and c form a right handed system and orthogonal in pairs.

65. Now, $r \cdot a = \alpha(a \cdot b \times c) + \beta(a \cdot c \times a) + \gamma(a \cdot a \times b)$
- $$= \alpha[a \cdot b \times c] + 0 + 0$$

Similarly, $r \cdot b = \beta[a \cdot b \times c]$

and $r \cdot c = \gamma[a \cdot b \times c]$

$$\therefore \frac{1}{2} r \cdot (a + b + c) = \frac{1}{2} [(r \cdot a) + (r \cdot b) + (r \cdot c)]$$

$$= \frac{1}{2} (\alpha + \beta + \gamma) [a \cdot b \times c]$$

$$= \frac{1}{2} (\alpha + \beta + \gamma) \times 2 = \alpha + \beta + \gamma$$

66. $[(a - b)(b - c)(c - a)]$

$$= (a - b) \cdot [(b - c) \times (c - a)]$$

$$= (a - b) \cdot [b \times c - b \times a - c \times c + c \times a]$$

$$= (a - b) \cdot [b \times c - b \times a + c \times a] \quad [\because c \times c = 0]$$

$$= a \cdot [b \times c] - a \cdot [b \times a] + a \cdot [c \times a]$$

$$= [a \cdot b \times c] - b \cdot [b \times c] + b \cdot [b \times a] - b \cdot [c \times a]$$

$$= [a \cdot b \times c] - 0 + 0 - 0 - [b \cdot c \times a]$$

$$= [a \cdot b \times c] - [a \cdot b \times c] = 0$$

67. Given, $r \cdot a = 0 \Rightarrow r \perp a$

$$r \cdot b = 0 \Rightarrow r \perp b$$

and $r \cdot c = 0 \Rightarrow r \perp c$

So, a , b and c are coplanar.

$$\therefore [a \cdot b \times c] = 0$$

68. Given, $a = x\hat{i} + 2\hat{j}$, $b = y\hat{j} + 3\hat{k}$ and $c = x\hat{i} + y\hat{j} + z\hat{k}$

Now, $a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & 2 & 0 \\ 0 & y & 3 \end{vmatrix}$

$$= (6 - 0)\hat{i} - (3x - 0)\hat{j} + (xy - 0)\hat{k}$$

$$= 6\hat{i} - 3x\hat{j} + xy\hat{k}$$

$$\Rightarrow 6\hat{i} - 3x\hat{j} + xy\hat{k} = 3\hat{i} - 3\hat{j} + \hat{k}$$

On equating the coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$, we get

$$z = 6, x = 1 \text{ and } xy = 1$$

$$\therefore y = 1$$

$$\Rightarrow a = \hat{i} + 2\hat{j}, b = \hat{j} + 3\hat{k} \text{ and } c = \hat{i} + \hat{j} + 6\hat{k}$$

$$\therefore [a \cdot b \times c] = \begin{vmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 1 & 1 & 6 \end{vmatrix} = 1(6 - 3) - 2(0 - 3) + 0$$

$$= 3 + 6 = 9$$

$$\begin{aligned}
 69. (u + v - w) \cdot [(u - v) \times (v - w)] \\
 &= (u + v - w) \cdot [v \times v - u \times w - v \times v + v \times w] \\
 &= (u + v - w) \cdot [u \times v - u \times w + v \times w] \quad [\because v \times v = 0] \\
 &= u \cdot (u \times v) - u \cdot (u \times w) + u \cdot (v \times w) + v \cdot (u \times v) \\
 &\quad - v \cdot (u \times w) + v \cdot (v \times w) - w \cdot (u \times v) \\
 &\quad + w \cdot (u \times w) - w \cdot (v \times w) \\
 &= 0 - 0 + [u \cdot v \cdot w] + 0 - [v \cdot u \cdot w] + 0 - [w \cdot u \cdot v] + 0 - 0 \\
 &= u \cdot (v \times w)
 \end{aligned}$$

$$70. a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{i} - 2\hat{j} + \hat{k}$$

$$\Rightarrow |a \times b| = 3 \text{ and } |c - a| = 2\sqrt{2}$$

On squaring both sides, we get

$$c^2 + a^2 - 2c \cdot a = 8$$

$$\Rightarrow c^2 + 9 - 2|c| = 8 \Rightarrow |c| = 1$$

$$\therefore |(a \times b) \times c| = |a \times b| |c| \sin 30^\circ = \frac{3}{2}$$

$$71. \text{ Now, } r \cdot a = r \cdot b$$

$$\Rightarrow r \cdot (a - b) = 0$$

$$\text{Similarly, } r \cdot (b - c) = 0$$

Since, r is parallel to $(a - b) \times (b - c)$.

So, r is parallel to $a \times b + c \times a + b \times c$.

$$72. \text{ Let the vertices be}$$

$$OA = \hat{i} + \hat{j} + \hat{k}, OB = 2\hat{i} + 3\hat{j},$$

$$OC = 3\hat{i} + 5\hat{j} - 2\hat{k}$$

$$\text{and } OD = \hat{k} - \hat{j}$$

$$\therefore AB = \hat{i} + 2\hat{j} - \hat{k}, BC = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$CD = -3\hat{i} - 6\hat{j} + 3\hat{k} = -3(\hat{i} + 2\hat{j} - \hat{k})$$

$$\text{and } DA = \hat{i} + 2\hat{j}$$

It is clear that, $|AB| \neq |BC| \neq |CA| \neq |DA|$

Also, $AB \parallel CD$

Hence, figure formed by four points is a trapezium.

$$73. \text{ Given, } a + b = -c$$

On squaring both sides, we get

$$(a + b)^2 = (-c)^2$$

$$\Rightarrow |a|^2 + |b|^2 + 2|a| \cdot |b| \cos \theta = |c|^2$$

$$\Rightarrow (3)^2 + (5)^2 + 2 \times 3 \times 5 \cos \theta = 49$$

$$\Rightarrow 30 \cos \theta = 15 \Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

$$74. \therefore |a \times b|^2 + |a \cdot b|^2 = 144$$

$$\Rightarrow |a|^2 \cdot |b|^2 (\sin^2 \theta + \cos^2 \theta) = 144$$

$$\Rightarrow 16|b|^2 = 144$$

$$\therefore |b| = 3$$

$$75. \text{ Let } T_1 = (a + b) \cdot p = a \cdot p + b \cdot p$$

$$= \frac{a \cdot (b \times c)}{[a \ b \ c]} + \frac{b \cdot (b \times c)}{[a \ b \ c]}$$

$$= \frac{[a \ b \ c]}{[a \ b \ c]} + \frac{[b \ b \ c]}{[a \ b \ c]} = 1 + 0 = 1$$

$$\text{Similarly, } T_2 = (b + c) \cdot q = 1$$

$$T_3 = (c + a) \cdot r = 1$$

$$\therefore T_1 + T_2 + T_3 = 3$$

$$76. [a \times b \ b \times c \ c \times a] = [a \ b \ c]^2 = 4^2 = 16$$

$$77. \text{ Given, } \hat{a} \times (\hat{b} \times \hat{c}) = \frac{1}{2} \hat{b}$$

$$\therefore (\hat{a} \cdot \hat{c}) \hat{b} - (\hat{a} \cdot \hat{b}) \hat{c} = \frac{\hat{b}}{2}$$

$$\Rightarrow \left(\hat{a} \cdot \hat{c} - \frac{1}{2} \right) \hat{b} - (\hat{a} \cdot \hat{b}) \hat{c} = 0$$

$$\therefore \hat{a} \cdot \hat{c} - \frac{1}{2} = 0$$

$$\Rightarrow \hat{a} \cdot \hat{c} = \frac{1}{2}$$

$$\Rightarrow |\hat{a}| |\hat{c}| \cos \theta = \frac{1}{2}$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

$$78. \text{ Let the required unit vector be } \hat{c} = x\hat{i} + y\hat{j}.$$

$$\therefore |\hat{c}| = 1$$

$$\Rightarrow x^2 + y^2 = 1$$

Also, $\hat{c}$ makes an angle 30° with the vector $\hat{i} + \hat{j}$.

$$\therefore \cos 30^\circ = \frac{x + y}{\sqrt{1^2 + 1^2}}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{x + y}{\sqrt{2}}$$

$$\Rightarrow x + y = \frac{\sqrt{6}}{2} \quad \dots(i)$$

Now, $\hat{c}$ makes an angle 60° with the vector $\hat{i} - \hat{j}$.

$$\therefore \cos 60^\circ = \frac{x - y}{\sqrt{1^2 + 1^2}}$$

$$\Rightarrow \frac{1}{2} = \frac{x - y}{2} \Rightarrow x - y = \frac{1}{\sqrt{2}} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = \frac{\sqrt{6} + \sqrt{2}}{4}$$

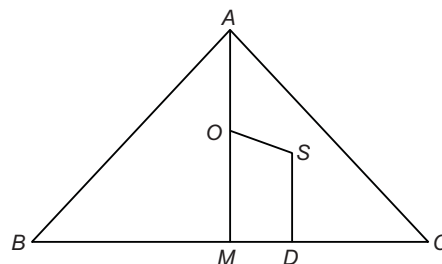
$$\text{and } y = \frac{\sqrt{6} - \sqrt{2}}{4}$$

$\therefore$ Required unit vector is

$$\hat{c} = \frac{1}{4} [(\sqrt{6} + \sqrt{2})\hat{i} + (\sqrt{6} - \sqrt{2})\hat{j}]$$

$$79. \text{ From the geometry, } 2SD = AO, \text{ where } D \text{ is the mid-point of } BC.$$

$$\therefore SA + SB + SC = SA + SD + DB + SD + DC$$



$$= SA + 2SD \quad [\because DB + DC = 0]$$

$$= SA + AO = SO$$

$$\therefore SA + SB + SC = SO$$

80. Required orthogonal unit vector

$$= \left[\cos\left(\frac{\pi}{2} + \alpha\right), \sin\left(\frac{\pi}{2} + \alpha\right) \right] \\ = (-\sin \alpha, \cos \alpha)$$

81. Given, area = $|a \times b| = 15$ sq units

If the sides are $(3a + 2b)$ and $(a + 3b)$, then

$\therefore$ Area of parallelogram

$$= |3a + 2b| \times |a + 3b| = 7|a \times b| \\ = 7 \times 15 \\ = 105 \text{ sq units}$$

82. The vector $\hat{c}$ perpendicular to $a = \hat{i} + \hat{j} + \hat{k}$ and $b = 2\hat{i} + \hat{j} + 3\hat{k}$ is

$$a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = 2\hat{i} - \hat{j} - \hat{k}$$

$$\therefore \text{Unit vector } (\hat{c}) = \frac{a \times b}{|a \times b|} = \frac{1}{\sqrt{6}}(2\hat{i} - \hat{j} - \hat{k})$$

83.

$$a \cdot \left\{ \frac{b \times c}{3b \cdot (c \times a)} \right\} - b \cdot \left\{ \frac{c \times a}{2c \cdot (a \times b)} \right\} = \frac{[a b c]}{3[a b c]} - \frac{[b c a]}{2[b c a]} \\ = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$$

84. Consider, $a \cdot (b \times c) = 0$

$$\Rightarrow b \cdot (c \times a) = 0$$

$$\Rightarrow c \cdot (a \times b) = 0$$

$$\therefore c \parallel a$$

85. Given, $a \perp b \Rightarrow a \cdot b = 0$

$$\text{Now, } a \times [a \times \{a \times (a \times b)\}]$$

$$= a \times \{a \times (a \cdot b)a - (a \cdot a)b\}$$

$$= a \times [a \times \{0 - (a \cdot a)b\}]$$

$$= a \times \{-(a \cdot a)(a \times b)\} = a \times \{(a \cdot a)(b \times a)\}$$

$$= (a \cdot a)\{a \times (b \times a)\}$$

$$= (a \cdot a)\{(a \cdot a)b - (a \cdot b)a\}$$

$$= (a \cdot a)\{(a \cdot a)b - 0\} = (a \cdot a)^2 b \quad [\because (a \cdot a) = |a|^2]$$

$$= (|a|^2)^2 b = |a|^4 b$$

86. Given, $|p + q| = 6$

$$\Rightarrow |p + q|^2 = 36$$

$$\Rightarrow p^2 + q^2 + 2p \cdot q = 36$$

$$\text{Similarly, } q^2 + r^2 + 2q \cdot r = 48$$

$$\text{and } r^2 + p^2 + 2r \cdot p = 16$$

On adding all, we get

$$2(p^2 + q^2 + r^2 + p \cdot q + q \cdot r + r \cdot p)$$

$$\Rightarrow 2(p^2 + q^2 + r^2) = 100 \quad [\because p \cdot q + q \cdot r + r \cdot p = 0]$$

$$\Rightarrow p^2 + q^2 + r^2 = 50$$

$$\Rightarrow |p + q + r|^2 = 50$$

$$\Rightarrow |p + q + r| = 5\sqrt{2}$$

$$87. (\hat{i} + \hat{j} + 2\hat{k}) \cdot \left(\frac{m\hat{i} + 2\hat{j} + 3\hat{k}}{\sqrt{13 + m^2}} \right) = 2$$

$$\Rightarrow m + 2 + 6 = 2\sqrt{13 + m^2}$$

$$\Rightarrow (m + 8)^2 = 4(13 + m^2)$$

$$\Rightarrow m^2 + 16m + 64 = 4m^2 + 52$$

$$\Rightarrow 3m^2 - 16m - 12 = 0$$

$$\Rightarrow (3m + 2)(m - 6) = 0$$

$$\therefore m = 6, -\frac{2}{3}$$

$$88. a \times b = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 1 \\ 1 & -2 & 2 \end{vmatrix} = \hat{i}(4 + 2) - \hat{j}(4 - 1) + \hat{k}(-4 - 2)$$

$$= 6\hat{i} - 3\hat{j} - 6\hat{k}$$

$$\Rightarrow |a \times b| = \sqrt{36 + 9 + 36} \\ = \sqrt{81} = 9$$

$$\therefore \text{Required vector} = \pm 6 \frac{a \times b}{|a \times b|} \\ = \pm \frac{6}{9}(6\hat{i} - 3\hat{j} - 6\hat{k}) \\ = \pm 2(2\hat{i} - \hat{j} - 2\hat{k})$$

$$89. |a \times b|^2 + (a \cdot b)^2 = |a|^2 |b|^2$$

$$\Rightarrow |a \times b|^2 = 25 \times 36 - (25)^2$$

$$= 25(36 - 25)$$

$$= 25 \times 11$$

$$\therefore |a \times b| = 5\sqrt{11}$$

$$90. \therefore \begin{vmatrix} 2 & 1 & 4 \\ 4 & -2 & 3 \\ 2 & -3 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow 2(2\lambda + 9) - 1(-4\lambda - 6) + 4(-12 + 4) = 0$$

$$\Rightarrow 4\lambda + 18 + 4\lambda + 6 - 48 + 16 = 0$$

$$\Rightarrow 8\lambda = 8$$

$$\therefore \lambda = 1$$

$$91. a \times c = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & -1 & -1 \end{vmatrix} = \hat{i}(-1 - 1) - \hat{j}(0 + 1) + \hat{k}(0 - 1)$$

$$= -2\hat{i} - \hat{j} - \hat{k}$$

$$\text{Given, } a \times b + c = 0$$

$$\Rightarrow a \times (a \times b) + a \times c = 0$$

$$\Rightarrow (a \cdot b)a - (a \cdot a)b = -a \times c$$

$$\Rightarrow 3a - 2b = -a \times c$$

$$\Rightarrow b = \frac{3a + a \times c}{2}$$

$$\Rightarrow b = \frac{3\hat{j} - 3\hat{k} - 2\hat{i} - \hat{j} - \hat{k}}{2}$$

$$= \frac{-2\hat{i} + 2\hat{j} - 4\hat{k}}{2} = -\hat{i} + \hat{j} - 2\hat{k}$$

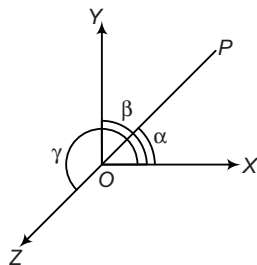
14

Three Dimensional Geometry

Direction Cosines

If α, β and γ are the angles which a directed line segment OP makes with the positive directions of the coordinate axes OX, OY and OZ respectively, then $\cos \alpha, \cos \beta$ and $\cos \gamma$ are known as the direction cosines of OP and are generally denoted by the letters l, m and n respectively.

i.e. $l = \cos \alpha, m = \cos \beta, n = \cos \gamma$



Properties of Direction Cosines

- i. If OP is a directed line segment with direction cosines l, m and n such that $OP = r$. Then, the coordinates of P are lr, mr and nr .
- ii. Sum of squares of direction cosines are always unity i.e. $l^2 + m^2 + n^2 = 1$.
- iii. Parallel lines have same direction cosines.
- iv. Direction cosines of a line are always unique.
- v. $0 \leq \alpha, \beta, \gamma \leq \pi$

Chapter Snapshot

- Direction Cosines
- Direction Ratios
- Angle between Two Vectors
- Straight Line in Space
- Angle between Two Lines
- Shortest Distance between Two Skew-lines
- Perpendicular Distance
- Reflection or Image of a Point in a Straight Line
- Plane
- Planes Bisecting the Angle between Two Given Planes
- Line and Plane
- Sphere

Direction Ratios

Let l, m, n be direction cosines of a line and a, b, c be three numbers such that $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$. Then,

direction ratios of the line are proportional to a, b and c .

Relation between Direction Cosines and Direction Ratios

If the direction ratios of a line are proportional to a, b and c , then its direction cosines are

$$l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}}$$

$$m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}}$$

and
$$n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- The direction cosines of a line are defined as the direction cosines of any directed line segment on it.
- Direction ratios of a line are not unique.
- The direction ratios of a line joining the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are $(x_2 - x_1), (y_2 - y_1), (z_2 - z_1)$ and direction cosines are $\frac{x_2 - x_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}, \frac{y_2 - y_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}, \frac{z_2 - z_1}{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}$.
- If $\mathbf{r} = a\hat{i} + b\hat{j} + c\hat{k}$ is a vector having direction cosines l, m and n , then

$$l = \frac{a}{|\mathbf{r}|}, m = \frac{b}{|\mathbf{r}|} \text{ and } n = \frac{c}{|\mathbf{r}|}$$

- If direction ratios of $\mathbf{r}$ are a, b and c , then

$$\mathbf{r} = \frac{|\mathbf{r}|}{\sqrt{a^2 + b^2 + c^2}} (a\hat{i} + b\hat{j} + c\hat{k})$$

Example 1. If a line OP makes with X -axis an angle of measure 120° and with Y -axis of measure 60° , then the angle made with the line with Z -axis is

- (a) 45° (b) 120°
(c) 60° (d) None of the above

Sol. (a) We have, $\alpha = 120^\circ$ and $\beta = 60^\circ$

$$\therefore \cos \alpha = \cos 120^\circ = -\frac{1}{2}$$

$$\text{and } \cos \beta = \cos 60^\circ = \frac{1}{2}$$

$$\text{But } \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\therefore \left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \cos^2 \gamma = 1$$

$$\Rightarrow \cos^2 \gamma = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow \cos \gamma = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \gamma = 45^\circ \text{ or } 135^\circ$$

Hence, the angle made with the line with Z -axis is 45° .

Example 2. The direction cosines of a line equally inclined to the axes are

(a) $\frac{1}{3}, \frac{1}{3}, \frac{1}{3}$ (b) $-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}$

(c) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ (d) None of these

Sol. (c) If a line makes angles α, β and γ with the axes, then we have $\alpha = \beta = \gamma$.

$$\therefore \cos \alpha = \cos \beta = \cos \gamma \Rightarrow l = m = n$$

$$\therefore l^2 + m^2 + n^2 = 1$$

$$\therefore l^2 + l^2 + l^2 = 1 \Rightarrow 3l^2 = 1$$

$$\Rightarrow l^2 = \frac{1}{3} \Rightarrow l = \pm \frac{1}{\sqrt{3}}$$

Hence, the direction cosines of a line are

$$\left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}}\right)$$

Example 3. If a line has the direction ratios $-18, 12, -4$, then its direction cosines are

(a) $\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$ (b) $\frac{-9}{13}, \frac{6}{13}, \frac{-2}{13}$

(c) $\frac{9}{13}, \frac{-6}{13}, \frac{-2}{13}$ (d) None of these

Sol. (a) If a, b and c are the direction ratios of a line, then direction cosines are

$$\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

Given direction ratios are $-18, 12, -4$.

Here, $a = -18, b = 12$ and $c = -4$, then direction cosines of a line are

$$\left(\frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}\right)$$

$$= \left(\frac{-18}{\sqrt{484}}, \frac{12}{\sqrt{484}}, \frac{-4}{\sqrt{484}}\right) = \left(\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}\right)$$

$$= \left(-\frac{9}{11}, \frac{6}{11}, \frac{-2}{11}\right)$$

Hence, the direction cosines are $-\frac{9}{11}, \frac{6}{11}$ and $\frac{-2}{11}$.

Example 4. A vector $\mathbf{r}$ is inclined at equal angles to OX, OY and OZ . If the magnitude of $\mathbf{r}$ is 6 units, then $\mathbf{r}$ is equal to

(a) $\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

(b) $-\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

(c) $-2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$

(d) None of the above

Sol. (c) Let r be inclined at an angle α to each axis, then

$$l = m = n = \cos \alpha$$

Since, $l^2 + m^2 + n^2 = 1$

$$\therefore \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1 \Rightarrow 3 \cos^2 \alpha = 1$$

If α is acute, then $l = m = n = \frac{1}{\sqrt{3}}, |r| = 6$

$$\therefore r = |r|(\hat{i} + \hat{j} + \hat{k})$$

$$= 6\left(\frac{1}{\sqrt{3}}\hat{i} + \frac{1}{\sqrt{3}}\hat{j} + \frac{1}{\sqrt{3}}\hat{k}\right) = 2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$$

If α is obtuse, then $l = m = n = -\frac{1}{\sqrt{3}}, |r| = 6$

$$\therefore r = |r|(\hat{i} + \hat{j} + \hat{k})$$

$$= 6\left(-\frac{1}{\sqrt{3}}\hat{i} - \frac{1}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{3}}\hat{k}\right) = -2\sqrt{3}(\hat{i} + \hat{j} + \hat{k})$$

➤ **Example 5.** If a line makes angles α, β, γ with the coordinate axes, then

- (a) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma - 1 = 0$
 (b) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma - 2 = 0$
 (c) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$
 (d) $\cos 2\alpha + \cos 2\beta + \cos 2\gamma + 2 = 0$

Sol. (c) Since, α, β, γ makes angles with coordinate axes.

$$\therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1 \quad [\text{by definition}]$$

$$\Rightarrow 2 \cos^2 \alpha + 2 \cos^2 \beta + 2 \cos^2 \gamma = 2$$

$$\Rightarrow 2 \cos^2 \alpha - 1 + 2 \cos^2 \beta - 1 + 2 \cos^2 \gamma - 1 = 2 - 3$$

$$\Rightarrow \cos 2\alpha + \cos 2\beta + \cos 2\gamma + 1 = 0$$

Angle between Two Vectors

If θ is the angle between two vectors whose direction cosines are l_1, m_1, n_1 and l_2, m_2, n_2 , then $\cos \theta = l_1 l_2 + m_1 m_2 + n_1 n_2$ and $\sin \theta$

$$= \sqrt{[(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2]}$$

- If $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$, then two vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2 are orthogonal.
- If $\frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$, then two vectors are parallel.

Angle in Terms of Direction Ratios

If θ is the angle between two vectors having direction ratios a_1, b_1, c_1 and a_2, b_2, c_2 , then

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Two vectors are orthogonal, when

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

and parallel, when $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Projection of the Join of Two Points on a Line

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then length of projection of PQ on a line whose direction cosines are l, m, n , is the given by

$$(x_2 - x_1)l + (y_2 - y_1)m + (z_2 - z_1)n.$$

➤ **Example 6.** The acute angle between two lines whose direction ratios are 2, 3, 6 and 1, 2, 2, is

(a) $\cos^{-1}\left(\frac{20}{21}\right)$ (b) $\cos^{-1}\left(\frac{18}{21}\right)$

(c) $\cos^{-1}\left(\frac{17}{21}\right)$ (d) None of these

Sol. (a) We have, $a_1 = 2, b_1 = 3, c_1 = 6, a_2 = 1, b_2 = 2, c_2 = 2$

If θ is the angle between two lines whose DR's are given, then

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{2 \times 1 + 3 \times 2 + 6 \times 2}{\sqrt{2^2 + 3^2 + 6^2} \sqrt{1^2 + 2^2 + 2^2}} = \frac{2 + 6 + 12}{7 \times 3} = \frac{20}{21}$$

$$\therefore \theta = \cos^{-1}\left(\frac{20}{21}\right)$$

➤ **Example 7.** If the projections of a line on the axes are 6, 2, 3, then its direction cosines are

(a) $\frac{6}{7}, -\frac{2}{7}, \frac{3}{7}$ (b) $\frac{6}{7}, \frac{2}{7}, -\frac{3}{7}$

(c) $-\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$ (d) $\frac{6}{7}, \frac{2}{7}, \frac{3}{7}$

Sol. (d) Suppose the line is drawn through the origin and its length is r .

If l, m, n are its direction cosines, then

$$lr = 6, mr = 2 \text{ and } nr = 3$$

$$\therefore l^2 r^2 = 36, m^2 r^2 = 4 \text{ and } n^2 r^2 = 9$$

On adding these, we get

$$l^2 r^2 + m^2 r^2 + n^2 r^2 = 36 + 4 + 9$$

$$\Rightarrow r^2(l^2 + m^2 + n^2) = 49$$

$$\Rightarrow r^2 = 49 \quad [\because l^2 + m^2 + n^2 = 1]$$

$$\Rightarrow r = 7$$

$$\therefore l = \frac{6}{7}, m = \frac{2}{7}, n = \frac{3}{7}$$

➤ **Example 8.** The points $(5, 2, 4), (6, -1, 2)$ and $(8, -7, k)$ are collinear, if k is equal to

(a) -2 (b) 2 (c) 3 (d) -1

Sol. (a) Let $A(5, 2, 4), B(6, -1, 2)$ and $C(8, -7, k)$ be the given points.

Then, direction ratios of AB are

$$(6 - 5), (-1 - 2), (2 - 4) \text{ i.e. } 1, -3, -2$$

and direction ratios of BC are

$$(8 - 6), (-7 + 1), (k - 2) \text{ i.e. } 2, -6, (k - 2)$$

Since, A, B and C are collinear.

$$\therefore \frac{2}{1} = \frac{-6}{-3} = \frac{k - 2}{-2}$$

On taking first and last terms, we get $\frac{2}{1} = \frac{k - 2}{-2}$

$$\Rightarrow k - 2 = -4$$

$$\therefore k = 2 - 4 = -2$$

- If projection of a line on X, Y and Z-axes are 6, 2 and 3 respectively, then direction cosines of the line is
 - $\left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$
 - $\left(\frac{7}{6}, \frac{7}{2}, \frac{7}{3}\right)$
 - $\left(\frac{6}{11}, \frac{2}{11}, \frac{3}{11}\right)$
 - None of these
- If the line segment adjoining the points A, B makes projection 1, 4, 3 on X, Y and Z-axes respectively. Then, the direction cosines of AB are
 - 1, 4, 3
 - $\frac{1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}$
 - $\frac{-1}{\sqrt{26}}, \frac{4}{\sqrt{26}}, \frac{3}{\sqrt{26}}$
 - $\frac{1}{\sqrt{26}}, \frac{-4}{\sqrt{26}}, \frac{3}{\sqrt{26}}$
- Direction cosines of the angle between two diagonals of cube is equal to
 - $\frac{2}{\sqrt{6}}$
 - $\frac{1}{3}$
 - $\frac{1}{2}$
 - None of these
- The direction cosines of the line passing through $P(2, 3, -1)$ and the origin, are
 - $\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$
 - $\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$
 - $\frac{-2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$
 - $\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}$
- If the direction cosines of a line are $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$, then
 - $0 < c < 1$
 - $c > 2$
 - $c = \pm \sqrt{2}$
 - $c = \pm \sqrt{3}$
- The direction cosines of the line which is perpendicular to the lines whose direction cosines are proportional to $(1, -1, 2)$ and $(2, 1, -1)$, are
 - $\frac{-2}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$
 - $-\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$
 - $-\frac{1}{\sqrt{35}}, -\frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}$
 - None of these
- If the direction cosines of two lines are proportional to $(2, 3, -6)$ and $(3, -4, 5)$, then the acute angle between them is
 - $\cos^{-1}\left(\frac{49}{36}\right)$
 - $\cos^{-1}\left(\frac{18\sqrt{2}}{35}\right)$
 - 96°
 - $\cos^{-1}\left(\frac{18}{35}\right)$
- A line makes acute angles α, β and λ with the coordinate axes such that $\cos \alpha \cos \beta = \cos \beta \cos \gamma = \frac{2}{9}$ and $\cos \gamma \cos \alpha = \frac{4}{9}$, then $\cos \alpha + \cos \beta + \cos \gamma$ is equal to
 - $\frac{25}{9}$
 - $\frac{5}{9}$
 - $\frac{5}{3}$
 - $\frac{2}{3}$
- If a line makes angles α, β, γ and δ with four diagonals of a cube, then the value of $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma + \sin^2 \delta$ is
 - $\frac{4}{3}$
 - $\frac{8}{3}$
 - $\frac{7}{3}$
 - 1
- A line makes the same angle θ with each of the X and Z-axes. If the angle β , which it makes with Y-axis, is such that $\sin^2 \beta = 3 \sin^2 \theta$, then $\cos^2 \theta$ equals
 - $\frac{2}{3}$
 - $\frac{1}{5}$
 - $\frac{3}{5}$
 - $\frac{2}{5}$

Straight Line in Space

Every equation of the first degree represents a plane. Two equations of the first degree are satisfied by the coordinates of every point on the line of intersection of the planes represented by them.

Therefore, the two equations of that line $ax + by + cz + d = 0$ and $a'x + b'y + c'z + d' = 0$ together represent a straight line.

Equation of Straight Line in Different Forms

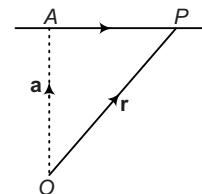
Equation of a Straight Line Passing through a Given Point and Parallel to a Given Vector

Equation of straight line passing through a point A with position vector $\mathbf{p}(x_1\hat{i} + y_1\hat{j} + z_1\hat{k})$ and parallel to a vector $\mathbf{q}(a\hat{i} + b\hat{j} + c\hat{k})$ is

$$\mathbf{r} = \mathbf{p} + \lambda \mathbf{q}$$

On putting the values of $\mathbf{r}$, $\mathbf{p}$ and $\mathbf{q}$, we get

$$x\hat{i} + y\hat{j} + z\hat{k} = (x_1\hat{i} + y_1\hat{j} + z_1\hat{k}) + \lambda (a\hat{i} + b\hat{j} + c\hat{k})$$



$$\therefore x - x_1 = \lambda a, y - y_1 = \lambda b, z - z_1 = \lambda c$$

$$\Rightarrow \frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

which is the required equation of line in cartesian form. Here, a, b and c are direction ratios. If l, m and n are direction cosines, then equation of straight line is

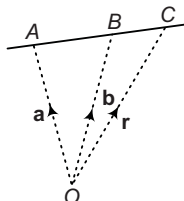
$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

This form of straight line is called symmetrical form.

Equation of Straight Line Passing through Two Points

The equation of a line passing through two points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$, is

$$\mathbf{r} = \mathbf{a} + \lambda(\mathbf{b} - \mathbf{a})$$



Equation of a straight line passing through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

- Intersection points of two planes also form a straight line, which is known as line of intersection but this is an unsymmetrical form of the straight line.
- The parametric equations of a line through (a_1, a_2, a_3) with direction cosines l, m and n are $x = a_1 + lr, y = a_2 + mr$ and $z = a_3 + nr$.
- The general coordinates of a point on a line is given by $(x_1 + lr, y_1 + mr, z_1 + nr)$, where r is the distance between point (x_1, y_1, z_1) and point whose coordinates is to be written.

Example 9. The vector and the cartesian equations of the line passing through the points $(5, 2, -4)$ and which is parallel to the vector $3\hat{i} + 2\hat{j} - 8\hat{k}$, are

(a) $(5\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$ and

$$\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z + 4}{-8}$$

(b) $(3\hat{i} + 2\hat{j} - 8\hat{k}) + \lambda(5\hat{i} + 2\hat{j} - 4\hat{k})$ and

$$\frac{x + 5}{3} = \frac{y + 2}{2} = \frac{z - 4}{-8}$$

(c) $(3\hat{i} + 2\hat{j} - 8\hat{k}) + \lambda(2\hat{i} + 4\hat{k})$ and

$$\frac{x + 5}{5} = \frac{y + 2}{2} = \frac{z - 4}{-4}$$

(d) None of the above

Sol. (a) We have, $\mathbf{r}_1 = 5\hat{i} + 2\hat{j} - 4\hat{k}$ and $\mathbf{b} = 3\hat{i} + 2\hat{j} - 8\hat{k}$

Therefore, the vector equation of the line is

$$\mathbf{r} = 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k})$$

Now, $\mathbf{r}$ is the position vector of any point $P(x, y, z)$ on the line.

$$\begin{aligned} \text{Therefore, } x\hat{i} + y\hat{j} - z\hat{k} &= 5\hat{i} + 2\hat{j} - 4\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 8\hat{k}) \\ &= (5 + 3\lambda)\hat{i} + (2 + 2\lambda)\hat{j} + (-4 - 8\lambda)\hat{k} \end{aligned}$$

Eliminating λ , we get

$$\frac{x - 5}{3} = \frac{y - 2}{2} = \frac{z + 4}{-8}$$

which is the equation of the line in cartesian form.

Example 10. The vector equation for the line passing through the points $(-1, 0, 2)$ and $(3, 4, 6)$, is

(a) $\hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$

(b) $2\hat{i} - \hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$

(c) $-\hat{i} + 2\hat{k} + \lambda(-\hat{i} + 4\hat{j} + 2\hat{k})$

(d) $-\hat{i} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 6\hat{k})$

Sol. (a) Let $\mathbf{a}$ and $\mathbf{b}$ be the position vectors of the points $A(-1, 0, 2)$ and $B(3, 4, 6)$.

Then, $\mathbf{a} = -\hat{i} + 2\hat{k}$

and $\mathbf{b} = 3\hat{i} + 4\hat{j} + 6\hat{k}$

Therefore, $\mathbf{b} - \mathbf{a} = 4\hat{i} + 4\hat{j} + 4\hat{k}$

Let $\mathbf{r}$ be the position vector of any point on the line.

Hence, the vector equation of the line is

$$\mathbf{r} = \hat{i} + 2\hat{k} + \lambda(4\hat{i} + 4\hat{j} + 4\hat{k})$$

Example 11. The vector equation of a line passing through a point with position vector $2\hat{i} - \hat{j} + \hat{k}$ and parallel to the line joining the points with position vectors $-\hat{i} + 4\hat{j} + \hat{k}$ and $\hat{i} + 2\hat{j} + 2\hat{k}$, is $\mathbf{r} = (2\hat{i} - \hat{j} + \hat{k}) + \lambda(x\hat{i} - 2\hat{j} + \hat{k})$, where x is equal to

(a) 0

(b) 1

(c) 2

(d) 4

Sol. (c) Let $\mathbf{a} = 2\hat{i} - \hat{j} + \hat{k}$, $\mathbf{b} = -\hat{i} + 4\hat{j} + \hat{k}$

and $\mathbf{c} = \hat{i} + 2\hat{j} + 2\hat{k}$

Then, the equation of the line will be

$$\begin{aligned} \mathbf{r} &= \mathbf{a} + \lambda(\mathbf{c} - \mathbf{b}) = (2\hat{i} - \hat{j} + \hat{k}) + \lambda\{(\hat{i} + 2\hat{j} + 2\hat{k}) \\ &\quad - (-\hat{i} + 4\hat{j} + \hat{k})\} \\ &= (2\hat{i} - \hat{j} + \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k}) \end{aligned}$$

Hence, $x = 2$

Angle between Two Lines

Vector Form

The angle between the two lines $\mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ is given by

$$\cos \theta = \frac{|\mathbf{b}_1 \cdot \mathbf{b}_2|}{|\mathbf{b}_1||\mathbf{b}_2|}$$

Cartesian Form

The angle between the two lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$$

and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ is given by

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Condition of Perpendicularity

The lines are perpendicular, if

$$\mathbf{b}_1 \cdot \mathbf{b}_2 = 0 \quad \text{or} \quad a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

Condition of Parallelism

The lines are parallel, if $\mathbf{b}_1 = \lambda \mathbf{b}_2$ for some scalar λ

$$\text{or } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

➤ **Example 12.** The angle between the pair of lines $\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $\mathbf{r} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$ is

- (a) 0 (b) $\pi/2$
(c) π (d) None of these

Sol. (d) Given lines are

$$\mathbf{r} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 4\hat{\mathbf{k}} + \lambda(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\text{and } \mathbf{r} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{k}} + \mu(3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})$$

Let the angle be θ , then

$$\begin{aligned} \cos \theta &= \frac{\mathbf{b}_1 \cdot \mathbf{b}_2}{|\mathbf{b}_1| |\mathbf{b}_2|} = \frac{(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}})}{\sqrt{1+4+4} \sqrt{9+4+36}} \\ &= \frac{3+4+12}{\sqrt{9} \sqrt{49}} = \frac{19}{21} \end{aligned}$$

$$\therefore \theta = \cos^{-1} \left(\frac{19}{21} \right)$$

➤ **Example 13.** The angle between the straight lines $\frac{x+1}{2} = \frac{y-2}{5} = \frac{z+3}{4}$ and $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-3}{-3}$ is

- (a) 45° (b) 30° (c) 60° (d) 90°

Sol. (d) The angle between the given lines is

$$\cos \theta = \frac{(2)(1) + (5)(2) + 4(-3)}{\sqrt{4+25+16} \sqrt{1+4+9}} = 0$$

$$\therefore \theta = 90^\circ$$

➤ **Example 14.** If the lines $6x-2=3y+1=2z-2$ and $\frac{x-2}{\lambda} = \frac{2y-5}{-3} = \frac{z+2}{0}$ are perpendicular,

then λ equals

- (a) 3 (b) 2
(c) -3 (d) 1

Sol. (a) The equations of the given lines can be rewritten as

$$\frac{x-1/3}{1} = \frac{y+1/3}{2} = \frac{z-1}{3} \quad \text{and} \quad \frac{x-2}{\lambda} = \frac{y-5/2}{-3/2} = \frac{z+2}{0}$$

If these lines are perpendicular, then

$$1 \times \lambda + 2 \times -\frac{3}{2} + 3 \times 0 = 0$$

$$\Rightarrow \lambda - 3 = 0 \Rightarrow \lambda = 3$$

Shortest Distance between Two Skew-lines

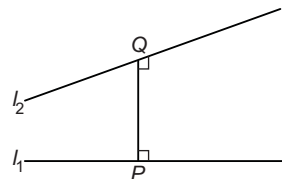
Skew-lines

Two straight lines in space which are neither parallel nor intersecting, are called skew-lines.

Thus, skew-lines are those lines which do not lie in the same plane.

Line of Shortest Distance

If l_1 and l_2 are two skew-lines, then there is one and only one line perpendicular to each of the lines l_1 and l_2 which is known as the line of shortest distance.



Shortest Distance

The shortest distance between two lines l_1 and l_2 is the distance PQ between the points P and Q , where the line of shortest distance intersects the two given lines.

Vector Form

The shortest distance between the lines

$\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ is given by

$$\frac{|(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Cartesian Form

The shortest distance between the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$$

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ is given by

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}}{\sqrt{(m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 + (l_1 m_2 - l_2 m_1)^2}}$$

Shortest Distance between Two Parallel Lines

The shortest distance between two parallel lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}$ is given by

$$\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$$

✎ If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ intersect, then the shortest distance between them is zero.

$$\therefore \frac{|(\mathbf{b}_1 \times \mathbf{b}_2) \cdot (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}_1 \times \mathbf{b}_2|} = 0$$

$$\Rightarrow \mathbf{b}_1 \times \mathbf{b}_2 \cdot (\mathbf{a}_2 - \mathbf{a}_1) = 0$$

$$\Rightarrow [\mathbf{b}_1, \mathbf{b}_2, (\mathbf{a}_2 - \mathbf{a}_1)] = 0$$

✎ **Example 15.** The shortest distance between the lines $\mathbf{r} = (4\hat{i} - \hat{j}) + \lambda (\hat{i} + 2\hat{j} - 3\hat{k})$ and $\mathbf{r} = (\hat{i} - \hat{j} + 2\hat{k}) + \mu (2\hat{i} + 4\hat{j} - 5\hat{k})$ is

- (a) 6 (b) $\frac{6}{\sqrt{5}}$
(c) $\frac{3}{\sqrt{5}}$ (d) $\frac{2}{\sqrt{5}}$

Sol. (b) We know that the shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ is given by

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|}$$

Here,

$$\mathbf{a}_1 = 4\hat{i} - \hat{j}, \mathbf{a}_2 = \hat{i} - \hat{j} + 2\hat{k}$$

and

$$\mathbf{b}_1 = \hat{i} + 2\hat{j} - 3\hat{k}, \mathbf{b}_2 = 2\hat{i} + 4\hat{j} - 5\hat{k}$$

$\therefore$

$$\mathbf{a}_2 - \mathbf{a}_1 = -3\hat{i} + 0\hat{j} + 2\hat{k}$$

and

$$\mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -3 \\ 2 & 4 & -5 \end{vmatrix} = 2\hat{i} - \hat{j} + 0\hat{k}$$

$$\Rightarrow (\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = (-3\hat{i} + 0\hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{j} + 0\hat{k}) = -6$$

and

$$|\mathbf{b}_1 \times \mathbf{b}_2| = \sqrt{4 + 1 + 0} = \sqrt{5}$$

$\therefore$

$$d = \frac{|(\mathbf{a}_2 - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)|}{|\mathbf{b}_1 \times \mathbf{b}_2|} = \frac{|-6|}{\sqrt{5}} = \frac{6}{\sqrt{5}}$$

✎ **Example 16.** The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is

- (a) $\sqrt{30}$ (b) $2\sqrt{30}$ (c) $5\sqrt{30}$ (d) $3\sqrt{30}$

Sol. (d) $SD = \frac{\begin{vmatrix} 6 & 15 & -3 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}}{\sqrt{(-4-2)^2 + (12+3)^2 + (6-3)^2}}$
 $= \frac{270}{\sqrt{270}} = \sqrt{270} = 3\sqrt{30}$

✎ **Example 17.** The distance between two parallel lines l_1 and l_2 whose equations are $\mathbf{r} = \hat{i} + 2\hat{j} - 4\hat{k} + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$ and $\mathbf{r} = 3\hat{i} + 3\hat{j} - 5\hat{k} + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$, is

- (a) $\frac{\sqrt{297}}{7}$ (b) $\frac{\sqrt{293}}{7}$
(c) $\frac{\sqrt{291}}{7}$ (d) $\frac{\sqrt{293}}{5}$

Sol. (b) Since, two lines are parallel, so we have

$$\mathbf{a}_1 = \hat{i} + 2\hat{j} - 4\hat{k}, \mathbf{a}_2 = 3\hat{i} + 3\hat{j} - 5\hat{k}$$

$$\text{and } \mathbf{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Therefore, the distance between the lines is given by

$$d = \frac{|\mathbf{b} \times (\mathbf{a}_2 - \mathbf{a}_1)|}{|\mathbf{b}|} = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 6 \\ 2 & 1 & -1 \end{vmatrix}}{\sqrt{4 + 9 + 36}}$$

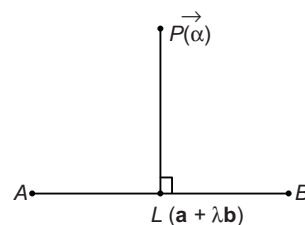
$$= \frac{|-9\hat{i} + 14\hat{j} - 4\hat{k}|}{\sqrt{49}} = \frac{\sqrt{293}}{\sqrt{7}}$$

Perpendicular Distance

Foot of Perpendicular from a Point on the Given Line

Vector Form

Let L be the foot of the perpendicular drawn from $P(\alpha)$ on the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$.



Since, $\mathbf{r}$ denotes the position vector of any point on the line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$, so the position vector of L will be $(\mathbf{a} + \lambda \mathbf{b})$.

$$\text{Direction ratios of } PL = \mathbf{a} - \vec{\alpha} + \lambda \mathbf{b}$$

Since, PL is perpendicular to $\mathbf{b}$.

$$\therefore (\mathbf{a} - \vec{\alpha} + \lambda \mathbf{b}) \cdot \mathbf{b} = 0 \Rightarrow (\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b} + \lambda \mathbf{b} \cdot \mathbf{b} = 0$$

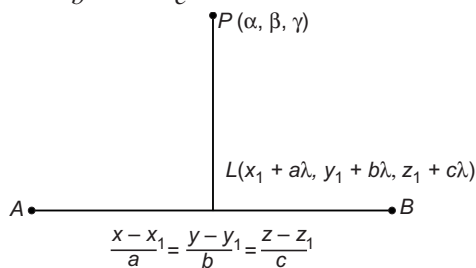
$$\Rightarrow \lambda = \frac{-(\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b}}{|\mathbf{b}|^2}$$

$$\text{Hence, position vector of } L \text{ is } \left\{ \mathbf{a} - \left(\frac{(\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \right) \mathbf{b} \right\},$$

which is the foot of the perpendicular.

Cartesian Form

Let $P(\alpha, \beta, \gamma)$ be the given point and $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ be the given line.



Here, the equation of line AB is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Let L be the foot of the perpendicular drawn from $P(\alpha, \beta, \gamma)$ on the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$.

Let the coordinates of L be

$$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$$

Then, the direction ratios of PL are

$$(x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma)$$

Direction ratios of AB are (a, b, c) .

Since, PL is perpendicular to AB .

$$\begin{aligned} \therefore a(x_1 + a\lambda - \alpha) + b(y_1 + b\lambda - \beta) + c(z_1 + c\lambda - \gamma) &= 0 \\ \Rightarrow \lambda &= \frac{a(\alpha - x_1) + b(\beta - y_1) + c(\gamma - z_1)}{a^2 + b^2 + c^2} \end{aligned}$$

Putting the value of λ in

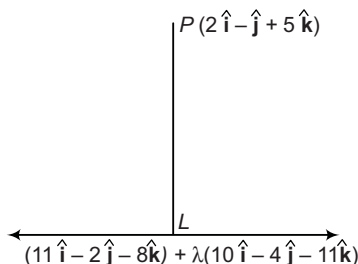
$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we get the foot of the perpendicular. Now, we can get distance PL using distance formula.

Example 18. The position vector of the foot of the perpendicular drawn from the point $2\hat{i} - \hat{j} + 5\hat{k}$ to the line $\mathbf{r} = (11\hat{i} - 2\hat{j} - 8\hat{k}) + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$, is

- (a) $\hat{i} + 3\hat{j} + 2\hat{k}$ (b) $-\hat{i} + 3\hat{j} - 2\hat{k}$
(c) $\hat{i} - 3\hat{j} - 2\hat{k}$ (d) $\hat{i} + 2\hat{j} + 3\hat{k}$

Sol. (d) Let L be the foot of the perpendicular drawn from $P(2\hat{i} - \hat{j} + 5\hat{k})$ on the line

$$\mathbf{r} = 11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k})$$



Let the position vector of L be

$$11\hat{i} - 2\hat{j} - 8\hat{k} + \lambda(10\hat{i} - 4\hat{j} - 11\hat{k}) = (11 + 10\lambda)\hat{i} + (-2 - 4\lambda)\hat{j} + (-8 - 11\lambda)\hat{k}$$

Then, $PL = (9 + 10\lambda)\hat{i} + (-1 - 4\lambda)\hat{j} + (-13 - 11\lambda)\hat{k}$

Since, PL is perpendicular to the given line which is parallel to vector $\mathbf{b} = 10\hat{i} - 4\hat{j} - 11\hat{k}$.

$$\begin{aligned} \therefore PL \cdot \mathbf{b} &= 0 \\ \Rightarrow 10(9 + 10\lambda) - 4(-1 - 4\lambda) - 11(-13 - 11\lambda) &= 0 \\ \Rightarrow \lambda &= -1 \end{aligned}$$

On putting the value of λ , we obtain the position vector of L as $\hat{i} + 2\hat{j} + 3\hat{k}$.

Example 19. The length of perpendicular from the point $(1, 2, 3)$ to the line

$$\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}, \text{ is}$$

- (a) 5 units (b) 7 units
(c) 4 units (d) None of these

Sol. (b) Let L be the foot of the perpendicular drawn from the point $P(1, 2, 3)$ to the given line.

Let the coordinates of L be

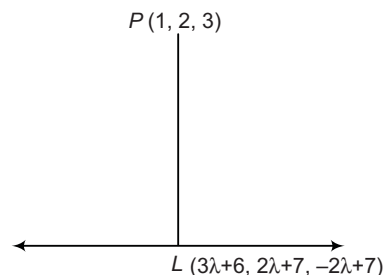
$$(3\lambda + 6, 2\lambda + 7, -2\lambda + 7) \quad \dots (i)$$

$\therefore$ Direction ratios of PL are proportional to

$$3\lambda + 6 - 1, 2\lambda + 7 - 2, -2\lambda + 7 - 3$$

i.e. $3\lambda + 5, 2\lambda + 5, -2\lambda + 4$

Direction ratios of the given line are proportional to $3, 2, -2$.



Since, PL is perpendicular to the given line.

$$\begin{aligned} \therefore 3(3\lambda + 5) + 2(2\lambda + 5) + (-2)(-2\lambda + 4) &= 0 \\ \Rightarrow \lambda &= -1 \end{aligned}$$

On putting $\lambda = -1$ in Eq. (i), we obtain that the coordinates of L as $(3, 5, 9)$.

$$\begin{aligned} \therefore PL &= \sqrt{(3-1)^2 + (5-2)^2 + (9-3)^2} \\ &= 7 \text{ units} \end{aligned}$$

Reflection or Image of a Point in a Straight Line

If the perpendicular PL from P on the given line is produced to Q such that $PL = QL$, then Q is known as the image or reflection of P in the given line. Also, L is the foot of the perpendicular or the projection of P on the line.

Vector Form

Let $P(\vec{\alpha})$ be the given point and $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ be the given line. Let PL be the perpendicular from P on the given line and Q be the image of P in $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. Let the position vector of L be $\mathbf{a} + \lambda\mathbf{b}$. Then,

$$PL = \mathbf{a} + \lambda\mathbf{b} - \vec{\alpha}$$

Since, PL is perpendicular to the given line.

$$\therefore PL \perp \mathbf{b} \Rightarrow PL \cdot \mathbf{b} = 0$$

$$\Rightarrow (\mathbf{a} + \lambda\mathbf{b} - \vec{\alpha}) \cdot \mathbf{b} = 0$$

$$\Rightarrow (\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b} + \lambda(\mathbf{b} \cdot \mathbf{b}) = 0 \Rightarrow \lambda = -\left\{ \frac{(\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\}$$

Therefore, the position vector of L is

$$\mathbf{a} + \lambda\mathbf{b} = \mathbf{a} - \left\{ \frac{(\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\} \mathbf{b}$$

Let $\vec{\alpha}'$ be the position vector of Q .

Since, L is the mid-point of PQ .

$$\therefore \frac{\vec{\alpha} + \vec{\alpha}'}{2} = \mathbf{a} - \left\{ \frac{(\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\} \mathbf{b}$$

$$\Rightarrow \vec{\alpha}' = 2\mathbf{a} - \left\{ \frac{2(\mathbf{a} - \vec{\alpha}) \cdot \mathbf{b}}{|\mathbf{b}|^2} \right\} \mathbf{b} - \vec{\alpha}$$

This gives the position vector of Q which is the image of P in the given line.

Cartesian Form

Let $P(\alpha, \beta, \gamma)$ be the given point and

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} \text{ be the given line.}$$

Let L be the foot of the perpendicular from P on the given line and Q be the image of P in the given line. Let the coordinates of L be

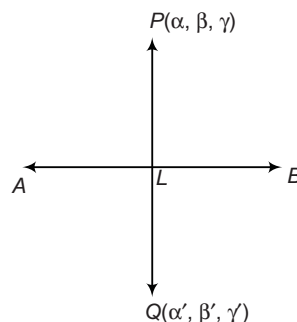
$(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$. Then, direction ratios of PL are

$$x_1 + a\lambda - \alpha, y_1 + b\lambda - \beta, z_1 + c\lambda - \gamma$$

Since, PL is perpendicular to the given line, whose direction ratios are a, b, c .

$$\therefore (x_1 + a\lambda - \alpha)a + (y_1 + b\lambda - \beta)b + (z_1 + c\lambda - \gamma)c = 0$$

$$\Rightarrow \lambda = -\frac{[a(x_1 - \alpha) + b(y_1 - \beta) + c(z_1 - \gamma)]}{a^2 + b^2 + c^2}$$



Substituting the value of λ in $(x_1 + a\lambda, y_1 + b\lambda, z_1 + c\lambda)$, we obtain coordinates of L .

Let the coordinates of Q be $(\alpha', \beta', \gamma')$. Since, L is the mid-point of PQ .

$$\therefore \frac{\alpha + \alpha'}{2} = x_1 + a\lambda, \frac{\beta + \beta'}{2} = y_1 + b\lambda$$

$$\text{and } \frac{\gamma + \gamma'}{2} = z_1 + c\lambda$$

$$\Rightarrow \alpha' = 2(x_1 + a\lambda) - \alpha, \beta' = 2(y_1 + b\lambda) - \beta$$

$$\text{and } \gamma' = 2(z_1 + c\lambda) - \gamma$$

➤ **Example 20.** The image of the point $(1, 6, 3)$

in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ has the coordinates

(a) $(1, 1, 7)$

(b) $(0, 1, 7)$

(c) $(1, 0, 7)$

(d) None of these

Sol. (c) Let $P(1, 6, 3)$ be the given point and L be the foot of the perpendicular from P to the given line. The coordinates of a general point on the given line are given by

$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \quad [\text{say}]$$

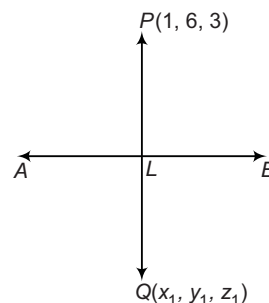
$$\text{i.e. } x = \lambda, y = 2\lambda + 1, z = 3\lambda + 2$$

Again, let the coordinates of L be

$$(\lambda, 2\lambda + 1, 3\lambda + 2) \quad \dots(i)$$

$$\therefore PL = (\lambda - 1)\hat{i} + (2\lambda - 5)\hat{j} + (3\lambda - 1)\hat{k}$$

The given line is parallel to the vector $\mathbf{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.



$$\Rightarrow PL \cdot \mathbf{b} = 0$$

$$\Rightarrow (\lambda - 1) \times 1 + (2\lambda - 5) \times 2 + (3\lambda - 1) \times 3 = 0 \Rightarrow \lambda = 1$$

So, the coordinates of L are $(1, 3, 5)$.

Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ in the given line. Then, L is the mid-point of PQ .

$$\therefore \frac{x_1 + 1}{2} = 1, \frac{y_1 + 6}{2} = 3 \text{ and } \frac{z_1 + 3}{2} = 5$$

$$\Rightarrow x_1 = 1, y_1 = 0 \text{ and } z_1 = 7$$

Hence, the image of $P(1, 6, 3)$ in the given line is $Q(1, 0, 7)$.

- The direction ratios of the line $x - y + z - 5 = 0$ and $x - 3y - 6 = 0$ are
 a $3, 1, -2$ b $2, -4, 1$
 c $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$ d $\frac{2}{\sqrt{41}}, \frac{-4}{\sqrt{41}}, \frac{1}{\sqrt{41}}$
- The equation of the straight line passing through the points $(4, -5, -2)$ and $(-1, 5, 3)$, is
 a $\frac{x-4}{1} = \frac{y+5}{-2} = \frac{z+2}{-1}$
 b $\frac{x+1}{1} = \frac{y-5}{2} = \frac{z-3}{-1}$
 c $\frac{x}{-1} = \frac{y}{5} = \frac{z}{3}$
 d $\frac{x}{4} = \frac{y}{-5} = \frac{z}{-2}$
- The angle between the lines $\frac{x}{1} = \frac{y}{0} = \frac{z}{-1}$ and $\frac{x}{3} = \frac{y}{4} = \frac{z}{5}$ is equal to
 a $\pi - \cos^{-1}\left(\frac{1}{5}\right)$ b $\cos^{-1}\left(\frac{1}{3}\right)$
 c $\cos^{-1}\left(\frac{1}{2}\right)$ d $\cos^{-1}\left(\frac{1}{4}\right)$
- The distance of the point $P(2, 3, 4)$ from the line $1 - x = \frac{y}{2} = \frac{1}{3}(1 + z)$ is
 a $\frac{1}{7}\sqrt{35}$ b $\frac{4}{7}\sqrt{35}$ c $\frac{2}{7}\sqrt{35}$ d $\frac{3}{7}\sqrt{35}$
- The shortest distance between the lines $1 + x = 2y = -12z$ and $x = y + 2 = 6z - 6$ is
 a 1 b 2 c 3 d 4
- The foot of the perpendicular from $(2, 4, -1)$ to the line $x + 5 = \frac{1}{4}(y + 3) = -\frac{1}{9}(z - 6)$ is
 a $(-4, 1, -3)$ b $(4, -1, -3)$
 c $(-4, -1, 3)$ d $(-4, -1, -3)$
- The shortest distance between the straight lines $\frac{x-6}{1} = \frac{2-y}{2} = \frac{z-2}{2}$ and $\frac{x+4}{3} = \frac{y}{-2} = \frac{1-z}{2}$ is
 a 9 b $\frac{25}{3}$ c $\frac{16}{3}$ d 4
- The point of intersection of the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-4}{5} = \frac{y-1}{2} = z$ is
 a $(0, 0, 0)$ b $(1, 1, 1)$
 c $(-1, -1, -1)$ d $(1, 2, 3)$
- If the lines $\frac{1-x}{3} = \frac{y-2}{2\alpha} = \frac{z-3}{2}$ and $\frac{x-1}{3\alpha} = y-1 = \frac{6-z}{5}$ are perpendicular, then the value of α is
 a $-\frac{10}{7}$ b $\frac{10}{7}$ c $-\frac{10}{11}$ d $\frac{10}{11}$
- If the straight lines $x = 1 + s, y = -3 - \lambda s, z = 1 + \lambda s$ and $x = \frac{t}{2}, y = 1 + t, z = 2 - t$ with parameters s and t respectively are coplanar, then λ equals
 a -2 b -1 c $-\frac{1}{2}$ d 0

Plane

A plane is a surface such that, if any two points are taken on it, the line segment joining them lies completely on the surface.

A plane is determined uniquely, if

- the normal to the plane and its distance from the origin is given i.e. the equation of a plane in normal form.
- it passes through a point and is perpendicular to a given direction.
- it passes through three given non-collinear points.

Equation of a Plane in Normal Form

Vector Form

The vector equation of a plane normal to unit vector $\hat{n}$ at a distance d from the origin is $\mathbf{r} \cdot \hat{n} = d$.

Cartesian Form

If l, m, n are direction cosines of the normal to a given plane which is at a distance p from the origin, then the equation of the plane is $lx + my + nz = p$.

➤ **Example 21.** The vector equation of the plane which is at a distance of 8 units from the origin and which is normal to the vector $2\hat{i} + \hat{j} + 2\hat{k}$, is

- (a) $\mathbf{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 8$ (b) $\mathbf{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24$
 (c) $\mathbf{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 4$ (d) None of these

Sol. (b) Here, $d = 8$ and $\mathbf{n} = 2\hat{i} + \hat{j} + 2\hat{k}$

$$\therefore \hat{n} = \frac{\mathbf{n}}{|\mathbf{n}|} = \frac{2\hat{i} + \hat{j} + 2\hat{k}}{\sqrt{4+1+4}} = \frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k}$$

Hence, the required equation of the plane is

$$\mathbf{r} \cdot \left(\frac{2}{3}\hat{i} + \frac{1}{3}\hat{j} + \frac{2}{3}\hat{k} \right) = 8$$

$$\Rightarrow \mathbf{r} \cdot (2\hat{i} + \hat{j} + 2\hat{k}) = 24$$

Equation of a Plane Passing through the Given Point

Vector Form

The vector equation of a plane passing through a point having position vector $\mathbf{a}$ and normal vector $\mathbf{n}$ is

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0 \quad \text{or} \quad \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Cartesian Form

The general equation of a plane passing through a given point (x_1, y_1, z_1) and normal to the plane whose direction ratios are (a, b, c) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

➤ **Example 22.** The vector equation of a plane passing through a point having position vector $2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and perpendicular to the vector $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$, is

$$(a) \mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 7 \quad (b) \mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = -7$$

$$(c) \mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 4 \quad (d) \text{None of these}$$

Sol. (b) We know that the vector equation of a plane passing through a point $\mathbf{a}$ and normal to $\mathbf{n}$ is

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$$

$$\text{or} \quad \mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

Here, $\mathbf{a} = 2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}$ and $\mathbf{n} = 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$

So, the equation of the required plane is

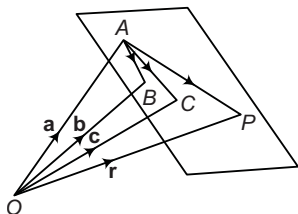
$$\mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 4\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\Rightarrow \mathbf{r} \cdot (2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = -7$$

Equation of a Plane Passing through the Three Points

Vector Form

Let A, B and C be the three given points with position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively and P be a point in the plane with position vector $\mathbf{r}$.



Then, equation of the plane is

$$[\mathbf{r} - \mathbf{a} \quad \mathbf{b} - \mathbf{a} \quad \mathbf{c} - \mathbf{a}] = 0$$

$$\text{or} \quad [\mathbf{r} \quad \mathbf{b} \quad \mathbf{c}] + [\mathbf{r} \quad \mathbf{a} \quad \mathbf{b}] + [\mathbf{r} \quad \mathbf{c} \quad \mathbf{a}] = [\mathbf{a} \quad \mathbf{b} \quad \mathbf{c}]$$

Cartesian Form

The equation of a plane passing through three given points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \\ x - x_3 & y - y_3 & z - z_3 \end{vmatrix} = 0$$

➤ **Example 23.** The equation of the plane passing through the points $(2, 5, -3), (-2, -3, 5)$ and $(5, 3, -3)$, is

$$(a) 2x + 3y + 4z = 14 \quad (b) 2x + 3y + 4z = 18$$

$$(c) 2x - 3y + 4z = 14 \quad (d) 2x + 3y - 4z = 18$$

Sol. (b) The equation of the plane passing through the points $(2, 5, -3), (-2, -3, 5)$ and $(5, 3, -3)$, is

$$\begin{vmatrix} x - 2 & y + 2 & z - 5 \\ -2 - 2 & -3 - 5 & 5 + 3 \\ 5 - 2 & 3 - 5 & -3 + 3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x - 2 & y + 2 & z - 5 \\ -4 & -8 & 8 \\ 3 & -2 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (x - 2)(16) - (y + 2)(-24) + (z - 5)(8 + 24) = 0$$

$$\Rightarrow 16x - 32 + 24y + 48 + 32z - 160 = 0$$

$$\Rightarrow 16x + 24y + 32z - 144 = 0$$

$$\Rightarrow 2x + 3y + 4z = 18$$

➤ **Example 24.** The vector equation of the plane passing through the points $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, is

$$(a) \mathbf{r} \cdot (9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) = -14$$

$$(b) \mathbf{r} \cdot (9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 14$$

$$(c) \mathbf{r} \cdot (3\hat{\mathbf{i}} + 9\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 14$$

(d) None of the above

Sol. (b) Let A, B , and C be the points with position vectors $\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}, 2\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$ respectively. Then,

$$\mathbf{AB} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$$

$$\text{and} \quad \mathbf{BC} = -\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 0\hat{\mathbf{k}}$$

A vector normal to the plane containing points A, B and C , is

$$\mathbf{n} = \mathbf{AB} \times \mathbf{BC} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -2 & 3 \\ -1 & 3 & 0 \end{vmatrix} = -9\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

The required plane passes through the point having position vector $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ and is normal to the

vector $\mathbf{n} = -9\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$

So, its vector equation is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \Rightarrow \mathbf{r} \cdot (9\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 14$$

Intercept Form of a Plane

The equation of a plane intercepting lengths a, b and c with X, Y and Z -axes respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

➤ **Example 25.** The equation of the plane whose intercepts on the coordinate axes are $-4, 2$ and 3 , is

- (a) $3x + 6y + 4z = 12$
 (b) $-3x + 6y + 4z = 12$
 (c) $-3x - 6y - 4z = 12$
 (d) None of the above

Sol. (b) We know that the equation of a plane whose intercepts on the coordinate axes are a, b and c respectively, is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Here, $a = -4, b = 2$ and $c = 3$

So, the equation of the required plane is

$$\frac{x}{-4} + \frac{y}{2} + \frac{z}{3} = 1$$

or $-3x + 6y + 4z = 12$

➤ **Example 26.** The intercepts of the plane $5x - 3y + 6z - 60 = 0$ on the coordinate axes are

- (a) $10, 20, -10$ (b) $10, -20, 12$
 (c) $12, -20, 10$ (d) $12, 20, -10$

Sol. (c) Given equation of the plane is

$$5x - 3y + 6z - 60 = 0$$

$$\Rightarrow 5x - 3y + 6z = 60$$

$$\Rightarrow \frac{x}{12} + \frac{y}{-20} + \frac{z}{10} = 1$$

Hence, the intercepts are $12, -20, 10$.

➤ **Example 27.** A plane meets the coordinate axes in A, B, C such that the centroid of $\triangle ABC$ is the point (p, q, r) . If the equation of the plane is

$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = k, \text{ then } k \text{ equals}$$

- (a) 1
 (b) 2
 (c) 3
 (d) None of the above

Sol. (c) Let the equation of the required plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots (i)$$

Then, the coordinates of A, B and C are $A(a, 0, 0), B(0, b, 0)$ and $C(0, 0, c)$ respectively.

So, the centroid of $\triangle ABC$ is $(a/3, b/3, c/3)$.

But the coordinates of the centroid are (p, q, r) .

$$\therefore p = \frac{a}{3}, q = \frac{b}{3} \text{ and } r = \frac{c}{3}$$

$$\Rightarrow a = 3p, b = 3q \text{ and } c = 3r$$

On substituting the values of a, b and c in Eq. (i), we obtain the required plane as

$$\frac{x}{3p} + \frac{y}{3q} + \frac{z}{3r} = 1$$

$$\Rightarrow \frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 3$$

Equation of a Plane in Parametric Form

Parametric Form

The equation of the plane passing through a point having position vector $\mathbf{a}$ and parallel to $\mathbf{b}$ and $\mathbf{c}$, is $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$, where λ and μ are scalars.

Non-parametric Form

The equation of the plane passing through a point having position vector $\mathbf{a}$ and parallel to vector $\mathbf{b}$ and $\mathbf{c}$, is

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}) = 0$$

or $\mathbf{r} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

or $[\mathbf{r} \ \mathbf{b} \ \mathbf{c}] = [\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$

➤ **Example 28.** The vector equation of the plane $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}}) + \lambda(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \mu(\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ is scalar product form, is

(a) $\mathbf{r} \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 7$

(b) $\mathbf{r} \cdot (5\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 7$

(c) $\mathbf{r} \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = -7$

(d) None the above

Sol. (a) Clearly, given plane passes through the point $\mathbf{a} = \hat{\mathbf{i}} - \hat{\mathbf{j}}$ and is parallel to the vectors $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}$.

So, it is perpendicular to the vector $\mathbf{n}$ which is given by

$$\mathbf{n} = \mathbf{b} \times \mathbf{c} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & 1 & 1 \\ 1 & -2 & 3 \end{vmatrix} = 5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$$

The vector equation of the plane in scalar product form is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$$

$$\Rightarrow \mathbf{r} \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = (\hat{\mathbf{i}} - \hat{\mathbf{j}}) \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$$

$$\Rightarrow \mathbf{r} \cdot (5\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}}) = 7$$

Equation of a Plane Parallel to a Given Plane

Vector Form

Since, parallel planes have the common normal. Therefore, equation of a plane parallel to the plane $\mathbf{r} \cdot \mathbf{n} = d_1$ is $\mathbf{r} \cdot \mathbf{n} = d_2$, where d_2 is a constant determined by the given condition.

Cartesian Form

Let $ax + by + cz + d = 0$ be a plane. Then, direction ratios of its normal are a, b, c . Since, parallel planes have common normal, therefore the direction ratios of the normal to the parallel plane are also a, b, c . Thus, the equation of a plane parallel to the plane $ax + by + cz + d = 0$ is $ax + by + cz + k = 0$, where k is an arbitrary constant and is determined by the given condition.

➤ **Example 29.** The equation of plane passing through the point $\hat{i} + \hat{j} + \hat{k}$ and parallel to the plane $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$, is

(a) $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$

(b) $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = -3$

(c) $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 3$

(d) None of the above

Sol. (c) The equation of a plane parallel to the plane $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 5$ is

$$\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = d \quad \dots (i)$$

Since, it passes through $\hat{i} + \hat{j} + \hat{k}$.

$$\therefore (\hat{i} + \hat{j} + \hat{k}) \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = d$$

$$\Rightarrow 2 - 1 + 2 = d \Rightarrow d = 3$$

On putting $d = 3$ in Eq. (i), we get

$$\mathbf{r} \cdot (2\hat{i} - \hat{j} + 2\hat{k}) = 3$$

which is the equation of the required plane.

Angle between Two Planes

The angle between two planes is defined as the angle between their normals.

Thus, the angle θ between the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

• If the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ are perpendicular, then $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular.

$$\therefore \mathbf{n}_1 \cdot \mathbf{n}_2 = 0$$

• If the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ are parallel, then $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel. Therefore, there exists a scalar λ such that $\mathbf{n}_1 = \lambda \mathbf{n}_2$.

If $a_1x + b_1y + c_1z + d_1 = 0$ and

$a_2x + b_2y + c_2z + d_2 = 0$ are cartesian equations of two planes, then vectors normal to them are

$$\mathbf{n}_1 = a_1\hat{i} + b_1\hat{j} + c_1\hat{k} \text{ and } \mathbf{n}_2 = a_2\hat{i} + b_2\hat{j} + c_2\hat{k}$$

Therefore, the angle θ between the planes is given by

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

$$\Rightarrow \cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

• If the planes are perpendicular, then $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular.

$$\therefore \mathbf{n}_1 \cdot \mathbf{n}_2 = 0$$

$$\Rightarrow a_1a_2 + b_1b_2 + c_1c_2 = 0$$

• If the planes are parallel, then $\mathbf{n}_1$ and $\mathbf{n}_2$ are parallel.

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

➤ **Example 30.** The angle between the planes $\mathbf{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 6$ and $\mathbf{r} \cdot (\hat{i} + \hat{j} + 2\hat{k}) = 5$, is

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{5\pi}{6}$

Sol. (a) We know that the angle between the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$ is given by

$$\cos \theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1| |\mathbf{n}_2|}$$

Here, $\mathbf{n}_1 = 2\hat{i} - \hat{j} + \hat{k}$ and $\mathbf{n}_2 = \hat{i} + \hat{j} + 2\hat{k}$.

$$\therefore \cos \theta = \frac{(2\hat{i} - \hat{j} + \hat{k}) \cdot (\hat{i} + \hat{j} + 2\hat{k})}{|2\hat{i} - \hat{j} + \hat{k}| |\hat{i} + \hat{j} + 2\hat{k}|} = \frac{1}{2}$$

$$\Rightarrow \theta = \pi/3$$

➤ **Example 31.** The angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$, is

(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) None of these

Sol. (b) Let θ be the angle between the planes $x + y + 2z = 9$ and $2x - y + z = 15$. Then,

$$\cos \theta = \frac{(1)(2) + (1)(-1) + (2)(1)}{\sqrt{1^2 + 1^2 + 2^2} \sqrt{2^2 + (-1)^2 + 1^2}} = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}$$

➤ **Example 32.** If the planes $2x - y + \lambda z - 5 = 0$ and $x + 4y + 2z - 7 = 0$ are perpendicular, then λ equals

(a) 1 (b) -1 (c) 2 (d) -2

Sol. (a) Direction ratios of normals to given planes are proportional to 2, -1, λ and 1, 4, 2 respectively.

Since, the planes are perpendicular.

$$\therefore 2 - 4 + 2\lambda = 0 \Rightarrow \lambda = 1$$

Plane Passing through Intersection of Two Planes

The intersection of two planes is always a straight line. e.g. XY -plane and XZ -plane intersect to form X -axis. The plane containing the line of intersection of two given planes is known as the plane passing through the intersection of two given planes. In the following discussion, we will obtain the equation of family of planes passing through the intersection of two given planes.

Vector Form

The equation of a plane passing through the intersection of the planes $\mathbf{r} \cdot \mathbf{n}_1 = d_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = d_2$, is

$$(\mathbf{r} \cdot \mathbf{n}_1 - d_1) + \lambda (\mathbf{r} \cdot \mathbf{n}_2 - d_2) = 0$$

$$\text{or } \mathbf{r} \cdot (\mathbf{n}_1 + \lambda \mathbf{n}_2) = d_1 + \lambda d_2$$

where, λ is an arbitrary constant.

Cartesian Form

The equation of a plane passing through the intersection of $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$ where, λ is a constant.

➤ **Example 33.** The vector equation of a plane passing through the intersection of the planes $\mathbf{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\mathbf{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$ and the point $(2, 1, -2)$, is

- (a) $\mathbf{r} \cdot (3\hat{i} + 2\hat{j}) = 8$ (b) $\mathbf{r} \cdot (2\hat{i} + 3\hat{j}) = 8$
(c) $\mathbf{r} \cdot (3\hat{i} + 2\hat{j}) + 8 = 0$ (d) None of these

Sol. (a) The equation of a plane passing through the intersection of $\mathbf{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) = 5$ and $\mathbf{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) = 3$, is given by

$$[\mathbf{r} \cdot (\hat{i} + 3\hat{j} - \hat{k}) - 5] + \lambda [\mathbf{r} \cdot (2\hat{i} - \hat{j} + \hat{k}) - 3] = 0$$

$$\Rightarrow \mathbf{r} \cdot [(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (-1 + \lambda)\hat{k}] - 5 - 3\lambda = 0 \dots (i)$$

If Eq. (i) passes through $(2, 1, -2)$, then the vector $2\hat{i} + \hat{j} - 2\hat{k}$ should satisfy it.

$$\therefore (2\hat{i} + \hat{j} - 2\hat{k}) \cdot [(1 + 2\lambda)\hat{i} + (3 - \lambda)\hat{j} + (-1 + \lambda)\hat{k}] - (5 + 3\lambda) = 0$$

$$\Rightarrow 2(1 + 2\lambda) + 1(3 - \lambda) - 2(-1 + \lambda) - (5 + 3\lambda) = 0$$

$$\Rightarrow -2\lambda + 2 = 0 \Rightarrow \lambda = 1$$

On putting $\lambda = 1$ in Eq. (i), we get the required equation of the plane as $\mathbf{r} \cdot (3\hat{i} + 2\hat{j} + 0\hat{k}) = 8$.

➤ **Example 34.** The equation of the plane containing the line of intersection of the planes $x + y + z - 6 = 0$ and $2x + 3y + 4z + 5 = 0$ and passing through the point $(1, 1, 1)$, is

- (a) $20x + 23y + 26z - 69 = 0$
(b) $20x + 26y + 23z - 69 = 0$
(c) $x + y + z - 3 = 0$
(d) $2x + 3y + 4z - 9 = 0$

Sol. (a) The equation of the plane containing the line of intersection of the given planes, is

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0 \dots (i)$$

If it passes through $(1, 1, 1)$, then

$$-3 + 14\lambda = 0$$

$$\Rightarrow \lambda = 3/14$$

On putting $\lambda = 3/14$ in Eq. (i), we get the equation of the required plane as $20x + 23y + 26z - 69 = 0$.

➤ **Example 35.** The equation of the plane which is perpendicular to the plane $5x + 3y + 6z + 8 = 0$ and which contains the line of intersection of the planes $x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$, is

- (a) $15x + 15y - 20z + 4 = 0$
(b) $51x + 15y - 50z + 173 = 0$
(c) $3x - 5y + 7 = 0$
(d) $3x + 5y - 5z + 9 = 0$

Sol. (b) The equation of a plane passing through the line of intersection of the planes

$x + 2y + 3z - 4 = 0$ and $2x + y - z + 5 = 0$, is given by

$$(x + 2y + 3z - 4) + \lambda(2x + y - z + 5) = 0$$

$$\Rightarrow x(1 + 2\lambda) + y(2 + \lambda) + z(3 - \lambda) - 4 + 5\lambda = 0 \dots (i)$$

which is perpendicular to the plane

$$5x + 3y + 6z + 8 = 0.$$

$$\therefore 5(1 + 2\lambda) + 3(2 + \lambda) + 6(3 - \lambda) = 0$$

$$\Rightarrow \lambda = -29/7$$

On putting $\lambda = -29/7$ in Eq. (i), we get the equation of the required plane as

$$-51x - 15y + 50z - 173 = 0$$

$$\text{or } 51x + 15y - 50z + 173 = 0$$

Distance of a Point from a Plane

Vector Form

The length of the perpendicular from a point P with position vector $\mathbf{a}$ to the plane $\mathbf{r} \cdot \mathbf{n} = d$, is given by

$$p = \frac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$$

Cartesian Form

The length of the perpendicular from a point $P(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$, is given by

$$\frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

➤ **Example 36.** The distance of the point $2\hat{i} + \hat{j} - \hat{k}$ from the plane $\mathbf{r} \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) = 9$, is

- (a) 13 (b) $\frac{13}{\sqrt{21}}$ (c) $\frac{15}{\sqrt{21}}$ (d) $\frac{-13}{\sqrt{21}}$

Sol. (b) We know that the perpendicular distance of a point p with position vector $\mathbf{a}$ from the plane $\mathbf{r} \cdot \mathbf{n} = d$, is given by

$$\frac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|}$$

Here, $\mathbf{a} = 2\hat{i} + \hat{j} - \hat{k}$, $\mathbf{n} = \hat{i} - 2\hat{j} + 4\hat{k}$ and $d = 9$

$\therefore$ Required distance

$$= \frac{|(2\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{i} - 2\hat{j} + 4\hat{k}) - 9|}{\sqrt{1 + 4 + 16}} = \frac{13}{\sqrt{21}}$$

➤ **Example 37.** The distance between the line $\mathbf{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $\mathbf{a} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$, is

- (a) $\frac{10}{9}$ (b) $\frac{10}{3\sqrt{3}}$
(c) $\frac{10}{3}$ (d) None of these

Sol. (b) The given line is $r = a + tb$, where
 $a = 2\hat{i} - 2\hat{j} + 3\hat{k}$, $b = \hat{i} - \hat{j} + 4\hat{k}$ and given plane is
 $r \cdot n = d$, where $n = \hat{i} + 5\hat{j} + \hat{k}$, $d = 5$.

Since, $b \cdot n = 1 - 5 + 4 = 0$

Given line is parallel to the given plane.

So, the distance between the line and the plane is equal to length of the perpendicular from the point $a = 2\hat{i} - 2\hat{j} + 3\hat{k}$ on the line to the given plane.

$$\therefore \text{Required distance} = \frac{|(2\hat{i} - 2\hat{j} + 3\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) - 5|}{\sqrt{1 + 25 + 1}} \\ = \frac{|2 - 10 + 3 - 5|}{\sqrt{27}} = \frac{10}{3\sqrt{3}}$$

➤ **Example 38.** The equation of the plane passing through the intersection of the planes $r \cdot (2\hat{i} + 6\hat{j}) + 12 = 0$ and $r \cdot (3\hat{i} - \hat{j} + 4\hat{k}) = 0$ at a unit distance from the origin, is

(a) $r \cdot (2\hat{i} + \hat{j} + 2\hat{k}) + 3 = 0$

(b) $r \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) + 3 = 0$

(c) $r \cdot (\hat{i} - 2\hat{j} - 2\hat{k}) + 3 = 0$

(d) $r \cdot (2\hat{i} + \hat{j} + 2\hat{k}) - 3 = 0$

Sol. (a) The equation of the plane passing through the intersection of the given planes, is

$$\{r \cdot (2\hat{i} + 6\hat{j}) + 12\} + \lambda \{r \cdot (3\hat{i} - \hat{j} + 4\hat{k})\} = 0 \\ \text{or } r \cdot \{(2 + 3\lambda)\hat{i} + (6 - \lambda)\hat{j} + 4\lambda\hat{k} + 12\} = 0 \quad \dots(i)$$

Since, it is a unit distance from the origin.

$$\therefore \frac{|12|}{\sqrt{(2 + 3\lambda)^2 + (6 - \lambda)^2 + 16\lambda^2}} = 1 \Rightarrow \lambda = \pm 2$$

Hence, the equations of the required planes are $r \cdot (2\hat{i} + \hat{j} + 2\hat{k}) + 3 = 0$ and $r \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) - 3 = 0$.

Distance between the Parallel Planes

Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_1x + b_1y + c_1z + d_2 = 0$ be two parallel planes. The distance between these planes can be obtained by taking any point $P(x_1, y_1, z_1)$ on anyone of the given planes and finding the perpendicular distance from $P(x_1, y_1, z_1)$ on the other. Also, distance between parallel planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ is given by

$$\frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$$

If $r \cdot n = d_1$ and $r \cdot n = d_2$ are vector equations in two parallel planes, then distance between them is given by $\frac{|d_1 - d_2|}{n}$.

➤ **Example 39.** The distance between the parallel planes $2x - y + 2z + 3 = 0$ and $4x - 2y + 4z + 5 = 0$, is

(a) $1/3$

(b) $2/6$

(c) $2/3$

(d) None of these

Sol. (d) Given equations of two planes are
 $2x - y + 2z + 3 = 0$ and $2x - y + 2z + \frac{5}{2} = 0$

$$\therefore \text{Required distance} = \frac{|3 - \frac{5}{2}|}{\sqrt{4 + 1 + 4}} = \frac{1}{6}$$

➤ **Example 40.** The distance between the parallel planes $r \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$ and $r \cdot (6\hat{i} - 9\hat{j} + 18\hat{k}) + 20 = 0$, is

(a) $\frac{2}{3}$

(b) $\frac{5}{3}$

(c) $\frac{4}{3}$

(d) $\frac{1}{3}$

Sol. (b) Given equations of two planes are

$$r \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = 5$$

and $r \cdot (2\hat{i} - 3\hat{j} + 6\hat{k}) = -\frac{20}{3}$

$$\therefore \text{Distance between the planes} = \frac{|5 - (-\frac{20}{3})|}{|2\hat{i} - 3\hat{j} + 6\hat{k}|} = \frac{5}{3}$$

➤ **Example 41.** If d_1 , d_2 and d_3 denote the distances of the plane $2x - 3y + 4z = 2$ from the planes $2x - 3y + 4z + 6 = 0$, $4x - 6y + 8z + 3 = 0$ and $2x - 3y + 4z - 6 = 0$ respectively. Then,

(a) $d_1 + 8d_2 + d_3 = 0$

(b) $d_1 + 16d_2 = 0$

(c) $8d_2 = d_1$

(d) $d_1 - 2d_2 + 3d_3 = \sqrt{29}$

Sol. (c) We have,

$$d_1 = \frac{|6 - 2|}{\sqrt{2^2 + (-3)^2 + 4^2}} = \frac{4}{\sqrt{29}}$$

$$d_2 = \frac{|3/2 - 2|}{\sqrt{2^2 + (-3)^2 + 4^2}} = \frac{1}{2\sqrt{29}}$$

and $d_3 = \frac{|-6 - 2|}{\sqrt{2^2 + (-3)^2 + 4^2}} = \frac{8}{\sqrt{29}}$

$$\therefore 8d_2 = d_1$$

Planes Bisecting the Angle between Two Given Planes

Vector Form

The equation of the planes bisecting the angles between the planes $r \cdot n_1 = d_1$ and $r \cdot n_2 = d_2$, are

$$\frac{|r \cdot n_1 - d_1|}{|n_1|} = \frac{|r \cdot n_2 - d_2|}{|n_2|}$$

or $\frac{r \cdot n_1 - d_1}{|n_1|} = \pm \frac{r \cdot n_2 - d_2}{|n_2|}$

or $r \cdot (n_1 \pm n_2) = \frac{d_1}{|n_1|} \pm \frac{d_2}{|n_2|}$

Cartesian Form

The equation of the planes bisecting the angles between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm \frac{(a_2x + b_2y + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}} \quad \dots(i)$$

Bisector of Acute/ Obtuse Angle

- Write the equations of the given planes such that their constant terms are positive.
- If $a_1a_2 + b_1b_2 + c_1c_2 > 0$, then origin lies in obtuse angle and hence positive sign in Eq. (i) gives the bisector of the obtuse angle.
- If $a_1a_2 + b_1b_2 + c_1c_2 < 0$, then origin lies in acute angle and hence positive sign in Eq. (i) gives the bisector of the acute angle.

➤ **Example 42.** The equation of the plane which bisects the angle between the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z - 3 = 0$ which contains the origin, is

(a) $33x - 13y + 32z + 45 = 0$

(b) $x - 3y + z - 5 = 0$

(c) $33x + 13y + 32z + 45 = 0$

(d) None of the above

Sol. (d) Equation of plane bisecting the angle containing origin is (making constant term of same sign)

$$\frac{-3x + 6y - 2z - 5}{\sqrt{3^2 + 6^2 + 2^2}} = + \frac{4x - 12y + 3z - 3}{\sqrt{4^2 + 12^2 + 3^2}}$$

$$\text{or } \frac{-3x + 6y - 2z - 5}{7} = \frac{4x - 12y + 3z - 3}{13}$$

$$\text{or } 67x - 162y + 47z + 44 = 0$$

Important Facts Related to Plane

- Image of a point in a plane** Image (x, y, z) (or reflection) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

- Foot** (x, y, z) of perpendicular drawn from a point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$$

- Any plane parallel to XY -plane is $z = \text{constant}$, similarly plane parallel to YZ -plane is $x = \text{constant}$ and plane parallel to ZX -plane is $y = \text{constant}$, where $x = 0$, $y = 0$ and $z = 0$ are respectively YZ , ZX and XY -planes.

- Any plane parallel to X -axis is of the form $by + cz = d$.

- Position of the points** Points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are on same side of plane $ax + by + cz + d = 0$, if $ax_1 + by_1 + cz_1 + d$ and $ax_2 + by_2 + cz_2 + d$ are of same sign. If they are of opposite sign, then the points are on the opposite sides.

- Equation of plane parallel to planes $ax + by + cz + d_1 = 0$ and $ax + by + cz + d_2 = 0$ and equidistance from them, is

$$ax + by + cz + \left(\frac{d_1 + d_2}{2} \right) = 0$$

- If **a, b, c** and **d** are coplanar, then

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = 0$$

- Plane passing through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

where,

$$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0$$

- Plane $ax + by + cz + d = 0$ intersecting a line segment joining $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ divides in the ratio $\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$.

- If this ratio is positive, then A and B are on opposite sides of the plane.
- If this ratio is negative, then A and B are on the same side of the plane.

➤ **Example 43.** The image of the point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$, is

(a) $(3, 5, 2)$

(b) $(-3, 5, 2)$

(c) $(3, 5, -2)$ (d) $(3, -5, 2)$

Sol. (b) Let (α, β, γ) be the image of the point $(1, 3, 4)$ in the plane $2x - y + z + 3 = 0$. Then,

$$\frac{\alpha - 1}{2} = \frac{\beta - 3}{-1} = \frac{\gamma - 4}{1} = - \frac{2(2 \times 1 - 3 \times 1 + 4 \times 1 + 3)}{2^2 + (-1)^2 + 1^2}$$

$$\Rightarrow \frac{\alpha - 1}{2} = \frac{\beta - 3}{-1} = \frac{\gamma - 4}{1} = -2$$

$$\Rightarrow \alpha = -3, \beta = 5, \gamma = 2$$

Thus, the image has the coordinates $(-3, 5, 2)$.

Work Book Exercise 14.3

14

Three Dimensional Geometry

- The intercepts of the plane $2x - 3y + 4z = 12$ on the coordinate axes are given by
 - $3, -2, 15$
 - $6, -4, 3$
 - $6, -4, -3$
 - $2, -3, 4$
- The equation of the plane passing through the mid-point of the line of join of the points $(1, 2, 3)$ and $(3, 4, 5)$ and perpendicular to it, is
 - $x + y + z = 9$
 - $x + y + z = -9$
 - $2x + 3y + 4z = 9$
 - $2x + 3y + 4z = -9$
- If θ is the angle between the planes $2x - y + z - 4 = 0$ and $x - 2y + z + 2 = 0$, then $\cos \theta$ is equal to
 - $\frac{2}{3}$
 - $\frac{3}{4}$
 - $\frac{4}{5}$
 - $\frac{5}{6}$
- Distance between two parallel planes $4x + 2y + 4z + 5 = 0$ and $2x + y + 2z - 8 = 0$ is
 - $\frac{7}{2}$
 - $\frac{2}{7}$
 - $-\frac{7}{2}$
 - $-\frac{2}{7}$
- If the plane $\frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$ cuts the coordinate axes in A, B, C , then the area of ΔABC is
 - $\sqrt{29}$ sq units
 - $\sqrt{41}$ sq units
 - $\sqrt{61}$ sq units
 - None of the above
- If $(2, -1, 3)$ is the foot of the perpendicular drawn from the origin to the plane, then the equation of the plane is
 - $2x + y - 3z + 6 = 0$
 - $2x - y + 3z - 14 = 0$
 - $2x - y + 3z - 13 = 0$
 - $2x + y + 3z - 10 = 0$
- The equation of the plane passing through the points $(0, 1, 2)$ and $(-1, 0, 3)$ and perpendicular to the plane $2x + 3y + z = 5$, is
 - $3x - 4y + 18z + 32 = 0$
 - $3x + 4y - 18x + 32 = 0$
 - $4x + 3y - 17z + 31 = 0$
 - $4x - 3y + z + 1 = 0$
- The reflection of the plane $2x - 3y + 4z - 3 = 0$ in the plane $x - y + z - 3 = 0$ is the plane
 - $4x - 3y + 2z - 15 = 0$
 - $x - 3y + 2z - 15 = 0$
 - $4x + 3y - 2z + 15 = 0$
 - None of the above
- The image of the point $(5, 4, 6)$ in the plane $x + y + 2z - 15 = 0$, is
 - $(3, 2, 2)$
 - $(2, 3, 2)$
 - $(2, 2, 3)$
 - $(-5, -4, -6)$
- A variable plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ at a unit distance from origin cuts the coordinate axes at A, B and C . If centroid (x, y, z) satisfies the equation $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = k$, then the value of k is
 - 9
 - 3
 - $1/9$
 - $1/3$

Line and Plane Equation of Plane Passing through a Given Line

- If equation of the line is given in symmetrical form as $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$, then equation of plane is

$$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0,$$
 where a, b, c are given by $al + bm + cn = 0$.
- If equation of line is given in general form as

$$a_1x + b_1y + c_1z + d_1 = 0$$
 and

$$a_2x + b_2y + c_2z + d_2 = 0$$
 then the equation of plane passing through this line is $(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$
- If the plane pass through parallel lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{b}$, then equation of the required plane is

$$[\mathbf{r} - \mathbf{a} \quad \mathbf{c} - \mathbf{a} \quad \mathbf{b}] = 0.$$

Example 44. The equation of the plane passing through the point $(0, 7, -7)$ and containing the line $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$, is

- $x + 7y + 7z = 0$
- $x + y + z = 7$
- $x + y + z = 0$
- $x + y - z = 14$

Sol. (c) Any plane passing through $(0, 7, -7)$ is

$$a(x-0) + b(y-7) + c(z+7) = 0 \quad \dots(i)$$

Eq. (i) contains the given line, it must pass through the point $(-1, 3, -2)$ and must be parallel to this line.

As Eq. (i) passes through $(-1, 3, -2)$, we get

$$a(-1-0) + b(3-7) + c(-2+7) = 0$$

$$\Rightarrow a + 4b - 5c = 0 \quad \dots(ii)$$

Also, Eq. (i) is parallel to the given line.

$$\therefore (-3)a + 2b + 1 \cdot c = 0$$

$$\Rightarrow -3a + 2b + c = 0 \quad \dots(iii)$$

From Eqs. (i) and (iii), we get

$$\frac{a}{14} = \frac{b}{14} = \frac{c}{14} \Rightarrow a = b = c = k \quad [\text{say}]$$

Putting $a = k, b = k$ and $c = k$ in Eq. (i), we get

$$kx + k(y-7) + k(z+7) = 0 \text{ or } x + y + z = 0$$

which is the required equation of the plane.

➤ **Example 45.** The point of intersection of the line $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4}$ and the plane

$$2x + 4y - z + 1 = 0 \text{ is}$$

$$(a) \left(-\frac{10}{3}, \frac{3}{2}, \frac{5}{3}\right) \quad (b) \left(-\frac{10}{3}, -\frac{3}{2}, \frac{5}{3}\right)$$

$$(c) \left(\frac{10}{3}, \frac{3}{2}, -\frac{5}{3}\right) \quad (d) \left(\frac{10}{3}, -\frac{3}{2}, \frac{5}{3}\right)$$

Sol. (d) Line: $\frac{x-1}{2} = \frac{y-2}{-3} = \frac{z+3}{4} = \lambda$

[say]

and plane: $2x + 4y - z + 1 = 0$

∴ Point of intersection of line and plane is

$$2(2\lambda + 1) + 4(-3\lambda + 2) - (4\lambda - 3) + 1 = 0$$

$$\Rightarrow 4\lambda + 2 - 12\lambda + 8 - 4\lambda + 3 + 1 = 0$$

$$\Rightarrow 12\lambda = 14$$

$$\Rightarrow \lambda = \frac{14}{12} = \frac{7}{6}$$

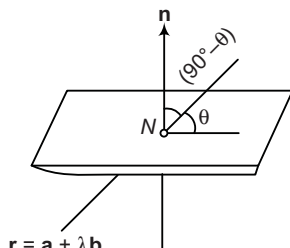
∴ Point of intersection of line and plane is $\left(\frac{10}{3}, -\frac{3}{2}, \frac{5}{3}\right)$.

Angle between a Line and a Plane

Let the equation of line be $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ and equation of plane be $\mathbf{r} \cdot \mathbf{n} = q$.

Let θ be the angle between a line and a plane, then

$$\sin \theta = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}| |\mathbf{n}|}$$



Angle between a line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and

plane $a_2x + b_2y + c_2z + d_2 = 0$, is given by

$$\sin \theta = \frac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

➤ If the line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ lies in the plane

$$a_2x + b_2y + c_2z + d_2 = 0, \text{ then } a_2x_1 + b_2y_1 + c_2z_1 + d_2 = 0$$

$$\text{and } a_1a_2 + b_1b_2 + c_1c_2 = 0.$$

Condition of Perpendicularity

If the line is perpendicular to the plane, then it is parallel to the normal to the plane. Therefore, $\mathbf{b}$ and $\mathbf{n}$ are parallel.

So, $\mathbf{b} \times \mathbf{n} = 0$ or $\mathbf{b} = \lambda \mathbf{n}$, for some scalar λ .

Condition of Parallelism

If the line is parallel to the plane, then it is perpendicular to the normal to the plane. Therefore, $\mathbf{b}$ and $\mathbf{n}$ are perpendicular.

$$\text{So, } \mathbf{b} \cdot \mathbf{n} = 0$$

➤ **Example 46.** The angle between the line

$$\frac{x+1}{2} = \frac{y}{3} = \frac{z-3}{6} \text{ and the plane } 10x + 2y - 11z = 3$$

is

$$(a) \frac{\pi}{2} \quad (b) \frac{\pi}{4} \quad (c) \frac{\pi}{6} \quad (d) \sin^{-1}\left(\frac{8}{21}\right)$$

Sol. (d) Let θ be the angle between the line and the plane, then

$$a = 2, b = 3, c = 6 \text{ and } a_1 = 10, b_1 = 2, c_1 = -11$$

$$\begin{aligned} \cos(90^\circ - \theta) &= \sin \theta = \frac{|aa_1 + bb_1 + cc_1|}{\sqrt{a^2 + b^2 + c^2} \cdot \sqrt{a_1^2 + b_1^2 + c_1^2}} \\ &= \frac{|(2)(10) + (3)(2) + (6)(-11)|}{\sqrt{2^2 + 3^2 + 6^2} \cdot \sqrt{10^2 + 2^2 + 11^2}} \\ &= \frac{|20 + 6 - 66|}{\sqrt{49} \cdot \sqrt{225}} = \frac{|-40|}{7 \cdot 15} \\ \Rightarrow \theta &= \sin^{-1}\left(\frac{8}{21}\right) \end{aligned}$$

➤ **Example 47.** Given line $L: \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-3}{-1}$

and the plane $\pi: x - 2y - z = 0$. Of the following assertions, the only one that is always true, is

- (a) L is perpendicular to π
- (b) L lies in π
- (c) L is parallel to π
- (d) None of the above

Sol. (b) Since, $3(1) + 2(-2) + (-1)(-3) = 3 - 4 + 3 = 0$

So, given line is perpendicular to the normal to the plane.

Given line is parallel to the given plane.

Also, $(1, -1, 3)$ lies on the plane $x - 2y - z = 0$ if

$$1 - 2(-1) - 3 = 0$$

i.e. $1 + 2 - 3 = 0$, which is true.

∴ L lies in plane π .

Coplanarity of Two Lines

Vector Form

If the lines $\mathbf{r} = \mathbf{a}_1 + \lambda \mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + \mu \mathbf{b}_2$ are coplanar, then

$$[\mathbf{a}_1 \ \mathbf{b}_1 \ \mathbf{b}_2] = [\mathbf{a}_2 \ \mathbf{b}_1 \ \mathbf{b}_2]$$

and the equation of the plane containing them is

$$[\mathbf{r} \ \mathbf{b}_1 \ \mathbf{b}_2] = [\mathbf{a}_1 \ \mathbf{b}_1 \ \mathbf{b}_2] \text{ or } [\mathbf{r} \ \mathbf{b}_1 \ \mathbf{b}_2] = [\mathbf{a}_2 \ \mathbf{b}_1 \ \mathbf{b}_2]$$

Cartesian Form

If the line $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar, then

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

and the equation of the plane containing them is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

or $\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

➤ **Example 48.** The equation of the plane containing the lines $\mathbf{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$

and $\mathbf{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$, is

(a) $\mathbf{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) + 14 = 0$

(b) $\mathbf{r} \cdot (3\hat{i} + 9\hat{j} + 2\hat{k}) = 14$

(c) $\mathbf{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14$

(d) None of the above

Sol. (c) Here, $a_1 = \hat{i} + \hat{j} - \hat{k}$, $a_2 = 4\hat{i} + 0\hat{j} - \hat{k}$, $b_1 = 3\hat{i} - \hat{j}$ and $b_2 = 2\hat{i} + 0\hat{j} + 3\hat{k}$.

The equation of the plane containing the given lines is

$$[\mathbf{r} \cdot \mathbf{b}_1 \mathbf{b}_2] = [\mathbf{a}_1 \mathbf{b}_1 \mathbf{b}_2] \Rightarrow \mathbf{r} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = -14$$

$$\Rightarrow \mathbf{r} \cdot (-3\hat{i} - 9\hat{j} + 2\hat{k}) = -14$$

$$[\therefore \mathbf{b}_1 \times \mathbf{b}_2 = -3\hat{i} - 9\hat{j} + 2\hat{k}]$$

$$\Rightarrow \mathbf{r} \cdot (3\hat{i} + 9\hat{j} - 2\hat{k}) = 14$$

➤ **Example 49.** The plane containing these two

lines $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and

$\frac{x-2}{1} = \frac{y-4}{4} = \frac{z-6}{7}$, is

(a) $x - 2y + z = 0$

(b) $x + 2y - z = 0$

(c) $x - 2y + z = 1$

(d) None of the above

Sol. (a) The equation of the plane containing the given lines is

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 4 & 7 \end{vmatrix} = 0 \Rightarrow x - 2y + z = 0$$

➤ **Example 50.** The equation of the plane

containing two lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-0}{3}$ and

$\frac{x}{-2} = \frac{y-2}{-3} = \frac{z+1}{-1}$, is

(a) $8x + y - 5z - 7 = 0$

(b) $8x + y + 5z - 7 = 0$

(c) $8x - y - 5z - 7 = 0$

(d) None of the above

Sol. (d) Given lines pass through $(1, -1, 0)$ and $(0, 2, -1)$ respectively and parallel to the vectors $2\hat{i} - \hat{j} + 3\hat{k}$ and $-2\hat{i} - 3\hat{j} - \hat{k}$.

$$\therefore [\mathbf{a}_2 - \mathbf{a}_1 \quad \mathbf{b}_1 \quad \mathbf{b}_2] = \begin{vmatrix} -1 & 3 & -1 \\ 2 & -1 & 3 \\ -2 & -3 & -1 \end{vmatrix} = -14 \neq 0$$

So, the given lines are non-coplanar.

Work Book Exercise 14.4

1 Angle between the line $\frac{x+1}{1} = \frac{y}{2} = \frac{z-1}{1}$ and a normal to the plane $x - y + z = 0$, is
a 0° b 30° c 45° d 90°

2 If the plane $2x - y + z = 0$ is parallel to the line $\frac{2x-1}{2} = \frac{2-y}{2} = \frac{z+1}{2}$, then the value of a is
a 4 b -4 c 2 d -2

3 The point of intersection of the line $\frac{x-1}{3} = \frac{y+2}{4} = \frac{z-3}{-2}$ and plane $2x - y + 3z - 1 = 0$, is
a $(10, -10, 3)$ b $(10, 10, -3)$
c $(-10, 10, 3)$ d None of these

4 If equation of plane passing through the line $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z-3}{4}$ and perpendicular to the plane $x + 2y + z = 12$, is given by $ax + by + cz + 4 = 0$, then
a $a = -8, b = 2, c = -5$
b $a = -9, b = -2, c = -5$
c $a = 9, b = -2, c = -5$
d None of the above

5 The distance of the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$, is
a 13 b 12
c 11 d 8

- 6 The equation of the line passing through the point $(3, 0, -4)$ and perpendicular to the plane $2x - 3y + 5z - 7 = 0$, is

a $\frac{x-2}{3} = \frac{y}{-3} = \frac{z+4}{5}$
 b $\frac{x-3}{2} = \frac{y}{-3} = \frac{z-4}{5}$
 c $\frac{x-3}{2} = \frac{-y}{3} = \frac{z+4}{5}$
 d $\frac{x+3}{2} = \frac{y}{3} = \frac{z-4}{5}$

- 7 The equation of the plane passing through the point $(2, -1, -3)$ and parallel to the lines $\frac{x-1}{3} = \frac{y+2}{2} = \frac{z}{-4}$ and $\frac{x}{2} = \frac{y-1}{-3} = \frac{z-2}{2}$, is

a $8x + 14y + 13z + 37 = 0$
 b $8x - 14y + 13z + 37 = 0$
 c $8x + 14y - 13z + 37 = 0$
 d $8x + 14y + 13z - 37 = 0$

- 8 The equation of line of intersection of planes $4x + 4y - 5z = 12$, $8x + 12y - 13z = 32$ can be written as

a $\frac{x-1}{2} = \frac{y+2}{-3} = \frac{z}{4}$ b $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z}{4}$
 c $\frac{x}{2} = \frac{y+1}{3} = \frac{z-2}{4}$ d $\frac{x}{2} = \frac{y}{3} = \frac{z-2}{4}$

- 9 If the plane $3x + y + 2z + 5 = 0$ is parallel to the line $\frac{3x-1}{2b} = 3 - y = \frac{z-1}{a}$, then the value of $3a + 3b$ is

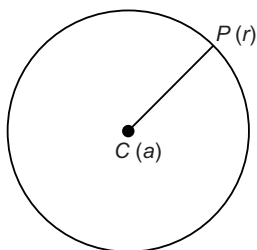
a $\frac{1}{2}$ b $\frac{3}{2}$ c 3 d 4

- 10 The distance of the point $(3, 8, 2)$ from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z = 0$, is

a 2 b 3 c 6 d 7

Sphere

A sphere is the locus of a point which moves in space in such a way that its distance from a fixed point always remains constant.



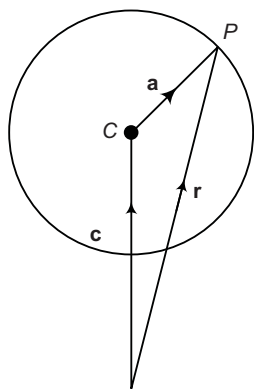
The fixed point is called the centre and the constant distance is called the radius of the sphere.

Equation of a Sphere

Vector Form

The vector equation of a sphere of radius r and centre having position vector $\mathbf{c}$, is $|\mathbf{r} - \mathbf{c}| = r$.

The vector equation of a sphere of radius a with centre at the origin, is $|\mathbf{r}| = a$.



Cartesian Form

The equation of a sphere with centre (a, b, c) and radius k , is given by

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = k^2$$

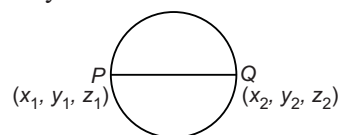
The equation of a sphere with centre at origin and radius k , is $x^2 + y^2 + z^2 = k^2$.

- (i) **General equation of a sphere** The general equation $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere with centre $(-u, -v, -w)$ and radius equal to $\sqrt{u^2 + v^2 + w^2 - d}$.

- (ii) **Equation of a sphere passes through four points** Equation of a sphere passing through four non-coplanar points (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4) , is

$$\begin{vmatrix} x^2 + y^2 + z^2 & x & y & z & 1 \\ x_1^2 + y_1^2 + z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2 + y_2^2 + z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2 + y_3^2 + z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2 + y_4^2 + z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

- (iii) **Equation of a sphere, the extremities of diameter being given** The equation of a sphere described on the joining of two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ as diameter, is given by



$$(x-x_1)(x-x_2) + (y-y_1)(y-y_2) + (z-z_1)(z-z_2) = 0$$

- (iv) **Condition of tangency** Condition for the plane $\mathbf{r} \cdot \mathbf{n} = d$ to touch the sphere $|\mathbf{r} - \mathbf{c}| = a$ is

$$\frac{|\mathbf{c} \cdot \mathbf{n} - d|}{|\mathbf{n}|} = a$$

- **Example 51.** Chord AB is a diameter of the sphere $|\mathbf{r} - 2\hat{i} - \hat{j} + 6\hat{k}| = \sqrt{18}$. If the coordinates of A are $(3, 2, -2)$, then the coordinates of B are
- (a) $(1, 0, 10)$ (b) $(1, 0, -10)$
(c) $(-1, 0, 10)$ (d) None of these

Sol. (b) The equation of the sphere is

$$|\mathbf{r} - 2\hat{i} - \hat{j} + 6\hat{k}| = \sqrt{18}$$

Its centre is at the point $(2\hat{i} + \hat{j} - 6\hat{k})$ i.e. at $(2, 1, -6)$.

Coordinates of A are $(3, 2, -2)$.

Let the coordinates of B be (α, β, γ) .

$$\text{Then, } \frac{3 + \alpha}{2} = 2, \frac{2 + \beta}{2} = 1 \text{ and } \frac{-2 + \gamma}{2} = -6$$

$$\Rightarrow \alpha = 1, \beta = 0, \gamma = -10$$

Therefore, coordinates of B are $(1, 0, -10)$.

Example 52. The equation of the sphere whose centre has the position vector $(3\hat{i} + 6\hat{j} - 4\hat{k})$ and which touches the plane $\mathbf{r} \cdot (2\hat{i} - 2\hat{j} - \hat{k}) = 10$, is

- (a) $|\mathbf{r} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = 4$
(b) $|\mathbf{r} - (3\hat{i} + 6\hat{j} + 4\hat{k})| = 4$
(c) $|\mathbf{r} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = 2$
(d) None of the above

Sol. (a) The centre of the sphere has the position vector

$$3\hat{i} + 6\hat{j} - 4\hat{k}$$

Radius of sphere = Distance of the centre

$$a = 3\hat{i} + 6\hat{j} - 4\hat{k}, \text{ from the plane } \mathbf{r} \cdot (2\hat{i} - 2\hat{j} - \hat{k}) = 10$$

$$\begin{aligned} &= \frac{|\mathbf{a} \cdot \mathbf{n} - d|}{|\mathbf{n}|} = \frac{|(3\hat{i} + 6\hat{j} - 4\hat{k}) \cdot (2\hat{i} - 2\hat{j} - \hat{k}) - 10|}{\sqrt{2^2 + (-2)^2 + (-1)^2}} \\ &= \frac{|6 - 12 + 4 - 10|}{\sqrt{4 + 4 + 1}} = \frac{|-12|}{3} = \frac{12}{3} = 4 \end{aligned}$$

So, centre of the sphere is $(3, 6, -4)$ and radius of the sphere is 4.

∴ Required equation of the sphere is

$$|\mathbf{r} - (3\hat{i} + 6\hat{j} - 4\hat{k})| = 4$$

- **Example 53.** The equation of the line passing through the point with position vector $2\hat{i} - 3\hat{j} - 5\hat{k}$ and perpendicular to the plane $\mathbf{r} \cdot (6\hat{i} - 3\hat{j} - 5\hat{k}) + 2 = 0$, is
- (a) $\mathbf{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + \lambda(-6\hat{i} + 3\hat{j} + 5\hat{k})$
(b) $\mathbf{r} = 2\hat{i} + 3\hat{j} - 5\hat{k} + \lambda(-6\hat{i} + 3\hat{j} + 5\hat{k})$
(c) $\mathbf{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + \lambda(6\hat{i} + 3\hat{j} + 5\hat{k})$
(d) None of the above

Sol. (a) Given equation of plane is $\mathbf{r} \cdot (6\hat{i} - 3\hat{j} - 5\hat{k}) = -2$.

$$\Rightarrow \mathbf{r} \cdot (-6\hat{i} + 3\hat{j} + 5\hat{k}) = 2$$

A vector normal to the plane is $-6\hat{i} + 3\hat{j} + 5\hat{k}$.

Direction ratios of normal to the plane are $-6, 3, 5$.

Since, the required line is perpendicular to the plane, therefore direction ratios of line are $-6, 3, 5$.

Hence, vector equation of the line passing through the point whose position vector is $2\hat{i} - 3\hat{j} - 5\hat{k}$ and having direction ratios $-6, 3, 5$, is

$$\mathbf{r} = 2\hat{i} - 3\hat{j} - 5\hat{k} + \lambda(-6\hat{i} + 3\hat{j} + 5\hat{k})$$

➤ **Example 54.** Radius of the circle

$$\mathbf{r}^2 + \mathbf{r} \cdot (2\hat{i} - 2\hat{j} - 4\hat{k}) - 19 = 0,$$

$$\mathbf{r} \cdot (\hat{i} - 2\hat{j} + 2\hat{k}) + 8 = 0, \text{ is}$$

- (a) 5
(b) 4
(c) 3
(d) 2

Sol. (b) Given circle is intersection of sphere

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0 \quad \dots(i)$$

$$\text{and plane } x - 2y + 2z + 8 = 0 \quad \dots(ii)$$

Centre of sphere is $(-1, 1, 2)$.

Length of the perpendicular from $(-1, 1, 2)$ upon Eq. (ii),

$$p = \frac{-1 - 2 + 4 + 8}{\sqrt{1 + 4 + 4}} = \frac{9}{3} = 3$$

$$\text{Radius of the sphere, } R = \sqrt{1 + 1 + 4 + 19} = 5$$

$$\begin{aligned} \therefore \text{Radius of the circle} &= \sqrt{R^2 - p^2} \\ &= \sqrt{25 - 9} \\ &= \sqrt{16} = 4 \end{aligned}$$

➤ **Example 55.** A sphere of constant radius k passes through origin and meets axes in A, B, C . The centroid of $\triangle ABC$ lies on the sphere

- (a) $9(x^2 + y^2 + z^2) = 4k^2$
(b) $3(x^2 + y^2 + z^2) = 4k^2$
(c) $x^2 + y^2 + z^2 = 4k^2$
(d) None of the above

Sol. (a) Let the coordinates of the points A, B , and C be $(a, 0, 0)$, $(0, b, 0)$, and $(0, 0, c)$, respectively.

Then, equation of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Radius of this sphere is equal to k .

$$\therefore a^2 + b^2 + c^2 = 4k^2 \quad \dots(i)$$

Coordinates (x, y, z) of the centroid of $\triangle ABC$ are given

$$\text{by } x = \frac{1}{3}a, y = \frac{1}{3}b, z = \frac{1}{3}c, \text{ so that}$$

$$a = 3x, b = 3y, c = 3z \quad \dots(ii)$$

Eliminating a, b, c from Eqs. (i) and (ii), we get

$$(3x)^2 + (3y)^2 + (3z)^2 = 4k^2$$

$$\Rightarrow 9(x^2 + y^2 + z^2) = 4k^2$$

Work Book Exercise 14.5

- The radius of sphere $x^2 + y^2 + z^2 = x + 2y + 3z$ is
 a $\frac{\sqrt{14}}{2}$ b $\sqrt{7}$ c $\frac{7}{2}$ d $\frac{\sqrt{7}}{2}$
- Let $(3, 4, -1)$ and $(-1, 2, 3)$ be the end points of a diameter of sphere. Then, the radius of the sphere is equal to
 a 1 b 2 c 3 d 9
- The centre of sphere passes through four points $(0, 0, 0)$, $(0, 2, 0)$, $(1, 0, 0)$ and $(0, 0, 4)$, is
 a $\left(-\frac{1}{2}, 1, 2\right)$ b $\left(1, \frac{1}{2}, 2\right)$
 c $\left(\frac{1}{2}, 1, 2\right)$ d $\left(\frac{1}{2}, 1, -2\right)$
- If $\mathbf{a}$ is a constant vector and p is a real constant with $|\mathbf{a}|^2 > p$, then the locus of a point with position vector $\mathbf{r}$ such that $|\mathbf{r}|^2 - 2\mathbf{r} \cdot \mathbf{a} + p = 0$, is
 a a sphere b an ellipse
 c a circle d a plane
- If a plane passes through a fixed point (q, p) and cut the axes in A, B, C . Then, the locus of the centre of the sphere $OABC$ is
 a $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 2$ b $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 1$
 c $\frac{p}{x} + \frac{q}{y} + \frac{r}{z} = 3$ d None of these
- The shortest distance from the point $(1, 2, -1)$ to the surface of the sphere $x^2 + y^2 + z^2 = 54$ is
 a $3\sqrt{6}$ b $2\sqrt{6}$
 c $\sqrt{6}$ d 2
- The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 1$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = -4$ is same as the intersection of one of the spheres and the plane is
 a $2x - y - z = 1$
 b $-2x + y + z = 1$
 c $2x - y + z = 1$
 d $2x + y + z = 1$
- If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid-point of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals
 a -2 b 2
 c -1 d 1
- The equation of the sphere whose centre is $(6, -1, 2)$ which touches the plane $2x - y + 2z - 2 = 0$, is
 a $x^2 + y^2 + z^2 - 12x + 2y - 4z - 16 = 0$
 b $x^2 + y^2 + z^2 - 12x + 2y - 4z = 0$
 c $x^2 + y^2 + z^2 - 12x + 2y - 4z + 16 = 0$
 d $x^2 + y^2 + z^2 - 12x + 2y - 4z + 6 = 0$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Let $P(3, 2, 6)$ be a point in space and Q be a point on the line

$$\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \mu(-3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}}).$$

Then, the value of μ for which the vector $\mathbf{PQ}$ is parallel to the plane $x - 4y + 3z = 1$, is

- (a) $1/4$ (b) $-1/4$
(c) $1/8$ (d) $-1/8$

Sol. Let O be the origin, as Q lies on the given line, $\mathbf{OQ} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}} + \mu(-3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 5\hat{\mathbf{k}})$

Also, $\mathbf{OP} = 3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}$

As PQ is parallel to the plane $x - 4y + 3z = 1$,

$$\therefore \mathbf{PQ} \cdot (\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 0$$

$$\Rightarrow (\mathbf{OQ} - \mathbf{OP}) \cdot (\hat{\mathbf{i}} - 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) = 0$$

$$\Rightarrow (1 + 4 + 6) + \mu(-3 - 4 + 15) - (3 - 8 + 18) = 0$$

$$\Rightarrow 11 + 8\mu - 13 = 0 \Rightarrow \mu = 1/4$$

Hence, (a) is the correct answer.

Ex 2. The plane $2x - y + 3z + 5 = 0$ is rotated through 90° about its line of intersection with the plane $5x - 4y - 2z + 1 = 0$. The equation of the plane in the new position is

- (a) $6x - 9y - 29z - 31 = 0$
(b) $27x - 24y - 26z - 13 = 0$
(c) $43x - 32y - 2z + 27 = 0$
(d) $26x - 43y - 15z - 165 = 0$

Sol. Equation of a plane passing through the line of intersection of the given plane is

$$2x - y + 3z + 5 + \lambda(5x - 4y - 2z + 1) = 0$$

$$\text{or } (2 + 5\lambda)x - (1 + 4\lambda)y + (3 - 2\lambda)z + 5 + \lambda = 0$$

This will be perpendicular to the plane

$$2x - y + 3z + 5 = 0, \text{ if}$$

$$2(2 + 5\lambda) + (1 + 4\lambda) + 3(3 - 2\lambda) = 0$$

$$\Rightarrow \lambda = -7/4$$

$\therefore$ The required equation of the plane is

$$4(2x - y + 3z + 5) - 7(5x - 4y - 2z + 1) = 0$$

$$\Rightarrow 27x - 24y - 26z - 13 = 0$$

Hence, (b) is the correct answer.

Ex 3. If the planes $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ pass through a line, then $a^2 + b^2 + c^2 + 2abc$ is equal to

- (a) 0 (b) 1
(c) 2 (d) 3

Sol. Given planes are

$$x - cy - bz = 0 \quad \dots (i)$$

$$cx - y + az = 0 \quad \dots (ii)$$

and $bx + ay - z = 0 \quad \dots (iii)$

Equation of plane passing through the line of intersection of planes (i) and (ii) may be taken as

$$(x - cy - bz) + \lambda(cx - y + az) = 0$$

$$\text{or } (1 + \lambda c)x - (c + \lambda)y + (a\lambda - b)z = 0 \quad \dots (iv)$$

Now, planes (iii) and (iv) are same.

$$\therefore \frac{1 + \lambda c}{b} = \frac{-(c + \lambda)}{a} = b - a\lambda$$

$$\Rightarrow \lambda = \frac{-(a + bc)}{ac + b}$$

$$\text{and } \lambda = -\frac{(ab + c)}{1 - a^2}$$

$$\therefore \frac{a + bc}{ac + b} = \frac{ab + c}{1 - a^2}$$

$$\Rightarrow a - a^3 + bc - a^2bc = a^2bc + ac^2 + ab^2 + bc$$

$$\Rightarrow a(2abc + c^2 + b^2 + a^2 - 1) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2abc = 1$$

Hence, (b) is the correct answer.

Ex 4. Let the pairs $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, $\mathbf{d}$ each determine a plane. Then, the planes are parallel, if

- (a) $(\mathbf{a} \times \mathbf{c}) \times (\mathbf{b} \times \mathbf{d}) = \mathbf{0}$
(b) $(\mathbf{a} \times \mathbf{c}) \cdot (\mathbf{b} \times \mathbf{d}) = \mathbf{0}$
(c) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$
(d) $(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$

Sol. Since, $\mathbf{a}$ and $\mathbf{b}$ are coplanar, therefore $\mathbf{a} \times \mathbf{b}$ is a vector perpendicular to the plane containing $\mathbf{a}$ and $\mathbf{b}$. Similarly, $\mathbf{c} \times \mathbf{d}$ is a vector perpendicular to the plane containing $\mathbf{c}$ and $\mathbf{d}$. Thus, the two planes will be parallel, if their normals, i.e. $(\mathbf{a} \times \mathbf{b})$ and $(\mathbf{c} \times \mathbf{d})$ are parallel.

$$\therefore (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = \mathbf{0}$$

Hence, (c) is the correct answer.

Ex 5. A variable plane passes through a fixed point (a, b, c) and meets the coordinate axes at P, Q, R . The locus of the point of intersection of the planes through P, Q, R parallel to the coordinate planes, is

(a) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$

(b) $ax + by + cz = 1$

(c) $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = -1$

(d) $ax + by + cz = -1$

Sol. Let the equation of plane be $\frac{z}{d'} + \frac{y}{b'} + \frac{x}{a'} = 1$.

This plane meets the coordinate axes in $A(a', 0, 0)$, $B(0, b', 0)$, $C(0, 0, c')$.

The plane passing through A, B, C and parallel to coordinates are

$$x = a', y = b' \text{ and } z = c'$$

Suppose these three planes intersect at the point (x_1, y_1, z_1) . Then,

$$x_1 = a', y_1 = b' \text{ and } z_1 = c' \quad \dots(i)$$

Since, the assuming plane passing through (a, b, c) .

$$\therefore \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'} = 1$$

$$\Rightarrow \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1 \quad [\text{from Eq. (i)}]$$

So, the locus of a point is

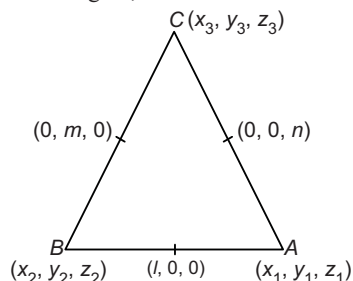
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$$

Hence, (a) is the correct answer.

Ex 6. In $\triangle ABC$, the mid-points of the sides AB, BC and CA are respectively $(l, 0, 0)$, $(0, m, 0)$ and $(0, 0, n)$. Then, $\frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2}$ is equal to

- (a) 2 (b) 4
(c) 8 (d) 16

Sol. From the figure,



$$x_1 + x_2 = 2l, y_1 + y_2 = 0, z_1 + z_2 = 0$$

$$x_2 + x_3 = 0, y_2 + y_3 = 2m, z_2 + z_3 = 0$$

$$\text{and } x_1 + x_3 = 0, y_1 + y_3 = 0, z_1 + z_3 = 2n$$

On solving, we get

$$x_1 = l, x_2 = l, x_3 = -l$$

$$y_1 = -m, y_2 = m, y_3 = m$$

and

$$z_1 = n, z_2 = -n, z_3 = n$$

So, coordinates are $A(l, -m, n)$, $B(l, m, -n)$ and $C(-l, m, n)$.

$$\begin{aligned} \therefore \frac{AB^2 + BC^2 + CA^2}{l^2 + m^2 + n^2} \\ = \frac{(4m^2 + 4n^2) + (4l^2 + 4n^2) + (4l^2 + 4m^2)}{l^2 + m^2 + n^2} = 8 \end{aligned}$$

Hence, (c) is the correct answer.

Ex 7. If the points $\hat{i} - \hat{j} + 3\hat{k}$ and $3\hat{i} + 3\hat{j} + 3\hat{k}$ are equidistant from the plane $\mathbf{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) + 9 = 0$, then they are

- (a) on the same sides of the plane
(b) parallel to the plane
(c) on the opposite sides of the plane
(d) None of the above

Sol. Given plane is $\mathbf{r} \cdot (5\hat{i} + 2\hat{j} - 7\hat{k}) = -9$

Length of the perpendicular from $\hat{i} - \hat{j} + 3\hat{k}$

$$= \frac{-9 - (\hat{i} - \hat{j} + 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k})}{\sqrt{25 + 4 + 49}}$$

$$= \frac{-9 - 5 + 2 + 21}{\sqrt{78}} = \frac{9}{\sqrt{78}}$$

Length of the perpendicular from $3\hat{i} + 3\hat{j} + 3\hat{k}$

$$= \frac{-9 - (3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (5\hat{i} + 2\hat{j} - 7\hat{k})}{\sqrt{78}}$$

$$= \frac{-9 - 15 - 6 + 21}{\sqrt{78}} = -\frac{9}{\sqrt{78}}$$

Thus, the length of the two perpendicular are equal in magnitude but opposite in sign. So, they are located on the opposite sides of the plane.

Hence, (c) is the correct answer.

Ex 8. If $|x_1| > |y_1| + |z_1|$, $|y_2| > |x_2| + |z_2|$, $|z_3| > |x_3| + |y_3|$, then $x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$, $x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$ and $x_3\hat{i} + y_3\hat{j} + z_3\hat{k}$ are

- (a) perpendicular
(b) collinear
(c) coplanar
(d) non-coplanar

Sol. If the given vectors are coplanar, then $\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0$.

The set of equations

$$x_1x + y_1y + z_1z = 0,$$

$$x_2x + y_2y + z_2z = 0$$

and

$$x_3x + y_3y + z_3z = 0$$

has a non-trivial solution.

Let the given set has a non-trivial solution x, y, z without loss of generality, we can assume that $x \geq y \geq z$.

For the given equation $x_1x + y_1y + z_1z = 0$, we have

$$x_1x = -y_1y - z_1z$$

$$\Rightarrow |x_1x| = |y_1y + z_1z| \leq |y_1y| + |z_1z|$$

$$\Rightarrow |x_1x| \leq |y_1x| + |z_1x| \Rightarrow |x_1| < |y_1| + |z_1|$$

which is a contradiction to the given inequality

i.e. $|x_1| > |y_1| + |z_1|$

Similarly, the other inequalities rule out the possibility of a non-trivial solution.

Therefore, the given equations have only a trivial solution.

So, the given vectors are non-coplanar.

Hence, (d) is the correct answer.

Ex 9. If the three planes $\mathbf{r} \cdot \mathbf{n}_1 = p_1$, $\mathbf{r} \cdot \mathbf{n}_2 = p_2$ and $\mathbf{r} \cdot \mathbf{n}_3 = p_3$ have a common line of intersection, then

$$p_1(\mathbf{n}_2 \times \mathbf{n}_3) + p_2(\mathbf{n}_3 \times \mathbf{n}_1) + p_3(\mathbf{n}_1 \times \mathbf{n}_2)$$

is equal to

- (a) 1 (b) 2
(c) 0 (d) -1

Sol. Equation of the plane passing through the line of intersection of the first two planes of

$$\mathbf{r} \cdot (\mathbf{n}_1 + \lambda \mathbf{n}_2) = p_1 + \lambda p_2, \text{ where } \lambda \text{ is a parameter.}$$

Since, three planes have a common line of intersection of the above equation should be identical with $\mathbf{r} \cdot \mathbf{n}_3 = p_3$ for some λ i.e. for some λ ,

$$\mathbf{n}_1 + \lambda \mathbf{n}_2 = k \mathbf{n}_3 \quad \dots (i)$$

$$\text{and } p_1 + \lambda p_2 = k p_3 \quad \dots (ii)$$

From Eq. (i),

$$\mathbf{n}_1 \times \mathbf{n}_3 + \lambda \mathbf{n}_2 \times \mathbf{n}_3 = 0$$

$$\text{and } \mathbf{n}_1 \times \mathbf{n}_2 = k \mathbf{n}_3 \times \mathbf{n}_2 \quad \dots (iii)$$

From Eq. (ii),

$$(p_1 + \lambda p_2)(\mathbf{n}_2 \times \mathbf{n}_3) = k p_3 (\mathbf{n}_2 \times \mathbf{n}_3) \\ = p_3 (\mathbf{n}_2 \times \mathbf{n}_1) \quad \dots (iv)$$

$$\Rightarrow p_1(\mathbf{n}_2 \times \mathbf{n}_3) + p_2 \lambda (\mathbf{n}_2 \times \mathbf{n}_3) + p_3 (\mathbf{n}_1 \times \mathbf{n}_2) = 0$$

$$\Rightarrow p_1(\mathbf{n}_2 \times \mathbf{n}_3) + p_2(\mathbf{n}_3 \times \mathbf{n}_1) + p_3(\mathbf{n}_1 \times \mathbf{n}_2) = 0 \\ [\text{from Eq. (iii)}]$$

Hence, (c) is the correct answer.

Ex 10. If a line L_1 passes through the point $3\hat{i}$ and is parallel to the vector $-\hat{i} + \hat{j} + \hat{k}$ and another line L_2 passes through the point $\hat{i} + \hat{j}$ and is parallel to the vector $\hat{i} + \hat{k}$, then point of intersection of the lines is

(a) $\hat{i} + 2\hat{j} + \hat{k}$

(b) $2\hat{i} + \hat{j} + \hat{k}$

(c) $\hat{i} - 2\hat{j} - \hat{k}$

(d) $\hat{i} - 2\hat{j} + \hat{k}$

Sol. Equation of L_1 is

$$\mathbf{r} = 3\hat{i} + \lambda(-\hat{i} + \hat{j} + \hat{k}) = (3 - \lambda)\hat{i} + \lambda\hat{j} + \lambda\hat{k}$$

Equation of L_2 is

$$\mathbf{r} = (\hat{i} + \hat{j}) + \mu(\hat{i} + \hat{k}) = (1 + \mu)\hat{i} + \hat{j} + \mu\hat{k}$$

For point of intersection, we get

$$\lambda = \mu = 1 \Rightarrow \mathbf{r} = 2\hat{i} + \hat{j} + \hat{k}$$

Hence, (b) is the correct answer.

Ex 11. If $OABC$ is a tetrahedron where O is the origin and A, B, C are three other vertices with position vectors $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ respectively, then the centre of sphere circumscribing the tetrahedron is given by the position vector

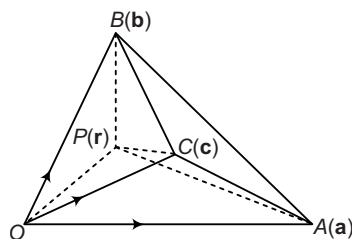
(a) $\frac{a^2(\mathbf{b} \times \mathbf{c}) + b^2(\mathbf{c} \times \mathbf{a}) + c^2(\mathbf{a} \times \mathbf{b})}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$

(b) $\frac{b^2(\mathbf{b} \times \mathbf{c}) + a^2(\mathbf{c} \times \mathbf{a}) + c^2(\mathbf{a} \times \mathbf{b})}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$

(c) $\frac{b^2(\mathbf{b} \times \mathbf{c}) + a^2(\mathbf{c} \times \mathbf{a}) + c^2(\mathbf{a} \times \mathbf{b})}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$

(d) $\frac{a^2(\mathbf{a} \times \mathbf{b}) + b^2(\mathbf{b} \times \mathbf{c}) + c^2(\mathbf{c} \times \mathbf{a})}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$

Sol. If the centre P is with position vector $\mathbf{r}$, then $\mathbf{a} - \mathbf{r} = \mathbf{PA}$, $\mathbf{b} - \mathbf{r} = \mathbf{PB}$, $\mathbf{c} - \mathbf{r} = \mathbf{PC}$, where $|\mathbf{PA}| = |\mathbf{PB}| = |\mathbf{PC}| = |\mathbf{OP}| = |\mathbf{r}|$



Consider $|\mathbf{a} - \mathbf{r}| = |\mathbf{r}|$

$$\Rightarrow (\mathbf{a} - \mathbf{r}) \cdot (\mathbf{a} - \mathbf{r}) = \mathbf{r} \cdot \mathbf{r}$$

$$\Rightarrow a^2 - 2\mathbf{a} \cdot \mathbf{r} + r^2 = r^2$$

$$\Rightarrow a^2 = 2\mathbf{a} \cdot \mathbf{r}$$

Similarly, $b^2 = 2\mathbf{b} \cdot \mathbf{r}$

and $c^2 = 2\mathbf{c} \cdot \mathbf{r}$

Since, $(\mathbf{b} \times \mathbf{c}), (\mathbf{c} \times \mathbf{a})$ and $(\mathbf{a} \times \mathbf{b})$ are non-coplanar.

$$\therefore \mathbf{r} = x(\mathbf{b} \times \mathbf{c}) + y(\mathbf{c} \times \mathbf{a}) + z(\mathbf{a} \times \mathbf{b})$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{r} = x\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) + y \cdot 0 + z \cdot 0 = x[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$$

$$\Rightarrow x = \frac{\mathbf{a} \cdot \mathbf{r}}{[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]} = \frac{a^2}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$$

Similarly, $y = \frac{b^2}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$ and $z = \frac{c^2}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$

Therefore, $\mathbf{r} = \frac{a^2(\mathbf{b} \times \mathbf{c}) + b^2(\mathbf{c} \times \mathbf{a}) + c^2(\mathbf{a} \times \mathbf{b})}{2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]}$

Hence, (a) is the correct answer.

Ex 12. Let the area of a given ΔABC be Δ . Points A_1, B_1, C_1 are the mid-points of the sides BC, CA and AB respectively. Point A_2 is the mid-point of CA_1 . Lines C_1A_1 and AA_2 meet the median BB_1 at points E and D respectively. If Δ_1 is the area of the quadrilateral A_1A_2DE , then $\frac{\Delta_1}{\Delta}$ is equal to

(a) $\frac{14}{64}$ (b) $\frac{11}{56}$ (c) $\frac{12}{52}$ (d) $\frac{13}{48}$

Sol. Let the position vectors of A, B and C be $\mathbf{0}, \mathbf{b}$ and $\mathbf{c}$ respectively. Then,

$$\mathbf{AC}_1 = \frac{\mathbf{b}}{2}, \mathbf{AB}_1 = \frac{\mathbf{c}}{2}, \mathbf{AA}_1 = \frac{\mathbf{b} + \mathbf{c}}{2}, \mathbf{AA}_2 = \frac{3\mathbf{c} + \mathbf{b}}{4}$$

Equations of the lines BB_1, AA_2 and C_1A_1 are respectively

$$\mathbf{r} = \mathbf{b} + \lambda_1 \left(\frac{\mathbf{c}}{2} - \mathbf{b} \right), \mathbf{r} = \lambda_2 \left(\frac{3\mathbf{c} + \mathbf{b}}{4} \right)$$

and $\mathbf{r} = \frac{\mathbf{b}}{2} + \lambda_3 \left(\frac{\mathbf{c}}{2} \right)$

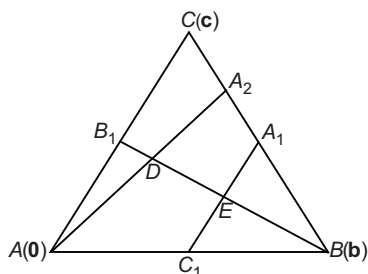
For point D , we have $\mathbf{b} + \lambda_1 \left(\frac{\mathbf{c}}{2} - \mathbf{b} \right) = \lambda_2 \left(\frac{3\mathbf{c} + \mathbf{b}}{4} \right)$

$$\Rightarrow \mathbf{b} \left(1 - \lambda_1 - \frac{\lambda_2}{4} \right) + \frac{\mathbf{c}}{4} (2\lambda_1 - 3\lambda_2) = \mathbf{0}$$

$$\Rightarrow 1 - \lambda_1 - \frac{\lambda_2}{4} = 0$$

$$\text{and } 2\lambda_1 - 3\lambda_2 = 0$$

$$\therefore \mathbf{AD} = \frac{1}{7}(\mathbf{b} + 3\mathbf{c})$$



For point E, we have

$$\mathbf{b} + \lambda_1 \left(\frac{\mathbf{c}}{2} - \mathbf{b} \right) = \frac{\mathbf{b}}{2} + \frac{\lambda_3}{2} \mathbf{c}$$

$$\Rightarrow \mathbf{b} \left(\frac{1}{2} - \lambda_1 \right) + \frac{\mathbf{c}}{2} (\lambda_1 - \lambda_3) = 0$$

$$\Rightarrow \lambda_1 = \lambda_3 = \frac{1}{2}$$

$$\Rightarrow \mathbf{AE} = \frac{2\mathbf{b} + \mathbf{c}}{4}$$

$$\text{Now, } \mathbf{EA}_2 = \frac{3\mathbf{c} + \mathbf{b} - 2\mathbf{b} - \mathbf{c}}{4} = \frac{2\mathbf{c} - \mathbf{b}}{4}$$

$$\text{and } \mathbf{DA}_1 = \frac{\mathbf{b} + \mathbf{c}}{2} - \frac{3\mathbf{c} + \mathbf{b}}{7} = \frac{5\mathbf{b} + \mathbf{c}}{14}$$

$\therefore$ Area of quadrilateral EA_1A_2D

$$= \frac{1}{2} |\mathbf{EA}_2 \times \mathbf{DA}_1|$$

$$= \frac{1}{112} |(2\mathbf{c} - \mathbf{b}) \times (5\mathbf{b} + \mathbf{c})|$$

$$= \frac{1}{112} |10\mathbf{c} \times \mathbf{b} - \mathbf{b} \times \mathbf{c}|$$

$$= \frac{11}{112} |\mathbf{c} \times \mathbf{b}| = \frac{11}{56} \cdot \frac{1}{2} |\mathbf{c} \times \mathbf{b}| = \frac{11}{56} \times \text{Area of } \triangle ABC$$

Thus, required ratio is $\frac{11}{56}$.

Hence, (b) is the correct answer.

Ex 13. The equation of a plane containing the lines $\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}_1$ and $\mathbf{r} = \mathbf{a}_2 + t\mathbf{b}_2$, where $[(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{b}_1 \times \mathbf{b}_2] = 0$, is

(a) $[(\mathbf{r} + \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)] = 1$

(b) $[(\mathbf{r} - \mathbf{b}_1) \cdot \mathbf{a} \cdot (\mathbf{b}_1 \times \mathbf{b}_2)] = 0$

(c) $[(\mathbf{r} - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)] = 0$

(d) None of the above

Sol. Given line passes through the points with position vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ and a normal to both the line is $\mathbf{b}_1 \times \mathbf{b}_2$ (the given condition $[(\mathbf{a}_2 - \mathbf{a}_1) \cdot \mathbf{b}_1 \times \mathbf{b}_2] = 0$ ensure that the lines are either parallel or intersecting). Thus, the required equation of the plane is

$$\mathbf{r} \cdot (\mathbf{b}_1 \times \mathbf{b}_2) = \mathbf{a}_1 \cdot (\mathbf{b}_1 \times \mathbf{b}_2)$$

or $[(\mathbf{r} - \mathbf{a}_1) \cdot (\mathbf{b}_1 \times \mathbf{b}_2)] = 0$

Hence, (c) is the correct answer.

Ex 14. The equation of the plane which passes through the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_1 = q_1$, $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ and is parallel to the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_3 = q_3$ and $\mathbf{r} \cdot \mathbf{n}_4 = q_4$, is

(a) $[\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4] (\mathbf{r} \cdot \mathbf{n}_1 - q_1) = [\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4] (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$

(b) $[\mathbf{n}_1 \ \mathbf{n}_2 \ \mathbf{n}_3] (\mathbf{r} \cdot \mathbf{n}_4 - q_4) = [\mathbf{n}_4 \ \mathbf{n}_3 \ \mathbf{n}_1] (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$

(c) $[\mathbf{n}_4 \ \mathbf{n}_3 \ \mathbf{n}_1] (\mathbf{r} \cdot \mathbf{n}_4 - q_4) = [\mathbf{n}_1 \ \mathbf{n}_2 \ \mathbf{n}_3] (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$

(d) None of the above

Sol. Equation of any plane passing through the intersection of $\mathbf{r} \cdot \mathbf{n}_1 = q_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = q_2$ is of the form

$$\mathbf{r} \cdot \mathbf{n}_1 + \lambda \mathbf{r} \cdot \mathbf{n}_2 = q_1 + \lambda q_2 \quad \dots (i)$$

where, λ is a parameter.

So, $\mathbf{n}_1 + \lambda \mathbf{n}_2$ is normal to the plane (i). Now, any plane parallel to the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n}_3 = q_3$ and $\mathbf{r} \cdot \mathbf{n}_4 = q_4$ is of the form

$$\mathbf{r} \cdot (\mathbf{n}_3 + \mu \mathbf{n}_4) = q_3 + \mu q_4$$

We must have $\mathbf{n}_1 + \lambda \mathbf{n}_2 = k(\mathbf{n}_3 + \mu \mathbf{n}_4)$ for some k

$$\Rightarrow [\mathbf{n}_1 + \lambda \mathbf{n}_2] \cdot [\mathbf{n}_3 \times \mathbf{n}_4] = 0$$

$$\Rightarrow [\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4] + \lambda [\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4] = 0$$

$$\Rightarrow \lambda = - \frac{[\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4]}{[\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4]}$$

On putting this value in Eq. (i), we get the equation of required plane as

$$\mathbf{r} \cdot \mathbf{n}_1 - q_1 = \frac{[\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4]}{[\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4]} (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$$

$$\Rightarrow [\mathbf{n}_2 \ \mathbf{n}_3 \ \mathbf{n}_4] (\mathbf{r} \cdot \mathbf{n}_1 - q_1) = [\mathbf{n}_1 \ \mathbf{n}_3 \ \mathbf{n}_4] (\mathbf{r} \cdot \mathbf{n}_2 - q_2)$$

Hence, (a) is the correct answer.

Ex 15. Let I be the incentre of $\triangle ABC$. Then, using vector method for any point P , $a(\mathbf{PA})^2 + b(\mathbf{PB})^2 + c(\mathbf{PC})^2$, where a, b and c have usual meanings, is equal to

(a) $a(\mathbf{IA})^2 + b(\mathbf{IB})^2 + c(\mathbf{IC})^2 + (a+b+c)(\mathbf{IP})^2$

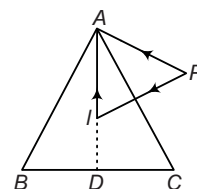
(b) $b(\mathbf{IA})^2 + c(\mathbf{IB})^2 + a(\mathbf{IC})^2 + (a+b+c)(\mathbf{IP})^2$

(c) $b(\mathbf{IA})^2 + c(\mathbf{IA})^2 + a(\mathbf{IC})^2 + (a+b+c)$

(d) $a(\mathbf{IA})^2 + b(\mathbf{IB})^2 + c(\mathbf{IC})^2 + (a+b+c)(\mathbf{IC})^2$

Sol. We have, $\mathbf{PI} + \mathbf{IA} = \mathbf{PA}$

$$\Rightarrow |\mathbf{PA}|^2 = |\mathbf{PI}|^2 + |\mathbf{IA}|^2 + 2\mathbf{PI} \cdot \mathbf{IA}$$



$$\Rightarrow a|\mathbf{PA}|^2 = a|\mathbf{PI}|^2 + a|\mathbf{IA}|^2 + 2\mathbf{PI} \cdot (a\mathbf{IA}) \quad \dots (i)$$

$$\text{Similarly, } b|\mathbf{PB}|^2 = b|\mathbf{PI}|^2 + b|\mathbf{IB}|^2 + 2\mathbf{PI} \cdot (b\mathbf{IB}) \quad \dots(ii)$$

$$\text{and } c|\mathbf{PC}|^2 = c|\mathbf{PI}|^2 + c|\mathbf{IC}|^2 + 2\mathbf{PI} \cdot (c\mathbf{IC}) \quad \dots(iii)$$

On adding Eqs. (i), (ii) and (iii), we get

$$\begin{aligned} a|\mathbf{PA}|^2 + b|\mathbf{PB}|^2 + c|\mathbf{PC}|^2 &= (a+b+c)|\mathbf{PI}|^2 + a|\mathbf{IA}|^2 + b|\mathbf{IB}|^2 + c|\mathbf{IC}|^2 \\ &\quad + 2\mathbf{PI} \cdot (a\mathbf{IA} + b\mathbf{IB} + c\mathbf{IC}) \end{aligned}$$

$$\Rightarrow a|\mathbf{PA}|^2 + b|\mathbf{PB}|^2 + c|\mathbf{PC}|^2 = (a+b+c)|\mathbf{PI}|^2 + a|\mathbf{IA}|^2 + b|\mathbf{IB}|^2 + c|\mathbf{IC}|^2$$

$$\{ \text{As, } a\mathbf{IA} + b\mathbf{IB} + c\mathbf{IC} = \mathbf{0} \}$$

Hence, (a) is the correct answer.

Ex 16. If P is any point on the circumcircle of $\triangle ABC$ other than the vertices, H is the orthocentre of $\triangle ABC$, M is the mid-point of PH and D is the mid-point of BC . Then,

- (a) AP is opposite side of DM
- (b) DM is parallel to AP
- (c) DM is perpendicular to AP
- (d) None of the above

Sol. Let S be the circumcentre of $\triangle ABC$.

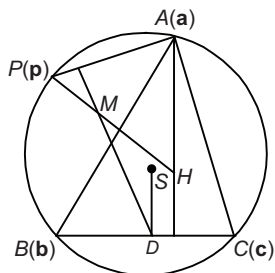
Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{p}$ be the position vectors of A, B, C and P respectively, with reference of origin S .

Then, $\mathbf{SA} = \mathbf{a}, \mathbf{SB} = \mathbf{b}, \mathbf{SC} = \mathbf{c}, \mathbf{SP} = \mathbf{p}$

and $|\mathbf{p}| = |\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = R = \text{circumradius}$

Now, $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{SA} + \mathbf{SB} + \mathbf{SC}$

$$= \mathbf{SA} + 2\mathbf{SD} = \mathbf{SA} + \mathbf{AH} = \mathbf{SH}$$



Position vector of $H = \mathbf{a} + \mathbf{b} + \mathbf{c}$

So, position vector of $M = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{p}}{2}$

$$\begin{aligned} \mathbf{DM} &= \text{Position vector of } M - \text{Position vector of } D \\ &= \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{p}}{2} - \frac{\mathbf{b} + \mathbf{c}}{2} = \frac{\mathbf{a} + \mathbf{p}}{2} \end{aligned}$$

Now,

$$\mathbf{DM} \cdot \mathbf{PA} = \left(\frac{\mathbf{a} + \mathbf{p}}{2} \right) \cdot \left(\frac{\mathbf{a} - \mathbf{p}}{2} \right) = \frac{|\mathbf{a}|^2 - |\mathbf{p}|^2}{2} = 0$$

$$\therefore \mathbf{DM} \perp \mathbf{AP}$$

Hence, (c) is the correct answer.

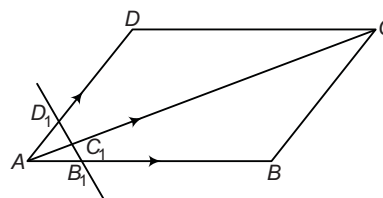
Ex 17. A straight line L cuts the lines AB, AC and AD of a parallelogram $ABCD$ at points B_1, C_1 and D_1 respectively. If $\mathbf{AB}_1 = \lambda_1 \mathbf{AB}$, $\mathbf{AD}_1 = \lambda_2 \mathbf{AD}$ and $\mathbf{AC}_1 = \lambda_3 \mathbf{AC}$, then $\frac{1}{\lambda_3}$ is equal to

- (a) $\frac{1}{\lambda_1} + \frac{1}{\lambda_2}$
- (b) $\frac{1}{\lambda_1} - \frac{1}{\lambda_2}$
- (c) $-\lambda_1 + \lambda_2$
- (d) $\lambda_1 + \lambda_2$

Sol. Let $\mathbf{AB} = \mathbf{a}, \mathbf{AD} = \mathbf{b}$, then $\mathbf{AC} = \mathbf{a} + \mathbf{b}$

Given, $\mathbf{AB}_1 = \lambda_1 \mathbf{a}, \mathbf{AD}_1 = \lambda_2 \mathbf{b}, \mathbf{AC}_1 = \lambda_3 (\mathbf{a} + \mathbf{b})$

$$\mathbf{B}_1\mathbf{D}_1 = \mathbf{AD}_1 - \mathbf{AB}_1 = \lambda_2 \mathbf{b} - \lambda_1 \mathbf{a}$$



Since, vectors $\mathbf{D}_1\mathbf{C}_1$ and $\mathbf{B}_1\mathbf{D}_1$ are collinear.

$$\therefore \mathbf{D}_1\mathbf{C}_1 = k \mathbf{B}_1\mathbf{D}_1 \text{ for some } k \in \mathbb{R}$$

$$\Rightarrow \mathbf{AC}_1 - \mathbf{AD}_1 = k \cdot \mathbf{B}_1\mathbf{D}_1$$

$$\Rightarrow \lambda_3 (\mathbf{a} + \mathbf{b}) - \lambda_2 \mathbf{b} = k \cdot (\lambda_1 \mathbf{b} - \lambda_1 \mathbf{a})$$

$$\Rightarrow \lambda_3 \mathbf{a} + (\lambda_3 - \lambda_2) \mathbf{b} = k \cdot \lambda_2 \mathbf{b} - k \cdot \lambda_1 \mathbf{a}$$

$$\text{Thus, } \lambda_3 = -k\lambda_1 \text{ and } \lambda_3 - \lambda_2 = k\lambda_2$$

$$\Rightarrow k = -\frac{\lambda_3}{\lambda_1} = \frac{\lambda_3 - \lambda_2}{\lambda_2}$$

$$\Rightarrow \lambda_1 \lambda_2 = \lambda_1 \lambda_3 + \lambda_2 \lambda_3$$

$$\Rightarrow \frac{1}{\lambda_3} = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}$$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 18. Which of following statement(s) is/are correct?

- (a) If $\mathbf{n} \cdot \mathbf{a} = 0, \mathbf{n} \cdot \mathbf{b} = 0$ and $\mathbf{n} \cdot \mathbf{c} = 0$ for some non-zero vectors $\mathbf{n}$, then $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$
- (b) There exists a vector making angles 30° and 45° with X and Y -axes
- (c) Locus of point for which $x = 3$ and $y = 4$ is a line parallel to the Z -axis whose distance from the Z -axis, is 5
- (d) The vertices of a regular tetrahedron are O, A, B and C , where O is the origin. The vector $\mathbf{OA} + \mathbf{OB} + \mathbf{OC}$ is perpendicular to the plane ABC

Sol. (a) Since, $\mathbf{n} \cdot \mathbf{a} = \mathbf{n} \cdot \mathbf{b} = \mathbf{n} \cdot \mathbf{c} = 0$

$\therefore \mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are coplanar.

$$\therefore [\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 0$$

(b) $\cos^2 30^\circ + \cos^2 45^\circ + \cos^2 \gamma = 1$

$$\therefore \sin^2 \gamma = \frac{3}{4} + \frac{1}{2} = \frac{5}{4} \text{ which is not possible.}$$

(c) Distance from Z -axis
 $= \sqrt{3^2 + 4^2} = 5$

(d) Since, $\mathbf{AB} \times \mathbf{BC}$ is perpendicular to the plane ABC .

$$\therefore \mathbf{AB} \times \mathbf{BC} = (\mathbf{OB} - \mathbf{OA}) \times (\mathbf{OC} - \mathbf{OB})$$

$$= \mathbf{OB} \times \mathbf{OC} - \mathbf{OA} \times \mathbf{OC} + \mathbf{OA} \times \mathbf{OB}$$

$$= \mathbf{OA} \times \mathbf{OB} + \mathbf{OB} \times \mathbf{OC} + \mathbf{OA} \times \mathbf{OB}$$

i.e. $\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$ is perpendicular to the plane ABC .

$$\therefore (\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (\mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}) \\ = \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{c}\} \mathbf{b} - \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{a}\} \mathbf{c}$$

$$\mathbf{b} + \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{c}\} \mathbf{b} - \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{a}\} \mathbf{c} \\ + \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{a}\} \mathbf{c} - \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot \mathbf{b}\} \mathbf{a} \\ = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} - \mathbf{c}) \mathbf{a} + (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{c} - \mathbf{a}) \mathbf{b} \\ + \{(\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} - \mathbf{b})\} \mathbf{c} \\ = (\mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{c} + \mathbf{b}^2 - \mathbf{c}^2) \mathbf{a} + (\mathbf{b} \cdot \mathbf{c} - \mathbf{b} \cdot \mathbf{a} + \mathbf{c}^2 - \mathbf{a}^2) \mathbf{b} \\ + (\mathbf{a}^2 - \mathbf{b}^2 + \mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b}) \mathbf{c} = \mathbf{0}$$

Thus, the statement is true.

Hence, (a), (c) and (d) are the correct answers.

Ex 19. Which of the following is/are correct about a tetrahedron?

- (a) Centroid of a tetrahedron lies on lines joining any vertex to the centroid of opposite face
- (b) Centroid of a tetrahedron lies on the lines joining the mid-point of the opposite faces
- (c) Distance of centroid from all the vertices are equal
- (d) None of the above

Sol. (a) Let $A(\mathbf{a})$, $B(\mathbf{b})$, $C(\mathbf{c})$ and $D(\mathbf{d})$ be the vertices of a tetrahedron, then centroid of the tetrahedron is

$$\frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$$

Centroid G_1 of the face BCD is $\frac{\mathbf{b} + \mathbf{c} + \mathbf{d}}{3}$.

Now, centroid of the tetrahedron G_1 divides AG_1 in the ratio 3 : 1.

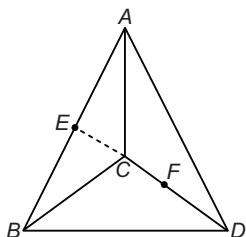
$$\text{i.e. } \frac{3(\mathbf{b} + \mathbf{c} + \mathbf{d}) + \mathbf{a}}{3 + 1} = \frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$$

$\therefore C$ lies on AG_1 .

(b) Let E be the mid-point of AB and F be the mid-point of CD .

$$\therefore \text{Position vector of } E \text{ is } \frac{\mathbf{a} + \mathbf{b}}{2}$$

$$\text{and position vector of } F \text{ is } \frac{\mathbf{c} + \mathbf{d}}{2}.$$



Mid-point of EF is $\frac{\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d}}{4}$ which is the centroid of the tetrahedron $ABCD$.

Hence, (a) and (b) are the correct answers.

Ex 20. The direction ratios of the line $x - y + z - 5 = 0 = x - 3y - 6$ are

- (a) 3, 1, -2
- (b) 2, -4, 1
- (c) $\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{-2}{\sqrt{14}}$
- (d) $\frac{2}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{1}{\sqrt{21}}$

Sol. Let the DR's of line be a, b, c .

Since, the line is perpendicular to both the planes.

$$\therefore a - b + c = 0$$

$$\Rightarrow a - 3b + 0 \cdot c = 0$$

$$\Rightarrow \frac{a}{3} = \frac{b}{1} = \frac{c}{-2}$$

Hence, (a) and (c) are the correct answers.

Ex 21. The equation of the line $x + y + z - 1 = 0$, $4x + y - 2z + 2 = 0$ written in the symmetrical form, is

- (a) $\frac{x+1}{1} = \frac{y-2}{-2} = \frac{z-0}{1}$
- (b) $\frac{x}{1} = \frac{y}{-2} = \frac{z-1}{1}$
- (c) $\frac{x+1/2}{1} = \frac{y-1}{-2} = \frac{z-1/2}{1}$
- (d) $\frac{x-1}{2} = \frac{y+2}{-1} = \frac{z-2}{2}$

Sol. We have, $x + y + z - 1 = 0$

$$\text{and } 4x + y - 2z + 2 = 0$$

$\therefore$ Direction ratios of the line are $(-3, 6, -3)$.

$$\text{i.e. } <1, -2, 1>$$

$$\text{Let } z = k, \text{ then } x = k - 1, y = 2 - 2k$$

i.e. $(k - 1, 2 - 2k, k)$ is any point on the line.

$\therefore (-1, 2, 0), (0, 0, 1), \left(-\frac{1}{2}, 1, \frac{1}{2}\right)$ and $(1, -2, 2)$ are points of the line.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 22. If the points $A(4, 5, 10)$, $B(2, 3, 4)$ and $C(1, 2, -1)$ are three vertices of a parallelogram $ABCD$, then

(a) vector equation of AB is $2\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$

(b) cartesian equation of BC is $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$

(c) coordinates of D are $(3, 4, 5)$

(d) $ABCD$ is a rectangle

Sol. Direction ratios of AB are $4 - 2, 5 - 3, 10 - 4$ or $2, 2, 6$. So, AB is parallel to the vector $\hat{i} + \hat{j} + 3\hat{k}$ and passes through $B(2, 3, 4)$, the vector $2\hat{i} + 3\hat{j} + 4\hat{k}$ and its equation is $2\hat{i} + 3\hat{j} + 4\hat{k} + \lambda(\hat{i} + \hat{j} + 3\hat{k})$.

Similarly, BC passes through the points $B(2, 3, 4)$ and its direction ratios are $2 - 1, 3 - 2, 4 + 1$ or $1, 1, 5$.

So, its cartesian equation is

$$\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{5}$$

Now, if D is (a, b, c) , then $ABCD$ is a parallelogram and mid-point of AC and BD is same.

$$\therefore \left(\frac{a+2}{2}, \frac{b+3}{2}, \frac{c+4}{2} \right) = \left(\frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right)$$

[diagonals of a parallelogram bisect each other]

$$\Rightarrow (a, b, c) = (3, 4, 5)$$

AB is not perpendicular to BC because

$$1 \times 1 + 1 \times 1 + 3 \times 5 \neq 0$$

So, $ABCD$ is not a rectangle.

Hence, (a), (b) and (c) are the correct answers.

Ex 23. The plane passing through the origin and containing the lines whose direction cosines are proportional to $1, -2, 2$ and $2, 3, -1$ passes through the point

- (a) $(1, -2, 2)$
- (b) $(2, 3, -1)$
- (c) $(3, 1, 1)$
- (d) $(4, 0, 7)$

Sol. Let the direction cosines of the normal to the plane be l, m, n . Since, the normal is perpendicular to both the lines lying in the plane, whose direction cosines are proportional to $1, -2, 2$ and $2, 3, -1$, we have

$$l - 2m + 2n = 0 \text{ and } 2l + 3m - n = 0$$

$$\Rightarrow \frac{l}{-4} = \frac{m}{5} = \frac{n}{7}$$

Since, the plane passes through the origin, therefore its equation is $lx + my + nz = 0$ or $4x - 5y - 7z = 0$ which clearly passes through the points given in options (a), (b) and (c).

Hence, (a), (b) and (c) are the correct answers.

Ex 24. The direction cosines of the lines bisecting the angle between the line whose direction

cosines are l_1, m_1, n_1 and l_2, m_2, n_2 and the angle between these lines is θ , are

- (a) $\frac{l_1 + l_2}{\cos \frac{\theta}{2}}, \frac{m_1 + m_2}{\cos \frac{\theta}{2}}, \frac{n_1 + n_2}{\cos \frac{\theta}{2}}$
- (b) $\frac{l_1 + l_2}{2 \cos \frac{\theta}{2}}, \frac{m_1 + m_2}{2 \cos \frac{\theta}{2}}, \frac{n_1 + n_2}{2 \cos \frac{\theta}{2}}$
- (c) $\frac{l_1 + l_2}{\sin \frac{\theta}{2}}, \frac{m_1 + m_2}{\sin \frac{\theta}{2}}, \frac{n_1 + n_2}{\sin \frac{\theta}{2}}$
- (d) $\frac{l_1 - l_2}{2 \sin \frac{\theta}{2}}, \frac{m_1 - m_2}{2 \sin \frac{\theta}{2}}, \frac{n_1 - n_2}{2 \sin \frac{\theta}{2}}$

Sol. Directions ratios of the bisector are

$$< l_1 + l_2, m_1 + m_2, n_1 + n_2 >$$

$$\begin{aligned} \therefore \sqrt{(l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2} \\ = \sqrt{2 + 2(l_1 l_2 + m_1 m_2 + n_1 n_2)} \\ = \sqrt{2 + 2 \cos \theta} = 2 \cos \frac{\theta}{2} \end{aligned}$$

$\therefore$ Direction cosines are

$$\left\langle \frac{l_1 + l_2}{2 \cos \frac{\theta}{2}}, \frac{m_1 + m_2}{2 \cos \frac{\theta}{2}}, \frac{n_1 + n_2}{2 \cos \frac{\theta}{2}} \right\rangle$$

Direction ratios of the other bisector are

$$< l_1 - l_2, m_1 - m_2, n_1 - n_2 >$$

$$\sqrt{(l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2} = 2 \sin \frac{\theta}{2}$$

$\therefore$ Direction cosines of the bisector are

$$\left\langle \frac{l_1 - l_2}{2 \sin \frac{\theta}{2}}, \frac{m_1 - m_2}{2 \sin \frac{\theta}{2}}, \frac{n_1 - n_2}{2 \sin \frac{\theta}{2}} \right\rangle$$

Hence, (b) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 25-31) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The correct choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Ex 25. Statement I A line L is perpendicular to the plane $3x - 4y + 5z = 10$.

Statement II Direction cosines of L are

$$< \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

Sol. Let $lx + my + nz = P$ be the equation of a plane in the normal form.

$\therefore$ DR's of the plane $3x - 4y + 5z = 10$ are $< 3, -4, 5 >$.

$$\Rightarrow \text{Direction cosines are } < \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} >$$

Hence, (a) is the correct answer.

Ex 26. The equation of two straight lines are

$$\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{-3}$$

and $\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z+3}{2}$.

Statement I The given lines are coplanar.

Statement II The equation $2x_1 - y_1 = 1$, $x_1 + 3y_1 = 4$, $3x_1 + 2y_1 = 5$ are consistent.

Sol. Any point on the first line is $(2x_1 + 1, x_1 - 3, -3x_1 - 2)$. Any point on the second line is $(y_1 - 2, 3y_1 + 1, 2y_1 - 3)$. If two lines are coplanar, then $2x_1 - y_1 = 1$, $x_1 + 3y_1 = 4$, $3x_1 + 2y_1 = 5$ are consistent. Hence, (a) is the correct answer.

Ex 27. Statement I The distance between the planes $4x - 5y + 3z = 5$ and $4x - 5y + 3z + 2 = 0$ is $\frac{3}{5\sqrt{2}}$.

Statement II The distance between $ax + by + cz + d_1 = 0$ and

$$ax + by + cz + d_2 = 0 \text{ is } \left| \frac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}} \right|.$$

Sol. Required distance = $\left| \frac{5 + 2}{\sqrt{50}} \right| = \frac{7}{5\sqrt{2}}$

Hence, (d) is the correct answer.

Ex 28. Given the line $L: \frac{x-1}{3} = \frac{y+1}{2} = \frac{z-3}{-1}$ and the plane $\pi: x - 2y - z = 0$.

Statement I L lies in π .

Statement II L is parallel to π .

Sol. $x = 1 + 3r$, $y = -1 + 2r$, $z = 3 - r$
 $\therefore 1 + 3r - 2(-1 + 2r) - 3 + r = 0$
 $\Rightarrow 3 \times 1 - 2 \times 2 + 1 \times 1 = 0$
 So, L is parallel to π .
 Hence, (d) is the correct answer.

Ex 29. Statement I Line $\frac{x-1}{3} = \frac{y-2}{11} = \frac{z+1}{11}$ lies in the plane $11x - 3z - 14 = 0$.

Statement II A straight line lies in a plane, if the line is parallel to the plane and a point of the line in the plane.

Sol. Statement I $(1, 2, -1)$ is a point on the line and $11 + 3 - 14 = 0$.

$\therefore$ The point lies on the plane $11x - 3z - 14 = 0$.

Further, $3 \times 11 + 11(-3) = 0$

$\therefore$ The line lies in the plane.

Statement II is also true.

Hence, (a) is the correct answer.

Ex 30. Statement I There exist two points on the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ which are at a distance of 2 units from point $(1, 2, -4)$.

Statement II Perpendicular distance of point $(1, 2, -4)$ from the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ is 1 unit.

Sol. Any point on the line $\frac{x-1}{1} = \frac{y}{-1} = \frac{z+2}{2}$ is

$$B(t+1, -t, 2t-2), t \in \mathbb{R}.$$

Also, AB is perpendicular to the line, where A is $(1, 2, -4)$.

$$\Rightarrow 1(t) - (-t-2) + 2(2t+2) = 0$$

$$\Rightarrow 6t + 6 = 0$$

$$\Rightarrow t = -1$$

$\therefore$ Point B is $(0, 1, -4)$.

$$\text{Thus, } AB = \sqrt{1+1+0} = \sqrt{2}$$

Hence, (c) is the correct answer.

Ex 31. Statement I Lines $\mathbf{r} = \hat{i} + \hat{j} - \hat{k} + \lambda(3\hat{i} - \hat{j})$ and $\mathbf{r} = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$ intersect.

Statement II If $\mathbf{b} \times \mathbf{d} = \mathbf{0}$, then lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$ do not intersect.

Sol. For the given lines, let $\mathbf{a}_1 = \hat{i} + \hat{j} - \hat{k}$, $\mathbf{a}_2 = 4\hat{i} - \hat{k}$, $\mathbf{b}_1 = 3\hat{i} - \hat{j}$ and $\mathbf{b}_2 = 2\hat{i} + 3\hat{k}$.

$$\text{Then, } [\mathbf{a}_2 - \mathbf{a}_1 \quad \mathbf{b}_1 \quad \mathbf{b}_2] = \begin{vmatrix} 4-1 & 0-1 & -1+1 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 0 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{vmatrix} = 0$$

So, the lines are coplanar. Also, vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ Thus, which the lines are directed, are not collinear.

Therefore, the lines intersect.

Thus, $\mathbf{b} \times \mathbf{d} = \mathbf{0}$, vectors $\mathbf{b}$ and $\mathbf{d}$ are collinear, therefore lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \lambda\mathbf{d}$ parallel not intersect. But this statement is not the correct explanation for Statement I.

Hence, (b) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 32-33) Two lines whose equations are $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3}$ lie in the same plane.

- Ex 32.** Point of intersection of the lines lies on
 (a) $3x + y + z = 20$ (b) $2x + y + z = 25$
 (c) $3x + 2y + z = 24$ (d) $x = y = z$

Sol. Let $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{4} = r_1$
 $\Rightarrow x = 3 + 2r_1, y = 2 + 3r_1, z = 1 + 4r_1$
 It will lie on $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3}$
 $\Rightarrow r_1 = 1$
 So, point of intersection is (5, 5, 5).
 Hence, (d) is the correct answer.

- Ex 33.** Angle between the plane containing both the lines and the plane $4x + y + 2z = 0$, is equal to
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$
 (c) $\frac{\pi}{6}$ (d) $\cos^{-1} \frac{2}{\sqrt{186}}$

Sol. Equation of plane containing both lines is
 $\begin{vmatrix} x-3 & y-2 & z-1 \\ 2 & 3 & 4 \\ 3 & 2 & 3 \end{vmatrix} = 0$
 $\Rightarrow (x-3)(1) + (y-2)(12-6) + (z-1)(4-9) = 0$
 $\Rightarrow x + 6y - 5z = 10$
 Thus, the angle is $\frac{\pi}{2}$.
 Hence, (b) is the correct answer.

Passage II (Ex. Nos. 34-35) A line L_1 passing through a point with position vector $\mathbf{p} = \hat{i} + 2\hat{j} + 3\hat{k}$ and parallel to $\mathbf{a} = \hat{i} + 2\hat{j} + 3\hat{k}$. Another line L_2 passing through a point with position vector $\mathbf{q} = 2\hat{i} + 3\hat{j} + \hat{k}$ and parallel to $\mathbf{b} = 3\hat{i} + \hat{j} + 2\hat{k}$.

- Ex 34.** Equation of plane equidistant from lines L_1 and L_2 is
 (a) $\mathbf{r} \cdot (\hat{i} - 7\hat{j} - 5\hat{k}) = 3$ (b) $\mathbf{r} \cdot (\hat{i} + 7\hat{j} + 5\hat{k}) = 3$
 (c) $\mathbf{r} \cdot (\hat{i} + 7\hat{j} - 5\hat{k}) = 9$ (d) $\mathbf{r} \cdot (\hat{i} + 7\hat{j} - 5\hat{k}) = 9$

Sol. Since, line L_1 is parallel to $\mathbf{a} = \hat{i} + 2\hat{j} + 3\hat{k}$
 and line L_2 is parallel to $\mathbf{b} = 3\hat{i} + \hat{j} + 2\hat{k}$
 Therefore, normal to the plane perpendicular to the lines L_1 and L_2 is $\mathbf{a} \times \mathbf{b} = (\hat{i} + 7\hat{j} - 5\hat{k})$ and plane passes through the point with position vector is $\frac{3}{2}\hat{i} + \frac{5}{2}\hat{j} + 2\hat{k}$.
 $\therefore$ Equation of plane is $\mathbf{r} \cdot (\hat{i} + 7\hat{j} - 5\hat{k}) = 9$.
 Hence, (c) is the correct answer.

Ex 35. Equation of a line passing through the point (2, -3, 2) and equally inclined to the lines L_1 and L_2 , is

- (a) $\frac{x-2}{2} = \frac{y-3}{-1} = \frac{z-2}{1}$
 (b) $\frac{x-2}{-2} = y+3 = z-2$
 (c) $\frac{x-2}{-4} = \frac{y+3}{3} = \frac{z-5}{2}$
 (d) $\frac{x+2}{4} = \frac{y+3}{3} = \frac{z-2}{-5}$

Sol. Angle bisectors of vectors $\mathbf{a}$ and $\mathbf{b}$ are
 $\mathbf{r}_1 = \frac{1}{\sqrt{14}}(2\hat{i} - \hat{j} - \hat{k})$ and $\mathbf{r}_2 = \frac{1}{\sqrt{14}}(4\hat{i} - 3\hat{j} - 5\hat{k})$.
 Thus, plane with either (2, -1, -1) or (4, -3, -5) as the direction ratios of normal and passing through (2, -3, 2), is the required plane.
 Hence, (b) is the correct answer.

Passage IV (Ex. Nos. 36-40) A conic section is obtained by the intersection of two inverted cones having same vertex with a plane. If a plane passes through the vertex, then its intersection with the cones either represents a pair of straight lines or a point depending upon whether it intersects the cones or not. If plane does not pass through vertex, then section may be hyperbola or circle depending upon whether plane is parallel to axis of the cone or perpendicular to it. If it has any other inclination, then intersection of plane and cone gives an ellipse.

- Ex 36.** If a variable point P moves such that the line passing through P and Q (0, 0, 2) makes an angle 60° with Z -axis, then locus of P is
 (a) $x^2 - y^2 - 3(z-2)^2 = 0$
 (b) $x^2 - y^2 + 3(z-2)^2 = 0$
 (c) $x^2 + y^2 - 3(z-2)^2 = 0$
 (d) $x^2 + y^2 + 3(z-2)^2 = 0$

Sol. Since, $P(x, y, z)$ lies on line through (0, 0, 2) making 60° with Z -axis.
 $\therefore \frac{x\hat{i} + y\hat{j} + (z-2)\hat{k} \cdot \hat{k}}{\sqrt{x^2 + y^2 + (z-2)^2}} = \cos 60^\circ$
 $\Rightarrow x^2 + y^2 - 3(z-2)^2 = 0$
 Hence, (c) is the correct answer.

- Ex 37.** The locus of intersection of locus of P with the plane $x + y + z = 1$, is
 (a) a pair straight lines (b) a hyperbola
 (c) a circle (d) an ellipse

Sol. Plane does not pass through (0, 0, 2) and it is neither parallel to axis nor generator.
 $\therefore$ Locus is an ellipse.
 Hence, (d) is the correct answer.

Ex 38. The locus of intersection of locus of P and the plane $x + y + z = 2$ is

- (a) a pair of straight lines
- (b) a hyperbola
- (c) a circle
- (d) a point

Sol. The plane passes through $(0, 0, 2)$ and its inclination with Z -axis is θ , where $\cos \theta = \frac{1}{\sqrt{3}}$ i.e. $\theta < 60^\circ$.

So, it represents a pair of straight lines.
Hence, (a) is the correct answer.

Ex 39. The locus of intersection of locus of P with $x + y = 2$ is

- (a) a straight line
- (b) a hyperbola
- (c) a circle
- (d) an ellipse

Sol. Since, this is parallel to Z -axis.
 $\therefore$ Locus is a hyperbola.
Hence, (b) is the correct answer.

Ex 40. The locus of intersection of locus of P with $2x + y + z = 2$ is

- (a) a straight line
- (b) a hyperbola
- (c) a circle
- (d) a point

Sol. The plane passes through $(0, 0, 2)$ and its inclination with Z -axis is θ , where $\cos \theta = \frac{1}{\sqrt{6}}$ i.e. $\theta > 60^\circ$.

So, it represents a point.
Hence, (d) is the correct answer.

Passage IV (Ex. Nos. 41-44) If $A(-2, 2, 3)$, $B(13, -3, 13)$ and L is a line through A .

Ex 41. If a point P moves in the space such that $3PA = 2PB$, then the locus of P is

- (a) $x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 = 0$
- (b) $x^2 + y^2 + z^2 - 28x + 12y + 10z - 247 = 0$
- (c) $x^2 + y^2 + z^2 + 28x - 12y - 10z + 247 = 0$
- (d) $x^2 + y^2 + z^2 - 28x + 12y - 10z + 247 = 0$

Sol. Let $P(x, y, z)$ be any point on the locus, then $3PA = 2PB$.

$$\begin{aligned} \therefore 9(PA)^2 &= 4(PB)^2 \\ \Rightarrow 9[(x+2)^2 + (y-2)^2 + (z-3)^2] &= 4[(x-13)^2 + (y+3)^2 + (z-13)^2] \\ \Rightarrow 5(x^2 + y^2 + z^2) + 140x - 60y + 50z - 1235 &= 0 \\ \Rightarrow x^2 + y^2 + z^2 + 28x - 12y + 10z - 247 &= 0 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 42. The coordinates of the point P which divides the join of A and B in the ratio $2:3$ internally, are

- (a) $(33/5, -2/5, 9)$
- (b) $(4, 0, 7)$
- (c) $(32/5, -12/5, 17/5)$
- (d) $(20, 0, 35)$

Sol. Required coordinates

$$\begin{aligned} &= \left(\frac{2 \times 13 + 3(-2)}{2+3}, \frac{2 \times (-3) + 3(2)}{2+3}, \frac{2 \times 13 + 3 \times 3}{2+3} \right) \\ &= (4, 0, 7) \end{aligned}$$

Hence, (b) is the correct answer.

Ex 43. The equation of a line L , perpendicular to the line AB , is

- (a) $\frac{x+2}{15} = \frac{y-2}{-5} = \frac{z-3}{10}$
- (b) $\frac{x-2}{3} = \frac{y+2}{13} = \frac{z+3}{2}$
- (c) $\frac{x+2}{3} = \frac{y-2}{13} = \frac{z-3}{2}$
- (d) $\frac{x-2}{15} = \frac{y+2}{-5} = \frac{z+3}{10}$

Sol. Direction ratios of AB are $13+2, -3-2, 13-3$ i.e. $15, -5, 10$.

Let the equation of the required line L be

$$\frac{x+2}{l} = \frac{y-2}{m} = \frac{z-3}{n}$$

Then, $15l - 5m + 10n = 0$

which is satisfied by option (a).

Hence, (a) is the correct answer.

Ex 44. The direction ratios of the normal to the plane passing through the origin and the points A and B , are

- (a) $15, -5, 10$
- (b) $11, -1, 16$
- (c) $3, 13, 2$
- (d) $7, 13, -4$

Sol. Let the equation of the plane passing through the origin be $ax + by + cz = 0$.

Since, it passes through $A(-2, 2, 3)$ and $B(13, -3, 13)$.

$$\therefore -2a + 2b + 3c = 0$$

$$\text{and } 13a - 3b + 13c = 0$$

$$\Rightarrow \frac{a}{26+9} = \frac{b}{39+26} = \frac{c}{-26+6}$$

$$\text{or } \frac{a}{35} = \frac{b}{65} = \frac{c}{-20}$$

$$\text{or } \frac{a}{7} = \frac{b}{13} = \frac{c}{-4}$$

Hence, (d) is the correct answer.

Type 5. Match the Columns

Ex 45. Match the statements of Column I with values of Column II.

Column I	Column II
A. If the line $\frac{x-1}{1} = \frac{y+1}{-2} = \frac{z+1}{\lambda}$ lies in the plane $3x - 2y + 5z = 0$, then λ is equal to	p. $\sin^{-1} \sqrt{\frac{6}{25}}$
B. If $(3, \lambda, \mu)$ is a point on the lines $2x + y + z - 3 = 0 = x - 2y + z - 1$, then $\lambda + \mu$ is equal to	q. $-\frac{7}{5}$
C. The angle between the line $x = y = z$ and the plane $4x - 3y + 5z = 2$ is	r. -3
D. The angle between the planes $x + y + z = 0$ and $3x - 4y + 5z = 0$ is	s. $\cos^{-1} \sqrt{\frac{8}{75}}$

Sol. A. $3 \cdot 1 - 2(-2) + 5(\lambda) = 0$

$$\Rightarrow \lambda = -\frac{7}{5}$$

B. Point $(3, \lambda, \mu)$ lies on

$$2x + y + z - 3 = x - 2y + z - 1 = 0$$

$$\Rightarrow 3 \cdot 2 + \lambda + \mu - 3 = 0$$

$$\text{and } 3 - 2\lambda + \mu - 1 = 0$$

$$\Rightarrow \lambda + \mu + 3 = 0$$

$$\text{and } 2\lambda - \mu - 2 = 0$$

$$\therefore \lambda + \mu = -3$$

$$C. \sin \theta = \frac{1 \cdot 4 + 1(-3) + 1 \cdot 5}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{16 + 9 + 25}}$$

$$\Rightarrow \sin \theta = \frac{6}{\sqrt{3} \sqrt{50}}$$

$$\therefore \theta = \sin^{-1} \sqrt{\frac{6}{25}}$$

$$D. \cos \theta = \frac{1 \cdot 3 + 1(-4) + 1 \cdot 5}{\sqrt{3} \sqrt{16 + 9 + 25}} = \frac{4}{\sqrt{3} \sqrt{50}}$$

$$\therefore \theta = \cos^{-1} \sqrt{\frac{8}{75}}$$

$$A \rightarrow q; B \rightarrow r; C \rightarrow p; D \rightarrow s$$

Ex 46. Match the statements of Column I with values of Column II.

Column I	Column II
A. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-1}{3} = \frac{y-3}{4} = \frac{z-5}{5}$ are	p. coincident
B. $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-3}{2} = \frac{y-5}{3} = \frac{z-7}{4}$ are	q. parallel and different
C. $\frac{x-2}{5} = \frac{y+3}{4} = \frac{z-2}{-2}$ and $\frac{x-7}{5} = \frac{y-1}{4} = \frac{z-2}{-2}$ are	r. skew
D. $\frac{x-3}{2} = \frac{y+2}{3} = \frac{z-4}{5}$ and $\frac{x-3}{3} = \frac{y-2}{2} = \frac{z-7}{5}$ are	s. intersecting at a point

Sol. A. Both the lines passes through the point $(7, 11, 15)$.

B. $\langle 2, 3, 4 \rangle$ are direction ratios of both the lines.

Also, the point $(1, 2, 3)$ is common to both.

$\therefore$ The lines are coincident.

C. $\langle 5, 4, -2 \rangle$ are direction ratios of both the lines.

$\therefore$ The lines are parallel.

$$\text{Also, } x = 2 + 5\lambda, y = -3 + 4\lambda, z = 5 - 2\lambda$$

$$\therefore \frac{2 + 5\lambda - 7}{5} = \frac{-3 + 4\lambda - 1}{4} = \frac{5 - 2\lambda - 2}{-2}$$

$$\text{i.e. } \lambda - 1 = \lambda - 1 = \frac{3 - 2\lambda}{-2}$$

$\therefore$ No value of λ exist.

Thus, the lines are parallel and different.

D. $\langle 2, 3, 5 \rangle$ and $\langle 3, 2, 5 \rangle$ are direction ratios of first and second lines respectively.

$\therefore$ The lines are not parallel.

$$\Rightarrow x = 3 + 2\lambda, y = -2 + 3\lambda, z = 4 + 5\lambda$$

$$\text{and } x = 3 + 3\mu, y = -2 + 2\mu, z = 7 + 5\mu$$

are parametric equations of the lines.

On solving $3 + 2\lambda = 3 + 3\mu$

$$\text{and } -2 + 3\lambda = -2 + 2\mu, \text{ we get } \lambda = \frac{12}{5}, \mu = \frac{8}{5}$$

Now, substituting these values in $4 + 5\lambda = 7 + 5\mu$, we get $4 + 12 = 7 + 8$ i.e. $16 = 15$ which is not true.

So, the lines do not intersect.

Hence, the lines are skew.

$$A \rightarrow s; B \rightarrow p; C \rightarrow q; D \rightarrow r$$

Ex 47. Match the statements of Column I with values of Column II.

Column I	Column II
A. $\frac{x-2}{3} = \frac{y-7}{4} = \frac{z+5}{2}$	p. perpendicular to the plane $3x + 4y + 2z = 1$
B. $\frac{x+1}{3} = \frac{y-3}{4} = \frac{z+7}{2}$	q. passes through $(2, 7, -5)$
C. $\frac{x-5}{1} = \frac{y+2}{3} = \frac{z-2}{4}$	r. direction cosines are $2/\sqrt{30}, 5/\sqrt{30}, 1/\sqrt{30}$
D. $\frac{x}{2} = \frac{y-2}{5} = \frac{z+6}{1}$	s. lies in the plane $7x - y - z = 35$

Sol. (A) and (B) passes through $(2, 7, -5)$, are parallel to the normal to the plane $3x + 4y + 2z = 1$ and hence is perpendicular to it.

C. Point $(5, -2, 2)$ lies on $\frac{x-5}{1} = \frac{y+2}{3} = \frac{z-2}{4}$ and

$7x - y - z = 35$ and the given line is perpendicular to normal to the plane $7x - y - z = 35$ as $1 \times 7 - 3 \times 1 - 4 \times 1 = 0$.

So, the given line lies in the plane $7x - y - z = 35$.

D. Direction cosines are $\frac{2}{\sqrt{4+25+1}}, \frac{5}{\sqrt{4+25+1}}, \frac{1}{\sqrt{4+25+1}}$

$$\frac{1}{\sqrt{4+25+1}} = \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{1}{\sqrt{30}}$$

and passes through $(2, 7, -5)$ as

$$\frac{2}{2} = \frac{7-2}{5} = \frac{-5+6}{1}$$

$$A \rightarrow p, q; B \rightarrow p, q; C \rightarrow s, D \rightarrow q, r$$

Type 6. Single Integer Answer Type Questions

Ex 48. If the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies exactly on the plane $2x - 4y + z = 7$, then the value of k is _____.

Sol. (7) Since, the point $(4, 2, k)$ must satisfy the plane.

$$\therefore 8 - 8 + k = 7 \\ \Rightarrow k = 7$$

Ex 49. If the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is $\lambda\sqrt{30}$ units, then the value of λ is _____.

Sol. (3) Given lines are

$$\mathbf{r} = (3, 8, 3) + \lambda(3, -1, 1)$$

$$\text{and } \mathbf{r} = (-3, -7, 6) + \mu(-3, 2, 4)$$

where, λ, μ are parameters.

$\therefore$ Shortest distance

$$= \frac{(-3-3, -7-8, 6-3) \cdot [(3, -1, 1) \times (-3, 2, 4)]}{|(3, -1, 1) \times (-3, 2, 4)|} \\ = \frac{(-6, -15, 3) \cdot (-6, -15, 3)}{\sqrt{36 + 225 + 9}} = \frac{(-6, -15, 3) \cdot (-6, -15, 3)}{\sqrt{270}} = \frac{36 + 225 + 9}{\sqrt{270}} = 3\sqrt{30} \text{ units}$$

$$\text{Hence, } \lambda = 3$$

Ex 50. If the perpendicular distance of the point $(6, 5, 8)$ from the Y -axis is 5λ units, then λ is equal to _____.

Sol. (2) Foot of perpendicular from $(6, 5, 8)$ on Y -axis is $(0, 5, 0)$.

$\therefore$ Required distance

$$= \sqrt{(6-0)^2 + (5-5)^2 + (8-0)^2} = 10 \text{ units}$$

$$\therefore 5\lambda = 10$$

$$\therefore \lambda = \frac{10}{5} = 2$$

Ex 51. A parallelopiped is formed by planes drawn through the points $(2, 4, 5)$ and $(5, 9, 7)$ parallel to the coordinate planes. The length of the diagonal of the parallelopiped is _____.

Sol. (7) The length of the edges are given by $a = 5 - 2 = 3$, $b = 9 - 4 = 5$, $c = 7 - 5 = 2$

$\therefore$ Length of the diagonal

$$= \sqrt{a^2 + b^2 + c^2} \\ = \sqrt{9 + 25 + 4} = 7 \text{ units}$$

Ex 52. If the lines $\frac{x-4}{15} = \frac{y-17}{9} = \frac{z-11}{8}$ and $\frac{x-15}{4} = \frac{y-9}{17} = \frac{z-8}{11}$ intersect at the point P , then square of the distance of P from the origin is λ , where $\lambda - 1390$ equals _____.

Sol. (8) Any point of $\frac{x-4}{15} = \frac{y-17}{9} = \frac{z-11}{8}$ is $(15r + 4, 9r + 17, 8r + 11)$ and any point on the line $\frac{x-15}{4} = \frac{y-9}{17} = \frac{z-8}{11}$ is $(4r' + 15, 17r' + 9, 11r' + 8)$

If these two points are same, then

$$15r + 4 = 4r' + 15,$$

$$9r + 17 = 17r' + 9$$

$$\text{and } 8r + 11 = 11r' + 8$$

$$\Rightarrow r = r' = 1$$

$\therefore$ The point of intersection is $P(19, 26, 19)$ and $(OP)^2 = (19)^2 + (26)^2 + (19)^2 = 1398$

Hence, $\lambda = 1398$

$$\therefore \lambda - 1390 = 8$$

Target Exercises

Type 1. Only One Correct Option

- If the direction cosines of a straight lines are $<k, k, k>$, then
 - $k > 0$
 - $0 < k < 1$
 - $k = 1$
 - $k = \frac{1}{\sqrt{3}}$ or $-\frac{1}{\sqrt{3}}$
- The lines $x = 1, y = 2$ is
 - parallel to X -axis
 - parallel to Y -axis
 - parallel to Z -axis
 - lies in a plane parallel to XY -plane
- The equation $(x-1)(x-2) = 0$ in three dimensional space represents a/an
 - pair of straight lines
 - sphere
 - pair of parallel planes
 - empty set
- A point (x, y, z) moves parallel to X -axis. Which of the three variables x, y, z remain fixed?
 - x
 - x and y
 - y and z
 - z and x
- The ratio in which the line joining $(2, 4, 5)$, $(3, 5, -4)$ is divided by the YZ -plane, is
 - $2 : 3$
 - $3 : 2$
 - $-2 : 3$
 - $4 : -3$
- The area of the triangle whose vertices are $(1, 2, 3)$, $(2, 5, -1)$ and $(-1, 1, 2)$, is
 - 150 sq units
 - 145 sq units
 - $\frac{\sqrt{155}}{2}$ sq units
 - $\frac{155}{2}$ sq units
- A line making angles 45° and 60° with the positive directions of X and Y -axes makes with the positive direction of Z -axis, an angle of
 - 60°
 - 120°
 - 60° or 120°
 - None of these
- The angle between a line with direction ratios $2 : 2 : 1$ and a line joining $(3, 1, 4)$ to $(7, 2, 12)$, is
 - $\cos^{-1}\left(\frac{2}{3}\right)$
 - $\cos^{-1}\left(\frac{3}{2}\right)$
 - $\tan^{-1}\left(-\frac{2}{3}\right)$
 - None of these
- Lines OA and OB are drawn from O with direction cosines proportional to $<1, -2, -1>$ and $<3, -2, 3>$ respectively. The direction ratios of the normal to the plane AOB are
 - $<4, 3, 2>$
 - $<4, -3, -2>$
 - $<-4, 3, -2>$
 - $<4, 3, -2>$
- In a three dimensional space, the equation $3x - 4y = 0$ represents
 - a plane containing Z -axis
 - a plane containing X -axis
 - a plane containing Y -axis
 - None of the above
- The equation $xy = 0$ in three dimensional space represents
 - a pair of straight lines
 - a plane
 - a pair of planes at right angles
 - a pair of parallel planes
- The lines $\mathbf{r} = (\hat{i} + \hat{j} + \hat{k})t + 3\hat{k}$ and $\mathbf{r} = (\hat{i} - 2\hat{j} + \hat{k})s + 3\hat{k}$
 - are skew
 - are parallel
 - intersect at right angles
 - None of these
- Equation of the line through $(1, 2, 3)$ and parallel to the plane $2x + 3y + z + 5 = 0$ are
 - $\frac{x-1}{-1} = \frac{y-2}{1} = \frac{z-3}{-1}$
 - $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{1}$
 - $\frac{x-1}{3} = \frac{y-2}{2} = \frac{z-3}{1}$
 - $\frac{x-1}{1} = \frac{y-2}{2} = \frac{z-3}{1}$
- The equation $\mathbf{r} = \lambda \hat{i}$ represents
 - the X -axis
 - the YOZ -plane
 - the Y -axis
 - the Z -axis
- If $P(\mathbf{p}), Q(\mathbf{q}), R(\mathbf{r})$ and $S(\mathbf{s})$ are four points such that $3\mathbf{p} + 8\mathbf{q} = 6\mathbf{r} + 5\mathbf{s}$, then the lines PQ and RS are
 - skew
 - intersecting
 - parallel
 - None of these
- If the projection of a line segment on the axes are $\sqrt{2}, 3\sqrt{2}, 4$ respectively, then the length of the segment is
 - 36
 - 6
 - 6
 - None of these
- The line $\frac{x-x_1}{0} = \frac{y-y_1}{1} = \frac{z-z_1}{2}$ is
 - at right angles to X -axis
 - at right angles to YZ -plane
 - is parallel to Y -axis
 - None of the above
- The lines $x = x_1, y = y_1$ is
 - parallel to X -axis
 - parallel to Y -axis
 - parallel to Z -axis
 - None of these

19. The lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z-3}{5}$ are
(a) skew (b) parallel
(c) intersecting (d) None of these
20. The lines $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$ and $\frac{x-x_1}{a'} = \frac{y-y_1}{b'} = \frac{z-z_1}{c'}$ are
(a) parallel (b) intersecting
(c) skew (d) at right angles
21. The lines $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x-1}{-2} = \frac{y-2}{-4} = \frac{z-3}{-6}$ are
(a) intersecting (b) skew
(c) parallel (d) coincident
22. The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is
(a) $\sqrt{30}$ (b) $2\sqrt{30}$ (c) $5\sqrt{30}$ (d) $3\sqrt{30}$
23. The direction ratios of the line which is perpendicular to the lines $\frac{x-7}{2} = \frac{y+17}{-3} = \frac{z-6}{1}$ and $\frac{x+5}{1} = \frac{y+3}{2} = \frac{z-4}{-2}$, are
(a) $\langle 4, 5, 7 \rangle$ (b) $\langle 4, -5, 7 \rangle$
(c) $\langle 4, -5, -7 \rangle$ (d) $\langle -4, 5, 7 \rangle$
24. The angle between the lines $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{2}$ and $\frac{x-1}{-\sqrt{3}-1} = \frac{y-1}{\sqrt{3}-1} = \frac{z-1}{4}$ is
(a) $\cos^{-1}\left(\frac{1}{65}\right)$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$
25. The shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-4}{4} = \frac{z-5}{5}$ is
(a) $\frac{1}{6}$ (b) $\frac{1}{\sqrt{6}}$
(c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{3}$
26. The distance between the lines $\mathbf{r} = \mathbf{a}_1 + t\mathbf{b}$ and $\mathbf{r} = \mathbf{a}_2 + s\mathbf{b}$ is
(a) $|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|$ (b) $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{b}|}$
(c) $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{a}_2 - \mathbf{a}_1|}$ (d) $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}|}{|\mathbf{a}_2 - \mathbf{a}_1| \cdot |\mathbf{b}|}$
27. The length of perpendicular from the point $(1, -1, 2)$ on the line $\frac{x+1}{2} = \frac{y-2}{-3} = \frac{z+2}{4}$, is
(a) $\sqrt{29}$ (b) $\sqrt{6}$
(c) $\sqrt{21}$ (d) None of these
28. The shortest distance between the lines $\mathbf{r} = \mathbf{a}_1 + \mathbf{b}_1 t$ and $\mathbf{r} = \mathbf{a}_2 + \mathbf{b}_2 s$ is
(a) $[\mathbf{a}_2 - \mathbf{a}_1 \cdot \mathbf{b}_1 \cdot \mathbf{b}_2]$ (b) $\frac{[\mathbf{a}_2 - \mathbf{a}_1 \cdot \mathbf{b}_1 \cdot \mathbf{b}_2]}{|\mathbf{b}_1 \times \mathbf{b}_2|}$
(c) $\frac{[\mathbf{a}_1 \cdot \mathbf{a}_2 \cdot \mathbf{b}_1 - \mathbf{b}_2]}{|\mathbf{a}_1 \times \mathbf{a}_2|}$ (d) None of these
29. The distance of the point $A(\mathbf{a}_2)$ from the line $\mathbf{r} = \mathbf{a}_1 + \mathbf{b}_1 t$ is
(a) $|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}_1|$ (b) $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}_1|}{|\mathbf{b}_1|}$
(c) $\frac{|(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{b}_1|}{|\mathbf{a}_2 - \mathbf{a}_1|}$ (d) None of these
30. The angle between the lines whose direction cosines are $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, -\frac{\sqrt{3}}{2}\right)$, is
(a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$
31. If the direction cosines l, m, n of two lines are connected by the relations $l + m + n = 0$ and $lm = 0$, then the angle between them is
(a) $\pi/3$
(b) $\pi/4$
(c) $\pi/2$
(d) 0
32. The equation of the plane passing through the point $(-1, 2, 0)$ and parallel to the lines $\frac{x}{3} = \frac{y+1}{0} = \frac{z-2}{-1}$ and $\frac{x-1}{1} = \frac{y+1}{2} = \frac{z+1}{-1}$, is
(a) $2x + 3y + 6z - 4 = 0$
(b) $x - 2y + 3z + 5 = 0$
(c) $x + y - 3z = 0$
(d) $x + y + 3z - 1 = 0$
33. If a plane meets the coordinate axes at A, B and C such that the centroid of the triangle is $(3, 3, 3)$. Then the equation of the plane is
(a) $x + y + z = 3$
(b) $x + y + z = 9$
(c) $3x + 3y + 3z = 1$
(d) $9x + 9y + 9z = 1$
34. The equation of the line passing through $(1, 1, 1)$ and perpendicular to the plane $2x + 3y - z - 5 = 0$, is
(a) $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{1}$ (b) $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{-1}$
(c) $\frac{x-1}{2} = \frac{y-1}{-1} = \frac{z-1}{1}$ (d) None of these
35. The distance between the planes $\mathbf{r} \cdot \hat{\mathbf{n}} = p_1$ and $\mathbf{r} \cdot (-\hat{\mathbf{n}}) = p_2$ is
(a) $p_1 + p_2$
(b) $|p_1 - p_2|$
(c) $p_2 - p_1$
(d) None of the above

36. Three given planes may
 (i) be all parallel.
 (ii) intersect at a point.
 (iii) intersect in a line.
 (iv) intersect taken two at a time in parallel lines.
 (v) be such that two of them are parallel and the third intersect them in parallel lines.

How many of the above possibilities are correct?

- (a) 4 (b) 5 (c) 2 (d) 3

37. If a plane meets the axes in A, B and C such that centroid of the $\triangle ABC$ is $(1, 2, 3)$. Then, the equation of the plane is

- (a) $x + \frac{y}{2} + \frac{z}{3} = 1$ (b) $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$
 (c) $x + 2y + 3z = 1$ (d) None of these

38. The locus of the equation $xy + yz = 0$ is

- (a) a pair of straight lines
 (b) a pair of parallel lines
 (c) a pair of perpendicular planes
 (d) None of the above

39. The direction cosines of a normal to the plane $2x - 3y - 6z + 14 = 0$ are

- (a) $\langle 2, -3, -6 \rangle$ (b) $\langle \frac{2}{7}, -\frac{3}{7}, -\frac{6}{7} \rangle$
 (c) $\langle -2, 3, 6 \rangle$ (d) None of these

40. If the plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1} \lambda$ with the X -axis, then λ is equal to

- (a) $\frac{2}{7}$ (b) $\frac{\sqrt{45}}{7}$
 (c) $-\frac{2}{7}$ (d) None of these

41. The equation of the plane parallel to the planes $2x - 3y + 6z - 7 = 0$ and $2x - 3y + 6z + 7 = 0$, is

- (a) $2x - 3y + 6z + 1 = 0$ (b) $2x - 3y + 6z - 1 = 0$
 (c) $2x - 3y + 6z = 0$ (d) None of these

42. The plane $x - 2y + z - 6 = 0$ and the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

are related as

- (a) parallel to the plane
 (b) at right angles to the plane
 (c) lies in the plane
 (d) meets the plane obliquely

43. The distance of the plane $\mathbf{r} \cdot \left(\frac{2}{7}\hat{\mathbf{i}} + \frac{3}{7}\hat{\mathbf{j}} - \frac{6}{7}\hat{\mathbf{k}} \right) = 1$

from the origin, is

- (a) 1 (b) 7
 (c) $\frac{1}{7}$ (d) None of these

44. If the plane $2x + 3y + 4z = 1$ meets the coordinate axes in A, B and C . Then, the centroid of $\triangle ABC$ is

- (a) $(2, 3, 4)$ (b) $\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \right)$
 (c) $\left(\frac{1}{6}, \frac{1}{9}, \frac{1}{12} \right)$ (d) $\left(\frac{1}{2}, \frac{3}{3}, \frac{4}{4} \right)$

45. What is the equation of the plane which passes through Z -axis and is perpendicular to the line $\frac{x-a}{\cos \theta} = \frac{y+2}{\sin \theta} = \frac{z-3}{0}$?

- (a) $x + y \tan \theta = 0$ (b) $y + x \tan \theta = 0$
 (c) $x \cos \theta - y \sin \theta = 0$ (d) $x \sin \theta - y \cos \theta = 0$

46. A plane passes through a fixed point (a, b, c) . The locus of the foot of the perpendicular to it from the origin is a sphere of radius

- (a) $\frac{3}{2}\sqrt{a^2 + b^2 + c^2}$ (b) $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$
 (c) $a^2 + b^2 + c^2$ (d) None of these

47. The angle between the line $\frac{x+1}{3} = \frac{y-1}{4} = \frac{z-2}{2}$ and

the plane $2x - 3y + z + 4 = 0$ is

- (a) $\cos^{-1} \left(-\frac{4}{\sqrt{406}} \right)$ (b) $\sin^{-1} \left(\frac{4}{\sqrt{406}} \right)$
 (c) $\sin^{-1} \left(-\frac{4}{\sqrt{406}} \right)$ (d) None of these

48. If $p_1 = 0$ and $p_2 = 0$ are two non-parallel planes, then the equation $p_1 + \lambda p_2 = 0$, $\lambda \in R$ represents the family of all planes through the line of section of the planes $P_1 = 0$ and $P_2 = 0$ except the plane

- (a) $p_1 = 0$
 (b) $p_2 = 0$
 (c) $p_1 + p_2 = 0$
 (d) $p_1 - p_2 = 0$

49. The equation of the plane passing through the intersection of the planes $x + 2y + 3z + 4 = 0$ and $4x + 3y + 2z + 1 = 0$ and the origin, is

- (a) $3x + 2y + z + 1 = 0$
 (b) $3x + 2y + z = 0$
 (c) $2x + 3y + z = 0$
 (d) $x + y + z = 0$

50. If $\mathbf{r} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \lambda (2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 4\hat{\mathbf{k}})$ and $\mathbf{r} \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}) = 3$ are the equations of a line and a plane respectively, then which of the following is true?

- (a) The line is perpendicular to the plane
 (b) The line lies in the plane
 (c) The line is parallel but does not lie in the plane
 (d) The line cuts the plane obliquely

51. The distance of the point $(2, 1, -1)$ from the plane $x - 2y + 4z = 9$ is

- (a) $\frac{\sqrt{13}}{21}$ (b) $\frac{13}{21}$
 (c) $\frac{13}{\sqrt{21}}$ (d) $\sqrt{\frac{13}{21}}$

52. The reflection of the point $(2, -1, 3)$ in the plane $3x - 2y - z = 9$ is

- (a) $\left(\frac{26}{7}, \frac{15}{7}, \frac{17}{7} \right)$ (b) $\left(\frac{26}{7}, -\frac{15}{7}, \frac{17}{7} \right)$
 (c) $\left(\frac{15}{7}, \frac{26}{7}, -\frac{17}{7} \right)$ (d) $\left(\frac{26}{7}, \frac{17}{7}, -\frac{15}{7} \right)$

53. The angle between the straight line $\frac{x-1}{2} = \frac{y+3}{-1} = \frac{z-5}{2}$ and the plane $4x - 2y + 4z = 9$ is
- (a) 60° (b) 90°
(c) 45° (d) 30°
54. The distance between the parallel planes $x + 2y - 3z = 2$ and $2x + 4y - 6z + 7 = 0$ is
- (a) $\frac{2}{\sqrt{14}}$ (b) $\frac{11}{\sqrt{56}}$
(c) $\frac{7}{\sqrt{56}}$ (d) None of these
55. The equation of the plane passing through the point $(1, 2, -3)$ which is parallel to the plane $3x - 5y + 2z = 11$, is given by
- (a) $3x - 5y + 2z - 13 = 0$
(b) $5x - 3y + 2z + 13 = 0$
(c) $3x - 2y + 5z + 13 = 0$
(d) $3x - 5y + 2z + 13 = 0$
56. The equation of the plane whose intercepts on the axes are thrice as long as those made by the plane $2x - 3y + 6z - 11 = 0$, is
- (a) $6x - 9y + 18z - 11 = 0$ (b) $2x - 3y + 6z - 33 = 0$
(c) $2x - 3y + 6z + 33 = 0$ (d) None of these
57. The equation of the plane passing through (α, β, γ) and parallel to the plane $x + y + z = 0$, is
- (a) $x + y + z + (\alpha + \beta + \gamma) = 0$ (b) $x + y + z = \alpha + \beta + \gamma$
(c) $x + y + z = \alpha\beta\gamma$ (d) None of these
58. The distance between the line $\mathbf{r} = 2\hat{i} - 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - \hat{j} + 4\hat{k})$ and the plane $\mathbf{r} \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 5$ is
- (a) $\frac{10}{3\sqrt{3}}$ (b) $\frac{10}{3}$
(c) $\frac{10}{9}$ (d) None of these
59. Freedom equations of a line are $x = x_1 + lr, y = y_1 + mr$ and $z = z_1 + nr$, where
- (a) r is the distance of (x, y, z) from (x_1, y_1, z_1)
(b) $|r|$ is the distance of (x, y, z) from (x_1, y_1, z_1)
(c) r is the distance of (x_1, y_1, z_1) from the origin
(d) None of the above
60. The line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and the plane $x - 2y + z = 0$
- (a) meet in a unique point
(b) do not meet at all
(c) meet in infinitely many points
(d) None of the above
61. The three planes $x + y = 0, y + z = 0$ and $x + z = 0$
- (a) meet in a unique point
(b) meet in a line
(c) meet taken two at a time in parallel lines
(d) None of the above
62. The equation of the line of intersection of the planes $\mathbf{r} \cdot \mathbf{n} = q, \mathbf{r} \cdot \mathbf{n}' = q'$ and passing through the point $\mathbf{a}$, is
- (a) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n} \times \mathbf{n}')$ (b) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n} - \mathbf{n}')$
(c) $\mathbf{r} = \mathbf{a} + \lambda(\mathbf{n} + \mathbf{n}')$ (d) None of these
63. The centre of the sphere passing through the points $(0, 3, 4), (0, 5, 0), (4, 0, 3)$ and $(-3, 4, 0)$, is
- (a) $(\frac{1}{4}, 3, \frac{7}{4})$ (b) $(0, 0, 0)$
(c) $(-4, 3, 0)$ (d) None of these
64. The equation $(\mathbf{r} - (\hat{i} + \hat{j})) \cdot (\mathbf{r} - (\hat{j} + \hat{k})) = 0$ represents
- (a) a pair of lines (b) a pair of planes
(c) a sphere (d) None of these
65. The radius of the circle $|\mathbf{r}| = 5, \mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$, is
- (a) 5 (b) 3
(c) 4 (d) None of these
66. The spheres $x^2 + y^2 + z^2 + x + y + z - 1 = 0$ and $x^2 + y^2 + z^2 + x + y + z - 5 = 0$
- (a) intersect in a plane (b) intersect in five points
(c) do not intersect (d) None of these
67. The equation of the sphere described on the join of $P(\mathbf{a})$ and $Q(\mathbf{b})$ as a diameter, is
- (a) $|\mathbf{r} - \mathbf{a}|^2 + |\mathbf{r} - \mathbf{b}|^2 = 0$
(b) $|\mathbf{r} - \mathbf{a}|^2 + |\mathbf{r} - \mathbf{b}|^2 = |\mathbf{a} - \mathbf{b}|^2$
(c) $|\mathbf{r} - \mathbf{a}| + |\mathbf{r} - \mathbf{b}| = 0$
(d) None of the above
68. The point equidistant from the points $(0, 0, 0), (1, 0, 0), (0, 2, 0)$ and $(0, 0, 3)$, is
- (a) $(1, 2, 3)$ (b) $(\frac{1}{2}, 1, \frac{3}{2})$
(c) $(-\frac{1}{2}, -1, -\frac{3}{2})$ (d) None of these
69. The locus of the equation $x^2 + y^2 + z^2 + 1 = 0$ is
- (a) an empty set (b) a degenerate sphere
(c) a sphere (d) a pair of planes
70. Centre of the sphere $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$ is
- (a) (x_2, y_2, z_2)
(b) $(\frac{x_1 - x_2}{2}, \frac{y_1 - y_2}{2}, \frac{z_1 - z_2}{2})$
(c) $(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2})$
(d) (x_1, y_1, z_1)
71. The plane of intersection of $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$ and $4x^2 + 4y^2 + 4z^2 + 4x + 4y + 4z - 1 = 0$ is
- (a) $4x + 4y + 4z + 9 = 0$ (b) $x + y + z + 9 = 0$
(c) $4x + 4y + 4z + 1 = 0$ (d) None of these

- 72.** In order that the sphere S_1 with the centre C_1 and radius r_1 may fully contain a sphere S_2 with centre C_2 and radius r_2 , we must have
 (a) $|C_1 C_2| < r_1 + r_2$ (b) $|C_1 C_2| < r_1 - r_2$
 (c) $|C_1 C_2| < 2(r_1 - r_2)$ (d) $|C_1 C_2| < \frac{1}{2}(r_1 + r_2)$
- 73.** The number of spheres of a given radius r and touching the coordinate axes, is
 (a) 4 (b) 6
 (c) 8 (d) None of these
- 74.** Position vector of the centre of the sphere $|\mathbf{r}|^2 + \mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - 6 = 0$ is
 (a) $\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ (b) $\frac{1}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
 (c) $\frac{\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}}{\sqrt{3}}$ (d) $-\frac{1}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$
- 75.** Equation of the sphere which passes through the points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$, is
 (a) $x^2 + y^2 + z^2 - ax - by - cz = 0$
 (b) $x^2 + y^2 + z^2 + ax + by + cz = 0$
 (c) $x^2 + y^2 + z^2 + ax - by + cz = 0$
 (d) $x^2 + y^2 + z^2 + ax - by - cz = 0$
- 76.** The equation of the sphere passing through the point $(1, 3, -2)$ and the circle $y^2 + z^2 = 25$, $x = 0$, is
 (a) $x^2 + y^2 + z^2 + 11x + 25 = 0$ (b) $x^2 + y^2 + z^2 - 11x + 25 = 0$
 (c) $x^2 + y^2 + z^2 + 11x - 25 = 0$ (d) $x^2 + y^2 + z^2 - 11x - 25 = 0$
- 77.** The shortest distance from the point $(1, 2, -1)$ to the surface of the sphere $x^2 + y^2 + z^2 = 24$, is
 (a) $3\sqrt{6}$ (b) $2\sqrt{6}$ (c) $\sqrt{6}$ (d) 2
- 78.** The equation of the sphere passing through the points $(0, 0, 0)$, $\left(\frac{a}{2}, 0, 0\right)$, $\left(0, \frac{b}{2}, 0\right)$ and $\left(0, 0, \frac{c}{2}\right)$ is
 (a) $x^2 + y^2 + z^2 - \frac{a}{2}x - \frac{b}{2}y - \frac{c}{2}z = 0$
 (b) $x^2 + y^2 + z^2 + \frac{a}{2}x + \frac{b}{2}y + \frac{c}{2}z = 0$
 (c) $x^2 + y^2 + z^2 - ax - by - cz = 0$
 (d) $x^2 + y^2 + z^2 + ax + by + cz = 0$
- 79.** The equation of the sphere concentric with the sphere $x^2 + y^2 + z^2 - 4x - 6y - 8z - 5 = 0$ and which passes through the origin, is
 (a) $x^2 + y^2 + z^2 - 4x - 6y - 8z = 0$
 (b) $x^2 + y^2 + z^2 - 6y - 8z = 0$
 (c) $x^2 + y^2 + z^2 = 0$
 (d) None of the above
- 80.** The angle between the plane $3x + 4y = 0$ and the line $x^2 + y^2 = 0$ is
 (a) 90° (b) 0° (c) 60° (d) 45°
- 81.** The equation of the sphere concentric with the sphere $x^2 + y^2 + z^2 - 4x - 6y - 8z - 5 = 0$ and which passes through the point $(0, 1, 0)$, is
 (a) $x^2 + y^2 + z^2 + 4x + 6y - 8z - 7 = 0$
 (b) $x^2 + y^2 + z^2 - 4x - 6y - 8z = 0$
 (c) $x^2 + y^2 + z^2 - 4x - 6y - 8z + 5 = 0$
 (d) $x^2 + y^2 + z^2 - 4x + 6y - 8z - 7 = 0$
- 82.** The equation of the sphere with centre $(3, 6, -4)$ and touching the plane $2x - 2y - z - 10 = 0$, is
 (a) $x^2 + y^2 + z^2 - 6x + 12y - 8z + 45 = 0$
 (b) $x^2 + y^2 + z^2 + 6x - 9y + 8z + 35 = 0$
 (c) $x^2 + y^2 + z^2 - 6x + 9y + 8z + 35 = 0$
 (d) $x^2 + y^2 + z^2 - 6x - 12y + 8z + 45 = 0$
- 83.** The equation $x^2 + y^2 + z^2 - 4x + 6y - 8z + 29 = 0$ represents
 (a) a sphere (b) the empty set
 (c) a point (d) None of these
- 84.** The centre of the sphere $x(x - x_1) + y(y - y_1) + z(z - z_1) = 0$ is
 (a) (x_1, y_1, z_1) (b) $(-x_1, -y_1, -z_1)$
 (c) $\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2}\right)$ (d) $\left(-\frac{x_1}{2}, -\frac{y_1}{2}, -\frac{z_1}{2}\right)$
- 85.** The position vector of the centre of the sphere $|\mathbf{r}|^2 + \mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) - 9 = 0$ is
 (a) $\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$ (b) $\frac{1}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$
 (c) $-\frac{1}{2}(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ (d) $-\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$
- 86.** The equation of the sphere circumscribing the tetrahedron whose faces are $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{a} + \frac{z}{a} = 1$, is
 (a) $x^2 + y^2 + z^2 = a^2$
 (b) $x^2 + y^2 + z^2 - 2ax - 2ay - 2az = 0$
 (c) $x^2 + y^2 + z^2 - ax - ay - az = a^2$
 (d) $x^2 + y^2 + z^2 - ax - ay - az = 0$
- 87.** The equation of the sphere whose centre is at origin and which touches the plane $\mathbf{r} \cdot \hat{\mathbf{n}} = p$, is
 (a) $|\mathbf{r}| = p$ (b) $|\mathbf{r}|^2 = p$
 (c) $|\mathbf{r}| = p^2$ (d) None of these
- 88.** A sphere through non-collinear points A , B and C is smallest when its centre is
 (a) at origin (b) at centroid of $\triangle ABC$
 (c) in the plane ABC (d) None of these
- 89.** There is one and only one sphere passing through
 (a) three points not in the same straight line
 (b) four points not in the same straight line
 (c) four points not in the same plane
 (d) None of the above

Type 2. More than One Correct Option

90. Consider the planes $3x - 6y + 2z + 5 = 0$ and $4x - 12y + 3z = 3$. Then, the plane $67x + 162y + 47z + 44 = 0$ bisects the angle between the planes which
- (a) contains origin (b) is acute
(c) is obtuse (d) None of these
91. The equation of line AB is $\frac{x}{2} = \frac{y}{-3} = \frac{z}{6}$. Through a point $P(1, 2, 5)$, line PN is drawn perpendicular to AB and line PQ is drawn parallel to the plane $3x + 4y + 5z = 0$ to meet AB is Q . Then,
- (a) coordinates of N are $\left(\frac{52}{49}, -\frac{78}{49}, \frac{156}{49}\right)$
(b) coordinates of Q are $\left(3, -\frac{9}{2}, 9\right)$
(c) the equation of PN is $\frac{x-1}{3} = \frac{y-2}{-176} = \frac{z-5}{-89}$
(d) coordinates of N are $\left(\frac{156}{49}, \frac{52}{49}, -\frac{78}{49}\right)$
92. Let PN be the perpendicular from the point $P(1, 2, 3)$ to XY -plane. If OP makes an angle α with positive direction of Z -axis and ON makes an angle β with the positive direction of X -axis, where O is the origin (α and β being acute angles), then
- (a) $\sin \alpha \sin \beta = \frac{2}{\sqrt{14}}$
(b) $\cos \alpha \cos \beta = \frac{1}{\sqrt{14}}$
(c) $\tan \alpha = \frac{\sqrt{5}}{3}$
(d) $\tan \beta = 2$
93. If the equation of a plane is $2x - y - 3z = 5$ and $A(1, 1, 1)$, $B(2, 1, -3)$, $C(1, -2, -2)$ and $D(-3, 1, 2)$ are four points. Then, which of the following line segment(s) is/are intersected by the plane?
- (a) AD (b) AB (c) AC (d) BC
94. Let $\mathbf{r} \cdot \mathbf{n}_1 = P_1$ and $\mathbf{r} \cdot \mathbf{n}_2 = P_2$ be two planes and $\mathbf{a}$ be a given point, then
- (a) the line passing through $\mathbf{a}$ and parallel to the line of intersection of given planes must be perpendicular to $\mathbf{n}_1 \times \mathbf{n}_2$
(b) the line passing through $\mathbf{a}$ and parallel to the line of intersection of given planes must be parallel to $\mathbf{n}_1 \times \mathbf{n}_2$
(c) the line passing through $\mathbf{a}$ and parallel to line of intersection of given planes must be $(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{n}_1 \times \mathbf{n}_2) = 0$
(d) the line passing through $\mathbf{a}$ and parallel to line of intersection of the given planes must be $(\mathbf{r} - \mathbf{a}) \times (\mathbf{n}_1 \times \mathbf{n}_2) = 0$
95. Let PM be the perpendicular from the point $P(1, 2, 3)$ to XY -plane. If OP makes an angle θ with the positive direction of Z -axis and OM makes an angle ϕ with the positive direction of X -axis, where O is the origin and θ, ϕ are acute angles, then
- (a) $\cos \theta \cos \phi = 1/\sqrt{14}$ (b) $\sin \theta \sin \phi = 2/\sqrt{14}$
(c) $\tan \phi = 2$ (d) $\tan \theta = \sqrt{5}/3$
96. A line with direction cosines proportional to $1, -5$ and -2 meets lines $x = y + 5 = z + 1$ and $x + 5 = 3y = 2z$. The coordinates of each of the points of intersection are given by
- (a) $(2, -3, 1)$ (b) $(1, 2, 3)$
(c) $(0, 5/3, 5/2)$ (d) $(3, -2, 2)$
97. If a rod of length 2 units whose one end is $(1, 0, -1)$ and other end touches the planes $x - 2y + 2z + 4 = 0$, then
- (a) the rod sweeps the figure whose volume is π cu units
(b) the area of the region which the rod traces on the plane is 2π
(c) the length of projection of the rod on the plane is $\sqrt{3}$ units
(d) the centre of the region which the rod traces on the plane, is $\left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3}\right)$

Type 3. Assertion and Reason

Directions (Q. Nos. 98-105) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The correct choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

98. **Statement I** Let $A(\hat{i} + \hat{j} + \hat{k})$ and $B(\hat{i} - \hat{j} + \hat{k})$ be two points, then $P(2\hat{i} + 3\hat{j} + \hat{k})$ lies exterior to the sphere with AB as one of its diameters.

Statement II If A and B are any two points and P is a point in space such that $\mathbf{PA} \cdot \mathbf{PB} > 0$, then the point P lies exterior to the space with AB as one of its diameters.

99. **Statement I** If $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then equation $\mathbf{r} \times (2\hat{i} + \hat{j} + 3\hat{k}) = 3\hat{i} + \hat{k}$ represents a straight line.

Statement II If $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then equation $\mathbf{r} \times (\hat{i} + 2\hat{j} - 3\hat{k}) = 2\hat{i} - \hat{j}$ represents a straight line.

- 100. Statement I** Let θ be the angle between the line $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z+2}{-2}$ and the plane $x + y - z = 5$.

$$\text{Then, } \theta = \sin^{-1} \frac{1}{\sqrt{51}}.$$

Statement II Angle between a straight line and a plane is the complement of angle between the line and normal to the plane.

- 101. Statement I** A point on the straight line $2x + 3y - 4z = 5$ and $3x - 2y + 4z = 7$ can be determined by taking $x = k$ and then solving the two equations for y and z , where k is any real number.

Statement II If $c' \neq kc$, then the straight line $ax + by + cz + d = 0$, $kax + kby + c'z + d' = 0$, does not intersect the plane $z = \alpha$, where α is any real number.

- 102.** Let the line L having equation $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{3}$ intersects the plane P , having equation $x - y + z = 5$ at the point A .

Statement I Equation of the line L' through the point A , lying in the plane P and having minimum inclination with line L , is

$$8x + y - 7z - 4 = 0 = x - y + z - 5$$

Statement II Line L' must be the projection of the line L in the plane P .

- 103. Statement I** The locus of a point which is equidistant from the points whose position vectors are $3\hat{i} - 2\hat{j} + 5\hat{k}$ and $\hat{i} + 2\hat{j} - \hat{k}$, is $\mathbf{r} \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 8$.

Statement II The locus of a point which is equidistant from the points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$, is $\left| \mathbf{r} - \frac{\mathbf{a} + \mathbf{b}}{2} \right| \cdot (\mathbf{a} - \mathbf{b}) = 0$.

- 104.** Consider the planes $3x - 6y - 2z = 15$ and $2x + y - 2z = 5$.

Statement I The parametric equations of the line of intersection of the given planes are $x = 3 + 14t$, $y = 1 + 2t$ and $z = 15t$ being the parameter.

Statement II The vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes.

- 105.** Consider the lines $L_1: \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$

$$\text{and } L_2: \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Statement I The shortest distance between the lines L_1 and L_2 is $3\sqrt{30}$.

Statement II If P is a point on L_1 and Q is a point on L_2 , then the shortest distance between L_1 and L_2 is equal to the length of the line segment PQ .

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 106-108) Consider a plane $x + y - z = 1$ and point $A(1, 2, -3)$. A line L has the equations $x = 1 + 3r$, $y = 2 - r$ and $z = 3 + 4r$.

- 106.** The coordinates of a point B of line L such that AB is parallel to the plane, is

- (a) $(10, -1, 15)$
(b) $(-5, 4, -5)$
(c) $(4, 1, 7)$
(d) $(-8, 5, -9)$

- 107.** The equation of the plane containing line L and point A has the equation

- (a) $x - 3y + 5 = 0$
(b) $x + 3y - 7 = 0$
(c) $3x - y - 1 = 0$
(d) $3x + y - 5 = 0$

- 108.** The distance between the points on the line which are at a distance of $4/\sqrt{3}$ from the plane, is

- (a) $4\sqrt{26}$ (b) 20
(c) $10\sqrt{13}$ (d) None of these

Passage II (Q. Nos. 109-111) Let $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ be two planes, where $d_1, d_2 > 0$. Then, origin lies in acute angle, if $a_1a_2 + b_1b_2 + c_1c_2 < 0$ and origin lies in obtuse angle, if $a_1a_2 + b_1b_2 + c_1c_2 > 0$.

Further, point (x_1, y_1, z_1) and origin both lie either in acute angle or in obtuse angle, if

$$(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) > 0.$$

One of (x_1, y_1, z_1) and origin lie in acute and the other in obtuse angle, if

$$(a_1x_1 + b_1y_1 + c_1z_1 + d_1)(a_2x_1 + b_2y_1 + c_2z_1 + d_2) < 0.$$

- 109.** Let the planes be $2x + 3y - 4z + 7 = 0$ and

$$x - 2y + 3z - 5 = 0. \text{ If a point } P \text{ is } (1, -2, 3), \text{ then}$$

- (a) O and P both lie in acute angle between the planes
(b) O and P both lie in obtuse angle
(c) O lies in acute angle and P lies in obtuse angle
(d) O lies in obtuse angle and P lies in acute angle

- 110.** Let the planes be $x + 2y - 3z + 5 = 0$ and

$$2x + y + 3z + 1 = 0. \text{ If a point } P \text{ is } (2, -1, 2), \text{ then}$$

- (a) O and P both lie in acute angle between the planes
(b) O and P both lie in obtuse angle
(c) O lies in acute angle and P lies in obtuse angle
(d) O lies in obtuse angle and P lies in acute angle

- 111.** Let the planes be $x + 2y - 3z + 2 = 0$ and

$$x - 2y + 3z + 7 = 0. \text{ If a point } P \text{ is } (1, 2, 2), \text{ then}$$

- (a) O and P both lie in acute angle between the planes
(b) O and P both lie in obtuse angle
(c) O lies in acute angle and P lies in obtuse angle
(d) O lies in obtuse angle and P lies in acute angle

Passage III (Q. Nos. 112-114) The line of greatest slope on an inclined plane P_1 is the line in the plane P_1 which is perpendicular to the line of intersection of the plane P_1 and a horizontal plane P_2 .

112. Assuming the plane $4x - 3y + 7z = 0$ to be horizontal, the direction cosines of the line of greatest slope in the plane $2x + y - 5z = 0$ are

- (a) $\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$ (b) $\frac{3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}$
(c) $\frac{-3}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{1}{\sqrt{11}}$ (d) None of these

113. The equation of a line of greatest slope as in above question can be

- (a) $\frac{x}{3} = \frac{y}{1} = \frac{z}{-1}$ (b) $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$
(c) $\frac{x}{-3} = \frac{y}{1} = \frac{z}{1}$ (d) $\frac{x}{1} = \frac{y}{3} = \frac{z}{-1}$

114. The coordinates of a point on the plane $2x + y - 5z = 0$, $2\sqrt{11}$ units away from the line of intersection of $2x + y - 5z = 0$ and $4x - 3y + 7z = 0$, are

- (a) (6, 2, -2) (b) (3, 1, -1) (c) (6, -2, 2) (d) (1, 3, -1)

Passage IV (Q. Nos. 115-117) In a parallelogram $OABC$ with position vectors of A is $3\hat{i} + 4\hat{j}$ and C is $4\hat{i} + 3\hat{j}$ with reference to O as origin. A point E is taken on the side BC which divides it in the ratio of 2 : 1. Also, the line segment AE intersects the line bisecting $\angle AOC$ internally at P . When CP extended meets AB at point F .

115. The position vector of P is

- (a) $\hat{i} + \hat{j}$ (b) $\frac{2}{3}(\hat{i} + \hat{j})$
(c) $\frac{13}{3}(\hat{i} + \hat{j})$ (d) $\frac{21}{5}(\hat{i} + \hat{j})$

116. The equation of line parallel to CP and passing through (2, 3, 4), is

- (a) $\frac{x-2}{1} = \frac{y-3}{5}, z=4$ (b) $\frac{x-2}{1} = \frac{y-3}{6}, z=4$
(c) $\frac{x-2}{2} = \frac{y-3}{5}, z=3$ (d) $\frac{x-2}{3} = \frac{y-3}{5}, z=3$

117. The equation of plane containing line AC and at a maximum distance from B , is

- (a) $\mathbf{r} \cdot (\hat{i} + \hat{j}) = 7$ (b) $\mathbf{r} \cdot (\hat{i} - \hat{j}) = 7$
(c) $\mathbf{r} \cdot (2\hat{i} - \hat{j}) = 7$ (d) $\mathbf{r} \cdot (3\hat{i} + 4\hat{j}) = 7$

Passage V (Q. Nos 118-120) Intersection of the sphere by a plane is called circular section.

- (i) If the plane intersect the sphere in more than one different points, then the section is called a circle.
(ii) If the circle of section is of greatest possible radius, then the circle is called great circle.
(iii) If the radius of circular section is zero, then the section is a point circle.
(iv) If the plane does not meet the sphere at all, then the section is an imaginary circle.

118. Sphere $x^2 + y^2 + z^2 = 4$ is intersected by the plane $2x + 3y + 6z + 7 = 0$ in

- (a) a great circle (b) a real circle but not great
(c) a point circle (d) an imaginary circle

119. Sphere $x^2 + y^2 + z^2 - 2x + 4y + 6z - 17 = 0$ is intersected by the plane $3x + 4y + 2z - 5 = 0$ in

- (a) a great circle (b) a real circle but not great
(c) a point circle (d) an imaginary circle

120. The sphere $x^2 + y^2 + z^2 + 2x + 6y - 8z - 1 = 0$ is intersected by the plane $x + 2y - 3z - 5 = 0$ in

- (a) a great circle (b) a real circle but not great
(c) a point circle (d) an imaginary circle

Type 5. Match the Columns

121. Match the statements of Column I with values of Column II.

Column I	Column II
A. Foot of perpendicular drawn from point (1, 2, 3) to the line $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{4}$ is	p. $\left(\frac{107}{29}, \frac{30}{29}, \frac{69}{29}\right)$
B. Image of a line from point (1, 2, 3) in the line $\frac{x-2}{2} = \frac{y-1}{3} = \frac{z-2}{4}$ is	q. $\left(\frac{88}{29}, \frac{125}{29}, \frac{69}{29}\right)$
C. Foot of perpendicular from the point (2, 3, 5) to the plane $2x + 3y - 4z + 17 = 0$ is	r. $\left(\frac{68}{29}, \frac{44}{29}, \frac{78}{29}\right)$
D. Image of the point (2, 5, 1) in the plane $3x - 2y + 4z - 5 = 0$ is	s. $\left(\frac{38}{29}, \frac{57}{29}, \frac{185}{29}\right)$

122. Match the statements of Column I with values of Column II.

Column I	Column II
A. If the line $x = 1 + a$, $y = 3 - \lambda a$, $z = 1 - \lambda a$ and $x = b$, $y = 1 + b$, $z = 2 - b$ are coplanar, then λ is	p. -3
B. If points (1, 2, 3) and (2, -1, 0) lie on the plane $2x + 3y - 2z = k$, then the integral value of k is	q. 2
C. The volume of the tetrahedron whose vertices are (3, 2, 3), (0, 3, 4), (6, 1, 4) and (6, 3, 2), is	r. 1
D. If (x_1, y_1, z_1) is the image of the point (1, 2, 3) by the plane $x + y + z + 3 = 0$, then z_1 equals	s. -2

123. Match the statements of Column I with values of Column II.

Column I	Column II
A. The vector equation of the plane perpendicular to the line $\frac{x-1}{1} = \frac{y-2}{-1} = \frac{z+1}{2}$ and passing through the point (3, 6, 2), is	p. $r \cdot (\hat{i} - 3\hat{j} - 2\hat{k}) = 3$
B. The vector equation of the plane passing through the point (5, -2, 4) and parallel to the plane $4x - 12y - 8z = 7$, is	q. $r \cdot (\hat{i} - \hat{j} - \hat{k}) = 2$
C. The vector equation of the plane containing $r = 2\hat{i} + \lambda(\hat{j} - \hat{k})$ and perpendicular to the plane $r \cdot (\hat{i} + \hat{k}) = 3$, is	r. $r \cdot (\hat{i} - \hat{j} - \hat{k}) = 0$
D. The vector equation of the plane containing the lines $\frac{x-1}{1} = \frac{y-1}{2} = \frac{z}{-1}$ and $\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{-2}$, is	s. $r \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 1$

Type 6. Single Integer Answer Type Questions

124. If the distance from the line $x = 2 + t$, $y = 1 + t$, $z = -\frac{1}{2} - \frac{1}{2}t$ to the plane $x + 2y + 6z = 10$ is $\frac{\lambda}{\sqrt{\mu}}$, then $5\lambda - \mu$ is _____.
125. If θ is the angle between the lines in which the planes $3x - 7y - 5z = 1$ and $5x - 13y + 3z + 2 = 0$ cuts the plane $8x - 11y + 2z = 0$, then $\sin \theta$ is _____.
126. The position vectors of the four angular points of a tetrahedron $OABC$ are (0, 0, 0), (0, 0, 2), (0, 4, 0) and (6, 0, 0) respectively. If a point P inside the tetrahedron is at the same distance r from the four plane faces of the tetrahedron, then the value of $9r$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. From a point $P(\lambda, \lambda, \lambda)$, perpendiculars PQ and PR are drawn respectively on the lines $y = x$, $z = 1$, $y = -x$ and $z = -1$. If P is such that $\angle QPR$ is a right angle, then the possible value(s) of λ is/are

[2014]
(a) $\sqrt{2}$ (b) 1 (c) -1 (d) $-\sqrt{2}$

2. If the perpendicular is drawn from point on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$, then

the foot of perpendicular lie on the line [2013]

- (a) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (b) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$
(c) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (d) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

3. If a line l passing through the origin is perpendicular to the lines

$$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, -\infty < t < \infty$$

$$l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, -\infty < s < \infty$$

Then, the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is/are [2013]

- (a) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (b) $(-1, -1, 0)$
(c) $(1, 1, 1)$ (d) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$

4. If two lines $L_1 : x = 5$, $\frac{y}{3-\alpha} = \frac{z}{-2}$ and

$$L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$$

are coplanar, then α can take

value(s) [2013]

- (a) 1 (b) 2 (c) 3 (d) 4

5. Consider the lines

$$L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}; L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$$

and the planes $P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ be the equation of the plane passing through the point of intersection of lines L_1 and L_2 perpendicular to the planes P_1 and P_2 . Match List I with List II and select the correct answer using the codes given below the lists. [2012]

List I		List II	
P.	$a =$	1.	13
Q.	$b =$	2.	-3
R.	$c =$	3.	1
S.	$d =$	4.	-2

Codes

P	Q	R	S	P	Q	R	S
(a) 3	2	4	1	(b) 1	3	4	2
(c) 3	2	1	4	(d) 2	4	1	3

6. The point P is the intersection of the straight line joining the points $Q(2, 3, 5)$ and $R(1, -1, 4)$ with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point $T(2, 1, 4)$ to QR , then the length of the line segment PS is [2012]
- (a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$
(c) 2 (d) $2\sqrt{2}$

7. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$, is [2012]
- (a) $5x - 11y + z = 17$ (b) $\sqrt{2}x + y = 3\sqrt{2} - 1$
(c) $x + y + z = \sqrt{3}$ (d) $x - \sqrt{2}y = 1 - \sqrt{2}$

8. If the straight lines $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is/are [2012]
- (a) $y + 2z = -1$ (b) $y + z = -1$
(c) $y - z = -1$ (d) $y - 2z = -1$

Directions (Q. Nos. 9-11) Consider the lines

$$L_1: \frac{x+1}{3} = \frac{y+2}{1} = \frac{z+1}{2}; L_2: \frac{x-2}{1} = \frac{y+2}{2} = \frac{z-3}{3}$$

[2012]

9. The unit vector perpendicular to both L_1 and L_2 is

- (a) $\frac{-\hat{i} + 7\hat{j} + 7\hat{k}}{\sqrt{99}}$
(b) $\frac{-\hat{i} - 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$
(c) $\frac{-\hat{i} + 7\hat{j} + 5\hat{k}}{5\sqrt{3}}$
(d) $\frac{7\hat{i} - 7\hat{j} - \hat{k}}{\sqrt{99}}$

10. The shortest distance between L_1 and L_2 is

- (a) 0 unit (b) $\frac{17}{\sqrt{3}}$ units
(c) $\frac{41}{5\sqrt{3}}$ units (d) $\frac{17}{5\sqrt{3}}$ units

11. The distance of the point $(1, 1, 1)$ from the plane passing through the point $(-1, -2, -1)$ and whose normal is perpendicular to both the lines L_1 and L_2 , is

- (a) $\frac{2}{\sqrt{75}}$ unit (b) $\frac{7}{\sqrt{75}}$ unit
(c) $\frac{13}{\sqrt{75}}$ units (d) $\frac{23}{\sqrt{75}}$ units

12. Equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to the plane

$$\text{containing the straight lines } \frac{x}{3} = \frac{y}{4} = \frac{z}{2} \text{ and}$$

$$\frac{x}{4} = \frac{y}{2} = \frac{z}{3} \text{ is}$$

[2010]

- (a) $x + 2y - 2z = 0$ (b) $3x + 2y - 2z = 0$
(c) $x - 2y + z = 0$ (d) $5x + 2y - 4z = 0$

13. If the distance of the point $P(1, -2, 1)$ from the plane $x + 2y - 2z = \alpha$, where $\alpha > 0$ is 5, then the foot of the perpendicular from P to the plane is

[2010]

- (a) $\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$ (b) $\left(\frac{4}{3}, -\frac{4}{3}, \frac{1}{3}\right)$
(c) $\left(\frac{1}{3}, \frac{2}{3}, -\frac{10}{3}\right)$ (d) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{5}{2}\right)$

14. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is

[2010]

- (a) 3 (b) 4
(c) 5 (d) 6

15. A line AB in three-dimensional space make angles 45° and 120° with the positive X -axis and the positive Y -axis respectively. If AB makes an acute angle θ with the positive Z -axis, then θ equals

[2010]

- (a) 30° (b) 45°
(c) 60° (d) 75°

16. **Statement I** The point $A(3, 1, 6)$ is the mirror image of the point $B(1, 3, 4)$ in the plane $x - y + z = 5$.

Statement II The plane $x - y + z = 5$ bisects the line segment joining $A(3, 1, 6)$ and $B(1, 3, 4)$. [2010]

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
(b) Statement I is true, Statement II is true, Statement II is not the correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

JEE Main/AIEEE

14

Three Dimensional Geometry

Target Exercises

17. Locus of the image of the point $(2, 3)$ in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0, k \in R$, is a [2015]
 (a) straight line parallel to X -axis
 (b) straight line parallel to Y -axis
 (c) circle of radius $\sqrt{2}$
 (d) circle of radius $\sqrt{3}$
18. The equation of the plane containing the lines $2x - 5y + z = 3, x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$, is [2015]
 (a) $2x + 6y + 12z = 13$ (b) $x + 3y + 6z = -7$
 (c) $x + 3y + 6z = 7$ (d) $2x + 6y + 12z = -13$
19. The distance of the point $(1, 0, 2)$ from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x + y + z = 16$, is [2015]
 (a) $2\sqrt{14}$ (b) 8
 (c) $3\sqrt{21}$ (d) 13
20. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$, is [2014]
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
 (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$
21. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line [2014]
 (a) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$ (b) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$
 (c) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$ (d) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$
22. If the lines $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z+1}{3}$ and $\frac{x+2}{2} = \frac{y-k}{3} = \frac{z}{4}$ are coplanar, then the value of k is [2013]
 (a) $\frac{11}{2}$ (b) $-\frac{11}{2}$ (c) $\frac{9}{2}$ (d) $-\frac{9}{2}$
23. A vector $\mathbf{n}$ is inclined to X -axis at 45° , to Y -axis at 60° and at an acute angle to Z -axis. If $\mathbf{n}$ is a normal to a plane passing through the point $(\sqrt{2}, -1, 1)$, then the equation of the plane is [2013]
 (a) $4\sqrt{2}x + 7y + z = 2$ (b) $\sqrt{2}x + y + z = 2$
 (c) $3\sqrt{2}x - 4y - 3z = 7$ (d) $\sqrt{2}x - y - z = 2$
24. Let Q be the foot of perpendicular from the origin to the plane $4x - 3y + z + 13 = 0$ and P be a point $(-1, 1, -6)$ on the plane. Then, length PQ is [2013]
 (a) $\sqrt{14}$ (b) $\sqrt{\frac{19}{2}}$ (c) $3\sqrt{\frac{7}{2}}$ (d) $\frac{3}{\sqrt{2}}$
25. The acute angle between two lines such that the direction cosines l, m, n of each of them satisfy the equations $l + m + n = 0$ and $l^2 + m^2 - n^2 = 0$, is [2013]
 (a) 15° (b) 30° (c) 60° (d) 45°
26. If the projections of a line segment on the X, Y and Z -axes in 3-dimensional space are 2, 3 and 6 respectively, then the length of the line segment is [2013]
 (a) 12 (b) 7 (c) 9 (d) 6
27. The equation of a plane through the line of intersection of the planes $x + 2y = 3, y - 2z + 1 = 0$ and perpendicular to the first plane, is [2013]
 (a) $2x - y - 10z = 9$ (b) $2x - y + 7z = 11$
 (c) $2x - y + 10z = 11$ (d) $2x - y - 9z = 10$
28. Let ABC be a triangle with vertices at points $A(2, 3, 5), B(-1, 3, 2)$ and $C(\lambda, 5, \mu)$ in three dimensional space. If the median through A is equally inclined with the axes, then (λ, μ) is equal to [2013]
 (a) $(10, 7)$ (b) $(7, 5)$ (c) $(7, 10)$ (d) $(5, 7)$
29. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is [2013]
 (a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$
30. If the lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-k}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then k can have [2013]
 (a) any value (b) exactly one value
 (c) exactly two values (d) exactly three values
31. If the lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then k is equal to [2012]
 (a) -1 (b) $\frac{2}{9}$ (c) $\frac{9}{2}$ (d) 0
32. An equation of a plane parallel to the plane $x - 2y + 2z - 5 = 0$ and at a unit distance from the origin, is [2012]
 (a) $x - 2y + 2z \pm 3 = 0$ (b) $x - 2y + 2z + 1 = 0$
 (c) $x - 2y + 2z - 1 = 0$ (d) $x - 2y + 2z + 5 = 0$
33. If the angle between the line $x = \frac{y-1}{2} = \frac{z-3}{\lambda}$ and the plane $x + 2y + 3z = 4$ is $\cos^{-1}\left(\frac{\sqrt{5}}{\sqrt{14}}\right)$, then λ is equal to [2011]
 (a) $\frac{3}{2}$ (b) $\frac{2}{5}$ (c) $\frac{5}{3}$ (d) $\frac{2}{3}$

34. Statement I The point $A(1, 0, 7)$ is the mirror image of the point $B(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

Statement II The line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ bisects the line segment joining $A(1, 0, 7)$ and $B(1, 6, 3)$.

[2011]

- (a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true, Statement II is false
(c) Statement I is false, Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

35. The length of the perpendicular drawn from the point $(3, -1, 11)$ to the line $\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ is

[2011]

- (a) $\sqrt{66}$ (b) $\sqrt{29}$ (c) $\sqrt{33}$ (d) $\sqrt{53}$

36. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along a straight line $x = y = z$, is

[2011]

- (a) $3\sqrt{5}$ (b) $10\sqrt{3}$
(c) $5\sqrt{3}$ (d) $3\sqrt{10}$

37. Let the line $\frac{x-2}{3} = \frac{y-1}{-5} = \frac{z+2}{2}$ lies in the plane $x + 3y - \alpha z + \beta = 0$. Then, (α, β) equals

[2009]

- (a) $(6, -17)$ (b) $(-6, 7)$ (c) $(5, -15)$ (d) $(-5, 15)$

38. The projections of a vector on the three coordinate axes are 6, -3, 2 respectively. The direction cosines of the vector are

[2009]

- (a) 6, -3, 2 (b) $\frac{6}{5}, -\frac{3}{5}, \frac{2}{5}$
(c) $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$ (d) $-\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$

39. The line passing through the points $(5, 1, a)$ and $(3, b, 1)$ crosses the YZ -plane at the point $(0, \frac{17}{2}, -\frac{13}{2})$. Then,

[2008]

- (a) $a = 8, b = 2$ (b) $a = 2, b = 8$
(c) $a = 4, b = 6$ (d) $a = 6, b = 4$

40. If the straight lines $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3}$ and $\frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2}$ intersect at a point, then the integer k is equal to

[2008]

- (a) -2 (b) -5 (c) 5 (d) 2

41. Let L be the line of intersection of the planes $2x + 3y + z = 1$ and $x + 3y + 2z = 2$. If L makes an angle α with the positive X -axis, then $\cos \alpha$ equals

[2007]

- (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$ (c) 1 (d) $\frac{1}{\sqrt{2}}$

42. If a line makes an angle of $\frac{\pi}{4}$ with the positive directions of each of X and Y -axes, then the angle that the line makes with the positive direction of the Z -axis is

[2007]

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

43. If $(2, 3, 5)$ is one end of a diameter of the sphere $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$, then the coordinates of the other end of the diameter are

[2007]

- (a) $(4, 9, -3)$ (b) $(4, -3, 3)$
(c) $(4, 3, 5)$ (d) $(4, 3, -3)$

44. The two lines $x = ay + b, z = cy + d$ and $x = a'y + b', z = c'y + d'$ are perpendicular to each other, if

[2006, 03]

- (a) $ad' + cd' = 1$
(b) $\frac{a}{d'} + \frac{c}{c'} = -1$
(c) $\frac{a}{d} + \frac{c}{c'} = 1$
(d) $ad' + cc' = -1$

45. The image of the point $(-1, 3, 4)$ in the plane $x - 2y = 0$ is

[2006]

- (a) $(15, 11, 4)$ (b) $(-\frac{17}{3}, -\frac{19}{3}, 1)$
(c) $(8, 4, 4)$ (d) $(\frac{9}{5}, -\frac{13}{5}, 4)$

46. If the plane $2ax - 3ay + 4az + 6 = 0$ passes through the mid-point of the line joining the centres of the spheres $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$ and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$, then a equals

[2005]

- (a) 2 (b) -2
(c) 1 (d) -1

47. If the angle θ between the line $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}z + 4 = 0$ is such that $\sin \theta = 1/3$. Then, value of λ is

[2005]

- (a) $-\frac{4}{3}$ (b) $\frac{3}{4}$
(c) $-\frac{3}{5}$ (d) $\frac{5}{3}$

48. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is

[2005]

- (a) 30° (b) 45°
(c) 90° (d) 0°

49. The plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z - 2 = 0$ in a circle of radius

[2005]

- (a) $\sqrt{2}$ (b) 2
(c) 1 (d) 3

50. A line makes the same angle θ with each of the X and Z -axes. If the angle β , which it makes with Y -axis, is such that $\sin^2 \beta = 3 \sin^2 \theta$, then $\cos^2 \theta$ equals [2004]
 (a) $\frac{2}{3}$ (b) $\frac{1}{5}$ (c) $\frac{3}{5}$ (d) $\frac{2}{5}$
51. Distance between two parallel planes $2x + y + 2z = 8$ and $4x + 2y + 4z + 5 = 0$ is [2004]
 (a) $\frac{3}{2}$ (b) $\frac{5}{2}$ (c) $\frac{7}{2}$ (d) $\frac{9}{2}$
52. A line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. The coordinates of each of the points of intersection are given by [2004]
 (a) $(3a, 3a, 3a)$, (a, a, a) (b) $(3a, 2a, 3a)$, (a, a, a)
 (c) $(3a, 2a, 3a)$, $(a, a, 2a)$ (d) $(2a, 3a, 3a)$, $(2a, a, a)$
53. If the straight lines $x = 1 + s$, $y = -3 - \lambda s$, $z = 1 + \lambda s$ and $x = \frac{t}{2}$, $y = 1 + t$, $z = 2 - t$, with parameters s and t respectively are coplanar, then λ equals [2004]
 (a) -2 (b) -1 (c) $-\frac{1}{2}$ (d) 0
54. The intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$ is the same as the intersection of one of the sphere and the plane [2004]
 (a) $x - y - z = 1$ (b) $x - 2y - z = 1$
 (c) $x - y - 2z = 1$ (d) $2x - y - z = 1$
55. The radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$ is cut by the plane $x + 2y + 2z + 7 = 0$, is [2003]
 (a) 1 (b) 2 (c) 3 (d) 4
56. The shortest distance from the plane $12x + 4y + 3z = 327$ to the sphere $x^2 + y^2 + z^2 + 4x - 2y - 6z = 155$, is [2003]
 (a) 26 (b) $11\frac{4}{13}$
 (c) 13 (d) 39
57. Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b, c and a', b', c' from the origin, then [2003]
 (a) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = 0$
 (b) $\frac{1}{a^2} + \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} + \frac{1}{b'^2} - \frac{1}{c'^2} = 0$
 (c) $\frac{1}{a^2} - \frac{1}{b^2} - \frac{1}{c^2} + \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$
 (d) $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$
58. A parallelepiped is formed by planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$, parallel to the coordinate planes. The length of a diagonal of the parallelepiped is [2002]
 (a) 7 units (b) $\sqrt{38}$ units
 (c) $\sqrt{155}$ units (d) None of these
59. The equation of the plane containing the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ is $a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$, where [2002]
 (a) $ax_1 + by_1 + cz_1 = 0$ (b) $al + bm + cn = 0$
 (c) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$ (d) $lx_1 + my_1 + nz_1 = 0$
60. The centre of the circle given by $\mathbf{r} \cdot (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}) = 15$ and $|\mathbf{r} - (\hat{\mathbf{j}} + 2\hat{\mathbf{k}})| = 4$, is [2002]
 (a) $(0, 1, 2)$ (b) $(1, 3, 4)$
 (c) $(-1, 3, 4)$ (d) None of these

Other Engineering Entrances

61. The angle between two diagonals of a cube is [Karnataka CET 2014]
 (a) $\cos^{-1}\left(\frac{1}{3}\right)$
 (b) 30°
 (c) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$
 (d) 45°
62. If the direction cosines of a vector of magnitude 3 are $\frac{2}{3}, \frac{-a}{3}, \frac{2}{3}$ and $a > 0$, then the vector is [Kerala CEE 2014]
 (a) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
 (b) $2\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
 (c) $\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
 (d) $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
 (e) $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$
63. The angle between a normal to the plane $2x - y + 2z - 1 = 0$ and the Z -axis is [Kerala CEE 2014]
 (a) $\cos^{-1}\left(\frac{1}{3}\right)$
 (b) $\sin^{-1}\left(\frac{2}{3}\right)$
 (c) $\cos^{-1}\left(\frac{2}{3}\right)$
 (d) $\sin^{-1}\left(\frac{1}{3}\right)$
 (e) $\sin^{-1}\left(\frac{3}{5}\right)$
64. If the direction cosines of two lines are given by $l + m + n = 0$ and $l^2 - 5m^2 + n^2 = 0$, then the angle between them is [EAMCET 2014]
 (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{3}$

65. If $A(3, 4, 5)$, $B(4, 6, 3)$, $C(-1, 2, 4)$ and $D(1, 0, 5)$ are such that the angle between the lines DC and AB is θ , then $\cos \theta$ is equal to [EAMCET 2014]
 (a) $\frac{7}{9}$ (b) $\frac{2}{9}$
 (c) $\frac{4}{9}$ (d) $\frac{5}{9}$
66. The angle between the lines, whose direction ratios are $(1, 1, 2)$, $(\sqrt{3}-1, -\sqrt{3}-1, 4)$, is [RPET 2014]
 (a) $\cos^{-1}\left(\frac{1}{65}\right)$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
67. The direction cosines of the line joining the points $(4, 3, -5)$ and $(-2, 1, -8)$, are [RPET 2014]
 (a) $\left(\frac{6}{7}, \frac{2}{7}, \frac{3}{7}\right)$ (b) $\left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$
 (c) $\left(\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$ (d) None of these
68. If two lines, whose equations are $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3}$ lie in the same plane. Then, the value of $\sin^{-1}(\sin \lambda)$ is equal to [BITSAT 2014]
 (a) 3 (b) $\pi - 3$
 (c) 4 (d) $\pi - 4$
69. A perpendicular is drawn from the point $P(2, 4, -1)$ to the line $\frac{x+5}{1} = \frac{y+3}{4} = \frac{z-6}{-9}$. The equation of the perpendicular from P to the given line is [Manipal 2014]
 (a) $\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$
 (b) $\frac{x+2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$
 (c) $\frac{x+2}{-6} = \frac{y-4}{3} = \frac{z+1}{2}$
 (d) $\frac{x+2}{6} = \frac{y+4}{3} = \frac{z+1}{2}$
70. Two lines $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = z$ intersect at a point, if k is equal to [VITEEE 2014]
 (a) $\frac{2}{9}$ (b) $\frac{1}{2}$
 (c) $\frac{9}{2}$ (d) $\frac{1}{6}$
71. The angle between the straight lines $x-1 = \frac{2y+3}{3} = \frac{z+5}{2}$ and $x=3r+2$, $y=-2r-1$, $z=2$, where r is a parameter, is [Kerala CEE 2014]
 (a) $\frac{\pi}{4}$ (b) $\cos^{-1}\left(\frac{-3}{\sqrt{182}}\right)$
 (c) $\sin^{-1}\left(\frac{-3}{\sqrt{182}}\right)$ (d) $\frac{\pi}{2}$
 (e) 0
72. If the plane $x + y + z = 1$ is rotated through an angle 90° about its line of intersection with the plane $x - 2y + 3z = 0$, then the new position of the plane is [BITSAT 2014]
 (a) $x - 5y + 4z = 1$ (b) $x - 5y + 4z = -1$
 (c) $x - 8y + 7z = 2$ (d) $x - 8y + 7z = -2$
73. If the points $(1, 2, 3)$ and $(2, -1, 0)$ lie on the opposite sides of the plane $2x + 3y - 2z = k$, then [VITEEE 2014]
 (a) $k < 1$ (b) $k > 2$
 (c) $k < 1$ or $k > 2$ (d) $1 < k < 2$
74. The volume of the tetrahedron included between the plane $3x + 4y - 5z - 60 = 0$ and the coordinate planes, is [VITEEE 2014]
 (a) 60 (b) 600 (c) 720 (d) 400
75. If a plane meets the coordinate axes at A, B and C such that the centroid of the triangle is $(1, 2, 4)$, then the equation of the plane is [VITEEE 2014]
 (a) $x + 2y + 4z = 12$ (b) $4x + 2y + z = 12$
 (c) $x + 2y + 4z = 3$ (d) $4x + 2y + z = 3$
76. The equation of the plane through the intersection of the planes $3x - y + 2z - 4 = 0$, $x + y + z - 2 = 0$ and the point $(2, 2, 1)$, is [Manipal 2014]
 (a) $7x + 5y + 4z + 8 = 0$ (b) $7x + 5y + 4z - 8 = 0$
 (c) $7x - 5y + 4z - 8 = 0$ (d) None of these
77. The image of the point with position vector $\hat{i} + 3\hat{k}$ in the plane $\mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$, is [Manipal 2014]
 (a) $\hat{i} + 2\hat{j} + \hat{k}$ (b) $\hat{i} + 2\hat{j} - \hat{k}$
 (c) $-\hat{i} - 2\hat{j} + \hat{k}$ (d) $\hat{i} - 2\hat{j} + \hat{k}$
78. Foot of the perpendicular drawn from the origin to the plane $2x - 3y + 4z = 29$, is [Kerala CEE 2014]
 (a) $(5, -1, 4)$ (b) $(7, -1, 3)$
 (c) $(5, -2, 3)$ (d) $(2, -3, 4)$
 (e) $(1, -3, 4)$
79. Equation of the plane through the mid-point of the line segment joining the points $P(4, 5, -10)$ and $Q(-1, 2, 1)$ and perpendicular to PQ , is [Kerala CEE 2014]
 (a) $\mathbf{r} \cdot \left(\frac{3}{2}\hat{i} + \frac{7}{2}\hat{j} - \frac{9}{2}\hat{k}\right) = 45$
 (b) $\mathbf{r} \cdot (-\hat{i} + 2\hat{j} + \hat{k}) = \frac{135}{2}$
 (c) $\mathbf{r} \cdot (5\hat{i} + 3\hat{j} - 11\hat{k}) + \frac{135}{2} = 0$
 (d) $\mathbf{r} \cdot (4\hat{i} + 5\hat{j} - 10\hat{k}) = 85$
 (e) $\mathbf{r} \cdot (5\hat{i} + 3\hat{j} - 11\hat{k}) = \frac{135}{2}$
80. The equation of the plane passing through the points $(2, 2, 1)$, $(9, 3, 6)$ and perpendicular to the plane $2x + 6y + 6z = 1$ is [GGSIU 2014]
 (a) $2x - 4y + 5z - 9 = 0$ (b) $3x + 4y - z - 5 = 0$
 (c) $3x + 4y - 5z - 9 = 0$ (d) $x + 4y - 9z - 3 = 0$

81. The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along a straight line $x = y = z = 2\sqrt{3}k$, then the value of k is [BITSAT 2014]
(a) 5 (b) 6 (c) $\sqrt{3}$ (d) 4
82. The line joining the points $(1, 1, 2)$ and $(3, -2, 1)$ meets the plane $3x + 2y + z = 6$ at the point [Manipal 2014]
(a) $(1, 1, 2)$ (b) $(3, -2, 1)$
(c) $(2, -3, 1)$ (d) $(3, 2, 1)$
83. Equation of the plane perpendicular to the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and passing through the point $(2, 3, 4)$, is [Karnataka CET 2014]
(a) $2x + 3y + z = 17$ (b) $x + 2y + 3z = 9$
(c) $3x + 2y + z = 16$ (d) $x + 2y + 3z = 20$
84. Equation of the line through the point $(2, 3, 1)$ and parallel to the line of intersection of the planes $x - 2y - z + 5 = 0$ and $x + y + 3z = 6$, is [Kerala CEE 2014]
(a) $\frac{x-2}{-5} = \frac{y-3}{-4} = \frac{z-1}{3}$
(b) $\frac{x-2}{5} = \frac{y-3}{-4} = \frac{z-1}{3}$
(c) $\frac{x-2}{5} = \frac{y-3}{4} = \frac{z-1}{3}$
(d) $\frac{x-2}{4} = \frac{y-3}{3} = \frac{z-1}{2}$
(e) $\frac{x-2}{-4} = \frac{y-3}{-3} = \frac{z-1}{2}$
85. The line $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is parallel to the plane [AMU 2014 Karnataka CET 2014]
(a) $3x + 4y + 5z = 7$
(b) $2x + 3y + 4z = 0$
(c) $x + y - z = 2$
(d) $2x + y - 2z = 0$
86. A plane which passes through the point $(3, 2, 0)$ and the line $\frac{(x-4)}{1} = \frac{(y-7)}{5} = \frac{(z-4)}{4}$, is [J&K CET 2014]
(a) $x + y + z = 1$
(b) $x + 2y - z = 1$
(c) $x - y + z = 1$
(d) $2x - y + z = 5$
87. Find the intersection of the spheres $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$. [J&K CET 2014]
(a) $x - y - z = 1$
(b) $x - 2y - z = 1$
(c) $x - y - 2z = 1$
(d) $2x - y - z = 1$
88. Find the radius of the circle in which the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z = 19$ is cut by the plane $x + 2y + 2z + 7 = 0$. [J&K CET 2014]
(a) 2 (b) 3 (c) 1 (d) 4
89. A straight line $\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}$ meets the plane $\mathbf{r} \cdot \hat{\mathbf{n}} = p$ in the point P whose position vector is [AMU 2014]
(a) $\mathbf{a} + \frac{\mathbf{n} \cdot \hat{\mathbf{n}}}{\mathbf{b} \cdot \hat{\mathbf{n}}} \mathbf{b}$ (b) $\mathbf{a} + \left(\frac{p - \mathbf{a} \cdot \hat{\mathbf{n}}}{\mathbf{b} \cdot \hat{\mathbf{n}}} \right) \mathbf{b}$
(c) $\mathbf{a} - \left(\frac{\mathbf{a} \cdot \hat{\mathbf{n}}}{\mathbf{b} \cdot \hat{\mathbf{n}}} \right) \mathbf{b}$ (d) None of these
90. If $\mathbf{a} = \hat{\mathbf{i}} + 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$. A vector in the plane of $\mathbf{a}$ and $\mathbf{b}$ whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is [AMU 2014]
(a) $4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$ (b) $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ (d) $4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 4\hat{\mathbf{k}}$
91. The shortest distance between the skew-lines $\mathbf{r} = (\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + t(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}})$ and $\mathbf{r} = (4\hat{\mathbf{i}} + 5\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) + t(2\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})$ is [EAMCET 2014]
(a) $\sqrt{6}$ (b) 3
(c) $2\sqrt{3}$ (d) $\sqrt{3}$
92. The angle between the lines $\mathbf{r} = (2\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + 3\hat{\mathbf{k}}) + \lambda(\hat{\mathbf{i}} + 4\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ and $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + \mu(\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - 3\hat{\mathbf{k}})$ is [EAMCET 2014]
(a) $\frac{\pi}{2}$
(b) $\cos^{-1}\left(\frac{9}{\sqrt{91}}\right)$
(c) $\cos^{-1}\left(\frac{7}{\sqrt{84}}\right)$
(d) $\frac{\pi}{3}$
93. The straight line $\mathbf{r} = (\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) + \lambda(2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ and the plane $\mathbf{r} \cdot (2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}) = 4$ are [GGSIPU 2014]
(a) perpendicular to each other
(b) parallel
(c) inclined at an angle 50°
(d) None of the above
94. The position vector of the point, where the line $\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}} + t(\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})$ meets the plane $\mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 5$, is [BITSAT 2013]
(a) $5\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}}$
(b) $5\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 3\hat{\mathbf{k}}$
(c) $2\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$
(d) $5\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$
95. If $\mathbf{a} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$ and $\mathbf{c} = \hat{\mathbf{i}} - \hat{\mathbf{j}} - \hat{\mathbf{k}}$ are three vectors. A vector $\mathbf{v}$ in the plane of $\mathbf{a}$ and $\mathbf{b}$ whose projection on $\mathbf{c}$ is $\frac{1}{\sqrt{3}}$, is given by [AMU 2013]
(a) $3\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$ (b) $3\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$
(c) $3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ (d) $-3\hat{\mathbf{i}} + \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$

96. The point of intersection of lines $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1}$ and $\frac{x+3}{-36} = \frac{y-3}{2} = \frac{z-6}{4}$ is [MP PET 2013]

- (a) (5, 7, -2) (b) (-3, 3, 6)
(c) (2, 10, 4) (d) $\left(21, \frac{5}{3}, \frac{10}{3}\right)$

97. Equation of the line passing through (2, -1, 1) and parallel to the line $\frac{x-5}{4} = \frac{y+2}{-3} = \frac{z}{5}$ is [OJEE 2013]

- (a) $\frac{x-2}{4} = \frac{y+1}{-3} = \frac{z-1}{5}$ (b) $\frac{x-2}{4} = \frac{y+1}{3} = \frac{z-1}{5}$
(c) $\frac{x-2}{-4} = \frac{y+1}{-3} = \frac{z-1}{5}$ (d) None of these

98. The vector equation $\mathbf{r} = \hat{i} - 2\hat{j} - \hat{k} + t(6\hat{j} - \hat{k})$ represents a straight line passing through the points [BITSAT 2012]

- (a) (0, 6, -1) and (1, -2, -1)
(b) (0, 6, -1) and (-1, -4, -2)
(c) (1, -2, -1) and (1, 4, -2)
(d) (1, -2, -1) and (0, -6, 1)

99. The shortest distance of the point (1, 2, 3) from X, Y and Z-axes, respectively are [MP PET 2011]

- (a) 1, 2, 3
(b) $\sqrt{5}, \sqrt{13}, \sqrt{10}$
(c) $\sqrt{10}, \sqrt{13}, \sqrt{5}$
(d) $\sqrt{13}, \sqrt{10}, \sqrt{5}$

100. If (1, -2, -2) and (0, 2, 1) are direction ratios of two lines, then the direction cosines of a perpendicular to both the lines are [GGSIPU 2011]

- (a) $\left(\frac{1}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$
(c) $\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$ (d) $\left(\frac{2}{\sqrt{14}}, -\frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

101. The plane $ax + by + cz = 1$ meets the coordinate axes at A, B and C. The centroid of $\triangle ABC$ is [MP PET 2011]

- (a) (3a, 3b, 3c) (b) $\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right)$
(c) $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ (d) (a, b, c)

102. The image of the point (1, 3, 4) with respect to the plane $2x - y + z + 3 = 0$ is [MP PET 2011]

- (a) (-1, 4, 3) (b) (-3, 5, 2)
(c) (1, 3, 4) (d) (-1, -3, -4)

103. An equation of the plane through the points (1, 0, 0), (0, 2, 0) and at a distance $\frac{6}{7}$ unit from the origin, is [Kerala CEE 2011]

- (a) $6x + 3y + z - 6 = 0$ (b) $6x + 3y + 2z - 6 = 0$
(c) $6x + 3y + z + 6 = 0$ (d) $6x + 3y + 2z + 6 = 0$
(e) $6x + 2y + 3z + 6 = 0$

104. The equation of the plane that contains the point (1, -1, 2) and is perpendicular to each of the planes $2x + 3y - 2z = 5$ and $x + 2y - 3z = 8$ is [AMU 2011]
(a) $5x + 4y - z = 7$ (b) $5x - 4y + z = 7$
(c) $-5x + 4y - z = 7$ (d) $5x - 4y - z = 7$

105. The equation of the plane perpendicular to Z-axis and passing through (2, -3, 5) is [J & K CET 2011]
(a) $x - 2 = 0$
(b) $y + 3 = 0$
(c) $z - 5 = 0$
(d) $2x - 3y + 5z + 4 = 0$

106. Find the shortest distance between the lines $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$ and $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$. [Guj CET 2011]

- (a) $2\sqrt{3}$ (b) $3\sqrt{3}$
(c) $4\sqrt{3}$ (d) None of these

107. The equation of a sphere having centre (1, 2, 3) and radius 3 units, is [Guj CET 2011]

- (a) $x^2 + y^2 + z^2 - 2x - 4y - 6z = 0$
(b) $x^2 + y^2 + z^2 - 2x - 4y - 6z + 5 = 0$
(c) $x^2 + y^2 + z^2 - 2x - 4y - 6z - 5 = 0$
(d) None of the above

108. The plane $2x - 2y + z + 5 = 0$ is a tangent to the sphere $(x-2)^2 + (y-2)^2 + (z-1)^2 = r^2$, if r equals [J&K CET 2011]

- (a) 1 (b) 2
(c) 4 (d) None of these

109. The angle between the line $\mathbf{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 4\hat{k})$ and the plane $\mathbf{r} \cdot (\hat{i} + 2\hat{j} - 2\hat{k}) = 0$ is [Kerala CEE 2011]

- (a) 0° (b) 60°
(c) 30° (d) 90°
(e) None of these

110. Find the distance between the planes $\mathbf{r} \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = 4$ and $\mathbf{r} \cdot (6\hat{i} - 3\hat{j} + 9\hat{k}) + 13 = 0$. [Guj CET 2011]

- (a) $\frac{5}{3\sqrt{14}}$ (b) $\frac{10}{3\sqrt{14}}$
(c) $\frac{25}{3\sqrt{14}}$ (d) None of these

111. The centre and radius of the sphere $\mathbf{r}^2 - 2\mathbf{r}(3\hat{i} + 4\hat{j} - 5\hat{k}) + 1 = 0$, are [GGSIPU 2011]

- (a) $3\hat{i} + 4\hat{j} - 5\hat{k}, 1$ (b) $-3\hat{i} - 4\hat{j} + 5\hat{k}, 7$
(c) $-3\hat{i} - 4\hat{j} + 5\hat{k}, 7$ (d) $3\hat{i} + 4\hat{j} - 5\hat{k}, 7$

112. Equation of the plane passing through the intersection of the planes $x + y + z = 6$, $2x + 3y + 4z + 5 = 0$ and the point (1, 1, 1), is [BITSAT 2010]

- (a) $20x + 23y + 26z - 69 = 0$
(b) $31x + 45y + 49z + 52 = 0$
(c) $8x + 5y + 2z - 69 = 0$
(d) $4x + 5y + 6z - 7 = 0$

- 113.** A plane makes intercepts a, b and c at A, B and C on the coordinate axes, respectively. If the centroid of $\triangle ABC$ is at $(3, 2, 1)$, then the equation of the plane is [Kerala CEE 2010]

(a) $x + 2y + 3z = 9$ (b) $2x - 3y - 6z = 18$
(c) $2x + 3y + 6z = 18$ (d) $2x + y + 6z = 18$
(e) $2x + 3y + 6z = 9$

- 114.** The equation of the perpendicular from the point (α, β, γ) to the plane $ax + by + cz + d = 0$, is [MP PET 2010]

(a) $\frac{x-a}{a\alpha} = \frac{y-b}{b\beta} = \frac{z-c}{c\gamma}$ (b) $\frac{x}{a} = \frac{y}{b} = \frac{z}{c}$
(c) $\frac{x-\alpha}{a} = \frac{y-\beta}{b} = \frac{z-\gamma}{c}$ (d) $\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}$

- 115.** The equation of the plane containing the lines $\frac{x-1}{2} = \frac{y+1}{-1} = \frac{z}{3}$ and $\frac{x}{2} = \frac{y-2}{-1} = \frac{z+1}{3}$, is [Kerala CEE 2010]

(a) $8x - y + 5z - 8 = 0$
(b) $8x + y - 5z - 7 = 0$
(c) $x - 8y + 3z + 6 = 0$
(d) $8x + y - 5z + 7 = 0$
(e) $x + y + z - 6 = 0$

- 116.** If the plane $3x + y + 2z + 6 = 0$ is parallel to the line $\frac{3x-1}{2b} = 3 - y = \frac{z-1}{a}$, then the value of $3a + 3b$ is [Kerala CEE 2010]
- (a) $1/2$ (b) $3/2$
(c) 3 (d) 4
(e) $5/2$

- 117.** The equation of the line passing through the point $(3, 0, -4)$ and perpendicular to the plane $2x - 3y + 5z - 7 = 0$, is [Kerala CEE 2010]

(a) $\frac{x-2}{3} = \frac{y}{-3} = \frac{z+4}{5}$
(b) $\frac{x-3}{2} = \frac{y}{-3} = \frac{z-4}{5}$
(c) $\frac{x-3}{2} = \frac{-y}{3} = \frac{z+4}{5}$
(d) $\frac{x+3}{2} = \frac{y}{3} = \frac{z-4}{5}$
(e) $\frac{x-2}{3} = \frac{y}{3} = \frac{z+4}{5}$

- 118.** The distance of the point $(3, 8, 2)$ from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z = 0$ is [OJEE 2010]
- (a) 2
(b) 3
(c) 6
(d) 7

- 119.** The vector equation of the straight line $\frac{1-x}{3} = \frac{y+1}{-2} = \frac{3-z}{-1}$ is [Kerala CEE 2010]
- (a) $\mathbf{r} = (\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - \hat{k})$
(b) $\mathbf{r} = (\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} - 2\hat{j} - \hat{k})$
(c) $\mathbf{r} = (3\hat{i} - 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + 3\hat{k})$
(d) $\mathbf{r} = (3\hat{i} + 2\hat{j} - \hat{k}) + \lambda(\hat{i} - \hat{j} + 3\hat{k})$
(e) $\mathbf{r} = (\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} + \hat{k})$

Answers

Work Book Exercise 14.1

1. (a) 2. (b) 3. (b) 4. (c) 5. (d) 6. (b) 7. (b) 8. (c) 9. (b) 10. (c)

Work Book Exercise 14.2

1. (a) 2. (a) 3. (a) 4. (d) 5. (b) 6. (a) 7. (b) 8. (c) 9. (a) 10. (a)

Work Book Exercise 14.3

1. (b) 2. (a) 3. (d) 4. (a) 5. (c) 6. (b) 7. (d) 8. (a) 9. (a) 10. (a)

Work Book Exercise 14.4

1. (a) 2. (b) 3. (b) 4. (c) 5. (a) 6. (c) 7. (a) 8. (b) 9. (b) 10. (d)

Work Book Exercise 14.5

1. (a) 2. (c) 3. (c) 4. (a) 5. (a) 6. (b) 7. (a) 8. (a) 9. (c)

Target Exercises

1. (d) 2. (c) 3. (c) 4. (c) 5. (c) 6. (c) 7. (c) 8. (a) 9. (d) 10. (a)
 11. (c) 12. (c) 13. (a) 14. (a) 15. (b) 16. (b) 17. (a) 18. (c) 19. (c) 20. (b)
 21. (d) 22. (d) 23. (a) 24. (c) 25. (b) 26. (b) 27. (d) 28. (b) 29. (c) 30. (c)
 31. (a) 32. (d) 33. (b) 34. (b) 35. (a) 36. (b) 37. (b) 38. (c) 39. (b) 40. (a)
 41. (c) 42. (a) 43. (a) 44. (c) 45. (a) 46. (b) 47. (b) 48. (b) 49. (b) 50. (b)
 51. (c) 52. (b) 53. (b) 54. (b) 55. (d) 56. (b) 57. (b) 58. (a) 59. (b) 60. (c)
 61. (a) 62. (a) 63. (b) 64. (c) 65. (c) 66. (c) 67. (b) 68. (b) 69. (a) 70. (c)
 71. (a) 72. (b) 73. (c) 74. (d) 75. (a) 76. (c) 77. (c) 78. (a) 79. (a) 80. (b)
 81. (c) 82. (d) 83. (c) 84. (c) 85. (c) 86. (d) 87. (a) 88. (b) 89. (c) 90. (a,b)
 91. (a,b,c) 92. (a,c,d) 93. (b,c) 94. (b,d) 95. (b,c,d) 96. (a, b) 97. (a,c) 98. (a) 99. (d) 100. (b)
 101. (b) 102. (a) 103. (a) 104. (d) 105. (c) 106. (d) 107. (b) 108. (d) 109. (b) 110. (c)
 111. (a) 112. (a) 113. (b) 114. (c) 115. (d) 116. (b) 117. (a) 118. (b) 119. (a) 120. (d)
 121. (*) 122. (*) 123. (*) 124. (4) 125. (1) 126. (6)

*

121. $A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$

122. $A \rightarrow s; B \rightarrow r; C \rightarrow q; D \rightarrow p$

123. $A \rightarrow s; B \rightarrow p; C \rightarrow q; D \rightarrow r$

Entrances Gallery

1. (c) 2. (d) 3. (b,d) 4. (a,d) 5. (a) 6. (a) 7. (a) 8. (b,c) 9. (b) 10. (d)
 11. (c) 12. (c) 13. (a) 14. (d) 15. (c) 16. (a) 17. (c) 18. (c) 19. (d) 20. (a)
 21. (a) 22. (a) 23. (b) 24. (c) 25. (c) 26. (b) 27. (c) 28. (c) 29. (c) 30. (c)
 31. (c) 32. (a) 33. (d) 34. (d) 35. (d) 36. (b) 37. (b) 38. (c) 39. (d) 40. (b)
 41. (a) 42. (d) 43. (a) 44. (d) 45. (d) 46. (b) 47. (d) 48. (c) 49. (c) 50. (c)
 51. (c) 52. (b) 53. (a) 54. (d) 55. (c) 56. (c) 57. (d) 58. (a) 59. (b) 60. (b)
 61. (a) 62. (b) 63. (c) 64. (d) 65. (c) 66. (c) 67. (a) 68. (d) 69. (a) 70. (c)
 71. (d) 72. (d) 73. (d) 74. (b) 75. (b) 76. (c) 77. (c) 78. (d) 79. (e) 80. (c)
 81. (a) 82. (b) 83. (d) 84. (a) 85. (d) 86. (c) 87. (d) 88. (b) 89. (b) 90. (a)
 91. (d) 92. (a) 93. (d) 94. (b) 95. (b) 96. (d) 97. (a) 98. (c) 99. (d)
 100. (b) 101. (c) 102. (b) 103. (b) 104. (d) 105. (c) 106. (d) 107. (b) 108. (b) 109. (a)
 110. (c) 111. (d) 112. (a) 113. (c) 114. (c) 115. (b) 116. (b) 117. (c) 118. (d) 119. (a)

Explanations

Target Exercises

- We must have, $k^2 + k^2 + k^2 = 1 \Rightarrow 3k^2 = 1$
 $\Rightarrow k = \pm \frac{1}{\sqrt{3}}$
- The line $x=1, y=2$ can be written as $\frac{x-1}{0} = \frac{y-2}{0} = \frac{z-0}{1}$, which is certainly parallel to Z-axis.
- $(x-1)(x-2)=0 \Leftrightarrow x=1, x=2$, which is a pair of parallel planes, both parallel to YOZ-plane.
- When a point (x, y, z) moves parallel to X-axis, both y and z -coordinates remain fixed.
- Let $A(2, 4, 5)$ and $B(3, 5, -4)$ and C be the point in which line AB meets the YZ -plane. Let C divide AB in the ratio $k:1$, then

$$C \equiv \left(\frac{3k+2}{k+1}, \frac{5k+4}{k+1}, \frac{-4k+5}{k+1} \right)$$

But C lies on YZ -plane, therefore

$$\frac{3k+2}{k+1} = 0 \Rightarrow k = -\frac{2}{3}$$
- Let the vertices be $A(1, 2, 3)$, $B(2, 5, -1)$ and $C(-1, 1, 2)$, then area of $\triangle ABC = \frac{1}{2} |\vec{AB} \times \vec{AC}|$

$$= \frac{1}{2} |(\hat{i} + 3\hat{j} - 4\hat{k}) \times (-2\hat{i} - \hat{j} - \hat{k})| = \frac{\sqrt{155}}{2} \text{ sq units}$$
- If γ is the required angle, then

$$\cos^2 45^\circ + \cos^2 60^\circ + \cos^2 \gamma = 1 \Rightarrow \cos^2 \gamma = \frac{1}{4}$$
- Hence, DR's of the line joining the points $(3, 1, 4)$ and $(7, 2, 12)$ are $\langle 7-3, 2-1, 12-4 \rangle$ i.e. $\langle 4, 1, 8 \rangle$.
 $\Rightarrow r = 60^\circ$ or 120°
 $\therefore$ Required angle is $\cos \theta = \frac{2}{3}$ or $\theta = \cos^{-1} \left(\frac{2}{3} \right)$.
- If $\langle a, b, c \rangle$ are direction ratios of a normal to the plane AOB , then $a-2b-c=0$ and $3a-2b+3c=0$

$$\Rightarrow \frac{a}{-6-2} = \frac{b}{-3-3} = \frac{c}{-2+6}$$

$$\Rightarrow \frac{a}{4} = \frac{b}{3} = \frac{c}{-2}$$
- Any point on the Z -axis i.e. $(0, 0, z)$ satisfies the given equation.
- $xy=0 \Leftrightarrow x=0$ or $y=0$. So, locus of $xy=0$ is the union of all points which lie in YOZ -plane or ZOX -plane. Hence, the equation $xy=0$ represents a pair of perpendicular planes.
- Direction numbers of the two lines are $\langle 1, 1, 1 \rangle$ and $\langle 1, -2, 1 \rangle$ and both the lines pass through the point $(0, 0, 3)$. Observe that $1 \cdot (1) + 1 \cdot (-2) + 1 \cdot (1) = 0$. Therefore, the lines intersect at right angles.

- DR's of the line given in option (a) are $\langle -1, 1, -1 \rangle$ and $2(-1) + 3(1) + 1(-1) = 0$. Hence, the required line is given in alternative (a), as it passes through the point $(1, 2, 3)$.
- The given equation is $r = \lambda \hat{i}$
 $\Rightarrow x\hat{i} + y\hat{j} + z\hat{k} = \lambda \hat{i} \Rightarrow x = \lambda, y = 0, z = 0$
 which is the equation of X -axis as they can be written as

$$\frac{x-0}{1} = \frac{y-0}{0} = \frac{z-0}{0}$$
- Given, $3p + 8q = 6r + 5s$

$$\Rightarrow \frac{3p+8q}{8+3} = \frac{6r+5s}{5+6}$$

 The point which divides PQ in ratio $8:3$ is the same as the point which divides RS in the ratio $5:6$. Hence, the line PQ and RS intersect.
- Required length

$$= \sqrt{\text{Sum of squares of the projections}}$$

$$= \sqrt{2^2 + 18^2 + 16^2} = \sqrt{366} = 6$$
- The given line is at right angled to X -axis as its DR's are $\langle 0, 1, 2 \rangle$.
- The equation of the line is $\frac{x-x_1}{0} = \frac{y-y_1}{0} = \frac{z-z_1}{1}$ and hence DR's of the line are $\langle 0, 0, 1 \rangle$, which means that the line parallel to the Z -axis.
- The given lines are not parallel and both the lines pass through the point $(1, 2, 3)$.
- Both lines contain the point (x_1, y_1, z_1) , therefore they are intersecting.
- Any point on the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is $(t, 2t, 3t)$. It lies in the second line, if $\frac{t-1}{-2} = \frac{2t-2}{-4} = \frac{3t-3}{-6}$ i.e. if $t-1 = 2t-2 = 3t-3$, which is true for all real t . Hence, the two lines are coincident.
- Given lines can be written as
 $r = a_1 + t b_1$ and $r = a_2 + s b_2$
 where, $a_1 = 3\hat{i} + 8\hat{j} + 3\hat{k}$, $a_2 = -3\hat{i} - 7\hat{j} + 6\hat{k}$,
 $b_1 = 3\hat{i} - \hat{j} + \hat{k}$ and $b_2 = -3\hat{i} + 2\hat{j} + 4\hat{k}$.
 $\therefore$ Required SD = $\frac{|[a_2 - a_1 \ b_1 \ b_2]|}{|b_1 \times b_2|} = 3\sqrt{30}$
- If $\langle a, b, c \rangle$ are direction ratios of a line perpendicular to a given lines, then
 $2a - 3b + c = 0$ and $a + 2b - 2c = 0$
 Now, $\frac{a}{6-2} = \frac{b}{1+4} = \frac{c}{4+3} \Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{7}$
 $\therefore$ DR's are $\langle 4, 5, 7 \rangle$.
- If θ is the required angle, then

$$\cos \theta = \frac{6}{\sqrt{24} \sqrt{6}}$$

$$= \frac{6}{12} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

25. Given, lines can be written as

$$r = \hat{i} + 2\hat{j} + 3\hat{k} + t(2\hat{i} + 3\hat{j} + 4\hat{k})$$

$$\text{and } r = 2\hat{i} + 4\hat{j} + 5\hat{k} + s(3\hat{i} + 4\hat{j} + 5\hat{k}).$$

∴ Required SD

$$= \frac{|\text{Scalar triple product of } b_1, b_2 \text{ and } a_2 - a_1|}{|b_1 \times b_2|}$$

$$\text{where } a_1 = \hat{i} + 2\hat{j} + 3\hat{k}, b_1 = 2\hat{i} + 3\hat{j} + 4\hat{k},$$

$$a_2 = 2\hat{i} + 4\hat{j} + 5\hat{k}, b_2 = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

∴ Required SD

$$= \frac{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}}{|(2\hat{i} + 3\hat{j} + 4\hat{k}) \times (3\hat{i} + 4\hat{j} + 5\hat{k})|} = \frac{1}{\sqrt{6}}$$

26. Note that the given lines are parallel. The required

$$\text{distance is } \frac{|(a_2 - a_1) \times b|}{|b|}, \text{ a standard formula.}$$

27. The point $(1, -1, 2)$ lies on the given line as

$$\frac{1+1}{2} = \frac{-1-2}{-3} = \frac{2+2}{4} \text{ is true.}$$

Hence, the length of perpendicular on the given line is 0.

28. Standard formula for shortest distance.

29. Standard formula.

30. Given that,

$$l_1 = \frac{\sqrt{3}}{4}, m_1 = \frac{1}{4}, n_1 = \frac{\sqrt{3}}{2}$$

$$\text{and } l_2 = \frac{\sqrt{3}}{4}, m_2 = \frac{1}{4}, n_2 = \frac{-\sqrt{3}}{2}$$

$$\therefore \cos \theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$$

$$= \left| \frac{\sqrt{3}}{4} \times \frac{\sqrt{3}}{4} + \frac{1}{4} \times \frac{1}{4} + \frac{\sqrt{3}}{2} \times \left(\frac{-\sqrt{3}}{2} \right) \right|$$

$$= \left| \frac{3}{16} + \frac{1}{16} - \frac{3}{4} \right| = \left| -\frac{2}{4} \right| = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{\pi}{3}$$

31. Given that,

$$l + m + n = 0 \quad \dots(i)$$

$$\text{and } lm = 0 \quad \dots(ii)$$

$$\text{From Eq. (i), } l = -(m + n)$$

$$\text{From Eq. (ii), } -(m + n)m = 0$$

$$\Rightarrow m, m + n = 0$$

$$\text{If } m = 0, l + m + n = 0, \text{ then}$$

$$\frac{l_1}{-1} = \frac{m_1}{0} = \frac{n_1}{1}$$

$$\text{and if } l + m + n = 0, \text{ then}$$

$$\frac{l_2}{0} = \frac{m_2}{-1} = \frac{n_2}{1}$$

$$\Rightarrow (l_1, m_1, n_1) = (-1, 0, 1)$$

$$\text{and } (l_2, m_2, n_2) = (0, -1, 1)$$

∴ Angle between them is given by

$$\cos \theta = \frac{0 + 0 + 1}{\sqrt{1 + 0 + 1} \sqrt{0 + 1 + 1}} = \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ = \frac{\pi}{3}$$

32. Any plane through given point $(-1, 2, 0)$ is

$$A(x + 1) + B(y - 2) + C(z - 0) = 0 \quad \dots(i)$$

If it is parallel to given lines, then

$$3A + 0B - C = 0$$

$$\text{and } A + 2B - C = 0$$

$$\Rightarrow \frac{A}{0 + 2} = \frac{B}{-1 + 3} = \frac{C}{6 - 0}$$

∴ Equation of the plane is

$$x + y + 3z = \lambda \text{ passing through } (-1, 2, 0).$$

$$\Rightarrow \lambda = 1$$

∴ Equation of plane is

$$x + y + 3z = 1$$

33. If $OA = a, OB = b$ and $OC = c$, then centroid of ΔABC is

$$\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right) = (3, 3, 3) \Rightarrow a = b = c = 9 \text{ and hence the}$$

equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\Rightarrow \frac{x}{9} + \frac{y}{9} + \frac{z}{9} = 1$$

$$\text{or } x + y + z = 9$$

34. DR's of the normal to the given plane are $\langle 2, 3, -1 \rangle$.

Hence, the required line is

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z-1}{-1}$$

35. The two planes are parallel and are at distances p_1

and p_2 from the origin and on opposite sides. Hence, the distance between them is $p_1 + p_2$.

36. All the given possibilities are correct.

37. If $OA = a, OB = b$ and $OC = c$, then the centroid of

$$\Delta ABC \text{ is } \left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3} \right) = (1, 2, 3)$$

$$\Rightarrow a = 3, b = 6, c = 9$$

∴ Equation of the plane is

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$$

38. $xy + yz = 0 \Leftrightarrow (x + z)y = 0 \Leftrightarrow y = 0$ or $x + z = 0$. The

equations $y = 0$ and $x + z = 0$ represent planes which are at right angles as their direction ratios of normal to the plane are $\langle 0, 1, 0 \rangle$ and $\langle 1, 0, 1 \rangle$ respectively.

39. Direction ratios of normal to the plane are

$\langle 2, -3, -6 \rangle$. Thus, DR's of a normal to the plane are $\langle 2, -3, -6 \rangle$. Therefore, direction cosines of a normal to the plane are

$$\left\langle \frac{2}{7}, -\frac{3}{7}, -\frac{6}{7} \right\rangle$$

or

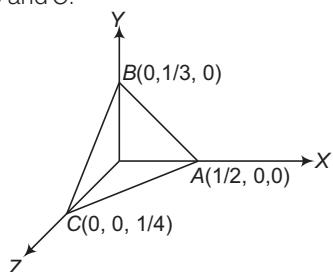
$$\left\langle -\frac{2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$$

40. DR's of a normal to the plane are $\langle 2, -3, 6 \rangle$ and

those of X-axis are $\langle 1, 0, 0 \rangle$. Thus, an angle between the plane and the line is given by

$$\sin^{-1} \left(\frac{2 \cdot 1 + (-3) \cdot 0 + 6 \cdot 0}{\sqrt{2^2 + (-3)^2 + 6^2} \sqrt{1^2 + 0^2 + 0^2}} \right) = \sin^{-1} \left(\frac{2}{7} \right)$$

41. The required plane is parallel to the given planes and lies mid way between them. If the other coefficients are same, then the constant term must be the arithmetic mean between the constant terms of the two planes.
42. Any point on the line is $(t, 2t, 3t)$. It lies in the given plane, if $t - 2(2t) + 3t - 6 = 0$ i.e. if $0t = 6$, which is not true for any real t . So, the line and plane do not meet i.e. the line is parallel to the plane.
43. The given equation is of the form $\mathbf{r} \cdot \hat{n} = p$, where $\hat{n}$ is a unit vector and $p > 0$ i.e. in the normal form.
44. Given plane $2x + 3y + 4z = 1$ meets the coordinate axes A, B and C .



$$\Rightarrow \frac{2x}{1} + \frac{3y}{1} + \frac{4z}{1} = 1$$

$$\Rightarrow \frac{x}{1/2} + \frac{y}{1/3} + \frac{z}{1/4} = 1$$

Then, centroid of the $\triangle ABC$

$$= \left(\frac{\frac{1}{2} + 0 + 0}{3}, \frac{0 + \frac{1}{3} + 0}{3}, \frac{0 + 0 + \frac{1}{4}}{3} \right)$$

$$= \left(\frac{1}{6}, \frac{1}{9}, \frac{1}{12} \right)$$

45. The plane is perpendicular to the line $\frac{x-a}{\cos \theta} = \frac{y+2}{\sin \theta} = \frac{z-3}{0}$.

So, the direction ratios of the normal of the plane are $\cos \theta, \sin \theta$ and 0 (i)

Now, the required plane passes through the Z -axis. So, the point $(0, 0, 0)$ lies on the plane. ... (ii)

From Eqs. (i) and (ii), we get equation of the plane as $\cos \theta (x - 0) + \sin \theta (y - 0) + 0 (z - 0) = 0$

$$\Rightarrow \cos \theta x + \sin \theta y = 0$$

$$\Rightarrow x + y \tan \theta = 0$$

46. Let the foot the perpendicular from the origin on the given plane be $P(\alpha, \beta, \gamma)$. Since, the plane passes through $A(a, b, c)$,

$$\therefore AP \perp OP$$

$$\Rightarrow AP \cdot OP = 0$$

$$\Rightarrow [(\alpha - a)\hat{i} + (\beta - b)\hat{j} + (\gamma - c)\hat{k}] \cdot (\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}) = 0$$

$$\Rightarrow \alpha(\alpha - a) + \beta(\beta - b) + \gamma(\gamma - c) = 0$$

Hence, the locus of (α, β, γ) is

$$x(x - a) + y(y - b) + z(z - c) = 0$$

$$\Rightarrow x^2 + y^2 + z^2 - ax - by - cz = 0$$

which is a sphere of radius $\frac{1}{2}\sqrt{a^2 + b^2 + c^2}$.

$$47. \sin \theta = \frac{|3 \times 2 + 4 \times (-3) + 2 \times 1|}{\sqrt{3^2 + 4^2 + 2^2} \sqrt{2^2 + (-3)^2 + (1)^2}}$$

$$= \frac{4}{\sqrt{29} \sqrt{14}}$$

$$= \frac{4}{\sqrt{406}}$$

$$\therefore \theta = \sin^{-1} \left(\frac{4}{\sqrt{406}} \right)$$

48. $p_1 + \lambda p_2 = 0$ is the family of all lines through the intersection of $p_1 = 0$ and $p_2 = 0$ except the plane $p_2 = 0$.

49. Any plane through the intersection of given plane is $(x + 2y + 3z + 4) + \lambda (4x + 3y + 2z + 1) = 0$... (i)

It contains $(0, 0, 0)$.

$$\Rightarrow 4 + \lambda = 0$$

$$\Rightarrow \lambda = -4$$

$\therefore$ Equation of the plane is $3x + 2y + z = 0$

50. The given line is

$$\frac{x-1}{2} = \frac{y-1}{1} = \frac{z-0}{4} \quad \dots (i)$$

and the given plane is

$$x + 2y - z = 3 \quad \dots (ii)$$

Any point on the line (i) is $(2t + 1, t + 1, 4t)$

This lies in the plane (ii), if

$$2t + 1 + 2(t + 1) - 4t - 3 = 0$$

$$\Rightarrow t = 0,$$

which is true for all real values of t . So, the line (i) lies in the plane (ii).

$$51. \text{ Required distance} = \frac{|2 - 2 \times 1 + 4 \times (-1) - 9|}{\sqrt{1^2 + 2^2 + 4^2}} = \frac{13}{\sqrt{21}}$$

$$52. \text{ Using image formula, } \frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c}$$

$$= \frac{-2(ax + by + cz + d)}{a^2 + b^2 + c^2}$$

$$\Rightarrow \frac{x_2 - 2}{3} = \frac{y_2 + 1}{-2} = \frac{z_2 - 3}{-1}$$

$$= -\frac{2(6 + 2 - 3 - 9)}{9 + 4 + 1}$$

$$\Rightarrow \frac{x_2 - 2}{3} = \frac{y_2 + 1}{-1} = \frac{z_2 - 3}{-1} = \frac{-2(-4)}{14}$$

$$\Rightarrow x_2 = \frac{26}{7}, y_2 = -\frac{15}{7}, z_2 = \frac{17}{7}$$

53. The DR's of the line are $\langle 2, -1, 2 \rangle$ and numbers of the plane are $\langle 4, -2, 4 \rangle$ which are proportional. Hence, the line is at right angles to the plane.

54. A point on the first plane can be taken as $(2, 0, 0)$ by taking $y = z = 0$ in the given plane.

Required distance between the given planes

= Distance of $(2, 0, 0)$ from the plane

$$2x + 4y - 6z + 7 = 0 = \frac{11}{\sqrt{56}}$$

55. Any plane parallel to the given plane is

$$3x - 5y + 2z + \lambda = 0 \quad \dots (i)$$

The point $(1, 2, -3)$ lies in it, if $\lambda = 13$.

56. If $D \neq 0$, then the equation of the plane whose intercepts on the axes are k times those made by the plane $Ax + By + Cz + D = 0$ is $Ax + By + Cz + kD = 0$.

57. The required plane is $1(x - \alpha) + 1(y - \beta) + 1(z - \gamma) = 0$.
 $\Rightarrow x + y + z = \alpha + \beta + \gamma$

58. As $(\hat{i} - \hat{j} + 4\hat{k}) \cdot (\hat{i} + 5\hat{j} + \hat{k}) = 0$.

Therefore, the line and the plane are parallel. A point on the line is $(2, -2, 3)$.

Required distance = Distance of $(2, -2, 3)$ from the given plane $x + 5y + z = 5$

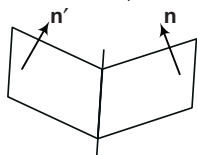
$$= \frac{|2 + 5(-2) + 3 - 5|}{\sqrt{1^2 + 5^2 + 1^2}} = \frac{10}{\sqrt{27}} = \frac{10}{3\sqrt{3}}$$

59. In the freedom equations of a line, $|r|$ is the distance of (x, y, z) from the fixed point (x_1, y_1, z_1) .

60. As, the plane passes through $(0, 0, 0)$ and DR'S of straight line and normal to the plane are perpendicular, so line lies in the plane.

61. The lines $x + y = 0 \Rightarrow x = -y$ and $y + z = 0 \Rightarrow z = -y$ meet in the line $\frac{x}{-1} = \frac{y}{1} = \frac{z}{1}$. Any point on this line is $(t, -t, t)$. This point lies in the plane $x + z = 0$, if $t + t = 0 \Rightarrow t = 0$. So, the three planes meet in a unique point namely $(0, 0, 0)$.

62. The line of intersection is parallel to the vector $n \times n'$.



i.e. $r \cdot n = q, r \cdot n' = q'$
 Let the line passes through a , then
 $r = a + \lambda (n \times n')$

63. Each of the given points is at a distance 5 from $(0, 0, 0)$. So, $(0, 0, 0)$ is the centre of the sphere through the given points.

64. The given equation is of the form $(r - a) \cdot (r - b) = 0$, which is the equation of a sphere in the diameter form.

65. The radius of the circle in reference $= \sqrt{5^2 - d^2}$, where d is the length of perpendicular from the centre $(0, 0, 0)$ of the sphere upon the plane.

$$r \cdot (\hat{i} + \hat{j} + \hat{k}) = 3\sqrt{3}$$

$$\text{or } r \cdot \left(\frac{1}{\sqrt{3}} \hat{i} + \frac{1}{\sqrt{3}} \hat{j} + \frac{1}{\sqrt{3}} \hat{k} \right) = 3$$

which is of the form $r \cdot \hat{n} = p$, where $p = 3$.

So, radius of the circle $= \sqrt{5^2 - 3^2} = 4$

66. As the given spheres are concentric and are of different radii, they do not have any point in common.

67. If $R(r)$ is any point on the sphere, then $\angle PRQ = 90^\circ$ and hence $PR^2 + QR^2 = PQ^2$
 $\Rightarrow |r - a|^2 + |r - b|^2 = |a - b|^2$

68. The required point is the centre of the sphere through the given points and the equation of that sphere is $x^2 + y^2 + z^2 - x - 2y - 3z = 0$. i.e. $\left(\frac{1}{2}, 1, \frac{3}{2}\right)$.

69. $x^2 + y^2 + z^2 + 1 = 0$ cannot be satisfied by any point as sum of squares of reals cannot be negative.

70. The given equation is in the diameter form. So, the ends of a diameter are $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$. Hence, the centre of the sphere is mid-point of AB .

71. First, write the second sphere as $x^2 + y^2 + z^2 + x + y + z - \frac{1}{4} = 0$.

Then, the plane of intersection is given by $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2$

$$- \left\{ x^2 + y^2 + z^2 + x + y + z - \frac{1}{4} \right\} = 0$$

$$\Rightarrow x + y + z + \frac{9}{4} = 0$$

72. Required condition is $|C_1 C_2| < r_1 - r_2$.

73. There is one sphere in each octant. So, there are 8 spheres of the same manner.

74. The given equation is $(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (x\hat{i} + y\hat{j} + z\hat{k}) + (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) - 6 = 0$

Its centre is $\left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$.

$$\therefore \text{PV of centre} = -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k}$$

75. Let the equation of sphere be $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

As it passes through $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$.

$$\therefore a^2 + 2ua + d = 0 \quad \dots(i)$$

$$b^2 + 2vb + d = 0 \quad \dots(ii)$$

$$c^2 + 2wc + d = 0 \quad \dots(iii)$$

$$\text{and } d = 0 \quad \dots(iv)$$

From Eqs. (i), (ii), (iii) and (iv), we get

$$u = -\frac{a}{2}, v = -\frac{b}{2}, w = -\frac{c}{2} \text{ and } d = 0$$

$\therefore$ Equation of the sphere is $x^2 + y^2 + z^2 - ax - by - cz = 0$

76. The given circle is $x = 0, y^2 + z^2 = 25$, which can be written as $x^2 + y^2 + z^2 - 25 = 0, x = 0$.

Equation of any sphere through this circle is

$$x^2 + y^2 + z^2 - 25 + \lambda x = 0 \quad \dots(i)$$

Since, the point $(1, 3, -2)$ lies on this sphere, therefore $\lambda = 11$.

77. Radius of the sphere is $2\sqrt{6}$ and distance of the given point from the centre of the sphere is $\sqrt{6}$.

So, the point lies within the sphere and its distance from the surface (along the radius which is shortest) is $2\sqrt{6} - \sqrt{6} = \sqrt{6}$.

78. The equation of the sphere through the points $(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$ and $(0, 0, 0)$ is $x^2 + y^2 + z^2 - \alpha x - \beta y - \gamma z = 0$

Take $\alpha = \frac{a}{2}$, $\beta = \frac{b}{2}$ and $\gamma = \frac{c}{2}$ to find the equation of the required sphere.

79. Equation of the sphere concentric with $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ and passing through the origin is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$.

80. The given plane is $3x + 4y = 0$, which passes through Z-axis i.e. the line $x^2 + y^2 = 0$. So, the given plane contains the given line.

81. The equation of sphere concentric with $x^2 + y^2 + z^2 - 4x - 6y - 8z - 5 = 0$ is $x^2 + y^2 + z^2 - 4x - 6y - 8z + \lambda = 0$

which passes through $(0, 1, 0)$.

$$\Rightarrow 1 - 6 + \lambda = 0 \quad \text{or} \quad \lambda = 5$$

$\therefore$ Required equation of sphere is

$$x^2 + y^2 + z^2 - 4x - 6y - 8z + 5 = 0.$$

82. Radius of the sphere = Distance of $(3, 6, -4)$ from the given plane = 4

Hence, required equation of the sphere is

$$(x - 3)^2 + (y - 6)^2 + (z + 4)^2 = (4)^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 6x - 12y + 8z + 45 = 0$$

83. Here, $u^2 + v^2 + w^2 - d = (-2)^2 + 3^2 + (-4)^2 - 29 = 0$
 $\therefore$ The equation represents a point.

84. The ends of a diameter are (x_1, y_1, z_1) and $(0, 0, 0)$.

Hence, the centre of the sphere

$$= \left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_1}{2} \right)$$

85. The given equation is

$$|r|^2 + r \cdot (\hat{i} + \hat{j} - \hat{k}) - 9 = 0$$

$$\Leftrightarrow |x\hat{i} + y\hat{j} + z\hat{k}|^2 = (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) - 9 = 0$$

$$\Leftrightarrow x^2 + y^2 + z^2 + x + y - z - 9 = 0$$

which is the equation of a sphere, whose centre is

$$\left(-\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right).$$

$$\therefore \text{PV of the centre is } -\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} + \frac{1}{2}\hat{k} = -\frac{1}{2}(\hat{i} + \hat{j} - \hat{k}).$$

86. The vertices of the tetrahedron are $(0, 0, 0)$, $(a, 0, 0)$, $(0, a, 0)$ and $(0, 0, a)$. The equation of the sphere through these points is $x^2 + y^2 + z^2 - ax - ay - az = 0$.

87. Since, the given plane touches the sphere. Therefore, radius of the sphere = Length of perpendicular from the centre i.e. $(0, 0, 0)$ upon the plane.

$$\therefore \text{Equation of the sphere is } |r| = p.$$

88. Sphere is smallest when centre is at centroid of $\triangle ABC$.

89. There is one and only one sphere through four points not in the same plane.

90. We have, $3x - 6y + 2z + 5 = 0$... (i)

$$\text{and } -4x + 12y - 3z + 3 = 0 \quad \dots \text{(ii)}$$

$$\therefore \frac{3x - 6y + 2z + 5}{\sqrt{9 + 36 + 4}} = \frac{-4x + 12y - 3z + 3}{\sqrt{16 + 144 + 9}}$$

which bisects the angle between the planes that contains the origin.

$$13(3x - 6y + 2z + 5) = 7(-4x + 12y - 3z + 3)$$

$$\Rightarrow 39x - 78y + 26z + 65 = -28x + 84y - 21z + 21$$

$$\Rightarrow 67x - 162y + 47z + 44 = 0$$

... (iii)

Let θ be the angle between Eqs. (i) and (iii), then find $\cos \theta$ and then we obtain $|\tan \theta| < 1$.

91. Equation of line AB is $\frac{x}{2} = \frac{y}{-3} = \frac{z}{6}$.

Its DR's are $\langle 2, -3, 6 \rangle$

(a) Let the coordinates be $\langle 2r, -3r, 6r \rangle$

Then, DR's of PN are $\langle 2r - 1, -3r - 2, 6r - 5 \rangle$

It is perpendicular to AB.

$$\therefore 2(2r - 1) - 3(-3r - 2) + 6(6r - 5) = 0$$

$$\Rightarrow 4r - 2 + 9r + 6 + 36r - 30 = 0$$

$$\Rightarrow 49r = 26$$

$$\Rightarrow r = \frac{26}{49}$$

$$\therefore \text{Coordinates of N are } \left(\frac{52}{49}, -\frac{78}{49}, \frac{156}{49} \right).$$

(b) Let the coordinates of Q be $\langle 2r, -3r, 6r \rangle$, then

DR's of PQ are $\langle 2r - 1, -3r - 2, 6r - 5 \rangle$.

Since, PQ is parallel to the plane.

$$\therefore 3(2r - 1) + 4(-3r - 2) + 5(6r - 5) = 0$$

$$\Rightarrow 6r - 3 - 12r - 8 + 30r - 25 = 0$$

$$\Rightarrow 24r = 36 \Rightarrow r = \frac{3}{2}$$

$$\therefore \text{Coordinates of Q are } \left(3, -\frac{9}{2}, 9 \right).$$

$$(c) \text{Equation of PN is } \frac{x-1}{3} = \frac{y-2}{-176} = \frac{z-5}{-89}$$

92. Let $OP = r$, then

$$\Rightarrow 1 = r \sin \alpha \cos \beta \quad \dots \text{(i)}$$

$$2 = r \sin \alpha \sin \beta \quad \dots \text{(ii)}$$

$$\text{and } 3 = r \cos \alpha \quad \dots \text{(iii)}$$

Squaring and adding Eqs. (i), (ii) and (iii), we get

$$r^2 = 14 \Rightarrow r = \pm \sqrt{14}$$

Using Eqs. (i), (ii), (iii) and (iv), we get the required result.

93. For A $(1, 1, 1)$, $2x - y - 3z - 5 = 2 - 1 - 3 - 5 < 0$

$$\text{For B } (2, 1, -3), 2x - y - 3z - 5 = 0 - 1 + 9 - 5 > 0$$

$$\text{For C } (1, -2, -2), 2x - y - 3z - 5 = 2 + 2 + 6 - 5 > 0$$

$$\text{For D } (-3, 1, 2), 2x - y - 3z - 5 = -6 - 1 - 6 - 5 < 0$$

A, D are on one side of the plane and B and C are on the other side, the line segments AB, AC, BD and CD intersect the plane.

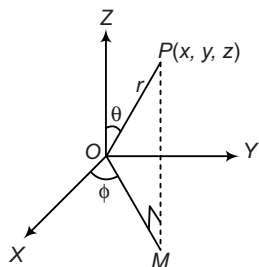
94. Since, n_1 and n_2 are normal to the given planes.

$\therefore$ The line is parallel to $n_1 \times n_2$.

$$\therefore (r - a) \times (n_1 \times n_2) = 0.$$

Also, $(r - a)$ is parallel to $n_1 \times n_2$.

95. If P be (x, y, z) , then from the figure,
 $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$ and $z = r \cos \theta$
 $1 = r \sin \theta \cos \phi$, $2 = r \sin \theta \sin \phi$ and $3 = r \cos \theta$



$$\Rightarrow 1^2 + 2^2 + 3^2 = r^2 \Rightarrow r = \pm \sqrt{14}$$

$$\therefore \sin \theta \cos \phi = \frac{1}{\sqrt{14}}, \sin \theta \sin \phi = \frac{2}{\sqrt{14}} \text{ and } \cos \theta = \frac{3}{\sqrt{14}}$$

[neglecting negative sign as θ and ϕ are acute]

$$\text{Now, } \frac{\sin \theta \sin \phi}{\sin \theta \cos \phi} = \frac{2}{1} \Rightarrow \tan \phi = 2$$

Also, $\tan \theta = \sqrt{5}/3$

96. Let the coordinates of the point(s) be a, b and c .
 Therefore, the equation of the line passing through (a, b, c) and whose direction ratios are $1, -5$ and -2 is
- $$\frac{x-a}{1} = \frac{y-b}{-5} = \frac{z-c}{-2} \quad \dots(i)$$

Line (i) intersects the line,

$$\frac{x}{1} = \frac{y+5}{1} = \frac{z+1}{1} \quad \dots(ii)$$

Therefore, these are coplanar

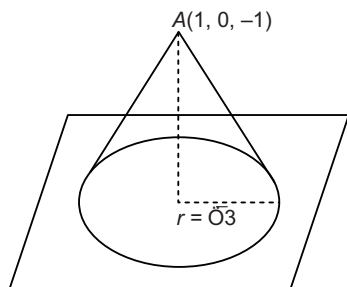
$$\therefore \begin{vmatrix} 1 & -5 & -2 \\ 1 & 1 & 1 \\ a & b+5 & c+1 \end{vmatrix} = 0 \text{ or } a + b - 2c + 3 = 0$$

Also, by using same procedure with the second equation, we get the condition

$$11a + 15b - 32c + 55 = 0.$$

Clearly, $(2, -3, 1)$ and $(1, 2, 3)$ satisfy above equations.

97. The rod sweeps out the figure which is a cone.



$$\therefore \text{Distance of point } A(1, 0, -1) \text{ from the plane} = \frac{|1 - 2 + 4|}{\sqrt{9}} = 1 \text{ unit}$$

The slant height l of the cone is 2 units.

Then, the radius of the base of the cone is

$$\sqrt{l^2 - 1} = \sqrt{4 - 1} = \sqrt{3}.$$

$$\therefore \text{Volume of the cone} = \frac{\pi}{3} (\sqrt{3})^2 \cdot 1 = \pi \text{ cu units}$$

Area of the circle on the plane which the rod traces is 3π .

Also, the centre of the circle is $Q(x, y, z)$.

$$\text{Then, } \frac{x-1}{1} = \frac{y-0}{-2} = \frac{z+1}{2} = \frac{-(1-0-2+4)}{1^2 + (-2)^2 + 2^2}$$

$$\text{or } Q(x, y, z) = \left(\frac{2}{3}, \frac{2}{3}, \frac{-5}{3} \right)$$

98. Statement I $PA \cdot PB = 9 > 0$

$\therefore P$ is exterior to the sphere.

Statement II is true (standard result).

99. Statement II $r \times (\hat{i} + 2\hat{j} - 3\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 2 & -3 \end{vmatrix}$

$$\hat{i}(-3y - 2z) - \hat{j}(-3x - z) + \hat{k}(2x - y)$$

$$\therefore -3y - 2z = 2, 3x + z = -1, 2x - y = 0$$

i.e. $-6x - 2z = 2, 3x + z = -1$

$$\therefore \text{Straight line } 2x - y = 0, 3x + z = -1$$

Statement I $r \times (2\hat{i} - \hat{j} + 3\hat{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 2 & -1 & 3 \end{vmatrix}$

$$= (3y + z)\hat{i} - (3x - 2z)\hat{j} + (-x - 2y)\hat{k}$$

$$\therefore 3y + z = 3, 3x - 2z = 0, -x - 2y = 1$$

$$\Rightarrow 3x - 2(3 - 3y) = 0$$

$$\Rightarrow 3x + 6y = 6$$

$$\Rightarrow x + 2y = 2$$

Now, $x + 2y = -1, x + 2y = 2$ are parallel planes.

$\therefore r \times (2\hat{i} - \hat{j} + 3\hat{k}) = 3\hat{i} + \hat{k}$ is not a straight line.

100. $\sin \theta = \left| \frac{2 - 3 + 2}{\sqrt{4 + 9 + 4} \sqrt{3}} \right| = \frac{1}{\sqrt{51}}$

$\therefore$ Statement I is true, Statement II is true by definition.

101. Statement I $3y - 4z = 5 - 2k - 2y + 4z = 7 - 3k$

$\therefore x = k, y = 12 - 5k$ and $z = \frac{31 - 13k}{4}$ is a point on the line for all real values of k .

Hence, Statement I is true.

Statement II Direction ratios of the straight line are $\langle bc' - kbc, kac - ac', 0 \rangle$ and direction ratios of normal to be plane $\langle 0, 0, 1 \rangle$.

$$\text{Now, } 0 \times (bc' - kbc) + 0 \times (kac - ac') + 1 \times 0 = 0.$$

$\therefore$ The straight line is parallel to the plane.

Hence, Statement II is true but does not explain Statement I.

102. Equation of the plane, perpendicular to the plane P and containing line L is $8x + y - 7z = 4$.

103. In Statement II, let r be the position vector of the point on the locus, then

$$|r - a| = |r - b| \Rightarrow (r - a)^2 = (r - b)^2$$

$$\Rightarrow |r|^2 + |a|^2 - 2r \cdot a = |r|^2 + |b|^2 - 2r \cdot b$$

$$\Rightarrow 2r \cdot (a - b) + |b|^2 - |a|^2 = 0$$

$$\Rightarrow 2r \cdot (a - b) + (b - a) \cdot (b + a) = 0$$

$$\Rightarrow \left(r - \frac{a+b}{2} \right) \cdot (a - b) = 0$$

Showing that Statement II is true using it for Statement I, we get the required locus as

$$\left[r - \frac{3\hat{i} - 2\hat{j} + 5\hat{k} + \hat{i} + 2\hat{j} - \hat{k}}{2} \right]$$

$$(3\hat{i} - 2\hat{j} + 5\hat{k} - (\hat{i} + 2\hat{j} - \hat{k})) = 0$$

$$\Rightarrow [r - (2\hat{i} + 2\hat{k})] \cdot (2\hat{i} - 4\hat{j} + 6\hat{k}) = 0$$

$$\Rightarrow r \cdot (\hat{i} - 2\hat{j} + 3\hat{k}) = 2 \times 1 - 0 \times 2 + 2 \times 3 = 8$$

Thus, Statement I is also true.

104. Equation of the line in Statement I can be written as

$$\frac{x-3}{14} = \frac{y-1}{2}$$

$$= \frac{z-0}{15} = t$$

If this represents the line of intersection of the given planes, then the point (3, 1, 0) which lies on the line must be on both the planes which is not true. So, Statement I is false. The direction ratios of the line of intersection of planes is

$$(-6)(-2) - (-2)(1), (-2)(2) - (-2)(3), (3)(1) - (2)(-6)$$

i.e. 14, 2, 15 showing that vector $14\hat{i} + 2\hat{j} + 15\hat{k}$ is parallel to the line of intersection of the given planes and hence the Statement II is true.

105. We can write,

$$L_1: r = 3\hat{i} + 8\hat{j} + 3\hat{k} + \lambda(3\hat{i} - \hat{j} + \hat{k}) = a + \lambda b$$

$$L_2: r = -3\hat{i} - 7\hat{j} + 6\hat{k} + \mu(-3\hat{i} + 2\hat{j} + 4\hat{k}) = c + \mu d$$

$$\text{where, } a = 3\hat{i} + 8\hat{j} + 3\hat{k}, c = -3\hat{i} - 7\hat{j} + 6\hat{k}$$

$$b = 3\hat{i} - \hat{j} + \hat{k}, d = -3\hat{i} + 2\hat{j} + 4\hat{k}$$

$$\therefore a - c = 6\hat{i} + 15\hat{j} - 3\hat{k} = 3(2\hat{i} + 5\hat{j} - \hat{k})$$

$$\text{and } b \times d = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$= -6\hat{i} - 15\hat{j} + 3\hat{k}$$

$$= -3(2\hat{i} + 5\hat{j} - \hat{k})$$

$$\therefore |b \times d| = 3\sqrt{4 + 25 + 1}$$

$$= 3\sqrt{30}$$

$$\text{Required shortest distance} = \frac{|(a - c) \cdot (b \times d)|}{|b \times d|}$$

$$= \frac{9(4 + 25 + 1)}{3\sqrt{30}}$$

$$= 3\sqrt{30}$$

So, Statement I is true, Statement II is false as $P(0, 9, 2)$ lies on L_1 and $Q(0, -9, 2)$ lies on L_2 such that $PQ = 18$, which is not equal to the shortest distance between the lines.

106. The line $\frac{x-1}{3} = \frac{y-2}{-1} = \frac{z-3}{4} = r$

Any point (say) $B \equiv (3r + 1, 2 - r, 3 + 4r)$ [on the line L]
 $AB = 3r, -r, 4r + 6$

Hence, AB is parallel to $x + y - z = 1$.

$$\Rightarrow 3r - r - 4r - 6 = 0$$

$$\text{or } r = -3$$

Hence, B is $(-8, 5, -9)$.

107. The equation of plane containing the line L is

$$A(x - 1) + B(y - 2) + C(z - 3) = 0,$$

$$\text{where } 3A - B + 4C = 0$$

Now, Eq. (i) also contains point A(1, 2, -3).

$$\therefore C = 0 \text{ and } 3A = B$$

the equation of plane is $x - 1 + 3(y - 2) = 0$

$$\text{or } x + 3y - 7 = 0$$

108. The distance of point $(1 + 3r, 2 - r, 3 + 4r)$ from the plane is

$$\frac{|1 + 3r + 2 - r - 3 - 4r - 1|}{\sqrt{1 + 1 + 1}} = \frac{|2r + 1|}{\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$\Rightarrow r = \frac{3}{2}, -\frac{5}{2}$$

So, the points are $A\left(\frac{11}{2}, \frac{1}{2}, \frac{10}{2}\right)$

$$\text{and } B\left(\frac{-13}{2}, \frac{9}{2}, \frac{-14}{2}\right) \Rightarrow AB = \sqrt{292}$$

109. Equation of the second plane is $-x + 2y - 3z + 5 = 0$

$$\Rightarrow 2(-1) + 3 \cdot 2 + (-4)(-3) > 0$$

Origin lies in obtuse angle.

$$(2 \times 1 + 3(-2) - 4 \times 3 + 5)(-1 + 2(-2) - 3 \times 3 + 5)$$

$$= (2 - 6 - 12 + 7)(-1 - 4 - 9 + 5) > 0$$

$\therefore P$ lies in obtuse angle.

110. $\therefore 1 \times 2 + 2 \times 1 - 3 \times 3 < 0$

$\therefore$ Origin lies in acute angle.

$$\text{Also, } (2 + 2(-1) - 3(2) + 5)(2 \times 2 - 1 + 3 \times 2 + 1)$$

$$= (-1)(10) < 0$$

$\therefore P$ lies in obtuse angle.

111. $\therefore 1 - 4 - 9 < 0$

$\therefore$ Origin lies in acute angle.

$$\text{Further, } (1 + 4 - 6 + 2)(1 - 4 + 6 + 7) > 0$$

$\therefore$ The point P lies in acute angle.

112. Plane P_1 is of the form $r \cdot n_1 = 0$, where $n_1 = (4, -3, 7)$.

Plane P_2 is of the form $r \cdot n_2 = 0$, where $n_2 = (2, 1 - 5)$.

The vector b along the line of intersection of planes is $n_1 \times n_2 = (4, 17, 5) = n_3$

Since, the line of greatest slope is perpendicular to n_3 and n_2 the vector along the line of greatest slope

$$= n_2 \times n_3 = (3, -1, 1) = n_4$$

$$\therefore \text{Unit vector } (n_4) = \hat{n} = \left(\frac{3}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}} \right)$$

which is equivalent to the direction cosines.

113. Since, $(0, 0, 0)$ is a point on both planes, it lies on the line of intersection.

Hence, the equation a line of greatest slope can be

$$\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$$

114. The point on the line $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$ at a distance

$2\sqrt{11}$ units from the origin is given by

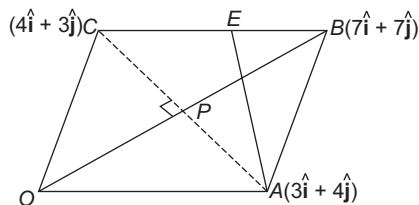
$$\frac{x}{3} = \frac{y}{-1} = \frac{z}{1} = \frac{2\sqrt{11}}{\sqrt{11}} = 2$$

$\therefore$ The point is $(6, -2, 2)$.

115. $OB = 7\hat{i} + 7\hat{j}$, $OE = 5\hat{i} + \frac{13}{3}\hat{j}$, and $OP = \frac{21}{5}(\hat{i} + \hat{j})$

116. Direction ratios of CP are (1, 6, 0), then equation of line passing through (2, 3, 4) and parallel to CP is

$$\frac{x-2}{1} = \frac{y-3}{6} = \frac{z-4}{0}$$



117. The plane containing line AC and at a maximum distance from B be perpendicular to the plane OABC. Since, OABC is rhombus, so OB must normal to the plane. Hence, equation of required plane is

$$[r - 4\hat{i} - 3\hat{j}] \cdot (\hat{i} + \hat{j}) = 0 \Rightarrow r \cdot (\hat{i} + \hat{j}) = 7$$

118. Distance of the centre (0, 0, 0) from the plane is

$$= \frac{7}{\sqrt{4 + 9 + 26}} = 1 < 2$$

119. Given centre is (1, -2, -3)
Clearly, it lies on
 $3x - 4y + 2z - 5 = 0$ { $3 + 8 - 6 - 5 = 0$ }
This section is a great circle.

120. Distance of the centre (-1, -3, 4) from the plane is

$$\left| \frac{-1 - 6 - 12 - 7}{\sqrt{1 + 4 + 9}} \right| = \frac{26}{\sqrt{14}} > 5 \text{ (radius)}$$

$\therefore$ The section is an imaginary circle.

121. A. Let the foot Q of perpendicular be
(2 + 2λ, 1 + 3λ, 2 + 4λ)
 $\therefore 2(2\lambda + 1) + 3(3\lambda - 1) + 4(4\lambda - 1) = 0$
 $29\lambda = 5\lambda = \frac{5}{29}$
 $\therefore$ Foot = $\left(\frac{68}{29}, \frac{44}{29}, \frac{78}{29}\right)$

B. Let the image be the point (a, b, c), then
 $\frac{1+a}{2} = \frac{68}{29}$, $\frac{2+b}{2} = \frac{44}{29}$ and $\frac{3+c}{2} = \frac{78}{29}$
i.e. $a = \frac{107}{29}$, $b = \frac{30}{29}$ and $c = \frac{69}{29}$

C. $\frac{x-2}{2} = \frac{y-3}{3} = \frac{z-5}{-4} = -\frac{4+9-20+17}{4+9+16} = -\frac{10}{29}$
 $\therefore a = \frac{38}{29}$, $b = \frac{57}{29}$ and $c = \frac{185}{29}$

D. $\frac{x-2}{3} = \frac{y-5}{-2} = \frac{z-1}{4} = -2 \cdot \frac{6-10+4-5}{29} = \frac{10}{29}$
 $\therefore x = 2 + \frac{30}{29} = \frac{88}{29}$, $y = 5 - \frac{20}{29} = \frac{125}{29}$
and $z = 1 + \frac{40}{29} = \frac{69}{29}$

122. A. The lines are $\frac{x-1}{1} = \frac{y+3}{-\lambda} = \frac{z-1}{\lambda}$
and $\frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{-1}$

Coplanarity implies = $\begin{vmatrix} -1 & 4 & 1 \\ 1 & -\lambda & \lambda \\ \frac{1}{2} & 1 & -1 \end{vmatrix} = 0$

$\Rightarrow \lambda = -2$

B. $2x + 3y - 2z - k = 0$

$\Rightarrow (2 + 6 - 6)(4 - 3 - k) < 0$

$\Rightarrow 1 < k < 2$

$\Rightarrow$ Integral value of $k = 1$

C. Volume = $\frac{1}{6} \begin{vmatrix} 0-3 & 3-2 & 4-3 \\ 6-3 & 1-2 & 4-3 \\ 6-3 & 3-2 & 2-3 \end{vmatrix} = \frac{1}{6} \times 12 = 2$

D. $\frac{x_1-1}{1} = \frac{y_1-2}{1} = \frac{z_1-3}{1} = \frac{-2(1+2+3+3)}{3} = -6$
 $\Rightarrow (x_1, y_1, z_1) = (-5, -4, -3)$
 $\therefore z_1 = -3$

123. A. The required plane passes through $3\hat{i} + 6\hat{j} + 2\hat{k}$ and perpendicular to the vector $\hat{i} - \hat{j} + 2\hat{k}$.

$\therefore r \cdot (3\hat{i} + 6\hat{j} + 2\hat{k})$ is perpendicular to $\hat{i} - \hat{j} + 2\hat{k}$.

Hence, the equation of the plane is
 $r \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 3 - 6 + 4 = 1$

B. The plane through (5, -2, 4) is parallel to the plane
 $4x - 12y - 8z = 7$

$\Rightarrow r \cdot (5\hat{i} - 2\hat{j} + 4\hat{k})$ is perpendicular to $4\hat{i} - 12\hat{j} - 8\hat{k}$
Hence, the equation of the plane is

$[r \cdot (5\hat{i} - 2\hat{j} + 4\hat{k})] \cdot (4\hat{i} - 12\hat{j} - 8\hat{k}) = 0$

i.e. $r \cdot (\hat{i} - 3\hat{j} - 2\hat{k}) = 3$

C. The first line passes through $2\hat{i}$ and parallel to $\hat{j} - \hat{k}$.

The given plane is perpendicular to $\hat{j} - \hat{k}$.

$\therefore$ The normal to the required plane is
 $(\hat{j} - \hat{k}) \times (\hat{i} + \hat{k}) = \hat{i} - \hat{j} - \hat{k}$

The equation of the required plane is
 $(r - 2\hat{i}) \cdot (\hat{i} + \hat{j} - \hat{k}) = 0$

i.e. $r \cdot (\hat{i} - \hat{j} - \hat{k}) = 2$

D. Since, the plane contains the lines

$\frac{x-1}{1} = \frac{y-1}{2} = \frac{z}{-1}$

and $\frac{x-1}{-1} = \frac{y-1}{1} = \frac{z}{-2}$

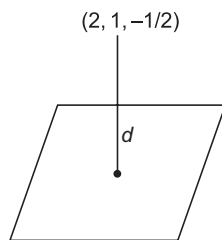
It passes the point $\hat{i} + \hat{j}$ and perpendicular to
 $(\hat{i} + 2\hat{j} - \hat{k}) \times (-\hat{i} + \hat{j} - 2\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$

i.e. $\hat{i} - \hat{j} - \hat{k}$.

$\therefore$ Its equation is $\{r - (\hat{i} + \hat{j})\} \cdot (\hat{i} - \hat{j} - \hat{k}) = 0$
 $\Rightarrow r \cdot (\hat{i} - \hat{j} - \hat{k}) = 0$

124. The line is $\frac{x-2}{1} = \frac{y-1}{1} = \frac{z+\frac{1}{2}}{-\frac{1}{2}} = t$... (i)

Line passes through $2\hat{i} + \hat{j} - \frac{1}{2}\hat{k}$ and is parallel to the vector $V = \hat{i} + \hat{j} - \frac{1}{2}\hat{k}$, vector normal to the plane $x + 2y + 6z = 10$ is $n = \hat{i} + 2\hat{j} + 6\hat{k}$.



$$\therefore V \cdot n = 1 + 2 - 3$$

$\Rightarrow$ Line (i) is parallel to the plane.

$$\therefore d = \frac{|2 + 2 - 3 - 10|}{\sqrt{1 + 4 + 36}} = \frac{9}{\sqrt{41}}$$

$$\Rightarrow \lambda = 9, \mu = 41$$

$$\Rightarrow 5\lambda - \mu = (5 \times 9) - 41 = 4$$

125. Vector v_1 along the line of intersection

$3x - 7y - 5z = 1$ and $8x - 11y + 2z = 0$, is given by

$$v_1 = n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -7 & -5 \\ 8 & -11 & 2 \end{vmatrix} = -23(3\hat{i} + 2\hat{j} - \hat{k})$$

Similarly, vector v_2 along the line of intersection of the plane $5x - 13y + 3z = 0$ and $8x - 11y + 2z = 0$ is

$$v_2 = n_3 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5 & -13 & 3 \\ 8 & -11 & 2 \end{vmatrix} = 7(\hat{i} + 2\hat{j} + 7\hat{k})$$

Now, $v_1 \cdot v_2 = 0 \Rightarrow$ Angle is $90^\circ \Rightarrow \sin 90^\circ = 1$

126. The given points are $O(0, 0, 0)$, $A(0, 0, 2)$, $B(0, 4, 0)$ and $C(6, 0, 0)$.

Here, three faces of tetrahedron XY, YZ, ZX -planes. are

Since, point P is equidistance from ZX, XY and YZ -planes, its coordinates are $P(r, r, r)$.

Equation of plane ABC is

$$2x + 3y + 6z = 12 \quad [\text{from intercept form}]$$

P is also at distance r from plane ABC .

$$\Rightarrow \frac{|2r + 3r + 6r - 12|}{\sqrt{4 + 9 + 36}} = r$$

$$\Rightarrow |11r - 12| = 7r$$

$$\Rightarrow 11r - 12 = \pm 7r$$

$$\Rightarrow r = \frac{12}{8}, 3$$

$$\therefore r = 2/3 \quad [\text{as } r < 2]$$

$$\text{Hence, } 9r = \frac{9 \times 2}{3} = 6$$

Entrances Gallery

1. Line L_1 given by $y = x, z = 1$ can be expressed

$$L_1: \frac{x}{1} = \frac{y}{1} = \frac{z-1}{0}$$

$$\frac{x}{1} = \frac{y}{1} = \frac{z-1}{0} = \alpha \quad [\text{say}]$$

$$\Rightarrow x = \alpha, y = \alpha \text{ and } z = 1$$

Let the coordinates of Q on L_1 be $(\alpha, \alpha, 1)$

Line L_2 is given by $y = -x, z = -1$ can be expressed as

$$L_2: \frac{x}{1} = \frac{y}{-1} = \frac{z+1}{0}$$

$$\frac{x}{1} = \frac{y}{-1} = \frac{z+1}{0} = \beta \quad [\text{say}]$$

$$\Rightarrow x = \beta, y = -\beta, z = -1$$

Let the coordinates of R on L_2 be $(\beta, -\beta, -1)$.

Direction ratios of PQ are $\lambda - \alpha, \lambda - \alpha, \lambda - 1$.

Now, $PQ \perp L_1$

$$\therefore 1(\lambda - \alpha) + 1(\lambda - \alpha) + 0(\lambda - 1) = 0 \Rightarrow \lambda = \alpha$$

$$\therefore Q(\lambda, \lambda, 1)$$

Direction ratios of PR are $\lambda - \beta, \lambda + \beta, \lambda + 1$.

Now, $PR \perp L_2$

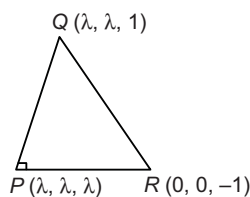
$$\therefore 1(\lambda - \beta) + (-1)(\lambda + \beta) + 0(\lambda + 1) = 0$$

$$\Rightarrow \lambda - \beta - \lambda - \beta = 0$$

$$\Rightarrow \beta = 0$$

$$\therefore R(0, 0, -1)$$

Now, $\angle QPR = 90^\circ$



$[a_1a_2 + b_1b_2 + c_1c_2 = 0, \text{ if two lines with}$

DR's $a_1, b_1, c_1; a_2, b_2, c_2$ are perpendicular]

$$\therefore (\lambda - \lambda)(\lambda - 0) + (\lambda - \lambda)(\lambda - 0) + (\lambda - 1)(\lambda + 1) = 0$$

$$\Rightarrow (\lambda - 1)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 1$$

$$\text{or } \lambda = -1$$

$\lambda = 1$, rejected as P and Q are different points.

$$\Rightarrow \lambda = -1$$

2. Let foot of perpendicular from $(2\lambda - 2, -\lambda - 1, 3\lambda)$ to

$x + y + z = 3$ be (x_2, y_2, z_2) .

$$\therefore \frac{x_2 - (2\lambda - 2)}{1} = \frac{y_2 - (-\lambda - 1)}{1}$$

$$= \frac{z_2 - (3\lambda)}{1}$$

$$= -\frac{(2\lambda - 2 - \lambda - 1 + 3\lambda - 3)}{1 + 1 + 1}$$

$$\Rightarrow x_2 - 2\lambda + 2 = y_2 + \lambda + 1$$

$$= z_2 - 3\lambda = 2 - \frac{4\lambda}{3}$$

$$\therefore x_2 = \frac{2\lambda}{3}, y_2 = 1 - \frac{7\lambda}{3}, z_2 = 2 + \frac{5\lambda}{3}$$

$$\Rightarrow \lambda = \frac{x_2 - 0}{\frac{2}{3}} = \frac{y_2 - 1}{\frac{-7}{3}} = \frac{z_2 - 2}{\frac{5}{3}}$$

$$\therefore \text{Foot of perpendicular lies on } \frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$$

3. Let $l : \frac{x-0}{a} = \frac{y-0}{b} = \frac{z-0}{c}$

which is perpendicular to

$$l_1 : (3\hat{i} - \hat{j} + 4\hat{k}) + t(\hat{i} + 2\hat{j} + 2\hat{k})$$

$$l_2 : (3\hat{i} + 3\hat{j} + 2\hat{k}) + s(2\hat{i} + 2\hat{j} + \hat{k})$$

$$\therefore \text{DR's of } l = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{i} + 3\hat{j} - 2\hat{k}$$

$$\therefore l : \frac{x}{-2} = \frac{y}{3} = \frac{z}{-2} = k_1$$

Now, $A(-2k_1, 3k_1, -2k_1)$

Since, A lies on l_1 .

$$\therefore -(2k_1)\hat{i} + (3k_1)\hat{j} - 2(k_1)\hat{k}$$

$$= (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}$$

$$\Rightarrow 3+t = -2k_1, -1+2t = 3k_1, 4+2t = -2k_1$$

$$\therefore k_1 = -1$$

$$\Rightarrow A(2, -3, 2)$$

Let any point on l_2 be $(3+2s, 3+2s, 2+s)$.

$$\therefore \sqrt{(2-3-2s)^2 + (-3-3-2s)^2 + (2-2-s)^2} = \sqrt{17}$$

$$\Rightarrow 9s^2 + 28s + 37 = 17$$

$$\Rightarrow 9s^2 + 28s + 20 = 0$$

$$\Rightarrow 9s^2 + 18s + 10s + 20 = 0$$

$$\Rightarrow (9s+10)(s+2) = 0$$

$$\therefore s = -2, -\frac{10}{9}$$

Hence, $(-1, -1, 0)$ and $(\frac{7}{9}, \frac{7}{9}, \frac{8}{9})$ are required points

4. Here,

$$x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$$

$$\Rightarrow \frac{x-5}{0} = \frac{y-0}{-(\alpha-3)} = \frac{z-0}{-2} \quad \dots (i)$$

and

$$x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$$

$$\frac{x-\alpha}{0} = \frac{y-0}{-1} = \frac{z-0}{2-\alpha} \quad \dots (ii)$$

$$\Rightarrow \begin{vmatrix} 5-\alpha & 0 & 0 \\ 0 & 3-\alpha & -2 \\ 0 & -1 & 2-\alpha \end{vmatrix} = 0$$

$$\Rightarrow (5-\alpha)[(3-\alpha)(2-\alpha) - 2] = 0$$

$$\Rightarrow (5-\alpha)(\alpha^2 - 5\alpha + 4) = 0$$

$$\Rightarrow (5-\alpha)(\alpha-1)(\alpha-4) = 0$$

$$\alpha = 1, 4, 5$$

5. Normal of plane $P : n = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix}$

$$= (-16)\hat{i} - (-42 - 6)\hat{j} + (32)\hat{k}$$

$$= -16\hat{i} + 48\hat{j} + 32\hat{k}$$

DR's of normal $n = \hat{i} - 3\hat{j} - 2\hat{k}$.

Point of intersection of L_1 and L_2 ,

$$2k_1 + 1 = k_2 + 4 \quad \text{and} \quad -k_1 = k_2 + 3$$

$$\Rightarrow k_1 = 2 \quad \text{and} \quad k_2 = 1$$

$\therefore$ Point of intersection is $(5, -2, -1)$.

$\therefore$ Equation of the plane is

$$1 \cdot (x-5) - 3(y+2) - 2(z+1) = 0$$

$$\Rightarrow x - 3y - 2z - 13 = 0$$

$$\Rightarrow x - 3y - 2z = 13$$

$$\therefore a = 1, b = -3, c = -2 \text{ and } d = 13$$

6. Equation of straight line QR is

$$\frac{x-2}{1-2} = \frac{y-3}{-1-3} = \frac{z-5}{4-5}$$

$$\Rightarrow \frac{x-2}{-1} = \frac{y-3}{-4} = \frac{z-5}{-1}$$

$$\Rightarrow \frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1} = \lambda \quad \dots (i)$$

$\therefore P(\lambda+2, 4\lambda+3, \lambda+5)$ must lie on $5x - 4y - z = 1$.

$$\Rightarrow 5(\lambda+2) - 4(4\lambda+3) - (\lambda+5) = 1$$

$$\Rightarrow 5\lambda + 10 - 16\lambda - 12 - \lambda - 5 = 1$$

$$\Rightarrow -7 - 12\lambda = 1$$

$$\therefore \lambda = -\frac{2}{3}$$

$$\text{or } P\left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right) \quad \dots (ii)$$

Again, we can assume S from Eq. (i), as $S(\mu+2, 4\mu+3, \mu+5)$

$\therefore$ DR's of $TS = (\mu+2-2, 4\mu+3-1, \mu+5-4)$

$$= (\mu, 4\mu+2, \mu+1)$$

and DR's of $QR = (1, 4, 1)$

Since, it is perpendicular.

$$\therefore 1(\mu) + 4(4\mu+2) + 1(\mu+1) = 0$$

$$\Rightarrow \mu = -\frac{1}{2}$$

$$\text{and } S\left(\frac{3}{2}, 1, \frac{9}{2}\right) \quad \dots (iii)$$

$\therefore$ Length of PS

$$= \sqrt{\left(\frac{3}{2} - \frac{4}{3}\right)^2 + \left(1 - \frac{1}{3}\right)^2 + \left(\frac{9}{2} - \frac{13}{3}\right)^2}$$

$$= \frac{1}{\sqrt{2}}$$

7. Equation of plane passing through intersection of two

planes $x + 2y + 3z = 2$ and $x - y + z = 3$ is

$$\Rightarrow (x + 2y + 3z - 2) + \lambda(x - y + z - 3) = 0$$

$$\Rightarrow (1+\lambda)x + (2-\lambda)y + (3+\lambda)z - (2+3\lambda) = 0$$

whose distance from $(3, 1, -1)$ is $\frac{2}{\sqrt{3}}$.

$$\Rightarrow \frac{|3(1+\lambda) + 1(2-\lambda) - 1(3+\lambda) - (2+3\lambda)|}{\sqrt{(1+\lambda)^2 + (2-\lambda)^2 + (3+\lambda)^2}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{|-2\lambda|}{\sqrt{3\lambda^2 + 4\lambda + 14}} = \frac{2}{\sqrt{3}}$$

$$\Rightarrow 3\lambda^2 = 3\lambda^2 + 4\lambda + 14$$

$$\Rightarrow \lambda = -\frac{7}{2}$$

$$\therefore \left(1 - \frac{7}{2}\right)x + \left(2 + \frac{7}{2}\right)y + \left(3 - \frac{7}{2}\right)z - \left(2 - \frac{21}{2}\right) = 0$$

$$\Rightarrow -\frac{5x}{2} + \frac{11y}{2} - \frac{1z}{2} + \frac{17}{2} = 0$$

$$\text{or } 5x - 11y + z - 17 = 0$$

8. Since, $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$
and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar.

$$\Rightarrow \begin{vmatrix} 2 & 0 & 0 \\ 2 & k & 2 \\ 5 & 2 & k \end{vmatrix} = 0$$

$$\Rightarrow k^2 = 4 \Rightarrow k = \pm 2$$

$$\therefore n_1 = b_1 \times d_1 = 6j - 6k, \text{ for } k = 2$$

$$\therefore n_2 = b_2 \times d_2 = 14j + 14k, \text{ for } k = -2$$

$$\frac{x}{\frac{2}{3}} = \frac{y-1}{\frac{-7}{3}} = \frac{z-2}{\frac{5}{3}} \Rightarrow \frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$$

So, the equation of planes are $(r-a) \cdot n_1 = 0$

$$\Rightarrow y - z = -1$$

$$\text{and } (r-a) \cdot n_2 = 0$$

$$\Rightarrow y + z = -1$$

9. The equation of given lines in vector form may be written as

$$L_1 : r = (-\hat{i} - 2\hat{j} - \hat{k}) + \lambda(3\hat{i} + \hat{j} + 2\hat{k})$$

$$L_2 : r = (2\hat{i} - 2\hat{j} + 3\hat{k}) + \mu(\hat{i} + 2\hat{j} + 3\hat{k})$$

Since, the vector perpendicular to both L_1 and L_2 .

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 1 & 2 & 3 \end{vmatrix} = -\hat{i} - 7\hat{j} + 5\hat{k}$$

$\therefore$ Required unit vector

$$= \frac{(-\hat{i} - 7\hat{j} + 5\hat{k})}{\sqrt{(-1)^2 + (-7)^2 + (5)^2}} = \frac{1}{5\sqrt{3}}(-\hat{i} - 7\hat{j} + 5\hat{k})$$

10. Shortest distance between L_1 and L_2

$$= \frac{|[(2-(-1))\hat{i} + (2-2)\hat{j} + \{3-(-1)\}\hat{k}] \cdot (-\hat{i} - 7\hat{j} + 5\hat{k})|}{5\sqrt{3}}$$

$$= \frac{|(3\hat{i} + 4\hat{k}) \cdot (-\hat{i} - 7\hat{j} + 5\hat{k})|}{5\sqrt{3}} = \frac{17}{5\sqrt{3}} \text{ units}$$

11. The equation of the plane passing through the point $(-1, -2, -1)$ and whose normal is perpendicular to both the given lines L_1 and L_2 may be written as

$$(x+1) + 7(y+2) - 5(z+1) = 0$$

$$\Rightarrow x + 7y - 5z + 10 = 0$$

Distance of the point $(1, 1, 1)$ from the plane

$$= \frac{|1 + 7 - 5 + 10|}{\sqrt{1 + 49 + 25}} = \frac{13}{\sqrt{75}} \text{ units}$$

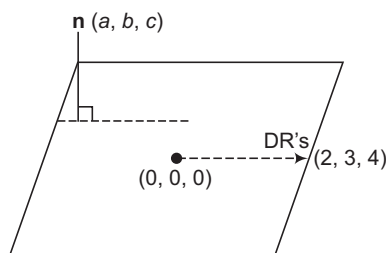
12. The DR's to normal of the plane containing $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$

$$\text{and } \frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

$$\Rightarrow n_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix} = (8\hat{i} - \hat{j} - 10\hat{k})$$

Also, equation of plane containing $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and

DR's of normal to be $n_2 = a\hat{i} + b\hat{j} + c\hat{k}$.



$$\Rightarrow ax + by + cz = 0 \quad \dots(i)$$

$$\text{where, } n_1 \cdot n_2 = 0 \quad \dots(ii)$$

$$\Rightarrow 8a - b - 10c = 0$$

$$\text{and } n_2 \perp (2\hat{i} + 3\hat{j} + 4\hat{k}) \quad \dots(iii)$$

$$\Rightarrow 2a + 3b + 4c = 0$$

From Eqs. (ii) and (iii), we get

$$\frac{a}{-1-10} = \frac{b}{8} = \frac{c}{-1}$$

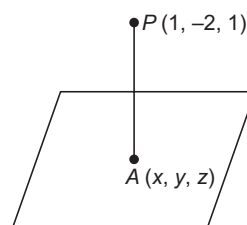
$$\Rightarrow \frac{a}{-4+30} = \frac{b}{-20-32} = \frac{c}{24+2}$$

$$\Rightarrow \frac{a}{26} = \frac{b}{-52} = \frac{c}{26}$$

$$\Rightarrow \frac{a}{1} = \frac{b}{-2} = \frac{c}{1} \quad \dots(iv)$$

From Eqs. (i) and (iv), required equation of plane is $x - 2y + z = 0$.

13. Distance of point P from plane = 5



$$x + 2y - 2z = \alpha$$

$$\Rightarrow 5 = \frac{|1 - 4 - 2 - \alpha|}{3}$$

$$\therefore \alpha = 10$$

Foot of the perpendicular is

$$\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-1}{-2} = \frac{5}{3}$$

$$x = \frac{8}{3}, y = \frac{4}{3} \text{ and } z = -\frac{7}{3}$$

Thus, the foot of the perpendicular is $A\left(\frac{8}{3}, \frac{4}{3}, -\frac{7}{3}\right)$.

14. Equation of the plane containing the given lines is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x-1)(-1) - (y-2)(-2) + (z-3)(-1) = 0$$

$$\Rightarrow -x + 1 + 2y - 4 - z + 3 = 0$$

$$\Rightarrow -x + 2y - z = 0 \quad \dots(i)$$

Given plane is

$$x - 2y + z = d \quad \dots(ii)$$

Eqs. (i) and (ii) are parallel.

Clearly, $A = 1$

Now, distance between plane

$$= \left| \frac{d}{\sqrt{1+4+1}} \right| = \sqrt{6}$$

$$\Rightarrow |d| = 6$$

15. We know that, $\cos^2 45^\circ + \cos^2 120^\circ + \cos^2 \theta = 1$

$$\Rightarrow \frac{1}{2} + \frac{1}{4} + \cos^2 \theta = 1$$

$$\Rightarrow \cos^2 \theta = \frac{1}{4}$$

$$\Rightarrow \cos \theta = \pm \frac{1}{2}$$

$$\Rightarrow \theta = 60^\circ \text{ or } 120^\circ$$

16. The image of the point (3, 1, 6) with respect to the plane $x - y + z = 5$ is

$$\frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-6}{1} = \frac{-2(3-1+6-5)}{1+1+1}$$

$$\Rightarrow \frac{x-3}{1} = \frac{y-1}{-1} = \frac{z-6}{1} = -2$$

$$\Rightarrow x = 3 - 2 = 1$$

$$y = 1 + 2 = 3$$

$$z = 6 - 2 = 4$$

which shows that Statement I is true.

We observe that the line segment joining the points $A(3, 1, 6)$ and $B(1, 3, 4)$ have direction ratios 2, -2, 2 which are proportional to 1, -1, 1 the direction ratios of the normal to the plane. Thus, Statement II is true.

Thus, Statements I and II are true and Statement II is correct explanation for Statement I.

17. Given line is $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$... (i)

This line will pass through the point of intersection of the lines

$$2x - 3y + 4 = 0 \quad \dots (ii)$$

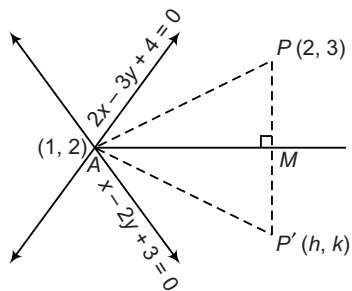
$$\text{and } x - 2y + 3 = 0 \quad \dots (iii)$$

On solving Eqs. (ii) and (iii), we get

$$x = 1, y = 2$$

$\therefore$ Point of intersection of lines (ii) and (iii) is (1, 2).

Let M be the mid-point of PP' , then AM is perpendicular bisector of PP' (where, A is the point of intersection of given lines).



$$\therefore AP = AP'$$

$$\Rightarrow \sqrt{(2-1)^2 + (3-2)^2} = \sqrt{(h-1)^2 + (k-2)^2}$$

$$\Rightarrow \sqrt{2} = \sqrt{h^2 + k^2 - 2h - 4k + 1 + 4}$$

$$\Rightarrow \sqrt{2} = \sqrt{h^2 + k^2 - 2h - 4k + 5}$$

$$\Rightarrow h^2 + k^2 - 2h - 4k + 5 = 2$$

$$\Rightarrow h^2 + k^2 - 2h - 4k + 3 = 0$$

Thus, the required locus is

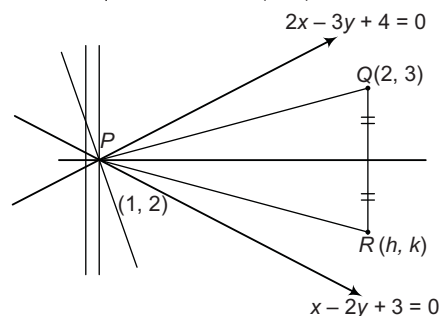
$$x^2 + y^2 - 2x - 4y + 3 = 0$$

which is a equation of circle with

$$\text{radius} = \sqrt{1+4-3} = \sqrt{2}$$

Aliter

$(2x - 3y + 4) + k(x - 2y + 3) = 0$ is family of lines passing through (1, 2). By congruency of triangles, we can prove that mirror image (h, k) and the point (2, 3) will be equidistant from (1, 2).



$\therefore$ Locus of (h, k) is $PR = PQ$

$$\Rightarrow (h-1)^2 + (k-2)^2 = (2-1)^2 + (3-2)^2$$

$$\text{or } (x-1)^2 + (y-2)^2 = 2$$

$\therefore$ Locus is a circle of radius $= \sqrt{2}$

18. Let equation of plane containing the lines $2x - 5y + z = 3$ and $x + y + 4z = 5$ be

$$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$$

$$\Rightarrow (2 + \lambda)x + (\lambda - 5)y + (4\lambda + 1)z - 3 - 5\lambda = 0 \quad \dots (i)$$

This plane is parallel to the plane $x + 3y + 6z = 1$.

$$\therefore \frac{2 + \lambda}{1} = \frac{\lambda - 5}{3} = \frac{4\lambda + 1}{6}$$

On taking first two equalities, we get

$$6 + 3\lambda = \lambda - 5$$

$$\Rightarrow 2\lambda = -11$$

$$\Rightarrow \lambda = -\frac{11}{2}$$

On taking last two equalities, we get

$$6\lambda - 30 = 3 + 12\lambda$$

$$\Rightarrow -6\lambda = 33$$

$$\Rightarrow \lambda = -\frac{11}{2}$$

So, the equation of required plane is

$$\left(2 - \frac{11}{2}\right)x + \left(-\frac{11}{2} - 5\right)y + \left(-\frac{44}{2} + 1\right)z$$

$$-3 + 5 \times \frac{11}{2} = 0$$

$$\Rightarrow -\frac{7}{2}x - \frac{21}{2}y - \frac{42}{2}z + \frac{49}{2} = 0$$

$$\Rightarrow x + 3y + 6z - 7 = 0$$

19. Given equation of line is

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = \lambda \quad [\text{say}] \dots (i)$$

and equation of plane is

$$x - y + z = 16 \quad \dots (ii)$$

Any point on the line (i) is

$$(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$$

Let this point be of intersection of the line and plane.

$$\therefore (3\lambda + 2) - (4\lambda - 1) + (12\lambda + 2) = 16$$

$$\Rightarrow 11\lambda + 5 = 16$$

$$\Rightarrow 11\lambda = 11$$

$$\Rightarrow \lambda = 1$$

$\therefore$ Point of intersection is (5, 3, 14).

Now, distance between the points (1, 0, 2) and (5, 3, 14)

$$= \sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2}$$

$$= \sqrt{16 + 9 + 144} = \sqrt{169} = 13$$

20. We know that, angle between two lines is

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\Rightarrow l + m + n = 0 \Rightarrow l = -(m + n)$$

$$\Rightarrow (m + n)^2 = l^2$$

$$\Rightarrow m^2 + n^2 + 2mn = m^2 + n^2$$

$$[\because l^2 = m^2 + n^2, \text{ given}]$$

$$\Rightarrow 2mn = 0$$

$$\text{When } m = 0 \Rightarrow l = -n$$

So, (l, m, n) is (1, 0, -1).

When n = 0, then l = -m

So, (l, m, n) is (1, -1, 0).

$$\therefore \cos \theta = \frac{1 + 0 + 0}{\sqrt{2} \times \sqrt{2}} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

21. Here, plane and line are parallel to each other. Equation of normal to the plane through the point (1, 3, 4) is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1} = k \quad [\text{say}]$$

Any point in this normal is (2k + 1, -k + 3, 4 + k).

Then, $\left(\frac{2k+1+1}{2}, \frac{3-k+3}{2}, \frac{4+k+4}{2}\right)$ lies on plane.

$$\Rightarrow 2(k+1) - \left(\frac{6-k}{2}\right) + \left(\frac{8+k}{2}\right) + 3 = 0$$

$$\Rightarrow k = -2$$

So, point through which this image pass is

$$(2k+1, 3-k, 4+k)$$

i.e. (2(-2) + 1, 3 + 2, 4 - 2) = (-3, 5, 2)

Hence, equation of image line is

$$\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$$

22. Since, the lines $\frac{x+1}{2} = \frac{y-1}{1} = \frac{z+1}{3}$

and $\frac{x+2}{2} = \frac{y-k}{3} = \frac{z}{4}$ are coplanar.

$$\therefore \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} -2+1 & k-1 & 0+1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & k-1 & 1 \\ 2 & 1 & 3 \\ 2 & 3 & 4 \end{vmatrix} = -(4-9) - (k-1)(8-6) + 1(6-2) = 0$$

$$\Rightarrow 5 - 2k + 2 + 4 = 0$$

$$\Rightarrow 2k = 11$$

$$\Rightarrow k = \frac{11}{2}$$

23. Since, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$$\Rightarrow \cos^2 45^\circ + \cos^2 60^\circ + \cos^2 \gamma = 1$$

$$\Rightarrow \cos^2 \gamma = 1 - \frac{1}{2} - \frac{1}{4}$$

$$\Rightarrow \cos^2 \gamma = 1 - \frac{3}{4} = \frac{1}{4}$$

$$\Rightarrow \cos \gamma = \frac{1}{2}$$

$\therefore$ Direction ratios of normal to the plane is

$$\langle \cos 45^\circ, \cos 60^\circ, \frac{1}{2} \rangle \text{ i.e. } \langle \frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2} \rangle$$

$\therefore$ Equation of plane passing through $(\sqrt{2}, -1, 1)$ is

$$\frac{1}{\sqrt{2}}(x - \sqrt{2}) + \frac{1}{2}(y + 1) + \frac{1}{2}(z - 1) = 0$$

$$\Rightarrow 2(x - \sqrt{2}) + \sqrt{2}(y + 1) + \sqrt{2}(z - 1) = 0$$

$$\Rightarrow \sqrt{2}(x - \sqrt{2}) + (y + 1) + (z - 1) = 0$$

$$\Rightarrow \sqrt{2}x - 2 + y + 1 + z - 1 = 0$$

$$\Rightarrow \sqrt{2}x + y + z = 2$$

24. Let foot of perpendicular Q(x, y, z) from O(0, 0, 0), then

$$\frac{x-0}{4} = \frac{y-0}{-3} = \frac{z-0}{1} = \frac{\{4(0) - 3(0) + 1(0) + 13\}}{4^2 + 3^2 + 1^3}$$

$$\Rightarrow \frac{x}{4} = \frac{y}{-3} = \frac{z}{1} = \frac{-13}{26} = -\frac{1}{2}$$

$$x = -2, y = \frac{3}{2}, z = -\frac{1}{2}$$

$$\therefore Q\left(-2, \frac{3}{2}, -\frac{1}{2}\right)$$

$$\therefore PQ = \sqrt{(-1+2)^2 + \left(1-\frac{3}{2}\right)^2 + \left(-6+\frac{1}{2}\right)^2}$$

$$= \sqrt{1 + \frac{1}{4} + \frac{121}{4}} = \frac{\sqrt{126}}{2} = 3\sqrt{\frac{7}{2}}$$

25. Since, $l + m + n = 0, l^2 + m^2 - n^2 = 0$

On solving, we get DC's of two lines are

$$\langle 0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \rangle \text{ and } \langle \frac{1}{\sqrt{2}}, 0, \frac{-1}{\sqrt{2}} \rangle$$

$$\therefore \theta = \frac{\pi}{3} \Rightarrow \theta = 60^\circ$$

26. Given that, the projections of a line segment on the X, Y and Z-axes in 3D-space are $lr = 2, mr = 3$ and $nr = 6$

$$\therefore (lr)^2 + (mr)^2 + (nr)^2 = (2)^2 + (3)^2 + (6)^2$$

$$\Rightarrow (l^2 + m^2 + n^2)r^2 = 4 + 9 + 36$$

$$\Rightarrow r^2 = 49 \Rightarrow r = 7$$

27. Intersection of two planes is

$$(x + 2y - 3) + \lambda(y - 2z + 1) = 0$$

$$\Rightarrow x + (2 + \lambda)y - 2\lambda z + \lambda - 3 = 0$$

$$\therefore 1(1) + 2(2 + \lambda) + 0(-2\lambda) = 0$$

$$\Rightarrow \lambda = -\frac{5}{2}$$

$\therefore$ Required equation of plane is $2x - y + 10z - 11 = 0$.

$$28. \text{ Centroid of } \triangle ABC, G = \left(\frac{2-1+\lambda}{3}, \frac{3+3+5}{3}, \frac{5+2+\mu}{3}\right)$$

$$= \left(\frac{1+\lambda}{3}, \frac{11}{3}, \frac{7+\mu}{3}\right)$$

Since, median is always passes through centroid.
So, they are equally inclined.

$$\begin{aligned}\therefore \frac{1+\lambda}{3} - 2 &= \frac{11}{3} - 3 = \frac{7+\mu}{3} - 5 \\ \Rightarrow \frac{\lambda-5}{3} &= \frac{\mu-8}{3} \\ \Rightarrow \lambda &= 7, \mu = 10\end{aligned}$$

29. Given planes are

$$\begin{aligned}2x + y + 2z - 8 &= 0 \\ \text{and} \quad 2x + y + 2z + \frac{5}{2} &= 0\end{aligned}$$

Distance between two planes

$$= \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2 + c^2}} = \frac{\left| -8 - \frac{5}{2} \right|}{\sqrt{2^2 + 1^2 + 2^2}} = \frac{\frac{21}{2}}{3} = \frac{7}{2}$$

30. Condition for two lines are coplanar, is

$$[a \ c \ b \ d] = 0$$

where, a and c are the points on a line and b, d are the direction ratios of the lines.

$$\therefore \begin{vmatrix} 2 & -1 & 3 & -4 & 4 & -5 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & -1 & -1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\begin{aligned}\Rightarrow 1(1+2k) + (1+k^2) - (2-k) &= 0 \\ \Rightarrow k^2 + 2k + k &= 0 \\ \Rightarrow k^2 + 3k &= 0 \\ \Rightarrow k &= 0, -3\end{aligned}$$

31. Let $L_1: \frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = p$

$$\text{and } L_2: \frac{x-3}{1} = \frac{y-k}{2} = \frac{z-0}{1} = q$$

Any point P on line L_1 is of type $P(2p+1, 3p-1, 4p+1)$ and any point Q on the L_2 is of type $Q(q+3, 2q+k, q)$.

Since, L_1 and L_2 are intersecting each other, hence both point P and Q should coincide at the point of intersection i.e. corresponding coordinates of P and Q should be same.

$$\begin{aligned}2p+1 &= q+3, 4p+1=q \\ \text{and} \quad 3p-1 &= 2q+k\end{aligned}$$

On solving $2p+1=q+3$ and $4p+1=q$, we get the values of p and q as

$$p = \frac{-3}{2} \quad \text{and} \quad q = -5$$

On substituting the values of p and q in the equation $3p-1=2q+k$, we get

$$3\left(\frac{-3}{2}\right) - 1 = 2(-5) + k$$

$$\Rightarrow k = \frac{9}{2}$$

32. Given plane $P: x-2y+2z-5=0$

Equation of family of planes parallel to the given plane P is

$$Q: x-2y+2z+d=0$$

Also, perpendicular distance of Q from origin is 1 unit.

$$\Rightarrow \frac{|0-2(0)+2(0)+d|}{\sqrt{1^2+2^2+2^2}} = 1$$

$$\Rightarrow \frac{|d|}{3} = 1$$

$$\Rightarrow d = \pm 3$$

Hence, the equation of the plane parallel to P and at unit distance from origin is

$$x-2y+2z \pm 3 = 0.$$

33. Angle between straight line

$r = a + \lambda b$ and plane $r \cdot \hat{n} = d$ is

$$\Rightarrow \sin \theta = \frac{b \cdot \hat{n}}{|b| |\hat{n}|}$$

$$\therefore \sin \theta = \frac{(\hat{i} + 2\hat{j} + \lambda \hat{k}) \cdot (\hat{i} + 2\hat{j} + 3\hat{k})}{\sqrt{1+4+\lambda^2} \sqrt{1+4+9}}$$

$$\Rightarrow \sin \theta = \frac{5+3\lambda}{\sqrt{\lambda^2+5} \sqrt{14}}$$

$$\text{Given, } \cos \theta = \frac{\sqrt{5}}{\sqrt{14}}$$

$$\therefore \sin \theta = \frac{3}{\sqrt{14}}$$

$$\Rightarrow \frac{3}{\sqrt{14}} = \frac{5+3\lambda}{\sqrt{\lambda^2+5} \sqrt{14}}$$

$$\Rightarrow 9(\lambda^2+5) = 9\lambda^2 + 30\lambda + 25$$

$$\Rightarrow 9\lambda^2 + 45 = 9\lambda^2 + 30\lambda + 25$$

$$\Rightarrow 30\lambda = 20$$

$$\therefore \lambda = \frac{2}{3}$$

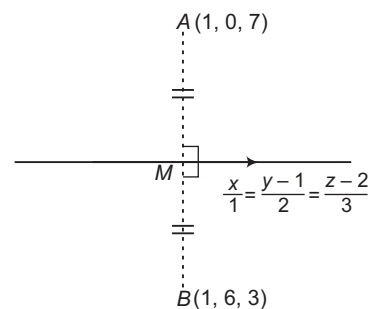
34. Mid-point of AB is $M(1, 3, 5)$

$$\text{which lies on } \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$

$$\text{as } \frac{1}{1} = \frac{3-1}{2} = \frac{5-2}{3}$$

$$\Rightarrow 1=1=1$$

$\therefore$ Statement II is true.



Also, directions of AB is

$$(1-1, 6-0, 3-7)$$

$$\text{i.e. } (0, 6, -4) \quad \dots (i)$$

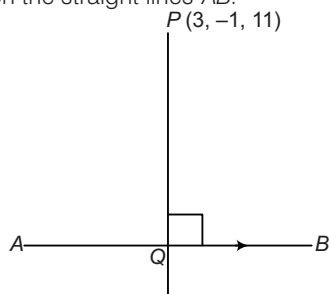
$$\text{and directions of straight line is } (1, 2, 3) \quad \dots (ii)$$

The two lines are perpendicular, if

$$0(1) + 6(2) - 4(3) = 12 - 12 = 0$$

$\therefore$ Statement I is true.

35. Let the coordinates of Q be $(2\lambda, 3\lambda + 2, 4\lambda + 3)$ is any point on the straight lines AB.



$\therefore$ DR's of PQ is $(2\lambda - 3, 3\lambda + 3, 4\lambda - 8)$ perpendicular to straight line AB

$$\frac{x}{2} = \frac{y-2}{3} = \frac{z-3}{4} = \lambda$$

having DR's $(2, 3, 4)$.

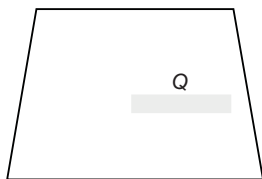
$$\begin{aligned} \Rightarrow 2(2\lambda - 3) + 3(3\lambda + 3) + 4(4\lambda - 8) &= 0 \\ \Rightarrow 4\lambda - 6 + 9\lambda + 9 + 16\lambda - 32 &= 0 \\ \Rightarrow 29\lambda - 29 &= 0 \\ \Rightarrow \lambda &= 1 \end{aligned}$$

$\Rightarrow$ Coordinates of Q are $(2, 5, 7)$.

$$\begin{aligned} \therefore |PQ| &= \sqrt{(3-2)^2 + (-1-5)^2 + (11-7)^2} \\ &= \sqrt{1 + 36 + 16} = \sqrt{53} \end{aligned}$$

36. Equation of PQ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda$$



$\therefore x = \lambda + 1, y = \lambda - 5, z = \lambda + 9$ lies on the plane

$$\begin{aligned} x - y + z &= 5 \\ \Rightarrow \lambda + 1 - \lambda + 5 + \lambda + 9 &= 5 \\ \Rightarrow \lambda &= -10 \\ \therefore \text{Coordinates of Q are } (-9, -15, -1). \end{aligned}$$

$$\Rightarrow |PQ| = \sqrt{(10)^2 + (10)^2 + (10)^2} = 10\sqrt{3}$$

37. DR's of given line are $(3, -5, 2)$.

DR's of normal to the plane are $(1, 3, -\alpha)$.

$\therefore$ Line is perpendicular to the normal.

$$\begin{aligned} \Rightarrow 3(1) - 5(3) + 2(-\alpha) &= 0 \\ \Rightarrow 3 - 15 - 2\alpha &= 0 \\ \Rightarrow 2\alpha &= -12 \Rightarrow \alpha = -6 \end{aligned}$$

Also, point $(2, 1, -2)$ lies on the plane.

$$\begin{aligned} \therefore 2 + 3 + 6(-2) + \beta &= 0 \\ \Rightarrow \beta &= 7 \\ \therefore (\alpha, \beta) &= (-6, 7) \end{aligned}$$

38. Projection of a vector on coordinate axes are

$x_2 - x_1, y_2 - y_1$ and $z_2 - z_1$ such that

$$x_2 - x_1 = 6, y_2 - y_1 = -3 \text{ and } z_2 - z_1 = 2.$$

$$\begin{aligned} \therefore \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ = \sqrt{36 + 9 + 4} = 7 \end{aligned}$$

Hence, DC's of the vector are $\frac{6}{7}, -\frac{3}{7}, \frac{2}{7}$.

39. Equation of line passing through $(5, 1, a)$ and $(3, b, 1)$ is

$$\frac{x-3}{5-3} = \frac{y-b}{1-b} = \frac{z-1}{a-1} \quad \dots(i)$$

Point $(0, \frac{17}{2}, -\frac{13}{2})$ satisfies Eq. (i), we get

$$-\frac{3}{2} = \frac{\frac{17}{2}-b}{1-b} = \frac{-\frac{13}{2}-1}{a-1}$$

$$\Rightarrow a-1 = \frac{\left(-\frac{15}{2}\right)}{\left(-\frac{3}{2}\right)} = 5$$

$$\Rightarrow a = 6$$

$$\text{Also, } -3(1-b) = 2\left(\frac{17}{2}-b\right)$$

$$\Rightarrow 3b-3 = 17-2b$$

$$\Rightarrow 5b = 20 \Rightarrow b = 4$$

40. Given, $\frac{x-1}{k} = \frac{y-2}{2} = \frac{z-3}{3} \quad \dots(i)$

$$\text{and } \frac{x-2}{3} = \frac{y-3}{k} = \frac{z-1}{2} \quad \dots(ii)$$

Since, lines intersect at a point. Then, shortest distance between them is zero.

$$\therefore \begin{vmatrix} k & 2 & 3 \\ 3 & k & 2 \\ 1 & 1 & -2 \end{vmatrix} = 0$$

$$\Rightarrow k(-2k-2) - 2(-6-2) + 3(3-k) = 0$$

$$\Rightarrow -2k^2 - 5k + 25 = 0$$

$$\Rightarrow 2k^2 + 5k - 25 = 0$$

$$\Rightarrow 2k^2 + 10k - 5k - 25 = 0$$

$$\Rightarrow 2k(k+5) - 5(k+5) = 0$$

$$\Rightarrow k = \frac{5}{2}, -5$$

Hence, integer value of k is -5 .

41. Let the direction cosines of line L be l, m, n . Since, the line intersect the given planes, then the normal to the planes are perpendicular to the line L .

$$\therefore 2l + 3m + n = 0 \quad \dots(i)$$

$$\text{and } l + 3m + 2n = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{l}{3} = \frac{m}{-3} = \frac{n}{3} = k \quad [\text{say}]$$

$$\therefore l^2 + m^2 + n^2 = 1$$

$$\therefore (3k)^2 + (-3k)^2 + (3k)^2 = 1$$

$$\Rightarrow 27k^2 = 1 \Rightarrow k = \frac{1}{3\sqrt{3}}$$

$$\therefore l = \frac{1}{\sqrt{3}} \Rightarrow \cos \alpha = \frac{1}{\sqrt{3}}$$

42. Since, a line makes an angle of $\frac{\pi}{4}$ with positive directions of each of X and Y -axes, therefore

$$\alpha = \frac{\pi}{4} \text{ and } \beta = \frac{\pi}{4}$$

We know that, $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$

$$\therefore \cos^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{4} + \cos^2 \gamma = 1$$

$$\Rightarrow \frac{1}{2} + \frac{1}{2} + \cos^2 \gamma = 1$$

$$\Rightarrow \cos^2 \gamma = 0$$

$$\therefore \gamma = \frac{\pi}{2}$$

43. Given equation of sphere is
 $x^2 + y^2 + z^2 - 6x - 12y - 2z + 20 = 0$
 whose coordinates of centre are (3, 6, 1).
 Since, one end of diameter are (2, 3, 5) and let the other end of diameter be (α, β, γ) , then
 $\frac{\alpha + 2}{2} = 3, \frac{\beta + 3}{2} = 6$ and $\frac{\gamma + 5}{2} = 1$
 $\Rightarrow \alpha = 4, \beta = 9$ and $\gamma = -3$
 Hence, the coordinates of other points are (4, 9, -3).

44. Given equations can be rewritten as
 $\frac{x-b}{a} = \frac{y-0}{1} = \frac{z-d}{c}$
 and
 $\frac{x-b'}{a'} = \frac{y-0}{1} = \frac{z-d'}{c'}$
 These lines will be perpendicular, if
 $aa' + cc' + 1 = 0$

45. We know that, the image (x, y, z) of a point (x_1, y_1, z_1) in a plane $ax + by + cz + d = 0$ is given by
 $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
 $= \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}$

Thus, the image of point $(-1, 3, 4)$ in a plane $x - 2y = 0$, is given by
 $\frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0}$
 $= \frac{-2[1 \times (-1) + (-2) \times 3 + 0 \times 4]}{1 + 4}$

$$\Rightarrow \frac{x+1}{1} = \frac{y-3}{-2} = \frac{z-4}{0} = \frac{-2(-7)}{5}$$

$$\Rightarrow x = \frac{14}{5} - 1 = \frac{9}{5}, y = -\frac{28}{5} + 3 = -\frac{13}{5} \text{ and } z = 4.$$

Hence, the image of point $(-1, 3, 4)$ is $(\frac{9}{5}, -\frac{13}{5}, 4)$.

46. Equations of given spheres are
 $x^2 + y^2 + z^2 + 6x - 8y - 2z = 13$... (i)
 and $x^2 + y^2 + z^2 - 10x + 4y - 2z = 8$... (ii)
 whose centres are $(-3, 4, 1)$ and $(5, -2, 1)$.
 Mid-point of $(-3, 4, 1)$ and $(5, -2, 1)$ is $(1, 1, 1)$.
 Since, the given plane $2ax - 3ay + 4az + 6 = 0$ passes through $(1, 1, 1)$.
 $\therefore 2a - 3a + 4a + 6 = 0$
 $\Rightarrow 3a = -6$
 $\Rightarrow a = -2$

47. Direction ratios of line are $(a_1, b_1, c_1) = (1, 2, 2)$
 and direction ratios of a plane are
 $(a_2, b_2, c_2) = (2, -1, \sqrt{\lambda})$
 $\therefore \sin \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$
 $= \frac{1 \times 2 + 2(-1) + 2 \times \sqrt{\lambda}}{\sqrt{1^2 + 2^2 + 2^2} \sqrt{2^2 + 1^2 + (\lambda)^2}}$

$$\Rightarrow \frac{1}{3} = \frac{2\sqrt{\lambda}}{3\sqrt{5+\lambda}}$$

$$\Rightarrow 5 + \lambda = 4\lambda$$

$$\therefore \lambda = \frac{5}{3}$$

48. The given equations of lines can be rewritten as

$$\frac{x}{3} = \frac{y}{2} = \frac{z}{-6} \text{ and } \frac{x}{2} = \frac{y}{-12} = \frac{z}{-3}$$

$\therefore$ Angle between the lines is

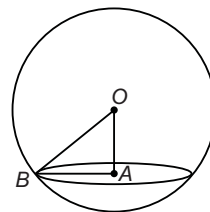
$$\theta = \cos^{-1} \left(\frac{3 \times 2 + 2(-12) - 6(-3)}{\sqrt{3^2 + 2^2 + (-6)^2} \sqrt{(2)^2 + (-12)^2 + (-3)^2}} \right)$$

$$= \cos^{-1}(0) = 90^\circ$$

49. Since, the centre of sphere

$$x^2 + y^2 + z^2 - x + z - 2 = 0 \text{ is } \left(\frac{1}{2}, 0, \frac{1}{2} \right)$$

and radius = $\sqrt{\frac{1}{4} + \frac{1}{4} + 2} = \frac{\sqrt{10}}{2}$



Distance from centre of sphere to the given plane,

$$OA = \left| \frac{1}{2} + \frac{1}{2} - 4 \right| = \frac{3}{\sqrt{6}}$$

So, radius of circle,

$$AB = \sqrt{OB^2 - OA^2} = \sqrt{\frac{10}{4} - \frac{9}{6}}$$

$$= \sqrt{\frac{30-18}{12}} = \sqrt{\frac{12}{12}} = 1$$

50. A line makes angle θ with X-axis and Z-axis and β with Y-axis.

$$\therefore l = \cos \theta, m = \cos \beta, n = \cos \theta$$

$$\therefore l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \cos^2 \theta + \cos^2 \beta + \cos^2 \theta = 1$$

$$\Rightarrow 2 \cos^2 \theta + \cos^2 \beta = 1$$

$$\Rightarrow 2 \cos^2 \theta = 1 - \cos^2 \beta$$

$$\Rightarrow 2 \cos^2 \theta = \sin^2 \beta \quad \dots (i)$$

But it is given that, $\sin^2 \beta = 3 \sin^2 \theta \quad \dots (ii)$

From Eqs. (i) and (ii), we get

$$3 \sin^2 \theta = 2 \cos^2 \theta$$

$$\Rightarrow 3(1 - \cos^2 \theta) = 2 \cos^2 \theta$$

$$\Rightarrow 3 = 5 \cos^2 \theta$$

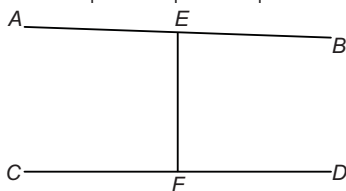
$$\Rightarrow \cos^2 \theta = \frac{3}{5}$$

51. As we know that, if two planes are parallel, then distance between them is the distance between $4x + 2y + 4z - 16 = 0$ and $4x + 2y + 4z + 5 = 0$.
 $\therefore$ Required distance

$$\left| \frac{5 + 16}{\sqrt{16 + 4 + 16}} \right| = \left| \frac{21}{\sqrt{36}} \right| = \frac{21}{6} = \frac{7}{2}$$

52. Let the equation of line AB be

$$\frac{x-0}{1} = \frac{y+a}{1} = \frac{z-0}{1} = k \quad [\text{say}]$$



Any point on the line is $F(k, k-a, k)$.

Also, the equation of other line CD is

$$\frac{x+a}{2} = \frac{y-0}{1} = \frac{z-0}{1} = \lambda \quad [\text{say}]$$

Any point on the line is $E(2\lambda - a, \lambda, \lambda)$.

Direction ratios of EF are

$$[(k-2\lambda+a), (k-a-\lambda), (k-\lambda)]$$

Since, it is given that direction ratios of EF are proportional to 2, 1, 2.

$$\therefore \frac{k-2\lambda+a}{2} = \frac{k-\lambda-a}{1} = \frac{k-\lambda}{2}$$

On solving first and second fractions, we get

$$k-2\lambda+a=2k-2\lambda-2a \Rightarrow k=3a \quad \dots(i)$$

On solving second and third fractions, we get

$$2k-2\lambda-2a=k-\lambda \Rightarrow k-\lambda=2a \Rightarrow \lambda=3a-2a \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \lambda=a$$

$\therefore$ Coordinates of E are $(3a, 2a, 3a)$ and coordinates of F are (a, a, a) .

53. Given, straight lines can be rewritten as

$$\frac{x-1}{1} = \frac{y+3}{-1} = \frac{z-1}{\lambda} = s$$

$$\text{and} \quad \frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{-2} = t$$

These two lines are coplanar, if

$$\begin{vmatrix} 1-0 & -3-1 & 1-2 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1 & -4 & -1 \\ 1 & -\lambda & \lambda \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$$\Rightarrow 1(2\lambda-2\lambda)+4(-2-\lambda)-1(2+\lambda)=0$$

$$\Rightarrow -8-4\lambda-2-\lambda=0$$

$$\Rightarrow -10=5\lambda \Rightarrow \lambda=-2$$

54. Given equation of two spheres are

$$x^2 + y^2 + z^2 + 7x - 2y - z - 13 = 0$$

$$\text{and} \quad x^2 + y^2 + z^2 - 3x + 3y + 4z - 8 = 0$$

If these spheres intersect, then $S - S' = 0$ represents the equation of common plane of intersection.

$$\therefore (x^2 + y^2 + z^2 + 7x - 2y - z - 13)$$

$$- (x^2 + y^2 + z^2 - 3x + 3y + 4z - 8) = 0$$

$$\Rightarrow 10x - 5y - 5z - 5 = 0$$

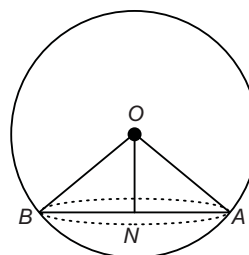
$$\Rightarrow 2x - y - z = 1$$

55. Given equation of sphere is

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0$$

whose centre is $(-1, 1, 2)$

$$\text{and} \quad \text{radius} = \sqrt{(-1)^2 + (1)^2 + (2)^2 + 19} = \sqrt{25} = 5$$



Also, equation of plane is

$$x + 2y + 2z + 7 = 0$$

Length of the perpendicular from centre O on the plane is

$$ON = \frac{-1 \times 1 + 1 \times 2 + 2 \times 2 + 7}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{12}{3} = 4$$

In $\triangle OAN$,

$$5^2 = 4^2 + AN^2$$

$$\Rightarrow AN^2 = 25 - 16$$

$$\Rightarrow AN^2 = 9$$

$$\therefore AN = 3$$

56. Given equations of sphere and plane are

$$x^2 + y^2 + z^2 + 4x - 2y - 6z - 155 = 0$$

$$\text{and} \quad 12x + 4y + 3z - 327 = 0$$

Centre of the sphere is $(-2, 1, 3)$

$$\text{and} \quad \text{radius} = \sqrt{4 + 1 + 9 + 155} = \sqrt{169} = 13$$

Length of the perpendicular from centre to the plane

$$= \frac{|-2 \times 12 + 1 \times 4 + 3 \times 3 - 327|}{\sqrt{(12)^2 + (4)^2 + 3^2}} = \frac{|-11 - 327|}{\sqrt{169}} = \frac{338}{13} = 26$$

Shortest distance between the plane and sphere

$$= 26 - \text{Radius of sphere}$$

$$= 26 - 13 = 13$$

57. Consider OX, OY, OZ and Ox, Oy, Oz are two system of rectangular axes.

Let their corresponding equations of plane be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(i)$$

$$\text{and} \quad \frac{x}{a'} + \frac{y}{b'} + \frac{z}{c'} = 1 \quad \dots(ii)$$

Since, length of perpendicular from origin to Eqs. (i) and (ii) must be same.

$$\therefore \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}} = \frac{1}{\sqrt{\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}}}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2}$$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} - \frac{1}{a'^2} - \frac{1}{b'^2} - \frac{1}{c'^2} = 0$$

58. A parallelopiped is formed by planes drawn through the points (2, 3, 5) and (5, 9, 7), parallel to the coordinate planes.

Let a, b, c be the lengths of edges, then

$$a = 5 - 2 = 3, b = 9 - 3 = 6 \text{ and } c = 7 - 5 = 2$$

So, the length of diagonal of a parallelopiped

$$\begin{aligned} &= \sqrt{a^2 + b^2 + c^2} \\ &= \sqrt{9 + 36 + 4} \\ &= \sqrt{49} = 7 \text{ units} \end{aligned}$$

59. The equation of plane containing the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \text{ is}$$

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

Since, normal to the plane is perpendicular to the given lines.

$$\therefore al + bm + cn = 0$$

60. The equation of a line through the centre $\hat{j} + 2\hat{k}$ and normal to the given plane is

$$r = \hat{j} + 2\hat{k} + \lambda(\hat{i} + 2\hat{j} + 2\hat{k}) \quad \dots(i)$$

This meets the plane at a point for which, we must have

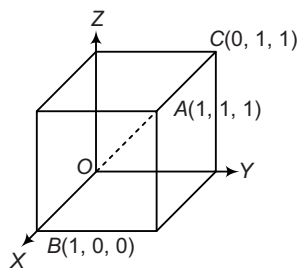
$$\begin{aligned} &[(\hat{j} + 2\hat{k}) + \lambda(\hat{i} + 2\hat{j} + 2\hat{k})] \cdot (\hat{i} + 2\hat{j} + 2\hat{k}) = 15 \\ \Rightarrow &2(1 + 2\lambda) + \lambda + 2(2 + 2\lambda) = 15 \\ \Rightarrow &6 + 9\lambda = 15 \\ \Rightarrow &\lambda = 1 \end{aligned}$$

On putting $\lambda = 1$ in Eq. (i), we get

$$r = \hat{i} + 3\hat{j} + 4\hat{k}$$

Hence, centre of circle is (1, 3, 4).

61. Let edge of a cube be 1 unit. The diagonals of a cube are OA and BC. So, DR's of diagonals OA and BC are (1, 1, 1) and (0 - 1, 1, 1) i.e. (-1, 1, 1).



Now, angle between the diagonals,

$$\begin{aligned} \cos \theta &= \frac{1(-1) + 1(1) + 1(1)}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{(-1)^2 + 1^2 + 1^2}} \\ &= \frac{1}{\sqrt{1^2 + 1^2 + 1^2} \sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}\sqrt{3}} = \frac{1}{3} \end{aligned}$$

$$\therefore \theta = \cos^{-1}(1/3)$$

62. Given direction cosines are $\frac{2}{3}, -\frac{a}{3}, \frac{2}{3}$.

Then, direction ratios are 2, -a, 2.

According to the question,

$$3 = \sqrt{2^2 + (-a)^2 + 2^2} \Rightarrow 9 = 8 + a^2$$

$$\Rightarrow a^2 = 1 \Rightarrow a = \pm 1$$

$$\Rightarrow a = 1$$

So, the required vector is $2\hat{i} - \hat{j} + 2\hat{k}$.

63. Given equation of plane is

$$2x - y + 2z - 1 = 0$$

So, DR's of normal to the given plane is (2, -1, 2).

$\therefore$ DC's of normal to the given plane is

$$\left(\frac{2}{\sqrt{2^2 + (-1)^2 + 2^2}}, \frac{-1}{\sqrt{2^2 + (-1)^2 + 2^2}}, \frac{2}{\sqrt{2^2 + (-1)^2 + 2^2}} \right)$$

$$\text{i.e. } \left(\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right)$$

Also, DC's of Z-axis is (0, 0, 1).

$\therefore$ Angle between normal to plane and Z-axis is

$$\cos \theta = \frac{2}{3} \times 0 - \frac{1}{3} \times 0 + \frac{2}{3} \times 1 = \frac{2}{3}$$

$$\therefore \theta = \cos^{-1}(2/3)$$

64. Given, direction cosines of two lines are

$$l + m + n = 0 \text{ and } l^2 - 5m^2 + n^2 = 0$$

$$\text{Also, } l^2 + m^2 + n^2 = 1$$

$$(l_1, m_1, n_1) = \left(-\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

$$\text{and } (l_2, m_2, n_2) = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}} \right)$$

$$\begin{aligned} \therefore \cos \theta &= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2} \sqrt{l_2^2 + m_2^2 + n_2^2}} \\ &= \frac{-\frac{2}{\sqrt{6}} \times \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{6}} \times \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}} \times \frac{2}{\sqrt{6}}}{1 \times 1} \\ &= \frac{-\frac{2}{6} + \frac{1}{6} - \frac{4}{6}}{1} = \frac{-\frac{5}{6}}{1} = -\frac{5}{6} \end{aligned}$$

$$\Rightarrow \cos \theta = -\frac{5}{6}$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\therefore \theta = 60^\circ = \frac{\pi}{3}$$

65. Given points are A(3, 4, 5), B(4, 6, 3), C(-1, 2, 4) and D(1, 0, 5). Now, DR's of DC = (-2, 2, -1)

DR's of AB = (1, 2, -2)

Let θ be the angle between AB and DC.

$$\begin{aligned} \therefore \cos \theta &= \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \\ &= \frac{-2 \times 1 + 2 \times 2 + (-1) \times (-2)}{\sqrt{(-2)^2 + (2)^2 + (-1)^2} \sqrt{(1)^2 + (2)^2 + (-2)^2}} \\ &= \frac{-2 + 4 + 2}{\sqrt{4 + 4 + 1} \sqrt{1 + 4 + 4}} = \frac{4}{3 \times 3} = \frac{4}{9} \end{aligned}$$

66. Since, angle between the lines whose direction ratios are (a_1, a_2, a_3) and (b_1, b_2, b_3) , is given by θ , where

$$\theta = \cos^{-1} \left(\frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} \sqrt{b_1^2 + b_2^2 + b_3^2}} \right)$$

Given direction ratios are (1, 1, 2) and $(\sqrt{3} - 1, -\sqrt{3} - 1, 4)$.

Hence, angle between the lines is

$$\begin{aligned} \theta &= \cos^{-1} \left\{ \frac{1 \cdot (\sqrt{3} - 1) + 1 \cdot (-\sqrt{3} - 1) + 2 \cdot 4}{\sqrt{1^2 + 1^2 + 2^2} \sqrt{(\sqrt{3} - 1)^2 + (-\sqrt{3} - 1)^2 + 4^2}} \right\} \\ &= \cos^{-1} \left\{ \frac{\sqrt{3} - 1 - \sqrt{3} - 1 + 8}{\sqrt{6} \sqrt{3 + 1 + 3 + 1 + 16}} \right\} + \cos^{-1} \left\{ \frac{6}{\sqrt{6} \sqrt{24}} \right\} \\ &= \cos^{-1} \left\{ \frac{6}{\sqrt{6} \sqrt{24}} \right\} = \cos^{-1} \left(\frac{1}{2} \right) = \cos^{-1} \left(\cos \frac{\pi}{3} \right) = \frac{\pi}{3} \end{aligned}$$

67. Let the points be $P = (4, 3, -5)$ and $Q = (-2, 1, -8)$.

$$\text{Now, } |PQ| = \sqrt{(-2-4)^2 + (1-3)^2 + (-8+5)^2} \\ = \sqrt{36 + 4 + 9} = \sqrt{49} = 7$$

∴ DC's of line PQ are

$$l = \frac{x_2 - x_1}{|PQ|}, m = \frac{y_2 - y_1}{|PQ|} \text{ and } n = \frac{z_2 - z_1}{|PQ|}$$

$$\therefore l = \frac{6}{7}, m = \frac{2}{7} \text{ and } n = \frac{3}{7}$$

68. Given lines are $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{\lambda}$... (i)

$$\text{and } \frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3} \text{ ... (ii)}$$

These lines lie in the same plane. So, both are coplanar.

$$\therefore \begin{vmatrix} 2 & 3 & \lambda \\ 3 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 2(-2+3) - 3(-3-3) + \lambda(-3-2) = 0$$

$$\Rightarrow 2 + 18 - 5\lambda = 0 \Rightarrow 5\lambda = 20 \Rightarrow \lambda = 4$$

$$\therefore \sin^{-1}(\sin \lambda) = \sin^{-1}[\sin 4] = \sin^{-1}[\sin(\pi - 4)] = \pi - 4$$

69. Obtain the foot of the perpendicular, we get $N(-4, 1, -3)$.

Equation of the line passing through $P(2, 4, -1)$ and $N(-4, 1, -3)$ is

$$\frac{x-2}{6} = \frac{y-4}{3} = \frac{z+1}{2}$$

70. We have, $\frac{x-1}{2} = \frac{y+1}{3} = \frac{z-1}{4} = r$

$$\Rightarrow x = 2r + 1, y = 3r - 1, z = 4r + 1$$

Since, the two lines intersect.

On putting above values in second line, we get

$$\frac{2r+1-3}{1} = \frac{3r-1-k}{2} = \frac{4r+1}{1}$$

Taking Ist and Illrd terms, we get

$$2r - 2 = 4r + 1 \Rightarrow r = -\frac{3}{2}$$

Also, taking IInd and Illrd terms, we get

$$3r - 1 - k = 8r + 2$$

$$\Rightarrow k = -5r - 3 = \frac{15}{2} - 3 = \frac{9}{2}$$

71. Given equation of lines are $x - 1 = \frac{2y+3}{3} = \frac{z+5}{2}$

$$\text{and } x = 3r + 2; y = -2r - 1; z = 2$$

$$\Rightarrow \frac{x-1}{1} = \frac{y+\frac{3}{2}}{\frac{3}{2}} = \frac{z+5}{2}$$

$$\text{and } \frac{x-2}{3} = r, \frac{y+1}{-2} = r, \frac{z-2}{0} = r$$

$$\Rightarrow \frac{x-1}{1} = \frac{y+\frac{3}{2}}{\frac{3}{2}} = \frac{z+5}{2} \text{ and } \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0}$$

$$\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z-2}{0}$$

So, DR's of Ist and IInd lines are $\left(1, \frac{3}{2}, 2\right)$ and $(3, -2, 0)$.

∴ The angle between two straight lines is

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \\ = \frac{1 \times 3 - \frac{3}{2} \times 2 + 2 \times 0}{\sqrt{1^2 + \left(\frac{3}{2}\right)^2 + 2^2} \cdot \sqrt{3^2 + 2^2 + 0^2}} \\ = \frac{3 - 3 + 0}{\sqrt{1 + \frac{9}{4} + 4} \cdot \sqrt{9 + 4}} = \frac{0}{\sqrt{\frac{29}{4}} \cdot \sqrt{13}}$$

$$\Rightarrow \cos \theta = 0$$

$$\Rightarrow \theta = \frac{\pi}{2}$$

72. The new position of plane is

$$(x - 2y + 3z) + \lambda(x + y + z - 1) = 0$$

$$\Rightarrow (1 + \lambda)x + (\lambda - 2)y + (3 + \lambda)z - \lambda = 0$$

Given that this is perpendicular to $x + y + z = 1$.

$$\therefore (1 + \lambda) \cdot 1 + (\lambda - 2) \cdot 1 + (3 + \lambda) \cdot 1 = 0$$

$$\Rightarrow 1 + \lambda + \lambda - 2 + 3 + \lambda = 0$$

$$\Rightarrow 3\lambda + 2 = 0$$

$$\Rightarrow \lambda = -\frac{2}{3}$$

Hence, the new position of the plane is

$$(x - 2y + 3z) - \frac{2}{3}(x + y + z - 1) = 0$$

$$\Rightarrow 3x - 2x - 6y - 2y + 9z - 2z + 2 = 0$$

$$\Rightarrow x - 8y + 7z = -2$$

73. Since, the points $(1, 2, 3)$ and $(2, -1, 0)$ lie on the opposite sides of the plane.

$$\therefore 2x + 3y - 2z - k = 0$$

$$\Rightarrow (2 + 6 - 6 - k)(4 - 3 - k) < 0$$

$$\Rightarrow (k - 1)(k - 2) < 0$$

$$\therefore 1 < k < 2$$

74. Given equation of the plane is

$$3x + 4y - 5z - 60 = 0$$

It can be written in the form

$$\frac{x}{20} + \frac{y}{15} + \frac{z}{-12} = 1$$

which meets the coordinate axes at the points $A(20, 0, 0)$, $B(0, 15, 0)$ and $C(0, 0, -12)$.

Since, the coordinates of the origin are $(0, 0, 0)$.

Therefore, volume of the tetrahedron OABC

$$= \frac{1}{6} \begin{vmatrix} 20 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & -12 \end{vmatrix} \\ = \frac{1}{6} |20 \times 15 \times (-12)| = 600$$

75. Let the equation of plane be $\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1$.

Then, $A(\alpha, 0, 0)$, $B(0, \beta, 0)$ and $C(0, 0, \gamma)$ are the points on the coordinate axes.

Since, the centroid of triangle is $(1, 2, 4)$.

$$\therefore \frac{\alpha}{3} = 1 \Rightarrow \alpha = 3 \\ \frac{\beta}{3} = 2 \Rightarrow \beta = 6$$

and $\frac{\gamma}{3} = 4 \Rightarrow \gamma = 12$

Hence, the equation of the plane is

$$\frac{x}{3} + \frac{y}{6} + \frac{z}{12} = 1 \Rightarrow 4x + 2y + z = 12$$

76. The equation of any plane through the intersection of the planes $3x - y + 2z - 4 = 0$ and $x + y + z - 2 = 0$ is

$$(3x - y + 2z - 4) + \lambda(x + y + z - 2) = 0 \quad \dots(i)$$

Since, the plane passes through the point $(2, 2, 1)$.

Therefore, this point will satisfy Eq. (i).

$$\therefore (3 \times 2 - 2 + 2 \times 1 - 4) + \lambda(2 + 2 + 1 - 2) = 0$$

$$\Rightarrow (6 - 4) + 3\lambda = 0$$

$$\Rightarrow 2 + 3\lambda = 0 \Rightarrow \lambda = \frac{-2}{3}$$

On substituting the value of λ in Eq. (i), we get the required plane as

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$$\Rightarrow 9x - 3y + 6z - 12 - 2x - 2y - 2z + 4 = 0$$

$$\Rightarrow 7x - 5y + 4z - 8 = 0$$

which is the required equation of the plane.

77. Here, the point is $(1, 0, 3)$ and equation of plane is $x + y + z = 1$.

Let α, β and γ be the image of plane.

$$\therefore \frac{\alpha - 1}{1} = \frac{\beta - 0}{1} = \frac{\gamma - 3}{1} = \frac{-2(1 + 0 + 3 - 1)}{1^2 + 1^2 + 1^2}$$

$$\Rightarrow \frac{\alpha - 1}{1} = \frac{\beta}{1} = \frac{\gamma - 3}{1} = \frac{-6}{3}$$

$$\Rightarrow \alpha = -1, \beta = -2, \gamma = 1$$

Hence, the image is $(-\hat{i} - 2\hat{j} + \hat{k})$.

78. Foot of the perpendicular drawn from the origin to the plane $ax + by + cz = d$ is

$$\left[\frac{ad}{a^2 + b^2 + c^2}, \frac{bd}{a^2 + b^2 + c^2}, \frac{cd}{a^2 + b^2 + c^2} \right]$$

Now, foot of perpendicular from the origin to the plane $2x - 3y + 4z = 29$ is

$$\left[\frac{2 \times 29}{2^2 + 3^2 + 4^2}, \frac{-3 \times 29}{2^2 + 3^2 + 4^2}, \frac{4 \times 29}{2^2 + 3^2 + 4^2} \right]$$

i.e. $(2, -3, 4)$

79. Coordinates of mid-point of $P(4, 5, -10)$ and $Q(-1, 2, 1)$ are

$$\left(\frac{4-1}{2}, \frac{5+2}{2}, \frac{-10+1}{2} \right) \text{ i.e. } \left(\frac{3}{2}, \frac{7}{2}, \frac{-9}{2} \right)$$

Now, DR's of PQ are $(-1 - 4, 2 - 5, 1 + 10)$

i.e. $(-5, -3, 11)$ or $(5, 3, -11)$

$\therefore$ Equation of plane passing through $\left(\frac{3}{2}, \frac{7}{2}, \frac{-9}{2} \right)$ and

having DR's $(5, 3, -11)$, is

$$5 \left(x - \frac{3}{2} \right) + 3 \left(y - \frac{7}{2} \right) - 11 \left(z + \frac{9}{2} \right) = 0$$

$$\Rightarrow 5x + 3y - 11z = \frac{15}{2} + \frac{21}{2} + \frac{99}{2}$$

$$\Rightarrow 5x + 3y - 11z = \frac{135}{2}$$

So, it is written in vector form as

$$r \cdot (5\hat{i} + 3\hat{j} - 11\hat{k}) = \frac{135}{2}$$

80. Clearly, the equation of plane passing through $(2, 2, 1)$ is given by

$$A(x - 2) + B(y - 2) + C(z - 1) = 0$$

where, A, B and C are direction ratios of normal to the plane.

Since, $(9, 3, 6)$ also lies on the plane

$$\therefore 7A + B + 5C = 0 \quad \dots(i)$$

Also, the plane is perpendicular to the plane

$$2x + 6y + 6z = 1$$

$$\therefore 2A + 6B + 6C = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\frac{A}{6-30} = \frac{-B}{42-10} = \frac{C}{42-2}$$

$$\Rightarrow \frac{A}{-24} = \frac{-B}{32} = \frac{C}{40}$$

$$\Rightarrow \frac{A}{-6} = \frac{B}{-8} = \frac{C}{10}$$

$\therefore$ Equation of required plane is given by

$$-6(x - 2) - 8(y - 2) + 10(z - 1) = 0$$

$$\Rightarrow -6x + 12 - 8y + 16 + 10z - 10 = 0$$

$$\Rightarrow -3x + 6 - 4y + 8 + 5z - 5 = 0$$

$$\Rightarrow 3x + 4y - 5z - 9 = 0$$

81. Given equation of plane is $x - y + z = 5$.

The distance measured along the line $x = y = z$.

So, the direction ratios of the given line is $(1, 1, 1)$.

$\therefore$ Equation of line PQ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda \quad [\text{say}]$$

$$\Rightarrow x = \lambda + 1, y = \lambda - 5, z = \lambda + 9$$

lies on the plane $x - y + z = 5$.

$$\Rightarrow \lambda + 1 - \lambda + 5 + \lambda + 9 = 5$$

$$\Rightarrow \lambda = -10$$

So, the coordinates of Q is $(-9, -15, -1)$ and the coordinates of P is $(1, -5, 9)$.

Now,

$$PQ = \sqrt{(1+9)^2 + (-5+15)^2 + (9+1)^2}$$

$$= \sqrt{100 + 100 + 100} = 10\sqrt{3}$$

$$\therefore 2\sqrt{3}k = 10\sqrt{3} \Rightarrow k = 5$$

82. The straight line joining the points $(1, 1, 2)$ and $(3, -2, 1)$ is

$$\frac{x-1}{2} = \frac{y-1}{-3} = \frac{z-2}{-1} = r \quad [\text{say}]$$

Then, the point is $(2r + 1, 1 - 3r, 2 - r)$

which lies on $3x + 2y + z = 6$

$$\therefore 3(2r + 1) + 2(1 - 3r) + 2 - r = 6$$

$$\Rightarrow r = 1$$

So, the required point is $(3, -2, 1)$.

83. Equation of plane passing through $(2, 3, 4)$ is

$$a(x - 2) + b(y - 3) + c(z - 4) = 0 \quad \dots(i)$$

Since, above plane is perpendicular to the line.

$$\therefore \frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$

Thus, normal to the plane is parallel to the line.

So, DR's of normal are $(1, 2, 3)$ i.e. $(a, b, c) = (1, 2, 3)$

From Eq. (i),

$$1(x - 2) + 2(y - 3) + 3(z - 4) = 0$$

$$\Rightarrow x + 2y + 3z = 20$$

84. Given equations of planes are

$$x - 2y - z + 5 = 0 \quad \text{and} \quad x + y + 3z = 6$$

Firstly, determine the intersection lines of two planes.

Let the DR's of intersection line be a, b and c .

Since, the normal to the given planes are perpendicular to the intersecting line.

$$\therefore a(1) + b(-2) + c(-1) = 0$$

$$\Rightarrow a - 2b - c = 0 \quad \dots(i)$$

$$\text{and} \quad a(1) + b(1) + c(3) = 0$$

$$\Rightarrow a + b + 3c = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{a}{-6+1} = \frac{b}{-1-3} = \frac{c}{1+2} \Rightarrow \frac{a}{-5} = \frac{b}{-4} = \frac{c}{3}$$

Since, the required line is passing through $(2, 3, 1)$ and parallel to the line of intersection.

$$\therefore \frac{x-2}{-5} = \frac{y-3}{-4} = \frac{z-1}{3}$$

85. We know that, if any line is parallel to the plane, then normal to the plane is perpendicular to the line.

$$\text{Given line is } \frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$$

So, DR's of a line are $(3, 4, 5)$.

Let us consider the equation of plane be

$$2x + y - 2z = 0$$

So, DR's of a plane are $(2, 1, -2)$.

$$\text{Now } 3 \times 2 + 4 \times 1 + 5 \times (-2) = 6 + 4 - 10 = 10 - 10 = 0$$

86. Given, plane passing through $(3, 2, 0)$ is

$$A(x-3) + B(y-2) + C(z-0) = 0 \quad \dots(i)$$

Plane (i) is passing through the line

$$\frac{x-4}{1} = \frac{y-7}{5} = \frac{z-4}{4}$$

$$\therefore A(4-3) + B(7-2) + C(4-0) = 0$$

$$\Rightarrow A + 5B + 4C = 0 \quad \dots(ii)$$

$$\text{Also, } A + 5B + 4C = 0 \quad \dots(iii)$$

Since, Eqs. (ii) and (iii) are same, so there are many planes which passes through the point $(3, 2, 0)$ and given line.

But from given options, point $(3, 2, 0)$ satisfies the plane $x - y + z = 1$.

So, the required equation of plane is $x - y + z = 1$.

87. We have two spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 7x - 2y - z = 13$$

$$\text{and } S_2 \equiv x^2 + y^2 + z^2 - 3x + 3y + 4z = 8$$

So, the intersection of S_1 and S_2 is given by $S_1 - S_2 = 0$

$$\therefore 7x + 3x - 2y - 3y - z - 4z = 13 - 8$$

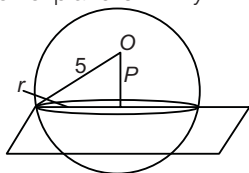
$$\Rightarrow 10x - 5y - 5z = 5$$

$$\Rightarrow 2x - y - z = 1 \quad [\text{dividing by 5}]$$

88. Given equation of sphere is

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 19 = 0 \quad \dots(i)$$

Its centre $= (-1, 1, 2)$, radius $= \sqrt{1+1+4+19} = \sqrt{25} = 5$ and equation of plane is $x + 2y + 2z + 7 = 0$. $\dots(ii)$



Length of the perpendicular from point $(-1, 1, 2)$ to the plane (ii)

$$= \frac{|-1+2+4+7|}{\sqrt{1+4+4}} = \frac{12}{3} = 4$$

$$\therefore \text{Radius of circle} = \sqrt{(5)^2 - (4)^2} = \sqrt{25-16} = \sqrt{9} = 3$$

89. The position vector of P of line $r = a + \lambda b$ meets the

$$\text{plane } r \cdot \hat{n} = p, \text{ is } a + \left(\frac{p - a \cdot \hat{n}}{b \cdot \hat{n}} \right) b.$$

90. Given vectors are $a = \hat{i} + 2\hat{j} + \hat{k}$, $b = \hat{i} - \hat{j} + \hat{k}$

$$\text{and } c = \hat{i} - \hat{j} - \hat{k}.$$

Any vector in the plane of a and b is

$$r = a + \lambda b = \hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$= (1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (1+\lambda)\hat{k}$$

The projection of r on c is $\frac{r \cdot c}{|c|} = \frac{1}{\sqrt{3}}$.

$$\therefore \frac{[(1+\lambda)\hat{i} + (2-\lambda)\hat{j} + (1+\lambda)\hat{k}] \cdot (\hat{i} - \hat{j} - \hat{k})}{\sqrt{1^2 + 1^2 + 1^2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \frac{1+\lambda-2+\lambda-1-\lambda}{\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \lambda-2=1 \Rightarrow \lambda=3$$

$$\therefore r = (1+3)\hat{i} + (2-3)\hat{j} + (1+3)\hat{k} = 4\hat{i} - \hat{j} + 4\hat{k}$$

91. Given skew-lines are

$$r_1 = (\hat{i} + 2\hat{j} + 3\hat{k}) + t(\hat{i} + 3\hat{j} + 2\hat{k})$$

$$\text{and } r_2 = (4\hat{i} + 5\hat{j} + 6\hat{k}) + t(2\hat{i} + 3\hat{j} + \hat{k})$$

$$\text{Here, } a_1 = \hat{i} + 2\hat{j} + 3\hat{k}, b_1 = \hat{i} + 3\hat{j} + 2\hat{k}$$

$$a_2 = 4\hat{i} + 5\hat{j} + 6\hat{k} \quad \text{and} \quad b_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\text{Now, } a_2 - a_1 = 4\hat{i} + 5\hat{j} + 6\hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$$

$$= 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\text{and } b_1 \times b_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (3-6)\hat{i} - (1-4)\hat{j} + (3-6)\hat{k}$$

$$= -3\hat{i} + 3\hat{j} - 3\hat{k}$$

$\therefore$ Shortest distance between skew-lines

$$= \frac{|(a_2 - a_1) \cdot (b_1 \times b_2)|}{|b_1 \times b_2|}$$

$$= \frac{|(3\hat{i} + 3\hat{j} + 3\hat{k}) \cdot (-3\hat{i} + 3\hat{j} - 3\hat{k})|}{\sqrt{(-3)^2 + (3)^2 + (-3)^2}}$$

$$= \frac{|-9+9-9|}{\sqrt{9+9+9}} = \frac{-9}{3\sqrt{3}} = \sqrt{3}$$

92. Given lines are $r_1 = 2\hat{i} + 3\hat{j} + 3\hat{k} + \lambda(\hat{i} + 4\hat{j} + 3\hat{k})$

$$\text{and } r_2 = (\hat{i} - \hat{j} + 2\hat{k}) + \mu(\hat{i} + 2\hat{j} - 3\hat{k})$$

Here, DR's of given lines are $(1, 4, 3)$ and $(1, 2, -3)$.

$\therefore$ Angle between these lines is

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{1 \times 1 + 4 \times 2 + 3 \times (-3)}{\sqrt{1^2 + 4^2 + 3^2} \sqrt{1^2 + 2^2 + (-3)^2}}$$

$$= \frac{1+8-9}{\sqrt{1+16+9} \sqrt{1+4+9}} = 0$$

$$\therefore \theta = \frac{\pi}{2}$$

93. Equations of straight line and the plane are

$$r = (\hat{i} - \hat{j} + \hat{k}) + \lambda (2\hat{i} + \hat{j} + 3\hat{k})$$

and $r \cdot (2\hat{i} + \hat{j} - \hat{k}) = 4$ respectively.

On comparing the equations with

$$r = a + \lambda b \text{ and } r \cdot n = d, \text{ we get}$$

$$b = 2\hat{i} + \hat{j} + 3\hat{k} \text{ and } n = 2\hat{i} + \hat{j} - \hat{k}$$

$\therefore$ Angle between line and plane is given by

$$\begin{aligned} \sin \theta &= \frac{|b \cdot n|}{|b||n|} \\ &= \frac{|2 \cdot 2 + 1 \cdot 1 + 3 \cdot (-1)|}{\sqrt{4+1+9}\sqrt{4+1+1}} \\ &= \frac{2}{\sqrt{14}\sqrt{6}} = \frac{1}{\sqrt{21}} \end{aligned}$$

$$\Rightarrow \theta = \sin^{-1}\left(\frac{1}{\sqrt{21}}\right)$$

94. Equation of line $r = \hat{i} - \hat{j} + \hat{k} + t(\hat{i} + \hat{j} - \hat{k})$ and equation of plane $r \cdot (\hat{i} + \hat{j} + \hat{k}) = 5$ are intersecting.

$$\therefore [(\hat{i} - \hat{j} + \hat{k}) + t(\hat{i} + \hat{j} - \hat{k})] \cdot [\hat{i} + \hat{j} + \hat{k}] = 5$$

$$\Rightarrow (1 - 1 + 1) + t(1 + 1 - 1) = 5 \Rightarrow t = 5 - 1 = 4$$

Put the value of t in equation of line, we get

$$r = (\hat{i} - \hat{j} + \hat{k}) + 4(\hat{i} + \hat{j} - \hat{k}) = 5\hat{i} + 3\hat{j} - 3\hat{k}$$

95. Here, $\frac{v \cdot c}{|c|} = \frac{1}{\sqrt{3}} \Rightarrow \frac{v \cdot c}{\sqrt{1+1+1}} = \frac{1}{\sqrt{3}} \Rightarrow \frac{v \cdot c}{\sqrt{3}} = \frac{1}{\sqrt{3}}$

$$\Rightarrow v \cdot c = 1 \Rightarrow v \cdot (\hat{i} - \hat{j} - \hat{k}) = 1 \quad \dots(i)$$

From option (b), $v = 3\hat{i} - \hat{j} + 3\hat{k}$

$$\therefore v \cdot c = (3\hat{i} - \hat{j} + 3\hat{k}) \cdot (\hat{i} - \hat{j} - \hat{k}) = 3 + 1 - 3 = 1$$

which satisfy Eq. (i).

$$\therefore v = 3\hat{i} - \hat{j} + 3\hat{k}$$

96. Let $\frac{x-5}{3} = \frac{y-7}{-1} = \frac{z+2}{1} = k$ [say]

Any point on the line is $P(3k+5, -k+7, k-2)$.

To determine the intersection point, equation second satisfying the point $P(3k+5, -k+7, k-2)$.

$$\therefore \frac{3k+5+3}{-36} = \frac{-k+7-3}{2} = \frac{k-2-6}{4}$$

$$\Rightarrow \frac{3k+8}{-36} = \frac{-k+4}{2} \Rightarrow k = \frac{16}{3}$$

$$\therefore \text{Point is } P\left(21, \frac{5}{3}, \frac{10}{3}\right).$$

97. Equation of line passing through (x_1, y_1, z_1) and parallel to the line $\frac{x-x_2}{a} = \frac{y-y_2}{b} = \frac{z-z_2}{c}$, is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

$$\therefore \text{Equation of line is } \frac{x-2}{4} = \frac{y+1}{-3} = \frac{z-1}{5}.$$

98. Equation of line passing through a and b is $a + t(b-a)$.

$$\text{Here, } b-a = 6\hat{j} - \hat{k}$$

$$\Rightarrow b = 6\hat{j} - \hat{k} + \hat{i} - 2\hat{j} - \hat{k} = \hat{i} + 4\hat{j} - 2\hat{k}$$

Hence, given line passes through $(1, -2, -1)$ and $(1, 4, -2)$.

99. Since, shortest distance of the point (a, b, c) from X, Y and Z-axes are respectively

$$\sqrt{b^2 + c^2}, \sqrt{a^2 + c^2}, \sqrt{a^2 + b^2}$$

Hence, the shortest distance of the point $(1, 2, 3)$ from X, Y and Z-axes respectively are $\sqrt{13}, \sqrt{10}, \sqrt{5}$.

100. If (a_1, b_1, c_1) and (a_2, b_2, c_2) are DR's of two lines, then DC's of a perpendicular to both the lines are

$$\frac{b_1c_2 - b_2c_1}{\sqrt{(b_1c_2 - b_2c_1)^2 + (a_2c_1 - a_1c_2)^2 + (a_1b_2 - a_2b_1)^2}},$$

$$\frac{a_2c_1 - a_1c_2}{\sqrt{(b_1c_2 - b_2c_1)^2 + (a_2c_1 - a_1c_2)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$\text{and } \frac{a_1b_2 - a_2b_1}{\sqrt{(b_1c_2 - b_2c_1)^2 + (a_2c_1 - a_1c_2)^2 + (a_1b_2 - a_2b_1)^2}}$$

Putting the values of a_1, b_1, c_1 and a_2, b_2, c_2 , we get

$$\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}$$

101. The plane $ax + by + cz = 1$ meets the coordinate axes at A, B and C. The coordinates of A, B and C are $\left(\frac{1}{a}, 0, 0\right), \left(0, \frac{1}{b}, 0\right)$ and $\left(0, 0, \frac{1}{c}\right)$, respectively.

Hence, the centroid of ΔABC is $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$.

102. The image of the point (x_1, y_1, z_1) with respect to the plane $ax + by + cz - d = 0$ is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = -2 \left(\frac{ax_1 + by_1 + cz_1 - d}{a^2 + b^2 + c^2} \right)$$

where, x, y and z are the images of the point.

Put the values of x_1, y_1, z_1 and a, b, c , we get

$$x = -3, y = 5 \text{ and } z = 2$$

103. An equation of plane is

$$lx + my + nz = d$$

where, l, m, n are direction cosines and d is perpendicular distance from origin.

$$\therefore d = \frac{6}{7}$$

$$\therefore lx + my + nz = \frac{6}{7} \quad \dots(i)$$

Since, Eq. (i) passes through the points $(1, 0, 0)$ and $(0, 2, 0)$.

$$\therefore l = \frac{6}{7} \text{ and } m = \frac{3}{7}$$

We know that, $l^2 + m^2 + n^2 = 1$

$$\Rightarrow \left(\frac{6}{7}\right)^2 + \left(\frac{3}{7}\right)^2 + n^2 = 1 \Rightarrow n = \frac{2}{7}$$

Hence, equation of the plane is

$$\frac{6}{7}x + \frac{3}{7}y + \frac{2}{7}z = \frac{6}{7}$$

$$\Rightarrow 6x + 3y + 2z - 6 = 0$$

104. Equation of plane contains the point (x_1, y_1, z_1) and perpendicular to each of the plane $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$, is

$$(b_1c_2 - b_2c_1)(x - x_1) + (a_2c_1 - a_1c_2)(y - y_1) + (a_1b_2 - a_2b_1)(z - z_1) = 0$$

Here, $x_1 = 1, y_1 = -1, z_1 = 2$

$$a_1 = 2, b_1 = 3, c_1 = -2$$

$$\text{and } a_2 = 1, b_2 = 2, c_2 = -3$$

Put these values, we get

$$(-9 + 4)(x - 1) + (-2 + 6)(y + 1) + (4 - 3)(z - 2) = 0$$

$$\Rightarrow 5x - 4y - z = 7$$

- 105.** The equation of the plane perpendicular to Z-axis and passing through (x_1, y_1, z_1) , is $z - z_1 = 0$.

$$\therefore z - 5 = 0$$

- 106.** Using the formula,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (a_1c_2 - a_2c_1)^2 + (a_1b_2 - a_2b_1)^2}}$$

$$= \frac{\begin{vmatrix} -4 & 16 & 0 \\ 2 & -7 & 5 \\ 2 & 1 & -3 \end{vmatrix}}{\sqrt{(21-5)^2 + (-6-10)^2 + (2+14)^2}}$$

$$\Rightarrow d = \frac{48}{\sqrt{3}} = 16\sqrt{3}$$

- 107.** Equation of sphere, whose centre $(1, 2, 3)$ and radius 3 units, is

$$(x - 1)^2 + (y - 2)^2 + (z - 3)^2 = (3)^2$$

$$\Rightarrow x^2 + y^2 + z^2 - 2x - 4y - 6z + 5 = 0$$

- 108.** Centre of sphere = $(2, 2, 1)$

Since, equation of plane $2x - 2y + z + 5 = 0$ is tangent of sphere.

$$\therefore r = \frac{|2(2) - 2(2) + 1 + 5|}{\sqrt{(2)^2 + (-2)^2 + (1)^2}}$$

$$= \frac{|4 - 4 + 6|}{\sqrt{9}} = \frac{6}{3} = 2$$

- 109.** Angle between the line $r = a + \lambda b$ and the plane $r \cdot n = d$ is

$$\sin \theta = \frac{|b \cdot n|}{|b||n|}$$

$$\therefore \sin \theta = \frac{|(2\hat{i} + 3\hat{j} + 4\hat{k}) \cdot (\hat{i} + 2\hat{j} - 2\hat{k})|}{\sqrt{(2)^2 + (3)^2 + (4)^2} \sqrt{(1)^2 + (2)^2 + (-2)^2}}$$

$$\Rightarrow \sin \theta = \frac{|2 + 6 - 8|}{\sqrt{29}\sqrt{9}}$$

$$\Rightarrow \sin \theta = 0$$

$$\therefore \theta = 0^\circ$$

- 110.** Given planes are $r \cdot (2\hat{i} - \hat{j} + 3\hat{k}) - 4 = 0$

$$\text{and } r \cdot (2\hat{i} - \hat{j} + 3\hat{k}) + \frac{13}{3} = 0$$

Since, they are parallel to each other.

$$\therefore \text{Required distance} = \frac{\left| \frac{13}{3} + 4 \right|}{\sqrt{2^2 + (-1)^2 + (3)^2}}$$

$$= \frac{25}{3\sqrt{14}}$$

- 111.** Centre of the given sphere

$$r^2 - 2r(3\hat{i} + 4\hat{j} - 5\hat{k}) + 1 = 0 \text{ is } (3\hat{i} + 4\hat{j} - 5\hat{k})$$

$$\text{and radius is } \sqrt{3^2 + 4^2 + 5^2 - 1} = 7$$

- 112.** Equation of required plane is

$$(x + y + z - 6) + \lambda(2x + 3y + 4z + 5) = 0$$

$$\text{which is passing through the point } (1, 1, 1).$$

$$\Rightarrow -3 + 14\lambda = 0$$

$$\Rightarrow \lambda = \frac{3}{14}$$

Hence, the required plane is $20x + 23y + 26z = 69$.

- 113.** Equation of plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

$$\text{Also, } 3 = \frac{a + 0 + 0}{3}$$

$$\Rightarrow a = 9$$

Similarly, $b = 6$ and $c = 3$

$\therefore$ Equation of required plane is

$$\frac{x}{9} + \frac{y}{6} + \frac{z}{3} = 1$$

$$\Rightarrow 2x + 3y + 6z = 18$$

- 114.** Equation of line which is passing through (α, β, γ) and perpendicular to the plane $ax + by + cz + d = 0$, is
- $$\frac{x - \alpha}{a} = \frac{y - \beta}{b} = \frac{z - \gamma}{c}$$

- 115.** Equation of plane is $a(x - 1) + b(y + 1) + cz = 0$

$\therefore$ plane is passing through $(1, -1, 0)$

Above plane also passing through $(0, 2, -1)$.

$$\therefore -a + 3b - c = 0$$

$$\text{Also, } 2a - b + 3c = 0$$

$$\Rightarrow \frac{a}{8} = \frac{b}{1} = \frac{c}{-5}$$

Hence, equation of plane is

$$8x + y - 5z - 7 = 0$$

- 116.** Given line can be rewritten as

$$\frac{x - \frac{1}{3}}{\frac{2b}{3}} = \frac{y - 3}{-1} = \frac{z - 1}{a}$$

and plane $3x + y + 2z + 6 = 0$ is parallel to the above line.

$$\therefore \frac{2b}{3} \cdot 3 + 1 \cdot (-1) + 2 \cdot a = 0$$

$$\Rightarrow 2a + 2b = 1$$

$$\Rightarrow 3a + 3b = \frac{3}{2}$$

- 117.** Direction ratios of normal to the given plane are $(2, -3, 5)$, which is the direction ratios of line passing through $(3, 0, -4)$.

$\therefore$ Equation of required line is

$$\frac{x - 3}{2} = \frac{y - 0}{-3} = \frac{z + 4}{5}$$

$$\Rightarrow \frac{x - 3}{2} = \frac{-y}{3} = \frac{z + 4}{5}$$

118. Equation of plane passing through $P(3, 8, 2)$ and parallel to $3x + 2y - 2z + 15 = 0$ is

$$3(x - 3) + 2(y - 8) - 2(z - 2) = 0$$

$\Rightarrow$

$$3x + 2y - 2z - 21 = 0$$

Given line is

$$\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3} = r$$

...(i)

[say]

Any point on the line is $Q(2r + 1, 4r + 3, 3r + 2)$.

Since, this point lies on the above plane.

$\therefore$

$$3(2r + 1) + 2(4r + 3) - 2(3r + 2) - 21 = 0$$

$\Rightarrow$

$$8r - 16 = 0$$

$\Rightarrow$

$$r = 2$$

So, coordinates of Q are $(5, 11, 8)$.

$\therefore$ Distance between P and Q

$$\begin{aligned} &= \sqrt{(5-3)^2 + (11-8)^2 + (8-2)^2} \\ &= \sqrt{4 + 9 + 36} = 7 \end{aligned}$$

119. Given line is $\frac{x-1}{-3} = \frac{y+1}{-2} = \frac{z-3}{1}$

So, line is passing through $(1, -1, 3)$ and having direction ratios $-3, -2, 1$ i.e. $3, 2, -1$.

$\therefore$ Vector equation of the line is

$$r = (\hat{i} - \hat{j} + 3\hat{k}) + \lambda(3\hat{i} + 2\hat{j} - \hat{k})$$

15

Linear Programming

Introduction

The main aim of any industrial organisation is to optimise the use of scarce resources which include machine, manpower, money, warehouse space, raw material, etc. One of the several tools used for this purpose is linear programming. The term ‘programming’ means planning and the term ‘linear’ implies that all equations or inequations involved in a problem would be linear. The technique of linear programming first invented by the Russian Mathematician L. V. Kantorovich and developed later by George B. Dantzig helps in choosing a particular option from amongst the several alternatives available for maximising profit or minimising costs. A linear programming problem which we shall abbreviate as ‘LPP’ involves a linear function of a number of variables in the form of equations or inequations, subject to certain conditions on the variables and the task is to maximise or minimise the function. In this chapter, our discussion will be confined to solve the problems in LPP in two variables only.

Linear Programming

Linear programming is a method of solving problems in which a quantity represented by a linear equation, often profit or cost, is to be maximised or minimised subject to condition expressed by a system of linear inequalities.

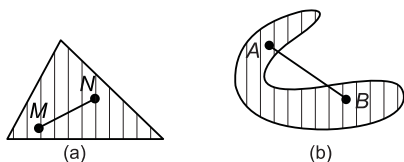
Some Definitions and Concepts Related to LPP

- (i) **Objective function** The linear function $ax + by + c$ whose maximum or minimum value is determined is called the objective function.
- (ii) The word optimise means to maximise or minimise.
- (iii) **Constraints** The system of inequations or equations involving the variables of a LPP which describe the conditions under which the optimisation is to be attained are called constraints. The constraints in a LPP involve the signs $\leq, =, \geq$.
- (iv) **Non-negativity restriction** These are the constraints which imply that the variables involved in a LPP are non-negative.
- (v) **Convex region** A region is said to be convex, if whenever you choose any two points in the region and draw the segment joining them, the segment is

Chapter Snapshot

- Introduction
- Linear Programming
- Some Definitions and Concepts Related to LPP
- General Linear Programming Problems
- Graphical Solution of Linear Inequations
- Graph of Linear Inequalities
- Formulation of a Linear Programming Problem
- Graphical Method for Solution of Linear Programming Problems
- Solution of Linear Equations by Determinant

contained in the region e.g. The shaded region in Fig. (a) is convex, while the shaded region in Fig. (b) is not.



- (vi) **Feasible region and feasible solution** The common region determined by all the constraints of an LPP is called a feasible region and every point in this region is called as feasible solution to LPP. The feasible region is always a convex set.
- (vii) **Infeasible solution** A solution of an LPP is an infeasible solution, if the system of constraints has no point which satisfies all the constraints and non-negativity restrictions.
- (viii) **Optimal feasible solution** A feasible solution of an LPP is said to be an optimal feasible solution, if it also optimises (maximises or minimises) the objective function.

➤ **Example 1.** Which of the following is not a convex set?

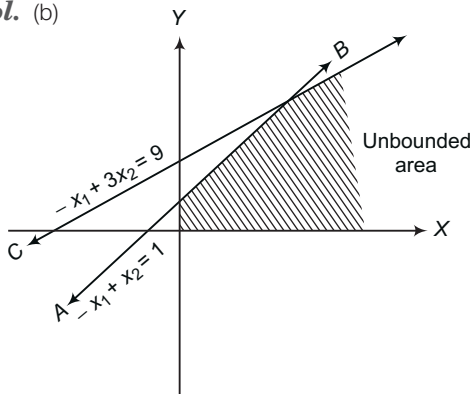
- (a) $\{(x, y) : 2x + 5y < 7\}$
 (b) $\{(x, y) : x^2 + y^2 \leq 4\}$
 (c) $\{x : |x| = 5\}$
 (d) $\{(x, y) : 3x^2 + 2y^2 \leq 6\}$

Sol. (c) A set S is called convex, if for all $x_1, x_2 \in S$ and $0 \leq \lambda \leq 1$, $\lambda x_1 + (1 - \lambda) x_2 \in S$. The set $\{x : |x| = 5\}$ is not a convex set, can be seen by drawing the region.

➤ **Example 2.** The constraints $-x_1 + x_2 \leq 1$, $-x_1 + 3x_2 \leq 9$ and $x_1, x_2 \geq 0$ defines on

- (a) bounded feasible space
 (b) unbounded feasible space
 (c) both bounded and unbounded feasible space
 (d) None of the above

Sol. (b)



General Linear Programming Problems

The mathematical statement of the general form of a linear programming problem (abbreviated LPP) may be written as follows:

Optimise (maximise or minimise)

$$Z = c_1x_1 + c_2x_2 + \dots + c_nx_n$$

subject to constraints are

$$\left. \begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &(\leq, =, \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &(\leq, =, \geq) b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &(\leq, =, \geq) b_m \end{aligned} \right\} \dots(i)$$

$$\text{and } x_1, x_2, \dots, x_n \geq 0 \quad \dots(ii)$$

where,

- (i) $x_1, x_2, \dots, x_n$ are the variables whose values we wish to determine and are called the decision or structural variables.
- (ii) The linear function Z which is to be maximised or minimised is called the objective function of the general LPP.
- (iii) The inequalities (i) are called the constraints of the general LPP.
- (iv) The set of inequalities (ii) is known as the set of non-negative restrictions of the general LPP.
- (v) The constant c_j ($j = 1, 2, \dots, n$) represents the contribution (profit or cost) to the objective function of the j th variable.
- (vi) The coefficients a_{ij} ($i = 1, 2, \dots, m; j = 1, 2, \dots, n$) are referred to as the technological or substitution coefficients.
- (vii) b_i ($i = 1, 2, \dots, m$) is the constant representing the requirement or availability of the j th constraint.
- (viii) The expression $(\leq, =, \geq)$ means that only one of the relationship in the set $(\leq, =, \geq)$ should hold for a particular constraint.

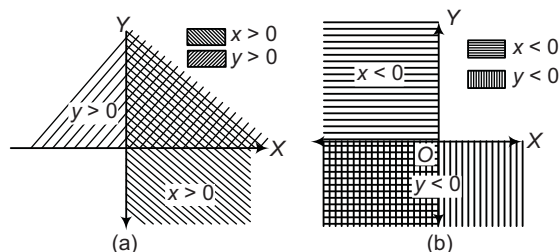
Graphical Solution of Linear Inequations

A knowledge of the graphical solution of linear inequalities is a pre-requisite to the study of problems of linear programming. We therefore, first tell you how to find the graphical solution of a system of linear inequations in two variables.

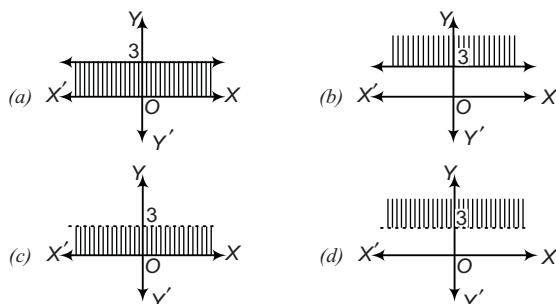
Graph of Linear Inequalities

Consider, inequalities of the form $ax + by > c$, $ax + by \geq c$, $ax + by < c$ and $ax + by \leq c$ (i.e. $4x + 3y > 2$, $3x - 5y \leq 13$, etc). ' $ax + by = c$ ' is called the associated equation of each of these inequalities.

The graph of an inequation in two variables is a region of the XY -plane. In Fig. (a), the area to the right of the X -axis represents the inequation $x > 0$ and the area above the X -axis represents the inequation $y > 0$. In Fig. (b), are shown the regions representing inequation $x < 0$ and $y < 0$. In Fig. (a) and Fig. (b), if we include the Y -axis in the graph, the shaded regions will then represent respectively the inequations $x \geq 0$ and $x \leq 0$. Similarly, on including the X -axis in the graph, we will get regions representing the inequations $y \geq 0$ and $y \leq 0$. The double shaded region represents the graph satisfying both the inequations.

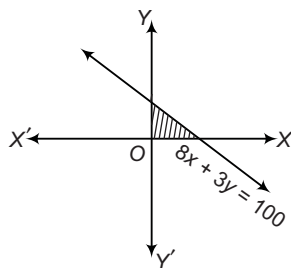


➤ **Example 3.** The graph of $y < 3$ is



Sol. (c) First draw the graph of $y = 3$. Since, given inequality is strict inequality. So, dotted line is drawn. Now, consider the point $O(0, 0)$, putting $y = 0$ in given inequality, we get $0 < 3$, which is true. $\therefore$ Solution region is the shaded region is below the line $y = 3$.

➤ **Example 4.** Which one of the following represent the shaded region in the figure?



(a) $8x + 3y \geq 100, x \geq 0, y \geq 0$

(b) $8x + 3y \leq 100, x \geq 0, y \geq 0$

(c) $8x + 3y \geq 100$

(d) $8x + 3y \leq 100$

Sol. (b) Consider, $8x + 3y = 100$. We observe that the shaded region and the origin lie on the same side of the line and $(0, 0)$ satisfies $8x + 3y \leq 100$. Therefore, we must have $8x + 3y \leq 100$ as linear inequality corresponding to the line $8x + 3y = 100$. Also, shaded region lies in the first quadrant only, therefore $x \geq 0, y \geq 0$. Hence, the linear equalities corresponding to the given solution are $8x + 3y \leq 100, x \geq 0, y \geq 0$.

Formulation of a Linear Programming Problem

The formulation of linear programming problem as a mathematical model involves the following steps:

- Step I** Identify the decision variables and assign symbols x and y to them. These decision variables are those quantities whose values we wish to determine.
- Step II** Identify all the constraints and express them as linear equations/inequations in terms of the decision variables. These constraints are the decision variables.
- Step III** Identify the objective function and express it as a linear function of decision variables. It might take the form of maximising profit or production or minimising cost.
- Step IV** Add the non-negativity restrictions on the decision variables, as in the physical problems. Negative values of decision variables have no valid interpretation.

➤ **Example 5.** A furniture dealer, deals in only tables and chairs. He can invest upto ₹ 50000 only and has a storage capacity of 100 pieces. The cost price of a table is ₹ 1200 and of a chair is ₹ 500. He can earn a profit of ₹ 180 on the sale of the table and ₹ 75 on the sale of one chair. Assuming that he can sell all the items he buys. If x is the number of tables and y be the number of chairs, then the given problem, mathematically reduces to,

(a) Maximise, $Z = 180x + 75y$
subject to constraints

$12x + 5y \leq 500, x + y \geq 100, x, y \geq 0$

(b) Maximise, $Z = 75x + 180y$
subject to constraints

$12x + 5y \geq 500, x + y \geq 100, x, y \geq 0$

(c) Maximise, $Z = 180x + 75y$
subject to constraints

$12x + 5y \leq 500, x + y \leq 100, x, y \geq 0$

(d) None of the above

Sol. (c) Let x and y denote the number of tables and chairs, respectively.

The cost of x tables = ₹ 1200 x
and cost of y chairs = ₹ 500 y
Since, total investment cannot be more than ₹ 50000.
Therefore, total cost = $1200x + 500y \leq 50,000$
[constraint]

Also, since storage capacity is for 100 pieces only.
Therefore, $x + y \leq 100$ [constraint]

Since, the number of chairs and tables cannot be negative,
 $x \geq 0, y \geq 0$ [non-negativity restrictions]

Now, profit on x tables = $180x$ and profit on y chairs = $75y$
The objective is to maximise the profit, therefore the objective function is $180x + 75y$.
Hence, the LPP is
Maximise $Z = 180x + 75y$
subject to the constraints $1200x + 500y \leq 50,000$
 $x + y \leq 100,$
 $x \geq 0, y \geq 0$

➤ **Example 6.** A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and sprayer. It takes 2h on the grinding/cutting machine and 3h on the sprayer to manufacture a pedestal lamp while, it takes 1h on the grinding/cutting machine and 2h on the sprayer to manufacture a shade. On any day, the sprayer is available for atmost 20h and the grinding/cutting machine for atmost 12h. The profit from the sale of a lamp is ₹ 50 and a shade is ₹ 30. Assuming that the manufacturer is able to sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit.

If x is number of lamps produced and y be the number of wooden shades produced. Then, the given problem, mathematically reduces to

- (a) Maximise $Z = 50x + 30y$
subject to constraints are
 $2x + y \leq 12, 3x + 2y \leq 20, x, y \geq 0$
- (b) Maximise $Z = 50x + 30y$
subject to constraints are
 $2x + y \geq 12, 3x + 2y \geq 20, x, y \geq 0$
- (c) Maximise $Z = 30x + 50y$
subject to constraints are
 $2x + y \geq 12, 3x + 2y \leq 20, x, y \geq 0$
- (d) None of the above

Sol. (a) For the sake of convenience, let us arrange the given information in a table as under:

	Lamp	Shade	Maximum number of hours
Grinding/cutting machine (number of hours)	2	1	12
Sprayer (number of hours)	3	2	20
Profit	₹ 50	₹ 30	

Let the number of lamps produced be x and the number of shades produced be y .

Then, total profit = ₹ $(50x + 30y)$. Since, the objective is to maximise the profit, so the objective function is
 $Z = 50x + 30y$

The constraints in the problem are $2x + y \leq 12$.
 $3x + 2y \leq 20$
and $x \geq 0, y \geq 0$ [non-negativity constraints]
Hence, the LPP is
Maximise $Z = 50x + 30y$
subject to the constraints are
 $2x + y \leq 12$
 $3x + 2y \leq 20$
 $x \geq 0$
 $y \geq 0$

Graphical Method for Solution of Linear Programming Problems

The graphical method for solving LPP involves only optimum solution of a LPP, if it exists, occurs at an extreme point (corner) of the feasible region. Feasible region consists of pair of value of (x, y) which satisfy all the constraints of the LPP.

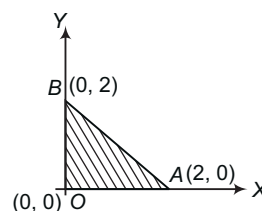
The graphical method involves three basic steps.

- Step I** Determine the feasible region. This region will contain the set of all points (x, y) that simultaneously satisfy all constraints (including non-negative restrictions).
- Step II** Determine the coordinates of each corner point of the feasible region and substitute the coordinates of the corner points into the objective function.
- Step III** The optimum solution occurs in a maximisation case at the corner point yielding the largest value of the objective function and in a minimisation case at the corner point yielding the smallest value of the objective function.

➤ **Example 7.** The point at which the maximum value of $3x + 2y$ subject to the constraints $x + y \leq 2, x \geq 0, y \geq 0$ is obtained, is

- (a) (0, 0) (b) (1.5, 1.5)
(c) (2, 0) (d) (0, 2)

Sol. (c) Graph of inequality $x + y \leq 2, x, y \geq 0$ is shown in the figure.



The corner points are $O(0, 0)$, $A(2, 0)$ and $B(0, 2)$. Here, the objective function is $Z = 3x + 2y$ and at $(0, 0)$, $(2, 0)$ and $(0, 2)$, its values are 0, 6 and 4. So, required point is $(2, 0)$ at which maximum value is attained.

➤ **Example 8.** The maximum value of $P = 6x + 8y$ subject to constraints

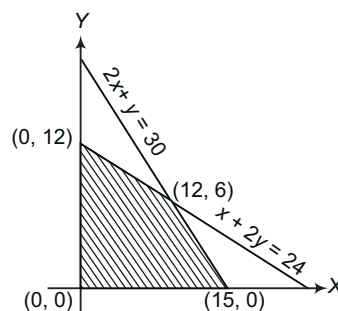
$2x + y \leq 30$, $x + 2y \leq 24$ and $x \geq 0$, $y \geq 0$ is

(a) 90 (b) 120 (c) 96 (d) 240

Sol. (b) Feasible region of constraints is shown in graph. Extreme points are, $(0, 0)$, $(15, 0)$, $(0, 12)$ and $(12, 6)$.

$$\begin{aligned} \therefore P &= 6x + 8y \\ \therefore P_{(12, 6)} &= 72 + 48 = 120 \end{aligned}$$

$$P_{(0, 0)} = 0, P_{(15, 0)} = 90, P_{(0, 12)} = 16$$



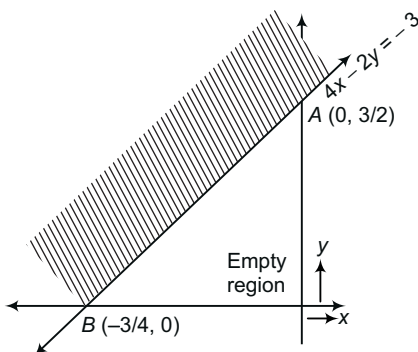
Clearly, maximum P is 120 at $(12, 6)$.

15

Linear Programming

Work Book Exercise

- Objective function of a LPP is
 - a constraint
 - a function to be optimised
 - a relation between the variables
 - None of the above
- Which of the following sets are not convex?
 - $\{(x, y) \mid 3 \leq x^2 + y^2 \leq 5\}$
 - $\{(x, y) \mid 3x^2 + 2y^2 \leq 6\}$
 - $\{(x, y) \mid 3x^2 + 4y^2 \leq 5\}$
 - $\{(x, y) \mid y \geq 2, y \leq 4\}$
- Shaded region is represented by



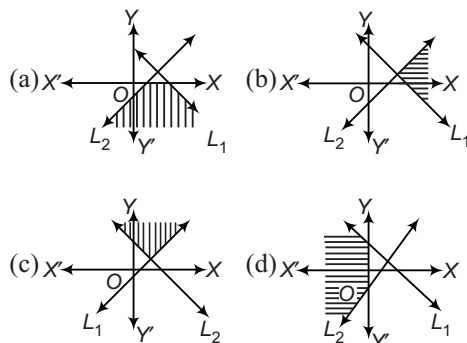
- $4x - 2y \leq 3$
 - $4x - 2y \geq -3$
 - $4x - 2y \geq 3$
 - $4x - 2y \leq -3$
- The shaded region for the inequality $y \leq x + 5y \leq 6$ is
 - to the non-origin side of $x + 5y = 6$
 - to the either side of $x + 5y = 6$
 - to the origin side of $x + 5y = 6$
 - to the neither side of $x + 5y = 6$
 - $Z = 30x + 20y$, $x + y \leq 8$, $x + 2y \geq 4$, $6x + 4y \geq 12$, $x \geq 0$, $y \geq 0$ has
 - unique solution
 - infinitely many solutions
 - minimum at $(4, 0)$
 - minimum 60 at point $(0, 3)$

- The maximum value of $Z = 5x + 3y$, subjected to the conditions $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x, y \geq 0$, is
 - $\frac{235}{19}$
 - $\frac{325}{19}$
 - $\frac{523}{19}$
 - $\frac{532}{19}$
- The region represented by the inequation system $x, y \geq 0$, $y \leq 6$, $x + y \leq 3$ is
 - unbounded in first quadrant
 - unbounded in first and second quadrants
 - bounded in first quadrant
 - None of the above
- A wholesale merchant wants to start the business of cereal with ₹ 24000. Wheat is ₹ 400 per quintal and rice is ₹ 600 per quintal. He has capacity to store 200 quintal cereal. He earns the profit ₹ 25 per quintal on wheat and ₹ 40 per quintal on rice. If he stores x quintal rice and y quintal wheat, then for maximum profit, the objective function is
 - $25x + 40y$
 - $40x + 25y$
 - $400x + 600y$
 - $10x + 24y$
- The coordinates of the point at which minimum value of $Z = 7x - 8y$ subject to the conditions $x + y - 20 \leq 0$, $y \geq 5$, $x \geq 0$, $y \geq 0$ is attained, is
 - $(20, 0)$
 - $(15, 5)$
 - $(0, 5)$
 - $(0, 20)$
- The maximum value of $\mu = 3x + 4y$, subjected to the conditions $x + y \leq 40$, $x + 2y \leq 60$, $x, y \geq 0$, is
 - 130
 - 140
 - 40
 - 20

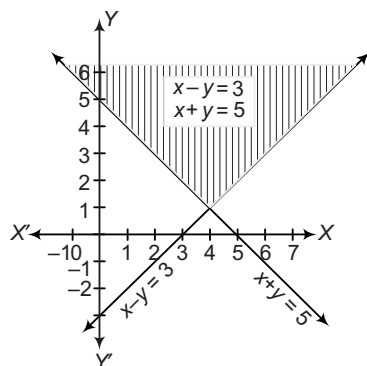
WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The graphical solution of linear inequalities $x + y \geq 5$ and $x - y \leq 3$, where $L_1 \equiv x + y = 5$ and $L_2 \equiv x - y = 3$, is



Sol. The graph of linear equation $x + y = 5$ is drawn in figure. We note that solution of inequality (i) is represented by the shaded region above the line $x + y = 5$, including the points on the line



On the same set of axes, we draw the graph of the equation $x - y = 3$ as shown in figure. Then, we note that inequality (ii) represents the shaded region above the line $x - y = 3$, including the points on the line.

Clearly, the double shaded region, common to the above two shaded regions is the required solution region of the given system of inequalities. Hence, (c) is the correct answer.

Ex 2. Let x and y be the number of tables and chairs respectively, on which a furniture dealer wants to make profit for the constraints

$$\text{Maximise } Z = 250x + 75y$$

subject to constraints are

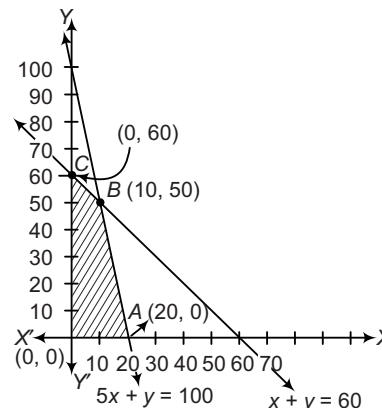
$$5x + y \leq 100$$

$$x + y \leq 60$$

$$x \geq 0$$

$$y \geq 0$$

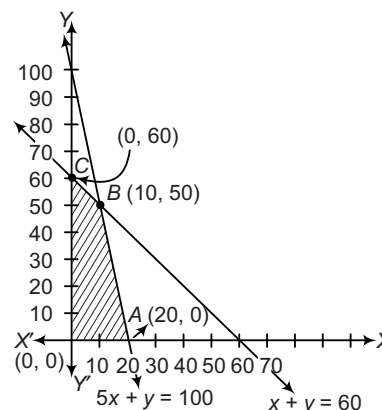
Consider, the following graph:



Then, the maximum profit to the dealer results from buying

- (a) 10 tables and 50 chairs
- (b) 50 tables and 10 chairs
- (c) 0 table and 60 chairs
- (d) 20 tables and 40 chairs

Sol. The graph of the given constraints is



The corner points (vertices) of the bounded (feasible) region are O , A , B and C and it is easy to find their coordinates as $(0, 0)$, $(20, 0)$, $(10, 50)$ and $(0, 60)$, respectively. Let us now compute the values of Z at these points.

Corner points	$Z = 250x + 75y$
$O(0, 0)$	0
$C(0, 60)$	4500
$B(10, 50)$	6250 (Maximum)
$A(20, 0)$	5000

We observe that the maximum profit to the dealer results from the investment strategy (10, 50), i.e. buying 10 tables and 50 chairs.

This method of solving linear programming problem is referred as corner point method.

Hence, (a) is the correct answer.

Ex 3. Consider the following statements.

- I. The objective function
maximise $Z = -x + 2y$
subject to the constraints
 $x \geq 3, x + y \geq 5, x + 2y \geq 6, x, y \geq 0$
has no maximum value.
- II. The objective function
maximise $Z = x + y$
subject to the constraints
 $x - y \leq -1, -x + y \leq 0, x, y \geq 0$
has no maximum value.

Choose the correct option.

- (a) Only I is true
- (b) Only II is true
- (c) Both I and II are true
- (d) Neither I nor II is true

Sol. I. Our problem is to maximise
 $Z = -x + 2y$... (i)
subject to the constraints are
 $x \geq 3$... (ii)
 $x + y \geq 5$... (iii)
 $x + 2y \geq 6$... (iv)
 $x \geq 0, y \geq 0$... (v)

Draw the graph of the line $x + y = 5$

x	0	5
y	5	0

Putting (0, 0) in the inequality $x + y \geq 5$, we have
 $0 + 0 \geq 5$
 $\Rightarrow 0 \geq 5$, which is false.

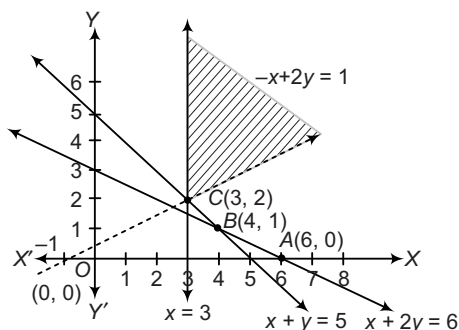
So, the half plane is away from the origin.

Draw the graph of line $x + 2y = 6$

x	0	6
y	3	0

Putting (0, 0) in the inequality $x + 2y \geq 6$, we get
 $0 + 2 \times 0 \geq 6$
 $\Rightarrow 0 \geq 6$, which is false.

So, the half plane is away from the origin.



Since, $x \geq 3, y \geq 0$

So, the feasible region lies in the first quadrant.

The point of intersection of lines $x = 3$

and $-x + 2y = 1$ is $C(3, 2)$.

and the point of intersection of lines

$x + 2y = 6$ and $x + y = 5$ is $B(4, 1)$.

It can be seen that the feasible region is unbounded.

The corner points of the feasible region are $A(6, 0)$, $B(4, 1)$ and $C(3, 2)$. The values of Z at these points are as follows:

Corner points	$Z = -x + 2y$
$A(6, 0)$	-6
$B(4, 1)$	-2
$C(3, 2)$	1 (Maximum)

As the feasible region is unbounded. Therefore, $Z = 1$ may or may not be the maximum value. For this, we graph the inequality $-x + 2y > 1$ and check whether the resulting half plane has points in common with the feasible region or not.

The resulting feasible region has points in common with the feasible region. Therefore, $Z = 1$ is not the maximum value. Hence, Z has no maximum value.

II. Our problem is to maximise

$Z = x + y$... (i)

subject to the constraints are

$x - y \leq -1$... (ii)

$-x + y \leq 0$... (iii)

$x \geq 0, y \geq 0$... (iv)

Draw the graph of the line $x - y = -1$

x	0	-1
y	1	0

Putting (0, 0) in the inequality $x - y \leq -1$, we get

$0 - 0 \leq -1$

$\Rightarrow 0 \leq -1$, which is false.

So, the half plane is away from the origin.

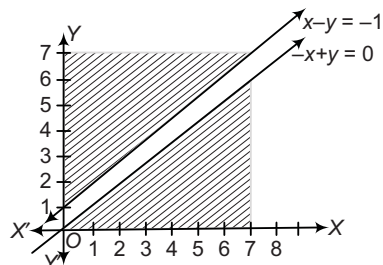
Draw the graph of the line $-x + y = 0$.

x	0	1
y	0	1

Putting (2, 0) in the inequality $-x + y \leq 0$, we get

$-2 + 0 \leq 0$

$\Rightarrow -2 \leq 0$, which is true.



So, the half plane is towards the origin.

Since, $x, y \geq 0$

So, the feasible region lies in the first quadrant.

From the above graph, it is clearly shown that there is no common region.

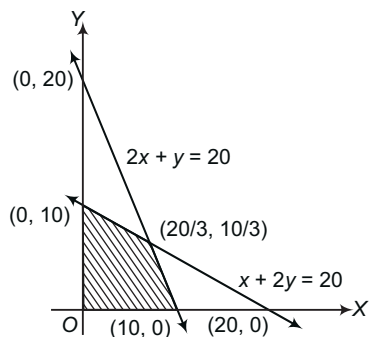
Hence, there is no feasible region and thus Z has no maximum value.

Hence, (c) is the correct answer.

- Ex 4.** The maximum value of $P = x + 3y$ such that $2x + y \leq 20$, $x + 2y \leq 20$, $x \geq 0$, $y \geq 0$, is
- 10
 - 60
 - 30
 - None of the above

Sol. The points in the feasible region are

$$(0, 0), (0, 10), \left(\frac{20}{3}, \frac{10}{3}\right) \text{ and } (10, 0).$$



Objective function $P = x + 3y$

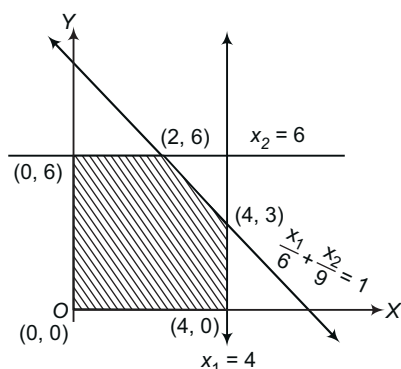
$$P_{(10, 0)} = 10, P_{(0, 10)} = 30, P_{\left(\frac{20}{3}, \frac{10}{3}\right)} = \frac{50}{3}$$

$\therefore$ Maximum value of P is 30 at $(0, 10)$.

Hence, (c) is the correct answer.

- Ex 5.** By graphical method, the solutions of linear programming problem maximise $Z = 3x_1 + 5x_2$ subject to constraints $3x_1 + 2x_2 \leq 18$, $x_1 \leq 4$, $x_2 \leq 6$, $x_1 \geq 0$, $x_2 \geq 0$, are
- $x_1 = 2, x_2 = 0, Z = 6$
 - $x_1 = 2, x_2 = 6, Z = 36$
 - $x_1 = 4, x_2 = 3, Z = 27$
 - $x_1 = 4, x_2 = 6, Z = 42$

Sol. We draw the graph of $x_1 \leq 4$, $x_2 \leq 6$, $\frac{x_1}{6} + \frac{x_2}{9} \leq 1$



Different points of convex polygon are

$(0, 0), (4, 0), (4, 3), (2, 6)$ and $(0, 6)$.

Maximum value of $Z = 3x_1 + 5x_2$ at

$x_1 = 2, x_2 = 6$ is $Z = 36$.

Hence, (b) is the correct answer.

- Ex 6.** A manufacturer produces nuts and bolts. It takes 1 h of work on machine A and 3 h on machine B to produce a package of nuts. It takes 3 h on machine A and 1 h on machine B to produce a package of bolts. He earns a profit of ₹ 17.50 per package on nuts and ₹ 7.0 per package on bolts.

If he operates his machines for atmost 12 h a day, then the number of packages of each produced each day so as to maximise his profit is

- 3 packages of nuts and 2 packages of bolts
- 3 packages of nuts and bolts each
- 2 packages of nuts and 3 packages of bolts
- None of the above

Sol. Let the manufacturer produces x nuts and y bolts.

We construct the following table:

Item	Number of packages	Machine A (in h)	Machine B (in h)	Profit (in ₹)
Nuts	x	$1x$	$3x$	$17.50x$
Bolts	y	$3x$	$1y$	$7y$
Total	$x + y$	$x + 3y$	$3x + y$	$17.50x + 7y$
Availability		12	12	

$\therefore$ Total profit, $Z = 17.5x + 7y$

i.e. maximise $Z = 17.5x + 7y$... (i)

subject to constraints are

$$x + 3y \leq 12 \quad \dots (ii)$$

$$3x + y \leq 12 \quad \dots (iii)$$

$$x \geq 0, y \geq 0 \quad \dots (iv)$$

Firstly, draw the graph of the line $x + 3y = 12$

x	0	12
y	4	0

Putting $(0, 0)$ in the inequality $x + 3y \leq 12$, we get

$$0 + 3 \times 0 \leq 12$$

$\Rightarrow 0 \leq 12$, which is true.

So, the half plane is towards the origin.

Since, $x, y \geq 0$

So, the feasible region lies in first quadrant.

Secondary, draw the graph of the line $3x + y = 12$

x	4	0
y	0	12

Putting $(0, 0)$ in the inequality $3x + y \leq 12$, we have

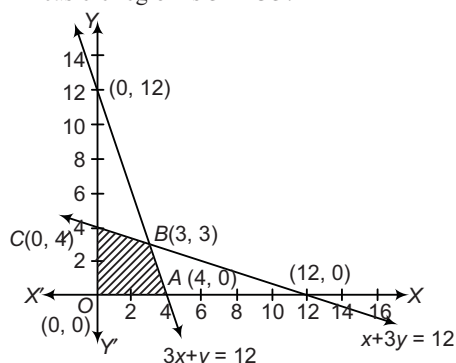
$$3 \times 0 + 0 \leq 12$$

$\Rightarrow 0 \leq 12$, which is true.

So, the half plane is towards the origin.

On solving equations $x + 3y = 12$ and $3x + y = 12$, we get B (3, 3).

∴ Feasible region is $OABCO$.



The corner points of the feasible region are $O(0,0)$, $A(4,0)$, $B(3,3)$ and $C(0,4)$. The values of Z at these points are as follow:

Corner points	$Z = 17.5x + 7y$
$O(0,0)$	0
$A(4,0)$	70
$B(3,3)$	73.50 (Maximum)
$C(0,4)$	28

The maximum value of Z is ₹ 73.50 at $(3,3)$.

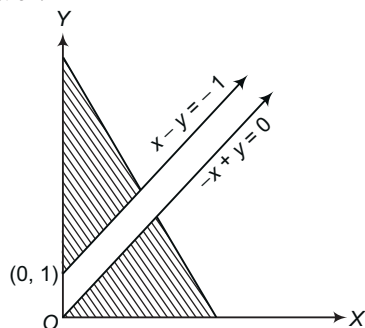
Thus, 3 packages of nuts and 3 packages of bolts should be produced each day to get the maximum profit ₹ 73.50.

Hence, (b) is the correct answer.

Ex 7. Maximum value of $Z = 3x + 4y$ subject to $x - y \leq -1$, $-x + y \leq 0$, $x, y \geq 0$ is given by

- (a) 1
(b) 4
(c) 6
(d) no feasible solution

Sol. We plot the lines $x - y = -1$ and $-x + y = 0$. The shaded portion denotes the region $x - y \leq -1$ and $-x + y \leq 0$. It is clear from the figure that there is no point in the first quadrant which satisfies the two constraints. Hence, the given LPP has no feasible solution.



Hence, (d) is the correct answer.

Ex 8. Every gram of wheat provides 0.1 g of proteins and 0.25 g of carbohydrates. The corresponding values of rice are 0.05 g and 0.5 g, respectively. Wheat costs ₹ 4 per kg and rice ₹ 6. The minimum daily requirements of proteins and carbohydrates for an average child are 50 g and 200 g respectively. Then, in

what quantities should wheat and rice be mixed in the daily diet to provide minimum daily requirements of proteins and carbohydrates at minimum cost?

- (a) 400, 200 (b) 300, 400
(c) 200, 400 (d) 400, 300

Sol. Suppose x g of wheat and y g of rice are mixed in the daily diet.

Since, every grams of wheat provides 0.1 g of proteins and every gram of rice gives 0.05 g of proteins. Therefore, x g of wheat and y g of rice will provide $0.1x + 0.05y$ g of proteins. But the minimum daily requirement of proteins is of 50 g.

$$\therefore 0.1x + 0.05y \geq 50$$

$$\Rightarrow \frac{x}{10} + \frac{y}{20} \geq 50$$

Similarly, x g of wheat and y g of rice will provide $0.25x + 0.5y$ of carbohydrates and the minimum daily requirement of carbohydrates is 200 g.

$$\therefore 0.25x + 0.5y \geq 200 \Rightarrow \frac{x}{4} + \frac{y}{2} \geq 200$$

Since, the quantities of wheat and rice cannot be negative.

$$\text{Therefore, } x \geq 0, y \geq 0$$

It is given that wheat costs ₹ 4 per kg and rice ₹ 6 per kg. So, x g of wheat and y g of rice will cost ₹ $\frac{4x}{1000} + \frac{6y}{1000}$.

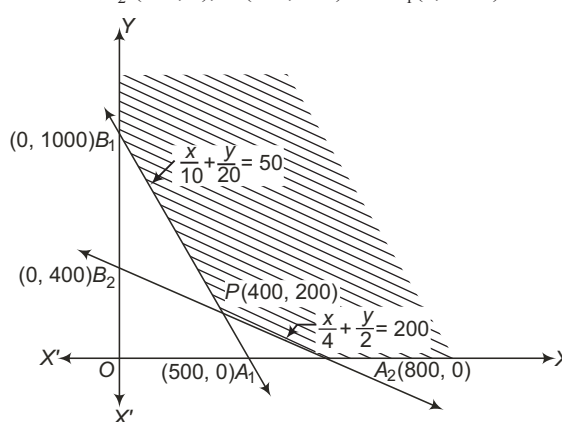
subject to the constraints are

$$\frac{x}{10} + \frac{y}{20} \geq 50,$$

$$\frac{x}{4} + \frac{y}{2} \geq 200 \text{ and } x \geq 0, y \geq 0$$

The solution set of the linear constraints is shaded in figure. The vertices of the shaded region are

$A_2(800,0)$, $P(400,200)$ and $B_1(0,1000)$.



The values of the objective function at these points are given in the following table:

Corner points	$Z = \frac{4x}{1000} + \frac{6y}{1000}$
$A_2(800,0)$	3.2
$P(400,200)$	2.8 (Minimum)
$B_1(0,1000)$	6

Clearly, Z is minimum for $x = 400$ and $y = 200$. The minimum diet cost is ₹ 2.8.

Hence, (a) is the correct answer.

Ex 9. An oil company required 13000, 20000 and 15000 barrels of high grade, medium grade and low grade oil, respectively. Refinery *A* produces 100, 300 and 200 barrels per day of high grade, medium grade and low grade oil, respectively, while refinery *B* produces 200, 400 and 100 barrels per day of high grade, medium grade and low grade oil, respectively.

If refinery *A* costs ₹ 400 per day and refinery *B* costs ₹ 300 per day to operate, then the days should each be run to minimise costs while satisfying requirements are

- (a) 30, 60 (b) 60, 30
(c) 40, 60 (d) 60, 40

Sol. The given data may be put in the following tabular form:

Refinery	High grade	Medium grade	Low grade	Cost per day
<i>A</i>	100	300	200	₹ 400
<i>B</i>	200	400	100	₹ 300
Minimum requirement	12000	20000	15000	

Suppose refineries *A* and *B* should run for x and y days respectively to minimise the total cost.

The mathematical form of the above is

Minimise $Z = 400x + 300y$

subject to constraints

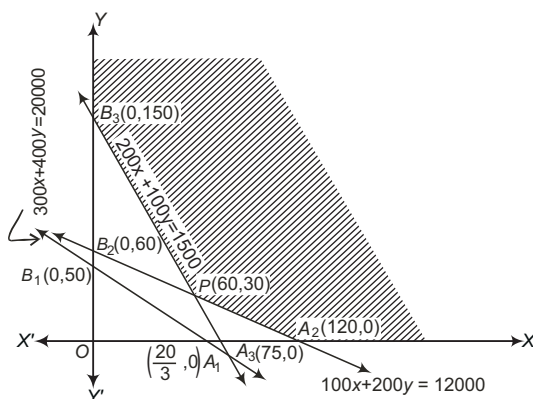
$$100x + 200y \geq 12000$$

$$300x + 400y \geq 20000$$

$$200x + 100y \geq 15000$$

and $x, y \geq 0$

The feasible region of the above LPP is represented by the shaded region in the given figure.



The corner points of the feasible region are $A_2(120, 0)$, $P(60, 30)$ and $B_3(0, 150)$. The value of the objective function at these points are given in the following table:

Corner points	$Z = 400x + 300y$
$A_2(120, 0)$	48000
$P(60, 30)$	33000 (Minimum)
$B_3(0, 150)$	45000

Clearly, Z is minimum when $x = 60$ and $y = 30$. Hence, the machine *A* should run for 60 days and the machine *B* should run for 30 days to minimise the cost while satisfying the constraints.

Hence, (b) is the correct answer.

Ex 10. A company produces soft drinks that has a contract which requires that a minimum of 80 units of the chemical *A* and 60 units of the chemical *B* to go into each bottle of the drink. The chemicals are available in a prepared mix from two different suppliers. Supplier *S* has a mix of 4 units of *A* and 2 units of *B* that costs ₹ 10, the supplier *T* has a mix of 1 unit of *A* and 1 unit of *B* that costs ₹ 3.

Then, the mixes from *S* and *T* should the company purchase to honour contract requirement and yet minimise cost, are

- (a) $S = 10, T = 40$
(b) $S = 40, T = 10$
(c) $S = 20, T = 80$
(d) $S = 10, T = 50$

Sol. The given data may be put in the following tabular form:

Supplier chemical	<i>S</i>	<i>T</i>	Minimum requirement
<i>A</i>	4	1	80
<i>B</i>	2	1	60
Cost per unit	₹ 10	₹ 4	

Suppose x units of mix are purchased from supplier *S* and y units are purchased from supplier *T*.

Total cost $Z = 10x + 4y$

Unit of chemical *A* per bottle = $4x + y$

But the minimum requirement of chemical *A* per bottle = 80

$$\therefore 4x + y \geq 80$$

Unit of chemical *B* per bottle = $2x + y$

But the minimum requirement of chemical *B* per bottle = 60

$$\therefore 2x + y \geq 60$$

Minimise $Z = 10x + 4y$

subject to constraints are

$$4x + y \geq 80$$

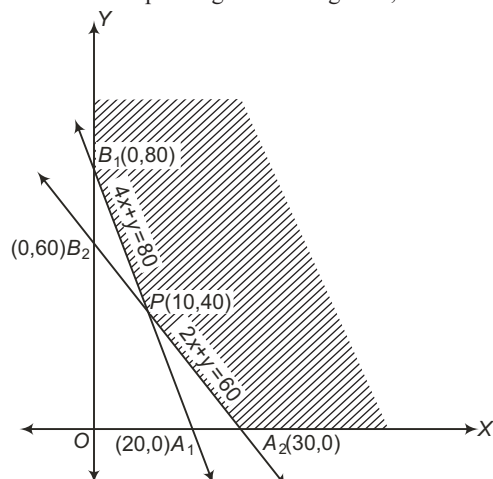
$$2x + y \geq 60$$

and $x \geq 0, y \geq 0$

Now, we find the feasible region which is the set of all points whose coordinates simultaneously satisfy all constraints including non-negativity restrictions. The shaded region in given figure represents the feasible region of the given LPP. The coordinates of the corner points of the feasible region are

$$A_2(30, 0), P(10, 40), B_1(0, 80).$$

These points are obtained by solving the equations of the corresponding intersecting lines, simultaneously.



The values of the objective function at these points are given in the following table:

Corner points	$Z = 10x + 4y$
$A_2(30, 0)$	300
$P(10, 40)$	260 (Minimum)
$B_1(0, 80)$	320

Clearly, Z is minimum at $(10, 40)$. Hence, $x = 10$, $y = 40$ is the optimal solution of the given LPP.

Hence, the cost per bottle is minimum when the company purchases 10 mixes from supplier S and 40 mixes from supplier T .

Hence, (a) is the correct answer.

Ex 11. A dealer wishes to purchase a number of fans and sewing machines. He has only ₹ 5760.00 to invest and has space for atmost 200 items. A fan costs him ₹ 360.00 and a sewing machine ₹ 240.00. His expectation is that he can sell a fan at a profit of ₹ 22.00 and a sewing machine at a profit of ₹ 18.00. Assuming that he can sell all the items that he can buy, how many fans and sewing machines he should buy to maximise his profit?

- (a) 8, 12 (b) 16, 0
(c) 0, 20 (d) 0, 22

Sol. Suppose the dealer buys x fans and y sewing machines. Since, the dealer has space for atmost 20 items.

Therefore, $x + y \leq 20$

A fan costs ₹ 360 and a sewing machine costs ₹ 240. Therefore, total cost of x fans and y machines is ₹ $(360x + 240y)$. But the dealer has only ₹ 5760 to invest. Therefore,

$$360x + 240y \leq 5760$$

Since, the dealer can sell all the items that he can buy and the profit on a fan is of ₹ 22 and on a sewing machine the profit is of ₹ 18. Therefore, total profit on selling x fans and y sewing machines is of ₹ $(22x + 18y)$.

Let Z denotes the total profit. Then, $Z = 22x + 18y$

Clearly, $x, y \geq 0$

Thus, the mathematical formulation of the given problem is

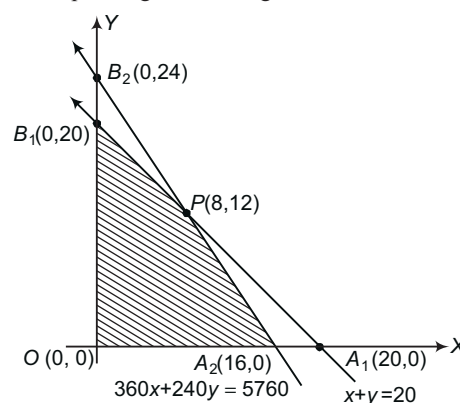
Maximise $Z = 22x + 18y$

subject to constraints are $x + y \leq 20$

$$360x + 240y \leq 5760 \quad \text{and} \quad x \geq 0, y \geq 0$$

To solve this LPP graphically, we first convert the inequations into equations and draw the corresponding lines. The feasible region of the LPP is shaded in the given figure. The corner points of the feasible region OA_2PB_1 are $O(0, 0)$, $A_2(16, 0)$, $P(8, 12)$ and $B_1(0, 20)$.

These points have been obtained by solving the corresponding intersecting lines, simultaneously.



The values of the objective function Z at corner points of the feasible region are given in the following table:

Corner points	$Z = 22x + 18y$
$O(0, 0)$	0
$A_2(16, 0)$	352
$P(8, 12)$	392 (Maximum)
$B_1(0, 20)$	360

Clearly, Z is maximum at $x = 8$ and $y = 12$. The maximum value of Z is 392.

Hence, the dealer should purchases 8 fans and 12 sewing machines to obtain the maximum profit under given conditions.

Hence, (a) is the correct answer.

Ex 12. A toy manufacturer produces two types of dolls; a basic version doll A and a deluxe version doll B . Each doll of type B takes twice as long to produce as one doll of type A . The company have time to make a maximum of 2000 dolls of type A per day, the supply of plastic is sufficient to produce 1500 dolls per day and each type requires equal amount of it.

The deluxe version, i.e. type B requires a fancy dress of which there are only 600 per day available. If the company makes a profit of ₹ 3 and ₹ 5 per doll, respectively, on doll A and B , then the number of each should be produced per day in order to maximise profit, is

- (a) 800, 500 (b) 500, 600
(c) 450, 450 (d) 1000, 500

Sol. Let x dolls of type A and y dolls of type B be produced per day to maximise the profit.

The mathematical form of the given LPP is as follows :

Maximise $Z = 3x + 5y$

subject to the constraints $x + 2y \leq 2000$,
 $x + y \leq 1500$

$y \leq 600$ and $x, y \geq 0$

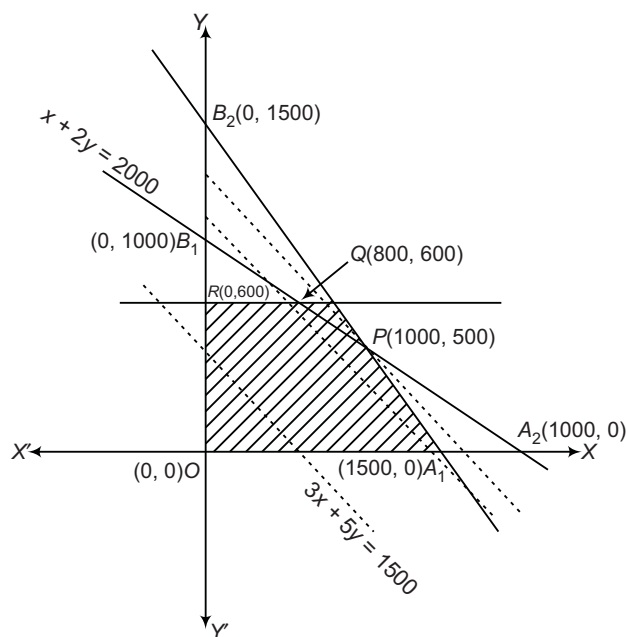
The set of all feasible solutions of the given LPP is represented by the feasible region shaded darkly in the given figure. The coordinates of the corner points of the feasible region are

$O(0, 0)$, $A_1(1500, 0)$, $P(1000, 500)$, $Q(800, 600)$

and $R(0, 600)$.

Now, to find a point or points in the feasible region which gives the maximum value of the objective function $Z = 3x + 5y$, let us give some value to Z , say 1500 and draw the dotted line $3x + 5y = 1500$ as shown in the given figure.

Now, draw lines parallel to the line $3x + 5y = 1500$ and obtain a line which is farthest for the origin and have atleast one point common to the feasible region. Clearly, $Z_1 = 3x + 5y$ is such a line. This line has only one point $P(1000, 500)$ common to the feasible region. Thus, $Z_1 = 3 \times 1000 + 5 \times 500 = 5500$ is the maximum value of Z and the optimal solution is $x = 1000$ and $y = 500$.



Hence, 1000 dolls of type A and 500 dolls of type B should be produced to maximise the profit and the maximum profit is ₹ 5500.

Hence, (d) is the correct answer.

Type 2. Assertion and Reason

Ex 13. Statement I The maximum value of
 $Z = 11x + 7y$

subject to the constraints are

$$2x + y \leq 6,$$

$$x \leq 2,$$

$$x \geq 0,$$

$$y \geq 0$$

occurs at the corner point $(0, 6)$.

Statement II If the feasible region of the given LPP is bounded, then the maximum and minimum values of the objective function occur at corner points.

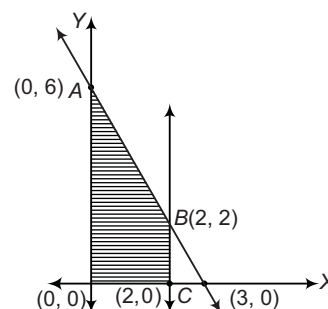
- (a) Statement I is true, Statement II is true;
Statement II is the correct explanation for Statement I
- (b) Statement I is true, Statement II is true;
Statement II is not the correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Sol. (a) The given LPP is maximise $Z = 11x + 7y$
subject to the constraints are

$$2x + y \leq 6, x \leq 2,$$

$$x \geq 0, y \geq 0$$

The corresponding graph of the LPP is given below:



From the above graph, we see that the shaded region is the feasible region $OABC$ which is bounded.

∴ The maximum value of the objective function Z occurs at the corner points.

The corner points are

$O(0, 0)$, $A(0, 6)$, $B(2, 2)$, $C(2, 0)$.

The values of Z at these corner points are given by

Corner points	$Z = 11x + 7y$
$(0, 0)$	0 (Minimum)
$(0, 6)$	42 (Maximum)
$(2, 2)$	36
$(2, 0)$	22

Thus, the maximum value of Z is 42 which occurs at the point $(0, 6)$.

The minimum and maximum values occur at corner points.

Hence, (a) is the correct answer.

Type 3. Linked Comprehension Based Questions

Passage I (Ex. Nos. 14-15) If a young man rides his motorcycle at 25 km/h, he has to spend ₹ 2 per km in petrol and if he rides it at 40 km/h, the petrol cost rises to ₹ 5 per km. He has ₹ 100 to spend on petrol and wishes to find the maximum distance, he can travel within one hour. If x and y denote the distance travelled by him (in km) at 25 km/h and 40 km/h, respectively.

Ex 14. The inequations represent the data, are

- (a) $2x + 5y \leq 100, \frac{x}{25} + \frac{y}{40} \geq 1, x \geq 0, y \geq 0$
 (b) $2x + 5y \geq 100, \frac{x}{25} + \frac{y}{40} \geq 1, x \geq 0, y \geq 0$
 (c) $2x + 5y \leq 100, \frac{x}{25} + \frac{y}{40} \leq 1, x \geq 0, y \geq 0$
 (d) $2x + 5y \leq 100, 25x + 40y \leq 1, x \geq 0, y \geq 0$

Sol. Time taken to cover x km = $\frac{x}{25}$ h

Time taken to cover y km = $\frac{y}{40}$ h

Since, he has only one hour.

$$\therefore \frac{x}{25} + \frac{y}{40} \leq 1$$

Total cost of petrol = $2x + 5y$

Since, he has only ₹ 100 to spend.

$$\therefore 2x + 5y \leq 100$$

$\therefore x$ and y are distances. So, x and y cannot be negative.

$$\therefore x \geq 0, y \geq 0$$

Hence, inequations represent above data are

$$2x + 5y \leq 100, \frac{x}{25} + \frac{y}{40} \leq 1, x \geq 0, y \geq 0$$

Hence, (c) is the correct answer.

Ex 15. The solution of the inequalities representing the data, is

- (a) unique (b) infinite
 (c) no solution (d) None of these

Sol. The inequations represent the data are

$$2x + 5y \leq 100 \quad \dots(i)$$

$$\frac{x}{25} + \frac{y}{40} \leq 1 \quad \dots(ii)$$

$$x \geq 0 \quad \dots(iii)$$

$$y \geq 0 \quad \dots(iv)$$

The lines corresponding to Eqs. (i) and (ii) are

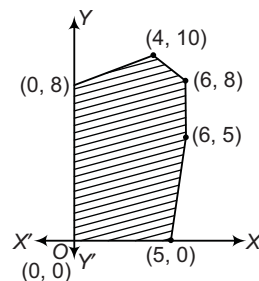
$$2x + 5y = 100 \quad \dots(v)$$

$$\frac{x}{25} + \frac{y}{40} = 1 \quad \dots(vi)$$

Line (v) cuts X -axis at (50, 0) and Y -axis at (0, 20).
 Line (vi) cuts X -axis at (25, 0) and Y -axis at (0, 40).
 Since, (0, 0) satisfies both Eqs. (i) and (ii). So, both inequality represent the region which contains the origin.

Hence, (b) is the correct answer.

Passage II (Ex. Nos. 16-18) The feasible solution for the LPP is shown as below.



Let $Z = 3x - 4y$ be the objective function.

Ex 16. Minimum of Z occurs at

- (a) (0, 0) (b) (0, 8)
 (c) (5, 0) (d) (4, 10)

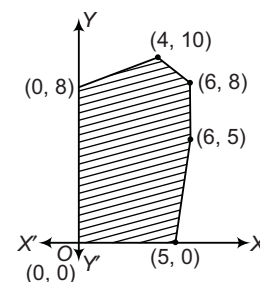
Ex 17. Maximum of Z occurs at

- (a) (5, 0) (b) (6, 5)
 (c) (6, 8) (d) (4, 10)

Ex 18. (Maximum value of Z + Minimum value of Z) is equal to

- (a) 13 (b) 1
 (c) -13 (d) -17

Solutions (Ex. Nos. 16-18) The feasible solution is



Given objective function is $Z = 3x - 4y$.

The values of Z at the corner points are given by

Corner points	Corresponding value of Z
(0, 0)	0
(0, 8)	-32 (Minimum)
(4, 10)	-28
(6, 8)	-14
(6, 5)	-2
(5, 0)	15 (Maximum)

16. From the above table, the minimum value of Z is -32 which occurs at (0, 8).
 Hence, (b) is the correct answer.

17. The maximum value of Z is 15 which occurs at (5, 0).
 Hence, (a) is the correct answer.

18. Maximum value of Z + Minimum value of Z
 $= 15 - 32 = -17$
 Hence, (d) is the correct answer.

Type 4. Match the Column

Ex 19. Match the terms of Column I with the terms of Column II.

Column I (Linear programming problem)	Column II (Solution)
A. Minimise $Z = -3x + 4y$ subject to $x + 2y \leq 8$, $3x + 2y \leq 12$, $x \geq 0$, $y \geq 0$	p. $\frac{235}{19}$
B. Maximise $Z = 5x + 3y$ subject to $3x + 5y \leq 15$, $5x + 2y \leq 10$, $x \geq 0$, $y \geq 0$	q. -12
C. Minimise $Z = 3x + 5y$ subject to $x + 3y \geq 3$, $x + y \geq 2$, $x, y \geq 0$	r. 18
D. Maximise $Z = 3x + 2y$ subject to $x + 2y \leq 10$, $3x + y \leq 15$, $x, y \geq 0$	s. 7
	t. No solution

Sol. A. Our problem is to minimise

$$Z = -3x + 4y \quad \dots(i)$$

subject to the constraints are

$$x + 2y \leq 8 \quad \dots(ii)$$

$$3x + 2y \leq 12 \quad \dots(iii)$$

$$x \geq 0, y \geq 0 \quad \dots(iv)$$

Draw the graph of the line $x + 2y = 8$.

x	0	8
y	4	0

Putting (0, 0) in the inequality $x + 2y \leq 8$, we get

$$0 + 0 \leq 8$$

$\Rightarrow 0 \leq 8$, which is true.

So, the half plane is towards the origin.

Since, $x, y \geq 0$

So, the feasible region lies in the first quadrant.

Draw the graph of the line $3x + 2y = 12$.

x	0	4
y	6	0

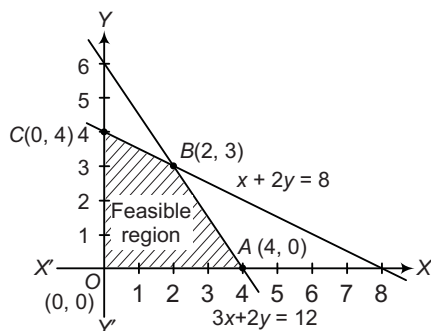
Putting (0, 0) in the inequality $3x + 2y \leq 12$, we get

$$3 \times 0 + 2 \times 0 \leq 12$$

$\Rightarrow 0 \leq 12$, which is true.

So, the half plane is towards the origin.

$\therefore$ Feasible region is $OABCO$.



On solving equations

$$x + 2y = 8 \quad \text{and} \quad 3x + 2y = 12, \text{ we get}$$

$$x = 2 \quad \text{and} \quad y = 3$$

$\therefore$ Intersection point B is (2, 3).

The corner points of the feasible region are $O(0, 0)$, $A(4, 0)$, $B(2, 3)$ and $C(0, 4)$. The values of Z at these points are as follows:

Corner points	$Z = -3x + 4y$
$O(0, 0)$	0
$A(4, 0)$	-12 (Minimum)
$B(2, 3)$	6
$C(0, 4)$	16

Therefore, the minimum value of Z is -12 at the point $A(4, 0)$.

B. Our problem is to maximise

$$Z = 5x + 3y \quad \dots(i)$$

subject to the constraints are

$$3x + 5y \leq 15 \quad \dots(ii)$$

$$5x + 2y \leq 10 \quad \dots(iii)$$

$$x \geq 0, y \geq 0 \quad \dots(iv)$$

Draw the graph of the line $3x + 5y = 15$.

x	0	5
y	3	0

Putting (0, 0) in the inequality $3x + 5y \leq 15$, we get

$$3 \times 0 + 5 \times 0 \leq 15$$

$\Rightarrow 0 \leq 15$, which is true.

So, the half plane is towards the origin.

Since, $x, y \geq 0$.

So, the feasible region lies in the first quadrant.

Draw the graph of the line $5x + 2y = 10$.

x	0	2
y	5	0

Putting (0, 0) in the inequality $5x + 2y \leq 10$, we get

$$5 \times 0 + 2 \times 0 \leq 10$$

$\Rightarrow 0 \leq 10$, which is true.

So, the half plane is towards the origin.

On solving given equations

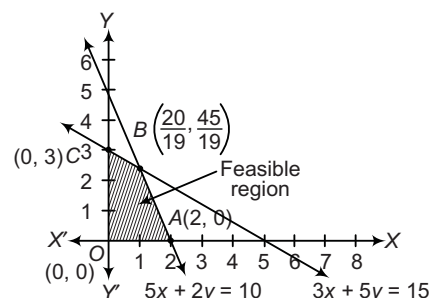
$$3x + 5y = 15$$

$$\text{and} \quad 5x + 2y = 10,$$

$$\text{we get} \quad x = \frac{20}{19}, y = \frac{45}{19}$$

Coordinates of point B are $\left(\frac{20}{19}, \frac{45}{19}\right)$.

$\therefore$ Feasible region is $OABCO$.



The corner points of the feasible region are $O(0, 0)$, $A(2, 0)$, $B\left(\frac{20}{19}, \frac{45}{19}\right)$ and $C(0, 3)$. The values of Z at these points are as follows:

Corner points	$Z = 5x + 3y$
$O(0, 0)$	0
$A(2, 0)$	10
$C(0, 3)$	9
$B\left(\frac{20}{19}, \frac{45}{19}\right)$	$\frac{235}{19}$ (Maximum)

Therefore, the maximum value of Z is $\frac{235}{19}$ at the point $B\left(\frac{20}{19}, \frac{45}{19}\right)$.

C. Our problem is to minimise

$$Z = 3x + 5y \quad \dots(i)$$

subject to the constraints are

$$x + 3y \geq 3 \quad \dots(ii)$$

$$x + y \geq 2 \quad \dots(iii)$$

$$x \geq 0, y \geq 0 \quad \dots(iv)$$

Draw the graph of the line $x + 3y = 3$.

x	0	3
y	1	0

Putting $(0, 0)$ in the inequality $x + 3y \geq 3$, we get

$$0 + 3 \times 0 \geq 3$$

$\Rightarrow 0 \geq 3$, which is false.

So, the half plane is away from the origin.

Since, $x, y \geq 0$

So, the feasible region lies in the first quadrant.

Now, draw the graph of the line $x + y = 2$.

x	0	2
y	2	0

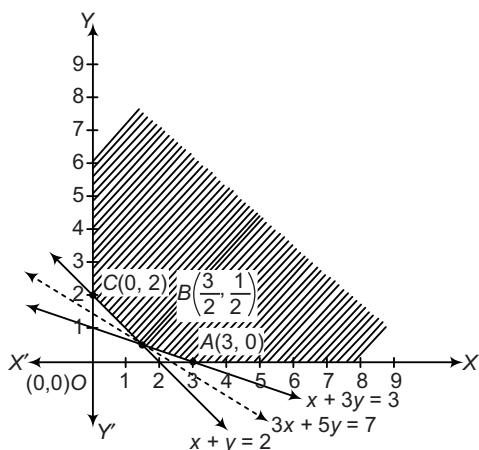
Putting $(0, 0)$ in the inequality $x + y \geq 2$, we get

$$0 + 0 \geq 2$$

$\Rightarrow 0 \geq 2$, which is false.

So, the half plane is away from the origin.

It can be seen that the feasible region is unbounded.



On solving equations $x + y = 2$ and $x + 3y = 3$, we get

$$x = \frac{3}{2}$$

and

$$y = \frac{1}{2}$$

$\therefore$ Intersection point B is $\left(\frac{3}{2}, \frac{1}{2}\right)$.

The corner points of the feasible region are $A(3, 0)$, $B\left(\frac{3}{2}, \frac{1}{2}\right)$ and $C(0, 2)$. The values of Z at these points are as follows:

Corner points	$Z = 3x + 5y$
$A(3, 0)$	9
$B\left(\frac{3}{2}, \frac{1}{2}\right)$	7 (Minimum)
$C(0, 2)$	10

As the feasible region is unbounded. Therefore, 7 may or may not be the minimum value of Z .

For this, we draw the graph of the inequality $3x + 5y < 7$ and check whether the resulting half plane has points in common with the feasible region or not.

It can be seen that the feasible region has no common point with $3x + 5y < 7$.

Therefore, the minimum value of Z is 7 at $\left(\frac{3}{2}, \frac{1}{2}\right)$.

D. Our problem is to maximise

$$Z = 3x + 2y \quad \dots(i)$$

subject to the constraints are

$$x + 2y \leq 10 \quad \dots(ii)$$

$$3x + y \leq 15 \quad \dots(iii)$$

$$x \geq 0, y \geq 0 \quad \dots(iv)$$

Draw the graph of the line $x + 2y = 10$.

x	0	10
y	5	0

Putting $(0, 0)$ in the inequality $x + 2y \leq 10$, we get

$$0 + 2 \times 0 \leq 10$$

$\Rightarrow 0 \leq 10$, which is true.

So, the half plane is towards the origin.

Since, $x, y \geq 0$.

So, the feasible region lies in the first quadrant.

Draw the graph of the line $3x + y = 15$.

x	0	5
y	15	0

Putting $(0, 0)$ in the inequality $3x + y \leq 15$, we get

$$3 \times 0 + 0 \leq 15$$

$\Rightarrow 0 \leq 15$, which is true.

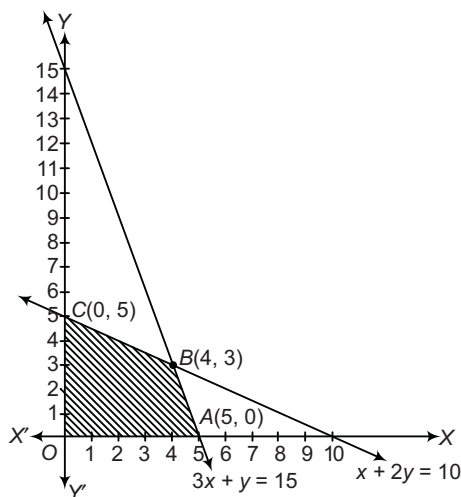
So, the half plane is towards the origin.

On solving equations $x + 2y = 10$ and $3x + y = 15$, we get

$$x = 4$$

and

$$y = 3$$



The corner points of the feasible region are $O(0, 0)$, $A(5, 0)$, $B(4, 3)$ and $C(0, 5)$. The values of Z at these points are as follows:

Corner points	$Z = 3x + 2y$
$O(0, 0)$	0
$A(5, 0)$	15
$B(4, 3)$	18 (Maximum)
$C(0, 5)$	10

Therefore, the maximum value of Z is 18 at the point $B(4, 3)$.

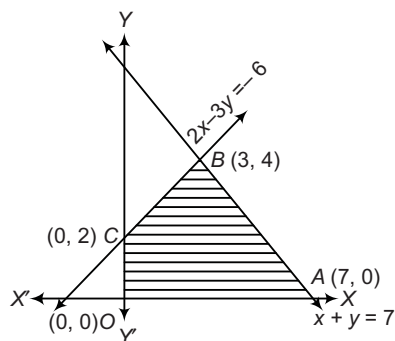
$A \rightarrow q$; $B \rightarrow p$; $C \rightarrow s$; $D \rightarrow r$

Type 5. Single Integer Answer Type Questions

Ex 20. The minimum value of the objective function $Z = 13x - 15y$ subject to the constraints are $x + y \leq 7$, $2x - 3y + 6 \geq 0$ and $x \geq 0$, $y \geq 0$ is λ . Then, $\lambda + 32$ is equal to _____.

Sol. (2) The given LPP is
Minimise $Z = 13x - 15y$
subject to the constraints are
 $x + y \leq 7$
 $2x - 3y + 6 \geq 0$
 $x, y \geq 0$

The graph of the given LPP is given below



From the above graph, we see that the shaded region is the feasible $OABCQ$ which is bounded. Now, the values of Z at the corner points are given by

Corner points	$Z = 13x - 15y$
$(0, 0)$	0
$(7, 0)$	91
$(3, 4)$	-21
$(0, 2)$	-30 (Minimum)

From the above table, we see that the minimum value of Z is -30.

Hence, $\lambda = -30 \Rightarrow \lambda + 32 = 2$

Ex 21. One kind of cake requires 200 g of flour and 25 g of fat and another kind of cake requires 100 g of flour and 50 g of fat. Then, the maximum number of cakes which can be made from 5 kg of flour and 1kg of fat assuming that there is no storage of the other ingredients used in making the cakes, is λ .

Then, the value of $\frac{\lambda}{6}$ is.

Sol. (5) Let x be the number of cakes of one kind and y be the number of cakes of other kind. Construct the following table:

Kind	Number of cakes	Flour required (in g)	Fat required (in g)
I	x	$200x$	$25x$
II	y	$100y$	$50y$
Total	$x + y$	$200x + 100y$	$25x + 50y$
Requirement		5000	1000

Our problem is to maximise $Z = x + y$... (i)
subject to the constraints are

$$200x + 100y \leq 5000$$

$$\Leftrightarrow 2x + y \leq 50 \quad \dots (ii)$$

$$25x + 50y \leq 1000 \Leftrightarrow x + 2y \leq 40 \quad \dots (iii)$$

$$\text{and } x, y \geq 0 \quad \dots (iv)$$

First, draw the graph of the line $2x + y = 50$.

x	0	25
y	50	0

Putting $(0, 0)$ in the inequality $2x + y \leq 50$, we get

$$2 \times 0 + 0 \leq 50$$

$$\Rightarrow 0 \leq 50, \text{ which is true.}$$

So, the half plane is towards the origin.

Secondly, draw the graph of the line $x + 2y = 40$.

x	0	40
y	20	0

Putting $(0, 0)$ in the inequality $x + 2y \leq 40$, we get

$$0 + 2 \times 0 \leq 40$$

$\Rightarrow$

$$0 \leq 40, \text{ which is true.}$$

So, the half plane is towards the origin.

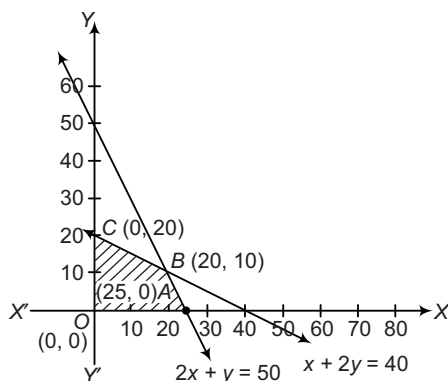
Since,

$$x, y \geq 0$$

So, the feasible region lies in the first quadrant.

On solving equations $2x + y = 50$ and $x + 2y = 40$, we get $B(20, 10)$

$\therefore$ Feasible region is $OABCO$.



The corner points of the feasible region are $O(0, 0)$, $A(25, 0)$, $B(20, 10)$ and $C(0, 20)$. The values of Z at these points are as follows:

Corner points	$Z = x + y$
$O(0, 0)$	0
$A(25, 0)$	25
$B(20, 10)$	30 (Maximum)
$C(0, 20)$	20

Thus, the maximum number of cakes that can be made is 30, i.e. 20 of one kind and 10 of the other kind.

Hence,

$$\lambda = 30$$

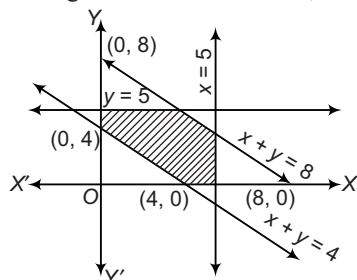
Then,

$$\frac{\lambda}{6} = 5$$

Target Exercises

Type 1. Only One Correct Option

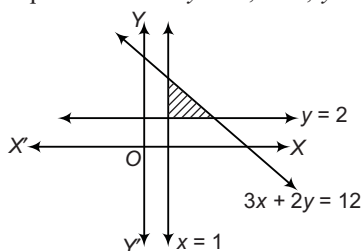
1. Linear inequalities for which the shaded region of the given figure is the solution set, are



- (a) $x + y \leq 8, x + y \leq 4, x \leq 5, y \leq 5, x \geq 0, y \geq 0$
 (b) $x + y \leq 8, x + y \geq 4, x \leq 5, y \leq 5, x \geq 0, y \geq 0$
 (c) $x + y \geq 8, x + y \geq 4, x \geq 5, y \geq 5, x \geq 0, y \geq 0$
 (d) None of the above

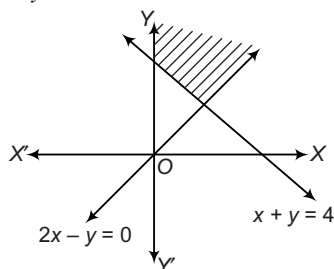
2. Which of the following is/are true?

I. The graphical solution of the system of inequalities $3x + 2y \leq 12, x \geq 1, y \geq 2$ is



II. The region represented by the solution set of the inequalities $2x + y \geq 6, 3x + 4y \leq 12$ is bounded.

III. The solution set of the inequalities $x + y \geq 4, 2x - y > 0$ is



- (a) Only I is true (b) I and II are true
 (c) I and III are true (d) only III is true

3. There is a factory located at each of the two places P and Q . From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C . The weekly requirements of the depots are respectively 5, 5 and 4 units commodity, while of the production capacity of the factories at P and

Q are 8 and 6 units, respectively. The cost of transportation per unit is given below:

From \ To	Cost (in ₹)		
	A	B	C
P	16	10	15
Q	10	12	10

Formulate the LPP, so that the units transported from each factory to each depot in such an order that the transportation cost is minimum.

- (a) Minimise $Z = x - 7y + 190$

subject to the constraints are

$$x + y \leq 8$$

$$x + y \geq 4$$

$$x \leq 5$$

$$y \leq 5$$

and $x, y \geq 0$

- (b) Minimise $Z = x + 7y + 190$

subject to the constraints are

$$x + y \leq 8$$

$$x + y \leq 5$$

and $x, y \geq 0$

- (c) Minimise $Z = x + 7y + 90$

subject to the constraints are

$$x + y \leq 8$$

$$x + y \leq 4$$

$$x \leq 5$$

$$y \leq 5$$

and $x, y \geq 0$

- (d) None of the above

4. A farmer mixes two brands P and Q of cattle feed.

Brand P , costing ₹ 250 per bag, contains 3 units of nutritional element A , 2.5 units of element B and 2 units of element C . Brand Q costing ₹ 200 per bag contains 1.5 units of nutritional element A , 11.25 units of elements B and 3 units of element C . The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units, respectively. Then, the minimum cost is

- (a) ₹ 1900 (b) ₹ 1850 (c) ₹ 1800 (d) ₹ 1950

5. A firm has to transport 1200 packages using large vans which can carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is ₹ 400 and each small van is ₹ 200. Not more than ₹ 3000 is to be spent on the job and the number of large vans cannot exceed the number of small vans.

Then, the minimum cost for the firm is

- (a) ₹ 2571.43 (b) ₹ 2571 (c) ₹ 2751.34 (d) ₹ 2157

6. A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for atmost 12 h whereas, machine III must be operated for atleast 5 h a day. She produces only two items M and N each requiring the use of all the three machines.

The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Items	Number of hours required on machines		
	I	II	III
M	1	2	1
N	2	1	1.25

She makes a profit of ₹ 600 and ₹ 400 on items M and N , respectively.

Then, to maximise the profit, number of units of item M , the manufacturer has to produce, is

- (a) 4
(b) 5
(c) 6
(d) 3
7. Anil wants to invest atmost ₹ 12000 in bonds A and B . According to the rules, he has to invest atleast ₹ 2000 in bond A and atleast ₹ 4000 in bond B . If the rate of interest in bond A is 8% per annum and in bond B is 10% per annum, then to maximise the interest, the investment in bond A and B are respectively
- (a) ₹ 10000 and ₹ 2000
(b) ₹ 2000 and ₹ 10000
(c) ₹ 6000 and ₹ 6000
(d) None of the above

8. The maximum value of Z , where $Z = 4x + 2y$ subject to constraints $4x + 2y \geq 46$, $x + 3y \leq 24$ and $x, y \geq 0$, is

- (a) 46
(b) 96
(c) 52
(d) None of these

9. If given constraints are $5x + 4y \geq 2$, $x \leq 6$ and $y \leq 7$, then the maximum value of the function $Z = x + 2y$, is

- (a) 13
(b) 14
(c) 15
(d) 20

10. The minimum value of the objective function $Z = 2x + 10y$ subject to constraints $x \geq 0$, $y \geq 0$, $x - y \geq 0$, $x - 5y \leq -5$, is

- (a) 10
(b) 15
(c) 12
(d) 8

11. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain atleast 8 units of vitamin A and 11 units of vitamin B. Food P costs ₹ 60/kg and food Q costs ₹ 80/kg. Food P contains 3 units/kg of vitamin A and 5 units/kg of vitamin B, while food Q contains 4 units/kg of vitamin A and 2 units/kg of vitamin B. Then, the minimum cost of the mixture is

- (a) ₹ 160
(b) ₹ 150
(c) ₹ 130
(d) ₹ 140

12. The linear programming problem

$$\text{Maximise } Z = x_1 + x_2$$

subject to constraints are

$$x_1 + 2x_2 \leq 2000, \quad x_1 + x_2 \leq 1500, \quad x_2 \leq 600$$

$$x_1 \geq 0 \text{ and } x_2 \geq 0 \text{ has}$$

- (a) no feasible solution
(b) unique optimal solution
(c) a finite number of optimal solutions
(d) infinite number of optimal solutions

Type 2. Assertion and Reason

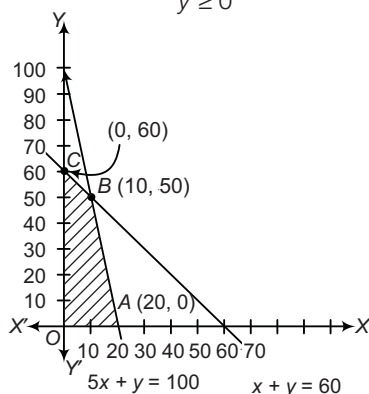
Directions (Q. Nos. 13-14) Consider, the graph of constraints stated as linear inequalities as below:

$$5x + y \leq 100 \quad \dots(i)$$

$$x + y \leq 60 \quad \dots(ii)$$

$$x \geq 0 \quad \dots(iii)$$

$$y \geq 0 \quad \dots(iv)$$



where, x and y are number of tables and chairs on which a furniture dealer wants to make his profit.

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

13. **Statement I** The points (10, 50), (0, 60) and (20, 0) are feasible solutions.

Statement II Points within and on the boundary of the feasible region represent feasible solutions of the constraints.

14. **Statement I** (25, 40) is an infeasible solution of the problem.

Statement II Any point inside the feasible region is called an infeasible solution.

Type 3. Linked Comprehension Based Questions

Passage (Q. Nos. 15-20) A furniture dealer deals in only two items — tables and chairs. He has ₹ 50000 to invest and has storage space of atmost 60 pieces. A table costs ₹ 2500 and a chair ₹ 500. He estimates that from the sale of one table, he can make a profit of ₹ 250 and that from the sale of one chair a profit of ₹ 75. He wants to know how many tables and chairs he should buy from the available money so as to maximise his total profit assuming that he can sell all the items which he buys.

15. The above stated optimisation problem is an example of
 (a) linear programming problem
 (b) functional programming problem
 (c) computer programming problem
 (d) None of the above
16. Which of the following are constraints for the given problem?
 (a) Maximum investment is ₹ 50000 and storage space is for maximum of 60 pieces
 (b) Minimum investment is ₹ 50000 and storage space is for minimum of 60 pieces
 (c) Maximum investment is ₹ 50000 and storage space is for minimum of 60 pieces
 (d) None of the above
17. If he buys tables only, then his profit will be
 (a) ₹ 500 (b) ₹ 5000
 (c) ₹ 7500 (d) ₹ 2500

18. Suppose he chooses to buy chairs only. Then, his maximum profit is
 (a) ₹ 4500 (b) ₹ 7500 (c) ₹ 4000 (d) ₹ 5700
19. If he choose 10 tables and 50 chairs, then his maximum profit is
 (a) ₹ 6200 (b) ₹ 6500 (c) ₹ 6250 (d) ₹ 6520
20. If x is the number of tables and y is the number of chairs, then the given problem, mathematically reduces to
 (a) maximise $Z = 250x + 75y$
 subject to the constraints are
 $5x + y \leq 100, x + y \leq 60$
 and $x, y \geq 0$
 (b) minimise $Z = 250x + 75y$
 subject to the constraints are
 $5x + y \leq 100, x + y \leq 60$
 and $x, y \geq 0$
 (c) minimise $Z = 250x + 75y$
 subject to the constraints are
 $5x + y \geq 100, x + y \geq 60$
 and $x, y \geq 0$
 (d) maximise $Z = 250x + 75y$
 subject to the constraints are
 $5x + y \geq 100, x + y \geq 60$
 and $x, y \geq 0$

Type 4. Match the Column

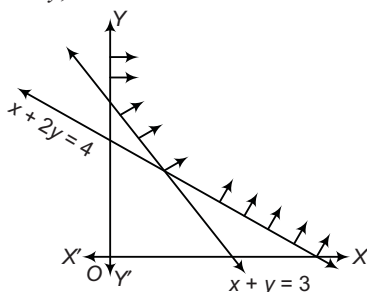
21. Match the linear inequalities given in Column I with solution set representing by graphs in Column II.

Column I	Column II
A. $2x - y > 1,$ $x - 2y < -1$	p.
B. $x + y \leq 6,$ $x + y \geq 4$	q.

Column I	Column II
C. $2x + y \geq 8,$ $x + 2y \geq 10$	r.
D. $x + y \leq 9,$ $y > x, x \geq 0$	s.

Type 5. Single Integer Answer Type Questions

22. The feasible region for an LPP is shown below. If $Z = 4x + y$, then the minimum value of Z is _____.



23. A factory manufactures two types of screws, A and B . Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 min on the automatic and 6 min on hand operated machines to get of screws A , while it takes 6 min on automatic and 3 min on the hand operated machines to manufacture a package to screws B .

Each machine is available for atmost 4 h on any day. The manufacturer can sell a package of screws A at a profit of ₹ 7 and screws B at a profit of ₹ 10.

Assuming that he can sell all the screws he manufactures.

If the maximum profit is λ . Then, $\left[\frac{\lambda}{100} \right]$, where $[]$ denotes the greatest integer function, is _____.

24. Two godowns A and B have grain capacity of 100 quintals and 50 quintals, respectively. They supply to 3 ration shops, D, E and F , whose requirements are 60, 50 and 40 quintals, respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per quintal (in ₹)		
From/To	A	B
D	6	4
E	3	2
F	2.50	3

Then, the minimum cost is k . Then, $(517 - k)$ is equal to _____.

Entrances Gallery

Other Engineering Entrances

1. A furniture dealer deals in only two items namely tables and chairs. He has ₹ 5000 to invest and space to store atmost 60 pieces. A table cost him ₹ 250 and a chair ₹ 60. He can sell a table at a profit of ₹ 15. Assume that, he can sell all the items that he produced. The number of constraints in the problem are

[BITSAT 2014]

- (a) 2 (b) 3
(c) 4 (d) 5

2. If $x + y \leq 2, x \geq 0, y \geq 0$ is the point at which maximum value of $3x + 2y$ attained will be

[Karnataka CET 2014]

- (a) (0, 2)
(b) (0, 0)
(c) (2, 0)
(d) $\left(\frac{1}{2}, \frac{1}{2} \right)$

3. Solve the linear programming problem.

Maximise $Z = x + 2y$

subject to the constraints $x - y \leq 10, 2x + 3y \leq 20$ and $x \geq 0, y \geq 0$

[J&K CET 2014]

- (a) $Z = 10$
(b) $Z = 30$
(c) $Z = 40$
(d) None of the above

4. Which of the following sets are not convex?

[AMU 2012]

- (a) $\{(x, y) : 8x^2 + 6y^2 \leq 24\}$
(b) $\{(x, y) : 6 \leq x^2 + y^2 \leq 36\}$
(c) $\{(x, y) : y \geq 3, y \geq 30\}$
(d) $\{(x, y) : x^2 \leq y\}$

5. Consider, the linear programming problem.

Maximise $Z = 4x + y$

subject to the constraints $x + y \leq 50, x + y \geq 100$ and $x, y \geq 0$. Then, maximum value of Z is

[AMU 2012]

- (a) 0 (b) 50
(c) 100 (d) Does not exist

6. The point which provides the solution of the linear programming problem, maximise $Z = 45x + 55y$ subject to the constraints $x, y \geq 0, 6x + 4y \leq 120$ and $3x + 10y \leq 180$ is

[MP PET 2011]

- (a) (15, 10) (b) (10, 15)
(c) (0, 18) (d) (20, 0)

7. Variables of the objective function of the linear programming problem are

[MP PET 2011]

- (a) zero
(b) zero or positive
(c) negative
(d) zero or negative

8. The area of the feasible region for the following constraints $3y + x \geq 3, x \geq 0$ and $y \geq 0$ will be [BITSAT 2011]
 (a) bounded
 (b) unbounded
 (c) convex
 (d) concave
9. The optimal value of the objective function is attained at the point is [BITSAT 2011]
 (a) given by intersection of inequations with axes only
 (b) given by intersection of inequations with X -axis only
 (c) given by corner points of the feasible region
 (d) None of the above
10. The maximum value of $Z = 9x + 13y$, subject to constraints $2x + 3y \leq 18, 2x + y \leq 10, x \geq 0$ and $y > 0$ is [MHT CET 2011]
 (a) 130 (b) 81
 (c) 79 (d) 99
11. Maximum value of $Z = 12x + 3y$, subject to constraints $x \geq 0, y \geq 0, x + y \leq 5$ and $3x + y \leq 9$ is [MP PET 2011]
 (a) 15 (b) 36 (c) 60 (d) 40
12. The maximum value of $Z = 10x + 6y$, subject to constraints $x \geq 0, y \geq 0, x + y \leq 12$ and $2x + y \leq 20$ is [AMU 2011]
 (a) 72 (b) 80 (c) 104 (d) 110
13. A vertex of a feasible region by the linear constraints $3x + 4y \leq 18, 2x + 3y \geq 3$ and $x, y \geq 0$ is [DCE 2011]
 (a) (0, 2) (b) (4.8, 0)
 (c) (0, 3) (d) None of these
14. For an LPP, minimise $Z = 2x + y$, subject to constraints $5x + 10y \leq 50, x + y \geq 1, y \leq 4$ and $x, y \geq 0$, then Z is equal to [DCE 2011]
 (a) 0 (b) 1
 (c) 2 (d) 12

Answers

Work Book Exercise

1. (b) 2. (a) 3. (b) 4. (c) 5. (d) 6. (a) 7. (c) 8. (b) 9. (d) 10. (b)

Target Exercises

1. (b) 2. (a) 3. (a) 4. (d) 5. (a) 6. (a) 7. (b) 8. (b) 9. (d) 10. (b)
 11. (a) 12. (d) 13. (a) 14. (c) 15. (a) 16. (a) 17. (b) 18. (a) 19. (c) 20. (a)
 21. (*) 22. (3) 23. (4) 24. (7)

*

21. $A \rightarrow p; B \rightarrow r; C \rightarrow s; D \rightarrow q$

Entrances Gallery

1. (c) 2. (c) 3. (d) 4. (d) 5. (d) 6. (b) 7. (b) 8. (b) 9. (c) 10. (c)
 11. (b) 12. (c) 13. (d) 14. (b)

Explanations

Target Exercises

1. (i) Consider, the line $x + y = 8$. We observe that the shaded region and origin lie on the same side of this line and $(0, 0)$ satisfies $x + y \leq 8$. Therefore, $x + y \leq 8$ is the linear inequality corresponding to the line $x + y = 8$.
- (ii) Consider, $x + y = 4$. We observe that shaded region and origin are on the opposite side of this line and $(0, 0)$ satisfies $x + y \geq 4$. Therefore, we must have $x + y \geq 4$ as linear inequalities corresponding to the line $x + y = 4$.
- (iii) Shaded portion lie below the line $y = 5$. So, $y \leq 5$ is the linear inequality corresponding to $y = 5$.
- (iv) Shaded portion lie on the left side of the line $x = 5$. So, $x \leq 5$ is the linear inequality corresponding to $x = 5$.
- (v) Also, the shaded region lies in the first quadrant only. Therefore, $x \geq 0, y \geq 0$
In view of (i), (ii), (iii), (iv) and (v) above the linear inequalities corresponding to the given solutions are $x + y \leq 8, x + y \geq 4, y \leq 5, x \leq 5, x \geq 0$ and $y \geq 0$.

2. I. The given system of inequalities are

$$3x + 2y \leq 12 \quad \dots(i)$$

$$x \geq 1 \quad \dots(ii)$$

$$\text{and} \quad y \geq 2 \quad \dots(iii)$$

Step I Consider, the given inequations as strict equations,

i.e. $3x + 2y = 12, x = 1, y = 2$

Step II Draw the table for $3x + 2y = 12$

(i.e. find the points on X-axis and Y-axis)

Step III Plot the graph

- (i) For $3x + 2y = 12$, use the table
- (ii) Graph of $x = 1$ will be a line parallel to Y-axis cutting X-axis at 1.
- (iii) Graph of $y = 2$ will be a line parallel to X-axis cutting Y-axis at 2.

Step IV Take a point $(0, 0)$ and put it in the given inequations (i), (ii) and (iii),

i.e. $0 + 0 \leq 12, 0 \leq 12$, which is true.

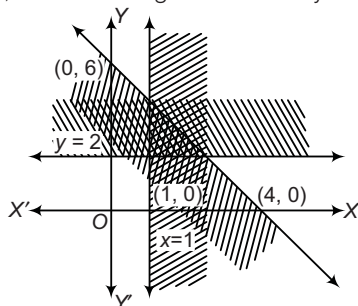
So, the shaded region will be towards the origin

$$0 \geq 1, \text{ which is false.}$$

So, the shaded region will be away from the origin

$$0 \geq 2, \text{ which is false.}$$

So, the shaded region will be away from the origin.



Thus, common shaded region shows the solution of the inequalities.

- II. The given system of inequalities are

$$2x + y \geq 6 \quad \dots(i)$$

$$\text{and} \quad 3x + 4y \leq 12 \quad \dots(ii)$$

Step I Consider, the given inequations as strict equations,

i.e. $2x + y = 6$

and $3x + 4y = 12$

Step II Find the points on the X-axis and Y-axis for

$$2x + y = 6$$

x	0	3
y	6	0

and $3x + 4y = 12$

x	0	4
y	3	0

Step III Plot the graph using the above tables.

Step IV Take a point $(0, 0)$ and put in the given inequations (i) and (ii),

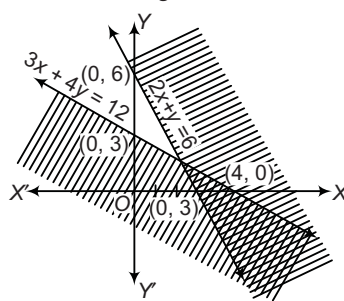
i.e. $0 + 0 \geq 6 \Rightarrow 0 \geq 6$, which is false.

So, the shaded region will be away from the origin

$$\text{and} \quad 0 + 0 \leq 12$$

$$\Rightarrow 0 \leq 12, \text{ which is true.}$$

So, the shaded region will be towards the origin.



Thus, common shaded region shows the solution of the inequality.

Since, common shaded region is not enclosed. So, it is not bounded.

- III. The given system of inequalities are

$$x + y \geq 4 \quad \dots(i)$$

$$\text{and} \quad 2x - y > 0 \quad \dots(ii)$$

Step I Consider, the given inequation as strict equations,

i.e. $x + y = 4, 2x - y = 0$

Step II Find the points on the X-axis and Y-axis for

$$x + y = 4$$

x	0	4
y	4	0

and

$$2x - y = 0$$

x	0	1
y	0	2

Step III Plot the graph using the above tables.

Step IV Take a point (0, 0) and put it in the inequation (i)

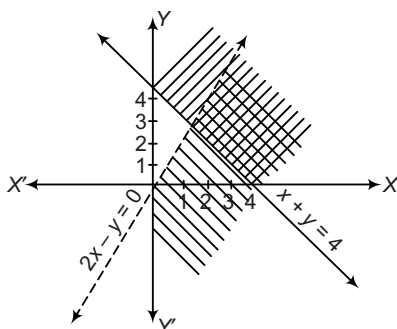
i.e. $0 + 0 \geq 4$, which is false.

So, the shaded region will be away from the origin.

Take a point (1, 0) and put it in the inequation (ii)

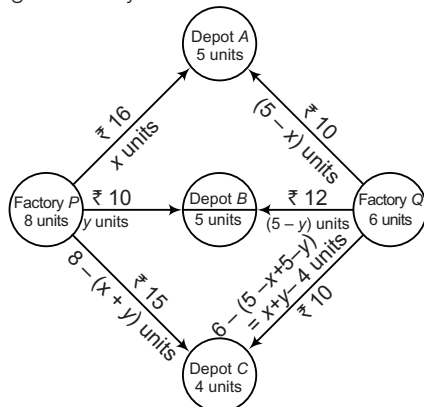
i.e. $2 - 0 > 0$, which is true.

So, the shaded region will be towards the point (1, 0).



Thus, common shaded region shows the solution of the inequalities.

3. The given information, can be expressed diagrammatically as follows:



Let the factory at P transports x units of commodity to depot at A and y units to depot at B. Since, the factory at P has the capacity of 8 units of the commodity.

Therefore, the left out $(8 - x - y)$ units will be transported to depot at C.

Since, the requirements are always non-negative quantities, therefore we have

$$x \geq 0, y \geq 0 \text{ and } 8 - x - y \geq 0$$

$$\Rightarrow x \geq 0, y \geq 0 \text{ and } x + y \leq 8$$

Since, weekly requirements of the depot at A is 5 units of the commodity and x units are transported from the factory at P. Therefore, the remaining $(5 - x)$ units are to be transported from the factory at Q.

Similarly, $(5 - y)$ units of the commodity will be transported from the factory at Q to the depot at B.

But the factory at Q has the capacity of 6 units only, therefore the remaining $6 - (5 - x + 5 - y) = x + y - 4$ units will be transported to the depot at C.

As, the requirements of the depots at A, B and C are always non-negative.

$$\therefore 5 - x \geq 0, 5 - y \geq 0 \text{ and } x + y - 4 \geq 0$$

$$\Rightarrow x \leq 5, y \leq 5 \text{ and } x + y \geq 4$$

The transportation cost from the factory at P to the depots at A, B and C are respectively, ₹16x, ₹10y and ₹15(8 - x - y). Similarly, the transportation cost from the factory at Q to the depots at A, B and C are respectively, ₹10(5 - x), ₹12(5 - y) and ₹10(x + y - 4).

$$\begin{aligned} \text{Therefore, the total transportation cost } Z \text{ is given by} \\ Z = 16x + 10y + 15(8 - x - y) + 10(5 - x) \\ + 12(5 - y) + 10(x + y - 4) \end{aligned}$$

$$= x - 7y + 190$$

Hence, the mathematical formulation of the given problem is as follows:

$$\text{Minimise } Z = x - 7y + 190$$

subject to the constraints are

$$x + y \leq 8$$

$$x + y \geq 4$$

$$x \leq 5$$

$$y \leq 5$$

$$\text{and } x \geq 0, y \geq 0$$

4. Let the farmer mixes x bags of brand P and y bags of brand Q. We construct the following table:

Brands	Numer of bags	Element A (units)	Element B (units)	Element C (units)	Cost
P	x	3x	2.5x	2x	250x
Q	y	1.5y	11.25y	3y	200y
Total	x + y	3x + 1.5y	2.5x + 11.25y	2x + 3y	250x + 200y
Minimum Requires		18	45	24	

So, our problem is minimise $Z = 250x + 200y$... (i)

subject to the constraints are

$$3x + 1.5y \geq 18 \Leftrightarrow 2x + y \geq 12 \quad \dots (ii)$$

$$2.5x + 11.25y \geq 45 \Leftrightarrow 2x + 9y \geq 36 \quad \dots (iii)$$

$$2x + 3y \geq 24 \quad \dots (iv)$$

$$\text{and } x \geq 0, y \geq 0 \quad \dots (v)$$

Firstly, draw the graph of the line $3x + 1.5y = 18$

x	6	0
y	0	12

Putting (0, 0) in the inequality $3x + 1.5y \geq 18$, we get

$$3 \times 0 + 1.5 \times 0 \geq 18$$

$$\Rightarrow 0 \geq 18, \text{ which is false.}$$

So, the half plane is away from the origin.

Secondly, draw the graph of the line

$$2.5x + 11.25y = 45$$

x	18	0
y	0	4

Putting (0, 0) in the inequality $2.5x + 11.25y \geq 45$,

$$\text{we get } 2.5 \times 0 + 11.25 \times 0 \geq 45$$

$$\Rightarrow 0 \geq 45, \text{ which is false.}$$

So, the half plane is away from the origin.

Thirdly, draw the graph of the line $2x + 3y = 24$

x	0	12
y	8	0

Putting $(0, 0)$ in the inequality $2x + 3y \geq 24$, we get

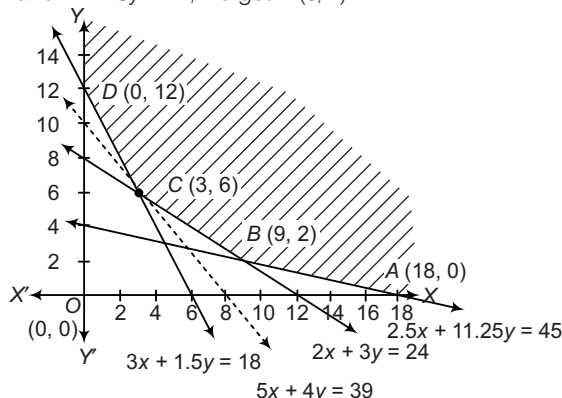
$$2 \times 0 + 3 \times 0 \geq 24 \Rightarrow 0 \geq 24, \text{ which is false.}$$

So, the half plane is away from the origin. Since, $x, y \geq 0$.

So, the feasible region lies in the first quadrant.

On solving equations $3x + 1.5y = 18$ and $2x + 3y = 24$, we get $C(3, 6)$.

Similarly, on solving equations $2.5x + 11.25y = 45$ and $2x + 3y = 24$, we get $B(9, 2)$.



The corner points of the feasible region are $A(18, 0)$, $B(9, 2)$, $C(3, 6)$ and $D(0, 12)$. The values of Z at these points are as follows:

Corner points	$Z = 250x + 200y$
$A(18, 0)$	4500
$B(9, 2)$	2650
$C(3, 6)$	1950 (Minimum)
$D(0, 12)$	2400

As the feasible region is unbounded, therefore 1950 may or may not be the minimum value of Z .

For this, we draw a graph of the inequality $250x + 200y < 1950$ or $5x + 4y < 39$ and check, whether the resulting half phase has points in common with the feasible region or not.

It can be seen that the feasible region has no common point with $5x + 4y < 39$.

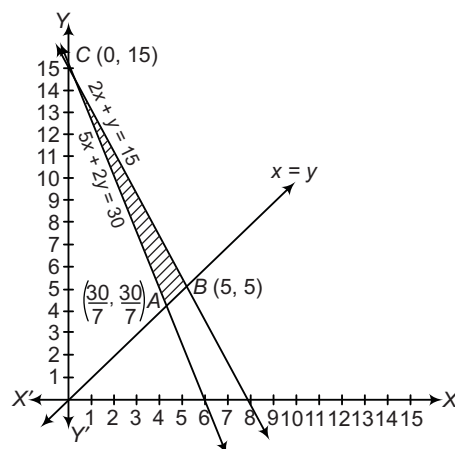
Therefore, the minimum value of Z is 1950 at $C(3, 6)$.

Thus, 3 bags of brand P and 6 bags of brand Q should be used in the mixture to minimise the cost to ₹ 1950.

5. Let x be the number of large vans and y be number of small vans, then the problem is to be minimised.

Thus, the given problem can be written mathematically as follows:

$$\begin{aligned} \text{Minimise } Z &= 400x + 200y \\ \text{subject to the constraints are} \\ 400x + 200y &\leq 3000 \\ 2x + y &\leq 15 \\ \text{and } x &\leq y \\ \Rightarrow 200x + 80y &\geq 1200 \\ 5x + 2y &\geq 30 \\ \text{and } x, y &\geq 0 \end{aligned}$$



The set of all feasible solutions of the above LPP is represented by the feasible region shaded in the above figure.

The coordinates of the corner points of the feasible region are $A\left(\frac{30}{7}, \frac{30}{7}\right)$, $B(5, 5)$ and $C(0, 15)$.

The values of Z at these points are as follows:

Corner points	$Z = 400x + 200y$
$A\left(\frac{30}{7}, \frac{30}{7}\right)$	2571.43 (Minimum)
$B(5, 5)$	3000
$C(0, 15)$	3000

Thus, the minimum cost for the firm is ₹ 2571.43

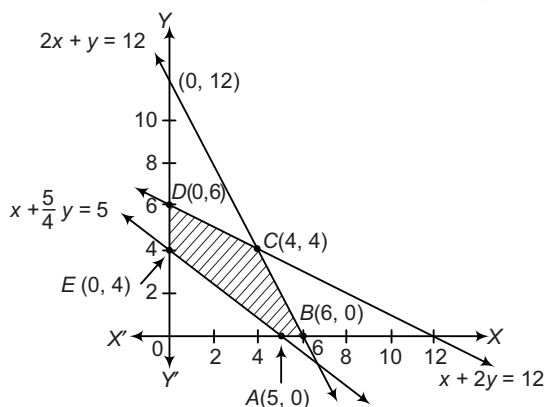
6. Let x and y be the number of items M and N , respectively.

Total profit on the production = ₹ $(600x + 400y)$

Mathematical formulation of the given problem is as follows:

$$\begin{aligned} \text{Maximise } Z &= 600x + 400y \\ \text{subject to the constraints are} \\ x + 2y &\leq 12 \quad (\text{constraint on machine I}) \quad \dots(i) \\ 2x + y &\leq 12 \quad (\text{constraint on machine II}) \quad \dots(ii) \\ x + (5/4)y &\geq 5 \quad (\text{constraint on machine III}) \quad \dots(iii) \\ \text{and } x &\geq 0, y \geq 0 \quad \dots(iv) \end{aligned}$$

Let us draw the graph of constraints (i) to (iv). $ABCDE$ is the feasible region (shaded) as shown in figure determined by the constraints (i) to (iv). Observe that the feasible region is bounded, coordinates of the corner points A, B, C, D and E are $(5, 0)$, $(6, 0)$, $(4, 4)$, $(0, 6)$ and $(0, 4)$, respectively.



Let us evaluate $Z = 600x + 400y$ at these corner points.

Corner points	$Z = 600x + 400y$
$A(5, 0)$	3000
$B(6, 0)$	3600
$C(4, 4)$	4000 (Maximum)
$D(0, 6)$	2400
$E(0, 4)$	1600

We see that the point $(4, 4)$ is given the maximum value of Z . Hence, the manufacturer has to produce 4 units of each item to get the maximum profit of ₹ 4000.

7. Let the investment in bond A and B are ₹ x and ₹ y , respectively.

Then, our problem is to

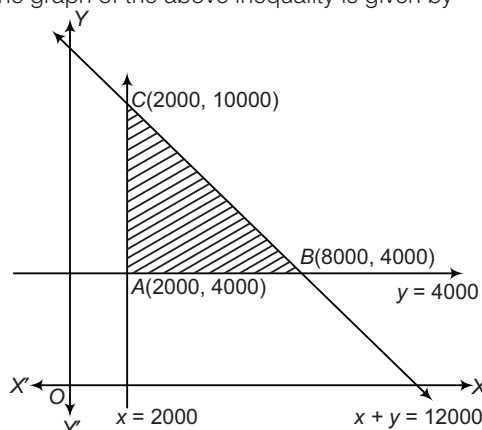
$$\text{maximise } Z = 0.08x + 0.10y$$

subject to the constraints are

$$x + y \leq 12000, x \geq 2000$$

$$y \geq 4000 \text{ and } y \geq 0$$

The graph of the above inequality is given by



In the above graph, shaded region is the feasible bounded region ABC.

Now, the values of Z at the corner points are as follows:

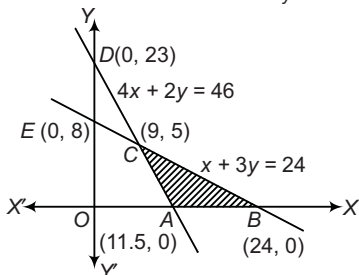
Corner points	$Z = 0.08x + 0.10y$
$A(2000, 4000)$	560
$B(8000, 4000)$	1040
$C(2000, 10000)$	1160 (Maximum)

From the above table, we see that the maximum value of Z is 1160, which occurs at the point $(2000, 10000)$.

Hence, to maximise the interest ₹ 2000 and ₹ 10000 must be invested in bond A and B, respectively.

8. Feasible region is ABCA

and maximise $Z = 4x + 2y$



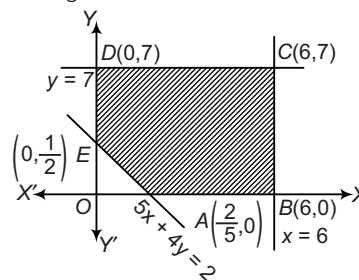
At point A $(11.5, 0)$, $Z = 4 \times 11.5 + 2 \times 0 = 46$

At point B $(24, 0)$, $Z = 4 \times 24 + 0 = 96$

and at point C $(9, 5)$, $Z = 4 \times 9 + 2 \times 5 = 46$

Hence, maximum value of Z is 96.

9. Feasible region is ABCDEA and $Z = x + 2y$



At point A $(\frac{2}{5}, 0)$, $Z = \frac{2}{5} + 0 = \frac{2}{5}$

At point B $(6, 0)$, $Z = 6 + 0 = 6$

At point C $(6, 7)$, $Z = 6 + 14 = 20$

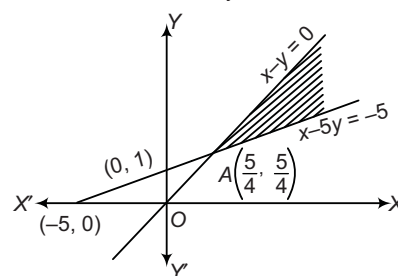
At point D $(0, 7)$, $Z = 0 + 2(7) = 14$

At point E $(0, \frac{1}{2})$, $Z = 0 + 2(\frac{1}{2}) = 1$

Hence, maximum value of Z is 20.

10. It is clear from the figure that the minimum value is at point A $(\frac{5}{4}, \frac{5}{4})$.

Given, $Z = 2x + 10y$



At point A $(\frac{5}{4}, \frac{5}{4})$, $z = 2(\frac{5}{4}) + 10(\frac{5}{4}) = 15$

11. Let Reshma mixes x kg of food P and y kg of food Q.

Construct the following table:

Food	Quantity	Vitamin (A)	Vitamin (B)	Cost (₹ per kg)
P	x kg	$3x$	$5x$	$60x$
Q	y kg	$4y$	$2y$	$80y$
Total		$3x + 4y$	$5x + 2y$	$60x + 80y$
Requirement		Atleast 8	Atleast 11	

The mixture must contain atleast 8 units of vitamin A and 11 units of vitamin B. Total cost Z of purchasing food is $Z = 60x + 80y$.

The mathematical formulation of the given problem is

$$\text{Minimise } Z = 60x + 80y \quad \dots(i)$$

subject to the constraints are

$$3x + 4y \geq 8 \quad \dots(ii)$$

$$5x + 2y \geq 11 \quad \dots(iii)$$

$$\text{and } x \geq 0, y \geq 0 \quad \dots(iv)$$

Draw the graph of the line $3x + 4y = 8$.

x	0	8/3
y	2	0

Putting (0, 0) in the inequality $3x + 4y \geq 8$, we get

$$3 \times 0 + 4 \times 0 \geq 8$$

$$0 \geq 8, \text{ which is false.}$$

So, the half plane is away from the origin.

Since, $x, y \geq 0$

So, the feasible region lies in the first quadrant.

Draw the graph of the line $5x + 2y = 11$.

x	0	11/5
y	11/2	0

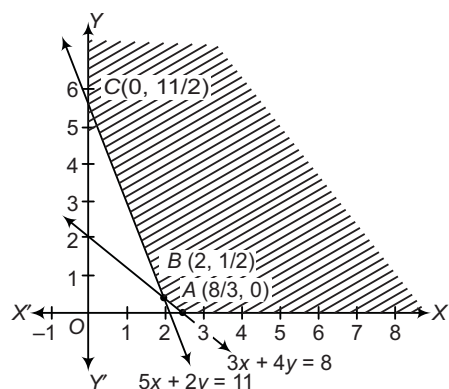
Putting (0, 0) in the inequality $5x + 2y \geq 11$, we have

$$5 \times 0 + 2 \times 0 \geq 11$$

$$\Rightarrow 0 \geq 11, \text{ which is false.}$$

So, the half plane is away from the origin.

It can be seen that the feasible region is unbounded.



On solving equations $3x + 4y = 8$ and $5x + 2y = 11$, we get $B\left(2, \frac{1}{2}\right)$.

The corner points of the feasible region are $A\left(\frac{8}{3}, 0\right)$,

$B\left(2, \frac{1}{2}\right)$ and $C\left(0, \frac{11}{2}\right)$.

The values of Z at these points are as follow:

Corner points	$Z = 60x + 80y$
$A\left(\frac{8}{3}, 0\right)$	160 (Minimum)
$B\left(2, \frac{1}{2}\right)$	160 (Minimum)
$C\left(0, \frac{11}{2}\right)$	440

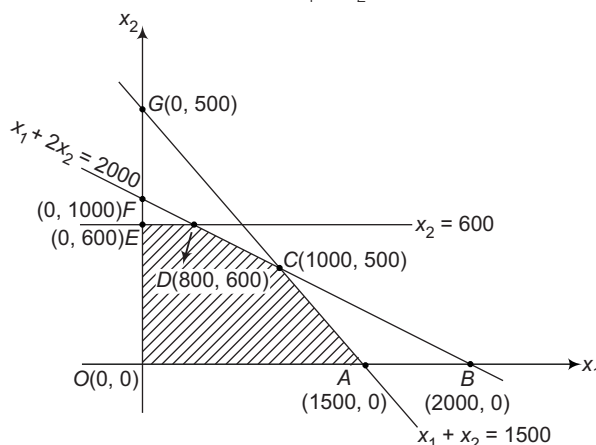
As the feasible region is unbounded, therefore 160 may or may not be the minimum value of Z . For this, we graph the inequality $60x + 80y < 160$ or $3x + 4y < 8$ and check whether the resulting half plane has points in common with the feasible region or not. It can be seen that the feasible region has no common point with $3x + 4y < 8$, therefore the minimum cost of the mixture will be ₹ 160 at line segment joining the points $A\left(\frac{8}{3}, 0\right)$ and $B\left(2, \frac{1}{2}\right)$.

12. Given constraints are

$$x_1 + 2x_2 \leq 2000, x_1 + x_2 \leq 1500, x_2 \leq 600 \text{ and } x_1 \geq 0$$

The feasible region is $OACDEO$.

$$\text{Given, } Z = x_1 + x_2$$



$$\text{At } O(0, 0), Z = 0 + 0 = 0$$

$$\text{At } A(1500, 0), Z = 1500 + 0 = 1500$$

$$\text{At } C(1000, 500), Z = 1000 + 500 = 1500$$

$$\text{At } D(800, 600), Z = 800 + 600 = 1400$$

$$\text{At } E(0, 600), Z = 0 + 600 = 600$$

Here, Z is maximum on the segment AC .

Hence, there are infinite optimal solutions.

13. **Feasible solutions** Points within and on the boundary of the feasible region represent feasible solutions of the constraints. In the given figure, every point within and on the boundary of the feasible region $OABC$ represents feasible solution to the problem. e.g. The point (10, 50) is a feasible solution of the problem and so are the points (0, 60), (20, 0). etc.

14. Any point outside the feasible region is called an infeasible solution. e.g. The point (25, 40) is an infeasible solution of the problem.

15. Linear programming problems are of much interest because of their wide applicability in industry, commerce, management, science, etc.

16. The dealer can invest his money in buying tables or chairs or combination of these. Further, he would earn different profits by following different investment strategies. There are certain over riding conditions or constraints viz., his investment is limited to a maximum of ₹ 50000 and so it is his storage space which is for a maximum of 60 pieces.

17. Suppose he decides to buy tables only and no chairs, so he can buy $50000 \div 2500$, i.e. 20 tables. His profit in this case will be ₹ (250×20) , i.e. ₹ 5000.

18. Suppose he chooses to buy chairs only and no tables. With his capital of ₹ 50000, he can buy $50000 \div 500$, i.e. 100 chairs. But he can store only 60 pieces. Therefore, he is forced to buy only 60 chairs which will give him a total profit of ₹ (60×75) , i.e. ₹ 4500.

19. He may choose to buy 10 tables and 50 chairs as he can store only 60 pieces. Total profit in this case would be ₹ $(10 \times 250 + 50 \times 75)$, i.e. ₹ 6250.

20. Let x be the number of tables and y be the number of chairs that the dealer buys. Obviously, x and y must be non-negative, i.e.

$$x \geq 0 \quad \dots(i)$$

$$y \geq 0 \quad \dots(ii)$$

[non-negative constraints]

The dealer is constrained by the maximum amount he can invest (here it is ₹ 50000) and by the maximum number of items he can store (here it is 60).

Stated mathematically,

$$2500x + 500y \leq 50000 \quad [\text{investment constraint}]$$

$$\text{or } 5x + y \leq 100 \quad \dots(iii)$$

$$\text{and } x + y \leq 60 \quad [\text{storage constraint}] \dots(iv)$$

The dealer wants to invest in such a way so as to maximise his profit, say Z which stated as a function of x and y is given by

$$Z = 250x + 75y \quad [\text{objective function}] \dots(v)$$

Mathematically, the given problems now reduce to

$$\text{Maximise } Z = 250x + 75y$$

subject to the constraints are

$$5x + y \leq 100, x + y \leq 60$$

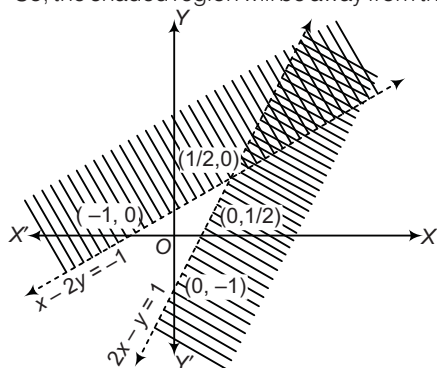
$$\text{and } x \geq 0, y \geq 0$$

21. A. The given system of inequalities

$$2x - y > 1 \quad \dots(i)$$

$$\text{and } x - 2y < -1 \quad \dots(ii)$$

So, the shaded region will be away from the origin.

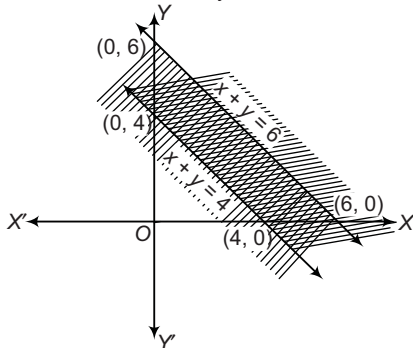


Thus, common shaded region shows the solution of the inequalities.

- B. The given system of inequalities

$$x + y \leq 6 \quad \dots(i)$$

$$\text{and } x + y \geq 4 \quad \dots(ii)$$



Take the inequality (i), (ii),

$$\text{i.e. } 0 + 0 \leq 6 \Rightarrow 0 \leq 6, \text{ which is true.}$$

So, the shaded region will be towards the origin.

Take the inequality (ii),

$$\text{i.e. } 0 + 0 \geq 4 \Rightarrow 0 \geq 4, \text{ which is false.}$$

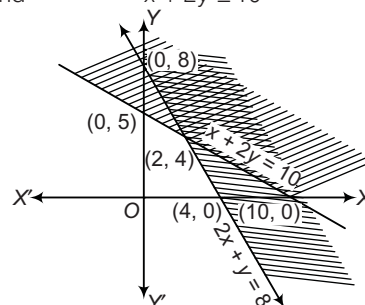
So, the shaded region will be away from the origin.

Thus, common shaded region shows the solution of the inequalities.

- C. The given system of inequalities is

$$2x + y \geq 8 \quad \dots(i)$$

$$\text{and } x + 2y \geq 10 \quad \dots(ii)$$



Thus, common shaded region shows the solution of the inequalities.

- D. The given system of inequalities is

$$x + y \leq 9 \quad \dots(i)$$

$$y > x \quad \dots(ii)$$

$$\text{and } x \geq 0 \quad \dots(iii)$$

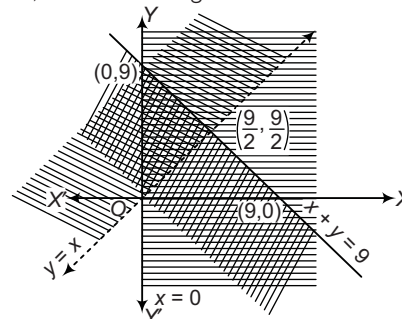
So, the shaded region will be towards the origin.

Take a point $(0, 1)$, put in $y > x$, $1 > 0$, which is true.

So, the shaded region will be towards the origin.

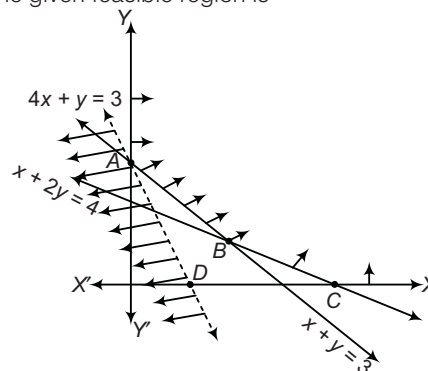
Take a point $(1, 0)$, put it in $x \geq 0$, $1 \geq 0$, which is true.

So, the shaded region will be towards the origin.



Thus, common shaded region shows the solution of the inequalities.

22. The given feasible region is



From the above graph, we see that ABC is the feasible region which is unbounded and its corner points are $A(0, 3)$, $B(2, 1)$ and $C(4, 0)$.

The given objective function is $Z = 4x + y$.
The values of Z at the corner points are as follows:

Corner points	$Z = 4x + y$
$A(0, 3)$	3 (Minimum)
$B(2, 1)$	9
$C(4, 0)$	16

From the above table, we see that the minimum value of Z is 3. Now, to check, if $Z = 3$ is the minimum value, we draw the line $4x + y = 3$.

Since, the open half plane $4x + y < 3$ does not intersect the feasible region at any point. Hence, minimum value of Z is 3.

23. Let the manufacturer produces x package of screws A and y package of screws B . We construct the following table:

Types of screw	Number of packages	Time on automatic machine (in min)	Time on hand machine (in min)	Profit (in ₹)
A	x	$4x$	$6x$	$7x$
B	y	$6y$	$3y$	$10y$
Total	$x + y$	$4x + 6y$	$6x + 3y$	$7x + 10y$
Availability		$4 \times 60 = 240$	$4 \times 60 = 240$	

The profit on a package of screws A is ₹ 7 and on the package of screws B is ₹ 10.

Our problem is to maximise $Z = 7x + 10y$... (i)
subject to the constraints are

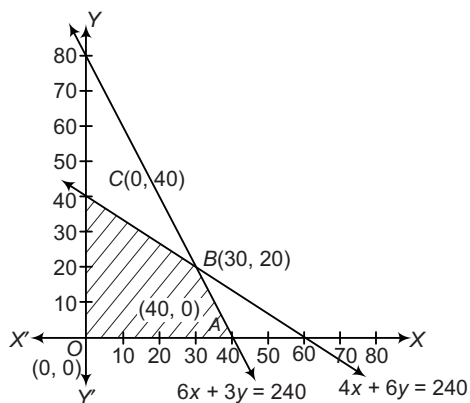
$$4x + 6y \leq 240 \Rightarrow 2x + 3y \leq 120 \quad \dots (ii)$$

$$6x + 3y \leq 240 \Rightarrow 2x + y \leq 80 \quad \dots (iii)$$

$$\text{and } x \geq 0, y \geq 0 \quad \dots (iv)$$

Firstly, draw the graph of the line $2x + 3y = 120$.

x	0	60
y	40	0



Putting $(0, 0)$ in the inequality $2x + 3y \leq 120$, we have
 $2 \times 0 + 3 \times 0 \leq 120$

$$\Rightarrow 0 \leq 120, \text{ which is true.}$$

So, the half plane is towards the origin.

Secondly, draw the graph of the line $2x + y = 80$.

x	40	0
y	0	80

Putting $(0, 0)$ in the inequality $2x + y \leq 80$, we get
 $2 \times 0 + 0 \leq 80$

$$\Rightarrow 0 \leq 80, \text{ which is true.}$$

So, the half plane is towards the origin. Since, $x, y \geq 0$

So, the feasible region lies in the first quadrant.

On solving equations $2x + 3y = 120$ and $2x + y = 80$, we get $B(30, 20)$.

$\therefore$ Feasible region is $OABCO$.

The corner points of the feasible region are $O(0, 0)$, $A(40, 0)$, $B(30, 20)$ and $C(0, 40)$. The values of Z at these points are as follows:

Corner points	$Z = 7x + 10y$
$O(0, 0)$	0
$A(40, 0)$	280
$B(30, 20)$	410 (Maximum)
$C(0, 40)$	400

The maximum value of Z is ₹ 410 at $B(30, 20)$.

Thus, the factory should produce 30 packages of screws A and 20 packages of screws B to get the maximum profit of ₹ 410.

Hence,

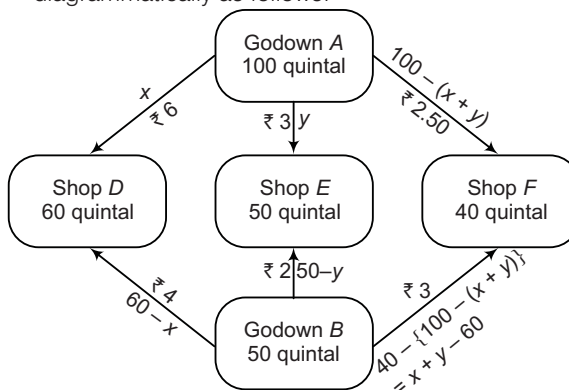
$$\lambda = 410$$

$$\left[\frac{\lambda}{100} \right] = \left[\frac{410}{100} \right] = 4$$

24. Let godown A supplies x and y quintals of grain to the shops D and E respectively.

Then, $(100 - x - y)$ will be supplied to shop F .

The requirement at shop D is 60 quintals, since x quintals are transported from godown A . Therefore, the remaining $(60 - x)$ quintals are transported from godown B . Similarly, $(50 - y)$ quintals and $40 - (100 - x - y) = (x + y - 60)$ quintals will be transported from godown B to shops E and F respectively. The given problem can be represented diagrammatically as follows:



Let Z be the total cost of transportation, then

$$Z = 6x + 3y + 2.50(100 - x - y) + 4(60 - x) + 2(50 - y) + 3\{(x + y) - 60\}$$

$$= 6x + 3y + 250 - 2.5x - 2.5y + 240 - 4x + 100 - 2y + 3x + 3y - 180$$

$$= 2.50x + 1.50y + 410 \quad \dots (i)$$

subject to the constraints are $60 - x \geq 0 \Leftrightarrow x \leq 60$... (ii)
 $50 - y \geq 0 \Rightarrow y \leq 50$... (iii)
 $100 - (x + y) \geq 0 \Rightarrow x + y \leq 100$... (iv)
 $x + y - 60 \geq 0 \Rightarrow x + y \geq 60$... (v)
 $x \geq 0, y \geq 0$... (vi)

Firstly, draw the graph of the line $x + y = 100$.

x	0	100
y	100	0

Putting (0, 0) in the inequality $x + y \leq 100$, we get
 $0 + 0 \leq 100 \Rightarrow 0 \leq 100$, which is true.

So, the half plane is towards the origin.

Secondly, draw the graph of the line $x = 60$.

Putting (0, 0) in the inequality $x \leq 60$, we get
 $0 \leq 60$, which is true.

So, the half plane is towards the origin.

Thirdly, draw the graph of the line $x + y = 60$.

x	0	60
y	60	0

Putting (0, 0) in the inequality $x + y \geq 60$, we get
 $0 + 0 \geq 60$

$\Rightarrow 0 \geq 60$, which is false.

So, the half plane is away from the origin.

Fourthly, draw the graph of the line $y = 50$.

Putting (0, 0) in the inequality $y \leq 50$, we get
 $0 \leq 50$, which is true.

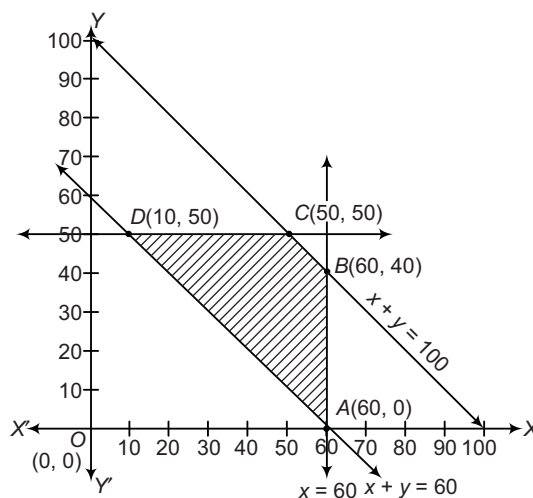
So, the half plane is towards the origin.

Since, $x, y \geq 0$.

So, the feasible region lies in the first quadrant.

The intersection points of the given lines are
 $A(60, 0), B(60, 40), C(50, 50)$ and $D(10, 50)$.

$\therefore$ Feasible region is ABCDA.



The corner points of the feasible region are $A(60, 0), B(60, 40), C(50, 50)$ and $D(10, 50)$. The values of Z at these points are as follows:

Corner points	$Z = 2.5x + 1.5y + 410$
$A(60, 0)$	560
$B(60, 40)$	620
$C(50, 50)$	610
$D(10, 50)$	510 (Minimum)

The minimum value of Z is 510 at $D(10, 50)$.

Thus, the amount of grain transported from A to D, E and F is 10 quintals, 50 quintals and 40 quintals respectively and from B to D, E and F is 50 quintals, 0 quintal and 0 quintal, respectively. The minimum cost is ₹ 510.

Hence,

$$k = 510$$

$$517 - k = 7$$

Entrances Gallery

1. If x tables and y chairs are purchased for maximum profit.

$$\text{Then, } x + y \leq 60, 5x + \frac{6y}{5} \leq 100, x \geq 0, y \geq 0$$

So, number of constraints are four.

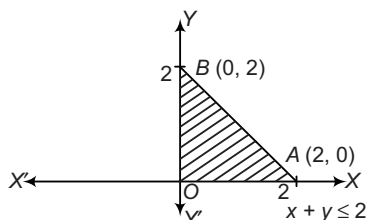
2. Given, $x + y \leq 2, x \geq 0$ and $y \geq 0$

$$\text{Let } Z = 3x + 2y$$

Now, table for $x + y = 2$

x	0	2	1
y	2	0	1

At (0, 0), $0 + 0 \leq 2 \Rightarrow 0 \leq 2$, which is true.



So, shaded portion is towards the origin.

$\therefore$ The corner points of shaded region are

$$O(0, 0), A(2, 0) \text{ and } B(0, 2).$$

At point $O(0, 0)$, $Z = 3(0) + 2(0) = 0$

At point $A(2, 0)$, $Z = 3(2) + 2(0) = 6$ (maximum)

At point $B(0, 2)$, $Z = 3(0) + 2(2) = 4$

Hence, maximum value of Z is 6 at point (2, 0).

3. We have to maximise,

$$Z = x + 2y$$

subject to the constraints are

$$x - y \leq 10$$

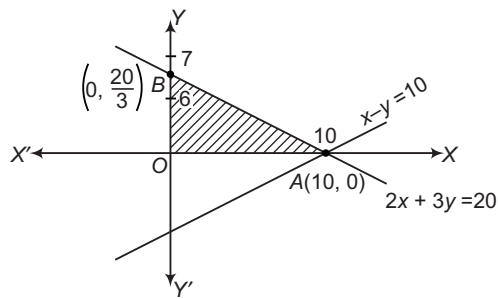
$$2x + 3y \leq 20$$

$$x \geq 0$$

$$y \geq 0$$

and

On taking given constraints as equations, we get the following graph:



Here, $OABO$ is the required feasible region whose corner points are $O(0, 0)$, $A(10, 0)$ and $B\left(0, \frac{20}{3}\right)$.

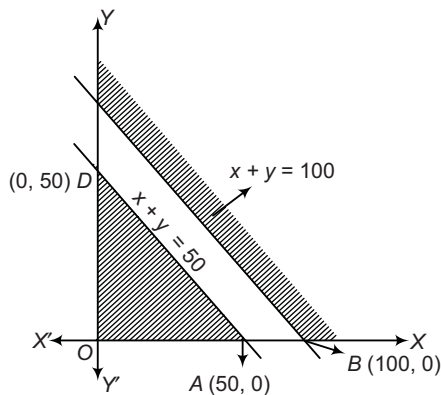
Corner points	$Z = x + 2y$
$O(0, 0)$	$Z = 0$
$A(10, 0)$	$Z = 10$
$B\left(0, \frac{20}{3}\right)$	$Z = 0 + 2 \times \frac{20}{3} = \frac{40}{3}$ (Maximum)

Hence, maximum value of Z is $\frac{40}{3}$, which is obtained at $B\left(0, \frac{20}{3}\right)$.

So, no option is correct.

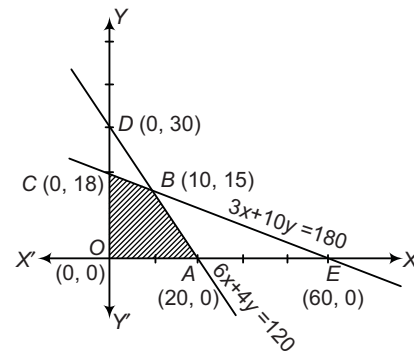
4. The graph of $y = x^2$ is a parabola. As, all the points satisfying the inequality, i.e. inside the parabola.

5. Given, maximise $Z = 4x + y$
and $x + y \leq 50$,
 $x + y \geq 100$,
 $x, y \geq 0$



Hence, it is clear from the graph that it is not bounded region, so maximum value cannot be determined.

6. Given, constraints are $6x + 4y \leq 120$... (i)
 $3x + 10y \leq 180$... (ii)
and $x, y \geq 0$... (iii)
Objective function $Z = 45x + 55y$ is to be maximised.

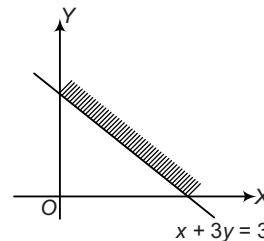


Here, $OABCO$ is the required feasible region, whose corner points are O, A, B and C .

Corner points	$Z = 45x + 55y$
$O(0, 0)$	0
$A(20, 0)$	900
$B(10, 15)$	1275 (Maximum)
$C(0, 18)$	990

Hence, the maximum value of Z is 1275 at $B(10, 15)$.

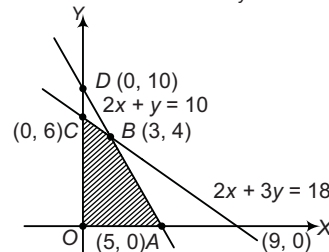
7. The values of variables of objective function of the LPP can never take negative values, therefore these variables are zero or positive.
8. Given, constraints are $3y + x \geq 3$, $x \geq 0$, $y \geq 0$



It is clear from the graph that, it is unbounded.

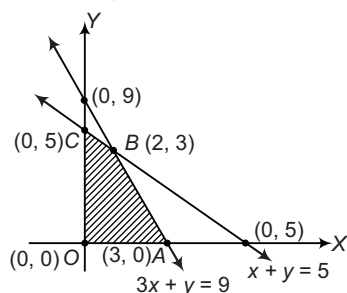
9. The optimal value of the objective function is attained at the points given by corner points of the feasible region.
10. Given constraints are $2x + 3y \leq 18$, $2x + y \leq 10$ and $x \geq 0$, $y \geq 0$.

The feasible region is $OABCO$.
Also, $Z = 9x + 13y$



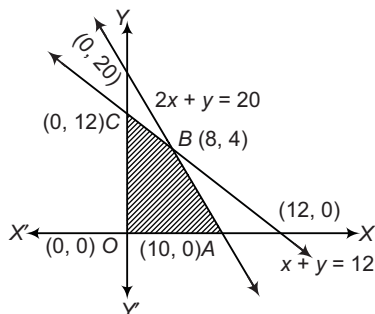
At $O(0, 0)$, $Z = 0$
At $A(5, 0)$, $Z = 45$
At $B(3, 4)$, $Z = 27 + 52 = 79$ (Maximum)
At $C(0, 6)$, $Z = 78$
Hence, maximum value of Z is 79.

11. Given constraints are $x \geq 0, y \geq 0, x + y \leq 5$ and $3x + y \leq 9$ and $Z = 12x + 3y$. Here, feasible region is $OABCO$.



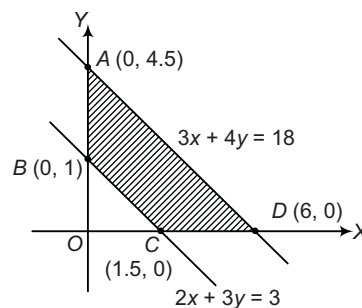
At point $O(0, 0)$, $Z = 12(0) + 3(0) = 0$
 At point $A(3, 0)$, $Z = 12(3) + 3(0) = 36$ (Maximum)
 At point $B(2, 3)$, $Z = 12(2) + 3(3) = 33$
 At point $C(0, 5)$, $Z = 12(0) + 3(5) = 15$
 Hence, maximum value of Z is 36.

12. Given, constraints are $x \geq 0, y \geq 0, x + y \leq 12, 2x + y \leq 20$. The feasible region is $OABCO$.

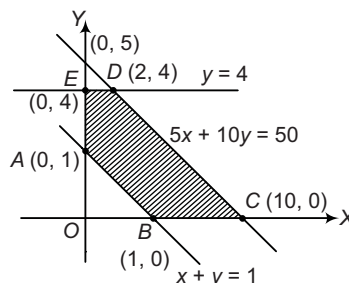


$\therefore Z = 10x + 6y$
 At point $O(0, 0)$, $Z = 10(0) + 6(0) = 0$
 At point $A(10, 0)$, $Z = 10(10) + 6(0) = 100$
 At point $B(8, 4)$, $Z = 10(8) + 6(4) = 104$ (Maximum)
 At point $C(0, 12)$, $Z = 10(0) + 6(12) = 72$
 Hence, maximum value of Z is 104.

13. Feasible region is $ABCD$ and vertices of the feasible region are $A(0, 4.5)$, $B(0, 1)$, $C(1.5, 0)$ and $D(6, 0)$.



14. Feasible region is $ABCDEA$ and vertices of the feasible region are $A(0, 1)$, $B(1, 0)$, $C(10, 0)$, $D(2, 4)$ and $E(0, 4)$.



$\therefore$ Minimise $Z = 2x + y$
 At point $A(0, 1)$, $Z = 2 \times 0 + 1 = 1$ (minimum)
 At point $B(1, 0)$, $Z = 2 \times 1 + 0 = 2$
 At point $C(10, 0)$, $Z = 2 \times 10 + 0 = 20$
 At point $D(2, 4)$, $Z = 2 \times 2 + 4 = 8$
 At point $E(0, 4)$, $Z = 2 \times 0 + 4 = 4$
 Thus, minimum value of objective function is 1 at $(0, 1)$.

16

Advanced Probability

Introduction

If an experiment is repeated under essentially homogeneous and similar conditions, in general we have two types of situations:

- (i) *The result, which is also called the outcome, is unique, i.e. certain.*

This type of phenomena are called deterministic or predictable. e.g. The velocity of a particle after time t is given by $v = u + at$, where u is the initial velocity and a is the acceleration.

This equation uniquely determines v at any time, if u and a are known.

- (ii) *The result is not unique but may be one of the several possible outcomes.*

The phenomena covered by this is called unpredictable or probabilistic.

e.g. In tossing of a coin, one is not sure whether a head or a tail will be obtained but the outcome must be tail or head.

Definitions of Various Terms

Random Experiment

It is an experiment in which

- (i) all the outcomes of the experiment are known in advance.
- (ii) the exact outcome of any specific performance of the experiment is unpredictable, i.e. not known in advance.

Sample Space

The set of all possible outcomes of an experiment is called the sample space, provided no two or more of these outcomes can occur simultaneously and exactly one of these outcomes must occur whenever the experiment is conducted. It is denoted by $n(S)$.

Event

Any particular performance of a random experiment is called a trial and outcome or combination of outcomes are termed as events.

e.g. If a coin is tossed repeatedly, the result is not unique. We may get any of the two faces, either head or tail.

Chapter Snapshot

- Introduction
- Definitions of Various Terms
- Probability
- Important Results Related to Probability
- Conditional Probability
- Multiplication Theorems on Probability
- Independent Events
- More on Independent Events
- Law of Total Probability
- Baye's Theorem
- Random Variable and its Distribution
- Mean and Variance of a Random Variable
- Binomial Distribution
- Mean and Variance of Binomial Distribution
- Relation between Mean and Variance
- Mode of Binomial Distribution
- Poisson Distribution
- Geometrical Probability
- Application based Results

Thus, tossing of a coin is a random experiment or trial and getting of a head or a tail is an event.

Event is called simple, if it corresponds to a single possible outcome of the experiment, otherwise it is known as a compound or composite event.

e.g. In tossing of a, the event of getting 6 is a simple event but the event of getting an even number is a composite event.

Types of Events

i. Equally likely events The given events are said to be equally likely, if none of them is expected to occur in preference to the other.
e.g. In throwing an unbiased die, all the six faces are equally likely to come.

ii. Mutually exclusive events Events are said to be exclusive or incompatible, if the happening of anyone of them precludes the happening of all the others, i.e. if no two or more of them can happen simultaneously in the same trial.

Or

Two or more events associated to a random experiment are said to be mutually exclusive events, if the occurrence of one of them prevents or denies the occurrence of all others.

If E_1 and E_2 are mutually exclusive, then $P(E_1 \cap E_2) = 0$.

e.g. In throwing a die, all the 6 faces numbered 1 to 6 are mutually exclusive, since if anyone of these faces comes, the possibility of others, in the same trial, is ruled out.

✎ If $E_1, E_2, \dots, E_n$ are mutually exclusive events, then $E_1 \cap E_2 \cap \dots \cap E_n = \phi$

iii. Exhaustive events The total number of possible outcomes of a random experiment is known as the exhaustive events or cases.

Or

Two or more events associated to a random experiment are exhaustive, if their union is the sample space, i.e. events $A_1, A_2, A_3, \dots, A_n$ associated to a random experiment with sample space are exhaustive, if

$$A_1 \cup A_2 \cup \dots \cup A_n = S$$

$$\text{i.e. } P(A_1 \cup A_2 \cup \dots \cup A_n) = 1$$

e.g. In throwing of two dice, the exhaustive number of cases is 6^2 , i.e. 36, since any of the numbers 1 to 6 on the first dice can be associated with any of the 6 numbers on the other dice.

✎ In throwing of n dice, the exhaustive number of cases is 6^n .

iv. Complement of an event If a sample space associated with a random experiment is S and event is E , then the complement of an event E is denoted by E' , E^c or $\bar{E}$, means event E does not occur.

$$E' = [n : n \in S, n \notin E]$$

$$(i) E \cup E' = S$$

$$(ii) E \cap E' = \phi$$

$$(iii) (E')' = E$$

e.g. When an unbiased die is thrown, then the sample space

$$S = [1, 2, 3, 4, 5, 6] \quad \text{and} \quad E = [1, 3, 5]$$

$$\text{Then,} \quad E' = [2, 4, 6]$$

Probability

The probability of an event E occur is the ratio of the number of cases in its favour to the total number of cases.

$$\begin{aligned} \therefore P(E) &= \frac{\text{Number of cases favourable to event } E}{\text{Total number of cases}} \\ &= \frac{n(E)}{n(S)} \end{aligned}$$

Probability of non-occurrence of event E is

$$P(\bar{E}) = 1 - P(E)$$

$$\text{and } 0 \leq P(E) \leq 1$$

If $P(E) = 1$, then event E is known as certain event and if $P(E) = 0$, then E is known as impossible event.

✎ **Example 1.** Two cards are drawn at random from a pack of 52 cards. The probability of getting atleast a spade and an ace, is

- (a) $\frac{1}{34}$ (b) $\frac{8}{221}$
(c) $\frac{1}{26}$ (d) $\frac{2}{51}$

Sol. (c) Total number of cases, $n(S) = {}^{52}C_2$

The number of selections of 1 spade and 1 ace from 3 aces or selections of the ace of spade and 1 other spade,

$$n(E) = {}^{13}C_1 \times {}^3C_1 + {}^{12}C_1 \times {}^1C_1 = 51$$

$$\therefore P(E) = \frac{51}{{}^{52}C_2} = \frac{1}{26}$$

Odds in Favour and Odds Against an Event

If a is the number of cases favourable to the event E , b is the number of cases favourable to the event $\bar{E}$, i.e. number of cases against to E . Then, odds in

favour of E are $a : b$ and odds against of E are $b : a$.
Then,

$$P(E) = \frac{a}{a+b}$$

and

$$P(\bar{E}) = \frac{b}{a+b}$$

Thus, odds in favour of an event $E = \frac{a}{b}$

$$\frac{\frac{a}{a+b}}{\frac{b}{a+b}} = \frac{P(E)}{P(\bar{E})}$$

and odds against an event $E = \frac{b}{a} = \frac{P(\bar{E})}{P(E)}$

➤ **Example 2.** What are the odds in favour of throwing atleast 8 in a single throw with two dice?

(a) 5 : 1

(b) 5 : 3

(c) 5 : 7

(d) 5 : 6

Sol. (c) Let S be the sample space and E be the event of occurrence of a total of atleast 8, then

$$n(S) = 6 \times 6 = 36 \quad \text{and} \quad n(E) = 15$$

$$\therefore P(E) = \frac{15}{36}$$

$$\Rightarrow P(\bar{E}) = 1 - P(E) = 1 - \frac{15}{36} = \frac{21}{36}$$

$$\text{Odd in favour of } E = P(E) : P(\bar{E}) = 5 : 7$$

Important Results Related to Probability

i. If E_1 and E_2 are events associated with a random experiment, then
 $P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$
[by addition theorem]

If E_1 and E_2 are mutually exclusive events, i.e. $E_1 \cap E_2 = \phi$, then

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) [\because P(\phi) = 0]$$

ii. If E_1, E_2 and E_3 are events associated with a random experiment, then
 $P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3) - P(E_1 \cap E_2) - P(E_2 \cap E_3) - P(E_1 \cap E_3) + P(E_1 \cap E_2 \cap E_3)$

If E_1, E_2 and E_3 are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup E_3) = P(E_1) + P(E_2) + P(E_3)$$

iii.

In general, $P(E_1 \cup E_2 \cup \dots \cup E_n)$

$$= \sum_{i=1}^n P(E_i) - \sum_{i < j} P(E_i \cap E_j) + \sum_{i < j < k} P(E_i \cap E_j \cap E_k) - \dots + (n/2)(-1)^{n-1} P(E_1 \cap E_2 \cap \dots \cap E_n)$$

If $E_1, E_2, \dots, E_n$ are mutually exclusive events, then

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = \sum_{i=1}^n P(E_i)$$

iv.

(a) $P(\bar{E}_1 \cap E_2) = P(E_2) - P(E_1 \cap E_2)$

(b) $P(E_1 \cap \bar{E}_2) = P(E_1) - P(E_1 \cap E_2)$

(c) $P[(E_1 \cap \bar{E}_2) \cup (\bar{E}_1 \cap E_2)] = P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$

(d) $P(\bar{E}_1 \cap \bar{E}_2) = 1 - P(E_1 \cup E_2)$

(e) $P(\bar{E}_1 \cup \bar{E}_2) = 1 - P(E_1 \cap E_2)$

v.

If $E_2 \subset E_1$, then

(a) $P(E_1 \cap \bar{E}_2) = P(E_1) - P(E_2)$

(b) $P(E_2) \leq P(E_1)$

vi.

Booley's inequality If events

$E_1, E_2, E_3, \dots, E_n$ are associated with a random experiment, then

(a) $P\left(\bigcap_{i=1}^n E_i\right) \geq \sum_{i=1}^n P(E_i) - (n-1)$

(b) $P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i)$

vii.

P (atleast two events from E_1, E_2 and E_3 occur)

$$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 2P(E_1 \cap E_2 \cap E_3)$$

viii.

P (two events from E_1, E_2 and E_3 occur)

$$= P(E_1 \cap E_2) + P(E_2 \cap E_3) + P(E_3 \cap E_1) - 3P(E_1 \cap E_2 \cap E_3)$$

ix.

P (only one event occur from E_1, E_2 and E_3)

$$= P(E_1) + P(E_2) + P(E_3) - 2P(E_1 \cap E_2) - 2P(E_2 \cap E_3) - 2P(E_3 \cap E_1) + 3P(E_1 \cap E_2 \cap E_3)$$

x.

If $E_1, E_2, E_3, \dots, E_n$ are exhaustive events, then

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = 1$$

- xi.* If $E_1, E_2, E_3, \dots, E_n$ are mutually exclusive and exhaustive events, then

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = P(E_1) + P(E_2) + \dots + P(E_n) = 1$$
- xii.* If $E_1, E_2, E_3, \dots, E_n$ are n events, then
 (a) $P(E_1 \cup E_2 \cup E_3 \cup \dots \cup E_n) \leq P(E_1) + P(E_2) + \dots + P(E_n)$
 (b) $P(E_1 \cup E_2 \cup \dots \cup E_n) \geq 1 - P(\bar{E}_1) - P(\bar{E}_2) - \dots - P(\bar{E}_n)$
- xiii.* If E_1 and E_2 are two events, then

$$P(E_1 \cap E_2) \leq P(E_1) \leq P(E_1 \cup E_2) \leq P(E_1) + P(E_2)$$
- xiv.* (a) P (exactly one of E_1 and E_2 occurs)

$$= P(E_1) + P(E_2) - 2P(E_1 \cap E_2)$$

$$= P(E_1 \cup E_2) - P(E_1 \cap E_2)$$

 (b) P (neither E_1 nor E_2)

$$= P(E'_1 \cap E'_2) = 1 - P(E_1 \cup E_2)$$

➤ **Example 3.** The probability that atleast one of the events A and B occurs is 0.6. If A and B occurs simultaneously with probability 0.2, then the value of $P(\bar{A}) + P(\bar{B})$ is

- (a) 1.8 (b) 1.2
 (c) 0.6 (d) None of these

Sol. (b) We have,

$$\begin{aligned} P(\text{atleast one of the events } A \text{ and } B \text{ occurs}) \\ \text{i.e. } P(A \cup B) &= 0.6 \\ \text{and } P(A \text{ and } B \text{ occur simultaneously}) \\ \text{i.e. } P(A \cap B) &= 0.2 \\ \text{Now, } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \Rightarrow 0.6 &= P(A) + P(B) - 0.2 \\ \Rightarrow 0.6 &= 1 - P(\bar{A}) + 1 - P(\bar{B}) - 0.2 \\ \Rightarrow 0.6 &= 2 - 0.2 - [P(\bar{A}) + P(\bar{B})] \\ \Rightarrow 0.6 &= 1.8 - [P(\bar{A}) + P(\bar{B})] \\ \Rightarrow P(\bar{A}) + P(\bar{B}) &= 1.8 - 0.6 = 1.2 \end{aligned}$$

Conditional Probability

Let A and B be two events associated with a random experiment. Then, the probability of occurrence of event A under the condition that B has already occurred and $P(B) \neq 0$, is called the conditional probability and it is denoted by $P\left(\frac{A}{B}\right)$.

Thus, we have

$$P\left(\frac{A}{B}\right) = \text{Probability of occurrence of } A \text{ given that } B \text{ has already occurred}$$

In fact, the meanings of symbols $P\left(\frac{A}{B}\right)$ and

$P\left(\frac{B}{A}\right)$ depend on the nature of the events A and B

and also on the nature of the random experiment.

These two symbols have the following meaning also:

$$P\left(\frac{A}{B}\right) = \text{Probability of occurrence of event } A$$

when event B occurs or probability of occurrence of event A with respect to event B .

$$\text{and } P\left(\frac{B}{A}\right) = \text{Probability of occurrence of event } B$$

when event A occurs or probability of occurrence of event B with respect to event A .

Multiplication Theorems on Probability

- (i) If A and B are two events associated with a random experiment, then

$$P(A \cap B) = P(A) P\left(\frac{B}{A}\right), \text{ if } P(A) \neq 0$$

$$\text{or } P(A \cap B) = P(B) P\left(\frac{A}{B}\right), \text{ if } P(B) \neq 0$$

- (ii) **Extension of multiplication theorem**

If $A_1, A_2, \dots, A_n$ are n events associated with a random experiment, then

$$P(A_1 \cap A_2 \cap A_3 \cap \dots \cap A_n) = P(A_1) P\left(\frac{A_2}{A_1}\right)$$

$$P\left(\frac{A_3}{A_1 \cap A_2}\right) \dots P\left(\frac{A_n}{A_1 \cap A_2 \cap \dots \cap A_{n-1}}\right)$$

$$\text{where, } P\left(\frac{A_i}{A_1 \cap A_2 \cap \dots \cap A_{i-1}}\right) \text{ represents}$$

the conditional probability of the occurrence of event A_i , given that the events

$A_1, A_2, \dots, A_{i-1}$ have already occurred.

➤ **Example 4.** A man draws a card from a pack of 52 cards and then replaces it. After shuffling the pack, he again draws a card. He repeats this activity number of times. The probability that he will draw a heart for the first time in the third draw, is

- (a) $\frac{9}{64}$ (b) $\frac{27}{64}$
 (c) $\frac{1}{4} \times \frac{{}^{39}C_2}{{}^{52}C_2}$ (d) None of these

Sol. (a) Let E be the event of drawing a heart. Clearly,

$$P(E) = \frac{{}^{13}C_1}{{}^{52}C_1} = \frac{1}{4}$$

$$\therefore \text{Required probability} = P(\bar{E} \bar{E} E) = P(\bar{E}) \cdot P(\bar{E}) \cdot P(E) \\ = \frac{3}{4} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{9}{64}$$

➤ **Example 5.** It has been found that if A and B play a game 12 times, A wins 6 times, B wins 4 times and they draw twice. A and B take part in a series of 3 games. The probability that they will win alternately, is

- (a) $\frac{5}{72}$ (b) $\frac{5}{36}$
(c) $\frac{19}{27}$ (d) None of these

Sol. (b) The probability of A winning in a game,

$$P(A) = \frac{6}{12} = \frac{1}{2}$$

The probability of B winning in a game, $P(B) = \frac{4}{12} = \frac{1}{3}$

$$\therefore \text{Required probability} = P(A \cap B \cap A) + P(B \cap A \cap B) \\ = P(A) \cdot P(B) \cdot P(A) + P(B) \cdot P(A) \cdot P(B) \\ = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{2} + \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{36}$$

➤ **Example 6.** A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one without replacement, then the probability of getting all white balls is

- (a) $\frac{1}{6}$ (b) $\frac{1}{898}$
(c) $\frac{1}{989}$ (d) None of these

Sol. (d) Let A, B, C and D denote events of getting a white ball in first, second, third and fourth draw, respectively. Then, required probability

$$= P(A \cap B \cap C \cap D) \\ = P(A)P\left(\frac{B}{A}\right)P\left(\frac{C}{A \cap B}\right)P\left(\frac{D}{A \cap B \cap C}\right) \dots (i)$$

Now, probability of drawing a white ball in first draw,

$$P(A) = \frac{5}{20} = \frac{1}{4}$$

When a white ball is drawn in the first draw, there are 19 balls left in the bag, out of which 4 are white.

$$\therefore P\left(\frac{B}{A}\right) = \frac{4}{19}$$

Since, the ball drawn is not replaced, therefore after drawing a white ball in second draw, there are 18 balls left in the bag, out of which 3 are white.

$$\therefore P\left(\frac{C}{A \cap B}\right) = \frac{3}{18} = \frac{1}{6}$$

After drawing a white ball in third draw, there are 17 balls left in the bag, out of which 2 are white.

$$\therefore P\left(\frac{D}{A \cap B \cap C}\right) = \frac{2}{17}$$

Hence, required probability

$$= P(A \cap B \cap C \cap D) \\ = P(A) \cdot P\left(\frac{B}{A}\right) \cdot P\left(\frac{C}{A \cap B}\right) \cdot P\left(\frac{D}{A \cap B \cap C}\right) \\ = \frac{1}{4} \times \frac{4}{19} \times \frac{1}{6} \times \frac{2}{17} = \frac{1}{969}$$

Independent Events

Two events A and B are said to be independent events, if the happening or non-happening of any random experiment does not affect the happening or non-happening of the other.

i.e. $P(A \cap B) = P(A)P(B)$

Otherwise, events are known as dependent events.

e.g. If we draw a card from a pack of well-shuffled cards and replace it before drawing the second card, the result of the second draw is independent of the first draw. But however, if the first card drawn is not replaced, then the second draw is dependent on the first draw.

- If E_1 and E_2 are independent events, then
 - E_1 and $\bar{E}_2$ are independent events.
 - $\bar{E}_1$ and E_2 are independent events.
 - $\bar{E}_1$ and $\bar{E}_2$ are independent events.
- Let the sample space associated to a random experiment is S , then ϕ and S are subsets of S . The event ϕ is known as impossible event and event S is known as certain event.

More on Independent Events

(i) If E_1 and E_2 are independent events, then

$$P\left(\frac{E_2}{E_1}\right) = P(E_2)$$

(ii) If $E_1, E_2, E_3, \dots, E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \dots \cup E_n) = 1 - P(E_1')P(E_2') \dots P(E_n')$$

(iii) If E_1 and E_2 are events such that $E_2 \neq \phi$, then

$$P\left(\frac{E_1}{E_2}\right) + P\left(\frac{E_1'}{E_2}\right) = 1$$

(iv) If E_1 and E_2 are events such that $E_1 \neq \phi$, then

$$P(E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) + P(E_1') \cdot P\left(\frac{E_2}{E_1'}\right)$$

(v) **Multiplication theorem on probability**

If E_1, E_2 and E_3 are three events such that $E_1 \neq \phi, E_1 E_2 \neq \phi$, then

$$P(E_1 \cap E_2 \cap E_3) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1 E_2}\right)$$

In general, if $E_1, E_2, \dots, E_n$ are n events such that $E_1 = \phi, E_1 E_2 \neq \phi, E_1 E_2 E_3 \neq \phi, \dots, E_1 E_2 \dots E_{n-1} = \phi$, then $P(E_1 \cap E_2 \cap \dots \cap E_n)$

$$= P(E_1) \cdot P\left(\frac{E_2}{E_1}\right) \cdot P\left(\frac{E_3}{E_1 E_2}\right) \dots P\left(\frac{E_n}{E_1 E_2 \dots E_{n-1}}\right)$$

(vi) If E_1 and E_2 are independent events, then

$$(a) P(E_1 \cap E_2) = P(E_1) \cdot P\left(\frac{E_2}{E_1}\right),$$

if $P(E_1) \neq 0$

$$(b) P(E_1 \cap E_2) = P(E_2) \cdot P\left(\frac{E_1}{E_2}\right),$$

if $P(E_2) \neq 0$

(vii) If E_1 and E_2 are independent events, then

$$P(E_1 \cap E_2) = P(E_1) \cdot P(E_2)$$

(viii) If $E_1, E_2, \dots, E_n$ are independent events, then

$$P(E_1 \cap E_2 \cap E_3 \cap E_4 \cap \dots \cap E_n) = P(E_1)P(E_2) \dots P(E_n)$$

(ix) **The events A and ϕ are always independent.**

$$\therefore P(A \cap \phi) = P(A) \cdot P(\phi) = 0$$

i.e. an event A and an impossible event (i.e. null event) can never occur together.

(x) **The events A and S are always independent.**

$$\therefore P(A \cap S) = P(A) \cdot P(S) = P(A)$$

i.e. occurrence of an event A along with sample space event is the event A itself.

(xi) **Mutually exclusive events** (none of which is an impossible event) **are not independent**

Let A and B be two non-impossible mutually exclusive events.

$$\therefore P(A \cup B) = P(A) + P(B) \quad \dots(i)$$

$$\text{Now, } P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

[by addition theorem]

$$\Rightarrow P(A \cap B) = 0 \quad [\text{using Eq. (i)}]$$

Since, A and B are non-impossible events.

$$\therefore P(A) \text{ and } P(B) \neq 0$$

$$\Rightarrow P(A) \cdot P(B) > 0$$

[$\because$ probability of an event is never negative]

$$\text{So, } P(A \cap B) \neq P(A) \cdot P(B)$$

$\therefore$ Events A and B are not independent.

Hence, mutually exclusive events (none of which is impossible event) are not independent.

(xii) **Non-impossible independent events are not mutually exclusive.**

Let A and B be non-impossible independent events, then

$$P(A \cap B) = P(A) \cdot P(B) \quad \dots(i)$$

Now,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

[by addition theorem]

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$$

...(ii)

Since, A and B are non-impossible events.

$$\therefore P(A) \text{ and } P(B) \neq 0$$

$$\Rightarrow P(A) \cdot P(B) \text{ exists finitely.}$$

Hence, $P(A \cup B) \neq P(A) + P(B)$ [from Eq.

(ii)]

Thus, the events are not mutually exclusive.

➤ **Example 7.** If A and B are two independent events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{5}$, then

$$(a) P(A \cup B) = \frac{3}{5} \quad (b) P\left(\frac{A}{B}\right) = \frac{1}{2}$$

$$(c) P\left(\frac{A}{A \cup B}\right) = \frac{5}{6} \quad (d) P\left(\frac{A \cap B}{A \cup B}\right) = 0$$

Sol. (a, b, c, d) Since, A and B are independent events.

$$\therefore P(A \cap B) = P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{5} = \frac{1}{10}$$

$$\text{and } P\left(\frac{A}{B}\right) = P(A) = \frac{1}{2}$$

So, option (b) is true.

$$\text{Now, } P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{1}{5} - \frac{1}{10} = \frac{3}{5}$$

So, option (a) is true.

Now,

$$P\left(\frac{A}{A \cup B}\right) = \frac{P[A \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A)}{P(A \cup B)} = \frac{\frac{1}{2}}{\frac{3}{5}} = \frac{5}{6}$$

$$\text{and } P\left(\frac{A \cap B}{A \cup B}\right) = P\left(\frac{A \cap B}{A \cup B}\right) = 0$$

So, options (c) and (d) are also hold good.

➤ **Example 8.** India play two matches each with West Indies and Australia. In any match, the probabilities of India getting 0, 1 and 2 points are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of India getting atleast 7 points, is

$$(a) 0.0875 \quad (b) \frac{1}{16}$$

$$(c) 0.1125 \quad (d) \text{None of these}$$

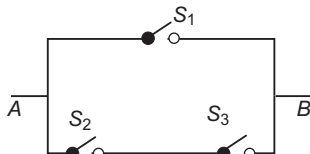
Sol. (a) $P(E_0) = 0.45, P(E_1) = 0.05, P(E_2) = 0.50$

For 7 points, there should be atleast three instances of 2 points and one of 1 or 2 points.

$$\therefore \text{Required probability} = P(E_2 E_2 E_2 E_2) + P(E_2 E_2 E_2 E_1)$$

$$\begin{aligned}
 &+ P(E_2 E_2 E_1 E_2) + P(E_2 E_1 E_2 E_2) + P(E_1 E_2 E_2 E_2) \\
 &= P(E_2)^4 + 4\{P(E_2)\}^3 \cdot P(E_1) \\
 &= (0.50)^4 + 4(0.50)^3 \cdot (0.05) = 0.0875
 \end{aligned}$$

➤ **Example 9.** An electrical system has open-closed switches S_1 , S_2 and S_3 as shown in figure.



The switches operate independently of one another and the current will flow from A to B either if S_1 is closed or if both S_2 and S_3 are closed. If $P(S_1) = P(S_2) = P(S_3) = 1/2$, then the probability that the circuit will work, is

- (a) $\frac{5}{8}$ (b) $\frac{3}{8}$ (c) $\frac{7}{8}$ (d) $\frac{1}{4}$

Sol. (a) Given, $P(S_1) = P(S_2) = P(S_3) = \frac{1}{2}$

Let E be the event that the current will flow. Then,

$$\begin{aligned}
 P(E) &= P((S_2 \cap S_3) \cup S_1) \\
 &= P(S_2 \cap S_3) + P(S_1) - P(S_1 \cap S_2 \cap S_3) \\
 &= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8}
 \end{aligned}$$

➤ **Example 10.** The odds against a certain event are 5:2 and the odds in favour of other event, independent of the former, are 6:5. Then, the probability that atleast one of the events will happen, is

- (a) $\frac{25}{77}$ (b) $\frac{52}{77}$
(c) $\frac{3}{7}$ (d) $\frac{5}{7}$

Sol. (b) $P(\text{first event does not happen}) = \frac{5}{5+2} = \frac{5}{7}$

$$P(\text{second event does not happen}) = \frac{5}{5+6} = \frac{5}{11}$$

$$\therefore P(\text{both the events fail to happen}) = \frac{5}{7} \times \frac{5}{11} = \frac{25}{77}$$

Therefore, the probability that atleast one of the events will happen is

$$1 - P(\text{none of two happens}) = 1 - \frac{25}{77} = \frac{52}{77}$$

➤ **Example 11.** Of three independent events, the chance that only the first occurs is a , that only the second occurs is b and only the third occurs is c . Then, the probabilities of occurrence of these three events, where x is a root of the equation $(a+x)(b+x)(c+x) = x^2$, are

- (a) $\frac{a}{a+x}, \frac{b}{b+x}, \frac{c}{c+x}$ (b) $\frac{b}{a+x}, \frac{a}{b+x}, \frac{c}{c+x}$
(c) $\frac{c}{a+x}, \frac{b}{b+x}, \frac{a}{c+x}$ (d) None of these

Sol. (a) Let A_1, A_2 and A_3 be three independent events. It is given that,

Probability of occurrence of A_1 only = a ,
Probability of occurrence of A_2 only = b
and probability of occurrence of A_3 only = c

$$\begin{aligned}
 \Rightarrow P(A_1 \cap \bar{A}_2 \cap \bar{A}_3) &= a, \\
 P(\bar{A}_1 \cap A_2 \cap \bar{A}_3) &= b \\
 \text{and } P(\bar{A}_1 \cap \bar{A}_2 \cap A_3) &= c \\
 \Rightarrow P(A_1) P(\bar{A}_2) P(\bar{A}_3) &= a \quad \dots(i) \\
 P(\bar{A}_1) P(A_2) P(\bar{A}_3) &= b \quad \dots(ii) \\
 \text{and } P(\bar{A}_1) P(\bar{A}_2) P(A_3) &= c \quad \dots(iii)
 \end{aligned}$$

Let $P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3) = x$

Then, from Eq.(i), we get

$$\begin{aligned}
 \frac{P(A_1)}{P(\bar{A}_1)} \cdot \frac{1}{x} &\Rightarrow \frac{P(A_1)}{P(\bar{A}_1)} = \frac{x}{a} \\
 \Rightarrow \frac{1 - P(A_1)}{P(A_1)} &= \frac{x}{a} \Rightarrow P(A_1) = \frac{a}{a+x}
 \end{aligned}$$

Similarly, from Eqs. (ii) and (iii), we get

$$P(A_2) = \frac{b}{b+x} \text{ and } P(A_3) = \frac{c}{c+x}$$

On multiplying Eqs. (i), (ii) and (iii), we get

$$\begin{aligned}
 P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot \{P(\bar{A}_1)P(\bar{A}_2)P(\bar{A}_3)\}^2 &= abc \\
 \Rightarrow P(A_1) \cdot P(A_2) \cdot P(A_3) \cdot x^2 &= abc \\
 \Rightarrow \frac{abc}{(a+x)(b+x)(c+x)} \cdot x^2 &= abc \\
 \Rightarrow x^2 &= (a+x)(b+x)(c+x)
 \end{aligned}$$

Thus, we have

$$P(A_1) = \frac{a}{a+x}, P(A_2) = \frac{b}{b+x} \text{ and } P(A_3) = \frac{c}{c+x}$$

where, x is a root of the equation

$$(a+x)(b+x)(c+x) = x^2$$

Work Book Exercise 16.1

1 A draws two cards at random from a pack of 52 cards. After returning them to the pack and shuffling it, B draws two cards at random. The probability that their draws contain exactly one common card, is

- a $\frac{25}{546}$ b $\frac{50}{663}$
c $\frac{25}{663}$ d None of these

2 A and B draw two cards each, one after another, from a pack of well-shuffled pack of 52 cards. The probability that all the four cards drawn are of the same suit, is

- a $\frac{44}{85 \times 49}$ b $\frac{11}{85 \times 49}$
c $\frac{13 \times 24}{17 \times 25 \times 49}$ d None of these

- 3 A fair coin is tossed repeatedly. The probability of getting a result in the fifth toss different from those obtained in the first four tosses, is
a $\frac{1}{2}$ **b** $\frac{1}{32}$ **c** $\frac{31}{32}$ **d** $\frac{1}{16}$
- 4 A card is drawn from a pack. The card is replaced and the pack is reshuffled. If this is done six times, then the probability that 2 hearts, 2 diamonds and 2 black cards are drawn, is
a $90 \cdot \left(\frac{1}{4}\right)^6$ **b** $\frac{45}{2} \cdot \left(\frac{3}{4}\right)^6$
c $\frac{90}{2^{10}}$ **d** None of these
- 5 For any two independent events E_1 and E_2 in a space S , $P[(E_1 \cup E_2) \cap (\bar{E}_1 \cap \bar{E}_2)]$ is equal to
a $\leq \frac{1}{4}$ **b** $> \frac{1}{4}$ **c** $\geq \frac{1}{2}$ **d** $> \frac{1}{2}$
- 6 A pack of cards consists of 15 cards numbered 1 to 15. Three cards are drawn at random with replacement. Then, the probability of getting two odd and one even numbered cards, is
a $\frac{348}{1125}$ **b** $\frac{398}{1125}$
c $\frac{448}{1125}$ **d** $\frac{498}{1125}$
- 7 Whenever horses a, b, c race together, their respective probabilities of winning the race are 0.3, 0.5 and 0.2, respectively. If they race three times, the probability that the same horse wins all the three races and the probability that a, b, c each wins one race, are respectively
a $\frac{8}{50}, \frac{9}{50}$ **b** $\frac{16}{100}, \frac{3}{100}$
c $\frac{12}{50}, \frac{15}{50}$ **d** $\frac{10}{50}, \frac{8}{50}$
- 8 Let A and B be two events. Suppose $P(A) = 0.4$, $P(B) = p$ and $P(A \cup B) = 0.7$. The value of p for which A and B are independent, is
a $\frac{1}{3}$ **b** $\frac{1}{4}$ **c** $\frac{1}{2}$ **d** $\frac{1}{5}$
- 9 One bag contains 3 white and 2 black balls and another contains 2 white and 3 black balls. A ball is drawn from the second bag and placed in the first, then a ball is drawn from the first bag and placed in the second. When the pair of the operations is repeated, the probability that the first bag will contain 5 white balls, is
a $\frac{1}{25}$ **b** $\frac{1}{125}$ **c** $\frac{1}{225}$ **d** $\frac{2}{15}$
- 10 5 out of 6 persons who usually work in an office, prefer coffee in the mid-morning, the other always drink tea. This morning of the usual 6, only 3 are present. The probability that one of them drinks tea, is
a $\frac{1}{2}$ **b** $\frac{1}{12}$ **c** $\frac{25}{72}$ **d** $\frac{5}{72}$
- 11 Nine cards are labelled 0, 1, 2, 3, 4, 5, 6, 7, 8. Two cards are drawn at random and put on a table in a successive order and then the resulting number is read, say 07 (seven), 14 (fourteen) and so on. The probability that the number is even, is
a $\frac{5}{9}$ **b** $\frac{4}{9}$
c $\frac{1}{2}$ **d** $\frac{2}{3}$
- 12 Two small squares on a chessboard are chosen at random. Probability that they have a common side, is
a $\frac{1}{3}$ **b** $\frac{1}{9}$
c $\frac{1}{18}$ **d** $\frac{5}{18}$
- 13 A bag contains n white and n black balls. Pairs of balls are drawn without replacement until the bag is empty. The probability that each pair consists of one white and one black ball, is
a $\frac{1}{2^n C_n}$ **b** $\frac{2^n}{2^n C_n}$
c $\frac{2^n}{n!}$ **d** $\frac{2^n}{(2n)!}$
- 14 If X speaks truth in 60% and Y in 50% of the cases. Then, the probability that they contradict each other narrating the same incident, is
a $\frac{1}{2}$ **b** $\frac{2}{3}$
c $\frac{1}{4}$ **d** None of these
- 15 If the probability of hitting a target by three marksmen are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Then, the probability that one and only one of them will hit the target when they fire simultaneously, is
a $\frac{23}{24}$ **b** $\frac{7}{24}$
c $\frac{11}{24}$ **d** None of these
- 16 From an urn containing six balls, 3 white and 3 black ones, a person selects at random an even number of balls (all the different ways of drawing an even number of balls are considered equally probable, irrespective of their number). Then, the probability that there will be the same number of black and white balls among them, is
a $\frac{4}{5}$ **b** $\frac{11}{15}$ **c** $\frac{11}{30}$ **d** $\frac{2}{5}$
- 17 One purse contains 6 copper coins and 1 silver coin; a second purse contains 4 copper coins. Five coins are drawn from the first purse and put into the second and then 2 coins are drawn from the second and put into the first. The probability that the silver coin is in the second purse, is
a $\frac{1}{2}$ **b** $\frac{4}{9}$ **c** $\frac{5}{9}$ **d** $\frac{2}{3}$

- 18** Fifteen coupons are numbered 1, 2, ..., 15 respectively. Seven coupons are selected at random one at a time with replacement. The probability that the largest number appearing on a selected coupon is 9, is

- a $\left(\frac{9}{10}\right)^6$ b $\left(\frac{8}{15}\right)^7$
c $\left(\frac{3}{5}\right)^7$ d None of these

- 19** A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are

examined, are all red, then the probability that the missing card is black, is

- a $\frac{1}{3}$ b $\frac{2}{3}$ c $\frac{1}{2}$ d $\frac{{}^{25}C_{13}}{{}^{51}C_{13}}$

- 20** In a bag, there are three tickets numbered 1, 2, 3. A ticket is drawn at random and put back and this is done four times. The probability that the sum of the numbers is even, is

- a $\frac{41}{81}$ b $\frac{39}{81}$
c $\frac{40}{81}$ d None of these

Law of Total Probability

Let in a random experiment, S be a sample space and $E_1, E_2, \dots, E_n$ be mutually exclusive and exhaustive events. If A is any event which occur with E_1 or E_2 or E_3 or ... or E_n , then

$$P(A) = P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right) + \dots + P(E_n) \cdot P\left(\frac{A}{E_n}\right) = \sum_{r=1}^n P(E_r) \cdot P\left(\frac{A}{E_r}\right)$$

Example 12. Two sets of candidates are competing for the positions of the board of directors of a company. The probabilities that the first and second sets will win, are 0.6 and 0.4, respectively. If the first set wins, the probability of introducing product is 0.8 and the corresponding probability, if the second set wins, is 0.3. What is the probability of the new product will be introduced?

- (a) 0.1 (b) 0.3 (c) 0.5 (d) 0.6

Sol. (d) Let $A_1(A_2)$ denotes the event that first (second) set wins and B be the event that a new product is introduced.

$$\therefore P(A_1) = 0.6, P(A_2) = 0.4$$

$$\text{Now, } P\left(\frac{B}{A_1}\right) = 0.8, P\left(\frac{B}{A_2}\right) = 0.3$$

$$\begin{aligned} \therefore P(B) &= P(B \cap A_1) + P(B \cap A_2) \\ &= P(A_1) \cdot P\left(\frac{B}{A_1}\right) + P(A_2) \cdot P\left(\frac{B}{A_2}\right) \\ &= 0.6 \times 0.8 + 0.4 \times 0.3 = 0.6 \end{aligned}$$

Baye's Theorem

If $E_1, E_2, \dots, E_n$ are mutually exclusive and exhaustive events with $P(E_i) \neq 0$; $i = 1, 2, \dots, n$, then for any event E which is a subset of $\bigcup_{i=1}^n E_i$ such that $P(E) > 0$,

$$P\left(\frac{E_i}{E}\right) = \frac{P(E_i) \cdot P\left(\frac{E}{E_i}\right)}{\sum_{i=1}^n P(E_i) \cdot P\left(\frac{E}{E_i}\right)}$$

The probabilities $P(E_1), P(E_2), \dots, P(E_i)$ are called prior probabilities because they exist before, we gain any information from the experiment itself.

$P\left(\frac{E_i}{E}\right)$; $i = 1, 2, \dots, n$ are called the likelihood probabilities.

The probability $P\left(\frac{E_i}{E}\right)$; $i = 1, 2, \dots, n$ are called posterior probabilities because they are determined after the results of the experiment are known.

Example 13. Box I contains 2 white and 3 red balls and box II contains 4 white and 5 red balls. One ball is drawn at random from one of the boxes and is found to be red. Then, the probability that it was from box II, is

- (a) $\frac{25}{32}$
(b) $\frac{21}{32}$
(c) $\frac{7}{32}$
(d) $\frac{15}{32}$

Sol. (a) Let A denotes the event that the drawn ball is red.

Let A_1 be the event that box I is selected and A_2 be the event that box II is selected.

$$\begin{aligned} \therefore P\left(\frac{A_2}{A}\right) &= \frac{P(A_2) \cdot P\left(\frac{A}{A_2}\right)}{P(A_1) \cdot P\left(\frac{A}{A_1}\right) + P(A_2) \cdot P\left(\frac{A}{A_2}\right)} \\ &= \frac{\frac{1}{2} \cdot \frac{5}{9}}{\frac{1}{2} \cdot \frac{5}{9} + \frac{1}{2} \cdot \frac{3}{5}} = \frac{25}{32} \end{aligned}$$

Example 14. The chances of defective screws in three boxes A, B, C are $\frac{1}{5}, \frac{1}{6}, \frac{1}{7}$ respectively. A box is selected at random and a screw drawn from it at random is found to be defective. Then, the probability that it came from box A , is

- (a) $\frac{16}{29}$ (b) $\frac{1}{15}$ (c) $\frac{27}{59}$ (d) $\frac{42}{107}$

Sol. (d) Let E_1, E_2 and E_3 denote the events of selecting boxes A, B, C respectively and A be the event that a screw selected at random is defective. Then,

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P\left(\frac{A}{E_1}\right) = \frac{1}{5}, P\left(\frac{A}{E_2}\right) = \frac{1}{6}, P\left(\frac{A}{E_3}\right) = \frac{1}{7}$$

By using Baye's theorem,

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \cdot P(A/E_1)}{P(E_1) \cdot P(A/E_1) + P(E_2) \cdot P(A/E_2) + P(E_3) \cdot P(A/E_3)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{6} + \frac{1}{3} \times \frac{1}{7}} = \frac{42}{107}$$

Example 15. A person goes to office either by car, scooter, bus or train, the probabilities of which being $\frac{1}{7}, \frac{3}{7}, \frac{2}{7}$ and $\frac{1}{7}$, respectively. The probability that he reaches office late, if he takes car, scooter, bus or train, is $\frac{2}{9}, \frac{1}{9}, \frac{4}{9}$ and $\frac{1}{9}$, respectively. If he reaches office in time, the probability that he travelled by car, is

- (a) $\frac{1}{5}$ (b) $\frac{1}{9}$
(c) $\frac{2}{11}$ (d) $\frac{1}{7}$

Sol. (d) Let C, S, B and T be the events of the person going by car, scooter, bus and train, respectively.

$$\text{Then, } P(C) = \frac{1}{7}, P(S) = \frac{3}{7}, P(B) = \frac{2}{7}, P(T) = \frac{1}{7}$$

Let L be the event that the person reaching the office in time. Then, $\bar{L}$ be the event that the person reaching the office in late.

$$P(L/C) = \frac{7}{9}, P(L/S) = \frac{8}{9}, P(L/B) = \frac{5}{9}, P(L/T) = \frac{8}{9}$$

$$\therefore P(C/L) = \frac{P(C) \cdot P(L/C)}{P(C) \cdot P(L/C) + P(S) \cdot P(L/S) + P(B) \cdot P(L/B) + P(T) \cdot P(L/T)}$$

$$= \frac{\frac{1}{7} \cdot \frac{7}{9}}{\frac{1}{7} \cdot \frac{7}{9} + \frac{3}{7} \cdot \frac{8}{9} + \frac{2}{7} \cdot \frac{5}{9} + \frac{1}{7} \cdot \frac{8}{9}}$$

$$= \frac{7}{49} = \frac{1}{7}$$

Example 16. Die A has 4 red and 2 white faces whereas die B has 2 red and 4 white faces. A coin is flipped once. If it shows a head, the game continues by throwing die A ; if it shows tail, then die B is to be used. If the probability that die A is used, is $\frac{32}{33}$ when it is given that red turns up every time in first n throws, then the value of n is

- (a) 4 (b) 3 (c) 5 (d) 6

Sol. (c) Let R be the event that a red face appears in each of the first n throws.

E_1 : Die A is used when head has already fallen

E_2 : Die B is used when tail has already fallen

$$\therefore P(R/E_1) = \left(\frac{2}{3}\right)^n \text{ and } P\left(\frac{R}{E_2}\right) = \left(\frac{1}{3}\right)^n$$

As per the given condition,

$$\frac{P(E_1) \cdot P(R/E_1)}{P(E_1) \cdot P(R/E_1) + P(E_2) \cdot P(R/E_2)} = \frac{32}{33}$$

$$\Rightarrow \frac{1/2(2/3)^n}{1/2(2/3)^n + 1/2(1/3)^n} = \frac{32}{33}$$

$$\Rightarrow \frac{2^n}{2^n + 1} = \frac{32}{33}$$

$$\therefore n = 5$$

Example 17. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn and are found to be diamonds. The probability of the lost card being a diamond, is

- (a) $\frac{11}{50}$ (b) $\frac{2}{25}$ (c) $\frac{9}{50}$ (d) $\frac{43}{50}$

Sol. (a) Let E_1 be the event that lost card is a diamond.

$\Rightarrow n(E_1) = 13$ and E_2 be the event that lost card is not a diamond

$\Rightarrow n(E_2) = 52 - 13 = 39$ and $n(S) = 52$

Then, E_1 and E_2 are mutually exclusive and exhaustive events.

$$\therefore P(E_1) = \frac{13}{52} = \frac{1}{4}$$

$$\text{and } P(E_2) = \frac{39}{52} = \frac{3}{4}$$

Let E be the event that two cards drawn from the remaining pack are diamonds.

When one diamond card is lost, there are 12 diamond cards out of 51 cards.

The cards can be drawn out of 12 diamond cards in ${}^{12}C_2$ ways.

Similarly, 2 diamond cards can be drawn out 51 cards in ${}^{51}C_2$ ways. The probability of getting two cards,

when one diamond card is lost, is given by $P\left(\frac{E}{E_1}\right)$.

$$\therefore P\left(\frac{E}{E_1}\right) = \frac{{}^{12}C_2}{{}^{51}C_2}$$

$$= \frac{12 \times 11}{51 \times 50} = \frac{12 \times 11}{51 \times 50}$$

$$= \frac{1 \times 2}{1 \times 2} = \frac{12 \times 11}{51 \times 50}$$

$$\text{and } P\left(\frac{E}{E_2}\right) = \frac{{}^{13}C_2}{{}^{51}C_2} = \frac{1 \times 2}{51 \times 50} = \frac{13 \times 12}{51 \times 50}$$

By using Baye's theorem, we obtain

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right) \cdot P(E_1)}{P\left(\frac{E}{E_1}\right) \cdot P(E_1) + P\left(\frac{E}{E_2}\right) \cdot P(E_2)}$$

$$= \frac{\frac{12 \times 11}{51 \times 50} \times \frac{1}{4}}{\frac{12 \times 11}{51 \times 50} \times \frac{1}{4} + \frac{13 \times 12}{51 \times 50} \times \frac{3}{4}}$$

$$= \frac{12 \times 11}{12 \times 11 + 13 \times 12 \times 3} = \frac{132}{132 + 468} = \frac{132}{600} = \frac{11}{50}$$

➤ **Example 18.** A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope, just two consecutive letters TA are visible. What is the probability that the letter came from TATANAGAR?

- (a) $\frac{2}{11}$ (b) $\frac{5}{11}$ (c) $\frac{7}{11}$ (d) $\frac{10}{11}$

Work Book Exercise 16.2

- Assume that the chances of a patient having a heart attack is 40%. It is also assumed that a meditation and yoga course reduces the risk of heart attack by 30% and prescription of certain drug reduces its chances by 25%. At a time, a patient can choose anyone of the two options with equal probabilities. It is given that after going through one of the two options, the patient selected at random suffers a heart attack. The probability that the patient followed a course of meditation and yoga, is
 a $\frac{3}{29}$ b $\frac{7}{29}$
 c $\frac{14}{29}$ d None of these
- If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F respectively and if $0 < P(F) < 1$, then
 a $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{F}\right) = 1$ b $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{\bar{F}}\right) = 1$
 c $P\left(\frac{\bar{E}}{F}\right) + P\left(\frac{E}{\bar{F}}\right) = 1$ d $P\left(\frac{E}{F}\right) + P\left(\frac{\bar{E}}{\bar{F}}\right) = 1$
- Probability that A speaks truth is $\frac{4}{5}$. A coin is tossed, A reports that a head appears. The probability that actually there was head, is
 a $\frac{4}{5}$ b $\frac{1}{2}$
 c $\frac{1}{5}$ d $\frac{2}{5}$

Sol. (c) Let E_1 be letter (i.e. envelope) has come from CALCUTTA.

E_2 be letter (i.e. envelope) has come from TATANAGAR and E be two consecutive letters (i.e. alphabet) TA are visible on envelope.

$$P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{2}$$

$$\Rightarrow P\left(\frac{E}{E_1}\right) = \frac{n(E \cap E_1)}{n(E_1)} = \frac{1}{7}$$

[∵ 7 pairs of consecutive letters are

CA, AL, LC, CU, UT, TT, TA]

$$P\left(\frac{E}{E_2}\right) = \frac{n(E \cap E_2)}{n(E_2)} = \frac{2}{8}$$

[∵ 8 pairs of consecutive letters are
 TA, AT, TA, AN, NA, AG, GA, AR]

By using Baye's theorem,

$$P\left(\frac{E_2}{E}\right) = \frac{P(E_2) \cdot P\left(\frac{E}{E_2}\right)}{P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_1) \cdot P\left(\frac{E}{E_1}\right)}$$

$$= \frac{\frac{1}{2} \cdot \frac{2}{8}}{\frac{1}{2} \cdot \frac{2}{8} + \frac{1}{2} \cdot \frac{1}{7}} = \frac{7}{11}$$

- A company has two plants to manufacture televisions. Plant I manufacture 70% of televisions and plant II manufacture 30%. At plant I, 80% of the televisions are rated as of standard quality and at plant II, 90% of the televisions are rated as of standard quality. A television is chosen at random and is found to be of standard quality. The probability that it has come from plant II, is
 a $\frac{17}{50}$ b $\frac{27}{83}$ c $\frac{3}{5}$ d $\frac{9}{83}$
- Three machines E_1, E_2 and E_3 in a certain factory produce 50%, 25% and 25%, respectively of the total daily output of electric tubes. It is known that 4% of the tubes produced one each of machines E_1 and E_2 are defective and that 5% of those produced on E_3 are defective. If one tube is picked up at random from a day's production, the probability that it is defective, is
 a 0.025 b 0.125 c 0.325 d 0.0425
- A pair of numbers is picked up randomly (without replacement) from the set [1, 2, 3, 5, 7, 11, 12, 13, 17, 19]. The probability that the number 11 was picked given that the sum of the numbers was even, is nearly
 a 0.1 b 0.125
 c 0.24 d 0.18

- 7 The probability that an automobile will be stolen and found within one week is 0.0006. The probability that an automobile will be stolen, is 0.0015. The probability that a stolen automobile will be found in one week, is
- a 0.3
b 0.4
c 0.5
d 0.6
- 8 A bag contains 3 red and 3 green balls and a person draws out 3 at random. He then drops 3 blue balls into the bag and again draws out 3 at random. The chance that the 3 later balls being all of different colours is
- a 15%
b 20%
c 27%
d 40%
- 9 When a missile is fired from a ship, the probability that it is intercepted, is $\frac{1}{3}$. The probability that the missile hits the target, given that it is not intercepted, is $\frac{3}{4}$. If three missiles are fired independently from the ship, then the probability that all three hits their targets, is
- a $\frac{1}{12}$
b $\frac{1}{8}$
c $\frac{3}{8}$
d $\frac{3}{4}$
- 10 There are two bags, one of which contains 3 black and 4 white balls, while the other contains 4 black and 3 white balls. A die is cast. If the face 1 or 3 turns up, a ball is taken from the first bag and if any other face turn up, a ball is chosen from the second bag. The probability of choosing a black ball is
- a $\frac{11}{21}$
b $\frac{10}{21}$
c $\frac{13}{21}$
d None of these

Random Variable and its Distribution

Let S be the sample space associated with a given random experiment. Then, a real valued function X which assigns to each event $w \in S$ to a unique real number $X(w)$ is called a random variable.

A random variable is a function that associate a unique numerical value with every outcome of an experiment. The value of the random variable will vary from trial-to-trial as the experiment is repeated.

A random variable is usually denoted by the capital letters X, Y, Z , etc.

e.g. A coin is tossed ten times. The random variable X is the number of tails that are noted. X can only take the values 0, 1, 2, ..., 10. So, X is a discrete random variable.

There are two types of random variable:

- (i) **Discrete random variable** If the range of the real function $X:U \rightarrow R$ is a finite set or an infinite set of real numbers, it is called a discrete random variable.
- (ii) **Continuous random variable** If the range of X is an interval (a, b) of R , then X is called a continuous random variable.

e.g. In tossing of two coins

$S = (HH, HT, TH, TT)$, let X denotes number of heads in tossing of two coins, then

$$X(HH) = 2, X(TH) = 1, X(TT) = 0$$

Probability Distribution

A random variable X takes values $X_1, X_2, \dots, X_n$ with respective probabilities $P_1, P_2, \dots, P_n$, then

X	X_1	X_2	$X_3 \dots$	X_n
$P(X)$	P_1	P_2	$P_3 \dots$	P_n

is known as the probability distribution of X .

Or

Probability distribution gives the values of the random variable along with the corresponding probabilities.

It satisfies the following conditions:

- (i) $0 \leq P(x_i) \leq 1$
(ii) $\sum P(x_i) = 1$

e.g. Probability distribution when two coins are tossed.

Let X denotes the number of heads occurred, then

$P(X=0)$ = Probability of occurrence of zero head

$$= P(TT) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

$P(X=1)$ = Probability of occurrence of one head

$$= P(HT) + P(TH) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{2}$$

$P(X=2)$ = Probability of occurrence of two heads

$$= P(HH) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$$

Thus, the probability distribution when two coins are tossed, is as given below:

X	0	1	2
$P(X)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

➤ **Example 19.** A random variable X has the following probability distribution. Then, the value of

X	0	1	2	3	4	5	6	7
$P(X)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

- (i) k (ii) $P(X < 3)$
 (iii) $P(X > 6)$ (iv) $P(0 < X < 3)$
 are respectively
 (a) $\frac{1}{10}, \frac{3}{10}, \frac{17}{100}$ and $\frac{3}{10}$ (b) $\frac{1}{10}, \frac{3}{10}, \frac{3}{10}$ and $\frac{17}{100}$
 (c) $\frac{17}{100}, \frac{1}{10}, \frac{3}{10}$ and $\frac{3}{10}$ (d) None of these

Sol. (a) (i) It is known that the sum of a probability distribution of random variable is one, i.e. $\sum P(X) = 1$, therefore
 $P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) = 1$
 $\Rightarrow 0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$
 $\Rightarrow 10k^2 + 9k - 1 = 0$
 $\Rightarrow 10k^2 + 10k - k - 1 = 0$
 $\Rightarrow 10k(k+1) - 1(k+1) = 0$
 $\Rightarrow (k+1)(10k-1) = 0$
 $\Rightarrow k+1 = 0$ or $10k-1 = 0$
 $\Rightarrow k = -1$ or $k = \frac{1}{10}$

But $k = -1$ is not possible as the probability of an event is never negative.

- $\therefore k = \frac{1}{10}$
 (ii) $P(X < 3) = P(0) + P(1) + P(2) = 0 + k + 2k$
 $= 0 + \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$ [put $k = \frac{1}{10}$]
 (iii) $P(X > 6) = P(7) = 7k^2 + k$
 $= \frac{7}{100} + \frac{1}{10} = \frac{17}{100}$ [put $k = \frac{1}{10}$]
 (iv) $P(0 < X < 3) = P(1) + P(2) = k + 2k$
 $= \frac{1}{10} + \frac{2}{10} = \frac{3}{10}$ [put $k = \frac{1}{10}$]

Mean and Variance of a Random Variable

If X is a discrete random variable which assumes values $x_1, x_2, x_3, \dots, x_n$ with respective probabilities $p_1, p_2, p_3, \dots, p_n$, then the mean $\bar{X}$ of X is defined as

$$\bar{X} = p_1 x_1 + p_2 x_2 + \dots + p_n x_n \Rightarrow \bar{X} = \sum_{i=1}^n p_i x_i$$

and variance of X is defined as

$$\begin{aligned} \text{var}(X) &= p_1 (x_1 - \bar{X})^2 + p_2 (x_2 - \bar{X})^2 \\ &\quad + \dots + p_n (x_n - \bar{X})^2 \\ &= \sum_{i=1}^n p_i (x_i - \bar{X})^2 \end{aligned}$$

where, $\bar{X} = \sum_{i=1}^n p_i x_i$ is the mean of x .

$$\Rightarrow \text{var}(X) = \sum_{i=1}^n p_i x_i^2 - \left(\sum_{i=1}^n p_i x_i \right)^2$$

The square root of the variance gives the standard deviation.

i.e. $\sqrt{\text{var}(X)} = \sqrt{\sigma^2} = \sigma$

- The mean of a random variable X is also known as its mathematical expectation or expected value and is denoted by $E(X)$.
- The variance and standard deviation of a random variable are always non-negative.

Important Results

i. **Variance** $V(X) = \sigma_x^2 = E(X^2) - \{E(X)\}^2$

where, $E(X^2) = \sum_{i=1}^n x_i^2 P(x_i)$

ii. **Standard Deviation**
 $\sqrt{V(X)} = \sigma_x = \sqrt{E(X^2) - \{E(X)\}^2}$

iii. If $Y = aX + b$, then

(a) $E(Y) = E(aX + b) = aE(X) + b$

(b) $\sigma_Y^2 = V(Y) = a^2 V(X) = a^2 \sigma_x^2$

(c) $\sigma_Y = \sqrt{V(Y)} = |a| \sigma_x$

iv. If $Z = aX^2 + bX + c$, then

$E(Z) = E(aX^2 + bX + c) = aE(X^2) + bE(X) + c$

➤ **Example 20.** A class has 15 students whose ages are 14, 17, 15, 14, 21, 17, 19, 20, 16, 18, 20, 17, 16, 19 and 20 yr. One student is selected in such a manner that each has the same chance of being of chosen and the age X of the selected student is recorded. What is the probability distribution of the random variable X ? Mean, variance and standard deviation of X , are respectively

- (a) 17.53, 4.8 and 2.19 (b) 2.19, 4.8 and 17.53
 (c) 17.53, 2.19 and 4.8 (d) None of these

Sol. (a) Here, total number of students = 15

The age of students in ascending order are 14, 14, 15, 16, 16, 17, 17, 17, 18, 19, 19, 20, 20, 20, 21.

Now, $P(X=14) = \frac{2}{15}, P(X=15) = \frac{1}{15},$
 $P(X=16) = \frac{2}{15}, P(X=17) = \frac{3}{15},$
 $P(X=18) = \frac{1}{15}, P(X=19) = \frac{2}{15},$
 $P(X=20) = \frac{3}{15}, P(X=21) = \frac{1}{15}$

Therefore, the probability distribution of random variable X is as follows:

X	14	15	16	17	18	19	20	21
Number of students	2	1	2	3	1	2	3	1
$P(X)$	$\frac{2}{5}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{3}{15}$	$\frac{1}{15}$

The third row gives the probability distribution of X .

$\therefore$ Mean of $X = \sum X P(X)$

$$= \frac{14 \times 2 + 15 \times 1 + 16 \times 2 + 17 \times 3 + 18 \times 1 + 19 \times 2 + 20 \times 3 + 21 \times 1}{15} = \frac{263}{15} = 17.53$$

Variance of $X = \sum X^2 P(X) - (\text{Mean})^2$

$$= \frac{[(14)^2 \times 2 + (15)^2 \times 1 + (16)^2 \times 2 + (17)^2 \times 3 + (18)^2 \times 1 + (19)^2 \times 2 + (20)^2 \times 3 + (21)^2 \times 1]}{15} - \left(\frac{263}{15}\right)^2$$

$$= \frac{[392 + 225 + 512 + 867 + 324 + 722 + 1200 + 441]}{15} - \left(\frac{263}{15}\right)^2$$

$$= \frac{4683}{15} - \left(\frac{263}{15}\right)^2 = 312.2 - 307.4 = 4.8$$

SD of $X = \sqrt{\text{Variance}} = \sqrt{4.8} = 2.19$

Binomial Distribution

Bernoulli Trial

In a random experiment, if there are two events, 'Success and Failure' and the sum of the probabilities of these two events is one, then any outcome of such experiment is known as a Bernoulli trial.

Let a Binomial experiment has probability of success p and that of failure q (i.e. $p + q = 1$). If E is an event and X is number of success, i.e. number of times event E occurs in n trials.

Then, probability distribution of Binomial distribution with parameters n and p , is given by

$$P(X = r) = \text{Probability of } r \text{ success in } n \text{ trials}$$

$$= {}^n C_r p^r q^{n-r}$$

$$= (r+1)\text{th term in the expansion of } (q + p)^n$$

It is written as

$$X \sim B(n, p) \quad \text{or} \quad X \sim Bi(n, p)$$

✎ The trials must meet the following requirements:

- > The total number of trials is fixed in advance.
- > There are just two outcomes of each trials, success and failure.
- > All the trials have the same probability of success.
- > The outcomes of all the trials are statistically independent.

➤ **Example 21.** A bag contains 14 balls of two colours, the number of balls of each colour being the same. 7 balls are drawn at random-one-by-one. The ball in hand is returned to the bag before each new draw. If the probability that

atleast 3 balls of each colour are drawn, is p , then

(a) $p > \frac{1}{2}$ (b) $p = \frac{1}{2}$ (c) $p < 1$ (d) $p < \frac{1}{2}$

Sol. (a) A ball drawn is of either colour. If drawing a ball of one colour is a success, then drawing ball of the other colour is a failure.

$\therefore$ The probability of success in one draw = $\frac{7}{14} = \frac{1}{2}$
and the probability of failure in one draw = $\frac{7}{14} = \frac{1}{2}$

$\therefore$ The probability that atleast 3 balls of each colour is drawn

$$= {}^7 C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^4 + {}^7 C_4 \cdot \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^3$$

$$= 2 \cdot \frac{7!}{(3!)(4!)} \cdot \frac{1}{2^7}$$

$$= 2 \cdot \frac{7 \cdot 6 \cdot 5}{6} \cdot \frac{1}{2^7} = \frac{35}{64} > \frac{1}{2}$$

➤ **Example 22.** A fair coin is tossed 100 times. The probability of getting tails an odd number of times, is

(a) $\frac{1}{2}$ (b) $\frac{1}{8}$
(c) $\frac{3}{8}$ (d) None of these

Sol. (a) Let X denotes the number of tails. Then, X is a Binomial variate with parameters $n = 100$ and $p = \frac{1}{2}$.

Therefore, $P(X = r) = {}^{100} C_r \left(\frac{1}{2}\right)^{100}$; $r = 0, 1, 2, \dots, 100$

$\therefore$ Required probability

$$= P(X = 1) + P(X = 3) + \dots + P(X = 99)$$

$$= \left(\frac{1}{2}\right)^{100} [{}^{100} C_1 + {}^{100} C_3 + \dots + {}^{100} C_{99}]$$

$$= \frac{1}{2^{100}} [2^{99}] = \frac{1}{2}$$

➤ **Example 23.** If X follows a Binomial distribution with parameters $n = 8$ and $p = \frac{1}{2}$, then

$P(|X - 4| \leq 2)$ is equal to

(a) $\frac{118}{128}$ (b) $\frac{119}{128}$
(c) $\frac{117}{128}$ (d) None of these

Sol. (b) We have,

$$P(|X - 4| \leq 2) = P(-2 \leq X - 4 \leq 2) = P(2 \leq X \leq 6)$$

$$= P(X = 2) + P(X = 3) + P(X = 4) + P(X = 5) + P(X = 6)$$

$$= {}^8 C_2 \left(\frac{1}{2}\right)^8 + {}^8 C_3 \left(\frac{1}{2}\right)^8 + {}^8 C_4 \left(\frac{1}{2}\right)^8 + {}^8 C_5 \left(\frac{1}{2}\right)^8 + {}^8 C_6 \left(\frac{1}{2}\right)^8$$

$$= \frac{1}{2^8} [28 + 56 + 70 + 56 + 28] = \frac{238}{2^8} = \frac{119}{128}$$

Mean and Variance of Binomial Distribution

Let $X \sim B(n, p)$, then $P(X = r) = {}^nC_r p^r q^{n-r}$

where, $r = 0, 1, 2, \dots, n$ and $p + q = 1$

$\therefore$ Mean, $\bar{X} = E(X) = np$

$\text{var}(X) = npq$

and standard deviation $= \sqrt{npq}$

Relation between Mean and Variance

Mean – Variance

$$= np - npq = np(1 - q) = np^2 > 0$$

$\Rightarrow$ Mean $>$ Variance

i.e. for Binomial variable X , value of mean is always greater than its variance.

Mode of Binomial Distribution

In Binomial distribution, the value of r for which $P(X = r)$ is maximum, is known as mode of Binomial distribution.

$$\therefore (n+1)p - 1 \leq r \leq (n+1)P$$

Example 24. In a box containing 100 bulbs, 10 are defective. The probability that out of a sample of 5 bulbs, none is defective, is

(a) 10^{-1} (b) $\left(\frac{1}{2}\right)^5$ (c) $\left(\frac{9}{10}\right)^5$ (d) $\frac{9}{10}$

Sol. (c) The repeated selection of defective bulbs from a box are Bernoulli trials. Let X denotes the number of defective bulbs out of a sample of 5 bulbs.

Now, probability of getting a defective bulb,

$$p = \frac{10}{100} = \frac{1}{10} \quad \text{and} \quad q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$$

Clearly, X has Binomial distribution with $n = 5$,

$$p = \frac{1}{10} \quad \text{and} \quad q = \frac{9}{10}$$

$$\therefore P(X = r) = {}^nC_r p^r q^{n-r} \\ = {}^5C_r \cdot \left(\frac{1}{10}\right)^r \cdot \left(\frac{9}{10}\right)^{5-r}$$

P (none of the bulb is defective)

$$= P(X = 0) = {}^5C_0 \cdot \left(\frac{1}{10}\right)^0 \cdot \left(\frac{9}{10}\right)^5 \\ = 1 \times 1 \times \left(\frac{9}{10}\right)^5 = \left(\frac{9}{10}\right)^5$$

Example 25. In a multiple choice examination with three possible answers for each of the five questions, what is the probability that a candidate would get four or more correct answers just by guessing?

(a) $\frac{3}{243}$ (b) $\frac{1}{243}$ (c) $\frac{25}{243}$ (d) $\frac{11}{243}$

Sol. (d) Let X denotes the number of correct answers given by the candidate. It is a case of Bernoulli trials with $n = 5$, where success is guessing a correct answer to a multiple choice question with 3 options.

$$\therefore p = \frac{1}{3} \quad \text{and} \quad q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

Clearly, X has a Binomial distribution with $n = 5$, $p = \frac{1}{3}$ and $q = \frac{2}{3}$.

$$P(X = r) = {}^5C_r \left(\frac{1}{3}\right)^r \cdot \left(\frac{2}{3}\right)^{5-r}$$

$\therefore$ Required probability $= P$ (four or more correct answers)

$$= P(X \geq 4) = P(4) + P(5) \\ = {}^5C_4 p^4 q + {}^5C_5 p^5 q^0 \\ = {}^5C_4 \left(\frac{1}{3}\right)^4 \cdot \frac{2}{3} + {}^5C_5 \left(\frac{1}{3}\right)^5 \cdot \left(\frac{2}{3}\right)^0 \\ = 5 \times \frac{2}{3} \times \frac{1}{3^4} + \frac{1}{3^5} = \frac{11}{3^5} = \frac{11}{243}$$

Work Book Exercise 16.3

1 Let X denotes the sum of the numbers obtained when two fair dice are rolled. Then, the variance and standard deviations of X are

a $\frac{31}{6}$ and $\sqrt{\frac{31}{6}}$ b $\frac{35}{6}$ and $\sqrt{\frac{35}{6}}$
c $\frac{17}{6}$ and $\sqrt{\frac{17}{6}}$ d None of these

2 A coin is tossed 7 times. The probability that he wins the toss on more occasions, is

a $\frac{1}{4}$ b $\frac{5}{8}$
c $\frac{1}{2}$ d None of these

3 From a box containing 20 tickets of value 1 to 20, four tickets are drawn one by one. After each draw, the ticket is replaced. The probability that the largest value of tickets drawn is 15, is

a $\left(\frac{3}{4}\right)^4$ b $\frac{27}{320}$
c $\frac{27}{1280}$ d None of these

4 Suppose X has a Binomial distribution $B\left(6, \frac{1}{2}\right)$, then which of the outcome is most likely?

a $X = 0$ and $X = 6$ b $X = 3$
c $X = 0$ d $X = 6$

- 5 A discrete random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
$P(X)$	C	$2C$	$2C$	$3C$	C^2	$2C^2$	$7C^2 + C$

The value of C and the mean of the distribution are

- a $\frac{1}{10}$ and 3.66 b $\frac{1}{20}$ and 2.66
c $\frac{1}{15}$ and 1.33 d None of these

- 6 The probability of guessing correctly atleast 8 out of 10 answers on a true-false examination, is

- a $\frac{7}{128}$
b $\frac{5}{128}$
c $\frac{9}{128}$
d None of the above

Poisson Distribution

It is limiting case of Binomial distribution. Let the number of events n be large ($n \rightarrow \infty$) and probability of success in each experiment be 0 and $np = \lambda$ (say) is finite, then

$$P(X = r) \text{ or } P(r) = \frac{e^{-\lambda} \lambda^r}{r!}, \text{ where } r = 0, 1, 2, \dots$$

and $\lambda = np$, where λ is known as parameter of Poisson distribution.

$$P(r+1) = \frac{\lambda}{r+1} P(r) \text{ is known as recurrence}$$

formula.

- $\sum_{r=0}^{\infty} P(r) = 1$
- If λ_1 and λ_2 are parameters of variables X and Y , then parameter of $(X + Y)$ will be $(\lambda_1 + \lambda_2)$.
- In Poisson distribution, mean = variance = $\lambda = np$.

➤ **Example 26.** If mean of a Poisson distribution of a random variable X is 2, then the value of $P(X > 1.5)$ is

- (a) $\frac{3}{e^2}$ (b) $\frac{3}{e}$ (c) $1 - \frac{3}{e}$ (d) $1 - \frac{3}{e^2}$

Sol. (d) Since, $P(X = r) = \frac{e^{-\lambda} \lambda^r}{r!}$ [where, $\lambda = \text{mean}$]

$$\begin{aligned} \therefore P(X = r > 1.5) &= P(2) + P(3) + \dots \\ &= 1 - P(X = r \leq 1) = 1 - P(0) - P(1) \\ &= 1 - \left(e^{-2} + \frac{e^{-2} \times 2}{1!} \right) = 1 - \frac{3}{e^2} \end{aligned}$$

Geometrical Probability

If the total number of outcomes of a trial in a random experiment is infinite, in such cases, the definition of probability is modified and the general expression for the probability P of occurrence of an event is given by

$$P = \frac{\text{Measure of the specified part of the region}}{\text{Measure of the whole region}}$$

where, measure means length or area or volume of the region, if we are dealing with one, two or three dimensional space respectively.

Application based Results

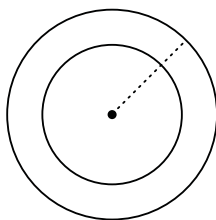
- i. When two dice are thrown, the number of ways of getting a total r is
- (a) $(r-1)$, if $2 \leq r \leq 7$
(b) $(13-r)$, if $8 \leq r \leq 12$
- ii. **Experiment with insertion of n letters in n addressed envelopes**
- (a) Probability of inserting all the n letters in right envelopes = $\frac{1}{n!}$
- (b) Probability that all letters does not in right envelope = $1 - \frac{1}{n!}$
- (c) Probability of keeping all the letters in wrong envelopes
- $$= \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^n}{n!}$$
- (d) Probability that exactly r letters are in right envelopes
- $$= \frac{1}{r!} \left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^{n-r} \frac{1}{(n-r)!} \right]$$
- iii. **Selection of shoes from a cupboard**
- (a) Out of n pair of shoes, if k shoes are selected at random, the probability that there is no pair, is
- $$P = \frac{{}^n C_k 2^k}{{}^{2n} C_k}$$
- (b) The probability that there is atleast one pair of shoes, is $(1 - p)$.
- iv. **Selection of squares from the chessboard**
- If r squares are selected from a chessboard, then probability that they lie on a diagonal, is
- $$\frac{4[{}^7 C_r + {}^6 C_r + \dots + {}^1 C_r] + 2({}^8 C_r)}{{}^{64} C_r}$$
- v. If A and B are two finite sets and a mapping is selected at random from the set of all mappings from A into B , then the probability that the mapping is

- (a) one-one function = $\frac{{}^n(B)P_n(A)}{{}^n(B)^{n(A)}}$
- (b) many-one function = $1 - \frac{{}^n(B)P_n(A)}{{}^n(B)^{n(A)}}$
- (c) constant function = $\frac{n(B)}{{}^n(B)^{n(A)}}$
- (d) one-one and onto function = $\frac{n(A)!}{{}^n(B)^{n(A)}}$

➤ **Example 27.** A point is selected at random from the interior of a circle. The probability that the point is closer to the centre than boundary of the circle, is

- (a) $\frac{3}{4}$ (b) $\frac{1}{2}$
- (c) $\frac{1}{4}$ (d) None of these

Sol. (c) In a circle, all the points which lie in the area of half the radius will be nearer to origin than to the boundary.



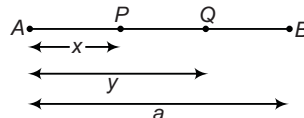
Let $n(S)$ be area of circle of radius r
and $n(E)$ be area of the circle of radius $\frac{r}{2}$.

$$\therefore \text{Required probability} = \frac{n(E)}{n(S)} = \frac{\pi \left(\frac{r}{2}\right)^2}{\pi r^2} = \frac{1}{4}$$

➤ **Example 28.** Two points are taken at random on the given straight line segment of length a . Then, the probability for the distance between them to exceed a given length c , where $0 < c < a$, is

- (a) $\left(1 + \frac{c}{a}\right)^2$ (b) $\left(1 - \frac{c}{a}\right)^2$
- (c) $\left(1 + \frac{a}{c}\right)^2$ (d) None of these

Sol. (b) Let AB be a straight line segment of length a and P, Q be two points on it such that $AP = x$ and $AQ = y$. Then, we have to find the probability for $|PQ| > c$, i.e. $|x - y| > c$. Clearly, $0 \leq x \leq a$ and $0 \leq y \leq a$

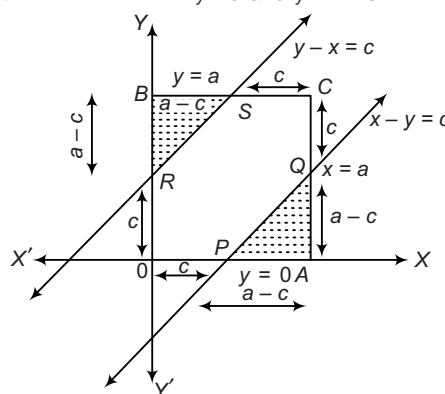


We observe that, there are two variables x and y satisfying $0 \leq x \leq a$ and $0 \leq y \leq a$. In the cartesian plane, the inequalities $0 \leq x \leq a$ and $0 \leq y \leq a$ represent the region enclosed by the square having $x = 0, y = 0, x = a$ and $y = a$ as its sides. Clearly, area enclosed by this square is a^2 .

Now, $|x - y| > c$

$$\Rightarrow x - y > c \text{ and } x - y < -c$$

$$\Rightarrow x - y > c \text{ and } y - x > c$$



In order to find $P(|x - y| > c)$, we require the area of the region enclosed by $x - y > c, y - x > c, 0 \leq x \leq a$ and $0 \leq y \leq a$. Clearly, shaded region represents the area enclosed.

We have,

$$\text{Area of the shaded region} = 2 \times \text{Area of triangle} = 2 \times \frac{1}{2} (a - c)^2 + \frac{1}{2} (a - c)^2 = (a - c)^2$$

Hence, required probability

$$= \frac{\text{Area enclosed by } |x - y| > c, 0 \leq x \leq a \text{ and } 0 \leq y \leq a}{\text{Area enclosed by } 0 \leq x \leq a, 0 \leq y \leq a}$$

$$= \frac{\text{Area of the shaded region}}{\text{Area of the square}} = \frac{(a - c)^2}{a^2} = \left(1 - \frac{c}{a}\right)^2$$

Work Book Exercise 16.4

- Ravish and Vineet decided to meet on New Year's day evening at a place in Cannaught Place between 7 pm and 8 pm. The first one to arrive waits for 15 min and then leave. Assuming that each is independently equally likely to arrive at any time during the hour (between 7 pm and 8 pm), then the probability that they meet is

a $\frac{5}{16}$ b $\frac{3}{16}$ c $\frac{3}{8}$ d $\frac{7}{16}$
- If a rod of length a is broken into three parts at random, then the probability that a triangle can be formed from these parts, is

a $\frac{1}{4}$ b $\frac{3}{4}$
 c $\frac{1}{2}$ d None of these
- If two numbers x and y are chosen such that $x \in [0, 4]$ and $y \in [0, 4]$, then the probability that the chosen number satisfy the inequation $y^2 \leq x$, is

a $\frac{1}{3}$ b $\frac{2}{3}$
 c $\frac{3}{5}$ d $\frac{3}{7}$
- With the vertices of a square as centres, four ellipses are drawn to pass through the centre and the middle points of its sides, the probability that a point of the square chosen at random will lie within any two of the ellipses, is

a $\frac{2\pi}{3}$
 b $\frac{2\pi}{3\sqrt{3}}$
 c $\frac{2\pi}{3\sqrt{3}} - 1$
 d None of the above
- The outcomes of an experiment are represented by points in the square bounded by $x = 0$, $y = 0$, $x = 2$ and $y = 2$ in XY -plane. If the probability is distributed uniformly, then the probability that $x^2 + y^2 > 1$ is

a $\frac{1}{4}$
 b $\frac{3}{4}$
 c $\frac{1}{2}$
 d None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If two events A and B are such that $P(A^C) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$, then $P[B / (A \cup B^C)]$ is equal to

- (a) $\frac{3}{4}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

Sol. Given that, $P(A^C) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$

$$\begin{aligned} \text{Then, } P[B / (A \cup B^C)] &= \frac{P[B \cap (A \cup B^C)]}{P(A \cup B^C)} \\ &= \frac{P[(B \cap A) \cup (B \cap B^C)]}{P(A \cup B^C)} \\ &= \frac{P(A \cap B)}{P(A) + P(B^C) - P(A \cap B^C)} \\ &= \frac{P(A) - P(A \cap B^C)}{1 - P(A^C) + 1 - P(B) - P(A \cap B^C)} \\ &= \frac{1 - 0.3 - 0.5}{1 - 0.3 + 1 - 0.4 - 0.5} = \frac{0.2}{0.8} = \frac{1}{4} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 2. The probability of the simultaneous occurrence of two events A and B is p . If the probability that exactly one of A, B occurs is q , then

- (a) $P(\bar{A}) + P(\bar{B}) = 1 + 2q - p$
 (b) $P(\bar{A}) + P(\bar{B}) = 1 - 2p - q$
 (c) $P\left(\frac{A \cap B}{A \cup B}\right) = \frac{p}{p + q}$
 (d) $P(\bar{A} \cap \bar{B}) = 2 - p + q$

Sol. We have,

$$\begin{aligned} P(A \cap B) &= p \text{ and } P(A) + P(B) - 2P(A \cap B) = q \\ \therefore P(A) + P(B) - 2p &= q \\ \Rightarrow P(A) + P(B) &= 2p + q \\ \Rightarrow 1 - P(\bar{A}) + 1 - P(\bar{B}) &= 2p + q \\ \Rightarrow P(\bar{A}) + P(\bar{B}) &= 2 - 2p - q \end{aligned}$$

So, options (a) and (b) are false.

$$\begin{aligned} \text{Now, } P\left(\frac{A \cap B}{A \cup B}\right) &= \frac{P\{(A \cap B) \cap (A \cup B)\}}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A) + P(B) - P(A \cap B)} \\ &= \frac{p}{2p + q - p} = \frac{p}{p + q} \end{aligned}$$

So, option (c) is correct.

$$\begin{aligned} \text{Finally, } P(\bar{A} \cap \bar{B}) &= P(\overline{A \cap B}) = 1 - P(A \cap B) \\ &= 1 - [P(A) + P(B) - P(A \cap B)] \\ &= 1 - [2p + q - p] = 1 - p - q \end{aligned}$$

So, option (d) is false.

Hence, (c) is the correct answer.

Ex 3. For two events A and B , it is given that

$$P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$$

$$\text{and } P\left(\frac{B}{A}\right) = \frac{1}{2}.$$

Then,

- (a) A and B are mutually exclusive events
 (b) A and B are dependent events
 (c) $P\left(\frac{A}{B}\right) = \frac{3}{4}$
 (d) None of the above

Sol. We have, $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$. This shows that A and B

are independent events. So, $P(B) = P\left(\frac{B}{A}\right) = \frac{1}{2}$

$$\text{Now, } P\left(\frac{A}{B}\right) = \frac{1}{4}$$

$$\Rightarrow P\left(\frac{\bar{A}}{B}\right) = 1 - \frac{1}{4} = \frac{3}{4}$$

Hence, (c) is the correct answer.

Ex 4. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then $P(B)$ is

- (a) $\frac{1}{5}$ (b) $\frac{1}{6}$ or $\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{5}{6}$

Sol. Let $P(A) = x$ and $P(B) = y$.

Since, A and B are independent events.

$$\begin{aligned} \therefore P(\bar{A} \cap B) &= \frac{2}{15} \\ \Rightarrow P(\bar{A}) P(B) &= \frac{2}{15} \\ \Rightarrow \{1 - P(A)\} P(B) &= \frac{2}{15} \\ \Rightarrow (1 - x) y &= \frac{2}{15} \\ \Rightarrow y - xy &= \frac{2}{15} \quad \dots(i) \\ \text{and } P(A \cap \bar{B}) &= \frac{1}{6} \end{aligned}$$

$$\begin{aligned}\Rightarrow P(A) P(\bar{B}) &= \frac{1}{6} \\ \Rightarrow x(1-y) &= \frac{1}{6} \\ \Rightarrow x-xy &= \frac{1}{6} \quad \dots(ii)\end{aligned}$$

On subtracting Eq. (i) from Eq. (ii), we get

$$\begin{aligned}x-y &= \frac{1}{30} \\ \Rightarrow x &= \frac{1}{30} + y\end{aligned}$$

On putting the value of x in Eq. (i), we get

$$\begin{aligned}y-y\left(\frac{1}{30}+y\right) &= \frac{2}{15} \\ \Rightarrow 30y^2-29y+4 &= 0 \\ \Rightarrow (6y-1)(5y-4) &= 0 \\ \Rightarrow y &= \frac{1}{6} \quad \text{or} \quad y = \frac{4}{5} \\ \Rightarrow P(B) &= \frac{1}{6} \\ \text{or} \quad P(B) &= \frac{4}{5}\end{aligned}$$

Hence, (b) is the correct answer.

Ex 5. A is one of 6 horses entered for a race and is to be ridden by one of two jockeys B and C . It is 2 : 1 that B rides A , in which case all the horses are equally likely to win. If C rides A , his chance of winning is trebled. What are the odds against winning of A ?

- (a) 5 : 13 (b) 5 : 18
(c) 13 : 5 (d) None of these

Sol. Let E = The event that horse A wins

E_1 = The event that jockey B rides horse A

E_2 = The event that jockey C rides horse A

According to question, odds in favour of $E_1 = 2 : 1$

$$\therefore P(E_1) = \frac{2}{3}$$

$$\text{and} \quad P\left(\frac{E}{E_1}\right) = \frac{1}{6}$$

[since, when B rides A , all six horses are equally likely to win]

$$\begin{aligned}P(E_2) &= 1 - P(E_1) \\ &= 1 - \frac{2}{3} = \frac{1}{3}\end{aligned}$$

$$\text{and} \quad P\left(\frac{E}{E_2}\right) = 3P\left(\frac{E}{E_1}\right) = \frac{1}{2}$$

Let $A_1 = E_1 \cap E$ and $A_2 = E_2 \cap E$

Now, required probability $P(E) = P(A_1) + P(A_2)$

$$\begin{aligned}&= P(E_1 \cap E) + P(E_2 \cap E) \\ &= P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) \\ &= \frac{2}{3} \cdot \frac{1}{6} + \frac{1}{3} \cdot \frac{1}{2} = \frac{5}{18}\end{aligned}$$

So that, odds against winning of A are 13 : 5.

Hence, (c) is the correct answer.

Ex 6. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of a ball drawn. A ball is again drawn at random. Then, the probability that the ball drawn now is white, is

- (a) $\frac{n}{m+n}$ (b) $\frac{m}{m+n}$
(c) $\frac{2n}{m+n}$ (d) None of these

Sol. Let w_1 = Ball drawn in the first draw is white,
 b_1 = Ball drawn in the first draw is black
and w_2 = Ball drawn in second draw is white.

$$\begin{aligned}\text{Then, } P(w_2) &= P(w_1) \cdot P\left(\frac{w_2}{w_1}\right) + P(b_1) \cdot P\left(\frac{w_2}{b_1}\right) \\ &= \left(\frac{m}{m+n}\right) \left(\frac{m+k}{m+n+k}\right) + \left(\frac{n}{m+n}\right) \left(\frac{m}{m+n+k}\right) \\ &= \frac{m}{m+n}\end{aligned}$$

Hence, (b) is the correct answer.

Ex 7. Pal's gardener is not dependable, the probability that he will forget to water the rose bush is $\frac{2}{3}$. The rose bush is in questionable condition. Any how if watered, the probability of its withering is $\frac{1}{2}$, if not

watered, the probability of its withering is $\frac{3}{4}$.

Pal went out to station and upon returning, he finds that the rose bush has withered. Then, the chance that the gardener did not water the rose bush, is

- (a) $\frac{2}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{3}{4}$

Sol. Let us suppose that the events

$\bar{A}$ = The rose bush has been watered,

A = The rose bush has not been watered

and B = The rose bush has been withered,

From the question, it is given that

$$P(\bar{A}) = \frac{2}{3}, P\left(\frac{B}{\bar{A}}\right) = \frac{1}{2}, P\left(\frac{B}{A}\right) = \frac{3}{4} \text{ and } P(A) = \frac{1}{3}$$

From Baye's theorem,

$$\begin{aligned}P\left(\frac{\bar{A}}{B}\right) &= \frac{P(\bar{A} \cap B)}{P(\bar{A} \cap B) + P(A \cap B)} \\ &= \frac{P(\bar{A}) \cdot P\left(\frac{B}{\bar{A}}\right)}{P(\bar{A}) \cdot P\left(\frac{B}{\bar{A}}\right) + P(A) \cdot P\left(\frac{B}{A}\right)} \\ &= \frac{\frac{2}{3} \times \frac{1}{2}}{\frac{2}{3} \times \frac{1}{2} + \frac{1}{3} \times \frac{3}{4}} = \frac{3}{4}\end{aligned}$$

Hence, (d) is the correct answer.

Ex 8. A box contains N coins, m of which are fair and the rest are biased. The probability of getting a head when a fair coin is tossed is $1/2$, while it is $2/3$ when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. Then, the probability that the coin drawn is fair, is

- (a) $\frac{9m}{8N+m}$ (b) $\frac{9m}{8N-m}$
(c) $\frac{9m}{8m-N}$ (d) $\frac{9m}{8m+N}$

Sol. E_1 : Coin is fair, E_2 : Coin is biased
and A : Second toss shows tail.

$$\begin{aligned} P\left(\frac{E_1}{A}\right) &= \frac{P\left(\frac{A}{E_1}\right)P(E_1)}{P\left(\frac{A}{E_1}\right)P(E_1) + P\left(\frac{A}{E_2}\right)P(E_2)} \\ &= \frac{\frac{m}{N} \cdot \frac{1}{2} \cdot \frac{1}{2}}{\frac{m}{N} \cdot \frac{1}{2} \cdot \frac{1}{2} + \frac{N-m}{N} \cdot \frac{2}{3} \cdot \frac{1}{3}} \\ &= \frac{9m}{9m+8N-8m} = \frac{9m}{8N+m} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 9. Let $P(x)$ denotes the probability of the occurrence of event x . Then, all those points $(x, y) = (P(A), P(B))$, in a plane which satisfy the conditions,

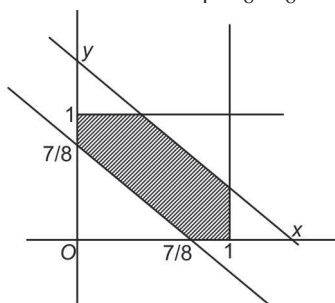
$$P(A \cup B) \geq \frac{3}{4} \text{ and } \frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8} \text{ implies}$$

- (a) $P(A) + P(B) < \frac{11}{8}$
(b) $P(A) + P(B) > \frac{11}{8}$
(c) $\frac{7}{8} \leq P(A) + P(B) \leq \frac{11}{8}$
(d) None of the above

$$\text{Sol. } P(A \cup B) \geq \frac{3}{4} \text{ and } \frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$$

$$\Rightarrow P(A) + P(B) - P(A \cap B) \geq \frac{3}{4}$$

$$\begin{aligned} \Rightarrow P(A) + P(B) &\geq \frac{3}{4} + P(A \cap B) \\ &= \frac{3}{4} + \frac{1}{8} = \frac{7}{8} \end{aligned}$$



We know that, $P(A \cup B) \leq 1$

$$P(A) + P(B) \leq 1 + P(A \cap B) \leq 1 + \frac{3}{8} = \frac{11}{8}$$

$$\Rightarrow x + y \leq \frac{11}{8}$$

$$\Rightarrow \frac{7}{8} \leq x + y \leq \frac{11}{8}$$

Also, $0 \leq P(A) \leq 1, 0 \leq P(B) \leq 1$

Hence, (c) is the correct answer.

Ex 10. Two dice are thrown simultaneously to get the coordinates of a point on XY -plane. Then, the probability that this point lies inside or on the region bounded by $|x| + |y| = 3$, is

- (a) $\frac{3}{14}$ (b) $\frac{2}{3}$ (c) $\frac{1}{12}$ (d) $\frac{4}{14}$

Sol. Any point on or inside the region bounded by $|x| + |y| = 3$ will have x and y -coordinates lying between -3 and $+3$ which will turn up as a throw of dice, i.e. $(1, 1)$, $(1, 2)$ and $(2, 1)$.

$$\text{So, probability is } \frac{3}{36} = \frac{1}{12}$$

Hence, (c) is the correct answer.

Ex 11. Suppose $f(x) = x^3 + ax^2 + bx + c$, where a, b and c are chosen respectively by throwing a die three times. Then, the probability that $f(x)$ is an increasing function, is

- (a) $\frac{4}{9}$ (b) $\frac{3}{8}$ (c) $\frac{2}{5}$ (d) $\frac{16}{34}$

$$\text{Sol. } f'(x) = 3x^2 + 2ax + b$$

$$y = f'(x) \text{ is increasing.}$$

$$\Rightarrow f'(x) \geq 0, \forall x$$

and for $f'(x) = 0$ should not form an interval.

$$\Rightarrow (2a)^2 - 4 \times 3 \times b \leq 0 \Rightarrow a^2 - 3b \leq 0$$

This is true for exactly 16 ordered pairs (a, b) , $1 \leq a, b \leq 6$ namely $(1, 1)$, $(1, 2)$, $(1, 3)$, $(1, 4)$, $(1, 5)$, $(1, 6)$, $(2, 2)$, $(2, 3)$, $(2, 4)$, $(2, 5)$, $(2, 6)$, $(3, 3)$, $(3, 4)$, $(3, 5)$, $(3, 6)$ and $(4, 6)$.

$$\text{Thus, required probability} = \frac{16}{36} = \frac{4}{9}$$

Hence, (a) is the correct answer.

Ex 12. An unbiased die, with faces numbered 1, 2, 3, 4, 5 and 6, is thrown n times and the list of n numbers shown up is noted. Then, the probability that, among the numbers 1, 2, 3, 4, 5 and 6, only three numbers appear in this list, is

$$(a) \frac{3^n \cdot {}^6C_3}{6^n}$$

$$(b) \frac{(3^n - 3 \cdot 2^n + 3) \cdot {}^6C_3}{6^n}$$

$$(c) \frac{(3^n - 3) \cdot {}^6C_3}{6^n}$$

(d) None of the above

Sol. Let us define a onto function F from $A : [r_1, r_2, \dots, r_n]$ to $B : [1, 2, 3]$, where $r_1, r_2, \dots, r_n$ are the readings of the n throws and 1, 2 and 3 are the numbers that appears in the n -throws. Number of such functions

$$M = N - [n(1) - n(2) + n(3)]$$

where, N = Total number of functions and

$n(t)$ = Number of functions having exactly t elements in the range

$$N = 3^n, n(1) = 3 \cdot 2^n, n(2) = 3, n(3) = 0$$

$$\Rightarrow M = (3^n - 3 \cdot 2^n + 3)$$

Hence, total number of favourable cases

$$= (3^n - 3 \cdot 2^n + 3) \cdot {}^6C_3$$

Total number of cases = 6^n

$$\therefore \text{Required probability} = \frac{(3^n - 3 \cdot 2^n + 3) \times {}^6C_3}{6^n}$$

Hence, (b) is the correct answer.

Ex 13. For a student to qualify, he must pass atleast two out of three exams. The probability that he will pass the 1st exam is p . If he fails in one of the exams, then the probability of his passing in the next exam is $\frac{p}{2}$, otherwise it

remains the same. Then, the probability that he will qualify, is

- (a) $2p^2 - p$ (b) $2p^2 - 2p$
(c) $2p^2 - p^3$ (d) None of these

Sol. Let E_1 : The event that the student will pass in the i th exam, $i = 1, 2, 3$.

E : The event that the student will qualify.

$$P(E) = P(E_1) \cdot P(E_2)$$

$$= p^2 + p \times (1-p) \frac{p}{2} + (1-p) \times \frac{p}{2} \times p$$

$$\Rightarrow P(E) = \frac{2p^2 + p^2 - p^3 + p^2 - p^3}{2} = 2p^2 - p^3$$

Hence, (c) is the correct answer.

Ex 14. Three natural numbers are taken at random from the set $A = \{x : 1 \leq x \leq 100, x \in N\}$. The probability that the AM of the numbers taken is 25, is

- (a) $\frac{{}^{77}C_2}{{}^{100}C_3}$ (b) $\frac{{}^{25}C_2}{{}^{100}C_3}$ (c) $\frac{{}^{74}C_{72}}{{}^{100}C_{97}}$ (d) $\frac{{}^{75}C_2}{{}^{100}C_3}$

Sol. Here, $n(S) = {}^{100}C_3$

As the AM of three numbers is 25, their sum = 75

$\therefore n(E)$ = The number of integral solution of

$$x_1 + x_2 + x_3 = 75$$

where, $x_1 \geq 1, x_2 \geq 1$ and $x_3 \geq 1$

= Coefficient of x^{75} in $(x + x^2 + x^3 + \dots)^3$

$$= \text{Coefficient of } x^{72} \text{ in } \left(\frac{1}{1-x} \right)^3$$

$$= \text{Coefficient of } x^{72} \text{ in } (1-x)^{-3}$$

$$= {}^{74}C_{72}$$

$$\therefore P(E) = \frac{{}^{74}C_{72}}{{}^{100}C_3} = \frac{{}^{74}C_{72}}{{}^{100}C_{97}}$$

Hence, (c) is the correct answer.

Ex 15. If the mean and variance of a Binomial variate X are 2 and 1 respectively, then the probability that X takes a value greater than 1, is

- (a) $\frac{2}{3}$ (b) $\frac{4}{5}$ (c) $\frac{7}{8}$ (d) $\frac{15}{16}$

Sol. Let n and p be the parameters of the Binomial distribution. Then, $np = 2$ and $npq = 1$.

$$\Rightarrow q = \frac{1}{2}, p = \frac{1}{2} \text{ and } n = 4$$

$$\begin{aligned} \therefore P(X \geq 1) &= 1 - P(X < 1) = 1 - P(X = 0) \\ &= 1 - {}^nC_0 p^0 q^{n-0} \\ &= 1 - {}^4C_0 \times \left(\frac{1}{2} \right)^4 = \frac{15}{16} \end{aligned}$$

Hence, (d) is the correct answer.

Ex 16. A die is thrown $2n+1$ times, $n \in N$. The probability that faces with even numbers show odd numbers of times, is

- (a) $\frac{2n+1}{4n+3}$ (b) less than $\frac{1}{2}$
(c) greater than $\frac{1}{2}$ (d) None of these

Sol. The probability of showing an even number in a throw

$$= \frac{3}{6} = \frac{1}{2}$$

$\therefore$ Required probability

$$\begin{aligned} &= {}^{2n+1}C_1 \cdot \frac{1}{2} \cdot \left(\frac{1}{2} \right)^{2n} + {}^{2n+1}C_3 \cdot \left(\frac{1}{2} \right)^3 \cdot \left(\frac{1}{2} \right)^{2n-2} \\ &\quad + \dots + {}^{2n+1}C_{2n+1} \left(\frac{1}{2} \right)^{2n+1} \\ &= ({}^{2n+1}C_1 + {}^{2n+1}C_3 + \dots + {}^{2n+1}C_{2n+1}) \left(\frac{1}{2} \right)^{2n+1} \\ &= 2^{2n} \cdot \left(\frac{1}{2} \right)^{2n+1} = \frac{1}{2} \end{aligned}$$

Hence, (d) is the correct answer.

Ex 17. If X follows a Binomial distribution with parameters $n=100$ and $p=\frac{1}{3}$, then $P(X=r)$

is maximum when r is equal to

- (a) 16 (b) 32
(c) 33 (d) None of these

Sol. Since, $(n+1)p = \frac{101}{3}$ is not an integer.

Therefore, $P(X=r)$ is maximum, when

$$r = \left\lceil \frac{101}{3} \right\rceil = 33$$

Hence, (c) is the correct answer.

Ex 18. An ordinary die is rolled in a certain number of times. The probability of getting an odd number 2 times is equal to the probability of getting an even number 3 times. Then, the probability of getting an odd number an odd number of times, is

- (a) $\frac{1}{32}$ (b) $\frac{5}{16}$
(c) $\frac{1}{2}$ (d) None of these

Sol. The probability of getting an odd number in a throw
 $= \frac{3}{6} = \frac{1}{2}$

The probability of getting an odd number 2 times in n trials
 $= {}^nC_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{n-2}$

Similarly, the probability of getting an even number 3 times in n trials
 $= {}^nC_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^{n-3}$

From the question,

$${}^nC_2 \cdot \left(\frac{1}{2}\right)^n = {}^nC_3 \cdot \left(\frac{1}{2}\right)^n \Rightarrow n = 5$$

$\therefore$ Required probability

$$= {}^5C_1 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^4 + {}^5C_3 \cdot \left(\frac{1}{2}\right)^3 \cdot \left(\frac{1}{2}\right)^2 + {}^5C_5 \cdot \left(\frac{1}{2}\right)^5$$

$$= (5 + 10 + 1) \cdot \frac{1}{2^5} = \frac{16}{32} = \frac{1}{2}$$

Hence, (c) is the correct answer.

Ex 19. If $a \in [-20, 0]$, then the probability that the equation $16x^2 + 8(a+5)x - 7a - 5 = 0$ has imaginary roots, is

- (a) $\frac{13}{20}$ (b) $\frac{20}{13}$
(c) $\frac{13}{24}$ (d) $\frac{15}{20}$

Sol. If the equation $16x^2 + 8(a+5)x - 7a - 5 = 0$ has imaginary roots, then

$$\begin{aligned} \Rightarrow & \text{Discriminant} < 0 \\ \Rightarrow & 64(a+5)^2 - 64(7a-5) < 0 \\ \Rightarrow & a^2 + 17a + 30 < 0 \\ \Rightarrow & (a+15)(a+2) < 0 \\ \Rightarrow & -15 < a < -2 \end{aligned}$$

$$\therefore \text{Required probability} = \frac{\int_{-15}^{-2} da}{\int_{-20}^{-2} da} = \frac{13}{20}$$

Hence, (a) is the correct answer.

Ex 20. A, B and C are contesting the election for the post of secretary of a club which does not allow ladies to become members. The probabilities of A, B and C winning the election are $\frac{1}{3}, \frac{2}{9}$ and $\frac{4}{9}$, respectively.

The probabilities of introducing the clause of admitting lady member to the club by A, B and C are 0.6, 0.7 and 0.5, respectively. The probability that ladies will be taken as members in the club after the election, is

- (a) $\frac{26}{45}$ (b) $\frac{5}{9}$
(c) $\frac{19}{45}$ (d) None of these

Sol. Let E_A = The event of A becoming secretary.

Similarly, E_B and E_C .

E_L = The event of admitting lady members.

Here, $P(E_A) = \frac{1}{3}, P(E_B) = \frac{2}{9}$ and $P(E_C) = \frac{4}{9}$

Clearly, E_A, E_B and E_C are mutually exclusive and exhaustive.

Also,

$$P\left(\frac{E_L}{E_A}\right) = 0.6, P\left(\frac{E_L}{E_B}\right) = 0.7 \text{ and } P\left(\frac{E_L}{E_C}\right) = 0.5$$

$\therefore$ Required probability

$$= P(E_A) \cdot P\left(\frac{E_L}{E_A}\right) + P(E_B) \cdot P\left(\frac{E_L}{E_B}\right) + P(E_C) \cdot P\left(\frac{E_L}{E_C}\right)$$

$$= \frac{1}{3} \times \frac{3}{5} + \frac{2}{9} \times \frac{7}{10} + \frac{4}{9} \times \frac{5}{10} = \frac{26}{45}$$

Hence, (a) is the correct answer.

Ex 21. If four dice are thrown together, then the probability that the sum of the numbers appearing on them is 13, is

- (a) $\frac{5}{216}$ (b) $\frac{11}{216}$
(c) $\frac{35}{324}$ (d) $\frac{11}{432}$

Sol. Total number of elementary events associated to the random experiments of throwing 4 dice is

$$6 \times 6 \times 6 \times 6 = 6^4$$

Favourable number of elementary events

$$= \text{Coefficient of } x^{13} \text{ in } (x + x^2 + x^3 + \dots + x^6)^4$$

$$= \text{Coefficient of } x^9 \text{ in } (1 + x + x^2 + \dots + x^5)^4$$

$$= \text{Coefficient of } x^9 \text{ in } \left(\frac{1-x^6}{1-x}\right)^4$$

$$= \text{Coefficient of } x^9 \text{ in } (1-x^6)^4 (1-x)^{-4}$$

$$= \text{Coefficient of } x^9 \text{ in } (1-4x^6 + \dots) (1-x)^{-4}$$

$$= \text{Coefficient of } x^9 \text{ in } (1-x)^{-4} - 4$$

$$= \text{Coefficient of } x^3 \text{ in } (1-x)^{-4}$$

$$= {}^{9+4-1}C_{4-1} - 4 \times {}^{3+4-1}C_{4-1}$$

$$[\because \text{coefficient of } x^n \text{ in } (1-x)^{-r} = {}^{n+r-1}C_{r-1}]$$

$$= {}^{12}C_3 - 4 \times {}^6C_3$$

$$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} - 4 \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 220 - 80 = 140$$

$$\text{So, the required probability} = \frac{140}{6^4} = \frac{35}{324}$$

Hence, (c) is the correct answer.

Ex 22. Six faces of a die are marked with the numbers 1, -1, 0, -2, 2 and 3. The die is thrown thrice. The probability that the sum of the numbers thrown is six, is

- (a) $\frac{1}{72}$ (b) $\frac{1}{12}$
(c) $\frac{5}{108}$ (d) $\frac{1}{36}$

Sol. Total number of elementary events = 6^3
Favourable number of elementary events
= Coefficient of x^6 in $(x + x^{-1} + x^0 + x^{-2} + x^2 + x^3)^3$
= Coefficient of x^6 in $\left(\frac{1 + x + x^2 + x^3 + x^4 + x^5}{x^2}\right)^3$
= Coefficient of x^{12} in $(1 + x + x^2 + x^3 + x^4 + x^5)^3$
= Coefficient of x^{12} in $\left(\frac{1 - x^6}{1 - x}\right)^3$
= Coefficient of x^{12} in $(1 - x^6)^3 (1 - x)^{-3}$
= Coefficient of x^{12} in $(1 - x)^{-3} - {}^3C_1 \cdot \text{Coefficient of } x^6 \text{ in } (1 - x)^{-3} + {}^3C_2 \cdot \text{Coefficient of } x^0 \text{ in } (1 - x)^{-3}$
= ${}^{12+3-1}C_{3-1} - {}^3C_1 \cdot {}^{6+3-1}C_{3-1} + {}^3C_2$
= ${}^{14}C_2 - {}^3C_1 \cdot {}^{6+3-1}C_{3-1} + {}^3C_2$
= ${}^{14}C_2 - {}^3C_1 \times {}^8C_2 + {}^3C_2 = 91 - 84 + 3 = 10$
So, required probability = $\frac{10}{6^3} = \frac{5}{108}$
Hence, (c) is the correct answer.

Ex 23. Given that the sum of two non-negative quantities is 200, the probability that their product is not less than $3/4$ times their greatest product value, is

- (a) $\frac{7}{16}$ (b) $\frac{101}{201}$ (c) $\frac{9}{16}$ (d) $\frac{10}{16}$

Sol. Let x and y be the two quantities. When the sum of two non-negative quantities is fixed, the product will be maximum when they are equal.
So, the greatest product = $xy = 10000$, where $x = y = 100$
Now, $xy \geq \frac{3}{4} \times 10000$
 $\Rightarrow xy \geq 7500$
 $\Rightarrow x(200 - x) \geq 7500$
 $\Rightarrow x^2 - 200x + 7500 \leq 0$
 $\Rightarrow (x - 50)(x - 150) \leq 0$
 $\Rightarrow 50 \leq x \leq 150$
So, favourable number of ways = 101
Total number of ways = 201
So, required probability = $\frac{101}{201}$
Hence, (b) is the correct answer.

Ex 24. Let A be a set containing n elements. A subset P of the set A is chosen at random. The set A is reconstructed by replacing the elements of P and another subset Q of A is chosen at random. The probability that $P \cap Q$ contains exactly m ($m < n$) elements, is

- (a) $\frac{3^{n-m}}{4^n}$ (b) $\frac{{}^nC_m \cdot 3^m}{4^n}$
(c) $\frac{{}^nC_m \cdot 3^{n-m}}{4^n}$ (d) None of these

Sol. We know that the number of subsets of a set containing n elements is 2^n .

Therefore, the number of ways of choosing P and Q is $2^n \times 2^n = 4^n$.

Out of n elements, m elements can be chosen in ${}^nC_m$ ways. If $P \cap Q$ contains exactly m elements, then from the remaining $n - m$ elements either an element belongs to P or Q but not both P and Q .

Suppose P contains r elements from the remaining $(n - m)$ elements. Then, Q may contain any number of elements from the remaining $(n - m) - r$ elements.

Therefore, P and Q can be chosen in ${}^{n-m}C_r \cdot 2^{(n-m)-r}$.

But r can vary from 0 to $(n - m)$. So, P and Q can be chosen in general in

$$\left(\sum_{r=0}^{n-m} {}^{(n-m)}C_r \cdot 2^{(n-m)-r} \right) {}^nC_m = (1 + 2)^{n-m} {}^nC_m = {}^nC_m \cdot 3^{n-m}$$

$$\text{So, required probability} = \frac{{}^nC_m \cdot 3^{n-m}}{4^n}$$

Hence, (c) is the correct answer.

Ex 25. Let X be a set containing n elements. If two subsets A and B of X are picked at random, the probability that A and B have the same number of elements, is

- (a) $\frac{{}^{2n}C_n}{2^{2n}}$ (b) $\frac{1}{{}^{2n}C_n}$
(c) $\frac{1 \cdot 3 \cdot 5 \dots (2n + 1)}{2^n n!}$ (d) $\frac{3^n}{4^n}$

Sol. Required probability

$$= \frac{\sum_{r=0}^n {}^nC_r \times {}^nC_r}{2^n \times 2^n} = \frac{{}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \dots + {}^nC_n^2}{4^n} = \frac{{}^{2n}C_n}{2^{2n}}$$

Hence, (a) is the correct answer.

Ex 26. $A_1, A_2, \dots, A_n$ are n independent events with $P(A_j) = \frac{1}{1+j}$ ($1 \leq j \leq n$). The probability that not one of $A_1, A_2, \dots, A_n$ occur, is

- (a) $\frac{n!}{(n+1)!}$ (b) $\frac{n}{(n+1)}$
(c) $\frac{1}{(n+1)!}$ (d) None of these

Sol. Required probability

$$\begin{aligned} &= P(\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n) \\ &= P(\bar{A}_1) P(\bar{A}_2) \dots P(\bar{A}_n) \\ &= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \dots \times \frac{n}{n+1} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \\ &\left[\because P(\bar{A}_j) = 1 - P(A_j) = 1 - \frac{1}{1+j} = \frac{j}{1+j} \right] \end{aligned}$$

Hence, (a) is the correct answer.

Ex 27. Three numbers are selected at random without replacement from the set of numbers $(1, 2, \dots, n)$. The conditional probability that the third number lies between the first two, if the first number is known to be smaller than the second, is

- (a) $\frac{1}{6}$ (b) $\frac{1}{3}$
(c) $\frac{1}{2}$ (d) None of these

Sol. Consider, the following events:

A = The first number is less than the second number.
 B = The third number lies between the first and the second.

We have to find $P\left(\frac{B}{A}\right)$.

$$\text{We have, } P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)}$$

The number of ways of choosing three numbers out of n is ${}^nC_3$. Let the selected numbers be a_1, a_2 and a_3 . Then, they satisfy exactly one of the following inequalities:

$$a_1 < a_2 < a_3, a_1 < a_3 < a_2, a_2 < a_1 < a_3, a_2 < a_3 < a_1, a_3 < a_1 < a_2, a_3 < a_2 < a_1$$

The total number of ways of selecting three numbers and then arranging them is $({}^nC_3 \times 3!)$.

$$\therefore P(A) = \frac{{}^nC_3 \times 3}{{}^nC_3 \times 3!} \text{ and } P(A \cap B) = \frac{{}^nC_3}{{}^nC_3 \times 3!}$$

$$\text{So, } P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{1}{3}$$

Hence, (b) is the correct answer.

Ex 28. Two numbers b and c are chosen at random with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. The probability that $x^2 + bx + c > 0, \forall x \in R$, is

- (a) $\frac{10}{27}$ (b) $\frac{31}{81}$ (c) $\frac{32}{81}$ (d) $\frac{11}{27}$

Sol. Since, b and c both can assume values from 1 to 9. So, total number of ways of choosing b and c is $9 \times 9 = 81$.

$$\text{Now, } x^2 + bx + c > 0, \forall x \in R$$

$$\Rightarrow D < 0$$

$$\text{i.e. } b^2 - 4c < 0$$

The following table shows the possible values of b and c for which $b^2 - 4c < 0$.

c	b	Total
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5

So, favourable number of ways = 32

$$\text{Thus, required probability} = \frac{32}{81}$$

Hence, (c) is the correct answer.

Ex 29. A bag contains $n+1$ coins. It is known that one of these coins shows heads on both sides, whereas the other coins are fair. One coin is selected at random and tossed. If the probability that toss results in heads is $\frac{7}{12}$,

then the value of n is

- (a) 3 (b) 4
(c) 5 (d) None of these

Sol. Let E_1 denotes the event 'a coin with two heads' is selected and E_2 denotes the event 'a fair coin is selected'. Let A be the event 'the toss result in heads'.

$$\text{Then, } P(E_1) = \frac{1}{n+1}, P(E_2) = \frac{n}{n+1}, P\left(\frac{A}{E_1}\right) = 1$$

$$\text{and } P\left(\frac{A}{E_2}\right) = \frac{1}{2}$$

$$\therefore P(A) = P(E_1) P\left(\frac{A}{E_1}\right) + P(E_2) P\left(\frac{A}{E_2}\right)$$

$$\Rightarrow \frac{7}{12} = \frac{1}{n+1} \times 1 + \frac{n}{n+1} \times \frac{1}{2} \left[\because P(A) = \frac{7}{12} \right]$$

$$\Rightarrow 12 + 6n = 7n + 7 \Rightarrow n = 5$$

Hence, (c) is the correct answer.

Ex 30. An experiment has 10 equally likely outcomes. Let A and B be two non-empty events of the experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is equal to

- (a) 2, 4 or 8 (b) 3, 6 or 9
(c) 4 or 8 (d) 5 or 10

Sol. Given, $P(A) = \frac{2}{5}$

For independent events, $P(A \cap B) = P(A) P(B)$

$$\Rightarrow P(A \cap B) \leq \frac{2}{5} \Rightarrow P(A \cap B) = \frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}$$

[maximum 4 outcomes may be in $P(A \cap B)$]

(i) When $P(A \cap B) = \frac{1}{10} \Rightarrow P(A)P(B) = \frac{1}{10}$

$\Rightarrow P(B) = \frac{1}{10} \times \frac{5}{2} = \frac{1}{4}$, not possible

(ii) When $P(A \cap B) = \frac{2}{10} \Rightarrow \frac{2}{5} \times P(B) = \frac{2}{10}$

$\Rightarrow P(B) = \frac{5}{10}$, outcomes of $B = 5$

(iii) When $P(A \cap B) = \frac{3}{10} \Rightarrow P(A)P(B) = \frac{3}{10}$

$\Rightarrow \frac{2}{5} \times P(B) = \frac{3}{10} \Rightarrow P(B) = \frac{3}{4}$, not possible

(iv) When $P(A \cap B) = \frac{4}{10} \Rightarrow P(A)P(B) = \frac{4}{10}$

$\Rightarrow P(B) = 1$, outcomes of $B = 10$

Hence, (d) is the correct answer.

Ex 31. If two cards are drawn successively with replacement from a well shuffled deck of 52 cards, then the mean of the number of aces is

- (a) $\frac{1}{13}$ (b) $\frac{3}{13}$
(c) $\frac{2}{13}$ (d) None of these

Sol. Let X denotes the number of aces.

Probability of selecting aces, $p = \frac{4}{52} = \frac{1}{13}$

Probability of not selecting aces, $q = 1 - \frac{1}{13} = \frac{12}{13}$

$P(X = 1) = 2 \times \left(\frac{1}{13}\right) \times \left(\frac{12}{13}\right) = \frac{24}{169}$

$P(X = 2) = \left(\frac{1}{13}\right)^2 \times \left(\frac{12}{13}\right)^0 = \frac{1}{169}$

Mean = $\sum P_i X_i = \frac{24}{169} + \frac{2}{169} = \frac{2}{13}$

Hence, (c) is the correct answer.

Ex 32. Three letters are written to three different persons and addresses on the three envelopes are also written. Without looking in the addresses, the letters are kept in these envelopes. The probability that all the letters are not placed into their right envelopes, is

- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$
(c) $\frac{1}{6}$ (d) $\frac{5}{6}$

Sol. Favourable ways = $3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right) = 2$

Total ways = $3!$

$\therefore$ Required probability = $\frac{2}{3!} = \frac{1}{3}$

Hence, (b) is the correct answer.

Ex 33. The probability that in a group of N (< 365) people, atleast two will have the same birthday, is

(a) $1 - \frac{(365)!}{(365 - N)!(365)!}$

(b) $\frac{(365)^N (365)!}{(365 - N)!} - 1$

(c) $1 - \frac{(365)^N (365)!}{(365!N)!}$

(d) $1 - \frac{(365)!}{(365 - N)!(365)^N}$

Sol. Let $\bar{A}$ be the event of different birthdays. Each can have birthday in 365 ways, so N persons can have their birthdays in 365^N ways. Number of ways in which all have different birthdays = ${}^{365}P_N$

$\therefore P(A) = 1 - P(\bar{A}) = 1 - \frac{{}^{365}P_N}{(365)^N}$
 $= 1 - \frac{(365)!}{(365)^N (365 - N)!}$

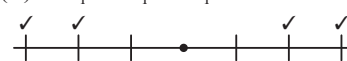
Hence, (d) is the correct answer.

Ex 34. There are 7 seats in a row. Three persons take seats in a row. Three persons take seats at random. The probability that the middle seat is always occupied and no two person are consecutive, is

- (a) $\frac{9}{70}$ (b) $\frac{9}{35}$
(c) $\frac{4}{35}$ (d) None of these

Sol. $n(S) = {}^7C_3 \times 3! = \frac{7 \cdot 6 \cdot 5}{6} \cdot 6 = 210$

$n(E) = {}^2C_1 \times {}^2C_1 \times {}^1C_1 \times 3!$



One has to sit at any one of the two marked seats on the left and the other has to sit at anyone of the two marked seats on the right.

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2 \times 2 \times 6}{210} = \frac{4}{35}$

Hence, (c) is the correct answer.

Ex 35. A second order determinant is written down at random using the numbers 1 and -1 , as elements. The probability that the value of the determinant is non-zero, is

- (a) $\frac{1}{2}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{1}{3}$

Sol. $n(S) = 2 \times 2 \times 2 \times 2 = 16$

As each of the four places can filled in 2 ways.

The zero determinants are $\begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}$, $\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix}$

$\begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}$, $\begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}$, $\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}$ and $\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$

$\therefore P(E) = \frac{8}{16} = \frac{1}{2}$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 36. A card selected at random from cards numbered as 00, 01, 02, ..., 99. An event is said to have occurred. If product of digits of the card number is 16. If card is selected 5 times with replacement each time, then the probability that the event occurs exactly three times, is

(a) ${}^5C_3 \left(\frac{3}{100}\right)^2 \left(\frac{97}{100}\right)^3$

(b) ${}^5C_3 \left(\frac{3}{100}\right)^3 \left(\frac{97}{100}\right)^2$

(c) ${}^5C_3 \left(\frac{0.3}{100}\right)^3 \left(\frac{9.7}{100}\right)^2$

(d) $10(0.03)^3 (0.97)^2$

Sol. Out of the numbers 00, 01, 02, 03, ..., 99, those numbers the product of whose digits is 16 are 28, 44, 82 i.e. only 3.

$$\therefore p = P(E) = \frac{3}{100}$$

$$\text{and } q = P(\bar{E}) = 1 - P(E) = 1 - p = 1 - \frac{3}{100} = \frac{97}{100}$$

$$\therefore \text{The Binomial distribution is } \left(\frac{97}{100} + \frac{3}{100}\right)^5.$$

$$\therefore \text{The probability that the event occurs exactly three times} = {}^5C_3 \left(\frac{3}{100}\right)^3 \left(\frac{97}{100}\right)^2 = 10 (0.03)^3 (0.97)^2$$

Hence, (b) and (d) are the correct answers.

Ex 37. The probabilities that a student passes in Mathematics, Physics and Chemistry are m , p and c respectively. Of these subject, a student has a 75% chance of passing in atleast one, a 50% chance of passing in atleast two and a 40% chance of passing in exactly two subjects. Which of the following relations are true?

(a) $p + m + c = \frac{19}{20}$ (b) $p + m + c = \frac{27}{20}$

(c) $pmc = \frac{1}{10}$ (d) $pmc = \frac{1}{4}$

Sol. Here, $P(M) = m$, $P(P) = p$, $P(C) = c$

The probability of passing in atleast one subject $= 1 - P(\bar{M}\bar{P}\bar{C})$

$$\therefore \frac{75}{100} = 1 - (1 - m)(1 - p)(1 - c)$$

$$\Rightarrow \frac{3}{4} = m + p + c - mp - mc - pc - mpc \quad \dots(i)$$

The probability of passing in atleast two subjects $= P(MPC) + P(M\bar{P}C) + P(\bar{M}PC) + P(\bar{M}\bar{P}C)$

$$\therefore \frac{1}{2} = mpc + mp(1 - c) + m(1 - p)c + (1 - m)pc$$

$$\therefore 2mpc = pm + mc + pc - \frac{1}{2} \quad \dots(ii)$$

The probability of passing in the exactly two subjects

$$= P(M\bar{P}C) + P(\bar{M}PC) + P(\bar{M}\bar{P}C)$$

$$\therefore \frac{2}{5} = mp(1 - c) + m(1 - p)c + (1 - m)pc$$

$$= mp + mc + pc - 3mpc \quad \dots(iii)$$

From Eqs. (ii) and (iii), we get

$$2mpc + \frac{1}{2} = \frac{2}{5} + 3mpc \Rightarrow mpc = \frac{1}{2} - \frac{2}{5} = \frac{1}{10}$$

From Eq. (ii), we get

$$\frac{1}{5} = mp + mc + pc - \frac{1}{2}$$

$$\Rightarrow mp + mc + pc = \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$$

From Eq. (i), we get

$$\frac{3}{4} = m + p + c - \frac{7}{10} + \frac{1}{10}$$

$$\therefore m + p + c = \frac{27}{20}$$

Hence, (b) and (c) are the correct answers.

Ex 38. A coin is tossed repeatedly. A and B call alternately for winning a prize of ₹ 30. One who calls correctly first wins the prize. A starts the call. Then, the expectation of

(a) A is ₹ 10

(b) B is ₹ 10

(c) A is ₹ 20

(d) B is ₹ 20

Sol. The probability of A winning $= P(A) + P(\bar{A}\bar{B}A) + \dots$

$$= \frac{1}{2} + \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{2}\right)\frac{1}{2} + \dots = \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \dots = \frac{1/2}{1 - \frac{1}{4}} = \frac{2}{3}$$

$$\therefore \text{The probability of } B \text{ winning} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\therefore \text{The expectation of } A = ₹ 30 \times \frac{2}{3} = ₹ 20$$

$$\text{and the expectation of } B = ₹ 30 \times \frac{1}{3} = ₹ 10$$

Hence, (b) and (c) are the correct answers.

Ex 39. If A and B are two events such that $P(B) \neq 1$, B^C denotes the event complementary to B , then

(a) $P\left(\frac{A}{B^C}\right) = \frac{P(A) - P(A \cap B)}{1 - P(B)}$

(b) $P(A \cap B) \geq P(A) + P(B) - 1$

(c) $P(A) > P\left(\frac{A}{B}\right)$ according as

$$P\left(\frac{A}{B^C}\right) > P(A)$$

(d) $P\left(\frac{A}{B^C}\right) + P\left(\frac{A^C}{B^C}\right) = 1$

Sol. Here, $P\left(\frac{A}{B^C}\right) = \frac{P(A \cap B^C)}{P(B^C)} = \frac{P(A) - P(A \cap B)}{1 - P(B)} \dots (i)$

and $1 \geq P(A) + P(B) - P(A \cap B)$
or $P(A \cap B) \geq P(A) + P(B) - 1 \dots (ii)$

Let $P(A) > P\left(\frac{A}{B}\right)$ or $P(A) > \frac{P(A \cap B)}{P(B)}$

$P(A) \cdot P(B) > P(A \cap B)$

To prove that, $P\left(\frac{A}{B^C}\right) > P(A)$

$\frac{P(A \cap B^C)}{P(B^C)} > P(A)$

$P(A) - P(A \cap B) > P(A)[1 - P(B)]$
 $- P(A \cap B) > - P(A) \cdot P(B)$

or $P(A) \cdot P(B) > P(A \cap B)$

From Eqs. (i) and (ii), we get

$P(A) > P\left(\frac{A}{B}\right) \Rightarrow P\left(\frac{A}{B^C}\right) > P(A) \dots (iii)$

$\Rightarrow P\left(\frac{A}{B^C}\right) + P\left(\frac{A}{B}\right) = \frac{P(A \cap B^C)}{P(B^C)} + \frac{P(A \cap B)}{P(B)}$
 $= \frac{P(A) - P(A \cap B) + 1 - P(A \cap B)}{P(B^C)} = \frac{P(B^C)}{P(B^C)} = 1$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 40. For any two events A and B defined on a sample space,

(a) $P\left(\frac{A}{B}\right) \geq \frac{P(A) + P(B) - 1}{P(B)}$, $P(B) \neq 0$ is

always true

(b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(c) $P(A \cup B) = 1 - P(A^C) \cdot P(B^C)$, if A and B are independent

(d) $P(A \cup B) = 1 - P(A^C) \cdot P(B^C)$, if A and B are disjoint

Sol. For A , $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A \cap B) \geq P(A) + P(B) - 1$

or $1 \geq P(A) + P(B) - P(A \cap B)$

or $1 \geq P(A \cup B)$

which is true

$\Rightarrow$ Option (a) is correct.

$\therefore$ (b) and (c) are obvious.

Hence, (a), (b) and (c) are the correct answers.

Ex 41. If E_1 and E_2 are two events such that

$P(E_1) = \frac{1}{4}$, $P\left(\frac{E_2}{E_1}\right) = \frac{1}{2}$ and $P\left(\frac{E_1}{E_2}\right) = \frac{1}{4}$, then

(a) E_1 and E_2 are independent

(b) E_1 and E_2 are exhaustive

(c) E_2 is twice as likely to occur as E_1

(d) probabilities of the events $E_1 \cap E_2$, E_1 and E_2 are in GP

Sol. $\therefore P\left(\frac{E_2}{E_1}\right) = \frac{P(E_1 \cap E_2)}{P(E_1)} \cdot \frac{1}{2} = \frac{P(E_1 \cap E_2)}{\frac{1}{4}}$
 $\Rightarrow P(E_1 \cap E_2) = \frac{1}{8} = P(E_2) \cdot P\left(\frac{E_1}{E_2}\right)$

We have, $P(E_1) = \frac{1}{4}$

$\Rightarrow P(E_2) = \frac{1}{2}$

Since, $P(E_1 \cap E_2) = \frac{1}{8} = P(E_1) \cdot P(E_2)$

$\therefore$ Events are independent.

Also, $P(E_1 \cup E_2) = \frac{1}{2} + \frac{1}{4} - \frac{1}{8} = \frac{5}{8}$

$\therefore E_1$ and E_2 are non-exhaustive.

Hence, (a), (c) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 42-46) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 42. Statement I If A and B are two events such that $0 < p(A)$, $p(B) < 1$, then

$P\left(\frac{A}{\bar{B}}\right) + P\left(\frac{\bar{A}}{\bar{B}}\right) = \frac{3}{2}$

Statement II If A and B are two events such that

$0 < P(A)$, $P(B) < 1$, then $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

and $P(\bar{B}) = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$.

Sol. Statement II $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$ [by definition]

$P(\bar{B}) = P\{(A \cup \bar{A}) \cap \bar{B}\}$

$= P\{(A \cap \bar{B}) \cup (\bar{A} \cap \bar{B})\} = P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})$

$\therefore$ Statement II is true.

Statement I

$P\left(\frac{A}{\bar{B}}\right) + P\left(\frac{\bar{A}}{\bar{B}}\right) = \frac{P(A \cap \bar{B})}{P(\bar{B})} + \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$
 $= \frac{P(A \cap \bar{B}) + P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{P(\bar{B})}{P(\bar{B})} = 1$

$\therefore$ Statement I is false.

Hence, (d) is the correct answer.

Ex 43. Statement I A card is drawn from a well shuffled ordinary deck of 52 playing cards. Let A be the event that 'card drawn is an ace' and B be the event that 'card drawn is a spade'. Then, the events A and B are independent.

Statement II Let A and B be two non-empty events. If $P\left(\frac{A}{B}\right) = P(A)$, then the events A and B are independent.

Sol. $P(A) = P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$\Rightarrow P(A \cap B) = P(A) \cdot P(B)$

$\therefore A$ and B are independent.

$\therefore$ Statement II is true.

$$P(A) = \frac{4}{52} = \frac{1}{13}$$

and $P(B) = \frac{1}{4}$

If the card drawn is known to be a spade, then it is an ace is the event $\frac{A}{B}$.

$$P\left(\frac{A}{B}\right) = \frac{1}{13} = P(A)$$

$\therefore A$ and B are independent.

Hence, (a) is the correct answer.

Ex 44. Statement I If $x + y + z = 5$ and $xy + yz + zx = 3$, ($x, y, z \in R$), then the probability for x is positive only, is $\frac{13}{16}$.

Statement II If $x + y + z = 5$ and $xy + yz + zx = 3$, then maximum and minimum value of x, y and z are same.

Sol. Since, maximum values of $x = \frac{13}{3}$

and minimum value of $x = -1$

$$\therefore \text{Required probability} = \frac{\int_0^{13/3} dx}{\int_{-1}^{13/3} dx} = \frac{\frac{13}{3}}{\frac{13}{3} + 1} = \frac{13}{16}$$

Hence, (b) is the correct answer.

Ex 45. Consider, a natural number x is chosen at random from the first 100 natural numbers.

Statement I The probability that $\frac{(x-10)(x-50)}{(x-30)} > 0$ is 0.69.

Statement II If A is an event, then $0 < P(A) < 1$.

Sol. Since, $\frac{(x-10)(x-50)}{(x-30)} > 0$

$\Rightarrow 10 < x < 30$ or $x > 50$

$\therefore x = 11, \dots, 29$

or $x = 51, 52, \dots, 100 \Rightarrow n(x) = 69$

$\therefore \text{Required probability} = \frac{69}{100} = 0.69$

Hence, (b) is the correct answer.

Ex 46. Statement I If 12 coins are thrown simultaneously, then probability of appearing exactly five head is equal to probability of appearing exactly 7 heads.

Statement II ${}^nC_r = {}^nC_s \Rightarrow$ either $r = s$ or $r + s = n$ and $P(H) = P(T)$ in a single trial.

Sol. Here, $p = q = \frac{1}{2}$

$\therefore$ Probability of appearing exactly five heads

$$= {}^{12}C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^7 = {}^{12}C_{12-5} \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^5$$

$$= {}^{12}C_7 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^5$$

which is equal to probability of appearing exactly seven heads.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 47-49) There are four boxes A_1, A_2, A_3 and A_4 . Box A_i has i cards and on each card, a number is printed, the numbers are from 1 to i . A box is selected randomly, the probability of selection of box A_i is $\frac{i}{10}$ and then a card is drawn. Let E_i represents the event that a card with number i is drawn.

Ex 47. $P(E_1)$ is equal to

- (a) $\frac{1}{5}$ (b) $\frac{1}{10}$ (c) $\frac{2}{5}$ (d) $\frac{1}{4}$

Sol. $P(E_1) = \frac{1}{10} \times 1 + \frac{2}{10} \times \frac{1}{2} + \frac{3}{10} \times \frac{1}{3} + \frac{4}{10} \times \frac{1}{4} = \frac{2}{5}$

Hence, (c) is the correct answer.

Ex 48. $P\left(\frac{A_3}{E_2}\right)$ is equal to

- (a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

Sol. $P\left(\frac{A_3}{E_2}\right) = \frac{\frac{3}{10} \times \frac{1}{3}}{\frac{2}{10} \times \frac{1}{2} + \frac{3}{10} \times \frac{1}{3} + \frac{4}{10} \times \frac{1}{4}} = \frac{1}{3}$

Hence, (b) is the correct answer.

Ex 49. Expectation of the number on the card is

- (a) 2 (b) 2.5 (c) 3 (d) 3.5

Sol. $P(E) = 2$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 50-53) A standard deck of playing card has 52 cards. There are four suits (club, diamond, heart and spade), each of which has thirteen numbered cards 2,...,9, 10, jack queen, king and ace.

In a game of card, each card is worth an amount of points. Each numbered card is worth its number (e.g. a 5 is worth 5 points); the jack, queen and king are each worth 10 points and the ace is worth your choice of either 1 point or 11 points. The object of the game is to have more points in your set of cards than your opponent without going over 21. Any set of cards with sum greater than 21 automatically loses.

Here's how the game is played. You and your opponent are each dealt two cards. Usually the first card for each player is dealt face down and the second card for each player is dealt face up. After the initial cards are dealt, the first player has the option of asking for another card or not taking any cards. The first player can keep asking for more cards until either he or she goes over 21. In this case, the player loses or stops at some number less than or equal to 21.

When the first player stops at some number less than or equal to 21, then the second player can take more cards until matching or exceeding the first player's number without going over 21. In this case, the second player wins or until going over 21, in which case the first player wins. We are to simplify the game a little and assume that all cards are dealt face up, so that all cards are visible. Assume your opponent is dealt cards and plays first.

Ex 50. The chance that the second card will be a heart and a jack, is

- (a) $\frac{4}{52}$ (b) $\frac{13}{52}$ (c) $\frac{17}{52}$ (d) $\frac{1}{52}$

Ex 51. The chance that the first card will be a heart or a jack, is

- (a) $\frac{13}{52}$ (b) $\frac{16}{52}$ (c) $\frac{17}{52}$ (d) $\frac{18}{52}$

Ex 52. Given that the first card is jack, the chance that it will be the heart, is

- (a) $\frac{1}{13}$ (b) $\frac{4}{13}$ (c) $\frac{1}{4}$ (d) $\frac{1}{3}$

Ex 53. Suppose your opponent is dealt a king and a 10 and you are dealt a queen and a 9. Being your opponent does not take any more cards and stays at 20. The chance you will win, if you are allowed to take as many cards as you need, is

- (a) $\frac{97}{564}$ (b) $\frac{25}{282}$ (c) $\frac{15}{188}$ (d) $\frac{1}{6}$

Sol. (Ex. Nos. 50-53)

P (2nd card is J of H)

$$= \frac{51}{52} \cdot \frac{1}{51} = \frac{1}{52}$$

There are 16 ways to get a jack or a hearts: get one of the heart (ace through king of hearts) or get one of the jack of clubs, jack of spades or jack of diamonds.

$$\text{Hence, } P(\text{jack or heart}) = \frac{16}{52}$$

$$P\left(\frac{H}{J}\right) = \frac{P(H \cap J)}{P(J)} = \frac{1}{4}$$

$$P(\text{win}) = P(\text{any } 2\text{'s or ace}) = \frac{4}{48} + \frac{4}{48} = \frac{1}{6}$$

50. (d) 51. (b) 52. (c) 53. (d)

Passage III (Ex. Nos. 54-56) The probability of happening of an event in one trial being known, then the probability of its happening exactly x times in n trials is given by ${}^nC_x q^{n-x} \cdot p^x$, where p = probability of happening the event

q = probability of not happening the event = $1 - p$

Now, ${}^nC_x q^{n-x} p^x$ is $(x + 1)$ th term in the expansion of $(q + p)^n$ whose expansion gives the happening of the event 0, 1, 2, ..., n times, respectively.

Ex 54. In four throws with a pair of dice, the chance of throwing doublets atleast twice is

- (a) $\frac{19}{144}$ (b) $\frac{125}{144}$
(c) $\frac{17}{144}$ (d) $\frac{18}{144}$

Sol. $P(2) + P(3) + P(4)$

$$= {}^4C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 + {}^4C_3 \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right) + {}^4C_4 \left(\frac{1}{6}\right)^4$$

$$= \frac{6 \times 25 + 20 + 1}{1296} = \frac{171}{1296} = \frac{19}{144}$$

Hence, (a) is the correct answer.

Ex 55. A man takes a forward step with probability 0.8 and backward step with probability 0.2. What is the probability that at the end of 9 steps he is exactly three steps away from starting point?

- (a) $\frac{69888}{5^8}$ (b) $\frac{5377}{5^8}$ (c) $\frac{5378}{5^8}$ (d) $\frac{5376}{5^8}$

Sol. $P(X = x) = {}^9C_x (0.2)^{9-x} (0.8)^x$

$$= {}^9C_6 (0.2)^3 (0.8)^6 + {}^9C_3 (0.2)^6 (0.8)^3$$

$$= {}^9C_6 (0.2)^3 (0.8)^6 + {}^9C_6 (0.2)^6 (0.8)^3$$

$$= {}^9C_6 (0.2)^3 (0.8)^3 [(0.8)^3 + (0.2)^3]$$

$$= {}^9C_6 (0.2)^3 (0.8)^3 \left[\frac{4^3}{5^3} + \frac{1}{5^3} \right]$$

$$= {}^9C_6 \left(\frac{1}{5}\right)^3 \left(\frac{4^3}{5^3}\right) \cdot \frac{65}{5^3} = \frac{69888}{5^8}$$

Hence, (a) is the correct answer.

Ex 56. Unbiased coin is tossed 6 times. The probability of getting atmost 4 heads, is

- (a) $\frac{7}{64}$ (b) $\frac{57}{64}$ (c) $\frac{21}{32}$ (d) $\frac{11}{32}$

Sol. $P(X = x) = {}^6C_x \left(\frac{1}{2}\right)^6 = 1 - P(X = 5) - P(X = 6)$

$$= 1 - {}^6C_5 \left(\frac{1}{2}\right)^6 - {}^6C_6 \left(\frac{1}{2}\right)^6$$

$$= 1 - \frac{6}{2^6} - \frac{1}{2^6} = 1 - \frac{7}{2^6} = 1 - \frac{7}{64}$$

$$= \frac{57}{64}$$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 57. Match the statements of Column I with values of Column II.

Column I	Column II
A. One ball is drawn from a bag containing 4 balls and is found to be white. The events that the bag contains '1 white', '2 white', '3 white' and '4 white' are equally likely. If the probability that all the balls are white $\frac{p}{15}$, then the value of p is	p. 9
B. From a set of 12 persons, if the number of different selection of committee, its chair person and its secretary (possibly same as chair person) is $13 \cdot 2^{10} m$, then the value of m is	q. 3
C. If $x, y, z > 0$ and $x + y + z = 1$, then the least value of $\frac{5x}{2-x} + \frac{5y}{2-y} + \frac{5z}{2-z}$ is	r. 12
D. If $\sum_{k=1}^{12} 12k \cdot {}^{12}C_k \cdot {}^{11}C_{k-1}$ is equal to $\frac{12 \times 21 \times 19 \times 17 \times \dots \times 3}{11!} \times 2^{12} \times p$, then the value of p is	s. 6

Sol. A. Let E_1, E_2, E_3, E_4 be the events that the bag contains 1 white, 2 white, 3 white, 4 white balls, respectively.

$$\text{Let } P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{1}{4}$$

Let W be the event that the ball drawn is white. Then,

$$P(W) = \sum P(E_i) P\left(\frac{W}{E_i}\right)$$

$$= \frac{1}{4} \left(\frac{1}{4} + \frac{2}{4} + \frac{3}{4} + \frac{4}{4} \right) = \frac{5}{8}$$

$$\text{Now, } P\left(\frac{E_4}{W}\right) = \frac{P(E_4) \cdot P\left(\frac{W}{E_4}\right)}{P(W)} = \left(\frac{\frac{1}{4}}{\frac{5}{8}}\right) = \frac{2}{5}$$

$$\therefore \frac{2}{5} = \frac{p}{15} \Rightarrow p = 6$$

$$\begin{aligned} \text{B. } & {}^{12}C_1 + {}^{12}C_2 ({}^2C_1 + 2 \cdot {}^2C_2) + {}^{12}C_3 ({}^3C_1 + 2 \cdot {}^3C_2) \\ & + \dots + {}^{12}C_{12} ({}^{12}C_1 + 2 \cdot {}^{12}C_2) \\ & = ({}^{12}C_1 + 2 \cdot {}^{12}C_2 + 3 \cdot {}^{12}C_3 + \dots + 12 \cdot {}^{12}C_{12}) \\ & + 2({}^{12}C_2 \cdot {}^2C_2 + {}^{12}C_3 \cdot {}^3C_2 + \dots + {}^{12}C_{12} \cdot {}^{12}C_2) \\ & = \sum_{r=1}^{12} r \cdot {}^{12}C_r + 2 \sum_{r=2}^{12} {}^{10}C_{r-2} \\ & = 12 \times 2^{11} + 12 \times 11 \times 2^{10} \end{aligned}$$

$$= 12 \times 2^{10} (2 + 11) = 13 \times 2^{10} \times 12$$

$$\Rightarrow 13 \times 2^{10} \times 12 = 13 \times 2^{10} \times m$$

$$\therefore m = 12$$

$$\begin{aligned} \text{C. } & \frac{5x}{2-x} + \frac{5y}{2-y} + \frac{5z}{2-z} \\ & = 5 \left[\frac{x}{2-x} + \frac{y}{2-y} + \frac{z}{2-z} \right] \\ & = 5 \left[\frac{x-2+2}{2-x} + \frac{y-2+2}{2-y} + \frac{z-2+2}{2-z} \right] \\ & = 5 \left[-3 + 2 \left(\frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z} \right) \right] \end{aligned}$$

$$\text{Now, } 2-x+2-y+2-z=5$$

$$\therefore \frac{5}{3} = - \frac{1}{\frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z}}$$

$$\text{i.e. } \frac{1}{2-x} + \frac{1}{2-y} + \frac{1}{2-z} \geq \frac{9}{5}$$

$$\text{Hence, } \frac{5x}{2-x} + \frac{5y}{2-y} + \frac{5z}{2-z} \geq 5 \left[-3 + 2 \cdot \frac{9}{5} \right] = 3$$

$\therefore$ Least value is 3.

$$\begin{aligned} \text{D. } & \sum_{k=1}^{12} 12k \cdot {}^{12}C_k \cdot {}^{11}C_{k-1} = 12^2 \sum_{k=1}^{12} ({}^{11}C_{k-1})^2 \\ & = 12^2 \cdot \frac{22!}{11!11!} \\ & = 12 \cdot \frac{23 \cdot 21 \cdot 19 \dots 3}{11!} \cdot 2^{12} \cdot 6 \end{aligned}$$

$$\therefore p = 6$$

$$A \rightarrow s; B \rightarrow r; C \rightarrow q; D \rightarrow s$$

Ex 58. A bag contains some white and some black balls, all combinations being equally likely. The total number of balls in the bag is 12. Four balls are drawn at random from the bag at random without replacement. Match the statements of Column I with values of Column II.

Column I	Column II
A. Probability that all the four balls are black, is	p. $\frac{14}{33}$
B. If the bag contains 10 black and 2 white balls, then the probability that all four balls are black, is	q. $\frac{1}{5}$
C. If all the four balls are black, then the probability, that the bag contains 10 black balls, is	r. $\frac{70}{429}$
D. Probability that two balls are black and two balls are white, is	s. $\frac{13}{165}$

Sol. Let E_i denotes the event that the bag contains i black and $(12 - i)$ white balls ($i = 0, 1, 2, \dots, 12$) and A denotes the event that the four balls drawn are all black. Then,

$$P(E_i) = \frac{1}{13} \quad (i = 0, 1, 2, \dots, 12)$$

$$P\left(\frac{A}{E_i}\right) = 0, \text{ for } i = 1, 2, 3$$

$$P\left(\frac{A}{E_i}\right) = \frac{{}^iC_4}{{}^{12-i}C_4}, \text{ for } i \geq 4$$

$$\begin{aligned} \text{A. } P(A) &= \sum_{i=0}^{12} P(E_i) P\left(\frac{A}{E_i}\right) \\ &= \frac{1}{13} \times \frac{1}{{}^{12}C_4} [{}^4C_4 + {}^5C_4 + \dots + {}^{12}C_4] \\ &= \frac{{}^{13}C_5}{{}^{13}C_4} = \frac{1}{5} \end{aligned}$$

$$\text{B. Clearly, } P\left(\frac{A}{E_{10}}\right) = \frac{{}^{10}C_4}{{}^{12}C_4} = \frac{14}{33}$$

C. By using Baye's theorem,

$$\begin{aligned} P\left(\frac{E_{10}}{A}\right) &= \frac{P(E_{10}) \cdot P\left(\frac{A}{E_{10}}\right)}{P(A)} \\ &= \frac{\frac{1}{13} \times \frac{14}{33}}{\frac{1}{5}} = \frac{70}{429} \end{aligned}$$

D. Let B denotes the probability of drawing 2 white and 2 black balls, then

$$P\left(\frac{B}{E_i}\right) = 0, \text{ if } i = 0, 1 \text{ or } 11, 12$$

$$P\left(\frac{B}{E_i}\right) = \frac{{}^iC_2 \times {}^{12-i}C_2}{{}^{12}C_4}, \text{ for } i = 2, 3, \dots, 10$$

$$\begin{aligned} \therefore P(B) &= \sum_{i=2}^{10} P(E_i) P\left(\frac{B}{E_i}\right) = \frac{1}{13} \times \frac{1}{{}^{12}C_4} \\ &\quad [{}^2C_2 \times {}^{10}C_2 + {}^3C_2 \times {}^9C_2 + \dots + {}^{10}C_2 \times {}^2C_2] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{13} \times \frac{1}{{}^{12}C_4} [2 \{ {}^2C_2 \times {}^{10}C_2 + {}^3C_2 \times {}^9C_2 + \dots + {}^5C_2 \\ &\quad + {}^7C_2 \} + {}^6C_2 \times {}^6C_2] \\ &= \frac{1}{13} \times \frac{1}{495} (1287) = \frac{1}{5} \\ &\text{A} \rightarrow \text{q}; \text{B} \rightarrow \text{p}; \text{C} \rightarrow \text{r}; \text{D} \rightarrow \text{q} \end{aligned}$$

Ex 59. Let A and B be two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{4}$. Now, match the statements of Column I with values of Column II.

Column I	Column II
A. $P(A \cup B)$ is equal to	p. $\frac{1}{12}$
B. $P\left(\frac{A}{A \cup B}\right)$ is equal to	q. $\frac{1}{2}$
C. $P\left(\frac{B}{A' \cap B'}\right)$ is equal to	r. $\frac{2}{3}$
D. $P\left(\frac{A'}{B'}\right)$ is equal to	s. 0

Sol. We have, $P(A \cap B) = P(A) \cdot P(B) = \frac{1}{12}$

$$\begin{aligned} \text{A. } P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ &= \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{1}{2} \end{aligned}$$

$$\text{B. } P\left(\frac{A}{A \cup B}\right) = \frac{P(A)}{P(A \cup B)} = \frac{2}{3}$$

$$\begin{aligned} \text{C. } P\left(\frac{B}{A' \cap B'}\right) &= \frac{P[B \cap (A' \cap B')]}{P(A' \cap B')} \\ &= \frac{P(\phi)}{P(A' \cap B')} = 0 \end{aligned}$$

$$\text{D. } P\left(\frac{A'}{B'}\right) = P(A') = \frac{2}{3}$$

A $\rightarrow$ q; B $\rightarrow$ r; C $\rightarrow$ s; D $\rightarrow$ r

Type 6. Single Integer Answer Type Questions

Ex 60. A die is thrown 7 times. If the probability that an odd number turns up atleast 4 times, is $\frac{a}{b}$, then $(a + b)$ is _____.

Sol. (3) Probability of success = $\frac{3}{6} = \frac{1}{2} \Rightarrow p = \frac{1}{2}, q = \frac{1}{2}$

For atleast four successes, required probability

$$\begin{aligned} &= {}^7C_4 \left(\frac{1}{2}\right)^4 \cdot \left(\frac{1}{2}\right)^3 + {}^7C_5 \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^2 \\ &\quad + {}^7C_6 \left(\frac{1}{2}\right)^6 \cdot \left(\frac{1}{2}\right)^1 + {}^7C_7 \left(\frac{1}{2}\right)^7 \\ &= \frac{35}{128} + \frac{21}{128} + \frac{7}{128} + \frac{1}{128} = \frac{64}{128} = \frac{1}{2} = \frac{a}{b} \end{aligned}$$

Hence, $a + b = 3$

Ex 61. There are four six faced dice such that each of two dice bears the numbers 0, 1, 2, 3, 4 and 5 and the other two dice are ordinary dice bearing numbers 1, 2, 3, 4, 5 and 6. If all the four dice are thrown, the probability that the total of numbers coming up on all the dice is 10, is $\frac{100a + 10b + c}{1296}$, then $a + b + c$ is _____.

Sol. (8) Total number of sample points in the sample space = $6^4 = 1296$

Number of sample points in favour of the event

= Coefficient of x^{10} in the expansion of

$$(1 + x + x^2 + \dots + x^5)^2 (x + x^2 + \dots + x^6)^2$$

= Coefficient of x^{10} in the expansion of

$$x^2(1 + x + x^2 + \dots + x^5)^4$$

$$\begin{aligned}
 &= \text{Coefficient of } x^8 \text{ in the expansion of } (1+x+x^2+\dots+x^5)^4 \\
 &= \text{Coefficient of } x^8 \text{ in the expansion of } \left(\frac{1-x^6}{1-x}\right)^4 \\
 &= \text{Coefficient of } x^8 \text{ in the expansion of } (1-x^6)^4 (1-x)^{-4} \\
 &= \text{Coefficient of } x^8 \text{ in the expansion of } (1-4x^6)(1-x)^{-4} \\
 &= 1 \times {}^{11}C_8 - 4 \times {}^5C_2 = 125 \\
 \therefore \text{ Required probability} &= \frac{125}{1296} \\
 &= \frac{1 \times 100 + 2 \times 10 + 5}{1296}
 \end{aligned}$$

Hence, $a + b + c = 8$

Ex 62. If m things are distributed among 'a' men and 'b' women, the probability that the number of things received by men is odd, is

$$\frac{1}{2} \left\{ \frac{(\lambda)^m - (\mu)^m}{(b+a)^m} \right\}, \text{ then } \frac{\lambda + \mu}{2b} \text{ is}$$

Sol. (1) A particular things is received by a man with probability $b = \frac{a}{a+b}$ and by a woman with probability $q = \frac{b}{a+b}$. If distributing a single object is an experiment, then this experiment is repeated m times.

$$\begin{aligned}
 &\text{The required probability} \\
 &= {}^mC_1 \cdot p \cdot q^{m-1} + {}^mC_3 \cdot p^3 \cdot q^{m-3} + {}^mC_5 \cdot p^5 \cdot q^{m-5} + \dots \\
 &= \frac{(q+p)^m - (q-p)^m}{2} = \frac{1}{2} \left[1 - \left(\frac{b-a}{b+a} \right)^m \right] \\
 &= \frac{1}{2} \left\{ \frac{(b+a)^m - (b-a)^m}{(b+a)^m} \right\} = \frac{1}{2} \left\{ \frac{\lambda^m - \mu^m}{(b+a)^m} \right\} \\
 \therefore \quad \frac{\lambda + \mu}{2b} &= 1
 \end{aligned}$$

Ex 63. A die is weighted such that the probability of rolling the face numbered n is proportional to n^2 ($n=1, 2, 3, 4, 5, 6$). The die is rolled twice, yielding the numbers a and b . The probability

that $a < b$ is p , then the value of $\left\lfloor \frac{2}{p} \right\rfloor$, where

$\lfloor \cdot \rfloor$ represents the greatest integer function, is

Sol. (5) $P(n) = kn^2$

Given,

$$P(1) = k; P(2) = 2^2k; P(3) = 3^2k; P(4) = 4^2k;$$

$$P(5) = 5^2k; P(6) = 6^2k$$

$$\therefore \text{ Total} = 91k = 1$$

$$\Rightarrow k = \frac{1}{91}$$

$$\therefore P(1) = \frac{1}{91}; P(2) = \frac{4}{91} \text{ and so on.}$$

Let three events A, B, C be defined as

$$A : a < b$$

$$B : a = b$$

$$C : a > b$$

By symmetry, $P(A) = P(C)$

$$\text{Also, } P(A) + P(B) + P(C) = 1 \quad \dots(i)$$

$$\begin{aligned}
 \text{Since, } P(B) &= \sum_{i=1}^6 [P(i)]^2 \\
 &= \left[\frac{1 + 16 + 81 + 256 + 625 + 1296}{91 \times 91} \right] \\
 &= \frac{2275}{91 \cdot 91} = \frac{25}{91}
 \end{aligned}$$

$$\text{Now, } 2P(A) + P(B) = 1$$

$$\Rightarrow P(A) = \frac{1}{2} [1 - P(B)] = \frac{33}{91}$$

$$\therefore p = \frac{33}{91}$$

$$\text{Hence, } \left\lfloor \frac{2}{p} \right\rfloor = \left\lfloor \frac{182}{33} \right\rfloor = 5$$

Ex 64. If A and B are two events such that $P(A) = 0.6$ and $P(B) = 0.8$ and the greatest value that $P\left(\frac{A}{B}\right)$ can have, is p , then the value of $8p$ is _____.

$$\text{Sol. (6) } P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{0.6}{0.8} = \frac{3}{4}$$

$$[\therefore \text{ maximum value of } P(A \cap B) = P(A) = 0.6]$$

$$\text{Hence, } p = \frac{3}{4} \Rightarrow 8p = 8 \cdot \frac{3}{4} = 6$$

Target Exercises

Type 1. Only One Correct Option

- If $P(A) = 0.65$, $P(B) = 0.80$, then $P(A \cap B)$ lies in the interval
 (a) $[0.30, 0.80]$ (b) $[0.35, 0.75]$
 (c) $[0.4, 0.70]$ (d) $[0.45, 0.65]$
- For two events A and B , if $P(A) = P\left(\frac{A}{B}\right) = \frac{1}{4}$ and $P\left(\frac{B}{A}\right) = \frac{1}{2}$, then
 (a) A and B are independent
 (b) $P\left(\frac{A'}{B}\right) = \frac{3}{4}$
 (c) $P\left(\frac{B'}{A}\right) = \frac{1}{2}$
 (d) All of the above
- Let E and F be two independent events such that $P(E) > P(F)$. The probability that both E and F happen, is $\frac{1}{12}$ and the probability that neither E nor F happens, is $\frac{1}{2}$, then
 (a) $P(E) = \frac{1}{3}$, $P(F) = \frac{1}{4}$
 (b) $P(E) = \frac{1}{2}$, $P(F) = \frac{1}{6}$
 (c) $P(E) = 1$, $P(F) = \frac{1}{12}$
 (d) $P(E) = \frac{1}{3}$, $P(F) = \frac{1}{2}$
- If A and B are two events such that $P(A) > 0$, $P(B) \neq 1$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ is equal to
 (a) $1 - P\left(\frac{A}{B}\right)$ (b) $1 - P\left(\frac{\bar{A}}{B}\right)$
 (c) $\frac{1 - P(A \cup B)}{P(\bar{B})}$ (d) $\frac{P(\bar{A})}{P(\bar{B})}$
- In a hurdle race, a runner has probability p of jumping over a specific hurdle. Given that in 5 trials, the runner succeeded 3 times, the conditional probability that the runner had succeeded in the first trial, is
 (a) $\frac{3}{5}$ (b) $\frac{2}{5}$
 (c) $\frac{1}{5}$ (d) None of these
- Three numbers are chosen at random without replacement from the set $A = \{x | 1 \leq x \leq 10, x \in N\}$. The probability that the minimum of the chosen numbers is 3 and maximum is 7, is
 (a) $\frac{1}{12}$ (b) $\frac{1}{15}$
 (c) $\frac{1}{40}$ (d) $\frac{39}{40}$
- A and B stand in ring along with 10 other persons. If the arrangement is at random, then the probability that there are exactly 3 persons between A and B , is
 (a) $\frac{1}{11}$ (b) $\frac{2}{11}$ (c) $\frac{3}{11}$ (d) $\frac{1}{12}$
- If E_1 denotes the events of coming sum 6 in throwing two dice and E_2 is the event of coming 2 in any one of the two, then $P\left(\frac{E_2}{E_1}\right)$ is
 (a) $1/5$ (b) $4/5$
 (c) $3/5$ (d) $2/5$
- If $6n$ tickets numbered 0, 1, 2, ..., $(6n - 1)$ are placed in a bag and three are drawn at random, then the probability that the sum of numbers on the ticket is $6n$, is
 (a) $\frac{3n}{{}^{6n}C_3}$ (b) $\frac{3n^2}{{}^{6n}C_3}$
 (c) $\frac{1}{2} \cdot \frac{3n}{{}^{6n}C_3}$ (d) $\frac{1}{2} \cdot \frac{3n^2}{{}^{6n}C_3}$
- A six-faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. The probability that the sum of two numbers thrown is even, is
 (a) $\frac{1}{12}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) $\frac{5}{9}$
- A coin is tossed 3 times in succession. If E is the event that there are atleast two heads and F is the event in which first throw is a head, then $P\left(\frac{E}{F}\right)$ is equal to
 (a) $\frac{3}{4}$ (b) $\frac{3}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{8}$

12. A laboratory blood test is 99% effective in detecting a certain disease when it is infact present. However, the test also yields a false positive result for 0.5% of the healthy person tested (i.e. if a healthy person is tested, then with probability 0.005, the test will imply he has the disease). If 0.1% of the population actually has the disease, then what is the probability that a person has disease given that his test result is positive?
- (a) $\frac{11}{133}$ (b) $\frac{22}{133}$
(c) $\frac{33}{133}$ (d) None of these
13. If $x_1, x_2, \dots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \dots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle number, is
- (a) $\frac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (b) $\frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$
(c) $\frac{{}^{19}C_2 \times {}^{31}C_2}{{}^{50}C_5}$ (d) None of these
14. Two dice are thrown together first and secondly three dice are thrown together. Then, the probability that the total in the first throw is 4 or more and at the same time the total in the second thrown is 6 or more, is
- (a) $\frac{1133}{1296}$ (b) $\frac{1137}{1296}$ (c) $\frac{1131}{1296}$ (d) $\frac{1000}{1296}$
15. The probability of a man hitting a target is $\frac{3}{4}$. He tries 5 times. The probability that the target will be hit atleast 3 times, is
- (a) $\frac{291}{364}$ (b) $\frac{371}{464}$
(c) $\frac{471}{502}$ (d) $\frac{459}{512}$
16. In a box containing 100 bulbs, 10 bulbs are defective. The probability that out of a sample of 5 bulbs, none is defective, is
- (a) 10^{-5} (b) 2^{-5} (c) $(0.9)^5$ (d) 0.9
17. The probability that Krishna will be alive 10 yr hence, is $\frac{7}{15}$ and the probability that Hari will be alive after 10 yr, is $\frac{7}{10}$. The probability that both Krishna and Hari will be alive 10 yr hence, is
- (a) $\frac{21}{150}$ (b) $\frac{24}{150}$ (c) $\frac{49}{150}$ (d) $\frac{56}{150}$
18. In 324 throws of 4 dice, the expected number of times 3 sixes occur is
- (a) 81 (b) 5 (c) 9 (d) 31
19. If in a distribution, each x is replaced by the corresponding value of $f(x)$, then the probability of getting x_i whose original probability is p_i , is
- (a) p_i (b) $f(p_i)$
(c) $f\left(\frac{1}{p_i}\right)$ (d) None of these
20. Let X and Y be two random variables. The relationship $E(XY) = E(X) \cdot E(Y)$ holds
- (a) always
(b) if $E(X + Y) = E(X) + E(Y)$ is true
(c) if X and Y are independent
(d) if X can be obtained from Y by a linear transformation
21. The chance of throwing a total of 3 or 5 or 11 with two dice is
- (a) $\frac{5}{36}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{19}{36}$
22. A fair die is tossed until a number greater than 4 appears. The probability that an even number of tosses shall be required, is
- (a) $\frac{1}{2}$ (b) $\frac{3}{5}$ (c) $\frac{1}{5}$ (d) $\frac{2}{3}$
23. A box contains 24 identical balls of which 12 are white and remaining black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw, is
- (a) $\frac{5}{64}$ (b) $\frac{27}{32}$ (c) $\frac{5}{32}$ (d) $\frac{1}{2}$
24. The probability that a man will live 10 more years, is $\frac{1}{4}$ and the probability that his wife will be live 10 more years, is $\frac{1}{3}$. Then, the probability that none of them will be alive after 10 yr, is
- (a) $\frac{5}{12}$ (b) $\frac{1}{2}$ (c) $\frac{7}{12}$ (d) $\frac{11}{12}$
25. A fair coin is tossed. If the result is a head, a pair of fair dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4, ..., 12, is picked up and the number on the card is noted. The probability that the noted number is 7 or 8, is
- (a) $\frac{193}{792}$ (b) $\frac{195}{792}$
(c) $\frac{191}{792}$ (d) None of these
26. A man throws a fair coin a number of times and gets 2 points for each head he throws and 1 point for each tail he throws. The probability that he gets exactly 6 points, is
- (a) $\frac{21}{32}$ (b) $\frac{23}{32}$ (c) $\frac{41}{64}$ (d) $\frac{43}{64}$

- 27.** One function is selected from all the functions $F: S \rightarrow S$, where $S = \{1, 2, 3, 4, 5, 6\}$. The probability that it is onto function, is
 (a) $\frac{5}{324}$ (b) $\frac{7}{324}$ (c) $\frac{5}{162}$ (d) $\frac{5}{81}$
- 28.** One hundred identical coins, each with probability p , of showing up a head, are tossed. If $0 < p < 1$ and the probability of heads on exactly 50 coins is equal to that of heads on exactly 51 coins, then the value of p is
 (a) $\frac{1}{2}$ (b) $\frac{49}{101}$ (c) $\frac{50}{101}$ (d) $\frac{51}{101}$
- 29.** Let the probability P_n that a family has exactly n children be αp^n , when $n \geq 1$ and $P_0 = 1 - \alpha p(1 + p + p^2 + \dots)$. Suppose that all sex distributions of n children have the same probability. If $k \geq 1$, then the probability that a family contains exactly k boys, is
 (a) $\frac{2\alpha}{(2-p)^{k+1}}$ (b) $\frac{p^k}{(2-p)^{k+1}}$
 (c) $\frac{2\alpha \cdot p^k}{(2-p)^{k+1}}$ (d) None of these
- 30.** A pair of dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7, is
 (a) $\frac{2}{5}$ (b) $\frac{1}{5}$
 (c) $\frac{3}{5}$ (d) None of these
- 31.** The probability of the occurrence of an event A is 0.5 and that of occurrence of another event B , is 0.3. If A and B are mutually exclusive events, then the probability that none of A and B will occur, is
 (a) 0.6 (b) 0.5
 (c) 0.7 (d) None of these
- 32.** Let A , B and C be three mutually independent events. Consider the two statements S_1 and S_2 .
 S_1 : A and $B \cup C$ are independent
 S_2 : A and $B \cap C$ are independent
 Then,
 (a) Both S_1 and S_2 are true
 (b) Only S_1 is true
 (c) Only S_2 is true
 (d) Neither S_1 nor S_2 is true
- 33.** Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three chosen vertices is an equilateral triangle, is
 (a) $\frac{1}{2}$ (b) $\frac{1}{5}$
 (c) $\frac{1}{10}$ (d) $\frac{1}{20}$
- 34.** A square is inscribed in a circle. If p_1 is the probability that a randomly chosen point of the circle lies within the square and p_2 is the probability that the point lies outside the square, then
 (a) $p_1 = p_2$ (b) $p_1 < p_2$
 (c) $p_1 > p_2$ and $p_1^2 - p_2^2 < \frac{1}{3}$ (d) None of these
- 35.** The probability that a person will hit a target in shooting practice, is 0.3. If he shoots 10 times, then the probability of his shooting the target, is
 (a) 1 (b) $1 - (0.7)^{10}$ (c) $(0.7)^{10}$ (d) $(0.3)^{10}$
- 36.** A sample space consists of three mutually independent and equally likely events. The probability of happening of each one of them is equal to
 (a) 0 (b) $\frac{1}{3}$
 (c) 1 (d) None of these
- 37.** The mean and variance of a binomial variate X are 2 and 1, respectively. The probability that X takes a value greater than 1, is
 (a) $\frac{1}{16}$ (b) $\frac{5}{16}$ (c) $\frac{11}{16}$ (d) $\frac{15}{16}$
- 38.** An ellipse of eccentricity $\frac{2\sqrt{2}}{3}$ is inscribed in a circle and a point within the circle is chosen at random. The probability that this point lies outside the ellipse, is
 (a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$
- 39.** A random variable X has the distribution
- | | | | |
|--------|-----|-----|-----|
| X | 2 | 3 | 4 |
| $P(X)$ | 0.3 | 0.4 | 0.3 |
- Then, variance of the distribution is
 (a) 0.6 (b) 0.7 (c) 0.77 (d) 1.55
- 40.** A box contains 3 white and 2 red balls. If we draw one ball and without replacing the first ball, then the probability of drawing red ball in the second draw, is
 (a) $\frac{8}{25}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{21}{25}$
- 41.** The mean and variance of a binomial distribution are $\frac{5}{4}$ and $\frac{15}{16}$ respectively, then value of p is
 (a) $\frac{1}{2}$ (b) $\frac{15}{16}$ (c) $\frac{1}{4}$ (d) $\frac{3}{4}$
- 42.** The mean and variance of a binomial distribution are 6 and 4, then n is
 (a) 9 (b) 12 (c) 18 (d) 10

- 43.** The probability that a value manufactured by a company will be defective, is 0.1. If 12 such values are manufactured, then the probability of exactly two defective is
(a) 0.230 (b) 0.0443
(c) 0.043 (d) 0.0284
- 44.** The Binomial distribution with mean 6 and variance 4 is
(a) $\left(\frac{2}{3} + \frac{1}{3}\right)^{18}$ (b) $\left(\frac{1}{2} + \frac{1}{2}\right)^{18}$
(c) $\left(\frac{4}{5} + \frac{1}{5}\right)^{18}$ (d) None of these
- 45.** Out of 2000 families with 4 children each, the number of families you expect to have 1 or 2 girls, is
(a) 950 (b) 650 (c) 1250 (d) 1050
- 46.** If X follows the Binomial distribution with parameters $n = 6$ and p and $9P(X = 4) = P(X = 2)$, then p is
(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$
- 47.** If probability of a defective bolt is 0.1, then mean and standard deviation of distribution of bolts in a total of 400, are
(a) 30, 3 (b) 40, 5 (c) 30, 4 (d) 40, 6
- 48.** The sum and product of the mean and variance of a binomial distribution are 24 and 128, respectively. The distribution is
(a) $\left(\frac{1}{7} + \frac{1}{8}\right)^{12}$ (b) $\left(\frac{1}{4} + \frac{3}{4}\right)^{16}$
(c) $\left(\frac{1}{6} + \frac{5}{6}\right)^{24}$ (d) $\left(\frac{1}{2} + \frac{1}{2}\right)^{32}$
- 49.** A die is thrown 100 times. Getting an even number is considered a success. The variance of the number of successes, is
(a) 10 (b) 20 (c) 25 (d) 50
- 50.** 10% of tools produced by a certain manufacturing process turn out to be defective. Assuming Binomial distribution, the probability of 2 defective in a sample of 10 tools chosen at random is
(a) 0.368 (b) 0.194
(c) 0.271 (d) None of these
- 51.** The standard deviation of Binomial distribution is
(a) pq (b) npq (c) $\sqrt{npq}$ (d) $\frac{npq}{6}$
- 52.** Let F denotes the set of all onto functions from $A = \{a_1, a_2, a_3, a_4\}$ to $B = \{x, y, z\}$. A function f is chosen at random from F . The probability that $f^{-1}(x)$ consists of exactly two elements, is
(a) $\frac{2}{3}$ (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) 0
- 53.** The value of m for Poisson's distribution is
(a) 0 (b) 1
(c) -1 (d) a finite number
- 54.** The mean of Poisson's distribution with parameter m is
(a) $\sqrt{m}$ (b) m^2 (c) $\frac{1}{m}$ (d) m
- 55.** In Poisson's distribution, $P(x)$ for $x = 0$ is 10%. The mean of distribution is
(a) 2.32026 (b) 3.308
(c) 1.875 (d) None of these
- 56.** For a Poisson variate x , $P(x = 2) = 3P(x = 3)$, then mean is
(a) 2 (b) -2 (c) 1 (d) 0.5
- 57.** Assuming distribution to be Poisson, the probability that atmost 5 defective bolts will be found in a box of 200 bolts, if it is known that 2% of such bolts are defective, is
(a) $e^{-5} \sum_{i=1}^5 \frac{4^i}{i!}$
(b) $e^{-4} \sum_{i=1}^4 \frac{4^i}{i!}$
(c) $e^{-4} \sum_{i=0}^5 \frac{4^i}{i!}$
(d) None of the above
- 58.** In a precision bombing attack, there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. The number of bombs which should be dropped to give a 99% chance or better of completely destroying the target, can be
(a) 12 (b) 11 (c) 10 (d) 13

Type 2. More than One Correct Option

- 59.** The letters of the word PROBABILITY are written down at random in a row. Let E_1 denotes the event that two I's are together and E_2 denotes the event that two B's are together, then

- (a) $P(E_1) = P(E_2) = \frac{2}{11}$ (b) $P(E_1 \cap E_2) = \frac{2}{55}$
(c) $P(E_1 \cup E_2) = \frac{18}{55}$ (d) $P(E_1/E_2) = \frac{1}{5}$

60. If the independent events A and B are such that $0 < P(A) < 1$ and $0 < P(B) < 1$, then which of the following option (s) is/are correct?

- (a) A and B are mutually exclusive
(b) A and $\bar{B}$ are independent
(c) A and $\bar{B}$ are independent

(d) $P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{\bar{B}}\right) = 1$

61. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$, then which of the following option (s) is/are correct?

- (a) $P(A \cup B) \geq \frac{2}{3}$
(b) $\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$
(c) $\frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$
(d) $P(A \cap \bar{B}) \leq \frac{1}{3}$

62. If A and B are two independent events such that $P(\bar{A} \cap B) = \frac{2}{15}$ and $P(A \cap \bar{B}) = \frac{1}{6}$, then $P(B)$ is

- (a) $\frac{4}{5}$ (b) $\frac{1}{6}$
(c) $\frac{1}{5}$ (d) $\frac{5}{6}$

Type 3. Assertion and Reason

Directions (Q. Nos. 66-69) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

66. Let A and B be two events such that $P(A) = \frac{3}{5}$ and

$P(B) = \frac{2}{3}$, then

Statement I $\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$

Statement II $\frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$

67. Consider an event for which probability of success is $1/2$.

63. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are

(a) $p_1 = \frac{1}{9}$ (b) $p_1 = \frac{1}{16}$ (c) $p_2 = \frac{1}{3}$ (d) $p_2 = \frac{1}{4}$

64. A bag contains b blue balls and r red balls. If two balls are drawn at random, then the probability of drawing two red balls is five times the probability of drawing two blue balls. Further more, the probability of drawing one ball of each colour is six times the probability of drawing two blue balls. Then,

(a) $b + r = 9$ (b) $br = 18$
(c) $|b - r| = 4$ (d) $\frac{b}{r} = 2$

65. The probability of the simultaneous occurrence of two events A and B is p . If the probability that exactly one of A, B occurs is q , then which of the following option (s) is/are correct?

- (a) $P(\bar{A}) + P(\bar{B}) = 2 + 2q - p$
(b) $P(\bar{A}) + P(\bar{B}) = 2 - 2p - q$
(c) $P(A \cap B / A \cup B) = \frac{p}{p + q}$
(d) $P(\bar{A} \cap \bar{B}) = 1 - p - q$

Statement I Probability that in n trials, there are r successes, where $r = 4k$ and k is an integer, is

$$\frac{1}{4} + \frac{1}{\frac{n}{2^{2+1}}} \cos\left(\frac{n\pi}{4}\right).$$

Statement II

$${}^nC_0 + {}^nC_4 + {}^nC_8 + \dots = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$$

68. Statement I There are 4 addressed envelopes and 4 letters for each one of them. The probability that no letter is mailed in its correct envelope is $\frac{3}{8}$.

Statement II The probability that all letters are not mailed in their correct envelope, is $\frac{23}{24}$.

69. Statement I If a fair coin is tossed 15 times, then the probability of getting head as many times in the first ten throws as in the last five, is $\frac{3003}{32768}$.

Statement II Sum of the series ${}^mC_r {}^nC_0 + {}^mC_{r-1} {}^nC_1 + \dots + {}^mC_0 {}^nC_r = {}^{m+n}C_r$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 70-72) Let S and T be two events defined on a sample space with probabilities $P(S) = 0.5$, $P(T) = 0.69$, $P\left(\frac{S}{T}\right) = 0.5$.

70. Events S and T are
 (a) mutually exclusive
 (b) independent
 (c) mutually exclusive and independent
 (d) neither mutually exclusive nor independent

71. The value of $P(S \text{ and } T)$ is
 (a) 0.3450 (b) 0.2500 (c) 0.6900 (d) 0.350

72. The value of $P(S \text{ or } T)$ is
 (a) 0.6900 (b) 1.19 (c) 0.8450 (d) 0

Passage II (Q. Nos. 73-75) If the squares of a 8×8 chessboard are painted either red or black at random.

73. The probability that not all the squares in any column are alternating in colour, is

- (a) $\left(1 - \frac{1}{2^7}\right)^8$ (b) $\frac{1}{2^{56}}$
 (c) $1 - \frac{1}{2^7}$ (d) None of these

74. The probability that the chessboard contains equal number of red and black squares, is

- (a) $\frac{{}^{64}C_{32}}{2^{64}}$ (b) $\frac{64!}{322^{64}!}$
 (c) $\frac{2^{32} - 1}{2^{64}}$ (d) None of these

75. The probability that all the squares in any column are of same colour and that of a row are of alternating colour, is

- (a) $\frac{1}{2^{64}}$ (b) $\frac{1}{2^{63}}$
 (c) $\frac{1}{2}$ (d) None of these

Passage III (Q. Nos. 76-78) A player A plays a game against a machine. At each round, he deposits one rupee in a slot and then flips a coin which has a probability (p) of showing a head. If the flipped coin show head, he gets back the rupee he deposited and one more rupee from the machine, else loses his rupee. Let A starts with ₹10 coins and $q = 1 - p$.

76. The probability that he will be drained out with all of his rupee coins exactly at the eleventh round, is
 (a) q^{11} (b) $1 - q^{11}$ (c) $pq^{10} + q^{11}$ (d) 0

77. The probability that all his money will be finished exactly at the twelfth round, is

- (a) q^{12} (b) $1 - q^{12}$
 (c) ${}^{10}C_1 pq^{11}$ (d) ${}^{12}C_2 p^2 q^{10}$

78. The probability that he is left with no money by the 14th round or earlier, is

- (a) $q^{10}(1 + 10pq + 45p^2q^2)$
 (b) $q^{14}(p^2q + 36pq + 7)$
 (c) $q^{12} + 3pq^{13} + 3p^{13}q + p^{12}$
 (d) $1 - {}^{10}C_1 pq^{11} - {}^{10}C_2 p^2 q^{12}$

Type 5. Match the Columns

79. Match the statements of Column I with values of Column II.

Column I	Column II
A. 7 white balls and 3 black balls are placed in a row at random. The probability that no two black balls are adjacent, is	p. $\frac{5}{11}$
B. 4 gentlemen and 4 ladies take seats at random round a table. The probability that they are sitting alternately, is	q. $\frac{1}{16}$
C. 10 different books and 2 different pens are given to 3 boys, so that each gets equal number of things. The probability that the same boy does not receive both the pens, is	r. $\frac{7}{15}$
D. A fair coin is tossed repeatedly. The probability of getting a result in the fifth toss different from those obtained in the first four tosses, is	s. $\frac{1}{35}$

80. Match the statements of Column I with values of Column II.

Column I	Column II
A. Six different balls are put in three different boxes, no box being empty. The probability of putting balls in boxes in equal number, is	p. $\frac{20}{27}$
B. Six letters are posted in 3 letter boxes. The probability that no letter box remains empty, is	q. $\frac{1}{6}$
C. Two persons A and B throw two dice each. If A throws a sum of 9, then the probability of B throwing a sum greater than A , is	r. $\frac{1}{3}$
D. If A and B are independent events and $P(A) = 0.3$ and $(A \cup B) = 0.8$, then $P(B)$ is equal to	s. $\frac{1}{4}$
	t. $\frac{2}{7}$

81. Match the statements of Column I with values of Column II.

Column I	Column II
A. The probability of a bomb hitting a bridge is $\frac{1}{2}$. Two direct hits are needed to destroy it. The number of bombs required so that the probability of the bridge being destroyed is greater than 0.9 can be	p. 4
B. A bag contains 2 red, 3 white and 5 black balls, a ball is drawn its colour is noted and replaced. The number of times, a ball can be drawn, so that the probability of getting a red ball for the first time is atleast $\frac{1}{2}$, is	q. 6

Column I	Column II
C. A drawer contains a mixture of red socks and blue socks, atmost 17 in all. It so happens that when two socks are selected randomly without replacement, there is a probability of exactly $\frac{1}{2}$ that both are red or both are blue. Then, number of red socks in the drawer can be	r. 7
D. There are two red, two blue, two white and certain number (greater than 0) of green socks in a drawer. If two socks are taken at random from the drawer without replacement, the probability that they are of the same colour is $\frac{1}{5}$, then the number of green socks are	s. 10

Type 6. Single Integer Answer Type Questions

82. An artillery target may be either at point A with probability $\frac{8}{9}$ or at point B with probability $\frac{1}{9}$. We

have 21 shells each of which can be fixed either at point A or B . Each shell may hit the target independently of the other shell with probability $\frac{1}{2}$.

If the number of shells must be fired at point A to hit the target with maximum probability is 1, then $\frac{\lambda}{3}$ is _____.

83. If the minimum number of tosses of a pair of dice, so that the probability of getting the sum of the numbers on the dice equal to 7 on atleast one toss, is greater than 0.95, is λ , then $(\lambda - 10)$ is _____.

(Given, $\log_{10} 2 = 0.3010$, $\log_{10} 3 = 0.4771$)

84. Two teams A and B play a tournament. The first one to win $(n+1)$ games, win the series. The probability that A wins a game, is p and that B wins a game, is q (no ties). If the value of $\sum_{r=0}^n {}^{n+r}C_n \cdot \frac{1}{2^{n+r}} = \frac{\lambda}{2}$, then λ is _____.

85. A speaks the truth '3 times out of 4' and B speaks the truth '2 times out of 3'. A die is thrown. Both assert that the number turned up is 2. The

probability of the truth of their assertion, is $\frac{a}{b}$, then $b - a$ is _____.

86. A bag contains 10 different balls. Five balls are drawn simultaneously and then replaced and then seven balls are drawn. If the probability that exactly three balls are common to the two drawn is p , then the value of $12p$ is _____.

87. A die is thrown three times. The chance that the highest number shown on the die is 4 is p , then the value of $\left[\frac{1}{p} \right]$, where $[]$ represents greatest integer function, is _____.

88. Two cards are drawn from a well-shuffled pack of 52 cards. The probability that one is a heart card and the other is a king, is p , then the value of $104p$ is _____.

89. Five different games are to be distributed among four children randomly. The probability that each child get atleast one game is p , then the value of $\left[\frac{1}{p} \right]$, where $[]$ represents the greatest integer function, is _____.

Entrances Gallery

16

JEE Advanced/IIT JEE

Advanced Probability

Target Exercises

1. Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is atleast one more than the number of girls ahead of her, is [2014]

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Directions (Q. Nos. 2-3) Box I contains three cards bearing numbers 1, 2, 3; box II contains five cards bearing numbers 1, 2, 3, 4, 5 and box III contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the i th box; $i=1, 2, 3$.

[2014]

2. The probability that $x_1 + x_2 + x_3$ is odd, is

(a) $\frac{29}{105}$ (b) $\frac{53}{105}$ (c) $\frac{57}{105}$ (d) $\frac{1}{2}$

3. The probability that x_1, x_2, x_3 are in an arithmetic progression, is

(a) $\frac{9}{105}$ (b) $\frac{10}{105}$ (c) $\frac{11}{105}$ (d) $\frac{7}{105}$

4. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$.

Then, the probability that the problem is solved correctly by atleast one of them, is [2013]

(a) $\frac{235}{256}$ (b) $\frac{21}{256}$ (c) $\frac{3}{256}$ (d) $\frac{253}{256}$

Directions (Q. Nos. 5-6) A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls. [2013]

5. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3 , then the probability that all 3 drawn balls are of the same colour, is

(a) $\frac{82}{648}$ (b) $\frac{90}{648}$
(c) $\frac{558}{648}$ (d) $\frac{566}{648}$

6. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, then the probability that these 2 balls are drawn from box B_2 , is

(a) $\frac{116}{181}$ (b) $\frac{126}{181}$
(c) $\frac{65}{181}$ (d) $\frac{55}{181}$

7. Four fair dice D_1, D_2, D_3 and D_4 each having six faces numbered 1, 2, 3, 4, 5 and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 , is [2012]

(a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{125}{216}$ (d) $\frac{127}{216}$

8. A ship is fitted with three engines E_1, E_2 and E_3 . The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational, atleast two of its engines must function. Let X denotes the event that the ship is operational and let X_1, X_2 and X_3 denote, respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is/are true? [2012]

(a) $P(X_1^C/X) = \frac{3}{16}$

(b) $P(\text{exactly two engines of the ship are functioning}) = \frac{7}{8}$

(c) $P(X/X_2) = \frac{5}{16}$

(d) $P(X/X_1) = \frac{7}{16}$

9. Let X and Y be two events such that $P(X/Y) = \frac{1}{2}, P(Y/X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$.

Which of the following is/are correct? [2012]

(a) $P(X \cup Y) = \frac{2}{3}$

(b) X and Y are independent

(c) X and Y are not independent

(d) $P(X^C \cap Y) = \frac{1}{3}$

10. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$.

If $P(T)$ denotes the probability of occurrence of the event T , then [2011]

(a) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$

(b) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(c) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$

(d) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$

Directions (Q. Nos. 11-12) Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls and U_2 contains only 1 white ball. A fair coin is tossed. If head appears, then 1 ball is drawn at random from U_1 and put into U_2 . However, if tail appears, then 2 balls are drawn at random from U_1 and put into U_2 . Now, 1 ball is drawn at random from U_2 . [2011]

11. The probability of the drawn ball from U_2 being white, is

- (a) $\frac{13}{30}$ (b) $\frac{23}{30}$
(c) $\frac{19}{30}$ (d) $\frac{11}{30}$

12. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin, is

- (a) $\frac{17}{23}$ (b) $\frac{11}{23}$ (c) $\frac{15}{23}$ (d) $\frac{12}{23}$

JEE Main/AIEEE

15. Let A and B be two events such that $P(\overline{A \cup B}) = \frac{1}{6}$,

$P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for the complement of the event A . Then, the events A and B are [2014, 08]

- (a) independent but not equally likely
(b) independent and equally likely
(c) mutually exclusive and independent
(d) equally likely but not independent

16. Let A , B and C try to hit a target simultaneously but independently. Their respective probabilities of hitting the targets are $\frac{3}{4}$, $\frac{1}{2}$ and $\frac{5}{8}$. The probability that the target is hit by A or B but not by C , is [2013]

- (a) $\frac{21}{64}$ (b) $\frac{7}{8}$
(c) $\frac{7}{32}$ (d) $\frac{9}{64}$

17. In two independent events, if the probability that exactly one of them occurs is $\frac{26}{49}$ and the probability that none of them occurs is $\frac{15}{49}$, then the probability of more probable of the two events is [2013]

- (a) $\frac{4}{7}$ (b) $\frac{6}{7}$
(c) $\frac{3}{7}$ (d) $\frac{5}{7}$

13. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1 , r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is [2010]

- (a) $\frac{1}{18}$ (b) $\frac{1}{9}$ (c) $\frac{2}{9}$ (d) $\frac{1}{36}$

14. A signal which can be green or red with probability $\frac{4}{5}$ and $\frac{1}{5}$ respectively, is received by station A and then transmitted to station B . The probability of each station receiving the signal correctly is $\frac{3}{4}$. If

the signal received at station B is green, then the probability that the original signal is green, is [2010]

- (a) $\frac{3}{5}$ (b) $\frac{6}{7}$ (c) $\frac{20}{23}$ (d) $\frac{9}{20}$

18. If two events A and B are mutually exclusive events such that $P(A) = \frac{3x+1}{3}$ and $P(B) = \frac{1-x}{4}$, then the set of possible values of x lies in the interval [2013]

- (a) $[0, 1]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$
(c) $\left[-\frac{1}{3}, \frac{5}{9}\right]$ (d) $\left[-\frac{7}{9}, \frac{4}{9}\right]$

19. A multiple choice examination has 5 questions. Each question has three alternative answers of which exactly one is correct. The probability that a student will get 4 or more correct answers just by guessing, is [2013]

- (a) $\frac{17}{3^5}$ (b) $\frac{13}{3^5}$ (c) $\frac{11}{3^5}$ (d) $\frac{10}{3^5}$

20. Three numbers are chosen at random without replacement from $\{1, 2, 3, \dots, 8\}$. The probability that their minimum is 3, given that their maximum is 6, is [2012]

- (a) $\frac{3}{8}$ (b) $\frac{1}{5}$ (c) $\frac{1}{4}$ (d) $\frac{2}{5}$

21. Consider, 5 independent Bernoulli's trials each with probability of success p . If the probability of atleast one failure is greater than or equal to $\frac{31}{32}$, then p lies in the interval [2011]

- (a) $\left(\frac{3}{4}, \frac{11}{12}\right]$ (b) $\left[0, \frac{1}{2}\right]$
(c) $\left(\frac{11}{12}, 1\right]$ (d) $\left[\frac{1}{2}, \frac{3}{4}\right]$

22. Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, \dots, 20\}$.

Statement I The probability that the chosen numbers when arranged in some order will form an AP, is $\frac{1}{85}$.

Statement II If the four chosen numbers form an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. [2010]

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

23. In a Binomial distribution $B\left(n, p = \frac{1}{4}\right)$, if the probability of atleast one success is greater than or equal to $\frac{9}{10}$, then n is greater than [2009]

- (a) $\frac{1}{\log_{10} 4 - \log_{10} 3}$
(b) $\frac{1}{\log_{10} 4 + \log_{10} 3}$
(c) $\frac{9}{\log_{10} 4 - \log_{10} 3}$
(d) $\frac{4}{\log_{10} 4 - \log_{10} 3}$

24. One ticket is selected at random from 50 tickets numbered 00, 01, 02, ..., 49. Then, the probability that the sum of the digits on the selected ticket is 8, given that the product of these digits is zero, equals [2009]

- (a) $\frac{1}{14}$ (b) $\frac{1}{7}$
(c) $\frac{5}{14}$ (d) $\frac{1}{50}$

25. It is given that the events A and B are such that $P(A) = \frac{1}{4}$, $P(A/B) = \frac{1}{2}$ and $P(B/A) = \frac{2}{3}$. Then, $P(B)$ is [2008]

- (a) $\frac{1}{2}$ (b) $\frac{1}{6}$
(c) $\frac{1}{3}$ (d) $\frac{2}{3}$

26. Two aeroplanes I and II bomb a target in succession. The probabilities of I and II scoring a hit correctly are 0.3 and 0.2, respectively. The second plane will bomb only, if the first misses the target. The probability that the target is hit by the second plane, is [2007]

- (a) 0.06 (b) 0.14
(c) 0.32 (d) 0.7

27. A random variable X has the probability distribution

X	1	2	3	4	5	6	7	8
$P(X)$	0.15	0.23	0.12	0.10	0.20	0.08	0.07	0.05

For the events $E = \{X \text{ is a prime number}\}$ and $F = \{X < 4\}$, $P(E \cup F)$ is [2004]

- (a) 0.87 (b) 0.77
(c) 0.35 (d) 0.50

28. Events A, B and C are mutually exclusive events such that $P(A) = \frac{3x+1}{3}$, $P(B) = \frac{1-x}{4}$ and $P(C) = \frac{1-2x}{2}$. Then, the set of possible values of x are in the interval [2003]

- (a) $\left[\frac{1}{3}, \frac{1}{2}\right]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$ (c) $\left[\frac{1}{3}, \frac{13}{3}\right]$ (d) $[0, 1]$

29. A problem in Mathematics is given to three students A, B, C and their respective probability of solving the problem is $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Probability that the problem is solved, is [2002]

- (a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

30. If A and B are two mutually exclusive events, then [2002]

- (a) $P(A) < P(\bar{B})$ (b) $P(A) > P(\bar{B})$
(c) $P(A) < P(B)$ (d) None of these

31. The probability of India winning a test match against West Indies is $\frac{1}{2}$ assuming independence from match-to-match. The probability that in a match series India's second win occurs at the third test, is [2002]

- (a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

32. A biased coin with probability p , $0 < p < 1$, of heads is tossed until a head appears for the first time. If the probability that the number of tosses required is even, is $\frac{2}{5}$, then p equals [2002]

- (a) $\frac{1}{3}$ (b) $\frac{2}{3}$
(c) $\frac{2}{5}$ (d) $\frac{3}{5}$

33. A fair die is tossed eight times. The probability that a third six is observed on the eight throw, is [2002]

- (a) $\frac{{}^7C_2 \times 5^5}{6^7}$ (b) $\frac{{}^7C_2 \times 5^5}{6^8}$
(c) $\frac{{}^7C_2 \times 5^5}{6^6}$ (d) None of these

- 34.** The probability of simultaneous occurrence of at least one of two events A and B is p . If the probability that exactly one of A, B occurs is q , then $P(A') + P(B')$ is equal to [BITSAT 2014]
 (a) $2 - 2p + q$ (b) $2 + 2p - q$
 (c) $3 - 3p + q$ (d) $2 - 4p + q$
- 35.** Out of 50 tickets numbered 00, 01, 02, ..., 49, one ticket is drawn randomly, the probability of the ticket having the product of its digits 7, given that the sum of the digits is 8, is [Manipal 2014]
 (a) $\frac{1}{14}$ (b) $\frac{3}{14}$
 (c) $\frac{1}{5}$ (d) None of these
- 36.** A and B are two events such that $P(A) \neq 0$, then $P\left(\frac{B}{A}\right)$, if (i) A is a subset of B (ii) $A \cap B = \phi$, are respectively [Karnataka CET 2014]
 (a) 1 and 1 (b) 0 and 1 (c) 0 and 0 (d) 1 and 0
- 37.** Let A and B be two events such that $P(A \cup B) = P(A) + P(B) - P(A)P(B)$. If $0 < P(A) < 1$ and $0 < P(B) < 1$, then $P(A \cup B)'$ is equal to [Kerala CEE 2014]
 (a) $1 - P(A)$ (b) $1 - P(A')$
 (c) $1 - P(A)P(B)$ (d) $[1 - P(A)]P(B')$
 (e) 1
- 38.** If events A, B are independent, $P(A') = \frac{2}{3}$ and $P(B') = \frac{2}{7}$, then $P(A \cap B)$ is equal to [Karnataka CET 2014]
 (a) $\frac{4}{21}$ (b) $\frac{5}{21}$
 (c) $\frac{1}{21}$ (d) $\frac{3}{21}$
- 39.** A student answers a multiple choice question with 5 alternatives, of which exactly one is correct. The probability that he knows the correct answer is p , $0 < p < 1$. If he does not know the correct answer, he randomly ticks one answer. Given that, he has answered the question correctly, the probability that he did not tick the answer randomly, is [WB JEE 2014]
 (a) $\frac{3p}{4p + 3}$
 (b) $\frac{5p}{3p + 2}$
 (c) $\frac{5p}{4p + 1}$
 (d) $\frac{4p}{3p + 1}$
- 40.** If A, B and C are mutually exclusive and exhaustive events of a random experiment such that $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$, then $P(A \cup C)$ equals [EAMCET 2014]
 (a) $\frac{10}{13}$ (b) $\frac{3}{13}$ (c) $\frac{6}{13}$ (d) $\frac{7}{13}$
- 41.** A candidate takes three tests in succession and the probability of passing the first test is p . The probability of passing each succeeding test is p or $\frac{p}{2}$ according as he passes or fails in the preceding one. The candidate is selected, if he passes at least two tests. The probability that the candidate is selected, is [EAMCET 2014]
 (a) $p^2(2 - p)$ (b) $p(2 - p)$
 (c) $p + p^2 + p^3$ (d) $p^2(1 - p)$
- 42.** If M and N are any two events, then the probability that exactly one of them occurs is (for an event set A , the complement is A^C) [GGSIIPS 2014]
 (a) $P(M) + P(N) - 2P(M \cup N)$
 (b) $P(M) + P(N) - P(M \cup N)$
 (c) $P(M^C) + P(N^C) - 2P(M^C \cup N^C)$
 (d) None of the above
- 43.** If $P(A \cup B) = 0.83$, $P(A) = 0.3$ and $P(B) = 0.6$, then the events will be [RPET 2013]
 (a) dependent (b) independent
 (c) Cannot say anything (d) None of these
- 44.** If X denotes the sum of the numbers obtained when two fair dice are rolled, then the variance and standard deviation of X are [VITEEE 2013]
 (a) $\frac{31}{6}$ and $\sqrt{\frac{31}{6}}$ (b) $\frac{35}{6}$ and $\sqrt{\frac{35}{6}}$
 (c) $\frac{17}{6}$ and $\sqrt{\frac{17}{6}}$ (d) $\frac{31}{6}$ and $\sqrt{\frac{35}{6}}$
- 45.** In a trial, the probability of success is twice the probability of failure. In six trials, the probability of at least four successes will be [Manipal 2013]
 (a) $\frac{496}{729}$ (b) $\frac{400}{729}$ (c) $\frac{500}{729}$ (d) $\frac{600}{729}$
- 46.** A box contains 100 bulbs, out of which 10 are defective. A sample of 5 bulbs is drawn. The probability that none is defective, is [Karnataka CET 2013]
 (a) $\frac{9}{10}$ (b) $\left(\frac{1}{10}\right)^5$
 (c) $\left(\frac{9}{10}\right)^5$ (d) $\left(\frac{1}{2}\right)^5$

47. If the mean and variance of a Binomial variate X are 8 and 4 respectively, then $P(X < 3)$ equals

[EAMCET 2013]

- (a) $\frac{265}{2^{15}}$ (b) $\frac{137}{2^{14}}$
(c) $\frac{137}{2^{16}}$ (d) $\frac{265}{2^{16}}$

48. A random variable X has the probability distribution given below:

X	1	2	3	4	5
$P(X)$	k	$2k$	$3k$	$2k$	k

Its variance is

[EAMCET 2013]

- (a) $\frac{16}{3}$ (b) $\frac{4}{3}$
(c) $\frac{5}{3}$ (d) $\frac{10}{3}$

49. If the mean and variance of a random variable X having a Binomial distribution are 4 and 2 respectively, then find the value of $P(X = 1)$.

[Guj CET 2013]

- (a) $\frac{1}{4}$ (b) $\frac{1}{16}$
(c) $\frac{1}{8}$ (d) $\frac{1}{32}$

50. A and B throw a die alternatively till one of them gets a six and wins the game. If A starts the game first, then the probability of A winning the game is

[MP PET 2012]

- (a) $\frac{3}{11}$ (b) $\frac{5}{11}$
(c) $\frac{6}{11}$ (d) $\frac{7}{11}$

51. If A and B are two events with $P(A^C) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$. Then,

$P\left[\frac{(B)}{(A \cup B^C)}\right]$ is equal to [WB JEE 2012]

- (a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

52. If A and B are two non-empty sets with B^C denoting the complement of set B such that $B \subset A$, then which of the following probability statements hold correct? [UP SEE 2011]

- (a) $P(A \cap B^C) = P(B) - P(A)$
(b) $P(A \cap B^C) = P(A) - P(B)$
(c) $P(B) \leq P(A)$
(d) $P(A) \leq P(B)$

53. If $f(x) = \frac{k}{2^x}$ is a probability distribution of a random variable X that can take on the values $x = 0, 1, 2, 3, 4$. Then, k is equal to [J&K CET 2011]

- (a) $\frac{16}{15}$ (b) $\frac{15}{16}$
(c) $\frac{31}{16}$ (d) None of these

54. The probability distribution of a random variable X is given below:

X	-5	-4	-3	-2	-1	0	1	2	3	4	5
$P(X)$	p	$2p$	$3p$	$4p$	$5p$	$7p$	$8p$	$9p$	$10p$	$11p$	$12p$

Then, the value of p is

[Kerala CEE 2010]

- (a) $\frac{1}{72}$ (b) $\frac{3}{73}$ (c) $\frac{5}{72}$ (d) $\frac{1}{74}$
(e) $\frac{1}{73}$

Answers

Work Book Exercise 16.1

1. (b)	2. (a)	3. (d)	4. (c)	5. (a)	6. (c)	7. (a)	8. (c)	9. (c)	10. (a)
11. (a)	12. (c)	13. (b)	14. (a)	15. (c)	16. (b)	17. (c)	18. (c)	19. (b)	20. (a)

Work Book Exercise 16.2

1. (c)	2. (a)	3. (a)	4. (b)	5. (d)	6. (c)	7. (b)	8. (c)	9. (b)	10. (a)

Work Book Exercise 16.3

1. (b)	2. (a)	3. (a)	4. (b)	5. (a)	6. (a)

Work Book Exercise 16.4

1. (d)	2. (a)	3. (a)	4. (c)	5. (b)

Target Exercises

1. (d)	2. (d)	3. (b)	4. (c)	5. (a)	6. (c)	7. (b)	8. (d)	9. (b)	10. (d)
11. (a)	12. (b)	13. (b)	14. (a)	15. (d)	16. (c)	17. (c)	18. (b)	19. (a)	20. (c)
21. (c)	22. (b)	23. (c)	24. (b)	25. (a)	26. (d)	27. (a)	28. (d)	29. (c)	30. (a)
31. (d)	32. (a)	33. (c)	34. (c)	35. (b)	36. (c)	37. (c)	38. (b)	39. (a)	40. (b)
41. (c)	42. (c)	43. (a)	44. (a)	45. (c)	46. (a)	47. (d)	48. (d)	49. (c)	50. (b)
51. (c)	52. (a)	53. (d)	54. (d)	55. (a)	56. (c)	57. (c)	58. (b)	59. (all)	60. (b,c,d)
61. (all)	62. (a,b)	63. (a,c)	64. (a,b)	65. (b,c,d)	66. (a)	67. (c)	68. (b)	69. (a)	70. (b)
71. (a)	72. (c)	73. (a)	74. (a)	75. (b)	76. (d)	77. (c)	78. (a)	79. (*)	80. (*)
81. (*)	82. (4)	83. (7)	84. (2)	85. (1)	86. (5)	87. (5)	88. (4)	89. (4)	

*

79. $A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$

80. $A \rightarrow q; B \rightarrow p; C \rightarrow q; D \rightarrow t$

81. $A \rightarrow r, s; B \rightarrow p; C \rightarrow p,q,r,s; D \rightarrow p$

Entrances Gallery

1. (a)	2. (b)	3. (c)	4. (a)	5. (a)	6. (d)	7. (a)	8. (b,d)	9. (a,b)	10. (a,d)
11. (b)	12. (d)	13. (c)	14. (c)	15. (a)	16. (a)	17. (a)	18. (c)	19. (c)	20. (b)
21. (b)	22. (c)	23. (a)	24. (a)	25. (c)	26. (c)	27. (b)	28. (a)	29. (a)	30. (a)
31. (b)	32. (a)	33. (b)	34. (a)	35. (c)	36. (d)	37. (d)	38. (b)	39. (c)	40. (d)
41. (a)	42. (d)	43. (a)	44. (b)	45. (a)	46. (c)	47. (c)	48. (b)	49. (d)	50. (c)
51. (a)	52. (b,c)	53. (d)	54. (a)						

Explanations

Target Exercises

1. $P(A \cap B) \leq \min \{P(A), P(B)\} = \min \{0.65, 0.80\} = 0.65$

$\therefore P(A \cap B) \leq 0.65$

Also, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$
 $\geq 0.65 + 0.80 - 1 = 0.45$

$\therefore 0.45 \leq P(A \cap B) \leq 0.65$

2. Given, $P\left(\frac{B}{A}\right) = \frac{1}{2}$

$\Rightarrow \frac{P(B \cap A)}{P(A)} = \frac{1}{2}$

$\Rightarrow P(B \cap A) = \frac{1}{2} \times \frac{1}{4} = \frac{1}{8}$

$P\left(\frac{A}{B}\right) = \frac{1}{4}$

$\Rightarrow \frac{P(A \cap B)}{P(B)} = \frac{1}{4}$

$\Rightarrow P(B) = 4P(A \cap B)$

$\Rightarrow P(B) = \frac{1}{2}$

$\therefore P(A \cap B) = \frac{1}{8} = \frac{1}{2} \cdot \frac{1}{4} = P(A) \cdot P(B)$

So, events A and B are independent.

Now, $P\left(\frac{A'}{B}\right) = \frac{P(A' \cap B)}{P(B)}$
 $= \frac{P(A') \cdot P(B)}{P(B)} = \frac{3}{4}$

and $P\left(\frac{B'}{A'}\right) = \frac{P(B' \cap A')}{P(A')}$
 $= \frac{P(B') \cdot P(A')}{P(A')} = \frac{1}{2}$

3. $P(E \cap F) = P(E) \cdot P(F) = \frac{1}{12}$... (i)

$P(E^C \cap F^C) = P(E^C) \cdot P(F^C) = \frac{1}{2}$

$\Rightarrow (1 - P(E))(1 - P(F)) = \frac{1}{2}$... (ii)

On solving Eqs. (i) and (ii), we get

$P(E) = \frac{1}{2}$ and $P(F) = \frac{1}{6}$ [$\because P(E) > P(F)$]

4. $P(\bar{A} / \bar{B}) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} = \frac{1 - P(A \cup B)}{P(\bar{B})}$

5. Let A denotes the event that the runner succeeds exactly 3 times out of five and B denotes the event that the runner succeeds on the first trial.

$P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$

But $P(B \cap A) = P$ (clearing succeeding in the first trial and exactly once in two other trials)

$P({}^4C_2 p^2 (1-p)^2) = 6p^2 (1-p)^2$

and $P(A) = {}^5C_3 p^3 (1-p)^2 = 10p^3 (1-p)^2$

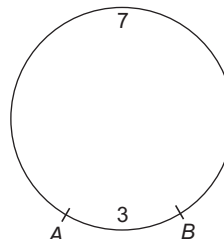
Thus, $P\left(\frac{B}{A}\right) = \frac{6p^2 (1-p)^2}{10p^3 (1-p)^2} = \frac{3}{5}$

6. $n(S) = {}^{10}C_3$ and $n(E) = {}^3C_1$ because on selecting 3, 7 and we have to select one from 4, 5 and 6.

$\therefore P(E) = \frac{{}^3C_1}{{}^{10}C_3} = \frac{1}{40}$

7. Number of ways of arranging 3 persons between A and B

$= 2 \times ({}^{10}P_3) \times 7! = 2 \times 10!$



Total number of arrangement of 12 persons is $11!$.

$\therefore$ Required probability $= \frac{2 \times 10!}{11!} = \frac{2}{11}$

8. E_1 occur as $(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)$

and E_2 occur as $(2, 1), (2, 2), (2, 3), (2, 4), (2, 5)$
 $(2, 6), (6, 2), (5, 2), (4, 2), (3, 2), (1, 2)$

$\therefore P\left(\frac{E_2}{E_1}\right) = \text{Probability of } E_2 \text{ when } E_1 \text{ has occurred} = \frac{2}{5}$

9. The total number of ways of selecting 3 tickets out of $6n$ numbers is ${}^{6n}C_3$. The sum of the numbers on three tickets drawn will be $6n$ in each of the following cases:

Lowest number	Ways of selection	Total number of ways
0	$(0, 1, 6n-1), (0, 2, 6n-2), \dots, (0, 3n-1, 3n+1)$	$(3n-1)$
1	$(1, 2, 6n-3), (1, 3, 6n-4), \dots, (1, 3n-1, 3n+1)$	$(3n-2)$
2	$(2, 3, 6n-5), (2, 4, 6n-6), \dots, (2, 3n-2, 3n)$	$(3n-4)$
3	$(3, 4, 6n-7), (3, 5, 6n-8), \dots, (3, 3n-2, 3n-1)$	$(3n-5)$
$\vdots$	$\vdots$	$\vdots$
$(2n-2)$	$(2n-2, 2n-1, 2n+3), (2n-2, 2n, 2n+2)$	2
$(2n-1)$	$(2n-1, 2n, 2n+1)$	1

Total number of ways of getting $6n$ as the sum

$= (3n-1) + (3n-2) + (3n-3) + (3n-5) + \dots + 2 + 1$
 $= (3n-1 + 3n-2) + (3n-4 + 3n-5) + \dots + (5+4) + (2+1)$

$= (6n-3) + (6n-9) + \dots + 9 + 3 = 3n^2$

$\therefore$ Required probability $= \frac{3n^2}{{}^{6n}C_3}$

10. Let probability for appearance of odd number be p .
Then, probability for appearance of even number is $2p$.

$$\therefore p + 2p = 1 \Rightarrow p = \frac{1}{3}$$

Sum of two numbers is even means either both numbers are even or odd.

$$\therefore \text{Required probability} = \frac{5}{9}$$

11. $P(E) = \frac{1}{2}$ [as E can occur as THH, HTH, HHT, HHH]

$$P(F) = \frac{1}{2} \quad [\text{as } F \text{ can occur as } HTH, HHT, HTT, HHH]$$

$$\text{Now, } P(E \cap F) = \frac{3}{8}$$

$$\therefore P\left(\frac{E}{F}\right) = \frac{P(E \cap F)}{P(F)} = \frac{3}{4}$$

12. Let E_1 be the event that the person has disease and E_2 be the event that the person is healthy.
Then, E_1 and E_2 are mutually exclusive and exhaustive.

$$\text{Moreover, } P(E_1) = 0.1\% = \frac{0.1}{100} = 0.001$$

$$\text{and } P(E_2) = 1 - 0.001 = 0.999$$

Let E be the event that test is positive.

$$P\left(\frac{E}{E_1}\right) = P(\text{result is positive given that person has disease})$$

$$= 99\% = \frac{99}{100} = 0.99$$

Probability that a person does not have disease and test result is positive.

$$\therefore P\left(\frac{E}{E_2}\right) = 0.5\% = \frac{0.5}{100} = 0.005$$

By using Baye's theorem, we obtain

$$\begin{aligned} P\left(\frac{E_1}{E}\right) &= \frac{P\left(\frac{E}{E_1}\right) \cdot P(E_1)}{P\left(\frac{E}{E_1}\right) \cdot P(E_1) + P\left(\frac{E}{E_2}\right) \cdot P(E_2)} \\ &= \frac{0.001 \times 0.99}{0.001 \times 0.99 + 0.999 \times 0.005} \\ &= \frac{0.00099}{0.00099 + 0.004995} \\ &= \frac{0.00099}{0.005985} + \frac{990}{5985} = \frac{110}{665} = \frac{22}{133} \end{aligned}$$

13. Here, $n(S) = {}^{50}C_5, n(E) = {}^{30}C_2 \times {}^{19}C_2$

$$\therefore P(E) = \frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$$

14. Consider the events

A = Getting a total of 4 or more in the throw of two dice

B = Getting a total of 6 or more in the simultaneous throw of three dice.

$$\therefore \text{Required probability} = P(A \cap B) = P(A) \cdot P(B) \dots (i)$$

For $P(A)$

Total number of elementary events = 6^2

Favourable events = 33

$$\therefore P(A) = \frac{33}{6^2} = \frac{11}{12}$$

For $P(B)$

Total number of elementary events = 6^3

Favourable number of elementary events

$$= 6^3 - \{\text{Sum of the coefficients of } x^3, x^4 \text{ and } x^5 \text{ in } (x + x^2 + \dots + x^6)^3\}$$

$$= 6^3 - (1 + 3 + 6) = 206$$

$$\therefore P(B) = \frac{206}{216} = \frac{103}{108}$$

$$\text{Hence, required probability} = \frac{11}{12} \times \frac{103}{108} = \frac{1133}{1296}$$

15. Using Binomial distribution, we have to find

$$P(3) + P(4) + P(5)$$

Their probability is

$${}^5C_3 \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right)^2 + {}^5C_4 \left(\frac{3}{4}\right)^4 \left(\frac{1}{4}\right)^1 + {}^5C_5 \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right)^0 = \frac{459}{512}$$

16. The probability of getting a defective bulb from the box, is $1/10$.

$$P(\text{none defective}) = {}^5C_0 \cdot \left(\frac{1}{10}\right)^0 \cdot \left(\frac{9}{10}\right)^5 = (0.9)^5$$

Hence, using Binomial distribution, the required probability is $(0.9)^5$.

17. The probability that both Krishna and Hari will be alive after 10 yr = $P(\text{both alive})$

$$= P(K \cap H) = P(K) \cdot P(H)$$

$$= \frac{7}{15} \cdot \frac{7}{10} = \frac{49}{150}$$

18. Probability of getting 3 sixes = ${}^4C_3 \left(\frac{1}{6}\right)^3 \times \left(\frac{5}{6}\right) = \frac{5}{324}$

$$\therefore \text{Expected number of times 3 sixes occur} = 324 \times \frac{5}{324} = 5$$

19. Since, only variate values are changed, therefore the probability of x_i and $f(x_i)$ are the same p_i .

20. $E(XY) = E(X) \cdot E(Y)$, if X and Y are independent.

21. 3 can be thrown as (1, 2), (2, 1).

5 can be thrown as (1, 4), (4, 1), (2, 3), (3, 2).

11 can be thrown as (5, 6), (6, 5).

$$\therefore \text{Required probability} = \frac{8}{36} = \frac{2}{9}$$

22. Let a number greater than 4 appear after exactly n tosses.

Then, the probability of this event

$$= \left(\frac{4}{6}\right)^n \left(\frac{2}{6}\right) = \left(\frac{2}{3}\right)^n \left(\frac{1}{3}\right)$$

Now, n can take values 0, 2, 4, 6, ...

Hence, the required probability

$$= \frac{1}{3} \left[1 + \frac{4}{9} + \frac{16}{81} + \dots \right] = \frac{3}{5}$$

23. To satisfy the condition of the problem, first 6 draws should give exactly 3 white balls and 7th draw should give a white ball.

Hence, the required probability

$$= {}^6C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right) = \frac{5}{32}$$

24. Let the name of the man be A and that of his wife be B. Then, $P(A \text{ will not be alive after 10 yr and } B \text{ will not be alive after 10 yr}) = P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B}) = \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

25. Probability of getting a total of 7 or 8 on a pair of dice, is $\frac{11}{36}$.

P (head shows up and number of 7 or 8 is noted)

$$= \frac{1}{2} \times \frac{11}{36}$$

Also, P (tail shows up and number 7 or 8 is noted)

$$= \frac{1}{2} \times \frac{2}{11}$$

Hence, the required probability = $\frac{11}{72} + \frac{1}{11} = \frac{193}{792}$

26. $P(HHH) + P(HHTT) + P(HTTTT) + P(TTTTTT)$
 $= \frac{1}{2^3} + {}^4C_2 \frac{1}{2^4} + {}^5C_1 \frac{1}{2^5} + \frac{1}{2^6} = \frac{43}{64}$

27. Total number of function is 6^6
 and number of onto function = 6!
 $\therefore$ Required probability = $\frac{6!}{6^6} = \frac{5}{324}$

28. We have, P (head on exactly 50 coins)
 $= P(\text{head on exactly 51 coins})$
 $\Rightarrow {}^{100}C_{50} \times p^{50} (1-p)^{50} = {}^{100}C_{51} \times p^{51} (1-p)^{49}$
 $\therefore p = \frac{51}{101}$

29. Here, $P_n = \alpha p^n, n \geq 1$
 and $P_0 = 1 - \alpha p(1 + p + p^2 + \dots)$

Consider the following events:

E_j = There are j children in the family; $j = 0, 1, 2, \dots, n$

A = There are exactly k boys in the family.

We have, $P(E_j) = p_j = \alpha p^j; j = 0, 1, 2, \dots, n$

$$\Rightarrow P\left(\frac{A}{E_j}\right) = \frac{{}^jC_k}{2^j}, j \geq k$$

$$\text{Now, } A = \bigcup_{j=k}^{\infty} (A \cap E_j)$$

$$\Rightarrow P(A) = P\left(\bigcup_{j=k}^{\infty} (A \cap E_j)\right)$$

$$\therefore P(A) = \sum_{j=k}^{\infty} P(A \cap E_j) = \sum_{j=k}^{\infty} P(E_j) P\left(\frac{A}{E_j}\right)$$

$$= \sum_{j=k}^{\infty} \alpha p^j \left(\frac{{}^jC_k}{2^j}\right) = \alpha \sum_{j=k}^{\infty} \left(\frac{p}{2}\right)^j \cdot {}^jC_k$$

$$= \alpha \sum_{r=0}^{\infty} {}^{k+r}C_r \left(\frac{p}{2}\right)^{k+r}$$

$$P(A) = \alpha \left(\frac{p}{2}\right)^k \sum_{r=0}^{\infty} {}^{k+r}C_r \left(\frac{p}{2}\right)^r$$

We have, ${}^{k+r}C_r = (-1)^r {}^{-(k+1)}C_r$

$$\therefore P(A) = \alpha \left(\frac{p}{2}\right)^k \sum_{r=0}^{\infty} {}^{-(k+1)}C_r \left(-\frac{p}{2}\right)^r$$

$$= \alpha \left(\frac{p}{2}\right)^k \left(1 - \frac{p}{2}\right)^{-(k+1)} = \alpha \left(\frac{p}{2}\right)^k \cdot \frac{2^{k+1}}{(2-p)^{k+1}}$$

30. The probability of getting a total of 5 on a pair of dice, is $\frac{1}{9}$ and that of getting a total of 7, is $\frac{1}{6}$. Also,

$P(\overline{57}) = \frac{13}{18}$ because there are 10 cases in favour of

getting a sum of 5 or 7. 5 can come before 7 in the following cases : 5 or $(\overline{57})$ 5 or $(\overline{57})(\overline{57})$ 5 or ...

Hence, the required probability is 2.5.

31. Since, the events A and B are mutually exclusive, so these events are not independent. Hence, the required probability cannot be found.

32. Given, $P(A \cap B) = P(A) \cdot P(B), P(B \cap C) = P(B) \cdot P(C),$

$$P(C \cap A) = P(C) \cdot P(A)$$

$$\text{and } P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

$$\text{We have, } P\{A \cap (B \cup C)\} = P\{(A \cap B) \cup (A \cap C)\}$$

$$= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C)$$

$$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$$

$$= P(A)\{P(B) + P(C) - P(B)P(C)\}$$

$$= P(A) \cdot \{P(B) + P(C) - P(B \cap C)\}$$

$$= P(A) \cdot P(B \cup C)$$

$\therefore A$ and $B \cup C$ are independent, i.e. S_1 is true

$$\text{Also, } P\{A \cap (B \cap C)\} = P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C)$$

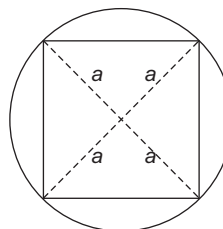
$$= P(A) \cdot P(B \cap C)$$

$\Rightarrow A$ and $B \cap C$ are independent, i.e. S_2 is true

Hence, S_1 and S_2 are true.

33. Let $ABCDEF$ be the hexagon. Three vertices of a triangle can be selected in ${}^6C_3$, i.e. 20 ways. The only possible equilateral triangles are ACE, BDF . Hence, the required probability is $1/10$.

34. Let a be the radius of the circle.



$\therefore$ The area of the inscribed square = $(a\sqrt{2})^2 = 2a^2$

$$\Rightarrow p_1 = \frac{2a^2}{\pi a^2} = \frac{2}{\pi}$$

$$\text{and } p_2 = \frac{\pi a^2 - 2a^2}{\pi a^2} = \frac{\pi - 2}{\pi}$$

$$\text{Now, } p_1 - p_2 = \frac{2}{\pi} - \left(\frac{\pi - 2}{\pi}\right) = \frac{4}{\pi} - 1 > 0$$

$$\Rightarrow \rho_1 > \rho_2$$

$$\text{Also, } (\rho_1 - \rho_2)(\rho_1 + \rho_2) = \left(\frac{4-\pi}{\pi}\right)(1)$$

$$\text{but } 3 < \pi < 4$$

$$\Rightarrow (\rho_1^2 - \rho_2^2) = \left(\frac{4-\pi}{\pi}\right) < \frac{1}{3}$$

$$\text{Hence, } (\rho_1^2 - \rho_2^2) < \frac{1}{3}$$

$$35. P(\text{shooting}) = 1 - P(X=0) = 1 - {}^{10}C_0 \cdot (0.3)^0 \cdot (0.7)^{10}$$

$$= 1 - (0.7)^{10}$$

$$36. \text{ Required probability} = 1$$

$$37. np = 2, npq = 1 \Rightarrow q = \frac{1}{2}, p = \frac{1}{2}, n = 4$$

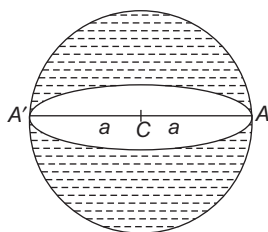
$$\therefore \text{ Required probability} = P(X > 1)$$

$$= 1 - P(X \leq 1)$$

$$= 1 - P(X=0) + P(X=1)$$

$$= 1 - \frac{5}{16} = \frac{11}{16}$$

$$38. \text{ Let ellipse be } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad [a > b]$$



$$\therefore \text{ Area of ellipse} = \pi ab = \pi a^2 \sqrt{1-e^2}$$

$$= \pi a^2 \sqrt{1-\frac{8}{9}} = \frac{\pi a^2}{3}$$

$$\text{Area of circle} = \pi a^2$$

$$\therefore \text{ Required probability} = \frac{\pi a^2 - \frac{\pi a^2}{3}}{\pi a^2} = \frac{2}{3}$$

$$39. \text{ Mean} = 2 \times (0.3) + 3 \times (0.4) + 4 \times (0.3) = 3$$

$$\therefore \text{ Variance, } \sigma_x^2 = 0.6$$

$$40. \text{ Case I The first ball is white and second is red.}$$

$$\text{Its probability} = \frac{3}{10}$$

$$\text{Case II The first ball is red and second is red.}$$

$$\text{Its probability} = \frac{1}{10}$$

$$\therefore \text{ Required probability} = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$$

$$41. \text{ Given, } np = \frac{5}{4}, npq = \frac{15}{16}$$

$$\Rightarrow p = \frac{1}{4}$$

$$\text{and } q = \frac{3}{4}$$

$$42. \text{ Given, } np = 6 \text{ and } npq = 4$$

$$\Rightarrow q = \frac{2}{3} \text{ and } p = \frac{1}{3}$$

$$\therefore n = 18$$

$$43. P(X=2) = {}^{12}C_2 (0.1)^2 (0.9)^{10} = 0.230$$

$$44. \text{ Given, } np = 6 \text{ and } npq = 4$$

$$\Rightarrow n = 18, p = \frac{1}{3}, q = \frac{2}{3}$$

$$\therefore (p+q)^{18} = \left(\frac{1}{3} + \frac{2}{3}\right)^{18}$$

$$45. \therefore p = q = \frac{1}{2}$$

$$\therefore p(1G) = {}^4C_1 \left(\frac{1}{2}\right)^4$$

$$\Rightarrow p(2G) = {}^4C_2 \left(\frac{1}{2}\right)^4$$

$$\Rightarrow p(1G \text{ or } 2G) = (4+6) \frac{1}{2^4} = \frac{5}{8}$$

$$\therefore \text{ Number of families} = 2000 \times \frac{5}{8} = 1250$$

$$46. \text{ Given, } 9P(X=4) = P(X=2)$$

$$\therefore 9 \cdot {}^6C_4 p^4 (1-p)^2 = {}^6C_2 p^2 (1-p)^4$$

$$\Rightarrow \frac{(1-p)^2}{p^2} = 9$$

$$\Rightarrow \frac{1-p}{p} = 3$$

$$\Rightarrow p = \frac{1}{4}$$

$$47. \text{ Given, } p = 0.1, n = 400, q = 0.9$$

$$\therefore \text{ Mean} = np = 40$$

$$\text{and SD } \sigma = \sqrt{npq} = 6$$

$$48. \text{ Given, } m + \sigma^2 = 24, m\sigma^2 = 128$$

$$\therefore m = 16 \text{ or } 8$$

$$m = 16 \Rightarrow \sigma^2 = 8$$

$$\text{and } m = 8 \Rightarrow \sigma^2 = 56$$

$$\text{Case I } np = 16, npq = 8$$

$$n = 32, p = q = \frac{1}{2}$$

$$\text{Case II } np = 8 \text{ and } npq = 56$$

$$\Rightarrow q = 7, \text{ which is not possible.}$$

$$49. \text{ Here, probability of success} = p = \frac{3}{6} = \frac{1}{2}$$

$$\therefore q = \frac{1}{2} \text{ and } n = 100$$

$$\therefore \text{ Variance} = npq = 25$$

$$50. n = 10, p = 0.1, q = 0.9, r = 2$$

$$\therefore P(X=2) = {}^{10}C_2 (0.1)^2 (0.9)^8 = 0.194$$

$$51. \text{ By definition, standard deviation} = \sqrt{npq}$$

$$52. \text{ Let us first count the number of elements in } F. \text{ Total number of functions from } A \text{ to } B \text{ is } 3^4 = 81.$$

$$\text{The number of functions which do not contain } x(y)[z] \text{ in its range, is } 2^4.$$

∴ The number functions which contain exactly two elements in the range, is $3 \cdot 2^4 = 48$.

The number of functions which contain exactly one element in its range, is 3.

Thus, the number of onto functions from A to B is $81 - 48 + 3 = 36$

[using principle of inclusion- exclusion]

∴ $n(F) = 36$

Let $f \in F$, then we count the number of ways in which $f^{-1}(x)$ consists of single element.

We can choose pre-image of x in 4 ways. The remaining 3 elements can be mapped onto in $\{y, z\}$ ways.

∴ $f^{-1}(x)$ will consist of exactly one element in $4 \times 6 = 24$ ways.

Thus, the probability of the required event is $\frac{24}{36} = \frac{2}{3}$

53. It could be any finite number.

54. Mean = m

55. Given, $P(0) = e^{-m} = \frac{10}{100} = \frac{1}{10}$

∴ $m = \log_e 10 = 2.32026$

56. $\frac{e^{-m} \cdot m^2}{2!} = 3e^{-m} \frac{m^3}{3!} \Rightarrow m = 1$

57. $p = \frac{2}{100}, n = 200$

∴ $m = np = 4$

Atmost 5 defective bolts = $p(0) + p(1) + \dots + p(5)$
 $= e^{-4} \sum_{i=0}^5 \frac{4^i}{i!}$

58. We have the probability that the bomb strikes the target is $p = \frac{1}{2}$. Let n be the number of bombs which

should be dropped to ensure 99% chance or better of completely destroying the target. Then, the probability that out of n bombs atleast two bombs strike the target is greater than 0.99. Let X denotes the number of bombs striking the target. Then,

$$P(X=r) = {}^nC_r \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{n-r} = {}^nC_r \left(\frac{1}{2}\right)^n, r = 0, 1, 2, \dots, n$$

We should have $P(X \geq 2) \geq 0.99$

$$\Rightarrow \{1 - P(X < 2)\} \geq 0.99$$

$$\Rightarrow 1 - \{P(X=0) + P(X=1)\} \geq 0.99$$

$$\Rightarrow 1 - \left\{(1+n) \frac{1}{2^n}\right\} \geq 0.99$$

$$\Rightarrow 0.01 \geq \frac{1+n}{2^n}$$

$$\Rightarrow 2^n > 100 + 100n \Rightarrow n \geq 11$$

Thus, the minimum number of bombs is 11.

59. (a) $P(E_1) = P(E_2) = \frac{10!}{2!11!} = \frac{2}{11}$

(b) $P(E_1 \cap E_2)$ = Probability of two I's are together and two B's are together

$$= \frac{9!}{11!} = \frac{4}{110} = \frac{2}{55}$$

$$(c) P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2) \\ = \frac{2}{11} + \frac{2}{11} - \frac{2}{55} = \frac{4}{11} - \frac{2}{55} = \frac{18}{55}$$

$$(d) P(E_1 / E_2) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{\frac{2}{55}}{\frac{2}{55}} = \frac{1}{11}$$

60. Since, A and B are independent events.

$$\therefore P(A \cap B) = P(A)P(B) \neq 0$$

So, A and B cannot be mutually exclusive events.

$$\begin{aligned} \text{Now, } P(A \cap \bar{B}) &= P(A) - P(A \cap B) \\ &= P(A) - P(A)P(B) \\ &= P(A)[1 - P(B)] \\ &= P(A)P(\bar{B}) \end{aligned}$$

$$\begin{aligned} \text{and } P(\bar{A} \cap \bar{B}) &= P(\overline{A \cup B}) \\ &= 1 - P(A \cup B) \\ &= 1 - [1 - P(\bar{A})P(\bar{B})] \\ &= P(\bar{A})P(\bar{B}) \end{aligned}$$

So, A and B as well as $\bar{A}$ and $\bar{B}$ are independent events.

$$\begin{aligned} \text{Finally, } P\left(\frac{A}{B}\right) + P\left(\frac{\bar{A}}{\bar{B}}\right) &= \frac{P(A \cap B)}{P(B)} + \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})} \\ &= \frac{P(A \cap B) + P(\bar{A} \cap \bar{B})}{P(B)} = \frac{P(B)}{P(B)} = 1 \end{aligned}$$

61. Since, $B \subseteq A \cup B$. Therefore,

$$P(B) \leq P(A \cup B) \Rightarrow P(A \cup B) \geq \frac{2}{3}$$

So, option (a) is correct.

$$\begin{aligned} \text{Now, } P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ \Rightarrow P(A \cap B) &\geq P(A) + P(B) - 1 \\ \Rightarrow P(A \cap B) &\geq \frac{3}{5} + \frac{2}{3} - 1 = \frac{4}{15} \end{aligned}$$

$$\begin{aligned} \text{Also, } A \cap B &\subseteq A \\ \Rightarrow P(A \cap B) &\leq P(A) \Rightarrow P(A \cap B) \leq \frac{3}{5} \end{aligned}$$

$$\therefore \frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$$

So, option (b) is correct.

$$\text{Now, } \frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$$

$$\Rightarrow \frac{4}{15} \leq P\left(\frac{A}{B}\right)P(B) \leq \frac{3}{5}$$

$$\Rightarrow \frac{4}{15} \leq \frac{2}{3}P\left(\frac{A}{B}\right) \leq \frac{3}{5}$$

$$\Rightarrow \frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$$

So, option (c) is correct.

$$\text{Finally, } P(A \cap \bar{B}) = P(A) - P(A \cap B) \leq \frac{3}{5} - \frac{4}{15} = \frac{1}{3}$$

So, option (d) is also correct.

62. Let $P(A) = x$ and $P(B) = y$. Since, A and B are independent events. Therefore,

$$\begin{aligned} P(\bar{A} \cap B) &= \frac{2}{15} \\ \Rightarrow P(\bar{A})P(B) &= \frac{2}{15} \\ \Rightarrow [1 - P(A)]P(B) &= \frac{2}{15} \\ \Rightarrow (1 - x)y &= \frac{2}{15} \\ \Rightarrow y - xy &= \frac{2}{15} \quad \dots(i) \end{aligned}$$

$$\begin{aligned} \text{and } P(A \cap \bar{B}) &= \frac{1}{6} \\ \Rightarrow P(A)P(\bar{B}) &= \frac{1}{6} \\ \Rightarrow x(1 - y) &= \frac{1}{6} \\ \Rightarrow x - xy &= \frac{1}{6} \quad \dots(ii) \end{aligned}$$

On subtracting Eq. (i) from Eq. (ii), we get

$$x - y = \frac{1}{30} \Rightarrow x = \frac{1}{30} + y$$

On putting the value of x in Eq. (i), we get

$$\begin{aligned} y - y\left(\frac{1}{30} + y\right) &= \frac{2}{15} \\ \Rightarrow 30y - y - 30y^2 &= 4 \\ \Rightarrow 30y^2 - 29y + 4 &= 0 \Rightarrow (6y - 1)(5y - 4) = 0 \\ \Rightarrow y &= \frac{1}{6} \text{ or } y = \frac{4}{5} \\ \Rightarrow P(B) &= \frac{1}{6} \text{ or } P(B) = \frac{4}{5} \end{aligned}$$

63. Let p_1, p_2 be the chances of happening of the first and second events, respectively. Then, according to the given conditions, we have

$$p_1 = p_2^2 \text{ and } \frac{1 - p_1}{p_1} = \left(\frac{1 - p_2}{p_2}\right)^3$$

$$\text{Hence, } \frac{1 - p_2^2}{p_2^2} = \left(\frac{1 - p_2}{p_2}\right)^3$$

$$\Rightarrow p_2(1 + p_2) = (1 - p_2)^2$$

$$\Rightarrow 3p_2 = 1 \Rightarrow p_2 = \frac{1}{3}$$

$$\text{and so } p_1 = \frac{1}{9}$$

64. Let the number of red and blue balls be r and b , respectively.

Then, the probability of drawing two red balls is

$$p_1 = \frac{{}^r C_2}{{}^{r+b} C_2} = \frac{r(r-1)}{(r+b)(r+b-1)}$$

The probability of drawing two blue balls is

$$p_2 = \frac{{}^b C_2}{{}^{r+b} C_2} = \frac{b(b-1)}{(r+b)(r+b-1)}$$

The probability of drawing one red and one blue ball is

$$p_3 = \frac{{}^r C_1 \times {}^b C_1}{{}^{r+b} C_2} = \frac{2br}{(r+b)(r+b-1)}$$

By hypothesis, $p_1 = 5p_2$ and $p_3 = 6p_2$

$$\begin{aligned} \therefore r(r-1) &= 5b(b-1) \text{ and } 2br = 6b(b-1) \\ \Rightarrow r &= 6, b = 3 \end{aligned}$$

65. We have, $P(A \cap B) = p$

and $P(A) + P(B) - 2P(A \cap B) = q$

$$\Rightarrow P(A) + P(B) - 2p = q$$

$$\Rightarrow P(A) + P(B) = 2p + q$$

$$\Rightarrow 1 - P(\bar{A}) + 1 - P(\bar{B}) = 2p + q$$

$$\Rightarrow P(\bar{A}) + P(\bar{B}) = 2 - 2p - q$$

So, option (b) is correct.

$$\text{Now, } P\left\{\frac{(A \cap B)}{(A \cup B)}\right\}$$

$$\begin{aligned} &= \frac{P[(A \cap B) \cap (A \cup B)]}{P(A \cup B)} = \frac{P(A \cap B)}{P(A \cup B)} \\ &= \frac{P(A \cap B)}{P(A) + P(B) - P(A \cap B)} \\ &= \frac{p}{2p + q - p} = \frac{p}{p + q} \end{aligned}$$

So, option (c) is correct.

$$\text{Finally, } P(\bar{A} \cap \bar{B}) = P(\bar{A \cup B})$$

$$= 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A \cap B)]$$

$$= 1 - [2p + q - p] = 1 - p - q$$

So, option (d) is correct.

66. $\therefore P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$

$$\therefore P(A \cap B) \geq \frac{3}{5} + \frac{2}{3} - 1$$

$$\Rightarrow P(A \cap B) \geq \frac{4}{15} \quad \dots(i)$$

$$\therefore P(A \cap B) \leq P(A)$$

$$\Rightarrow P(A \cap B) \leq \frac{3}{5} \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5} \quad \dots(iii)$$

$$\text{From Eq. (iii), } \frac{4}{15P(B)} \leq \frac{P(A \cap B)}{P(B)} \leq \frac{3}{5P(B)}$$

$$\Rightarrow \frac{2}{5} \leq P\left(\frac{A}{B}\right) \leq \frac{9}{10}$$

67. According to Statement I, the required probability is

$$\begin{aligned} & {}^n C_0 \left(\frac{1}{2}\right)^n + {}^n C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{n-4} + {}^n C_8 \left(\frac{1}{2}\right)^8 \left(\frac{1}{2}\right)^{n-8} + \dots \\ &= ({}^n C_0 + {}^n C_4 + {}^n C_8 + \dots) \left(\frac{1}{2}\right)^n \end{aligned}$$

Now consider the Binomial expansion,

$$(1 + x)^n = {}^n C_0 + {}^n C_1 x + {}^n C_2 x^2 + \dots$$

On putting $x = i$, where $i = \sqrt{-1}$, we get

$$(1 + i)^n = ({}^n C_0 - {}^n C_2 + {}^n C_4 - \dots) + i({}^n C_1 - {}^n C_3 + {}^n C_5 - \dots)$$

$$\Rightarrow \left[\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right) \right]^n = ({}^n C_0 - {}^n C_2 + {}^n C_4 - \dots)$$

$$+ i({}^n C_1 - {}^n C_3 + {}^n C_5 - \dots)$$

$$\Rightarrow {}^n C_0 - {}^n C_2 + {}^n C_4 - \dots = 2^{n/2} \cos \frac{n\pi}{4}$$

Also we know that,

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \dots = 2^{n-1}$$

$$\Rightarrow 2({}^nC_0 + {}^nC_4 + {}^nC_8 + \dots) = 2^{n-1} + 2^{n/2} \cos \frac{n\pi}{4}$$

Hence, the required probability is

$$\frac{1}{4} + \frac{1}{2^{n/2+1}} \cos\left(\frac{n\pi}{4}\right)$$

68. The total number of cases, $n(S) = 4!$.

Let E be the event that no letter is mailed in its correct envelope. Then, the favourable number of cases is

$$4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right) = 9$$

Hence, the required probability is

$$P(E) = \frac{9}{24} = \frac{3}{8}$$

Also, the probability that all the letters are placed in the correct envelope is $\frac{1}{24}$.

Hence, the probability that all the letters are not placed in the correct envelope is $\frac{23}{24}$.

Hence, Statement II is correct but does not explain Statement I.

69. In Binomial theorem, we have proved Statement II. Now, there may be 0, 1, 2, 3, 4 or 5 heads in the last five throws and the same can be for the first 10 throws. The number of cases thus may be given by

$$m = {}^5C_0 {}^{10}C_0 + {}^5C_1 {}^{10}C_1 + {}^5C_2 {}^{10}C_2 + {}^5C_3 {}^{10}C_3 + {}^5C_4 {}^{10}C_4 + {}^5C_5 {}^{10}C_5$$

$$= {}^5C_0 {}^{10}C_0 + {}^5C_1 {}^{10}C_9 + {}^5C_2 {}^{10}C_8 + {}^5C_3 {}^{10}C_7 + {}^5C_4 {}^{10}C_6 + {}^5C_5 {}^{10}C_5$$

$$= {}^{10+5}C_{10} = {}^{15}C_{10} = 3003$$

The total number of ways (N) is $2^{15} = 32768$.

Hence, the required probability is $\frac{m}{N} = \frac{3003}{32768}$.

70. $P\left(\frac{S}{T}\right) = \frac{P(S \cap T)}{P(T)}$

$$\Rightarrow 0.5 = \frac{P(S \cap T)}{0.69}$$

$$\Rightarrow P(S \cap T) = 0.5 \times 0.69 = P(S)P(T)$$

Therefore, S and T are independent.

71. $\therefore P(S \text{ and } T) = P(S)P(T) = 0.69 \times 0.5 = 0.345$

72. $P(S \text{ or } T) = P(S) + P(T) - P(S \cap T)$

$$= 0.5 + 0.69 - 0.345 = 0.8450$$

73. The total number of ways of painting first column when colours are not alternating is $2^8 - 2$. The total number of ways when no column has alternating colours is $(2^8 - 2)^8 / 2^{64}$

$$= \left(\frac{2^8 - 2}{2^8}\right)^8 = \left(1 - \frac{1}{2^7}\right)^8$$

74. The number of ways that the square has equal number of red and black squares, is ${}^{64}C_{32}$. Hence, the required probability is $\frac{{}^{64}C_{32}}{2^{64}}$.

75. This is possible only when the columns are alternating red and black. Hence, the required probability is $\frac{2}{2^{64}} = \frac{1}{2^{63}}$.

76. If A got atleast one head during the first 10 rounds, then he will not be drained out at the 11th round.

$\therefore$ Required probability = 0

77. To finish at the 12th round, he must have exactly 1 head in the first 10 rounds and a tail at the 11th and the 12th round.

$\therefore$ Required probability = ${}^{10}C_1 p q^{11}$.

78. Again, he cannot be drained at the 13th round, if A gets atleast 2 heads in the first 10 rounds. To finish at the 14th round, A must get exactly 2 heads in the first 10 rounds and a tail on all the rounds from 11th to 14th.

$\therefore$ Probability = ${}^{10}C_2 p^2 q^{12}$

Hence, required probability

$$= q^{10} = {}^{10}C_1 p q^{11} + {}^{10}C_2 p^2 q^{12}$$

$$= q^{10}(1 + 10pq + 45p^2 q^2)$$

79. A. $\begin{matrix} B & B & B & B & B & B & B & B \\ | & | & | & | & | & | & | & | \\ W & W & W & W & \dots & W & W & W \end{matrix}$

$$n(S) = \frac{10!}{(7!)(3!)}$$

$$n(E) = {}^8C_3 = \frac{8!}{(3!)(5!)}$$

because there are 8 places for 3 black balls.

$$\therefore P(E) = \frac{\frac{(3!)(5!)}{10!}}{\frac{(7!)(3!)}{10!}} = \frac{(8!)(7!)}{(10!)(5!)} = \frac{7 \cdot 6}{10 \cdot 9} = \frac{7}{15}$$

- B. $n(S) = 7!$, $n(E) = (3!) \times (4!)$

[$\because$ after making 4 gentlemen sit in $3!$ ways, 4 ladies can sit in $4!$ ways in between the gentlemen]

$$\therefore P(E) = \frac{(3!) \times (4!)}{7!} = \frac{6}{7 \times 6 \times 5} = \frac{1}{35}$$

- C. $n(S) = {}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$n(E) = n(S)$ - The number of ways in which one boy gets both the pens

$$= n(S) - {}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)$$

$$\therefore P(E) = 1 - \frac{{}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)}{{}^{12}C_4 \times {}^8C_4 \times {}^4C_4} = 1 - \frac{6}{11} = \frac{5}{11}$$

- D. Required probability

$$= P(E E E E \bar{E}) + P(\bar{E} \bar{E} \bar{E} \bar{E} E)$$

$$= \{P(E)\}^4 \cdot P(\bar{E}) + \{P(\bar{E})\}^4 \cdot P(E) = \frac{1}{16}$$

80. A. Number of ways of putting the balls in boxes

$$= \frac{6!}{2!2!2!} + \frac{6!3!}{1!1!2!4!} + \frac{6!3!}{1!2!3!}$$

Number of ways of putting the balls, so that each

$$\text{box gets two balls is } \frac{6!}{2!2!2!}$$

$\therefore$ Required probability = $\frac{1}{6}$

- B. Required probability = $\frac{3^6 - {}^3C_1 2^6 + 3}{3^6} = \frac{20}{27}$
- C. The number of ways B can throw 9 is 4.
The number of ways A can throw greater than 9 is 6.
 $\therefore$ Required probability = $\frac{4 \times 6}{4 \times 36} = \frac{1}{6}$
- D. $P(A \cup B') = P(A) + P(B') - P(A \cap B')$
 $\therefore 0.8 = 0.3 + P(B') - P(A) \cdot P(B')$
 $\Rightarrow 0.5 = P(B')[1 - P(A)]$
 $\Rightarrow P(B') = \frac{0.5}{0.7} = \frac{5}{7}$
 $\therefore P(B) = \frac{2}{7}$

81. A. $P(\text{success}) = \frac{1}{2}$; $P(\text{failure}) = \frac{1}{2}$

Suppose n bombs are to be dropped. Let E be the event that the Bridge is destroyed. Then,
 $P(E) = 1 - P(0 \text{ or } 1 \text{ success})$

$$= 1 - \left[\left(\frac{1}{2} \right)^n + {}^nC_1 \frac{1}{2} \left(\frac{1}{2} \right)^{n-1} \right]$$

$$= 1 - \left(\frac{1}{2^n} + \frac{n}{2^n} \right) \geq 0.9$$

$$\Rightarrow \frac{1}{10} \geq \frac{n+1}{2^n} \quad \text{or} \quad \frac{2^n}{10(n+1)} \geq 1$$

Which is possible when $n=7$ and 10 .

- B. The bag contains 2 red, 3 white and 5 black balls.
Hence, $P(S) = \frac{1}{5}$; $P(F) = \frac{4}{5}$. Let E be the event of getting a red ball.

$$P(E) = P(S \text{ or } FS \text{ or } FFS \text{ or } \dots) \geq \frac{1}{2}$$

$$\therefore P(F)^n \leq \frac{1}{2}; \left(\frac{4}{5} \right)^n \leq \frac{1}{2}$$

The value of n consistent is 4.

- C. Let there be x red socks and y blue socks and $x > y$. Then,

$$\frac{{}^xC_2 + {}^yC_2}{{}^{x+y}C_2} = \frac{1}{2}$$

$$\text{or} \quad \frac{x(x-1) + y(y-1)}{(x+y)(x+y-1)} = \frac{1}{2}$$

On multiplying both sides by $2(x+y)(x+y-1)$ and expanding, we get

$$2x^2 - 2x + 2y^2 - 2y = x^2 + 2xy + y^2 - x - y$$

On rearranging, we have

$$x^2 - 2xy + y^2 = x + y$$

$$\Rightarrow (x-y)^2 = x+y$$

$$\Rightarrow |x-y| = \sqrt{x+y}$$

$$\text{Now, } x+y \leq 17 \Rightarrow x-y \leq \sqrt{17}$$

As $x-y$ must be an integer, so

$$x-y = 4$$

$$\therefore x+y = 16$$

On adding both together dividing by 2 yields $x \leq 10$.

- D. Let the number of green socks be $x > 0$. Let E be the event that two socks drawn are of the same colour.

$$P(E) = P(RR \text{ or } BB \text{ or } WW \text{ or } GG)$$

$$= \frac{3}{6+x} + \frac{{}^xC_2}{{}^{6+x}C_2}$$

$$= \frac{6}{(x+6)(x+5)} + \frac{x(x-1)}{(x+6)(x+5)}$$

$$= \frac{1}{5}$$

$$\Rightarrow 5(x^2 - x + 6) = x^2 + 11x + 30$$

$$\Rightarrow 4x^2 - 16x = 0$$

$$\therefore x = 4$$

82. Let E denotes the event that the target is hit when x shells are fired at point A . Let E_1 and E_2 denote the events that the artillery targets are at points A and B .

$$\text{We have, } P(E_1) = \frac{8}{9}, P(E_2) = \frac{1}{9}$$

$$\Rightarrow P\left(\frac{E}{E_1}\right) = 1 - \left(\frac{1}{2}\right)^x$$

$$\text{and } P\left(\frac{E}{E_2}\right) = 1 - \left(\frac{1}{2}\right)^{21-x}$$

$$\text{Now, } P(E) = \frac{8}{9} \left[1 - \left(\frac{1}{2}\right)^x \right] + \frac{1}{9} \left[1 - \left(\frac{1}{2}\right)^{21-x} \right]$$

$$\Rightarrow \frac{d}{dx} [P(E)] = \frac{8}{9} \left(\frac{1}{2}\right)^x \log 2 + \frac{1}{9} \left[-\left(\frac{1}{2}\right)^{21-x} \log 2 \right]$$

$$\text{Now, we must have } \frac{d}{dx} [P(E)] = 0$$

$$\Rightarrow x = 12, \text{ also } \frac{d^2}{dx^2} [P(E)] < 0$$

$$\text{Hence, } P(E) \text{ is maximum, when } x = 12 = \lambda. \therefore \frac{\lambda}{3} = 4$$

83. $n(S) = 36$

Let E be the event of getting the sum of digits on the die equal to 7, then $n(E) = 6$

$$P(E) = \frac{6}{36} = \frac{1}{6} = p$$

$$\text{and } P(E') = q = \frac{5}{6}$$

Probability of not throwing the sum 7 in first m trials = q^m

$$\therefore P(\text{atleast one 7 in } m \text{ throws}) = 1 - q^m = 1 - \left(\frac{5}{6}\right)^m$$

According to the question,

$$1 - \left(\frac{5}{6}\right)^m > 0.95$$

$$\Rightarrow \left(\frac{5}{6}\right)^m > 0.05$$

$$\Rightarrow m [\log_{10} 5 - \log_{10} 6] < \log_{10} 1 - \log_{10} 20$$

$$\therefore m > 16.44$$

So, the least number of trials = 17

$$\text{Hence, } \lambda = 17$$

$$\therefore \lambda - 10 = 7$$

84. A wins the series, if out of first $n + r$ games, A wins n games, $0 \leq r \leq n$ and wins $(n + r + 1)$ th game.

$$\therefore P(A) = \left(\sum_{r=0}^n {}^{n+r}C_n \cdot q^r \cdot p^n \right) \cdot p$$

[where, $p + q = 1$]

$$\text{Similarly, } P(B) = \sum_{r=0}^n ({}^{n+r}C_n) \cdot q^{n+1} \cdot p^r$$

$$\text{Now, } P(A) + P(B) = 1$$

$$\Rightarrow \sum_{r=0}^n [q^r \cdot p^{n+1} + q^{n+1} \cdot p^r] {}^{n+r}C_n = 1 \quad \dots(i)$$

$$\text{Now, put } p = q = \frac{1}{2}$$

$$\text{From Eq. (i), } \sum_{r=0}^n ({}^{n+r}C_n) \cdot \frac{1}{2^{n+r}} = 1 = \frac{\lambda}{2}$$

$$\Rightarrow \lambda = 2$$

85. Let A and B be the events 'A speaks the truth' and 'B speaks the truth', respectively. Let C be the event 'the number turned up is not 2 but both agree to the same conclusion that the die has turned up 2'. Then,

$$P(A) = \frac{3}{4}, P(B) = \frac{2}{3}$$

$$\text{and } P(C) = \frac{1}{5} \times \frac{1}{5}$$

There are following two hypothesis:

- (i) the die turns up 2.
- (ii) the die does not turns up 2.

Let there be the events E_1 and E_2 respectively, then

$$P(E_1) = \frac{1}{6}, P(E_2) = \frac{5}{6} \quad [\text{priori probabilities}]$$

Now, let E be the event 'the statement made by A and B agree to the same conclusion.'

$$\text{Then, } P\left(\frac{E}{E_1}\right) = P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{2}{3} = \frac{1}{2}$$

$$P\left(\frac{E}{E_2}\right) = P(\bar{A}) \cdot P(\bar{B}) \cdot P(C)$$

$$= \frac{1}{4} \cdot \frac{1}{3} \cdot \frac{1}{5} = \frac{1}{300}$$

$$\text{Thus, } P(E) = P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right)$$

$$= \frac{1}{6} \times \frac{1}{2} + \frac{5}{6} \times \frac{1}{300} = \frac{31}{360}$$

$$\therefore P\left(\frac{E_1}{E}\right) = \frac{P(E_1) \cdot P\left(\frac{E}{E_1}\right)}{P(E)}$$

$$= \frac{30}{31} = \frac{a}{b}$$

$$\text{Hence, } b - a = 31 - 30 = 1$$

86. The number of ways of drawing 7 balls (second draws) = ${}^{10}C_7$

For each set of 7 balls of the second draw, 3 must be common to the set of 5 balls of the first draw, i.e. 2 other balls can be drawn in ${}^3C_2$ ways.

Thus, for each set of 7 balls of the second draw, there are ${}^7C_3 \times {}^3C_2$ ways of making the first draw so that there are 3 balls common.

Hence, the probability of having three balls in common is

$$p = \frac{{}^7C_3 \times {}^3C_2}{{}^{10}C_7}$$

$$= \frac{5}{12}$$

$$\therefore 12p = 5$$

87. Highest number in three throws is 4.

$\Rightarrow$ Atleast one of the throws must be equal to 4.

Number of ways when three blocks are filled from

$$\{1, 2, 3, 4\} = 4^3$$

$\therefore$ Number of ways when filled from $\{1, 2, 3\} = 3^3$

Required number of ways = $4^3 - 3^3$

$$\text{Hence, probability} = \frac{4^3 - 3^3}{6^3}$$

$$\Rightarrow p = \frac{37}{216}$$

$$\Rightarrow \frac{1}{p} = \frac{216}{37} = 5.837$$

$$\therefore \left[\frac{1}{p} \right] = 5$$

88. Let event A : Card is of heart but not of king (12 cards)

Event B : King but not heart (3 cards)

Event C : Heart and king (1 card)

$\therefore$ Required probability,

$$P(E) = \frac{{}^{12}C_1 \cdot {}^3C_1 + {}^3C_1 \cdot {}^1C_1 + {}^1C_1 \cdot {}^{12}C_1}{{}^{52}C_2} = \frac{2}{52}$$

$$\therefore 104p = 4$$

89. Total ways of distribution = $n(S) = 4^5$

Total ways of distribution, so that each child get atleast one game,

$$n(E) = 4^5 - {}^4C_1 3^5 + {}^4C_2 2^5 - {}^4C_3$$

$$= 1024 - 4 \times 243 + 6 \times 32 - 4$$

$$= 240$$

$$\text{Required probability} = \frac{n(E)}{n(S)}$$

$$= \frac{240}{4^5} = \frac{15}{64}$$

$$\therefore \left[\frac{1}{p} \right] = 4$$

1. Total number of ways to arrange 3 boys and 2 girls are 5!

According to the given condition, following cases may arise:

B G G B B
G G B B B
G B G B B
G B B G B
B G B G B

So, number of favourable ways = $5 \times 3! \times 2! = 60$

$$\therefore \text{Required probability} = \frac{60}{120} = \frac{1}{2}$$

2. As $x_1 + x_2 + x_3$ is odd.

So, all may be odd or one of them is odd and other two are even.

$\therefore$ Required probability

$$= \frac{{}^2C_1 \times {}^3C_1 \times {}^4C_1 + {}^1C_1 \times {}^2C_1 \times {}^4C_1 + {}^2C_1 \times {}^2C_1 \times {}^3C_1 + {}^1C_1 \times {}^3C_1 \times {}^3C_1}{{}^3C_1 \times {}^5C_1 \times {}^7C_1}$$

$$= \frac{24 + 8 + 12 + 9}{105} = \frac{53}{105}$$

3. $\therefore x_1, x_2, x_3$ are in AP.

$$\therefore x_1 + x_3 = 2x_2$$

So, $x_1 + x_3$ should be even number.

Either both x_1 and x_3 are odd or both are even.

$$\therefore \text{Required probability} = \frac{{}^2C_1 \times {}^4C_1 + {}^1C_1 \times {}^3C_1}{{}^3C_1 \times {}^5C_1 \times {}^7C_1}$$

$$= \frac{11}{105}$$

4. Probability that the problem is solved correctly by atleast one of them = $1 - P(\text{Problem is not solved by all})$

$\therefore P(\text{problem is solved}) = 1 - P(\text{problem is not solved})$

$$= 1 - P(\bar{A}) \cdot P(\bar{B}) \cdot P(\bar{C}) \cdot P(\bar{D})$$

$$= 1 - \left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{7}{8} \right) = 1 - \frac{21}{256} = \frac{235}{256}$$

5. Required probability = $P(\text{all are white}) + P(\text{all are red}) + P(\text{all are black})$

$$= \frac{1}{6} \times \frac{2}{9} \times \frac{3}{12} + \frac{3}{6} \times \frac{3}{9} \times \frac{4}{12} + \frac{2}{6} \times \frac{4}{9} \times \frac{5}{12}$$

$$= \frac{6}{648} + \frac{36}{648} + \frac{40}{648} = \frac{82}{648}$$

6. Let A : One ball is white and other is red

E_1 : Both balls are from box B_1

E_2 : Both balls are from box B_2

E_3 : Both balls are from box B_3

$\therefore$ Required probability = $P\left(\frac{E_2}{A}\right)$

$$= \frac{P\left(\frac{A}{E_2}\right) \cdot P(E_2)}{P\left(\frac{A}{E_1}\right) \cdot P(E_1) + P\left(\frac{A}{E_2}\right) \cdot P(E_2) + P\left(\frac{A}{E_3}\right) \cdot P(E_3)}$$

$$= \frac{\frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} \times \frac{1}{3}}{\frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} \times \frac{1}{3} + \frac{{}^2C_1 \times {}^3C_1}{{}^9C_2} \times \frac{1}{3} + \frac{{}^3C_1 \times {}^4C_1}{{}^{12}C_2} \times \frac{1}{3}}$$

$$= \frac{\frac{1}{6}}{\frac{1}{5} + \frac{1}{6} + \frac{2}{11}} = \frac{55}{181}$$

7. Sample space = $6 \times 6 \times 6 \times 6 = 6^4$ favourable events

Case I First, we should select one number for D_4 which appears on all, i.e. ${}^6C_1 \times 1$.

Case II For D_4 , there are ${}^6C_1$ ways. Now, it appears on anyone of D_1, D_2, D_3 , i.e. ${}^3C_1 \times 1$.

For other two, there are 5×5 ways.

$$\Rightarrow {}^6C_1 \times {}^3C_1 \times 1 \times 5 \times 5$$

Case III For D_4 , there are ${}^6C_1$ ways. Now, it appears on any two of D_1, D_2, D_3 .

$$\text{i.e. } {}^3C_2 \times 1^2$$

For other one, there are 5 ways.

$$\text{i.e. } {}^6C_1 \times {}^3C_2 \times 1^2 \times 5 \text{ ways}$$

Thus, required probability

$$= \frac{{}^6C_1 + {}^6C_1 \times {}^3C_1 \times 5^2 + {}^6C_1 \times {}^3C_2 \times 5}{6^4}$$

$$= \frac{6(1 + 75 + 15)}{6^4} = \frac{91}{216}$$

8. Given, $P(E_1) = \frac{1}{2}, P(E_2) = \frac{1}{4}, P(E_3) = \frac{1}{4}$

$$\therefore P(X) = \left\{ P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) \right. \\ \left. + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3) \right\}$$

$$= \left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \right) + \left(\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} \right)$$

$$= \frac{1}{4}$$

- (a) $P(X_1^C / X)$

$$= P\left(\frac{X_1^C \cap X}{P(X)}\right) = \frac{P(\bar{E}_1 \cap E_2 \cap E_3)}{P(X)} = \frac{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4}}{\frac{1}{4}} = \frac{1}{8}$$

- (b) P (exactly two engines of the ship are functioning)

$$= \frac{P(E_1 \cap E_2 \cap \bar{E}_3) + P(E_1 \cap \bar{E}_2 \cap E_3) + P(\bar{E}_1 \cap E_2 \cap E_3)}{P(X)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4}}{\frac{1}{4}} = \frac{7}{8}$$

- (c) $P\left(\frac{X}{X_2}\right) = \frac{P(X \cap X_2)}{P(X_2)}$

$$= \frac{P(\text{ship is operating with } E_2 \text{ function})}{P(X_2)}$$

$$= \frac{P(E_1 \cap E_2 \cap \bar{E}_3) + P(\bar{E}_1 \cap E_2 \cap E_3) + P(E_1 \cap E_2 \cap E_3)}{P(E_2)}$$

$$= \frac{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4}}{\frac{1}{4}} = \frac{5}{8}$$

$$(d) P(X/X_1) = \frac{P(X \cap X_1)}{P(X_1)} = \frac{\frac{1}{2} \cdot \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} \cdot \frac{3}{4}}{1/2} = \frac{7}{16}$$

9. Here, $P\left(\frac{X}{Y}\right) = \frac{1}{2}$, $P\left(\frac{Y}{X}\right) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$

$$\therefore P\left(\frac{X}{Y}\right) = \frac{P(X \cap Y)}{P(Y)} \Rightarrow \frac{1}{2} = \frac{1/6}{P(Y)}$$

$$\Rightarrow P(Y) = \frac{1}{3} \quad \dots(i)$$

$$P\left(\frac{Y}{X}\right) = \frac{1}{3}$$

$$\Rightarrow \frac{P(X \cap Y)}{P(X)} = \frac{1}{3}$$

$$\Rightarrow \frac{1}{6} = \frac{1}{3} P(X)$$

$$\therefore P(X) = \frac{1}{2} \quad \dots(ii)$$

Now, $P(X \cup Y) = P(X) + P(Y) - P(X \cap Y)$
 $= \frac{1}{2} + \frac{1}{3} - \frac{1}{6} = \frac{2}{3} \quad \dots(iii)$

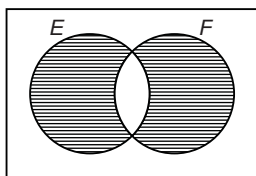
$$P(X \cap Y) = \frac{1}{6}$$

and $P(X) \cdot P(Y) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

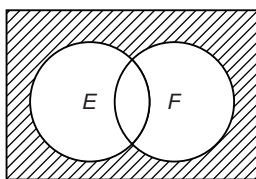
$$\Rightarrow P(X \cap Y) = P(X) \cdot P(Y)$$

$\therefore$ Independent event,
 $P(X^c \cap Y) = P(Y) - P(X \cap Y)$
 $= \frac{1}{3} - \frac{1}{6} = \frac{1}{6}$

10.



$$P(E \cup F) - P(E \cap F) = \frac{11}{25} \quad \dots(i)$$



Neither of them occurs = $\frac{2}{25}$
 $\Rightarrow P(\bar{E} \cap \bar{F}) = \frac{2}{25} \quad \dots(ii)$

From Eq. (i), $P(E) + P(F) - 2P(E \cap F) = \frac{11}{25} \quad \dots(iii)$

From Eq. (ii), $[1 - P(E)][1 - P(F)] = \frac{2}{25}$
 $\Rightarrow 1 - P(E) - P(F) + P(E) \cdot P(F) = \frac{2}{25} \quad \dots(iv)$

From Eqs. (iii) and (iv), we get

$$P(E) + P(F) = \frac{7}{5} \quad \text{and} \quad P(E) \cdot P(F) = \frac{12}{25}$$

$$\therefore P(E) \cdot \left\{ \frac{7}{5} - P(E) \right\} = \frac{12}{25}$$

$$\Rightarrow [P(E)]^2 - \frac{7}{5}P(E) + \frac{12}{25} = 0$$

$$\Rightarrow \left[P(E) - \frac{3}{5} \right] \left[P(E) - \frac{4}{5} \right] = 0$$

$$\therefore P(E) = \frac{3}{5} \quad \text{or} \quad \frac{4}{5}$$

and $P(F) = \frac{4}{5} \quad \text{or} \quad \frac{3}{5}$

11. Now, probability of the drawn ball from U_2 being white is

$$P(\text{white} / U_2) = P(H) \cdot \left\{ \frac{{}^3C_1}{{}^5C_1} \times \frac{{}^2C_1}{{}^2C_1} + \frac{{}^2C_1}{{}^5C_1} \times \frac{{}^1C_1}{{}^2C_1} \right\} + P(T) \left\{ \frac{{}^3C_2}{{}^5C_2} \times \frac{{}^3C_2}{{}^3C_1} + \frac{{}^2C_2}{{}^5C_2} \times \frac{{}^1C_1}{{}^3C_1} + \frac{{}^3C_1 \cdot {}^2C_1}{{}^5C_2} \times \frac{{}^2C_1}{{}^3C_1} \right\}$$

$$P(\text{white} / U_2) = \frac{1}{2} \left\{ \frac{3}{5} \times 1 + \frac{2}{5} \times \frac{1}{2} \right\}$$

$$+ \frac{1}{2} \left\{ \frac{3}{10} \times 1 + \frac{1}{10} \times \frac{1}{3} + \frac{6}{10} \times \frac{2}{3} \right\} = \frac{23}{30}$$

12. P (head appeared/white from U_2)

$$= P(H) \cdot \left\{ \frac{{}^3C_1}{{}^5C_1} \times \frac{{}^2C_1}{{}^2C_1} \times \frac{{}^2C_1}{{}^5C_1} \times \frac{{}^1C_1}{{}^2C_1} \right\}$$

$$= \frac{23/30}{23/30}$$

$$= \frac{1}{2} \cdot \left\{ \frac{3}{5} \times 1 + \frac{2}{5} \times \frac{1}{2} \right\} = \frac{12}{23}$$

13. Sample space A dice is thrown thrice,

$$n(S) = 6 \times 6 \times 6$$

Favourable events $\omega^1 + \omega^2 + \omega^3 = 0$, i.e. (r_1, r_2, r_3)

are ordered 3-triples

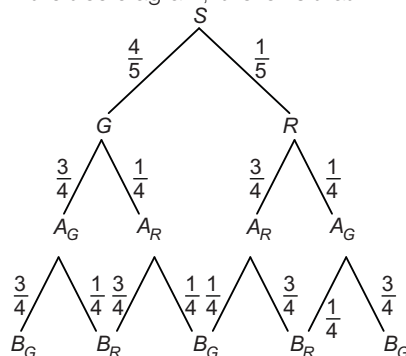
which can take values,

$$\begin{aligned} & (1, 2, 3), (1, 5, 3), (4, 2, 3), (4, 5, 3) \\ & (1, 2, 6), (1, 5, 6), (4, 2, 6), (4, 5, 6) \end{aligned}$$

i.e. 8 ordered pairs and each can be arranged in $3!$ ways = 6

$$\therefore n(E) = 8 \times 6 \Rightarrow P(E) = \frac{8 \times 6}{6 \times 6 \times 6} = \frac{2}{9}$$

14. From the tree diagram, it follows that



$$P(B_G) = \frac{46}{80}$$

$$P(B_G / G) = \frac{10}{16} = \frac{5}{8}$$

$$\therefore P(B_G \cap G) = \frac{5}{8} \times \frac{4}{5} = \frac{1}{2}$$

$$\text{and } P(G / B_G) = \frac{1/2}{P(B_G)} = \frac{1}{2} \times \frac{80}{46} = \frac{20}{23}$$

$$15. \text{ Given, } P(\overline{A \cup B}) = \frac{1}{6}, P(A \cap B) = \frac{1}{4}, P(\overline{A}) = \frac{1}{4}$$

$$\therefore P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{6} = \frac{5}{6}$$

$$\text{and } P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{5}{6} = \frac{3}{4} + P(B) - \frac{1}{4}$$

$$\Rightarrow P(B) = \frac{1}{3}$$

A and B are not equally likely.

$$\therefore P(A \cap B) = P(A) \cdot P(B) = \frac{1}{4}$$

So, events are independent.

$$16. P(A \cup B \cap \overline{C}) = P(A \cup B) \times P(\overline{C})$$

$$= [P(A) + P(B) - P(A \cap B)] \times P(\overline{C})$$

$$= \left(\frac{3}{4} + \frac{1}{2} - \frac{3}{8} \right) \times \frac{3}{8}$$

$$\left[\because P(A \cap B) = P(A) \cdot P(B) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{8} \right]$$

$$= \frac{7}{8} \cdot \frac{3}{8} = \frac{21}{64}$$

$$17. P(A \cap \overline{B}) + P(\overline{A} \cap B) = \frac{26}{49}$$

$$\Rightarrow P(A)P(\overline{B}) + P(\overline{A})P(B) = \frac{26}{49} \quad \dots(i)$$

$$\text{Also, } P(\overline{A} \cap \overline{B}) = \frac{15}{49}$$

$$\therefore P(\overline{A})P(\overline{B}) = \frac{15}{49}$$

$$\text{On solving, } P(A) = \frac{4}{7}, P(B) = \frac{2}{7}$$

$$\therefore P(A) = \frac{4}{7}$$

$$18. \therefore 0 \leq \frac{3x+1}{3} \leq 1, 0 \leq \frac{1-x}{4} \leq 1$$

$$\text{and } 0 \leq \frac{3x+1}{3} + \frac{1-x}{4} \leq 1$$

$$\Rightarrow 0 \leq 3x+1 \leq 3, 0 \leq 1-x \leq 4, 0 \leq \frac{12x+4+3-3x}{12} \leq 1$$

$$\Rightarrow -1 \leq 3x \leq 2, 0 \leq 1-x \leq 4, 0 \leq 9x+7 \leq 12$$

$$\Rightarrow -\frac{1}{3} \leq x \leq \frac{2}{3}, -1 \leq -x \leq 3, -7 \leq 9x \leq 5$$

$$\Rightarrow -\frac{1}{3} \leq x \leq \frac{2}{3}, -3 \leq x \leq 1, -\frac{7}{9} \leq x \leq \frac{5}{9}$$

$$\therefore -\frac{1}{3} \leq x \leq \frac{5}{9}$$

$$19. \text{ Since, probability of guessing a correct answer, } p = \frac{1}{3}$$

$$\text{and probability of guessing a wrong answer, } q = \frac{2}{3}$$

$\therefore$ The probability of guessing 4 or more correct answers

$$= {}^5C_4 \left(\frac{1}{3} \right)^4 \cdot \frac{2}{3} + {}^5C_5 \left(\frac{1}{3} \right)^5$$

$$= 5 \cdot \frac{2}{3^5} + \frac{1}{3^5} = \frac{11}{3^5}$$

20. Let A denotes the event that the minimum of the three selected numbers is 3 and B denotes the event that the maximum of the three selected numbers is 6. If the maximum number is 6, then

(i) 6 should be one of the three selected numbers (1 way)

(ii) The remaining two numbers should be less than 6 (i.e. any 2 from 1 to 5) (${}^5C_2$ ways)

$$\therefore P(B) = \frac{1 \times {}^5C_2}{{}^6C_3} = \frac{10}{8}$$

$$\therefore P(B) = \frac{1 \times {}^5C_2}{{}^6C_3} = \frac{10}{8}$$

Similarly, $P(A \cap B)$ = The probability that the minimum number is 3 and the maximum number is 6. If the minimum number is 3 and the maximum number is 6, then

(a) 6 should be one of the 3 selected numbers (1 way)

(b) 3 should be one of the three selected numbers (1 way)

(c) The remaining 1 number should lie between 3 and 6 (i.e. anyone of 4 and 5) (2 ways)

$$\therefore P(A \cap B) = \frac{1 \times 1 \times {}^2C_1}{{}^6C_3}$$

$$\therefore P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2}{10} = \frac{1}{5}$$

21. We have, $n = 5$ and $r \geq 1$

$$\therefore P(X=r) = {}^nC_r p^r q^{n-r}, P(X \geq 1) = 1 - P(X=0)$$

$$= 1 - {}^5C_0 \cdot p^5 \cdot q^0 \geq \frac{31}{32}$$

$$\Rightarrow p^5 \leq 1 - \frac{31}{32} = \frac{1}{32}$$

$$\therefore p \leq \frac{1}{2} \text{ and } p \geq 0$$

$$\Rightarrow p \in \left[0, \frac{1}{2} \right]$$

$$22. n(S) = {}^{20}C_4$$

Statement I When common difference is 1, then total number of cases = 17

When common difference is 2, then total number of cases = 14

When common difference is 3, then total number of cases = 11

When common difference is 4, then total number of cases = 8

When common difference is 5, then total number of cases = 5

When common difference is 6, then total number of cases = 2

Hence, required probability

$$= \frac{17 + 14 + 11 + 8 + 5 + 2}{{}^{20}C_4} = \frac{57}{4845} = \frac{1}{85}$$

23. According to the given condition,

$$1 - \left(\frac{3}{4}\right)^n \geq \frac{9}{10}$$

$$\Rightarrow \left(\frac{3}{4}\right)^n \leq 1 - \frac{9}{10} = \frac{1}{10}$$

$$\Rightarrow \left(\frac{4}{3}\right)^n \geq 10$$

$$\Rightarrow n(\log_{10} 4 - \log_{10} 3) \geq \log_{10} 10 = 1$$

$$\therefore n \geq \frac{1}{\log_{10} 4 - \log_{10} 3}$$

24. $S = \{00, 01, 02, \dots, 49\}$

Let A be the event that sum of the digits on the selected ticket is 8, then $A = \{08, 17, 26, 35, 44\}$

Let B be the event that the product of the digits is zero.

$$\therefore B = \{00, 01, 02, 03, \dots, 09, 10, 20, 30, 40\}$$

$$\therefore A \cap B = \{08\}$$

$\therefore$ Hence, required probability

$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{50}}{\frac{14}{50}} = \frac{1}{14}$$

25. Given that, $P(A) = \frac{1}{4}$, $P\left(\frac{A}{B}\right) = \frac{1}{2}$

and $P\left(\frac{B}{A}\right) = \frac{2}{3}$

We know that, $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$... (i)

and $P\left(\frac{B}{A}\right) = \frac{P(B \cap A)}{P(A)}$... (ii)

From Eqs. (i) and (ii), we get

$$P(B) = \frac{P\left(\frac{B}{A}\right) \cdot P(A)}{P\left(\frac{A}{B}\right)} = \frac{\left(\frac{2}{3}\right) \cdot \left(\frac{1}{4}\right)}{\left(\frac{1}{2}\right)} = \frac{1}{3}$$

26. Let the events be A = Ist aeroplane hit the target

B = IInd aeroplane hit the target

and their corresponding probabilities are

$$P(A) = 0.3 \quad \text{and} \quad P(B) = 0.2$$

$$\Rightarrow P(\bar{A}) = 0.7 \quad \text{and} \quad P(\bar{B}) = 0.8$$

$$\therefore \text{Required probability} = P(\bar{A})P(B) + P(\bar{A})P(\bar{B})P(\bar{A})P(B) + \dots$$

$$= (0.7)(0.2) + (0.7)(0.8)(0.7)(0.2) + \dots$$

$$= 0.14 [1 + (0.56) + (0.56)^2 + \dots]$$

$$= 0.14 \left(\frac{1}{1 - 0.56} \right) = \frac{0.14}{0.44} = 0.32$$

27. Given, $E = \{X \text{ is a prime number}\} = \{2, 3, 5, 7\}$

$$\therefore P(E) = P(X=2) + P(X=3) + P(X=5) + P(X=7)$$

$$\Rightarrow P(E) = 0.23 + 0.12 + 0.20 + 0.07 = 0.62$$

$$\text{and } F = \{X < 4\} = \{1, 2, 3\}$$

$$\Rightarrow P(F) = P(X=1) + P(X=2) + P(X=3)$$

$$\Rightarrow P(F) = 0.15 + 0.23 + 0.12 = 0.5$$

$$\text{and } E \cap F = \{X \text{ is prime number as well as } < 4\} = \{2, 3\}$$

$$P(E \cap F) = P(X=2) + P(X=3)$$

$$= 0.23 + 0.12$$

$$= 0.35$$

$\therefore$ Required probability,

$$P(E \cup F) = P(E) + P(F) - P(E \cap F)$$

$$= 0.62 + 0.5 - 0.35 = 0.77$$

28. Since, $0 \leq P(A) \leq 1$, $0 \leq P(B) \leq 1$, $0 \leq P(C) \leq 1$

$$\text{and } 0 \leq P(A) + P(B) + P(C) \leq 1$$

$$\therefore 0 \leq \frac{3x+1}{3} \leq 1$$

$$\Rightarrow -\frac{1}{3} \leq x \leq \frac{2}{3} \quad \dots (i)$$

$$0 \leq \frac{1-x}{4} \leq 1$$

$$\Rightarrow -3 \leq x \leq 1 \quad \dots (ii)$$

$$0 \leq \frac{1-2x}{2} \leq 1$$

$$\Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2} \quad \dots (iii)$$

$$\text{and } 0 \leq \frac{3x+1}{3} + \frac{1-x}{4} + \frac{1-2x}{2} \leq 1$$

$$\Rightarrow 0 \leq 13 - 3x \leq 12$$

$$\Rightarrow \frac{1}{3} \leq x \leq \frac{13}{3} \quad \dots (iv)$$

$$\text{From Eqs. (i), (ii), (iii) and (iv), } \frac{1}{3} \leq x \leq \frac{1}{2}$$

29. Since, the probabilities of solving the problem by A , B and C are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively.

$\therefore$ Probability that the problem is not solved

$$= P(\bar{A})P(\bar{B})P(\bar{C})$$

$$= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)$$

$$= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4}$$

Hence, the probability that the problem is solved

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

30. Since, A and B are two mutually exclusive events.

$$\therefore A \cap B = \phi$$

$$\Rightarrow \text{Either } A \subseteq \bar{B} \quad \text{or} \quad B \subseteq \bar{A}$$

$$\Rightarrow P(A) \leq P(\bar{B}) \quad \text{or} \quad P(B) \leq P(\bar{A})$$

31. Let A_1 , A_2 and A_3 be the events of match winning in first, second and third matches, respectively, whose probabilities are

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{2}$$

$\therefore$ Required probability

$$= P(A_1 A_2 A_3) + P(\bar{A}_1 A_2 A_3)$$

$$= P(A_1)P(\bar{A}_2)P(A_3) + P(\bar{A}_1)P(A_2)P(A_3)$$

$$= \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^3$$

$$= \frac{1}{8} + \frac{1}{8} = \frac{1}{4}$$

32. Let the probability of getting a head is p and not getting a head is q .

Since, head appears first time in an even throw 2 or 4 or 6...

$$\begin{aligned}\therefore \frac{2}{5} &= qp + q^3p + q^5p + \dots \\ \Rightarrow \frac{2}{5} &= \frac{qp}{1-q^2} \\ \Rightarrow \frac{2}{5} &= \frac{(1-p)p}{1-(1-p)^2} \quad [\because q = 1-p] \\ \Rightarrow \frac{2}{5} &= \frac{1-p}{2-p} \\ \Rightarrow 4-2p &= 5-5p \\ \Rightarrow p &= \frac{1}{3}\end{aligned}$$

33. Probability of getting success, $p = \frac{1}{6}$ and probability of failure, $q = \frac{5}{6}$

$$\begin{aligned}\therefore \text{Required probability} &= {}^7C_2 \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^5 \times \frac{1}{6} \\ &= \frac{{}^7C_2 \times 5^5}{6^8}\end{aligned}$$

34. Since, $P(\text{exactly one of } A, B \text{ occurs}) = q$ [given]

$$\begin{aligned}\therefore P(A \cup B) - P(A \cap B) &= q \\ \Rightarrow p - P(A \cap B) &= q \\ \Rightarrow P(A \cap B) &= p - q \\ \Rightarrow 1 - P(A' \cup B') &= p - q \\ \Rightarrow P(A' \cup B') &= 1 - p + q \\ \Rightarrow P(A') + P(B') - P(A' \cap B') &= 1 + q - p \\ \Rightarrow P(A') + P(B') &= (1 - p + q) + [1 - P(A \cup B)] \\ &= (1 - p + q) + (1 - p) = 2 - 2p + q\end{aligned}$$

35. Total number of cases $= {}^{50}C_1 = 50$

Let A be the event of selecting ticket with sum of digits 8.

Favourable cases to A are $\{08, 17, 26, 35, 44\}$.

Let B be the event of selecting ticket with product of its digits 7.

Favourable cases to B is only $\{17\}$.

$$\text{Now, } P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{1/50}{5/50} = \frac{1}{5}$$

36. Given, $P(A) \neq 0$

(i) If A is subset of B .

Then, $P(A \cap B) = P(A)$

$$\Rightarrow P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{P(A)}{P(A)} = 1$$

(ii) If $A \cap B = \phi$, then $P(A \cap B) = 0$

$$\therefore P\left(\frac{B}{A}\right) = \frac{P(A \cap B)}{P(A)} = \frac{0}{P(A)} = 0$$

37. Given, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$$\begin{aligned}\therefore P(A \cup B) &= 1 - P(A \cap B) \\ &= 1 - [P(A) + P(B) - P(A)P(B)] \\ &= 1[1 - P(A)] - P(B)[1 - P(A)] \\ &= [1 - P(A)][1 - P(B)] \\ &= [1 - P(A)]P(B')\end{aligned}$$

38. Given, $P(A') = \frac{2}{3}$ and $P(B') = \frac{2}{7}$

Since, events A and B are independent.

$$\begin{aligned}\therefore P(A \cap B) &= P(A) \cdot P(B) = [1 - P(A')][1 - P(B')] \\ &= \left(1 - \frac{2}{3}\right)\left(1 - \frac{2}{7}\right) = \frac{1}{3} \times \frac{5}{7} = \frac{5}{21}\end{aligned}$$

39. Let E_1 = Student does not know the answer

E_2 = Student knows the answer

and E = Student answer correctly

$$\begin{aligned}\therefore P(E_1) &= 1 - p \\ \Rightarrow P(E_2) &= p \\ \Rightarrow P\left(\frac{E}{E_2}\right) &= 1 \quad \text{and} \quad P\left(\frac{E}{E_1}\right) = \frac{1}{5}\end{aligned}$$

$$\begin{aligned}\therefore P\left(\frac{E_2}{E}\right) &= \frac{P(E_2)P\left(\frac{E}{E_2}\right)}{P(E_1)P\left(\frac{E}{E_1}\right) + P(E_2)P\left(\frac{E}{E_2}\right)} \\ &= \frac{p(1)}{(1-p)\frac{1}{5} + p(1)} = \frac{p}{1-p+5p} = \frac{5p}{1+4p}\end{aligned}$$

40. Given, $P(B) = \frac{3}{2}P(A)$ and $P(C) = \frac{1}{2}P(B)$

Since, A, B and C are exclusive events.

$$\begin{aligned}\therefore P(A) + P(B) + P(C) &= 1 \\ \Rightarrow P(A) + \frac{3}{2}P(A) + \frac{1}{2} \times \frac{3}{2}P(A) &= 1 \\ \Rightarrow P(A)\left(1 + \frac{3}{2} + \frac{3}{4}\right) &= 1 \\ \Rightarrow \frac{13}{4}P(A) &= 1 \\ \Rightarrow P(A) &= \frac{4}{13} \\ \Rightarrow P(C) &= \frac{1}{2} \times \frac{3}{2}P(A) \\ &= \frac{3}{4} \times \frac{4}{13} = \frac{3}{13}\end{aligned}$$

Also, A, B and C are mutually exclusive.

$$\begin{aligned}\therefore P(A \cap B) &= P(B \cap C) = P(C \cap A) = 0 \\ \Rightarrow P(A \cup C) &= P(A) + P(C) - 0 \\ &= \frac{4}{13} + \frac{3}{13} \\ &= \frac{7}{13}\end{aligned}$$

41. Required probability = Probability of passing two tests

+ Probability of passing all three tests

$= P(\text{passing Ist and IInd tests and fail in the IIIRD test})$

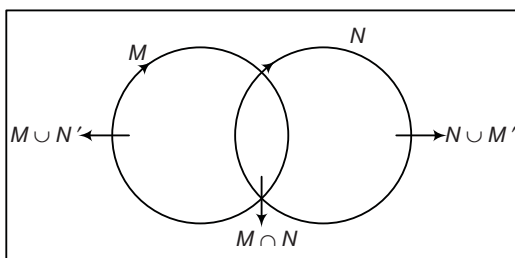
$+ P(\text{passing Ist test, fail in IInd test and passing in IIIRD test})$

$+ P(\text{fail in Ist test, passing in IInd and IIIRD tests})$

$+ P(\text{passing in all three tests})$

$$\begin{aligned}&= ppq + pq \times \frac{p}{2} + q \times \frac{p}{2} \times p + ppp \\ &= p^2(1-p) + p(1-p)\frac{p}{2} + (1-p)\frac{p^2}{2} + p^3 \\ &= p^2\left(1-p + \frac{1}{2} - \frac{p}{2} + \frac{1}{2} - \frac{p}{2} + p\right) \\ &= p^2(2-p)\end{aligned}$$

42. We have two events M and N .



$$\begin{aligned} \text{Clearly, } P(\text{exactly one of them occurs}) &= P(M - N) + P(N - M) \\ &= P(M) - P(M \cap N) + P(N) - P(M \cap N) \\ &= P(M) + P(N) - 2P(M \cap N) \end{aligned}$$

None of the options is correct.

43. Given, $P(A \cup B) = 0.83$, $P(A) = 0.3$ and $P(B) = 0.6$

$$\begin{aligned} P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\ \Rightarrow 0.83 &= 0.3 + 0.6 - P(A \cap B) \end{aligned}$$

$$\Rightarrow P(A \cap B) = 0.9 - 0.83 = 0.07$$

$$\text{Now, } P(A) \cdot P(B) = 0.3 \times 0.6 = 0.18$$

$$\therefore P(A) \cdot P(B) \neq P(A \cap B)$$

Hence, events are dependent.

44. Let X denotes the sum of the numbers obtained, when two fair dice are rolled. So, X may have values 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 and 12.

$$P(X=2) = P\{(1,1)\} = \frac{1}{36}$$

$$P(X=3) = P\{(1,2), (2,1)\} = \frac{2}{36}$$

$$P(X=4) = \frac{3}{36}; P(X=5) = \frac{4}{36}$$

$$P(X=6) = \frac{5}{36}; P(X=7) = \frac{6}{36}; P(X=8) = \frac{5}{36}$$

$$P(X=9) = \frac{4}{36}; P(X=10) = \frac{3}{36}; P(X=11) = \frac{2}{36}$$

$$\text{and } P(X=12) = \frac{1}{36}$$

$\therefore$ Probability distribution table is given below:

X	2	3	4	5	6	7	8	9	10	11	12
$P(X)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Mean, $\bar{X} = \sum XP(X)$

$$\begin{aligned} &= \frac{2 \times 1 + 3 \times 2 + 4 \times 3 + 5 \times 4 + 6 \times 5 + 7 \times 6}{36} \\ &= \frac{174}{36} = 4.83 \end{aligned}$$

$$= \frac{252}{36} = 7$$

Variance $= \sum X^2 P(X) - (\bar{X})^2$

$$\begin{aligned} &= \frac{2^2 \times 1 + 3^2 \times 2 + 4^2 \times 3 + 5^2 \times 4 + 6^2 \times 5 + 7^2 \times 6}{36} \\ &= \frac{1974}{36} - 49 = \frac{1974 - 1764}{36} \end{aligned}$$

$$\begin{aligned} &= \frac{210}{36} = \frac{35}{6} \end{aligned}$$

$$\begin{aligned} \therefore \text{Variance} &= \frac{35}{6} \\ \text{Hence, SD} &= \sqrt{\frac{35}{6}} \end{aligned}$$

45. Let the probability of success and failure be p and q , respectively.

$$\text{Then, } p = 2q \text{ and } q + p = 1$$

$$\Rightarrow 3q = 1 \Rightarrow q = \frac{1}{3}$$

$$\therefore p = \frac{2}{3}$$

Hence, required probability

$$\begin{aligned} &= {}^6C_4 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^2 + {}^6C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right) + {}^6C_6 \left(\frac{2}{3}\right)^6 \\ &= \frac{496}{729} \end{aligned}$$

46. Let probability of defective bulb, $p = \frac{10}{100} = \frac{1}{10} = 0.1$

and probability of non-defective bulb, $q = 1 - 0.1 = 0.9$

Here, $n = 5$

$$\begin{aligned} \therefore P(\text{none is defective}) &= P(X=0) = {}^5C_0 (0.1)^0 (0.9)^5 \\ &= 1 \times (0.9)^5 = \left(\frac{9}{10}\right)^5 \end{aligned}$$

47. Given mean of Binomial variable, $np = 8$

and variance of Binomial variable, $npq = 4$

$$\therefore q = \frac{1}{2} \text{ and } p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

$$\text{and } n \left(\frac{1}{2}\right) = 8 \Rightarrow n = 16$$

$$\therefore P(X < 3) = P(X=0) + P(X=1) + P(X=2)$$

$$\begin{aligned} &= {}^{16}C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^{16-0} + {}^{16}C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{16-1} \\ &\quad + {}^{16}C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{16-2} \end{aligned}$$

$$= 1 \left(\frac{1}{2}\right)^{16} + 16 \left(\frac{1}{2}\right)^{16} + 120 \left(\frac{1}{2}\right)^{16}$$

$$= \frac{137}{2^{16}}$$

48. Given distribution is

X	1	2	3	4	5
$P(X)$	k	$2k$	$3k$	$2k$	k

$\therefore$ Variance

$$\begin{aligned} &= \sum x_i^2 p - (\sum x_i p)^2 = (1k + 8k + 27k + 32k + 25k) \\ &\quad - (k + 4k + 9k + 8k + 5k)^2 \end{aligned}$$

$$\begin{aligned} &= (93k) - (27k)^2 = \left(93 \times \frac{1}{9}\right) - \left(27 \times \frac{1}{9}\right)^2 \\ &\quad \left[\because \sum p = 1 \therefore k = \frac{1}{9} \right] \end{aligned}$$

$$= \frac{93}{9} - 9 = \frac{93 - 81}{9} = \frac{12}{9} = \frac{4}{3}$$

49. Given mean, $np = 4$

and variance, $npq = 2$

... (i)

... (ii)

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{npq}{np} = \frac{2}{4}$$

$$\Rightarrow q = \frac{1}{2}$$

$$\therefore p = \frac{1}{2}$$

$$\text{From Eq. (i), } n = \frac{4}{1/2} = 8$$

$$\text{Now, } P(X=1) = {}^nC_1 p^1 q^{n-1}$$

$$= {}^8C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^7$$

$$= \frac{8!}{1!7!} \times \frac{1}{2^8} = \frac{8}{2^8} = \frac{1}{32}$$

50. Let S denotes the success of getting a 6 and F denotes the failure of not getting a 6.

$$\text{Thus, } P(S) = \frac{1}{6}, P(F) = \frac{5}{6}$$

$\therefore P(A \text{ win}) = P(S) + P(FFS) + P(FFFS) + \dots$ so on.

$$= \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{5}{6} \cdot \frac{1}{6} + \dots$$

$$= \frac{1}{6} + \left(\frac{5}{6}\right)^2 \cdot \frac{1}{6} + \left(\frac{5}{6}\right)^4 \cdot \frac{1}{6} + \dots = \frac{\frac{1}{6}}{1 - \frac{25}{36}} = \frac{6}{11}$$

51. Given, $P(A^C) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^C) = 0.5$

$$\therefore P(A^C) = 0.3$$

$$\Rightarrow P(A) = 1 - P(A^C) = 0.7$$

$$\text{and } P(B) = 0.4$$

$$\Rightarrow P(B^C) = 1 - P(B) = 0.6$$

$$\text{Consider, } P(A \cap B^C) = P(A) - P(A \cap B)$$

$$\Rightarrow 0.5 = 0.7 - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = 0.2$$

$$\text{Now, } P\left[\frac{B}{(A \cup B^C)}\right] = \frac{P[B \cap (A \cup B^C)]}{P(A \cup B^C)}$$

$$= \frac{P[(B \cap A) \cup (B \cap B^C)]}{P(A) + P(B^C) - P(A \cap B^C)}$$

$$= \frac{P(B \cap A)}{0.7 + 0.6 - 0.5} = \frac{0.2}{0.8} = \frac{1}{4}$$

52. Since, $B \subset A$

$$\therefore P(A \cap B^C) = P(A) - P(B)$$

$$\text{and } P(B) \leq P(A)$$

53. We have, $f(x) = \frac{k}{2^x}$; $x = 0, 1, 2, 3, 4$

Since, $f(x)$ is a probability distribution of a random variable. Therefore we have,

$$\sum_{x=0}^4 f(x) = 1$$

$$\Rightarrow \sum_{x=0}^4 \left(\frac{k}{2^x}\right) = 1$$

$$\Rightarrow k \sum_{x=0}^4 \frac{1}{2^x} = 1$$

$$\Rightarrow k \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4}\right) = 1$$

$$\Rightarrow k \left(\frac{16 + 8 + 4 + 2 + 1}{2^4}\right) = 1$$

$$\Rightarrow k \left(\frac{31}{16}\right) = 1$$

$$\therefore k = \frac{16}{31}$$

54. $\therefore$ Sum of probabilities = 1

$$\Rightarrow p + 2p + 3p + 4p + 5p + 7p + 8p + 9p + 10p + 11p + 12p = 1$$

$$\Rightarrow 72p = 1$$

$$\therefore p = \frac{1}{72}$$

SOLVED PAPER 2015

JEE Advanced

Paper ①

Section 1 (Maximum Marks : 16)

- This section contains four questions.
- The answer to each question is a single digit integer ranging from 0 to 9 both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- **Marking scheme**
+ 4 If the bubble corresponding to the answer is darkened.
0 In all other cases.

1. The minimum number of times a fair coin needs to be tossed, so that the probability of getting atleast two heads is atleast 0.96, is

2. Let $f: R \rightarrow R$ be a function defined by $f(x) = \begin{cases} [x], & x \leq 2 \\ 0, & x > 2 \end{cases}$ where $[x]$ denotes the greatest integer less than or equal to x .

If $I = \int_{-1}^2 \frac{xf(x^2)}{2+f(x+1)} dx$, then the value of $(4I - 1)$ is

3. A cylindrical container is to be made from certain solid material with the following constraints : It has a fixed inner volume of $V \text{ mm}^3$, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular

disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum, when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is

4. Let $F(x) = \int_x^{x^2 + \frac{\pi}{6}} 2\cos^2 t \, dt$ for all $x \in R$ and $f: \left[0, \frac{1}{2}\right] \rightarrow [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

Section 2 (Maximum Marks : 36)

- This section contains nine questions.
- Each question has four options (a), (b), (c) and (d). One or more than one of these four options are correct.
- For each question, darken the bubble(s) corresponding to all the correct options in the ORS.
- **Marking scheme**
+ 4 If only the bubble(s) corresponding to all the correct options are darkened.
0 If none of the bubbles is darkened.
-2 In all the other cases.

5. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statement(s) is/are true?

- (a) $y(-4) = 0$
(b) $y(-2) = 0$
(c) $y(x)$ has a critical point in the interval $(-1, 0)$
(d) $y(x)$ has no critical point in the interval $(-1, 0)$

6. Consider the family of all circles whose centres lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are the functions of x, y and y' (here, $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$), then which of the following statement(s) is/are true?

- (a) $P = y + x$ (b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$ (d) $P - Q = x + y - y' - (y')^2$

1078 JEE Advanced Solved Paper 2015

7. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$.

$$\text{Let } f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

and $h(x) = e^{|x|}$ for all $x \in \mathbb{R}$. Let $(f \circ h)(x)$ denotes $f\{h(x)\}$ and $(h \circ f)(x)$ denotes $h\{f(x)\}$. Then, which of the following is/are true?

- (a) f is differentiable at $x = 0$ (b) h is differentiable at $x = 0$
(c) $f \circ h$ is differentiable at $x = 0$ (d) $h \circ f$ is differentiable at $x = 0$

8. Let $f(x) = \sin \left[\frac{\pi}{6} \sin \left(\frac{\pi}{2} \sin x \right) \right]$ for all $x \in \mathbb{R}$ and

$$g(x) = \frac{\pi}{2} \sin x \text{ for all } x \in \mathbb{R}. \text{ Let } (f \circ g)(x) \text{ denotes } f\{g(x)\}$$

and $(g \circ f)(x)$ denotes $g\{f(x)\}$. Then, which of the following is/are true?

- (a) Range of f is $\left[-\frac{1}{2}, \frac{1}{2} \right]$
(b) Range of $f \circ g$ is $\left[-\frac{1}{2}, \frac{1}{2} \right]$
(c) $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$
(d) There is an $x \in \mathbb{R}$ such that $(g \circ f)(x) = 1$

9. Let ΔPQR be a triangle. Let $\mathbf{a} = \mathbf{QR}$, $\mathbf{b} = \mathbf{RP}$ and $\mathbf{c} = \mathbf{PQ}$. If $|\mathbf{a}| = 12$, $|\mathbf{b}| = 4\sqrt{3}$ and $\mathbf{b} \cdot \mathbf{c} = 24$, then which of the following is/are true?

- (a) $\frac{|\mathbf{c}|^2}{2} - |\mathbf{a}| = 12$
(b) $\frac{|\mathbf{c}|^2}{2} + |\mathbf{a}| = 30$
(c) $|\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| = 48\sqrt{3}$
(d) $\mathbf{a} \cdot \mathbf{b} = -72$

10. Let X and Y be two arbitrary, 3×3 , non-zero, skew-symmetric matrices and Z be an arbitrary, 3×3 , non-zero, symmetric matrix. Then, which of the following matrices is/are skew-symmetric?

- (a) $Y^3 Z^4 - Z^4 Y^3$
(b) $X^{44} + Y^{44}$
(c) $X^4 Z^3 - Z^3 X^4$
(d) $X^{23} + Y^{23}$

11. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha?$$

- (a) -4 (b) 9
(c) -9 (d) 4

12. In $\mathbb{R}^3$, consider the planes $P_1: y = 0$ and $P_2: x + z = 1$. Let P_3 be a plane, different from P_1 and P_2 , which passes through the intersection of P_1 and P_2 . If the distance of the point $(0, 1, 0)$ from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relation(s) is/are true?

- (a) $2\alpha + \beta + 2\gamma + 2 = 0$ (b) $2\alpha - \beta + 2\gamma + 4 = 0$
(c) $2\alpha + \beta - 2\gamma - 10 = 0$ (d) $2\alpha - \beta + 2\gamma - 8 = 0$

13. In $\mathbb{R}^3$, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1: x + 2y - z + 1 = 0$ and $P_2: 2x - y + z - 1 = 0$. Let M be the locus of the foot of the perpendiculars drawn from the points on L to the plane P_1 . Which of the following point(s) lie(s) on M ?

- (a) $\left(0, -\frac{5}{6}, -\frac{2}{3} \right)$ (b) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6} \right)$
(c) $\left(-\frac{5}{6}, 0, \frac{1}{6} \right)$ (d) $\left(-\frac{1}{3}, 0, \frac{2}{3} \right)$

Section 3 (Maximum Marks : 16)

- This section contains two questions.
- Each question contains two columns, Column I and Column II.
- Column I has four entries (A), (B), (C) and (D).
- Column II has five entries (P), (Q), (R), (S) and (T).
- Match the entries in Column I with the entries in Column II.
- One or more entries in Column I may match with one or more entries in Column II.
- The ORS contains a 4×5 matrix whose layout will be similar to the one shown below:

(A)	(P)	(Q)	(R)	(S)	(T)
(B)	(P)	(Q)	(R)	(S)	(T)
(C)	(P)	(Q)	(R)	(S)	(T)
(D)	(P)	(Q)	(R)	(S)	(T)

- For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D).

Marking scheme

For each entry in Column I

+2 If only the bubble(s) corresponding to all the correct matches are darkened.

0 If none of the bubbles is darkened.

−1 In all other cases.

14.

	Column I		Column II
A.	In R^2 , if the magnitude of the projection vector of the vector $\alpha \hat{i} + \beta \hat{j}$ on $\sqrt{3} \hat{i} + \hat{j}$ is $\sqrt{3}$ and if $\alpha = 2 + \sqrt{3} \beta$, then possible value(s) of $ \alpha $ is/are	P.	1
B.	Let a and b be real numbers such that the function $f(x) = \begin{cases} -3ax^2 - 2, & x < 1 \\ bx + a^2, & x \geq 1 \end{cases}$ is differentiable for all $x \in R$. Then, possible value(s) of a is/are	Q.	2
C.	Let $\omega (\neq 1)$ be a complex cube root of unity. If $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$, then the possible value(s) of n is/are	R.	3
D.	Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is in arithmetic progression, then the value(s) of $ q - 2a $ is/are	S.	4
		T.	5

15.

	Column I		Column II
A.	In ΔXYZ , let a, b and c be the lengths of the sides opposite to the angles X, Y and Z , respectively. If $2(a^2 - b^2) = c^2$ and $\lambda = \frac{\sin(X - Y)}{\sin Z}$, then possible value(s) of n for which $\cos(n\pi\lambda) = 0$, is/are	P.	1
B.	In ΔXYZ , let a, b and c be the lengths of the sides opposite to the angles X, Y and Z , respectively. If $1 + \cos 2X - 2\cos 2Y = 2\sin X \sin Y$, then possible value(s) of $\frac{a}{b}$ is/are	Q.	2
C.	In R^2 , let $\sqrt{3} \hat{i} + \hat{j}$, $\hat{i} + \sqrt{3} \hat{j}$ and $\beta \hat{i} + (1 - \beta) \hat{j}$ be the position vectors of X, Y and Z with respect to the origin O , respectively. If the distance of Z from the bisector of the acute angle of $\mathbf{OX}$ with $\mathbf{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $ \beta $ is/are	R.	3
D.	Suppose that $F(\alpha)$ denotes the area of the region bounded by $x = 0$, $x = 2$, $y^2 = 4x$ and $y = \alpha x - 1 + \alpha x - 2 + \alpha x$, where $\alpha \in \{0, 1\}$. Then, the value(s) of $F(\alpha) + \frac{8}{3}\sqrt{2}$, when $\alpha = 0$ and $\alpha = 1$, is/are	S.	5
		T.	6

Paper 2

Section 1 (Maximum Marks : 16)

- This section contains four questions.
- The answer to each question is a single digit integer ranging from 0 to 9 both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- Marking scheme**
 - + 4 If the bubble corresponding to the answer is darkened.
 - 0 In all other cases.

1. Let m and n be two positive integers greater than 1. If

$$\lim_{\alpha \rightarrow 0} \left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m} \right) = -\left(\frac{e}{2}\right), \text{ then the value of } \frac{m}{n} \text{ is}$$

2. If $\alpha = \int_0^1 (e^{9x+3 \tan^{-1} x}) \left(\frac{12+9x^2}{1+x^2} \right) dx$, where $\tan^{-1} x$ takes

only principal values, then the value of $\left(\log_e |1 + \alpha| - \frac{3\pi}{4} \right)$ is

3. Let $f: R \rightarrow R$ be a continuous odd function, which vanishes exactly at one point and $f(1) = \frac{1}{2}$. Suppose that

$$F(x) = \int_{-1}^x f(t) dt \quad \text{for all } x \in [-1, 2] \quad \text{and}$$

$$G(x) = \int_{-1}^x t |f(t)| dt \quad \text{for all } x \in [-1, 2]. \quad \text{If}$$

$$\lim_{x \rightarrow 1} \frac{F(x)}{G(x)} = \frac{1}{14}, \text{ then the value of } f\left(\frac{1}{2}\right) \text{ is}$$

4. Suppose that $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ are three non-coplanar vectors in R^3 . Let the components of a vector $\mathbf{s}$ along $\mathbf{p}, \mathbf{q}$ and $\mathbf{r}$ be 4, 3 and 5, respectively. If the components of this vector $\mathbf{s}$ along $(-\mathbf{p} + \mathbf{q} + \mathbf{r})$, $(\mathbf{p} - \mathbf{q} + \mathbf{r})$ and $(-\mathbf{p} - \mathbf{q} + \mathbf{r})$ are x, y and z respectively, then the value of $2x + y + z$ is

Section 2 (Maximum Marks : 16)

- This section contains four questions.
- Each question has four options (a), (b), (c) and (d). One or more than one of these four options are correct.
- For each question, darken the bubble(s) corresponding to all the correct options in the ORS.
- Marking scheme**
 - + 4 If only the bubble(s) corresponding to all the correct options are darkened.
 - 0 If none of the bubbles is darkened.
 - 2 In all other cases.

5. Let $f, g: [-1, 2] \rightarrow R$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points $-1, 0$ and 2 be as given in the following table:

	$x = -1$	$x = 0$	$x = 2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$, the function $(f - 3g)''$ never vanishes. Then, the correct statement(s) is/are

- (a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$
- (b) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$
- (c) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$
- (d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$
6. Let $f(x) = 7 \tan^8 x + 7 \tan^6 x - 3 \tan^4 x - 3 \tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then, the correct expression(s) is/are

$$(a) \int_0^{\pi/4} x f(x) dx = \frac{1}{12}$$

$$(b) \int_0^{\pi/4} f(x) dx = 0$$

$$(c) \int_0^{\pi/4} x f(x) dx = \frac{1}{6}$$

$$(d) \int_0^{\pi/4} f(x) dx = 1$$

7. Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$. If $m \leq \int_{1/2}^1 f(x) dx \leq M$, then the possible values of m and M are

$$(a) m = 13, M = 24$$

$$(b) m = 2.6, M = 3.9$$

$$(c) m = -11, M = 0$$

$$(d) m = 1, M = 12$$

8. The option(s) with the values of a and L that satisfy the

$$\text{equation } \frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^4 at) dt}{\int_0^{\pi} e^t (\sin^6 at + \cos^4 at) dt} = L, \text{ is/are}$$

$$(a) a = 2, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

$$(b) a = 2, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$$

$$(c) a = 4, L = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

$$(d) a = 4, L = \frac{e^{4\pi} + 1}{e^{\pi} + 1}$$

Section 3 (Maximum Marks : 32)

- This section contains two paragraphs.
- Based on each paragraph, there will be two questions
- Each question has four options (a), (b), (c) and (d). One or more than one of these four options are correct.
- For each question, darken the bubbles(s) corresponding to all the correct options in the ORS.

Marking scheme

- +4 If only the bubble(s) corresponding to all the correct options are darkened.
- 0 If none of the bubbles is darkened.
- 2 In all other cases.

Paragraph 1

Let $F : R \rightarrow R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1, 3)$. Let $f(x) = xF(x)$ for all $x \in R$.

9. The correct statement(s) is/are
- $f'(1) < 0$
 - $f(2) < 0$
 - $f'(x) \neq 0$ for any $x \in (1, 3)$
 - $f'(x) = 0$ for some $x \in (1, 3)$
10. If $\int_1^3 x^2 F'(x) dx = -12$ and $\int_1^3 x^3 F''(x) dx = 40$, then the correct expression(s) is/are
- $9f'(3) + f'(1) - 32 = 0$
 - $\int_1^3 f(x) dx = 12$
 - $9f'(3) - f'(1) + 32 = 0$
 - $\int_1^3 f(x) dx = -12$

Paragraph 2

Let n_1 and n_2 be the number of red and black balls, respectively in box I. Let n_3 and n_4 be the number of red and black balls, respectively in box II.

11. One of the two boxes, box I and box II was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II, is $\frac{1}{3}$, then the correct option(s)

with the possible values of n_1, n_2, n_3 and n_4 is/are

- $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$
- $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$
- $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$
- $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

12. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is $\frac{1}{3}$, then the correct option(s) with the

possible values of n_1 and n_2 is/are

- $n_1 = 4$ and $n_2 = 6$
- $n_1 = 2$ and $n_2 = 3$
- $n_1 = 10$ and $n_2 = 20$
- $n_1 = 3$ and $n_2 = 6$

Answer with Explanations

Paper 1

1. (8) Using Binomial distribution,

$$\begin{aligned} P(X \geq 2) &= 1 - P(X=0) - P(X=1) \\ &= 1 - \left(\frac{1}{2}\right)^n - \left[{}^nC_1 \cdot \left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^{n-1} \right] \\ &= 1 - \frac{1}{2^n} - {}^nC_1 \cdot \frac{1}{2^n} = 1 - \left(\frac{1+n}{2^n}\right) \end{aligned}$$

Given, $P(X \geq 2) \geq 0.96$

$$\therefore 1 - \frac{(n+1)}{2^n} \geq \frac{24}{25} \Rightarrow \frac{n+1}{2^n} \leq \frac{1}{25}$$

$$\therefore n = 8$$

2. (0) Here, $f(x) = \begin{cases} [x], & x \leq 2 \\ 0, & x > 2 \end{cases}$

$$\begin{aligned} \therefore I &= \int_{-1}^2 \frac{x f(x^2)}{2 + f(x+1)} dx \\ &= \int_{-1}^0 \frac{x f(x^2)}{2 + f(x+1)} dx + \int_0^1 \frac{x f(x^2)}{2 + f(x+1)} dx \\ &\quad + \int_1^{\sqrt{2}} \frac{x f(x^2)}{2 + f(x+1)} dx + \int_{\sqrt{2}}^{\sqrt{3}} \frac{x f(x^2)}{2 + f(x+1)} dx \\ &\quad + \int_{\sqrt{3}}^2 \frac{x f(x^2)}{2 + f(x+1)} dx \\ &= \int_{-1}^0 0 dx + \int_0^1 0 dx + \int_1^{\sqrt{2}} \frac{x \cdot 1}{2+0} dx + \int_{\sqrt{2}}^{\sqrt{3}} 0 dx + \int_{\sqrt{3}}^2 0 dx \end{aligned}$$

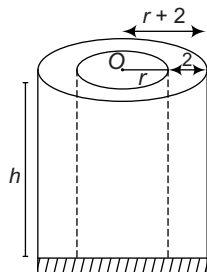
$$\left[\begin{array}{l} \because -1 < x < 0 \Rightarrow 0 < x^2 < 1 \Rightarrow [x^2] = 0, \\ 0 < x < 1 \Rightarrow 0 < x^2 < 1 \Rightarrow [x^2] = 0, \\ 1 < x < \sqrt{2} \Rightarrow \begin{cases} 1 < x^2 < 2 \Rightarrow [x^2] = 1 \\ 2 < x+1 < 1+\sqrt{2} \Rightarrow f(x+1) = 0, \end{cases} \\ \sqrt{2} < x < \sqrt{3} \Rightarrow 2 < x^2 < 3 \Rightarrow f(x^2) = 0, \\ \text{and } \sqrt{3} < x < 2 \Rightarrow 3 < x^2 < 4 \Rightarrow f(x^2) = 0 \end{array} \right]$$

$$\Rightarrow I = \int_1^{\sqrt{2}} \frac{x}{2} dx = \left[\frac{x^2}{4} \right]_1^{\sqrt{2}} = \frac{1}{4} (2-1) = \frac{1}{4}$$

$$\therefore 4I = 1$$

$$\Rightarrow 4I - 1 = 0$$

3. (4) Here, volume of cylindrical container, $V = \pi r^2 h$... (i)
and let volume of the material used be T .



$$\therefore T = \pi [(r+2)^2 - r^2] h + \pi (r+2)^2 \times 2$$

$$\Rightarrow T = \pi [(r+2)^2 - r^2] \cdot \frac{V}{\pi r^2} + 2\pi (r+2)^2$$

$[\because V = \pi r^2 h \Rightarrow h = \frac{V}{\pi r^2}]$

$$\Rightarrow T = V \left(\frac{r+2}{r} \right)^2 + 2\pi (r+2)^2 - V$$

On differentiating w.r.t. r , we get

$$\frac{dT}{dr} = 2V \cdot \left(\frac{r+2}{r} \right) \cdot \left(\frac{-2}{r^2} \right) + 4\pi (r+2)$$

$$\text{At } r = 10, \frac{dT}{dr} = 0$$

$$\text{Now, } 0 = (r+2) \cdot 4 \left(\pi - \frac{V}{r^3} \right)$$

$$\Rightarrow \frac{V}{r^3} = \pi, \text{ where } r = 10$$

$$\Rightarrow \frac{V}{1000} = \pi \text{ or } \frac{V}{250\pi} = 4$$

4. (3) Since, $F'(a) + 2$ is the area bounded by $x=0$, $y=0$, $y=f(x)$ and $x=a$.

$$\therefore \int_0^a f(x) dx = F'(a) + 2$$

Using Newton-Leibnitz formula,

$$f(a) = F''(a) \text{ and } f(0) = F''(0) \quad \dots (i)$$

$$\text{Given, } F(x) = \int_x^{x^2 + \pi/6} 2 \cos^2 t dt$$

On differentiating,

$$F'(x) = 2 \cos^2 \left(x^2 + \frac{\pi}{6} \right) \cdot 2x - 2 \cos^2 x \cdot 1$$

Again differentiating,

$$\begin{aligned} F''(x) &= 4 \left\{ \cos^2 \left(x^2 + \frac{\pi}{6} \right) - 2x \cos \left(x^2 + \frac{\pi}{6} \right) \sin \left(x^2 + \frac{\pi}{6} \right) 2x \right\} \\ &\quad + \{ 4 \cos x \cdot \sin x \} \\ &= 4 \left\{ \cos^2 \left(x^2 + \frac{\pi}{6} \right) - 4x^2 \cos \left(x^2 + \frac{\pi}{6} \right) \sin \left(x^2 + \frac{\pi}{6} \right) \right\} \\ &\quad + 2 \sin 2x \end{aligned}$$

$$\therefore F''(0) = 4 \left\{ \cos^2 \left(\frac{\pi}{6} \right) \right\} = 3$$

$$\therefore f(0) = 3$$

5. (a, c) Here, $(1 + e^x)y' + ye^x = 1$

$$\Rightarrow \frac{dy}{dx} + e^x \cdot \frac{dy}{dx} + ye^x = 1$$

$$\Rightarrow dy + e^x dy + ye^x dx = dx$$

$$\Rightarrow dy + d(e^x y) = dx$$

On integrating both sides, we get

$$y + e^x y = x + C$$

Given, $y(0) = 2$
 $\Rightarrow 2 + e^0 \cdot 2 = 0 + C \Rightarrow C = 4$
 $\therefore y(1 + e^x) = x + 4 \Rightarrow y = \frac{x + 4}{1 + e^x}$

Now at $x = -4$, $y = \frac{-4 + 4}{1 + e^{-4}} = 0$

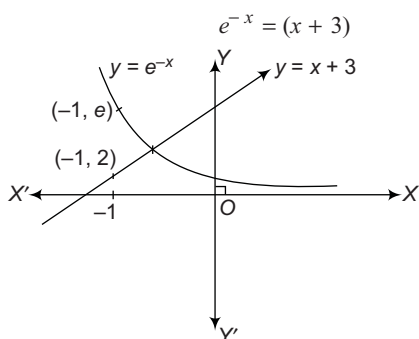
$\therefore y(-4) = 0$

For critical points, $\frac{dy}{dx} = 0$

i.e. $\frac{dy}{dx} = \frac{(1 + e^x) \cdot 1 - (x + 4)e^x}{(1 + e^x)^2} = 0$

$\Rightarrow e^x(x + 3) - 1 = 0$

or



Clearly, the intersection point lies between $(-1, 0)$.
 $\therefore y(x)$ has a critical point in the interval $(-1, 0)$.

6. (b, c) Since, centre lies on $y = x$.

$\therefore$ Equation of circle is

$$x^2 + y^2 - 2ax - 2ay + c = 0$$

On differentiating, we get

$$2x + 2yy' - 2a - 2ay' = 0$$

$$\Rightarrow x + yy' - a - ay' = 0$$

$$\Rightarrow a = \frac{x + yy'}{1 + y'}$$

Again differentiating, we get

$$0 = \frac{(1 + y')[1 + yy'' + (y')^2] - (x + yy')(y'')}{(1 + y')^2}$$

$$\Rightarrow (1 + y')[1 + (y')^2 + yy''] - (x + yy')(y'') = 0$$

$$\Rightarrow 1 + y'[(y')^2 + y' + 1] + y''(y - x) = 0$$

On comparing with $Py'' + Qy' + 1 = 0$, we get

$$P = y - x \text{ and } Q = (y')^2 + y' + 1$$

7. (a, d)

(a) Here, $f(x) = \begin{cases} g(x), & x > 0 \\ 0, & x = 0 \\ -g(x), & x < 0 \end{cases}$

$$f'(x) = \begin{cases} g'(x), & x \geq 0 \\ -g'(x), & x < 0 \end{cases}$$

$\therefore$ Option (a) is correct.

(b) $h(x) = e^{|x|} = \begin{cases} e^x, & x \geq 0 \\ e^{-x}, & x < 0 \end{cases} \Rightarrow h'(x) = \begin{cases} e^x, & x \geq 0 \\ -e^{-x}, & x < 0 \end{cases}$

$$\Rightarrow h'(0^+) = 1 \text{ and } h'(0^-) = -1$$

So, $h(x)$ is not differentiable at $x = 0$.

$\therefore$ Option (b) is not correct.

(c) $(f \circ h)(x) = f\{h(x)\}$, as $h(x) > 0$

$$= \begin{cases} g(e^x), & x \geq 0 \\ g(e^{-x}), & x < 0 \end{cases}$$

$$\Rightarrow (f \circ h)'(x) = \begin{cases} e^x g'(e^x), & x \geq 0 \\ -e^{-x} g'(e^{-x}), & x < 0 \end{cases}$$

$$\Rightarrow (f \circ h)'(0^+) = g'(1), (f \circ h)'(0^-) = -g'(1)$$

So, $(f \circ h)(x)$ is not differentiable at $x = 0$.

$\therefore$ Option (c) is not correct.

(d) $(h \circ f)(x) = e^{|f(x)|} = \begin{cases} e^{g(x)}, & x \neq 0 \\ e^0 = 1, & x = 0 \end{cases}$

$$\begin{aligned} \text{Now, } (h \circ f)'(0) &= \lim_{h \rightarrow 0} \frac{e^{g(x)} - 1}{x} = \lim_{h \rightarrow 0} \frac{e^{g(x)} - 1}{|g(x)|} \cdot \frac{|g(x)|}{x} \\ &= \lim_{h \rightarrow 0} \frac{e^{g(x)} - 1}{|g(x)|} \cdot \lim_{h \rightarrow 0} \frac{|g(x) - 0|}{|x|} \cdot \lim_{h \rightarrow 0} \frac{|x|}{x} \\ &= 1 \cdot g'(0) \cdot \lim_{h \rightarrow 0} \frac{|x|}{x} = 0, \text{ as } g'(0) = 0 \end{aligned}$$

$\therefore$ Option (d) is correct.

8. (a, b, c)

(a) $f(x) = \sin \left[\frac{\pi}{6} \sin \left(\frac{\pi}{2} \sin x \right) \right], x \in \mathbb{R}$
 $= \sin \left(\frac{\pi}{6} \sin \theta \right), \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right], \text{ where } \theta = \frac{\pi}{2} \sin x$
 $= \sin \alpha, \alpha \in \left[-\frac{\pi}{6}, \frac{\pi}{6} \right], \text{ where } \alpha = \frac{\pi}{6} \sin \theta$

$$\therefore f(x) \in \left[-\frac{1}{2}, \frac{1}{2} \right]$$

Hence, range of $f(x) \in \left[-\frac{1}{2}, \frac{1}{2} \right]$

So, option (a) is correct.

(b) $f\{g(x)\} = f(t), t \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \Rightarrow f(t) \in \left[-\frac{1}{2}, \frac{1}{2} \right]$

$\therefore$ Option (b) is correct.

(c) $\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \lim_{x \rightarrow 0} \frac{\sin \left[\frac{\pi}{6} \sin \left(\frac{\pi}{2} \sin x \right) \right]}{\frac{\pi}{2} (\sin x)}$
 $= \lim_{x \rightarrow 0} \frac{\sin \left[\frac{\pi}{6} \sin \left(\frac{\pi}{2} \sin x \right) \right]}{\frac{\pi}{6} \sin \left(\frac{\pi}{2} \sin x \right)} \cdot \frac{\frac{\pi}{6} \sin \left(\frac{\pi}{2} \sin x \right)}{\left(\frac{\pi}{2} \sin x \right)}$
 $= 1 \times \frac{\pi}{6} \times 1 = \frac{\pi}{6}$

$\therefore$ Option (c) is correct.

1084 JEE Advanced Solved Paper 2015

(d) $g\{f(x)\} = 1$

$$\Rightarrow \frac{\pi}{2} \sin \{f(x)\} = 1$$

$$\Rightarrow \sin \{f(x)\} = \frac{2}{\pi} \quad \dots(i)$$

But $f(x) \in \left[-\frac{1}{2}, \frac{1}{2}\right] \subset \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

$$\therefore \sin \{f(x)\} \in \left[-\frac{1}{2}, \frac{1}{2}\right] \quad \dots(ii)$$

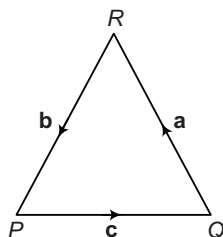
$$\Rightarrow \sin \{f(x)\} \neq \frac{2}{\pi} \quad [\text{from Eqs. (i) and (ii)}]$$

i.e. No solution.

$\therefore$ Option (d) is not correct.

9. (a, c, d) Given, $|\mathbf{a}| = 12$, $|\mathbf{b}| = 4\sqrt{3}$

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$$



$$\Rightarrow \mathbf{a} = -(\mathbf{b} + \mathbf{c})$$

We have, $|\mathbf{a}|^2 = |\mathbf{b} + \mathbf{c}|^2$

$$\Rightarrow |\mathbf{a}|^2 = |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2\mathbf{b} \cdot \mathbf{c}$$

$$\Rightarrow 144 = 48 + |\mathbf{c}|^2 + 48$$

$$\Rightarrow |\mathbf{c}|^2 = 48$$

$$\Rightarrow |\mathbf{c}| = 4\sqrt{3}$$

Also, $|\mathbf{c}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2 + 2\mathbf{a} \cdot \mathbf{b}$

$$\Rightarrow 48 = 144 + 48 + 2\mathbf{a} \cdot \mathbf{b}$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} = -72$$

$\therefore$ Option (d) is correct.

Also, $\mathbf{a} \times \mathbf{b} = \mathbf{c} \times \mathbf{a}$

$$\Rightarrow \mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a} = 2\mathbf{a} \times \mathbf{b}$$

$$\begin{aligned} \Rightarrow |\mathbf{a} \times \mathbf{b} + \mathbf{c} \times \mathbf{a}| &= 2|\mathbf{a} \times \mathbf{b}| \\ &= 2\sqrt{|\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a} \cdot \mathbf{b})^2} \\ &= 2\sqrt{(144)(48) - (-72)^2} \\ &= 2(12)\sqrt{48 - 36} \\ &= 48\sqrt{3} \end{aligned}$$

$\therefore$ Option (c) is correct.

Also, $\frac{|\mathbf{c}|^2}{2} - |\mathbf{a}| = 24 - 12 = 12$

$\therefore$ Option (a) is correct.

and $\frac{|\mathbf{c}|^2}{2} + |\mathbf{a}| = 24 + 12 = 36$

$\therefore$ Option (b) is not correct.

10. (c, d) Given, $X^T = -X$, $Y^T = -Y$, $Z^T = Z$

(a) Let $P = Y^3 Z^4 - Z^4 Y^3$

$$\begin{aligned} \text{Then, } P^T &= (Y^3 Z^4)^T - (Z^4 Y^3)^T \\ &= (Z^T)^4 (Y^T)^3 - (Y^T)^3 (Z^T)^4 \\ &= -Z^4 Y^3 + Y^3 Z^4 = P \end{aligned}$$

$\therefore$ P is symmetric matrix.

(b) Let $P = X^{44} + Y^{44}$

$$\text{Then, } P^T = (X^T)^{44} + (Y^T)^{44} = X^{44} + Y^{44} = P$$

$\therefore$ P is symmetric matrix.

(c) Let $P = X^4 Z^3 - Z^3 X^4$

$$\begin{aligned} \text{Then, } P^T &= (X^4 Z^3)^T - (Z^3 X^4)^T \\ &= (Z^T)^3 (X^T)^4 - (X^T)^4 (Z^T)^3 \\ &= Z^3 X^4 - X^4 Z^3 = -P \end{aligned}$$

$\therefore$ P is skew-symmetric matrix.

(d) Let $P = X^{23} + Y^{23}$

$$\text{Then, } P^T = (X^T)^{23} + (Y^T)^{23} = -X^{23} - Y^{23} = -P$$

$\therefore$ P is skew-symmetric matrix.

11. (b, c) Given determinant could be expressed as product of two determinants.

$$\text{i.e. } \begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \begin{vmatrix} 1+2\alpha+\alpha^2 & 1+4\alpha+4\alpha^2 & 1+6\alpha+9\alpha^2 \\ 4+4\alpha+\alpha^2 & 4+8\alpha+4\alpha^2 & 4+12\alpha+9\alpha^2 \\ 9+6\alpha+\alpha^2 & 9+12\alpha+4\alpha^2 & 9+18\alpha+9\alpha^2 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \begin{vmatrix} 1 & \alpha & \alpha^2 \\ 4 & 2\alpha & \alpha^2 \\ 9 & 3\alpha & \alpha^2 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \cdot \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$$\Rightarrow -8\alpha^3 = -648\alpha$$

$$\Rightarrow \alpha^3 - 81\alpha = 0$$

$$\Rightarrow \alpha(\alpha^2 - 81) = 0$$

$$\therefore \alpha = 0, \pm 9$$

12. (b, d) Here, $P_3 : (x+z-1) + \lambda y = 0$

i.e. $P_3 : x + \lambda y + z - 1 = 0$

whose distance from $(0, 1, 0)$ is 1.

$$\therefore \frac{|0 + \lambda + 0 - 1|}{\sqrt{1 + \lambda^2 + 1}} = 1$$

$$\Rightarrow |\lambda - 1| = \sqrt{\lambda^2 + 2}$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 = \lambda^2 + 2$$

...(i)

$$\Rightarrow \lambda = -\frac{1}{2}$$

$\therefore$ Equation of P_3 is $2x - y + 2z - 2 = 0$.

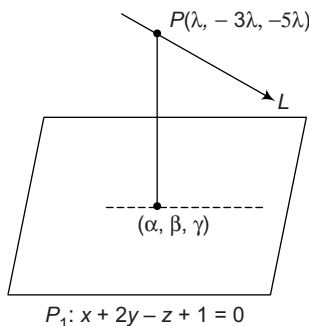
$\therefore$ Distance from (α, β, γ) is 2.

$$\therefore \frac{|2\alpha - \beta + 2\gamma - 2|}{\sqrt{4 + 1 + 4}} = 2$$

$$\Rightarrow 2\alpha - \beta + 2\gamma - 2 = \pm 6$$

$$\Rightarrow 2\alpha - \beta + 2\gamma = 8 \text{ and } 2\alpha - \beta + 2\gamma = -4$$

13. (a, b) Since, L is at constant distance from two planes P_1 and P_2 .



Therefore, L is parallel to the line through intersection of P_1 and P_2 .

$$\begin{aligned} \therefore \text{DR's of } L &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{vmatrix} \\ &= \hat{i}(2-1) - \hat{j}(1+2) + \hat{k}(-1-4) \\ &= \hat{i} - 3\hat{j} - 5\hat{k} \end{aligned}$$

$\therefore$ DR's of L are $(1, -3, -5)$ passing through $(0, 0, 0)$.

Now, equation of L is

$$\frac{x-0}{1} = \frac{y-0}{-3} = \frac{z-0}{-5}$$

$$\text{For any point on } L, \frac{x}{1} = \frac{y}{-3} = \frac{z}{-5} = \lambda \quad [\text{say}]$$

$$\text{i.e. } P(\lambda, -3\lambda, -5\lambda)$$

If (α, β, γ) is foot of perpendicular from P on P_1 , then

$$\frac{\alpha - \lambda}{1} = \frac{\beta + 3\lambda}{2} = \frac{\gamma + 5\lambda}{-1} = k \quad [\text{say}]$$

$$\Rightarrow \alpha = \lambda + k, \beta = 2k - 3\lambda, \gamma = -k - 5\lambda$$

which satisfy $P_1: x + 2y - z + 1 = 0$

$$\Rightarrow (\lambda + k) + 2(2k - 3\lambda) - (-k - 5\lambda) + 1 = 0$$

$$\Rightarrow k = -\frac{1}{6}$$

$$\therefore x = -\frac{1}{6} + \lambda, y = -\frac{1}{3} - 3\lambda, z = \frac{1}{6} - 5\lambda$$

which satisfy options (a) and (b).

14. $A \rightarrow P, Q; B \rightarrow P, Q; C \rightarrow P, Q, S, T; D \rightarrow Q, T$

A. Projection of $(\alpha\hat{i} + \beta\hat{j})$ on $(\sqrt{3}\hat{i} + \hat{j}) = \sqrt{3}$

$$\Rightarrow \frac{|\sqrt{3}\alpha + \beta|}{2} = \sqrt{3}$$

$$\Rightarrow \sqrt{3}\alpha + \beta = \pm 2\sqrt{3} \quad \dots(i)$$

$$\text{and } \alpha - \sqrt{3}\beta = 2 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\alpha = 2, -1$$

$$\therefore |\alpha| = 2, 1$$

B. Since, if any function is differentiable, then it is continuous for all x .

By continuity at $x = 1$,

$$-3a - 2 = b + a^2 \quad \dots(i)$$

$$\text{By differentiability, } f'(x) = \begin{cases} -6ax, & x < 1 \\ b, & x > 1 \end{cases}$$

$$\therefore -6a = b \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$-3a - 2 = -6a + a^2$$

$$\Rightarrow a^2 - 3a + 2 = 0$$

$$\Rightarrow a = 1, 2$$

$$C. (3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3}$$

$$+ (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$$

$$\Rightarrow \{(-3 + 2\omega + 3\omega^2)\omega\}^{4n+3} + \{-3 + 2\omega + 3\omega^2\}\omega^2\}^{4n+3}$$

$$+ \{(-3 + 2\omega + 3\omega^2)\}^{4n+3} = 0$$

$$\Rightarrow (-3 + 2\omega + 3\omega^2)^{4n+3} \{\omega^{4n+3} + (\omega^2)^{4n+3} + 1\} = 0$$

$$\Rightarrow \omega^n + \omega^{2n} + 1 = 0$$

which is true only when n is not a multiple of 3.

$$\therefore n = 1, 2, 4, 5$$

$$D. \text{ Here, } \frac{2ab}{a+b} = 4$$

$$\text{and } 2(5-a) = b-5$$

$$\Rightarrow b = 15 - 2a \quad \dots(i)$$

$$\text{Now, } 2ab = 4(a+b)$$

$$\Rightarrow 2a(15-2a) = 4(a+15-2a)$$

$$\Rightarrow 15a - 2a^2 = 30 - 2a$$

$$\Rightarrow 2a^2 - 17a + 30 = 0$$

$$\Rightarrow a = 5/2, 6$$

$$\text{Hence, } |q - 2a| = |10 - 2a| = 5 \text{ or } 2$$

15. $A \rightarrow P, R, S; B \rightarrow P; C \rightarrow P, Q; D \rightarrow S, T$

$$A. \text{ Since, } 2(a^2 - b^2) = c^2$$

$$\therefore 2(\sin^2 x - \sin^2 y) = \sin^2 z \quad [\text{using sine law}]$$

$$\Rightarrow 2\sin(x-y) \cdot \sin(x+y) = \sin^2 z$$

$$\Rightarrow 2\sin(x-y) \cdot \sin z = \sin^2 z \quad [\because x+y+z=\pi]$$

$$\Rightarrow \frac{\sin(x-y)}{\sin z} = \frac{1}{2} = \lambda$$

$$\text{Also, } \cos(n\pi\lambda) = 0$$

$$\Rightarrow \cos\left(\frac{n\pi}{2}\right) = 0$$

$$\therefore n = 1, 3, 5$$

1086 JEE Advanced Solved Paper 2015

B. $1 + \cos 2X - 2 \cos 2Y = 2 \sin X \cdot \sin Y$

$$\Rightarrow 1 + 1 - 2 \sin^2 X - 2(1 - 2 \sin^2 Y) = 2 \sin X \cdot \sin Y$$

$$\Rightarrow -2a^2 + 4b^2 = 2ab$$

$$\Rightarrow a^2 + ab - 2b^2 = 0$$

$$\Rightarrow \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 2 = 0$$

$$\Rightarrow \left(\frac{a}{b} + 2\right)\left(\frac{a}{b} - 1\right) = 0$$

$$\Rightarrow \frac{a}{b} = 1, -2$$

C. $\mathbf{OX} = \sqrt{3}\hat{i} + \hat{j}, \mathbf{OY} = \hat{i} + \sqrt{3}\hat{j}, \mathbf{OZ} = \beta\hat{i} + (1-\beta)\hat{j}$

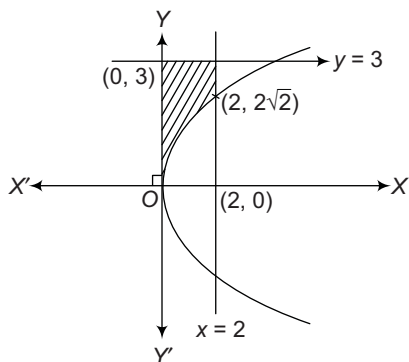
Angle bisector of $\mathbf{OX}$ and $\mathbf{OY}$ is along the line $y = x$ and its distance from $(\beta, 1-\beta)$ is

$$\frac{|\beta - (1-\beta)|}{\sqrt{2}} = \frac{3}{\sqrt{2}}$$

$$\Rightarrow 2\beta - 1 = \pm 3$$

$$\therefore \beta = 2, -1 \text{ or } |\beta| = 1, 2$$

D. Area bounded by $x = 0, x = 2, y^2 = 4x$



When $\alpha = 0$, then

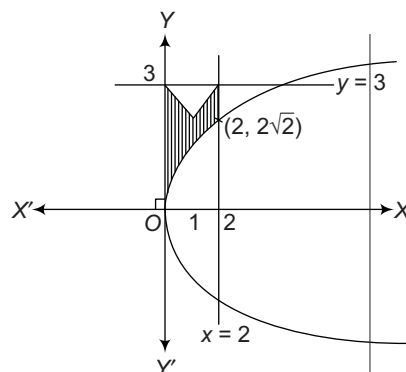
Area bounded, $F(\alpha) = 2 \times 3 - \int_0^2 \sqrt{4x} \, dx$

$$= 6 - 2 \left[\frac{x^{3/2}}{3/2} \right]_0^2 = 6 - \frac{4}{3} \cdot (2^{3/2} - 0) = 6 - \frac{8}{3}\sqrt{2}$$

$$\therefore F(\alpha) + \frac{8\sqrt{2}}{3} = 6 \quad \dots(i)$$

When $\alpha = 1$, then

$$y = |x-1| + |x-2| + x = \begin{cases} 3x-3, & x \geq 2 \\ x+1, & 1 < x < 2 \\ 3-x, & x \leq 1 \end{cases}$$



$\therefore$ Area bounded,

$$F(\alpha) = (6-1) - \int_0^2 2\sqrt{x} \, dx \Rightarrow F(\alpha) = 5 - \frac{8}{3}\sqrt{2}$$

$$\therefore F(\alpha) + \frac{8}{3}\sqrt{2} = 5 \quad \dots(ii)$$

Hence, values of $F(\alpha) + \frac{8}{3}\sqrt{2}$ are 5 and 6

[from Eqs. (i) and (ii)]

Paper 2

$$\begin{aligned}
 1. (2) \text{ Given, } \lim_{\alpha \rightarrow 0} \left[\frac{e^{\cos(\alpha^n)} - e}{\alpha^m} \right] &= -\frac{e}{2} \\
 \Rightarrow \lim_{\alpha \rightarrow 0} \frac{e^{\cos(\alpha^n)} - 1}{\cos(\alpha^n) - 1} \cdot \frac{\cos(\alpha^n) - 1}{\alpha^m} &= -\frac{e}{2} \\
 \Rightarrow \lim_{\alpha \rightarrow 0} e \left\{ \frac{e^{\cos(\alpha^n)} - 1}{\cos(\alpha^n) - 1} \right\} \cdot \lim_{\alpha \rightarrow 0} \frac{-2 \sin^2 \frac{\alpha^n}{2}}{\alpha^m} &= -e/2 \\
 \Rightarrow e \times 1 \times (-2) \lim_{\alpha \rightarrow 0} \frac{\sin^2 \left(\frac{\alpha^n}{2} \right)}{\frac{\alpha^{2n}}{4}} \cdot \frac{\alpha^{2n}}{4 \alpha^m} &= -\frac{e}{2} \\
 \Rightarrow e \times 1 \times (-2) \times 1 \times \lim_{\alpha \rightarrow 0} \frac{\alpha^{2n-m}}{4} &= -\frac{e}{2}
 \end{aligned}$$

For this to be exists,

$$2n - m = 0 \Rightarrow \frac{m}{n} = 2$$

$$2. (9) \text{ Here, } \alpha = \int_0^1 e^{(9x+3 \tan^{-1} x)} \left(\frac{12+9x^2}{1+x^2} \right) dx$$

$$\text{Put } 9x + 3 \tan^{-1} x = t$$

$$\Rightarrow \left(9 + \frac{3}{1+x^2} \right) dx = dt$$

$$\therefore \alpha = \int_0^{9+3\pi/4} e^t dt = [e^t]_0^{9+3\pi/4} = e^{9+3\pi/4} - 1$$

$$\Rightarrow \log_e |1 + \alpha| = 9 + \frac{3\pi}{4}$$

$$\Rightarrow \log_e |\alpha + 1| - \frac{3\pi}{4} = 9$$

$$3. (7) \text{ Here, } \lim_{x \rightarrow 1} \frac{F(x)}{G(x)} = \frac{1}{14}$$

$$\Rightarrow \lim_{x \rightarrow 1} \frac{F'(x)}{G'(x)} = \frac{1}{14} \quad [\text{using L' Hospital's rule}] \dots (i)$$

$$\text{As } F(x) = \int_{-1}^x f(t) dt \Rightarrow F'(x) = f(x) \dots (ii)$$

$$\text{and } G(x) = \int_{-1}^x t |f\{f(t)\}| dt$$

$$\Rightarrow G'(x) = x |f\{f(x)\}| \dots (iii)$$

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 1} \frac{F(x)}{G(x)} &= \lim_{x \rightarrow 1} \frac{F'(x)}{G'(x)} = \lim_{x \rightarrow 1} \frac{f(x)}{x |f\{f(x)\}|} \\
 &= \frac{f(1)}{1 |f\{f(1)\}|} = \frac{1/2}{|f(1/2)|} \dots (iv)
 \end{aligned}$$

$$\text{Given, } \lim_{x \rightarrow 1} \frac{F(x)}{G(x)} = \frac{1}{14}$$

$$\therefore \frac{\frac{1}{2}}{\left| f\left(\frac{1}{2}\right) \right|} = \frac{1}{14}$$

$$\Rightarrow \left| f\left(\frac{1}{2}\right) \right| = 7$$

$$4. (9) \text{ Here, } s = 4p + 3q + 5r \dots (i)$$

$$\text{and } s = (-p + q + r)x + (p - q + r)y + (-p - q + r)z \dots (ii)$$

$$\therefore 4p + 3q + 5r = p(-x + y - z) + q(x - y - z) + r(x + y + z)$$

On comparing both sides, we get

$$-x + y - z = 4, x - y - z = 3 \text{ and } x + y + z = 5$$

On solving above equations, we get

$$x = 4, y = \frac{9}{2}, z = -\frac{7}{2}$$

$$\therefore 2x + y + z = 8 + \frac{9}{2} - \frac{7}{2} = 9$$

$$5. (b, c) \text{ Let } F(x) = f(x) - 3g(x)$$

$$\therefore F(-1) = 3, F(0) = 3$$

$$\text{and } F(2) = 3$$

So, $F'(x)$ will vanish atleast twice in $(-1, 0) \cup (0, 2)$.

$\therefore F''(x) > 0$ or < 0 , $\forall x \in (-1, 0) \cup (0, 2)$

Hence, $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$ and one solution in $(0, 2)$.

$$6. (a, b) \text{ Here, } f(x) = 7 \tan^8 x + 7 \tan^6 x - 3 \tan^4 x - 3 \tan^2 x$$

$$\text{for all } x \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$$

$$\begin{aligned}
 \therefore f(x) &= 7 \tan^6 x \sec^2 x - 3 \tan^2 x \sec^2 x \\
 &= (7 \tan^6 x - 3 \tan^2 x) \sec^2 x
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \int_0^{\pi/4} x f(x) dx &= \int_0^{\pi/4} x (7 \tan^6 x - 3 \tan^2 x) \sec^2 x dx \\
 &= [x (\tan^7 x - \tan^3 x)]_0^{\pi/4} \\
 &\quad - \int_0^{\pi/4} 1 (\tan^7 x - \tan^3 x) dx \\
 &= 0 - \int_0^{\pi/4} \tan^3 x (\tan^4 x - 1) dx \\
 &= - \int_0^{\pi/4} \tan^3 x (\tan^2 x - 1) \sec^2 x dx
 \end{aligned}$$

$$\text{Put } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$\begin{aligned}
 \therefore \int_0^{\pi/4} x f(x) dx &= - \int_0^1 t^3 (t^2 - 1) dt \\
 &= - \int_0^1 (t^5 - t^3) dt = \left[-\frac{t^6}{6} + \frac{t^4}{4} \right]_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, } \int_0^{\pi/4} f(x) dx &= \int_0^{\pi/4} (7 \tan^6 x - 3 \tan^2 x) \sec^2 x dx \\
 &= \int_0^1 (7t^6 - 3t^2) dt = [t^7 - t^3]_0^1 = 0
 \end{aligned}$$

$$7. (b) \text{ Here, } f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$$

$$\therefore \frac{192x^3}{3} \leq f'(x) \leq \frac{192x^3}{2}$$

On integrating between the limits $\frac{1}{2}$ to x , we get

$$\int_{1/2}^x \frac{192x^3}{3} dx \leq \int_{1/2}^x f'(x) dx \leq \int_{1/2}^x \frac{192x^3}{2} dx$$

1088 JEE Advanced Solved Paper 2015

$$\Rightarrow \frac{192}{12} \left(x^4 - \frac{1}{16} \right) \leq f(x) - f\left(\frac{1}{2}\right) \leq 24x^4 - \frac{3}{2}$$

$$\Rightarrow 16x^4 - 1 \leq f(x) \leq 24x^4 - \frac{3}{2}$$

Again integrating between the limits $\frac{1}{2}$ to 1, we get

$$\int_{1/2}^1 (16x^4 - 1) dx \leq \int_{1/2}^1 f(x) dx \leq \int_{1/2}^1 \left(24x^4 - \frac{3}{2} \right) dx$$

$$\Rightarrow \left[\frac{16x^5}{5} - x \right]_{1/2}^1 \leq \int_{1/2}^1 f(x) dx \leq \left[\frac{24x^5}{5} - \frac{3}{2}x \right]_{1/2}^1$$

$$\Rightarrow \left(\frac{11}{5} + \frac{2}{5} \right) \leq \int_{1/2}^1 f(x) dx \leq \left(\frac{33}{10} + \frac{6}{10} \right)$$

$$\Rightarrow 2.6 \leq \int_{1/2}^1 f(x) dx \leq 3.9$$

8. (a, c) Let $I_1 = \int_0^{4\pi} e^t (\sin^6 at + \cos^6 at) dt$

$$\begin{aligned} &= \int_0^\pi e^t (\sin^6 at + \cos^6 at) dt + \int_\pi^{2\pi} e^t (\sin^6 at + \cos^6 at) dt \\ &\quad + \int_{2\pi}^{3\pi} e^t (\sin^6 at + \cos^6 at) dt + \int_{3\pi}^{4\pi} e^t (\sin^6 at + \cos^6 at) dt \end{aligned}$$

$$\therefore I_1 = I_2 + I_3 + I_4 + I_5 \quad \dots(i)$$

Now, $I_3 = \int_\pi^{2\pi} e^t (\sin^6 at + \cos^6 at) dt$

Put $t = \pi + t$

$$\Rightarrow dt = dt$$

$$\begin{aligned} \therefore I_3 &= \int_0^\pi e^{\pi+t} (\sin^6 at + \cos^6 at) dt \\ &= e^\pi \cdot I_2 \end{aligned} \quad \dots(ii)$$

Now, $I_4 = \int_{2\pi}^{3\pi} e^t (\sin^6 at + \cos^6 at) dt$

Put $t = 2\pi + t \Rightarrow dt = dt$

$$\begin{aligned} \therefore I_4 &= \int_0^\pi e^{t+2\pi} (\sin^6 at + \cos^6 at) dt \\ &= e^{2\pi} \cdot I_2 \end{aligned} \quad \dots(iii)$$

and $I_5 = \int_{3\pi}^{4\pi} e^t (\sin^6 at + \cos^6 at) dt$

Put $t = 3\pi + t$

$$\begin{aligned} \therefore I_5 &= \int_0^\pi e^{3\pi+t} (\sin^6 at + \cos^6 at) dt \\ &= e^{3\pi} \cdot I_2 \end{aligned} \quad \dots(iv)$$

From Eqs. (i), (ii), (iii) and (iv), we get

$$\begin{aligned} I_1 &= I_2 + e^\pi \cdot I_2 + e^{2\pi} \cdot I_2 + e^{3\pi} \cdot I_2 \\ &= (1 + e^\pi + e^{2\pi} + e^{3\pi}) I_2 \end{aligned}$$

$$\begin{aligned} \therefore L &= \frac{\int_0^{4\pi} e^t (\sin^6 at + \cos^6 at) dt}{\int_0^\pi e^t (\sin^6 at + \cos^6 at) dt} \\ &= (1 + e^\pi + e^{2\pi} + e^{3\pi}) \\ &= \frac{1 \cdot (e^{4\pi} - 1)}{e^\pi - 1} \text{ for } a \in R \end{aligned}$$

9. (a, b, c) According to the given data,

$$F(x) < 0, \forall x \in (1, 3)$$

We have, $f(x) = xF(x)$

$$\Rightarrow f'(x) = F(x) + xF'(x) \quad \dots(i)$$

$$\Rightarrow f'(1) = F(1) + F'(1) < 0 \quad [\text{given } F(1)=0 \text{ and } F'(x) < 0]$$

Also, $f(2) = 2F(2) < 0 \quad [\text{using } F(x) < 0, \forall x \in (1, 3)]$

$$\begin{aligned} \text{Now, } f'(x) &= F(x) + xF'(x) < 0 \\ &[\text{using } F(x) < 0, \forall x \in (1, 3)] \end{aligned}$$

$$\Rightarrow f'(x) < 0$$

10. (a, b, c) Given, $\int_1^3 x^2 F'(x) dx = -12$

$$\Rightarrow [x^2 F(x)]_1^3 - \int_1^3 2x \cdot F(x) dx = -12$$

$$\Rightarrow 9F(3) - F(1) - 2 \int_1^3 f(x) dx = -12$$

$$[\because xF(x) = f(x), \text{ given}]$$

$$\Rightarrow -36 - 0 - 2 \int_1^3 f(x) dx = -12$$

$$\therefore \int_1^3 f(x) dx = -12$$

and $\int_1^3 x^3 F''(x) dx = 40$

$$\Rightarrow [x^3 F'(x)]_1^3 - \int_1^3 3x^2 F'(x) dx = 40$$

$$\Rightarrow [x^2 (xF'(x))]_1^3 - 3 \times (-12) = 40$$

$$\Rightarrow \{x^2 \cdot [f'(x) - F(x)]\}_1^3 = 4$$

$$\Rightarrow 9[f'(3) - F(3)] - [f'(1) - F(1)] = 4$$

$$\Rightarrow 9[f'(3) + 4] - [f'(1) - 0] = 4$$

$$\Rightarrow 9f'(3) - f'(1) = -32$$

11. (a, b)

n_1	Red	n_3	Red
n_2	Black	n_4	Black
Box I		Box II	

Let A = Drawing red ball

$$\therefore P(A) = P(B_1) \cdot P(A/B_1) + P(B_2) \cdot P(A/B_2)$$

$$= \frac{1}{2} \left(\frac{n_1}{n_1 + n_2} \right) + \frac{1}{2} \left(\frac{n_3}{n_3 + n_4} \right)$$

Given, $P(B_2/A) = \frac{1}{3}$

$$\Rightarrow \frac{P(B_2) \cdot P(B_2 \cap A)}{P(A)} = \frac{1}{3}$$

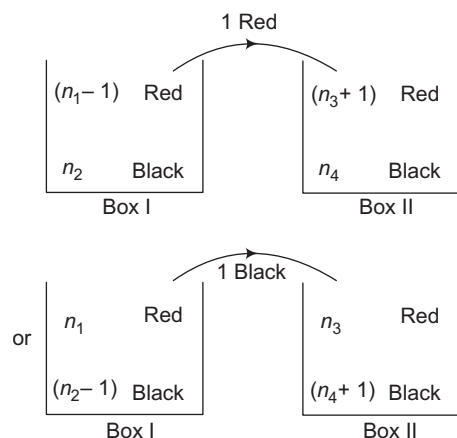
$$\Rightarrow \frac{\frac{1}{2} \left(\frac{n_3}{n_3 + n_4} \right)}{\frac{1}{2} \left(\frac{n_1}{n_1 + n_2} \right) + \frac{1}{2} \left(\frac{n_3}{n_3 + n_4} \right)} = \frac{1}{3}$$

$$\Rightarrow \frac{\frac{n_3(n_1 + n_2)}{n_1(n_3 + n_4) + n_3(n_1 + n_2)}}{1} = \frac{1}{3}$$

$$\Rightarrow \frac{n_3(n_1 + n_2)}{n_1(n_3 + n_4) + n_3(n_1 + n_2)} = \frac{1}{3}$$

Now, check options, then clearly options (a) and (b) satisfy.

12. (c, d)



$$\therefore P(\text{drawing red ball from } B_1) = \frac{1}{3}$$

$$\Rightarrow \left(\frac{n_1 - 1}{n_1 + n_2 - 1} \right) \left(\frac{n_1}{n_1 + n_2} \right) + \left(\frac{n_2}{n_1 + n_2} \right) \left(\frac{n_1}{n_1 + n_2 - 1} \right) = \frac{1}{3}$$

$$\Rightarrow \frac{n_1^2 + n_1 n_2 - n_1}{(n_1 + n_2)(n_1 + n_2 - 1)} = \frac{1}{3}$$

Clearly, options (c) and (d) satisfy.

For More Material Join: @JEEAdvanced_2024

JEE Main 2016 (Solved)

- If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \operatorname{adj} A = AA^T$, then $5a + b$ is equal to
 (a) -1 (b) 5 (c) 4 (d) 13
- For $x \in \mathbb{R}$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then
 (a) g is not differentiable at $x = 0$
 (b) $g'(0) = \cos(\log 2)$
 (c) $g'(0) = -\cos(\log 2)$
 (d) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$
- Consider $f(x) = \tan^{-1} \left(\sqrt{\frac{1+\sin x}{1-\sin x}} \right)$, $x \in \left(0, \frac{\pi}{2} \right)$.
 A normal to $y = f(x)$ at $x = \frac{\pi}{6}$ also passes through the point
 (a) $(0, 0)$ (b) $\left(0, \frac{2\pi}{3} \right)$
 (c) $\left(\frac{\pi}{6}, 0 \right)$ (d) $\left(\frac{\pi}{4}, 0 \right)$
- A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then
 (a) $2x = (\pi + 4)r$ (b) $(4 - \pi)x = \pi r$
 (c) $x = 2r$ (d) $2x = r$
- The integral $\int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx$ is equal to
 (a) $\frac{-x^5}{(x^5 + x^3 + 1)^2} + C$ (b) $\frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$
 (c) $\frac{x^5}{2(x^5 + x^3 + 1)^2} + C$ (d) $\frac{-x^{10}}{2(x^5 + x^3 + 1)^2} + C$
 where, C is an arbitrary constant.
- The area (in sq units) of the region $\{(x, y) : y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x, x \geq 0, y \geq 0\}$ is
 (a) $\pi - \frac{4}{3}$ (b) $\pi - \frac{8}{3}$ (c) $\pi - \frac{4\sqrt{2}}{3}$ (d) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$
- If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)dx = x dy$, then $f\left(-\frac{1}{2}\right)$ is equal to
 (a) $-\frac{2}{5}$ (b) $-\frac{4}{5}$ (c) $\frac{2}{5}$ (d) $\frac{4}{5}$
- If the line, $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane, $lx + my - z = 9$, then $l^2 + m^2$ is equal to
 (a) 26 (b) 18 (c) 5 (d) 2
- Let $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$ and $\hat{\mathbf{c}}$ be three unit vectors such that $\hat{\mathbf{a}} \times (\hat{\mathbf{b}} \times \hat{\mathbf{c}}) = \frac{\sqrt{3}}{2}(\hat{\mathbf{b}} + \hat{\mathbf{c}})$. If $\hat{\mathbf{b}}$ is not parallel to $\hat{\mathbf{c}}$, then the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ is
 (a) $\frac{3\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

JEE Advanced 2016 (Solved)

Paper I

1. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all $x > 0$, is

(Single Option Correct)

- (a) $\frac{1}{64}$ (b) $\frac{1}{32}$ (c) $\frac{1}{27}$ (d) $\frac{1}{25}$

2. A computer producing factory has only two plants T_1 and T_2 . Plant T_1 produces 20% and plant T_2 produces 80% of the total computers produced. 7% of computers produced in the factory turn out to be defective. It is known that $P(\text{computer turns out to be defective, given that it is produced in plant } T_1) = 10P(\text{computer turns out to be defective, given that it is produced in plant } T_2)$, where $P(E)$ denotes the probability of an event E . A computer produced in the factory is randomly selected and it does not turn out to be defective. Then, the probability that it is produced in plant T_2 , is

(Single Option Correct)

- (a) $\frac{36}{73}$ (b) $\frac{47}{79}$ (c) $\frac{78}{93}$ (d) $\frac{75}{83}$

3. A solution curve of the differential equation $(x^2 + xy + 4x + 2y + 4)\frac{dy}{dx} - y^2 = 0, x > 0$, passes through the point $(1, 3)$. Then, the solution curve

(One or More Than One Option Correct)

- (a) intersects $y = x + 2$ exactly at one point
(b) intersects $y = x + 2$ exactly at two points
(c) intersects $y = (x + 2)^2$
(d) does not intersect $y = (x + 3)^2$

4. Consider a pyramid $OPQRS$ located in the first octant ($x \geq 0, y \geq 0, z \geq 0$) with O as origin, and OP and OR along the X -axis and the Y -axis, respectively. The base $OPQR$ of the pyramid is a square with $OP = 3$. The point S is directly above the mid-point T of diagonal OQ such that $TS = 3$. Then,

(One or More Than One Option Correct)

- (a) the acute angle between OQ and OS is $\frac{\pi}{3}$
(b) the equation of the plane containing the ΔOQS is $x - y = 0$
(c) the length of the perpendicular from P to the plane containing the ΔOQS is $\frac{3}{\sqrt{2}}$
(d) the perpendicular distance from O to the straight line containing RS is $\sqrt{\frac{15}{2}}$

5. In a ΔXYZ , let x, y, z be the lengths of sides opposite to the angles X, Y, Z respectively and $2s = x + y + z$. If

$$\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} \text{ and area of incircle of the } \Delta XYZ \text{ is } \frac{8\pi}{3}, \text{ then}$$

(One or More Than One Option Correct)

- (a) area of the ΔXYZ is $6\sqrt{6}$
(b) the radius of circumcircle of the ΔXYZ is $\frac{35}{6}\sqrt{6}$
(c) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
(d) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$

6. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q . The normal to the circle at P intersects a line drawn through Q parallel to RS at point E . Then, the locus of E passes through the point(s)

(One or More Than One Option Correct)

- (a) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}} \right)$ (b) $\left(\frac{1}{4}, \frac{1}{2} \right)$ (c) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}} \right)$ (d) $\left(\frac{1}{4}, -\frac{1}{2} \right)$

7. Let $f: \mathbb{R} \rightarrow \mathbb{R}, g: \mathbb{R} \rightarrow \mathbb{R}$ and $h: \mathbb{R} \rightarrow \mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2, g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in \mathbb{R}$. Then,

(One or More Than One Option Correct)

- (a) $g'(2) = \frac{1}{15}$ (b) $h'(1) = 666$
(c) $h(0) = 16$ (d) $h(g(3)) = 36$

8. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, where $\alpha \in \mathbb{R}$. Suppose $Q = [q_{ij}]$ is a

matrix such that $PQ = kI$, where $k \in \mathbb{R}, k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$,

then (One or More Than One Option Correct)

- (a) $\alpha = 0, k = 8$ (b) $4\alpha - k + 8 = 0$
(c) $\det(P \operatorname{adj}(Q)) = 2^9$ (d) $\det(Q \operatorname{adj}(P)) = 2^{13}$

9. The total number of distincts $x \in \mathbb{R}$ for which

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10 \text{ is}$$

(Single Digit Integer)

10. The total number of distincts $x \in [0, 1]$ for which

$$\int_0^x \frac{t^2}{1+t^4} dt = 2x - 1 \text{ is}$$

(Single Digit Integer)

Paper II

1. Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3.

If $Q = [q_{ij}]$ is a matrix, such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}}$ equals

- (a) 52 (b) 103 (c) 201 (d) 205
(Single Option Correct)

2. Area of the region $\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$ is equal to
(a) $\frac{1}{6}$ (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{5}{3}$
(Single Option Correct)

3. Let P be the image of the point $(3, 1, 7)$ with respect to the plane $x - y + z = 3$. Then, the equation of the plane passing through P and containing the straight line $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$ is
(a) $x + y - 3z = 0$ (b) $3x + z = 0$
(c) $x - 4y + 7z = 0$ (d) $2x - y = 0$
(Single Option Correct)

4. Let $a, b \in R$ and $f: R \rightarrow R$ be defined by $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$. Then, f is
(a) differentiable at $x = 0$, if $a = 0$ and $b = 1$
(b) differentiable at $x = 1$, if $a = 1$ and $b = 0$
(c) not differentiable at $x = 0$, if $a = 1$ and $b = 0$
(d) not differentiable at $x = 1$, if $a = 1$ and $b = 1$
(One or More Than One Option Correct)

5. Let $f: R \rightarrow (0, \infty)$ and $g: R \rightarrow R$ be twice differentiable functions such that f'' and g'' are continuous functions on R . Suppose $f''(2) = g(2) = 0$, $f'(2) \neq 0$ and $g'(2) \neq 0$. If $\lim_{x \rightarrow 2} \frac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

- (a) f has a local minimum at $x = 2$
(b) f has a local maximum at $x = 2$
(c) $f''(2) > f(2)$
(One or More Than One Option Correct)

(d) $f(x) - f''(x) = 0$, for atleast one $x \in R$

6. Let $\hat{u} = u_1 \hat{i} + u_2 \hat{j} + u_3 \hat{k}$ be a unit vector in R^3 and $\hat{w} = \frac{1}{\sqrt{6}}(\hat{i} + \hat{j} + 2\hat{k})$. Given that there exists a vector v in R^3 , such that $|\hat{u} + v| = 1$ and $\hat{w} \cdot (\hat{u} + v) = 1$.

Which of the following statement(s) is/are correct?

(One or More Than One Option Correct)

- (a) There is exactly one choice for such v
(b) There are infinitely many choices for such v
(c) If $\hat{u}$ lies in the XY -plane, then $|u_1| = |u_2|$
(d) If $\hat{u}$ lies in the XY -plane, then $2|u_1| = |u_3|$

7. Let $a, \lambda, \mu \in R$. Consider the system of linear equations

$$ax + 2y = \lambda \text{ and } 3x - 2y = \mu.$$

Which of the following statement(s) is/are correct?

- (a) If $a = -3$, then the system has infinitely many solutions for all values of λ and μ
(b) If $a \neq -3$, then the system has a unique solution for all values of λ and μ
(c) If $\lambda + \mu = 0$, then the system has infinitely many solutions for $a = -3$
(d) If $\lambda + \mu \neq 0$, then the system has no solution for $a = -3$

8. Let $f: \left[-\frac{1}{2}, 2\right] \rightarrow R$ and $g: \left[-\frac{1}{2}, 2\right] \rightarrow R$ be functions defined by $f(x) = [x^2 - 3]$ and $g(x) = |x|f(x) + |4x - 7|f(x)$, where $[y]$ denotes the greatest integer less than or equal to y for $y \in R$. Then,

(One or More Than One Option Correct)

- (a) f is discontinuous exactly at three points in $\left[-\frac{1}{2}, 2\right]$
(b) f is discontinuous exactly at four points in $\left[-\frac{1}{2}, 2\right]$
(c) g is not differentiable exactly at four points in $\left(-\frac{1}{2}, 2\right)$
(d) g is not differentiable exactly at five points in $\left(-\frac{1}{2}, 2\right)$

SOLUTIONS

JEE Main

1. (b) Given, $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A \text{ adj } A = AA^T$

Clearly, $A(\text{adj } A) = |A| I_2$ [$\because$ if A is square matrix of order n , then $A(\text{adj } A) = (\text{adj } A) \cdot A = |A| I_n$]

$$= \begin{vmatrix} 5a & -b \\ 3 & 2 \end{vmatrix} I_2 = (10a + 3b) I_2$$

$$= (10a + 3b) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} \quad \dots(i)$$

$$\text{and } AA^T = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix} = \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix} \quad \dots(ii)$$

$$\therefore A(\text{adj } A) = AA^T$$

$$\therefore \begin{bmatrix} 10a + 3b & 0 \\ 0 & 10a + 3b \end{bmatrix} = \begin{bmatrix} 25a^2 + b^2 & 15a - 2b \\ 15a - 2b & 13 \end{bmatrix}$$

[using Eqs. (i) and (ii)]

$$\Rightarrow 15a - 2b = 0 \Rightarrow a = \frac{2b}{15} \quad \dots(iii)$$

$$\text{and } 10a + 3b = 13 \quad \dots(iv)$$

On substituting the value of 'a' from Eq. (iii) in Eq. (iv), we get

$$10 \cdot \left(\frac{2b}{15}\right) + 3b = 13 \Rightarrow \frac{20b + 45b}{15} = 13 \Rightarrow \frac{65b}{15} = 13$$

$$\Rightarrow b = 3$$

Now, substituting the value of b in Eq. (iii), we get $5a = 2$

Hence, $5a + b = 2 + 3 = 5$

2. (d) Given, system of linear equation is

$$x + \lambda y - z = 0; \lambda x - y - z = 0; x + y - \lambda z = 0$$

Note that, given system will have a non-trivial solution only if

$$\text{determinant of coefficient matrix is zero, i.e. } \begin{vmatrix} 1 & \lambda & -1 \\ \lambda & -1 & -1 \\ 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow 1(\lambda + 1) - \lambda(-\lambda^2 + 1) - 1(\lambda + 1) = 0$$

$$\Rightarrow \lambda + 1 + \lambda^3 - \lambda - \lambda - 1 = 0 \Rightarrow \lambda^3 - \lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - 1) = 0 \Rightarrow \lambda = 0 \text{ or } \lambda = \pm 1$$

Hence, given system of linear equation has a non-trivial solution for exactly three values of λ .

3. (b) We have, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, $x \in R$

Note that, for $x \rightarrow 0$, $\log 2 > \sin x$

$$\therefore f(x) = \log 2 - \sin x$$

$$\Rightarrow g(x) = \log 2 - \sin(f(x)) \\ = \log 2 - \sin(\log 2 - \sin x)$$

Clearly, $g(x)$ is differentiable at $x = 0$ as $\sin x$ is differentiable.

$$\text{Now, } g'(x) = -\cos(\log 2 - \sin x) \cdot (-\cos x) \\ = \cos x \cdot \cos(\log 2 - \sin x)$$

$$\Rightarrow g'(0) = 1 \cdot \cos(\log 2)$$

4. (b) We have, $f(x) = \tan^{-1} \sqrt{\frac{1 + \sin x}{1 - \sin x}}$, $x \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow f(x) = \tan^{-1} \sqrt{\frac{\left(\cos \frac{x}{2} + \sin \frac{x}{2}\right)^2}{\left(\cos \frac{x}{2} - \sin \frac{x}{2}\right)^2}} \\ = \tan^{-1} \left(\frac{\cos \frac{x}{2} + \sin \frac{x}{2}}{\cos \frac{x}{2} - \sin \frac{x}{2}} \right) \\ \left(\because \cos \frac{x}{2} > \sin \frac{x}{2} \text{ for } 0 < \frac{x}{2} < \frac{\pi}{4} \right)$$

$$= \tan^{-1} \left(\frac{1 + \tan \frac{x}{2}}{1 - \tan \frac{x}{2}} \right) = \tan^{-1} \left[\tan \left(\frac{\pi}{4} + \frac{x}{2} \right) \right] = \frac{\pi}{4} + \frac{x}{2}$$

$$\Rightarrow f'(x) = \frac{1}{2} \Rightarrow f'\left(\frac{\pi}{6}\right) = \frac{1}{2}$$

Now, equation of normal at $x = \frac{\pi}{6}$ is given by

$$\left(y - f\left(\frac{\pi}{6}\right) \right) = -2 \left(x - \frac{\pi}{6} \right) \\ \Rightarrow \left(y - \frac{\pi}{3} \right) = -2 \left(x - \frac{\pi}{6} \right) \\ \left[\because f\left(\frac{\pi}{6}\right) = \frac{\pi}{4} + \frac{\pi}{12} = \frac{4\pi}{12} = \frac{\pi}{3} \right] \\ \text{which passes through } \left(0, \frac{2\pi}{3} \right).$$

5. (c) According to given information, we have

Perimeter of square + Perimeter of circle = 2 units

$$\Rightarrow 4x + 2\pi r = 2$$

$$\Rightarrow r = \frac{1 - 2x}{\pi} \quad \dots(i)$$

Now, let A be the sum of the areas of the square and the circle. Then,

$$A = x^2 + \pi r^2 = x^2 + \pi \frac{(1 - 2x)^2}{\pi^2}$$

$$\Rightarrow A(x) = x^2 + \frac{(1 - 2x)^2}{\pi}$$

Now, for minimum value of $A(x)$, $\frac{dA}{dx} = 0$

$$\Rightarrow 2x + \frac{2(1 - 2x)}{\pi} \cdot (-2) = 0 \Rightarrow x = \frac{2 - 4x}{\pi}$$

$$\Rightarrow \pi x + 4x = 2 \Rightarrow x = \frac{2}{\pi + 4} \quad \dots(ii)$$

Now, from Eq. (i), we get

$$r = \frac{1 - 2 \cdot \frac{2}{\pi + 4}}{\pi} = \frac{\pi + 4 - 4}{\pi(\pi + 4)} = \frac{1}{\pi + 4} \quad \dots(iii)$$

From Eqs. (ii) and (iii), we get $x = 2r$

$$6. (b) \text{ Let } I = \int \frac{2x^{12} + 5x^9}{(x^5 + x^3 + 1)^3} dx \\ = \int \frac{2x^{12} + 5x^9}{x^{15} (1 + x^{-2} + x^{-5})^3} dx = \int \frac{2x^{-3} + 5x^{-6}}{(1 + x^{-2} + x^{-5})^3} dx$$

Now, put $1 + x^{-2} + x^{-5} = t$

$$\Rightarrow (-2x^{-3} - 5x^{-6}) dx = dt$$

$$\Rightarrow (2x^{-3} + 5x^{-6}) dx = -dt$$

$$\therefore I = - \int \frac{dt}{t^3} = - \int t^{-3} dt$$

$$= - \frac{t^{-3+1}}{-3+1} + C = \frac{1}{2t^2} + C = \frac{x^{10}}{2(x^5 + x^3 + 1)^2} + C$$

7. (b) Given equations of curves are

$$y^2 = 2x \quad \dots(i)$$

which is a parabola with vertex (0, 0) and axis parallel to X-axis.

$$\text{And } x^2 + y^2 = 4x \quad \dots(ii)$$

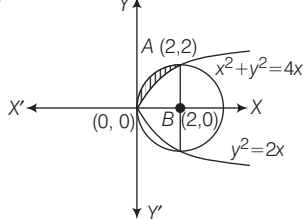
which is a circle with centre (2, 0) and radius = 2

On substituting $y^2 = 2x$ in Eq. (ii), we get

$$x^2 + 2x = 4x \Rightarrow x^2 = 2x \Rightarrow x = 0 \text{ or } x = 2$$

$$\Rightarrow y = 0 \text{ or } y = \pm 2 \quad [\text{using Eq. (i)}]$$

Now, the required area is the area of shaded region, i.e.



$$\begin{aligned} \text{Required area} &= \frac{\text{Area of a circle}}{4} - \int_0^2 \sqrt{2x} \, dx \\ &= \frac{\pi(2)^2}{4} - \sqrt{2} \int_0^2 x^{1/2} dx = \pi - \sqrt{2} \left[\frac{x^{3/2}}{3/2} \right]_0^2 \\ &= \pi - \frac{2\sqrt{2}}{3} [2\sqrt{2} - 0] = \left(\pi - \frac{8}{3} \right) \text{ sq unit} \end{aligned}$$

8. (d) Given differential equation is

$$y(1 + xy) dx = x dy$$

$$\Rightarrow y dx + xy^2 dx = x dy$$

$$\Rightarrow \frac{x dy - y dx}{y^2} = x dx$$

$$\Rightarrow -\frac{(y dx - x dy)}{y^2} = x dx$$

$$\Rightarrow -d\left(\frac{x}{y}\right) = x dx$$

On integrating both sides, we get

$$-\frac{x}{y} = \frac{x^2}{2} + C \quad \dots(i)$$

$\therefore$ It passes through (1, -1).

$$\therefore 1 = \frac{1}{2} + C \Rightarrow C = \frac{1}{2}$$

$$\text{Now, from Eq. (i)} \quad -\frac{x}{y} = \frac{x^2}{2} + \frac{1}{2}$$

$$\Rightarrow x^2 + 1 = -\frac{2x}{y} \Rightarrow y = -\frac{2x}{x^2 + 1}$$

$$\therefore f\left(-\frac{1}{2}\right) = \frac{4}{5}$$

9. (d) Since, the line $\frac{x-3}{2} = \frac{y+2}{-1} = \frac{z+4}{3}$ lies in the plane

$lx + my - z = 9$, therefore we have

$$2l - m - 3 = 0$$

[$\therefore$ normal will be perpendicular to the line]

$$\Rightarrow 2l - m = 3 \quad \dots(i)$$

and $3l - 2m + 4 = 9$

[$\therefore$ point (3, -2, -4) lies on the plane]

$$\Rightarrow 3l - 2m = 5 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$l = 1 \text{ and } m = -1$$

$$\therefore 1^2 + m^2 = 2$$

10. (d) Given, $|\hat{a}| = |\hat{b}| = |\hat{c}| = 1$

$$\text{and } \hat{a} \times (\hat{b} \times \hat{c}) = \frac{\sqrt{3}}{2} (\hat{b} + \hat{c})$$

$$\text{Now, consider } \hat{a} \times (\hat{b} \times \hat{c}) = \frac{\sqrt{3}}{2} (\hat{b} + \hat{c})$$

$$\Rightarrow (\hat{a} \cdot \hat{c}) \hat{b} - (\hat{a} \cdot \hat{b}) \hat{c} = \frac{\sqrt{3}}{2} \hat{b} + \frac{\sqrt{3}}{2} \hat{c}$$

On comparing, we get

$$\hat{a} \cdot \hat{b} = -\frac{\sqrt{3}}{2} \Rightarrow |\hat{a}| |\hat{b}| \cos \theta = -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \cos \theta = -\frac{\sqrt{3}}{2} \quad [\because |\hat{a}| = |\hat{b}| = 1]$$

$$\Rightarrow \cos \theta = \cos \left(\pi - \frac{\pi}{6} \right) \Rightarrow \theta = \frac{5\pi}{6}$$

JEE Advanced

Paper I

1. (c) Here, to find the least value of $\alpha \in \mathbb{R}$, for which

$$4\alpha x^2 + \frac{1}{x} \geq 1, \text{ for all } x > 0.$$

i.e. to find the minimum value of α when $y = 4\alpha x^2 + \frac{1}{x}$; $x > 0$ attains minimum value of α .

$$\therefore \frac{dy}{dx} = 8\alpha x - \frac{1}{x^2} \quad \dots(i)$$

$$\text{Now, } \frac{d^2y}{dx^2} = 8\alpha + \frac{2}{x^3} \quad \dots(ii)$$

$$\text{When } \frac{dy}{dx} = 0, \text{ then } 8x^3\alpha = 1$$

At $x = \left(\frac{1}{8\alpha}\right)^{1/3}$, $\frac{d^2y}{dx^2} = 8\alpha + 16\alpha = 24\alpha$, Thus, y attains minimum when

$$x = \left(\frac{1}{8\alpha}\right)^{1/3}; \alpha > 0.$$

$$\therefore y \text{ attains minimum when } x = \left(\frac{1}{8\alpha}\right)^{1/3}.$$

$$\text{i.e. } 4\alpha \left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} \geq 1$$

$$\Rightarrow \alpha^{1/3} + 2\alpha^{1/3} \geq 1$$

$$\Rightarrow 3\alpha^{1/3} \geq 1$$

$$\Rightarrow \alpha \geq \frac{1}{27}$$

Hence, the least value of α is $\frac{1}{27}$.

2. (c) Let $x = P$ (computer turns out to be defective, given that it is produced in plant T_2)

$$\Rightarrow x = P\left(\frac{D}{T_2}\right) \quad \dots(i)$$

where, D = Defective computer

$\therefore P$ (computer turns out to be defective given that is produced in plant T_1) = $10x$

$$\text{i.e. } P\left(\frac{D}{T_1}\right) = 10x \quad \dots(\text{ii})$$

$$\text{Also, } P(T_1) = \frac{20}{100} \text{ and } P(T_2) = \frac{80}{100}$$

$$\text{Given, } P(\text{defective computer}) = \frac{7}{100}$$

$$\text{i.e. } P(D) = \frac{7}{100}$$

Using law of total probability,

$$P(D) = P(T_1) \cdot P\left(\frac{D}{T_1}\right) + P(T_2) \cdot P\left(\frac{D}{T_2}\right)$$

$$\therefore \frac{7}{100} = \left(\frac{20}{100}\right) \cdot 10x + \left(\frac{80}{100}\right) \cdot x$$

$$\Rightarrow 7 = (280)x \Rightarrow x = \frac{1}{40} \quad \dots(\text{iii})$$

$$\therefore P\left(\frac{D}{T_2}\right) = \frac{1}{40} \text{ and } P\left(\frac{D}{T_1}\right) = \frac{10}{40}$$

$$\Rightarrow P\left(\frac{\bar{D}}{T_2}\right) = 1 - \frac{1}{40} = \frac{39}{40}$$

$$\text{and } P\left(\frac{\bar{D}}{T_1}\right) = 1 - \frac{10}{40} = \frac{30}{40} \quad \dots(\text{iv})$$

Using Baye's theorem,

$$\begin{aligned} P\left(\frac{T_2}{D}\right) &= \frac{P(T_2 \cap D)}{P(T_1 \cap D) + P(T_2 \cap D)} \\ &= \frac{P(T_2) \cdot P\left(\frac{D}{T_2}\right)}{P(T_1) \cdot P\left(\frac{D}{T_1}\right) + P(T_2) \cdot P\left(\frac{D}{T_2}\right)} \\ &= \frac{\frac{80}{100} \cdot \frac{1}{40}}{\frac{20}{100} \cdot \frac{10}{40} + \frac{80}{100} \cdot \frac{1}{40}} = \frac{78}{93} \end{aligned}$$

$$3. (a, d) \text{ Given, } (x^2 + xy + 4x + 2y + 4) \frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow [(x^2 + 4x + 4) + y(x + 2)] \frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow [(x + 2)^2 + y(x + 2)] \frac{dy}{dx} - y^2 = 0$$

Put $x + 2 = X$ and $y = Y$, then

$$(X^2 + XY) \frac{dY}{dX} - Y^2 = 0$$

$$\Rightarrow X^2 dY + XY dY - Y^2 dX = 0$$

$$\Rightarrow X^2 dY + Y(X dY - Y dX) = 0$$

$$\Rightarrow -\frac{dY}{Y} = \frac{X dY - Y dX}{X^2}$$

$$\Rightarrow -d(\log |Y|) = d\left(\frac{Y}{X}\right)$$

On integrating both sides, we get

$$-\log |Y| = \frac{Y}{X} + C, \text{ where } x + 2 = X \text{ and } y = Y$$

$$\Rightarrow -\log |y| = \frac{y}{x + 2} + C \quad \dots(\text{i})$$

Since, it passes through the point (1, 3).

$$\therefore -\log 3 = \frac{3}{1 + 2} + C$$

$$\Rightarrow C = -1 - \log 3 = -(\log e + \log 3) = -\log 3e$$

$\therefore$ Eq. (i) becomes

$$\log |y| + \frac{y}{x + 2} - \log (3e) = 0$$

$$\Rightarrow \log \left(\frac{|y|}{3e}\right) + \frac{y}{x + 2} = 0 \quad \dots(\text{ii})$$

Now, to check option (a), $y = x + 2$ intersects the curve.

$$\Rightarrow \log \left(\frac{|x + 2|}{3e}\right) + \frac{x + 2}{x + 2} = 0$$

$$\Rightarrow \log \left(\frac{|x + 2|}{3e}\right) = -1$$

$$\Rightarrow \frac{|x + 2|}{3e} = e^{-1} = \frac{1}{e}$$

$$\Rightarrow |x + 2| = 3 \text{ or } x + 2 = \pm 3$$

$$\therefore x = 1, -5 \text{ (rejected), as } x > 0$$

$$\therefore x = 1 \text{ only one solution.}$$

[given]

Thus, (a) is the correct answer.

To check option (c), we have

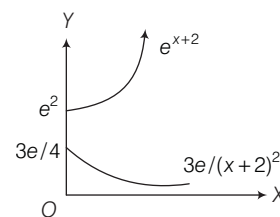
$$y = (x + 2)^2 \text{ and } \log \left(\frac{|y|}{3e}\right) + \frac{y}{x + 2} = 0$$

$$\Rightarrow \log \left[\frac{|x + 2|^2}{3e}\right] + \frac{(x + 2)^2}{x + 2} = 0$$

$$\Rightarrow \log \left[\frac{|x + 2|^2}{3e}\right] = -(x + 2)$$

$$\Rightarrow \frac{(x + 2)^2}{3e} = e^{-(x + 2)} \text{ or } (x + 2)^2 \cdot e^{x + 2} = 3e$$

$$\Rightarrow e^{x + 2} = \frac{3e}{(x + 2)^2}$$



Clearly, they have no solution.

To check option (d), $y = (x + 3)^2$

$$\text{i.e. } \log \left[\frac{|x + 3|^2}{3e}\right] + \frac{(x + 3)^2}{(x + 2)} = 0$$

To check the number of solutions.

$$\text{Let } g(x) = 2 \log(x + 3) + \frac{(x + 3)^2}{(x + 2)} - \log(3e)$$

$$\begin{aligned}\therefore g'(x) &= \frac{2}{x+3} + \left(\frac{(x+2) \cdot 2(x+3) - (x+3)^2 \cdot 1}{(x+2)^2} \right) - 0 \\ &= \frac{2}{x+3} + \frac{(x+3)(x+1)}{(x+2)^2}\end{aligned}$$

Clearly, when $x > 0$, then, $g'(x) > 0$

$\therefore g(x)$ is increasing, when $x > 0$.

Thus, when $x > 0$, then $g(x) > g(0)$

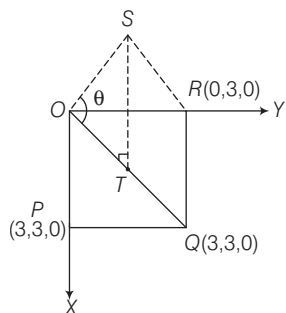
$$g(x) > \log\left(\frac{3}{e}\right) + \frac{9}{4} > 0$$

Hence, there is no solution.

Thus, option (d) is true.

4. (b, c, d) Given, square base $OP = OR = 3$

$$\therefore P(3, 0, 0), R(0, 3, 0)$$



Also, mid-point of OQ is $T\left(\frac{3}{2}, \frac{3}{2}, 0\right)$.

Since, S is directly above the mid-point T of diagonal OQ and $ST = 3$.

$$\text{i.e. } S\left(\frac{3}{2}, \frac{3}{2}, 3\right)$$

Here, DR's of $OQ(3, 3, 0)$ and DR's of $OS\left(\frac{3}{2}, \frac{3}{2}, 3\right)$.

$$\therefore \cos \theta = \frac{\frac{9}{2} + \frac{9}{2}}{\sqrt{9+9+0} \sqrt{\frac{9}{4} + \frac{9}{4} + 9}} = \frac{9}{\sqrt{18} \cdot \sqrt{\frac{27}{2}}} = \frac{1}{\sqrt{3}}$$

$\therefore$ Option (a) is incorrect.

Now, equation of the plane containing the ΔOQS is

$$\begin{vmatrix} x & y & z \\ 3 & 3 & 0 \\ 3/2 & 3/2 & 3 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x & y & z \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$$

$$\Rightarrow x(2-0) - y(2-0) + z(1-1) = 0$$

$$\Rightarrow 2x - 2y = 0 \text{ or } x - y = 0$$

$\therefore$ Option (b) is correct.

Now, length of the perpendicular from $P(3, 0, 0)$ to the plane containing ΔOQS is

$$\frac{|3-0|}{\sqrt{1+1}} = \frac{3}{\sqrt{2}}$$

$\therefore$ Option (c) is correct.

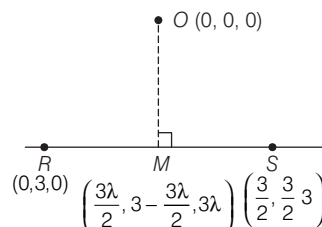
Here, equation of RS is

$$\frac{x-0}{3/2} = \frac{y-3}{-3/2} = \frac{z-0}{3} = \lambda$$

$$\Rightarrow x = \frac{3}{2}\lambda, y = -\frac{3}{2}\lambda + 3, z = 3\lambda$$

To find the distance from $O(0, 0, 0)$ to RS .

Let M be the foot of perpendicular.



$$\therefore \overline{OM} \perp \overline{RS} \Rightarrow \overline{OM} \cdot \overline{RS} = 0$$

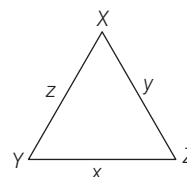
$$\Rightarrow \frac{9\lambda}{4} - \frac{3}{2}\left(3 - \frac{3\lambda}{2}\right) + 3(3\lambda) = 0 \Rightarrow \lambda = \frac{1}{3}$$

$$\therefore M\left(\frac{1}{2}, \frac{5}{2}, 1\right) \Rightarrow OM = \sqrt{\frac{1}{4} + \frac{25}{4} + 1} = \sqrt{\frac{30}{4}} = \sqrt{\frac{15}{2}}$$

$\therefore$ Option (d) is correct.

5. (a, c, d) Given a ΔXYZ , where $2s = x + y + z$

$$\text{and } \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$$



$$\therefore \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{3s - (x+y+z)}{4+3+2} = \frac{s}{9}$$

$$\text{or } \frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2} = \frac{s}{9} = \lambda \text{ (let)}$$

$$\Rightarrow s = 9\lambda, s = 4\lambda + x, s = 3\lambda + y \text{ and } s = 2\lambda + z$$

$$\therefore s = 9\lambda, x = 5\lambda, y = 6\lambda, z = 7\lambda$$

$$\text{Now, } \Delta = \sqrt{s(s-x)(s-y)(s-z)} \quad [\text{heron's formula}]$$

$$= \sqrt{9\lambda \cdot 4\lambda \cdot 3\lambda \cdot 2\lambda} = 6\sqrt{6}\lambda^2 \quad \dots(i)$$

$$\text{Also, } \pi r^2 = \frac{8\pi}{3} \Rightarrow r^2 = \frac{8}{3} \quad \dots(ii)$$

$$\text{and } R = \frac{xyz}{4\Delta} = \frac{(5\lambda)(6\lambda)(7\lambda)}{4 \cdot 6\sqrt{6}\lambda^2} = \frac{35\lambda}{4\sqrt{6}} \quad \dots(iii)$$

$$\text{Now, } r^2 = \frac{8}{3} = \frac{\Delta^2}{S^2} = \frac{216\lambda^4}{81\lambda^2}$$

$$\Rightarrow \frac{8}{3} = \frac{8}{3}\lambda^2 \quad [\text{from Eq. (ii)}]$$

$$\Rightarrow \lambda = 1$$

$$(a) \Delta XYZ = 6\sqrt{6}\lambda^2 = 6\sqrt{6}$$

$\therefore$ Option (a) is correct.

$$(b) \text{ Radius of circumcircle } (R) = \frac{35}{4\sqrt{6}}\lambda = \frac{35}{4\sqrt{6}}$$

∴ Option (b) is incorrect.

$$\begin{aligned} \text{(c) Since, } r &= 4R \sin \frac{X}{2} \cdot \sin \frac{Y}{2} \cdot \sin \frac{Z}{2} \\ \Rightarrow \frac{2\sqrt{2}}{\sqrt{3}} &= 4 \cdot \frac{35}{4\sqrt{6}} \sin \frac{X}{2} \cdot \sin \frac{Y}{2} \cdot \sin \frac{Z}{2} \\ \Rightarrow \frac{4}{35} &= \sin \frac{X}{2} \cdot \sin \frac{Y}{2} \cdot \sin \frac{Z}{2} \end{aligned}$$

∴ Option (c) is correct.

$$\begin{aligned} \text{(d) } \sin^2\left(\frac{X+Y}{2}\right) &= \cos^2\left(\frac{Z}{2}\right), \text{ as } \frac{X+Y}{2} = 90^\circ - \frac{Z}{2} \\ &= \frac{s(s-z)}{xy} = \frac{9 \times 2}{5 \times 6} = \frac{3}{5} \end{aligned}$$

∴ Option (d) is correct.

6. (b, c) As, $g(f(x)) = x$

Thus, $g(x)$ is inverse of $f(x)$.

$$\Rightarrow g(f(x)) = x$$

$$\Rightarrow g'(f(x)) \cdot f'(x) = 1$$

$$\therefore g'(f(x)) = \frac{1}{f'(x)} \quad \dots(i)$$

[where, $f'(x) = 3x^2 + 3$]

When $f(x) = 2$, then

$$x^3 + 3x + 2 = 2$$

$$\Rightarrow x = 0$$

i.e. when $x = 0$, then $f(x) = 2$

$$\therefore g'(f(x)) = \frac{1}{3x^2 + 3} \text{ at } (0, 2)$$

$$\Rightarrow g'(2) = \frac{1}{3}$$

∴ Option (a) is incorrect.

Now, $h(g(g(x))) = x$

$$\Rightarrow h(g(g(f(x)))) = f(x)$$

$$\Rightarrow h(g(x)) = f(x) \quad \dots(ii)$$

As $g(f(x)) = x$

$$\therefore h(g(3)) = f(3) = 3^3 + 3(3) + 2 = 38$$

∴ Option (d) is incorrect.

From Eq. (ii),

$$h(g(x)) = f(x)$$

$$\Rightarrow h(g(f(x))) = f(f(x))$$

$$\Rightarrow h(x) = f(f(x)) \quad \dots(iii)$$

[using $g(f(x)) = x$]

$$\Rightarrow h'(x) = f'(f(x)) \cdot f'(x) \quad \dots(iv)$$

Putting $x = 1$, we get

$$\begin{aligned} h'(1) &= f'(f(1)) \cdot f'(1) \\ &= (3 \times 36 + 3) \times (6) \\ &= 111 \times 6 = 666 \end{aligned}$$

∴ Option (b) is correct.

Putting $x = 0$ in Eq. (iii), we get

$$\begin{aligned} h(0) &= f(f(0)) \\ &= f(2) = 8 + 6 + 2 = 16 \end{aligned}$$

∴ Option (c) is correct.

$$7. \text{ (a) Here, } f'(x) = 2 - \frac{f(x)}{x}$$

$$\text{or } \frac{dy}{dx} + \frac{y}{x} = 2 \quad [\text{i.e. linear differential equation in } y]$$

$$\text{Integrating Factor, IF} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$$\therefore \text{Required solution is } y \cdot (\text{IF}) = \int Q(\text{IF}) dx + C$$

$$\Rightarrow y(x) = \int 2(x) dx + C$$

$$\Rightarrow yx = x^2 + C$$

$$\therefore y = x + \frac{C}{x} \quad [\because C \neq 0, \text{ as } f(1) \neq 1]$$

$$\text{(a) } \lim_{x \rightarrow 0^+} f'\left(\frac{1}{x}\right) = \lim_{x \rightarrow 0^+} (1 - Cx^2) = 1$$

∴ Option (a) is correct.

$$\text{(b) } \lim_{x \rightarrow 0^+} x f\left(\frac{1}{x}\right) = \lim_{x \rightarrow 0^+} (1 + Cx^2) = 1$$

∴ Option (b) is incorrect.

$$\text{(c) } \lim_{x \rightarrow 0^+} x^2 f'(x) = \lim_{x \rightarrow 0^+} (x^2 - C) = -C \neq 0$$

∴ Option (c) is incorrect.

$$\text{(d) } f(x) = x + \frac{C}{x}, C \neq 0$$

$$\text{For } C > 0, \lim_{x \rightarrow 0^+} f(x) = \infty$$

∴ Function is not bounded in $(0, 2)$.

∴ Option (d) is incorrect.

$$8. \text{ (b, c) Here, } P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$$

$$\text{Now, } |P| = 3(5\alpha) + 1(-3\alpha) - 2(-10) = 12\alpha + 20 \quad \dots(i)$$

$$\therefore \text{adj}(P) = \begin{bmatrix} 5\alpha & 2\alpha & -10 \\ -10 & 6 & 12 \\ -\alpha & -(3\alpha + 4) & 2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 5\alpha & -10 & -\alpha \\ 2\alpha & 6 & -3\alpha - 4 \\ -10 & 12 & 2 \end{bmatrix} \quad \dots(ii)$$

As,

$$PQ = kI$$

$$\Rightarrow |P| |Q| = |kI|$$

$$\Rightarrow |P| |Q| = k^3$$

$$\Rightarrow |P| \left(\frac{k^2}{2} \right) = k^3 \quad \left[\text{given, } |Q| = \frac{k^2}{2} \right]$$

$$\Rightarrow |P| = 2k \quad \dots(iii)$$

$$\therefore PQ = ki$$

$$\therefore Q = kp^{-1}I$$

$$= k \cdot \frac{\text{adj } P}{|P|} = \frac{k(\text{adj } P)}{2k} \quad [\text{from Eq. (iii)}]$$

$$= \frac{\text{adj } P}{2} = \frac{1}{2} \begin{bmatrix} 5\alpha & -10 & -\alpha \\ 2\alpha & 6 & -3\alpha - 4 \\ -10 & 12 & 2 \end{bmatrix}$$

$$\begin{aligned} \therefore q_{23} &= \frac{-3\alpha - 4}{2} \quad \left[\text{given, } q_{23} = -\frac{k}{8} \right] \\ \Rightarrow -\frac{(3\alpha + 4)}{2} &= -\frac{k}{8} \\ \Rightarrow (3\alpha + 4) \times 4 &= k \\ \Rightarrow 12\alpha + 16 &= k \quad \dots(\text{iv}) \\ \text{From Eq. (iii), } |P| &= 2k \\ \Rightarrow 12\alpha + 20 &= 2k \quad [\text{from Eq. (i)}] \dots(\text{v}) \\ \text{On solving Eqs. (iv) and (v), we get} \\ \alpha &= -1 \text{ and } k = 4 \quad \dots(\text{vi}) \\ \therefore 4\alpha - k + 8 &= -4 - 4 + 8 = 0 \end{aligned}$$

$\therefore$ Option (b) is correct.

Now, $|P \text{ adj } (Q)| = |P| |\text{adj } Q|$

$$= 2k \left(\frac{k^2}{2} \right)^2 = \frac{k^5}{2} = \frac{2^{10}}{2} = 2^9$$

$\therefore$ Option (c) is correct.

9. (2) Given,
$$\begin{vmatrix} x & x^2 & 1+x^3 \\ 2x & 4x^2 & 1+8x^3 \\ 3x & 9x^2 & 1+27x^3 \end{vmatrix} = 10$$

$$\Rightarrow x \cdot x^2 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 2 & 4 & 1+8x^3 \\ 3 & 9 & 1+27x^3 \end{vmatrix} = 10$$

Apply $R_2 \rightarrow R_2 - 2R_1$ and $R_3 \rightarrow R_3 - 3R_1$, we get

$$x^3 \begin{vmatrix} 1 & 1 & 1+x^3 \\ 0 & 2 & -1+6x^3 \\ 0 & 6 & -2+24x^3 \end{vmatrix} = 10 \Rightarrow x^3 \cdot \begin{vmatrix} 2 & 6x^3-1 \\ 6 & 24x^3-2 \end{vmatrix} = 10$$

$$\Rightarrow x^3(48x^3 - 4 - 36x^3 + 6) = 10$$

$$\Rightarrow 12x^6 + 2x^3 = 10$$

$$\Rightarrow 6x^6 + x^3 - 5 = 0 \Rightarrow x^3 = \frac{5}{6}, -1$$

$$\therefore x = \left(\frac{5}{6} \right)^{1/3}, -1$$

Hence, the number of real solutions is 2.

10. (1) Let $f(x) = \int_0^x \frac{t^2}{1+t^4} dt$

$$\Rightarrow f'(x) = \frac{x^2}{1+x^4} > 0, \text{ for all } x \in [0, 1]$$

$\therefore f(x)$ is increasing.

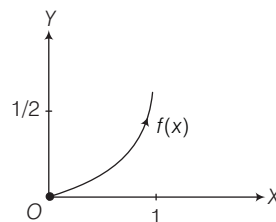
At $x = 0$, $f(0) = 0$ and at $x = 1$,

$$f(1) = \int_0^1 \frac{t^2}{1+t^4} dt$$

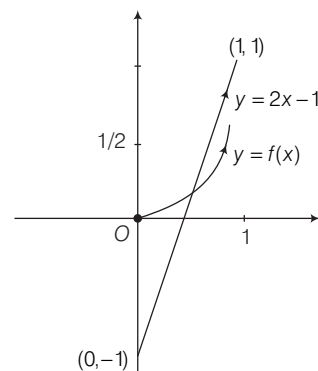
Because, $0 < \frac{t^2}{1+t^4} < \frac{1}{2}$

$$\Rightarrow \int_0^1 0 \cdot dt < \int_0^1 \frac{t^2}{1+t^4} dt < \int_0^1 \frac{1}{2} \cdot dt \Rightarrow 0 < f(1) < \frac{1}{2}$$

Thus, $f(x)$ can be plotted as



$\therefore y = f(x)$ and $y = 2x - 1$ can be shown as



From the graph, the total number of distinct solutions for $x \in (0, 1] = 1$. [as they intersect only at one point]

Paper II

1. (b) Here, $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$

$$\begin{aligned} \therefore P^2 &= \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 4+4 & 1 & 0 \\ 16+32 & 4+4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 4 \times 2 & 1 & 0 \\ 16(1+2) & 4 \times 2 & 1 \end{bmatrix} \quad \dots(\text{i}) \end{aligned}$$

$$\begin{aligned} \text{and } P^3 &= \begin{bmatrix} 1 & 0 & 0 \\ 4 \times 2 & 1 & 0 \\ 16(1+2) & 4 \times 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 4 \times 3 & 1 & 0 \\ 16(1+2+3) & 4 \times 3 & 1 \end{bmatrix} \quad \dots(\text{ii}) \end{aligned}$$

$$\text{From symmetry, } P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 4 \times 50 & 1 & 0 \\ 16(1+2+3+\dots+50) & 4 \times 50 & 1 \end{bmatrix}$$

$$\therefore P^{50} - Q = I \quad [\text{given}]$$

$$\therefore \begin{bmatrix} 1 - q_{11} & -q_{12} & -q_{13} \\ 200 - q_{21} & 1 - q_{22} & -q_{23} \\ 16 \times \frac{50}{2} (51) - q_{31} & 200 - q_{32} & 1 - q_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow 200 - q_{21} = 0, \frac{16 \times 50 \times 51}{2} - q_{31} = 0,$$

$$200 - q_{32} = 0$$

$$\therefore q_{21} = 200, q_{32} = 200, q_{31} = 20400$$

$$\text{Thus, } \frac{q_{31} + q_{32}}{q_{21}} = \frac{20400 + 200}{200} = \frac{20600}{200} = 103$$

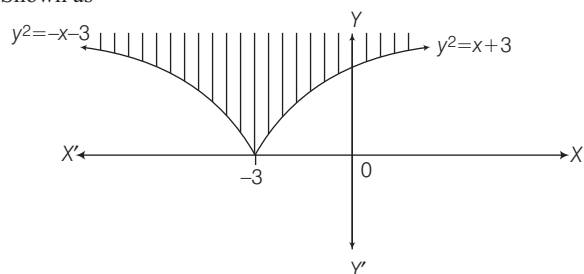
2. (c) Here, $\{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$

$$\therefore y \geq \sqrt{x+3}$$

$$\Rightarrow y \geq \begin{cases} \sqrt{x+3}, & \text{when } x \geq -3 \\ \sqrt{-x-3}, & \text{when } x \leq -3 \end{cases}$$

$$\text{or } y^2 \geq \begin{cases} x+3, & \text{when } x \geq -3 \\ -3-x, & \text{when } x \leq -3 \end{cases}$$

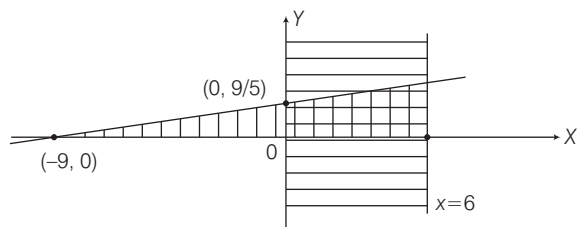
Shown as



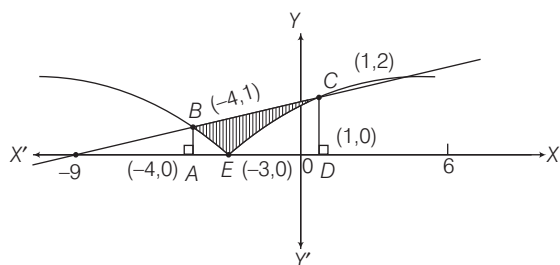
Also, $5y \leq (x+9) \leq 15$

$\Rightarrow (x+9) \geq 5y$ and $x \leq 6$

Shown as



$\therefore \{(x, y) \in R^2 : y \geq \sqrt{|x+3|}, 5y \leq (x+9) \leq 15\}$



$\therefore$ Required area = Area of trapezium ABCD
 - Area of ABE under parabola
 - Area of CDE under parabola

$$= \frac{1}{2} (1+2)(5) - \int_{-4}^{-3} \sqrt{-(x+3)} dx - \int_{-3}^1 \sqrt{x+3} dx$$

$$= \frac{15}{2} - \left[\frac{(-3-x)^{3/2}}{-\frac{3}{2}} \right]_{-4}^{-3} - \left[\frac{(x+3)^{3/2}}{\frac{3}{2}} \right]_{-3}^1$$

$$= \frac{15}{2} + \frac{2}{3} [0-1] - \frac{2}{3} [8-0]$$

$$= \frac{15}{2} - \frac{2}{3} - \frac{16}{3}$$

$$= \frac{15}{2} - \frac{18}{3} = \frac{3}{2}$$

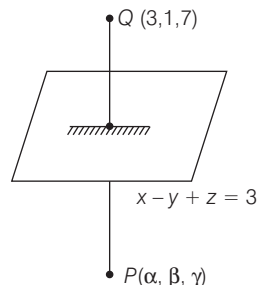
3. (c) Let image of $Q(3, 1, 7)$ w.r.t. $x - y + z = 3$ be $P(\alpha, \beta, \gamma)$.

$$\therefore \frac{\alpha - 3}{1} = \frac{\beta - 1}{-1} = \frac{\gamma - 7}{1}$$

$$= \frac{-2(3-1+7-3)}{1^2 + (-1)^2 + (1)^2}$$

$$\Rightarrow \alpha - 3 = 1 - \beta = \gamma - 7 = -4$$

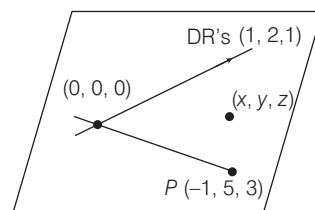
$$\therefore \alpha = -1, \beta = 5, \gamma = 3$$



Hence, the image of $Q(3, 1, 7)$ is $P(-1, 5, 3)$.

To find equation of plane passing through $P(-1, 5, 3)$ and

containing $\frac{x}{1} = \frac{y}{2} = \frac{z}{1}$



$$\Rightarrow \begin{vmatrix} x-0 & y-0 & z-0 \\ 1-0 & 2-0 & 1-0 \\ -1-0 & 5-0 & 3-0 \end{vmatrix} = 0$$

$$\Rightarrow x(6-5) - y(3+1) + z(5+2) = 0$$

$$\therefore x - 4y + 7z = 0$$

4. (a,b) Here, $f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$

If $x^3 - x \geq 0$

$$\Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$$

$$x^3 - x \leq 0$$

$$\Rightarrow \cos|x^3 - x| = \cos(x^3 - x)$$

$$\therefore \cos(|x^3 - x|) = \cos(x^3 - x), \forall x \in R \quad \dots(i)$$

Again, if $x^3 + x \geq 0$

$$\Rightarrow |x| \sin(|x^3 + x|) = x \sin(x^3 + x)$$

$$x^3 + x \leq 0$$

$$\Rightarrow |x| \sin(|x^3 + x|) = -x \sin(-(x^3 + x))$$

$$\therefore |x| \sin(|x^3 + x|) = x \sin(x^3 + x), \forall x \in R \quad \dots(ii)$$

$$\Rightarrow f(x) = a \cos(|x^3 - x|) + b|x| \sin(|x^3 + x|)$$

$$\therefore f(x) = a \cos(x^3 - x) + b|x| \sin(x^3 + x) \quad \dots(iii)$$

which is clearly sum and composition of differential functions.

Hence, $f(x)$ is always continuous and differentiable.

5. (a,d) Here, $\lim_{x \rightarrow 2} \frac{f(x) \cdot g(x)}{f'(x) \cdot g'(x)} = 1$

$$\Rightarrow \lim_{x \rightarrow 2} \frac{f(x) g'(x) + f'(x) g(x)}{f''(x) g'(x) + f'(x) g''(x)} = 1$$

[using L' Hospital's rule]

$$\Rightarrow \frac{f(2) g'(2) + f'(2) g(2)}{f''(2) g'(2) + f'(2) g''(2)} = 1$$

$$\Rightarrow \frac{f(2) g'(2)}{f''(2) g'(2)} = 1 \quad [\because f'(2) = g(2) = 0]$$

$$\Rightarrow f(2) = f''(2) \quad \dots (i)$$

$$\therefore f(x) - f''(x) = 0, \text{ for atleast one } x \in R.$$

$\Rightarrow$ option (d) is correct.

Also, $f: R \rightarrow (0, \infty)$

$$\Rightarrow f(2) > 0$$

$$\therefore f''(2) = f(2) > 0 \quad [\text{from Eq. (i)}]$$

$$\text{Since, } f'(2) = 0 \text{ and } f''(2) > 0$$

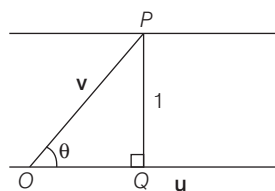
$$\therefore f(x) \text{ attains local minimum at } x = 2.$$

$\Rightarrow$ option (a) is correct.

6. (b, c) Let θ be the angle between $\hat{u}$ and $\mathbf{v}$.

$$\therefore |\hat{u} \times \mathbf{v}| = 1 \Rightarrow |\hat{u}| |\mathbf{v}| \sin \theta = 1$$

$$\therefore |\mathbf{v}| \sin \theta = 1 \quad [\because |\hat{u}| = 1] \quad \dots(i)$$



Clearly, there may be infinite vectors $\mathbf{OP} = \hat{\mathbf{v}}$, such that P is always 1 unit distance from $\hat{\mathbf{u}}$.

$\therefore$ option (b) is correct.

Again, let ϕ be the angle between $\hat{\mathbf{w}}$ and $\hat{\mathbf{u}} \times \mathbf{v}$.

$$\therefore \hat{\mathbf{w}} \cdot (\hat{\mathbf{u}} \times \mathbf{v}) = 1$$

$$\Rightarrow |\hat{\mathbf{w}}| |\hat{\mathbf{u}} \times \mathbf{v}| \cos \phi = 1$$

$$\Rightarrow \cos \phi = 1 \Rightarrow \phi = 0$$

$$\text{Thus, } \hat{\mathbf{w}} = \hat{\mathbf{u}} \times \mathbf{v}$$

Now, if $\hat{\mathbf{u}}$ lies in XY -plane, then

$$\hat{\mathbf{u}} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & u_2 & 0 \\ v_1 & v_2 & v_3 \end{vmatrix} \text{ or } \hat{\mathbf{u}} = u_1 \hat{\mathbf{i}} + u_2 \hat{\mathbf{j}}$$

$$\therefore \hat{\mathbf{w}} = (u_2 v_3) \hat{\mathbf{i}} - (u_1 v_3) \hat{\mathbf{j}} + (u_1 v_2 - v_1 u_2) \hat{\mathbf{k}} = \frac{1}{\sqrt{6}} (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\therefore u_2 v_3 = \frac{1}{\sqrt{6}}, u_1 v_3 = \frac{-1}{\sqrt{6}} \quad x=1$$

$$\Rightarrow \frac{u_2 v_3}{u_1 v_3} = -1 \text{ or } |u_1| = |u_2|$$

$\therefore$ option (c) is correct.

Now, if $\hat{\mathbf{u}}$ lies in XZ -plane, then $\hat{\mathbf{u}} = u_1 \hat{\mathbf{i}} + u_3 \hat{\mathbf{k}}$

$$\therefore \hat{\mathbf{u}} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ u_1 & 0 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$$\Rightarrow \hat{\mathbf{w}} = (-v_2 u_3) \hat{\mathbf{i}} - (u_1 v_3 - u_3 v_1) \hat{\mathbf{j}} + (u_1 v_2) \hat{\mathbf{k}}$$

$$\Rightarrow \hat{\mathbf{w}} = \frac{1}{\sqrt{6}} (\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}})$$

$$\Rightarrow -v_2 u_3 = \frac{1}{\sqrt{6}} \text{ and } u_1 v_2 = \frac{2}{\sqrt{6}}$$

$$\therefore |u_2| = 2 |u_3|$$

$\therefore$ option (d) is wrong.

7. (b,c,d) Here, $ax + 2y = \lambda$

$$\text{and } 3x - 2y = \mu$$

For $a = -3$, above equations will be parallel or coincident, i.e.

parallel for $\lambda + \mu \neq 0$ and coincident, if $\lambda + \mu = 0$ and if

$a \neq -3$, equations are intersecting, i.e. unique solution.

8. (b,c) Here,

$$f(x) = [x^2 - 3] = [x^2] - 3 = \begin{cases} -3, & -1/2 \leq x < 1 \\ -2, & 1 \leq x < \sqrt{2} \\ -1, & \sqrt{2} \leq x < \sqrt{3} \\ 0, & \sqrt{3} \leq x < 2 \\ 1, & x = 2 \end{cases}$$

$$\text{and } g(x) = |x| f(x) + |4x - 7| f(x)$$

$$= (|x| + |4x - 7|) f(x)$$

$$= (|x| + |4x - 7|) [x^2 - 3]$$

$$= \begin{cases} (-x - 4x - 7)(-3), & -1/2 \leq x < 1 \\ (x - 4x + 7)(-3), & 0 \leq x < 1 \\ (x - 4x + 7)(-2), & 1 \leq x < \sqrt{2} \\ (x - 4x + 7)(-1), & \sqrt{2} \leq x < \sqrt{3} \\ (x - 4x + 7)(0), & \sqrt{3} \leq x < 7/4 \\ (x + 4x - 7)(0), & 7/4 \leq x < 2 \\ (x + 4x - 7)(1), & x = 2 \end{cases}$$

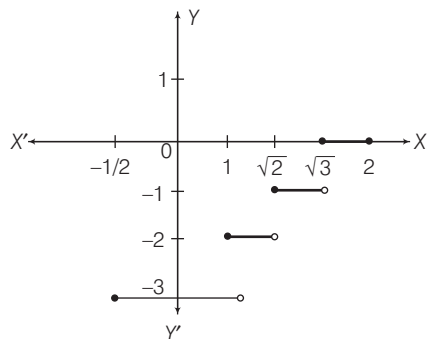
12

JEE Main and Advanced Solved Papers 2016

$$\therefore g(x) = \begin{cases} 15x + 21, & -1/2 \leq x < 0 \\ 9x - 21, & 0 \leq x < 1 \\ 6x - 14, & 1 \leq x < \sqrt{2} \\ 3x - 7, & \sqrt{2} \leq x < \sqrt{3} \\ 0, & \sqrt{3} \leq x < 2 \\ 5x - 7, & x = 2 \end{cases}$$

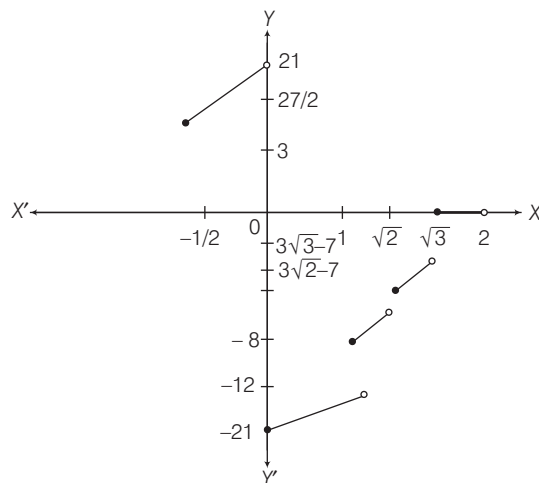
Now, the graphs of $f(x)$ and $g(x)$ are shown below.

Graph for $f(x)$



Clearly, $f(x)$ is discontinuous at 4 points.
 $\therefore$ Option (b) is correct.

Graph for $g(x)$



Clearly, $g(x)$ is not differentiable at 4 points, when
 $x \in (-1/2, 2)$.
 $\therefore$ Option (c) is correct.

**JEE Main & Adv./BITSAT/KERALA CEE
KCET/AP & TS EAMCET/VIT/MHT CET**

SOLVED PAPERS 2017

JEE Main

- If S is the set of distinct values of b for which the following system of linear equations
 $x + y + z = 1$, $x + ay + z = 1$ and $ax + by + z = 0$
 has no solution, then S is
 (a) an infinite set
 (b) a finite set containing two or more elements
 (c) singleton set
 (d) a empty set
- If 20 m of wire is available for fencing off a flower-bed in the form of a circular sector, then the maximum area (in sq m) of the flower-bed is
 (a) 12.5 (b) 10 (c) 25 (d) 30
- The area (in sq units) of the region
 $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is
 (a) $\frac{59}{12}$ (b) $\frac{3}{2}$ (c) $\frac{7}{3}$ (d) $\frac{5}{2}$
- If the image of the point $P(1, -2, 3)$ in the plane
 $2x + 3y - 4z + 22 = 0$ measured parallel to the line
 $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ is Q , then PQ is equal to
 (a) $3\sqrt{5}$ (b) $2\sqrt{42}$ (c) $\sqrt{42}$ (d) $6\sqrt{5}$
- For $x \in \left(0, \frac{1}{4}\right)$, if the derivative of
 $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then $g(x)$ equals
 (a) $\frac{9}{1+9x^3}$ (b) $\frac{3x\sqrt{x}}{1-9x^3}$ (c) $\frac{3x}{1-9x^3}$ (d) $\frac{3}{1+9x^3}$
- If $(2 + \sin x) \frac{dy}{dx} + (y + 1) \cos x = 0$ and $y(0) = 1$, then
 $y\left(\frac{\pi}{2}\right)$ is equal to
 (a) $\frac{1}{3}$ (b) $-\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) $\frac{4}{3}$
- If $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then $\text{adj}(3A^2 + 12A)$ is equal to
 (a) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (b) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (c) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$ (d) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$
- The distance of the point $(1, 3, -7)$ from the plane passing through the point $(1, -1, -1)$ having normal perpendicular to both the lines
 $\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$ and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$, is
 (a) $\frac{20}{\sqrt{74}}$ units (b) $\frac{10}{\sqrt{83}}$ units (c) $\frac{5}{\sqrt{83}}$ units (d) $\frac{10}{\sqrt{74}}$ units
- Let $I_n = \int \tan^n x \, dx$ ($n > 1$). If
 $I_4 + I_6 = a \tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to
 (a) $\left(-\frac{1}{5}, 1\right)$ (b) $\left(\frac{1}{5}, 0\right)$ (c) $\left(\frac{1}{5}, -1\right)$ (d) $\left(-\frac{1}{5}, 0\right)$
- The function $f: R \rightarrow \left[-\frac{1}{2}, \frac{1}{2}\right]$ defined as $f(x) = \frac{x}{1+x^2}$ is
 (a) invertible (b) injective but not surjective
 (c) surjective but not injective (d) neither injective nor surjective
- Let $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$, $\mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}}$ and $\mathbf{c}$ be a vector such that
 $|\mathbf{c} - \mathbf{a}| = 3$, $|(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$ and the angle between $\mathbf{c}$ and $\mathbf{a} \times \mathbf{b}$ is 30° . Then, $\mathbf{a} \cdot \mathbf{c}$ is equal to
 (a) $\frac{25}{8}$ (b) 2 (c) 5 (d) $\frac{1}{8}$
- The normal to the curve $y(x-2)(x-3) = x+6$ at the point, where the curve intersects the Y -axis passes through the point
 (a) $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$ (c) $\left(\frac{1}{2}, -\frac{1}{3}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$
- A box contains 15 green and 10 yellow balls. If 10 balls are randomly drawn one-by-one with replacement, then the variance of the number of green balls drawn is
 (a) $\frac{12}{5}$ (b) 6 (c) 4 (d) $\frac{6}{25}$
- $\int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$ is equal to
 (a) -2 (b) 2
 (c) 4 (d) -1

2 Solved Papers 2017

JEE Advanced

Paper 1

1. Let X and Y be two events such that $P(X) = \frac{1}{3}$,

$$P(X|Y) = \frac{1}{2} \text{ and } P(Y|X) = \frac{2}{5}. \text{ Then}$$

(One or More Than One Correct Option)

- (a) $P(Y) = \frac{4}{15}$ (b) $P(X|Y) = \frac{1}{2}$
(c) $P(X \cup Y) = \frac{2}{5}$ (d) $P(X \cap Y) = \frac{1}{5}$

2. Let $f: R \rightarrow (0, 1)$ be a continuous function. Then, which of the following function(s) has (have) the value zero at some point in the interval $(0, 1)$?

(One or More Than One Correct Option)

- (a) $e^x - \int_0^x f(t) \sin t \, dt$ (b) $f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t \, dt$
(c) $x - \int_0^{\frac{\pi}{2}-x} f(t) \cos t \, dt$ (d) $x^9 - f(x)$

3. Let $[x]$ be the greatest integer less than or equals to x . Then, at which of the following point(s) the function $f(x) = x \cos(\pi(x + [x]))$ is discontinuous ?

(One or More Than One Correct Option)

- (a) $x = -1$ (b) $x = 1$ (c) $x = 0$ (d) $x = 2$

4. Which of the following is(are) NOT the square of a 3×3 matrix with real entries?

- (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (c) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(One or More Than One Correct Option)

5. For a real number α , if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$

(Single Digit Integer)

6. Let $f: R \rightarrow R$ be a differentiable function such that

$$f(0) = 0, f\left(\frac{\pi}{2}\right) = 3 \text{ and } f'(0) = 1.$$

$$\text{If } g(x) = \int_x^{\frac{\pi}{2}} [f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t f(t)] dt$$

$$\text{for } x \in \left(0, \frac{\pi}{2}\right], \text{ then } \lim_{x \rightarrow 0} g(x) =$$

(Single Digit Integer)

Paragraph Based Questions

Directions (Q.Nos. 7-9) by appropriately matching the information given in the three columns of the following table.

Let $f(x) = x + \log_e x - x \log_e x$, $x \in (0, \infty)$

Column 1 contains information about zeros of $f(x)$, $f'(x)$ and $f''(x)$.

Column 2 contains information about the limiting behaviour of $f(x)$, $f'(x)$ and $f''(x)$ at infinity.

Column 3 contains information about increasing/decreasing nature of $f(x)$ and $f'(x)$.

Column-1	Column-2	Column-3
(I) $f(x) = 0$ for some $x \in (1, e^2)$	(i) $\lim_{x \rightarrow \infty} f(x) = 0$	(P) f is increasing in $(0, 1)$
(II) $f'(x) = 0$ for some $x \in (1, e)$	(ii) $\lim_{x \rightarrow \infty} f(x) = -\infty$	(Q) f is decreasing in (e, e^2)
(III) $f'(x) = 0$ for some $x \in (0, 1)$	(iii) $\lim_{x \rightarrow \infty} f'(x) = -\infty$	(R) f' is increasing in $(0, 1)$
(IV) $f''(x) = 0$ for some $x \in (1, e)$	(iv) $\lim_{x \rightarrow \infty} f''(x) = 0$	(S) f' is decreasing in (e, e^2)

7. Which of the following options is the only INCORRECT combination?

- (a) (I) (iii) (P) (b) (II) (iv) (Q)
(c) (II) (iii) (P) (d) (III) (i) (R)

8. Which of the following options is the only CORRECT combination?

- (a) (I) (ii) (R) (b) (III) (iv) (P) (c) (II) (iii) (S) (d) (IV) (i) (S)

9. Which of the following options is the only CORRECT combination?

- (a) (III) (iii) (R) (b) (IV) (iv) (S) (c) (II) (ii) (Q) (d) (I) (i) (P)

Paper 2

1. If $f: R \rightarrow R$ is a twice differentiable function such that

$$f'''(x) > 0 \text{ for all } x \in R, \text{ and } f\left(\frac{1}{2}\right) = \frac{1}{2}, f(1) = 1, \text{ then}$$

(Single Correct Option)

- (a) $f'(1) \leq 0$ (b) $f'(1) > 1$
(c) $0 < f'(1) \leq \frac{1}{2}$ (d) $\frac{1}{2} < f'(1) \leq 1$

2. If $y = y(x)$ satisfies the differential equation

$$8\sqrt{x}(\sqrt{9+\sqrt{x}}) dy = \left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1} dx, \quad x > 0 \text{ and}$$

$$y(0) = \sqrt{7}, \text{ then } y(256) =$$

(Single Correct Option)

- (a) 16 (b) 3
(c) 9 (d) 80

3. How many 3×3 matrices M with entries from $\{0, 1, 2\}$ are there, for which the sum of the diagonal entries of $M^T M$ is 5 ? (Single Correct Option)

(a) 198 (b) 162 (c) 126 (d) 135

4. Three randomly chosen nonnegative integers x, y and z are found to satisfy the equation $x + y + z = 10$. Then the probability that z is even, is (Single Correct Option)

(a) $\frac{1}{2}$ (b) $\frac{36}{55}$ (c) $\frac{6}{11}$ (d) $\frac{5}{11}$

5. Let O be the origin and let PQR be an arbitrary triangle. The point S is such that

$$\begin{aligned} \mathbf{OP} \cdot \mathbf{OQ} + \mathbf{OR} \cdot \mathbf{OS} &= \mathbf{OR} \cdot \mathbf{OP} + \mathbf{OQ} \cdot \mathbf{OS} \\ &= \mathbf{OQ} \cdot \mathbf{OR} + \mathbf{OP} \cdot \mathbf{OS} \end{aligned}$$

Then the triangle PQR has S as its (Single Correct Option)

(a) centroid (b) orthocentre (c) incentre (d) circumcentre

6. The equation of the plane passing through the point $(1, 1, 1)$ and perpendicular to the planes $2x + y - 2z = 5$ and $3x - 6y - 2z = 7$ is (Single Correct Option)

(a) $14x + 2y - 15z = 1$ (b) $-14x + 2y + 15z = 3$
(c) $14x - 2y + 15z = 27$ (d) $14x + 2y + 15z = 31$

7. If $f: R \rightarrow R$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in R$, and $f(0) = 1$ then (One or More Than One Correct Option)

(a) $f(x) > e^{2x}$ in $(0, \infty)$ (b) $f'(x) < e^{2x}$ in $(0, \infty)$
(c) $f(x)$ is increasing in $(0, \infty)$ (d) $f(x)$ is decreasing in $(0, \infty)$

8. If $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(x+1)} dx$, then

(One or More Than One Correct Option)

(a) $I > \log_e 99$ (b) $I < \log_e 99$ (c) $I < \frac{49}{50}$ (d) $I > \frac{49}{50}$

9. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in R^2 : x^3 \leq y \leq x, 0 \leq x \leq 1\}$ into two equal parts, then (One or More Than One Correct Option)

(a) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (b) $\alpha^4 + 4\alpha^2 - 1 = 0$

(c) $\frac{1}{2} < \alpha < 1$ (d) $0 < \alpha \leq \frac{1}{2}$

10. If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t) dt$, then

(One or More Than One Correct Option)

(a) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (b) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

(c) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (d) $g'\left(\frac{\pi}{2}\right) = -2\pi$

11. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then

(One or More Than One Correct Option)

(a) $f(x)$ attains its minimum at $x = 0$
(b) $f(x)$ attains its maximum at $x = 0$
(c) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$
(d) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

Paragraph Based Questions (Q. Nos. 12-13)

Let O be the origin and $\mathbf{OX}, \mathbf{OY}, \mathbf{OZ}$ be three unit vectors in the directions of the sides $\mathbf{QR}, \mathbf{RP}, \mathbf{PQ}$ respectively, of a triangle PQR .

12. If the triangle varies, then the minimum value of $\cos(P+Q) + \cos(Q+R) + \cos(R+P)$ is

(a) $-\frac{3}{2}$ (b) $\frac{3}{2}$ (c) $\frac{5}{3}$ (d) $-\frac{5}{3}$

13. $|\mathbf{OX} \times \mathbf{OY}| =$

(a) $\sin(P+Q)$ (b) $\sin(P+R)$ (c) $\sin(Q+R)$ (d) $\sin 2R$

BITSAT

1. Let $A = \{1, 2, 3, 4, 5\}$ and R be a relation defined by

$R = \{(x, y) : x, y \in A, x + y = 5\}$. Then, R is

(a) reflexive and symmetric but not transitive
(b) an equivalence relation
(c) symmetric but neither reflexive nor transitive
(d) neither reflexive nor symmetric but transitive

2. Let $A = [-1, 1]$ and $f: A \rightarrow A$ be defined as $f(x) = x|x|$ for all $x \in A$, then $f(x)$ is

(a) many-one and into function
(b) one-one and into function
(c) many-one and into function
(d) one-one and onto function

3. The length of perpendicular drawn from the point $(2, 3, 4)$ to line $\frac{4-x}{2} = \frac{y}{6} = \frac{1-z}{3}$, is

(a) $\frac{3}{7}\sqrt{101}$ (b) $\frac{2}{7}\sqrt{101}$ (c) $\frac{2}{7}\sqrt{103}$ (d) $\frac{3}{7}\sqrt{103}$

4. The image of the point $(1, 6, 3)$ in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is

(a) $(-1, 0, 7)$ (b) $(-1, 0, -7)$ (c) $(1, 0, 7)$ (d) $(2, 0, 7)$

5. The distances of the point $(1, -5, 9)$ from the plane $x + y + z = 5$ measured along a straight line $x = y = z$ is $2\sqrt{3}k$, then the value of k is

(a) 5 (b) 6 (c) $\sqrt{3}$ (d) 4

4 Solved Papers 2017

6. If $f(x) = \begin{cases} e^x, & x \leq 0 \\ |1-x|, & x > 0 \end{cases}$, then
 (a) $f(x)$ is differentiable at $x = 0$
 (b) $f(x)$ is continuous at $x = 0, 1$
 (c) $f(x)$ is differentiable at $x = 1$
 (d) None of the above
7. If a function $f: R \rightarrow R$ satisfy the equation $f(x+y) = f(x) + f(y)$, $\forall x, y$ and the function $f(x)$ is continuous at $x = 0$, then
 (a) $f(x)$ is continuous for all positive real values of x
 (b) $f(x)$ is continuous for all x
 (c) $f(x) = 0$ for all x
 (d) None of the above
8. The value of $f(0)$, so that the function $f(x) = \frac{2x - \sin^{-1} x}{2x + \tan^{-1} x}$ is continuous at each point in its domain, is
 (a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$
9. Consider the greatest integer function, defined by $f(x) = [x]$, $0 \leq x < 2$. Then
 (a) f is derivable at $x = 1$ (b) f is not derivable at $x = 1$
 (c) f is derivable at $x = 2$ (d) None of these
10. Let $f(x) = -2x^3 + 21x^2 - 60x + 41$, then
 (a) $f(x)$ is decreasing in $(-\infty, 1)$
 (b) $f(x)$ is decreasing in $(-\infty, 2)$
 (c) $f(x)$ is increasing in $(-\infty, 1)$
 (d) $f(x)$ is increasing in $(-\infty, 2)$
11. Rolle's theorem is not applicable for the function $f(x) = |x|$ in the interval $[-1, 1]$ because
 (a) $f'(1)$ does not exist (b) $f'(-1)$ does not exist
 (c) $f(x)$ is discontinuous at $x = 0$
 (d) $f'(0)$ does not exist
12. The area under the curve $y = |\cos x - \sin x|$, $0 \leq x \leq \frac{\pi}{2}$ and above X -axis, is
 (a) $2\sqrt{2}$ (b) $2\sqrt{2} - 2$ (c) $2\sqrt{2} + 2$ (d) 0
13. The solution of $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$, is
 (a) $\log \left[1 + \tan \left(\frac{x+y}{2} \right) \right] + C = 0$
 (b) $\log \left[1 + \tan \left(\frac{x+y}{2} \right) \right] = x + C$
 (c) $\log \left[1 - \tan \left(\frac{x+y}{2} \right) \right] = x + C$
 (d) None of the above
14. The area enclosed by the curves $y = x^3$ and $y = \sqrt{x}$ is
 (a) $\frac{5}{3}$ sq units (b) $\frac{5}{4}$ sq units (c) $\frac{5}{12}$ sq units (d) $\frac{12}{5}$ sq units
15. Let $\hat{a}$ and $\hat{b}$ be two non-collinear unit vectors. If $u = \hat{a} - (\hat{a} \cdot \hat{b})\hat{b}$ and $v = \hat{a} \times \hat{b}$, then $|v|$ is equal to
 (a) $|u|$ (b) $|u| + |v \cdot \hat{a}|$
 (c) $2|v|$ (d) $|v| + u \cdot (\hat{a} \times \hat{b})$
16. The maximum value of $z = 9x + 13y$ subject to constraints $2x + 3y \leq 18$, $2x + y \leq 10$, $x \geq 0$, $y \geq 0$ is
 (a) 130 (b) 81 (c) 79 (d) 99
17. If $\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$ and $f(1) = 2$,
 $f(p+q) = f(p) \cdot f(q)$, $\forall p, q \in R$, then
 $x^{f(1)} + y^{f(2)} + z^{f(3)} - \frac{(x+y+z)}{x^{f(1)} + y^{f(2)} + z^{f(3)}}$ is equal to
 (a) 0 (b) 1 (c) 2 (d) 3
18. If $\int \frac{(\sqrt{x})^5 dx}{(\sqrt{x})^7 + x^6} = \lambda \log \left(\frac{x^9}{x^9 + 1} \right) + C$, then $a + \lambda$ equal to
 (a) 2 (b) > 2 (c) < 2 (d) > 3

KERALA CEE

1. $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$ is equal to
 (a) 0 (b) $-\pi$ (c) $3\pi/2$ (d) $\pi/2$ (e) $\pi/4$
2. The minimum value of the function $\max(x, x^2)$ is equal to
 (a) 0 (b) 1 (c) 2 (d) $1/2$ (e) $3/2$
3. Let $f(x+y) = f(x)f(y)$ for all x and y . If $f(0) = 1$, $f(3) = 3$ and $f'(0) = 11$, then $f'(3)$ is equal to
 (a) 11 (b) 22 (c) 33 (d) 44 (e) 55
4. The order of the differential equation $\left(\frac{d^3 y}{dx^3} \right)^2 + \left(\frac{d^2 y}{dx^2} \right)^2 + \left(\frac{dy}{dx} \right)^5 = 0$ is
 (a) 3 (b) 4 (c) 1 (d) 5 (e) 6

5. $\int_{2016}^{2017} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4033 - x}} dx$ is equal to
(a) $1/4$ (b) $3/2$ (c) $2017/2$ (d) $1/2$ (e) 508
6. The solution of $dy/dx + y \tan x = \sec x$, $y(0) = 0$ is
(a) $y \sec x = \tan x$ (b) $y \tan x = \sec x$
(c) $\tan x = y \tan x$ (d) $x \sec x = \tan y$
(e) $y \cot x = \sec x$
7. If the vectors $2\hat{i} + 2\hat{j} + 6\hat{k}$, $2\hat{i} + \lambda\hat{j} + 6\hat{k}$ and $2\hat{i} - 3\hat{j} + \hat{k}$ are coplanar, then the value of λ is
(a) -10 (b) 1 (c) 0 (d) 10 (e) 2
8. The distance between $(2, 1, 0)$ and $2x + y + 2z + 5 = 0$ is
(a) 10 (b) $10/3$ (c) $10/9$ (d) 5 (e) 1
9. The equation of the plane that passes through the points $(1, 0, 2)$, $(-1, 1, 2)$, $(5, 0, 3)$ is
(a) $x + 2y - 4z + 7 = 0$ (b) $x + 2y - 3z + 7 = 0$
(c) $x - 2y + 4z + 7 = 0$ (d) $2y - 4z - 7 + x = 0$
(e) $x + 2y + 3z + 7 = 0$
10. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$ and $|\mathbf{a}| = 7, |\mathbf{b}| = 5, |\mathbf{c}| = 3$, then the angle between $\mathbf{c}$ and $\mathbf{b}$ is
(a) $\pi/3$ (b) $\pi/6$ (c) $\pi/4$ (d) π (e) 0
11. Let $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$, where $a > 0$. The minimum of f is attained at a point q and the maximum is attained at a point p . If $p^3 = q$, then a is equal to
(a) 1 (b) 3 (c) 2 (d) $\sqrt{2}$ (e) $1/2$
12. $\int_{-1}^1 \max\{x, x^3\} dx$ is equal to
(a) $3/4$ (b) $1/4$ (c) $1/2$ (d) 1 (e) 0
13. If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, then $f''(x)$ is equal to
(a) $x^3 + 6x^2$ (b) $6x^3$ (c) $3x$ (d) $6x^2$ (e) 0
14. $\int \frac{x^2}{1 + (x^3)^2} dx$ is equal to
(a) $\tan^{-1} x^2 + C$ (b) $2/3 \tan^{-1} x^3 + C$
(c) $1/3 \tan^{-1}(x^3) + C$ (d) $1/2 \tan^{-1} x^2 + C$
(e) $\tan^{-1} x^3 + C$
15. Let $f_n(x)$ be the n th derivative of $f(x)$. The least value of n so that $f_n = f_{n+1}$, where $f(x) = x^2 + e^x$ is
(a) 4 (b) 5 (c) 2 (d) 3 (e) 6
16. The value of $\cos\left(\tan^{-1}\left(\frac{3}{4}\right)\right)$ is
(a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{3}{4}$ (d) $\frac{2}{5}$ (e) 0
17. The angle between the pair of lines $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$ and $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$ is
(a) $\cos^{-1}\left(\frac{21}{9\sqrt{38}}\right)$ (b) $\cos^{-1}\left(\frac{23}{9\sqrt{38}}\right)$
(c) $\cos^{-1}\left(\frac{24}{9\sqrt{38}}\right)$ (d) $\cos^{-1}\left(\frac{25}{9\sqrt{38}}\right)$
(e) $\cos^{-1}\left(\frac{26}{9\sqrt{38}}\right)$
18. Let $\mathbf{a}$ be a unit vector. If $(\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} + \mathbf{a}) = 12$, then the magnitude of $\mathbf{x}$ is
(a) $\sqrt{8}$ (b) $\sqrt{9}$ (c) $\sqrt{10}$
(d) $\sqrt{13}$ (e) $\sqrt{12}$
19. Let $f(x+y) = f(x)f(y)$ and $f(x) = 1 + \sin(3x)g(x)$, where g is differentiable. The $f'(x)$ is equal to
(a) $3f(x)$ (b) $g(0)$ (c) $f(x)g(0)$ (d) $3g(x)$ (e) $3f(x)g(0)$
20. The roots of the equation $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$ are
(a) $1, 2$ (b) $-1, 2$ (c) $-1, -2$ (d) $1, -2$ (e) $1, 1$
21. Standard deviation of first n odd natural numbers is
(a) $\sqrt{n}$ (b) $\sqrt{\frac{(n+2)(n+1)}{3}}$ (c) $\sqrt{\frac{n^2-1}{3}}$
(d) n (e) $2n$
22. $\begin{vmatrix} 1 & 1 & 1 \\ p & q & r \\ p & q & r+1 \end{vmatrix}$ is equal to
(a) $q-p$ (b) $q+p$ (c) q (d) p (e) 0
23. Let $A = \begin{bmatrix} 5 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 20 & 5 \\ -1 & 0 \end{bmatrix}$. If $4A + 5B - C = 0$, then C is
(a) $\begin{bmatrix} 5 & 25 \\ -1 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 20 & 5 \\ -1 & 0 \end{bmatrix}$ (c) $\begin{bmatrix} 5 & -1 \\ 0 & 25 \end{bmatrix}$ (d) $\begin{bmatrix} 5 & 25 \\ -1 & 5 \end{bmatrix}$
(e) $\begin{bmatrix} 0 & 5 \\ 5 & 25 \end{bmatrix}$
24. If $U = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$, then U^{-1} is
(a) U^T (b) U (c) I (d) 0 (e) U^2
25. If $A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$, then A^{-1} is
(a) A^T (b) A^2 (c) A (d) I (e) 0

6 Solved Papers 2017

26. If $\begin{pmatrix} x+y & x-y \\ 2x+z & x+z \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, then the values of x , y and z are respectively

- (a) 0, 0, 1 (b) 1, 1, 0 (c) -1, 0, 0 (d) 0, 0, 0 (e) 1, 1, 1

27. $\begin{pmatrix} 7 & 1 & 5 \\ 8 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is equal to

- (a) $\begin{pmatrix} 16 \\ 27 \end{pmatrix}$ (b) $\begin{pmatrix} 27 \\ 16 \end{pmatrix}$ (c) $\begin{pmatrix} 15 \\ 16 \end{pmatrix}$ (d) $\begin{pmatrix} 16 \\ 15 \end{pmatrix}$ (e) $\begin{pmatrix} 16 \\ 16 \end{pmatrix}$

28. If $\begin{pmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 4 & a \end{pmatrix}$ is singular, then the value of a is

- (a) $a = -6$ (b) $a = 5$ (c) $a = -5$ (d) $a = 6$ (e) $a = 0$

29. If $\begin{pmatrix} 1 & 2 & -3 \\ 0 & 4 & 5 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, then (x, y, z) is equal to

- (a) (1, 6, 6) (b) (1, -6, 1) (c) (1, 1, 6) (d) (6, -1, 1) (e) (-1, 6, 1)

30. If $A = \begin{pmatrix} 1 & 5 \\ 0 & 2 \end{pmatrix}$, then

- (a) $A^2 - 2A + 2I = 0$ (b) $A^2 - 3A + 2I = 0$
(c) $A^2 - 5A + 2I = 0$ (d) $2A^2 - A + I = 0$
(e) $A^2 + 3A + 2I = 0$

31. If $\begin{pmatrix} 2x+y & x+y \\ p-q & p+q \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then (x, y, p, q) equals

- (a) 0, 1, 0, 0 (b) 0, -1, 0, 0 (c) 1, 0, 0, 0 (d) 0, 1, 0, 1
(e) 1, 0, 1, 0

32. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-w^2 & w^2 \\ 1 & w & w^4 \end{vmatrix}$, where $w \neq 1$ is a complex

number such that $w^3 = 1$. Then Δ equals

- (a) $3w + w^2$ (b) $3w^2$ (c) $3(w = w)^2$
(d) $-3w^2$ (e) $3w^2 + 1$

33. If $\begin{vmatrix} 3i & -9i & 1 \\ 2 & 9i & -1 \\ 10 & 9 & i \end{vmatrix} = x + iy$, then

- (a) $x = 1, y = 1$ (b) $x = 0, y = 1$ (c) $x = 1, y = 0$ (d) $x = 0, y = 0$
(e) $x = -1, y = 0$

34. If $A = \begin{vmatrix} 4 & k & k \\ 0 & k & k \\ 0 & 0 & k \end{vmatrix}$ and $\det(A) = 256$, then $|k|$ equals

- (a) 4 (b) 5 (c) 6 (d) 7 (e) 8

35. If $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, then $A^n + nI$ is equal to

- (a) I (b) nA (c) $I + nA$ (d) $I - nA$ (e) $nA - I$

36. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, then A^{2017} is equal to

- (a) $2^{2015}A$ (b) $2^{2016}A$ (c) $2^{2014}A$ (d) $2^{2017}A$ (e) $2^{2020}A$

37. If $\int_0^x f(t) dt = x^2 + e^x$ ($x > 0$), then $f(1)$ is equal to

- (a) $1 + e$ (b) $2 + e$ (c) $3 + e$ (d) e (e) 0

38. $\int \frac{x+1}{x^{1/2}} dx =$

- (a) $-x^{3/2} + x^{1/2} + C$ (b) $x^{1/2}$
(c) $x^{3/2} + 2x^{1/2} + C$ (d) $x^{3/2} + x^{1/2} + C$
(e) $x^{3/2}$

39. $\int \frac{dx}{e^x + e^{-x} + 2}$ is equal to

- (a) $\frac{1}{e^x + 1} + C$ (b) $\frac{-1}{e^x + 1} + C$ (c) $\frac{1}{1 + e^{-x}} + C$ (d) $\frac{1}{e^{-x} - 1} + C$
(e) $\frac{1}{e^x - 1} + C$

40. $\int_{-1}^0 \frac{dx}{x^2 + x + 2}$ is equal to

- (a) $\pi/4$ (b) $\pi/2$ (c) π (d) 0 (e) $-\pi$

Karnataka CET

1. If an LPP admits optimal solution at two consecutive vertices of a feasible region, then

- (a) the LPP under consideration is not solvable
(b) the LPP under consideration must be reconstructed
(c) the required optimal solution is at the mid-point of the line joining two points
(d) the optimal solution occurs at every point on the line joining these two points

2. Let $\Delta = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$ and

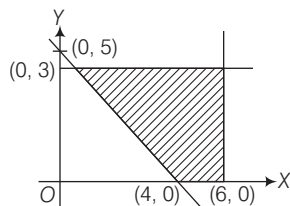
$\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$, then $\begin{vmatrix} Ax & By & Cy \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$

- (a) $\Delta_1 = 2\Delta$ (b) $\Delta_1 = -\Delta$ (c) $\Delta_1 = \Delta$ (d) $\Delta_1 \neq \Delta$

3. The function $f(x) = x^2 + 2x - 5$ is strictly increasing in the interval

(a) $(-\infty, -1)$ (b) $[-1, \infty)$
(c) $(-\infty, -1]$ (d) $(-1, \infty)$

4. The shaded region in the figure is the solution set of the inequations



(a) $5x + 4y \leq 20, x \leq 6, y \leq 3, x \geq 0, y \geq 0$
(b) $5x + 4y \geq 20, x \geq 6, y \leq 3, x \geq 0, y \geq 0$
(c) $5x + 4y \geq 20, x \leq 6, y \leq 3, x \geq 0, y \geq 0$
(d) $5x + 4y \geq 20, x \leq 6, y \geq 3, x \geq 0, y \geq 0$

5. The point on the curve $y^2 = x$ where the tangent makes an angle of $\pi/4$ with X -axis is

(a) (4, 2) (b) $(\frac{1}{2}, \frac{1}{4})$ (c) $(\frac{1}{4}, \frac{1}{2})$ (d) (1, 1)

6. $\int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$

(a) 1 (b) 0 (c) $\frac{\pi}{2}$ (d) $-\frac{\pi}{2}$

7. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors, then angle between $\mathbf{a}$ and $\mathbf{b}$ for $\sqrt{3}\mathbf{a} - \mathbf{b}$ to be unit vector is

(a) 45° (b) 60° (c) 90° (d) 30°

8. $\int_{-5}^5 |x + 2| dx$ is equal to

(a) 28 (b) 30
(c) 29 (d) 27

9. $\int_0^{\pi/2} \frac{1}{a^2 \cdot \sin^2 x + b^2 \cdot \cos^2 x} dx$

(a) $\frac{\pi}{2ab}$ (b) $\frac{\pi b}{4a}$ (c) $\frac{\pi a}{2b}$ (d) $\frac{\pi a}{4b}$

10. $\int \sqrt{x^2 + 2x + 5} dx$ is equal to

(a) $\frac{1}{2}(x+1)\sqrt{x^2+2x+5} + 2\log|x+1+\sqrt{x^2+2x+5}| + C$
(b) $(x+1)\sqrt{x^2+2x+5} + \frac{1}{2}\log|x+1+\sqrt{x^2+2x+5}| + C$
(c) $(x+1)\sqrt{x^2+2x+5} + 2\log|x+1+\sqrt{x^2+2x+5}| + C$
(d) $(x+1)\sqrt{x^2+2x+5} - 2\log|x+1+\sqrt{x^2+2x+5}| + C$

11. The range of $\sec^{-1} x$ is

(a) $[0, \pi] - \left\{\frac{\pi}{2}\right\}$ (b) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
(c) $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (d) $[0, \pi]$

12. $\int \frac{(x+3)e^x}{(x+4)^2} dx$ is equal to

(a) $\frac{e^x}{(x+4)^2} + C$ (b) $\frac{e^x}{(x+3)} + C$
(c) $\frac{1}{(x+4)^2} + C$ (d) $\frac{e^x}{(x+4)} + C$

13. If $y = \begin{vmatrix} f(x) & g(x) & h(x) \\ l & m & n \\ a & b & c \end{vmatrix}$, then $\frac{dy}{dx}$ is equal to

(a) $\begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l & m & n \\ a & b & c \end{vmatrix}$ (b) $\begin{vmatrix} l & m & n \\ f(x) & g(x) & h(x) \\ a & b & c \end{vmatrix}$
(c) $\begin{vmatrix} f'(x) & l & a \\ g'(x) & m & b \\ h'(x) & n & c \end{vmatrix}$ (d) $\begin{vmatrix} l & m & n \\ a & b & c \\ f'(x) & g'(x) & h'(x) \end{vmatrix}$

14. General solution of differential equations $\frac{dy}{dx} + y = 1 (y \neq 1)$ is

(a) $\log \left| \frac{1}{1-y} \right| = x + C$ (b) $\log |1-y| = x + C$
(c) $\log |1+y| = x + C$ (d) $\log \left| \frac{1}{1-y} \right| = -x + C$

15. If A is a square matrix of order 3×3 , then $|KA|$ is equal to

(a) $K^2|A|$ (b) $K|A|$ (c) $3K|A|$ (d) $K^3|A|$

16. If $A = \frac{1}{\pi} \begin{vmatrix} \sin^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & \cot^{-1}(\pi x) \end{vmatrix}$

$B = \begin{vmatrix} -\cos^{-1}(\pi x) & \tan^{-1}\left(\frac{x}{\pi}\right) \\ \sin^{-1}\left(\frac{x}{\pi}\right) & -\tan^{-1}(\pi x) \end{vmatrix}$, then $A - B$ is

(a) 0 (b) $\frac{1}{2}I$ (c) I (d) $2I$

17. The probability distribution of X is

X	0	1	2	3
$P(X)$	0.3	k	$2k$	$2k$

The value of k is

(a) 0.7 (b) 0.3 (c) 1 (d) 0.14

18. The degree of the differential equation

$\left[1 + \left(\frac{dy}{dx} \right)^2 \right]^2 = \frac{d^2 y}{dx^2}$ is

(a) 3 (b) 2 (c) 1 (d) 4

8 Solved Papers 2017

19. $\int \frac{\cos 2x - \cos 2\theta}{\cos x - \cos \theta} dx$ is equal to
 (a) $2(\sin x + x \cos \theta) + C$ (b) $2(\sin x - x \cos \theta) + C$
 (c) $2(\sin x + 2x \cos \theta) + C$ (d) $2(\sin x - 2x \cos \theta) + C$
20. If $\mathbf{a} = 2\hat{i} + \lambda\hat{j} + \hat{k}$ and $\mathbf{b} = \hat{i} + 2\hat{j} + 3\hat{k}$ are orthogonal, then value of λ is
 (a) $3/2$ (b) 1 (c) 0 (d) $-5/2$
21. If $f(x) = \begin{cases} kx^2 & \text{if } x \leq 2 \\ 3 & \text{if } x > 2 \end{cases}$ is continuous at $x = 2$, then the value of k is
 (a) $3/4$ (b) 4 (c) $4/3$ (d) 3
22. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$, then the value of $\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$ is equal to
 (a) $3/2$ (b) 1
 (c) 3 (d) $-3/2$
23. Let $f: R \rightarrow R$ be defined by $f(x) = x^4$, then
 (a) f may be one-one and onto
 (b) f is neither one-one nor onto
 (c) f is one-one and onto
 (d) f is one-one but not onto
24. The plane $2x - 3y + 6z - 11 = 0$ makes an angle $\sin^{-1}(\alpha)$ with X -axis, the value of α is equal to
 (a) $\frac{\sqrt{2}}{3}$ (b) $\frac{2}{7}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{3}{7}$
25. $\int_{0.2}^{3.5} [x] dx$ is equal to
 (a) 3.5 (b) 4 (c) 4.5 (d) 3
26. If $y = \tan^{-1}\left(\frac{\sin x + \cos x}{\cos x - \sin x}\right)$, then $\frac{dy}{dx}$ is equal to
 (a) $1/2$ (b) 0 (c) $\pi/4$ (d) 1
27. The integrating factor of the differential equation $x \cdot \frac{dy}{dx} + 2y = x^2$ is ($x \neq 0$)
 (a) x (b) $\log |x|$ (c) x^2 (d) $e^{\log x}$
28. The derivative of $\cos^{-1}(2x^2 - 1)$ w.r.t. $\cos^{-1}x$ is
 (a) $1 - x^2$ (b) $\frac{2}{x}$ (c) $\frac{-1}{2\sqrt{1-x^2}}$ (d) 2
29. The rate of change of volume of a sphere with respect to its surface area when the radius is 4 cm is
 (a) $4 \text{ cm}^3/\text{cm}^2$ (b) $6 \text{ cm}^3/\text{cm}^2$ (c) $2 \text{ cm}^3/\text{cm}^2$ (d) $8 \text{ cm}^3/\text{cm}^2$
30. $\int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx$ is equal to
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{3}$
31. The area of the region bounded by the curve $y = x^2$ and the line $y = 16$
 (a) $\frac{64}{3}$ sq units (b) $\frac{32}{3}$ sq units
 (c) $\frac{256}{3}$ sq units (d) $\frac{128}{3}$ sq units
32. Area of the region bounded by the curve $y = \cos x$, $x = 0$ and $x = \pi$ is
 (a) 4 sq units (b) 3 sq units
 (c) 1 sq units (d) 2 sq units
33. The value of c in mean value theorem for the function $f(x) = x^2$ in $[2, 4]$ is
 (a) 3 (b) $7/2$ (c) 4 (d) 2
34. If $\tan^{-1}x + \tan^{-1}y = \frac{4\pi}{5}$, then $\cot^{-1}x + \cot^{-1}y$ is equal to
 (a) $\frac{2\pi}{5}$ (b) $\frac{\pi}{5}$ (c) $\frac{3\pi}{5}$ (d) π
35. If a matrix A is both symmetric and skew symmetric, then
 (a) A is diagonal matrix (b) A is a zero matrix
 (c) A is scalar matrix (d) A is square matrix
36. The distance of the point $(-2, 4, -5)$ from the line $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ is
 (a) $\sqrt{\frac{37}{10}}$ (b) $\frac{37}{10}$ (c) $\frac{\sqrt{37}}{10}$ (d) $\frac{37}{\sqrt{10}}$

Andhra Pradesh EAMCET

1. Given below is the distribution of a random variable X

$X = x$	1	2	3	4
$P(X = x)$	λ	2λ	3λ	4λ

If $\alpha = P(X < 3)$ and $\beta = P(X > 2)$, then $\alpha : \beta =$

- (a) $2 : 5$ (b) $3 : 4$
 (c) $4 : 5$ (d) $3 : 7$

2. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is defined by

$$f(x) = \begin{cases} x-1, & \text{for } x \leq 1 \\ 2-x^2, & \text{for } 1 < x \leq 3 \\ x-10, & \text{for } 3 < x < 5 \\ 2x, & \text{for } x \geq 5 \end{cases}$$

then the set of points of discontinuity of f is

- (a) $\mathbb{R} - \{1, 5\}$ (b) $\{1, 3, 5\}$ (c) $\{1, 5\}$ (d) $\mathbb{R} - \{1, 3, 5\}$

3. If a non-zero vector is parallel to the line of intersection of the plane determined by the vectors $\hat{j} - \hat{k}$, $3\hat{j} - 2\hat{k}$ and the plane determined by the vectors $2\hat{i} + 3\hat{j}$, $\hat{i} - 3\hat{j}$, then the angle between the vectors $\mathbf{a}$ and $\hat{i} + \hat{j} + \hat{k}$ is
 (a) $\sin^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\pm \frac{2}{\sqrt{3}}\right)$
 (c) $\tan^{-1}\sqrt{3}$ (d) $\cos^{-1}\left(\pm \frac{1}{\sqrt{3}}\right)$
4. For $x^2 - 4 \neq 0$, the value of $\frac{d}{dx} \left[\log \left\{ e^x \left(\frac{x-2}{x+2} \right)^{3/4} \right\} \right]$ at $x = 3$ is
 (a) $\frac{8}{5}$ (b) 2 (c) 1 (d) $\frac{8e^3}{5}$
5. If $y = \frac{\sinh^{-1} x}{\sqrt{1+x^2}}$ then $(1+x^2)y_2 + 3xy_1 + y =$
 (a) 2 (b) 1 (c) -1 (d) 0
6. The area (in sq. units) enclosed between the curves $y = x^2$ and $y = |x|$ is
 (a) $\frac{2}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{3}$ (d) 1
7. $\int \frac{5x^2 + 3}{x^2(x^2 - 2)} dx =$
 (a) $\frac{13}{2\sqrt{2}} \log \left| \frac{\sqrt{2}-x}{\sqrt{2}+x} \right| + \frac{3}{2x} + C$ (b) $\frac{13}{4\sqrt{2}} \log \left| \frac{x+\sqrt{2}}{x-\sqrt{2}} \right| + \frac{3}{2x} + C$
 (c) $\frac{13}{4\sqrt{2}} \log \left| \frac{x-\sqrt{2}}{x+\sqrt{2}} \right| + \frac{3}{2x} + C$ (d) $\frac{5}{3\sqrt{2}} \log \left| \frac{x+\sqrt{2}}{x-\sqrt{2}} \right| + \frac{3}{5}x + C$
8. If $y = \tan^{-1} \left\{ \frac{x}{1+\sqrt{1-x^2}} \right\} + \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}} \right\}$, then $\frac{dy}{dx} =$
 (a) $\frac{1-2x}{2\sqrt{1-x^2}}$ (b) $\frac{1-2x}{x\sqrt{1-x^2}}$ (c) $\frac{2x+1}{x\sqrt{1-x}}$ (d) $\frac{2-x}{2\sqrt{1-x^2}}$
9. The equation of the plane through (4,4,0) and perpendicular to the planes $2x + y + 2z + 3 = 0$ and $3x + 3y + 2z - 8 = 0$ is
 (a) $4x + 3y + 3z = 28$ (b) $4x - 2y - 3z = 8$
 (c) $4x + 2y + 3z = 24$ (d) $4x + 2y - 3z = 24$
10. The solution of the equation $(x - 4y^3) \frac{dy}{dx} - y = 0, (y > 0)$ is
 (a) $x = y^3 + cy$ (b) $x + 2y^3 = cy$
 (c) $y = x^3 + cx$ (d) $y + 2x^3 = cx$
11. A bag P contains 5 white marbles and 3 black marbles. Four marbles are drawn at random from P and are put in an empty bag Q . If a marble drawn at random from Q is found to be black then the probability that all the three black marbles in P are transferred to the bag Q is.
 (a) $\frac{1}{7}$ (b) $\frac{6}{7}$ (c) $\frac{1}{8}$ (d) $\frac{7}{8}$
12. $\operatorname{sech}^{-1} \left(\frac{1}{\sqrt{2}} \right) + \operatorname{cosech}^{-1}(-1) =$
 (a) 0 (b) $\sqrt{2} + 1$ (c) $\sqrt{2}$ (d) $\sqrt{2} - 1$
13. If the points having the position vectors $3\hat{i} - 2\hat{j} - \hat{k}$, $2\hat{i} + 3\hat{j} - 4\hat{k}$, $-\hat{i} + \hat{j} + 2\hat{k}$ and $4\hat{i} + 5\hat{j} + \lambda\hat{k}$ are coplanar, then $\lambda =$
 (a) $-\frac{146}{17}$ (b) 8 (c) -8 (d) $\frac{146}{17}$
14. ΔABC is formed by $A(1, 8, 4)$, $B(0, -11, 4)$ and $C(2, -3, 1)$. If D is the foot of the perpendicular from A to BC . Then the coordinates of D are
 (a) $(-4, 5, 2)$ (b) $(4, 5, -2)$ (c) $(4, -5, 2)$ (d) $(4, -5, -2)$
15. For the function $f(x) = (x-1)(x-2)$ defined on $\left[0, \frac{1}{2}\right]$, the value of c satisfying Lagrange's mean value theorem is
 (a) $\frac{1}{5}$ (b) $\frac{1}{3}$ (c) $\frac{1}{7}$ (d) $\frac{1}{4}$
16. A container is the shape of an inverted cone. Its height is 6 m and radius is 4 m at the top. If it is filled with water at the rate of $3\text{ m}^3/\text{min}$ then the rate of change of height of water (in m/min) when the water level is 3 m is
 (a) $\frac{3}{4\pi}$ (b) $\frac{2}{9\pi}$ (c) 16π (d) 2π
17. If the mean and variance of a binomial variate X are $\frac{4}{3}, \frac{8}{9}$ respectively, then $P(X = 2) =$
 (a) $\frac{4}{27}$ (b) $\frac{16}{81}$ (c) $\frac{8}{27}$ (d) $\frac{8}{81}$
18. If $\cot(\cos^{-1} x) = \sec \left\{ \tan^{-1} \left(\frac{a}{\sqrt{b^2 - a^2}} \right) \right\} : b > a$, then $x =$
 (a) $\frac{b}{\sqrt{2b^2 - a^2}}$ (b) $\frac{\sqrt{b^2 - a^2}}{ab}$ (c) $\frac{a}{\sqrt{2b^2 - a^2}}$ (d) $\frac{\sqrt{b^2 - a^2}}{a}$
19. In ΔABC , L, M, N are points on BC, CA, AB respectively, dividing them in the ratio $1 : 2, 2 : 3, 3 : 5$. If the point K divides AB in the ratio $5 : 3$, then $\frac{|AL + BM + CN|}{|CK|} =$
 (a) $\frac{5}{8}$ (b) $\frac{2}{5}$ (c) $\frac{3}{5}$ (d) $\frac{1}{15}$

10 Solved Papers 2017

20. If $\int_0^{10} f(x) dx = 5$, then $\sum_{k=1}^{10} \int_0^1 f(k-1+x) dx =$
 (a) 50 (b) 10 (c) 5 (d) 20
21. $\int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx =$
 (a) $\frac{-1}{e} \log |x^e + e^x| + C$ (b) $-e \log |x^e + e^x| + C$
 (c) $\frac{1}{e} \log |x^e + e^x| + C$ (d) $e \log |x^e + e^x| + C$
22. If $\Delta = \begin{vmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ -1 & -\cos \theta & 1 \end{vmatrix}$, then Δ lies in the interval
 (a) $[2, 4]$ (b) $(2, 4)$ (c) $[1, 4]$ (d) $[-1, 1]$
23. If **a**, **b** and **c** are non-zero vectors such that **a** and **b** are not perpendicular to each other, then the vector **r** which is perpendicular to **a** and satisfying **r** $\times$ **b** = **c** $\times$ **b** is
 (a) $\frac{(a \times b) \times c}{c \cdot a}$ (b) $\frac{b \times (a \times c)}{b \cdot c}$ (c) $\frac{(b \times c) \times a}{a \cdot b}$ (d) $\frac{(c \times b) \times a}{a \cdot c}$
24. If a, b, c are non-zero real numbers and if the equations $(a-1)x = y+z$, $(b-1)y = z+x$, $(c-1)z = x+y$ have a non-trivial solution, then $ab+bc+ca =$
 (a) $a^2b^2c^2$ (b) 0 (c) abc (d) $a+b+c$
25. If a cylindrical vessel of given volume V with no lid on the top is to be made from a sheet of metal, then the radius (r) and height (h) of the vessel so that the metal sheet used is minimum is
 (a) $r = 3\sqrt{\frac{\pi}{V}}$, $h = 3\sqrt{\frac{\pi}{V}}$ (b) $r = \sqrt{\pi V}$, $h = \sqrt{\pi V}$
 (c) $r = 3\sqrt{\frac{V}{\pi}}$, $h = 3\sqrt{\frac{V}{\pi}}$ (d) $r = \sqrt{\frac{V}{\pi}}$, $h = \sqrt{\frac{V}{\pi}}$
26. $\int \frac{x + \sin x}{1 + \cos x} dx =$
 (a) $x \tan \frac{x}{2} + C$ (b) $x \sin \frac{x}{2} + \cos \frac{x}{2} + C$
 (c) $x \tan \frac{x}{2} + \sec \frac{x}{2} + C$ (d) $x \sec \frac{x}{2} + \tan \frac{x}{2} + C$
27. If $I_n = \int \frac{\sin nx}{\cos x} dx$, then $I_n =$
 (a) $\frac{-2}{n-1} \cos(n-1)x - I_{n-2}$ (b) $\frac{2}{n-1} \cos(n-1)x + I_{n-2}$
 (c) $\frac{-2}{n+1} \sin(n+1)x - I_{n-2}$ (d) $\frac{-2}{n+1} \cos(n-1)x - I_{n-2}$
28. Let N be the set of all natural numbers, Z be the set of all integers and $\sigma : N \rightarrow Z$ be defined by

$$\sigma(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$
 Then
 (a) σ is onto but not one-one (b) σ is one-one but not onto
 (c) σ is neither one-one nor (d) σ is one-one and onto
29. The differential equation corresponding to the family of circles in the plane touching the Y -axis at the origin, is
 (a) $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$ (b) $\frac{dy}{dx} = \frac{2xy}{x^2 + y^2}$
 (c) $\frac{dy}{dx} = \frac{x^2 - y^2}{2xy}$ (d) $\frac{dy}{dx} = \frac{x^2 + y^2}{2xy}$
30. If a system of three linear equations in three unknowns, which is in the matrix equation form of $AX = D$, is inconsistent, then $\frac{\text{rank of } A}{\text{rank of } AD}$ is
 (a) Less than one (b) Greater than or equal to one
 (c) One (d) Greater than one
31. The triad (x, y, z) of real number such that
 $(3\hat{i} - \hat{j} + 2\hat{k}) = (2\hat{i} + 3\hat{j} - \hat{k})x + (\hat{i} - 2\hat{j} + 2\hat{k})y + (-2\hat{i} + \hat{j} - 2\hat{k})z$ is
 (a) $(-2, 5, 3)$ (b) $(2, -5, 3)$ (c) $(2, 5, 3)$ (d) $(2, 5, -3)$
32. If the volume of the tetrahedron formed by the coterminal edges a, b and c is 4, then the volume of the parallelepiped formed by the coterminal edges **a** $\times$ **b**, **b** $\times$ **c** and **c** $\times$ **a** is
 (a) 576 (b) 48 (c) 16 (d) 144

Telangana State EAMCET

1. Match the following

	List I		List II
I.	$\int_{-1}^1 x x dx$	(a)	$\frac{\pi}{2}$
II.	$\int_0^{\frac{\pi}{2}} \left(1 + \log \left(\frac{4+3\sin x}{4+3\cos x} \right) \right) dx$	(b)	$\int_0^a f(x) dx$
III.	$\int_0^a f(x) dx$	(c)	$\int_0^a [f(x) + f(-x)] dx$

	List I		List II
IV.	$\int_{-a}^a f(x) dx$	(d)	0
		(e)	$\int_0^a f(a-x) dx$
	I II III IV		I II III IV
	(a) d a e c		(b) d a c b
	(c) d c a e		(d) a d b c

2. If f is differentiable, $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$, $f(3) = 3$, $f'(0) = 11$, then $f'(3) =$
 (a) $\frac{3}{11}$ (b) $\frac{11}{3}$ (c) 8 (d) 33

3. $\int_0^{\pi} \frac{x dx}{4\cos^2 x + 9\sin^2 x} =$
 (a) $\frac{\pi^2}{12}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi^2}{6}$ (d) $\frac{\pi^2}{3}$

4. The probability distribution of a random variable X is given below

$X = k$	0	1	2	3	4
$P(X = k)$	0.1	0.4	0.3	0.2	0

The variance of X is

- (a) 1.6 (b) 0.24 (c) 0.84 (d) 0.75

5. If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 4 \end{bmatrix}$, $A = B + C$, $B = B^T$ and $C = -C^T$,

then $C =$

(a) $\begin{bmatrix} 0 & 0.5 & 0 \\ -0.5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & -0.5 & 0 \end{bmatrix}$
 (c) $\begin{bmatrix} 0 & -0.5 & 0.5 \\ 0.5 & 0 & 0 \\ -0.5 & 0 & 0 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & 0.5 & 0 \\ -0.5 & 0 & 0.5 \\ 0 & -0.5 & 0 \end{bmatrix}$

6. If $\hat{i}$ is a unit vector, then $|\mathbf{a} \times \hat{i}|^2 + |\mathbf{a} \times \hat{j}|^2 + |\mathbf{a} \times \hat{k}|^2 =$
 (a) 2 (b) 4 (c) 1 (d) 0

7. If $\int e^{2x} f'(x) dx = g(x)$, then $\int (e^{2x} f(x) + e^{2x} f'(x)) dx =$
 (a) $\frac{1}{2} [e^{2x} f(x) - g(x)] + C$ (b) $\frac{1}{2} [e^{2x} f(x) + g(x)] + C$
 (c) $\frac{1}{2} [e^{2x} f(2x) + g(x)] + C$ (d) $\frac{1}{2} [e^{2x} f'(x) + g(x)] + C$

8. $\int x^4 e^{2x} dx =$

(a) $\frac{e^{2x}}{4} (2x^4 - 4x^3 + 6x^2 - 6x + 3) + C$
 (b) $\frac{e^{2x}}{2} (2x^4 - 4x^3 + 6x^2 - 6x + 3) + C$
 (c) $\frac{e^{2x}}{8} (2x^4 + 4x^3 + 6x^2 + 6x + 3) + C$
 (d) $-\frac{e^{2x}}{4} (2x^4 + 4x^3 + 6x^2 + 6x + 3) + C$

9. If $\mathbf{a} = x\hat{i} + y\hat{j} + z\hat{k}$, then

$(\mathbf{a} \times \hat{i}) \cdot (\hat{i} + \hat{j}) + (\mathbf{a} \times \hat{j}) \cdot (\hat{j} + \hat{k}) + (\mathbf{a} \times \hat{k}) \cdot (\hat{k} + \hat{i}) =$

- (a) $x - y + z$ (b) $x + y + z$ (c) $x + y - z$ (d) $-x + y + z$

10. Let $f: (-1, 1) \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. If $g(x) = (f(2f(x) + 2))^2$, then $g'(0) =$
 (a) 0 (b) -2 (c) 4 (d) -4

11. The solution of $\frac{dy}{dx} = \frac{x+y}{x-y}$ is

(a) $\tan^{-1}\left(\frac{y}{x}\right) = \log \sqrt{x^2 + y^2} + C$
 (b) $\tan^{-1}\left(\frac{y}{x}\right) = \log \sqrt{x^2 - y^2} + C$
 (c) $\sin^{-1}\left(\frac{y}{x}\right) = \log \sqrt{x^2 + y^2} + C$
 (d) $\cos^{-1}\left(\frac{y}{x}\right) = \log \sqrt{x^2 - y^2} + C$

12. The solution of $(y - 3x^2)dx + xdy = 0$ is

(a) $y(x) = \sin x + \frac{1}{x^2} + C$ (b) $y(x) = \cos x - \frac{1}{x^2} + C$
 (c) $y(x) = x^2 + \frac{C}{x}$ (d) $y(x) = \sqrt{x} + \frac{C}{x}$

13. A point on the plane that passes through the points (1, -1, 6), (0, 0, 7) and perpendicular to the plane $x - 2y + z = 6$ is

- (a) (1, -1, 2) (b) (1, 1, 2) (c) (-1, 1, 2) (d) (1, 1, -2)

14. If the slope of the tangent of the curve $y = ax^3 + bx + 4$ at (2, 14) is 21, then the values of a and b respectively

- (a) 2, -3 (b) 3, -2 (c) -3, -2 (d) 2, 3

15. The probability distribution of a random variable X is given below

x	1	2	3	4	5	6
$P(X = x)$	a	a	a	b	b	0.3

If mean of X is 4.2, then a and b are respectively equal to

- (a) 0.3, 0.2 (b) 0.1, 0.4 (c) 0.1, 0.2 (d) 0.2, 0.1

16. The equation of tangent to the curve $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ at

the point (a, b) is

(a) $\frac{x}{a} = -\frac{y}{b}$ (b) $\frac{x}{a} + \frac{y}{b} = 2$ (c) $\frac{x}{a} = \frac{y}{b}$ (d) $\frac{x}{a} + \frac{y}{b} = n$

17. If $\Delta = \begin{vmatrix} 1 & 5 & 6 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{vmatrix}$ and $\Delta' = \begin{vmatrix} 1 & 0 & 1 \\ 3 & 0 & 3 \\ 4 & 6 & 100 \end{vmatrix}$, then

- (a) $\Delta^2 - 3\Delta' = 0$
 (b) $(\Delta + \Delta')^2 - 3(\Delta + \Delta') + 2 = 0$
 (c) $(\Delta + \Delta')^2 + 3(\Delta + \Delta') + 5 = 0$
 (d) $\Delta + 3\Delta' + 1 = 0$

18. $\int \frac{dx}{x(x^4 + 1)} =$

(a) $\frac{1}{4} \log \left(\frac{x^4 + 1}{x^4} \right) + C$ (b) $\frac{1}{4} \log \left(\frac{x^4}{x^4 + 1} \right) + C$
 (c) $\frac{1}{4} \log (x^4 + 1) + C$ (d) $\frac{1}{4} \log \left(\frac{x^4}{x^4 + 2} \right) + C$

12 Solved Papers 2017

19. $\sin^{-1} \frac{\sqrt{3}}{2} + \sin^{-1} \sqrt{\frac{2}{3}} =$

- (a) $\sin^{-1} \frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}}$ (b) $\pi - \sin^{-1} \left(\frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}} \right)$
 (c) $-\pi - \sin^{-1} \left(\frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}} \right)$ (d) $\pi + \sin^{-1} \left(\frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}} \right)$

20. If the slope of the tangent of the circle $S \equiv x^2 + y^2 - 13 = 0$ at $(2, 3)$ is m , then the point $\left(m, \frac{-1}{m}\right)$ is

- (a) an external point with respect to the circle $S = 0$
 (b) an internal point with respect to the circle $S = 0$
 (c) the centre of the circle $S = 0$
 (d) a point on the circle $S = 0$

21. $f: (-\infty, 0] \rightarrow [0, \infty)$ is defined as $f(x) = x^2$. The domain and range of its inverse is

- (a) Domain of $(f^{-1}) = [0, \infty)$, Range of $(f^{-1}) = (-\infty, 0]$
 (b) Domain of $(f^{-1}) = [0, \infty)$, Range of $(f^{-1}) = (-\infty, \infty]$
 (c) Domain of $(f^{-1}) = [0, \infty)$, Range of $(f^{-1}) = (0, \infty)$
 (d) f^{-1} does not exist

22. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$ and

$(\mathbf{a}, \mathbf{b}) = \frac{\pi}{3}$, then $|\mathbf{a} \times \mathbf{b}| + |\mathbf{b} \times \mathbf{c}| + |\mathbf{c} \times \mathbf{a}| =$

- (a) $\frac{3}{2}$ (b) 0
 (c) $\frac{3\sqrt{3}}{2}$ (d) 3

23. The differential equation of the simple harmonic motion given by $x = A \cos(nt + \alpha)$ is

- (a) $\frac{d^2x}{dt^2} - n^2x = 0$ (b) $\frac{d^2x}{dt^2} + n^2x = 0$
 (c) $\frac{dx}{dt} - \frac{d^2x}{dt^2} = 0$ (d) $\frac{d^2x}{dt^2} - \frac{dx}{dt} + nx = 0$

24. If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and α is the angle between them, then $\mathbf{a} + \mathbf{b}$ is a unit vector when $\cos \alpha =$

- (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $-\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{2}$

25. If rank of $\begin{pmatrix} x & x & x \\ x & x^2 & x \\ x & x & x+1 \end{pmatrix}$ is 1, then

- (a) $x = 0$ or $x = 1$ (b) $x = 1$
 (c) $x = 0$ (d) $x \neq 0$

26. If the vectors $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$, $\mathbf{b} = \hat{i} - \hat{j} + 2\hat{k}$ and $\mathbf{c} = x\hat{i} + (x-2)\hat{j} - \hat{k}$ are coplanar, then $x =$

- (a) 1 (b) 2 (c) 0 (d) -2

27. Let $\mathbf{a} = 2\hat{i} + \hat{j} - 3\hat{k}$ and $\mathbf{b} = \hat{i} + 3\hat{j} + 2\hat{k}$. Then the volume of the parallelepiped having coterminous edges as $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$, where $\mathbf{c}$ is the vector perpendicular to the plane of $\mathbf{a}, \mathbf{b}$ and $|\mathbf{c}| = 2$ is

- (a) $2\sqrt{195}$ (b) 24 (c) $\sqrt{200}$ (d) $\sqrt{195}$

28. The local maximum of $y = x^3 - 3x^2 + 5$ is attained at

- (a) $x = 0$ (b) $x = 2$ (c) $x = 1$ (d) $x = -1$

29. If $f(x) = \begin{cases} \sin x & \text{if } x \leq 0 \\ x^2 + a^2 & \text{if } 0 < x < 1 \\ bx + 2 & \text{if } 1 \leq x \leq 2 \\ 0 & \text{if } x > 2 \end{cases}$ is continuous on $\mathbb{R}$, then

$a + b + ab =$
 (a) -2 (b) 0 (c) 2 (d) -1

30. If $\cosh^{-1} x = 2 \log_e (\sqrt{2} + 1)$, then $x =$

- (a) 1 (b) 2 (c) 4 (d) 3

31. If $\int f(x) \cos x \, dx = \frac{1}{2} (f(x))^2 + C$ and $f(0) = 0$, then

$f'(0) =$
 (a) 1 (b) -1 (c) 0 (d) 2

VIT

1. If the slope of the curve $y = \frac{ax}{b-x}$ at the point $(1, 1)$ is 2,

then

- (a) $a = 1, b = -2$ (b) $a = -1, b = 2$
 (c) $a = 1, b = 2$ (d) None of these

2. If the distance between the plane $Ax - 2y + z = d$ and the plane containing the lines $\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$ and

$\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$ is $\sqrt{6}$, then $|d|$ is equal to

- (a) 3 (b) 4 (c) 6 (d) 1

3. If the lines $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect, then the value of k is

- (a) $3/2$ (b) $7/2$ (c) $-2/7$ (d) $-3/2$

4. The differential equation $\frac{dy}{dx} + \frac{y^2}{x^2} = y/x$ has the solution

- (a) $x = y(\log x + C)$ (b) $y = x(\log y + C)$
 (c) $x = (y + C)\log x$ (d) $y = (x + C)\log y$

5. If P is a 3×3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is 3×3 identity, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that PX is equal to
- (a) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (b) X (c) $2X$ (d) $-X$
6. The value of $\int_{-2}^3 |1 - x^2| dx$, is
- (a) $\frac{1}{3}$ (b) $14/3$ (c) $7/3$ (d) $28/3$
7. $\int \frac{1}{\tan x + \cot x + \sec x + \operatorname{cosec} x} dx$ is equal to
- (a) $\frac{1}{2}(\sin x + \cos x + x) + C$ (b) $\frac{1}{2}(\sin x - \cos x - x) + C$
 (c) $\frac{1}{2}(\cos x - x + \sin x) + C$ (d) None of the above
8. If $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} dx = k \int_0^1 \frac{\log(1+x)}{1+x^2} dx$, then k is equal to
- (a) 4 (b) 8 (c) π (d) 2π
9. The equation of the curve in which the portion of the tangent included between the coordinate axes is bisected at the point of contact, is
- (a) a parabola (b) an ellipse
 (c) a circle (d) a hyperbola
10. The solution of $\cos(x+y) dy = dx$, is
- (a) $y = \tan\left(\frac{x+y}{2}\right) + C$ (b) $y = \cos^{-1}(y/x)$
 (c) $y = x \sec(y/x) + C$ (d) None of the above
11. The degree of the differential equation satisfying $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$, is
- (a) 1 (b) 2 (c) 3 (d) None
12. If the tangent at $(1, 1)$ on $y^2 = x(2-x)^2$ meets the curve again at P , then P is
- (a) $(4, 4)$ (b) $(-1, 2)$ (c) $(9/4, 3/8)$ (d) None of these
13. Let E' denote the complement of an event E . Let E, F, G be pairwise independent events such that $P(G) > 0$ and $P(E \cap F \cap G) = 0$. Then, $P(E' \cap F' / G)$ equals
- (a) $P(E') + P(F')$ (b) $P(E') - P(F')$
 (c) $P(E') - P(F)$ (d) $P(E) - P(F')$
14. An unbiased coin is tossed n times. Let X denotes the number of times head occurs. If $P(X = 4)$, $P(X = 5)$ and $P(X = 6)$ are in AP, then the value of n can be
- (a) 7, 14 (b) 10, 14
 (c) 12, 7 (d) None of these
15. Let X and Y be two events such that $P(X/Y) = 1/2$, $P(Y/X) = 1/3$ and $P(X \cap Y) = 1/6$. Which of the following is incorrect?
- (a) $P(X \cup Y) = 2/3$ (b) X and Y are independent
 (c) $P(X^C \cap Y) = 1/6$ (d) X and Y are not independent
16. If a, b, c are non-coplanar unit vectors such that $a \times (b \times c) = \frac{b+c}{\sqrt{2}}$, then the angle between a and b is
- (a) $\frac{3\pi}{4}$ (b) $\pi/4$ (c) $\pi/2$ (d) π
17. If a, b, c and a', b', c' form a reciprocal system of vectors, then $a \cdot a' + b \cdot b' + c \cdot c'$ is equal to
- (a) 0 (b) 1 (c) 2 (d) 3
18. If a, b, c are three non-zero vectors which are pairwise non-collinear. Also, $a + 3b$ is collinear with c and $b + 2c$ is collinear with a , then $a + 3b + 6c$ is
- (a) c (b) 0 (c) $a + c$ (d) a
19. If the relation between direction ratios of two lines are given by $a + b + c = 0$ and $2ab + 2ac - bc = 0$, then the angle between the lines is
- (a) π (b) $\frac{2\pi}{3}$ (c) $\pi/2$ (d) $\pi/3$
20. If $f(x) = \begin{cases} ax^2 - b & , a \leq x < 1 \\ 2 & , x = 1 \\ x + 1 & , 1 < x \leq 2 \end{cases}$ Then, the value of the pair (a, b) for which $f(x)$ cannot be continuous at $x = 1$, is
- (a) $(2, 0)$ (b) $(1, -1)$
 (c) $(4, 2)$ (d) $(1, 1)$
21. The area between the curve $y = 2x^4 - x^2$, the X -axis and the ordinates of two minima of the curve is
- (a) $7/120$ (b) $9/120$
 (c) $11/120$ (d) $13/120$
22. The range of the function $f(x) = \tan \sqrt{\frac{\pi^2}{9} - x^2}$ is
- (a) $[0, \sqrt{3}]$ (b) $(0, \sqrt{3})$
 (c) $[0, \sqrt{3})$ (d) $(0, \sqrt{3}]$
23. If $\sin^{-1} x + \sin^{-1}(1-x) = \cos^{-1} x$, then x equals
- (a) 1, -1 (b) 1, 0
 (c) 0, $1/2$ (d) None of these
24. The equation of the plane through the intersection of the planes $x + y + z = 1$ and $2x + 3y - z + 4 = 0$ and parallel to X -axis, is
- (a) $y - 3z + 6 = 0$ (b) $3y - z + 6 = 0$
 (c) $y + 3z + 6 = 0$ (d) $3y - 2z + 6 = 0$

14 Solved Papers 2017

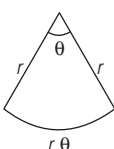
MHT CET

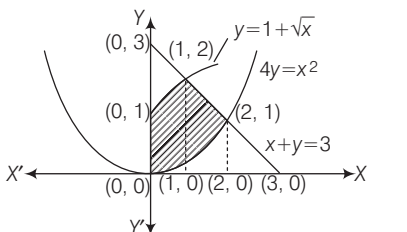
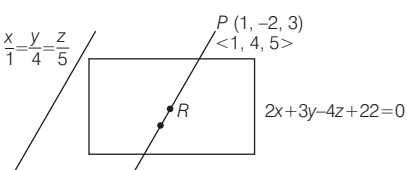
1. The objective function $z = 4x_1 + 5x_2$, subject to $2x_1 + x_2 \geq 7$, $2x_1 + 3x_2 \leq 15$, $x_2 \leq 3$, $x_1, x_2 \geq 0$ has minimum value at the point
 - (a) on X-axis
 - (b) on Y-axis
 - (c) at the origin
 - (d) on the line parallel to X-axis
2. The maximum value of $f(x) = \frac{\log x}{x}$ ($x \neq 0, x \neq 1$) is
 - (a) e
 - (b) $\frac{1}{e}$
 - (c) e^2
 - (d) $\frac{1}{e^2}$
3. $\int_0^1 x \tan^{-1} x \, dx =$
 - (a) $\frac{\pi}{4} + \frac{1}{2}$
 - (b) $\frac{\pi}{4} - \frac{1}{2}$
 - (c) $\frac{1}{2} - \frac{\pi}{4}$
 - (d) $-\frac{\pi}{4} - \frac{1}{2}$
4. If $g(x)$ is the inverse function of $f(x)$ and $f'(x) = \frac{1}{1+x^4}$, then $g'(x)$ is
 - (a) $1 + [g(x)]^4$
 - (b) $1 - [g(x)]^4$
 - (c) $1 + [f(x)]^4$
 - (d) $\frac{1}{1 + [g(x)]^4}$
5. The inverse of the matrix $\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$ is
 - (a) $-\frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 2 & -3 \end{bmatrix}$
 - (b) $-\frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$
 - (c) $-\frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$
 - (d) $-\frac{1}{3} \begin{bmatrix} -3 & 0 & 0 \\ -3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$
6. If $\int \frac{1}{\sqrt{9-16x^2}} dx = \alpha \sin^{-1}(\beta x) + c$, then $\alpha + \frac{1}{\beta} =$
 - (a) 1
 - (b) $\frac{7}{12}$
 - (c) $\frac{19}{12}$
 - (d) $\frac{9}{12}$
7. If the function $f(x) = \begin{cases} \left[\tan\left(\frac{\pi}{4} + x\right) \right]^{\frac{1}{x}} & \text{for } x \neq 0 \\ K & \text{for } x = 0 \end{cases}$ is continuous at $x = 0$, then $K = ?$
 - (a) e
 - (b) e^{-1}
 - (c) e^2
 - (d) e^{-2}
8. For an invertible matrix A if $A(\text{adj } A) = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$, then $|A| =$
 - (a) 100
 - (b) -100
 - (c) 10
 - (d) -10
9. The solution of the differential equation $\frac{dy}{dx} = \tan\left(\frac{y}{x}\right) + \frac{y}{x}$ is
 - (a) $\cos\left(\frac{y}{x}\right) = cx$
 - (b) $\sin\left(\frac{y}{x}\right) = cx$
 - (c) $\cos\left(\frac{y}{x}\right) = cy$
 - (d) $\sin\left(\frac{y}{x}\right) = cy$
10. If $x = f(t)$ and $y = g(t)$ are differentiable functions of t , then $\frac{d^2 y}{dx^2}$ is
 - (a) $\frac{f'(t) \cdot g''(t) - g'(t) \cdot f''(t)}{[f'(t)]^3}$
 - (b) $\frac{f'(t) \cdot g''(t) - g'(t) \cdot f''(t)}{[f'(t)]^2}$
 - (c) $\frac{g'(t) \cdot f''(t) - f'(t) \cdot g''(t)}{[f'(t)]^3}$
 - (d) $\frac{g'(t) \cdot f''(t) + f'(t) \cdot g''(t)}{[f'(t)]^3}$
11. The equation of line equally inclined to coordinate axes and passing through $(-3, 2, -5)$ is
 - (a) $\frac{x+3}{1} = \frac{y-2}{1} = \frac{z+5}{1}$
 - (b) $\frac{x+3}{-1} = \frac{y-2}{1} = \frac{z+5}{-1}$
 - (c) $\frac{x+3}{-1} = \frac{y-2}{1} = \frac{z+5}{1}$
 - (d) $\frac{x+3}{-1} = \frac{y-2}{1} = \frac{z+5}{-1}$
12. If $\int_0^{\frac{\pi}{2}} \log \cos x \, dx = \frac{\pi}{2} \log\left(\frac{1}{2}\right)$, then $\int_0^{\frac{\pi}{2}} \log \sec x \, dx =$
 - (a) $\frac{\pi}{2} \log\left(\frac{1}{2}\right)$
 - (b) $1 - \frac{\pi}{2} \log\left(\frac{1}{2}\right)$
 - (c) $1 + \frac{\pi}{2} \log\left(\frac{1}{2}\right)$
 - (d) $\frac{\pi}{2} \log 2$
13. If the angle between the planes $\mathbf{r} \cdot (m\hat{i} - \hat{j} + 2\hat{k}) + 3 = 0$ and $\mathbf{r} \cdot (2\hat{i} - m\hat{j} - \hat{k}) - 5 = 0$ is $\frac{\pi}{3}$, then $m =$
 - (a) 2
 - (b) ± 3
 - (c) 3
 - (d) -2
14. If the origin and the points $P(2, 3, 4)$, $Q(1, 2, 3)$ and $R(x, y, z)$ are coplanar, then
 - (a) $x - 2y - z = 0$
 - (b) $x + 2y + z = 0$
 - (c) $x - 2y + z = 0$
 - (d) $2x - 2y + z = 0$
15. Let $\square PQRS$ be a quadrilateral. If M and N are the mid-points of the sides PQ and RS respectively, then $\overrightarrow{PS} + \overrightarrow{QR} =$
 - (a) $3 \overrightarrow{MN}$
 - (b) $4 \overrightarrow{MN}$
 - (c) $2 \overrightarrow{MN}$
 - (d) $2 \overrightarrow{NM}$
16. If vector $\mathbf{r}$ with direction cosines l, m, n is equally inclined to the coordinate axes, then the total number of such vectors is
 - (a) 4
 - (b) 6
 - (c) 8
 - (d) 2
17. If $\int \frac{1}{(x^2+4)(x^2+9)} dx = A \tan^{-1} \frac{x}{2} + B \tan^{-1} \left(\frac{x}{3}\right) + C$, then $A - B =$
 - (a) $\frac{1}{6}$
 - (b) $\frac{1}{30}$
 - (c) $-\frac{1}{30}$
 - (d) $-\frac{1}{6}$
18. If α and β are roots of the equation $x^2 + 5|x| - 6 = 0$, then the value of $|\tan^{-1} \alpha - \tan^{-1} \beta|$ is
 - (a) $\frac{\pi}{2}$
 - (b) 0
 - (c) π
 - (d) $\frac{\pi}{4}$

19. If $x = a\left(t - \frac{1}{t}\right)$, $y = a\left(t + \frac{1}{t}\right)$, where t is the parameter, then $\frac{dy}{dx} = ?$
 (a) $\frac{y}{x}$ (b) $\frac{-x}{y}$ (c) $\frac{x}{y}$ (d) $\frac{-y}{x}$
20. The point on the curve $y = \sqrt{x-1}$, where the tangent is perpendicular to the line $2x + y - 5 = 0$ is
 (a) (2, -1) (b) (10, 3) (c) (2, 1) (d) (5, -2)
21. If $\int \sqrt{\frac{x-5}{x-7}} dx = A \sqrt{x^2 - 12x + 35} + \log|x - 6 + \sqrt{x^2 - 12x + 35}| + C$, then $A =$
 (a) -1 (b) $\frac{1}{2}$ (c) $-\frac{1}{2}$ (d) 1
22. At random variable $X \sim B(n, p)$, if values of mean and variance of X are 18 and 12 respectively, then total number of possible values of X are
 (a) 54 (b) 55 (c) 12 (d) 18
23. The area of the region bounded by the lines $y = 2x + 1$, $y = 3x + 1$ and $x = 4$ is
 (a) 16 sq unit (b) $\frac{121}{3}$ sq unit (c) $\frac{121}{6}$ sq unit (d) 8 sq unit
24. A box contains 6 pens, 2 of which are defective. Two pens are taken randomly from the box. If r.v. X : number of defective pens obtained, then standard deviation of $X =$
 (a) $\pm \frac{4}{3\sqrt{5}}$ (b) $\frac{8}{3}$ (c) $\frac{16}{45}$ (d) $\frac{4}{3\sqrt{5}}$
25. If the volume of spherical ball is increasing at the rate of $4\pi \text{ cm}^3/\text{s}$, then the rate of change of its surface area when the volume is $288\pi \text{ cm}^3$, is
 (a) $\frac{4}{3}\pi \text{ cm}^2/\text{s}$ (b) $\frac{2}{3}\pi \text{ cm}^2/\text{s}$ (c) $4\pi \text{ cm}^2/\text{s}$ (d) $2\pi \text{ cm}^2/\text{s}$
26. If $f(x) = \begin{cases} \log(\sec^2 x)^{\cot^2 x} & \text{for } x \neq 0 \\ K & \text{for } x = 0 \end{cases}$ is continuous at $x = 0$, then K is
 (a) e^{-1} (b) 1 (c) e (d) 0
27. The differential equation of all parabolas whose axis is Y -axis, is
 (a) $x \frac{d^2 y}{dx^2} - \frac{dy}{dx} = 0$ (b) $x \frac{d^2 y}{dx^2} + \frac{dy}{dx} = 0$
 (c) $\frac{d^2 y}{dx^2} - y = 0$ (d) $\frac{d^2 y}{dx^2} - \frac{dy}{dx} = 0$
28. $\int_0^3 [x] dx = \dots\dots\dots$ where $[x]$ is greatest integer function.
 (a) 3 (b) 0 (c) 2 (d) 1
29. The objective function of LPP defined over the convex set attains its optimum value at
 (a) atleast two of the corner points
 (b) all the corner points
 (c) atleast one of the corner points
 (d) None of the corner points
30. If the inverse of the matrix $\begin{bmatrix} \alpha & 14 & -1 \\ 2 & 3 & 1 \\ 6 & 2 & 3 \end{bmatrix}$ does not exist, then the value of α is
 (a) 1 (b) -1 (c) 0 (d) -2
31. If $f(x) = x$ for $x \leq 0$ and $f(x) = 0$ for $x > 0$, then $f(x)$ at $x = 0$ is
 (a) continuous but not differentiable
 (b) not continuous but differentiable
 (c) continuous and differentiable
 (d) not continuous and not differentiable
32. The equation of the plane through $(-1, 1, 2)$, whose normal makes equal acute angles with coordinate axes is
 (a) $\mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ (b) $\mathbf{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 6$
 (c) $\mathbf{r} \cdot (3\hat{i} - 3\hat{j} + 3\hat{k}) = 2$ (d) $\mathbf{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 3$
33. If the distance of points $2\hat{i} + 3\hat{j} + \lambda\hat{k}$ from the plane $\mathbf{r} \cdot (3\hat{i} + 2\hat{j} + 6\hat{k}) = 13$ is 5 units, then $\lambda =$
 (a) $6 - \frac{17}{3}$ (b) $6, \frac{17}{3}$ (c) $-6, -\frac{17}{3}$ (d) $-6, \frac{17}{3}$
34. The value of $\cos^{-1}\left(\cot\left(\frac{\pi}{2}\right)\right) + \cos^{-1}\left(\sin\left(\frac{2\pi}{3}\right)\right)$ is
 (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) π
35. The particular solution of the differential equation $x dy + 2y dx = 0$, when $x = 2$, $y = 1$ is
 (a) $xy = 4$ (b) $x^2 y = 4$ (c) $xy^2 = 4$ (d) $x^2 y^2 = 4$
36. For the following distribution function $F(x)$ of a r.v. X .
- | | | | | | | |
|--------|-----|------|------|------|------|---|
| x | 1 | 2 | 3 | 4 | 5 | 6 |
| $F(x)$ | 0.2 | 0.37 | 0.48 | 0.62 | 0.85 | 1 |
- $P(3 < x \leq 5) =$
 (a) 0.48 (b) 0.37 (c) 0.27 (d) 1.47
37. The lines $\frac{x-1}{2} = \frac{y+1}{2} = \frac{z-1}{4}$ and $\frac{x-3}{1} = \frac{y-k}{2} = \frac{z}{1}$ intersect each other at point
 (a) $(-2, -4, 5)$ (b) $(-2, -4, -5)$ (c) $(2, 4, -5)$ (d) $(2, -4, -5)$
38. $\int \frac{\sec^8 x}{\csc x} dx =$
 (a) $\frac{\sec^8 x}{8} + c$ (b) $\frac{\sec^7 x}{7} + c$ (c) $\frac{\sec^6 x}{6} + c$ (d) $\frac{\sec^9 x}{9} + c$

Answer with Explanations

JEE Main

1. (d) $\therefore \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 1 \end{vmatrix}$
- $$= 1(a-b) - 1(1-a) + 1(b-a^2)$$
- $$= -(a-1)^2$$
- $$\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ 0 & b & 1 \end{vmatrix}$$
- $$= 1(a-b) - 1(1) + 1(b) = -(a-1)$$
- $$\Delta_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ a & 0 & 1 \end{vmatrix}$$
- $$= 1(1) - 1(1-a) + 1(0-a) = 0$$
- and $\Delta_3 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & a & 1 \\ a & b & 0 \end{vmatrix}$
- $$= 1(-b) - 1(-a) + 1(b-a^2)$$
- $$= -a(a-1)$$
- For $a=1$
- $$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$$
- Δ for $b=1$ only
- $$x+y+z=1, x+y+z=2$$
- and $x+y+z=0$
- i.e. no solution ($\therefore$ RHS is not equal)
- Hence, for no solution $b=1$ only
2. (c) Total length $= 2r + r\theta = 20$
- $$\Rightarrow \theta = \frac{20-2r}{r}$$
- 
- Now, area of flower-bed,
- $$A = \frac{1}{2} r^2 \theta \Rightarrow A = \frac{1}{2} r^2 \left(\frac{20-2r}{r} \right)$$
- $$\Rightarrow A = 10r - r^2 \Rightarrow \frac{dA}{dr} = 10 - 2r$$
- For maxima or minima, put $\frac{dA}{dr} = 0$.
- $$\Rightarrow 10 - 2r = 0 \Rightarrow r = 5$$
- $$\therefore A_{\max} = \frac{1}{2} (5)^2 \left[\frac{20-2(5)}{5} \right]$$
- $$= \frac{1}{2} \times 25 \times 2 = 25 \text{ sq m}$$
3. (d) Required area
- $$= \int_0^1 (1 + \sqrt{x}) dx$$
- $$+ \int_1^2 (3-x) dx - \int_0^2 \frac{x^2}{4} dx$$

- 
- $$= \left[x + \frac{x^{3/2}}{3/2} \right]_0^1 + \left[3x - \frac{x^2}{2} \right]_1^2 - \left[\frac{x^3}{12} \right]_0^2$$
4. (b) Any line parallel to $\frac{x}{1} = \frac{y}{4} = \frac{z}{5}$ and passing through $P(1, -2, 3)$ is
- 
- $$\frac{x-1}{1} = \frac{y+2}{4} = \frac{z-3}{5} = \lambda \text{ (say)}$$
- Any point on above line can be written as
- $$(\lambda + 1, 4\lambda - 2, 5\lambda + 3)$$
- $\therefore$ Coordinates of R are
- $$(\lambda + 1, 4\lambda - 2, 5\lambda + 3)$$
- Since, point R lies on the above plane.
- $$\therefore 2(\lambda + 1) + 3(4\lambda - 2) - 4(5\lambda + 3) + 22 = 0$$
- $$\Rightarrow \lambda = 1$$
- So, point R is (2, 2, 8).
- Now,
- $$PR = \sqrt{(2-1)^2 + (2+2)^2 + (8-3)^2}$$
- $$= \sqrt{42}$$
- $\therefore PQ = 2PR = 2\sqrt{42}$
5. (a) Let $y = \tan^{-1} \left(\frac{6x\sqrt{x}}{1-9x^3} \right)$
- $$= \tan^{-1} \left[\frac{2 \cdot (3x^{3/2})}{1 - (3x^{3/2})^2} \right] = 2 \tan^{-1} (3x^{3/2})$$
- $$\therefore \frac{dy}{dx} = 2 \cdot \frac{1}{1 + (3x^{3/2})^2} \cdot 3 \times \frac{3}{2} (x)^{1/2}$$
- $$= \frac{9}{1 + 9x^3} \cdot \sqrt{x} \Rightarrow g(x) = \frac{9}{1 + 9x^3}$$
6. (a) We have,
- $$\Rightarrow \frac{dy}{dx} + \frac{\cos x}{2 + \sin x} y = \frac{-\cos x}{2 + \sin x}$$
- which is a linear differential equation.
- $$\therefore \text{IF} = e^{\int \frac{\cos x}{2 + \sin x} dx} = e^{\log(2 + \sin x)}$$
- $$= 2 + \sin x$$
- $\therefore$ Required solution is given by

- $$y \cdot (2 + \sin x) = \int \frac{-\cos x}{2 + \sin x} \cdot (2 + \sin x) dx + C$$
- $$\Rightarrow y(2 + \sin x) = -\sin x + C$$
- Also, $y(0) = 1$
- $$\therefore 1(2 + \sin 0) = -\sin 0 + C \Rightarrow C = 2$$
- $$\therefore y = \frac{2 - \sin x}{2 + \sin x} \Rightarrow y\left(\frac{\pi}{2}\right) = \frac{2 - \sin \frac{\pi}{2}}{2 + \sin \frac{\pi}{2}} = \frac{1}{3}$$
7. (b) Now, $A^2 = A \cdot A$
- $$= \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$$
- $$= \begin{bmatrix} 4 + 12 & -6 - 3 \\ -8 - 4 & 12 + 1 \end{bmatrix} = \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix}$$
- Now, $3A^2 + 12A = 3 \begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} + 12 \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$
- $$= \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix} + \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$
- $$= \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$
- $$\therefore \text{adj}(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$$
8. (b) Given, equations of lines are
- $$\frac{x-1}{1} = \frac{y+2}{-2} = \frac{z-4}{3}$$
- and $\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z+7}{-1}$
- Let $\mathbf{n}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$
- and $\mathbf{n}_2 = 2\hat{i} - \hat{j} - \hat{k}$
- $\therefore$ Any vector $\mathbf{n}$ perpendicular to both $\mathbf{n}_1, \mathbf{n}_2$ is given by $\mathbf{n} = \mathbf{n}_1 \times \mathbf{n}_2$
- $$\Rightarrow \mathbf{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 3 \\ 2 & -1 & -1 \end{vmatrix} = 5\hat{i} + 7\hat{j} + 3\hat{k}$$
- $\therefore$ Equation of a plane passing through $(1, -1, -1)$ and perpendicular to $\mathbf{n}$ is given by
- $$5(x-1) + 7(y+1) + 3(z+1) = 0$$
- $$\Rightarrow 5x + 7y + 3z + 5 = 0$$
- $\therefore$ Required distance
- $$= \frac{|5 + 21 - 21 + 5|}{\sqrt{5^2 + 7^2 + 3^2}} = \frac{10}{\sqrt{83}} \text{ units}$$
9. (b) $\therefore I_n + I_{n+2} = \int \tan^n x dx + \int \tan^{n+2} x dx$
- $$= \int \tan^n x (1 + \tan^2 x) dx$$

$$= \int \tan^n x \sec^2 x \, dx = \frac{\tan^{n+1} x}{n+1} + C$$

Put $n = 4$, we get

$$10. (c) \text{ Now, } f\left(\frac{1}{x}\right) = \frac{\frac{1}{x}}{1 + \frac{1}{x^2}} = \frac{x}{1 + x^2} = f(x)$$

$$\therefore f\left(\frac{1}{2}\right) = f(2) \text{ or } f\left(\frac{1}{3}\right) = f(3) \text{ and so on.}$$

So, $f(x)$ is many-one function.

$$\text{Again, let } y = f(x) \Rightarrow y = \frac{x}{1 + x^2}$$

$$\Rightarrow y + x^2 y = x \Rightarrow yx^2 - x + y = 0$$

As, $x \in \mathbb{R}$

$$\therefore (-1)^2 - 4(y)(y) \geq 0 \Rightarrow 1 - 4y^2 \geq 0$$

$$\Rightarrow y \in \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\therefore \text{Range} = \text{Codomain} = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

So, $f(x)$ is surjective.

$$11. (b) \text{ We have, } \mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 2\hat{\mathbf{k}}$$

$$\Rightarrow |\mathbf{a}| = \sqrt{4 + 1 + 4} = 3$$

$$\text{and } \mathbf{b} = \hat{\mathbf{i}} + \hat{\mathbf{j}} \Rightarrow |\mathbf{b}| = \sqrt{1 + 1} = \sqrt{2}$$

$$\text{Now, } |\mathbf{c} - \mathbf{a}| = 3 \Rightarrow |\mathbf{c} - \mathbf{a}|^2 = 9$$

$$\Rightarrow (\mathbf{c} - \mathbf{a}) \cdot (\mathbf{c} - \mathbf{a}) = 9$$

$$\Rightarrow |\mathbf{c}|^2 + |\mathbf{a}|^2 - 2\mathbf{c} \cdot \mathbf{a} = 9 \quad \dots(i)$$

$$\text{Again, } |(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}| = 3$$

$$\Rightarrow |\mathbf{a} \times \mathbf{b}| |\mathbf{c}| \sin 30^\circ = 3 \Rightarrow |\mathbf{c}| = \frac{6}{|\mathbf{a} \times \mathbf{b}|}$$

$$\text{But } \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -2 \\ 1 & 1 & 0 \end{vmatrix} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{j}} + \hat{\mathbf{k}}$$

$$\therefore |\mathbf{c}| = \frac{6}{\sqrt{4 + 4 + 1}} = 2 \quad \dots(ii)$$

Using Eqs. (i) and (ii)

$$12. (b) \text{ The point of intersection is } (0, 1).$$

$$\text{Now, } y = \frac{x + 6}{(x - 2)(x - 3)}$$

$$\Rightarrow \frac{dy}{dx} =$$

$$\frac{1(x - 2)(x - 3) - (x + 6)(x - 3 + x - 2)}{(x - 2)^2(x - 3)^2}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(0, 1)} = \frac{6 + 30}{4 \times 9} = \frac{36}{36} = 1$$

$\therefore$ Equation of normal at $(0, 1)$ is given by

$$y - 1 = \frac{-1}{1}(x - 0) \Rightarrow x + y - 1 = 0$$

$$\text{which passes through the point } \left(\frac{1}{2}, \frac{1}{2}\right).$$

$$13. (a) \therefore \text{ Total number of balls} = 15 + 10 = 25$$

$$P(\text{green balls}) = \frac{15}{25} = \frac{3}{5} = p =$$

Probability of success

$$P(\text{yellow balls}) = \frac{10}{25} = \frac{2}{5} = q =$$

Probability of unsucccess and $n = 10 =$
Number of trials.

$$\therefore \text{Variance} = npq = 10 \times \frac{3}{5} \times \frac{2}{5} = \frac{12}{5}$$

$$14. (b) \text{ Let } I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1 + \cos x}$$

$$= \int_{\pi/4}^{3\pi/4} \frac{1 - \cos x}{1 - \cos^2 x} dx$$

$$= \int_{\pi/4}^{3\pi/4} \frac{1 - \cos x}{\sin^2 x} dx$$

$$= \int_{\pi/4}^{3\pi/4} (\operatorname{cosec}^2 x - \operatorname{cosec} x \cot x) dx$$

$$= [-\cot x + \operatorname{cosec} x]_{\pi/4}^{3\pi/4}$$

$$= [(1 + \sqrt{2}) - (-1 + \sqrt{2})] = 2$$

JEE Advanced

Paper 1

$$1. (a, b) P(X) = \frac{1}{3}$$

$$P\left(\frac{X}{Y}\right) = \frac{P(X \cap Y)}{P(Y)} = \frac{1}{2}$$

$$P\left(\frac{Y}{X}\right) = \frac{P(X \cap Y)}{P(X)} = \frac{2}{5}$$

$$P(X \cap Y) = \frac{2}{15} \Rightarrow P(Y) = \frac{4}{15}$$

$$P\left(\frac{X'}{Y}\right) = \frac{P(Y) - P(X \cap Y)}{P(Y)}$$

$$= \frac{\frac{4}{15} - \frac{2}{15}}{\frac{4}{15}} = \frac{1}{2}$$

$$P(X \cup Y) = \frac{1}{3} + \frac{4}{15} - \frac{2}{15} = \frac{7}{15}$$

$$2. (c, d) (a) \because e^x \in (1, e) \text{ in } (0, 1) \text{ and}$$

$$\int_0^x f(t) \sin t \, dt \in (0, 1) \text{ in } (0, 1)$$

$$\therefore e^x - \int_0^x f(t) \sin t \, dt \text{ cannot be zero.}$$

$$(b) f(x) + \int_0^{\frac{\pi}{2}} f(t) \sin t \, dt \text{ always positive}$$

$$(c) \text{ Let } h(x) = x - \int_0^{\pi-x} f(t) \cos t \, dt,$$

$$h(0) = - \int_0^{\frac{\pi}{2}} f(t) \cos t \, dt < 0$$

$$h(1) = 1 - \int_0^{\frac{\pi-1}{2}} f(t) \cos t \, dt > 0$$

$$(d) \text{ Let } g(x) = x^9 - f(x)$$

$$g(0) = -f(0) < 0$$

$$g(1) = 1 - f(1) > 0$$

Substitute the value of a in option (a), (b), (c) and (d).

$$3. (a, b, d) f(x) = x \cos(\pi(x + [x]))$$

At $x = 0$

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} x \cos(\pi(x + [x])) = 0$$

$$\text{and } f(x) = 0$$

$\therefore$ It is continuous at $x = 0$ and clearly discontinuous at other integer points.

$$4. (a, c) \text{ For a matrix to be square of other matrix its determinant should be positive.}$$

(a) and (c) $\rightarrow$ Correct

(b) and (d) $\rightarrow$ Incorrect

$$5. (1) \begin{vmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{vmatrix} = 0$$

$$\Rightarrow \alpha^4 - 2\alpha^2 + 1 = 0 \Rightarrow \alpha^2 = 1$$

$$\Rightarrow \alpha = \pm 1$$

But $\alpha = 1$ not possible

$$\therefore \alpha = -1$$

$$\text{Hence, } 1 + \alpha + \alpha^2 = 1$$

$$6. (2) g(x)$$

$$= \int_{\pi/2}^x (f'(t) \operatorname{cosec} t - \cot t \operatorname{cosec} t f(t)) dt$$

$$\therefore g(x) = f\left(\frac{\pi}{2}\right) \operatorname{cosec} \frac{\pi}{2} - f(x) \operatorname{cosec} x$$

$$\Rightarrow g(x) = 3 - \frac{f(x)}{\sin x}$$

$$\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \left(\frac{3 \sin x - f(x)}{\sin x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{3 \cos x - f'(x)}{\cos x} = \frac{3 - 1}{1} = 2$$

$$7. (d) \quad 8. (c) \quad 9. (c)$$

Paper 2

$$1. (b) f'(x) \text{ is increasing}$$

$$\text{For some } x \text{ in } \left(\frac{1}{2}, 1\right)$$

$$f'(x) = 1 \quad [\text{LMVT}]$$

$$\therefore f'(1) > 1$$

$$2. (b) \frac{dy}{dx} = \frac{1}{8\sqrt{x}\sqrt{9+\sqrt{x}}\sqrt{4+\sqrt{9+\sqrt{x}}}}$$

$$\Rightarrow y = \sqrt{4 + \sqrt{9 + \sqrt{x}}} + c$$

$$\text{Now, } y(0) = \sqrt{7} + c \Rightarrow c = 0$$

$$y(256) = \sqrt{4 + \sqrt{9 + 16}} = \sqrt{4 + 5} = 3$$

$$3. (a) \text{ Sum of diagonal entries of } M^T M \text{ is } \sum a_i^2.$$

$$\sum_{i=1}^9 a_i^2 = 5$$

18 Solved Papers 2017

Possibilities

I. 2, 1, 0, 0, 0, 0, 0, 0, which gives $\frac{9!}{7!}$

matrices

II. 1, 1, 1, 1, 1, 0, 0, 0, 0, which gives $\frac{9!}{4! \times 5!}$ matrices

Total matrices = $9 \times 8 + 9 \times 7 \times 2 = 198$

4. (c) Sample space $\rightarrow {}^{12}C_2$

Number of possibilities for z is even.

$$z = 0 \Rightarrow {}^{11}C_1 \Rightarrow z = 2 \Rightarrow {}^9C_1$$

$$z = 4 \Rightarrow {}^7C_1 \Rightarrow z = 6 \Rightarrow {}^5C_1$$

$$z = 8 \Rightarrow {}^3C_1 \Rightarrow z = 10 \Rightarrow {}^1C_1$$

Total = 36

$$\therefore \text{Probability} = \frac{36}{66} = \frac{6}{11}$$

5. (b) $\mathbf{OP} \cdot \mathbf{OQ} + \mathbf{OR} \cdot \mathbf{OS} = \mathbf{OR} \cdot \mathbf{OP} + \mathbf{OQ} \cdot \mathbf{OS}$

$$\Rightarrow \mathbf{OP}(\mathbf{OQ} - \mathbf{OR}) + \mathbf{OS}(\mathbf{OR} - \mathbf{OQ}) = 0$$

$$\Rightarrow (\mathbf{OR} - \mathbf{OS})(\mathbf{OQ} - \mathbf{OR}) = 0$$

$$\Rightarrow \mathbf{SP} \cdot \mathbf{RQ} = 0$$

$$\text{Similarly } \mathbf{SR} \cdot \mathbf{PQ} = 0 \text{ and } \mathbf{SQ} \cdot \mathbf{PR} = 0$$

$\therefore S$ is orthocentre.

6. (d) Let the equation of plane be $ax + by + cz = 1$. Then

$$a + b + c = 1 \Rightarrow 2a + b - 2c = 0$$

$$3a - 6b - 2c = 0$$

$$\Rightarrow a = 7b \Rightarrow c = \frac{15b}{2}$$

$$b = \frac{2}{31}, a = \frac{14}{31}, c = \frac{15}{31}$$

$$\therefore 14x + 2y + 15z = 31$$

7. (a,c) $f'(x) > 2f(x)$

$$\Rightarrow \frac{dy}{y} > 2dx \Rightarrow \int_1^{f(x)} \frac{dy}{y} > 2 \int_0^x dx$$

$$\ln(f(x)) > 2x \Rightarrow f(x) > e^{2x}$$

Also, as $f'(x) > 2f(x)$

$$\therefore f'(x) > 2e^{2x} > 0$$

8. (b,d) $I = \sum_{k=1}^{98} \int_k^{k+1} \frac{(k+1)}{x(x+1)} dx$

$$\text{Clearly, } I > \sum_{k=1}^{98} \int_k^{k+1} \frac{(k+1)}{(x+1)^2} dx$$

$$\Rightarrow I > \sum_{k=1}^{98} (k+1) \int_k^{k+1} \frac{1}{(x+1)^2} dx$$

$$\Rightarrow I > \sum_{k=1}^{98} (-(k+1)) \left[\frac{1}{k+2} - \frac{1}{k+1} \right]$$

$$\Rightarrow I > \sum_{k=1}^{98} \frac{1}{k+2}$$

$$\Rightarrow I > \frac{1}{3} + \dots + \frac{1}{100} > \frac{98}{100} \Rightarrow I > \frac{49}{50}$$

$$\text{Also, } I < \sum_{k=1}^{98} \int_k^{k+1} \frac{k+1}{x(k+1)} dx$$

$$= \sum_{k=1}^{98} [\log_e(k+1) - \log_e k]$$

$$I < \log_e 99$$

$$9. (a,c) \int_0^1 (x - x^3) dx = 2 \int_0^{\alpha} (x - x^3) dx$$

$$\frac{1}{4} = 2 \left(\frac{\alpha^2}{2} - \frac{\alpha^4}{4} \right) \Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0$$

$$\alpha^2 = \frac{4 - \sqrt{16 - 8}}{4} \quad (\because \alpha \in (0, 1))$$

$$\alpha^2 = 1 - \frac{1}{\sqrt{2}}$$

$$10. (*) g(x) = \int_{\sin x}^{\sin 2x} \sin^{-1}(t) dt$$

$$g'(x) = 2 \cos 2x \sin^{-1}(\sin 2x) - \cos x \sin^{-1}(\sin x)$$

$$g'\left(\frac{\pi}{2}\right) = -2 \sin^{-1}(0) = 0$$

$$g'\left(-\frac{\pi}{2}\right) = -2 \sin^{-1}(0) = 0$$

No option is matching.

$$11. (b,c) f(x) = \begin{vmatrix} \cos 2x & \cos 2x & \sin 2x \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$$

$$\cos 2x(\cos^2 x + \sin^2 x) - \cos 2x$$

$$(-\cos^2 x + \sin^2 x) + \sin 2x(-\sin 2x)$$

$$= \cos 2x + \cos 4x$$

$$f'(x) = -2 \sin 2x - 4 \sin 4x$$

$$= -2 \sin 2x(1 + 4 \cos 2x)$$

$$\text{At } x = 0 \Rightarrow f'(x) = 0$$

$$\text{and } f(x) = 2$$

$$\text{Also, } f'(x) = 0$$

$$\Rightarrow \sin 2x = 0$$

$$\text{or } \cos 2x = \frac{-1}{4}$$

$$\Rightarrow x = \frac{n\pi}{2} \text{ or } \cos 2x = -\frac{1}{4}$$

12. (a)

13. (a)

BITSAT

1. (c) We first write the elements of the set R .

$$\text{i.e. } R = \{(1, 4), (4, 1), (2, 3), (3, 2)\}$$

Since, $(1, 1) \notin R \Rightarrow R$ is not reflexive.

Now as, $(1, 4) \in R \Rightarrow (4, 1) \in R$ and

$(2, 3) \in R$

$$\Rightarrow (3, 2) \in R$$

So, R is symmetric

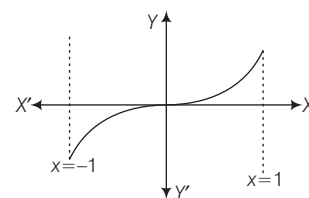
Now, $(1, 4) \in R$ and $(4, 1) \in R$

$$\nRightarrow (1, 1) \in R$$

Thus, R is not transitive.

Hence, R is symmetric but neither reflexive nor transitive.

$$2. (d) f(x) = x|x| = \begin{cases} x^2, & x \geq 0 \\ -x^2, & x < 0 \end{cases}$$



Since, $-1 \leq x \leq 1$, therefore

$$-1 \leq f(x) \leq 1$$

$\therefore$ Function is one-one and onto.

3. (a) Let P be the foot of the perpendicular drawn from $A(2, 3, 4)$ to the given line l .

$$\text{Then, } \frac{x-4}{-2} = \frac{y}{6} = \frac{z-1}{-3}$$

Now, any point on the line l is given by

$$x = 4 - 2\lambda, y = 6\lambda, z = 1 - 3\lambda$$

The coordinates of P are

$$(4 - 2\lambda, 6\lambda, 1 - 3\lambda)$$

The direction ratios of AP are

$$(4 - 2\lambda - 2, 6\lambda - 3, 1 - 3\lambda - 4)$$

$$\text{i.e. } (2 - 2\lambda, 6\lambda - 3, -3 - 3\lambda)$$

And the direction ratios of l are $-2, 6$ and -3 .

Given, $AP \perp l$

$$\therefore -2(2 - 2\lambda) + 6(6\lambda - 3) - 3(-3 - 3\lambda) = 0$$

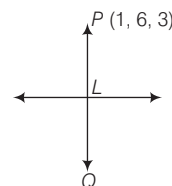
$$\Rightarrow \lambda = \frac{13}{49}$$

$$\therefore AP^2 = (4 - 2\lambda - 2)^2 + (6\lambda - 3)^2 + (1 - 3\lambda - 4)^2 = 22 - 26\lambda + 49\lambda^2$$

$$\text{Put } \lambda = \frac{13}{49}, \text{ we get}$$

$$AP^2 = \frac{909}{49} \Rightarrow AP = \frac{3}{7} \sqrt{101}$$

4. (c) Let $P(1, 6, 3)$ be the given point and L be the foot of perpendicular from P to the given line. The coordinates of a general point on the given line are



$$\frac{x-0}{1} = \frac{y-1}{2} = \frac{z-2}{3} = \lambda \text{ (say)}$$

$$\text{i.e. } x = \lambda, y = 2\lambda + 1, z = 3\lambda + 2$$

If the coordinates of L are

$$(\lambda, 2\lambda + 1, 3\lambda + 2), \text{ then the direction ratios of } PL \text{ are } (\lambda - 1, 2\lambda - 5, 3\lambda - 1).$$

Since, the direction ratios of given line which is perpendicular to PL , are $1, 2$ and 3 . Therefore,

$$(\lambda - 1)1 + (2\lambda - 5)2 + (3\lambda - 1)3 = 0, \text{ which gives } \lambda = 1.$$

Hence, coordinates of L are $(1, 3, 5)$.

Let $Q(x_1, y_1, z_1)$ be the image of $P(1, 6, 3)$ in the given line. Then, L is the mid-point of PQ .

$$\text{Therefore, } \frac{x_1 + 1}{2} = 1, \frac{y_1 + 6}{2} = 3,$$

$$\frac{z_1 + 3}{2} = 5$$

$$\Rightarrow x_1 = 1, y = 0, z = 7$$

Hence, the image of $(1, 6, 3)$ in the given line is $(1, 0, 7)$.

5. (a) Given equation of plane is $x + y + z = 5$.

The distance measured along the line $x = y = z$.

Direction ratios of the given line is $(1, 1, 1)$.

So, the equation of line PQ is

$$\frac{x-1}{1} = \frac{y-6}{1} = \frac{z-3}{1} = \lambda$$

$$\Rightarrow x = \lambda + 1, y = \lambda - 5, z = \lambda + 9$$

lies on the plane $x + y + z = 5$

$$\therefore (\lambda + 1) - (\lambda + 5) + (\lambda + 9) = 5$$

$$\Rightarrow \lambda = -10$$

The coordinate of Q is $(-9, -15, -1)$

and the coordinate of P is $(1, -5, 9)$

$$PQ = \sqrt{(10)^2 + (10)^2 + (10)^2} = 10\sqrt{3}$$

$$\therefore 2\sqrt{3}k = 10\sqrt{3} \Rightarrow k = 5$$

6. (b) $f(x) = \begin{cases} e^x, & x \leq 0 \\ 1 - x, & 0 < x \leq 1 \\ x - 1, & x > 1 \end{cases}$

$$Rf'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} -\left[\frac{h-1}{h}\right] = -1$$

$$Lf'(0) = \lim_{h \rightarrow 0} \frac{f(0-h) - f(0)}{-h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{-h} - 1}{-h} = 1$$

So, it is not differentiable at $x = 0$.

Similarly, it is not differentiable at $x = 1$ but it is continuous at $x = 0$ and 1 .

7. (b) $\therefore \lim_{x \rightarrow 0} f(x) = f(0)$

Let a be any point

Now, at $x = a$,

$$\lim_{x \rightarrow a} f(x) = \lim_{h \rightarrow 0} f(a+h) = \lim_{h \rightarrow 0} f(a) + \lim_{h \rightarrow 0} f(h)$$

$$= f(a) + f(0) = f(a+0) = f(a)$$

$\therefore f(x)$ is continuous at $x = a$, where a is any arbitrary point.

8. (a) Since, $x, \sin^{-1} x, \tan^{-1} x$ are continuous functions, so the function f is clearly continuous at each point of its domain except possibly at $x = 0$.

So, for f to be continuous at $x = 0$

$$f(0) = \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{2 - \frac{\sin^{-1} x}{x}}{2 + \frac{\tan^{-1} x}{x}} = \frac{1}{3}$$

9. (b) The given function is

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < 2 \end{cases}$$

$$\text{Now, LHD} = \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1}$$

$$= \lim_{x \rightarrow 1^-} \frac{0 - 1}{x - 1}$$

Which does not exist

$\therefore f$ is not derivable at $x = 1$.

10. (b) 11. (d)

12. (b) Required area

$$= \int_0^{\pi/2} |\cos x - \sin x| dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx$$

$$+ \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$= [\sin x + \cos x]_0^{\pi/4} + [-\cos x - \sin x]_{\pi/4}^{\pi/2}$$

$$= (\sqrt{2} - 1) - (1 - \sqrt{2}) = 2\sqrt{2} - 2$$

13. (b) Put $x + y = v$ and $1 + \frac{dy}{dx} = \frac{dv}{dx}$

Therefore, the differential equation reduces to

$$\frac{dv}{dx} = (1 + \cos v) + \sin v$$

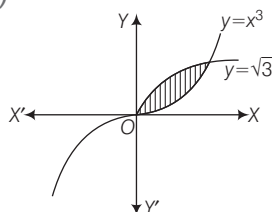
$$= 2 \cos^2 \frac{v}{2} + 2 \sin \frac{v}{2} \cos \frac{v}{2}$$

$$= 2 \cos^2 \frac{v}{2} \left(1 + \tan \frac{v}{2}\right)$$

$$\Rightarrow \int \frac{\sec^2 \frac{v}{2}}{2 \left[1 + \tan \left(\frac{v}{2}\right)\right]} dv = \int dx$$

$$\therefore \log \left[1 + \tan \left(\frac{x+y}{2}\right)\right] = x + C$$

14. (c)



$$\therefore A = \left| \int_0^1 (x^3 - \sqrt{x}) dx \right|$$

15. (a) $v = \hat{a} \times \hat{b} = |\hat{a}| |\hat{b}| \sin \theta \hat{n} = \sin \theta \hat{n}$

Where, θ is the angle between a and b .

$$[\because |\hat{a}| = 1 = |\hat{b}|]$$

$$\therefore [V] = \sin \theta$$

$$\begin{aligned} \text{Now, } u &= \hat{a} - (\hat{a} \cdot \hat{b}) \hat{b} = \hat{a} - (\cos \theta) \hat{b} \\ \Rightarrow |u|^2 &= (\hat{a} - (\cos \theta) \hat{b}) \cdot (\hat{a} - (\cos \theta) \hat{b}) \\ &= a^2 + \cos^2 \theta b^2 - 2 \cos \theta ab \\ &= 1 + \cos^2 \theta - 2 \cos^2 \theta \\ &= 1 - \cos^2 \theta = \sin^2 \theta \end{aligned}$$

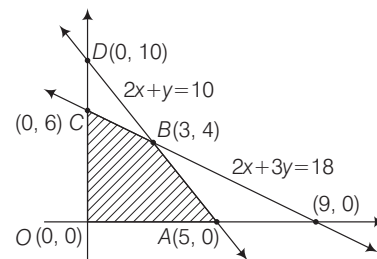
$$\therefore |u| = \sin \theta$$

$$\text{Thus } |v| = |u|$$

16. (c) Given, constraints are $2x + 3y \leq 18, 2x + y \leq 10, x \geq 0, y \geq 0$.

The feasible region is $OABCO$.

$$\text{Also, } z = 9x + 13y.$$



$$\text{At } O(0,0), z = 0$$

$$\text{At } A(5,0), z = 45$$

$$\text{At } B(3,4), z = 27 + 52 = 79$$

$$\text{At } C(0,6), z = 78$$

$\therefore$ Maximum value of z is 79.

17. (c) $\therefore -\frac{\pi}{2} \leq \sin^{-1} x \leq \frac{\pi}{2}$,

$$-\frac{\pi}{2} \leq \sin^{-1} y \leq \frac{\pi}{2}$$

$$\text{and } -\frac{\pi}{2} \leq \sin^{-1} z \leq \frac{\pi}{2}$$

Given that,

$$\sin^{-1} x + \sin^{-1} y + \sin^{-1} z = \frac{3\pi}{2}$$

Which is possible only when

$$\sin^{-1} x = \sin^{-1} y = \sin^{-1} z = \frac{\pi}{2}$$

$$\Rightarrow x = y = z = 1$$

$$\text{Put } p = q = 1$$

$$\text{Then, } f(2) = f(1) f(1) = 2 \times 2 = 4$$

$$\text{and put } p = 1, q = 2$$

$$\text{then, } f(3) = f(1) f(2)$$

$$= 2 \cdot 2^2 = 8$$

$$\therefore x^{f(1)} + y^{f(2)} + z^{f(3)}$$

$$= \frac{x + y + z}{x^{f(1)} + y^{f(2)} + z^{f(3)}}$$

18. (b) Let $I = \int \frac{dx}{(\sqrt{x})^2 + (\sqrt{x})^7}$

$$= \int \frac{dx}{(\sqrt{x})^7 \left(\frac{1}{(\sqrt{x})^5} + 1 \right)}$$

$$\text{Put } 1 + \frac{1}{(\sqrt{x})^5} = t$$

20 Solved Papers 2017

KERALA CEE

1. (e) Let

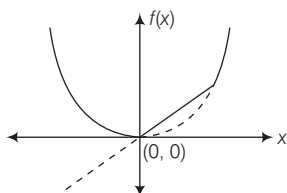
$$I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(i)$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

2. (a) Let $f(x) = \max\{x, x^2\}$



$\therefore$ Minimum value of $f(x) = 0$.

3. (c) We have,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(3) = \lim_{h \rightarrow 0} \frac{f(3+h) - f(3)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(3)f(h) - f(3+0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(3)f(h) - f(3)f(0)}{h}$$

$$= f(3) \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$= f(3) \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$= f(3)f'(0) = 3 \times 11 = 33$$

4. (a)

5. (d) Let

$$I = \int_{2016}^{2017} \frac{\sqrt{x}}{\sqrt{x} + \sqrt{4033-x}} dx \quad \dots(i)$$

$$\therefore I = \int_{2016}^{2017} \frac{\sqrt{4033-x}}{\sqrt{4033-x} + \sqrt{x}} dx$$

$$+ \int_{2016}^{2017} \frac{\sqrt{4033-(4033-x)}}{\sqrt{4033-x} + \sqrt{x}} dx$$

$$\Rightarrow I = \int_{2016}^{2017} \frac{\sqrt{4033-x}}{\sqrt{4033-x} + \sqrt{x}} dx \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

6. (a) Here, $I.F = e^{\int \tan x dx}$

$$= e^{\log \sec x} = \sec x$$

$\therefore$ The solution is given by

$$y \cdot \sec x = \int \sec x \cdot \sec x dx + C$$

$$y \sec x = \tan x + C \quad \dots(i)$$

Now, $y = 0$, when $x = 0$,

$$\therefore 0 = 0 + C \quad [\text{From Eq. (i)}]$$

$$\Rightarrow C = 0$$

Putting $C = 0$ in Eq. (i), we get

$$y \sec x = \tan x$$

$$7. (e) \therefore \begin{vmatrix} 2 & 2 & 6 \\ 2 & \lambda & 6 \\ 2 & -3 & 1 \end{vmatrix} = 0$$

8. (b)

$$9. (a) \text{ Here, } \begin{vmatrix} x-1 & y-0 & z-2 \\ -1-1 & 1-0 & 2-2 \\ 5-1 & 0-0 & 3-2 \end{vmatrix} = 0$$

10. (a) We have,

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0 \Rightarrow \mathbf{b} + \mathbf{c} = -\mathbf{a}$$

$$\Rightarrow |\mathbf{b} + \mathbf{c}| = |-\mathbf{a}| \Rightarrow |\mathbf{b} + \mathbf{c}| = |\mathbf{a}|$$

$$\Rightarrow |\mathbf{b} + \mathbf{c}|^2 = |\mathbf{a}|^2$$

$$\Rightarrow (\mathbf{b} + \mathbf{c}) \cdot (\mathbf{b} + \mathbf{c}) = |\mathbf{a}|^2$$

$$\Rightarrow |\mathbf{b}|^2 + |\mathbf{c}|^2 + 2|\mathbf{b}||\mathbf{c}|\cos\theta = |\mathbf{a}|^2$$

11. (*) $f'(x) = 6x^2 - 18ax + 12a^2$

For maximum or minimum, $f'(x) = 0$

$$\Rightarrow 6x^2 - 18ax + 12a^2 = 0$$

$$x = \frac{18a \pm \sqrt{324a^2 - 288a^2}}{2 \times 6}$$

$$= \frac{18a \pm \sqrt{36a^2}}{12} = \frac{18a \pm 6a}{12} = 2a, a$$

Now, $f''(x) = 12x - 18a$

$$\text{At } x = 2a, f''(x) = 24a - 18a = 6a > 0, \text{ maxima}$$

$$\therefore p = f(2a) = 2 \times 8a^3 - 36a^3 + 24a^3 + 1 = 4a^3 + 1$$

$$\text{At } x = a, f''(x) = 12 \times a - 18a = -6a < a, \text{ minima}$$

$$\therefore q = f(a) = 2a^3 - 9a^3 + 12a^3 + 1 = 5a^3 + 1$$

Also given $p^3 = q$

$$\therefore (4a^3 + 1)^3 = (5a^3 + 1)$$

$$\Rightarrow a = 0, \text{ but } a > 0 \text{ (given)}$$

$$12. (b) \text{ Let } I = \int_{-1}^1 \max\{x, x^3\} dx$$

$$= \int_{-1}^0 \max\{x, x^3\} dx + \int_0^1 \max\{x, x^3\} dx$$

$$= \int_{-1}^0 x^3 dx + \int_0^1 x dx$$

13. (d) We have,

$$f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = x \begin{vmatrix} 1 & x & x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$$

$$= x^2 \begin{vmatrix} 1 & x & x \\ 1 & 2x & 3x \\ 0 & 2 & 6 \end{vmatrix} = x^2 \begin{vmatrix} 1 & x & x \\ 0 & x & 2x \\ 0 & 2 & 6 \end{vmatrix}$$

14. (c) Put $x^3 = t \Rightarrow 3x^2 dx = dt$

$$\therefore I = \frac{1}{3} \int \frac{dt}{1+t^2} = \frac{1}{3} \tan^{-1} t + C$$

$$= \frac{1}{3} \tan^{-1}(x^3) + C$$

15. (d) We have,

$$f(x) = x^2 + e^x \Rightarrow f_1(x) = 2x + e^x$$

$$\Rightarrow f_2(x) = 2 + e^x$$

$$\Rightarrow f_3(x) = e^x \Rightarrow f_4(x) = e^x$$

$$\text{Since, } f_5(x) = f_4(x)$$

$\therefore$ Last value of n is 3.

16. (a) We have,

$$\cos\left(\tan^{-1} \frac{3}{4}\right) = \cos\left(\cos^{-1} \frac{1}{\sqrt{1+(3/4)^2}}\right)$$

17. (e) dr' of given lines are $< 2, 5, -3 >$

and $< -1, 8, 4 >$ respectively.

$$\therefore \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{(2)(-1) + (5)(8) + (-3)(4)}{\sqrt{(2)^2 + (5)^2 + (-3)^2} \sqrt{(-1)^2 + (8)^2 + (4)^2}}$$

18. (d) We have, $|\mathbf{a}| = 1$

$$\text{Now, } (\mathbf{x} - \mathbf{a}) \cdot (\mathbf{x} + \mathbf{a}) = 12$$

$$\Rightarrow \mathbf{x} \cdot \mathbf{x} + \mathbf{x} \cdot \mathbf{a} - \mathbf{a} \cdot \mathbf{x} - \mathbf{a} \cdot \mathbf{a} = 12$$

$$\Rightarrow |\mathbf{x}|^2 - |\mathbf{a}|^2 = 12$$

$$\Rightarrow |\mathbf{x}|^2 - 1 = 12$$

$$\Rightarrow |\mathbf{x}|^2 = 13$$

$$\Rightarrow |\mathbf{x}| = \sqrt{13}$$

$$19. (c) f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{f(x)f(h) - f(x)}{h}$$

$$= f(x) \lim_{h \rightarrow 0} \left(\frac{1 + \sin 3h(g(h)) - 1}{h} \right)$$

$$= f(x) \lim_{h \rightarrow 0} \frac{\sin 3h}{3h} \lim_{h \rightarrow 0} g(h)$$

$$= f(x) \times 1 \times g(0) = f(x)g(0)$$

20. (b) On applying $C_1 \rightarrow C_1 + C_2 + C_3$, we get

$$\begin{vmatrix} x+1 & 1 & 1 \\ x+1 & x-1 & 1 \\ x+1 & 1 & x-1 \end{vmatrix} = 0$$

$$\Rightarrow (x+1) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

On applying,

$R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$, we get

$$\Rightarrow (x+1) \begin{vmatrix} 0 & 2-x & 0 \\ 0 & x-2 & 2-x \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

21. (c) Standard deviation,

$$\sigma = \sqrt{\frac{\sum x_i^2}{N} - (\bar{x})^2}$$

$$\text{Here, } \bar{x} = \frac{\sum x_i}{N} = \frac{1+3+5+\dots+(2n-1)}{n}$$

$$= \frac{\frac{n}{2} [1+2n-1]}{n} = n$$

Again,
 $\sum x_i^2 = 1^2 + 3^2 + 5^2 + \dots (2n-1)^2$
 $= \sum (2n-1)^2$
 $= \sum (4n^2 - 4n + 1)$
 $= 4\sum n^2 - 4\sum n + \sum 1$
 $= \frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n$
 $= \frac{n}{3}[4n^2 - 1]$

$\therefore \sigma = \sqrt{\frac{n(4n^2 - 1)}{3n} - n^2}$
 $= \sqrt{\frac{n^2 - 1}{3}}$

22. (a) On applying,
 $C_1 \rightarrow C_1 - C_2, C_2 \rightarrow C_2 - C_3$, we get

$$= \begin{vmatrix} 0 & 0 & 1 \\ p-q & q-r & r \\ p-q & q-r-1 & r+1 \end{vmatrix}$$

$$= (p-q) \begin{vmatrix} 0 & 0 & 1 \\ 1 & q-r & r \\ 1 & q-r-1 & r+1 \end{vmatrix}$$

$$= (p-q) \cdot 1 [q-r-1 - q+r]$$

$$= q-p$$

23. (*) $\therefore C = 4A + 5B$

24. (a) $\therefore U^{-1} = \frac{1}{|U|} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$= \frac{1}{1} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Again, $U^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$\therefore U^{-1} = U^T$

25. (a) We have,

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore |A| = -(-1)[(1)(-1)-0] = -1$

$\therefore A^{-1} = \frac{1}{|A|} \text{adj } A$

$$= \frac{1}{-1} \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^T = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} = A^T$$

26. (a) 27. (b)

28. (d) $\begin{vmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 1 & 4 & a \end{vmatrix} = 0$

29. (d) We have,

$$\Rightarrow \begin{bmatrix} x+2y-3z \\ 4y+5z \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow x+2y-3z=1, 4y+5z=1, z=1$

On solving above equations, we get
 $x=6, y=-1, z=1$

30. (b) We have,

Now,
 $A^2 = A \cdot A = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 15 \\ 0 & 4 \end{bmatrix}$

31. (a) $\therefore 2x + y = 1 \dots (i)$

$x + y = 1 \dots (ii)$

$p - q = 0 \dots (iii)$

$p + q = 0 \dots (iv)$

On solving Eqs. (i) and (ii), we get

$x=0, y=1$

and on solving Eqs. (iii) and (iv), we get

$p=q=0$

32. (b) We have,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-w^2 & w^2 \\ 1 & w & w^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & w & w^2 \\ 1 & w & w \end{vmatrix}$$

$$= 2w^2 - (-w^2) = 3w^2$$

33. (d) $\begin{vmatrix} 3i+2 & 0 & 0 \\ 2 & 9i & -1 \\ 10 & 9 & i \end{vmatrix} = x + iy$

$[\therefore R_1 \rightarrow R_1 + R_2]$

$\Rightarrow (3i+2)[9i^2+9] = x + iy$

34. (e)

35. (c) Now,

$$A^2 = A \cdot A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = A^2 \cdot A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$\therefore A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}$

Now,

$$A^n + nI = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} + n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+n & 0 \\ n & 1+n \end{bmatrix}$$

Again, $I + nA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1+n & 0 \\ n & 1+n \end{bmatrix}$

$\therefore A^n + nI = I + nA$

36. (b) We have,

Now $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = 2A$$

Again, $A^3 = A^2 \cdot A$
 $= \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = 2^2 \cdot A$

$\therefore A^n = 2^{n-1}A$

$\therefore A^{2017} = 2^{2016}A$

37. (b) We have $\int_0^x f(t) dt = x^2 + e^x$

Using Leibnitz Rule,

$f(x) = 2x + e^x \Rightarrow f(1) = 2 + e$

38. (*)

39. (b) Let $= \int \frac{e^x}{e^{2x} + 2e^x + 1} dx$

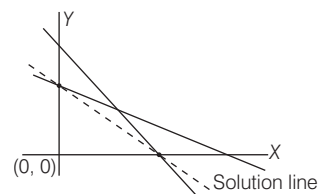
Put $e^x = t \Rightarrow e^x dx = dt$

$\therefore I = \int \frac{dt}{t^2 + 2t + 1} = \int \frac{dt}{(t+1)^2}$
 $= \frac{-1}{t+1} + C = \frac{-1}{e^x + 1} + C$

40. (*) Let $I = \int_{-1}^0 \frac{dx}{x^2 + x + 2}$
 $= \int_{-1}^0 \frac{dx}{\left(x + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{7}}{2}\right)^2}$

KCET

1. (d) If an LPP admits optimal solution at two consecutive vertices of a feasible region, then the optimal solution occurs at every point on the line joining these two points.



2. (c) Now, $\Delta_1 = \begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$

On applying

$$C_1 \rightarrow xC_1, C_2 \rightarrow yC_2, C_3 \rightarrow zC_3,$$

$$= \frac{1}{xyz} \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

$$= \frac{xyz}{xyz} \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} Ax & By & Cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

22 Solved Papers 2017

3. (b) We have,

$$f(x) = x^2 + 2x - 5$$

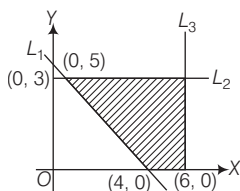
$$\Rightarrow f'(x) = 2x + 2 = 2(x + 1)$$

$$\therefore f'(x) \geq 0 \Rightarrow 2(x + 1) \geq 0$$

$$\Rightarrow x + 1 \geq 0 \Rightarrow x \geq -1$$

$\therefore f(x)$ is increasing on $[-1, \infty)$.

4. (c)



Equation of line L_1 is given by

$$y - 0 = \frac{5 - 0}{0 - 4}(x - 4)$$

$$\Rightarrow 5x + 4y = 20$$

According to the shaded position

$$5x + 4y \geq 20$$

Now, equation of line L_2 is $y = 3$.

$$y \leq 3$$

Again, equation of line L_3 is $x = 6$.

$$x \leq 6$$

Also, so $x \geq 0, y \geq 0$.

5. (c) We have, $y^2 = x$... (i)

$$\therefore 2y \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y}$$

$$\therefore \text{Slope of tangent} = \frac{1}{2y} \quad \dots \text{(ii)}$$

Now, tangent makes an angle of $\pi/4$ with X-axis

$$\therefore \text{Slope of tangent} = \tan \frac{\pi}{4} = 1 \quad \dots \text{(iii)}$$

From Eqs. (ii) and (iii), we get

$$\frac{1}{2y} = 1 \Rightarrow y = \frac{1}{2}$$

6. (c) Let $I = \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{\sin x} + 1}$

$$\dots \text{(i)}$$

$$\Rightarrow I = \int_{-\pi/2}^{\pi/2} \frac{dx}{e^{-\sin x} + 1}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{e^{\sin x}}{e^{\sin x} + 1} dx \quad \dots \text{(ii)}$$

On adding Eqs. (i) and (ii), we get

7. (d) Here $|\mathbf{a}| = |\mathbf{b}| = |\sqrt{3}\mathbf{a} - \mathbf{b}| = 1$

$$\text{Now, } |\sqrt{3}\mathbf{a} - \mathbf{b}| = 1$$

$$\Rightarrow |\sqrt{3}\mathbf{a} - \mathbf{b}|^2 = 1$$

$$\Rightarrow (\sqrt{3}\mathbf{a} - \mathbf{b}) \cdot (\sqrt{3}\mathbf{a} - \mathbf{b}) = 1$$

$$\Rightarrow 3|\mathbf{a}|^2 + |\mathbf{b}|^2 - 2\sqrt{3}\mathbf{a} \cdot \mathbf{b} = 1$$

$$\Rightarrow 3(1)^2 + (1)^2 - 2\sqrt{3}|\mathbf{a}||\mathbf{b}|\cos\theta = 1$$

8. (c) Let $I = \int_{-5}^5 |x + 2| dx$

$$= \int_{-5}^{-2} |x + 2| dx + \int_{-2}^5 |x + 2| dx$$

$$= -\int_{-5}^{-2} (x + 2) dx + \int_{-2}^5 (x + 2) dx$$

$$= -\left[\left(\frac{4}{2} - 4\right) - \left(\frac{25}{2} - 10\right)\right] + \left[\left(\frac{25}{2} + 10\right) - \left(\frac{4}{2} - 4\right)\right]$$

$$= -\left[-2 - \frac{5}{2}\right] + \left[\frac{45}{2} + 2\right]$$

$$= 2 + \frac{5}{2} + \frac{45}{2} + 2 = 29$$

9. (a) $I = \int_0^{\pi/2} \frac{1}{a^2 \sin^2 x + b^2 \cos^2 x} dx$

$$= \int_0^{\pi/2} \frac{\sec^2 x}{a^2 \tan^2 x + b^2} dx$$

$$\text{Put } \tan x = t$$

$$\Rightarrow \sec^2 x dx = dt$$

$$\therefore I = \int_0^{\infty} \frac{dt}{a^2 t^2 + b^2} = \frac{1}{a^2} \int_0^{\infty} \frac{dt}{t^2 + \left(\frac{b}{a}\right)^2}$$

10. (a) Let $I = \int \sqrt{x^2 + 2x + 5} dx$

$$= \int \sqrt{(x + 1)^2 + 4} dx$$

11. (a) The range of $\sec^{-1} x$

$$= [0, \pi] - \left\{\frac{\pi}{2}\right\}$$

12. (d) Let $I = \int \frac{x + 3}{(x + 4)^2} e^x dx$

$$= \int e^x \left[\frac{x + 4 - 1}{(x + 4)^2} \right] dx$$

$$= \int e^x \frac{1}{x + 4} dx + \int e^x \left(\frac{-1}{(x + 4)^2} \right) dx$$

$$= \frac{1}{x + 4} e^x - \int e^x \left(\frac{-1}{(x + 4)^2} \right) dx +$$

$$\int e^x \left(-\frac{1}{(x + 4)^2} \right) dx$$

13. (a, c, d) We have,

$$\therefore \frac{dy}{dx} = \begin{vmatrix} \frac{d}{dx} f(x) & \frac{d}{dx} g(x) & \frac{d}{dx} h(x) \\ a & b & c \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{d}{dx} a & \frac{d}{dx} b & \frac{d}{dx} c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ \frac{d}{dx} a & \frac{d}{dx} b & \frac{d}{dx} c \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ a & b & c \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ 0 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ a & b & c \end{vmatrix}$$

$$= \begin{vmatrix} l & m & n \\ a & b & c \\ f'(x) & g'(x) & h'(x) \end{vmatrix} = \begin{vmatrix} f'(x) & l & a \\ g'(x) & m & b \\ h'(x) & n & c \end{vmatrix}$$

14. (a) 15. (d)

$$16. (c) \text{ Hint } A = \frac{1}{\pi} \left[\sin^{-1}(\pi x) \cot^{-1}(\pi x) - \tan^{-1}\left(\frac{x}{\pi}\right) \sin^{-1}\left(\frac{x}{\pi}\right) \right]$$

$$B = \cos^{-1}(\pi x) \tan^{-1}(\pi x) - \sin^{-1}\left(\frac{x}{\pi}\right) \tan^{-1}\left(\frac{x}{\pi}\right)$$

17. (d) $\therefore \Sigma P(x) = 1$

18. (c)

$$19. (a) I = \int \frac{(2\cos^2 x - 1) - (2\cos^2 \theta - 1)}{\cos x - \cos \theta} dx$$

$$= 2 \int \frac{\cos^2 x - \cos^2 \theta}{\cos x - \cos \theta} dx$$

$$= 2 \int \frac{(\cos x - \cos \theta)(\cos x + \cos \theta)}{\cos x - \cos \theta} dx$$

20. (d) We have,

Since, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal

$$\therefore \mathbf{a} \cdot \mathbf{b} = 0$$

21. (a) Since, $f(x)$ is continuous at $x = 2$

$$\therefore \text{LHL (at } x = 2) = \text{RHL (at } x = 2)$$

$$\Rightarrow \lim_{x \rightarrow 2^-} kx^2 = \lim_{x \rightarrow 2^+} 3$$

$$\Rightarrow \lim_{h \rightarrow 0} k(2 - h)^2 = 3$$

22. (d) We have, $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 1$

Also, we have

$$\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$$

$$\therefore \mathbf{a} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot 0$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 0$$

$$\Rightarrow |\mathbf{a}|^2 + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 0$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = -|\mathbf{a}|^2 = -1 \quad \dots \text{(i)}$$

Similarly,

$$\mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{c} = -1 \quad \dots \text{(ii)}$$

$$\text{and } \mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{b} = -1 \quad \dots \text{(iii)}$$

On adding Eqs. (i), (ii) and (iii), we get

23. (b) We have, $f(x) = x^4$

$$\text{Let } f(x_1) = f(x_2)$$

$$\Rightarrow x_1^4 = x_2^4 \Rightarrow x_1 = \pm x_2$$

$\therefore f(x)$ is not one-one.

Again, co-domain of $f(x) = R$

and Range of $f(x) = [0, \infty)$

$\therefore$ Co-domain $\neq$ Range

$\therefore f(x)$ is not onto.

24. (b) Dr' of the normal to the plane

$$= \langle 2, -3, 6 \rangle$$

Also, dr' of X-axis = $\langle 1, 0, 0 \rangle$

If θ is the angle between plane and X-axis, then

$$\sin \theta = \frac{2 \times 1 + (-3) \times 0 + 6 \times 0}{\sqrt{(2)^2 + (-3)^2 + (6)^2} \sqrt{(1)^2 + (0)^2 + (0)^2}}$$

25. (c) Let $I = \int_{0.2}^{3.5} [x] dx$

$$= \int_{0.2}^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx + \int_3^{3.5} [x] dx$$

$$= \int_{0.2}^1 0 dx + \int_1^2 1 \cdot dx + \int_2^3 2 \cdot dx + \int_3^{3.5} 3 \cdot dx$$

$$= 0 + [x]_1^2 + [2x]_2^3 + [3x]_3^{3.5}$$

$$= (2-1) + (6-4) + (10.5-9)$$

$$= 1 + 2 + 1.5 = 4.5$$

26. (d) We have, $y = \tan^{-1} \left(\frac{\sin x + \cos x}{\cos x - \sin x} \right)$

$$= \tan^{-1} \left(\frac{1 + \tan x}{1 - \tan x} \right)$$

$$= \tan^{-1} \left(\frac{\tan \pi/4 + \tan x}{1 - \tan \pi/4 \tan x} \right)$$

27. (c)

28. (d) Let $u = \cos^{-1}(2x^2 - 1)$ and $v = \cos^{-1}x$

Now, $u = \cos^{-1}(2x^2 - 1)$

Put $x = \cos \theta$

$$\therefore u = \cos^{-1}(2\cos^2 \theta - 1)$$

$$= \cos^{-1}(\cos 2\theta) = 2\theta = 2\cos^{-1}x$$

Again, $\frac{du}{dv} = \left(\frac{\frac{du}{dx}}{\frac{dv}{dx}} \right)$

$$= \frac{\frac{d}{dx}(2\cos^{-1}x)}{\frac{d}{dx}(\cos^{-1}x)} = \frac{\left(-\frac{2}{\sqrt{1-x^2}} \right)}{\left(\frac{-1}{\sqrt{1-x^2}} \right)} = 2$$

29. (c) Let the radius of the sphere be a .

$$\therefore \text{Volume, } V = \frac{4}{3} \pi a^3$$

$$\Rightarrow \frac{dV}{da} = \frac{4}{3} \pi (3a^2) = 4\pi a^2$$

Again, surface area, $s = 4\pi a^2$

$$\Rightarrow \frac{ds}{da} = 4\pi(2a) = 8\pi a$$

$$\therefore \frac{dV}{ds} = \frac{(dV/da)}{(ds/da)} = \frac{4\pi a^2}{8\pi a} = \frac{a}{2} = \frac{4}{2}$$

$$= 2 \text{ cm}^3/\text{cm}^2$$

30. (a) Let $I = \int_0^{\pi/2} \frac{\tan^7 x}{\cot^7 x + \tan^7 x} dx \dots (i)$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\tan^7(\pi/2 - x)}{\cot^7(\pi/2 - x) + \tan^7(\pi/2 - x)} dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cot^7 x}{\tan^7 x + \cot^7 x} dx \dots (ii)$$

31. (c) Required area $= 2 \int_0^{16} x dy$

$$= 2 \int_0^{16} \sqrt{y} dy = 2 \left[\frac{y^{3/2}}{3/2} \right]_0^{16}$$

$$= \frac{4}{3} [16^{3/2} - 0^{3/2}]$$

$$= \frac{4}{3} \times 64 = \frac{256}{3} \text{ sq units.}$$

32. (d) Required Area $= \int_0^{\pi} |\cos x| dx$

$$= 2 \int_0^{\pi/2} |\cos x| dx = 2 \int_0^{\pi/2} \cos x dx$$

33. (a) For mean value theorem

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\therefore f(x) = x^2 \Rightarrow f'(x) = 2x$$

$$\therefore 2c = \frac{f(4) - f(2)}{4 - 2}$$

34. (b) We have,

$$\Rightarrow \frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5}$$

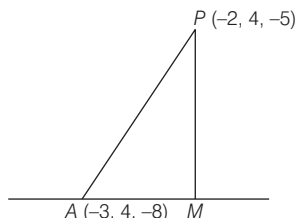
$$\Rightarrow \cot^{-1}x + \cot^{-1}y = \pi - \frac{4\pi}{5} = \frac{\pi}{5}$$

35. (b) Let A be a matrix which is both symmetric and skew symmetric.

$\therefore A' = A$ and $A' = -A$

This is possible only when A is zero matrix.

36. (a) The line passes through $A(-3, 4, -8)$ and is parallel to the vector $\mathbf{b} = 3\hat{i} + 5\hat{j} + 6\hat{k}$



Let M be the foot of the perpendicular from $P(-2, 4, -5)$ on the given line.

We have

$$\mathbf{AP} = -\hat{i} + 0\hat{j} - 3\hat{k}$$

$$\Rightarrow |\mathbf{AP}| = \sqrt{1 + 0 + 9} = \sqrt{10}$$

Clearly, $AM = \text{Projection of } \mathbf{AP} \text{ on } \mathbf{b}$

$$\Rightarrow AM = \frac{|\mathbf{AP} \cdot \mathbf{b}|}{|\mathbf{b}|}$$

$$= \frac{|(-\hat{i} - 3\hat{k}) \cdot (3\hat{i} + 5\hat{j} + 6\hat{k})|}{|3\hat{i} + 5\hat{j} + 6\hat{k}|}$$

AP EAMCET

1. (d) Clearly

$$\alpha = P(x < 3) = P(x = 1) + P(x = 2) = 3\lambda$$

$$\text{and } \beta = P(x > 2) = P(x = 3) + P(x = 4) = 7\lambda$$

$$\therefore \alpha : \beta = 3 : 7$$

2. (c) Clearly, $f(x)$ will be continuous in the intervals $(-\infty, 1)$, $(1, 3)$, $(3, 5)$ and $(5, \infty)$

Now, let us check the continuity at $x = 1, 3$ and 5 .

Here,

$$(i) \lim_{x \rightarrow 1^-} f(x) = 1 - 1 = 0$$

$$\text{and } \lim_{x \rightarrow 1^+} f(x) = 2 - 1 = 1$$

therefore f is not continuous at $x = 1$

$$(ii) \lim_{x \rightarrow 3^-} f(x) = 2 - 9 = -7 \text{ and}$$

$$f(3) = \lim_{x \rightarrow 3^+} f(x) = 3 - 10 = -7,$$

therefore f is continuous at $x = 3$

$$(iii) \lim_{x \rightarrow 5^-} f(x) = 5 - 10 = -5 \text{ and,}$$

$$\lim_{x \rightarrow 5^+} f(x) = 2 \times 5 = 10,$$

therefore f is not continuous at $x = 5$.

Thus, points of discontinuity of f is $\{1, 5\}$.

3. (d) Normal to plane P_1 is

$$\mathbf{n}_1 = [(\hat{j} - \hat{k}) \times (3\hat{j} - 2\hat{k})]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 0 & 3 & -2 \end{vmatrix} = \hat{i}$$

Normal to the plane P_2 is

$$\mathbf{n}_2 = [(2\hat{i} + 3\hat{j}) \times (\hat{i} - 3\hat{j})]$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 0 \\ 1 & -3 & 0 \end{vmatrix} = -9\hat{k}$$

Since, $\mathbf{a}$ is parallel to the line of intersection of planes P_1 and P_2 , therefore $\mathbf{a}$ is parallel to $\pm(\mathbf{n}_1 \times \mathbf{n}_2)$ i.e. $\mathbf{a}$ is parallel to $9\hat{j}$.

4. (a) Let $f(x) = \log \left\{ e^x \left(\frac{x-2}{x+2} \right)^{\frac{3}{4}} \right\}$

$$= \log e^x + \log \left(\frac{x-2}{x+2} \right)^{\frac{3}{4}}$$

$$= x \log e + \frac{3}{4} \log \left(\frac{x-2}{x+2} \right)$$

$$= x + \frac{3}{4} \log \left(\frac{x-2}{x+2} \right) \quad [\because \log e = 1]$$

Now, $\frac{d}{dx} (f(x)) = 1 + \frac{3}{4} \frac{d}{dx} \log \left(\frac{x-2}{x+2} \right)$

$$= 1 + \frac{3}{4} \left[\frac{x+2}{x-2} \cdot \frac{(x+2) \cdot 1 - (x-2) \cdot 1}{(x+2)^2} \right]$$

5. (d) We have, $\sqrt{1+x^2} y = \sinh^{-1} x$

$$\Rightarrow \sqrt{1+x^2} y_1 + y \frac{1}{2\sqrt{1+x^2}} \cdot 2x$$

24 Solved Papers 2017

$$= \frac{1}{\sqrt{1+x^2}}$$

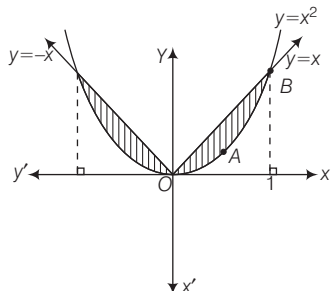
$$(1+x^2)y_1 + xy = 1$$

$$(1+x^2)y_2 + y_1 \cdot 2x + xy_1 + y = 0$$

$$\Rightarrow (1+x^2)y_2 + 3xy_1 + y = 0$$

6. (c) Given equation of curves are
 $y = x^2$... (i)

and $y = |x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$... (ii)



Clearly, required area = 2 (Area of region OABO)

$$= 2 \int_0^1 (x - x^2) dx = 2 \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1$$

7. (c) Let $I = \int \frac{5x^2 + 3}{x^2(x^2 - 2)} dx$

Now, consider $\frac{5x^2 + 3}{x^2(x^2 - 2)}$ and take

$$x^2 = y.$$

Then, $\frac{5x^2 + 3}{x^2(x^2 - 2)} = \frac{5y + 3}{y(y - 2)}$

Now, put $\frac{5y + 3}{y(y - 2)} = \frac{A}{y} + \frac{B}{y - 2}$

$$\Rightarrow 5y + 3 = (y - 2)A + yB$$

$$\Rightarrow 5y + 3 = y(A + B) - 2A$$

$$\Rightarrow A + B = 5 \text{ and } 3 = -2A$$

$$\Rightarrow A = -\frac{3}{2} \text{ and } B = \frac{13}{2}$$

Thus, $\frac{5x^2 + 3}{x^2(x^2 - 2)} = -\frac{3}{2} \frac{1}{x^2} + \frac{13}{2} \frac{1}{x^2 - 2}$

$$\therefore I = -\frac{3}{2} \int \frac{dx}{x^2} + \frac{13}{2} \int \frac{dx}{x^2 - 2}$$

8. (a) On substituting $x = \cos 2\theta$, we get

$$y = \tan^{-1} \left\{ \frac{\cos 2\theta}{1 + \sqrt{\sin^2 2\theta}} \right\}$$

$$+ \sin \left\{ 2 \tan^{-1} \sqrt{\frac{1 - \cos 2\theta}{1 + \cos 2\theta}} \right\}$$

$$\Rightarrow y = \tan^{-1} \left\{ \frac{\cos 2\theta}{1 + \sin 2\theta} \right\}$$

$$+ \sin \left\{ 2 \tan^{-1} \sqrt{\frac{2 \sin^2 \theta}{2 \cos^2 \theta}} \right\}$$

$$\Rightarrow y = \tan^{-1} \left\{ \frac{\cos^2 \theta - \sin^2 \theta}{(\cos \theta + \sin \theta)^2} \right\}$$

$$+ \sin \{ 2 \tan^{-1} (\tan \theta) \}$$

$$\Rightarrow y = \tan^{-1} \left\{ \frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} \right\} + \sin 2\theta$$

$$\Rightarrow y = \tan^{-1} \left\{ \frac{1 - \tan \theta}{1 + \tan \theta} \right\} + \sin 2\theta$$

$$\Rightarrow y = \tan^{-1} \left(\tan \left(\frac{\pi}{4} - \theta \right) \right) + \sin 2\theta$$

$$\Rightarrow y = \frac{\pi}{4} - \theta + \sin 2\theta$$

$$\Rightarrow y = \frac{\pi}{4} - \frac{1}{2} \cos^{-1} x + \sqrt{1 - x^2}$$

9. (b)

10. (b) Given differential equation is

$$(x - 4y^3) \frac{dy}{dx} - y = 0, (y > 0)$$

$$\Rightarrow (x - 4y^3) \frac{dy}{dx} = y$$

$$\Rightarrow \frac{x - 4y^3}{y} = \frac{dx}{dy}$$

$$\Rightarrow \frac{dx}{dy} = \frac{x}{y} - 4y^2$$

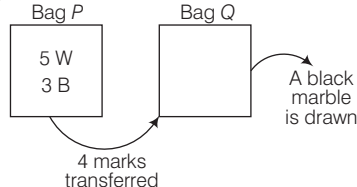
$$\Rightarrow \frac{dx}{dy} - \frac{x}{y} = -4y^2, \text{ which is a}$$

linear differential equations of the form $\frac{dx}{dy} + Px = Q$

Now, I.F. = $e^{\int P dy} = e^{\int -\frac{1}{y} dy} = e^{-\log y}$
 $= e^{\log y^{-1}} = \frac{1}{y}$

and its solution is given by

11. (a)



Let E_1 be the event that 1W and 3 B marbles are transferred.

E_2 be the event that 2W and 2 B marbles are transferred.

E_3 be the event that 3W and 1 B marbles are transferred.

E_4 be the event that 4W and 0 B marbles are transferred

and A be the event that a black marble is drawn from bag Q.

Then, $P(E_1) = \frac{{}^5C_1 \times {}^3C_3}{{}^8C_4} = \frac{5}{{}^8C_4}$

$$P(E_2) = \frac{{}^5C_2 \times {}^3C_2}{{}^8C_4} = \frac{30}{{}^8C_4}$$

$$P(E_3) = \frac{{}^5C_3 \times {}^3C_1}{{}^8C_4} = \frac{30}{{}^8C_4}$$

$$P(E_4) = \frac{{}^5C_4 \times {}^3C_0}{{}^8C_4} = \frac{5}{{}^8C_4}$$

$$P(A / E_1) = \frac{3}{4}$$

$$P(A / E_2) = \frac{2}{4}$$

$$P(A / E_3) = \frac{1}{4}$$

$$P(A / E_4) = 0$$

Required probability = $P(E_1 / A)$

$$= \frac{P(E_1) \cdot P(A / E_1)}{(P(E_1) \cdot P(A / E_1) + P(E_2) \cdot P(A / E_2) + P(E_3) \cdot P(A / E_3) + P(E_4) \cdot P(A / E_4))}$$

12. (a) Consider,

$$\operatorname{sech}^{-1} \left(\frac{1}{\sqrt{2}} \right) + \operatorname{cosech}^{-1}(-1)$$

$$= \log_e \left(\frac{1 + \sqrt{1 - 1/2}}{1/\sqrt{2}} \right)$$

$$+ \log_e \left(\frac{1 - \sqrt{1 + 1}}{-1} \right)$$

$$= \log_e \left(\frac{\sqrt{2} + 1}{\frac{\sqrt{2}}{1}} \right) + \log_e (\sqrt{2} - 1)$$

$$= \log_e (\sqrt{2} + 1) + \log_e (\sqrt{2} - 1)$$

$$= \log_e ((\sqrt{2})^2 - 1) [\because$$

$$\log m + \log n = \log(m \times n)]$$

$$= \log_e (2 - 1) = \log_e 1 = 0$$

13. (a) 14. (b)

15. (d) We have a function

$$f(x) = (x - 1)(x - 2)$$

$$= x^2 - 3x + 2 \text{ defined on } \left[0, \frac{1}{2} \right].$$

According to LMV theorem,

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f\left(\frac{1}{2}\right) - f(0)}{\frac{1}{2} - 0}$$

$$\Rightarrow 2c - 3 = \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) - (-1)(-2)}{\frac{1}{2}}$$

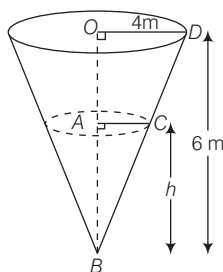
$$\Rightarrow 2c - 3 = 2 \left(\frac{3}{4} - 2 \right)$$

$$\Rightarrow 2c - 3 = -\frac{5}{2}$$

$$\Rightarrow 4c - 6 = -5 \Rightarrow c = \frac{1}{4} \in \left(0, \frac{1}{2} \right)$$

Thus, required value of c is $\frac{1}{4}$.

16. (a) Let V be the volume, r be the radius and h be the height of cone at any time t . Then



We have, $\frac{dv}{dt} = 3\text{m}^3/\text{min}$

We have to find, $\frac{dh}{dt}$ when $h = 3\text{m}$

Clearly, $V = \frac{1}{3}\pi r^2 h$... (i)

Since $\triangle BOD \sim \triangle BAC$ [By AA-similarity criterion]

$$\therefore \frac{BO}{BA} = \frac{OD}{AC} \Rightarrow \frac{6}{h} = \frac{4}{r}$$

$$\Rightarrow r = \frac{2}{3}h$$

Now, from Eq. (i), we get

$$V = \frac{1}{3}\pi \left(\frac{2}{3}h\right)^2 h = \frac{4}{27}\pi h^3$$

$$\Rightarrow \frac{dV}{dt} = \frac{4}{27}\pi 3h^2 \frac{dh}{dt}$$

$$\Rightarrow 3 = \frac{4}{27}\pi 3h^2 \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{3 \times 27}{4\pi 3h^2}$$

$$\Rightarrow \left. \frac{dh}{dt} \right|_{h=3} = \frac{3 \times 27}{4\pi 27} = \frac{3}{4\pi} \text{ m/min}$$

17. (c)

18. (a) We have,

$$\cot(\cos^{-1} x) = \sec \left\{ \tan^{-1} \left(\frac{a}{\sqrt{b^2 - a^2}} \right) \right\}$$

$$\Rightarrow \cot \left(\cot^{-1} \frac{x}{\sqrt{1-x^2}} \right) = \sec$$

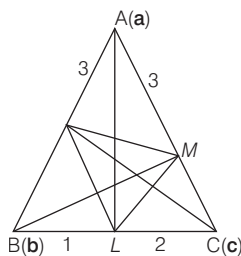
$$\left\{ \sec^{-1} \sqrt{1 + \left(\frac{a}{\sqrt{b^2 - a^2}} \right)^2} \right\}$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \sqrt{\frac{b^2 - a^2 + a^2}{b^2 - a^2}}$$

$$\Rightarrow \frac{x}{\sqrt{1-x^2}} = \frac{b}{\sqrt{b^2 - a^2}}$$

19. (d) We have

In $\triangle ABC$, L, M, N are points on BC, CA and AB respectively, dividing in the ratio $1:2$, $2:3$ and $3:5$.



$$\text{Now, } L = \frac{2b+c}{3}, \quad M = \frac{2a+c}{5}$$

$$N = \frac{3b+5a}{8}, \quad K = \frac{5b+3a}{8}$$

$$AL = \frac{2b+c}{3} - a = \frac{2b+c-3a}{3}$$

$$BM = \frac{2a+c}{5} - b = \frac{2a+c-5b}{5}$$

$$CN = \frac{3b+5a}{8} - c = \frac{3b+5a-8c}{8}$$

$$CK = \frac{5b+3a}{8} - c = \frac{5b+3a-8c}{8}$$

$$\therefore \frac{AL+BM+CN}{CK} =$$

$$\frac{\frac{2b+c-3a}{3} + \frac{2a+c-5b}{5} + \frac{3b+5a-8c}{8}}{\frac{5b+3a-8c}{8}}$$

$$= \frac{(5b+3a-8c)}{15(5b+3a-8c)} = \frac{1}{15}$$

20. (c) Let $I = \int_0^1 f(k-1+x) dx$

Now, put $k-1+x = t \Rightarrow dx = dt$

Also, when $x = 0$ then $t = k-1$

and when $x = 1$ then $t = k$

$$\therefore I = \int_{k-1}^k f(t) dt$$

$$= \int_{k-1}^k f(x) dx \quad [\because t \text{ is an}$$

dummy variable]

$$\text{Now, } \sum_{k=1}^{10} \int_0^1 f(k-1+x) dx$$

$$= \sum_{k=1}^{10} \int_{k-1}^k f(x) dx$$

$$= \int_0^1 f(x) dx + \int_1^2 f(x) dx + \dots + \int_9^{10} f(x) dx$$

$$= \int_0^{10} f(x) dx = 5$$

21. (c) Let $I = \int \frac{x^{e-1} + e^{x-1}}{x^e + e^x} dx$

Now, put $x^e + e^x = t$

22. (a) We have

$$\Delta = \begin{vmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ -1 & -\cos \theta & 1 \end{vmatrix}$$

On applying $R_3 \rightarrow R_3 + R_1$, we get

$$\Delta = \begin{vmatrix} 1 & \cos \theta & 1 \\ -\cos \theta & 1 & \cos \theta \\ 0 & 0 & 2 \end{vmatrix}$$

On expanding along R_3 , we get

$$\Delta = 2(1 + \cos^2 \theta)$$

$$\therefore 0 \leq \cos^2 \theta \leq 1$$

$$\therefore 1 \leq 1 + \cos^2 \theta \leq 2$$

$$\Rightarrow 2 \leq 2(1 + \cos^2 \theta) \leq 4$$

Thus, $\Delta \in [2, 4]$

23. (c)

24. (c) Now this system will have non-trivial solution, if

$$\begin{vmatrix} a-1 & -1 & -1 \\ 1 & -(b-1) & 1 \\ 1 & 1 & -(c-1) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a-1 & -1 & -1 \\ 1 & 1-b & 1 \\ 1 & 1 & 1-c \end{vmatrix} = 0$$

25. (c) Given, $V =$ volume of cylindrical vessel $= \pi r^2 h$... (i)

Let S be the area of metal sheet used to form a cylindrical vessel. Then

$$S = 2\pi rh + \pi r^2$$

$$= 2\pi r \frac{V}{\pi r^2} + \pi r^2 \quad [\text{using Eq. (i)}]$$

$$\Rightarrow S(r) = \frac{2V}{r} + \pi r^2$$

For minimum of $S(r)$, put $S'(r) = 0$

$$\Rightarrow \frac{-2V}{r^2} + 2\pi r = 0 \Rightarrow \frac{2V}{r^2} = 2\pi r$$

$$\Rightarrow r^3 = \frac{V}{\pi} \Rightarrow r = \sqrt[3]{\frac{V}{\pi}}$$

On substituting $r = \left(\frac{V}{\pi}\right)^{1/3}$ in Eq. (i), we get

$$V = \pi \left(\frac{V}{\pi}\right)^{2/3} \cdot h \Rightarrow h = \frac{V}{\pi} \times \frac{\pi^{2/3}}{V^{2/3}}$$

$$\Rightarrow h = \frac{V^{1/3}}{\pi^{1/3}} = \sqrt[3]{\frac{V}{\pi}}$$

Thus, required radius is $r = \sqrt[3]{\frac{V}{\pi}}$ and

height is $h = \sqrt[3]{\frac{V}{\pi}}$.

26. (a) Let $I = \int \frac{x + \sin x}{1 + \cos x} dx$

$$= \int \frac{x + \sin x}{2\cos^2 x/2} dx$$

$$= \frac{1}{2} \int x \sec^2 \frac{x}{2} dx +$$

$$\int \frac{2\sin x/2 \cos x/2}{2\cos^2 x/2} dx$$

26 Solved Papers 2017

$$\begin{aligned}
 &= \frac{1}{2} \int x \sec^2 \frac{x}{2} dx + \int \tan \frac{x}{2} dx \\
 &= \frac{1}{2} \left[x \frac{\tan \frac{x}{2}}{1/2} - \int \frac{\tan x / 2}{1/2} dx \right] \\
 &\quad + \int \tan \frac{x}{2} dx \\
 &= x \tan \frac{x}{2} - \int \tan \frac{x}{2} dx + \int \tan \frac{x}{2} dx \\
 &= x \tan \frac{x}{2} + C
 \end{aligned}$$

27. (a) We have, $I_n = \int \frac{\sin nx}{\cos x} dx$

$$\begin{aligned}
 &= \int \frac{\sin(nx - x + x)}{\cos x} dx \\
 &= \int \frac{\sin(n-1)x + x}{\cos x} dx \\
 &\quad \sin(n-1)x \cos x \\
 &= \int \frac{+ \cos(n-1)x \sin x}{\cos x} dx \\
 &= \int \sin(n-1)x dx \\
 &\quad + \int \frac{\cos(n-1)x \sin x}{\cos x} dx \\
 &\quad + \int \frac{\cos(n-1)x \sin x}{\cos x} dx \\
 &= \int \sin(n-1)x dx \\
 &\quad + \frac{1}{2} \int \frac{2 \sin x \cos(n-1)x}{\cos x} dx \\
 &= -\frac{\cos(n-1)x}{n-1} \\
 &\quad \sin(x + (n-1)x) \\
 &\quad + \frac{1}{2} \int \frac{+ \sin(x - (n-1)x)}{\cos x} dx \\
 &= -\frac{\cos(n-1)x}{(n-1)} \\
 &\quad + \frac{1}{2} \int \frac{\sin nx + \sin(2-n)x}{\cos x} dx \\
 &= -\frac{\cos(n-1)x}{(n-1)} \\
 &\quad + \frac{1}{2} \int \frac{\sin nx}{\cos x} dx - \frac{1}{2} \int \frac{\sin(n-2)x}{\cos x} dx \\
 &= -\frac{\cos(n-1)x}{(n-1)} + \frac{1}{2} I_n - \frac{1}{2} I_{n-2} \\
 &\Rightarrow \left(1 - \frac{1}{2}\right) I_n = -\frac{\cos(n-1)x}{(n-1)} - \frac{1}{2} I_{n-2} \\
 &\Rightarrow I_n = \frac{-2}{n-1} \cos(n-1)x - I_{n-2}
 \end{aligned}$$

28. (d)

29. (a) Equation of family of circles, which touch the Y-axis at origin is
 $(x - a)^2 + y^2 = a^2$

30. (a) Given matrix form of system of three linear equations in three unknowns is

$$AX = D$$

We know that the above system of linear equations will be inconsistent if rank of $A \neq$ rank of AD and rank of $AD >$ rank of A
 $[\because \text{rank of augmented matrix} \geq \text{rank of coefficient matrix}]$
 $\Rightarrow \frac{\text{rank of } A}{\text{rank of } AD} < 1$

31. (c)

32. (a)

TS EAMCET

1. (a) I. $\int_{-1}^1 x|x| dx = 0$
 $[\because x|x| \text{ is an odd function}]$

II. Let
 $I = \int_0^{\pi/2} \left[1 + \left(\log \frac{4 + 3\sin x}{4 + 3\cos x} \right) \right] dx$
 $I = \int_0^{\pi/2} \left[1 + \log \frac{4 + 3\sin(\frac{\pi}{2} - x)}{4 + 3\cos(\frac{\pi}{2} - x)} \right] dx$
 $I = \int_0^{\pi/2} \left[1 + \left(\log \frac{4 + 3\cos x}{4 + 3\sin x} \right) \right] dx$
 $\Rightarrow 2I = \int_0^{\pi/2} \left(2 + \log \frac{4 + 3\sin x}{4 + 3\cos x} + \log \frac{4 + 3\cos x}{4 + 3\sin x} \right) dx$
 $\Rightarrow 2I = \int_0^{\pi/2} (2 + \log 1) dx$
 $= \int_0^{\pi/2} 2 dx = \pi$
 $\Rightarrow I = \frac{\pi}{2}$

III. $\int_0^a f(x) dx = \int_0^a f(a-x) dx$

IV. $\int_{-a}^a f(x) dx$
 $= \int_0^a f(x) dx + \int_0^a f(-x) dx$
 $= \int_0^a [f(x) + f(-x)] dx$

2. (d)

3. (a) We have,

$$\begin{aligned}
 I &= \int_0^{\pi} \frac{x dx}{4 \cos^2 x + 9 \sin^2 x} \\
 \Rightarrow I &= \int_0^{\pi} \frac{(\pi - x) dx}{4 \cos^2(\pi - x) + 9 \sin^2(\pi - x)} \\
 \Rightarrow I &= \int_0^{\pi} \frac{(\pi - x) dx}{4 \cos^2 x + 9 \sin^2 x}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow 2I &= \int_0^{\pi} \frac{\pi dx}{4 \cos^2 x + 9 \sin^2 x} \\
 \Rightarrow 2I &= \int_0^{\pi} \frac{\pi \sec^2 x dx}{4 + 9 \tan^2 x} \\
 \Rightarrow 2I &= \int_0^{\pi/2} \frac{2\pi \sec^2 x dx}{4 + 9 \tan^2 x} \\
 \Rightarrow I &= \frac{\pi}{9} \int_0^{\pi/2} \frac{\sec^2 x dx}{\frac{4}{9} + \tan^2 x}
 \end{aligned}$$

Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$x = 0, t = 0, x = \frac{\pi}{2}, t = \infty$$

$$\begin{aligned}
 \Rightarrow I &= \frac{\pi}{9} \int_0^{\infty} \frac{dt}{\left(\frac{2}{3}\right)^2 + t^2} \\
 \Rightarrow I &= \frac{\pi}{9} \times \frac{3}{2} \left[\tan^{-1} \frac{3t}{2} \right]_0^{\infty} \\
 \Rightarrow I &= \frac{\pi}{9} \times \frac{3}{2} \times \frac{\pi}{2} = \frac{\pi^2}{12}
 \end{aligned}$$

4. (c)

5. (b)

6. (a) We have,

$$\begin{aligned}
 &|\mathbf{a} \times \hat{\mathbf{i}}|^2 + |\mathbf{a} \times \hat{\mathbf{j}}|^2 + |\mathbf{a} \times \hat{\mathbf{k}}|^2 \\
 &(|\mathbf{a}| |\hat{\mathbf{i}}| \sin \alpha)^2 + (|\mathbf{a}| |\hat{\mathbf{j}}| \sin \beta)^2 + (|\mathbf{a}| |\hat{\mathbf{k}}| \sin \gamma)^2 \\
 &= \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma \\
 &\quad [\because |\mathbf{a}| = |\hat{\mathbf{i}}| = |\hat{\mathbf{j}}| = |\hat{\mathbf{k}}| = 1] \\
 &= 1 - \cos^2 \alpha + 1 - \cos^2 \beta + 1 - \cos^2 \gamma \\
 &= 3 - (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma) \\
 &= 3 - 1 = 2 \\
 &\quad [\because \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1]
 \end{aligned}$$

7. (a) We have, $\int e^{2x} f'(x) dx = g(x)$

$$\begin{aligned}
 \text{Let } I &= \int (e^{2x} f(x) + e^{2x} f'(x)) dx \\
 &= f(x) \int e^{2x} dx - \int f'(x) \int e^{2x} dx dx \\
 &\quad + \int e^{2x} f'(x) dx \\
 &= \frac{f(x) e^{2x}}{2} - \frac{1}{2} \int e^{2x} f'(x) dx \\
 &\quad + \int e^{2x} f'(x) dx \\
 &= \frac{e^{2x}}{2} f(x) - \frac{1}{2} \int e^{2x} f'(x) dx \\
 &= \frac{1}{2} [e^{2x} f(x) - \int e^{2x} f'(x) dx] \\
 &= \frac{1}{2} [e^{2x} f(x) - g(x)] + C
 \end{aligned}$$

8. (a) Let $I = \int x^4 e^{2x} dx$

$$\begin{aligned}
 \Rightarrow I &= x^4 \int e^{2x} dx \\
 &\quad - \int \frac{dx^4}{dx} \cdot \int e^{2x} \cdot dx \cdot dx + C \\
 &= \frac{x^4}{2} \cdot e^{2x} - \frac{1}{2} \int 4x^3 e^{2x} dx + C
 \end{aligned}$$

$$\Rightarrow \frac{x^4 e^{2x}}{2} - 2 \left[\frac{x^3 e^{2x}}{2} - \int \frac{3x^2 e^{2x}}{2} dx \right] + C$$

$$\Rightarrow \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + 3 \left[\frac{x^2 e^{2x}}{2} - \frac{1}{2} \int 2x e^{2x} dx \right] + C$$

$$\Rightarrow \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - 3 \left[\frac{x e^{2x}}{2} - \int \frac{e^{2x}}{2} dx \right] + C$$

$$\Rightarrow \frac{x^4 e^{2x}}{2} - x^3 e^{2x} + \frac{3}{2} x^2 e^{2x} - \frac{3}{2} x e^{2x} + \frac{3 e^{2x}}{4} + C$$

$$\Rightarrow \frac{e^{2x}}{4} [2x^4 - 4x^3 + 6x^2 - 6x + 3] + C$$

9. (b) 10. (d)

11. (a) Put $y = vx$

$$\frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\therefore v + x \frac{dv}{dx} = \frac{x + vx}{x - vx} = \frac{1 + v}{1 - v}$$

$$\Rightarrow \frac{1 - v}{1 + v^2} dv = \frac{dx}{x}$$

$$\therefore \int \frac{1}{1 + v^2} dv - \frac{1}{2} \int \frac{2v}{1 + v^2} dv = \int \frac{dx}{x}$$

$$\Rightarrow \tan^{-1} v - \frac{1}{2} \log |1 + v^2| = \log x + C$$

12. (c) 13. (b)

14. (a) The curve $y = ax^3 + bx + 4$ is passes through (2, 14).

$$\therefore 14 = a(2)^3 + b(2) + 4$$

$$\Rightarrow 14 = 8a + 2b + 4$$

$$\Rightarrow 5 = 4a + b \quad \dots (i)$$

Slope of tangent to the curve $y = ax^3 + bx + 4$

$$\text{i.e. } \frac{dy}{dx} = 3ax^2 + b$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{(2, 14)} = 3a(2)^2 + b$$

$$\Rightarrow 21 = 12a + b \quad \dots (ii)$$

$$\left[\therefore \frac{dy}{dx} = 21 \right]$$

Solving Eqs. (i) and (ii), we get $a = 2, b = -3$

15. (c)

16. (b) We have, $\left(\frac{x}{a} \right)^n + \left(\frac{y}{b} \right)^n = 2$

On differentiating w.r.t. x , we get

$$\frac{nx^{n-1}}{a^n} + \frac{ny^{n-1}}{b^n} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-b^n x^{n-1}}{a^n y^{n-1}}$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{(a, b)} = \frac{-b^n a^{n-1}}{a^n b^{n-1}} = \frac{-b}{a}$$

Equations of tangent at (a, b) is

$$y - b = \frac{-b}{a} (x - a)$$

$$\Rightarrow ay - ab = -bx + ab$$

$$\Rightarrow bx + ay = 2ab$$

$$\Rightarrow \frac{bx}{ab} + \frac{ay}{ab} = \frac{2ab}{ab}$$

$$\Rightarrow \frac{x}{a} + \frac{y}{b} = 2$$

17. (b)

$$18. (b) \int \frac{dx}{x(x^4 + 1)} = \int \frac{x^4 + 1 - x^4}{x(x^4 + 1)} dx$$

$$= \int \frac{1}{x} dx - \int \frac{x^3}{x^4 + 1} dx$$

$$= \log |x| - \int \frac{x^3}{x^4 + 1} dx + C$$

$$\text{Let } x^4 + 1 = t \Rightarrow x^3 dx = dt$$

$$\Rightarrow x^3 dx = \frac{1}{4}$$

$$19. (b) \text{ We have, } \sin^{-1} \frac{\sqrt{3}}{2} + \sin^{-1} \frac{\sqrt{2}}{3}$$

$$= \pi - \sin^{-1} \left[\frac{\sqrt{3}}{2} \sqrt{1 - \left(\frac{\sqrt{2}}{3} \right)^2} + \sqrt{\frac{2}{3}} \sqrt{1 - \left(\frac{\sqrt{3}}{2} \right)^2} \right]$$

$$\text{As } 0 < \frac{\sqrt{3}}{2} = 0.866, \sqrt{\frac{2}{3}} = 0.816 \leq 1 \text{ and}$$

$$\left(\frac{\sqrt{3}}{2} \right)^2 + \left(\sqrt{\frac{2}{3}} \right)^2 = 1.4167 > 1 \quad [\therefore \sin^{-1} x + \sin^{-1} y = \pi - \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}), \text{ if } 0 < x, y \leq 1 \text{ and } x^2 + y^2 > 1]$$

$$= \pi - \sin^{-1} \left[\frac{\sqrt{3}}{2} \sqrt{1 - \frac{2}{9}} + \sqrt{\frac{2}{3}} \sqrt{1 - \frac{3}{4}} \right]$$

$$= \pi - \sin^{-1} \left[\frac{\sqrt{3}}{2} \sqrt{\frac{7}{9}} + \sqrt{\frac{2}{3}} \sqrt{\frac{1}{4}} \right]$$

$$= \pi - \sin^{-1} \left[\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{3}} + \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{1}{2} \right]$$

$$= \pi - \sin^{-1} \left[\frac{\sqrt{3} + \sqrt{2}}{2\sqrt{3}} \right]$$

20. (b) Given equations of circle is $S \equiv x^2 + y^2 - 13 = 0 \quad \dots (i)$

On differentiating w.r.t. x , we get

$$2x + 2y \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}$$

Now, slope of tangent,

$$m = \frac{dy}{dx} \Big|_{\text{at } (2, 3)} = \frac{-2}{3}$$

$$\Rightarrow m = \frac{-2}{3}$$

$$\therefore \left(m, -\frac{1}{m} \right) = \left(-\frac{2}{3}, \frac{3}{2} \right)$$

On substituting this point in LHS Eq. (i), we get

$$\text{LHS} = \frac{4}{9} + \frac{9}{4} - 13 = \frac{16 + 81 - 468}{36} < 0$$

$\therefore \left(m, -\frac{1}{m} \right)$ is an internal point with respect to the circle $S = 0$.

21. (a) 22. (c)

23. (b) Given,

$$x = A \cos(nt + \alpha) \quad \dots (i)$$

On differentiating x w.r.t. 't', we get

$$\frac{dx}{dt} = A \frac{d}{dt} \cos(nt + \alpha)$$

$$= -A \sin(nt + \alpha) \frac{d}{dt} (nt + \alpha)$$

$$= -A \sin(nt + \alpha) (n)$$

$$\Rightarrow \frac{dx}{dt} = -A n \sin(nt + \alpha)$$

$$\Rightarrow \frac{dx}{dt} = -\frac{x n \sin(nt + \alpha)}{\cos(nt + \alpha)} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \frac{dx}{dt} = -n x \tan(nt + \alpha) \quad \dots (ii)$$

Again, differentiating w.r.t. 't', we get

$$\frac{d^2 x}{dt^2} = - \left[n x \frac{d}{dt} \tan(nt + \alpha) + \tan(nt + \alpha) \frac{d}{dt} n x \right]$$

$$= - \left[n x \sec^2(nt + \alpha) \frac{d}{dt} (nt + \alpha) + \tan(nt + \alpha) \frac{d}{dt} n x \right]$$

$$= - [n x \sec^2(nt + \alpha) n + n \tan(nt + \alpha) \times -n x \tan(nt + \alpha)] \quad [\text{from Eq. (ii)}]$$

$$= - [n^2 x \sec^2(nt + \alpha) - n^2 x \tan^2(nt + \alpha)]$$

$$= -n^2 x [\sec^2(nt + \alpha) - \tan^2(nt + \alpha)]$$

$$= -n^2 x \times 1 \quad [\because \sec^2 \theta - \tan^2 \theta = 1]$$

$$\Rightarrow \frac{d^2 x}{dt^2} = -n^2 x \Rightarrow \frac{d^2 x}{dt^2} + n^2 x = 0$$

24. (a)

$$25. (c) A = \begin{pmatrix} x & x & x \\ x & x^2 & x \\ x & x & x+1 \end{pmatrix}$$

Since, rank $A = 1$, therefore atleast one determinant of order 1 should be

28 Solved Papers 2017

non-zero and all the determinants of order 2 and 3 should be zero.

If $x = 0$, then $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, which

have non-zero determinant of order 1 only.

If $x = 1$, then $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$, which

have a non-zero determinant of order

2, nearly $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}$

$\therefore x$ can take value 0 only.

26. (d)

27. (a) We have, $\mathbf{a} = 2\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$

$$\mathbf{b} = \hat{\mathbf{i}} + 3\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$$

Clearly, $\mathbf{c}$ is parallel to $\mathbf{a} \times \mathbf{b}$

Here, $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 1 & -3 \\ 1 & 3 & 2 \end{vmatrix}$

$$= \hat{\mathbf{i}}(11) - \hat{\mathbf{j}}(7) + \hat{\mathbf{k}}(5)$$

$$= 11\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}} = \mathbf{d} \text{ (say)}$$

Now, $\hat{\mathbf{d}} = \frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{11\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}}}{\sqrt{195}}$

Thus, $\mathbf{c} = |\mathbf{c}| \hat{\mathbf{d}} = \frac{2}{\sqrt{195}} (11\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 5\hat{\mathbf{k}})$

Hence, volume of the parallelepiped = $|\mathbf{a} \cdot \mathbf{b} \cdot \mathbf{c}|$

$$= \begin{vmatrix} 2 & 1 & -3 \\ 11 & -7 & 5 \\ \frac{22}{\sqrt{195}} & \frac{-14}{\sqrt{195}} & \frac{10}{\sqrt{195}} \end{vmatrix}$$

$$= \frac{2}{\sqrt{195}} \begin{vmatrix} 2 & 1 & -3 \\ 11 & -7 & 5 \end{vmatrix}$$

$$= \frac{2}{\sqrt{195}} [2(29) - 1(-17) - 3(-40)]$$

$$= \frac{2}{\sqrt{195}} [58 + 17 + 120] = \frac{2}{\sqrt{195}} \times 195$$

$$= 2\sqrt{195}$$

28. (a) Given,

$$y = x^3 - 3x^2 + 5 \quad \dots (i)$$

On differentiating both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = 3x^2 - 6x \quad \dots (ii)$$

For local maxima or local minima, put

$$\frac{dy}{dx} = 0$$

$$\Rightarrow 3x^2 - 6x = 0$$

$$\Rightarrow 3x(x - 2) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 2$$

Now, differentiating Eq. (ii) w.r.t. 'x' we get

$$\frac{d^2y}{dx^2} = 6x - 6$$

$$\Rightarrow \left(\frac{d^2y}{dx^2} \right)_{x=0} = -6 < 0$$

$\therefore x = 0$ is a point of local maxima.

$$\text{and } \left(\frac{d^2y}{dx^2} \right)_{x=2} = 6 \times 2 - 6$$

$$= 12 - 6 = 6 > 0$$

$\therefore x = 2$ is a point of local minima.

29. (d) Given,

$$f(x) = \begin{cases} \sin x, & \text{if } x \leq 0 \\ x^2 + a^2, & \text{if } 0 < x < 1 \\ bx + 2, & \text{if } 1 \leq x \leq 2 \\ 0, & \text{if } x > 2 \end{cases} \text{ is}$$

continuous on IR,

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x)$$

$$\text{and } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \sin x = \lim_{x \rightarrow 0^+} x^2 + a^2$$

$$\text{and } \lim_{x \rightarrow 2^-} bx + 2 = \lim_{x \rightarrow 2^+} 0$$

$$\Rightarrow 0 = 0 + a^2 \text{ and } 2b + 2 = 0$$

$$\Rightarrow a = 0 \text{ and } b = -1$$

$$\text{Now, } a + b + ab = 0 + (-1) + 0 = -1$$

30. (d)

31. (a) We have,

$$\int f(x) \cos x \, dx = \frac{1}{2} (f(x))^2 + C$$

On differentiating w.r.t. 'x', we get

$$f(x) \cos x = \frac{1}{2} \times 2f(x) \cdot f'(x)$$

$$\Rightarrow f(x) \cos x = f(x) \cdot f'(x)$$

$$\Rightarrow f'(x) = \cos x$$

$$\Rightarrow f'(0) = \cos 0$$

$$\Rightarrow f'(0) = 1 \quad [\because \cos 0 = 1]$$

VIT

1. (c)

2. (c)

3. (b) So, given lines will intersect, if

$$\begin{vmatrix} 3-1 & k+2 & 0-1 \\ 2 & 3 & 4 \\ 1 & 2 & 1 \end{vmatrix} = 0$$

4. (a)

5. (d) We have, $P^T = 2P + I$

$$\Rightarrow (P^T)^T = (2P + I)^T$$

$$\Rightarrow P = 2P^T + I$$

$$\Rightarrow P = 2(2P + I) + I$$

$$\Rightarrow P = 4P + 3I$$

$$\Rightarrow 3P + 3I = 0 \Rightarrow P + I = 0$$

$$\Rightarrow P = -I \Rightarrow PX = -IX$$

$$\Rightarrow PX = -X$$

6. (d) We have,

$$|1 - x^2| = |x^2 - 1|$$

$$= \begin{cases} -(x^2 - 1), & \text{if } -1 < x < 1 \\ x^2 - 1, & \text{otherwise} \end{cases}$$

$$\therefore I = \int_{-2}^3 |1 - x^2| \, dx$$

$$\Rightarrow I = \int_{-2}^{-1} (x^2 - 1) \, dx$$

$$+ \int_{-1}^1 -(x^2 - 1) \, dx + \int_1^3 (x^2 - 1) \, dx$$

$$\Rightarrow I = \left[\frac{x^3}{3} - x \right]_{-2}^{-1} - \left[\frac{x^3}{3} - x \right]_{-1}^1 + \left[\frac{x^3}{3} - x \right]_1^3$$

$$\Rightarrow I = [(-1/3 + 1) - (-8/3 + 2)] - \left[\left(\frac{1}{3} - 1 \right) - (-1/3 + 1) \right] + \left[\left(\frac{27}{3} - 3 \right) - (1/3 - 1) \right]$$

$$\Rightarrow I = \frac{4}{3} + 4/3 + \frac{20}{3} = \frac{28}{3}$$

7. (b) Let

$$I = \int \frac{1}{\tan x + \cot x + \sec x + \operatorname{cosec} x} \, dx.$$

Then, $I = \int \frac{\sin x \cos x}{1 + \sin x + \cos x} \, dx$

$$\Rightarrow I = \frac{1}{2} \int \frac{(1 + 2\sin x \cos x - 1)}{1 + \sin x + \cos x} \, dx$$

$$= \frac{1}{2} \int \frac{(\sin x + \cos x)^2 - 1}{1 + \sin x + \cos x} \, dx$$

$$(\sin x + \cos x + 1)$$

$$I = \frac{1}{2} \int \frac{(\sin x + \cos x - 1)}{1 + \sin x + \cos x} \, dx$$

$$\Rightarrow I = \frac{1}{2} \int (\sin x + \cos x - 1) \, dx$$

$$\Rightarrow I = \frac{1}{2} (-\cos x + \sin x - x) + C$$

8. (b) We have, $\int_0^\infty \frac{\log(1+x^2)}{1+x^2} \, dx$

$$= k \int_0^1 \frac{\log(1+x)}{1+x^2} \, dx$$

$$\Rightarrow \int_0^{\pi/2} \log \sec^2 \theta \, d\theta$$

$$= k \int_0^{\pi/4} \log(1 + \tan \theta) \, d\theta,$$

where $x = \tan \theta$

$$\begin{aligned} &\Rightarrow -2 \int_0^{\pi/2} \log \cos \theta d\theta \\ &= k \int_0^{\pi/4} \log(1 + \tan \theta) d\theta \\ &\Rightarrow -2X - \frac{\pi}{2} \log_e 2 = k \left(\frac{\pi}{8} \log 2 \right) \\ &\Rightarrow k = 8 \end{aligned}$$

9. (d) 10. (a)

11. (a) We have,

$$a(x - y) = \sqrt{1 - y^2} + \sqrt{1 - x^2}$$

On putting $x = \sin A$, $y = \sin B$, we get

$$\begin{aligned} \cos A + \cos B &= a(\sin A - \sin B) \\ \Rightarrow \cot \frac{A+B}{2} &= a \\ \Rightarrow A - B &= 2\cot^{-1} a \\ \Rightarrow \sin^{-1} x - \sin^{-1} y &= 2\cot^{-1} a \end{aligned}$$

On differentiating w.r.t. x , we get

$$\begin{aligned} \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}} \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= \frac{\sqrt{1-y^2}}{\sqrt{1-x^2}} \end{aligned}$$

12. (c)

13. (c) Clearly,

$$\begin{aligned} P(E' \cap F' \cap G) &= \frac{P(E' \cap F' \cap G)}{P(G)} \\ &= \frac{P(G) - P\{(E \cup F) \cap G\}}{P(G)} \\ &= \frac{P(G) - P(E \cap G) - P(F \cap G) + P(E \cap F \cap G)}{P(G)} \\ &= \frac{P(G) - P(E)P(G) - P(F)P(G) + 0}{P(G)} \\ &= 1 - P(E) - P(F) = P(E') - P(F) \end{aligned}$$

14. (a)

15. (d) We have,

$$\begin{aligned} P(X/Y) &= 1/2, P(Y/X) = 1/3 \\ \text{and } P(X \cap Y) &= 1/6 \\ \therefore P(X) &= \frac{P(X \cap Y)}{P(Y/X)} = \frac{1/6}{1/3} = 1/2 \\ P(Y) &= \frac{P(X \cap Y)}{P(X/Y)} = \frac{1/6}{1/2} = 1/3 \end{aligned}$$

Clearly, $P(X \cap Y) = P(X)P(Y)$

So, X and Y are independent events.

$$\begin{aligned} \therefore P(X \cup Y) &= 1 - P(\bar{X})P(\bar{Y}) \\ &= 1 - 1/2 \times 2/3 = 2/3 \\ \text{and } P(X^c \cap Y) &= P(Y) - P(X \cap Y) \\ &= 1/3 - 1/6 = 1/6 \end{aligned}$$

16. (a) We have,

$$\begin{aligned} a \times (b \times c) &= \frac{b+c}{\sqrt{2}} \\ \Rightarrow (a \cdot c)b - (a \cdot b)c &= \frac{1}{\sqrt{2}}b + \frac{1}{\sqrt{2}}c \\ \Rightarrow \left\{ a \cdot c - \frac{1}{\sqrt{2}} \right\} \end{aligned}$$

$$\begin{aligned} b - \left\{ a \cdot b + \frac{1}{\sqrt{2}} \right\} c &= 0 \\ \Rightarrow 0a + \left\{ a \cdot c - \frac{1}{\sqrt{2}} \right\} b &+ \left\{ -(a \cdot b) - \frac{1}{\sqrt{2}} \right\} c = 0 \\ \Rightarrow a \cdot c - \frac{1}{\sqrt{2}} = 0 \text{ and } -(a \cdot b) - \frac{1}{\sqrt{2}} = 0 \\ &[\because a, b, c \text{ are non-coplanar}] \\ \text{Now, } a \cdot b &= -\frac{1}{\sqrt{2}} \\ \Rightarrow |a||b|\cos \theta &= -\frac{1}{\sqrt{2}} \\ \Rightarrow \cos \theta &= -\frac{1}{\sqrt{2}} \\ &[\because a, b, c \text{ are unit vectors}] \\ \Rightarrow \theta &= \frac{3\pi}{4} \end{aligned}$$

17. (d) We have,

$$\begin{aligned} a' &= \frac{b \times c}{[abc]}, b' = \frac{c \times a}{[abc]}, c' = \frac{a \times b}{[abc]} \\ \text{or, } a' &= \lambda(b \times c), b' = \lambda(c \times a) \text{ and } \\ c' &= \lambda(a \times b) \end{aligned}$$

$$\text{Where, } \lambda = \frac{1}{[abc]}$$

$$a \cdot a' = b \cdot b' = c \cdot c' = 1$$

$$\therefore a \cdot a' + b \cdot b' + c \cdot c' = 1 + 1 + 1 = 3$$

18. (b)

19. (b) We have,

$$\begin{aligned} a + b + c &= 0 \\ \text{and } 2ab + 2ac - bc &= 0 \\ \Rightarrow a &= -(b + c) \\ \text{and } 2a(b + c) - bc &= 0 \\ \Rightarrow -2(b + c)^2 - bc &= 0 \\ \Rightarrow 2b^2 + 5bc + 2c^2 &= 0 \\ \Rightarrow (2b + c)(b + 2c) &= 0 \\ \Rightarrow 2b + c = 0 \text{ or } b + 2c &= 0 \\ \text{If } 2b + c = 0, \text{ then } a &= -(b + c) \\ \Rightarrow a &= b \\ \therefore a = b \text{ and } c = -2b &\Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{-2} \\ \text{If } b + 2c = 0, \text{ then } a &= -(b + 1) \Rightarrow a = c \\ \therefore a = c \text{ and } b = -2c &\Rightarrow \frac{a}{1} = \frac{b}{-2} = \frac{c}{1} \end{aligned}$$

20. (d) For $f(x)$ to be continuous at $x = 1$, we must have

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1^+} f(x) = f(1) \\ \text{Now, } \lim_{x \rightarrow 1^-} f(x) &= \lim_{h \rightarrow 0} f(1 - h) \\ &= \lim_{h \rightarrow 0} a(1 - h)^2 - b = a - b \\ \text{and } \lim_{x \rightarrow 1^+} f(x) &= \lim_{h \rightarrow 0} f(1 + h) \\ &= \lim_{h \rightarrow 0} 1 + h + 1 = 2 \end{aligned}$$

Therefore, $f(x)$ is not continuous at $x = 1$, if $(a - b) \neq 2$

21. (a) We have, $y = 2x^4 - x^2$

$$\Rightarrow \frac{dy}{dx} = 8x^3 - 2x$$

On putting $\frac{dy}{dx} = 0$, we get $x = 0$,
 $x = \pm 1/2$

$$\text{Now, } \frac{d^2y}{dx^2} = 24x^2 - 2$$

It can be easily seen that y is minimum for $x = \pm 1/2$

Thus, required area A is given by

$$\begin{aligned} A &= \left| \int_{-1/2}^{1/2} y dx \right| = \left| \int_{-1/2}^{1/2} (2x^4 - x^2) dx \right| \\ \Rightarrow A &= \left| 2 \int_0^1 (2x^4 - x^2) dx \right| \\ &= \left| 2 \left[\frac{2x^5}{5} - \frac{x^3}{3} \right]_0^1 \right| = \frac{7}{120} \end{aligned}$$

22. (a) Clearly, $f(x)$ is defined for

$$x \in [-\pi/3, \pi/3].$$

Since, $\tan x$ is an increasing function

in $[0, \pi/2]$ and $0 \leq \frac{\pi^2}{9} - x^2 \leq \frac{\pi^2}{9}$ for

$$x \in [-\pi/3, \pi/3]$$

$$\Rightarrow 0 \leq \sqrt{\frac{\pi^2}{9} - x^2} \leq \pi/3 \text{ for}$$

$$x \in [-\pi/3, \pi/3]$$

Also, $f(-x) = f(x)$ for all

$x \in [-\pi/3, \pi/3]$. Therefore, Range

$$(f) = [f(-\pi/3), f(0)] = [\tan 0, \tan \pi/3] = [0, \sqrt{3}]$$

23. (c) We have,

$$\begin{aligned} \sin^{-1} x + \sin^{-1}(1 - x) &= \cos^{-1} x \\ \Rightarrow \sin\{\sin^{-1} x + \sin^{-1}(1 - x)\} &= \sin(\cos^{-1} x) \\ \Rightarrow x\sqrt{1 - (1 - x)^2} + \sqrt{1 - x^2}(1 - x) &= \sqrt{1 - x^2} \\ \Rightarrow x\sqrt{1 - (1 - x)^2} &= x\sqrt{1 - x^2} \\ \Rightarrow x = 0 \text{ or } 2x - x^2 &= 1 - x^2 \\ \Rightarrow x = 0 \text{ or } x = 1/2 \end{aligned}$$

24. (a)

MHT CET

1. (a)

2. (b) Given, $f(x) = \frac{\log x}{x}$... (i)

On differentiating both sides w.r.t. x , we get

$$f'(x) = \frac{x \cdot \frac{1}{x} - \log x \cdot 1}{x^2} = \frac{1 - \log x}{x^2}$$

For maximum or minimum value, put $f'(x) = 0$

$$\frac{1 - \log x}{x^2} = 0 \Rightarrow \log x = 1 \Rightarrow x = e$$

30 Solved Papers 2017

On substituting $x = e$ in Eq. (i), we get

$$f(e) = \frac{\log e}{e} = \frac{1}{e} < 0$$

$\therefore$ Maximum value of $f(x)$ is $\frac{1}{e}$ at $x = e$.

$$\begin{aligned} 3. (b) \text{ Let } I &= \int_0^1 x \tan^{-1} x \, dx \\ &= [\tan^{-1} x \int x \, dx]_0^1 \\ &\quad - \int_0^1 \left(\frac{d}{dx} \tan^{-1} x \int x \, dx \right) dx \\ &= \left[\tan^{-1} x \cdot \frac{x^2}{2} \right]_0^1 - \int_0^1 \left(\frac{1}{1+x^2} \cdot \frac{x^2}{2} \right) dx \\ &= \left(\frac{1}{2} \tan^{-1} 1 - 0 \right) - \frac{1}{2} \int_0^1 \frac{1+x^2-1}{1+x^2} dx \\ &= \frac{1}{2} \cdot \frac{\pi}{4} - \frac{1}{2} \int_0^1 \left(1 - \frac{1}{1+x^2} \right) dx \\ &= \frac{\pi}{8} - \frac{1}{2} [x - \tan^{-1} x]_0^1 \\ &= \frac{\pi}{8} - \frac{1}{2} \left(1 - \tan^{-1} 1 \right) \end{aligned}$$

4. (a) Given, $g(x) = f'(x)$

$$f(g(x)) = x$$

On differentiating both sides w.r.t. 'x', we get

$$\begin{aligned} f'(g(x)) \cdot g'(x) &= 1 \\ \therefore \frac{1}{1+(g(x))^4} \cdot g'(x) &= 1 \\ [\because f'(x) = \frac{1}{1+x^4} \text{ (given)}] \\ \Rightarrow g'(x) &= 1 + [g(x)]^4 \end{aligned}$$

5. (b) 6. (a)

7. (c) We have, $f(x)$

$$= \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{1/x} = K$$

Since, $f(x)$ is continuous at $x = 0$, then

$$\begin{aligned} f(0) &= \lim_{x \rightarrow 0} f(x) \\ &= \lim_{x \rightarrow 0} \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{1/x} \\ \Rightarrow K &= \lim_{x \rightarrow 0} \left[\frac{1 + \tan x}{1 - \tan x} \right]^{1/x} \quad [1^\infty \text{ form}] \\ &= e^{\lim_{x \rightarrow 0} \left[\frac{1 + \tan x}{1 - \tan x} \right] \cdot \frac{1}{x}} \\ &= e^{\lim_{x \rightarrow 0} \left(\frac{2 \tan x}{1 - \tan^2 x} \right) \cdot \frac{1}{x}} \\ &= e^{2 \lim_{x \rightarrow 0} \frac{\tan x}{x} \cdot \lim_{x \rightarrow 0} \frac{1}{1 - \tan x}} \\ &= e^{\left[\because \lim_{x \rightarrow 0} \frac{\tan x}{x} = 1 \right]} \end{aligned}$$

$$\therefore K = e^{2 \cdot 1 \cdot \left(\frac{1}{1-0} \right)} = e^2$$

8. (c)

9. (b) Given, $\frac{dy}{dx} = \tan \left(\frac{y}{x} \right) + \left(\frac{y}{x} \right) \dots (i)$

On putting, $y = Vx$

$$\Rightarrow \frac{dy}{dx} = V + x \frac{dV}{dx} \text{ in Eq. (i), we get}$$

$$V + x \frac{dV}{dx} = \tan V + V$$

$$\Rightarrow x \frac{dV}{dx} = \tan V \Rightarrow \frac{1}{\tan V} dV = \frac{1}{x} dx$$

10. (a) Given, $x = f(t)$ and $y = g(t)$

On differentiating both sides w.r.t. 't', we get

$$\frac{dx}{dt} = f'(t) \text{ and } \frac{dy}{dt} = g'(t)$$

$$\text{We know that, } \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \Rightarrow \frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

$$\begin{aligned} \frac{d^2 y}{dx^2} &= \frac{f'(t) \cdot g''(t) - g'(t) \cdot f''(t)}{(f'(t))^2} \cdot \frac{dt}{dx} \\ &= \frac{f'(t) \cdot g''(t) - g'(t) \cdot f''(t)}{(f'(t))^2} \cdot \frac{1}{f'(t)} \\ &= \frac{f'(t) \cdot g''(t) - g'(t) \cdot f''(t)}{(f'(t))^3} \end{aligned}$$

11. (b)

12. (d) Given,

$$\int_0^{\pi/2} \log \cos x \, dx = \frac{\pi}{2} \log \frac{1}{2} \dots (i)$$

$$\begin{aligned} \text{Let } I &= \int_0^{\pi/2} \log \sec x \, dx \\ &= \int_0^{\pi/2} \log \left(\frac{1}{\cos x} \right) dx \end{aligned}$$

$$\begin{aligned} &= \int_0^{\pi/2} \log (\cos x)^{-1} dx \\ &= - \int_0^{\pi/2} \log (\cos x) dx \\ &= - \int_0^{\pi/2} \log (\cos x) dx \\ &\quad [\because \log m^{-n} = -n \log m] \\ &= - \frac{\pi}{2} \log \left(\frac{1}{2} \right) \left(\because \log \left(\frac{1}{a} \right) = -\log a \right) \\ &= \frac{\pi}{2} \log 2 \end{aligned}$$

13. (c) Let $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the planes

$$\mathbf{F} \cdot (m\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) + 3 = 0$$

and $\mathbf{r} \cdot (2\hat{\mathbf{i}} - m\hat{\mathbf{j}} - \hat{\mathbf{k}}) - 5 = 0$, respectively.

Also, $\theta = \frac{\pi}{3}$ be the angle between the given planes.

$$\text{We know that, } \cos \theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|}$$

$$\begin{aligned} & (m\hat{\mathbf{i}} - \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (2\hat{\mathbf{i}} - m\hat{\mathbf{j}} - \hat{\mathbf{k}}) \\ \therefore \cos \frac{\pi}{3} &= \frac{(2m - m^2 - 2)}{\sqrt{m^2 + (-1)^2 + 2^2} \sqrt{2^2 + (-m)^2 + (-1)^2}} \end{aligned}$$

$$\Rightarrow \frac{1}{2} = \frac{2m + m - 2}{\sqrt{m^2 + 1 + 4} \sqrt{4 + m^2 + 1}} \quad \left[\because \cos \frac{\pi}{3} = \frac{1}{2} \right]$$

$$= \frac{3m-2}{\sqrt{(m^2+5)^2}} \Rightarrow \pm \frac{1}{2} = \frac{3m-2}{m^2+5}$$

$$\Rightarrow \pm (m^2+5) = 6m-4$$

14. (c)

15. (c)

16. (c) Given vector $\mathbf{r}$ with direction cosines l, m, n is equally inclined to the coordinate axes.

$$\therefore l = m = n \dots (i)$$

$$\therefore l^2 + m^2 + n^2 = 1$$

$$l^2 + l^2 + l^2 = 1 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 3l^2 = 1 \Rightarrow l^2 = \frac{1}{3}$$

$$\Rightarrow l = \pm \frac{1}{\sqrt{3}}$$

$$\therefore l = m = n = \pm \frac{1}{\sqrt{3}}$$

Now, vector

$$\mathbf{r} = |\mathbf{r}| \left(\pm \frac{1}{\sqrt{3}} \hat{\mathbf{i}} + \pm \frac{1}{\sqrt{3}} \hat{\mathbf{j}} + \pm \frac{1}{\sqrt{3}} \hat{\mathbf{k}} \right)$$

Since, each has 2 choices i.e.

$$l = m = n = \frac{1}{\sqrt{3}}$$

$\therefore$ Total number of such vectors = $2^3 = 8$

17. (a) Let $I = \int \frac{1}{(x^2+4)(x^2+9)} dx$

$$\begin{aligned} &= \int \frac{1}{(x^2+9-x^2-4)} \left[\frac{1}{x^2-4} - \frac{1}{x^2+9} \right] dx \\ &= \int \frac{1}{5} \left(\frac{1}{x^2+4} - \frac{1}{x^2+9} \right) dx \end{aligned}$$

18. (a) Given α and β be the roots of the equation

$$x^2 + 5|x| - 6 = 0$$

Now, $|x|^2 + 5|x| - 6 = 0$

$$|x|^2 + 6|x| - |x| - 6 = 0$$

[by factorisation]

$$|x| (|x| + 6) - 1 (|x| + 6) = 0$$

$$(|x| + 6) (|x| - 1) = 0$$

$$|x| = -6 \text{ or } |x| = 1$$

(Since, modulus cannot be giving negative values)

$$\therefore |x| = 1 \Rightarrow x = \pm 1$$

So, $\alpha = 1$ and $\beta = -1$

$$\therefore \text{Now, } |\tan^{-1} \alpha - \tan^{-1} \beta| = |\tan^{-1} 1 - \tan^{-1} (-1)|$$

$$= \left| \frac{\pi}{4} - \left(-\frac{\pi}{4} \right) \right| = \left| \frac{\pi}{4} + \frac{\pi}{4} \right| = \left| \frac{\pi}{2} \right|$$

19. (c) Given, $x = a\left(t - \frac{1}{t}\right)$

and $y = a\left(t + \frac{1}{t}\right)$

Now,

$$y^2 - x^2 = a^2 \left[\left(t + \frac{1}{t}\right)^2 - \left(t - \frac{1}{t}\right)^2 \right]$$

$$= a^2 \left[t^2 + \frac{1}{t^2} + 2 - t^2 - \frac{1}{t^2} + 2 \right]$$

$$\Rightarrow y^2 - x^2 = 4a^2$$

20. (c) Let slope of the curve $y = \sqrt{x-1}$ is m_1 and slope of the line $2x + y - 5 = 0$ is m_2

Now, $m_1 = \frac{dy}{dx} = \frac{d}{dx} \sqrt{x-1}$

and $m_2 = \frac{dy}{dx} = \frac{d}{dx} (5-2x)$

$$\Rightarrow m_1 = \frac{1}{2\sqrt{x-1}} \text{ and } m_2 = -2$$

21. (d) Let $I = \int \sqrt{\frac{x-5}{x-7}} dx$

$$= \int \sqrt{\frac{(x-5)(x-5)}{(x-7)(x-5)}} dx$$

$$= \int \frac{x-5}{\sqrt{x^2-5x-7x+35}} dx$$

$$= \frac{1}{2} \int \frac{2x-10}{\sqrt{x^2-12x+35}} dx$$

$$= \frac{1}{2} \int \frac{2x-12+2}{\sqrt{x^2-12x+35}} dx$$

$$= \frac{1}{2} \int \frac{2x-12}{\sqrt{x^2-12x+35}} dx +$$

$$\int \frac{1}{\sqrt{x^2-12x+35}} dx$$

$$= \sqrt{x^2-12x+35}$$

$$+ \int \frac{1}{\sqrt{(x^2-12x+36-1)}} dx + C$$

$$= \sqrt{x^2-12x+35}$$

$$+ \int \frac{1}{\sqrt{(x-6)^2-1^2}} dx + C$$

$$I = \sqrt{x^2-12x+35} + \log$$

$$\left| x-6 + \sqrt{x^2-12x+35} \right| + C$$

$$\therefore I = A\sqrt{x^2-12x+35} +$$

$$\log |x-6 + \sqrt{x^2-12x+35} + C$$

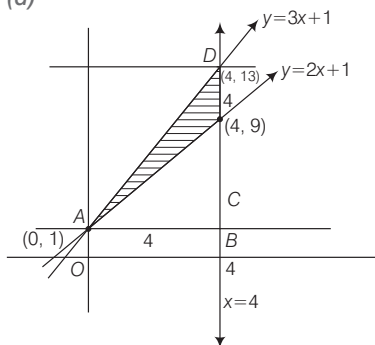
$$= 1 \cdot \sqrt{x^2-12x+35} + \log$$

$$|x-6 + \sqrt{x^2-12x+35} + C$$

On comparing both sides, we get $A=1$

22. (b)

23. (d)



$$\text{Area of shaded region} = \frac{1}{2} \begin{vmatrix} 0 & 1 & 1 \\ 4 & 9 & 1 \\ 4 & 13 & 1 \end{vmatrix}$$

24. (d) Let x be the number of defective pens.

Two pens are taken randomly from the box.

$\therefore x$ can take values 0, 1, 2

$$p(x=0) = \frac{{}^4C_2}{{}^6C_2} = \frac{4 \times 3}{6 \times 5} = \frac{2}{5} = \frac{6}{15}$$

$$p(x=1) = \frac{{}^2C_1 \times {}^4C_1}{{}^6C_2} = \frac{2 \times 4 \times 2 \times 1}{6 \times 5} = \frac{8}{15}$$

$$p(x=2) = \frac{{}^2C_2}{{}^6C_2} = \frac{1 \times 2 \times 1}{6 \times 5} = \frac{1}{15}$$

x	p	$x_i p_i$	$x_i^2 p_i$
0	$\frac{6}{15}$	0	0
1	$\frac{8}{15}$	$\frac{8}{15}$	$\frac{8}{15}$
2	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$

$$E(x) = \frac{10}{15} \text{ and } = \frac{2}{3}$$

$$E(x^2) = \frac{12}{15} = \frac{4}{5}$$

$$\text{Standard deviation} = \sqrt{E(x^2) - [E(x)]^2}$$

$$\text{Standard deviation} = \sqrt{\left(\frac{4}{5}\right) - \left(\frac{2}{3}\right)^2}$$

25. (a) Let V and r be the volume and radius of spherical ball, respectively.

$$\text{Volume of spherical ball} = \frac{4}{3} \pi r^3$$

$$\Rightarrow V = \frac{4}{3} \pi r^3 \quad \dots (i)$$

$$\Rightarrow 288\pi = \frac{4}{3} \pi r^3$$

$$\Rightarrow 288 = \frac{4}{3} r^3$$

$$\Rightarrow r^3 = 72 \times 3 = 8 \times 27$$

$$\Rightarrow r^3 = 2^3 \times 3^3$$

$$\Rightarrow r = 6$$

Now, $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\therefore 4\pi = 4\pi r^2 \frac{dr}{dt}$$

$$\Rightarrow 1 = (6)^2 \frac{dr}{dt} \quad [\because r=6]$$

$$\Rightarrow \frac{dr}{dt} = \frac{1}{36}$$

Now, surface area of spherical ball,

$$(s) = 4\pi r^2$$

$$\Rightarrow s = 4\pi r^2$$

On differentiating both sides, w.r.t. 't', we get

$$\frac{ds}{dt} = 4 \times 2\pi r \frac{dr}{dt}$$

$$= 8 \times \pi \times 6 \times \frac{1}{36} \quad \left[\because r=6 \text{ and } \frac{dr}{dt} = \frac{1}{36} \right]$$

$$\Rightarrow \frac{ds}{dt} = \frac{4\pi}{3} \text{ cm}^2/\text{s}$$

26. (b) Given,

$$f(x) = \begin{cases} \log(\sec^2 x)^{\cot^2 x}, & \text{for } x \neq 0 \\ k, & \text{for } x = 0 \end{cases}$$

is continuous at $x=0$

$$f(0) = \lim_{x \rightarrow 0} \log(\sec^2 x)^{\cot^2 x}$$

$$= \lim_{x \rightarrow 0} \cot^2 x \log(\sec^2 x) \quad [\because$$

$$\log m^n = n \log m]$$

$$= \lim_{x \rightarrow 0} \frac{1}{\tan^2 x} \log(1 + \tan^2 x)$$

$$[\because \sec^2 x = 1 + \tan^2 x \text{ and } \cot x = \frac{1}{\tan x}]$$

$$= \lim_{x \rightarrow 0} \frac{\log(1 + \tan^2 x)}{\tan^2 x}$$

$$= \lim_{\tan x \rightarrow 0} \frac{\log(1 + \tan^2 x)}{\tan^2 x} = 1 \quad [\because$$

As $x \rightarrow 0$, $\tan x \rightarrow 0$]

27. (a) Axis of parabola = Y axis

and vertex of parabola is (0, k).

$\therefore$ Equation of parabola is

$$(x-0)^2 = 4a(y-k)$$

$$x^2 = 4ay - 4ak$$

On differentiate both sides w.r.t. 'x', we get

$$2x = 4a \frac{dy}{dx} \Rightarrow x = 2a \frac{dy}{dx}$$

$$\therefore \frac{1}{2a} = \frac{1}{x} \frac{dy}{dx}$$

On differentiating both sides w.r.t. 'x', we get

$$\frac{d}{dx} \left(\frac{1}{x} \cdot \frac{dy}{dx} \right) = \frac{d}{dx} \left(\frac{1}{2a} \right)$$

$$\Rightarrow \frac{1}{x} \cdot \frac{d^2 y}{dx^2} + \frac{dy}{dx} \left(-\frac{1}{x^2} \right) = 0$$

$$\therefore x \frac{d^2 y}{dx^2} - \frac{dy}{dx} = 0$$

32 Solved Papers 2017

28. (a) Let $I = \int_0^3 [x] dx$
 $= \int_0^1 0 dx + \int_1^2 1 dx + \int_2^3 2 dx$
 $= [x]_1^2 + 2[x]_2^3$
 $= (2-1) + 2(3-2) = 1 + 2 = 3$

29. (c) 30. (d)

31. (a) Given, $f(x) = \begin{cases} x, & \text{for } x \leq 0 \\ 0, & \text{for } x > 0 \end{cases}$

For continuity at $x=0$

$$\text{LHS at } x=0 \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x$$

$$\lim_{h \rightarrow 0^-} (0-h) = 0 \text{ and RHL at } x=0$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 0 = 0$$

$$\text{Also, } f(0) = 0$$

$$\therefore \text{LHL} = \text{RHL} = f(0)$$

Hence, $f(x)$ is continuous at $x=0$

For differentiability at $x=0$

$$f'(x) = 1 \text{ for } x \leq 0, 0 \text{ for } x > 0$$

$\therefore f(x)$ is not differentiable at $x=0$

32. (a) The equation of plane passing through $\mathbf{a} = -\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}$ and perpendicular to $\mathbf{n} = \hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}$ is

$$\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n} \Rightarrow (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = (-\hat{\mathbf{i}} + \hat{\mathbf{j}} + 2\hat{\mathbf{k}}) \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}})$$

$$= (-1 + 1 + 2)$$

$$\mathbf{r} \cdot (\hat{\mathbf{i}} + \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 2$$

33. (a) Given point be $(2, 3, \lambda)$ and equation of the plane be

$$\mathbf{r} \cdot (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = 13$$

$$\therefore (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) \cdot (3\hat{\mathbf{i}} + 2\hat{\mathbf{j}} + 6\hat{\mathbf{k}}) = 13$$

$$[\therefore \mathbf{r} \cdot (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})]$$

$$\Rightarrow 3x + 2y + 6z = 13$$

$$\text{or } 3x + 2y + 6z - 13 = 0$$

Now, distance of the plane from the point $(2, 3, \lambda)$ is

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$5 = \frac{|3 \times 2 + 2 \times 3 + 6 \times \lambda - 13|}{\sqrt{3^2 + 2^2 + 6^2}}$$

[given, $d=5$]

$$\pm 5 = \frac{6 + 6 + 6\lambda - 13}{\sqrt{9 + 4 + 36}} \Rightarrow \pm 5 = \frac{6\lambda - 1}{\sqrt{49}}$$

$$\therefore 5 = \frac{|6\lambda - 1|}{7}$$

$$\Rightarrow \pm 35 = 6\lambda - 1$$

$$\Rightarrow 35 = 6\lambda - 1 \text{ or } -35 = 6\lambda - 1$$

$$\Rightarrow \lambda = 6, \lambda = -\frac{17}{3}$$

34. (a) We have,

$$\cos^{-1}\left(\cot \frac{\pi}{2}\right) + \cos^{-1}\left(\sin \frac{2\pi}{3}\right)$$

$$= \cos^{-1}(0) + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$\left[\because \cot \frac{\pi}{2} = 0 \text{ and } \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2} \right]$$

$$= \cos^{-1}\left(\cos \frac{\pi}{2}\right) + \cos^{-1}\left(\cos \frac{\pi}{6}\right)$$

$$\left[\because \cos \frac{\pi}{2} = 0 \text{ and } \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2} \right]$$

$$= \frac{\pi}{2} + \frac{\pi}{6}, \frac{3\pi + \pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$$

35. (b) We have, $x dy + 2y dx = 0$

$$\Rightarrow \frac{dy}{dy} + \frac{2dx}{x} = 0$$

[dividing both sides by xy]

On Integrating both sides, we get

$$\int \frac{1}{y} dy + 2 \int \frac{1}{x} dx = \log C$$

$$\Rightarrow \log y + 2 \log x = \log C$$

$$[\therefore n \log m = \log m^n]$$

$$\therefore \log yx^2 = \log C$$

$$[\therefore \log m + \log n = \log mn]$$

$$\Rightarrow yx^2 = C$$

When $x = 2$ and $y = 1$, then

$$C = 1 \times 2^2 = 4$$

$\therefore$ Particular solution is $x^2 y = 4$.

36. (b)

x	1	2	3	4	5	6
f(x)	0.2	0.37	0.48	0.62	0.85	1
p(x)	0.2	0.17	0.11	0.14	0.23	0.15

$$p(3 < x \leq 5) = p(x=4) + p(x=5)$$

$$= 0.14 + 0.23 = 0.37$$

37. (b) Given lines are

$$\frac{x-1}{2} = \frac{y+1}{2} = \frac{z-1}{4} = \lambda \text{ (say)} \quad \dots(i)$$

$$\text{and } \frac{x-3}{1} = \frac{y-6}{2} = \frac{z}{1} \quad \dots(ii)$$

Any point on the line (i) is

$$P(2\lambda + 1, 2\lambda - 1, 4\lambda + 1)$$

Put $x = \lambda + 1, y = 2\lambda - 1$

and $z = 4\lambda + 1$

In Eq. (ii), we get

$$\frac{2\lambda + 1 - 3}{1} = \frac{2\lambda - 1 - 6}{2} = \frac{4\lambda + 1}{1}$$

Taking first two terms, we get

$$4\lambda - 4 = 2\lambda - 7$$

$$\Rightarrow 2\lambda = -3$$

Put $\lambda = -\frac{3}{2}$ in Eq. (ii), we get

Point of intersection is

$$P\left[\left(2 \times -\frac{3}{2} + 1, 2 \times -\frac{3}{2} - 1\right) \times 4 \times \left(-\frac{3}{2} + 1\right)\right]$$

$$\equiv P(-2, -4, -5)$$

38. (b) Let $I = \int \frac{\sec^8 x}{\csc x} dx$

$$= \int \frac{\sin x}{\cos^8 x} dx$$

$$= \int \tan x \cdot \sec^7 x dx$$

$$\Rightarrow I = \int \sec^6 x \cdot \sec x \tan x dx$$

On putting $\sec x = t$

$$\Rightarrow \sec x \cdot \tan x dx = dt,$$

we get

$$I = \int t^6 dt = \frac{t^7}{7} + c$$

$$= \frac{\sec^7 x}{7} + c \quad [\therefore t = \sec x]$$

Engineering Entrances Solved Papers 2018

**JEE Main & Advanced/BITSAT/KCET/Andhra Pradesh
& Telangana State (EAMCET)/VIT/MHT CET**

JEE Main

1. If $\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix} = (A+Bx)(x-A)^2$, then the

ordered pair (A, B) is equal to

- (a) $(-4, -5)$ (b) $(-4, 3)$
(c) $(-4, 5)$ (d) $(4, 5)$

2. If the system of linear equations

$$x + ky + 3z = 0$$

$$3x + ky - 2z = 0$$

$$2x + 4y - 3z = 0$$

has a non-zero solution (x, y, z) , then $\frac{xz}{y^2}$ is equal to

- (a) -10 (b) 10 (c) -30 (d) 30

3. Let $S = \{t \in \mathbf{R} : f(x) = |x - \pi| (e^{|x|} - 1) \sin|x| \text{ is not differentiable at } t\}$. Then, the set S is equal to

- (a) $\emptyset$ (an empty set) (b) $\{0\}$
(c) $\{\pi\}$ (d) $\{0, \pi\}$

4. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then the value of b is

- (a) 6 (b) $\frac{7}{2}$ (c) 4 (d) $\frac{9}{2}$

5. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in \mathbf{R} - \{-1, 0, 1\}$. If

$h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is

- (a) 3 (b) -3 (c) $-2\sqrt{2}$ (d) $2\sqrt{2}$

6. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β ($\alpha < \beta$) be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then, the area (in sq units) bounded by the curve $y = (g \circ f)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$, is

- (a) $\frac{1}{2}(\sqrt{3} - 1)$ (b) $\frac{1}{2}(\sqrt{3} + 1)$
(c) $\frac{1}{2}(\sqrt{3} - \sqrt{2})$ (d) $\frac{1}{2}(\sqrt{2} - 1)$

7. Let $y = y(x)$ be the solution of the differential equation

$$\sin x \frac{dy}{dx} + y \cos x = 4x, x \in (0, \pi).$$

If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to

- (a) $\frac{4}{9\sqrt{3}}\pi^2$ (b) $\frac{-8}{9\sqrt{3}}\pi^2$
(c) $-\frac{8}{9}\pi^2$ (d) $-\frac{4}{9}\pi^2$

8. If L_1 is the line of intersection of the planes $2x - 2y + 3z - 2 = 0$, $x - y + z + 1 = 0$ and L_2 is the line of intersection of the planes $x + 2y - z - 3 = 0$, $3x - y + 2z - 1 = 0$, then the distance of the origin from the plane, containing the lines L_1 and L_2 is

- (a) $\frac{1}{4\sqrt{2}}$ (b) $\frac{1}{3\sqrt{2}}$
(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{\sqrt{2}}$

9. The length of the projection of the line segment joining the points $(5, -1, 4)$ and $(4, -1, 3)$ on the plane, $x + y + z = 7$ is

- (a) $\frac{2}{\sqrt{3}}$ (b) $\frac{2}{3}$
(c) $\frac{1}{3}$ (d) $\sqrt{\frac{2}{3}}$

10. Let $\mathbf{u}$ be a vector coplanar with the vectors $\mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}$ and $\mathbf{b} = \hat{j} + \hat{k}$. If $\mathbf{u}$ is perpendicular to $\mathbf{a}$ and $\mathbf{u} \cdot \mathbf{b} = 24$, then $|\mathbf{u}|^2$ is equal to

- (a) 336 (b) 315 (c) 256 (d) 84

11. A bag contains 4 red and 6 black balls. A ball is drawn at random from the bag, its colour is observed and this ball along with two additional balls of the same colour are returned to the bag. If now a ball is drawn at random from the bag, then the probability that this drawn ball is red, is

- (a) $\frac{3}{10}$ (b) $\frac{2}{5}$ (c) $\frac{1}{5}$ (d) $\frac{3}{4}$

2 Engineering Entrances Solved Papers 2018

JEE Advanced

Paper 1

1. Let $P_1 : 2x + y - z = 3$ and $P_2 : x + 2y + z = 2$ be two planes. Then, which of the following statement(s) is (are) TRUE? **(One or More than One Option Correct)**

- (a) The line of intersection of P_1 and P_2 has direction ratios 1, 2, -1
 (b) The line $\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$ is perpendicular to the line of intersection of P_1 and P_2
 (c) The acute angle between P_1 and P_2 is 60°
 (d) If P_3 is the plane passing through the point (4, 2, -2) and perpendicular to the line of intersection of P_1 and P_2 , then the distance of the point (2, 1, 1) from the plane P_3 is $\frac{2}{\sqrt{3}}$

2. For every twice differentiable function $f : R \rightarrow [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) There exist $r, s \in R$, where $r < s$, such that f is one-one on the open interval (r, s)
 (b) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$
 (c) $\lim_{x \rightarrow \infty} f(x) = 1$
 (d) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

3. Let $f : R \rightarrow R$ and $g : R \rightarrow R$ be two non-constant differentiable functions. If $f'(x) = (e^{f(x)-g(x)}) g'(x)$ for all $x \in R$ and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) $f(2) < 1 - \log_e 2$ (b) $f(2) > 1 - \log_e 2$
 (c) $g(1) > 1 - \log_e 2$ (d) $g(1) < 1 - \log_e 2$

4. Let $f : [0, \infty) \rightarrow R$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t) dt$ for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) The curve $y = f(x)$ passes through the point (1, 2)
 (b) The curve $y = f(x)$ passes through the point (2, -1)
 (c) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$
 (d) The area of the region $\{(x, y) \in [0, 1] \times R : f(x) \leq y \leq \sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

5. The number of real solutions of the equation

$$\sin^{-1} \left(\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2} \right)^i \right) = \frac{\pi}{2} - \cos^{-1} \left(\sum_{i=1}^{\infty} \left(-\frac{x}{2} \right)^i - \sum_{i=1}^{\infty} (-x)^i \right) \text{ lying in the interval } \left(-\frac{1}{2}, \frac{1}{2} \right) \text{ is } \dots\dots\dots$$

(Here, the inverse trigonometric functions $\sin^{-1} x$ and $\cos^{-1} x$ assume values in $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$ and $[0, \pi]$, respectively.) **(Numerical Value Based Question)**

6. Let $\mathbf{a}$ and $\mathbf{b}$ be two unit vectors such that $\mathbf{a} \cdot \mathbf{b} = 0$. For some $x, y \in R$, let $\mathbf{c} = x\mathbf{a} + y\mathbf{b} + (\mathbf{a} \times \mathbf{b})$. If $|\mathbf{c}| = 2$ and the vector $\mathbf{c}$ is inclined at the same angle α to both $\mathbf{a}$ and $\mathbf{b}$, then the value of $8 \cos^2 \alpha$ is

(Numerical Value Based Question)

7. A farmer F_1 has a land in the shape of a triangle with vertices at $P(0, 0)$, $Q(1, 1)$ and $R(2, 0)$. From this land, a neighbouring farmer F_2 takes away the region which lies between the sides PQ and a curve of the form $y = x^n$ ($n > 1$). If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR , then the value of n is

(Numerical Value Based Question)

Paper 2

8. For any positive integer n , define $f_n : (0, \infty) \rightarrow R$ as $f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{1}{1+(x+j)(x+j-1)} \right)$ for all $x \in (0, \infty)$.

(Here, the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2} \right)$). Then, which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) $\sum_{j=1}^5 \tan^2(f_j(0)) = 55$
 (b) $\sum_{j=1}^{10} (1+f'_j(0)) \sec^2(f_j(0)) = 10$
 (c) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \tan(f_n(x)) = \frac{1}{n}$
 (d) For any fixed positive integer n , $\lim_{x \rightarrow \infty} \sec^2(f_n(x)) = 1$

9. Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in R$ and the system of equations (in real variables)

$$\begin{aligned} -x + 2y + 5z &= b_1 \\ 2x - 4y + 3z &= b_2 \\ x - 2y + 2z &= b_3 \end{aligned}$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one

solution for each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$?

(One or More than One Option Correct)

Engineering Entrances Solved Papers 2018 3

- (a) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$
 (b) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$
 (c) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$
 (d) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

10. Let $f: (0, \pi) \rightarrow \mathbb{R}$ be a twice differentiable function such that $\lim_{t \rightarrow x} \frac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x$ for all $x \in (0, \pi)$.

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$
 (b) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$
 (c) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$
 (d) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

11. The value of the integral $\int_0^{1/2} \frac{1 + \sqrt{3}}{((x+1)^2(1-x)^6)^{1/4}} dx$ is

(Numerical Value Based Question)

12. Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is

(Numerical Value Based Question)

13. Let X be a set with exactly 5 elements and Y be a set with exactly 7 elements. If α is the number of one-one functions from X to Y and β is the number of onto functions from Y to X , then the value of $\frac{1}{5!}(\beta - \alpha)$ is

(Numerical Value Based Question)

14. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2 + 5y)(5y - 2)$, then the value of $\lim_{x \rightarrow -\infty} f(x)$ is

(Numerical Value Based Question)

15. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation $f(x+y) = f(x)f'(y) + f'(x)f(y)$ for all $x, y \in \mathbb{R}$.

Then, the value of $\log_e(f(4))$ is

(Numerical Value Based Question)

16. Let P be a point in the first octant, whose image Q in the plane $x + y = 3$ (that is, the line segment PQ is perpendicular to the plane $x + y = 3$ and the mid-point of

PQ lies in the plane $x + y = 3$) lies on the Z -axis. Let the distance of P from the X -axis be 5. If R is the image of P in the XY -plane, then the length of PR is

(Numerical Value Based Question)

17. Consider the cube in the first octant with sides OP , OQ and OR of length 1, along the X -axis, Y -axis and Z -axis, respectively, where $O(0, 0, 0)$ is the origin. Let $S\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

be the centre of the cube and T be the vertex of the cube opposite to the origin O such that S lies on the diagonal OT . If $\mathbf{p} = \mathbf{SP}$, $\mathbf{q} = \mathbf{SQ}$, $\mathbf{r} = \mathbf{SR}$ and $\mathbf{t} = \mathbf{ST}$, then the value of $|(\mathbf{p} \times \mathbf{q}) \times (\mathbf{r} \times \mathbf{t})|$ is

(Numerical Value Based Question)

18. Let $E_1 = \{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\}$ and

$$E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$$

(Here, the inverse trigonometric function $\sin^{-1} x$ assumes values in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.) Let $f: E_1 \rightarrow \mathbb{R}$ be the function

defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$ and $g: E_2 \rightarrow \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

List-I	List-II
P. The range of f is	1. $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	2. $(0, 1)$
R. The domain of f contains	3. $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	4. $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

(Matching Type Question)

The correct option is

- (a) $P \rightarrow 4$; $Q \rightarrow 2$; $R \rightarrow 1$; $S \rightarrow 1$
 (b) $P \rightarrow 3$; $Q \rightarrow 3$; $R \rightarrow 6$; $S \rightarrow 5$
 (c) $P \rightarrow 4$; $Q \rightarrow 2$; $R \rightarrow 1$; $S \rightarrow 6$
 (d) $P \rightarrow 4$; $Q \rightarrow 3$; $R \rightarrow 6$; $S \rightarrow 5$

4 Engineering Entrances Solved Papers 2018

19. Let $f_1 : R \rightarrow R$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow R$, $f_3 : (-1, e^{\pi/2} - 2) \rightarrow R$

and $f_4 : R \rightarrow R$ be functions defined by

- (i) $f_1(x) = \sin(\sqrt{1-e^{-x^2}})$,
 (ii) $f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1} x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,
 (iii) $f_3(x) = [\sin(\log_e(x+2))]$ where for $t \in R$, $[t]$ denotes the greatest integer less than or equal to t ,
 (iv) $f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

List-I	List-II
P. The function f_1 is	1. NOT continuous at $x = 0$
Q. The function f_2 is	2. continuous at $x=0$ and NOT differentiable at $x=0$
R. The function f_3 is	3. differentiable at $x=0$ and its derivative is NOT continuous at $x=0$
S. The function f_4 is	4. differentiable at $x=0$ and its derivative is continuous at $x=0$

(Matching Type Question)

The correct option is

- (a) $P \rightarrow 2$; $Q \rightarrow 3$; $R \rightarrow 1$; $S \rightarrow 4$
 (b) $P \rightarrow 4$; $Q \rightarrow 1$; $R \rightarrow 2$; $S \rightarrow 3$
 (c) $P \rightarrow 4$; $Q \rightarrow 2$; $R \rightarrow 1$; $S \rightarrow 3$
 (d) $P \rightarrow 2$; $Q \rightarrow 1$; $R \rightarrow 4$; $S \rightarrow 3$

BITSAT

1. If $P = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$, $abc = P^T P = I$, then the value of

$a^3 + b^3 + c^3$ is

- (a) 2 (b) 1
 (c) 0 (d) 5

2. If $x^2 = \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 1 \\ \sin \theta & \cos \theta & 2 \end{vmatrix}$, then the value of

$4x^2 + x \sin \frac{3\pi}{2} + 5$ is

- (a) $13 - \sqrt{2}$ (b) $13 + \sqrt{2}$
 (c) $\sqrt{2} - 13$ (d) Both (a) and (b)

3. The inverse of the function $f(x) = \log_a(x + \sqrt{x^2 + 1})$ (where, $a < 0$, $a \neq 1$) is

- (a) $\frac{1}{2}(a^x - a^{-x})$ (b) not defined for all x
 (c) defined for $x > 0$ (d) None of these

4. The value of $S = \sum_{n=1}^{\infty} \tan^{-1} \frac{2n}{n^4 + n^2 + 2}$ is equal to

- (a) $\frac{\pi}{2}$ (b) π
 (c) $\frac{\pi}{4}$ (d) None of these

5. Normals drawn to $y^2 = 4ax$ at the points where it is intersected by the line $y = mx + c$ intersected at P . Coordinates of foot of the another normal drawn to the parabola from the point 'P' is

- (a) $\left(\frac{a}{m^2}, -\frac{2a}{m}\right)$ (b) $\left(\frac{9}{m}, -\frac{6a}{m}\right)$
 (c) $(am^2, -2am)$ (d) $\left(\frac{4a}{m^2}, -\frac{4a}{m}\right)$

6. If the sum of squares of distances of a point from the planes $x + y + z = 0$, $x - z = 0$ and $x - 2y + z = 0$ is p^2 , then locus of the point is

- (a) $x^2 + z^2 = p^2$ (b) $x^2 + 2xy + y^2 + z^2 = p^2$
 (c) $x + y + z = p^2$ (d) $x^2 + y^2 + z^2 = p^2$

7. Line $\frac{(x+1)}{\lambda} = y - 1 = \frac{(z+2)}{-4}$ is perpendicular to $2x + 2y - 8z + 5 = 0$, then λ is

- (a) 1 (b) -4 (c) -5 (d) -3

8. If the planes $\mathbf{r}(2\hat{i} - \lambda\hat{j} + 3\hat{k}) = 0$ and $\mathbf{r} \cdot (\lambda\hat{i} + 5\hat{j} - \hat{k}) = 5$ are perpendicular to each other, then the value of $\lambda^2 + \lambda$ is

- (a) 0 (b) 2 (c) 1 (d) 3

9. Solution of the differential equation

$\frac{dy}{dx} = \sin(x+y) + \cos(x+y)$ is equal to

- (a) $\log\left(2 + \sec \frac{x+y}{2}\right) = x + C$ (b) $\log(1 + \tan(x+y)) = x + C$
 (c) $\log\left(1 + \tan \frac{x+y}{2}\right) = y + C$ (d) $\log\left(1 + \tan \frac{x+y}{2}\right) = x + C$

10. An inverted conical flask is being filled with water at the rate of $3 \text{ cm}^3/\text{sec}$. The height of the flask is 10 cm and the radius of the base is 5 cm. How fast is the water level rising when the level is 4 cm?

- (a) $\frac{4}{3}\pi \text{ cm/sec}$ (b) $\frac{3}{4\pi} \text{ cm/sec}$
 (c) $\frac{3\pi}{4} \text{ cm/sec}$ (d) $\frac{4}{3\pi} \text{ cm/sec}$

11. The equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin is

- (a) $y = 2(x-1)$ (b) $y = 2(e^x - x - 1)$
 (c) $y = 2(e^x - 1)$ (d) $y = 2(e^x x - 1)$

Engineering Entrances Solved Papers 2018 5

12. Let $f(x) = \begin{cases} x^p \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $f(x)$ is continuous but

not differentiable at $x = 0$, if

- (a) $p < 0$ (b) $p = 0$ (c) $0 < p \leq 1$ (d) $p \geq 1$

13. The solution of the differential equation $\frac{d^2 y}{dx^2} = \sin 3x + e^x + x^2$ when $y_1(0) = 1$ and $y(0) = 0$, is

- (a) $-\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$
 (b) $-\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x$
 (c) $-\frac{\cos 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x + 1$
 (d) None of the above

14. For which interval the given function $f(x) = -2x^3 - 9x^2 - 12x + 1$ is decreasing?

- (a) $(-2, \infty)$ (b) $(-2, -1)$
 (c) $(-\infty, -1)$ (d) $(-\infty, -2)$ or $(-1, \infty)$

15. If θ is the angle between the vectors $4(\hat{i} - \hat{k})$ and $\hat{i} + \hat{j} + \hat{k}$, then $(\sin \theta + \cos \theta)$ equals to

- (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

16. In a $\triangle ABC$, D, E, F are the mid-points of the sides BC, CA and AB respectively, the vector $\overrightarrow{AD}$ is equal to

- (a) $\overrightarrow{BE} + \overrightarrow{CF}$ (b) $\overrightarrow{BE} - \overrightarrow{CF}$
 (c) $\overrightarrow{CF} - \overrightarrow{BE}$ (d) $-\overrightarrow{BE} - \overrightarrow{CF}$

17. If h is the altitude of a parallelopiped determined by the vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and the base is taken to be the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ where $\mathbf{a} = \hat{i} + \hat{j} + \hat{k}$, $\mathbf{b} = 2\hat{i} + 4\hat{j} - \hat{k}$ and $\mathbf{c} = \hat{i} + \hat{j} + 3\hat{k}$, then the value of $19h^2$ is

- (a) 19 (b) 16
 (c) 8 (d) None of these

18. The mean and variance of a Binomial distribution ($\overrightarrow{BD}$) for 3 trials is 2.7, then the $\overrightarrow{BD}$ is given by

- (a) $(0.2 + 0.8)^5$ (b) $(0.3 + 0.7)^5$
 (c) $(0.4 + 0.6)^5$ (d) None of these

19. Let $P(x) = \int \frac{dx}{e^x + 8e^{-x} + 4e^{-3x}}$,

$$Q(x) = \int \frac{dx}{e^{3x} + 8e^x + 4e^{-x}} \text{ and } R(x) = P(x) - 2Q(x).$$

If $R(x) = \frac{1}{2}A \left(\frac{B + 2e^{-x}}{C} \right) + K$, then the value of (A, B, C)

is

- (a) $(\tan^{-1}, 2, e^x)$ (b) $(\tan^{-1}, e^x, 2)$
 (c) $\left(\tan^{-1}, \frac{1}{2}, \frac{1}{e^x}\right)$ (d) $\left(\tan^{-1}, \frac{1}{e^x}, \frac{1}{2}\right)$

20. The value of $\int_0^1 \cot^{-1}(1 - x + x^2) dx$ is

- (a) $\log 2$ (b) $\frac{\pi}{2} - \log 2$ (c) $\frac{\pi}{2} + \log 2$ (d) $-\log 2$

21. The area of the region included between the curves $x^2 + y^2 = a^2$ and $\sqrt{|x|} + \sqrt{|y|} = \sqrt{a}$ ($a > 0$), is

- (a) $\left(\pi - \frac{2}{3}\right)a^2$ sq units (b) $\left(\frac{2}{3} - \pi\right)a^2$ sq units
 (c) $\frac{2}{3}\pi a^2$ sq units (d) $\left(\pi + \frac{2}{3}\right)a^2$ sq units

22. Let A and B are two independent events. If the probability that both A and B occur together is $\frac{1}{6}$ and the probability that neither of them occurs is $\frac{1}{3}$, then the probability of occurrence of A is

- (a) 0 or 1 (b) $\frac{1}{2}$ or $\frac{1}{3}$ (c) $\frac{1}{2}$ or $\frac{1}{4}$ (d) $\frac{1}{3}$ or $\frac{1}{4}$

23. In a test an examiner either guesses or copies or knows the answer to a multiple choice question with 4 choices. The probability that he/she makes a guess is $\frac{1}{3}$. The probability that he/she copies the answer is $\frac{1}{6}$. If the probability that the answer is correct, given that he/she copied, it is $\frac{1}{8}$, then the probability that he/she knows the answer to a question given that he/she correctly answered it, is

- (a) $\frac{27}{29}$ (b) $\frac{26}{29}$ (c) $\frac{25}{29}$ (d) $\frac{24}{29}$

24. The maximum value of $Z = 4x + 2y$ subject to constraints $2x + 3y \leq 18$, $x + y \geq 10$ and $x, y \geq 0$ is

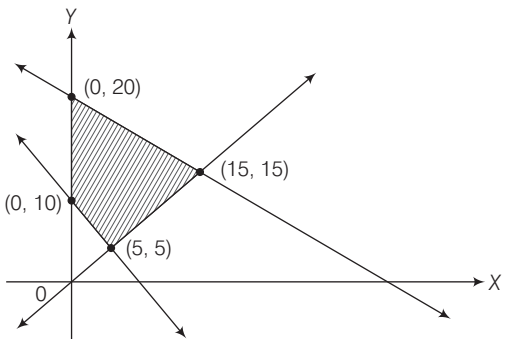
- (a) 20 (b) 36
 (c) 40 (d) None of these

25. The coordinates of the point at which minimum value of $Z = 7x - 8y$ subject to constraints $x + y - 20 \leq 0$, $y \geq 5$, $x \geq 0$, $y \geq 0$ is attained, is

- (a) (20, 0) (b) (15, 5) (c) (0, 5) (d) (0, 20)

6 Engineering Entrances Solved Papers 2018

KCET

1. $\int_0^1 \frac{dx}{e^x + e^{-x}}$ is equal to
 - (a) $\frac{\pi}{4} - \tan^{-1}(e)$
 - (b) $\tan^{-1}(e) - \frac{\pi}{4}$
 - (c) $\tan^{-1}(e) + \frac{\pi}{4}$
 - (d) $\tan^{-1}(e)$
2. $\int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$ is equal to
 - (a) $\frac{1}{\sqrt{2}} \tan^{-1} \frac{\sqrt{2}}{\sqrt{3}}$
 - (b) $\frac{2}{\sqrt{2}} \tan^{-1} \left(\frac{3}{\sqrt{2}} \right)$
 - (c) $\frac{\sqrt{2}}{2} \tan^{-1} \left(\frac{3}{2} \right)$
 - (d) $\frac{\sqrt{2}}{2} \tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$
3. The area of the region bounded by the curve $y = \cos x$ between $x = 0$ and $x = \pi$ is
 - (a) 1 sq unit
 - (b) 4 sq units
 - (c) 2 sq units
 - (d) 3 sq units
4. The area bounded by the line $y = x$, X -axis and ordinates $x = -1$ and $x = 2$ is
 - (a) $\frac{3}{2}$
 - (b) $\frac{5}{2}$
 - (c) 2
 - (d) 3
5. The degree and the order of the differential equation $\frac{d^2 y}{dx^2} = 3\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ respectively are
 - (a) 2 and 3
 - (b) 3 and 2
 - (c) 2 and 2
 - (d) 3 and 3
6. The solution of the differential equation $x \frac{dy}{dx} - y = 3$ represents a family of
 - (a) straight lines
 - (b) circles
 - (c) parabolas
 - (d) ellipses
7. The integrating factor of $\frac{dy}{dx} + y = \frac{1+y}{x}$ is
 - (a) xe^x
 - (b) $xe^{1/x}$
 - (c) $\frac{e^x}{x}$
 - (d) $\frac{x}{e^x}$
8. If $|\vec{a} \times \vec{b}|^2 + |\vec{a} \cdot \vec{b}|^2 = 144$ and $|\vec{a}| = 4$, then the value of $|\vec{b}|$ is
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 4
9. If $\vec{a}$ and $\vec{b}$ are mutually perpendicular unit vectors, then $(3\vec{a} + 2\vec{b}) \cdot (5\vec{a} - 6\vec{b})$ is equal to
 - (a) 5
 - (b) 3
 - (c) 6
 - (d) 12
10. If the vector $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are coplanar ($a \neq b \neq c \neq 1$), then the value of $abc - (a + b + c)$ is equal to
 - (a) 2
 - (b) -2
 - (c) 0
 - (d) -1
11. If $\vec{a} = \hat{i} + \lambda\hat{j} + 2\hat{k}$; $\vec{b} = \mu\hat{i} + \hat{j} - \hat{k}$ are orthogonal and $|\vec{a}| = |\vec{b}|$, then (λ, μ) is equal to
 - (a) $\left(\frac{1}{4}, \frac{7}{4}\right)$
 - (b) $\left(\frac{7}{4}, \frac{1}{4}\right)$
 - (c) $\left(\frac{1}{4}, \frac{9}{4}\right)$
 - (d) $\left(\frac{-1}{4}, \frac{9}{4}\right)$
12. The image of the point (1, 6, 3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$ is
 - (a) (1, 0, 7)
 - (b) (7, 0, 1)
 - (c) (2, 7, 0)
 - (d) (-1, -6, -3)
13. The angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$ is
 - (a) 0°
 - (b) 45°
 - (c) 90°
 - (d) 30°
14. The value of k such that the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies on the plane $2x - 4y + z = 7$ is
 - (a) -7
 - (b) 4
 - (c) -4
 - (d) 7
15. The locus represented by $xy + yz = 0$ is
 - (a) a pair of perpendicular lines
 - (b) a pair of parallel lines
 - (c) a pair of parallel planes
 - (d) a pair of perpendicular planes
16. The feasible region of an LPP is shown in the figure. If $z = 3x + 9y$, then the minimum value of z occurs at
 
 - (a) (5, 5)
 - (b) (0, 10)
 - (c) (0, 20)
 - (d) (15, 15)
17. For the LPP, maximize $z = x + 4y$ subject to the constraints $x + 2y \leq 2$, $x + 2y \geq 8$, $x, y \geq 0$
 - (a) $z_{\max} = 4$
 - (b) $z_{\max} = 8$
 - (c) $z_{\max} = 16$
 - (d) has no feasible solution
18. For the probability distribution given by

$X = x_i$	0	1	2
P_i	$\frac{25}{36}$	$\frac{5}{18}$	$\frac{1}{36}$

 the standard deviation (σ) is
 - (a) $\sqrt{\frac{1}{3}}$
 - (b) $\frac{1}{3} \sqrt{\frac{5}{2}}$
 - (c) $\sqrt{\frac{5}{36}}$
 - (d) None of these
19. A bag contains 17 tickets numbered from 1 to 17. A ticket is drawn at random, then another ticket is drawn without replacing the first one. The probability that both the tickets may show even numbers is
 - (a) $\frac{7}{34}$
 - (b) $\frac{8}{17}$
 - (c) $\frac{7}{16}$
 - (d) $\frac{7}{17}$

Engineering Entrances Solved Papers 2018 7

- 20.** A flashlight has 10 batteries out of which 4 are dead. If 3 batteries are selected without replacement and tested, then the probability that all 3 are dead is
 (a) $\frac{1}{30}$ (b) $\frac{2}{8}$ (c) $\frac{1}{15}$ (d) $\frac{1}{10}$
- 21.** Let $f, g: R \rightarrow R$ be two functions defined as $f(x) = |x| + x$ and $g(x) = |x| - x \forall x \in R$. Then $(f \circ g)(x)$ for $x < 0$ is
 (a) 0 (b) $4x$ (c) $-4x$ (d) $2x$
- 22.** A set S is having 6 distinct elements. The number of distinct functions from S to S which are not bijections is
 (a) $6! - 6$ (b) $6^6 - 6$ (c) $6^6 - 6!$ (d) $6!$
- 23.** If $\sin^{-1} x + \cos^{-1} y = \frac{2\pi}{5}$, then $\cos^{-1} x + \sin^{-1} y$ is
 (a) $\frac{2\pi}{5}$ (b) $\frac{3\pi}{5}$ (c) $\frac{4\pi}{5}$ (d) $\frac{3\pi}{10}$
- 24.** The value of the expression $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is
 (a) $2 - \sqrt{5}$ (b) $\sqrt{5} - 2$ (c) $\frac{\sqrt{5}-2}{2}$ (d) $5 - \sqrt{2}$
- 25.** If $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$, then $A^n = 2^k A$, where k is equal to
 (a) 2^{n-1} (b) $n+1$ (c) $n-1$ (d) $2(n-1)$
- 26.** If $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$, then the values of x and y respectively are
 (a) $-3, -1$ (b) $1, 3$ (c) $3, 1$ (d) $-1, 3$
- 27.** If $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$, then AA' is equal to
 (a) A (b) zero matrix (c) A' (d) I
- 28.** If $x, y, z \in R$, then the value of determinant

$$\begin{vmatrix} (5^x + 5^{-x})^2 & (5^x - 5^{-x})^2 & 1 \\ (6^x + 6^{-x})^2 & (6^x - 6^{-x})^2 & 1 \\ (7^x + 7^{-x})^2 & (7^x - 7^{-x})^2 & 1 \end{vmatrix}$$
 is
 (a) 10 (b) 12 (c) 1 (d) 0
- 29.** The value of determinant

$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$$
 is
 (a) $a^3 + b^3 + c^3$ (b) $3abc$
 (c) $a^3 + b^3 + c^3 - 3abc$ (d) $a^3 + b^3 + c^3 + 3abc$
- 30.** If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of a triangle whose area is k square units, then

$$\begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}$$
 is
 (a) $32k^2$ (b) $16k^2$ (c) $64k^2$ (d) $48k^2$
- 31.** Let A be a square matrix of order 3×3 , then $|5A|$ is equal to
 (a) $5|A|$ (b) $125|A|$ (c) $25|A|$ (d) $15|A|$
- 32.** If $f(x) = \begin{cases} \sqrt{1+kx} - \sqrt{1-kx} & \text{if } -1 \leq x < 0 \\ \frac{x}{x-1} & \text{if } 0 \leq x \leq 1 \end{cases}$ is continuous at $x = 0$, then the value of k is
 (a) $k = 1$ (b) $k = -1$ (c) $k = 0$ (d) $k = 2$
- 33.** If $\cos y = x \cos(a+y)$ with $\cos a \neq \pm 1$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{\sin a}{\cos^2(a+y)}$ (b) $\frac{\cos^2(a+y)}{\sin a}$
 (c) $\frac{\cos a}{\sin^2(a+y)}$ (d) $\frac{\cos^2(a+y)}{\cos a}$
- 34.** If $f(x) = |\cos x - \sin x|$, then $f'\left(\frac{\pi}{6}\right)$ is equal to
 (a) $-\frac{1}{2}(1 + \sqrt{3})$ (b) $\frac{1}{2}(1 + \sqrt{3})$ (c) $-\frac{1}{2}(1 - \sqrt{3})$ (d) $\frac{1}{2}(1 - \sqrt{3})$
- 35.** If $y = \sqrt{x + \sqrt{x + \sqrt{x + \dots}}}$, then $\frac{dy}{dx}$ is equal to
 (a) $\frac{1}{y^2 - 1}$ (b) $\frac{1}{2y + 1}$ (c) $\frac{2y}{y^2 - 1}$ (d) $\frac{1}{2y - 1}$
- 36.** If $f(x) = \begin{cases} \frac{\log_e x}{x-1} & ; x \neq 1 \\ k & ; x = 1 \end{cases}$ is continuous at $x = 1$, then the value of k is
 (a) e (b) 1 (c) -1 (d) 0
- 37.** Approximate change in the volume V of a cube of side x metres caused by increasing the side by 3% is
 (a) $0.09 x^3 \text{ m}^3$ (b) $0.03 x^3 \text{ m}^3$
 (c) $0.06 x^3 \text{ m}^3$ (d) $0.04 x^3 \text{ m}^3$
- 38.** The maximum value of $\left(\frac{1}{x}\right)^x$ is
 (a) e (b) e^e (c) $e^{1/e}$ (d) $\left(\frac{1}{e}\right)^{1/e}$
- 39.** $f(x) = x^x$ has stationary point at
 (a) $x = e$ (b) $x = \frac{1}{e}$ (c) $x = 1$ (d) $x = \sqrt{e}$
- 40.** The maximum area of a rectangle inscribed in the circle $(x+1)^2 + (y-3)^2 = 64$ is
 (a) 64 sq units (b) 72 sq units (c) 128 sq units (d) 8 sq units
- 41.** $\int \frac{1}{1+e^x} dx$ is equal to
 (a) $\log_e \left(\frac{e^x + 1}{e^x} \right) + C$ (b) $\log_e \left(\frac{e^x - 1}{e^x} \right) + C$
 (c) $\log_e \left(\frac{e^x}{e^x + 1} \right) + C$ (d) $\log_e \left(\frac{e^x}{e^x - 1} \right) + C$

8 Engineering Entrances Solved Papers 2018

42. $\int \frac{1}{\sqrt{3-6x-9x^2}} dx$ is equal to

- (a) $\sin^{-1}\left(\frac{3x+1}{2}\right) + C$
 (b) $\sin^{-1}\left(\frac{3x+1}{6}\right) + C$
 (c) $\frac{1}{3}\sin^{-1}\left(\frac{3x+1}{2}\right) + C$
 (d) $\sin^{-1}\left(\frac{2x+1}{3}\right) + C$

43. $\int e^{\sin x} \cdot \left(\frac{\sin x + 1}{\sec x}\right) dx$ is equal to

- (a) $\sin x \cdot e^{\sin x} + C$ (b) $\cos x \cdot e^{\sin x} + C$
 (c) $e^{\sin x} + C$ (d) $e^{\sin x} (\sin x + 1) + C$

44. $\int_{-2}^2 |x \cos \pi x| dx$ is equal to

- (a) $\frac{8}{\pi}$ (b) $\frac{4}{\pi}$ (c) $\frac{2}{\pi}$ (d) $\frac{1}{\pi}$

AP EAMCET

1. For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, $A^{-1} =$

- (a) A (b) A^2 (c) A^3 (d) A^4

2. If $A = \begin{bmatrix} k/2 & 0 & 0 \\ 0 & l/3 & 0 \\ 0 & 0 & m/4 \end{bmatrix}$ and $A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$,

then $k + l + m =$

- (a) 1 (b) 9 (c) 14 (d) 29

3. If A and B are the two real values of k for which the system of equations $x + 2y + z = 1$, $x + 3y + 4z = k$, $x + 5y + 10z = k^2$ is consistent, then $A + B =$

- (a) 3 (b) 4 (c) 5 (d) 7

4. If $13e^{i \tan^{-1} \frac{5}{12}} = a + ib$, then the ordered pair $(a, b) =$

- (a) (12, 5) (b) (5, 12) (c) (24, 10) (d) (10, 24)

5. If $\frac{x^2 + 5x + 7}{(x-3)^3} + \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$, then the equation of the line having slope A and passing through the point (B, C) is

- (a) $x + y - 20 = 0$ (b) $x - y + 20 = 0$
 (c) $x + y + 20 = 0$ (d) $x - y - 20 = 0$

6. Let x, y be real numbers such that $x \neq y$ and $xy \neq 1$. If $ax + b \sec(\tan^{-1} x) = c$ and $ay + b \sec(\tan^{-1} y) = c$, then $\frac{x+y}{1-xy} =$

- (a) $\frac{2ab}{a^2 - b^2}$ (b) $\frac{2ac}{a^2 + c^2}$ (c) $\frac{2ab}{a^2 + b^2}$ (d) $\frac{2ac}{a^2 - c^2}$

7. $\tanh^{-1} \frac{1}{2} + \coth^{-1} 3 =$

- (a) $\log \sqrt{6}$ (b) $\log 6$
 (c) $-\log \sqrt{6}$ (d) $-\log 6$

8. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three non-coplanar vectors. The vector equation of a line which passes through the point of intersection of two lines, one joining the points

$\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}$, $-\mathbf{a} - 2\mathbf{b} - 3\mathbf{c}$ and the other joining the points $-4\mathbf{c}$, $6\mathbf{a} - 4\mathbf{b} + 4\mathbf{c}$ is

- (a) $r = 2a - 4b + 3c + \mu(a - 6b + 4c)$
 (b) $r = 3a + 6b - c + \mu(a + 2b + c)$
 (c) $r = 2a + 3b - c + \mu(a + b - c)$
 (d) $r = -2b + 3c + \mu(a - 4b + 3c)$

9. If $\mathbf{a} = \hat{i} - 2\hat{j} - 3\hat{k}$, $\mathbf{b} = 2\hat{i} + \hat{j} - \hat{k}$, $\mathbf{c} = \hat{i} + 3\hat{j} - 2\hat{k}$, then $[(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})] \times (\mathbf{c} \times \mathbf{a}) \times (\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) =$

- (a) 160000 (b) -8000
 (c) 400 (d) -40

10. If $\mathbf{a} = \hat{i} + 2\hat{j} + 3\hat{k}$, $\mathbf{b} = -\hat{i} + 2\hat{j} + \hat{k}$, $\mathbf{c} = \hat{i} + 2\hat{j} - 2\hat{k}$, $\mathbf{n}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and θ is the angle between $\mathbf{c}$ and $\mathbf{n}$ then $\sin \theta =$

- (a) $\frac{\sqrt{2}}{3}$ (b) $\frac{\sqrt{2}}{3\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{2}$

11. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are mutually perpendicular vectors of the same magnitude, then the cosine of the angle between $\mathbf{a}$ and $\mathbf{a} + \mathbf{b} + \mathbf{c}$ is

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}$

12. If $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ are non-coplanar vectors and the four points with position vectors $2\mathbf{a} + 3\mathbf{b} - \mathbf{c}$, $\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}$, $3\mathbf{a} + 4\mathbf{b} + 2\mathbf{c}$ and $k\mathbf{a} - 6\mathbf{b} + 6\mathbf{c}$ are coplanar, then $k =$

- (a) 0 (b) 1 (c) 2 (d) 3

13. A man is known to speak the truth 2 out of 3 times. If he throws a die and reports that it is six, then the probability that it is actually five, is

- (a) $\frac{3}{8}$ (b) $\frac{1}{7}$ (c) $\frac{2}{7}$ (d) $\frac{4}{5}$

14. If the probability function of a random variable X is defined by $P(X = k) = a \left(\frac{k+1}{2^k}\right)$ for $k = 0, 1, 2, 3, 4, 5$,

then the probability that X takes a prime value is

- (a) $\frac{13}{20}$ (b) $\frac{23}{60}$ (c) $\frac{11}{20}$ (d) $\frac{19}{60}$

15. If X is a binomial variate with mean 6 and variance 2, then the value of $P(5 \leq X \leq 7)$ is

- (a) $\frac{4762}{6561}$ (b) $\frac{4672}{6561}$ (c) $\frac{5264}{6561}$ (d) $\frac{5462}{6561}$

Engineering Entrances Solved Papers 2018 9

- 16.** A plane cuts the coordinate axes X, Y, Z at A, B, C respectively such that the centroid of the $\triangle ABC$ is $(6, 6, 3)$. Then the equation of that plane is
 (a) $x + y + z - 6 = 0$ (b) $x + 2y + z - 18 = 0$
 (c) $2x + y + z - 18 = 0$ (d) $x + y + 2z - 18 = 0$
- 17.** If the foot of the perpendicular drawn from the origin to a plane is $(1, 2, 3)$, then a point on that plane is
 (a) $(3, 2, 1)$ (b) $(7, 2, 1)$
 (c) $(7, 3, -1)$ (d) $(6, -3, 4)$
- 18.** If a function f defined by $f(x) = \begin{cases} \frac{1 - \sqrt{2} \sin x}{\pi - 4x}, & \text{if } x \neq \frac{\pi}{4} \\ k, & \text{if } x = \frac{\pi}{4} \end{cases}$ is continuous at $x = \pi/4$, then $k =$
 (a) $\frac{1}{4}$ (b) 1 (c) $-\frac{1}{4}$ (d) 2
- 19.** The derivative of $f(x) = x^{\tan^{-1} x}$ with respect to $g(x) = \sec^{-1} \left(\frac{1}{2x^2 - 1} \right)$ is
 (a) $\frac{1}{2} \sqrt{1 - x^2} x^{\tan^{-1} x} \left[\frac{\log x}{1 + x^2} + \frac{\tan^{-1} x}{x} \right]$
 (b) $-\frac{1}{2} \sqrt{1 - x^2} x^{\tan^{-1} x} [\log(\tan^{-1} x) + x(1 + x^2) \tan^{-1} x]$
 (c) $\frac{-2 \tan^{-1} x \left[\frac{\log x}{1 + x^2} + \frac{\tan^{-1} x}{x} \right]}{\sqrt{1 - x^2}}$
 (d) $-\frac{1}{2} \sqrt{1 - x^2} x^{\tan^{-1} x} \left[\frac{\log x}{1 + x^2} + \frac{\tan^{-1} x}{x} \right]$
- 20.** If $x = 3 \cos t$ and $y = 4 \sin t$, then $\frac{d^2 y}{dx^2}$ at the point $(x_0, y_0) = \left(\frac{3}{2} \sqrt{2}, 2\sqrt{2} \right)$, is
 (a) $\frac{4\sqrt{2}}{9}$ (b) $-\frac{4\sqrt{2}}{9}$ (c) $\frac{8\sqrt{2}}{9}$ (d) $-\frac{8\sqrt{2}}{9}$
- 21.** If $y = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\frac{\sqrt{a-b} \tan \frac{x}{2}}{\sqrt{a+b}} \right]$, then $\frac{d^2 y}{dx^2} \Big|_{x=\pi/2} =$
 (a) $\frac{b}{2a^2}$ (b) $\frac{b}{a^2}$ (c) $\frac{2b}{a}$ (d) $\frac{b^2}{2a}$
- 22.** If $f(x) = x^3 + ax^2 + bx + 5 \sin^2 x$ is an increasing function on R , then
 (a) $a^2 - 3b - 15 < 0$ (b) $a^2 - 3b + 15 > 0$
 (c) $a^2 - 3b - 15 > 0$ (d) $a^2 + 3b + 15 > 0$
- 23.** The approximate value of $\cos 31^\circ$ is (Take $1^\circ = 0.0174$)
 (a) 0.7521 (b) 0.866
 (c) 0.7146 (d) 0.8573
- 24.** If x and y are two positive numbers such that $x + y = 32$, then the minimum value of $x^2 + y^2$ is,
 (a) 500 (b) 256 (c) 1024 (d) 512
- 25.** The constant 'c' of Lagrange's mean value theorem for the function $f(x) = \frac{2x+3}{4x-1}$ defined on $[1, 2]$ is
 (a) $\frac{1+\sqrt{15}}{3}$ (b) $\frac{1+\sqrt{21}}{4}$ (c) $\frac{5}{3}$ (d) $\frac{3}{2}$
- 26.** $\int \frac{\sin 2x dx}{\sin^4 x + \cos^4 x} = \tan^{-1}(f(x)) + c$, then $f\left(\frac{\pi}{3}\right) =$
 (a) 1 (b) 2 (c) 3 (d) $1/3$
- 27.** $\int \left(\frac{\log x - 1}{1 + (\log x)^2} \right)^2 dx =$
 (a) $\frac{\log x}{1 + (\log x)^2} + c$ (b) $\frac{x}{x^2 + 1} + c$
 (c) $\frac{x}{1 + (\log x)^2} + c$ (d) $\frac{-x}{1 + (\log x)^2} + c$
- 28.** $\int \frac{dx}{x^3 + 3x^2 + 2x} =$
 (a) $\log|x| + \log \left| \frac{x+2}{x+1} \right| + c$
 (b) $\log|x| - \log|x+1| + \log|x+2| + c$
 (c) $\frac{1}{2} [\log|x| + \log|x+1| + \log|x+2|] + c$
 (d) $\frac{1}{2} \log \left(\frac{x^2 + 2x}{(x+1)^2} \right) + c$
- 29.** For $n \geq 2$, If $I_n = \int \sec^n x dx$, then $I_4 - \frac{2}{3} I_2 =$
 (a) $\sec^2 x \tan x + c$ (b) $\frac{1}{3} \sec^2 x \tan x + c$
 (c) $\frac{2}{3} \sec^2 x \tan x + c$ (d) $\frac{1}{2} \log|\sec x + \tan x| + c$
- 30.** $\lim_{n \rightarrow \infty} \left(\frac{\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \dots + n\sqrt{n}}{n^{5/2}} \right) =$
 (a) 1 (b) $\frac{5}{2}$ (c) 0 (d) $\frac{2}{5}$
- 31.** $\int_0^{\alpha/3} \frac{f(x)}{f(x) + f\left(\frac{\alpha - 3x}{3}\right)} dx =$
 (a) $\frac{2\alpha}{3}$ (b) $\frac{\alpha}{2}$ (c) $\frac{\alpha}{3}$ (d) $\frac{\alpha}{6}$
- 32.** The area (in sq. units) of the region bounded by the X -axis and the curve $y = 1 - x - 6x^2$ is
 (a) $\frac{125}{216}$ (b) $\frac{125}{512}$ (c) $\frac{25}{216}$ (d) $\frac{25}{512}$
- 33.** If m and n are respectively the order and degree of the differential equation of the family of parabolas with focus at the origin and $-axis$ as its axis, then $mn - m + n =$
 (a) 1 (b) 4 (c) 3 (d) 2

10 Engineering Entrances Solved Papers 2018

TS EAMCET

- Let $f: R \rightarrow R, g: R \rightarrow R$ be differentiable functions such that $(f \circ g)(x) = x$. If $f(x) = 2x + \cos x + \sin^2 x$, then the value of $\sum_{n=1}^{99} g(1 + (2n-1)\pi)$ is
 (a) 1250π (b) $(99)^2 \frac{\pi}{2}$ (c) $(99)^2 \pi$ (d) 2500π
- Let $A = \begin{bmatrix} \frac{1}{6} & \frac{-1}{3} & \frac{-1}{6} \\ \frac{-1}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{-1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix}$. If $A^{2016l} + A^{2017m} + A^{2018n} = \frac{1}{\alpha} A$, for every $l, m, n \in N$, then the value of α is
 (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$
- Let $l, m, n \in R$ and $A = \begin{bmatrix} 1 & r & r^2 & l \\ r & r^2 & 1 & m \\ r^2 & 1 & r & n \end{bmatrix}$. Then, the set of all real values of r for which the rank of is 3, is
 (a) $(0, \infty)$ (b) R (c) $R - \{1\}$ (d) $R - \{0\}$
- The following system of equations

$$x + y + z = 9, \quad 2x + 5y + 7z = 52$$
 and $x + 7y + 11z = 77$ has
 (a) no solution (b) exactly 2 solution
 (c) only one solution (d) infinitely many solutions
- If $\frac{x^4 + x^3 + 2x^2 - 2x + 1}{x^3 + x^2} = P(x) + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$, then $A + B + C =$
 (a) $P(0)$ (b) $P(2)$ (c) $P(3)$ (d) $P(4)$
- $2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{5\sqrt{2}}{7} + 2 \tan^{-1} \frac{1}{8} =$
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{8}$
- Let $3\hat{i} + \hat{j} - \hat{k}$ be the position vector of a point B . Let A be a point on the line which is passing through B and parallel to the vector $2\hat{i} - \hat{j} + 2\hat{k}$. If $|\mathbf{BA}| = 18$, then the position vector of A is
 (a) $-9\hat{i} + 7\hat{j} - 13\hat{k}$ (b) $-9\hat{i} + 3\hat{j} + 12\hat{k}$
 (c) $9\hat{i} - 3\hat{j} + 2\hat{k}$ (d) $3\hat{i} - \hat{j} + 7\hat{k}$
- The vector that is parallel to the vector $2\hat{i} - 2\hat{j} - 4\hat{k}$ and coplanar with the vectors $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$ is
 (a) $\hat{i} - \hat{k}$ (b) $\hat{i} + \hat{j} - \hat{k}$
 (c) $\hat{i} - \hat{j} - 2\hat{k}$ (d) $3\hat{i} + 3\hat{j} + 6\hat{k}$
- A line L is passing through the point A whose position vector is $\hat{i} + 2\hat{j} - 3\hat{k}$ and parallel to the vector $2\hat{i} + \hat{j} + 2\hat{k}$. A plane π is passing through the points $\hat{i} + \hat{j} + \hat{k}, \hat{i} - \hat{j} - \hat{k}$ and parallel to the vector $\hat{i} - 2\hat{j}$. Then, the point where this plane π meets the line L is
 (a) $\frac{1}{3}(-7\hat{i} + \hat{j} - 19\hat{k})$ (b) $7\hat{i} + \hat{j} - 19\hat{k}$
 (c) $3\hat{i} + 3\hat{j} - \hat{k}$ (d) $2\hat{i} - \hat{j} + \hat{k}$
- If the position vectors of three points A, B, C respectively are $\hat{i} + 2\hat{j} + \hat{k}, 2\hat{i} - \hat{j} + 2\hat{k}$ and $\hat{i} + \hat{j} + 2\hat{k}$, then the perpendicular distance of the point C from the line AB is
 (a) $\sqrt{\frac{3}{11}}$ (b) $\sqrt{\frac{4}{11}}$ (c) $\sqrt{\frac{6}{11}}$ (d) $\sqrt{\frac{8}{11}}$
- The volume of a tetrahedron whose vertices are $4\hat{i} + 5\hat{j} + \hat{k}, -\hat{j} + \hat{k}, 3\hat{i} + 9\hat{j} + 4\hat{k}$ and $-2\hat{i} + 4\hat{j} + 4\hat{k}$ is (in cubic units)
 (a) $\frac{14}{3}$ (b) 5 (c) 6 (d) 30
- If the vectors $\mathbf{b}, \mathbf{c}, \mathbf{d}$ are not coplanar, then the vector $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$ is
 (a) parallel to $\mathbf{a}$ (b) parallel to $\mathbf{b}$
 (c) parallel to $\mathbf{c}$ (d) perpendicular to $\mathbf{a}$
- 5 persons entered a lift cabin on the ground floor of a 7 floor house. Suppose, that each of them independently and with equal probability can leave the cabin at any floor beginning with the first. The probability of all the 5 persons leaving the cabin at different floors, is
 (a) $\frac{360}{2401}$ (b) $\frac{5}{54}$
 (c) $\frac{5}{18}$ (d) $\frac{5!}{7!}$
- A company produces 10000 items per day. On a particular day 2500 items were produced on machine A, 3500 on machine B and 4000 on machine (c) The probability that an item produced by the machines A, B, C to be defective is respectively 2%, 3% and 5%. If one item is selected at random from the output and is found to be defective, then the probability that it was produced machine C, is
 (a) $\frac{10}{71}$ (b) $\frac{16}{71}$ (c) $\frac{40}{71}$ (d) $\frac{21}{71}$
- An executive in a company makes on an average 5 telephone calls per hour at a cost of ` 2 per call. The probability that in any hour the cost of the calls exceeds a sum of 4 is
 (a) $\frac{2e^4 - 35}{2e^5}$ (b) $\frac{2e^5 - 37}{2e^5}$
 (c) $1 - \frac{37}{e^4}$ (d) $1 - (18.5)e^5$

Engineering Entrances Solved Papers 2018 11

16. The lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $mn + nl + lm = 0$ are

- (a) perpendicular if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$
 (b) perpendicular if $\sqrt{a} + \sqrt{b} + \sqrt{c} = 0$
 (c) parallel if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$
 (d) parallel to $a + b + c = 0$

17. If the plane passing through the points (1, 2, 3), (2, 3, 1) and (3, 1, 2) is $ax + by + cz = 1$, then $a + 2b + 3c =$

- (a) 0 (b) 1 (c) 6 (d) 18

18. If $f(x) = \begin{cases} \frac{x-2}{|x-2|} + a, & x < 2 \\ a + b, & x = 2 \\ \frac{x-2}{|x-2|} + b, & x > 2 \end{cases}$

is continuous at $x = 2$, then $a + b =$

- (a) 2 (b) 1 (c) 0 (d) -1

19. If $y = \frac{(\sin^{-1} x)^2}{2}$, then $(1 - x^2)y_2 - xy_1 =$

- (a) y (b) $2y$ (c) 1 (d) 2

20. If the relative errors in the base radius and the height of a cone are same and equal to 0.02, then the percentage error in the volume of that cone is

- (a) 2 (b) 4 (c) 6 (d) 8

21. The normal at a point θ to the curve $x = a(1 + \cos \theta)$, $y = a \sin \theta$ always passes through the fixed point

- (a) (0, a) (b) (2a, 0) (c) (a, 0) (d) (a, a)

22. Let $f(x)$ be continuous on $[0, 6]$ and differentiable on $(0, 6)$. Let $f(0) = 12$ and $f(6) = -4$.

If $g(x) = \frac{f(x)}{x+1}$, then for some Lagrange's constant

$c \in (0, 6)$, $g'(c) =$

- (a) $-\frac{44}{3}$ (b) $-\frac{22}{21}$
 (c) $\frac{32}{21}$ (d) $-\frac{44}{21}$

23. If (α, β) and (γ, δ) where $\alpha < \gamma$, are the turning points of $f(x) = 2x^3 - 15x^2 + 36x - 8$, then $\alpha - \gamma - \beta + \delta =$

- (a) 0 (b) -2 (c) 2 (d) 1

24. The height of a cylinder of the greatest volume that can be inscribed in a sphere of radius 3 is

- (a) $3\sqrt{3}$ (b) $2\sqrt{3}$ (c) $\sqrt{3}$ (d) $\sqrt{2}$

25. $\int \frac{dx}{(e^x + e^{-x})^2} =$

- (a) $\frac{1}{2(e^{2x} + 1)} + c$ (b) $-\frac{1}{2(e^{2x} + 1)} + c$
 (c) $\frac{1}{3(e^{2x} + 1)} + c$ (d) $-\frac{1}{(e^{2x} + 1)} + c$

26. $\int_0^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2018}}} =$

- (a) π (b) $\frac{3\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$

27. If $\int \frac{x}{(x^2 + 1)(x - 1)} dx = A \log|x^2 + 1| + B \tan^{-1} x + C \log|x - 1| + d$, then $A + B + C =$

- (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{3}{4}$ (d) $\frac{5}{4}$

28. $\int_0^{\pi/2} \frac{\cos^3 x}{\sin x + \cos x} dx =$

- (a) $\frac{\pi - 1}{2}$ (b) $\frac{\pi - 1}{4}$ (c) $\frac{1 + \pi}{4}$ (d) $\frac{\pi - 3}{4}$

29. $\int_0^3 (2 + x^2) dx =$

- (a) $\lim_{n \rightarrow \infty} \frac{1}{n} \left[2n + \frac{1^2 + 2^2 + \dots + (3n)^2}{n^2} \right]$
 (b) $\lim_{n \rightarrow \infty} \frac{1}{n} \left[3n + \frac{1^2 + 2^2 + \dots + 6n^2}{n^2} \right]$
 (c) $\lim_{n \rightarrow \infty} \frac{1}{n} \left[6n + \frac{1^2 + 2^2 + \dots + 9n^2}{n^2} \right]$
 (d) $\lim_{n \rightarrow \infty} \frac{1}{n} \left[3n + \frac{1^2 + 2^2 + \dots + 3n^2}{n^2} \right]$

30. The area enclosed (in square units) by the curve $y = x^4 - x^2$, the X -axis and the vertical lines passing through the two minimum points of the curve is

- (a) $\frac{48\sqrt{2}}{5}$ (b) $\frac{5}{48\sqrt{2}}$
 (c) $\frac{7}{60\sqrt{2}}$ (d) $\frac{7}{30\sqrt{2}}$

31. The differential equation corresponding to the family of circles having centres on X -axis and passing through the origin is

- (a) $y^2 + x^2 + \frac{dy}{dx} = 0$ (b) $y^2 - x^2 + \frac{dy}{dx} = 0$
 (c) $y^2 + x^2 + 2xy \frac{dy}{dx} = 0$ (d) $y^2 - x^2 - 2xy \frac{dy}{dx} = 0$

32. The general solution of the differential equation $(x^2 + xy)y' = y^2$ is

- (a) $e^{y/x} = cx$ (b) $e^{-y/x} = cy$
 (c) $e^{-y/x} = cxy$ (d) $e^{-2y/x} = cy$

33. At any point on a curve, the slope of the tangent is equal to the sum of abscissa and the product of ordinate and abscissa of that point. If the curve passes through (0, 1), then the equation of the curve is

- (a) $y = 2e^{\frac{x^2}{2}} - 1$ (b) $y = 2e^{x^2}$
 (c) $y = e^{-x^2}$ (d) $y = 2e^{-x^2} - 1$

12 Engineering Entrances Solved Papers 2018

VIT

1. If the sum of the squares of the intercepts on the axes cut off by the tangent to the curve $x^{1/3} + y^{1/3} = a^{1/3}$ (with $a > 0$) at $\left(\frac{a}{8}, \frac{a}{8}\right)$ is 2, then a has the value
 (a) 1 (b) 2 (c) 4 (d) 8
2. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then, the value of the determinant $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 - \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$ is
 (a) 3ω (b) $3\omega(\omega - 1)$ (c) $3\omega^2$ (d) $3\omega(1 - \omega)$
3. If $A^3 = O$, then $I + A + A^2$ equals
 (a) $I - A$ (b) $(I - A)^{-1}$
 (c) $(I + A)^{-1}$ (d) I
4. If A is a singular matrix, then $\text{adj } A$ is
 (a) non-singular (b) singular
 (c) symmetric (d) not defined
5. If in a $\triangle ABC$, $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$, then the value of $\sin^2 A + \sin^2 B + \sin^2 C$ is
 (a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) $\frac{3\sqrt{3}}{2}$ (d) 1
6. If the angle θ between the line $\frac{x+1}{1} = \frac{y-1}{2} = \frac{z-2}{2}$ and the plane $2x - y + \sqrt{\lambda}z + 4 = 0$ is such that $\sin \theta = \frac{1}{3}$, the value of λ is
 (a) $-\frac{4}{3}$ (b) $\frac{3}{4}$ (c) $-\frac{3}{5}$ (d) $\frac{5}{3}$
7. The lines $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-1}{4}$ and $\frac{x-1}{k} = \frac{y-4}{2} = \frac{z-5}{1}$ are coplanar, then the value of k are
 (a) 3, -3 (b) 0, -1 (c) 1, -1 (d) 0, -3
8. The general solution of $y^2 dx + (x^2 - xy + y^2) dy = 0$, is
 (a) $\tan^{-1} \frac{x}{y} + \log y + C = 0$
 (b) $2 \tan^{-1} \frac{x}{y} + \log x + C = 0$
 (c) $\log(y + \sqrt{x^2 + y^2}) + \log y + C = 0$
 (d) $\log y = \tan^{-1} \frac{y}{x} + C$
9. If A and B are two non-singular matrices which commute, then $(A(A+B)^{-1}B)^{-1}(AB)$ is equal to
 (a) $A+B$ (b) $A^{-1}+B$ (c) $A^{-1}+B^{-1}$ (d) $A+B^{-1}$
10. The value of $\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1} \right)^2 + \left(\frac{x-1}{x+1} \right)^2 - 2 \right]^{1/2} dx$ is
 (a) 0 (b) $4 \log \left(\frac{4}{3} \right)$ (c) $-4 \log \left(\frac{4}{3} \right)$ (d) 1
11. $\int \frac{1}{\sqrt{e^{5x}} \sqrt[4]{(e^{2x} + e^{-2x})^3}} dx$ is equal to
 (a) $(1 + e^{-4x})^{1/4} + C$ (b) $-(1 + e^{-4x})^{1/4} + C$
 (c) $(1 - e^{-4x})^{1/4} + C$ (d) $-(1 - e^{-4x})^{1/4} + C$
12. $\int_0^2 [x^2 - x + 1] dx$, where $[.]$ denotes the greatest integer function is equal to
 (a) $\frac{\sqrt{5}}{2}$ (b) $\frac{5 + \sqrt{5}}{2}$
 (c) $\frac{5 - \sqrt{5}}{2}$ (d) $\frac{\sqrt{5} - \sqrt{5}}{2}$
13. The solution of $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$, is
 (a) $xe^{\tan^{-1} y} = \frac{1}{2} e^{2 \tan^{-1} y} + C$ (b) $ye^{\tan^{-1} y} = \frac{1}{2} e^{2 \tan^{-1} y} + C$
 (c) $xe^{2 \tan^{-1} y} = 2e^{\tan^{-1} y} + C$ (d) $ye^{2 \tan^{-1} y} = 2e^{\tan^{-1} y} + C$
14. The order and degree of the differential equation $\ln \left(\frac{dy}{dx} \right) = ax + by$ is
 (a) 0, 1 (b) 1, 0
 (c) 1, 1 (d) None of these
15. If $y = x^3 - ax^2 + 48x + 7$ is an increasing function for all real values of x , then a lies in
 (a) $(-14, 14)$ (b) $(-12, 12)$
 (c) $(-16, 16)$ (d) $(-21, 21)$
16. A coin is tossed $2n$ times. The chance that the number of times one gets head is not equal to the number of times one gets tail is
 (a) $2^n C_n \left(\frac{1}{2} \right)^{2n}$ (b) $1 - {}^{2n}C_n$
 (c) $1 - {}^{2n}C_n \left(\frac{1}{4^n} \right)$ (d) None of these
17. If $\vec{a}$ and $\vec{b}$ are two unit vectors inclined at an angle θ such that $|\vec{a} + \vec{b}| < 1$, then
 (a) $\frac{2\pi}{3} < \theta < \frac{4\pi}{3}$ (b) $\theta < \frac{\pi}{3}$ (c) $\theta > \frac{2\pi}{3}$ (d) $\theta = \frac{\pi}{2}$
18. Let $\vec{a} = 2\hat{i} + \hat{j} - 2\hat{k}$ and $\vec{b} = \hat{i} + \hat{j}$. If $\vec{c}$ is a vector such that $\vec{a} \cdot \vec{c} = |\vec{c}|$, $|\vec{c} - \vec{a}| = 2\sqrt{2}$ and the angle between $\vec{a} \times \vec{b}$ and $\vec{c}$ is 30° . Then, $|(\vec{a} \times \vec{b}) \times \vec{c}|$ is equal to
 (a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) 2 (d) 3

Engineering Entrances Solved Papers 2018 13

19. If the volume of the parallelopiped formed by the vectors $\vec{a}, \vec{b}, \vec{c}$ as three coterminus edges is 27 cu units, then the volume of the parallelopiped having, $\alpha = \vec{a} + 2\vec{b} - \vec{c}$, $\beta = \vec{a} - \vec{b}$ and $\gamma = \vec{a} - \vec{b} - \vec{c}$ as three coterminus edges is
- (a) 27 cu units (b) 9 cu units
(c) 81 cu units (d) None of these
20. The angle between the lines whose direction cosines are given by the equations $3l + m + 5n = 0$, $6mn - 2nl + 5lm = 0$, is
- (a) $\sin^{-1}\left(-\frac{1}{6}\right)$ (b) $\cos^{-1}\left(-\frac{1}{6}\right)$
(c) $\tan^{-1}\left(-\frac{1}{6}\right)$ (d) None of these
21. If $f: R \rightarrow R$ is defined by $f(x) = \begin{cases} \frac{\cos 3x - \cos x}{x^2}, & x \neq 0 \\ \lambda, & x = 0 \end{cases}$ and if f is continuous at $x = 0$, then λ is equal to
- (a) -2 (b) -4
(c) -6 (d) -8
22. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is
- (a) 1 (b) 2
(c) $2\sqrt{2}$ (d) 4
23. If $\alpha \leq \sin^{-1} x + \cos^{-1} x + \tan^{-1} x \leq \beta$, then
- (a) $\alpha = \frac{\pi}{4}, \beta = \frac{3\pi}{4}$ (b) $\alpha = -\pi, \beta = 2\pi$
(c) $\alpha = 0, \beta = \pi$ (d) None of these
24. If two lines, whose equations are $\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3}$ lie in the same plane. Then, the value of $\sin^{-1} \sin \lambda$ is
- (a) 3 (b) $\pi - 3$
(c) 4 (d) $\pi - 4$

MHT CET

1. If $\int_0^k \frac{dx}{2+18x^2} = \frac{\pi}{24}$, then the value of k is
- (a) 3 (b) 4
(c) $\frac{1}{3}$ (d) $\frac{1}{4}$
2. $\int \frac{1}{\sin x \cdot \cos^2 x} dx =$
- (a) $\sec x + \log |\sec x + \tan x| + C$
(b) $\sec x \cdot \tan x + C$
(c) $\sec x + \log |\sec x - \tan x| + C$
(d) $\sec x + \log |\operatorname{cosec} x - \cot x| + C$
3. If $\log_{10} \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 2$, then $\frac{dy}{dx} =$
- (a) $\frac{x}{y}$ (b) $-\frac{y}{x}$ (c) $-\frac{x}{y}$ (d) $\frac{y}{x}$
4. If $f(x) = \begin{cases} x^2 + \alpha & \text{for } x \geq 0 \\ 2\sqrt{x^2 + 1} + \beta & \text{for } x < 0 \end{cases}$ is continuous at $x = 0$ and $f\left(\frac{1}{2}\right) = 2$, then $\alpha^2 + \beta^2$ is
- (a) 3 (b) $\frac{8}{25}$ (c) $\frac{25}{8}$ (d) $\frac{1}{3}$
5. If $y = (\tan^{-1} x)^2$, then $(x^2 + 1)^2 \frac{d^2 y}{dx^2} + 2x(x^2 + 1) \frac{dy}{dx} =$
- (a) 4 (b) 2
(c) 1 (d) 0
6. If $A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$, then $(A^2 - 5A) A^{-1} =$
- (a) $\begin{bmatrix} 4 & 2 & 3 \\ -1 & 4 & 2 \\ 1 & 2 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} -4 & 2 & 3 \\ -1 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$
(c) $\begin{bmatrix} -4 & -1 & 1 \\ 2 & -4 & 2 \\ 3 & 2 & -1 \end{bmatrix}$ (d) $\begin{bmatrix} -1 & -2 & 1 \\ 4 & -2 & -3 \\ 1 & 4 & -2 \end{bmatrix}$
7. The equation of line passing through $(3, -1, 2)$ and perpendicular to the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$ and $\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$ is
- (a) $\frac{x+3}{2} = \frac{y+1}{3} = \frac{z-2}{2}$ (b) $\frac{x-3}{3} = \frac{y+1}{2} = \frac{z-2}{2}$
(c) $\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-2}{2}$ (d) $\frac{x-3}{2} = \frac{y+1}{2} = \frac{z-2}{3}$
8. If $\int \frac{dx}{\sqrt{16-9x^2}} = A \sin^{-1}(Bx) + C$, then $A + B =$
- (a) $\frac{9}{4}$ (b) $\frac{19}{4}$ (c) $\frac{3}{4}$ (d) $\frac{13}{12}$
9. $\int e^x \left[\frac{2 + \sin 2x}{1 + \cos 2x} \right] dx =$
- (a) $e^x \tan x + C$ (b) $e^x + \tan x + C$
(c) $2e^x \tan x + C$ (d) $e^x \tan 2x + C$

14 Engineering Entrances Solved Papers 2018

- 10.** A coin is tossed three times. If X denotes the absolute difference between the number of heads and the number of tails, then $P(X = 1) =$
 (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{1}{6}$ (d) $\frac{3}{4}$
- 11.** The area of the region bounded by $x^2 = 4y$, $y = 1$, $y = 4$ and the Y -axis lying in the first quadrant is square units.
 (a) $\frac{22}{3}$ (b) $\frac{28}{3}$ (c) 30 (d) $\frac{27}{4}$
- 12.** If $f(x) = \frac{e^{x^2} - \cos x}{x^2}$, for $x \neq 0$ is continuous at $x = 0$, then value of $f(0)$ is
 (a) $\frac{2}{3}$ (b) $\frac{5}{2}$ (c) 1 (d) $\frac{3}{2}$
- 13.** If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors having magnitudes 1, 2, 3 respectively, then $[\vec{a} + \vec{b} + \vec{c}, \vec{b} - \vec{a}, \vec{c}] =$
 (a) 0 (b) 6 (c) 12 (d) 18
- 14.** If planes $x - cy - bz = 0$, $cx - y + az = 0$ and $bx + ay - z = 0$ pass through a straight line, then $a^2 + b^2 + c^2$
 (a) $1 - abc$ (b) $abc - 1$
 (c) $1 - 2abc$ (d) $2abc - 1$
- 15.** A die is thrown four times. The probability of getting perfect square in at least one throw is
 (a) $\frac{16}{81}$ (b) $\frac{65}{81}$ (c) $\frac{23}{81}$ (d) $\frac{58}{81}$
- 16.** $\int_0^{\pi/4} x \cdot \sec^2 x \, dx =$
 (a) $\frac{\pi}{4} + \log \sqrt{2}$ (b) $\frac{\pi}{4} - \log \sqrt{2}$
 (c) $1 + \log \sqrt{2}$ (d) $1 - \frac{1}{2} \log 2$
- 17.** If $x = e^\theta (\sin \theta - \cos \theta)$, $y = e^\theta (\sin \theta + \cos \theta)$, then $\frac{dy}{dx}$ at $\theta = \frac{\pi}{4}$ is
 (a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{2}$
- 18.** If $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$, then $x =$
 (a) -1 (b) $\frac{1}{3}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$
- 19.** Matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix}$, then the value of $a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$ is
 (a) 1 (b) 13 (c) -1 (d) -13
- 20.** If $f(x) = \frac{x}{x^2 + 1}$ is increasing function, then the value of x lies in
 (a) R (b) $(-\infty, -1)$
 (c) $(1, \infty)$ (d) $(-1, 1)$
- 21.** The minimum value of the function $f(x) = x \log x$ is
 (a) $-\frac{1}{e}$ (b) $-e$
 (c) $\frac{1}{e}$ (d) e
- 22.** If $X \sim B(n, p)$ with $n = 10$, $p = 0.4$, then $E(X^2) =$
 (a) 4 (b) 2.4 (c) 3.6 (d) 18.4
- 23.** The general solution of differential equation $\frac{dx}{dy} = \cos(x + y)$ is
 (a) $\tan\left(\frac{x+y}{2}\right) = y + C$ (b) $\tan\left(\frac{x+y}{2}\right) = x + C$
 (c) $\cot\left(\frac{x+y}{2}\right) = y + C$ (d) $\cot\left(\frac{x+y}{2}\right) = x + C$
- 24.** If planes $\vec{r} \cdot (p\hat{i} - \hat{j} + 2\hat{k}) + 3 = 0$ and $\vec{r} \cdot (2\hat{i} - p\hat{j} - \hat{k}) - 5 = 0$ include angle $\frac{\pi}{3}$, then the value of p is
 (a) 1, -3 (b) -1, 3 (c) -3 (d) 3
- 25.** The order of the differential equation of all parabolas, whose latus rectum is $4a$ and axis parallel to the X -axis, is
 (a) one (b) four (c) three (d) two
- 26.** If a line makes angle 120° and 60° with the positive directions of X and Z -axes respectively, then the angle made by the line with positive Y -axis is
 (a) 150° (b) 60° (c) 135° (d) 120°
- 27.** L and M are two points with position vectors $2\vec{a} - \vec{b}$ and $\vec{a} + 2\vec{b}$, respectively. The position vector of the point N which divides the line segment LM in the ratio 2 : 1 externally is
 (a) $3\vec{b}$ (b) $4\vec{b}$ (c) $5\vec{b}$ (d) $3\vec{a} + 4\vec{b}$

Answer with Explanations

JEE Main

1. (c) Given,

$$\begin{vmatrix} x-4 & 2x & 2x \\ 2x & x-4 & 2x \\ 2x & 2x & x-4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

⇒ Apply $C_1 \rightarrow C_1 + C_2 + C_3$

$$\begin{vmatrix} 5x-4 & 2x & 2x \\ 5x-4 & x-4 & 2x \\ 5x-4 & 2x & x-4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

Taking common $(5x - 4)$ from C_1 , we get

$$(5x - 4) \begin{vmatrix} 1 & 2x & 2x \\ 1 & x-4 & 2x \\ 1 & 2x & x-4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

Apply $R_2 \rightarrow R_2 - R_1$ and $R_3 \rightarrow R_3 - R_1$

$$\therefore (5x - 4) \begin{vmatrix} 1 & 2x & 0 \\ 0 & -x-4 & 0 \\ 0 & 0 & -x-4 \end{vmatrix}$$

$$= (A + Bx)(x - A)^2$$

Expanding along C_1 , we get

$$(5x - 4)(x + 4)^2 = (A + Bx)(x - A)^2$$

Equating, we get

$$A = -4 \text{ and } B = 5$$

2. (b) We have,

$$x + ky + 3z = 0; 3x + ky - 2z = 0; 2x + 4y - 3z = 0$$

System of equation has non-zero solution, if

$$\begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 4 & -3 \end{vmatrix} = 0$$

$$\Rightarrow (-3k + 8) - k(-9 + 4) + 3(12 - 2k) = 0$$

$$\Rightarrow -3k + 8 + 9k - 4k + 36 - 6k = 0$$

$$\Rightarrow -4k + 44 = 0 \Rightarrow k = 11$$

Let $z = \lambda$, then we get

$$x + 11y + 3\lambda = 0 \quad \dots(i)$$

$$3x + 11y - 2\lambda = 0 \quad \dots(ii)$$

$$\text{and } 2x + 4y - 3\lambda = 0 \quad \dots(iii)$$

Solving Eqs. (i) and (ii), we get

$$x = \frac{5\lambda}{2}, y = \frac{-\lambda}{2}, z = \lambda$$

$$\Rightarrow \frac{xz}{y^2} = \frac{5\lambda^2}{2 \times \left(-\frac{\lambda}{2}\right)^2} = 10$$

3. (a) We have,

$$f(x) = |x - \pi|(e^{|x|} - 1)\sin|x|$$

$$f(x) = \begin{cases} (x - \pi)(e^{-x} - 1)\sin x, & x < 0 \\ -(x - \pi)(e^x - 1)\sin x, & 0 \leq x < \pi \\ (x - \pi)(e^x - 1)\sin x, & x \geq \pi \end{cases}$$

We check the differentiability at $x = 0$ and π .

We have,

$$f'(x) = \begin{cases} (x - \pi)(e^{-x} - 1)\cos x + (e^{-x} - 1)\sin x \\ \quad + (x - \pi)\sin xe^{-x}(-1), & x < 0 \\ -[(x - \pi)(e^x - 1)\cos x + (e^x - 1)\sin x \\ \quad + (x - \pi)\sin xe^x], & 0 < x < \pi \\ (x - \pi)(e^x - 1)\cos x + (e^x - 1)\sin x \\ \quad + (x - \pi)\sin xe^x, & x > \pi \end{cases}$$

$$\text{Clearly, } \lim_{x \rightarrow 0^-} f'(x) = 0 = \lim_{x \rightarrow 0^+} f'(x)$$

$$\text{and } \lim_{x \rightarrow \pi^-} f'(x) = 0 = \lim_{x \rightarrow \pi^+} f'(x)$$

∴ f is differentiable at $x = 0$ and $x = \pi$

Hence, f is differentiable for all x .

4. (d) We have, $y^2 = 6x$

$$\Rightarrow 2y \frac{dy}{dx} = 6 \Rightarrow \frac{dy}{dx} = \frac{3}{y}$$

$$\text{Slope of tangent at } (x_1, y_1) \text{ is } m_1 = \frac{3}{y_1}$$

$$\text{Also, } 9x^2 + by^2 = 16$$

$$\Rightarrow 18x + 2by \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-9x}{by}$$

$$\text{Slope of tangent at } (x_1, y_1) \text{ is } m_2 = \frac{-9x_1}{by_1}$$

Since, these are intersection at right angle.

$$\therefore m_1 m_2 = -1 \Rightarrow \frac{27x_1}{by_1^2} = 1$$

$$\Rightarrow \frac{27x_1}{6bx_1} = 1 \quad [\because y_1^2 = 6x_1]$$

$$\Rightarrow b = \frac{9}{2}$$

5. (d) We have,

$$f(x) = x^2 + \frac{1}{x^2} \text{ and } g(x) = x - \frac{1}{x}$$

$$\Rightarrow h(x) = \frac{f(x)}{g(x)}$$

$$\therefore h(x) = \frac{x^2 + \frac{1}{x^2}}{x - \frac{1}{x}} = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{x - \frac{1}{x}}$$

$$\Rightarrow h(x) = \left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}}$$

$$x - \frac{1}{x} > 0,$$

$$\left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in [2\sqrt{2}, \infty)$$

$$x - \frac{1}{x} < 0,$$

$$\left(x - \frac{1}{x}\right) + \frac{2}{x - \frac{1}{x}} \in (-\infty, 2\sqrt{2}]$$

∴ Local minimum value is $2\sqrt{2}$.

6. (a) We have,

$$\Rightarrow 18x^2 - 9\pi x + \pi^2 = 0$$

$$\Rightarrow 18x^2 - 6\pi x - 3\pi x + \pi^2 = 0$$

$$(6x - \pi)(3x - \pi) = 0$$

$$\Rightarrow x = \frac{\pi}{6}, \frac{\pi}{3}$$

Now, $\alpha < \beta$

$$\alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$$

Given, $g(x) = \cos x^2$ and $f(x) = \sqrt{x}$

$$y = g(Of(x))$$

$$\therefore y = g(f(x)) = \cos x$$

Area of region bounded by $x = \alpha$, $x = \beta$, $y = 0$ and curve $y = g(f(x))$ is

$$A = \int_{\pi/6}^{\pi/3} \cos x \, dx$$

$$A = [\sin x]_{\pi/6}^{\pi/3}$$

$$A = \sin \frac{\pi}{3} - \sin \frac{\pi}{6} = \frac{\sqrt{3}}{2} - \frac{1}{2}$$

$$A = \left(\frac{\sqrt{3} - 1}{2}\right)$$

7. (c) We have,

$$\sin x \frac{dy}{dx} + y \cos x = 4x$$

$$\Rightarrow \frac{dy}{dx} + y \cot x = 4x \operatorname{cosec} x$$

This is a linear differential equation of form

$$\frac{dy}{dx} + Py = Q$$

where $P = \cot x$, $Q = 4x \operatorname{cosec} x$

$$\text{Now, } IF = e^{\int P dx} = e^{\int \cot x dx}$$

$$= e^{\log \sin x} = \sin x$$

Solution of the differential equation is

$$y \cdot \sin x = \int 4x \operatorname{cosec} x \sin x dx + C$$

$$\Rightarrow y \sin x = \int 4x dx + C = 2x^2 + C$$

Put $x = \frac{\pi}{2}$, $y = 0$, we get

16 Engineering Entrances Solved Papers 2018

$$C = -\frac{\pi^2}{2}$$

$$\Rightarrow y \sin x = 2x^2 - \frac{\pi^2}{2}$$

$$\text{Put } x = \frac{\pi}{6}$$

$$\therefore y\left(\frac{1}{2}\right) = 2\left(\frac{\pi^2}{36}\right) - \frac{\pi^2}{2}$$

$$\Rightarrow y = \frac{\pi^2}{9} - \pi^2 \Rightarrow y = -\frac{8\pi^2}{9}$$

Alternate Method

We have, $\sin x \frac{dy}{dx} + y \cos x = 4x$, which

can be written as $\frac{d}{dx}(\sin x \cdot y) = 4x$

On integrating both sides, we get

$$\int \frac{d}{dx}(\sin x \cdot y) \cdot dx = \int 4x \cdot dx$$

$$\Rightarrow y \cdot \sin x = \frac{4x^2}{2} + C$$

$$\Rightarrow y \cdot \sin x = 2x^2 + C$$

Now, as $y = 0$ when $x = \frac{\pi}{2}$

$$\therefore C = -\frac{\pi^2}{2} \Rightarrow y \cdot \sin x = 2x^2 - \frac{\pi^2}{2}$$

Now, putting $x = \frac{\pi}{6}$, we get

$$y\left(\frac{1}{2}\right) = 2\left(\frac{\pi^2}{36}\right) - \frac{\pi^2}{2}$$

$$\Rightarrow y = \frac{\pi^2}{9} - \pi^2 = -\frac{8\pi^2}{9}$$

8. (b) L_1 is the line of intersection of the plane $2x - 2y + 3z - 2 = 0$ and $x - y + z + 1 = 0$ and L_2 is the line of intersection of the plane $x + 2y - z - 3 = 0$ and $3x - y + 2z - 1 = 0$

Since L_1 is parallel to $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{vmatrix} = \hat{i} + \hat{j}$

L_2 is parallel to $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\hat{i} - 5\hat{j} - 7\hat{k}$

Also, L_2 passes through $\left(\frac{5}{7}, \frac{8}{7}, 0\right)$

[put $z = 0$ in last two planes]

So, equation of plane is

$$\begin{vmatrix} x - \frac{5}{7} & y - \frac{8}{7} & z \\ 1 & 1 & 0 \\ 3 & -5 & -7 \end{vmatrix} = 0$$

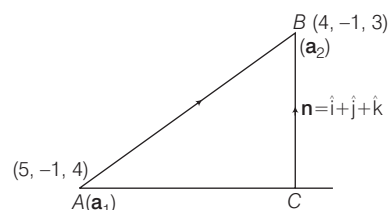
$$\Rightarrow 7x - 7y + 8z + 3 = 0$$

Now, perpendicular distance from origin is

$$\left| \frac{3}{\sqrt{7^2 + 7^2 + 8^2}} \right| = \frac{3}{\sqrt{162}} = \frac{1}{3\sqrt{2}}$$

9. (d) **Key idea** length of projection of the line segment joining $\mathbf{a}_1$ and $\mathbf{a}_2$ on the plane $\mathbf{r} \cdot \mathbf{n} = d$ is $\frac{(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{n}}{|\mathbf{n}|}$

Length of projection the line segment joining the points (5, -1, 4) and (4, -1, 3) on the plane $x + y + z = 7$ is



$$AC = \frac{(\mathbf{a}_2 - \mathbf{a}_1) \times \mathbf{n}}{|\mathbf{n}|}$$

$$= \frac{|(-\hat{i} - \hat{k}) \times (\hat{i} + \hat{j} + \hat{k})|}{|i + j + k|}$$

$$AC = \frac{|\hat{i} - \hat{k}|}{\sqrt{3}} \Rightarrow AC = \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2}}{3}$$

Alternative Method

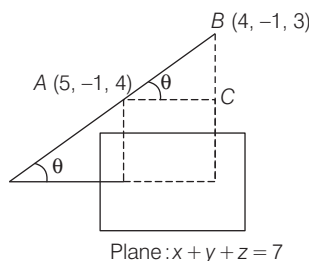
Clearly, DR's of AB are 4 - 5, -1 + 1, 3 - 4, i.e. -1, 0, -1

and DR's of normal to plane are 1, 1, 1.

Now, let θ be the angle between the line and plane, then θ is given by

$$\sin \theta = \frac{|-1 + 0 - 1|}{\sqrt{(-1)^2 + (-1)^2} \sqrt{1^2 + 1^2 + 1^2}}$$

$$= \frac{2}{\sqrt{2} \sqrt{3}} = \frac{\sqrt{2}}{\sqrt{3}}$$



$$\Rightarrow \sin \theta = \frac{\sqrt{2}}{\sqrt{3}}$$

$$\Rightarrow \cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \frac{2}{3}} = \frac{1}{\sqrt{3}}$$

Clearly, length of projection

$$= AB \cos \theta = \sqrt{2} \cdot \frac{1}{\sqrt{3}} \quad [\because AB = \sqrt{2}]$$

$$= \frac{\sqrt{2}}{\sqrt{3}}$$

10. (a) **Key idea** If any vector $\mathbf{x}$ is coplanar with the vector $\mathbf{y}$ and $\mathbf{z}$, then $\mathbf{x} = \lambda \mathbf{y} + \mu \mathbf{z}$

Here, $\mathbf{u}$ is coplanar with $\mathbf{a}$ and $\mathbf{b}$.

$$\therefore \mathbf{u} = \lambda \mathbf{a} + \mu \mathbf{b}$$

Dot product with $\mathbf{a}$, we get

$$\mathbf{u} \cdot \mathbf{a} = \lambda(\mathbf{a} \cdot \mathbf{a}) + \mu(\mathbf{b} \cdot \mathbf{a})$$

$$\Rightarrow 0 = 14\lambda + 2\mu \quad \dots(i)$$

$$[\because \mathbf{a} = 2\hat{i} + 3\hat{j} - \hat{k}, \mathbf{b} = \hat{j} + \hat{k}, \mathbf{u} \cdot \mathbf{a} = 0]$$

Dot product with $\mathbf{b}$, we get

$$\mathbf{u} \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b}) + \mu(\mathbf{b} \cdot \mathbf{b})$$

$$24 = 2\lambda + 2\mu \quad \dots(ii) \quad [\because \mathbf{u} \cdot \mathbf{b} = 24]$$

Solving Eqs. (i) and (ii), we get

$$\lambda = -2, \mu = 14$$

Dot product with $\mathbf{u}$, we get

$$|\mathbf{u}|^2 = \lambda(\mathbf{u} \cdot \mathbf{a}) + \mu(\mathbf{u} \cdot \mathbf{b})$$

$$|\mathbf{u}|^2 = -2(0) + 14(24)$$

$$\Rightarrow |\mathbf{u}|^2 = 336$$

11. (b) **Key idea** Use the theorem of total probability

Let E_1 = Event that first ball drawn is red

E_2 = Event that first ball drawn is black

A = Event that second ball drawn is red

$$P(E_1) = \frac{4}{10}, P\left(\frac{A}{E_1}\right) = \frac{6}{12}$$

$$\Rightarrow P(E_2) = \frac{6}{10}, P\left(\frac{A}{E_2}\right) = \frac{4}{12}$$

By law of total probability

$$P(A) = P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)$$

$$= \frac{4}{10} \times \frac{6}{12} + \frac{6}{10} \times \frac{4}{12}$$

$$= \frac{24 + 24}{120} = \frac{48}{120} = \frac{2}{5}$$

JEE Advanced

1. (c, d) We have,

$$P_1: 2x + y - z = 3$$

$$\text{and } P_2: x + 2y + z = 2$$

$$\text{Here, } \vec{n}_1 = 2\hat{i} + \hat{j} - \hat{k}$$

$$\text{and } \vec{n}_2 = \hat{i} + 2\hat{j} + \hat{k}$$

(a) Direction ratio of the line of

intersection of P_1 and P_2 is $\theta \vec{n}_1 \times \vec{n}_2$

$$\text{i.e. } \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (1 + 2)\hat{i} - (2 + 1)\hat{j} + (4 - 1)\hat{k}$$

$$= 3(\hat{i} - \hat{j} + \hat{k})$$

Hence, statement a is false.

(b) We have,

$$\frac{3x-4}{9} = \frac{1-3y}{9} = \frac{z}{3}$$

$$\Rightarrow \frac{x-\frac{4}{3}}{3} = \frac{\left(y-\frac{1}{3}\right)}{-3} = \frac{z}{3}$$

This line is parallel to the line of intersection of P_1 and P_2 .

Hence, statement (b) is false.

- (c) Let acute angle between P_1 and P_2 be θ .

We know that,

$$\cos \theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}$$

$$= \frac{(2\hat{i} + \hat{j} - \hat{k}) \cdot (\hat{i} + 2\hat{j} + \hat{k})}{|2\hat{i} + \hat{j} - \hat{k}| |\hat{i} + 2\hat{j} + \hat{k}|}$$

$$= \frac{2+2-1}{\sqrt{6} \times \sqrt{6}} = \frac{1}{2}$$

$$\theta = 60^\circ$$

Hence, statement (c) is true.

- (d) Equation of plane passing through the point $(4, 2, -2)$ and perpendicular to the line of intersection of P_1 and P_2 is

$$3(x-4) - 3(y-2) + 3(z+2) = 0$$

$$\Rightarrow 3x - 3y + 3z - 12 + 6 + 6 = 0$$

$$\Rightarrow x - y + z = 0$$

Now, distance of the point $(2, 1, 1)$ from the plane

$$x - y + z = 0 \text{ is}$$

$$D = \frac{|2-1+1|}{\sqrt{1+1+1}} = \frac{2}{\sqrt{3}}$$

Hence, statement (d) is true.

2. (a, b, d) We have,

$$(f(0))^2 + (f'(0))^2 = 85$$

and $f: R \rightarrow [-2, 2]$

- (a) Since, f is twice differentiable function, so f is continuous function.

$\therefore$ This is true for every continuous function.

Hence, we can always find $x \in (r, s)$, where $f(x)$ is one-one.

$\therefore$ This statement is true.

- (b) By L.M.V.T

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$\Rightarrow |f'(c)| = \left| \frac{f(b) - f(a)}{b - a} \right|$$

$$\Rightarrow |f'(x_0)| = \left| \frac{f(0) - f(-4)}{0 - (-4)} \right|$$

$$= \left| \frac{f(0) - f(-4)}{4} \right|$$

Range of f is $[-2, 2]$

$$\therefore -4 \leq f(0) - f(-4) \leq 4$$

$$\Rightarrow 0 \leq \left| \frac{f(0) - f(-4)}{4} \right| \leq 1$$

Hence, $|f'(x_0)| \leq 1$.

Hence, statement is true.

- (c) As no function is given, then we

$$\text{assume } f(x) = 2\sin\left(\frac{\sqrt{85}x}{2}\right)$$

$$\therefore f'(x) = \sqrt{85} \cos\left(\frac{\sqrt{85}x}{2}\right)$$

$$\text{Now, } (f(0))^2 + (f'(0))^2 = (2\sin 0)^2 + (\sqrt{85}\cos 0)^2$$

$$(f(0))^2 + (f'(0))^2 = 85$$

and $\lim_{x \rightarrow \infty} f(x)$ does not exist.

Hence, statement is false.

- (d) From option b, $|f'(x_0)| \leq 1$

and $x_0 \in (-4, 0)$

$$\therefore (f'(x_0))^2 \leq 1$$

Hence,

$$g(x_0) = (f(x_0))^2 + (f'(x_0))^2 \leq 4 + 1$$

$$[\because f(x_0) \in [-2, 2]]$$

$$\Rightarrow g(x_0) \leq 5$$

Now, let $p \in (-4, 0)$ for which

$$g(p) = 5$$

Similarly, let q be smallest positive number $q \in (0, 4)$

such that $g(q) = 5$

Hence, by Rolle's theorem is (p, q) $g'(c) = 0$ for $\alpha \in (-4, 4)$ and since $g(x)$ is greater than 5 as we move from $x = p$ to $x = q$

and $f(x)^2 \leq 4$

$$\Rightarrow (f'(x))^2 \geq 1 \text{ in } (p, q)$$

Thus, $g'(c) = 0$

$$\Rightarrow f'f + f'f'' = 0$$

So, $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$

Hence, statement is true.

3. (b, c) We have,

$$f'(x) = e^{(f(x) - g(x))} g'(x) \quad \forall x \in R$$

$$\Rightarrow f'(x) = \frac{e^{f(x)}}{e^{g(x)}} g'(x)$$

$$\Rightarrow \frac{f'(x)}{e^{f(x)}} = \frac{g'(x)}{e^{g(x)}}$$

$$\Rightarrow e^{-f(x)} f'(x) = e^{-g(x)} g'(x)$$

On integrating both side, we get

$$e^{-f(x)} = e^{-g(x)} + C$$

At $x = 1$

$$e^{-f(1)} = e^{-g(1)} + C$$

$$e^{-1} = e^{-g(1)} + C [\because f(1) = 1] \quad \dots(i)$$

At $x = 2$

$$e^{-f(2)} = e^{-g(2)} + C$$

$$\Rightarrow e^{-f(2)} = e^{-1} + C [\because g(2) = 1]$$

$\dots(ii)$

From Eqs. (i) and (ii)

$$e^{-f(2)} = 2e^{-1} - e^{-g(1)} \quad \dots(iii)$$

$$\Rightarrow e^{-f(2)} > 2e^{-1}$$

We know that, e^{-x} is decreasing

$$\therefore -f(2) < \log_e 2 - 1$$

$$f(2) > 1 - \log_e 2$$

$$\Rightarrow e^{-g(1)} + e^{-f(2)} = 2e^{-1} \quad [\text{from Eq. (iii)}]$$

$$\Rightarrow e^{-g(1)} < 2e^{-1}$$

$$-g(1) < \log_e 2 - 1$$

$$\Rightarrow g(1) > -\log_e 2$$

4. (b, c) We have,

$$f(x) = 1 - 2x + \int_0^x e^x - t f(t) dt$$

On multiplying e^{-x} both sides, we get

$$e^{-x} f(x) = e^{-x} - 2xe^{-x} + \int_0^x e^{-t} f(t) dt$$

On differentiating both side w.r.t. x , we get

$$e^{-x} f'(x) - e^{-x} f(x) = -e^{-x} - 2e^{-x} + 2xe^{-x} + e^{-x} f(x)$$

$$\Rightarrow f'(x) - 2f(x) = 2x - 3$$

[dividing both sides by e^{-x}]

$$\text{Let } f(x) = y$$

$$\Rightarrow f'(x) = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} - 2y = 2x - 3$$

which is linear differential equation of the form $\frac{dy}{dx} + Py = Q$. Here, $P = -2$ and

$$Q = 2x - 3.$$

$$\text{Now, IF} = e^{\int P dx} = e^{\int -2 dx} = e^{-2x}$$

$\therefore$ Solution of the given differential equation is

$$y \cdot e^{-2x} = \int (2x - 3) e^{-2x} dx + C$$

$$y \cdot e^{-2x} = \frac{-(2x - 3) \cdot e^{-2x}}{2}$$

$$+ 2 \int \frac{e^{-2x}}{2} dx + C$$

[by using integration by parts]

$$\Rightarrow y \cdot e^{-2x} = \frac{-(2x - 3) e^{-2x}}{2} - \frac{e^{-2x}}{2} + C$$

$$\Rightarrow y = (1 - x) + Ce^{2x}$$

On putting $x = 0$ and $y = 1$, we get

$$1 = 1 + C \Rightarrow C = 0$$

18 Engineering Entrances Solved Papers 2018

$$\therefore y = 1 - x$$

$y = 1 - x$ passes through $(2, -1)$

Now, area of region bounded by curve $y = \sqrt{1 - x^2}$ and $y = 1 - x$ is shown as

$$\begin{aligned} \therefore \text{Area of shaded region} &= \text{Area of 1st quadrant of a circle} \\ &\quad - \text{Area of } \triangle OAB \\ &= \frac{\pi}{4}(1)^2 - \frac{1}{2} \times 1 \times 1 \\ &= \frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4} \end{aligned}$$

Hence, options b and c are correct.

5. (2) We have,

$$\begin{aligned} \sin^{-1} \left(\sum_{i=1}^{\infty} x^{i+1} - x \sum_{i=1}^{\infty} \left(\frac{x}{2} \right)^i \right) \\ &= \frac{\pi}{2} - \cos^{-1} \left(\sum_{i=1}^{\infty} \left(\frac{-x}{2} \right)^i - \sum_{i=1}^{\infty} (-x)^i \right) \\ \Rightarrow \sin^{-1} \left[\frac{x^2}{1-x} - \frac{x \cdot \frac{x}{2}}{1 - \frac{x}{2}} \right] \\ &= \frac{\pi}{2} - \cos^{-1} \left[\frac{-x}{2} \cdot \frac{2}{1+x} - \frac{(-x)}{1+x} \right] \\ \left[\because \sum_{i=1}^{\infty} x^{i+1} = x^2 + x^3 + x^4 + \dots = \frac{x^2}{1-x} \right] \\ \text{using sum of infinite terms of GP} \end{aligned}$$

$$\begin{aligned} \Rightarrow \sin^{-1} \left[\frac{x^2}{1-x} - \frac{x^2}{2-x} \right] \\ &= \frac{\pi}{2} - \cos^{-1} \left[\frac{x}{1+x} - \frac{x}{2+x} \right] \\ \Rightarrow \sin^{-1} \left[\frac{x^2}{1-x} - \frac{x^2}{2-x} \right] \\ &= \sin^{-1} \left(\frac{x}{1+x} - \frac{x}{2+x} \right) \\ \left[\because \sin^{-1} x = \frac{\pi}{2} - \cos^{-1} x \right] \end{aligned}$$

$$\begin{aligned} \Rightarrow \frac{x^2}{1-x} - \frac{x^2}{2-x} &= \frac{x}{1+x} - \frac{x}{2+x} \\ \Rightarrow x^2 \left(\frac{2-x-1+x}{(1-x)(2-x)} \right) &= x \left(\frac{2+x-1-x}{(1+x)(2+x)} \right) \\ \Rightarrow \frac{x}{2-3x+x^2} &= \frac{1}{2+3x+x^2} \end{aligned}$$

or $x = 0$

$$\Rightarrow x^3 + 3x^2 + 2x = x^2 - 3x + 2$$

$$\Rightarrow x^3 + 2x^2 + 5x - 2 = 0 \text{ or } x = 0$$

$$\text{Let } f(x) = x^3 + 2x^2 + 5x - 2$$

$$f'(x) = 3x^2 + 4x + 5$$

$$f'(x) > 0, \forall x \in \mathbb{R}$$

$\therefore x^3 + 2x^2 + 5x - 2$ has only one real roots

Therefore, total number of real solution is 2.

6. (3) We have,

$$\vec{c} = x\vec{a} + y\vec{b} + \vec{a} \times \vec{b} \text{ and}$$

$$\vec{a} \cdot \vec{b} = 0$$

$$|\vec{a}| = |\vec{b}| = 1 \text{ and } |\vec{c}| = 2$$

Also, given $\vec{c}$ is inclined on $\vec{a}$ and $\vec{b}$ with same angle α

$$\therefore \vec{a} \cdot \vec{c} = x|\vec{a}|^2 + y(\vec{a} \cdot \vec{b}) + \vec{a} \cdot (\vec{a} \times \vec{b})$$

$$|\vec{a}| |\vec{c}| \cos \alpha = x + 0 + 0$$

$$x = 2 \cos \alpha$$

Similarly,

$$|\vec{b}| |\vec{c}| \cos \alpha = 0 + y + 0$$

$$\Rightarrow y = 2 \cos \alpha$$

$$|\vec{c}|^2 = x^2 + y^2 + |\vec{a} \times \vec{b}|^2$$

$$4 = 8 \cos^2 \alpha + |\vec{a}|^2 |\vec{b}|^2 \sin^2 90^\circ$$

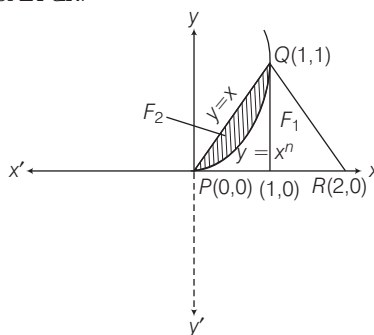
$$4 = 8 \cos^2 \alpha + 1$$

$$\Rightarrow 8 \cos^2 \alpha = 3$$

7. (4) We have,

$$y = x^n, n > 1$$

$\therefore P(0, 0)$ $Q(1, 1)$ and $R(2, 0)$ are vertices of $\triangle PQR$.



$\therefore$ Area of shaded region = 30% of area of $\triangle PQR$

$$\Rightarrow \int_0^1 (x - x^n) dx = \frac{30}{100} \times \frac{1}{2} \times 2 \times 1$$

$$\Rightarrow \left[\frac{x^2}{2} - \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{3}{10}$$

$$\Rightarrow \left(\frac{1}{2} - \frac{1}{n+1} \right) = \frac{3}{10}$$

$$\Rightarrow \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10} = \frac{2}{10} = \frac{1}{5}$$

$$\Rightarrow n+1 = 5 \Rightarrow n = 4$$

8. (d) We have,

$$f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{1}{1+(x+j)(x+j-1)} \right)$$

for all $x \in (0, \infty)$

$$\Rightarrow f_n(x) = \sum_{j=1}^n \tan^{-1} \left(\frac{(x+j) - (x+j-1)}{1+(x+j)(x+j-1)} \right)$$

$$\Rightarrow f_n(x) = \sum_{j=1}^n [\tan^{-1}(x+j) - \tan^{-1}(x+j-1)]$$

$$\begin{aligned} \Rightarrow f_n(x) &= (\tan^{-1}(x+1) - \tan^{-1}x) \\ &\quad + (\tan^{-1}(x+2) - \tan^{-1}(x+1)) \\ &\quad + (\tan^{-1}(x+3) - \tan^{-1}(x+2)) \\ &\quad + \dots + (\tan^{-1}(x+n) - \tan^{-1}(x+n-1)) \end{aligned}$$

$$\Rightarrow f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$$

This statement is false as $x \neq 0$, i.e., $x \in (0, \infty)$

(b) This statement is also false as $0 \notin (0, \infty)$

$$(c) f_n(x) = \tan^{-1}(x+n) - \tan^{-1}x$$

$$\lim_{x \rightarrow \infty} \tan(f_n(x)) = \lim_{x \rightarrow \infty} \tan(\tan^{-1}(x+n) - \tan^{-1}x)$$

$$\begin{aligned} \Rightarrow \lim_{x \rightarrow \infty} \tan(f_n(x)) &= \lim_{x \rightarrow \infty} \tan \left(\tan^{-1} \frac{n}{1+nx+x^2} \right) \\ &= \lim_{x \rightarrow \infty} \frac{n}{1+nx+x^2} = 0 \end{aligned}$$

$\therefore$ (c) statement is false.

$$(d) \lim_{x \rightarrow \infty} \sec^2(f_n(x))$$

$$= \lim_{x \rightarrow \infty} (1 + \tan^2 f_n(x))$$

$$= 1 + \lim_{x \rightarrow \infty} \tan^2(f_n(x))$$

$$= 1 + 0 = 1$$

$\therefore$ (d) statement is true

9. (a, d) We have,

$$-x + 2y + 5z = b_1$$

$$2x - 4y + 3z = b_2$$

$$x - 2y + 2z = b_3$$

$$\therefore D = \begin{vmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{vmatrix}$$

$$\text{and } D_1 = D_2 = D_3 = 0$$

$$\Rightarrow D_1 = \begin{vmatrix} b_1 & 2 & 5 \\ b_2 & -4 & 3 \\ b_3 & -2 & 2 \end{vmatrix}$$

$$= -2b_1 - 14b_2 + 26b_3 = 0$$

$$\Rightarrow b_1 + 7b_2 = 13b_3 \quad \dots(i)$$

$$(a) D = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 1 & 2 & 6 \end{vmatrix}$$

$$= 1(24 - 10) + 1(10 - 12) \\ = 14 - 2 = 12 \neq 0$$

Here, $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3 .

$$(b) D = \begin{vmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{vmatrix} \\ = 1(-6 + 6) - 1(-15 + 12) \\ + 3(-5 + 4) = 0$$

For atleast one solution

$$D_1 = D_2 = D_3 = 0$$

$$\text{Now, } D_1 = \begin{vmatrix} b_1 & 1 & 3 \\ b_2 & 2 & 6 \\ b_3 & -1 & -3 \end{vmatrix} = b_1(-6 + 6) \\ - b_2(-3 + 3) + b_3(6 - 6) \\ = 0$$

$$D_2 = \begin{vmatrix} 1 & b_1 & 3 \\ 5 & b_2 & 6 \\ -2 & b_3 & -3 \end{vmatrix} \\ = -b_1(-15 + 12) + b_2(-3 + 6) \\ - b_3(6 - 15) \\ = 3b_1 + 3b_2 + 9b_3 = 0 \\ \Rightarrow b_1 + b_2 + 3b_3 = 0$$

not satisfies the Eq. (i)

$\therefore$ It has no solution.

$$(c) D = \begin{vmatrix} -1 & 2 & -5 \\ 2 & -4 & 10 \\ 1 & -2 & 5 \end{vmatrix} \\ = -1(-20 + 20) - 2(10 - 10) \\ - 5(-4 + 4) \\ = 0$$

Here, $b_2 = -2b_1$ and $b_3 = -b_1$ satisfies the Eq. (i)

Planes are parallel.

$$(d) D = \begin{vmatrix} 1 & 2 & 5 \\ 2 & 0 & 3 \\ 1 & 4 & -5 \end{vmatrix} \\ = 1(0 - 12) - 2(-10 - 3) + 5(8 - 0) \\ = 54$$

$D \neq 0$

$\therefore$ It has unique solution for any b_1, b_2, b_3 .

10. (b, c, d) Given,

$$\lim_{t \rightarrow x} \frac{f(x) \sin t - f(t) \sin x}{t - x} = \sin^2 x$$

Using L'Hospital rules

$$\lim_{t \rightarrow x} \frac{f(x) \cos t - f'(t) \sin x}{1} = \sin^2 x$$

$$\Rightarrow f(x) \cos x - f'(x) \sin x = \sin^2 x$$

$$\Rightarrow f'(x) \sin x - f(x) \cos x = -\sin^2 x$$

$$\Rightarrow \frac{f'(x) \sin x - f(x) \cos x}{\sin^2 x} = -1$$

$$\Rightarrow d \left(\frac{f(x)}{\sin x} \right) = -1$$

On integrating, we get

$$\frac{f(x)}{\sin x} = -x + C$$

$$\Rightarrow f(x) = -x \sin x + C \sin x$$

$$\text{It is given that } x = \frac{\pi}{6}, f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$$

$$\therefore f\left(\frac{\pi}{6}\right) = -\frac{\pi}{6} \sin \frac{\pi}{6} + C \sin \frac{\pi}{6} \\ = -\frac{\pi}{12} = -\frac{\pi}{12} + \frac{1}{2}C$$

$$\Rightarrow C = 0$$

$$\therefore f(x) = -x \sin x$$

$$(a) f(x) = -x \sin x$$

$$f\left(\frac{\pi}{4}\right) = -\frac{\pi}{4} \sin \frac{\pi}{4} = -\frac{\pi}{4\sqrt{2}} \text{ false}$$

$$(b) f(x) = -x \sin x$$

$$\sin x > x - \frac{x^3}{6}, \forall x \in (0, \pi)$$

$$\Rightarrow -x \sin x < -x^2 + \frac{x^4}{6}$$

$$\Rightarrow f(x) < \frac{x^4}{6} - x^2, \forall x \in (0, \pi)$$

It is true

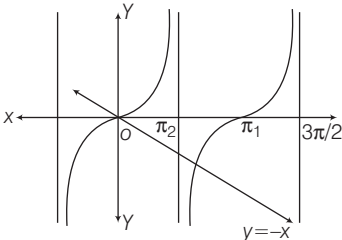
$$(c) f(x) = -x \sin x$$

$$f'(x) = -\sin x - x \cos x$$

$$f''(x) = 0$$

$$\Rightarrow -\sin x - x \cos x = 0$$

$$\tan x = -x$$



$\Rightarrow$ There exists $\alpha \in (0, \pi)$ for which

$$f'(\alpha) = 0$$

It is true

$$(d) f(x) = -x \sin x$$

$$f'(x) = -\sin x - x \cos x$$

$$f''(x) = -2 \cos x + x \sin x$$

$$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}, f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$

$$\therefore f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$$

It is true.

$$11. (2) \text{ Let } I = \int_0^{1/2} \frac{1 + \sqrt{3}}{[(x+1)^2(1-x)^6]^{1/4}} dx$$

$$\Rightarrow I = \int_0^{1/2} \frac{1 + \sqrt{3}}{(1-x)^2 \left[\frac{1-x}{1+x} \right]^{1/4}} dx$$

$$\text{Put } \frac{1-x}{1+x} = t \Rightarrow \frac{-2 dx}{(1+x)^2} = dt$$

$$\text{when } x=0, t=1, x=\frac{1}{2}, t=\frac{1}{3}$$

$$\therefore I = \int_1^{1/3} \frac{(1+\sqrt{3}) dt}{-2t^{5/4}}$$

$$\Rightarrow I = \frac{-(1+\sqrt{3})}{2} \left[\frac{-2}{\sqrt{t}} \right]_1^{1/3}$$

$$\Rightarrow I = (1+\sqrt{3})(\sqrt{3}-1)$$

$$\Rightarrow I = 3 - 1 = 2$$

$$12. (4) \text{ Let } \text{Det}(P) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) \\ + a_3(b_1c_2 - b_2c_1)$$

Now, maximum value of $\text{Det}(P) = 6$

If $a_1 = 1, a_2 = -1, a_3 = 1,$

$$b_2c_3 = b_1c_3 = b_1c_2 = 1$$

$$\text{and } b_3c_2 = b_3c_1 = b_2c_1 = -1$$

But it is not possible as

$$(b_2c_3)(b_3c_1)(b_1c_2) = -1$$

$$\text{and } (b_1c_3)(b_3c_2)(b_2c_1) = 1$$

i.e., $b_1b_2b_3c_1c_2c_3 = 1$ and -1

Similar contradiction occurs when

$$a_1 = 1, a_2 = 1, a_3 = 1,$$

$$b_2c_1 = b_3c_1 = b_1c_2 = 1$$

$$\text{and } b_3c_2 = b_1c_3 = b_1c_2 = -1$$

Now, for value to be 5 one of the terms

must be zero but that will make 2

terms zero which means answer

cannot be 5

$$\text{Now, } \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 4$$

Hence, maximum value is 4.

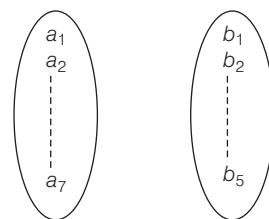
13. (119) Given, X has exactly 5 elements and Y has exactly 7 elements.

$$\therefore n(X) = 5 \text{ and } n(Y) = 7$$

Now, number of one-one functions from X to Y is

$$\alpha = {}^7P_5 = {}^7C_5 \times 5!$$

Number of onto functions from Y to X is β



$$1, 1, 1, 1, 3 \text{ or } 1, 1, 1, 2, 2$$

$$\therefore \beta = \frac{7!}{3!4!} \times 5! + \frac{7!}{(2!)^3 3!} \times 5! \\ = ({}^7C_3 + 3 \cdot {}^7C_3) 5! \\ = 4 \times {}^7C_3 \times 5!$$

20 Engineering Entrances Solved Papers 2018

$$\therefore \frac{\beta - \alpha}{5!} = \frac{(4 \times {}^7C_3 - {}^7C_5) 5!}{5!}$$

$$= 4 \times 35 - 21 = 140 - 21 = 119$$

14. (0.4) We have,

$$\frac{dy}{dx} = (2 + 5y)(5y - 2)$$

$$\Rightarrow \frac{dy}{25y^2 - 4} = dx$$

$$\Rightarrow \frac{1}{25} \left(\frac{dy}{y^2 - \frac{4}{25}} \right) = dx$$

On integrating both sides, we get

$$\frac{1}{25} \int \frac{dy}{y^2 - \left(\frac{2}{5}\right)^2} = \int dx$$

$$\Rightarrow \frac{1}{25} \times \frac{1}{2 \times \frac{2}{5}} \log \left| \frac{y - 2/5}{y + 2/5} \right| = x + C$$

$$\Rightarrow \log \left| \frac{5y - 2}{5y + 2} \right| = 20(x + C)$$

$$\Rightarrow \left| \frac{5y - 2}{5y + 2} \right| = A e^{20x} \quad [\because e^{20C} = A]$$

when $x = 0 \Rightarrow y = 0$, then $A = 1$

$$\therefore \left| \frac{5y - 2}{5y + 2} \right| = e^{20x}$$

$$\lim_{x \rightarrow -\infty} \left| \frac{5f(x) - 2}{5f(x) + 2} \right| = \lim_{x \rightarrow -\infty} e^{20x}$$

$$\Rightarrow \lim_{n \rightarrow -\infty} \left| \frac{5f(x) - 2}{5f(x) + 2} \right| = 0$$

$$\Rightarrow \lim_{n \rightarrow -\infty} 5f(x) - 2 = 0$$

$$\Rightarrow \lim_{n \rightarrow -\infty} f(x) = \frac{2}{5} = 0.4$$

15. (2) Given,

$$f(x + y) = f(x)f'(y) + f'(x)f(y),$$

$\forall x, y \in R$

and $f(0) = 1$

Put $x = y = 0$, we get

$$f(0) = f(0)f'(0) + f'(0)f(0)$$

$$\Rightarrow 1 = 2f'(0) \Rightarrow f'(0) = \frac{1}{2}$$

Put $x = x$ and $y = 0$, we get

$$f(x) = f(x)f'(0) + f'(x)f(0)$$

$$\Rightarrow f(x) = \frac{1}{2}f(x) + f'(x)$$

$$\Rightarrow f'(x) = \frac{1}{2}f(x) \Rightarrow \frac{f'(x)}{f(x)} = \frac{1}{2}$$

On integrating, we get

$$\log f(x) = \frac{1}{2}x + C$$

$$\Rightarrow f(x) = A e^{\frac{1}{2}x}, \text{ where } e^C = A$$

If $f(0) = 1$, then $A = 1$

Hence, $f(x) = e^{\frac{1}{2}x}$

$$\Rightarrow \log_e f(x) = \frac{1}{2}x$$

$$\Rightarrow \log_e f(4) = \frac{1}{2} \times 4 = 2$$

16. (8) Let $P(\alpha, \beta, \gamma)$ and R is image of P in the XY -plane.

$$\therefore R(\alpha, \beta, -\gamma)$$

Also, Q is the image of P in the plane $x + y = 3$

$$\therefore \frac{x - \alpha}{1} = \frac{y - \beta}{1} = \frac{z - \gamma}{0}$$

$$= \frac{-2(\alpha + \beta - 3)}{2}$$

$$x = 3 - \beta, y = 3 - \alpha, z = \gamma$$

Since, Q lies on Z -axis

$$\therefore \beta = 3, \alpha = 3, z = \gamma$$

$$\therefore P(3, 3, \gamma)$$

Given, distance of P from X -axis be 5

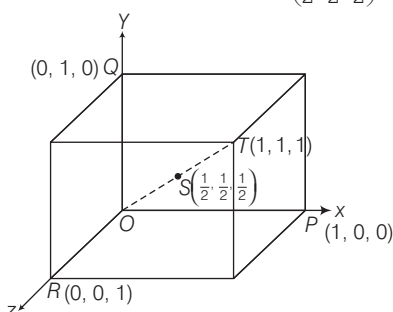
$$\therefore 5 = \sqrt{3^2 + \gamma^2}$$

$$25 - 9 = \gamma^2 \Rightarrow \gamma = \pm 4$$

$$\text{Then, } PR = |2\gamma| = |2 \times 4| = 8$$

17. (0.5) Here, $P(1, 0, 0)$, $Q(0, 1, 0)$,

$$R(0, 0, 1), T(1, 1, 1) \text{ and } S = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$



$$\text{Now, } \vec{p} = \vec{SP} = \vec{OP} - \vec{OS}$$

$$= \left(\frac{1}{2}\hat{i} - \frac{1}{2}\hat{j} - \frac{1}{2}\hat{k}\right)$$

$$= \frac{1}{2}(\hat{i} - \hat{j} - \hat{k})$$

$$\vec{q} = \vec{SQ} = \frac{1}{2}(-\hat{i} + \hat{j} - \hat{k})$$

$$\vec{r} = \vec{SR} = \frac{1}{2}(-\hat{i} - \hat{j} + \hat{k})$$

$$\text{and } \vec{t} = \vec{ST} = \frac{1}{2}(\hat{i} + \hat{j} + \hat{k})$$

$$\vec{p} \times \vec{q} = \frac{1}{4} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & -1 \\ -1 & 1 & -1 \end{vmatrix} = \frac{1}{4}(2\hat{i} + 2\hat{j})$$

$$\text{and } \vec{r} \times \vec{t} = \frac{1}{4} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{4}(-2\hat{i} + 2\hat{j})$$

Now,

$$(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t}) = \frac{1}{16} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 0 \\ -2 & 2 & 0 \end{vmatrix}$$

$$= \frac{1}{16}(8\hat{k}) = \frac{1}{2}\hat{k}$$

$$\therefore |(\vec{p} \times \vec{q}) \times (\vec{r} \times \vec{t})| = \left|\frac{1}{2}\hat{k}\right| = \frac{1}{2} = 0.5$$

18. (a) We have,

$$E_1 = \left\{ x \in R : x \neq 1 \text{ and } \frac{x}{x-1} > 0 \right\}$$

$$\therefore \frac{x}{x-1} > 0$$

$$\begin{array}{c} + \quad - \quad + \\ \hline 0 \quad 1 \end{array}$$

$$E_1 = x \in (-\infty, 0) \cup (1, \infty)$$

and

$$E_2 = \left\{ x \in E_1 : \sin^{-1} \left(\log_e \left(\frac{x}{x-1} \right) \right) \right. \\ \left. \text{is a real number} \right\}$$

$$E_2 = -1 \leq \log_e \frac{x}{x-1} \leq 1$$

$$\Rightarrow e^{-1} \leq \frac{x}{x-1} \leq e$$

$$\text{Now, } \frac{x}{x-1} \geq e^{-1} \Rightarrow \frac{x}{x-1} - \frac{1}{e} \geq 0$$

$$\Rightarrow \frac{ex - x + 1}{e(x-1)} \geq 0$$

$$\Rightarrow \frac{x(e-1) + 1}{(x-1)e} \geq 0$$

$$\Rightarrow x \in \left(-\infty, \frac{1}{1-e}\right] \cup (1, \infty)$$

$$\text{Also, } \frac{x}{x-1} \leq e \Rightarrow \frac{(e-1)x - e}{x-1} \geq 0$$

$$\Rightarrow x \in (-\infty, 1) \cup \left[\frac{e}{e-1}, \infty\right)$$

$$\text{So, } E_2 = \left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$$

$\therefore$ The domain of f and g are

$$\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$$

and Range of $\frac{x}{x-1}$ is $R^+ - \{1\}$

$\Rightarrow$ Range of f is $R - \{0\}$ or $(-\infty, 0) \cup (0, \infty)$

Range of g is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$ or

$$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$$

Now, $P \rightarrow 4, Q \rightarrow 2, R \rightarrow 1, S \rightarrow 1$

Hence, option (a) is correct answer.

19. (d)

(i) Given,

$$f_1: R \rightarrow R \text{ and } f_1(x) = \sin(\sqrt{1-e^{-x^2}})$$

$\therefore f_1(x)$ is continuous at $x = 0$

Now, $f_1'(x)$

$$= \cos \sqrt{1-e^{-x^2}} \cdot \frac{1}{2\sqrt{1-e^{-x^2}}} (2xe^{-x^2})$$

At $x = 0$

$f_1'(x)$ does not exist.

$\therefore f_1(x)$ is not differential at $x = 0$

Hence, option (2) for P .

$$(ii) \text{ Given, } f_2(x) = \begin{cases} \frac{|\sin x|}{\tan^{-1} x}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$$

$$\Rightarrow f_2(x) = \begin{cases} \frac{-\sin x}{\tan^{-1} x} & x < 0 \\ \frac{\sin x}{\tan^{-1} x} & x > 0 \\ 1 & x = 0 \end{cases}$$

Clearly, $f_2(x)$ is not continuous at $x = 0$.

$\therefore$ Option (1) for Q .

(iii) Given, $f_3(x) = [\sin(\log_e(x+2))]$, where $[\]$ is G.I.F.

and $f_3: (-1, e^{\pi/2} - 2) \rightarrow R$

It is given

$$-1 < x < e^{\pi/2} - 2$$

$$\Rightarrow -1 + 2 < x + 2 < e^{\pi/2} - 2 + 2$$

$$\Rightarrow 1 < x + 2 < e^{\pi/2}$$

$$\Rightarrow \log_e 1 < \log_e(x+2) < \log_e e^{\pi/2}$$

$$\Rightarrow 0 < \log_e(x+2) < \frac{\pi}{2}$$

$$\Rightarrow \sin 0 < \sin \log_e(x+2) < \sin \frac{\pi}{2}$$

$$\Rightarrow 0 < \sin \log_e(x+2) < 1$$

$$\therefore [\sin \log_e(x+2)] = 0$$

$$\therefore f_3(x) = 0, f_3'(x) = f_3''(x) = 0$$

It is differentiable and continuous at $x = 0$.

$\therefore$ Option (4) for R

$$(iv) \text{ Given, } f_4(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\text{Now, } \lim_{x \rightarrow 0} f_4(x) = \lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x}\right) = 0$$

$$f_4'(x) = 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$$\text{For } x = 0, f_4'(x) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h}$$

$$\Rightarrow f_4'(x) = \lim_{h \rightarrow 0} \frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h}$$

$$\Rightarrow f_4'(x) = \lim_{h \rightarrow 0} h \sin\left(\frac{1}{h}\right) = 0$$

Thus,

$$f_4'(x) = \begin{cases} 2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Again, $\lim_{x \rightarrow 0}$

$$f'(x) = \lim_{x \rightarrow 0} \left(2x \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)\right)$$

does not exist.

Since, $\lim_{x \rightarrow 0} \cos\left(\frac{1}{x}\right)$ does not

exist.

Hence, $f'(x)$ is not continuous at $x = 0$.

$\therefore$ Option (3) for S .

BITSAT

$$1. (a) \text{ Given, } P = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

$$\therefore P^T = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$$

Also given, $P^T P = I$

$$\therefore |P^T P| = |I| = 1$$

$$\text{or } \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = 1$$

$$\Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = \pm 1$$

$$\text{or } 3abc - a^3 - b^3 - c^3 = \pm 1$$

$$\text{or } 3 \mp 1 = a^3 + b^3 + c^3$$

$$\therefore a^3 + b^3 + c^3 = 2 \text{ or } 4$$

$\therefore$ Option (a) is correct.

$$2. (d) \text{ Given, } x^2 = \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 1 \\ \sin \theta & \cos \theta & 2 \end{vmatrix}$$

$$\text{On expanding } \begin{vmatrix} \sin \theta & \cos \theta & 0 \\ -\cos \theta & \sin \theta & 1 \\ \sin \theta & \cos \theta & 2 \end{vmatrix}$$

along C_3 , we get

$$x^2 = 0 - 1 \begin{vmatrix} \sin \theta & \cos \theta \\ \sin \theta & \cos \theta \end{vmatrix} + 2 \begin{vmatrix} \sin \theta & \cos \theta \\ -\cos \theta & \sin \theta \end{vmatrix}$$

$$= -1(\sin \theta \cos \theta - \sin \theta \cos \theta) + 2(\sin^2 \theta + \cos^2 \theta)$$

$$= -1 \times 0 + 2 \times 1$$

$$\Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2}$$

If $x = \sqrt{2}$, then

$$4x^2 + x \sin \frac{3\pi}{2} + 5$$

$$= 4 \times (\sqrt{2})^2 - \sqrt{2} + 5$$

$$= 8 - \sqrt{2} + 5 = (13 - \sqrt{2})$$

If $x = -\sqrt{2}$, then

$$4x^2 + x \sin \frac{3\pi}{2} + 5$$

$$= 4 \times (-\sqrt{2})^2 - \sqrt{2} + 5$$

$$= 8 + \sqrt{2} + 5 = (13 + \sqrt{2})$$

3. (a) Let $f(x) = y$, then

$$a^y = x + \sqrt{x^2 + 1}$$

$$\Rightarrow a^{-y} = \frac{1}{x + \sqrt{x^2 + 1}}$$

$$\Rightarrow a^{-y} = \frac{x - \sqrt{x^2 + 1}}{-1} \quad [\text{rationalising}]$$

$$\therefore a^y - a^{-y} = 2x \Rightarrow x = \frac{1}{2}(a^y - a^{-y})$$

$$\therefore f^{-1}(y) = \frac{1}{2}(a^y - a^{-y})$$

$$[\because f(x) = y \Rightarrow x = f^{-1}(y)]$$

$$\Rightarrow f^{-1}(x) = \frac{1}{2}(a^x - a^{-x})$$

[replacing y by x]

Hence, option (a) is correct.

$$4. (c) \text{ Given, } S = \sum_{n=1}^{\infty} \tan^{-1} \frac{2n}{n^4 + n^2 + 2} \dots (i)$$

$$\text{Let } n^4 + n^2 + 1$$

$$= [(n^2)^2 + 1^2 + 2(n^2)(1)] - n^2$$

$$= (n^2 + 1)^2 - n^2$$

$$= (n^2 + n + 1)(n^2 - n + 1) \dots (ii)$$

$$\text{Let } u_n = \tan^{-1} \left(\frac{2n}{n^4 + n^2 + 2} \right)$$

$$= \tan^{-1} \frac{2n}{1 + (n^4 + n^2 + 1)}$$

$$= \tan^{-1} \frac{(n^2 + n + 1) - (n^2 - n + 1)}{1 + (n^2 + n + 1)(n^2 - n + 1)}$$

$$u_n = \tan^{-1}(n^2 + n + 1) - \tan^{-1}(n^2 - n + 1) \dots (iii)$$

On putting $n = 1, 2, 3, \dots$ successively in Eq. (iii), we get

$$u_1 = \tan^{-1} 3 - \tan^{-1} 1$$

22 Engineering Entrances Solved Papers 2018

$$\begin{aligned} u_2 &= \tan^{-1} 7 - \tan^{-1} 3 \\ u_3 &= \tan^{-1} 13 - \tan^{-1} 7 \\ &\dots\dots\dots \\ u_n &= \tan^{-1}(n^2 + n + 1) \\ &\quad - \tan^{-1}(n^2 - n + 1) \end{aligned}$$

On adding vertically, we get

$$\sum_{n=1}^{\infty} u_n = \tan^{-1}(n^2 + n + 1) - \tan^{-1} 1$$

$$S = \lim_{n \rightarrow \infty} \sum_{n=1}^{\infty} u_n \quad [\text{from Eq. (i)}]$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \tan^{-1}(n^2 + n + 1) - \tan^{-1} 1 \\ &= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \end{aligned}$$

5. (d) Let $y = mx + c$, intersect $y^2 = 4ax$ at $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$

Then,

$$\frac{2}{t_1 + t_2} = m \Rightarrow t_1 + t_2 = \frac{2}{m}$$

Let the foot of another normal be $C(at_3^2, 2at_3)$.

Then, $t_1 + t_2 + t_3 = 0$

$$t_3 = (t_1 + t_2) = -\frac{2}{m}$$

Thus, other foot is $\left(\frac{4a}{m^2}, \frac{-4a}{m}\right)$.

Hence, option (d) is correct.

Given,

6. (d) Let distances of a point $p(x, y, z)$ from the planes

$$x + y + z = 0, x - z = 0 \text{ and}$$

$$x - 2y + z = 0 \text{ are}$$

$$\frac{x + y + z}{\sqrt{3}}, \frac{x - z}{\sqrt{2}} \text{ and } \frac{x - 2y + z}{\sqrt{6}}$$

respectively, then

the sum of the squares of distances, is as

$$\Rightarrow \left(\frac{x + y + z}{\sqrt{3}}\right)^2 + \left(\frac{x - z}{\sqrt{2}}\right)^2 + \left(\frac{x - 2y + z}{\sqrt{6}}\right)^2 = p^2$$

$$\Rightarrow 2(x + y + z)^2 + 3(x - z)^2 + (x - 2y + z)^2 = 6p^2$$

$$\Rightarrow 2x^2 + 2y^2 + 2z^2 + 4xy + 4yz + 4zx + 3x^2$$

$$+ 3z^2 - 6xz + x^2 + 4y^2 + z^2 - 4xy - 4yz + 2xz = 6p^2$$

$$\Rightarrow 6x^2 + 6y + 6z^2 = 6p^2$$

$$\Rightarrow x^2 + y^2 + z^2 = p^2$$

Hence, option (d) is correct.

7. (a) When line

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1}$$

is perpendicular to plane $ax + by + cz + d = 0$, then

$$\frac{a_1}{a} = \frac{-b_1}{b} = \frac{c_1}{c}$$

$$\therefore \frac{\lambda}{2} = \frac{1}{2} = \frac{-4}{-8} \Rightarrow \lambda = 1$$

Hence, option (a) is correct.

8. (a) Since, given planes are perpendicular to each other, i.e. its normal are perpendicular

$$\therefore 2(\lambda) - \lambda(5) + 3(-1) = 0$$

$$\Rightarrow -3\lambda - 3 = 0 \Rightarrow \lambda = -1$$

$$\therefore \lambda^2 + \lambda = (-1)^2 - 1 = 0$$

Hence, option (a) is correct.

9. (d) Put $x + y = v$ and $\frac{dv}{dx} = 1 + \frac{dy}{dx}$ in the given differential equation.

$$\therefore \frac{dv}{dx} = 1 + \sin v + \cos v$$

$$\Rightarrow \frac{dv}{2\cos^2 \frac{v}{2} + 2\sin \frac{v}{2} \cos \frac{v}{2}} = dx$$

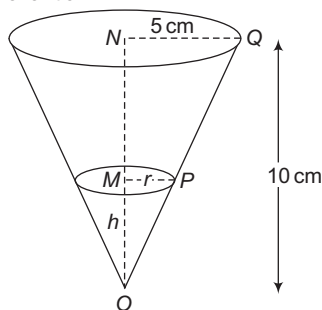
$$\Rightarrow \frac{\frac{1}{2} \sec^2 \frac{v}{2}}{1 + \tan \frac{v}{2}} dv = dx$$

$$\Rightarrow \log\left(1 + \tan \frac{x + y}{2}\right) = x + C$$

[by integration]

Hence, option (d) is correct.

10. (b) Let depth of water at time t be h and the radius of the base of water level be r .



From similar $\triangle OMP$ and $\triangle ONQ$, we have

$$\frac{OM}{ON} = \frac{PM}{QN} \Rightarrow \frac{h}{10} = \frac{r}{5}$$

$$\Rightarrow r = \frac{5h}{10} \Rightarrow r = \frac{h}{2}$$

$$\text{We have, } V = \frac{1}{3} \pi r^2 h = \frac{1}{3} \pi \left(\frac{h}{2}\right)^2 h$$

$$\Rightarrow V = \frac{\pi h^3}{12}$$

On differentiating both sides, we get

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3h^2 \frac{dh}{dt} = \frac{\pi h^2}{4} \frac{dh}{dt}$$

$$\text{Given, } \frac{dV}{dt} = 3 \text{ cm}^3/\text{sec} \text{ when } h = 4 \text{ cm,}$$

so we get

$$3 = \frac{\pi \times 4^2}{4} \frac{dh}{dt}$$

$$\Rightarrow \frac{dh}{dt} = \frac{3}{4\pi} \text{ cm/sec}$$

Hence, the water level is rising at

$$\frac{3}{4\pi} \text{ cm/sec(c)}$$

$\therefore$ Option (b) is correct.

11. (b) We have,

$$\frac{dy}{dx} = y + 2x \Rightarrow \frac{dy}{dx} - y = 2x$$

$$\text{Now, IF} = e^{-\int 1 dx} = e^{-x}$$

$$y \cdot e^{-x} = \int 2x e^{-x} dx + k, k \text{ be the}$$

constant of integration

$$= 2[x \int e^{-x} dx - \int 1 \cdot (-e^{-x}) dx] + k$$

[using integration by parts]

$$\Rightarrow y \cdot e^{-x} = -2xe^{-x} - 2e^{-x} + k \dots(i)$$

As curve (i) passes through (0, 0)

$$\therefore 0 = 0 - 2 + k$$

$$\Rightarrow k = 2$$

Thus, the curve is

$$ye^{-x} = -2xe^{-x} - 2e^{-x} + 2$$

$$\therefore y = 2(e^x - x - 1)$$

Hence, option (b) is correct.

12. (c) $f(0) = 0$

For $f(x)$ to be continuous at $x = 0$

$$\lim_{x \rightarrow 0} f(x) = 0$$

$$\therefore \lim_{x \rightarrow 0} x^p \sin \frac{1}{x} = 0$$

This possible only when $p > 0 \dots(ii)$

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{h^p \sin \frac{1}{h} - 0}{h}$$

$$= \lim_{h \rightarrow 0} h^{p-1} \sin \frac{1}{h}$$

$$\Rightarrow f'(0) \text{ will exist only when } p > 0$$

$\therefore f(x)$ will not be differentiable if

$$p \leq 1 \dots(iii)$$

From Eqs. (i) and (ii), for $f(x)$ to be not differentiable but continuous at $x = 0$, possible values of p are given by $0 < p \leq 1$. Hence option (c) is correct.

13. (a) Integrating the given differential equation, we have

$$\frac{dy}{dx} = \frac{-\cos 3x}{3} + e^x + \frac{x^3}{3} + C_1$$

But $y_1(0) = 1$

$$\text{So, } 1 = \left(-\frac{1}{3}\right) + 1 + C_1 \Rightarrow C_1 = \frac{1}{3}$$

$$\therefore \frac{dy}{dx} = \frac{-\cos 3x}{3} + e^x + \frac{x^3}{3} + \frac{1}{3}$$

Again integrating, we get

$$y = -\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x + C_2$$

But $y(0) = 0$, so $0 = 0 + 1 + C_2$

$$\Rightarrow C_2 = -1$$

$$\text{Thus, } y = -\frac{\sin 3x}{9} + e^x + \frac{x^4}{12} + \frac{1}{3}x - 1$$

Hence, option (a) correct.

14. (d) Given, $f(x) = -2x^3 - 9x^2 - 12x + 1$

$$\Rightarrow f'(x) = -6x^2 - 18x - 12$$

To be decreasing $f'(x) < 0$

$$\Rightarrow -6x^2 - 18x - 12 < 0$$

$$\Rightarrow x^2 + 3x + 2 > 0$$

$$\Rightarrow (x+2)(x+1) > 0$$

Therefore, either $x < -2$ or $x > -1$

$$\Rightarrow x \in (-1, \infty) \text{ or } (-\infty, -2)$$

Hence, option (d) is correct.

15. (c) Let $\mathbf{a} = 4\hat{i} - 4\hat{k}$ and $\mathbf{b} = \hat{i} + \hat{j} + \hat{k}$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$

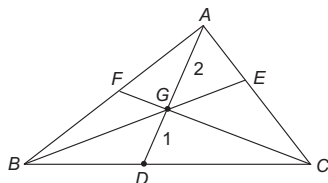
$$\begin{aligned} \therefore \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{(4\hat{i} - 4\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})}{|\mathbf{a}| |\mathbf{b}|} \\ &= \frac{4 + 0 - 4}{|\mathbf{a}| |\mathbf{b}|} = 0 \end{aligned}$$

$$\cos \theta = \cos 90^\circ \Rightarrow \theta = 90^\circ$$

$$\therefore \sin \theta + \cos \theta = \cos 90^\circ + \sin 90^\circ = 0 + 1 = 1$$

Hence, option (c) is correct.

16. (d)



$$\begin{aligned} \therefore \vec{AD} &= 3\vec{GD} \\ &= 3 \cdot \frac{1}{2}(\vec{GB} + \vec{GC}) \\ &= \frac{3}{2} \left(\frac{2}{3}\vec{EB} + \frac{2}{3}\vec{FC} \right) \\ &= -\vec{BE} - \vec{CF} \end{aligned}$$

Hence, option (d) is correct.

17. (c) Volume of the parallelepiped

$$= |[\mathbf{a} \mathbf{b} \mathbf{c}]|$$

$\Rightarrow$ (area of the base parallelogram)

$$\times h = |[\mathbf{a} \mathbf{b} \mathbf{c}]|$$

$$\Rightarrow |\mathbf{a} \times \mathbf{b}| h = |[\mathbf{a} \mathbf{b} \mathbf{c}]|$$

$$\Rightarrow |-5\hat{i} + 3\hat{j} + 2\hat{k}| h = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & -1 \\ 1 & 1 & 3 \end{vmatrix}$$

$$\Rightarrow \sqrt{38} h = 4 \Rightarrow h = \frac{4}{\sqrt{38}}$$

$$\Rightarrow 38h^2 = 4^2 \Rightarrow 19h^2 = 8$$

Hence, option (c) is correct.

18. (b) Mean (m) for BD = np

and variance (σ^2) for BD = npq

Given, $np + npq = 2.7$ and $n = 3$

$$\therefore np(1 + q) = 2.7$$

$$\Rightarrow p(1 + q) = \frac{2.7}{3}$$

$$\Rightarrow p(1 + q) = 0.9$$

$$\Rightarrow (1 - q)(1 + q) = 0.9$$

$[\because \text{for BD probability standard deviation (SD), } p + q = 1]$

$$\Rightarrow 1 - q^2 = 0.9$$

$$\Rightarrow q^2 = 0.1 \Rightarrow q = \pm 0.3$$

$$\therefore q = 0.3$$

$[\because q = -0.3 \text{ can't possible as } 0 \leq q \leq 1]$

$$\Rightarrow p = 0.7 \text{ and } q = 0.3$$

$$\therefore (b)(d) = (0.3 + 0.7)^5$$

Hence, option (b) is correct.

19. (b) We have,

$$R(x) = \int \frac{dx}{e^x + 8e^{-x} + 4e^{-3x}}$$

$$-2 \int \frac{dx}{e^{3x} + 8e^x + 4e^{-x}}$$

$$\Rightarrow R(x) = \int \frac{e^x (e^{2x} - 2)}{e^{4x} + 8e^{2x} + 4} dx$$

On substituting

$$e^x = t \Rightarrow e^x dx = dt,$$

we get

$$R(t) = \int \frac{(t^2 - 2) dt}{t^4 + 8t^2 + 4}$$

$$= \int \frac{(1 - 2t^{-2}) dt}{(t + 2t^{-1})^2 + 4}$$

$$= \frac{1}{2} \tan^{-1} \left(\frac{t + 2t^{-1}}{2} \right) + K$$

$$\Rightarrow R(x) = \frac{1}{2} \tan^{-1} \left(\frac{e^x + 2e^{-x}}{2} \right) + K$$

Hence, $(A, B, C) = (\tan^{-1}, e^x, 2)$

$\therefore$ Option (b) is correct.

$$20. (b) \cot^{-1}(1 - x + x^2) = \tan^{-1} \left(\frac{1}{1 - x + x^2} \right)$$

$$= \tan^{-1} \left(\frac{1}{1 - x(1 - x)} \right)$$

$$= \tan^{-1} \left(\frac{x + (1 - x)}{1 - x(1 - x)} \right)$$

$$\Rightarrow \cot^{-1}(1 - x + x^2)$$

$$= \tan^{-1} x - \tan^{-1}(1 - x)$$

$$\therefore \int_0^1 \cot^{-1}(1 - x + x^2) dx$$

$$= \int_0^1 \tan^{-1} x dx - \int_0^1 \tan^{-1}(1 - x) dx$$

$$= \int_0^1 \tan^{-1} x dx + \int_0^1 \tan^{-1} x dx$$

$$\left[\because \int_0^a f(x) dx = - \int_0^a f(a - x) dx \right]$$

$$= 2 \int_0^1 \tan^{-1} x dx$$

On evaluating by integration by parts, we have

$$= 2 \left\{ [\tan^{-1} x \cdot x]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx \right\}$$

$$= 2 \left\{ \frac{\pi}{4} - \left[\frac{1}{2} \ln(1 + x^2) \right]_0^1 \right\}$$

$$= 2 \left[\frac{\pi}{4} - \frac{1}{2} \log 2 \right] = \frac{\pi}{2} - \log 2$$

$$\text{Hence, } \int_0^1 \cot^{-1}(1 - x + x^2) dx$$

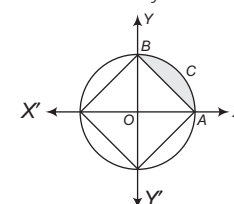
$$= \frac{\pi}{2} - \log 2$$

So, option (b) is correct.

21. (a) Given curves are

$$\sqrt{|x|} + \sqrt{|y|} = \sqrt{a} \quad \dots(i)$$

$$\text{and } x^2 + y^2 = a^2 \quad \dots(ii)$$



Now, required areas = 4 [shaded area in the first quadrant]

$$= 4 \left[\frac{\pi a^2}{4} - \int_0^a (\sqrt{a} - \sqrt{x})^2 dx \right]$$

[from Equ. (i) and (ii)]

$$= 4 \left[\frac{\pi a^2}{4} - \int_0^a (a + x - 2\sqrt{a} \sqrt{x}) dx \right]$$

$$= 4 \left[\frac{\pi a^2}{4} - 4 \left[ax + \frac{x^2}{2} - \frac{4}{3} \sqrt{a} x^{3/2} \right]_0^a \right]$$

$$= \left(\pi - \frac{2}{3} \right) a^2 \text{ sq units}$$

Hence, option (a) is correct.

24 Engineering Entrances Solved Papers 2018

22. (b) $P(A \cap B) = \frac{1}{6}$
 and $P(A^C \cap B^C) = \frac{1}{3}$
 Now, $P(A \cup B)^C = P(A^C \cap B^C) = \frac{1}{3}$
 $\Rightarrow 1 - P(A \cup B) = \frac{1}{3}$
 $\Rightarrow P(A \cup B) = \frac{2}{3}$
 But $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
 $\Rightarrow P(A) + P(B) = \frac{5}{6} \quad \dots(i)$
 $\because A$ and B are independent events
 $\therefore P(A \cap B) = P(A)P(B)$
 $\Rightarrow P(A)P(B) = \frac{1}{6}$
 $[P(A) - P(B)]^2 = [P(A) + P(B)]^2 - 4P(A)P(B)$
 $= \frac{25}{36} - \frac{4}{6} = \frac{1}{36}$
 $\Rightarrow P(A) - P(B) = \pm \frac{1}{6} \quad \dots(ii)$

On solving Eqs. (i) and (ii), we get

$$P(A) = \frac{1}{2} \text{ or } \frac{1}{3}$$

Hence, option (b) is correct.

23. (d) Let A be the event that the examinee gives the correct answer.
 Let G, C, K stand for guessing, copying and knowing, respectively.
 Given, $P(C) = \frac{1}{6}, P(G) = \frac{1}{3}$
 and $P\left(\frac{A}{C}\right) = \frac{1}{8}$
 $\Rightarrow P(C) + P(G) + P(K) = 1$
 $\Rightarrow \frac{1}{6} + \frac{1}{3} + P(K) = 1$
 $\therefore P(K) = \frac{1}{2}$

Also, $P\left(\frac{A}{K}\right) = 1$, for if the examinee knows, he/she will correctly answer it and $P\left(\frac{A}{G}\right) = \frac{1}{4}$, since there are four choices.

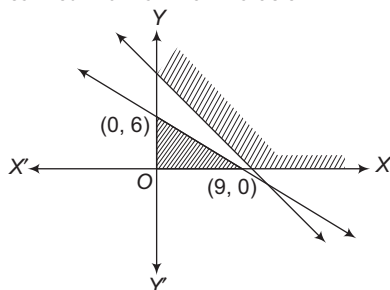
Now, total probability

$$\begin{aligned} P(A) &= P(G)P\left(\frac{A}{G}\right) + P(C)P\left(\frac{A}{C}\right) + P(K)P\left(\frac{A}{K}\right) \\ &= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{6} \cdot \frac{1}{8} + \frac{1}{2} \cdot 1 \\ &= \frac{1}{12} + \frac{1}{48} + \frac{1}{2} = \frac{4 + 1 + 24}{48} = \frac{29}{48} \end{aligned}$$

$$\begin{aligned} \text{Now, } P\left(\frac{K}{A}\right) &= \frac{P(K)P\left(\frac{A}{K}\right)}{P(A)} \\ &= \frac{\frac{1}{2} \cdot 1}{\frac{29}{48}} = \frac{1}{2} \cdot \frac{48}{29} = \frac{24}{29} \end{aligned}$$

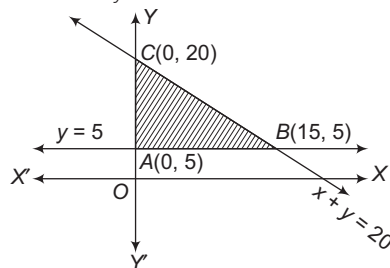
Hence, option (d) is correct.

24. (d) From the figure, it is clear that there is no common area. So, we cannot find maximum value of z .



Hence, option (d) is correct.

25. (d) Feasible region is $ABCA$ and $Z = 7x - 8y$



at $A(0, 5)$, $Z = 7 \times 0 - 8 \times 5 = -40$
 at $B(15, 5)$, $Z = 7(15) - 8(5) = 65$
 at $C(0, 20)$, $Z = 7(0) - 8(20) = -160$
 Hence, the minimum value of Z is attained at point $(0, 20)$.
 So, option (d) is correct.

KCET

1. (b) We have, $I = \int_0^1 \frac{dx}{e^x + e^{-x}}$
 $\Rightarrow I = \int_0^1 \frac{e^x}{1 + e^{2x}} dx \quad \dots(i)$
 put $e^x = t$ and $e^x dx = dt$, where if $x = 0, t = 1$ and if $x = 1, t = e$
 $\therefore I = \int_1^e \frac{dt}{1 + t^2}$
 $\Rightarrow I = [\tan^{-1} t]_1^e$
 $\Rightarrow I = \tan^{-1} e - \tan^{-1} 1$
 $\Rightarrow I = \tan^{-1} e - \frac{\pi}{4}$

2. (a) We have, $I = \int_0^{1/2} \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

Put $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$

$$x = 0, \theta = 0, x = \frac{1}{2}, \theta = \frac{\pi}{6}$$

$$\therefore I = \int_0^{\pi/6} \frac{\cos \theta d\theta}{(1 + \sin^2 \theta) \cos \theta}$$

$$\therefore I = \int_0^{\pi/6} \frac{d\theta}{1 + \sin^2 \theta}$$

Divide by $\cos^2 \theta$, we get

$$I = \int_0^{\pi/6} \frac{\sec^2 \theta d\theta}{\sec^2 \theta + \tan^2 \theta}$$

$$\Rightarrow I = \int_0^{\pi/6} \frac{\sec^2 \theta d\theta}{1 + 2\tan^2 \theta}$$

$$\Rightarrow I = \frac{1}{2} \int_0^{\pi/6} \frac{\sec^2 \theta d\theta}{\left(\frac{1}{\sqrt{2}}\right)^2 + \tan^2 \theta}$$

Put $\tan \theta = t, \sec^2 \theta d\theta = dt$

$$\theta = 0, t = 0 \quad \theta = \frac{\pi}{6}, t = \frac{1}{\sqrt{3}}$$

$$\therefore I = \frac{1}{2} \int_0^{1/\sqrt{3}} \frac{dt}{\left(\frac{1}{\sqrt{2}}\right)^2 + t^2}$$

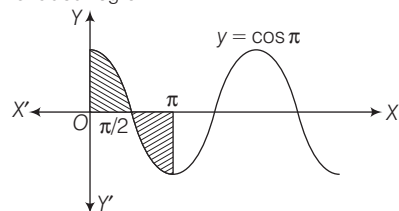
$$\Rightarrow I = \frac{1}{2} (\sqrt{2} \tan^{-1} \sqrt{2} t)_0^{1/\sqrt{3}}$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \left[\tan^{-1} \left(\frac{\sqrt{2}}{\sqrt{3}} \right) - 0 \right]$$

$$\Rightarrow I = \frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}}{\sqrt{3}} \right)$$

3. (c) Area of region bounded by the curve

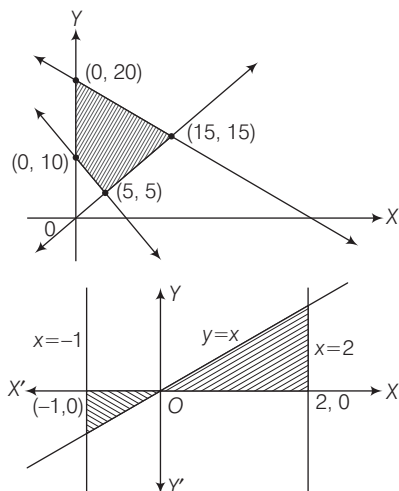
$y = \cos x$ between $x = 0$ and $x = \pi$ is shaded region.



Required area of shaded region

$$\begin{aligned} &= \int_0^{\pi/2} \cos x dx + \left| \int_{\pi/2}^{\pi} \cos x dx \right| \\ &= [\sin x]_0^{\pi/2} + |[\sin x]_{\pi/2}^{\pi}| \\ &= \left[\sin \frac{\pi}{2} - \sin 0 \right] + \left| \sin \pi - \sin \frac{\pi}{2} \right| \\ &= (1 - 0) + |(0 - 1)| \\ &= 1 + 1 = 2 \text{ sq units} \end{aligned}$$

4. (b) Given line is
 $y = x$, X -axis, $x = -1$ and $x = 2$



$$\text{Required area} = \left| \int_{-1}^0 x \, dx \right| + \int_0^2 x \, dx$$

$$= \left[\frac{x^2}{2} \right]_{-1}^0 + \left[\frac{x^2}{2} \right]_0^2$$

$$= \left[0 - \frac{1}{2} \right] + (2 - 0)$$

$$= \frac{1}{2} + 2 = \frac{5}{2} \text{ sq units}$$

5. (b) We have,

$$\frac{d^2 y}{dx^2} = 3 \sqrt{1 + \left(\frac{dy}{dx} \right)^2}$$

Cubing both sides, we get

$$\left(\frac{d^2 y}{dx^2} \right)^3 = 1 + \left(\frac{dy}{dx} \right)^2$$

Clearly, degree and order of given differential equation is 3 and 2 respectively.

6. (a) We have,

$$x \frac{dy}{dx} - y = 3 \Rightarrow x \, dy = (y + 3) \, dx$$

$$\Rightarrow \frac{dy}{y + 3} = \frac{dx}{x}$$

On integrating,

$$\int \frac{dy}{y + 3} = \int \frac{dx}{x}$$

$$\Rightarrow \log(y + 3) = \log x + \log c$$

$$\Rightarrow \log(y + 3) = \log cx \Rightarrow y + 3 = cx$$

which represent the family of straight line.

7. (c) We have,

$$\frac{dy}{dx} + y = \frac{1 + y}{x}$$

$$\Rightarrow \frac{dy}{dx} + y - \frac{y}{x} = \frac{1}{x}$$

$$\Rightarrow \frac{dy}{dx} + \left(1 - \frac{1}{x} \right) y = \frac{1}{x}$$

Here, $P = 1 - \frac{1}{x}$ and $Q = \frac{1}{x}$

$$\therefore \text{I.F.} = e^{\int P \, dx} = e^{\int \left(1 - \frac{1}{x} \right) dx} = e^{(x - \log x)}$$

$$= \frac{e^x}{e^{\log x}} = \frac{e^x}{x}$$

8. (c) We know that,

$$|\mathbf{a} \times \mathbf{b}|^2 + |\mathbf{a} \cdot \mathbf{b}|^2 = |\mathbf{a}|^2 |\mathbf{b}|^2$$

$$\Rightarrow 144 = (4)^2 |\mathbf{b}|^2$$

$$[\because |\mathbf{a}|^2 = 4, |\mathbf{a} \times \mathbf{b}|^2 + (\mathbf{a} \cdot \mathbf{b})^2 = 144]$$

$$\Rightarrow |\mathbf{b}|^2 = \frac{144}{16} = 9 \Rightarrow |\mathbf{b}| = 3$$

9. (b) Given, $|\mathbf{a}| = |\mathbf{b}| = 1$ and $\mathbf{a} \cdot \mathbf{b} = 0$

$$\text{Now, } (3\mathbf{a} + 2\mathbf{b}) \cdot (5\mathbf{a} - 6\mathbf{b})$$

$$= 15|\mathbf{a}|^2 - 18\mathbf{a} \cdot \mathbf{b} + 10\mathbf{a} \cdot \mathbf{b} - 12|\mathbf{b}|^2$$

$$= 15 - 0 + 0 - 12 = 3$$

10. (b) Given, $a\hat{i} + \hat{j} + \hat{k}$, $\hat{i} + b\hat{j} + \hat{k}$ and $\hat{i} + \hat{j} + c\hat{k}$ are coplanar vector

$$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow a(bc - 1) - 1(c - 1) + 1(1 - b) = 0$$

$$\Rightarrow abc - a - c + 1 + 1 - b = 0$$

$$\Rightarrow abc - (a + b + c) = -2$$

11. (a) We have,

$$\mathbf{a} = \hat{i} + \lambda\hat{j} + 2\hat{k} \text{ and } \mathbf{b} = \mu\hat{i} + \hat{j} - \hat{k} \text{ are orthogonal}$$

$$\therefore \mathbf{a} \cdot \mathbf{b} = 0$$

$$\Rightarrow (\hat{i} + \lambda\hat{j} + 2\hat{k}) \cdot (\mu\hat{i} + \hat{j} - \hat{k}) = 0$$

$$\Rightarrow \mu + \lambda - 2 = 0 \dots (i)$$

$$\text{Also, } |\mathbf{a}| = |\mathbf{b}|$$

$$\therefore 1 + \lambda^2 + 4 = \mu^2 + 1 + 1$$

$$\Rightarrow \mu^2 - \lambda^2 = 3 \Rightarrow (\mu + \lambda)(\mu - \lambda) = 3$$

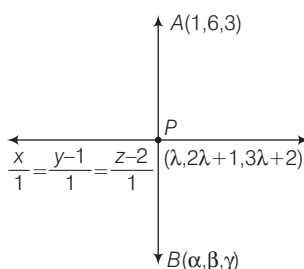
$$\Rightarrow 2(\mu - \lambda) = 3 \quad [\because \text{from Eq (i)}]$$

$$\Rightarrow \mu - \lambda = \frac{3}{2} \dots (ii)$$

Solving Eqs, (i) and (ii), we get

$$\mu = \frac{7}{4} \text{ and } \lambda = \frac{1}{4} \therefore (\lambda, \mu) = \left(\frac{1}{4}, \frac{7}{4} \right)$$

12. (a) Let any point P lie on the line



$$\frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{1} \text{ is}$$

$$P(\lambda, 2\lambda + 1, 3\lambda + 2).$$

Direction ratio of line PA is

$$(\lambda - 1, 2\lambda - 5, 3\lambda - 1)$$

Since, line PA is perpendicular to the given line.

$$\therefore 1(\lambda - 1) + 2(2\lambda - 5) + 3(3\lambda - 1) = 0$$

$$\Rightarrow \lambda - 1 + 4\lambda - 10 + 9\lambda - 3 = 0$$

$$\Rightarrow 14\lambda = 14 \Rightarrow \lambda = 1$$

point $P(1, 3, 5)$

P is the mid-point of AB

$$\therefore 1 = \frac{\alpha + 1}{2} \Rightarrow \alpha = 1$$

$$3 = \frac{\beta + 6}{2} \Rightarrow \beta = 0$$

$$5 = \frac{\gamma + 3}{2} \Rightarrow \gamma = 7$$

$\therefore$ Image of the point $(1, 6, 3)$ in the line

$$\frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{1} \text{ is } (1, 0, 7).$$

13. (c) Given lines are

$$2x = 3y = -z \text{ and } 6x = -y = -4z$$

$$\Rightarrow \frac{x}{\frac{1}{2}} = \frac{y}{\frac{1}{3}} = \frac{z}{-1} \text{ and } \frac{x}{\frac{1}{6}} = \frac{y}{-1} = \frac{z}{-\frac{1}{4}}$$

$$\text{Here, } a_1 = \frac{1}{2}, b_1 = \frac{1}{3}, c_1 = -1$$

$$\text{and } a_2 = \frac{1}{6}, b_2 = -1, c_2 = -\frac{1}{4}$$

$$\cos \theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2)}}$$

$$= \frac{\frac{1}{2} \times \frac{1}{6} + \frac{1}{3}(-1) + (-1)\left(-\frac{1}{4}\right)}{\left(\frac{1}{4} + \frac{1}{9} + 1\right)\left(\frac{1}{36} + 1 + \frac{1}{16}\right)}$$

$$= \frac{\frac{1}{12} - \frac{1}{3} + \frac{1}{4}}{\sqrt{\left(\frac{1}{4} + \frac{1}{9} + 1\right)\left(\frac{1}{36} + 1 + \frac{1}{16}\right)}}$$

$$\Rightarrow \cos \theta = 0^\circ \Rightarrow \theta = 90^\circ$$

14. (d) Given line, $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$

lie on the plane $2x - 4y + z = 7$

$$\therefore \text{Point } (4, 2, k) \text{ lie on the plane}$$

$$2x - 4y + z = 7$$

$$\Rightarrow 2(4) - 4(2) + k = 7 \Rightarrow k = 7$$

15. (d) We have,

$$xy + yz = 0 \Rightarrow y(x + z) = 0$$

$$\Rightarrow y = 0 \text{ and } x + z = 0$$

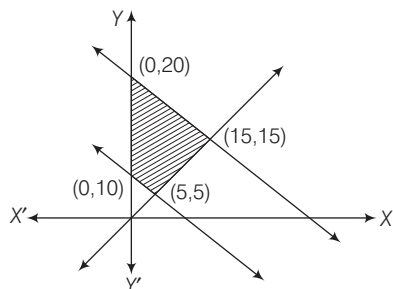
Direction ratio of plane are $(0, 1, 0)$ and $(1, 0, 1)$

$$\therefore \cos \theta = 0$$

Hence, $xy + yz = 0$ represent a pair of perpendicular plane.

26 Engineering Entrances Solved Papers 2018

16. (a) We have, $z = 3x + 9y$

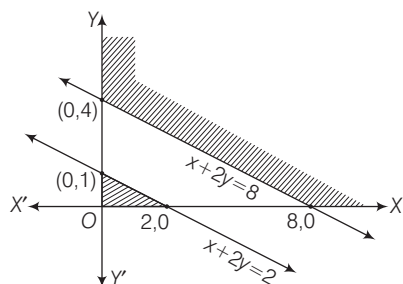


Corner point	$z = 3x + 9y$
(5, 5)	$15 + 45 = 60$
(0, 10)	$0 + 90 = 90$
(0, 20)	$0 + 180 = 180$
(15, 15)	$45 + 135 = 80$

Minimum value of z is 60 at (5, 5).

17. (d) We have,
Maximize $z = x + 4y$
Subject to constraints
 $x + 2y \leq 2$
 $x + 2y \geq 0$
 $x, y \geq 0$

The graph of inequalities are



$\therefore$ Clearly from graph there has no feasible solution.

18. (b) Probability distribution given by

$X = x_i$	P_i	$P_i x_i$	$P_i x_i^2$
0	$\frac{25}{36}$	0	0
1	$\frac{5}{18}$	$\frac{5}{18}$	$\frac{5}{18}$
2	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{4}{36}$
$\Sigma P_i x_i = \frac{1}{3} \quad \Sigma P_i x_i^2 = \frac{7}{18}$			

Standard deviation

$$\begin{aligned}\sigma &= \sqrt{\Sigma P_i x_i^2 - (\Sigma P_i x_i)^2} \\ &= \sqrt{\frac{7}{18} - \left(\frac{1}{3}\right)^2} = \sqrt{\frac{7}{18} - \frac{1}{9}} \\ &= \sqrt{\frac{5}{18}} = \frac{1}{3}\sqrt{\frac{5}{2}}\end{aligned}$$

19. (a) A bag contains 17 tickets numbered from 1 to 17.

Probability that both the ticket may show even numbers when draw one-by-one without replacement is

$$= \frac{8}{17} \times \frac{7}{16} = \frac{7}{34}$$

20. (a) Given flashlight has 10 batteries out of which 4 are dead

$$\text{Required probability} = \frac{4}{10} \times \frac{3}{9} \times \frac{2}{8} = \frac{1}{30}$$

21. (a) We have, $f(x) = |x| + x$

$$\text{and } g(x) = |x| - x$$

$$\therefore (fog)(x) = f(g(x))$$

$$= f(|x| - x)$$

$$= f(-x - x)$$

$$= f(-2x)$$

$$= |-2x| + (-2x) \quad [\because x < 0]$$

$$= 2x - 2x = 0$$

22. (c) We have, $n(A) = 6$

Total number of distinct function from A to A is 6^6

and total number of one-one functions $= {}^6P_6 = 6!$

$\therefore$ Total number of distinct function from A to $A = 6^6 - 6!$

23. (b) We have,

$$\sin^{-1} x + \cos^{-1} y = \frac{2\pi}{5}$$

$$\Rightarrow \frac{\pi}{2} - \cos^{-1} x + \frac{\pi}{2} - \sin^{-1} y = \frac{2\pi}{5}$$

$$\Rightarrow \pi - (\cos^{-1} x + \sin^{-1} y) = \frac{2\pi}{5}$$

$$\Rightarrow \cos^{-1} x + \sin^{-1} y = \pi - \frac{2\pi}{5}$$

$$\Rightarrow \cos^{-1} x + \sin^{-1} y = \frac{3\pi}{5}$$

24. (b) We know that,

$$\tan\left(\frac{x}{2}\right) = \frac{1 - \cos x}{\sin x}$$

$$\therefore \tan\left(\frac{1}{2} \cos^{-1} \frac{2}{\sqrt{5}}\right) = \frac{1 - \cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)}$$

$$= \frac{1 - 2/\sqrt{5}}{\sin(\sin^{-1}(1/\sqrt{5}))}$$

$$= \frac{1 - 2/\sqrt{5}}{1/\sqrt{5}} = \sqrt{5} - 2$$

25. (d) Given, $A = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$

$$\text{Now, } A^2 = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 8 & -8 \\ -8 & 8 \end{bmatrix}$$

$$\Rightarrow A^2 = 2^2 \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$\Rightarrow A^2 = 2^2 A$$

$$\text{Now, } A^3 = A^2 \cdot A \Rightarrow A^3 = 2^2 \cdot A^2$$

$$\Rightarrow A^3 = 2^4 \cdot A \Rightarrow A^4 = A^3 \cdot A$$

$$\Rightarrow A^4 = 2^4 \cdot A^2 \Rightarrow A^4 = 2^6 A$$

$$\Rightarrow A^5 = A^4 \cdot A \Rightarrow A^5 = 2^6 A^2$$

$$A^5 = 2^8 A$$

$$\therefore A^n = 2^{2(n-1)} A$$

Hence, $k = 2(n-1)$

26. (d) We have,

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

$$\therefore \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} -2 \\ 6 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\therefore x = -1 \text{ and } y = 3$$

27. (d) Given, $A = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix}$

$$A' = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$\therefore AA' = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$\Rightarrow AA' = \begin{bmatrix} \cos^2 \alpha + \sin^2 \alpha & 0 \\ 0 & \sin^2 \alpha + \cos^2 \alpha \end{bmatrix}$$

$$\Rightarrow AA' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow AA' = I$$

28. (d) Let

$$A = \begin{bmatrix} (5^x + 5^{-x})^2 & (5^x - 5^{-x})^2 & 1 \\ (6^x + 6^{-x})^2 & (6^x - 6^{-x})^2 & 1 \\ (7^x + 7^{-x})^2 & (7^x - 7^{-x})^2 & 1 \end{bmatrix}$$

$$C_1 \rightarrow C_1 - C_2$$

$$A = \begin{bmatrix} (5^x + 5^{-x})^2 - (5^x - 5^{-x})^2 & (5^x - 5^{-x})^2 & 1 \\ (6^x + 6^{-x})^2 - (6^x - 6^{-x})^2 & (6^x - 6^{-x})^2 & 1 \\ (7^x + 7^{-x})^2 - (7^x - 7^{-x})^2 & (7^x - 7^{-x})^2 & 1 \end{bmatrix}$$

$$A = \begin{vmatrix} 4 & (5^x - 5^{-x})^2 & 1 \\ 4 & (6^x - 6^{-x})^2 & 1 \\ 4 & (7^x - 7^{-x})^2 & 1 \end{vmatrix}$$

$$A = 4 \begin{vmatrix} 1 & (5^x - 5^{-x})^2 & 1 \\ 1 & (6^x - 6^{-x})^2 & 1 \\ 1 & (7^x - 7^{-x})^2 & 1 \end{vmatrix}$$

$$[\because (a+b)^2 - (a-b)^2 = 4ab]$$

$$A = 0 \quad [\because C_1 \text{ and } C_3 \text{ are identical}]$$

$$29. (*) \text{ Let } \Delta = \begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$$

Applying $C_3 \rightarrow C_2 + C_3$, we get

$$\Delta = \begin{vmatrix} a-b & b+c & a+b+c \\ b-a & c+a & a+b+c \\ c-a & a+b & a+b+c \end{vmatrix}$$

$$= (a+b+c) \begin{vmatrix} a-b & b+c & 1 \\ b-a & c+a & 1 \\ c-a & a+b & 1 \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1$ and

$$R_3 \rightarrow R_3 - R_1$$

$$= (a+b+c) \begin{vmatrix} a-b & b+c & 1 \\ 2b & a-b & 0 \\ c-2a+b & a-c & 0 \end{vmatrix}$$

$$= (a+b+c) [1(2b(a-c) - (a-b)(c-2a+b))]$$

$$= (a+b+c) [2ab - 2bc$$

$$- (ac - 2a^2 + ab - bc + 2ab - b^2)]$$

$$= (a+b+c) [-bc - ac - ab + 2a^2 + b^2]$$

None of the option is correct.

30. (c) We have,

$(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the vertices of triangle whose area is k

$$\therefore \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = k$$

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 2k$$

$$\text{Now, } \begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}^2$$

$$= \begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & 4 \\ x_2 & y_2 & 4 \\ x_3 & y_3 & 4 \end{vmatrix}$$

$$= 16 \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$= 16 \cdot 2k \cdot 2k = 64k^2$$

31. (b) Let A be a square matrix of order 3.

$$\therefore |5A| = 5^3 |A| = 125 |A|$$

32. (b) We have,

$$f(x) = \begin{cases} \sqrt{1+kx} - \sqrt{1-kx} & \text{if } -1 \leq x < 0 \\ \frac{x}{2x+1} & \text{if } 0 \leq x \leq 1 \end{cases}$$

Since this function is continuous at $x=0$.

$$\therefore \lim_{x \rightarrow 0^-} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} = \frac{2(0)+1}{0-1}$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{(1+kx) - (1-kx)}{x(\sqrt{1+kx} + \sqrt{1-kx})} = -1$$

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{2k}{(\sqrt{1+kx} + \sqrt{1-kx})} = -1$$

$$\therefore k = -1$$

33. (b) We have

$$\cos y = x \cos(a+y)$$

$$\Rightarrow \frac{\cos y}{\cos(a+y)} = x$$

On differentiating w.r.t. x , we get

$$\left(\frac{\cos(a+y)(-\sin y) + \cos y \sin(a+y)}{\cos^2(a+y)} \right) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin(a+y) \cos y - \cos(a+y) \sin y}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$$

34. (a) We have,

$$f(x) = |\cos x - \sin x|$$

$$f(x) = \begin{cases} \cos x - \sin x; & 0 < x \leq \frac{\pi}{4} \\ \sin x - \cos x & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

$$f'(x) = \begin{cases} -\sin x - \cos x; & 0 < x \leq \frac{\pi}{4} \\ \cos x + \sin x; & \frac{\pi}{4} < x < \frac{\pi}{2} \end{cases}$$

$$f'\left(\frac{\pi}{6}\right) = -\sin\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right)$$

$$= -\frac{1}{2} - \frac{\sqrt{3}}{2} = -\frac{(\sqrt{3}+1)}{2}$$

35. (d) We have,

$$y = \sqrt{x + \sqrt{x + \sqrt{x + \dots \infty}}}$$

$$\Rightarrow y = \sqrt{x+y} \Rightarrow y^2 = x+y$$

$$\Rightarrow y^2 - y = x$$

On differentiating both sides, we get

$$(2y-1) \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{2y-1}$$

36. (b) We have,

$$f(x) = \begin{cases} \frac{\log_e x}{x-1}; & x \neq 1 \\ k & x = 1 \end{cases}$$

Since this function is continuous at $x=1$

$$\therefore \lim_{x \rightarrow 1} f(x) = k \Rightarrow \lim_{x \rightarrow 1} \frac{\log_e x}{x-1} = k$$

Apply L'Hospital's rule,

$$\lim_{x \rightarrow 1} \frac{1}{x} = k \Rightarrow 1 = k$$

$$\therefore k = 1$$

37. (a) Volume of cube, $V = x^3$

$$\frac{dV}{dx} = 3x^2$$

given $dx = 3\%$ of $x = 0.03x$

Approximate change in volume V

$$\text{i.e. } dV = \left(\frac{dV}{dx} \right) dx$$

$$\Rightarrow dV = 3x^2(0.03)x m^3$$

$$\Rightarrow dV = 0.09 x^3 m^3$$

Hence, approximate change in volume V is $0.09 x^3 m^3$.

38. (c) We have,

$$f(x) = \left(\frac{1}{x} \right)^x \Rightarrow f(x) = x^{-x}$$

$$\Rightarrow f(x) = e^{-x \log x}$$

$$\Rightarrow f'(x) = e^{-x \log x} (-1 - \log x)$$

$$\Rightarrow f'(x) = -x^{-x} (1 + \log x)$$

For maxima or minima

$$f'(x) = 0$$

$$\therefore -x^{-x} (1 + \log x) = 0$$

$$\Rightarrow \log x = -1 \Rightarrow x = e^{-1}$$

$$\therefore f(x) \text{ has maximum value at } x = \frac{1}{e}$$

$$f\left(\frac{1}{e}\right) = \left(\frac{i}{1/e} \right)^{1/e} = (e)^{1/e}$$

Hence, maximum value of $\left(\frac{1}{x} \right)^x$ is $(e)^{1/e}$.

39. (b) We have,

$$f(x) = x^x \Rightarrow f(x) = e^{x \log x}$$

$$\Rightarrow f'(x) = e^{x \log x} (1 + \log x)$$

For stationary point,

$$f'(x) = 0$$

$$\Rightarrow e^{x \log x} (1 + \log x) = 0$$

$$\Rightarrow \log x = -1 \Rightarrow x = \frac{1}{e}$$

$\therefore f(x) = x^x$ has stationary point at

$$x = \frac{1}{e}$$

28 Engineering Entrances Solved Papers 2018

40. (c) We know that maximum area of rectangle inscribed in any circle of radius r is $2r^2$.

Here, equation of circle is

$$(x+1)^2 + (y-3)^2 = 64$$

∴ Radius of given circle is 8

∴ Maximum area of rectangle inscribed in the given circle is $2 \times (8)^2 = 2 \times 64 = 128$ sq units

41. (c) We have,

$$I = \int \frac{1}{1+e^x} dx$$

$$\Rightarrow I = \int \frac{e^{-x}}{e^{-x} + 1} dx$$

Put $e^{-x} + 1 = t \Rightarrow -e^{-x} dx = dt$

$$\therefore I = -\int \frac{dt}{t} \Rightarrow I = -\log t + C$$

$$\Rightarrow I = \log \frac{1}{t} + C$$

$$\Rightarrow I = \log \left(\frac{1}{e^{-x} + 1} \right) + C$$

$$\Rightarrow I = \log \left(\frac{e^x}{e^x + 1} \right) + C$$

42. (c) We have,

$$I = \int \frac{1}{\sqrt{3-6x-9x^2}} dx$$

$$I = \int \frac{1}{\sqrt{3-(9x^2+6x+1)+1}} dx$$

$$I = \int \frac{1}{\sqrt{(2)^2 - (3x+1)^2}} dx$$

Put $3x+1 = t \Rightarrow 3 dx = dt$

$$\therefore I = \frac{1}{3} \int \frac{dt}{\sqrt{(2)^2 - t^2}}$$

$$\Rightarrow I = \frac{1}{3} \sin^{-1} \left(\frac{t}{2} \right) + C$$

$$\Rightarrow I = \frac{1}{3} \sin^{-1} \left(\frac{3x+1}{2} \right) + C$$

43. (a) We have,

$$I = \int e^{\sin x} \left(\frac{\sin x + 1}{\sec x} \right) dx$$

$$\Rightarrow I = \int e^{\sin x} \cdot (\sin x + 1) \cos x dx$$

Put $\sin x = t \Rightarrow \cos x dx = dt$

$$\therefore I = \int e^t (t+1) dt$$

$$[\because \int e^x \{f(x) + f'(x)\} dx = e^x f(x) + C]$$

$$\Rightarrow I = t e^t + C$$

$$\Rightarrow I = \sin x \cdot e^{\sin x} + C$$

44. (a) Let $I = \int_{-2}^2 |x \cos \pi x| dx$

$$\Rightarrow I = 2 \int_0^2 x \cos \pi x dx$$

$$\Rightarrow I = 2 \left[\int_0^{1/2} x \cos \pi x dx - \int_{1/2}^{3/2} x \cos \pi x dx + \int_{3/2}^2 x \cos \pi x dx \right]$$

$$\Rightarrow I = 2 \left[\left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_0^{1/2} - \left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_{1/2}^{3/2} + \left[\frac{x \sin \pi x}{\pi} + \frac{\cos \pi x}{\pi^2} \right]_{3/2}^2 \right]$$

$$\Rightarrow I = 2 \left[\left(\frac{1}{2\pi} + 0 \right) - \left(0 + \frac{1}{\pi^2} \right) - \left\{ \left(\frac{-3}{2\pi} + 0 \right) - \left(\frac{1}{2\pi} + 0 \right) \right\} + \left(0 + \frac{1}{\pi^2} \right) - \left(\frac{-3}{2\pi} + 0 \right) \right]$$

$$\Rightarrow I = 2 \left[\frac{1}{2\pi} - \frac{1}{\pi^2} + \frac{3}{2\pi} + \frac{1}{2\pi} + \frac{1}{\pi^2} + \frac{3}{2\pi} \right]$$

$$\Rightarrow I = \frac{8}{\pi}$$

AP EAMCET

1. (c) Given, $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

Now, $|A| = \begin{vmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{vmatrix}$

$$= 3(-3+4) + 3(2-0) + 4(-2-0)$$

$$= 3 + 6 - 8 = 1 \neq 0$$

Here,

$$A_{11} = 1, A_{12} = -2, A_{13} = -2$$

$$A_{21} = -1, A_{22} = 3, A_{23} = 3$$

$$A_{31} = 0, A_{32} = -4, A_{33} = -3$$

$$\therefore \text{adj}(A) = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Now,

$$A^{-1} = \frac{\text{adj}(A)}{|A|} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \quad \dots(i)$$

$$A^2 = A \cdot A$$

$$= \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix}$$

$$\therefore A^3 = A^2 \cdot A$$

$$= \begin{bmatrix} 3 & -4 & 4 \\ 0 & -1 & 0 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get $A^{-1} = A^3$

2. (d) Given,

$$A = \begin{bmatrix} k/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & m/4 \end{bmatrix}$$

$$\text{and } A^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}$$

Since, $AA^{-1} = I$

$$\Rightarrow \begin{bmatrix} \frac{k}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{m}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \frac{k}{4} & 0 & 0 \\ 0 & \frac{1}{9} & 0 \\ 0 & 0 & \frac{m}{16} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

On comparing, we get

$$\frac{k}{4} = 1, \frac{1}{9} = 1 \text{ and } \frac{m}{16} = 1$$

$$\Rightarrow k = 4, l = 9, m = 16$$

$$\therefore k + l + m = 4 + 9 + 16 = 29$$

3. (a) Given system of equation is,

$$\begin{aligned} x + 2y + z &= 1 & x + 3y + 4z &= k \\ x + 5y + 10z &= k^2 \end{aligned}$$

$$\therefore D = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 10 \end{vmatrix}$$

$$= 1(30-20) - 2(10-4) + 1(5-3) = 10 - 12 + 2 = 0$$

Since, $D = 0$

∴ Given system of equation is consistent.

Therefore, $D_1 = 0$

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ k & 3 & 4 \\ k^2 & 5 & 10 \end{vmatrix}$$

$$\Rightarrow 1(30-20) - 2(10k-4k^2) + (5k-3k^2) = 0$$

$$\Rightarrow 10 - 20k + 8k^2 + 5k - 3k^2 = 0$$

$$\Rightarrow 5k^2 - 15k + 10 = 0$$

$$\Rightarrow k^2 - 3k + 2 = 0$$

$$\Rightarrow (k-2)(k-1) = 0$$

$$\Rightarrow k = 2, 1$$

Hence, the real values of k i.e.

$$A = 2 \text{ and } B = 1$$

$$\therefore A + B = 2 + 1 = 3$$

4. (a) We have,

$$13e^{i \tan^{-1} \frac{5}{12}} = a + ib$$

$$\Rightarrow 13 \left[\cos \left(\tan^{-1} \frac{5}{12} \right) + i \sin \left(\tan^{-1} \frac{5}{12} \right) \right] = a + ib$$

$$\Rightarrow 13 \left[\cos \left(\cos^{-1} \frac{12}{13} \right) + i \sin \left(\sin^{-1} \frac{5}{13} \right) \right] = a + ib$$

$$\Rightarrow 13 \left[\frac{12}{13} + i \frac{5}{13} \right] = a + ib$$

$$\Rightarrow 12 + 5i = a + ib$$

$$\therefore a = 12, b = 5$$

$$\therefore (a, b) = (12, 5)$$

5. (b) We have,

$$\frac{x^2 + 5x + 7}{(x-3)^3} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3}$$

$$\Rightarrow x^2 + 5x + 7$$

$$= A(x-3)^2 + B(x-3) + C \quad \dots(i)$$

Put $x = 3$ in Eq. (i), we get

$$9 + 15 + 7 = C \Rightarrow C = 31$$

Put $x = 0$ in Eq. (i), we get

$$7 = 9A - 3B + 31$$

$$\Rightarrow 9A - 3B = -24$$

$$\Rightarrow 3A - B = -8 \quad \dots(ii)$$

Put $x = 1$ in Eq. (i), we get

$$1 + 5 + 7 = 4A - 2B + 31$$

$$\Rightarrow 13 = 4A - 2B + 31$$

$$\Rightarrow 4A - 2B = -18$$

$$\Rightarrow 2A - B = -9 \quad \dots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$A = 1 \text{ and } B = 11$$

$\therefore$ Equation of line having slope A and passing through the points (B, C) is

$$y - 31 = 1(x - 11)$$

$$y - 31 = x - 11$$

$$x - y + 20 = 0$$

6. (d) Let $x = \tan \theta$, then we have

$$a \tan \theta + b \sec \theta = c$$

$$\Rightarrow a \sin \theta + b = c \cos \theta$$

$$\Rightarrow c \cos \theta - a \sin \theta = b$$

$$\text{Let } a^2 + c^2 = r^2$$

So, for some α ,

$$c = r \cos \alpha,$$

$$a = r \sin \alpha$$

$$\text{Also, } \tan \alpha = \frac{a}{c}$$

$$\text{Thus, } \cos \alpha \cos \theta - \sin \alpha \sin \theta = \frac{b}{r}$$

$$\Rightarrow \cos(\alpha + \theta) = \frac{b}{r}$$

$$\Rightarrow \alpha + \theta = \pm \cos^{-1} \frac{b}{r}$$

Let $\alpha + \theta$ be the positive solution and $\alpha + \phi$ the negative solution, where

$$y = \tan \phi$$

$$\therefore \alpha + \phi = -(\alpha + \theta)$$

$$\Rightarrow -2\alpha = \theta + \phi$$

$$\Rightarrow \tan(-2\alpha) = \tan(\theta + \phi)$$

$$\Rightarrow \frac{-2 \tan \alpha}{1 - \tan^2 \alpha} = \tan(\theta + \phi)$$

$$\Rightarrow \frac{-2a/c}{1 - a^2/c^2} = \frac{x+y}{1-xy}$$

$$\Rightarrow \frac{2ac}{a^2 - c^2} = \frac{x+y}{1-xy}$$

$$7. (a) \tan h^{-1} \left(\frac{1}{2} \right) + \cot h^{-1}(3)$$

$$= \frac{1}{2} \ln \left(\frac{1 + (1/2)}{1 - (1/2)} \right) + \frac{1}{2} \ln \left(\frac{3+1}{3-1} \right)$$

$$= \frac{1}{2} \log \left(\frac{(3/2)}{(1/2)} \right) + \frac{1}{2} \log \left(\frac{4}{2} \right)$$

$$= \frac{1}{2} \log 3 + \frac{1}{2} \log 2$$

$$= \log \sqrt{3} + \log \sqrt{2}$$

$$= \log(\sqrt{3} \cdot \sqrt{2}) = \log \sqrt{6}.$$

8. (b) Equation of line joining the points,

$$\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}, -\mathbf{a} - 2\mathbf{b} - 3\mathbf{c}$$

$$\mathbf{r} = (\mathbf{a} + 2\mathbf{b} - 5\mathbf{c}) + \lambda_1 (2\mathbf{a} + 4\mathbf{b} - 2\mathbf{c}) \dots(i)$$

Similarly, equation of the line joining the points $-4\mathbf{c}, 6\mathbf{a} - 4\mathbf{b} + 4\mathbf{c}$ is

$$\mathbf{r} = -4\mathbf{c} + \lambda_2 (6\mathbf{a} - 4\mathbf{b} + 8\mathbf{c}) \dots(ii)$$

Now, for point of intersection of lines (i) and (ii)

$$(2\lambda_1 + 1)\mathbf{a} + (4\lambda_1 + 2)\mathbf{b} + (-2\lambda_1 - 5)\mathbf{c}$$

$$= (6\lambda_2)\mathbf{a} + (-4\lambda_2)\mathbf{b} + (8\lambda_2 + 4)\mathbf{c}$$

On comparing

$$2\lambda_1 + 1 = 6\lambda_2 \quad \dots(iii)$$

$$4\lambda_1 + 2 = -4\lambda_2 \quad \dots(iv)$$

$$\text{and } -2\lambda_1 - 5 = 8\lambda_2 + 4 \quad \dots(v)$$

From Eqs. (iii), (iv) and (v)

$$\lambda_1 = -\frac{1}{2} \text{ and } \lambda_2 = 0$$

So, intersection points is $-4\mathbf{c}$ and for $\mu = -3$ the line $\mathbf{r} = (3\mathbf{a} + 6\mathbf{b} - \mathbf{c}) + \mu$

$(\mathbf{a} + 2\mathbf{b} + \mathbf{c})$ passes through point $-4\mathbf{c}$ only.

$$9. (a) \text{ We have, } \mathbf{a} = \hat{i} - 2\hat{j} - 3\hat{k}$$

$$\mathbf{b} = 2\hat{i} + \hat{j} - \hat{k}$$

$$\mathbf{c} = \hat{i} + 3\hat{j} - 2\hat{k}$$

Now,

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c}) = [\mathbf{a} \mathbf{b} \mathbf{c}] \mathbf{b} - [\mathbf{a} \mathbf{b} \mathbf{c}] \mathbf{c}$$

$$(\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a}) = [\mathbf{b} \mathbf{c} \mathbf{a}] \mathbf{c} - [\mathbf{b} \mathbf{c} \mathbf{a}] \mathbf{a}$$

$$(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b}) = [\mathbf{c} \mathbf{a} \mathbf{b}] \mathbf{a} - [\mathbf{c} \mathbf{a} \mathbf{b}] \mathbf{b}$$

$$\text{Now, } [\mathbf{a} \mathbf{b} \mathbf{c}] = \begin{vmatrix} 1 & -2 & -3 \\ 2 & 1 & -1 \\ 1 & 3 & -2 \end{vmatrix}$$

$$= 1(-2+3) + 2(-4+1) - 3(6-1)$$

$$= 1 - 6 - 15 = -20$$

$$\therefore [(\mathbf{a} \times \mathbf{b}) \times (\mathbf{b} \times \mathbf{c})] (\mathbf{b} \times \mathbf{c}) \times (\mathbf{c} \times \mathbf{a})$$

$$= [-20 \mathbf{b} - 20 \mathbf{c} - 20 \mathbf{a}]$$

$$= \begin{vmatrix} -40 & -20 & 20 \\ -20 & -60 & 40 \\ -20 & 40 & 60 \end{vmatrix}$$

$$= -40[-3600 - 1600] + 20[-1200 + 800] + 20[-800 - 1200]$$

$$= 208000 - 8000 - 40000 = 160000.$$

10. (b) We have, $\mathbf{n} \perp \mathbf{a}$ and $\mathbf{n} \perp \mathbf{b}$

$$\therefore \mathbf{n} = \mathbf{a} \times \mathbf{b}$$

$$= \begin{vmatrix} i & j & k \\ 1 & 2 & 3 \\ -1 & 2 & 1 \end{vmatrix} = -4i - 4j + 4k$$

$$\text{Again, } \sin \theta = \frac{|\mathbf{n} \times \mathbf{c}|}{|\mathbf{n}| |\mathbf{c}|}$$

$$\text{Now, } \mathbf{n} \times \mathbf{c} = \begin{vmatrix} i & j & k \\ -4 & -4 & 4 \\ 1 & 2 & -2 \end{vmatrix}$$

$$= -4\hat{j} - 4\hat{k}$$

$$\therefore \sin \theta = \frac{|-4\hat{j} - 4\hat{k}|}{|-4\hat{i} - 4\hat{j} + 4\hat{k}| |\hat{i} + 2\hat{j} - 2\hat{k}|}$$

$$= \frac{\sqrt{(-4)^2 + (-4)^2}}{\sqrt{(-4)^2 + (-4)^2 + (4)^2} \sqrt{(1)^2 + (2)^2 + (-2)^2}}$$

$$= \frac{\sqrt{16+16}}{\sqrt{16+16+16} \sqrt{1+4+4}}$$

$$= \frac{4\sqrt{2}}{(4\sqrt{3}) \times 3} = \frac{\sqrt{2}}{3\sqrt{3}}$$

11. (b) Let $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = \lambda$

Now,

$$|\mathbf{a} + \mathbf{b} + \mathbf{c}|^2 = (\mathbf{a} + \mathbf{b} + \mathbf{c}) \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c})$$

30 Engineering Entrances Solved Papers 2018

$$\begin{aligned}
 &= |a|^2 + |b|^2 + |c|^2 + 2(a \cdot b + b \cdot c + c \cdot a) \\
 &= \lambda^2 + \lambda^2 + \lambda^2 + 2(0 + 0 + 0) \\
 &= 3\lambda^2 \quad [\because a \perp b, b \perp c, c \perp a] \\
 \therefore |a + b + c| &= \sqrt{3} \lambda \\
 \text{Now, let } \theta &\text{ be angle between } a \text{ and } a + b + c \\
 \therefore \cos \theta &= \frac{a \cdot (a + b + c)}{|a| |a + b + c|} \\
 &= \frac{|a|^2}{|a| |a + b + c|} \\
 &= \frac{\lambda^2}{\lambda + \sqrt{3}\lambda} = \frac{1}{\sqrt{3}}.
 \end{aligned}$$

12. (b) Let $A(2a + 3b - c)$, $B(a - 2b + 3c)$

$$C(3a + 4b - 2c) \text{ and } D(ka - 6b + 6c)$$

$$\therefore AB = -a - 5b + 4c$$

$$AC = a + b - c$$

$$AD = (k - 2)a - 9b + 7c$$

Since, A, B, C, D are coplanar

$$\therefore \begin{vmatrix} -1 & -5 & 4 \\ 1 & 1 & -1 \\ k-2 & -9 & 7 \end{vmatrix} = 0$$

$$\Rightarrow (k-2)(5-4) + 9(1-4) + 7(-1+5) = 0$$

$$\Rightarrow k - 2 - 27 + 28 = 0$$

$$\Rightarrow k - 1 = 0 \Rightarrow k = 1.$$

13. (b) Let the events

E_1 = six occurs

E_2 = six does not occurs

A = the mean reports that it is six.

We have,

$$P(E_1) = \frac{1}{6}, P(E_2) = \frac{5}{6}$$

$$\text{Now, } P\left(\frac{A}{E_1}\right) = \frac{2}{3}, P\left(\frac{A}{E_2}\right) = \frac{1}{3}$$

$$\therefore P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \times P\left(\frac{A}{E_1}\right)}{P(E_1) \times P\left(\frac{A}{E_1}\right) + P(E_2) \times P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{6} \times \frac{2}{3}}{\frac{1}{6} \times \frac{2}{3} + \frac{5}{6} \times \frac{1}{3}} = \frac{2}{7}$$

The probability that it is actually six = $\frac{2}{7}$

The probability that it is not actually six

$$= 1 - \frac{2}{7} = \frac{5}{7}$$

The probability that is actually five

$$= \frac{1}{5} \times \frac{5}{7} = \frac{1}{7}.$$

14. (b) We have,

$$P(x = k) = a \left(\frac{k+1}{2^k} \right)$$

We know that,

$$\Sigma P(x = k) = 1$$

$$\Rightarrow P(x=0) + P(x=1) + P(x=2) + P(x=3) + P(x=4) + P(x=5) = 1$$

$$\Rightarrow a + \frac{2a}{2} + \frac{3a}{4} + \frac{4a}{8} + \frac{5a}{16} + \frac{6a}{32} = 1$$

$$\Rightarrow a + a + \frac{3a}{4} + \frac{1a}{2} + \frac{5a}{16} + \frac{3a}{16} = 1$$

$$\Rightarrow \frac{16a + 16a + 12a + 8a + 5a + 3a}{16} = 1$$

$$\Rightarrow 60a = 16 \Rightarrow a = \frac{16}{60} \Rightarrow a = \frac{4}{15}$$

$\therefore$ Required probability

$$= P(x=2) + P(x=3) + P(x=4)$$

$$= \frac{3a}{4} + \frac{1a}{2} + \frac{3a}{16}$$

$$= \frac{12a + 8a + 3a}{16} = \frac{23a}{16} = \frac{23}{16} \times \frac{4}{15} = \frac{23}{60}$$

15. (b) We have,

$$np = 6 \text{ and } npq = 2$$

$$\Rightarrow 6 \times q = 2 \Rightarrow q = \frac{1}{3}$$

$$\Rightarrow p = 1 - q = 1 - \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow n \times \frac{2}{3} = 6 \Rightarrow n = 9$$

$$\therefore P(5 \leq x \leq 7) = P(x=5)$$

$$= {}^9C_5 \left(\frac{2}{3}\right)^5 \left(\frac{1}{3}\right)^4 + {}^9C_6 \left(\frac{2}{3}\right)^6 \left(\frac{1}{3}\right)^3$$

$$+ {}^9C_7 \left(\frac{2}{3}\right)^7 \left(\frac{1}{3}\right)^2$$

$$= \frac{1}{3^9} \left[\frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4 \times 3 \times 2 \times 1} \times 32 + \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} \times 64 \right]$$

$$+ \frac{9 \times 8 \times 7 \times 2 \times 4 + 3 \times 8 \times 7 \times 32}{3^9 + 9 \times 4 \times 128}$$

$$= \frac{1344 + 1792 + 1536}{6561} = \frac{4672}{6561}$$

16. (d) Let the equation of the plane is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

By the definition of the centroid

$$\left(\frac{a}{3}, \frac{b}{3}, \frac{c}{3}\right) = (6, 6, 3)$$

Therefore,

$$a = 18, b = 18, c = 9$$

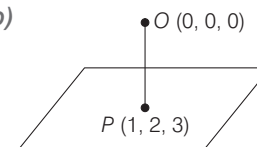
The equation of the plane becomes

$$\frac{x}{18} + \frac{y}{18} + \frac{z}{9} = 1 \Rightarrow \frac{x + y + 2z}{18} = 1$$

$$\Rightarrow x + y + 2z = 18$$

$$\Rightarrow x + y + 2z - 18 = 0$$

17. (b)



DR's of

$$OP = \langle 1 - 0, 2 - 0, 3 - 0 \rangle = \langle 1, 2, 3 \rangle$$

Since, OP is perpendicular to the plane, therefore OP is normal to the plane.

$\therefore$ Equation of plane passing through $(1, 2, 3)$ and having $d'r \langle 1, 2, 3 \rangle$ of its normal is

$$1(x-1) + 2(y-2) + 3(z-3) = 0$$

$$\Rightarrow x + 2y + 3z - 14 = 0$$

$\therefore (7, 2, 1)$ lies on the given plane.

18. (a) We have,

$$f(x) = \begin{cases} \frac{1 - \sqrt{2} \sin x}{\pi - 4x}, & x \neq \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$$

Since, $f(x)$ is continuous at

$$x = \frac{\pi}{4}$$

$$\therefore f(\pi/4) = \lim_{x \rightarrow \pi/4} f(x)$$

$$\Rightarrow k = \lim_{x \rightarrow \pi/4} \frac{1 - \sqrt{2} \sin x}{\pi - 4x}$$

$$\Rightarrow k = \lim_{x \rightarrow \pi/4} \frac{-\sqrt{2} \cos x}{-4}$$

(using L' hospital Rule)

$$\Rightarrow k = \frac{\sqrt{2}}{4} \times \frac{1}{\sqrt{2}} \Rightarrow k = \frac{1}{4}$$

19. (d) We have,

$$f(x) = x^{\tan^{-1} x}$$

$$\Rightarrow \log f(x) = \tan^{-1} x \log x$$

$$\therefore \frac{1}{f(x)} \cdot \frac{d}{dx} f(x) = \frac{1}{1+x^2} \log x + \frac{\tan^{-1} x}{x}$$

$$\Rightarrow \frac{df(x)}{dx} = x^{\tan^{-1} x} \left[\frac{\log x}{1+x^2} + \frac{\tan^{-1} x}{x} \right]$$

$$\text{Also, } g(x) = \sec^{-1} \left(\frac{1}{2x^2 - 1} \right)$$

$$= \cos^{-1} (2x^2 - 1) = 2\cos^{-1} x$$

$$\therefore \frac{d}{dx} g(x) = \frac{-2}{\sqrt{1-x^2}}$$

$$\begin{aligned} \therefore \frac{df(x)}{d \cdot g(x)} &= \frac{\frac{df(x)}{dx}}{\frac{dg(x)}{dx}} \\ &= \frac{x^{\tan^{-1} x} \left[\frac{\log x}{1+x^2} + \frac{\tan^{-1} x}{x} \right]}{\frac{-2}{\sqrt{1-x^2}}} \\ &= \frac{-1}{2} \sqrt{1-x^2} \cdot x^{\tan^{-1} x} \left[\frac{\log x}{1+x^2} + \frac{\tan^{-1} x}{x} \right] \end{aligned}$$

20. (d) We have,

$$\begin{aligned} x &= 3 \cos t \text{ and } y = 4 \sin t \\ \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} &= \cos^2 t + \sin^2 t \\ \Rightarrow \frac{x^2}{9} + \frac{y^2}{16} &= 1 \\ \Rightarrow \frac{2x}{9} + \frac{2y}{16} \frac{dy}{dx} &= 0 \\ \Rightarrow \frac{dy}{dx} &= \frac{-16}{9} \frac{x}{y} \\ \Rightarrow \frac{d^2 y}{dx^2} &= \frac{-16}{9} \left[\frac{1 \cdot y - x \frac{dy}{dx}}{y^2} \right] \\ \Rightarrow \frac{d^2 y}{dx^2} &= \frac{-16}{9} \left[\frac{y - x \left(\frac{-16x}{9y} \right)}{y^2} \right] \\ &= \frac{-16}{9} \left[\frac{9y^2 + 16x^2}{9y^3} \right] = \frac{-16}{9} \times \frac{144}{9y^3} \\ &\quad \left[\because \frac{x^2}{9} + \frac{y^2}{16} = 1 \right] \\ &= \frac{-16 \times 144}{81 \times (2\sqrt{2})^3} \left[\because (x_0, y_0) = \left(\frac{3\sqrt{2}}{2}, 2\sqrt{2} \right) \right] \\ &= \frac{-16 \times 144}{81 \times 16\sqrt{2}} = \frac{-144}{81 \times \sqrt{2}} = \frac{-16}{9\sqrt{2}} = \frac{-8\sqrt{2}}{9} \end{aligned}$$

21. (b) We have,

$$\begin{aligned} y &= \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left[\sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right] \\ \Rightarrow \frac{dy}{dx} &= \left[\frac{2}{\sqrt{a^2 - b^2}} \cdot \frac{1}{1 + \left(\frac{a-b}{a+b} \right) \tan^2 \frac{x}{2}} \right] \\ &\quad \cdot \left[\sqrt{\frac{a-b}{a+b}} \cdot \sec^2 \frac{x}{2} \cdot \frac{1}{2} \right] \\ &= \frac{\sec^2 x / 2}{a+b} \cdot \frac{a+b}{(a+b) + (a-b) \tan^2 x / 2} \\ &= \frac{\sec^2 x / 2}{a+b + a \tan^2 \frac{x}{2} - b \tan^2 x / 2} \end{aligned}$$

$$\begin{aligned} &= \frac{\sec^2 x / 2}{a \left(1 + \tan^2 \frac{x}{2} \right) + b \left(1 - \tan^2 \frac{x}{2} \right)} \\ &= \frac{\sec^2 x / 2}{\left(1 + \tan^2 \frac{x}{2} \right)} \left[a + b \frac{1 - \tan^2 x / 2}{1 + \tan^2 x / 2} \right] \\ \frac{dy}{dx} &= \frac{1}{a + b \cos x} \\ \therefore \frac{d^2 y}{dx^2} &= \frac{-1}{(a + b \cos x)^2} \cdot (-b \sin x) \\ \Rightarrow \frac{d^2 y}{dx^2} &= \frac{b \sin x}{(a + b \cos x)^2} \\ \therefore \frac{d^2 y}{dx^2} \Big|_{x=\frac{\pi}{2}} &= \frac{b \sin \pi / 2}{\left(a + b \cos \frac{\pi}{2} \right)^2} = \frac{b}{a^2} \end{aligned}$$

22. (*) $\because f'(x) > 0$

$$\begin{aligned} \Rightarrow 3x^2 + 2ax + b + 10 \sin x \cos x &> 0 \\ \Rightarrow 3x^2 + 2ax + b + 5 \sin 2x &> 0 \\ \Rightarrow 3x^2 + 2ax + b - 5 &> 0 \\ \Rightarrow 3x^2 + 2ax + (b-5) &> 0 \\ \text{Here, } a > 0, a < 0 \\ \therefore (2a)^2 - 4 \times 3 \times (b-5) &< 0 \\ 4a^2 - 12(b-5) &< 0 \\ a^2 - 3(b-5) &< 0 \\ a^2 - 3b + 15 &< 0. \end{aligned}$$

23. (d) Let $y = \cos(x)$, $x = 30^\circ$ and $x + \Delta x = 31^\circ$

$$\text{Let us find } (y)_{x=\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$$

Then, $\Delta x = 1^\circ = 0.0174$ radian.

Consider the given function,

$$y = f(x) = \cos(x)$$

Differentiating w.r.t. x ,

$$\frac{dy}{dx} = -\sin(x)$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{x=\frac{\pi}{6}} = -\sin \frac{\pi}{6}$$

$$\Rightarrow \left(\frac{dy}{dx} \right)_{x=\frac{\pi}{6}} = -\frac{1}{2}$$

Let Δy be the change in y due to the change Δx in x .

$$\begin{aligned} \therefore \Delta y &= \frac{dy}{dx} \times \Delta x \\ &= \left(-\frac{1}{2} \right) \times 0.0174 \\ &= (-0.5) \times 0.0174 \approx -0.0087 \\ \therefore f(36^\circ) &= y + \Delta y \\ &= \frac{\sqrt{3}}{2} - 0.0087 \end{aligned}$$

$$\begin{aligned} &= \frac{1.732}{2} - 0.0087 \\ &= 0.8660 - 0.0087 \\ &= 0.8573 \end{aligned}$$

24. (d) Let, $s = x^2 + y^2$

$$= x^2 + (32 - x)^2 \quad [\because x + y = 32]$$

$$\begin{aligned} \therefore \frac{ds}{dx} &= 2x + 2(32 - x)(-1) \\ &= 2x - 2(32 - x) \\ &= 2x - 64 + 2x \\ &= 4x - 64 \end{aligned}$$

For maxima or minima,

$$\frac{ds}{dx} = 0$$

$$\begin{aligned} \Rightarrow 4x - 64 &= 0 \Rightarrow x = 16 \\ \text{and } y &= 32 - x = 32 - 16 = 16 \end{aligned}$$

$$\text{Again, } \frac{d^2 s}{dx^2} = 4 = + \text{ive}$$

$\therefore s$ is minimum when $x = y = 16$

$\therefore$ Minimum value of

$$s = 16^2 + 16^2 = 256 + 256 = 512.$$

25. (b) Using mean value theorem, we have

$$\begin{aligned} f'(c) &= \frac{f(2) - f(1)}{2 - 1} \\ \Rightarrow \frac{-14}{(4c - 1)^2} &= \frac{1 - (5/3)}{1} \\ \Rightarrow \frac{-14}{(4c - 1)^2} &= -\frac{2}{3} \\ \Rightarrow (4c - 1)^2 &= 21 \\ \Rightarrow (4c - 1) &= \pm \sqrt{21} \\ \Rightarrow 4c &= 1 + \sqrt{21} \text{ or } 4c = 1 - \sqrt{21} \\ \Rightarrow c &= \frac{1 + \sqrt{21}}{4} \end{aligned}$$

$$\left[\because c = \frac{1 - \sqrt{21}}{4} \notin [1, 2] \right]$$

$$\begin{aligned} 26. (c) \text{ Let } I &= \int \frac{\sin 2x \, dx}{\sin^4 x + \cos^4 x} \\ &= \int \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x} \, dx \end{aligned}$$

Dividing by $\cos^2 x$ in numerator and denominator, we get

$$= \int \frac{2 \tan x \sec^2 x}{1 + (\tan^2 x)^2} \, dx$$

Again let $\tan^2 x = t$

$$\begin{aligned} \therefore I &= \int \frac{dt}{1 + t^2} \\ &= \tan^{-1} t + C \\ &= \tan^{-1}(\tan^2 x) + C \end{aligned}$$

32 Engineering Entrances Solved Papers 2018

By comparing with $\tan^{-1} f(x) + C$, we get

$$f(x) = \tan^2 x$$

$$\therefore f\left(\frac{\pi}{3}\right) = \left\{\tan\left(\frac{\pi}{3}\right)\right\}^2 = (\sqrt{3})^2 = 3.$$

27. (c) We have,

$$\begin{aligned} I &= \int \left\{ \frac{\log x - 1}{1 + (\log x)^2} \right\}^2 dx \\ &= \int e^t \frac{(t-1)^2}{(t^2+1)^2} dt, \text{ where } t = \log x \\ &= \int e^t \frac{t^2 + 1 - 2t}{(t^2+1)^2} dt \\ &= \int e^t \left\{ \frac{1}{t^2+1} + \frac{-2t}{(t^2+1)^2} \right\} dt \\ &= \frac{e^t}{t^2+1} + C = \frac{x}{(\log x)^2 + 1} + C. \end{aligned}$$

28. (d) Let $I = \int \frac{dx}{x^3 + 3x^2 + 2x}$

$$= \int \frac{dx}{x(x^2 + 3x + 2)}$$

$$= \int \frac{dx}{x(x+1)(x+2)}$$

Let $\frac{1}{x(x+1)(x+2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x+2}$

$$\Rightarrow 1 = A(x+1)(x+2) + Bx(x+2) + Cx(x+1)$$

Put $x = 0$, we get

$$A = \frac{1}{2}$$

Put $x = -1$, we get

$$B = -1$$

Put $x = -2$, we get

$$C = \frac{1}{2}$$

$$\begin{aligned} \therefore I &= \int \left(\frac{1}{2x} - \frac{1}{x+1} + \frac{1}{2(x+2)} \right) dx \\ &= \frac{1}{2} \log x - \log(x+1) + \frac{1}{2} \log(x+2) + C \\ &= \frac{1}{2} [\log x - 2 \log(x+1) + \log(x+2)] + C \\ &= \frac{1}{2} \log \left| \frac{x(x+2)}{(x+1)^2} \right| + C \\ &= \frac{1}{2} \log \left| \frac{x^2 + 2x}{(x+1)^2} \right| + C. \end{aligned}$$

29. (b) We have,

$$I_n = \int \sec^n x \, dx$$

$$\therefore I_2 = \int \sec^2 x \, dx = \tan x + c_1$$

$$\text{and } I_4 = \int \sec^4 x \, dx$$

$$= \int \sec^2 x \cdot \sec^2 x \, dx$$

$$= \int (\tan^2 x + 1) \sec^2 x \, dx$$

$$= \frac{\tan^3 x}{3} + \tan x + c_2$$

$$\therefore I_4 = \frac{2}{3} I_2$$

$$= \frac{\tan^3 x}{3} + \tan x + c_2 - \frac{2}{3} \tan x - \frac{2c_1}{3}$$

$$= \frac{1}{3} \tan^3 x + \frac{1}{3} \tan x + c$$

$$\left[\text{where } c_2 - \frac{2}{3} c_1 = c \right]$$

$$= \frac{1}{3} \tan x (\tan^2 x + 1) + c$$

$$= \frac{1}{3} \tan x \sec^2 x + c.$$

30. (d) We have,

$$\lim_{x \rightarrow \infty} \left[\frac{\sqrt{1} + 2\sqrt{2} + 3\sqrt{3} + \dots + n\sqrt{n}}{n^{5/2}} \right]$$

$$= \lim_{x \rightarrow \infty} \sum_{r=1}^n \left(\frac{r}{n} \right)^{3/2} \cdot \frac{1}{n}$$

$$= \int_0^1 x^{3/2} dx = \left[\frac{x^{5/2}}{5/2} \right]_0^1 = \frac{2}{5}.$$

31. (d) Let

$$I = \int_0^{\alpha/3} \frac{f(x)}{f(x) + f\left(\frac{\alpha-3x}{3}\right)} dx \quad \dots (i)$$

$\Rightarrow$

$$I = \int_0^{\alpha/3} \frac{f\left(\frac{\alpha}{3} - x\right)}{f\left(\frac{\alpha}{3} - x\right) + f\left(\frac{\alpha-3\left(\frac{\alpha}{3} - x\right)}{3}\right)} dx$$

$$\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right]$$

$$\Rightarrow I = \int_0^{\alpha/3} \frac{f\left(\frac{\alpha-3x}{3}\right)}{f\left(\frac{\alpha-3x}{3}\right) + f(x)} dx \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2I = \int_0^{\alpha/3} 1 \, dx \Rightarrow 2I = [x]_0^{\alpha/3}$$

$$\Rightarrow 2I = \frac{\alpha}{3} \Rightarrow I = \frac{\alpha}{6}.$$

32. (a) We have,

$$y = 1 - x - 6x^2$$

$$\Rightarrow y = -[6x^2 + x - 1]$$

$$\Rightarrow y = -6 \left[x^2 + \frac{x}{6} - \frac{1}{6} \right]$$

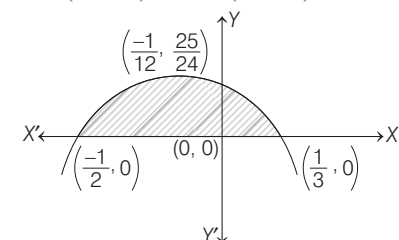
$$\Rightarrow y = -6 \left[\left(x + \frac{1}{12} \right)^2 - \frac{1}{144} - \frac{1}{6} \right]$$

$$\Rightarrow y = -6 \left(x + \frac{1}{12} \right)^2 + \frac{1}{24} + 1$$

$$\Rightarrow y = -6 \left(x + \frac{1}{12} \right)^2 + \frac{25}{24}$$

$$\Rightarrow 6 \left(x + \frac{1}{12} \right)^2 = - \left(y - \frac{25}{24} \right)$$

$$\Rightarrow \left(x + \frac{1}{12} \right)^2 = - \frac{1}{6} \left(y - \frac{25}{24} \right)$$



$$= \int_{-1/2}^{1/3} (1 - x - 6x^2) dx$$

$$= \left[x - \frac{x^2}{2} - 2x^3 \right]_{-1/2}^{1/3}$$

$$= \left(\frac{1}{3} - \frac{1}{18} - \frac{2}{27} \right) - \left(-\frac{1}{2} - \frac{1}{8} + \frac{1}{4} \right)$$

$$= \left(\frac{18-3-4}{54} \right) - \left(\frac{-4-1+2}{8} \right)$$

$$= \frac{11}{54} + \frac{3}{8} = \frac{44+81}{216} = \frac{125}{216} \text{ sq unit}$$

33. (c) The equation of the family of parabolas with focus at the origin and X-axis as its axis is given by

$$y^2 = 4a(x+a) = 4ax + 4a^2 \dots (i)$$

$$\therefore 2y \frac{dy}{dx} = 4a \Rightarrow a = \frac{1}{2} y \frac{dy}{dx} \dots (ii)$$

From Eqs. (i) and (ii), we have

$$y^2 = 2xy \frac{dy}{dx} + y^2 \left(\frac{dy}{dx} \right)^2$$

$$\therefore \text{order} = m = 1 \text{ and degree} = n = 2$$

$$\therefore mn - m + n = 1 \times 2 - 1 + 2 = 3.$$

TS EAMCET

1. (b) $\because f \circ g(x) = x$ and

$$f(x) = 2x + \cos x + \sin^2 x$$

$\therefore g(x)$ is inverse function of $f(x)$.

$$\text{So, } 1 + (2n-1)\pi = 2x + \cos x + \sin^2 x$$

$$\Rightarrow 1 + (2n-1)\pi = 2x + \cos x + 1 - \cos^2 x$$

$$\Rightarrow 2x + \cos x - \cos^2 x = (2n-1)\pi \dots (i)$$

$$\text{For } n = 1, x = \frac{\pi}{2}$$

$$\begin{aligned} n=2, x &= \frac{3\pi}{2} \\ n=3, x &= \frac{7\pi}{2} \\ &\vdots \\ n=n, x &= \frac{2n-1}{2}\pi \end{aligned}$$

$$\begin{aligned} \text{So, } \sum_{n=1}^{99} g(1 + (2n-1)\pi) &= \frac{\pi}{2} + \frac{3\pi}{2} + \frac{7\pi}{2} + \dots + \text{upto 99 terms} \\ &= \frac{\pi}{2} [1 + 3 + 7 + \dots + \text{upto 99 terms}] \\ &= (99)^2 \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} 2. (b) \therefore A &= \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \end{bmatrix} \\ &= 6 \begin{bmatrix} 1 & -2 & -1 \\ -2 & 4 & 2 \\ -1 & 2 & 1 \end{bmatrix} \end{aligned}$$

$$A^p = A, \text{ for every } p \in N$$

$$\text{So, } A^{2016l} = A^{2017m} = A^{2018n}$$

$$= A, \text{ for every } l, m, n \in N.$$

$$\begin{aligned} \text{So, } A^{2016l} + A^{2017m} + A^{2018n} &= 3A \\ &= \frac{1}{\alpha} A \quad (\text{given}) \end{aligned}$$

$$\Rightarrow \alpha = \frac{1}{3}$$

3. (c) $\therefore A$ is 3×4 matrix, so rank of $A \leq 3$.

$\therefore$ It is given that rank of $A = 3$

$$\therefore \begin{vmatrix} 1 & r & r^2 \\ r & r^2 & 1 \\ r^2 & 1 & r \end{vmatrix} \neq 0$$

$$\Rightarrow r^3 + r^3 + r^3 - r^6 - 1 - r^3 \neq 0$$

$$\Rightarrow -(r^3 - 1)^2 \neq 0 \Rightarrow r \neq 1$$

$$\therefore r \in R - \{1\}$$

4. (d) The given system of equations

$$x + y + z = 9$$

$$2x + 5y + 7z = 52$$

$$\text{and } x + 7y + 11z = 77$$

$$\therefore \Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 1 & 7 & 11 \end{vmatrix}$$

$$= 55 + 7 + 14 - 5 - 49 - 22 = 0$$

$$\Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 77 & 7 & 11 \end{vmatrix}$$

$$= 495 + 539 + 364 - 385 - 441 - 572 = 0$$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 1 & 77 & 11 \end{vmatrix} = 0$$

$$\text{and } \Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 1 & 7 & 77 \end{vmatrix} = 0$$

$$\therefore \Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$\therefore$ System of equations have infinitely many solutions.

$$\begin{aligned} 5. (c) \frac{x^4 + x^3 + 2x^2 - 2x + 1}{x^3 + x^2} &= x + \frac{2x^2 - 2x + 1}{x^3 + x^2} \\ &= x + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} \quad [\text{let}] \\ \Rightarrow 2x^2 - 2x + 1 &= Ax(x+1) + B(x+1) + Cx^2 \end{aligned}$$

By comparing the coefficients of x^2 , x and constant terms.

$$A + C = 2, A + B = -2 \text{ and } B = 1$$

$$\Rightarrow A = -3, B = 1 \text{ and } C = 5$$

$$\text{So, } A + B + C = 3 \text{ and } P(x) = x$$

$$\text{So, } A + B + C = P(3)$$

$$\begin{aligned} 6. (b) 2 \tan^{-1} \frac{1}{5} + \sec^{-1} \frac{5\sqrt{2}}{7} + 2 \tan^{-1} \frac{1}{8} &= 2 \left[\tan^{-1} \frac{1}{5} + \tan^{-1} \frac{1}{8} \right] + \sec^{-1} \frac{5\sqrt{2}}{7} \\ &= 2 \tan^{-1} \frac{\frac{1}{5} + \frac{1}{8}}{1 - \frac{1}{40}} + \sec^{-1} \frac{5\sqrt{2}}{7} \\ &= 2 \tan^{-1} \frac{13}{39} + \sec^{-1} \frac{5\sqrt{2}}{7} \\ &= 2 \tan^{-1} \frac{1}{3} + \tan^{-1} \frac{1}{7} \\ &\quad \left[\because \sec^{-1} \frac{5\sqrt{2}}{7} = \tan^{-1} \frac{1}{7} \right] \\ &= \tan^{-1} \frac{(2/3)}{1 - (1/9)} + \tan^{-1} \frac{1}{7} \end{aligned}$$

$$\begin{aligned} &= \tan^{-1} \frac{6}{8} + \tan^{-1} \frac{1}{7} = \tan^{-1} \frac{\frac{3}{4} + \frac{1}{7}}{1 - \left(\frac{3}{4} \times \frac{1}{7}\right)} \\ &= \tan^{-1} \left(\frac{21 + 4}{28 - 3} \right) = \tan^{-1}(1) = \frac{\pi}{4} \end{aligned}$$

7. (a) $\therefore \mathbf{BA}$ is parallel to $2\hat{i} - \hat{j} + 2\hat{k}$, so

$$\mathbf{BA} = \pm |\mathbf{BA}| \frac{2\hat{i} - \hat{j} + 2\hat{k}}{\sqrt{2^2 + 1^2 + 2^2}}$$

$$= \pm \frac{18}{3} (2\hat{i} - \hat{j} + 2\hat{k})$$

$$= \pm (12\hat{i} - 6\hat{j} + 12\hat{k})$$

$$\therefore \mathbf{OA} - \mathbf{OB} = \pm (12\hat{i} - 6\hat{j} + 12\hat{k})$$

$$\Rightarrow \mathbf{OA} = (3\hat{i} + \hat{j} - \hat{k}) \pm (12\hat{i} - 6\hat{j} + 12\hat{k})$$

$$= 15\hat{i} - 5\hat{j} + 11\hat{k} \text{ or } -9\hat{i} + 7\hat{j} - 13\hat{k}$$

8. (c) Let the required vector is

$$a\hat{i} + b\hat{j} + c\hat{k}$$

$\therefore$ The above vector is parallel to $2\hat{i} - 2\hat{j} - 4\hat{k}$, so

$$\frac{a}{2} = \frac{b}{-2} = \frac{c}{-4} = k \quad (\text{Let}) \quad \dots(i)$$

and coplanar with the vectors $\hat{i} + \hat{j}$ and $\hat{j} + \hat{k}$, so

$$\begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ a & b & c \end{vmatrix} = 0$$

$$\Rightarrow 1(c - b) + 1(a) = 0$$

$$\Rightarrow a - b + c = 0 \quad \dots(ii)$$

Now, only option $\hat{i} - \hat{j} - 2\hat{k}$ satisfy the relation Eqs. (i) and (ii).

9. (a) Equation of given line is

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z+3}{2} = r \quad (\text{let}) \quad \dots(i)$$

The point on the line is

$$P(2r+1, r+2, 2r-3)$$

Now, equation of plane is

$$(\mathbf{r} - (\hat{i} + \hat{j} + \hat{k})) \cdot \{(\hat{i} + \hat{j} + \hat{k}) - (\hat{i} - \hat{j} - \hat{k})\} \times (\hat{i} - 2\hat{j}) = 0$$

$$\Rightarrow (\mathbf{r} - (\hat{i} + \hat{j} + \hat{k})) \cdot [(2\hat{j} + 2\hat{k}) \times (\hat{i} - 2\hat{j})] = 0$$

$$\Rightarrow (\mathbf{r} - (\hat{i} + \hat{j} + \hat{k})) \cdot (4\hat{i} + 2\hat{j} - 2\hat{k}) = 0$$

$$\Rightarrow 4x + 2y - 2z = 4 \quad \dots(ii)$$

$\therefore$ Point P is point of intersection of line Eq. (i) and plane Eq. (ii).

$$\therefore 4(2r+1) + 2(r+2) - 2(2r-3) = 4$$

$$\Rightarrow r = -\frac{5}{3}$$

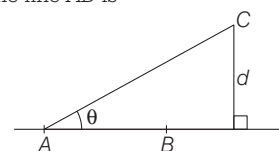
$$\text{So, point } P \text{ is } \left(-\frac{7}{3}, \frac{1}{3}, -\frac{19}{3}\right)$$

$$\text{or } \frac{1}{3}(-7\hat{i} + \hat{j} - 19\hat{k})$$

10. (c) The given vectors $A(\hat{i} + 2\hat{j} + \hat{k})$,

$B(2\hat{i} - \hat{j} + 2\hat{k})$, $C(\hat{i} + \hat{j} + 2\hat{k})$, so the

perpendicular distance of point C from the line AB is



$$|\mathbf{AC}| \sin \theta = \frac{|\mathbf{AB} \times \mathbf{AC}|}{|\mathbf{AB}|} = d \quad (\text{let})$$

34 Engineering Entrances Solved Papers 2018

$$\therefore \mathbf{AB} = \hat{i} - 3\hat{j} + \hat{k}, \mathbf{AC} = -\hat{j} + \hat{k}$$

$$\text{So, } d = \frac{\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 1 \\ 0 & -1 & 1 \end{vmatrix}}{\sqrt{1+9+1}} = \frac{|-2\hat{i} - \hat{j} - \hat{k}|}{\sqrt{11}} = \frac{\sqrt{4+1+1}}{\sqrt{11}} = \sqrt{\frac{6}{11}}$$

11. (b) Given vertices of a tetrahedron are $A(4\hat{i} + 5\hat{j} + \hat{k})$, $B(-\hat{j} + \hat{k})$

$$C(3\hat{i} + 9\hat{j} + 4\hat{k}), D(-2\hat{i} + 4\hat{j} + 4\hat{k})$$

$$\text{So, } \mathbf{AB} = -4\hat{i} - 6\hat{j}$$

$$\mathbf{AC} = -\hat{i} + 4\hat{j} + 3\hat{k}$$

$$\mathbf{AD} = -6\hat{i} - \hat{j} + 3\hat{k}$$

Now, volume of the given tetrahedron is

$$V = \frac{1}{6} \begin{vmatrix} -4 & -6 & 0 \\ -1 & 4 & 3 \\ -6 & -1 & 3 \end{vmatrix} = \frac{1}{6} \times |[-4(12+3) + 6(-3+18) + 0]| = \frac{1}{6} |[-60 + 90]| = \frac{30}{6} = 5 \text{ cubic units}$$

12. (a) The vectors

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) = [\mathbf{a} \mathbf{c} \mathbf{d}] \mathbf{b} - [\mathbf{b} \mathbf{c} \mathbf{d}] \mathbf{a}$$

$$(\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) = [\mathbf{a} \mathbf{d} \mathbf{b}] \mathbf{c} - [\mathbf{c} \mathbf{d} \mathbf{b}] \mathbf{a}$$

$$\text{and } (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c}) =$$

$$[\mathbf{a} \mathbf{d} \mathbf{c}] \mathbf{b} - [\mathbf{a} \mathbf{d} \mathbf{b}] \mathbf{c}$$

$$\therefore (\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) + (\mathbf{a} \times \mathbf{c}) \times (\mathbf{d} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{d}) \times (\mathbf{b} \times \mathbf{c})$$

$$= [\mathbf{a} \mathbf{c} \mathbf{d}] \mathbf{b} - [\mathbf{b} \mathbf{c} \mathbf{d}] \mathbf{a}$$

$$+ [\mathbf{a} \mathbf{d} \mathbf{b}] \mathbf{c} - [\mathbf{c} \mathbf{d} \mathbf{b}] \mathbf{a}$$

$$+ [\mathbf{a} \mathbf{d} \mathbf{c}] \mathbf{b} - [\mathbf{a} \mathbf{d} \mathbf{b}] \mathbf{c}$$

$$[\because [\mathbf{a} \mathbf{c} \mathbf{d}] = -[\mathbf{a} \mathbf{d} \mathbf{c}]]$$

$$= -2[\mathbf{b} \mathbf{c} \mathbf{d}] \mathbf{a}$$

$\therefore$ Given vector is parallel to $\mathbf{a}$.

13. (b) From ground floor to 7 floor, a person can leave the lift cabin in six floor (beginning with the first floor). Since, 5 persons leaving the cabin at different floors, so number of way to do so is ${}^6P_5$.

And total number of possible ways = 6^5

$$\text{So, required probability} = \frac{{}^6P_5}{6^5} = \frac{6 \times 5 \times 4 \times 3 \times 2}{6 \times 6 \times 6 \times 6 \times 6} = \frac{5}{54}$$

14. (c) Total number of items produced in a day = 10000.

Let E_1, E_2, E_3 be the events of selecting a item produced by Machines A, B, C respectively. Then,

$$P(E_1) = \frac{2500}{10000} = \frac{25}{100} = \frac{5}{20}$$

$$P(E_2) = \frac{3500}{10000} = \frac{35}{100} = \frac{7}{20}$$

$$P(E_3) = \frac{4000}{10000} = \frac{40}{100} = \frac{8}{20}$$

Let E be the event of selecting a defective item. Then,

$$P\left(\frac{E}{E_1}\right) = \frac{2}{100}, P\left(\frac{E}{E_2}\right) = \frac{3}{100},$$

$$P\left(\frac{E}{E_3}\right) = \frac{5}{100}$$

$$\text{So, required probability} = P\left(\frac{E_3}{E}\right)$$

$$= \frac{P(E_3) \cdot P\left(\frac{E}{E_3}\right)}{P(E_1) \cdot P\left(\frac{E}{E_1}\right) + P(E_2) \cdot P\left(\frac{E}{E_2}\right) + P(E_3) \cdot P\left(\frac{E}{E_3}\right)}$$

$$= \frac{\left(\frac{8}{20} \times \frac{5}{100}\right)}{\left(\frac{5}{20} \times \frac{2}{100}\right) + \left(\frac{7}{20} \times \frac{3}{100}\right) + \left(\frac{8}{20} \times \frac{5}{100}\right)}$$

$$= \frac{40}{10 + 21 + 40} = \frac{40}{71}$$

15. (b) The number of telephone calls per hour is a random variable with mean = 5.

The required probability is given by

$$P(r > 2) = 1 - P(r \leq 2) = 1 - \sum_{r=0}^2 \frac{e^{-5} \cdot 5^r}{r!} = 1 - \left[\frac{5^0}{0!} + \frac{5^1}{1!} + \frac{5^2}{2!} \right] \frac{1}{e^5} = 1 - \frac{37}{2e^5} = \frac{2e^5 - 37}{2e^5}$$

16. (a) Given relations

$$al + bm + cn = 0 \quad \dots(i)$$

$$\text{and } mn + nl + lm = 0 \quad \dots(ii)$$

From relations Eqs. (i) and (ii),

$$\left(\frac{al + bm}{-c} \right) (l + m) + lm = 0,$$

[by eliminating 'n']

$$\Rightarrow al^2 + (a + b - c)lm + bm^2 = 0$$

$$\Rightarrow a\left(\frac{l}{m}\right)^2 + (a + b - c)\left(\frac{l}{m}\right) + b = 0$$

$\dots(iii)$

$$\text{Let roots of Eq. (iii) is } \frac{l_1}{m_1} \text{ and } \frac{l_2}{m_2},$$

$$\therefore \frac{l_1 l_2}{m_1 m_2} = \frac{b}{a} \quad \dots(iv)$$

$$\text{Similarly, } \frac{l_1 l_2}{n_1 n_2} = \frac{c}{a} \quad \dots(v)$$

$$\text{So, } \frac{l_1 l_2}{\frac{1}{a}} = \frac{m_1 m_2}{\frac{1}{b}} = \frac{n_1 n_2}{\frac{1}{c}}$$

[from Eqs. (iv) and (v)]

If lines whose direction cosines given by the relation Eqs. (i) and (ii) are

$$\text{perpendicular if } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

17. (b) If the plane $ax + by + cz = 1$, passes through (1, 2, 3), then $a + 2b + 3c = 1$.

$$18. (c) \text{ Given, } f(x) = \begin{cases} \frac{x-2}{|x-2|} + a, & x < 2 \\ a + b, & x = 2 \\ \frac{x-2}{|x-2|} + b, & x > 2 \end{cases}$$

If $f(x)$ is continuous at $x = 2$, then

$$-1 + a = a + b = 1 + b$$

$$\text{So, } a = 1, b = -1$$

$$\therefore a + b = 0$$

19. (c) Given, $y = \frac{(\sin^{-1} x)^2}{2}$

$$\Rightarrow 2y_1 = 2(\sin^{-1} x) \times \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \sqrt{1-x^2} y_1 = \sin^{-1} x$$

$$\Rightarrow \sqrt{1-x^2} y_2 = \frac{2x}{2\sqrt{1-x^2}}$$

$$y_1 = \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow (1-x^2)y_2 - xy_1 = 1$$

20. (c) Given that relative errors

$$\frac{\Delta r}{r} = \frac{\Delta h}{h} = 0.02$$

$$\text{For cone, the volume is } V = \frac{1}{3} \pi r^2 h$$

$$\text{percentage error } \frac{\Delta V}{V} \times 100 = \left(2 \frac{\Delta r}{r} + \frac{\Delta h}{h} \right) \times 100 = 6$$

21. (c) Given curve is

$$x = a(1 + \cos \theta), y = a \sin \theta$$

$$\Rightarrow (x-a)^2 + y^2 = a^2, \text{ is a circle.}$$

The normal to the circle always passes through the centre of the circle.

So, normal at a point to the given curve passes through centre (a, 0).

22. (d) According to Lagrange's Mean value theorem

$$g'(c) = \frac{g(6) - g(0)}{6 - 0} = \frac{f(6) - f(0)}{6 + 1 - 0 + 1}$$

$$= \frac{-\frac{4}{7} - 12}{6} = -\frac{88}{42} = -\frac{44}{21}$$

23. (b) For turning point $f'(x) = 0$

$$\Rightarrow 6x^2 - 30x + 36 = 0$$

$$\Rightarrow (x - 2)(x - 3) = 0$$

$$\Rightarrow x = 2, 3$$

$$\text{and } f(2) = 20 \text{ and } f(3) = 19$$

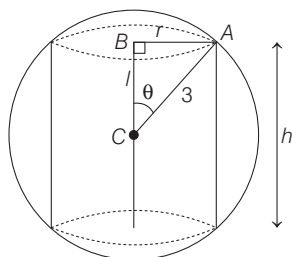
$$\text{So, } (\alpha, \beta) = (2, 20) \text{ and } (\gamma, \delta) = (3, 19)$$

$$\text{So, } \alpha - \gamma - \beta + \delta$$

$$= 2 - 3 - 20 + 19 = -2$$

24. (b) Since volume of required cylinder

$$(V) = \pi r^2 h \quad \dots(i)$$



$$\therefore l^2 + r^2 = 9$$

$$\Rightarrow r^2 = 9 - l^2$$

$$\text{and } h = 2l$$

$$\text{So, } V = \pi(9 - l^2)h = \pi\left(9 - \frac{h^2}{4}\right)h$$

$$\text{For maximum volume } \frac{dV}{dh} = 0$$

$$\Rightarrow \pi\left(9 - \frac{3h^2}{4}\right) = 0 \Rightarrow h = 2\sqrt{3}$$

25. (b) $I = \int \frac{dx}{(e^x + e^{-x})^2} = \int \frac{e^{2x}}{(e^{2x} + 1)^2} dx$

$$\text{Let } (e^{2x} + 1) = t \Rightarrow 2e^{2x} dx = dt$$

$$\text{So, } I = \frac{1}{2} \int \frac{dt}{t^2} = -\frac{1}{2t} + c$$

$$= -\frac{1}{2(e^{2x} + 1)} + c$$

26. (d) $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1 + (\tan x)^{\sqrt{2018}}}$

$$= \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{\sqrt{2018}}}{(\cos x)^{\sqrt{2018}} + (\sin x)^{\sqrt{2018}}} dx \quad \dots(i)$$

By using property

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{\sqrt{2018}}}{(\sin x)^{\sqrt{2018}} + (\cos x)^{\sqrt{2018}}} dx \quad \dots(ii)$$

By adding Eqs. (i) and (ii),

$$2I = \int_0^{\frac{\pi}{2}} \frac{(\sin x)^{\sqrt{2018}} + (\cos x)^{\sqrt{2018}}}{(\sin x)^{\sqrt{2018}} + (\cos x)^{\sqrt{2018}}} dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} 1 dx = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$$

27. (c) Let $\frac{x}{(x^2 + 1)(x - 1)} = \frac{P}{x - 1} + \frac{Qx + R}{x^2 + 1}$

$$\Rightarrow x = P(x^2 + 1) + (Qx + R)(x - 1)$$

On comparing the coefficients of different terms

$$P + Q = 0 \quad \dots(i)$$

$$R - Q = 1 \quad \dots(ii)$$

$$\text{and } P - R = 0 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii),

$$P = R = \frac{1}{2} \text{ and } Q = -\frac{1}{2}$$

$$\text{So, } I = \int \frac{x}{(x^2 + 1)(x - 1)} dx$$

$$= \frac{1}{2} \int \frac{dx}{x - 1} + \frac{1}{2} \int \frac{-x + 1}{x^2 + 1} dx$$

$$= \frac{1}{2} \log |x - 1| - \frac{1}{4} \int \frac{2x dx}{x^2 + 1}$$

$$+ \frac{1}{2} \int \frac{dx}{x^2 + 1}$$

$$= \frac{1}{2} \log |x - 1| - \frac{1}{4} \log |x^2 + 1|$$

$$+ \frac{1}{2} \tan^{-1}(x) + d$$

$$\text{So, } A = -\frac{1}{4}, B = \frac{1}{2}, C = \frac{1}{2}$$

$$\therefore A + B + C = -\frac{1}{4} + \frac{1}{2} + \frac{1}{2} = \frac{3}{4}$$

28. (b) Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos^3 x}{\sin x + \cos x} dx \quad \dots(i)$

By using property

$$\int_0^a f(x) dx = \int_0^a f(a - x) dx$$

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x}{\cos x + \sin x} dx \quad \dots(ii)$$

By adding Eqs. (i) and (ii),

$$2I = \int_0^{\frac{\pi}{2}} \frac{\sin^3 x + \cos^3 x}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{(\sin x + \cos x)(\sin^2 x + \cos^2 x - \sin x \cos x)}{\sin x + \cos x} dx$$

$$= \int_0^{\frac{\pi}{2}} (1 - \sin x \cos x) dx$$

$$2I = \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{2} \sin 2x\right) dx$$

$$= \frac{\pi}{2} - \frac{1}{2} \left(\frac{-\cos 2x}{2} \right)_{\frac{\pi}{2}}^0 = \frac{\pi}{2} + \frac{1}{4}(-1 - 1)$$

$$\Rightarrow 2I = \frac{\pi}{2} - \frac{1}{2} \Rightarrow I = \frac{\pi - 1}{4}$$

29. (c) $\int_0^3 (2x + x^2) dx = \int_0^3 f(x) dx$

$$\text{where, } f(x) = (2 + x^2)$$

$$\text{Now, } \int_0^3 f(x) dx = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{3n} f\left(\frac{r}{n}\right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{3n} \left[2 + \left(\frac{r}{n}\right)^2 \right]$$

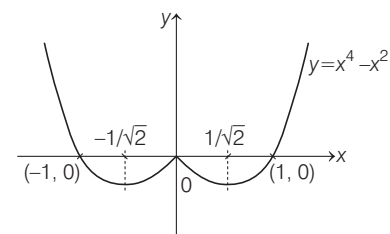
$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[(2 \times 3n) + \frac{1^2 + 2^2 + 3^2 + \dots + (3n)^2}{n^2} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[6n + \frac{1^2 + 2^2 + 3^2 + \dots + 9n^2}{n^2} \right]$$

30. (d) For points of minimum

$$\frac{dy}{dx} = 0 \Rightarrow 4x^3 - 2x = 0$$

$$\Rightarrow x = \pm \frac{1}{\sqrt{2}}$$



$$\text{The required area} = 2 \left| \int_0^{\frac{1}{\sqrt{2}}} (x^4 - x^2) dx \right|$$

$$= 2 \left[\frac{x^5}{5} - \frac{x^3}{3} \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= 2 \left| \frac{1}{5 \times 4\sqrt{2}} - \frac{1}{3 \times 2\sqrt{2}} \right|$$

$$= 2 \left| \frac{3 - 10}{60\sqrt{2}} \right| = \frac{14}{60\sqrt{2}} = \frac{7}{30\sqrt{2}}$$

31. (d) Equation of family of circles having centres on X-axis and passing through the origin is

$$x^2 + y^2 + 2ax = 0 \quad \dots(i)$$

On differentiation

$$2x + 2y \frac{dy}{dx} + 2a = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), on the elimination of $2a$

$$x^2 + y^2 + \left(-2x - 2y \frac{dy}{dx}\right)x = 0$$

36 Engineering Entrances Solved Papers 2018

$$\Rightarrow x^2 + y^2 - 2x^2 - 2xy \frac{dy}{dx} = 0$$

$$\Rightarrow y^2 - x^2 - 2xy \frac{dy}{dx} = 0$$

32. (b) Given differential equation

$$\frac{dy}{dx} = \frac{y^2}{x^2 + xy}$$

put $y = vx$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

Now, the differential equation will be

$$v + x \frac{dv}{dx} = \frac{v^2 x^2}{x^2 + vx^2}$$

$$\Rightarrow v + x \frac{dv}{dx} = \frac{v^2}{1 + v}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v^2}{1 + v} - v = \frac{v^2 - v - v^2}{1 + v}$$

$$= -\frac{v}{1 + v}$$

$$\Rightarrow \frac{1 + v}{v} dv = -\frac{dx}{x}$$

$$\Rightarrow \log v + v = -\log x - \log c$$

$$\Rightarrow \log\left(\frac{y}{x}\right) + \left(\frac{y}{x}\right) = -\log(cx)$$

$$\log\left(\frac{y}{x} \times cx\right) = -\frac{y}{x}$$

$$\Rightarrow cy = e^{-\frac{y}{x}} \Rightarrow e^{\frac{y}{x}} = cy$$

33. (a) According to the question,

$$\frac{dy}{dx} = x + xy \Rightarrow \frac{dy}{dx} = xdx$$

$$\Rightarrow \log|1 + y| = \frac{x^2}{2} + c \quad \dots(i)$$

$\therefore$ The curve (i) passes through the point (0, 1)

$$\therefore \log 2 = c$$

$\therefore$ The equation of required curve is

$$\log(1 + y) = \frac{x^2}{2} + \log 2$$

$$\Rightarrow \frac{1 + y}{2} = e^{\frac{x^2}{2}} \Rightarrow y = 2e^{\frac{x^2}{2}} - 1$$

VIT

1. (c) We have,

$$\frac{1}{x^3} + \frac{1}{y^3} = \frac{1}{a^3}$$

$$\Rightarrow \frac{1}{3}x^{-\frac{2}{3}} + \frac{1}{3}y^{-\frac{2}{3}} \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{x^{-\frac{2}{3}}}{y^{-\frac{2}{3}}} = -\frac{y^{\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$\therefore \left(\frac{dy}{dx}\right)_{\left(\frac{a}{8}, \frac{a}{8}\right)} = -1$$

The equation of the tangent at $\left(\frac{a}{8}, \frac{a}{8}\right)$

is given by

$$y - \frac{a}{8} = -1\left(x - \frac{a}{8}\right) \Rightarrow x + y - \frac{a}{4} = 0$$

The x and y intercepts of this line on the coordinate axes are each equal to $\frac{a}{4}$. So, we have

$$\left(\frac{a}{4}\right)^2 + \left(\frac{a}{4}\right)^2 = 2 \Rightarrow \left(\frac{a}{4}\right)^2 = 1$$

$$\Rightarrow \frac{a}{4} = 1 \quad (\because a > 0)$$

$$\Rightarrow a = 4$$

2. (b) We have,

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 - \omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$[\because 1 + \omega + \omega^2 = 0 \text{ and } \omega^4 = \omega^3 \cdot \omega = \omega]$$

On applying, $R_1 \rightarrow R_1 + R_2 + R_3$

$$= \begin{vmatrix} 3 & 1 + \omega + \omega^2 & 1 + \omega + \omega^2 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= 3[\omega^2 - \omega^4] = 3(\omega^2 - \omega) = 3\omega(\omega - 1)$$

3. (b) We know that,

$$I^3 - A^3 = (I - A)(I^2 + A^2 + IA)$$

$$\Rightarrow I - A^3 = (I - A)(I + A^2 + A)$$

$$[\because I^3 = I^2 = I \text{ and } IA = A]$$

$$\Rightarrow I - 0 = (I - A)(I + A^2 + A)$$

$$[\text{given, } A^3 = 0]$$

$$\Rightarrow (I + A + A^2) = \frac{I}{I - A} = (I - A)^{-1}$$

4. (b) We have,

$$|A| = 0$$

Let A be a square matrix of order n.

$$\text{Then, } |\text{adj } A| = |A|^{n-1}$$

$$\Rightarrow |\text{adj } A| = 0 \quad [\because |A| = 0]$$

$\therefore$ adj A is a singular matrix.

5. (a) We have,

$$\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$$

$$\Rightarrow (c^2 - ab) - a(c - a) + b(b - c) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$\Rightarrow \frac{1}{2} \{(a-b)^2 + (b-c)^2 + (c-a)^2\} = 0$$

$$\Rightarrow a - b = 0, b - c = 0, c - a = 0$$

$$\Rightarrow a = b = c$$

$\therefore \Delta ABC$ is an equilateral triangle.

$$\Rightarrow A = B = C = \frac{\pi}{3}$$

$$\therefore \sin^2 A + \sin^2 B + \sin^2 C$$

$$= \sin^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{3} + \sin^2 \frac{\pi}{3}$$

$$= \frac{3}{4} + \frac{3}{4} + \frac{3}{4} = \frac{9}{4}$$

6. (d) The given line is parallel to the vector $\vec{b} = \hat{i} + 2\hat{j} + 2\hat{k}$ and the given plane is normal to the vector $\vec{n} = 2\hat{i} - \hat{j} + \sqrt{\lambda}\hat{k}$

$$\therefore \sin \theta = \frac{|\vec{b} \cdot \vec{n}|}{|\vec{b}| |\vec{n}|}$$

$$\Rightarrow \frac{1}{3} = \frac{2 - 2 + 2\sqrt{\lambda}}{3 \times \sqrt{\lambda + 5}}$$

$$\Rightarrow \sqrt{\lambda + 5} = 2\sqrt{\lambda} \Rightarrow \lambda = \frac{5}{3}$$

7. (d) We know that the lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and}$$

$$\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2} \text{ are coplanar,}$$

$$\text{if } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

So, the given lines will be coplanar iff

$$\begin{vmatrix} 1 - 2 & 4 - 3 & 5 - 4 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -1 & 1 & 1 \\ 1 & 1 & -k \\ k & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow -1(1 + 2k) - 1(1 + k^2) + 1(2 - k) = 0$$

$$\Rightarrow -1 - 2k - 1 - k^2 + 2 - k = 0$$

$$\Rightarrow k^2 + 3k = 0$$

$$\Rightarrow k(k + 3) = 0$$

$$\Rightarrow k = 0, -3$$

8. (d) We have,

$$y^2 dx + (x^2 - xy + y^2) dy = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y^2}{x^2 - xy + y^2}$$

On putting $y = vx$ and $\frac{dy}{dx} = v + x \frac{dv}{dx}$,

we get

$$v + x \frac{dv}{dx} = -\frac{v^2}{1 - v + v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{-v - v^3}{v^2 - v + 1}$$

$$\Rightarrow \frac{v^2 - v + 1}{v(v^2 + 1)} dv = -\frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{v} - \frac{1}{v^2 + 1} \right) dv = -\frac{dx}{x}$$

On integrating, we get

$$\log v - \tan^{-1} v = -\log x + C$$

$$\Rightarrow \log \left(\frac{y}{x} \right) - \tan^{-1} \frac{y}{x} = -\log x + C$$

$$\Rightarrow \log y = \tan^{-1} \frac{y}{x} + C$$

9. (a) $(A(A+B)^{-1}B)^{-1}$

$$= B^{-1}(A+B)A^{-1} = B^{-1}AA^{-1} + B^{-1}BA^{-1}$$

$$= B^{-1}I + IA^{-1}$$

$$[\because AA^{-1} = I \text{ and } B^{-1}B = I]$$

$$= B^{-1} + A^{-1}$$

$$[\because B^{-1}I = B^{-1} \text{ and } IA^{-1} = A^{-1}]$$

$$\therefore (A(A+B)^{-1}B)^{-1}(AB)$$

$$= (B^{-1} + A^{-1})AB = B^{-1}AB + A^{-1}AB$$

$$= B^{-1}(BA) + A^{-1}AB \quad [\because AB = BA]$$

$$= IA + IB = A + B$$

10. (b) Let

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\left(\frac{x+1}{x-1} \right)^2 + \left(\frac{x-1}{x+1} \right)^2 - 2 \right] dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left[\left[\frac{x+1}{x-1} - \frac{x-1}{x+1} \right]^2 \right] dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{x+1}{x-1} - \frac{x-1}{x+1} \right| dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left| \frac{4x}{x^2-1} \right| dx$$

$$= 2 \int_0^{\frac{1}{2}} \left| \frac{4x}{x^2-1} \right| dx = 2 \int_0^{\frac{1}{2}} \left(-\frac{4x}{x^2-1} \right) dx$$

$$\left[\because \left| \frac{4x}{x^2-1} \right| = -\frac{4x}{x^2-1}, 0 \leq x \leq \frac{1}{2} \right]$$

$$= -4 \left[\log |x^2 - 1| \right]_0^{\frac{1}{2}} = 4 \log \left(\frac{4}{3} \right)$$

11. (b) Let $I = \int \frac{1}{\sqrt{e^{5x}} \sqrt[4]{(e^{2x} + e^{-2x})^3}} dx$

$$= \int \frac{1}{\sqrt{e^{5x}} \cdot e^{\frac{3}{2}} (1 + e^{-4x})^{\frac{3}{4}}} dx$$

$$= \int \frac{1}{e^{4x} (1 + e^{-4x})^{\frac{3}{4}}} dx$$

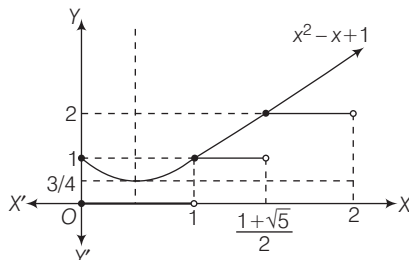
$$\text{Again let } 1 + e^{-4x} = t$$

$$\therefore -4e^{-4x} dx = dt$$

$$\therefore 7 = -\frac{1}{4} \int \frac{dt}{t^{\frac{3}{4}}} = -\frac{1}{4} \cdot \frac{t^{\frac{1}{4}}}{\frac{1}{4}} + C$$

$$= -(1 + e^{-4x})^{\frac{1}{4}} + C$$

12. (c) The graph of $[x^2 - x + 1]$ is as shown below



$$\therefore I = \int_0^2 [x^2 - x + 1] dx$$

$$= \int_0^1 0 dx + \int_1^{\frac{1+\sqrt{5}}{2}} 1 dx + \int_{\frac{1+\sqrt{5}}{2}}^2 2 dx$$

$$= \left(\frac{1+\sqrt{5}}{2} - 1 \right) + 2 \left(2 - \left(\frac{1+\sqrt{5}}{2} \right) \right)$$

$$= \frac{5-\sqrt{5}}{2}$$

13. (a) We have,

$$(1 + y^2) \frac{dx}{dy} + x = e^{\tan^{-1} y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{e^{\tan^{-1} y}}{1+y^2}$$

$$\therefore \text{IF} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

Hence, solution is given by

$$xe^{\tan^{-1} y} = \int \frac{e^{\tan^{-1} y} e^{\tan^{-1} y}}{1+y^2} dy + C$$

$$\Rightarrow xe^{\tan^{-1} y} = \frac{1}{2} e^{2 \tan^{-1} y} + C$$

14. (c) We have,

$$\ln \left(\frac{dy}{dx} \right) = ax + by \Rightarrow \frac{dy}{dx} = e^{ax+by}$$

$\therefore$ Order = 1 and degree = 1

15. (b) We have,

$$y = x^3 - ax^2 + 48x + 7$$

$$\therefore \frac{dy}{dx} = 3x^2 - 2ax + 48 > 0$$

$[\because y \text{ is an increasing function}]$

$\therefore$ Discriminant, $D < 0$

$$\Rightarrow 4a^2 - 4 \times 3 \times 48 < 0$$

$$\Rightarrow a^2 - 144 < 0 \Rightarrow a \in (-12, 12)$$

16. (c) Required probability

= 1 - Probability of getting equal number heads and tails

= 1 - Probability of getting n heads and n tails

$$= 1 - {}^{2n}C_n \left(\frac{1}{2} \right)^n \left(\frac{1}{2} \right)^{2n-n} = 1 - {}^{2n}C_n \left(\frac{1}{4} \right)^n$$

17. (a) We have

$$|\vec{a} + \vec{b}| < 1 \Rightarrow |\vec{a} + \vec{b}|^2 < 1$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) < 1$$

$$\Rightarrow 1 + 1 + 2\cos \theta < 1$$

$$\Rightarrow 2 + 2\cos \theta < 1 \Rightarrow \cos \theta < -\frac{1}{2}$$

$$\Rightarrow \frac{2\pi}{3} < \theta < \frac{4\pi}{3}$$

18. (b) We have, $\vec{a} \times \vec{b} = 2\hat{i} - 2\hat{j} + \hat{k}$

$$\therefore |(\vec{a} \times \vec{b}) \times \vec{c}| = |\vec{a} \times \vec{b}| |\vec{c}| \sin 30^\circ$$

$$= \frac{3}{2} |\vec{c}| \quad \dots (i)$$

$$\text{Now, } |\vec{c} - \vec{a}| = 2\sqrt{2} \quad [\text{given}]$$

$$\Rightarrow |\vec{c} - \vec{a}|^2 = 8$$

$$\Rightarrow |\vec{c}|^2 + |\vec{a}|^2 - 2(\vec{a} \cdot \vec{c}) = 8$$

$$\Rightarrow |\vec{c}|^2 + 9 - 2|\vec{c}| = 8$$

$$\Rightarrow |\vec{c}|^2 - 2|\vec{c}| + 1 = 0 \Rightarrow |\vec{c}| = 1$$

$$\text{Hence, } |(\vec{a} \times \vec{b}) \times \vec{c}| = \frac{3}{2}$$

[from Eq. (i)]

19. (c) We have, $|\vec{a} \vec{b} \vec{c}| = 27$ cu units

$$\text{Now, } [\vec{a} \vec{b} \vec{c}] = \begin{vmatrix} 1 & 2 & -1 \\ 1 & -1 & 0 \\ 1 & -1 & 1 \end{vmatrix} [\vec{a} \vec{b} \vec{c}]$$

$$\Rightarrow [\vec{a} \vec{b} \vec{c}] = -3 [\vec{a} \vec{b} \vec{c}]$$

$$\therefore \text{Required volume} = |[\vec{a} \vec{b} \vec{c}]|$$

$$= 3 |[\vec{a} \vec{b} \vec{c}]| = 3 \times 27 = 81 \text{ cu units}$$

20. (b) The given equations are

$$3l + m + 5n = 0 \quad \dots (i)$$

$$\text{and } 6mn - 2nl + 5lm = 0 \quad \dots (ii)$$

From Eq. (i), we get $m = -3l - 5n$

On putting $m = -3l - 5n$ in Eq. (ii), we get

$$6(-3l - 5n)n - 2nl + 5l(-3l - 5n) = 0$$

$$\Rightarrow 30n^2 + 45ln + 15l^2 = 0$$

$$\Rightarrow 2n^2 + 3ln + l^2 = 0$$

$$\Rightarrow (n + l)(2n + l) = 0$$

$$\Rightarrow l = -n \text{ or } l = -2n$$

When, $l = -n$, then $m = -2n$

and when $l = -2n$, then $m = n$

$\therefore$ DR's of two lines are

$$-n, -2n, n \text{ and } -2n, n, n$$

$$\Rightarrow +1, +2, -1 \text{ and } -2, 1, 1$$

$\therefore$ Angle between the two lines,

38 Engineering Entrances Solved Papers 2018

$$\cos \theta = \frac{1 \times (-2) + 2 \times 1 + (-1) \times 1}{\sqrt{1^2 + 2^2 + (-1)^2} \sqrt{(-2)^2 + 1^2 + 1}} \\ = -\frac{1}{6}$$

$$\Rightarrow \theta = \cos^{-1} \left(-\frac{1}{6} \right)$$

21. (b) If f is continuous at $x=0$, then

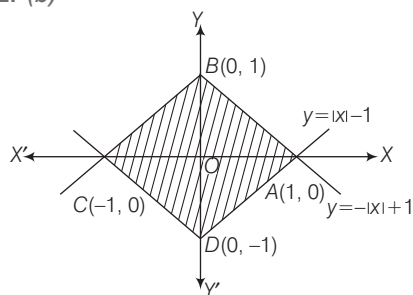
$$\lim_{x \rightarrow 0} f(x) = f(0) \\ \Rightarrow \lim_{x \rightarrow 0} \frac{\cos 3x - \cos x}{x^2} = \lambda$$

$$\Rightarrow \lim_{x \rightarrow 0} -\frac{2 \sin 2x \sin x}{x^2} = \lambda$$

$$\Rightarrow -4 \lim_{x \rightarrow 0} \left[\frac{\sin 2x}{2x} \right] \left[\frac{\sin x}{x} \right] = \lambda$$

$$\Rightarrow \lambda = -4 \times 1 \times 1 = -4$$

22. (b)



$$A = 4 (\text{Area of } \triangle CAB)$$

$$\Rightarrow A = 4 \int_0^1 y \, dx$$

$$\Rightarrow A = 4 \int_0^1 (-x + 1) \, dx$$

$$\Rightarrow A = 4 \left[-\frac{x^2}{2} + x \right]_0^1 = 2 \text{ sq units}$$

23. (a) We have,

$$\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2} \text{ for all } x \in [-1, 1]$$

$$\text{Also, } -\frac{\pi}{4} \leq \tan^{-1} x \leq \frac{\pi}{4} \text{ for all } x \in [-1, 1]$$

$$\therefore -\frac{\pi}{4} + \frac{\pi}{2} \leq \sin^{-1} x + \cos^{-1} x + \tan^{-1} x$$

$$\leq \frac{\pi}{4} + \frac{\pi}{2}$$

$$\Rightarrow \frac{\pi}{4} \leq \sin^{-1} x + \cos^{-1} x + \tan^{-1} x \leq \frac{3\pi}{4}$$

24. (d) Given lines are

$$\frac{x-3}{2} = \frac{y-2}{3} = \frac{z-1}{\lambda} \quad \dots(i)$$

$$\text{and } \frac{x-2}{3} = \frac{y-3}{2} = \frac{z-2}{3} \quad \dots(ii)$$

These lines lie in the same plane.

So, both are coplanar.

$$\therefore \begin{vmatrix} 2 & 3 & \lambda \\ 3 & 2 & 3 \\ 3-1 & 2-3 & 1-2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 3 & \lambda \\ 3 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = 0$$

$$\Rightarrow 2(-2+3) - 3(-3-3) + \lambda(-3-2) = 0$$

$$\Rightarrow \lambda = 4$$

$$\therefore \sin^{-1} \sin \lambda = \sin^{-1} \sin 4$$

$$= \sin^{-1} \sin(\pi - 4) = \pi - 4$$

MHT CET

1. (c) We have,

$$\int_0^k \frac{dx}{2 + 18x^2} = \frac{\pi}{24}$$

$$\Rightarrow \frac{1}{18} \int_0^k \frac{dx}{\left(\frac{1}{3}\right)^2 + x^2} = \frac{\pi}{24}$$

$$\Rightarrow \frac{1}{18} \times \frac{1}{1/3} [\tan^{-1} 3x]_0^k = \frac{\pi}{24}$$

$$\Rightarrow [\tan^{-1} 3k - \tan^{-1} 0] = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1} 3k = \frac{\pi}{4}$$

$$\Rightarrow 3k = \tan \frac{\pi}{4} \Rightarrow 3k = 1 \Rightarrow k = \frac{1}{3}$$

2. (d) We have,

$$I = \int \frac{dx}{\sin x \cdot \cos^2 x}$$

$$\Rightarrow I = \int \frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} dx$$

$$\Rightarrow I = \int \frac{\sin^2 x}{\sin x \cos^2 x} dx + \int \frac{\cos^2 x}{\sin x \cos^2 x} dx$$

$$\Rightarrow I = \int \sec x \tan x \, dx + \int \operatorname{cosec} x \, dx$$

$$\Rightarrow I = \sec x + \log |\operatorname{cosec} x - \cot x| + C$$

3. (d) Given,

$$\log_{10} \left(\frac{x^3 - y^3}{x^3 + y^3} \right) = 2$$

$$\Rightarrow \frac{x^3 - y^3}{x^3 + y^3} = 10^2 = 100$$

$$\Rightarrow x^3 - y^3 = 100(x^3 + y^3)$$

$$\Rightarrow 101 y^3 = -99 x^3$$

On differentiating both sides w.r.t. x , we get

$$101 \times 3y^2 \frac{dy}{dx} = -99 \cdot (3x^2)$$

$$\Rightarrow 101 y^2 \frac{dy}{dx} = -99 x^2$$

On multiplying by x both sides, we get

$$\Rightarrow 101 xy^2 \frac{dy}{dx} = -99 x^3$$

$$\Rightarrow \frac{dy}{dx} = \frac{-99 x^3}{101 xy^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{101 y^3}{101 xy^2} \quad [\because -99x^3 = 101 y^3]$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

4. (c) We have,

$$f(x) = \begin{cases} x^2 + \alpha, & x \geq 0 \\ 2\sqrt{x^2 + 1} + \beta, & x < 0 \end{cases}$$

is continuous at $x=0$.

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0^-} x^2 + \alpha = \lim_{x \rightarrow 0^+} 2\sqrt{x^2 + 1} + \beta \\ = \alpha \quad [\because f(0) = \alpha]$$

$$\Rightarrow \alpha = 2 + \beta$$

$$\Rightarrow \alpha - \beta = 2 \quad \dots(i)$$

$$\text{Given, } f\left(\frac{1}{2}\right) = 2$$

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2 + \alpha \quad \left[\because \frac{1}{2} > 0\right]$$

$$\Rightarrow 2 = \frac{1}{4} + \alpha \Rightarrow \alpha = \frac{7}{4}$$

On putting the value of α in Eq. (i), we get

$$\beta = \frac{7}{4} - 2 = -\frac{1}{4}$$

$$\text{Hence, } \alpha^2 + \beta^2 = \left(\frac{7}{4}\right)^2 + \left(-\frac{1}{4}\right)^2 \\ = \frac{49 + 1}{16} = \frac{50}{16} = \frac{25}{8}$$

5. (a) We have, $y = (\tan^{-1} x)^2$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{2 \tan^{-1} x}{1 + x^2}$$

$$\Rightarrow (1 + x^2) \frac{dy}{dx} = 2 \tan^{-1} x$$

On squaring both sides, we get

$$(1 + x^2)^2 \left(\frac{dy}{dx}\right)^2 = 4(\tan^{-1} x)^2$$

$$\Rightarrow (1 + x^2)^2 \left(\frac{dy}{dx}\right)^2 = 4y$$

$$[\because y = (\tan^{-1} x)^2]$$

Again, differentiating w.r.t. x , we get

$$(1 + x^2)^2 \left(2 \frac{dy}{dx} \cdot \frac{d^2 y}{dx^2} \right) \\ + 2(1 + x^2)(2x) \left(\frac{dy}{dx} \right)^2 \\ = 4 \frac{dy}{dx}$$

On dividing both sides by $2 \frac{dy}{dx}$, we get

$$(1 + x^2)^2 \left(\frac{d^2 y}{dx^2} \right) + 2x(1 + x^2) \frac{dy}{dx} = 4$$

6. (b) We have,

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix}$$

Now, $(A^2 - 5A)A^{-1}$

$$= A^2 \cdot A^{-1} - 5A \cdot A^{-1} = A - 5I$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 1 & 2 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \\ = \begin{bmatrix} -4 & 2 & 3 \\ -1 & -4 & 2 \\ 1 & 2 & -1 \end{bmatrix}$$

7. (c) Direction ratio of the line

perpendicular to the lines $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$

and $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ is $\alpha(\vec{b}_1 \times \vec{b}_2)$

$\therefore$ Direction ratio of line perpendicular to the lines

$$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(2\hat{i} - 2\hat{j} + \hat{k})$$

and $\vec{r} = (2\hat{i} + \hat{j} - 3\hat{k}) + \mu(\hat{i} - 2\hat{j} + 2\hat{k})$

$$\text{is } \alpha \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -2 & 1 \\ 1 & -2 & 2 \end{vmatrix}$$

$$= \alpha[(-4 + 2)\hat{i} - (4 - 1)\hat{j} + (-4 + 2)\hat{k}] \\ = \alpha[-2\hat{i} - 3\hat{j} - 2\hat{k}]$$

Now, equation of line passing through $(3, -1, 2)$ and parallel to $-2\hat{i} - 3\hat{j} - 2\hat{k}$ is

$$\vec{r} = 3\hat{i} - \hat{j} + 2\hat{k} + \beta(2\hat{i} + 3\hat{j} + 2\hat{k})$$

Hence, cartesian form of the above equation is

$$\frac{x-3}{2} = \frac{y+1}{3} = \frac{z-2}{2}$$

8. (d) We have,

$$\int \frac{dx}{\sqrt{16-9x^2}} = A \sin^{-1}(Bx) + C$$

$$\Rightarrow \frac{1}{3} \int \frac{dx}{\sqrt{(4/3)^2 - x^2}} = A \sin^{-1}(Bx) + C$$

$$\Rightarrow \frac{1}{3} \sin^{-1} \frac{3x}{4} + C = A \sin^{-1}(Bx) + C$$

On comparing both sides, we get

$$A = \frac{1}{3} \text{ and } B = \frac{3}{4}$$

$$\therefore A + B = \frac{1}{3} + \frac{3}{4} = \frac{13}{12}$$

$$9. (a) \text{ Let } I = \int e^x \left(\frac{2 + \sin 2x}{1 + \cos 2x} \right) dx$$

$$\Rightarrow I = \int e^x \left(\frac{2}{1 + \cos 2x} + \frac{\sin 2x}{1 + \cos 2x} \right) dx$$

$$\Rightarrow I = \int e^x \left[\frac{2}{2\cos^2 x} + \frac{2\sin x \cos x}{2\cos^2 x} \right] dx$$

$$\Rightarrow I = \int e^x [\sec^2 x + \tan x] dx$$

$$\Rightarrow I = \int e^x (\tan x + \sec^2 x) dx$$

$$\Rightarrow I = e^x \tan x + C$$

$$[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + C]$$

10. (d) Given,

A coin is tossed three times

and X = absolute difference between the number of heads and number of tails.

Now, $X = 1$ when exactly two head or two tail comes

$$\therefore P(X = 1) = \frac{3}{8} + \frac{3}{8} = \frac{6}{8} = \frac{3}{4}$$

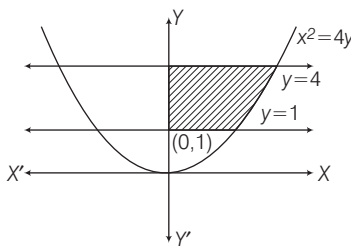
$$\left[\because \text{probability of exactly two head} = \frac{3}{8} \right. \\ \left. \text{and probability of exactly two tails} = \frac{3}{8} \right]$$

11. (b) We have,

$$x^2 = 4y$$

$$y = 1$$

$$y = 4$$



and the Y-axis lying in the first quadrant.

$$\therefore \text{Required area} = \int_1^4 \sqrt{4y} dy$$

$$= 2 \left[\frac{2}{3} y^{\frac{3}{2}} \right]_1^4 = \frac{4}{3} [4^{\frac{3}{2}} - 1]$$

$$= \frac{4}{3} (8 - 1) = \frac{4}{3} \times 7 = \frac{28}{3}$$

12. (d) We have,

$$f(x) = \frac{e^{x^2} - \cos x}{x^2}$$

is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} \frac{e^{x^2} - \cos x}{x^2}$$

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} \frac{2xe^{x^2} + \sin x}{2x}$$

[apply L'Hospital rule]

$$\Rightarrow f(0) = \lim_{x \rightarrow 0} \frac{2xe^{x^2}}{2x} + \lim_{x \rightarrow 0} \frac{\sin x}{2x} \\ = 1 + \frac{1}{2} = \frac{3}{2}$$

13. (c) We have,

$\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular vectors.

$$\therefore \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{a} \cdot \vec{c} = 0$$

$$\text{and } |\vec{a}| = 1, |\vec{b}| = 2, |\vec{c}| = 3$$

Now, $[\vec{a} + \vec{b} + \vec{c}, \vec{b} - \vec{a}, \vec{c}]$

$$= (\vec{a} + \vec{b} + \vec{c}) \cdot ((\vec{b} - \vec{a}) \times \vec{c})$$

$$= (\vec{a} + \vec{b} + \vec{c}) \cdot [(\vec{b} \times \vec{c}) - (\vec{a} \times \vec{c})]$$

$$= \vec{a} \cdot (\vec{b} \times \vec{c}) - \vec{a} \cdot (\vec{a} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{c}) - \vec{b} \cdot (\vec{a} \times \vec{c})$$

$$+ \vec{c} \cdot (\vec{b} \times \vec{c}) - \vec{c} \cdot (\vec{a} \times \vec{c})$$

$$= [\vec{a} \vec{b} \vec{c}] - [\vec{a} \vec{a} \vec{c}] + [\vec{b} \vec{b} \vec{c}] - [\vec{b} \vec{a} \vec{c}]$$

$$+ [\vec{c} \vec{b} \vec{c}] - [\vec{c} \vec{a} \vec{c}]$$

$$= [\vec{a} \vec{b} \vec{c}] - 0 + 0 + [\vec{a} \vec{b} \vec{c}] + 0 - 0$$

$$= 2[\vec{a} \vec{b} \vec{c}]$$

$$= 2\vec{a} \cdot (\vec{b} \times \vec{c}) = 2\vec{a} \cdot \left(|\vec{b}| |\vec{c}| \sin \frac{\pi}{2} \hat{n} \right)$$

$$= 2\vec{a} \cdot (2 \times 3 \times 1 \times \hat{n}) = 12\vec{a} \cdot \hat{n}$$

$$= 12 |\vec{a}| |\hat{n}| \cos 0^\circ = 12 \times 1 \times 1 \times 1 = 12$$

Hence, option (c) is correct.

14. (c) We have,

$$x - cy - bz = 0$$

$$cx - y + az = 0$$

$$\text{and } bx + ay - z = 0$$

Since, these planes are passing through a straight line.

$$\therefore \begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow (1 - a^2) + c(-c - ab) - b(ac + b) = 0$$

$$\Rightarrow 1 - a^2 - c^2 - abc - abc - b^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 = 1 - 2abc$$

15. (b) A die is thrown four time

$$\therefore n = 4$$

p (probability of getting perfect square i.e. 1, 4)

$$\text{Here, } p = \frac{2}{6} = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

$\therefore$ Required probability

$$= P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - {}^4C_0 \left(\frac{1}{3} \right)^0 \left(\frac{2}{3} \right)^4 = 1 - \frac{16}{81} = \frac{65}{81}$$

16. (b) Let

$$I = \int_0^{\pi/4} x \sec^2 x dx$$

$$\Rightarrow I = [x \int \sec^2 x dx]_0^{\pi/4}$$

$$- \int_0^{\pi/4} \left(\frac{dx}{dx} \int \sec^2 x dx \right) dx$$

[integration by parts]

40 Engineering Entrances Solved Papers 2018

$$\begin{aligned} \Rightarrow I &= [x \tan x]_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx \\ \Rightarrow I &= \left[\frac{\pi}{4} \tan \frac{\pi}{4} - 0 \right] - [\log \sec x]_0^{\pi/4} \\ \Rightarrow I &= \frac{\pi}{4} - \left(\log \sec \frac{\pi}{4} - \log \sec 0 \right) \\ I &= \frac{\pi}{4} - \log \sqrt{2} - \log 1 \\ \Rightarrow I &= \frac{\pi}{4} - \log \sqrt{2} \end{aligned}$$

17. (a) Given,

$$\begin{aligned} x &= e^\theta (\sin \theta - \cos \theta) \\ \text{and } y &= e^\theta (\sin \theta + \cos \theta) \\ \Rightarrow \frac{dx}{d\theta} &= e^\theta (\sin \theta - \cos \theta + \cos \theta + \sin \theta) \\ &= 2e^\theta \sin \theta \\ \text{and } \frac{dy}{d\theta} &= e^\theta (\sin \theta + \cos \theta + \cos \theta - \sin \theta) \\ &= 2e^\theta \cos \theta \\ \therefore \frac{dy}{dx} &= \frac{d\theta}{dx/d\theta} = \frac{2e^\theta \cos \theta}{2e^\theta \sin \theta} = \cot \theta \\ \Rightarrow \left(\frac{dy}{dx} \right)_{\theta=\frac{\pi}{4}} &= \cot \frac{\pi}{4} = 1 \end{aligned}$$

18. (c) We have,

$$\begin{aligned} \tan^{-1} 2x + \tan^{-1} 3x &= \frac{\pi}{4} \\ \Rightarrow \tan^{-1} \left(\frac{2x+3x}{1-6x^2} \right) &= \frac{\pi}{4} \\ \Rightarrow \frac{5x}{1-6x^2} &= \tan \frac{\pi}{4} \Rightarrow \frac{5x}{1-6x^2} = 1 \\ \Rightarrow 6x^2 + 5x - 1 &= 0 \\ \Rightarrow 6x^2 + 6x - x - 1 &= 0 \\ \Rightarrow (6x-1)(x+1) &= 0 \\ \Rightarrow x &= \frac{1}{6}, x \neq -1 \end{aligned}$$

19. (c) We have,

$$\begin{aligned} A &= \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 5 \\ 2 & 4 & 7 \end{bmatrix} \\ a_{31} &= 2, a_{32} = 4, a_{33} = 7 \\ \text{and } A_{31} &= \begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = 10 - 3 = 7 \\ A_{32} &= - \begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} = -(5-3) = -2 \\ A_{33} &= \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1 - 2 = -1 \\ \therefore a_{31} A_{31} + a_{32} A_{32} + a_{33} A_{33} &= 2(7) + 4(-2) + 7(-1) \\ &= 14 - 8 - 7 = -1 \end{aligned}$$

$$\begin{aligned} 20. (d) \text{ Given, } f(x) &= \frac{x}{x^2+1} \\ \Rightarrow f'(x) &= \frac{(x^2+1) \times 1 - x(2x)}{(x^2+1)^2} \\ &= \frac{(x^2+1-2x^2)}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} \end{aligned}$$

Since, $f(x)$ is increasing function.

$$\begin{aligned} \therefore f'(x) > 0 &\Rightarrow \frac{1-x^2}{(x^2+1)^2} > 0 \\ \Rightarrow 1-x^2 > 0 &\Rightarrow x^2 < 1 \Rightarrow -1 < x < 1 \\ \therefore x &\in (-1, 1) \\ \text{Hence, option (d) is correct.} \end{aligned}$$

21. (a) We have,

$$\begin{aligned} f(x) &= x \log x \\ f'(x) &= 1 + \log x \\ \text{For maxima or minima, put } f'(x) &= 0 \\ \therefore 1 + \log x &= 0 \\ \Rightarrow \log x &= -1 \Rightarrow x = e^{-1} \\ \text{Now, } f''(x) &= \frac{1}{x} \Rightarrow f''(e^{-1}) = \frac{1}{e^{-1}} = e > 0 \\ \therefore f(x) &\text{ is decreasing.} \\ \text{Hence, minimum value of } f(x) &\text{ at } x = e^{-1} \text{ is} \\ f(e^{-1}) &= e^{-1} \log e^{-1} = \frac{1}{e} (-\log e) = -\frac{1}{e} \end{aligned}$$

22. (d) Given, $n = 10$ and $p = 0.4$

$$\begin{aligned} \therefore q &= 1 - p = 1 - 0.4 = 0.6 \\ E(x) &= np = 10(0.4) = 4 \\ \sigma^2 &= npq = 10(0.4)(0.6) = 2.4 \end{aligned}$$

We know that,

$$\begin{aligned} \sigma^2 &= E(X^2) - (E(X))^2 \\ \therefore E(X^2) &= \sigma^2 + (E(X))^2 \\ \Rightarrow E(X^2) &= 2.4 + (4)^2 = 2.4 + 16 = 18.4 \end{aligned}$$

23. (a) We have,

$$\begin{aligned} \frac{dx}{dy} &= \cos(x+y) \\ \text{Put, } x+y &= v \\ \Rightarrow \frac{dx}{dy} + 1 &= \frac{dv}{dy} \Rightarrow \frac{dx}{dy} = \frac{dv}{dy} - 1 \\ \therefore \text{Equation becomes} \\ \frac{dv}{dy} - 1 &= \cos v \\ \Rightarrow \frac{dv}{dy} &= 1 + \cos v \\ \Rightarrow \frac{dv}{1 + \cos v} + dy &\Rightarrow \frac{dv}{2 \cos^2 \frac{v}{2}} = dy \end{aligned}$$

On integrating, we get

$$\frac{1}{2} \int \sec^2 \frac{v}{2} dv = \int dy$$

$$\begin{aligned} \Rightarrow \tan \frac{v}{2} &= y + C \\ \Rightarrow \tan \left(\frac{x+y}{2} \right) &= y + C \quad [\because v = x + y] \end{aligned}$$

24. (d) We have,

$$\begin{aligned} \vec{r} \cdot (\hat{p}\hat{i} - \hat{j} + 2\hat{k}) + 3 &= 0 \\ \text{and } \vec{r} \cdot (2\hat{i} - \hat{p}\hat{j} - \hat{k}) - 5 &= 0 \\ \text{Here, } \vec{n}_1 &= \hat{p}\hat{i} - \hat{j} + 2\hat{k} \\ \text{and } \vec{n}_2 &= 2\hat{i} - \hat{p}\hat{j} - \hat{k} \\ \text{Since, } \cos \theta &= \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} \\ \therefore \cos \frac{\pi}{3} &= \frac{(\hat{p}\hat{i} - \hat{j} + 2\hat{k}) \cdot (2\hat{i} - \hat{p}\hat{j} - \hat{k})}{\sqrt{p^2+1+4} \sqrt{4+p^2+1}} \\ &\quad \left[\text{given, } \theta = \frac{\pi}{3} \right] \\ \frac{1}{2} &= \frac{2p+p-2}{p^2+5} \end{aligned}$$

$$\begin{aligned} \Rightarrow p^2 + 5 &= 6p - 4 \\ \Rightarrow p^2 - 6p + 9 &= 0 \\ \Rightarrow (p-3)^2 &= 0 \\ \Rightarrow p &= 3 \end{aligned}$$

25. (d) Equation of all parabola, whose axis is parallel to X-axis and have latusrectum $4a$, is $(y-K)^2 = 4a(x-h)^2$. Here, we have two effective constants, K and h . So, we have to differentiate it twice.

Hence, order of differential equation is 2.

26. (c) Let angles α, β, γ makes positive direction of X, Y and Z-axes, respectively.

$$\begin{aligned} \therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma &= 1 \\ \text{Here, } \alpha &= 120^\circ \text{ and } \gamma = 60^\circ \\ \therefore \cos^2 120^\circ + \cos^2 \beta + \cos^2 60^\circ &= 1 \\ \Rightarrow \left(\frac{1}{2} \right)^2 + \cos^2 \beta + \left(\frac{1}{2} \right)^2 &= 1 \\ \Rightarrow \cos^2 \beta &= 1 - \frac{1}{2} = \frac{1}{2} \\ \Rightarrow \cos \beta &= \frac{\pm 1}{\sqrt{2}} \\ \Rightarrow \beta &= 135^\circ \end{aligned}$$

27. (c) Given,

$$\begin{aligned} L &= 2\vec{a} - \vec{b} \\ \text{and } M &= \vec{a} + 2\vec{b} \\ \text{Since, } N &\text{ divides } LM \text{ in the ratio } 2:1 \\ \text{externally.} \\ \therefore \text{Position vector of } N &= \frac{2(\vec{a} + 2\vec{b}) - 1(2\vec{a} - \vec{b})}{2-1} \\ &= \frac{2\vec{a} + 4\vec{b} - 2\vec{a} + \vec{b}}{1} = 5\vec{b} \end{aligned}$$

Engineering Entrance Questions 2019-20

JEE Main & Adv., BITSAT, AP & TS EAMCET, MHT-CET, WBJEE

Matrix

- Let α be a root of the equation $x^2 + x + 1 = 0$ and the matrix $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$, then the matrix A^{31} is equal to **JEE Main 2020**
(a) A^3 (b) I_3 (c) A^2 (d) A
- The number of all 3×3 matrices A , with entries from the set $\{-1, 0, 1\}$ such that the sum of the diagonal elements of AA^T is 3, is **JEE Main 2020**
- If $A = \begin{pmatrix} 2 & 2 \\ 9 & 4 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, the $10A^{-1}$ is equal to **JEE Main 2020**
(a) $6I - A$ (b) $A - 6I$
(c) $4I - A$ (d) $A - 4I$
- Let $A = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}$ ($\alpha \in R$) such that $A^{32} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then, a value of α is **JEE Main 2019**
(a) $\frac{\pi}{32}$ (b) 0 (c) $\frac{\pi}{64}$ (d) $\frac{\pi}{16}$
- If $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$, then the inverse of $\begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$ is **JEE Main 2019**
(a) $\begin{bmatrix} 1 & 0 \\ 12 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$
- The rank of the matrix $\begin{bmatrix} 3 & 5 & -1 & 4 \\ 2 & 1 & 3 & -2 \\ 8 & 11 & 1 & 6 \\ -7 & -14 & 6 & -14 \end{bmatrix}$ is **TS EAMCET 2019**
(a) 1 (b) 2 (c) 3 (d) 4
- Let A be a square matrix of order 3 whose all entries are 1 and let I_3 be the identity matrix of order 3. Then, the matrix $A - 3I_3$ is **WB JEE 2019**
(a) invertible
(b) orthogonal
(c) non-invertible
(d) real Skew Symmetric matrix

Determinant

- If the system of linear equations $2x + 2ay + az = 0$, $2x + 3by + bz = 0$, $2x + 4cy + cz = 0$ where $a, b, c \in R$ are non-zero and distinct; has a non-zero solution, then **JEE Main 2020**
(a) $a + b + c = 0$
(b) $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.
(c) a, b, c are in A.P.
(d) a, b, c are in G.P.
- Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two 3×3 real matrices such that $b_{ij} = (3)^{(i+j-2)} a_{ji}$, where $i, j = 1, 2, 3$. If the determinant of B is 81, then the determinant of A is **JEE Main 2020**
(a) $\frac{1}{9}$ (b) 3 (c) $\frac{1}{81}$ (d) $\frac{1}{3}$
- If the system of linear equations, $x + y + z = 6$, $x + 2y + 3z = 10$, $3x + 2y + \lambda z = \mu$ has more than two solutions, then $\mu - \lambda^2$ is equal to **JEE Main 2020**
- If for some α and β in R , the intersection of the following three planes $x + 4y - 2z = 1$, $x + 7y - 5z = \beta$, $x + 5y + \alpha z = 5$ is a line in R^3 , then $\alpha + \beta$ is equal to **JEE Main 2020**
(a) 0 (b) 10 (c) -10 (d) 2
- If the matrices $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}$, $B = \text{adj } A$ and $C = 3A$, then $\frac{|\text{adj } B|}{|C|}$ is equal to **JEE Main 2020**
(a) 16 (b) 2 (c) 8 (d) 72
- Let $a - 2b + c = 1$. If $f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$, then **JEE Main 2020**
(a) $f(-50) = 501$ (b) $f(50) = 1$
(c) $f(-50) = -1$ (d) $f(50) = -501$
- If the system of linear equations $x - 2y + kz = 1$, $2x + y + z = 2$, $3x - y - kz = 3$ has a solution (x, y, z) , $z \neq 0$, then (x, y) lies on the straight line whose equation is **JEE Main 2019**
(a) $3x - 4y - 4 = 0$
(b) $3x - 4y - 1 = 0$
(c) $4x - 3y - 4 = 0$
(d) $4x - 3y - 1 = 0$
- If $\Delta_1 = \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}$, $x \neq 0$, then for all $\theta \in \left(0, \frac{\pi}{2}\right)$ **JEE Main 2019**
(a) $\Delta_1 + \Delta_2 = -2(x^3 + x - 1)$
(b) $\Delta_1 - \Delta_2 = -2x^3$
(c) $\Delta_1 + \Delta_2 = -2x^3$
(d) $\Delta_1 - \Delta_2 = x(\cos 2\theta - \cos 4\theta)$
- Let $d \in R$, and $A = \begin{bmatrix} -2 & 4+d & (\sin \theta) - 2 \\ 1 & (\sin \theta) + 2 & d \\ 5 & (2\sin \theta) - d & (-\sin \theta) + 2 + 2d \end{bmatrix}$, $\theta \in [0, 2\pi]$. If the minimum value of $\det(A)$ is 8, then a value of d is **JEE Main 2019**
(a) -5 (b) -7
(c) $2(\sqrt{2} + 1)$ (d) $2(\sqrt{2} + 2)$
- Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{bmatrix}$, where $b > 0$. Then, the minimum value of $\frac{\det(A)}{b}$ is **JEE Adv. 2019**

- (a) $-\sqrt{3}$ (b) $-2\sqrt{3}$
(c) $2\sqrt{3}$ (d) $\sqrt{3}$
- 11.** If system of equation $ax + y + z = a$, $x + by + z = b$ and $x + y + cz = c$ is inconsistent, then which of the following is correct? **BITSAT 2019**
(a) $abc - a - b - c + 2 = 0$
(b) $abc - a - b - c + 3 = 0, a = 1$
(c) $abc - a - b - c + 3 = 0$
(d) $abc - a - b - c + 2 = 0, a \neq 1, b \neq 1, c \neq 1$
- 12.** Let $a, b, c \in \mathbb{R}^+$ and the system of equations $(1-a)x + y + z = 0$, $x + (1-b)y + z = 0$ and $x + y + (1-c)z = 0$ has infinitely many solutions, the minimum value of 'abc' is **BITSAT 2019**
(a) $3\sqrt{3}$ (b) 9 (c) 27 (d) 3
- 13.** If $abc \neq 0$ and the system of equations $x + 7ay + 2az = 0$, $x + 6by + 2bz = 0$, $x + 5cy + 2cz = 0$ has a non-trivial solution, then a, b, c are in **AP EAMCET 2019**
(a) harmonic progression
(b) geometric progression
(c) arithmetic progression
(d) arithmetic-geometric progression
- 14.** A value of θ for which the following system of equations has a non-trivial solution is **TS EAMCET 2019**
(4 sin θ) $x - 3y - z = 0$, $x - (6 \cos 2\theta)$
 $y + z = 0$, $3x - 12y + 4z = 0$
(a) $\tan^{-1}\left(\frac{1}{2}\right)$ (b) $\frac{\pi}{4}$
(c) $\sin^{-1}\left(\frac{3}{16}\right)$ (d) $\frac{\pi}{12}$
- 15.** Consider the following system of equations in matrix form
$$\begin{pmatrix} 1 \\ 2 \\ \lambda \end{pmatrix} \begin{pmatrix} 1 & 2 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O$$

Then which one of the following statements is true? **TS EAMCET 2019**
(a) $\forall \lambda \in (-\infty, \infty)$, the given system has non trivial solution
(b) $\forall \lambda \in (-\infty, \infty)$, the given system has only trivial solution
- (c) For $\lambda \neq 0$, the given system does not have any solution
(d) For $\lambda = 0$, the given system is inconsistent
- 16.** If $\begin{vmatrix} x^2 + 3x & x + 1 & x - 3 \\ x - 1 & 2 - x & x + 4 \\ x - 3 & x - 3 & 3x \end{vmatrix} = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, then $(a_1 + a_3) + 2(a_0 + a_2 + a_4) =$ **TS EAMCET 2019**
(a) -1 (b) 0
(c) 1 (d) -29
- 17.** Suppose $n > 1$ and A is a $n \times n$ non singular matrix such that $|\text{Adj } A| = \text{Adj } (\text{Adj } A)$. Then the matrix whose rank is n , is **MHTCET 2019**
(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix}$ (b) $\begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 2 & 0 & -1 \\ 3 & 4 & 1 & 2 \\ -2 & 3 & 2 & 5 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & 4 & -1 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$

Relations and Functions

- 1.** If $g(x) = x^2 + x - 1$ and $(g \circ f)$
 $(x) = 4x^2 - 10x + 5$, then $f\left(\frac{5}{4}\right)$ is equal to **JEE Main 2020**
(a) $-\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $-\frac{3}{2}$ (d) $\frac{1}{2}$
- 2.** Let $f : (1, 3) \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is **JEE Main 2020**
(a) $\left(\frac{2}{5}, \frac{3}{5}\right] \cup \left(\frac{3}{4}, \frac{4}{5}\right]$ (b) $\left(\frac{2}{5}, \frac{4}{5}\right]$
(c) $\left(\frac{3}{5}, \frac{4}{5}\right]$ (d) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$
- 3.** Let $A = \{x \in \mathbb{R} : x \text{ is not a positive integer}\}$. Define a function $f : A \rightarrow \mathbb{R}$ as $f(x) = \frac{2x}{x-1}$, then f is **JEE Main 2019**
(a) injective but not surjective
(b) not injective
(c) surjective but not injective
(d) neither injective nor surjective
- 4.** Let a function $f : (0, \infty) \rightarrow (0, \infty)$ be defined by $f(x) = \left|1 - \frac{1}{x}\right|$. Then, f is **JEE Main 2019**
(a) injective only
(b) both injective as well as surjective
(c) not injective but it is surjective
(d) neither injective nor surjective
- 5.** Let $f(x) = a^x (a > 0)$ be written as $f(x) = f_1(x) + f_2(x)$, where $f_1(x)$ is an even function and $f_2(x)$ is an odd function. **JEE Main 2019**
Then $f_1(x+y) + f_1(x-y)$ equals
(a) $2f_1(x+y) + f_1(x-y)$ equal
(b) $2f_1(x+y) \cdot f_2(x-y)$
(c) $2f_1(x) \cdot f_2(y)$
(d) $2f_1(x) \cdot f_1(y)$
- 6.** Let $f(x) = x^2$, $x \in \mathbb{R}$. For any $A \subseteq \mathbb{R}$, define $g(A) = \{x \in \mathbb{R} : f(x) \in A\}$. If $S = [0, 4]$, then which one of the following statements is not true? **JEE Main 2019**
(a) $f(g(S)) = S$ (b) $g(f(S)) \neq S$
(c) $g(f(S)) = g(S)$ (d) $f(g(S)) \neq f(S)$
- 7.** For $x \in \mathbb{R}$, let $[x]$ denote the greatest integer $\leq x$, then the sum of the series $\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \dots + \left[-\frac{1}{3} - \frac{99}{100}\right]$ is **JEE Main 2019**
(a) -153 (b) -133
(c) -131 (d) -135
- 8.** For $x \in \left(0, \frac{3}{2}\right)$, let $f(x) = \sqrt{x}$, $g(x) = \tan x$ and $h(x) = \frac{1-x^2}{1+x^2}$. If $\phi(x) = ((hof)og)(x)$, then $\phi\left(\frac{\pi}{3}\right)$ is equal to **JEE Main 2019**
(a) $\tan \frac{\pi}{12}$ (b) $\tan \frac{11\pi}{12}$
(c) $\tan \frac{7\pi}{12}$ (d) $\tan \frac{5\pi}{12}$
- 9.** Let $D = \mathbb{R} - \{0, 1\}$ and $f : D \rightarrow D$, $g : D \rightarrow D$ and $h : D \rightarrow D$ be three functions defined by $f(x) = \frac{1}{x}$; $g(x) = 1 - x$ and $h(x) = \frac{1}{1-x}$.
If $j : D \rightarrow D$ is such that $(gofof)(x) = f(x)$ for all $x \in D$, then which one of the following is $j(x)$? **AP EAMCET 2019**
(a) $(fog)(x)$
(b) $f(x)$
(c) $g(x)$
(d) $(goh)(x)$

- 10.** Let S, T, U be three non-void sets and $f: S \rightarrow T, g: T \rightarrow U$ be so that $gof: S \rightarrow U$ is surjective. Then, WB JEE 2019
 (a) g and f are both surjective
 (b) g is surjective, f may not be so
 (c) f is surjective, g may not be so
 (d) f and g both may not be surjective
- 11.** Let $f: R \rightarrow R$ be defined by $f(x) = x^2 - \frac{x^2}{1+x^2}$ for all $x \in R$. Then, WB JEE 2019
 (a) f is one-one but not onto mapping
 (b) f is onto but not one-one mapping
 (c) f is both one-one and onto
 (d) f is neither one-one nor onto
- 12.** If $f(x) = 3x + 6, g(x) = 4x + k$ and $fog(x) = gof(x)$ then $k = \dots\dots$ MHT CET 2019
 (a) -9 (b) 18
 (c) $\frac{1}{9}$ (d) 9

Inverse Trigonometric Functions

- 1.** If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to JEE Main 2019
 (a) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (b) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$
 (c) $\tan^{-1}\left(\frac{9}{14}\right)$ (d) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$
- 2.** If $\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha$, where $-1 \leq x \leq 1, -2 \leq y \leq 2, x \leq \frac{y}{2}$, then for all $x, y, 4x^2 - 4xy \cos \alpha + y^2$ is equal to JEE Main 2019
 (a) $2\sin^2 \alpha$ (b) $4\cos^2 \alpha + 2x^2 y^2$
 (c) $4\sin^2 \alpha$ (d) $4\sin^2 \alpha - 2x^2 y^2$
- 3.** The value of $\sin^{-1}\left(\frac{12}{13}\right) - \sin^{-1}\left(\frac{3}{5}\right)$ is equal to JEE Main 2019
 (a) $\pi - \sin^{-1}\left(\frac{63}{65}\right)$ (b) $\frac{\pi}{2} - \sin^{-1}\left(\frac{56}{65}\right)$
 (c) $\frac{\pi}{2} - \cos^{-1}\left(\frac{9}{65}\right)$ (d) $\pi - \cos^{-1}\left(\frac{33}{65}\right)$
- 4.** If α and β are the roots of the quadratic equation $3x^2 - 16x + 5 = 0$, then $\tan^{-1} \alpha + \tan^{-1} \beta - \tan^{-1}\left(\frac{\alpha + \beta}{1 - \alpha\beta}\right) =$ TS EAMCET 2019
 (a) 0 (b) π (c) $\frac{\pi}{2}$ (d) $-\pi$
- 5.** If $4\sin^{-1} x + 6\cos^{-1} x = 3\pi$ then $x = \dots\dots$ MHT CET 2019
 (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) 0 (d) $-\frac{1}{2}$

Continuity and Differentiability

- 1.** Let the function, $f: [-7, 0] \rightarrow R$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$. If $f(-7) = -3$ and $f'(x) \leq 2$, for all $x \in (-7, 0)$, then for all such functions $f, f(-1) + f(0)$ lies in the interval JEE Main 2020
 (a) $(-\infty, 11]$ (b) $[-3, 11]$
 (c) $[-6, 20]$ (d) $(-\infty, 20]$
- 2.** If the function f defined on $\left(-\frac{1}{3}, \frac{1}{3}\right)$ by $f(x) = \begin{cases} \frac{1}{x} \log_e \left(\frac{1+3x}{1-2x} \right), & \text{when } x \neq 0 \\ k, & \text{when } x = 0 \end{cases}$ is continuous, then k is equal to JEE Main 2020
- 3.** If $f(x) = \begin{cases} \frac{\sin(a+2)x + \sin x}{x} & ; x < 0 \\ b & ; x = 0 \\ \frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}} & ; x > 0 \end{cases}$ is continuous at $x = 0$, then $a + 2b$ is equal to JEE Main 2020
 (a) -2 (b) 1
 (c) 0 (d) -1
- 4.** Let f and g be differentiable functions on R such that fog is the identity function. If for some $a, b \in R, g'(a) = 5$ and $g(a) = b$, then $f'(b)$ is equal to JEE Main 2020
 (a) $\frac{1}{5}$ (b) 5 (c) $\frac{2}{5}$ (d) 1
- 5.** Let $f: [-1, 3] \rightarrow R$ be defined as $f(x) = \begin{cases} |x| + [x], & -1 \leq x < 1 \\ x + |x|, & 1 \leq x < 2 \\ x + [x], & 2 \leq x \leq 3, \end{cases}$ where, $[t]$ denotes the greatest integer less than or equal to t . Then, f is discontinuous at JEE Main 2019
 (a) four or more points
 (b) only two points
 (c) only three points
 (d) only one point
- 6.** Let $f(x) = 15 - |x - 10|; x \in R$. Then, the set of all values of x , at which the function, $g(x) = f(f(x))$ is not differentiable, is JEE Main 2019
 (a) $\{5, 10, 15, 20\}$ (b) $\{5, 10, 15\}$
 (c) $\{10\}$ (d) $\{10, 15\}$
- 7.** If $f(x) = [x] - \left[\frac{x}{4}\right], x \in R$ where $[x]$ denotes the greatest integer function, then JEE Adv. 2019
 (a) $\lim_{x \rightarrow 4+} f(x)$ exists but $\lim_{x \rightarrow 4-} f(x)$ does not exist
- (b) f is continuous at $x = 4$
 (c) Both $\lim_{x \rightarrow 4-} f(x)$ and $\lim_{x \rightarrow 4+} f(x)$ exist but are not equal
 (d) $\lim_{x \rightarrow 4} f(x)$ exists but $\lim_{x \rightarrow 4+} f(x)$ does not exist
- 8.** Let $f: R \rightarrow R$ be the function defined by $f(x) = \begin{cases} 5, & \text{if } x \leq 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \leq x < 5 \\ 30, & \text{if } x \geq 5 \end{cases}$ then f is AP EAMCET 2019
 (a) continuous if $a = 5$ and $b = 5$
 (b) continuous if $a = 0$ and $b = 5$
 (c) continuous if $a = -5$ and $b = 10$
 (d) not continuous for any values of a and b
- 9.** Let $[x]$ denote the greatest integer less than or equal to x . Then the number of points where the function $y = [x] + |1 - x|, -1 \leq x \leq 3$ is not differentiable, is AP EAMCET 2019
 (a) 1 (b) 2
 (c) 3 (d) 4
- 10.** The set of values of x for which the function $f(x) = \log \left(\frac{x-1}{x+2} \right)$ is continuous, is TS EAMCET 2019

- (a) R
 (b) $(-\infty, -2) \cup (0, \infty)$
 (c) $(-\infty, -2) \cup (1, \infty)$
 (d) $(-2, -1)$

11. If the function $f: R \rightarrow R$, defined by

$$f(x) = \begin{cases} 5 - 3x, & \text{if } x \leq \frac{5}{3} \\ x^2 - 3x + 20, & \text{if } x > \frac{5}{3} \end{cases}, \text{ then 'f'}$$

is TS EAMCET 2019

- (a) continuous at $x = \frac{5}{3}$

- (b) differentiable at $x = \frac{5}{3}$
 (c) differentiable at $x = 2$
 (d) discontinuous at $x = -2$

12. Which of the following function is not continuous at $x = 0$? MHT CET 2019

- (a) $f(x) = (1 + 2x)^{1/x}, x \neq 0$
 $\quad \quad \quad = e^2, \quad x = 0$
 (b) $f(x) = \sin x - \cos x, x \neq 0$
 $\quad \quad \quad = -1, \quad x = 0$
 (c) $f(x) = \frac{e^{1/x} - 1}{e^{1/x} + 1}, x \neq 0$
 $\quad \quad \quad = -1, \quad x = 0$

$$(d) f(x) = \frac{e^{5x} - e^{2x}}{\sin 3x}, \quad x \neq 0$$

$$= 1, \quad x = 0$$

13. If the function

$$f(x) = \frac{(e^{kx} - 1) \tan kx}{4x^2}, \quad x \neq 0$$

$$= 16, \quad x = 0$$

is continuous at $x = 0$, then $k = \dots\dots$ MHT CET 2019

- (a) $\pm \frac{1}{8}$ (b) ± 4
 (c) ± 2 (d) ± 8

Differentiation

1. If $y(\alpha) = \sqrt{2 \left(\frac{\tan \alpha + \cot \alpha}{1 + \tan^2 \alpha} \right) + \frac{1}{\sin^2 \alpha}}$,
 $\alpha \in \left(\frac{3\pi}{4}, \pi \right)$, then $\frac{dy}{d\alpha}$ at $\alpha = \frac{5\pi}{6}$ is JEE Main 2020
 (a) $-\frac{1}{4}$ (b) $\frac{4}{3}$ (c) -4 (d) 4

2. Let $y = y(x)$ be a function of x satisfying
 $y \sqrt{1 - x^2} = k - x \sqrt{1 - y^2}$ where k is a
 constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$. Then $\frac{dy}{dx}$ at
 $x = \frac{1}{2}$, is equal to JEE Main 2020
 (a) $\frac{\sqrt{5}}{2}$ (b) $-\frac{\sqrt{5}}{2}$ (c) $\frac{2}{\sqrt{5}}$ (d) $-\frac{\sqrt{5}}{4}$

3. Let
 $f(x) = (\sin(\tan^{-1} x) + \sin(\cot^{-1} x))^2 - 1$,
 $|x| > 1$. If $\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx}(\sin^{-1}(f(x)))$ and
 $y(\sqrt{3}) = \frac{\pi}{6}$, then $y(-\sqrt{3})$ is equal to JEE Main 2020
 (a) $\frac{5\pi}{6}$ (b) $-\frac{\pi}{6}$
 (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{3}$

4. If $2y = \left(\cot^{-1} \left(\frac{\sqrt{3} \cos x + \sin x}{\cos x - \sqrt{3} \sin x} \right) \right)^2$,

- $x \in \left(0, \frac{\pi}{2} \right)$ then $\frac{dy}{dx}$ is equal to JEE Main 2019
 (a) $\frac{\pi}{6} - x$ (b) $x - \frac{\pi}{6}$
 (c) $\frac{\pi}{3} - x$ (d) $2x - \frac{\pi}{3}$

5. Let $f(x) = \log_e(\sin x)$, $(0 < x < \pi)$ and
 $g(x) = \sin^{-1}(e^{-x})$, $(x \geq 0)$. If α is a
 positive real number such that
 $a = (fog)'(\alpha)$ and $b = (fog)(\alpha)$, then JEE Main 2019
 (a) $a\alpha^2 - b\alpha - a = 0$
 (b) $a\alpha^2 - b\alpha - a = 1$
 (c) $a\alpha^2 + b\alpha - a = -2\alpha^2$
 (d) $a\alpha^2 + b\alpha + a = 0$

6. If $e^y + xy = e$, the ordered pair
 $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2} \right)$ at $x = 0$ is equal to JEE Main 2019
 (a) $\left(\frac{1}{e}, -\frac{1}{e^2} \right)$ (b) $\left(-\frac{1}{e}, \frac{1}{e^2} \right)$
 (c) $\left(\frac{1}{e}, \frac{1}{e^2} \right)$ (d) $\left(-\frac{1}{e}, -\frac{1}{e^2} \right)$

7. Let K be the set of all real values of x ,
 where the function
 $f(x) = \sin |x| - |x| + 2(x - \pi) \cos |x|$
 is not differentiable. Then, the set K is
 equal to JEE Main 2019
 (a) $\{0\}$ (b) $\emptyset$ (an empty set)
 (c) $\{\pi\}$ (d) $\{0, \pi\}$

8. If $f(x) = \cot^{-1} \left(\frac{x^x - x^{-x}}{2} \right)$, then $f'(1) =$ AP EAMCET 2019
 (a) $-\log 2$ (b) $\log 2$ (c) 1 (d) -1

9. If $\sqrt{1 - x^6} + \sqrt{1 - y^6} = a(x^3 - y^3)$, then
 $y^2 \frac{dy}{dx} =$ AP EAMCET 2019
 (a) $\sqrt{\frac{1 - y^6}{1 - x^6}}$ (b) $x \sqrt{\frac{1 - y^6}{1 - x^6}}$
 (c) $x^2 \sqrt{\frac{1 - y^6}{1 - x^6}}$ (d) $\frac{1}{x^2} \sqrt{\frac{1 - y^6}{1 - x^6}}$

10. If $y = f(x)$ is twice differentiable
 function such that at a point P ,
 $\frac{dy}{dx} = 4$, $\frac{d^2y}{dx^2} = -3$, then $\left(\frac{d^2x}{dy^2} \right)_P =$ AP EAMCET 2019
 (a) $\frac{64}{3}$ (b) $\frac{16}{3}$
 (c) $\frac{3}{16}$ (d) $\frac{3}{64}$

11. If $f(x) = \cos^{-1} \left[\frac{1 - (\log x)^2}{1 + (\log x)^2} \right]$, then
 $f'(e) = \dots$ MHT CET 2019
 (a) $\frac{1}{e}$ (b) $\frac{2}{e^2}$
 (c) $\frac{2}{e}$ (d) 1

Application of Derivatives

1. The value of c in the Lagrange's mean
 value theorem for the function
 $f(x) = x^3 - 4x^2 + 8x + 11$, when
 $x \in [0, 1]$ is JEE Main 2020
 (a) $\frac{\sqrt{7} - 2}{3}$ (b) $\frac{2}{3}$
 (c) $\frac{4 - \sqrt{5}}{3}$ (d) $\frac{4 - \sqrt{7}}{3}$

2. Let $f(x) = x \cos^{-1}(-\sin |x|)$,
 $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, then which of the
 following is true? JEE Main 2020
 (a) f' is decreasing in $\left(-\frac{\pi}{2}, 0 \right)$ and
 increasing in $\left(0, \frac{\pi}{2} \right)$

- (b) f' is increasing in $\left(-\frac{\pi}{2}, 0 \right)$ and
 decreasing in $\left(0, \frac{\pi}{2} \right)$
 (c) f is not differentiable at $x = 0$
 (d) $f'(0) = -\frac{\pi}{2}$

3. Let S be the set of all functions $f : [0, 1] \rightarrow \mathbb{R}$, which are continuous on $[0, 1]$ and differentiable on $(0, 1)$. Then for every f in S , there exists a $c \in (0, 1)$, depending on f , such that **JEE Main 2020**

(a) $\frac{f(1) - f(c)}{1 - c} = f'(c)$
 (b) $|f(c) - f(1)| < |f'(c)|$
 (c) $|f(c) + f(1)| < (1 + c) |f'(c)|$
 (d) $|f(c) - f(1)| < (1 - c) |f'(c)|$

4. A spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness that melts at a rate of $50 \text{ cm}^3/\text{min}$. When the thickness of ice is 5 cm, then the rate (in cm/min .) at which of the thickness of ice decreases, is **JEE Main 2020**

(a) $\frac{5}{6\pi}$ (b) $\frac{1}{54\pi}$ (c) $\frac{1}{36\pi}$ (d) $\frac{1}{18\pi}$

5. Let $f : [0, 2] \rightarrow \mathbb{R}$ be a twice differentiable function such that $f''(x) > 0$, for all $x \in (0, 2)$. If $\phi(x) = f(x) + f(2 - x)$, then ϕ is **JEE Main 2019**
 (a) increasing on $(0, 1)$ and decreasing on $(1, 2)$
 (b) decreasing on $(0, 2)$
 (c) decreasing on $(0, 1)$ and increasing on $(1, 2)$
 (d) increasing on $(0, 2)$

6. If the tangent to the curve $y = \frac{x}{x^2 - 3}$, $x \in \mathbb{R}$, ($x \neq \pm \sqrt{3}$), at a point

$(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line $2x + 6y - 11 = 0$, then **JEE Main 2019**

(a) $|6\alpha + 2\beta| = 19$ (b) $|6\alpha + 2\beta| = 9$
 (c) $|2\alpha + 6\beta| = 19$ (d) $|2\alpha + 6\beta| = 11$

7. If $f(x) = (2k + 1)x - 3 - ke^{-x} + 2e^x$ is monotonically increasing for all $x \in \mathbb{R}$, then the least value of k is **AP EAMCET 2019**
 (a) 1 (b) 0 (c) $-\frac{1}{2}$ (d) -1

8. If the function $f(x) = ax^3 + bx^2 + 11x - 6$ satisfies the conditions of Rolle's theorem in $[1, 3]$ and $f'\left(2 + \frac{1}{\sqrt{3}}\right) = 0$, then $a + b =$ **AP EAMCET 2019**
 (a) -5 (b) -3 (c) 4 (d) 7

Maxima and Minima

1. If $f(a + b + 1 - x) = f(x)$, for all x , where a and b are fixed positive real numbers, then

$\frac{1}{a + b} \int_a^b x(f(x) + f(x + 1))dx$ is equal to **JEE Main 2020**

(a) $\int_{a+1}^{b+1} f(x + 1)dx$
 (b) $\int_{a+1}^{b+1} f(x)dx$
 (c) $\int_{a-1}^{b-1} f(x + 1)dx$
 (d) $\int_{a-1}^{b-1} f(x)dx$

2. Let a function $f : [0, 5] \rightarrow \mathbb{R}$ be continuous, $f(1) = 3$ and F be defined as:

$F(x) = \int_1^x t^2 g(t)dt$, where $g(t) = \int_1^t f(u)du$.

Then for the function F , the point $x = 1$ is **JEE Main 2020**

- (a) not a critical point
 (b) a point of inflection
 (c) a point of local maxima
 (d) a point of local minima

3. If S_1 and S_2 are respectively the sets of local minimum and local maximum points of the function, $f(x) = 9x^4 + 12x^3 - 36x^2 + 25$, $x \in \mathbb{R}$, then **JEE Main 2019**

(a) $S_1 = \{-2\}$; $S_2 = \{0, 1\}$
 (b) $S_1 = \{-2, 0\}$; $S_2 = \{1\}$
 (c) $S_1 = \{-2, 1\}$; $S_2 = \{0\}$
 (d) $S_1 = \{-1\}$; $S_2 = \{0, 2\}$

4. If $f(x)$ is a non-zero polynomial of degree four, having local extreme points at $x = -1, 0, 1$, then the set $S = \{x \in \mathbb{R} : f(x) = f(0)\}$ contains exactly **JEE Main 2019**

- (a) four rational numbers
 (b) two irrational and two rational numbers
 (c) four irrational numbers
 (d) two irrational and one rational number

5. The maximum value of the function

$f(x) = \tan\left(x + \frac{2\pi}{3}\right)$

$-\tan\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in $\left[-\frac{5\pi}{12}, \frac{-\pi}{3}\right]$ is **AP EAMCET 2019**
 (a) $\frac{11\sqrt{2}}{6}$ (b) $\frac{11\sqrt{3}}{6}$ (c) 3 (d) 1

6. For $a > 0$, if the function $f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$ attains its maximum value at p and minimum value at q such that $p^2 = q$, then $a =$ **AP EAMCET 2019**

(a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4

7. If the function $f : [-1, 1] \rightarrow \mathbb{R}$ defined by $f(x) = \begin{cases} 2^x + 1, & \text{for } x \in [-1, 0) \\ 1, & \text{for } x = 0 \\ 2^x - 1, & \text{for } x \in (0, 1] \end{cases}$ then,

in $[-1, 1]$, $f(x)$ has **AP EAMCET 2019**

- (a) a maximum
 (b) a minimum
 (c) both maximum and minimum
 (d) neither maximum nor minimum

Indefinite Integrals

1. The integral $\int \frac{dx}{(x + 4)^{8/7}(x - 3)^{6/7}}$ is equal to (where C is a constant of integration) **JEE Main 2020**

(a) $-\left(\frac{x - 3}{x + 4}\right)^{-1/7} + C$
 (b) $\frac{1}{2}\left(\frac{x - 3}{x + 4}\right)^{3/7} + C$
 (c) $\left(\frac{x - 3}{x + 4}\right)^{1/7} + C$
 (d) $-\frac{1}{13}\left(\frac{x - 3}{x + 4}\right)^{-13/7} + C$

2. If $\int \frac{d\theta}{\cos^2 \theta (\tan 2\theta + \sec 2\theta)} = \lambda \tan \theta + 2 \log_e |f(\theta)| + C$ where C is a constant of integration, then the ordered pair $(\lambda, f(\theta))$ is equal to **JEE Main 2020**

(a) $(1, 1 + \tan \theta)$ (b) $(1, 1 - \tan \theta)$
 (c) $(-1, 1 + \tan \theta)$ (d) $(-1, 1 - \tan \theta)$

3. If $\int x^5 e^{-4x^3} dx = \frac{1}{48} e^{-4x^3} f(x) + C$, where C is a constant of integration, then $f(x)$ is equal to **JEE Main 2019**

(a) $-4x^3 - 1$ (b) $4x^3 + 1$
 (c) $-2x^3 - 1$ (d) $-2x^3 + 1$

4. $\int \frac{\sin 5x}{\sin \frac{x}{2}} dx$ is equal to **JEE Main 2019**
 (where, C is a constant of integration)

(a) $2x + \sin x + 2 \sin 2x + C$
 (b) $x + 2 \sin x + 2 \sin 2x + C$
 (c) $x + 2 \sin x + \sin 2x + C$
 (d) $2x + \sin x + \sin 2x + C$

5. If $\int \frac{dx}{(x^2 - 2x + 10)^2} = A \left(\tan^{-1} \left(\frac{x-1}{3} \right) + \frac{f(x)}{x^2 - 2x + 10} \right) + C$,

where, C is a constant of integration, then JEE Main 2019

(a) $A = \frac{1}{27}$ and $f(x) = 9(x-1)$

(b) $A = \frac{1}{81}$ and $f(x) = 3(x-1)$

(c) $A = \frac{1}{54}$ and $f(x) = 3(x-1)$

(d) $A = \frac{1}{54}$ and $f(x) = 9(x-1)^2$

6. Let $\alpha \in (0, \pi/2)$ be fixed. If the integral $\int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx = A(x) \cos 2\alpha + B(x)$

$\sin 2\alpha + C$, where C is a constant of integration, then the functions $A(x)$ and $B(x)$ are respectively JEE Adv. 2019

(a) $x + \alpha$ and $\log_e |\sin(x + \alpha)|$

(b) $x - \alpha$ and $\log_e |\sin(x - \alpha)|$

(c) $x - \alpha$ and $\log_e |\cos(x - \alpha)|$

(d) $x + \alpha$ and $\log_e |\sin(x - \alpha)|$

7. The integral $\int \left(1 + x - \frac{1}{x} \right) e^{x + \frac{1}{x}} dx$ is equal to BITSAT 2019

(a) $(x-1)e^{x + \frac{1}{x}} + C$ (b) $xe^{x + \frac{1}{x}} + C$

(c) $(x+1)e^{x + \frac{1}{x}} + C$ (d) $-xe^{x + \frac{1}{x}} + C$

8. If $f'(x) = f(x) + \int_0^1 f(x) dx$, $f(0) = 1$, then $f(x) =$ BITSAT 2019

(a) $\frac{2e^x}{3-e} + \frac{1-e}{3-e}$ (b) $\frac{e^x}{3-e} + \frac{1-e}{1-e}$

(c) $\frac{3e^x}{2-e} + \frac{1+e}{1-e}$ (d) $\frac{3e^x}{2-e} + \frac{1-e}{3+e}$

9. If $\int \tan(x - \alpha) \tan(x + \alpha) \cdot \tan 2x dx = p \log |\sec 2x| + q \log |\sec(x + \alpha)| + r \log |\sec(x - \alpha)| + c$ then $p + q + r = \dots\dots\dots$ MHT CET 2019

(a) $\frac{-3}{2}$ (b) $\frac{-5}{2}$ (c) $\frac{5}{2}$ (d) $\frac{3}{2}$

10. $\int \frac{x^2 + 1}{x^4 - x^2 + 1} dx = \dots\dots\dots$ MHT CET 2019

(a) $\tan^{-1} \left(\frac{x^2 + 1}{2} \right) + c$

(b) $\tan^{-1}(x^2) + c$

(c) $\tan^{-1}(2x^2 - 1) + c$

(d) $\tan^{-1} \left(\frac{x^2 - 1}{x} \right) + c$

11. $\int \frac{x^3 + 2x}{x^4 + 4} dx =$ TS EAMCET 2019

(a) $\frac{1}{2} \left[\tan^{-1} \left(\frac{x^2}{2} \right) + \log \left(\frac{\sqrt{x^4 + 4}}{2} \right) \right] + C$

(b) $\frac{1}{2} \tan^{-1} \left(\frac{x^2 + 2}{2x} \right) + C$

(c) $\frac{1}{2} \left[\tan^{-1} \left(\frac{x^2}{2} \right) - \log \left(\frac{\sqrt{x^4 + 4}}{4} \right) \right] + C$

(d) $\frac{1}{\sqrt{2}} \tan^{-1} \left(\frac{x^2 + 1}{\sqrt{2}x} \right) + C$

12. If $\frac{2x+7}{(x^2+4)(x^2+9)(x^2+16)} = \frac{Ax+l}{x^2+4} + \frac{Bx+m}{x^2+9} + \frac{Cx+n}{x^2+16}$, then

$\frac{1}{A} + \frac{1}{B} + \frac{1}{C} =$ TS EAMCET 2019

(a) 0 (b) 27 (c) $\frac{105}{2}$ (d) $\frac{109}{2}$

13. If $\int \cos x \cdot \cos 2x \cdot \cos 5x dx = A \sin 2x + B \sin 4x + C \sin 6x + D \sin 8x + k$ (where k is the arbitrary constant of integration), then $\frac{1}{B} + \frac{1}{C} =$ TS EAMCET 2019

(a) $\frac{1}{A} - \frac{1}{D}$ (b) $\frac{1}{A} + \frac{1}{D}$

(c) 1 (d) 0

14. $\int \sqrt{1 + 2 \cot x (\cot x + \operatorname{cosec} x)} dx =$ TS EAMCET 2019

(a) $2 \log \left| \sin \frac{x}{2} \right| + c$

(b) $2 \log \left| \cos \frac{x}{2} \right| + c$

(c) $\log \left| \sin \frac{x}{2} + \cos \frac{x}{2} \right| + c$

(d) $2 \log |\sin x + \cos x| + c$

15. If $\int e^{\alpha x} \left(\frac{1 - \beta \sin x}{1 - \cos x} \right) dx = -e^x \cot \frac{x}{2} + c$, then $\frac{\alpha^2 + \beta^2}{2\alpha\beta} =$ TS EAMCET 2019

(a) -1 (b) 1
(c) 2 (d) -2

16. $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx =$ AP EAMCET 2019

(a) $2 \tan^{-1} \left(\sqrt{\frac{1+x+x^2}{x}} \right) + c$

(b) $\tan^{-1} \left(\sqrt{\frac{1+x+x^2}{x}} \right) + c$

(c) $\tan^{-1} \left(\sqrt{\frac{x}{1+x+x^2}} \right) + c$

(d) $\tan^{-1} \left(\sqrt{\frac{1+x^2}{x}} \right) + c$

17. If $I(x) = \int x^2 (\log x)^2 dx$ and $I(1) = 0$, then $I(x) =$ AP EAMCET 2019

(a) $\frac{x^3}{18} [8(\log x)^2 - 3 \log x] + \frac{7}{18}$

(b) $\frac{x^3}{27} [9(\log x)^2 + 6 \log x] - \frac{2}{27}$

(c) $\frac{x^3}{27} [9(\log x)^2 - 6 \log x + 2] - \frac{2}{27}$

(d) $\frac{x^3}{27} [9(\log x)^2 - 6 \log x - 2] + \frac{2}{27}$

Definite Integrals

1. If $f(a+b+1-x) = f(x)$, for all x , where a and b are fixed positive real numbers, then $\frac{1}{a+b} \int_a^b x(f(x) + f(x+1)) dx$ is equal to JEE Main 2020

(a) $\int_{a+1}^{b+1} f(x+1) dx$ (b) $\int_{a+1}^{b+1} f(x) dx$

(c) $\int_{a-1}^{b-1} f(x+1) dx$ (d) $\int_{a-1}^{b-1} f(x) dx$

2. If θ_1 and θ_2 be respectively the smallest and the largest values of θ in $(0, 2\pi) - \{\pi\}$ which satisfy the equation, $2 \cot^2 \theta - \frac{5}{\sin \theta} + 4 = 0$, then

$\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta$ is equal to JEE Main 2020

(a) $\frac{\pi}{3} + \frac{1}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{2\pi}{3}$ (d) $\frac{\pi}{9}$

3. If for all real triplets (a, b, c) , $f(x) = a + bx + cx^2$; then $\int_0^1 f(x) dx$ is equal to JEE Main 2020

(a) $2 \left\{ 3f(1) + 2f\left(\frac{1}{2}\right) \right\}$

(b) $\frac{1}{3} \left\{ f(0) + f\left(\frac{1}{2}\right) \right\}$

- (c) $\frac{1}{2} \left\{ f(1) + 3f\left(\frac{1}{2}\right) \right\}$
 (d) $\frac{1}{6} \left\{ f(0) + f(1) + 4f\left(\frac{1}{2}\right) \right\}$
4. If $\int_0^{\pi/3} \frac{\tan \theta}{\sqrt{2k \sec \theta}} d\theta = 1 - \frac{1}{\sqrt{2}}$, ($k > 0$), then the value of k is **JEE Main 2019**
 (a) 1 (b) $\frac{1}{2}$
 (c) 2 (d) 4
5. The value of the integral $\int_{-2}^2 \frac{\sin^2 x}{\left[\frac{x}{\pi} \right] + \frac{1}{2}} dx$ (where, $[x]$ denotes the greatest integer less than or equal to x) is **JEE Main 2019**
 (a) $4 - \sin 4$ (b) 4
 (c) $\sin 4$ (d) 0
6. The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x] + [\sin x] + 4}$, where $[t]$ denotes the greatest integer less than or equal to t , is **JEE Main 2019**
 (a) $\frac{1}{12} (7\pi - 5)$ (b) $\frac{1}{12} (7\pi + 5)$
 (c) $\frac{3}{10} (4\pi - 3)$ (d) $\frac{3}{20} (4\pi - 3)$
7. The integral $\int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x (\tan^5 x + \cot^5 x)}$ equals **JEE Main 2019**
 (a) $\frac{1}{5} \left(\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{3\sqrt{3}} \right) \right)$
- (b) $\frac{1}{20} \tan^{-1} \left(\frac{1}{9\sqrt{3}} \right)$
 (c) $\frac{1}{10} \left(\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{9\sqrt{3}} \right) \right)$
 (d) $\frac{\pi}{40}$
8. Let $f(x) = \int_0^x g(t) dt$, where g is a non-zero even function. If $f(x+5) = g(x)$, then $\int_0^x f(t) dt$ equals **JEE Main 2019**
 (a) $5 \int_{x+5}^5 g(t) dt$ (b) $\int_{x+5}^{x+5} g(t) dt$
 (c) $2 \int_5^{x+5} g(t) dt$ (d) $\int_{x+5}^5 g(t) dt$
9. $\lim_{n \rightarrow \infty} \left(\frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \dots + \frac{(2n)^{1/3}}{n^{4/3}} \right)$ is equal to **JEE Main 2019**
 (a) $\frac{4}{3} (2)^{4/3}$ (b) $\frac{3}{4} (2)^{4/3} - \frac{4}{3}$
 (c) $\frac{3}{4} (2)^{4/3} - \frac{3}{4}$ (d) $\frac{4}{3} (2)^{3/4}$
10. The value of $\int_0^{2\pi} [\sin 2x (1 + \cos 3x)] dx$, where $[t]$ denotes the greatest integer function, is **JEE Main 2019**
 (a) $-\pi$ (b) 2π (c) -2π (d) π
11. A value of α such that $\int_{\alpha+1}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)} = \log_e \left(\frac{9}{8} \right)$ is **JEE Main 2019**
 (a) -2 (b) $\frac{1}{2}$
 (c) $-\frac{1}{2}$ (d) 2
12. $\int_{\log_e 2}^x \frac{dt}{\sqrt{e^t - 1}} = \frac{\pi}{6} \Rightarrow x =$ **AP EAMCET 2019**
 (a) $2 \cdot \log_e 2$ (b) $3 \cdot \log_e 2$
 (c) $4 \cdot \log_e 2$ (d) $8 \cdot \log_e 2$
13. The value of the integral $\int_{-1}^1 \left\{ \frac{x^{2015}}{e^{|x|}(x^2 + \cos x)} + \frac{1}{e^{|x|}} \right\} dx$ is equal to **WB JEE 2019**
 (a) 0 (b) $1 - e^{-1}$
 (c) $2e^{-1}$ (d) $2(1 - e^{-1})$
14. The value of the integration $\int_{-\pi/4}^{\pi/4} \left(\lambda |\sin x| + \frac{\mu \sin x}{1 + \cos x} + \gamma \right) dx$ is independent of λ only, independent of μ only, independent of γ only, depends on λ, μ and γ **WB JEE 2019**
15. $\int_{\pi/18}^{4\pi/9} \frac{2\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx = \dots$ **MHCET 2019**
 (a) $\frac{7\pi}{36}$ (b) $\frac{5\pi}{36}$ (c) $\frac{7\pi}{18}$ (d) $\frac{5\pi}{18}$

Area Bounded by Curves

1. The area of the region, enclosed by the circle $x^2 + y^2 = 2$ which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is **JEE Main 2020**
 (a) $\frac{1}{3} (12\pi - 1)$ (b) $\frac{1}{6} (12\pi - 1)$
 (c) $\frac{1}{6} (24\pi - 1)$ (d) $\frac{1}{3} (6\pi - 1)$
2. The area (in sq. units) of the region $\{(x, y) \in R^2 \mid 4x^2 \leq y \leq 8x + 12\}$ is **JEE Main 2020**
 (a) $\frac{124}{3}$ (b) $\frac{125}{3}$ (c) $\frac{127}{3}$ (d) $\frac{128}{3}$
3. For $a > 0$, let the curves $C_1 : y^2 = ax$ and $C_2 : x^2 = ay$ intersect at origin O and a point P . Let the line $x = b$ ($0 < b < a$) intersect the chord OP and the x -axis at points Q and R , respectively. If the line $x = b$ bisects the area bounded by the curves, C_1 and C_2 , and the area of $\Delta OQR = \frac{1}{2}$, then 'a' satisfies the equation **JEE Main 2020**
- (a) $x^6 + 6x^3 - 4 = 0$
 (b) $x^6 - 12x^3 + 4 = 0$
 (c) $x^6 - 6x^3 + 4 = 0$
 (d) $x^6 - 12x^3 - 4 = 0$
4. Given, $f(x) = \begin{cases} x & , 0 \leq x < \frac{1}{2} \\ \frac{1}{2} & , x = \frac{1}{2} \\ 1-x & , \frac{1}{2} < x \leq 1 \end{cases}$ and $g(x) = \left(x - \frac{1}{2} \right)^2$, $x \in R$. Then the area (in sq. units) of the region bounded by the curves $y = f(x)$ and $y = g(x)$ between the lines, $2x = 1$ and $2x = \sqrt{3}$, is **JEE Main 2020**
 (a) $\frac{1}{2} + \frac{\sqrt{3}}{4}$ (b) $\frac{1}{3} + \frac{\sqrt{3}}{4}$
 (c) $\frac{1}{2} - \frac{\sqrt{3}}{4}$ (d) $\frac{\sqrt{3}}{4} - \frac{1}{3}$
5. The area of the region $A = \{(x, y); 0 \leq y \leq x \mid x| + 1 \text{ and } -1 \leq x \leq 1\}$ in sq. units, is **JEE Main 2019**
- (a) 2 (b) $\frac{4}{3}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$
6. The area (in sq units) of the region bounded by the curve $x^2 = 4y$ and the straight line $x = 4y - 2$ is **JEE Main 2019**
 (a) $\frac{7}{8}$ (b) $\frac{9}{8}$ (c) $\frac{5}{4}$ (d) $\frac{3}{4}$
7. Let $S(\alpha) = \{(x, y) : y^2 \leq x, 0 \leq x \leq \alpha\}$ and $A(\alpha)$ is area of the region $S(\alpha)$. If for $\lambda, 0 < \lambda < 4$, $A(\lambda) : A(4) = 2 : 5$, then λ equals **JEE Main 2019**
 (a) $2 \left(\frac{4}{25} \right)^{\frac{1}{3}}$ (b) $4 \left(\frac{2}{5} \right)^{\frac{1}{3}}$
 (c) $4 \left(\frac{4}{25} \right)^{\frac{1}{3}}$ (d) $2 \left(\frac{2}{5} \right)^{\frac{1}{3}}$
8. The region represented by $|x - y| \leq 2$ and $|x + y| \leq 2$ is bounded by a **JEE Main 2019**
 (a) rhombus of side length 2 units
 (b) rhombus of area $8\sqrt{2}$ sq units
 (c) square of side length $2\sqrt{2}$ units
 (d) square of area 16 sq units

9. The area (in sq units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is
JEE Main 2019
- (a) $\frac{3}{2}$ (b) $\log_e 2 + \frac{3}{2}$
(c) $\frac{1}{2}$ (d) $\frac{3}{2} - \frac{1}{\log_e 2}$

10. The area of the region $\{(x, y) : xy \leq 8, 1 \leq y \leq x^2\}$ is
JEE Adv. 2019
- (a) $8 \log_e 2 - \frac{14}{3}$ (b) $8 \log_e 2 - \frac{7}{3}$
(c) $16 \log_e 2 - \frac{14}{3}$ (d) $16 \log_e 2 - 6$
11. Area bounded by two branches of the curve $(y - x)^2 = x^3$ and $x = 1$ is
BITSAT 2019

- a. $\frac{5}{4}$ sq unit b. $\frac{3}{2}$ sq unit
c. $\frac{1}{4}$ sq unit d. $\frac{4}{5}$ sq unit

12. Area of the region (in square units) bounded by the curve $y = x^2 + 4$ and the line $y = 5x - 2$ is
TS EAMCET 2019
- (a) $\frac{1}{2}$ (b) $\frac{1}{12}$ (c) 2 (d) $\frac{1}{6}$

Differential Equations

1. If $y = y(x)$ is the solution of the differential equation, $e^y \left(\frac{dy}{dx} - 1 \right) = e^x$ such that $y(0) = 0$, then $y(1)$ is equal to
JEE Main 2020
- (a) $2 + \log_e 2$ (b) $2e$
(c) $1 + \log_e 2$ (d) $\log_e 2$
2. Let $y = y(x)$ be the solution curve of the differential equation, $(y^2 - x) \frac{dy}{dx} = 1$, satisfying $y(0) = 1$. This curve intersects the x -axis at a point whose abscissa is
JEE Main 2020
- (a) 2 (b) $2 - e$
(c) $-e$ (d) $2 + e$
3. If $\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$; $y(1) = 1$; then a value of x satisfying $y(x) = e$ is
JEE Main 2020
- (a) $\frac{1}{2}\sqrt{3}e$ (b) $\sqrt{3}e$ (c) $\sqrt{2}e$ (d) $\frac{e}{\sqrt{2}}$
4. Let $y = y(x)$ be the solution of the differential equation, $(x^2 + 1)^2 \frac{dy}{dx} + 2x(x^2 + 1)y = 1$ such that $y(0) = 0$. If $\sqrt{a} y(1) = \frac{\pi}{32}$, then the value of 'a' is
JEE Main 2019
- (a) $\frac{1}{4}$ (b) $\frac{1}{2}$
(c) 1 (d) $\frac{1}{16}$
5. If $\cos x \frac{dy}{dx} - y \sin x = 6x$, $\left(0 < x < \frac{\pi}{2}\right)$ and $y\left(\frac{\pi}{3}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to
JEE Main 2019
- (a) $\frac{\pi^2}{2\sqrt{3}}$ (b) $-\frac{\pi^2}{2\sqrt{3}}$ (c) $-\frac{\pi^2}{4\sqrt{3}}$ (d) $-\frac{\pi^2}{2}$
6. If $y = y(x)$ is the solution of the differential equation $\frac{dy}{dx} = (\tan x - y) \sec^2 x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

such that $y(0) = 0$, then $y\left(-\frac{\pi}{4}\right)$ is equal to
JEE Main 2019

(a) $\frac{1}{e} - 2$ (b) $\frac{1}{2} - e$ (c) $2 + \frac{1}{e}$ (d) $e - 2$

7. Let $y = y(x)$ be the solution of the differential equation, $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, such that $y(0) = 1$. Then
JEE Main 2019
- (a) $y'\left(\frac{\pi}{4}\right) - y'\left(-\frac{\pi}{4}\right) = \pi - \sqrt{2}$
(b) $y'\left(\frac{\pi}{4}\right) + y'\left(-\frac{\pi}{4}\right) = -\sqrt{2}$
(c) $y\left(\frac{\pi}{4}\right) + y\left(-\frac{\pi}{4}\right) = \frac{\pi^2}{2} + 2$
(d) $y\left(\frac{\pi}{4}\right) - y\left(-\frac{\pi}{4}\right) = \sqrt{2}$

8. The solution of the differential equation, $\frac{dy}{dx} = (x - y)^2$, when $y(1) = 1$, is
JEE Main 2019
- (a) $\log_e \left| \frac{2 - y}{2 - x} \right| = 2(y - 1)$
(b) $-\log_e \left| \frac{1 + x - y}{1 - x + y} \right| = x + y - 2$
(c) $\log_e \left| \frac{2 - x}{2 - y} \right| = x - y$
(d) $-\log_e \left| \frac{1 - x + y}{1 + x - y} \right| = 2(x - 1)$

9. Solution of the equation $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ is
BITSAT 2019
- a. $e^y = e^x - 1 + ce^{-e^x}$
b. $e^{y-x} = -1 + ce^{-e^x}$
c. $e^x + e^y = ce^{-e^x}$
d. None of these

10. The general solution of the differential equation $yy' = x \left[\frac{y^2}{x^2} + \frac{\phi\left(\frac{y^2}{x^2}\right)}{\phi'\left(\frac{y^2}{x^2}\right)} \right]$, where

ϕ is an arbitrary function, is
AP EAMCET 2019

- (a) $x \phi\left(\frac{y^2}{x^2}\right) = cy$ (b) $x^2 \phi\left(\frac{y^2}{x^2}\right) = c$
(c) $x^2 \phi\left(\frac{y^2}{x^2}\right) = cy^2$ (d) $\phi\left(\frac{y^2}{x^2}\right) = cx^2$

11. The solution of the differential equation $\frac{dy}{dx} = 1 - \cos(y - x) \cot(y - x)$ is
TS EAMCET 2019
- (a) $x \tan(y - x) = c$
(b) $x = \tan(y - x) + c$
(c) $x = \sec(y - x) + c$
(d) $x + \sec(y - x) = c$

12. If m and n are the order and degree of the differential equation of the family of parabolas with focus at the origin and X -axis as its axis, then
TS EAMCET 2019
- $mn - m + n =$
- (a) 1 (b) 2
(c) 3 (d) 4

13. The solution of the differential equation $\frac{dy}{dx} + \frac{x}{y} \cdot \frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} = 0$ is
TS EAMCET 2019
- (a) $x^2 + y^2 + 3 \log(x^2 + y^2) = c$
(b) $x^2 + 3xy - 3 \log(x^2 + y^2 + 2) = c$
(c) $x^2 + 2y^2 - 3 \log(x^2 + y^2 + 2) = c$
(d) $-x^2 - 2y^2 - 3 \log(x^2 + y^2) = c$

14. If $x \log x \frac{dy}{dx} + y = \log x^2$ and $y(e) = 0$, then $y(e^2) =$
TS EAMCET 2019
- (a) 0 (b) 1
(c) $\frac{1}{2}$ (d) $\frac{3}{2}$

Vector Algebra

1. A vector $\mathbf{a} = \alpha \hat{i} + 2\hat{j} + \beta \hat{k}$ ($\alpha, \beta \in \mathbb{R}$) lies in the plane of the vectors, $\mathbf{b} = \hat{i} + \hat{j}$ and $\mathbf{c} = \hat{i} - \hat{j} + 4\hat{k}$. If $\mathbf{a}$ bisects the angle between $\mathbf{b}$ and $\mathbf{c}$, then
- JEE Main 2020
- (a) $\mathbf{a} \cdot \hat{i} + 3 = 0$ (b) $\mathbf{a} \cdot \hat{k} + 2 = 0$
(c) $\mathbf{a} \cdot \hat{i} + 1 = 0$ (d) $\mathbf{a} \cdot \hat{k} + 4 = 0$

2. Let $\mathbf{a}, \mathbf{b}$ and $\mathbf{c}$ be three unit vectors such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$. If $\lambda = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}$ and $\mathbf{d} = \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a}$, then the ordered pair, $(\lambda, \mathbf{d})$ is equal to
- JEE Main 2020
- (a) $\left(\frac{3}{2}, 3\mathbf{b} \times \mathbf{c}\right)$ (b) $\left(-\frac{3}{2}, 3\mathbf{c} \times \mathbf{b}\right)$
(c) $\left(\frac{3}{2}, 3\mathbf{a} \times \mathbf{c}\right)$ (d) $\left(-\frac{3}{2}, 3\mathbf{a} \times \mathbf{b}\right)$

3. Let the volume of a parallelepiped whose coterminal edges are given by $\mathbf{u} = \hat{i} + \hat{j} + \lambda \hat{k}$, $\mathbf{v} = \hat{i} + \hat{j} + 3\hat{k}$ and $\mathbf{w} = 2\hat{i} + \hat{j} + \hat{k}$ be 1 cu. unit. If θ be the angle between the edges $\mathbf{u}$ and $\mathbf{w}$, then $\cos \theta$ can be
- JEE Main 2020
- (a) $\frac{5}{3\sqrt{3}}$ (b) $\frac{7}{6\sqrt{3}}$
(c) $\frac{7}{6\sqrt{6}}$ (d) $\frac{5}{7}$

4. Let $\mathbf{a} = 2\hat{i} + \lambda_1 \hat{j} + 3\hat{k}$, $\mathbf{b} = 4\hat{i} + (3 - \lambda_2)\hat{j} + 6\hat{k}$ and $\mathbf{c} = 3\hat{i} + 6\hat{j} + (\lambda_3 - 1)\hat{k}$ be three vectors such that $\mathbf{b} = 2\mathbf{a}$ and $\mathbf{a}$ is perpendicular to $\mathbf{c}$. Then a possible value of $(\lambda_1, \lambda_2, \lambda_3)$ is
- JEE Main 2019
- (a) (1, 3, 1) (b) (1, 5, 1)
(c) $\left(-\frac{1}{2}, 4, 0\right)$ (d) $\left(\frac{1}{2}, 4, -2\right)$

5. Let $\alpha = (\lambda - 2)\mathbf{a} + \mathbf{b}$ and $\beta = (4\lambda - 2)\mathbf{a} + 3\mathbf{b}$ be two given

vectors where vectors $\mathbf{a}$ and $\mathbf{b}$ are non-collinear. The value of λ for which vectors α and β are collinear, is

JEE Main 2019

(a) 4 (b) -3 (c) 3 (d) -4

6. If $4\hat{i} + 7\hat{j} + 8\hat{k}$, $2\hat{i} + 3\hat{j} + 4\hat{k}$, $2\hat{i} + 5\hat{j} + 7\hat{k}$ are respectively the position vectors of the vertices A, B, C of $\triangle ABC$, then the position vector of the point where the bisector of angle A meet BC is
- AP EAMCET 2019
- (a) $2\hat{i} + \frac{13}{3}\hat{j} + 2\hat{k}$ (b) $2\hat{i} - \frac{13}{3}\hat{j} + 6\hat{k}$
(c) $2\hat{i} + 13\hat{j} + 6\hat{k}$ (d) $2\hat{i} + \frac{13}{3}\hat{j} + 6\hat{k}$

7. If $\mathbf{OA} = 6\hat{i} + 3\hat{j} - 4\hat{k}$, $\mathbf{OB} = 2\hat{j} + \hat{k}$, $\mathbf{OC} = 5\hat{i} - \hat{j} + 2\hat{k}$ are the coterminal edges of a parallelepiped, then the height of the parallelepiped drawn from the vertex A is
- TS EAMCET 2019
- (a) $\frac{85}{3}$ (b) $\frac{5}{\sqrt{32}}$ (c) $\frac{85}{\sqrt{257}}$ (d) $\frac{17}{\sqrt{6}}$

8. If the vectors $x\hat{i} - 3\hat{j} + 7\hat{k}$ and $\hat{i} + y\hat{j} - z\hat{k}$ are collinear then the value of $\frac{xy^2}{z}$ is equal
- MHT CET 2019
- (a) $\frac{9}{7}$ (b) $\frac{-9}{7}$ (c) $\frac{-7}{9}$ (d) $\frac{7}{9}$

9. $\mathbf{a} = 4\hat{i} + 3\hat{j}$ and $\mathbf{b}$ are two vectors in XOY plane and $\mathbf{a}$ is perpendicular to $\mathbf{b}$. A vector $\mathbf{c}$ lying in the same plane and having projections 1 and 2 respectively $\mathbf{a}$ and $\mathbf{b}$ is
- TS EAMCET 2019
- (a) $2\hat{i} - \hat{j}$ (b) $2\hat{i} + \hat{j}$
(c) $\hat{i} + 2\hat{j}$ (d) $2\hat{i} + 2\hat{j}$

10. Let L be the line parallel to the vector $\sqrt{2}\hat{i} - 5\hat{j} + 3\hat{k}$ and passing through the point A given by $\hat{i} + 2\hat{j} - 3\hat{k}$. If the

distance between A and a point P on the line L is 18 units, then the position vector of such a point P is

TS EAMCET 2019

(a) $(1 - 3\sqrt{2})\hat{i} + 17\hat{j} - 12\hat{k}$
(b) $(1 + 3\sqrt{2})\hat{i} + 17\hat{j} + 12\hat{k}$
(c) $(1 + 3\sqrt{2})\hat{i} - 17\hat{j} - 12\hat{k}$
(d) $(1 - 3\sqrt{2})\hat{i} - 17\hat{j} + 12\hat{k}$

11. If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors such that $\mathbf{b} \times \mathbf{c} = \mathbf{a}$; $\mathbf{c} \times \mathbf{a} = \mathbf{b}$; $\mathbf{a} \times \mathbf{b} = \mathbf{c}$, then which of the following is not TRUE?
- MHT CET 2019
- (a) $|\mathbf{a}| - |\mathbf{b}| = 0$ (b) $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 2$
(c) $[\mathbf{a} \ \mathbf{b} \ \mathbf{c}] = 1$ (d) $|\mathbf{a}| |\mathbf{b}| |\mathbf{c}| = 1$

12. Let $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ be three unit vectors such that $\hat{\alpha} \times (\hat{\beta} \times \hat{\gamma}) = \frac{1}{2}(\hat{\beta} + \hat{\gamma})$ where $\hat{\alpha} \times (\hat{\beta} \times \hat{\gamma}) = (\hat{\alpha} \cdot \hat{\gamma})\hat{\beta} - (\hat{\alpha} \cdot \hat{\beta})\hat{\gamma}$. If $\hat{\beta}$ is not parallel to $\hat{\gamma}$, then the angle between $\hat{\alpha}$ and $\hat{\beta}$ is
- WB JEE 2019
- (a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{2\pi}{3}$

13. The position vectors of the points A, B, C and D are $3\hat{i} - 2\hat{j} - \hat{k}$, $2\hat{i} - 3\hat{j} + 2\hat{k}$, $5\hat{i} - \hat{j} + 2\hat{k}$ and $4\hat{i} - \hat{j} - \lambda\hat{k}$, respectively. If the points A, B, C and D lie on a plane, the value of λ is
- WB JEE 2019
- (a) 0 (b) 1
(c) 2 (d) -4

14. If $\mathbf{a} + \mathbf{b}, \mathbf{b} + \mathbf{c}$ and $\mathbf{c} + \mathbf{a}$ are coterminal edges of a parallelepiped then its volume is
- WB JEE 2019
- (a) $3[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (b) 0
(c) $2[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$ (d) $4[\mathbf{a} \ \mathbf{b} \ \mathbf{c}]$

Three Dimensional Geometry

1. Let P be a plane passing through the points $(2, 1, 0)$, $(4, 1, 1)$ and $(5, 0, 1)$ and R be any point $(2, 1, 6)$. Then the image of R in the plane P is
- JEE Main 2020
- (a) (6, 5, 2)
(b) (4, 3, 2)
(c) (6, 5, -2)
(d) (3, 4, -2)

2. The mirror image of the point $(1, 2, 3)$ in a plane is $\left(-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}\right)$. Which of the following points lies on this plane?
- JEE Main 2020
- (a) (1, -1, 1) (b) (-1, -1, 1)
(c) (1, 1, 1) (d) (-1, -1, -1)

3. The equation of the plane containing the straight line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and

perpendicular to the plane containing the straight lines $\frac{x}{3} = \frac{y}{4} = \frac{z}{2}$ and

$\frac{x}{4} = \frac{y}{2} = \frac{z}{3}$ is

JEE Main 2019

(a) $5x + 2y - 4z = 0$
(b) $x + 2y - 2z = 0$
(c) $3x + 2y - 3z = 0$
(d) $x - 2y + z = 0$

4. The plane which bisects the line segment joining the points $(-3, -3, 4)$ and $(3, 7, 6)$ at right angles, passes through which one of the following points? **JEE Main 2019**
 (a) $(4, -1, 7)$ (b) $(2, 1, 3)$
 (c) $(-2, 3, 5)$ (d) $(4, 1, -2)$
5. On which of the following lines lies the point of intersection of the line, $\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1}$ and the plane, $x + y + z = 2$? **JEE Main 2019**
 (a) $\frac{x-4}{1} = \frac{y-5}{1} = \frac{z-5}{-1}$
 (b) $\frac{x+3}{3} = \frac{4-y}{3} = \frac{z+1}{-2}$
 (c) $\frac{x-2}{2} = \frac{y-3}{2} = \frac{z+3}{3}$
 (d) $\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+4}{-5}$
6. If the point $(2, \alpha, \beta)$ lies on the plane which passes through the points $(3, 4, 2)$ and $(7, 0, 6)$ and is perpendicular to the plane $2x - 5y = 15$, then $2\alpha - 3\beta$ is equal to **JEE Main 2019**
 (a) 17 (b) 7 (c) 5 (d) 12
7. The equation of a plane containing the line of intersection of the planes $2x - y - 4 = 0$ and $y + 2z - 4 = 0$ and passing through the point $(1, 1, 0)$ is **JEE Main 2019**
 (a) $x - 3y - 2z = -2$
 (b) $2x - z = 2$
 (c) $x - y - z = 0$
 (d) $x + 3y + z = 4$

8. The vector equation of the plane through the line of intersection of the planes $x + y + z = 1$ and $2x + 3y + 4z = 5$, which is perpendicular to the plane $x - y + z = 0$ is **JEE Main 2019**
 (a) $\mathbf{r} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{k}}) - 2 = 0$
 (b) $\mathbf{r} \times (\hat{\mathbf{i}} + \hat{\mathbf{k}}) + 2 = 0$
 (c) $\mathbf{r} \times (\hat{\mathbf{i}} - \hat{\mathbf{k}}) + 2 = 0$
 (d) $\mathbf{r} \cdot (\hat{\mathbf{i}} - \hat{\mathbf{k}}) + 2 = 0$
9. A plane passing through the points $(0, -1, 0)$ and $(0, 0, 1)$ and making an angle $\frac{\pi}{4}$ with the plane $y - z + 5 = 0$, also passes through the point **JEE Main 2019**
 (a) $(\sqrt{2}, 1, 4)$ (b) $(-\sqrt{2}, 1, -4)$
 (c) $(-\sqrt{2}, -1, -4)$ (d) $(\sqrt{2}, -1, 4)$
10. If $Q(0, -1, -3)$ is the image of the point P in the plane $3x - y + 4z = 2$ and R is the point $(3, -1, -2)$, then the area (in sq units) of ΔPQR is **JEE Main 2019**
 (a) $\frac{\sqrt{91}}{2}$ (b) $2\sqrt{13}$ (c) $\frac{\sqrt{91}}{4}$ (d) $\frac{\sqrt{65}}{2}$
11. The angle between the lines whose direction cosine satisfy the equations $l + m + n = 0$ and $l^2 = m^2 + n^2$ is **BITSAT 2019**
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{2}$
12. The straight line $x + y + 1 = 0$ bisects an angle between a pair of lines, of which one is $2x - 3y + 4 = 0$.

Then the equation of the other line in that pair is **AP EAMCET 2019**

- (a) $2x + 3y + 4 = 0$
 (b) $x - y + 1 = 0$
 (c) $5x - 5y + 9 = 0$
 (d) $3x - 2y + 5 = 0$
13. A plane is making intercepts 2, 3, 4 on X, Y and Z -axes respectively. Another plane is passing through the point $(-1, 6, 2)$ and is perpendicular to the line joining the points $(1, 2, 3)$ and $(-2, 3, 4)$. Then angle between the two planes is **AP EAMCET 2019**
 (a) 90° (b) $\cos^{-1} \sqrt{\frac{12}{61}}$
 (c) $\cos^{-1} \sqrt{\frac{11}{61}}$ (d) $\cos^{-1} \sqrt{\frac{5}{6}}$
14. The direction ratios of the normal to the plane passing through the points $(1, 2, -3), (-1, -2, 1)$ and parallel to $\frac{x-2}{2} = \frac{y+1}{3} = \frac{z}{4}$ is **WB JEE 2019**
 (a) $(2, 3, 4)$ (b) $(14, -8, -1)$
 (c) $(-2, 0, -3)$ (d) $(1, -2, -3)$
15. The equation of the plane passing through the point $\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ and perpendicular to the line of intersection of the planes $\mathbf{r} \cdot (3\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}) = 1$ and $\mathbf{r} \cdot (\hat{\mathbf{i}} + 4\hat{\mathbf{j}} - 2\hat{\mathbf{k}}) = 2$, is **AP EAMCET 2019**
 (a) $\mathbf{r} \cdot (-2\hat{\mathbf{i}} - 5\hat{\mathbf{j}} + \hat{\mathbf{k}}) = 0$
 (b) $\mathbf{r} \cdot (\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 4\hat{\mathbf{k}}) = 0$
 (c) $\mathbf{r} \cdot (2\hat{\mathbf{i}} - 7\hat{\mathbf{j}} - 13\hat{\mathbf{k}}) = 1$
 (d) $\mathbf{r} \cdot (-2\hat{\mathbf{i}} + 7\hat{\mathbf{j}} + 13\hat{\mathbf{k}}) = 0$

Advanced Probability

1. An unbiased coin is tossed 5 times. Suppose that a variable X is assigned the value k when k consecutive heads are obtained for $k = 3, 4, 5$, otherwise X takes the value -1 . Then the expected value of X , is **JEE Main 2020**
 (a) $-\frac{3}{16}$ (b) $\frac{3}{16}$
 (c) $\frac{1}{8}$ (d) $-\frac{1}{8}$
2. In a workshop, there are five machines and the probability of any one of them to be out of service on a day is $\frac{1}{4}$. If the probability that at most two machines will be out of service on the same day is $\left(\frac{3}{4}\right)^3 k$, then k is equal to **JEE Main 2020**

- (a) 4 (b) $\frac{17}{4}$
 (c) $\frac{17}{8}$ (d) $\frac{17}{2}$
3. Let A and B be two independent events such that $P(A) = \frac{1}{3}$ and $P(B) = \frac{1}{6}$. Then, which of the following is TRUE? **JEE Main 2020**
 (a) $P(A/(A \cup B)) = \frac{1}{4}$ (b) $P(A'/B') = \frac{1}{3}$
 (c) $P(A/B) = \frac{1}{3}$ (d) $P(A/B) = \frac{2}{3}$
4. In a box, there are 20 cards, out of which 10 are labelled as A and the remaining 10 are labelled as B . Cards are drawn at random, one after the other and with replacement, till a second A -card is obtained. The

probability that the second A -card appears before the third B -card is: **JEE Main 2020**

- (a) $\frac{15}{16}$ (b) $\frac{9}{16}$
 (c) $\frac{13}{16}$ (d) $\frac{11}{16}$
5. A random variable X has the following probability distribution **JEE Main 2020**
- | X | 1 | 2 | 3 | 4 | 5 |
|--------|-------|------|-----|------|--------|
| $P(X)$ | K^2 | $2K$ | K | $2K$ | $5K^2$ |
- Then $P(X > 2)$ is equal to
 (a) $\frac{1}{6}$ (b) $\frac{23}{36}$
 (c) $\frac{1}{36}$ (d) $\frac{7}{12}$
6. A pot contain 5 red and 2 green balls. At random a ball is drawn from this

pot. If a drawn ball is green then put a red ball in the pot and if a drawn ball is red, then put a green ball in the pot, while drawn ball is not replace in the pot. Now we draw another ball randomly, the probability of second ball to be red is

JEE Main 2019

- (a) $\frac{27}{49}$ (b) $\frac{26}{49}$ (c) $\frac{21}{49}$ (d) $\frac{32}{49}$

- 7.** Two integers are selected at random from the set $\{1, 2, \dots, 11\}$. Given that the sum of selected numbers is even, the conditional probability that both the numbers are even is

JEE Main 2019

- (a) $\frac{2}{5}$ (b) $\frac{1}{2}$ (c) $\frac{7}{10}$ (d) $\frac{3}{5}$

- 8.** Assume that each born child is equally likely to be a boy or a girl. If two families have two children each, then the conditional probability that all children are girls given that at least two are girls; is

JEE Main 2019

- (a) $\frac{1}{17}$ (b) $\frac{1}{12}$ (c) $\frac{1}{10}$ (d) $\frac{1}{11}$

- 9.** For an initial screening of an admission test, a candidate is given fifty problems to solve. If the probability that the candidate can solve any problem is $\frac{4}{5}$, then the probability that he is unable to solve less than two problem is

JEE Main 2019

- (a) $\frac{201}{5} \left(\frac{1}{5}\right)^{49}$ (b) $\frac{316}{25} \left(\frac{4}{5}\right)^{48}$
(c) $\frac{54}{5} \left(\frac{4}{5}\right)^{49}$ (d) $\frac{164}{25} \left(\frac{1}{5}\right)^{48}$

- 10.** Three numbers are chosen at random without replacement from $\{1, 2, 3, \dots, 8\}$. The probability that their minimum is 3, given that their maximum is 6, is

BITSAT 2019

- a. $\frac{3}{8}$ b. $\frac{1}{5}$
c. $\frac{1}{4}$ d. $\frac{2}{5}$

- 11.** An envelope is known to have come from either 'LONDON' or 'CLIFTON'. On the postal mark only two successive letters ON are legible. The probability that the envelope comes from LONDON is

JEE Main 2019

- (a) $\frac{12}{17}$ (b) $\frac{5}{17}$
(c) $\frac{3}{17}$ (d) $\frac{2}{5}$

- 12.** If two events E_1, E_2 are such that $P(E_1 \cup E_2) = \frac{5}{8}$,

$P(\bar{E}_1) = \frac{3}{4}, P(E_2) = \frac{1}{2}$, then E_1 and E_2 are

TS EAMCET 2019

- (a) independent events (b) mutually exclusive events
(c) exhaustive events (d) not independent events

- 13.** An examination is attempted by 5000 graduates, 2000 post-graduates and 1000 doctorate holders. The probabilities that a graduate, a post graduate and a doctorate holder will pass the examination are $\frac{2}{3}, \frac{3}{4}, \frac{4}{4}$ respectively. If one of the examine

passed the examination, then the probability that he is a post-graduate is

TS EAMCET 2019

- (a) $\frac{45}{169}$ (b) $\frac{100}{169}$
(c) $\frac{24}{169}$ (d) $\frac{5}{64}$

- 14.** The probability function of a random variable X is given by $P(X = k) = Ck^2$, where c is a constant and $k \in \{0, 1, 2, 3, 4\}$. If σ^2 is the variance of X and μ is the mean of X , then $\sigma^2 + \mu^2 =$

AP EAMCET 2019

- (a) 3.33 (b) 11.8
(c) $\frac{1}{30}$ (d) 354

- 15.** The following table shows the probability of selecting the boxes A, B and C and the number of balls of different colours contained in them.

AP EAMCET 2019

Box	Number of balls			Probability
	White	Green	Red	
A	1	2	3	$\frac{1}{2}$
B	2	3	1	$\frac{1}{3}$
C	3	1	2	$\frac{1}{6}$

- (a) $\frac{1}{13}$ (b) $\frac{6}{13}$
(c) $\frac{5}{13}$ (d) $\frac{7}{13}$

- 16.** If a random variable X has the probability distribution given by $P(X = 0) = 3C^3, P(X = 2) = 5C - 10C^2$ and $P(X = 4) = 4C - 1$, then the variance of that distribution is

AP EAMCET 2019

- (a) $\frac{68}{9}$ (b) $\frac{22}{9}$
(c) $\frac{612}{81}$ (d) $\frac{128}{81}$

- 17.** A box contains 30 toys of same size in which 10 toys are white and all the remaining toys are blue. A toy is drawn at random from the box and it is replaced in the box after noting down its colour. If 5 toys are drawn in this way, then the probability of getting atmost 2 white toys is

AP EAMCET 2019

- (a) $\left(\frac{6}{9}\right)^2$ (b) $\left(\frac{8}{9}\right)^2$
(c) $\left(\frac{7}{9}\right)^2$ (d) $\left(\frac{2}{3}\right)^5$

- 18.** It is observed that 25% of the cases related to child labour reported to the police station are solved. If 6 new cases are reported, then the probability that atleast 5 of them will be solved is

MHT CET 2019

- (a) $\left(\frac{1}{4}\right)^6$ (b) $\frac{19}{1024}$ (c) $\frac{19}{2048}$ (d) $\frac{19}{4096}$

Solutions with Explanation

Matrix

1. (a) It is given that ' α ' is the root of equation $x^2 + x + 1 = 0$, so $\alpha^2 + \alpha + 1 = 0$ and $\alpha^3 = 1$.

$$\text{Now, the matrix } A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \quad [\because \alpha^3 = 1]$$

$$\therefore A^2 = \left(\frac{1}{\sqrt{3}}\right)^2 \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \alpha^2 & \alpha \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 1+1+1 & 1+\alpha+\alpha^2 & 1+\alpha^2+\alpha \\ 1+\alpha+\alpha^2 & 1+\alpha^2+\alpha^4 & 1+\alpha^3+\alpha^3 \\ 1+\alpha^2+\alpha & 1+\alpha^3+\alpha^3 & 1+\alpha^4+\alpha^2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \quad [\because \alpha^3 = 1 \text{ and } \alpha^2 + \alpha + 1 = 0]$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore A^4 = A^2 \cdot A^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$\therefore A^8 = I = A^{16} = A^{20} = A^{24} = A^{28}$$

$$\therefore A^{31} = A^{28} \cdot A^3 = I \cdot A^3 = A^3 \Rightarrow A^{31} = A^3$$

2. (672) Let a 3×3 matrix $A = [a_{ij}]_{3 \times 3}$,

$i, j \in \{1, 2, 3\}$ So, sum of diagonal elements of AA^T

$$= \text{trace of } AA^T = \text{tr}([a_{ij}][a_{ji}])$$

$$= a_{11}^2 + a_{12}^2 + a_{13}^2 + a_{21}^2 + a_{22}^2 + a_{23}^2 + a_{31}^2 + a_{32}^2 + a_{33}^2 = 3 \text{ (given)}$$

$$\therefore a_{ij} \in \{-1, 0, 1\}$$

So, possible combinations are

$$0, 0, 0, 0, 0, 0, 1, 1, 1$$

$$0, 0, 0, 0, 0, 0, -1, 1, 1$$

$$0, 0, 0, 0, 0, 0, -1, -1, 1$$

$$0, 0, 0, 0, 0, 0, -1, -1, -1$$

$\therefore$ Number of such matrices such that $\text{tr}(AA^T) = 3$ is

$$= {}^9C_6 \times 2 \times 2 \times 2 = \frac{9 \times 8 \times 7}{3 \times 2} \times 2 \times 2 \times 2$$

$$= 84 \times 8 = 672$$

Hence, answer is 672.

3. (b) Given matrix $A = \begin{bmatrix} 2 & 2 \\ 9 & 4 \end{bmatrix}$

$\therefore$ Characteristic equation of matrix A

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & 2 \\ 9 & 4-\lambda \end{vmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - 18 = 0$$

$$\lambda^2 - 6\lambda - 10 = 0$$

Each square matrix satisfies the characteristic equation.

$$\Rightarrow A^2 - 6A - 10I = 0$$

On multiplying both sides by A^{-1} , we get

$$AI - 6I - 10A^{-1} = 0 \quad (\because AA^{-1} = I)$$

$$\Rightarrow 10A^{-1} = A - 6I$$

Hence, option (b) is correct.

4. (c) Given, matrix $A = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$

$$\therefore A^2 = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \alpha - \sin^2 \alpha & -\cos \alpha \sin \alpha - \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha + \cos \alpha \sin \alpha & -\sin^2 \alpha + \cos^2 \alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$$

Similarly,

$$A^n = \begin{bmatrix} \cos(n\alpha) & -\sin(n\alpha) \\ \sin(n\alpha) & \cos(n\alpha) \end{bmatrix}, n \in \mathbb{N}$$

$$\Rightarrow A^{32} = \begin{bmatrix} \cos(32\alpha) & -\sin(32\alpha) \\ \sin(32\alpha) & \cos(32\alpha) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(given)

So, $\cos(32\alpha) = 0$ and $\sin(32\alpha) = 1$

$$\Rightarrow 32\alpha = \frac{\pi}{2} \Rightarrow \alpha = \frac{\pi}{64}$$

5. (b) Given

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \dots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2+1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2+1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3+2+1 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \dots \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & (n-1) + (n-2) + \dots + 3 + 2 + 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{n(n-1)}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 78 \\ 0 & 1 \end{bmatrix}$$

Since, both matrices are equal, so equating corresponding element, we get

$$\frac{n(n-1)}{2} = 78 \Rightarrow n(n-1) = 156$$

$$= 13 \times 12 = 13(13-1)$$

$$\Rightarrow n = 13$$

$$\text{So, } A = \begin{bmatrix} 1 & 13 \\ 0 & 1 \end{bmatrix} = A^{-1} = \begin{bmatrix} 1 & -13 \\ 0 & 1 \end{bmatrix}$$

[$\therefore$ if $|A| = 1$ and $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then

$$A^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}]$$

6. (b) We have,

$$\begin{bmatrix} 3 & 5 & -1 & 4 \\ 2 & 1 & 3 & -2 \\ 8 & 11 & 1 & 6 \\ -7 & -14 & 6 & -14 \end{bmatrix}$$

$$[R_1 \rightarrow R_1 - R_2]$$

$$= \begin{bmatrix} 1 & 4 & -4 & 6 \\ 2 & 1 & 3 & -2 \\ 8 & 11 & 1 & 6 \\ -7 & -14 & 6 & -14 \end{bmatrix}$$

$$[R_2 \rightarrow R_2 - 2R_1, R_3 \rightarrow R_3 - 8R_1,$$

$$R_4 \rightarrow R_4 + 7R_1]$$

$$\begin{bmatrix} 1 & 4 & -4 & 6 \\ 0 & -7 & 11 & -14 \\ 0 & -21 & 33 & -42 \\ 0 & 14 & -22 & 28 \end{bmatrix}$$

$$[R_3 \rightarrow R_3 - 3R_2]$$

$$\begin{bmatrix} 1 & 4 & -4 & 6 \\ 0 & -7 & 11 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 14 & -22 & 28 \end{bmatrix}$$

$$[R_3 \leftrightarrow R_4]$$

$$\begin{bmatrix} 1 & 4 & -4 & 6 \\ 0 & -7 & 11 & -14 \\ 0 & 14 & -22 & 28 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[R_3 \rightarrow R_3 + 2R_2]$$

$$\begin{bmatrix} 1 & 4 & -4 & 6 \\ 0 & -7 & 11 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Number of non-zero rows = 2

$\therefore$ Rank of matrices $\cong 2$

7. (c) Given, that $A - 3I_3$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= [-6 + 3 + 3] = 0$$

Now, $\det = 0$

So, matrix $A - 3I_3$ is non-invertible matrix.

Determinant

1. (b) Given system of linear equations

$$2x + 2ay + az = 0 \dots(i)$$

$$2x + 3by + bz = 0 \dots(ii),$$

$$\text{and } 2x + 4cy + cz = 0 \dots(iii)$$

where $a, b, c \in \mathbb{R}$ are non-zero and distinct, has a non-zero solution, means

$$\Delta = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 2a & a \\ 2 & 3b & b \\ 2 & 4c & c \end{vmatrix} = 0$$

On applying $C_1 \rightarrow \frac{C_1}{2}$, we get

$$\Rightarrow \begin{vmatrix} 1 & 2a & a \\ 1 & 3b & b \\ 1 & 4c & c \end{vmatrix} = 0$$

On applying $R_2 \rightarrow R_2 - R_1$ and

$R_3 \rightarrow R_3 - R_1$, we get

$$\Rightarrow \begin{vmatrix} 1 & 2a & a \\ 0 & 3b - 2a & b - a \\ 0 & 4c - 2a & c - a \end{vmatrix} = 0$$

Expansion with respect to first column

$$\Rightarrow (3b - 2a)(c - a) - (b - a)(4c - 2a) = 0$$

$$\Rightarrow$$

$$(3bc - 3ba - 2ac + 2a^2) - (4bc - 2ba - 4ac$$

$$+ 2a^2) = 0$$

$$\Rightarrow -bc - ab + 2ac = 0$$

$$\Rightarrow 2ac = ab + bc$$

$$\Rightarrow \frac{2}{b} = \frac{1}{c} + \frac{1}{a} \quad [\text{on dividing by } abc]$$

$$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c} \text{ are in A.P.}$$

2. (a) It is given that there are two matrices of order 3×3 each, $A = [a_{ij}]$ and $B = [b_{ij}]$, such that $b_{ij} = 3^{i+j-2} a_{ij}$

$\therefore$ Matrix

$$B = \begin{bmatrix} 3^{1+1-2} a_{11} & 3^{1+2-2} a_{12} & 3^{1+3-2} a_{13} \\ 3^{2+1-2} a_{21} & 3^{2+2-2} a_{22} & 3^{2+3-2} a_{23} \\ 3^{3+1-2} a_{31} & 3^{3+2-2} a_{32} & 3^{3+3-2} a_{33} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 3a_{12} & 3^2 a_{13} \\ 3a_{21} & 3^2 a_{22} & 3^3 a_{23} \\ 3^2 a_{31} & 3^3 a_{32} & 3^4 a_{33} \end{bmatrix}$$

$$= (3 \times 3^2) \times (3 \times 3^2) \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$= 3^6 \cdot A$$

$$\Rightarrow A = \frac{1}{3^6} B \Rightarrow \det(A) = \frac{1}{3^6} \det(B)$$

$$\Rightarrow \det(A) = \frac{81}{3^6} (\because \det(B) = 81 \text{ (given)})$$

$$= \frac{1}{3^2} = \frac{1}{9}$$

3. (13) The given system of linear equations

$$x + y + z = 6,$$

$$x + 2y + 3z = 10,$$

and $3x + 2y + \lambda z = \mu$ has more than two solutions, so system must have infinite number of solutions.

Now, on putting $z = 0$ in above equation, we get

$$x + y = 6 \dots(i)$$

$$x + 2y = 10 \dots(ii)$$

$$\text{and } 3x + 2y = \mu \dots(iii)$$

On solving Eqs. (i) and (ii), we are getting

$$y = 4 \text{ and } x = 2$$

From Eq. (iii), we get

$$\mu = 3(2) + 2(4) = 14$$

Now, on putting $y = 0$ and $\mu = 14$, in the given system of linear Eq., we get

$$x + z = 6 \dots(iv)$$

$$x + 3z = 10 \dots(v)$$

$$\text{and } 3x + \lambda z = 14 \dots(vi)$$

On solving Eqs. (iv) and (v), we get $z = 2$ and $x = 4$

From Eq. (vi), we get

$$3(4) + \lambda(2) = 14$$

$$\lambda = 1$$

$$\therefore \mu - \lambda^2 = 14 - 1 = 13$$

4. (b) It is given that the three planes

$$x + 4y - 2z = 1,$$

$$x + 7y - 5z = \beta$$

$$\text{and } x + 5y + \alpha z = 5$$

having intersection is a line in $\mathbb{R}^3$, so

$$\Delta = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 4 & -2 \\ 1 & 7 & -5 \\ 1 & 5 & \alpha \end{vmatrix} = 0$$

$$\Rightarrow 1(7\alpha + 25) - 4(\alpha + 5) - 2(5 - 7) = 0$$

$$\Rightarrow 7\alpha + 25 - 4\alpha - 20 + 4 = 0$$

$$\Rightarrow 3\alpha + 9 = 0$$

$$\Rightarrow \alpha = -3$$

$$\text{and } \Delta_3 = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 4 & 1 \\ 1 & 7 & \beta \\ 1 & 5 & 5 \end{vmatrix} = 0$$

$$\Rightarrow 1(35 - 5\beta) - 4(5 - \beta) + 1(5 - 7) = 0$$

$$\Rightarrow 35 - 5\beta - 20 + 4\beta - 2 = 0$$

$$\Rightarrow \beta = 13$$

$$\therefore \alpha + \beta = -3 + 13 = 10$$

Hence, option (b) is correct.

5. (c) Given matrices

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{bmatrix}, B = \text{adj}(A)$$

and

$$C = 3A$$

$$\text{Now, } |\text{adj}(B)| = |\text{adj}(\text{adj}(A))|$$

$$[\because B = \text{adj}(A)]$$

and $|\text{adj}(\text{adj}(A))| = |A|^{(n-1)^2}$, where n is the order of square matrix A .

$$\therefore |\text{adj}(\text{adj}(A))| = |A|^{(3-1)^2} = |A|^4$$

$$\text{and } |C| = |3A| = 3^3 |A| = 27 |A|$$

$\therefore |KA| = K^n |A|$, where K is a scalar and n is the order of square matrix A .

$$\therefore \frac{|\text{adj } B|}{|C|} = \frac{|A|^4}{27 |A|} = \frac{|A|^3}{27}$$

$$\therefore |A| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \\ 1 & -1 & 3 \end{vmatrix}$$

$$= 1(9 + 4) - 1(3 - 4) + 2(-1 - 3)$$

$$= 13 + 1 - 8 = 6$$

$$\text{So, } \frac{|\text{adj } B|}{|C|} = \frac{|A|^3}{27} = \frac{6^3}{27} = 2^3 = 8$$

Hence, option (c) is correct.

$$6. (b) \text{ Given, } f(x) = \begin{vmatrix} x+a & x+2 & x+1 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$$

On applying $R_1 \rightarrow R_1 - 2R_2 + R_3$, we get

$$f(x) = \begin{vmatrix} a-2b+c & 0 & 0 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$$

$$\therefore a - 2b + c = 1 \text{ so, (given)}$$

$$f(x) = \begin{vmatrix} 1 & 0 & 0 \\ x+b & x+3 & x+2 \\ x+c & x+4 & x+3 \end{vmatrix}$$

On applying $R_2 \rightarrow R_2 - R_3$, we get

$$f(x) = \begin{vmatrix} 1 & 0 & 0 \\ b-c & -1 & -1 \\ x+c & x+4 & x+3 \end{vmatrix}$$

On applying $C_3 \rightarrow C_3 - C_2$, we get

$$f(x) = \begin{vmatrix} 1 & 0 & 0 \\ b-c & -1 & 0 \\ x+c & x+4 & -1 \end{vmatrix} = 1$$

$$\text{So, } f(50) = 1$$

Hence, option (b) is correct.

7. (c) Given system of linear equations

$$x - 2y + kz = 1 \dots(i)$$

$$2x + y + z = 2 \dots(ii)$$

$$\text{and } 3x - y - kz = 3 \dots(iii)$$

has a solution (x, y, z) , $z \neq 0$.

On adding Eqs. (i) and (iii), we get

$$x - 2y + kz + 3x - y - kz = 1 + 3$$

$$4x - 3y = 4$$

$$\Rightarrow 4x - 3y - 4 = 0$$

This is the required equation of the straight line in which point (x, y) lies.

8. (c) Given determinants are

$$\Delta_1 = \begin{vmatrix} x & \sin \theta & \cos \theta \\ -\sin \theta & -x & 1 \\ \cos \theta & 1 & x \end{vmatrix}$$

$$= -x^3 + \sin \theta \cos \theta - \sin \theta \cos \theta$$

$$+ x \cos^2 \theta - x + x \sin^2 \theta$$

$$= -x^3$$

$$\text{and } \Delta_2 = \begin{vmatrix} x & \sin 2\theta & \cos 2\theta \\ -\sin 2\theta & -x & 1 \\ \cos 2\theta & 1 & x \end{vmatrix}, x \neq 0$$

$$= -x^3 \text{ (similarly as } \Delta_1)$$

So, according to options, we get

$$\Delta_1 + \Delta_2 = -2x^3$$

9. (a) Given,

$$A = \begin{bmatrix} -2 & 4+d & (\sin \theta) - 2 \\ 1 & (\sin \theta) + 2 & d \\ 5 & (2\sin \theta) - d & (-\sin \theta) + 2 + 2d \end{bmatrix}$$

$\therefore$

$$|A| = \begin{vmatrix} -2 & 4+d & (\sin \theta) - 2 \\ 1 & (\sin \theta) + 2 & d \\ 5 & (2\sin \theta) - d & (-\sin \theta) + 2 + 2d \end{vmatrix}$$

$$= 1[(4+d)d - (\sin \theta + 2)(\sin \theta - 2)]$$

(expanding along R_3)

$$= (d+2)^2 - \sin^2 \theta$$

Note that $|A|$ will be minimum if $\sin^2 \theta$ is maximum i.e. if $\sin^2 \theta$ takes value 1.

$$\therefore |A|_{\min} = 8,$$

$$\text{therefore } (d+2)^2 - 1 = 8$$

$$\Rightarrow (d+2)^2 = 9$$

$$\Rightarrow d+2 = \pm 3$$

$$\Rightarrow d = 1, -5$$

10. (c) Given matrix

$$A = \begin{bmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{bmatrix}, b > 0$$

So, $\det$

$$(A) = |A| = \begin{vmatrix} 2 & b & 1 \\ b & b^2 + 1 & b \\ 1 & b & 2 \end{vmatrix}$$

$$= 2[2b^2 + 2 - b^2] - b^2 - 1$$

$$= 2b^2 + 4 - b^2 - 1 = b^2 + 3$$

$$\Rightarrow \frac{\det(A)}{b} = \frac{b^2 + 3}{b} = b + \frac{3}{b}$$

Now, by AM $\geq$ GM, we get

$$b + \frac{3}{b} \geq \left(b \times \frac{3}{b}\right)^{1/2} \quad \{\because b > 0\}$$

$$\Rightarrow b + \frac{3}{b} \geq 2\sqrt{3}$$

$$\text{So, minimum value of } \frac{\det(A)}{b} = 2\sqrt{3}$$

11. (d) We have,

$$ax + y + z = a \quad \dots (i)$$

$$x + by + z = b \quad \dots (ii)$$

$$x + y + cz = c \quad \dots (iii)$$

are inconsistent.

$$\therefore \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \text{ and } \begin{vmatrix} a & 1 & 1 \\ b & b & 1 \\ c & 1 & c \end{vmatrix} \neq 0$$

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$abc - a - b - c + 2 = 0$$

$$\text{and } \begin{vmatrix} a & 1 & 1 \\ b & b & 1 \\ c & 1 & c \end{vmatrix} \neq 0$$

It is possible only when $a \neq 1, b \neq 1$ and $c \neq 1$

12. (c) We have,

$$(1-a)x + y + z = 0$$

$$x + (1-b)y + z = 0$$

$$x + y + (1-c)z = 0$$

has many solutions

$$\therefore \begin{vmatrix} 1-a & 1 & 1 \\ 1 & 1-b & 1 \\ 1 & 1 & 1-c \end{vmatrix} = 0$$

$$\Rightarrow ab + ac + bc = abc$$

$$\therefore \frac{ab + bc + ac}{3} \geq (a^2 b^2 c^2)^{1/3} = (abc)^{1/3} \geq 3$$

$$\therefore \text{Minimum of } abc = (3)^3 = 27$$

13. (a) Given, system of linear equation,

$abc \neq 0$, is

$$x + 7ay + 2az = 0, x + 6by + 2bz = 0$$

and $x + 5cy + 2cz = 0$ has a non-zero trivial solution.

$$\text{So, } A = 0 \Rightarrow \begin{vmatrix} 1 & 7a & 2a \\ 1 & 6b & 2b \\ 1 & 5c & 2c \end{vmatrix} = 0$$

$$\Rightarrow 1(12bc - 10bc) - 7a(2c - 2b) + 2a(5c - 6b) = 0$$

$$\Rightarrow 2bc - 4ac + 2ab = 0$$

$$\Rightarrow 2ac = bc + ab$$

$$\Rightarrow b = \frac{2ac}{a+c} \text{ or } \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$$

Then, a, b, c are in Harmonic Progression.

Hence, option (a) is correct.

14 (c) We have,

$$(4 \sin \theta)x - 3y + z = 0$$

$$x - (6 \cos 2\theta)y + z = 0$$

$$3x - 12y + 4z = 0$$

System of equation has a non trivial solution

$$\therefore \begin{vmatrix} 4 \sin \theta & -3 & 1 \\ 1 & -6 \cos 2\theta & 1 \\ 3 & -12 & 4 \end{vmatrix} = 0$$

$$R_1 \rightarrow 4R_1 - R_3$$

$$\begin{vmatrix} 16 \sin \theta - 3 & 0 & 0 \\ 3 & -6 \cos 2\theta & 1 \\ 3 & -12 & 4 \end{vmatrix} = 0$$

$$= (16 \sin \theta - 3)(-24 \cos 2\theta + 12) = 0$$

$$= (16 \sin \theta - 3)(1 - 2 \cos 2\theta) = 0$$

$$\Rightarrow 16 \sin \theta = 3 \text{ or } 2 \cos 2\theta = 1$$

$$\theta = \sin^{-1} \frac{3}{16} \text{ or } \theta = \frac{\pi}{6}$$

$$\therefore \theta = \sin^{-1} \frac{3}{16}$$

15. (a) We have,

$$\begin{bmatrix} 1 \\ 2 \\ \lambda \end{bmatrix} \begin{bmatrix} 1 & 2 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 2 & \lambda \\ 2 & 4 & 2\lambda \\ \lambda & 2\lambda & \lambda^2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\text{Let } A = \begin{bmatrix} 1 & 2 & \lambda \\ 2 & 4 & 2\lambda \\ \lambda & 2\lambda & \lambda^2 \end{bmatrix}$$

$$|A| = 1(4\lambda^2 - 4\lambda^2) - 2(2\lambda^2 - 2\lambda^2)$$

$$+ \lambda(4\lambda - 4\lambda)$$

$$|A| = 0, \forall \lambda \in R$$

$\therefore$ The given system has non trivial solution.

16. (a) We have,

$$\begin{bmatrix} x^2 + 3x & x+1 & x-3 \\ x-1 & 2-x & x+4 \\ x-3 & x-3 & 3x \end{bmatrix}$$

$$= a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$$

Put $x = 1$ both sides, we get

$$\begin{bmatrix} 4 & 2 & -2 \\ 0 & 1 & 5 \\ -2 & -2 & 3 \end{bmatrix} = a_0 + a_1 + a_2 + a_3 + a_4$$

$$4(3+10) - 2(10+2) = a_0 + a_1 + a_2$$

$$+ a_3 + a_4$$

$$52 - 24 = a_0 + a_1 + a_2 + a_3 + a_4$$

$$\Rightarrow a_0 + a_1 + a_2 + a_3 + a_4 = 28 \quad \dots (i)$$

Put $x = -1$ both sides, we get

$$\begin{bmatrix} -2 & 0 & -4 \\ -2 & 3 & 3 \\ -4 & -4 & -3 \end{bmatrix} = a_0 - a_1 + a_2 - a_3 + a_4$$

$$\Rightarrow a_0 - a_1 + a_2 - a_3 + a_4 = -86 \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2(a_0 + a_2 + a_4) = -58$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2(a_1 + a_3) = 28 + 86 = 114$$

$$\Rightarrow a_1 + a_3 = 57$$

$$\therefore (a_1 + a_3) + 2(a_0 + a_2 + a_4)$$

$$= 57 - 58 = -1$$

17. (a) Given,

$$|\text{adj } A| = |\text{adj}(\text{adj } A)|$$

$$|A|^{n-1} = |A|^{(n-1)^2}$$

$$\Rightarrow n-1 = (n-1)^2$$

$$\Rightarrow n-1 = 1$$

$$\Rightarrow n = 2 \quad |\because n > 1|$$

The matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & 7 & 8 \end{bmatrix} \text{ has rank} = 2$$

Relations and Functions

1. (a) Since the $g(x) = x^2 + x - 1$ and $\text{gof}(x) = 4x^2 - 10x + 5$ both are quadratic equations, therefore $f(x)$ must be a linear equation.

Now, let $f(x) = ax + b$,

So, $\text{gof}(x) = 4x^2 - 10x + 5$

$$\Rightarrow g(ax + b) = 4x^2 - 10x + 5$$

$$\Rightarrow (ax + b)^2 + (ax + b) - 1$$

$$= 4x^2 - 10x + 5$$

$$[\because g(x) = x^2 + x - 1]$$

$$\Rightarrow a^2x^2 + a(2b+1)x + (b^2+b-1) = 4x^2 - 10x + 5$$

On comparing the coefficients of terms, we get

$$a^2 = 4 \quad \dots(i)$$

$$a(2b+1) = -10 \quad \dots(ii)$$

$$b^2 + b - 1 = 5 \quad \dots(iii)$$

From Eq. (i), we have $a = \pm 2$

From Eq. (iii), we have $b = -3, 2$

and the equation is getting satisfied by $a = -2$ and $b = 2$, so $f(x) = -2x + 2$

$$\therefore f\left(\frac{5}{4}\right) = -2\left(\frac{5}{4}\right) + 2 = -\frac{5}{2} + 2 = -\frac{1}{2}$$

2. (d) The given function $f : (1, 3) \rightarrow R$,

defined by $f(x) = \frac{x[x]}{1+x^2}$

$$= \begin{cases} \frac{x(1)}{1+x^2}, & x \in (1, 2) \\ \frac{x(2)}{1+x^2}, & x \in [2, 3) \end{cases}$$

$$\therefore f(x) = \begin{cases} \frac{x}{1+x^2}, & x \in (1, 2) \\ \frac{2x}{1+x^2}, & x \in [2, 3) \end{cases}$$

is a decreasing function, so

$$\lim_{h \rightarrow 0} f(3-h) \rightarrow \frac{2(3)}{1+9} = \frac{3}{5}$$

$$f(2) = \frac{4}{5}$$

Similarly, $f(x \rightarrow 2^-)$ tends to $\frac{2}{5}$ and

$f(x \rightarrow 1^+)$ tends to $\frac{1}{2}$.

So range of the given function 'f' is

$$\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$$

Hence, option (d) is correct.

3. (a) We have a function $f : A \rightarrow R$ defined as

$$f(x) = \frac{2x}{x-1}$$

One-one Let $x_1, x_2 \in A$ such that

$$f(x_1) = f(x_2)$$

$$\Rightarrow \frac{2x_1}{x_1-1} = \frac{2x_2}{x_2-1}$$

$$\Rightarrow 2x_1x_2 - 2x_1 = 2x_1x_2 - 2x_2$$

$$\Rightarrow x_1 = x_2$$

Thus, $f(x_1) = f(x_2)$ has only one solution, $x_1 = x_2$

$\therefore f(x)$ is one-one (injective)

$$\text{Onto Let } x = 2, \text{ then } f(2) = \frac{2 \times 2}{2-1} = 4$$

But $x = 2$ is not in the domain, and $f(x)$ is one-one function

$\therefore f(x)$ can never be 4.

Similarly, $f(x)$ can not take many values.

Hence, $f(x)$ is into (not surjective).

4. (d) We have,

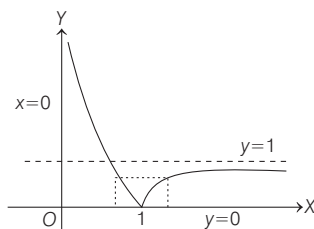
$$f(x) = \frac{|x-1|}{|x|} = \begin{cases} -\frac{(x-1)}{x}, & \text{if } 0 < x \leq 1 \\ \frac{x-1}{x}, & \text{if } x > 1 \end{cases}$$

$$= \begin{cases} \frac{1}{x} - 1, & \text{if } 0 < x \leq 1 \\ 1 - \frac{1}{x}, & \text{if } x > 1 \end{cases}$$

Now, let us draw the graph of $y = f(x)$

Note that when $x \rightarrow 0$, then $f(x) \rightarrow \infty$, when $x = 1$, then $f(x) = 0$, and when $x \rightarrow \infty$, then $f(x) \rightarrow 1$

Clearly, $f(x)$ is not injective because if $f(x) < 1$, then f is many one, as shown in figure.



Also, $f(x)$ is not surjective because range of $f(x)$ is $[0, \infty)$ and but in problem co-domain is $(0, \infty)$, which is wrong.

$\therefore f(x)$ is neither injective nor surjective

5. (d) Given, function $f(x) = a^x, a > 0$ is written as sum of an even and odd functions $f_1(x)$ and $f_2(x)$ respectively.

$$\text{Clearly, } f_1(x) = \frac{a^x + a^{-x}}{2} \text{ and } f_2(x) = \frac{a^x - a^{-x}}{2}$$

So, $f_1(x+y) + f_1(x-y)$

$$= \frac{1}{2}[a^{x+y} + a^{-(x+y)}] + \frac{1}{2}[a^{x-y} + a^{-(x-y)}]$$

$$= \frac{1}{2}\left[a^x a^y + \frac{1}{a^x a^y} + \frac{a^y}{a^x} + \frac{a^x}{a^y}\right]$$

$$= \frac{1}{2}\left(a^x + \frac{1}{a^x}\right)\left(a^y + \frac{1}{a^y}\right)$$

$$= 2\left(\frac{a^x + a^{-x}}{2}\right)\left(\frac{a^y + a^{-y}}{2}\right)$$

$$= 2f_1(x) \cdot f_1(y)$$

6. (c) Given, functions $f(x) = x^2, x \in R$

and $g(A) = \{x \in R : f(x) \in A\}; A \subseteq R$

Now, for $S = [0, 4]$

$$g(S) = \{x \in R : f(x) \in S = [0, 4]\}$$

$$= \{x \in R : x^2 \in [0, 4]\}$$

$$= \{x \in R : x \in [-2, 2]\}$$

$$\Rightarrow g(S) = [-2, 2]$$

$$\text{So, } f(g(S)) = [0, 4] = S$$

Now, $f(S) = \{x^2 : x \in S = [0, 4]\} = [0, 16]$ and

$$g(f(S)) = \{x \in R : f(x) \in f(S) = [0, 16]\}$$

$$= \{x \in R : f(x) \in [0, 16]\}$$

$$= \{x \in R : x^2 \in [0, 16]\}$$

$$= \{x \in R : x \in [-4, 4]\} = [-4, 4]$$

From above, it is clear that $g(f(S)) = g(S)$.

7. (b) $\therefore [-x] = -[x] - 1$, if $x \notin \text{Integer}$,

$$\text{and } [x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \dots +$$

$$\left[x + \frac{n-1}{n}\right] = [nx], n \in N.$$

So given series

$$\left[-\frac{1}{3}\right] + \left[-\frac{1}{3} - \frac{1}{100}\right] + \left[-\frac{1}{3} - \frac{2}{100}\right] + \dots$$

$$\dots + \left[-\frac{1}{3} - \frac{99}{100}\right]$$

$$\begin{aligned}
 &= \left(-\left[\frac{1}{3}\right] - 1\right) + \left(-\left[\frac{1}{3} + \frac{1}{100}\right] - 1\right) \\
 &+ \left(-\left[\frac{1}{3} + \frac{2}{100}\right] - 1\right) + \dots \\
 &\quad + \left(-\left[\frac{1}{3} + \frac{99}{100}\right] - 1\right) \\
 &= (-1) \times 100 - \left[\frac{1}{3} \times 100\right] \\
 &= -100 - 33 = -133
 \end{aligned}$$

8. (b) Given, for $x \in (0, 3/2)$, functions

$$f(x) = \sqrt{x} \quad \dots (i)$$

$$g(x) = \tan x \quad \dots (ii)$$

$$\text{and } h(x) = \frac{1-x^2}{1+x^2} \quad \dots (iii)$$

Also given,

$$\phi(x) = ((hof)og)(x) = (hof)(g(x))$$

$$= h(f(g(x))) = h(f(\tan x))$$

$$= h(\sqrt{\tan x}) = \frac{1 - (\sqrt{\tan x})^2}{1 + (\sqrt{\tan x})^2}$$

$$= \frac{1 - \tan x}{1 + \tan x} = \tan\left(\frac{\pi}{4} - x\right)$$

$$\text{Now, } \phi\left(\frac{\pi}{3}\right) = \tan\left(\frac{\pi}{4} - \frac{\pi}{3}\right)$$

$$= \tan\left(\frac{3\pi - 4\pi}{12}\right) = \tan\left(-\frac{\pi}{12}\right)$$

$$= -\tan\left(\frac{\pi}{12}\right) = \tan\left(\pi - \frac{\pi}{12}\right) = \tan\left(\frac{11\pi}{12}\right)$$

9. (c) We have,

$$(gojof)(x) = f(x) \text{ for all } x \in D$$

$$\Rightarrow g((jof)(x)) = \frac{1}{x}$$

$$\Rightarrow 1 - (jof)(x) = \frac{1}{x}$$

$$\Rightarrow 1 - j(f(x)) = \frac{1}{x}$$

$$\Rightarrow 1 - j\left(\frac{1}{x}\right) = \frac{1}{x}$$

$$\Rightarrow j\left(\frac{1}{x}\right) = 1 - \frac{1}{x}$$

$$\Rightarrow j(x) = 1 - x = g(x)$$

10. (b) We have,

$$gof : S \rightarrow U \text{ is an onto function.}$$

Let Z be any arbitrary element such that $Z \in U$.

Now, $gof : S \rightarrow U$ is onto

$$\Rightarrow gof(x) = Z, \text{ for } x \in S$$

$$\Rightarrow g(f(x)) = Z$$

$$\Rightarrow g(y) = Z, \text{ where } y = f(x) \in T \text{ for all } Z \in U$$

$\therefore$ For all $Z \in U$, there exists $y = f(x) \in T$ such that $g(y) = Z$

$\therefore g : T \rightarrow U$ is an onto function.

11. (d) We have,

$$f(x) = x^2 - \frac{x^2}{1+x^2}$$

$$\therefore f(-x) = (-x)^2 - \frac{(-x)^2}{1+(-x)^2}$$

$$= x^2 - \frac{x^2}{1+x^2} = f(x)$$

$$\therefore f(-x) = f(x).$$

So, $f(x)$ is many one

$$\begin{aligned}
 \text{Again, } f(x) &= x^2 - \frac{x^2}{1+x^2} \\
 &= \frac{x^2 + x^4 - x^2}{1+x^2} = \frac{x^4}{1+x^2}
 \end{aligned}$$

$\therefore$ Range of $f(x)$ will be $[0, \infty)$

But, co-domain of $f(x) = R$

$\therefore$ Range $\neq$ co-domain. So, $f(x)$ is not onto.

12. (d) We have,

$$f(x) = 3x + 6, g(x) = 4x + k$$

Since, $fog(x) = gof(x)$

$$\Rightarrow f(g(x)) = g(f(x))$$

$$\Rightarrow f(4x + k) = g(3x + 6)$$

$$\Rightarrow 3(4x + k) + 6 = 4(3x + 6) + k$$

$$\Rightarrow 3k - k = 24 - 6$$

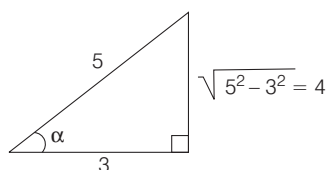
$$\Rightarrow 2k = 18$$

$$\Rightarrow k = 9$$

Inverse Trigonometric Functions

1. (d) Given, $\alpha = \cos^{-1}\left(\frac{3}{5}\right)$ and $\beta = \tan^{-1}\left(\frac{1}{3}\right)$

where $0 < \alpha, \beta < \frac{\pi}{2}$

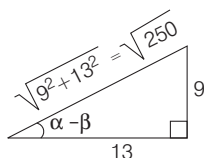


$$\text{Clearly, } \alpha = \tan^{-1} \frac{4}{3}$$

$$\text{So, } \alpha - \beta = \tan^{-1} \frac{4}{3} - \tan^{-1} \frac{1}{3}$$

$$= \tan^{-1} \left(\frac{\frac{4}{3} - \frac{1}{3}}{1 + \left(\frac{4}{3} \times \frac{1}{3}\right)} \right) = \tan^{-1} \frac{1}{1 + \frac{4}{9}}$$

$$= \tan^{-1} \frac{9}{13}$$



$$= \sin^{-1} \frac{9}{\sqrt{9^2 + 13^2}} = \sin^{-1} \frac{9}{\sqrt{250}}$$

$$= \sin^{-1} \left(\frac{9}{5\sqrt{10}} \right)$$

2. (c) Given equation is

$$\cos^{-1} x - \cos^{-1} \frac{y}{2} = \alpha, \text{ where } -1 \leq x \leq 1,$$

$$-2 \leq y \leq 2 \text{ and } x \leq \frac{y}{2}$$

$$\therefore \cos^{-1} \left(x \frac{y}{2} + \sqrt{1-x^2} \sqrt{1-(y/2)^2} \right) = \alpha$$

$$[\because \cos^{-1} x - \cos^{-1} y = \cos^{-1} (xy + \sqrt{1-x^2} \sqrt{1-y^2})]$$

$$|x|, |y| \leq 1 \text{ and } x + y \geq 0]$$

$$\Rightarrow \frac{xy}{2} + \sqrt{1-x^2} \sqrt{1-(y/2)^2} = \cos \alpha$$

$$\Rightarrow \sqrt{1-x^2} \sqrt{1-(y/2)^2} = \cos \alpha - \frac{xy}{2}$$

On squaring both sides, we get

$$(1-x^2) \left(1 - \frac{y^2}{4} \right)$$

$$= \cos^2 \alpha + \frac{x^2 y^2}{4} - 2 \frac{xy}{2} \cos \alpha$$

$$\Rightarrow 4x^2 - 4xy \cos \alpha + y^2 = 4 \sin^2 \alpha$$

3. (b) We have,

$$\sin^{-1} \left(\frac{12}{13} \right) - \sin^{-1} \left(\frac{3}{5} \right)$$

$$= \sin^{-1} \left(\frac{12}{13} \sqrt{1 - \left(\frac{3}{5}\right)^2} - \frac{3}{5} \sqrt{1 - \left(\frac{12}{13}\right)^2} \right)$$

$$[\because \sin^{-1} x - \sin^{-1} y = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2})]$$

if $x^2 + y^2 \leq 1$ or

if $xy > 0$ and $x^2 + y^2 > 1 \forall x, y \in [-1, 1]$

$$= \sin^{-1} \left(\frac{12}{13} \times \frac{4}{5} - \frac{3}{5} \times \frac{12}{13} \right) = \sin^{-1} \left(\frac{48-36}{65} \right)$$

$$= \sin^{-1} \left(\frac{33}{65} \right) = \cos^{-1} \sqrt{1 - \left(\frac{33}{65}\right)^2}$$

$$= \cos^{-1} \left(\frac{56}{65} \right) = \frac{\pi}{2} - \sin^{-1} \left(\frac{56}{65} \right)$$

4. (b) α, β are roots of equation

$$3x^2 - 16x + 5 = 0.$$

$$\therefore \alpha + \beta = \frac{16}{3}, \alpha\beta = \frac{5}{3}$$

Now,

$$\tan^{-1} \alpha + \tan^{-1} \beta - \tan^{-1} \left(\frac{\alpha + \beta}{1 - \alpha\beta} \right)$$

$$= \pi + \tan^{-1}\left(\frac{\alpha + \beta}{1 - \alpha\beta}\right) - \tan^{-1}\left(\frac{\alpha + \beta}{1 - \alpha\beta}\right)$$

$$[\because \alpha\beta > 1]$$

$$= \pi$$

Continuity and Differentiability

1. (d) If function $f : [-7, 0] \rightarrow \mathbb{R}$ be continuous on $[-7, 0]$ and differentiable on $(-7, 0)$, then according to LMVT, we have
- $$\frac{f(-1) - f(-7)}{(-1) - (-7)} = f'(x) \leq 2$$

$$\forall x \in (-7, -1)$$

$$\Rightarrow \frac{f(-1) - f(-7)}{6} \leq 2$$

$$\Rightarrow f(-1) \leq 9 \quad \dots(i)$$

Similarly

$$\frac{f(0) - f(-7)}{0 - (-7)} = f'(x) \leq 2, \forall x \in (-7, 0)$$

$$\Rightarrow \frac{f(0) - f(-7)}{7} \leq 2 \Rightarrow f(0) \leq 11 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$f(-1) + f(0) \leq 20$$

$\therefore f(-1) + f(0)$ lies in the interval $(-\infty, 20]$.

2. (05) The given function 'f' defined on

$$\left(-\frac{1}{3}, \frac{1}{3}\right) \text{ by}$$

$$f(x) = \begin{cases} \frac{1}{x} \log_e \left(\frac{1+3x}{1-2x} \right), & \text{where } x \neq 0 \\ k, & \text{where } x = 0 \end{cases}$$

is continuous, then

$$k = \lim_{x \rightarrow 0} \frac{1}{x} \log_e \left(\frac{1+3x}{1-2x} \right)$$

$$= \lim_{x \rightarrow 0} \left[\frac{\log_e(1+3x)}{x} - \frac{\log_e(1-2x)}{x} \right]$$

$$= 3 - (-2) = 5$$

$$\Rightarrow k = 5$$

3. (c) It is given that, the function

$$f(x) = \begin{cases} \frac{\sin(a+2)x + \sin x}{x} & ; x < 0 \\ b & ; x = 0 \\ \frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}} & ; x > 0 \end{cases}$$

is continuous at $x = 0$, then

$$\lim_{x \rightarrow 0^-} f(x) = f(0) = \lim_{x \rightarrow 0^+} f(x)$$

$$\therefore b = \lim_{x \rightarrow 0^+} f(x)$$

$$= \lim_{x \rightarrow 0^+} \frac{(x+3x^2)^{1/3} - x^{1/3}}{x^{4/3}}$$

$$= \lim_{x \rightarrow 0^+} \frac{(1+3x)^{1/3} - 1}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{1 + \frac{1}{3}(3x) + (\text{higher degree terms in } x) - 1}{x}$$

5. (c) We have, $4\sin^{-1}x + 6\cos^{-1}x = 3\pi$

$$\Rightarrow 4\sin^{-1}x + 6\left(\frac{\pi}{2} - \sin^{-1}x\right) = 3\pi$$

$$\Rightarrow 4\sin^{-1}x + 3\pi - 6\sin^{-1}x = 3\pi$$

$$= \lim_{x \rightarrow 0^+} [1 + (\text{higher degree terms of } x)]$$

$$= 1 \text{ (on applying limit)}$$

$$\Rightarrow b = 1$$

Now, $f(0) = \lim_{x \rightarrow 0^+} f(x)$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{\sin(a+2)x + \sin x}{x} = 1$$

$$[\because f(0) = b = 1]$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \frac{2\sin\left(\frac{a+3}{2}x\right)\cos\frac{ax}{2}}{x} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0^+} \left[2 \times \frac{\sin\left(\frac{a+3}{2}x\right)}{\left(\frac{a+3}{2}x\right)} \cdot \left(\frac{a+3}{2}\right) \cdot \cos\frac{ax}{2} \right] = 1$$

$$\Rightarrow a + 3 = 1 \Rightarrow a = -2$$

So, $a + 2b = -2 + 2(1) = 0$

4. (a) It is given that

$$f \circ g(x) = x, \text{ then}$$

$$g \circ f(x) = x$$

$$\Rightarrow g'(f(x))f'(x) = 1$$

On putting $x = b$, we get

$$g'(f(b))f'(b) = 1$$

$$\Rightarrow f'(b) = \frac{1}{g'(f(b))}$$

$$g \circ f(x) = x$$

$$\Rightarrow g(f(b)) = b \quad (\text{on putting } x = b)$$

$$\Rightarrow g(f(b)) = g(a) \quad (\because g(a) = b)$$

$$\Rightarrow f(b) = a$$

$$\therefore f'(b) = \frac{1}{g'(a)} = \frac{1}{5} \quad (\text{given})$$

Hence, option (a) is correct.

5. (c) Given function $f : [-1, 3] \rightarrow \mathbb{R}$ is defined as

$$f(x) = \begin{cases} |x| + [x], & -1 \leq x < 1 \\ x + |x|, & 1 \leq x < 2 \\ x + [x], & 2 \leq x \leq 3 \end{cases}$$

$$= \begin{cases} -x - 1, & -1 \leq x < 0 \\ x, & 0 \leq x < 1 \\ 2x, & 1 \leq x < 2 \\ x + 2, & 2 \leq x < 3 \\ 6, & x = 3 \end{cases}$$

$[\because \text{if } n \leq x < n+1, \forall n \in \text{Integer}, [x] = n]$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = -1 \neq f(0) \quad [\because f(0) = 0]$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = 1 \neq f(1) \quad [\because f(1) = 2]$$

$$\Rightarrow -2\sin^{-1}x = 0$$

$$\Rightarrow \sin^{-1}x = 0$$

$$\Rightarrow x = \sin(0) = 0$$

$$\therefore \lim_{x \rightarrow 2^-} f(x) = 4 = f(2) = \lim_{x \rightarrow 2^+} f(x) = 4$$

$$[\because f(2) = 4]$$

and $\lim_{x \rightarrow 3^-} f(x) = 5 \neq f(3)$

$$[\because f(3) = 6]$$

$\therefore$ Function $f(x)$ is discontinuous at points 0, 1 and 3.

6. (b) Given function is

$$f(x) = 15 - |x - 10|, x \in \mathbb{R} \text{ and}$$

$$g(x) = f(f(x))$$

$$= f(15 - |x - 10|)$$

$$= 15 - |15 - |x - 10|| - 10|$$

$$= 15 - |5 - |x - 10||$$

$$= \begin{cases} 15 - |5 - (x - 10)|, & x \geq 10 \\ 15 - |5 + (x - 10)|, & x < 10 \end{cases}$$

$$= \begin{cases} 15 + (x - 5) = 10 + x, & x < 5 \\ 15 - (x - 5) = 20 - x, & 5 \leq x < 10 \\ 15 + (x - 15) = x, & 10 \leq x < 15 \\ 15 - (x - 15) = 30 - x, & x \geq 15 \end{cases}$$

From the above definition it is clear that $g(x)$ is not differentiable at $x = 5, 10, 15$.

7. (b) Given function $f(x) = [x] - \left[\frac{x}{4}\right], x \in \mathbb{R}$

$$\text{Now, } \lim_{x \rightarrow 4^+} f(x) = \lim_{h \rightarrow 0} \left[(4+h) - \left[\frac{4+h}{4} \right] \right]$$

$$= \lim_{h \rightarrow 0} (4 - 1) = 3$$

$$\text{and } \lim_{x \rightarrow 4^-} f(x) = \lim_{h \rightarrow 0} \left[(4-h) - \left[\frac{4-h}{4} \right] \right]$$

$$[\because \text{put } x = 4 - h, \text{ when } x \rightarrow 4^- \text{ then } h \rightarrow 0]$$

$$= \lim_{h \rightarrow 0} (3 - 0) = 3$$

$$\text{and } f(4) = [4] - \left[\frac{4}{4} \right] = 4 - 1 = 3$$

$$\therefore \lim_{x \rightarrow 4^-} f(x) = f(4) = \lim_{x \rightarrow 4^+} f(x) = 3$$

So, function $f(x)$ is continuous at $x = 4$.

8. (d) Given function $f : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$f(x) = \begin{cases} 5, & \text{if } x \leq 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \leq x < 5 \\ 30, & \text{if } x \geq 5 \end{cases}$$

If f is continuous at $x = 1$, then

$$a + b = 5 \quad \dots(i)$$

If f is continuous at $x = 3$, then

$$a + 3b = b + 15$$

$$\Rightarrow a + 2b = 15 \quad \dots(ii)$$

and if f is continuous at $x = 5$,

then

$$b + 25 = 30 \\ \Rightarrow b = 5 \quad \dots(iii)$$

From Eqs. (ii) and (iii), we get

$$a = 5 \quad \dots(iv)$$

but $a = 5$ and $b = 5$ doesn't satisfy the Eq. (i).

so, $f: R \rightarrow R$ is not continuous for any values of a and b .

Hence, option (d) is correct.

9. (d) The given function $y = [x] + |1 - x|$ have point of discontinuity at $x = 0, 1, 2$ and 3 for $-1 \leq x \leq 3$.

So function $y = [x] + |1 - x|$ is not differentiable at 4 points. Hence, option (d) is correct.

10. (c) We have, $f(x) = \log \left(\frac{x-1}{x+2} \right)$

$f(x)$ is continuous of

$$\frac{x-1}{x+2} > 0$$

$$\Rightarrow (x-1)(x+2) > 0$$

$$\therefore x \in (-\infty, -2) \cup (1, \infty)$$

- 11 (c) We have,

$$f(x) = \begin{cases} 5 - 3x, & \text{of } x \leq \frac{5}{3} \\ x^2 - 3x + 20, & \text{of } x > \frac{5}{3} \end{cases} \\ \lim_{x \rightarrow \frac{5}{3}} f(x) = 5 - 3 \left(\frac{5}{3} \right) = 0$$

$$\lim_{x \rightarrow \frac{5}{3}} f(x) = \lim_{x \rightarrow \frac{5}{3}} x^2 - 3x + 20 \\ = \left(\frac{5}{3} \right)^2 - 3 \left(\frac{5}{3} \right) + 20 = \frac{160}{9}$$

$$\lim_{x \rightarrow \frac{5}{3}} f(x) \neq \lim_{x \rightarrow \frac{5}{3}} f(x)$$

$$\therefore f(x) \text{ is discontinuous of } x = \frac{5}{3}$$

Hence $f(x)$ is derivative at $x = 2$.

12. (c) Option (a), Here, $f(0) = e^2$

$$\text{and } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (1 + 2x)^{1/x} = e^2 \\ [\because \lim_{x \rightarrow 0} (1 + \lambda x)^{1/x} = e^\lambda]$$

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x) = e^2$$

$\Rightarrow f(x)$ is continuous at $x = 0$.

Option (b), Here, $f(0) = -1$

$$\text{and } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} (\sin x - \cos x) \\ = \sin 0 - \cos 0 \\ = -1$$

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x) = -1$$

$\Rightarrow f(x)$ is continuous at $x = 0$

Option (c),

Here, $f(0) = -1$

$$\text{and } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1} \\ = \lim_{x \rightarrow 0} \frac{1 - e^{-1/x}}{1 + e^{-1/x}} = 1$$

$\therefore f(0) \neq \lim_{x \rightarrow 0} f(x) \Rightarrow f(x)$ is not continuous option (d),

Here $f(0) = 1$

$$\text{and } \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{e^{5x} - e^{2x}}{\sin 3x} \times \frac{3x}{3x} \\ = \lim_{x \rightarrow 0} \frac{e^{5x} - e^{2x}}{3x} \times \lim_{x \rightarrow 0} \frac{3x}{\sin 3x} \\ \left(1 + \frac{5x}{1!} + \frac{(5x)^2}{2!} + \dots \right) - \\ \left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \dots \right) \\ = \lim_{x \rightarrow 0} \frac{\left(1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \dots \right)}{3x} \\ \left(\because \lim_{x \rightarrow 0} \frac{3x}{\sin 3x} = 1 \right) \\ = \lim_{x \rightarrow 0} \frac{\left(\frac{3x}{1!} + \frac{21x^2}{2!} + \dots \right)}{3x} = 1$$

$$\therefore f(0) = \lim_{x \rightarrow 0} f(x) = 1 \\ \Rightarrow f(x) \text{ is continuous at } x = 0.$$

- 13 (d) We have function,

$$f(x) = \begin{cases} \frac{(e^{kx} - 1) \tan kx}{4x^2}, & x \neq 0 \\ 16, & x = 0 \end{cases}$$

is continuous at $x = 0$

$$\therefore \lim_{x \rightarrow 0} f(x) = f(0)$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{(e^{kx} - 1) \tan kx}{4x^2} = 16 \quad \left(\frac{0}{0} \text{ form} \right)$$

$$\Rightarrow \lim_{x \rightarrow 0} \left(\frac{e^{kx} - 1}{kx} \right) \left(\frac{\tan kx}{kx} \right) \times \frac{k^2}{4} = 16$$

$$\Rightarrow \frac{k^2}{4} = 16 \Rightarrow k^2 = 64 \Rightarrow k = \pm 8$$

Differentiation

1. (d) It is given that

$$y(\alpha) = \sqrt{2 \left(\frac{\tan \alpha + \cot \alpha}{1 + \tan^2 \alpha} \right) + \frac{1}{\sin^2 \alpha}}$$

$$\alpha \in \left(\frac{3\pi}{4}, \pi \right)$$

$$\Rightarrow y(\alpha) = \sqrt{2 \cot \alpha \left(\frac{\tan^2 \alpha + 1}{1 + \tan^2 \alpha} \right) + \frac{1}{\sin^2 \alpha}} \\ = \sqrt{2 \cot \alpha + \operatorname{cosec}^2 \alpha} \\ = \sqrt{2 \cot \alpha + 1 + \cot^2 \alpha} \\ = \sqrt{(1 + \cot \alpha)^2} = |1 + \cot \alpha|$$

$$\because \cot \alpha \in (-\infty, -1), \text{ for } \alpha \in \left(\frac{3\pi}{4}, \pi \right)$$

$$\therefore y(\alpha) = -(1 + \cot \alpha)$$

$$[\because |x| = -x, \text{ for } x < 0]$$

$$\therefore \frac{dy}{d\alpha} = -(0 - \operatorname{cosec}^2 \alpha) = \operatorname{cosec}^2 \alpha$$

$$\text{So, } \left. \frac{dy}{d\alpha} \right|_{\alpha = \frac{5\pi}{6}} = \operatorname{cosec}^2 \left(\frac{5\pi}{6} \right)$$

$$= \operatorname{cosec}^2 \left(\pi - \frac{\pi}{6} \right)$$

$$= \operatorname{cosec}^2 \left(\frac{\pi}{6} \right) = (2)^2 = 4$$

2. (b) Given functional relation is

$$y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$$

On differentiating both sides w.r.t. x , we get

$$\frac{-2yx}{2\sqrt{1-x^2}} + \sqrt{1-x^2} \frac{dy}{dx} \\ = 0 + \frac{2xy \frac{dy}{dx}}{2\sqrt{1-y^2}} - \sqrt{1-y^2}$$

$$\Rightarrow \frac{dy}{dx} \left\{ \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \right\} \\ = \left\{ \frac{yx}{\sqrt{1-x^2}} - \sqrt{1-y^2} \right\}$$

$$\Rightarrow \frac{dy}{dx} \left\{ \frac{\sqrt{1-x^2} \sqrt{1-y^2} - xy}{\sqrt{1-y^2}} \right\} \\ = \frac{xy - \sqrt{1-x^2} \sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\therefore y \left(\frac{1}{2} \right) = -\frac{1}{4} \quad (\text{given})$$

$$\therefore \left. \frac{dy}{dx} \right|_{x=\frac{1}{2}} = -\frac{\sqrt{1-\left(-\frac{1}{4}\right)^2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = -\frac{\sqrt{\frac{16-1}{4}}}{\sqrt{\frac{4-1}{4}}} \\ = -\frac{\sqrt{15}}{2\sqrt{3}} = -\frac{\sqrt{5}}{2}$$

3. (b) It is given that

$$f(x) = (\sin(\tan^{-1} x) + \sin(\cot^{-1} x))^2 - 1$$

$$= \left[\sin \left(\sin^{-1} \left(\frac{x}{\sqrt{1+x^2}} \right) \right) \right]^2 \\ + \sin \left(\sin^{-1} \left(\frac{1}{\sqrt{1+x^2}} \right) \right) - 1$$

$$= \left(\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}} \right)^2 - 1$$

$$= \frac{(x+1)^2}{1+x^2} - 1 = \frac{x^2 + 1 + 2x - 1 - x^2}{1+x^2}$$

$$\Rightarrow f(x) = \frac{2x}{1+x^2} \quad \dots(i) \text{ for } |x| > 1$$

and, also given that $\frac{dy}{dx} = \frac{1}{2} \frac{d}{dx} (\sin^{-1}(f(x)))$

$$\Rightarrow 2y = \sin^{-1}(f(x)) + C$$

(on integrating both sides)

$$\Rightarrow 2y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) + C, \text{ for } |x| > 1$$

$$\therefore \sin^{-1} \frac{2x}{1+x^2} = \begin{cases} 2 \tan^{-1} x, & |x| \leq 1 \\ \pi - 2 \tan^{-1} x, & x > 1 \\ -\pi - 2 \tan^{-1} x, & x < -1 \end{cases}$$

$$\therefore \text{at } x = \sqrt{3}; y = \frac{\pi}{6} \text{ (given)}$$

$$\therefore 2 \frac{\pi}{6} = \pi - 2 \tan^{-1}(\sqrt{3}) + C \Rightarrow C = 0$$

Then at $x = -\sqrt{3}$

$$\begin{aligned} 2y &= -\pi - 2 \tan^{-1}(-\sqrt{3}) \\ &= -\pi + \frac{2\pi}{3} = -\frac{\pi}{3} \end{aligned}$$

$$\Rightarrow y = -\frac{\pi}{6}$$

Hence, option (b) is correct.

4. (b) Given expression is

$$\begin{aligned} 2y &= \left(\cot^{-1} \left(\frac{\sqrt{3} \cos x + \sin x}{\cos x - \sqrt{3} \sin x} \right) \right)^2 \\ &= \left(\cot^{-1} \left(\frac{\sqrt{3} \cot x + 1}{\cot x - \sqrt{3}} \right) \right)^2 \end{aligned}$$

[dividing each term of numerator and denominator by $\sin x$]

$$= \left(\cot^{-1} \left(\frac{\cot \frac{\pi}{6} \cot x + 1}{\cot x - \cot \frac{\pi}{6}} \right) \right)^2 \left[\because \cot \frac{\pi}{6} = \sqrt{3} \right]$$

$$= \left(\cot^{-1} \left(\cot \left(\frac{\pi}{6} - x \right) \right) \right)^2$$

$$\left[\because \cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A} \right]$$

$$= \begin{cases} \left(\frac{\pi}{6} - x \right)^2, & 0 < x < \frac{\pi}{6} \\ \left(\pi + \left(\frac{\pi}{6} - x \right) \right)^2, & \frac{\pi}{6} < x < \frac{\pi}{2} \\ \left[\because \cot^{-1}(\cot \theta) = \begin{cases} \pi + \theta, & -\pi < \theta < 0 \\ \theta, & 0 < \theta < \pi \\ \theta - \pi, & \pi < \theta < 2\pi \end{cases} \right] \end{cases}$$

$$\Rightarrow 2y = \begin{cases} \left(\frac{\pi}{6} - x \right)^2, & 0 < x < \frac{\pi}{6} \\ \left(\frac{7\pi}{6} - x \right)^2, & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$$

$$\Rightarrow \frac{dy}{dx} = \begin{cases} x - \frac{\pi}{6}, & 0 < x < \frac{\pi}{6} \\ x - \frac{7\pi}{6}, & \frac{\pi}{6} < x < \frac{\pi}{2} \end{cases}$$

5. (b) Given functions, $f(x) = \log_e(\sin x)$,
($0 < x < \pi$) and $g(x) = \sin^{-1}(e^{-x})$, $x \geq 0$.

Now, $f \circ g(x) = f(g(x)) = f(\sin^{-1}(e^{-x}))$

$$= \log_e(\sin(\sin^{-1}(e^{-x})))$$

$$= \log_e(e^{-x}) \quad \{ \because \sin(\sin^{-1} x) = x, \text{ if } x \in [-1, 1] \}$$

$$= -x \quad \dots(i)$$

$$\text{and } (f \circ g)'(x) = \frac{d}{dx}(-x) = -1 \quad \dots(ii)$$

According to the question,

$$\therefore a = (f \circ g)'(\alpha) = -1 \text{ [from Eq. (ii)]}$$

$$\text{and } b = (f \circ g)(\alpha) = -(\alpha) \text{ [from Eq. (i)]}$$

for a positive real value ' α '.

Since, the value of $a = -1$ and $b = -\alpha$, satisfy the quadratic equation (from the given options)

$$a\alpha^2 - b\alpha - a = 1.$$

6. (b) **Key Idea** Differentiating the given equation twice w.r.t. ' x '.

Given equation is

$$e^y + xy = e \quad \dots(i)$$

On differentiating both sides w.r.t. x , we get

$$e^y \frac{dy}{dx} + x \frac{dy}{dx} + y = 0 \quad \dots(ii)$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{e^y + x} \right) \quad \dots(iii)$$

Again differentiating Eq. (ii) w.r.t. ' x ', we get

$$\begin{aligned} e^y \frac{d^2y}{dx^2} + e^y \left(\frac{dy}{dx} \right)^2 + x \frac{d^2y}{dx^2} \\ + \frac{dy}{dx} + \frac{dy}{dx} = 0 \quad \dots(iv) \end{aligned}$$

Now, on putting $x = 0$ in Eq. (i), we get

$$e^y = e^1$$

$$\Rightarrow y = 1$$

On putting $x = 0$, $y = 1$ in Eq. (iii), we get

$$\frac{dy}{dx} = -\frac{1}{e+0} = -\frac{1}{e}$$

Now, on putting $x = 0$, $y = 1$ and $\frac{dy}{dx} = -\frac{1}{e}$

in Eq. (iv), we get

$$\begin{aligned} e^1 \frac{d^2y}{dx^2} + e^1 \left(-\frac{1}{e} \right)^2 + 0 \left(\frac{d^2y}{dx^2} \right) + \left(-\frac{1}{e} \right) \\ + \left(-\frac{1}{e} \right) = 0 \end{aligned}$$

$$\Rightarrow \frac{d^2y}{dx^2} \Big|_{(0,1)} = \frac{1}{e^2}$$

$$\text{So, } \left(\frac{dy}{dx}, \frac{d^2y}{dx^2} \right) \text{ at } (0, 1) \text{ is } \left(-\frac{1}{e}, \frac{1}{e^2} \right)$$

7. (b) We have,

$$f(x) = \sin|x| - |x| + 2(x - \pi) \cos|x|$$

$$f(x) =$$

$$\begin{cases} -\sin x + x + 2(x - \pi) \cos x, & \text{if } x < 0 \\ \sin x - x + 2(x - \pi) \cos x, & \text{if } x \geq 0 \end{cases}$$

$$[\because \sin(-\theta) = -\sin \theta \text{ and } \cos(-\theta) = \cos \theta]$$

$$\therefore f'(x) =$$

$$\begin{cases} -\cos x + 1 + 2 \cos x - 2(x - \pi) \sin x; & \text{if } x < 0 \\ \cos x - 1 + 2 \cos x - 2(x - \pi) \sin x, & \text{if } x > 0 \end{cases}$$

Clearly, $f(x)$ is differentiable everywhere except possibly at $x = 0$

[$\because f'(x)$ exist for $x < 0$ and $x > 0$]

Here,

$$\begin{aligned} Rf'(0) &= \lim_{x \rightarrow 0^+} (3 \cos x - 1 - 2(x - \pi) \sin x) \\ &= 3 - 1 - 0 = 2 \end{aligned}$$

$$\text{and } Lf'(0) = \lim_{x \rightarrow 0^-} (\cos x + 1 - 2(x - \pi) \sin x)$$

$$= 1 + 1 - 0 = 2$$

$$\therefore Rf'(0) = Lf'(0)$$

So, $f(x)$ is differentiable at all values of x .

$$\Rightarrow K = \phi$$

8. (d) Given,

$$f(x) = \cot^{-1} \left(\frac{x^x - x^{-x}}{2} \right)$$

$$\text{Let } y = \cot^{-1} \left(\frac{x^{2x} - 1}{2 \cdot x^x} \right)$$

$$\text{Put } x^x = \tan \theta$$

$$\therefore y = \cot^{-1} \left(\frac{\tan^2 \theta - 1}{2 \tan \theta} \right)$$

$$= \cot^{-1}(-\cot 2\theta)$$

$$y = \pi - \cot^{-1}(\cot 2\theta)$$

$$y = \pi - 2\theta$$

$$y = \pi - 2 \tan^{-1}(x^x)$$

$$\therefore \frac{dy}{dx} = -\frac{2}{1+x^{2x}} \cdot x^x (1 + \log x)$$

$$\begin{aligned} \text{So, } \frac{dy}{dx} \Big|_{x=1} &= \frac{-2}{1+(1)^2} \cdot 1^1 (1 + \log 1) \\ &= -\frac{2}{2} \cdot 1 = -1 \end{aligned}$$

9. (c) Given equation is

$$\sqrt{1-x^6} + \sqrt{1-y^6} = a(x^3 - y^3)$$

$$\Rightarrow \frac{\sqrt{1-x^6} + \sqrt{1-y^6}}{x^3 - y^3} = a$$

On differentiating both sides w.r.t. x , we get

$$\begin{aligned} \left[(x^3 - y^3) \left(\frac{-6x^5}{2\sqrt{1-x^6}} - \frac{6y^5}{2\sqrt{1-y^6}} \frac{dy}{dx} \right) \right. \\ \left. - (\sqrt{1-x^6} + \sqrt{1-y^6}) \left(3x^2 - 3y^2 \frac{dy}{dx} \right) \right] \\ \frac{dy}{dx} = 0 \end{aligned}$$

$$\Rightarrow \left(y^2 (\sqrt{1-x^6} + \sqrt{1-y^6}) - \frac{y^5 (x^3 - y^3)}{\sqrt{1-y^6}} \right) \frac{dy}{dx}$$

$$= x^2 (\sqrt{1-x^6} + \sqrt{1-y^6}) + \frac{x^5}{\sqrt{1-x^6}} (x^3 - y^3)$$

$$\begin{aligned} &\Rightarrow y^2 \frac{dy}{dx} \\ &\left[\frac{\sqrt{1-x^6} + \sqrt{1-y^6} + (1-y^6) - y^3 x^3 + y^6}{\sqrt{1-y^6}} \right] \\ &= x^2 \left[\frac{(1-x^6) + \sqrt{1-y^6} \sqrt{1-x^6} + x^6 - x^3 y^3}{\sqrt{1-x^6}} \right] \\ &\Rightarrow y^2 \frac{dy}{dx} \left[\frac{\sqrt{1-x^6} \sqrt{1-y^6} + 1 - x^3 y^3}{\sqrt{1-y^6}} \right] \\ &= x^2 \left[\frac{1 + \sqrt{1-y^6} \sqrt{1-x^6} - x^3 y^3}{\sqrt{1-x^6}} \right] \\ &\Rightarrow y^2 \frac{dy}{dx} = x^2 \sqrt{\frac{1-y^6}{1-x^6}} \end{aligned}$$

Hence, option (c) is correct.

10. (d)

$$\begin{aligned} \therefore \frac{d^2 x}{dy^2} &= \frac{d}{dy} \left(\frac{dx}{dy} \right) = \frac{dx}{dy} \frac{d}{dx} \left(\frac{1}{\left(\frac{dy}{dx} \right)} \right) \\ &= \left(\frac{dx}{dy} \right) \left(\frac{-\frac{d^2 y}{dx^2}}{\left(\frac{dy}{dx} \right)^2} \right) \\ &= - \left(\frac{d^2 y}{dx^2} \right) \left(\frac{dy}{dx} \right)^3 \\ \therefore \left(\frac{d^2 x}{dy^2} \right)_p &= - \frac{(-3)}{(4)^3} = \frac{3}{64} \end{aligned}$$

Hence, option (d) is correct.

11. (a) We have

$$\begin{aligned} f(x) &= \cos^{-1} \left[\frac{1 - (\log x)^2}{1 + (\log x)^2} \right] \\ f(x) &= 2 \tan^{-1} (\log x) \\ \left[\because \cos^{-1} \left(\frac{1 - x^2}{1 + x^2} \right) = 2 \tan^{-1} x \right] \\ f'(x) &= \frac{2}{1 + (\log x)^2} \times \frac{1}{x} \\ f'(e) &= \frac{2}{1 + 1} \times \frac{1}{e} \\ &= \frac{1}{e} \end{aligned}$$

Application of Derivatives

1. (d) Given function

$$f(x) = x^3 - 4x^2 + 8x + 11,$$

when $x \in [0, 1]$ is a continuous function in interval $x \in [0, 1]$ and differentiable in interval $x \in (0, 1)$, so according to Lagrange's mean value theorem for $x = c \in (0, 1)$

$$f'(c) = \frac{f(1) - f(0)}{1 - 0}$$

$$\begin{aligned} \Rightarrow (3x^2 - 8x + 8)_{x=c} &= \frac{(1 - 4 + 8 + 11) - 11}{1 - 0} \\ &= \frac{4 - \sqrt{7}}{3} \end{aligned}$$

$$\Rightarrow 3c^2 - 8c + 8 = 5$$

$$\Rightarrow 3c^2 - 8c + 3 = 0$$

$$\Rightarrow c = \frac{8 - \sqrt{64 - 36}}{6} \quad [\because c \in (0, 1)]$$

$$= \frac{4 - \sqrt{7}}{3}$$

2. (a) Given function

$$f(x) = x \cos^{-1}(-\sin|x|),$$

$$x \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$= x(\pi - \cos^{-1}(\sin|x|))$$

$$\{ \because \cos^{-1}(-x) = \pi - \cos^{-1}x \}$$

$$= x \left[\pi - \left(\frac{\pi}{2} - \sin^{-1}|\sin|x| \right) \right]$$

$$\{ \because \cos^{-1}x = \frac{\pi}{2} - \sin^{-1}x \}$$

$$= x \left[\frac{\pi}{2} + |x| \right] \{ \because \sin^{-1} \sin x = x \}$$

$$= \begin{cases} x \left(\frac{\pi}{2} - x \right), & x \in \left(-\frac{\pi}{2}, 0 \right) \\ x \left(\frac{\pi}{2} + x \right), & x \in \left(0, \frac{\pi}{2} \right) \end{cases}$$

$$\text{So, } f'(x) = \begin{cases} \frac{\pi}{2} - 2x, & x \in \left(-\frac{\pi}{2}, 0 \right) \\ \frac{\pi}{2} + 2x, & x \in \left(0, \frac{\pi}{2} \right) \end{cases}$$

$$\Rightarrow f''(x) = \begin{cases} -2, & x \in \left(-\frac{\pi}{2}, 0 \right) \\ 2, & x \in \left(0, \frac{\pi}{2} \right) \end{cases}$$

$$\therefore f' \text{ is decreasing in } \left(-\frac{\pi}{2}, 0 \right) \text{ and increasing in } \left(0, \frac{\pi}{2} \right).$$

Hence, option (a) is correct.

3. (*) Since, the functions $f : [0, 1] \rightarrow R$ which are continuous on $[0, 1]$ and differentiable on $(0, 1)$.

If f is a constant function then options (b), (c) and (d) are incorrect.

According to LMVT, for $c' \in (c, 1)$

$$f'(c') = \frac{f(1) - f(c)}{1 - c}$$

but $c' \neq c$, so option (a) is also incorrect.

4. (d) It is given that, a spherical iron ball of 10 cm radius is coated with a layer of ice of uniform thickness, let the thickness is 'x' cm, then volume of the ball is

$$V = \frac{4}{3} \pi (10 + x)^3$$

On differentiating w.r.t. 't', we get

$$\frac{dV}{dt} = 4\pi(10 + x)^2 \frac{dx}{dt}, \quad \dots (i)$$

where t is time in min.

$$\text{It is given, the } \frac{dV}{dt} = -50 \text{ cm}^3 / \text{min},$$

Now when x is 5 cm, then

$$-50 = 4\pi(10 + 5)^2 \frac{dx}{dt} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \frac{dx}{dt} = -\frac{50}{4\pi(225)} = -\frac{1}{18\pi} \text{ cm/min}$$

Negative sign indicates the thickness of ice layer decreases with time.

Hence, option (d) is correct.

5. (c) Given, $\phi(x) = f(x) + f(2 - x)$,

$$\forall x \in (0, 2)$$

$$\Rightarrow \phi'(x) = f'(x) - f'(2 - x) \quad \dots (i)$$

Also, we have $f''(x) > 0 \forall x \in (0, 2)$

$\Rightarrow f'(x)$ is a strictly increasing function

$$\forall x \in (0, 2).$$

Now, for $\phi(x)$ to be increasing,

$$\phi'(x) \geq 0$$

$$\Rightarrow f'(x) - f'(2 - x) \geq 0 \quad [\text{using Eq. (i)}]$$

$$\Rightarrow f'(x) \geq f'(2 - x) \Rightarrow x > 2 - x$$

$[\because f' \text{ is a strictly increasing function}]$

$$\Rightarrow 2x > 2 \Rightarrow x > 1$$

Thus, $\phi(x)$ is increasing on $(1, 2)$.

Similarly, for $\phi(x)$ to be decreasing,

$$\phi'(x) \leq 0$$

$$\Rightarrow f'(x) - f'(2 - x) \leq 0 \quad [\text{using Eq. (i)}]$$

$$\Rightarrow f'(x) \leq f'(2 - x)$$

$$\Rightarrow x < 2 - x$$

$$\Rightarrow 2x < 2 \Rightarrow x < 1$$

Thus, $\phi(x)$ is decreasing on $(0, 1)$.

6. (a) Equation of given curve is

$$y = \frac{x}{x^2 - 3}, \quad x \in R, (x \neq \pm \sqrt{3}) \quad \dots (i)$$

On differentiating Eq. (i) w.r.t. x , we get

$$\frac{dy}{dx} = \frac{(x^2 - 3) - x(2x)}{(x^2 - 3)^2} = \frac{(-x^2 - 3)}{(x^2 - 3)^2}$$

It is given that tangent at a point $(\alpha, \beta) \neq (0, 0)$ on it is parallel to the line

$$2x + 6y - 11 = 0.$$

$$\therefore \text{Slope of this line} = -\frac{2}{6} = \frac{dy}{dx}\bigg|_{(\alpha, \beta)}$$

$$\Rightarrow -\frac{\alpha^2 + 3}{(\alpha^2 - 3)^2} = -\frac{1}{3}$$

$$\Rightarrow \alpha = 3 \text{ or } -3, \quad [\because \alpha \neq 0]$$

Now, from Eq. (i),

$$\beta = \frac{\alpha}{\alpha^2 - 3} \Rightarrow \beta = \frac{3}{9 - 3} \text{ or } \frac{-3}{9 - 3} \\ = \frac{1}{2} \text{ or } -\frac{1}{2}$$

Maxima and Minima

1. (c) Let the integral

$$I = \frac{1}{a+b} \int_a^b x(f(x) + f(x+1))dx \quad \dots(i)$$

On applying property

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)dx, \text{ we get}$$

$$I = \frac{1}{a+b} \int_a^b (a+b-x)(f(a+b-x) + f(a+b-x+1))dx \\ = \frac{1}{a+b} \int_a^b (a+b-x)(f(x+1) + f(x))dx \quad \dots(ii)$$

$$\left[\because f(a+b+1-x) = f(x) \right] \\ \left[\because f(a+b-x) = f(x+1) \right]$$

On adding integrals Eqs. (i) and (ii), we get

$$2I = \frac{a+b}{a+b} \int_a^b (f(x) + f(x+1))dx \\ \Rightarrow I = \frac{1}{2} \left[\int_a^b f(x)dx + \int_a^b f(x+1)dx \right] \\ = \frac{1}{2} \left[\int_a^b f(x)dx + \int_a^{b+1} f(x)dx \right] \\ = \frac{1}{2} \left[\int_a^b f(x)dx + \int_a^b f(x)dx \right] \\ I = \int_a^b f(x)dx$$

Put $x = t + 1$, then upper limit $t = b - 1$ and lower limit $t = a - 1$ and $dx = dt$, so

$$I = \int_{a-1}^{b-1} f(t+1)dt = \int_{a-1}^{b-1} f(x+1)dx$$

2. (d) It is given, $F(x) = \int_1^x t^2 g(t)dt$, then

$$F'(x) = x^2 g(x) \text{ (by Newton-Leibniz rule)}$$

$$\therefore g(t) = \int_1^t f(u)du \Rightarrow g(1) = 0$$

$$\therefore F'(x=1) = 1^2 \cdot g(1) = 0$$

$$\text{Now, } F''(x) = 2x g(x) + x^2 g'(x)$$

$$= 2x \int_1^x f(u)du + x^2 f(x)$$

$$\text{Now, } F''(1) = 2 \int_1^1 f(u)du + f(1)$$

7. (b) Given,

$$f(x) = (2k+1)x - 3 - ke^{-x} + 2e^x$$

Since, $f(x)$ is monotonically increasing for all $x \in R$.

$$\text{So, } f'(x) \geq 0$$

$$(2k+1) + ke^{-x} + 2e^x \geq 0$$

$$\Rightarrow e^{-x}((2k+1)e^x + k + 2e^{2x}) \geq 0$$

or

$$(2k+1)e^x + k + 2e^{2x} \geq 0$$

$$\Rightarrow 2e^x(e^x + k) + 1(e^x + k) \geq 0$$

$$\Rightarrow (2e^x + 1)(e^x + k) \geq 0$$

$$= 2 \times 0 + 3 \quad (\because f(1) = 3 \text{ given}) \\ = 3$$

$$\therefore F''(1) = 3 > 0 \text{ and } F'(1) = 0$$

$\therefore$ The point $x=1$ is point of local minima for the function F .

Hence, option (d) is correct.

3. (c) Given function is

$$f(x) = 9x^4 + 12x^3 - 36x^2 + 25 = y \text{ (let)}$$

$$\text{For maxima or minima put } \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = 36x^3 + 36x^2 - 72x = 0$$

$$\Rightarrow x^3 + x^2 - 2x = 0$$

$$\Rightarrow x[x^2 + x - 2] = 0$$

$$\Rightarrow x[x^2 + 2x - x - 2] = 0$$

$$\Rightarrow x[x(x+2) - 1(x+2)] = 0$$

$$\Rightarrow x(x-1)(x+2) = 0$$

$$\Rightarrow x = -2, 0, 1$$

By sign method, we have following

$$\begin{array}{ccccccc} & - & + & - & + & & \\ & -2 & 0 & 1 & & & \end{array}$$

Since, $\frac{dy}{dx}$ changes it's sign from negative to positive at $x = -2$ and 1 , so $x = -2, 1$ are points of local minima. Also, $\frac{dy}{dx}$ changes it's

sign from positive to negative at $x = 0$, so $x = 0$ is point of local maxima.

$$\therefore S_1 = \{-2, 1\} \text{ and } S_2 = \{0\}.$$

4. (d) The non-zero four degree polynomial

$f(x)$ has extremum points at $x = -1, 0, 1$, so we can assume

$$f'(x) = a(x+1)(x-0)(x-1) = ax(x^2-1)$$

where, a is non-zero constant.

$$f'(x) = ax^3 - ax$$

$$\Rightarrow f(x) = \frac{a}{4}x^4 - \frac{a}{2}x^2 + C$$

[integrating both sides]

where, C is constant of integration.

$$\text{Now, since } f(x) = f(0)$$

$$\Rightarrow e^x + k \geq 0 \text{ or } k \geq -e^x$$

Hence, least value of k is zero.

8. (a) Given, $f(x) = ax^3 + bx^2 + 11x - 6$

satisfies the Rolle's theorem in $[1, 3]$. So,

$$f(1) = 0 \text{ and } f(3) = 0$$

$$\Rightarrow a + b + 5 = 0 \quad \dots(i)$$

$$\text{and } 27a + 9b + 27 = 0$$

$$\Rightarrow 9a + 3b + 9 = 0$$

$$\Rightarrow 3a + b + 3 = 0 \quad \dots(ii)$$

From Eq. (i), we get $a + b = -5$

$$\Rightarrow \frac{a}{4}x^4 - \frac{a}{2}x^2 + C = C \Rightarrow \frac{x^4}{4} = \frac{x^2}{2}$$

$$\Rightarrow x^2(x^2 - 2) = 0 \Rightarrow x = -\sqrt{2}, 0, \sqrt{2}$$

Thus, $f(x) = f(0)$ has one rational and two irrational roots.

5. (b) Given,

$$f(x) = \tan\left(x + \frac{2\pi}{3}\right)$$

$$- \tan\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$$

$$\therefore \tan A - \tan B = \frac{\sin(A-B)}{\cos A \cos B}$$

$$\therefore \tan\left(x + \frac{2\pi}{3}\right) - \tan\left(x + \frac{\pi}{6}\right)$$

$$= \frac{\sin \frac{\pi}{2}}{\cos\left(x + \frac{2\pi}{3}\right) \cos\left(x + \frac{\pi}{6}\right)}$$

$$\text{Now, } f(x) = \frac{1 \times 2}{2 \cos\left(x + \frac{2\pi}{3}\right) \cos\left(x + \frac{\pi}{6}\right)}$$

$$+ \cos(x + 30^\circ) \\ = \frac{2}{\cos(150^\circ + 2x) + \cos 90^\circ} + \cos(x + 30^\circ)$$

$$= \cos(x + 30^\circ) - \frac{2}{\cos(2x - 30^\circ)}$$

Differentiate w.r.t. x , we get

$$f'(x) = -\sin(x + 30^\circ) - \frac{4 \sin(2x - 30^\circ)}{\cos^2(2x - 30^\circ)}$$

Since, it lie $-75^\circ < x < -60^\circ$

then $(30^\circ + x) \in [-45^\circ, -30^\circ]$

and $(2x - 30^\circ) \in [-180^\circ, -150^\circ]$

$\therefore f'(x) > 0$ for all $-75^\circ < x < -60^\circ$

So, $f_{\max}$ at $x = -60^\circ$

$$f(-60^\circ) = \tan(60^\circ) - \tan(-30^\circ) + \cos(-30^\circ)$$

$$= \sqrt{3} + \frac{\sqrt{3}}{3} + \frac{\sqrt{3}}{2}$$

$$= \frac{6\sqrt{3} + 2\sqrt{3} + 3\sqrt{3}}{6} = \frac{11\sqrt{3}}{6}$$

6. (b) Given,

$$f(x) = 2x^3 - 9ax^2 + 12a^2x + 1$$

has maximum value at P and minimum value at Q .

$$\text{So, } f'(p) = 0 \text{ and } f'(q) = 0$$

Now, $f'(x) = 6x^2 - 18ax + 12a^2$ has roots p and q

$$\therefore p + q = \frac{18a}{6} = 3a$$

$$\text{and } pq = \frac{12a^2}{6} = 2a^2$$

on solving both equations, we get

$$p = a \text{ and } q = 2a$$

$$\text{Given, } p^2 = q$$

$$\therefore a^2 = 2a \Rightarrow a^2 - 2a = 0$$

$$\Rightarrow a(a - 2) = 0 \Rightarrow a = 2 \quad (\because a \neq 0)$$

7. (d) The given function $f: [-1, 1] \rightarrow R$

$$\text{defined by } f(x) = \begin{cases} 2^x + 1, & \text{for } x \in [-1, 0) \\ 1, & \text{for } x = 0 \\ 2^x - 1, & \text{for } x \in (0, 1] \end{cases}$$

The given function $f(x)$ in $x \in (-1, 0)$ strictly increasing and $f(0^-) \rightarrow 2$

but $f(0) = 1$

and in interval $x \in (0, 1)$, again it is strictly increasing, but $f(0^+) = 0$

So, function has neither maximum nor minimum. Hence, option (d) is correct.

Indefinite Integrals

1. (c) The given integral,

$$I = \int \frac{dx}{(x+4)^{8/7} (x-3)^{6/7}}$$

$$= \int \frac{\frac{dx}{(x+4)^2}}{\frac{(x+4)^{8/7} (x-3)^{6/7}}{(x+4)^2}} = \int \frac{\frac{dx}{(x+4)^2}}{\left(\frac{x-3}{x+4}\right)^{6/7}}$$

$$\text{Now, let } \frac{x-3}{x+4} = t^7$$

$$\Rightarrow \frac{(x+4) - (x-3)}{(x+4)^2} dx = 7t^6 dt$$

$$\Rightarrow \frac{7}{(x+4)^2} dx = 7t^6 dt$$

$$\text{So, } I = \int \frac{t^6 dt}{t^6} = \int dt = t + c$$

$$\therefore I = \left(\frac{x-3}{x+4}\right)^{1/7} + C$$

Hence, option (c) is correct.

2. (c) Given integral,

$$I = \int \frac{d\theta}{\cos^2 \theta (\tan 2\theta + \sec 2\theta)}$$

$$= \int \frac{\sec^2 \theta \cos 2\theta}{(\sin 2\theta + 1)} d\theta$$

$$= \int \frac{\sec^2 \theta (1 - \tan^2 \theta)}{(2 \tan \theta + 1 + \tan^2 \theta)} d\theta$$

$$= \int \frac{\sec^2 \theta (1 - \tan \theta)}{1 + \tan \theta} d\theta$$

$$\text{Let, } \tan \theta = t$$

$$\Rightarrow \sec^2 \theta d\theta = dt$$

$$\text{Then, } I = \int \frac{1-t}{1+t} dt$$

$$= \int \left(-1 + \frac{2}{1+t}\right) dt$$

$$= -t + 2 \log_e |1+t| + C$$

$$= -\tan \theta + 2 \log_e |1 + \tan \theta| + C$$

$$= \lambda \tan \theta + 2 \log_e |f(\theta)| + C \text{ (given)}$$

$$\therefore (\lambda, f(\theta)) = (-1, 1 + \tan \theta)$$

(on comparing)

Hence, option (c) is correct.

3. (a) Given,

$$\int x^5 e^{-4x^3} dx = \frac{1}{48} e^{-4x^3} f(x) + C$$

In LHS, put $x^3 = t$

$$\Rightarrow 3x^2 dx = dt$$

$$\text{So, } \int x^5 e^{-4x^3} dx = \frac{1}{3} \int t e^{-4t} dt$$

$$= \frac{1}{3} \left[t \frac{e^{-4t}}{-4} - \int \frac{e^{-4t}}{-4} dt \right] = \frac{1}{3} \left[\frac{te^{-4t}}{-4} + \frac{e^{-4t}}{-16} \right] + C$$

$$= -\frac{1}{48} e^{-4t} [4t + 1] + C$$

$$= -\frac{e^{-4x^3}}{48} [4x^3 + 1] + C \quad [\because t = x^3]$$

$$\therefore f(x) = -1 - 4x^3$$

$$4. (c) \text{ Let } I = \int \frac{\sin \frac{5x}{2}}{\sin \frac{x}{2}} dx = \int \frac{2 \sin \frac{5x}{2} \cos \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}} dx$$

[multiplying by $2 \cos \frac{x}{2}$ in numerator and denominator]

$$= \int \frac{\sin 3x + \sin 2x}{\sin x} dx$$

$$= \int \frac{(3 \sin x - 4 \sin^3 x) + 2 \sin x \cos x}{\sin x} dx$$

$$= \int (3 - 4 \sin^2 x + 2 \cos x) dx$$

$$= \int [3 - 2(1 - \cos 2x) + 2 \cos x] dx$$

$$= \int [1 + 2 \cos 2x + 2 \cos x] dx$$

$$= x + 2 \sin x + \sin 2x + C$$

$$5. (c) \text{ Let } I = \int \frac{dx}{(x^2 - 2x + 10)^2} = \int \frac{dx}{((x-1)^2 + 3^2)^2}$$

$$\text{Now, put } x-1 = 3 \tan \theta \Rightarrow dx = 3 \sec^2 \theta d\theta$$

$$\text{So, } I = \int \frac{3 \sec^2 \theta d\theta}{(3^2 \tan^2 \theta + 3^2)^2} = \int \frac{3 \sec^2 \theta d\theta}{3^4 \sec^4 \theta}$$

$$= \frac{1}{27} \int \cos^2 \theta d\theta = \frac{1}{27} \int \frac{1 + \cos 2\theta}{2} d\theta$$

$$\left[\because \cos^2 \theta = \frac{1 + \cos 2\theta}{2} \right]$$

$$= \frac{1}{54} \int (1 + \cos 2\theta) d\theta = \frac{1}{54} \left(\theta + \frac{\sin 2\theta}{2} \right) + C$$

$$= \frac{1}{54} \tan^{-1} \left(\frac{x-1}{3} \right) + \frac{1}{108} \left(\frac{2 \tan \theta}{1 + \tan^2 \theta} \right) + C$$

$$\left[\because \sin 2\theta = \frac{2 \tan \theta}{1 + \tan^2 \theta} \right]$$

$$= \frac{1}{54} \tan^{-1} \left(\frac{x-1}{3} \right) + \frac{1}{54} \frac{\left(\frac{x-1}{3} \right)}{1 + \left(\frac{x-1}{3} \right)^2} + C$$

$$= \frac{1}{54} \tan^{-1} \left(\frac{x-1}{3} \right) + \frac{1}{18} \left(\frac{x-1}{(x-1)^2 + 3^2} \right) + C$$

$$= \frac{1}{54} \tan^{-1} \left(\frac{x-1}{3} \right) + \frac{1}{18} \left(\frac{x-1}{x^2 - 2x + 10} \right) + C$$

$$= \frac{1}{54} \left[\tan^{-1} \left(\frac{x-1}{3} \right) + \frac{3(x-1)}{x^2 - 2x + 10} \right] + C$$

It is given that

$$I = A \left[\tan^{-1} \left(\frac{x-1}{3} \right) + \frac{f(x)}{x^2 - 2x + 10} \right] + C$$

On comparing, we get $A = \frac{1}{54}$ and

$$f(x) = 3(x-1)$$

$$6. (b) \text{ Let } I = \int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx,$$

$$\alpha \in \left(0, \frac{\pi}{2} \right) = \int \frac{\frac{\sin x}{\cos x} + \frac{\sin \alpha}{\cos \alpha}}{\frac{\sin x}{\cos x} - \frac{\sin \alpha}{\cos \alpha}} dx$$

$$= \int \frac{\sin x \cos \alpha + \sin \alpha \cos x}{\sin x \cos \alpha - \sin \alpha \cos x} dx$$

$$= \int \frac{\sin(x + \alpha)}{\sin(x - \alpha)} dx$$

$$\text{Now, put } x - \alpha = t$$

$$\Rightarrow dx = dt, \text{ so}$$

$$I = \int \frac{\sin(t + 2\alpha)}{\sin t} dt$$

$$= \int \frac{\sin t \cos 2\alpha + \sin 2\alpha \cos t}{\sin t} dt$$

$$\begin{aligned}
 &= \int \left(\cos 2\alpha + \sin 2\alpha \frac{\cos t}{\sin t} \right) dt \\
 &= t (\cos 2\alpha) + (\sin 2\alpha) \log_e |\sin t| + C \\
 &= (x - \alpha) \cos 2\alpha + (\sin 2\alpha) \log_e |\sin (x - \alpha)| + C \\
 &= A(x) \cos 2\alpha + B(x) \sin 2\alpha + C \text{ (given)} \\
 \text{Now on comparing, we get} \\
 A(x) = x - \alpha \text{ and } B(x) = \log_e |\sin (x - \alpha)|
 \end{aligned}$$

7. (b) We have,

$$\begin{aligned}
 I &= \int \left(1 + x - \frac{1}{x} \right) e^{x + \frac{1}{x}} dx \\
 I &= \int e^{x + \frac{1}{x}} dx + \int \left(x - \frac{1}{x} \right) e^{x + \frac{1}{x}} dx \\
 I &= \int e^{x + \frac{1}{x}} dx + \int x \left(1 - \frac{1}{x^2} \right) e^{x + \frac{1}{x}} dx \\
 I &= \int e^{x + \frac{1}{x}} dx + x e^{x + \frac{1}{x}} - \int e^{x + \frac{1}{x}} dx \\
 I &= x e^{x + \frac{1}{x}} + C
 \end{aligned}$$

8. (a) Given,

$$f'(x) = f(x) + \int_0^1 f(x) dx, f(0) = 1$$

...(i)

$$\Rightarrow f''(x) = f'(x) + 0 \Rightarrow \frac{f''(x)}{f'(x)} = 1$$

$$\Rightarrow \int \frac{f''(x)}{f'(x)} dx = \int dx$$

$$\Rightarrow \log f'(x) = x + C$$

$$\Rightarrow f'(x) = A e^x$$

$$\Rightarrow f(x) = A e^x + K$$

$$f(0) = A + K = 1 \quad \dots \text{(ii)}$$

$$\therefore A e^x = A e^x + K + \int_0^1 (A e^x + k) dx$$

$$\Rightarrow A(e - 1) + 2k = 0 \quad \dots \text{(iii)}$$

From Eqs. (ii) and (iii),

we get

$$A = \frac{2}{3 - e}, k = \frac{1 - e}{3 - e}$$

$$\therefore f(x) = \frac{2e^x}{3 - e} + \frac{1 - e}{3 - e}$$

9. (a) We have,

$$\begin{aligned}
 &\int \tan(x - \alpha) \tan(x + \alpha) \tan 2x dx \\
 &= p \log |\sec 2x| + q \log |\sec(x + \alpha)| \\
 &\quad + r \log |\sec(x - \alpha)| + C
 \end{aligned}$$

$$\text{Here, } \tan 2x = \tan((x - \alpha) + (x + \alpha))$$

$$= \tan(x - \alpha) + \tan(x + \alpha)$$

$$1 - \tan(x - \alpha) \tan(x + \alpha)$$

$$\Rightarrow \tan 2x - \tan(x - \alpha) \tan(x + \alpha)$$

$$= \tan(x - \alpha) + \tan(x + \alpha)$$

$$\Rightarrow \tan(x - \alpha) \tan(x + \alpha) \tan 2x$$

$$\Rightarrow \tan 2x - \tan(x - \alpha) - \tan(x + \alpha)$$

$$\therefore \int \tan(x - \alpha) \tan(x + \alpha) \tan 2x dx$$

$$= \int (\tan 2x - \tan(x - \alpha) - \tan(x + \alpha)) dx$$

$$\begin{aligned}
 &= \frac{\log |\sec 2x|}{2} - \log |\sec(x - \alpha)| \\
 &\quad - \log |\sec(x + \alpha)| + C
 \end{aligned}$$

$$= \frac{1}{2} \log |\sec 2x| + (-1) \log |\sec(x + \alpha)|$$

$$+ (-1) \log |\sec(x - \alpha)| + C$$

$$\therefore p = \frac{1}{2}, q = -1 \text{ and } r = -1$$

$$\therefore p + q + r = \frac{1}{2} + (-1) + (-1)$$

$$= \frac{1}{2} - 2 = -\frac{3}{2}$$

10. (d) Let $I = \int \frac{x^2 + 1}{x^4 - x^2 + 1} dx$

$$\begin{aligned}
 &= \int \frac{1 + \frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}} dx = \int \frac{1 + \frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 1}
 \end{aligned}$$

$$\text{put } x - \frac{1}{x} = t \left(1 + \frac{1}{x^2} \right) dx = dt$$

$$\therefore I = \int \frac{dt}{t^2 + 1} = \tan^{-1}(t) + c$$

$$= \tan^{-1}\left(x - \frac{1}{x}\right) + c = \tan^{-1}\left(\frac{x^2 - 1}{x}\right) + c$$

11. (a) Let

$$I = \int \frac{x^3 + 2x}{x^4 + 4} dx$$

$$I = \int \frac{x^3}{x^4 + 4} dx + \int \frac{2x}{x^4 + 4} dx$$

$$I = \frac{1}{4} \int \frac{4x^3}{x^4 + 4} dx + \int \frac{dt}{t^2 + 4}$$

$$[\because \text{put } x^2 = t]$$

$$I = \frac{1}{4} \log |x^4 + 4| + \frac{1}{2} \tan^{-1} \frac{t}{2} + c$$

$$I = \frac{1}{4} \tan^{-1} \frac{x^2}{2} + \frac{1}{2} \log \frac{\sqrt{x^4 + 4}}{2} + c$$

$$I = \frac{1}{2} \left[\tan^{-1} \left(\frac{x^2}{2} \right) + \log \frac{\sqrt{x^4 + 4}}{2} + c \right]$$

12. (d) We have,

$$\frac{2x + 7}{(x^2 + 4)(x^2 + 9)(x^2 + 16)}$$

$$= \frac{Ax + l}{x^2 + 4} + \frac{Bx + m}{x^2 + 9} + \frac{Cx + n}{x^2 + 16}$$

$$\begin{aligned}
 2x + 7 &= (Ax + l)(x^2 + 9)(x^2 + 16) + \\
 &\quad (Bx + m)(x^2 + 4)(x^2 + 16) + \\
 &\quad (Cx + n)(x^2 + 4)(x^2 + 9)
 \end{aligned}$$

$$\text{Put } x = 2i$$

$$4i + 7 = 120Ai + 60l$$

$$\Rightarrow \frac{l}{A} = \frac{60}{2}$$

$$\text{Put } x = 3i$$

$$6(i) + 7 = -105Bi - 35m$$

$$\Rightarrow \frac{1}{B} = \frac{-35}{2}$$

$$\text{Put } x = 4i$$

$$8i + 7 = 336Ci + 84n$$

$$\Rightarrow \frac{1}{C} = \frac{84}{2}$$

$$\therefore \frac{1}{A} + \frac{1}{B} + \frac{1}{C} = \frac{60 - 35 + 84}{2} = \frac{109}{2}$$

13. (b) Given,

$$\begin{aligned}
 &\int \cos x \cdot \cos 2x \cdot \cos 5x dx \\
 &= \frac{1}{2} \int 2 \cos x \cos 5x \cos 2x dx \\
 &= \frac{1}{2} \int \{ \cos(5x + x) + \cos(5x - x) \} \cos 2x dx \\
 &= \frac{1}{2} \int (\cos 6x + \cos 4x) \cos 2x dx \\
 &= \frac{1}{4} \int (2 \cos 6x \cos 2x + 2 \cos 2x \cos 4x) dx \\
 &= \frac{1}{4} \int (\cos 8x + \cos 4x + \cos 6x + \cos 2x) dx \\
 &= \frac{1}{4} \left[\frac{\sin 8x}{8} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \frac{\sin 2x}{2} \right] + k \\
 &= \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24} + \frac{\sin 8x}{32} + k
 \end{aligned}$$

On comparing,

$$A = \frac{1}{8}, B = \frac{1}{16}, C = \frac{1}{24} \text{ and } D = \frac{1}{32}$$

$$\therefore \frac{1}{B} + \frac{1}{C} = 16 + 24 = 40$$

$$\text{Now, } \frac{1}{A} + \frac{1}{D} = 8 + 32 = 40$$

$$\therefore \frac{1}{B} + \frac{1}{C} = \frac{1}{A} + \frac{1}{D}$$

14. (a) We have,

$$\begin{aligned}
 I &= \int \sqrt{1 + 2 \cot x (\cot x + \operatorname{cosec} x)} dx \\
 I &= \int \sqrt{1 + \cot^2 x + 2 \operatorname{cosec} x \cot x + \cot^2 x} dx
 \end{aligned}$$

$$I = \int \sqrt{(\operatorname{cosec} x + \cot x)^2} dx$$

$$I = \int (\operatorname{cosec} x + \cot x) dx$$

$$I = \log(1 - \cos x) + c$$

$$\Rightarrow I = \log \left(2 \sin^2 \frac{x}{2} \right) + c$$

$$\Rightarrow I = 2 \log \left| \sin \frac{x}{2} \right| + c$$

15. (b) We have,

$$\int e^{\alpha x} \left(\frac{1 - \beta \sin x}{1 - \cos x} \right) dx = -e^x \cot \frac{x}{2} + c$$

On differentiating by both sides, we get

$$\begin{aligned}
 e^{\alpha x} \left(\frac{1 - \beta \sin x}{1 - \cos x} \right) \\
 = -e^x \left[\cot \frac{x}{2} - \frac{1}{2} \operatorname{cosec}^2 \frac{x}{2} \right]
 \end{aligned}$$

$$= -e^x \left(\frac{\cos \frac{x}{2}}{\sin \frac{x}{2}} - \frac{1}{2 \sin^2 \frac{x}{2}} \right)$$

$$= -e^x \left[\frac{2 \sin \frac{x}{2} \cos \frac{x}{2} - 1}{2 \sin^2 \frac{x}{2}} \right]$$

$$\Rightarrow e^{\alpha x} \left(\frac{1 - \beta \sin x}{1 - \cos x} \right) = e^x \left(\frac{1 - \sin x}{1 - \cos x} \right)$$

$$\therefore \alpha = 1, \beta = 1$$

$$\text{Hence, } \frac{\alpha^2 + \beta^2}{2\alpha\beta} = \frac{1+1}{2} = 1$$

16. (a) $\int \frac{x-1}{(x+1)\sqrt{x^3+x^2+x}} dx$

$$= \int \frac{x^2-1}{(x+1)^2 x \sqrt{1+\frac{1}{x}+x}} dx$$

$$= \int \frac{x^2-1}{\left(1+1+\frac{1}{x}+x\right)x^2 \sqrt{1+\frac{1}{x}+x}} dx$$

$$= \int \frac{1-\frac{1}{x^2}}{\left[1+\left(1+\frac{1}{x}+x\right)\right] \sqrt{1+\frac{1}{x}+x}} dx$$

Let $1+\frac{1}{x}+x=t^2 \Rightarrow \left(1-\frac{1}{x^2}\right)dx = 2t dt$

$$= \int \frac{2t dt}{(1+t^2)t} = 2 \int \frac{dt}{1+t^2}$$

$$= 2 \tan^{-1}(t) + c$$

$$= 2 \tan^{-1} \left(\sqrt{\frac{1+x+x^2}{x}} \right) + C$$

Hence, option (a) is correct.

17. (c) Given integral

$$I(x) = \int x^2 (\log x)^2 dx$$

$$= \frac{x^3}{3} (\log x)^2 - \int \frac{x^3}{3} \frac{2 \log x}{x} dx$$

[by integration by parts]

$$= \frac{x^3}{3} (\log x)^2 - \frac{2}{3} \left[\frac{x^3}{3} (\log x) - \int \frac{x^3}{3} \left(\frac{1}{x} \right) dx \right]$$

$$= \frac{x^3}{3} (\log x)^2 - \frac{2}{3} \left[\frac{x^3}{3} (\log x) - \frac{1}{3} \frac{x^3}{3} \right] + C$$

$$= \frac{x^3}{27} [9(\log x)^2 - 6(\log x) + 2] + C$$

$\therefore I(1) = 0$

$\therefore \frac{2}{27} + C = 0$

$\Rightarrow C = -\frac{2}{27}$

$\therefore I(x) = \frac{x^3}{27} [9(\log x)^2 - 6(\log x) + 2] - \frac{2}{27}$

Hence, option (c) is correct.

Definite Integrals

1. (c) Let the integral

$$I = \frac{1}{a+b} \int_a^b x(f(x) + f(x+1)) dx \quad \dots(i)$$

On applying property

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx, \text{ we get}$$

$$I = \frac{1}{a+b} \int_a^b (a+b-x)(f(a+b-x) + f(a+b-x+1)) dx$$

$$= \frac{1}{a+b} \int_a^b (a+b-x)(f(x+1) + f(x)) dx \dots(ii)$$

$$\left[\begin{array}{l} \therefore f(a+b+1-x) = f(x) \\ \therefore f(a+b-x) = f(x+1) \end{array} \right]$$

On adding integrals Eqs. (i) and (ii), we get

$$2I = \frac{a+b}{a+b} \int_a^b (f(x) + f(x+1)) dx$$

$$\Rightarrow I = \frac{1}{2} \left[\int_a^b f(x) dx + \int_a^b f(x+1) dx \right]$$

$$= \frac{1}{2} \left[\int_a^b f(x) dx + \int_a^b (a+b-x+1) dx \right]$$

$$= \frac{1}{2} \left[\int_a^b f(x) dx + \int_a^b f(x) dx \right]$$

$$I = \int_a^b f(x) dx$$

Put $x = t + 1$, then upper limit $t = b - 1$ and lower limit $t = a - 1$ and $dx = dt$, so

$$I = \int_{a-1}^{b-1} f(t+1) dt$$

$$= \int_{a-1}^{b-1} f(x+1) dx$$

2. (b) Given trigonometric equation for $\theta \in (0, 2\pi) - \{\pi\}$ is

$$2 \cot^2 \theta - \frac{5}{\sin \theta} + 4 = 0$$

$$\Rightarrow 2 \cos^2 \theta - 5 \sin \theta + 4 \sin^2 \theta = 0$$

$$\Rightarrow 2 \sin^2 \theta - 5 \sin \theta + 2 = 0$$

$$\Rightarrow 2 \sin^2 \theta - 4 \sin \theta - \sin \theta + 2 = 0$$

$$\Rightarrow 2 \sin \theta (\sin \theta - 2) - 1(\sin \theta - 2) = 0$$

$$\Rightarrow (\sin \theta - 2)(2 \sin \theta - 1) = 0$$

$\therefore \sin \theta \in [-1, 1] - \{0\}$, for $\theta \in (0, 2\pi) - \{\pi\}$

$\therefore \sin \theta - 2 \neq 0$

So, $\sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$

$\therefore \theta_1 = \frac{\pi}{6}$ and $\theta_2 = \frac{5\pi}{6}$

Now, the integral $\int_{\theta_1}^{\theta_2} \cos^2 3\theta d\theta$

$$= \int_{\pi/6}^{5\pi/6} \left(\frac{1 + \cos 6\theta}{2} \right) d\theta$$

$$= \frac{1}{2} \left[\theta + \frac{\sin 6\theta}{6} \right]_{\pi/6}^{5\pi/6}$$

$$= \frac{1}{2} \left(\frac{5\pi}{6} - \frac{\pi}{6} \right) + \frac{1}{12} (0 - 0)$$

$$= \frac{1}{2} \left(\frac{4\pi}{6} \right) = \frac{\pi}{3}$$

3. (d) It is given that $f(x) = a + bx + cx^2$, then

$$\int_0^1 f(x) dx = \int_0^1 (a + bx + cx^2) dx$$

$$= \left[ax + \frac{b}{2} x^2 + \frac{c}{3} x^3 \right]_0^1 = a + \frac{b}{2} + \frac{c}{3}$$

$$= \frac{6a + 3b + 2c}{6}$$

$\therefore f(0) = a, f(1) = a + b + c$

and $4f\left(\frac{1}{2}\right) = 4a + 2b + c$

$$\therefore \int_0^1 f(x) dx = \frac{6a + 3b + 2c}{6}$$

$$= \frac{1}{6} \left\{ f(0) + f(1) + 4f\left(\frac{1}{2}\right) \right\}$$

Hence, option (d) is correct.

4. (c) We have, $\int_0^{\pi/3} \frac{\tan \theta}{\sqrt{2k \sec \theta}} d\theta = 1 - \frac{1}{\sqrt{2}}$

($k > 0$)

Let $I = \int_0^{\pi/3} \frac{\tan \theta}{\sqrt{2k \sec \theta}} d\theta$

$$= \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{\tan \theta}{\sqrt{\sec \theta}} d\theta$$

$$= \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{(\sin \theta)}{(\cos \theta) \sqrt{\frac{1}{\cos \theta}}} d\theta$$

$$= \frac{1}{\sqrt{2k}} \int_0^{\pi/3} \frac{\sin \theta}{\sqrt{\cos \theta}} d\theta$$

Let $\cos \theta = t \Rightarrow -\sin \theta d\theta = dt$

$\Rightarrow \sin \theta d\theta = -dt$

for lower limit, $\theta = 0 \Rightarrow t = \cos 0 = 1$

for upper limit, $\theta = \frac{\pi}{3} \Rightarrow t = \cos \frac{\pi}{3} = \frac{1}{2}$

$$\Rightarrow I = \frac{1}{\sqrt{2k}} \int_1^{1/2} \frac{-dt}{\sqrt{t}} = \frac{-1}{\sqrt{2k}} \int_1^{1/2} t^{-1/2} dt$$

$$= -\frac{1}{\sqrt{2k}} \left(\frac{t^{-1/2+1}}{-1/2+1} \right)_1^{1/2} = -\frac{1}{\sqrt{2k}} [2\sqrt{t}]_1^{1/2}$$

$$= -\frac{2}{\sqrt{2k}} \left[\sqrt{\frac{1}{2}} - \sqrt{1} \right]$$

$$= \frac{2}{\sqrt{2k}} \left(1 - \frac{1}{\sqrt{2}} \right)$$

$$\therefore I = 1 - \frac{1}{\sqrt{2}} \quad (\text{given})$$

$$\therefore \frac{2}{\sqrt{2k}} \left(1 - \frac{1}{\sqrt{2}} \right) = 1 - \frac{1}{\sqrt{2}} \Rightarrow \frac{2}{\sqrt{2k}} = 1$$

$$\Rightarrow 2 = \sqrt{2k}$$

$$\Rightarrow 2k = 4 \Rightarrow k = 2$$

5. (d) Let $I = \int_{-2}^2 \frac{\sin^2 x}{\frac{1}{2} + \left[\frac{x}{\pi} \right]} dx$

Also, let $f(x) = \frac{\sin^2 x}{\frac{1}{2} + \left[\frac{x}{\pi} \right]}$

Then, $f(-x) = \frac{\sin^2(-x)}{\frac{1}{2} + \left[\frac{-x}{\pi} \right]}$

(replacing x by $-x$)

$$= \frac{\sin^2 x}{\frac{1}{2} + \left(-1 - \left[\frac{x}{\pi} \right] \right)}$$

$$\Rightarrow f(-x) = -\frac{\sin^2 x}{\frac{1}{2} + \left[\frac{x}{\pi} \right]} = -f(x)$$

i.e. $f(x)$ is odd function

$$\therefore I = 0$$

$$\left[\because \int_{-a}^a f(x) dx = \begin{cases} 0, & \text{if } f(x) \text{ is odd function} \\ 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is even function} \end{cases} \right]$$

6. (d) Let $I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{[x] + [\sin x] + 4}$

$$= \int_{-\frac{\pi}{2}}^{-1} \frac{dx}{[x] + [\sin x] + 4} + \int_{-1}^0 \frac{dx}{[x] + [\sin x] + 4}$$

$$+ \int_0^1 \frac{dx}{[x] + [\sin x] + 4} + \int_1^{\frac{\pi}{2}} \frac{dx}{[x] + [\sin x] + 4}$$

$$\therefore [x] = \begin{cases} -2, & -\pi/2 < x < -1 \\ -1, & -1 \leq x < 0 \\ 0, & 0 \leq x < 1 \\ 1, & 1 \leq x < \pi/2 \end{cases}$$

$$\text{and } [\sin x] = \begin{cases} -1, & -\pi/2 < x < -1 \\ -1, & -1 < x < 0 \\ 0, & 0 < x < 1 \\ 0, & 1 < x < \pi/2 \end{cases}$$

[$\because$ For $x < 0, -1 \leq \sin x$

< 0 and for $x > 0, 0 < \sin x \leq 1$]

$$\text{So, } I = \int_{-\frac{\pi}{2}}^{-1} \frac{dx}{-2-1+4} + \int_{-1}^0 \frac{dx}{-1-1+4} + \int_0^1 \frac{dx}{0+0+4} + \int_1^{\frac{\pi}{2}} \frac{dx}{1+0+4}$$

$$= \int_{-\frac{\pi}{2}}^{-1} \frac{dx}{1} + \int_{-1}^0 \frac{dx}{2} + \int_0^1 \frac{dx}{4} + \int_1^{\frac{\pi}{2}} \frac{dx}{5}$$

$$= \left(-1 + \frac{\pi}{2} \right) + \frac{1}{2} (0+1) + \frac{1}{4} (1-0)$$

$$+ \frac{1}{5} \left(\frac{\pi}{2} - 1 \right)$$

$$= \left(-1 + \frac{1}{2} + \frac{1}{4} - \frac{1}{5} \right) + \left(\frac{\pi}{2} + \frac{\pi}{10} \right)$$

$$= \frac{-20+10+5-4}{20} + \frac{5\pi+\pi}{10}$$

$$= -\frac{9}{20} + \frac{3\pi}{5} = \frac{3}{20} (4\pi-3)$$

7. (c) Let $I = \int_{\pi/6}^{\pi/4} \frac{dx}{\sin 2x (\tan^5 x + \cot^5 x)}$

$$= \int_{\pi/6}^{\pi/4} \frac{(1 + \tan^2 x) \tan^5 x}{2 \tan x (\tan^{10} x + 1)} dx$$

$$\left[\because \sin 2x = \frac{2 \tan x}{1 + \tan^2 x} \right]$$

$$= \frac{1}{2} \int_{\pi/6}^{\pi/4} \frac{\tan^4 x \sec^2 x}{(\tan^{10} x + 1)} dx$$

Put $\tan^5 x = t$

$$[\because \sec^2 x = 1 + \tan^2 x]$$

$$\Rightarrow 5 \tan^4 x \sec^2 x dx = dt$$

x	$\frac{\pi}{6}$	$\frac{\pi}{4}$
t	$\left(\frac{1}{\sqrt{3}} \right)^5$	1

$$\therefore I = \frac{1}{2} \cdot \frac{1}{5} \int_{(1/\sqrt{3})^5}^1 \frac{dt}{t^2 + 1} = \frac{1}{10} (\tan^{-1}(t))_{(1/\sqrt{3})^5}^1$$

$$= \frac{1}{10} \left(\tan^{-1}(1) - \tan^{-1} \left(\frac{1}{9\sqrt{3}} \right) \right)$$

$$= \frac{1}{10} \left(\frac{\pi}{4} - \tan^{-1} \left(\frac{1}{9\sqrt{3}} \right) \right)$$

8. (d) Given, $f(x) = \int_0^x g(t) dt$

On replacing x by $(-x)$, we get

$$f(-x) = \int_0^{-x} g(t) dt$$

Now, put $t = -u$, so

$$f(-x) = - \int_0^x g(-u) du$$

$$= - \int_0^x g(u) du = -f(x)$$

[$\because g$ is an even function]

$$\Rightarrow f(-x) = -f(x)$$

$\Rightarrow f$ is an odd function.

Now, it is given that $f(x+5) = g(x)$

$$\therefore f(5-x) = g(-x) = g(x) = f(x+5)$$

[$\because g$ is an even function]

$$\Rightarrow f(5-x) = f(x+5) \quad \dots(i)$$

$$\text{Let } I = \int_0^x f(t) dt$$

$$\text{Put } t = u+5 \Rightarrow t-5 = u \Rightarrow dt = du$$

$$\therefore I = \int_{-5}^{x-5} f(u+5) du = \int_{-5}^{x-5} g(u) du$$

Put $u = -t \Rightarrow du = -dt$, we get

$$I = - \int_5^{5-x} g(-t) dt = \int_{5-x}^5 g(t) dt$$

[$\because - \int_a^b f(x) dx = \int_b^a f(x) dx$ and g is an even function]

$$I = \int_{5-x}^5 f'(t) dt$$

[by Leibnitz rule $f'(x) = g(x)$]

$$= f(5) - f(5-x) = f(5) - f(5+x) \quad [\text{from Eq. (i)}]$$

$$= \int_{5+x}^5 f'(t) dt = \int_{5+x}^5 g(t) dt$$

9. (b) Let $p = \lim_{n \rightarrow \infty}$

$$\left(\frac{(n+1)^{1/3}}{n^{4/3}} + \frac{(n+2)^{1/3}}{n^{4/3}} + \dots + \frac{(2n)^{1/3}}{n^{4/3}} \right)$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{(n+r)^{1/3}}{n^{4/3}}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(1 + \frac{r}{n} \right)^{1/3} \frac{n^{1/3}}{n^{4/3}}$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \left(1 + \frac{r}{n} \right)^{1/3}$$

Now, as per integration as limit of sum.

$$\text{Let } \frac{r}{n} = x \text{ and } \frac{1}{n} = dx \quad [: n \rightarrow \infty]$$

Then, upper limit of integral is 1 and lower limit of integral is 0.

$$\text{So, } p = \int_0^1 (1+x)^{1/3} dx$$

$$\left[\because \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx \right]$$

$$= \left[\frac{3}{4} (1+x)^{4/3} \right]_0^1$$

$$= \frac{3}{4} (2^{4/3} - 1) = \frac{3}{4} (2)^{4/3} - \frac{3}{4}$$

10. (a) Given integral

$$I = \int_0^{2\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$$

$$= \int_0^{\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$$

$$+ \int_{\pi}^{2\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$$

$$= I_1 + I_2 \text{ (let)} \quad \dots (i)$$

Now, $I_2 = \int_{\pi}^{2\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$

let $2\pi - x = t$, upper limit $t=0$ and lower limit $t=\pi$

and $dx = -dt$

So, $I_2 = - \int_{\pi}^0 [-\sin 2x \cdot (1 + \cos 3x)] dx$
 $= \int_0^{\pi} [-\sin 2x \cdot (1 + \cos 3x)] dx \quad \dots(ii)$

$\therefore I = \int_0^{\pi} [\sin 2x \cdot (1 + \cos 3x)] dx$
 $+ \int_0^{\pi} [-\sin 2x \cdot (1 + \cos 3x)] dx$
 [from Eqs. (i) and (ii)]
 $= \int_0^{\pi} (-1) dx \quad [\because [x] + [-x] = -1, x \notin \text{Integer}]$

11. (a) Let $I = \int_{\alpha}^{\alpha+1} \frac{dx}{(x+\alpha)(x+\alpha+1)}$
 $= \int_{\alpha}^{\alpha+1} \frac{(x+\alpha+1) - (x+\alpha)}{(x+\alpha)(x+\alpha+1)} dx$
 $= \int_{\alpha}^{\alpha+1} \left(\frac{1}{x+\alpha} - \frac{1}{x+\alpha+1} \right) dx$
 $= [\log_e(x+\alpha) - \log_e(x+\alpha+1)]_{\alpha}^{\alpha+1}$
 $= \left[\log_e \left(\frac{x+\alpha}{x+\alpha+1} \right) \right]_{\alpha}^{\alpha+1}$
 $= \log_e \frac{2\alpha+1}{2\alpha+2} - \log_e \frac{2\alpha}{2\alpha+1}$
 $= \log_e \left(\frac{9}{8} \right) \quad (\text{given})$

$\Rightarrow \frac{(2\alpha+1)^2}{4\alpha(\alpha+1)} = \frac{9}{8}$

$\Rightarrow 8[4\alpha^2 + 4\alpha + 1] = 36(\alpha^2 + \alpha)$

$\Rightarrow 8\alpha^2 + 8\alpha + 2 = 9\alpha^2 + 9\alpha$

$\Rightarrow \alpha^2 + \alpha - 2 = 0$

$\Rightarrow (\alpha+2)(\alpha-1) = 0 \Rightarrow \alpha = 1, -2$

From the options we get $\alpha = -2$

12. (a) Given integral

$I = \int_{\log_e 2}^x \frac{dt}{\sqrt{e^t - 1}} = \frac{\pi}{6} \quad (\text{given})$

$\Rightarrow \int_{\log_e 2}^x \frac{e^{-t/2}}{\sqrt{1 - (e^{-t/2})^2}} dt = \frac{\pi}{6}$

put $e^{-t/2} = u$, at $t = \log_e 2, u = \frac{1}{\sqrt{2}}$ and at

$t = x, u = e^{-x/2}$ and $e^{-t/2} dt = -2 du$

so, $I = \int_{\frac{1}{\sqrt{2}}}^{e^{-x/2}} \frac{-2 du}{\sqrt{1 - u^2}} = \frac{\pi}{6}$

$\Rightarrow [-2 \sin^{-1} u]_{\frac{1}{\sqrt{2}}}^{e^{-x/2}} = \frac{\pi}{6}$
 $\Rightarrow \sin^{-1}(e^{-x/2}) - \frac{\pi}{4} = -\frac{\pi}{12}$

$\Rightarrow \sin^{-1}(e^{-x/2}) = \frac{\pi}{6}$

$\Rightarrow \sin^{-1}(e^{-x/2}) = \frac{\pi}{6}$

$\Rightarrow e^{-x/2} = \sin \frac{\pi}{6} = \frac{1}{2}$

$\Rightarrow -\frac{x}{2} = \log_e \left(\frac{1}{2} \right) = -\log_e(2)$

$\Rightarrow x = 2 \cdot \log_e(2)$

Hence, option (a) is correct.

13. (d) Let $I = \int_{-1}^1 \left\{ \frac{x^{2015}}{e^{|x|}(x^2 + \cos x)} + \frac{1}{e^{|x|}} \right\} dx$

$= \int_{-1}^1 \frac{x^{2015}}{e^{|x|}(x^2 + \cos x)} dx + \int_{-1}^1 \frac{1}{e^{|x|}} dx$

Let $f(x) = \frac{x^{2015}}{e^{|x|}(x^2 + \cos x)}$ and $g(x) = \frac{1}{e^{|x|}}$

Again, $f(-x) = \frac{(-x)^{2015}}{e^{-|x|}((-x)^2 + \cos(-x))}$
 $= \frac{-x^{2015}}{e^{|x|}(x^2 + \cos x)} = -f(x)$

and $g(-x) = \frac{1}{e^{|-x|}} = \frac{1}{e^{|x|}} = g(x)$

$\therefore f(x)$ is odd function and $g(x)$ is even function.

$\therefore \int_{-1}^1 f(x) dx = 0$ and $\int_{-1}^1 g(x) dx = 2 \int_0^1 g(x) dx$

$\therefore I = 2 \int_0^1 \frac{1}{e^{|x|}} dx = 2(1 - e^{-1})$

14. (b) Let

$I = \int_{-\pi/4}^{\pi/4} \left(\lambda |\sin x| + \frac{\mu \sin x}{1 + \cos x} + \gamma \right) dx$
 $= \int_{-\pi/4}^{\pi/4} (\lambda |\sin x| + \gamma) dx + \mu \int_{-\pi/4}^{\pi/4} \frac{\sin x}{1 + \cos x} dx$

Let $f(x) = \frac{\sin x}{1 + \cos x}$
 $\Rightarrow f(-x) = \frac{\sin(-x)}{1 + \cos(-x)}$
 $= \frac{-\sin x}{1 + \cos x} = -f(x)$

$\therefore f(x)$ is an odd function.

So, $\int_{-\pi/4}^{\pi/4} \left(\frac{\sin x}{1 + \cos x} \right) dx = 0$

$\therefore I = \int_{-\pi/4}^{\pi/4} (\lambda |\sin x| + \gamma) dx$

$\therefore I$ is independent of μ .

15. (c) Let $I = \int_{\pi/18}^{4\pi/18} \frac{2\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx$
 $= 2 \int_{\pi/18}^{4\pi/18} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \quad \dots(i)$
 $= 2$

$\int_{\pi/18}^{4\pi/18} \frac{\sqrt{\sin \left(\frac{4\pi}{9} + \frac{\pi}{18} - x \right)}}{\sqrt{\sin \left(\frac{4\pi}{9} + \frac{\pi}{18} - x \right)} + \sqrt{\cos \left(\frac{4\pi}{9} + \frac{\pi}{18} - x \right)}} dx$
 $\left(\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right)$

$\Rightarrow I = 2 \int_{\pi/18}^{4\pi/18} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \quad \dots(ii)$

On adding Eqs. (i) and (ii), we get

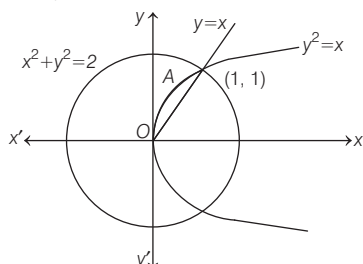
$2I = 2 \int_{\pi/18}^{4\pi/18} dx = 2 \left[x \right]_{\pi/18}^{4\pi/18}$
 $= 2 \left[\frac{4\pi}{9} - \frac{\pi}{18} \right] = 2 \left(\frac{7\pi}{18} \right)$

$\Rightarrow I = \frac{7\pi}{18}$

Area Bounded by Curve

1. (b) Let the area of the region, enclosed by the parabola $y^2 = x$ and straight line $y = x$ is

$$A = \int_0^1 (\sqrt{x} - x) dx$$



$$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2} \right]_0^1 = \left(\frac{1}{3/2} - \frac{1}{2} \right) = \left(\frac{2}{3} - \frac{1}{2} \right) = \frac{4-3}{6} = \frac{1}{6}$$

$\therefore$ Area of circle having radius $r = \sqrt{2}$ unit is $\pi r^2 = 2\pi$

$\therefore$ The area of the region, enclosed by the circle $x^2 + y^2 = 2$, which is not common to the region bounded by the parabola $y^2 = x$ and the straight line $y = x$, is $2\pi - \frac{1}{6} = \frac{1}{6}(12\pi - 1)$.

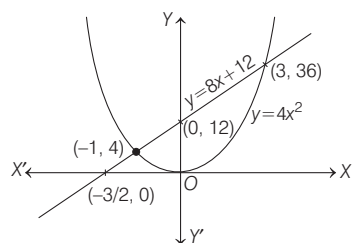
2. (d) The area of region $\{(x, y) \in \mathbb{R}^2 \mid 4x^2 \leq y \leq 8x + 12\}$ is area of region bounded by curves $y = 4x^2$ and $y = 8x + 12$.

To get the point of intersection of curves, on eliminating y , we get

$$4x^2 = 8x + 12$$

$$\Rightarrow x^2 - 2x - 3 = 0$$

$$\Rightarrow x = -1, 3$$

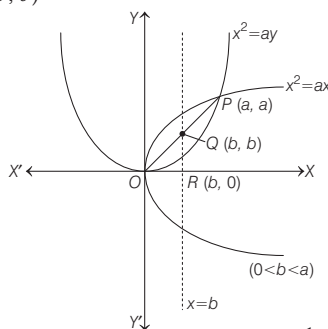


$$\text{So, required area} = \int_{-1}^3 (8x + 12 - 4x^2) dx$$

$$= \left[\frac{8x^2}{2} + 12x - \frac{4x^3}{3} \right]_{-1}^3 = \{(4 \times 9) + (12 \times 3) - (4 \times 9)\} - \left\{ 4 - 12 + \frac{4}{3} \right\}$$

$$= 36 + \frac{20}{3} = \frac{128}{3} \text{ sq. units}$$

3. (b) Given curves $C_1 : y^2 = ax$, and $C_2 : x^2 = ay$, ($a > 0$), intersect each other at origin 'O' and a point $P(a, a)$. $\therefore O(0, 0)$, Q and $P(a, a)$ are collinear. So, $Q(b, b)$



Now, as it is given area of $\Delta OQR = \frac{1}{2}$

$$\Rightarrow \frac{1}{2} b^2 = \frac{1}{2} \Rightarrow b = 1 \quad (\because b > 0)$$

$\therefore$ The line $x = b$ bisects the area bounded by the curves, C_1 and C_2 , so

$$\begin{aligned} \int_0^1 (\sqrt{a}\sqrt{x} - \frac{x^2}{a}) dx &= \int_1^a (\sqrt{a}\sqrt{x} - \frac{x^2}{a}) dx \\ \Rightarrow \left[\sqrt{a} \frac{x^{3/2}}{3/2} - \frac{1}{a} \frac{x^3}{3} \right]_0^1 &= \left[\sqrt{a} \frac{x^{3/2}}{3/2} - \frac{1}{a} \frac{x^3}{3} \right]_1^a \\ \Rightarrow \frac{2\sqrt{a}}{3} - \frac{1}{3a} &= \frac{2a^{3/2}}{3} - \frac{a^2}{3} - \frac{2\sqrt{a}}{3} + \frac{1}{3a} \\ \Rightarrow \frac{4\sqrt{a}}{3} &= \frac{a^2}{3} + \frac{2}{3a} \Rightarrow 4a\sqrt{a} = a^3 + 2 \\ \Rightarrow 16a^3 &= a^6 + 4 + 4a^3 \\ \Rightarrow a^6 - 12a^3 + 4 &= 0 \end{aligned}$$

$\therefore a$ satisfies the equation $x^6 - 12x^3 + 4 = 0$

Hence, option (b) is correct.

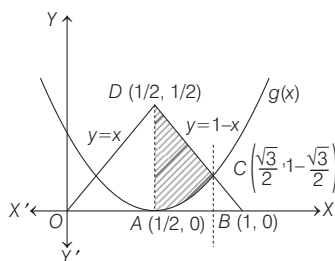
4. (d) On drawing the given curves

$$y = f(x) = \begin{cases} x, & 0 \leq x < \frac{1}{2} \\ \frac{1}{2}, & x = \frac{1}{2} \\ 1-x, & \frac{1}{2} < x \leq 1 \end{cases} \text{ and}$$

$$y = g(x) = \left(x - \frac{1}{2} \right)^2, \text{ we have}$$

Coordinate of points

$$A\left(\frac{1}{2}, 0\right), B(1, 0), C\left(\frac{\sqrt{3}}{2}, 1 - \frac{\sqrt{3}}{2}\right) \text{ and } D\left(\frac{1}{2}, \frac{1}{2}\right)$$



So, required area = area of shaded region

$$\begin{aligned} &= \int_{\frac{1}{2}}^1 \left[1-x - \left(x - \frac{1}{2} \right)^2 \right] dx \\ &= \int_{\frac{1}{2}}^1 \left(1-x - x^2 + x - \frac{1}{4} \right) dx \\ &= \int_{\frac{1}{2}}^1 \left(\frac{3}{4} - x^2 \right) dx = \left(\frac{3}{4}x - \frac{x^3}{3} \right)_{\frac{1}{2}}^1 \\ &= \frac{3\sqrt{3}}{8} - \frac{3\sqrt{3}}{24} - \frac{3}{8} + \frac{1}{24} = \left(\frac{\sqrt{3}}{4} - \frac{1}{3} \right) \text{ sq. units} \end{aligned}$$

Hence, option (d) is correct.

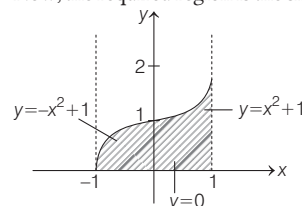
5. (a) We have,

$$A = \{(x, y) : 0 \leq y \leq x \mid |x| + 1 \text{ and } -1 \leq x \leq 1\}$$

When $x \geq 0$, then $0 \leq y \leq x^2 + 1$

and when $x < 0$, then $0 \leq y \leq -x^2 + 1$

Now, the required region is the shaded region.



$[\because y = x^2 + 1 \Rightarrow x^2 = (y-1)$, parabola with vertex (0, 1) and $y = -x^2 + 1$

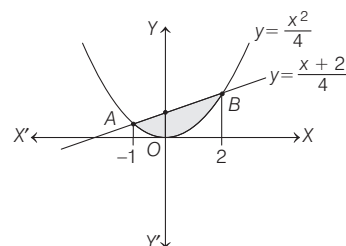
$$\Rightarrow x^2 = -(y-1),$$

parabola with vertex (0, 1) but open downward]

We need to calculate the shaded area, which is equal to

$$\begin{aligned} &\int_{-1}^0 (-x^2 + 1) dx + \int_0^1 (x^2 + 1) dx \\ &= \left(0 - \left[-\frac{(-1)^3}{3} + (-1) \right] \right) + \left(\left[\frac{1}{3} + 1 \right] - 0 \right) \\ &= -\left(\frac{1}{3} - 1 \right) + \frac{4}{3} = \frac{2}{3} + \frac{4}{3} = 2 \end{aligned}$$

6. (b) Given equation of curve is $x^2 = 4y$, which represent a parabola with vertex (0, 0) and it open upward.



Now, let us find the points of intersection of $x^2 = 4y$ and $4y = x + 2$

For this consider, $x^2 = x + 2$

$$\Rightarrow x^2 - x - 2 = 0$$

$$\Rightarrow (x - 2)(x + 1) = 0$$

$$\Rightarrow x = -1, x = 2$$

When $x = -1$, then $y = \frac{1}{4}$

and when $x = 2$, then $y = 1$

Thus, the points of intersection are

$A(-1, \frac{1}{4})$ and $B(2, 1)$.

Now, required area = area of shaded region

$$= \int_{-1}^2 \{y(\text{line}) - y(\text{parabola})\} dx$$

$$= \int_{-1}^2 \left(\frac{x+2}{4} - \frac{x^2}{4} \right) dx$$

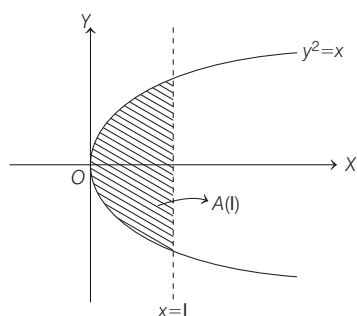
$$= \frac{1}{4} \left[\left(2 + 4 - \frac{8}{3} \right) - \left(\frac{1}{2} - 2 + \frac{1}{3} \right) \right]$$

$$= \frac{1}{4} \left[8 - \frac{1}{2} - 3 \right] = \frac{1}{4} \left[5 - \frac{1}{2} \right] = \frac{9}{8} \text{ sq units.}$$

7. (c) Given,

$S(\alpha) = \{(x, y) : y^2 \leq x, 0 \leq x \leq \alpha\}$ and

$A(\alpha)$ is area of the region $S(\alpha)$



Clearly, $A(\lambda) = 2 \int_0^\lambda \sqrt{x} dx$

$$= 2 \left[\frac{x^{3/2}}{3/2} \right]_0^\lambda = \frac{4}{3} \lambda^{3/2}$$

Since, $\frac{A(\lambda)}{A(4)} = \frac{2}{5}$, ($0 < \lambda < 4$)

$$\Rightarrow \frac{\lambda^{3/2}}{4^{3/2}} = \frac{2}{5}$$

$$\Rightarrow \left(\frac{\lambda}{4} \right)^3 = \left(\frac{2}{5} \right)^2$$

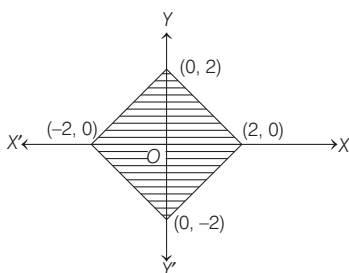
$$\Rightarrow \frac{\lambda}{4} = \left(\frac{4}{25} \right)^{1/3}$$

$$\Rightarrow \lambda = 4 \left(\frac{4}{25} \right)^{1/3}$$

8. (c) The given inequalities are

$|x - y| \leq 2$ and $|x + y| \leq 2$.

On drawing, the above inequalities, we get a square



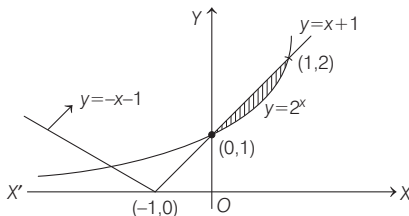
Now, the area of shaded region is equal to the area of a square having side length $\sqrt{(2-0)^2 + (0-2)^2} = 2\sqrt{2}$ units.

9. (d) Given, equations of curves

$y = 2^x$

and $y = |x + 1| = \begin{cases} x + 1, & x \geq -1 \\ -x - 1, & x < -1 \end{cases}$

$\therefore$ The figure of above given curves is



In first quadrant, the above given curves intersect each other at (1, 2).

So, the required area = $\int_0^1 ((x + 1) - 2^x) dx$

$$= \left[\frac{x^2}{2} + x - \frac{2^x}{\log_e 2} \right]_0^1$$

$$\left[\because \int a^x dx = \frac{a^x}{\log_e a} + C \right]$$

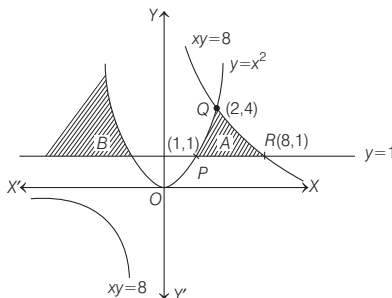
$$= \left[\frac{1}{2} + 1 - \frac{2}{\log_e 2} + \frac{1}{\log_e 2} \right]$$

$$= \frac{3}{2} - \frac{1}{\log_e 2}$$

10. (c) The given region

$\{(x, y) : xy \leq 8, 1 \leq y \leq x^2\}$.

From the figure, region A and B satisfy the given region, but only A is bounded region, so area of bounded region



$$A = \int_1^2 (x^2 - 1) dx + \int_2^8 \left(\frac{8}{x} - 1 \right) dx$$

[$\because$ Points $P(1, 1)$, $Q(2, 4)$ and $R(8, 1)$]

$$= \left[\frac{x^3}{3} - x \right]_1^2 + \left[8 \log |x| - x \right]_2^8$$

$$= \left(\frac{8}{3} - 2 - \frac{1}{3} + 1 \right) + 8 \log 8 - 8 - 8 \log 2 + 2$$

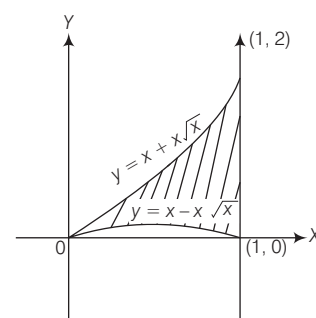
$$= -\frac{14}{3} + 16 \log 2$$

$$= 16 \log 2 - \frac{14}{3}$$

11. (d) Given, curve is $(y - x)^2 = x^3$

$$\Rightarrow y - x = \pm x\sqrt{x}$$

$$\Rightarrow y = x \pm x\sqrt{x}$$



$\therefore$ Area bounded

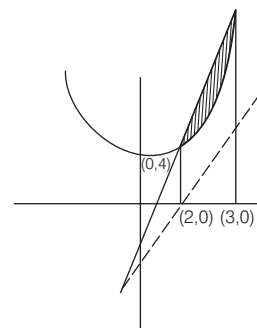
$$= \left| \int_0^1 \{(x + x\sqrt{x}) - (x - x\sqrt{x})\} dx \right|$$

$$= \left| 2 \int_0^1 x^{3/2} dx \right| = \frac{4}{5} \text{ sq unit}$$

12. (d) We have,

$y = x^2 + 4$, $y = 5x - 2$

Solving equation, we get



(2, 8) and (3, 13)

Area of shaded region

$$\int_2^3 ((5x - 2) - (x^2 + 4)) dx$$

$$= \int_2^3 (5x - x^2 - 6) dx = \left[\frac{5x^2}{2} - \frac{x^3}{3} - 6x \right]_2^3$$

$$= \left[\left(\frac{45}{2} - 9 - 18 \right) - \left(10 - \frac{8}{3} - 12 \right) \right] = \frac{1}{6}$$

Differential Equations

1. (c) The given differential equation

$$e^y \left(\frac{dy}{dx} - 1 \right) = e^x$$

$$\Rightarrow \frac{dy}{dx} - 1 = e^{x-y}$$

$$\text{Let } x - y = t$$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore -\frac{dt}{dx} = e^t$$

$$\Rightarrow -\int \frac{dt}{e^t} = dx \Rightarrow e^{-t} = x + c$$

$$\Rightarrow e^{-(x-y)} = x + c$$

$$\Rightarrow e^{y-x} = x + c$$

$$\therefore y(0) = 0, \text{ so } e^{0-0} = 0 + c \Rightarrow c = 1$$

$$\therefore e^{y-x} = x + 1$$

$$\Rightarrow y = x + \log_e(x + 1)$$

$$\text{So, } y(1) = 1 + \log_e 2$$

2. (b) Given differential equation

$$(y^2 - x) \frac{dy}{dx} = 1$$

$$\Rightarrow \frac{dx}{dy} + x = y^2 \text{ is a linear differential equation.}$$

$$\therefore \text{I.F.} = e^{\int 1 dy} = e^y$$

$$\therefore \text{Solution of given differential equation is}$$

$$xe^y = \int y^2 e^y dy + C$$

$$= y^2 e^y - 2 \int y e^y dy + C$$

$$= y^2 e^y - 2 \{ y e^y - \int e^y dy \} + C$$

$$\Rightarrow xe^y = y^2 e^y - 2 y e^y + 2 e^y + C$$

$$\therefore y(0) = 1$$

$$\Rightarrow 0e' = 1e' - 2e' + 2e' + C$$

$$\Rightarrow C = -e$$

$$\therefore \text{Particular solution of given differential equation is}$$

$$xe^y = y^2 e^y - 2 y e^y + 2 e^y - e \quad \dots(i)$$

$$\therefore \text{Curve (i) intersects the X-axis, then } y = 0$$

$$\therefore xe^0 = 2 - e$$

$$\Rightarrow x = 2 - e$$

$$\text{Therefore, abscissa of point of intersection of curve (i) and X-axis is } (2 - e).$$

3. (b) Given differential equation,

$$\frac{dy}{dx} = \frac{xy}{x^2 + y^2}$$

$$\text{Put } y = vx$$

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{then, } v + x \frac{dv}{dx} = \frac{v}{1 + v^2}$$

$$\Rightarrow x \frac{dv}{dx} = \frac{v - v - v^3}{1 + v^2}$$

$$\Rightarrow \left(\frac{1}{v^3} + \frac{1}{v} \right) dv = -\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{2v^2} + \log_e |v| = -\log_e |x| + C$$

$$(\text{on integration both sides})$$

$$\Rightarrow -\frac{x^2}{2y^2} + \log_e |y| = C \quad \left(\because v = \frac{y}{x} \right)$$

$$\therefore y(1) = 1$$

$$\therefore C = -\frac{1}{2}$$

$$\text{Now, as } y = e, \text{ then}$$

$$-\frac{x^2}{2e^2} + 1 = -\frac{1}{2}$$

$$\Rightarrow \frac{3}{2} = \frac{x^2}{2e^2} \Rightarrow x^2 = 3e^2 \Rightarrow x = \sqrt{3}e$$

$$\text{Hence, option (b) is correct.}$$

4. (d) Given differential equation is

$$(x^2 + 1)^2 \frac{dy}{dx} + 2x(x^2 + 1)y = 1$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{2x}{1 + x^2} \right) y = \frac{1}{(1 + x^2)^2}$$

$$[\text{dividing each term by } (1 + x^2)^2] \quad \dots(i)$$

$$\text{This is a linear differential equation of the form } \frac{dy}{dx} + P \cdot y = Q$$

$$= e^{\ln(1 + x^2)} = (1 + x^2)$$

$$\text{and required solution of differential Eq. (i) is given by}$$

$$y \cdot (\text{I.F.}) = \int Q(\text{I.F.}) dx + C$$

$$\Rightarrow y(1 + x^2) = \int \frac{1}{(1 + x^2)^2} (1 + x^2) dx + C$$

$$\Rightarrow y(1 + x^2) = \int \frac{dx}{1 + x^2} + C$$

$$\Rightarrow y(1 + x^2) = \tan^{-1}(x) + C$$

$$\therefore y(0) = 0$$

$$\therefore C = 0$$

$$\therefore y(1 + x^2) = \tan^{-1} x \quad [\because C = 0]$$

$$\Rightarrow \sqrt{a} y = \sqrt{a} \left(\frac{\tan^{-1} x}{1 + x^2} \right)$$

$$[\text{multiplying both sides by } \sqrt{a}]$$

$$\text{Now, at } x = 1$$

$$\sqrt{a} y(1) = \sqrt{a} \left(\frac{\tan^{-1}(1)}{1 + 1} \right)$$

$$= \sqrt{a} \frac{\frac{\pi}{4}}{2} = \frac{\sqrt{a} \pi}{8} = \frac{\pi}{32} \text{ (given)}$$

$$\therefore \sqrt{a} = \frac{1}{4} \Rightarrow a = \frac{1}{16}$$

5. (b) Given differential equation

$$\cos x \frac{dy}{dx} - (\sin x)y = 6x$$

$$\Rightarrow \frac{dy}{dx} - (\tan x)y = \frac{6x}{\cos x}, \text{ which is the linear differential equation of the form}$$

$$\frac{dy}{dx} + Px = Q,$$

$$\text{where } P = -\tan x \text{ and } Q = \frac{6x}{\cos x}$$

$$\text{So, I.F.} = e^{-\int \tan x dx} = e^{-\log(\sec x)} = \cos x$$

$$\therefore \text{Required solution of differential equation is}$$

$$y(\cos x) = \int (6x) \frac{\cos x}{\cos x} dx + C$$

$$= 6 \frac{x^2}{2} + C = 3x^2 + C$$

$$\text{Given, } y\left(\frac{\pi}{3}\right) = 0$$

$$\text{So, } 0 = 3\left(\frac{\pi}{3}\right)^2 + C \Rightarrow C = -\frac{\pi^2}{3}$$

$$\therefore y(\cos x) = 3x^2 - \frac{\pi^2}{3}$$

$$\text{Now, at } x = \frac{\pi}{6}$$

$$y\left(\frac{\sqrt{3}}{2}\right) = 3 \frac{\pi^2}{36} - \frac{\pi^2}{3} = -\frac{\pi^2}{4}$$

$$\Rightarrow y = -\frac{\pi^2}{2\sqrt{3}}$$

6. (d) Given differential equation

$$\frac{dy}{dx} = (\tan x - y) \sec^2 x$$

$$\Rightarrow \frac{dy}{dx} + (\sec^2 x)y = \sec^2 x \tan x,$$

$$\text{which is linear differential equation of the form}$$

$$\frac{dy}{dx} + Py = Q,$$

$$\text{I.F.} = e^{\int \sec^2 x dx} = e^{\tan x}$$

$$\text{So, solution of given differential equation is}$$

$$y(e^{\tan x}) = \int e^{\tan x} \cdot \sec^2 x \tan x dx + C$$

$$\text{Let } \tan x = t \Rightarrow \sec^2 x dx = dt$$

$$ye^{\tan x} = \int e^t \cdot t dt + C = te^t - \int e^t dt + C$$

$$[\text{using integration by parts method}]$$

$$= e^t (t - 1) + C$$

$$\Rightarrow y \cdot e^{\tan x} = e^{\tan x} (\tan x - 1) + C$$

$$[\because t = \tan x]$$

$$\therefore y(0) = 0$$

$$\Rightarrow C = 1$$

$$\therefore y \cdot e^{\tan x} = e^{\tan x} (\tan x - 1) + 1$$

$$\text{Now, at } x = -\frac{\pi}{4}$$

$$ye^{-1} = e^{-1} (-1 - 1) + 1$$

$$\Rightarrow ye^{-1} = -2e^{-1} + 1 \Rightarrow y = e - 2$$

7. (a) Given differential equation is

$$\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x, \text{ which is linear differential equation in the form of } \frac{dy}{dx} + Py = Q.$$

Here, $P = \tan x$ and $Q = 2x + x^2 \tan x$

$$\therefore \text{IF} = e^{\int \tan x \, dx} = e^{\log_e (\sec x)} = \sec x$$

Now, solution of linear differential equation is given as

$$y \times \text{IF} = \int (Q \times \text{IF}) dx + C$$

$$\therefore y(\sec x) = \int (2x + x^2 \tan x) \sec x \, dx + C$$

$$= \int (2x \sec x) \, dx + \int x^2 \sec x \tan x \, dx + C$$

$$\begin{aligned} \therefore \int x^2 \sec x \tan x \, dx \\ = x^2 \sec x - \int (2x \sec x) \, dx \end{aligned}$$

Therefore, solution is

$$\begin{aligned} y \sec x &= 2 \int x \sec x \, dx + x^2 \sec x \\ &\quad - 2 \int x \sec x \, dx + C \\ \Rightarrow y \sec x &= x^2 \sec x + C \quad \dots(i) \end{aligned}$$

$$\therefore y(0) = 1 \Rightarrow 1(1) = 0(1) + C \Rightarrow C = 1$$

$$\text{Now, } y = x^2 + \cos x \quad [\text{from Eq. (i)}]$$

$$\text{and } y' = 2x - \sin x$$

According to options,

$$y' \left(\frac{\pi}{4} \right) - y' \left(-\frac{\pi}{4} \right) = \left(2 \left(\frac{\pi}{4} \right) - \frac{1}{\sqrt{2}} \right)$$

$$- \left(2 \left(-\frac{\pi}{4} \right) + \frac{1}{\sqrt{2}} \right) = \pi - \sqrt{2}$$

$$\begin{aligned} \text{and } y' \left(\frac{\pi}{4} \right) + y' \left(-\frac{\pi}{4} \right) &= \left(2 \left(\frac{\pi}{4} \right) - \frac{1}{\sqrt{2}} \right) \\ &\quad + \left(2 \left(-\frac{\pi}{4} \right) + \frac{1}{\sqrt{2}} \right) = 0 \end{aligned}$$

$$\begin{aligned} \text{and } y \left(\frac{\pi}{4} \right) + y \left(-\frac{\pi}{4} \right) &= \frac{\pi^2}{16} + \frac{1}{\sqrt{2}} \\ &\quad + \frac{\pi^2}{16} + \frac{1}{\sqrt{2}} = \frac{\pi^2}{4} + \sqrt{2} \end{aligned}$$

$$\begin{aligned} \text{and } y \left(\frac{\pi}{4} \right) - y \left(-\frac{\pi}{4} \right) &= \frac{\pi^2}{16} + \frac{1}{\sqrt{2}} - \frac{\pi^2}{16} \\ &\quad - \frac{1}{\sqrt{2}} = 0 \end{aligned}$$

8. (d) We have, $\frac{dy}{dx} = (x - y)^2$ which is a differentia equation of the form

$$\frac{dy}{dx} = f(ax + by + c)$$

$$\text{Put } x - y = t$$

$$\Rightarrow 1 - \frac{dy}{dx} = \frac{dt}{dx}$$

$$\Rightarrow \frac{dy}{dx} = 1 - \frac{dt}{dx}$$

$$\Rightarrow 1 - \frac{dt}{dx} = t^2 \quad \left[\because \frac{dy}{dx} = (x - y)^2 \right]$$

$$\Rightarrow \frac{1}{2} \log_e \left(\frac{1+t}{1-t} \right) = x + C$$

$$\left[\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log_e \left| \frac{a+x}{a-x} \right| + C \right]$$

$$\Rightarrow \frac{1}{2} \log_e \left(\frac{1+x-y}{1-x+y} \right) = x + C$$

$$[\because t = x - y]$$

Since, $y = 1$ when $x = 1$, therefore

$$\Rightarrow C = -1 \quad [\because \log 1 = 0]$$

$$\therefore \frac{1}{2} \log_e \left(\frac{1+x-y}{1-x+y} \right) = x - 1$$

$$\Rightarrow -\log_e \left| \frac{1-x+y}{1+x-y} \right| = 2(x-1)$$

$$[\because \log \frac{1}{x} = \log x^{-1} = -\log x]$$

9. (a) We have,

$$\frac{dy}{dx} = e^{x-y} (e^x - e^y) \Rightarrow e^y \frac{dy}{dx} + e^y e^x = e^{2x}$$

$$\text{Put } e^y = v$$

$$\therefore e^y \frac{dy}{dx} = \frac{dv}{dx} \Rightarrow \frac{dv}{dx} + ve^x = e^{2x}$$

This is linear equation

$$\text{I.F} = e^{\int e^x dx} = e^{e^x}$$

$\therefore$ Required solution is

$$v \cdot e^{e^x} = \int e^{e^x} \cdot e^{2x} dx + C$$

$$\Rightarrow e^y = e^x - 1 + ce^{-e^x}$$

10. (d) Given, differential equation

$$yy' = x \left[\frac{y^2}{x^2} + \frac{\phi \left(\frac{y^2}{x^2} \right)}{\phi' \left(\frac{y^2}{x^2} \right)} \right]$$

where ϕ is an arbitrary function.

$$\Rightarrow \frac{y}{x} \frac{dy}{dx} = \frac{y^2}{x^2} + \frac{\phi \left(\frac{y^2}{x^2} \right)}{\phi' \left(\frac{y^2}{x^2} \right)}$$

Let $y = vx$, where v is function of x .

$$\Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$$

$$\text{so } v \left(v + x \frac{dv}{dx} \right) = v^2 + \frac{\phi(v^2)}{\phi'(v^2)}$$

$$\Rightarrow vx \frac{dv}{dx} = \frac{\phi(v^2)}{\phi'(v^2)}$$

$$\Rightarrow \int v \frac{\phi(v^2)}{\phi'(v^2)} dv = \int \frac{dx}{x}$$

$$\Rightarrow \frac{1}{2} \log \phi(v^2) = \log x + \log \frac{c}{2}$$

$$= \phi(v^2) = x^2 c$$

$$\Rightarrow \phi \left(\frac{y^2}{x^2} \right) = x^2 c$$

11. (d) We have,

$$\frac{dy}{dx} = 1 - \cos(y-x) \cot(y-x)$$

$$\text{Put } y - x = v \Rightarrow \frac{dy}{dx} = 1 + \frac{dv}{dx}$$

$$\therefore 1 + \frac{dv}{dx} = 1 - \cos v \cot v$$

$$\Rightarrow \frac{dv}{dx} = -\frac{\cos^2 v}{\sin v}$$

$$\Rightarrow -\int \frac{\sin v}{\cos^2 v} dv = \int dx$$

$$\Rightarrow -\int \sec v \tan v dv = dx$$

$$\Rightarrow -\sec v = x + c \Rightarrow x + \sec(y-x) = c$$

12. (c) Equation of family of parabola with focus at origin and the X-axis its axis is

$$y^2 = -4a(x-a) \quad \dots(i)$$

$$\Rightarrow y^2 = -4ax + 4a^2$$

$$\Rightarrow 2yy' = -4a$$

$$\Rightarrow a = -\frac{yy'}{2}$$

Putting the value of a in Eq. (i), we get

$$y^2 = 2yy' \left(x + \frac{yy'}{2} \right)$$

$$\Rightarrow y \left(\frac{dy}{dx} \right)^2 + 2x \frac{dy}{dx} - y = 0$$

$$m = \text{order} = 1$$

$$n = \text{degree} = 2$$

$$\therefore mn - m + n = 2 - 1 + 2 = 3$$

13. (c) We have,

$$\frac{dy}{dx} + \frac{x}{y} \left(\frac{x^2 + y^2 - 1}{2(x^2 + y^2) + 1} \right) = 0$$

$$\Rightarrow 2y(x^2 + y^2) dy + y dy + x(x^2 + y^2)$$

$$dx - xdx = 0$$

$$\Rightarrow 2y(x^2 + y^2 + 2 - 2) dy + ydy +$$

$$x(x^2 + y^2 + 2 - 2) dx - xdx = 0$$

$$\Rightarrow 2ydy + xdx = 3 \left(\frac{xdx + ydy}{x^2 + y^2 + 2} \right)$$

$$\Rightarrow 2ydy + xdx = \frac{3}{2} \left(\frac{2xdx + 2ydy}{x^2 + y^2 + 2} \right)$$

$$\Rightarrow 2ydy + xdx = \frac{3}{2} d(\log(x^2 + y^2 + 2))$$

On integrating both sides, we get

$$2y^2 + x^2 = 3 \log(x^2 + y^2 + 2) + c$$

$$x^2 + 2y^2 - 3 \log|x^2 + y^2 + 2| = c$$

14. (d) We have,

$$x \log x \frac{dy}{dx} + y = \log x^2$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2 \log x}{x \log x}$$

$$\Rightarrow \frac{dy}{dx} + \frac{y}{x \log x} = \frac{2}{x}$$

$$\text{IF} = e^{\int \frac{1}{x \log x} dx} = e^{\log(\log x)}$$

IF = log x

Solution of different equation is

$$y \log x = \int \frac{2 \log x}{x} dx + c$$

$$y \log x = (\log x)^2 + c$$

put $x = e$, $y = 0$ we get

$$0 = 1 + c \Rightarrow c = -1$$

$$\therefore y \log x = (\log x)^2 - 1$$

put $x = e^2$

$$y \log e^2 = (\log e^2)^2 - 1$$

$$2y = 4 - 1 \Rightarrow y = \frac{3}{2}$$

$$\therefore y(e^2) = \frac{3}{2}$$

Vectors Algebra

1. (b) Given vectors $\mathbf{b} = \hat{i} + \hat{j}$ and

$\mathbf{c} = \hat{i} - \hat{j} + 4\hat{k}$. So, vector bisects the angle

between $\mathbf{b}$ and $\mathbf{c}$ is $\mathbf{a} = \lambda \left(\frac{\mathbf{b}}{|\mathbf{b}|} + \frac{\mathbf{c}}{|\mathbf{c}|} \right)$

$$\begin{aligned} \Rightarrow \mathbf{a} &= \lambda \left(\frac{\hat{i} + \hat{j}}{\sqrt{2}} + \frac{\hat{i} - \hat{j} + 4\hat{k}}{\sqrt{18}} \right) \\ &= \frac{\lambda}{3\sqrt{2}} ((3\hat{i} + 3\hat{j}) + (\hat{i} - \hat{j} + 4\hat{k})) \\ &= \frac{\lambda}{3\sqrt{2}} (4\hat{i} + 2\hat{j} + 4\hat{k}) \\ &= \frac{\lambda}{3\sqrt{2}} (2\hat{i} + \hat{j} + 2\hat{k}) \end{aligned}$$

On comparing with given

$\mathbf{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k}$, $(\alpha, \beta \in R)$ and

$$\frac{\lambda}{3\sqrt{2}} (4\hat{i} + 2\hat{j} + 4\hat{k}), \text{ we have}$$

$$\lambda = 3\sqrt{2}, \text{ so } \alpha = 4 \text{ and } \beta = 4$$

which not satisfy the given options.

Now, on comparing with

$$\frac{\lambda}{3\sqrt{2}} (2\hat{i} + \hat{j} + 2\hat{k}) \text{ and } \mathbf{a} = \alpha\hat{i} + 2\hat{j} + \beta\hat{k},$$

we have

$$\lambda = \frac{3\sqrt{2}}{2}, \text{ so } \alpha = 1 \text{ and } \beta = -2$$

Now, since $\mathbf{a} \cdot \hat{k} = -2 \Rightarrow \mathbf{a} \cdot \hat{k} + 2 = 0$.

2. (d) Given three unit vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$

such that $\mathbf{a} + \mathbf{b} + \mathbf{c} = 0$... (i)

If we do dot product in Eq. (i) with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then we get Relations

$$\mathbf{a} \cdot \mathbf{a} + \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 0$$

$$\Rightarrow \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = -1, \mathbf{b} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{c} = -1 \text{ and}$$

$$\mathbf{c} \cdot \mathbf{a} + \mathbf{c} \cdot \mathbf{b} = -1$$

$\therefore$ On adding above three obtained relations, we get

$$2(\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a}) = -3$$

$$(\because \mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a})$$

$$\Rightarrow \lambda = \mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{c} + \mathbf{c} \cdot \mathbf{a} = -\frac{3}{2}$$

If we do cross product in Eq. (i) with $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, then we get relation

$$0 + \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$$

$$\Rightarrow \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c} = 0$$

$$\text{and } \mathbf{b} \times \mathbf{a} + \mathbf{b} \times \mathbf{c} = 0$$

$$\text{and } \mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b} = 0$$

$$\therefore \mathbf{a} \times \mathbf{b} + \mathbf{b} \times \mathbf{c} + \mathbf{c} \times \mathbf{a} = 3(\mathbf{a} \times \mathbf{b})$$

$$= 3(\mathbf{b} \times \mathbf{c})$$

$$= 3(\mathbf{c} \times \mathbf{a})$$

$$\therefore \mathbf{d} = 3\mathbf{a} \times \mathbf{b} = 3\mathbf{b} \times \mathbf{c} = 3\mathbf{c} \times \mathbf{a}$$

From the given options the ordered pair,

$$(\lambda, \mathbf{d}) = \left(-\frac{3}{2}, 3\mathbf{a} \times \mathbf{b} \right)$$

3. (b) Since, the volume of parallelepiped whose coterminal edges are

$$\mathbf{u} = \hat{i} + \hat{j} + \lambda\hat{k}, \mathbf{v} = \hat{i} + \hat{j} + 3\hat{k} \text{ and}$$

$$\mathbf{w} = 2\hat{i} + \hat{j} + \hat{k} \text{ is}$$

$$[\mathbf{u} \ \mathbf{v} \ \mathbf{w}] = \begin{vmatrix} 1 & 1 & \lambda \\ 1 & 1 & 3 \\ 2 & 1 & 1 \end{vmatrix} = 1 \text{ cubic unit (given)}$$

$$\Rightarrow |1(1-3) - 1(1-6) + \lambda(1-2)| = 1$$

$$\Rightarrow |-2 + 5 - \lambda| = 1 \Rightarrow |\lambda - 3| = 1$$

$$\Rightarrow \lambda - 3 = \pm 1 \Rightarrow \lambda = 2, 4$$

Since, angle between $\mathbf{u}$ and $\mathbf{w}$ is θ , so

$$\cos \theta = \frac{|2 + 1 + \lambda|}{\sqrt{1+1+\lambda^2} \sqrt{4+1+1}}$$

$$= \frac{|\lambda + 3|}{\sqrt{\lambda^2 + 2\sqrt{6}}}$$

$$\text{for } \lambda = 2, \cos \theta = \frac{5}{6}$$

$$\text{for } \lambda = 4, \cos \theta = \frac{7}{6\sqrt{3}}$$

Hence, option (b) is correct.

4. (c) We have, $\mathbf{a} = 2\hat{i} + \lambda_1\hat{j} + 3\hat{k}$;

$$\mathbf{b} = 4\hat{i} + (3 - \lambda_2)\hat{j} + 6\hat{k}$$

$$\text{and } \mathbf{c} = 3\hat{i} + 6\hat{j} + (\lambda_3 - 1)\hat{k},$$

$$\text{such that } \mathbf{b} = 2\mathbf{a}$$

$$\text{Now, } \mathbf{b} = 2\mathbf{a}$$

$$\Rightarrow 4\hat{i} + (3 - \lambda_2)\hat{j} + 6\hat{k} = 2(2\hat{i} + \lambda_1\hat{j} + 3\hat{k})$$

$$\Rightarrow 4\hat{i} + (3 - \lambda_2)\hat{j} + 6\hat{k} = 4\hat{i} + 2\lambda_1\hat{j} + 6\hat{k}$$

$$\Rightarrow (3 - 2\lambda_1 - \lambda_2)\hat{j} = 0$$

$$\Rightarrow 3 - 2\lambda_1 - \lambda_2 = 0$$

$$\Rightarrow 2\lambda_1 + \lambda_2 = 3 \quad \dots (i)$$

Also, as $\mathbf{a}$ is perpendicular to $\mathbf{c}$, therefore $\mathbf{a} \cdot \mathbf{c} = 0$

$$\Rightarrow (2\hat{i} + \lambda_1\hat{j} + 3\hat{k}) \cdot (3\hat{i} + 6\hat{j} + (\lambda_3 - 1)\hat{k}) = 0$$

$$\Rightarrow 6 + 6\lambda_1 + 3(\lambda_3 - 1) = 0$$

$$\Rightarrow 6\lambda_1 + 3\lambda_3 + 3 = 0$$

$$\Rightarrow 2\lambda_1 + \lambda_3 = -1 \quad \dots (ii)$$

Now, from Eq. (i), $\lambda_2 = 3 - 2\lambda_1$ and from Eq. (ii)

$$\lambda_3 = -2\lambda_1 - 1$$

$$\therefore (\lambda_1, \lambda_2, \lambda_3) \equiv (\lambda_1, 3 - 2\lambda_1, -2\lambda_1 - 1)$$

If $\lambda_1 = -\frac{1}{2}$, then $\lambda_2 = 4$, and $\lambda_3 = 0$

Thus, a possible value of

$$(\lambda_1, \lambda_2, \lambda_3) = \left(-\frac{1}{2}, 4, 0 \right)$$

5. (d) Two vectors $\mathbf{c}$ and $\mathbf{d}$ are said to be collinear if we can write $\mathbf{c} = \lambda\mathbf{b}$ for some non-zero scalar λ .

Let the vectors $\alpha = (\lambda - 2)\mathbf{a} + \mathbf{b}$

and $\beta = (4\lambda - 2)\mathbf{a} + 3\mathbf{b}$ are

collinear, where $\mathbf{a}$ and $\mathbf{b}$ are non-collinear.

$\therefore$ We can write

$$\alpha = k\beta, \text{ for some } k \in R - \{0\}$$

$$\Rightarrow (\lambda - 2)\mathbf{a} + \mathbf{b} = k[(4\lambda - 2)\mathbf{a} + 3\mathbf{b}]$$

$$\Rightarrow [(\lambda - 2) - k(4\lambda - 2)]\mathbf{a} + (1 - 3k)\mathbf{b} = 0$$

Now, as $\mathbf{a}$ and $\mathbf{b}$ are non-collinear, therefore they are linearly independent and hence

$$(\lambda - 2) - k(4\lambda - 2) = 0 \text{ and } 1 - 3k = 0$$

$$\Rightarrow \lambda - 2 = k(4\lambda - 2) \text{ and } 3k = 1$$

$$\Rightarrow \lambda - 2 = \frac{1}{3}(4\lambda - 2) \left[\because 3k = 1 \Rightarrow k = \frac{1}{3} \right]$$

$$\Rightarrow 3\lambda - 6 = 4\lambda - 2 \Rightarrow \lambda = -4$$

6. (d) As we know bisector of angle A divides the BC in ratio $c : b$.

where c is length of side $AB = \sqrt{4 + 16 + 16} = 6$

and b is length of side $AC = \sqrt{4 + 4 + 1} = 3$

$\therefore$ Position vector of the point where the bisector of angle A meet BC is

$$\frac{6(2\hat{i} + 5\hat{j} + 7\hat{k}) + 3(2\hat{i} + 3\hat{j} + 4\hat{k})}{6 + 3}$$

$$= \frac{18\hat{i} + 39\hat{j} + 54\hat{k}}{9} = 2\hat{i} + \frac{13}{3}\hat{j} + 6\hat{k}$$

Hence, option (d) is correct.

7. (d) We have,

$$\mathbf{OA} = 6\hat{i} + 3\hat{j} - 4\hat{k}, \mathbf{OB} = 2\hat{j} + \hat{k}$$

$$\mathbf{OC} = 5\hat{i} - \hat{j} + 2\hat{k}$$

$$\mathbf{OA} \cdot (\mathbf{OB} \times \mathbf{OC}) = \begin{vmatrix} 6 & 3 & -4 \\ 0 & 2 & 1 \\ 5 & -1 & 2 \end{vmatrix}$$

$$= 6(4 + 1) - 3(-5) - 4(-10)$$

$$\mathbf{OA} \cdot (\mathbf{OB} \times \mathbf{OC}) = 30 + 15 + 40 = 85$$

$$\mathbf{OB} \times \mathbf{OC} = \begin{vmatrix} 1 & 5 & k \\ 0 & 2 & 1 \\ 5 & -1 & 2 \end{vmatrix} = 5\hat{i} + 5\hat{j} - 10\hat{k}$$

$$|\mathbf{OB} \times \mathbf{OC}| = \sqrt{25 + 25 + 100} = 5\sqrt{6}$$

Height of parallelepiped drawn from A

$$= \frac{\mathbf{OA} \cdot (\mathbf{OB} \times \mathbf{OC})}{|\mathbf{OB} \times \mathbf{OC}|} = \frac{85}{5\sqrt{6}} = \frac{17}{\sqrt{6}}$$

8. (b) Given, vectors $x\hat{i} - 3\hat{j} + 7\hat{k}$ and $\hat{i} + y\hat{j} - z\hat{k}$ are collinear

$$\therefore \frac{x}{1} = \frac{-3}{y} = \frac{7}{-z} = k$$

$$\Rightarrow x = k, -3 = ky \text{ and } 7 = -kz$$

$$\text{Now, } \frac{xy^2}{z} = \frac{k\left(-\frac{3}{k}\right)^2}{\left(-\frac{7}{k}\right)} = \frac{\frac{9}{k}}{-\frac{7}{k}} = \frac{-9}{7}$$

9. (a) Given, $\mathbf{a} = 4\hat{i} + 3\hat{j}$ and $\mathbf{a} \cdot \mathbf{b} = 0$

$$\frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|} = 1, \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|} = 2$$

$$\mathbf{a} \cdot \mathbf{b} = 0$$

$$\therefore \mathbf{b} = 3\hat{i} - 4\hat{j}$$

$$\text{Let } \mathbf{c} = x\hat{i} + y\hat{j}$$

$$\therefore \frac{(4\hat{i} + 3\hat{j}) \cdot (x\hat{i} + y\hat{j})}{5} = 1$$

$$= 4x + 3y = 5 \quad \dots (i)$$

$$\text{and } \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}|} = 2$$

$$\therefore 3x - 4y = 10 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 2, y = -1$$

$$\therefore \mathbf{c} = 2\hat{i} - \hat{j}$$

10. (a) A line L passing through $\hat{i} + 2\hat{j} - 3\hat{k}$ and parallel to vector $\sqrt{2}\hat{i} - 5\hat{j} + 3\hat{k}$ is $\mathbf{r} = \hat{i} + 2\hat{j} - 3\hat{k} + \lambda(\sqrt{2}\hat{i} - 5\hat{j} + 3\hat{k})$

Let any point P lie on line is

$$(\sqrt{2}\lambda + 1, -5\lambda + 2, 3\lambda - 3)$$

$$\text{point } A = \hat{i} + 2\hat{j} - 3\hat{k}$$

$$\text{Given } PA = 18$$

$$\therefore (\sqrt{2}\lambda + 1)^2 + (-5\lambda + 2)^2 + (3\lambda - 3)^2 = 18^2$$

$$2\lambda^2 + 25\lambda^2 + 9\lambda^2 = 18^2$$

$$\lambda^2 = \frac{18^2}{36} = 9$$

$$\lambda = 3, -3$$

$$\therefore \text{Point } P = (3\sqrt{2} + 1)\hat{i} - 13\hat{j} + 6\hat{k}$$

$$\text{or } (1 - 3\sqrt{2})\hat{i} + 17\hat{j} - 12\hat{k}$$

11. (b) Given, $\mathbf{a} \times \mathbf{b} = \mathbf{c}$

$$\mathbf{b} \times \mathbf{c} = \mathbf{a}$$

$$\mathbf{c} \times \mathbf{a} = \mathbf{b}$$

$$(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{a}$$

$$(\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{b}$$

$$\mathbf{a}^2 - \mathbf{b}^2 = (\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a} - (\mathbf{c} \times \mathbf{a}) \cdot \mathbf{b}$$

$$= [\mathbf{b} \mathbf{c} \mathbf{a}] - [\mathbf{c} \mathbf{a} \mathbf{b}] = 0$$

$$\therefore |\mathbf{a}|^2 - |\mathbf{b}|^2 = 0$$

$$|\mathbf{a}| = |\mathbf{b}| = 0$$

We know that,

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{b} \times \mathbf{c}| = |\mathbf{c} \times \mathbf{a}| = |\mathbf{a} \mathbf{b} \mathbf{c}|^2$$

$$|\mathbf{a} \mathbf{b} \mathbf{c}| = |\mathbf{a} \mathbf{b} \mathbf{c}|^2$$

$$|\mathbf{a} \mathbf{b} \mathbf{c}|[-|\mathbf{a} \mathbf{b} \mathbf{c}| - 1] = 0$$

$$|\mathbf{a} \mathbf{b} \mathbf{c}| = 1$$

$$|\mathbf{a}||\mathbf{b}||\mathbf{c}| = 1$$

$$|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| \neq 2$$

12. (d) Given,

$$|\hat{\alpha}| = |\hat{\beta}| = |\hat{\gamma}| = 1$$

$$\text{and } \hat{\alpha} \times (\hat{\beta} \times \hat{\gamma}) = \frac{1}{2}(\hat{\beta} + \hat{\gamma})$$

$$\text{Now, } \hat{\alpha} \times (\hat{\beta} \times \hat{\gamma}) = \frac{1}{2}(\hat{\beta} + \hat{\gamma})$$

$$\Rightarrow (\hat{\alpha} \cdot \hat{\gamma})\hat{\beta} - (\hat{\alpha} \cdot \hat{\beta})\hat{\gamma} = \frac{1}{2}\hat{\beta} + \frac{1}{2}\hat{\gamma}$$

On comparing we get,

$$-(\hat{\alpha} \cdot \hat{\beta}) = \frac{1}{2} \Rightarrow |\hat{\alpha}| |\hat{\beta}| \cos \theta = -\frac{1}{2}$$

$$\Rightarrow \cos \theta = -\frac{1}{2} \quad [\because |\hat{\alpha}| = |\hat{\beta}| = 1]$$

$$\Rightarrow \theta = 120^\circ \text{ or } \frac{2\pi}{3}$$

13. (d) The position vectors of the points

$$A, B, C \text{ and } D \text{ are } 3\hat{i} - 2\hat{j} - \hat{k},$$

$$2\hat{i} - 3\hat{j} + 2\hat{k}, 5\hat{i} - \hat{j} + 2\hat{k} \text{ and } 4\hat{i} - \hat{j} - \lambda\hat{k}$$

respectively.

$$\mathbf{BA} = (3\hat{i} - 2\hat{j} - \hat{k}) - (2\hat{i} - 3\hat{j} + 2\hat{k}) = \hat{i} + \hat{j} - 3\hat{k}$$

$$\mathbf{CA} = (3\hat{i} - 2\hat{j} - \hat{k}) - (5\hat{i} - \hat{j} + 2\hat{k}) = -2\hat{i} - \hat{j} - 3\hat{k}$$

$$\mathbf{DA} = (3\hat{i} - 2\hat{j} - \hat{k}) - (4\hat{i} - \hat{j} + \lambda\hat{k}) = -\hat{i} - \hat{j} - (1 + \lambda)\hat{k}$$

These are coplanar, so

$$\begin{vmatrix} 1 & 1 & -3 \\ -2 & -1 & -3 \\ -1 & -1 & -(1 + \lambda) \end{vmatrix} = 0$$

$$= \begin{vmatrix} -1 & -1 & 3 \\ 2 & 1 & 3 \\ 1 & 1 & 1 + \lambda \end{vmatrix} = 0$$

$$-1[1 + \lambda - 3] + 1[2 + 2\lambda - 3] + 3[2 - 1] = 0$$

$$\Rightarrow -1[\lambda - 2] + 1[2\lambda - 1] + 3 = 0$$

$$\Rightarrow -\lambda + 2 + 2\lambda - 1 + 3 = 0$$

$$\Rightarrow \lambda + 4 = 0 \Rightarrow \lambda = -4$$

14. (c) We have $\mathbf{a} + \mathbf{b}$, $\mathbf{b} + \mathbf{c}$ and $\mathbf{c} + \mathbf{a}$ are coterminal edge of a parallelepiped then its volume is $[\mathbf{a} + \mathbf{b} \mathbf{b} + \mathbf{c} \mathbf{c} + \mathbf{a}] = 2[\mathbf{a} \mathbf{b} \mathbf{c}]$

$$\mathbf{n}_1 = \mathbf{b}_1 \times \mathbf{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 4 & 2 \\ 4 & 2 & 3 \end{vmatrix}$$

$$= \hat{i}(12 - 4) - \hat{j}(9 - 8) + \hat{k}(6 - 16) = 8\hat{i} - \hat{j} - 10\hat{k}$$

Let P_2 be the plane containing the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$ and perpendicular to plane P_1 .

For the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{4}$, the direction vector

is $\mathbf{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$ and it passes through the point with position vector $\mathbf{a} = 0\hat{i} + 0\hat{j} + 0\hat{k}$.

$\therefore P_2$ is perpendicular to P_1 , therefore $\mathbf{n}_1$ and $\mathbf{b}$ lies along the plane.

Also, P_2 also passes through the point with position vector $\mathbf{a}$.

$\therefore$ Equation of plane P_2 is given by

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{n}_1 \times \mathbf{b}) = 0$$

Three Dimensional Geometry

1. (c) Equation of plane passing through the points $(2, 1, 0)$, $(4, 1, 1)$ and $(5, 0, 1)$ is

$$\begin{vmatrix} x-2 & y-1 & z-0 \\ 4-2 & 1-1 & 1-0 \\ 5-2 & 0-1 & 1-0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & y-1 & z \\ 2 & 0 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0$$

$$\Rightarrow (x-2)(0+1) - (y-1)(2-3) + z(-2-0) = 0$$

$$\Rightarrow x - 2 + y - 1 - 2z = 0$$

$$\Rightarrow x + y - 2z - 3 = 0 \quad \dots (i)$$

Now, let the image of point $R(2, 1, 6)$ w.r.t. plane (1) is (x_1, y_1, z_1) , then

$$\frac{x_1 - 2}{1} = \frac{y_1 - 1}{1} = \frac{z_1 - 6}{-2} = -2 \left(\frac{2 + 1 - 12 - 3}{1^2 + 1^2 + (-2)^2} \right)$$

$$\Rightarrow \frac{x_1 - 2}{1} = \frac{y_1 - 1}{1} = \frac{z_1 - 6}{-2} = \frac{2 \times 12}{6} = 4$$

$$\Rightarrow x_1 = 6, y_1 = 5, z_1 = -2$$

$\therefore$ Image of point $R(2, 1, 6)$ w.r.t. plane (1) is $(6, 5, -2)$.

2. (a) Since, the mirror image of the point

$$A(1, 2, 3) \text{ in a plane is } B\left(-\frac{7}{3}, -\frac{4}{3}, -\frac{1}{3}\right),$$

then mid-point of AB lies in the plane and dr 's of line AB is the dr 's, of normal to the plane.

3. (d) Let P_1 be the plane containing the lines

$$\frac{x}{3} = \frac{y}{4} = \frac{z}{2} \text{ and } \frac{x}{4} = \frac{y}{2} = \frac{z}{3}$$

For these two lines, direction vectors are

$$\mathbf{b}_1 = 3\hat{i} + 4\hat{j} + 2\hat{k} \text{ and } \mathbf{b}_2 = 4\hat{i} + 2\hat{j} + 3\hat{k}.$$

A vector along the normal to the plane P_1 is given by

$$\Rightarrow \begin{vmatrix} x-0 & y-0 & z-0 \\ 8 & -1 & -10 \\ 2 & 3 & 4 \end{vmatrix} = 0$$

$$\Rightarrow x(-4+30) - y(32+20) + z(24+2) = 0$$

$$\Rightarrow 26x - 52y + 26z = 0$$

$$\Rightarrow x - 2y + z = 0$$

4. (d) Let the given points be $A(-3, -3, 4)$

and $B(3, 7, 6)$.

Then, mid-point of line joining A, B is

$$P\left(\frac{-3+3}{2}, \frac{-3+7}{2}, \frac{4+6}{2}\right) = P(0, 2, 5)$$

$\therefore$ The required plane is perpendicular

bisector of line joining A, B , so direction ratios of normal to the plane is proportional to the direction ratios of line joining A, B .

So, the direction ratios of normal to the plane are 6, 10, 2.

$[\therefore \text{DR's of } AB \text{ are } 3+3, 7+3, 6-4, \text{ i.e. } 6, 10, 2]$

Now, equation of plane is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$6(x - 0) + 10(y - 2) + 2(z - 5) = 0$$

$[\therefore P(0, 2, 5) \text{ line on the plane}]$

$$\Rightarrow 3x + 5y - 10 + z - 5 = 0$$

$$\Rightarrow 3x + 5y + z = 15 \dots (i)$$

On checking all the options, the option (4, 1, -2)

satisfy the equation of plane (i).

5. (d) Given equation of line is

$$\frac{x-4}{2} = \frac{y-5}{2} = \frac{z-3}{1} = r \text{ (let) } \dots (i)$$

$$\Rightarrow x = 2r + 4; y = 2r + 5 \text{ and } z = r + 3$$

$\therefore$ General point on the line (i) is

$$P(2r + 4, 2r + 5, r + 3)$$

So, the point of intersection of line (i) and plane $x + y + z = 2$ will be of the form $P(2r + 4, 2r + 5, r + 3)$ for some $r \in R$

$$\Rightarrow (2r + 4) + (2r + 5) + (r + 3) = 2$$

$[\therefore \text{the point will lie on the plane}]$

$$\Rightarrow 5r = -10 \Rightarrow r = -2$$

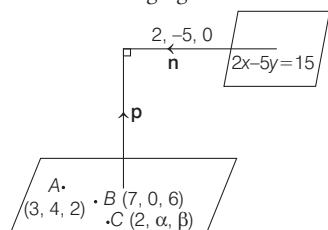
So, the point of intersection is $P(0, 1, 1)$

[putting $r = -2$ in $(2r + 4, 2r + 5, r + 3)$]

Now, on checking the options, we get

$$\frac{x-1}{1} = \frac{y-3}{2} = \frac{z+4}{-5} \text{ contain the point } (0, 1, 1)$$

6. (b) According to given information, we have the following figure.



From figure, it is clear that

$$(\mathbf{AB} \times \mathbf{BC}) = \mathbf{p} \text{ and } \mathbf{n} = 2\hat{i} - 5\hat{j} + 0\hat{k}$$

$$\therefore \mathbf{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -4 & 4 \\ -5 & \alpha & \beta - 6 \end{vmatrix}$$

$$[\therefore \mathbf{AB} = (7-3)\hat{i} + (0-4)\hat{j} + (6-2)\hat{k}$$

$$= 4\hat{i} - 4\hat{j} + 4\hat{k} \text{ and}$$

$$\mathbf{BC} = (2-7)\hat{i} + (\alpha-0)\hat{j} + (\beta-6)\hat{k}$$

$$= -5\hat{i} + \alpha\hat{j} + (\beta-6)\hat{k}]$$

$$\Rightarrow \mathbf{p} = (24 - 4\alpha - 4\beta)\hat{i} + \hat{j}(4 - 4\beta)$$

$$+ \hat{k}(4\alpha - 20)$$

Now, as the planes are perpendicular, therefore $\mathbf{p} \cdot \mathbf{n} = 0$

$$\Rightarrow ((24 - 4\alpha - 4\beta)\hat{i} + (4 - 4\beta)\hat{j}$$

$$+ (4\alpha - 20)\hat{k}) \cdot$$

$$(2\hat{i} - 5\hat{j} + 0\hat{k}) = 0$$

$$\Rightarrow 2(24 - 4\alpha - 4\beta) - 5(4 - 4\beta) + 0 = 0$$

$$\Rightarrow 8(6 - \alpha - \beta) - 4(5 - 5\beta) = 0$$

$$\Rightarrow 12 - 2\alpha - 2\beta - 5 + 5\beta = 0$$

$$\Rightarrow 2\alpha - 3\beta = 7$$

7. (c) Equations of given planes are

$$2x - y - 4 = 0 \dots (i)$$

$$\text{and } y + 2z - 4 = 0 \dots (ii)$$

Now, equation of family of planes passes through the line of intersection of given planes (i) and (ii) is

$$(2x - y - 4) + \lambda(y + 2z - 4) = 0 \dots (iii)$$

According to the question,

Plane (iii) passes through the point $(1, 1, 0)$, so

$$(2 - 1 - 4) + \lambda(1 + 0 - 4) = 0$$

$$\Rightarrow -3 - 3\lambda = 0$$

$$\Rightarrow \lambda = -1$$

Now, equation of required plane can be obtained by putting $\lambda = -1$ in the equation of plane (iii).

$$\Rightarrow (2x - y - 4) - 1(y + 2z - 4) = 0$$

$$\Rightarrow x - y - z = 0$$

8. (d) Since, equation of planes passes through the line of intersection of the planes

$$x + y + z = 1 \text{ and } 2x + 3y + 4z = 5, \text{ is}$$

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (1 + 2\lambda)x + (1 + 3\lambda)y + (1 + 4\lambda)z$$

$$- (1 + 5\lambda) = 0 \dots (i)$$

$\therefore$ The plane (i) is perpendicular to the plane $x - y + z = 0$.

$$\therefore (1 + 2\lambda) - (1 + 3\lambda) + (1 + 4\lambda) = 0$$

$[\therefore \text{if plane } a_1x + b_1y + c_1z + d_1 = 0 \text{ is perpendicular to plane}$

$$a_2x + b_2y + c_2z + d_2 = 0, \text{ then}$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 0]$$

$$\Rightarrow 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3} \dots (ii)$$

So, the equation of required plane, is

$$\left(1 - \frac{2}{3}\right)x + \left(1 - \frac{3}{3}\right)y + \left(1 - \frac{4}{3}\right)$$

$$z - \left(1 - \frac{5}{3}\right) = 0$$

$$\Rightarrow \frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0 \Rightarrow x - z + 2 = 0$$

Now, vector form, is $\mathbf{r} \cdot (\hat{i} - \hat{k}) + 2 = 0$

9. (a) Let the equation of plane is

$$ax + by + cz = d \dots (i)$$

Since plane (i) passes through the points $(0, -1, 0)$ and $(0, 0, 1)$, then

$$-b = d \text{ and } c = d$$

$\therefore$ Equation of plane becomes $ax - dy + dz = d \dots (ii)$

$\therefore$ The plane (ii) makes an angle of $\frac{\pi}{4}$ with the

plane $y - z + 5 = 0$.

$$\cos \frac{\pi}{4} = \left| \frac{-d - d}{\sqrt{a^2 + d^2 + d^2} \sqrt{1 + 1}} \right|$$

$[\therefore \text{The angle between the two planes } a_1x + b_1y + c_1z + d = 0 \text{ and}$

$$a_2x + b_2y + c_2z + d = 0 \text{ is}$$

$$\cos \theta = \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{|-2d|}{\sqrt{a^2 + 2d^2} \sqrt{2}}$$

$$\Rightarrow \sqrt{a^2 + 2d^2} = |-d|$$

$$\Rightarrow a^2 + 2d^2 = 4d^2 \text{ [squaring both sides]}$$

$$\Rightarrow a^2 = 2d^2 \Rightarrow a = \pm \sqrt{2}d$$

So, the Eq. (ii) becomes

$$\pm \sqrt{2}x - y + z = 1 \dots (i)$$

Now, from options $(\sqrt{2}, 1, 4)$ satisfy the plane $-\sqrt{2}x - y + z = 1$

10. (a) Given, equation of plane $3x - y + 4z = 2$

$\dots (i)$ and the point $Q(0, -1, -3)$ is the image of point P in the plane (i), so point P is also image of point Q w.r.t. plane (i).

Let the coordinates of point P is (x_1, y_1, z_1) , then

$$\frac{x_1 - 0}{3} = \frac{y_1 + 1}{-1} = \frac{z_1 + 3}{4} = -2 \frac{[3(0) - 1(-1) + 4(-3) - 2]}{3^2 + (-1)^2 + 4^2}$$

$[\therefore \text{image of the point } (x_1, y_1, z_1) \text{ in the plane } ax + by + cz + d = 0 \text{ is } (x, y, z), \text{ where}$

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = \frac{-2(ax_1 + by_1 + cz_1 + d)}{a^2 + b^2 + c^2}]$$

$$\Rightarrow \frac{x_1 - 0}{3} = \frac{y_1 + 1}{-1} = \frac{z_1 + 3}{4}$$

$$= 2 \frac{(1 - 12 - 2)}{26} = \frac{26}{26} = 1$$

$$\Rightarrow P(x_1, y_1, z_1) = (3, -2, 1)$$

Now, area of ΔPQR , where point $R(3, -1, -2)$

$$= \frac{1}{2} \left| \overrightarrow{PQ} \times \overrightarrow{PR} \right|$$

$$= \frac{1}{2} \left| (-3\hat{i} + \hat{j} - 4\hat{k}) \times (0\hat{i} + \hat{j} - 3\hat{k}) \right|$$

$$= \frac{1}{2} \left| \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 1 & -4 \\ 0 & 1 & -3 \end{vmatrix} \right| = \frac{1}{2} \left| \hat{i} - 9\hat{j} - 3\hat{k} \right|$$

$$= \frac{1}{2} \sqrt{1 + 81 + 9} = \frac{\sqrt{91}}{2} \text{ sq units}$$

11. (a) We have,

$$l + m + n = 0 \quad \dots(i)$$

$$\text{and } l^2 = m^2 + n^2 \quad \dots(ii)$$

Putting the value of l in Eq. (ii), we get

$$(m + n)^2 = m^2 + n^2$$

$$\Rightarrow m^2 + n^2 + 2mn = m^2 + n^2$$

$$\Rightarrow mn = 0$$

$$\Rightarrow m = 0, n = 0$$

$$\text{when } m = 0, l = -n$$

$$\therefore \text{Direction cosines are } \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \right)$$

$$\text{When } n = 0, l = -m$$

$$\therefore \text{Direction cosines are } \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0 \right)$$

Angle between lines are

$$\cos \theta = \left(\frac{-1}{\sqrt{2}} \times -\frac{1}{\sqrt{2}} \right) + \left(0 \times \frac{1}{\sqrt{2}} \right) + \left(\frac{1}{\sqrt{2}} \times 0 \right)$$

$$\cos \theta = \frac{1}{2}, \theta = \pi/3$$

12. (d) Point of intersection of lines

$$x + y + 1 = 0 \text{ and } 2x - 3y + 4 = 0 \text{ is}$$

$$A\left(-\frac{7}{5}, \frac{2}{5}\right)$$

Now, Let a point $P(-2, 0)$ on the line $2x - 3y + 4 = 0$ and the image of point $P(-2, 0)$ w.r.t. line $x + y + 1 = 0$ lying on the other line in the bisector pair. If the coordinate of image of point $P(-2, 0)$ w.r.t. line $x + y + 1 = 0$ is (h, k) , then

$$\frac{h + 2}{1} = \frac{k - 0}{1} = -2 \times \frac{-2 + 1}{2} = 1$$

$$\Rightarrow h = -1 \text{ and } k = 1$$

$$\Rightarrow (h, k) = (-1, 1)$$

Now, equation of required bisector of angle is line joining the points $A\left(-\frac{7}{5}, \frac{2}{5}\right)$ and

$(-1, 1)$ is

$$\frac{y - 1}{\frac{3}{5}} = \frac{x + 1}{\frac{2}{5}} \Rightarrow 2y - 2 = 3x + 3$$

$$\Rightarrow 3x - 2y + 5 = 0$$

Hence, option (d) is correct.

13. (c) Given, X- intercept $(a) = 2$

$$\text{Y- intercept } (b) = 3$$

$$\text{Z- intercept } (c) = 4$$

$$\therefore \text{Equation of the plane is } \frac{x}{2} + \frac{y}{3} + \frac{z}{4} = 1$$

$$6x + 4y + 3z = 12 \dots(i)$$

$$\text{Given, points are } A = (-1, 6, 2)$$

$$B = (1, 2, 3)$$

$$C = (-2, 3, 4)$$

$$\text{DR's of } \overrightarrow{BC} = (-2, -1, 3 - 2, 4 - 3)$$

$$= (-3, 1, 1)$$

$\therefore$ Equation of plane passing through $A(-1, 6, 2)$ and having DR's $(-3, 1, 1)$ is given by

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$-3(x + 1) + 1(y - 6) + 1(z - 2) = 0$$

Angle between the planes (i) and (ii) is

$$\cos \theta = \frac{|a_1 a_2 + b_1 b_2 + c_1 c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2} \sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$= \frac{|(6)(-3) + (4)(1) + (3)(1)|}{\sqrt{36 + 16 + 9} \sqrt{9 + 1 + 1}}$$

$$\cos \theta = \sqrt{\frac{11}{61}}$$

$$\therefore \theta = \cos^{-1} \sqrt{\frac{11}{61}}$$

$\therefore$ Hence, option (c) is correct.

14. (b) Direction ratios of line joining the

points $(1, 2, -3)$ and $(-1, -2, 1)$ is

$$(-1 - 1, -2 - 2, 1 + 3)$$

$$\text{i.e. } (-2, -4, 4)$$

Let the direction ratios of the normal to the plane is (a, b, c) , then

$$2a + 3b + 4c = 0 \quad \dots(i)$$

$$\text{and } -2a - 4b + 4c = 0 \quad \dots(ii)$$

By Eqs. (i) and (ii), we get

$$\frac{a}{3(4) - (4)(-4)} = \frac{b}{(-2)(4) - (-2)(4)}$$

$$= \frac{c}{(2)(-4) - (-2)(3)}$$

$$\Rightarrow \frac{a}{12 + 16} = \frac{b}{-8 - 8} = \frac{c}{-8 + 6}$$

$$\Rightarrow \frac{a}{28} = \frac{b}{-16} = \frac{c}{-2}$$

$$\Rightarrow \frac{a}{14} = \frac{b}{-8} = \frac{c}{-1}$$

Hence, direction ratios are $(14, -8, -1)$.

15. (c) Equation of plane passes through the

point $(\hat{i} + 2\hat{j} - \hat{k}) = (1, 2, -1)$ having

direction ratios to normal are a, b, c is

$$a(x - 1) + b(y - 2) + c(z + 1) = 0 \quad \dots(i)$$

$\therefore$ Plane (i) is perpendicular to line of intersection of planes $\mathbf{r} \cdot (3\hat{i} - \hat{j} + \hat{k}) = 1$

$$\text{and } \mathbf{r} \cdot (\hat{i} + 4\hat{j} - 2\hat{k}) = 2$$

$$\therefore 3a - b + c = 0$$

$$\text{and } a + 4b - 2c = 0$$

$$\Rightarrow \frac{a}{2} = \frac{b}{-7} = \frac{c}{-13}$$

So, equation of required plane is

$$2(x - 1) - 7(y - 2) - 13(z + 1) = 0$$

$$\Rightarrow 2x - 7y - 13z - 2 + 14 - 13 = 0$$

$$\Rightarrow 2x - 7y - 13z = 1$$

$$\text{or } \mathbf{r} \cdot (2\hat{i} - 7\hat{j} - 13\hat{k}) = 1$$

Hence, option (c) is correct

Advanced Probability

1. (c) If k represents the number of consecutive heads and $P(k)$ is the corresponding probability while an unbiased coin is tossed 5 times, then

k	0	1	2	3	4	5
$P(k)$	$\frac{1}{2^5}$	$\frac{12}{2^5}$	$\frac{11}{2^5}$	$\frac{5}{2^5}$	$\frac{2}{2^5}$	$\frac{1}{2^5}$

Now, the expected value of variable X is

$$(-1)\frac{1}{2^5} + (-1)\frac{12}{2^5} + (-1)\frac{11}{2^5} + (3)\frac{5}{2^5}$$

$$+ (4)\frac{2}{2^5} + (5)\frac{1}{2^5}$$

$$= \frac{1}{2^5} [-1 - 12 - 11 + 15 + 8 + 5]$$

$$= \frac{28 - 24}{2^5} = \frac{4}{2^5} = \frac{1}{2^3} = \frac{1}{8}$$

2. (c) It is given that out of the five machines in a workshop, the probability of any one of them to be out of service on a day is $\frac{1}{4}$

So, the probability that at most two machines will be out of service on the same day = probability that no machine is out of service

+ probability that exactly one machine is out of service + probability that exactly two of machines are out of service

$$\left(1 - \frac{1}{4}\right)^5 + {}^5C_1 \frac{1}{4} \left(1 - \frac{1}{4}\right)^4$$

$$+ {}^5C_2 \left(\frac{1}{4}\right)^2 \left(1 - \frac{1}{4}\right)^3 = \left(\frac{3}{4}\right)^3 k \text{ (given)}$$

$$\Rightarrow \left(\frac{3}{4}\right)^5 + \frac{5}{4} \left(\frac{3}{4}\right)^4 + 10 \frac{3^3}{4^3} = \left(\frac{3}{4}\right)^3 k$$

$$\Rightarrow k = \left(\frac{3}{4}\right)^2 + \frac{15}{4^2} + \frac{10}{4^2}$$

$$= \frac{9 + 15 + 10}{16} = \frac{34}{16}$$

$$\Rightarrow k = \frac{17}{8}$$

3. (c) It is given that A and B are two independent events such that $P(A) = \frac{1}{3}$ and

$$P(B) = \frac{1}{6}, \text{ then } P(A \cap B) = P(A) \cdot P(B) = \frac{1}{18}$$

$$\text{Now, } P(A|A \cup B) = \frac{P(A \cap (A \cup B))}{P(A \cup B)}$$

$$= \frac{P(A)}{P(A) + P(B) - P(A \cap B)}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} - \frac{1}{18}} = \frac{\frac{1}{3}}{\frac{6 + 3 - 1}{18}} = \frac{6}{8} = \frac{3}{4}$$

$$\text{and } P(A|B') = \frac{P(A' \cap B')}{P(B')} = \frac{P((A \cup B)')}{P(B')}$$

$$= \frac{1 - P(A \cup B)}{1 - P(B)} = \frac{1 - \frac{6 + 3 - 1}{18}}{1 - \frac{1}{6}} = \frac{\frac{10}{18}}{\frac{5}{6}} = \frac{2}{3}$$

$$\text{and } P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A)P(B)}{P(B)} = \frac{1}{3}$$

$$\text{Hence, option (c) is correct.}$$

4. (d) According to given information, the required probability is equals to

$$P(A \cap A) + P(A \cap B \cap A) + P(B \cap A \cap A) + P(B \cap B \cap A \cap A) + P(B \cap A \cap B \cap A) + P(A \cap B \cap B \cap A)$$

$$= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\right)$$

$$= \frac{1}{4} + \frac{2}{8} + \frac{3}{16} = \frac{4 + 4 + 3}{16} = \frac{11}{16}$$

Hence, option (d) is correct.

5. (b) Given probability distribution for a random variable x is

X	1	2	3	4	5
$P(X)$	K^2	$2K$	K	$2K$	$5K^2$

$$\text{Then, } P(X > 2) = 1 - \{P(X=1) + P(X=2)\}$$

$$= 1 - K^2 - 2K$$

$$\therefore \sum_{x=1}^5 P(x) = 1 \Rightarrow 6K^2 + 5K = 1$$

$$\Rightarrow 6K^2 + 6K - K - 1 = 0$$

$$\Rightarrow 6K(K+1) - 1(K+1) = 0$$

$$\Rightarrow K = \frac{1}{6}, -1 \quad \{\because P(x) \geq 0, \text{ so } K \neq -1\}$$

$$\Rightarrow K = \frac{1}{6}$$

$$\therefore P(X > 2) = 1 - \frac{1}{36} - \frac{2}{6} = \frac{36 - 1 - 12}{36} = \frac{23}{36}$$

Hence, option (b) is correct.

6. (d) $P(G1)$ = Probability that 1st ball drawn is green = $\frac{2}{7}$

$$P(R1) = \text{Probability that 1st ball drawn in red} = \frac{5}{7}$$

$$P(R) = \text{Probability that 2nd ball drawn is red.}$$

$$= P(G1) \cdot \frac{6}{7} + P(R1) \cdot \frac{4}{7}$$

$$= \frac{2}{7} \times \frac{6}{7} + \frac{5}{7} \times \frac{4}{7} = \frac{32}{49}$$

7. (a) In $\{1, 2, 3, \dots, 11\}$ there are 5 even numbers and 6 odd numbers. The sum even is possible only when both are odd or both are even.

Let A be the event that denotes both numbers are even and B be the event that denotes sum of numbers is even.

$$\text{Then, } n(A) = {}^5C_2 \text{ and } n(B) = {}^5C_2 + {}^6C_2$$

Required probability

$$P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{{}^5C_2 / {}^{11}C_2}{{}^6C_2 + {}^5C_2}$$

$$= \frac{{}^5C_2}{{}^6C_2 + {}^5C_2} = \frac{10}{15 + 10} = \frac{2}{5}$$

8. (d) Let event B is being boy while event G being girl.

$$\text{According to the question, } P(B) = P(G) = \frac{1}{2}$$

Now, required conditional probability that all children are girls given that at least two are girls, is

$$= \frac{\text{All 4 girls}}{(\text{All 4 girls}) + (\text{exactly 3 girls} + 1 \text{ boy}) + (\text{exactly 2 girls} + 2 \text{ boys})}$$

$$= \frac{\left(\frac{1}{2}\right)^4}{\left(\frac{1}{2}\right)^4 + {}^4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + {}^4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2}$$

$$= \frac{1}{1 + 4 + 6} = \frac{1}{11}$$

9. (c) Given that, there are 50 problems to solve in an admission test and probability that the candidate can solve any problem is

$\frac{4}{5} = q$ (say). So, probability that the candidate cannot solve a problem is $p = 1 - q = 1 - \frac{4}{5} = \frac{1}{5}$.

Now, let X be a random variable which denotes the number of problems that the candidate is unable to solve. Then, X follows binomial distribution with parameters $n = 50$ and $p = \frac{1}{5}$.

Now, according to binomial probability distribution concept

$$P(X = r) = {}^{50}C_r \left(\frac{1}{5}\right)^r \left(\frac{4}{5}\right)^{50-r}, \quad r = 0, 1, \dots, 50$$

$\therefore$ Required probability

$$= P(X < 2) = P(X = 0) + P(X = 1)$$

$$= {}^{50}C_0 \left(\frac{4}{5}\right)^{50} + {}^{50}C_1 \frac{4^{49}}{5^{50}}$$

$$= \left(\frac{4}{5}\right)^{49} \left(\frac{4}{5} + \frac{50}{5}\right) = \frac{54}{5} \left(\frac{4}{5}\right)^{49}$$

10. (b) Let A be the event that maximum is 6.

B be the event that minimum is 3.

$$P(A) = \frac{{}^5C_2}{{}^8C_3} \text{ (number less than 6 is 5)}$$

$$P(B) = \frac{{}^5C_2}{{}^8C_3} \text{ (number greater than 3 is 5)}$$

$$P(A \cap B) = \frac{{}^2C_1}{{}^8C_3}$$

$\therefore$ Required probability

$$= P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{{}^2C_1}{{}^5C_2} = \frac{2}{10} = \frac{1}{5}$$

11. (a) According to the given information, the number of two successive letters in word LONDON is 5, similarly in word CLIFTON is 6.

Let event E_1 is selecting word LONDON and event E_2 is selecting word CLIFTON.

$$\text{So, } P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}$$

and the event A is getting successive letter ON.

$$\text{Then, } P(A/E_1) = \frac{2}{5} \text{ and } P(A/E_2) = \frac{1}{6}$$

$\therefore$ Required probability,

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1) \cdot P\left(\frac{A}{E_1}\right)}{P(E_1) \cdot P\left(\frac{A}{E_1}\right) + P(E_2) \cdot P\left(\frac{A}{E_2}\right)}$$

$$= \frac{\frac{1}{2} \times \frac{2}{5}}{\frac{1}{2} \times \frac{2}{5} + \left(\frac{1}{2} \times \frac{1}{6}\right)} = \frac{\frac{2}{5}}{\frac{2}{5} + \frac{1}{6}}$$

$$= \frac{2 \times 6}{12 + 5} = \frac{12}{17}$$

Hence, option (a) is correct.

12. (a) Given,

$$P(E_1 \cup E_2) = \frac{5}{8}, P(\bar{E}_1) = \frac{3}{4}, P(E_2) = \frac{1}{2}$$

$$P(E_1 \cap E_2) = P(E_1) + P(E_2) - P(E_1 \cup E_2)$$

$$P(E_1 \cap E_2) = \frac{1}{4} + \frac{1}{2} - \frac{5}{8}$$

$$P(E_1 \cap E_2) = \frac{1}{8}$$

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$$

$\therefore E_1$ and E_2 are independent events.

13. (a) Consider the event

E_1 = graduate holder

E_2 = post-graduate holders

E_3 = doctorate holders

A = passed examination

$$P(E_1) = \frac{5}{8}, P(E_2) = \frac{2}{8}, P(E_3) = \frac{1}{8}$$

$$P(A/E_1) = \frac{2}{3}, P(A/E_2) = \frac{3}{4}$$

$$P(A/E_3) = 4/5$$

$\therefore$ Required probability

$$= P(E_2/A)$$

$$= \frac{P(E_2) \times P(A/E_2)}{P(E_1) \times P(A/E_1) + P(E_2) \times P(A/E_2) + P(E_3) \times P(A/E_3)}$$

$$= \frac{\frac{2}{8} \times 3/4}{\frac{5}{8} \times 2/3 + \frac{2}{8} \times 3/4 + \frac{1}{8} \times 4/5} = \frac{45}{169}$$

14. (b) Given probability function

$$P(X=k) = Ck^2, \text{ where } C \text{ is a constant and}$$

$$K \in \{0, 1, 2, 3, 4\}$$

$$\therefore \sigma^2 = E(x^2) - \mu^2$$

$$\Rightarrow \sigma^2 + \mu^2 = E(x^2) = \sum(x_i^2) P(x_i)$$

$$= 0(C(0)^2) + 1(C(1)^2) + 4(C(2)^2) + 9(C(3)^2) + 16(C(4)^2)$$

$$= C + 16C + 81C + 256C$$

$$= 354C$$

...(i)

$$\therefore \sum P(x_i) = 1 \Rightarrow C + 4C + 9C + 16C = 1$$

$$\Rightarrow 30C = 1 \Rightarrow C = \frac{1}{30}$$

Put the value of C , in Eq. (i), we get

$$\text{So, } \sigma^2 + \mu^2 = \frac{354}{30} = 11.8.$$

Hence, option (b) is correct.

15. (a) Probability of green ball $P(G)$

$$= \frac{1}{2} \times \frac{2}{6} + \frac{1}{3} \times \frac{3}{6} + \frac{1}{6} \times \frac{1}{6} = \frac{1}{6} + \frac{1}{6} + \frac{1}{36}$$

$$= \frac{6 + 6 + 1}{36} = \frac{13}{36}$$

Let probability of drawn ball is green comes

from bag C is $P\left(\frac{C}{G}\right)$ then

$$P\left(\frac{C}{G}\right) = \frac{P\left(\frac{G}{C}\right) \times P(C)}{P(G)} = \frac{\frac{1}{6} \times \frac{1}{6}}{\frac{13}{36}} = \frac{1}{36} \times \frac{36}{13}$$

$$= \frac{1}{13}$$

16. (d) Given,

$$P(X=0) = 3C^3$$

$$P(X=2) = 5C - 10C^2$$

$$\text{and } P(X=4) = 4C - 1$$

We know that,

$$\sum P(X) = 1$$

$$\Rightarrow 3C^3 + (5C - 10C^2) + (4C - 1) = 1$$

$$\Rightarrow 3C^3 - 10C^2 + 9C - 2 = 0$$

$$\Rightarrow (C-1)(3C^2 - 7C + 2) = 0$$

$$\Rightarrow (C-1)(3C-1)(C-2) = 0$$

$$\Rightarrow C = 1, \frac{1}{3}, 2$$

$$\therefore C = \frac{1}{3}$$

Now,

X	0	2	4
$P(X)$	$\frac{1}{9}$	$\frac{5}{9}$	$\frac{1}{3}$

$$\text{Hence, variance} = \sum X_p^2 - (\sum X_p)^2$$

$$= \left(0^2 \times \frac{1}{9} + 4 \times \frac{5}{9} + 16 \times \frac{1}{3}\right) - \left(\frac{10}{9} + \frac{4}{3}\right)^2$$

$$= \left(\frac{20}{9} + \frac{16}{3}\right) - \left(\frac{66}{27}\right)^2$$

$$= \frac{60 + 144}{27} - \frac{484}{81}$$

$$= \frac{204}{27} - \frac{484}{81}$$

$$= \frac{612 - 484}{81} = \frac{128}{81}$$

17. (b) Required probability

= No white ball + one white ball

+ two white ball

$$= \left(\frac{2}{3}\right)^5 + {}^5C_1 \left(\frac{2}{3}\right)^4 \left(\frac{1}{3}\right)^1 + {}^5C_2 \left(\frac{2}{3}\right)^3 \left(\frac{1}{3}\right)^2$$

$$= \left(\frac{1}{3}\right)^5 [2^5 + 5 \cdot 2^4 + 10 \cdot 2^3]$$

$$= \left(\frac{1}{3}\right)^5 [32 + 80 + 80]$$

$$= \frac{192}{3^5} = \frac{64 \times 3}{3^5}$$

$$= \frac{64}{3^4} = \frac{8 \times 8}{3^2 \times 3^2} = \left(\frac{8}{9}\right)^2$$

18. (d) Here, $n = 6$

$$p = 25\% = \frac{1}{4}$$

$$q = 1 - p = 3/4$$

Let X be the random variable, which denotes the number of cases that police station are solved.

$\therefore$ According to question,

$$P(X=5) + P(X=6) = {}^6C_5 (p)^5 (q)^{6-5} + {}^6C_6 (p)^6 (q)^{6-6}$$

$$= {}^6C_1 \left(\frac{1}{4}\right)^5 \left(\frac{3}{4}\right)^1 + {}^6C_0 \left(\frac{1}{4}\right)^6 \left(\frac{3}{4}\right)^0$$

$$= \frac{1}{(4)^6} [6 \times 3 + 1]$$

$$= \frac{19}{4096}$$