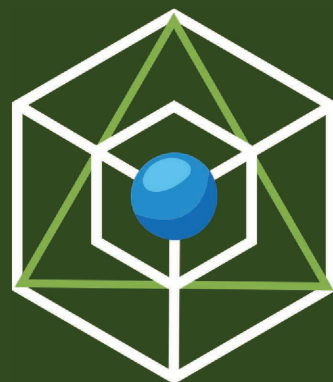


COMPLETE STUDY PACK FOR ENGINEERING ENTRANCES

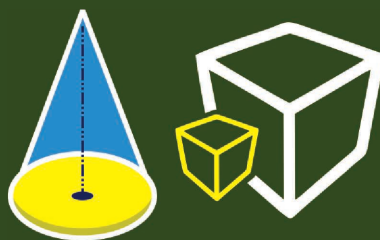
 arihant



OBJECTIVE MATHEMATICS

USEFUL FOR

JEE Main & Advanced,
BITSAT, MHT-CET,
Andhra Pradesh &
Telangana State (EAMCET),
Karnataka CET, VIT,
Kerala CEE & WB JEE etc.



More than
5000+
Questions
of All Types

Volume 1



Amit M. Agarwal

For More Material Join: @JEEAdvanced_2024

COMPLETE STUDY PACK FOR
**ENGINEERING
ENTRANCES**

**OBJECTIVE
MATHEMATICS**

Volume 1

For More Material Join: @JEEAdvanced_2024

COMPLETE STUDY PACK FOR
ENGINEERING
ENTRANCES

OBJECTIVE
MATHEMATICS

Volume 1

Amit M. Agarwal



ARIHANT PRAKASHAN (SERIES), MEERUT



Arihant Prakashan (Series), Meerut

All Rights Reserved

卐 © **AUTHOR**

No part of this publication may be re-produced, stored in a retrieval system or by any means, electronic, mechanical, photocopying, recording, scanning, web or otherwise without the written permission of the publisher. Arihant has obtained all the information in this book from the sources believed to be reliable and true. However, Arihant or its editors or authors or illustrators don't take any responsibility for the absolute accuracy of any information published and the damage or loss suffered thereupon.

All disputes subject to Meerut (UP) jurisdiction only.

卐 **Administrative & Production Offices**

Regd. Office

'Ramchhaya' 4577/15, Agarwal Road, Darya Ganj, New Delhi - 110002
Tele: 011- 47630600, 43518550

Head Office

Kalindi, TP Nagar, Meerut (UP) - 250002
Tel: 0121-7156203, 7156204

卐 **Sales & Support Offices**

Agra, Ahmedabad, Bengaluru, Bareilly, Chennai, Delhi, Guwahati,
Hyderabad, Jaipur, Jhansi, Kolkata, Lucknow, Nagpur & Pune.

卐 **ISBN** 978-93-25299-12-2

卐 **PO No :** TXT-XX-XXXXXXX-X-XX

Published by Arihant Publications (India) Ltd.

For further information about the books published by Arihant, log on to
www.arihantbooks.com or e-mail at info@arihantbooks.com

Follow us on



PREFACE

Engineering offers the most exciting and fulfilling of careers. As a Engineer you can find satisfaction by serving the society through your knowledge of technology. Although the number of Engineering colleges imparting quality education and training has significantly increased after independence in the country, but simultaneous increase in the number of serious aspirants has made the competition difficult, it is no longer easy to get a seat in a prestigious Engineering college today.

For success, you require an objective approach of the study. This does not mean you 'prepare' yourself for just 'objective questions'. Objective Approach means more than that. It could be defined as that approach through which a student is able to master the concepts of the subject and also the skills required to tackle the questions asked in different entrances such as JEE Main & Advanced, as well other regional Engineering entrances. These two-volume books on Mathematics 'Objective Mathematics (Vol.1 & 2)' fill the needs of such books in the market in Mathematics and are borne out of my experience of teaching Mathematics to Engineering aspirants.

The plan of the presentation of the subject matter in the books is as follows

- The whole chapter has been divided under logical topic heads to cover the syllabi of JEE Main & Advanced and various Engineering entrances in India.
- The Text develops the concepts in an easy going manner, taking the help of the examples from the day-to-day life.
- Important points of the topics have been highlighted in the text. Under Notes, some extra points regarding the topics have been given to enrich the students.
- The Solved Examples make the students learn the basic problem solving skills in Mathematics. Very detailed explanations have been provided to make the students skilled in systematically tackling the problems.
- The answers / solutions to all the questions have been provided.
- The Objective Questions have been divided according to their types Single correct option, More than One, Assertion-Reason, Matching Type, Integer Type, Passage Based, etc. which can take the students to a level required for various Engineering entrances in the present scenario.
- Entrance Corner includes the Previous Years' Questions asked in JEE Main & Advanced and other various Engineering entrances. At the end of the book, JEE Main & Advanced & Other Regional Entrances Solved Papers have been given.

I would open-heartedly welcome the suggestions for the further improvements of this book (Vol.1) from the students and teachers.

Amit M. Agarwal

CONTENTS

1. SETS	1-19	• Introduction • Set • Notations • Representation of Sets • Types of Sets • Venn Diagram • Operations on Sets • Laws of Algebra of Sets • Formulae to Solve Practical Problems on Union and Intersection of Sets
2. FUNDAMENTALS OF RELATION AND FUNCTION	20-40	• Ordered Pair • Cartesian Product of Sets • Properties of Cartesian Product of Sets • Relation • Representation of Relation • Domain and Range of Relations • Some Particular Types of Relations • Inverse Relation • Composition of Relations • Functions or Mappings • Difference between Relation and Function • Domain, Codomain and Range of a Function • Equal Functions • Classification of Functions • Algebra of Real Functions • Composition of Functions
3. SEQUENCE AND SERIES	41-108	• Introduction • Arithmetic Progression (AP) • Geometric Progression (GP) • Harmonic Progression (HP) • Arithmetico-Geometric Progression (AGP) • Some Special Series
4. COMPLEX NUMBERS	109-181	• The Real Number System • Modulus of a Real Number • Imaginary Number • Complex Number • Algebra of Complex Numbers • Conjugate of a Complex Number • Modulus of a Complex Number • Argument (or Amplitude) of a Complex Number • Various Forms of a Complex Number • De-Moivre's Theorem • Roots of Unity • Geometrical Applications of Complex Numbers • Loci in Complex Plane • Logarithm of Complex Numbers
5. INEQUALITIES AND QUADRATIC EQUATION	182-268	• Inequality • Generalised Method of Intervals for Solving Inequalities by Wavy Curve Method (Line Rule) • Absolute Value of a Real Number • Logarithms • Arithmetico-Geometric Mean Inequality • Quadratic Equation with Real Coefficients • Formation of a Polynomial Equation from Given Roots • Symmetric Function of the Roots

- Transformation of Equations
- Common Roots
- Quadratic Expression and its Graph
- Maximum and Minimum Values of Rational Expression
- Location of the Roots of a Quadratic Equation
- Algebraic Interpretation of Rolle's Theorem
- Condition for Resolution into Linear Factors
- Some Application of Graphs to Find the Roots of Equations

6. PERMUTATION AND COMBINATION 269-332

- Fundamental Principles of Counting (FPC)
- Factorial
- Exponent of Prime p in Factorial n
- Representation of Symbols ${}^nP_r$ and ${}^nC_r$
- Some Basic Arrangements and Selections
- Summation of Numbers (3 different ways)
- Permutations under Certain Conditions
- Circular Permutations
- Geometrical Applications of ${}^nC_r$
- Selection of One or More Objects
- Number of Divisors and the Sum of the Divisors of a Given Natural Number
- Division of Objects into Groups
- Derrangements
- Number of Integral Solutions of Linear Equations and Inequations

7. MATHEMATICAL INDUCTION 333-347

- Introduction
- Statement
- Principle of Mathematical Induction
- Algorithm for Mathematical Induction
- Types of Problems

8. BINOMIAL THEOREM 348-417

- Binomial Theorem for Positive Integral Index
- Multinomial Theorem
- Greatest Coefficient
- Greatest Term
- R-f Factor Relation
- Divisibility Problems
- Properties of Binomial Coefficients
- Binomial Theorem for any Index
- Approximation
- Exponential Series
- Logarithmic Series

9. TRIGONOMETRIC FUNCTIONS AND EQUATIONS 418-511

- Introduction
- Measure of Angles
- Systems of Measurement of Angles
- Trigonometric Ratios
- Trigonometric Function
- Graph of Trigonometric Functions
- Trigonometrical Identities
- Trigonometric Ratios of Allied Angles
- Trigonometrical Ratios of Compound Angles
- Trigonometric Ratios of Multiples of an Angle
- Maximum and Minimum Values of Trigonometrical Expressions
- Trigonometric Equations
- General Solution of Trigonometric Equations
- Solution of Trigonometric Inequality

10. PROPERTIES OF TRIANGLES, HEIGHTS AND DISTANCES 512-589

- Introduction
- Relation between the Sides and Angles of Triangle

- Trigonometric Ratios of Half Angles of a Triangle
- Area of a Triangle
- Conditional Identities
- Solution of Triangles
- Circles Connected with Triangle
- The Orthocentre and the Pedal Triangle
- Cyclic Quadrilateral
- Regular Polygon
- Heights and Distances
- Some Important Properties of Triangles
- Some Properties Related to Circle

11. CARTESIAN SYSTEM OF RECTANGULAR COORDINATES 590-626

- Introduction
- Coordinate System
- Distance Formulae
- Applications of Distance Formula
- Section Formulae
- Area of a Triangle
- Area of a Quadrilateral
- Some Standard Points of a Triangle
- Locus
- Transformation of Axes

12. STRAIGHT LINE AND PAIR OF STRAIGHT LINES 627-694

- Straight Line
- Angle between Two Lines
- Point of Intersection of Two Lines
- Image of a Point with Respect to a Line
- Family of Lines through the Intersection of Two Given Lines

- Locus and its Equation
- Combined Equation of a Pair of Straight Lines
- Bisectors of the Angle between the Lines Given by a Homogeneous Equation
- General Equation of Second Degree
- Equations of the Angle Bisectors
- Distance between the Pair of Parallel Lines

13. CIRCLE 695-791

- Introduction
- Standard Equation of a Circle
- Circle Passing through Three Points
- Position of a Point with respect to a Circle
- Intersection of a Straight Line and a Circle
- Equation of Tangent
- Normal to a Circle
- Pair of Tangents
- Director Circle
- Pole and Polar
- Diameter of a Circle
- Angle of Intersection of Two Circles
- Family of Circles
- Coaxial System of Circles
- Limiting Points

14. PARABOLA 792-853

- Conic Sections
- Parabola
- Other Standard Forms of Parabola
- Position of a Point with respect to a Parabola
- Intersection of a Line and a Parabola
- Equation of Tangent
- Angle of Intersection of Two Parabolas
- Equation of Normal to Parabola

- Number of Normals and Conormal Points
- Combined Equation of Pair of Tangents
- Director Circle
- Diameter of a Parabola
- Pole and Polar of a Parabola
- Lengths of Tangent, Subtangent, Normal and Subnormal

15. ELLIPSE

854-918

- Introduction
- Position of a Point with respect to an Ellipse
- Equation of the Chord
- Intersection of a Line and an Ellipse
- Tangent
- Combined Equation of the Pair of Tangents
- Director Circle
- Normal
- Number of Normals and Conormal Points
- Pole and Polar
- Conjugate Lines
- Diameter

16. HYPERBOLA

919-971

- Introduction
- Conjugate Hyperbola
- Position of a Point with respect to a Hyperbola
- Intersection of a Line and a Hyperbola
- Tangent to a Hyperbola
- Director Circle
- Normals to a Hyperbola
- Equation of the Pair of Tangents
- Equations of Chord
- Pole and Polar

- Conjugate points
- Conjugate Lines
- Diameter
- Asymptotes
- Rectangular Hyperbola
- Tangent to a Rectangular Hyperbola
- Normals to a Rectangular Hyperbola

17. INTRODUCTION TO THREE

DIMENSIONAL (3D) GEOMETRY

972-986

- Coordinate Axes and Coordinate Planes in Three Dimensional Space
- Coordinates of a Point in Space
- Distance between Two Points
- Section Formulae
- Centroid of a Triangle

**18. INTRODUCTION TO LIMITS
& DERIVATIVES**

987-1040

- Limits
- Existence of Limit
- Algebra of Limits
- Evaluation of Limits by Using L' Hospital's Rule
- Evaluation of Algebraic Limits
- Evaluation of Trigonometric Limits
- Evaluation of Exponential and Logarithmic Limits
- Evaluation of Exponential Limits of the Form 1^∞
- Sandwich Theorem for Evaluating Limits
- Some Useful Expansions
- Use of Newton-Leibnitz's Formula in Evaluating the Limits
- Derivative

- Geometrical Meaning of a Derivative
- Derivative from First Principle
- Differentiation of Some Important Functions
- Algebra of Derivative of Functions
- Chain Rule
- Logarithmic Differentiation

19. MATHEMATICAL REASONING 1041-1057

- Statements or Propositions
- Use of Venn Diagrams in Checking Truth and Falsity of Statements
- Truth Table
- Logical Connectives/Operators
- Quantifiers and Quantified Statements
- Negation of a Quantified Statement
- Logical Equivalence
- Negation of a Compound Statement
- Converse, Inverse and Contrapositive of an Implication
- Tautologies and Contradictions
- Algebra of Statements
- Duality

20. STATISTICS 1058-1094

- Measures of Central Tendency
- Measures of Dispersion
- Skewness
- Some Results to be Remembered
- Correlation Analysis
- Characteristics of Correlation Coefficient
- Regression Analysis
- Properties of Regression Coefficients
- Properties of Lines of Regression

21. FUNDAMENTALS OF PROBABILITY 1095-1132

- Introduction
- Some Basic Definitions
- Event
- Important Events
- Algebra of Events
- Probability
- Geometrical Probability
- Addition Theorem of Probability
- Independent Events
- Boole's Inequality

• JEE Advanced Solved Paper 2015	1135-1140
• JEE Main & Advanced Solved Papers 2016	1-12
• JEE Main & Advanced/ BITSAT/Kerala CEE/ KCET/AP & TS EAMCET/ VIT/MHT CET Solved Papers 2017	1-32
• JEE Main & Advanced/ BITSAT/ KCET/AP & TS EAMCET/ VIT/MHT CET Solved Papers 2018	1-35
• JEE Main & Advanced/ BITSAT/ AP & TS EAMCET/ MHT CET/WB JEE Solved Papers 2019-20	1-31

1

Sets

Introduction

In our mathematical language, everything in this universe whether living or non-living is called an object. If we consider a collection of objects given in such a way that it is possible to tell beyond doubt, whether a given object is in the collection under consideration or not, then such a collection of objects is called a well-defined collection of objects.

Set

A set is a well-defined collection of objects. By 'well-defined collection of objects', it means that we can definitely decide whether a given particular object belongs to a given collection or not. The objects, elements and members of a set are synonymous terms.

- ☞ **Example 1.** *The set of intelligent students in a class is*
- | | |
|------------------|-----------------------------------|
| (a) a null set | (b) a well-defined collection |
| (c) a finite set | (d) not a well-defined collection |

Sol. (d) Since, the criterion for determining the intelligence of a student may vary from person to person, so this is not a well-defined collection of objects.

- ☞ **Example 2.** *Which of the following is the collection of first five prime numbers?*
- | | |
|---------------------------|--------------------------|
| (a) $\{1, 2, 3, 5, 7\}$ | (b) $\{2, 3, 5, 7, 11\}$ |
| (c) $\{3, 5, 7, 11, 13\}$ | (d) $\{1, 2, 3, 4, 5\}$ |

Sol. (b) It is clear that first five prime numbers are $\{2, 3, 5, 7, 11\}$.

Notations

Sets are usually denoted by capital letters A, B, C etc., and their elements by small letters a, b, c , etc.

Let A be any set of objects and let a be a member of A , then we write $a \in A$ and read it as 'a belongs to A ' or 'a is an element of A ' or 'a is member of A '. If a is not an object of A , then we write $a \notin A$ and read as 'a does not belong to A ' or 'a is not an element of A ' or 'a is not a member of A '.

Representation of Sets

A set is often represented in one of the following two forms :

(i) Roster form or Tabular form	(ii) Set builder form

Chapter Snapshot

- Introduction
- Set
- Notations
- Representation of Sets
- Types of Sets
- Venn Diagram
- Operations on Sets
- Laws of Algebra of Sets
- Formulae to Solve Practical Problems on Union and Intersection of Sets

Roster Form or Tabular Form

In this form, a set is described by listing its elements, separated by commas, within braces $\{ \}$.

e.g. The set of months of a year which have thirty days, in roster form can be described as

$\{\text{April, June, September, November}\}$

➤ **Example 3.** The roster form of set $A = \{x : x \text{ is a positive integer and divisor of } 9\}$

(a) $\{3, 6, 9\}$ (b) $\{1, 5, 9\}$ (c) $\{1, 3, 9\}$ (d) $\{2, 3, 9\}$

Sol. (c) Since, x is a positive integer and a divisor of 9. So, x can take values 1, 3, 9.

∴ Roster form of the set A is $\{1, 3, 9\}$.

Set Builder Form

In this form, instead of listing all the elements of a set we describe the set by some special property (properties) satisfied by all of its elements and write it as

$A = \{x : P(x) \text{ holds}\} = \{x | x \text{ has the property } P(x)\}$ and read it as ' A is the set of all elements of x such that x has the property P '. The symbol ':' or '|' stands for 'such that'.

➤ **Example 4.** If set $A = \{1, 2, 3, 4\}$, then it can be written in set builder form as

(a) $A = \{x : x \in N \text{ and } x \leq 5\}$
 (b) $A = \{x : x \in N \text{ and } x < 5\}$
 (c) $A = \{x : x \in N \text{ and } 1 < x < 5\}$
 (d) $A = \{x : x \in N \text{ and } x < 4\}$

Sol. (b) Clearly, $A = \{x : x \in N \text{ and } x < 5\}$ describes the set $A = \{1, 2, 3, 4\}$

Types of Sets

Empty Set

A set which does not contain any element is called an empty set or null set or void set and is denoted by the symbol ϕ or $\{ \}$.

➤ **Example 5.** The set $\{x : x \in N, 3 < x < 4\}$ is equal to

(a) $\{3, 4\}$ (b) $\{4\}$ (c) ϕ (d) $\{3\}$

Sol. (c) Since, $x \in N$ and $3 < x < 4$, so the given set has no element.

➤ **Example 6.** The set $\{x : x^2 + 1 = 0 \text{ and } x \in R\}$ is equal to

(a) $\{-1, 1\}$ (b) $\{1\}$
 (c) $\{-1\}$ (d) ϕ

Sol. (d) We know that there is no real number x such that $x^2 + 1 = 0$.

- The set $\{0\}$ is not an empty set as it contains the element 0 (zero).
- The set $\{\phi\}$ is not a null set. It is a set containing one element ϕ .
- A set which has at least one element is called a non-empty set.

Singleton Set

A set consisting of only one element is called a singleton set.

➤ **Example 7.** A set $A = \{x : x \in N \text{ and } 2 < x < 4\}$ is called

(a) null set
 (b) infinite set
 (c) singleton set
 (d) None of the above

Sol. (c) $\{3\}$; Since, $x \in N$ and $2 < x < 4 \Rightarrow x = 3$

Finite Set

A set which is empty or having a definite number of elements is called a finite set.

➤ **Example 8.** A set $\{x | x \in N \text{ and } 1 \leq x \leq 10\}$ is called

(a) infinite set (b) finite set
 (c) singleton set (d) null set

Sol. (b) Since, elements of the given set is $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Cardinal Number of a Finite Set

The number of distinct elements contained in a finite set A is called its cardinal number and is denoted by $n(A)$.

e.g. If $A = \{1, 3, 5, 7\}$, then $n(A) = 4$.

➤ **Example 9.** The number of elements in a set $\{x : x \text{ are all vowels}\}$ is

(a) 6 (b) 26
 (c) 5 (d) 2

Sol. (c) $A = \{a, e, i, o, u\}$
 So, $n(A) = 5$

Infinite Set

A set which is not finite is called an infinite set. In other words, a set in which the process of counting of elements does not come to an end is called an infinite set.

➤ **Example 10.** A set of all points in a plane is called

(a) empty set (b) finite set
 (c) infinite set (d) None of these

Sol. (c) Points in a plane cannot be counted.



Equal Sets

Two sets A and B are said to be equal, if every element of A is also an element of B and every element of B is also an element of A . Thus, if $x \in A \Rightarrow x \in B$ and $y \in B \Rightarrow y \in A$, then A and B are equal sets and we can write $A = B$.

➤ **Example 11.** The set $A = \{x : x \in N \text{ and } 1 < x < 7\}$ is equal to

- (a) $\{2, 3, 4, 5, 6\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{2, 3, 4, 5, 6, 7\}$ (d) $\{1, 2, 3, 4, 5, 6, 7\}$

Sol. (a) Whenever, we have to show that two sets A and B are equal, show that $A \subseteq B$ and $B \subseteq A$, then $A = B$.

Equivalent Sets

Two finite sets A and B are said to be equivalent, if $n(A) = n(B)$.

Clearly, equal sets are equivalent, but equivalent sets need not to be equal.

Equivalence of two sets is denoted by the symbol ' $\sim$ '. Thus, if A and B are equivalent sets, we write $A \sim B$ which is read as ' A is equivalent to B '.

➤ **Example 12.** The set $\{1, 2, 3, 4, 5\}$ is equivalent to

- (a) $\{x : x \in \text{all vowels}\}$ (b) $\{2, 3, 4\}$
(c) $\{1, 2, 4, 5\}$ (d) $\{1, 2, 3\}$

Sol. (a)

Subset and Superset

The set B is said to be subset of set A , if every element of set B is also an element of set A . Symbolically, we write it as, $B \subseteq A$ or $A \supseteq B$, where A is superset of B .

(i) $B \subseteq A$ is read as B is contained in A or B is subset of A or A is superset of B .

(ii) $A \supseteq B$ is read as A contains B or B is a subset of A .

Evidently, if A and B are two sets such that $A \in B \Rightarrow x \in A$, then B is subset of A . The symbol ' $\Rightarrow$ ' stands for 'implies'. We read it as ' x belongs to B ' implies that ' x belongs to A '.

➤ **Example 13.** The set $A = \{1, 2, 3\}$ is the subset of

- (a) $\{1, 2, 4, 5\}$ (b) $\{1, \{2, 3\}, 4, 5\}$
(c) $\{1, 2, 3, 7, 8\}$ (d) $\{\{1, 2\}, 3, 4\}$

Sol. (c)

Proper Subset

The set B is said to be a proper subset of set A , if every element of set B is an element of A , whereas every element of A is not an element of B .

We write it as $B \subset A$ and read it as B is a proper subset of A . Thus, B is a proper subset of A , if every element of B is an element of A and there is atleast one element in A which is not in B .

Observe that $A \subseteq A$ i.e. every set is a subset of itself, but not a proper subset.

➤ **Example 14.** The set $\{1, 2\}$ is the proper subset of

- (a) $\{1, 2\}$ (b) $\{1, 3, 4\}$
(c) $\{1, 2, 3\}$ (d) $\{\{1, 2\}, 3, 4\}$

Sol. (c)

Intervals as Subsets of R

Let $a, b \in R$ and $a < b$, then

- An open interval denoted by (a, b) is the set of all real numbers such that $(a, b) = \{x \in R : a < x < b\}$
- A closed interval denoted by $[a, b]$ is the set of all real numbers such that $[a, b] = \{x \in R : a \leq x \leq b\}$
- Interval closed at one end and open at the other are given by $[a, b) = \{x \in R : a \leq x < b\}$ and $(a, b] = \{x \in R : a < x \leq b\}$

➤ **Example 15.** The subset of R as intervals $\{x : x \in R, -12 < x < -10\}$ is

- (a) $(-12, -10]$ (b) $[-12, -10]$
(c) $(-12, -10)$ (d) $[-12, -10)$

Sol. (c) The given subset is belong to open interval.
 $\therefore \{x : x \in R, -12 < x < -10\}$ is $(-12, -10)$.

➤ **Example 16.** Let R be set of points inside a rectangle of sides a and b ($a, b > 1$) with two sides along the positive direction of X -axis and Y -axis, then

- (a) $R = \{(x, y) : 0 \leq x \leq a, 0 \leq y < b\}$
(b) $R = \{(x, y) : 0 \leq x < a, 0 \leq y \leq b\}$
(c) $R = \{(x, y) : 0 \leq x \leq a, 0 < y < b\}$
(d) $R = \{(x, y) : 0 < x < a, 0 < y < b\}$

Sol. (d) Since, (x, y) both lies in first quadrant.

$\therefore x, y > 0$ and less than side a and b because R lies inside the rectangle.

$\therefore R = \{(x, y) : 0 < x < a, 0 < y < b\}$

Power Set

The set formed by all the subsets of a given set A is called the power set of A . It is usually denoted by $P(A)$.

➤ **Example 17.** If set $A = \{1, 2, 3\}$, then $P(A)$ is equal to

- (a) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}\}$
(b) $\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}\}$
(c) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$
(d) $\{\{1\}, \{2\}, \{3\}, \{1, 2, 3\}\}$

Sol. (c)

Some Results on Subsets

- (i) Every set is a subset of itself.
- (ii) The empty set is a subset of every set.
- (iii) The total number of subset of finite set containing n elements is 2^n .

Comparable Sets

Two sets A and B are said to be comparable, if one of them is a subset of the other i.e. either $A \subseteq B$ or $B \subseteq A$.

➤ **Example 18.** The set $\{1, 2, 3\}$ is comparable with

- (a) $\{1, 2, 4, 5\}$ (b) $\{\{1, 2\}, 4, 3\}$
- (c) $\{\{1, 3\}, 2, 4\}$ (d) $\{1, 2, 3, 4\}$

Sol. (d) Since, $\{1, 2, 3\}$ is the subset of $\{1, 2, 3, 4\}$.

Universal Set

In any discussion of set theory, there is always a set that contains all the sets under consideration i.e. it is a superset of each of the given sets. Such a set is called the universal set and is denoted by U .

➤ **Example 19.** Which of the following set is the universal set of sets $A = \{2, 4, 5\}$, $B = \{1, 3, 5\}$?

$C = \{3, 5, 7, 11\}$, $D = \{2, 4, 8, 10\}$

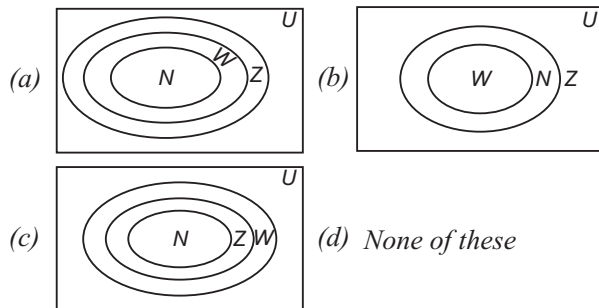
- (a) $\{1, 2, 3, 5, 6, 7, 8, 9, 10\}$
- (b) $\{1, 2, 3, 4, 5, 6, 7, 8, 10\}$
- (c) $\{1, 2, 3, 4, 5, 7, 8, 10, 11\}$
- (d) $\{1, 2, 3, 4, 6, 7, 8, 10, 11\}$

Sol. (c)

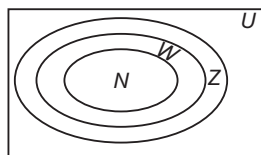
Venn Diagram

Venn diagrams are the diagrams which represent the relationship between sets.

➤ **Example 20.** The set of natural numbers is a subset of whole numbers which is a subset of integers. Then, correct representation of it through Venn diagram is



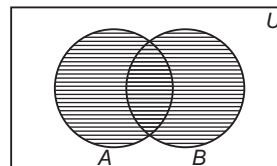
Sol. (a)



Operations on Sets

Now, we introduce some operations on sets to construct new sets from the given ones.

- i. **Union of sets** The union of two sets A and B , denoted by $A \cup B$ is the set of all those elements, each one of which is either in A or in B or in both A and B .



Thus, $A \cup B = \{x : x \in A \text{ or } x \in B\}$

Clearly $x \in A \cup B$

$\Rightarrow x \in A$

or $x \in B$

and $x \notin A \cup B$

$\Rightarrow x \notin A$ and $x \notin B$

In the figure, the shaded part represents, $A \cup B$.

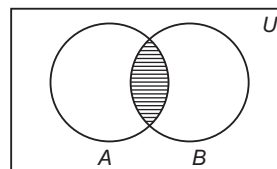
It is evident that $A \subseteq A \cup B$, $B \subseteq A \cup B$.

➤ **Example 21.** If two sets $A = \{1, 2, 3\}$ and $B = \{1, 3, 5, 7\}$, then $A \cup B$ is equal to

- (a) $\{1, 2, 3, 7\}$ (b) $\{1, 2, 3, 5, 7, 8\}$
- (c) $\{1, 2, 3, 5, 7\}$ (d) $\{1, 2, 3, 4, 5, 6, 7\}$

Sol. (c)

- ii. **Intersection of sets** The intersection of two sets A and B , denoted by $A \cap B$ is the set of all elements, common to both A and B .



Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

Clearly, $x \in A \cap B$

$\Rightarrow x \in A$ and $x \in B$

and $x \notin A \cap B \Rightarrow x \in A$ or $x \in B$

In the figure, the shaded part represents $A \cap B$.

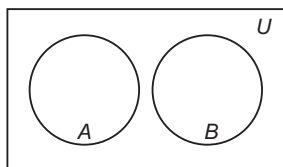
It is evident that $A \cap B \subseteq A$, $A \cap B \subseteq B$.

➤ **Example 22.** If two sets $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 5\}$, then $A \cap B$ is equal to

- (a) $\{1, 3\}$ (b) $\{2, 4\}$
- (c) $\{3, 4\}$ (d) $\{1, 6\}$

Sol. (b) Since, 2 and 4 are the common elements in sets A and B .

- iii. **Disjoint sets** Two sets A and B are said to be disjoint, if $A \cap B = \phi$. If $A \cap B \neq \phi$, then A and B are said to be intersecting or overlapping sets.



➤ **Example 23.** Given that,

$$A = \{1, 2, 3, 4, 5, 6\}$$

$$B = \{7, 8, 9, 10, 11\}$$

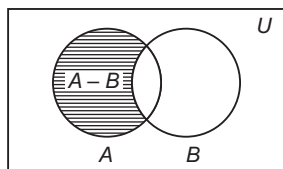
$$C = \{6, 8, 10, 12, 14\}.$$

Which of the following pair of sets are disjoint sets?

- (a) A and B
 (b) B and C
 (c) C and A
 (d) None of the above

Sol. (a)

- iv. **Difference of sets** If A and B are two sets, then their difference $A - B$ is the set of all those elements of A which do not belong to B .



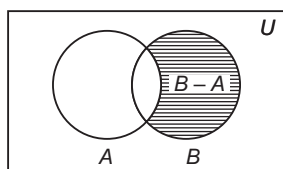
Thus, $A - B = \{x : x \in A \text{ and } x \notin B\}$

Clearly, $x \in A - B$

$$\Rightarrow x \in A \text{ and } x \notin B$$

In the figure, the shaded part represents $A - B$.

Similarly, the difference $B - A$ is the set of all those elements of B that do not belong to A i.e.



$$B - A = \{x : x \in B \text{ and } x \notin A\}$$

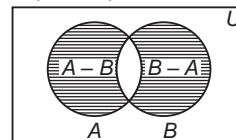
➤ **Example 24.** Given that two sets

$A = \{1, 3, 5, 7, 9\}$ and $B = \{2, 3, 5, 7, 11\}$, then $A - B$ is

- (a) $\{1, 9\}$
 (b) $\{2, 11\}$
 (c) $\{3, 5, 7\}$
 (d) $\{1, 2, 3, 5, 7, 9, 11\}$

Sol. (a) $\{1, 9\}$, since $1, 9 \notin B$

- v. **Symmetric difference of two sets** The symmetric difference of two sets A and B is the set $(A - B) \cup (B - A)$ and is denoted by $A \Delta B$.



$$\text{Thus, } A \Delta B = (A - B) \cup (B - A)$$

$$= \{x : x \notin A \cap B\}$$

The shaded part represents $A \Delta B$.

➤ **Example 25.** If set $A = \{1, 3, 5, 7, 9\}$ and set

$B = \{2, 3, 5, 7, 11\}$, then $A \Delta B$ is equal to

- (a) $\{3, 5, 7\}$ (b) $\{1, 2\}$
 (c) $\{9, 11\}$ (d) $\{1, 2, 9, 11\}$

Sol. (d) $A \Delta B = (A - B) \cup (B - A)$

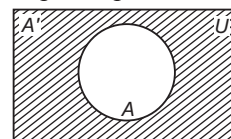
$$= \{1, 9\} \cup \{2, 11\} = \{1, 2, 9, 11\}$$

- vi. **Complement of a set** Let U be the universal set and $A \subset U$, then the complement of A , denoted by A' or $U - A$ is defined as

$$A' = \{x : x \in U \text{ and } x \notin A\}$$

Clearly, $x \in A' \Leftrightarrow x \notin A$

The shaded part represents A' .



➤ **Example 26.** Given that $U = \{x : x \text{ is a letter in English alphabet}\}$ and $A = \{x : x \text{ is a vowel}\}$, then

A' is equal to

- (a) ϕ (b) U
 (c) $\{x : x \in \text{all consonants}\}$ (d) None of these

Sol. (c)

- vii. **Some results on complement** The following results are the direct consequences of the definition of the complement of the set.

- (a) $U' = \phi$
 (b) $\phi' = \{x \in U : x \notin \phi\} = U$
 (c) $(A')' = \{x \in U : x \notin A'\} = \{x \in U : x \in A\} = A$
 (d) $A \cap A' = \{x \in U : x \in A\} \cap \{x \in U : x \notin A\} = \phi$
 (e) $A \cup A' = \{x \in U : x \in A\} \cup \{x \in U : x \notin A\} = U$

Laws of Algebra of Sets

In this article, we shall state and prove some fundamental laws of algebra of sets.

1. **Idempotent laws** For any set A , we have

- (i) $A \cup A = A$ (ii) $A \cap A = A$

2. **Identity laws** For any set A , we have ϕ and U as identity elements for union and intersection, respectively.

Proof

- (i) $A \cup \phi = \{x : x \in A \text{ or } x \in \phi\} = \{x : x \in A\} = A$
 (ii) $A \cap U = \{x : x \in A \text{ and } x \in U\} = \{x : x \in A\} = A$

3. **Commutative laws** For any two sets A and B , we have

- (i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

i.e. union and intersection are commutative.

Proof

Recall that two sets X and Y are equal iff $X \subseteq Y$ if every element of X belongs to Y .

- (i) Let x be an arbitrary element of $A \cup B$. Then,

$$\begin{aligned} x &\in A \cup B \\ \Rightarrow x &\in A \text{ or } x \in B \\ \Rightarrow x &\in B \text{ or } x \in A \\ \Rightarrow x &\in B \cup A \\ \therefore A \cup B &\subseteq B \cup A \end{aligned}$$

Similarly, $B \cup A \subseteq A \cup B$

Hence, $A \cup B = B \cup A$

- (ii) Let x be an arbitrary element of $A \cap B$. Then,

$$\begin{aligned} x &\in A \cap B \\ \Rightarrow x &\in A \text{ and } x \in B \\ \Rightarrow x &\in B \text{ and } x \in A \\ \Rightarrow x &\in B \cap A \\ A \cap B &\subseteq B \cap A \end{aligned}$$

Similarly, $B \cap A \subseteq A \cap B$

Hence, $A \cap B = B \cap A$

4. **Associative laws** If A , B and C are any three sets, then

- (i) $(A \cup B) \cup C = A \cup (B \cup C)$

- (ii) $A \cap (B \cap C) = (A \cap B) \cap C$

i.e. union and intersection are associative.

Proof

- (i) Let x be an arbitrary element of $(A \cup B) \cup C$.

$$\begin{aligned} \text{Then, } x &\in (A \cup B) \cup C \\ \Rightarrow x &\in (A \cup B) \text{ or } x \in C \\ \Rightarrow (x \in A \text{ or } x \in B) \text{ or } x \in C \\ \Rightarrow x &\in A \text{ or } (x \in B \text{ or } x \in C) \\ \Rightarrow x &\in A \text{ or } x \in (B \cup C) \\ \Rightarrow x &\in A \cup (B \cup C) \end{aligned}$$

$\therefore (A \cup B) \cup C \subseteq A \cup (B \cup C)$

Similarly, $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

Hence, $(A \cup B) \cup C = A \cup (B \cup C)$

- (ii) Let x be an arbitrary element of $A \cap (B \cap C)$.

$$\begin{aligned} \text{Then, } x &\in A \cap (B \cap C) \\ \Rightarrow x &\in A \text{ and } x \in (B \cap C) \\ \Rightarrow x &\in A \text{ and } (x \in B \text{ and } x \in C) \\ \Rightarrow (x \in A \text{ and } x \in B) \text{ and } x \in C \\ \Rightarrow x &\in (A \cap B) \text{ and } x \in C \\ \Rightarrow x &\in (A \cap B) \cap C \\ \therefore A \cap (B \cap C) &\subseteq (A \cap B) \cap C \end{aligned}$$

Similarly, $(A \cap B) \cap C \subseteq A \cap (B \cap C)$

Hence, $A \cap (B \cap C) = (A \cap B) \cap C$

5. **Distributive laws** If A , B and C are any three sets, then

- (i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

- (ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

i.e. union and intersection are distributive over intersection and union, respectively.

Proof

- (i) Let x be an arbitrary element of $A \cup (B \cap C)$.

$$\begin{aligned} \text{Then, } x &\in A \cup (B \cap C) \\ \Rightarrow x &\in A \text{ or } x \in (B \cap C) \\ \Rightarrow x &\in A \\ \text{or } x &\in (B \cap C) \\ \Rightarrow (x \in B \text{ and } x \in C) \\ \text{and } x &\in A \\ \Rightarrow x &\in (A \cap B) \text{ and } x \in (A \cap C) \\ \Rightarrow x &\in (A \cap B) \cup (A \cap C) \\ \therefore A \cup (B \cap C) &\subseteq (A \cap B) \cup (A \cap C) \end{aligned}$$

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cup (B \cap C)$

Hence, $A \cup (B \cap C) = (A \cap B) \cup (A \cap C)$

- (ii) Let x be an arbitrary element of $A \cap (B \cup C)$.

$$\begin{aligned} \text{Then, } x &\in A \cap (B \cup C) \\ \Rightarrow x &\in A \\ \text{and } x &\in (B \cup C) \\ \Rightarrow x &\in A \\ \text{and } x &\in (B \cup C) \\ \Rightarrow (x \in B \text{ or } x \in C) \\ \Rightarrow (x \in A \text{ and } x \in B) \\ \text{or } (x \in A \text{ and } x \in C) \\ \Rightarrow x &\in (A \cap B) \\ \text{or } x &\in (A \cap C) \\ \Rightarrow x &\in (A \cap B) \cup (A \cap C) \\ \therefore A \cap (B \cup C) &\subseteq (A \cap B) \cup (A \cap C) \end{aligned}$$

Similarly, $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

Hence, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

6. **De-Morgan's laws** If A and B are any two sets, then

$$(i) (A \cup B)' = A' \cap B'$$

$$(ii) (A \cap B)' = A' \cup B'$$

Proof

(i) Let x be an arbitrary element of $(A \cup B)'$.

$$\text{Then, } x \in (A \cup B)'$$

$$\Rightarrow x \notin (A \cup B)$$

$$\Rightarrow x \notin A$$

$$\text{and } x \notin B$$

$$\Rightarrow x \in A'$$

$$\text{and } x \in B'$$

$$\Rightarrow x \in A' \cap B'$$

$$\therefore (A \cup B)' \subseteq A' \cap B'$$

Again, let y be an arbitrary element of $A' \cap B'$. Then, $y \in A' \cap B'$

$$\Rightarrow y \in A'$$

$$\text{and } y \in B'$$

$$\Rightarrow y \notin A$$

$$\text{and } y \notin B$$

$$\Rightarrow y \notin A \cap B$$

$$\Rightarrow y \in (A \cap B)'$$

$$\therefore A' \cap B' \subseteq (A \cap B)'$$

$$\text{Hence, } (A \cap B)' = A' \cap B'$$

(ii) Let x be an arbitrary element of $(A \cap B)'$.

$$\text{Then, } x \in (A \cap B)'$$

$$\Rightarrow x \notin (A \cap B)$$

$$\Rightarrow x \notin A$$

$$\text{and } x \notin B$$

$$\Rightarrow x \in A'$$

$$\text{or } x \in B'$$

$$\Rightarrow x \in A' \cup B'$$

$$\Rightarrow (A \cap B)' \subseteq A' \cup B'$$

Again, let y be an arbitrary element of $A' \cup B'$. Then,

$$\Rightarrow y \in (A' \cup B')$$

$$\Rightarrow y \in A'$$

$$\text{or } y \in B'$$

$$\Rightarrow y \notin A \text{ or } y \notin B$$

$$\Rightarrow y \notin (A \cap B)$$

$$\Rightarrow y \in (A \cap B)'$$

$$\therefore A' \cup B' \subseteq (A \cap B)'$$

$$\text{Hence, } (A \cap B)' = A' \cup B'$$

Formulae to Solve Practical Problems on Union and Intersection of Sets

Let A, B, C be any finite sets and U be the finite universal sets, then

$$(i) n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$(ii) \text{ If } (A \cap B) = \phi, \text{ then } n(A \cup B) = n(A) + n(B)$$

$$(iii) n(A - B) = \text{Number of elements which belong to only } A \text{ of sets } A \text{ and } B$$

$$= n(A) - n(A \cap B) = n(A \cup B) - n(B)$$

$$(iv) n(A \Delta B) = \text{Number of elements which belong to exactly one of } A \text{ or } B$$

$$= n(A) + n(B) - 2n(A \cap B)$$

$$(v) n(A \cup B \cup C) = n(A) + n(B) + n(C)$$

$$- n(A \cap B) - n(B \cap C) - n(A \cap C) + n(A \cap B \cap C)$$

$$(vi) \text{ Number of elements in exactly two of the sets } A, B, C$$

$$= n(A \cap B) + n(B \cap C) + n(C \cap A) - 3n(A \cap B \cap C)$$

$$(vii) \text{ Number of elements in exactly one of the sets } A, B, C$$

$$= n(A) + n(B) + n(C) - 2n(A \cap B) - 2n(B \cap C) - 2n(A \cap C) + 3n(A \cap B \cap C)$$

$$(viii) n(A' \cup B') = n\{(A \cap B)'\} = n(U) - n(A \cap B)$$

$$(ix) n(A' \cap B') = n\{(A \cup B)'\} = n(U) - n(A \cup B)$$

➤ **Example 27.** In a group of 50 people, 35 speak Hindi, 25 speak both English and Hindi and all the people speak atleast one of two languages. The number of people who speak 'English', 'English but not Hindi' are respectively

$$(a) 40, 15$$

$$(b) 15, 40$$

$$(c) 35, 15$$

$$(d) 25, 15$$

Sol. (a) Let H denotes the set of people speaking Hindi and E denotes the set of people speaking English.

$$\text{We have, } n(H \cup E) = 50, n(H) = 35, n(H \cap E) = 25$$

Also, we have

$$n(H \cup E) = n(H) + n(E) - n(H \cap E)$$

$$\Rightarrow 50 = 35 + n(E) - 25$$

$$\Rightarrow n(E) = 75 - 35 = 40$$

which is the number of people who speak English.

We have to find the people speak only English

$$\text{i.e. } n(E - H) = n(H \cup E) - n(H) = 50 - 35 = 15$$

- 8

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17\}$,

$B = \{2, 4, \dots, 18\}$ and N is the universal set, then $A' \cup \{(A \cup B) \cap B'\}$ is

- (a) A (b) N
(c) B (d) None of these

Sol. We have, $(A \cup B) \cap B' = A - (A \cap B)$

Since, A and B are disjoint sets.

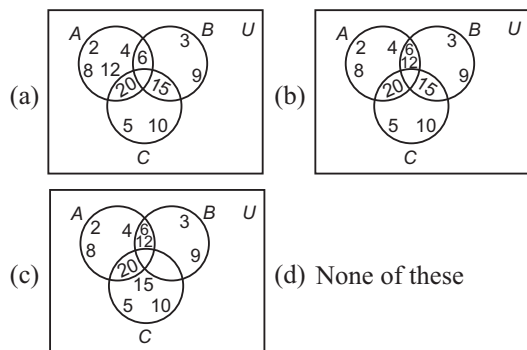
$$\therefore A \cap B = \phi$$

$$\Rightarrow (A \cup B) \cap B' = A$$

$$\therefore \{(A \cup B) \cap B'\} \cup A' = A \cup A' = N$$

Hence, (b) is the correct answer.

Ex 2. Let A, B and C are subsets of universal set U . If $A = \{2, 4, 6, 8, 12, 20\}$, $B = \{3, 6, 9, 12, 15\}$, $C = \{5, 10, 15, 20\}$ and U is the set of all whole numbers. Then, the correct Venn diagram is



Sol. Given, $A = \{2, 4, 6, 8, 12, 20\}$
 $B = \{3, 6, 9, 12, 15\}$
 $C = \{5, 10, 15, 20\}$

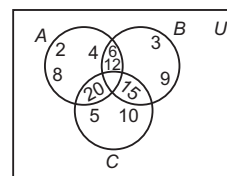
Now,

$$A \cap B = \{2, 4, 6, 8, 12, 20\} \cap \{3, 6, 9, 12, 15\} = \{6, 12\}$$

$$B \cap C = \{3, 6, 9, 12, 15\} \cap \{5, 10, 15, 20\} = \{15\}$$

$$C \cap A = \{5, 10, 15, 20\} \cap \{2, 4, 6, 8, 12, 20\} = \{20\}$$

$$\text{and } A \cap B \cap C = \phi$$



Hence, (b) is the correct answer.

Ex 3. Let A and B have 3 and 6 elements, respectively. What can be minimum number of elements in $A \cup B$?

- (a) 3 (b) 6
(c) 9 (d) 18

Sol. Note that $A \cup B$ will contain minimum number of elements, if $A \subset B$

$$\text{So that, } n(A \cup B) = n(B) = 6$$

Hence, (b) is the correct answer.

Ex 4. Two finite sets have m and n elements. The total number of subsets of the first set is 56 more than total number of subsets of second set. The values of m and n are

- (a) 7 and 6 (b) 6 and 3
(c) 5 and 1 (d) 8 and 7

Sol. Given, $2^m - 2^n = 56$

By hit and trial method, we get $m = 6$ and $n = 3$

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 5. Let A and B be any two sets. Which of the following are correct?

- (a) $(A - B) \cap B = \phi$
(b) $(A - B) \cup B = \phi$
(c) $(A - B) \cap A = A \cap B'$
(d) $(A - B) \cup B = A \cup B$

Sol. We have,

$$\begin{aligned} \text{(a) } (A - B) \cap B &= (A \cap B') \cap B \quad [\because A - B = A \cap B'] \\ &= A \cap (B' \cap B) \\ &= A \cap \phi \quad [\because B' \cap B = \phi] \\ &= \phi \end{aligned}$$

$$\begin{aligned} \text{(b) } (A - B) \cup B &= (A \cap B') \cup B \\ &= (A \cup B) \cap (B' \cup B) \\ &= (A \cup B) \cap U \quad [\because B' \cup B = U] \\ &= A \cup B \end{aligned}$$

$$\text{(c) } (A - B) \cap A = (A \cap B') \cap A = A \cap B'$$

$$\text{(d) } (A - B) \cup B = A \cup B$$

Hence, (a), (c) and (d) are the correct answers.

Ex 6. If A and B are two sets such that $n(A) = 70$,

$$n(B) = 60, n(A \cup B) = 110, \text{ then}$$

- (a) $n(A - B) = 40$
(b) $n(B - A) = 40$
(c) $n(A \cap B) = 20$
(d) $n(A - B) = n(A \cap B')$

Sol. We have, $n(A) = 70, n(B) = 60, n(A \cup B) = 110$

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\begin{aligned} \Rightarrow n(A \cap B) &= n(A) + n(B) - n(A \cup B) \\ &= 70 + 60 - 110 \\ &= 20 \end{aligned}$$

$$\begin{aligned} \text{(a) } n(A - B) &= n(A \cap B') = n(A) - n(A \cap B) \\ &= 70 - 20 = 50 \end{aligned}$$

$$\begin{aligned} \text{(b) } n(B - A) &= n(B \cap A') \\ &= n(B) - n(A \cap B) \\ &= 60 - 20 = 40 \end{aligned}$$

Hence, (b), (c) and (d) are the correct answers.

1 Type 3. Assertion and Reason

Directions (Ex. Nos. 7-9) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Ex 7. Statement I If A and B are disjoint sets, then $(A' \cup B')' = \phi$.

Statement II $(A \cup B)' = A' \cap B'$
[De-Morgan's law]

Sol. $(A' \cup B')' = (A')' \cap (B')' = A \cap B = \phi$, as A and B are disjoint.

$\therefore$ Statement I is true, Statement II is true.

Also, Statement II is a correct explanation of Statement I.

Hence, (a) is the correct answer.

Ex 8. Statement I If A and B are any two non-empty sets, then

$$(A - B) \cup (B - A) = (A \cup B) - (A \cap B)$$

Statement II $(A - B) = A \cap B'$

$$\text{and } (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$$

Sol. $(A - B) \cup (B - A) = (A \cap B') \cup (B \cap A')$
[$\because (A - B) = A \cap B'$]
 $= \{(A \cap B') \cup B\} \cap \{(A \cap B') \cup A'\}$ [distributive law]
 $= \{(A \cup B) \cap (B' \cup B)\} \cap \{(A \cup A') \cap (B' \cup A')\}$
 $= \{(A \cup B) \cap U\} \cap \{U \cap B' \cup A'\}$ [U = Universal set]
 $= (A \cup B) \cap (B' \cup A') = (A \cup B) \cap (A \cap B)'$
 $= (A \cup B) - (A \cap B)$

So, both statements are true and Statement II is correct explanation of Statement I.

Hence, (a) is the correct answer.

Ex 9. Statement I If $A' \cup B = U$, then $A \subseteq B$.

Statement II $A \subseteq B \Leftrightarrow B^C \subset A^C$

Sol. $A' \cup B = U$, iff $B = A$ [$\because A \cup A' = U$]
 $\Rightarrow A = B \Rightarrow A \subseteq B$
Also, $A \subseteq B \Rightarrow B^C \subset A^C$

So, both statements are true but Statement II is not the correct explanation of Statement I.

Hence, (b) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage (Ex. Nos. 10-12) In a town of 10000 families, it was found that 40% families buy newspaper A, 20% families buy newspaper B and 10% families buy newspaper C. 5% families buy A and B, 3% buy B and C and 4% buy A and C. If 2% families buy all the three news papers, then

Ex 10. The number of families which buy newspaper A only, is

- (a) 3300 (b) 4000 (c) 1400 (d) 2000

Ex 11. The number of families which buy newspaper none of A, B and C, is

- (a) 4400 (b) 4000 (c) 1000 (d) 3300

Ex 12. The number of families which buy newspaper B only, is

- (a) 1400 (b) 2500
- (c) 3000 (d) 2000

Sol. (Ex. Nos. 10-12) Let P , Q and R be the sets of families buying newspapers A, B and C respectively. Let U be the universal set. Then,

$$n(U) = 10000, n(P) = 40\% \text{ of } 10000 = 4000$$

$$n(Q) = 20\% \text{ of } 10000 = 2000,$$

$$n(R) = 10\% \text{ of } 10000 = 1000$$

$$n(P \cap Q) = 5\% \text{ of } 10000 = 500$$

$$n(Q \cap R) = 3\% \text{ of } 10000 = 300$$

$$n(R \cap P) = 4\% \text{ of } 10000 = 400$$

$$n(P \cap Q \cap R) = 2\% \text{ of } 10000 = 200$$

10. The number of families which buy newspaper A only
 $= n(P \cap Q' \cap R') = n(P \cap (Q \cup R)')$
 $= n(P) - n[P \cap (Q \cup R)]$

$$\begin{aligned} & [\because n(A \cap B') = n(A) - n(A \cap B)] \\ &= n(P) - n[P \cap Q] \cup (P \cap R) \\ &= n(P) - [n(P \cap Q) + n(P \cap R) - n(P \cap Q \cap R)] \\ &= 4000 - [500 + 400 - 200] = 3300 \end{aligned}$$

Hence, (a) is the correct answer.

11. The number of families which buy newspaper none of A, B and C

$$\begin{aligned} &= n(P \cup Q \cup R)' \\ &= n(U) - n(P \cup Q \cup R) \\ &= n(U) - [n(P) + n(Q) + n(R) - n(P \cap Q) \\ &\quad - n(Q \cap R) - n(P \cap R) + n(P \cap Q \cap R)] \\ &= 10000 - [4000 + 2000 + 1000 - 500 \\ &\quad - 300 - 400 + 200] \\ &= 10000 - 6000 \\ &= 4000 \end{aligned}$$

Hence, (b) is the correct answer.

12. The number of families which buy newspaper B only
 $= n(Q \cap P' \cap R') = n(Q \cap (P \cup R)')$

$$\begin{aligned} &= n[Q \cap (P \cup R)'] \\ &= n(Q) - n[Q \cap (P \cup R)] \\ &= n(Q) - n[(Q \cap P) \cup (Q \cap R)] \\ &= n(Q) - [n(P \cap Q) + n(Q \cap R) - n(P \cap Q \cap R)] \\ &= 2000 - [500 + 300 - 200] = 2000 - 600 = 1400 \end{aligned}$$

Hence, (a) is the correct answer.

Type 5. Match the Column

Ex 13. Match the following sets for all sets A and B .

Column I		Column II	
A.	If $A \subset B$, then $A \cap B'$	p.	$A \cup B$
B.	If $A \subset B$, then $A' \cup B$	q.	ϕ
C.	$(A - B) \cup A$	r.	U
D.	$(A - B) \cup B$	s.	A

Sol. A. Since, $A \subset B$, so there is no element in A , which does not belong to B .

$$\therefore A - B = \phi \Rightarrow A \cap B' = \phi$$

B. Since, $A \subset B$

$$\therefore A \cap B' = \phi$$

$$\Rightarrow (A \cap B')' = \phi'$$

$$\Rightarrow A' \cup B = U$$

C. $(A - B) \cup A = A$ [$\because (A - B) \subset A$]

$$\begin{aligned} \text{D. } (A - B) \cup B &= (A \cap B') \cup B \\ &= (A \cup B) \cap (B' \cup B) \\ &= (A \cup B) \cap U \\ &= A \cup B \end{aligned}$$

$A \rightarrow q; B \rightarrow r, C \rightarrow s; D \rightarrow p$

Type 6. Single Integer Answer Type Questions

Ex 14. If $U = \{x : x^5 - 6x^4 + 11x^3 - 6x^2 = 0\}$,
 $A = \{x : x^2 - 5x + 6 = 0\}$ and
 $B = \{x : x^2 - 3x + 2 = 0\}$, then $n(A \cap B)'$
 is equal to _____.

Sol. (3) We have, $U = \{x : x^5 - 6x^4 + 11x^3 - 6x^2 = 0\}$
 $= \{0, 1, 2, 3\}$

$$A = \{x : x^2 - 5x + 6 = 0\} = \{2, 3\}$$

$$\text{and } B = \{x : x^2 - 3x + 2 = 0\} = \{1, 2\}$$

$$\therefore A \cap B = \{2\}$$

$$\text{Hence, } (A \cap B)' = U - (A \cap B)$$

$$= \{0, 1, 2, 3\} - \{2\} = \{0, 1, 3\}$$

$$\therefore n(A \cap B)' = 3$$

Ex 15. If $A = \{x : x \text{ is an even natural number}\}$
 and $B = \{x : x \text{ is a prime number}\}$, then
 $n(A \cap B)$ is equal to _____.

Sol. (1) We have,

$$A = \{x : x \text{ is an even natural number}\}$$

$$= \{2, 4, 6, 8, \dots\}$$

$$\text{and } B = \{x : x \text{ is a prime number}\}$$

$$= \{2, 3, 5, 7, 11, \dots\}$$

$$A \cap B = \{2\}$$

$$n(A \cap B) = 1$$

Target Exercises

Type 1. Only One Correct Option

- The set $\{x : x \text{ is a positive integer less than 6 and } 3^x - 1 \text{ is an even number}\}$ in roster form is
 (a) $\{1, 2, 3, 4, 5\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
 (c) $\{2, 4, 6\}$ (d) $\{1, 3, 5\}$
- The set $A = \{x : x^4 - x^3 - x^2 = 0 \text{ and } x \in N\}$ represents
 (a) a null set (b) a singleton set
 (c) an infinite set (d) None of these
- If $A = \{1, 2, 3\}$, $B = \{x \in R : x^2 - 2x + 1 = 0\}$, $C = \{1, 2, 3\}$ and $D = \{x \in R : x^3 - 6x^2 + 11x - 6 = 0\}$, then the equal sets are
 (a) A and B (b) A and C
 (c) A , B and C (d) A , C and D
- Which of the following is true?
 (a) $a \in \{\{a\}, b\}$ (b) $\{b, c\} \subset \{a, \{b, c\}\}$
 (c) $\{a, b\} \subset \{a, \{b, c\}\}$ (d) None of these
- Let P be the set of prime numbers and $S = \{t : 2^t - 1 \text{ is a prime}\}$. Then,
 (a) $S \subset P$ (b) $P \subset S$
 (c) $S = P$ (d) None of these
- If $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{2, 4, 6, 8\}$ and $B = \{2, 3, 5, 7\}$, then $(A \cup B)'$, $(A' \cap B')$, $(A \Delta B)$ is equal to
 (a) $\{1, 9\}$, $\{2, 8\}$, $\{3, 4, 5, 6, 7, 8\}$
 (b) $\{1, 9\}$, $\{1, 9\}$, $\{3, 4, 5, 6, 7, 8\}$
 (c) $\{1, 9\}$, $\{1, 9\}$, $\{5, 6, 7, 8\}$
 (d) None of the above
- If $A = \{x : x \text{ is a multiple of 3}\}$ and $B = \{x : x \text{ is a multiple of 5}\}$, then $A - B$ is equal to
 (a) $\overline{A} \cap B$ (b) $A \cap \overline{B}$ (c) $\overline{A} \cap \overline{B}$ (d) $\overline{A \cap B}$
- Which of the following is not correct?
 (a) $A \subseteq A'$ if and only if $A = \phi$
 (b) $A' \subseteq A$ if and only if $A = X$, where X is the universal set
 (c) If $A \cup B = A \cup C$, then $B = C$
 (d) $B = C$ if and only if $A \cup B = A \cup C$ and $A \cap B = A \cap C$
- The set $(A \cup B \cup C) \cap (A \cap B' \cap C')' \cap C'$ is equal to
 (a) $B \cap C'$ (b) $A \cap C$
 (c) $B' \cap C'$ (d) None of these
- If $A \cup B = A \cup C$ and $A \cap B = A \cap C$, then
 (a) $B = C$ only when $A \subseteq B$
 (b) $B = C$ only when $A \subseteq C$
 (c) $B = C$
 (d) None of the above
- If $aN = \{an : n \in N\}$, $bN = \{nb : n \in N\}$, $cN = \{cN : n \in N\}$ and $bN \cap cN = dN$, where $a, b, c \in N$ and b, c are coprime, then
 (a) $b = cd$ (b) $c = bd$
 (c) $d = bc$ (d) None of these
- If $A = \{\theta : 2\cos^2 \theta + \sin \theta \leq 2\}$ and $B = \left\{\theta : \frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}\right\}$, then $A \cap B$ is equal to
 (a) $\left\{\theta : \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6}\right\}$
 (b) $\left\{\theta : \pi \leq \theta \leq \frac{3\pi}{2}\right\}$
 (c) $\left\{\theta : \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6} \text{ or } \pi \leq \theta \leq \frac{3\pi}{2}\right\}$
 (d) None of the above
- Let $A = \{x : x \text{ is a digit in the number } 3591\}$, $B = \{x : x \in N, x < 10\}$. Which of the following is false?
 (a) $A \cap B = \{1, 3, 5, 9\}$ (b) $A - B = \phi$
 (c) $B - A = \{2, 4, 6, 7, 8\}$ (d) $A \cup B = \{1, 2, 3, 5, 9\}$
- Let A and B be two sets, such that $A \cup B = A$. Then, $A \cap B$ is equal to
 (a) ϕ (b) B
 (c) A (d) None of these
- Let A and B be two sets, then $(A \cup B)' \cup (A' \cap B)$ is equal to
 (a) A' (b) A
 (c) B' (d) None of these
- Let U be the universal set and $A \cup B \cup C = U$. Then, $\{(A - B) \cup (B - C) \cup (C - A)\}'$ is equal to
 (a) $A \cup B \cup C$
 (b) $A \cup (B \cap C)$
 (c) $A \cap B \cap C$
 (d) $A \cap (B \cup C)$
- If A and B are two sets, then $(A - B) \cup (B - A) \cup (A \cap B)$ is equal to
 (a) $A \cup B$ (b) $A \cap B$
 (c) A (d) B'
- If $A = \{y : y = 2x, x \in N\}$, $B = \{y : y = 2x - 1, x \in N\}$, then $(A \cap B)'$ is
 (a) A (b) B (c) ϕ (d) U
- If $n(A) = 4$ and $n(B) = 7$, then the minimum and maximum value of $n(A \cup B)$ are, respectively
 (a) 4 and 11 (b) 4 and 7
 (c) 7 and 11 (d) None of these



Sets

20. Let X be the universal set for sets A and B . If $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B')$ is equal to 300, provided (X) is equal to
(a) 600 (b) 700
(c) 800 (d) 900
21. Suppose $A_1, A_2, \dots, A_{30}$ are thirty sets each with five elements and $B_1, B_2, \dots, B_n$ are n sets each with three elements. Let $\bigcup_{i=1}^{30} A_i = \bigcup_{j=1}^n B_j = S$. Assume that each element of S belongs to exactly 10 of the A_i 's and exactly 9 of B_j 's. The value of n must be
(a) 30 (b) 40
(c) 45 (d) 50
22. If there are three athletic teams in a school, 21 are in the basketball team, 26 in hockey team and 29 in the football team. 14 play hockey and basketball, 15 play hockey and football, 12 play football and basketball and 8 play all the games. The total number of members is
(a) 42 (b) 43
(c) 45 (d) None of these
23. In a certain town, 25% families own a phone and 15% own a car, 65% families own neither a phone nor a car. 2000 families own both a car and a phone. Consider the following statements in this regard :
I. 10% families own both a car and a phone.
II. 35% families own either a car or a phone.
III. 40000 families live in the town.
Which of the above statements are correct?
(a) I and II (b) I and III
(c) II and III (d) I, II and III
24. 20 teachers of a school either teach Mathematics or Physics. 12 of them teach Mathematics, while 4 teach both the subject. Then, the number of teachers teaching Physics only is
(a) 12 (b) 8
(c) 16 (d) None of these
25. A survey shows that 63% of the Americans like cheese whereas 76% like apples. If $x\%$ of the Americans like both cheese and apples, then we have
(a) $x \geq 39$ (b) $x \leq 63$
(c) $39 \leq x \leq 63$ (d) None of these

Type 2. More than One Correct Option

26. If $A = \{1, 5, 7, 9\}$, $B = \{2, 5\}$, then $A - B$ is equal to
(a) $\{1, 7, 9\}$
(b) $\{1, 7, 9, 2\}$
(c) $\overline{A \cap B}$
(d) $A \cap \overline{B}$
27. If $X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$, then
(a) the smallest set of X is $\{3, 5, 9\}$
(b) the smallest set of X is $\{2, 3, 5, 9\}$
(c) the largest set of X is $\{1, 2, 3, 5, 9\}$
(d) the largest set of X is $\{2, 3, 4, 9\}$
28. In a group of 50 students, the number of students studying French, English, Sanskrit were found to be as follows
French = 17; English = 13; Sanskrit = 15;
French and English = 09; English and Sanskrit = 4;
French and Sanskrit = 5; English, French and Sanskrit = 3. The number of students who study
(a) French only is 6
(b) Sanskrit only is 8
(c) French and Sanskrit but not English is 2
(d) atleast one of the three language is 30

Type 3. Assertion and Reason

Directions (Q. Nos. 29-31) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

29. **Statement I** If $A = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21\}$, $B = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$ and N is the universal set, then $A' \cup ((A \cup B) \cap B') = A$
Statement II $(A \cup B) \cap B'$ is always A .
30. **Statement I** If A, B and C are any three non-empty sets, then $A - (B \cup C) = (A - B) \cap (A - C)$
Statement II $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$
31. **Statement I** If A, B and C any three sets and U is universal set, such that $n(U) = 700$, $n(A) = 200$, $n(B) = 300$ and $n(A \cap B) = 100$, then $n(A' \cap B') = 300$.
Statement II $n(A' \cap B') = n(U) - n(A \cap B)$

Type 4. Linked Comprehension Based Questions

Passage (Q. Nos. 32-34) Out of 100 students; 15 passed in English, 12 passed in Mathematics, 8 in Science, 6 in English and Mathematics, 7 in Mathematics and Science, 4 in English and Science, 4 in all the three passed.

32. The number of students passed in English and Mathematics but not in Science is
(a) 3 (b) 2 (c) 4 (d) 5

33. The number of students only passed in Mathematics is
(a) 5 (b) 4
(c) 3 (d) 2

34. The number of students only passed in more than one subject is
(a) 9 (b) 3
(c) 2 (d) 1

Type 5. Match the Column

35. Match the following sets for all sets A , B and C :

Column I	Column II
A. $[(A' \cup B') - A]'$	p. $A - B$
B. $[B' \cup (B' - A)]'$	q. A
C. $A - (A \cap B)$	r. B
D. $(A - B) \cap (C - B)$	s. $(A \cap C) - B$

Type 6. Single Integer Answer Type Questions

36. If $A = \{a, b, c\}$, then the number of proper subsets of A is equal to _____ .
37. If $A = \{x \in C : x^2 = 1\}$ and $B = \{x \in C : x^4 = 1\}$, then number of elements in $(A - B)$ is _____ .
38. If A and B are two sets containing 3 and 6 elements, respectively. The maximum number of elements in $A \cup B$ is equal to _____ .
39. A college awarded 38 medals in football, 15 in basketball and 20 in cricket. If these medals went to a total of 58 men and only three men got medals in all the three sports, how many received medals in exactly two of three sports?

Entrances Gallery

JEE Advanced/IIT JEE

1. Let $P = \{\theta : \sin \theta - \cos \theta = \sqrt{2} \cos \theta\}$ and $Q = \{\theta : \sin \theta + \cos \theta = \sqrt{2} \sin \theta\}$ be two sets. Then,
(a) $P \subset Q$ and $Q - P \neq \phi$ (b) $Q \subset P$ [2011]
(c) $P \not\subset Q$ (d) $P = Q$
2. Let $S = \{1, 2, 3, 4\}$. The total number of unordered pairs of disjoint subsets of S is equal to [2010]
(a) 25 (b) 34
(c) 42 (d) 41

JEE Main/AIEEE

3. Let A and B be two sets containing four and two elements respectively. Then, the number of subsets of the set $A \times B$, each having atleast three elements, is [2015]
(a) 219 (b) 256 (c) 275 (d) 510
4. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to [2014]
(a) N (b) $Y - X$
(c) X (d) Y
5. If A , B and C are three sets, such that $A \cap B = A \cap C$ and $A \cup B = A \cup C$, then [2009]
(a) $A = C$
(b) $B = C$
(c) $A \cap B = \phi$
(d) $A = B$

Other Engineering Entrances



Sets

6. In a class of 60 students, 25 students play cricket and 20 students play tennis and 10 students both the games, then the number of students who play neither is [Karnataka CET 2014]
(a) 45 (b) 0 (c) 25 (d) 35

7. The set $A = \{x : |2x + 3| < 7\}$ is equal to the set [Karnataka CET 2014]
(a) $D = \{x : 0 < (x + 5) < 7\}$ (b) $B = \{x : -3 < x < 7\}$
(c) $E = \{x : -7 < x < 7\}$ (d) $C = \{x : 13 < 2x < 4\}$

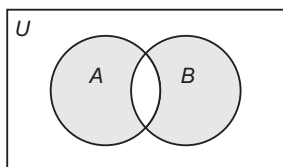
8. There is a group of 265 persons who like either singing or dancing or painting. In this group 200 like singing, 110 like dancing and 55 like painting. If 60 persons like both singing and dancing, 30 like both singing and painting and 10 like all three activities, then the number of persons who like only dancing and painting is [WB JEE 2014]
(a) 10 (b) 20 (c) 30 (d) 40

9. Let $X_n = \left\{ z = x + iy : |z|^2 \leq \frac{1}{n} \right\}$ for all integers $n \geq 1$.

Then, $\bigcap_{n=1}^{\infty} X_n$ is [WB JEE 2014]

- (a) a singleton set
(b) not a finite set
(c) an empty set
(d) a finite set with more than one element

10. The shaded region in the figure represents [Kerala CEE 2014]



- (a) $A \cap B$ (b) $A \cup B$ (c) $B - A$ (d) $A - B$
(e) $(A - B) \cup (B - A)$

11. If $n(A) = 1000$, $n(B) = 500$ and if $n(A \cap B) \geq 1$ and $n(A \cup B) = p$, then [Kerala CEE 2012]
(a) $500 \leq p \leq 1000$ (b) $1001 \leq p \leq 1498$
(c) $1000 \leq p \leq 1499$ (d) $999 \leq p \leq 1499$
(e) $1000 \leq p \leq 1498$

12. Out of 64 students, the number of students taking Mathematics is 45 and number of students taking both Mathematics and Biology is 10. Then, the number of students taking only Biology is [OJEE 2012]
(a) 18 (b) 19
(c) 20 (d) None of these

13. There are 100 students in a class. In the examination, 50 of them failed in Mathematics, 45 failed in Physics, 40 failed in Biology and 32 failed in exactly two of the three subjects. Only one student passed in all the subjects. Then, the number of students failing in all the three subjects is [WB JEE 2012]
(a) 12 (b) 4
(c) 2 (d) Cannot be determined

14. The set $A = \{x : x \in R, x^2 = 16 \text{ and } 2x = 6\}$ is equal to [BITSAT 2011]
(a) ϕ (b) $\{14, 3, 4\}$ (c) $\{3\}$ (d) $\{4\}$

15. Let A and B be two sets, then $(A \cup B)' \cap (A' \cap B)$ is equal to [GGSIU 2011]
(a) A' (b) A
(c) B' (d) None of these

16. Let $A = \{1, 2\}$, $B = \{\{1\}, \{2\}\}$, $C = \{\{1, 2\}\}$. Then which of the following relation is true? [J&K CET 2011]
(a) $A = B$ (b) $B \subseteq C$
(c) $A \in C$ (d) $D \subset C$

17. 25 people employed for programme A , 50 people for programme B , 10 people for both. So, number of employee employed for only A is [OJEE 2011]
(a) 15 (b) 20 (c) 35 (d) 40

Target Exercises

Answers

Work Book Exercise

1. (b) 2. (c) 3. (b) 4. (a) 5. (c) 6. (c) 7. (b) 8. (b) 9. (b) 10. (a)

Target Exercises

1. (a) 2. (a) 3. (d) 4. (d) 5. (a) 6. (b) 7. (b) 8. (c) 9. (a) 10. (c)
11. (c) 12. (c) 13. (d) 14. (b) 15. (a) 16. (c) 17. (a) 18. (d) 19. (c) 20. (b)
21. (c) 22. (b) 23. (c) 24. (b) 25. (c) 26. (a,d) 27. (a,c) 28. (a,c,d) 29. (c) 30. (b)
31. (c) 32. (b) 33. (c) 34. (a) 35. (*) 36. (7) 37. (0) 38. (9) 39. (9)

*A $\rightarrow$ q; B $\rightarrow$ r; C $\rightarrow$ p; D $\rightarrow$ s

Entrances Gallery

1. (d) 2. (d) 3. (a) 4. (d) 5. (b) 6. (c) 7. (a) 8. (a) 9. (a) 10. (e)
11. (c) 12. (b) 13. (c) 14. (a) 15. (d) 16. (c) 17. (a)

Explanations

Target Exercises

1. Since, $3^x - 1$ is an even number for all $x \in \mathbb{Z}^+$. So, the given set in roster form is $\{1, 2, 3, 4, 5\}$.

2. Given equation can be written as

$$x^2(x^2 - x - 1) = 0 \Rightarrow x = 0, \frac{1 \pm \sqrt{5}}{2}$$

$\therefore$ There is no natural value of x , which satisfies

$$x^4 - x^3 - x^2 = 0$$

3. $A = \{1, 2, 3\}$

$$B = \{x \in \mathbb{R} : x^2 - 2x + 1 = 0\} = \{1\}$$

$$C = \{1, 2, 3\}$$

$$D = \{x \in \mathbb{R} : x^3 - 6x^2 + 11x - 6 = 0\} = \{1, 2, 3\}$$

$\therefore A, C$ and D are equal.

4. a is not an element of $\{\{a\}, b\}$

$$\therefore a \notin \{\{a\}, b\}$$

$\{b, c\}$ is the element of $\{a, \{b, c\}\}$

$$\therefore \{b, c\} \in \{a, \{b, c\}\}$$

$$b \in \{a, b\} \text{ but } b \notin \{a, \{b, c\}\}$$

$$\therefore \{a, b\} \not\subset \{a, \{b, c\}\}$$

5. Let $x \notin P$

$\Rightarrow x$ is a composite number.

Let us now assume that $x \in S$

$$\Rightarrow 2^x - 1 = m \quad [\text{where, } m \text{ is a prime number}]$$

$$\Rightarrow 2^x = m + 1$$

which is not true for all composite numbers, say for $x = 4$ because $2^4 = 16$ which cannot be equal to the sum of any prime number m and 1.

Thus, we arrive at a contradiction

$$\Rightarrow x \notin S$$

$$\text{So, } S \subset P$$

6. Given, $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$$A = \{2, 4, 6, 8\}$$

$$\text{and } B = \{2, 3, 5, 7\}$$

$$\text{Now, } A' = U - A$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 4, 6, 8\}$$

$$= \{1, 3, 5, 7, 9\}$$

$$\text{and } B' = U - B$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\} - \{2, 3, 5, 7\}$$

$$= \{1, 4, 6, 8, 9\}$$

$$(i) A \cup B = \{2, 4, 6, 8\} \cup \{2, 3, 5, 7\}$$

$$= \{2, 3, 4, 5, 6, 7, 8\}$$

$$\therefore (A \cup B)' = U - A \cup B$$

$$= \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$- \{2, 3, 4, 5, 6, 7, 8\}$$

$$= \{1, 9\}$$

$$(ii) (A' \cap B') = \{1, 3, 5, 7, 9\} \cap \{1, 4, 6, 8, 9\}$$

$$= \{1, 9\}$$

$$(iii) \text{ Now, } A - B = \{2, 4, 6, 8\} - \{2, 3, 5, 7\}$$

$$= \{4, 6, 8\}$$

$$\text{and } B - A = \{2, 3, 5, 7\} - \{2, 4, 6, 8\}$$

$$= \{3, 5, 7\}$$

$$\therefore A \Delta B = (A - B) \cup (B - A)$$

$$= \{4, 6, 8\} \cup \{3, 5, 7\} = \{3, 4, 5, 6, 7, 8\}$$

7. $A - B = A \cap \bar{B}$ is a general result.

8. Options (a) and (b) are trivially true. Option (c) is incorrect because if $A = \{2, 3, 4\}$, $B = \{3\}$, $C = \{4\}$, then $A \cup B = A \cup C$ but $B \neq C$. Option (d) is also correct.

9. $(A \cup B \cup C) \cap (A \cap B' \cap C') \cap C'$

$$= (A \cup B \cup C) \cap (A' \cup B \cup C) \cap C'$$

$$= [(A \cap A') \cup (B \cup C)] \cap C'$$

$$= (\phi \cup B \cup C) \cap C'$$

$$= (B \cap C') \cup (C \cap C')$$

$$= (B \cap C') \cup \phi = B \cap C'$$

10. Let $x \in B \Rightarrow x \in A \cup B \Rightarrow x \in A \cup C$

$$\text{Case I } x \in A$$

$$\therefore x \in A \cap B \text{ or } x \in A \cap C \text{ or } x \in C$$

$$\therefore B \subseteq C$$

$$\text{Case II } x \in C$$

$$\therefore x \in B \Rightarrow x \in C \text{ or } B \subseteq C$$

$$\text{Similarly, } C \subseteq B$$

$$\therefore B = C$$

11. We have, $bN = \{nb : n \in \mathbb{N}\}$, $cN = \{cn : n \in \mathbb{N}\}$

$$\text{and } dN = \{dn : n \in \mathbb{N}\}$$

$$\text{Also, } bN \cap cN = dN$$

$$\therefore bc \in bN \cap cN \text{ or } bc \in dN$$

$$\therefore bc = d \quad [\text{as } b \text{ and } c \text{ are coprime}]$$

12. Let $2\cos^2\theta + \sin\theta \leq 2$ and $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$

$$\Rightarrow 2 - 2\sin^2\theta + \sin\theta \leq 2$$

$$\Rightarrow 2\sin^2\theta - \sin\theta \geq 0 \Rightarrow \sin\theta(2\sin\theta - 1) \geq 0$$

$$\Rightarrow \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6} \text{ or } \pi \leq \theta \leq \frac{3\pi}{2}$$

$$\therefore A \cap B = \left\{ \theta : \frac{\pi}{2} \leq \theta \leq \frac{5\pi}{6} \text{ or } \pi \leq \theta \leq \frac{3\pi}{2} \right\}$$

13. We have, $A = \{1, 3, 5, 9\}$

$$\text{and } B = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

$$\text{Now, find } A \cup B, A \cap B, A - B, B - A$$

$\therefore$ Option (d) is false.

14. $A \cup B = A \Rightarrow B \subseteq A \Rightarrow A \cap B = B$

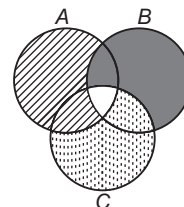
15. $(A \cup B)' \cup (A' \cap B) = (A' \cap B') \cup (A' \cap B)$

$$= (A' \cup A') \cap (A' \cup B) \cap (B' \cup A') \cap (B' \cup B)$$

$$= A' \cap \{A' \cup (B \cap B')\} \cap U = A' \cap (A' \cup \phi) \cap U$$

$$= A' \cap A' \cap U = A' \cap U = A'$$

16. $(A - B) \cup (B - C) \cup (C - A)$ is represented by the shaded portion in the figure. The unshaded portion is $A \cap B \cap C$.



$$\therefore \{(A - B) \cup (B - C) \cup (C - A)\}' = A \cap B \cap C$$

17. Draw the Venn diagram. From the Venn diagram,
 $(A - B) \cup (B - A) \cup (A \cap B) = A \cup B$

18. $A = \{2, 4, 6, 8, \dots\}$, $B = \{1, 3, 5, 7, \dots\}$
 $A \cap B = \phi$
 $(A \cap B)' = U$

19. $n(A \cup B)$ is minimum when $A \subseteq B$.

In this case, $A \cup B = B$
 and we have $n(A \cup B) = n(B) = 7$
 $n(A \cup B)$ is maximum when $A \cap B = \phi$.
 In this case, $n(A \cup B) = n(A) + n(B)$
 $= 4 + 7 = 11$

20. We have, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
 $\therefore n(A \cup B) = 200 + 300 - 100 = 400$
 Also, $n(A' \cap B') = n((A \cup B)') = n(X) - n(A \cup B)$
 $\Rightarrow 300 = n(X) - 400 \Rightarrow n(X) = 700$

21. Since, $S = \bigcup_{i=1}^{30} A_i$ and each element of S is in $10A_i$'s.

We have, $n(S) = \frac{1}{10} \sum_{i=1}^{30} n(A_i) = \frac{1}{10} (30 \times 5) = 15$

Also, $S = \bigcup_{j=1}^{30} B_j$ and each element of S is in $9 B_j$'s.

We have, $n(S) = \frac{1}{9} \sum_{j=1}^n n(B_j)$

$\Rightarrow 15 = \frac{1}{9} (n \times 3) \Rightarrow n = 45$

22. Let B, H, F be the sets of members in the basketball team, hockey team, football team, respectively.

$\therefore n(B) = 21, n(H) = 26, n(F) = 29$
 $n(H \cap B) = 14, n(H \cap F) = 15$
 $n(F \cap B) = 12, n(B \cap H \cap F) = 8$
 $\therefore n(B \cup H \cup F) = n(B) + n(H) + n(F)$
 $- n(B \cap H) - n(H \cap F) - n(B \cap F)$
 $+ n(B \cap H \cap F)$
 $= 21 + 26 + 29 - 14 - 15 - 12 + 8 = 43$

23. Let X, P, C denote the sets of all families, families owning phone, families owning car, respectively.

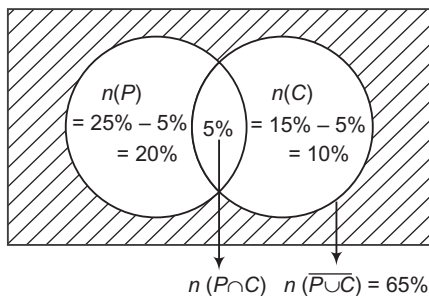
Let total number of families be k .
 Since, $n(P \cap C) = \frac{k \times 5}{100} = \frac{k}{20} \Rightarrow 2000 = \frac{k}{20}$

$\Rightarrow k = 40000$

$\therefore$ Statement III is correct.

Now, $n(C \cap P) = 2000 = \frac{2000}{40000} \times 100\% = 5\%$

$\therefore$ Statement I is incorrect.



$\therefore$ It is clear from the Venn diagram that Statement II is correct.

$\therefore$ The correct answer is (c).

24. $n(M \cup P) = 20, n(M) = 12, n(M \cap P) = 4$

$\therefore n(M \cup P) = n(M) + n(P) - n(M \cap P)$
 $\Rightarrow 20 = 12 + n(P) - 4$
 $\therefore n(P) = 12$

So, the required number
 $= n(P) - n(M \cap P) = 12 - 4 = 8$

25. $n(X) = 100, n(C) = 63, n(A) = 76, n(C \cap A) = x$

Now, $n(C \cup A) = n(C) + n(A) - n(C \cap A)$
 $\therefore n(C \cup A) = 63 + 76 - x = 139 - x$

$\therefore 139 - x \leq 100$ or $x \geq 39$

Also, $C \cap A \subseteq C$ and $C \cap A \subseteq A$

$\therefore n(C \cap A) \leq n(C)$

and $n(C \cap A) \leq n(A)$

$\Rightarrow n(C \cap A) \leq 63$

and $n(C \cap A) \leq 76$

$\Rightarrow n(C \cap A) \leq 63$ i.e. $x \leq 63$

$\therefore 39 \leq x \leq 63$

26. Given, $A = \{1, 5, 7, 9\}, B = \{2, 5\}$

$A - B$ = Set of all those elements of A , which do not belong to B

$= \{1, 7, 9\}$

Also, $A - B = A \cap \bar{B}$ is a general result.

Hence, options (a) and (d) are correct.

27. Given, $X \cup \{1, 2\} = \{1, 2, 3, 5, 9\}$

The smallest set of $X = \{1, 2, 3, 5, 9\} - \{1, 2\} = \{3, 5, 9\}$

The largest set of X = The set which contains atleast

elements 3, 5, 9

$= \{1, 2, 3, 5, 9\}$

28. Given, $n(F) = 17, n(E) = 13, n(S) = 15, n(F \cap E) = 9,$

$n(E \cap S) = 4, n(F \cap S) = 5, n(E \cap F \cap S) = 3$

(a) Number of students study only French

$= n(F) - n(F \cap E) - n(F \cap S) + n(F \cap E \cap S)$
 $= 17 - 9 - 5 + 3 = 6$

(b) Number of students study only Sanskrit

$= n(S) - n(S \cap F) - n(S \cap E) + n(F \cap E \cap S)$
 $= 15 - 5 - 4 + 3 = 9$

(c) $n(F \cap S \cap \bar{E}) = n(F \cap S) - n(F \cap E \cap S)$

$= 5 - 3 = 2$

(d) $\therefore n(F \cup E \cup S) = n(F) + n(E) + n(S)$

$- n(F \cap E) - n(E \cap S) - n(S \cap F)$
 $+ n(F \cap E \cap S)$
 $= 17 + 13 + 15 - 9 - 4 - 5 + 3 = 30$

29. In Statement I, $A \cap B = \phi$

$\Rightarrow (A \cup B) \cap B' = A$

$A' \cup ((A \cup B) \cap B') = A' \cup A = U$

Statement I is true and Statement II is false.

Option (c) is true.

30. $A - (B \cup C) = A \cap (B \cup C)' = A \cap (B' \cap C')$

$= (A \cap B') \cap (A \cap C')$

$= (A - B) \cap (A - C)$

Statement I is true.

$A \cap (B \Delta C) = A \cap \{(B - C) \cup (C - B)\}$

$= \{A \cap (B - C)\} \cup \{A \cap (C - B)\}$

$= \{(A \cap B) - (A \cap C)\} \cup \{(A \cap C) - (A \cap B)\}$

$= (A \cap B) \Delta (A \cap C)$

Statement II is true.

Option (b) is true.

$$\begin{aligned} 31. \text{ Since, } n(A' \cap B') &= n(U) - n(A \cup B) \\ &= n(U) - \{n(A) + n(B) - n(A \cap B)\} \\ &= 700 - \{200 + 300 - 100\} \\ &= 300 \end{aligned}$$

$\therefore$ Statement I is true and Statement II is false.
Option (c) is true.

Solutions (Q. Nos. 32-34)

Let U, E, M and S be denote the total number of students passed in English, passed in Mathematics and passed in Science, respectively.

$$\begin{aligned} \text{Here, } n(U) &= 100, n(E) = 15, n(M) = 12, \\ n(S) &= 8, n(E \cap M) = 6, n(M \cap S) = 7, \\ n(E \cap S) &= 4 \text{ and } n(E \cap M \cap S) = 4 \end{aligned}$$

32. The number of students passed in English and Mathematics but not in Science

$$\begin{aligned} &= n(E \cap M \cap \bar{S}) \\ &= n(E \cap M) - n(E \cap M \cap S) = 6 - 4 = 2 \end{aligned}$$

33. The number of students only passed in Mathematics

$$\begin{aligned} &= n(M \cap \bar{E} \cap \bar{S}) \\ &= n(M) - n(M \cap E) - n(M \cap S) + n(M \cap E \cap S) \\ &= 12 - 6 - 7 + 4 = 16 - 13 = 3 \end{aligned}$$

34. The number of students only passed in more than one subject

$$\begin{aligned} &= n(M \cap E) + n(M \cap S) + n(S \cap E) - 2n(M \cap E \cap S) \\ &= 6 + 7 + 4 - 2(4) = 17 - 8 = 9 \end{aligned}$$

35. A. $[(A' \cup B') - A]' = [(A \cap B)' \cap A']'$

$$= (A \cap B) \cup A = A$$

B. $[B' \cup (B' - A)]' = B' \cup (B' \cap A')'$

$$\begin{aligned} &= [B' \cup (B \cup A)]' \\ &= [B \cap (B \cup A)] = B \end{aligned}$$

$$\begin{aligned} \text{C. } A - (A \cap B) &= A \cap (A \cap B)' \\ &= A \cap (A' \cup B') \\ &= (A \cap A') \cup (A \cap B') \\ &= \phi \cup (A \cap B') \\ &= A \cap B' = A - B \end{aligned}$$

$$\begin{aligned} \text{D. } (A - B) \cap (C - B) &= (A \cap B') \cap (C \cap B') \\ &= (A \cap C) \cap B' = (A \cap C) - B \end{aligned}$$

36. We know that the total number of proper subsets of a finite set containing n elements is $2^n - 1$.

$$\therefore \text{ Number of proper subsets of } A = 2^3 - 1 = 7$$

37. $A = \{-1, 1\}, B = \{-i, i, -1, 1\}$

$$A - B = \phi$$

$$\therefore n(A - B) = 0$$

38. The maximum number of elements in $A \cup B$ iff

$$n(A \cap B) = 0$$

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B) = 3 + 6 = 9$$

39. We have, $n(F) = 38, n(B) = 15, n(C) = 20,$

$$n(F \cup B \cup C) = 58, n(F \cap B \cap C) = 3$$

$$\begin{aligned} \text{Now, } n(F \cup B \cup C) &= n(F) + n(B) + n(C) \\ &\quad - n(F \cap B) - n(F \cap C) - n(B \cap C) \\ &\quad + n(F \cap B \cap C) \end{aligned}$$

$$\begin{aligned} \Rightarrow n(F \cap B) + n(B \cap C) + n(F \cap C) &= 38 + 15 + 20 + 3 - 58 = 18 \end{aligned}$$

Now, number of men who received medals in exactly two of the three sports

$$\begin{aligned} &= n(F \cap B) + n(B \cap C) + n(F \cap C) - 3n(F \cap B \cap C) \\ &= 18 - 3 \times 3 = 18 - 9 = 9 \end{aligned}$$

Entrances Gallery

$$\begin{aligned} 1. \text{ Since, } \cos \theta (\sqrt{2} + 1) &= \sin \theta \\ \Rightarrow \tan \theta &= \sqrt{2} + 1 \\ \text{and } \sin \theta (\sqrt{2} - 1) &= \cos \theta \\ \Rightarrow \tan \theta &= \frac{1}{\sqrt{2} - 1} \times \frac{\sqrt{2} + 1}{\sqrt{2} + 1} = (\sqrt{2} + 1) \\ \therefore P &= Q \end{aligned}$$

2. Let $A \cap B = \phi, A, B \subset S$

$$\begin{aligned} \therefore \text{Total number of unordered pairs of disjoint subsets of } S &= \frac{3^4 + 1}{2} = 41 \end{aligned}$$

3. Given, $n(A) = 4, n(B) = 2$

$$\begin{aligned} \Rightarrow n(A \times B) &= 8 \\ \text{Total number of subsets of set } (A \times B) &= 2^8 \end{aligned}$$

Number of subsets of set $(A \times B)$ having no element (i.e. ϕ) = 1

Number of subsets of set $(A \times B)$ having one element = ${}^8C_1$

Number of subsets of set $(A \times B)$ having two elements = ${}^8C_2$

$\therefore$ Number of subsets having atleast three elements

$$\begin{aligned} &= 2^8 - (1 + {}^8C_1 + {}^8C_2) \\ &= 2^8 - 1 - 8 - 28 = 2^8 - 37 \\ &= 256 - 37 = 219 \end{aligned}$$

4. We have, $X = \{4^n - 3n - 1 : n \in N\}$

$$X = \{0, 9, 54, 243, \dots\} \quad [\text{put } n = 1, 2, 3, \dots]$$

$$Y = \{9(n-1) : n \in N\}$$

$$Y = \{0, 9, 18, 27, \dots\} \quad [\text{put } n = 1, 2, 3, \dots]$$

It is clear that $X \subset Y$

$$\therefore X \cup Y = Y$$

5. Since, $A \cap B = A \cap C$

$$\text{and } A \cup B = A \cup C \Rightarrow B = C$$

6. Let the number of students play cricket = C

Number of students play tennis = T

and total number of students = S

$$\therefore n(S) = 60, n(C) = 25, n(T) = 20 \text{ and } n(C \cap T) = 10$$

$$\begin{aligned} \text{Now, } n(C \cup T) &= n(C) + n(T) - n(C \cap T) \\ &= 25 + 20 - 10 = 35 \end{aligned}$$

$\therefore$ The number of students who play neither game

$$\begin{aligned} &= n(C \cup T)' = n(S) - n(C \cup T) \\ &= 60 - 35 = 25 \end{aligned}$$

7. Given, set $A = \{x : |2x + 3| < 7\}$

$$\text{Now, } |2x + 3| < 7$$

$$\Rightarrow -7 < 2x + 3 < 7$$

$$\Rightarrow -7 - 3 < 2x < 7 - 3$$

$$\Rightarrow -10 < 2x < 4$$

$$\Rightarrow -5 < x < 2$$

$$\Rightarrow 0 < (x + 5) < 7$$

8. Let D, P, S denote dancing, painting and singing respectively.

$$\begin{aligned}\therefore n(D \cup P \cup S) &= 265, \\ n(S) &= 200, n(D) = 110, n(P) = 55, \\ n(S \cap D) &= 60, n(S \cap P) = 30 \\ \text{and } n(D \cap P \cap S) &= 10 \\ \therefore n(D \cup P \cup S) &= n(D) + n(P) + n(S) - n(D \cap P) \\ &\quad - n(P \cap S) - n(S \cap D) + n(D \cap P \cap S) \\ \therefore 265 &= 110 + 55 + 200 - n(D \cap P) \\ &\quad - 30 - 60 + 10 \\ \Rightarrow 265 &= 285 - n(D \cap P) \\ \Rightarrow n(D \cap P) &= 20\end{aligned}$$

$$\begin{aligned}\therefore \text{Number of persons who like dancing and painting} \\ &= n(D \cap P) - n(D \cap P \cap S) \\ &= 20 - 10 \\ &= 10\end{aligned}$$

9. Given, $X_n = \left\{ z = x + iy : |z|^2 \leq \frac{1}{n} \right\} = \left\{ x^2 + y^2 \leq \frac{1}{n} \right\}$

$$\begin{aligned}\therefore X_1 &= \{x^2 + y^2 \leq 1\}, X_2 = \left\{x^2 + y^2 \leq \frac{1}{2}\right\} \\ X_3 &= \left\{x^2 + y^2 \leq \frac{1}{3}\right\}, \dots, X_\infty = \{x^2 + y^2 \leq 0\}\end{aligned}$$

$$\therefore \bigcap_{n=1}^{\infty} X_n = X_1 \cap X_2 \cap \dots \cap X_\infty = \{x^2 + y^2 = 0\}$$

Hence, $\bigcap_{n=1}^{\infty} X_n$ is a singleton set.

10. Given, figure clearly represents
 $(A - B) \cup (B - A)$

11. Given, $n(A) = 1000, n(B) = 500, n(A \cap B) \geq 1, n(A \cup B) = p$

$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow 1 \leq n(A \cap B) \leq 500$$

$$\text{Hence, } p \leq 1000 + 500 - 1 = 1499$$

$$\text{and } p \geq 1000 + 500 - 500 = 1000$$

$$\therefore 1000 \leq p \leq 1499$$

12. $n(M) = 45, n(M \cap B) = 10, n(M \cup B) = 64$

$$\Rightarrow n(B) = n(M \cup B) - n(M) + n(M \cap B)$$

$$= 64 - 45 + 10 = 29$$

$$\Rightarrow n(\text{only } B) = n(B) - n(M \cap B) = 29 - 10 = 19$$

13. $n(M) = 50 = \text{Number of students failed in Mathematics}$

$$n(P) = 45, n(B) = 40$$

$$\therefore n(M \cap P) + n(M \cap B) + n(P \cap B)$$

$$- 3n(M \cap P \cap B) = 32 \dots (i)$$

We have, to find $n(M \cap P \cap B)$.

$$\text{Total number of students} = 100 \quad [\text{given}]$$

$$\text{Clearly, } n(M \cup P \cup B) = 99$$

$$\Rightarrow n(M) + n(P) + n(B) - \{n(M \cap P) + n(M \cap B)$$

$$+ n(P \cap B)\} + n(M \cap P \cap B) = 99$$

$$\Rightarrow 50 + 45 + 40 - \{32 + 3n(M \cap P \cap B)\}$$

$$+ n(M \cap P \cap B) = 99 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 135 - 32 - 2n(M \cap P \cap B) = 99$$

$$\Rightarrow 2n(M \cap P \cap B) = 4$$

$$\Rightarrow n(M \cap P \cap B) = 2$$

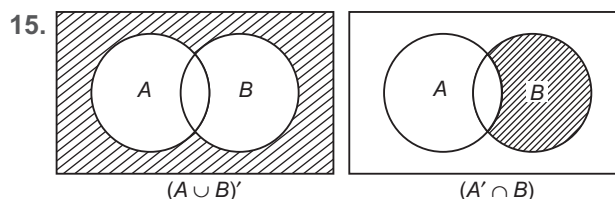
14. Since, $x^2 = 16$

$$\Rightarrow x = \pm 4 \quad \text{and} \quad 2x = 6$$

$$\Rightarrow x = 3$$

Hence, no value of x is satisfied.

$$\therefore A = \phi$$



$$\therefore (A \cup B)' \cap (A' \cap B) = \phi$$

16. $A = \{1, 2\} \in \{\{1, 2\}\} = C$

$$\therefore A \in C$$

17. $n(A - B) = n(A) - n(A \cap B)$

$$= 25 - 10 = 15$$

2

Fundamentals of Relation and Function

Ordered Pair

Two elements a and b listed in a specific order form an ordered pair, denoted by (a, b) . In an ordered pair (a, b) ; a is regarded as the first element and b is the second element. It is evident from the definition that

$$(a, b) \neq (b, a)$$

Equality of Ordered Pairs

Two ordered pairs (a_1, b_1) and (a_2, b_2) are equal iff

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2$$

$$\text{i.e.} \quad (a_1, b_1) = (a_2, b_2)$$

$$\Rightarrow \quad a_1 = a_2 \quad \text{and} \quad b_1 = b_2$$

Thus, it is evident from the definition that, $(1, 2) \neq (2, 1)$ and $(1, 1) \neq (2, 2)$.

➤ **Example 1.** If $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$, find the values of x and y .

(a) 2 and 1

(b) 2 and 2

(c) -2 and 1

(d) 1 and 1

Sol. (a) Given, $\left(\frac{x}{3} + 1, y - \frac{2}{3}\right) = \left(\frac{5}{3}, \frac{1}{3}\right)$

$$\Rightarrow \quad \frac{x}{3} + 1 = \frac{5}{3} \quad \text{and} \quad y - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow \quad \frac{x}{3} = \frac{5}{3} - 1 \quad \text{and} \quad y = \frac{1}{3} + \frac{2}{3}$$

$$x = 2 \quad \text{and} \quad y = 1$$

Cartesian Product of Sets

Let A and B be two non-empty sets. The cartesian product of A and B denoted by $A \times B$ is defined as the set of all ordered pairs (a, b) , where $a \in A$ and $b \in B$.

Symbolically, $A \times B = \{(a, b); a \in A \text{ and } b \in B\}$.

Chapter Snapshot

- Ordered Pair
- Cartesian Product of Sets
- Properties of Cartesian Product of Sets
- Relation
- Representation of Relation
- Domain and Range of Relations
- Some Particular Types of Relations
- Inverse Relation
- Composition of Relations
- Functions or Mappings
- Difference between Relation and Function
- Domain, Codomain and Range of a Function
- Equal Functions
- Classification of Functions
- Algebra of Real Functions
- Composition of Functions

If there are three sets A, B, C and $a \in A, b \in B, c \in C$, then we form an ordered triplet (a, b, c) . The set of all ordered triplets (a, b, c) is called the cartesian product of these sets A, B and C ,

i.e. $A \times B \times C = \{(a, b, c) : a \in A, b \in B, c \in C\}$

Number of Elements in the Cartesian Product of Two Sets

If A and B are two finite sets, then

$$n(A \times B) = n(A) \times n(B)$$

- If either A or B is an infinite set, then $A \times B$ is an infinite set.
- If A, B, C are finite sets, then $n(A \times B \times C) = n(A) \times n(B) \times n(C)$.

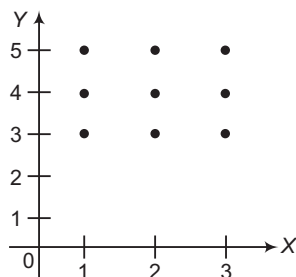
➤ **Example 2.** Let $A = \{1, 2, 3\}$ and $B = \{x : x \in N, x \text{ is a prime number less than } 5\}$, then $n(A \times B)$ is
(a) 3 (b) 4 (c) 5 (d) 6

Sol. (d) We have, $A = \{1, 2, 3\}$
and $B = \{x : x \in N, x \text{ is a prime number less than } 5\}$
 $= \{2, 3\}$
 $\therefore n(A) = 3$ and $n(B) = 2$
 $\Rightarrow n(A \times B) = n(A) \times n(B) = 3 \times 2 = 6$

Graphical Representation of Cartesian Product of Two Sets

Let A and B be two non-empty subsets of R . Then, the graph of $A \times B$ is the set of all points in the plane represented by the ordered pairs of $A \times B$.

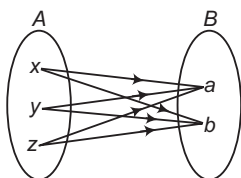
Let $A = \{1, 2, 3\}$, $B = \{3, 4, 5\}$, then $A \times B = \{(1, 3), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 3), (3, 4), (3, 5)\}$



Diagrammatic Representation of Cartesian Product of Two Sets

In order to represent $A \times B$ by an arrow diagram, we first draw Venn diagrams representing sets A and B one opposite to the other as shown in figure. Now, we draw line segments starting from each element of A and terminating to each element of set B .

If $A = \{x, y, z\}$ and $B = \{a, b\}$, then following figure gives the arrow diagram of $A \times B$.



Properties of Cartesian Product of Sets

If A, B and C are three sets, then

- $A \times (B \cup C) = (A \times B) \cup (A \times C)$
- $A \times (B \cap C) = (A \times B) \cap (A \times C)$
- $A \times (B - C) = (A \times B) - (A \times C)$
- $A \times B = B \times A \Leftrightarrow A = B$
- If $A \subseteq B \Rightarrow A \times A \subseteq (A \times B) \cap (B \times A)$
- If $A \subseteq B \Rightarrow A \times C \subseteq B \times C$
- If $A \subseteq B$ and $C \subseteq D \Rightarrow A \times C \subseteq B \times D$
- $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$
- $A \times (B' \cup C')' = (A \times B) \cap (A \times C)$
- $A \times (B' \cap C')' = (A \times B) \cup (A \times C)$
- If A and B have n common elements, then $A \times B$ and $B \times A$ will have n^2 common elements.

➤ **Example 3.** If $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$, then $A \times (B \cap C)$ is
(a) $\{(1, 3), (2, 3), (3, 3)\}$ (b) $\{(1, 4), (2, 4), (3, 4)\}$
(c) $\{(4, 1), (4, 2), (4, 3)\}$ (d) $\{(3, 4), (3, 5), (3, 6)\}$

Sol. (b) We have, $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$
Now, $B \cap C = \{4\}$
 $\therefore A \times (B \cap C) = \{(1, 4), (2, 4), (3, 4)\}$

Relation

Let A and B be two non-empty sets. Then, a relation R from A to B is a subset of $A \times B$.

Thus, R is a relation from A to $B \Rightarrow R \subseteq A \times B$. If R is a relation from a non-empty set A to a non-empty set B and if $(a, b) \in R$, then we write aRb which is read as ' a is related to b by the relation R '. If $(a, b) \notin R$, then we write $a \not R b$ and we say that ' a is not related to b by the relation R '.

➤ **Example 4.** Let $A = \{1, 2, 3\}$ and $B = \{-1, 0, 1\}$, then which of the following is not a relation from A to B ?

- $R = \{(1, -1), (2, 0), (3, 1)\}$
- $R = \{(1, -1), (1, 0), (2, 0), (2, 1)\}$
- $R = \{(2, 0), (2, 1), (1, -1), (3, 1), (3, 0)\}$
- $R = \{(1, -1), (2, 0), (1, 3)\}$

Sol. (d) $\because 3 \notin B$

$\therefore \{(1, -1), (2, 0), (1, 3)\}$ is not a relation from A to B .

➤ **Example 5.** Let $A = \{1, 2, 3\}$, we define

$$R_1 = \{(1, 2), (3, 2), (1, 3)\},$$

$$R_2 = \{(1, 3), (3, 6), (2, 1), (1, 2)\}.$$

Then, on A

(a) R_1 is relation and R_2 is not

(b) R_1 and R_2 are relations

(c) R_1 and R_2 are not relations

(d) None of the above

Sol. (a) $R_1 \subseteq A \times A$, so R_1 is a relation on A . But $(3, 6) \notin A \times A$, so $R_2 \not\subseteq A \times A$. Hence, R_2 is not a relation on A .

Representation of Relation

Roster Form

In this form, a relation is represented by the set of all ordered pairs belonging to R .

e.g. The relation R defined on the set of natural number as $\{(a, b) : b \text{ is square of } a\}$, then roster form of this relation is

$$\{(1, 1), (2, 4), (3, 9), (4, 16), \dots\}$$

Set Builder Form

In this form, a relation is represented as

$$R = \{(a, b) : a \in A, b \in B \text{ and } a, b \text{ satisfy the rule which associates } a \text{ and } b\}$$

e.g. In set builder form, the relation

$$R = \{(2, 8), (3, 27), (5, 125), (7, 343)\}$$
 is written as

$$R = \{(x, x^3) : x \text{ is a prime number less than } 10\}$$

➤ **Example 6.** Let $A = \{3, 5\}$, $B = \{7, 11\}$ and $R = \{(a, b) : a + b \text{ is odd } a \in A, b \in B\}$.

Then, R in roster form is

$$(a) \{(3, 7), (5, 11)\} \quad (b) \{(3, 11), (5, 7)\}$$

$$(c) \phi \quad (d) \text{None of these}$$

Sol. (c) We have, $A = \{3, 5\}$ and $B = \{7, 11\}$. Here, elements of A and B are odd numbers. Sum of two odd numbers is always even. So, there is no element that belongs to R . Hence, $R = \phi$

Domain and Range of Relations

Let R be a relation from A to B . The domain of R is the set of all those elements $a \in A$ such that $(a, b) \in R$ for some $b \in B$.

$$\therefore \text{Domain of } R = \{a \in A : (a, b) \in R\}$$

and range of R is the set of all those elements $b \in B$ such that $(a, b) \in R$ for some $a \in A$.

$$\therefore \text{Range of } R = \{b \in B : (a, b) \in R\}$$

Here, B is called the codomain of R .

e.g. Let $A = \{1, 2, 3\}$ and $B = \{3, 5, 6\}$

$$aRb \Leftrightarrow a < b$$

$$\text{Then, } R = \{(1, 3), (1, 5), (1, 6), (2, 3), (2, 5), (2, 6),$$

$$(3, 5), (3, 6)\}$$

$\therefore$ Domain of $R = \{1, 2, 3\}$, range of $R = \{3, 5, 6\}$ and codomain of $R = \{3, 5, 6\}$

➤ Let A and B be two non-empty finite sets having p and q elements respectively, then the total number of relations from A to $B = 2^{pq}$.

➤ **Example 7.** Find the domain and range of the relation R defined by

$$R = \{(x, x+5) : x \in \{0, 1, 2, 3, 4, 5\}\}$$

$$(a) \{0, 1, 2, 3, 4, 5\}, \{5, 6, 7, 8, 9, 10\}$$

$$(b) \{1, 2, 3, 4, 5\}, \{5, 6, 7, 8, 9, 10\}$$

$$(c) \{0, 1, 2, 3, 4, 5\}, \{6, 7, 8, 9, 10\}$$

$$(d) \text{None of the above}$$

Sol. (a) Given, $R = \{(x, x+5) : x \in \{0, 1, 2, 3, 4, 5\}\}$

Putting $x = 0, 1, 2, 3, 4, 5$, we get

$$R = \{(0, 5), (1, 6), (2, 7), (3, 8), (4, 9), (5, 10)\}$$

$$\Rightarrow \text{Domain} = \{0, 1, 2, 3, 4, 5\}$$

$$\text{Now, } y = x + 5 = 5, 6, 7, 8, 9, 10 \text{ [after putting the value of } x]$$

$$\Rightarrow \text{Range} = \{5, 6, 7, 8, 9, 10\}$$

➤ **Example 8.** If R is a relation on a finite set having n elements, then the number of relations on A is

$$(a) 2^n \quad (b) 2^{n^2} \quad (c) n^2 \quad (d) n^n$$

Sol. (b) We have, a set having n elements

$$\text{i.e. } n(A) = n$$

$$\text{and } n(A \times A) = n(A) \times n(A) = n \times n = n^2$$

$$\therefore \text{Number of relations on } A = 2^{n(A \times A)} = 2^{n^2}$$

Some Particular Types of Relations

i. **Void relation** Since, $\phi \subset A \times A$, it follows that ϕ is a relation on A , called the empty or void relation.

➤ **Example 9.** Consider the relation R on the set $A = \{2, 4, 6\}$, $R = \{(x, y) : |x - y| \text{ is odd } x, y \in A\}$

Sol. We observe that, $|x - y|$ is always an even number for every element of A . Therefore, $(x, y) \notin R, \forall x, y \in A$.

$$\Rightarrow R \text{ is a void relation on } A.$$

➤ **Example 10.** A relation R on a set A is called an empty relation, if

(a) no element of A is related to any element of A

(b) every element of A is related to every element of A

(c) some elements of A are related to some element of A

(d) None of the above

Sol. (a) A relation R on a set A is called an empty relation, if it has no element, i.e. no ordered pair, i.e. no element of A is related to any element of A .

ii. **Universal relation** Since, $A \times A \subseteq A \times A$, it follows that $A \times A$ is a relation on A , called the universal relation.

➤ **Example 11.** Consider the relation R on set $A = \{1, 2, 3, 4, 5\}$ defined by $R = \{(x, y) : |x - y| \geq 0, x, y \in A\}$.

Sol. We observed that, $|x - y| \geq 0, \forall x, y \in A$
 $\Rightarrow (x, y) \in R, \forall (x, y) \in A \times A$
 Each element of set A is related to every element of set A
 $\Rightarrow R = A \times A$
 $\Rightarrow R$ is universal relation on set A .

iii. **Identity relation** The relation $I_A = \{(a, a) : \forall a \in A\}$ is called the identity relation on A .

➤ **Example 12.** If $A = \{1, 2, 3\}$, then define the identity relation on A .

Sol. The identity relation on A is given by $I_A = \{(1, 1), (2, 2), (3, 3)\}$

Inverse Relation

If R is a relation on A , then the relation R^{-1} on A , defined by $R^{-1} = \{(b, a) : (a, b) \in R\}$ is called an inverse relation on A .

Clearly, domain $(R^{-1}) = \text{range}(R)$
 and range $(R^{-1}) = \text{domain}(R)$

➤ **Example 13.** If $A = \{1, 2, 3\}$ and $R = \{(1, 2), (2, 2), (3, 1), (3, 2)\}$, then define inverse relation of R on A .

Sol. R , being a subset of $A \times A$, is a relation on A .

Clearly, $1R2; 2R2; 3R1$ and $3R2$.

Domain $(R) = \{1, 2, 3\}$

and range $(R) = \{2, 1\}$

Also, $R^{-1} = \{(2, 1), (2, 2), (1, 3), (2, 3)\}$

Domain $(R^{-1}) = \{2, 1\}$

and range $(R^{-1}) = \{1, 2, 3\}$

Composition of Relations

Let $R \subseteq A \times B, S \subseteq B \times C$ be two relations. Then, composition of the relations R and S denoted by $SoR \subseteq A \times C$ and is defined by $(a, c) \in (SoR)$ iff $\exists b \in B$ such that $(a, b) \in R, (b, c) \in S$.

➤ **Example 14.** If $A = \{1, 2, 3\}, B = \{a, b, c, d\}, C = \{\alpha, \beta, \gamma\}, R \subseteq (A \times B) = \{(1, a), (1, c), (2, d)\}, S \subseteq (B \times C) = \{(a, \alpha), (a, \gamma), (a, \beta)\}$, then what is the composite of relations R and S ?

Sol. $SoR \subseteq (A \times C) = \{(1, \alpha), (1, \gamma), (1, \beta)\}$

➤ One should be careful in computing the relation RoS . Actually, SoR starts with R and RoS starts with S .

In general, $SoR \neq RoS$

Also, $(SoR)^{-1} = R^{-1}oS^{-1}$, known as reversal rule.

Work Book Exercise 2.1

1 If A and B are two sets, then $A \times B = B \times A$, if and only if

- a $A \subset B$ b $B \subset A$
 c $A = B$ d None of these

2 If the set A has 3 elements and the set $B = \{1, 3, 4, 5\}$, then the number of elements in $(A \times B)$ is

- a 11 b 12 c 13 d 15

3 Let $A = \{1, 2\}, B = \{1, 2, 3, 4\}, C = \{5, 6\}$ and $D = \{5, 6, 7, 8\}$

Following statements are given below:

- I. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
 II. $A \times C$ is a subset of $B \times D$.

Which of the above statement is correct?

- a Only I b Only II
 c Both I and II d None of these

4 Let R be a relation from a set A to a set B , then

- a $R = A \cup B$ b $R = A \cap B$
 c $R \subseteq A \times B$ d $R \subseteq B \times A$

5 Let R be the relation on Z defined by $R = \{(a, b) : a, b \in Z, a - b \text{ is an integer}\}$. Find the domain and range of R .

- a Z, Z b Z^+, Z
 c Z, Z^- d None of these

6 Let a relation R be defined by $R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$, then RoR is equal to

- a $\{(1, 5), (1, 6), (3, 6)\}$
 b $\{(1, 4), (1, 5), (3, 6)\}$
 c $\{(1, 5), (1, 6), (3, 7)\}$
 d $\{(1, 4), (1, 5), (3, 7)\}$

7 If the relation $R : A \rightarrow B$, where $A = \{1, 2, 3, 4\}$ and $B = \{1, 3, 5\}$ is defined by $R = \{(x, y) : x < y, x \in A, y \in B\}$, then $R^{-1}oR$ is

- a $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
 b $\{(3, 1), (5, 1), (5, 2), (5, 3), (5, 4)\}$
 c $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$
 d None of the above

8 If R is a relation from set $A = \{2, 4, 5\}$ to set $B = \{1, 2, 3, 4, 6, 8\}$ defined by $xRy \Leftrightarrow x$ divides y , where $x \in A, y \in B$. Then, R^{-1} is

- a $\{(2, 2), (2, 4), (2, 6), (2, 8), (4, 4), (4, 8)\}$
 b $\{(2, 2), (1, 4), (1, 2), (1, 5), (2, 4), (4, 4)\}$
 c $\{(2, 2), (4, 2), (6, 2), (8, 2), (4, 4), (8, 4)\}$
 d None of the above

Functions or Mappings

Let A and B be two non-empty sets. Then, a function f from set A to B is a rule which associates elements of set A to elements of set B such that all elements of set A are associated to elements of set B in unique way.

If f associates $x \in A$ to $y \in B$, then we say that y is the image of the element x and denote it by $f(x)$ and write as $y = f(x)$. The element x is called the pre-image or inverse image of y .

A function is denoted by $f : A \rightarrow B$ or $A \xrightarrow{f} B$.

- **Example 15.** Let $A = \{1, 2, 3\}$, $B = \{2, 3, 4\}$ and f_1, f_2 and f_3 be three subsets of $A \times B$ as given
- $$f_1 = \{(1, 2), (2, 3), (3, 4)\}$$
- $$f_2 = \{(1, 2), (1, 3), (2, 3), (3, 4)\}$$
- $$f_3 = \{(1, 3), (2, 4)\}. \text{ Then, which is/are function(s)?}$$

Sol. Obviously, f_1 is a function from A to B but f_2 and f_3 are not functions from A to B , f_2 is not a function from A to B because $1 \in A$ has two images 2 and 3 and f_3 is not a function from A to B , because $3 \in A$ has no image in B .

- **Example 16.** If $x, y \in \{1, 2, 3, 4\}$, then which of the following is/are function(s) in the given set?
- $$(a) f_1 = \{(x, y) : y = x + 1\}$$
- $$(b) f_2 = \{(x, y) : x + y > 4\}$$
- $$(c) f_3 = \{(x, y) : y < x\}$$
- $$(d) f_4 = \{(x, y) : x + y = 5\}$$

Sol. (d) We have,

$$f_1 = \{(1, 2), (2, 3), (3, 4)\}$$

$$f_2 = \{(1, 4), (4, 1), (2, 3), (3, 2), (2, 4), (4, 2), (3, 4), (4, 3)\}$$

$$f_3 = \{(2, 1), (3, 1), (4, 1), (3, 2), (4, 2), (4, 3)\}$$

$$f_4 = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$$

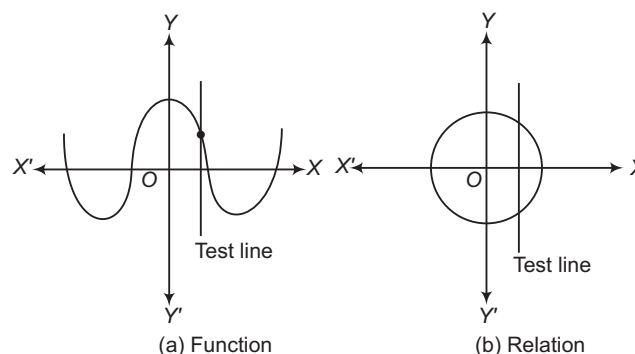
Here, f_4 is a function but f_1, f_2 and f_3 are not functions. f_1 is not a function because 4 has no image. f_2, f_3 are not functions because 3, 4 have more than one images.

- There may exist some elements in set B which are not the images of any element in set A .
- To each and every independent element in A , there corresponds one and only one image in B .
- Every function is a relation but every relation may or may not be a function.
- The number of functions from a finite set A into finite set $B = [n(B)]^{[n(A)]}$.

Difference between Relation and Function

Geometrically, if we draw a Vertical Parallel Line (VPL), i.e. any line which is parallel to Y -axis ($x = a$),

then if this line intersects the graph of the expression in more than one point, then the expression is a relation else, if it intersects at only one point, the expression is a function. e.g.



- **Example 17.** Let $A = \{1, 2, 3, 4\}$, $B = \{1, 5, 9, 11, 15, 16\}$ and $f = \{(1, 5), (2, 9), (3, 1), (4, 5), (2, 11)\}$, then
- f is a relation from A to B
 - f is a function from A to B
 - Both (a) and (b)
 - None of the above

Sol. (a) Since, first elements of the ordered pairs in f belong to A and second elements of the ordered pairs belong to B , so f is relation from A to B .

Now, 2 has two different images 9 and 11, so f is not a function.

Domain, Codomain and Range of a Function

Let $f : A \rightarrow B$. Then, the set A is known as the domain of f and the set B is known as the codomain of f . The set of all f -images of elements of A is known as the image of A or image set of A under f and is denoted by $f(A)$.

$$\text{Thus, } f(A) = \{f(x) : x \in A\} = \text{Range of } f$$

$$\text{Clearly, } f(A) \subseteq B$$

- **Example 18.** Let N be the set of all natural numbers, then the rule $f : N \rightarrow N$ such that $f(x) = x^2, \forall x \in N$. Write domain, range and codomain of the given function.

Sol. Given, $f(x) = x^2$

$$f(1) = (1)^2 = 1; f(2) = 2^2 = 4;$$

$$f(3) = 3^2 = 9 \text{ and so on.}$$

$$\therefore \text{Domain } (f) = N$$

$$\text{Range } (f) = \{1, 4, 9, \dots\}$$

$$\text{and codomain } (f) = N$$

➤ **Example 19.** The domain of the real function

$$f(x) = \frac{1}{\sqrt{4-x^2}} \text{ is}$$

- (a) the set of all real numbers
(b) the set of all positive real numbers
(c) $(-2, 2)$
(d) $[-2, 2]$

Sol. (c) We have, $f(x) = \frac{1}{\sqrt{4-x^2}}$

$$\begin{aligned} f(x) \text{ is defined iff } 4-x^2 > 0 \\ \Rightarrow x^2 - 4 < 0 \\ \Rightarrow (x+2)(x-2) < 0 \\ \Rightarrow -2 < x < 2 \\ \therefore \text{Domain of } f = (-2, 2) \end{aligned}$$

➤ **Example 20.** Let $f = \left\{ \left(x, \frac{x^2}{1+x^2} \right) : x \in R \right\}$ be a

function from R into R . The range of f is

- (a) $[0, \infty)$ (b) $(0, 1)$
(c) $[0, 1)$ (d) $[1, \infty)$

Sol. (c) We have, $f(x) = \frac{x^2}{1+x^2}$

Domain of $f = R$

Range of f : Let $f(x) = y$, then

$$y = \frac{x^2}{1+x^2} \Rightarrow y + x^2 y = x^2$$

$$\Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \sqrt{\frac{y}{1-y}}$$

Clearly, x will take real values iff $\frac{y}{1-y} \geq 0$

$$\Rightarrow \frac{y}{y-1} \leq 0 \Rightarrow 0 \leq y < 1$$

$\therefore$ Range of $f = [0, 1)$

Equal Functions

Two functions f and g are said to be equal iff

- domain of f = domain of g .
- codomain of f = codomain of g .
- $f(x) = g(x)$ for every x belonging to their common domain.

If two functions f and g are equal, then we write $f = g$.

e.g. Let $A = \{1, 2\}$, $B = \{3, 6\}$ and $f : A \rightarrow B$ given by $f(x) = x^2 + 2$ and $g : A \rightarrow B$ given by $g(x) = 3x$. Then, we observe that f and g have the same domain and codomain. Also, we have

$$f(1) = 3 = g(1)$$

and $f(2) = 6 = g(2)$

Hence, $f = g$

$\therefore f$ and g are equal functions.

➤ **Example 21.** The domain for which the functions $f(x) = 2x^2 - 1$ and $g(x) = 1 - 3x$ are equal, is

- (a) $\{1, 2\}$ (b) $\{-2, 2\}$ (c) $\left\{-2, \frac{1}{2}\right\}$ (d) $\{0, 1\}$

Sol. (c) We have, $f(x) = 2x^2 - 1$

and $g(x) = 1 - 3x$

$$f(x) = g(x)$$

$$\Rightarrow 2x^2 - 1 = 1 - 3x$$

$$\Rightarrow 2x^2 + 3x - 2 = 0$$

$$\Rightarrow (x+2)(2x-1) = 0$$

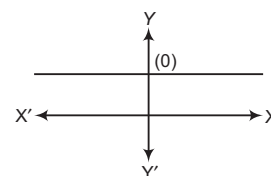
$$\Rightarrow x = -2, \frac{1}{2}$$

Thus, $f(x)$ and $g(x)$ are equal on the set $\left\{-2, \frac{1}{2}\right\}$.

Classification of Functions

Constant Function

A function which does not change as its parameters vary, i.e. the function whose rate of change is zero. In short, a constant function is a function that always gives or returns the same value.



or

Let k be a constant, then function $f(x) = k$, $\forall x \in R$ is known as constant function.

Domain of $f(x) = R$ and Range of $f(x) = \{k\}$

Polynomial Function

The function $y = f(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$, where $a_0, a_1, a_2, \dots, a_n$ are real coefficients and n is a non-negative integer, is known as a polynomial function. If $a_0 \neq 0$, then degree of polynomial function is n .

Domain of $f(x) = R$

and range varies from function to function.

Rational Function

If $P(x)$ and $Q(x)$ are polynomial functions, $Q(x) \neq 0$, then function $f(x) = \frac{P(x)}{Q(x)}$ is known as

rational function.

Domain of $f(x) = R - \{x : Q(x) = 0\}$

and range varies from function to function.

Irrational Function

The function containing one or more terms having non-integral rational powers of x is called irrational function.

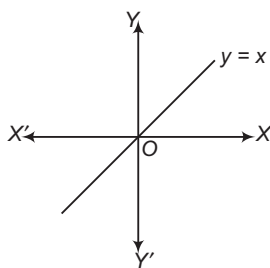
e.g. $y = f(x) = \frac{5x^{3/2} - 7x^{1/2}}{x^{1/2} - 1}$

Domain varies from function to function.

Identity Function

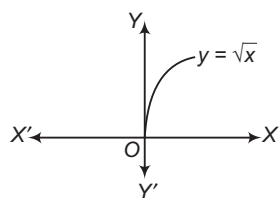
Function $f(x) = x$, $\forall x \in R$ is known as identity function. It is a straight line passing through origin and having slope unity.

Domain of $f(x) = R$
and range of $f(x) = R$



Square Root Function

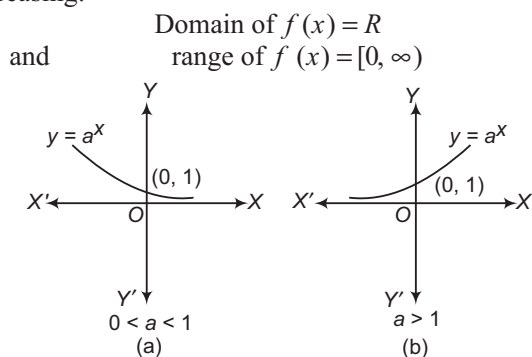
The function that associates every positive real number x to $+\sqrt{x}$ is called the square root function, i.e. $f(x) = +\sqrt{x}$.



Range of $f(x) = [0, \infty)$

Exponential Function

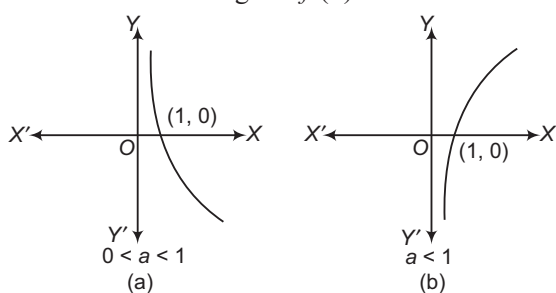
A function of the form $f(x) = a^x$, a is a positive real number, is an exponential function. The value of the function depends upon the value of a . For $0 < a < 1$, function is decreasing and for $a > 1$, function is increasing.



Logarithmic Function

Function $f(x) = \log_a x$, ($x, a > 0$) and $a \neq 1$, is known as logarithmic function.

Domain of $f(x) = (0, \infty)$
and range of $f(x) = R$

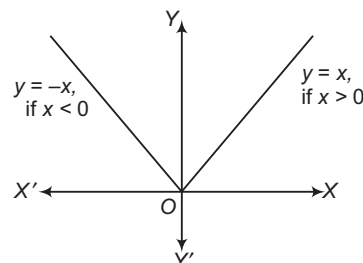


Modulus Function

Function $y = f(x) = |x|$ is known as modulus function.

$$y = f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Domain of $f(x) = x \in R$
and range of $f(x) = [0, \infty)$



Properties of Modulus Function

- i. For any real number x , $\sqrt{x^2} = |x|$.
- ii. If a and b are positive real numbers, then
 - (a) $x^2 \leq a^2 \Leftrightarrow |x| \leq a \Leftrightarrow -a \leq x \leq a$
 - (b) $x^2 \geq a^2 \Leftrightarrow |x| \geq a \Leftrightarrow x \leq -a \text{ or } x \geq a$
 - (c) $a^2 \leq x^2 \leq b^2 \Leftrightarrow a \leq |x| \leq b$
 $\Leftrightarrow x \in [-b, -a] \cup [a, b]$
- iii. $|x \pm y| \leq |x| + |y|$
- iv. $|x + y| \geq ||x| - |y||$

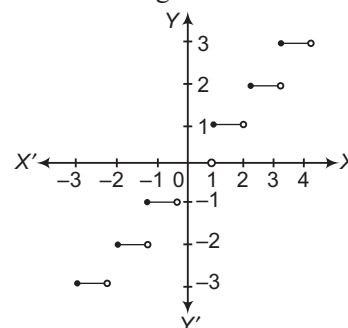
Greatest Integer Function

For any real number x , the greatest integer function $[x]$ is equal to greatest integer less than or equal to x .

In general, if n is an integer and x is any number satisfying $n \leq x < n+1$, then $[x] = n$. It is also known as integral part function.

e.g. If $2 \leq x < 3$, then $[x] = 2$

Domain = R
Range = I



Properties of Greatest Integer Function

If n is an integer and x is any real number between n and $n+1$, then

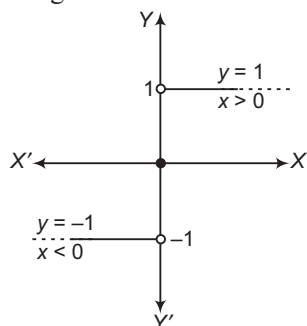
- i. $[-n] = -[n]$
- ii. $[x+n] = [x] + n$
- iii. $[-x] = -[x] - 1$, x is not an integer.
- iv. $[x+y] \geq [x] + [y]$
- v. $[x] > n \Rightarrow x \geq n+1$
- vi. $[x] < n \Rightarrow x < n$
- vii. $[x+y] = [x] + [y+x-[x]]$, $\forall x, y \in R$
- viii. $[x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \dots + \left[x + \frac{n-1}{n}\right] = [nx]$,
 $n \in N$

Signum Function

The function defined by

$$f(x) = \frac{|x|}{x} = \begin{cases} -1, & x < 0 \\ 0, & x = 0 \\ 1, & x > 0 \end{cases}$$

is called the signum function.



Domain = R

Range = $\{-1, 0, 1\}$

➤ **Example 22.** Which of the following is/are the polynomial function?

I. $f(x) = x^3 - x^2 + 2$, $\forall x \in R$

II. $f(x) = x^4 + \sqrt{2}x$, $\forall x \in R$

III. $f(x) = x^{2/3} + 2x$, $\forall x \in R$

(a) I, II (b) II, III (c) I, II, III (d) Only I

Sol. (a) A function $f : R \rightarrow R$ is said to be polynomial function if each of x in R ,

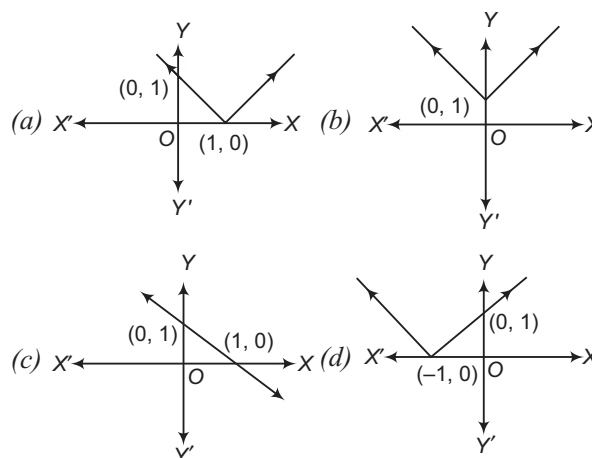
$$f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n, \text{ where } x \text{ is positive integer and } a_0, a_1, a_2, \dots, a_n \in R.$$

$\therefore$ I and II are polynomial functions whereas III is not a polynomial function because degree of x in III is not an integer.

➤ **Example 23.** The function f is defined by

$$f(x) = \begin{cases} 1-x, & x < 0 \\ 1, & x = 0 \\ x+1, & x > 0 \end{cases}$$

Then, the graph of $f(x)$ is



Sol. (b) We have, $f(x) = \begin{cases} 1-x, & x < 0 \\ 1, & x = 0 \\ x+1, & x > 0 \end{cases}$

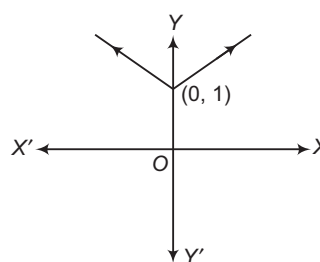
Let $f(x) = y$

$$\therefore x + y = 1, \quad x < 0$$

$$y = 1, \quad x = 0$$

$$\text{and } y - x = 1, \quad x > 0$$

$\therefore$ Graph of this function is



Algebra of Real Functions

Let $f : X \rightarrow R$ and $g : X \rightarrow R$ be any two real functions, where $X \subset R$.

(i) Addition of two real functions

$$(f+g)(x) = f(x) + g(x)$$

(ii) Subtraction of two real functions

$$(f-g)(x) = f(x) - g(x)$$

(iii) Multiplication by a scalar

$$(\alpha \cdot f)(x) = \alpha \cdot f(x), \alpha, x \in R$$

(iv) Multiplication of two real functions

$$fg(x) = f(x) \cdot g(x)$$

(v) Quotient of two real functions

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

➤ **Example 24.** Let f and g be two real functions defined by $f(x) = \frac{1}{x+4}$, $x \neq -4$ and

$g(x) = (x+4)^3$, then f/g is equal to

(a) $(x+4)^2$ (b) $\frac{1}{(x+4)^4}$ (c) $\frac{1}{(x+4)^3}$ (d) $\frac{1}{(x+4)^2}$

Sol. (b) We have, $f(x) = \frac{1}{x+4}$ and $g(x) = (x+4)^3$

Now, $\frac{f}{g} = \frac{\frac{1}{x+4}}{(x+4)^3} = \frac{1}{(x+4)^4}$, $x \neq -4$

➤ **Example 25.** Let $f(x) = x$, $g(x) = \frac{1}{x}$ and

$h(x) = f(x) \cdot g(x)$, then $h(x) = 1$

(a) $\forall x \in R$ (b) $\forall x \in Q$
(c) $\forall x \in R - Q$ (d) $\forall x \in R, x \neq 0$

Sol. (d) We have, $f(x) = x$, $\forall x \in R$

and $g(x) = \frac{1}{x}$, $\forall x \in R - \{0\}$

Domain of $f(x) \cdot g(x) = R - \{0\}$

$\therefore f(x) \cdot g(x) = x \times \frac{1}{x} = 1$, $\forall x \in R - \{0\}$

i.e. $x \in R, x \neq 0$

Composition of Functions

Let A, B and C be three non-empty sets. Let $f: A \rightarrow B$ and $g: B \rightarrow C$. Since, $f: A \rightarrow B$, for each $x \in A$, there exists a unique element $g[f(x)]$ of C . Thus, for each $x \in A$, there is associated a unique element $g[f(x)]$ of C . Thus, from f and g , we can define a new function from A to C . This function is called the product or composite of f and g , denoted by gof and defined by

$(gof): A \rightarrow C$ such that $(gof)(x) = g[f(x)]$, $\forall x \in A$.

➤ **Example 26.** If $f: R \rightarrow R$ and $g: R \rightarrow R$ are defined by $f(x) = 2x + 3$ and $g(x) = x^2 + 7$, then values of x such that $g\{f(x)\} = 8$ are

(a) 1, 2 (b) -1, 2 (c) -1, -2 (d) 1, -2

Sol. (c) We have, $f(x) = 2x + 3$ and $g(x) = x^2 + 7$

Now, $g\{f(x)\} = (f(x))^2 + 7 = 8$

$\Rightarrow (2x + 3)^2 + 7 = 8$

$\Rightarrow (2x + 3)^2 = 1 \Rightarrow 2x + 3 = \pm 1$

$\Rightarrow 2x = -4$ and $2x = -2 \Rightarrow x = -2, x = -1$

➤ **Example 27.** If $f(x) = \log\left(\frac{1+x}{1-x}\right)$ and

$g(x) = \frac{3x + x^3}{1 + 3x^2}$, then $f(g(x))$ is equal to

(a) $f(3x)$ (b) $[f(x)]^3$ (c) $3f(x)$ (d) $-f(x)$

Sol. (c) We have, $f(x) = \log\left(\frac{1+x}{1-x}\right)$

and $g(x) = \frac{3x + x^3}{1 + 3x^2}$

Now, $f\{g(x)\} = \log\left(\frac{1+g(x)}{1-g(x)}\right)$

$= \log\left[\frac{1 + \frac{3x + x^3}{1 + 3x^2}}{1 - \frac{3x + x^3}{1 + 3x^2}}\right] = \log\left[\frac{1 + 3x^2 + 3x + x^3}{1 + 3x^2 - 3x - x^3}\right]$

$= \log\left(\frac{1+x}{1-x}\right)^3 = 3 \log\left(\frac{1+x}{1-x}\right)$

$\Rightarrow f\{g(x)\} = 3f(x)$

Properties of Composition of Functions

- i. The composition of functions is not commutative.
i.e. $fog \neq gof$
- ii. The composition of functions is associative.
i.e. $fo(goh) = (fog)oh$
- iii. The composition of any function with the identity function is the function itself.
i.e. if $f: A \rightarrow B$, then $foI_A = I_B of = f$

Work Book Exercise 2.2

1 Find the range of the function $f(x) = x^2 + 2$.

- a $(-2, 2)$ b $[2, \infty)$
c $[3, \infty)$ d None of these

2 If the function $f(x) = \frac{a^x + a^{-x}}{2}$, ($a > 2$), then

$f(x+y) + f(x-y)$ is equal to

- a $2f(x)f(y)$ b $f(x)f(y)$
c $\frac{f(x)}{f(y)}$ d None of these

3 Domain and range of $f(x) = \frac{|x-3|}{x-3}$ are,

respectively

- a $R, [-1, 1]$ b $R - \{3\}, \{1, -1\}$
c R^T, R d None of these

4 The domain of the function $f(x) = \sqrt{2 - 2x - x^2}$ is

- a $-\sqrt{3} \leq x \leq \sqrt{3}$
b $(-1 - \sqrt{3}) \leq x \leq (-1 + \sqrt{3})$
c $-2 \leq x \leq 2$
d $(-2 - \sqrt{3}) \leq x \leq (-2 + \sqrt{3})$

5 If $f(x) = \sin^2 x$ and the composite function $g\{f(x)\} = |\sin x|$, then the function $g(x)$ is equal to

- a $\sqrt{x-1}$ b $\sqrt{x}$ c $\sqrt{x+1}$ d $-\sqrt{x}$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The composite mapping $f \circ g$ of the maps $f: R \rightarrow R, f(x) = \sin x, g: R \rightarrow R, g(x) = x^2$ is
(a) $\sin x + x^2$ (b) $(\sin x)^2$ (c) $\sin x^2$ (d) $\frac{\sin x}{x^2}$

Sol. We have,

$$(f \circ g)(x) = f(g(x)) = f(x^2) = \sin x^2$$

Hence, (c) is the correct answer.

Ex 2. If R is a relation ' $<$ ' from $A = \{1, 2, 3, 4\}$ to $B = \{1, 3, 5\}$, i.e. $(a, b) \in R$ iff $a < b$, then RoR^{-1} is
(a) $\{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$
(b) $\{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$
(c) $\{(3, 3), (3, 5), (5, 3), (5, 5)\}$
(d) $\{(3, 3), (3, 4), (4, 5)\}$

Sol. We have,

$$R = \{(1, 3), (1, 5), (2, 3), (2, 5), (3, 5), (4, 5)\}$$

$$\therefore R^{-1} = \{(3, 1), (5, 1), (3, 2), (5, 2), (5, 3), (5, 4)\}$$

$$\text{Now, } RoR^{-1} = \{(3, 3), (3, 5), (5, 3), (5, 5)\}$$

Hence, (c) is the correct answer.

Ex 3. The range of the function $f(x) = \frac{\sin(\pi[x^2+1])}{x^4+1}$ where, $[]$ is greatest integer function, is
(a) $[0, 1]$ (b) $[-1, 1]$
(c) $\{0\}$ (d) None of these

Sol. Since, $[x^2+1]$ is an integer.

$$\therefore \sin(\pi[x^2+1]) = 0 \Rightarrow f(x) = \frac{\sin(\pi[x^2+1])}{x^4+1} = 0$$

Now, range of $f = R_f = \{0\}$

Hence, (c) is the correct answer.

Ex 4. The simplified form of

$$f(x) = |x-2| + |2-x|, -3 \leq x \leq 3 \text{ is}$$

(a) $\begin{cases} 2x, & 2 \leq x \leq 3 \\ 4, & -2 \leq x < 2 \\ -2x, & -3 \leq x < -2 \end{cases}$ (b) $\begin{cases} -2x, & 2 \leq x \leq 3 \\ 4, & -2 \leq x < 2 \\ 2x, & -3 \leq x < -2 \end{cases}$

(c) $\begin{cases} 3x, & 2 \leq x \leq 3 \\ 4, & -2 \leq x < 2 \\ -2x, & -3 \leq x < -2 \end{cases}$ (d) None of these

Sol. $\therefore |x-2| = \begin{cases} x-2, & x \geq 2 \\ -(x-2), & x < 2 \end{cases}$

and $|x+2| = \begin{cases} (x+2), & x \geq -2 \\ -(x+2), & x < -2 \end{cases}$

$$\Rightarrow f(x) = \begin{cases} (x-2) + (2+x), & 2 \leq x \leq 3 \\ -(x-2) + (x+2), & -2 \leq x < 2 \\ -(x-2) - (x+2), & -3 \leq x < -2 \end{cases}$$

$$= \begin{cases} x-2+2+x, & 2 \leq x \leq 3 \\ -x+2+x+2, & -2 \leq x < 2 \\ -x+2-x-2, & -3 \leq x < -2 \end{cases} = \begin{cases} 2x, & 2 \leq x \leq 3 \\ 4, & -2 \leq x < 2 \\ -2x, & -3 \leq x < -2 \end{cases}$$

Hence, (a) is the correct answer.

Ex 5. Let A and B be two sets such that $n(A \times B) = 6$. If three elements of $A \times B$ are $(3, 2), (7, 5), (8, 5)$, then
(a) $A = \{3, 7, 8\}$ (b) $B = \{2, 5, 7\}$
(c) $B = \{3, 5\}$ (d) None of these

Sol. $\because (3, 2), (7, 5), (8, 5) \in A \times B$

$$\therefore 3, 7, 8 \in A \text{ and } 2, 5 \in B$$

$$\text{Also, } n(A \times B) = 6 = 3 \times 2$$

$$\therefore A = \{3, 7, 8\} \text{ and } B = \{2, 5\}$$

Hence, (a) is the correct answer.

Ex 6. If $f(x) = \frac{x-1}{x+1}$, then $f(2x)$ is equal to

(a) $\frac{3f(x)+1}{f(x)+3}$ (b) $\frac{f(x)+1}{f(x)+3}$
(c) $\frac{f(x)+3}{f(x)+1}$ (d) None of these

Sol. $\frac{3f(x)+1}{f(x)+3} = \frac{3\left(\frac{x-1}{x+1}\right)+1}{\frac{x-1}{x+1}+3}$

$$= \frac{3x-3+x+1}{x-1+3x+3} = \frac{4x-2}{4x+2} = f(2x)$$

Hence, (a) is the correct answer.

Ex 7. Let $f: (4, 6) \rightarrow (6, 8)$ be a function defined by $f(x) = x + \left[\frac{x}{2}\right]$, where $[]$ denotes the greatest integer function, then $f^{-1}(x)$ is equal to

(a) $x - \left[\frac{x}{2}\right]$ (b) $-x - 2$ (c) $x - 2$ (d) $\frac{1}{x + \left[\frac{\pi}{2}\right]}$

Sol. Since, $x \in (4, 6)$

$$\therefore f(x) = x + \left[\frac{x}{2}\right] = x + 2$$

Let $f(x) = y$

$$\therefore y = x + 2 \Rightarrow x = y - 2$$

$$\therefore f^{-1}(x) = x - 2$$

Hence, (c) is the correct answer.

Ex 8. Find the range of $f(x) = 1 + 3 \cos 2x$.

- (a) $[2, 3]$
 (b) $[2, 4]$
 (c) $[-2, 4]$
 (d) None of the above

Sol. $f(x) = 1 + 3 \cos 2x$

$$\begin{aligned} \because -1 &\leq \cos 2x \leq 1 \\ \Rightarrow -3 &\leq 3 \cos 2x \leq 3 \\ \Rightarrow 1 - 3 &\leq 1 + 3 \cos 2x \leq 3 + 1 \\ \Rightarrow -2 &\leq f(x) \leq 4 \\ \therefore \text{Range} &= [-2, 4] \end{aligned}$$

Hence, (c) is the correct answer.

Ex 9. Find the domain of the function

$$f(x) = \frac{1}{\sqrt{x+|x|}}$$

- (a) R (b) R^+ (c) R^- (d) $R + R$

Sol. Given, $f(x) = \frac{1}{\sqrt{x+|x|}}$, for the function to be defined,

the denominator should not be equal to 0.

$$x + |x| > 0 \quad [\because x + |x| \neq 0]$$

$$\Rightarrow |x| > -x$$

It is true for all R^+ .

$[R^+$ means set of all positive real numbers]

$\therefore$ Domain of the given function = R^+

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 10. Let $f, g: R \rightarrow R$ be defined, respectively by

$$f(x) = x + 1 \text{ and } g(x) = 2x - 3. \text{ Then,}$$

$$(a) (f+g)(x) = 3x - 2 \quad (b) (f-g)(2) = 6$$

$$(c) \left(\frac{f}{g}\right)(2) = 3 \quad (d) (f-g)(x) = 4 - x$$

Sol. We have, $f(x) = x + 1$ and $g(x) = 2x - 3$

$$(f+g)(x) = x + 1 + 2x - 3 = 3x - 2$$

$$(f-g)(x) = x + 1 - 2x + 3 = 4 - x$$

$$\left(\frac{f}{g}\right)(x) = \frac{x+1}{2x-3}$$

$$\frac{f}{g}(2) = \frac{3}{4-3} = 3$$

$$(f-g)(2) = 4 - 2 = 2$$

Hence, (a), (c) and (d) are the correct answers.

Ex 11. If $n(A) = 10$ and $n(B) = 5$, then

$$(a) n(A \times B) = 50$$

$$(b) n(A \times B) = n(B \times A)$$

$$(c) \text{ number of relations from } A \text{ to } B \text{ is } 2^{10}$$

$$(d) \text{ All of the above}$$

Sol. Given, $n(A) = 10$, $n(B) = 5$

$$n(A \times B) = n(A) \times n(B)$$

$$= 10 \times 5 = 50$$

$$n(B \times A) = n(B) \times n(A)$$

$$= 5 \times 10$$

$$= 50$$

$$\therefore \text{Total number of relations from } A \text{ to } B \text{ is } 2^{n(A \times B)} = 2^{50}$$

Hence, (a) and (b) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 12-13) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

(a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I

(c) Statement I is true, Statement II is false

(d) Statement I is false, Statement II is true

Ex 12. Let $A = \{1, 2, 3\}$, we define

$$R_1 = \{(1, 3), (1, 4), (2, 1), (2, 2), (2, 3)\} \text{ and}$$

$$R_2 = \{(1, 1), (1, 2), (1, 3)\}. \text{ Then,}$$

Statement I R_1 is not a relation and R_2 is relation on A .

Statement II If R is relation on A , then $R \subseteq A \times A$.

Sol. If R_1 is a relation on A , then $R_1 \subseteq A \times A$

$$\text{Here, } R_1 \text{ contains } (1, 4) \notin A \times A \Rightarrow R_1 \not\subseteq A \times A$$

$\therefore R_1$ is not a relation.

Whereas $R_2 \subset A \times A \Rightarrow R_2$ is relation.

Hence, (a) is the correct answer.

Ex 13. Statement I $f(x) = \frac{1}{1-x}$, $x \neq 0, 1$, then the

graph of the function $y = f(f(f(x)))$, $x > 1$ is a straight line.

Statement II If $f(x) = \frac{1}{1-x}$

$$\Rightarrow f[f\{f(x)\}] = x, \forall x \in R$$

$$\text{Sol. } f\{f(x)\} = \frac{1}{1-f(x)} = \frac{1}{1-\frac{1}{1-x}} = \frac{x-1}{x}, \forall x \in R - \{0, 1\}$$

$$\therefore f[f\{f(x)\}] = \frac{1}{1-f\{f(x)\}} = \frac{1}{1-\frac{x-1}{x}}$$

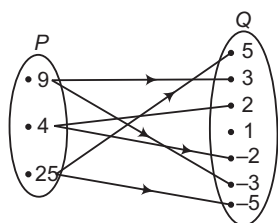
$$= x, \forall x \in R - \{0, 1\}$$

$\Rightarrow$ Statement I is true, Statement II is false.

Hence, (c) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage (Ex. Nos. 14-15) The figure shows a relation R between the sets P and Q .



Ex 14. The relation R in set builder form is

- (a) $\{(x, y) : x \in P, y \in Q\}$
- (b) $\{(x, y) : x \in Q, y \in P\}$
- (c) $\{(x, y) : x \text{ is the square of } y, x \in P, y \in Q\}$
- (d) $\{(x, y) : y \text{ is the square of } x, x \in P, y \in Q\}$

Sol. It is obvious that the relation R is 'x is the square of y'.
In set builder form, $R = \{(x, y) : x \text{ is the square of } y, x \in P, y \in Q\}$

Hence, (c) is the correct answer.

Ex 15. The relation R in roster form is

- (a) $\{(9, 3), (4, 2), (25, 5)\}$
- (b) $\{(9, -3), (4, -2), (25, -5)\}$
- (c) $\{(9, -3), (9, 3), (4, -2), (4, 2), (25, -5), (25, 5)\}$
- (d) None of the above

Sol. In roster form,

$R = \{(9, 3), (9, -3), (4, 2), (4, -2), (25, 5), (25, -5)\}$
Hence, (c) is the correct answer.

Type 5. Match the Columns

Ex 16. Let $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$.
Then, match the following in Column I with the sets of ordered pairs in Column II.

Column I	Column II
A. $A \times (B \cap C)$	p. $\{(1, 4), (2, 4), (3, 4)\}$
B. $(A \times B) \cap (A \times C)$	q. $\{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
C. $A \times (B \cup C)$	r. $\{(3, 4)\}$
D. $(A \times B) \cup (A \times C)$	
E. $(A \cap B) \times (B \cap C)$	

Sol. Given, $A = \{1, 2, 3\}$, $B = \{3, 4\}$ and $C = \{4, 5, 6\}$

- A. $B \cap C = \{4\}$
 $\therefore A \times (B \cap C) = \{(1, 4), (2, 4), (3, 4)\}$
- B. $A \times B = \{(1, 3), (1, 4), (2, 3), (2, 4), (3, 3), (3, 4)\}$
 $A \times C = \{(1, 4), (1, 5), (1, 6), (2, 4), (2, 5), (2, 6), (3, 4), (3, 5), (3, 6)\}$
 $\therefore (A \times B) \cap (A \times C) = \{(1, 4), (2, 4), (3, 4)\}$
- C. $B \cup C = \{3, 4, 5, 6\}$
 $\therefore A \times (B \cup C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
- D. $(A \times B) \cup (A \times C) = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), (3, 3), (3, 4), (3, 5), (3, 6)\}$
- E. $A \cap B = \{3\}$, $B \cap C = \{4\}$
 $\therefore (A \cap B) \times (B \cap C) = \{(3, 4)\}$
 $A \rightarrow p; B \rightarrow q; C \rightarrow r; D \rightarrow q; E \rightarrow r$

Ex 17. Let $A = \{1, 2, 3, 4, \dots, 14\}$. A relation R from A to A is defined by $R = \{(x, y) : 3x - y = 0, \text{ where } x, y \in A\}$. Then, match the terms of Column I with the terms of Column II and choose the correct option from the codes given below:

Column I	Column II
A. In roster form, the relation R is	p. $\{1, 2, 3, 4\}$
B. The domain of R is	q. $\{3, 6, 9, 12\}$
C. The range of R is	r. $\{1, 2, 3, 4, \dots, 14\}$
D. The codomain of R is	s. $\{(1, 3), (2, 6), (3, 9), (4, 12)\}$

Sol. We have, $3x - y = 0 \Rightarrow y = 3x$

For $x = 1, y = 3 \in A$
 $x = 2, y = 6 \in A$
 $x = 3, y = 9 \in A$
 $x = 4, y = 12 \in A$
 $x = 5, y = 15 \notin A$

- A. In roster form, $R = \{(1, 3), (2, 6), (3, 9), (4, 12)\}$
- B. Domain of R = Set of first elements of ordered pairs in R
 $= \{1, 2, 3, 4\}$
- C. Range of R = Set of second elements of ordered pairs in R
 $= \{3, 6, 9, 12\}$
- D. Codomain of R is the set A .
 $A \rightarrow s; B \rightarrow p; C \rightarrow q; D \rightarrow r$

Type 6. Single Integer Answer Type Questions

Ex 18. If $n(A) = 4$, $n(B) = 3$ and $n(A \times B \times C) = 24$, then $n(C)$ is equal to _____.

Sol. (2) We have, $n(A) = 4$, $n(B) = 3$
and $n(A \times B \times C) = 24$
 $\Rightarrow n(A) \times n(B) \times n(C) = 24$
 $4 \times 3 \times n(C) = 24$
 $n(C) = 2$

Ex 19. Let $A = \{x, y, z\}$ and $B = \{1, 2\}$. Then, the number of relations from A to B is 2^k , then k is equal to _____.

Sol. (6) We have, $n(A) = 3$, $n(B) = 2$
 $n(A \times B) = 6$

Total number of relations from A to $B = 2^{n(A \times B)} = 2^6$
 $\therefore k = 6$

Target Exercises

Type 1. Only One Correct Option

- If two sets A and B are having 99 elements in common, then the number of elements common to each of the sets $A \times B$ and $B \times A$ are
(a) 2^{99} (b) 99^2 (c) 100 (d) 18
- Let $n(A) = m$ and $n(B) = n$. Then, the total number of non-empty relations that can be defined from A to B is
(a) m^n (b) $n^m - 1$ (c) $mn - 1$ (d) $2^{mn} - 1$
- Let $X = \{1, 2, 3, 4, 5\}$ and $Y = \{1, 3, 5, 7, 9\}$. Which of the following is not a relation from X to Y ?
(a) $R_1 = \{(x, y) : y = x + 2, x \in X, y \in Y\}$
(b) $R_2 = \{(1, 1), (2, 1), (3, 3), (4, 3), (5, 5)\}$
(c) $R_3 = \{(1, 1), (1, 3), (3, 5), (3, 7), (5, 7)\}$
(d) $R_4 = \{(1, 3), (2, 5), (2, 4), (7, 9)\}$
- A relation R from C to R is defined by xRy iff $|x| = y$. Which of the following is correct?
(a) $(2 + 3i)R13$ (b) $3R(-3)$
(c) $(1 + i)R2$ (d) $iR1$
- The relation R defined on the set $A = \{1, 2, 3, 4, 5\}$ by $R = \{(x, y) : |x^2 - y^2| < 16\}$ is given by
(a) $\{(1, 1), (2, 1), (3, 1), (4, 1), (2, 3)\}$
(b) $\{(2, 2), (3, 2), (4, 2), (2, 4)\}$
(c) $\{(3, 3), (4, 3), (5, 4), (3, 4)\}$
(d) None of the above
- A relation R is defined in the set Z of integers as follows $(x, y) \in R$ iff $x^2 + y^2 = 9$. Which of the following is false?
(a) $R = \{(0, 3), (0, -3), (3, 0), (-3, 0)\}$
(b) Domain of $R = \{-3, 0, 3\}$
(c) Range of $R = \{-3, 0, 3\}$
(d) None of the above
- If $R = \{(x, y) : x, y \in Z, x^2 + y^2 \leq 4\}$ is a relation in Z , then domain of R is
(a) $\{0, 1, 2\}$ (b) $\{-2, -1, 0\}$
(c) $\{-2, -1, 0, 1, 2\}$ (d) None of these
- Let R be a relation in N defined by $R = \{(x, y) : x + 2y = 8\}$. The range of R is
(a) $\{2, 4, 6\}$ (b) $\{1, 2, 3\}$
(c) $\{1, 2, 3, 4, 6\}$ (d) None of these
- Let $A = \{1, 2, 3\}$, $B = \{1, 3, 5\}$. If relation R from A to B is given by $\{(1, 3), (2, 5), (3, 3)\}$, then R^{-1} is
(a) $\{(3, 3), (3, 1), (5, 3)\}$
(b) $\{(1, 3), (2, 5), (3, 3)\}$
(c) $\{(1, 3), (5, 2)\}$
(d) None of the above
- R is a relation from $\{11, 12, 13\}$ to $\{8, 10, 12\}$ defined by $y = x - 3$. The relation R^{-1} is
(a) $\{(11, 8), (13, 10)\}$
(b) $\{(8, 11), (10, 13)\}$
(c) $\{(8, 11), (9, 12), (10, 13)\}$
(d) None of the above
- Let a relation R be defined by $R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$. The relation $R^{-1} \circ R$ is given by
(a) $\{(1, 1), (4, 4), (7, 4), (4, 7), (7, 7)\}$
(b) $\{(1, 1), (4, 4), (4, 7), (7, 4), (7, 7), (3, 3)\}$
(c) $\{(1, 5), (1, 6), (3, 6)\}$
(d) None of the above
- If R is a relation on the set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ given by $xRy \Leftrightarrow y = 3x$, then RoR^{-1} is
(a) $\{(1, 3), (2, 6), (3, 9)\}$
(b) $\{(1, 1), (2, 2), (3, 3)\}$
(c) $\{(3, 3), (6, 6), (9, 9)\}$
(d) None of the above
- If $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations, then $(SoR)^{-1}$ is equal to
(a) $S^{-1} \circ R^{-1}$ (b) RoS
(c) $R^{-1} \circ S^{-1}$ (d) None of these
- If R is a relation from a set A to the set B and S is a relation from B to C , then the relation SoR
(a) is from C to A (b) is from A to C
(c) does not exist (d) None of these
- Let $A = [-1, 1]$, $B = [-1, 1]$, $C = [0, \infty)$. If $R_1 = \{(x, y) \in A \times B : x^2 + y^2 = 1\}$ and $R_2 = \{(x, y) \in A \times C : x^2 + y^2 = 1\}$, then
(a) R_1 defines a function from A into B
(b) R_2 defines a function from A into C
(c) R_1 defines a function from A onto B
(d) R_2 defines a function from A onto C
- The relation f is defined by $f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$ and the relation g is defined by $g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$. Which of the following relation is a function?
(a) f
(b) g
(c) f, g
(d) None of the above

- 17.** The domain of $F(x) = \frac{\log_2(x+3)}{x^2 + 3x + 2}$ is
 (a) $R - \{-1, -2\}$ (b) $(-2, \infty)$
 (c) $R - \{-1, -2, -3\}$ (d) $(-3, \infty) - \{-1, -2\}$
- 18.** The domain and range of the function f given by $f(x) = 2 - |x - 5|$ is
 (a) Domain = R^+ , Range = $(-\infty, 1]$
 (b) Domain = R , Range = $(-\infty, 2]$
 (c) Domain = R , Range = $(-\infty, 2)$
 (d) Domain = R^+ , Range = $(-\infty, 2]$
- 19.** The range of the function $f(x) = \frac{x-2}{2-x}$ when $x \neq 2$ is
 (a) R (b) $R - \{1\}$ (c) $\{-1\}$ (d) $R - \{-1\}$
- 20.** The range of the function $f(x) = |x|$ is
 (a) $(0, \infty)$ (b) $(-\infty, 0)$
 (c) $[0, \infty)$ (d) None of these
- 21.** If f and g are two functions defined as $f(x) = x + 2$, $x \leq 0$; $g(x) = 3$, $x \geq 0$, then the domain of $f + g$ is
 (a) $\{0\}$ (b) $[0, \infty)$ (c) $(-\infty, \infty)$ (d) $(-\infty, 0)$
- 22.** If $[x]^2 - 5[x] + 6 = 0$, where $[.]$ denotes the greatest integer function, then
 (a) $x \in [3, 4]$ (b) $x \in (2, 3]$ (c) $x \in [2, 3]$ (d) $x \in [2, 4)$
- 23.** If $f(x) = 1 - \frac{1}{x}$, then $f\left\{f\left(\frac{1}{x}\right)\right\}$ is
 (a) $\frac{1}{x}$ (b) $\frac{1}{1+x}$ (c) $\frac{x}{x-1}$ (d) $\frac{1}{x-1}$
- 24.** Let $f(x) = \frac{x-1}{x+1}$, then $f\{f(x)\}$ is
 (a) $\frac{1}{x}$ (b) $-\frac{1}{x}$ (c) $\frac{1}{x+1}$ (d) $\frac{1}{x-1}$
- 25.** Two functions f and g are said to commute, if $(f \circ g)(x) = (g \circ f)(x)$, $\forall x$, then which one of the following functions are commute?
 (a) $f(x) = x^3$, $g(x) = x + 1$
 (b) $f(x) = \sqrt{x}$, $g(x) = \cos x$
 (c) $f(x) = x^m$, $g(x) = x^n$, $m \neq n$, $m, n \in I$ (I is the set of all integers)
 (d) $f(x) = x - 1$, $g(x) = x^2 + 1$
- 26.** If $f(x) = x^2 + 1$, then the value of $(f \circ f)(x)$ is equal to
 (a) $x^4 + 2x^2 + 2$ (b) $x^4 + 2x^2 - 2$
 (c) $x^4 + x^2 + 1$ (d) None of these
- 27.** If $f(x)$ is defined on $[0, 1]$ by the rule $f(x) = \begin{cases} x, & \text{if } x \in Q \\ 1-x, & \text{if } x \notin Q \end{cases}$, then for $x \in [0, 1]$
 (a) $(f \circ f)(x) = 1 + x$ (b) $(f \circ f)(x) = x$
 (c) $(f \circ f)(x) = 1 - x$ (d) None of these
- 28.** If $f(x) = \begin{cases} x^3 + 1 & x < 0 \\ x^2 + 1 & x \geq 0 \end{cases}$, $g(x) = \begin{cases} (x-1)^{1/3}, & x < 1 \\ (x-1)^{1/2}, & x \geq 1 \end{cases}$, then $(g \circ f)(x)$ is equal to
 (a) $x, \forall x \in R$ (b) $x - 1, \forall x \in R$
 (c) $x + 1, \forall x \in R$ (d) None of these
- 29.** If $f(x) = [x]$ and $g(x) = x - [x]$, then which of the following is the zero function?
 (a) $(f + g)(x)$ (b) $(fg)(x)$
 (c) $(f - g)(x)$ (d) $(f \circ g)(x)$
- 30.** If $f(x) = \frac{x}{x-1}$, $x \neq 1$, then $(\underbrace{f \circ f \circ \dots \circ f}_{19 \text{ times}})(x)$ is equal to
 (a) $\frac{x}{x-1}$ (b) $\left(\frac{x}{x-1}\right)^{19}$ (c) $\frac{19x}{x-1}$ (d) x
- 31.** Let $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$, $x \in R - \{0\}$, then $f(x)$ is equal to
 (a) x^2 (b) $x^2 - 1$
 (c) $x^2 - 2$, when $|x| \geq 2$ (d) None of these
- 32.** Let $f: [0, 1] \rightarrow [0, 1]$ and $g: [0, 1] \rightarrow [0, 1]$ be two functions defined by $f(x) = \frac{1-x}{1+x}$ and $g(x) = 4x(1-x)$, then $(f \circ g)(x)$ is equal to
 (a) $\frac{8x(1-x)}{(1+x)^2}$ (b) $\frac{4(1-x)}{1+x}$
 (c) $\frac{1-4x+4x^2}{1+4x-4x^2}$ (d) None of these
- 33.** Let $f(x) = |x - 1|$, $x \in R$, then
 (a) $f(x^2) = (f(x))^2$ (b) $f(x + y) = f(x) + f(y)$
 (c) $f(|x|) = |f(x)|$ (d) None of these

Type 2. More than One Correct Option

- 34.** If $A = \{a, b, c, d\}$, $B = \{1, 2, 3\}$, which of the following sets of ordered pairs are not relation from A to B ?
 (a) $\{(a, 1), (a, 3)\}$
 (b) $\{(b, 1), (c, 2), (d, 1)\}$
 (c) $\{(a, 2), (b, 3), (3, b)\}$
 (d) $\{(a, 1), (b, 2), (3, c)\}$
- 35.** The relation R defined in $A = \{1, 2, 3\}$ by aRb , if $|a^2 - b^2| \leq 5$. Which of the following is true?
 (a) $R = \{(1, 1), (2, 2), (3, 3), (2, 1), (1, 2), (2, 3), (3, 2)\}$
 (b) $R^{-1} = R$
 (c) Domain of $R = \{1, 2, 3\}$
 (d) Range of $R = \{5\}$

Type 3. Assertion and Reason

Directions (Q. Nos. 36-37) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

36. Statement I The domain of the real function f defined by $f(x) = \sqrt{x-1}$ is $R - \{1\}$.

Statement II The range of the function defined by $f(x) = \sqrt{x-1}$ is $[0, \infty)$.

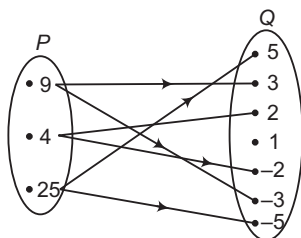
37. Statement I Let f and g be two real functions given by $f = \{(0, 1), (2, 0), (3, -4), (4, 2), (5, 1)\}$ and $g = \{(1, 0), (2, 2), (3, -1), (4, 4), (5, 3)\}$.

Then, domain of $f \cdot g$ is given by $\{2, 3, 4, 5\}$.

Statement II Let f and g be two real functions. Then, $(f \cdot g)(x) = f\{g(x)\}$.

Type 4. Linked Comprehension Based Questions

Passage (Q. Nos. 38-39) The figure shows a relation R between the sets P and Q .



38. The domain of the relations R is

- (a) $\{4, 9, 25\}$
(b) $\{2, 3, 5\}$
(c) $\{-2, -3, -5\}$
(d) $\{-4, -9, -25\}$

39. The range of the relation R is

- (a) $\{2, 3, 4\}$
(b) $\{-2, -3, -5\}$
(c) $\{-5, -3, -2, 2, 3, 5\}$
(d) $\{-5, -3, -2, 1, 2, 3, 5\}$

Type 5. Match the Column

40. Match the terms of Column I with the terms of Column II.

Column I	Column II
A. If $P = \{x : x < 3, x \in N\}$ and $Q = \{x : x \leq 2, x \in N\}$, then $(P \cup Q) \times (P \cap Q)$ is equal to	p. $\{(0, 8), (8, 0), (0, -8), (-8, 0)\}$
B. If $A = \{x : x \in W, x < 2\}$, $B = \{x : x \in N, 1 < x < 5\}$ and $C = \{3, 5\}$, where W is the set of whole numbers, then $A \times (B \cap C)$ is equal to	q. $\{(-2, 3), (-1, 5), (0, 7), (1, 9), (2, 11)\}$
C. If $R = \{(x, y) : x \text{ and } y \text{ are integers and } x^2 + y^2 = 64\}$ is a relation, then R in roster form is	r. $\{(0, 3), (1, 3)\}$
D. If $R = \{(x, y) : y = 2x + 7, x \in Z \text{ and } -2 \leq x \leq 2\}$ is a relation, then R is	s. $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$

Type 6. Single Integer Answer Type Questions

- 41.** Let A and B be two sets having 3 elements in common. If $n(A) = 5$ and $n(B) = 4$, then $n[(A \times B) \cap (B \times A)]$ is equal to _____.
- 42.** Let $f = \{(1, 1), (2, 3), (0, -1), (-1, -3)\}$ be a linear function from Z to Z , such that $f(x) = kx - 1$, then k is equal to _____.

Entrances Gallery

AIEEE

- The domain of the function $f(x) = \frac{1}{\sqrt{|x| - x}}$ is [2011]
 - $(0, \infty)$
 - $(-\infty, 0)$
 - $(-\infty, \infty) - \{0\}$
 - $(-\infty, \infty)$
- Let for $a \neq a_1 \neq 0$, $f(x) = ax^2 + bx + c$, $g(x) = a_1x^2 + b_1x + c_1$ and $p(x) = f(x) - g(x)$. If $p(x) = 0$ only for $x = -1$ and $p(-2) = 2$, then the value of $p(2)$ is [2011]
 - 18
 - 3
 - 9
 - 6
- Let $f: N \rightarrow Y$ be a function defined as $f(x) = 4x + 3$ where $Y = \{y \in N : y = 4x + 3 \text{ for some } x \in N\}$. If f is invertible, then its inverse is [2008]
 - $g(y) = \frac{y-3}{4}$
 - $g(y) = \frac{3y+4}{3}$
 - $g(y) = 4 + \frac{y+3}{4}$
 - $g(y) = \frac{y+3}{4}$
- A real valued function $f(x)$ satisfies the functional equation $f(x-y) = f(x)f(y) - f(a-x)f(a+y)$, where a is a given constant and $f(0) = 1$, $f(2a-x)$ is equal to [2005]
 - $f(-x)$
 - $f(a) + f(a-x)$
 - $f(x)$
 - $-f(x)$
- If $f: R \rightarrow R$ satisfies $f(x+y) = f(x) + f(y)$, $\forall x, y \in R$ and $f(1) = 7$, then $\sum_{r=1}^n f(r)$ is [2003]
 - $\frac{7n}{2}$
 - $\frac{7(n+1)}{2}$
 - $7n(n+1)$
 - $\frac{7n(n+1)}{2}$

Other Engineering Entrances

- Let $A = \{1, 2, 3, 4\}$ and R be the relation on A defined by $\{(a, b) : a, b \in A, a \times b \text{ is an even number}\}$, then the range of R is [J&K CET 2014]
 - $\{1, 2, 4\}$
 - $\{2, 4\}$
 - $\{2, 3, 4\}$
 - $\{1, 2, 4\}$
- The domain of the function $f(x) = \frac{(x^2 + 1)}{x^2 - 3x + 3}$ is [J&K CET 2014]
 - $R - \{1, 2\}$
 - $R - \{1, 4\}$
 - R
 - $R - \{1\}$
- Let $A = \{1, 2, 3, 4, 5\}$. The domain of the relation on A defined by $R = \{(x, y) : y = 2x - 1\}$ is [J&K CET 2014]
 - $\{1, 2, 3\}$
 - $\{1, 2\}$
 - $\{1, 3, 5\}$
 - $\{2, 4\}$
- If $f(x) = \sqrt{x}$ and $g(x) = 2x - 3$, then domain of $(f \circ g)(x)$ is [Kerala CEE 2014]
 - $(-\infty, -3)$
 - $(-\infty, -\frac{3}{2})$
 - $[-\frac{3}{2}, 0]$
 - $[0, \frac{3}{2}]$
 - $[\frac{3}{2}, \infty)$
- If $f(x) = 3 - x$, $-4 \leq x \leq 4$, then the domain of $\log_e(f(x))$ is [J&K CET 2011]
 - $[-4, 4]$
 - $(-\infty, 3]$
 - $(-\infty, 3)$
 - $[-4, 3)$
- If $f(x) = \frac{x+2}{3x-1}$, then $f\{f(x)\}$ is [Kerala CEE 2014]
 - x
 - $-x$
 - $\frac{1}{x}$
 - $-\frac{1}{x}$
 - 0
- The range of the function $f(x) = \log_e(3x^2 + 4)$ is equal to [Kerala CEE 2012]
 - $[\log_e 2, \infty)$
 - $[\log_e 3, \infty)$
 - $[2 \log_e 3, \infty)$
 - $[0, \infty)$
 - $[2 \log_e 2, \infty)$
- Let R be the set of real numbers and the functions $f: R \rightarrow R$ and $g: R \rightarrow R$ be defined by $f(x) = x^2 + 2x - 3$ and $g(x) = x + 1$. Then, the value of x for which $g(f(x)) = f(g(x))$ is [WB JEE 2012]
 - 1
 - 0
 - 1
 - 2
- Let $A = \{x, y, z\}$ and $B = \{a, b, c, d\}$. Which one of the following is not a relation from A to B ? [Kerala CEE 2011]
 - $\{(x, a), (x, c)\}$
 - $\{(y, c), (y, d)\}$
 - $\{(z, a), (z, d)\}$
 - $\{(z, b), (y, b), (a, d)\}$
 - $\{(x, c)\}$
- If $f(x) = 3 - x$, $-4 \leq x \leq 4$, then the domain of $\log_e(f(x))$ is [J&K CET 2011]
 - $[-4, 4]$
 - $(-\infty, 3]$
 - $(-\infty, 3)$
 - $[-4, 3)$

15. Let $f = \{(0, -1), (-1, 3), (2, 3), (3, 5)\}$ be a function from Z to Z defined by $f(x) = ax + b$. Then, [AMU 2011]

- (a) $a = 1, b = -2$
(b) $a = 2, b = 1$
(c) $a = 2, b = -1$
(d) $a = 1, b = 2$

16. If $f(x) = x^2 - 1$ and $g(x) = (x + 1)^2$, then $(gof)(x)$ is [Kerala CEE 2011]

- (a) $(x + 1)^4 - 1$
(b) $x^4 - 1$
(c) x^4
(d) $(x + 1)^4$
(e) $(x - 1)^4 - 1$

17. Let R be the set of real numbers and the mapping $f : R \rightarrow R$ and $g : R \rightarrow R$ be defined by $f(x) = 5 - x^2$ and $g(x) = 3x - 4$, then the value of $(f \circ g)(-1)$ is [WB JEE 2010]

- (a) -44 (b) -54 (c) -32 (d) -64

18. Let $A = \{1, 0, 1, 2\}$, $B = \{4, 2, 0, 2\}$ and $f, g : A \rightarrow B$ be functions defined by $f(x) = x^2 - x$ and $g(x) = 2 \left| x - \frac{1}{2} \right| - 1$. Then, [AMU 2010]

- (a) $f = g$
(b) $f = 2g$
(c) $g = 2f$
(d) None of the above

Answers

Work Book Exercise 2.1

1. (c) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a) 7. (c) 8. (c)

Work Book Exercise 2.2

1. (b) 2. (a) 3. (b) 4. (b) 5. (b)

Target Exercises

1. (b) 2. (d) 3. (d) 4. (d) 5. (d) 6. (d) 7. (c) 8. (b) 9. (d) 10. (b)
11. (b) 12. (b) 13. (c) 14. (b) 15. (d) 16. (a) 17. (d) 18. (b) 19. (c) 20. (c)
21. (a) 22. (d) 23. (c) 24. (b) 25. (c) 26. (a) 27. (b) 28. (a) 29. (d) 30. (a)
31. (c) 32. (c) 33. (d) 34. (c,d) 35. (a,b,c) 36. (d) 37. (c) 38. (a) 39. (c) 40. (*)
41. (9) 42. (2)

* $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow p$; $D \rightarrow q$

Entrances Gallery

1. (b) 2. (a) 3. (a) 4. (d) 5. (d) 6. (b) 7. (c) 8. (a) 9. (e) 10. (a)
11. (e) 12. (a) 13. (d) 14. (d) 15. (c) 16. (c) 17. (a) 18. (a)

Explanations

Target Exercises

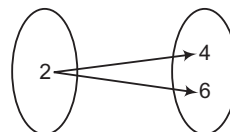
1. $n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)]$
 $= n(A \cap B) \times n(B \cap A)$
 $= 99 \times 99 = 99^2$
2. Given, $n(A) = m$ and $n(B) = n$
 Now, total number of relations from A to $B = 2^{mn}$
 $\therefore$ Total number of non-empty relations from A to B
 $= 2^{mn} - 1$
3. R_1 is a relation from X to Y because $R_1 \subseteq X \times Y$.
 R_2 is a relation from X to Y because $R_2 \subseteq X \times Y$.
 R_3 is a relation from X to Y because $R_3 \subseteq X \times Y$.
 R_4 is not a relation from X to Y because $(2, 4), (7, 9) \notin X \times Y$.
4. $|i| = \sqrt{0^2 + (1)^2} = \sqrt{1} = 1$
 $\therefore iR1$ is correct.
5. We have, $R = \{(x, y) : |x^2 - y^2| < 16\}$
 $\therefore R = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4),$
 $(3, 1), (3, 2), (3, 3), (3, 4), (4, 1), (4, 2),$
 $(4, 3), (4, 4), (4, 5), (5, 4), (5, 5)\}$
6. $x^2 + y^2 = 9 \Rightarrow y = 9 - x^2$
 $\Rightarrow y = \pm \sqrt{9 - x^2}$
 $\therefore R = \{(0, 3), (0, -3), (3, 0), (-3, 0)\}$
 Domain of $R = \{x : (x, y) \in R\} = \{0, 3, -3\}$
 Range of $R = \{y : (x, y) \in R\} = \{3, -3, 0\}$
7. We have, $R = \{(x, y) : x, y \in \mathbb{Z}, x^2 + y^2 \leq 4\}$
 $R = \{(0, 0), (0, -1), (0, 1), (0, -2), (0, 2), (-1, 0), (1, 0),$
 $(1, 1), (1, -1), (-1, 1), (-1, -1), (2, 0), (-2, 0)\}$
 $\therefore$ Domain of $R = \{x : (x, y) \in R\} = \{0, -1, 1, -2, 2\}$
8. $R = \{(x, y) : x + 2y = 8, x, y \in \mathbb{N}\}$
 $x + 2y = 8 \Rightarrow y = \frac{8 - x}{2}$
 $R = \{(2, 3), (4, 2), (6, 1)\}$
 $\therefore$ Range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$
9. We have, $R = \{(1, 3), (2, 5), (3, 3)\}$
 $\therefore R^{-1} = \{(y, x) : (x, y) \in R\} = \{(3, 1), (5, 2), (3, 3)\}$
10. We have, $R = \{(x, y) : y = x - 3, x = 11 \text{ or } 12 \text{ or } 13,$
 $y = 8 \text{ or } 10 \text{ or } 12\}$
 $= \{(11, 8), (13, 10)\}$
 $\therefore R^{-1} = \{(y, x) : (x, y) \in R\}$
 $= \{(8, 11), (10, 13)\}$
11. We have, $R = \{(4, 5), (1, 4), (4, 6), (7, 6), (3, 7)\}$
 $R^{-1} = \{(5, 4), (4, 1), (6, 4), (6, 7), (7, 3)\}$
 $\therefore R^{-1} \circ R = \{(4, 4), (1, 1), (4, 7), (7, 4), (7, 7), (3, 3)\}$
12. We have, set $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$
 $xRy \Rightarrow y = 3x$
 $\therefore R = \{(1, 3), (2, 6), (3, 9)\}$

and $R^{-1} = \{(3, 1), (6, 2), (9, 3)\}$
 $\therefore R \circ R^{-1} = \{(1, 1), (2, 2), (3, 3)\}$

13. SoR is a relation from A to C .
 $\therefore (SoR)^{-1}$ is a relation from C to A .
 R^{-1} is a relation from B to A .
 S^{-1} is a relation from C to B .
 $\therefore R^{-1} \circ S^{-1}$ is a relation from C to A .
 Let $(c, a) \in (SoR)^{-1}$
 $\therefore (a, c) \in SoR$
 $\Rightarrow \exists b \in B : (a, b) \in R \text{ and } (b, c) \in S$
 $\Rightarrow (b, a) \in R^{-1} \text{ and } (c, b) \in S^{-1}$
 $\Rightarrow (c, a) \in R^{-1} \circ S^{-1}$
 $\therefore (SoR)^{-1} \subseteq R^{-1} \circ S^{-1}$
 Conversely, let $(c, a) \in R^{-1} \circ S^{-1}$
 $\Rightarrow \exists b \in B : (c, b) \in S^{-1} \text{ and } (b, a) \in R^{-1}$
 $\Rightarrow (b, c) \in S \text{ and } (a, b) \in R$
 $\Rightarrow (a, c) \in SoR \Rightarrow (c, a) \in (SoR)^{-1}$
 $\therefore R^{-1} \circ S^{-1} \subseteq (SoR)^{-1}$
 Combining, we get $(SoR)^{-1} = R^{-1} \circ S^{-1}$

14. Since, $R \subseteq A \times B$ and $S \subseteq B \times C$, we have $SoR \subseteq A \times C$
 $\therefore SoR$ is a relation from A to C .
15. $R_2 = \{(x, y) \in A \times C : x^2 + y^2 = 1\}$
 $x^2 + y^2 = 1 \Rightarrow y = \pm \sqrt{1 - x^2}$
 $\Rightarrow y = \sqrt{1 - x^2} \quad [\because y \geq 0]$
 $\therefore R_2$ is a function from A onto C .

16. Given, $f(x) = \begin{cases} x^2, & 0 \leq x \leq 3 \\ 3x, & 3 \leq x \leq 10 \end{cases}$
 $\therefore f(x) = x^2$ is well-defined in $0 \leq x \leq 3$
 and $f(x) = 3x$ is also well-defined in $3 \leq x \leq 10$
 At $x = 3, f(x) = x^2$
 $\Rightarrow f(3) = 3^2 = 9$
 At $x = 3, f(x) = 3x$
 $\Rightarrow f(3) = 3 \times 3 = 9$
 $\Rightarrow f(x)$ is defined at $x = 3$. Hence, f is a function.
 $g(x) = \begin{cases} x^2, & 0 \leq x \leq 2 \\ 3x, & 2 \leq x \leq 10 \end{cases}$
 $\therefore g(x) = x^2$ is well-defined in $0 \leq x \leq 2$
 and $g(x) = 3x$ is also well-defined in $2 \leq x \leq 10$



But at $x = 2, g(x) = x^2$
 $\Rightarrow g(2) = 2^2 = 4$
 At $x = 2, g(x) = 3x$
 $\Rightarrow g(2) = 3 \times 2 = 6$
 Therefore, $g(x)$ is not defined at $x = 2$.
 Hence, g is not a function.

17. We have, $F(x) = \frac{\log_2(x+3)}{x^2+3x+2}$

$\therefore F(x)$ is defined, if $x+3 > 0$
and $x^2+3x+2 \neq 0$

$\Rightarrow F(x)$ is defined, if $x > -3$

and $x \neq -1, -2$

$\Rightarrow$ Domain of $F(x) = (-3, \infty) - \{-1, -2\}$

18. Given, $f(x) = 2 - |x - 5|$

Domain of $f(x)$ is defined for all real values of x .

$\therefore |x - 5| \geq 0 \Rightarrow -|x - 5| \leq 0$

$\Rightarrow 2 - |x - 5| \leq 2 \Rightarrow f(x) \leq 2$

Hence, range of $f(x)$ is $(-\infty, 2]$.

19. $f(x) = \frac{x-2}{2-x} = -\frac{x-2}{x-2} = -1$, if $x \neq 2$

$\therefore$ Range of $f = \{-1\}$

20. $f(x) = |x| \Rightarrow R(f) = [0, \infty)$

21. $D(f+g) = D(f) \cap D(g)$

$(-\infty, 0] \cap [0, \infty) = \{0\}$

22. Given, $[x]^2 - 5[x] + 6 = 0$

$\Rightarrow [x]^2 - 3[x] - 2[x] + 6 = 0$

$\Rightarrow [x]([x] - 3) - 2([x] - 3) = 0$

$\Rightarrow ([x] - 3)([x] - 2) = 0$

$\Rightarrow [x] = 3$ or $[x] = 2$

$\Rightarrow x \in [3, 4)$ or $x \in [2, 3)$

$\therefore x \in [2, 4)$

23. $f\left\{f\left(\frac{1}{x}\right)\right\} = f\left(1 - \frac{1}{(1/x)}\right) = f(1 - x) = 1 - \frac{1}{1-x}$
 $= \frac{1-x-1}{1-x} = \frac{x}{x-1}$

24. $f\{f(x)\}d = f\left(\frac{x-1}{x+1}\right) = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1}$
 $= \frac{\frac{x-1-x-1}{x+1}}{\frac{x-1+x+1}{x+1}} = \frac{-2}{2x} = -\frac{1}{x}$

25. $(fog)(x) = (gof)(x)$ only when

$f(x) = x^m$

and

$g(x) = x^n$

As

$f\{g(x)\} = (x^n)^m = x^{nm}$

and

$g\{f(x)\} = (x^m)^n = x^{nm}$

$\therefore$

$f\{g(x)\} = g\{f(x)\}$

26. $(fof)(x) = f\{f(x)\} = f(x^2+1) = (x^2+1)^2+1$
 $= x^4+2x^2+2$

27. Let $x \in [0, 1]$

Case I $x \in \mathbb{Q}$

$\therefore f(x) = x$

and $(fof)(x) = f\{f(x)\} = f(x) = x$ $[\because x \in \mathbb{Q}]$

Case II $x \notin \mathbb{Q}$

$\therefore f(x) = 1-x$

and $(fof)(x) = f\{f(x)\} = f(1-x)$

$= 1 - (1-x) = x$ $[\because 1-x \notin \mathbb{Q}]$

Now, in both cases, $(fof)(x) = x$

28. Let $x < 0$

$\therefore (gof)(x) = g\{f(x)\} = g(x^3+1)$
 $= [(x^3+1)-1]^{1/3} [\because x < 0 \Rightarrow x^3+1 < 1]$
 $= (x^3)^{1/3} = x$

Let $x \geq 0$

$\therefore (gof)(x) = g\{f(x)\} = g(x^2+1) = [(x^2+1)-1]^{1/2}$
 $[\because x \geq 0 \Rightarrow x^2+1 \geq 1]$
 $= (x^2)^{1/2} = |x| = x$ $[\because x \geq 0]$

$\therefore (gof)(x) = x, \forall x \in \mathbb{R}$

29. $(f+g)(x) = f(x) + g(x) = [x] + (x-[x]) = x \neq 0$

$(fg)(x) = f(x) \cdot g(x) = [x](x-[x]) \neq 0$

$(f-g)(x) = f(x) - g(x) = [x] - (x-[x])$

$= 2[x] - x \neq 0$

$(fog)(x) = f(g(x)) = f(x-[x]) = [x-[x]] = 0$

$[\because x-[x] \in [0, 1)]$

30. $(fof)x = f\left(\frac{x}{x-1}\right) = \frac{\frac{x}{x-1}}{\frac{x}{x-1}-1} = x$

$\Rightarrow (fofof)x = f(fof)(x) = f(x) = \frac{x}{x-1}$

$\therefore (fofof \dots 19 \text{ times})(x) = \frac{x}{x-1}$

31. $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$

Let $z = x + \frac{1}{x}$

$\therefore f(z) = z^2 - 2$

Replacing z by x , we get

$f(x) = x^2 - 2$, when $|x| \geq 2$

32. $f\{g(x)\} = \frac{1-g(x)}{1+g(x)} = \frac{1-4(x)(1-x)}{1+4x(1-x)}$
 $= \frac{1-4x+4x^2}{1+4x-4x^2}$

33. We have, $f(x) = |x-1|$

(a) $|x^2-1| = (|x-1|)^2$, which is not true.

(b) $|x+y-1| = |x-1| + |y-1|$, which is not true.

(c) $||x|-1| = (|x-1|)$, which is not true.

34. $\{(a, 1), (a, 3)\} \subseteq A \times B$

$\therefore$ This is a relation.

$\{(b, 1), (c, 2), (d, 1)\} \subseteq A \times B$

$\therefore$ This is a relation.

$(3, b) \notin A \times B$

$\therefore (a, 2), (b, 3), (3, b)$ is not a relation from A to B .

$(3, c) \notin A \times B$

$\therefore \{(a, 1), (b, 2), (3, c)\}$ is not a relation from A to B .

35. $\therefore R = \{(1, 1), (1, 2), (2, 1), (2, 2), (2, 3), (3, 2), (3, 3)\}$

$R^{-1} = \{(y, x) : (x, y) \in R\}$

$= \{(1, 1), (2, 1), (1, 2), (2, 2), (3, 2), (2, 3), (3, 3)\} = R$

Domain of $R = \{x : (x, y) \in R\} = \{1, 2, 3\}$

and range of $R = \{y : (x, y) \in R\} = \{1, 2, 3\}$

36. We have, $f(x) = \sqrt{x-1}$
 $f(x)$ is defined, if $x-1 \geq 0$
 i.e. $x \geq 1$
 $\therefore$ Domain of $f = [1, \infty)$
 Hence, Statement I is false.
 Let $f(x) = y$
 Then, $y = \sqrt{x-1}$
 $\Rightarrow y^2 = x-1$
 $\Rightarrow x = y^2 + 1$
 $\therefore$ Range of $f = [0, \infty)$ $[\because x \geq 1]$
37. Domain of $f = \{0, 2, 3, 4, 5\}$
 Domain of $g = \{1, 2, 3, 4, 5\}$
 Domain of $f \cdot g$ = Domain of $f \cap$ Domain of $g = \{2, 3, 4, 5\}$
 Hence, Statement I is true.
 If f and g are two real functions.
 Then, $(f \cdot g)(x) = f(x) \cdot g(x)$
 Hence, Statement II is false.
38. Domain of R = Set of first elements of ordered pairs in R
 $= \{4, 9, 25\}$

39. Range of R = Set of second elements of ordered pairs in R
 $= \{-5, 5, -3, 3, -2, 2\}$
40. A. $P = \{1, 2\}, Q = \{1, 2\}$
 $P \cup Q = \{1, 2\}, P \cap Q = \{1, 2\}$
 $\therefore (P \cup Q) \times (P \cap Q) = \{(1, 1), (1, 2), (2, 1), (2, 2)\}$
 B. $A = \{0, 1\}, B = \{2, 3, 4\}, C = \{3, 5\}$
 $B \cap C = \{3\}$
 $A \times (B \cap C) = \{(0, 3), (1, 3)\}$
 C. $R = \{(0, 8), (0, -8), (8, 0), (-8, 0)\}$
 D. $R = \{(-2, 3), (-1, 5), (0, 7), (1, 9), (2, 11)\}$
41. Given, $n(A) = 5, n(B) = 4, n(A \cap B) = 3$
 $n[(A \times B) \cap (B \times A)] = n[(A \cap B) \times (B \cap A)]$
 $= n(A \cap B) \times n(B \cap A) = 3 \times 3 = 9$
42. Since, $f(x)$ is a linear function, $f(x) = mx + c$
 Also, $f(1) = m + c = 1$
 and $f(0) = c = -1$
 This gives $m = 2$
 and $f(x) = 2x - 1$
 $\therefore k = 2$

Entrances Gallery

1. $y = \frac{1}{\sqrt{|x|} - x}$
 For domain, $|x| - x > 0$
 $\Rightarrow |x| > x$
 This is possible, only if $x < 0$.
 $\therefore x \in (-\infty, 0)$
2. Given, $p(x) = f(x) - g(x)$ has only one root -1 .
 $\therefore p(x) = (a - a_1)x^2 + (b - b_1)x + (c - c_1) \dots (i)$
 $\Rightarrow \frac{-(b - b_1)}{(a - a_1)} = -2 \dots (ii)$
 and $\frac{(c - c_1)}{(a - a_1)} = 1 \dots (iii)$
 Also, $p(-2) = 2$
 $\Rightarrow 4(a - a_1) - 2(b - b_1) + (c - c_1) = 2$
 $\Rightarrow 4(a - a_1) - 4(a - a_1) + (a - a_1) = 2$
 $\Rightarrow (a - a_1) = 2$
 Substituting $(a - a_1) = 2$ in Eq. (i) and Eq. (ii),
 $(b - b_1) = 4, (c - c_1) = 2$
 $\therefore p(2) = 4(a - a_1) + 2(b - b_1) + (c - c_1)$
 $[from Eq. (i)]$
 $= 8 + 8 + 2 = 18$
3. Since, $Y = \{y \in Y : y = 4x + 3 \text{ for some } x \in N\}$
 $\therefore Y = \{7, 11, \dots, \infty\}$
 Now, $y = 4x + 3$
 $\Rightarrow x = \frac{y-3}{4}$
 Inverse of $f(x)$ is $g(y) = \frac{y-3}{4}$.
4. Given, $f(x - y) = f(x)f(y) - f(a - x)f(a + y)$
 Let $x = 0 = y$
 $\Rightarrow f(0) = \{f(0)\}^2 - \{f(a)\}^2$
5. $\sum_{r=1}^n f(r) = f(1) + f(2) + f(3) + \dots + f(n)$
 $= f(1) + 2f(1) + 3f(1) + \dots + nf(1)$
 $[since, f(x + y) = f(x) + f(y)]$
 $= (1 + 2 + 3 + \dots + n)f(1) = f(1) \sum n$
 $= \frac{7n(n+1)}{2} [\because f(1) = 7, given]$
6. Since, $A = \{1, 2, 3, 4\}$ and R be a relation on A defined by $\{(a, b); a, b \in A, a \times b \text{ is an even number}\}$.
 Thus, R be the set of even numbers.
 $\therefore$ Range of $R = \{2, 4\}$
7. $\therefore f(x) = \frac{(x^2 + 1)}{(x^2 - 3x + 3)}$
 $\Rightarrow (x^2 + 1)$ is defined for all $x \in R$.
 Also, $x^2 - 3x + 3$ has imaginary roots.
 $\therefore x^2 - 3x + 3$ is also defined for all real numbers,
 i.e. $x \in R$.
 Hence, domain of the function is R .
8. Since, $A = \{1, 2, 3, 4, 5\}$
 $\therefore$ Relation from A to A
 $R = \{(x, y) : y = 2x - 1\} = \{(1, 1), (2, 3), (3, 5)\}$
 $\Rightarrow$ Domain $= \{1, 2, 3\}$
9. Given, $f(x) = \sqrt{x}$
 and $g(x) = 2x - 3$
 $\therefore (fog)(x) = f\{g(x)\}$
 $= f(2x - 3) = \sqrt{2x - 3}$

For domain of $(fog)(x)$,

$$2x - 3 \geq 0 \Rightarrow 2x \geq 3$$

$$\Rightarrow x \geq \frac{3}{2}$$

$$\therefore x \in \left[\frac{3}{2}, \infty \right)$$

10. Given, $f(x) = \frac{x+2}{3x-1}$

$$\begin{aligned} \therefore f\{f(x)\} &= f\left(\frac{x+2}{3x-1}\right) = \frac{\frac{x+2}{3x-1} + 2}{3\left(\frac{x+2}{3x-1}\right) - 1} \\ &= \frac{x+2+6x-2}{3x+6-(3x-1)} \\ &= \frac{7x}{7} = x \end{aligned}$$

11. Given, $f(x) = \log_e(3x^2 + 4)$

Let $y = \log_e(3x^2 + 4)$

$$\Rightarrow 3x^2 + 4 = e^y$$

$$\Rightarrow x = \sqrt{\frac{e^y - 4}{3}}$$

Here, $\frac{e^y - 4}{3} \geq 0$

$$\Rightarrow e^y \geq 4$$

$$\Rightarrow y \geq 2 \log_e 2$$

12. Given, $f(x) = x^2 + 2x - 3$

and $g(x) = x + 1$

$$\therefore f\{g(x)\} = g\{f(x)\}$$

$$\Rightarrow f(x+1) = g(x^2 + 2x - 3)$$

$$\Rightarrow (x+1)^2 + 2(x+1) - 3 = x^2 + 2x - 3 + 1$$

$$\Rightarrow x^2 + 2x + 1 + 2x - 1 = x^2 + 2x - 2$$

$$\Rightarrow 2x = -2$$

$$\therefore x = -1$$

13. $\{(z, b), (y, b), (a, d)\}$ is not a relation from A to B because $a \notin A$.

14. Given, $f(x) = 3 - x$, here $-4 \leq x \leq 4$

To find domain of $\log_e \{f(x)\} = \log_e(3 - x)$

For $\log_e \{f(x)\}$ to be defined,

$$3 - x > 0 \Rightarrow 3 > x$$

$$\Rightarrow x < 3$$

$$\Rightarrow \text{Domain} = [-4, 3)$$

15. Here, $f(0) = 0 + b \Rightarrow -1 = b$

and $f(-1) = -a + b$

$$\Rightarrow -3 = -a + b$$

$$\Rightarrow -a = -3 + 1$$

$$\Rightarrow a = 2$$

16. $(gof)(x) = g\{f(x)\} = g(x^2 - 1) = (x^2 - 1 + 1)^2 = x^4$

17. $(fog)(-1) = f\{g(-1)\} = f(-7) = 5 - 49 = -44$

18. We have, $A = \{1, 0, 1, 2\}$

$$B = \{4, 2, 0, 2\}$$

and $f, g : A \rightarrow B$

Now, $f(x) = x^2 - x$

$$\Rightarrow f(0) = 0 - 0 = 0$$

and $f(1) = 1 - 1 = 0$

and $f(2) = 2^2 - 2 = 2$

Also, $g(x) = 2 \left| x - \frac{1}{2} \right| - 1$

$$\Rightarrow g(0) = 2 \left| \frac{-1}{2} \right| - 1 = 1 - 1 = 0$$

and $g(1) = 2 \left| 1 - \frac{1}{2} \right| - 1 = 1 - 1 = 0$

and $g(2) = 2 \left| 2 - \frac{1}{2} \right| - 1$

$$= \frac{2 \cdot 3}{2} - 1 = 2$$

Thus, $f(x) = g(x), \forall x$

Hence, $f = g$

3

Sequence and Series

Introduction

Sequence Sequence is a function whose domain is the set N of natural numbers.

Real Sequence A sequence whose range is a subset of R , is called a real sequence.

Progression It is not necessary that the terms of a sequence always follow a certain pattern or they are described by some explicit formula for the n th term. Those sequences whose terms follow certain patterns, are called progressions.

Series By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, a_4, \dots, a_n, \dots$ is a sequence, then the expression $a_1 + a_2 + a_3 + \dots + a_n + \dots$ is the corresponding series.

A series is finite or infinite according as corresponding sequence is finite or infinite, i.e. the number of terms in the corresponding sequence is finite or infinite.

➤ **Example 1.** The first 3 terms of the sequence which is defined by $a_n = \frac{n}{n^2 + 1}$,

are

- (a) $2, \frac{5}{2}, \frac{10}{3}$ (b) $\frac{1}{2}, \frac{2}{5}, \frac{3}{10}$ (c) $\frac{2}{5}, \frac{1}{2}, 3$ (d) None of these

Sol. (b) On putting $n = 1, 2, 3$ in $a_n = \frac{n}{n^2 + 1}$, we get

$$a_1 = \frac{1}{1^2 + 1} = \frac{1}{2}, \quad a_2 = \frac{2}{2^2 + 1} = \frac{2}{5}$$

and $a_3 = \frac{3}{3^2 + 1} = \frac{3}{10}$

Hence, the first 3 terms are $\frac{1}{2}, \frac{2}{5}$ and $\frac{3}{10}$.

Arithmetic Progression (AP)

A sequence is called an arithmetic progression, if the difference of a term and the previous term is always same, i.e.

$$a_{n+1} - a_n = \text{Constant } (d), \forall n \in N$$

The constant difference, generally denoted by d is called the common difference.

Chapter Snapshot

- Introduction
- Arithmetic Progression (AP)
- Geometric Progression (GP)
- Harmonic Progression (HP)
- Arithmetico-Geometric Progression (AGP)
- Some Special Series

➤ **Example 2.** Show that the sequence $\langle a_n \rangle$ defined by $a_n = 4n + 5$ is an AP. Also, find its common difference.

Sol. We have, $a_n = 4n + 5$

On replacing n by $(n + 1)$, we get

$$a_{n+1} = 4(n + 1) + 5 = 4n + 9$$

$$\text{Now, } a_{n+1} - a_n = (4n + 9) - (4n + 5)$$

$$\text{Clearly, } a_{n+1} - a_n = 4, \forall n \in N$$

So, the given sequence is an AP with common difference 4.

Properties of an Arithmetic Progression

i. If a is the first term and d is the common difference of an AP, then its n th term or general term a_n is given by

$$a_n = a + (n - 1)d$$

➤ **Example 3.** If the sequence 9, 12, 15, 18, ... is an AP, then its general term is

$$(a) 3n + 1 \quad (b) 3n + 2 \quad (c) 3n + 4 \quad (d) 3n + 6$$

Sol. (d) We have, $d = (12 - 9) = 3$

As the given sequence is an AP with common difference 3 and first term 9.

$\therefore$ General term = n th term

$$= a + (n - 1)d = 9 + (n - 1)3 = 3n + 6$$

➤ **Example 4.** Let T_r be the r th term of an AP for $r = 1, 2, 3, \dots$, if for some positive integers m, n , we have $T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then T_{mn} is equal to

$$(a) \frac{1}{mn} \quad (b) \frac{1}{m} + \frac{1}{n} \quad (c) 1 \quad (d) 0$$

Sol. (c) Let a be the first term and d be the common difference.

$$\therefore T_m = a + (m - 1)d = \frac{1}{n} \quad \dots(i)$$

$$\text{and } T_n = a + (n - 1)d = \frac{1}{m} \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$(m - n)d = \frac{1}{n} - \frac{1}{m} = \frac{m - n}{mn}$$

$$\Rightarrow d = \frac{1}{mn}$$

$$\begin{aligned} \text{Again, } T_{mn} &= a + (mn - 1)d \\ &= a + (mn - n + n - 1)d \\ &= a + (n - 1)d + (mn - n)d \\ &= T_n + n(m - 1)\frac{1}{mn} \\ &= \frac{1}{m} + \frac{(m - 1)}{m} = 1 \end{aligned}$$

ii. A sequence is an AP iff its n th term is of the form $An + B$, i.e. a linear expression in n . The common difference in such a case is A , i.e. the coefficient of n .

➤ **Example 5.** If the sequence $\langle a_n \rangle$ is defined by $a_n = 3n + 2$, then difference of any two consecutive terms is

$$(a) 2 \quad (b) 3 \quad (c) 4 \quad (d) 5$$

Sol. (b) Since, n th term is of the form $An + B$, therefore it is an AP and the common difference is coefficient of n i.e. 3.

iii. If a constant is added to or subtracted from each term of an AP, then the resulting sequence is also an AP with the same common difference.

➤ **Example 6.** If a, b and c are in AP, then which of the following will also be in AP?

$$(a) \sqrt{3} + a, \sqrt{3} + b, \sqrt{3} + c$$

$$(b) a - 1, b - 2, c - 3$$

$$(c) a + 1, b + 2, c + 3$$

$$(d) a - k, b - 2k, c - 3k$$

Sol. (a) $a + \sqrt{3}, b + \sqrt{3}$ and $c + \sqrt{3}$ are in AP because each term of the given AP is being added by a constant number $\sqrt{3}$.

iv. If each term of a given AP is multiplied or divided by a non-zero constant k , then the resulting sequence is also an AP with common difference kd or d/k respectively, where d is the common difference of the given AP.

➤ **Example 7.** If a, b and c are in AP, then prove that

$$a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{c} + \frac{1}{a}\right), c\left(\frac{1}{a} + \frac{1}{b}\right) \text{ are in AP.}$$

Sol. Since, a, b and c are in AP.

On dividing each term by abc , the resulting sequence

$$\frac{a}{abc}, \frac{b}{abc} \text{ and } \frac{c}{abc} \text{ are in AP.}$$

$$\Rightarrow \frac{1}{bc}, \frac{1}{ac} \text{ and } \frac{1}{ab} \text{ are in AP.}$$

$$\Rightarrow \frac{ab + bc + ca}{bc}, \frac{ab + bc + ca}{ac}, \frac{ab + bc + ca}{ab} \text{ are in AP}$$

[on multiplying each term by $ab + bc + ca$]

$$\Rightarrow \frac{ab + bc + ca}{bc} - 1, \frac{ab + bc + ca}{ac} - 1, \frac{ab + bc + ca}{ab} - 1$$

are in AP.

[on adding -1 to each term]

$$\Rightarrow \frac{ab + ac}{bc}, \frac{ab + bc}{ca}, \frac{bc + ca}{ab} \text{ are in AP.}$$

$$\Rightarrow a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{c} + \frac{1}{a}\right), c\left(\frac{1}{a} + \frac{1}{b}\right) \text{ are in AP.}$$

Hence proved.

v. In a finite AP, the sum of the terms equidistant from the beginning and end is always same and is equal to the sum of first and last term, i.e.

$$a_k + a_{n(k-1)} = a_1 + a_n \text{ for all } k = 1, 2, 3, \dots, (n - 1).$$

➤ **Example 8.** If an AP is given by 3, a_2 , ..., 20, then which of the following is the sum of second and second last terms?

- (a) 20 (b) 3 (c) 23 (d) 22

Sol. (c) As we know, $a_k + a_{n-(k-1)} = a_1 + a_n$

On putting $k = 2$, we get

$$a_2 + a_{n-1} = a_1 + a_n = 3 + 20 = 23$$

➤ **Example 9.** If $a_1, a_2, a_3, \dots, a_{2n+1}$ are in AP, then $(a_{2n+1} + a_1) + (a_{2n} + a_2) + \dots + (a_{n+2} + a_n)$ is equal to

- (a) $2na_n$ (b) $2na_{n-1}$
(c) $2na_{n+1}$ (d) None of these

Sol. (c) Clearly, $a_1 + a_{2n+1} = a_2 + a_{2n} = a_3 + a_{2n-1}$

$$= \dots = a_{n+2} + a_n$$

and

$$a_r = a_1 + (r-1)d$$

$$\begin{aligned} \therefore (a_{2n+1} + a_1) + (a_{2n} + a_2) + \dots + (a_{n+2} + a_n) \\ = n(a_{2n+1} + a_1) = n(2a_1 + 2nd) \\ = 2n(a_1 + nd) = 2na_{n+1} \end{aligned}$$

vi. Three numbers a, b and c are in AP iff
 $2b = a + c$

➤ **Example 10.** If a, b and c are in AP, then

$$a + \frac{1}{bc}, b + \frac{1}{ca} \text{ and } c + \frac{1}{ab} \text{ are in}$$

- (a) HP (b) AP
(c) GP (d) None of these

Sol. (b) $a + \frac{1}{bc}, b + \frac{1}{ca}$ and $c + \frac{1}{ab}$ are in AP, if and only if

$$\begin{aligned} \left(b + \frac{1}{ca}\right) - \left(a + \frac{1}{bc}\right) &= \left(c + \frac{1}{ab}\right) - \left(b + \frac{1}{ca}\right) \\ \Rightarrow (b-a) + \frac{(b-a)}{abc} &= (c-b) + \frac{(c-b)}{abc} \\ \Rightarrow (b-a)\{abc+1\} &= (c-b)\{abc+1\} \\ \Rightarrow b-a &= c-b \\ \Rightarrow 2b &= a+c \end{aligned}$$

which is true, as given a, b and c are in AP.

vii. If the terms of an AP are chosen at regular intervals, then they form an AP.

➤ **Example 11.** If $T_1, T_2, T_3, T_4, T_5, T_6, T_7, \dots$ are in AP with common difference 2. Then, the difference of any two consecutive terms of the sequence $T_1, T_4, T_7, \dots$ is

- (a) 3 (b) 4 (c) 6 (d) 8

Sol. (c) Clearly, $T_1, T_4, T_7, \dots$ are in AP.

$$\text{Now, } T_4 - T_1 = T_1 + 3 \times 2 - T_1 = 6$$

viii. If a_n, a_{n+1} and a_{n+2} are any three consecutive terms of an AP, then
 $2a_{n+1} = a_n + a_{n+2}$

➤ **Example 12.** Four numbers are in arithmetic progression. The sum of first and last term is 8 and the product of both middle terms is 15. The least number of the sequence is

- (a) 4 (b) 3 (c) 2 (d) 1

Sol. (d) Let A_1, A_2, A_3 and A_4 be four numbers in AP.

$$\text{Given, } A_1 + A_4 = 8 \quad \dots (i)$$

$$\text{and } A_2 A_3 = 15 \quad \dots (ii)$$

$$\text{We know, } A_2 + A_3 = A_1 + A_4$$

$$\therefore A_2 + A_3 = 8 \quad \dots (iii)$$

On solving Eqs. (ii) and (iii), we get

$$A_2 = 3, A_3 = 5 \text{ or } A_2 = 5, A_3 = 3$$

$$\therefore 2A_2 = A_1 + A_3 \quad [\text{by property}]$$

$$\Rightarrow 2 \times 3 = A_1 + 5 \text{ or } 2 \times 5 = A_1 + 3$$

$$\Rightarrow A_1 = 1 \text{ or } A_1 = 7$$

$$\text{Similarly, } A_4 = 7 \text{ or } A_4 = 1$$

Hence, least number of sequence is 1.

Selection of Terms in an AP

Sometimes, it is required to select a finite number of terms in AP. It is always convenient, if we select the terms in the following manner:

Selecting Odd Number of Terms of AP

i. Selecting 3 terms of AP

$$a - d, a, a + d$$

ii. Selecting 5 terms of AP

$$a - 2d, a - d, a, a + d, a + 2d$$

➤ **Example 13.** If the sum of three numbers in AP is -3 and their product is 8, then the numbers are

- (a) 2, 1, 4 (b) 2, -1 , -4
(c) 4, 2, 1 (d) None of these

Sol. (b) Let the numbers be $a - d, a$ and $a + d$.

$$\text{Given, sum} = -3$$

$$\Rightarrow (a - d) + a + (a + d) = -3$$

$$\Rightarrow 3a = -3 \Rightarrow a = -1$$

$$\text{and product} = 8$$

$$\Rightarrow (a - d)a(a + d) = 8$$

$$\Rightarrow a(a^2 - d^2) = 8$$

$$\Rightarrow (-1)(1 - d^2) = 8$$

$$\Rightarrow d^2 = 9$$

$$\Rightarrow d = \pm 3$$

If $d = 3$, then the numbers are $-4, -1, 2$.

If $d = -3$, then the numbers are $2, -1, -4$.

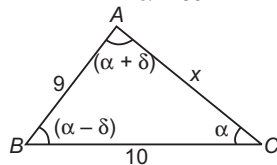
➤ **Example 14.** In a triangle, the lengths of the two larger sides are 10 and 9, respectively. If the angles are in AP, then the length of the third side can be

- (a) $\sqrt{91}$ (b) $3\sqrt{3}$
(c) 5 (d) None of these

Sol. (d) Since, angles of a triangle are in AP.

$$\therefore \alpha + (\alpha - \delta) + (\alpha + \delta) = 180^\circ$$

$$\Rightarrow \alpha = 60^\circ$$



Using cosine law in $\triangle ACB$,

$$\cos 60^\circ = \frac{(10)^2 + (x)^2 - 9^2}{2 \cdot (10) \cdot (x)}$$

$$\Rightarrow \frac{1}{2} = \frac{19 + x^2}{20x}$$

$$\Rightarrow x^2 - 10x + 19 = 0$$

$$\Rightarrow x = 5 \pm \sqrt{6}$$

Selecting Even Number of Terms of AP

- i. Selecting 2 terms of AP
 $a - d, a + d$
- ii. Selecting 4 terms of AP
 $a - 3d, a - d, a + d, a + 3d$

- It should be noted that in case of odd number of terms, the middle term is a and the common difference is d while in case of an even number of terms the middle terms are $a - d, a + d$ and the common difference is $2d$.
- If sum of the numbers is not given, then the numbers can be taken as $a, a + d, a + 2d, a + 3d, \dots$

Example 15. If the sum of four numbers of AP is 20, then which of the following is first term of the given AP?

- (a) 5 (b) 4 (c) 6 (d) 10

Sol. (a) Let the four numbers in AP be $a - 3d, a - d, a + d, a + 3d$.

$$\text{Now, } a - 3d + a - d + a + d + a + 3d = 20$$

$$\Rightarrow 4a = 20 \Rightarrow a = 5$$

Example 16. Divide 32 into four parts which are in AP such that the ratio of product of extremes to the product of means is 7:15.

- (a) 2, 6, 10, 14 (b) 3, 9, 15, 18
(c) 4, 12, 20, 28 (d) 5, 15, 25, 35

Sol. (a) Let the four parts be $a - 3d, a - d, a + d$ and $a + 3d$.

$$\therefore \text{Sum} = 32$$

$$\Rightarrow (a - 3d) + (a - d) + (a + d) + (a + 3d) = 32$$

$$\Rightarrow 4a = 32 \Rightarrow a = 8$$

$$\text{and } \frac{(a - 3d)(a + 3d)}{(a - d)(a + d)} = \frac{7}{15}$$

$$\Rightarrow \frac{a^2 - 9d^2}{a^2 - d^2} = \frac{7}{15}$$

$$\Rightarrow \frac{64 - 9d^2}{64 - d^2} = \frac{7}{15}$$

$$\Rightarrow 128d^2 = 512$$

$$\Rightarrow d^2 = 4 \Rightarrow d = \pm 2$$

Thus, the four parts are 2, 6, 10, 14.

Sum of n Terms of an AP

The sum S_n of n terms of an AP with first term a and common difference d is given by

$$S_n = \frac{n}{2} [2a + (n-1)d]$$

$$\text{Also, } S_n = \frac{n}{2} [a + l]$$

where, l = last term $= a + (n-1)d$

Example 17. Which of the following is the sum of 20 terms of an AP 1, 4, 7, 10, ...?

- (a) 690 (b) 590
(c) 591 (d) 691

Sol. (b) Let a be the first term and d be the common difference of the given AP. Then, $a = 1$ and $d = 3$.

We have to find the sum of 20 terms of the given AP.

On putting $a = 1, d = 3$ and $n = 20$ in $S_n = \frac{n}{2} [2a + (n-1)d]$, we get

$$S_{20} = \frac{20}{2} [2 \times 1 + (20-1) \times 3]$$

$$= 10 \times 59 = 590$$

Example 18. If S_r denotes the sum of the first r terms of an AP, then $\frac{S_{3r} - S_{r-1}}{S_{2r} - S_{2r-1}}$ is equal to

- (a) $2r - 1$ (b) $2r + 1$
(c) $4r + 1$ (d) $2r + 3$

Sol. (b) $\frac{S_{3r} - S_{r-1}}{S_{2r} - S_{2r-1}}$

$$= \frac{\frac{3r}{2} [2a + (3r-1)d] - \frac{(r-1)}{2} [2a + (r-2)d]}{\frac{2r}{2} [2a + (2r-1)d] - \frac{(2r-1)}{2} [2a + (2r-2)d]}$$

$$= \frac{2a(2r+1) + d(8r^2-2)}{2a + d(4r-2)}$$

$$= \frac{(2r+1)[2a + 2(2r-1)d]}{[2a + 2(2r-1)d]} = (2r+1)$$

A sequence is an AP if and only if the sum of its n terms is of the form $An^2 + Bn$, where A, B are constants. In such a case, the common difference of the AP is $2A$.

Example 19. If the sum of n terms of an AP is $3n^2 + 5n$, then the common difference is

- (a) 2 (b) 3
(c) 5 (d) 6

Sol. (d) Since, $S_n = 3n^2 + 5n$ is of the form $An^2 + Bn$.

Therefore, the common difference is $2A = 6$.

Arithmetic Mean of Two Given Numbers

Let a and b be two given numbers, then the arithmetic mean of a and b is a number A such that a, A, b are in AP.

Now, a, A, b are in AP.

$$\Rightarrow 2A = a + b \Rightarrow A = \frac{a+b}{2}$$

Thus, the arithmetic mean of two numbers a and b is given by $A = \frac{a+b}{2}$.

Arithmetic Mean of n Given Numbers

Let $a, a_2, \dots, a_n$ be n given numbers, then the arithmetic mean of $a_1, a_2, a_3, \dots, a_n$ is given by

$$A = \frac{a_1 + a_2 + a_3 + \dots + a_n}{n}$$

Insertion of n Arithmetic Means between a and b

If between two given quantities a and b , we have to insert n quantities $A_1, A_2, \dots, A_n$ such that $a, A_1, A_2, \dots, A_n, b$ form an AP, then we say that $A_1, A_2, \dots, A_n$ are n arithmetic means between a and b .

Let d be the common difference of this AP. Clearly, it contains $(n+2)$ terms.

$$\therefore b = (n+2)\text{th term}$$

$$\Rightarrow b = a + (n+1)d \Rightarrow d = \frac{b-a}{n+1}$$

Now, $A_1 = a + d, A_2 = a + 2d, \dots$ and $A_n = a + nd$

Sum of n AM's between a and b is $n\left(\frac{a+b}{2}\right)$ i.e.

$$A_1 + A_2 + \dots + A_n = nA$$

Example 20. Three arithmetic means between 3 and 19 are

- (a) 7, 11, 15 (b) 7, 10, 13
(c) 7, 12, 17 (d) None of these

Sol. (a) Let A_1, A_2, A_3 be three AM's between 3 and 19. Then, 3, $A_1, A_2, A_3, 19$ are in AP whose common difference is

$$d = \frac{19-3}{3+1} = 4$$

$$\therefore \begin{aligned} A_1 &= 3 + d = 3 + 4 = 7 \\ A_2 &= 3 + 2d = 3 + 8 = 11 \\ A_3 &= 3 + 3d = 3 + 12 = 15 \end{aligned}$$

Hence, the required AM's are 7, 11, 15.

Example 21. If $(2n+r)r, n \in N, r \in N$ is expressed the sum of k consecutive odd natural numbers, then k is equal to

- (a) r (b) n (c) $r+1$ (d) $n+1$

Sol. (a) $(2n+r)r = (n+r)^2 - n^2$
 $= \{1 + 3 + 5 + \dots \text{to } (n+r) \text{ terms}\}$
 $- \{1 + 3 + 5 + \dots \text{to } n \text{ terms}\}$
 as $(n+r)^2 - n^2 = \{1 + 3 + 5 + \dots (n+r) \text{ terms}\}$
 $- \{1 + 3 + 5 + \dots n \text{ terms}\}$
 $\therefore$ Sum of r consecutive odd natural numbers,
 $k = r$

Example 22. The sum of the products of the numbers $\pm 1, \pm 2, \pm 3, \pm 4, \pm 5$ taking two at a time is

- (a) 165
(b) -55
(c) 55
(d) None of the above

Sol. (b) $(1 - 1 + 2 - 2 + \dots + 5 - 5)^2$
 $= 1^2 + 1^2 + 2^2 + 2^2 + \dots + 5^2 + 5^2 + 2S$
 where, S is the required number.
 or $0 = 2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 2S$
 $\therefore 2S = \frac{-5(5+1)(10+1)}{6} = -55$

Example 23. Observe that $1^3 = 1, 2^3 = 3 + 5, 3^3 = 7 + 9 + 11, 4^3 = 13 + 15 + 17 + 19$. Then, n^3 as a similar series is

- (a) $\left[2\left\{\frac{n(n-1)}{2} + 1\right\} - 1\right] + \left[2\left\{\frac{(n+1)n}{2} + 1\right\} + 1\right]$
 $+ \dots + \left[2\left\{\frac{(n+1)n}{2} + 1\right\} + 2n - 3\right]$
 (b) $(n^2 + n + 1) + (n^2 + n + 3) + (n^2 + n + 5)$
 $+ \dots + (n^2 + 3n - 1)$
 (c) $(n^2 - n + 1) + (n^2 - n + 3) + (n^2 - n + 5)$
 $+ \dots + (n^2 + n - 1)$
 (d) None of the above

Sol. (c) $1^3 = 1 \cdot (1-1) + 1, 2^3 = (2 \cdot 1 + 1) + 5$

$$3^3 = (3 \cdot 2 + 1) + 9 + 11$$

$$4^3 = (4 \cdot 3 + 1) + 15 + 17 + 19, \text{ etc.}$$

$$\therefore n^3 = \{n \cdot (n-1) + 1\} + \dots$$

next term being 2 more than the previous

$$= (n^2 - n + 1) + (n^2 - n + 3) + \dots + (n^2 + n - 1)$$

Example 24. Along a road lies an odd number of stones placed at intervals of 10 m. These stones have to be assembled around the middle stone. A person can carry only one stone at a time. A man carried out the job starting with the stone in the middle, carrying stones in succession, thereby covering a distance of 4.8 km. Then, the number of stones is

- (a) 15 (b) 29
(c) 31 (d) 35

Sol. (c) Let there be $2n+1$ stone i.e. n stones on each side of the middle stone. The man will run 20 m, to pick up the first stone and return, 40 m for the second stone and so on. So, he runs $(n/2) \cdot \{2 \times 20 + (n-1)20\} = 10n(n+1)$ m to pick up the stones on one side and hence $20n(n+1)$ m to pick up all the stones.

$$\therefore 20n(n+1) = 4800 \text{ or } n = 15$$

Hence, there are $2n+1 = 31$ stones.

- 1 If the n th term of an AP 5, 8, 11, ... is 320, then n is equal to
105 104 106 112
- 2 If the 5th term of an AP is 11 and the 9th term is 7, then the 14th term is
-1 2 1 0
- 3 If $p-1$, $p+3$ and $3p-1$ are in AP, then p is equal to
a 4 b -4 c 2 d -2
- 4 The 15th term of sequence $x-7$, $x-2$ and $x+3$ is
 $x+63$ $x-63$ $63-x$ $63+x$
- 5 If the sum of three numbers of an AP is 24, then middle term is
6 8 3 2
- 6 The sum of the integers from 1 to 100 that are divisible by 2 or 5, is
3000 3050
3600 None of these
- 7 If N which is the set of natural numbers, is partitioned into subsets $S_1 = \{1\}$, $S_2 = \{2, 3\}$, $S_3 = \{4, 5, 6\}$, ..., then the sum of the members in S_{50} is
62255 62525
65225 62555
- 8 If $\frac{3+5+7+\dots+n \text{ terms}}{5+8+11+\dots+10 \text{ terms}} = 7$, then the value of n is
35 36 37 40
- 9 The sum of the first ten terms of an AP is four times the sum of the first five terms, then the ratio of the first term to the common difference is
 $\frac{1}{2}$ 2 $\frac{1}{4}$ 4
- 10 If p, q and r are in AP, then $p^3 + r^3 - 8q^3$ is equal to
 $-6pqr$ $4pqr$
 $2pqr$ None of these
- 11 Consider an AP with first term a and common difference d . Let S_k denotes the sum of first k terms. If $\frac{S_{kx}}{S_x}$ is independent of x , then
 $a = 2d$ $a = d$
 $2a = d$ None of these
- 12 If $a_1, a_2, a_3, \dots$ are in AP such that
 $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$,
then $a_1 + a_2 + a_3 + \dots + a_{23} + a_{24}$ is equal to
909 75 750 900
- 13 If the numbers a, b, c, d, e are in AP, then the value of $a - 4b + 6c - 4d + e$ is
1 2 0 $16c$
- 14 Sum of n terms of the series
 $\frac{1}{\sqrt{2} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{8}} + \frac{1}{\sqrt{8} + \sqrt{11}} + \dots$ is
 $\sqrt{3n+2} - \sqrt{2}$ $\frac{1}{3}(\sqrt{3n+2} - \sqrt{2})$
 $\sqrt{3n+2} + \sqrt{2}$ None of these

Geometric Progression (GP)

A sequence of non-zero numbers is called a geometric progression, if the ratio of a term and the term preceding to it is always a constant quantity.

The constant ratio is called the common ratio of the GP.

In other words, a sequence $a_1, a_2, a_3, \dots, a_n, \dots$ is called a geometric progression, if

$$\frac{a_{n+1}}{a_n} = \text{Constant}, \forall n \in N$$

Example 25. Show that the sequence given by $a_n = 3(2^n)$, $\forall n \in N$ is a GP. Also, find its common ratio.

Sol. We have, $a_n = 3(2^n)$ and $a_{n+1} = 3(2^{n+1})$

$$\text{Now, } \frac{a_{n+1}}{a_n} = \frac{3(2^{n+1})}{3(2^n)} = 2$$

Clearly, $\frac{a_{n+1}}{a_n} = 2$ (constant), $\forall n \in N$. So, the given sequence is a GP with common ratio 2.

Example 26. Which of the following is a GP?

(a) 2, 4, 8, 16, ...

(b) $\frac{1}{9}, \frac{-1}{27}, \frac{1}{81}, \frac{-1}{243}, \dots$

(c) 0.01, 0.0001, 0.000001, ...

(d) All of the above

Sol. (d)

(a) We have, $a_1 = 2, \frac{a_2}{a_1} = 2, \frac{a_3}{a_2} = 2, \frac{a_4}{a_3} = 2$ and so on

(b) We observe, $a_1 = \frac{1}{9}, \frac{a_2}{a_1} = \frac{-1}{3},$

$\frac{a_3}{a_2} = \frac{-1}{3}, \frac{a_4}{a_3} = \frac{-1}{3}$ and so on.

(c) We have, $a_1 = 0.01, \frac{a_2}{a_1} = 0.01,$

$\frac{a_3}{a_2} = 0.01, \frac{a_4}{a_3} = 0.01$ and so on.

It is observed that in each case, every term except the first term bears a constant ratio to the term immediately preceding it.

Thus, options (a), (b) and (c) all are in GP.

General Term of a GP

If a is the first term of a GP and r is the common ratio, then its n th term (general term) t_n is given by

$$t_n = ar^{n-1}$$

➤ **Example 27.** The n th and 10th terms of a GP 5, 25, 125, ..., are respectively

- (a) $5^{n-1}, 5^9$ (b) $5^{n-2}, 5^8$
(c) $5^n, 5^{10}$ (d) None of these

Sol. (c) Here, $a = 5$ and $r = 5$. Thus, the n th term of the given GP is given by $a_n = ar^{n-1}$

$$\Rightarrow a_n = 5(5)^{n-1} = 5^n$$

Now, for $n = 10$, we have $a_{10} = 5^{10}$

➤ **Example 28.** The third term of a geometric progression is 4. The product of the first five terms is

- (a) 4^3 (b) 4^5
(c) 4^4 (d) None of these

Sol. (b) Here, $t_3 = 4 \Rightarrow ar^2 = 4$

$$\therefore \text{Product of first five terms} = a \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4 \\ = a^5 r^{10} = (ar^2)^5 = 4^5$$

Selection of Terms in GP

Sometimes, it is required to select a finite number of terms in GP. It is always convenient, if we select the terms in the following manner:

Number of terms	Terms	Common ratio
3	$\frac{a}{r}, a, ar$	r
4	$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$	r^2
5	$\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$	r

If the product of the numbers is not given, then the numbers can be taken as $a, ar, ar^2, ar^3, \dots$

➤ **Example 29.** If the sum of three numbers in GP is 38 and their product is 1728, then numbers are

- (a) 8, 12, 18 (b) 8, 16, 32
(c) 18, 12, 8 (d) None of these

Sol. (a, c) Let the three numbers be $\frac{a}{r}, a$ and ar .

$$\text{Then, product} = 1728 \\ \Rightarrow \frac{a}{r} \cdot a \cdot ar = 1728$$

$$\Rightarrow a^3 = 1728$$

$$\Rightarrow a = 12$$

$$\therefore \text{Sum} = 38$$

$$\therefore \frac{a}{r} + a + ar = 38 \Rightarrow a \left(\frac{1}{r} + 1 + r \right) = 38$$

$$\Rightarrow 12 \left(\frac{1+r+r^2}{r} \right) = 38$$

$$\Rightarrow 6 + 6r + 6r^2 = 19r \\ \Rightarrow 6r^2 - 13r + 6 = 0 \\ \Rightarrow (3r-2)(2r-3) = 0 \\ \Rightarrow r = 3/2 \text{ and } 2/3$$

Hence, on putting the values of a and r , the required numbers are 8, 12, 18 or 18, 12, 8.

➤ **Example 30.** If four numbers in GP are such that the third number is greater than the first by 9 and the second number is greater than the fourth by 18, then the numbers are

- (a) 3, 6, 12, 24 (b) -3, 6, -12, 24
(c) 3, -6, 12, -24 (d) None of these

Sol. (c) Let the four numbers be a, ar, ar^2 and ar^3 .

$$\text{It is given that, } ar^2 - a = 9$$

$$\text{and } ar - ar^3 = 18$$

$$\Rightarrow a(r^2 - 1) = 9 \quad \dots(i)$$

$$\text{and } ar(1 - r^2) = 18 \quad \dots(ii)$$

On dividing Eq. (ii) by Eq. (i), we get

$$\frac{ar(1-r^2)}{a(r^2-1)} = \frac{18}{9} \Rightarrow r = -2$$

On putting $r = -2$ in $a(r^2 - 1) = 9$, we get $a = 3$

Hence, the numbers are 3, -6, 12, -24.

Properties of Geometric Progressions

i. If all the terms of a GP is multiplied or divided by the same non-zero constant, then it remains a GP with the same common ratio.

➤ **Example 31.** If 5, 25, 125, ... are in GP, then 1, 5, 25, ... are in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. (b) $\because 5, 25, 125, \dots$ are in GP.

On dividing each term by 5, we get the following sequence 1, 5, 25, ..., which is again a GP as here

$$a_1 = 1, \frac{a_2}{a_1} = 5, \frac{a_3}{a_2} = 5 \text{ and so on.}$$

ii. The reciprocals of the terms of a given GP form a GP.

➤ **Example 32.** If $\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \dots$ are in GP,

then $2, 2^2, 2^3, \dots$ are in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. (b) On reciprocating each term of a GP, we get the following sequence $2, 2^2, 2^3, 2^4, \dots$ which is again a GP as

$$\text{here } a_1 = 2, \frac{a_2}{a_1} = 2, \frac{a_3}{a_2} = 2 \text{ and so on.}$$

iii. If each term of a GP is raised to the same power, then the resulting sequence also form a GP.

➤ **Example 33.** If a, b and c are in GP, then $a^{1/2}, b^{1/2}, c^{1/2}$ are in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. (b) Since, a, b and c are in GP.

$$\frac{b}{a} = \frac{c}{b} \Rightarrow \left(\frac{b}{a}\right)^{1/2} = \left(\frac{c}{b}\right)^{1/2} \Rightarrow \frac{b^{1/2}}{a^{1/2}} = \frac{c^{1/2}}{b^{1/2}}$$

Hence, $a^{1/2}, b^{1/2}$ and $c^{1/2}$ are in GP.

iv. In a finite GP, the product of the terms equidistant from the beginning and the end is always same and is equal to the product of the first and the last terms.

➤ **Example 34.** If $2, a, 2^3, b$ and 2^5 are in GP, then ab is

- (a) 32 (b) 64 (c) 128 (d) 55

Sol. (b) Since, $2, a, 2^3, b$ and 2^5 are in GP.

∴ Product of a and b = Product of first and last terms
 $= 2 \cdot 2^5 = 2^6 = 64$

U. Three non-zero numbers a, b and c are in GP iff $b^2 = ac$.

➤ **Example 35.** If $\sum_{n=1}^n n, \frac{\sqrt{10}}{3} \cdot \sum_{n=1}^n n^2$ and $\sum_{n=1}^n n^3$

are in GP, then the value of n is

- (a) 2 (b) 3
(c) 4 (d) None of these

Sol. (c) Since, $\sum_{n=1}^n n, \frac{\sqrt{10}}{3} \cdot \sum_{n=1}^n n^2$ and $\sum_{n=1}^n n^3$ are in GP.

$$\Rightarrow \frac{n(n+1)}{2}, \frac{\sqrt{10}}{3} \cdot \frac{n(n+1)(2n+1)}{6}, \frac{n^2(n+1)^2}{4} \text{ are in GP.}$$

$$\Rightarrow \frac{10 \cdot n^2(n+1)^2(2n+1)^2}{9 \cdot 36} = \frac{n(n+1)}{2} \cdot \frac{n^2(n+1)^2}{4}$$

$$\Rightarrow 20(2n+1)^2 = 81n(n+1)$$

$$\Rightarrow n^2 + n - 20 = 0 \Rightarrow n = 4$$

vi. If the terms of a given GP are chosen at regular intervals, then the new sequence so formed also form a GP.

➤ **Example 36.** If $1, 5, 25, 125, 625, \dots$ forms a GP, then the sequence consisting odd term of the given GP, is

- (a) AP (b) GP (c) HP (d) None of these

Sol. (b) Clearly, the sequence of odd terms of the given GP is $1, 25, 625, \dots$, which is a GP as every term except the first term bears a constant ratio to the term immediately preceding it.

vii. If $a_1, a_2, a_3, \dots, a_n, \dots$ are in GP of non-zero non-negative terms, then $\log a_1, \log a_2, \dots, \log a_n, \dots$ are in AP and vice-versa.

➤ **Example 37.** If $3, 3^2, 3^3$ are in GP, then $\log 3, \log 9$ and $\log 27$ are in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. (a) Since, $3, 9$ and 27 are in GP.

$$\therefore 9^2 = 3 \times 27$$

$$\Rightarrow \log 9^2 = \log(3 \times 27)$$

$$\Rightarrow 2 \log 9 = \log 3 + \log 27$$

Hence, $\log 3, \log 9$ and $\log 27$ are in AP.

➤ **Example 38.** If $\log\left(\frac{5c}{a}\right), \log\left(\frac{3b}{5c}\right)$ and

$\log\left(\frac{a}{3b}\right)$ are in AP, where a, b and c are in GP,

then a, b, c are the lengths of sides of
 (a) an isosceles triangle (b) an equilateral triangle
 (c) a scalene triangle (d) None of these

Sol. (d) Since, $\log\left(\frac{5c}{a}\right), \log\left(\frac{3b}{5c}\right)$ and $\log\left(\frac{a}{3b}\right)$ are in AP.

$$\Rightarrow \frac{5c}{a}, \frac{3b}{5c} \text{ and } \frac{a}{3b} \text{ are in GP.}$$

$$\Rightarrow \left(\frac{3b}{5c}\right)^2 = \frac{5c}{a} \cdot \frac{a}{3b}$$

$$\Rightarrow 3b = 5c$$

$$\text{Also, } b^2 = ac$$

$$\therefore 9ac = 25c^2$$

$$\Rightarrow 9a = 25c$$

$$\Rightarrow \frac{9a}{5} = 5c = 3b$$

$$\Rightarrow \frac{a}{5} = \frac{b}{3} = \frac{c}{9/5}$$

$$\Rightarrow b + c < a$$

Since, sum of two sides is smaller than the third side.
 So, triangle is not formed.

➤ **Example 39.** A circle of radius r is inscribed in a square. The mid-points of sides of the square have been connected by line segment and a new square resulted. The sides of the resulting square were also connected by segment so that a new square was obtained and so on, then the radius of the circle inscribed in the n th square is

(a) $\left[2^{\frac{1-n}{2}}\right] r$

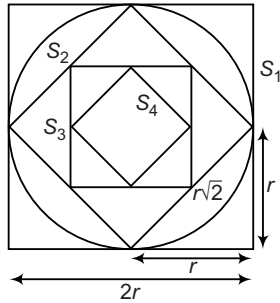
(b) $\left[2^{\frac{3-3n}{2}}\right] r$

(c) $\left[2^{-\frac{n}{2}}\right] r$

(d) $\left[2^{-\frac{5-3n}{2}}\right] r$

Sol. (a) Side of square $S_1 = 2r$

Side of square $S_2 = r\sqrt{2}$



$$= \frac{2r}{2} \left(\frac{1}{\sqrt{2}} \right)^{2-1} = 2r \left[\frac{1}{\sqrt{2}} \right]^{2-1}$$

Side of square $S_3 = 2r \left(\frac{1}{\sqrt{2}} \right)^{3-1} = 2r \left(\frac{1}{\sqrt{2}} \right)^2$ and so on.

Side of square $S_n = 2r \left(\frac{1}{\sqrt{2}} \right)^{n-1}$

$\therefore$ Radius $= r(2^{-1/2})^{n-1} = r(2^{(1-n)/2})$ and so on.

Side of square $S_n = r(2^{-1/2})^{n-1} = r[2^{(1-n)/2}]$

Sum of n Terms of a GP

i. The sum of n terms of a GP with first term a and common ratio r is given by

$$S_n = \begin{cases} \frac{a(1-r^n)}{1-r}, & |r| < 1 \\ \frac{a(r^n-1)}{r-1}, & |r| > 1 \\ an, & r = 1 \end{cases}$$

➤ **Example 40.** The sum of 7 terms of a GP

3, 6, 12, ... is

- (a) 200
(b) 320
(c) 381
(d) None of the above

Sol. (c) Here, $a = 3$ and $r = 2$

$$\begin{aligned} \therefore S_7 &= a \left(\frac{r^7-1}{r-1} \right) = 3 \left(\frac{2^7-1}{2-1} \right) \\ &= 3(128-1) \\ &= 381 \end{aligned}$$

➤ **Example 41.** If

$$S_n = \sum_{r=1}^n \frac{1+2+2^2+\dots \text{upto } r \text{ terms}}{2^r}, \text{ then } S_n \text{ is}$$

equal to

- (a) $2^n - (n+1)$ (b) $1 - \frac{1}{2^n}$
(c) $n-1 + \frac{1}{2^n}$ (d) $2^n - 1$

$$\begin{aligned} \text{Sol. (c)} S_n &= \sum_{r=1}^n \frac{2^r-1}{2^r} = \sum_{r=1}^n \left(1 - \frac{1}{2^r} \right) \\ &= n - \left(\frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n} \right) \\ &= n - \frac{1}{2} \left(1 - \frac{1}{2^n} \right) = n - 1 + \frac{1}{2^n} \end{aligned}$$

ii. If l is the last term of a GP, then $l = ar^{n-1}$.

$$\Rightarrow S_n = \frac{a-lr}{1-r} \quad \text{or} \quad S_n = \frac{lr-a}{r-1}, r \neq 1$$

➤ **Example 42.** In a GP, if $\{a_n\}$, $a_1 = 3$, $a_n = 96$ and $S_n = 189$, then the value of r is

- (a) 5 (b) 2 (c) 3 (d) 8

Sol. (b) Here, $a_1 = 3$, $a_n = l = 96$ and $S_n = 189$

$$\therefore S_n = \frac{a-lr}{1-r} \Rightarrow 189 = \frac{3-96r}{1-r}$$

$$\Rightarrow 189 - 189r = 3 - 96r \Rightarrow 186 = 93r$$

$$\Rightarrow r = 2$$

iii. If $|r| < 1$, then $\lim_{n \rightarrow \infty} r^n = 0$. Therefore, the sum S

of an infinite GP with common ratio r satisfying $|r| < 1$ is given by

$$S = \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} a \left(\frac{1-r^n}{1-r} \right) \Rightarrow S = \frac{a}{1-r}$$

Thus, the sum S of an infinite GP with first term a and common ratio r ($-1 < r < 1$) is given

$$\text{by} \quad S = \frac{a}{1-r}$$

➤ **Example 43.** The sum to infinity of the GP

$$\frac{-5}{4}, \frac{5}{16}, \frac{-5}{64}, \dots, \text{ is}$$

- (a) 1 (b) 0
(c) -1 (d) None of these

Sol. (c) The given GP has first term $(a) = \frac{-5}{4}$ and the

common ratio $(r) = \frac{-1}{4}$. Also, $|r| < 1$

Hence, the sum to infinity is given by

$$S = \frac{a}{1-r} = \frac{\frac{-5}{4}}{1 - \left(\frac{-1}{4} \right)} = -1$$

➤ **Example 44.** If $S_k = \lim_{n \rightarrow \infty} \sum_{i=0}^n \frac{1}{(k+1)^i}$, then

$\sum_{k=1}^n k S_k$ is equal to

- (a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$ (c) $\frac{n(n+2)}{2}$ (d) $\frac{n(n+3)}{2}$

Sol. (d) $S_k = 1 + \frac{1}{k+1} + \frac{1}{(k+1)^2} + \dots = \frac{1}{1 - \frac{1}{k+1}} = \frac{k+1}{k}$

$\therefore k S_k = k + 1$

Now, $\sum_{k=1}^n k S_k = \sum_{k=1}^n (k+1) = 2 + 3 + \dots + (n+1)$
 $= \frac{n}{2} [2 + (n+1)] = \frac{n(n+3)}{2}$

Geometric Mean of Two Given Numbers

Let a and b be two given numbers. Then, the geometric mean of a and b is a number G , such that a, G, b are in GP.

Now, a, G, b in GP.

$\Rightarrow \frac{G}{a} = \frac{b}{G} \Rightarrow G^2 = ab \Rightarrow G = \sqrt{ab}$

This, the geometric mean of a and b is given by $G = \sqrt{ab}$.

Geometric Mean of n Given Numbers

Let $a_1, a_2, a_3, \dots, a_n$ be n given numbers, then the geometric mean of $a_1, a_2, \dots, a_n$ is given by

$G = (a_1 a_2 \dots a_n)^{1/n}$

Insertion of n Geometric Means between a and b

Let a and b be two given numbers. If n numbers $G_1, G_2, \dots, G_n$ are inserted between a and b such that the sequence $a, G_1, G_2, \dots, G_n, b$ is a GP. Then, the numbers $G_1, G_2, \dots, G_n$ are known as n geometric means (GM's) between a and b .

Now, sequence $a, G_1, G_2, \dots, G_n, b$ is a GP consisting of $(n+2)$ terms. Let r be the common ratio of this GP.

Then, $b = (n+2)\text{th term} = ar^{n+1}$

$\Rightarrow r^{n+1} = \frac{b}{a} \Rightarrow r = \left(\frac{b}{a}\right)^{1/(n+1)}$

$\therefore G_1 = ar = a \left(\frac{b}{a}\right)^{1/(n+1)}$

$G_2 = ar^2 = a \left(\frac{b}{a}\right)^{2/(n+1)}$

$\vdots$

$G_n = ar^n = a \left(\frac{b}{a}\right)^{n/(n+1)}$

Product of n GM's between a and b , i.e. $G_1 \times G_2 \times G_3 \times \dots \times G_n = (ab)^{n/2} = G^n$

Example 45. The five geometric means between 576 and 9 are

- (a) 192, 96, 48, 24, 12 (b) 288, 144, 72, 36, 18
 (c) 208, 104, 52, 26, 13 (d) None of these

Sol. (b) Let G_1, G_2, G_3, G_4 and G_5 be five geometric means between $a = 576$ and $b = 9$.

Then, 576, $G_1, G_2, G_3, G_4, G_5, 9$ are in GP with common ratio r given by

$r = \left(\frac{9}{576}\right)^{\frac{1}{5+1}} = \left(\frac{1}{64}\right)^{\frac{1}{6}} = \frac{1}{2} \quad \left[\because r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}} \right]$

$\therefore G_1 = ar = 576 \times \frac{1}{2} = 288, G_2 = ar^2 = 576 \times \frac{1}{4} = 144$

$G_3 = ar^3 = 576 \times \frac{1}{8} = 72, G_4 = ar^4 = 576 \times \frac{1}{16} = 36$

and $G_5 = ar^5 = 576 \times \frac{1}{32} = 18$

Hence, 288, 144, 72, 36, 18 are the required geometric means between 576 and 9.

Example 46. If a is the AM of b and c and the two geometric means are G_1 and G_2 , then $G_1^3 + G_2^3$ is equal to

- (a) abc (b) $2abc$ (c) $3abc$ (d) $5abc$

Sol. (b) It is given that a is the AM of b and c .

$\therefore a = \frac{b+c}{2} \Rightarrow b+c = 2a \quad \dots(i)$

Since, G_1 and G_2 are two geometric means between b and c .

Therefore, b, G_1, G_2, c are in GP with common ratio

$r = \left(\frac{c}{b}\right)^{\frac{1}{3}}$

Now, $G_1 = br = b \left(\frac{c}{b}\right)^{1/3} = c^{1/3} b^{2/3}$

and $G_2 = br^2 = b \left(\frac{c}{b}\right)^{2/3} = b^{1/3} c^{2/3}$

$\Rightarrow G_1^3 = b^2 c$ and $G_2^3 = bc^2$

$\Rightarrow G_1^3 + G_2^3 = b^2 c + bc^2 = bc(b+c)$

$\Rightarrow G_1^3 + G_2^3 = 2abc$ [from Eq. (i)]

Example 47. If the arithmetic mean and geometric mean of a and b are A and G respectively, then the value of $A - G$ will be

- (a) $\frac{a-b}{2}$ (b) $\frac{a+b}{2}$ (c) $\left[\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right]^2$ (d) $\frac{2ab}{a+b}$

Sol. (c) Arithmetic mean of a and $b = A = \frac{a+b}{2}$

Geometric mean of a and $b = G = \sqrt{ab}$

$\therefore A - G = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2}$

$= \frac{(\sqrt{a})^2 + (\sqrt{b})^2 - 2\sqrt{a}\sqrt{b}}{2} = \left[\frac{\sqrt{a}-\sqrt{b}}{\sqrt{2}}\right]^2$

Example 48. In the sequence 1, 2, 2, 4, 4, 4, 4, 8, 8, 8, 8, 8, ..., where n consecutive terms have the value n , then 1025th term is

- (a) 2^9 (b) 2^{10} (c) 2^{11} (d) 2^8

Sol. (b) Let the 1025th term be 2^n . Then,

$1 + 2 + 4 + 8 + \dots + 2^{n-1} < 1025 < 1 + 2 + 4 + 8 + \dots + 2^n$

$\therefore 2^n - 1 < 1025 < 2^{n+1} - 1$ or $2^n < 1026 < 2^{n+1}$

$\Rightarrow n = 10$

Hence, 1025th term is 2^{10} .

Work Book Exercise 3.2

3

Sequence and Series

- The common ratio of a GP having 10th term and 1st term equal to 1536 and -3 , is
2 1 -2 2
- Any geometric series can be summed till infinite number of terms by the formula
 $S_{\infty} = \frac{a}{1-r}; r \in R$ $S_{\infty} = \frac{a}{1-r}; |r| < 1$
 $S_{\infty} = \frac{a}{1-r}; |r| > 1$ None of these
- If $x, 2x+2$ and $3x+3$ are in GP, then the 4th term is
27 -27 13.5 -13.5
- If 1, x, y, z and 2 are in GP, then xyz is equal to
4 2
8 None of these
- Which term of the sequence $2, 1, 2^{-1}, 4^{-1}, 8^{-1}, \dots$ is 128^{-1} ?
9th 8th
7th 10th
- If $a_1, a_2, \dots, a_n$ are in AP, then $\log a_1, \log a_2, \dots, \log a_n$ are in
GP
not GP
AP
None of the above
- If the sum of the first 10 terms of a certain GP is equal to 244 times the sum of first 5 terms. Then, the common ratio is
4 7
2 3
- If $a, 2b, 3c$ are in AP and a, b, c (distinct) are in GP, then the common ratio of a GP is
1 3 $1/3$ 2
- If a GP having an even number of terms and the sum of all terms is five times the sum of terms occupying odd places, then the common ratio will be
3 5 4 $1/4$
- For $n \geq 3$, the n th roots of unity form
an HP an AP
a GP None of these
- If $1 + a + a^2 + a^3 + \dots + a^n = (1+a)(1+a^2)(1+a^4)$, then n is equal to
3 5 7 9
- The rational number $2.357357357 \dots$ is
 $\frac{2355}{10001}$ $\frac{2379}{999}$ $\frac{2355}{999}$ $\frac{2379}{10001}$
- $g^{1/3} \cdot g^{1/9} \cdot g^{1/27} \dots$ is equal to
9 3 81 $\sqrt[3]{81}$
- If b_1, b_2 and b_3 ($b_1 > 0$) are three successive terms of a GP with common ratio r , then the possible value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds, is given by
4 2 2.5
- If S denotes the sum to infinity and S_n denotes the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$, such that $S - S_n < \frac{1}{1000}$, then the least value of n is
8 9 10 11

Harmonic Progression (HP)

A sequence $a_1, a_2, \dots, a_n, \dots$ of non-zero numbers is called a harmonic progression, if the sequence $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n}, \dots$ is an arithmetic progression.

The sequence $1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots$ are in HP, because the sequence $1, 3, 5, 7, \dots$ are in AP.

Harmonic Mean of Two Given Numbers

If a and b are two non-zero numbers, then the harmonic mean of a and b is a number H such that the sequence a, H and b are in HP.

Now, a, H and b are in HP.

$$\Rightarrow \frac{1}{a}, \frac{1}{H} \text{ and } \frac{1}{b} \text{ are in AP.}$$

$$\Rightarrow \frac{2}{H} = \frac{1}{a} + \frac{1}{b} \Rightarrow H = \frac{2ab}{a+b}$$

Thus, the harmonic mean H of two numbers a and b is given by

$$H = \frac{2ab}{a+b}$$

Harmonic Mean of n Numbers

If $a_1, a_2, a_3, \dots, a_n$ are n non-zero numbers, then the harmonic mean H of these numbers is given by

$$\frac{1}{H} = \frac{\frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}}{n}$$

➤ **Example 49.** Which of the following is harmonic mean of numbers $\frac{1}{3}$ and $\frac{1}{7}$?

- (a) 1 (b) $\frac{1}{5}$ (c) $\frac{1}{10}$ (d) 5

Sol. (b) $\therefore H = \frac{2ab}{a+b}$

On putting $a = \frac{1}{3}$ and $b = \frac{1}{7}$, we get

$$H = \frac{2 \left(\frac{1}{3} \right) \left(\frac{1}{7} \right)}{\frac{1}{3} + \frac{1}{7}} = \frac{1}{5}$$

➤ **Example 50.** The harmonic mean of the roots of the equation

$$(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0 \text{ is}$$

- (a) 2 (b) 4 (c) 6 (d) 8

Sol. (b) Let α and β be the roots of given quadratic equation. Then,

$$\alpha + \beta = \frac{4 + \sqrt{5}}{5 + \sqrt{2}} \quad \text{and} \quad \alpha\beta = \frac{8 + 2\sqrt{5}}{5 + \sqrt{2}}$$

Let H be the harmonic mean between α and β , then

$$H = \frac{2\alpha\beta}{\alpha + \beta} = \frac{16 + 4\sqrt{5}}{4 + \sqrt{5}} = 4$$

Insertion of n Harmonic Means between a and b

The n numbers $H_1, H_2, \dots, H_n$ are said to be harmonic means between a and b , if $a, H_1, H_2, \dots, H_n, b$ are in HP, i.e. if $\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \dots, \frac{1}{H_n}, \frac{1}{b}$ are in AP. Let d

be the common difference of this AP. Then,

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d$$

$$\Rightarrow d = \frac{1}{n+1} \left(\frac{1}{b} - \frac{1}{a} \right) = \frac{a-b}{(n+1)ab}$$

$$\text{Thus, } \frac{1}{H_1} = \frac{1}{a} + \frac{a-b}{(n+1)ab}$$

$$\Rightarrow \frac{1}{H_2} = \frac{1}{a} + \frac{2(a-b)}{(n+1)ab}$$

$$\vdots$$

$$\frac{1}{H_n} = \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}$$

➤ **Example 51.** Insert two harmonic means between 3 and 5.

- (a) $\frac{30}{11}, \frac{30}{13}$ (b) $\frac{45}{13}, \frac{45}{11}$
(c) $\frac{30}{12}, \frac{45}{14}$ (d) None of these

Sol. (b) $a = 3, b = 5$ and $n = 2$

$$\therefore \frac{1}{H_1} = \frac{1}{3} + \frac{(3-5)}{(2+1)(15)} = \frac{13}{45} \Rightarrow H_1 = \frac{45}{13}$$

$$\text{and } \frac{1}{H_2} = \frac{1}{3} + \frac{2(3-5)}{(2+1)(15)} = \frac{11}{45} \Rightarrow H_2 = \frac{45}{11}$$

Properties of Arithmetic, Geometric and Harmonic Means between Two Given Numbers

Let A, G and H be arithmetic, geometric and harmonic means of two positive numbers a and b . Then,

$$A = \frac{a+b}{2}, G = \sqrt{ab} \quad \text{and} \quad H = \frac{2ab}{a+b}$$

i. These three means possess the following properties: $A \geq G \geq H$

➤ **Example 52.** The minimum value of $4^x + 4^{1-x}$, $x \in R$, is

- (a) 2 (b) 4
(c) 1 (d) None of these

Sol. (b) $\therefore AM \geq GM$ for positive numbers.

$$\text{So, } \frac{4^x + \frac{4}{4^x}}{2} \geq \sqrt{4^x \cdot \frac{4}{4^x}} = 2$$

$$\Rightarrow 4^x + 4^{1-x} \geq 4$$

➤ **Example 53.** If $\sqrt{3}, A$ and $\sqrt{2}$ are in AP, then $\frac{\sqrt{3} + \sqrt{2}}{2}$ is greater than or equal to

- (a) $\sqrt{5}$ (b) $\sqrt{6}$
(c) $\sqrt{8}$ (d) None of these

Sol. (b) $\therefore \sqrt{3}, A$ and $\sqrt{2}$ are in AP.

$$\therefore A = \frac{\sqrt{3} + \sqrt{2}}{2} \text{ is the AM of } \sqrt{3} \text{ and } \sqrt{2}.$$

$$\text{Thus, } \frac{\sqrt{3} + \sqrt{2}}{2} \geq \sqrt{\sqrt{3} \cdot \sqrt{2}} = \sqrt{6}$$

➤ **Example 54.** If the product of n positive numbers is unity, then their sum is

- (a) a positive integer (b) divisible by n
(c) equal to $n + \frac{1}{n}$ (d) never less than n

Sol. (d) Since, product of n positive numbers is unity.

$$\therefore x_1 \cdot x_2 \cdot x_3 \dots x_n = 1 \quad \dots(i)$$

Using $AM \geq GM$, we have

$$\frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \geq (x_1 \cdot x_2 \cdot x_3 \dots x_n)^{1/n}$$

$$\Rightarrow x_1 + x_2 + \dots + x_n \geq n(1)^{1/n}$$

Hence, sum of n positive numbers is never less than n .

ii. A, G, H form a GP i.e. $G^2 = AH$

➤ **Example 55.** If AM of two terms is 9 and HM is 36, then GM will be

- (a) 18 (b) 12
(c) 16 (d) None of these

Sol. (a) $\therefore (AM)(HM) = (GM)^2$

$$\therefore 9 \cdot 36 = (GM)^2 \Rightarrow GM = 18$$

- iii. If A, G and H are arithmetic, geometric and harmonic means between three given numbers a, b and c , then the equation having a, b, c as its roots, is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

➤ **Example 56.** If α, β and γ are the roots of the equation $x^3 - 6x^2 + 11x - 6 = 0$. Then, HM of the roots is

- (a) 6 (b) 11
(c) $\frac{11}{6}$ (d) $\frac{18}{11}$

Sol. (d) We know that, if α, β and γ are the roots and A, G, H are AM, GM and HM of the roots, then equation is

$$x^3 - 3Ax^2 + \frac{3G^3}{H}x - G^3 = 0$$

Given, $x^3 - 6x^2 + 11x - 6 = 0$

On comparing, we get

$$3A = 6 \quad \dots(i)$$

$$\frac{3G^3}{H} = 11 \quad \dots(ii)$$

and $G^3 = 6 \quad \dots(iii)$

From Eqs. (ii) and (iii), we get

$$H = \frac{18}{11}$$

Arithmetico-Geometric Progression (AGP)

A sequence of the form

$$a, (a+d)r, (a+2d)r^2, \dots, [a+(n-1)d]r^{n-1}, \dots$$

is called an arithmetico-geometric progression.

Arithmetico-Geometric Series

Let $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \dots$ be an arithmetico-geometric progression.

Then, $a + (a+d)r + (a+2d)r^2 + (a+3d)r^3 + \dots$ is an arithmetico-geometric series.

➤ **Example 57.** The 8th term of the sequence

1, 4, 12, 32, 80, ... is

- (a) 1000 (b) 1200
(c) 1024 (d) None of these

Sol. (c) The given sequence can be rewritten as

$$1, (1+1) \cdot 2, (1+2 \cdot 1) \cdot 2^2, (1+3 \cdot 1) \cdot 2^3, \dots$$

Clearly, the sequence is an arithmetico-geometric progression with $a = 1, d = 1$ and $r = 2$.

$$\therefore \text{8th term} = (a + 7d) \cdot r^7$$

$$= (1 + 7 \cdot 1) \cdot 2^7 = 8 \cdot 2^7$$

$$= 2^3 \cdot 2^7 = 2^{10} = 1024$$

Sum of n Terms of an Arithmetico-Geometric Progression

The sum of n terms of an arithmetico-geometric progression $a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \dots$ is given by

$$S_n = \begin{cases} \frac{a}{1-r} + dr \frac{(1-r^{n-1})}{(1-r)^2} - \frac{\{a+(n-1)d\}r^n}{1-r}, & \text{when } r \neq 1 \\ \frac{n}{2}[2a+(n-1)d], & \text{when } r = 1 \end{cases}$$

Proof Let S_n denotes the sum of n terms of the given sequence.

$$\begin{aligned} \text{Then, } S_n &= a + (a+d)r + (a+2d)r^2 \\ &\quad + \dots + \{a+(n-1)d\}r^{n-1} \dots(i) \\ \Rightarrow rS_n &= ar + (a+d)r^2 + \dots + \{a+(n-2)d\}r^{n-1} \\ &\quad + \{a+(n-1)d\}r^n \dots(ii) \end{aligned}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\begin{aligned} S_n - rS_n &= a + [dr + dr^2 + \dots + dr^{n-1}] \\ &\quad - \{a+(n-1)d\}r^n \end{aligned}$$

$$\begin{aligned} \Rightarrow S_n(1-r) &= a + dr \left(\frac{1-r^{n-1}}{1-r} \right) \\ &\quad - \{a+(n-1)d\}r^n; r \neq 1 \\ \Rightarrow S_n &= \frac{a}{1-r} + dr \frac{1-r^{n-1}}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}; r \neq 1 \end{aligned}$$

For $r = 1$,

$$\begin{aligned} S_n &= a + (a+d) + (a+2d) + \dots + [a+(n-1)d] \\ &= \frac{n}{2}[2a+(n-1)d] \end{aligned}$$

➤ **Example 58.** The sum of 10 terms of the sequence 1, 6, 27, 108, ... is

- (a) 280481 (b) 280482
(c) 280484 (d) 280483

Sol. (d) The given sequence can be rewritten as

$$1, (1+1) \cdot 3, (1+2 \cdot 1) \cdot 3^2, (1+3 \cdot 1) \cdot 3^3, \dots$$

which is an arithmetico-geometric progression with $a = 1, d = 1$ and $r = 3$.

$$\begin{aligned} \text{Hence, } S_{10} &= \frac{a}{1-r} + dr \frac{1-r^9}{(1-r)^2} - \frac{[a+9d]r^{10}}{1-r} \\ &= \frac{1}{-2} + 3 \frac{(1-3^9)}{(1-3)^2} - \frac{[1+9]3^{10}}{(1-3)} \\ &= \frac{-1}{2} + \frac{(3-3^{10})}{4} + \frac{3^{10} \cdot 10}{2} \\ &= \frac{-2+3-3^{10}+20 \cdot 3^{10}}{4} \\ &= \frac{1+19 \cdot 3^{10}}{4} = 280483 \end{aligned}$$

3

Objective Mathematics Vol. 1

Sum of an Infinite Arithmetico-Geometric Progression

Let $|r| < 1$, then $r^n, r^{n-1} \rightarrow 0$ as $n \rightarrow \infty$ and it can also be shown that $n \cdot r^n \rightarrow 0$ as $n \rightarrow \infty$. So, from

$$S_n = \frac{a}{1-r} + dr \frac{(1-r^{n-1})}{(1-r)^2} - \frac{[a + (n-1)d]r^n}{1-r}, \text{ we get}$$

$$S_n \rightarrow \frac{a}{1-r} + \frac{dr}{(1-r)^2} \text{ as } n \rightarrow \infty$$

In other words, when $|r| < 1$, then sum to infinity of an arithmetico-geometric series is

$$S_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

➤ **Example 59.** The sum of

$$0.2 + 0.004 + 0.00006 + 0.0000008 + \dots \text{ is}$$

- (a) $\frac{200}{891}$
 (b) $\frac{2000}{9801}$
 (c) $\frac{1000}{9801}$
 (d) None of the above

Sol. (b) Sum = $\frac{2}{10} + \frac{4}{10^3} + \frac{6}{10^5} + \frac{8}{10^7} + \dots$
 $= \frac{2}{10} \left(1 + \frac{2}{10^2} + \frac{3}{10^4} + \dots \right) = \frac{2}{10} S_1$

Clearly, S_1 is an arithmetico-geometric series with $a = 1$, $d = 1$ and $r = \frac{1}{10^2}$.

$$\therefore S_1 = \frac{a}{1-r} + \frac{dr}{(1-r)^2}$$

$$\begin{aligned} &= \frac{1}{1 - \frac{1}{10^2}} + \frac{\frac{1}{10^2}}{\left(1 - \frac{1}{10^2}\right)^2} \\ &= \frac{10^2}{10^2 - 1} + \frac{10^4}{10^2(10^2 - 1)^2} \\ &= 10^2 \left[\frac{10^2 - 1 + 1}{(10^2 - 1)^2} \right] = \frac{10^4}{(10^2 - 1)^2} \\ \therefore S &= \frac{2}{10} \times \frac{10^4}{(10^2 - 1)^2} = \frac{2000}{9801} \end{aligned}$$

Aliter

Let $S = \frac{2}{10} + \frac{4}{10^3} + \frac{6}{10^5} + \frac{8}{10^7} + \dots$

$$\therefore S - \frac{S}{10^2} = \frac{2}{10} + \frac{2}{10^3} + \frac{2}{10^5} + \dots$$

$$\Rightarrow \frac{99S}{100} = \frac{\frac{2}{10}}{1 - \frac{1}{10^2}} = \frac{2}{10} \times \frac{10^2}{10^2 - 1} = \frac{20}{99}$$

$$\Rightarrow S = \frac{20 \times 100}{(99)^2} = \frac{2000}{9801}$$

➤ **Example 60.** Find the value of the expression

$$\sum_{i=1}^n \sum_{j=1}^i \sum_{k=1}^j 1$$

Sol. $\sum_{i=1}^n \sum_{j=1}^i \sum_{k=1}^j 1 = \sum_{i=1}^n \sum_{j=1}^i j$

$$\begin{aligned} &= \frac{1}{2} \left[\sum_{i=1}^n i^2 + \sum_{i=1}^n i \right] \\ &= \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right] \\ &= \frac{n(n+1)}{2} [2n+1+3] \\ &= \frac{n(n+1)(n+2)}{6} \end{aligned}$$

Work Book Exercise 3.3

1 The first term of HP having k th term equal to $\frac{1}{a + (k-1)d}$, is

- a a b $\frac{1}{a}$ c $\frac{1}{k}$ d k

2 If a, b, c and d are distinct positive real numbers in HP, then

- a $ab > cd$ b $ac > bd$
 c $ad > bc$ d None of these

3 The 4th term of the sequence $3, \frac{3}{2}, 1, \dots$ is

- a $\frac{3}{4}$
 b $\frac{4}{3}$
 c $\frac{2}{3}$
 d None of the above

4 If $|x| < 1$, then $1 + 3x + 5x^2 + 7x^3 + \dots$ is equal to

- a $\frac{1+x}{(1-x)^2}$ b $\frac{2+x}{(1-x)^2}$
 c $\frac{x}{(1-x)^2}$ d None of these

5 The sum of n terms of the series

$$1 + 2 \left(1 + \frac{1}{n} \right) + 3 \left(1 + \frac{1}{n} \right)^2 + \dots \text{ is}$$

- a $n^2 + 1$ b $n(n+1)$ c $n \left(1 + \frac{1}{n} \right)^2$ d n^2

6 If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd) + p(b^2 + c^2 + d^2) \leq 0$, then a, b, c and d are in

- a GP b AP
 c HP d None of these

➤ **Example 63.** The sum of n terms of the series $5 + 7 + 13 + 31 + 85 + \dots$ is

$$(a) \frac{1}{2} (3^n + 8n - 1) \quad (b) \frac{1}{2} (3^n + 7n + 1)$$

$$(c) \frac{1}{2} (3^n + 5n) \quad (d) \frac{1}{2} (3^n + 7n)$$

Sol. (a) The given series is $5 + 7 + 13 + 31 + 85 + \dots$
The differences of successive terms are 2, 6, 18, 54, which are in GP with common ratio i.e. 3.

$$\therefore T_n = a(3)^{n-1} + bn + c$$

On putting $n = 1, 2, 3$, we get

$$T_1 = a + b + c = 5 \quad \dots(i)$$

$$T_2 = 3a + 2b + c = 7 \quad \dots(ii)$$

$$\text{and } T_3 = 9a + 3b + c = 13 \quad \dots(iii)$$

On solving Eqs. (i), (ii) and (iii), we get

$$a = 1, b = 0 \text{ and } c = 4$$

$$\therefore T_n = 3^{n-1} + 4$$

$$\begin{aligned} \therefore S_n &= \sum T_n = \sum (3^{n-1} + 4) \\ &= \sum 3^{n-1} + \sum 4 = \frac{3^n - 1}{3 - 1} + 4n \\ &= \frac{3^n - 1}{2} + 4n = \frac{1}{2} (3^n + 8n - 1) \end{aligned}$$

➤ **Example 64.** Find the sum to n terms

$$3 + 7 + 13 + 21 + \dots$$

$$\text{Sol. Let } S = 3 + 7 + 13 + 21 + \dots + T_n \quad \dots(i)$$

$$\text{and } S = 3 + 7 + 13 + \dots + T_{n-1} + T_n \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\begin{aligned} T_n &= 3 + 4 + 6 + 8 + \dots + (T_n - T_{n-1}) \\ &= 3 + \frac{n-1}{2} [8 + (n-2)2] = 3 + (n-1)(n+2) \\ &= n^2 + n + 1 \end{aligned}$$

$$\text{Hence, } S = \sum (n^2 + n + 1)$$

$$\begin{aligned} &= \sum n^2 + \sum n + \sum 1 \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n \\ &= \frac{n}{3} (n^2 + 3n + 5) \end{aligned}$$

➤ **Example 65.** Find the sum to n terms

$$1 + 4 + 10 + 22 + \dots$$

$$\text{Sol. Let } S = 1 + 4 + 10 + 22 + \dots + T_n \quad \dots(i)$$

$$\text{and } S = 1 + 4 + 10 + \dots + T_{n-1} + T_n \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\begin{aligned} T_n &= 1 + (3 + 6 + 12 + \dots + T_n - T_{n-1}) \\ \Rightarrow T_n &= 1 + 3 \left(\frac{2^{n-1} - 1}{2 - 1} \right) \end{aligned}$$

$$\Rightarrow T_n = 3 \cdot 2^{n-1} - 2$$

$$\text{So, } S = \sum T_n = 3 \sum 2^{n-1} - \sum 2$$

$$\begin{aligned} &= 3 \cdot \left(\frac{2^n - 1}{2 - 1} \right) - 2n \\ &= 3 \cdot 2^n - 2n - 3 \end{aligned}$$

Second Way

We can find n th term of the series by the following steps:

i. Denote the n th term by T_n and the sum of the series upto n terms by S_n .

ii. Rewrite the given series with each term shifted by one place to the right.

iii. By subtracting the later series from the former, find T_n .

iv. From T_n , S_n can be found by appropriate summation.

➤ **Example 66.** The sum of n terms of the series $1 + 2 + 5 + 12 + 25 + 46 + \dots$ is

$$(a) \frac{1}{12} n(n-1)(n^2 - 3n + 8)$$

$$(b) \frac{1}{24} n(n+1)(n^2 - 3n + 8)$$

$$(c) \frac{1}{12} n(n+1)(n^2 - 3n + 8)$$

(d) None of the above

Sol. (c) Let the sum of the series be S_n and n th term of the series be T_n . Then,

$$S_n = 1 + 2 + 5 + 12 + 25 + 46 + \dots + T_{n-1} + T_n \quad \dots(i)$$

$$S_n = 1 + 2 + 5 + 12 + 25 + \dots + T_{n-1} + T_n \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\begin{aligned} 0 &= 1 + 1 + 3 + 7 + 13 + 21 + \dots + (T_n - T_{n-1}) - T_n \\ \Rightarrow T_n &= 1 + 1 + 3 + 7 + 13 + 21 + \dots + t_{n-1} + t_n \quad \dots(iii) \end{aligned}$$

[since, n th term of T_n is t_n]

$$\Rightarrow T_n = 1 + 1 + 3 + 7 + 13 + \dots + t_{n-1} + t_n \quad \dots(iv)$$

On subtracting Eq. (iv) from Eq. (iii), we get

$$0 = 1 + 0 + 2 + 4 + 6 + 8 + \dots + (t_n - t_{n-1}) - t_n$$

$$\Rightarrow t_n = 1 + 2 + 4 + 6 + 8 + \dots + (n-1) \text{ terms}$$

$$= 1 + [2 + 4 + 6 + 8 + \dots + (n-2) \text{ terms}]$$

$$= 1 + \frac{n-2}{2} [2 \cdot 2 + (n-2-1)2]$$

$$= 1 + (n-2)(2 + n - 3)$$

$$= n^2 - 3n + 3$$

$$\therefore T_n = \sum t_n = \sum n^2 - 3 \sum n + 3n$$

$$= \frac{n(n+1)(2n+1)}{6} - 3 \cdot \frac{n(n+1)}{2} + 3n$$

$$= \frac{n}{3} (n^2 - 3n + 5)$$

$$\therefore S_n = \frac{1}{3} \sum n^3 - \sum n^2 + 5 \sum n$$

$$= \frac{1}{3} \left[\frac{n(n+1)}{2} \right]^2 - \frac{1}{6} \cdot \frac{n(n+1)(2n+1)}{1} + \frac{5}{3} \cdot \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{6 \times 2} [n^2 + n - 4n - 2 + 10]$$

$$= \frac{1}{12} \cdot n(n+1)(n^2 - 3n + 8)$$

➤ **Example 67.** Find the sum to n terms of the series $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots$

Sol. Let T_r be the general term of the series.

$$\text{So, } T_r = r(r+1)$$

To express $t_r = f(r) - f(r-1)$ multiply and divide t_r by $[(r+2) - (r-1)]$.

$$\begin{aligned} \text{So, } T_r &= \frac{r}{3} (r+1) [(r+2) - (r-1)] \\ &= \frac{1}{3} [r(r+1)(r+2) - (r-1)r(r+1)] \end{aligned}$$

$$\text{Let } f(r) = \frac{1}{3} r(r+1)(r+2)$$

$$\text{So, } T_r = [f(r) - f(r-1)]$$

$$\text{Now, } S = \sum_{r=1}^n T_r = T_1 + T_2 + T_3 + \dots + T_n$$

$$T_1 = \frac{1}{3} [1 \cdot 2 \cdot 3 - 0]$$

$$T_2 = \frac{1}{3} [2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3]$$

$$T_3 = \frac{1}{3} [3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4]$$

$$\vdots$$

$$T_n = \frac{1}{3} [n(n+1)(n+2) - (n-1)n(n+1)]$$

$$\therefore S = \frac{1}{3} n(n+1)(n+2)$$

Hence, the sum of series is $f(n) - f(0)$.

➤ **Example 68.** Find the sum to n terms of the series given below

$$\frac{4}{1 \cdot 2 \cdot 3} + \frac{5}{2 \cdot 3 \cdot 4} + \frac{6}{3 \cdot 4 \cdot 5} + \dots$$

$$\text{Sol. Let } T_r = \frac{r+3}{r(r+1)(r+2)}$$

$$= \frac{1}{(r+1)(r+2)} + \frac{3}{r(r+1)(r+2)}$$

$$= \left[\frac{1}{r+1} - \frac{1}{r+2} \right] + \frac{3}{2} \left[\frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)} \right]$$

$$\therefore S = \left[\frac{1}{2} - \frac{1}{n+2} \right] + \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$= \frac{5}{4} - \frac{1}{n+2} \left[1 + \frac{3}{2(n+1)} \right]$$

$$= \frac{5}{4} - \frac{1}{2(n+1)(n+2)} [2n+5]$$

V_n Method

To find the sum of the series of the forms

$$i. \quad \frac{1}{a_1 a_2 a_3 \dots a_r} + \frac{1}{a_2 a_3 \dots a_{r+1}} + \dots + \frac{1}{a_n a_{n+1} \dots a_{n+r-1}}$$

$$ii. \quad a_1 a_2 \dots a_r + a_2 a_3 \dots a_{r+1} + \dots + a_n a_{n+1} \dots a_{n+r-1}$$

where, $a_1, a_2, a_3 \dots a_n \dots$ are in AP.

Solution of Form (i) Let d be the common difference of an AP. Then, $a_n = a_1 + (n-1)d$

Let sum of the series and n th term be denoted by S_n and T_n , respectively. Then,

$$S_n = \frac{1}{a_1 a_2 \dots a_r} + \frac{1}{a_2 a_3 \dots a_{r+1}} + \dots + \frac{1}{a_n a_{n+1} \dots a_{n+r-1}}$$

$$T_n = \frac{1}{a_n a_{n+1} a_{n+2} \dots a_{n+r-2} a_{n+r-1}} \dots (i)$$

$$\text{Let } V_n = \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-2} a_{n+r-1}} \dots (ii)$$

[avoiding first term for V_n i.e. a_n in T_n]

$$V_{n-1} = \frac{1}{a_n a_{n+1} \dots a_{n+r-3} a_{n+r-2}}$$

$$\therefore V_n - V_{n-1} = \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-2} a_{n+r-1}} - \frac{1}{a_n a_{n+1} \dots a_{n+r-2}}$$

$$V_n - V_{n-1} = \frac{a_n - a_{n+r-1}}{a_n a_{n+1} a_{n+2} \dots a_{n+r-2} a_{n+r-1}}$$

$$V_n - V_{n-1} = T_n (a_n - a_{n+r-1}) \quad [\text{from Eq. (i)}]$$

$$= T_n \{ [a_1 + (n-1)d] - [a_1 + (n+r-2)d] \}$$

$$= d(1-r)T_n$$

$$\therefore T_n = \frac{(V_n - V_{n-1})}{d(1-r)}$$

$$\Rightarrow T_n = \frac{1}{d(r-1)} \{V_{n-1} - V_n\}$$

Putting $n = 1, 2, 3, 4, \dots, n$, we get

$$T_1 = \frac{1}{d(r-1)} (V_0 - V_1)$$

$$T_2 = \frac{1}{d(r-1)} (V_1 - V_2)$$

$$T_3 = \frac{1}{d(r-1)} (V_2 - V_3)$$

$$\vdots$$

$$T_n = \frac{1}{d(r-1)} (V_{n-1} - V_n)$$

Adding the above equations, we get

$$T_1 + T_2 + T_3 + \dots + T_n = \frac{1}{(r-1)d} (V_0 - V_n)$$

$$\therefore S_n = \frac{1}{(r-1)d} \left\{ \frac{1}{a_1 a_2 \dots a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-1}} \right\}$$

Hence, the sum of n terms is

$$S_n = \frac{1}{(r-1)(a_2 - a_1)} \left\{ \frac{1}{a_1 a_2 \dots a_{r-1}} - \frac{1}{a_{n+1} a_{n+2} \dots a_{n+r-1}} \right\}$$

Corollary If $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP, then

For $r = 2$

$$\begin{aligned} \frac{1}{a_1 a_2} + \frac{1}{a_2 a_3} + \dots + \frac{1}{a_n a_{n+1}} \\ = \frac{1}{(a_2 - a_1)} \left(\frac{1}{a_1} - \frac{1}{a_{n+1}} \right) \\ = \frac{a_{n+1} - a_1}{(a_2 - a_1) a_1 a_{n+1}} \\ = \frac{a_1 + (n+1-1)d - a_1}{da_1 a_{n+1}} \\ = \frac{nd}{da_1 a_{n+1}} = \frac{n}{a_1 a_{n+1}} \end{aligned}$$

For $r = 3$

$$\begin{aligned} \frac{1}{a_1 a_2 a_3} + \frac{1}{a_2 a_3 a_4} + \dots + \frac{1}{a_n a_{n+1} a_{n+2}} \\ = \frac{1}{2(a_2 - a_1)} \left[\frac{1}{a_1 a_2} - \frac{1}{a_{n+1} a_{n+2}} \right] \end{aligned}$$

For $r = 4$

$$\begin{aligned} \frac{1}{a_1 a_2 a_3 a_4} + \frac{1}{a_2 a_3 a_4 a_5} \\ + \dots + \frac{1}{a_n a_{n+1} a_{n+2} a_{n+3}} \\ = \frac{1}{3(a_2 - a_1)} \left[\frac{1}{a_1 a_2 a_3} - \frac{1}{a_{n+1} a_{n+2} a_{n+3}} \right] \end{aligned}$$

Solution of Form (ii) Let S_n be the sum and T_n be the n th term of the series, then

$$S_n = a_1 a_2 \dots a_r + a_2 a_3 \dots a_{r+1} + \dots + a_n a_{n+1} \dots a_{n+r-1}$$

$$\therefore T_n = a_n a_{n+1} a_{n+2} \dots a_{n+r-2} a_{n+r-1}$$

$$\text{Let } V_n = a_n a_{n+1} a_{n+2} \dots a_{n+r-2} a_{n+r-1} a_{n+r} \quad [\text{taking one extra term in } T_n \text{ for } V_n]$$

$$\therefore V_n - 1 = a_{n-1} a_n a_{n+1} \dots a_{n+r-3} a_{n+r-2} a_{n+r-1}$$

$$\Rightarrow V_n - V_{n-1} = a_n a_{n+1} a_{n+2} \dots$$

$$\begin{aligned} & a_{n+r-1} (a_{n+r} - a_{n-1}) \quad [\text{from Eq. (i)}] \\ & = T_n \{([a_1 + (n+r-1)d] - [a_1 + (n-2)d])\} \\ & = (r+1)d T_n \end{aligned}$$

$$\Rightarrow T_n = \frac{1}{(r+1)d} (V_n - V_{n-1})$$

Putting $n = 1, 2, 3, \dots, n$, we get

$$T_1 = \frac{1}{(r+1)d} (V_1 - V_0)$$

$$T_2 = \frac{1}{(r+1)d} (V_2 - V_1)$$

$$T_3 = \frac{1}{(r+1)d} (V_3 - V_2)$$

$$\vdots \quad \vdots \quad \vdots$$

$$T_n = \frac{1}{(r+1)d} (V_n - V_{n-1})$$

Adding the above equations, we get

$$T_1 + T_2 + T_3 + \dots + T_n = \frac{1}{(r+1)d} (V_n - V_0)$$

$$\therefore S_n = \frac{1}{(r+1)(a_2 - a_1)} (a_n a_{n+1} \dots a_{n+r} - a_0 a_1 a_2 \dots a_r) \quad [\because a_0 = a_1 - d]$$

Hence, the sum of n terms is

$$S_n = \frac{1}{(r+1)(a_2 - a_1)} (a_n a_{n+1} \dots a_{n+r} - a_0 a_1 a_2 \dots a_r)$$

Corollary If $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP, then

For $r = 2$

$$\begin{aligned} a_1 a_2 + a_2 a_3 + \dots + a_n a_{n+1} \\ = \frac{1}{3(a_2 - a_1)} (a_n a_{n+1} a_{n+2} - a_0 a_1 a_2) \end{aligned}$$

For $r = 3$

$$\begin{aligned} a_1 a_2 a_3 + a_2 a_3 a_4 + \dots + a_n a_{n+1} a_{n+2} \\ = \frac{1}{4(a_2 - a_1)} (a_n a_{n+1} a_{n+2} a_{n+3} - a_0 a_1 a_2 a_3) \end{aligned}$$

Example 69. The sum of n terms of the series

$$\frac{4}{1 \cdot 2 \cdot 3} + \frac{5}{2 \cdot 3 \cdot 4} + \frac{6}{3 \cdot 4 \cdot 5} + \dots \text{ is}$$

$$(a) \frac{n+3}{n(n+1)(n+2)}$$

$$(b) \frac{n(n+1)}{n(n+1)(n+2)}$$

$$(c) \frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}$$

$$(d) \text{ None of the above}$$

Sol. (c) Here, $T_n = \frac{n+3}{n(n+1)(n+2)}$

$$\Rightarrow T_n = \frac{1}{(n+1)(n+2)} + \frac{3}{n(n+1)(n+2)}$$

$$\Rightarrow T_n = \left[\frac{1}{n+1} - \frac{1}{n+2} \right] + \frac{3}{2} \left[\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore T_1 = \left(\frac{1}{2} - \frac{1}{3} \right) + \frac{3}{2} \left(\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} \right)$$

$$T_2 = \left(\frac{1}{3} - \frac{1}{4} \right) + \frac{3}{2} \left(\frac{1}{2 \cdot 3} - \frac{1}{3 \cdot 4} \right)$$

and so on.

Now, $S = T_1 + T_2 + T_3 + \dots$

$$\Rightarrow S = \left[\frac{1}{2} - \frac{1}{n+2} \right] + \frac{3}{2} \left[\frac{1}{2} - \frac{1}{(n+1)(n+2)} \right]$$

$$\therefore S = \frac{5}{4} - \frac{2n+5}{2(n+1)(n+2)}$$

➤ **Example 70.** If $\sum_{r=1}^n T_r = \frac{n}{8} (n+1)(n+2)(n+3)$,

then find $\sum_{r=1}^n \frac{1}{T_r}$.

$$\begin{aligned} \text{Sol. } \therefore T_n &= S_n - S_{n-1}, \quad \sum_{r=1}^n T_r - \sum_{r=1}^{n-1} T_r \\ &= \frac{n(n+1)(n+2)(n+3)}{8} - \frac{(n-1)n(n+1)(n+2)}{8} \end{aligned}$$

$$= \frac{n(n+1)(n+2)}{8} [(n+3) - (n-1)]$$

$$= \frac{n(n+1)(n+2)}{8} (4) = \frac{n(n+1)(n+2)}{2}$$

$$\Rightarrow \frac{1}{T_n} = \frac{2}{n(n+1)(n+2)} \quad \dots(i)$$

$$\therefore V_n = \frac{2}{(n+1)(n+2)}$$

$$\text{Then, } V_{n-1} = \frac{2}{n(n+1)}$$

$$\therefore V_n - V_{n-1} = \frac{2}{(n+1)(n+2)} - \frac{2}{n(n+1)}$$

$$= -\frac{4}{n(n+1)(n+2)} = -2 \left(\frac{1}{T_n} \right) \quad [\text{from Eq. (i)}]$$

$$\therefore \frac{1}{T_n} = -\frac{1}{2} (V_n - V_{n-1})$$

Putting $n = 1, 2, 3, \dots, n$, we get

$$\frac{1}{T_1} = -\frac{1}{2} (V_1 - V_0)$$

$$\frac{1}{T_2} = -\frac{1}{2} (V_2 - V_1)$$

$$\frac{1}{T_3} = -\frac{1}{2} (V_3 - V_2)$$

$$\vdots$$

$$\frac{1}{T_n} = -\frac{1}{2} (V_n - V_{n-1})$$

$$\therefore \frac{1}{T_1} + \frac{1}{T_2} + \frac{1}{T_3} + \dots + \frac{1}{T_n} = -\frac{1}{2} (V_n - V_0)$$

$$\Rightarrow \sum_{r=1}^n \frac{1}{T_r} = -\frac{1}{2} \left[\frac{2}{(n+1)(n+2)} - \frac{2}{2} \right]$$

$$= \frac{(n+1)(n+2) - 2}{2(n+1)(n+2)}$$

$$= \frac{n^2 + 3n}{2(n+1)(n+2)}$$

Some Important Series

$$(i) e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

$$(ii) \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \frac{x^n}{n} + \dots$$

$$(iii) (1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + \dots, (-1 < x < 1)$$

Work Book Exercise 3.4

1 The sum of n terms of the series $2^2 + 4^2 + 6^2 + \dots$ is

- a $\frac{n(n+1)(2n+1)}{3}$ b $\frac{2n(n+1)(2n+1)}{3}$
c $\frac{n(n+1)(2n+1)}{6}$ d $\frac{n(n+1)(2n+1)}{9}$

2 The sum of n terms of $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \dots$ is

- a $\frac{n(n+1)(n+2)(n+3)}{4}$
b $\frac{2n(n+1)(n+2)(n+3)}{3}$
c $\frac{(n+2)(n+1)(n+3)}{4}$
d $\frac{n(n-1)(n-2)(n-3)}{4}$

3 The sum of n terms of the series

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots \text{ is}$$

- a $\frac{n}{(n+1)(n+2)}$ b $\frac{1}{n} - \frac{1}{n+1}$
c $\frac{n}{n+1}$ d None of these

4 The sum of n terms of the series

$$\frac{3}{1^2} + \frac{5}{1^2 + 2^2} + \frac{7}{1^2 + 2^2 + 3^2} + \dots \text{ is}$$

- a $\frac{n+1}{n}$ b $\frac{n}{n+1}$
c $\frac{6n}{n+1}$ d $\frac{6(n-1)}{n}$

5 If $\sum_{r=1}^n T_r = (3^n - 1)$, then find the sum of $\sum_{r=1}^n \frac{1}{T_r}$.

- a $\frac{3}{4} \left[1 - \left(\frac{1}{3} \right)^n \right]$ b $\frac{3}{2} (3^n - 1)$
c $\frac{3}{4} (1 - 3^n)$ d $\frac{1}{2} \left[1 - \left(\frac{1}{3} \right)^n \right]$

6 The sum of n terms of the series $5 + 7 + 13 + 31 + 85 + \dots$ is

- a $\frac{1}{2} n(2n + 1)$
b $\frac{1}{2} (3^n + 8n - 1)$
c $\frac{1}{2} (3^{2n} - 2n)$
d None of the above

7 The sum of the series $\sum_{k=1}^{360} \left(\frac{1}{k \sqrt{k+1} + (k+1) \sqrt{k}} \right)$

is

- a $\frac{18}{19}$ b $\frac{15}{19}$
c $\frac{17}{19}$ d $\frac{16}{19}$

8 The sum of the series $\sum_{r=1}^n \frac{r}{(r+1)!}$, where

$n! = 1 \cdot 2 \cdot 3 \dots n$, is

- a $\frac{n}{(n+1)!}$ b $1 - \frac{1}{(n+1)!}$
c $\frac{1}{(n+1)!}$ d $\frac{n+1}{n!}$

9 The sum of first n terms of the series $1^3 + 3 \times 2^2 + 3^3 + 3 \times 4^2 + 5^3 + 3 \times 6^2 + \dots$, when n is even, is

- a $\frac{n+1}{8} (n^3 + 7n^2 - 3n - 1)$ b $\frac{n}{8} (n^3 + 7n^2 - 4n + 1)$
c $\frac{n}{8} (n^3 + 4n^2 + 10n + 8)$ d None of these

10 The sum of first n terms of the series

$$\frac{1}{1+1^2+1^4} + \frac{2}{1+2^2+2^4} + \frac{3}{1+3^2+3^4} + \dots \text{ is}$$

- a $\frac{1}{4} \left(1 - \frac{1}{n^2 + n + 1} \right)$ b $\frac{1}{n^2 + n + 1}$
c $\frac{1}{n + n^2 + n^4}$ d $\frac{1}{2} \left(1 - \frac{1}{n^2 + n + 1} \right)$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If S_1, S_2 and S_3 denote the sum of first n_1, n_2 and n_3 terms respectively of an AP, then $\frac{S_1}{n_1}(n_2 - n_3) + \frac{S_2}{n_2}(n_3 - n_1) + \frac{S_3}{n_3}(n_1 - n_2)$ is

equal to

- (a) 0
(b) 1
(c) $S_1 S_2 S_3$
(d) $n_1 n_2 n_3$

Sol. We have, $S_1 = \frac{n_1}{2}[2a + (n_1 - 1)d]$

$$\Rightarrow \frac{2S_1}{n_1} = 2a + (n_1 - 1)d$$

$$S_2 = \frac{n_2}{2}[2a + (n_2 - 1)d]$$

$$\Rightarrow \frac{2S_2}{n_2} = 2a + (n_2 - 1)d$$

and $S_3 = \frac{n_3}{2}[2a + (n_3 - 1)d]$

$$\Rightarrow \frac{2S_3}{n_3} = 2a + (n_3 - 1)d$$

$$\begin{aligned} \therefore \frac{2S_1}{n_1}(n_2 - n_3) + \frac{2S_2}{n_2}(n_3 - n_1) + \frac{2S_3}{n_3}(n_1 - n_2) \\ = [2a + (n_1 - 1)d](n_2 - n_3) + [2a + (n_2 - 1)d](n_3 - n_1) \\ + [2a + (n_3 - 1)d](n_1 - n_2) \\ = 0 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 2. Let a and b be two given numbers. If A denotes the single AM and S denotes the sum of n AM's between a and b , then S/A depends on

- (a) n, a, b (b) n, b (c) n, a (d) n

Sol. $\therefore A$ is the single AM between a and b .

$$\therefore A = \frac{a + b}{2}$$

Let $A_1, A_2, \dots, A_n$ be n AM's between a and b .

$\therefore a, A_1, A_2, \dots, A_n, b$ is an AP with common difference

$$d = \frac{b - a}{n + 1}$$

$$\text{Now, } S = A_1 + A_2 + \dots + A_n = \frac{n}{2}(A_1 + A_n)$$

$$= \frac{n}{2}(a + d + b - d) = \frac{n}{2}(a + b)$$

$$\therefore S = n \left(\frac{a + b}{2} \right) = nA$$

$$\Rightarrow \frac{S}{A} = n$$

Hence, (d) is the correct answer.

Ex 3. The sum of the products of $2n$ numbers $\pm 1, \pm 2, \pm 3, \dots, \pm n$ taking two at a time is

- (a) $-\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$
(c) $-\frac{n(n+1)(2n+1)}{6}$ (d) None of these

Sol. We have, $(1 - 1 + 2 - 2 + 3 - 3 + \dots + n - n)^2$
 $= 1^2 + 1^2 + 2^2 + 2^2 + \dots + n^2 + n^2 + 2S$

where, S is the required sum.

$$\Rightarrow 0 = 2(1^2 + 2^2 + \dots + n^2) + 2S$$

$$\Rightarrow S = -(1^2 + 2^2 + \dots + n^2) = -\frac{n(n+1)(2n+1)}{6}$$

Hence, (c) is the correct answer.

Ex 4. The number of terms common to two AP's 3, 7, 11, ..., 407 and 2, 9, 16, ..., 709 is

- (a) 21 (b) 28
(c) 14 (d) None of these

Sol. By inspection, first common term to both the sequences is 23. Second common term = 51

Third term = 79 and so on.

These numbers form an AP 23, 51, 79, ...

$$\therefore T_{15} = 23 + 14(28) = 23 + 392 = 415 > 407$$

$$\text{and } T_{14} = 23 + 13(28) = 387 < 407$$

$\therefore$ Number of common terms = 14

Hence, (c) is the correct answer.

Aliter

Let $t_m = t_n$
 $\Rightarrow 3 + (m - 1)4 = 2 + (n - 1)7$

$$\Rightarrow 4m - 1 = 7n - 5$$

$$\Rightarrow 4(m + 1) = 7n$$

$$\Rightarrow \frac{m + 1}{7} = \frac{n}{4} = \lambda,$$

where, $m \leq 102$

and $n \leq 102$

$$\therefore m = 7\lambda - 1, n = 4\lambda$$

$$\lambda \leq 14 \frac{5}{7} \text{ and } \lambda \leq 25 \frac{2}{4}$$

$$\therefore \lambda \leq 14, \text{ number of common terms} = 14$$

Hence, (c) is the correct answer.

Ex 5. The sum of all the numbers of the form n^3 which lie between 100 and 10000, is

- (a) 43261
(b) 53261
(c) 63261
(d) None of the above

Sol. The smallest and the largest numbers between 100 and 10000, which can be written in the form n^3 , are

$$5^3 = 125 \text{ and } 21^3 = 9261$$

∴ Required sum

$$= 5^3 + 6^3 + 7^3 + \dots + 21^3 = \sum_{n=1}^{21} n^3 - \sum_{n=1}^4 n^3$$

$$= \left[\frac{1}{4} n^2 (n+1)^2 \right]_{n=21} - \left[\frac{1}{4} n^2 (n+1)^2 \right]_{n=4}$$

$$= 53361 - 100 = 53261$$

Hence, (b) is the correct answer.

Ex 6. If $1, \log_y x, \log_z y$ and $-15 \log_x z$ are in AP, then

- (a) $z^3 = x$ (b) $x = y^{-2}$
 (c) $z^{-2} = y$ (d) None of these

Sol. Let d be the common difference.

Then, $\log_y x = 1 + d \Rightarrow x = y^{1+d}$
 $\log_z y = 1 + 2d \Rightarrow y = z^{1+2d}$

and $-15 \log_x z = 1 + 3d$

$\Rightarrow z = x^{-(1+3d)/15}$

∴ $x = y^{1+d} = z^{(1+2d)(1+d)}$
 $= x^{-(1+d)(1+2d)(1+3d)/15}$

$\Rightarrow (1+d)(1+2d)(1+3d) = -15$

$\Rightarrow 6d^3 + 11d^2 + 6d + 16 = 0$

$\Rightarrow (d+2)(6d^2 - d + 8) = 0$

$\Rightarrow d = -2$

∴ $x = y^{1+d} = y^{-1}, y = z^{1+2d} = z^{-3}$

and $x = (z^{-3})^{-1} = z^3$

Hence, (a) is the correct answer.

Ex 7. $L = \lim_{n \rightarrow \infty} \frac{1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \dots \text{ upto } n \text{ terms}}{n(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots \text{ upto } n \text{ terms})}$

is equal to

- (a) $\frac{3}{4}$ (b) $\frac{1}{4}$
 (c) $\frac{1}{2}$ (d) $\frac{5}{4}$

Sol. Numerator = $\sum_{r=2}^{n+1} (r-1)r(r+1)$

$$= \sum_{r=2}^{n+1} (r^3 - r) = \sum_{r=1}^{n+1} (r^3 - r)$$

$$= \frac{1}{4} (n+1)^2 (n+2)^2 - \frac{1}{2} (n+1)(n+2)$$

Also, $\sum_{r=1}^n r(r+1) = \sum_{r=1}^n [(r+1)^2 - (r+1)]$

$$= \sum_{s=2}^{n+1} (s^2 - s) = \sum_{s=1}^{n+1} (s^2 - s)$$

$$= \frac{1}{6} (n+1)(n+2)(2n+3) - \frac{1}{2} (n+1)(n+2)$$

Thus,

$$L = \lim_{n \rightarrow \infty} \frac{\frac{1}{4} \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{2}{n}\right)^2 - \frac{1}{2} \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}{\frac{1}{6} \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(2 + \frac{3}{n}\right) - \frac{1}{2} \left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right)}$$

$$= \frac{1}{4} \times \frac{6}{2} = \frac{3}{4}$$

Hence, (a) is the correct answer.

Ex 8. If three positive real numbers a, b and c are in AP such that $abc = 4$, then the minimum possible value of b is

- (a) $2^{3/2}$ (b) $2^{2/3}$ (c) $2^{1/3}$ (d) $2^{5/2}$

Sol. Let d be the common difference of an AP, then

$$4 = abc = (b-d)b(b+d)$$

$$= b(b^2 - d^2)$$

$$\Rightarrow b^3 = 4 + bd^2 \geq 4 \quad [\because b > 0, d^2 \geq 0]$$

$$\Rightarrow b \geq 2^{2/3}$$

Thus, minimum possible value of b is $2^{2/3}$, that is the case when $d = 0$.

Hence, (b) is the correct answer.

Ex 9. If the lengths of sides of a right triangle are in AP, then the sines of the acute angle are

- (a) $\frac{3}{5}, \frac{4}{5}$ (b) $\sqrt{\frac{2}{3}}, \sqrt{\frac{1}{3}}$
 (c) $\sqrt{\frac{\sqrt{5}-1}{2}}, \sqrt{\frac{\sqrt{5}+1}{2}}$ (d) $\sqrt{\frac{\sqrt{3}-1}{2}}, \sqrt{\frac{\sqrt{3}+1}{2}}$

Sol. Let the sides of the triangle be $a-d, a, a+d$, where $a > d > 0$. By Pythagoras theorem, we have

$$(a+d)^2 = (a-d)^2 + a^2$$

$$\Rightarrow 4ad = a^2 \Rightarrow d = \frac{a}{4}$$

We have, $\sin \theta = \frac{a}{a+d} = \frac{4}{5}$

and $\sin(90^\circ - \theta) = \frac{3}{5}$

Hence, (a) is the correct answer.

Ex 10. If $S_n = \sum_{r=1}^n t_r = \frac{1}{6} n(2n^2 + 9n + 13)$, then

$\sum_{r=1}^n \sqrt{t_r}$ is equal to

- (a) $\frac{1}{2} n(n+1)$ (b) $\frac{1}{2} n(n+2)$
 (c) $\frac{1}{2} n(n+3)$ (d) $\frac{1}{2} n(n+5)$

Sol. We have, $t_n = S_n - S_{n-1}, \forall n \geq 1$

$$\therefore t_n = \frac{1}{6} n(2n^2 + 9n + 13)$$

$$- \frac{1}{6} (n-1) \{2(n-1)^2 + 9(n-1) + 13\}$$

$$= \frac{1}{6} [2\{(n)^3 - (n-1)^3\} + 9\{n^2 - (n-1)^2\}$$

$$+ 13(n - n + 1)]$$

$$= \frac{1}{6} [6n^2 - 6n + 2 + 9(2n-1) + 13]$$

$$= \frac{1}{6} (6n^2 + 12n + 6) = (n+1)^2$$

$$\therefore \sum_{r=1}^n \sqrt{t_r} = \sum_{r=1}^n (r+1) = \frac{1}{2} (n+1)(n+2) - 1$$

$$= \frac{1}{2} n(n+3)$$

Hence, (c) is the correct answer.

- Ex 11.** Let a_n be the n th term of an AP. If $\sum_{r=1}^{10^{99}} a_{2r} = 10^{100}$ and $\sum_{r=1}^{10^{99}} a_{2r-1} = 10^{99}$, then the common difference of an AP is
(a) 1 (b) 9 (c) 10 (d) 10^{99}

Sol. Let d be the common difference of an AP, then

$$\begin{aligned} a_{2r} &= a_{2r-1} + d \\ \Rightarrow \sum_{r=1}^{10^{99}} a_{2r} &= \sum_{r=1}^{10^{99}} (a_{2r-1} + d) \\ \Rightarrow \sum_{r=1}^{10^{99}} a_{2r} &= \sum_{r=1}^{10^{99}} a_{2r-1} + 10^{99} d \\ \Rightarrow 10^{100} &= 10^{99} + 10^{99} d \\ \Rightarrow d &= \frac{10^{100} - 10^{99}}{10^{99}} = \frac{10^{99}(10-1)}{10^{99}} = 9 \end{aligned}$$

Hence, (b) is the correct answer.

- Ex 12.** Let $S(n)$ denotes the sum of first n terms of an AP. Then, the value of $\frac{S(3n)}{S(2n) - S(n)}$ is
(a) 3 (b) $\frac{1}{3}$
(c) 9 (d) None of these

Sol. $S(3n) = \frac{3n}{2} [2a + (3n-1)d]$
 $S(2n) = \frac{2n}{2} [2a + (2n-1)d]$
 $S(n) = \frac{n}{2} [2a + (n-1)d]$
 $\therefore S(2n) - S(n) = \frac{n}{2} [2\{2a + (2n-1)d\} - 2a - (n-1)d]$
 $= \frac{n}{2} [2a + (3n-1)d] = \frac{S(3n)}{3}$
 $\therefore \frac{S(3n)}{S(2n) - S(n)} = 3$

Hence, (a) is the correct answer.

- Ex 13.** The lengths of three unequal edges of a rectangular solid block are in GP. The volume of the block is 216 cm^3 and the total surface area is 252 cm^2 . The length of the longest edge is
(a) 12 cm (b) 6 cm (c) 18 cm (d) 3 cm

Sol. Let the edges be $\frac{a}{r}$, a and ar , where $r > 1$.

According to the question,

$$\begin{aligned} \frac{a}{r} \cdot a \cdot ar &= 216 \Rightarrow a^3 = 216 \Rightarrow a = 6 \\ \text{and } 2 \left(\frac{a}{r} \cdot a + a \cdot ar + \frac{a}{r} \cdot ar \right) &= 252 \\ \therefore \frac{1}{r} + r + 1 &= \frac{7}{2} \Rightarrow r = \frac{1}{2}, 2 \end{aligned}$$

Since, $a = 6$ and $r = 2$, so the longest side = $ar = 12$
Hence, (a) is the correct answer.

- Ex 14.** The sum of infinite terms of a decreasing GP is equal to the greatest value of the function $f(x) = x^3 + 3x - 9$ in the interval $[-2, 3]$ and the difference between the first two terms is $f'(0)$. Then, the common ratio of a GP is

- (a) $-\frac{2}{3}$
(b) $\frac{4}{3}$
(c) $\frac{2}{3}$
(d) $-\frac{4}{3}$

Sol. Let the GP be $a, ar, ar^2, \dots$ ($0 < r < 1$).

According to the question,

$$\frac{a}{1-r} = 3^3 + 3 \cdot 3 - 9 \quad \dots(i)$$

$$\left[\begin{array}{l} f'(x) = 3x^2 + 3 > 0. \text{ So } f(x) \text{ is monotonically} \\ \text{increasing and } f(3) \text{ is the greatest value in } [-2, 3] \end{array} \right]$$

Also, $f'(0) = 3$
 $\therefore a - ar = 3 \quad \dots(ii)$

On solving Eqs. (i) and (ii), i.e. $a = 27(1-r)$ and $a(1-r) = 3$, we get $r = \frac{2}{3}, \frac{4}{3}$ but $r < 1$

Hence, (c) is the correct answer.

- Ex 15.** The value of

$$\sum_{r=1}^n \frac{1}{\sqrt{a+rx} + \sqrt{a+(r-1)x}}$$

(a) $\frac{n}{\sqrt{a} + \sqrt{a+nx}}$
 (b) $\frac{\sqrt{a+nx} + \sqrt{a}}{x}$
 (c) $\frac{n(\sqrt{a+nx} - a)}{x}$
 (d) None of the above

Sol. Let $t_r = \frac{1}{\sqrt{a+rx} + \sqrt{a+(r-1)x}}$
 $= \frac{\sqrt{a+rx} - \sqrt{a+(r-1)x}}{a+rx - a - (r-1)x}$
 $= \frac{1}{x} [\sqrt{a+rx} - \sqrt{a+(r-1)x}]$
 $\therefore t_1 + t_2 + \dots + t_n = \frac{1}{x} [\{\sqrt{a+x} - \sqrt{a}\} + \{\sqrt{a+2x} - \sqrt{a+x}\} + \dots + \{\sqrt{a+nx} - \sqrt{a+(n-1)x}\}]$
 $= \frac{1}{x} [\sqrt{a+nx} - \sqrt{a}]$
 $= \frac{a+nx - a}{x(\sqrt{a+nx} + \sqrt{a})}$
 $= \frac{n}{\sqrt{a} + \sqrt{a+nx}}$

Hence, (a) is the correct answer.

Ex 16. If $\sum_{k=1}^n \left(\sum_{m=1}^k m^2 \right) = an^4 + bn^3 + cn^2 + dn + e$, then

- (a) $a = \frac{1}{12}$ (b) $b = \frac{1}{2}$
(c) $d = \frac{1}{5}$ (d) $e = 1$

Sol. $\sum_{k=1}^n \left(\sum_{m=1}^k m^2 \right) = \sum_{k=1}^n \frac{k(k+1)(2k+1)}{6}$
 $= \frac{1}{6} \sum_{k=1}^n (2k^3 + 3k^2 + k)$
 $= \frac{1}{3} \cdot \left\{ \frac{n(n+1)}{2} \right\}^2 + \frac{1}{2} \left[\frac{n(n+1)(2n+1)}{6} \right] + \frac{1}{6} \left[\frac{n(n+1)}{2} \right]$
 $a = \text{Coefficient of } n^4 = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$
Hence, (a) is the correct answer.

Ex 17. The value of $(0.2)^{\log_{\sqrt{5}} \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right)}$ is
(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 4

Sol. We have, $\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots = \frac{\frac{1}{4}}{1 - \frac{1}{2}} = \frac{1}{2} = 2^{-1}$
 $\therefore (0.2)^{\log_{\sqrt{5}} \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots \right)} = \left(\frac{1}{5} \right)^{\log_{\sqrt{5}} 2^{-1}}$
 $= (5^{-1})^{\frac{-1}{2} \log_5 2} = 5^{2 \log_5 2} = 5^{\log_5 (2^2)} = 2^2 = 4$
Hence, (d) is the correct answer.

Ex 18. If $x + y + z = 1$ and x, y and z are positive numbers such that $(1-x)(1-y)(1-z) \geq kxyz$, then k is equal to
(a) 2 (b) 4
(c) 8 (d) 16

Sol. $\therefore$ AM $\geq$ GM

$$\therefore \frac{y+z}{2} \geq \sqrt{yz} \quad \dots(i)$$

$$\frac{z+x}{2} \geq \sqrt{zx} \quad \dots(ii)$$

and $\frac{x+y}{2} \geq \sqrt{xy} \quad \dots(iii)$

On multiplying Eqs. (i), (ii) and (iii), we get

$$\frac{(y+z)(z+x)(x+y)}{8} \geq xyz$$

$$\text{or } (1-x)(1-y)(1-z) \geq 8xyz$$

Hence, (c) is the correct answer.

Ex 19. If $a + b + c = 3$ and $a > 0, b > 0, c > 0$, then the greatest value of $a^2 b^3 c^2$ is

- (a) $\frac{3^{10} \cdot 2^4}{7^7}$ (b) $\frac{3^9 \cdot 2^4}{7^7}$
(c) $\frac{3^8 \cdot 2^4}{7^7}$ (d) None of these

Sol. On taking AM and GM of 7 numbers

$$\frac{a}{2}, \frac{a}{2}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2}, \text{ we get}$$

$$\frac{2 \cdot \frac{a}{2} + 3 \cdot \frac{b}{3} + 2 \cdot \frac{c}{2}}{7} \geq \left\{ \left(\frac{a}{2} \right)^2 \left(\frac{b}{3} \right)^3 \left(\frac{c}{2} \right)^2 \right\}^{\frac{1}{7}}$$

$$\Rightarrow \frac{3}{7} \geq \left(\frac{a^2 b^3 c^2}{2^2 \cdot 3^3 \cdot 2^2} \right)^{\frac{1}{7}}$$

$$\Rightarrow \frac{3^7}{7^7} \geq \frac{a^2 b^3 c^2}{2^2 \cdot 3^3 \cdot 2^2}$$

$$\Rightarrow a^2 b^3 c^2 \leq \frac{3^{10} \cdot 2^4}{7^7}$$

$$\therefore \text{Greatest value of } a^2 b^3 c^2 = \frac{3^{10} \cdot 2^4}{7^7}$$

Hence, (a) is the correct answer.

Ex 20. If a, b and c are digits, then the rational number represented by $0.cababab\dots$, is

- (a) $\frac{99c + ba}{990}$ (b) $\frac{99c + 10a + b}{99}$
(c) $\frac{99c + 10a + b}{990}$ (d) None of these

Sol. Let $R = 0.cababab\dots$

$$= 0.c + 0.0ab + 0.000ab + \dots$$

$$= \frac{c}{10} + \frac{ab}{10^3} + \frac{ab}{10^5} + \dots$$

$$= \frac{c}{10} + \left[\frac{ab}{10^3} \right] \left[\frac{1}{1 - \frac{1}{10^2}} \right] = \frac{c}{10} + \left[\frac{ab}{10^3} \times \frac{10^2}{99} \right]$$

$$= \frac{c}{10} + \frac{ab}{990} = \frac{99c + ab}{990} = \frac{99c + 10a + b}{990}$$

Hence, (c) is the correct answer.

Aliter

$$\text{Let } R = 0.cababab\dots$$

$$\Rightarrow 10R = c.ababab\dots \quad \dots(i)$$

$$\text{and } 10^3 R = cab.ababab\dots \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$10^3 R - 10R = cab - c$$

$$\Rightarrow R = \frac{cab - c}{990}$$

$$\Rightarrow R = \frac{100c + 10a + b - c}{990}$$

$$\Rightarrow R = \frac{99c + 10a + b}{990}$$

Hence, (c) is the correct answer.

Ex 21. If $f(n) = \left[\frac{1}{2} + \frac{n}{100} \right]$, where $[x]$ denotes the integral part of x , then the value of $\sum_{n=1}^{100} f(n)$ is

- (a) 50 (b) 51
(c) 1 (d) None of these

Sol. $f(1) = \left[\frac{1}{2} + \frac{1}{100} \right] = 0, \dots,$
 $\vdots$
 $f(49) = \left[\frac{1}{2} + \frac{49}{100} \right] = \left[\frac{99}{100} \right] = 0$
 $f(50) = \left[\frac{1}{2} + \frac{50}{100} \right] = 1, f(51) = \left[\frac{1}{2} + \frac{51}{100} \right] = 1,$
 $\vdots$
 $f(100) = \left[\frac{1}{2} + \frac{100}{100} \right] = 1$
 $\therefore \sum_{n=1}^{100} f(n) = \underbrace{(0 + 0 + \dots + 0)}_{49 \text{ terms}} + \underbrace{(1 + 1 + \dots + 1)}_{51 \text{ terms}} = 51$

Hence, (b) is the correct answer.

Ex 22. $A_r, r = 1, 2, 3, \dots, n$ are n points on the parabola $y^2 = 4x$ in the first quadrant. If $A_r = (x_r, y_r)$, where $x_1, x_2, x_3, \dots, x_n$ are in GP and $x_1 = 1, x_2 = 2$, then y_n is equal to

(a) $2^{\frac{n+1}{2}}$ (b) 2^{n+1} (c) $(\sqrt{2})^{n+1}$ (d) $2^{n/2}$

Sol. $y = 2\sqrt{x}$, being in the first quadrant.

The sequence of x -coordinates is 1, 2, 4, 8, ...

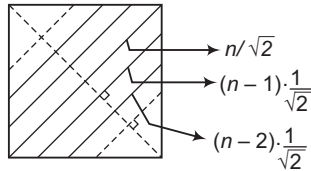
$\therefore$ The sequence of y -coordinates is
 $2, 2\sqrt{2}, 2\sqrt{4}, 2\sqrt{8}, \dots$

where, the common ratio is $\sqrt{2}$.

$\therefore y_n = 2 \cdot (\sqrt{2})^{n-1} = 2^{\frac{n+1}{2}}$

Hence, (a) is the correct answer.

Ex 23. In the given square, a diagonal is drawn and parallel line segments joining points on the adjacent sides are drawn on both sides of the diagonal. The length of the diagonal is $n\sqrt{2}$ cm. If the distance between consecutive line segments is $\frac{1}{\sqrt{2}}$ cm, then the sum of the lengths of all possible line segments and the diagonal is



(a) $n(n+1)\sqrt{2}$ cm (b) n^2 cm
 (c) $n(n+2)$ cm (d) $n^2\sqrt{2}$ cm

Sol. Lengths of line segments on one side of the diagonal are $\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, \dots, (n-1)\sqrt{2}$.

$\therefore$ Required sum
 $= 2\{\sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + \dots + (n-1)\sqrt{2}\} + n\sqrt{2}$
 $= 2\sqrt{2}\{1 + 2 + 3 + \dots + (n-1)\} + n\sqrt{2}$
 $= 2\sqrt{2} \cdot \frac{n(n-1)}{2} + n\sqrt{2} = n^2\sqrt{2}$

Hence, (d) is the correct answer.

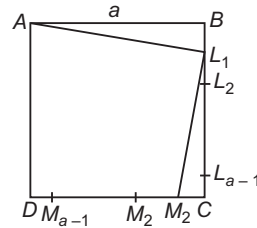
Ex 24. $ABCD$ is a square of length $a, a \in N, a > 1$.

Let $L_1, L_2, L_3, \dots, L_{a-1}$ be points on BC such that $BL_1 = L_1L_2 = L_2L_3 = \dots = 1$ and $M_1, M_2, M_3, \dots, M_{a-1}$ be points on CD such that $CM_1 = M_1M_2 = M_2M_3 = \dots = 1$. Then,

$\sum_{n=1}^{a-1} (AL_n^2 + L_nM_n^2)$ is equal to

- (a) $\frac{1}{2}a(a-1)^2$
 (b) $\frac{1}{2}a(a-1)(4a-1)$
 (c) $\frac{1}{2}(a-1)(2a-1)(4a-1)$
 (d) None of the above

Sol.



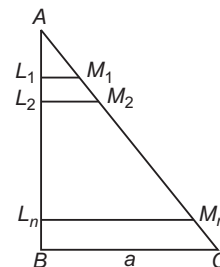
$AL_1^2 + L_1M_1^2 = (a^2 + 1^2) + \{(a-1)^2 + 1^2\}$
 $AL_2^2 + L_2M_2^2 = (a^2 + 2^2) + \{(a-2)^2 + 2^2\}$
 $\vdots$
 $AL_{a-1}^2 + L_{a-1}M_{a-1}^2 = a^2 + (a-1)^2 + \{1^2 + (a-1)^2\}$
 $\therefore$ Required sum
 $= (a-1)a^2 + \{1^2 + 2^2 + \dots + (a-1)^2\}$
 $+ 2\{1^2 + 2^2 + \dots + (a-1)^2\}$
 $= (a-1)a^2 + 3 \cdot \frac{(a-1)a(2a-1)}{6}$
 $= a(a-1) \left\{ a + \frac{2a-1}{2} \right\} = \frac{a(a-1)(4a-1)}{2}$

Hence, (b) is the correct answer.

Ex 25. ABC is a right angled triangle in which $\angle B = 90^\circ$ and $BC = a$. If n points $L_1, L_2, \dots, L_n$ on AB are such that AB is divided in $n+1$ equal parts and $L_1M_1, L_2M_2, \dots, L_nM_n$ are line segments parallel to BC and $M_1, M_2, \dots, M_n$ are on AC , then the sum of the lengths of $L_1M_1, L_2M_2, \dots, L_nM_n$ is

- (a) $\frac{a(n+1)}{2}$ (b) $\frac{a(n-1)}{2}$
 (c) $\frac{an}{2}$ (d) None of these

Sol.



$$\begin{aligned} \therefore \frac{AL_1}{AB} &= \frac{L_1M_1}{BC} \\ \therefore \frac{1}{n+1} &= \frac{L_1M_1}{a} \\ \Rightarrow L_1M_1 &= \frac{a}{n+1} \quad \text{and} \quad \frac{AL_2}{AB} = \frac{L_2M_2}{BC} \\ \Rightarrow \frac{2}{n+1} &= \frac{L_2M_2}{a} \\ \Rightarrow L_2M_2 &= \frac{2a}{n+1} \quad \text{and so on.} \\ \therefore \text{Required sum} &= \frac{a}{n+1} + \frac{2a}{n+1} + \frac{3a}{n+1} + \dots + \frac{na}{n+1} \\ &= \frac{a}{n+1} (1 + 2 + \dots + n) = \frac{a}{n+1} \cdot \frac{n(n+1)}{2} = \frac{an}{2} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 26. If a, b, c and d are in HP, then $ab + bc + cd$ is equal to

- (a) ad (b) $2ad$
(c) $3ad$ (d) None of these

Sol. Since, a, b, c and d are in HP, therefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ and $\frac{1}{d}$ are in AP.

$$\therefore \frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b} = \frac{1}{d} - \frac{1}{c} = k \quad [\text{say}]$$

$$\Rightarrow \frac{1}{b} - \frac{1}{a} = k, a - b = kab \quad \dots(i)$$

$$\frac{1}{c} - \frac{1}{b} = k, b - c = kbc \quad \dots(ii)$$

$$\text{and} \quad \frac{1}{d} - \frac{1}{c} = k, c - d = kcd \quad \dots(iii)$$

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{1}{d} - \frac{1}{a} = 3k, a - d = k(ab + bc + cd)$$

$$\text{or} \quad a - d = 3kad, a - d = k(ab + bc + cd)$$

$$\therefore 3ad = (ab + bc + cd)$$

Hence, (c) is the correct answer.

Ex 27. If a, b and c are in HP, then $\frac{a}{b+c}, \frac{b}{c+a}$ and

$$\frac{c}{a+b}$$

- (a) AP (b) GP
(c) HP (d) None of these

Sol. $\therefore a, b$ and c are in HP.

$$\therefore \frac{1}{a}, \frac{1}{b} \text{ and } \frac{1}{c} \text{ are in AP.}$$

$$\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c} \text{ are in AP}$$

$$\Rightarrow 1 + \frac{b+c}{a}, 1 + \frac{c+a}{b}, 1 + \frac{a+b}{c} \text{ are in AP}$$

$$\Rightarrow \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c} \text{ are in AP.}$$

$$\Rightarrow \frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in HP.}$$

Hence, (c) is the correct answer.

Ex 28. If $x > 1, y > 1$ and $z > 1$ are in GP, then

$$\frac{1}{1+\ln x}, \frac{1}{1+\ln y} \text{ and } \frac{1}{1+\ln z} \text{ are in}$$

- (a) AP (b) GP
(c) HP (d) None of these

Sol. Since, x, y and z are in GP.

$$\therefore y^2 = xz$$

Taking log on both sides, we get

$$2 \ln y = \ln x + \ln z$$

$$\Rightarrow 2 + 2 \ln y = (1 + \ln x) + (1 + \ln z)$$

$$\Rightarrow 2(1 + \ln y) = (1 + \ln x) + (1 + \ln z)$$

$$\therefore 1 + \ln x, 1 + \ln y \text{ and } 1 + \ln z \text{ are in AP.}$$

$$\Rightarrow \frac{1}{1+\ln x}, \frac{1}{1+\ln y} \text{ and } \frac{1}{1+\ln z} \text{ are in HP.}$$

Hence, (c) is the correct answer.

Ex 29. If $a, a_1, a_2, a_3, \dots, a_{2n}, b$ are in AP and $a, g_1, g_2, \dots, g_{2n}, b$ are in GP and h is the HM of a and b , then

$$\frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + \dots + \frac{a_n + a_{n+1}}{g_n g_{n+1}} \text{ is}$$

equal to

- (a) $2nh$ (b) $\frac{n}{h}$ (c) nh (d) $\frac{2n}{h}$

Sol. Since, $a, a_1, a_2, a_3, \dots, a_{2n}, b$ are in AP.

$$\therefore a + b = a_1 + a_{2n} = a_2 + a_{2n-1} = \dots = a_n + a_{n+1}$$

Also, $a, g_1, g_2, \dots, g_{2n}, b$ are in GP.

$$\therefore ab = g_1 g_{2n} = g_2 g_{2n-1} = \dots = g_n g_{n+1}$$

Now, the given expression

$$\begin{aligned} \frac{a_1 + a_{2n}}{g_1 g_{2n}} + \frac{a_2 + a_{2n-1}}{g_2 g_{2n-1}} + \dots + \frac{a_n + a_{n+1}}{g_n g_{n+1}} \\ = \frac{n(a+b)}{ab} = \frac{2n}{h} \quad \left[\because h = \frac{2ab}{a+b} \right] \end{aligned}$$

Hence, (d) is the correct answer.

Ex 30. If $0.272727\dots, x$ and $0.727272\dots$ are in HP, then x must be

- (a) rational (b) integer
(c) irrational (d) None of these

Sol. Let $A = 0.272727\dots$

$$\Rightarrow 10^2 A = 27.2727\dots$$

$$\Rightarrow A = \frac{27}{99} = \frac{3}{11}$$

$$\text{Similarly, } 0.727272\dots = \frac{8}{11}$$

Since, $0.272727\dots, x$ and $0.727272\dots$ are in HP.

$$\Rightarrow \frac{3}{11}, x, \frac{8}{11} \text{ are in HP.}$$

$$\text{Now, } x = \frac{2 \cdot \frac{3}{11} \cdot \frac{8}{11}}{\frac{3}{11} + \frac{8}{11}} = \frac{48}{121}$$

$\therefore x$ is rational.

Hence, (a) is the correct answer.

Ex 31. The arithmetic mean, geometric mean and harmonic mean between two positive real quantities are themselves in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. Relation between arithmetic mean (A), geometric mean (G) and harmonic mean (H) is given by $G^2 = AH$ or $G = \sqrt{AH}$. Clearly, A, G and H are in GP.
Hence, (b) is the correct answer.

Ex 32. A person purchases 1 kg tomatoes from each of the 4 places at the rate of 1 kg, 2 kg, 3 kg and 4 kg per rupee respectively. On the average, he has purchased x kg tomatoes per rupee, then the value of x is

- (a) 2 (b) 2.5
(c) 1.92 (d) None of these

Sol. Since, we are given rate per rupees, harmonic mean will give the correct answer.

$$\therefore \text{HM} = \frac{4}{\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}} = \frac{4 \times 12}{12 + 6 + 4 + 3} = \frac{48}{25} = 1.92 \text{ kg per rupee}$$

Hence, (c) is the correct answer.

Ex 33. If m is the root of the equation $(1-ab)x^2 - (a^2 + b^2)x - (1+ab) = 0$ and m harmonic means are inserted between a and b , then the difference between the last and the first of the means equals

- (a) $ab(a-b)$ (b) $a(b-a)$
(c) $ab(b-a)$ (d) $b-a$

Sol. According to the question,

$$(1-ab)m^2 - (a^2 + b^2)m - (1+ab) = 0$$

$$\Rightarrow m(a^2 + b^2) + (m^2 + 1)ab = m^2 - 1 \quad \dots(i)$$

Now, H_1 = First HM between a and b

$$= \frac{(m+1)ab}{a+mb} \quad \text{and} \quad H_m = \frac{(m+1)ab}{b+ma}$$

$$\therefore H_m - H_1 = (m+1)ab \left[\frac{1}{b+ma} - \frac{1}{a+mb} \right]$$

$$= (m+1)ab \left[\frac{(m-1)(b-a)}{(b+ma)(a+mb)} \right]$$

$$= \frac{(m^2-1)ab(b-a)}{m(a^2+b^2) + (m^2+1)ab}$$

$$= \frac{(m^2-1)ab(b-a)}{m^2-1} \quad [\text{from Eq. (i)}]$$

$$= ab(b-a)$$

Hence, (c) is the correct answer.

Ex 34. If a, b and c are in AP and a, mb, c are in GP, then a, m^2b and c are in

- (a) HP (b) GP
(c) AP (d) None of these

Sol. Since, a, b and c are in AP.

$$\therefore 2b = a + c \quad \dots(i)$$

Since, a, mb and c are in GP.

$$\therefore mb = \sqrt{ac} \Rightarrow m^2b^2 = ac \quad \dots(ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{2b}{m^2b^2} = \frac{a+c}{ac}$$

$$\text{i.e.} \quad \frac{2}{m^2b} = \frac{1}{a} + \frac{1}{c}$$

$$\Rightarrow a, m^2b \text{ and } c \text{ are in HP.}$$

Hence, (a) is the correct answer.

Ex 35. The AM, HM and GM between two numbers are $\frac{144}{15}$, 15 and 12 but not necessarily in this order. Then, HM, GM and AM respectively are

- (a) $\frac{144}{15}$, 12, 15 (b) $\frac{144}{15}$, 15, 12
(c) 15, 12, $\frac{144}{15}$ (d) 12, 15, $\frac{144}{15}$

Sol. Since, $A > G > H$, i.e. $H < G < A$

$$\therefore \text{Required numbers are } \frac{144}{15}, 12, 15, \text{ respectively.}$$

Hence, (a) is the correct answer.

Ex 36. Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ are the roots of the equation $Bx^2 - 6x + 1 = 0$, then values of A and B such that α, β, γ and δ are in HP, are

- (a) $A = 3, B = 8$ (b) $A = -3, B = 8$
(c) $A = 3, B = -8$ (d) None of these

Sol. α, β, γ and δ are in HP $\Rightarrow \frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ and $\frac{1}{\delta}$ are in AP.

Let d be the common difference of this AP.

Now, α and γ are roots of $Ax^2 - 4x + 1 = 0$.

$$\therefore \frac{\alpha + \gamma}{\alpha\gamma} = \frac{4/A}{1/A} = 4 \Rightarrow \frac{1}{\alpha} + \frac{1}{\gamma} = 4$$

$$\text{i.e.} \quad \frac{1}{\alpha} + \frac{1}{\alpha} + 2d = 4 \Rightarrow \frac{1}{\alpha} + d = 2 \quad \dots(i)$$

Since, β and δ are roots of $Bx^2 - 6x + 1 = 0$.

$$\therefore \frac{\beta + \delta}{\beta\delta} = \frac{1}{\beta} + \frac{1}{\delta} = \frac{6/B}{1/B} = 6$$

$$\text{or} \quad \frac{1}{\alpha} + d + \frac{1}{\alpha} + 3d = 6$$

$$\Rightarrow \frac{1}{\alpha} + 2d = 3 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{1}{\alpha} = 1, d = 1$$

$$\therefore \frac{1}{\alpha} = 1, \frac{1}{\beta} = 2, \frac{1}{\gamma} = 3 \text{ and } \frac{1}{\delta} = 4$$

$$\text{Since,} \quad \frac{1}{\alpha\gamma} = A \Rightarrow A = 3$$

$$\text{Also,} \quad \frac{1}{\beta\delta} = B \Rightarrow B = 8$$

$$\text{Thus,} \quad A = 3 \text{ and } B = 8$$

Hence, (a) is the correct answer.

Ex 37. If $\log(x+z) + \log(x+z-2y) = 2 \log(x-z)$, then x , y and z are in

- (a) GP (b) AP
(c) HP (d) None of these

Sol. Given, $\log(x+z) + \log(x+z-2y) = 2 \log(x-z)$
 $\Rightarrow \log[(x+z-2y)(x+z)] = \log(x-z)^2$
 $\Rightarrow (x+z-2y)(x+z) = (x-z)^2$
 $\Rightarrow x^2 + 2xz + z^2 - 2yz - 2xy = x^2 + z^2 - 2xz$
 $\Rightarrow 4xz = 2yz + 2xy \Rightarrow 2xz = y(x+z)$
 $\therefore y = \frac{2xz}{x+z}$

So, x , y and z are in HP.

Hence, (c) is the correct answer.

Ex 38. If $H_1, H_2, \dots, H_n$ are n harmonic means between a and b ($a \neq b$), then value of $\frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b}$ is

- (a) $n+1$ (b) $n-1$
(c) $2n$ (d) $2n+3$

Sol. As $a, H_1, H_2, \dots, H_n, b$ are in HP.

$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \dots, \frac{1}{H_n}, \frac{1}{b}$ are in AP.

Let d be the common difference of an AP, then

$$\frac{1}{b} = \frac{1}{a} + (n+1)d$$

$$\Rightarrow d = \frac{1}{n+1} \cdot \frac{a-b}{ab}$$

Thus, $\frac{1}{H_1} = \frac{1}{a} + d$ and $\frac{1}{H_n} = \frac{1}{b} - d$

$\Rightarrow \frac{a}{H_1} = 1 + ad$ and $\frac{b}{H_n} = 1 - bd$

Now, $\frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = \frac{1 + \frac{a}{H_1}}{1 - \frac{a}{H_1}} + \frac{1 + \frac{b}{H_n}}{1 - \frac{b}{H_n}}$

$$= \frac{1 + 1 + ad}{1 - 1 - ad} + \frac{1 + 1 - bd}{1 - 1 + bd}$$

$$= \frac{2 + ad}{-ad} + \frac{2 - bd}{bd} = \frac{2a - abd - 2b - abd}{abd}$$

$$= \frac{2[(a-b) - abd]}{abd}$$

$$= \frac{2[(n+1)dab - abd]}{abd} = 2n$$

Hence, (c) is the correct answer.

Ex 39. If $a_1 = 0$ and $a_1, a_2, a_3, \dots, a_n$ are real numbers such that $|a_i| = |a_{i-1} + 1|$ for all i , then the AM of the numbers $a_1, a_2, \dots, a_n$ has value x , where

- (a) $x \leq -\frac{1}{2}$ (b) $x \geq -\frac{1}{2}$
(c) $x < -\frac{1}{2}$ (d) None of these

Sol. We have, $|a_i| = |a_{i-1} + 1|$

$$\Rightarrow a_i^2 = a_{i-1}^2 + 2a_{i-1} + 1$$

Putting $i = 1, 2, 3, \dots, n+1$, we get

$$a_1^2 = 0$$

$$a_2^2 = a_1^2 + 2a_1 + 1$$

$$a_3^2 = a_2^2 + 2a_2 + 1$$

$$\vdots$$

$$a_n^2 = a_{n-1}^2 + 2a_{n-1} + 1$$

$$a_{n+1}^2 = a_n^2 + 2a_n + 1$$

On adding, we get $\sum_{i=1}^{n+1} a_i^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i=1}^n a_i + n$

$$\Rightarrow 2 \sum_{i=1}^n a_i = -n + a_{n+1}^2 \geq -n$$

$$\Rightarrow \frac{a_1 + a_2 + \dots + a_n}{n} \geq -\frac{1}{2}$$

According to the question,

$$x = \frac{a_1 + a_2 + \dots + a_n}{n} \Rightarrow x \geq -\frac{1}{2}$$

Hence, (b) is the correct answer.

Ex 40. The sum of first n terms of the series $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots$ is $\frac{n(n+1)^2}{2}$, when n is even. When n is odd, the

sum is

- (a) $\frac{n^2(n+1)}{2}$ (b) $\frac{n(n+1)^2}{2}$
(c) $\left[\frac{n(n+1)^2}{2} \right]$ (d) $\frac{n(n+1)}{2}$

Sol. When n is odd, last term will be n^2 .

Then, the sum is

$$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots + 2(n-1)^2 + n^2$$

$$= \frac{(n-1)n^2}{2} + n^2 \left[\text{replacing } n \text{ by } n-1 \text{ in } \frac{n(n+1)^2}{2} \right]$$

$$= \frac{n^3 - n^2 + 2n^2}{2}$$

$$= \frac{n^3 + n^2}{2} = \frac{n^2(n+1)}{2}$$

Hence, (a) is the correct answer.

Ex 41. If three positive real numbers a, b, c are in AP such that $abc = 4$, then the minimum possible value of b is

- (a) $2^{3/2}$ (b) $2^{2/3}$ (c) $2^{1/3}$ (d) $2^{5/2}$

Sol. Let d be the common difference of an AP, then $4 = abc = (b-d)b(b+d) = b(b^2 - d^2)$

$$\Rightarrow b^3 = 4 + bd^2 \geq 4 \quad [\because b > 0, d^2 \geq 0]$$

$$\Rightarrow b \geq 2^{2/3}$$

Thus, minimum possible value of b is $2^{2/3}$, that is the case when $d = 0$.

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 42. If a, b and c are in HP, then

- (a) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in HP
 (b) $\frac{2}{b} = \frac{1}{b-a} + \frac{1}{b-c}$
 (c) $a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in GP
 (d) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in HP

Sol. Since, a, b and c are in HP.

So, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.
 $\Rightarrow \frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in AP.
 $\Rightarrow \frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in AP.

[on subtracting 1 from each term]

$\Rightarrow \frac{b+c}{a} - 1, \frac{c+a}{b} - 1, \frac{a+b}{c} - 1$ are in AP.
 $\Rightarrow \frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP.

Thus, $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in HP.

and $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are also in HP.

Also we have, $b = \frac{2ac}{a+c}$

So, $\frac{1}{b-a} + \frac{1}{b-c} = \frac{2b-(a+c)}{(b-a)(b-c)}$
 $= \frac{2b-(a+c)}{b^2-b(a+c)+ac}$
 $= \frac{2b-\frac{2ac}{b}}{b^2-b\cdot\frac{2ac}{b}+ac}$
 $= \frac{2}{b} \cdot \frac{b^2-ac}{b^2-ac} = \frac{2}{b}$

Lastly, $\left(a - \frac{b}{2}\right)\left(c - \frac{b}{2}\right) = ac - \frac{b}{2}(a+c) + \frac{b^2}{4}$
 $= ac - \frac{b}{2} \cdot \frac{2ac}{b} + \frac{b^2}{4} = \frac{b^2}{4}$

$\therefore a - \frac{b}{2}, \frac{b}{2}, c - \frac{b}{2}$ are in GP.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 43. For a positive integer n , let

$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{(2^n)-1}$. Then,

- (a) $a(n) < n$ (b) $a(n) > \frac{n}{2}$
 (c) $a(2n) > n$ (d) $a(2n) < 2n$

Sol. We have,

$a(n) = 1 + \left(\frac{1}{2} + \frac{1}{3}\right) + \left(\frac{1}{4} + \dots + \frac{1}{7}\right) + \left(\frac{1}{8} + \dots + \frac{1}{15}\right)$
 $+ \dots + \left(\frac{1}{2^{n-1}} + \frac{1}{2^{n-1}+1} + \dots + \frac{1}{(2^n)-1}\right)$
 $< 1 + \frac{2}{2} + \frac{4}{4} + \frac{8}{8} + \dots + \frac{2^{n-1}}{2^{n-1}} = n$

Thus, $a(n) < n \Rightarrow a(2n) < 2n$

Now, $a(n) = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right)$
 $+ \dots + \left(\frac{1}{2^{n-1}} + \dots + \frac{1}{2^n}\right) - \frac{1}{2^n}$
 $> \left(1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \dots + \frac{2^{n-1}}{2^n} - \frac{1}{2^n}\right)$
 $\geq \underbrace{\frac{1}{2} + \frac{1}{2} + \dots + \frac{1}{2}}_{n \text{ times}} = \frac{n}{2}$

Thus, $a(n) > \frac{n}{2} \Rightarrow a(2n) > n$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 44. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then

- (a) $a = b = c$
 (b) a, b and $-\frac{1}{2}c$ are in GP
 (c) a, b and c are in GP
 (d) $-\frac{1}{2}a, b$ and c are in GP

Sol. From the given condition,

$2b = a + c$ and $b^2 = \frac{2a^2c^2}{a^2+c^2}$

On eliminating b , we get $\left(\frac{a+c}{2}\right)^2 = \frac{2a^2c^2}{a^2+c^2}$

$\Rightarrow (a^2+c^2)^2 + 2ac(a^2+c^2) - 8a^2c^2 = 0$

$\Rightarrow (a^2+c^2+4ac)(a^2+c^2-2ac) = 0$

$\Rightarrow [(a+c)^2+2ac](a-c)^2 = 0$

$\Rightarrow 4\left(b^2 + \frac{1}{2}ac\right)(a-c)^2 = 0$

$\Rightarrow a-c=0$ or $b^2 = -\frac{1}{2}ac$

If $a=c$, we get $a=b=c$ and a, b, c are in GP.

If $b^2 = -\frac{1}{2}ac$, then either $a, b, -\frac{1}{2}c$ are in GP

or $-\frac{1}{2}a, b, c$ are in GP.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 45. Let $a_n = \underbrace{(111\dots 1)}_{n \text{ times}}$, then

- (a) a_{912} is not prime (b) a_{951} is not prime
 (c) a_{480} is not prime (d) a_{91} is not prime

Sol. As a_{912} , a_{951} and a_{480} are divisible by 3, none of them is prime. For a_{91} , we have

$$\begin{aligned} a_{91} &= \frac{1}{9} \underbrace{(99 \dots 9)}_{91 \text{ times}} = \frac{1}{9} (10^{91} - 1) \\ &= \frac{1}{9} [(10^7)^{13} - 1] \\ &= \left[\frac{(10^7)^{13} - 1}{10^7 - 1} \right] \left[\frac{10^7 - 1}{10 - 1} \right] \\ &= [(10^7)^{12} + (10^7)^{11} + \dots + 10^7 + 1] \\ &\quad \times [10^6 + 10^5 + \dots + 10 + 1] \end{aligned}$$

$\Rightarrow a_{91}$ is not prime.

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 46. If a, b, c and d are four unequal positive numbers which are in AP, then

$$\begin{aligned} \text{(a)} \quad \frac{1}{a} + \frac{1}{d} &= \frac{1}{b} + \frac{1}{c} & \text{(b)} \quad \frac{1}{a} + \frac{1}{d} &< \frac{1}{b} + \frac{1}{c} \\ \text{(c)} \quad \frac{1}{a} + \frac{1}{d} &> \frac{1}{b} + \frac{1}{c} & \text{(d)} \quad \frac{1}{b} + \frac{1}{c} &> \frac{4}{a+d} \end{aligned}$$

Sol. Let $b = a + p$, $c = a + 2p$, $d = a + 3p$

$$\begin{aligned} \frac{\frac{1}{a} + \frac{1}{d}}{\frac{1}{b} + \frac{1}{c}} &= \frac{\frac{1}{a} + \frac{1}{a+3p}}{\frac{1}{a+p} + \frac{1}{a+2p}} \\ &= \frac{(a+p)(a+2p)}{a(a+3p)} \\ &= \frac{a^2 + 3ap + 2p^2}{a^2 + 3ap} > 1 \\ \Rightarrow \quad \frac{1}{a} + \frac{1}{d} &> \frac{1}{b} + \frac{1}{c} \\ \therefore \left(\frac{1}{b} + \frac{1}{c} \right) (a+d) &= \left(\frac{1}{a+p} + \frac{1}{a+2p} \right) (a+a+3p) \\ &= \frac{(2a+3p)^2}{a^2 + 3ap + 2p^2} \\ &= 4 + \frac{p^2}{a^2 + 3ap + 2p^2} > 4 \end{aligned}$$

Hence, (c) and (d) are the correct answers.

Ex 47. The sum of the series

$$\begin{aligned} &\tan^{-1} \left(\frac{1}{2} \right) + \tan^{-1} \left(\frac{2}{9} \right) + \tan^{-1} \left(\frac{1}{8} \right) \\ &\quad + \tan^{-1} \left(\frac{2}{25} \right) + \tan^{-1} \left(\frac{1}{18} \right) + \dots \text{ is} \\ \text{(a)} \quad &\tan^{-1} (3) & \text{(b)} \quad \cot^{-1} \left(\frac{1}{3} \right) \\ \text{(c)} \quad &\tan^{-1} \left(\frac{1}{3} \right) & \text{(d)} \quad \text{None of these} \end{aligned}$$

Sol. Let $T_r = \tan^{-1} \left(\frac{\sqrt{2}}{r+1} \right)$, then

$$\begin{aligned} S_n &= \sum_{r=1}^n \tan^{-1} \left(\frac{2}{r^2 + 2r + 1} \right) = \sum_{r=1}^n \tan^{-1} \frac{r+2-r}{1+r(r+2)} \\ &= \sum_{r=1}^n (\tan^{-1} (r+2) - \tan^{-1} r) \\ S_n &= \tan^{-1} (n+2) + \tan^{-1} (n+1) - \tan^{-1} 2 - \tan^{-1} 1 \\ &= \tan^{-1} \left(\frac{3n^2 + 7n}{n^2 + 9n + 10} \right) \\ \therefore S_\infty &= \tan^{-1} (3) = \cot^{-1} \left(\frac{1}{3} \right) \end{aligned}$$

Hence, (a) and (b) are the correct answers.

Ex 48. If a, b and c are three unequal positive quantities in HP, then

$$\begin{aligned} \text{(a)} \quad a^{100} + c^{100} &> 2b^{100} & \text{(b)} \quad a^3 + c^3 &> 2b^3 \\ \text{(c)} \quad a^5 + c^5 &> 2b^5 & \text{(d)} \quad a^2 + c^2 &> 2b^2 \end{aligned}$$

Sol. Since, b is the HM of a and c and their GM $= \sqrt{ac}$

$$\begin{aligned} \therefore \quad &\text{GM} > \text{HM} \\ \Rightarrow \quad &\sqrt{ac} > b \\ \text{and AM of } a^n \text{ and } c^n &> \text{GM of } a^n \text{ and } c^n \\ \frac{a^n + c^n}{2} &> \sqrt{a^n c^n} \\ a^n + c^n &> 2(\sqrt{ac})^n > 2b^n \end{aligned}$$

Put $n = 100, 3, 5, 2$

Hence, (a), (b), (c) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 49-52) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- Statement I is true, Statement II is false
- Statement I is false, Statement II is true

Ex 49. Statement I If $a + 2b + 3c = 1$ and $a > 0, b > 0, c > 0$, then the greatest value of $a^3 b^2 c$ is $\frac{1}{5184}$.

Statement II There exists an AP such that sum upto its n terms is given by $S_n = an^3 + bn^2 + cn + d$.

Sol. We can use AM $\geq$ GM to have,

$$\frac{\frac{3a}{3} + 2b + \frac{c}{1/3}}{6} \geq \left(\frac{a^3}{27} \cdot b^2 \cdot 3c \right)^{1/6}$$

$$\Rightarrow \left(\frac{1}{6}\right)^6 \geq \frac{a^3 b^2 c}{9} \quad \text{or} \quad a^3 b^2 c \leq \frac{9}{6^6} \Rightarrow a^3 b^2 c \leq \frac{1}{5184}$$

Also, sum upto n terms of an AP is always a quadratic expression in n . Clearly, Statement I is true and Statement II is false.

Hence, (c) is the correct answer.

Ex 50. Statement I If a, b and c are non-zero real numbers such that $3(a^2 + b^2 + c^2 + 1) = 2(a + b + c + ab + bc + ca)$, then a, b and c are in AP as well as in GP.

Statement II A series is an AP as well as a GP, if all the terms in the series are equal and non-zero.

Sol. We have, $3(a^2 + b^2 + c^2 + 1) - 2(a + b + c + ab + bc + ca) = 0$

$$\Rightarrow (a-1)^2 + (b-1)^2 + (c-1)^2 + (a-b)^2 + (b-c)^2 + (c-a)^2 = 0$$

$$\Rightarrow a = b = c = 1$$

Hence, (a) is the correct answer.

Ex 51. Statement I Given, 1, 2, 4, 8,... is a GP; 4, 8, 16, 32,... is a GP, then 1 + 4, 2 + 8, 4 + 16, 8 + 32,... is also a GP.

Statement II If general term of a GP with common ratio r be T_{k+1} and general term of another GP with common ratio r be T'_{k+1} , then the series whose general term

$T''_{k+1} = T_{k+1} + T'_{k+1}$ is also a GP with common ratio r .

Sol. Let $T_{k+1} = ar^k$ and $T'_{k+1} = br^k$

$$\therefore T''_{k+1} = ar^k + br^k = (a+b)r^k$$

$\therefore T''_{k+1}$ is general term of a GP with common ratio r .

Hence, (a) is the correct answer.

Ex 52. Statement I If one AM, A and two GM's, p and q are inserted between any two numbers, then $p^3 + q^3 = 2Apq$.

Statement II If x, y and z are in GP, then $y^2 = xz$.

Sol. Statement I a, A and b are in AP.

$$\Rightarrow 2A = a + b \quad \dots(i)$$

Since, a, p, q and b are in GP.

$$\therefore pq = ab \quad \dots(ii)$$

Let common ratio of GP be r .

$$\therefore b = ar^3$$

$$\Rightarrow r = \left(\frac{b}{a}\right)^{1/3}$$

$$\therefore p = ar$$

$$\Rightarrow p = a \left(\frac{b}{a}\right)^{1/3} \Rightarrow p^3 = a^2 b \quad \dots(iii)$$

$$\text{and } q = ar^2 \Rightarrow q = a \left(\frac{b}{a}\right)^{2/3}$$

$$\Rightarrow q^3 = ab^2 \quad \dots(iv)$$

From Eqs. (i), (ii), (iii) and (iv), we get

$$p^3 + q^3 = 2Apq$$

Statement II is obviously true.

Hence, (b) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 53-55) We know that, if $a_1, a_2, \dots, a_n$ are in HP, then $\frac{1}{a_1}, \frac{1}{a_2}, \dots, \frac{1}{a_n}$ are in AP and

vice-versa. If $a_1, a_2, \dots, a_n$ are in AP with common difference d , then for any $b (b > 0)$, the numbers $b^{a_1}, b^{a_2}, b^{a_3}, \dots, b^{a_n}$ are in GP with common ratio b^d . If $a_1, a_2, \dots, a_n$ are positive and in GP with common ratio (r) , then for any base $b (b > 0)$, $\log_b a_1, \log_b a_2, \dots, \log_b a_n$ are in AP with common difference $\log_b r$.

Ex 53. If x, y and z are respectively the p th, q th and r th terms of an AP as well as of a GP, then $x^{y-z} \cdot y^{z-x} \cdot z^{x-y}$ is equal to

- (a) p (b) q (c) r (d) 1

Sol. Let the base be taken as e . Again, let, x, y and z be terms of a GP with common ratio R , then $\ln x, \ln y, \ln z$ are terms of AP with common difference $\ln R$. Also, let x, y and z be terms of AP with common difference d .

Now, $x - y = (p - q)d$ etc., and

$$\ln x - \ln y = (p - q) \ln R \text{ etc.}$$

$$\text{Let } E = x^{y-z} \cdot y^{z-x} \cdot z^{x-y}$$

$$\begin{aligned} \therefore \ln E &= (y-z) \ln x + (z-x) \ln y + (x-y) \ln z \\ &= (q-r) d \ln x + (r-p) d \ln y + (p-q) d \ln z \\ &= d [p(\ln z - \ln y) + q(\ln x - \ln z) + r(\ln y - \ln x)] \\ &= d \ln r [p(r-q) + q(p-r) + r(q-p)] = 0 \end{aligned}$$

$$\Rightarrow E = 1$$

Hence, (d) is the correct answer.

Ex 54. If a, b, c, d are in GP and $a^x = b^y = c^z = d^v$, then x, y, z, v are in

- (a) AP (b) GP
(c) HP (d) None of these

Sol. Let a, b, c, d be in GP with common ratio r then $\ln a, \ln b, \ln c, \ln d$ are in AP with common difference $\ln r$. Also, $a^x = b^y = c^z = d^v$

$$\Rightarrow x \ln a = y \ln b = z \ln c = v \ln d$$

$$\Rightarrow x \ln a = y (\ln a + \ln r) = z (\ln a + 2 \ln r) = v (\ln a + 3 \ln r)$$

$$\Rightarrow \frac{x-z}{z} = 2 \cdot \frac{x-y}{y} \quad \text{and} \quad \frac{y-v}{v} = 2 \cdot \frac{y-z}{z}$$

$$\Rightarrow \frac{1}{x} + \frac{1}{z} = \frac{2}{y} \quad \text{and} \quad \frac{1}{y} + \frac{1}{v} = \frac{2}{z}$$

So, x, y, z and v are in HP.

Hence, (c) is the correct answer.

3

Ex 55. If p, q and r are in AP, then p th, q th and r th terms of any GP are in
(a) AP (b) GP (c) HP (d) AGP

Sol. Since, p, q and r are in AP.

$\therefore b^p, b^q, b^r$ are in GP for any b ($b > 0$)

$\Rightarrow b^{p-1}, b^{q-1}, b^{r-1}$ are in GP.

$\Rightarrow ab^{p-1}, ab^{q-1}, ab^{r-1}$ are in GP for any a ($a \neq 0$)

So, p th, q th and r th terms of any GP are in GP.

Hence, (b) is the correct answer.

Passage II (Ex. Nos. 56-60) Let $A_1, A_2, A_3, \dots, A_m$ be arithmetic means between -2 and 1027 and $G_1, G_2, G_3, \dots, G_n$ be geometric means between 1 and 1024 . Product of geometric means is 2^{45} and sum of arithmetic means is 1025×171 .

Ex 56. The value of n is

- (a) 7 (b) 9
(c) 11 (d) None of these

Sol. $G_1 G_2 \dots G_n = (\sqrt[n]{1 \times 1024})^n = 2^{5n}$

$\therefore 2^{5n} = 2^{45} \Rightarrow n = 9$

Hence, (b) is the correct answer.

Ex 57. The value of m is

- (a) 340 (b) 342 (c) 344 (d) 346

Sol. $\therefore A_1 + A_2 + A_3 + \dots + A_{m-1} + A_m$
 $= 1025 \times 171$

$\therefore m \left(\frac{-2 + 1027}{2} \right) = 1025 \times 171$

$\Rightarrow m = 342$

Hence, (b) is the correct answer.

Ex 58. The value of $G_1 + G_2 + G_3 + \dots + G_n$ is

- (a) 1022 (b) 2044
(c) 512 (d) None of these

Sol. $\therefore n = 9$

$\therefore r = (1024)^{\frac{1}{9+1}} = 2$

So, $G_1 = 2$ and $r = 2$

Now, $G_1 + G_2 + \dots + G_n = \frac{2(2^9 - 1)}{2 - 1}$

$= 1024 - 2 = 1022$

Hence, (a) is the correct answer.

Ex 59. The common difference of the progression

$A_1, A_3, A_5, \dots, A_{m-1}$ is

- (a) 6 (b) 3 (c) 2 (d) 1

Sol. Common difference of sequence $A_1, A_2, \dots, A_m$

$$= \frac{1027 + 2}{342 + 1} = 3$$

$\therefore$ Common difference of sequence $A_1, A_3, A_5, \dots, A_{m-1}$ is 6.

Hence, (a) is the correct answer.

Ex 60. The numbers $2A_{171}, G_5^2 + 1$ and $2A_{172}$ are in

- (a) AP
(b) GP
(c) HP
(d) AGP

Sol. We have, $A_{171} + A_{172} = -2 + 1027 = 1025$

$$\therefore \frac{2A_{171} + 2A_{172}}{2} = 1025$$

Also, $G_5 = 1 \times 2^5 = 32$

Now, $G_5^2 = 1024$

i.e. $G_5^2 + 1 = 1025$

So, $2A_{171}, G_5^2 + 1$ and $2A_{172}$ are in AP.

Hence, (a) is the correct answer.

Passage III (Ex. Nos. 61-63) There are two sets A and B each of which consists of three numbers in AP whose sum is 15 and where D and d are the common difference such that $D - d = 1$. If $\frac{p}{q} = \frac{7}{8}$, where p and q

are the product of the numbers respectively and $d > 0$, in the two sets.

Ex 61. The value of p is

- (a) 100 (b) 120
(c) 105 (d) 110

Ex 62. The value of q is

- (a) 100 (b) 120
(c) 105 (d) 110

Ex 63. The value of $D + d$ is

- (a) 1 (b) 2
(c) 3 (d) 4

Sol. (Ex. Nos. 61-63) Let numbers in set A be $a - D, a, a + D$ and in set B be $b - d, b, b + d$.

$$3a = 3b = 15$$

$$\Rightarrow a = b = 5$$

Now, set $A = \{5 - D, 5, 5 + D\}$

and set $B = \{5 - d, 5, 5 + d\}$

where, $D = d + 1$

$$\therefore \frac{p}{q} = \frac{5(25 - D^2)}{5(25 - d^2)} = \frac{7}{8}$$

$$\Rightarrow 25(8 - 7) = 8(d + 1)^2 - 7d^2$$

$$\Rightarrow d = -17, 1 \text{ but } d > 0$$

$$\Rightarrow d = 1$$

So, numbers in set A are $\{3, 5, 7\}$ and in set B are $\{4, 5, 6\}$.

Now, $p = 3 \times 5 \times 7 = 105$

and $q = 4 \times 5 \times 6 = 120$

Hence, value of $D + d = 3$

61. (c)

62. (b)

63. (c)

Passage IV (Ex. Nos. 64-66) Four different integers form an increasing AP. One of these numbers is equal to the sum of the squares of the other three numbers.

Ex 64. The smallest number is

- (a) -2 (b) 0 (c) -1 (d) 2

Ex 65. The common difference of the four numbers is

- (a) 2 (b) 1
(c) 3 (d) 4

Ex 66. The sum of all the four numbers is

- (a) 10 (b) 8
(c) 2 (d) 6

Sol. (Ex. Nos. 64-66) Let four integers be $a - d, a, a + d$ and $a + 2d$, where a and d are integers and $d > 0$.

$$\therefore a + 2d = (a - d)^2 + a^2 + (a + d)^2$$

$$\Rightarrow 2d^2 - 2d + 3a^2 - a = 0$$

$$d = \frac{1}{2} [1 \pm \sqrt{1 + 2a - 6a^2}] \dots (i)$$

Since, d is positive integer.

$$\therefore 1 + 2a - 6a^2 > 0$$

$$\Rightarrow 6a^2 - 2a - 1 < 0$$

$$\Rightarrow \frac{1 - \sqrt{7}}{6} < a < \frac{1 + \sqrt{7}}{6}$$

$\therefore a$ is an integer.

$\therefore a = 0$, put in Eq. (i),

$$d = 1 \text{ or } 0 \text{ but } d > 0$$

$$\therefore d = 1$$

Now, the four numbers are -1, 0, 1, 2 and their sum is 2.

64. (c)

65. (b)

66. (c)

Type 5. Match the Columns

Ex 67. Match the statements of Column I with values of Column II.

Column I	Column II
A. Suppose that $F(n+1) = \frac{2F(n)+1}{2}$ for $n = 1, 2, 3, \dots$ and $F(1) = 2$. Then, $F(101)$ equals	p. 42
B. If $a_1, a_2, a_3, \dots, a_{21}$ are in AP and $a_3 + a_5 + a_{11} + a_{17} + a_{19} = 10$, then the value of $\sum_{i=1}^{21} a_i$ is	q. 1620
C. 10th term of the series $S = 1 + 5 + 13 + 29 + \dots$ is	r. 52
D. The sum of all two digit numbers which are not divisible by 2 or 3, is	s. 2045

Sol. A. $F(n+1) = \frac{2F(n)+1}{2} = F(n) + \frac{1}{2}$

$\therefore F(1), F(2), F(3), \dots$ is an AP with common difference $\frac{1}{2}$. Thus, $F(101) = 2 + 100 \times \frac{1}{2} = 52$

B. $a_1 + 2d + a_1 + 4d + a_1 + 10d + a_1 + 16d + a_1 + 18d = 5a_1 + 50d = 5(a_1 + 10d) = 10$ i.e. $a_1 + 10d = 2$
Now, $\sum_{i=1}^{21} a_i = \frac{21}{2} [2a_1 + 20d] = 21(a_1 + 10d) = 42$

C. $S = 1 + 5 + 13 + 29 + \dots + t_{10} \dots (i)$

$S = 1 + 5 + 13 + \dots + t_9 + t_{10} \dots (ii)$

On subtracting Eq. (ii) from Eq. (i), we get

$t_{10} = 1 + 4 + 8 + 16 + \dots$ upto 10 terms
 $= 1 + (4 + 8 + 16 + \dots$ upto 9 terms) $= 2045$

D. Sum of all two digit numbers $= \frac{90}{2} (10 + 99) = (45) (109)$

Sum of all two digit numbers divisible by 2

$= \frac{45}{2} (10 + 98) = (45) (54)$

Sum of all two digit numbers divisible by 3

$= \frac{30}{2} (12 + 99) = 15(111)$

Sum of all two digit numbers divisible by 6

$= \frac{15}{2} (12 + 96) = 15(54)$

$\therefore$ Required sum

$= 45(109) + 15(54) - (45)(54) - 15(111) = 1620$

A $\rightarrow$ r; B $\rightarrow$ p; C $\rightarrow$ s; D $\rightarrow$ q

Ex 68. Match the statements of Column I with values of Column II.

Column I	Column II
A. The arithmetic mean of two positive numbers is 6 and their geometric mean G and harmonic mean H satisfy $G^2 + 3H = 48$, then the product of number is	p. $\frac{240}{77}$
B. The sum of the series $\frac{5}{1^2 \cdot 4^2} + \frac{11}{4^2 \cdot 7^2} + \frac{17}{7^2 \cdot 10^2} + \dots$ is	q. 32
C. If the first two terms of a harmonic progression are $\frac{1}{2}$ and $\frac{1}{3}$, then the harmonic mean of the first four terms is	r. $\frac{1}{3}$

Sol. A. $a + b = 12$

$\therefore ab + \frac{6ab}{a+b} = 48$

$\Rightarrow ab + \frac{ab}{2} = 48$

$\therefore ab = 32$

B. $S = \frac{5}{1^2 \cdot 4^2} + \frac{11}{4^2 \cdot 7^2} + \frac{17}{7^2 \cdot 10^2} + \dots$

$\Rightarrow 3S = \frac{3 \cdot 5}{1^2 \cdot 4^2} + \frac{3 \cdot 11}{4^2 \cdot 7^2} + \frac{3 \cdot 17}{7^2 \cdot 10^2} + \dots$

$$\begin{aligned}\Rightarrow 3S &= \frac{(4-1) \cdot (4+1)}{1^2 \cdot 4^2} + \frac{(7-4)(7+4)}{4^2 \cdot 7^2} \\ &\quad + \frac{(10-7)(10+7)}{7^2 \cdot 10^2} + \dots \\ \Rightarrow 3S &= \frac{4^2-1^2}{1^2 \cdot 4^2} + \frac{7^2-4^2}{4^2 \cdot 7^2} + \frac{10^2-7^2}{7^2 \cdot 10^2} + \dots \\ \Rightarrow 3S &= 1 - \frac{1}{4^2} + \frac{1}{4^2} - \frac{1}{7^2} + \frac{1}{7^2} - \frac{1}{10^2} + \dots\end{aligned}$$

$$\begin{aligned}\Rightarrow 3S &= 1 \\ \Rightarrow S &= \frac{1}{3}\end{aligned}$$

$$\begin{aligned}\text{C. HM of } \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \\ &= \frac{4}{\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5}} = \frac{240}{77}\end{aligned}$$

$$A \rightarrow q; B \rightarrow r; C \rightarrow p$$

Type 6. Single Integer Answer Type Questions

Ex 69. The largest term of the sequence $\frac{1}{503}, \frac{4}{524}, \frac{9}{581}, \frac{16}{692}, \dots$ is T_p . Then, the value of p is _____.

$$\begin{aligned}\text{Sol. (7)} \quad T_n &= \frac{n^2}{500 + 3n^3} \\ \Rightarrow \frac{dT_n}{dn} &= \frac{(500 + 3n^3) \cdot 2n - n^2 \cdot 9n^2}{(500 + 3n^3)^2} \\ &= \frac{n(1000 - 3n^3)}{(500 + 3n^3)^2}\end{aligned}$$

For maximum of T_n , put

$$\frac{dT_n}{dn} = 0$$

$$\Rightarrow n = \left(\frac{1000}{3}\right)^{1/3}$$

$$\text{Now, } 6 < \left(\frac{1000}{3}\right)^{1/3} < 7$$

Hence, T_7 is the largest term, so largest term in the given sequence is $\frac{49}{1529}$.

$$\begin{aligned}\text{Ex 70. For } x > 1, \text{ evaluate } &\frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} \\ &+ \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \dots\end{aligned}$$

$$\begin{aligned}\text{Sol. (1) Let } S &= \frac{x}{x+1} + \frac{x^2}{(x+1)(x^2+1)} \\ &\quad + \frac{x^4}{(x+1)(x^2+1)(x^4+1)} + \dots\end{aligned}$$

Now, we can write

$$\begin{aligned}\frac{x}{x+1} &= 1 - \frac{1}{x+1} \\ \frac{x^2}{(x+1)(x^2+1)} &= \frac{1}{x+1} - \frac{1}{(x+1)(x^2+1)}\end{aligned}$$

$$\begin{aligned}\frac{x^4}{(x+1)(x^2+1)(x^4+1)} &= \frac{1}{(x+1)(x^2+1)} \\ &\quad - \frac{1}{(x+1)(x^2+1)(x^4+1)} \\ &\vdots \\ \frac{x^{2^{n-1}}}{(x+1)(x^2+1) \dots (x^{2^{n-1}}+1)} &= \frac{1}{(x+1)(x^2+1) \dots (x^{2^{n-2}}+1)} \\ &\quad - \frac{1}{(x+1)(x^2+1) \dots (x^{2^{n-1}}+1)} \\ S_n &= 1 - \frac{1}{(x+1)(x^2+1) \dots (x^{2^{n-1}}+1)} \\ S_\infty &= 1\end{aligned}$$

Ex 71. If $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots$ and $\frac{f(x)}{1-x} = b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots$ where, $a_0 = 1$, $b_1 = 3$ and $b_{10} = k^{11} - 1$, then k is _____.

Given that $a_0, a_1, a_2, \dots$ are in GP.

$$\begin{aligned}\text{Sol. (2)} \quad \because f(x) &= a_0 + a_1x + a_2x^2 + \dots + a_nx^n + \dots \\ \text{and } f(x)(1-x)^{-1} &= b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots, \\ \therefore f(x) \{1+x+x^2+\dots+x^n+\dots\} \\ &= b_0 + b_1x + b_2x^2 + \dots + b_nx^n + \dots\end{aligned}$$

On comparing the coefficient of x^n both sides, we get

$$\begin{aligned}a_0 + a_1 + a_2 + \dots + a_{n-1} + a_n &= b_n \\ b_0 &= a_0 \quad \text{and} \quad b_1 = a_0 + a_1 \\ \Rightarrow b_0 &= 1 \quad \text{and} \quad 3 = a_0 + a_1 \Rightarrow a_1 = 2\end{aligned}$$

Since, $a_0, a_1, a_2, \dots$ are in GP.

So, first term is 1 and common ratio is 2.

$$\begin{aligned}\text{Now, } b_{10} &= a_0 + a_1 + a_2 + \dots + a_{10} \\ &= 1 + 2 + 2^2 + \dots + 2^{10} = 2^{11} - 1\end{aligned}$$

Target Exercises

Type 1. Only One Correct Option

- The next term of the sequence 1, 5, 14, 30, 55, ... is
(a) 91 (b) 85 (c) 90 (d) 95
- The next term of the sequence 1, 3, 6, 10, ... is
(a) 16 (b) 13 (c) 15 (d) 14
- In a certain AP, 5 times the 5th term is equal to 8 times the 8th term, then 13th term is
(a) -13 (b) -12 (c) -1 (d) 0
- If $a_1 = a_2 = 2$, $a_n = a_{n-1} - 1$ ($n > 2$), then a_5 is
(a) 0 (b) -2 (c) -1 (d) 1
- The second term of an AP is $(x - y)$ and the fifth term is $(x + y)$, then its first term is
(a) $x - \frac{1}{3}y$ (b) $x - \frac{2}{3}y$ (c) $x - \frac{4}{3}y$ (d) $x - \frac{5}{3}y$
- If a, b and c are the sides of a $\triangle ABC$ are in AP, then $\cot \frac{C}{2}$ is equal to
(a) $3 \tan \frac{A}{2}$ (b) $3 \tan \frac{B}{2}$
(c) $3 \cot \frac{A}{2}$ (d) $3 \cot \frac{B}{2}$
- If a, b, c, d, e and f are in AP, then $e - c$ is equal to
(a) $2(c - a)$ (b) $2(d - c)$
(c) $2(f - d)$ (d) $d - c$
- If $\cos x = b$, then for what b do the roots of the equation form an AP?
(a) -1 (b) $\frac{1}{2}$
(c) $\frac{\sqrt{3}}{2}$ (d) None of these
- If $\log_3 2, \log_3 (2^x - 5)$ and $\log_3 (2^x - 7/2)$ are in AP, then x is equal to
(a) 2 (b) 3 (c) 4 (d) 2, 3
- If the p th term of an AP is q and the q th term is p , then the r th term is
(a) $q - p + r$ (b) $p - q + r$
(c) $p + q + r$ (d) $p + q - r$
- The consecutive terms $\frac{1}{1+\sqrt{x}}, \frac{1}{1-x}, \frac{1}{1-\sqrt{x}}$ of a series are in
(a) HP (b) GP
(c) AP (d) AGP
- If $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in AP, then $\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) \left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)$ is equal to
(a) $\frac{4}{ac} - \frac{3}{b^2}$ (b) $\frac{b^2 - ac}{a^2 b^2 c^2}$
(c) $\frac{4}{ac} - \frac{1}{b^2}$ (d) None of these
- If 7th and 13th terms of an AP are 34 and 64 respectively, then its 18th term is
(a) 87 (b) 88 (c) 89 (d) 90
- If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in AP, then the common difference will be
(a) ± 1 (b) ± 2 (c) ± 3 (d) ± 4
- Given that a, b and c are in AP. The determinant $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$ in its simplest form is equal to
(a) $x^3 + 3ax + 7c$ (b) 0
(c) 15 (d) $10x^2 + 5x + 2c$
- If $a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{c} + \frac{1}{a}\right)$ and $c\left(\frac{1}{a} + \frac{1}{b}\right)$ are in AP, then
(a) $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in AP (b) a, b and c are in HP
(c) $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in GP (d) a, b and c are in AP
- If a, b and c are in AP, then $(a + 2b - c)(2b + c - a)(c + a - b)$ equals
(a) $\frac{abc}{2}$ (b) abc (c) $2abc$ (d) $4abc$
- If a, b and c are in arithmetic progression, then $\frac{1}{\sqrt{a} + \sqrt{b}}, \frac{1}{\sqrt{a} + \sqrt{c}}$ and $\frac{1}{\sqrt{b} + \sqrt{c}}$ are in
(a) AP (b) GP
(c) HP (d) None of these
- If $\frac{1}{p+q}, \frac{1}{q+r}$ and $\frac{1}{r+p}$ are in AP, then
(a) p, q and r are in AP
(b) q^2, p^2 and r^2 are in AP
(c) p^2, q^2 and r^2 are in AP
(d) p, q and r are in GP

- 20.** The number of numbers lying between 100 and 500 that are divisible by 7 but not by 21, is
 (a) 57 (b) 19
 (c) 38 (d) None of these
- 21.** If the first, second and last terms of an AP are a, b, c respectively, then the sum is
 (a) $\frac{(a+b)(a+c-2b)}{2(b-a)}$ (b) $\frac{(b+c)(a+b-2c)}{2(b-a)}$
 (c) $\frac{(a+c)(b+c-2a)}{2(b-a)}$ (d) None of these
- 22.** The maximum sum of the series
 $20 + 19\frac{1}{3} + 18\frac{2}{3} + 18 + \dots$ is
 (a) 310 (b) 290
 (c) 320 (d) None of these
- 23.** The sum of all natural numbers less than 200 that are divisible neither by 3 nor by 5, is
 (a) 10730 (b) 10732
 (c) 15375 (d) None of these
- 24.** The four numbers are in AP, such that their sum is 50 and greatest of them is 4 times the least, are
 (a) 7, 11, 15, 19 (b) 6, 11, 16, 21
 (c) 5, 10, 15, 20 (d) None of these
- 25.** If the sum of four integers in AP is 24 and their product is 945, then the numbers are
 (a) 3, 5, 7, 9 (b) 5, 8, 11, 14
 (c) 4, 8, 12, 16 (d) None of these
- 26.** If the sum of any number of terms in a sequence is always three times the squared number of these terms, then the sequence is
 (a) an AP (b) a GP
 (c) an HP (d) None of these
- 27.** The sum of all two digit numbers which when divided by 4, yield unity as remainder, is
 (a) 1100 (b) 1200
 (c) 1210 (d) None of these
- 28.** The sum of numbers of three digits which are divisible by 7, is
 (a) 70336 (b) 70331
 (c) 70330 (d) None of these
- 29.** The sum of all odd numbers of four digits which are divisible by 9, is
 (a) 2754000 (b) 2753000
 (c) 2752000 (d) None of these
- 30.** The sum of positive terms of the series
 $10 + 9\frac{4}{7} + 9\frac{1}{7} + \dots$ is
 (a) $\frac{352}{7}$ (b) $\frac{437}{7}$
 (c) $\frac{852}{7}$ (d) None of these
- 31.** The maximum sum of an AP 40, 38, 36, 34, ... is
 (a) 390 (b) 420
 (c) 460 (d) None of these
- 32.** A man arranges to pay off a debt of ₹ 3600 by 40 annual instalments which are in AP. When 30 of the instalments are paid, he dies leaving one-third of the debt unpaid. The value of 8th instalment is
 (a) ₹ 35 (b) ₹ 50
 (c) ₹ 65 (d) None of these
- 33.** A polygon has 25 sides, the lengths of which starting from the smallest side are in AP. If the perimeter of the polygon is 2100 cm and the length of the largest side 20 times that of the smallest, then the length of the smallest side and the common difference of an AP are
 (a) 8 cm, $6\frac{1}{3}$ cm (b) 6 cm, $6\frac{1}{3}$ cm
 (c) 8 cm, $5\frac{1}{3}$ cm (d) None of these
- 34.** A club consists of members whose ages are in AP, the common difference being 3 months. If the youngest member of the club is just 7 yr old and the sum of the ages of all the members is 250 yr, then the number of members in the club is
 (a) 15 (b) 25 (c) 20 (d) 30
- 35.** The sum of the series 2, 5, 8, 11, ... is 60100, then n is equal to
 (a) 100 (b) 200
 (c) 150 (d) 250
- 36.** If the sum of n terms of an AP is $3n^2 + 5n$, then which of its terms is 164?
 (a) 26th (b) 27th
 (c) 28th (d) None of these
- 37.** The first and last terms of an AP are 1 and 11. If the sum of its terms is 36, then the number of terms will be
 (a) 5 (b) 6 (c) 7 (d) 8
- 38.** Sum of n terms of the series
 $\sqrt{2} + \sqrt{8} + \sqrt{18} + \sqrt{32} + \dots$ is
 (a) $\frac{n(n+1)}{2}$ (b) $2n(n+1)$
 (c) $\frac{n(n+1)}{\sqrt{2}}$ (d) 1
- 39.** The sum of n terms of the series
 $\frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \dots$ is
 (a) $\frac{\sqrt{2n+1}}{2}$ (b) $\frac{-1 + \sqrt{2n+1}}{2}$
 (c) $\frac{1}{\sqrt{2n+1}}$ (d) $\sqrt{2n-1}$

40. The sum of all two digit odd numbers is
(a) 2475 (b) 2530 (c) 4905 (d) 5049
41. If the sum of p terms of an AP is q and that of q terms is p , then the sum of $p + q$ terms will be
(a) 0 (b) $p - q$
(c) $p + q$ (d) $-(p + q)$
42. If the sum of n terms of an AP is $3n^2 - n$ and its common difference is 6, then its first term is
(a) 2 (b) 3 (c) 1 (d) 4
43. If $S_n = nP + \frac{n}{2}(n-1)Q$, where S_n denotes the sum of first n terms of an AP, then the common difference is
(a) $P + Q$ (b) $2P + 3Q$
(c) $2Q$ (d) Q
44. If the sum of first n terms of a series is $5n^2 + 2n$, then its second term is
(a) 16 (b) 17 (c) $\frac{27}{14}$ (d) $\frac{56}{15}$
45. If the n th term of a series is $\left(\frac{n}{x}\right) + y$, then sum of r terms will be
(a) $\left\{\frac{r(r+1)}{2x}\right\} + ry$ (b) $\left\{\frac{r(r-1)}{2x}\right\}$
(c) $\left\{\frac{r(r-1)}{2x}\right\} - xy$ (d) $\left\{\frac{r(r+1)}{2x}\right\} - rx$
46. If S_n denotes the sum of n terms of an AP, then $S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$ is equal to
(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) 2
47. If $\frac{a + be^y}{a - be^y} = \frac{b + ce^y}{b - ce^y} = \frac{c + de^y}{c - de^y}$, then a, b, c and d are in
(a) AP (b) GP
(c) HP (d) None of these
48. In a GP, if $(m + n)$ th term is p and $(m - n)$ th term is q , then its m th term is
(a) 0 (b) pq (c) $\sqrt{pq}$ (d) $\frac{1}{2}(p + q)$
49. If $\frac{1}{2} \operatorname{cosec}^2 \theta, 2 \cot \theta, \sec \theta, 0 < \theta < \frac{\pi}{2}$ are in GP, then θ is equal to
(a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) None of these
50. If 10th term of a GP is 9 and 4th term is 4, then its 7th term is
(a) 6 (b) 14 (c) $\frac{27}{14}$ (d) $\frac{56}{15}$
51. If 4th, 7th and 10th terms of a GP are p, q and r respectively, then
(a) $p^2 = q^2 + r^2$ (b) $p^2 = qr$
(c) $q^2 = pr$ (d) $r^2 = p^2 + q^2$
52. If 6th term of a GP is 32 and its 8th term is 128, then the value of the common ratio is
(a) -1 (b) 2
(c) 4 (d) -4
53. If x_1, x_2, x_3 as well as y_1, y_2, y_3 are in GP with same common ratio, then the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3)
(a) lie on a straight line
(b) lie on an ellipse
(c) lie on a circle
(d) are vertices of a triangle
54. If $a^x = b^y = c^z$ and a, b, c are in GP, then x, y and z are in
(a) AP (b) GP
(c) HP (d) None of these
55. If p, q, r are in AP and x, y, z are in GP, then $x^{q-r} \cdot y^{r-p} \cdot z^{p-q}$ is equal to
(a) 1 (b) 2
(c) -1 (d) None of these
56. The sum of three numbers in GP is 56. If we subtract 1, 7, 21 from these numbers in that order, we obtain an AP. Then, three numbers are
(a) 8, 16, 32 (b) 10, 18, 26
(c) 9, 16, 23 (d) None of these
57. If the sides of a right angled triangle are in GP, then the cosine of the greater acute angle is
(a) $\frac{2}{1 + \sqrt{5}}$ (b) $\frac{1}{1 - \sqrt{5}}$
(c) $\frac{1 + \sqrt{5}}{2}$ (d) None of these
58. The consecutive numbers of a three digit number form a GP. If we subtract 792 from this number, then we get a number consisting of the same digits written in the reverse order and if we increase the second digit of the required number by 2, then resulting number forms an AP. The number is
(a) 139 (b) 193
(c) 931 (d) None of these
59. Three non-zero numbers a, b and c are in AP. If increasing a by 1 or increasing c by 2, the numbers are in GP, then b equals
(a) 10 (b) 14
(c) 12 (d) 16
60. The sum of the series $0.4 + 0.004 + 0.00004 + \dots$ is
(a) $\frac{11}{25}$ (b) $\frac{41}{100}$ (c) $\frac{40}{99}$ (d) $\frac{2}{5}$

61. If $x > 0$, then the sum of the series $e^{-x} - e^{-2x} + e^{-3x} - \dots$ is
- (a) $\frac{1}{1 - e^{-x}}$ (b) $\frac{1}{e^x - 1}$
 (c) $\frac{1}{1 + e^{-x}}$ (d) $\frac{1}{1 + e^x}$
62. The n th term of a GP is 128 and the sum to its n terms is 255. If its common ratio is 2, then its first term is
- (a) 1 (b) 3
 (c) 8 (d) None of these
63. The sum of $(x+2)^{n-1} + (x+2)^{n-2} + \dots + (x+1)^{n-1}$ is equal to
- (a) $(x+2)^n - (x+1)^n$
 (b) $(x+2)^{n-1} - (x+1)^{n-1}$
 (c) $(x+2)^n - (x+1)^n$
 (d) None of the above
64. If $(a, b), (c, d), (e, f)$ are the vertices of a triangle such that a, c, e are in GP with common ratio r and b, d, f are in GP with common ratio s , then the area of the triangle is
- (a) $\frac{ab}{2} (r+1)(s+2)(s+r)$
 (b) $\frac{ab}{2} (r-1)(s-2)(s-r)$
 (c) $\frac{ab}{2} (r-1)(s-1)(s-r)$
 (d) $\frac{ab}{2} (r+1)(s+1)(s-r)$
65. If sum of all terms of an infinite GP is $\frac{1}{5}$ times the sum of odd terms. Then, common ratio is
- (a) 2 (b) 3
 (c) $-\frac{4}{5}$ (d) 5
66. The sum of 10 terms of the series $\sqrt{2} + \sqrt{6} + \sqrt{18} + \dots$ is
- (a) $121(\sqrt{6} + \sqrt{2})$
 (b) $243(\sqrt{3} + 1)$
 (c) $\frac{121}{\sqrt{3} - 1}$
 (d) $242(\sqrt{3} - 1)$
67. If $(1.05)^{50} = 11.658$, then $\sum_{n=1}^{49} (1.05)^n$ equals
- (a) 208.34 (b) 212.12
 (c) 212.16 (d) 213.16
68. If the second term of a GP is 2 and the sum of its infinite terms is 8, then its first term is
- (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) 2 (d) 4
69. Let a_n be the n th term of a GP of positive numbers. If $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$, such that $\alpha \neq \beta$, then the common ratio is
- (a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$
 (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{\beta}{\alpha}}$
70. If $f(x)$ is a function satisfying $f(x+y) = f(x)f(y)$, $\forall x, y \in \mathbb{N}$ such that $f(1) = 3$ and $\sum_{x=1}^n f(x) = 120$. Then, the value of n is
- (a) 4 (b) 5
 (c) 6 (d) None of these
71. The value of $0.42\overline{3}$ is
- (a) $\frac{419}{999}$ (b) $\frac{419}{990}$
 (c) $\frac{423}{1000}$ (d) None of these
72. If $x = 1 + y + y^2 + \dots$, then y is
- (a) $\frac{x}{x-1}$ (b) $\frac{x}{1-x}$ (c) $\frac{x-1}{x}$ (d) $\frac{1-x}{x}$
73. The sum of first four terms of a GP is $12(1 - \sqrt{5})$. If the common ratio is $-\sqrt{5}$, then the first term of a GP is
- (a) 1 (b) 2
 (c) 3 (d) 4
74. The length of a side of a square is a metre. A second square is formed by joining the mid-points of these square. Then, a third square is formed by joining the mid-points of the second square and so on. Then, the sum of the area of the squares which carried upto infinity, is
- (a) a^2 (b) $2a^2$ (c) $3a^2$ (d) $4a^2$
75. The sum of an infinite geometric series is 3. A series which is formed by squares of its terms also have sum equal to 3. Then, the sequence corresponding to the series will be
- (a) $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \dots$ (b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$
 (c) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \dots$ (d) $1, -\frac{1}{3}, \frac{1}{9}, -\frac{1}{27}, \dots$
76. The product of $x^{1/2} \cdot x^{1/4} \cdot x^{1/8} \dots$ equals
- (a) 0 (b) 1 (c) x (d) ∞
77. Let $S_1, S_2, \dots$ be squares such that for each $n \geq 1$, the length of a side of S_n equals the length of the diagonal of S_{n+1} . If the length of a side of S_1 is 10 cm and the area of S_n less than 1 sq cm, then the least value of n is
- (a) 7 (b) 8 (c) 9 (d) 10

- 78.** Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in GP, then integral values of p and q respectively are
 (a) $-2, -32$ (b) $-2, 3$
 (c) $-6, 3$ (d) $-6, -32$
- 79.** Sum of the series $(1+x) + (1+x+x^2) + (1+x+x^2+x^3) + \dots$ n terms is
 (a) $\frac{1}{1-x} \left[n - \frac{x^2(1-x^n)}{1-x} \right]$ (b) $\frac{1}{1-x} \left[n - \frac{x^3(1-x^n)}{1-x} \right]$
 (c) $\frac{1}{1-x} \left[n - \frac{x(1-x^n)}{1-x} \right]$ (d) None of these
- 80.** Sum of the series $9 + 99 + 999 + \dots$ upto n terms is
 (a) $\frac{1}{9} (10^n - 9n - 10)$
 (b) $\frac{1}{9} (10^{n+1} - 9n - 10)$
 (c) $\frac{1}{9} (10^{n+1} + 9n - 10)$
 (d) None of the above
- 81.** Sum of the series $0.5 + 0.55 + 0.555 + \dots$ upto n terms is
 (a) $\frac{5}{9} \left[n - \frac{2}{9} \left(1 - \frac{1}{10^n} \right) \right]$ (b) $\frac{3}{9} \left[3 - \frac{1}{9} \left(1 - \frac{1}{10^n} \right) \right]$
 (c) $\frac{7}{9} \left[n - \frac{2}{9} \left(1 - \frac{1}{10^n} \right) \right]$ (d) $\frac{5}{9} \left[n - \frac{1}{9} \left(1 - \frac{1}{10^n} \right) \right]$
- 82.** A number consists of three digits in GP. If the sum of extreme digits is greater than twice middle digit by 1 and the sum of the left hand and middle digit is two-third of the sum of the middle and right hand digits, then the number is
 (a) 4, 6, 9 (b) 3, 7, 6
 (c) 4, 6, 8 (d) None of these
- 83.** If one geometric mean G and two arithmetic means p and q are inserted between two numbers, then G^2 is equal to
 (a) $(3p - q)(3q - p)$ (b) $(2p - q)(2q - p)$
 (c) $(4p - q)(4q - p)$ (d) None of these
- 84.** If in a GP of $3n$ terms, S_1 denotes the sum of the first n terms, S_2 denotes the sum of the second block of n terms and S_3 denotes the sum of the last n terms, then S_1, S_2 and S_3 are in
 (a) AP (b) GP
 (c) HP (d) None of these
- 85.** The product of $(32) \cdot (32)^{1/6} \cdot (32)^{1/36} \dots$ is equal to
 (a) 16 (b) 64 (c) 32 (d) 0
- 86.** The sum of the series $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \dots$ is
 (a) $\frac{16}{25}$ (b) $\frac{11}{8}$ (c) $\frac{35}{16}$ (d) $\frac{8}{11}$
- 87.** The sum of the series $1 + 2x + 3x^2 + 4x^3 + \dots$ (where, x lies between 0 and 1, i.e. $0 < x < 1$) is
 (a) $\frac{1}{1+x}$ (b) $\frac{1}{1-x}$ (c) $\frac{1}{1-2x}$ (d) $\frac{1}{(1-x)^2}$
- 88.** The sum of the series $1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + 100 \cdot 2^{99}$ is
 (a) $100 \cdot 2^{100} + 1$ (b) $99 \cdot 2^{100} + 1$
 (c) $99 \cdot 2^{99} - 1$ (d) $100 \cdot 2^{100} - 1$
- 89.** The sum of the series $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots$ upto 16 terms is
 (a) 246 (b) 346 (c) 446 (d) 546
- 90.** The sum of $\frac{1 \cdot 2}{1^3} + \frac{2 \cdot 3}{1^3 + 2^3} + \frac{3 \cdot 4}{1^3 + 2^3 + 3^3} + \dots$ upto n term is
 (a) $\frac{n-1}{2}$ (b) $\frac{n}{n+1}$ (c) $\frac{n+1}{n+2}$ (d) $\frac{n+1}{n}$
- 91.** The sum of the series $S = 1^2 - 2^2 + 3^2 - 4^2 + \dots - 2002^2 + 2003^2$ is
 (a) 2007006 (b) 1005004
 (c) 2000506 (d) None of these
- 92.** The value of n for which $704 + \frac{1}{2}(704) + \frac{1}{4}(704) + \dots$ n terms $= 1984 - \frac{1}{2}(1984) + \frac{1}{4}(1984) - \dots$ upto terms, is
 (a) 5 (b) 3 (c) 4 (d) 10
- 93.** The positive integer n for which $2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + n \cdot 2^n = 2^{n+10}$, is
 (a) 510 (b) 511 (c) 512 (d) 513
- 94.** If $1^2 + 2^2 + 3^2 + \dots + 2003^2 = (2003)(4007)(334)$ and $(1)(2003) + (2)(2002) + (3)(2001) + \dots + (2003)(1) = (2003)(334)(x)$, then x equals
 (a) 2005 (b) 2004
 (c) 2003 (d) 2001
- 95.** If $x > 0$ and $\log_2 x + \log_2(\sqrt{x}) + \log_2(\sqrt[4]{x}) + \log_2(\sqrt[8]{x}) + \log_2(\sqrt[16]{x}) + \dots = 4$, then x equals
 (a) 2 (b) 3 (c) 4 (d) 5
- 96.** If $x_1, x_2, \dots, x_n$ are n non-zero real numbers such that $(x_1^2 + x_2^2 + \dots + x_{n-1}^2)(x_2^2 + x_3^2 + \dots + x_n^2) \leq (x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n)^2$, then $x_1, x_2, \dots, x_n$ are in
 (a) AP (b) GP
 (c) HP (d) None of these

97. If $\sum_{r=1}^n r^4 = f(n)$, then $\sum_{r=1}^n (2r-1)^4$ is equal to

- (a) $f(2n) - 16f(n)$ (b) $f(2n) - 7f(n)$
(c) $f(2n-1) - 8f(n)$ (d) None of these

98. The value of $\lim_{n \rightarrow \infty} \sum_{r=1}^n \tan^{-1} \left(\frac{1}{2r^2} \right)$ is

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) 1 (d) None of these

99. If $1^3 = 1, 2^3 = 3 + 5, 3^3 = 7 + 9 + 11$

and $4^3 = 13 + 15 + 17 + 19$,

then n^3 as a similar series is

- (a) $\left[2 \left\{ \frac{n(n-1)}{2} + 1 \right\} - 1 \right] + \left[2 \left\{ \frac{(n+1)n}{2} + 1 \right\} + 1 \right]$
 $+ \dots + \left[2 \left\{ \frac{(n+1)n}{2} + 1 \right\} + 2n - 3 \right]$
 (b) $(n^2 + n + 1) + (n^2 + n + 3) + (n^2 + n + 5)$
 $+ \dots + (n^2 + 3n - 1)$
 (c) $(n^2 - n + 1) + (n^2 - n + 3) + (n^2 - n + 5)$
 $+ \dots + (n^2 + n - 1)$
 (d) None of the above

100. $\sum_{r=1}^n \frac{r^2 - r - 1}{(r+1)!}$ is equal to

- (a) $\frac{n}{(n+1)!}$
 (b) $\frac{-1}{(n+1)(n-1)!}$
 (c) $\frac{n}{(n+1)!} - 1$
 (d) None of the above

101. Solution set for $(\sqrt{2+\sqrt{2}})^x + (\sqrt{2-\sqrt{2}})^x = 2 \cdot 2^{x/4}$ is

- (a) $\{2\}$
 (b) $\{0\}$
 (c) $[0, 2]$
 (d) None of the above

102. If $x^2 - (a+b+c)x + (ab+bc+ca) = 0$ has

imaginary roots, where $a, b, c \in R^+$, then $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$

- (a) can be sides of a triangle
 (b) can't be sides of a triangle
 (c) nothing can be said
 (d) None of the above

103. If first three terms of sequence $\frac{1}{16}, a, b, \frac{1}{6}$ are in

geometric series and the last three terms are in harmonic series, then the values of a and b will be

- (a) $a = -\frac{1}{4}, b = 1$ (b) $a = \frac{1}{12}, b = \frac{1}{9}$
 (c) Both (a) and (b) are true (d) None of these

104. If the roots of the equation

$a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, then a, b and c are in

- (a) AP (b) GP
 (c) HP (d) None of these

105. If AM between two numbers is 5 and their GM is 4, then the HM will be

- (a) $\frac{16}{5}$ (b) $\frac{14}{5}$
 (c) $\frac{11}{5}$ (d) None of these

106. If $\log(a+c), \log(c-a)$ and $\log(a-2b+c)$ are in AP, then

- (a) a, b and c are in AP (b) a^2, b^2 and c^2 are in AP
 (c) a, b and c are in GP (d) a, b and c are in HP

107. If $\cos(x-y), \cos x$ and $\cos(x+y)$ are in HP, then $\cos x \sec \left(\frac{y}{2} \right)$ is equal to

- (a) $\pm \sqrt{2}$ (b) $\pm \frac{1}{\sqrt{2}}$
 (c) ± 2 (d) None of these

108. $\log_3 2, \log_6 2$ and $\log_{12} 2$ are in

- (a) AP (b) GP
 (c) HP (d) None of these

109. If a, b and c are in HP, then correct statement is

- (a) $a^2 + c^2 > b^2$ (b) $a^2 + c^2 > 2b^2$
 (c) $a^2 + c^2 < 2b^2$ (d) $a^2 + c^2 = 2b^2$

110. The AM, GM and HM of two numbers are x, y and z , respectively. Then, which of the following is true?

- (a) $z < y < x$ (b) $y < x < z$ (c) $x < y < z$ (d) $z < x < y$

111. If a, b and c are in HP, then the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through a fixed point and that point is

- (a) $(-1, -2)$ (b) $(-1, 2)$ (c) $(1, -2)$ (d) $\left(1, -\frac{1}{2}\right)$

112. Let $a_1, a_2, \dots, a_{10}$ be in AP and $h_1, h_2, \dots, h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, then $a_4 h_7$ is

- (a) 2 (b) 3 (c) 5 (d) 6

113. If x, y and z are in HP, where $z > y > x$. Then, value of $\log(x+z) + \log(x-2y+z)$ is

- (a) $2 \log(y-z)$ (b) $2 \log(z-x)$
 (c) $4 \log(x-z)$ (d) $\log(x-z)$

114. If $H_1, H_2, H_3, \dots, H_{2n+1}$ are in HP, then

$\sum_{i=1}^{2n} (-1)^i \left(\frac{H_i + H_{i+1}}{H_i - H_{i+1}} \right)$ is equal to

- (a) $2n-1$ (b) $2n+1$ (c) $2n$ (d) $2n+2$

Type 2. More than One Correct Option

115. For the AP given by $a_1, a_2, \dots, a_n, \dots$, the equation satisfied is

- (a) $a_1 + 2a_2 + a_3 = 0$
 (b) $a_1 - 2a_2 + a_3 = 0$
 (c) $a_1 + 3a_2 - 3a_3 - a_4 = 0$
 (d) $a_1 - 4a_2 + 6a_3 - 4a_4 + a_5 = 0$

116. If sum of the first three consecutive terms of an AP is 9 and the sum of their squares is 35. Then, sum to n terms of the series is

- (a) $n(n+1)$ (b) n^2
 (c) $n(4-n)$ (d) $n(6-n)$

117. If $a_1, a_2, \dots, a_n$ are in AP with common difference d ,

$$\text{then } \cot^{-1} \left[\frac{1+a_1a_2}{d} \right] + \cot^{-1} \left[\frac{1+a_2a_3}{d} \right] + \cot^{-1} \left[\frac{1+a_3a_4}{d} \right] + \dots + \cot^{-1} \left[\frac{1+a_na_{n-1}}{d} \right]$$

is equal to

- (a) $\tan^{-1} a_n - \tan^{-1} a_1$ (b) $\cot^{-1} a_1 + \cot^{-1} a_n$
 (c) $\cot^{-1} a_1 - \cot^{-1} a_n$ (d) None of these

118. If a, b and c are three terms of an AP such that $a \neq b$, then $\frac{b-c}{a-b}$ may be equal to

- (a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 1 (d) 2

119. The roots of the equation

$x^5 - 40x^4 + Px^3 + Qx^2 + Rx - S = 0$ are in GP. If the sum of their reciprocals is 10, then the value of S can be equal to

- (a) 32 (b) $-\frac{1}{32}$
 (c) -32 (d) $\frac{1}{32}$

120. Consider an AP $a_1, a_2, \dots, a_n, \dots$ and the GP $b_1, b_2, \dots, b_n, \dots$ such that $a_1 = b_1 = 1$, $a_9 = b_9$ and $\sum_{r=1}^9 a_r = 369$, then

- (a) $b_6 = 27$ (b) $b_7 = 27$ (c) $b_8 = 81$ (d) $b_9 = 81$

121. If $\sin \beta$ is the geometric mean between $\sin \alpha$ and $\cos \alpha$, then $\cos 2\beta$ is equal to

- (a) $2\sin^2 \left(\frac{\pi}{4} - \alpha \right)$ (b) $2\cos^2 \left(\frac{\pi}{4} - \alpha \right)$
 (c) $2\cos^2 \left(\frac{\pi}{4} + \alpha \right)$ (d) $2\sin^2 \left(\frac{\pi}{4} + \alpha \right)$

122. If the first and $(2n+1)$ th terms of an AP, a GP and an HP of positive terms are equal and their $(n+1)$ th terms are a, b and c respectively, then

- (a) $a = b = c$ (b) $a \geq b \geq c$
 (c) $a + c = 2b$ (d) $ac = b^2$

Type 3. Assertion and Reason

Directions (Q. Nos. 123-129) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choice are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

123. If positive numbers x, y and z satisfy $x + y + z = 1$, then

Statement I $\left[\frac{(1-2x) + (1-2y) + (1-2z)}{3} \right] \geq (1-2x)(1-2y)(1-2z)^{1/3}$

Statement II For any three positive numbers a, b and c , $AM \geq GM$.

124. Statement I If $\frac{a}{a_1}, \frac{b}{b_1}$ and $\frac{c}{c_1}$ are in AP, then a_1, b_1 and c_1 are in GP.

Statement II If $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ have a common root and $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in AP, then a_1, b_1 and c_1 are in GP.

125. Statement I If all terms of a series with positive terms are smaller than 10^{-5} , then the sum of the series upto infinity will be finite.

Statement II If $S_n > \frac{n}{10^5}$, then $\lim_{n \rightarrow \infty} S_n$ is infinite.

126. Statement I If three positive numbers in GP represent sides of a triangle, then the common ratio of a GP must lie between $\frac{\sqrt{5}-1}{2}$ and $\frac{\sqrt{5}+1}{2}$.

Statement II Three positive real numbers can form a triangle, if sum of any two numbers is greater than the third number.

127. Statement I There exists an AP, whose three terms are $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5}$.

Statement II There does not exist distinct real numbers p, q and r satisfying

$$\sqrt{2} = A + (p-1)d$$

$$\sqrt{3} = A + (q-1)d \quad \text{and} \quad \sqrt{5} = A + (r-1)d$$

128. Statement I $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0, \forall x > 0$.

Statement II Every sequence whose n th term contains $n!$ in the denominator converges to zero.

129. Statement I If the sequence $\{a_n\}$ is monotonically increasing, then the sequence $\{S_n\}$, where $S_n = \frac{a_1 + a_2 + \dots + a_n}{n}$ is monotonically increasing too.

Statement II If $\{a_n\}$ is monotonically increasing, then the expression $na_{n+1} - (a_1 + a_2 + \dots + a_n)$ is positive.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 130-132) If A, G and H are respectively arithmetic, geometric and harmonic means between a and b , both being unequal and positive, then

$$A = \frac{a+b}{2} \Rightarrow a+b = 2A$$

$$G = \sqrt{ab} \Rightarrow ab = G^2$$

and $H = \frac{2ab}{a+b} \Rightarrow G^2 = AH$

From above discussion, we can say that a and b are the roots of the equation $x^2 - 2Ax + G^2 = 0$.

Now, quadratic equations $x^2 - Px + Q = 0$ and $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ have both root common and satisfy the relation $b = \frac{2ac}{(a+c)}$, where a, b

and c are real numbers.

130. The value of $\frac{a(b-c)}{b(c-a)}$ is

- (a) -2 (b) 2 (c) -1/2 (d) 1/2

131. The value of $[P]$, where, $[\]$ denotes the greatest integer function, is

- (a) -2 (b) -1
(c) 2 (d) 1

132. The value of $[2P - Q]$, where $[\]$ denotes the greatest integer function, is

- (a) 2 (b) 3
(c) 5 (d) 6

Passage II (Q. Nos. 133-135) Let the sequence $a_1, a_2, a_3, \dots, a_n$ be in GP. If the area bounded by the parabolas $y^2 = 4a_n x$ and $y^2 = 4a_n(a_n - x)$ is A_n .

133. The sequence $A_1, A_2, A_3, \dots$, lies in

- (a) AP
(b) GP
(c) HP
(d) None of the above

134. For $a_n = 1$, the point of intersection of the curves are

- (a) $\left(\frac{1}{2}, \sqrt{2}\right)$ (b) $\left(\frac{4}{3}, \sqrt{2}\right)$ (c) $\left(\frac{8}{3}, \sqrt{2}\right)$ (d) $\left(\frac{16}{3}, \sqrt{2}\right)$

135. For $a_n = 1$, the area of region bounded by both the curves is

- (a) $4\sqrt{2}$ sq units (b) $\frac{4\sqrt{2}}{3}$ sq units
(c) $\frac{8\sqrt{2}}{3}$ sq units (d) $\frac{16\sqrt{2}}{3}$ sq units

Type 5. Match the Columns

136. Let a, b and c be positive integers such that $a + b + c = n$. Match the statements of Column I with values of Column II.

Column I	Column II
A. $(a^a b^b c^c)^{1/n}$ is less than or equal to	p. $\frac{ab + bc + ca}{n}$
B. $(a^b b^c c^a)^{1/n}$ is less than or equal to	q. $\frac{a^2 + b^2 + c^2}{n}$
C. $(a^c b^a c^b)^{1/n}$ is less than or equal to	r. abc
D. $(a^a b^b c^c)^{1/n} + (a^b b^c c^a)^{1/n} + (a^c b^a c^b)^{1/n}$ is less than or equal to	s. n

137. Match the statements of Column I with values of Column II.

Column I	Column II
A. The AM of two positive numbers a and b exceed their GM by $3/2$ and the GM exceed their HM by $6/5$ such that $a + b = \alpha, a - b = \beta$, then	p. $\alpha + \beta^2 = 96$
B. The AM of two positive numbers a and b exceed their GM by 2 and HM is one-fifth of the greater of a and b such that $a + b = \alpha, a - b = \beta$, then	q. $\alpha + \beta^2 = 74$
C. The HM of two positive numbers a and b is 4, their AM is A and GM G satisfy the relation $2A + G^2 = 27$. If $a + b = \alpha, a - b = \beta$, then	r. $\alpha^2 + \beta = 234$
	s. $\alpha^2 + \beta = 84$
	t. $1 + \left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha}{\beta}\right)^2 + \dots + \left(\frac{\alpha}{\beta}\right)^5 = 364$

Type 6. Single Integer Answer Type Questions

138. If a sequence $a_1, a_2, a_3, \dots, a_n$ of real numbers is such that $a_1 = 0$, $|a_2| = |a_1 + 1|$, $|a_3| = |a_2 + 1|$, ..., $|a_n|$ is equal to $|a_{n-1} + 1|$, where the arithmetic mean of $a_1, a_2, \dots, a_n$ cannot be less than $-\lambda/\mu$, then find the value of $\lambda + \mu$.
139. If $(m+1)$ th, $(n+1)$ th and $(r+1)$ th terms of an AP are in GP and m, n, r are in HP, where the ratios of the common difference to the first term in the AP is $-\lambda/n$, then find the value of λ .
140. Find the HCF of the minimum non-negative values of a, b and c , given that the equation $x^4 + ax^3 + bx^2 + cx + 1 = 0$ has only real roots.
141. If K is a positive integer such that $36 + K, 300 + K$ and $596 + K$ are the squares of three consecutive terms of an arithmetic progression, then find $(K - 920)$.

Entrances Gallery

JEE Advanced/IIT JEE

- Let a, b and c be positive integers such that b/a is an integer. If a, b and c are in geometric progression and the arithmetic mean of a, b, c is $b + 2$, then the value of $\frac{a^2 + a - 14}{a + 1}$ is _____. [2014]
- The value of $\cot \left\{ \sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right\}$ is _____. [2013]
 - $\frac{23}{25}$
 - $\frac{25}{23}$
 - $\frac{23}{24}$
 - $\frac{24}{23}$
- If $S_n = \sum_{k=1}^n (-1)^{\frac{k(k+1)}{2}} k^2$, then S_n can take value(s) _____. [2013]
 - 1056
 - 1088
 - 1120
 - 1332
- If $a_1, a_2, a_3, \dots$ are in a harmonic progression with $a_1 = 5$ and $a_{20} = 25$. Then, least positive integer n for which $a_n < 0$, is _____. [2012]
 - 22
 - 23
 - 24
 - 25
- Let $a_1, a_2, a_3, \dots, a_{100}$ be an arithmetic progression with $a_1 = 3$ and $S_p = \sum_{i=1}^p a_i, 1 \leq p \leq 100$. For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n , then a_2 is _____. [2011]
- The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is _____. [2011]
- If $S_k, k = 1, 2, \dots, 100$, denotes the sum of the infinite geometric series, whose first term is $\frac{k-1}{k!}$ and the common ratio is $1/k$. Then, the value of $\frac{100^2}{100!} + \sum_{k=1}^{100} (k^2 - 3k + 1) S_k$ is _____. [2010]
- Let $a_1, a_2, a_3, \dots, a_{11}$ be real numbers satisfying $a_1 = 15, 27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, \dots, 11$. If $\frac{a_1^2 + a_2^2 + \dots + a_{11}^2}{11} = 90$, then the value of $\frac{a_1 + a_2 + \dots + a_{11}}{11}$ is _____. [2010]

JEE Main/AIEEE

9. If m is the AM of two distinct real numbers l and n ($l, n > 1$) and G_1, G_2 and G_3 are three geometric means between l and n , then $G_1^4 + 2G_2^4 + G_3^4$ equals [2015]
 (a) $4l^2mn$ (b) $4lm^2n$ (c) $4lmn^2$ (d) $4l^2m^2n^2$
10. The sum of first 9 terms of the series $\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots$ is [2015]
 (a) 71 (b) 96 (c) 142 (d) 192
11. Let α and β be the roots of equation $px^2 + qx + r = 0$, $p \neq 0$. If p, q and r are in AP and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is [2014]
 (a) $\frac{\sqrt{61}}{9}$ (b) $\frac{2\sqrt{17}}{9}$ (c) $\frac{\sqrt{34}}{9}$ (d) $\frac{2\sqrt{13}}{9}$
12. If $(10)^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \dots + 10(11)^9 = k(10)^9$, then k is equal to [2014]
 (a) $\frac{121}{10}$ (b) $\frac{441}{100}$ (c) 100 (d) 110
13. Three positive numbers form an increasing GP. If the middle term in this GP is doubled, then new numbers are in AP. Then, the common ratio of a GP is [2014]
 (a) $\sqrt{2} + \sqrt{3}$ (b) $3 + \sqrt{2}$
 (c) $2 - \sqrt{3}$ (d) $2 + \sqrt{3}$
14. The sum of first 20 terms of the sequence 0.7, 0.77, 0.777, ... is [2013]
 (a) $\frac{7}{81}(179 - 10^{-20})$ (b) $\frac{7}{9}(99 - 10^{-20})$
 (c) $\frac{7}{81}(179 + 10^{-20})$ (d) $\frac{7}{9}(99 + 10^{-20})$
15. If 100 times the 100th term of an AP with non-zero common difference equal to 50 times its 50th term, then the 150th term of this AP is [2012]
 (a) -150 (b) 150 times its 50th term
 (c) 150 (d) 0
16. **Statement I** The sum of the series $1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \dots + (361 + 380 + 400)$ is 8000.
Statement II $\sum_{k=1}^n [k^3 - (k-1)^3] = n^3$ for any natural number n . [2012]
 (a) Statement I is false, Statement II is true
 (b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (d) Statement I is true, Statement II is false
17. A man saves ₹ 200 in each of the first three months of his service. In each of the subsequent months, his savings increases by ₹ 40 more than the savings of immediately previous month. His total saving from the start of service will be ₹ 11040 after [2011]
 (a) 19 months (b) 20 months
 (c) 21 months (d) 18 months
18. A person is to count 4500 currency notes. Let a_n denotes the number of notes he counts in the n th minute. If $a_1 = a_2 = \dots = a_{10} = 150$ and $a_{10}, a_{11}, \dots$ are in AP with common difference -2, then the time taken by him to count all notes, is [2010]
 (a) 24 min (b) 34 min (c) 125 min (d) 135 min
19. The sum of the series $1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \dots$ is [2009]
 (a) 3 (b) 4 (c) 6 (d) 2
20. The first two terms of a geometric progression add upto 12. The sum of the third and fourth terms is 48. If the terms of the geometric progression are alternately positive and negative, then the first term is [2008]
 (a) -4 (b) -12 (c) 12 (d) 4
21. In a geometric progression consisting of positive terms, each term equals the sum of the next two terms. Then, the common ratio of this progression equals [2007]
 (a) $\frac{1}{2}(1 - \sqrt{5})$ (b) $\frac{1}{2}\sqrt{5}$ (c) $\sqrt{5}$ (d) $\frac{1}{2}(\sqrt{5} - 1)$
22. Let $a_1, a_2, a_3, \dots$ be terms of an AP. If $\frac{a_1 + a_2 + \dots + a_p}{a_1 + a_2 + \dots + a_q} = \frac{p^2}{q^2}$, $p \neq q$, then $\frac{a_6}{a_{21}}$ equals [2006]
 (a) $\frac{7}{2}$ (b) $\frac{2}{7}$ (c) $\frac{11}{41}$ (d) $\frac{41}{11}$
23. If $a_1, a_2, \dots, a_n$ are in HP, then the expression $a_1a_2 + a_2a_3 + \dots + a_{n-1}a_n$ is equal to [2006]
 (a) $(n-1)(a_1 - a_n)$ (b) na_1a_n
 (c) $(n-1)a_1a_n$ (d) $n(a_1 - a_n)$
24. If $x = \sum_{n=0}^{\infty} a^n$, $y = \sum_{n=0}^{\infty} b^n$, $z = \sum_{n=0}^{\infty} c^n$, where a, b, c are in AP and $|a| < 1, |b| < 1, |c| < 1$, then x, y and z are in [2005]
 (a) HP (b) AGP (c) AP (d) GP
25. The sum of the series $1 + \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} + \frac{1}{64 \cdot 6!} + \dots$ is [2005]
 (a) $\frac{e+1}{2\sqrt{e}}$ (b) $\frac{e-1}{2\sqrt{e}}$
 (c) $\frac{e+1}{\sqrt{e}}$ (d) $\frac{e-1}{\sqrt{e}}$

- 26.** Let T_r be the r th term of an AP, whose first term is a and common difference is d . If for some positive integers $m, n, m \neq n, T_m = \frac{1}{n}$ and $T_n = \frac{1}{m}$, then $a - d$ equals [2004]
 (a) 0 (b) 1 (c) $\frac{1}{mn}$ (d) $\frac{1}{m} + \frac{1}{n}$
- 27.** The sum of the first n terms of the series $1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots$ is $\frac{n(n+1)^2}{2}$, where n is even. When n is odd, then the sum is [2004]
 (a) $\frac{3n(n+1)}{2}$ (b) $\frac{n^2(n+1)}{2}$
 (c) $\frac{n(n+1)^2}{4}$ (d) $\left[\frac{n(n+1)}{2}\right]^2$
- 28.** The sum of the series $\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots$ is [2004]
 (a) $\frac{(e^2 - 1)}{2}$ (b) $\frac{(e - 1)^2}{2e}$ (c) $\frac{(e^2 - 1)}{2e}$ (d) $\frac{(e^2 - 2)}{e}$
- 29.** The sum of the series $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \dots$ is [2003]
 (a) $2 \log_e 2$ (b) $\log_e 2 - 1$ (c) $\log_e 2$ (d) $\log_e \left(\frac{4}{e}\right)$
- 30.** If $1, \log_3 \sqrt{(3^{1-x} + 2)}, \log_3 (4 \cdot 3^x - 1)$ are in AP. Then, x equals [2002]
 (a) $\log_3 4$ (b) $1 - \log_3 4$ (c) $1 - \log_4 3$ (d) $\log_4 3$
- 31.** The value of $2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \dots$ is [2002]
 (a) $\frac{1}{2}$ (b) 2
 (c) $\frac{3}{2}$ (d) 4
- 32.** If term of a GP is 2, then the product of its 9 terms is [2002]
 (a) 256 (b) 512
 (c) 1024 (d) None of the above
- 33.** $\sum_{n=0}^{\infty} \frac{(\log_e x)^n}{n!}$ is equal to [2002]
 (a) $\log_e x$ (b) x
 (c) $\log_x e$ (d) None of these
- 34.** $e^{(x-1) - \frac{1}{2}(x-1)^2 + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \dots}$ is equal to [2002]
 (a) $\log(x-1)$
 (b) $\log x$
 (c) x
 (d) None of the above

Other Engineering Entrances

- 35.** If V_r denotes the sum of the first r terms of an arithmetic progression (AP), whose first term is r and the common difference is $(2r-1)$. Then, the sum of $V_1 + V_2 + \dots + V_n$ is [AMU 2014]
 (a) $\frac{1}{12} n(n+1)(3n^2 - n + 1)$
 (b) $\frac{1}{12} n(n+1)(3n^2 + n + 2)$
 (c) $\frac{1}{2} n(2n^2 - n + 1)$
 (d) $\frac{1}{3} (2n^3 - 2n + 3)$
- 36.** If a, b and c are positive numbers in a GP, then the roots of the quadratic equation $(\log_e a)x^2 - (2 \log_e b)x + (\log_e c) = 0$ are [WB JEE 2014]
 (a) -1 and $\frac{\log_e c}{\log_e a}$ (b) 1 and $-\frac{\log_e c}{\log_e a}$
 (c) 1 and $\log_a c$ (d) -1 and $\log_c a$
- 37.** If $f(x) = x + 1/2$, then the number of real values of x for which the three unequal terms $f(x), f(2x), f(4x)$ are in HP, is [WB JEE 2014]
 (a) 1 (b) 0 (c) 3 (d) 2
- 38.** If p, q, r and s are positive real numbers such that $p + q + r + s = 2$, then $M = (p+q)(r+s)$ satisfies the relation [BITSAT 2014]
 (a) $0 < M \leq 1$ (b) $1 \leq M \leq 2$ (c) $2 \leq M \leq 3$ (d) $3 \leq M \leq 4$
- 39.** Let α and β denote the cube roots of unity other than 1 and $\alpha \neq \beta$. If $S = \sum_{n=0}^{302} (-1)^n \left(\frac{\alpha}{\beta}\right)^n$, then the value of S is [WB JEE 2014]
 (a) either -2ω or $-2\omega^2$ (b) either -2ω or $2\omega^2$
 (c) either 2ω or $-2\omega^2$ (d) either 2ω or $2\omega^2$
- 40.** If α and β are the roots of the equation $x^2 - px + q = 0$, then the value of $(\alpha + \beta)x - \left(\frac{\alpha^2 + \beta^2}{2}\right)x^2 + \left(\frac{\alpha^3 + \beta^3}{3}\right)x^3 + \dots$ is [BITSAT 2014]
 (a) $\log(1 - px + qx^2)$ (b) $\log(1 + px - qx^2)$
 (c) $\log(1 + px + qx^2)$ (d) None of the above
- 41.** The sum of the series $\log_4 2 - \log_8 2 + \log_{16} 2 - \dots$ is [BITSAT 2014]
 (a) e^2 (b) $\log_e 2 + 1$
 (c) $\log_e 3 - 2$ (d) $1 - \log_e 2$
- 42.** If $\frac{e^x}{1-x} = B_0 + B_1x + B_2x^2 + \dots + B_nx^n + \dots$, then the value of $B_n - B_{n-1}$ is [BITSAT 2014]
 (a) 1 (b) $\frac{1}{n}$
 (c) $\frac{1}{n!}$ (d) None of these

- 43.** If $a = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$, $b = \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!}$ and $c = \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!}$, then the value of $a^3 + b^3 + c^3 - 3abc$ is [BITSAT 2014]
 (a) 1 (b) 0
 (c) -1 (d) -2
- 44.** For every real number x , let $f(x) = \frac{x}{1!} + \frac{3}{2!}x^2 + \frac{7}{3!}x^3 + \frac{15}{4!}x^4 + \dots$. Then, the equation $f(x) = 0$ has [WB JEE 2014]
 (a) no real solution
 (b) exactly one real solution
 (c) exactly two real solutions
 (d) infinite number of real solutions
- 45.** Let S denotes the sum of the series $1 + \frac{8}{2!} + \frac{21}{3!} + \frac{40}{4!} + \frac{65}{5!} + \dots$, then [WB JEE 2014]
 (a) $S < 8$ (b) $S > 12$ (c) $8 < S < 12$ (d) $S = 8$
- 46.** In an arithmetical progression $a_1, a_2, a_3, \dots$, sum $(S) = a_1^2 - a_2^2 + a_3^2 - a_4^2 + \dots - a_{2k}^2$ is equal to [BITSAT 2013]
 (a) $\frac{k}{2k-1}(a_1^2 - a_{2k}^2)$ (b) $\frac{2k}{k-1}(a_{2k}^2 - a_1^2)$
 (c) $\frac{k}{k+1}(a_1^2 - a_{2k}^2)$ (d) None of these
- 47.** If $\log_x ax$, $\log_x bx$ and $\log_x cx$ are in AP, where a, b, c, x belong to $(1, \infty)$, then a, b and c are in [UP SEE 2013]
 (a) AP (b) HP
 (c) GP (d) None of these
- 48.** Six numbers are in AP such that their sum is 3. The first term is 4 times the third term. Then, the fifth term is [WB JEE 2012]
 (a) -15 (b) -3
 (c) 9 (d) -4
- 49.** If $\log_{10} 2$, $\log_{10} (2^x - 1)$ and $\log_{10} (2^x + 3)$ are in AP, then x is equal to [AMU 2012]
 (a) $\log_2 5$ (b) $\log_2(-1)$ (c) $\log_2\left(\frac{1}{5}\right)$ (d) $\log_5 2$
- 50.** $0.2 + 0.22 + 0.222 + \dots$ upto n terms is equal to [BITSAT 2012]
 (a) $\left(\frac{2}{9}\right) - \left(\frac{2}{81}\right)(1 - 10^{-n})$ (b) $n - \left(\frac{1}{9}\right)(1 - 10^{-n})$
 (c) $\left(\frac{2}{9}\right) \left[n - \left(\frac{1}{9}\right)(1 - 10^{-n}) \right]$ (d) $\left(\frac{2}{9}\right)$
- 51.** The sum of the series $1 + \frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12} + \dots$ is [WB JEE 2013]
 (a) $\sqrt{\frac{2}{3}}$ (b) $\sqrt{\frac{3}{2}}$
 (c) $\sqrt{\frac{3}{2}}$ (d) $\sqrt{\frac{1}{3}}$
- 52.** If $H_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$, then the value of $S_n = 1 + \frac{3}{2} + \frac{5}{3} + \dots + \frac{2n-1}{n}$ is [AMU 2012]
 (a) $H_n + 2n$ (b) $n - 1 + H_n$ (c) $H_n - 2n$ (d) $2n - H_n$
- 53.** If AM and HM between two numbers are 27 and 12 respectively, then their GM is [BITSAT 2011]
 (a) 9 (b) 18
 (c) 24 (d) 36
- 54.** If x, y and z are in geometric progression, then $\log_x 10, \log_y 10$ and $\log_z 10$ are in [UP SEE 2011]
 (a) AP (b) GP
 (c) HP (d) None of these
- 55.** The value of $4 + 2(1+2)\log 2 + \frac{2(1+2^2)}{2!}(\log 2)^2 + \frac{2(1+2^3)}{3!}(\log 2)^3 + \dots$ is [UP SEE 2011]
 (a) 10 (b) 12
 (c) $\log(3^2 \cdot 4^2)$ (d) $\log(2^2 \cdot 3^2)$
- 56.** The value of $\frac{2}{1!} + \frac{2+4}{2!} + \frac{2+4+6}{3!} + \dots$ is [BITSAT 2011]
 (a) e (b) $2e$
 (c) $3e$ (d) None of these
- 57.** If the sum to first n terms of an AP 2, 4, 6, ... is 240, then the value of n is [BITSAT 2010]
 (a) 14 (b) 15
 (c) 16 (d) 17
- 58.** If sum of an infinite geometric series is $\frac{4}{3}$ and its 1st term is $\frac{3}{4}$, then its common ratio is [BITSAT 2010]
 (a) $\frac{7}{16}$ (b) $\frac{9}{16}$ (c) $\frac{1}{9}$ (d) $\frac{7}{9}$
- 59.** The value of n for which $\frac{x^{n+1} + y^{n+1}}{x^n + y^n}$ is the geometric mean of x and y , is [WB JEE 2010]
 (a) $n = -\frac{1}{2}$ (b) $n = \frac{1}{2}$ (c) $n = 1$ (d) $n = -1$
- 60.** GM and HM of two numbers are 10 and 8 respectively. The numbers are [WB JEE 2010]
 (a) 5, 20 (b) 4, 25
 (c) 2, 50 (d) 1, 100
- 61.** Sum of n terms of the series $1^3 + 3^3 + 5^3 + 7^3 + \dots$ is [WB JEE 2010]
 (a) $n^2(2n^2 - 1)$ (b) $n^3(n - 1)$
 (c) $n^3 + 8n + 4$ (d) $2n^4 + 3n^2$
- 62.** The value of $\frac{2}{3!} + \frac{4}{5!} + \frac{6}{7!} + \dots$ is [WB JEE 2010]
 (a) $e^{1/2}$ (b) e^{-1} (c) e (d) $e^{-1/3}$

Answers

Work Book Exercise 3.1

1. (c) 2. (b) 3. (a) 4. (a) 5. (b) 6. (b) 7. (b) 8. (a) 9. (a) 10. (a)
11. (c) 12. (d) 13. (c) 14. (b)

Work Book Exercise 3.2

1. (c) 2. (b) 3. (d) 4. (d) 5. (a) 6. (b) 7. (d) 8. (c) 9. (c) 10. (c)
11. (c) 12. (c) 13. (b) 14. (a) 15. (d)

Work Book Exercise 3.3

1. (b) 2. (c) 3. (a) 4. (a) 5. (a) 6. (a) 7. (a) 8. (b) 9. (a) 10. (a)

Work Book Exercise 3.4

1. (b) 2. (a) 3. (c) 4. (c) 5. (a) 6. (b) 7. (a) 8. (b) 9. (c) 10. (d)

Target Exercises

1. (a) 2. (c) 3. (d) 4. (c) 5. (d) 6. (a) 7. (b) 8. (a) 9. (b) 10. (d)
11. (c) 12. (a) 13. (c) 14. (c) 15. (b) 16. (d) 17. (d) 18. (a) 19. (b) 20. (c)
21. (c) 22. (a) 23. (b) 24. (c) 25. (a) 26. (a) 27. (c) 28. (a) 29. (a) 30. (c)
31. (b) 32. (c) 33. (a) 34. (b) 35. (b) 36. (b) 37. (b) 38. (c) 39. (b) 40. (a)
41. (d) 42. (a) 43. (d) 44. (b) 45. (a) 46. (a) 47. (b) 48. (c) 49. (c) 50. (a)
51. (c) 52. (b) 53. (a) 54. (c) 55. (a) 56. (a) 57. (a) 58. (c) 59. (c) 60. (c)
61. (d) 62. (a) 63. (c) 64. (c) 65. (c) 66. (a) 67. (c) 68. (d) 69. (a) 70. (a)
71. (b) 72. (c) 73. (b) 74. (b) 75. (a) 76. (c) 77. (b) 78. (a) 79. (a) 80. (b)
81. (d) 82. (a) 83. (b) 84. (b) 85. (b) 86. (c) 87. (d) 88. (b) 89. (c) 90. (b)
91. (a) 92. (a) 93. (d) 94. (a) 95. (c) 96. (b) 97. (a) 98. (b) 99. (c) 100. (b)
101. (b) 102. (a) 103. (c) 104. (c) 105. (a) 106. (d) 107. (a) 108. (c) 109. (b) 110. (a)
111. (c) 112. (d) 113. (b) 114. (c) 115. (b,d) 116. (b,d) 117. (a,c) 118. (c,d) 119. (a,c) 120. (b,d)
121. (a,c) 122. (b,d) 123. (b) 124. (a) 125. (d) 126. (a) 127. (d) 128. (c) 129. (a) 130. (c)
131. (c) 132. (b) 133. (b) 134. (a) 135. (b) 136. (*) 137. (**) 138. (3) 139. (2) 140. (2)
141. (5)

* $A \rightarrow q, r$; $B \rightarrow p, q, r$; $C \rightarrow p, q, r$; $D \rightarrow s$

** $A \rightarrow p, r$; $B \rightarrow q$; $C \rightarrow s, t$

Entrances Gallery

1. (4) 2. (b) 3. (a,d) 4. (d) 5. (3) 6. (8) 7. (4) 8. (0) 9. (b) 10. (b)
11. (d) 12. (c) 13. (d) 14. (c) 15. (d) 16. (b) 17. (c) 18. (b) 19. (a) 20. (b)
21. (d) 22. (c) 23. (c) 24. (a) 25. (a) 26. (a) 27. (b) 28. (b) 29. (d) 30. (b)
31. (b) 32. (b) 33. (b) 34. (c) 35. (b) 36. (c) 37. (a) 38. (a) 39. (a) 40. (c)
41. (d) 42. (c) 43. (a) 44. (b) 45. (c) 46. (a) 47. (c) 48. (d) 49. (a) 50. (c)
51. (b) 52. (d) 53. (b) 54. (c) 55. (b) 56. (c) 57. (b) 58. (a) 59. (a) 60. (a)
61. (a) 62. (b)

Explanations

Target Exercises

- We have,

$$5 = 2^2 + 1,$$

$$14 = 3^2 + 5,$$

$$30 = 4^2 + 14,$$

$$55 = 5^2 + 30,$$
 i.e. square of term is added to previous term.
 $\therefore$ Next term $= 6^2 + 55 = 36 + 55 = 91$
- We have,

$$3 = 1 + 2$$

$$6 = 3 + 3$$

$$10 = 6 + 4$$
 $\therefore$ Next term $= 10 + 5 = 15$
- Given, $5T_5 = 8T_8$

$$\Rightarrow 5(a + 4d) = 8(a + 7d)$$

$$\therefore 3a + 36d = 0$$

$$\Rightarrow a + 12d = 0$$

$$\Rightarrow T_{13} = 0$$
- We have, $a_3 = a_2 - 1 = 2 - 1 = 1$
 $a_4 = a_3 - 1 = 1 - 1 = 0$
 $a_5 = a_4 - 1 = 0 - 1 = -1$
- Let a and d be the first term and common difference respectively of given AP. Then,

$$T_2 = x - y$$

$$\Rightarrow a + d = x - y$$
 and

$$T_5 = x + y$$

$$\Rightarrow a + 4d = x + y$$

$$\Rightarrow a = x - \frac{5}{3}y$$
- Given a, b and c are in AP.

$$\Rightarrow 2b = a + c$$

$$\Rightarrow 3b = a + b + c = 2s$$
 Now, $\cot \frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$$= \sqrt{\frac{2s(2s-2c)}{(2s-2a)(2s-2b)}} = \sqrt{\frac{3b(2s-2c)}{(2s-2a)(3b-2b)}}$$

$$= \sqrt{\frac{3(s-c)}{s-a}} = 3\sqrt{\frac{s-c}{3(s-a)}}$$

$$= 3\sqrt{\frac{b(s-c)}{3b(s-a)}} = 3\sqrt{\frac{(2s-2b)(s-c)}{2s(s-a)}}$$

$$= 3\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$= 3 \tan \frac{A}{2}$$
- Let x be the common difference of AP a, b, c, d, e, f .
 Then,

$$e - c = (a + 4x) - (a + 2x)$$

$$= 2x = 2(d - c)$$
- From given options, if we consider $b = -1, \cos x = b$
 $\Rightarrow \cos x = -1$, which is satisfied for $x = \pi, 3\pi, 5\pi, \dots$,
 which form an AP with common difference 2π . For other values of b , the roots of the equation are not in AP.

- Given, $\log_3 2, \log_3(2^x - 5)$ and $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP.

$$\therefore 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$$

$$\Rightarrow 2^x = 8 \text{ or } 2^x = 4$$

$$\Rightarrow x = 3 \text{ or } x = 2$$
 Neglecting $x = 2$ as $(2^x - 5) < 0$ for $x = 2$
 $\Rightarrow x = 3$
- Given, $T_p = q \Rightarrow a + (p-1)d = q$... (i)
 and $T_q = p \Rightarrow a + (q-1)d = p$... (ii)
 On subtracting Eq. (ii) from Eq. (i), we get

$$\Rightarrow (p-q)d = -(p-q) \Rightarrow d = -1$$
 On putting $d = -1$ in Eq. (i), we get

$$a + (p-1)(-1) = q$$

$$\Rightarrow a = p + q - 1$$

$$\therefore T_r = a + (r-1)d$$

$$= (p + q - 1) + (r-1)(-1)$$

$$= p + q - 1 - r + 1 = p + q - r$$
- Here,

$$\frac{1}{1-x} - \frac{1}{1+\sqrt{x}} = \frac{\sqrt{x}}{1-x}$$
 and

$$\frac{1}{1-\sqrt{x}} - \frac{1}{1-x} = \frac{\sqrt{x}}{1-x}$$
 Hence, these terms are in AP.
- Since, $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in AP.

$$\therefore \frac{1}{a} + \frac{1}{c} = \frac{2}{b}$$
 ... (i)
 Now, $\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right)\left(\frac{1}{b} + \frac{1}{c} - \frac{1}{a}\right)$

$$= \left\{\frac{1}{a} + \frac{1}{b} - \left(\frac{2}{b} - \frac{1}{a}\right)\right\}\left\{\frac{1}{b} + \frac{1}{c} - \left(\frac{2}{b} - \frac{1}{c}\right)\right\}$$

$$\quad \quad \quad \text{[from Eq. (i)]}$$

$$= \left(\frac{2}{a} - \frac{1}{b}\right)\left(\frac{2}{c} - \frac{1}{b}\right) = \frac{4}{ac} - \frac{2}{b}\left(\frac{1}{a} + \frac{1}{c}\right) + \frac{1}{b^2}$$

$$= \frac{4}{ac} - \frac{2}{b}\left(\frac{2}{b}\right) + \frac{1}{b^2} = \frac{4}{ac} - \frac{3}{b^2}$$
- Let a and d be the first term and common difference, respectively of given AP, then

$$a_7 = 34 \Rightarrow a + 6d = 34$$
 ... (i)
 and $a_{13} = 64 \Rightarrow a + 12d = 64$... (ii)
 On solving Eqs. (i) and (ii), we get

$$a = 4, d = 5$$

$$\therefore a_{18} = a + 17d = 4 + 85 = 89$$
- Let roots of given equation in AP be $a-d, a$ and $a+d$.
 Then, $(a-d) + a + (a+d) = 12$

$$\Rightarrow a = 4$$
 Also, $(a-d)a(a+d) = 28$

$$\Rightarrow a(a^2 - d^2) = 28 \Rightarrow 4(16 - d^2) = 28$$

$$\Rightarrow 16 - d^2 = 7 \Rightarrow d^2 = 9 \Rightarrow d = \pm 3$$

15.
$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}$$

Applying $C_1 \rightarrow C_1 - C_2$

$$= \begin{vmatrix} -1 & x+2 & x+a \\ -1 & x+3 & x+b \\ -1 & x+4 & x+c \end{vmatrix}$$

$$= (-1) \begin{vmatrix} 1 & x+2 & x+a \\ 1 & x+3 & x+b \\ 1 & x+4 & x+c \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$

$$= (-1) \begin{vmatrix} 1 & x+2 & x+a \\ 0 & 1 & b-a \\ 0 & 2 & c-a \end{vmatrix}$$

$$= (-1) [c + a - 2b] = 0$$

[since, a, b and c are in AP, so $2b = a + c$]

16. Given $a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{c} + \frac{1}{a}\right)$ and $c\left(\frac{1}{a} + \frac{1}{b}\right)$ are in AP.

$$\Rightarrow \frac{a(b+c)}{bc}, \frac{b(c+a)}{ca}, \frac{c(a+b)}{ab} \text{ are in AP.}$$

$$\Rightarrow (b-a)(cb+ca+ab) = (c-b)(ac+ab+cb)$$

$$\Rightarrow b-a = c-b$$

$\Rightarrow a, b$ and c are in AP.

17. Given a, b and c are in AP.

$$\therefore 2b = a + c \quad \dots(i)$$

$$\text{Now, } (a+2b-c)(2b+c-a)(c+a-b)$$

$$= (a+a+c-c)(a+c+c-a)(2b-b) \text{ [from Eq. (i)]}$$

$$= (2a)(2c)(b) = 4abc$$

18. We know, $\frac{1}{\sqrt{a} + \sqrt{b}}, \frac{1}{\sqrt{a} + \sqrt{c}}$ and $\frac{1}{\sqrt{b} + \sqrt{c}}$ are in AP, if

$$\frac{1}{\sqrt{a} + \sqrt{c}} - \frac{1}{\sqrt{a} + \sqrt{b}} = \frac{1}{\sqrt{b} + \sqrt{c}} - \frac{1}{\sqrt{a} + \sqrt{b}}$$

$$\text{i.e. } \frac{\sqrt{b} - \sqrt{c}}{\sqrt{a} + \sqrt{b}} = \frac{\sqrt{a} - \sqrt{b}}{\sqrt{b} + \sqrt{c}}$$

$$\text{i.e. } b - c = a - b, \text{ which is true.}$$

19. Given, $\frac{1}{p+q}, \frac{1}{q+r}$ and $\frac{1}{r+p}$ are in AP.

$$\Rightarrow \frac{1}{q+r} - \frac{1}{p+q} = \frac{1}{r+p} - \frac{1}{q+r}$$

$$\Rightarrow p^2 - r^2 = q^2 - p^2$$

So, q^2, p^2 and r^2 are in AP.

20. The numbers between 100 and 500 that are divisible by 7 are 105, 112, 119, 126, 133, 140, 147, ..., 483, 490, 497.

Let such numbers be n .

$$\text{Then, } 497 = 105 + (n-1) \times 7$$

$$\Rightarrow n = 57$$

The number between 100 and 500 that are divisible by 21 are 105, 126, 147, ..., 483.

Let such numbers be m .

$$\text{Then, } 483 = 105 + (m-1) \times 21$$

$$\Rightarrow m = 19$$

$$\therefore \text{Required number} = n - m = 57 - 19 = 38$$

21. We have, first term = a

$$\therefore T_1 = a$$

$$\text{and second term} = b$$

$$\therefore T_2 = b$$

Then, common difference

$$d = T_2 - T_1 = b - a$$

$$\text{Also, last term} = c \Rightarrow c = a + (n-1)d$$

$$\Rightarrow n = \frac{c-a+d}{d}$$

$$\Rightarrow n = \frac{(b+c-2a)}{(b-a)} \quad [\because d = b-a]$$

$\therefore$ Sum of n terms,

$$S_n = \frac{n}{2}(a+l) = \frac{(b+c-2a)(a+c)}{2(b-a)}$$

22. The given series is arithmetic whose first term is 20 and

$$\text{common difference is } -\frac{2}{3}.$$

As the common difference is negative, the terms will become negative after some stage. So, the sum is maximum, if only positive terms are added.

$$\text{Now, } t_n = 20 + (n-1)\left(-\frac{2}{3}\right) \geq 0,$$

$$\text{if } 60 - 2(n-1) \geq 0 \Rightarrow 62 \geq 2n \Rightarrow 31 \geq n$$

So, the first 31 terms are non-negative.

$\therefore$ Maximum sum,

$$S_{31} = \frac{31}{2} \left\{ 2 \times 20 + (31-1) \left(-\frac{2}{3} \right) \right\}$$

$$= \frac{31}{2} \{ 40 - 20 \} = 310$$

23. Required sum

$$= (1+2+3+\dots+199) - (3+6+9+\dots+198)$$

$$- (5+10+15+\dots+195) + (15+30+45+\dots+195)$$

$$= \frac{199}{2}(1+199) - \frac{66}{2}(3+198)$$

$$- \frac{39}{2}(5+195) + \frac{13}{2}(15+195)$$

$$= 199 \times 100 - 33 \times 201 - 39 \times 100 + 13 \times 105 = 10732$$

24. Let the four numbers in AP be

$$\alpha - 3\beta, \alpha - \beta, \alpha + \beta, \alpha + 3\beta.$$

$$\text{Given, } \alpha - 3\beta + \alpha - \beta + \alpha + \beta + \alpha + 3\beta = 50$$

$$\Rightarrow 4\alpha = 50 \Rightarrow \alpha = \frac{25}{2}$$

$$\text{and } \alpha + 3\beta = 4(\alpha - 3\beta)$$

$$\Rightarrow 3\alpha = 15\beta$$

$$\therefore \beta = \frac{\alpha}{5} = \frac{25}{5 \times 2} = \frac{5}{2}$$

Hence, the four numbers are 5, 10, 15 and 20.

25. Let the four numbers in AP be

$$\alpha - 3\beta, \alpha - \beta, \alpha + \beta \text{ and } \alpha + 3\beta.$$

$$\text{Given, } \alpha - 3\beta + \alpha - \beta + \alpha + \beta + \alpha + 3\beta = 24$$

$$\Rightarrow 4\alpha = 24$$

$$\text{and } (\alpha - 3\beta)(\alpha - \beta)(\alpha + \beta)(\alpha + 3\beta) = 945$$

$$\Rightarrow (\alpha^2 - 9\beta^2)(\alpha^2 - \beta^2) = 945$$

$$\Rightarrow (36 - 9\beta^2)(36 - \beta^2) = 945$$

$$\Rightarrow \beta^4 - 40\beta^2 + 144 = 105$$

$$\Rightarrow \beta^4 - 40\beta^2 + 39 = 0$$

$$\Rightarrow \beta^4 - \beta^2 - 39\beta^2 + 39 = 0$$

$$\Rightarrow (\beta^2 - 1)(\beta^2 - 39) = 0$$

Since, numbers are integers.

$$\therefore \beta^2 \neq 39$$

$$\therefore \beta^2 - 1 = 0$$

$$\Rightarrow \beta = \pm 1$$

Hence, the four integers are 3, 5, 7 and 9.

26. Given, $S_n = 3n^2$

$$\therefore T_n = S_n - S_{n-1}$$

$$= 3n^2 - 3(n-1)^2 = 6n - 3$$

On putting $n = 1, 2, 3, \dots$, the sequence is 3, 9, 15, 21, ... which is clearly an AP.

27. The first two digit number which when divided by 4 leaves remainder 1 is $4 \cdot 3 + 1 = 13$ and last is $4 \cdot 24 + 1 = 97$.

Thus, we have to find the sum of the series

$$13 + 17 + 21 + \dots + 97$$

which is an AP.

$$\therefore 97 = 13 + (n-1)4$$

$$\Rightarrow n = 22$$

$$\text{and } S_n = \frac{n}{2} [a + l] = 11[13 + 97]$$

$$= 11 \times 110 = 1210$$

28. The least and the greatest numbers of three digits divisible by 7 are 105 and 994, respectively.

So, it is required to find the sum of the series

$$105 + 112 + 119 + \dots + 994$$

Here, $a = 105, d = 7, a_n = 994$

$$\text{Now, } a_n = a + (n-1)d$$

$$\Rightarrow 994 = 105 + (n-1) \times 7$$

$$\Rightarrow 994 - 105 = 7(n-1)$$

$$\Rightarrow 889 = 7(n-1)$$

$$\Rightarrow n-1 = \frac{889}{7}$$

$$\therefore n-1 = 127$$

$$\Rightarrow n = 127 + 1 = 128$$

$$\therefore \text{Sum} = \frac{n}{2} [2a + (n-1)d]$$

$$= \frac{128}{2} [2 \times 105 + (128-1) \times 7]$$

$$= 64(210 + 889)$$

$$= 64 \times 1099$$

$$= 70336$$

29. The odd numbers of four digits which are divisible by 9 are 1017, 1035, ..., 9999.

These are in AP with common difference 18.

$$a = 1017, d = 18 \text{ and } l = 9999$$

$$\therefore n\text{th term, } a_n = a + (n-1)d$$

$$\Rightarrow 9999 = 1017 + (n-1) \times 18$$

$$\Rightarrow 18n = 9999 - 999 = 9000$$

$$\Rightarrow n = 500$$

$$\therefore S_n = \frac{n}{2} (a_1 + a_n)$$

$$= \frac{500}{2} (1017 + 9999)$$

$$= 250 \times 11016$$

$$= 2754000$$

30. Here, $a = 10$ and $d = -\frac{3}{7}$

$$\text{Then, } t_n = 10 + (n-1) \left(-\frac{3}{7} \right)$$

Now, t_n is positive, if

$$10 + (n-1) \left(-\frac{3}{7} \right) \geq 0$$

$$\Rightarrow 70 - 3(n-1) \geq 0$$

$$\Rightarrow 73 \geq 3n \Rightarrow 24 \frac{1}{3} \geq n$$

So, first 24 terms are positive.

Now, sum of the positive terms,

$$S_{24} = \frac{24}{2} \left[2 \times 10 + 23 \times \left(-\frac{3}{7} \right) \right]$$

$$= 12 \left[20 - \frac{69}{7} \right] = \frac{852}{7}$$

31. Let n th term be the first negative term.

Then, n th term, $t_n < 0$

$$\therefore 40 + (n-1)(-2) < 0 \Rightarrow 42 - 2n < 0$$

$$\Rightarrow 2n > 42 \Rightarrow n > 21$$

$\therefore$ Least value of $n = 22$

Hence, first 21 terms of an AP are non-negative.

Since, sum will be maximum, if no negative terms is taken.

$$\therefore \text{Maximum sum, } S_{21} = \frac{21}{2} [2 \cdot 40 + (20)(-2)]$$

$$= \frac{21}{2} \times 40 = 420$$

32. Let the first instalment be a and common difference of an AP be d .

Given, 3600 = Sum of 40 terms

$$\Rightarrow 3600 = \frac{40}{2} [2a + (40-1)d]$$

$$\Rightarrow 3600 = 20 [2a + 39d]$$

$$\Rightarrow 180 = 2a + 39d \quad \dots(i)$$

After 30 instalments, one-third of the debt is unpaid.

$$\text{Hence, } \frac{3600}{3} = 1200 \text{ is unpaid and } 2400 \text{ is paid.}$$

$$\text{Now, } 2400 = \frac{30}{2} [2a + (30-1)d]$$

$$\Rightarrow 160 = 2a + 29d \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$20 = 10d$$

$$\therefore d = 2$$

From Eq. (i),

$$180 = 2a + 39 \cdot 2$$

$$\Rightarrow 2a = 180 - 78 = 102$$

$$\therefore a = 51$$

Now, the value of 8th instalment

$$= a + (8-1)d = 51 + 7 \cdot 2 = ₹ 65$$

33. Let a be the length of the smallest side and d be the common difference.

Here, $n = 25$ and $S_{25} = 2100$

$$\text{Now, } S_n = \frac{n}{2} [2a + (n-1)d]$$

$$\Rightarrow 2100 = \frac{25}{2} [2a + (25-1)d]$$

$$\Rightarrow a + 12d = 84 \quad \dots(i)$$

The largest side = 25th side

$$= a + (25 - 1)d = a + 24d$$

$$\therefore a + 24d = 20a \quad [\text{given}] \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$a = 8, d = 6\frac{1}{3}$$

$$34. S_n = \frac{n}{2} [2a + (n - 1)d]$$

Here, $a = 1\text{st term} = 7 \text{ yr}$, $d = 3 \text{ months} = \frac{1}{4} \text{ yr}$

$$\Rightarrow S_n = 250 \text{ yr}$$

$$250 = \frac{n}{2} \left[2 \times 7 + (n - 1) \times \frac{1}{4} \right]$$

$$\Rightarrow 250 = \frac{n}{2} \left(\frac{n + 55}{4} \right)$$

$$\Rightarrow 2000 = n^2 + 55n$$

$$\Rightarrow n^2 + 55n - 2000 = 0$$

$$\Rightarrow (n - 25)(n + 80) = 0$$

$$\Rightarrow n = 25$$

$\therefore$ Number of members in the club = 25

$$35. S_n = \frac{n}{2} [2a + (n - 1)d]$$

$$\Rightarrow 60100 = \frac{n}{2} [4 + (n - 1)(3)]$$

$$\Rightarrow \frac{n}{2} [3n + 1] = 60100$$

$$\Rightarrow n = 200$$

$$36. T_n = S_n - S_{n-1}$$

$$= [3n^2 + 5n] - [3(n - 1)^2 + 5(n - 1)]$$

$$= [3n^2 + 5n] - [3(n^2 - 2n + 1) + 5(n - 1)]$$

$$= 6n + 2$$

$$\text{Let } T_n = 164$$

$$\text{Then, } 164 = 6n + 2 \Rightarrow 6n = 162 \Rightarrow n = 27$$

Hence, 164 is 27th term.

$$37. \text{ Here, } a = 1, l = 11 \text{ and } S_n = 36$$

$$\therefore S_n = \frac{n}{2} (a + l)$$

$$\therefore 36 = \frac{n}{2} (1 + 11) = 6n \Rightarrow n = 6$$

$$38. \text{ Given series is } \sqrt{2} + 2\sqrt{2} + 3\sqrt{2} + 4\sqrt{2} + \dots$$

$$= \sqrt{2} [1 + 2 + 3 + 4 + \dots \text{ upto } n \text{ terms}]$$

$$= \sqrt{2} (\Sigma n) = \sqrt{2} \left[\frac{n(n + 1)}{2} \right] \quad \left[\because \Sigma n = \frac{n(n + 1)}{2} \right]$$

$$\therefore S_n = \frac{n(n + 1)}{\sqrt{2}}$$

$$39. \text{ Let } S_n = \frac{1}{\sqrt{1} + \sqrt{3}} + \frac{1}{\sqrt{3} + \sqrt{5}} + \frac{1}{\sqrt{5} + \sqrt{7}} + \dots + \frac{1}{\sqrt{2n-1} + \sqrt{2n+1}}$$

$$= \left(\frac{\sqrt{3} - 1}{2} \right) + \left(\frac{\sqrt{5} - \sqrt{3}}{2} \right) + \left(\frac{\sqrt{7} - \sqrt{5}}{2} \right) + \dots + \left(\frac{\sqrt{2n+1} - \sqrt{2n-1}}{2} \right)$$

$$= \frac{\sqrt{2n+1} - 1}{2}$$

40. Two digit odd numbers are 11, 13, ..., 99. These clearly form an AP with $a = 11$, $d = 2$ and $n = 45$.

$\therefore$ Required sum

$$= \frac{n}{2} (a + l) = \frac{45}{2} (11 + 99) = 2475$$

$$41. S_p - S_q = 2a(p - q) + (p^2 - q^2 - p + q)d = 2(q - p)$$

$$\Rightarrow 2a + (p + q - 1)d = -2$$

$$\therefore S_{p+q} = \frac{(p+q)}{2} [2a + (p+q-1)d]$$

$$= \frac{-2(p+q)}{2}$$

$$= -(p+q)$$

$$42. \text{ Given, } S_n = 3n^2 - n$$

If t_n is the n th term of an AP, then

$$t_n = S_n - S_{n-1}$$

$$= (3n^2 - n) - [3(n-1)^2 - (n-1)] = 6n - 4$$

$$\therefore t_1 = 6 - 4 = 2$$

$$43. \text{ We have, } S_n = \frac{n}{2} [2p + (n-1)Q]$$

By definition, $d = Q$

$$44. \text{ Given, } S_n = 5n^2 + 2n$$

$$\Rightarrow S_1 = 5 + 2 = 7$$

$$\text{and } S_2 = 5(4) + 2(2) = 24$$

$$\therefore T_2 = S_2 - S_1$$

$$= 24 - 7 = 17$$

$$45. \text{ Given, } t_n = \frac{n}{x} + y$$

$$\Rightarrow t_1 = \frac{1}{x} + y \quad \text{and} \quad t_2 = \frac{2}{x} + y$$

$$\Rightarrow d = t_2 - t_1 = \frac{1}{x}$$

where, d is common difference of given AP.

$$\therefore S_r = \frac{r}{2} [2a_1 + (r-1)d]$$

$$= \frac{r(r+1)}{2x} + ry \quad \left[\because a_1 = \frac{1}{x} + y \right]$$

$$46. S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n$$

$$= (S_{n+3} - S_{n+2}) - 2(S_{n+2} - S_{n+1}) + (S_{n+1} - S_n)$$

$$= T_{n+3} - 2T_{n+2} + T_{n+1}$$

$$= (T_{n+3} - T_{n+2}) - (T_{n+2} - T_{n+1})$$

$$= d - d = 0$$

$$47. \frac{a + be^y}{a - be^y} = \frac{b + ce^y}{b - ce^y} = \frac{c + de^y}{c - de^y}$$

Applying componendo and dividendo rule, we get

$$\frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$\Rightarrow a, b, c$ and d are in GP.

$$48. \text{ Given, } T_{m+n} = p \text{ and } T_{m-n} = q$$

$$\Rightarrow ar^{m+n-1} = p \quad \dots (i)$$

$$\text{and } ar^{m-n-1} = q \quad \dots (ii)$$

On multiplying Eqs. (i) and (ii), we get

$$a^2 r^{2m-2} = pq$$

$$\Rightarrow (ar^{m-1})^2 = pq \Rightarrow ar^{m-1} = \sqrt{pq}$$

$$\Rightarrow T_m = \sqrt{pq}$$

49. Since, $1/2 \operatorname{cosec}^2 \theta$, $2 \cot \theta$ and $\sec \theta$ are in GP.

$$\therefore \frac{1}{2} \operatorname{cosec}^2 \theta \cdot \sec \theta = 4 \cot^2 \theta$$

$$\Rightarrow \cos \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$$

50. Let a and r be the first term and the common ratio respectively of GP.

$$\text{Then, } T_{10} = 9 \Rightarrow ar^9 = 9$$

$$\text{and } T_4 = 4 \Rightarrow ar^3 = 4$$

$$\therefore T_7 = ar^6 = (ar^9 \cdot ar^3)^{1/2} = (9 \times 4)^{1/2} = 6$$

51. Let A and R be the first term and the common ratio respectively of given GP. Then,

$$T_4 = p \Rightarrow AR^3 = p$$

$$T_7 = q \Rightarrow AR^6 = q$$

$$\text{and } T_{10} = r \Rightarrow AR^9 = r$$

$$\text{Now, } (AR^3)(AR^9) = A^2 R^{12} = (AR^6)^2 \Rightarrow pr = q^2$$

52. Let a and r be the first term and the common ratio respectively of GP.

$$\text{Then, } a_6 = 32 \Rightarrow ar^5 = 32 \quad \dots(i)$$

$$\text{and } a_8 = 128 \Rightarrow ar^7 = 128 \quad \dots(ii)$$

$$\text{On dividing Eq. (ii) by Eq. (i), we get}$$

$$r^2 = 4 \Rightarrow r = \pm 2$$

53. Let r be the common ratio of a GP, then

$$\text{Area formed by three points}$$

$$= \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_1 r & y_1 r & 1 \\ x_1 r^2 & y_1 r^2 & 1 \end{vmatrix} = \frac{1}{2} x_1 y_1 \begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix} = 0$$

So, points are collinear, i.e. lie on a straight line.

54. Let $a^x = b^y = c^z = k$

$$\Rightarrow a = k^{1/x}, b = k^{1/y} \text{ and } c = k^{1/z}$$

$$\text{Since, } a, b \text{ and } c \text{ are in GP.}$$

$$\therefore b^2 = ac \Rightarrow (k^{1/y})^2 = k^{1/x} k^{1/z}$$

$$\Rightarrow k^{2/y} = k^{1/x + 1/z} \Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

$$\Rightarrow \frac{1}{x}, \frac{1}{y} \text{ and } \frac{1}{z} \text{ are in AP.}$$

$$\Rightarrow x, y \text{ and } z \text{ are in HP.}$$

55. Let d be the common difference of an AP and $R (\neq 0)$ be the common ratio of a GP. Then,

$$q = p + d, r = p + 2d \text{ and } y = xR, z = xR^2$$

$$\therefore q - r = -d, r - p = 2d \text{ and } p - q = -d$$

$$\therefore x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = x^{-d} \cdot (xR)^{2d} \cdot (xR^2)^{-d}$$

$$= (x^{-d} \cdot x^{2d} \cdot x^{-d}) (R^{2d} \cdot R^{-2d})$$

$$= (x^{-d+2d-d}) \cdot (R^{2d-2d})$$

$$= x^0 \cdot R^0 = 1 \times 1 = 1$$

56. Let the three numbers in GP be a, ar and ar^2 .

$$\therefore a + ar + ar^2 = 56 \quad [\text{given}] \dots(i)$$

$$\text{On subtracting 1, 7, 21 from the numbers, we get}$$

$$a - 1, ar - 7, ar^2 - 21, \text{ which are given to be in AP.}$$

$$\therefore (ar - 7) - (a - 1) = (ar^2 - 21) - (ar - 7)$$

$$\Rightarrow ar - a - 6 = ar^2 - ar - 14$$

$$\Rightarrow a - 2ar + ar^2 = 8 \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$3ar = 48 \Rightarrow a = \frac{16}{r} \quad \dots(iii)$$

On substituting $a = \frac{16}{r}$ in Eq. (i), we get

$$\frac{16}{r} + 16 + 16r = 56$$

$$\Rightarrow 16r^2 - 40r + 16 = 0$$

$$\Rightarrow 2r^2 - 5r + 2 = 0$$

$$\Rightarrow (r - 2)(2r - 1) = 0$$

$$\therefore r = 2, \frac{1}{2}$$

If $r = 2$, then from Eq. (iii), $a = 8$ and the numbers are 8, 16, 32.

If $r = \frac{1}{2}$, then from Eq. (iii), $a = 32$ and the numbers are 32, 16, 8.

57. Let the sides of the right angled triangle be a, ar and ar^2 , out of which ar^2 is the hypotenuse, then $r > 1$.

$$\text{Now, } a^2 r^4 = a^2 + a^2 r^2$$

$$\Rightarrow r^4 - r^2 - 1 = 0 \Rightarrow r^2 = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore r > 1$$

$$\therefore r^2 > 1$$

$$\Rightarrow r^2 = \frac{1 + \sqrt{5}}{2}$$

Let $\angle C$ be the greater acute angle.

$$\text{Now, } \cos C = \frac{a}{ar^2} = \frac{1}{r^2} = \frac{2}{1 + \sqrt{5}}$$

58. Let the three digits be a, ar and ar^2 .

According to the hypothesis,

$$100a + 10ar + ar^2 - 792 = 100ar^2 + 10ar + a$$

$$\Rightarrow a(1 - r^2) = 8 \quad \dots(i)$$

and $a, ar + 2, ar^2$ are in AP.

$$\therefore 2(ar + 2) = a + ar^2$$

$$\Rightarrow a(r^2 - 2r + 1) = 4 \quad \dots(ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{a(1 - r^2)}{a(r^2 - 2r + 1)} = \frac{8}{4}$$

$$\Rightarrow \frac{(1 + r)(1 - r)}{(r - 1)^2} = 2 \Rightarrow \frac{r + 1}{1 - r} = 2$$

$$\therefore r = 1/3$$

From Eq. (i), $a = 9$

Thus, digits are 9, 3, 1 and so the required number is 931.

59. Given a, b and c are in AP.

$$\Rightarrow 2b = a + c \quad \dots(i)$$

Also, given $a + 1, b, c$ are in GP and $a, b, c + 2$ are also in GP.

$$b^2 = (a + 1)c \quad \dots(ii)$$

$$\text{and } b^2 = a(c + 2) \quad \dots(iii)$$

On solving Eqs. (i), (ii) and (iii), we get

$$a = 8, b = 12 \text{ and } c = 16$$

$$\text{Hence, } b = 12$$

60. The given series is a GP with $a = 0.4$.

$$r = \frac{1}{100}$$

$$\therefore S = \frac{a}{1-r} = \frac{40}{99} \quad \left[\because S_{\infty} = \frac{a}{1-r} \right]$$

61. The given series is a GP with

$$a = e^{-x}, r = \frac{-e^{-2x}}{e^{-x}} = -e^{-x}$$

$$\therefore S = \frac{a}{1-r} = \frac{e^{-x}}{1+e^{-x}} = \frac{1}{1+e^x} \quad [\because -e^{-x} < 1, \text{ for } x > 0]$$

62. We have, $\frac{128r-a}{r-1} = 255 \Rightarrow \frac{256-a}{2-1} = 255 \quad [\because r = 2]$
- $$\Rightarrow 256 - a = 255 \Rightarrow a = 1$$

63. Clearly, $\frac{(x+2)^n - (x+1)^n}{(x+2) - (x+1)}$
- $$= (x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2 + \dots + (x+1)^{n-1}$$
- $\therefore$ Required sum $= (x+2)^n - (x+1)^n$
- $$[\because (x+2) - (x+1) = 1]$$

64. Here, $c = ar, e = ar^2, d = bs, f = bs^2$

$$\therefore \text{Area of triangle} = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ c & d & 1 \\ e & f & 1 \end{vmatrix}$$

$$= \frac{1}{2} ab(s-r)(s-1)(r-1)$$

65. Let S denotes the sum of all terms and S_1 denotes the sum of odd terms.

$$S_1 = \frac{a}{1-r^2} \quad (\text{i.e. sum of odd terms})$$

Given, $S = \frac{1}{5} \cdot S_1 \Rightarrow \frac{a}{1-r} = \frac{1}{5} \cdot \frac{a}{1-r^2}$

$$\Rightarrow 1+r = \frac{1}{5} \Rightarrow r = -\frac{4}{5}$$

66. The given series is a GP with $a = \sqrt{2}, r = \sqrt{3} > 1$

$$\therefore S_{10} = \frac{a(r^{10} - 1)}{r - 1} = \frac{\sqrt{2}[(\sqrt{3})^{10} - 1]}{\sqrt{3} - 1}$$

$$= \frac{\sqrt{2}}{2} (242)(\sqrt{3} + 1) = 121(\sqrt{6} + \sqrt{2})$$

67. $\sum_{n=1}^{49} (1.05)^n = 1.05 + (1.05)^2 + (1.05)^3 + \dots + (1.05)^{49}$
- $$= \frac{1.05[(1.05)^{49} - 1]}{1.05 - 1}$$
- $$= 212.16$$

68. Let a and r be the first term and the common ratio respectively of given infinite GP. Then,

$$ar = 2 \quad \dots (i)$$

Also, $\frac{a}{1-r} = 8 \quad \dots (ii)$

From Eq. (i), $r = \frac{2}{a}$

On putting the value of r in Eq. (ii), we get

$$a = 4$$

69. Let a and r be the first term and the common ratio respectively of given GP. Then,

$$\alpha = \sum_{n=1}^{100} a_{2n}$$

$$\Rightarrow \alpha = \frac{ar(1-r^{200})}{(1-r^2)} \quad \left[\because S_n = \frac{a(1-r^n)}{1-r} \right]$$

$$\Rightarrow \beta = \frac{a(1-r^{200})}{1-r^2}$$

$$\Rightarrow \frac{\alpha}{\beta} = r$$

70. Given, $f(x+y) = f(x)f(y), \forall x, y \in N$

For any $x \in N, f(x) = [f(1)]^x = 3^x \quad [\because f(1) = 3]$

$$\therefore \sum_{x=1}^n f(x) = 120$$

$$\Rightarrow \sum_{x=1}^n 3^x = 120$$

$$\Rightarrow 3^1 + 3^2 + 3^3 + \dots + 3^n = 120$$

$$\Rightarrow 3^n - 1 = 80$$

$$\Rightarrow 3^n = 81 = 3^4 \Rightarrow n = 4$$

71. $0.4\overline{23} = 0.4 + 0.023 + 0.00023 + \dots$

$$= \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \frac{23}{10^7} + \dots = \frac{419}{990}$$

72. Given, $x = 1 + y + y^2 + \dots$

$$\Rightarrow x = \frac{1}{1-y} \Rightarrow 1-y = \frac{1}{x}$$

$$\Rightarrow y = 1 - \frac{1}{x} = \frac{x-1}{x}$$

73. Let the GP be a, ar, ar^2 and ar^3 .

Then, $a(1+r+r^2+r^3)$

$$= a \times \left[\frac{1-r^4}{1-r} \right]$$

$$= 12(1-\sqrt{5})$$

$$= \frac{12(1+\sqrt{5})(1-\sqrt{5})}{1+\sqrt{5}}$$

$$= \frac{-48}{1+\sqrt{5}}$$

$$\Rightarrow \frac{a(1-25)}{1+\sqrt{5}} = \frac{-48}{1+\sqrt{5}}$$

$$\Rightarrow a = 2$$

74. Sum of areas of all the squares

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \dots$$

$$= \frac{a^2}{1-\frac{1}{2}} = 2a^2$$

75. Let a be the first term and r ($|r| < 1$) be the common ratio of infinite GP. Then,

$$\frac{a}{1-r} = 3 \text{ and } \frac{a^2}{1-r^2} = 3 \quad \dots (i)$$

Also, $\frac{a^2}{(1-r)^2} = 9 \quad \dots (ii)$

On solving Eqs. (i) and (ii), we get

$$a = \frac{3}{2}$$

and

$$r = \frac{1}{2}$$

Hence, the sequence corresponding to the series will

$$\text{be } \frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \dots$$

76. Given product = $x^{\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots} = x^{\frac{\left(\frac{1}{2}\right)}{1 - \left(\frac{1}{2}\right)}} = x$

77. $\therefore$ Length of a side of S_n = Length of a diagonal of S_{n+1}
 $\Rightarrow$ Length of a side of $S_n = \sqrt{2}$ (Length of a side of S_{n+1})
 $\Rightarrow \frac{\text{Length of a side of } S_{n+1}}{\text{Length of a side of } S_n} = \frac{1}{\sqrt{2}}, \forall n \geq 1$
 $\Rightarrow$ Side of $S_1, S_2, \dots, S_n$ form a GP with common ratio $\frac{1}{\sqrt{2}}$ and first term 10.

$$\therefore \text{Side of } S_n = 10 \left(\frac{1}{\sqrt{2}} \right)^{n-1} = \frac{10}{2^{\left(\frac{n-1}{2}\right)}}$$

$$\Rightarrow \text{Area of } S_n = (S_n)^2 = \frac{100}{2^{n-1}}$$

$$\therefore \text{Area of } S_n < 1 \quad [\text{given}]$$

$$\Rightarrow \frac{100}{2^{n-1}} < 1$$

$$\Rightarrow 2^{n-1} > 100$$

$$\Rightarrow n - 1 \geq 7$$

$$\Rightarrow n \geq 8$$

78. Let r be the common ratio of the given GP. Then, terms of GP are $\alpha, \alpha r, \alpha r^2$ and αr^3 .

According to the question,

$$\alpha(1+r) = 1, \alpha^2 r = p, \alpha r^2(1+r) = 4, \alpha^2 r^5 = q$$

On solving, we get $(p, q) = (-2, -32)$

79. Consider, $(1+x) + (1+x+x^2) + (1+x+x^2+x^3) + \dots$ upto n terms

$$= \frac{1-x^2}{1-x} + \frac{1-x^3}{1-x} + \frac{1-x^4}{1-x} + \dots \text{ upto } n \text{ terms}$$

$$= \frac{1}{1-x} [(1+1+1+\dots \text{ to } n \text{ terms})$$

$$- (x^2 + x^3 + x^4 + \dots \text{ upto } n \text{ terms})]$$

$$= \frac{1}{1-x} \left[n - \frac{x^2(1-x^n)}{1-x} \right]$$

80. We have, $9 + 99 + 999 + \dots$ upto n terms

$$= (10-1) + (10^2-1) + (10^3-1) + \dots \text{ upto } n \text{ terms}$$

$$= (10 + 10^2 + 10^3 + \dots \text{ upto } n \text{ terms}) - n$$

$$= \frac{10(10^n - 1)}{10 - 1} - n$$

$$= \frac{10(10^n - 1)}{9} - n$$

$$= \frac{1}{9} (10^{n+1} - 9n - 10)$$

81. Let $S_n = 0.5 + 0.55 + 0.555 + \dots$ upto n terms

$$= \frac{5}{9} [0.9 + 0.99 + 0.999 + \dots \text{ upto } n \text{ terms}]$$

$$= \frac{5}{9} \left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{100}\right) + \left(1 - \frac{1}{1000}\right) + \dots \right]$$

$$= \frac{5}{9} \left[(1 + 1 + 1 + \dots) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots \right) \right]$$

$$= \frac{5}{9} \left[n - \frac{1}{10} \cdot \frac{\left(1 - \frac{1}{10^n}\right)}{1 - \frac{1}{10}} \right] = \frac{5}{9} \left[n - \frac{1}{10} \times \frac{10}{9} \left(1 - \frac{1}{10^n}\right) \right]$$

$$= \frac{5}{9} \left[n - \frac{1}{9} \left(1 - \frac{1}{10^n}\right) \right]$$

82. Let the three digits be a, ar and ar^2 .

$$\text{Given, } a + ar^2 = 2ar + 1$$

$$\Rightarrow a(r^2 - 2r + 1) = 1$$

$$\Rightarrow a(r-1)^2 = 1 \quad \dots (i)$$

$$\text{Also, given } a + ar = \frac{2}{3}(ar + ar^2)$$

$$\Rightarrow 3a(1+r) = 2ra(1+r)$$

$$\Rightarrow (1+r)(3-2r) = 0 \quad [\because a \neq 0]$$

$$\therefore r = \frac{3}{2}, -1$$

$$\text{If } r = \frac{3}{2}, \text{ then from Eq. (i), } a = \frac{1}{(r-1)^2} = \frac{1}{\left(\frac{3}{2}-1\right)^2} = 4$$

If $r = -1$, then from Eq. (i), $a = \frac{1}{4}$, which is not possible, as a is an integer.

$$\text{Hence, } a = 4, ar = 4 \cdot \frac{3}{2} = 6 \text{ and } ar^2 = 4 \cdot \frac{9}{4} = 9$$

83. Let the two numbers be a and b , then

$$G = \sqrt{ab} \text{ or } G^2 = ab$$

Also, p and q are two AM's between a and b .

$\therefore a, p, q$ and b are in AP.

$$\text{Now, } p - a = q - p \text{ and } q - p = b - q$$

$$\therefore a = 2p - q \text{ and } b = 2q - p$$

$$\text{Hence, } G^2 = ab = (2p - q)(2q - p)$$

84. Let the $3n$ terms of a GP be $a, ar, ar^2, \dots, ar^{n-1}, ar^n, ar^{n+1}, ar^{n+2}, \dots, ar^{2n-1}, ar^{2n}, ar^{2n+1}, ar^{2n+2}, \dots, ar^{3n-1}$.

$$\text{Then, } S_1 = a + ar + ar^2 + \dots + ar^{n-1}$$

$$= \frac{a(1-r^n)}{1-r}$$

$$S_2 = ar^n + ar^{n+1} + ar^{n+2} + \dots + ar^{2n-1}$$

$$= \frac{ar^n(1-r^n)}{1-r}$$

$$S_3 = ar^{2n} + ar^{2n+1} + ar^{2n+2} + \dots + ar^{3n-1}$$

$$= \frac{ar^{2n}(1-r^n)}{1-r}$$

$$\text{Now, } (S_2)^2 = a^2 r^{2n} \frac{(1-r^n)^2}{(1-r)^2}$$

$$= \frac{a(1-r^n)}{1-r} \cdot ar^{2n} \frac{(1-r^n)}{1-r} = S_1 S_3$$

Hence, S_1, S_2 and S_3 are in GP.

$$85. (32)(32)^{1/6}(32)^{1/36} \dots = 32^{1+\frac{1}{6}+\frac{1}{36}+\dots} = 32^{\left(\frac{1}{1-\frac{1}{6}}\right)} = 32^{6/5} = (2^5)^{6/5} = 2^6 = 64$$

$$86. \text{ Let } S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \dots$$

$$\Rightarrow \frac{1}{5}S = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \dots$$

On subtracting, we get

$$\frac{4}{5}S = 1 + \frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \dots$$

$$\Rightarrow S = \frac{7}{4} \times \frac{5}{4} = \frac{35}{16}$$

$$87. \text{ Let } S = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$xS = x + 2x^2 + 3x^3 + \dots$$

On subtracting, we get

$$(1-x)S = 1 + x + x^2 + x^3 + \dots$$

$$\Rightarrow (1-x)S = \frac{1}{1-x} \Rightarrow S = \frac{1}{(1-x)^2}$$

$$88. \text{ Let } S = 1 + 2 \cdot 2 + 3 \cdot 2^2 + 4 \cdot 2^3 + \dots + 100 \cdot 2^{99}$$

$$2S = 1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + 99 \cdot 2^{99} + 100 \cdot 2^{100}$$

On subtracting, we get

$$-S = 1 + (1 \cdot 2 + 1 \cdot 2^2 + 1 \cdot 2^3 + \dots \text{ to } 99 \text{ terms} - 100 \cdot 2^{100})$$

$$= 1 + 2(2^{99} - 1) - 100 \cdot 2^{100}$$

$$\Rightarrow S = 99 \cdot 2^{100} + 1$$

$$89. \text{ Here, } T_n = \frac{\Sigma n^3}{\frac{n}{2}[2 \times 1 + (n-1)2]} = \frac{\frac{n^2(n+1)^2}{4}}{\frac{n^2}{2}} = \frac{n(n+1)}{2}$$

$$\Rightarrow S_{16} = \frac{1}{4} \sum_{k=1}^{16} (k+1)^2 = 446$$

$$90. T_n = \frac{2 \cdot 2}{\Sigma n^3} = \frac{4}{\frac{n^2(n+1)^2}{4}} = \frac{16}{n^2(n+1)^2}$$

$$= \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore S_n = 1 - \frac{1}{n+1} = \frac{n}{n+1}$$

$$91. \text{ We can write } S \text{ as}$$

$$S = (1-2)(1+2) + (3-4)(3+4) + \dots + (2001-2002)(2001+2002) + 2003^2$$

$$= -[1+2+3+4+\dots+2002] + 2003^2$$

$$= -\frac{1}{2}(2002)(2003) + (2003)^2 = 2007006$$

$$92. \text{ According to the given condition,}$$

$$(704)(2) \left\{ 1 - \frac{1}{2^n} \right\} = \frac{1984(2)}{3} \left\{ 1 - \frac{(-1)^n}{2^n} \right\}$$

$$\Rightarrow 128 = \frac{2112}{2^n} - 1984 \frac{(-1)^n}{2^n}$$

If n is odd, then we get $2^n = 32 \Rightarrow n = 5$

If n is even, then we get $128 = \frac{128}{2^n} \Rightarrow n = 0$

$$93. \text{ We have, } 2^{n+10} = 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \dots + n \cdot 2^n$$

$$\Rightarrow 2(2^{n+10}) = 2 \cdot 2^3 + 3 \cdot 2^4 + \dots + (n-1) \cdot 2^n + n \cdot 2^{n+1}$$

On subtracting, we get

$$-2^{n+10} = 2 \cdot 2^2 + 2^3 + 2^4 + \dots + 2^n - n \cdot 2^{n+1}$$

$$= 8 + \frac{8(2^{n-2} - 1)}{2-1} - n \cdot 2^{n+1}$$

$$= 8 + 2^{n+1} - 8 - n \cdot 2^{n+1} = 2^{n+1} - (n)2^{n+1}$$

$$\Rightarrow 2^{10} = 2n - 2$$

$$\Rightarrow n = 513$$

$$94. \text{ Let } S = (1)(2003) + (2)(2002) + (3)(2001) + \dots + (2003)(1)$$

$$\text{ and } K = 1^2 + 2^2 + 3^2 + \dots + 2003^2$$

On adding, we get

$$S + K = (2004)[1 + 2 + 3 + \dots + 2003]$$

$$\Rightarrow (2003)(334)(x) + (2003)(4007)(334) = (2004)(1002)(2003)$$

$$\Rightarrow x = 2005$$

$$95. \text{ We can write the given equation as}$$

$$\log_2 x^{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\dots} = 4$$

$$\Rightarrow \log_2(x^2) = 4 \Rightarrow x^2 = 2^4 \Rightarrow x = 4$$

$$96. \text{ We shall make use of the identity}$$

$$(a_1^2 + a_2^2 + \dots + a_m^2)(b_1^2 + b_2^2 + \dots + b_m^2) - (a_1b_1 + a_2b_2 + \dots + a_mb_m)^2$$

$$= (a_1b_2 - a_2b_1)^2 + (a_1b_3 - a_3b_1)^2 + \dots + (a_{m-1}b_m - a_mb_{m-1})^2$$

Thus, $(x_1^2 + x_2^2 + \dots + x_{n-1}^2)(x_2^2 + x_3^2 + \dots + x_n^2) - (x_1x_2 + x_2x_3 + \dots + x_{n-1}x_n)^2 \leq 0$

$$\Rightarrow (x_1x_3 - x_2x_2)^2 + (x_1x_4 - x_3x_3)^2 + \dots + (x_{n-2}x_n - x_{n-1}x_{n-1})^2 \leq 0$$

As $x_1, x_2, \dots, x_n$ are real, this is possible if and only if

$$x_1x_3 - x_2^2 = x_2x_4 - x_3^2 = \dots = x_{n-2}x_n - x_{n-1}^2 = 0$$

$$\Rightarrow \frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_4} = \dots = \frac{x_n}{x_{n-1}}$$

$\Rightarrow x_1, x_2, \dots, x_n$ are in GP.

$$97. \text{ We have,}$$

$$\sum_{r=1}^n (2r-1)^4 = \sum_{r=1}^{2n} r^4 - \sum_{r=1}^n (2r)^4$$

$$= f(2n) - 16f(n)$$

$$98. \text{ We have, } \frac{1}{2r^2} = \frac{2}{4r^2} = \frac{(2r+1) - (2r-1)}{1 + (2r+1)(2r-1)}$$

$$\tan^{-1}\left(\frac{1}{2r^2}\right) = \tan^{-1}\left\{\frac{(2r+1) - (2r-1)}{1 + (2r+1)(2r-1)}\right\}$$

$$= \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$$

$$\Rightarrow \sum_{r=1}^n \tan^{-1}\left(\frac{1}{2r^2}\right) = \sum_{r=1}^n \{\tan^{-1}(2r+1) - \tan^{-1}(2r-1)\}$$

$$= \tan^{-1}(2n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}(2n+1) - \frac{\pi}{4}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \sum_{r=1}^n \tan^{-1}\left(\frac{1}{2r^2}\right) = \lim_{n \rightarrow \infty} \left\{ \tan^{-1}(2n+1) - \frac{\pi}{4} \right\}$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

99. $1^3 = 1 \cdot (1-1) + 1$
 $2^3 = (2 \cdot 1 + 1) + 1$
 $3^3 = (3 \cdot 2 + 1) + 9 + 11$
 $4^3 = (4 \cdot 3 + 1) + 15 + 17 + 19$, etc
 $\therefore n^3 = \{n \cdot (n-1) + 1\} + \dots$, in which next term being 2 more than the previous
 $\therefore n^3 = (n^2 - n + 1) + (n^2 - n + 3) + \dots + (n^2 + n - 1)$

100. $\sum_{r=1}^n \frac{r^2 - r - 1}{(r+1)!} = \sum_{r=1}^n \left(\frac{r-1}{r!} - \frac{r}{(r+1)!} \right) = \frac{-n}{(n+1)!}$
 $= -\frac{n}{(n+1)n \cdot (n-1)!}$
 $= -\frac{1}{(n+1)(n-1)!}$

101. Applying AM $\geq$ GM

Since, AM = GM

$\therefore (\sqrt{2} + \sqrt{2})^x = (\sqrt{2} - \sqrt{2})^x$ i.e. for $x = 0$

[$\because$ AM = GM iff $a = b$]

102. As, $D < 0$

$\Rightarrow (a+b+c)^2 - 4(ab+bc+ca) < 0$
 $\Rightarrow (a-b+c)^2 < 4ac$
 $\Rightarrow -2\sqrt{ac} < a-b+c$
 $\Rightarrow (\sqrt{a} + \sqrt{c}) > \sqrt{b}$

103. Given $\frac{1}{16}$, a and b are in GP.

$\Rightarrow a^2 = \frac{b}{16} \dots (i)$

Also, a , b and $\frac{1}{6}$ are in HP.

$\Rightarrow b = \frac{2 \times a \times \frac{1}{6}}{a + \frac{1}{6}} = \frac{2a}{6a+1} \dots (ii)$

On solving Eqs. (i) and (ii), we get

$a = \frac{1}{12}, -\frac{1}{4}$

and $b = \frac{1}{9}, 1$

104. Since, $a(b-c) + b(c-a) + c(a-b) = 0$

$\therefore x = 1$ is a root of the given equation.

Since, both the roots are equal, therefore the other root is also 1.

$\therefore$ Their sum = $1+1=2$

$\Rightarrow \frac{-b(c-a)}{a(b-c)} = 2$

$\Rightarrow -bc + ab = 2ab - 2ac$

$\Rightarrow 2ac = b(a+c)$

$\therefore b = \frac{2ac}{a+c}$

Hence, a , b and c are in HP.

105. Let two numbers be a and b . Then,

$\frac{a+b}{2} = 5 \Rightarrow a+b=10$

Also, $\sqrt{ab} = 4 \Rightarrow ab=16$

$\therefore$ HM between a and $b = \frac{2ab}{a+b} = \frac{2(16)}{10} = \frac{16}{5}$

106. Given, $\log(a+c)$, $\log(c-a)$ and $\log(a-2b+c)$ are in AP.

$\Rightarrow 2\log(c-a) = \log(a+c) + \log(a-2b+c)$

$\Rightarrow (c-a)^2 = (a+c)(a-2b+c)$

$\Rightarrow (c-a)^2 = (a+c)^2 - 2b(a+c)$

$\Rightarrow 2b(a+c) = (a+c)^2 - (c-a)^2 = 4ac$

$\Rightarrow b = \frac{2ac}{a+c}$

$\Rightarrow a, b$ and c are in HP.

107. Given, $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in HP.

$\Rightarrow \cos x = \frac{2\cos(x-y)\cos(x+y)}{\cos(x-y) + \cos(x+y)}$

$\Rightarrow \cos x = \frac{2(\cos^2 x - \sin^2 y)}{2\cos x \cos y}$

$\Rightarrow \cos^2 x = 1 + \cos y = 2\cos^2 \frac{y}{2}$

$\Rightarrow \cos^2 x \sec^2 \frac{y}{2} = 2$

$\Rightarrow \cos x \sec \frac{y}{2} = \pm \sqrt{2}$

108. $\log_2 6 = \log_2(3 \times 2) = \log_2 3 + \log_2 2 = 1 + \log_2 3$

and $\log_2 12 = \log_2(2^2 \times 3)$

$= \log_2 3 + 2\log_2 2$

$= 2 + \log_2 3$

Since, $\log_2 3, 1 + \log_2 3, 2 + \log_2 3$ are in AP.

$\Rightarrow \log_2 3, \log_2 6, \log_2 12$ are in AP.

$\Rightarrow \log_3 2, \log_6 2, \log_{12} 2$ are in HP.

109. Given a, b and c are in HP $\Rightarrow b$ is HM of a and c

$\Rightarrow \frac{a+c}{2} > b$ [$\because$ AM $>$ HM]

Now, $\frac{a^2+c^2}{2} > \left(\frac{a+c}{2}\right)^2$

$\Rightarrow \frac{a^2+c^2}{2} > b^2$

$\Rightarrow a^2+c^2 > 2b^2$

110. $\therefore$ AM $>$ GM $>$ HM

$\therefore x > y > z \Rightarrow z < y < x$

111. Given a, b and c are in HP.

$\Rightarrow \frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

$\Rightarrow \frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

$\Rightarrow \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$

Hence, the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ passes through a fixed point $(1, -2)$.

112. Given, $a_{10} = 3 \Rightarrow a_1 + 9d = 3$

$\Rightarrow 2 + 9d = 3$

[$\because a_1 = 2$]

$\Rightarrow d = \frac{1}{9}$

$\therefore a_4 = a_1 + 3d = 2 + \frac{1}{3} = \frac{7}{3}$

$h_{10} = 3 \Rightarrow \frac{1}{h_{10}} = \frac{1}{3}$

$$\Rightarrow D = -\frac{1}{54}$$

$$\therefore \frac{1}{h_7} = \frac{1}{h_1} + 6D$$

$$= \frac{1}{2} - \frac{1}{9}$$

$$= \frac{7}{18}$$

$$\Rightarrow h_7 = \frac{18}{7}$$

$$\therefore a_4 h_7 = \frac{7}{3} \times \frac{18}{7} = 6$$

113. Since, x, y and z are in HP.

$$\therefore y = \frac{2xz}{x+z}$$

$$\Rightarrow x - 2y + z = x + z - \frac{4xz}{x+z}$$

$$= \frac{(x+z)^2 - 4xz}{x+z}$$

$$= \frac{(z-x)^2}{x+z}$$

$$\Rightarrow (x+z)(x-2y+z) = (z-x)^2$$

$$\Rightarrow \log(x+z) + \log(x-2y+z) = 2\log(z-x)$$

114. Let $\frac{1}{H_{i+1}} - \frac{1}{H_i} = K$

$$\therefore \sum_{i=1}^{2n} (-1)^i \left(\frac{H_i + H_{i+1}}{H_i - H_{i+1}} \right)$$

$$= \sum_{i=1}^{2n} \frac{(-1)^i}{K} \left(\frac{1}{H_{i+1}} + \frac{1}{H_i} \right) = 2n$$

115. Let the given AP be $a_i, a_i + d, a_i + 2d, \dots$

By substituting the value, we can find that only options (b) and (d) are correct.

116. Let the AP be $a, a + d, a + 2d, a + 3d, \dots$

$$\text{Given that, } 3a + 3d = 9$$

$$\Rightarrow a + d = 3$$

$$\text{and } a^2 + (a+d)^2 + (a+2d)^2 = 35$$

$$\Rightarrow a^2 + (3)^2 + (3+d)^2 = 35$$

$$\Rightarrow (3-d)^2 + 3^2 + (3+d)^2 = 35$$

$$\Rightarrow d = \pm 2$$

$$\text{and } a = 1, 5$$

$$\therefore S_n = \frac{n}{2} [2a + (n-1)d]$$

where $d = 2, a = 1$

$$\Rightarrow S_n = n^2$$

When $d = -2, a = 5$, then $S_n = n(6-n)$

117. $\cot^{-1} \left[\frac{1+a_1 a_2}{a_2 - a_1} \right] + \cot^{-1} \left[\frac{1+a_2 a_3}{a_3 - a_2} \right]$

$$+ \cot^{-1} \left[\frac{1+a_3 a_4}{a_4 - a_3} \right] + \dots + \cot^{-1} \left[\frac{1+a_n a_{n-1}}{a_n - a_{n-1}} \right]$$

$$= \cot^{-1} a_1 - \cot^{-1} a_2 + \cot^{-1} a_2 - \cot^{-1} a_3$$

$$+ \dots + \cot^{-1} a_{n-1} - \cot^{-1} a_n$$

$$= \cot^{-1} a_1 - \cot^{-1} a_n = \tan^{-1} a_n - \tan^{-1} a_1$$

118. If a, b and c are p th, q th and r th terms of an AP, then $\frac{b-c}{a-b}$ is always a rational number.

119. Let the roots be a_1, a_2, a_3, a_4, a_5 , then

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_4} + \frac{1}{a_5} = 10$$

$$\Rightarrow \frac{\sum a_1 a_2 a_3 a_4}{a_1 a_2 a_3 a_4 a_5} = 10$$

$$\Rightarrow 10S = \sum a_1 a_2 a_3 a_4$$

Since, a_1, a_2, a_3, a_4, a_5 are in GP, then consider a_1, a_2, a_3, a_4 and a_5 as a, ar, ar^2, ar^3, ar^4

$$\Rightarrow 10S = 10a^5 r^{10} = a^3 r^6 (a + ar + ar^2 + ar^3 + ar^4)$$

$$\Rightarrow 10a^5 r^{10} = a^3 r^6 (40) \quad [\because \text{sum of roots} = 40]$$

$$\Rightarrow a^2 r^4 = 4$$

$$\Rightarrow ar^2 = \pm 2$$

$$\Rightarrow S = a^3 r^6 (4) = (\pm 2)^3 (4) = \pm 32$$

120. Since, $a_1, a_2, a_3, \dots, a_n, \dots$ are in AP and $b_1, b_2, \dots, b_n, \dots$ are in GP.

$$\Rightarrow 1, (1+d), \dots, [1+(n-1)d] \dots \text{ are in AP}$$

$$\Rightarrow 1, r, r^2, \dots, r^{n-1}, \dots \text{ are in GP.}$$

Also, $a_9 = b_9 \Rightarrow 1 + 8d = r^8$

$$\therefore S_9 = \frac{9}{2} [2 + (9-1)d] = 369$$

$$\Rightarrow 2 + 8d = 41 \times 2$$

$$\Rightarrow d = 10$$

$$\Rightarrow r^8 = 81$$

$$\Rightarrow r^8 = (\sqrt{3})^8$$

$$\Rightarrow r = \sqrt{3}$$

$$\therefore b_7 = (\sqrt{3})^6 = 9 \times 3 = 27$$

and $b_9 = 81$

121. Given, $\sin \beta = \sqrt{\sin \alpha \cos \alpha}$

$$\Rightarrow \sin^2 \beta = \sin \alpha \cos \alpha$$

Now, $\cos 2\beta = 1 - 2\sin^2 \beta = 1 - 2\sin \alpha \cos \alpha$

$$= (\sin \alpha - \cos \alpha)^2$$

$$= 2\sin^2 \left(\frac{\pi}{4} - \alpha \right)$$

or $2\cos^2 \left(\frac{\pi}{4} + \alpha \right)$

122. Since, a is AM between 1st and $(2n+1)$ th terms, b is GM between 1st and $(2n+1)$ th terms and c is HM between 1st and $(2n+1)$ th terms.

$$\Rightarrow AM \geq GM \geq HM \quad \text{and} \quad AM \times HM = (GM)^2$$

123. If each of x, y and z is less than 1, then Statement I is obviously true.

Also, $1 - 2x + 1 - 2y + 1 - 2z = 3 - 2(x + y + z) = 1$,
The sum of the three given numbers is positive also at must one of x, y and z can be more than 1.

If one of x, y and z is more than or equal to $\frac{1}{2}$, then their product is less than equal to zero, hence still remains true.

Statement II is always true but it does not explain Statement I.

124. Since, $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ have common root.

$$(a_1c - ac_1)^2 = 4(ab_1 - ba_1)(bc_1 - cb_1) \quad \dots(i)$$

If $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in AP, then

$$\frac{ba_1 - ab_1}{a_1b_1} = \frac{cb_1 - c_1b}{b_1c_1} = k$$

and $\frac{ca_1 - ac_1}{a_1c_1} = 2k$

On putting these values in Eq. (i), we get

$$c_1a_1 = b_1^2$$

125. Statement I is false, since each term of the series $\frac{1}{10^6} + \frac{1}{10^6} + \frac{1}{10^6} + \frac{1}{10^6} + \dots$ is smaller than 10^{-5} but its sum upto infinity is infinity.

Statement II is true, since $\lim_{n \rightarrow \infty} \frac{n}{10^5}$ is not finite as $n \rightarrow \infty$

126. Statement I can be proved by taking the intersection of the inequalities.

$$a > 0, ar > 0, ar^2 > 0$$

$$\text{at } a + ar > ar^2, ar + ar^2 > a, ar^2 + a > ar$$

The inequalities follow from reason.

127. If we could show that Statement II is true, then Statement I will be false. Indeed if Statement II is false, then

$$\sqrt{2} - \sqrt{3} = (p - q)d \quad \dots(i)$$

$$\text{and } \sqrt{3} - \sqrt{5} = (q - r)d \quad \dots(ii)$$

On dividing Eq. (i) by Eq. (ii) we get

$$\frac{\sqrt{2} - \sqrt{3}}{\sqrt{3} - \sqrt{5}} = \frac{p - q}{q - r}$$

$$\Rightarrow \text{Rational} = \text{Irrational}$$

Hence, Statement I is false and Statement II is true.

128. Statement I is true, since for any $x > 0$, we can choose sufficiently larger n such that $\frac{x^n}{n!}$ is small. Statement II is

false, since $\frac{(n!)^2}{n!}$ contains $n!$ in the denominator but diverges to ∞ .

129. We can show that Statement I is true and follows from Statement II. Indeed

$$\begin{aligned} S_{n+1} - S_n &= \frac{a_1 + a_2 + \dots + a_n + a_{n+1}}{n+1} - \frac{a_1 + a_2 + \dots + a_n}{n} \\ &= \frac{na_{n+1} - (a_1 + a_2 + \dots + a_n)}{n(n+1)} \\ &= \frac{(a_{n+1} - a_1) + (a_{n+1} - a_2) + \dots + (a_{n+1} - a_n)}{n(n+1)} > 0 \\ &[\because a_{n+1} > a_1, a_{n+1} > a_2, \dots, \text{etc.}] \end{aligned}$$

Solutions (Q. Nos. 130-132)

$$\text{Given, } a(b-c)x^2 + b(c-a)x + c(a-b) = 0 \quad \dots(i)$$

Since, $a(b-c) + b(c-a) + c(a-b) = 0$, therefore $x = 1$ is a root of Eq. (i).

Let other root be α , then

$$\begin{aligned} 1 \times \alpha &= \frac{c(a-b)}{a(b-c)} \\ &= \frac{c \left(a - \frac{2ac}{a+c} \right)}{a \left(\frac{2ac}{a+c} - c \right)} = 1 \end{aligned}$$

$$\therefore \alpha = 1$$

Hence, both roots of Eq. (i) are 1, 1.

Then, roots of $x^2 - Px + Q = 0$ are also 1, 1.

$$\text{Now, } 1 + 1 = P, 1 \cdot 1 = Q$$

$$\therefore P = 2, Q = 1$$

$$\therefore [P] = [2] = 2$$

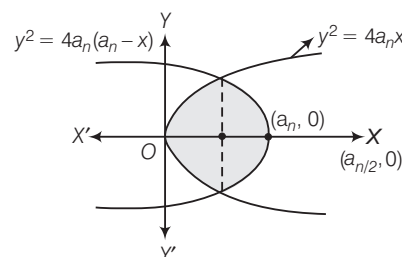
$$\text{Now, } [2P - Q] = [4 - 1] = 3$$

$\therefore$ Roots of Eq. (i) are 1, 1, then

$$1 + 1 = -\frac{b(c-a)}{a(b-c)}$$

$$\Rightarrow \frac{a(b-c)}{b(c-a)} = -\frac{1}{2}$$

$$133. A_n = 2 \int_0^{a_n/2} 2\sqrt{a_n} \sqrt{x} dx + 2 \int_{a_n/2}^{a_n} 2\sqrt{a_n} \sqrt{a_n - x} dx$$



$$\begin{aligned} &= 4\sqrt{a_n} \cdot \frac{2}{3} \left[x^{3/2} \right]_0^{a_n/2} - 4\sqrt{a_n} \cdot \frac{2}{3} \left[(a_n - x)^{3/2} \right]_{a_n/2}^{a_n} \\ &= \frac{8}{3} \sqrt{a_n} \left(\frac{a_n \sqrt{a_n}}{2\sqrt{2}} \right) - \frac{8}{3} \sqrt{a_n} \left(0 - \frac{a_n \sqrt{a_n}}{2\sqrt{2}} \right) \\ &= \frac{2 \cdot 8 a_n^2}{3 \cdot 2\sqrt{2}} = \frac{4\sqrt{2}}{3} a_n^2 \text{ sq units} \end{aligned}$$

134. We have, $y^2 = 4x$ and $y^2 = 4 - 4x$

$$\Rightarrow 8x = 4$$

$$\Rightarrow x = \frac{1}{2}$$

$$\text{and } y = \pm \sqrt{2}$$

135. For $a_n = 1$,

$$A_n = \frac{4\sqrt{2}}{3} \text{ sq units}$$

136. $a^a b^b c^c = (aa \dots a \text{ times}) (bb \dots b \text{ times}) (cc \dots c \text{ times})$

Applying AM-GM inequality,

$$\begin{aligned} &(a^a b^b c^c)^{1/n} \\ &\leq \frac{[a + a + \dots + a \text{ times}] + [b + b + \dots + b \text{ times}] + [c + c + \dots + c \text{ times}]}{n} \\ &= \frac{a^2 + b^2 + c^2}{n} \end{aligned}$$

Similarly, $(a^b b^c c^a)^{1/n} \leq \frac{ab + bc + ca}{n} \leq \frac{a^2 + b^2 + c^2}{n}$

Also, $(a^c b^a c^b)^{1/n} \leq \frac{ac + ba + cb}{n} \leq \frac{a^2 + b^2 + c^2}{n}$

So, $(a^a b^b c^c)^{1/n} + (a^b b^c c^a)^{1/n} + (a^c b^a c^b)^{1/n} \leq \frac{(a+b+c)^2}{n} = n$

Further, $a < n \Rightarrow a^a < a^n$
 $\therefore (a^a b^b c^c) < (abc)^n \Rightarrow (a^a b^b c^c)^{1/n} < (abc)$

137. A. $\therefore A = G + \frac{3}{2}$... (i)

and $G = H + \frac{6}{5}$... (ii)

$\therefore G^2 = AH$

$\Rightarrow G^2 = \left(G + \frac{3}{2}\right)\left(G - \frac{6}{5}\right)$ [from Eqs. (i) and (ii)]

$\therefore G = 6, A = \frac{15}{2}$ [from Eq. (i)]

Since, a and b are the roots of $x^2 - 15x + 36 = 0$.

On solving, we get $a = 12, b = 3$ or $a = 3, b = 12$

$\therefore \alpha = 15, \beta = 9$

$\Rightarrow \alpha + \beta^2 = 15 + 81 = 96$

and $\alpha^2 + \beta = 225 + 9 = 234$

B. $\therefore A = G + 2$... (i)

and $H = \frac{1}{5}b$ [if $b > a$]

$\therefore G^2 = AH = \frac{bA}{5}$

$\Rightarrow 5ab = bA$

$\Rightarrow A = 5a$

$\Rightarrow \frac{a+b}{2} = 5a$

$\Rightarrow b = 9a$

From Eq. (i), $5a = \sqrt{ab} + 2$

$\Rightarrow 5a = 3a + 2$

$\therefore a = 1$ and $b = 9$

$\Rightarrow \alpha = a + b = 10$

and $\beta = |a - b| = 8$

$\therefore \alpha + \beta^2 = 10 + 64 = 74$

C. $\therefore H = 4$

$\Rightarrow 2A + G^2 = 27$

$\Rightarrow 2A + AH = 27$ [$\because G^2 = AH$]

$\Rightarrow 6A = 27$

$\Rightarrow A = \frac{9}{2}$

and $G^2 = \frac{9}{2} \times 4 = 18$

Since, a and b are the roots of $x^2 - 9x + 18 = 0$.

$\therefore a = 6, b = 3$ or $a = 3, b = 6$

Now, $\alpha = a + b = 9, \beta = |a - b| = 3$

$\therefore \alpha^2 + \beta = 81 + 3 = 84$

and $1 + \left(\frac{\alpha}{\beta}\right) + \left(\frac{\alpha}{\beta}\right)^2 + \left(\frac{\alpha}{\beta}\right)^3 + \left(\frac{\alpha}{\beta}\right)^4 + \left(\frac{\alpha}{\beta}\right)^5$

$= 1 + 3 + 3^2 + 3^3 + 3^4 + 3^5$

$= 364$

138. Let us add one more number a_{n+1} to the given sequence.

The number a_{n+1} is such that $|a_{n+1}| = |a_n + 1|$. On squaring all the numbers, we have

$a_1^2 = 0$

$a_2^2 = a_1^2 + 2a_1 + 1$

$a_3^2 = a_2^2 + 2a_2 + 1$

$a_4^2 = a_3^2 + 2a_3 + 1$

$\vdots$

$a_n^2 = a_{n-1}^2 + 2a_{n-1} + 1$

$a_{n+1}^2 = a_n^2 + 2a_n + 1$

On adding the above equalities, we get

$a_1^2 + a_2^2 + \dots + a_n^2 + a_{n+1}^2$

$= a_1^2 + a_2^2 + \dots + a_n^2$

$+ 2(a_1 + a_2 + \dots + a_n) + n$

$\Rightarrow 2(a_1 + a_2 + \dots + a_n) = -n + a_{n+1}^2 \geq -n$

$\Rightarrow \frac{a_1 + a_2 + \dots + a_n}{n} \geq -\frac{1}{2} = -\frac{\lambda}{\mu}$

$\therefore \lambda + \mu = 3$

139. Let a be the first term and d be the common difference of an AP. Again, let x, y, z be the $(m+1)$ th, $(n+1)$ th and $(r+1)$ th terms of an AP. Then, $x = a + md, y = a + nd$ and $z = a + rd$.

Since, x, y and z are in GP.

$\therefore y^2 = xz$ i.e. $(a + nd)^2 = (a + rd)(a + md)$

So, $\frac{d}{a} = \frac{r + m - 2n}{n^2 - rm}$

Now, m, n and r are in HP.

$\Rightarrow \frac{2}{n} = \frac{1}{m} + \frac{1}{r}$

$\Rightarrow \frac{2}{n} = \frac{m+r}{mr}$

Hence, $\frac{d}{a} = \frac{2\left(\frac{r+m}{2} - n\right)}{n\left(n - \frac{rm}{n}\right)} = \frac{2\left(\frac{mr}{n} - n\right)}{n\left(n - \frac{rm}{n}\right)} = \frac{-2}{n} = -\frac{\lambda}{n}$

$\Rightarrow \lambda = 2$

140. Let $f(x) = x^4 + ax^3 + bx^2 + cx + 1$

As a, b and c are non-negative, no root of the equation $f(x) = 0$ can be positive. Further as $f(0) \neq 0$, all the roots of the equation, say x_1, x_2, x_3 and x_4 are negative.

We have,

$\sum x_1 = -a, \sum x_1 x_2 = b, \sum x_1 x_2 x_3 = -c$ and $x_1 x_2 x_3 x_4 = 1$

Using $AM \geq GM$ for positive numbers $-x_1, -x_2, -x_3$ and $-x_4$, we get

$\frac{a}{4} \geq 1 \Rightarrow a \geq 4$

Using $AM \geq GM$ for positive numbers $x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4$ and $x_3 x_4$, we get

$\frac{b}{6} \geq 1 \Rightarrow b \geq 6$

Finally, using $AM \geq GM$ for positive numbers

$-x_1 x_2 x_3, -x_1 x_2 x_4, -x_1 x_3 x_4$ and $-x_2 x_3 x_4$, we get

$\frac{c}{4} \geq 1 \Rightarrow c \geq 4$

$\therefore \text{HCF of } \{a, b, c\} = \text{HCF of } \{4, 6, 4\} = 2$

141. Let the three consecutive terms be
 $a-d, a, a+d$, where $d > 0$
 Then, $a^2 - 2ad + d^2 = 36 + K$... (i)
 $a^2 = 300 + K$... (ii)
 and $a^2 + 2ad + d^2 = 596 + K$... (iii)
 On subtracting Eq. (i) from Eq. (ii) we get
 $d(2a - d) = 264$... (iv)
 On subtracting Eq. (ii) from Eq. (iii), we get
 $d(2a + d) = 296$... (v)

Again, subtracting Eq. (iv) from Eq. (v), we get
 $2d^2 = 32 \Rightarrow d^2 = 16 \Rightarrow d = 4$
 [$\therefore d = -4$, rejected]

From Eq. (iv),
 $4(2a - 4) = 264$
 $\Rightarrow 2a - 4 = 66$
 $\Rightarrow 2a = 70 \Rightarrow a = 35$
 $\therefore K = 35^2 - 300 = 1225 - 300 = 925$
 $\Rightarrow K - 920 = 5$

Entrances Gallery

1. $\frac{b}{a} = \frac{c}{b} = (\text{integer})$
 $\Rightarrow b^2 = ac \Rightarrow c = \frac{b^2}{a}$... (i)
 Also, $\frac{a+b+c}{3} = b+2$
 $\Rightarrow a+b+c = 3b+6$
 $\Rightarrow a-2b+c = 6$
 $\Rightarrow a-2b+\frac{b^2}{a} = 6$ [from Eq. (i), $c = \frac{b^2}{a}$]
 $\Rightarrow 1 - \frac{2b}{a} + \frac{b^2}{a^2} = \frac{6}{a}$
 $\Rightarrow \left(\frac{b}{a} - 1\right)^2 = \frac{6}{a}$
 $\Rightarrow a = 6 \text{ only}$
 $\Rightarrow \frac{a^2 + a - 14}{a + 1} = 4$

2. $\cot \left[\sum_{n=1}^{23} \cot^{-1} \left(1 + \sum_{k=1}^n 2k \right) \right]$
 $= \cot \left(\sum_{n=1}^{23} \cot^{-1} (1 + 2 + 4 + 6 + 8 + \dots + 2n) \right)$
 $= \cot \left(\sum_{n=1}^{23} \cot^{-1} \{1 + n(n+1)\} \right)$
 $= \cot \left(\sum_{n=1}^{23} \tan^{-1} \frac{1}{1 + n(n+1)} \right)$
 $= \cot \left(\sum_{n=1}^{23} \tan^{-1} \left\{ \frac{n+1-n}{1 + n(n+1)} \right\} \right)$
 $= \cot \left\{ \sum_{n=1}^{23} [\tan^{-1}(n+1) - \tan^{-1} n] \right\}$
 $= \cot [(\tan^{-1} 2 - \tan^{-1} 1) + (\tan^{-1} 3 - \tan^{-1} 2)$
 $+ (\tan^{-1} 4 - \tan^{-1} 3) + \dots + (\tan^{-1} 24 - \tan^{-1} 23)]$
 $= \cot \{ \tan^{-1} 24 - \tan^{-1} 1 \}$
 $= \cot \left(\tan^{-1} \frac{24-1}{1+24 \cdot 1} \right) = \cot \left(\tan^{-1} \frac{23}{25} \right)$
 $= \cot \left(\cot^{-1} \frac{25}{23} \right)$
 $= \frac{25}{23}$

3. $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} \cdot k^2$
 $= -(1)^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 + \dots$
 $= (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2) + \dots$
 $= 2 \{ \underbrace{(4 + 12 + 20 + \dots)}_{n \text{ terms}} + \underbrace{(6 + 14 + 22 + \dots)}_{n \text{ terms}} \}$
 $= 2 \left[\frac{n}{2} \{ 2 \times 4 + (n-1)8 \} + \frac{n}{2} \{ 2 \times 6 + (n-1)8 \} \right]$
 $= 2[n(4 + 4n - 4) + n(6 + 4n - 4)]$
 $= 2[4n^2 + 4n^2 + 2n] = 4n(4n + 1)$

Here, $1056 = 32 \times 33$, $1088 = 32 \times 34$,
 $1120 = 32 \times 35$, $1332 = 36 \times 37$

Hence, 1056 and 1332 are possible answers.

4. Here, $a_1 = 5$, $a_{20} = 25$ for HP
 $\therefore \frac{1}{a} = 5$
 and $\frac{1}{a+19d} = 25$
 $\Rightarrow \frac{1}{5} + 19d = \frac{1}{25}$
 $\Rightarrow 19d = \frac{1}{25} - \frac{1}{5} = -\frac{4}{25}$
 $\therefore d = \frac{-4}{19 \times 25}$
 $\therefore a_n < 0 \Rightarrow \frac{1}{a + (n-1)d} < 0$
 $\Rightarrow \frac{1}{5} + (n-1)d < 0$
 $\Rightarrow \frac{1}{5} - \frac{4}{19 \times 25} (n-1) < 0 \Rightarrow (n-1) > \frac{95}{4}$
 $\Rightarrow n > 1 + \frac{95}{4} \Rightarrow n > 24.75$

Hence, the least positive value of n is 25.

5. Given, $a_1 = 3$, $m = 5n$ and $a_1, a_2, \dots, a_{100}$ are in AP.
 Also, $\frac{S_m}{S_n} = \frac{S_{5n}}{S_n}$ is independent of n .

Consider $\frac{S_m}{S_n} = \frac{\frac{5n}{2} [2 \times 3 + (5n-1)d]}{\frac{n}{2} [2 \times 3 + (n-1)d]}$
 $= \frac{5 \{ (6-d) + 5nd \}}{(6-d) + nd}$

For independent of n , put $6 - d = 0$

$$\Rightarrow d = 6$$

$$\therefore a_2 = a_1 + d = 3 + 6 = 9$$

Also, if $d = 0$, then $\frac{S_m}{S_n}$ is independent of n .

$$\therefore a_2 = 3$$

6. Using AM $\geq$ GM,

$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}}{8}$$

$$\geq (a^{-5} \cdot a^{-4} \cdot a^{-3} \cdot a^{-3} \cdot a^{-3} \cdot 1 \cdot a^8 \cdot a^{10})^{1/8}$$

$$\Rightarrow a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10} \geq 8 \cdot 1$$

Hence, the minimum value is 8.

$$7. \text{ We have, } S_k = \frac{k-1}{1-\frac{1}{k}} = \frac{1}{(k-1)!}$$

$$\text{Now, } (k^2 - 3k + 1)S_k = \{(k-2)(k-1)-1\} \times \frac{1}{(k-1)!}$$

$$= \frac{1}{(k-3)!} - \frac{1}{(k-1)!}$$

$$\Rightarrow \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k|$$

$$= 1 + 1 + 2 - \left(\frac{1}{99!} + \frac{1}{98!}\right)$$

$$= 4 - \frac{100^2}{100!}$$

$$\Rightarrow \frac{100^2}{100!} + \sum_{k=1}^{100} |(k^2 - 3k + 1)S_k| = 4$$

8. Since, $a_k = 2a_{k-1} - a_{k-2}$

So, $a_1, a_2, \dots, a_{11}$ are in AP.

$$\therefore \frac{a_1^2 + a_2^2 + \dots + a_{11}^2}{11} = \frac{11a_1^2 + 35 \times 11d^2 + 10 \times 11a_1d}{11}$$

$$= 90$$

$$\Rightarrow 225 + 35d^2 + 150d = 90$$

$$\Rightarrow 35d^2 + 150d + 135 = 0$$

$$\Rightarrow d = -3, -\frac{9}{7}$$

$$\text{Given, } a_2 < \frac{27}{2}$$

$$\therefore d = -3 \text{ and } d \neq -\frac{9}{7}$$

$$\Rightarrow \frac{a_1 + a_2 + \dots + a_{11}}{11} = \frac{1}{2} [30 - 10 \times 3] = 0$$

9. Given, m is the AM of l and n .

$$\therefore l + n = 2m \quad \dots(i)$$

and G_1, G_2, G_3 are geometric means between l and n .

$$\therefore l, G_1, G_2, G_3, n \text{ are in GP.}$$

Let r be the common ratio of this GP.

$$\therefore G_1 = lr$$

$$G_2 = lr^2$$

$$G_3 = lr^3$$

$$n = lr^4$$

$$\Rightarrow r = \left(\frac{n}{l}\right)^{\frac{1}{4}}$$

$$\text{Now, } G_1^4 + 2G_2^4 + G_3^4 = (lr)^4 + 2(lr^2)^4 + (lr^3)^4$$

$$= l^4 \times r^4 (1 + 2r^4 + r^8)$$

$$= l^4 \times r^4 (r^4 + 1) = l^4 \times \frac{n}{l} \left(\frac{n+l}{l}\right)^2$$

$$= ln \times 4lm^2 = 4lm^2n$$

10. Given, series is

$$\frac{1^3}{1} + \frac{1^3 + 2^3}{1+3} + \frac{1^3 + 2^3 + 3^3}{1+3+5} + \dots \infty$$

Let T_n be the n th term of the given series.

$$\therefore T_n = \frac{1^3 + 2^3 + 3^3 + \dots + n^3}{1 + 3 + 5 + \dots \text{ to } n \text{ terms}}$$

$$= \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^2} = \frac{(n+1)^2}{4}$$

$$\therefore S_9 = \sum_{n=1}^9 \frac{(n+1)^2}{4} = \frac{1}{4} [2^2 + 3^2 + \dots + 10^2] + 1^2 - 1^2]$$

$$= \frac{1}{4} \left[\frac{10(10+1)(20+1)}{6} - 1 \right] = \frac{384}{4} = 96$$

11. Given, α and β are roots of $px^2 + qx + r = 0$, $p \neq 0$.

$$\therefore \alpha + \beta = \frac{-q}{p}, \alpha\beta = \frac{r}{p} \quad \dots(i)$$

Since, p, q and r are in AP.

$$\therefore 2q = p + r \quad \dots(ii)$$

$$\text{Also, } \frac{1}{\alpha} + \frac{1}{\beta} = 4$$

$$\Rightarrow \frac{\alpha + \beta}{\alpha\beta} = 4$$

$$\Rightarrow \alpha + \beta = 4\alpha\beta \Rightarrow \frac{-q}{p} = \frac{4r}{p} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow q = -4r$$

On putting the value of q in Eq. (ii), we get

$$2(-4r) = p + r$$

$$\Rightarrow p = -9r$$

$$\text{Now, } \alpha + \beta = \frac{-q}{p} = \frac{4r}{p} = \frac{4r}{-9r} = -\frac{4}{9}$$

$$\text{and } \alpha\beta = \frac{r}{p} = \frac{r}{-9r} = -\frac{1}{9}$$

$$\therefore (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{16}{81} + \frac{4}{9} = \frac{16 + 36}{81}$$

$$\Rightarrow (\alpha - \beta)^2 = \frac{52}{81} \Rightarrow |\alpha - \beta| = \frac{2}{9} \sqrt{13}$$

$$12. \text{ Given, } k \cdot 10^9 = 10^9 + 2(11)^1(10)^8 + 3(11)^2(10)^7 + \dots + 10(11)^9$$

$$\Rightarrow k = 1 + 2\left(\frac{11}{10}\right) + 3\left(\frac{11}{10}\right)^2 + \dots + 10\left(\frac{11}{10}\right)^9 \quad \dots(i)$$

$$\Rightarrow \left(\frac{11}{10}\right)k = 1\left(\frac{11}{10}\right) + 2\left(\frac{11}{10}\right)^2 + \dots + 9\left(\frac{11}{10}\right)^9 + 10\left(\frac{11}{10}\right)^{10} \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$k\left(1 - \frac{11}{10}\right) = 1 + \frac{11}{10} + \left(\frac{11}{10}\right)^2 + \dots + \left(\frac{11}{10}\right)^9 - 10\left(\frac{11}{10}\right)^{10}$$

$$\Rightarrow k\left(\frac{10-11}{10}\right) = \frac{1\left[\left(\frac{11}{10}\right)^{10} - 1\right]}{\left(\frac{11}{10} - 1\right)} - 10\left(\frac{11}{10}\right)^{10}$$

$$[\because \text{in GP, sum of } n \text{ terms} = \frac{a(r^n - 1)}{r - 1}, \text{ when } r > 1]$$

$$\Rightarrow -k = 10\left[10\left(\frac{11}{10}\right)^{10} - 10 - 10\left(\frac{11}{10}\right)^{10}\right]$$

$$\therefore k = 100$$

13. Let a, ar and ar^2 be in GP ($r > 1$).

On multiplying middle term by 2, then the numbers $a, 2ar$ and ar^2 are in AP.

$$\Rightarrow 4ar = a + ar^2$$

$$\Rightarrow r^2 - 4r + 1 = 0$$

$$\Rightarrow r = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\Rightarrow r = 2 + \sqrt{3} \quad [\because \text{GP is increasing}]$$

14. Let $S = \frac{7}{10} + \frac{77}{10^2} + \frac{777}{10^3} + \dots$ upto 20 terms

$$= \frac{7}{9} \left[\frac{9}{10} + \frac{99}{100} + \frac{999}{1000} + \dots \text{ upto 20 terms} \right]$$

$$= \frac{7}{9} \left[\left(1 - \frac{1}{10}\right) + \left(1 - \frac{1}{10^2}\right) + \left(1 - \frac{1}{10^3}\right) + \dots \text{ upto 20 terms} \right]$$

$$= \frac{7}{9} \left[(1 + 1 + \dots \text{ upto 20 terms}) - \left(\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \dots \text{ upto 20 terms} \right) \right]$$

$$= \frac{7}{9} \left[20 - \frac{1}{10} \left(\frac{1 - \left(\frac{1}{10}\right)^{20}}{1 - \frac{1}{10}} \right) \right]$$

$$= \frac{7}{9} \left[20 - \frac{1}{10} \times \frac{10}{9} \left\{ 1 - \left(\frac{1}{10}\right)^{20} \right\} \right]$$

$$= \frac{7}{9} \left[20 - \frac{10}{9} (1 - 10^{-20}) \right] = \frac{7}{81} (180 - 1 + 10^{-20})$$

$$= \frac{7}{81} (179 + 10^{-20})$$

15. Here, $T_{100} = a + (100 - 1)d = a + 99d$

$$T_{50} = a + (50 - 1)d = a + 49d$$

$$T_{150} = a + (150 - 1)d = a + 149d$$

Now, according to the given condition,

$$100 \times T_{100} = 50 \times T_{50}$$

$$\Rightarrow 100(a + 99d) = 50(a + 49d)$$

$$\Rightarrow 2(a + 99d) = (a + 49d)$$

$$\Rightarrow a + 149d = 0$$

$$\Rightarrow T_{150} = 0$$

16. **Statement I** Let $S = (1) + (1 + 2 + 4) + (4 + 6 + 9)$

$$+ \dots + (361 + 380 + 400)$$

$$S = (0 + 0 + 1) + (1 + 2 + 4) + (4 + 6 + 9)$$

$$+ \dots + (361 + 380 + 400)$$

Now, we can clearly observe the first elements in each bracket.

In second bracket, the first element is $1 = 1^2$

In third bracket, the first element is $4 = 2^2$

In fourth bracket, the first element is $9 = 3^2$

In last bracket, the first element is $361 = 19^2$

Hence, we can conclude that there are 20 brackets in all.

Also, in each of the bracket, there are 3 terms out of which the first and last terms are perfect squares of consecutive integers and the middle term is their product.

Now, the general term of the series is

$$T_r = (r-1)^2 + (r-1)r + (r)^2$$

So, the sum of n terms of the series is

$$S_n = \sum_{r=1}^n \{(r-1)^2 + (r-1)r + (r)^2\}$$

$$\Rightarrow S_n = \sum_{r=1}^n \left\{ \frac{r^3 - (r-1)^3}{r - (r-1)} \right\}$$

$$\Rightarrow S_n = \sum_{r=1}^n \{r^3 - (r-1)^3\}$$

$$\text{Now, let } S_n = \sum_{k=1}^n \{k^3 - (k-1)^3\}$$

On substituting the value of k and rearranging the terms, we get

$$S_n = -0^3 + (1^3 - 1^3) + \dots + [(n-1)^3 - (n-1)^3] + n^3$$

$$\Rightarrow S_n = n^3$$

Since, the number of terms is 20, hence substituting $n = 20$, we get

$$S_{20} = 8000$$

Statement II We have already proved in the Statement I that

$$S_n = \sum_{k=1}^n \{k^3 - (k-1)^3\} = n^3$$

17. Let the time taken to save ₹ 11040 be $(n + 3)$ months.

For first 3 months, he save ₹ 200 each month.

In $(n + 3)$ months, his total savings is

$$3 \times 200 + \frac{n}{2} [2(240) + (n-1) \times 40] = 11040$$

$$\Rightarrow 600 + 20n(n+1) = 11040$$

$$\Rightarrow 20n(n+1) = 10440$$

$$\Rightarrow n(n+1) = 522$$

$$\Rightarrow n^2 + 11n - 522 = 0$$

$$\Rightarrow n^2 + 29n - 18n - 522 = 0$$

$$\Rightarrow n(n+29) - 18(n+29) = 0$$

$$\Rightarrow n = 18$$

$$\text{or } n = -29$$

$$\therefore n = 18 \quad [\text{neglecting } n = -29]$$

Hence, total time = $n + 3$

$$= 18 + 3 = 21 \text{ months}$$

18. Number of notes that the person counts in 10 min

$$= 10 \times 150 = 1500$$

We have, $a_{10}, a_{11}, a_{12}, \dots$ are in AP with common difference -2 .

Let n be the time taken to count remaining 3000 notes, then

$$\frac{n}{2} [2 \times 148 + (n-1) \times (-2)] = 3000$$

$$\Rightarrow n^2 - 149n + 3000 = 0$$

$$\Rightarrow (n-24)(n-125) = 0$$

$$\Rightarrow n = 24, 125$$

Hence, total time taken by the person to count all notes
 $= 10 + 24 = 34$ min

$$[\because n \neq 125 \text{ as } a_{135} = 148 + 124(-2) < 0]$$

19. Let $S = 1 + \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \dots$

$$\Rightarrow S - 1 = \frac{2}{3} + \frac{6}{3^2} + \frac{10}{3^3} + \frac{14}{3^4} + \dots \quad \dots(i)$$

$$\Rightarrow \frac{S-1}{3} = \frac{2}{3^2} + \frac{6}{3^3} + \frac{10}{3^4} + \frac{14}{3^5} + \dots \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\frac{2}{3}(S-1) = \frac{2}{3} + \frac{4}{3^2} + \frac{4}{3^3} + \frac{4}{3^4} + \dots$$

$$\Rightarrow S - 1 = 1 + \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \dots$$

$$\Rightarrow S = 2 + \frac{\frac{2}{3}}{1 - \frac{1}{3}} = 2 + 1 = 3$$

20. Since, $a + ar = a(1+r) = 12 \quad \dots(i)$

and $ar^2 + ar^3 = ar^2(1+r) = 48 \quad \dots(ii)$

From Eqs. (i) and (ii), we get

$$r^2 = 4 \Rightarrow r = -2$$

On putting the value of r in Eq. (i), we get

$$a = -12$$

21. Since, each term is equal to the sum of the next two terms.

$$\therefore ar^{n-1} = ar^n + ar^{n+1} \Rightarrow 1 = r + r^2$$

$$\Rightarrow r^2 + r - 1 = 0$$

$$r = \frac{\sqrt{5}-1}{2} \quad \left[\because r \neq \frac{-\sqrt{5}-1}{2} \right]$$

22. Since, $\frac{a_1 + a_2 + \dots + a_p}{a_1 + a_2 + \dots + a_q} = \frac{p^2}{q^2}$

$$\therefore \frac{\frac{p}{2} [2a_1 + (p-1)d]}{\frac{q}{2} [2a_1 + (q-1)d]} = \frac{p^2}{q^2}$$

$$\Rightarrow \frac{(2a_1 - d) + pd}{(2a_1 - d) + qd} = \frac{p}{q}$$

$$\Rightarrow (2a_1 - d)(p - q) = 0$$

$$\Rightarrow a_1 = \frac{d}{2} \quad [\because p \neq q]$$

$$\text{Now, } \frac{a_6}{a_{21}} = \frac{a_1 + 5d}{a_1 + 20d} = \frac{\frac{d}{2} + 5d}{\frac{d}{2} + 20d} = \frac{11d}{41d} = \frac{11}{41}$$

23. Since, $a_1, a_2, a_3, \dots, a_n$ are in HP.

$$\therefore \frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n} \text{ are in AP.}$$

Let d be the common difference of AP.

$$\therefore \frac{1}{a_2} - \frac{1}{a_1} = d$$

$$\Rightarrow a_1 - a_2 = a_1 a_2 d$$

Similarly, $a_2 - a_3 = a_2 a_3 d$

$$\vdots \quad \vdots \quad \vdots$$

$$a_{n-1} - a_n = a_{n-1} a_n d$$

On adding all the equations, we get

$$a_1 - a_n = d \{a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n\} \quad \dots(i)$$

Also, $\frac{1}{a_n} - \frac{1}{a_1} = (n-1)d$

$$\Rightarrow d = \frac{a_1 - a_n}{a_1 a_n (n-1)}$$

On putting the value of d in Eq. (i), we get

$$a_1 - a_n = \frac{a_1 - a_n}{a_1 a_n (n-1)} \{a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n\}$$

$$\Rightarrow a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n = a_1 a_n (n-1)$$

24. Given that, $x = \sum_{n=0}^{\infty} a^n, y = \sum_{n=0}^{\infty} b^n, z = \sum_{n=0}^{\infty} c^n$

$$\Rightarrow x = 1 + a + a^2 + \dots = \frac{1}{1-a} \quad \dots(i)$$

Similarly, $y = \frac{1}{1-b} \quad \dots(ii)$

and $z = \frac{1}{1-c} \quad \dots(iii)$

Now, a, b and c are in AP.

$\Rightarrow -a, -b$ and $-c$ are in AP.

$\Rightarrow 1-a, 1-b$ and $1-c$ are also in AP.

$$\Rightarrow \frac{1}{1-a}, \frac{1}{1-b} \text{ and } \frac{1}{1-c} \text{ are in HP.}$$

$\Rightarrow x, y, z$ are in HP.

Aliter

From Eqs. (i), (ii) and (iii),

$$x = \frac{1}{1-a}, y = \frac{1}{1-b} \text{ and } z = \frac{1}{1-c}$$

$$\Rightarrow a = \frac{x-1}{x}, b = \frac{y-1}{y} \text{ and } c = \frac{z-1}{z}$$

Since, a, b and c are in AP.

$$\therefore 2b = a + c$$

$$\Rightarrow 2 \left(\frac{y-1}{y} \right) = \frac{x-1}{x} + \frac{z-1}{z}$$

$$\Rightarrow 2 - \frac{2}{y} = 1 - \frac{1}{x} + 1 - \frac{1}{z}$$

$$\Rightarrow \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$$

Hence, x, y and z are in HP.

25. We know that,

$$\frac{e^x + e^{-x}}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

On putting $x = \frac{1}{2}$ in both sides, we get

$$\frac{e^{1/2} + e^{-1/2}}{2} = 1 + \left(\frac{1}{2} \right)^2 \frac{1}{2!} + \left(\frac{1}{2} \right)^4 \frac{1}{4!} + \dots$$

$$\Rightarrow \frac{e + 1}{2\sqrt{e}} = 1 + \frac{1}{4 \cdot 2!} + \frac{1}{16 \cdot 4!} + \frac{1}{64 \cdot 6!} + \dots$$

26. Given, $T_m = \frac{1}{n}$
 $\Rightarrow a + (m-1)d = \frac{1}{n}$... (i)
 and $T_n = \frac{1}{m}$
 $\Rightarrow a + (n-1)d = \frac{1}{m}$... (ii)
 On solving Eqs. (i) and (ii), we get
 $a = d = \frac{1}{mn}$
 $\therefore a - d = 0$

27. Given that the sum of n terms of given series is $\frac{n(n+1)^2}{2}$, if n is even.

Let n be odd, i.e. $n = 2m + 1$
 Then, $S_{2m+1} = S_{2m} + (2m+1)$ th term
 $= \frac{(n-1)n^2}{2} + n$ th term
 $= \frac{(n-1)n^2}{2} + n^2$
 $= \frac{n^2}{2} \left(\frac{n-1+2}{2} \right)$
 $= \frac{(n+1)n^2}{2}$

28. We know that,
 $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$... (i)
 and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots$... (ii)

On adding Eqs. (i) and (ii), we get
 $e + e^{-1} = 2 + \frac{2}{2!} + \frac{2}{4!} + \dots$
 $\Rightarrow \frac{e^2 + 1}{e} - 2 = \frac{2}{2!} + \frac{2}{4!} + \dots$
 $\Rightarrow \frac{e^2 + 1 - 2e}{e} = 2 \left(\frac{1}{2!} + \frac{1}{4!} + \dots \right)$
 $\Rightarrow \frac{(e-1)^2}{2e} = \frac{1}{2!} + \frac{1}{4!} + \dots$

29. Consider, $\frac{1}{1 \cdot 2} - \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} - \dots$
 $= \left(1 - \frac{1}{2}\right) - \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) - \dots$
 $= 1 - 2 \cdot \frac{1}{2} + 2 \cdot \frac{1}{3} - 2 \cdot \frac{1}{4} + \dots$
 $= 2 \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots\right) - 1$
 $= 2 \log(1+1) - 1$
 $= \log 2^2 - \log e = \log \frac{4}{e}$

30. Since, $1, \log_3 \sqrt{3^{1-x} + 2}$ and $\log_3 (4 \cdot 3^x - 1)$ are in AP.
 $\therefore 2 \log_3 (3^{1-x} + 2)^{1/2} = \log_3 3 + \log_3 (4 \cdot 3^x - 1)$
 $\Rightarrow \log_3 (3^{1-x} + 2) = \log_3 3(4 \cdot 3^x - 1)$
 $\Rightarrow 3^{1-x} + 2 = 12 \cdot 3^x - 3$
 $\therefore \frac{3}{t} + 2 = 12t - 3$ [let $3^x = t$]

$\Rightarrow 12t^2 - 5t - 3 = 0$
 $\Rightarrow (3t+1)(4t-3) = 0$
 $\Rightarrow t = -\frac{1}{3}, \frac{3}{4}$
 $\Rightarrow 3^x = \frac{3}{4}$ [$\because 3^x$ cannot be negative]
 $\Rightarrow \log_3 \left(\frac{3}{4}\right) = x$
 $\Rightarrow x = 1 - \log_3 4$

31. Let $S = 2^{1/4} \cdot 4^{1/8} \cdot 8^{1/16} \dots$
 $= 2^{1/4} \cdot 2^{2/8} \cdot 2^{3/16} \dots$
 $= 2^{1/4} \left[1 + \frac{2}{2} + \frac{3}{2^2} + \dots \right] = 2^{1/4} (S_1)$

where, $S_1 = 1 + \frac{2}{2} + \frac{3}{2^2} + \dots$

It is an infinite arithmetico-geometric progression.

$\therefore S_1 = \frac{a}{1-r} + \frac{d \cdot r}{(1-r)^2}$
 $= \frac{1}{1-\frac{1}{2}} + \frac{1 \cdot \frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2 + 2 = 4$
 $\therefore S = 2^{1/4(4)} = 2$

32. Since, 5th term of a GP = 2
 $\therefore ar^4 = 2$... (i)
 where, a and r are the first term and common ratio respectively of a GP.
 Now, required product
 $= a \times ar \times ar^2 \times ar^3 \times ar^4 \times ar^5 \times ar^6 \times ar^7 \times ar^8$
 $= a^9 r^{36} = (ar^4)^9$
 $= 2^9 = 512$ [from Eq. (i)]

33. $\sum_{n=0}^{\infty} \frac{(\log_e x)^n}{n!} = 1 + \frac{\log_e x}{1!} + \frac{(\log_e x)^2}{2!} + \dots$
 $= e^{\log_e x} = x$

34. $e^{(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \dots} = e^{\log(1+x-1)} = x$

35. V_r = Sum of first r terms of an AP whose first term is r and common difference $(2r-1)$
 $= \frac{r}{2} [2r + (r-1)(2r-1)]$
 $= \frac{r}{2} [2r + 2r^2 - 3r + 1]$
 $= \frac{r}{2} [2r^2 - r + 1]$
 $= \frac{1}{2} [2r^3 - r^2 + r]$

Now, $\sum_{r=1}^n V_r = \frac{1}{2} \left[2 \sum_{r=1}^n r^3 - \sum_{r=1}^n r^2 + \sum_{r=1}^n r \right]$
 $= \frac{1}{2} \left[2 \cdot \left(\frac{n(n+1)}{2} \right)^2 - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right]$
 $= \frac{1}{2} \left[\frac{n^2(n+1)^2}{2} - \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right]$

$$\begin{aligned} &= \frac{n(n+1)}{4} \left[n(n+1) - \frac{(2n+1)}{3} + 1 \right] \\ &= \frac{n(n+1)}{4} \left[\frac{3n^2 + 3n - 2n - 1 + 3}{3} \right] \\ &= \frac{n(n+1)}{12} (3n^2 + n + 2) \end{aligned}$$

36. Since, a, b and c are in GP.

$$\therefore b^2 = ac$$

Given equation is

$$(\log_e a) x^2 - (2 \log_e b) x + (\log_e c) = 0 \quad \dots(i)$$

On putting $x = 1$ in Eq. (i), we get

$$\log_e a - 2 \log_e b + \log_e c = 0$$

$$\Rightarrow 2 \log_e b = \log_e a + \log_e c$$

$$\Rightarrow \log_e b^2 = \log_e ac$$

$$\Rightarrow b^2 = ac, \text{ which is true.}$$

Hence, one of the roots of given equation is 1.

Let another root be α .

$$\therefore \text{Sum of roots, } 1 + \alpha = \frac{2 \log_e b}{\log_e a} = \frac{\log_e b^2}{\log_e a}$$

$$\begin{aligned} \Rightarrow \alpha &= \frac{\log_e ac}{\log_e a} - 1 = \frac{(\log_e a + \log_e c)}{\log_e a} - 1 \\ &= \frac{\log_e c}{\log_e a} = \log_a c \end{aligned}$$

Hence, roots are 1 and $\log_a c$.

37. Given, $f(x) = x + \frac{1}{2} = \frac{2x+1}{2}$

$$\therefore f(2x) = 2x + \frac{1}{2} \Rightarrow f(2x) = \frac{4x+1}{2}$$

$$\text{and } f(4x) = 4x + \frac{1}{2}$$

$$\Rightarrow f(4x) = \frac{8x+1}{2}$$

Since, $f(x), f(2x)$ and $f(4x)$ are in HP.

$$\therefore \frac{1}{f(x)}, \frac{1}{f(2x)} \text{ and } \frac{1}{f(4x)} \text{ are in AP.}$$

$$\Rightarrow \frac{1}{f(2x)} = \frac{\frac{1}{f(x)} + \frac{1}{f(4x)}}{2}$$

$$\Rightarrow \frac{2}{4x+1} = \frac{\frac{2}{2x+1} + \frac{2}{8x+1}}{2}$$

$$\Rightarrow \frac{2}{4x+1} = \frac{10x+2}{(2x+1)(8x+1)}$$

$$\Rightarrow (2x+1)(8x+1) = (5x+1)(4x+1)$$

$$\Rightarrow 16x^2 + 10x + 1 = 20x^2 + 9x + 1$$

$$\Rightarrow 4x^2 - x = 0$$

$$\Rightarrow x(4x-1) = 0$$

$$\Rightarrow x = \frac{1}{4} \quad [\because x \neq 0]$$

Hence, one real value of x for which the three unequal terms are in HP.

38. $\therefore \text{AM} \geq \text{GM}$

$$\therefore \frac{(p+q) + (r+s)}{2} \geq \sqrt{(p+q)(r+s)}$$

$$\Rightarrow \frac{2}{2} \geq \sqrt{M}$$

$$\Rightarrow \sqrt{M} \leq 1 \Rightarrow M \leq 1$$

$$\text{Also, } (p+q)(r+s) > 0 \quad [\because p, q, r, s > 0]$$

$$\therefore M > 0$$

$$\text{Hence, } 0 < M \leq 1$$

39. **Case I** Let $\alpha = \omega$ and $\beta = \omega^2$

$$\begin{aligned} \therefore S &= \sum_{n=0}^{302} (-1)^n \left(\frac{\omega}{\omega^2} \right)^n = \sum_{n=0}^{302} (-1)^n (\omega^2)^n \\ &= 1 - \omega^2 + \omega^4 - \omega^6 + \omega^8 - \omega^{10} + \omega^{12} \\ &\quad + \dots + \omega^{600} - \omega^{602} + \omega^{604} \\ &= 1 - \omega^2 + \omega - 1 + \omega^2 - \omega + 1 + \dots + 1 - \omega^2 + \omega \\ &= 0 + \dots + 1 - \omega^2 + \omega \\ &= -\omega^2 - \omega^2 = -2\omega^2 \quad [\because 1 + \omega + \omega^2 = 0] \end{aligned}$$

Case II Let $\alpha = \omega^2$ and $\beta = \omega$

$$\begin{aligned} \therefore S &= \sum_{n=0}^{302} (-1)^n \left(\frac{\omega^2}{\omega} \right)^n = \sum_{n=0}^{302} (-1)^n \left(\frac{\omega^4}{\omega^3} \right)^n \\ &= \sum_{n=0}^{302} (-1)^n (\omega)^n \\ &= 1 - \omega + \omega^2 - \omega^3 + \omega^4 - \omega^5 + \omega^6 \\ &\quad - \dots + \omega^{300} - \omega^{301} + \omega^{302} \\ &= 1 - \omega + \omega^2 - 1 + \omega - \omega^2 + 1 - \dots + 1 - \omega + \omega^2 \\ &= 0 + \dots + 1 + \omega^2 - \omega = -\omega - \omega = -2\omega \end{aligned}$$

$$\begin{aligned} 40. \text{ Given, } (\alpha + \beta)x - \left(\frac{\alpha^2 + \beta^2}{2} \right)x^2 + \left(\frac{\alpha^3 + \beta^3}{3} \right)x^3 + \dots \\ &= \left[\alpha x - \frac{1}{2} (\alpha x)^2 + \frac{1}{3} (\alpha x)^3 - \dots \right] \\ &\quad + \left[\beta x - \frac{1}{2} (\beta x)^2 + \frac{1}{3} (\beta x)^3 - \dots \right] \\ &= \log(1 + \alpha x) + \log(1 + \beta x) \\ &= \log[1 + (\alpha + \beta)x + \alpha\beta x^2] \end{aligned}$$

$$\text{Now, } \alpha + \beta = p \text{ and } \alpha\beta = q$$

$$\begin{aligned} \text{Hence, } (\alpha + \beta)x - \left(\frac{\alpha^2 + \beta^2}{2} \right)x^2 + \left(\frac{\alpha^3 + \beta^3}{3} \right)x^3 + \dots \\ &= \log(1 + px + qx^2) \end{aligned}$$

41. Given, $\log_4 2 - \log_8 2 + \log_{16} 2 - \dots$

$$\begin{aligned} &= \frac{1}{\log_2 4} - \frac{1}{\log_2 8} + \frac{1}{\log_2 16} - \dots \\ &= \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \dots \\ &= 1 - \left[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots \right] \\ &= 1 - \log_e 2 \end{aligned}$$

42. We have, $e^x = (1-x)(B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1} + B_nx^n + \dots)$

By the expansion of e^x , we get

$$\begin{aligned} &1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^n}{n!} + \dots \\ &= (1-x)(B_0 + B_1x + B_2x^2 + \dots + B_{n-1}x^{n-1} + B_nx^n + \dots) \end{aligned}$$

On equating the coefficient of x^n both sides, we get

$$B_n - B_{n-1} = \frac{1}{n!}$$

43. We have, $a = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!}$, $b = \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!}$
 and $c = \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!}$
 Now, $a + b + c = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} + \sum_{n=1}^{\infty} \frac{x^{3n-2}}{(3n-2)!} + \sum_{n=1}^{\infty} \frac{x^{3n-1}}{(3n-1)!}$
 $= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = e^x$
 $a + b\omega + c\omega^2 = 1 + \omega x + \frac{\omega^2 x^2}{2!} + \frac{\omega^3 x^3}{3!} + \dots = e^{\omega x}$
 and $a + b\omega^2 + c\omega = e^{\omega^2 x}$, ω is an imaginary cube root of unity.
 Now, $a^3 + b^3 + c^3 - 3abc$
 $= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$
 $= e^x \cdot e^{\omega x} \cdot e^{\omega^2 x} = e^{x(1 + \omega + \omega^2)} = e^{0 \cdot x} = 1$

44. Given, $f(x) = \frac{x}{1!} + \frac{3}{2!}x^2 + \frac{7}{3!}x^3 + \frac{15}{4!}x^4 + \dots$
 $= \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$
 $= 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$
 $= 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$
 $= 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \dots$

$\Rightarrow f(x) = e^{2x} - e^x$
 Put $f(x) = 0$, we get
 $e^{2x} - e^x = 0$
 $\Rightarrow e^x(e^x - 1) = 0$
 $\Rightarrow e^x = 0$ or $e^x = 1$
 $\Rightarrow x = 0$
 Hence, exactly one real solution exists.

45. We have, $S = 1 + \frac{8}{2!} + \frac{21}{3!} + \frac{40}{4!} + \frac{65}{5!} + \dots$
 Let $S_1 = 1 + 8 + 21 + 40 + 65 + \dots + T_n$... (i)
 and $S_1 = 1 + 8 + 21 + 40 + \dots + T_{n-1} + T_n$... (ii)
 On subtracting Eq. (ii) from Eq. (i), we get
 $0 = 1 + 7 + 13 + 19 + 25 + \dots - T_n$
 $T_n = 1 + 7 + 13 + 19 + 25 + \dots$ upto n terms
 $= \frac{n}{2} [2(1) + (n-1)6]$
 $= n[1 + 3(n-1)] = n(3n-2)$
 $\therefore S = \sum \frac{n(3n-2)}{n!} = \sum \frac{3n-2}{(n-1)!}$
 $= \sum \frac{3n-3+1}{(n-1)!}$
 $\Rightarrow S = \sum \frac{3}{(n-2)!} + \sum \frac{1}{(n-1)!} = 3e + e = 4e$
 $\left[\because e = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots \right]$

We know that, $2 < e < 3$

$\therefore 8 < 4e < 12$
 $\Rightarrow 8 < S < 12$

46. Since, $a_1, a_2, \dots, a_n$ are in AP, therefore
 $a_2 - a_1 = a_3 - a_2 = \dots = a_{2k} - a_{2k-1} = d$ [say]
 Now, $a_1^2 - a_2^2 = (a_1 - a_2)(a_1 + a_2)$
 $= -d(a_1 + a_2)$
 $a_3^2 - a_4^2 = -d(a_3 + a_4)$
 $a_{2k-1}^2 - a_{2k}^2 = -d(a_{2k-1} + a_{2k})$

On adding, we get
 $S = -d(a_1 + a_2 + \dots + a_{2k})$
 $= -d \left[\frac{2k}{2}(a_1 + a_{2k}) \right]$
 $= -dk(a_1 + a_{2k}) = \frac{-dk}{(a_1 - a_{2k})}(a_1^2 - a_{2k}^2)$
 $= \frac{-dk(a_1^2 - a_{2k}^2)}{[(a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \dots + (a_{2k-2} - a_{2k-1}) + (a_{2k-1} - a_{2k})]}$
 $= \frac{-dk(a_1^2 - a_{2k}^2)}{(-d)(2k-1)} = \frac{k}{2k-1}(a_1^2 - a_{2k}^2)$

47. Given, $\log_x(ax), \log_x(bx)$ and $\log_x(cx)$ are in AP.
 $\Rightarrow 1 + \log_x a, 1 + \log_x b, 1 + \log_x c$ are in AP.
 $\Rightarrow \log_x a, \log_x b, \log_x c$ are in AP.
 $\therefore \frac{\log a}{\log x} + \frac{\log c}{\log x} = 2 \frac{\log b}{\log x}$
 $\Rightarrow \log a + \log c = 2 \log b \Rightarrow ac = b^2$

48. Let the six numbers in AP be
 $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$
 $\therefore a - 5d + a - 3d + a - d + a + d + a + 3d + a + 5d = 3[\because \text{sum} = 3]$
 $\Rightarrow 6a = 3$
 $\Rightarrow a = \frac{1}{2}$

Also, $T_1 = 4T_3$, where T_1, T_3 are respectively first and third terms of an AP.

$\Rightarrow a - 5d = 4(a - d) \Rightarrow d = -3a = -\frac{3}{2}$
 So, the fifth term $= a + 3d$
 $= \frac{1}{2} + 3\left(-\frac{3}{2}\right) = \frac{1}{2} - \frac{9}{2} = -4$

49. Since, $\log_{10} 2, \log_{10} (2^x - 1)$ and $\log_{10} (2^x + 3)$ are in AP.

$\therefore \log_{10} (2^x - 1) = \frac{\log_{10} 2 + \log_{10} (2^x + 3)}{2}$
 $\Rightarrow 2 \log_{10} (2^x - 1) = \log_{10} 2 + \log_{10} (2^x + 3)$
 $\Rightarrow \log_{10} (2^x - 1)^2 = \log_{10} 2(2^x + 3)$
 $\Rightarrow (2^x - 1)^2 = 2(2^x + 3)$
 $\Rightarrow 2^{2x} + 1 - 2^{x+1} = 2^{x+1} + 6$
 $\Rightarrow 2^{2x} + 1 - 2 \cdot 2^{x+1} = 6$
 $\Rightarrow 2^{2x} - 4 \cdot 2^x - 5 = 0$

Now, let $2^x = t$
 $\Rightarrow t^2 - 4t - 5 = 0$
 $\Rightarrow (t - 5)(t + 1) = 0$
 $\Rightarrow t = 5$ or $t = -1$
 $\Rightarrow 2^x = 5$

[neglecting $2^x = -1$ as 2^x is always positive]

$\Rightarrow x \log_2 2 = \log_2 5$
 $\Rightarrow x = \log_2 5$

$$\begin{aligned}
 50. S_n &= 0.2 + 0.22 + 0.222 + \dots \text{ upto } n \text{ terms} \\
 &= 2 [0.1 + 0.11 + 0.111 + \dots \text{ upto } n \text{ terms}] \\
 &= \frac{2}{9} [0.9 + 0.99 + 0.999 + \dots \text{ upto } n \text{ terms}] \\
 &= \frac{2}{9} [(1-0.1) + \{1-(0.1)^2\} + \{1-(0.1)^3\} \\
 &\quad + \dots \text{ upto } n \text{ terms}]
 \end{aligned}$$

$$= \frac{2}{9} [n - \{(0.1) + (0.1)^2 + (0.1)^3 + \dots \text{ upto } n \text{ terms}\}]$$

$$= \frac{2}{9} \left[n - (0.1) \frac{1 - (0.1)^n}{1 - 0.1} \right] = \frac{2}{9} \left[n - \frac{1}{9} (1 - 10^{-n}) \right]$$

$$\begin{aligned}
 51. 1 + \frac{1}{3} + \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12} + \dots \\
 = 1 + \frac{1}{3} + \frac{1 \cdot 3}{1 \cdot 2 \cdot 2} \left(\frac{2}{3} \right)^2 + \frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3 \cdot 2} \left(\frac{2}{3} \right)^3 + \dots \\
 = 1 + \frac{1}{2} \cdot \frac{2}{3} + \frac{1 \cdot 2}{2!} \left(\frac{2}{3} \right)^2 \\
 \quad + \frac{1 \cdot 2 \cdot 1}{3!} \left(\frac{2}{3} \right)^3 + \dots \\
 = \left(1 - \frac{2}{3} \right)^{-1/2} = 3^{1/2} = \sqrt{3}
 \end{aligned}$$

$$\begin{aligned}
 52. \text{ We have, } H_n &= 1 + \frac{1}{2} + \dots + \frac{1}{n} \\
 \text{Consider, } S_n &= 1 + \frac{3}{2} + \frac{5}{3} + \dots + \frac{2n-1}{n} \\
 &= 1 + \left(2 - \frac{1}{2} \right) + \left(2 - \frac{1}{3} \right) + \dots + \left(2 - \frac{1}{n} \right) \\
 &= \underbrace{[1 + \{2 + 2 + \dots + 2\}]}_{(n-1) \text{ times}} - \left[\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right] \\
 &= [1 + 2(n-1)] - \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right] + 1 \\
 &= 2n - \left[1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right] = 2n - H_n
 \end{aligned}$$

$$\begin{aligned}
 53. \text{ Let the numbers be } a \text{ and } b. \\
 \therefore \frac{a+b}{2} = 27 \text{ and } \frac{2ab}{a+b} = 12 \\
 \Rightarrow a+b = 54 \text{ and } 2ab = 12(a+b) \\
 \Rightarrow 2ab = 12(54) \\
 \Rightarrow ab = 6(54) = 324 \\
 \Rightarrow \sqrt{ab} = 18 \\
 \text{Thus, GM} = 18
 \end{aligned}$$

$$\begin{aligned}
 54. \text{ Since, } x, y \text{ and } z \text{ are in GP.} \\
 \therefore y^2 = xz \\
 \text{Now, taking } \log_{10} \text{ on both sides, we get} \\
 2 \log_{10} y = \log_{10} x + \log_{10} z \\
 \Rightarrow 2 \left(\frac{1}{\log_y 10} \right) = \frac{1}{\log_x 10} + \frac{1}{\log_z 10} \\
 \Rightarrow \frac{1}{\log_x 10}, \frac{1}{\log_y 10}, \frac{1}{\log_z 10} \text{ are in AP.} \\
 \Rightarrow \log_x 10, \log_y 10 \text{ and } \log_z 10 \text{ are in HP.}
 \end{aligned}$$

$$\begin{aligned}
 55. 4 + 2(1+2) \log 2 + \frac{2(1+2^2)(\log 2)^2}{2!} \\
 \quad + \frac{2(1+2^3)(\log 2)^3}{3!} + \dots \\
 = 2 \left(1 + \log 2 + \frac{(\log 2)^2}{2!} + \dots \right) \\
 \quad + 2 \left(1 + 2 \log 2 + \frac{(2 \log 2)^2}{2!} + \dots \right) \\
 = 2(e^{\log 2}) + 2(e^{2 \log 2}) \\
 = 2 \times 2 + 2e^{\log 4} \\
 = 4 + 2 \times 4 = 12
 \end{aligned}$$

56. Let T_n be the n th term of the given series.

$$\begin{aligned}
 \therefore T_n &= \frac{n(n+1)}{n!} = \frac{n+1}{(n-1)!} = \frac{(n-1)+2}{(n-1)!} \\
 &= \frac{1}{(n-2)!} + \frac{2}{(n-1)!} \\
 \therefore S &= \sum_{n=1}^{\infty} T_n = \sum_{n=2}^{\infty} \frac{1}{(n-2)!} + 2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \\
 &= e + 2e = 3e
 \end{aligned}$$

57. Given, sum of n terms of an AP = 240

$$\begin{aligned}
 \therefore \frac{n}{2} [2 \times 2 + (n-1) \times 2] &= 240 \\
 \Rightarrow n(2+n-1) &= 240 \\
 \Rightarrow n(n+1) &= 15 \times 16 \\
 \Rightarrow n &= 15
 \end{aligned}$$

58. Given, $S_{\infty} = \frac{4}{3}$ and $a = \frac{3}{4}$

Let r be the common ratio.

$$\begin{aligned}
 \therefore \frac{a}{1-r} &= \frac{4}{3} \\
 \Rightarrow \frac{\frac{3}{4}}{1-r} &= \frac{4}{3} \\
 \Rightarrow \frac{16-9}{12} &= \frac{4}{3}r \\
 \Rightarrow \frac{7}{12} &= \frac{4}{3}r \\
 \Rightarrow r &= \frac{7}{16}
 \end{aligned}$$

59. We have, $\frac{x^{n+1} + y^{n+1}}{x^n + y^n} = \sqrt{xy}$, for some n

$$\begin{aligned}
 \Rightarrow x^{n+1} + y^{n+1} &= (xy)^{1/2} (x^n + y^n) \\
 \Rightarrow x^n \cdot x + y^n \cdot y &= x^n \cdot x^{1/2} y^{1/2} + y^n \cdot x^{1/2} y^{1/2} \\
 \Rightarrow x^n (x - \sqrt{xy}) + y^n (y - \sqrt{xy}) &= 0 \\
 \Rightarrow x^n \cdot \sqrt{x} (\sqrt{x} - \sqrt{y}) + y^n \cdot y^n (\sqrt{y} - \sqrt{x}) &= 0 \\
 \Rightarrow (\sqrt{x} - \sqrt{y}) (x^n \cdot \sqrt{x} - y^n \cdot \sqrt{y}) &= 0 \\
 \text{For } x \neq y, \quad x^n \sqrt{x} &= y^n \sqrt{y} \\
 \Rightarrow \left(\frac{x}{y} \right)^{n+\frac{1}{2}} &= 1 \\
 \Rightarrow n + \frac{1}{2} &= 0 \\
 \Rightarrow n &= -\frac{1}{2}
 \end{aligned}$$

60. Let the numbers be a and b .

$$\therefore \sqrt{ab} = 10$$

$$\Rightarrow ab = 100$$

$$\text{and } \frac{2ab}{a+b} = 8$$

$$\Rightarrow \frac{200}{a+b} = 8$$

$$\Rightarrow a+b = 25$$

On solving Eqs. (i) and (ii), we get

$$a = 5 \quad \text{and} \quad b = 20$$

$$\text{or } a = 20 \quad \text{and} \quad b = 5$$

61. $T_n = (2n-1)^3 = 8n^3 - 1^3 - 3 \cdot 2n \cdot 1(2n-1)$

$$= 8n^3 - 1 - 12n^2 + 6n$$

$$= 8n^3 - 12n^2 + 6n - 1$$

$$\therefore S_n = \sum T_n$$

$$= 8 \sum n^3 - 12 \sum n^2 + 6 \sum n - \sum 1$$

...(i)

...(ii)

$$\begin{aligned} &= 8 \left[\frac{n(n+1)}{2} \right]^2 - 12 \left[\frac{n(n+1)(2n+1)}{6} \right] \\ &\quad + 6 \left[\frac{n(n+1)}{2} \right] - n \\ &= 2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n \\ &= n(n+1)[2n(n+1) - 2(2n+1) + 3] - n \\ &= n(n+1)[2n^2 + 2n - 4n - 2 + 3] - n \\ &= n(n+1)[2n^2 - 2n + 1] - n \\ &= n(n+1) \cdot 2n(n-1) + n(n+1) - n \\ &= 2n^2(n^2-1) + n^2 = n^2(2n^2-1) \end{aligned}$$

62. Here, $T_n = \frac{2n}{(2n+1)!} = \frac{1}{2n!} - \frac{1}{(2n+1)!}$

$$\sum T_n = \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots$$

$$= 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \dots = e^{-1}$$

4

Complex Numbers

The Real Number System

Natural Numbers (N) The numbers which are used for counting are known as natural numbers (also known as set of positive integers), i.e. $N = \{1, 2, 3, \dots\}$.

Whole Numbers (W) If '0' is included in the set of natural numbers, then we get the set of whole numbers, i.e.
 $W = \{0, 1, 2, \dots\} = \{N\} + \{0\}$.

Integers (Z or I) If negative natural numbers are included in the set of whole numbers, then we get set of integers, i.e.
 Z or $I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$.

Rational Numbers (Q) The numbers which are in the form of $\frac{p}{q}$, (where $p, q \in I$, $q \neq 0$), are called as rational numbers.
 e.g. $\frac{2}{3}, 3, \frac{1}{3}, 0.76, 1.2333, \frac{22}{7}$ etc.

Irrational Numbers (Q') The numbers which are not rational, i.e. which cannot be expressed in $\frac{p}{q}$ form or whose decimal part is non-terminating, non-repeating but which may represent magnitude of physical quantities are called irrational numbers.
 e.g. $\sqrt{2}, 5^{1/3}, \pi, e$, etc.

Real Numbers (R) The set of rational and irrational numbers is called a set of real numbers, i.e.

$$N \subset W \subset Z \subset Q \subset R$$

Chapter Snapshot

- The Real Number System
- Modulus of a Real Number
- Imaginary Number
- Complex Number
- Algebra of Complex Numbers
- Conjugate of a Complex Number
- Modulus of a Complex Number
- Argument (or Amplitude) of a Complex Number
- Various Forms of a Complex Number
- De-Moivre's Theorem
- Roots of Unity
- Geometrical Applications of Complex Numbers
- Loci in Complex Plane
- Logarithm of Complex Numbers

- The real number system is totally ordered for any two numbers $a, b \in \mathbb{R}$. We must say, either $a < b$ or $b < a$ or $a = b$.
- All real numbers can be represented by points on a straight line. This line is called as number line.
- Division by zero is meaningless.
- Number zero is neither positive nor negative but it is an even number.
- Square of a real number is always non-negative.
- An integer is said to be even, if it is divisible by 2, otherwise it is an odd number.
- Number '0' is an additive identity.
- The magnitude of a physical quantity may be expressed as a real number times, a standard unit.
- Number '1' is multiplicative identity.
- A positive integer p is called prime, if its only divisors are ± 1 and $\pm p$.
- Between two real numbers, there lie infinite real numbers.
- Infinity (∞) is the concept of the number greater than greatest you can imagine. It is not a number, it is just a concept, so we do not associate equality with it.

➤ **Example 1.** The value of $\frac{10^{25}}{\infty}$ is

- (a) 1 (b) 2 (c) 10^{50} (d) 0

Sol. (d) As we know, ∞ is the number greater than greatest we imagine. Also, the value of 1 upon ∞ is tending to zero. Hence, $\frac{10^{25}}{\infty} = 0$.

Intervals

Let a, x, b are real numbers, so that

$x \in [a, b] \Rightarrow a \leq x \leq b$, $[a, b]$ is known as the closed interval a, b .

$x \in (a, b) \Rightarrow a < x < b$, (a, b) is known as the open interval a, b .

$x \in (a, b] \Rightarrow a < x \leq b$, $(a, b]$ is known as open, closed interval a, b .

$x \in [a, b) \Rightarrow a \leq x < b$, $[a, b)$ is known as closed, open interval a, b .

➤ **Example 2.** If $a = \sqrt{3} + 1$, $b = 2\sqrt{2}$, then the value of $\frac{a}{b}$ lies in the interval

- (a) $(-1, 0)$ (b) $(0, 1)$ (c) $[0.5, 1.5]$ (d) $(0, 1)$

Sol. (b, c, d) Consider $\frac{a}{b} = \frac{\sqrt{3} + 1}{2\sqrt{2}} = 0.966$

Clearly, $0.966 \in (0, 1)$, $(0, 1]$ and $[0.5, 1.5]$.

Modulus of a Real Number

The modulus of a real number x is defined as follows:

$$|x| = \begin{cases} x, & \text{when } x > 0 \\ 0, & \text{when } x = 0 \\ -x, & \text{when } x < 0 \end{cases}$$

$$|x - a| = \begin{cases} x - a, & \text{when } x \geq a \\ -(x - a), & \text{when } x < a \end{cases}$$

➤ **Example 3.** If $f(x) = |x - 2| + |x| + |x + 3|$, then the value of $f(x)$ for $x \leq -3$ is

- (a) $3x - 1$
(b) $-3x + 1$
(c) $x - 3$
(d) $-(3x + 1)$

Sol. (d) $\because |x - 2| = -(x - 2) = 2 - x$ for $x \leq -3$
and $|x| = -x$ for $x \leq -3$
and $|x + 3| = -(x + 3)$ for $x \leq -3$
 $\therefore f(x) = 2 - x - x - x - 3$
 $= -3x - 1$
 $= -(3x + 1)$

Imaginary Number

Square root of a negative real number is imaginary number. While solving equation $x^2 + 1 = 0$, we get $x = \pm \sqrt{-1}$ which is imaginary. So, the quantity $\sqrt{-1}$ is denoted by 'i' called 'iota'. Thus, $i = \sqrt{-1}$.

e.g. $\sqrt{-2}, \sqrt{-3}, \sqrt{-4}, \dots$ may expressed as $i\sqrt{2}, i\sqrt{3}, 2i, \dots$

Integral powers of iota As we have seen

$$i = \sqrt{-1}, \text{ so } i^2 = -1, i^3 = -i \text{ and } i^4 = 1.$$

Hence, $n \in \mathbb{N}$, $i^n = i, -1, -i, 1$ attains four values according to the value of n , so

$$\begin{aligned} i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \\ i^{4n} \text{ or } i^{4n+4} &= 1 \end{aligned}$$

In other words,

$$i^n = \begin{cases} (-1)^{n/2}, & \text{if } n \text{ is an even integer} \\ (-1)^{\frac{n-1}{2}} i, & \text{if } n \text{ is an odd integer} \end{cases}$$

- $i^2 = \sqrt{-1} \times \sqrt{-1} \neq \sqrt{1}$
- $\sqrt{-a} \times \sqrt{-b} \neq \sqrt{ab}$, so for two real numbers a and b but $\sqrt{a} \cdot \sqrt{b} = \sqrt{a \cdot b}$ possible, if both a, b are non-negative.
- i is neither positive, zero nor negative. Due to this reason, order relations are not defined for imaginary numbers.
- The sum of any four consecutive powers of i is zero, i.e. $i^{4n+1} + i^{4n+2} + i^{4n+3} + i^{4n+4} = 0$

➤ **Example 4.** The value of i^{2014} is

- (a) i
(b) $-i$
(c) 1
(d) -1

Sol. (d) Consider $i^{2014} = (i^2)^{1007}$
 $= (-1)^{1007} = -1$

➤ **Example 5.** If $a < 0$, $b > 0$, then $\sqrt{a} \cdot \sqrt{b}$ is equal to

- (a) $-\sqrt{|a| \cdot b}$ (b) $\sqrt{|a| \cdot b} \cdot i$
(c) $\sqrt{|a| \cdot b}$ (d) None of these

Sol. (b) As we can only multiply the positive values in square root.

$$\therefore \sqrt{a} \sqrt{b} = \sqrt{-|a|} \sqrt{b}, \text{ as } a < 0 \text{ and } b > 0$$

$$\text{i.e. } \sqrt{-1} \cdot \sqrt{|a|} \sqrt{b} = i \sqrt{|a|} \sqrt{b} = \sqrt{|a| \cdot b} \cdot i$$

➤ **Example 6.** The value of the sum

$$\sum_{n=1}^{13} (i^n + i^{n+1}), \text{ where } i = \sqrt{-1} \text{ is}$$

- (a) i (b) $i - 1$ (c) $-i$ (d) 0

Sol. (b) Since, the sum of any four consecutive powers of i , is zero.

$$\therefore \sum_{n=1}^{13} (i^n + i^{n+1}) = \sum_{n=1}^{13} i^n + \sum_{n=1}^{13} i^{n+1} \quad \dots (i)$$

$$= (i + i^2 + i^3 + \dots + i^{13}) + (i^2 + i^3 + i^4 + \dots + i^{14})$$

$$= i - 1 \quad [\text{from Eq. (i)}]$$

➤ **Example 7.** The value of

$$\frac{2^n}{(1+i)^{2n}} + \frac{(1+i)^{2n}}{2^n}, n \in I, \text{ is equal to}$$

- (a) 0 (b) 2
(c) $\{1 + (-1)^n\} \cdot i^n$ (d) None of these

Sol. (c) Here, $\frac{2^n}{(1+i)^{2n}} + \frac{(1+i)^{2n}}{2^n}$

$$= \frac{2^n}{(1+i^2+2i)^n} + \frac{(1+i^2+2i)^n}{2^n}$$

$$= \frac{2^n}{(2i)^n} + \frac{(2i)^n}{2^n} = \frac{1}{i^n} + i^n$$

$$= \frac{i^n}{i^{2n}} + i^n = \frac{i^n}{(-1)^n} + i^n = i^n \left\{ \frac{1}{(-1)^n} + 1 \right\}$$

$$= i^n \cdot \{(-1)^n + 1\}$$

➤ **Example 8.** If $i = \sqrt{-1}$, then the number of values of $i^n + i^{-n}$ for different $n \in I$ is

- (a) 3 (b) 2 (c) 4 (d) 1

Sol. (a) As $i^n + i^{-n}$ can be written as

$$x = i^n + \frac{1}{i^n} = \frac{i^{2n} + 1}{i^n}$$

If $n = 4$, $x = \frac{1+1}{1} = 2$

If $n = 5$, $x = \frac{i^2+1}{i} = 0$

If $n = 6$, $x = \frac{1+1}{-1} = -2$

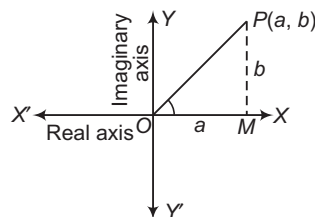
If $n = 7$, $x = \frac{i^2+1}{-i} = 0$

and so on.

Which shows there exist three different solutions for $n \in I$.

Complex Number

The complex number $z = a + ib = (a, b)$ can be represented by a point P , whose coordinates are referred to rectangular axes XOX' and YOY' , which are called real and imaginary axis respectively. The plane formed by rectangular axes is called argand plane or argand diagram or complex plane or Gaussian plane.



- A number of the form $z = x + iy = \text{Re } z + i \text{Im } z$ is called a complex number.
- Two complex numbers are said to be equal, if and only if their real parts and imaginary parts are separately equal,
i.e. $a + ib = c + id \Leftrightarrow a = c \text{ and } b = d$.
- The complex numbers do not possess the property of order, i.e.
 $x + iy < (\text{or}) > c + id$ is not defined.
- A complex number z is purely real, if $\text{Im}(z) = 0$ and said to be purely imaginary, if $\text{Re}(z) = 0$.
The complex number $0 = 0 + i \cdot 0$ is both purely real and purely imaginary.

➤ **Example 9.** If $z = (-5i) \left(\frac{1}{8}i \right)$, then $\text{Im}(z)$ is

equal to

- (a) 1 (b) 0
(c) -1 (d) None of these

Sol. (b) Consider $z = (-5i) \left(\frac{1}{8}i \right)$

Let us first express z in the format $a + ib$, then

$$z = \frac{-5}{8}i^2 = -\frac{5}{8}(-1) = \frac{5}{8} + i \cdot 0 \Rightarrow \text{Im}(z) = 0$$

➤ **Example 10.** If $4x + i(3x - y) = 3 + i(-6)$, where x and y are real numbers, then the value of x and y respectively are

- (a) $3, 33$ (b) $\frac{3}{5}, \frac{33}{5}$
(c) $\frac{3}{4}, \frac{33}{4}$ (d) None of these

Sol. (c) We have,

$$4x + i(3x - y) = 3 + i(-6) \quad \dots (i)$$

Equating the real and imaginary parts of Eq. (i), we get

$$4x = 3 \text{ and } 3x - y = -6,$$

which on solving simultaneously, we get

$$x = \frac{3}{4}$$

and

$$y = \frac{33}{4}$$

Algebra of Complex Numbers

Addition of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then

$$\begin{aligned} z_1 + z_2 &= x_1 + iy_1 + x_2 + iy_2 \\ &= (x_1 + x_2) + i(y_1 + y_2) \end{aligned}$$

$$\Rightarrow \operatorname{Re}(z_1 + z_2) = \operatorname{Re}(z_1) + \operatorname{Re}(z_2)$$

$$\text{and } \operatorname{Im}(z_1 + z_2) = \operatorname{Im}(z_1) + \operatorname{Im}(z_2)$$

Properties of Addition of Complex Numbers

$$(a) z_1 + z_2 = z_2 + z_1 \quad [\text{commutative law}]$$

$$(b) z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3 \quad [\text{associative law}]$$

$$(c) z + 0 = 0 + z \quad (\text{where, } 0 = 0 + i0) \quad [\text{additive identity law}]$$

➤ **Example 11.** The value of $3(7 + i7) + i(7 + i7)$ is

$$(a) 15 + 27i \quad (b) 14 + 28i \quad (c) 14 - 28i \quad (d) 14 + 23i$$

$$\begin{aligned} \text{Sol. (b) We have, } 3(7 + i7) + i(7 + i7) &= 21 + 21i + 7i + 7i^2 \\ &= 21 + 28i - 7 \quad [\because i^2 = -1] \\ &= 14 + 28i \end{aligned}$$

Subtraction of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 + iy_2) \\ &= (x_1 - x_2) + i(y_1 - y_2) \end{aligned}$$

$$\Rightarrow \operatorname{Re}(z_1 - z_2) = \operatorname{Re}(z_1) - \operatorname{Re}(z_2)$$

$$\text{and } \operatorname{Im}(z_1 - z_2) = \operatorname{Im}(z_1) - \operatorname{Im}(z_2)$$

➤ **Example 12.** The value of

$$\left[\left(\frac{1}{3} + i\frac{7}{3} \right) + \left(4 + i\frac{1}{3} \right) \right] - \left(-\frac{4}{3} + i \right) \text{ is}$$

$$(a) \frac{5}{3} - i\frac{17}{3}$$

$$(b) \frac{17}{3} - i\frac{5}{3}$$

$$(c) \frac{17}{3} + i\frac{5}{3}$$

$$(d) \frac{17}{5} - i\frac{4}{3}$$

$$\begin{aligned} \text{Sol. (c) } \left(\frac{1}{3} + i\frac{7}{3} + 4 + i\frac{1}{3} \right) + \frac{4}{3} - i &= \left(\frac{1}{3} + 4 + \frac{4}{3} \right) + i \left(\frac{7}{3} + \frac{1}{3} - 1 \right) \\ &= \left(\frac{1}{3} + \frac{4}{1} + \frac{4}{3} \right) + i \left(\frac{7}{3} + \frac{1}{3} - \frac{3}{3} \right) \\ &= \frac{(1 + 12 + 4)}{3} + i \left(\frac{7 + 1 - 3}{3} \right) \\ &= \frac{17}{3} + i\frac{5}{3} \end{aligned}$$

Multiplication of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ be two complex numbers, then

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1)(x_2 + iy_2) \\ &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ \Rightarrow z_1 \cdot z_2 &= [\operatorname{Re}(z_1)\operatorname{Re}(z_2) - \operatorname{Im}(z_1)\operatorname{Im}(z_2)] \\ &\quad + i[\operatorname{Re}(z_1)\operatorname{Im}(z_2) + \operatorname{Re}(z_2)\operatorname{Im}(z_1)] \end{aligned}$$

Properties of Multiplication of Complex Numbers

$$(a) z_1 \cdot z_2 = z_2 \cdot z_1 \quad [\text{commutative law}]$$

$$(b) (z_1 \cdot z_2)z_3 = z_1(z_2 \cdot z_3) \quad [\text{associative law}]$$

$$(c) \text{ If } z_1 \cdot z_2 = 1 = z_2 \cdot z_1, \text{ then } z_1 \text{ and } z_2 \text{ are multiplicative inverse of each other.}$$

$$(d) (i) z_1(z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3 \quad [\text{left distribution law}]$$

$$(ii) (z_2 + z_3)z_1 = z_2 \cdot z_1 + z_3 \cdot z_1 \quad [\text{right distribution law}]$$

➤ **Example 13.** The real values of x and y , if

$$\frac{(1+i)x - 2i}{(3+i)} + \frac{(2-3i)y + i}{(3-i)} = i, \text{ are respectively}$$

$$(a) 3, -1$$

$$(b) 3, 1$$

$$(c) -3, 1$$

$$(d) -3, -1$$

$$\text{Sol. (a) } \frac{(1+i)x - 2i}{(3+i)} + \frac{(2-3i)y + i}{(3-i)} = i$$

$$\begin{aligned} \Rightarrow \{ (1+i)x - 2i \} (3-i) + \{ (2-3i)y + i \} (3+i) &= i(3+i)(3-i) \\ &= i(3+i)(3-i) \\ \Rightarrow (1+i)(3-i)x - 2i(3-i) + (2-3i)(3+i)y &+ i(3+i) = 10i \\ \Rightarrow (4+2i)x - 6i - 2 + (9-7i)y + 3i - 1 &= 10i \\ \Rightarrow (4x - 2 + 9y - 1) + i(2x - 6 - 7y + 3) &= 10i \\ \Rightarrow (4x + 9y - 3) + i(2x - 7y - 3) &= 10i \end{aligned}$$

On equating real and imaginary parts on both sides, we get

$$4x + 9y = 3 \quad \dots (i)$$

$$\text{and } x - 7y = 13 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get $x = 3, y = -1$

➤ **Example 14.** The multiplicative inverse of $4 - 3i$ is

$$(a) \frac{4}{25} - \frac{3i}{25}$$

$$(b) \frac{4}{25} + \frac{3i}{25}$$

$$(c) \frac{4}{16} + \frac{3i}{25}$$

$$(d) \text{ None of these}$$

$$\text{Sol. (b) Let } z = 4 - 3i$$

Then, its multiplicative inverse is

$$\begin{aligned} \frac{1}{z} &= \frac{1}{4-3i} = \frac{1}{4-3i} \times \frac{4+3i}{4+3i} = \frac{4+3i}{16-9i^2} \\ &= \frac{4+3i}{16+9} \quad [\because (a-b)(a+b) = a^2 - b^2] \\ &= \frac{4+3i}{25} \quad [\because i^2 = -1] \\ &= \frac{4}{25} + \frac{3i}{25} \end{aligned}$$

Division of Complex Numbers

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ ($\neq 0$) be two complex numbers, then

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{1}{x_2^2 + y_2^2} [(x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2)]$$

➤ **Example 15.** The value of $\frac{z_1}{z_2}$, where $z_1 = 2 + 3i$

and $z_2 = 1 + 2i$, is

- (a) $\frac{8}{5} + \frac{1}{5}i$ (b) $\frac{8}{5} - \frac{1}{5}i$
(c) $\frac{1}{5} - \frac{8}{5}i$ (d) None of these

Sol. (b) $\because z_1 = 2 + 3i$ and $z_2 = 1 + 2i$

$$\therefore \frac{z_1}{z_2} = \frac{1}{1 + 2i} = \frac{1 - 2i}{(1 + 2i)(1 - 2i)} = \frac{1 - 2i}{1 - 4i^2} = \frac{1 - 2i}{5} = \frac{1}{5} - \frac{2}{5}i$$

$$\text{Then, } \frac{z_1}{z_2} = z_1 \cdot z_2^{-1} = (2 + 3i) \left(\frac{1}{5} - \frac{2}{5}i \right) = \left(\frac{2}{5} + \frac{6}{5} \right) + i \left(-\frac{4}{5} + \frac{3}{5} \right) = \frac{8}{5} - \frac{1}{5}i$$

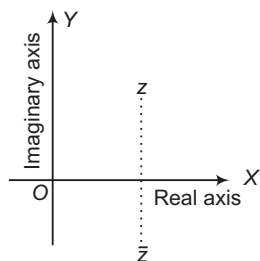
Aliter

Here, $x_1 = 2$, $y_1 = 3$, $x_2 = 1$ and $y_2 = 2$.

$$\therefore \frac{z_1}{z_2} = \frac{1}{x_2^2 + y_2^2} \{ (x_1 x_2 + y_1 y_2) + i(x_2 y_1 - x_1 y_2) \} = \frac{1}{1^2 + 2^2} \{ (2 \times 1 + 3 \times 2) + i(2 \times 3 - 1 \times 2) \} = \frac{1}{5} \{ (2 + 6) + i(6 - 2) \} = \frac{8}{5} - \frac{1}{5}i$$

Conjugate of a Complex Number

Geometrically, the conjugate of z is the reflection of point image of z in the real axis.



$$\therefore z = x + iy \text{ and } \bar{z} = x - iy.$$

$$\text{e.g. If } z = 3 + 4i, \text{ then } \bar{z} = 3 - 4i.$$

Properties of Conjugate

$\bar{\bar{z}}$ is the mirror image of z along real axis.

- (i) $\overline{(\bar{z})} = z$
(ii) $z = \bar{z} \Leftrightarrow z$ is purely real
(iii) $z = -\bar{z} \Leftrightarrow z$ is purely imaginary.

$$(iv) \operatorname{Re}(z) = \operatorname{Re}(\bar{z}) = \frac{z + \bar{z}}{2}$$

$$(v) \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

$$(vi) \overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$$

$$(vii) \overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$$

$$(viii) \overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

$$(ix) \overline{\left(\frac{z_1}{z_2} \right)} = \left(\frac{\bar{z}_1}{\bar{z}_2} \right), (z_2 \neq 0)$$

$$(x) z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2 \operatorname{Re}(\bar{z}_1 z_2) = 2 \operatorname{Re}(z_1 \bar{z}_2)$$

$$(xi) \bar{z}^n = (\bar{z})^n = \overline{(z^n)}$$

$$(xii) \text{ If } z = f(z_1), \text{ then } \bar{z} = f(\bar{z}_1)$$

➤ **Example 16.** If $z_1 = 9y^2 - 4 - 10ix$ and

$z_2 = 8y^2 + 20i$, where $z_1 = \bar{z}_2$, then $z = x + iy$ is equal to

- (a) $-2 + 2i$
(b) $-2 \pm 2i$
(c) $-2 \pm i$
(d) None of the above

Sol. (b) Given, $z_1 = \bar{z}_2$

$$\Rightarrow 9y^2 - 4 - 10ix = 8y^2 + 20i$$

$$\Rightarrow (y^2 - 4) - 10i(x + 2) = 0$$

Since, complex number is zero.

$$\Rightarrow y^2 - 4 = 0 \text{ and } x + 2 = 0$$

$$\therefore y = \pm 2$$

$$\text{and } x = -2$$

$$\text{Thus, } z = x + iy = -2 \pm 2i$$

➤ **Example 17.** If $(1 + i)z = (1 - i)\bar{z}$, then z is

- (a) $x(1 - i)$, $x \in R$
(b) $x(1 + i)$, $x \in R$
(c) $\frac{x}{1 + i}$, $x \in R^+$
(d) None of the above

Sol. (a) $(1 + i)(x + iy) = (1 - i)(x - iy)$

$$\Rightarrow (x - y) + i(x + y) = (x - y) - i(x + y)$$

$$\Rightarrow x + y = 0$$

$$\therefore z = x - ix = x(1 - i)$$

Modulus of a Complex Number

Let $z = x + iy$ be any complex number. Then, $|z| = \sqrt{x^2 + y^2}$ is called the modulus of the complex number z , where modulus $|z|$ represents distance of z from origin.

e.g. If $z = 3 + 2i$ is a complex number, then

$$|z| = \sqrt{3^2 + 2^2} = \sqrt{9 + 4} = \sqrt{13}$$

➤ **Example 18.** If z is a complex number satisfying the relation $|z+1| = z+2(1+i)$, then z is

- (a) $\frac{1}{2}(1+4i)$
 (b) $\frac{1}{2}(3+4i)$
 (c) $\frac{1}{2}(1-4i)$
 (d) $\frac{1}{2}(3-4i)$

Sol. (c) Let $z = x + iy$

$$\begin{aligned} \therefore |x + iy + 1| &= x + iy + 2(1+i) \\ \Rightarrow \sqrt{(x+1)^2 + y^2} &= (x+2) + i(y+2) \\ \Rightarrow \sqrt{(x+1)^2 + y^2} &= x+2 \quad \text{and} \quad y+2=0 \\ \Rightarrow (x+1)^2 + 4 &= (x+2)^2 \quad \text{and} \quad y=-2 \\ \Rightarrow 2x+5 &= 4x+4 \quad \text{and} \quad y=-2 \\ \Rightarrow x &= \frac{1}{2} \quad \text{and} \quad y=-2 \\ \Rightarrow z &= \frac{1}{2}(1-4i) \end{aligned}$$

➤ **Example 19.** If $z = 1 + i \tan \alpha$, where

$\pi < \alpha < \frac{3\pi}{2}$, then $|z|$ is equal to

- (a) $\sec \alpha$
 (b) $-\sec \alpha$
 (c) $\operatorname{cosec} \alpha$
 (d) None of the above

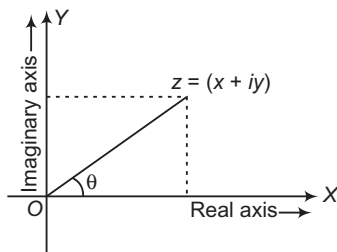
Sol. (b) As $z = 1 + i \tan \alpha$

$$\begin{aligned} \therefore |z| &= \sqrt{1 + \tan^2 \alpha} = |\sec \alpha| \\ \Rightarrow |z| &= -\sec \alpha, \text{ as } \pi < \alpha < \frac{3\pi}{2} \end{aligned}$$

Argument (or Amplitude) of a Complex Number

Argument of $z = \theta$

Argument of z is not unique. General value of argument of z is $2n\pi + \theta$.

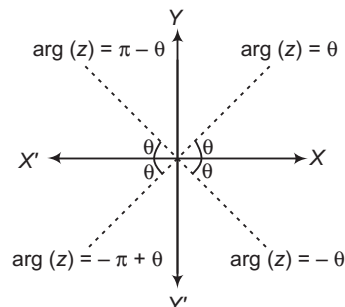


Principal Value of Argument

The value of θ of the argument, which satisfies the inequality $-\pi < \theta \leq \pi$ is called the principal value of the argument.

Principal values of the argument are $\theta, \pi - \theta, -\pi + \theta, -\theta$ according as the complex number lies on the Ist, IInd, IIIrd or IVth quadrant.

Here, $\theta = \tan^{-1} \frac{|y|}{|x|}$, where $z = x + iy$



e.g. $\arg(1+i) = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}$
 $\arg(1-i) = \tan^{-1} \left(\frac{-1}{1} \right) = -\frac{\pi}{4}$
 $\arg(-1-i) = \tan^{-1} \left(\frac{-1}{-1} \right) = -\pi + \frac{\pi}{4}$
 $\arg(-1+i) = \tan^{-1} \left(\frac{1}{-1} \right) = \pi - \frac{\pi}{4}$

- Argument of 0 is not defined.
- If $z_1 = z_2$, then $|z_1| = |z_2|$ and $\arg(z_1) = \arg(z_2)$
- Argument of purely imaginary number is $\frac{\pi}{2}$ or $-\frac{\pi}{2}$
- Argument of purely real number is 0 or π .

➤ **Example 20.** The modulus and argument of the complex number $\frac{1+2i}{1-3i}$ is

- (a) $\frac{1}{\sqrt{2}}, \frac{3\pi}{4}$
 (b) $\frac{1}{\sqrt{2}}, -\frac{3\pi}{4}$
 (c) $\frac{1}{2}, \frac{3\pi}{4}$
 (d) None of the above

Sol. (a) Given, $z = \frac{1+2i}{1-3i}$

$$\begin{aligned} \therefore z &= \frac{1+2i}{1-3i} \times \frac{1+3i}{1+3i} = \frac{1+3i+2i+6i^2}{1^2-(3i)^2} \\ &= \frac{1+5i+6(-1)}{1-9i^2} \quad [\because (a+b)(a-b)=a^2-b^2] \\ &= \frac{1+5i-6}{1-9i^2} \quad [\because i^2 = -1] \\ &= \frac{1+5i-6}{1+9} = \frac{-5+5i}{10} = \frac{-1+i}{2} \end{aligned}$$

$$\Rightarrow z = -\frac{1}{2} + \frac{1}{2}i$$

$$\therefore |z| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} \quad [\because |a + ib| = \sqrt{a^2 + b^2}]$$

$$= \sqrt{\frac{1}{4} + \frac{1}{4}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore \text{Now, } \tan \theta = \left| \frac{\frac{1}{-2}}{\frac{1}{-2}} \right| \quad \left[\because \theta = \tan^{-1} \left(\frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \right) \right]$$

$$\Rightarrow \tan \theta = 1 = \tan \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$$

Since, the real part of z is negative and imaginary part of z is positive, so the point lies in IInd quadrant.

$$\therefore \arg(z) = \pi - \theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

Hence, modulus = $\frac{1}{\sqrt{2}}$

and $\arg(z) = \frac{3\pi}{4}$

Work Book Exercise 4.1

- 1 If $\frac{x-3}{3+i} + \frac{y-3}{3-i} = i$, where $x, y \in R$, then

- a $x=2$ and $y=-8$ b $x=-2$ and $y=8$
c $x=-2$ and $y=-8$ d $x=2$ and $y=8$

- 2 What is the real part of $(1+i)^{50}$?

- a 0 b 2^{25}
c -2^{25} d -2^{50}

- 3 The complex number z satisfies $z + |z| = 2 + 8i$. Then, the value of $|z|$ is

- a 10 b 13 c 17 d 23

- 4 If $z + z^3 = 0$, then which of the following must be true on the complex plane?

- a $\operatorname{Re}(z) < 0$ b $\operatorname{Re}(z) = 0$
c $\operatorname{Im}(z) = 0$ d $z^4 = 1$

- 5 The sequences $S = i + 2i^2 + 3i^3 + \dots$ upto 100 terms simplifies to, where $i = \sqrt{-1}$

- a $50(1-i)$ b $25i$
c $25(1+i)$ d $100(1-i)$

- 6 Given $i = \sqrt{-1}$, then the value of the sum

$$\frac{1}{1+i} + \frac{1}{1-i} + \frac{1}{-1+i} + \frac{1}{-1-i} + \frac{2}{1+i} + \frac{2}{1-i} + \frac{2}{-1+i} + \frac{2}{-1-i} + \frac{3}{1+i} + \frac{3}{1-i} + \frac{3}{-1+i} + \frac{3}{-1-i} + \dots + \frac{n}{1+i} + \frac{n}{1-i} + \frac{n}{-1+i} + \frac{n}{-1-i} \text{ is}$$

- a $2n^2 + 2n$ b $2in^2 + 2in$
c $(1+i)n^2$ d None of these

- 7 Let $i = \sqrt{-1}$. The product of the real part of the roots of $z^2 - z = 5 - 5i$ is

- a -25 b -6 c -5 d 25

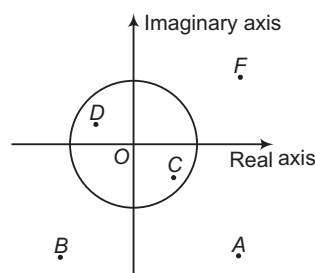
- 8 Number of complex numbers z satisfying $z^3 = \bar{z}$ is

- a 1 b 2
c 4 d 5

- 9 Number of real solution of the equation, $z^3 + iz - 1 = 0$ is

- a zero b one
c two d three

- 10 The diagram shows several numbers in the complex plane. The circle is the unit circle centered at the origin. One of these numbers is the reciprocal of F , which is



- a A b B c C d D

- 11 Identify the incorrect statement.

- a No non-zero complex number z satisfies the equation $\bar{z} = -4z$
b $z = \bar{z}$ implies that z is purely real
c $z = -\bar{z}$ implies that z is purely imaginary
d If z_1, z_2 are the roots of the quadratic equation $az^2 + bz + c = 0$ such that $\operatorname{Im}(z_1 z_2) \neq 0$, then a, b, c must be real numbers

- 12 If $z = (3 + 7i)(p + iq)$, where $p, q \in I - \{0\}$, is purely imaginary, then minimum value of $|z|^2$ is

- a 0 b 58
c $\frac{3364}{3}$ d 3364

- 13 Consider two complex numbers α and β as

$$\alpha = \left(\frac{a+bi}{a-bi} \right)^2 + \left(\frac{a-bi}{a+bi} \right)^2, \text{ where } a, b \in R \text{ and}$$

$$\beta = \frac{z-1}{z+1}, \text{ where } |z| = 1, \text{ then}$$

- a both α and β are purely real
b both α and β are purely imaginary
c α is purely real and β is purely imaginary
d β is purely real and α is purely imaginary

- 14 If z is a complex number having the argument θ , $0 < \theta < \frac{\pi}{2}$ and satisfying the equality $|z - 3i| = 3$.

Then, $\cot \theta - \frac{6}{z}$ is equal to

- a 1 b -1 c i d $-i$

Various Forms of a Complex Number

Polar Form

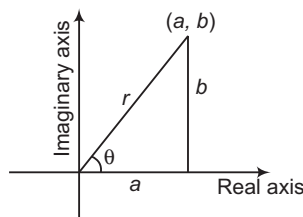
Let $z = a + ib$ be any complex number, then by taking

$$a = r \cos \theta \quad \text{and} \quad b = r \sin \theta$$

We have,

$$z = a + ib = r(\cos \theta + i \sin \theta)$$

(known as polar form)



where, $r = |z|$ and θ = Principal value of argument of z .

➤ **Example 21.** The polar form of the complex number $\frac{-16}{1 + i\sqrt{3}}$ is

- (a) $4 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$ (b) $\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$
 (c) $8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$ (d) None of these

Sol. (c) Given complex number = $\frac{-16}{1 + i\sqrt{3}}$

$$= \frac{-16}{1 + i\sqrt{3}} \times \frac{(1 - i\sqrt{3})}{(1 - i\sqrt{3})}$$

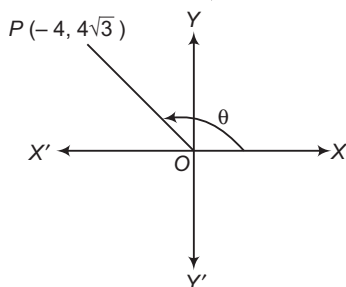
$$= \frac{-16(1 - i\sqrt{3})}{1 - (i\sqrt{3})^2}$$

$$= \frac{-16(1 - i\sqrt{3})}{1 + 3}$$

$$= \frac{-16(1 - i\sqrt{3})}{4}$$

$$= -4(1 - i\sqrt{3})$$

$$= -4 + i4\sqrt{3}$$



Let $-4 = r \cos \theta$, $4\sqrt{3} = r \sin \theta$

By squaring and adding, we get

$$16 + 48 = r^2(\cos^2 \theta + \sin^2 \theta)$$

which gives, $r^2 = 64$, i.e. $r = 8$

Hence, $\cos \theta = \frac{-1}{2}$
 $\sin \theta = \frac{\sqrt{3}}{2}$
 $\Rightarrow \theta = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

Thus, the required polar form is

$$8 \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

➤ **Example 22.** Let z and w be two non-zero complex numbers, such that $|z| = |w|$ and $\arg(z) + \arg(w) = \pi$. Then, z is equal to

- (a) w (b) $-w$ (c) $-\bar{w}$ (d) $\bar{w}$

Sol. (c) Given $|z| = |w| = r$ and $\arg(w) = \theta$

Also, $\arg(z) + \arg(w) = \pi$

$\Rightarrow \arg(z) = \pi - \theta$

Now, $z = r[\cos(\pi - \theta) + i \sin(\pi - \theta)]$
 $= r[-\cos \theta + i \sin \theta]$
 $= -r(\cos \theta + i \sin \theta) = -\bar{w}$

Eulerian Form of a Complex Number

We have,

$$e^{i\theta} = \cos \theta + i \sin \theta \quad \text{and} \quad e^{-i\theta} = \cos \theta - i \sin \theta.$$

These two are called Euler's notations.

Let z be any complex number, such that $|z| = r$ and $\arg(z) = \theta$. Then, in polar form, z can be written as

$$z = r(\cos \theta + i \sin \theta)$$

Using Euler's notations, we have

$$z = re^{i\theta}$$

This form of z is known as the Eulerian form.

➤ **Example 23.** Express the following complex numbers in Eulerian form.

- (i) $1 + i$ (ii) $-2 + 2i$ (iii) $-1 - i\sqrt{3}$

Sol. (i) Given, $z = 1 + i$.

Then, $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

Let θ be the argument of z . Then, $\tan \theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4}$

So, Eulerian form of z is $\sqrt{2}e^{i\frac{\pi}{4}}$.

(ii) Given, $z = -2 + 2i$

Then, $r = |z| = \sqrt{(-2)^2 + 2^2} = 2\sqrt{2}$

Let θ be the argument of z . Then,

$$\tan \theta = \frac{2}{-2} = -1 \Rightarrow \theta = \frac{3\pi}{4}$$

So, Eulerian form of z is $2\sqrt{2}e^{i\frac{3\pi}{4}}$.

(iii) Given, $z = -1 - i\sqrt{3}$

Then, $r = |z| = \sqrt{(-1)^2 + (-\sqrt{3})^2} = 2$

Let θ be the argument of z . Then,

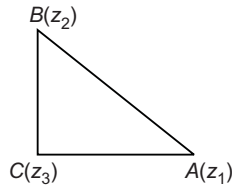
$$\tan \theta = \frac{-\sqrt{3}}{-1} = \sqrt{3} \Rightarrow \theta = \frac{2\pi}{3}$$

So, Eulerian form of z is $2e^{-i\frac{2\pi}{3}}$.

➤ **Example 24.** Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C . Show that

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2).$$

Sol. In an isosceles ΔABC , $AC = BC$ and BC perpendicular to AC . It means that AC is rotated through angle $\frac{\pi}{2}$ to occupy the position BC .



We have,

$$\frac{z_2 - z_3}{z_1 - z_3} = e^{i\pi/2} = i$$

$$\begin{aligned} \Rightarrow z_2 - z_3 &= i(z_1 - z_3) \\ \Rightarrow z_2^2 + z_3^2 - 2z_2z_3 &= -(z_1^2 + z_3^2 - 2z_1z_3) \\ \Rightarrow z_2^2 + z_3^2 - 2z_1z_2 &= 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2 \\ &= 2(z_1 - z_3)(z_3 - z_2) \\ \Rightarrow (z_1 - z_2)^2 &= 2(z_1 - z_3)(z_3 - z_2) \end{aligned}$$

Properties of Modulus of Complex Number

- i. $|z| \geq 0 \Rightarrow |z| = 0$ iff $z = 0$ and $|z| > 0$ iff $z \neq 0$
- ii. $-|z| \leq \operatorname{Re}(z) \leq |z|$ and $-|z| \leq \operatorname{Im}(z) \leq |z|$
- iii. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
- iv. $z\bar{z} = |z|^2$
- v. $|z_1 z_2| = |z_1| |z_2|$
In general, $|z_1 z_2 z_3 \dots z_n| = |z_1| |z_2| |z_3| \dots |z_n|$
- vi. $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$, ($z_2 \neq 0$)
- vii. $|z_1 \pm z_2| \leq |z_1| + |z_2|$
In general, $|z_1 \pm z_2 \pm z_3 \pm \dots \pm z_n| \leq |z_1| + |z_2| + |z_3| + \dots + |z_n|$
- viii. $|z_1 \pm z_2| \geq ||z_1| - |z_2||$
- ix. $|z^n| = |z|^n$
- x. $||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|$
Thus, $|z_1| + |z_2|$ is the greatest possible value of $|z_1 + z_2|$ and $||z_1| - |z_2||$ is the least possible value of $|z_1 + z_2|$.
- xi. $|z_1 \pm z_2|^2 = (z_1 \pm z_2)(\bar{z}_1 \pm \bar{z}_2)$
 $= |z_1|^2 + |z_2|^2 \pm (z_1 \bar{z}_2 + \bar{z}_1 z_2)$
 $= |z_1|^2 + |z_2|^2 \pm 2\operatorname{Re}(z_1 \bar{z}_2)$

$$\begin{aligned} \text{xii. } |z_1 + z_2|^2 &= |z_1|^2 + |z_2|^2 \\ \Leftrightarrow \frac{z_1}{z_2} &\text{ is purely imaginary.} \end{aligned}$$

$$\text{xiii. } |z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

$$\begin{aligned} \text{xiv. } |az_1 - bz_2|^2 + |bz_1 + az_2|^2 \\ = (a^2 + b^2)(|z_1|^2 + |z_2|^2), \text{ where } a, b \in \mathbb{R}. \end{aligned}$$

➤ If z is unimodular, then $|z| = 1$. Now, if $f(z)$ is a unimodular, then it is always be expressed as $f(z) = \cos \theta + i \sin \theta$, $\theta \in \mathbb{R}$.

• Square root of $z = a + ib$ is given by

$$\sqrt{z} = \pm \left\{ \sqrt{\frac{|z|+a}{2}} + i \sqrt{\frac{|z|-a}{2}} \right\}, a = \operatorname{Re}(z)$$

To find the square root of $a - ib$, replace i by $-i$ in the above result.

• If $x, y \in \mathbb{R}$, then

$$\begin{aligned} \sqrt{x+iy} + \sqrt{x-iy} &= \sqrt{2(\sqrt{x^2+y^2}+x)} \\ \sqrt{x+iy} - \sqrt{x-iy} &= i\sqrt{2(\sqrt{x^2+y^2}-x)} \end{aligned}$$

➤ **Example 25.** If $x - iy = \sqrt{\frac{a-ib}{c-id}}$, then

$(x^2 + y^2)^2$ is equal to

- (a) $\frac{a^2 - b^2}{c^2 - d^2}$
- (b) $\frac{a^2 + b^2}{c^2 + d^2}$
- (c) $\frac{a^2 + b^2}{c^2 - d^2}$
- (d) None of these

Sol. (b) Given, $x - iy = \sqrt{\frac{a-ib}{c-id}}$

$$\Rightarrow x + i(-y) = \left(\frac{a-ib}{c-id} \right)^{1/2}$$

On taking modulus both sides, we get

$$|x + i(-y)| = \left| \left(\frac{a-ib}{c-id} \right)^{1/2} \right|$$

$$\Rightarrow \sqrt{x^2 + (-y)^2} = \left| \frac{a-ib}{c-id} \right|^{1/2}$$

$$[\because |x + iy| = \sqrt{x^2 + y^2} \text{ and } |z^n| = |z|^n]$$

$$\Rightarrow \sqrt{x^2 + y^2} = \left| \frac{a-ib}{c-id} \right|^{1/2}$$

On squaring both sides, we get

$$x^2 + y^2 = \left| \frac{a-ib}{c-id} \right|$$

$$\Rightarrow x^2 + y^2 = \frac{|a-ib|}{|c-id|} \quad \left[\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$$

$$\Rightarrow (x^2 + y^2) = \frac{\sqrt{a^2 + b^2}}{\sqrt{c^2 + d^2}} \quad [\because |x - iy| = \sqrt{x^2 + y^2}]$$

On squaring both sides, we get

$$(x^2 + y^2)^2 = \frac{a^2 + b^2}{c^2 + d^2}$$

➤ **Example 26.** For a complex number z , the minimum value of $|z| + |z - 2|$ is

- (a) 1 (b) 2
(c) 3 (d) None of these

Sol. (b) By using $|z_1| + |z_2| \geq |z_1 - z_2|$
We have, $|z| + |z - 2| \geq |z - (z - 2)|$
 $\therefore |z| + |z - 2| \geq 2$

➤ **Example 27.** If $|z_1 - 1| < 1$, $|z_2 - 2| < 2$ and

- $|z_3 - 3| < 3$, then $|z_1 + z_2 + z_3|$
(a) is less than 6 (b) is more than 3
(c) is less than 12 (d) lies between 6 and 12

Sol. (c) $|z_1 + z_2 + z_3| = |(z_1 - 1) + (z_2 - 2) + (z_3 - 3) + 6|$
 $\leq |z_1 - 1| + |z_2 - 2| + |z_3 - 3| + 6$
 $< 1 + 2 + 3 + 6 = 12$

➤ **Example 28.** If α, β are two complex numbers, then $|\alpha|^2 + |\beta|^2$ is equal to

- (a) $\frac{1}{2}(|\alpha + \beta|^2 - |\alpha - \beta|^2)$ (b) $\frac{1}{2}(|\alpha + \beta|^2 + |\alpha - \beta|^2)$
(c) $|\alpha + \beta|^2 + |\alpha - \beta|^2$ (d) None of these

Sol. (b) $|\alpha + \beta|^2 = (\alpha + \beta)(\overline{\alpha + \beta}) = (\alpha + \beta)(\overline{\alpha} + \overline{\beta})$
 $= \alpha\overline{\alpha} + \beta\overline{\beta} + \alpha\overline{\beta} + \beta\overline{\alpha}$
 $= |\alpha|^2 + |\beta|^2 + \alpha\overline{\beta} + \beta\overline{\alpha}$... (i)
 $|\alpha - \beta|^2 = (\alpha - \beta)(\overline{\alpha - \beta}) = \alpha\overline{\alpha} + \beta\overline{\beta} - \alpha\overline{\beta} - \beta\overline{\alpha}$
 $= |\alpha|^2 + |\beta|^2 - \alpha\overline{\beta} - \beta\overline{\alpha}$... (ii)

Adding Eqs. (i) and (ii), we get

$$|\alpha|^2 + |\beta|^2 = \frac{1}{2}\{|\alpha + \beta|^2 + |\alpha - \beta|^2\}$$

➤ **Example 29.** If z_1 and z_2 are two complex numbers, such that $|z_1| < 1 < |z_2|$, then prove that

$$\left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1.$$

Sol. Given, $|z_1| < 1$ and $|z_2| > 1$... (i)

Then, to prove

$$\left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1 \quad \left[\because \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} \right]$$

$$\Rightarrow |1 - z_1 \bar{z}_2| < |z_1 - z_2| \quad \dots (ii)$$

On squaring both sides, we get

$$(1 - z_1 \bar{z}_2)(1 - \bar{z}_1 z_2) < (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) \quad (\because |z|^2 = z\bar{z})$$

$$\Rightarrow 1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 < z_1 \bar{z}_1 - z_1 \bar{z}_2 - z_2 \bar{z}_1 + z_2 \bar{z}_2$$

$$\Rightarrow 1 + |z_1|^2 |z_2|^2 < |z_1|^2 + |z_2|^2$$

$$\Rightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2 |z_2|^2 < 0$$

$$\Rightarrow (1 - |z_1|^2)(1 - |z_2|^2) < 0 \quad \dots (iii)$$

which is true by Eq. (i) as $|z_1| < 1$ and $|z_2| > 1$.

$$\therefore (1 - |z_1|^2) > 0$$

$$\text{and } (1 - |z_2|^2) < 0$$

$\therefore$ Eq. (iii) is true, whenever Eq. (i) is true.

$$\Rightarrow \left| \frac{1 - z_1 \bar{z}_2}{z_1 - z_2} \right| < 1 \quad \text{Hence proved.}$$

➤ **Example 30.** The value of $\sqrt{-8 - 6i}$ is equal to
(a) $1 \pm 3i$ (b) $\pm(1 - 3i)$ (c) $\pm(1 + 3i)$ (d) $\pm(3 - i)$

Sol. (b) Given, $\sqrt{-8 - 6i} = x + iy = z$

$$\text{Then, } \sqrt{z} = \pm \left[\sqrt{\frac{|z| + x}{2}} - i \sqrt{\frac{|z| - x}{2}} \right] \quad [\because \operatorname{Im}(z) < 0]$$

$$\Rightarrow \sqrt{z} = \pm \left[\sqrt{\frac{10 - 8}{2}} - i \sqrt{\frac{10 + 8}{2}} \right] \quad [\because |z| = \sqrt{x^2 + y^2}]$$

$$\Rightarrow \sqrt{z} = \pm(1 - 3i)$$

➤ **Example 31.** The value of

$$(4 + 3\sqrt{-20})^{1/2} + (4 - 3\sqrt{-20})^{1/2} \text{ is}$$

(a) ± 6 (b) 0 (c) $\pm\sqrt{5}$ (d) ± 3

Sol. (a) We may write, $(4 + 3\sqrt{-20}) = (4 + 6i\sqrt{5})$

$$\text{Let } (4 + 3\sqrt{-20})^{1/2} = (x + iy)$$

$$\text{Then, } (4 + 6i\sqrt{5})^{1/2} = (x + iy)$$

$$\Rightarrow 4 + 6i\sqrt{5} = (x^2 - y^2) + (2xy)i$$

$$\Rightarrow x^2 + y^2 = 4 \quad \text{and} \quad 2xy = 6\sqrt{5}$$

$$\therefore x^2 + y^2 = \sqrt{(x^2 - y^2)^2 + 4x^2 y^2}$$

$$= \sqrt{(16 + 180)} = \sqrt{196} = 14$$

On solving the equations $x^2 + y^2 = 14$ and $x^2 - y^2 = 4$, we get

$$x^2 = 9 \quad \text{and} \quad y^2 = 5$$

$$\therefore x = \pm 3 \quad \text{and} \quad y = \pm\sqrt{5}$$

Since, $xy > 0$, it follows that x and y are of the same sign.

$$\therefore x = 3, y = \sqrt{5} \quad \text{or} \quad x = -3, y = -\sqrt{5}$$

$$\text{So, } (4 + 3\sqrt{-20})^{1/2} = (4 + 6i\sqrt{5})^{1/2} = \pm(3 + \sqrt{5}i) \quad \dots (i)$$

$$\therefore (4 - 3\sqrt{-20})^{1/2} = \pm(3 - \sqrt{5}i) \quad \dots (ii)$$

$$\text{Hence, } (4 + 3\sqrt{-20})^{1/2} + (4 - 3\sqrt{-20})^{1/2} = \pm 6$$

[on adding Eqs. (i) and (ii)]

Aliter

$$\text{Here, } x = 4, y = 6\sqrt{5}$$

$$\therefore = (4 + 3\sqrt{-20})^{1/2} + (4 - 3\sqrt{-20})^{1/2}$$

$$= \sqrt{2 \{ \sqrt{4^2 + (6\sqrt{5})^2} + 4 \}} = \sqrt{2(\sqrt{16 + 180} + 4)}$$

$$= \sqrt{2(14 + 4)} = \sqrt{36} = \pm 6$$

Properties of Arguments

i. $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2) + 2k\pi$
($k = 0$ or 1 or -1)

In general,

$$\arg(z_1 z_2 z_3 \dots z_n) = \arg(z_1) + \arg(z_2) + \arg(z_3) + \dots + \arg(z_n) + 2k\pi \quad (k = 0 \text{ or } 1 \text{ or } -1)$$

ii. $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2k\pi$
($k = 0$ or 1 or -1)

iii. $\arg\left(\frac{z}{\bar{z}}\right) = 2\arg(z) + 2k\pi$ ($k = 0$ or 1 or -1)

iv. $\arg(z^n) = n\arg(z) + 2k\pi$ ($k = 0$ or 1 or -1)

v. If $\arg\left(\frac{z_2}{z_1}\right) = \theta$, then $\arg\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta$, where $k \in I$.

- vi. $\arg(\bar{z}) = -\arg(z) = \arg\left(\frac{1}{z}\right)$
- vii. If $\arg(z) = 0 \Rightarrow z$ is real.
- viii. $\arg(z_1 \bar{z}_2) = \arg(z_1) - \arg(z_2)$
- ix. $|z_1 + z_2| = |z_1 - z_2| \Rightarrow \arg(z_1) - \arg(z_2) = \frac{\pi}{2}$
- x. $|z_1 + z_2| = |z_1| + |z_2| \Rightarrow \arg(z_1) = \arg(z_2)$
- xi. If $|z_1| \leq 1, |z_2| \leq 1$, then
 (a) $|z_1 - z_2|^2 \leq (|z_1| - |z_2|)^2 + [\arg(z_1) - \arg(z_2)]^2$
 (b) $|z_1 + z_2|^2 \geq (|z_1| + |z_2|)^2 - [\arg(z_1) - \arg(z_2)]^2$
- xii. $|z_1 \pm z_2|^2 = |z_1|^2 + |z_2|^2 \pm 2|z_1||z_2| \cos(\theta_1 - \theta_2)$
- xiii. $z_1 \bar{z}_2 + \bar{z}_1 z_2 = 2|z_1||z_2| \cos(\theta_1 - \theta_2)$
 where, $\theta_1 = \arg(z_1)$ and $\theta_2 = \arg(z_2)$

✎ Proper value of k must be chosen, so that RHS of (i), (ii), (iii) and (iv) lies in $(-\pi, \pi)$.

➤ **Example 32.** If $z = \frac{\sqrt{3} + i}{\sqrt{3} - i}$, then the

fundamental amplitude of z is

- (a) $-\frac{\pi}{3}$ (b) $\frac{\pi}{3}$
 (c) $\frac{\pi}{6}$ (d) None of these

Sol. (b) $\arg(z) = \arg\left(\frac{\sqrt{3} + i}{\sqrt{3} - i}\right)$
 $= \arg(\sqrt{3} + i) - \arg(\sqrt{3} - i)$
 $= \tan^{-1} \frac{1}{\sqrt{3}} - \tan^{-1} \left(\frac{-1}{\sqrt{3}}\right) = \frac{\pi}{6} + \frac{\pi}{6} = \frac{\pi}{3}$

➤ **Example 33.** The value of $\arg(i\omega) + \arg(i\omega^2)$, where $i = \sqrt{-1}$ and $\omega = \sqrt[3]{1} = \text{non-real}$, is

- (a) 0 (b) $\frac{\pi}{2}$
 (c) π (d) None of these

Sol. (c) $\arg(i\omega) + \arg(i\omega^2) = \arg(i^2 \cdot \omega^3) = \arg(-1) = \pi$

➤ **Example 34.** If z_1, z_2 and z_3, z_4 are two pairs of conjugate complex numbers, then

$\arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right)$ equals to

- (a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{2}$ (d) π

Sol. (a) We have, $z_2 = \bar{z}_1$ and $z_4 = \bar{z}_3$
 $\therefore z_1 z_2 = |z_1|^2$ and $z_3 z_4 = |z_3|^2$

$$\text{Now, } \arg\left(\frac{z_1}{z_4}\right) + \arg\left(\frac{z_2}{z_3}\right) = \arg\left(\frac{z_1 z_2}{z_4 z_3}\right)$$

$$= \arg\left(\frac{|z_1|^2}{|z_3|^2}\right) = \arg\left(\left|\frac{z_1}{z_3}\right|^2\right) = 0$$

[$\because$ argument of positive real number is zero]

De-Moivre's Theorem

- (a) If $n \in I$ (the set of integers), then
 $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$.
- (b) If $n \in Q$ (the set of rational numbers), then
 $\cos n\theta + i \sin n\theta$ is one of the values of
 $(\cos \theta + i \sin \theta)^n$.

Remark

- (i) The theorem is also true for $(\cos \theta - i \sin \theta)$, i.e.
 $(\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$, because
 $(\cos \theta - i \sin \theta)^n = [\cos(-\theta) + i \sin(-\theta)]^n$
 $= \cos(n(-\theta)) + i \sin(n(-\theta))$
 $= \cos(-n\theta) + i \sin(-n\theta)$
 $= \cos n\theta - i \sin n\theta$

(ii) $\frac{1}{\cos \theta + i \sin \theta} = (\cos \theta + i \sin \theta)^{-1} = \cos \theta - i \sin \theta$

(iii) If $z = (\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 + i \sin \theta_2) \dots (\cos \theta_n + i \sin \theta_n)$

Then, $z = \cos(\theta_1 + \theta_2 + \dots + \theta_n) + i \sin(\theta_1 + \theta_2 + \dots + \theta_n)$

- (iv) If $z = r(\cos \theta + i \sin \theta)$ and n is a positive integer, then

$$z^{1/n} = r^{1/n} \left[\cos\left(\frac{2k\pi + \theta}{n}\right) + i \sin\left(\frac{2k\pi + \theta}{n}\right) \right],$$

where $k = 0, 1, 2, \dots, (n-1)$

- ✎ • $(\sin \theta \pm i \cos \theta)^n \neq \sin n\theta \pm i \cos n\theta$
 • $(\sin \theta + i \cos \theta)^n = \left[\cos\left(\frac{\pi}{2} - \theta\right) + i \sin\left(\frac{\pi}{2} - \theta\right) \right]^n$
 $= \cos\left(\frac{n\pi}{2} - n\theta\right) + i \sin\left(\frac{n\pi}{2} - n\theta\right)$
 • $(\cos \theta + i \sin \phi)^n \neq \cos n\theta + i \sin n\phi$

➤ **Example 35.** If $z = \frac{\cos \theta + i \sin \theta}{\cos \theta - i \sin \theta}$, $\frac{\pi}{4} < \theta < \frac{\pi}{2}$.

Then, $\arg(z)$ is

- (a) 2θ (b) $2\theta - \pi$
 (c) $\pi + 2\theta$ (d) None of these

Sol. (a) $z = (\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta$

where, $\frac{\pi}{2} < 2\theta < \pi$

$\therefore$ Clearly, $\arg(z) = 2\theta$ $\left[\because \frac{\pi}{2} < \arg(z) < \pi \right]$

➤ **Example 36.** If $x_r = \cos\left(\frac{\pi}{2^r}\right) + i \sin\left(\frac{\pi}{2^r}\right)$, then

the value of $x_1 x_2 x_3 \dots \infty$ is

- (a) -1 (b) 1
(c) 0 (d) None of these

Sol. (a) $\therefore x_r = \cos\left(\frac{\pi}{2^r}\right) + i \sin\left(\frac{\pi}{2^r}\right)$

$$\therefore x_1 = \cos\frac{\pi}{2} + i \sin\frac{\pi}{2}$$

$$x_2 = \cos\frac{\pi}{2^2} + i \sin\frac{\pi}{2^2}$$

$$\vdots$$

$$\Rightarrow x_1 x_2 x_3 \dots = \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right) \left(\cos\frac{\pi}{2^2} + i \sin\frac{\pi}{2^2}\right) \dots$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots\right) + i \sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots\right)$$

$$= \cos\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots\right) + i \sin\left(\frac{\pi}{2} + \frac{\pi}{2^2} + \dots\right)$$

$$= \cos\pi + i \sin\pi = -1$$

➤ **Example 37.** $\frac{(\cos\theta + i \sin\theta)^4}{(\sin\theta + i \cos\theta)^5}$ is equal to

- (a) $\cos\theta - i \sin\theta$ (b) $\cos 9\theta - i \sin 9\theta$
(c) $\sin\theta - i \cos\theta$ (d) $\sin 9\theta - i \cos 9\theta$

Sol. (d) $\frac{(\cos\theta + i \sin\theta)^4}{(\sin\theta + i \cos\theta)^5} = \frac{(\cos\theta + i \sin\theta)^4}{i^5 \left(\frac{1}{i} \sin\theta + \cos\theta\right)^5}$

$$= \frac{(\cos\theta + i \sin\theta)^4}{i(\cos\theta - i \sin\theta)^5} = \frac{(\cos\theta + i \sin\theta)^4}{i(\cos\theta + i \sin\theta)^{-5}}$$

$$= \frac{1}{i} (\cos\theta + i \sin\theta)^9 = \sin 9\theta - i \cos 9\theta$$

➤ **Example 38.** If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$,

then

- (a) $\operatorname{Re}(z) = 0$ (b) $\operatorname{Im}(z) = 0$
(c) $\operatorname{Re}(z) > 0, \operatorname{Im}(z) > 0$ (d) $\operatorname{Re}(z) > 0, \operatorname{Im}(z) < 0$

Sol. (b) Given,

$$z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$$

$$= \left[\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)\right]^5 + \left[\cos\left(\frac{\pi}{6}\right) - i \sin\left(\frac{\pi}{6}\right)\right]^5$$

$$= \cos\frac{5\pi}{6} + i \sin\frac{5\pi}{6} + \cos\frac{5\pi}{6} - i \sin\frac{5\pi}{6} = 2\cos\frac{5\pi}{6}$$

Hence, $\operatorname{Im}(z) = 0$.

➤ **Example 39.** The product of all the values of $\left(\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right)^{3/4}$ is

- (a) -1 (b) 1 (c) $3/2$ (d) $-1/2$

Sol. (b) Given, $\left[\cos\frac{\pi}{3} + i \sin\frac{\pi}{3}\right]^{3/4} = [\cos\pi + i \sin\pi]^{1/4}$

Since, the expression has only 4 different roots, therefore on putting $n = 0, 1, 2, 3$ in

$$\cos\left[\frac{2n\pi + \pi}{4}\right] + i \sin\left[\frac{2n\pi + \pi}{4}\right] \text{ and multiplying them,}$$

$$\text{we get } \left[\cos\frac{\pi}{4} + i \sin\frac{\pi}{4}\right] \left[\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right]$$

$$\left[\cos\frac{5\pi}{4} + i \sin\frac{5\pi}{4}\right] \left[\cos\frac{7\pi}{4} + i \sin\frac{7\pi}{4}\right]$$

$$= \left[\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right] \left[\frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right] \left[\frac{-1}{\sqrt{2}} + i \frac{1}{\sqrt{2}}\right] \left[\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}}\right]$$

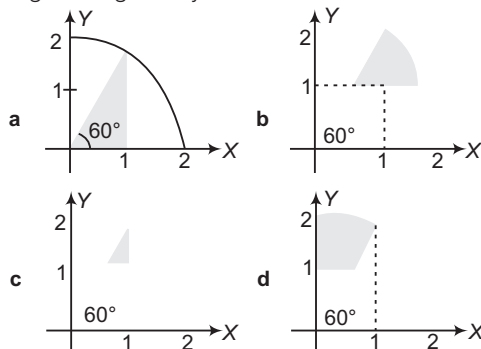
$$= \left(-\frac{1}{2} - \frac{1}{2}\right) \left(-\frac{1}{2} - \frac{1}{2}\right) = (-1)(-1) = 1$$

Work Book Exercise 4.2

1 The maximum and minimum values of $|z + 1|$, when $|z + 3| \leq 3$ are

- a (5, 0) b (6, 0) c (7, 1) d (5, 1)

2 The region represented by inequalities $\arg z \leq \frac{\pi}{3}, |z| \leq 2, \operatorname{Im}(z) \geq 1$ in the argand diagram is given by



3 If $\frac{1+2i}{2+i} = r(\cos\theta + i \sin\theta)$, then

- a $r = 1, \theta = \tan^{-1} \frac{3}{4}$ b $r = \sqrt{5}, \theta = \tan^{-1} \frac{4}{3}$
c $r = 1, \theta = \tan^{-1} \frac{4}{3}$ d None of these

4 If z_1 and z_2 are two non-zero complex numbers, such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to

- a $-\pi$ b $-\frac{\pi}{2}$ c 0 d $\frac{\pi}{2}$

5 If $z = 1 - \sin\alpha + i \cos\alpha$, where $\alpha \in \left(0, \frac{\pi}{2}\right)$, then

the modulus and the principal value of the argument of z are respectively

- a $\sqrt{2(1 - \sin\alpha)}, \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$ b $\sqrt{2(1 - \sin\alpha)}, \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$
c $\sqrt{2(1 + \sin\alpha)}, \left(\frac{\pi}{4} + \frac{\alpha}{2}\right)$ d $\sqrt{2(1 + \sin\alpha)}, \left(\frac{\pi}{4} - \frac{\alpha}{2}\right)$

6 The expression $\left[\frac{1 + \sin \frac{\pi}{8} + i \cos \frac{\pi}{8}}{1 + \sin \frac{\pi}{8} - i \cos \frac{\pi}{8}} \right]^8$ is equal to

- a 1 b -1 c i d -i

7 If $z_r = \cos \frac{2r\pi}{5} + i \sin \frac{2r\pi}{5}$, $r = 0, 1, 2, 3, 4, \dots$, then

$z_1 z_2 z_3 z_4 z_5$ is equal to

- a -1 b 0
c 1 d None of these

8 If $z_n = \cos \frac{\pi}{(2n+1)(2n+3)} + i \sin \frac{\pi}{(2n+1)(2n+3)}$,

then $\lim_{n \rightarrow \infty} (z_1 \cdot z_2 \cdot z_3 \dots z_n)$ is equal to

- a $\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$ b $\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}$
c $\cos \frac{\pi}{12} + i \sin \frac{\pi}{12}$ d None of these

9 If z_1, z_2 are two complex numbers and a, b are two real numbers, then $|az_1 - bz_2|^2 + |bz_1 + az_2|^2$ is equal to

- a $(a+b)^2[|z_1|^2 + |z_2|^2]$
b $(a+b)[|z_1|^2 + |z_2|^2]$
c $(a^2 - b^2)[|z_1|^2 + |z_2|^2]$
d $(a^2 + b^2)[|z_1|^2 + |z_2|^2]$

10 All real numbers x , which satisfy the inequality $|1 + 4i - 2^{-x}| \leq 5$, where $i = \sqrt{-1}$, $x \in R$ are

- a $[-2, \infty)$ b $(-\infty, 2]$
c $[0, \infty)$ d $[-2, 0]$

11 Square root of $x^2 + \frac{1}{x^2} - \frac{4}{i} \left(x - \frac{1}{x} \right) - 6$, where

$x \in R$ is equal to

- a $\pm \left(x - \frac{1}{x} + 2i \right)$ b $\pm \left(x - \frac{1}{x} - 2i \right)$
c $\pm \left(x + \frac{1}{x} + 2i \right)$ d $\pm \left(x + \frac{1}{x} - 2i \right)$

12 The minimum value of $|1+z| + |1-z|$, where z is a complex number, is

- a 2 b $\frac{3}{2}$
c 1 d 0

13 If $\arg(z+a) = \frac{\pi}{6}$ and $\arg(z-a) = \frac{2\pi}{3}$, $a \in R^+$, then

- a z is independent of a b $|a| = |z+a|$
c $z = a$, c is $\frac{\pi}{6}$ d $z = a$, c is $\frac{\pi}{3}$

14 Let z be a complex number satisfying the equation $(z^3 + 3)^2 = -16$, then $|z|$ has the value equal to

- a $5^{1/2}$ b $5^{1/3}$ c $5^{2/3}$ d 5

15 For $z_1 = \sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$, $z_2 = \sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$, $z_3 = \sqrt[6]{\frac{1+i}{\sqrt{3}-i}}$,

which of the following holds good?

- a $\Sigma |z_i|^2 = \frac{3}{2}$
b $|z_1|^4 + |z_2|^4 = |z_3|^{-8}$
c $\Sigma |z_i|^3 + |z_2|^3 = |z_3|^{-6}$
d $|z_1|^4 + |z_2|^4 = |z_3|^8$

Roots of Unity

Cube Roots of Unity

Let $x = \sqrt[3]{1}$

$\Rightarrow x^3 - 1 = 0$

$\Rightarrow (x-1)(x^2+x+1) = 0$

Therefore, $x = 1, \frac{-1+i\sqrt{3}}{2}, \frac{-1-i\sqrt{3}}{2}$

If second root is represented by ω , then third root will be ω^2 . Therefore, cube roots of unity are 1, ω , ω^2 and ω , ω^2 are called the imaginary cube roots of unity.

Properties

- i. $1 + \omega^r + \omega^{2r} = \begin{cases} 0, & \text{if } r \text{ is not a multiple of } 3 \\ 3, & \text{if } r \text{ is a multiple of } 3 \end{cases}$
- ii. $\omega^3 = 1$ or $\omega^{3r} = 1$
- iii. $\omega^{3r+1} = \omega$, $\omega^{3r+2} = \omega^2$
- iv. It always forms an equilateral triangle.

Important Identities

- (i) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$
- (ii) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$
- (iii) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$
- (iv) $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$
- (v) $x^2 + y^2 = (x + iy)(x - iy)$
- (vi) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$
- (vii) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega^2)$
- (viii) $x^2 + y^2 + z^2 - xy - yz - zx$
 $= (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$
or $(x\omega + y\omega^2 + z)(x\omega^2 + y\omega + z)$
or $(x\omega + y + z\omega^2)(x\omega^2 + y + z)$
- (ix) $x^3 + y^3 + z^3 - 3xyz$
 $= (x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$
- (x) Two points $P(z_1)$ and $Q(z_2)$ lie on the same side or opposite side of the line $\bar{a}z + a\bar{z} + b$ accordingly as $\bar{a}z_1 + a\bar{z}_1 + b$ and $\bar{a}z_2 + a\bar{z}_2 + b$ have same sign or opposite sign.

➤ **Example 40.** If $x^2 - x + 1 = 0$, then the value of

$$\sum_{n=1}^5 \left(x^n + \frac{1}{x^n} \right)^2 \text{ is}$$

- (a) 8
(b) 10
(c) 12
(d) None of the above

Sol. (a) $x^2 - x + 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{3}i}{2} = -\omega, -\omega^2$

$$\begin{aligned} \therefore \sum_{n=1}^5 \left(x^{2n} + \frac{1}{x^{2n}} + 2 \right) &= \left(x^2 + \frac{1}{x^2} + 2 \right) + \left(x^4 + \frac{1}{x^4} + 2 \right) + \left(x^6 + \frac{1}{x^6} + 2 \right) \\ &\quad + \left(x^8 + \frac{1}{x^8} + 2 \right) + \left(x^{10} + \frac{1}{x^{10}} + 2 \right) \\ &= (\omega^2 + \omega^4 + \omega^6 + \omega^8 + \omega^{10}) \\ &\quad + \left(\frac{1}{\omega^2} + \frac{1}{\omega^4} + \frac{1}{\omega^6} + \frac{1}{\omega^8} + \frac{1}{\omega^{10}} \right) + 10 \\ &= -1 - 1 + 10 = 8 \end{aligned}$$

[$\because x = -\omega$ or $-\omega^2$]

➤ **Example 41.** If $3^{49}(x + iy) = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i \right)^{100}$ and

$x = ky$, then k is

- (a) $-\frac{1}{3}$ (b) $\sqrt{3}$
(c) $-\sqrt{3}$ (d) $-\frac{1}{\sqrt{3}}$

Sol. (d) As $3^{49}(x + iy) = \left(\frac{3}{2} + \frac{\sqrt{3}}{2}i \right)^{100} = \left[i\sqrt{3} \left(\frac{1 - i\sqrt{3}}{2} \right) \right]^{100}$

$$\Rightarrow 3^{49}(x + iy) = i^{100} \cdot 3^{50} \cdot (-\omega)^{100}$$

$$\Rightarrow 3^{49}(x + iy) = 3^{50} \cdot \omega = 3^{50} \cdot \left(-\frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$$

$$\therefore x + iy = -\frac{3}{2} + \frac{3\sqrt{3}}{2}i$$

$$\Rightarrow x = -\frac{1}{\sqrt{3}}y$$

$$\therefore k = -\frac{1}{\sqrt{3}}$$

➤ **Example 42.** If $z^2 - z + 1 = 0$, then $z^n - z^{-n}$, where n is a multiple of 3, is

- (a) $2(-1)^n$
(b) 0
(c) $(-1)^{n+1}$
(d) None of the above

Sol. (b) As $z^2 - z + 1 = 0 \Rightarrow z = -\omega, -\omega^2$

$$\begin{aligned} \therefore z^n - z^{-n} &= (-1)^n \omega^n - (-1)^n \omega^{-n} \\ &= (-1)^n (\omega^n - \omega^{-n}), \text{ if } n \text{ is a multiple of } 3. \\ &= (-1)^n (1 - 1) = 0 \end{aligned}$$

n th Roots of Unity

Let $z = 1^{1/n}$. Then,

$$z = (\cos 0^\circ + i \sin 0^\circ)^{1/n}$$

$$z = (\cos 2r\pi + i \sin 2r\pi)^{1/n}, r \in \mathbb{Z}$$

$$\Rightarrow z = \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n}, r = 0, 1, 2, \dots, (n-1)$$

[using De-Moivre's theorem]

$$\Rightarrow z = e^{\frac{i2r\pi}{n}}, r = 0, 1, 2, \dots, (n-1)$$

$$\Rightarrow z = \{e^{i2\pi/n}\}^r, r = 0, 1, 2, \dots, (n-1)$$

$$z = \alpha^r, \alpha = e^{\frac{i2\pi}{n}}, r = 0, 1, 2, \dots, (n-1)$$

Thus, n th roots of unity are $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$, where

$$\alpha = e^{\frac{i2\pi}{n}} = \cos \frac{2\pi}{n} + i \sin \frac{2\pi}{n}$$

Properties of n th Roots of Unity

- i. n th roots of unity form a GP with common ratio $e^{\frac{i2\pi}{n}}$.
- ii. Sum of n th roots of unity is always zero.
- iii. Sum of p th powers of n th roots of unity is zero, if p is not a multiple of n .
- iv. Sum of p th powers of n th roots of unity is n , if p is a multiple of n .
- v. Product of n th roots of unity is $(-1)^{n-1}$.
- vi. n th roots of unity lie on the unit circle $|z| = 1$ and divide its circumference into n equal parts.

➤ **Example 43.** If $z_1, z_2, z_3, \dots, z_n$ are n th roots of unity, then for $k = 1, 2, \dots, n$

$$(a) |z_k| = k |z_{k+1}| \quad (b) |z_{k+1}| = k |z_k|$$

$$(c) |z_{k+1}| = |z_k| + |z_{k+1}| \quad (d) |z_k| = |z_{k+1}|$$

Sol. (d) The n th roots of unity are given by

$$z_k = e^{\frac{i2\pi(k-1)}{n}}, k = 1, 2, \dots, n$$

$$\therefore |z_k| = \left| e^{\frac{i2\pi(k-1)}{n}} \right| = 1, \forall k = 1, 2, \dots, n$$

$$\Rightarrow |z_k| = |z_{k+1}|, \forall k = 1, 2, \dots, n$$

➤ **Example 44.** If n is a positive integer greater than unity and z is a complex number satisfying the equation $z^n = (z+1)^n$, then

- (a) $\operatorname{Re}(z) < 0$ (b) $\operatorname{Re}(z) > 0$
(c) $\operatorname{Re}(z) = 0$ (d) None of these

Sol. (a) We have, $z^n = (1+z)^n \Rightarrow \left(\frac{z}{z+1}\right)^n = 1$

$$\Rightarrow \frac{z}{z+1} = 1^{1/n}$$

$\Rightarrow \frac{z}{z+1}$ is n th root of unity.

$$\Rightarrow \left|\frac{z}{z+1}\right| = 1 \Rightarrow \frac{|z|}{|z+1|} = 1$$

$$\Rightarrow |z| = |z+1|$$

$$\Rightarrow x + \frac{1}{2} = 0 \quad [\text{taking } z = x + iy]$$

$$\Rightarrow x = -\frac{1}{2} \Rightarrow \operatorname{Re}(z) < 0$$

Example 45. If α is non-real and $\alpha = \sqrt[n]{1}$, then the value of $2^{|1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}|}$ is equal to

- (a) 4 (b) 2
(c) 1 (d) None of these

Sol. (a) $\because \alpha^5 = 1$

$$\therefore |1+\alpha+\alpha^2+\alpha^{-2}-\alpha^{-1}| = |1+\alpha+\alpha^2+\alpha^3-\alpha^4|$$

$$= |1+\alpha+\alpha^2+\alpha^3+\alpha^4-2\alpha^4|$$

$$= \left|\frac{1-\alpha^5}{1-\alpha} - 2\alpha^4\right| = |2\alpha^4| = 2|\alpha|^4 = 2 \times 1 = 2$$

$$\therefore 2^2 = 4$$

Work Book Exercise 4.3

1 If ω is a non-real cube root of unity, then the expression $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$ is equal to

- a 0 b 3 c 1 d 2

2 $x^{3m} + x^{3n-1} + x^{3r-2}$, where $m, n, r \in N$, is divisible by

- a $x^2 - x + 1$ b $x^2 + x + 1$
c $x^2 + x - 1$ d None of these

3 If ω is a non-real cube root of unity, then $\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2} + \frac{2+3\omega+\omega^2}{3+\omega+2\omega^2}$ is equal to

- a -1 b 2ω c 0 d -2ω

4 If $(\sqrt{3}+i)^n = (\sqrt{3}-i)^n$, $n \in N$, then least value of n is

- a 3 b 4
c 6 d None of these

5 If $x^3 - 1 = 0$ has the non-real complex roots α, β , then the value of $(1+2\alpha+\beta)^3 - (3+3\alpha+5\beta)^3$ is

- a -7 b 6
c -5 d 0

6 If $(\sqrt{3}-i)^n = 2^n$, $n \in I$, the set of integers, then n is a multiple of

- a 6 b 10 c 9 d 12

7 If z is a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then $|z|$ is equal to

- a $\frac{1}{2}$ b $\frac{3}{4}$
c 1 d None of these

8 If z is a non-real root of $\sqrt[3]{-1}$, then $z^{86} + z^{175} + z^{289}$ is equal to

- a 0 b -1 c 3 d 1

9 Non-real complex number z satisfying the equation $z^3 + 2z^2 + 3z + 2 = 0$ are

- a $\frac{-1 \pm \sqrt{-7}}{2}$ b $\frac{1+\sqrt{7}i}{2}, \frac{1-\sqrt{7}i}{2}$
c $-i, \frac{-1+\sqrt{7}i}{2}, \frac{-1-\sqrt{7}i}{2}$ d None of these

10 If α is the non-real n th root of unity, then $1 + 3\alpha + 5\alpha^2 + \dots + (2n-1)\alpha^{n-1}$ is equal to

- a $\frac{2n}{1-\alpha}$ b $\frac{n}{1-\alpha}$
c $\frac{n}{2(1-\alpha)}$ d None of these

11 $\sqrt{-1-\sqrt{-1-\sqrt{-1}\dots\infty}}$ is equal to, where ω is the imaginary cube root of unity and $i = \sqrt{-1}$.

- a ω or ω^2 b $-\omega$ or $-\omega^2$
c $1+i$ or $1-i$ d $-1+i$ or $-1-i$

12 If $\alpha = e^{i2\pi/n}$, then $(11-\alpha)(11-\alpha^2)\dots(11-\alpha^{n-1})$ is equal to

- a 11^{n-1} b $\frac{11^n-1}{10}$ c $\frac{11^{n-1}-1}{10}$ d $\frac{11^{n-1}-1}{11}$

13 The complex number w satisfying the equation $\omega^3 = 8i$ and lying in the IInd quadrant on the complex plane is

- a $-\sqrt{3}+i$ b $-\frac{\sqrt{3}}{2}+\frac{1}{2}i$
c $-2\sqrt{3}+i$ d $-\sqrt{3}+2i$

14 Let z be a complex number satisfying the equation $z^6 + z^3 + 1 = 0$. If this equation has a root $re^{i\theta}$ with $90^\circ < \theta < 180^\circ$, then the value of θ is

- a 100° b 110° c 160° d 170°

15 If ω is an imaginary cube root of unity, then the value of $(p+q)^3 + (p\omega+q\omega^2)^3 + (p\omega^2+q\omega)^3$ is

- a p^3+q^3
b $3(p^3+q^3)$
c $3(p^3+q^3)-pq(p+q)$
d $3(p^3+q^3)+pq(p+q)$

16 If $z^2 - z + 1 = 0$, then the value of

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \dots + \left(z^{24} + \frac{1}{z^{24}}\right)^2 \text{ is equal to}$$

- a 24 b 32
c 48 d None of these

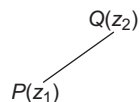
- 17 If $p = a + b\omega + c\omega^2$, $q = b + c\omega + a\omega^2$ and $r = c + a\omega + b\omega^2$, where $a, b, c \neq 0$ and ω is the complex cube root of unity, then
 a $p + q + r = a + b + c$
 b $p^2 + q^2 + r^2 = a^2 + b^2 + c^2$
 c $p^2 + q^2 + r^2 = -2(pq + qr + rp)$
 d None of the above
- 18 If a and b are imaginary cube roots of unity, then $\alpha^n + \beta^n$ is equal to
 a $2\cos\frac{2n\pi}{3}$
 b $\cos\frac{2n\pi}{3}$
 c $2i\sin\frac{2n\pi}{3}$
 d $i\sin\frac{2n\pi}{3}$
- 19 If the six solutions of $x^6 = -64$ are written in the form $a + bi$, where a and b are real, then the product of those solutions with $a > 0$ is
 a 4 b 8 c 16 d 64
- 20 If $\cos\theta + i\sin\theta$ is a root of the equation $x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_{n-1}x + a_n = 0$, then the value of $\sum_{r=1}^n a_r \cos r\theta$ is
 a 0
 b 1
 c -1
 d None of the above
- 21 If ω is a complex n th root of unity, then $\sum_{r=1}^n (ar + b)\omega^{r-1}$ is equal to
 a $\frac{n(n+1)a}{2}$
 b $\frac{nb}{1-n}$
 c $\frac{na}{n-1}$
 d None of these
- 22 If α, β respectively are the fifth and fourth non-real roots of unity, then the value of $(1+\alpha)(1+\beta)(1+\alpha^2)(1+\beta^2)(1+\beta^3)(1+\alpha^3)$ is
 a 0
 b $(1+\alpha+\alpha^2)(1-\beta^2)$
 c $(1+\alpha)(1+\beta+\beta^2)$
 d 1
- 23 When the polynomial $5x^3 + Mx + N$ is divided by $x^2 + x + 1$, the remainder is 0. The value of $(M+N)$ is equal to
 a -3 b 5 c -5 d 15
- 24 If z and w are two complex numbers simultaneously satisfying the equations, $z^3 + w^5 = 0$ and $z^2 \cdot \bar{w}^4 = 1$, then
 a z and w both are purely real
 b z is purely real and w is purely imaginary
 c w is purely real and z is purely imaginary
 d z and w both are imaginary
- 25 Number of ordered pairs (z, ω) of the complex numbers z and ω satisfying the system of equations, $z^3 + \omega^7 = 0$ and $z^5 \cdot \omega^{11} = 1$ is
 a 7 b 5 c 3 d 2
- 26 If $1, z_1, z_2, z_3, \dots, z_{n-1}$ is the n th roots of unity and w is a non-real complex cube root of unity, then the product of $\prod_{r=1}^{n-1} (\omega - z_r)$ is cannot be equal to
 a 0 b 1 c -1 d $1 + \omega$
27. If $Z_r, r = 1, 2, 3, \dots, 50$ are the roots of the equation $\sum_{r=0}^{50} (Z)^r = 0$, then the value of $\sum_{r=1}^{50} \frac{1}{Z_r - 1}$ is
 a -85 b -25 c 25 d 75

Geometrical Applications of Complex Numbers

Basic Concepts in Geometry

Distance formula The distance between two points $P(z_1)$ and $Q(z_2)$ is given by

$$PQ = |z_2 - z_1| = |\text{affix of } Q - \text{affix of } P|$$



For any complex number z ,

$$|z| = |z - 0| = |z - (0 + i0)|$$

Thus, modulus of a complex number z represented by a point in the argand plane is its distance from origin.

➤ **Example 46.** Length of the line segment joining the points $-1 - i$ and $2 + 3i$ is
 (a) -5 (b) 15 (c) 5 (d) 25

Sol. (c) Let $z_1 = -1 - i$ and $z_2 = 2 + 3i$

$$\begin{aligned} \text{Then, required distance} &= |z_2 - z_1| \\ &= |2 + 3i + 1 + i| \\ &= 5 \end{aligned}$$

Section formulae If $R(z)$ divides the line segment joining $P(z_1)$ and $Q(z_2)$ in the ratio $m_1 : m_2$ ($m_1, m_2 > 0$), then

$$(i) \text{ For internal division, } z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}$$

$$(ii) \text{ For external division, } z = \frac{m_1 z_2 - m_2 z_1}{m_1 - m_2}$$

If $R(z)$ is the mid-point of PQ , then affix of R is

$$\frac{z_1 + z_2}{2}$$

➤ **Example 47.** If A, B, C are three points in the argand plane representing the complex numbers z_1, z_2, z_3 such that $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in \mathbb{R}$, then

the distance of A from the line BC is

- (a) λ
 (b) $\frac{\lambda}{\lambda + 1}$
 (c) 1
 (d) 0

Sol. (d) As $z_1 = \frac{\lambda z_2 + z_3}{\lambda + 1}$

which shows z_1 divides z_2, z_3 in the ratio of $1 : \lambda$.

Thus, the points are collinear.

∴ Distance of A from line BC is zero.

➤ **Example 48.** Find the relation, if z_1, z_2, z_3, z_4 are the affixes of the vertices of a parallelogram taken in order.

Sol. As the diagonals of a parallelogram bisect each other, therefore affix of the mid-point of AC is same as the affix of the mid-point of BD .

$$\text{i.e.} \quad \frac{z_1 + z_3}{2} = \frac{z_2 + z_4}{2}$$

$$\Rightarrow z_1 + z_3 = z_2 + z_4$$

Equation of the Straight Line

Equation of the Line Passing through the Points z_1 and z_2

Let z be any point on the line joining z_1 and z_2 , then

$$\arg \frac{z - z_1}{z_2 - z_1} = \pi \text{ or } 0$$

$$\Rightarrow \frac{z - z_1}{z_2 - z_1} \text{ must be real.}$$

∴ Required equation is

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \Rightarrow \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1 \bar{z}_2 - z_2 \bar{z}_1 = 0$$

General Equation of a Line

$\bar{a}z + a\bar{z} + b = 0$, represents a straight line, where b is a real number and a is a complex number.

Parametric Equation of a Line

$z = z_1 + t(z_2 - z_1)$, where t is real parameter

$= (1 - t)z_1 + tz_2$, represents the complete line through z_1 and z_2 .

➤ **Example 49.** Find the general equation of line joining the points $z_1 = (1 + i)$ and $z_2 = (1 - i)$.

Sol. Clearly, the equation of a line is given by

$$z(\bar{z}_1 - \bar{z}_2) - \bar{z}(z_1 - z_2) + z_1 \bar{z}_2 - z_2 \bar{z}_1 = 0$$

where, $z_1 = 1 + i$ and $z_2 = 1 - i$

On substituting the values of z_1 and z_2 , we get

$$z(1 - i - 1 - i) - \bar{z}(1 + i - 1 + i) + (1 + i)(1 + i) - (1 - i)(1 - i) = 0$$

$$\Rightarrow z(-2i) - \bar{z}(2i) + (1 - 1 + 2i) - (1 - 1 - 2i) = 0$$

$$\Rightarrow -2iz - 2i\bar{z} + 4i = 0$$

$$\Rightarrow z + \bar{z} - 2 = 0, \text{ which is the required equation.}$$

Condition of Collinearity

Three points z_1, z_2 and z_3 are collinear, if

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0$$

➤ **Example 50.** If z_1, z_2, z_3 are three complex numbers such that $5z_1 - 13z_2 + 8z_3 = 0$, then prove that

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

Sol. $5z_1 - 13z_2 + 8z_3 = 0$

$$\Rightarrow \frac{5z_1 + 8z_3}{5 + 8} = z_2$$

$\Rightarrow z_1, z_2$ and z_3 are collinear.

$$\Rightarrow \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0 \quad [\text{condition of collinear points}]$$

Hence proved.

Length of Perpendicular

The length of perpendicular from a point z_1 to $\bar{a}z + a\bar{z} + b = 0$ is given by

$$\frac{|\bar{a}z_1 + a\bar{z}_1 + b|}{2|a|}$$

➤ **Example 51.** The length of perpendicular from $P(2 - 3i)$ to the line $(3 + 4i)z + (3 - 4i)\bar{z} + 9 = 0$ is equal to

- (a) 9
 (b) $\frac{9}{4}$
 (c) $\frac{9}{2}$
 (d) None of these

Sol. (c) Let PM be the required length, then

$$PM = \frac{|(2 - 3i)(3 + 4i) + (3 - 4i)(2 + 3i) + 9|}{2|3 - 4i|}$$

$$= \frac{45}{10} = \frac{9}{2}$$

Slope of a Line

Slope of the Line Segment Joining Two Points

If A, B represent complex numbers z_1, z_2 in the argand plane, then the complex slope of AB is defined by

$$\frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

and real slope is defined by $\frac{\operatorname{Re}(z_2 - z_1)}{\operatorname{Im}(z_2 - z_1)}$.

Slope of Line $\bar{a}z + a\bar{z} + b = 0$

The complex slope of the line

$$\bar{a}z + a\bar{z} + b = 0 \text{ is } \frac{-a}{\bar{a}} = \frac{-\text{Coefficient of } \bar{z}}{\text{Coefficient of } z}$$

and real slope of the line $\bar{a}z + a\bar{z} + b = 0$ is

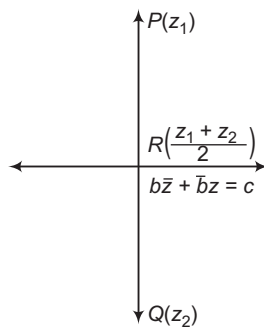
$$-\frac{\operatorname{Re}(a)}{\operatorname{Im}(a)} = \frac{-i(a + \bar{a})}{(a - \bar{a})}$$

- If w_1 and w_2 are the complex slope of two lines on the argand plane, then the lines are
 - (a) perpendicular, if $w_1 + w_2 = 0$
 - (b) parallel, if $w_1 = w_2$
- The equation of a line parallel to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} + \bar{a}z + \lambda = 0$, where $\lambda \in \mathbb{R}$.
- The equation of a line perpendicular to the line $a\bar{z} + \bar{a}z + b = 0$ is $a\bar{z} - \bar{a}z + i\lambda = 0$, where $\lambda \in \mathbb{R}$.

Example 52. If a point z_1 is the reflection of a point z_2 through the line $b\bar{z} + \bar{b}z = c$, $b \neq 0$ in the argand plane, then $\bar{b}z_2 + b\bar{z}_1$ is equal to

- (a) $4c$
- (b) $2c$
- (c) c
- (d) None of these

Sol. (c) If $P(z_1)$ is the reflection of $Q(z_2)$ through the line $b\bar{z} + \bar{b}z = c$ in the argand plane. Then,



$R\left(\frac{z_1 + z_2}{2}\right)$ lies on the line.

$$b\left(\frac{\bar{z}_1 + \bar{z}_2}{2}\right) + \bar{b}\left(\frac{z_1 + z_2}{2}\right) = c$$

$$\Rightarrow b\bar{z}_1 + \bar{b}z_1 + b\bar{z}_2 + \bar{b}z_2 = 2c \quad \dots (i)$$

Since, PQ is perpendicular to the line $b\bar{z} + \bar{b}z = c$. Therefore,

Slope of PQ + Slope of the line = 0

$$\Rightarrow \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} + \left(\frac{-b}{\bar{b}}\right) = 0$$

$$\begin{aligned} \Rightarrow & \bar{b}(z_2 - z_1) - b(\bar{z}_2 - \bar{z}_1) = 0 \\ \Rightarrow & \bar{b}z_2 - \bar{b}z_1 - b\bar{z}_2 + b\bar{z}_1 = 0 \quad \dots (ii) \end{aligned}$$

Adding Eqs. (i) and (ii), we get

$$2(b\bar{z}_1 + \bar{b}z_2) = 2c \Rightarrow b\bar{z}_1 + \bar{b}z_2 = c$$

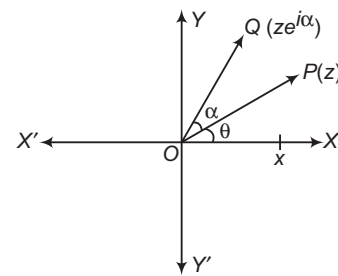
Concept of Rotation

In this section, we shall learn about the effect of multiplication of a complex number by $e^{i\alpha}$ which will also be interpreted geometrically.

Complex Number as a Rotating Arrow in the Argand Plane

1. Let $z = r(\cos \theta + i \sin \theta) = re^{i\theta}$ be a complex number, represented by a point P in the argand plane. Then,

$$OP = r \text{ and } \angle XOP = \theta$$



$$\text{Now, } ze^{i\alpha} = re^{i\theta} \cdot e^{i\alpha} = re^{i(\theta + \alpha)}$$

This shows that $ze^{i\alpha}$ is the complex number whose modulus is r and argument $\theta + \alpha$. Clearly, $ze^{i\alpha}$ is represented by a point Q in the argand plane such that $OQ = r$ and $\angle XOQ = \theta + \alpha$.

In other words, to obtain the point representing $ze^{i\alpha}$, we rotate OP through angle α in anti-clockwise sense. Thus, multiplication by $e^{i\alpha}$ to z rotates the vector OP in anti-clockwise sense through an angle α and vice-versa. Similarly, multiplication of z with $e^{-i\alpha}$ will rotate the vector OP in clockwise sense.

Remark Let z_1 and

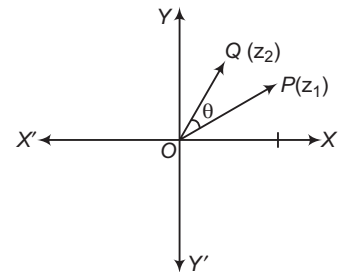
z_2 be two complex numbers represented by points P and Q in the argand plane, such that $\angle POQ = \theta$. Then, $z_1 e^{i\theta}$ is vector of magnitude

$|z_1| = OP$ along OQ and $\frac{z_1 e^{i\theta}}{|z_1|}$ is a unit vector

along OQ . Consequently, $|z_2| \cdot \frac{z_1 e^{i\theta}}{|z_1|}$ is a vector of

magnitude $|z_2| = OQ$ along OQ .

$$\text{i.e. } z_2 = \frac{|z_2|}{|z_1|} z_1 e^{i\theta} \Rightarrow z_2 = \frac{|z_2|}{|z_1|} \cdot z_1 e^{i\theta}$$



➤ **Example 53.** The point represented by the complex number $2 - i$ is rotated about origin through an angle $\frac{\pi}{2}$ in the clockwise direction, the new position of point is

- (a) $1 + 2i$ (b) $-1 - 2i$
(c) $2 + i$ (d) $-1 + 2i$

Sol. (b) Here, $z = 2 - i$

Let z_1 be the required complex number.

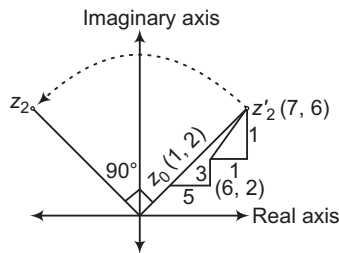
$$\therefore z_1 = ze^{-i\pi/2}$$

$$\begin{aligned}\therefore z_1 &= (2 - i) \left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right] \\ &= (2 - i) \left(\cos\frac{\pi}{2} - i \sin\frac{\pi}{2} \right) \\ &= (2 - i)(0 - i) \\ &= -(2i - i^2) = -2i - 1\end{aligned}$$

➤ **Example 54.** A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from origin by 3 units to reach a point z_1 . From z_1 , the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\pi/2$ in anti-clockwise direction on a circle with centre at origin to reach a point z_2 . The point z_2 is given by

- (a) $6 + 7i$ (b) $-7 + 6i$
(c) $7 + 6i$ (d) $-6 + 7i$

Sol. (d)



$$\begin{aligned}z_2' &= (6 + \sqrt{2} \cos 45^\circ, 5 + \sqrt{2} \sin 45^\circ) \\ &= (7, 6) = 7 + 6i\end{aligned}$$

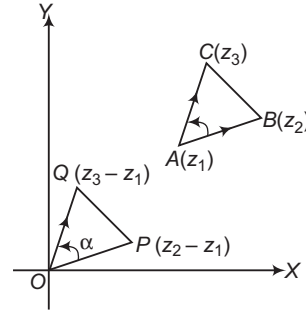
By rotation about $(0, 0)$,

$$\begin{aligned}\frac{z_2}{z_2'} &= e^{i\pi/2} \Rightarrow z_2 = z_2' \left(e^{i\pi/2} \right) \\ z_2 &= (7 + 6i) \left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2} \right) \\ &= (7 + 6i)(i) = -6 + 7i\end{aligned}$$

2. Let z_1, z_2 and z_3 be the vertices of a ΔABC described in anti-clockwise sense. Draw OP and OQ parallel and equal to AB and AC , respectively. Then, point P is $z_2 - z_1$ and Q is $z_3 - z_1$. If OP is rotated through $\angle \alpha$ in anti-clockwise sense, it coincides with OQ .

Then,

$$\begin{aligned}\frac{z_3 - z_1}{z_2 - z_1} &= \frac{OQ}{OP} (\cos \alpha + i \sin \alpha) \\ &= \frac{CA}{BA} e^{i\alpha} \\ &= \frac{|z_3 - z_1|}{|z_2 - z_1|} e^{i\alpha}\end{aligned}$$



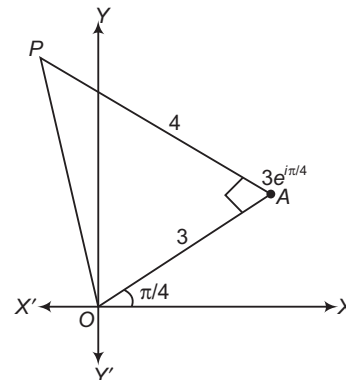
$$\left[\begin{aligned} &\because \Delta OPQ \text{ and } \Delta ABC \text{ are congruent,} \\ &\therefore \frac{OQ}{OP} = \frac{CA}{BA} \end{aligned} \right]$$

or $\text{amp} \left(\frac{z_3 - z_1}{z_2 - z_1} \right) = \alpha$

➤ **Example 55.** A man walks a distance of 3 units from the origin towards the North-East ($N 45^\circ E$) direction. From there, he walks a distance of 4 units towards the North-West ($N 45^\circ W$) direction to reach a point P . Then, the position of P in the argand plane is

- (a) $3e^{i\pi/4} + 4i$ (b) $(3 - 4i)e^{i\pi/4}$
(c) $(4 + 3i)e^{i\pi/4}$ (d) $(3 + 4i)e^{i\pi/4}$

Sol. (d) Let $OA = 3$ units, so that the complex number associated with A is $3e^{i\pi/4}$. If z is the complex number associated with P , then

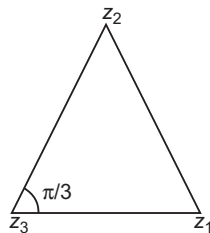


$$\begin{aligned}\frac{z - 3e^{i\pi/4}}{0 - 3e^{i\pi/4}} &= \frac{4}{3} e^{-i\pi/2} \\ &= -\frac{4i}{3}\end{aligned}$$

$$\begin{aligned}\Rightarrow 3z - 9e^{i\pi/4} &= 12e^{i\pi/4} \\ \Rightarrow z &= (3 + 4i)e^{i\pi/4}\end{aligned}$$

- **Example 56.** The complex numbers z_1, z_2 and z_3 satisfying $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$ are the vertices of a triangle which is
- (a) of area zero (b) right angled isosceles
(c) equilateral (d) obtuse angled isosceles

Sol. (c) $\frac{z_1 - z_3}{z_2 - z_3} = \frac{1 - i\sqrt{3}}{2}$
 $= \frac{(1 - i\sqrt{3})(1 + i\sqrt{3})}{2(1 + i\sqrt{3})} = \frac{1 - i^2 3}{2(1 + i\sqrt{3})}$
 $= \frac{4}{2(1 + i\sqrt{3})} = \frac{2}{(1 + i\sqrt{3})}$
 $\Rightarrow \frac{z_2 - z_3}{z_1 - z_3} = \frac{1 + i\sqrt{3}}{2}$
 $= \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$



$\Rightarrow \left| \frac{z_2 - z_3}{z_1 - z_3} \right| = 1$ and $\arg \left(\frac{z_2 - z_3}{z_1 - z_3} \right) = \frac{\pi}{3}$

Hence, the triangle is an equilateral.

Area of Triangle

- (i) Area of the triangle with vertices z_1, z_2 and z_3 is $\left| \sum \left[\frac{(z_2 - z_3) |z_1|^2}{4iz_1} \right] \right|$ sq unit.
- (ii) The area of triangle whose vertices are z, iz and $z + iz$ is $\frac{1}{2} |z|^2$.
- (iii) The area of triangle whose vertices are $z, \omega z$ and $z + \omega z$ is $\frac{\sqrt{3}}{4} |z|^2$.

- **Example 57.** If the area of the triangle on the complex plane formed by the points z, iz and $z + iz$ is 50 sq units, then $|z|$ is
- (a) 5 (b) 10
(c) 15 (d) None of these

Sol. (b) We know that, the area of the triangle formed by the points z, iz and $z + iz$ is $\frac{1}{2} |z|^2$.

$\therefore \frac{1}{2} |z|^2 = 50$

$\Rightarrow |z| = 10$

- **Example 58.** If the area of the triangle on the complex plane formed by complex numbers $z, \omega z$ and $z + \omega z$ is $4\sqrt{3}$ sq units, then $|z|$ is
- (a) 4 (b) 2 (c) 6 (d) 3

Sol. (a) The area of the triangle formed by $z, \omega z$ and $z + \omega z$ is $\frac{\sqrt{3}}{4} |z|^2$.

$\therefore \frac{\sqrt{3}}{4} |z|^2 = 4\sqrt{3}$

$\Rightarrow |z| = 4$

Applications of Triangle

1. **Centroid** The centroid of the triangle (in the argand plane) formed by z_1, z_2 and z_3 is given by

$$\frac{1}{3} (z_1 + z_2 + z_3)$$

2. **Incentre** The incentre of the triangle (in the argand plane), formed by z_1, z_2 and z_3 is $\frac{az_1 + bz_2 + cz_3}{a + b + c}$, where

$$a = |z_2 - z_3|, b = |z_3 - z_1|, c = |z_1 - z_2|$$

3. **Excentres** The excentres of the triangle (in the argand plane), for meet by z_1, z_2 and z_3 are given by

(i) $I_1 = \frac{-az_1 + bz_2 + cz_3}{-a + b + c}$

(ii) $I_2 = \frac{az_1 - bz_2 + cz_3}{a - b + c}$

(iii) $I_3 = \frac{az_1 + bz_2 - cz_3}{a + b - c}$

where, $a = |z_2 - z_3|, b = |z_3 - z_1|$ and $c = |z_1 - z_2|$

4. **Circumcentre** The circumcentre of the triangle (in the argand plane), formed by

$$z_1, z_2, z_3 \text{ is given by } \frac{\sum z_1 \bar{z}_1 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)}$$

5. **Orthocentre** The orthocentre of the triangle (in the argand plane), formed by z_1, z_2, z_3 is given by

$$\frac{\sum z_1^2 (\bar{z}_2 - \bar{z}_3) + \sum |z_1|^2 (z_2 - z_3)}{\sum (z_1 \bar{z}_2 - \bar{z}_1 z_2)}$$

- **Example 59.** If z_1, z_2 and z_3 are affixes of the vertices A, B and C , respectively of a $\triangle ABC$ having centroid at G such that $z = 0$ is the mid-point of AG , then

- (a) $z_1 + z_2 + z_3 = 0$ (b) $z_1 + 4z_2 + z_3 = 0$
(c) $z_1 + z_2 + 4z_3 = 0$ (d) $4z_1 + z_2 + z_3 = 0$

Sol. (d) The affix of G is $\frac{z_1 + z_2 + z_3}{3}$. Since, $z = 0$ is the

mid-point of AG . Therefore, affix of the mid-point of AG is 0.

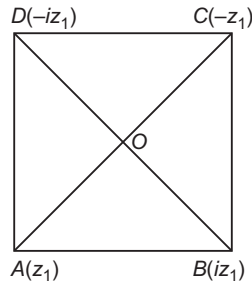
$$\Rightarrow \frac{\frac{z_1 + z_2 + z_3}{3} + z_1}{1 + 1} = 0 \Rightarrow 4z_1 + z_2 + z_3 = 0$$

- **Example 60.** The centre of a square $ABCD$ is at the origin and point A is represented by z_1 . The centroid of $\triangle BCD$ is represented by

- (a) $\frac{z_1}{3}$ (b) $-\frac{z_1}{3}$
(c) $\frac{iz_1}{3}$ (d) $-\frac{iz_1}{3}$

Sol. (b) The affixes of the vertices B, C and D are $iz_1 - z_1$ and $-iz_1$, respectively. Therefore, the affix of the centroid of $\triangle BCD$ is

$$\frac{iz_1 - z_1 - iz_1}{3} = -\frac{z_1}{3}$$



➤ **Example 61.** Let $P(e^{i\theta_1}), Q(e^{i\theta_2})$ and $R(e^{i\theta_3})$ be the vertices of $\triangle PQR$ in the argand plane. Then, the orthocentre of the $\triangle PQR$ is

- (a) $e^{i(\theta_1 + \theta_2 + \theta_3)}$ (b) $\frac{2}{3}e^{i(\theta_1 + \theta_2 + \theta_3)}$
 (c) $e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}$ (d) None of these

Sol. (c) We have, $|e^{i\theta_1}| = |e^{i\theta_2}| = |e^{i\theta_3}| = 1$
 $\Rightarrow OP = OQ = OR = 1$, where O is the origin.
 $\Rightarrow$ Origin O is the circumcentre of $\triangle PQR$.
 The affix of the centroid is

$$\frac{1}{3}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3})$$

Let z be the affix of the orthocentre. Since, centroid divides the segment joining circumcentre and orthocentre in the ratio 1:2.

$$\therefore \frac{1}{3}(e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}) = \frac{1 \cdot z + 2 \cdot 0}{1 + 2}$$

$$\Rightarrow z = e^{i\theta_1} + e^{i\theta_2} + e^{i\theta_3}$$

6. Equilateral Triangle

- (i) The triangle whose vertices are the points z_1, z_2 and z_3 on the argand plane, is an equilateral triangle, if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\text{or } \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

- (ii) If the complex numbers z_1, z_2 and z_3 are the vertices of an equilateral triangle and z_0 is the circumcentre of the triangle, then $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

7. Isosceles Triangle

- (i) If z_1, z_2 and z_3 are the vertices of a right angled isosceles triangle, then

$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

- (ii) If z_1, z_2 and z_3 are the vertices of an isosceles triangle, right angled at z_2 , then

$$z_1^2 + z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$$

➤ **Example 62.** Prove that the complex numbers z_1, z_2 and the origin form an equilateral triangle only, if $z_1^2 + z_2^2 - z_1z_2 = 0$.

Sol. If z_1, z_2 and z_3 form an equilateral triangle, then

$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$\Rightarrow z_1^2 + z_2^2 + 0^2 = z_1z_2 + z_2 \cdot 0 + 0 \cdot z_1$$

$$\Rightarrow z_1^2 + z_2^2 = z_1z_2 \Rightarrow z_1^2 + z_2^2 - z_1z_2 = 0$$

Hence proved.

➤ **Example 63.** Let z_1 and z_2 be two complex

numbers such that $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1$, then

- (a) z_1, z_2 are collinear
 (b) z_1, z_2 and the origin form a right angled triangle
 (c) z_1, z_2 and the origin form an equilateral triangle
 (d) None of the above

Sol. (c) We have, $\frac{z_1}{z_2} + \frac{z_2}{z_1} = 1 \Rightarrow z_1^2 + z_2^2 = z_1z_2$

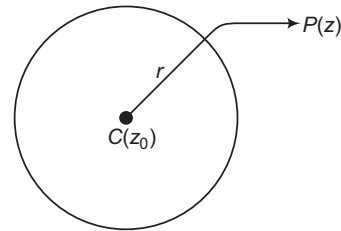
$$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1, \text{ where } z_3 = 0$$

$\Rightarrow z_1, z_2$ and the origin form an equilateral triangle.

Circle

Equation of a Circle

- (i) The equation of a circle whose centre is at point having affix z_0 and radius r , is $|z - z_0| = r$.



- (ii) If the centre of the circle is at origin and radius r , then its equation is $|z| = r$.
 (iii) $|z - z_0| < r$ represents interior of a circle $|z - z_0| = r$ and $|z - z_0| > r$ represents exterior of the circle $|z - z_0| = r$.
 (iv) **General equation of a circle** The general equation of the circle is $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, where a is complex number and $b \in \mathbb{R}$.
 $\therefore$ Centre and radius are $-a$ and $\sqrt{|a|^2 - b}$, respectively.

- (v) **Equation of circle in diametric form** If end points of diameter represented by $A(z_1)$ and $B(z_2)$ and $P(z)$ is any point on the circle, then $(z - z_1)(\bar{z} - \bar{z}_2) + (z - z_2)(\bar{z} - \bar{z}_1) = 0$ which is required equation of circle in diametric form.

➤ **Example 64.** A circle whose radius is r and centre z_0 , then the equation of the circle is

- (a) $z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$
 (b) $z\bar{z} + z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$
 (c) $z\bar{z} - z\bar{z}_0 + \bar{z}z_0 - z_0\bar{z}_0 = r^2$
 (d) None of the above

Sol. (a) Equation of circle $|z - z_0|^2 = r^2$
 $\Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2 \Rightarrow (z - z_0)(\bar{z} - \bar{z}_0) = r^2$
 $z\bar{z} - z\bar{z}_0 - \bar{z}z_0 + z_0\bar{z}_0 = r^2$

➤ **Example 65.** The set of values of k for which the equation $z\bar{z} + (-3 + 4i)\bar{z} - (3 + 4i)z + k = 0$ represents a circle is

- (a) $(-\infty, 25)$ (b) $(25, \infty)$ (c) $(5, \infty)$ (d) $(-\infty, 5)$

Sol. (a) We have,

$$z\bar{z} + (-3 + 4i)\bar{z} - (3 + 4i)z + k = 0$$

This equation represents a circle with centre

$$a + (3 - 4i) \text{ and Radius } = \sqrt{(-3 - 4i)^2 - k} = \sqrt{25 - k}$$

For circle to exist, we must have

$$25 - k > 0 \Rightarrow k < 25$$

Hence, the given equation will represent a circle if $k < 25$.

Loci in Complex Plane

If z is a variable point and z_1, z_2 are two fixed points in the argand plane, then

- (i) $|z - z_1| = |z - z_2|$
 $\Rightarrow$ Locus of z is the perpendicular bisector of the line segment joining z_1 and z_2 .
 (ii) $|z - z_1| + |z - z_2| = k$, if $|k| > |z_1 - z_2|$
 $\Rightarrow$ Locus of z is an ellipse.
 (iii) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
 $\Rightarrow$ Locus of z is the line segment joining z_1 and z_2
 (iv) $|z - z_1| - |z - z_2| = |z_1 - z_2|$
 $\Rightarrow$ Locus of z is a straight line joining z_1 and z_2 but z does not lie between z_1 and z_2 .
 (v) $|z - z_1| - |z - z_2| = k$, where $|k| < |z_1 - z_2|$
 $\Rightarrow$ Locus of z is a hyperbola.
 (vi) $|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$
 $\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the extremities of diameter.
 (vii) $|z - z_1| = k|z - z_2|$, ($k \neq 1$)
 $\Rightarrow$ Locus of z is a circle.
 (viii) $\arg\left(\frac{z - z_1}{z - z_2}\right) = \alpha$ (fixed)
 $\Rightarrow$ Locus of z is a segment of circle.
 (ix) $\arg\left(\frac{z - z_1}{z - z_2}\right) = \pm \pi/2$

$\Rightarrow$ Locus of z is a circle with z_1 and z_2 as the vertices of diameter.

$$(x) \arg\left(\frac{z - z_1}{z - z_2}\right) = 0 \text{ or } \pi$$

$\Rightarrow$ Locus of z is a straight line passing through z_1 and z_2 .

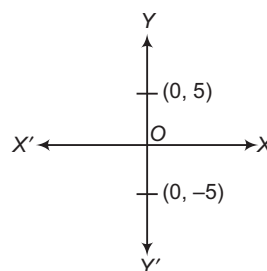
➤ **Example 66.** The complex numbers $z = x + iy$

which satisfy the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$, lie on

- (a) the X-axis
 (b) the straight line $y = 5$
 (c) a circle passing through the origin
 (d) None of the above

Sol. (a) Given, $\left|\frac{z - 5i}{z + 5i}\right| = 1 \Rightarrow |z - 5i| = |z + 5i|$

[if $|z - z_1| = |z - z_2|$, then it is a perpendicular bisector of z_1 and z_2]



$\therefore$ Perpendicular bisector of $(0, 5)$ and $(0, -5)$ is X-axis.

➤ **Example 67.** The equation

$$|z - i| + |z + i| = k, k > 0$$

can represent an ellipse, if k^2 is

- (a) < 1 (b) < 2
 (c) > 4 (d) None of these

Sol. (c) $|z - z_1| + |z - z_2| = k$ represents ellipse, if $|k| > |z_1 - z_2|$

Thus, $|z - i| + |z + i| = k$ represents ellipse, if

$$|k| > |i + i| \text{ or } |k| > |2i|$$

$$\therefore |k| > 2 \text{ or } k^2 > 4$$

➤ **Example 68.** The equation $|z + i| - |z - i| = k$ represents a hyperbola if

- (a) $-2 < k < 2$ (b) $k > 2$
 (c) $0 < k < 2$ (d) None of these

Sol. (a) $|z - z_1| - |z - z_2| = k$, represents hyperbola,

if $|k| < |z_1 - z_2|$

Thus, $|z + i| - |z - i| = k$, represents hyperbola, if

$$|k| < |i + i| \text{ or } |k| < 2$$

$$\Rightarrow -2 < k < 2$$

➤ **Example 69.** If $z = 1 - t + i\sqrt{t^2 + t + 2}$, where t is a real parameter. The locus of z in the argand plane is

- (a) a hyperbola (b) an ellipse
 (c) a straight line (d) None of these

Sol. (a) $x + iy = 1 - t + i\sqrt{t^2 + t + 2}$
 $\Rightarrow x = 1 - t, y = \sqrt{t^2 + t + 2}$
 Eliminating t , $y^2 = t^2 + t + 2$
 $= (1 - x)^2 + 1 - x + 2$
 $= \left(x - \frac{3}{2}\right)^2 + \frac{7}{4}$

or $y^2 - \left(x - \frac{3}{2}\right)^2 = \frac{7}{4}$

which is a hyperbola.

➤ **Example 70.** Identify the locus of z , if

$\bar{z} = \bar{a} + \frac{r^2}{z - a}, > 0.$

Sol. $\bar{z} = \bar{a} + \frac{r^2}{z - a} \Rightarrow \bar{z} - \bar{a} = \frac{r^2}{z - a}$

$\Rightarrow (z - a)(\bar{z} - \bar{a}) = r^2$

$\Rightarrow |z - a|^2 = r^2$

$\Rightarrow |z - a| = r$

Hence, locus of z is circle having centre a and radius r .

➤ **Example 71.** If the equation

$|z - z_1|^2 + |z - z_2|^2 = k$ represents the equation of a circle, where $z_1 = 2 + 3i$, $z_2 = 4 + 3i$ are the extremities of a diameter, then the value of k is

(a) $\frac{1}{4}$

(b) 4

(c) 2

(d) None of these

Sol. (b) As z_1 and z_2 are the extremities of diameter.

$\Rightarrow |z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow k = |z_1 - z_2|^2 = |2 + 3i - 4 - 3i|^2 = |-2|^2 = 4$

➤ **Example 72.** If $|z + 1| = \sqrt{2}|z - 1|$, then the locus described by the point z in the argand diagram is a

(a) straight line

(b) circle

(c) parabola

(d) None of these

Sol. (b) $|z + 1| = \sqrt{2}|z - 1|$

Putting $z = x + iy \Rightarrow |x + iy + 1| = \sqrt{2}|x + iy - 1|$

$\Rightarrow |(x + 1) + iy| = \sqrt{2}|(x - 1) + iy|$

$\Rightarrow (x + 1)^2 + y^2 = 2[(x - 1)^2 + y^2]$

$\Rightarrow x^2 + y^2 - 6x + 1 = 0$

which is the equation of a circle.

➤ **Example 73.** The locus of z given by $\left|\frac{z-1}{z-i}\right| = 1$

is

(a) a circle

(b) an ellipse

(c) a straight line

(d) a parabola

Sol. (c) $\left|\frac{z-1}{z-i}\right| = 1 \Rightarrow |z-1| = |z-i|$

$\Rightarrow \frac{|(x-1) + iy|}{\sqrt{(x-1)^2 + y^2}} = \frac{|x + i(y-1)|}{\sqrt{x^2 + (y-1)^2}}$

$\Rightarrow 2x = 2y$

or $x - y = 0$

which is the equation of a straight line.

➤ **Example 74.** If $z = x + iy$ and

$|z - 2 + i| = |z - 3 - i|$, then locus of z is

(a) $2x + 4y - 5 = 0$

(b) $2x - 4y - 5 = 0$

(c) $x + 2y = 0$

(d) $x - 2y + 5 = 0$

Sol. (a) $|z - 2 + i| = |z - 3 - i|$

$\Rightarrow |(x-2) + i(y+1)| = |(x-3) + i(y-1)|$

$\Rightarrow \sqrt{(x-2)^2 + (y+1)^2} = \sqrt{(x-3)^2 + (y-1)^2}$

$\Rightarrow x^2 + 4 - 4x + y^2 + 1 + 2y = x^2 + 9 - 6x + y^2 + 1 - 2y$

$\Rightarrow 2x + 4y - 5 = 0$

➤ **Example 75.** If $\bar{z} = \bar{z}_0 + A(z - z_0)$, where A is a constant, then prove that locus of z is a straight line.

Sol. $\bar{z} = \bar{z}_0 + A(z - z_0)$

$\Rightarrow Az - \bar{z} - Az_0 + \bar{z}_0 = 0 \quad \dots(i)$

$\Rightarrow \bar{A}\bar{z} - z - \bar{A}\bar{z}_0 + z_0 = 0 \quad \dots(ii)$

Adding Eqs. (i) and (ii), we get

$(A - 1)z + (\bar{A} - 1)\bar{z} - (\bar{A}\bar{z}_0 + Az_0) + z_0 + \bar{z}_0 = 0$

This is of the form

$\bar{a}z + a\bar{z} + b = 0$, where

$a = \bar{A} - 1$ and $b = -(\bar{A}\bar{z}_0 + Az_0) + z_0 + \bar{z}_0 \in \mathbb{R}$

Hence, locus of z is a straight line.

➤ **Example 76.** Plot the region represented by

$\frac{\pi}{3} \leq \arg\left(\frac{z+1}{z-1}\right) \leq \frac{2\pi}{3}$ in the argand plane.

Sol. Let us take

$\arg\left(\frac{z+1}{z-1}\right) = \frac{2\pi}{3}$, clearly z

lies on the minor arc of the circle passing through $(1, 0)$ and $(-1, 0)$.

Similarly, $\arg\left(\frac{z+1}{z-1}\right) = \frac{\pi}{3}$

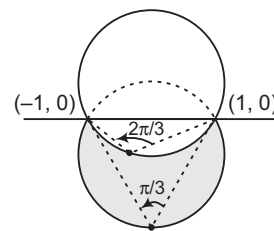
means that z is lying on the major arc of the circle passing through $(1, 0)$ and $(-1, 0)$.

Now, if we take any point in the region included between the two arcs, say $P_1(z_1)$, we get

$\frac{\pi}{3} \leq \arg\left(\frac{z+1}{z-1}\right) \leq \frac{2\pi}{3}$

Thus, $\frac{\pi}{3} \leq \arg\left(\frac{z+1}{z-1}\right) \leq \frac{2\pi}{3}$ represents the shaded region

excluding the points $(1, 0)$ and $(-1, 0)$.



Some Important Results

- i. Four points z_1, z_2, z_3 and z_4 in anti-clockwise order will be concyclic, if and only if
- $$\theta = \arg \left(\frac{z_2 - z_4}{z_1 - z_4} \right) = \arg \left(\frac{z_2 - z_3}{z_1 - z_3} \right)$$
- $$\Rightarrow \arg \left(\frac{z_2 - z_4}{z_1 - z_4} \right) - \arg \left(\frac{z_2 - z_3}{z_1 - z_3} \right) = 2n\pi, n \in I$$
- $$\Rightarrow \arg \left[\left(\frac{z_2 - z_4}{z_1 - z_4} \right) \left(\frac{z_1 - z_3}{z_2 - z_3} \right) \right] = 2n\pi$$
- $$\Rightarrow \left(\frac{z_2 - z_4}{z_1 - z_4} \right) \left(\frac{z_1 - z_3}{z_2 - z_3} \right) \text{ is real and positive.}$$

Points z_1, z_2, z_3 and z_4 (not necessarily in order) will be concyclic,

if $\left(\frac{z_2 - z_4}{z_1 - z_4} \right) \left(\frac{z_1 - z_3}{z_2 - z_3} \right)$ is positive or negative.

- Example 77. Show that the points $3 + 4i, 3 - 4i, -4 + 3i, -4 - 3i$ are concyclic.

Sol. Let $z_1 = 3 + 4i, z_2 = 3 - 4i, z_3 = -4 + 3i$
and $z_4 = -4 - 3i$

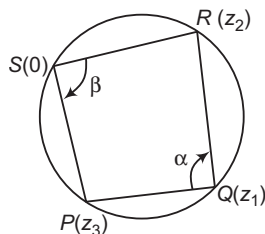
Then,

$$\begin{aligned} \left(\frac{z_2 - z_4}{z_1 - z_4} \right) \left(\frac{z_1 - z_3}{z_2 - z_3} \right) &= \frac{(7 - i)(7 + i)}{(7 + 7i)(7 - 7i)} \\ &= \frac{50}{49 + 49} = \frac{25}{49} \\ &= \text{a real number} \end{aligned}$$

Hence, the given points are concyclic.

- Example 78. If z_1, z_2 and z_3 are complex numbers, such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, then show that the points represented by z_1, z_2 and z_3 lie on a circle passing through the origin.

Sol.



$$\begin{aligned} \frac{2}{z_1} &= \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1} \\ \Rightarrow \frac{z_2 - z_1}{z_1 z_2} &= \frac{z_1 - z_3}{z_3 z_1} \Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3} \\ \Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) &= \arg \left(-\frac{z_2}{z_3} \right) \\ \Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) &= \pi + \arg \left(\frac{z_2}{z_3} \right) \end{aligned}$$

$$\Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) = \pi - \arg \left(\frac{z_3}{z_2} \right)$$

$$\Rightarrow \alpha = \pi - \beta \Rightarrow \alpha + \beta = \pi$$

Hence, the points are concyclic.

- ii. If $\left| z + \frac{1}{z} \right| = a$, then the greatest and least values of $|z|$ are respectively $\frac{a + \sqrt{a^2 + 4}}{2}$ and $\frac{-a + \sqrt{a^2 + 4}}{2}$

- Example 79. If $\left| z + \frac{4}{z} \right| = a$, then find the greatest and least values of $|z|$.

Sol. $\left| \frac{z}{2} + \frac{2}{z} \right| = \frac{a}{2}$. Now, put $\frac{z}{2} = w$

$$\therefore \left| w + \frac{1}{w} \right| = \frac{a}{2}$$

$$\text{Now, greatest value of } |w| = \frac{\frac{a}{2} + \sqrt{\frac{a^2}{4} + 4}}{2} \text{ and } |z| = 2|w|$$

$$\text{Greatest value of } |z| = \frac{a}{2} + \sqrt{\frac{a^2}{4} + 4}$$

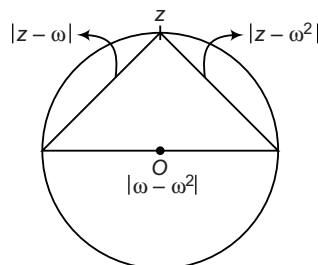
$$\text{Similarly, we can find least value of } |z| = -\frac{a}{2} + \sqrt{\frac{a^2}{4} + 4}$$

- Example 80. Let z_1 and z_2 be two non-real complex cube roots of unity and $|z - z_1|^2 + |z - z_2|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter, then the value of λ is
- (a) 4 (b) 3 (c) 2 (d) $\sqrt{2}$

Sol. (b) $|z - \omega|^2 + |z - \omega^2|^2 - \lambda$ could be shown as given

below :

$$\begin{aligned} \Rightarrow \lambda &= |\omega - \omega^2|^2 \\ &= \left| \frac{-1 + i\sqrt{3}}{2} - \frac{-1 - i\sqrt{3}}{2} \right|^2 \\ &= \left| \frac{-1 + i\sqrt{3} + 1 + i\sqrt{3}}{2} \right|^2 \\ &= |i\sqrt{3}|^2 = 3 \end{aligned}$$



Logarithm of Complex Numbers

Let $z = \alpha + i\beta = re^{i(\theta + 2n\pi)}$

$$\begin{aligned} \log z &= \log(re^{i(\theta + 2n\pi)}) = \log r + i(\theta + 2n\pi) \\ &= \log |z| + i \arg z + 2n\pi i \end{aligned}$$

If we put $n = 0$, we get principal value of $\log z$.

$\therefore$ Principal value of $\log z = \log |z| + i \arg z$.

➤ **Example 81.** The value of i^i is

- (a) $e^{-\pi/2}$ (b) $e^{\pi/2}$
(c) $e^{\pi/4}$ (d) None of these

Sol. (a) Let $z = i^i$, $\log_e z = \log_e i^i = i \log_e i = i \log_e e^{i\pi/2}$

$$= i^2 \frac{\pi}{2} \log_e e = -\frac{\pi}{2}$$

$$\Rightarrow z = e^{-\pi/2}$$

➤ **Example 82.** If $z = i \log_e (2 - \sqrt{3})$, then the value of $\cos z$ is

- (a) 2 (b) -2 (c) 2i (d) -2i

Sol. (a) We know that, $\cos z = \frac{e^{iz} + e^{-iz}}{2}$

$$= \frac{e^{i^2 \log_e (2 - \sqrt{3})} + e^{-i^2 \log_e (2 - \sqrt{3})}}{2}$$

$$= \frac{e^{\log_e (2 - \sqrt{3})^{-1}} + e^{\log_e (2 - \sqrt{3})}}{2}$$

$$= \frac{(2 - \sqrt{3})^{-1} + (2 - \sqrt{3})}{2}$$

$$= \frac{1}{2} \left[\frac{1}{2 - \sqrt{3}} + 2 - \sqrt{3} \right] = \frac{1}{2} \left[\frac{2 + \sqrt{3}}{4 - 3} + (2 - \sqrt{3}) \right]$$

$$= \frac{1}{2} [2 + \sqrt{3} + 2 - \sqrt{3}] = 2$$

Work Book Exercise 4.4

1 $\left| \frac{z-1}{z+1} \right| = 1$, represents

- a a circle b an ellipse
c a straight line d None of these

2 $|z-4| < |z-2|$, represents the region given by

- a $\operatorname{Re}(z) > 0$ b $\operatorname{Re}(z) < 0$
c $\operatorname{Re}(z) > 2$ d None of these

3 If $2z_1 - 3z_2 + z_3 = 0$, then z_1, z_2, z_3 are represented by

- a three vertices of a triangle
b three collinear points
c three vertices of a rhombus
d None of the above

4 If $z = x + iy$, such that $|z+1| = |z-1|$ and

$$\operatorname{amp} \left(\frac{z-1}{z+1} \right) = \frac{\pi}{4}, \text{ then}$$

- a $x = \sqrt{2} + 1, y = 0$
b $x = 0, y = \sqrt{2} + 1$
c $x = 0, y = \sqrt{2} - 1$
d $x = \sqrt{2} - 1, y = 0$

5 If $z^8 = (z-1)^8$, then the roots of this equation are

- a collinear
b concyclic
c the vertices of irregular polygon
d None of the above

6 The equation $z\bar{z} + a\bar{z} + \bar{a}z + b = 0$, $b \in R$ represents a circle, if

- a $|a|^2 = b$ b $|a|^2 \geq b$
c $|a|^2 < b$ d None of these

7 If $z = (\lambda + 3) + i\sqrt{3 - \lambda^2}$, where $|\lambda| < \sqrt{3}$, then locus of z is

- a circle b parabola
c line d None of these

8 If a point P denoting the complex number z moves on the complex plane such that $|\operatorname{Re}(z)| + |\operatorname{Im}(z)| = 1$, then the locus of z is

- a a square b a circle
c two intersecting lines d a line

9 The figure formed by four points $1 + 0i, -1 + 0i, 3 + 4i$ and $\frac{25}{-3 - 4i}$ on the argand plane is

- a a parallelogram but not a rectangle
b a trapezium which is not equilateral
c a cyclic quadrilateral
d None of the above

10 If $i = \sqrt{-1}$, then define a sequence of complex number by $z_1 = 0, z_{n+1} = z_n^2 + i$ for $n \geq 1$. In the complex plane, how far from the origin is z_{111} ?

- a 1 b $\sqrt{2}$ c $\sqrt{3}$ d $\sqrt{110}$

11 The complex numbers whose real and imaginary parts are integers and satisfy the relation $z\bar{z}^3 + z^3\bar{z} = 350$, forms a rectangle on the argand plane, the length of whose diagonal is

- a 5 units b 10 units c 15 units d 25 units

12 The locus of z , for $\arg z = -\frac{\pi}{3}$ is

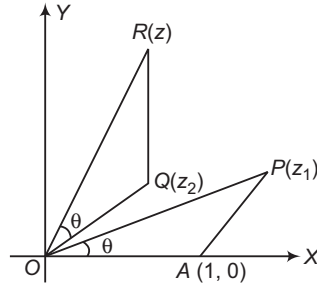
- a same as the locus of z for $\arg z = \frac{2\pi}{3}$
b same as the locus of z for $\arg z = \frac{\pi}{3}$
c the part of the straight line $\sqrt{3}x + y = 0$ with $y < 0, x > 0$
d the part of the straight line $\sqrt{3}x + y = 0$ with $y > 0, x < 0$

13 If z_1 and z_2 are two complex numbers and $\arg \frac{z_1 + z_2}{z_1 - z_2} = \frac{\pi}{2}$ but $|z_1 + z_2| \neq |z_1 - z_2|$, then the figure formed by the points represented by $0, z_1, z_2$ and $z_1 + z_2$ is

- a a parallelogram but not a rectangle or a rhombus
b a rectangle but not a square
c a rhombus but not a square
d a square

14 z_1 and z_2 are two distinct points in an argand plane. If $a|z_1| = b|z_2|$, where $a, b \in R$, then the point $\frac{az_1}{bz_2} + \frac{bz_2}{az_1}$ is a point on the

- a line segment $[-2, 2]$ of the real axis
b line segment $[-2, 2]$ of the imaginary axis
c unit circle $|z| = 1$
d the line with $\arg z = \tan^{-1} 2$

- 15** The roots z_1, z_2, z_3 of the equation $z^3 + 3\alpha z^2 + 3\beta z + \gamma = 0$ correspond to the points A, B and C on the complex plane. Then, the triangle is equilateral, if
a $\alpha^2 = \beta$ **b** $\alpha = \beta^2$
c $\alpha^2 = 3\beta^2$ **d** $3\alpha^2 = \beta^2$
- 16** If $z_1^2 + z_2^2 + 2z_1z_2 \cos \theta = 0$, then the points represented by z_1, z_2 and the origin form
a equilateral triangle **b** right angled triangle
c isosceles triangle **d** None of these
- 17** Complex numbers z_1, z_2, z_3 are the vertices A, B and C , respectively of an isosceles right angled triangle, right angled at C . Then, $(z_1 - z_2)^2$ is equal to
a $2(z_1 - z_3)(z_3 - z_2)$ **b** $\frac{1}{2}(z_1 + z_3)(z_3 - z_1)$
c $(z_1 - z_3)(z_3 - z_2)$ **d** None of these
- 18** A, B and C are points represented by complex numbers z_1, z_2 and z_3 . If the circumcentre of the $\triangle ABC$ is at the origin and the altitude AD of the triangle meets the circumcircle again at P , then the complex number representing point P is
a $z = \frac{z_1z_2}{z_3}$ **b** $z = -\frac{z_3z_2}{z_1}$
c $z = -\frac{z_1z_2}{z_3}$ **d** $z = \frac{z_3z_2}{z_1}$
- 19** If z_1 and z_2 are the roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex number. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, then p^2 is equal to
a $4q \cos^2 \left(\frac{\alpha}{2} \right)$ **b** $2q \cos^2 \alpha$
c $q \cos^2 \frac{\alpha}{4}$ **d** None of these
- 20** If $\operatorname{Re} \left(\frac{z+4}{2z-i} \right) = \frac{1}{2}$, then z is represented by a point lying on
a a circle **b** an ellipse
c a straight line **d** None of these
- 21** Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + \sqrt{3}i$ and z_1, z_2 and z_3 are in the clockwise sense, then
a $z_2 = 1 - \sqrt{3}i, z_3 = 2$
b $z_2 = 2, z_3 = 1 - \sqrt{3}i$
c $z_2 = -1 + \sqrt{3}i, z_3 = -2$
d None of the above
- 22** Suppose z_1, z_2 and z_3 are the vertices of an equilateral triangle circumscribing in the circle $|z| = 2$. If $z_1 = 1 + \sqrt{3}i$ and z_1, z_2 and z_3 are in the anti-clockwise sense, then z_2 is
a $1 - \sqrt{3}i$
b 2
c $\frac{1}{2}(1 - \sqrt{3}i)$
d None of the above
- 23** The straight line $(1+2i)z + (2i-1)\bar{z} = 10i$ on the complex plane, has intercept on the imaginary axis equal to
a 5 **b** $\frac{5}{2}$ **c** $-\frac{5}{2}$ **d** -5
- 24** If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $z + \frac{1}{z}$ is equal to
a $\frac{5}{3}$ **b** $\frac{8}{3}$
c $\frac{11}{3}$ **d** None of these
- 25.** If $z_1 = \frac{a}{1-i}, z_2 = \frac{b}{2+i}, z_3 = a-bi$, for $a, b \in \mathbb{R}$ and $z_1 - z_2 = 1$, then the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand plane is given by
a $\frac{1}{9}(1+7i)$ **b** $\frac{1}{3}(1+7i)$ **c** $\frac{1}{3}(1-3i)$ **d** $\frac{1}{9}(1-3i)$
- 26.** Let $\lambda \in \mathbb{R}$. If the origin and the non-real roots of $2z^2 + 2z + \lambda = 0$ form three vertices of an equilateral triangle in the argand plane, then λ is
a 1 **b** $2/3$
c 2 **d** -1
- 27** If a, b, c and u, v, w are complex numbers representing the vertices of two triangles, such that $c = (1-r)a + rb$ and $w = (1-r)u + rv$. When r is a complex number, then the two triangles
a have the same area **b** are similar
c are congruent **d** None of these
- 28.** On the complex plane, $\triangle OAP$ and $\triangle OQR$ are similar and $I(OA) = 1$. If the points P and Q denote the complex numbers z_1 and z_2 , then the complex numbers z denoted by the point R is given by
a z_1z_2 **b** $\frac{z_1}{z_2}$
c $\frac{z_2}{z_1}$ **d** $\frac{z_1 + z_2}{z_2}$
- 
- 29** Intercept made by the circle $zz + \alpha z + \bar{\alpha} \bar{z} + r = 0$ on the real axis on complex plane, is
a $\sqrt{(\alpha + \bar{\alpha}) - r}$ **b** $\sqrt{(\alpha + \bar{\alpha})^2 - 2r}$
c $\sqrt{(\alpha + \bar{\alpha})^2 + r}$ **d** $\sqrt{(\alpha + \bar{\alpha})^2 - 4r}$
- 30** The equation of the radical axis of the two circles represented by the equations, $|z-2| = 3$ and $|z-2-3i| = 4$ on the complex plane is
a $3y+1=0$
b $3y-1=0$
c $2y-1=0$
d None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $a = \cos \alpha + i \sin \alpha$, $b = \cos \beta + i \sin \beta$,
 $c = \cos \gamma + i \sin \gamma$ and $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$, then
 $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha)$ is equal
 to
 (a) $\frac{3}{2}$ (b) $-\frac{3}{2}$
 (c) 0 (d) 1

Sol. $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} = 1$
 $\Rightarrow \frac{\cos \alpha}{\cos \beta} + \frac{\cos \beta}{\cos \gamma} + \frac{\cos \gamma}{\cos \alpha} = 1$,
 where, $\text{cis } \theta$ represents $\cos \theta + i \sin \theta$.
 $\Rightarrow \text{cis}(\alpha - \beta) + \text{cis}(\beta - \gamma) + \text{cis}(\gamma - \alpha) = 1$
 Equation real parts of both sides
 $\cos(\alpha - \beta) + \cos(\beta - \gamma) + \cos(\gamma - \alpha) = 1$
 Hence, (d) is the correct answer.

Ex 2. The locus of the centre of a circle which
 touches the circles $|z - z_1| = a$ and $|z - z_2| = b$
 externally (z, z_1 and z_2 are complex numbers)
 will be
 (a) an ellipse (b) a hyperbola
 (c) a circle (d) None of these

Sol. Let $A(z_1), B(z_2)$ be the centres of given circles and P be
 the centre of the variable circle which touches given
 circles externally, then
 $|AP| = a + r$ and $|BP| = b + r$,
 where r is the radius of the variable circle.
 On subtraction, we get
 $|AP| - |BP| = a - b$
 $\Rightarrow ||AP| - |BP|| = |a - b|$, a constant.
 Hence, locus of P is
 (i) right bisector AB , if $a = b$.
 (ii) a hyperbola, if $|a - b| < |AB| = |z_2 - z_1|$.
 (iii) an empty set, if $|a - b| > |AB| = |z_2 - z_1|$.
 (iv) set of all points on line AB except those which lie
 between A and B , if $|a - b| = |AB| \neq 0$.
 Hence, (d) is the correct answer.

Ex 3. If $\arg(\bar{z}_1) = \arg(z_2)$, then
 (a) $z_2 = kz_1^{-1}, k > 0$ (b) $z_2 = kz_1, k > 0$
 (c) $|z_2| = |\bar{z}_1|$ (d) None of these

Sol. $\bar{z}_1 = \frac{z_1 \bar{z}_1}{z_1} = |z_1|^2 z_1^{-1}$
 $\Rightarrow \arg(z_1^{-1}) = \arg(\bar{z}_1) = \arg(z_2)$
 $\Rightarrow z_2 = kz_1^{-1} (k > 0)$
 Hence, (a) is the correct answer.

Ex 4. If $\frac{\tan \alpha - i \left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \right)}{1 + 2i \sin \frac{\alpha}{2}}$ is purely
 imaginary, then α is given by
 (a) $n\pi + \frac{\pi}{4}$ (b) $n\pi - \frac{\pi}{4}$
 (c) $(2n + 1)\pi$ (d) $2n\pi + \frac{\pi}{4}$

Sol. $\text{Re} \left(\frac{\tan \alpha - i \left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \right)}{1 + 2i \sin \frac{\alpha}{2}} \right) = 0$
 $\Rightarrow \text{Re} \left(\frac{\left\{ \tan \alpha - i \left(\sin \frac{\alpha}{2} + \cos \frac{\alpha}{2} \right) \right\} \left(1 - 2i \sin \frac{\alpha}{2} \right)}{1 + 4 \sin^2 \frac{\alpha}{2}} \right) = 0$
 $\Rightarrow \text{Re} \left\{ \frac{\tan \alpha - 2 \sin \frac{\alpha}{2} \left(\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} \right)}{1 + 4 \sin^2 \frac{\alpha}{2}} - i \left(\frac{\tan \alpha \cdot 2 \sin \frac{\alpha}{2} + \sin \frac{\alpha}{2} + \cos \frac{\alpha}{2}}{1 + 4 \sin^2 \frac{\alpha}{2}} \right) \right\} = 0$
 $\Rightarrow \tan \alpha = 2 \sin^2 \frac{\alpha}{2} + \sin \alpha$
 $\Rightarrow \frac{\sin \alpha}{\cos \alpha} = \sin \alpha + 1 - \cos \alpha$
 $\Rightarrow \sin \alpha = \sin \alpha \cdot \cos \alpha + \cos \alpha - \cos^2 \alpha$
 $\Rightarrow \sin \alpha (1 - \cos \alpha) = \cos \alpha (1 - \cos \alpha)$
 $\Rightarrow \sin \alpha = \cos \alpha, \cos \alpha = 1$
 $\Rightarrow \alpha = n\pi + \frac{\pi}{4}$
 or $\alpha = 2n\pi, n \in \mathbb{I}$
 Hence, (a) is the correct answer.

Ex 5. The expression $\tan \left[i \log \left(\frac{a - ib}{a + ib} \right) \right]$ reduces to
 (a) $\frac{ab}{a^2 + b^2}$ (b) $\frac{2ab}{a^2 - b^2}$
 (c) $\frac{ab}{a^2 - b^2}$ (d) $\frac{2ab}{a^2 + b^2}$

Sol. Given, $\tan \left[i \log \left(\frac{a - ib}{a + ib} \right) \right]$
 Let $a + ib = re^{i\theta}, r^2 = a^2 + b^2$
 $\Rightarrow a - ib = re^{-i\theta}, \tan \theta = \frac{b}{a}$

$$\Rightarrow \frac{a-ib}{a+ib} = e^{-2i\theta}$$

$$i \log \left(\frac{a-ib}{a+ib} \right) = i \log(e^{-2i\theta}) = 2\theta$$

$$\Rightarrow \tan \left[i \log \left(\frac{a-ib}{a+ib} \right) \right] = \tan 2\theta$$

$$= \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$= \frac{2b/a}{1 - \frac{b^2}{a^2}} = \frac{2ab}{a^2 - b^2}$$

Hence, (b) is the correct answer.

- Ex 6.** If $z_1 = a + ib$ and $z_2 = c + id$ are complex numbers such that $|z_1| = |z_2| = 1$ and $\operatorname{Re}(z_1 \bar{z}_2) = 0$, then the pair of complex numbers $a + ic = w_1$ and $b + id = w_2$ satisfies
- (a) $|w_1| \neq 1$ (b) $|w_2| \neq 1$
 (c) $\operatorname{Re}(w_1 \bar{w}_2) = 0$ (d) None of these

Sol. $z_1 = a + ib, z_2 = c + id$

$$|z_1| = |z_2| = 1$$

$$\Rightarrow a^2 + b^2 = c^2 + d^2 = 1$$

$$w_1 = a + ic, w_2 = b + id$$

$$z_1 \bar{z}_2 = (a + ib)(c - id)$$

$$= (ac + bd) + i(bc - ad)$$

As $\operatorname{Re}(z_1 \bar{z}_2) = 0$

$$\Rightarrow ac + bd = 0 \Rightarrow ac = -bd$$

$$w_1 \bar{w}_2 = (a + ic)(b - id)$$

$$= (ab + cd) + i(bc - ad)$$

We have, $a^2 + b^2 = c^2 + d^2$

$$\Rightarrow a^2 - c^2 = d^2 - b^2$$

$$\Rightarrow a^2 - c^2 + 2i ac = d^2 - b^2 - 2i bd \quad [\because ac = -bd]$$

$$\Rightarrow (a + ic)^2 = (d - ib)^2$$

$$\Rightarrow a + ic = (d - ib) \quad \text{or} \quad -d + ib$$

$$\Rightarrow a = d \quad \text{and} \quad c = -b \quad \text{or} \quad a = -d, b = c$$

$$\Rightarrow c^2 + d^2 = b^2 + d^2$$

$$a^2 + c^2 = a^2 + b^2$$

$$\Rightarrow a^2 + c^2 = 1, b^2 + d^2 = 1$$

$$\Rightarrow |w_1| = |w_2| = 1$$

Also, $ab + cd = -cd + cd = 0$

$$\Rightarrow \operatorname{Re}(w_1 \bar{w}_2) = 0$$

Hence, (c) is the correct answer.

- Ex 7.** If $\left| \frac{z_1}{z_2} \right| = 1$ and $\arg(z_1 z_2) = 0$, then

- (a) $z_1 = z_2$ (b) $|z_2|^2 = z_1 z_2$
 (c) $z_1 z_2 = 1$ (d) None of these

Sol. Let $z_1 = r_1 (\cos \theta_1 + i \sin \theta_1)$, then $\left| \frac{z_1}{z_2} \right| = 1$

$$\Rightarrow |z_1| = |z_2|$$

$$\Rightarrow |z_1| = |z_2| = r_1$$

Now, $\arg(z_1 z_2) = 0$

$$\Rightarrow \arg(z_1) + \arg(z_2) = 0$$

$$\Rightarrow \arg(z_2) = -\theta_1$$

Therefore, $z_2 = r_1 [\cos(-\theta_1) + i \sin(-\theta_1)]$

$$= r_1 (\cos \theta_1 - i \sin \theta_1) = \bar{z}_1$$

$$\Rightarrow \bar{z}_2 = \overline{(\bar{z}_1)} = z_1 \Rightarrow |z_2|^2 = z_1 z_2$$

Hence, (b) is the correct answer.

- Ex 8.** If z is a complex number, then $z^2 + \bar{z}^2 = 2$ represents
- (a) a circle (b) a straight line
 (c) a hyperbola (d) an ellipse

Sol. Let $z = x + iy$, then

$$z^2 + \bar{z}^2 = 2$$

$$\Rightarrow (x + iy)^2 + (x - iy)^2 = 2$$

$$\Rightarrow x^2 - y^2 = 1$$

which represents a hyperbola.

Hence, (c) is the correct answer.

- Ex 9.** If $\frac{1-i\alpha}{1+i\alpha} = A + iB$, then $A^2 + B^2$ equals

- (a) 1 (b) α^2 (c) -1 (d) $-\alpha^2$

Sol. $A + iB = \frac{1-i\alpha}{1+i\alpha} \Rightarrow A - iB = \frac{1+i\alpha}{1-i\alpha}$

$$\Rightarrow (A + iB)(A - iB) = \frac{(1-i\alpha)(1+i\alpha)}{(1+i\alpha)(1-i\alpha)} = 1$$

$$\Rightarrow A^2 + B^2 = 1$$

Hence, (a) is the correct answer.

- Ex 10.** If $|z_1| = |z_2|$ and $\arg(z_1) + \arg(z_2) = \pi/2$, then

- (a) $z_1 z_2$ is purely real
 (b) $z_1 z_2$ is purely imaginary
 (c) $(z_1 + z_2)^2$ is purely real
 (d) $\arg(z_1^{-1}) + \arg(z_2^{-1}) = \frac{\pi}{2}$

Sol. Let $|z_1| = |z_2| = r$

$$\Rightarrow z_1 = r(\cos \theta + i \sin \theta)$$

and $z_2 = r \left[\cos \left(\frac{\pi}{2} - \theta \right) + i \sin \left(\frac{\pi}{2} - \theta \right) \right]$

$$\Rightarrow z_1 z_2 = r^2 i, \text{ which is purely imaginary.}$$

$$z_1 + z_2 = r[(\cos \theta + \sin \theta) + i(\cos \theta + \sin \theta)]$$

$$\Rightarrow (z_1 + z_2)^2 = 2r^2 \cdot (\cos \theta + \sin \theta)^2 \cdot i$$

which is purely imaginary.

Also, $\arg(z_1^{-1}) + \arg(z_2^{-1}) = -\frac{\pi}{2}$

Hence, (b) is the correct answer.

- Ex 11.** The value of the expression

$$2 \left(1 + \frac{1}{\omega} \right) \left(1 + \frac{1}{\omega^2} \right) + 3 \left(2 + \frac{1}{\omega} \right) \left(2 + \frac{1}{\omega^2} \right)$$

$$+ 4 \left(3 + \frac{1}{\omega} \right) \left(3 + \frac{1}{\omega^2} \right) + \dots + (n+1) \left(n + \frac{1}{\omega} \right) \left(n + \frac{1}{\omega^2} \right),$$

where ω is an imaginary cube root of unity, is

- (a) $\frac{n(n^2 + 2)}{3}$ (b) $\frac{n(n^2 - 2)}{3}$
 (c) $\frac{n^2(n+1)^2 + 4n}{4}$ (d) None of these

Sol. $t_n = (n+1)\left(n + \frac{1}{\omega}\right)\left(n + \frac{1}{\omega^2}\right)n^3 + n^2\left(\frac{1}{\omega^2} + \frac{1}{\omega} + 1\right) + n\left(1 + \frac{1}{\omega^2} + \frac{1}{\omega}\right) + 1$
 $= n^3 + n^2(\omega + \omega^2 + 1) + n(\omega + \omega^2 + 1) + 1 = n^3 + 1$
 $\therefore S_n = \sum_{r=1}^n t_r = \sum_{r=1}^n (r^3 + 1) = \frac{n^2(n+1)^2}{4} + n$

Hence, (c) is the correct answer.

Ex 12. If z_1 and z_2 are two complex numbers satisfying the equation $\left|\frac{z_1 + iz_2}{z_1 - iz_2}\right| = 1$, then $\frac{z_1}{z_2}$

- is
 (a) purely real
 (b) of unit modulus
 (c) purely imaginary
 (d) None of the above

Sol. $(z_1 + iz_2)(\bar{z}_1 - i\bar{z}_2) = (z_1 - iz_2)(\bar{z}_1 + i\bar{z}_2)$
 $\Rightarrow \bar{z}_1 z_2 = z_1 \bar{z}_2$
 $\Rightarrow \frac{z_1}{z_2} = \frac{\bar{z}_1}{\bar{z}_2}$
 $\Rightarrow \frac{z_1}{z_2}$ is purely real.

Hence, (a) is the correct answer.

Ex 13. If $z = -2 + 2\sqrt{3}i$, then $z^{2n} + 2^{2n}z^n + 2^{4n}$ may be equal to
 (a) 2^{2n}
 (b) 0
 (c) $3 \cdot 4^{2n}$, n is multiple of 3
 (d) None of the above

Sol. $z = -2 + 2\sqrt{3}i = 4\omega$
 $\therefore z^{2n} + 2^{2n}z^n + 2^{4n} = 4^{2n}\omega^{2n} + 2^{2n} \cdot 4^n \cdot \omega^n + 2^{4n}$
 $= 4^{2n}[\omega^{2n} + \omega^n + 1]$
 $= 0$, if n is not a multiple of 3.
 $= 3 \cdot 4^{2n}$, if n is a multiple of 3.

Hence, (c) is the correct answer.

Ex 14. The complex number $z = 1 + i$ is rotated through an angle $3\pi/2$ in anti-clockwise direction about the origin and stretched by additional $\sqrt{2}$ units, then the new complex number is
 (a) $-\sqrt{2} - \sqrt{2}i$
 (b) $\sqrt{2} - \sqrt{2}i$
 (c) $2 - \sqrt{2}i$
 (d) None of the above

Sol. If z_1 is the new complex number, then

$$|z_1| = |z| + \sqrt{2} = 2\sqrt{2}$$

Also, $\frac{z_1}{z} = \frac{|z_1|}{|z|} \cdot e^{i3\pi/2}$

$$\Rightarrow z_1 = z \cdot 2 \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$$

$$= 2(1+i)(0-i) = -2i + 2 = 2(1-i)$$

Hence, (d) is the correct answer.

Ex 15. If $2\cos\theta = x + \frac{1}{x}$ and $2\cos\phi = y + \frac{1}{y}$, then

- (a) $\frac{x}{y} + \frac{y}{x} = 2\cos(\theta + \phi)$
 (b) $x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$
 (c) $\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta + n\phi)$
 (d) $xy + \frac{1}{xy} = 2\cos(\theta - \phi)$

Sol. $2\cos\theta = x + \frac{1}{x}$, $2\cos\phi = y + \frac{1}{y}$

$$\Rightarrow x^2 - 2x\cos\theta + 1 = 0$$

$$\Rightarrow x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\Rightarrow x = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

Similarly, $y = e^{\pm i\phi}$

$$\Rightarrow \frac{x}{y} + \frac{y}{x} = 2\cos(\theta - \phi)$$

$$x^m y^n + \frac{1}{x^m y^n} = 2\cos(m\theta + n\phi)$$

$$\frac{x^m}{y^n} + \frac{y^n}{x^m} = 2\cos(m\theta - n\phi)$$

$$xy + \frac{1}{xy} = 2\cos(\theta + \phi)$$

Hence, (b) is the correct answer.

Ex 16. The complex numbers z_1 and z_2 are such that $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part, then $\left(\frac{z_1 + z_2}{z_1 - z_2}\right)$ may be

- (a) zero
 (b) real and positive
 (c) real and negative
 (d) purely imaginary

Sol. Given, $|z_1| = |z_2|$

$$\operatorname{Re}(z_1) > 0, \operatorname{Im}(z_2) < 0$$

$$\operatorname{Re}\left(\frac{z_1 + z_2}{z_1 - z_2}\right) = \frac{1}{2} \left(\frac{z_1 + z_2}{z_1 - z_2} + \frac{\bar{z}_1 + \bar{z}_2}{\bar{z}_1 - \bar{z}_2} \right)$$

$$= \frac{1}{2} \left(\frac{(z_1 + z_2)(\bar{z}_1 - \bar{z}_2) + (\bar{z}_1 + \bar{z}_2)(z_1 - z_2)}{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} \right)$$

$$= \frac{1}{2} \left(\frac{z_1 \bar{z}_1 - z_1 \bar{z}_2 + z_2 \bar{z}_1 - z_2 \bar{z}_2 + z_1 \bar{z}_1 + z_1 \bar{z}_2}{|z_1 - z_2|^2} \right)$$

$$= \frac{1}{2} \left(\frac{2|z_1|^2 - |z_2|^2}{|z_1 - z_2|^2} \right) = 0$$

$$\Rightarrow \left(\frac{z_1 + z_2}{z_1 - z_2} \right) \text{ is purely imaginary.}$$

Hence, (d) is the correct answer.

- Ex 17.** If $\left| \frac{z_1 z - z_2}{z_1 z + z_2} \right| = k$, ($z_1, z_2 \neq 0$), then
- for $k \neq 1$, locus z is a straight line
 - for $k \notin \{1, 0\}$, z lies on a circle
 - for $k \neq 0$, z represents a point
 - for $k \neq 1$, z lies on the perpendicular bisector of the line segment joining $\frac{z_2}{z_1}$ and $-\frac{z_2}{z_1}$

Sol. Given, $\left| \frac{z_1 z - z_2}{z_1 z + z_2} \right| = k \Rightarrow \left| \frac{z - \frac{z_2}{z_1}}{z + \frac{z_2}{z_1}} \right| = k$

Clearly, if $k \neq 0, 1$ then z would lie on a circle.

If $k = 1$, z would lie on a perpendicular bisector of the line segment joining $\frac{z_2}{z_1}$ and $-\frac{z_2}{z_1}$.

If $k = 0$, z represents a point.

Hence, (b) is the correct answer.

- Ex 18.** If $A(z_1)$, $B(z_2)$ and $C(z_3)$ are the vertices of a $\triangle ABC$ in which $\angle ABC = \frac{\pi}{4}$ and $\frac{AB}{BC} = \sqrt{2}$, then the value of z_2 is equal to
- $z_3 + i(z_1 + z_3)$
 - $z_3 - i(z_1 - z_3)$
 - $z_3 + i(z_1 - z_3)$
 - None of the above

Sol. Given, $\frac{AB}{BC} = \sqrt{2}$

Considering the rotation about B , we get

$$\frac{z_1 - z_2}{z_3 - z_2} = \frac{|z_1 - z_2|}{|z_3 - z_2|} \cdot e^{i\pi/4} = \frac{AB}{BC} \cdot e^{i\pi/4}$$

$$= \sqrt{2} \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}} \right) = 1 + i$$

$$\Rightarrow z_1 - z_2 = (1 + i)(z_3 - z_2)$$

$$\Rightarrow z_1 - (1 + i)z_3 = z_2(1 - 1 - i)$$

$$\Rightarrow iz_2 = -z_1 + (1 + i)z_3$$

$$\Rightarrow z_2 = iz_1 - i(1 + i)z_3$$

$$\therefore z_2 = z_3 + i(z_1 - z_3)$$

Hence, (c) is the correct answer.

- Ex 19.** If $z_1, z_2 \in C$, $z_1^2 + z_2^2 \in R$, $z_1(z_1^2 - 3z_2^2) = 2$ and $z_2(3z_1^2 - z_2^2) = 11$, then the value of $z_1^2 + z_2^2$ is
- 5
 - 6
 - 10
 - 12

Sol. Here, $z_1(z_1^2 - 3z_2^2) = 2 \quad \dots(i)$

$$z_2(3z_1^2 - z_2^2) = 11 \quad \dots(ii)$$

On multiplying Eq. (ii) by i and adding it to Eq. (i), we get

$$z_1^3 - 3z_2^2 z_1 + i(3z_1^2 z_2 - z_2^3) = 2 + 11i$$

$$\Rightarrow (z_1 + iz_2) = 2 + 11i \quad \dots(iii)$$

Again multiplying Eq. (ii) by i and subtracting it from Eq. (i), we get

$$z_1^3 - 3z_2^2 z_1 - i(3z_1^2 z_2 - z_2^3) = 2 - 11i$$

$$\Rightarrow (z_1 - iz_2)^3 = 2 - 11i \quad \dots(iv)$$

On multiplying Eqs. (iii) and (iv), we get

$$(z_1^2 + z_2^2)^3 = 4 + 121 \Rightarrow z_1^2 + z_2^2 = 5$$

Hence, (a) is the correct answer.

- Ex 20.** $\sqrt{1 - c^2} = nc - 1$ and $z = e^{i\theta}$, then $\frac{c}{2n} (1 + nz) \left(1 + \frac{n}{z} \right)$ is equal to
- $1 - c \cos \theta$
 - $1 + 2c \cos \theta$
 - $1 + c \cos \theta$
 - $1 - 2c \cos \theta$

Sol. Here, $\sqrt{1 - c^2} = nc - 1$

$$\Rightarrow 1 - c^2 = n^2 c^2 - 2nc + 1$$

$$\therefore \frac{c}{2n} = \frac{1}{1 + n^2} \quad \dots(i)$$

$$\text{or } \frac{c}{2n} (1 + nz) \left(1 + \frac{n}{z} \right) = \frac{1}{1 + n^2} \left\{ 1 + n^2 + n \left(z + \frac{1}{z} \right) \right\}$$

$$= \frac{1}{1 + n^2} \{ 1 + n^2 + n \cdot (2 \cos \theta) \}$$

$$= \frac{(1 + n^2) + 2nc \cos \theta}{1 + n^2}$$

$$= 1 + \left(\frac{2n}{1 + n^2} \right) \cos \theta$$

[using Eq. (i)]

$$= 1 + c \cos \theta$$

Hence, (c) is the correct answer.

- Ex 21.** Consider an ellipse having its foci at $A(z_1)$ and $B(z_2)$ in the argand plane. If the eccentricity of the ellipse is e and it is known that origin is an interior point of the ellipse, then

$$(a) e \in \left(0, \frac{|z_1 + z_2|}{|z_1| + |z_2|} \right)$$

$$(b) e \in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|} \right)$$

$$(c) e \in \left(0, \frac{|z_1 + z_2|}{|z_1| - |z_2|} \right)$$

(d) Can't be determined

Sol. If $P(z)$ is any point on the ellipse. Then, equation of the ellipse is

$$|z - z_1| + |z - z_2| = \frac{|z_1 - z_2|}{e} \quad \dots(i)$$

If we replace z by z_1 or z_2 , Eq. (i) becomes $|z_1 - z_2|$.

For $P(z)$ to lie on ellipse, we have

$$|z - z_1| + |z - z_2| < \frac{|z_1 - z_2|}{e}$$

It is given that origin is an interior point of the ellipse.

$$\Rightarrow |0 - z_1| + |0 - z_2| < \frac{|z_1 - z_2|}{e}$$

$$\Rightarrow e \in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|}\right)$$

Hence, (b) is the correct answer.

Ex 22. If $|z - 2 - i| = |z| \left| \sin \left(\frac{\pi}{4} - \arg(z) \right) \right|$, then

locus of z is

- (a) a pair of straight lines
- (b) a circle
- (c) a parabola
- (d) an ellipse

Sol. We have, $|(x - 2) + i(y - 1)| = |z| \left| \frac{1}{\sqrt{2}} \cos \theta - \frac{1}{\sqrt{2}} \sin \theta \right|$

where, $\theta = \arg(z)$

$$\sqrt{(x - 2)^2 + (y - 1)^2} = \frac{1}{\sqrt{2}} |x - y|$$

which is a parabola.

Hence, (c) is the correct answer.

Ex 23. $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{100}$ are all the 100th roots of unity. The numerical value of $\sum_{1 \leq i < j \leq 100} (\alpha_i \alpha_j)^5$ is

- (a) 20
- (b) 0
- (c) $(20)^{1/20}$
- (d) None of these

$$\begin{aligned} \text{Sol. } \sum_{1 \leq i < j \leq 100} \alpha_i^5 \alpha_j^5 &= (\alpha_1^5 + \alpha_2^5 + \alpha_3^5 + \alpha_4^5 + \alpha_5^5 + \dots)^2 \\ &\quad - (\alpha_1^{10} + \alpha_2^{10} + \alpha_3^{10} + \alpha_4^{10} + \alpha_5^{10} + \dots) \\ &= 0 - 0 = 0 \end{aligned}$$

$$\text{Because } (\alpha_1^r + \alpha_2^r + \dots + \alpha_{100}^r) = \begin{cases} 100, & \text{if } r = 100k \\ 0, & \text{if } r \neq 100k \end{cases}$$

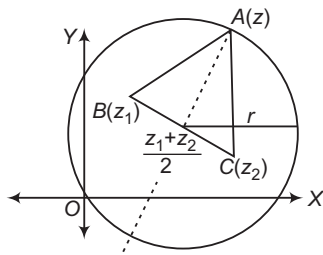
Hence, (b) is the correct answer.

Ex 24. The maximum area of the triangle formed by the complex coordinates z, z_1 and z_2 , which satisfy the relations $|z - z_1| = |z - z_2|$,

$$\left| z - \left(\frac{z_1 + z_2}{2} \right) \right| \leq r, \text{ where } r > \frac{|z_1 - z_2|}{2}, \text{ is}$$

- (a) $\frac{1}{2} |z_1 - z_2|^2$
- (b) $\frac{1}{2} |z_1 - z_2| r$
- (c) $\frac{1}{2} |z_1 - z_2|^2 r^2$
- (d) $\frac{1}{2} |z_1 - z_2| r^2$

Sol.



By the given conditions, the area of $\triangle ABC$ is

$$\frac{1}{2} |z_1 - z_2| r$$

Hence, (b) is the correct answer.

Ex 25. Locus of z , if

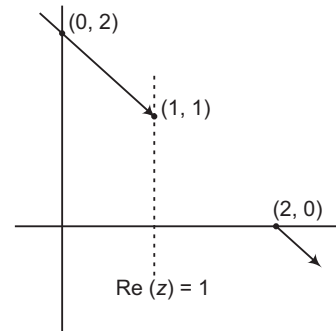
$$\arg[z - (1 + i)] = \begin{cases} \frac{3\pi}{4}, & \text{when } |z| \leq |z - 2| \\ -\frac{\pi}{4}, & \text{when } |z| > |z - 2| \end{cases}, \text{ is}$$

- (a) straight line passing through $(2, 0)$
- (b) straight line passing through $(2, 0), (1, 1)$
- (c) a line segment
- (d) a set of two rays

Sol. The given equation is written as

$$\arg[z - (1 + i)] = \begin{cases} \frac{3\pi}{4}, & \text{when } x \leq 2 \\ -\frac{\pi}{4}, & \text{when } x > 2 \end{cases}$$

The locus is a set of two rays.



Hence, (d) is the correct answer.

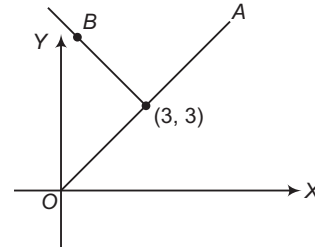
Ex 26. If $A = \left\{ z \mid \arg(z) = \frac{\pi}{4} \right\}$ and

$$B = \left\{ z \mid \arg(z - 3 - 3i) = \frac{2\pi}{3} \right\}, z \in \mathbb{C}. \text{ Then,}$$

$n(A \cap B)$ is equal to

- (a) 1
- (b) 2
- (c) 3
- (d) 0

Sol. We can observe that, $3 + 3i \in A$ but $\notin B$.



$$\therefore n(A \cap B) = 0$$

Hence, (d) is the correct answer.

Ex 27. Dividing $f(z)$ by $z - i$, we obtain the remainder i and dividing it by $z + i$, we get the remainder $1 + i$. The remainder upon the division of $f(z)$ by $z^2 + 1$, is

- (a) $\frac{1}{2}(z+1)+i$ (b) $\frac{1}{2}(iz+1)+i$
(c) $\frac{1}{2}(iz-1)+i$ (d) $\frac{1}{2}(z+i)+1$

Type 2. More than One Correct Option

Ex 28. If $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ are complex numbers such that $|z_1| = 1$, $|z_2| = 2$ and $\operatorname{Re}(z_1 z_2) = 0$, then the pair of complex numbers $\omega_1 = a_1 + \frac{ia_2}{2}$ and $\omega_2 = 2b_1 + ib_2$, satisfy

- (a) $|\omega_1| = 1$ (b) $|\omega_2| = 2$
(c) $\operatorname{Re}(\omega_1 \omega_2) = 0$ (d) $\operatorname{Im}(\omega_1 \omega_2) = 0$

Sol. $a_1^2 + b_1^2 = 1$, $a_2^2 + b_2^2 = 4$ and $a_1 a_2 = b_1 b_2$

$$a_2^2 + b_2^2 = 4a_1^2 + 4b_1^2$$

$$(a_2 + 2ia_1)^2 = (2b_1 + ib_2)^2 \Rightarrow a_2 = \pm 2b_1$$

$$|\omega_1|^2 = a_1^2 + a_1^2 + \frac{a_2^2}{4} = a_1^2 + b_1^2 = 1$$

$$\Rightarrow |\omega_1| = 1 \text{ and } |\omega_2|^2 = 4b_1^2 + b_2^2 = 4$$

$$\Rightarrow |\omega_2| = 2$$

$$\operatorname{Re}(\omega_1 \omega_2) = 2a_1 b_1 - \frac{a_2 b_2}{2} = 0$$

$$\operatorname{Im}(\omega_1 \omega_2) = a_1 b_2 + a_2 b_1 = 2a_1^2 + 2b_1^2 = 2$$

Hence, (a), (b) and (c) are the correct answers.

Ex 29. If from a point P representing the complex number z_1 on the curve $|z| = 2$, pair of tangents are drawn to the curve $|z| = 1$, meeting at point $Q(z_2)$ and $R(z_3)$, then

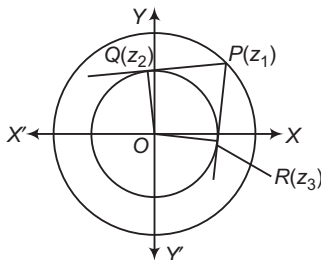
(a) complex number $\frac{z_1 + z_2 + z_3}{3}$ will lie on the curve $|z| = 1$

$$(b) \left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) \left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) = 9$$

$$(c) \arg \left(\frac{z_2}{z_3} \right) = \frac{2\pi}{3}$$

(d) orthocentre and circumcentre of ΔPQR will coincide

Sol. Options (a) and (d) are true as PQR is an equilateral triangle, so orthocentre, circumcentre and centroid will coincide.



Sol. $f(z) = g(z)(z-i)(z+i) + az + b$; where $a, b \in \mathbb{C}$

$$f(i) = i \Rightarrow ai + b = i \quad \dots(i)$$

$$f(-i) = 1 + i \Rightarrow a(-i) + b = 1 + i \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$a = \frac{i}{2}, b = \frac{1}{2} + i$$

$$\text{Hence, required remainder} = az + b = \frac{1}{2}iz + \frac{1}{2} + i$$

Hence, (b) is the correct answer.

$$(b) \left| \frac{z_1 + z_2 + z_3}{3} \right| = 1 \Rightarrow |z_1 + z_2 + z_3|^2 = 9$$

$$\Rightarrow (z_1 + z_2 + z_3)(\bar{z}_1 + \bar{z}_2 + \bar{z}_3) = 9$$

$$\Rightarrow \left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) \left(\frac{4}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) = 9$$

$$(c) \angle QOR = 120^\circ$$

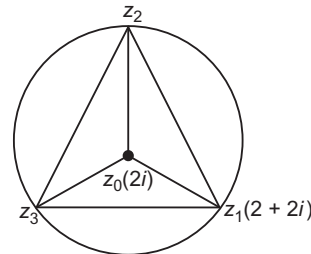
Hence, (a), (b), (c) and (d) are the correct answers.

Ex 30. One vertex of the triangle of maximum area that can be inscribed in the curve $|z - 2i| = 2$, is $2 + 2i$, remaining vertices is/are

$$(a) -1 + i(2 + \sqrt{3}) \quad (b) -1 - i(2 + \sqrt{3})$$

$$(c) -1 + i(2 - \sqrt{3}) \quad (d) -1 - i(2 - \sqrt{3})$$

Sol. Clearly, the inscribed triangle is equilateral.



$$\Rightarrow \frac{z_2 - z_0}{z_1 - z_0} = e^{i\frac{2\pi}{3}}, \frac{z_3 - z_0}{z_1 - z_0} = e^{-i\frac{2\pi}{3}}$$

$$\Rightarrow z_2 = -1 + i(2 + \sqrt{3})$$

$$\text{and } z_3 = -1 + i(2 - \sqrt{3})$$

Hence, options (a) and (c) are the correct answers.

Ex 31. Let z be a complex number and a be a real parameter, such that $z^2 + az + a^2 = 0$, then

(a) locus of z is a pair of straight lines

(b) locus of z is a circle

$$(c) \arg(z) = \pm \frac{2\pi}{3}$$

$$(d) |z| = |a|$$

Sol. $z^2 + az + a^2 = 0$

$$\Rightarrow z = a\omega, a\omega^2$$

where, ω is non-real root of cube unity.

$\Rightarrow$ Locus of z is a pair of straight lines

$$\text{and } \arg(z) = \arg(a) + \arg(\omega) \text{ or } \arg(a) + \arg(\omega^2)$$

$$\Rightarrow \arg(z) = \pm \frac{2\pi}{3}$$

$$\text{Also, } |z| = |a||\omega| \text{ or } |a||\omega^2| \Rightarrow |z| = |a|$$

Hence, (a), (c) and (d) are the correct answers.

Type 3. Assertion and Reason

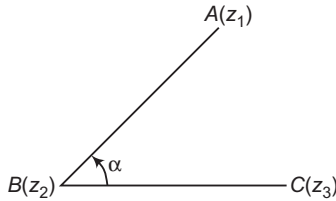
Directions (Ex. Nos. 32-36) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

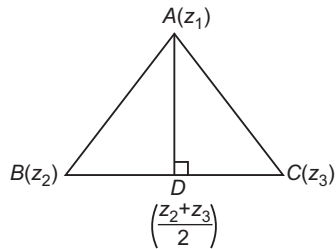
Ex 32. Statement I If $A(z_1)$, $B(z_2)$, $C(z_3)$ are the vertices of an equilateral $\triangle ABC$, then

$$\arg \left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2} \right) = \frac{\pi}{4}$$

Statement II If $\angle B = \alpha$, then $\frac{z_1 - z_2}{z_3 - z_2} = \frac{AB}{BC} e^{i\alpha}$ or $\arg \left(\frac{z_1 - z_2}{z_3 - z_2} \right) = \alpha$



$$\text{Sol. } \therefore \arg \left(\frac{z_2 + z_3 - 2z_1}{z_3 - z_2} \right) = \arg \left(\frac{\frac{z_2 + z_3 - z_1}{2} - z_1}{z_3 - z_2} \right)$$



$$= \frac{\pi}{2} \quad [\text{as } AD \perp BC]$$

Hence, (d) is the correct answer.

Ex 33. Statement I If $x + \frac{1}{x} = 1$ and

$$p = x^{4000} + \frac{1}{x^{4000}} \text{ and } q \text{ is the digit at unit place}$$

in the number $2^{2^n} + 1$, $n \in \mathbb{N}$ and $n > 1$, then the value of $p + q = 8$.

Statement II ω , ω^2 are the roots of $x + \frac{1}{x} = -1$, $x^3 + \frac{1}{x^3} = 2$.

$$\text{Sol. } x + \frac{1}{x} = 1 \Rightarrow x^2 - x + 1 = 0$$

$$\therefore x = -\omega, -\omega^2$$

$$\text{Now, for } x = -\omega, p = \omega^{4000} + \frac{1}{\omega^{4000}} = \omega + \frac{1}{\omega} = -1$$

$$\text{Similarly, for } x = -\omega^2, p = -1$$

$$\text{For } n > 1, 2^n = 4k$$

$$\therefore (2)^{2^n} = 2^{4k} = (16)^k = a \text{ number with last } = 6$$

$$\Rightarrow q = 6 + 1 = 7$$

$$\text{Hence, } p + q = -1 + 7 = 6$$

Hence, (d) is the correct answer.

Ex 34. Statement I If z_1, z_2, z_3 are complex numbers represent the points A, B, C such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$. Then, A, B, C passes through origin.

Statement II If $2z_2 = z_1 + z_3$, then z_1, z_2, z_3 are concyclic.

$$\text{Sol. } \frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3} \Rightarrow \frac{1}{z_1} - \frac{1}{z_2} = \frac{1}{z_3} - \frac{1}{z_1}$$

$$\Rightarrow \frac{z_2 - z_1}{z_1 z_2} = \frac{z_1 - z_3}{z_1 z_3}$$

$$\Rightarrow \frac{z_2 - z_1}{z_3 - z_1} = -\frac{z_2}{z_3}$$

$$\Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) = \arg \left(-\frac{z_2}{z_3} \right)$$

$$\Rightarrow \arg \left(\frac{z_2 - z_1}{z_3 - z_1} \right) = \pi - \arg \left(\frac{z_2}{z_3} \right) = \pi - \arg \left(\frac{z_2}{z_3} \right)$$

$$\Rightarrow \angle CAB = \pi - \angle COB \Rightarrow \angle CAB + \angle COB = \pi$$

$\Rightarrow$ Points O, A, B, C are concyclic.

Hence, (b) is the correct answer.

Ex 35. Statement I $3 + ix^2y$ and $x^2 + y + 4i$ are complex conjugate numbers, then $x^2 + y^2 = 4$.

Statement II If sum and product of two complex numbers is real, then they are conjugate complex number.

$$\text{Sol. If } 3 + ix^2y \text{ and } x^2 + y + 4i \text{ are conjugate, then}$$

$$x^2y = -4 \quad \text{and} \quad x^2 + y = 3$$

$$\Rightarrow x^2 = 4, y = -1 \Rightarrow x^2 + y^2 = 5$$

Hence, (d) is the correct answer.

Ex 36. Statement I If $|z| < \sqrt{2} - 1$, then $|z^2 + 2z \cos \alpha| < 1$

Statement II $|z_1 + z_2| \leq |z_1| + |z_2|$, also $|\cos \alpha| \leq 1$.

$$\text{Sol. } |z^2 + 2z \cos \alpha| < |z|^2 + |2z \cos \alpha|$$

$$< |z|^2 + 2|z| |\cos \alpha|$$

$$< (\sqrt{2} - 1)^2 + 2(\sqrt{2} - 1) < 1$$

$$[\because |\cos \alpha| \leq 1]$$

Hence, (a) is the correct answer.

4

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 37-39) In argand plane, $|z|$ represents the distance of a point z from the origin. In general, $|z_1 - z_2|$ represents the distance between two points z_1 and z_2 . Also, for a general moving point z in argand plane, if $\arg(z) = \theta$, then $z = |z|e^{i\theta}$, where $e^{i\theta} = \cos \theta + i \sin \theta$.

Ex 37. The equation $|z - z_1| + |z - z_2| = 10$, if $z_1 = 3 + 4i$ and $z_2 = -3 - 4i$ represents
(a) point circle (b) ordered pair (0, 0)
(c) ellipse (d) None of these

Sol. As $|z_1 - z_2| = |6 + 8i| = 10$
 $\therefore |z - z_1| + |z - z_2| = |z_1 - z_2|$
 $\Rightarrow z$ lies between z_1 and z_2 .
i.e. a line segment.
Hence, (d) is the correct answer.

Ex 38. $\|z - z_1| - |z - z_2|\| = t$, $t > 10$ where t is a real parameter, always represents
(a) ellipse (b) hyperbola
(c) circle (d) None of these

Sol. Given equation can represent a hyperbola, since
 $|z_1 - z_2| = 10$
 $\Rightarrow z$ lies on hyperbola.
Hence, (b) is the correct answer.

Ex 39. If $|z - (3 + 2i)| = \left| z \cos \left(\frac{\pi}{4} - \arg(z) \right) \right|$, then locus of z is a/an
(a) circle (b) parabola
(c) ellipse (d) hyperbola

Sol. Given, $|z - (3 + 2i)| = |(x - 3) + i(y - 2)|$
 $= |z| \left| \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \right|$, where $\theta = \arg(z)$
 $\Rightarrow \sqrt{(x - 3)^2 + (y - 2)^2} = \frac{1}{\sqrt{2}} |x + y|$
which is a parabola.
Hence, (b) is the correct answer.

Passage II (Ex. Nos. 40-42) The complex slope of a line passing through two points represented by complex numbers z_1 and z_2 is defined by $\frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$ and we shall

denote by ω . If z_0 is complex number and c is a real number, then $z_0 \bar{z} + \bar{z}_0 z = c$ represents a straight line. Its complex slope is $-\frac{z_0}{\bar{z}_0}$. Now, consider two lines

$$\bar{\alpha}z + \alpha\bar{z} + i\beta = 0 \dots (i) \quad \text{and} \quad \bar{a}z + a\bar{z} + b = 0 \dots (ii)$$

where, α, β and a, b are complex constants and let their complex slopes be denoted by ω_1 and ω_2 , respectively.

Ex 40. If the lines are inclined at an angle of 120° to each other, then
(a) $\omega_2 \bar{\omega}_1 = \omega_1 \bar{\omega}_2$ (b) $\omega_2 \bar{\omega}_1^2 = \omega_1 \bar{\omega}_2^2$
(c) $\omega_1^2 = \omega_2^2$ (d) $\omega_1 + 2\omega_2 = 0$

Sol. $\omega_1 = \omega_2 e^{i\frac{4\pi}{3}}$
 $\Rightarrow \omega_1^3 = \omega_2^3 \Rightarrow \omega_1^3 \bar{\omega}_1^2 \bar{\omega}_2^2 = \omega_2^3 \bar{\omega}_1^2 \bar{\omega}_2^2$
 $\Rightarrow \omega_1 |\omega_1|^2 \bar{\omega}_2^2 = \omega_2 |\omega_2|^2 \bar{\omega}_1^2$
 $\Rightarrow \omega_1 \bar{\omega}_1^2 = \omega_2 \bar{\omega}_1^2$, as $|\omega_1| = |\omega_2|$
 $\therefore \omega_2 \bar{\omega}_1^2 = \omega_1 \bar{\omega}_2^2$
Hence, (b) is the correct answer.

Ex 41. Which of the following must be true?
(a) a must be pure imaginary
(b) β must be pure imaginary
(c) a must be real
(d) β must be imaginary

Sol. Since, $i\beta$ is real.
 $\therefore \beta$ is pure imaginary.
Hence, (b) is the correct answer.

Ex 42. If line (i) makes an angle of 45° with real axis, then $(1 + i) \left(-\frac{2\alpha}{\alpha} \right)$ is
(a) $2\sqrt{2}$ (b) $2\sqrt{2}i$ (c) $2(1 - i)$ (d) $-2(1 + i)$

Sol. $-\frac{\alpha}{\alpha} = e^{\pm i\frac{\pi}{2}} = \pm i$
 $\therefore (1 + i) \left(-\frac{2\alpha}{\alpha} \right) = \pm 2(-1 + i)$
Hence, (c) is the correct answer.

Passage III (Ex. Nos. 43-45) Consider ΔABC in argand plane. Let $A(0)$, $B(1)$ and $C(1 + i)$ be its vertices and M be the mid-point of CA . Let z be a variable complex number in the plane. Let us another variable complex number defined as, $u = z^2 + 1$.

Ex 43. Locus of u , when z is on BM , is a/an
(a) circle (b) parabola
(c) ellipse (d) hyperbola

Ex 44. Axis of locus of u , when z is on BM , is
(a) real axis (b) imaginary axis
(c) $z + \bar{z} = 2$ (d) $z - \bar{z} = 2i$

Ex 45. Directrix of locus of z , when z is on BM , is
(a) real axis (b) imaginary axis
(c) $z + \bar{z} = 2$ (d) $z - \bar{z} = 2i$

Sol. (Ex. Nos. 43-45)
 $BM \equiv y - 0 = -1(x - 1)$
 $x + y = 1$
 $\therefore \sqrt{u - 1} = t + i(1 - t)$
 $u = 2t + 2i t(1 - t)$
 $x = 2t$ and $y = 2t(1 - t)$, where $u = x + iy$
 $(x - 1)^2 = -2 \left(y - \frac{1}{2} \right)$, which is parabola.
Axis is $x = 1$, i.e. $z + \bar{z} = 2$
Directrix is $y = 1$, i.e. $z - \bar{z} = 2i$

43. (b) **44.** (c) **45.** (d)

Type 5. Match the Columns

Ex 46. Match the following :

Column I	Column II
A. Locus of the point z satisfying the equation $\operatorname{Re}(z^2) = \operatorname{Re}(z + \bar{z})$, is a/an	p. circle
B. Locus of the point z satisfying the equation $ z - z_1 + z - z_2 = \lambda$, $\lambda \in R^+$ and $\lambda < z_1 - z_2 $, is a/an	q. straight line
C. Locus of the point satisfying the equation $\left \frac{2z-i}{z+1} \right = m$, where $i = \sqrt{-1}$, is a/an	r. ellipse and $m \in R^+$
D. If $ z = 25$, then the points representing the complex number $-1 + 75\bar{z}$, is a/an	s. rectangular hyperbola

Sol. A. Put $z = x + iy$

$$\therefore \operatorname{Re}(x + iy)^2 = \operatorname{Re}(x + iy + x - iy)$$

$$x^2 - y^2 = 2x$$

$$\text{or } x^2 - y^2 - 2x = 0$$

Rectangular hyperbola, eccentricity $= \sqrt{2}$

B. For ellipse, $\lambda > |z_1 - z_2|$

and for straight line,

$$\lambda = |z_1 - z_2|$$

$$C. \therefore \left| \frac{2z-i}{z+1} \right| = m$$

$$\Rightarrow \left| \frac{z - \frac{i}{2}}{z+1} \right| = \frac{m}{2}$$

For $m = 2$,

$$\Rightarrow \left| \frac{z - \frac{i}{2}}{z+1} \right| = 1$$

$$\Rightarrow \left| z - \frac{i}{2} \right| = |z + 1|, \text{ i.e. straight line}$$

D. Given, $|z| = 25$

$$\text{Let } z_1 = -1 + 75\bar{z}$$

$$\therefore |75\bar{z}| = |z_1 + 1| \text{ or } |75\bar{z}| = |z_1 + 1|$$

$$\Rightarrow 75 \times 25 = |z_1 + 1|$$

$$\therefore |z_1 - (-1)| = 1875$$

$\Rightarrow z_1$ lies on circle.

Ex. 47. If z_1, z_2, z_3, z_4 are the roots of the equation

$$z^4 + z^3 + z^2 + z + 1 = 0, \text{ then}$$

Column I	Column II
A. $\left \sum_{i=1}^4 z_i^4 \right $ is equal to	p. 0
B. $\sum_{i=1}^4 z_i^5$ is equal to	q. 4
C. $\prod_{i=1}^4 (z_i + 2)$ is equal to	r. 1
D. Least value of $[z_1 + z_2]$, where $[]$ represents greatest integer function, is	s. 11

Sol. The given equation is $\frac{z^5 - 1}{z - 1} = 0$, which means that

z_1, z_2, z_3, z_4 are four out of five roots of unity except 1.

$$A. z_1^4 + z_2^4 + z_3^4 + z_4^4 + 1^4 = 0$$

$$\Rightarrow \left| \sum_{i=1}^4 z_i^4 \right| = 1$$

$$B. z_1^5 + z_2^5 + z_3^5 + z_4^5 + 1 = 5$$

$$\Rightarrow \sum_{i=1}^4 z_i^5 = 4$$

$$C. z^4 + z^3 + z^2 + z + 1$$

$$= (z - z_1)(z - z_2)(z - z_3)(z - z_4)$$

Putting $z = -2$ on both the sides, we get

$$\prod_{i=1}^4 (z_i + 2) = 11$$

$$D. |z_1 + z_2| = \sqrt{2 + 2\cos 144^\circ} \text{ for minimum}$$

$$= 2\cos 72^\circ = \frac{\sqrt{5} - 1}{2}$$

whose greatest integer is 0.

$A \rightarrow r; B \rightarrow q; C \rightarrow s; D \rightarrow p$

Type 6. Single Integer Answer Type Questions

Ex 48. Consider an equilateral triangle having vertices at points

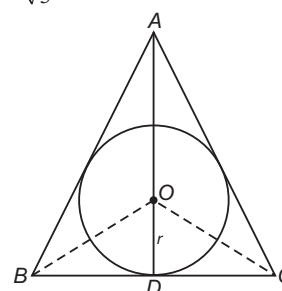
$$A \left(\frac{2}{\sqrt{3}} e^{\frac{i\pi}{2}} \right), B \left(\frac{2}{\sqrt{3}} e^{\frac{-i\pi}{6}} \right) \text{ and}$$

$$C \left(\frac{2}{\sqrt{3}} e^{\frac{-i5\pi}{6}} \right). \text{ If } P(z) \text{ is any point on its}$$

incircle, then $AP^2 + BP^2 + CP^2$ is equal to

_____.

Sol. (5) $A(z_1) = \frac{2i}{\sqrt{3}}$



$$B(z_2) = \frac{2}{\sqrt{3}} \left\{ \frac{\sqrt{3}}{2} - i \frac{1}{2} \right\} = 1 - \frac{i}{\sqrt{3}}$$

$$C(z_3) = \frac{2}{\sqrt{3}} \left\{ -\frac{\sqrt{3}}{2} - \frac{i}{2} \right\} = -1 - \frac{i}{\sqrt{3}}$$

Radius of incircle of $\triangle ABC$, i.e. $r = \frac{1}{\sqrt{3}}$ unit.

Hence, any point on incircle i.e. $P(z)$ is

$$\frac{1}{\sqrt{3}} \cos \alpha, \frac{1}{\sqrt{3}} \sin \alpha \text{ i.e. } \frac{1}{\sqrt{3}} (\cos \alpha + i \sin \alpha)$$

Solving for $|AP|^2 + |BP|^2 + |CP|^2$, we get

$$AP^2 + BP^2 + CP^2 = 5$$

Ex 49. Let $A_1, A_2, \dots, A_n$ be the vertices of a regular polygon of n sides in a circle of radius unity and

$$a = |A_1 A_2|^2 + |A_1 A_3|^2 + \dots + |A_1 A_n|^2,$$

$$b = |A_1 A_2| |A_1 A_3| \dots |A_1 A_n|, \text{ then } \frac{a}{b} = \text{---}.$$

Sol. (2) Let us assume that O is the centre of the polygon and $z_0, z_1, \dots, z_{n-1}$ represent the affixes of $A_1, A_2, \dots, A_n$ such that

$$z_0 = 1, z_1 = \alpha, z_2 = \alpha^2, \dots, z_{n-1} = \alpha^{n-1}, \text{ where } \alpha = e^{\frac{i2\pi}{n}}$$

$$\text{Now, } |A_1 A_r|^2 = |\alpha^r - 1|^2 = |1 - \alpha^r|^2$$

$$= \left| 1 - \cos \frac{2r\pi}{n} - i \sin \frac{2r\pi}{n} \right|^2$$

$$= \left(1 - \cos \frac{2r\pi}{n} \right)^2 + \left(\sin \frac{2r\pi}{n} \right)^2 = 2 - 2 \cos \frac{2r\pi}{n}$$

$$\therefore \sum_{r=1}^n |A_1 A_r|^2 = \sum_{r=1}^n \left(2 - 2 \cos \frac{2r\pi}{n} \right)$$

$$= 2(n-1) \left\{ \cos \frac{2\pi}{n} + \cos \frac{4\pi}{n} + \dots + \cos \frac{2(n-1)\pi}{n} \right\}$$

$$= 2(n-1) - 2 \cdot \text{Real part of } (\alpha + \alpha^2 + \dots + \alpha^{n-1})$$

$$= 2(n-1) - 2(1)$$

$$[\text{since, } \{1 + \alpha + \alpha^2 + \dots + \alpha^{n-1} = 0\}]$$

$$\therefore |A_1 A_2|^2 + |A_1 A_3|^2 + \dots + |A_1 A_n|^2 = 2n$$

$$\text{Also, let } E = |A_1 A_2| |A_1 A_3| \dots |A_1 A_n|$$

$$= |1 - \alpha| |1 - \alpha^2| |1 - \alpha^3| \dots |1 - \alpha^{n-1}|$$

$$= (1 - \alpha)(1 - \alpha^2)(1 - \alpha^3) \dots (1 - \alpha^{n-1})$$

Since, $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ are the roots of $z^n - 1 = 0$.

$$\Rightarrow (z-1)(z-\alpha)(z-\alpha^2) \dots (z-\alpha^{n-1}) = \frac{z^n - 1}{z - 1}$$

$$\Rightarrow (z-\alpha)(z-\alpha^2) \dots (z-\alpha^{n-1}) = z^n - 1$$

$$= 1 + z + z^2 + \dots + z^{n-1}$$

Substituting $z = 1$, we have

$$(1 - \alpha)(1 - \alpha^2) \dots (1 - \alpha^{n-1}) = n$$

$$|1 - \alpha| |1 - \alpha^2| \dots |1 - \alpha^{n-1}| = n$$

Hence, the value of $\frac{a}{b} = \frac{2n}{n} = 2$

Target Exercises

Type 1. Only One Correct Option

- The value of $\frac{i^{592} + i^{590} + i^{588} + i^{586} + i^{584}}{i^{582} + i^{580} + i^{578} + i^{576} + i^{574}} - 1$ is
 - (a) -1
 - (b) -2
 - (c) -3
 - (d) -4
- $i^{57} + \frac{1}{i^{125}}$, when simplified has the value
 - (a) 0
 - (b) $2i$
 - (c) $-2i$
 - (d) 2
- $i^n + i^{n+1} + i^{n+2} + i^{n+3}$ is equal to
 - (a) 1
 - (b) -1
 - (c) 0
 - (d) None of these
- $1 + i^2 + i^4 + i^6 + \dots + i^{2n}$ is
 - (a) positive
 - (b) negative
 - (c) 0
 - (d) Can't be determined
- The value of $\left[i^{19} + \left(\frac{1}{i} \right)^{25} \right]^2$ is
 - (a) 4
 - (b) -4
 - (c) 2
 - (d) -2
- If n is any positive integer, then the value of $\frac{i^{4n+1} - i^{4n-1}}{2}$ equals
 - (a) 1
 - (b) -1
 - (c) i
 - (d) $-i$
- For a positive integer n , the expression $(1-i)^n \left(1 - \frac{1}{i} \right)^n$ equals
 - (a) 0
 - (b) $2i^n$
 - (c) 2^n
 - (d) 4^n
- If the number $\frac{(1-i)^n}{(1+i)^{n-2}}$ is real and positive, then n is
 - (a) any integer
 - (b) 2λ
 - (c) $4\lambda + 1$
 - (d) None of these
- The smallest positive integer n for which $\left(\frac{1+i}{1-i} \right)^n = -1$ is
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 4
- The smallest positive number n for which $(1+i)^{2n} = (1-i)^{2n}$ is
 - (a) 4
 - (b) 8
 - (c) 2
 - (d) 12
- The complex number $\frac{1+2i}{1-i}$ lies in the
 - (a) I quadrant
 - (b) II quadrant
 - (c) III quadrant
 - (d) IV quadrant
- The value of $\frac{(1-i)^3}{1-i^3}$ is equal to
 - (a) i
 - (b) -1
 - (c) 1
 - (d) -2
- $\left(\frac{1+i}{1-i} \right)^2 + \left(\frac{1-i}{1+i} \right)^2$ is equal to
 - (a) $2i$
 - (b) $-2i$
 - (c) -2
 - (d) 2
- The value of $\operatorname{Re} \left[\frac{(1+i)^2}{3-i} \right]$ is equal to
 - (a) $-\frac{1}{5}$
 - (b) $\frac{1}{5}$
 - (c) $\frac{1}{10}$
 - (d) $-\frac{1}{10}$
- If $a^2 + b^2 = 1$, then $\frac{(1+b+ia)}{(1+b-ia)}$ is equal to
 - (a) 1
 - (b) 2
 - (c) $b+ia$
 - (d) $a+ib$
- If $b+ic = (1+a)z$ and $a^2 + b^2 + c^2 = 1$, then $\frac{1+iz}{1-iz}$ equals
 - (a) $\frac{a-ib}{1-c}$
 - (b) $\frac{a-ib}{1+c}$
 - (c) $\frac{a+ib}{1-c}$
 - (d) $\frac{a+ib}{1+c}$
- If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$, then
 - (a) $x = 3, y = 1$
 - (b) $x = 1, y = 3$
 - (c) $x = 0, y = 3$
 - (d) $x = 0, y = 0$
- $\sqrt{2}i$ equals
 - (a) $1+i$
 - (b) $1-i$
 - (c) $-\sqrt{2}i$
 - (d) None of these
- If z is a complex number such that $|z| \neq 0$ and $\operatorname{Re}(z) = 0$, then
 - (a) $\operatorname{Re}(z^2) = 0$
 - (b) $\operatorname{Im}(z^2) = 0$
 - (c) $\operatorname{Re}(z^2) = \operatorname{Im}(z^2)$
 - (d) None of these
- If $(x+iy)^{1/3} = a+ib$, then $\frac{x}{a} + \frac{y}{b}$ is equal to
 - (a) $2(a^2 - b^2)$
 - (b) $4(a^2 - b^2)$
 - (c) $8(a^2 - b^2)$
 - (d) None of these

- 21.** If $8iz^3 + 12z^2 - 18z + 27i = 0$, then
 (a) $|z| = \frac{3}{2}$ (b) $|z| = \frac{2}{3}$ (c) $|z| = 1$ (d) $|z| = \frac{3}{4}$
- 22.** If $z = 1 + i$, then the multiplicative inverse of z^2 is
 (a) $1 - i$ (b) $\frac{i}{2}$ (c) $-\frac{i}{2}$ (d) $2i$
- 23.** If $\left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 = x + iy$, then (x, y) is equal to
 (a) $(0, 2)$ (b) $(-2, 0)$
 (c) $(0, -2)$ (d) None of these
- 24.** The multiplicative inverse of $(6 + 5i)^2$ is
 (a) $\frac{11}{60} - \frac{60}{61}i$ (b) $\frac{11}{61} - \frac{60}{61}i$
 (c) $\frac{9}{61} - \frac{60}{61}i$ (d) None of these
- 25.** The multiplicative inverse of $\frac{3+4i}{4-5i}$ is
 (a) $-\frac{8}{25} + \frac{31}{25}i$ (b) $\frac{8}{25} - \frac{31}{25}i$
 (c) $-\frac{8}{25} - \frac{31}{25}i$ (d) None of these
- 26.** If $z = x + iy$ lies in III quadrant, then $\frac{\bar{z}}{z}$ also lies in
 III quadrant, if
 (a) $x > y > 0$ (b) $x < y < 0$
 (c) $y < x < 0$ (d) $y > x > 0$
- 27.** The conjugate complex number of $\frac{2-i}{(1-2i)^2}$ is
 (a) $\frac{2}{25} + \frac{11}{25}i$ (b) $\frac{2}{25} - \frac{11}{25}i$
 (c) $-\frac{2}{25} + \frac{11}{25}i$ (d) $-\frac{2}{25} - \frac{11}{25}i$
- 28.** $\left|(1+i)\left(\frac{2+i}{3+i}\right)\right|$ is equal to
 (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$ (c) 1 (d) -1
- 29.** The value of $|\sqrt{2}i - \sqrt{-2}i|$ is
 (a) 2 (b) $\sqrt{2}$ (c) 0 (d) $2\sqrt{2}$
- 30.** If $\frac{1-ix}{1+ix} = a - ib$ and $a^2 + b^2 = 1$, where a and b are real, then x is equal to
 (a) $\frac{2a}{(1+a)^2 + b^2}$ (b) $\frac{2b}{(1+a)^2 + b^2}$
 (c) $\frac{2a}{(1+b)^2 + a^2}$ (d) $\frac{2b}{(1+b)^2 + a^2}$
- 31.** The modulus of $\frac{(3+2i)^2}{(4-3i)}$ is
 (a) $\frac{13}{5}$ (b) $\frac{11}{5}$ (c) $\frac{9}{5}$ (d) $\frac{7}{5}$
- 32.** If $x + iy = \frac{u+iv}{u-iv}$, then $x^2 + y^2$ is equal to
 (a) 1 (b) -1
 (c) 0 (d) None of these
- 33.** If $(1+i)(1+2i)(1+3i)\dots(1+ni) = \alpha + i\beta$, then $2 \cdot 5 \cdot 10 \dots (1+n^2)$ is equal to
 (a) $\alpha - i\beta$ (b) $\alpha^2 - \beta^2$
 (c) $\alpha^2 + \beta^2$ (d) None of these
- 34.** The principal argument of the complex number $\frac{(1+i)^5(1+\sqrt{3}i)^2}{-2i(-\sqrt{3}+i)}$ is
 (a) $\frac{19\pi}{12}$ (b) $-\frac{7\pi}{12}$ (c) $-\frac{5\pi}{12}$ (d) $\frac{5\pi}{12}$
- 35.** The complex number $i + \sqrt{3}$ in polar form can be written as
 (a) $\frac{1}{\sqrt{2}}\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)$ (b) $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
 (c) $\frac{1}{2}\left(\sin\frac{\pi}{6} + i\cos\frac{\pi}{6}\right)$ (d) $4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$
- 36.** If z is a complex number, then
 (a) $|z^2| > |z|^2$ (b) $|z^2| = |z|^2$ (c) $|z^2| < |z|^2$ (d) $|z^2| \geq |z|^2$
- 37.** If $z = x + iy$, x and y are real, then $|x| + |y| \leq k|z|$, where k is equal to
 (a) 1 (b) $\sqrt{2}$
 (c) $\sqrt{3}$ (d) None of these
- 38.** If z is any complex number such that $|z+4| \leq 3$, then the least value and greatest value of $|z+1|$ are
 (a) 1, 6 (b) 0, 6
 (c) 2, 8 (d) None of these
- 39.** If z satisfies $|z+1| < |z-2|$, then $w = 3z + 2 + i$ satisfies
 (a) $|w+1| < |w-8|$ (b) $|w+1| < |w-7|$
 (c) $|w+\bar{w}| > 7$ (d) $|w+5| < |w-4|$
- 40.** For any complex number z , the minimum value of $|z| + |z-1|$ is
 (a) 1 (b) 0 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$
- 41.** The greatest positive argument of complex number satisfying $|z-4| = \operatorname{Re}(z)$, is
 (a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$
 (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{4}$
- 42.** The square roots of $-2 + 2\sqrt{3}i$ are
 (a) $\pm(1 + \sqrt{3}i)$ (b) $\pm(1 - \sqrt{3}i)$
 (c) $\pm(-1 + \sqrt{3}i)$ (d) None of these
- 43.** If $z = a + ib$, where $a > 0, b > 0$, then
 (a) $|z| \geq \frac{1}{\sqrt{2}}(a-b)$ (b) $|z| \geq \frac{1}{\sqrt{2}}(a+b)$
 (c) $|z| < \frac{1}{\sqrt{2}}(a+b)$ (d) None of these

44. If for the complex numbers z_1 and z_2 , $|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2 = k(1 - |z_1|^2)(1 - |z_2|^2)$, then k is equal to
(a) 1 (b) -1 (c) 2 (d) 4
45. The range of real number α for which the equation $z + \alpha|z - 1| + 2i = 0$ has a solution, is
(a) $\left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right]$ (b) $\left[-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right]$
(c) $\left[0, \frac{\sqrt{5}}{2}\right]$ (d) $\left(-\infty, -\frac{\sqrt{5}}{2}\right] \cup \left[\frac{\sqrt{5}}{2}, \infty\right)$
46. If $|z| = 2$ and locus of $5z - 1$ is the circle having radius a and $z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta = 0$, then $|z_1| : |z_2|$ is equal to
(a) $a : 1$ (b) $2a : 1$
(c) $a : 10$ (d) None of these
47. The maximum value of $|z|$ when z satisfies the condition $\left|z + \frac{2}{z}\right| = 2$ is
(a) $\sqrt{3} - 1$ (b) $\sqrt{3} + 1$
(c) $\sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$
48. If $z_1 \neq z_2$ and $|z_1 + z_2| = \left|\frac{1}{z_1} + \frac{1}{z_2}\right|$, then
(a) at least one of z_1, z_2 is unimodular
(b) $z_1 \cdot z_2$ is unimodular
(c) $z_1 \cdot z_2$ is non-unimodular
(d) None of the above
49. If $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the n , n th roots of unity and z_1, z_2 are any two complex numbers, then $\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2$ is equal to
(a) $n[|z_1|^2 + |z_2|^2]$ (b) $(n-1)[|z_1|^2 + |z_2|^2]$
(c) $(n+1)[|z_1|^2 + |z_2|^2]$ (d) None of these
50. If z satisfies the equation $|z| - z = 1 + 2i$, then z is equal to
(a) $\frac{3}{2} + 2i$ (b) $\frac{3}{2} - 2i$
(c) $2 - \left(\frac{3}{2}\right)i$ (d) $2 + \left(\frac{3}{2}\right)i$
51. If $\sqrt{3} + i = (a + ib)(c + id)$, then $\tan^{-1} \frac{b}{a} + \tan^{-1} \frac{d}{c}$ is equal to
(a) $\frac{\pi}{2} + 2n\pi$, for some integer n
(b) $-\frac{\pi}{3} + n\pi$, for some integer n
(c) $\frac{\pi}{6} + n\pi$, for some integer n
(d) $-\frac{\pi}{6} + 2n\pi$, for some integer n
52. If z is any complex number satisfying $|z - 1| = 1$, then which of the following is correct?
(a) $\arg(z - 1) = 2 \arg(z)$
(b) $2 \arg(z) = 2/3 \arg(z^2 - z)$
(c) $\arg(z - 1) = \arg(z + 1)$
(d) $\arg(z) = 2 \arg(z + 1)$
53. If $z = -1$, then the principal value of $\arg(z^{2/3})$ is equal to
(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ or 2π or $\frac{10\pi}{3}$
(c) $\frac{5\pi}{3}$ (d) π
54. If $z_1 = 8 + 4i, z_2 = 6 + 4i$ and $\arg\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$, then z satisfies
(a) $|z - 7 - 4i| = 1$ (b) $|z - 7 - 5i| = \sqrt{2}$
(c) $|z - 4i| = 8$ (d) $|z - 7i| = \sqrt{18}$
55. If $\arg(z) < 0$, then $\arg(-z) - \arg(z)$ is equal to
(a) π (b) $-\pi$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$
56. If z is a point on the argand plane such that $|z - 1| = 1$, then $\frac{z - 2}{z}$ is equal to
(a) $\tan(\arg z)$ (b) $\cot(\arg z)$
(c) $i \tan(\arg z)$ (d) None of these
57. If $C^2 + S^2 = 1$, then $\frac{1 + C + iS}{1 + C - iS}$ is equal to
(a) $C + iS$ (b) $C - iS$ (c) $S + iC$ (d) $S - iC$
58. The imaginary part of $(z - 1)(\cos \alpha - i \sin \alpha) + (z - 1)^{-1} \times (\cos \alpha + i \sin \alpha)$ is zero, if
(a) $|z - 1| = 2$ (b) $\arg(z - 1) = 2\alpha$
(c) $\arg(z - 1) = \alpha$ (d) $|z| = 1$
59. The value of $\frac{\left(\sin \frac{\pi}{8} + i \cos \frac{\pi}{8}\right)^8}{\left(\sin \frac{\pi}{8} - i \cos \frac{\pi}{8}\right)^8}$ is
(a) -1 (b) 0 (c) 1 (d) $2i$
60. If $a = \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3}$, then the value of $\left(\frac{1+a}{2}\right)^{3n}$ is
(a) $(-1)^n$ (b) $\frac{(-1)^n}{2^{3n}}$ (c) $\frac{1}{2^{3n}}$ (d) $(-1)^n + 1$
61. The value of $\left(\frac{1+i}{\sqrt{2}}\right)^8 + \left(\frac{1-i}{\sqrt{2}}\right)^8$ is equal to
(a) 4 (b) 6 (c) 8 (d) 2
62. The value of $\sum_{k=1}^{10} \left(\sin \frac{2\pi k}{11} - i \cos \frac{2\pi k}{11}\right)$ is
(a) 1 (b) -1
(c) i (d) $-i$

63. The value of $\frac{4(\cos 75^\circ + i \sin 75^\circ)}{0.4(\cos 30^\circ + i \sin 30^\circ)}$ is
 (a) $\frac{10}{\sqrt{2}}(1+i)$ (b) $\frac{10}{\sqrt{2}}(1-i)$
 (c) $\frac{5}{\sqrt{2}}(1+i)$ (d) None of these
64. The value of $\frac{(\cos 2\theta - i \sin 2\theta)^7 (\cos 3\theta + i \sin 3\theta)^{-5}}{(\cos 4\theta + i \sin 4\theta)^{12} (\cos 5\theta + i \sin 5\theta)^{-6}}$ is
 (a) $\cos 33\theta + i \sin 33\theta$ (b) $\cos 33\theta - i \sin 33\theta$
 (c) $\cos 47\theta + i \sin 47\theta$ (d) $\cos 47\theta - i \sin 47\theta$
65. $(\cos 2\theta + i \sin 2\theta)^{-5} (\cos 3\theta - i \sin 3\theta)^6$
 $(\sin \theta - i \cos \theta)^3$ is equal to
 (a) $\cos 25\theta + i \sin 25\theta$
 (b) $\cos 25\theta - i \sin 25\theta$
 (c) $\sin 25\theta + i \cos 25\theta$
 (d) $\sin 25\theta - i \cos 25\theta$
66. The principal value of the $\arg(z)$ and $|z|$ of the complex number $z = \frac{(1 + \cos \theta + i \sin \theta)^5}{(\cos \theta + i \sin \theta)^3}$ are
 (a) $-\frac{\theta}{2}, 32 \cos^5 \frac{\theta}{2}$ (b) $\frac{\theta}{2}, 32 \cos^5 \frac{\theta}{2}$
 (c) $-\frac{\theta}{2}, 16 \cos^4 \frac{\theta}{2}$ (d) None of these
67. If $z = \cos \theta + i \sin \theta$, then
 (a) $z^n + \frac{1}{z^n} = 2 \cos n\theta$ (b) $z^n + \frac{1}{z^n} = 2^n \cos n\theta$
 (c) $z^n - \frac{1}{z^n} = 2^n i \sin n\theta$ (d) $z^n - \frac{1}{z^n} = (2i)^n \sin n\theta$
68. If $z = \cos \theta + i \sin \theta$, then $\frac{z^{2n} - 1}{z^{2n} + 1}$, where n is an integer, is equal to
 (a) $i \cot n\theta$ (b) $i \tan n\theta$ (c) $\tan n\theta$ (d) $\cot n\theta$
69. If $a = \cos 2\alpha + i \sin 2\alpha$, $b = \cos 2\beta + i \sin 2\beta$,
 $c = \cos 2\gamma + i \sin 2\gamma$ and $d = \cos 2\delta + i \sin 2\delta$, then
 $\sqrt{abcd} + \frac{1}{\sqrt{abcd}}$ is equal to
 (a) $\sqrt{2} \cos(\alpha + \beta + \gamma + \delta)$ (b) $2 \cos(\alpha + \beta + \gamma + \delta)$
 (c) $\cos(\alpha + \beta + \gamma + \delta)$ (d) None of these
70. $e^{2mi \cot^{-1} p} \cdot \left(\frac{pi+1}{pi-1}\right)^m$ is equal to
 (a) 0 (b) 1
 (c) -1 (d) None of these
71. If $\alpha = \cos\left(\frac{8\pi}{11}\right) + i \sin\left(\frac{8\pi}{11}\right)$, then
 $\operatorname{Re}(\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$ is equal to
 (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
 (c) 0 (d) None of these
72. If $\omega (\neq 1)$ is cube root of unity satisfying
 $\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$ and
 $\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = 2\omega$, then the value of
 $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$ is equal to
 (a) 2 (b) -2
 (c) $-1 + \omega^2$ (d) None of these
73. If α and β are the roots of the equation $x^2 - 2x + 4 = 0$, then the value of $\alpha^6 + \beta^6$ is
 (a) 64 (b) 128
 (c) 256 (d) None of these
74. If z is any complex number such that $z + \frac{1}{z} = 1$, then the value of $z^{99} + \frac{1}{z^{99}}$ is
 (a) 1 (b) -1
 (c) 2 (d) -2
75. The value of $(-1 + \sqrt{-3})^{62} + (-1 - \sqrt{-3})^{62}$ is
 (a) 2^{62} (b) 2^{64}
 (c) -2^{62} (d) 0
76. If ω is a cube root of unity, then
 $(3 + 5\omega + 3\omega^2)^2 (3 + 3\omega + 5\omega^2)^2$ is equal to
 (a) 4 (b) 0
 (c) -4 (d) None of these
77. The value of $\left(\frac{1+i\sqrt{3}}{1-i\sqrt{3}}\right)^6 + \left(\frac{1-i\sqrt{3}}{1+i\sqrt{3}}\right)^6$ is
 (a) 2 (b) -2 (c) 1 (d) 0
78. If $z_1 = \sqrt{3} + i\sqrt{3}$ and $z_2 = \sqrt{3} + i$, then the complex number $\left(\frac{z_1}{z_2}\right)^{50}$ lies in the quadrant number
 (a) I (b) II (c) III (d) IV
79. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, then xyz is equal to
 (a) $(a+b)^3$ (b) $a^3 + b^3$
 (c) $a^3 - b^3$ (d) $(a+b)^3 + 3ab(a+b)$
80. If 1, ω and ω^2 are the cube roots of unity, then the value of $(1+\omega)^3 - (1+\omega^2)^3$ is
 (a) 2ω (b) 2 (c) -2 (d) 0
81. If 1, ω and ω^2 are the three cube roots of unity and α, β and γ are the roots of $p, p < 0$, then for any x, y and z , the expression $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$ equals
 (a) 1
 (b) ω
 (c) ω^2
 (d) None of the above

- 82.** The value of the expression $1(2-\omega)(2-\omega^2) + 2(3-\omega)(3-\omega^2) + \dots + (n-1)(n-\omega)(n-\omega^2)$, where ω is an imaginary cube root of unity, is
- (a) $\left(\frac{n(n+1)}{2}\right)^2$ (b) $\left(\frac{n(n+1)}{2}\right)^2 - n$
 (c) $\left(\frac{n(n+1)}{2}\right)^2 + n$ (d) None of these
- 83.** If α and β are the complex cube roots of unity, then $\alpha^3 + \beta^3 + \alpha^{-2}\beta^{-2}$ is equal to
- (a) 0 (b) 3
 (c) -3 (d) None of these
- 84.** If ω is a complex cube root of unity, then $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$ is equal to
- (a) 12 (b) 14
 (c) 16 (d) None of these
- 85.** If $x = \omega - \omega^2 - 2$, then the value of $x^4 + 3x^3 + 2x^2 - 11x - 6$ is
- (a) 1 (b) -1
 (c) 2 (d) None of these
- 86.** If 1, ω and ω^2 are the three cube roots of unity, then $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)\dots$ to $2n$ factors is equal to
- (a) 1 (b) -1
 (c) 0 (d) None of these
- 87.** If $x = a + b + c$, $y = a\alpha + b\beta + c$ and $z = a\beta + b\alpha + c$ where, α and β are complex cube roots of unity, then xyz is equal to
- (a) $2(a^3 + b^3 + c^3)$ (b) $2(a^3 - b^3 - c^3)$
 (c) $a^3 + b^3 + c^3 - 3abc$ (d) $a^3 - b^3 - c^3$
- 88.** The common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{1985} + z^{100} + 1 = 0$ are
- (a) -1, ω
 (b) -1, ω^2
 (c) ω , ω^2
 (d) None of the above
- 89.** The values of $(16)^{1/4}$ are
- (a) $\pm 2, \pm 2i$
 (b) $\pm 4, \pm 4i$
 (c) $\pm 1, \pm i$
 (d) None of the above
- 90.** If p, q and r are positive integers and ω is an imaginary cube root of unity and $f(x) = x^{3p} + x^{3q+1} + x^{3r+2}$, then $f(\omega)$ is equal to
- (a) ω
 (b) $-\omega^2$
 (c) 0
 (d) None of the above
- 91.** If $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ are the n , n th roots of unity, then $\sum_{i=1}^{n-1} \frac{1}{2-\alpha^i}$ is equal to
- (a) $\frac{(n-2)2^{n-1}+1}{2^n-1}$ (b) $(n-2) \cdot 2^n$
 (c) $\frac{(n-2) \cdot 2^{n-1}}{2^n-1}$ (d) None of these
- 92.** The triangle formed by the points $1, \frac{1+i}{\sqrt{2}}$ and i as vertices in the argand diagram is a/an
- (a) scalene (b) equilateral
 (c) isosceles (d) right angled
- 93.** If the points represented by complex numbers $z_1 = a + ib$, $z_2 = a' + ib'$ and $z_1 - z_2$ are collinear, then
- (a) $ab' + a'b = 0$ (b) $ab' - a'b = 0$
 (c) $ab + a'b' = 0$ (d) $ab - a'b' = 0$
- 94.** If $\alpha + i\beta = \tan^{-1} z$, $z = x + iy$ and α is constant, then the locus of z is
- (a) $x^2 + y^2 + 2x \cot 2\alpha = 1$ (b) $(x^2 + y^2) \cot 2\alpha = 1 + x$
 (c) $x^2 + y^2 + 2y \tan 2\alpha = 1$ (d) $x^2 + y^2 + 2x \sin 2\alpha = 1$
- 95.** The equation $|z + 1 - i| = |z + i - 1|$ represents
- (a) a straight line (b) a circle
 (c) a parabola (d) a hyperbola
- 96.** If the roots of $(z-1)^n = i(z+1)^n$ are plotted in the argand plane, they as
- (a) on a parabola (b) concyclic
 (c) collinear (d) the vertices of a triangle
- 97.** Locus of the point z satisfying the equation $|iz - 1| + |z - i| = 2$ is
- (a) a straight line (b) a circle
 (c) an ellipse (d) a pair of straight lines
- 98.** For $x_1, x_2, y_1, y_2 \in R$, if $0 < x_1 < x_2$, $y_1 = y_2$ and $z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$ and $z_3 = \frac{1}{2}(z_1 + z_2)$, then z_1, z_2, z_3 satisfy
- (a) $|z_1| = |z_2| = |z_3|$ (b) $|z_1| < |z_2| < |z_3|$
 (c) $|z_1| > |z_2| > |z_3|$ (d) $|z_1| < |z_3| < |z_2|$
- 99.** The equation not representing a circle is given by
- (a) $\operatorname{Re}\left(\frac{1+z}{1-z}\right) = 0$
 (b) $z\bar{z} + iz - i\bar{z} + 1 = 0$
 (c) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$
 (d) $\left|\frac{z-1}{z+1}\right| = 1$
- 100.** If z_1, z_2 and z_3 are three complex numbers in AP, then they lie on
- (a) a circle (b) a straight line
 (c) a parabola (d) an ellipse

- 101.** If the area of the triangle on the argand plane formed by the complex numbers $-z$, iz , $z - iz$ is 600 sq units, then $|z|$ is equal to
(a) 10 (b) 20
(c) 30 (d) None of these
- 102.** The complex numbers given by $1 - 3i$, $4 + 3i$ and $3 + i$ represent the vertices of
(a) a right angled triangle
(b) an isosceles triangle
(c) an equilateral triangle
(d) None of the above
- 103.** If $\operatorname{Re}\left(\frac{z + 2i}{z + 4}\right) = 0$, then z lies on a circle with centre
(a) $(-2, -1)$ (b) $(-2, 1)$
(c) $(2, -1)$ (d) $(2, 1)$
- 104.** If $\operatorname{Im}\left(\frac{z + 2i}{z + 2}\right) = 0$, then z lies on the curve
(a) $x^2 + y^2 + 2x + 2y = 0$
(b) $x^2 + y^2 - 2x = 0$
(c) $x + y + 2 = 0$
(d) None of the above
- 105.** In the argand plane, all the complex numbers satisfying $|z - 4i| + |z + 4i| = 10$ lie on
(a) a straight line (b) a circle
(c) an ellipse (d) a parabola
- 106.** If $z = x + iy$ and a is a real number such that $|z - ai| = |z + ai|$, then locus of z is
(a) X -axis
(b) Y -axis
(c) $x = y$
(d) $x^2 + y^2 = 1$
- 107.** If P represents $z = x + iy$ in the argand plane and $|z - 1|^2 + |z + 1|^2 = 4$, then the locus of P is
(a) $x^2 + y^2 = 2$
(b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 = 4$
(d) $x + y = 2$
- 108.** All the roots of the equation $a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 3$, where $|a_i| \leq 1$, $i = 1, 2, 3, 4$, lie outside the circle with centre origin and radius
(a) 1
(b) $\frac{1}{3}$
(c) $\frac{2}{3}$
(d) None of the above
- 109.** The region of argand diagram defined by $|z - 1| + |z + 1| \leq 4$ is
(a) interior of an ellipse
(b) exterior of a circle
(c) interior and boundary of an ellipse
(d) None of the above
- 110.** The locus of the complex number z in an argand plane satisfying the inequality $\log_{1/2} \left(\frac{|z - 1| + 4}{3|z - 1| - 2} \right) > 1$ (where, $|z - 1| \neq \frac{2}{3}$) is
(a) a circle
(b) interior of a circle
(c) exterior of a circle
(d) None of the above
- 111.** The closest distance of the origin from a curve given as $a\bar{z} + \bar{a}z + a\bar{a} = 0$ (a is a complex number), is
(a) 1 (b) $\frac{|a|}{2}$
(c) $\frac{\operatorname{Re}(a)}{|a|}$ (d) $\frac{\operatorname{Im}(a)}{|a|}$
- 112.** Let a be a complex number such that $|a| < 1$ and $z_1, z_2, \dots, z_n$ be the vertices of a polygon such that $z_k = 1 + a + a^2 + \dots + a^k$, then the vertices of the polygon lie within the circle
(a) $|z - a| = \frac{1}{|1 - a|}$
(b) $|z - 1| = \frac{1}{|1 - a|}$
(c) $\left| z - \frac{1}{1 - a} \right| = \frac{1}{|1 - a|}$
(d) None of these
- 113.** If z_1, z_2, z_3 and z_4 are the four complex numbers represented by the vertices of a quadrilateral taken in order such that $z_1 - z_4 = z_2 - z_3$ and $\operatorname{amp} \left(\frac{z_4 - z_1}{z_2 - z_1} \right) = \frac{\pi}{2}$, then the quadrilateral is a
(a) square
(b) rhombus
(c) cyclic quadrilateral
(d) None of the above
- 114.** The locus of z satisfying $\operatorname{Im}(z^2) = 4$ is
(a) a circle
(b) a rectangular hyperbola
(c) a pair of straight lines
(d) None of the above
- 115.** The curve represented by $\operatorname{Re}(z^2) = 4$ is
(a) a parabola
(b) an ellipse
(c) a circle
(d) a rectangular hyperbola
- 116.** A point z is equidistant from three distinct points z_1, z_2, z_3 in argand plane. If z, z_1 and z_2 are collinear, then $\arg \left(\frac{z_3 - z_1}{z_3 - z_2} \right)$ will be $(z_1, z_2, z_3$ in anti-clockwise sense)
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) $\frac{2\pi}{3}$

Type 2. More than One Correct Option

- 117.** If z_1, z_2, z_3 and z_4 are roots of the equation $a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$ where, a_0, a_1, a_2, a_3 and a_4 are real, then
 (a) $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$ are also roots of equation
 (b) z_1 is equal to atleast one of $\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4$
 (c) $-\bar{z}_1, -\bar{z}_2, -\bar{z}_3, -\bar{z}_4$ are also roots of equation
 (d) None of the above
- 118.** If $z^3 + (3 + 2i)z + (-1 + ia) = 0$ has one real root, then the value of a lies in the interval ($a \in R$)
 (a) $(-2, 1)$ (b) $(-1, 0)$
 (c) $(0, 1)$ (d) $(-2, 3)$
- 119.** The real value of θ for which the expression $\frac{i + i \cos \theta}{1 - 2i \cos \theta}$ is a real number, is
 (a) $2n\pi + \frac{\pi}{2}, n \in I$ (b) $2n\pi - \frac{\pi}{2}, n \in I$
 (c) $2n\pi \pm \frac{\pi}{2}, n \in I$ (d) $2n\pi \pm \frac{\pi}{4}, n \in I$
- 120.** If α is a complex constant such that $\alpha z + z + \bar{\alpha} = 0$ has a real root, then
 (a) $\alpha + \bar{\alpha} = 1$
 (b) $\alpha + \bar{\alpha} = 0$
 (c) $\alpha + \bar{\alpha} = -1$
 (d) the absolute value of the real root is 1
- 121.** If $|z_1| = |z_2| = 1$ and $\arg z_1 + \arg z_2 = 0$, then
 (a) $z_1 z_2 = 1$
 (b) $z_1 + z_2 = 0$
 (c) $z_1 = \bar{z}_2$
 (d) None of the above
- 122.** If $|z_1| = 15$ and $|z_2 - 3 - 4i| = 5$, then
 (a) $|z_1 - z_2|_{\min} = 5$ (b) $|z_1 - z_2|_{\min} = 10$
 (c) $|z_1 - z_2|_{\max} = 20$ (d) $|z_1 - z_2|_{\max} = 25$
- 123.** If $|z - (1/z)| = 1$, then
 (a) $|z|_{\max} = \frac{1 + \sqrt{5}}{2}$ (b) $|z|_{\min} = \frac{\sqrt{5} - 1}{2}$
 (c) $|z|_{\max} = \frac{\sqrt{5} - 1}{2}$ (d) $|z|_{\min} = \frac{\sqrt{5} + 1}{\sqrt{2}}$
- 124.** If z_1 and z_2 are two complex numbers ($z_1 \neq z_2$) satisfying $|z_1^2 - z_2^2| = |\bar{z}_1^2 + \bar{z}_2^2 - 2\bar{z}_1 \bar{z}_2|$, then
 (a) $\frac{z_1}{z_2}$ is purely imaginary (b) $\frac{z_1}{z_2}$ is purely real
 (c) $|\arg z_1 - \arg z_2| = \pi$ (d) $|\arg z_1 - \arg z_2| = \frac{\pi}{2}$
- 125.** If $|z - 1| = 1$, then
 (a) $\arg \{(z - 1)/z\}$ can be equal to $-\pi/4$
 (b) $(z - 2)/z$ is purely imaginary number
 (c) $(z - 2)/z$ is purely real number
 (d) If $\arg(z) = \theta$, where $z \neq 0$ and θ is acute, then $1 - 2/z = i \tan \theta$
- 126.** Let complex number z of the form $x + iy$ satisfy $\arg \left(\frac{3z - 6 - 3i}{2z - 8 - 6i} \right) = \frac{\pi}{4}$ and $|z - 3 + i| = 3$. Then, the ordered pairs (x, y) are
 (a) $\left(4 - \frac{4}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}} \right)$ (b) $\left(4 + \frac{4}{\sqrt{5}}, 1 - \frac{2}{\sqrt{5}} \right)$
 (c) $(6, -1)$ (d) $(0, -1)$
- 127.** If $z = \omega, \omega^2$, where ω is a non-real complex cube root of unity, are two vertices of an equilateral triangle in the argand plane, then the third vertex may be represented by
 (a) $z = 1$ (b) $z = 0$
 (c) $z = -2$ (d) $z = -1$
- 128.** If $\arg(z_1 z_2) = 0$ and $|z_1| = |z_2| = 1$, then
 (a) $z_1 + z_2 = 0$ (b) $z_1 z_2 = 1$
 (c) $z_1 = \bar{z}_2$ (d) None of these
- 129.** If z is complex number satisfying $z + z^{-1} = 1$, then $z^n + z^{-n}, n \in N$ has the value
 (a) $2(-1)^n$, when n is a multiple of 3
 (b) $(-1)^{n-1}$, when n is not a multiple of 3
 (c) $(-1)^{n+1}$, when n is a multiple of 3
 (d) 0, when n is not a multiple of 3
- 130.** Let z_1, z_2 be two complex numbers represented by points on the circle $|z| = 1$ and $|z| = 2$ respectively, then
 (a) $\max |2z_1 + z_2| = 4$ (b) $\min |z_1 - z_2| = 1$
 (c) $\left| z_2 + \frac{1}{z_1} \right| \leq 3$ (d) None of these
- 131.** $ABCD$ is a square, vertices being taken in the anti-clockwise sense. If A represents the complex number z and the intersection of the diagonals is the origin, then
 (a) B represents the complex number iz
 (b) D represents the complex number iz
 (c) B represents the complex number iz
 (d) D represents the complex number $-iz$
- 132.** If z_0, z_1 represent point P, Q on the locus $|z - 1| = 1$ and the line segment PQ subtend an angle $\frac{\pi}{2}$ at the point $z = 1$, then z_1 is equal to
 (a) $1 + i(z_0 - 1)$ (b) $\frac{i}{z_0 - 1}$
 (c) $1 - i(z_0 - 1)$ (d) $i(z_0 - 1)$
- 133.** If $|z_1| = |z_2| = |z_3| = 1$ and z_1, z_2 and z_3 are represented by the vertices of an equilateral triangle, then
 (a) $z_1 + z_2 + z_3 = 0$ (b) $z_1 z_2 z_3 = 1$
 (c) $z_1 z_2 + z_2 z_3 + z_3 z_1 = 0$ (d) None of these

Type 3. Assertion and Reason

Direction (Q. Nos. 134-144) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I.
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I.
(c) Statement I is true, Statement II is false.
(d) Statement I is false, Statement II is true.

134. Consider $|z_1| = 1, |z_2| = 2$ and $|z_3| = 3$.

Statement I If $|z_1 + 2z_2 + 3z_3| = 6$, then the value of $|z_2 z_3 + 8z_3 z_1 + 27z_1 z_2|$ is 36.

Statement II $|z_1 + z_2 + z_3| \leq |z_1| + |z_2| + |z_3|$

135. Statement I $7 + i > 5 + i$

Statement II Cancellation laws does not hold true in complex numbers.

136. Statement I If z_1 and z_2 are two complex numbers

such that $|z_1| = |z_2| + |z_1 - z_2|$, then $\text{Im}\left(\frac{z_1}{z_2}\right) = 0$.

Statement II $\arg(z) = 0 \Rightarrow z$ is purely real.

137. Statement I If $z + \frac{1}{z} = 1$ ($z \neq 0$ is a complex number), then the maximum value of $|z|$ is $\frac{\sqrt{5}+1}{2}$.

Statement II On the locus $z + \frac{1}{z} = 1$, the farthest distance from origin is $\frac{\sqrt{5}+1}{2}$.

138. Statement I The number of complex numbers z satisfying $|z|^2 + a|z| + b = 0$ ($a, b \in \mathbb{R}$) is atmost 2.

Statement II A quadratic equation in which all the coefficients are non-zero can have atmost two roots.

139. Statement I If $\cos(1-i) = a + ib$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, then $a = \frac{1}{2}\left(e + \frac{1}{e}\right)\cos 1$, $b = \frac{1}{2}\left(e - \frac{1}{e}\right)\sin 1$.

Statement II $e^{i\theta} = \cos \theta + i \sin \theta$

140. Statement I The product of all values of $(\cos \alpha + i \sin \alpha)^{3/5}$ is $\cos 3\alpha + i \sin 3\alpha$.

Statement II The product of fifth roots of unity is 1.

141. Statement I The locus of the centre of a circle which touches the circles $|z - z_1| = a$ and $|z - z_2| = b$ externally (z, z_1 and z_2 are complex numbers) will be hyperbola.

Statement II $|z - z_1| - |z - z_2| < |z_2 - z_1|$
 $\Rightarrow z$ lies on hyperbola.

142. Statement I Consider an ellipse having its foci at $A(z_1)$ and $B(z_2)$ in the argand plane. If the eccentricity of the ellipse is e and it is known that origin is an interior point of the ellipse, then

$$e \in \left(\frac{|z_1 + z_2|}{|z_1| + |z_2|}, \frac{|z_1 - z_2|}{|z_1| + |z_2|} \right)$$

Statement II If z_0 is the point interior to curve

$$\begin{aligned} |z - z_1| + |z - z_2| &= \lambda \\ |z_0 - z_1| + |z_0 - z_2| &< \lambda \end{aligned}$$

143. Statement I The equation $|z - i| + |z + i| = k$, $k > 0$, can represent an ellipse, if $k > 2i$.

Statement II $|z - z_1| + |z - z_2| = k$, represents an ellipse, if $|k| > |z_1 - z_2|$.

144. Statement I The equation $z\bar{z} + \bar{a}z + a\bar{z} + \lambda = 0$, where a is a complex number represents a circle in argand plane, if λ is real.

Statement II The radius of the circle $z\bar{z} + \bar{a}z + a\bar{z} + \lambda = 0$ is $\sqrt{a\bar{a} - \lambda}$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 145-147) Consider the complex numbers z_1 and z_2 satisfying the relation $|z_1 + z_1|^2 = |z_1|^2 + |z_2|^2$.

145. Complex number $z_1 \bar{z}_2$ is

- (a) purely real (b) purely imaginary
(c) zero (d) None of these

146. Complex number z_1 / z_2 is

- (a) purely real
(b) purely imaginary
(c) zero
(d) None of the above

147. One of the possible argument of complex number $i(z_1 / z_2)$ is

- (a) $\frac{\pi}{2}$ (b) $-\frac{\pi}{2}$ (c) 0 (d) None of these

Passage II (Q. Nos. 148-150) If z satisfy the relation $|z - \{\alpha^2 - 7\alpha + 11 + i\}| = 1$, $\alpha \in \mathbb{R}$.

Also, $\arg(z) \geq \frac{\pi}{2}$ is satisfied by atleast one z .

148. The values of α lies in the interval

- (a) $[2, 6]$ (b) $\left[1, \frac{7}{2}\right]$ (c) $[2, 5]$ (d) $[-4, -2]$

149. Maximum value of $|z - i|$ is

- (a) $\frac{7}{2}$ (b) $\frac{4}{9}$ (c) $\frac{5}{9}$ (d) $\frac{9}{4}$

150. Maximum value of $\arg(z)$, for which $|z - i|$ is maximum, is

- (a) $\pi - \tan^{-1}\left(\frac{4}{9}\right)$ (b) $\pi - \tan^{-1}\left(\frac{9}{4}\right)$
(c) $\pi - \tan^{-1}\left(\frac{1}{9}\right)$ (d) $\pi - \tan^{-1}\left(\frac{4}{5}\right)$

Passage III (Q. Nos. 151-153) On the sides AB and BC of a $\triangle ABC$, squares are drawn with centres D and E such that points C and D lies on the same side of line AB

AB and points A and E lies in the opposite side of line BC . If A, B and C are represent by the complex numbers $1, \omega$ and ω^2 respectively.

151. Angle between AC and DE is equal to

- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$

152. The length of DE is

- (a) $\frac{\sqrt{3}}{2}$ (b) $\sqrt{3}$ (c) $\sqrt{2}$ (d) $\sqrt{6}$

153. The length of AE is

- (a) $\frac{3 - \sqrt{3}}{2}$ (b) $\frac{3 + \sqrt{3}}{2}$ (c) $3 - \sqrt{3}$ (d) $3 + \sqrt{3}$

Type 5. Match the Columns

154. Match the following:

Column I	Column II
A. $\arg\left(\frac{z^2 - 1}{z^2 + 1}\right) = 0; z \neq \pm i, \pm 1$	P. Portions of a line
B. $\ z - \cos^{-1} \cos 12 - z - \sin^{-1} \sin 12\ = 8(\pi - 3)$	q. Point of intersection of hyperbola
C. $z^2 + k_1 = i z ^2 + k_2; k_1 \neq k_2 \in \mathbb{R} - \{0\}$	r. Pair of open rays
D. $\left z - 1 - \sin^{-1} \frac{1}{\sqrt{3}}\right + \left z + \cos^{-1} \frac{1}{\sqrt{3}} - \frac{\pi}{2}\right = 1$	s. line segment

155. For the equation $z^6 - 6z + 20 = 0$, match the items of Column I with that of Column II.

Column I	Column II
A. The number of the roots in the first quadrant can be	p. 1
B. The number of the roots in the second quadrant can be	q. 2
C. The number of the roots in the third quadrant can be	r. 3
D. The number of the roots in the fourth quadrant can be	s. 4

156. Match the following:

Column I	Column II
A. $z^4 - 1 = 0$	p. $z = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$
B. $z^4 + 1 = 0$	q. $z = \cos \frac{\pi}{8} - i \sin \frac{\pi}{8}$
C. $iz^4 + 1 = 0$	r. $z = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$
D. $iz^4 - 1 = 0$	s. $z = \cos 0 + i \sin 0$

157. Match the following:

Column I	Column II
A. $\sqrt{-1 - \sqrt{-1 - \sqrt{\dots \infty}}}$ is	p. 0
B. If $z_i, i = 1, 2, \dots, 6$ are the vertices of the six sided regular polygon inscribed in a circle $ z = 2$, then $\sum_{i=1}^6 z_i^2$ is equal to	q. $\frac{n}{\omega - 1}$
C. If $\omega_k, k \in \mathbb{I}; 0 \leq k \leq n - 1$ are the n th roots of unity, then the maximum value of $(n!)^{1/n} \omega^{(n-1)/2}$ is	r. ω
D. If z_1, z_2, z_3 are the affixes points and A, B and C are lying on circle centred at origin. If the altitude is drawn from vertex A to base BC , such that it meets the circumcircle at $P(z)$, then $zz_1 + z_2 z_3$ is	s. ω^2

Type 6. Single Integer Answer Type Questions

158. If $x = a + bi$ is a complex number such that $x^2 = 3 + 4i$ and $x^3 = 2 + 11i$, where $i = \sqrt{-1}$, then $(a + b)$ equals _____.

159. If the complex numbers z is simultaneously satisfy equations $\left|\frac{z - 12}{z - 8i}\right| = \frac{5}{3}, \left|\frac{z - 4}{z - 8}\right| = 1$, then $\operatorname{Re}(z)$ is _____.

160. If $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is $\frac{\lambda}{3}$, where λ is _____.

161. If z is a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then $|z|$ is equal to _____.

162. $ABCD$ is a rhombus. Its diagonals AC and BD intersect at the point M and satisfy $BD = 2AC$. Its points D and M represent the complex numbers $1 + i$ and $2 - i$ respectively. Then, the complex number represented by A is z , has $\operatorname{Re}(z) = \{\lambda_1, \lambda_2\}$, then the value of $\lambda_1 + \lambda_2$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Match the following : [2014]

Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i \sin\left(\frac{2k\pi}{10}\right); k = 1, 2, \dots, 9$.

Column I	Column II
A. For each z_k , there exists a z_j such that $z_k \cdot z_j = 1$	p. True
B. There exists $k \in \{1, 2, \dots, 9\}$, such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers	q. False
C. $\frac{ 1 - z_1 1 - z_2 \dots 1 - z_9 }{10}$ equals	r. 1
D. $1 - \sum_{k=1}^9 \cos\left(\frac{2k\pi}{10}\right)$ equals	s. 2

Codes

A	B	C	D	A	B	C	D
(a) p	q	s	r	(b) q	p	r	s
(c) p	q	r	s	(d) q	p	s	r

2. Let complex numbers α and $\frac{1}{\alpha}$ lie on the circles

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \quad \text{and} \\ (x - x_0)^2 + (y - y_0)^2 = 4r^2, \quad \text{respectively. If}$$

$z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha|$ equals [2013]

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

3. Let $w = \frac{\sqrt{3} + i}{2}$ and $P = \{w^n : n = 1, 2, 3, \dots\}$. Further

$$H_1 = \left\{ z \in C : \operatorname{Re}(z) > \frac{1}{2} \right\} \quad \text{and}$$

$$H_2 = \left\{ z \in C : \operatorname{Re}(z) < \frac{1}{2} \right\}, \text{ where } C \text{ is the set of all}$$

complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1 O z_2$ equals [2013]

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{6}$ (c) $\frac{2\pi}{3}$ (d) $\frac{5\pi}{6}$

Passage (Q. Nos. 4-5) Let $S = S_1 \cap S_2 \cap S_3$

where, $S_1 = \{z \in C : |z| < 4\}$,

$$S_2 = \left\{ z \in C : \operatorname{Im} \left[\frac{z - 1 + \sqrt{3}i}{1 - \sqrt{3}i} \right] > 0 \right\}$$

and $S_3 = \{z \in C : \operatorname{Re} z > 0\}$.

4. The area of S equals [2013]

- (a) $\frac{10\pi}{3}$ (b) $\frac{20\pi}{3}$ (c) $\frac{16\pi}{3}$ (d) $\frac{32\pi}{3}$

5. $\min_{z \in S} |1 - 3i - z|$ equals [2013]

- (a) $\frac{2 - \sqrt{3}}{2}$ (b) $\frac{2 + \sqrt{3}}{2}$ (c) $\frac{3 - \sqrt{3}}{2}$ (d) $\frac{3 + \sqrt{3}}{2}$

6. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then, a cannot take the value [2012]

- (a) -1 (b) $\frac{1}{3}$
(c) $\frac{1}{2}$ (d) $\frac{3}{4}$

7. If z is any complex number satisfying $|z - 3 - 2i| \leq 2$, then the minimum value of $|2z - 6 + 5i|$ is [2011]

8. Let $\omega = e^{\frac{2i\pi}{3}}$ and a, b, c, x, y, z be non-zero complex numbers, such that $a + b + c = x$, $a + b\omega + c\omega^2 = y$ and $a + b\omega^2 + c\omega = z$. Then, the value of $\frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is [2011]

9. Let z_1 and z_2 be two distinct complex numbers and $z = (1 - t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If $\arg(w)$ denotes the principal argument of a non-zero complex number w , then [2010]

- (a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(b) $\arg(z - z_1) = \arg(z - z_2)$
(c) $\left| \frac{z - z_1}{z_2 - z_1} - \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0$
(d) $\arg(z - z_1) = \arg(z_2 - z_1)$

10. Match the statement in Column I with those in Column II.

Here, z takes the values in the complex plane and $\operatorname{Im} z$ and $\operatorname{Re} z$ denote, respectively the imaginary part and the real part of z . [2010]

Column I	Column II
A. The set of point z satisfying $ z - i z = z + i z $ is contained in or equal to	p. an ellipse with eccentricity $\frac{4}{5}$
B. The set of point z satisfying $ z + 4 + z - 4 = 10$ is contained in or equal to	q. the set of point z satisfying $\operatorname{Im} z = 0$
C. If $ \omega = 2$, then the set of point $z = \omega - 1/\omega$ is contained in or equal to	r. the set of point z satisfying $ \operatorname{Im}(z) \leq 1$
D. If $ \omega = 1$, then the set of point $z = \omega + 1/\omega$ is contained in or equal to	s. the set of point z satisfying $ \operatorname{Re}(z) \leq 1$
	t. the set of point z satisfying $ \operatorname{Im} z \leq 5$

JEE Main/AIEEE



11. A complex number z is said to be unimodular, if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then, the point z_1 lies on a [2015]
 (a) straight line parallel to X -axis
 (b) straight line parallel to Y -axis
 (c) circle of radius 2
 (d) circle of radius $\sqrt{2}$.
12. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{z}\right|$ [2014]
 (a) is equal to $\frac{5}{2}$
 (b) lies in the interval $(1, 2)$
 (c) is strictly greater than $\frac{5}{2}$
 (d) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$
13. If z is a complex number of unit modulus and argument θ , then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals [2013]
 (a) $-\theta$ (b) $\frac{\pi}{2} - \theta$ (c) θ (d) $\pi - \theta$
14. If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies [2012]
 (a) either on the real axis or on a circle passing through the origin
 (b) on a circle with centre at the origin
 (c) either on the real axis or on a circle not passing through the origin
 (d) on the imaginary axis
15. Let α and β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line $\operatorname{Re}(z) = 1$, then it is necessary that [2012]
 (a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
 (c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$
16. If $\omega (\neq 1)$ is a cube root of unity and $(1 + \omega)^7 = A + B\omega$. Then, (A, B) equals [2011]
 (a) $(1, 1)$ (b) $(1, 0)$
 (c) $(-1, 1)$ (d) $(0, 1)$
17. The number of complex numbers z such that $|z-1| = |z+1| = |z-i|$ equals [2010]
 (a) 0 (b) 1
 (c) 2 (d) ∞
18. If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ is equal to [2010]
 (a) -2 (b) -1
 (c) 1 (d) 2
19. If $\left|z - \frac{4}{z}\right| = 2$, then the maximum value of $|z|$ is equal to [2009]
 (a) $\sqrt{3} + 1$ (b) $\sqrt{5} + 1$ (c) 2 (d) $2 + \sqrt{2}$
20. The conjugate of a complex number is $\frac{1}{i-1}$. Then, that complex number is [2008]
 (a) $-\frac{1}{i-1}$ (b) $\frac{1}{i+1}$ (c) $-\frac{1}{i+1}$ (d) $\frac{1}{i-1}$
21. If $|z+4| \leq 3$, then the maximum value of $|z+1|$ is [2007]
 (a) 4 (b) 10 (c) 6 (d) 0
22. The value of $\sum_{k=1}^{10} \left(\sin \frac{2k\pi}{11} + i \cos \frac{2k\pi}{11}\right)$ is [2006]
 (a) 1 (b) -1 (c) $-i$ (d) i
23. If $z^2 + z + 1 = 0$, where z is complex number, then the value of $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2 + \dots + \left(z^6 + \frac{1}{z^6}\right)^2$ is [2006]
 (a) 54 (b) 6 (c) 12 (d) 18
24. If the cube roots of unity are $1, \omega, \omega^2$, then the roots of the equation $(x-1)^3 + 8 = 0$, are [2005]
 (a) $-1, 1+2\omega, 1+2\omega^2$ (b) $-1, 1-2\omega, 1-2\omega^2$
 (c) $-1, -1, -1$ (d) $-1, -1+2\omega, -1-2\omega^2$
25. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg(z_1) - \arg(z_2)$ is equal to [2005]
 (a) $-\frac{\pi}{2}$ (b) 0 (c) $-\pi$ (d) $\frac{\pi}{2}$
26. If $w = \frac{z}{z - \frac{i}{3}}$ and $|w| = 1$, then z lies on [2005]
 (a) a parabola (b) a straight line
 (c) a circle (d) an ellipse
27. Let z and w be two complex numbers such that $\bar{z} + i\bar{w} = 0$ and $\arg(zw) = \pi$. Then, $\arg(z)$ equals [2004]
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
 (c) $\frac{3\pi}{4}$ (d) $\frac{5\pi}{4}$
28. If $z = x - iy$ and $z^{1/3} = p + iq$, then $\left(\frac{x}{p} + \frac{y}{q}\right) / (p^2 + q^2)$ is equal to [2004]
 (a) 1 (b) -1 (c) 2 (d) -2

29. If $|z^2 - 1| = |z|^2 + 1$, then z lies on [2004]
 (a) the real axis (b) the imaginary axis
 (c) a circle (d) an ellipse
30. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then, [2003]
 (a) $a^2 = b$ (b) $a^2 = 2b$ (c) $a^2 = 3b$ (d) $a^2 = 4b$
31. If z and w are two non-zero complex numbers such that $|zw| = 1$, and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then $\bar{z}(w)$ is equal to [2003]
 (a) 1 (b) -1 (c) i (d) $-i$
32. If $\left(\frac{1+i}{1-i}\right)^x = 1$, then [2003]
 (a) $x = 4n$, where n is any positive integer
 (b) $x = 2n$, where n is any positive integer
 (c) $x = 4n + 1$, where n is any positive integer
 (d) $x = 2n + 1$, where n is any positive integer
33. If ω is an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals [2002]
 (a) 128ω
 (b) -128ω
 (c) $128\omega^2$
 (d) $-128\omega^2$

Other Engineering Entrances

34. If α and β are two different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ is equal to [Karnataka CET 2014]
 (a) $1/2$ (b) 0
 (c) -1 (d) 1
35. If $z = \frac{(\sqrt{3} + i)^3 (3i + 4)^2}{(8 + 6i)^2}$, then $|z|$ is equal to [Kerala CEE 2014]
 (a) 8 (b) 2 (c) 5
 (d) 4 (e) 10
36. Let $w \neq \pm 1$ be complex number. If $|w| = 1$ and $z = \frac{w-1}{w+1}$, then $\operatorname{Re}(z)$ is equal to [Kerala CEE 2014]
 (a) 1 (b) $\frac{1}{|w+1|}$ (c) $\operatorname{Re}(w)$ (d) 0
 (e) $w + \bar{w}$
37. The value of $|z|^2 + |z-3|^2 + |z-i|^2$ is minimum, when z equals [WB JEE 2014]
 (a) $2 - \frac{2}{3}i$ (b) $45 + 3i$
 (c) $1 + \frac{i}{3}$ (d) $1 - \frac{i}{3}$
38. Convert $(i+1)/\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$ in polar form. [J&K CET 2014]
 (a) $\cos(\pi/4) + i\sin(\pi/4)$
 (b) $\cos(\pi/2) - i\sin(\pi/2)$
 (c) $\sqrt{2}[\cos(\pi/4) + i\sin(\pi/4)]$
 (d) $\sqrt{2}[\cos(\pi/2) + i\sin(\pi/2)]$
39. Let $z_1 \neq z_2$ and $|z_1| = |z_2|$. If z_1 has positive real part and z_2 has negative imaginary part. Then, $\frac{z_2 + z_1}{z_1 - z_2}$ may be [Manipal 2014]
 (a) 0 (b) real and positive
 (c) real and negative (d) None of these
40. If $z = e^{2\pi i/3}$, then $1 + z + 3z^2 + 2z^3 + 2z^4 + 3z^5$ is equal to [Kerala CEE 2014]
 (a) $-3e^{\pi i/3}$ (b) $3e^{\pi i/3}$ (c) $3e^{2\pi i/3}$
 (d) $-3e^{2\pi i/3}$ (e) 0
41. If ω, ω^2 are the cube roots of unity, then roots of equation $(x-1)^3 + 5 = 0$ are [RPET 2014]
 (a) $-5, -5\omega, -5\omega^2$
 (b) $-4, 1-5\omega, 1-5\omega^2$
 (c) $6, 1-5\omega, 1+5\omega^2$
 (d) None of the above
42. The complex number $z = x + iy$, which satisfies the equation $\left|\frac{z-3i}{z+3i}\right| = 1$, lie on [BITSAT 2014]
 (a) the X -axis
 (b) the straight line $y = 3$
 (c) a circle passing through the origin
 (d) None of the above
43. If the complex numbers z_1, z_2 and z_3 denote the vertices of an isosceles triangle, right angled at z_1 , then $(z_1 - z_2)^2 + (z_1 - z_3)^2$ is equal to [Kerala CEE 2014]
 (a) 0 (b) $(z_2 + z_3)^2$ (c) 2
 (d) 3 (e) $(z_2 - z_3)^2$
44. Let z_1 and z_2 be two fixed complex numbers in the argand plane and z be an arbitrary point satisfying $|z - z_1| + |z - z_2| = 2|z_1 - z_2|$. Then, the locus of z will be [WB JEE 2014]
 (a) an ellipse
 (b) a straight line joining z_1 and z_2
 (c) a parabola
 (d) a bisector of the line segment joining z_1 and z_2
45. Let z_1 be a fixed point on the circle of radius 1 centred at the origin in the argand plane and $z_1 \neq \pm 1$. Consider an equilateral triangle inscribed in the circle with z_1, z_2 and z_3 as the vertices taken in the counter clockwise direction. Then, $z_1 z_2 z_3$ is equal to [WB JEE 2014]
 (a) z_1^2
 (b) z_1^3
 (c) z_1^4
 (d) z_1



46. Suppose that z_1, z_2 and z_3 are three vertices of an equilateral triangle in the argand plane. Let $\alpha = \frac{1}{2}(\sqrt{3} - i)$ and β be a non-zero complex numbers. The points $\alpha z_1 + \beta, \alpha z_2 + \beta, \alpha z_3 + \beta$ will be [WB JEE 2014]
- (a) the vertices of an equilateral triangle
(b) the vertices of an isosceles triangle
(c) collinear
(d) the vertices of a scalene triangle
47. In the argand plane, the distinct roots of $1 + z + z^3 + z^4 = 0$ (z is a complex number) represent vertices of [WB JEE 2014]
- (a) a square
(b) an equilateral triangle
(c) a rhombus
(d) a rectangle
48. If $z_1 = 2\sqrt{2}(1 + i)$ and $z_2 = 1 + i\sqrt{3}$, then $z_1^2 z_2^3$ is equal to [Kerala CEE 2013]
- (a) $128i$ (b) $64i$ (c) $-64i$
(d) $-128i$ (e) 256
49. The value of $(1 + i)^3 + (1 - i)^3$ is equal to [J&K CET 2013]
- (a) 1 (b) -2 (c) 0 (d) -4
50. If $z(2 - i2\sqrt{3})^2 = i(\sqrt{3} + i)^4$, then amplitude of z is [UP SEE 2013]
- (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{6}$ (d) None of these
51. If z_1, z_2 and z_3 are complex numbers such that $|z_1| = |z_2| = |z_3| = \left| \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right| = 1$, then $|z_1 + z_2 + z_3|$ is [AMU 2013]
- (a) 3 (b) 1
(c) greater than 3 (d) less than 1
52. If $(\sqrt{3}i + 1)^{100} = 2^{99}(a + ib)$, then $a^2 + b^2$ is equal to [RPET 2013]
- (a) 4 (b) 3
(c) 2 (d) 0
53. The value of $(1 + \sqrt{3}i)^4 + (1 - \sqrt{3}i)^4$ is [RPET 2013]
- (a) -16 (b) 16
(c) 14 (d) -14
54. If the fourth roots of unity are z_1, z_2, z_3 and z_4 , then $z_1^2 + z_2^2 + z_3^2 + z_4^2$ is equal to [Karnataka CET 2013]
- (a) 0 (b) 2
(c) 3 (d) None of these
55. Among the complex number z satisfying condition $|z + 1 - i| \leq 1$, the number having the least positive argument is [OJEE 2013]
- (a) $1 - i$
(b) $1 + i$
(c) $-i$
(d) None of the above
56. If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2| + |z_1 - z_2|$, then [Manipal 2012]
- (a) $\operatorname{Im}\left(\frac{z_1}{z_2}\right) = 0$ (b) $\operatorname{Re}\left(\frac{z_1}{z_2}\right) = 0$
(c) $\operatorname{Re}\left(\frac{z_1}{z_2}\right) = \operatorname{Im}\left(\frac{z_1}{z_2}\right)$ (d) None of these
57. If the conjugate of $(x + iy)(1 - 2i)$ is $1 + i$, then [Karnataka CET 2012]
- (a) $x - iy = \frac{1 - i}{1 - 2i}$ (b) $x + iy = \frac{1 - i}{1 - 2i}$
(c) $x = \frac{1}{5}$ (d) $x = -\frac{1}{5}$
58. If $\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)^{50} = 3^{25}(x + iy)$, where x and y are real, then the ordered pair (x, y) is [WB JEE 2012]
- (a) $(-3, 0)$ (b) $(0, 3)$
(c) $(0, -3)$ (d) $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
59. If $|z - \bar{z}| + |z + \bar{z}| = 2$, then z lies on [AMU 2012]
- (a) a circle (b) a square (c) an ellipse (d) a line
60. If $2x = 3 + 5i$, then the value of $2x^3 + 2x^2 - 7x + 72$ is [MP PET 2011]
- (a) 4 (b) -4
(c) 8 (d) -8
61. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$, then $\frac{z_1}{z_2}$ is [MP PET 2011]
- (a) purely real
(b) purely imaginary
(c) zero of purely imaginary
(d) neither real nor imaginary
62. The value of $\left| \frac{1 + i\sqrt{3}}{\left(1 + \frac{1}{i + 1}\right)^2} \right|$ is [Karnataka CET 2011]
- (a) 20 (b) 9
(c) $\frac{5}{4}$ (d) $\frac{4}{5}$
63. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\arg\left(\frac{z_1}{z_2}\right)$ is [Kerala CEE 2011]
- (a) 0 (b) $-\pi$ (c) $-\frac{\pi}{2}$
(d) $\frac{\pi}{2}$ (e) π
64. If $\omega \neq 1$ is a cube root of unity, then the sum of the series $S = 1 + 2\omega + 3\omega^2 + \dots + 3n\omega^{3n-1}$ is [WB JEE 2011]
- (a) $\frac{3n}{\omega - 1}$ (b) $3n(\omega - 1)$
(c) $\frac{\omega - 1}{3n}$ (d) 0

65. $\left[\frac{1 + \cos\left(\frac{\theta}{2}\right) - i \sin\left(\frac{\theta}{2}\right)}{1 + \cos\left(\frac{\theta}{2}\right) + i \sin\left(\frac{\theta}{2}\right)} \right]^{4n}$ is equal to [UP SEE 2011]
 (a) $\cos n\theta - i \sin n\theta$ (b) $\cos n\theta + i \sin n\theta$
 (c) $\cos 2n\theta - i \sin 2n\theta$ (d) $\cos 2n\theta + i \sin 2n\theta$
66. If $x = \frac{-3 + i\sqrt{3}}{2}$ is a complex number, then the value of $(x^2 + 3x)^2 (x^2 + 3x + 1)$ is [UP SEE 2011]
 (a) $-\frac{9}{8}$ (b) 6
 (c) -18 (d) 36
67. If $(x + iy)^{1/3} = 2 + 3i$, then $3x + 2y$ is equal to [Kerala CEE 2010]
 (a) -20 (b) -60 (c) -120
 (d) 60 (e) 156
68. The modulus of the complex number z such that $|z + 3 - i| = 1$ and $\arg(z) = \pi$, is equal to [Kerala CEE 2010]
 (a) 1 (b) 2 (c) 9
 (d) 4 (e) 3
69. If $z_1, z_2, \dots, z_n$ are complex numbers such that $|z_1| = |z_2| = \dots = |z_n| = 1$, then $|z_1 + z_2 + \dots + z_n|$ is equal to [Kerala CEE 2010]
 (a) $|z_1 z_2 z_3 \dots z_n|$ (b) $|z_1| + |z_2| + \dots + |z_n|$
 (c) $\left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right|$ (d) n
 (e) $\sqrt{n}$
70. If $z_1 = \sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$ and $z_2 = \sqrt{3} \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$, then $|z_1 z_2|$ is [Kerala CEE 2010]
 (a) 6 (b) $\sqrt{2}$
 (c) $\sqrt{6}$ (d) $\sqrt{3}$
 (e) $\sqrt{2} + \sqrt{3}$
71. If $z = r(\cos \theta + i \sin \theta)$, then the value of $\frac{z}{\bar{z}} + \frac{\bar{z}}{z}$ is [Kerala CEE 2010]
 (a) $\cos 2\theta$ (b) $2 \cos 2\theta$
 (c) $2 \cos \theta$ (d) $2 \sin \theta$
 (e) $2 \sin 2\theta$
72. If $z = \frac{4}{1-i}$, then $\bar{z}$ is (where, $\bar{z}$ is complex conjugate of z) [WB JEE 2010]
 (a) $2(1+i)$ (b) $(1+i)$
 (c) $\frac{2}{1-i}$ (d) $\frac{4}{1-i}$
73. If $-\pi < \arg(z) < -\frac{\pi}{2}$, then $\arg(\bar{z}) - \arg(-\bar{z})$ is [WB JEE 2010]
 (a) π
 (b) $-\pi$
 (c) $\pi/2$
 (d) $-\pi/2$
74. The value of $\frac{\cos 30^\circ + i \sin 30^\circ}{\cos 60^\circ - i \sin 60^\circ}$ is equal to [VITEEE 2010]
 (a) i (b) $-i$
 (c) $\frac{1 + \sqrt{3}i}{2}$ (d) $\frac{1 - \sqrt{3}i}{2}$

Answers

Work Book Exercise 4.1

1. (b)	2. (a)	3. (c)	4. (b)	5. (a)	6. (d)	7. (b)	8. (d)	9. (a)	10. (c)
11. (d)	12. (d)	13. (c)	14. (c)						

Work Book Exercise 4.2

1. (a)	2. (b)	3. (a)	4. (c)	5. (a)	6. (b)	7. (c)	8. (b)	9. (d)	10. (d)
11. (a)	12. (a)	13. (d)	14. (b)	15. (b)					

Work Book Exercise 4.3

1. (c)	2. (b)	3. (b)	4. (c)	5. (a)	6. (d)	7. (c)	8. (b)	9. (a)	10. (d)
11. (a)	12. (b)	13. (b)	14. (c)	15. (b)	16. (c)	17. (c)	18. (a)	19. (a)	20. (c)
21. (c)	22. (a)	23. (c)	24. (a)	25. (d)	26. (c)	27. (b)			

Work Book Exercise 4.4

1. (c)	2. (d)	3. (b)	4. (b)	5. (a)	6. (b)	7. (a)	8. (a)	9. (c)	10. (b)
11. (b)	12. (c)	13. (c)	14. (a)	15. (a)	16. (c)	17. (a)	18. (b)	19. (a)	20. (c)
21. (a)	22. (d)	23. (a)	24. (b)	25. (a)	26. (b)	27. (b)	28. (a)	29. (d)	30. (b)

Target Exercises

1. (b)	2. (a)	3. (c)	4. (d)	5. (b)	6. (c)	7. (c)	8. (c)	9. (b)	10. (c)
11. (b)	12. (d)	13. (c)	14. (a)	15. (c)	16. (d)	17. (d)	18. (a)	19. (b)	20. (b)
21. (a)	22. (c)	23. (c)	24. (d)	25. (c)	26. (c)	27. (d)	28. (c)	29. (a)	30. (b)
31. (a)	32. (a)	33. (c)	34. (c)	35. (b)	36. (b)	37. (b)	38. (b)	39. (a)	40. (a)
41. (d)	42. (a)	43. (b)	44. (a)	45. (a)	46. (c)	47. (b)	48. (b)	49. (a)	50. (b)
51. (c)	52. (a)	53. (b)	54. (b)	55. (a)	56. (c)	57. (a)	58. (c)	59. (c)	60. (b)
61. (d)	62. (c)	63. (a)	64. (d)	65. (c)	66. (a)	67. (a)	68. (b)	69. (b)	70. (b)
71. (b)	72. (a)	73. (b)	74. (d)	75. (c)	76. (c)	77. (a)	78. (a)	79. (b)	80. (d)
81. (c)	82. (b)	83. (b)	84. (c)	85. (a)	86. (a)	87. (c)	88. (c)	89. (a)	90. (c)
91. (a)	92. (c)	93. (b)	94. (a)	95. (a)	96. (c)	97. (a)	98. (d)	99. (d)	100. (b)
101. (b)	102. (d)	103. (a)	104. (c)	105. (c)	106. (a)	107. (b)	108. (c)	109. (c)	110. (c)
111. (b)	112. (c)	113. (c)	114. (b)	115. (d)	116. (a)	117. (a,b)	118. (a,b,d)	119. (a,b,c)	120. (a,c,d)
121. (a,c)	122. (a,d)	123. (a,b)	124. (a,d)	125. (a,b,d)	126. (a,b)	127. (a,c)	128. (b,c)	129. (a,b)	130. (a,b,c)
131. (a,d)	132. (a,c)	133. (a,b)	134. (b)	135. (d)	136. (a)	137. (a)	138. (d)	139. (a)	140. (b)
141. (d)	142. (d)	143. (d)	144. (a)	145. (b)	146. (b)	147. (c)	148. (c)	149. (d)	150. (a)
151. (c)	152. (a)	153. (b)	154. (*)	155. (**)	156. (***)	157. (****)	158. (3)	159. (6)	160. (8)
161. (1)	162. (4)								

* $A \rightarrow r$; $B \rightarrow p$; $C \rightarrow q$; $D \rightarrow s$

** $A \rightarrow p, q$; $B \rightarrow p, q$; $C \rightarrow p, q$; $D \rightarrow p, q$

*** $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow p$; $D \rightarrow q$

**** $A \rightarrow r, s$; $B \rightarrow p$; $C \rightarrow q$; $D \rightarrow p$

Entrances Gallery

1. (c)	2. (c)	3. (c,d)	4. (b)	5. (c)	6. (d)	7. (5)	8. (3)	9. (a,c,d)	10. (*)
11. (c)	12. (b)	13. (c)	14. (a)	15. (c)	16. (a)	17. (b)	18. (c)	19. (b)	20. (c)
21. (c)	22. (c)	23. (c)	24. (b)	25. (b)	26. (b)	27. (c)	28. (d)	29. (b)	30. (c)
31. (d)	32. (a)	33. (d)	34. (d)	35. (b)	36. (d)	37. (c)	38. (d)	39. (a)	40. (a)
41. (b)	42. (a)	43. (a)	44. (a)	45. (b)	46. (a)	47. (b)	48. (d)	49. (d)	50. (a)
51. (b)	52. (a)	53. (a)	54. (a)	55. (d)	56. (a)	57. (b)	58. (d)	59. (a)	60. (a)
61. (b)	62. (d)	63. (a)	64. (a)	65. (c)	66. (c)	67. (c)	68. (e)	69. (c)	70. (c)
71. (b)	72. (d)	73. (a)	74. (a)						

* $A \rightarrow q$; $B \rightarrow p, t$; $C \rightarrow p$; $D \rightarrow s, q$

Explanations

Target Exercises

$$1. \frac{i^{584} (i^8 + i^6 + i^4 + i^2 + 1)}{i^{574} (i^8 + i^6 + i^4 + i^2 + 1)} - 1 = \frac{i^{584}}{i^{574}} - 1 = i^{10} - 1 = -1 - 1 = -2$$

$$2. i^{57} + \frac{1}{i^{125}} = (i^4)^{14} i + \frac{1}{(i^4)^{31} i} = i + \frac{1}{i} = i - i = 0$$

$$3. \text{ We have, } i^n + i^{n+1} + i^{n+2} + i^{n+3} \\ = i^n (1 + i + i^2 + i^3) \\ = i^n (1 + i - 1 - i) \quad [\because i^3 = i^2 \cdot i = (-1) \cdot i = -i] \\ = i^n (0) = 0$$

$$4. \text{ Given expression } = 1 + i^2 + i^4 + i^6 + \dots + i^{2n} \\ = 1 - 1 + 1 - 1 + \dots + (-1)^n \\ \text{which cannot be determined unless } n \text{ is known.}$$

$$5. \text{ We have, } \left[i^{19} + \left(\frac{1}{i} \right)^{25} \right]^2 = \left[i^{16} \cdot i^3 + \frac{1}{i^{24} \cdot i} \right]^2 \\ = \left[i^2 \cdot i + \frac{1}{i} \right]^2 = (-i - i)^2 \\ = (-2i)^2 = 4i^2 = -4$$

$$6. \text{ We have, } \frac{i^{4n+1} - i^{4n-1}}{2} \\ = \frac{i^{4n} \cdot i - i^{4n} \cdot i^{-1}}{2} = \frac{i - \frac{1}{i}}{2} \quad [\because i^{4n} = (i^4)^n = (1)^n = 1] \\ = \frac{i^2 - 1}{2i} = \frac{-1 - 1}{2i} = \frac{-2}{2i} = \frac{1}{-i} = i$$

$$7. (1-i)^n \left(1 - \frac{1}{i} \right)^n = (1-i)^n (1+i)^n \\ = (1+1)^n = 2^n$$

$$8. \frac{(1-i)^n}{(1+i)^{n-2}} = \frac{(1-i)^n}{(1+i)^{n-2}} \times \frac{(1-i)^{n-2}}{(1-i)^{n-2}} \\ = 2 \frac{(-i)^{n-1}}{2} = 2 \frac{(-1)^{\frac{n-1}{2}}}{2} \\ \text{This is positive and real, if } \frac{n-1}{2} \text{ is even.}$$

$$\text{Let } \frac{n-1}{2} = 2\lambda \\ \therefore n = 4\lambda + 1$$

$$9. \left(\frac{1+i}{1-i} \right)^n = \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i} \right)^n = \left(\frac{1-1+2i}{1+1} \right)^n \\ = i^n = -1, \text{ if } n = 2$$

$$10. (1+i)^{2n} = (1-i)^{2n} \Rightarrow 1 = (-1)^n \\ \therefore \text{The smallest value of } n \text{ is } 2.$$

$$11. \frac{1+2i}{1-i} = \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{1+3i-2}{2} = -\frac{1}{2} + \frac{3}{2}i \\ \therefore \frac{1+2i}{1-i} \text{ lies in II quadrant}$$

$$12. \frac{(1-i)^3}{1-i^3} = \frac{1-3i+3i^2-i^3}{1+i} = \frac{-2-2i}{1+i} = -2$$

$$13. \text{ We have, } \left(\frac{1+i}{1-i} \right)^2 + \left(\frac{1-i}{1+i} \right)^2 \\ = \frac{(1+i)^2}{(1-i)^2} + \frac{(1-i)^2}{(1+i)^2} = \frac{2i}{-2i} + \frac{-2i}{2i} \\ \left[\begin{array}{l} \because (1+i)^2 = 1+i^2+2i=2i \\ \text{and } (1-i)^2 = 1+i^2-2i=-2i \end{array} \right] \\ = -1-1 = -2$$

$$14. \text{ Re} \left[\frac{(1+i)^2}{3-i} \right] = \text{Re} \left[\left(\frac{2i}{3-i} \right) \left(\frac{3+i}{3+i} \right) \right] \\ = \text{Re} \left[\frac{6i-2}{9+1} \right] \\ = \text{Re} \left[-\frac{2}{10} + \frac{6}{10}i \right] = -\frac{1}{5}$$

$$15. \text{ Given that, } a^2 + b^2 = 1 \\ \therefore \frac{1+b+ia}{1+b-ia} = \frac{(1+b+ia)(1+b+ia)}{(1+b-ia)(1+b+ia)} \\ = \frac{(1+b)^2 - a^2 + 2ia(1+b)}{1+b^2+2b+a^2} \\ = \frac{(1-a^2)+2b+b^2+2ia(1+b)}{2(1+b)} \\ = \frac{2b^2+2b+2ia(1+b)}{2(1+b)} \\ = b+ia$$

$$16. \text{ We have, } b+ic = (1+a)z \\ \therefore z = \frac{b+ic}{1+a} \Rightarrow iz = \frac{ib-c}{1+a} \\ \Rightarrow \frac{1+iz}{1-iz} = \frac{1+a-c+ib}{1+a+c-ib} \\ = \frac{(1+a-c)+ib}{(1+a+c)-ib} \times \frac{(1+a+c)+ib}{(1+a+c)+ib} \\ = \frac{a^2+a+iab+ib}{1+a+ac+c} \\ = \frac{(a+1)(a+ib)}{(a+1)(1+c)} = \frac{a+ib}{1+c}$$

$$17. x+iy = 6i(3i^2+3)+3i(4i+20)+1(12-60i) \\ = 0-12+60i+12-60i = 0+0i \\ \therefore x=0, y=0$$

$$18. \sqrt{2i} = \sqrt{1+i^2+2i} = \sqrt{(1+i)^2} = 1+i$$

$$19. \text{ Let } z = x+iy, \text{ where } x, y \in \mathbb{R} \\ \therefore x^2+y^2 \neq 0 \text{ and } x=0 \\ \therefore z = 0+iy = iy, y \neq 0 \\ \Rightarrow z^2 = -y^2 \\ \Rightarrow \text{Im}(z^2) = 0$$

$$20. \text{ We have, } (x+iy)^{1/3} = a+ib \\ \text{On cubing both sides, we get} \\ x+iy = (a+ib)^3 \\ = a^3 + 3a^2(ib) + 3a(ib)^2 + (ib)^3$$

$$\begin{aligned}
 &= a^3 + 3a^2bi + 3ab^2i^2 + i^3b^3 \\
 &= a^3 + 3a^2bi - 3ab^2 - ib^3 \quad [\because i^3 = i^2 \cdot i = -i] \\
 &= a(a^2 - 3b^2) + ib(3a^2 - b^2) \\
 \therefore x &= a(a^2 - 3b^2) \\
 \text{and } y &= b(3a^2 - b^2) \\
 \Rightarrow \frac{x}{a} &= a^2 - 3b^2 \quad \text{and} \quad \frac{y}{b} = 3a^2 - b^2 \\
 \therefore \frac{x}{a} + \frac{y}{b} &= 4(a^2 - b^2)
 \end{aligned}$$

21. We have,

$$\begin{aligned}
 8iz^3 + 12z^2 - 18z + 27i &= 0 \\
 \Rightarrow 4z^2(2iz + 3) + 9i(2iz + 3) &= 0 \\
 \Rightarrow (2iz + 3)(4z^2 + 9i) &= 0 \\
 \Rightarrow 2iz + 3 = 0 \quad \text{or} \quad 4z^2 + 9i = 0 \\
 \therefore |z| &= \frac{3}{2}
 \end{aligned}$$

22. Multiplicative inverse of z^2

$$\frac{1}{z^2} = \frac{1}{(1+i)^2} = \frac{1}{2i} = -\frac{i}{2}$$

23. Since, $\frac{1+i}{1-i} = i$ and $\frac{1-i}{1+i} = -i$

$$\begin{aligned}
 \therefore \left(\frac{1+i}{1-i}\right)^3 - \left(\frac{1-i}{1+i}\right)^3 &= x + iy \\
 \Rightarrow i^3 - (-i)^3 &= x + iy \\
 \Rightarrow -i + i^3 &= x + iy \\
 \Rightarrow -i - i &= x + iy \\
 x + iy &= 0 - 2i
 \end{aligned}$$

Hence, $(x, y) = (0, -2)$.

24. Let $z = (6 + 5i)^2 = 36 + 2(6)(5i) + 25i^2$

$$\begin{aligned}
 &= 36 + 60i - 25 = 11 + 60i \\
 \text{Then, } \bar{z} &= 11 - 60i \\
 \text{and } |z| &= \sqrt{(11)^2 + (60)^2} \\
 &= \sqrt{121 + 3600} \\
 &= \sqrt{3721} = 61
 \end{aligned}$$

Multiplicative inverse of

$$z = \frac{\bar{z}}{|z|^2} = \frac{11 - 60i}{(61)^2} = \frac{11}{(61)^2} - \frac{60}{(61)^2}i$$

$$\begin{aligned}
 25. \text{ Let } z &= \frac{3+4i}{4-5i} = \frac{3+4i}{4-5i} \times \frac{4+5i}{4+5i} \\
 &= \frac{12+15i+16i+20i^2}{16-25i^2} \\
 &= \frac{12+31i-20}{16+25} \\
 &= \frac{8}{41} + \frac{31}{41}i
 \end{aligned}$$

$$\text{Then, } \bar{z} = -\frac{8}{41} - \frac{31}{41}i$$

$$\begin{aligned}
 \text{and } |z| &= \sqrt{\left(-\frac{8}{41}\right)^2 + \left(-\frac{31}{41}\right)^2} \\
 &= \sqrt{\frac{64+961}{(41)^2}} \\
 &= \sqrt{\frac{1025}{(41)^2}} = \sqrt{\frac{25 \times 41}{(41)^2}} = \frac{5}{\sqrt{41}}
 \end{aligned}$$

$\therefore$ Multiplicative inverse of

$$\begin{aligned}
 z &= \frac{\bar{z}}{|z|^2} = \frac{-\frac{8}{41} - \frac{31}{41}i}{\frac{25}{41}} = \frac{-8-31i}{25} \\
 &= -\frac{8}{25} - \frac{31}{25}i
 \end{aligned}$$

26. Let $z = x + iy$, where $x < 0, y < 0$

$$\begin{aligned}
 \therefore \frac{\bar{z}}{z} &= \frac{x-iy}{x+iy} = \frac{(x-iy)^2}{(x^2+y^2)} = \frac{x^2-y^2}{x^2+y^2} - \left(\frac{2xy}{x^2+y^2}\right)i \\
 \frac{\bar{z}}{z} &\text{ lies in III quadrant, if } x^2 - y^2 < 0 \text{ and } -2xy < 0
 \end{aligned}$$

$$\begin{aligned}
 &x < 0, y < 0 \\
 \Rightarrow -2xy &< 0 \\
 \text{Also, } x^2 - y^2 &< 0, \text{ if } (x+y)(x-y) < 0 \\
 \text{If } x - y > 0, \text{ if } x > y & \quad [\because x, y < 0 \Rightarrow x + y < 0] \\
 \therefore \frac{\bar{z}}{z} &\text{ lies in III quadrant, if } y < x < 0.
 \end{aligned}$$

27. Let $z = \frac{2-i}{(1-2i)^2}$

$$\begin{aligned}
 \therefore z &= \frac{2-i}{-3-4i} = \frac{-2+i}{3+4i} \times \frac{3-4i}{3-4i} \\
 &= -\frac{2}{25} + \frac{11}{25}i \\
 \therefore \bar{z} &= -\frac{2}{25} - \frac{11}{25}i
 \end{aligned}$$

$$\begin{aligned}
 28. \left|(1+i)\left(\frac{2+i}{3+i}\right)\right| &= \frac{|1+i| \times |2+i|}{|3+i|} \\
 &= \frac{\sqrt{2} \times \sqrt{5}}{\sqrt{10}} = 1
 \end{aligned}$$

$$\begin{aligned}
 29. \sqrt{2i} - \sqrt{-2i} &= \sqrt{(1+i)^2} - \sqrt{(1-i)^2} \\
 &= (1+i) - (1-i) = 2i \\
 \therefore |\sqrt{2i} - \sqrt{-2i}| &= |2i| = \sqrt{0+4} = 2
 \end{aligned}$$

30. $\frac{1-ix}{1+ix} = a - ib$

$$\begin{aligned}
 \Rightarrow \frac{(1-ix)(1-ix)}{(1+ix)(1-ix)} &= a - ib \\
 \Rightarrow \frac{1-x^2-2ix}{1+x^2} &= \frac{a-ib}{1} \\
 \Rightarrow \frac{1-x^2}{1+x^2} = a \quad \text{and} \quad \frac{2x}{1+x^2} &= b
 \end{aligned}$$

Now, x can be written as

$$\begin{aligned}
 x &= \frac{\frac{2x}{1+x^2}}{\frac{2}{1+x^2}} = \frac{\frac{2x}{1+x^2}}{\frac{1-x^2}{1+x^2} + 1} \\
 &= \frac{b}{1+a} = \frac{2b}{1+1+2a} \\
 &= \frac{2b}{1+(a^2+b^2)+2a} = \frac{2b}{(1+a)^2+b^2}
 \end{aligned}$$

$$\begin{aligned}
 31. \left|\frac{(3+2i)^2}{(4-3i)}\right| &= \frac{|3+2i|^2}{|4-3i|} \\
 &= \frac{(\sqrt{9+4})^2}{(\sqrt{16+9})} = \frac{13}{5}
 \end{aligned}$$

32. We have, $x + iy = \frac{u + iv}{u - iv}$... (i)

We know that, when two complex numbers are equal their conjugates are also equal.

$\therefore x - iy = \frac{u - iv}{u + iv}$... (ii)

On multiplying Eqs. (i) and (ii), we get

$$(x + iy)(x - iy) = \frac{u + iv}{u - iv} \times \frac{u - iv}{u + iv}$$

$$\Rightarrow x^2 + y^2 = \frac{u^2 - i^2 v^2}{u^2 - i^2 v^2}$$

$$\therefore x^2 + y^2 = \frac{u^2 + v^2}{u^2 + v^2} = 1$$

33. Taking modulus and squaring on both sides, we get

$$|1 + i|^2 \cdot |1 + 2i|^2 \cdot |1 + 3i|^2 \dots |1 + ni|^2 = |\alpha + i\beta|^2$$

$$\Rightarrow (1 + 1) \cdot (1 + 4) \cdot (1 + 9) \dots (1 + n^2) = \alpha^2 + \beta^2$$

$$\Rightarrow 2 \cdot 5 \cdot 10 \dots (1 + n^2) = \alpha^2 + \beta^2$$

34.
$$\frac{(1 + i)^5 (1 + \sqrt{3}i)^2}{-2i(-\sqrt{3} + i)}$$

$$= \frac{(\sqrt{2})^5 \left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^5 2^2 \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^2}{2i2\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)}$$

$$\Rightarrow \text{Argument} = \frac{5\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{2} + \frac{\pi}{6} = \frac{9\pi}{12}$$

Therefore, the principal argument is $-\frac{5\pi}{12}$.

35. $i + \sqrt{3} = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2\left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)$

36. $|z^2| = |z z| = |z| |z| = |z|^2$

37. For every $a \in \mathbb{R}$, $|a| = \sqrt{a^2}$

$$\therefore |a|^2 = a^2$$

Now, $(|x| - |y|)^2 \geq 0$

$$\Rightarrow |x|^2 + |y|^2 - 2|x||y| \geq 0$$

$$\Rightarrow 2|x||y| \leq |x|^2 + |y|^2$$

$$\Rightarrow |x|^2 + |y|^2 + 2|x||y| \leq |x|^2 + 2|y|^2$$

$$\Rightarrow (|x| + |y|)^2 \leq 2(x^2 + y^2)$$

$$\Rightarrow (|x| + |y|)^2 \leq 2|z|^2$$

$$\therefore |x| + |y| \leq \sqrt{2}|z|$$

38. $|z + 1| = |(z + 4) - 3| \leq |z + 4| + |-3| \leq 3 + 3 = 6$

$$\therefore \text{Greatest value of } |z + 1| = 6$$

Also, $|z + 1| \geq |z + 4| - |-3| \geq 0$

$$\therefore 0 \leq |z + 1| \leq 6$$

39. $|z + 1| < |z - 2|$

$$\Rightarrow |3z + 3| < |3z - 6|$$

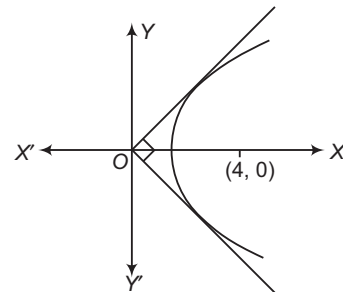
$$\Rightarrow |w + 1 - i| < |w - 8 - i|$$

$$\Rightarrow |w + 1| < |w - 8|$$

40. $|z| + |z - 1| \geq |z - z + 1| \geq 1$

41. Now, $|z - 4| = \text{Re}(z)$

$$\Rightarrow \sqrt{(x - 4)^2 + y^2} = x$$



$$\Rightarrow x^2 - 8x + 16 + y^2 = x^2$$

$$\Rightarrow y^2 = 8(x - 2)$$

The given relation represents the part of parabola with focus (4, 0) lying above X-axis and the imaginary axis as the directrix. The two tangents from directrix are at right angle. Hence, the greatest positive argument of z is $\frac{\pi}{4}$.

42. Let $a + bi$ be a square root of $-2 + 2\sqrt{3}i$.

$$\therefore (a + bi)^2 = -2 + 2\sqrt{3}i$$

$$\Rightarrow a^2 + 2abi + i^2 b^2 = -2 + 2\sqrt{3}i$$

$$\Rightarrow (a^2 - b^2) + 2abi = -2 + 2\sqrt{3}i$$

$$\therefore a^2 - b^2 = -2 \quad \dots (i)$$

$$\text{and } 2ab = 2\sqrt{3} \quad \dots (ii)$$

Now, $(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2 b^2$

$$= (-2)^2 + (2\sqrt{3})^2$$

$$= 4 + 12 = 16$$

$$\therefore a^2 + b^2 = \sqrt{16} = 4 \quad \dots (iii)$$

On solving Eqs. (i) and (iii), we get

$$2a^2 = 2 \text{ or } a^2 = 1$$

$$\therefore a = \pm 1$$

$$\Rightarrow 2b^2 = 6 \text{ or } b^2 = 3$$

$$\therefore b = \pm \sqrt{3}$$

From Eq. (ii), $ab = \sqrt{3}$, which is positive.

Either $a = 1, b = \sqrt{3}$ or $a = -1, b = -\sqrt{3}$

Hence, the two square roots are $1 + \sqrt{3}i$ and $-1 - \sqrt{3}i$ i.e. $\pm(1 + \sqrt{3}i)$.

Aliter

$$\sqrt{z} = \pm \left(\sqrt{\frac{|z| + a}{2}} + i \sqrt{\frac{|z| - a}{2}} \right)$$

$$= \pm \left(\sqrt{\frac{4 - 2}{2}} + i \sqrt{\frac{4 + 2}{2}} \right) = \pm (1 + i\sqrt{3})$$

43. As, $(a - b)^2 \geq 0, a^2 + b^2 \geq 2ab \quad \dots (i)$

But $|z| = \sqrt{a^2 + b^2}$

From Eq. (i), we get $|z|^2 \geq 2ab$

$$\therefore |z|^2 + a^2 + b^2 \geq a^2 + b^2 + 2ab$$

$$\Rightarrow |z|^2 + |z|^2 \geq (a + b)^2$$

$$\Rightarrow 2|z|^2 \geq (a + b)^2$$

$$\Rightarrow \sqrt{2}|z| \geq a + b, \text{ as } |z| \text{ is positive}$$

$$\therefore |z| \geq \frac{1}{\sqrt{2}}(a + b)$$

44. We have, $|1 - \bar{z}_1 z_2|^2 - |z_1 - z_2|^2$
 $= (1 - \bar{z}_1 z_2)(1 - \overline{\bar{z}_1 z_2}) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ [$\because z \bar{z} = |z|^2$]
 $= (1 - \bar{z}_1 z_2)(1 - \bar{\bar{z}_1} \bar{\bar{z}_2}) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$
 $[\because \bar{\bar{z}_1} = z_1 \text{ and } \bar{\bar{z}_2} = z_2]$
 $= (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$ [$\because \bar{\bar{z}_1} = z_1$]
 $= 1 - z_1 \bar{z}_2 - \bar{z}_1 z_2 + z_1 \bar{z}_1 z_2 \bar{z}_2 - z_1 \bar{z}_1 + z_1 \bar{z}_2 + \bar{z}_1 z_2 - z_2 \bar{z}_2$
 $= 1 + |z_1|^2 |z_2|^2 - |z_1|^2 - |z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$
 $\therefore \quad \quad \quad k=1$

45. Let $z = x + iy$
 We have, $z + \alpha |z - 1| + 2i = 0$
 $\Rightarrow x + i(y + 2) + \alpha \sqrt{(x - 1)^2 + y^2} = 0$
 On equating real and imaginary parts
 $y + 2 = 0 \Rightarrow y = -2$
 and $x + \alpha \sqrt{(x - 1)^2 + 4} = 0$
 $\therefore x^2 = \alpha^2 (x^2 - 2x + 5)$
 or $(1 - \alpha^2)x^2 + 2\alpha^2 x - 5\alpha^2 = 0$
 Since, x is real.
 $\therefore D = B^2 - 4AC \geq 0$
 $\Rightarrow 4\alpha^4 + 20\alpha^2(1 + \alpha^2) \geq 0$
 $\Rightarrow -4\alpha^4 + 5\alpha^2 \geq 0$
 $\Rightarrow 4\alpha^2 \left(\alpha^2 - \frac{5}{4} \right) \leq 0$
 $\therefore \frac{-\sqrt{5}}{2} \leq \alpha \leq \frac{\sqrt{5}}{2}$

46. Given, $|z| = 2$
 Let $\alpha = 5z - 1 \Rightarrow |\alpha + 1| = 5|z| = 5(2) = 10$
 $\Rightarrow |\alpha + 1| = 10$
 $\therefore$ Locus of α , i.e. $5z - 1$ is the circle having centre at -1 and radius 10.
 $\therefore a = 10$
 Again, $z_1^2 + z_2^2 - 2z_1 z_2 \cos \theta = 0$
 $\Rightarrow \left(\frac{z_1}{z_2} \right)^2 - 2 \frac{z_1}{z_2} \cos \theta + 1 = 0$
 $\Rightarrow \frac{z_1}{z_2} = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} = \cos \theta \pm i \sin \theta$
 $\Rightarrow \left| \frac{z_1}{z_2} \right| = |\cos \theta \pm i \sin \theta| = 1 = \frac{a}{10}$
 $\therefore |z_1| : |z_2| = a : 10$

47. We have, $|z| = \left| z + \frac{2}{z} - \frac{2}{z} \right| \leq \left| z + \frac{2}{z} \right| + \left| \frac{2}{z} \right|$
 $\Rightarrow |z| \leq 2 + \frac{2}{|z|} \Rightarrow |z|^2 \leq 2|z| + 2$
 $\Rightarrow |z|^2 - 2|z| + 1 \leq 1 + 2$
 $\Rightarrow (|z| - 1)^2 \leq 3 \Rightarrow -\sqrt{3} \leq |z| - 1 \leq \sqrt{3}$
 $\Rightarrow 1 - \sqrt{3} \leq |z| \leq 1 + \sqrt{3}$
 So, the maximum value of $|z|$ is $1 + \sqrt{3}$.

48. We have, $|z_1 + z_2| = \left| \frac{1}{z_1} + \frac{1}{z_2} \right| = \left| \frac{z_1 + z_2}{z_1 z_2} \right|$
 $\Rightarrow |z_1 + z_2| \left(1 - \frac{1}{|z_1 z_2|} \right) = 0$
 $\Rightarrow |z_1 z_2| = 1 \quad [\because z_1 \neq -z_2]$

49. Since, $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the n th roots of unity.
 $\therefore \sum_{k=0}^{n-1} \omega^k = 0$ and $\sum_{k=0}^{n-1} (\bar{\omega})^k = 0$
 Now, $\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 = \sum_{k=0}^{n-1} (z_1 + \omega^k z_2) \{ \bar{z}_1 + (\bar{\omega})^k \bar{z}_2 \}$
 $= \sum_{k=0}^{n-1} [z_1 \bar{z}_1 + z_1 \bar{z}_2 (\bar{\omega})^k + \bar{z}_1 z_2 \omega^k + z_2 \bar{z}_2 (\omega)^k (\bar{\omega})^k]$
 $= \sum_{k=0}^{n-1} |z_1|^2 + \sum_{k=0}^{n-1} z_1 \bar{z}_2 (\bar{\omega})^k + \sum_{k=0}^{n-1} \bar{z}_1 z_2 \omega^k + \sum_{k=0}^{n-1} |z_2|^2$
 $= n|z_1|^2 + 0 + 0 + n|z_2|^2 = n(|z_1|^2 + |z_2|^2)$

50. Let $z = x + iy$
 $\therefore |z - z| = 1 + 2i$
 $\Rightarrow \sqrt{x^2 + y^2} - (x + iy) = 1 + 2i$
 $\Rightarrow (\sqrt{x^2 + y^2} - x) + i(-y) = 1 + 2i$
 $\Rightarrow \sqrt{x^2 + y^2} - x = 1$
 and $y = -2$
 If $y = -2, \sqrt{x^2 + 4} - x = 1$
 $\Rightarrow x^2 + 4 = (1 + x)^2$
 $\Rightarrow 2x = 3$ or $x = \frac{3}{2}$
 $\therefore z = x + iy = \frac{3}{2} - 2i$

51. $(\sqrt{3} + i) = (a + ib)(c + id)$
 On taking argument both sides, we get
 $\tan^{-1} \frac{1}{\sqrt{3}} = \tan^{-1} \frac{b}{a} + \tan^{-1} \frac{d}{c}$
 $\Rightarrow \tan^{-1} \frac{b}{a} + \tan^{-1} \frac{d}{c} = n\pi + \frac{\pi}{6}, n \in \mathbb{Z}$
 52. Let $z - 1 = e^{i\theta}$ [$\because |z - 1| = 1$]
 $\Rightarrow z = 1 + \cos \theta + i \sin \theta$
 $= 2 \cos^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}$
 $= 2 \cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)$
 $\therefore \arg(z) = \frac{\theta}{2} = \frac{1}{2} \arg(z - 1)$ or $\arg(z - 1) = 2 \arg(z)$

53. $\arg(z^{2/3}) = \frac{2}{3} \arg(-1)$
 $= \frac{2}{3} \pi, \frac{2}{3} \cdot 3\pi, \frac{2}{3} \cdot 5\pi = \frac{2\pi}{3}, 2\pi, \frac{10\pi}{3}$

54. $\frac{z - z_1}{z - z_2} = \frac{(x + iy) - (8 + 4i)}{(x + iy) - (6 + 4i)} = \frac{(x - 8) + i(y - 4)}{(x - 6) + i(y - 4)}$
 We have, $\arg \left(\frac{z - z_1}{z - z_2} \right) = \frac{\pi}{4}$
 $\Rightarrow \arg(z - z_1) - \arg(z - z_2) = \frac{\pi}{4}$
 $\Rightarrow \tan^{-1} \frac{y - 4}{x - 8} - \tan^{-1} \frac{y - 4}{x - 6} = \frac{\pi}{4}$
 $\Rightarrow \frac{2(y - 4)}{x^2 + y^2 - 14x - 8y + 64} = 1$
 $\Rightarrow x^2 + y^2 - 14x - 10y + 72 = 0$
 $\Rightarrow (x - 7)^2 + (y - 5)^2 = (\sqrt{2})^2$
 $\therefore |z - (7 + 5i)| = \sqrt{2}$

55. $\arg(-z) = \pi - \theta = \pi + (-\theta) = \pi + \arg(z)$
 $\therefore \arg(-z) - \arg(z) = \pi$

56. Since, $|z - 1| = 1$

Let $z - 1 = \cos \theta + i \sin \theta$

Then, $z - 2 = \cos \theta + i \sin \theta - 1$

$$= -2 \sin^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2i \sin \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \quad \dots(i)$$

and

$$z = 1 + \cos \theta + i \sin \theta$$

$$= 2 \cos^2 \frac{\theta}{2} + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2}$$

$$= 2 \cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right) \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{z-2}{z} = i \tan \frac{\theta}{2} = i \tan(\arg z) \left[\because \arg z = \frac{\theta}{2}, \text{ from Eq. (ii)} \right]$$

57. We have, $C^2 + S^2 = 1$

Let $C = \cos \theta$, $S = \sin \theta$

$$\therefore \frac{1+C+iS}{1+C-iS} = \frac{1+\cos \theta + i \sin \theta}{1+\cos \theta - i \sin \theta}$$

$$= \frac{\cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)}{\cos \frac{\theta}{2} \left(\cos \frac{\theta}{2} - i \sin \frac{\theta}{2} \right)}$$

$$= \frac{\left(\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right)^2}{1} = \cos \theta + i \sin \theta$$

$$= C + iS$$

58. Let $z - 1 = r(\cos \theta + i \sin \theta) = re^{i\theta}$

$$\therefore \text{Given expression} = re^{i\theta} \cdot e^{-i\alpha} + \frac{1}{re^{i\theta}} e^{i\alpha}$$

$$= re^{i(\theta-\alpha)} + \frac{1}{r} e^{-i(\theta-\alpha)}$$

Since, imaginary part of given expression is zero, we have

$$r \sin(\theta - \alpha) - \frac{1}{r} \sin(\theta - \alpha) = 0$$

$$r^2 - 1 = 0 \Rightarrow r^2 = 1$$

$$\Rightarrow r = 1 \Rightarrow |z - 1| = 1$$

$$\text{or } \sin(\theta - \alpha) = 0 \Rightarrow (\theta - \alpha) = 0$$

$$\Rightarrow \theta = \alpha \Rightarrow \arg(z - 1) = \alpha$$

59. $\frac{\left(\sin \frac{\pi}{8} + i \cos \frac{\pi}{8} \right)^8}{\left(\sin \frac{\pi}{8} - i \cos \frac{\pi}{8} \right)^8} = \left(\cos \frac{\pi}{8} + i \sin \frac{\pi}{8} \right)^{-16}$

$$= \cos 2\pi - i \sin 2\pi = 1$$

60. $\frac{1+a}{2} = \frac{1}{2} \left(1 + \cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right)$

$$= \frac{1}{2} \cdot 2 \cos \frac{2\pi}{3} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$= -\frac{1}{2} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)$$

$$\therefore \left(\frac{1+a}{2} \right)^{3n} = \left(-\frac{1}{2} \right)^{3n} \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right)^{3n} = \frac{(-1)^n}{2^{3n}}$$

61. We have, $\left(\frac{1+i}{\sqrt{2}} \right)^8 + \left(\frac{1-i}{\sqrt{2}} \right)^8$

$$= \left[\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]^8 + \left[\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right]^8$$

$$= \cos 2\pi + i \sin 2\pi + \cos 2\pi - i \sin 2\pi$$

$$= 2 \cos 2\pi = 2(1) = 2 \quad [\text{using De-Moivre's theorem}]$$

62. We have, $\sum_{k=1}^{10} \left(\sin \frac{2\pi k}{11} - i \cos \frac{2\pi k}{11} \right)$

$$= \sum_{k=1}^{10} \left(-i^2 \sin \frac{2\pi k}{11} - i \cos \frac{2\pi k}{11} \right)$$

$$= -i \sum_{k=1}^{10} \left(\cos \frac{2\pi k}{11} + i \sin \frac{2\pi k}{11} \right) = -i \sum_{k=1}^{10} e^{i \frac{2\pi k}{11}}$$

$$= -i \left[\sum_{k=1}^{10} e^{i \frac{2\pi k}{11}} - 1 \right]$$

$$= -i (\text{Sum of 11th roots of unity} - 1)$$

$$= -i(0 - 1) = i$$

63. We have, $\frac{4(\cos 75^\circ + i \sin 75^\circ)}{0.4(\cos 30^\circ + i \sin 30^\circ)}$

$$= \frac{10 \{ \cos(75^\circ - 30^\circ) + i \sin(75^\circ - 30^\circ) \}}{\cos^2 30^\circ + \sin^2 30^\circ}$$

$$= 10(\cos 45^\circ + i \sin 45^\circ) = \frac{10}{\sqrt{2}}(1 + i)$$

64. Using De-Moivre's theorem, the given expression

$$= \frac{(\cos \theta + i \sin \theta)^{-14} (\cos \theta + i \sin \theta)^{-15}}{(\cos \theta + i \sin \theta)^{48} (\cos \theta + i \sin \theta)^{-30}}$$

$$= (\cos \theta + i \sin \theta)^{-47} = \cos 47\theta - i \sin 47\theta$$

65. We have,

$$\sin \theta - i \cos \theta = -i^2 \sin \theta - i \cos \theta$$

$$= -i(\cos \theta + i \sin \theta)$$

$\therefore$ The given expression, using De-Moivre's theorem

$$= (-i)^3 [\cos(-25\theta) + i \sin(-25\theta)]$$

$$= i [\cos 25\theta - i \sin 25\theta]$$

$$= \sin 25\theta + i \cos 25\theta$$

66. We have, $z = \frac{(1 + \cos \theta + i \sin \theta)^5}{(\cos \theta + i \sin \theta)^3}$

$$= \frac{\left\{ 1 + 2 \cos^2 \frac{\theta}{2} - 1 + 2i \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right\}^5}{\cos 3\theta + i \sin 3\theta}$$

$$= \frac{32 \cos^5 \frac{\theta}{2} \left\{ \cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right\}^5}{\cos 3\theta + i \sin 3\theta}$$

$$= 32 \cos^5 \frac{\theta}{2} \left\{ \cos \frac{5\theta}{2} + i \sin \frac{5\theta}{2} \right\} \{ \cos 3\theta - i \sin 3\theta \}$$

$$= 32 \cos^5 \frac{\theta}{2} \left[\cos \left\{ \frac{5\theta}{2} - 3\theta \right\} + i \sin \left\{ \frac{5\theta}{2} - 3\theta \right\} \right]$$

$$= 32 \cos^5 \frac{\theta}{2} \left[\cos \left(-\frac{\theta}{2} \right) + i \sin \left(-\frac{\theta}{2} \right) \right]$$

Hence, the modulus and argument of z are $32 \left[\cos^5 \frac{\theta}{2} \right]$ and $\left(-\frac{\theta}{2} \right)$, respectively.

67. We have, $\frac{1}{z} = \frac{1}{\cos \theta + i \sin \theta} = \cos \theta - i \sin \theta$
 $\therefore z^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$,
 and $\frac{1}{z^n} = (\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$
 Hence, $z^n + \frac{1}{z^n} = 2 \cos n\theta$ and $z^n - \frac{1}{z^n} = 2i \sin n\theta$

68. We have,
 $\frac{z^{2n} - 1}{z^{2n} + 1} = \frac{(\cos \theta + i \sin \theta)^{2n} - 1}{(\cos \theta + i \sin \theta)^{2n} + 1}$
 $= \frac{\cos 2n\theta + i \sin 2n\theta - 1}{\cos 2n\theta + i \sin 2n\theta + 1}$
 [using De-Moivre's theorem]
 $= \frac{(1 - 2 \sin^2 n\theta) + 2i \sin n\theta \cos n\theta - 1}{(2 \cos^2 n\theta - 1) + 2i \sin n\theta \cos n\theta + 1}$
 $= \frac{i \sin n\theta \cos n\theta + i^2 \sin^2 n\theta}{\cos^2 n\theta + i \sin n\theta \cos n\theta} \quad [\because i^2 = -1]$
 $= \frac{i \sin n\theta (\cos n\theta + i \sin n\theta)}{\cos n\theta (\cos n\theta + i \sin n\theta)} = i \tan n\theta$

69. We have, $abcd = \cos (2\alpha + 2\beta + 2\gamma + 2\delta)$
 $+ i \sin (2\alpha + 2\beta + 2\gamma + 2\delta)$
 $\therefore \sqrt{abcd} = [\cos (2\alpha + 2\beta + 2\gamma + 2\delta)$
 $+ i \sin (2\alpha + 2\beta + 2\gamma + 2\delta)]^{1/2}$
 or $\sqrt{abcd} = \cos (\alpha + \beta + \gamma + \delta)$
 $+ i \sin (\alpha + \beta + \gamma + \delta) \dots (i)$
 [using De-Moivre's theorem]
 $\therefore \frac{1}{\sqrt{abcd}} = \cos (\alpha + \beta + \gamma + \delta)$
 $- i \sin (\alpha + \beta + \gamma + \delta) \dots (ii)$
 On adding Eqs. (i) and (ii), we get
 $\sqrt{abcd} + \frac{1}{\sqrt{abcd}} = 2 \cos (\alpha + \beta + \gamma + \delta)$

70. Let $\cot^{-1} p = \theta$, then $p = \cot \theta$
 $\therefore e^{2mi \cot^{-1} p} \cdot \left(\frac{pi + 1}{pi - 1} \right)^m = e^{2mi\theta} \cdot \left(\frac{i \cot \theta + 1}{i \cot \theta - 1} \right)^m$
 $= e^{2mi\theta} \cdot \left(\frac{i (\cot \theta - i)}{i (\cot \theta + i)} \right)^m = e^{2mi\theta} \cdot \left(\frac{\cot \theta - i}{\cot \theta + i} \right)^m$
 $= e^{2mi\theta} \cdot \left(\frac{\cos \theta - i \sin \theta}{\cos \theta + i \sin \theta} \right)^m = e^{2mi\theta} \cdot \left(\frac{e^{-i\theta}}{e^{i\theta}} \right)^m$
 $= e^{2mi\theta} (e^{-2i\theta})^m = e^0 = 1$

71. We have, $\alpha = \cos \left(\frac{8\pi}{11} \right) + i \sin \left(\frac{8\pi}{11} \right) = e^{\frac{8\pi i}{11}}$
 $\text{Re} (\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$
 $= \frac{\alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5}{2}$
 $= \frac{[-1 + (1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5)$
 $+ \bar{\alpha} + \bar{\alpha}^2 + \bar{\alpha}^3 + \bar{\alpha}^4 + \bar{\alpha}^5]}{2}$
 $= \frac{-1 + 0}{2} \quad [\text{sum of 11 and 11th roots of unity}]$
 $= -\frac{1}{2}$

72. We have, $\frac{1}{a + \omega} + \frac{1}{b + \omega} + \frac{1}{c + \omega} = 2\omega^2 = \frac{2}{\omega}$
 and $\frac{1}{a + \omega^2} + \frac{1}{b + \omega^2} + \frac{1}{c + \omega^2} = 2\omega = \frac{2}{\omega}$
 $\therefore \omega$ and ω^2 are roots of the equation
 $\frac{1}{a + x} + \frac{1}{b + x} + \frac{1}{c + x} = \frac{2}{x}$

When $x = 1$,
 $\frac{1}{a + 1} + \frac{1}{b + 1} + \frac{1}{c + 1} = \frac{2}{1} = 2$

73. $x^2 - 2x + 4 = 0 \Rightarrow x = 1 + \sqrt{3}i = 2 \left(\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \right)$
 $= -2\omega - 2\omega^2 \Rightarrow \alpha^6 + \beta^6 = 128$

74. $z + \frac{1}{z} = 1$
 $\Rightarrow z = -\omega \quad \text{or} \quad -\omega^2$
 $\therefore z^{99} + \frac{1}{z^{99}} = (-1) + \frac{1}{(-1)} = -2$

75. $(-1 + \sqrt{-3})^{62} + (-1 - \sqrt{-3})^{62}$
 $= 2^{62} \omega^{62} + 2^{62} (\omega^2)^{62}$
 $= 2^{62} [(\omega^3)^{20} \omega^2 + (\omega^3)^{41} \omega]$
 $= 2^{62} [(\omega^2 + \omega)] = -2^{62}$

76. Given expression
 $= \{3(1 + \omega^2) + 5\omega\}^2 + \{3(1 + \omega) + 5\omega^2\}^2$
 $= (-3\omega + 5\omega)^2 + (-3\omega^2 + 5\omega^2)^2$
 $= 4\omega^2 + 4\omega = -4$

77. $\left(\frac{\omega^2}{\omega} \right)^6 + \left(\frac{\omega}{\omega^2} \right)^6 = 2$, using $\omega = \frac{-1 + i\sqrt{3}}{2}$ and
 $\omega^2 = \frac{-1 - i\sqrt{3}}{2}$

78. Here, $\left(\frac{z_1}{z_2} \right)^2 = \frac{-3i\omega}{2}$
 $\Rightarrow \left(\frac{z_1}{z_2} \right)^{50} = \left[\left(\frac{z_1}{z_2} \right)^2 \right]^{25} = \left(\frac{-3i\omega}{2} \right)^{25} = -\left(\frac{3}{2} \right)^{25} i\omega$
 $= \left(\frac{3}{2} \right)^{25} \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right)$
 $\therefore \left(\frac{z_1}{z_2} \right)^{50}$ lies in I quadrant.

79. $\therefore xyz = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$
 $= (a + b)(a^2 - ab + b^2) = a^3 + b^3$

80. $(1 + \omega)^3 - (1 + \omega^2)^3 = (-\omega^2)^3 - (-\omega)^3$
 $= -\omega^6 + \omega^3 = -1 + 1 = 0$

81. $p^{1/3} = (-p)^{1/3} (-1)^{1/3}$
 $= -(-p)^{1/3}, -(-p)^{1/3} \omega, -(-p)^{1/3} \omega^2$
 $= -q, -q\omega, -q\omega^2$, where $q = (-p)^{1/3}$
 Let $\alpha = -q, \beta = -q\omega$ and $\gamma = -q\omega^2$
 $\therefore \frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha} = \frac{-q(x + y\omega + z\omega^2)}{-q(x\omega + y\omega^2 + z)}$
 $= \frac{x\omega^3 + y\omega^4 + z\omega^2}{x\omega + y\omega^2 + z} = \omega^2$

82. Given expression

$$\begin{aligned} &= \sum_{k=2}^n (k-1)(k-\omega)(k-\omega^2) \\ &= \sum_{k=2}^n (k-1)(k^2+k+1) = \sum_{k=2}^n (k^3-1) \\ &= (2^3+3^3+\dots+n^3)-(n-1) \\ &= (\Sigma n^3-1^3)-n+1 \\ &= \left(\frac{n(n+1)}{2}\right)^2 - n \end{aligned}$$

83. Let $\alpha = \omega$ and $\beta = \omega^2$

$$\begin{aligned} \text{Now, } \alpha^3 + \beta^3 + \alpha^{-2}\beta^{-2} &= \omega^3 + \omega^6 + \frac{1}{\omega^2} \cdot \frac{1}{\omega^4} \\ &= \omega^3 + (\omega^3)^2 + \frac{1}{(\omega^3)^2} = 1 + (1)^2 + \frac{1}{(1)^2} = 3 \quad [\because \omega^3 = 1] \end{aligned}$$

84. We have,

$$\begin{aligned} &(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16}) \\ &= (1+\omega^2-\omega)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega) \\ &\quad [\because \omega^4 = \omega^3 \cdot \omega = \omega; \omega^8 = (\omega^3)^2 \cdot \omega^2; \\ &\quad \omega^{16} = (\omega^3)^5 \cdot \omega = \omega \text{ and } \omega^3 = 1] \\ &= (-\omega-\omega)(-\omega^2-\omega^2)(-\omega-\omega)(-\omega^2-\omega^2) \\ &= (-2\omega)(-2\omega^2)(-2\omega)(-2\omega^2) = 16 \cdot \omega^6 \\ &= 16(\omega^3)^2 \\ &= 16(1)^2 = 16 \end{aligned}$$

85. We have,

$$\begin{aligned} x &= \omega - \omega^2 - 2 \\ \text{or } x+2 &= \omega - \omega^2 \\ \text{On squaring, } x^2+4x+4 &= \omega^2 + \omega^4 - 2\omega^3 \\ &= \omega^2 + \omega^3 \cdot \omega - 2\omega^3 \\ &= \omega^2 + \omega - 2 \quad [\because \omega^2 = 1] \\ &= -1-2 \\ &= -3 \\ \Rightarrow x^2+4x+7 &= 0 \\ \text{On dividing } x^4+3x^3+2x^2-11x-6 &\text{ by } x^2+4x+7, \\ \text{we get} \\ x^4+3x^3+2x^2-11x-6 &= (x^2+4x+7)(x^2-x-1)+1 \\ &= (0)(x^2-x-1)+1 \\ &= 0+1=1 \end{aligned}$$

86. We have,

$$\begin{aligned} &(1+\omega)(1+\omega)^2(1+\omega^4)(1+\omega^8)\dots \text{ to } 2n \text{ factors} \\ &= (1+\omega)(1+\omega^2)(1+\omega^3 \cdot \omega)(1+\omega^6 \cdot \omega^2)\dots \text{ to } 2n \text{ factors} \\ &= (1+\omega)(1+\omega^2)(1+\omega)(1+\omega^2)\dots \text{ to } 2n \text{ factors} \\ &\quad [\because \omega^3 = \omega^6 = \dots = 1] \\ &= [(1+\omega)(1+\omega)\dots \text{ to } n \text{ factors}] \\ &\quad [(1+\omega^2)(1+\omega^2)\dots \text{ to } n \text{ factors}] \\ &= (1+\omega)^n (1+\omega^2)^n = [(1+\omega)(1+\omega^2)]^n \\ &= (1+\omega+\omega^2+\omega^3)^n = (0+1)^n = 1 \\ &\quad [\because 1+\omega+\omega^2 = 0, \omega^3 = 1] \end{aligned}$$

87. We have, $\alpha = \omega$ and $\beta = \omega^2$

$$\begin{aligned} \text{Then, } xyz &= (a+b+c)(a\omega+b\omega^2+c)(a\omega^2+b\omega+c) \\ &= (a+b+c)(a^2+b^2+c^2-ab-bc-ca) \\ &= a^3+b^3+c^3-3abc \end{aligned}$$

88. We have, $z^3+2z^2+2z+1=0$

$$\Rightarrow (z+1)(z^2+z+1)=0$$

Its roots are $-1, \omega$ and ω^2 . The root $z = -1$ does not satisfy the equation, $z^{1985} + z^{100} + 1 = 0$ but $z = \omega$ and $z = \omega^2$ satisfy it.

Hence, ω and ω^2 are the common roots.

89. We have, $(16)^{1/4} = (2^4)^{1/4} = 2(1)^{1/4}$

$$= 2(\cos 0 + i \sin 0)^{1/4}$$

$$= 2 \left\{ \cos \frac{1}{4}(2k\pi + 0) + i \sin \frac{1}{4}(2k\pi + 0) \right\}, k = 0, 1, 2, 3$$

$$= 2 \times 1, 2 \times i, 2 \times -1, 2 \times -i, = \pm 2, \pm 2i$$

90. We have, $f(\omega) = \omega^{3p} + \omega^{3q+1} + \omega^{3r+2}$

$$\begin{aligned} &= \omega^{3p} + \omega^{3q} \cdot \omega + \omega^{3r} \cdot \omega^2 \\ &= (\omega^3)^p + (\omega^3)^q \cdot \omega + (\omega^3)^r \cdot \omega^2 \\ &= 1 + \omega + \omega^2 = 0 \quad [\because \omega^3 = 1] \end{aligned}$$

91. Since $1, \alpha, \alpha^2, \dots, \alpha^{n-1}$ are n and n th roots of unity.

$$\therefore x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)\dots(x-\alpha^{n-1})$$

$$\Rightarrow \log(x^n - 1) = \log(x-1) + \log(x-\alpha) + \dots + \log(x-\alpha^{n-1})$$

On differentiating both sides w.r.t. x , we get

$$\frac{nx^{n-1}}{x^n - 1} = \frac{1}{x-1} + \frac{1}{x-\alpha} + \frac{1}{x-\alpha^2} + \dots + \frac{1}{x-\alpha^{n-1}}$$

Putting $x=2$, we get

$$\frac{n2^{n-1}}{2^n - 1} = \frac{1}{1} + \frac{1}{2-\alpha} + \frac{1}{2-\alpha^2} + \dots + \frac{1}{2-\alpha^{n-1}}$$

$$\therefore \frac{n \cdot 2^{n-1}}{2^n - 1} - 1 = \sum_{i=1}^{n-1} \frac{1}{2-\alpha^i}$$

$$\begin{aligned} \text{Hence, } \sum_{i=1}^{n-1} \frac{1}{2-\alpha^i} &= \frac{n \cdot 2^{n-1} - 2^n + 1}{2^n - 1} \\ &= \frac{(n-2)2^{n-1} + 1}{2^n - 1} \end{aligned}$$

92. The vertices of the triangle are $A(1, 0), B(1/\sqrt{2}, 1/\sqrt{2})$ and $C(0, 1)$.

$$\begin{aligned} \therefore AB^2 &= 2 - \sqrt{2} \\ BC^2 &= 2 - \sqrt{2} \\ AC^2 &= 1 + 1 = 2 \\ AB &= BC \end{aligned}$$

93. By definition,

$z_1 = x_1 + iy_1, z_2 = x_2 + iy_2$ and $z_3 = x_3 + iy_3$ are collinear, if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} a & b & 1 \\ a' & b' & 1 \\ a-a' & b-b' & 1 \end{vmatrix} = 0$$

$$\Rightarrow ab' = a'b$$

94. Let $z = x + iy$, where $x, y \in \mathbb{R}$

$$\therefore \alpha + i\beta = \tan^{-1}(x + iy)$$

$$\Rightarrow \alpha - i\beta = \tan^{-1}(x - iy)$$

$$\begin{aligned} \Rightarrow 2\alpha &= \tan^{-1}(x + iy) + \tan^{-1}(x - iy) \\ &= \tan^{-1} \frac{2x}{1 - x^2 - y^2} \end{aligned}$$

$$\therefore x^2 + y^2 + 2x \cot 2\alpha = 1$$

95. $|z - z_1| = |z - z_2|$ represents a perpendicular bisector.

96. We have, $(z - 1)^n = i(z + 1)^n$

$$\begin{aligned} \therefore |z - 1|^n &= |i| |z + 1|^n \\ \Rightarrow |z - 1|^n &= |z + 1|^n \\ \Rightarrow |z - 1| &= |z + 1| \\ \Rightarrow |z - 1|^2 &= |z + 1|^2 \\ \Rightarrow (x - 1)^2 + y^2 &= (x + 1)^2 + y^2 \\ \Rightarrow 4x &= 0 \Rightarrow x = 0 \end{aligned}$$

$\therefore$ The roots lies on the Y-axis.

97. Let $z = x + iy$, where $x, y \in R$

$$\begin{aligned} \text{We have, } |iz - 1| + |z - i| &= 2 \\ \Rightarrow |ix - y - 1| + |x + iy - i| &= 2 \\ \Rightarrow \sqrt{(-y - 1)^2 + x^2} + \sqrt{x^2 + (y - 1)^2} &= 2 \\ \Rightarrow x^2 &= 0 \text{ or } x = 0 \end{aligned}$$

$\therefore$ The locus of z is a straight line.

98. Clearly, $|z_1| < |z_3| < |z_2|$, as z_3 is mid-point of z_1 and z_2 .

99. (a) $\operatorname{Re} \left(\frac{1+z}{1-z} \right) = 0 \Rightarrow \arg \left(\frac{1+z}{1-z} \right) = \frac{\pi}{2}$

This represents a circle.

(b) $z\bar{z} + iz - \bar{z} + 1 = 0$ represent a circle.

(c) $\arg \left(\frac{z-1}{z+1} \right) = \frac{\pi}{2}$

This represents a circle.

(d) $\left| \frac{z-1}{z+1} \right| = 1 \Rightarrow |z-1| = |z+1|$

This represents a straight line.

100. We have, $z_2 = \frac{z_1 + z_3}{2}$

The point representing z_2 divided the line segment joining points representing z_1 and z_3 in the ratio 1:1.
 $\therefore$ The points lies on a line.

101. Area of the triangle on the argand plane formed by complex numbers $-z, iz, z - iz$ is $\frac{3}{2} |z|^2$.

$$\therefore \frac{3}{2} |z|^2 = 600 \Rightarrow |z| = 20$$

102. Let $z_1 = 1 - 3i, z_2 = 4 + 3i$ and $z_3 = 3 + i$. Then, z_3 divides the line segment joining z_1 and z_2 in the ratio 2:1 internally. So, the points z_1, z_2 and z_3 are collinear.

103. Let $z = x + iy$

$$\begin{aligned} \text{Then, } \frac{z+2i}{z+4} &= \frac{x+iy+2i}{x+iy+4} = \frac{x+i(y+2)}{(x+4)+iy} \\ &= \frac{[x+i(y+2)][(x+4)-iy]}{(x+4)^2+y^2} \\ &= \frac{(x^2+4x+y^2+2y)+i(2x+4y+8)}{(x+4)^2+y^2} \end{aligned}$$

$$\text{Since, } \operatorname{Re} \left(\frac{z+2i}{z+4} \right) = 0 \Rightarrow x^2 + y^2 + 4x + 2y = 0,$$

which represents a circle with centre $(-2, -1)$.

104. Let $z = x + iy$

$$\text{Then, } \frac{z+2i}{z+2} = \frac{x+iy+2i}{x+iy+2} = \frac{x+(y+2)i}{(x+2)+iy}$$

$$\begin{aligned} &= \frac{[x+(y+2)i][(x+2)-iy]}{(x+2)^2+y^2} \\ &= \frac{(x^2+y^2+2x+2y)+i(2x+2y+4)}{(x+2)^2+y^2} \end{aligned}$$

$$\text{Since, } \operatorname{Im} \left(\frac{z+2i}{z+2} \right) = 0 \Rightarrow x + y + 2 = 0$$

which represents a straight line.

105. We have,

$$\begin{aligned} |z - 4i| + |z + 4i| &= 10 \\ \Rightarrow |z - (0 + 4i)| + |z - (0 - 4i)| &= 10 \end{aligned}$$

This represents an ellipse.

$$|4i + 4i| < 10 \text{ i.e. } 10 > 8$$

106. We have, $|z - ai| = |z + ai|$

$$\begin{aligned} \Rightarrow |x + i(y-a)|^2 &= |x + i(y+a)|^2 \\ \Rightarrow x^2 + (y-a)^2 &= x^2 + (y+a)^2 \\ \Rightarrow 4ay &= 0; y = 0, \text{ which is X-axis.} \end{aligned}$$

107. We have, $|z-1|^2 + |z+1|^2 = 4$

$$\begin{aligned} \Rightarrow |(x-1)+iy|^2 + |(x+1)+iy|^2 &= 4 \\ \Rightarrow (x-1)^2 + y^2 + (x+1)^2 + y^2 &= 4 \\ \Rightarrow 2(x^2+1) + 2y^2 &= 4 \\ \therefore \text{The locus of } P \text{ is } x^2 + y^2 &= 1. \end{aligned}$$

108. We have, $a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 3$

$$\begin{aligned} \Rightarrow |3| &= |a_1 z^3 + a_2 z^2 + a_3 z + a_4| \\ \Rightarrow 3 &\leq |a_1 z^3| + |a_2 z^2| + |a_3 z| + |a_4| \\ \Rightarrow 3 &\leq |a_1| |z|^3 + |a_2| |z|^2 + |a_3| |z| + |a_4| \\ \Rightarrow 3 &\leq |z|^3 + |z|^2 + |z| + 1 \quad [\because |a_i| \leq 1] \\ \Rightarrow 3 &\leq 1 + |z| + |z|^2 + |z|^3 < 1 + |z| + |z|^2 + |z|^3 + \dots \infty \\ \Rightarrow 3 &< 1 + |z| + |z|^2 + |z|^3 + \dots \infty \\ \Rightarrow 3 &< \frac{1}{1-|z|} \Rightarrow 1-|z| < \frac{1}{3} \\ \Rightarrow \frac{2}{3} - |z| &< 0 \\ \therefore |z| &> \frac{2}{3} \end{aligned}$$

109. We have, $|z-1| + |z+1| \leq 4$

$$\begin{aligned} \Rightarrow \sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2} &\leq 4 \\ \text{where, } z &= x + iy \\ \Rightarrow &= A + B \leq 4 \quad \dots(i) \end{aligned}$$

$$\text{where, } A = \sqrt{(x-1)^2 + y^2}$$

$$\text{and } B = \sqrt{(x+1)^2 + y^2}$$

$$\begin{aligned} \text{But } [(x-1)^2 + y^2] - [(x+1)^2 + y^2] &= -4x \\ \text{i.e. } A^2 - B^2 &= -4x \quad \dots(ii) \end{aligned}$$

On dividing Eq. (ii) by Eq. (i), we get

$$\begin{aligned} \frac{\sqrt{(x-1)^2 + y^2} - \sqrt{(x+1)^2 + y^2}}{\sqrt{(x-1)^2 + y^2} + \sqrt{(x+1)^2 + y^2}} &\leq -x \\ \Rightarrow 2\sqrt{(x-1)^2 + y^2} &\leq 4 - x \\ \Rightarrow 3x^2 + 4y^2 &\leq 12 \end{aligned}$$

[squaring and simplifying]

$$\text{or } \frac{x^2}{4} + \frac{y^2}{3} \leq 1$$

which represents the interior and boundary of an ellipse.

110. We have, $\log_{1/2} \left(\frac{|z-1|+4}{3|z-1|-2} \right) > 1 = \log_{1/2} \left(\frac{1}{2} \right)$

$$\Rightarrow \frac{|z-1|+4}{3|z-1|-2} < \frac{1}{2} < 1$$

$[\because \log_a x$ is a decreasing function, if $a < 1$]

$$\Rightarrow |z-1|+4 < 3|z-1|-2$$

$$\Rightarrow 2|z-1| > 6 \Rightarrow |z-1| > 3$$

which is an exterior of a circle.

111. The closest distance = length of the perpendicular from the origin on the line $a\bar{z} + \bar{a}z + a\bar{a} = 0$

$$= \frac{a(0) + \bar{a}|0| + a\bar{a}}{2|a|} = \frac{|a|^2}{2|a|} = \frac{|a|}{2}$$

112. We have, $z_k = 1 + a + a^2 + \dots + a^k = \frac{1-a^{k+1}}{1-a}$

$$\Rightarrow z_k = \frac{1}{1-a} - \frac{a^{k+1}}{1-a}$$

$$\Rightarrow \left| z_k - \frac{1}{1-a} \right| = \frac{|a|^{k+1}}{|1-a|} < \frac{1}{|1-a|} \quad [\because |a| < 1]$$

$\therefore$ Vertices of the polygon $z_1, z_2, \dots, z_n$ lie within the circle

$$\left| z - \frac{1}{1-a} \right| = \frac{1}{|1-a|}$$

113. We have, $z_1 - z_4 = z_2 - z_3$ or $\frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$ i.e. the diagonals bisect each other.

$\therefore$ It is a parallelogram.

Also, $\arg \left(\frac{z_4 - z_1}{z_2 - z_1} \right) = \frac{\pi}{2}$

$\Rightarrow$ Angle at z_1 is a right angle.

$\therefore$ It is a rectangle and hence, a cyclic quadrilateral.

114. We have, $\text{Im}(z^2) = 4$

$$\Rightarrow \text{Im}[(x^2 - y^2) + 2ixy] = 4 \quad [\text{putting } z = x + iy]$$

$\Rightarrow 2xy = 4$ or $xy = 2$, which is a rectangular hyperbola.

115. We have, $\text{Re}(z^2) = 4$

$$\Rightarrow \text{Re}[(x^2 - y^2) + 2ixy] = 4 \quad [\text{putting } z = x + iy]$$

$$\Rightarrow x^2 - y^2 = 4$$

which is a rectangular hyperbola.

116. $|z - z_1| = |z - z_2| = |z - z_3|$

$\therefore z$ must be the mid-point of z_1 and z_2 and since $|z - z_1| = |z - z_2|$ and z, z_1 and z_2 are collinear.

$\Rightarrow z$ is circumcentre of Δ formed by z_1, z_2 and z_3 .

$$\therefore \arg \left(\frac{z_3 - z_1}{z_3 - z_2} \right) = \pm \frac{\pi}{2} \quad [\text{neglecting -ve value}]$$

117. $a_0 z^4 + a_1 z^3 + a_2 z^2 + a_3 z + a_4 = 0$

$$\Rightarrow a_0 \bar{z}^4 + a_1 \bar{z}^3 + a_2 \bar{z}^2 + a_3 \bar{z} + a_4 = 0$$

[taking conjugate on both sides]

$$\Rightarrow a_0 (\bar{z}^4) + a_1 (\bar{z}^3) + a_2 (\bar{z}^2) + a_3 (\bar{z}) + a_4 = 0$$

$\therefore \bar{z}$ is a root of the equation if z is a root. So, option (a) is correct.

Also, if z_1 is real, $z_1 = \bar{z}_1$.

If z_1 is non-real complex, then $\bar{z}_1$ is also a root because imaginary root occurs in conjugate pairs. So, option (b) is also correct.

118. Let $z = \alpha$ be a real root. Then,

$$\alpha^3 + (3+2i)\alpha + (-1+ia) = 0$$

$$\Rightarrow (\alpha^3 + 3\alpha - 1) - i(a + 2\alpha) = 0$$

$$\Rightarrow \alpha^3 + 3\alpha - 1 = 0$$

and $\alpha = -a/2$

$$\Rightarrow -\frac{a^3}{8} - \frac{3a}{2} - 1 = 0$$

$$\Rightarrow a^3 + 12a + 8 = 0$$

$$\text{Let } f(a) = a^3 + 12a + 8$$

$$\therefore f(-1) < 0, f(0) > 0, f(-2) < 0, f(1) > 0 \text{ and } f(3) = 0$$

119. We have,

$$\frac{1+i\cos\theta}{1-2i\cos\theta} = \frac{(1+i\cos\theta)(1+2i\cos\theta)}{(1-2i\cos\theta)(1+2i\cos\theta)} = \frac{(1-2\cos^2\theta) + i3\cos\theta}{1+4\cos^2\theta}$$

Thus, $\frac{(1+i\cos\theta)}{(1-2i\cos\theta)}$ is a real number, if $\cos\theta = 0$

$$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{2}$$

where, n is an integer.

120. Let $z = c$ be a real root.

$$\text{Then, } \alpha c^2 + c + \bar{\alpha} = 0 \quad \dots(i)$$

$$\text{Let } \alpha = p + iq$$

$$\text{Then, } (p+iq)c^2 + c + p - iq = 0$$

$$\Rightarrow pc^2 + c + p = 0$$

$$\text{and } qc^2 - q = 0$$

$$\Rightarrow c = \pm 1 \quad [\because q \neq 0]$$

From Eq. (i), we get $\alpha \pm 1 + \bar{\alpha} = 0$

$$\text{Also, } |c| = 1$$

121. Let $z_2 = r_2 (\cos\theta_2 + i\sin\theta_2)$

$$\Rightarrow |z_2| = r_2$$

$$\text{Also, } \arg(z_1) + \arg(z_2) = 0$$

$$\Rightarrow \arg(z_1) = -\arg(z_2) = -\theta_2$$

$$\therefore z_1 = r_2 [\cos(-\theta_2) + i\sin(-\theta_2)]$$

$$= r_2 [\cos\theta_2 - i\sin\theta_2]$$

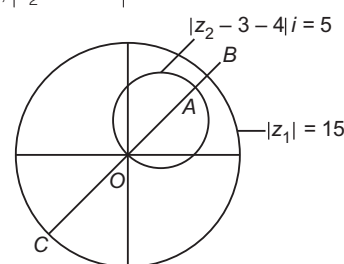
$$\Rightarrow z_1 = \bar{z}_2$$

$$\Rightarrow z_1 = \frac{1}{z_2}$$

$$\therefore z_1 z_2 = 1$$

122. We have,

$$|z_1| = 15, |z_2 - 3 - 4i| = 5$$



Minimum value of $|z_1 - z_2|$,

$$AB = OB - OA = 15 - 10 = 5$$

Maximum value of $|z_1 - z_2|$,

$$\begin{aligned} CA &= OA + OC \\ &= 10 + 15 \\ &= 25 \end{aligned}$$

123. Given, $\left|z - \frac{1}{z}\right| = 1$

$$\Rightarrow 1 \geq \left|z - \frac{1}{z}\right| \Rightarrow -1 \leq z - \frac{1}{z} \leq 1$$

$$\Rightarrow -|z| \leq |z|^2 - 1 \leq |z|$$

From $|z|^2 - 1 \geq |z|$, we get

$$|z|^2 + |z| - 1 \geq 0$$

$$\Rightarrow |z| \geq \frac{-1 + \sqrt{5}}{2} \quad \dots(i)$$

From $|z|^2 - 1 \leq |z|$, we get

$$|z|^2 - |z| - 1 \leq 0$$

$$\Rightarrow \frac{1 - \sqrt{5}}{2} \leq |z| \leq \frac{1 + \sqrt{5}}{2} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\Rightarrow \frac{-1 + \sqrt{5}}{2} \leq |z| \leq \frac{1 + \sqrt{5}}{2}$$

$$\Rightarrow |z|_{\min} = \frac{\sqrt{5} - 1}{2}, |z|_{\max} = \frac{1 + \sqrt{5}}{2}$$

124. $|z_1^2 - z_2^2| = |\bar{z}_1^2 - \bar{z}_2^2 - 2\bar{z}_1\bar{z}_2|$

$$\Rightarrow |z_1 - z_2||z_1 + z_2| = |\bar{z}_1 - \bar{z}_2|^2$$

$$\Rightarrow |z_1 + z_2| = |\bar{z}_1 + \bar{z}_2|$$

$$|z_1 + z_2| = |z_1 - z_2| \Rightarrow \left|\frac{z_1}{z_2} + 1\right| = \left|\frac{z_1}{z_2} - 1\right|$$

$$\Rightarrow \frac{z_1}{z_2} \text{ lies on } \perp \text{ bisector of } 1 \text{ and } -1.$$

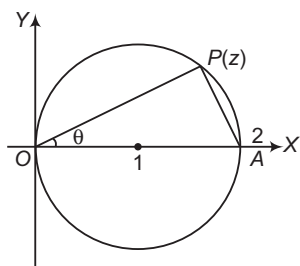
$$\Rightarrow \frac{z_1}{z_2} \text{ lies on imaginary axis.}$$

$$\Rightarrow \frac{z_1}{z_2} \text{ is purely imaginary.}$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = \pm \frac{\pi}{2}$$

$$\therefore |\arg(z_1) - \arg(z_2)| = \frac{\pi}{2}$$

125. Since, $\arg((z-1-i)/z)$ is the angle subtended by the chord joining the points O and $1+i$ at the circumference of the circle $|z-1|=1$, so it is equal to $-\pi/4$. The line joining the points $z=0$ and $z=2+0i$ is the diameter.



$$\therefore \arg \frac{z-2}{z} = \pm \frac{\pi}{2} \Rightarrow \frac{z-2}{z-0} \text{ is purely imaginary.}$$

$$\text{We have, } \angle POA = \frac{\pi}{2}$$

$$\Rightarrow \arg\left(\frac{2-z}{0-z}\right) = \frac{\pi}{2} \Rightarrow \frac{z-2}{z} = \frac{AP}{OP} i$$

$$\text{Now, in } \triangle OAP, \tan \theta = \frac{AP}{OP} \text{ Thus, } z-2=0$$

126. Given, $z = x + iy$

$$\begin{aligned} \text{Now, } \frac{3z-6-3i}{2z-8-6i} &= \frac{3(x+iy)-6-3i}{2(x+iy)-8-6i} \\ &= \frac{[3x-6+i(3y-3)][(2x-8)-i(2y-6)]}{[(2x-8)+i(2y-6)][(2x-8)-i(2y-6)]} \\ &= \frac{6x^2-36x+48+6y^2-24y+18+i[6xy-6x-24y+24-6xy+12y+18x-36]}{(2x-8)^2+(2y-6)^2} \\ &= \frac{6x^2+6y^2-36x-24y+66}{(2x-8)^2+(2y-6)^2} \\ &\quad + \frac{(12x-12y-12)}{(2x-8)^2+(2y-6)^2} = a+ib \text{ [say]} \end{aligned}$$

$$\text{Since, } \arg(a+ib) = \frac{\pi}{4}$$

$$\therefore \tan \frac{\pi}{4} = \frac{b}{a} \Rightarrow a=b$$

$$\Rightarrow 6x^2+6y^2-36x-24y+66=12x-12y-12$$

$$\Rightarrow 6x^2+6y^2-48x-12y+78=8$$

$$\Rightarrow x^2+y^2-8x-2y+13=0 \quad \dots(i)$$

$$\text{Again, } |z-3-i|=3 \Rightarrow |x+iy-3-i|=3$$

$$\Rightarrow (x-3)^2+(y+1)^2=9$$

$$\Rightarrow x^2+y^2-6x+2y+1=0 \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$-2x-4y+12=0$$

$$\Rightarrow x=-2y+6 \quad \dots(iii)$$

Putting the value of x in Eq. (ii), we get

$$(-2y+6)^2+y^2-6(-2y+6)+2y+1=0$$

$$\Rightarrow 5y^2-10y+1=0$$

$$\therefore y = \frac{10 \pm 4\sqrt{5}}{10} = 1 \pm \frac{2}{\sqrt{5}}$$

$$\therefore x = -2y+6 = 4 \mp \frac{4}{\sqrt{5}}$$

$$\therefore z = x+iy = 4 \mp \frac{4}{\sqrt{5}} + i \left(1 \pm \frac{2}{\sqrt{5}}\right)$$

Thus, the order pairs (x, y) satisfying the given equations such that $x = x + iy$ and $\left(4 - \frac{4}{\sqrt{5}}, 1 + \frac{2}{\sqrt{5}}\right)$

$$\text{and } \left(4 + \frac{4}{\sqrt{5}}, 1 - \frac{2}{\sqrt{5}}\right).$$

127. Clearly, we have to find it for real z . Let $z = x$.

$$\text{Then, } |x-w| = |x-w^2| = |w-w^2|$$

$$\Rightarrow \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} = \left|\frac{-1+\sqrt{3}i}{2} - \frac{-1-\sqrt{3}i}{2}\right|^2 = 3$$

$$\Rightarrow x + \frac{1}{2} = \pm \frac{3}{2}$$

$$\Rightarrow x = 1, -2$$

128. $\arg(z_1 z_2) = 0 \Rightarrow \arg(z_1) + \arg(z_2) = 0$

$$\therefore \arg(z_1) = -\arg(z_2) = \arg(\bar{z}_2)$$

$$\therefore |z_1| = |z_2|, \text{ we get } |z_1| = |\bar{z}_2|$$

$$\text{So, } z_1 = \bar{z}_2$$

$$\text{Also, } z_1 z_2 = \bar{z}_2 z_2 = |z_2|^2 = 1, \text{ because } |z_2| = 1$$

129. $z + z^{-1} = 1 \Rightarrow z^2 - z + 1 = 0$

$$\therefore z = \frac{1 \pm \sqrt{3}i}{2} = -\omega, -\omega^2$$

$$\therefore z^n + z^{-n} = (-\omega)^n + (-\omega)^{-n} \text{ or } (-\omega^2)^n + (-\omega^2)^{-n}$$

$$= (-1)^n \omega^n + (-1)^{-n} \cdot \frac{1}{\omega^n}$$

or $= (-1)^n \omega^{2n} + (-1)^{-n} \cdot \frac{1}{\omega^{2n}}$

$$= (-1)^n \left(\omega^n + \frac{1}{\omega^n} \right)$$

or $= (-1)^n \left(\omega^{2n} + \frac{1}{\omega^{2n}} \right)$

$$= (-1)^n \left(\omega^n + \frac{1}{\omega^n} \right) \text{ because } \omega^{3n} = 1$$

$$= (-1)^n \cdot (\omega^n + \omega^{2n})$$

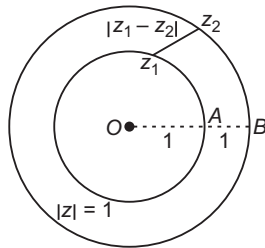
$$= (-1)^n \cdot (1 + 1) \text{ or } (-1)^n \cdot (-1)$$

Depending on whether n is a multiple of 3 or not.

130. $|2z_1 + z_2| \leq |2z_1| + |z_2|$

$$= 2|z_1| + |z_2| = 2 \times 1 + 2 = 4$$

From the figure, $|z_1 - z_2|$ is the least, when $0, z_1, z_2$ are collinear.

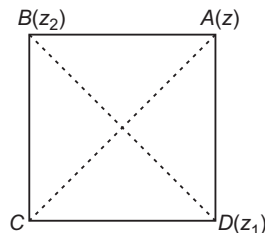


Then, $|z_1 - z_2| = 1$

Again, $\left| z_2 + \frac{1}{z_1} \right| \leq |z_2| + \left| \frac{1}{z_1} \right|$

$$= 2 + \left| \frac{1}{z_1} \right| = 2 + \frac{1}{1} = 3$$

131. $OB = OD = OA = |z|$



If $\text{amp}(z) = \theta$, then $\text{amp}(z_1) = \theta - \frac{\pi}{2}$,

$\text{amp}(z_2) = \theta + \frac{\pi}{2}$

and $z = |z| \{ \cos \theta + i \sin \theta \}$

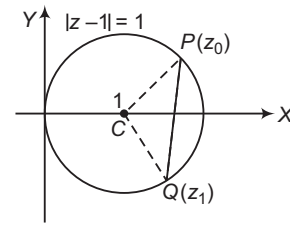
$$\therefore z_1 = |z| \left\{ \cos \left(\theta - \frac{\pi}{2} \right) + i \sin \left(\theta - \frac{\pi}{2} \right) \right\}$$

$$= |z| \{ \sin \theta - i \cos \theta \}$$

$$= |z| (-i) (\cos \theta + i \sin \theta) = -iz$$

$$z_2 = |z| \left\{ \cos \left(\theta + \frac{\pi}{2} \right) + i \sin \left(\theta + \frac{\pi}{2} \right) \right\} = iz$$

132. $\therefore |z_1 - 1| = 1, |z_0 - 1| = 1$



$$\therefore \left| \frac{z_1 - 1}{z_0 - 1} \right| = 1$$

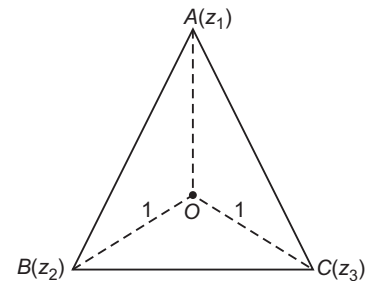
$$\angle QCP = \frac{\pi}{2}$$

$$\Rightarrow \text{amp} \left(\frac{z_1 - 1}{z_0 - 1} \right) = \frac{\pi}{2}, -\frac{\pi}{2}$$

$$\therefore \frac{z_1 - 1}{z_0 - 1} = 1 \cdot \left(\cos \frac{\pm \pi}{2} + i \sin \frac{\pm \pi}{2} \right) = i, -i$$

$$\therefore z_1 - 1 = (z_0 - 1)i, z_1 - 1 = -(z_0 - 1)i$$

133. Given, $|z_1| = |z_2| = |z_3|$



$\Rightarrow$ The vertices are at equal distances from the origin $z = 0$.

$\therefore$ The origin is at the centroid of the equilateral triangle.

$$\therefore \frac{z_1 + z_2 + z_3}{3} = 0$$

$$\therefore z_1 + z_2 + z_3 = 0$$

with OA as real axis,

$$z_1 = 1, z_2 = 1(\cos 120^\circ + i \sin 120^\circ),$$

$$z_3 = 1(\cos 120^\circ - i \sin 120^\circ)$$

$$\therefore z_1 z_2 z_3 = 1 \cdot \left(-\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right)$$

$$= \left(-\frac{1}{2} \right)^2 + \left(\frac{\sqrt{3}}{2} \right)^2$$

$$= 1$$

134. $|z_2 z_3 + 8z_3 z_1 + 27z_1 z_2| = \left| z_1 z_2 z_3 \left(\frac{1}{z_1} + \frac{8}{z_2} + \frac{27}{z_3} \right) \right|$

$$= |z_1| |z_2| |z_3| \left| \frac{1}{z_1} + \frac{8}{z_2} + \frac{27}{z_3} \right|$$

$$= |z_1| |z_2| |z_3| \left| \frac{\bar{z}_1}{|z_1|^2} + \frac{8\bar{z}_2}{|z_2|^2} + \frac{27\bar{z}_3}{|z_3|^2} \right|$$

$$= |z_1| |z_2| |z_3| \left| \frac{\bar{z}_1}{1} + \frac{8\bar{z}_2}{4} + \frac{27\bar{z}_3}{9} \right|$$

$$= |z_1| |z_2| |z_3| |z_1 + 2z_2 + 3z_3|$$

$$= |z_1| |z_2| |z_3| |z_1 + 2z_2 + 3z_3|$$

$$= 1 \cdot 2 \cdot 3 \cdot 6 = 36$$

- 135.** Statement I is false, since there is no order relation in the set of complex numbers.
Cancellation laws, $a + c > b + c$
 $\Rightarrow a > b$ does not hold true in complex numbers, therefore Statement II is true.

- 136.** Given that, $\arg(z) = 0$

$\Rightarrow z$ is purely real.

$\therefore$ Statement II is true.

Also, $|z_1| = |z_2| + |z_1 - z_2|$

$$\begin{aligned} \therefore (|z_2 - z_1|)^2 &= (|z_1| - |z_2|)^2 \\ \Rightarrow &= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2) \\ &= |z_1|^2 + |z_2|^2 - 2|z_1||z_2| \end{aligned}$$

$$\Rightarrow \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0$$

$$\Rightarrow \arg(z_1) - \arg(z_2) = 0$$

$$\Rightarrow \arg\left(\frac{z_1}{z_2}\right) = 0 \Rightarrow \frac{z_1}{z_2} \text{ is purely real.}$$

$$\Rightarrow \operatorname{Im}\left(\frac{z_1}{z_2}\right) = 0$$

Hence, Statement I and Statement II are true and Statement II is correct explanation of Statement I.

- 137.** We will show that Statement I is true and follows from Statement II.

Indeed

$$\begin{aligned} |z| &= \left| z + \frac{1}{z} - \frac{1}{z} \right| \leq \left| z + \frac{1}{z} \right| + \left| \frac{1}{z} \right| \leq 1 + \frac{1}{|z|} \\ \Rightarrow |z|^2 - |z| - 1 &\leq 0 \\ \Rightarrow |z| &\in \left[\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right] \end{aligned}$$

$$\text{But as } |z| > 0, \text{ we have } |z| \in \left(0, \frac{1 + \sqrt{5}}{2} \right)$$

$$\Rightarrow \text{Maximum value of } |z| \text{ is } \frac{1 + \sqrt{5}}{2}.$$

- 138.** Statement I is false, since $|z| = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$, if $a^2 > 4b$

and $|z|$ is a positive number c , then $|z| = c$
 $\Rightarrow z = c(\cos \theta + i \sin \theta)$

$\Rightarrow$ Infinite complex numbers satisfy the given equation. Statement II is true.

A quadratic can have more than two roots, if all coefficients are zero.

- 139.** $\therefore e^{i\theta} = \cos \theta + i \sin \theta$

$$\Rightarrow e^{-i\theta} = \cos \theta - i \sin \theta$$

$$\therefore \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\text{Now, } \cos(1-i) = \frac{e^{i(1-i)} + e^{-i(1-i)}}{2} = \frac{e^{(1+i)} + e^{-(1+i)}}{2}$$

$$= \frac{e(e^i) + e^{-1}(e^{-i})}{2}$$

$$= \frac{e(\cos 1 + i \sin 1) + e^{-1}(\cos 1 - i \sin 1)}{2}$$

$$= \frac{1}{2} \left(e + \frac{1}{e} \right) \cos 1 + \frac{i}{2} \left(e - \frac{1}{e} \right) \sin 1$$

$$\therefore a = \frac{1}{2} \left(e + \frac{1}{e} \right) \cos 1, b = \frac{1}{2} \left(e - \frac{1}{e} \right) \sin 1$$

- 140.** We have,

$$(\cos \theta + i \sin \theta)^{3/5} = (\cos 3\theta + i \sin 3\theta)^{1/5}$$

$$= [\cos(2r\pi + 3\theta) + i \sin(2r\pi + 3\theta)]^{1/5}$$

where, $r = 0, 1, 2, 3, 4$

$$= e^{i\left(\frac{2r\pi + 3\theta}{5}\right)}, r = 0, 1, 2, 3, 4$$

Hence, product of all values of $(\cos \theta + i \sin \theta)^{3/5}$

$$\begin{aligned} &= e^{i\frac{3\theta}{5}} e^{i\left(\frac{2\pi + 3\theta}{5}\right)} e^{i\left(\frac{4\pi + 3\theta}{5}\right)} e^{i\left(\frac{6\pi + 3\theta}{5}\right)} e^{i\left(\frac{8\pi + 3\theta}{5}\right)} \\ &= e^{i3\theta + 4\pi i} = e^{4\pi i} e^{i3\theta} \\ &= \cos 3\theta + i \sin 3\theta \end{aligned}$$

Also, product of roots of the equation $x^5 - 1 = 0$ is 1.

Hence, Statement II is true, but it is not a correct explanation of Statement I.

- 141.** Let $A(z_1)$ and $B(z_2)$ be the centres of the given circles and P be the centre of the variable circle, which touches given circles externally, then

$$|AP| = a + r \text{ and } |BP| = b + r$$

where, r is the radius of the variable circle.

On subtraction, we get

$$\begin{aligned} |AP| - |BP| &= |a - b| \\ \Rightarrow |AP - BP| &= |a - b| \text{ is a constant.} \end{aligned}$$

Hence, locus of P is

(i) a right bisector of AB , if $a = b$

(ii) a hyperbola if $|a - b| < |AB| = |z_2 - z_1|$

(iii) an empty set, if $|a - b| > |AB| = |z_2 - z_1|$

(iv) set of all points on line AB except those which lie between A and B , if $|a - b| = |AB| \neq 0$

Thus, Statement I is false and Statement II is true.

- 142.** If $P(z)$ is any point on the ellipse, then equation of the ellipse is

$$|z - z_1| + |z - z_2| = \frac{|z_1 - z_2|}{e} \quad \dots(i)$$

For $P(z)$ to lie in ellipse, we have

$$|z - z_1| + |z - z_2| < \frac{|z_1 - z_2|}{e}$$

It is given that origin is an interior point of the ellipse.

$$\Rightarrow |0 - z_1| + |0 - z_2| < \frac{|z_1 - z_2|}{e}$$

$$\therefore e \in \left(0, \frac{|z_1 - z_2|}{|z_1| + |z_2|} \right)$$

Hence, Statement I is false and Statement II is true.

- 143.** As, we know $|z - z_1| + |z - z_2| = k$ represents an ellipse, if $|k| > |z_1 - z_2|$. Thus, $|z - i| + |z + i| = k$ represents an ellipse, if $|k| > |i + i|$ or $|k| > 2$.

$\therefore$ Statement I is false and Statement II is true.

- 144.** The equation can be rewritten as

$$z\bar{z} + \bar{a}z + a\bar{z} + a\bar{a} = a\bar{a} - \lambda$$

$$\Rightarrow (z + a)(\bar{z} + \bar{a}) = a\bar{a} - \lambda$$

$$\Rightarrow |z + a| = \sqrt{a\bar{a} - \lambda}$$

Since, $a\bar{a}$ is real, λ should be real.

$\sqrt{a\bar{a} - \lambda}$ represents radius of the circle.

Hence, Statement I and Statement II both are true and Statement II is the correct explanation for Statement I.

Solutions (Q. Nos. 145-147)

Given that,

$$\begin{aligned} & |z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \\ \Rightarrow & |z_1|^2 + |z_2|^2 + z_1 \bar{z}_2 + \bar{z}_1 z_2 = |z_1|^2 + |z_2|^2 \\ \Rightarrow & z_1 \bar{z}_2 + \bar{z}_1 z_2 = 0 \quad \dots (i) \\ \Rightarrow & \frac{z_1}{z_2} + \frac{\bar{z}_1}{\bar{z}_2} = 0 \quad [\text{dividing by } z_2 \bar{z}_2] \\ \Rightarrow & \frac{z_1}{z_2} + \left(\frac{z_1}{z_2} \right) = 0 \quad \dots (ii) \end{aligned}$$

145. From Eq. (i), $z_2 \bar{z}_2$ is purely imaginary.

146. From Eq. (ii), z_1/z_2 is purely imaginary

147. Also, $i(z_1/z_2)$ is purely real. Hence, its possible arguments are 0 and π .

148. Clearly, according to the least possibility
 $\alpha^2 - 7\alpha + 11 \leq 1 \Rightarrow \alpha \in [2, 5]$

149. $|(z-i) - (\alpha^2 - 7\alpha + 11)| = 1$

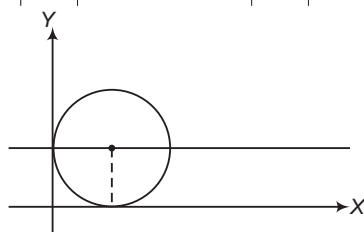
$$\therefore |(z-i) - (\alpha^2 - 7\alpha + 11)| \geq |z-i| - |\alpha^2 - 7\alpha + 11|$$

[using $|z_1 - z_2| \geq |z_1| - |z_2|$]

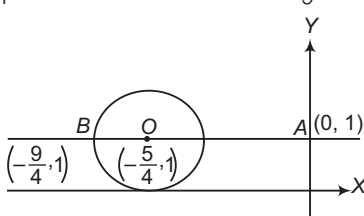
$$\Rightarrow |\alpha^2 - 7\alpha + 11| + 1 \geq |z-i|$$

[since, $\alpha^2 - 7\alpha + 11 \geq -5/4$]

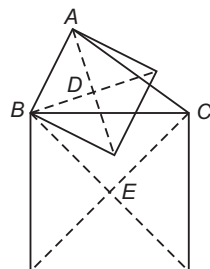
$$\therefore |z-i| \leq 1 + 5/4 \quad \text{or} \quad |z-i| \leq 9/4$$



150. $AB = \frac{9}{4}$ and $\arg(z) = \pi - \tan^{-1} \frac{4}{9}$



151. $z_D = \frac{z_B - z_A i}{1-i}$, $z_E = \frac{z_B - z_C i}{1-i}$



$\therefore$ Angle between AC and DE

$$= \arg \left(\frac{z_C - z_A}{z_E - z_D} \right) - \arg \left(\frac{z_C - z_A}{z_A - z_C} \frac{1-i}{i} \right) = \frac{\pi}{4}$$

152. $DE = \frac{(z_A - z_C)i}{1-i} = \frac{|1-\omega^2|}{\sqrt{2}} = \frac{\sqrt{3}}{2}$

153. $z_E = \frac{z_B - z_C i}{1-i} = \frac{\omega - \omega^2 i}{1-i}$
 $= \frac{(\omega + \omega^2) + i(\omega - \omega^2)}{2} = \frac{-1 - \sqrt{3}}{2}$

$$\therefore AE = 1 + \frac{1 + \sqrt{3}}{2} = \frac{3 + \sqrt{3}}{2}$$

154. A. $\arg \left(\frac{z^2 - 1}{z^2 + 1} \right) = 0; z \neq \pm i$

$$= \frac{z^2 - 1}{z^2 + 1} = \frac{\bar{z}^2 - 1}{\bar{z}^2 + 1}$$

$$\Rightarrow z - \bar{z} = 0, z + \bar{z} = 0, y = 0, x = 0$$

Locus of z is portion of pair of lines $xy = 0$

$$\left[\because \left(\frac{z^2 - 1}{z^2 + 1} \right) > 0 \right]$$

B. Given,

$$|z - \cos^{-1} \cos 12| - |z - \sin^{-1} \sin 12| = 8(\pi - 3)$$

$$\text{Since, } |\cos^{-1} \cos 12 - \sin^{-1} \sin 12| = 8(\pi - 3)$$

$\therefore$ Locus of z is portion of a line joining z_1 and z_2 except the segment between z_1 and z_2 .

C. $z^2 - i|z|^2 = k_2 - k_1$

$$\therefore x^2 - y^2 + 2ixy - i\lambda_1 = \lambda_2$$

$$\Rightarrow x^2 - y^2 = \lambda_2 \quad \text{and} \quad xy = \frac{\lambda_1}{2}$$

$\therefore$ Locus of z is point of intersection of hyperbola.

D. Given, $\left| z - 1 - \sin^{-1} \frac{1}{\sqrt{3}} \right| + \left| z + \cos^{-1} \frac{1}{\sqrt{3}} - \frac{\pi}{2} \right| = 1$

$$\text{since, } 1 + \sin^{-1} \frac{1}{\sqrt{3}} + \cos^{-1} \frac{1}{\sqrt{3}} - \frac{\pi}{2} = 1$$

$$\therefore |z - z_1| + |z - z_2| = |z_1 + z_2|$$

$\Rightarrow$ Locus of z is the segment joining z_1 and z_2 .

155. There are no real roots of the equation $z^6 - 6z + 20 = 0$.

If $x + iy$ is a root, then $x - iy$ is also a root.

Let the roots be $x_1 \pm iy_1, x_2 \pm iy_2, x_3 \pm iy_3$

$$\text{Sum of the roots} = 2(x_1 + x_2 + x_3) = 0$$

$$\Rightarrow x_1 + x_2 + x_3 = 0$$

$\Rightarrow$ One of x_1, x_2, x_3 is negative and other two are positive or vice-versa.

$\Rightarrow$ The number of roots in each of the quadrant is either 1 or 2.

156. A. $z^4 - 1 = 0 \Rightarrow z^4 = 1 = \cos 0 + i \sin 0$

$$\Rightarrow z = (\cos 0 + i \sin 0)^{1/4} = \cos 0 + i \sin 0$$

B. $z^4 + 1 = 0 \Rightarrow z^4 = -1 = \cos \pi + i \sin \pi$

$$\Rightarrow z = (\cos \pi + i \sin \pi)^{1/4} = \cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$$

C. $iz^4 + 1 = 0 \Rightarrow z^4 = i = \cos \frac{\pi}{2} + i \sin \frac{\pi}{2}$

$$\Rightarrow z = \left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2} \right)^{1/4} = \cos \frac{\pi}{8} + i \sin \frac{\pi}{8}$$

D. $iz^4 - 1 = 0$

$$\Rightarrow z^4 = -i = \cos \frac{\pi}{2} - i \sin \frac{\pi}{2}$$

$$\Rightarrow z = \left(\cos \frac{\pi}{2} - i \sin \frac{\pi}{2} \right)^{1/4} = \cos \frac{\pi}{8} - i \sin \frac{\pi}{8}$$

157. A. Let $z = \sqrt{1-z} \Rightarrow z^2 + z + 1 = 0$

$$\Rightarrow z = \omega, \omega^2$$

B. Here, $z_1 + z_2 + z_3 + \dots + z_6 = 0 \dots (i)$

$$z_2 = z_1 e^{(i2\pi)/n}, \text{ if } e^{(i2\pi)/n} = \alpha \Rightarrow z_2 = z_1 \alpha$$

$$\text{Similarly, } z_3 = z_1 \alpha^2, z_4 = z_1 \alpha^3, z_5 = z_1 \alpha^4, z_6 = z_1 \alpha^5$$

On squaring and adding and then using Eq. (i), we get

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = 0$$

$$C. \frac{1+2\omega+3\omega^2+4\omega^3+\dots+n\omega^{n-1}}{n} \geq \left\{ n! \omega^{\frac{n(n+1)}{2}} \right\}^{1/n}$$

[$\because$ AM $\geq$ GM]

Now,

$$E = 1+2\omega+3\omega^2+4\omega^3+\dots+n\omega^{n-1}+\dots$$

$$\omega E = \frac{\omega+2\omega^2+3\omega^3+\dots+(n-1)\omega^{n-1}+n\omega^n}{(1-\omega)E = 1+\omega+\omega^2+\dots+\omega^{n-1}-n\omega^n}$$

$$\Rightarrow E = \frac{0-n(1)}{1-\omega}$$

$$\therefore E = \frac{n}{\omega-1}$$

D. Let $x^2 + y^2 = 1$

$$\text{where, } z_1 = i, z_2 = 1, z_3 = -1$$

$$\therefore z = -i$$

$$\Rightarrow zz_1 + z_2 z_3 = 1 - 1 = 0$$

158. Consider,

$$x = \frac{x^3}{x^2} = \frac{2+11i}{3+4i} \times \frac{3-4i}{3-4i} = \frac{50+25i}{25}$$

$$= 2+i$$

$$\therefore a+b=2+1=3$$

159. Here, $\left| \frac{z-4}{z-8} \right| = 1 \Rightarrow \left| \frac{x-4+iy}{x-8+iy} \right| = 1$

$$\Rightarrow (x-4)^2 + y^2 = (x-8)^2 + y^2$$

$$\Rightarrow x = 6$$

$$\text{With } x = 6, \left| \frac{z-12}{z-8i} \right| = \frac{5}{3}$$

$$\Rightarrow y^2 - 25y + 136 = 0$$

$$\Rightarrow 9(36+y^2) = 25[36+(y-8)^2]$$

$$\Rightarrow y = 17, 8$$

Thus, the required numbers are $z = 6 + 17i, 6 + 8i$.

Hence, the value of $\text{Re}(z)$ is 6.

160. $\left| z + \frac{1}{z} \right| \geq \left| z - \frac{1}{z} \right| = \left| z - \frac{1}{|z|} \right| = \left| z - \frac{1}{|z|} \right|$ as $|z| \geq 3$

$$\text{Let } f(x) = x - \frac{1}{x} \text{ for } x \geq 3, f'(x) = 1 + \frac{1}{x^2} > 0$$

$\Rightarrow f(x)$ is increasing function, so $f(x)_{\min}$ at $x = 3$ is

$$\left| z + \frac{1}{z} \right|_{\min} = \left| z - \frac{1}{|z|} \right|_{|z|=3} = \frac{8}{3}$$

Hence, λ is equal to 8.

161. Consider

$$z^4 + z^3 + 2z^2 + z + 1 = 0$$

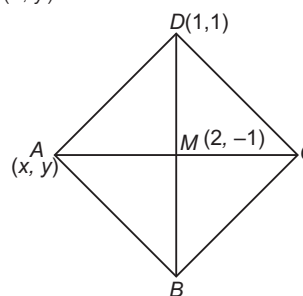
$$\Rightarrow z^4 + z^3 + z^2 + z^2 + z + 1 = 0$$

$$\Rightarrow z^2(z^2 + z + 1) + (z^2 + z + 1) = 0$$

$$\Rightarrow (z^2 + z + 1)(z^2 + 1) = 0$$

$$\therefore z = i, -i, \omega, \omega^2, \text{ for each } |z| = 1$$

162. Let A be (x, y).



$$\text{It is given that, } BD = 2AC$$

$$\Rightarrow MD = 2AM$$

Also, DM is perpendicular to AM .

$$\Rightarrow (1-2)^2 + (1+1)^2 = 4[(x-2)^2 + (y+1)^2] \dots (i)$$

$$\text{and } \frac{y+1}{x-2} \cdot \frac{1+1}{1-2} = -1$$

$$\Rightarrow 2(y+1) = x-2$$

With $x-2 = 2(y+1)$, Eq. (i) can be written as

$$(y+1)^2 = \frac{1}{4}$$

$$\Rightarrow y = -\frac{1}{2}, -\frac{3}{2}$$

$$\Rightarrow x = 3, 1$$

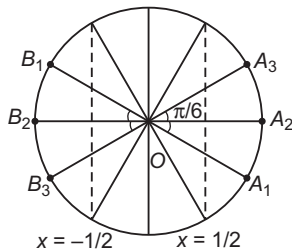
$$\therefore \lambda_1 + \lambda_2 = 4$$

Entrances Gallery

1. A. z_k is 10th root of unity $\Rightarrow \bar{z}_k$ will also be 10th root of unity. Take z_j as $\bar{z}_k$.
- B. $z_1 \neq 0$, take $z = \frac{z_k}{z_1}$, we can always find z .
- C. $z^{10} - 1 = (z - 1)(z - z_1) \dots (z - z_9)$
 $\Rightarrow (z - z_1)(z - z_2) \dots (z - z_9) = 1 + z + z^2 + \dots + z^9, \forall z \in \text{complex number}$
 Put $z = 1$
 $(1 - z_1)(1 - z_2) \dots (1 - z_9) = 10$
- D. $1 + z_1 + z_2 + \dots + z_9 = 0$
 $\Rightarrow \text{Re}(1) + \text{Re}(z_1) + \dots + \text{Re}(z_9) = 0$
 $\Rightarrow \text{Re}(z_1) + \text{Re}(z_2) + \dots + \text{Re}(z_9) = -1$
 $\therefore 1 - \sum_{k=1}^9 \cos \frac{2k\pi}{10} = 2$

2. $OB = |\alpha|$
 $OC = \frac{1}{|\alpha|} = \frac{1}{|\alpha|}$
 In $\triangle OBD$,
 $\cos \theta = \frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|}$
 In $\triangle OCD$,
 $\cos \theta = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$
 $\Rightarrow \frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|} = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$
 $\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

3. $w = \frac{\sqrt{3} + i}{2} = e^{i\pi/6}$



So, $w^n = e^{i\left(\frac{n\pi}{6}\right)}$
 Now, for z_1 , $\cos \frac{n\pi}{6} > \frac{1}{2}$
 and for z_2 , $\cos \frac{n\pi}{6} < -\frac{1}{2}$

Possible position of z_1 are A_1, A_2, A_3 , whereas of z_2 are B_1, B_2, B_3 (as shown in the figure).

So, possible value of $\angle z_1 O z_2$ according to the given options is $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$.

4. Area of

$$S = S_1 \cap S_2 \cap S_3$$

$$= \frac{\pi \times 4^2}{4} + \frac{4^2 \times \pi}{6}$$

$$= 4^2 \pi \left\{ \frac{1}{4} + \frac{1}{6} \right\}$$

$$= \frac{20\pi}{3}$$

5. Distance of $(-3, \sqrt{3})$ from $y + \sqrt{3}x = 0$

$$> \left| \frac{-3 + \sqrt{3} \times 1}{2} \right| > \frac{3 - \sqrt{3}}{2}$$

6. Given, is $z^2 + z + 1 - a = 0$

Clearly, this equation do not have real roots, if

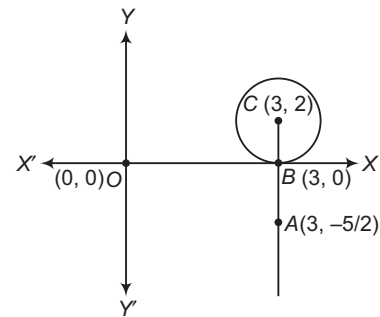
$$D < 0$$

$$\Rightarrow 1 - 4(1 - a) < 0$$

$$\Rightarrow 4a < 3$$

$$\therefore a < \frac{3}{4}$$

7. Length $AB = \frac{5}{2}$



$\therefore$ Minimum value = 5

8. The expression may not attain integral value for all a, b, c .

If we consider $a = b = c$, then

$$x = 3a$$

$$y = a(1 + \omega + \omega^2) = 0$$

$$z = a(1 + \omega^2 + \omega) = 0$$

$$\therefore |x|^2 + |y|^2 + |z|^2 = 9|a|^2$$

$$\therefore \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \frac{9}{3} = 3$$

$$\text{As, } |x|^2 + |y|^2 + |z|^2$$

$$= (a + b + c)(\bar{a} + \bar{b} + \bar{c}) + (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega) + (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)$$

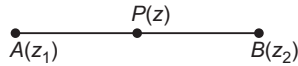
$$= 3(|a|^2 + |b|^2 + |c|^2) + a\bar{b}(1 + \omega + \omega^2) + \bar{a}b(1 + \omega + \omega^2) + b\bar{c}(1 + \omega + \omega^2) + \bar{b}c(1 + \omega + \omega^2) + a\bar{c}(1 + \omega + \omega^2) + \bar{a}c(1 + \omega + \omega^2)$$

$$= 3(|a|^2 + |b|^2 + |c|^2) \quad [\text{using } 1 + \omega + \omega^2 = 0]$$

$$\therefore \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = 3$$

9. Given, $z = (1-t)z_1 + tz_2$

$$\begin{aligned} \Rightarrow \frac{z - z_1}{z_2 - z_1} &= t \\ \Rightarrow \arg\left(\frac{z - z_1}{z_2 - z_1}\right) &= 0 \quad \dots(i) \\ \Rightarrow \arg(z - z_1) &= \arg(z_2 - z_1) \\ \Rightarrow \frac{z - z_1}{z_2 - z_1} &= \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \\ \Rightarrow \left| \frac{z - z_1}{z_2 - z_1} \right| &= \left| \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \right| = 0 \end{aligned}$$



$$\Rightarrow AP + PB = AB$$

$$\Rightarrow |z - z_1| + |z - z_2| = |z_1 - z_2|$$

10. A. $\left| \frac{z}{|z|} - i \right| = \left| \frac{z}{|z|} + i \right|, z \neq 0$

$\frac{z}{|z|}$ is unimodular complex number
and lies on perpendicular bisector of i and $-i$
 $\Rightarrow \frac{z}{|z|} = \pm 1 \Rightarrow z = \pm 1|z| \Rightarrow a$ is a real number
 $\Rightarrow \operatorname{Im}(z) = 0$

B. $|z + 4| + |z - 4| = 10$
 z lies on an ellipse, whose focus are $(4, 0)$ and $(-4, 0)$ and length of major axis is 10.
 $\Rightarrow 2ae = 8$ and $2a = 10$
 $\Rightarrow e = 4/5$
 $|\operatorname{Re}(z)| \leq 5$

C. $\therefore |\omega| = 2 \Rightarrow w = 2(\cos \theta + i \sin \theta)$
 $x + iy = 2(\cos \theta + i \sin \theta) - \frac{1}{2}(\cos \theta - i \sin \theta)$
 $= \frac{3}{2} \cos \theta + i \frac{5}{2} \sin \theta$
 $\Rightarrow \frac{x^2}{(3/2)^2} + \frac{y^2}{(5/2)^2} = 1$
 $e^2 = 1 - \frac{9/4}{25/4}$
 $= 1 - \frac{9}{25} = \frac{16}{25}$
 $\Rightarrow e = \frac{4}{5}$

D. $|\omega| = 1$
 $\Rightarrow x + iy = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta$
 $x + iy = 2 \cos \theta$
 $|\operatorname{Re}(z)| \leq 1, |\operatorname{Im}(z)| = 0$

11. Given, z_2 is not unimodular i.e. $|z_2| \neq 1$

and $\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular.

$$\begin{aligned} \Rightarrow \left| \frac{z_1 - 2z_2}{2 - z_1\bar{z}_2} \right| &= 1 \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2 \\ \Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) &= (2 - z_1\bar{z}_2)(2 - \bar{z}_1z_2) \quad [\because z\bar{z} = |z|^2] \\ \Rightarrow |z_1|^2 + 4|z_2|^2 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 &= 4 + |z_1|^2|z_2|^2 - 2\bar{z}_1z_2 - 2z_1\bar{z}_2 \end{aligned}$$

$$\begin{aligned} \Rightarrow (|z_2|^2 - 1)(|z_1|^2 - 4) &= 0 \\ \therefore |z_2| &\neq 1 \\ \therefore |z_1| &= 2 \\ \text{Let } z_1 &= x + iy \Rightarrow x^2 + y^2 = (2)^2 \\ \therefore \text{Point } z_1 &\text{ lies on a circle of radius 2.} \end{aligned}$$

12. $\therefore |z| \geq 2$

$$\therefore \left| z + \frac{1}{2} \right| \geq \left| z \right| - \left| \frac{1}{2} \right| \geq 2 - \frac{1}{2} \geq \frac{3}{2}$$

Hence, minimum distance between z and $\left(-\frac{1}{2}, 0\right)$ is $\frac{3}{2}$.

13. Given, $|z| = 1, \arg z = \theta$

$$\begin{aligned} \therefore z &= e^{i\theta} \\ \text{But } \bar{z} &= \frac{1}{z} \\ \therefore \arg\left(\frac{1+z}{1+\frac{1}{z}}\right) &= \arg(z) = \theta \end{aligned}$$

14. Given, a complex number $\frac{z^2}{z-1}$ ($z \neq 1$) is purely real.

To find the locus of the complex number z .
Since, $\frac{z^2}{z-1}$ ($z \neq 1$) is purely real.

$$\begin{aligned} \therefore \frac{z^2}{z-1} &= \frac{\bar{z}^2}{\bar{z}-1} \\ \Rightarrow z^2(\bar{z}-1) &= \bar{z}^2(z-1) \\ \Rightarrow z^2\bar{z} - z^2 &= \bar{z}^2z - \bar{z}^2 \\ \Rightarrow z\bar{z}z - z^2 &= \bar{z}z\bar{z} - \bar{z}^2 \\ \Rightarrow |z|^2z - z^2 &= \bar{z}|z|^2 - \bar{z}^2 \end{aligned}$$

On rearranging the terms, we get

$$\begin{aligned} |z|^2z - \bar{z}|z|^2 &= \bar{z}^2 - z^2 \\ \Rightarrow |z|^2(z - \bar{z}) &= (\bar{z} - z)(\bar{z} + z) \\ \Rightarrow |z|^2(z - \bar{z}) - (z - \bar{z})(\bar{z} + z) &= 0 \\ \Rightarrow (z - \bar{z})(|z|^2 - (\bar{z} + z)) &= 0 \end{aligned}$$

Either $(z - \bar{z}) = 0$

or $[|z|^2 - (\bar{z} + z)] = 0$

Now, $z = \bar{z}$

$\Rightarrow$ Locus of z is real axis and

$$(|z|^2 - (\bar{z} + z)) = 0 \Rightarrow z\bar{z} - (\bar{z} + z) = 0$$

Locus of z is a circle passing through the origin.

Aliter

Put $z = x + iy$, then

$$\begin{aligned} \frac{z^2}{z-1} &= \frac{(x+iy)^2}{(x+iy)-1} = \frac{(x^2-y^2)+i(2xy)}{(x-1)+iy} \\ &= \frac{(x^2-y^2)+i(2xy)}{(x-1)+iy} \times \frac{(x-1)-iy}{(x-1)-iy} \end{aligned}$$

Since, $\frac{z^2}{z-1}$ ($z \neq 1$) is purely real, hence its imaginary

part should be equal to zero.

$$\Rightarrow (x^2 - y^2)(-y) + (2xy)(x - 1) = 0$$

$$\Rightarrow y(x^2 - y^2 + 2x - 2x^2) = 0$$

$$\Rightarrow y(x^2 + y^2 - 2x) = 0$$

Either $y = 0$ or $x^2 + y^2 - 2x = 0$

Now, $y = 0$, locus of z is real axis, and $x^2 + y^2 - 2x = 0$, locus of z is a circle passing through the origin.
Locus of z is either real axis or a circle passing through the origin.

15. Let $z = x + iy$, given $\operatorname{Re}(z) = 1$

$$\therefore x = 1 \Rightarrow z = 1 + iy$$

Since, complex roots are conjugate of each other.

$$\therefore z = 1 + iy \text{ and } 1 - iy \text{ are two roots of } z^2 + \alpha z + \beta = 0$$

$$\text{Product of roots} = \beta$$

$$\Rightarrow (1 + iy)(1 - iy) = \beta$$

$$\therefore \beta = 1 + y^2 \geq 1 \Rightarrow \beta \in (1, \infty)$$

16. $\because (1 + \omega)^7 = A + B\omega$, we know $1 + \omega + \omega^2 = 0$

$$\therefore 1 + \omega = -\omega^2$$

$$\Rightarrow (-\omega^2)^7 = A + B\omega$$

$$\Rightarrow -\omega^{14} = A + B\omega \quad [\because \omega^{14} = \omega^{12} \cdot \omega^2 = \omega^2]$$

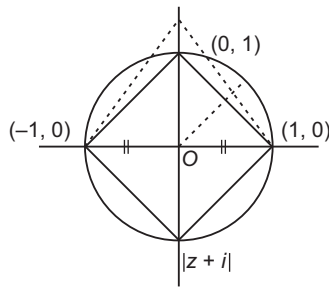
$$\Rightarrow -\omega^2 = A + B\omega$$

$$\Rightarrow 1 + \omega = A + B\omega$$

On comparing

$$A = 1, B = 1$$

17. We have, $|z - 1| = |z + 1| = |z - i|$



Clearly, z is the circumcentre of the triangle formed by the vertices $(1, 0)$, $(0, 1)$ and $(-1, 0)$, which is unique. Hence, the number of complex number z is one.

18. Since, α and β are roots of the equation $x^2 - x + 1 = 0$.

$$\Rightarrow \alpha + \beta = 1, \alpha\beta = 1 \Rightarrow x = \frac{1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow x = \frac{1 + \sqrt{3}i}{2} \text{ or } \frac{1 - \sqrt{3}i}{2}$$

$$\Rightarrow x = -\omega \text{ or } -\omega^2$$

Thus, $\alpha = -\omega^2$, then $\beta = -\omega$

or $\alpha = -\omega$, then $\beta = -\omega^2$ [where, $\omega^3 = 1$]

$$\begin{aligned} \text{Hence, } \alpha^{2009} + \beta^{2009} &= (-\omega)^{2009} + (-\omega^2)^{2009} \\ &= -[(\omega^3)^{669} \cdot \omega^2 + (\omega^3)^{1337} \cdot \omega] \\ &= -[\omega^2 + \omega] = -(-1) = 1 \end{aligned}$$

19. $|z| = \left| \left(z - \frac{4}{z} \right) + \frac{4}{z} \right|$

$$\Rightarrow |z| \leq \left| z - \frac{4}{z} \right| + \frac{4}{|z|}$$

$$\Rightarrow |z| \leq 2 + \frac{4}{|z|}$$

$$\Rightarrow |z|^2 - 2|z| - 4 \leq 0$$

$$\Rightarrow |z| \leq \sqrt{5} + 1$$

$\therefore$ Maximum value of $|z|$ is $\sqrt{5} + 1$.

For minimum value of $|z|$, we take

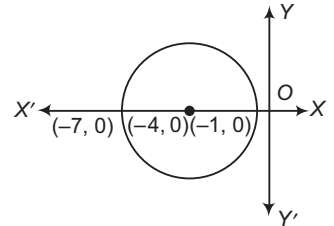
$$\left| z - \frac{4}{z} \right| - \left| \frac{4}{z} \right| \leq \left| \left(z - \frac{4}{z} \right) + \frac{4}{z} \right|$$

from which we get $|z| \geq \sqrt{5} - 1$

20. Let $z = \frac{1}{i - 1}$

$$\therefore \bar{z} = \overline{\left(\frac{1}{i - 1} \right)} = \frac{1}{-i - 1} = -\frac{1}{i + 1}$$

21. From the argand diagram, maximum value of $|z + 1|$ is 6.



Aliter

$$|z + 1| = |z + 4 - 3|$$

$$\leq |z + 4| + |-3| \leq 6$$

Thus, maximum value of $|z + 1|$ is 6.

22. $\sum_{k=1}^{10} \left(\sin \frac{2k\pi}{11} + i \cos \frac{2k\pi}{11} \right)$

$$= i \sum_{k=1}^{10} \left(\cos \frac{2k\pi}{11} - i \sin \frac{2k\pi}{11} \right)$$

$$= i \sum_{k=1}^{10} \left(e^{\frac{2k\pi}{11}} \right)$$

$$= i \left\{ \sum_{k=0}^{10} \left(e^{\frac{2k\pi}{11}} \right) - 1 \right\} = -i$$

23. Given, equation is $z^2 + z + 1 = 0$

$$\Rightarrow z = \omega, \omega^2$$

$$\begin{aligned} \text{Now, } \left(z + \frac{1}{z} \right)^2 + \left(z^2 + \frac{1}{z^2} \right)^2 + \left(z^3 + \frac{1}{z^3} \right)^2 \\ + \left(z^4 + \frac{1}{z^4} \right)^2 + \left(z^5 + \frac{1}{z^5} \right)^2 + \left(z^6 + \frac{1}{z^6} \right)^2 \\ = (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + (\omega^3 + \omega^{-3})^2 \\ + (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + (\omega^6 + \omega^{-6})^2 \\ = (-1)^2 + (-1)^2 + (1 + 1)^2 + (-1)^2 + (-1)^2 + (1 + 1)^2 \\ = 1 + 1 + 4 + 1 + 1 + 4 = 12 \end{aligned}$$

24. Given that, $(x - 1)^3 + 8 = 0$

$$\Rightarrow (x - 1)^3 = (-2)^3$$

$$\Rightarrow \left(\frac{x - 1}{-2} \right)^3 = 1$$

$$\Rightarrow \left(\frac{x - 1}{-2} \right) = (1)^{1/3}$$

Cube roots of $\left(\frac{x - 1}{-2} \right)$ are $1, \omega, \omega^2$.

Cube roots of $(x - 1)$ are $-2, -2\omega$ and $-2\omega^2$.

$\therefore$ Cube roots of x are $-1, 1 - 2\omega$ and $1 - 2\omega^2$.

25. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\text{Given, } |z_1 + z_2| = |z_1| + |z_2|$$

$$\begin{aligned} \therefore \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \\ = \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \end{aligned}$$

On squaring both sides,

$$\begin{aligned} &= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2 \\ &= x_1^2 + y_1^2 + x_2^2 + y_2^2 + 2\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \\ \Rightarrow x_1x_2 + y_1y_2 &= \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} \end{aligned}$$

Again, squaring both sides, we get

$$\begin{aligned} \Rightarrow &= x_1^2x_2^2 + y_1^2y_2^2 + 2x_1x_2y_1y_2 \\ &= x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + y_1^2x_2^2 \end{aligned}$$

$$\Rightarrow (x_1y_2 - y_1x_2)^2 = 0$$

$$\Rightarrow \frac{y_1}{x_1} = \frac{y_2}{x_2}$$

$$\Rightarrow \tan^{-1}\left(\frac{y_1}{x_1}\right) = \tan^{-1}\left(\frac{y_2}{x_2}\right)$$

$$\Rightarrow \arg(z_1) = \arg(z_2)$$

$$\therefore \arg(z_1) - \arg(z_2) = 0$$

Aliter

$$\begin{aligned} \text{Given, } |z_1 + z_2| &= |z_1| + |z_2| \\ &= |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1\bar{z}_2) \\ &\quad [\text{squaring both sides}] \\ &= |z_1|^2 + |z_2|^2 + 2|z_1||z_2| \end{aligned}$$

$$\Rightarrow \operatorname{Re}(z_1\bar{z}_2) = |z_1||z_2|$$

$$\Rightarrow |z_1||z_2|\cos(\theta_1 - \theta_2) = |z_1||z_2|$$

$$\Rightarrow \theta_1 - \theta_2 = 0$$

$$\therefore \arg(z_1) - \arg(z_2) = 0$$

$$26. \text{ Given, } w = \frac{z}{z - \frac{i}{3}} \text{ and } |w| = 1$$

$$\Rightarrow \left| \frac{z}{z - \frac{i}{3}} \right| = 1 \Rightarrow |z| = \left| z - \frac{i}{3} \right|$$

$$\Rightarrow z \text{ lies on perpendicular bisector of } (0, 0) \text{ and } \left(0, \frac{1}{3}\right).$$

Hence, z lies on a straight line.

$$27. \text{ Given, } \bar{z} + i\bar{w} = 0$$

$$\Rightarrow \bar{z} = -i\bar{w}$$

$$\Rightarrow z = iw$$

$$\Rightarrow w = -iz$$

$$\text{and } \arg(zw) = \pi$$

$$\Rightarrow \arg(-iz^2) = \pi$$

$$\Rightarrow \arg(-i) + 2\arg(z) = \pi$$

$$\Rightarrow -\frac{\pi}{2} + 2\arg(z) = \pi \quad \left[\because \arg(-i) = -\frac{\pi}{2} \right]$$

$$\therefore \arg(z) = \frac{3\pi}{4}$$

$$28. \text{ Since, } z^{1/3} = p + iq$$

$$\therefore z = (p + iq)^3 = p^3 - iq^3 + 3p^2q - 3pq^2$$

$$\text{Given that, } z = x - iy \Rightarrow x - iy = p^3 - 3pq^2 + i(3p^2q - q^3)$$

$$\Rightarrow x = p^3 - 3pq^2$$

$$\text{and } -y = 3p^2q - q^3$$

$$\Rightarrow \frac{x}{p} = p^2 - 3q^2$$

$$\text{and } \frac{y}{q} = -3p^2 + q^2$$

$$\therefore \frac{x}{p} + \frac{y}{q} = -2p^2 - 2q^2$$

$$\Rightarrow \frac{1}{(p^2 + q^2)} \left(\frac{x}{p} + \frac{y}{q} \right) = -2$$

29. Using the relation, if

$$|z_1 + z_2| = |z_1| + |z_2|$$

$$\text{Then, } \arg(z_1) = \arg(z_2)$$

$$\text{Since, } |z^2 + (-1)| = |z^2| + |-1|$$

$$\text{Then, } \arg(z^2) = \arg(-1)$$

$$\Rightarrow 2\arg(z) = \pi \quad [\because \arg(-1) = \pi]$$

$$\Rightarrow \arg(z) = \frac{\pi}{2}$$

$\Rightarrow z$ lies on Y -axis (imaginary axis).

30. Since, origin, z_1 and z_2 are the vertices of an equilateral triangle, then

$$z_1^2 + z_2^2 = z_1z_2$$

$$\Rightarrow (z_1 + z_2)^2 = 3z_1z_2 \quad \dots(i)$$

Again, z_1, z_2 are the roots of the equation

$$z^2 + az + b = 0$$

$$\text{Then, } z_1 + z_2 = -a \text{ and } z_1z_2 = b$$

On putting these values in Eq. (i), we get

$$(-a)^2 = 3b \Rightarrow a^2 = 3b$$

$$31. \text{ Let } z = r_1 e^{i\theta} \Rightarrow w = r_2 e^{i\phi} \Rightarrow \bar{z} = r_1 e^{-i\theta}$$

$$\text{Given, } |zw| = 1$$

$$\Rightarrow |r_1 e^{i\theta} \cdot r_2 e^{i\phi}| = 1$$

$$\Rightarrow r_1 r_2 = 1 \quad \dots(i)$$

$$\text{and } \arg(z) - \arg(w) = \frac{\pi}{2}$$

$$\Rightarrow \theta - \phi = \frac{\pi}{2} \quad \dots(ii)$$

$$\text{Now, } \bar{z}w = r_1 e^{-i\theta} \cdot r_2 e^{i\phi}$$

$$= r_1 r_2 e^{-i(\theta - \phi)}$$

$$= 1 \cdot e^{i\pi/2} \quad [\text{from Eqs. (i) and (ii)}]$$

$$= \cos \frac{\pi}{2} - i \sin \frac{\pi}{2}$$

$$\therefore \bar{z}w = -i$$

$$\begin{aligned} 32. \text{ Now, } \left(\frac{1+i}{1-i} \right)^x &= \left[\frac{(1+i)(1+i)}{(1-i)(1+i)} \right]^x \\ &= \left[\frac{(1+i)^2}{1-i^2} \right]^x = \left[\frac{1-1+2i}{2} \right]^x \end{aligned}$$

$$\Rightarrow \left(\frac{1+i}{1-i} \right)^x = (i)^x = 1 \quad [\text{given}]$$

$$\Rightarrow (i)^x = (i)^{4n}$$

where, n is any positive integer.

$$\therefore x = 4n$$

$$33. \text{ Now, } (1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7 \quad [\because 1 + \omega + \omega^2 = 0]$$

$$= (-2\omega^2)^7$$

$$= -2^7 \cdot \omega^{14}$$

$$= -128(\omega^3)^4 \omega^2$$

$$= -128\omega^2 \quad [\because \omega^3 = 1]$$

34. Given, $|\beta| = 1$

$$\therefore \frac{|\beta - \alpha|}{|1 - \bar{\alpha}\beta|} = \frac{|\beta - \alpha|}{|\beta\bar{\beta} - \bar{\alpha}\beta|} \quad [\because 1 = \beta\bar{\beta}]$$

$$= \frac{1}{|\beta|} \frac{|\beta - \alpha|}{|\beta - \alpha|} = \frac{1}{(1)} \times 1 = 1 \quad [\because |\bar{z}| = |z|]$$

35. Given, $z = \frac{(\sqrt{3} + i)^3(3i + 4)^2}{(8 + 6i)^2}$

$$\therefore |z| = \frac{|(\sqrt{3} + i)^3(3i + 4)^2|}{|(8 + 6i)^2|}$$

$$= \frac{|(\sqrt{3} + i)^3| |(3i + 4)^2|}{|8 + 6i|^2} \quad \left[\because \frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|} \right]$$

$$= \frac{|\sqrt{3} + i|^3 |3i + 4|^2}{|8 + 6i|^2} \quad [\because |z^n| = |z|^n]$$

$$= \frac{(\sqrt{3+1})^3 (\sqrt{9+16})^2}{(\sqrt{64+36})^2} = \frac{(2)^3 (5)^2}{(10)^2} = \frac{10^2 \cdot 2}{(10)^2} = 2$$

36. Given, $|w| = 1$ and $z = \frac{w-1}{w+1}$

$$\Rightarrow z(w+1) = w-1 \Rightarrow w(z-1) = -1-z$$

$$\Rightarrow w = \frac{1+z}{1-z} \Rightarrow |w| = \left| \frac{1+z}{1-z} \right|$$

$$\Rightarrow 1 = \frac{|1+z|}{|1-z|} \Rightarrow |1-z| = |1+z|$$

On squaring both sides, we get

$$1^2 + |z|^2 - 2\operatorname{Re}(z) = 1^2 + |z|^2 - 2\operatorname{Re}(z)$$

$$\Rightarrow 4\operatorname{Re}(z) = 0$$

$$\therefore \operatorname{Re}(z) = 0$$

37. Let $z = x + iy$

$$\therefore |z|^2 + |z-3|^2 + |z-i|^2$$

$$= |x+iy|^2 + |(x-3)+iy|^2 + |x+i(y-1)|^2$$

$$= x^2 + y^2 + (x-3)^2 + y^2 + x^2 + (y-1)^2$$

$$= x^2 + y^2 + x^2 - 6x + 9 + y^2 + x^2 + y^2 + 1 - 2y$$

$$= 3x^2 + 3y^2 - 6x - 2y + 10$$

$$= 3(x^2 - 2x + 1) + 3\left(y^2 - \frac{2}{3}y + \frac{1}{9}\right) + 10 - 3 - \frac{1}{3}$$

$$= 3(x-1)^2 + 3\left(y - \frac{1}{3}\right)^2 + \frac{20}{3}$$

It is minimum, when $x-1=0$ and $y-\frac{1}{3}=0$

$$\therefore x=1 \quad \text{and} \quad y=\frac{1}{3}$$

$$\therefore z = 1 + \frac{1}{3}i$$

38.
$$\frac{1+i}{\left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4}\right)} = \frac{\sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)}{e^{-i\pi/4}}$$

$$= \frac{\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)}{e^{-i\pi/4}}$$

$$= \sqrt{2}e^{i\pi/4 + i\pi/4} = \sqrt{2}e^{2i\pi/4} = \sqrt{2}e^{i\pi/2}$$

$$= \sqrt{2}\left[\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right]$$

39. Let $z_1 = x + iy$ and $z_2 = p - iq$, where $x, q > 0$

$$\text{Given, } |z_1| = |z_2|$$

$$\Rightarrow x^2 + y^2 = p^2 + q^2 \quad \dots(i)$$

$$\therefore \frac{z_1 + z_2}{z_1 - z_2} = \frac{(x+p) + i(y-q)}{(x-p) + i(y+q)}$$

$$= \frac{-2i(xq + yp)}{(x-p)^2 + (y+q)^2} \quad \dots(ii)$$

If $xq + yp \neq 0$, then it is purely imaginary and if

$$xq + yp = 0 \quad \text{or} \quad \frac{x}{p} = -\frac{y}{q} = \lambda \quad [\text{say}]$$

$$\Rightarrow x = p\lambda, y = -q\lambda$$

From Eq. (i), we get

$$p^2 + q^2 = \lambda^2(p^2 + q^2)$$

$$\Rightarrow \lambda^2 = 1$$

$$\Rightarrow \lambda = \pm 1$$

For $\lambda = -1$ and $z_1 \neq z_2$, but $|z_1| = |z_2|$

In this case, Eq. (ii) is zero.

40. Given, $z = e^{2\pi/3}$

$$\therefore 1 + z + 3z^2 + 2z^3 + 2z^4 + 2z^4 + 3z^5$$

$$= 1 + e^{2\pi/3} + 3(e^{2\pi/3})^2 + 2(e^{2\pi/3})^3$$

$$+ 2(e^{2\pi/3})^4 + 3(e^{2\pi/3})^5$$

$$= 1 + \left[\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right]$$

$$+ 3 \left[\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{8\pi}{3}\right) \right]$$

$$+ 2 \left[\cos 2\pi + i \sin 2\pi \right] + 2 \left[\cos \frac{8\pi}{3} + i \sin \frac{8\pi}{3} \right]$$

$$+ 3 \left[\cos\left(\frac{10\pi}{3}\right) + i \sin\left(\frac{10\pi}{3}\right) \right]$$

$$= 1 + [\cos 120^\circ + i \sin 120^\circ] + 3[\cos 240^\circ$$

$$+ i \sin 240^\circ] + 2[\cos 360^\circ + i \sin 360^\circ]$$

$$+ 2[\cos 480^\circ + i \sin 480^\circ] + 3[\cos 600^\circ + i \sin 600^\circ]$$

$$= 1 + \left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right] + 3 \left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right]$$

$$+ 2[1 + 0] + 2[\cos 120^\circ + i \sin 120^\circ]$$

$$+ 3[\cos 120^\circ - i \sin 120^\circ]$$

$$= 3 - 2 - \sqrt{2} + 2 \left[-\frac{1}{2} + \frac{\sqrt{3}}{2}i \right] + 3 \left[-\frac{1}{2} - \frac{\sqrt{3}}{2}i \right]$$

$$= 1 - \sqrt{3}i + \left(\frac{-5}{2}\right) - \frac{\sqrt{3}}{2}i = -\frac{3}{2} - \frac{3\sqrt{3}}{2}i$$

$$= -3 \left[\frac{1}{2} + \frac{\sqrt{3}}{2}i \right] = -3e^{i\pi/3}$$

41. Given, $(x-1)^3 = -5$

$$(x-1) = -5, -5\omega, -5\omega^2$$

$$\Rightarrow x = 1-5, 1-5\omega, 1-5\omega^2$$

$$\Rightarrow x = -4, 1-5\omega, 1-5\omega^2$$

42. Given, $\left| \frac{z-3i}{z+3i} \right| = 1$

$$\Rightarrow |z-3i| = |z+3i|$$

[if $|z-z_1| = |z-z_2|$, then it is perpendicular bisector of z_1 and z_2]

Hence, perpendicular bisector of $(0, 3)$ and $(0, -3)$ is X-axis.

$$\begin{aligned} 43. (z_3 - z_1) &= (z_2 - z_1)(\cos 90^\circ + i \sin 90^\circ) \\ \Rightarrow (z_3 - z_1) &= i(z_2 - z_1) \\ \Rightarrow (z_3 - z_1)^2 &= -(z_2 - z_1)^2 \\ \Rightarrow (z_3 - z_1)^2 + (z_2 - z_1)^2 &= 0 \end{aligned}$$

Aliter

$$\begin{aligned} \text{Let } z_1 &= 0, z_2 = 1 - i \text{ and } z_3 = 1 + i \\ \text{Now, } (z_1 - z_2)^2 + (z_1 - z_3)^2 &= [0 - (1 - i)]^2 + [0 - (1 + i)]^2 \\ &= (1 - i)^2 + (1 + i)^2 \\ &= 1^2 + i^2 - 2i + 1^2 + i^2 + 2i \\ &= 1 - 1 + 1 - 1 = 0 \end{aligned}$$

44. We know that,
 $|z - z_1| + |z - z_2| = k$ will represent an ellipse, if
 $|z_1 - z_2| < k$.
Hence, the equation $|z - z_1| + |z - z_2| = 2|z_1 - z_2|$,
represents an ellipse.

45. Given, $z_1 \neq \pm 1$

Since, z_2 and z_3 can be obtained by rotating vector
representing through $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$, respectively.

$$\begin{aligned} \therefore z_2 &= z_1 \omega \\ \text{and } z_3 &= z_1 \omega^2 \\ \therefore z_1 z_2 z_3 &= z_1 \times z_1 \omega \times z_1 \omega^2 \\ &= z_1^3 \omega^3 = z_1^3 \quad [\because \omega^3 = 1] \end{aligned}$$

46. Since, z_1, z_2 and z_3 are the vertices of an equilateral
triangle, therefore

$$\begin{aligned} |z_1 - z_2| &= |z_2 - z_3| \\ &= |z_3 - z_1| = k \quad [\text{say}] \\ \text{Also, } \alpha &= \frac{1}{2}(\sqrt{3} + i) \\ \Rightarrow |\alpha| &= \frac{1}{2}\sqrt{3+1} = \frac{1}{2} \times 2 = 1 \\ \text{Let } A &= \alpha z_1 + \beta, B = \alpha z_2 + \beta \\ \text{and } C &= \alpha z_3 + \beta \\ \text{Now, } |AB| &= |\alpha z_2 + \beta - (\alpha z_1 + \beta)| \\ &= |\alpha(z_2 - z_1)| \\ &= |\alpha| |z_2 - z_1| \\ &= 1 |z_2 - z_1| = |z_2 - z_1| \\ &= |z_2 - z_1| = k \end{aligned}$$

Similarly, $BC = CA = k$

Hence, the points $\alpha z_1 + \beta, \alpha z_2 + \beta$ and $\alpha z_3 + \beta$ are the
vertices of an equilateral triangle.

47. Given, equation is

$$\begin{aligned} 1 + z + z^3 + z^4 &= 0. \\ \Rightarrow (1 + z) + z^3(1 + z) &= 0 \\ \Rightarrow (1 + z)(1 + z^3) &= 1 \\ \Rightarrow z = -1, z^3 = -1 \Rightarrow z = -1 \\ \therefore z &= -1, -\omega, -\omega^2 \end{aligned}$$

Hence, roots are in cubic roots of unity.

Hence, these roots are the vertices of an equilateral
triangle.

48. Given, $z_1 = 2\sqrt{2}(1 + i)$

$$\begin{aligned} \text{and } z_2 &= 1 + i\sqrt{3} \\ \therefore z_1^2 z_2^3 &= [2\sqrt{2}(1 + i)]^2 [1 + i\sqrt{3}]^3 \\ &= 8(1^2 + i^2 + 2i)[1^3 + (i\sqrt{3})^3] + 3 \cdot 1 \cdot i\sqrt{3}(1 + i\sqrt{3}) \end{aligned}$$

$$\begin{aligned} &= 8(1 - 1 + 2i)[1 - 3\sqrt{3}i + 3\sqrt{3}i(1 + i\sqrt{3})] \\ &= 8(2i)[1 - 9] \\ &= 16i(-8) = -128i \end{aligned}$$

$$\begin{aligned} 49. 1 + i^3 + 3i(1 + i) + 1 - i^3 - 3i(1 - i) \\ &= 2 + 3i + 3i^2 - 3i + 3i^2 \\ &= 2 + 6i^2 \\ &= 2 - 6 = -4 \end{aligned}$$

$$\begin{aligned} 50. z &= \frac{i(\sqrt{3} + i)^4}{4(1 - \sqrt{3}i)^2} = \frac{i}{4} \cdot \frac{(2 + 2\sqrt{3}i)^2}{-2 - 2\sqrt{3}i} = \frac{i(-2 + 2\sqrt{3}i)}{2(-1 - \sqrt{3}i)} \\ &= \frac{i(-2 + \sqrt{3}i)}{(-1 - \sqrt{3}i)} \times \frac{(-1 + i\sqrt{3})}{(-1 + i\sqrt{3})} \\ &= \frac{i(-2 - 2i\sqrt{3})}{4} = \left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) \end{aligned}$$

$$\therefore \text{Amplitude} = -\frac{\pi}{6}$$

$$\begin{aligned} 51. \because |z_1| &= |z_2| = |z_3| = 1 \\ \Rightarrow |z_1|^2 &= |z_2|^2 = |z_3|^2 = 1 \\ \therefore z_1 \bar{z}_1 &= z_2 \bar{z}_2 = z_3 \bar{z}_3 = 1 \\ \Rightarrow z_1 &= \frac{1}{\bar{z}_1}, z_2 = \frac{1}{\bar{z}_2} \\ \text{and } z_3 &= \frac{1}{\bar{z}_3} \end{aligned}$$

$$\begin{aligned} \therefore |z_1 + z_2 + z_3| &= \left| \frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} \right| \\ &= 1 = 1 \end{aligned}$$

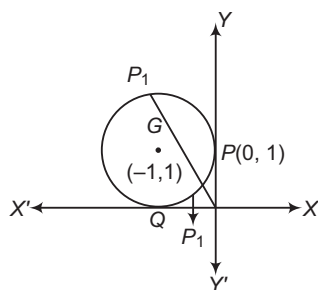
$$\begin{aligned} 52. \text{ Given, } (\sqrt{3}i + 1)^{100} &= 2^{99}(a + ib) \\ \therefore 2^{100} \left(\frac{\sqrt{3}}{2}i + \frac{1}{2} \right)^{100} &= 2^{99}(a + ib) \quad \left[\because \omega^2 = \frac{-1 - \sqrt{3}i}{2} \right] \\ \Rightarrow 2(-\omega^2)^{100} &= a + ib \\ \Rightarrow 2 \times \omega^{200} &= a + ib \\ \Rightarrow 2 \times (\omega^3)^{66} \times \omega^2 &= a + ib \\ \Rightarrow 2 \times (1)^{66} \times \left(\frac{-1 - \sqrt{3}i}{2} \right) &= a + ib \\ \Rightarrow -1 - \sqrt{3}i &= a + ib \\ \therefore a^2 + b^2 &= 4 \end{aligned}$$

$$\begin{aligned} 53. (1 + \sqrt{3}i)^4 + (1 - \sqrt{3}i)^4 &= (-2\omega^2)^4 + (-2\omega)^4 \\ &= 16\omega^8 + 16\omega^4 \\ &= 16(\omega^3)^2 \omega^2 + 16(\omega^3)\omega \\ &= 16(\omega^2 + \omega) = 16(-1) = -16 \quad [\because 1 + \omega + \omega^2 = 0] \end{aligned}$$

$$\begin{aligned} 54. \therefore 1^{1/4} &= (\cos \theta + i \sin \theta)^{1/4} \\ &= (\cos 2\pi r + i \sin 2\pi r)^{1/4} \\ &= \cos \frac{\pi r}{2} + i \sin \frac{\pi r}{2} \end{aligned}$$

$$\begin{aligned} \text{where, } r &= 0, 1, 2, 3 \\ \therefore 1^{1/4} &= 1, i, -1, -i \\ \therefore \text{Required value} &= 1^2 + i^2 + (-1)^2 + (-i)^2 \\ &= 1 - 1 + 1 - 1 \\ &= 0 \end{aligned}$$

55. From figure, it is clear that argument $\geq 90^\circ$.



The given equation $|z - (-1 + i)| \leq 1$ represents the points inside and on the boundary of the circle, centred at $(-1, 1)$ and whose radius is 1. It lies in 2nd quadrant. The point $(0, 1)$, i.e. i lies on it such that, it has least argument.

$\therefore$ The required complex number is i .

56. $(|z_1| - |z_2|)^2 = |z_1 - z_2|^2$
 $\Rightarrow |z_1|^2 + |z_2|^2 - 2|z_1||z_2| = |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 = 2n\pi + \theta_2$

Thus, if $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$

and $z_2 = r_2(\cos \theta_1 + i \sin \theta_1)$

$\Rightarrow z_1 = r_1(\cos \theta_2 + i \sin \theta_2)$

and $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$

$\Rightarrow \frac{z_1}{z_2} = \frac{r_1}{r_2}$

$\Rightarrow \operatorname{Im}\left(\frac{z_1}{z_2}\right) = 0$

57. Given, $(x + iy)(1 - 2i) = 1 + i$

$\Rightarrow (x + iy)(1 - 2i) = 1 + i$

$\Rightarrow (x - iy)(1 + 2i) = 1 + i \quad \dots(i)$

$\Rightarrow x - iy = \frac{1 + i}{1 + 2i}$

$\Rightarrow \overline{x - iy} = \overline{\left(\frac{1 + i}{1 + 2i}\right)} = \frac{1 - i}{1 - 2i} \quad \dots(ii)$

$\Rightarrow x + iy = \frac{1 - i}{1 - 2i}$

58. We have, $\left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)^{50} = 3^{25}(x + iy)$

$\Rightarrow (\sqrt{3})^{50} \left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)^{50} = 3^{25}(x + iy)$

$\Rightarrow 3^{25} \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\right)^{50} = 3^{25}(x + iy)$

$\Rightarrow \cos 50 \frac{\pi}{6} + i \sin 50 \frac{\pi}{6} = x + iy$

$\Rightarrow \cos 25 \frac{\pi}{3} + i \sin 25 \frac{\pi}{3} = x + iy$

$\Rightarrow \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} = x + iy$

$\Rightarrow \frac{1}{2} + i\frac{\sqrt{3}}{2} = x + iy$

$\therefore (x, y) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

59. Locus of z is a circle passing through origin.

60. Given, $2x = 3 + 5i$

$\Rightarrow 8x^3 = 27 - 125i + 27 + 5i - 25 \times 9$

$\Rightarrow 2x^3 = \frac{-198 + 10i}{4} \quad \dots(i)$

Also, $4x^2 = 9 - 25 + 30i$

$\Rightarrow 2x^2 = \frac{-16 + 30i}{2} = -8 + 15i \quad \dots(ii)$

$\therefore 2x^3 + 2x^2 - 7x + 72$
 $= \frac{-198 + 10i}{4} + (-8 + 15i) - x + 72$

[form Eqs. (i) and (ii)]

$= \frac{58 + 70i - 28x}{4}$

$= \frac{58 + 70i - 14(3 + 5i)}{4}$

$= \frac{16}{4} = 4$

61. We have, $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2\operatorname{Re}(z_1 \bar{z}_2) = |z_1|^2 + |z_2|^2$

$\Rightarrow \operatorname{Re}(z_1 \bar{z}_2) = 0$

$\Rightarrow z_1 \bar{z}_2 + z_2 \bar{z}_1 = 0$

$\Rightarrow \frac{z_1}{z_2} = -\frac{\bar{z}_1}{\bar{z}_2}$
 $= -\left(\frac{\bar{z}_1}{z_2}\right)$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

62. Let $z = \frac{1 + i\sqrt{3}}{(i + 2)^2} = \frac{(1 + i\sqrt{3})(2i)}{3 + 4i}$

$= \frac{(1 + i\sqrt{3})(2i)(3 - 4i)}{25}$

$\Rightarrow |z| = \frac{|1 + i\sqrt{3}||2i||3 - 4i|}{|25|}$
 $= \frac{2 \times 2 \times 5}{25} = \frac{4}{5}$

63. We have,

$|z_1 + z_2|^2 = (|z_1|^2 + |z_2|^2 + 2|z_1||z_2|)$

$\Rightarrow |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \theta_1 = \theta_2$

$\Rightarrow \arg(z_1) = \arg(z_2)$

$\therefore \arg\left(\frac{z_1}{z_2}\right) = 0$

64. Given,

$S = 1 + 2\omega + 3\omega^2 + \dots + 3n\omega^{3n-1}$

$\therefore S\omega = \omega + 2\omega^2 + 2\omega^2 + \dots + (3n - 1)\omega^{3n-1} + 3n\omega^{2n}$

$\Rightarrow S(1 - \omega) = 1 + \omega + \omega^2 + \dots + \omega^{3n-1} - 3n\omega^{3n} = 0 - 3n$

$\Rightarrow S = -\frac{3n}{1 - \omega} = \frac{3n}{\omega - 1}$

$$\begin{aligned}
 65. \text{ Let } z &= \left[\frac{2 \cos^2 \left(\frac{\theta}{4} \right) - i 2 \sin \left(\frac{\theta}{4} \right) \cos \left(\frac{\theta}{4} \right)}{2 \cos^2 \left(\frac{\theta}{4} \right) + 2i \sin \left(\frac{\theta}{4} \right) \cos \left(\frac{\theta}{4} \right)} \right]^{4n} \\
 &= \left[\frac{\cos \left(\frac{\theta}{4} \right) - i \sin \left(\frac{\theta}{4} \right)}{\cos \left(\frac{\theta}{4} \right) + i \sin \left(\frac{\theta}{4} \right)} \right]^{4n} \\
 &= \frac{\cos n\theta - i \sin n\theta}{\cos n\theta + i \sin n\theta} = \frac{(\cos \theta + i \sin \theta)^{-n}}{(\cos \theta + i \sin \theta)^n} \\
 &= (\cos \theta + i \sin \theta)^{-2n} = \cos 2n\theta - i \sin 2n\theta
 \end{aligned}$$

$$\begin{aligned}
 66. \text{ Given, } x &= \frac{-3 + i\sqrt{3}}{2} \\
 \Rightarrow x &= \frac{-1 + i\sqrt{3}}{2} - 1 = \omega - 1 \\
 \therefore (x^2 + 3x)^2 (x^2 + 3x + 1) &= [(\omega - 1)^2 + 3(\omega - 1)]^2 [(\omega - 1)^2 + 3(\omega - 1) + 1] \\
 &= (\omega^2 + \omega - 2)^2 (\omega^2 + \omega - 1) \\
 &= (-1 - 2)^2 (-1 - 1) = -18 \quad [\because 1 + \omega + \omega^2 = 0]
 \end{aligned}$$

$$\begin{aligned}
 67. \text{ Given, } (x + iy)^{1/3} &= 2 + 3i \\
 \Rightarrow x + iy &= (2 + 3i)^3 \\
 &= 8 + 36i + 54i^2 + 27i^3 \\
 &= -46 + 9i
 \end{aligned}$$

On equating real and imaginary parts from both sides, we get

$$\begin{aligned}
 x &= -46, y = 9 \\
 \therefore 3x + 2y &= -138 + 18 = -120
 \end{aligned}$$

$$\begin{aligned}
 68. \text{ Let } z &= x + iy \\
 \therefore |z + 3 - i| &= |(x + 3) + i(y - 1)| = 1 \\
 \Rightarrow \sqrt{(x + 3)^2 + (y - 1)^2} &= 1 \quad \dots(i) \\
 \therefore \arg z = \pi &\Rightarrow \tan^{-1} \frac{y}{x} = \pi
 \end{aligned}$$

$$\Rightarrow \frac{y}{x} = \tan \pi = 0 \Rightarrow y = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\begin{aligned}
 x &= -3, y = 0 \\
 \therefore z &= -3 \\
 \Rightarrow |z| &= |-3| = 3
 \end{aligned}$$

$$\begin{aligned}
 69. \text{ Given, } |z_1| &= |z_2| = \dots = |z_n| = 1 \\
 \Rightarrow |z_1|^2 &= |z_2|^2 = \dots = |z_n|^2 = 1 \\
 \Rightarrow z_1 \bar{z}_1 &= z_2 \bar{z}_2 = \dots = z_n \bar{z}_n = 1 \\
 \Rightarrow \bar{z}_1 &= \frac{1}{z_1}, \bar{z}_2 = \frac{1}{z_2}, \dots, \bar{z}_n = \frac{1}{z_n} \quad \dots(i)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } |z_1 + z_2 + \dots + z_n| &= |\bar{z}_1 + \bar{z}_2 + \dots + \bar{z}_n| \\
 &= \left| \frac{1}{z_1} + \frac{1}{z_2} + \dots + \frac{1}{z_n} \right| \quad [\text{from Eq. (i)}]
 \end{aligned}$$

$$\begin{aligned}
 70. \therefore |z_1| &= \sqrt{2} \text{ and } |z_2| = \sqrt{3} \\
 \therefore |z_1 z_2| &= |z_1| |z_2| = \sqrt{6}
 \end{aligned}$$

$$71. \frac{z}{\bar{z}} + \frac{\bar{z}}{z} = \frac{re^{i\theta}}{re^{-i\theta}} + \frac{re^{-i\theta}}{re^{i\theta}} = e^{i2\theta} + e^{-i2\theta} = 2 \cos 2\theta$$

$$\begin{aligned}
 72. z &= \frac{4(1+i)}{2} = 2(1+i) = \frac{4}{1+i} \\
 \therefore \bar{z} &= \frac{4}{1-i}
 \end{aligned}$$

$$\begin{aligned}
 73. \text{ If } \arg(z) &= -\pi + \theta \\
 \Rightarrow \arg(\bar{z}) &= \pi - \theta \\
 \arg(-\bar{z}) &= -\theta \\
 \therefore \arg(\bar{z}) - \arg(-\bar{z}) &= \pi - \theta - (-\theta) \\
 &= \pi - \theta + \theta = \pi
 \end{aligned}$$

$$\begin{aligned}
 74. \frac{\cos 30^\circ + i \sin 30^\circ}{\cos 60^\circ - i \sin 60^\circ} &= (\cos 30^\circ + i \sin 30^\circ)(\cos 60^\circ + i \sin 60^\circ) \\
 &= \cos 90^\circ + i \sin 90^\circ = i
 \end{aligned}$$

5

Inequalities and Quadratic Equation

Inequality

An equation having signs $>$, $<$, $\geq$ or $\leq$ at the place of sign of equality ($=$) is known as inequality or inequation.

or

Let a and b be two real numbers. If $a - b$ is negative, we say that a is less than b ($a < b$) and if $a - b$ is positive, then a is greater than b ($a > b$).

Properties of Inequalities

We shall learn about some elementary properties of inequalities which will be used in the subsequent discussion.

- i. If $a > b$ and $b > c$, then $a > c$. Generally, if $a_1 > a_2, a_2 > a_3, \dots, a_{n-1} > a_n$, then $a_1 > a_n$
- ii. If $a > b$, then $a \pm c > b \pm c, \forall c \in R$
- iii. If $a > b$, then for $m > 0, am > bm, \frac{a}{m} > \frac{b}{m}$ and for $m < 0, bm > am, \frac{b}{m} > \frac{a}{m}$
- iv. If $a > b > 0$, then

(a) $a^2 > b^2$	(b) $ a > b $	(c) $\frac{1}{a} < \frac{1}{b}$

 and if $a < b < 0$, then

(a) $a^2 > b^2$	(b) $ a > b $	(c) $\frac{1}{a} > \frac{1}{b}$
- v. If $a < 0 < b$, then

(a) $a^2 > b^2$, if $ a > b $	(b) $a^2 < b^2$, if $ a < b $
- vi. If $a < x < b$ and a, b are positive real numbers, then

$$a^2 < x^2 < b^2$$
- vii. If $a < x < b$ and a is negative number and b is positive number, then

(a) $0 \leq x^2 < b^2$, if $ b > a $	(b) $0 \leq x^2 \leq a^2$, if $ a > b $

Chapter Snapshot

- Inequality
- Generalised Method of Intervals for Solving Inequalities by Wavy Curve Method (Line Rule)
- Absolute Value of a Real Number
- Logarithms
- Arithmetico-Geometric Mean Inequality
- Quadratic Equation with Real Coefficients
- Formation of a Polynomial Equation from Given Roots
- Symmetric Function of the Roots
- Transformation of Equations
- Common Roots
- Quadratic Expression and its Graph
- Maximum and Minimum Values of Rational Expression
- Location of the Roots of a Quadratic Equation
- Algebraic Interpretation of Rolle's Theorem
- Condition for Resolution into Linear Factors
- Some Application of Graphs to Find the Roots of Equations

- viii. If $x \in [a, b] \Rightarrow \begin{cases} \frac{1}{x} \in \left[\frac{1}{b}, \frac{1}{a} \right], \\ \text{if } a \text{ and } b \text{ have same sign} \\ \frac{1}{x} \in \left(-\infty, \frac{1}{a} \right] \cup \left[\frac{1}{b}, \infty \right), \\ \text{if } a \text{ and } b \text{ have opposite} \\ \text{signs} \end{cases}$
- ix. If $\frac{a}{b} > 0$, then
(a) $a > 0$, if $b > 0$ (b) $a < 0$, if $b < 0$
- x. If $a_i > b_i > 0$, where $i = 1, 2, 3, \dots, n$, then
 $a_1 a_2 a_3 \dots a_n > b_1 b_2 b_3 \dots b_n$
- xi. If $a_i > b_i$, where $i = 1, 2, 3, \dots, n$, then
 $a_1 + a_2 + a_3 + \dots + a_n > b_1 + b_2 + \dots + b_n$
- xii. If $0 < a < 1$ and n is a positive rational number, then
(a) $0 < a^n < 1$ (b) $a^{-n} > 1$
- xiii. If a and b are positive real numbers such that $a < b$ and if n is any positive rational number, then
(a) $a^n < b^n$ (b) $a^{-n} > b^{-n}$
(c) $a^{1/n} < b^{1/n}$
- xiv. If $a > 1$ and n is any positive rational number, then
(a) $a^n > 1$ (b) $0 < a^{-n} < 1$
- xv. If $0 < a < 1$ and m, n are positive rational numbers, then
(a) $m > n \Rightarrow a^m < a^n$
(b) $m < n \Rightarrow a^m > a^n$
- xvi. If $a > 1$ and m, n are positive rational numbers, then
(a) $m > n \Rightarrow a^m > a^n$
(b) $m < n \Rightarrow a^m < a^n$

➤ **Example 1.** If a, b and c are positive real numbers such that $a < b < c$, then show that

$$\frac{a^2}{c} < \frac{a^2 + b^2 + c^2}{a + b + c} < \frac{c^2}{a}.$$

Sol. We have, $c > b > a > 0$
 $\Rightarrow c^2 > b^2 > a^2 > 0$
 Now, $a < b < c$
 $\Rightarrow 3a < a + b + c < 3c$
 $\Rightarrow \frac{1}{3c} < \frac{1}{a + b + c} < \frac{1}{3a}$... (i)

and $a^2 < b^2 < c^2$
 $\Rightarrow 3a^2 < a^2 + b^2 + c^2 < 3c^2$... (ii)

From Eqs. (i) and (ii), we get

$$\frac{3a^2}{3c} < \frac{a^2 + b^2 + c^2}{a + b + c} < \frac{3c^2}{3a}$$

$$\Rightarrow \frac{a^2}{c} < \frac{a^2 + b^2 + c^2}{a + b + c} < \frac{c^2}{a}$$

➤ **Example 2.** The value of x for which $12x - 6 < 0$, $12 - 3x < 0$, is

- (a) ϕ (b) R
 (c) $R - \{0\}$ (d) None of these

Sol. (a) $12x - 6 < 0$

$$\Rightarrow 12x < 6 \Rightarrow x < \frac{1}{2}$$

and $12 - 3x < 0$... (i)

$$\Rightarrow 12 < 3x$$

$$\Rightarrow 4 < x$$

$$\Rightarrow x > 4$$
 ... (ii)

So, there is no real number x satisfying both the inequalities (i) and (ii).

Hence, the given system of inequalities has no solution.

➤ **Example 3.** The value of x for which

$$\frac{x-3}{4} - x < \frac{x-1}{2} - \frac{x-2}{3}, 2-x > 2x-8$$

- (a) $\left[-1, \frac{10}{3}\right]$ (b) $\left(-1, \frac{10}{3}\right)$
 (c) R (d) None of these

Sol. (b) We have, $\frac{x-3}{4} - x < \frac{x-1}{2} - \frac{x-2}{3}$

$$\Rightarrow 3x - 9 - 12x < 6x - 6 - 4x + 8$$

$$\Rightarrow -11x < 11 \Rightarrow -x < 1$$

$$\Rightarrow x > -1$$
 ... (i)

and $2 - x > 2x - 8$

$$\Rightarrow -3x > -10$$

$$\Rightarrow x < \frac{10}{3}$$
 ... (ii)

From Eqs. (i) and (ii), the solution of the given system of inequalities is given by $x \in \left(-1, \frac{10}{3}\right)$.

Generalised Method of Intervals for Solving Inequalities by Wavy Curve Method (Line Rule)

$$\text{Let } F(x) = (x - a_1)^{k_1} (x - a_2)^{k_2} \dots (x - a_{n-1})^{k_{n-1}} (x - a_n)^{k_n}$$

where, $k_1, k_2, \dots, k_n \in \mathbb{Z}$ and $a_1, a_2, \dots, a_n$ are fixed real numbers satisfying the condition

$$a_1 < a_2 < a_3 < \dots < a_{n-1} < a_n$$

For solving $F(x) > 0$ or $F(x) < 0$, consider the following algorithm:

- We mark the numbers $a_1, a_2, \dots, a_n$ on the number axis and put the plus sign in the interval on the right of the largest of these numbers, i.e. on the right of a_n .
- Then, we put the plus sign in the interval on the left of a_n , if k_n is an even number and the minus sign, if k_n is an odd number. In the next interval, we put a sign according to the following rule:
- When passing through the point a_{n-1} the polynomial $F(x)$ changes sign, if k_{n-1} is an odd number. Then, we consider the next interval and put a sign in it using the same rule.
- Thus, we consider all the intervals. The solution of the inequality $F(x) > 0$ is the union of all intervals in which we have put the plus sign and the solution of the inequality $F(x) < 0$ is the union of all intervals in which we have put the minus sign.

Solution of Rational Algebraic Inequation

If $P(x)$ and $Q(x)$ are polynomial in x , then the inequation

$$\frac{P(x)}{Q(x)} > 0, \frac{P(x)}{Q(x)} < 0, \frac{P(x)}{Q(x)} \geq 0$$

and $\frac{P(x)}{Q(x)} \leq 0$

are known as rational algebraic inequations.

To solve these inequations, we use the sign method as explained in the following algorithm :

Algorithm

- Step I** Obtain $P(x)$ and $Q(x)$.
- Step II** Factorise $P(x)$ and $Q(x)$ into linear factors.
- Step III** Make the coefficient of x positive in all factors.
- Step IV** Obtain critical points by equating all factors to zero.
- Step V** Plot the critical points on the number line. If there are n critical points, then they divide the number line into $(n+1)$ regions.
- Step VI** In the right most region the expression $\frac{P(x)}{Q(x)}$ bears positive sign and in other regions the expression bears positive and negative signs depending on the exponents of the factors.

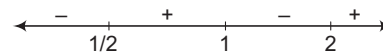
➤ **Example 4.** Solve $(x-1)(x-2)(1-2x) > 0$.

Sol. We have, $(x-1)(x-2)(1-2x) > 0$

$$\Rightarrow -(x-1)(x-2)(2x-1) > 0$$

$$\Rightarrow (x-1)(x-2)(2x-1) < 0$$

On number line mark $x = \frac{1}{2}, 1, 2$



When $x > 2$, all factors $(x-1)$, $(2x-1)$ and $(x-2)$ are positive.

Hence, $(x-1)(x-2)(2x-1) > 0$ for $x > 2$.

Now, put positive and negative sign alternatively as shown in figure.

Hence, solution set of $(x-1)(x-2)(1-2x) > 0$ or

$$(x-1)(x-2)(2x-1) < 0 \text{ is } \left(-\infty, \frac{1}{2}\right) \cup (1, 2).$$

➤ **Example 5.** Solve $\frac{2x-3}{3x-5} \geq 3$.

Sol. $\frac{2x-3}{3x-5} \geq 3$

$$\Rightarrow \frac{2x-3}{3x-5} - 3 \geq 0$$

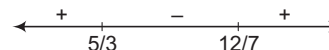
$$\Rightarrow \frac{2x-3-9x+15}{3x-5} \geq 0$$

$$\Rightarrow \frac{-7x+12}{3x-5} \geq 0$$

$$\Rightarrow \frac{7x-12}{3x-5} \leq 0$$

$$\Rightarrow (7x-12)(3x-5) \leq 0$$

Sign scheme of $(7x-12)(3x-5)$ is as follows :



$$\Rightarrow x \in \left[\frac{5}{3}, \frac{12}{7}\right]$$

✎ $x = \frac{5}{3}$ is not included in the solutions as at $x = \frac{5}{3}$ denominator becomes zero.

➤ **Example 6.** The value of x for which

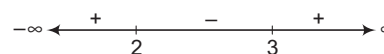
$$(x-2)^3(x-3) < 0, \text{ is}$$

(a) $(2, 3)$ (b) $[2, 3)$ (c) $(0, 3)$ (d) $(2, 3]$

Sol. (a) $(x-2)^3(x-3) < 0$

$$\Rightarrow (x-2)(x-3) < 0$$

[as $(x-2)^2$ is positive for all real values of $x \neq 2$]



Here, interval is open as $x = 2, 3$ do not satisfy inequality.

i.e. $2 < x < 3$ or $x \in (2, 3)$

Work Book Exercise 5.1

5

Inequalities and Quadratic Equation

- 1 The value of x satisfying the inequalities $x + \frac{1}{x} \geq 2$ hold
- (0, ∞) R ϕ $[0, \infty)$
- 2 $\frac{x^2}{x-1} \geq 0$
- (1, ∞) $[1, \infty)$
- $\{0\} \cup (1, \infty)$ None of these
- 3 $(x-2)^4(x-3)^3(x-4)^2(1-x) \leq 0$
- (1, 3)
- $(-\infty, 1) \cup (3, \infty)$
- $(-\infty, 1] \cup [3, \infty)$
- None of the above

- 4 If $c < d$, $x^2 + (c+d)x + cd < 0$

$$\begin{aligned} &(-d, -c] \\ &(-d, -c) \\ &R \\ &\phi \end{aligned}$$

- 5 $\frac{(2x-1)(x-1)^4(x-2)^4}{(x-2)(x-4)^4} \leq 0$
- $\left[\frac{1}{2}, 2\right)$
- R
- ϕ
- $\left(\frac{1}{2}, 2\right)$

Absolute Value of a Real Number

The absolute value of a real number ' x ' is denoted by $|x|$ and defined by $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

- (i) $|x|$ is also defined as $\sqrt{x^2}$
 - (ii) If x is positive, then $\sqrt{x^2} = x$
 - (iii) If x is negative, then $\sqrt{x^2} = -x$
- e.g. $\sqrt{9} = \sqrt{(-3)^2} = \sqrt{3^2} = 3$
- or $\sqrt{9} = \pm 3$
- (i) $a \leq |a|$
 - (ii) $|ab| = |a||b|$
 - (iii) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
 - (iv) $|a+b| \leq |a| + |b|$ [triangle inequality]
 - (v) $|a-b| \geq ||a| - |b||$ [triangle inequality]
 - (vi) $|a+b| = |a| + |b|$, iff $ab \geq 0$
 - (vii) $|a+b| = ||a| - |b||$, iff $ab \leq 0$

Inequations Containing Absolute Values

By definition,

$$|x| < a \Rightarrow -a < x < a \quad (a > 0)$$

$$|x| \leq a \Rightarrow -a \leq x \leq a$$

$$|x| > a \Rightarrow x < -a \text{ and } x > a$$

$$|x| \geq a \Rightarrow x \leq -a \text{ and } x \geq a$$

$$a \leq |x| \leq b \Rightarrow x \in [-b, -a] \cup [a, b]$$

where, $a, b > 0$

Forms of the Inequations Containing Absolute Values

Form 1 The inequation of the form

$f(|x|) < g(x)$ is equivalent to the collection of systems

$$\begin{cases} f(x) < g(x), & \text{if } x \geq 0 \\ f(-x) < g(x), & \text{if } x < 0 \end{cases}$$

Form 2 The inequation of the form

$|f(x)| < g(x)$, is equivalent to the systems

$$\begin{cases} f(x) < g(x), & \text{if } g(x) > 0 \\ -f(x) < g(x), & \text{if } g(x) > 0 \end{cases}$$

Form 3 The inequation of the form

$|f(x)| > g(x)$, is equivalent to the systems

$$\begin{cases} f(x) > g(x), & \text{if } g(x) < 0 \\ -f(x) > g(x), & \text{if } g(x) < 0 \end{cases}$$

Form 4 The inequation of the form

$|f(|x|)| \geq g(x)$ is equivalent to the collection of systems

$$\begin{cases} |f(x)| \geq g(x), & \text{if } x \geq 0 \\ |f(-x)| \geq g(x), & \text{if } x < 0 \end{cases}$$

Form 5 The inequation of the form

$|f(x)| \geq |g(x)|$ is equivalent to the collection of system

$$f^2(x) \geq g^2(x)$$

Form 6 The inequation of the form

$$h(x, |f(x)|) \geq g(x)$$

is equivalent to the collection of systems

$$\begin{cases} h\{x, f(x)\} \geq g(x), & \text{if } f(x) \geq 0 \\ h\{x, -f(x)\} \geq g(x), & \text{if } f(x) < 0 \end{cases}$$

➤ **Example 7.** The solution set of $|3 - 4x| \geq 9$ is

- (a) $\left(-\infty, \frac{3}{2}\right] \cup [3, \infty)$
 (b) $\left(-\infty, -\frac{3}{2}\right] \cup [3, \infty)$
 (c) $(-\infty, 2) \cup [2, \infty)$
 (d) None of the above

Sol. (b) We have,

$$\begin{aligned} |3 - 4x| &\geq 9 \\ \Rightarrow 3 - 4x &\leq -9 \quad \text{or} \quad 3 - 4x \geq 9 \\ &\quad [\text{since, } |x| \geq a \Rightarrow x \leq -a \text{ or } x \geq a] \\ \Rightarrow -4x &\leq -12 \quad \text{or} \quad -4x \geq 6 \\ \Rightarrow x &\geq 3 \quad \text{or} \quad x \leq -\frac{3}{2} \quad [\text{dividing both sides by } -4] \\ \Rightarrow x &\in \left(-\infty, -\frac{3}{2}\right] \cup [3, \infty) \end{aligned}$$

➤ **Example 8.** The solution set of $\frac{|x+3|+x}{x+2} > 1$ is

- (a) $[-5, -2] \cup [-1, \infty)$ (b) $[-5, -2) \cup [-1, \infty)$
 (c) $(-5, -2) \cup (-1, \infty)$ (d) None of these

Sol. (c) We have, $\frac{|x+3|+x}{x+2} > 1$

$$\begin{aligned} \Rightarrow \frac{|x+3|+x}{x+2} - 1 &> 0 \\ \Rightarrow \frac{|x+3|-2}{x+2} &> 0 \end{aligned}$$

Case I When $x+3 \geq 0$, i.e. $x \geq -3$

$$\begin{aligned} \therefore \frac{|x+3|-2}{x+2} &> 0 \\ \Rightarrow \frac{x+3-2}{x+2} &> 0 \\ \Rightarrow \frac{x+1}{x+2} &> 0 \\ \Rightarrow \{x+1 > 0 \text{ and } x+2 > 0\} \text{ or } \{x+1 < 0 \text{ and } x+2 < 0\} \\ \Rightarrow \{x > -1 \text{ and } x > -2\} \text{ or } \{x < -1 \text{ and } x < -2\} \\ \Rightarrow x > -1 \text{ or } x < -2 \\ \Rightarrow x \in (-1, \infty) \text{ or } x \in (-\infty, -2) \\ \Rightarrow x \in (-3, -2) \cup (-1, \infty) \quad [\because x \geq -3] \dots (i) \end{aligned}$$

Case II When $x+3 < 0$, i.e. $x < -3$

$$\begin{aligned} \therefore \frac{|x+3|-2}{x+2} &> 0 \\ \Rightarrow \frac{-x-3-2}{x+2} &> 0 \\ \Rightarrow \frac{-(x+5)}{x+2} &> 0 \\ \Rightarrow \frac{(x+5)}{x+2} &< 0 \\ \Rightarrow \{x+5 < 0 \text{ and } x+2 > 0\} \text{ or } \{x+5 > 0 \text{ and } x+2 < 0\} \\ \Rightarrow (x < -5 \text{ and } x > -2), \text{ which is not possible} \\ \text{or } \{x > -5 \text{ and } x < -2\} \\ \Rightarrow x \in (-5, -2) \dots (ii) \end{aligned}$$

From Eqs. (i) and (ii), we get

$$x \in (-5, -2) \cup (-1, \infty)$$

➤ **Example 9.** Solve the inequation

$$\left|1 - \frac{|x|}{1+|x|}\right| \geq \frac{1}{2}$$

Sol. The given inequation is equivalent to the collection of systems.

$$\begin{aligned} \Rightarrow \begin{cases} 1 - \frac{x}{1+x} \geq \frac{1}{2}, & \text{if } x \geq 0 \\ 1 + \frac{x}{1-x} \geq \frac{1}{2}, & \text{if } x < 0 \end{cases} &\Rightarrow \begin{cases} \frac{1}{1+x} \geq \frac{1}{2}, & \text{if } x \geq 0 \\ \frac{1}{1-x} \geq \frac{1}{2}, & \text{if } x < 0 \end{cases} \\ \Rightarrow \begin{cases} \frac{1}{1+x} \geq \frac{1}{2}, & \text{if } x \geq 0 \\ \frac{1}{1-x} \geq \frac{1}{2}, & \text{if } x < 0 \end{cases} &\Rightarrow \begin{cases} \frac{1-x}{1+x} \geq 0, & \text{if } x \geq 0 \\ \frac{1+x}{1-x} \geq 0, & \text{if } x < 0 \end{cases} \\ \Rightarrow \begin{cases} \frac{1-x}{1+x} \geq 0, & \text{if } x \geq 0 \\ \frac{1+x}{1-x} \geq 0, & \text{if } x < 0 \end{cases} &\Rightarrow \begin{cases} \frac{x-1}{x+1} \leq 0, & \text{if } x \geq 0 \\ \frac{x+1}{x-1} \leq 0, & \text{if } x < 0 \end{cases} \end{aligned}$$

For $\frac{x-1}{x+1} \leq 0$, if $x \geq 0$, then

$$0 \leq x \leq 1 \dots (i)$$

For

$$\frac{x+1}{x-1} \leq 0, \text{ if } x < 0$$

$\therefore$

$$-1 \leq x < 0 \dots (ii)$$

From Eqs. (i) and (ii), the solution of the given equation is $x \in [-1, 1]$.

➤ **Example 10.** The value of x , $|x+3| > |2x-1|$ is

- (a) $\left(-\frac{2}{3}, 4\right)$ (b) $\left(-\frac{2}{3}, \infty\right)$
 (c) $(0, 1)$ (d) None of these

Sol. (a) Given, $|x+3| > |2x-1|$

On squaring both sides, we get

$$\begin{aligned} |x+3|^2 &> |2x-1|^2 \\ \Rightarrow \{(x+3) - (2x-1)\} \{(x+3) + (2x-1)\} &> 0 \\ \Rightarrow \{(-x+4)(3x+2)\} &> 0 \\ \Rightarrow x \in \left(-\frac{2}{3}, 4\right) \end{aligned}$$

Logarithms

Let there be a number $a > 0, \neq 1$. A number p is called the logarithm of a number x to the base a , if $a^p = x$ and is written as $p = \log_a x$. Obviously x must be positive.

- If $x < 0$, $\log_a x$ is imaginary and if $x = 0$, $\log_a x$ does not exist.
- $\log_a x$ exists if and only if $x, a > 0$ and $a \neq 1$.

Properties of Logarithms

- i. $a^{\log_a x} = x; a \neq 0, \pm 1, x > 0$
- ii. $a^{\log_b x} = x^{\log_b a}; a > 0, b > 0, \neq 1, x > 0$
- iii. $\log_a a = 1, \log_a 1 = 0; a > 0, \neq 1$
- iv. $\log_a x = \frac{1}{\log_x a}; x, a > 0, \neq 1$
- v. $\log_a x = \log_b x \cdot \log_a b = \frac{\log_b x}{\log_b a}; a, b > 0, \neq 1, x > 0$
- vi. For $m, n > 0, a > 0, \neq 1$
 - (a) $\log_a (m \cdot n) = \log_a m + \log_a n$
 - (b) $\log_a \left(\frac{m}{n}\right) = \log_a m - \log_a n$
 - (c) $\log_a (m^x) = x \log_a m$
- vii. For $x > 0, a > 0, \neq 1$
 - (a) $\log_{a^n} (x) = \left(\frac{1}{n}\right) \log_a x$
 - (b) $\log_{a^n} (x^m) = \left(\frac{m}{n}\right) \log_a x$
- viii.
 - (a) $\log_a x > 0$, iff $x > 1, a > 1$ or $0 < x < 1, 0 < a < 1$
 - (b) $\log_a x < 0$, iff $x > 1, 0 < a < 1$ or $0 < x < 1, a > 1$
- ix. For $x > y > 0$,
 - (a) $\log_a x > \log_a y$, if $a > 1$
 - (b) $\log_a x < \log_a y$, if $0 < a < 1$

- x. If $a > 1$, then
 - (a) $\log_a x > p \Rightarrow x > a^p$
 - (b) $0 < \log_a x < p \Rightarrow 0 < x < a^p$
- xi. If $0 < a < 1$, then
 - (a) $\log_a x > p \Rightarrow 0 < x < a^p$
 - (b) $0 < \log_a x < p \Rightarrow a^p < x < 1$

➤ **Example 11.** The value of x , $\log_e (x - 3) < 1$ is
 (a) (0, 3) (b) (0, e) (c) (0, e + 3) (d) (3, 3 + e)

Sol. (d) From definition of logarithms

$$\begin{aligned} x - 3 &> 0 \\ \Rightarrow x &> 3 \quad \dots(i) \end{aligned}$$

Also, $e > 1$ given inequality may written as

$$x - 3 < (e)^1 \Rightarrow x < 3 + e$$

$$\Rightarrow x \in (3, 3 + e)$$

➤ **Example 12.** The value of x , $\log_{1/2} x \geq \log_{1/3} x$ is
 (a) (0, 1] (b) (0, 1)
 (c) [0, 1] (d) None of these

Sol. (a) **Case I** When $x \neq 1$ and $x > 0$

$$\begin{aligned} \log_{1/2} x &\geq \log_{1/3} x \\ \Rightarrow \log_x 2 &\geq \log_x 3, \text{ when } x \neq 1 \end{aligned}$$

which is possible, only if $0 < x < 1$.

Case II When $x = 1$

$$\log_{1/2} x = \log_{1/3} x, \text{ i.e. equality holds.}$$

Combining the above cases,

$$0 < x \leq 1 \text{ or } x \in (0, 1]$$

Work Book Exercise 5.2

Solve for x ,

- 1 $|x + 4| > 5$
 (−∞, 1) (−∞, 9)
 R − [−9, 1] None of these
- 2 $|x + 2| < 4$
 (−6, 2) (−6, 0) (−6, 2] (0, 2)
- 3 $\log_2 x > 0$
 (0, ∞) (−∞, 0) (1, ∞) (4, ∞)
- 4 $\log_4 x > 1$
 (0, ∞) (4, ∞) (−4, ∞) (−∞, 4)
- 5 $\log_{0.2} (x + 5) > 0$
 (−5, −4) (−5, 4) (0, 4) (0, 5)
- 6 $\log_x 0.5 > 2$
 $\left(\sqrt{\frac{1}{2}}, 1\right)$ (−∞, 1) (−1, 0) (−1, 1)

7 $(x - 1)^2 > 9$

$$\begin{aligned} &(-\infty, -2) \cup (4, \infty) \\ &(4, \infty) \\ &(-\infty, 2) \\ &(-\infty, -2] \cup (4, \infty) \end{aligned}$$

8 $(x + 1)^2 < 25$

$$\begin{aligned} &(-6, 4] \quad (-6, 4) \\ &(4, 6) \quad (-4, 6) \end{aligned}$$

9 $\log_x (x + 7) < 0$

$$(0, 1) \quad (-\infty, 1) \quad (-1, 0) \quad (-1, 1)$$

10 $\left|\frac{x^2 + 6}{5x}\right| \geq 1$

$$\begin{aligned} &(-\infty, -3) \\ &(-\infty, -3) \cup (3, \infty) \\ &R \\ &(-\infty, -3] \cup [-2, 0) \cup (0, 2] \cup [3, \infty) \end{aligned}$$

Arithmetico-Geometric Mean Inequality

If $a_1, a_2, \dots, a_n$ are n distinct positive real numbers, then

$$\frac{a_1 + a_2 + \dots + a_n}{n} > (a_1 a_2 \dots a_n)^{1/n}$$

i.e. AM > GM

➤ If $a_1 = a_2 = \dots = a_n$, then AM = GM. Thus, equality occurs when all quantities are equal.

➤ **Example 13.** If a, b and c are three distinct positive real numbers, then

$$(a) (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) > 9$$

$$(b) (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) < 9$$

$$(c) (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9$$

$$(d) (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq 9$$

Sol. (a) We have, AM > GM

$$\begin{aligned} \therefore \frac{a + b + c}{3} &> (abc)^{1/3} \text{ and } \frac{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}{3} > \left(\frac{1}{abc} \right)^{1/3} \\ \Rightarrow a + b + c &> 3(abc)^{1/3} \text{ and } \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) > \frac{3}{(abc)^{1/3}} \\ \Rightarrow (a + b + c) &\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) > 9 \end{aligned}$$

Weighted AM and GM Inequalities

If $a_1, a_2, \dots, a_n$ are n positive real numbers and $m_1, m_2, \dots, m_n$ are n positive rational numbers, then

$$\frac{m_1 a_1 + m_2 a_2 + \dots + m_n a_n}{m_1 + m_2 + \dots + m_n} > (a_1^{m_1} \cdot a_2^{m_2} \dots a_n^{m_n})^{\frac{1}{m_1 + m_2 + \dots + m_n}}$$

i.e. weighted AM > weighted GM

➤ **Example 14.** If a, b and c are distinct positive integers, then $ax^{b-c} + bx^{c-a} + cx^{a-b}$ is

$$(a) > 2(a + b + c)$$

$$(b) R$$

$$(c) > (a + b + c)$$

$$(d) < (a + b + c)$$

Sol. (c) We have,

$$\begin{aligned} \frac{a \cdot x^{b-c} + b \cdot x^{c-a} + c \cdot x^{a-b}}{a + b + c} &> \\ \frac{\{ (x^{b-c})^a (x^{c-a})^b (x^{a-b})^c \}^{1/a+b+c}}{a + b + c} &> \\ \frac{ax^{b-c} + bx^{c-a} + cx^{a-b}}{a + b + c} &> \\ \frac{\{ x^{a(b-c) + b(c-a) + c(a-b)} \}^{1/a+b+c}}{a + b + c} &> \\ \Rightarrow ax^{b-c} + bx^{c-a} + cx^{a-b} &> (a + b + c) \end{aligned}$$

➤ **Example 15.** If the equation

$x^4 - 4x^3 + ax^2 + bx + 1 = 0$ has four positive roots, then a, b are

$$(a) a = 4, b = 6$$

$$(b) a = -4, b = 6$$

$$(c) a = 2, b = 3$$

$$(d) a = 6, b = -4$$

Sol. (d) Let α, β, γ and δ be four roots of the given equation.

Then, $\alpha + \beta + \gamma + \delta = 4$ and $\alpha\beta\gamma\delta = 1$

$\Rightarrow$ AM of $\alpha, \beta, \gamma, \delta$ = GM of $\alpha, \beta, \gamma, \delta \Rightarrow \alpha = \beta = \gamma = \delta$

$\Rightarrow \alpha = \beta = \gamma = \delta = 1$ [$\because \alpha + \beta + \gamma + \delta = 4$, also $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = a$, $\alpha\beta\gamma + \alpha\beta\delta + \beta\gamma\delta + \alpha\gamma\delta = -b$]

$$\therefore x^4 - 4x^3 + ax^2 + bx + 1 = x^4 - 4x^3 + 6x^2 - 4x + 1$$

$$\Rightarrow a = 6 \text{ and } b = -4$$

Two Important Inequalities

i. If $a_1, a_2, \dots, a_n$ are n positive distinct real numbers, then

$$(a) \frac{a_1^m + a_2^m + \dots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)^m,$$

if $m < 0$ or $m > 1$

$$(b) \frac{a_1^m + a_2^m + \dots + a_n^m}{n} < \left(\frac{a_1 + a_2 + \dots + a_n}{n} \right)^m$$

if $0 < m < 1$.

i.e. the arithmetic mean of the m th power of n positive quantities is greater than m th power of their arithmetic mean except when m is a positive proper fraction known as m th power mean.

(c) If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are rational numbers and m is a rational number, then

$$\frac{b_1 a_1^m + b_2 a_2^m + \dots + b_n a_n^m}{b_1 + b_2 + \dots + b_n} > \left(\frac{b_1 a_1 + b_2 a_2 + \dots + b_n a_n}{b_1 + b_2 + \dots + b_n} \right)^m,$$

if $m < 0$ or $m > 1$ and

$$\frac{b_1 a_1^m + b_2 a_2^m + \dots + b_n a_n^m}{b_1 + b_2 + \dots + b_n}$$

$$< \left(\frac{b_1 a_1 + b_2 a_2 + \dots + b_n a_n}{b_1 + b_2 + \dots + b_n} \right)^m, \text{ if } 0 < m < 1$$

ii. If $a_1, a_2, a_3, \dots, a_n$ are distinct positive real numbers and p, q, r are natural numbers, then

$$\begin{aligned} \frac{a_1^{p+q+r} + a_2^{p+q+r} + \dots + a_n^{p+q+r}}{n} &> \\ > \left(\frac{a_1^p + a_2^p + \dots + a_n^p}{n} \right) \left(\frac{a_1^q + a_2^q + \dots + a_n^q}{n} \right) \left(\frac{a_1^r + a_2^r + \dots + a_n^r}{n} \right) \end{aligned}$$

➤ **Example 16.** If $m > 1$ and $n \in \mathbb{N}$, then

$$(a) \frac{1^m + 2^m + 3^m + \dots + n^m}{n} > \left(\frac{n+1}{2}\right)^m$$

$$(b) \frac{1^m + 2^m + 3^m + \dots + n^m}{n} < \left(\frac{n+1}{2}\right)^m$$

$$(c) \frac{1^m + 2^m + 3^m + \dots + n^m}{n} \geq 1$$

$$(d) \frac{1^m + 2^m + 3^m + \dots + n^m}{n} \leq 1$$

Sol. (a) We know that for $m > 1$,
AM of m th power $>$ m th power of AM

$$\therefore \frac{1^m + 2^m + 3^m + \dots + n^m}{n} > \left(\frac{1 + 2 + 3 + \dots + n}{n}\right)^m$$

$$\Rightarrow \frac{1^m + 2^m + 3^m + \dots + n^m}{n} > \left(\frac{n+1}{2}\right)^m$$

Some Other Standard Inequalities

i. Weierstrass Inequality

(a) If $a_1, a_2, \dots, a_n$ are n positive real numbers, then for $n \geq 2$,
 $(1 + a_1)(1 + a_2) \dots (1 + a_n) > 1 + a_1 + a_2 + \dots + a_n$

(b) If $a_1, a_2, \dots, a_n$ are positive numbers less than unity, then
 $(1 - a_1)(1 - a_2) \dots (1 - a_n) > 1 - a_1 - a_2 - \dots - a_n$

ii. Cauchy-Schwartz Inequality

If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are $2n$ real numbers, then

$$(a_1 b_1 + a_2 b_2 + \dots + a_n b_n)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^2 + b_2^2 + \dots + b_n^2)$$

with the equality holding if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n}$$

iii. Tchebychev's Inequality

If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are real numbers such that
 $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n$ and $b_1 \leq b_2 \leq \dots \leq b_n$, then

$$n(a_1 b_1 + a_2 b_2 + \dots + a_n b_n) \geq (a_1 + a_2 + \dots + a_n)(b_1 + b_2 + \dots + b_n)$$

or

$$\frac{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}{n} \geq \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right) \left(\frac{b_1 + b_2 + \dots + b_n}{n}\right)$$

➤ **Example 17.** If none of $b_1, b_2, \dots, b_n$ is zero,

then $\left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n}\right)^2$ is

$$(a) \geq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$(b) \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$(c) > (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$(d) < (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

Sol. (b) By Cauchy-Schwartz's inequality, we have

$$(a_1 b_1^{-1} + a_2 b_2^{-1} + \dots + a_n b_n^{-1})^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$\Rightarrow \left(\frac{a_1}{b_1} + \frac{a_2}{b_2} + \dots + \frac{a_n}{b_n}\right)^2 \leq (a_1^2 + a_2^2 + \dots + a_n^2) \left(\frac{1}{b_1^2} + \frac{1}{b_2^2} + \dots + \frac{1}{b_n^2}\right)$$

Applications of Inequalities to Find the Greatest and Least Values

i. If $x_1, x_2, \dots, x_n$ are n positive variables such that $x_1 + x_2 + \dots + x_n = c$ (constant), then the product $x_1 x_2 \dots x_n$ is greatest when

$$x_1 = x_2 = \dots = x_n = \frac{c}{n} \text{ and the greatest value is } \left(\frac{c}{n}\right)^n.$$

ii. If $x_1, x_2, \dots, x_n$ are positive variables such that $x_1 x_2 \dots x_n = c$ (constant), then the sum $x_1 + x_2 + \dots + x_n$ is least when $x_1 = x_2 = \dots = x_n = c^{1/n}$ and the least value of the sum is $n(c^{1/n})$.

iii. If $x_1, x_2, \dots, x_n$ are variables and $m_1, m_2, \dots, m_n$ are positive real numbers such that $x_1 + x_2 + \dots + x_n = c$ (constant), then $x_1^{m_1} \cdot x_2^{m_2} \dots x_n^{m_n}$ is greatest, when

$$\frac{x_1}{m_1} = \frac{x_2}{m_2} = \dots = \frac{x_n}{m_n} = \frac{x_1 + x_2 + \dots + x_n}{m_1 + m_2 + \dots + m_n}$$

➤ **Example 18.** The greatest value of $x^2 y^3$ when $3x + 4y = 5$, is

$$(a) \frac{3}{8} \quad (b) \frac{3}{16} \quad (c) \frac{6}{5} \quad (d) \frac{8}{3}$$

Sol. (b) Let $P = x^2 y^3$. Clearly, P is the product of 5 factors such that two of them are equal to x and the remaining 3 are equal to y .

$$\text{Now, } 3x + 4y = 5$$

$$\Rightarrow 2\left(\frac{3x}{2}\right) + 3\left(\frac{4y}{3}\right) = 5$$

$$\Rightarrow \frac{3x}{2} + \frac{3x}{2} + \frac{4y}{3} + \frac{4y}{3} + \frac{4y}{3} = 5$$

$$\Rightarrow \left(\frac{16}{3} x^2 y^3\right) \leq \left(\frac{5}{5}\right)^5 \Rightarrow x^2 y^3 \leq \frac{3}{16}$$

or maximum of $x^2 y^3 = 3/16$

- 1 If a, b and c are all positive real numbers, which one of the following is true?

$$(b+c)(c+a)(a+b) \geq 8abc$$

$$(b+c)(c+a)(a+b) < 8abc$$

$$\left(\frac{a}{e} + \frac{b}{f} + \frac{c}{g}\right)\left(\frac{e}{a} + \frac{f}{b} + \frac{g}{c}\right) < 9$$

$$(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) < 4$$

- 2 If $a_1, a_2, \dots, a_n$ and $b_1, b_2, \dots, b_n$ are any two sets of positive real numbers, then

$$\left(\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + \dots + \frac{a_n^2}{b_n}\right)^2$$

$$\leq (a_1^4 + a_2^4 + \dots + a_n^4)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$\geq (a_1^4 + a_2^4 + \dots + a_n^4)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$< (a_1^4 + a_2^4 + \dots + a_n^4)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

$$> (a_1^4 + a_2^4 + \dots + a_n^4)(b_1^{-2} + b_2^{-2} + \dots + b_n^{-2})$$

- 3 If a and b are two positive quantities whose sum is λ , then the minimum value of $\sqrt{\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)}$ is

$$\lambda - \frac{1}{\lambda} \quad \lambda - \frac{2}{\lambda} \quad 1 + \frac{2}{\lambda} \quad 1 + \frac{1}{\lambda}$$

- 4 If $a^2x^4 + b^2y^4 = c^6$, then the greatest value of xy is

$$\frac{c^3}{2ab} \quad \frac{c^3}{4ab} \quad \frac{c^3}{2\sqrt{ab}} \quad \frac{c^3}{\sqrt{2ab}}$$

- 5 Which of the following is true, where $a, b, c > 0$?

$$2(a^3 + b^3 + c^3) \geq bc(b+c) + ca(c+a) + ab(a+b)$$

$$\frac{a^3 + b^3 + c^3}{3} > \frac{(a+b+c) \cdot (a^2 + b^2 + c^2)}{9}$$

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} < \frac{1}{2}(a+b+c)$$

$$\frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} < \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Quadratic Equation with Real Coefficients

An equation of the form

$$ax^2 + bx + c = 0 \quad \dots(i)$$

where, $a \neq 0, a, b, c \in R$ is called a quadratic equation with real coefficients.

The quantity $D = b^2 - 4ac$ is known as the discriminant of the quadratic equation in (i) whose roots are given by

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

The nature of the roots is as given below:

- If a quadratic equation in x has more than two roots, then it is an identity in x that is $a = b = c = 0$.
- If $a = 1$ and $b, c \in I$ and the roots are rational numbers, then these roots must be integers.
- The roots are of the form $p \pm \sqrt{q}$ ($p, q \in Q$), iff a, b and c are rational and D is not a perfect square.
- The roots are real and distinct, iff $D > 0$.
- The roots are real and equal, iff $D = 0$.
- The roots are complex with non-zero imaginary part, iff $D < 0$.
- The roots are rational, iff a, b and c are rational and D is a perfect square.

- If α is a root of the equation $f(x) = 0$, then the polynomial $f(x)$ is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of $f(x)$ and conversely.
- Every equation of n th degree ($n \geq 1$) has exactly n roots and if the equation has more than n roots, it is an identity.
- If the coefficients of the equation $f(x) = 0$ are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root, i.e. imaginary roots occur in conjugate pairs.
- If the coefficients in the equation are all rational and $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root, where $\alpha, \beta \in Q$ and β is not a perfect square.
- If there is any two real numbers ' a ' and ' b ' such that $f(a)$ and $f(b)$ are of opposite signs, then $f(x) = 0$ must have at least one real root between ' a ' and ' b '.
- Every equation $f(x) = 0$ of odd degree has at least one real root of a sign opposite to that of its last term.

Example 19. The number of values of a for which $(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a^2 - 4 = 0$ is an identity in x , is

- (a) 0 (b) 2
(c) 1 (d) 3

Sol. (b) It is an identity in x , if

$$a^2 - 3a + 2 = 0, a^2 - 5a + 6 = 0, a^2 - 4 = 0$$

$$\Rightarrow a = 1, 2 \text{ and } a = 2, 3 \text{ and } a = 2, -2$$

$$\therefore \text{Equation is identity, if } a = 2$$

➤ **Example 20.** If $\cos \theta$, $\sin \phi$ and $\sin \theta$ are in GP, then roots of $x^2 + 2\cot \phi \cdot x + 1 = 0$ are always

- (a) equal (b) real
(c) imaginary (d) greater than 1

Sol. (b) As $\cos \theta$, $\sin \phi$ and $\sin \theta$ are in GP.

$$\Rightarrow \sin^2 \phi = \cos \theta \cdot \sin \theta$$

$$\Rightarrow \cos 2\phi = 1 - \sin 2\theta \quad \dots(i)$$

Also, discriminant, $D = 4\cot^2 \phi - 4 = \frac{4\cos 2\phi}{\sin^2 \phi}$

$$= \frac{4(1 - \sin 2\theta)}{\sin^2 \phi} > 0; \text{ as } \sin 2\theta < 1 \text{ and } \sin^2 \phi > 0$$

$\Rightarrow$ Roots are real.

Also, when $\cos \theta = \sin \theta$

$\Rightarrow$ Three numbers are equal which is a special case i.e. the series forms both AP and GP.

➤ **Example 21.** If l , m and n are real, $l \neq m$, then the roots of the equation

$$(l - m)x^2 - 5(l + m)x - 2(l - m) = 0 \text{ are}$$

- (a) real and equal (b) complex
(c) real and unequal (d) None of these

Sol. (c) Discriminant of the given equation is

$$D = 25(l + m)^2 + 8(l - m)^2$$

As, $l \neq m$, $(l - m)^2 > 0$.

Also, $(l + m)^2 \geq 0$

Thus, $D > 0$

Hence, roots are real and unequal.

➤ **Example 22.** The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is

- (a) 4 (b) 1 (c) 3 (d) 2

Sol. (a) Since, $|x|^2 - 3|x| + 2 = 0$

$$\Rightarrow (|x| - 1)(|x| - 2) = 0 \Rightarrow |x| = 1, 2$$

$$\therefore x = 1, -1, 2, -2$$

Hence, four real solutions exist.

➤ **Example 23.** Let α and β be the roots of the equation $(x - a)(x - b) = c$, $c \neq 0$. Then, the roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are

- (a) a, c (b) b, c
(c) a, b (d) $a + c, b + c$

Sol. (c) Given α and β are the roots of $(x - a)(x - b) - c = 0$

$$\Rightarrow (x - a)(x - b) - c = (x - \alpha)(x - \beta)$$

$$\Rightarrow (x - a)(x - b) = (x - \alpha)(x - \beta) + c$$

$\Rightarrow a$ and b are the roots of equation

$$(x - \alpha)(x - \beta) + c = 0$$

➤ **Example 24.** If the roots of the equation $ax^2 + x + b = 0$ are real, then the roots of the equation $x^2 - 4\sqrt{ab}x + 1 = 0$ will be

- (a) rational (b) irrational
(c) real (d) imaginary

Sol. (d) Since, $ax^2 + x + b = 0$ has real roots

$$\Rightarrow (1)^2 - 4ab \geq 0$$

$$\Rightarrow -4ab \geq -1$$

$$\text{or } 4ab \leq 1 \quad \dots(i)$$

Now, second equation is $x^2 - 4\sqrt{ab}x + 1 = 0$

$$\therefore D = 16ab - 4$$

$$\text{From Eq. (i), } D < 0$$

Hence, roots are imaginary.

➤ **Example 25.** Let a, b and c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is the root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$, then the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies

$$(a) \gamma = \frac{\alpha + \beta}{2} \quad (b) \gamma = \alpha + \frac{\beta}{2}$$

$$(c) \gamma = \alpha \quad (d) \alpha < \gamma < \beta$$

Sol. (d) Since, α is a root of $a^2x^2 + bx + c = 0$

$$\Rightarrow a^2\alpha^2 + b\alpha + c = 0 \quad \dots(i)$$

and β is a root of $a^2x^2 - bx - c = 0$

$$\Rightarrow a^2\beta^2 - b\beta - c = 0 \quad \dots(ii)$$

$$\text{Let } f(x) = a^2x^2 + 2bx + 2c$$

$$\therefore f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c$$

$$= a^2\alpha^2 - 2a^2\alpha^2 = -a^2\alpha^2 \quad [\text{from Eq. (i)}]$$

$$\text{and } f(\beta) = a^2\beta^2 + 2b\beta + 2c$$

$$= a^2\beta^2 + 2a^2\beta^2 = 3a^2\beta^2 \quad [\text{from Eq. (ii)}]$$

$$\Rightarrow f(\alpha)f(\beta) < 0$$

$\therefore f(x)$ must have a root lying in the open interval (α, β) .

$$\therefore \alpha < \gamma < \beta$$

➤ **Example 26.** Find the values of a for which

$$4^t - (a - 4)2^t + \frac{9}{4}a < 0, \forall t \in (1, 2).$$

Sol. Let $2^t = x$ and $f(x) = x^2 - (a - 4)x + \frac{9}{4}a$

$$\text{We want } f(x) < 0, \forall x \in (2^1, 2^2)$$

$$\text{i.e. } \forall x \in (2, 4)$$

Since, we want $f(x) < 0, \forall x \in (2, 4)$, one of the $f(x)$ should be less than 2 and the other must be greater than 4.

i.e. $f(2) < 0$ and $f(4) < 0, a < -48$ and $a > \frac{128}{7}$ which is not possible.

Hence, no such a exists.

➤ **Example 27.** The roots of $ax^2 + bx + c = 0$, where $a \neq 0$ and coefficients are real, are non-real complex and $a + c < b$. Then,

$$(a) 4a + c > 2b \quad (b) 4a + c < 2b$$

$$(c) 4a + c = 2b \quad (d) \text{None of these}$$

Sol. (b) Since, $ax^2 + bx + c = 0$ has non-real.

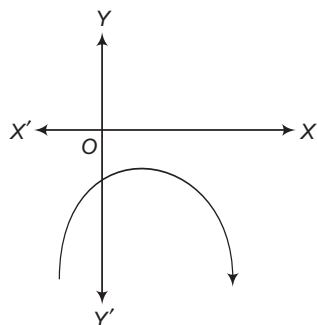
$$\text{Thus, either } f(x) = ax^2 + bx + c$$

$$\Rightarrow f(x) > 0 \text{ or } f(x) < 0, \text{ for all } x$$

$$\text{As } a + c < b \Rightarrow f(-1) < 0$$

$$\Rightarrow a - b + c < 0 \text{ or } a + c < 0$$

$$\therefore f(x) < 0, \text{ for all } x.$$



$$\text{Thus, for all } x \in \mathbb{R}, ax^2 + bx + c < 0$$

$$\text{or for } x = -2, 4a - 2b + c < 0$$

$$\Rightarrow 4a + c < 2b$$

Formation of a Polynomial Equation from Given Roots

If $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ are the roots of an n th degree equation, then the equation is

$$x^n - S_1x^{n-1} + S_2x^{n-2} - S_3x^{n-3} + \dots + (-1)^n S_n = 0$$

where, S_k denotes the sum of the products of roots taking k roots at a time.

Particular Cases

- i. **Quadratic equation** If α and β are the roots of a quadratic equation, then the equation is

$$x^2 - S_1x + S_2 = 0$$
 i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$
- ii. **Cubic equation** If α, β and γ are the roots of a cubic equation, then the equation is

$$x^3 - S_1x^2 + S_2x - S_3 = 0$$
 i.e. $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0$

➤ **Example 28.** If a and b are rational and b is not a perfect square, then the quadratic equation with rational coefficients whose one root is $\frac{1}{a + \sqrt{b}}$,

is

$$(a) x^2 - 2ax + (a^2 - b) = 0$$

$$(b) (a^2 - b)x^2 - 2ax + 1 = 0$$

$$(c) (a^2 - b)x^2 - 2bx + 1 = 0$$

(d) None of the above

Sol. (b) As irrational roots always occur in pairs,

$$\text{i.e. the roots of given equation are } \frac{1}{a + \sqrt{b}}, \frac{1}{a - \sqrt{b}}.$$

$$\text{Thus, sum of roots} = \frac{2a}{a^2 - b}$$

$$\text{and product of roots} = \frac{1}{a^2 - b}$$

Hence, the quadratic equation

$$x^2 - \left(\frac{2a}{a^2 - b}\right)x + \left(\frac{1}{a^2 - b}\right) = 0$$

$$\Rightarrow (a^2 - b)x^2 - 2ax + 1 = 0$$

➤ **Example 29.** The quadratic equation whose roots are the AM and HM of the roots of the equation $x^2 + 7x - 1 = 0$ is

$$(a) 14x^2 + 14x - 45 = 0 \quad (b) 14x^2 - 14x + 14 = 0$$

$$(c) 14x^2 + 45x - 14 = 0 \quad (d) \text{None of these}$$

Sol. (c) Here, $x^2 + 7x - 1 = 0$ has $\alpha + \beta = -7$ and $\alpha\beta = -1$
 $\therefore$ Quadratic equation's roots are AM and HM of the roots of the equation $x^2 + 7x - 1 = 0$

$$\Rightarrow \alpha' = \frac{\alpha + \beta}{2} \text{ and } \beta' = \frac{2\alpha\beta}{\alpha + \beta}$$

$$\text{or } \alpha' = -\frac{7}{2} \text{ and } \beta' = \frac{-2}{-7} = \frac{2}{7}$$

$\therefore$ Required equation is $x^2 - (\alpha' + \beta')x + \alpha'\beta' = 0$

$$\Rightarrow x^2 - \left(-\frac{45}{14}\right)x - 1 = 0$$

$$\text{or } 14x^2 + 45x - 14 = 0$$

Symmetric Function of the Roots

A function of α and β is said to be a symmetric function, if it remains unchanged when α and β are interchanged.

e.g. $\alpha^2 + \beta^2 + 2\alpha\beta$ is a symmetric function of α and β whereas $\alpha^2 - \beta^2 + 3\alpha\beta$ is not a symmetric function of α and β .

In order to find the value of a symmetric function of α and β , express the given function in terms of $\alpha + \beta$ and $\alpha\beta$. The following results may be useful:

$$i. \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$ii. \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$$

$$iii. \alpha^4 + \beta^4 = (\alpha^3 + \beta^3)(\alpha + \beta) - \alpha\beta(\alpha^2 + \beta^2)$$

$$iv. \alpha^5 + \beta^5 = (\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$$

$$v. |\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$vi. \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$$

$$vii. \alpha^3 - \beta^3 = (\alpha - \beta)[(\alpha + \beta)^2 - \alpha\beta]$$

$$= (\alpha - \beta)^3 + 3\alpha\beta(\alpha - \beta)$$

$$viii. \alpha^4 - \beta^4 = (\alpha + \beta)(\alpha - \beta)(\alpha^2 + \beta^2)$$

➤ **Example 30.** If α and β are the roots of $ax^2 + 2bx + c = 0$, then $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ is equal to

(a) $\frac{4b^2 - 2ac}{ac}$ (b) $\frac{4b^2 - 4ac}{ac}$
 (c) $\frac{2b^2 - 2ac}{ac}$ (d) $\frac{2b^2 - 4ac}{ac}$

Sol. (a) Since, α and β are the roots of equation

$$ax^2 + 2bx + c = 0$$

$$\therefore \alpha + \beta = \frac{-2b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\text{Now, } \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$\Rightarrow \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{(4b^2/a^2) - 2c/a}{c/a} = \frac{4b^2 - 2ac}{ac}$$

➤ **Example 31.** If α and β are the roots of the equation $px^2 + qx + r = 0$, then the value of $\alpha^3\beta + \beta^3\alpha$ is

(a) $\frac{r}{p^3}$ (b) $\frac{r(q^2 - 2pr)}{p^3}$
 (c) $\frac{r(q^2 + 2pr)}{p^3}$ (d) $\frac{(q^2 - 2pr)}{p^3}$

Sol. (b) Since, α and β are the roots of the equation $px^2 + qx + r = 0$, therefore $\alpha + \beta = -\frac{q}{p}$ and $\alpha\beta = \frac{r}{p}$.

$$\begin{aligned} \therefore \alpha^3\beta + \beta^3\alpha &= \alpha\beta(\alpha^2 + \beta^2) \\ &= (\alpha\beta)[(\alpha + \beta)^2 - 2\alpha\beta] \\ &= \left(\frac{r}{p}\right)\left(\frac{q^2}{p^2} - \frac{2r}{p}\right) = \frac{r(q^2 - 2pr)}{p^3} \end{aligned}$$

Transformation of Equations

Let α and β be the roots of the equation $ax^2 + bx + c = 0$, then the equation

- Whose roots are $\alpha + k, \beta + k$ is obtained by replacing x by $x - k$ in the given equation.
- Whose roots are $\alpha - k, \beta - k$ is obtained by replacing x by $x + k$ in the given equation.
- Whose roots are $\alpha k, \beta k$ is obtained by multiplying the coefficients of x^2, x and constant term by k^0, k^1, k^2 respectively in the given equation.
- Whose roots are $\frac{\alpha}{k}, \frac{\beta}{k}$ is obtained by multiplying the coefficients of x^2, x and constant term by k^2, k^1, k^0 respectively in the given equation.

v. To obtain an equation whose roots are reciprocals of the roots of a given equation is obtained by replacing x by $1/x$ in the given equation.

vi. To transform an equation to another equation whose roots are negative of the roots of a given equation, replace x by $-x$.

vii. To transform an equation to another equation whose roots are square of the roots of a given equation, replace x by $\sqrt{x}$.

viii. To transform an equation to another equation whose roots are cubes of the roots of a given equation replace x by $x^{1/3}$.

➤ **Example 32.** Form an equation whose roots are cubes of the roots of equation $ax^3 + bx^2 + cx + d = 0$.

Sol. Replacing x by $x^{1/3}$ in the given equation, we get

$$\begin{aligned} a(x^{1/3})^3 + b(x^{1/3})^2 + c(x^{1/3}) + d &= 0 \\ \Rightarrow ax + d &= -(bx^{2/3} + cx^{1/3}) \\ \Rightarrow (ax + d)^3 &= -(bx^{2/3} + cx^{1/3})^3 \\ \Rightarrow a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 &= -\{b^3x^2 + c^3x + 3bcx(bx^{2/3} + cx^{1/3})\} \\ \Rightarrow a^3x^3 + 3a^2dx^2 + 3ad^2x + d^3 &= -\{b^3x^2 + c^3x - 3bcx(ax + d)\} \\ \Rightarrow a^3x^3 + x^2(3a^2d - 3abc + b^3) &+ x(3ad^2 - 3bcd + c^3) + d^3 = 0 \end{aligned}$$

which is the required equation.

➤ **Example 33.** If the roots of $a_1x^2 + b_1x + c_1 = 0$ are α_1, β_1 and that of $a_2x^2 + b_2x + c_2 = 0$ are α_2, β_2 such that $\alpha_1\alpha_2 = \beta_1\beta_2 = 1$, then

(a) $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(b) $\frac{a_1}{c_2} = \frac{b_1}{b_2} = \frac{c_1}{a_2}$

(c) $a_1a_2 = b_1b_2 = c_1c_2$

(d) None of the above

Sol. (b) Roots of the second equation are reciprocal of the first.

$$\text{As } \alpha_2 = \frac{1}{\alpha_1}$$

$$\text{and } \beta_2 = \frac{1}{\beta_1}$$

$\therefore$ Replacing x by $\frac{1}{x}$ in $a_1x^2 + b_1x + c_1 = 0$, we get

$$c_1x^2 + b_1x + a_1 = 0 \quad \dots(i)$$

$$\text{and } a_2x^2 + b_2x + c_2 = 0 \quad [\text{given}] \dots(ii)$$

Eqs. (i) and (ii) are same equations.

$$\therefore \frac{c_1}{a_2} = \frac{b_1}{b_2} = \frac{a_1}{c_2}$$

➤ **Example 34.** If α, β and γ are the roots of the equation $x(1+x^2) + x^2(6+x) + 2 = 0$, then the value of $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ is

- (a) -3 (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) None of these

Sol. (c) Since, $2x^3 + 6x^2 + x + 2 = 0$ has roots α, β and γ .
So, $2x^3 + x^2 + 6x + 2 = 0$ has roots α^{-1}, β^{-1} and γ^{-1}

[writing coefficients in reverse order, since roots are reciprocal]

Hence, sum of the roots = $-\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$

$$\therefore \alpha^{-1} + \beta^{-1} + \gamma^{-1} = -\frac{1}{2}$$

Descartes Rule of Signs for the Roots of a Polynomial

Rule 1 The maximum number of positive real roots of a polynomial equation

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n = 0$$

is the number of changes of the signs of coefficients from positive to negative and negative to positive. For instance, in the equation $x^3 + 3x^2 + 7x - 11 = 0$ the signs of coefficients are

+ + + -

As there is just one change of sign, the number of positive roots of $x^3 + 3x^2 + 7x - 11 = 0$ is at most 1.

Rule 2 The maximum number of negative roots of the polynomial equation $f(x) = 0$ is the number of changes from positive to negative and negative to positive in the signs of coefficients of the equation $f(-x) = 0$.

Common Roots

Let $a_1 x^2 + b_1 x + c_1 = 0$ and $a_2 x^2 + b_2 x + c_2 = 0$ be two quadratic equations such that $a_1, a_2 \neq 0$ and $a_1 b_2 \neq a_2 b_1$.

i. **When one common root** If α is the common root of these equations, then $(b_1 c_2 - b_2 c_1)(a_1 b_2 - a_2 b_1) = (c_1 a_2 - c_2 a_1)^2$ which is the condition for roots of two quadratic equations to be common.

ii. **When two common roots** The required condition is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

➤ **Example 35.** If $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root and a, b and c are non-zero real numbers, then find the value of $(a^3 + b^3 + c^3) / abc$.

Sol. Given that, $ax^2 + bx + c = 0$

and $bx^2 + cx + a = 0$ have a common root. Hence,

$$\begin{aligned} (bc - a^2)^2 &= (ab - c^2)(ac - b^2) \\ \Rightarrow b^2 c^2 + a^4 - 2a^2 bc &= a^2 bc - ab^3 - ac^3 + b^2 c^2 \\ \Rightarrow a^4 + ab^3 + ac^3 &= 3a^2 bc \\ \Rightarrow \frac{a^3 + b^3 + c^3}{abc} &= 3 \end{aligned}$$

➤ **Example 36.** If the equation $x^2 + 5x + 8 = 0$ and $ax^2 + bx + c = 0$; $a, b, c \in R$ have a common root, then $a : b : c$ is

- (a) $1 : 5 : 6$ (b) $1 : 5 : 8$
(c) $8 : 5 : 1$ (d) None of these

Sol. (b) Since, the quadratic equation $x^2 + 5x + 8 = 0$ has imaginary roots. So, equation $ax^2 + bx + c = 0$ will have both roots same as the equation $x^2 + 5x + 8 = 0$.

$$\text{Thus, } \frac{a}{1} = \frac{b}{5} = \frac{c}{8} \Rightarrow a = \lambda, b = 5\lambda, c = 8\lambda$$

Hence, the required ratio is $1 : 5 : 8$.

Work Book Exercise 5.4

- If α and β are the roots of the equation $2x^2 - 3x - 6 = 0$, then equation whose roots are $\alpha^2 + 2, \beta^2 + 2$ is
 $4x^2 + 49x + 118 = 0$ $4x^2 - 49x + 118 = 0$
 $4x^2 - 49x - 118 = 0$ $x^2 - 49x + 118 = 0$
- If $2 + i\sqrt{3}$ is a root of $x^2 + px + q = 0$, where p and q are real, then (p, q) is equal to
 $(-4, 7)$ $(4, -7)$
 $(-7, 4)$ $(4, 7)$

- The equations $ax^2 + bx + a = 0$, $x^3 - 2x^2 + 2x - 1 = 0$ have two roots in common. Then, $a + b$ must be equal to
 1 -1
 0 None of these
- If $|x^2| + |x| - 2 = 0$, then the value of x is equal to
 2
 -2
 ± 1
None of the above

- 5 If $|x - 2| + |x - 9| = 7$, then the set of values of x is

$\{2, 9\}$ $(2, 7)$ $\{2\}$ **d** $[2, 9]$

- 6 If a, b and c are three distinct positive real numbers, then the number of real roots of $ax^2 + 2b|x| + c = 0$ is

0 1 2 4

- 7 If $\sin \alpha, \sin \beta$ and $\cos \alpha$ are in GP, then roots of $x^2 + 2x \cot \beta + 1 = 0$ are always

equal real
imaginary greater than 1

- 8 If $x^2 + ax + b$ is an integer for every integer x , then

d is always an integer but b need not be an integer
 b is always an integer but d need not be an integer
Cannot be discussed
 a and b are always integers

- 9 Let $a > 0, b > 0$ and $c > 0$. Then, both the roots of the equation $ax^2 + bx + c = 0$

are real and negative have negative real part
are rational numbers None of these

- 10 If p, q and r are real and $p \neq q$, then the roots of the equation

$$(p - q)x^2 + 5(p + q)x - 2(p - q) = 0 \text{ are}$$

real and equal complex
real and unequal None of these

- 11 If α and β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$, then the equation $x^2 - 4qx + 2q^2 - r = 0$ has always

two real roots
two positive roots
two negative roots
one positive and one negative root

- 12 The real roots of the equation

$$|x|^3 - 3x^2 + 3|x| - 2 = 0 \text{ are}$$

0, 2 ± 1 ± 2 1, 2

- 13 If α and β are the roots of the quadratic equation $6x^2 - 6x + 1 = 0$, then $\frac{1}{2}(a + b\alpha + c\alpha^2 + d\alpha^3)$

$$+ \frac{1}{2}(a + b\beta + c\beta^2 + d\beta^3) \text{ is equal to}$$

$$\frac{12d + 6c + 4b + a}{12}$$

$$12a + 6b + 4c + 9d$$

$$\frac{1}{12}(12a + 6b + 4c + 3d)$$

None of these

- 14 Both the roots of the equation

$$(x - b)(x - c) + (x - a)(x - c) + (x - a)(x - b) = 0$$

are always

positive negative
real None of these

Quadratic Expression and its Graph

Let a, b and c be real numbers and $a \neq 0$. Then, $f(x) = ax^2 + bx + c$ is known as a quadratic expression or a quadratic polynomial in x . In this section, we shall discuss the graph of the quadratic polynomial, i.e. the curve whose equation is $y = ax^2 + bx + c, a \neq 0$.

Consider, $y = ax^2 + bx + c$

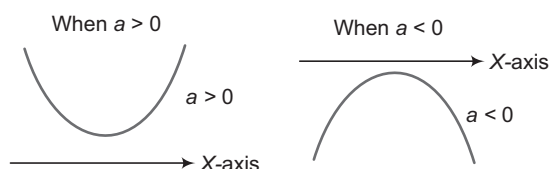
$$= a \left[\left(x + \frac{b}{2a} \right)^2 + \frac{4ac - b^2}{4a^2} \right]$$

$$= a \left[\left(x + \frac{b}{2a} \right)^2 - \frac{D}{4a^2} \right]$$

$$\Rightarrow \left(y + \frac{D}{4a} \right) = a \left(x + \frac{b}{2a} \right)^2$$

Thus, $y = f(x)$ represents a parabola.

The parabola opens upwards or downwards accordingly as $a > 0$ or $a < 0$



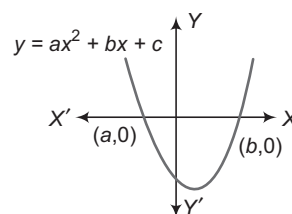
Thus, we may draw the following conclusions:

- (i) **The parabola will intersect the X-axis in two distinct points iff $D > 0$**

In this case, the parabola cuts X-axis at $(\alpha, 0)$ and $(\beta, 0)$, where $\alpha = \frac{-b - \sqrt{D}}{2a}$ and $\beta = \frac{-b + \sqrt{D}}{2a}$

For $a > 0$, we have

$$y_{\min} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}, y_{\max} \rightarrow \infty$$

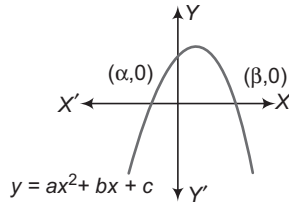


$$\text{Also, } y = f(x) \text{ is } \begin{cases} < 0, & \text{if } \alpha < x < \beta \\ = 0, & \text{if } x = \alpha, \beta \\ > 0, & \text{if } x < \alpha \text{ or } x > \beta \end{cases}$$

For $a < 0$, we have

$$y_{\max} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}, y_{\min} \rightarrow -\infty$$

and $y = f(x)$ is $\begin{cases} < 0, & \text{if } x < \alpha \text{ or } x > \beta \\ = 0, & \text{if } x = \alpha, \beta \\ > 0, & \text{if } \alpha < x < \beta \end{cases}$



(ii) The parabola will just touch the X -axis at one point iff $D = 0$

In this case, the parabola touches X -axis at $(\alpha, 0)$, when

$$\alpha = \beta = \frac{-b}{2a}$$

For $a > 0$, we have

$$y_{\min} = 0$$

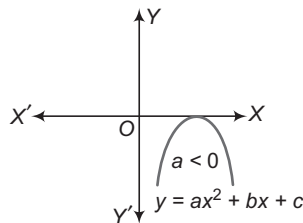
at $x = \frac{-b}{2a}, y_{\max} \rightarrow \infty$

and $y = f(x)$ is $\begin{cases} > 0, & \text{if } x \neq \alpha \\ = 0, & \text{if } x = \alpha \end{cases}$

For $a < 0$, we have

$$y_{\max} = 0 \text{ at } x = \frac{-b}{2a}, y_{\min} \rightarrow -\infty$$

and $y = f(x) = \begin{cases} < 0, & \text{if } x \neq \alpha \\ 0, & \text{if } x = \alpha \end{cases}$



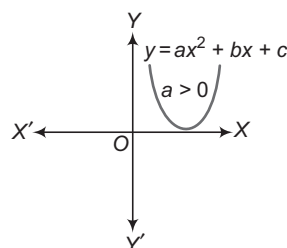
(iii) The parabola will not intersect X -axis iff $D < 0$

In this case, the parabola remains either completely above X -axis or completely below X -axis according as $a > 0$ or $a < 0$.

For $a > 0$, we have

$$y_{\min} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}, y_{\max} \rightarrow \infty$$

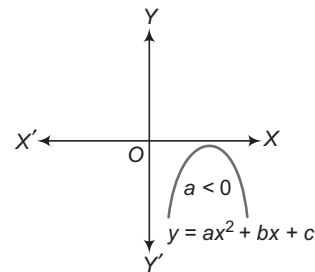
and $y = f(x) > 0, \forall x$



For $a < 0$, we have

$$y_{\max} = \frac{-D}{4a} \text{ at } x = \frac{-b}{2a}, y_{\min} \rightarrow -\infty$$

and $y = f(x) < 0, \forall x$



$\Rightarrow ax^2 + bx + c > 0, \forall x \in \mathbb{R}$, iff $a > 0$ and $D < 0$ and $ax^2 + bx + c < 0, \forall x \in \mathbb{R}$, iff $a < 0$ and $D < 0$.

Example 37. What is the minimum height of any point on the curve $y = x^2 - 4x + 6$ above the X -axis?

Sol. Here, $a = 1 > 0$ and $D = 16 - 4(6) = -8 < 0$

Therefore, the parabola remains completely above X -axis and minimum value of any point on the curve is given by $\frac{-D}{4a}$.

$$\text{i.e. } y_{\min} = \frac{-D}{4a} = \frac{8}{4} = 2$$

Hence, the minimum height is 2.

Example 38. The roots of $ax^2 + bx + c = 0$, where $a \neq 0$ and coefficients are real, are non-real complex and $a + c < b$. Then,

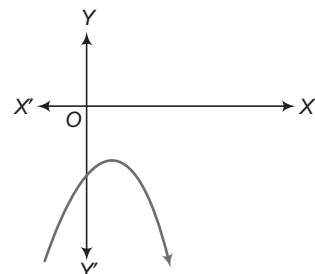
- (a) $4a + c > 2b$
- (b) $4a + c < 2b$
- (c) $4a + c = 2b$
- (d) None of the above

Sol. (b) Since, $f(x) = ax^2 + bx + c = 0$ has non-real roots.

Thus, either

$$f(x) > 0 \text{ or } f(x) < 0, \forall x$$

$$\text{As } a + c < b \Rightarrow f(-1) < 0$$



$$\text{i.e. } a - b + c < 0 \text{ or } a + c < b$$

$$\therefore f(x) < 0, \forall x$$

Thus, for all $x \in \mathbb{R}; ax^2 + bx + c < 0$.

$$\text{Now, for } x = -2; 4a - 2b + c < 0$$

$$\Rightarrow 4a + c < 2b$$

Solution of Quadratic Inequations

Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$ and $a \neq 0$.

Then, $f(x) \geq 0$, $f(x) < 0$, $f(x) \leq 0$, $f(x) > 0$ are quadratic inequations. The set of all real values of x , which satisfy an inequation is called its solution set.

The following example will make it clear:

➤ **Example 39.** Solve the inequation $-x^2 + 3x + 4 < 0$.

Sol. We have,

$$\begin{aligned} & -x^2 + 3x + 4 < 0 \\ \Rightarrow & x^2 - 3x - 4 > 0 \\ \Rightarrow & (x - 4)(x + 1) > 0 \\ & \begin{array}{c} + \quad \quad - \quad \quad + \\ \hline -1 \quad \quad 4 \end{array} \\ \Rightarrow & x < -1 \\ \text{or} & x > 4 \\ \Rightarrow & x \in (-\infty, -1) \cup (4, \infty) \end{aligned}$$

➤ **Frequently used inequalities**

- $(x - a)(x - b) < 0 \Rightarrow x \in (a, b)$, where $a < b$
- $(x - a)(x - b) > 0 \Rightarrow x \in (-\infty, a) \cup (b, \infty)$, where $a < b$
- $x^2 \leq a^2 \Rightarrow x \in [-a, a]$
- $x^2 \geq a^2 \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$
- If $ax^2 + bx + c < 0$, ($a > 0$)
 $\Rightarrow x \in (\alpha, \beta)$, where α and β ($\alpha < \beta$) are roots of the equation $ax^2 + bx + c = 0$
- If $ax^2 + bx + c > 0$, ($a > 0$)
 $\Rightarrow x \in (-\infty, \alpha) \cup (\beta, \infty)$, where α and β ($\alpha < \beta$) are roots of the equation $ax^2 + bx + c = 0$.

Maximum and Minimum Values of Rational Expression

Here, we shall find the values attained by a rational expression of the form $\frac{a_1x^2 + b_1x + c_1}{a_2x^2 + b_2x + c_2}$ for real values of x .

The following algorithm will explain the procedure:

Algorithm

- Step I** Equate the given rational expression to y .
- Step II** Obtain a quadratic equation in x by simplifying the expression in Step I.

Step III Obtain the discriminant of the quadratic equation in Step II.

Step IV Put discriminant ≥ 0 and solve the inequation for y . The values of y so obtained determines the set of values attained by the given rational expression.

➤ **Example 40.** If x is real, then prove that the values of $\frac{x^2 - 3x + 4}{x^2 + 3x + 4}$ lie between $\frac{1}{7}$ and 7.

Sol. Let $y = \frac{x^2 - 3x + 4}{x^2 + 3x + 4}$. Then,

$$\begin{aligned} & x^2(y - 1) + 3x(y + 1) + 4(y - 1) = 0 \\ \text{Now, if } & D \geq 0 \\ \Rightarrow & 9(y + 1)^2 - 16(y - 1)^2 \geq 0 \\ \Rightarrow & -7y^2 + 50y - 7 \geq 0 \\ \Rightarrow & 7y^2 - 50y + 7 \leq 0 \\ \Rightarrow & (7y - 1)(y - 7) \leq 0 \\ \Rightarrow & \frac{1}{7} \leq y \leq 7 \end{aligned}$$

Hence, the given expression lies between $\frac{1}{7}$ and 7.

➤ **Example 41.** If $f(x) = \frac{x^2 - 1}{x^2 + 1}$ for every real

- number x , then the minimum value of f
- (a) does not exist because f is unbounded
- (b) is not attained even though f is bounded
- (c) is equal to 1
- (d) is equal to -1

Sol. (d) Let $y = \frac{x^2 - 1}{x^2 + 1}$

$$\begin{aligned} \Rightarrow & yx^2 + y - x^2 + 1 = 0 \\ \Rightarrow & x^2(y - 1) + y + 1 = 0 \\ \text{Now, for real values of } x \text{ consider } & D \geq 0 \\ \Rightarrow & -4(y^2 - 1) \geq 0 \\ \Rightarrow & y^2 - 1 \leq 0 \\ \Rightarrow & -1 \leq y \leq 1 \end{aligned}$$

Thus, minimum value of $f(x)$ is -1.

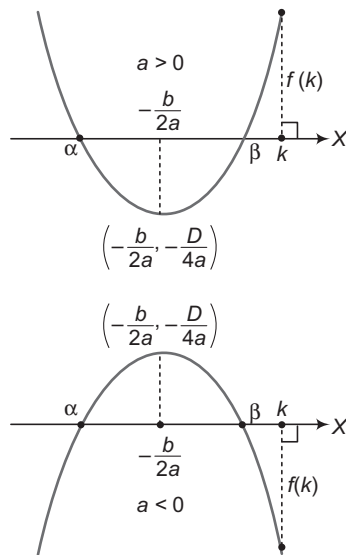
Location of the Roots of a Quadratic Equation

Here, we shall discuss various conditions satisfied by the coefficients of a quadratic equation when the location of its roots is given.

Let $f(x) = ax^2 + bx + c$, $a, b, c \in R$, $a \neq 0$ and α, β be the roots of $f(x) = 0$. Suppose $k, k_1, k_2 \in R$ and $k_1 < k_2$.

Then, the following hold good :

i. **Conditions for a number k** (If both the roots of $f(x) = 0$ are less than k)



- (a) $D \geq 0$ (roots may be equal)
- (b) $af(k) > 0$
- (c) $k > -\frac{b}{2a}$, where $\alpha \leq \beta$

➤ **Example 42.** Find the values of m for which both roots of equation $x^2 - mx + 1 = 0$ are less than unity.

Sol. Let $f(x) = x^2 - mx + 1$, as both roots of $f(x) = 0$ are less than 1, we can take $D \geq 0$, $af(1) > 0$

and $-\frac{b}{2a} < 1$.

Case I Consider, $D \geq 0$

$$(-m)^2 - 4 \cdot 1 \cdot 1 \geq 0$$

$$\Rightarrow (m+2)(m-2) \geq 0$$

$$\Rightarrow m \in (-\infty, -2] \cup [2, \infty) \quad \dots(i)$$

Case II Consider, $af(1) > 0$

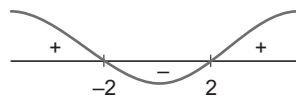
$$1 \cdot (1 - m + 1) > 0$$

$$\Rightarrow m - 2 < 0$$

$$\Rightarrow m < 2$$

$$\Rightarrow m \in (-\infty, 2) \quad \dots(ii)$$

Case III Consider, $-\frac{b}{2a} < 1$



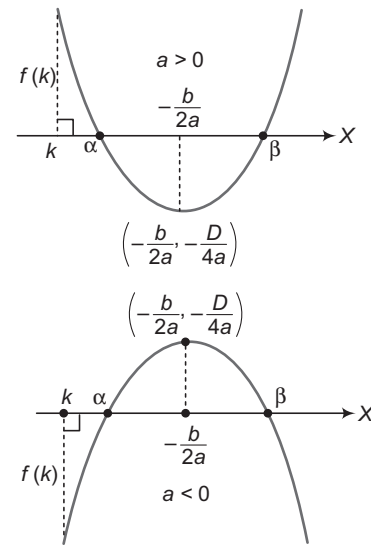
$$\frac{m}{2} < 1$$

$$\Rightarrow m < 2$$

$$\Rightarrow m \in (-\infty, 2) \quad \dots(iii)$$

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in (-\infty, -2]$.

ii. **Conditions for a number k** (If both the roots of $f(x) = 0$ are greater than k)



- (a) $D \geq 0$ (roots may be equal)
- (b) $af(k) > 0$
- (c) $k < -\frac{b}{2a}$, where $\alpha \leq \beta$

➤ **Example 43.** For what values of $m \in \mathbb{R}$, both roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ exceed 3?

Sol. Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$.

as both roots of $f(x) = 0$ are greater than 3, we can take $D \geq 0$, $af(3) > 0$ and $-\frac{b}{2a} > 3$.

Case I Consider, $D \geq 0$

$$(-6m)^2 - 4 \cdot 1 \cdot (9m^2 - 2m + 2) \geq 0 \Rightarrow 8m - 8 \geq 0$$

$$\therefore m \geq 1 \text{ or } m \in [1, \infty) \quad \dots(i)$$

Case II Consider, $af(3) > 0$

$$1 \cdot (9 - 18m + 9m^2 - 2m + 2) > 0$$

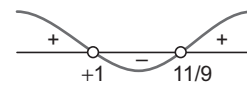
$$\Rightarrow 9m^2 - 20m + 11 > 0$$

$$\Rightarrow (9m - 11)(m - 1) > 0$$

$$\Rightarrow \left(m - \frac{11}{9}\right)(m - 1) > 0$$

$$\Rightarrow m \in (-\infty, 1) \cup \left(\frac{11}{9}, \infty\right) \quad \dots(ii)$$

Case III Consider, $-\frac{b}{2a} > 3$

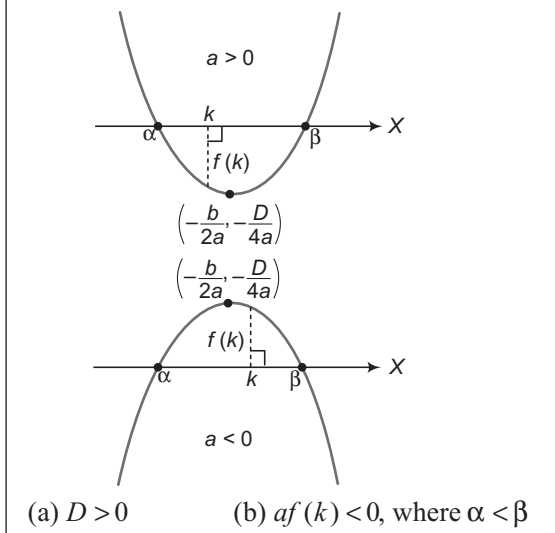


$$\frac{6m}{2} > 3 \Rightarrow m > 1$$

$$\Rightarrow m \in (1, \infty) \quad \dots(iii)$$

Hence, the values of m satisfying Eqs. (i), (ii) and (iii) at the same time are $m \in \left(\frac{11}{9}, \infty\right)$.

iii. **Conditions for a number k** (If k lies between the roots of $f(x) = 0$)



→ **Example 44.** Find all values of p , so that 6 lies between roots of the equation $x^2 + 2(p-3)x + 9 = 0$.

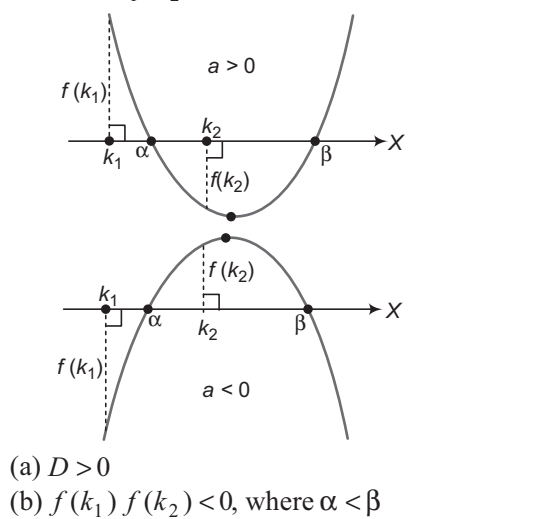
Sol. Let $f(x) = x^2 + 2(p-3)x + 9$ as 6 lies between the roots of $f(x) = 0$, we can take $D > 0$ and $af(6) < 0$.

Case I Consider, $D > 0$
 $\{2(p-3)\}^2 - 4 \cdot 1 \cdot 9 > 0$
 $\Rightarrow (p-3)^2 - 9 > 0 \Rightarrow p(p-6) > 0$
 $\Rightarrow p \in (-\infty, 0) \cup (6, \infty)$... (i)

Case II Consider, $af(6) < 0$
 $1 \cdot \{36 + 12(p-3) + 9\} < 0$
 $\Rightarrow 12p + 9 < 0 \Rightarrow p + \frac{3}{4} < 0$
 $\Rightarrow p \in (-\infty, -3/4)$... (ii)

Hence, the values of p satisfying Eqs. (i) and (ii) at the same time are $p \in (-\infty, -3/4)$.

iv. **Conditions for numbers k_1 and k_2** (If exactly one root of $f(x) = 0$ lies in the interval (k_1, k_2)).



→ **Example 45.** Find the values of m for which exactly one root of the equation $x^2 - 2mx + m^2 - 1 = 0$ lies in the interval $(-2, 4)$.

Sol. Let $f(x) = x^2 - 2mx + m^2 - 1$ as exactly one root of $f(x) = 0$ lies in the interval, we can take $D > 0$ and $f(-2)f(4) < 0$.

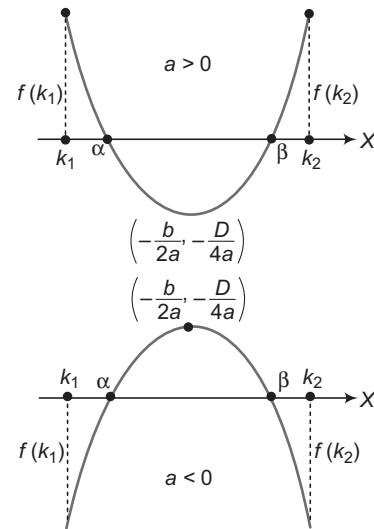
Case I Consider, $D > 0$
 $(-2m)^2 - 4 \cdot 1 \cdot (m^2 - 1) > 0$
 $\Rightarrow 4 > 0$
 $\therefore m \in \mathbb{R}$... (i)

Case II Consider, $f(-2)f(4) < 0$

 $(4 + 4m + m^2 - 1)(16 - 8m + m^2 - 1) < 0$
 $\Rightarrow (m^2 + 4m + 3)(m^2 - 8m + 15) < 0$
 $\Rightarrow (m+1)(m+3)(m-3)(m-5) < 0$
 $\Rightarrow (m+3)(m+1)(m-3)(m-5) < 0$
 $\therefore m \in (-3, -1) \cup (3, 5)$... (ii)

Hence, the values of m satisfying Eqs. (i) and (ii) at the same time are $m \in (-3, -1) \cup (3, 5)$.

v. **Conditions for numbers k_1 and k_2** (If both roots of $f(x) = 0$ are confined between k_1 and k_2)



(a) $D \geq 0$ (roots may be equal)
 (b) $af(k_1) > 0$
 (c) $af(k_2) > 0$
 (d) $k_1 < -\frac{b}{2a} < k_2$, where $\alpha \leq \beta$ and $k_1 < k_2$

→ **Example 46.** Find all values of a for which the equation $4x^2 - 2x + a = 0$ has two roots in the interval $(-1, 1)$.

Sol. Let $f(x) = 4x^2 - 2x + a$ as both roots of the equation $f(x) = 0$ lie between $(-1, 1)$, we can take $D \geq 0$, $af(-1) > 0$, $af(1) > 0$ and $-1 < \frac{1}{4} < 1$.

Case I Consider, $D \geq 0$

$$(-2)^2 - 4 \cdot 4 \cdot a \geq 0 \Rightarrow a \leq \frac{1}{4}$$

$$\Rightarrow a \in \left(-\infty, \frac{1}{4}\right] \quad \dots(i)$$

Case II Consider, $af(-1) > 0$

$$4(4 + 2 + a) > 0$$

$$\Rightarrow a > -6 \Rightarrow a \in (-6, \infty) \quad \dots(ii)$$

Case III Consider, $af(1) > 0$

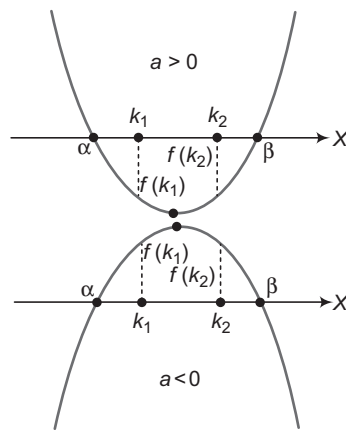
$$4(4 - 2 + a) > 0$$

$$\Rightarrow a > -2 \Rightarrow a \in (-2, \infty) \quad \dots(iii)$$

Hence, the values of a satisfying Eqs. (i), (ii) and (iii) at the same time are $a \in \left(-2, \frac{1}{4}\right]$.

vi.

Conditions for numbers k_1 and k_2 (If k_1 and k_2 lie between the roots of $f(x) = 0$)



(a) $D > 0$

(b) $af(k_1) < 0$

(c) $af(k_2) < 0$, where $\alpha < \beta$

Example 47. Find the values of a for which one root of equation $(a-5)x^2 - 2ax + a-4 = 0$ is smaller than 1 and the other greater than 2.

Sol. Let $f(x) = (a-5)x^2 - 2ax + a-4$ ($a \neq 5$) as 1 and 2 lie between the roots of $f(x) = 0$, we can take $D > 0$, $(a-5)f(1) < 0$ and $(a-5)f(2) < 0$.

Case I Consider, $D > 0$

$$(-2a)^2 - 4 \cdot (a-5) \cdot (a-4) > 0$$

$$\Rightarrow 9a - 20 > 0 \Rightarrow a \in (20/9, \infty) \quad \dots(i)$$

Case II Consider, $(a-5)f(1) < 0$

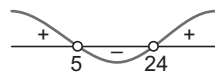
$$(a-5)(a-5-2a+a-4) < 0$$

$$\Rightarrow (a-5)(-9) < 0$$

$$\Rightarrow a-5 > 0$$

$$\Rightarrow a \in (5, \infty) \quad \dots(ii)$$

Case III Consider, $(a-5)f(2) < 0$



$$(a-5)\{4(a-5)-4a+a-4\} < 0$$

$$\Rightarrow (a-5)(a-24) < 0$$

$$\Rightarrow a \in (5, 24) \quad \dots(iii)$$

Hence, the values of a satisfying Eqs. (i), (ii) and (iii) at the same time are $a \in (5, 24)$.

Example 48. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3, then

(a) $a < 2$

(b) $2 \leq a \leq 3$

(c) $3a < a \leq 4$

(d) $a > 4$

Sol. (a) Let $f(x) = x^2 - 2ax + a^2 + a - 3$

Since, both roots are less than 3.

i.e. $\alpha < 3, \beta < 3$

$$\text{Sum (S)} = \alpha + \beta < 6$$

$$\Rightarrow \frac{\alpha + \beta}{2} < 3 \Rightarrow \frac{2a}{2} < 3$$

$$\Rightarrow a < 3 \quad \dots(i)$$

Again, product $(P) = \alpha\beta$

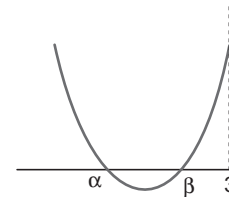
$$\Rightarrow P < 9 \Rightarrow \alpha\beta < 9$$

$$\Rightarrow a^2 + a - 3 < 9$$

$$\Rightarrow a^2 + a - 12 < 0$$

$$\Rightarrow (a-3)(a+4) < 0$$

$$\Rightarrow -4 < a < 3 \quad \dots(ii)$$



Again, $D = B^2 - 4AC \geq 0$

$$\Rightarrow (-2a)^2 - 4 \cdot 1 \cdot (a^2 + a - 3) \geq 0$$

$$\Rightarrow 4a^2 - 4a^2 - 4a + 12 \geq 0$$

$$\Rightarrow -4a + 12 \geq 0$$

$$\Rightarrow a \leq 3 \quad \dots(iii)$$

Again,

$$af(3) > 0$$

$$\Rightarrow 1[(3)^2 - 2a(3) + a^2 + a - 3] > 0$$

$$\Rightarrow 9 - 6a + a^2 + a - 3 > 0$$

$$\Rightarrow a^2 - 5a + 6 > 0$$

$$\Rightarrow (a-2)(a-3) > 0$$

$$\therefore a \in (-\infty, 2) \cup (3, \infty) \quad \dots(iv)$$

From Eqs. (i), (ii), (iii) and (iv), we get

$$a \in (-4, 2)$$

Algebraic Interpretation of Rolle's Theorem

Let $f(x)$ be a polynomial having α and β as its roots such that $\alpha < \beta$. Then, $f(\alpha) = f(\beta) = 0$. Also, a polynomial function is everywhere continuous and differentiable. Thus, $f(x)$ satisfies all the three conditions of Rolle's theorem. Consequently, there exists $\gamma \in (\alpha, \beta)$ such that $f'(\gamma) = 0$, i.e. $f'(x) = 0$ at $x = \gamma$. In other words, $x = \gamma$ is a root of $f'(x) = 0$. Thus, algebraically Rolle's theorem can be interpreted as follows:

Between any two roots of a polynomial $f(x)$, there is always a root of its derivative $f'(x)$.

➤ **Example 49.** If $a + b + c = 0$, then the quadratic equation $3ax^2 + 2bx + c = 0$ has

- (a) atleast one root in $(0, 1)$
- (b) both imaginary roots
- (c) one root in $(1, 2)$, other in $(-1, 0)$
- (d) None of the above

Sol. (a) Consider the function $f(x) = ax^3 + bx^2 + cx$
 $f(x)$ being a polynomial function is continuous and differentiable, $\forall x \in R$

$$\begin{aligned} f(0) &= 0, f(1) = a + b + c = 0 & [\text{given}] \\ \Rightarrow f(0) &= f(1) \end{aligned}$$

Thus, $f(x)$ satisfies all conditions of Rolle's theorem. So, there exists a point $k \in (0, 1)$ such that $f'(k) = 0$.

Condition for Resolution into Linear Factors

The quadratic function

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is resolvable into linear rational factors iff

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\text{i.e.} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

➤ **Example 50.** For what values of m can the expression $2x^2 + mxy + 3y^2 - 5y - 2$ be expressed as the product of two linear factors?

Sol. Comparing the given equation with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have

$$a = 2, h = \frac{m}{2}, b = 3, c = -2, f = -\frac{5}{2} \text{ and } g = 0$$

The given expression is resolvable into linear factors, if $abc + 2hxy - af^2 - bg^2 - ch^2 = 0$

$$\Rightarrow -12 - \frac{25}{2} + 2 \left(\frac{m^2}{4} \right) = 0$$

$$\Rightarrow m^2 = 49$$

$$\therefore m = \pm 7$$

Some Application of Graphs to Find the Roots of Equations

Here, we shall discuss some examples to find the roots of equations with the help of graphs.

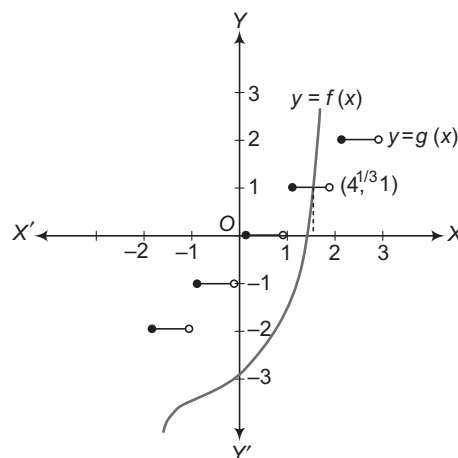
➤ **Example 51.** If $x^3 - [x] = 3$, where $[x]$ denotes the greatest integer less than or equal to x , then

- (a) $x \in \{4^{1/3}\}$
- (b) $x \in R$
- (c) Cannot be discussed
- (d) None of the above

Sol. (a) We have,

$$x^3 - [x] = 3 \Rightarrow x^3 - 3 = [x]$$

$$\text{Let } f(x) = x^3 - 3 \text{ and } g(x) = [x]$$



Solving the given equations means finding the x -coordinates of the points of intersection of the curves $y = f(x)$ and $y = g(x)$.

If $1 < x < 2$, we have $g(x) = 1$ and $f(x) = x^3 - 3$

At the point of intersection of the two curves, we have

$$x^3 - 3 = 1 \Rightarrow x = 4^{1/3}$$

Hence, $x = 4^{1/3}$ is the solution of the equation

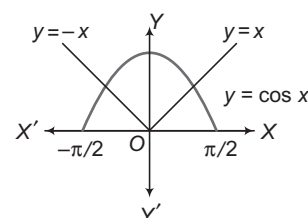
$$x^3 - [x] = 3$$

➤ **Example 52.** The number of solutions of the equation $|x| = \cos x$ are

- (a) one
- (b) two
- (c) three
- (d) zero

Sol. (b) Here, $|x| = \cos x$

Thus, to find value of x for which both curves have point of intersections.



Clearly, there are two points of intersection of $y = |x|$ and $y = \cos x$.

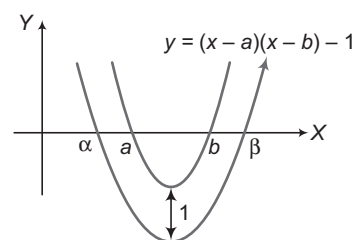
Hence, there are two real solutions.

➤ **Example 53.** If $b > a$, then the equation

$(x - a)(x - b) - 1 = 0$ has

- (a) both roots in (a, b)
- (b) both roots in $(-\infty, a)$
- (c) both roots in $(b, +\infty)$
- (d) one root in $(-\infty, a)$ and the other in (b, ∞)

Sol. (d)



From graph, it is clear that one of the roots of $(x - a)(x - b) - 1 = 0$ lies in $(-\infty, a)$ and other lies in (b, ∞) .

- 1 The number of values of k for which $\{x^2 - (k-2)x + k^2\}\{x^2 + kx + (2k-1)\}$ is a perfect square, is
 - a 1
 - b 2
 - c 0
 - d None of these
- 2 If $x^2 - 4x + \log_{1/2} a = 0$ does not have two distinct real roots, then maximum value of a is
 - a $\frac{1}{4}$
 - b $\frac{1}{16}$
 - c $-\frac{1}{4}$
 - d None of these
- 3 If $c > 0$ and $4a + c < 2b$, then $ax^2 - bx + c = 0$ has a root in the interval
 - a $(0, 2)$
 - b $(2, 4)$
 - c $(0, 1)$
 - d $(-2, 0)$
- 4 Conditions on a and b for which $x^2 - ax - b^2$ is less than zero for atleast one positive x are
 - a $a - 3 > 0, b < 0$
 - b $a - 3 > 0, b > 0$
 - c $a, b \in R$
 - d Cannot be discussed
- 5 The set of possible values of λ for which $x^2 - (\lambda^2 - 5\lambda + 5)x + (2\lambda^2 - 3\lambda - 4) = 0$ has roots, whose sum and product are both less than 1, is
 - a $\left(-1, \frac{5}{2}\right)$
 - b $(1, 4)$
 - c $\left[1, \frac{5}{2}\right]$
 - d $\left(1, \frac{5}{2}\right)$
- 6 If $ax^2 + bx + 6 = 0$ does not have two distinct real roots, then the least value of $3a + b$ is
 - a $\frac{2}{1}$
 - b $-\frac{2}{-1}$
 - c $-\frac{1}{2}$
 - d None of these
- 7 If x is real, then the least value of the expression $\frac{x^2 - 6x + 5}{x^2 + 2x + 1}$ is
 - a -1
 - b $-\frac{1}{2}$
 - c $-\frac{1}{3}$
 - d None of these
- 8 If the quadratic equation $\alpha x^2 + \beta x + a^2 + b^2 + c^2 - ab - bc - ca = 0$ has imaginary roots, then
 - a $2(\alpha - \beta) + (a - b)^2 + (b - c)^2 + (c - a)^2 > 0$
 - b $2(\alpha - \beta) + (a - b)^2 + (b - c)^2 + (c - a)^2 < 0$
 - c $2(\alpha - \beta) + (a - b)^2 + (b - c)^2 + (c - a)^2 = 0$
 - d None of the above
- 9 Let S denotes the set of real values of d for which the roots of the equation $x^2 - ax - a^2 = 0$ exceeds d , then S equals
 - a $(-\infty, 0)$
 - b $\left(-2, -\frac{1}{2}\right)$
 - c $\left(-\frac{1}{2}, \frac{1}{4}\right)$
 - d null set
- 10 $x^4 - 4x - 1 = 0$ has
 - a exactly two positive real roots
 - b exactly two negative real roots
 - c exactly two real roots
 - d None of the above
- 11 The roots α and β of the quadratic equation are $ax^2 + bx + c = 0$ are real and of opposite sign. Then, the roots of the equation $\alpha(x - \beta)^2 + \beta(x - \alpha)^2 = 0$ are
 - a positive
 - b negative
 - c real and of opposite sign
 - d imaginary
- 12 The graph of curve $x^2 = 3x - y - 2$ is
 - a between the lines $x = 1$ and $x = \frac{3}{2}$
 - b between the lines $x = 1$ and $x = 2$
 - c strictly below the line $4y = 1$
 - d None of the above
- 13 If $0 < a < 5, 0 < b < 5$ and $\frac{x^2 + 5}{2} = x - 2 \cos(a + bx)$ is satisfied for atleast one real x , then the greatest value of $a + b$ is
 - a π
 - b $\frac{\pi}{2}$
 - c 3π
 - d 4π
- 14 If the roots of $ax^2 - bx - c = 0$ change by the same quantity, then the expression in a, b and c that does not change, is
 - a $\frac{b^2 - 4ac}{a^2}$
 - b $\frac{b - 4c}{a}$
 - c $\frac{b^2 + 4ac}{a^2}$
 - d None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If $0x^2 + x + |x| + 1 \leq 0$, then x lies in

- (a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) R (d) ϕ

Sol. $x^2 + x + |x| + 1 \leq 0$ gives two cases due to $|x|$.

Case I When $x \geq 0$... (i)

$$x^2 + 2x + 1 \leq 0 \quad \text{or} \quad (x+1)^2 \leq 0$$

$$\Rightarrow x + 1 = 0$$

$$\therefore x = -1 \text{ but } x \geq 0 \quad [\text{from Eq. (i)}]$$

$\therefore$ No value of $x \geq 0$.

Case II When $x < 0$... (ii)

$$x^2 + x - x + 1 \leq 0$$

$$\Rightarrow x^2 + 1 \leq 0$$

which is not true for any value of $x < 0$

$\therefore$ No value of $x < 0$

From above two cases no value of x is possible

i.e. $x \in \phi$.

Hence, (d) is the correct answer.

Ex 2. If $|x-1| + |x| + |x+1| \geq 6$, then x lies in

- (a) $(-\infty, 2]$ (b) $(-\infty, -2] \cup [2, \infty)$
(c) R (d) ϕ

Sol. $|x-1| + |x| + |x+1| \geq 6$, gives four cases

Case I When $x < -1$... (i)

$$\text{Then, } -(x+1) - x - (x-1) \geq 6$$

$$\Rightarrow -x - 1 - x - x + 1 \geq 6$$

$$\Rightarrow -3x \geq 6 \text{ or } x \leq -2 \quad \dots (ii)$$

From Eqs. (i) and (ii), $x \leq -2$

Case II When $-1 \leq x \leq 0$... (iii)

$$\Rightarrow x + 1 - x - (x-1) \geq 6$$

$$\Rightarrow 1 - x + 1 \geq 6 \Rightarrow x \leq -4 \quad \dots (iv)$$

$\therefore$ No value of x . [from Eqs. (iii) and (iv)]

Case III When $0 \leq x \leq 1$... (v)

$$\Rightarrow x + 1 + x - (x-1) \geq 6$$

$$\Rightarrow x \geq 4 \quad \dots (vi)$$

No solution, using Eqs. (v) and (vi).

Case IV When $x > 1$... (vii)

$$\Rightarrow x + 1 + x + x - 1 \geq 6$$

$$\Rightarrow 3x \geq 6 \text{ or } x \geq 2 \quad \dots (viii)$$

From Eqs. (vii) and (viii), $x \geq 2$

Thus, from above four cases,

$$x \leq -2 \quad \text{or} \quad x \geq 2$$

$$\Rightarrow x \in (-\infty, -2] \cup [2, \infty)$$

Aliter

As discussed above, solution exists only when all are positive or negative:

$$\text{Thus, } |x+1| + |x| + |x-1| \geq 6$$

$$\Rightarrow -(x+1) - (x) - (x-1) \geq 6$$

$$\text{or } (x+1) + x + (x-1) \geq 6$$

$$\Rightarrow -3x \geq 6$$

$$\text{or } 3x \geq 6$$

$$\Rightarrow x \leq -2$$

$$\text{or } x \geq 2$$

$$\Rightarrow x \in (-\infty, -2] \cup [2, \infty)$$

Hence, (b) is the correct answer.

Ex 3. If $a_1, a_2, \dots, a_n$ are positive real numbers whose product is a fixed real number c , then the minimum value of $a_1 + a_2 + \dots + a_{n-1} + 2a_n$ is

(a) $n(2c)^{1/n}$

(b) $(n+1)c^{1/n}$

(c) $2nc^{1/n}$

(d) $(n+1)(2c)^{1/n}$

Sol. We have,

$$\frac{1}{n}(a_1 + a_2 + \dots + a_{n-1} + 2a_n)$$

$$\geq [a_1 a_2 \dots a_{n-1} (2a_n)]^{1/n} = (2c)^{1/n} \quad [\text{using AM} \geq \text{GM}]$$

$$\Rightarrow a_1 + a_2 + \dots + a_{n-1} + 2a_n \geq n(2c)^{1/n}$$

Thus, minimum value of $a_1 + a_2 + \dots + a_{n-1} + 2a_n$ is $n(2c)^{1/n}$.

Hence, (a) is the correct answer.

Ex 4. Solution set of

$$\log_3 (x^2 - 2) < \log_3 \left(\frac{3}{2}|x| - 1 \right) \text{ is}$$

(a) $(-\sqrt{2}, -1)$

(b) $(-2, -\sqrt{2})$

(c) $(-\sqrt{2}, 2)$

(d) None of these

Sol. We have,

$$x^2 - 2 > 0, \frac{3}{2}|x| - 1 > 0$$

$$\text{and } x^2 - 2 < \frac{3}{2}|x| - 1$$

$$\Rightarrow x^2 > 2, |x| > \frac{2}{3} \quad \text{and} \quad |x|^2 - \frac{3}{2}|x| - 1 < 0$$

$$\Rightarrow |x| > \sqrt{2} \quad \text{and} \quad |x| < 2$$

$$\Rightarrow x \in (-2, -\sqrt{2}) \cup (\sqrt{2}, 2)$$

Hence, (d) is the correct answer.

Ex 5. Solution set of the inequality

$$\frac{1}{2^x - 1} > \frac{1}{1 - 2^{x-1}} \text{ is}$$

(a) $(1, \infty)$

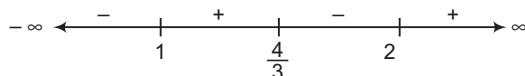
(b) $\left[0, \log_2 \left(\frac{4}{3} \right) \right]$

(c) $(-1, \infty)$

(d) $\left[0, \log_2 \left(\frac{4}{3} \right) \right] \cup (1, \infty)$

Sol. Put $2^x = t$, then $t > 0$. The given inequality becomes

$$\begin{aligned} & \frac{1}{t-1} > \frac{2}{2-t} \\ \Rightarrow & \frac{1}{t-1} - \frac{2}{2-t} > 0 \\ \Rightarrow & \frac{2-t-2t+2}{(t-1)(2-t)} > 0 \\ \Rightarrow & \frac{4-3t}{(t-1)(2-t)} > 0 \\ \Rightarrow & \frac{(3t-4)}{(t-1)(t-2)} > 0 \end{aligned}$$



$$\begin{aligned} \Rightarrow & 1 < t < \frac{4}{3} \quad \text{or} \quad t > 2 \\ \Rightarrow & 1 < 2^x < \frac{4}{3} \quad \text{or} \quad 2^x > 2 \\ \Rightarrow & 0 < x < \log_2\left(\frac{4}{3}\right) \quad \text{or} \quad x > 1 \end{aligned}$$

Hence, (d) is the correct answer.

Ex 6. Given, $n^4 < 10^n$ for a fixed positive integer $n \geq 2$, then

- (a) $(n+1)^4 < 10^{n+1}$ (b) $(n+1)^4 > 10^{n+1}$
(c) $(n-1)^4 < 10^{n+2}$ (d) None of these

Sol. We have,

$$\begin{aligned} \left(\frac{n+1}{n}\right)^4 &= \left(1 + \frac{1}{n}\right)^4 \leq \left(1 + \frac{1}{2}\right)^4 = \left(\frac{3}{2}\right)^4 < 10 \quad [\because n \geq 2] \\ \Rightarrow (n+1)^4 &\leq n^4 \cdot 10 < 10^n \cdot 10 = 10^{n+1} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 7. If x_1, x_2, x_3 and x_4 are four positive real numbers such that $x_1 + \frac{1}{x_2} = 4, x_2 + \frac{1}{x_3} = 1,$

$$x_3 + \frac{1}{x_4} = 4 \text{ and } x_4 + \frac{1}{x_1} = 1, \text{ then}$$

- (a) $x_1 = x_3$ and $x_2 = x_4$ (b) $x_2 = x_4$ but $x_1 \neq x_3$
(c) $x_1 x_2 = 1, x_3 x_4 \neq 1$ (d) $x_3 x_4 = 1, x_1 x_2 \neq 1$

Sol. Using AM $\geq$ GM,

$$\begin{aligned} x_1 + \frac{1}{x_2} &\geq 2\sqrt{\frac{x_1}{x_2}}; x_2 + \frac{1}{x_3} \geq 2\sqrt{\frac{x_2}{x_3}} \\ x_3 + \frac{1}{x_4} &\geq 2\sqrt{\frac{x_3}{x_4}}; x_4 + \frac{1}{x_1} \geq 2\sqrt{\frac{x_4}{x_1}} \\ \Rightarrow \left(x_1 + \frac{1}{x_2}\right) \left(x_2 + \frac{1}{x_3}\right) \left(x_3 + \frac{1}{x_4}\right) \left(x_4 + \frac{1}{x_1}\right) &\geq 2^4 \end{aligned}$$

$$\text{But } \left(x_1 + \frac{1}{x_2}\right) \left(x_2 + \frac{1}{x_3}\right) \left(x_3 + \frac{1}{x_4}\right) \left(x_4 + \frac{1}{x_1}\right) = 16$$

$$\therefore x_1 = \frac{1}{x_2}, x_2 = \frac{1}{x_3}, x_3 = \frac{1}{x_4}, x_4 = \frac{1}{x_1}$$

$$\Rightarrow x_1 = 2, x_2 = \frac{1}{2}, x_3 = 2, x_4 = \frac{1}{2}$$

Hence, (a) is the correct answer.

Ex 8. If roots of the equation

$$x^4 - 8x^3 + bx^2 + cx + 16 = 0$$

are positive, then

- (a) $b = 8, c = 2$ (b) $c = -32, b = -24$
(c) $b = 24, c = -32$ (d) $c = 32, b = 24$

Sol. If x_1, x_2, x_3 and x_4 are four roots of given equation, then

$$x_1 + x_2 + x_3 + x_4 = 8 \text{ and } x_1 x_2 x_3 x_4 = 16$$

As AM $\geq$ GM, we get

$$2 = \frac{1}{4} (x_1 + x_2 + x_3 + x_4) \geq (x_1 x_2 x_3 x_4)^{1/4} = 2$$

i.e. AM = GM

$$\text{Thus, } x_1 = x_2 = x_3 = x_4 = 2$$

$$\therefore x^4 - 8x^3 + bx^2 + cx + 16 = (x-2)^4$$

$$\Rightarrow b = 24 \text{ and } c = -32$$

Hence, (c) is the correct answer.

Ex 9. If a, b and c are three positive real numbers such that $a + b \geq c$, then

- (a) $\frac{a}{1+a} + \frac{b}{1+b} \geq \frac{c}{1+c}$ (b) $\frac{a}{1+a} + \frac{b}{1+b} < \frac{c}{1+c}$
(c) $\frac{a}{1+a} + \frac{b}{1+b} > \frac{c}{1+c}$ (d) None of these

Sol. We have,

$$\begin{aligned} \frac{a}{1+a} + \frac{b}{1+b} &\geq \frac{a}{1+a+b} + \frac{b}{1+a+b} \\ &= \frac{a+b}{1+a+b} = \frac{1}{\frac{1}{a+b} + 1} \end{aligned}$$

Since, $a + b \geq c$, we get

$$\frac{1}{a+b} \leq \frac{1}{c} \Rightarrow 1 + \frac{1}{a+b} \leq 1 + \frac{1}{c} \Rightarrow \frac{1}{\frac{1}{a+b} + 1} \geq \frac{1}{\frac{1}{c} + 1}$$

$$\text{Thus, } \frac{a}{1+a} + \frac{b}{1+b} \geq \frac{1}{\frac{1}{c} + 1} \geq \frac{1}{\frac{1}{c} + 1} = \frac{c}{1+c}$$

Hence, (a) is the correct answer.

Ex 10. If $a_i > 0$ for $i = 1, 2, \dots, n$ and $a_1 a_2 \dots a_n = 1$, then minimum value of $(1 + a_1)(1 + a_2) \dots (1 + a_n)$ is

- (a) $2^{n/2}$ (b) 2^n
(c) 2^{2n} (d) 1

Sol. We have,

$$\frac{1}{2}(1 + a_1) \geq \sqrt{1 \cdot a_1} = \sqrt{a_1} \quad [\text{using A.M} \geq \text{G.M.}]$$

$$\frac{1}{2}(1 + a_2) \geq \sqrt{1 \cdot a_2} = \sqrt{a_2}$$

$$\vdots$$

$$\frac{1}{2}(1 + a_n) \geq \sqrt{1 \cdot a_n} = \sqrt{a_n}$$

On multiplying above inequalities, we get

$$\frac{1}{2^n} (1 + a_1)(1 + a_2) \dots (1 + a_n) \geq \sqrt{a_1 a_2 \dots a_n} = 1$$

$$\Rightarrow (1 + a_1)(1 + a_2) \dots (1 + a_n) \geq 2^n$$

Hence, (b) is the correct answer.

Ex 11. If a, b and c are the sides of a triangle, then

$$\frac{a}{b+c-a} + \frac{b}{c+a-b} + \frac{c}{a+b-c}$$

(a) ≤ 3 (b) ≥ 3
(c) ≥ 2 (d) ≤ 2

Sol. As a, b and c are sides of a triangle, $b+c-a, c+a-b, a+b-c > 0$.

$$\text{Let } x = b+c-a, y = c+a-b \text{ and } z = a+b-c$$

$$\Rightarrow y+z=2a, z+x=2b \text{ and } x+y=2c$$

Thus, LHS of the inequality

$$= \frac{y+z}{2x} + \frac{z+x}{2y} + \frac{x+y}{2z}$$

$$= \frac{1}{2} \left(\frac{y}{x} + \frac{x}{y} + \frac{z}{x} + \frac{x}{z} + \frac{y}{z} + \frac{z}{y} \right)$$

$$\geq \frac{6}{2} \left(\frac{y}{x} \cdot \frac{x}{y} \cdot \frac{z}{x} \cdot \frac{x}{z} \cdot \frac{y}{z} \cdot \frac{z}{y} \right)^{1/6} = 3 \quad [\because \text{AM} \geq \text{GM}]$$

Hence, (b) is the correct answer.

Ex 12. If $a+b+c=6$, then

$$\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1}$$

(a) ≤ 9 (b) ≥ 9 (c) > 9 (d) < 9

Sol. By the Cauchy-Schwartz's inequality,

$$(\sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1})^2$$

$$\leq (1+1+1)(4a+1+4b+1+4c+1)$$

$$= 3[4(a+b+c)+3] = (3)(27)$$

$$\Rightarrow \sqrt{4a+1} + \sqrt{4b+1} + \sqrt{4c+1} \leq 9$$

Hence, (a) is the correct answer.

Ex 13. If a, b and $c \in R$, then the square root of $a^2+b^2+c^2-bc-ca-ab$ is greater than or equal to

(a) $\frac{\sqrt{3}}{2} \max\{|b-c|, |c-a|, |a-b|\}$
(b) $\frac{3}{2} \max\{|b-c|, |c-a|, |a-b|\}$
(c) $\max\{|b-c|, |c-a|, |a-b|\}$
(d) $\frac{\sqrt{3}}{4} \max\{|b-c|, |c-a|, |a-b|\}$

Sol. We have, $a^2+b^2+c^2-bc-ca-ab$

$$= \frac{1}{2} [(b^2+c^2-2bc) + (c^2+a^2-2ca) + (a^2+b^2-2ab)]$$

$$= \frac{1}{2} [(b-c)^2 + (c-a)^2 + (a-b)^2] \geq 0$$

Also, $a^2+b^2+c^2-bc-ca-ab - \frac{3}{4}(b-c)^2$

$$= \frac{1}{4} [4a^2+4b^2+4c^2-4bc-4ca-4ab - 3(b^2+c^2-2bc)]$$

$$= \frac{1}{4} [4a^2+b^2+c^2+2bc-4a(c+b)]$$

$$= \frac{1}{4} [4a^2 + (b+c)^2 - 4a(b+c)]$$

$$= \frac{1}{4} [2a - (b+c)]^2 \geq 0$$

Similarly, $a^2+b^2+c^2-bc-ca-ab \geq \frac{\sqrt{3}}{2} |c-a|$

and $a^2+b^2+c^2-bc-ca-ab \geq \frac{\sqrt{3}}{2} |a-b|$

$$\Rightarrow a^2+b^2+c^2-bc-ca-ab \geq \frac{\sqrt{3}}{2} \max\{|b-c|, |c-a|, |a-b|\}$$

Hence, (a) is the correct answer.

Ex 14. If a, b and c are three distinct positive real numbers, then the number of real roots of $ax^2+2b|x|-c=0$ is

- (a) 4
(b) 2
(c) 0
(d) None of the above

Sol. $a|x|^2+2b|x|-c=0$

$$\therefore |x| = \frac{-2b \pm \sqrt{4b^2+4ac}}{2}$$

$$= -b \pm \sqrt{b^2+ac}$$

Since, a, b and c are positive. So, $|x| = -b + \sqrt{b^2+ac}$

$\therefore x$ has two real values, neglecting

$$|x| = -b - \sqrt{b^2+ac}, \text{ as } |x| \geq 0$$

Hence, (b) is the correct answer.

Ex 15. If the ratio of the roots of $\lambda x^2 + \mu x + \nu = 0$ is equal to the ratio of the roots of $x^2 + x + 1 = 0$, then λ, μ and ν are in

- (a) AP
(b) GP
(c) HP
(d) None of the above

Sol. As ratio of roots for $\lambda x^2 + \mu x + \nu = 0$ and $x^2 + x + 1 = 0$ are equal.

$$\therefore \frac{\alpha}{\beta} = \frac{\alpha'}{\beta'}, \text{ where } \alpha \text{ and } \beta \text{ are roots of } \lambda x^2 + \mu x + \nu = 0$$

and α', β' are roots of $x^2 + x + 1 = 0$

Now, ω and ω^2 are the roots of $x^2 + x + 1 = 0$

$$\therefore \frac{\alpha}{\beta} = \frac{\omega}{\omega^2} = \frac{1}{\omega} \Rightarrow \beta = \omega\alpha$$

and $\alpha + \beta = -\frac{\mu}{\lambda}, \alpha\beta = \frac{\nu}{\lambda}$

$$\Rightarrow \alpha(1+\omega) = -\frac{\mu}{\lambda}, \alpha^2\omega = \frac{\nu}{\lambda}$$

$$\Rightarrow -\alpha\omega^2 = -\frac{\mu}{\lambda}, \alpha^2\omega = \frac{\nu}{\lambda}$$

$$\Rightarrow \frac{\mu^2}{\lambda^2} = \frac{\nu}{\lambda} \text{ or } \mu^2 = \lambda\nu$$

Hence, (b) is the correct answer.

Ex 16. If $x^2 - 2r \cdot p_r x + r = 0$; $r = 1, 2, 3$ are three quadratic equations of which each pair has exactly one root common, then the number of solutions of the triplet (p_1, p_2, p_3) is
(a) 2 (b) 1 (c) 9 (d) 27

Sol. $x^2 - 2p_1x + 1 = 0$ has roots α, β
 $x^2 - 4p_2x + 2 = 0$ has roots α, γ
 $x^2 - 6p_3x + 3 = 0$ has roots β, γ
 $\Rightarrow \alpha + \beta = 2p_1, \alpha + \gamma = 4p_2, \beta + \gamma = 6p_3, \alpha\beta = 1,$
 $\alpha\gamma = 2, \beta\gamma = 3$
 $\therefore \gamma - \beta = 4p_2 - 2p_1$ and $\alpha(\gamma - \beta) = 1$
 $\Rightarrow \alpha = \frac{1}{4p_2 - 2p_1}$
 Also, $\frac{\beta}{\gamma} = \frac{1}{2} \Rightarrow 2\beta = \gamma$
 $\Rightarrow \beta \cdot 2\beta = 3$
 $\Rightarrow \beta = \pm \sqrt{\frac{3}{2}}$
 $\therefore \frac{1}{4p_2 - 2p_1} \left(\pm \sqrt{\frac{3}{2}} \right) = 1 \quad \dots(i)$
 Again, $\frac{1}{\alpha} + \frac{1}{\beta} = 2p_1, \frac{1}{\alpha} + \frac{1}{\gamma} = 2p_2$
 and $\frac{1}{\gamma} = -p_1 + p_2 + p_3$
 $\therefore \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = p_1 + p_2 + p_3$
 $\Rightarrow \frac{1}{\alpha} = p_1 + p_2 - p_3$
 $\frac{1}{\beta} = p_1 - p_2 + p_3$
 $\frac{1}{\gamma} = -p_1 + p_2 + p_3$
 So, $\frac{\gamma}{\beta} = 2 = \frac{p_1 - p_2 + p_3}{-p_1 + p_2 + p_3}$
 and $\frac{\alpha}{\gamma} = 3 = \frac{-p_1 + p_2 + p_3}{p_1 + p_2 - p_3}$
 Thus, we get $\frac{p_1}{7} = \frac{p_2}{4} = \frac{p_3}{9} = t$.
 Using this in Eq. (i), we get two values of x .
 Hence, (a) is the correct answer.

Ex 17. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that $\min f(x) > \max g(x)$, then relation between b and c is

- (a) no relation (b) $0 < c < \frac{b}{2}$
 (c) $|c| < \sqrt{2}|b|$ (d) $|c| > \sqrt{2}|b|$

Sol. We have, $f(x) = (x + b)^2 + 2c^2 - b^2$
 $\Rightarrow \min f(x) = 2c^2 - b^2$
 and $g(x) = b^2 + c^2 - (x + c)^2$
 $\Rightarrow \max g(x) = b^2 + c^2$

Thus, $\min f(x) > \max g(x)$
 $\Rightarrow 2c^2 - b^2 > b^2 + c^2$
 $\Rightarrow c^2 > 2b^2$
 $|c| > \sqrt{2}|b|$

Hence, (d) is the correct answer.

Ex 18. If $a \in R$ and the equation $(a - 2)(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where, $[x]$ denotes the greatest integer $\leq x$) has no integral solution and has exactly one solution in $(2, 3)$, then a lies in the interval
 (a) $(-1, 2)$ (b) $(0, 1)$
 (c) $(-1, 0)$ (d) $(2, 3)$

Sol. Let $y = x - [x]$. Then, equation $(a - 2)(x - [x])^2 + 2(x - [x]) + a^2 = 0 \quad \dots(i)$
 can be written as $f(y) = (a - 2)y^2 + 2y + a^2 = 0 \quad \dots(ii)$
 As x cannot be an integer, $y = x - [x] \neq 0$
 When $2 < x < 3, [x] = 2$
 $\Rightarrow 0 < x - [x] < 1$ i.e. $0 < y < 1$
 Since, Eq. (i) has exactly one solution in the interval $(2, 3)$, Eq. (ii) has exactly one solution in the interval $(0, 1)$. This is possible, if $f(0) \cdot f(1) < 0$
 For otherwise the Eq. (ii) has either no or two solutions in $(0, 1)$.
 $\Rightarrow a^2 \{a - 2 + 2 + a^2\} < 0$
 $\Rightarrow a(a + 1) < 0 \quad [\because a^2 > 0]$
 $\Rightarrow -1 < a < 0$ or $a \in (-1, 0)$
 Hence, (c) is the correct answer.

Ex 19. If a, b and c are all distinct, then

$a \frac{(x - b)(x - c)}{(a - b)(a - c)} + b \frac{(x - c)(x - a)}{(b - c)(b - a)} + c \frac{(x - a)(x - b)}{(c - a)(c - b)} - x$ is equal to
 (a) 0 (b) abc
 (c) $a + b + c$ (d) None of these

Sol. Let $f(x) = a \frac{(x - b)(x - c)}{(a - b)(a - c)} + b \frac{(x - c)(x - a)}{(b - c)(b - a)} + c \frac{(x - a)(x - b)}{(c - a)(c - b)} - x$
 We have, $f(a) = 0, f(b) = 0$ and $f(c) = 0$, i.e. f is a trinomial which has three distinct zeroes but $f(x)$ is quadratic, so it could not have more than two zeroes.
 $\Rightarrow f(x) \equiv 0$
 Hence, $a \frac{(x - b)(x - c)}{(a - b)(a - c)} + b \frac{(x - a)(x - c)}{(b - a)(b - c)} + c \frac{(x - a)(x - b)}{(c - a)(c - b)} \equiv 0, \forall x \in R$
 Hence, (a) is the correct answer.

Ex 20. If x is real, then the maximum value of

$$y = 2(a-x)(x + \sqrt{x^2 + b^2})$$

(a) $a^2 + b^2$ (b) a^2
(c) $\sqrt{a^2 + b^2}$ (d) $a\sqrt{a^2 + b^2}$

Sol. Let $t = x + \sqrt{x^2 + b^2}$

$$\Rightarrow \frac{1}{t} = \frac{1}{x + \sqrt{x^2 + b^2}} = \frac{\sqrt{x^2 + b^2} - x}{b^2}$$

$$\therefore t - \frac{b^2}{t} = 2x \quad \text{and} \quad t + \frac{b^2}{t} = 2\sqrt{x^2 + b^2}$$

$$\text{Thus, } 2(a-x)(x + \sqrt{x^2 + b^2}) = \left(2a - t + \frac{b^2}{t}\right)(t)$$

$$\begin{aligned} &= 2at - t^2 + b^2 \\ &= a^2 + b^2 - (a^2 - 2at + t^2) \\ &= a^2 + b^2 - (a-t)^2 \end{aligned}$$

$$\text{Therefore, } y = 2(a-x)(x + \sqrt{x^2 + b^2}) \leq a^2 + b^2$$

So, the maximum value of y is $a^2 + b^2$. This value is attained at $t = a$ or $x = \frac{a^2 - b^2}{2a}$.

Hence, (a) is the correct answer.

Ex 21. If $a > 0$, $a \neq 1$, then the equation

$$2 \log_x a + \log_{ax} a + 3 \log_{a^2x} a = 0$$

(a) exactly one real root
(b) two real roots
(c) no real root
(d) infinite number of real roots

Sol. The equation $2 \log_x a + \log_{ax} a + 3 \log_{a^2x} a = 0$ can be written as

$$\frac{2 \log a}{\log x} + \frac{\log a}{\log(ax)} + \frac{3 \log a}{\log(a^2x)} = 0 \quad \dots(i)$$

As $a > 0$ and $a \neq 1$, $\log a \neq 0$, Eq. (i) can be written as

$$\frac{2}{y} + \frac{1}{b+y} + \frac{3}{2b+y} = 0$$

where, $b = \log a$ and $y = \log x$

$$\Rightarrow \frac{2(b+y)(2b+y) + y(2b+y)}{2(b+y)(2b+y) + y(2b+y)} = 0$$

$$\Rightarrow 4b^2 + 11by + 6y^2 = 0$$

$$\Rightarrow y = \frac{-11b \pm \sqrt{121b^2 - 96b^2}}{12} = -\frac{4b}{3}, -\frac{b}{2}$$

$$\Rightarrow \log x = -\frac{4}{3} \log a \quad \text{or} \quad -\frac{1}{2} \log a$$

$$\Rightarrow x = a^{-4/3}, a^{-1/2}$$

Hence, (b) is the correct answer.

Ex 22. The set of values of p for which the roots of the equation $3x^2 + 2x + p(p-1) = 0$ are of opposite sign, is

(a) $(-\infty, 0)$ (b) $(0, 1)$
(c) $(1, \infty)$ (d) $(0, \infty)$

Sol. Since, the roots of the given equation are of opposite sign.

$\therefore$ Product of the roots < 0

$$\Rightarrow \frac{p(p-1)}{3} < 0 \Rightarrow p(p-1) < 0 \Rightarrow p \in (0, 1)$$

Hence, (b) is the correct answer.

Ex 23. If $p, q \in \{1, 2, 3, 4\}$, then the number of equations of the form $px^2 + qx + 1 = 0$ having real roots is

(a) 15 (b) 9 (c) 7 (d) 8

Sol. For real roots, we have

$$D \geq 0 \Rightarrow q^2 - 4p \geq 0 \Rightarrow q^2 \geq 4p$$

$$\text{If } p = 1, \text{ then } q^2 \geq 4 \Rightarrow q^2 \geq 4 \Rightarrow q = 2, 3, 4$$

$$\text{If } p = 2, \text{ then } q^2 \geq 8 \Rightarrow q^2 \geq 8 \Rightarrow q = 3, 4$$

$$\text{If } p = 3, \text{ then } q^2 \geq 12 \Rightarrow q^2 \geq 12 \Rightarrow q = 4$$

$$\text{If } p = 4, \text{ then } q^2 \geq 16 \Rightarrow q^2 \geq 16 \Rightarrow q = 4$$

Thus, we see that there are 7 cases.

Hence, (c) is the correct answer.

Ex 24. If p and q are the roots of the equation

$$x^2 + px + q = 0, \text{ then}$$

(a) $p = 1, q = -2$
(b) $p = 0, q = 1$
(c) $p = -2, q = 0$
(d) $p = -2, q = 1$

Sol. Since, p and q are roots of the equation

$$x^2 + px + q = 0 \Rightarrow p + q = -p \text{ and } pq = q$$

$$pq = q \Rightarrow q = 0 \text{ or } p = 1$$

$$\text{If } q = 0, \text{ then } p = 0 \text{ and if } p = 1, \text{ then } q = -2$$

Hence, (a) is the correct answer.

Ex 25. If $x^2 + 6x - 27 > 0$ and $x^2 - 3x - 4 < 0$, then

(a) $x > 3$ (b) $x < 4$
(c) $3 < x < 4$ (d) $x = \frac{7}{2}$

$$\text{Sol. } x^2 + 6x - 27 > 0 \quad \text{and} \quad x^2 - 3x - 4 < 0$$

$$\Rightarrow (x+9)(x-3) > 0 \quad \text{and} \quad (x-4)(x+1) < 0$$

$$\Rightarrow x \in (-\infty, -9) \cup (3, \infty) \quad \text{and} \quad (-1 < x < 4)$$

$$\Rightarrow 3 < x < 4$$

Hence, (c) is the correct answer.

Ex 26. If $x^2 + ax + b$ is an integer for every integer x , then

(a) a is always an integer but b need not be an integer
(b) b is always an integer but a need not be an integer
(c) $a + b$ is always an integer
(d) Cannot be discussed

Sol. Let $f(x) = x^2 + ax + b$

$$\text{Clearly, } f(0) = b \Rightarrow b \text{ is an integer.}$$

$$\text{Now, } f(1) = 1 + a + b$$

$$\Rightarrow a + b \text{ is an integer.}$$

Hence, (c) is the correct answer.

Ex 27. Sum of the real roots of the equation $x^2 + 5|x| + 6 = 0$ is equal to

- (a) 5 (b) 10
(c) -5 (d) Does not exist

Sol. Since, x^2 , $5|x|$ and 6 are positive, so $x^2 + 5|x| + 6 = 0$ does not have any real root. Therefore, sum does not exist. Hence, (d) is the correct answer.

Ex 28. The number of real solutions of the equation

$$\frac{6-x}{x^2-4} = 2 + \frac{x}{(x+2)}$$

- (a) two (b) one
(c) zero (d) None of these

Sol. $\frac{6-x}{x^2-4} = 2 + \frac{x}{(x+2)}$
 $\Rightarrow \frac{6-x}{x^2-4} = \frac{2x+4+x}{(x+2)}$
 $\Rightarrow \frac{6-x}{x^2-4} = \frac{3x+4}{x+2}$
 $\Rightarrow 6-x = \frac{(3x+4)(x+2)(x-2)}{x+2}$
 $\Rightarrow 6-x = (3x+4)(x-2)$ [where, $x+2 \neq 0$]
 $\Rightarrow 3x^2 - x - 14 = 0, x \neq -2$
 $\Rightarrow x = -2, \frac{7}{3}, x \neq -2$
 $\therefore$ Only one solution
Hence, (b) is the correct answer.

Ex 29. For real θ , value of $\sin^2 \theta + \cos^4 \theta$ lies in the interval

- (a) $[1, 2]$ (b) $\left[\frac{3}{4}, 1\right]$
(c) $\left[\frac{1}{4}, \frac{5}{16}\right]$ (d) None of these

Sol. Since, $0 \leq \cos^2 \theta \leq 1$
 $\Rightarrow \cos^4 \theta \leq \cos^2 \theta$
 $\therefore \sin^2 \theta + \cos^4 \theta \leq \sin^2 \theta + \cos^2 \theta \leq 1$... (i)
Also, $\sin^2 \theta + \cos^4 \theta = \sin^2 \theta + (1 - \sin^2 \theta)^2$
 $= \sin^4 \theta - \sin^2 \theta + 1$
 $= \left(\sin^2 \theta - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$... (ii)

From Eqs. (i) and (ii), we get
 $\frac{3}{4} \leq \sin^2 \theta + \cos^4 \theta \leq 1$

Hence, (b) is the correct answer.

Ex 30. If $0 < a < b < c$ and the roots α and β of the equation $ax^2 + bx + c = 0$ are imaginary, then

- (a) $|\alpha| = |\beta|$ (b) $|\alpha| < 1$
(c) $|b| < 1$ (d) None of these

Sol. Since, roots are imaginary, therefore $b^2 - 4ac < 0$.

The roots α and β are given by

$$\alpha = \frac{-b + i\sqrt{4ac - b^2}}{2a}$$

$$\text{and } \beta = \frac{-b - i\sqrt{4ac - b^2}}{2a}$$

Therefore, $|\alpha| = |\beta|$

$$\text{Further, } |\alpha| = \sqrt{\frac{b^2}{4a^2} + \frac{4ac - b^2}{4a^2}} = \sqrt{\frac{c}{a}}$$

$$\Rightarrow |\alpha| > 1 \quad [\because c > a]$$

Hence, (a) is the correct answer.

Ex 31. The number of real solutions of the equation

$$\left(\frac{9}{10}\right)^x = -3 + x - x^2$$

- (a) 0 (b) 1
(c) 2 (d) None of these

Sol. Let $f(x) = -3 + x - x^2$. Then, $f(x) < 0, \forall x$, because coefficient of $x^2 < 0$ and $D < 0$. Thus, LHS of the given equation is always positive whereas the RHS is always less than zero. The given equation has no solution. Hence, (a) is the correct answer.

Ex 32. Roots of the equation $ax^3 + bx^2 + cx + d = 0$ remain unchanged by increasing each coefficient by one unit, then

- (a) $a = b = c = d \neq 0$ (b) $a = b \neq c = d \neq 0$
(c) $a \neq b = c = d \neq 0$ (d) $a = b = c \neq d \neq 0$

Sol. $ax^3 + bx^2 + cx + d$
and $(a+1)x^3 + (b+1)x^2 + (c+1)x + (d+1) = 0$
must be identical.

$$\Rightarrow \frac{a+1}{a} = \frac{b+1}{b} = \frac{c+1}{c} = \frac{d+1}{d}$$

$$\Rightarrow a = b = c = d \neq 0$$

Hence, (a) is the correct answer.

Ex 33. Complete set of values of a such that

$$\frac{x^2 - x}{1 - ax}$$
 attains all real values, is

- (a) $[1, \infty)$ (b) $(0, 4]$
(c) $(0, 1]$ (d) None of these

$$\text{Sol. } y = \frac{x^2 - x}{1 - ax} \Rightarrow x^2 - x = y - axy$$

$$\Rightarrow x^2 + x(ay - 1) - y = 0$$

$$\text{Since, } x \text{ is real } \Rightarrow (ay - 1)^2 + 4y \geq 0$$

$$\Rightarrow a^2 y^2 + 2y(2 - a) + 1 \geq 0, \forall y \in \mathbb{R}$$

$$\Rightarrow a^2 > 0, 4(2 - a)^2 - 4a^2 \leq 0$$

$$\Rightarrow 4 + a^2 - 4a - a^2 \leq 0$$

$$\Rightarrow a^2 > 0, 4a \geq 4$$

$$\Rightarrow a \geq 1$$

$$\Rightarrow a \in [1, \infty)$$

Hence, (a) is the correct answer.

Ex 34. If the equations $ax^2 - 2bx + c = 0$, $bx^2 - 2cx + a = 0$ and $cx^2 - 2ax + b = 0$ all have only positive roots, then

- (a) $a = b = c$ (b) $a \neq b \neq c$
(c) $a \neq b = c$ (d) $a = b \neq c$

Sol. Clearly, $\frac{b}{a} > 0, \frac{c}{a} > 0, \frac{c}{b} > 0, \frac{a}{b} > 0, \frac{a}{c} > 0, \frac{b}{c} > 0$,

Thus, a, b and c all have same signs.

Also, $b^2 \geq ac, c^2 \geq ab, a^2 \geq bc$

These last three inequalities can hold simultaneously if and only if $a = b = c$.

Hence, (a) is the correct answer.

Ex 35. If $a \in \mathbb{R}^-$ and $a \neq -2$, then the equation

$$x^2 + a|x| + 1 = 0$$

- (a) cannot have any real root
(b) must have exactly two real roots
(c) must have either exactly two real roots or no real root
(d) must have either four real roots or no real root

Sol. $x^2 + a|x| + 1 = 0$

$$\Rightarrow |x|^2 + a|x| + 1 = 0$$

$$|x| = \frac{-a \pm \sqrt{a^2 - 4}}{2}$$

$\because a < 0$ and not equal to -2 .

$$\Rightarrow \frac{-a + \sqrt{a^2 - 4}}{2} \text{ and } \frac{-a - \sqrt{a^2 - 4}}{2} \text{ both are}$$

positive.

Thus, there are four roots, if $a < -2$, else no real root.

Hence, (d) is the correct answer.

Ex 36. If $x^2 + ax + 1$ is a factor of $ax^3 + bx + c$, then

- (a) $b + a + a^2 = 0, a = c$
(b) $b - a + a^2 = 0, a = c$
(c) $b + a - a^2 = 0, a = c$
(d) None of the above

Sol. $x^2 + ax + 1$ must divide $ax^3 + bx + c$.

$$\text{Now, } \frac{ax^3 + bx + c}{x^2 + ax + 1}$$

$$= a(x - a) + \frac{(b - a + a^3)x + c + a^2}{x^2 + ax + 1}$$

$$\Rightarrow (b - a + a^3)x + (c + a^2) = 0, \forall x$$

$$\Rightarrow b - a + a^3 = 0 \text{ and } c + a^2 = 0$$

Hence, (d) is the correct answer.

Ex 37. If $x^2 - (a - 3)x + a = 0$ has atleast one positive root, then

- (a) $a \in (-\infty, 0) \cup [7, 9]$
(b) $a \in (-\infty, 0) \cup [7, \infty)$
(c) $a \in (-\infty, 0] \cup [9, \infty)$
(d) None of the above

Sol. $x^2 - (a - 3)x + a = 0$,

$$D = (a - 3)^2 - 4a = a^2 - 10a + 9$$

$$D \geq 0$$

$$\Rightarrow (a - 9)(a - 1) \geq 0$$

$$\Rightarrow a \in (-\infty, 1] \cup [9, \infty)$$

Case I When both roots are positive.

$$D \geq 0$$

$$\Rightarrow (a - 9)(a - 1) \geq 0$$

$$\Rightarrow D \geq 0, a > 0, a > 3$$

$$\Rightarrow a \in [9, \infty) \quad \dots(i)$$

Case II When exactly one root is positive.

$$\Rightarrow a \leq 0 \quad \dots(ii)$$

Thus, from Eqs. (i) and (ii),

$$a \in (-\infty, 0] \cup [9, \infty)$$

Hence, (c) is the correct answer.

Ex 38. If roots of $x^2 - (a - 3)x + a = 0$ are such that both of them is greater than 2, then

- (a) $a \in [7, 9]$ (b) $a \in [7, \infty)$
(c) $a \in [9, 10]$ (d) $a \in [7, 9]$

Sol. $x^2 - (a - 3)x + a = 0$

$$\Rightarrow D = (a - 3)^2 - 4a$$

$$= a^2 - 10a + 9 = (a - 1)(a - 9)$$

When both roots are greater than 2, then

$$D \geq 0, f(2) > 0, -\frac{B}{2A} > 2$$

$$\Rightarrow (a - 1)(a - 9) \geq 0; 4 - (a - 3)2 + a > 0; \frac{a - 3}{2} > 2$$

$$\Rightarrow a \in (-\infty, 1] \cup [9, \infty); a < 10; a > 7$$

$$a \in [9, 10]$$

Hence, (c) is the correct answer.

Ex 39. If the quadratic equation $ax^2 + bx + 6 = 0$ does not have real roots and $b \in \mathbb{R}^+$, then

$$(a) a > \max \left\{ \frac{b^2}{24}, b - 6 \right\}$$

$$(b) a < \max \left\{ \frac{b^2}{24}, b - 6 \right\}$$

$$(c) a > \min \left\{ \frac{b^2}{24}, b - 6 \right\}$$

$$(d) a < \min \left\{ \frac{b^2}{24}, b - 6 \right\}$$

Sol. $ax^2 + bx + 6 = 0$, roots are not real.

$$\Rightarrow D < 0 \Rightarrow b^2 - 24a < 0 \Rightarrow a > \frac{b^2}{24}$$

i.e. a is positive. $\dots(i)$

$$\text{Also, } f(-1) > 0$$

$$\Rightarrow a - b + 6 > 0$$

$$\Rightarrow a > b - 6 \quad \dots(ii)$$

$$\Rightarrow a > \max \left\{ \frac{b^2}{24}, b - 6 \right\} \text{ [from Eqs. (i) and (ii)]}$$

Hence, (a) is the correct answer.

Ex 40. Consider the equation $x^2 + 2x - n = 0$, where $n \in N$ and $n \in [5, 100]$. Total number of different values of n so that the given equation has integral roots, is

- (a) 4 (b) 8 (c) 3 (d) 6

Sol. $x^2 + 2x - n = 0 \Rightarrow (x+1)^2 = n+1$

$$\Rightarrow x = -1 \pm \sqrt{n+1}$$

Thus, $n+1$ should be a perfect square.

Since, $n \in [5, 100] \Rightarrow n+1 \in [6, 100]$

Number of perfect squares between 1 and 100 is 10.

Thus, n can take $10-2$, i.e. 8 different values.

Hence, (b) is the correct answer.

Ex 41. If $\log_2(ax^2 + x + a) \geq 1, \forall x \in R$, then exhaustive set of values of a is

- (a) $\left(0, 1 + \frac{\sqrt{5}}{2}\right)$ (b) $\left(1 - \frac{\sqrt{5}}{2}, 1 + \frac{\sqrt{5}}{2}\right)$
(c) $\left(0, 1 - \frac{\sqrt{5}}{2}\right)$ (d) $\left(1 + \frac{\sqrt{5}}{2}, \infty\right)$

Sol. $\log_2(ax^2 + x + a) \geq 1, \forall x \in R$

$$\Rightarrow ax^2 + x + a \geq 2, \forall x \in R$$

$$\Rightarrow ax^2 + x + (a-2) \geq 0, \forall x \in R$$

$$\Rightarrow a > 0, 1-4a(a-2) \leq 0$$

$$\Rightarrow 4a^2 - 8a - 1 \geq 0$$

$$\Rightarrow a > 0, a \in \left(-\infty, 1 - \frac{\sqrt{5}}{2}\right) \cup \left(1 + \frac{\sqrt{5}}{2}, \infty\right)$$

$$\Rightarrow a \in \left(1 + \frac{\sqrt{5}}{2}, \infty\right)$$

Hence, (d) is the correct answer.

Ex 42. If $x^2 - x + a - 3 < 0$ for atleast one negative value of x , then complete set of values of ' a ' is

- (a) $(-\infty, 4)$
(b) $(-\infty, 2)$
(c) $(-\infty, 3)$
(d) $(-\infty, 1)$

Sol. The equation $x^2 - x + a - 3 = 0$ must have atleast one negative root.

For real roots, $D \geq 0 \Rightarrow 1-4(a-3) \geq 0$

$$\Rightarrow a \leq \frac{13}{4} \Rightarrow a \in \left(-\infty, \frac{13}{4}\right] \quad \dots(i)$$

Both root will be non-negative, if

$$D \geq 0, a-3 \geq 0$$

$$\Rightarrow a \leq \frac{13}{4}, a \geq 3$$

$$\Rightarrow a \in \left[3, \frac{13}{4}\right] \quad \dots(ii)$$

Thus, equation will have atleast one negative root, if

$$a \in \left(-\infty, \frac{13}{4}\right] \cup \left[3, \frac{13}{4}\right] \quad [\text{from Eqs. (i) and (ii)}]$$

$$\Rightarrow a \in (-\infty, 3)$$

Hence, (c) is the correct answer.

Ex 43. If $16 - x^2 > |x - a|$ is to be satisfied by atleast one negative value of x , then complete set of values of a is

- (a) $\left(-16, \frac{65}{4}\right)$ (b) $\left(-8, \frac{65}{2}\right)$
(c) $(-16, 16)$ (d) $(-8, 8)$

Sol. $16 - x^2 > |x - a|$

$$\Rightarrow x^2 - 16 < x - a < 16 - x^2$$

$$\Rightarrow x^2 - 16 - x < -a < 16 - x^2 - x$$

$$\Rightarrow x + 16 - x^2 > a > -16 + x^2 + x$$

$$\Rightarrow \frac{65}{4} - \left(x - \frac{1}{2}\right)^2 > a > -16 + x^2 + x; x \leq 0$$

$$\text{Since, } x \in R^- \Rightarrow a \in \left(-16, \frac{65}{4}\right)$$

Hence, (a) is the correct answer.

Ex 44. If $x_1, x_2, x_3, \dots, x_n$ are the roots of $x^n + ax + b = 0$, then the value of $(x_1 - x_2)(x_1 - x_3)(x_1 - x_4) \dots (x_1 - x_n)$ is

- (a) $nx_1 + b$ (b) $n(x_1)^{n-1}$
(c) $n(x_1)^{n-1} + a$ (d) $n(x_1)^{n-1} + b$

Sol. $x^n + x + b = (x - x_1)(x - x_2) \dots (x - x_n)$

$$\Rightarrow (x - x_2)(x - x_3) \dots (x - x_n) = \frac{x^n + ax + b}{(x - x_1)}$$

$$\Rightarrow (x_1 - x_2)(x_1 - x_3) \dots (x_1 - x_n) = \lim_{x \rightarrow x_1} \frac{x^n + ax + b}{x - x_1} = n(x_1)^{n-1} + a$$

Hence, (c) is the correct answer.

Ex 45. If $x^2 + (a-b)x + (1-a-b) = 0$, where $a, b \in R$, then the values of a for which equation has unequal roots for all values of b , is given by

- (a) $a < 1$ (b) $a > 1$
(c) $a \in R$ (d) None of these

Sol. For unequal roots, $D > 0$

$$\Rightarrow (a-b)^2 - 4(1-a-b) > 0$$

$$\Rightarrow b^2 + b(4-2a) + a^2 + 4a - 4 > 0$$

For the above quadratic inequation to be true, $\forall b \in R$, the discriminant of its corresponding equation should be less than zero.

$$\text{i.e. } (4-2a)^2 - 4(a^2 + 4a - 4) < 0$$

$$\Rightarrow -32a + 32 < 0$$

$$\Rightarrow a > 1$$

Hence, (b) is the correct answer.

Ex 46. Let a, b and c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Then, $a^3x^2 + abcx + c^3 = 0$ has roots

- (a) $\alpha^2\beta, \beta^2\alpha$ (b) α, β^2
(c) $\alpha^2\beta, \beta\alpha$ (d) $\alpha^3\beta, \beta^3\alpha$

Sol. Dividing the equation, $a^3x^2 + abcx + c^3 = 0$ by c^2 , we

$$\text{get } a\left(\frac{ax}{c}\right)^2 + b\left(\frac{ax}{c}\right) + c = 0$$

$$\Rightarrow \frac{ax}{c} = \alpha, \beta \text{ are roots}$$

$$\Rightarrow x = \frac{c}{a}\alpha \text{ and } x = \frac{c}{a}\beta$$

$$\Rightarrow x = \alpha^2\beta \text{ and } \alpha\beta^2 \text{ are roots of above equation.}$$

Hence, (a) is the correct answer.

Ex 47. The values of 'a' for which $x^2 + ax + \sin^{-1}(x^2 - 4x + 5) + \cos^{-1}(x^2 - 4x + 5) = 0$ has atleast one solution, is

- (a) -2 (b) -2 + π
(c) - $\frac{\pi}{4}$ (d) -2 - $\frac{\pi}{4}$

Sol. Since, $\sin^{-1}\theta$ is defined only when

$$-1 \leq \theta \leq 1$$

$$\Rightarrow -1 \leq x^2 - 4x + 5 \leq 1$$

$$\Rightarrow x^2 - 4x + 4 \leq 0 \text{ and } x^2 - 4x + 6 \geq 0$$

$$\Rightarrow (x-2)^2 \leq 0 \text{ and } (x-2)^2 + 2 \geq 0$$

$$\Rightarrow x = 2 \text{ is only solution}$$

Putting $x = 2$, we get

$$4 + 2a + \frac{\pi}{2} = 0$$

$$\therefore a = -2 - \frac{\pi}{4}$$

Hence, (d) is the correct answer.

Ex 48. The set of values of x satisfying $\log_{|\sin x|}|\cos x|$

$$+ \log_{|\cos x|}|\sin x| = 2, \text{ when } x \in \left(0, \frac{\pi}{2}\right), \text{ is}$$

- (a) $\left\{\frac{\pi}{4}\right\}$ (b) $\left\{\frac{\pi}{3}\right\}$ (c) $\left(0, \frac{\pi}{4}\right)$ (d) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Sol. Here, $\log_{|\sin x|}|\cos x| + \frac{1}{\log_{|\sin x|}|\cos x|} = 2$

$$\Rightarrow y + \frac{1}{y} = 2, \text{ where } y = \log_{|\sin x|}|\cos x|$$

$$\Rightarrow y^2 - 2y + 1 = 0$$

$$\Rightarrow y = 1$$

$$\therefore \log_{|\sin x|}|\cos x| = 1$$

$$\Rightarrow |\cos x| = |\sin x|$$

$$\Rightarrow |\tan x| = 1$$

$$\Rightarrow x = \frac{\pi}{4}$$

Hence, (a) is the correct answer.

Ex 49. If $a, b, c \in I$ and $ax^2 + bx + c = 0$ has irrational root and $\lambda \in Q = \frac{p}{q}$, where $f(x) = ax^2 + bx + c$,

then $|f(\lambda)|$

- (a) $\geq \frac{1}{q^2}$ (b) $< \frac{1}{q^2}$

(c) Cannot be discussed (d) None of these

Sol. Let $f(x) = ax^2 + bx + c$

Since, coefficients are integers and one root is irrational, then both the roots are irrational.

$$\therefore \text{For any } \lambda \in Q, f(\lambda) \neq 0$$

$$\Rightarrow f(\lambda) > 0 \text{ or } f(\lambda) < 0, \text{ but } f(\lambda) \neq 0$$

$$\therefore |f(\lambda)| > 0$$

$$\text{or } \left| a\frac{p^2}{q^2} + b\frac{p}{q} + c \right| > 0$$

$$\text{or } \frac{1}{q^2} |ap^2 + bpq + cq^2| > 0 \quad \dots(i)$$

As $a, b, c, p, q \in \text{integers}$

$$\therefore |ap^2 + bpq + cq^2| \geq 1 \quad \dots(ii)$$

$$\Rightarrow |f(\lambda)| = \frac{1}{q^2} |ap^2 + bpq + cq^2| \geq \frac{1}{q^2}$$

[from Eqs. (i) and (ii)]

Hence, (a) is the correct answer.

Ex 50. The equation $x^2 \cdot 2^{|x-3|+4} + 2^{x-1}$
 $= x^2 \cdot 2^{x+1} + 2^{|x-3|+2}$ has

- (a) no solution (b) two solutions
(c) $x = \pm \frac{1}{2}$ and $x \geq 3$ (d) $x \geq 3$

Sol. Here, $x^2 \cdot 2^{|x-3|+4} + 2^{x-1} = x^2 \cdot 2^{x+1} + 2^{|x-3|+2}$

If $x - 3 \geq 0$

$$\Rightarrow x^2 \cdot 2^{x-3+4} + 2^{x-1} = x^2 \cdot 2^{x+1} + 2^{x-3+2}$$

$$\Rightarrow x^2 \cdot 2^{x+1} + 2^{x-1} = x^2 \cdot 2^{x+1} + 2^{x-1}$$

which is always true for $x \geq 3$.

Again, if $x - 3 < 0$

$$x^2 \cdot 2^{-(x-3)+4} + 2^{x-1} = x^2 \cdot 2^{x+1} + 2^{-(x-3)+2}$$

$$\Rightarrow x^2 \cdot 2^{7-x} + 2^{x-1} = x^2 \cdot 2^{x+1} + 2^{5-x}$$

$$\Rightarrow x^2 \cdot 2^{7-x} - 2^{5-x} = x^2 \cdot 2^{x+1} - 2^{x-1}$$

$$\Rightarrow 2^{5-x}(x^2 \cdot 2^2 - 1) = 2^{x-1}(x^2 \cdot 2^2 - 1)$$

$$\Rightarrow (2^{5-x} - 2^{x-1})(4x^2 - 1) = 0$$

$$\Rightarrow x^2 = \frac{1}{4} \text{ and } 5 - x = x - 1$$

$$\Rightarrow x = \pm \frac{1}{2} \text{ and } x = 3$$

$$\therefore \text{Solution set is } x = \pm \frac{1}{2} \text{ and } x \geq 3$$

Hence, (c) is the correct answer.

Ex 51. For any real value of $\theta \neq \pi$, the value of the

$$\text{expression } y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta} \text{ is}$$

- (a) $-1 \leq y \leq 2$ (b) $y < 0$ and $y > 2$
(c) $-1 \leq y \leq 1$ (d) $y \geq 1$

Sol. Here, $y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta}$

$$\Rightarrow (y-1)\cos^2 \theta + y\cos \theta + 1 = 0$$

$$\Rightarrow \cos \theta = -1 \text{ and } \cos \theta = \frac{1}{1-y}$$

$$\therefore -1 < \frac{1}{1-y} < 1$$

$$\Rightarrow \frac{1}{1-y} + 1 > 0 \quad \text{and} \quad \frac{1}{1-y} - 1 < 0$$

$$\Rightarrow \frac{2-y}{1-y} > 0 \quad \text{and} \quad \frac{y}{1-y} < 0$$

$$\Rightarrow y < 0 \quad \text{and} \quad y > 2$$

Hence, (b) is the correct answer.

Ex 52. If the roots of the equation $x^2 + ax + b = 0$ are c and d , then one of the roots of the equation $x^2 + (2c + a)x + (c^2 + ac + b^2) = 0$ is
(a) c (b) $d - c$ (c) $2d$ (d) $2c$

Sol. Here, $f(x) = x^2 + ax + b$, then

$$\begin{aligned} f(x+c) &= (x+c)^2 + a(x+c) + b \\ &= x^2 + (2c+a)x + c^2 + ac + b \end{aligned}$$

which shows roots of $f(x)$ are transformed to $(x-c)$, i.e. roots of $f(x+c) = 0$ are $c-c$ and $d-c$.

Thus, $x^2 + (2c+a)x + c^2 + ac + b^2 = 0$ has roots 0 and $(d-c)$.

Hence, (b) is the correct answer.

Ex 53. Number of integers, which satisfy the inequality $\frac{(16)^{1/x}}{(2^{x+3})} > 1$, is equal to
(a) infinite (b) 0 (c) 1 (d) 4

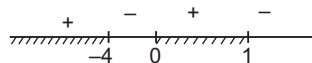
Sol. Here, $\frac{(16)^{1/x}}{2^{x+3}} > 1$

$$\Rightarrow \frac{2^{4/x}}{2^{x+3}} > 1 \quad \text{or} \quad 2^{\frac{4}{x}-x-3} > 1$$

$$\text{i.e.} \quad 2^{\frac{4}{x}-x-3} > 2^0 \Rightarrow \frac{4}{x} - x - 3 > 0$$

$$\Rightarrow -\frac{(x^2 + 3x - 4)}{x} > 0 \quad \text{or} \quad \frac{-(x+4)(x-1)}{x} > 0$$

Using number line rule,



$$\Rightarrow x \in (-\infty, -4) \cup (0, 1)$$

Hence, (a) is the correct answer.

Ex 54. Number of non-negative integral solution of equation $y^4 + 6xy^2 - 8x = 0$ is equal to
(a) 1 (b) 2 (c) 3 (d) infinite

Sol. Here, $x = \frac{y^4}{8-6y^2}$ but x is non-negative.

$$\text{So,} \quad \frac{y^4}{8-6y^2} \geq 0$$

$$\Rightarrow 8-6y^2 > 0 \quad \text{or} \quad y^2 < \frac{4}{3}$$

i.e. $y = 0, 1$ [as y is non-negative integers]

For $y = 0$, we get $x = 0$

For $y = 1$, we get $x = \frac{1}{2} \notin \text{integer}$

$\therefore$ Only one non-negative integral solution.

Hence, (a) is the correct answer.

Ex 55. For what values of $K \in R$ the expression $2x^2 + Kxy + 3y^2 - 5y - 2$ can be expressed as $(a_1x + b_1y + c_1) \cdot (a_2x + b_2y + c_2)$?

(a) $-3, -4$

(b) $2, 3$

(c) $3, 4$

(d) $7, -7$

Sol. As, $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ can be expressed as product of two linear factors, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$\Rightarrow 2x^2 + Kxy + 3y^2 - 5y - 2$ can be expressed as $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2)$, if

$$2(3)(-2) + 2\left(-\frac{5}{2}\right)(0)\left(\frac{K}{2}\right) - 2\left(-\frac{5}{2}\right)^2 - 3(0)^2 - (-2)\left(\frac{K}{2}\right)^2 = 0$$

$$\Rightarrow K^2 - 49 = 0 \Rightarrow K = 7, -7$$

Hence, (d) is the correct answer.

Ex 56. The number of triangles formed by the lines represented by $x^3 - x^2 - x - 2 = 0$ and $xy^2 + 2xy + 4x - 2y^2 - 4y - 8 = 0$ is

(a) 1

(b) 2

(c) 3

(d) None of these

Sol. Here, $x^3 - x^2 - x - 2 = 0$

$$\Rightarrow (x-2)(x^2 + x + 1) = 0 \Rightarrow x = 2 \quad \dots(i)$$

$$\text{and } xy^2 + 2xy + 4x - 2y^2 - 4y - 8 = 0$$

$$\Rightarrow y^2(x-2) + 2y(x-2) + 4(x-2) = 0$$

$$\Rightarrow (x-2)(y^2 + 2y + 4) = 0$$

$$\Rightarrow x = 2 \quad \dots(ii)$$

Both the equation represent same line. So, number of triangles is zero.

Hence, (d) is the correct answer.

Ex 57. If $x^2 - (a+b+c)x + (ab+bc+ca) = 0$ has imaginary roots, where $a, b, c \in R^+$, then $\sqrt{a}, \sqrt{b}$ and $\sqrt{c}$

(a) can be sides of triangle

(b) cannot be side of triangle

(c) Nothing can be said

(d) None of the above

Sol. Here, $x^2 - (a+b+c)x + (ab+bc+ca) = 0$ has imaginary roots.

$$\Rightarrow D < 0 \quad \text{or} \quad (a+b+c)^2 - 4(ab+bc+ca) < 0$$

$$\Rightarrow (a^2 + b^2 + c^2 - 2ab - 2bc + 2ac) < 4ac$$

$$\Rightarrow (a-b+c)^2 < 4ac$$

$$\Rightarrow -2\sqrt{ac} < a-b+c$$

$$\Rightarrow (a+c+2\sqrt{ac}) > b$$

$$\Rightarrow (\sqrt{a} + \sqrt{c})^2 > b \quad \text{or} \quad \sqrt{a} + \sqrt{c} > \sqrt{b}$$

$$\text{Similarly, } \sqrt{b} + \sqrt{c} > \sqrt{a} \quad \text{and} \quad \sqrt{a} + \sqrt{b} > \sqrt{c}$$

$$\Rightarrow \sqrt{a}, \sqrt{b} \text{ and } \sqrt{c} \text{ can be sides of triangle.}$$

Hence, (a) is the correct answer.

Ex 64. If α and β are the roots of the equation $x^2 + px - \frac{1}{2p^2} = 0$, where $p \in R$, then the

minimum value of $\alpha^4 + \beta^4$ is

- (a) 2 (b) $2 + \sqrt{2}$ (c) $2\sqrt{2}$ (d) $2 - \sqrt{2}$

Sol. Here, $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$

$$\begin{aligned} &= \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2 \\ &= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{2p^4} \\ &= p^4 + \frac{1}{2p^4} + 2 \\ &= \left(p^2 - \frac{1}{\sqrt{2}p^2}\right)^2 + 2 + \sqrt{2} \geq 2 + \sqrt{2} \end{aligned}$$

Thus, minimum value of $\alpha^4 + \beta^4$ is $2 + \sqrt{2}$.

Hence, (b) is the correct answer.

Ex 65. If α and β are roots of the equation $x^2 + ax + b = 0$, then maximum value of the expression $-(x^2 + ax + b) - \frac{(\alpha - \beta)^2}{4}$ will be

- (a) $\frac{a^2 - 4b}{4}$ (b) $\frac{b^2 - 4a}{4}$
(c) 0 (d) None of these

Sol. Here, $y = -(x^2 + ax + b)$

$$y \text{ is maximum i.e. } y_{\max} = -\frac{D}{4a} = -\frac{(a^2 - 4b)}{4}$$

$$\therefore y_{\max} = \frac{4b - a^2}{4} = -\frac{(\alpha - \beta)^2}{4}$$

$$\Rightarrow \text{Maximum value of } -(x^2 + ax + b) - \frac{(\alpha - \beta)^2}{4} = 0$$

Hence, (c) is the correct answer.

Ex 66. If $a, b, c > 0$ and $a(1-b) > \frac{1}{4}$, $b(1-c) > \frac{1}{4}$ and $c(1-a) > \frac{1}{4}$, then

- (a) never possible (b) always true
(c) Cannot be discussed (d) None of these

Sol. If the given inequalities hold, then

$$a(1-b)b(1-c)c(1-a) > \frac{1}{64} \quad \dots (i)$$

For $x > 0$, we have

$$x(1-x) = x - x^2$$

$$= \frac{1}{4} - \left(\frac{1}{2} - x\right)^2 \leq \frac{1}{4} \quad \left[\because \left(\frac{1}{2} - x\right)^2 \geq 0\right]$$

$$\Rightarrow a(1-a)b(1-b)c(1-c) \leq \frac{1}{64} \quad \dots (ii)$$

Eqs. (i) and (ii) cannot hold simultaneously.

Hence, (a) is the correct answer.

Ex 67. $x^4 - 4x - 1 = 0$

- (a) all real roots (b) all imaginary roots
(c) exactly two real roots (d) None of these

Sol. Let $f(x) = x^4 - 4x - 1$

$$f'(x) = 4x^3 - 4 = 4(x-1)(x^2 + x + 1)$$

So, $f(x)$ decrease in $(-\infty, 1)$ and increases in $(1, \infty)$.

But $f(1) = -4$, so $f(x) = 0$ has only two real roots in which one will be positive and one will be negative. (as $f(0)$ is negative).

Hence, (c) is the correct answer.

Ex 68. If an unknown polynomial is divided by $(x-1)$ and $(x-2)$, we obtain the remainder 2 and 1 respectively, then the remainder resulting from the division of this polynomial by $(x-1)(x-2)$ is

- (a) 3 (b) -3 (c) $3-x$ (d) $3-2x$

Sol. Let the polynomial be $P(x)$, when divided by $(x-1)$ leaves remainder 2 $\Rightarrow P(1) = 2$ and when $P(x)$ divided by $(x-2)$ leaves remainder 1 $\Rightarrow P(2) = 1$. When $P(x)$ is divided by $(x-1)(x-2)$, could be given by

$$P(x) = \{q(x)\}(x-1)(x-2) + r(x)$$

where, $q(x)$ is quotient and $r(x)$ is remainder.

$$\text{i.e. } r(x) = ax + b$$

$$\Rightarrow P(x) = (x-1)(x-2)q(x) + (ax+b) \quad \dots (i)$$

$$\therefore P(1) \Rightarrow 0 + a + b = 2$$

$$\text{and } P(2) \Rightarrow 0 + 2a + b = 1$$

On solving, we get $a = -1, b = 3$

$$\therefore \text{Remainder} = -x + 3$$

Hence, (c) is the correct answer.

Type 2. More than One Correct Option

Ex 69. If $3^{x+2} - 9^{-1/x} > 0$, then the interval of x can be

- (a) $x \in (0, \infty)$ (b) $x \in (0, 250)$
(c) $x \in R$ (d) $x \in (-250, 250)$

$$\text{Sol. } 3^{x+2} > 9^{-1/x} \Rightarrow x+2 > -\frac{2}{x} \Rightarrow \frac{x^2 + 2x + 2}{x} > 0$$

$$\frac{(x+1)^2 + 1}{x} > 0$$

$$\Rightarrow x > 0$$

Hence, (a) and (b) are the correct answers.

Ex 70. If $ax^2 - bx + c = 0$ has two distinct roots lying

in the interval $(0, 1)$, $a, b, c \in N$. Then,

- (a) $\log_5 abc = 1$
(b) $\log_6 abc = 2$
(c) $\log_5 abc = 3$
(d) $\log_6 abc = 4$

$$\text{Sol. } \alpha + \beta = \frac{b}{a}, \alpha\beta = \frac{c}{a}, 0 < \alpha\beta < 1$$

$$\Rightarrow 0 < (1-\alpha) \cdot (1-\beta) < 1$$

$$\begin{aligned} \Rightarrow 0 < \alpha(1-\alpha) &\leq \frac{1}{4} \\ \text{and } 0 < \beta(1-\beta) &\leq \frac{1}{4} \\ 0 < \alpha\beta(1-(\alpha+\beta)+\alpha\beta) &\leq \frac{1}{16} \\ 0 < \frac{c}{a}\left(1-\frac{b}{a}+\frac{c}{a}\right) &\leq \frac{1}{16} \\ 0 < c(a-b+c) &< \frac{a^2}{16} \\ \Rightarrow a=b, c=1 \\ \Rightarrow a^2 &> 16 \\ \Rightarrow a &> 4 \\ \Rightarrow a &= 5 \\ \Rightarrow b &= 5 \\ \Rightarrow \min(abc) &= 25 \end{aligned}$$

Hence, (b), (c) and (d) are the correct answers.

Ex 71. If $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ ($a, b, c \in R$) have a common non-real root, then
 (a) $|b| < 2|a|$
 (b) $|b| < 2|c|$
 (c) $a = \pm c$
 (d) $a = c$

Sol. $D_1 = b^2 - 4ac < 0$, $D_2 = b^2 - 4ac < 0$, as the root is non-real.

$\Rightarrow$ Both roots will be common.

$$\Rightarrow \frac{a}{c} = \frac{b}{b} = \frac{c}{a} = 1$$

$$\Rightarrow a = c$$

$$\text{Now, } b^2 - 4ac < 0$$

$$\Rightarrow b^2 - 4a^2 < 0$$

$$\Rightarrow |b| < 2|a|$$

Hence, (a), (b) and (d) are the correct answers.

Ex 72. Consider the equation $x^2 + x - a = 0$, $a \in N$.

If equation has integral roots, then

$$(a) a = 2$$

$$(b) a = 6$$

$$(c) a = 12$$

$$(d) a = 20$$

Sol. Discriminant, $D = 1 + 4a$

$\Rightarrow 1 + 4a$ should be a perfect square.

As $1 + 4a$ is always odd.

$$\Rightarrow 1 + 4a = (2\lambda + 1)^2, \lambda \in I^+$$

$$\Rightarrow a = \lambda(\lambda + 1)$$

Hence, (a), (b), (c) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 73-76) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

Ex 73. Statement I The quadratic equation $(a-b)x^2 + (b-c)x + (c-a) = 0$ has one root $x = 1$.

Statement II If sum of the coefficients in a quadratic equation vanishes, then its one root is $x = 1$.

Sol. Statement I $a - b + b - c + c - a = 0$

If quadratic equation $ax^2 + bx + c = 0$ has one root $x = 1$, then $a + b + c = 0$

Sum of coefficients = 0

Hence, (a) is the correct answer.

Ex 74. Statement I The equation $(x-p)(x-r) + \lambda(x-q)(x-s) = 0$, $p < q < r < s$ has non-real roots, if $\lambda > 0$.

Statement II The equation $ax^2 + bx + c = 0$, $a, b, c \in R$ has non-real roots, if $b^2 - 4ac < 0$.

Sol. Statement I

$$\text{Let } f(x) = (x-p)(x-r) + \lambda(x-q)(x-s)$$

If $\lambda > 0$, then $f(p) > 0$, $f(q) < 0$, $f(r) < 0$ and $f(s) > 0$
 $f(x) = 0$ has one real root between p and q and other real root between r and s .

Statement II Obviously true.

Hence, (d) is the correct answer.

Ex 75. Statement I If roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c < 1$.

Statement II If a, b and c are odd integers, then the roots of the equation $4abcx^2 + (b^2 - 4ac)x - b = 0$ are real and distinct.

Sol. Statement I Given equation $x^2 - bx + c = 0$

Let α and β be two roots such that $|\alpha - \beta| = 1$.

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1 \Rightarrow b^2 - 4c = 1$$

Statement II Given equation

$$4abcx^2 + (b^2 - 4ac)x - b = 0$$

$$\therefore D = (b^2 - 4ac)^2 + 16ab^2c$$

$$\Rightarrow D = (b^2 + 4ac)^2 > 0$$

So, roots are real and unequal.

Hence, (d) is the correct answer.

Ex 76. Statement I If one root is $\sqrt{5} - \sqrt{2}$, then the equation of lowest degree with rational coefficient is $x^4 - 14x^2 + 9 = 0$.

Statement II For a polynomial equation with rational coefficient irrational roots occur in pairs.

Sol. Given a root of the equation is

$$x = \sqrt{5} - \sqrt{2}$$

On squaring both sides, we get

$$x^2 = 5 + 2 - 2\sqrt{10} \Rightarrow 7 - x^2 = 2\sqrt{10}$$

Again squaring both sides, we get

$$49 + x^4 - 14x^2 = 40 \Rightarrow x^4 - 14x^2 + 9 = 0$$

For polynomial equation with rational coefficients irrational roots occurs in pairs.

Hence, (a) is the correct answer.

Ex 77. Statement I The number of values of a for which $(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a^2 - 4 = 0$ is an identity in x , is 2.

Statement II If $a = b = c = 0$, then equation $ax^2 + bx + c = 0$ is an identity in x .

Sol. Statement I $a^2 - 3a + 2 = 0$

$$\Rightarrow a = 1, 2$$

$$a^2 - 5a + 6 = 0$$

$$\Rightarrow a = 2, 3$$

$$a^2 - 4 = 0$$

$$\Rightarrow a = \pm 2$$

$a = 2$ is the only solution. Hence, Statement I is false.

Statement II is true by definition.

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 78-80) Let $f(x) = x^2 + b_1x + c_1$, $g(x) = x^2 + b_2x + c_2$, real roots of $f(x) = 0$ be α, β and real roots of $g(x) = 0$ be $\alpha + \delta, \beta + \delta$. Least value of $f(x)$ be $-\frac{1}{4}$. Least value of $g(x)$ occurs at $x = \frac{7}{2}$.

Ex 78. The least value of $g(x)$ is

- (a) -1 (b) $-\frac{1}{2}$
 (c) $-\frac{1}{4}$ (d) $-\frac{1}{3}$

Ex 79. The value of b_2 is

- (a) 6 (b) -7
 (c) 8 (d) 0

Ex 80. The roots of $g(x) = 0$ are

- (a) 3, 4 (b) -3, 4
 (c) 3, -4 (d) -3, -4

Sol. (Q. Nos. 78-80)

$$\begin{aligned} (\beta - \alpha) &= (\beta + \delta) - (\alpha + \delta) \Rightarrow (\beta + \alpha)^2 - 4\alpha\beta \\ &= [(\beta + \delta) + (\alpha + \delta)]^2 - 4(\beta + \delta)(\alpha + \delta) - 4c_1 \\ &= (-b_2)^2 - 4c_2 \end{aligned}$$

$$\begin{aligned} D_1 &= D_2 \quad \dots(i) \\ \text{Least value of } f(x) &= -\frac{D_1}{4} = -\frac{1}{4} \Rightarrow D_1 = 1 \end{aligned}$$

$$\begin{aligned} \text{and } D_2 &= 1 \\ \therefore \text{Least value of } g(x) &= -\frac{D_2}{4} = -\frac{1}{4} \end{aligned}$$

$$\text{Least value of } g(x) \text{ occurs at } x = -\frac{b_2}{2} = \frac{7}{2} \Rightarrow b_2 = -7$$

$$\begin{aligned} b_2^2 - 4c_2 &= D_2 \Rightarrow 49 - 4c_2 = 1 \\ \Rightarrow \frac{48}{4} &= c_2 \Rightarrow c_2 = 12 \\ x^2 - 7x + 12 &= 0 \Rightarrow x = 3, 4 \end{aligned}$$

78. (c) **79.** (b) **80.** (a)

Passage II (Ex. Nos. 81-83) If roots of the equation $x^4 - 12x^3 + bx^2 + cx + 81 = 0$ are positive.

Ex 81. Value of b is

- (a) -54 (b) 54 (c) 27 (d) -27

Ex 82. Value of c is

- (a) 108 (b) -108 (c) 54 (d) -54

Ex 83. Root of equation $2bx + c = 0$ is

- (a) -1/2 (b) 1/2 (c) 1 (d) -1

Sol. (Q. Nos. 81-83)

Let roots of $x^4 - 12x^3 + bx^2 + cx + 81 = 0$ be $\alpha, \beta, \gamma, \delta$.

$$\Rightarrow \alpha + \beta + \gamma + \delta = 12 \text{ and } \alpha\beta\gamma\delta = 81$$

$$\frac{\alpha + \beta + \gamma + \delta}{4} \geq (\alpha\beta\gamma\delta)^{1/4}$$

$$\text{Since, } \frac{\alpha + \beta + \gamma + \delta}{4} = 3 \text{ and } (\alpha\beta\gamma\delta)^{1/4} = 3$$

$$\therefore \alpha = \beta = \gamma = \delta \Rightarrow \alpha = \beta = \gamma = \delta = 3$$

$$\text{Now, } b = \sum \alpha\beta = 6 \times 9 = 54$$

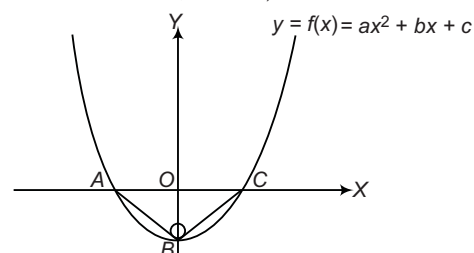
$$c = -\sum \alpha\beta\gamma = -4 \times 27 = -108$$

$$\Rightarrow 2b + c = 0$$

$$\therefore 1 \text{ is a root of equation } 2bx + c = 0.$$

81. (b) **82.** (b) **83.** (c)

Passage III (Ex. Nos. 84-86) In the given figure vertices of $\triangle ABC$ lie on $y = f(x) = ax^2 + bx + c$. The $\triangle ABC$ is right angled isosceles triangle whose hypotenuse $AC = 4\sqrt{2}$ units, then



Ex 84. $y = f(x)$ is given by

(a) $y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$ (b) $y = \frac{x^2}{2} - 2$
(c) $y = x^2 - 8$ (d) $y = x^2 - 2\sqrt{2}$

Sol. $\therefore AC = 4\sqrt{2}$

$\therefore AB = BC = \frac{4\sqrt{2}}{\sqrt{2}} = 4$ units

and $OB = \sqrt{4^2 - (2\sqrt{2})^2} = 2\sqrt{2}$

$\therefore A(-2\sqrt{2}, 0), B(2\sqrt{2}, 0), C(0, -2\sqrt{2})$

$\therefore y = ax^2 + bx + c$ passes through A, B and C , we get

$y = \frac{x^2}{2\sqrt{2}} - 2\sqrt{2}$

Hence, (a) is the correct answer.

Ex 85. Minimum value of $y = f(x)$ is

(a) $2\sqrt{2}$ (b) $-2\sqrt{2}$ (c) 2 (d) -2

Sol. Minimum value of $y = \frac{x^2}{2\sqrt{2}}$ at $x = 0$ is $-2\sqrt{2}$.

Hence, (b) is the correct answer.

Ex 86. Number of integral value of k for which $k/2$ lies between the roots of $f(x) = 0$, is

(a) 9 (b) 10 (c) 11 (d) 12

Sol. $\therefore$ Roots of $f(x) = 0$

i.e. $\frac{x^2}{2\sqrt{2}} - 2\sqrt{2} = 0 \Rightarrow x = \pm 2\sqrt{2}$

$\therefore$ Number of integral value, of k for which $k/2$ lies in $(-2\sqrt{2}, 2\sqrt{2})$ is 11.

Hence, (c) is the correct answer.

Type 5. Match the Columns

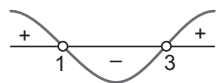
Ex 87. Match the statements of Column I with values of Column II.

Column I	Column II
A. The equation $x^3 - 6x^2 + 9x + \lambda = 0$ has exactly one root in $(1, 3)$, then $[\lambda + 1]$ is (where, $[\]$ denotes the greatest integer function)	p. -3
B. If $-3 < \frac{x^2 - \lambda x - 2}{x^2 + x + 1} < 2, \forall x \in R$, then $[\lambda]$ can be (where $[\]$ denotes the greatest integer function)	q. -2
C. If $x^2 + \lambda x + 1 = 0$ and $(b - c)x^2 + (c - a)x + (a - b) = 0$ have both the roots common, then $[\lambda - 1]$ is (where, $[\]$ denotes the greatest integer function)	r. -1
D. If N is the number of solutions of the equation $ x - 4 - x -2x = 4$, then the value of $-N$ is	s. 0

Sol. A. Let $f(x) = x^3 - 6x^2 + 9x + \lambda$

$\therefore f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$

$\therefore f'(x) = 0$ in $(1, 3)$



But $f(1) = 4 + \lambda$ and $f(3) = \lambda$

For $f(x) = 0$ to have exactly one root in $(1, 3)$

$f(1)$ and $f(3)$ should have opposite signs

$\therefore f(1)f(3) < 0$

$\Rightarrow \lambda(\lambda + 4) < 0 \Rightarrow -3 < \lambda + 1 < 0$

$\therefore -3 < \lambda + 1 < 0$

$\Rightarrow [\lambda + 1] = -3, -2, -1, 0$

B. $\therefore -3 < \frac{x^2 - \lambda x - 2}{x^2 + x + 1} < 2$

$\Rightarrow -3x^2 - 3x - 3 < x^2 - \lambda x - 2 < 2x^2 + 2x + 2$

$[\because x^2 + x + 1 > 0, \forall x \in R]$

From inequality (i),

$4x^2 - (\lambda - 3)x + 1 > 0$

$\Rightarrow (\lambda - 3)^2 - 4 \cdot 1 \cdot 4 < 0$

$\Rightarrow -4 < \lambda + 2 < 4$

$\Rightarrow -1 < \lambda < 7 \quad \dots(i)$

From inequality (ii),

$x^2 + (\lambda + 2)x + 4 > 0$

$\therefore (\lambda + 2)^2 - 4 \cdot 1 \cdot 4 < 0$

$\Rightarrow -4 < \lambda + 2 < 4$

$\Rightarrow -6 < \lambda < 2 \quad \dots(ii)$

From Eqs. (i) and (ii), we get $-1 < \lambda < 2$

$\therefore [\lambda] = -1, 0, 1$

C. $\therefore x = 1$ satisfies

$(b - c)x^2 + (c - a)x + (a - b) = 0$

$\therefore x = 1$ satisfies $x^2 + \lambda x + 1 = 0$

Then, $1 + \lambda + 1 = 0$

$\Rightarrow [\lambda - 1] = [-2 - 1] = -3$

D. $\therefore |x^2 - x - 6| = x + 2$

$\Rightarrow |(x - 3)(x + 2)| = x + 2$

$\Rightarrow |x - 3||x + 2| = x + 2$

$\Rightarrow \begin{cases} (x - 3)(x + 2) = (x + 2), & x < -2 \\ -(x - 3)(x + 2) = (x + 2), & -2 \leq x < 3 \\ (x - 3)(x + 2) = x + 2, & x > 3 \end{cases}$

$\Rightarrow \begin{cases} x = 4, & x < -2 \\ x = -2, 2, & -2 \leq x < 3 \\ x = 4, & x > 3 \end{cases}$

Hence, $x = -2, 2, 4$

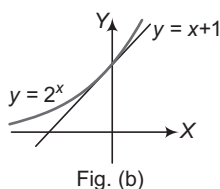
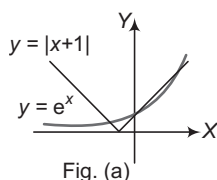
$\therefore N = 3$

$A \rightarrow p, q, r, s; B \rightarrow r, s; C \rightarrow p; D \rightarrow p$

Ex 88. Match the statements of Column I with the values of Column II.

Column I	Column II
A. Number of real solutions of $ x + 1 = e^x$ is	p. 2
B. The number of non-negative real roots of $2^x - x - 1 = 0$ is equal to	q. 3
C. If p and q are the roots of the quadratic equation $x^2 - (\alpha - 2)x - \alpha - 1 = 0$, then minimum value of $p^2 + q^2$ is equal to	r. 6
D. If α and β are the roots of $2x^2 + 7x + c = 0$ and $ \alpha^2 - \beta^2 = \frac{7}{4}$, then c is equal to	s. 5

Sol.



Type 6. Single Integer Answer Type Questions

Ex 89. If the quadratic polynomial,

$$y = (\cot \alpha)x^2 + 2(\sqrt{\sin \alpha})x + \frac{1}{2}\tan \alpha,$$

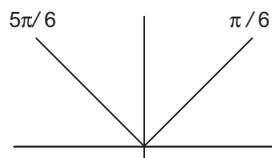
$\alpha \in [0, 2\pi]$, can take negative values for all $x \in \mathbb{R}$ and the value of $\alpha \in \left(\frac{5\pi}{\lambda}, \pi\right)$, then the value of λ is _____.

Sol. (6) $b^2 - 4ac < 0$ and $a < 0$

Hence, $\cot \alpha < 0$ i.e. $\alpha \in 2\text{nd}$ and 4th quadrants and

$$4 \sin \alpha - 2 \tan \alpha \cot \alpha < 0$$

$$2 \sin \alpha < 1 \Rightarrow \sin \alpha < \frac{1}{2}$$



$\therefore \alpha \in 2\text{nd}$ and 4th quadrant

$\sin \alpha \geq 0$ for $\sqrt{\sin \alpha}$ to exist.

$\therefore \alpha \in 2\text{nd}$ quadrant

Hence, $\alpha \in \left(\frac{5\pi}{6}, \pi\right)$

Ex 90. Number of quadratic equations with real roots which remain unchanged even after squaring their roots, is _____.

Sol. (3) $\alpha\beta = \alpha^2 \beta^2$... (i)

and $\alpha^2 + \beta^2 = \alpha + \beta$... (ii)

So, $\alpha\beta(1 - \alpha\beta) = 0 \Rightarrow \alpha = 0$ or $\beta = 0$ or $\alpha\beta = 1$

If $\alpha = 0$, then from Eq. (ii), $\beta = 0$ or $\beta = 1$

$\Rightarrow$ Roots are $(0, 0)$ or $(0, 1)$

If $\beta = 0$, then $\alpha = 0$ or $\beta = 1$

A. Number of solutions is 3. [fig. (a)]

B. $2^x - x - 1 = 0$

Consider $2^x = x + 1$

$\therefore$ There are two solutions.

$x = 0, 1$ [both are non-negative] [fig. (b)]

C. $p + q = \alpha - 2$

$$\Rightarrow pq = -\alpha - 1$$

$$\therefore p^2 + q^2 = (\alpha - 2)^2 - 2(-\alpha - 1)$$

$$= \alpha^2 - 4\alpha + 4 + 2\alpha + 2$$

$$= \alpha^2 - 2\alpha + 6 = (\alpha - 1)^2 + 5$$

$\therefore$ Least value of $p^2 + q^2 = 5$

D. $\alpha + \beta = -\frac{7}{2}, \alpha\beta = \frac{c}{2}$

$$\therefore \frac{7}{4} = |\alpha^2 - \beta^2| = |(\alpha + \beta)(\alpha - \beta)| \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} = \frac{7}{2}$$

$$\sqrt{\frac{49}{4} - 2c} \Rightarrow \frac{7}{4} = \frac{7}{4} \sqrt{49 - 8c}$$

$$\Rightarrow 49 - 8c = 1 \Rightarrow c = 6$$

$A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow r$

If $\beta = \frac{1}{\alpha}$, then $\alpha^2 + \frac{1}{\alpha^2} = \alpha + \frac{1}{\alpha}$

$$\Rightarrow \left[\alpha + \frac{1}{\alpha}\right]^2 - 2 = \alpha + \frac{1}{\alpha}$$

$$\therefore t^2 - t - 2 = 0 \Rightarrow (t - 2)(t + 1) = 0$$

$$\Rightarrow t = 2 \text{ or } t = -1$$

If $t = 2$, then $\alpha = 1$ and $\beta = 1$. If $t = -1$, then roots are imaginary (ω or ω^2).

So, the number of quadratic equation is 3.

Ex 91. The number of non-zero solutions of the equation $x^2 - 5x - (\text{sgn } x)6 = 0$ is _____.

Sol. (1) If $x = 0$, we have $x^2 - 5x = 0 \Rightarrow x = 0$ or 5

$\Rightarrow$ No solution. If $x > 0$, we have $x^2 - 5x - 6 = 0$

$$\Rightarrow (x - 6)(x + 1) = 0 \Rightarrow x = 6 \text{ is the solution.}$$

If $x < 0$, we have $x^2 - 5x + 6 = 0$

$$\Rightarrow (x - 3)(x - 2) = 0 \Rightarrow \text{No solution}$$

$\therefore$ Only one solution.

Ex 92. The number of solutions for the equation $\log_4(2x^2 + x + 1) - \log_2(2x - 1) = 1$ is _____.

Sol. (1) $\log_4(2x^2 + x + 1) - \log_2(2x - 1) = 1$

$$\Rightarrow \frac{\log_e(2x^2 + x + 1)}{\log_e(4)} - \frac{\log_e(2x - 1)}{\log_e 2} = 1$$

$$\Rightarrow \log_e \frac{2x^2 + x + 1}{(2x - 1)^2} = \log_e 4 \Rightarrow (2x^2 + x + 1)$$

$$= 4(4x^2 - 4x + 1) \Rightarrow 14x^2 - 17x + 3 = 0$$

$$\Rightarrow (14x - 3)(x - 1) = 0 \Rightarrow x = \frac{3}{14} \Rightarrow x = 1$$

But $x = \frac{3}{14}$ does not lie in the domain of function

Hence, $x = 1$ is the only solution.

Target Exercises

Type 1. Only One Correct Option

- The least integer satisfying $49.4 - \left(\frac{27-x}{10}\right) < 47.4 - \left(\frac{27-9x}{10}\right)$ is
(a) 2 (b) 3
(c) 4 (d) None of these
- Let $f(x) = ax^2 + bx + c$, $a \neq 0$. Suppose, $f(-1) < 1$, $f(1) > -1$ and $f(3) < -4$. Then,
(a) Cannot be discussed (b) $b + 1 > 0$
(c) $b + 1 < 0$ (d) b is positive real
- If a and b are positive and unequal, then
(a) $a^4 + b^4 > a^3b + ab^3$ (b) $a^4 + b^4 < a^3b + ab^3$
(c) $a^3 + b^3 < a^2b + ab^2$ (d) None of these
- The inequality $\frac{2}{x} < 3$ is true, when x belongs to
(a) $[2/3, \infty)$ (b) $(-\infty, 2/3]$
(c) $(2/3, \infty) \cup (-\infty, 0)$ (d) None of these
- $\frac{x+4}{x-3} < 2$ is satisfied when x satisfies
(a) $(-\infty, 3) \cup (10, \infty)$ (b) $(3, 10)$
(c) $(-\infty, 3) \cup [10, \infty)$ (d) None of these
- Solution of $(x-1)^2(x+4) < 0$ is
(a) $(-\infty, 1)$ (b) $(-\infty, -4)$
(c) $(-1, 4)$ (d) $(1, 4)$
- Solution of $(2x+1)(x-3)(x+7) < 0$ is
(a) $(-\infty, -7) \cup (-1/2, 3)$ (b) $(-\infty, -7) \cup (1/2, 3)$
(c) $(-\infty, 7) \cup (-1/2, 3)$ (d) $(-\infty, -7) \cup (3, \infty)$
- The set of values of x which satisfy the inequations $5x + 2 < 3x + 8$ and $\frac{x+2}{x-1} < 4$ is
(a) $(-\infty, 1)$ (b) $(2, 3)$
(c) $(-\infty, 3)$ (d) $(-\infty, 1) \cup (2, 3)$
- Solution of $|3x+2| < 1$ is
(a) $[-1, -1/3]$ (b) $\{-1/3, -1\}$
(c) $(-1, -1/3)$ (d) None of these
- Solution of $|3-x| = x-3$ is
(a) $x < 3$ (b) $x > 3$ (c) $x \geq 3$ (d) $x \leq 3$
- Solution of $|2x-3| < |x+2|$ is
(a) $(-\infty, 1/3)$ (b) $(1/3, 5)$
(c) $(5, \infty)$ (d) $(-\infty, 1/3) \cup (5, \infty)$
- Solution of $0 < |3x+1| < \frac{1}{3}$ is
(a) $(-4/9, -2/9)$
(b) $[-4/9, -2/9]$
(c) $(-4/9, -2/9) - \{-1/3\}$
(d) $[-4/9, -2/9] - \{-1/3\}$
- Solution of $|x-1| \geq |x-3|$ is
(a) $x \leq 2$ (b) $x \geq 2$
(c) $[1, 3]$ (d) None of these
- Solution of $||x|-1| < |1-x|$, $x \in R$ is
(a) $(-1, 1)$ (b) $(0, \infty)$
(c) $(-1, \infty)$ (d) None of these
- Solution of $\left|\frac{1}{x} - 2\right| < 4$ is
(a) $(-\infty, -1/2)$ (b) $(1/6, \infty)$
(c) $(-1/2, 1/6)$ (d) $(-\infty, -1/2) \cup (1/6, \infty)$
- Solution of $\left|1 + \frac{3}{x}\right| > 2$ is
(a) $(0, 3]$ (b) $[-1, 0)$
(c) $(-1, 0) \cup (0, 3)$ (d) None of these
- Solution of $|x^2 - 10| \leq 6$ is
(a) $(2, 4)$ (b) $(-4, -2)$
(c) $(-4, -2) \cup (2, 4)$ (d) $[-4, -2] \cup [2, 4]$
- Solution of $\left|x + \frac{1}{x}\right| > 2$ is
(a) $R - \{0\}$ (b) $R - \{-1, 0, 1\}$
(c) $R - \{1\}$ (d) $R - \{-1, 1\}$
- The solution set of $\left|\frac{x+1}{x}\right| + |x+1| = \frac{(x+1)^2}{|x|}$ is
(a) $\{x | x \geq 0\}$ (b) $\{x | x > 0\} \cup \{-1\}$
(c) $\{-1, 1\}$ (d) $\{x | x \geq 1 \text{ or } x \leq -1\}$
- The set of real values of x satisfying $|x-1| \leq 3$ and $|x-1| \geq 1$ is
(a) $[2, 4]$ (b) $(-\infty, 2] \cup [4, \infty)$
(c) $[-2, 0] \cup [2, 4]$ (d) None of these
- Solution of the inequality $x > \sqrt{1-x}$ is given by
(a) $]-\infty, (-1-\sqrt{5})/2[$
(b) $](\sqrt{5}-1)/2, \infty[$
(c) $]-\infty, (-1-\sqrt{5})/2[\cup](\sqrt{5}-1)/2, \infty[$
(d) $](\sqrt{5}-1)/2, 1[$

- 22.** The product of all the solutions of the equation $|(x-2)|^2 - 3|x-2| + 2 = 0$ is
 (a) 2 (b) -4
 (c) 0 (d) None of these
- 23.** Which of the following statement(s) is/are correct?
 (a) $a > b \Rightarrow ax > bx, x \neq 0, (a, b, x \in R)$
 (b) $|x| > |y| \Rightarrow x > y (x, y \in R)$
 (c) $a > b \Rightarrow \frac{1}{a} > \frac{1}{b}$
 (d) $a > b \Rightarrow a + c > b + c$
- 24.** Solution of $|x-1| + |x-2| + |x-3| \geq 6$ is
 (a) $[0, 4]$ (b) $(-\infty, -2) \cup [4, \infty)$
 (c) $(-\infty, 0] \cup [4, \infty)$ (d) None of these
- 25.** If $\log_4 5 = a$ and $\log_5 6 = b$, then $\log_3 2$ is equal to
 (a) $\frac{1}{2a+1}$ (b) $\frac{1}{2b+1}$
 (c) $2ab+1$ (d) $\frac{1}{2ab-1}$
- 26.** If $\log_5 a \cdot \log_a x = 2$, then x is equal to
 (a) 125 (b) a^2
 (c) 25 (d) None of these
- 27.** The number of solutions of $\log_2 (x+5) = 6-x$ is
 (a) 2 (b) 0
 (c) 3 (d) None of these
- 28.** The solution set of $\log_2 |4-5x| > 2$ is
 (a) $\left(\frac{8}{5}, +\infty\right)$ (b) $\left(\frac{4}{5}, \frac{8}{5}\right)$
 (c) $(-\infty, 0) \cup \left(\frac{8}{5}, +\infty\right)$ (d) None of these
- 29.** The set of real values of x for which $\log_{0.2} \frac{x+2}{x} \leq 1$ is
 (a) $\left(-\infty, -\frac{5}{2}\right] \cup (0, +\infty)$ (b) $\left[\frac{5}{2}, +\infty\right)$
 (c) $(-\infty, -2) \cup (0, +\infty)$ (d) None of these
- 30.** The number of real values of the parameter k for which $(\log_{16} x)^2 - \log_{16} x + \log_{16} k = 0$, with real coefficients will have exactly one solution, is
 (a) 2 (b) 1
 (c) 4 (d) None of these
- 31.** $x^{\log_x a \times \log_a y \times \log_y z}$ is equal to
 (a) x (b) y
 (c) z (d) None of these
- 32.** The number of zeroes coming immediately after the decimal point in the value of $(5)^{25}$ is (given, $\log_{10} 2 = 0.30103$)
 (a) 16
 (b) 17
 (c) 18
 (d) None of the above
- 33.** If $\log x \log y \log z = (y-z)(z-x)(x-y)$, then
 (a) $x^y \cdot y^z \cdot z^x = 1$
 (b) $x^x \cdot y^y \cdot z^z = 1$
 (c) $\sqrt[3]{x} \cdot \sqrt[3]{y} \cdot \sqrt[3]{z} = 1$
 (d) None of the above
- 34.** The solution set of the inequation $\log_{1/3} (x^2 + x + 1) + 1 > 0$ is
 (a) $(-\infty, -2) \cup (1, +\infty)$ (b) $[-1, 2]$
 (c) $(-2, 1)$ (d) $(-\infty, +\infty)$
- 35.** Let $f(x) = \sqrt{\log_{10} x^2}$. The set of all values of x for which $f(x)$ is real, is
 (a) $[-1, 1]$ (b) $[1, +\infty)$
 (c) $(-\infty, -1]$ (d) $(-\infty, -1] \cup [1, +\infty)$
- 36.** If $x_n > x_{n-1} > \dots > x_2 > x_1 > 1$, then the value of $\log_{x_1} \log_{x_2} \log_{x_3} \dots \log_{x_n} x_n^{x_{n-1}^{x_{n-2}^{x_{n-3}^{x_{n-4}^{x_{n-5}^{x_{n-6}^{x_{n-7}^{x_{n-8}^{x_{n-9}^{x_{n-10}^{x_{n-11}^{x_{n-12}^{x_{n-13}^{x_{n-14}^{x_{n-15}^{x_{n-16}^{x_{n-17}^{x_{n-18}^{x_{n-19}^{x_{n-20}^{x_{n-21}^{x_{n-22}^{x_{n-23}^{x_{n-24}^{x_{n-25}^{x_{n-26}^{x_{n-27}^{x_{n-28}^{x_{n-29}^{x_{n-30}^{x_{n-31}^{x_{n-32}^{x_{n-33}^{x_{n-34}^{x_{n-35}^{x_{n-36}^{x_{n-37}^{x_{n-38}^{x_{n-39}^{x_{n-40}^{x_{n-41}^{x_{n-42}^{x_{n-43}^{x_{n-44}^{x_{n-45}^{x_{n-46}^{x_{n-47}^{x_{n-48}^{x_{n-49}^{x_{n-50}^{x_{n-51}^{x_{n-52}^{x_{n-53}^{x_{n-54}^{x_{n-55}^{x_{n-56}^{x_{n-57}^{x_{n-58}^{x_{n-59}^{x_{n-60}^{x_{n-61}^{x_{n-62}^{x_{n-63}^{x_{n-64}^{x_{n-65}^{x_{n-66}^{x_{n-67}^{x_{n-68}^{x_{n-69}^{x_{n-70}^{x_{n-71}^{x_{n-72}^{x_{n-73}^{x_{n-74}^{x_{n-75}^{x_{n-76}^{x_{n-77}^{x_{n-78}^{x_{n-79}^{x_{n-80}^{x_{n-81}^{x_{n-82}^{x_{n-83}^{x_{n-84}^{x_{n-85}^{x_{n-86}^{x_{n-87}^{x_{n-88}^{x_{n-89}^{x_{n-90}^{x_{n-91}^{x_{n-92}^{x_{n-93}^{x_{n-94}^{x_{n-95}^{x_{n-96}^{x_{n-97}^{x_{n-98}^{x_{n-99}^{x_{n-100}^{x_{n-101}^{x_{n-102}^{x_{n-103}^{x_{n-104}^{x_{n-105}^{x_{n-106}^{x_{n-107}^{x_{n-108}^{x_{n-109}^{x_{n-110}^{x_{n-111}^{x_{n-112}^{x_{n-113}^{x_{n-114}^{x_{n-115}^{x_{n-116}^{x_{n-117}^{x_{n-118}^{x_{n-119}^{x_{n-120}^{x_{n-121}^{x_{n-122}^{x_{n-123}^{x_{n-124}^{x_{n-125}^{x_{n-126}^{x_{n-127}^{x_{n-128}^{x_{n-129}^{x_{n-130}^{x_{n-131}^{x_{n-132}^{x_{n-133}^{x_{n-134}^{x_{n-135}^{x_{n-136}^{x_{n-137}^{x_{n-138}^{x_{n-139}^{x_{n-140}^{x_{n-141}^{x_{n-142}^{x_{n-143}^{x_{n-144}^{x_{n-145}^{x_{n-146}^{x_{n-147}^{x_{n-148}^{x_{n-149}^{x_{n-150}^{x_{n-151}^{x_{n-152}^{x_{n-153}^{x_{n-154}^{x_{n-155}^{x_{n-156}^{x_{n-157}^{x_{n-158}^{x_{n-159}^{x_{n-160}^{x_{n-161}^{x_{n-162}^{x_{n-163}^{x_{n-164}^{x_{n-165}^{x_{n-166}^{x_{n-167}^{x_{n-168}^{x_{n-169}^{x_{n-170}^{x_{n-171}^{x_{n-172}^{x_{n-173}^{x_{n-174}^{x_{n-175}^{x_{n-176}^{x_{n-177}^{x_{n-178}^{x_{n-179}^{x_{n-180}^{x_{n-181}^{x_{n-182}^{x_{n-183}^{x_{n-184}^{x_{n-185}^{x_{n-186}^{x_{n-187}^{x_{n-188}^{x_{n-189}^{x_{n-190}^{x_{n-191}^{x_{n-192}^{x_{n-193}^{x_{n-194}^{x_{n-195}^{x_{n-196}^{x_{n-197}^{x_{n-198}^{x_{n-199}^{x_{n-200}^{x_{n-201}^{x_{n-202}^{x_{n-203}^{x_{n-204}^{x_{n-205}^{x_{n-206}^{x_{n-207}^{x_{n-208}^{x_{n-209}^{x_{n-210}^{x_{n-211}^{x_{n-212}^{x_{n-213}^{x_{n-214}^{x_{n-215}^{x_{n-216}^{x_{n-217}^{x_{n-218}^{x_{n-219}^{x_{n-220}^{x_{n-221}^{x_{n-222}^{x_{n-223}^{x_{n-224}^{x_{n-225}^{x_{n-226}^{x_{n-227}^{x_{n-228}^{x_{n-229}^{x_{n-230}^{x_{n-231}^{x_{n-232}^{x_{n-233}^{x_{n-234}^{x_{n-235}^{x_{n-236}^{x_{n-237}^{x_{n-238}^{x_{n-239}^{x_{n-240}^{x_{n-241}^{x_{n-242}^{x_{n-243}^{x_{n-244}^{x_{n-245}^{x_{n-246}^{x_{n-247}^{x_{n-248}^{x_{n-249}^{x_{n-250}^{x_{n-251}^{x_{n-252}^{x_{n-253}^{x_{n-254}^{x_{n-255}^{x_{n-256}^{x_{n-257}^{x_{n-258}^{x_{n-259}^{x_{n-260}^{x_{n-261}^{x_{n-262}^{x_{n-263}^{x_{n-264}^{x_{n-265}^{x_{n-266}^{x_{n-267}^{x_{n-268}^{x_{n-269}^{x_{n-270}^{x_{n-271}^{x_{n-272}^{x_{n-273}^{x_{n-274}^{x_{n-275}^{x_{n-276}^{x_{n-277}^{x_{n-278}^{x_{n-279}^{x_{n-280}^{x_{n-281}^{x_{n-282}^{x_{n-283}^{x_{n-284}^{x_{n-285}^{x_{n-286}^{x_{n-287}^{x_{n-288}^{x_{n-289}^{x_{n-290}^{x_{n-291}^{x_{n-292}^{x_{n-293}^{x_{n-294}^{x_{n-295}^{x_{n-296}^{x_{n-297}^{x_{n-298}^{x_{n-299}^{x_{n-300}^{x_{n-301}^{x_{n-302}^{x_{n-303}^{x_{n-304}^{x_{n-305}^{x_{n-306}^{x_{n-307}^{x_{n-308}^{x_{n-309}^{x_{n-310}^{x_{n-311}^{x_{n-312}^{x_{n-313}^{x_{n-314}^{x_{n-315}^{x_{n-316}^{x_{n-317}^{x_{n-318}^{x_{n-319}^{x_{n-320}^{x_{n-321}^{x_{n-322}^{x_{n-323}^{x_{n-324}^{x_{n-325}^{x_{n-326}^{x_{n-327}^{x_{n-328}^{x_{n-329}^{x_{n-330}^{x_{n-331}^{x_{n-332}^{x_{n-333}^{x_{n-334}^{x_{n-335}^{x_{n-336}^{x_{n-337}^{x_{n-338}^{x_{n-339}^{x_{n-340}^{x_{n-341}^{x_{n-342}^{x_{n-343}^{x_{n-344}^{x_{n-345}^{x_{n-346}^{x_{n-347}^{x_{n-348}^{x_{n-349}^{x_{n-350}^{x_{n-351}^{x_{n-352}^{x_{n-353}^{x_{n-354}^{x_{n-355}^{x_{n-356}^{x_{n-357}^{x_{n-358}^{x_{n-359}^{x_{n-360}^{x_{n-361}^{x_{n-362}^{x_{n-363}^{x_{n-364}^{x_{n-365}^{x_{n-366}^{x_{n-367}^{x_{n-368}^{x_{n-369}^{x_{n-370}^{x_{n-371}^{x_{n-372}^{x_{n-373}^{x_{n-374}^{x_{n-375}^{x_{n-376}^{x_{n-377}^{x_{n-378}^{x_{n-379}^{x_{n-380}^{x_{n-381}^{x_{n-382}^{x_{n-383}^{x_{n-384}^{x_{n-385}^{x_{n-386}^{x_{n-387}^{x_{n-388}^{x_{n-389}^{x_{n-390}^{x_{n-391}^{x_{n-392}^{x_{n-393}^{x_{n-394}^{x_{n-395}^{x_{n-396}^{x_{n-397}^{x_{n-398}^{x_{n-399}^{x_{n-400}^{x_{n-401}^{x_{n-402}^{x_{n-403}^{x_{n-404}^{x_{n-405}^{x_{n-406}^{x_{n-407}^{x_{n-408}^{x_{n-409}^{x_{n-410}^{x_{n-411}^{x_{n-412}^{x_{n-413}^{x_{n-414}^{x_{n-415}^{x_{n-416}^{x_{n-417}^{x_{n-418}^{x_{n-419}^{x_{n-420}^{x_{n-421}^{x_{n-422}^{x_{n-423}^{x_{n-424}^{x_{n-425}^{x_{n-426}^{x_{n-427}^{x_{n-428}^{x_{n-429}^{x_{n-430}^{x_{n-431}^{x_{n-432}^{x_{n-433}^{x_{n-434}^{x_{n-435}^{x_{n-436}^{x_{n-437}^{x_{n-438}^{x_{n-439}^{x_{n-440}^{x_{n-441}^{x_{n-442}^{x_{n-443}^{x_{n-444}^{x_{n-445}^{x_{n-446}^{x_{n-447}^{x_{n-448}^{x_{n-449}^{x_{n-450}^{x_{n-451}^{x_{n-452}^{x_{n-453}^{x_{n-454}^{x_{n-455}^{x_{n-456}^{x_{n-457}^{x_{n-458}^{x_{n-459}^{x_{n-460}^{x_{n-461}^{x_{n-462}^{x_{n-463}^{x_{n-464}^{x_{n-465}^{x_{n-466}^{x_{n-467}^{x_{n-468}^{x_{n-469}^{x_{n-470}^{x_{n-471}^{x_{n-472}^{x_{n-473}^{x_{n-474}^{x_{n-475}^{x_{n-476}^{x_{n-477}^{x_{n-478}^{x_{n-479}^{x_{n-480}^{x_{n-481}^{x_{n-482}^{x_{n-483}^{x_{n-484}^{x_{n-485}^{x_{n-486}^{x_{n-487}^{x_{n-488}^{x_{n-489}^{x_{n-490}^{x_{n-491}^{x_{n-492}^{x_{n-493}^{x_{n-494}^{x_{n-495}^{x_{n-496}^{x_{n-497}^{x_{n-498}^{x_{n-499}^{x_{n-500}^{x_{n-501}^{x_{n-502}^{x_{n-503}^{x_{n-504}^{x_{n-505}^{x_{n-506}^{x_{n-507}^{x_{n-508}^{x_{n-509}^{x_{n-510}^{x_{n-511}^{x_{n-512}^{x_{n-513}^{x_{n-514}^{x_{n-515}^{x_{n-516}^{x_{n-517}^{x_{n-518}^{x_{n-519}^{x_{n-520}^{x_{n-521}^{x_{n-522}^{x_{n-523}^{x_{n-524}^{x_{n-525}^{x_{n-526}^{x_{n-527}^{x_{n-528}^{x_{n-529}^{x_{n-530}^{x_{n-531}^{x_{n-532}^{x_{n-533}^{x_{n-534}^{x_{n-535}^{x_{n-536}^{x_{n-537}^{x_{n-538}^{x_{n-539}^{x_{n-540}^{x_{n-541}^{x_{n-542}^{x_{n-543}^{x_{n-544}^{x_{n-545}^{x_{n-546}^{x_{n-547}^{x_{n-548}^{x_{n-549}^{x_{n-550}^{x_{n-551}^{x_{n-552}^{x_{n-553}^{x_{n-554}^{x_{n-555}^{x_{n-556}^{x_{n-557}^{x_{n-558}^{x_{n-559}^{x_{n-560}^{x_{n-561}^{x_{n-562}^{x_{n-563}^{x_{n-564}^{x_{n-565}^{x_{n-566}^{x_{n-567}^{x_{n-568}^{x_{n-569}^{x_{n-570}^{x_{n-571}^{x_{n-572}^{x_{n-573}^{x_{n-574}^{x_{n-575}^{x_{n-576}^{x_{n-577}^{x_{n-578}^{x_{n-579}^{x_{n-580}^{x_{n-581}^{x_{n-582}^{x_{n-583}^{x_{n-584}^{x_{n-585}^{x_{n-586}^{x_{n-587}^{x_{n-588}^{x_{n-589}^{x_{n-590}^{x_{n-591}^{x_{n-592}^{x_{n-593}^{x_{n-594}^{x_{n-595}^{x_{n-596}^{x_{n-597}^{x_{n-598}^{x_{n-599}^{x_{n-600}^{x_{n-601}^{x_{n-602}^{x_{n-603}^{x_{n-604}^{x_{n-605}^{x_{n-606}^{x_{n-607}^{x_{n-608}^{x_{n-609}^{x_{n-610}^{x_{n-611}^{x_{n-612}^{x_{n-613}^{x_{n-614}^{x_{n-615}^{x_{n-616}^{x_{n-617}^{x_{n-618}^{x_{n-619}^{x_{n-620}^{x_{n-621}^{x_{n-622}^{x_{n-623}^{x_{n-624}^{x_{n-625}^{x_{n-626}^{x_{n-627}^{x_{n-628}^{x_{n-629}^{x_{n-630}^{x_{n-631}^{x_{n-632}^{x_{n-633}^{x_{n-634}^{x_{n-635}^{x_{n-636}^{x_{n-637}^{x_{n-638}^{x_{n-639}^{x_{n-640}^{x_{n-641}^{x_{n-642}^{x_{n-643}^{x_{n-644}^{x_{n-645}^{x_{n-646}^{x_{n-647}^{x_{n-648}^{x_{n-649}^{x_{n-650}^{x_{n-651}^{x_{n-652}^{x_{n-653}^{x_{n-654}^{x_{n-655}^{x_{n-656}^{x_{n-657}^{x_{n-658}^{x_{n-659}^{x_{n-660}^{x_{n-661}^{x_{n-662}^{x_{n-663}^{x_{n-664}^{x_{n-665}^{x_{n-666}^{x_{n-667}^{x_{n-668}^{x_{n-669}^{x_{n-670}^{x_{n-671}^{x_{n-672}^{x_{n-673}^{x_{n-674}^{x_{n-675}^{x_{n-676}^{x_{n-677}^{x_{n-678}^{x_{n-679}^{x_{n-680}^{x_{n-681}^{x_{n-682}^{x_{n-683}^{x_{n-684}^{x_{n-685}^{x_{n-686}^{x_{n-687}^{x_{n-688}^{x_{n-689}^{x_{n-690}^{x_{n-691}^{x_{n-692}^{x_{n-693}^{x_{n-694}^{x_{n-695}^{x_{n-696}^{x_{n-697}^{x_{n-698}^{x_{n-699}^{x_{n-700}^{x_{n-701}^{x_{n-702}^{x_{n-703}^{x_{n-704}^{x_{n-705}^{x_{n-706}^{x_{n-707}^{x_{n-708}^{x_{n-709}^{x_{n-710}^{x_{n-711}^{x_{n-712}^{x_{n-713}^{x_{n-714}^{x_{n-715}^{x_{n-716}^{x_{n-717}^{x_{n-718}^{x_{n-719}^{x_{n-720}^{x_{n-721}^{x_{n-722}^{x_{n-723}^{x_{n-724}^{x_{n-725}^{x_{n-726}^{x_{n-727}^{x_{n-728}^{x_{n-729}^{x_{n-730}^{x_{n-731}^{x_{n-732}^{x_{n-733}^{x_{n-734}^{x_{n-735}^{x_{n-736}^{x_{n-737}^{x_{n-738}^{x_{n-739}^{x_{n-740}^{x_{n-741}^{x_{n-742}^{x_{n-743}^{x_{n-744}^{x_{n-745}^{x_{n-746}^{x_{n-747}^{x_{n-748}^{x_{n-749}^{x_{n-750}^{x_{n-751}^{x_{n-752}^{x_{n-753}^{x_{n-754}^{x_{n-755}^{x_{n-756}^{x_{n-757}^{x_{n-758}^{x_{n-759}^{x_{n-760}^{x_{n-761}^{x_{n-762}^{x_{n-763}^{x_{n-764}^{x_{n-765}^{x_{n-766}^{x_{n-767}^{x_{n-768}^{x_{n-769}^{x_{n-770}^{x_{n-771}^{x_{n-772}^{x_{n-773}^{x_{n-774}^{x_{n-775}^{x_{n-776}^{x_{n-777}^{x_{n-778}^{x_{n-779}^{x_{n-780}^{x_{n-781}^{x_{n-782}^{x_{n-783}^{x_{n-784}^{x_{n-785}^{x_{n-786}^{x_{n-787}^{x_{n-788}^{x_{n-789}^{x_{n-790}^{x_{n-791}^{x_{n-792}^{x_{n-793}^{x_{n-794}^{x_{n-795}^{x_{n-796}^{x_{n-797}^{x_{n-798}^{x_{n-799}^{x_{n-800}^{x_{n-801}^{x_{n-802}^{x_{n-803}^{x_{n-804}^{x_{n-805}^{x_{n-806}^{x_{n-807}^{x_{n-808}^{x_{n-809}^{x_{n-810}^{x_{n-811}^{x_{n-812}^{x_{n-813}^{x_{n-814}^{x_{n-815}^{x_{n-816}^{x_{n-817}^{x_{n-818}^{x_{n-819}^{x_{n-820}^{x_{n-821}^{x_{n-822}^{x_{n-823}^{x_{n-824}^{x_{n-825}^{x_{n-826}^{x_{n-827}^{x_{n-828}^{x_{n-829}^{x_{n-830}^{x_{n-831}^{x_{n-832}^{x_{n-833}^{x_{n-834}^{x_{n-835}^{x_{n-836}^{x_{n-837}^{x_{n-838}^{x_{n-839}^{x_{n-840}^{x_{n-841}^{x_{n-842}^{x_{n-843}^{x_{n-844}^{x_{n-845}^{x_{n-846}^{x_{n-847}^{x_{n-848}^{x_{n-849}^{x_{n-850}^{x_{n-851}^{x_{n-852}^{x_{n-853}^{x_{n-854}^{x_{n-855}^{x_{n-856}^{x_{n-857}^{x_{n-858}^{x_{n-859}^{x_{n-860}^{x_{n-861}^{x_{n-862}^{x_{n-863}^{x_{n-864}^{x_{n-865}^{x_{n-866}^{x_{n-867}^{x_{n-868}^{x_{n-869}^{x_{n-870}^{x_{n-871}^{x_{n-872}^{x_{n-873}^{x_{n-874}^{x_{n-875}^{x_{n-876}^{x_{n-877}^{x_{n-878}^{x_{n-879}^{x_{n-880}^{x_{n-881}^{x_{n-882}^{x_{n-883}^{x_{n-884}^{x_{n-885}^{x_{n-886}^{x_{n-887}^{x_{n-888}^{x_{n-889}^{x_{n-890}^{x_{n-891}^{x_{n-892}^{x_{n-893}^{x_{n-894}^{x_{n-895}^{x_{n-896}^{x_{n-897}^{x_{n-898}^{x_{n-899}^{x_{n-900}^{x_{n-901}^{x_{n-902}^{x_{n-903}^{x_{n-904}^{x_{n-905}^{x_{n-906}^{x_{n-907}^{x_{n-908}^{x_{n-909}^{x_{n-910}^{x_{n-911}^{x_{n-912}^{x_{n-913}^{x_{n-914}^{x_{n-915}^{x_{n-916}^{x_{n-917}^{x_{n-918}^{x_{n-919}^{x_{n-920}^{x_{n-921}^{x_{n-922}^{x_{n-923}^{x_{n-924}^{x_{n-925}^{x_{n-926}^{x_{n-927}^{x_{n-928}^{x_{n-929}^{x_{n-930}^{x_{n-931}^{x_{n-932}^{x_{n-933}^{x_{n-934}^{x_{n-935}^{x_{n-936}^{x_{n-937}^{x_{n-938}^{x_{n-939}^{x_{n-940}^{x_{n-941}^{x_{n-942}^{x_{n-943}^{x_{n-944}^{x_{n-945}^{x_{n-946}^{x_{n-947}^{x_{n-948}^{x_{n-949}^{x_{n-950}^{x_{n-951}^{x_{n-952}^{x_{n-953}^{x_{n-954}^{x_{n-955}^{x_{n-956}^{x_{n-957}^{x_{n-958}^{x_{n-959}^{x_{n-960}^{x_{n-961}^{x_{n-962}^{x_{n-963}^{x_{n-964}^{x_{n-965}^{x_{n-966}^{x_{n-967}^{x_{n-968}^{x_{n-969}^{x_{n-970}^{x_{n-971}^{x_{n-972}^{x_{n-973}^{x_{n-974}^{x_{n-975}^{x_{n-976}^{x_{n-977}^{x_{n-978}^{x_{n-979}^{x_{n-980}^{x_{n-981}^{x_{n-982}^{x_{n-983}^{x_{n-984}^{x_{n-985}^{x_{n-986}^{x_{n-987}^{x_{n-988}^{x_{n-989}^{x_{n-990}^{x_{n-991}^{x_{n-992}^{x_{n-993}^{x_{n-994}^{x_{n-995}^{x_{n-996}^{x_{n-997}^{x_{n-998}^{x_{n-999}^{x_{n-1000}^{x_{n-1001}^{x_{n-1002}^{x_{n-1003}^{x_{n-1004}^{x_{n-1005}^{x_{n-1006}^{x_{n-1007}^{x_{n-1008}^{x_{n-1009}^{x_{n-1010}^{x_{n-1011}^{x_{n-1012}^{x_{n-1013}^{x_{n-1014}^{x_{n-1015}^{x_{n-1016}^{x_{n-1017}^{x_{n-1018}^{x_{n-1019}^{x_{n-1020}^{x_{n-1021}^{x_{n-1022}^{x_{n-1023}^{x_{n-1024}^{x_{n-1025}^{x_{n-1026}^{x_{n-1027}^{x_{n-1028}^{x_{n-1029}^{x_{n-1030}^{x_{n-1031}^{x_{n-1032}^{x_{n-1033}^{x_{n-1034}^{x_{n-1035}^{x_{n-1036}^{x_{n-1037}^{x_{n-1038}^{x_{n-1039}^{x_{n-1040}^{x_{n-1041}^{x_{n-1042}^{x_{n-1043}^{x_{n-1044}^{x_{n-1045}^{x_{n-1046}^{x_{n-1047}^{x_{n-1048}^{x_{n-1049}^{x_{n-1050}^{x_{n-1051}^{x_{n-1052}^{x_{n-1053}^{x_{n-1054}^{x_{n-1055}^{x_{n-1056}^{x_{n-1057}^{x_{n-1058}^{x_{n-1059}^{x_{n-1060}^{x_{n-1061}^{x_{n-1062}^{x_{n-1063}^{x_{n-1064}^{x_{n-1065}^{x_{n-1066}^{x_{n-1067}^{x_{n-1068}^{x_{n-1069}^{x_{n-1070}^{x_{n-1071}^{x_{n-1072}^{x_{n-1073}^{x_{n-1074}^{x_{n-1075}^{x_{n-1076}^{x_{n-1077}^{x_{n-1078}^{x_{n-1079}^{x_{n-1080}^{x_{n-1081}^{x_{n-1082}^{x_{n-1083}^{x_{n-1084}^{x_{n-1085}^{x_{n-1086}^{x_{n-1087}^{x_{n-1088}^{x_{n-1089}^{x_{n-1090}^{x_{n-1091}^{x_{n-1092}^{x_{n-1093}^{x_{n-1094}^{x_{n-1095}^{x_{n-1096}^{x_{n-1097}^{x_{n-1098}^{x_{n-1099}^{x_{n-1100}^{x_{n-1101}^{x_{n-1102}^{x_{n-1103}^{x_{n-1104}^{x_{n-1105}^{x_{n-1106}^{x_{n-1107}^{x_{n-1108}^{x_{n-1109}^{x_{n-1110}^{x_{n-1111}^{x_{n-1112}^{x_{n-1113}^{x_{n-1114}^{x_{n-1115}^{x_{n-1116}^{x_{n-1117}^{x_{n-1118}^{x_{n-1119}^{x_{n-1120}^{x_{n-1121}^{x_{n-1122}^{x_{n-1123}^{x_{n-1124}^{x_{n-1125}^{x_{n-1126}^{x_{n-1127}^{x_{n-1128}^{x_{n-1129}^{x_{n-1130}^{x_{n-1131}^{x_{n-1132}^{x_{n-1133}^{x_{n-1134}^{x_{n-1135}^{x_{n-1136}^{x_{n-1137}^{x_{n-1138}^{x_{n-1139}^{x_{n-1140}^{x_{n-1141}^{x_{n-1142}^{x_{n-1143}^{x_{n-1144}^{x_{n-1145}^{x_{n-1146}^{x_{n-1147}^{x_{n-1148}^{x_{n-1149}^{x_{n-1150}^{x_{n-1151}^{x_{n-1152}^{x_{n-1153}^{x_{n-1154}^{x_{n-1155}^{x_{n-1156}^{x_{n-1157}^{x_{n-1158}^{x_{n-1159}$

- 43.** For positive real numbers a, b and c , which one of the following holds?
 (a) $a^2 + b^2 + c^2 \geq bc + ca + ab$
 (b) $(b + c)(c + a)(a + b) \leq 8abc$
 (c) $\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq 3$
 (d) $a^3 + b^3 + c^3 \geq abc$
- 44.** For positive real numbers a, b and c , which of the following holds?
 (a) $a + b + c > 3 \Rightarrow a^2 + b^2 + c^2 > 3$
 (b) $a^6 + b^6 \leq 12a^2b^2 - 64$
 (c) $a + b + c = \alpha \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{9}{\alpha}$
 (d) None of the above
- 45.** For positive real numbers a, b and c , such that $a + b + c = p$, which one holds?
 (a) $(p - a)(p - b)(p - c) \geq \frac{8}{27}p^3$
 (b) $(p - a)(p - b)(p - c) \geq 8abc$
 (c) $\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq p$
 (d) None of the above
- 46.** The minimum value of $P = bcx + cay + abz$, when $xyz = abc$, is
 (a) $3abc$ (b) $6abc$
 (c) abc (d) $4abc$
- 47.** The greatest value of $P = a^2b^3c^4$, when $a + b + c = 18$, is
 (a) $P = 4^4 \cdot 6^4 \cdot 8^4$ (b) $P = 4^2 \cdot 6^3 \cdot 8^4$
 (c) $P = 3^2 \cdot 6^3 \cdot 8^4$ (d) $P = 4^2 \cdot 6^2 \cdot 8^2$
- 48.** If $x_1, x_2, \dots, x_n$ are any real numbers and n is any positive integer, then
 (a) $n \sum_{i=1}^n x_i^2 < \left(\sum_{i=1}^n x_i \right)^2$ (b) $n \sum_{i=1}^n x_i^2 \geq \left(\sum_{i=1}^n x_i \right)^2$
 (c) $\sum_{i=1}^n x_i^2 \geq n \left(\sum_{i=1}^n x_i \right)^2$ (d) None of these
- 49.** If x, y and z are positive real numbers such that $x + y + z = 2$, then
 (a) $(2 - x)(2 - y)(2 - z) \geq 8xyz$
 (b) $x^{-1} + y^{-1} + z^{-1} \geq 1/2$
 (c) $(2 - x)(2 - y)(2 - z) < 8xyz$
 (d) None of the above
- 50.** For non-negative real numbers such that $a_1 + a_2 + \dots + a_n = p$ and $q = \sum_{i < j} a_i a_j$, then
 (a) $q \leq \frac{1}{2}p^2$ (b) $q > \frac{1}{4}p^2$
 (c) $q < \frac{p}{2}$ (d) $q > \frac{p^2}{2}$
- 51.** The polynomial $(ax^2 + bx + c)(ax^2 - dx - c)$, $ac \neq 0$, has
 (a) four real zeroes
 (b) atleast two real zeroes
 (c) atmost two real zeroes
 (d) no real zeroes
- 52.** If a, b and c are non-zero, unequal rational numbers, then the roots of the equation $abc^2x^2 + (3a^2 + b^2)cx - 6a^2 - ab + 2b^2 = 0$ are
 (a) rational (b) imaginary
 (c) irrational (d) None of these
- 53.** The equation $(a + 2)x^2 + (a - 3)x = 2a - 1$, $a \neq -2$ has roots rational for
 (a) all rational values of a except $a = -2$
 (b) all real values of a except $a = -2$
 (c) rational values of $a > \frac{1}{2}$
 (d) None of the above
- 54.** If the absolute value of the difference of roots of the equation $x^2 + px + 1 = 0$ exceeds $\sqrt{3p}$, then
 (a) $p < -1$ or $p > 4$ (b) $p > 4$
 (c) $-1 < p < 4$ (d) $0 \leq p < 4$
- 55.** If $\frac{1}{4 - 3i}$ is a root of $ax^2 + bx + 1 = 0$, where a and b are real, then
 (a) $a = 25, b = -8$ (b) $a = 25, b = 8$
 (c) $a = 5, b = 4$ (d) None of these
- 56.** If $a \in R, b \in R$, then the factors of the expression $a(x^2 - y^2) - bxy$ are
 (a) real and different (b) real and identical
 (c) complex (d) None of these
- 57.** The number of positive integral values of k for which $(16x^2 + 12x + 39) + k(9x^2 - 2x + 11)$ is a perfect square, is
 (a) two (b) zero
 (c) one (d) None of these
- 58.** For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is the square of the other, then p is equal to
 (a) $\frac{1}{3}$ (b) 1
 (c) 3 (d) $\frac{2}{3}$
- 59.** Let α and β be the roots of the quadratic equation $x^2 + px + p^3 = 0$ ($p \neq 0$). If (α, β) is a point on the parabola $y^2 = x$, then the roots of the quadratic equation are
 (a) 4, -2
 (b) -4, -2
 (c) 4, 2
 (d) -4, 2

- 60.** If a, b and c are positive real numbers, then the number of real roots of the equation $ax^2 + b|x| + c = 0$ is
(a) 2 (b) 4
(c) 0 (d) None of these
- 61.** If α and β are the roots of $ax^2 + bx + c = 0$ and $\alpha + k, \beta + k$ are the roots of $px^2 + qx + r = 0$, then $\frac{b^2 - 4ac}{q^2 - 4pr}$ is equal to
(a) $\left(\frac{p}{a}\right)^2$ (b) 1 (c) $\left(\frac{a}{p}\right)^2$ (d) 0
- 62.** If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $m : n$, then
(a) $mnb^2 = ac(m + n)^2$ (b) $b^2(m + n) = mn$
(c) $m + n = b^2mn$ (d) $c^2(mn) = ab(m + n)^2$
- 63.** If $x + \frac{1}{x} = 5$, then $\left(x^3 + \frac{1}{x^3}\right) - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)$ is equal to
(a) 0 (b) 5
(c) -5 (d) 10
- 64.** The value of $\sqrt{6 + \sqrt{6 + \sqrt{6 + \dots \infty}}}$ is
(a) 3 (b) 6
(c) -2 (d) -4
- 65.** The sum of the roots of the equation $x^2 + px + q = 0$ is equal to the sum of their squares, then
(a) $p^2 - q^2 = 0$ (b) $p^2 + q^2 = 2q$
(c) $p^2 + p = 2q$ (d) None of these
- 66.** The roots of the equation $(q - r)x^2 + (r - p)x + (p - q) = 0$ are
(a) $\frac{r - p}{q - r}, 1$ (b) $\frac{p - q}{q - r}, 1$
(c) $\frac{q - r}{p - q}, 1$ (d) $\frac{r - p}{p - q}, 1$
- 67.** A lad was asked his age by his friend. The lad said, 'The number you get when you subtract 25 times my age from twice the square of my age will be thrice your age.' If the friend's age is 14, then age of the lad is
(a) 21 (b) 28
(c) 14 (d) 25
- 68.** If the roots of the equation $\frac{x^2 - bx}{ax - c} = \frac{\lambda - 1}{\lambda + 1}$ are such that $\alpha + \beta = 0$, then the value of λ is
(a) $\frac{a - b}{a + b}$ (b) c
(c) $\frac{1}{c}$ (d) $\frac{a + b}{a - b}$
- 69.** If the roots of the equation $(a^2 + b^2)y^2 - 2(ac + bd)y + c^2 + d^2 = 0$ are equal, then
(a) $ab = dc$ (b) $ac = bd$
(c) $ad + bc = 0$ (d) $\frac{a}{b} = \frac{c}{d}$
- 70.** If p, q and r are positive and are in AP, then the roots of the quadratic equation $px^2 + qx + r = 0$ are real for
(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (b) $\left|\frac{p}{r} - 7\right| < 4\sqrt{3}$
(c) all p and r (d) no p and r
- 71.** The roots of the equation $\frac{2x + 1}{9} + \frac{x^2 + 7}{x^2 - 7} = \frac{2x + 17}{9}$ are
(a) 3, -3 (b) 5, -5
(c) $\sqrt{3}, -\sqrt{3}$ (d) $\sqrt{5}, -\sqrt{5}$
- 72.** If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then a, b and c are in
(a) AP (b) GP
(c) HP (d) None of these
- 73.** If α and β are the roots of the equation $x^2 + x\sqrt{\alpha} + \beta = 0$, then the values of α and β are
(a) $\alpha = 1$ and $\beta = -1$ (b) $\alpha = 1$ and $\beta = -2$
(c) $\alpha = 2$ and $\beta = 1$ (d) $\alpha = 2$ and $\beta = -2$
- 74.** If α and β are the roots of the equation $8x^2 - 3x + 27 = 0$, then the value of $\left(\frac{\alpha^2}{\beta}\right)^{1/3} + \left(\frac{\beta^2}{\alpha}\right)^{1/3}$ is
(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{7}{2}$ (d) 4
- 75.** If the expression $a^2(b^2 - c^2)x^2 + b^2(c^2 - a^2)x + c^2(a^2 - b^2)$ is a perfect square, then
(a) a, b and c are in AP (b) a^2, b^2 and c^2 are in AP
(c) a^2, b^2 and c^2 are in HP (d) a^2, b^2 and c^2 are in GP
- 76.** The values of m for which the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots, are
(a) 0, 3 (b) 1 (c) 2 (d) 3
- 77.** If α is a root of $4x^2 + 2x - 1 = 0$, then the other root are
(a) $3\alpha^3 - 4\alpha$ (b) $4\alpha^3 - 3\alpha$
(c) $3\alpha^3 + 4\alpha$ (d) $4\alpha^3 + 3\alpha$
- 78.** If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ are equal, then a, b and c are in
(a) AP (b) GP
(c) HP (d) None of these

79. If the ratio of the roots of $lx^2 + nx + n = 0$ is $p : q$, then
- (a) $\sqrt{\frac{q}{p}} + \sqrt{\frac{p}{q}} + \sqrt{\frac{l}{n}} = 0$
 (b) $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 0$
 (c) $\sqrt{\frac{q}{p}} + \sqrt{\frac{p}{q}} + \sqrt{\frac{l}{n}} = 1$
 (d) $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} = 1$
80. The roots of the equation $(x - a)(x - b) = abx^2$ are always
- (a) real (b) imaginary
 (c) Cannot be discussed (d) None of these
81. The equation $\sqrt{x + 3 - 4\sqrt{x - 1}} + \sqrt{x + 8 - 6\sqrt{x - 1}} = 1$ has
- (a) no solution (b) one solution
 (c) two solutions (d) more than two solutions
82. If the sum of the roots of the equation $ax^2 + bx + c = 0$ is equal to the sum of the reciprocals of their squares, then bc^2 , ca^2 and ab^2 are in
- (a) AP (b) GP
 (c) HP (d) None of these
83. If $\sin \theta$ and $\cos \theta$ are the roots of the equation $ax^2 + bx + c$, then
- (a) $(a - c)^2 = b^2 - c^2$ (b) $(a - c)^2 = b^2 + c^2$
 (c) $(a + c)^2 = b^2 - c^2$ (d) $(a + c)^2 = b^2 + c^2$
84. A car travels 25 km an hour faster than a bus for a journey of 500 km. If the bus takes 10 h more than the car, then the speeds of the bus and the car are
- (a) 25 km/h, 40 km/h (b) 25 km/h, 50 km/h
 (c) 25 km/h, 60 km/h (d) None of these
85. If $(ax^2 + bx + c)y + a'x^2 + b'x + c' = 0$, then the condition that x may be rational function of y , is
- (a) $(ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$
 (b) $(ab' - a'b)^2 = (ac' - a'c)(bc' - b'c)$
 (c) $(bc' - b'c)^2 = (ab' - a'b)(ac' - a'c)$
 (d) None of the above
86. If $a(b - c)x^2 + b(c - a)xy + c(a - b)y^2$ is a perfect square, then a , b and c are in
- (a) AP (b) GP
 (c) HP (d) None of the above
87. If α and β are the roots of $ax^2 + c = bx$, then the equation $(a + cy)^2 = b^2y$ in y has the roots
- (a) α^{-1}, β^{-1} (b) α^2, β^2
 (c) $\alpha\beta^{-1}, \alpha^{-1}\beta$ (d) α^{-2}, β^{-2}
88. If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}, \frac{c}{b}$ are in
- (a) AP (b) GP
 (c) HP (d) None of these
89. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + (8 + 2\sqrt{5}) = 0$ is
- (a) 2 (b) 4
 (c) 7 (d) 8
90. If $x^2 - 3x + 2$ is a factor of $x^4 - px^2 + q$, then p and q are
- (a) 2, 3 (b) 4, -5
 (c) 5, 4 (d) 0, 0
91. If $9^x - 4 \times 3^{x+2} + 3^5 = 0$, then the solution set is
- (a) {1, 2} (b) {2, 3} (c) {2, 4} (d) {1, 3}
92. If the roots of the equation $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$ are equal in magnitude and opposite in sign, then
- (a) $p + q = r$
 (b) $p + q = 2r$
 (c) product of roots = $\frac{1}{2}(p^2 + q^2)$
 (d) None of the above
93. The solution set of $\left| \frac{x+1}{x} \right| + |x+1| = \frac{(x+1)^2}{|x|}$ is
- (a) $\{x \mid x \geq 0\}$ (b) $\{x \mid x > 0\} \cup \{-1\}$
 (c) $\{-1, 1\}$ (d) $\{x \mid x \geq 1 \text{ or } x \leq -1\}$
94. $\{x \in R : |x - 2| = x^2\}$ is equal to
- (a) $[-1, 2]$ (b) $[1, 2]$
 (c) $[-1, -2]$ (d) $\{-2, 1\}$
95. Number of solutions of an equation $|x|^2 - 3|x| + 2 = 0$ will be
- (a) 4 (b) 1 (c) 3 (d) 2
96. In the equation $4^{x+2} = 2^{x+3} + 48$, the value of x will be
- (a) $-\frac{3}{2}$ (b) -2
 (c) -3 (d) 1
97. The real values of x which satisfy the equation $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ are
- (a) ± 2 (b) $\pm \sqrt{2}$
 (c) $\pm 2, \pm \sqrt{2}$ (d) $2, \sqrt{2}$
98. The equation $125^x + 45^x = 2 \cdot (27)^x$ has
- (a) no solution
 (b) one solution
 (c) two solutions
 (d) more than two solutions

- 99.** If α and β are the roots of the equation $x^2 - px + q = 0$ and $\alpha > 0, \beta > 0$, then the value of $\alpha^{1/4} + \beta^{1/4}$ is $(p + 6\sqrt{q} + 4q^{1/4}\sqrt{p + 2\sqrt{q}})^k$, where k is equal to
(a) 1 (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$
- 100.** The solutions of the equation $(3|x| - 3)^2 = |x| + 7$ which belongs to the domain of definition of the function $y = \sqrt{x(x-3)}$, are given by
(a) $\frac{1}{9}, -2$ (b) $-\frac{1}{9}, 2$ (c) $\pm \frac{1}{9}, \pm 2$ (d) $-\frac{1}{9}, -2$
- 101.** The roots of the equation $2^{x+2} \cdot 3^{\frac{3x}{x-1}} = 9$ are given by
(a) $\log_2\left(\frac{2}{3}\right), -2$ (b) $3, -3$
(c) $-2, 1 - \log_2 3$ (d) $1 - \log_2 3, 2$
- 102.** If $f(x) = x - [x], x (\neq 0) \in R$, where $[x]$ is greatest integer less than or equal to x , then the number of solutions of $f(x) + f\left(\frac{1}{x}\right) = 1$ is
(a) 0 (b) 1 (c) 2 (d) infinite
- 103.** The number of real solutions of the equation $27^{1/x} + 12^{1/x} = 2 \cdot 8^{1/x}$ is
(a) 1 (b) 2 (c) infinite (d) 0
- 104.** If α and β ($\alpha < \beta$) are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then
(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$
- 105.** If $x^2 - 2x + \sin^2 \alpha = 0$, then
(a) $x \in [-1, 1]$ (b) $x \in [0, 2]$
(c) $x \in [-2, 2]$ (d) $x \in [1, 2]$
- 106.** The set of values of m for which the roots of the equation $x^2 - (m+1)x + m + 4 = 0$ are real and negative consists of all m such that
(a) $-3 < m \leq -1$ (b) $-4 < m \leq -3$
(c) $-3 \leq m \leq 5$ (d) $-3 \geq m$ or $m \geq 5$
- 107.** If $a \leq 0$, then the roots of the equation $x^2 - 2a|x-a| - 3a^2 = 0$ are
(a) $(1 - \sqrt{2})a, (\sqrt{6} - 1)a$
(b) $(-1 + \sqrt{6})a, (1 + \sqrt{2})a$
(c) $(1 + \sqrt{2})a, 0$
(d) $-(1 \pm \sqrt{6})a, (1 \pm \sqrt{2})a$
- 108.** For $a \geq 0$, the roots of the equation $x^2 - 2a|x-a| - 3a^2 = 0$ are given by
(a) $-a(\sqrt{6} + 1)$ (b) $a(\sqrt{6} + 1)$
(c) $a(1 + \sqrt{2}), a(-1 - \sqrt{6})$ (d) $a(1 + \sqrt{2})$
- 109.** The value of x for which $2^{x/2} + (\sqrt{2} + 1)^x = (5 + 2\sqrt{2})^{x/2}$, is
(a) 1 (b) 0
(c) 2 (d) None of these
- 110.** If $[x]^2 = [x + 2]$, where $[x]$ is the greatest integer less than or equal to x , then x must be such that
(a) $x = 2, -1$ (b) $[-1, 0) \cup [2, 3)$
(c) $x \in [-1, 0)$ (d) None of these
- 111.** Number of real roots of the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ is
(a) 1 (b) 0
(c) 2 (d) None of these
- 112.** The roots of the equation $x^{\sqrt{x}} = \sqrt{x^x}$ are
(a) 0 and 4 (b) 0 and 1
(c) 0, 1 and 4 (d) 1 and 4
- 113.** The quadratic equation $8\sec^2 \theta - 6\sec \theta + 1 = 0$ has
(a) infinitely many roots (b) exactly two roots
(c) exactly four roots (d) no roots
- 114.** If $\log_2 x + \log_x 2 = \frac{10}{3} = \log_2 y + \log_y 2$ and $x \neq y$, then $x + y^3$ is equal to
(a) 10 (b) $\frac{65}{8}$
(c) $\frac{37}{6}$ (d) None of these
- 115.** The number of solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is
(a) 0 (b) 1
(c) 2 (d) infinitely many
- 116.** Let $D = a^2 + b^2 + c^2$, where a, b being consecutive integers and $c = ab$, then $\sqrt{D}$ is
(a) always an even integer
(b) always an odd integer
(c) always irrational
(d) Can't be discussed
- 117.** The constant term of the quadratic expression $\sum_{k=2}^n \left(x - \frac{1}{k-1}\right) \left(x - \frac{1}{k}\right)$, as $n \rightarrow \infty$ is
(a) -1 (b) 0
(c) 1 (d) None of the above
- 118.** The system $y^{(x^2+7x+12)} = 1$ and $x + y = 6, y > 0$ has
(a) no solution (b) one solution
(c) two solutions (d) more than two solutions
- 119.** The curve $y = (\lambda + 1)x^2 + 2$ intersects the curve $y = \lambda x + 3$ in exactly one point, if λ equals
(a) $\{-2, 2\}$ (b) $\{1\}$ (c) $\{-2\}$ (d) $\{2\}$

- 120.** If the expression $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$ is a perfect square, then
 (a) $a = b = c$ (b) $a = \pm b = \pm c$
 (c) $a = b \neq c$ (d) None of these
- 121.** Sum the non-real roots of $(x^2 + x - 2)(x^2 + x - 3) = 12$ is
 (a) -1 (b) 1 (c) -6 (d) 6
- 122.** The number of irrational roots of the equation $4x/(x^2 + x + 3) + 5x/(x^2 - 5x + 3) = -3/2$ is
 (a) 4 (b) 0 (c) 1 (d) 2
- 123.** If α and β are the roots of the equation $x^2 - 2x + 3 = 0$. Then, the equation whose roots are $P = \alpha^3 - 3\alpha^2 + 5\alpha - 2$ and $Q = \beta^3 - \beta^2 + \beta + 5$, is
 (a) $x^2 + 3x + 2 = 0$ (b) $x^2 - 3x - 2 = 0$
 (c) $x^2 - 3x + 2 = 0$ (d) None of these
- 124.** If α and β are the roots of the equation $2x^2 - 35x + 2 = 0$, then the value of $(2\alpha - 35)^3 (2\beta - 35)^3$ is
 (a) 8 (b) 1
 (c) 64 (d) None of these
- 125.** The sum of values of x satisfying the equation $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$ is
 (a) 3 (b) 0
 (c) 2 (d) None of these
- 126.** If α and β are the roots of the equation $ax^2 + bx + c = 0$, then the value of $(a\alpha^2 + c)/(a\alpha + b) + (a\beta^2 + c)/(a\beta + b)$ is
 (a) $\frac{b(b^2 - 2ac)}{4a}$ (b) $\frac{b^2 - 4ac}{2a}$
 (c) $\frac{b(b^2 - 2ac)}{a^2c}$ (d) None of these
- 127.** If α and β are the roots of the equation $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + q = 0$, then the roots of $x^2 - 4qx + 2q^2 - r = 0$ are always
 (a) both non-real (b) both positive
 (c) both negative (d) opposite in sign
- 128.** Suppose A, B and C are defined as $A = a^2b + ab^2 - a^2c - ac^2$, $B = b^2c + bc^2 - a^2b - ab^2$ and $C = a^2c + ac^2 - b^2c - bc^2$, where $a > b > c > 0$ and the equation $Ax^2 + Bx + C = 0$ has equal roots, then a, b and c are in
 (a) AP (b) GP (c) HP (d) AGP
- 129.** If α and β are the roots of $ax^2 + c = bx$, then the equation $(a + cy)^2 = b^2y$ in y has the roots
 (a) $\alpha\beta^{-1}, \alpha^{-1}\beta$ (b) α^{-2}, β^{-2}
 (c) α^{-1}, β^{-1} (d) α^2, β^2
- 130.** If α and β are the roots of the equation $x^2 + px - 1/(2p^2) = 0$, where $p \in R$. Then, the minimum value of $\alpha^4 + \beta^4$ is
 (a) $2\sqrt{2}$ (b) $2 - \sqrt{2}$
 (c) 2 (d) $2 + \sqrt{2}$
- 131.** If x_1 and x_2 are the roots of $ax^2 + bx + c = 0$ and $x_1x_2 < 0$, then the roots of $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$ are
 (a) real and of opposite sign (b) negative
 (c) positive (d) non-real
- 132.** The number of integral values of a for which the quadratic equation $(x + a)(x + 1991) + 1 = 0$ has integral roots, is
 (a) 3 (b) 0
 (c) 1 (d) 2
- 133.** The number of real solutions of the equation $(9/10)^x = -3 + x - x^2$ is
 (a) 2 (b) 0
 (c) 1 (d) None of these
- 134.** If x, y, z and t are real numbers $x^2 + y^2 = 9, z^2 + t^2 = 4$ and $xt - yz = 6$. Then, the greatest value of $P = xz$ is
 (a) 2 (b) 3 (c) 4 (d) 6
- 135.** If $x^2 + ax - 3x - (a + 2) = 0$ has real and distinct roots, then minimum value of $(a^2 + 1)/(a^2 + 2)$ is
 (a) 1 (b) 0
 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
- 136.** The value of the expression $x^4 - 8x^3 + 18x^2 - 8x + 2$, when $x = 2 + \sqrt{3}$, is
 (a) 2 (b) 1 (c) 0 (d) 3
- 137.** $x^{3^n} + y^{3^n}$ is divisible by $x + y$, if
 (a) n is any integer ≥ 0
 (b) n is an odd positive integer
 (c) n is an even positive integer
 (d) n is a rational number
- 138.** If $z_0 = \alpha + i\beta, i = \sqrt{-1}$, then the roots of the cubic equation $x^3 - 2(1 + \alpha)x^2 + (4\alpha + \alpha^2 + \beta^2)x - 2(\alpha^2 + \beta^2) = 0$ are
 (a) $2, z_0, \bar{z}_0$ (b) $1, z_0, -z_0$
 (c) $2, z_0, -\bar{z}_0$ (d) $2, -z_0, \bar{z}_0$
- 139.** If α, β and γ are the roots of $x^3 + 64 = 0$, then the equation whose roots are $\left(\frac{\alpha}{\beta}\right)^2$ and $\left(\frac{\alpha}{\gamma}\right)^2$, is
 (a) $x^2 - 4x + 16 = 0$ (b) $x^2 + x + 1 = 0$
 (c) $x^2 + 4x + 16 = 0$ (d) $x^2 - x + 1 = 0$

- 140.** If $(x-1)^3$ is a factor of $x^4 + ax^3 + bx^2 + cx - 1$, then the other factor is
 (a) $x-3$ (b) $x+1$
 (c) $x+2$ (d) None of these
- 141.** Let $\alpha + i\beta, \alpha, \beta \in R$ be a root of the equation $x^3 + qx + r = 0$, $q, r \in R$. The cubic equation is independent of α and β whose one root is 2α , is
 (a) $x^3 + qx - r = 0$ (b) $x^3 + qx + 2r = 0$
 (c) $x^3 + qx + r = 0$ (d) $x^3 - qx + r = 0$
- 142.** If α, β are the roots of $x^2 + px + q = 0$ and $x^{2n} + p^n x^n + q^n = 0$ and $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ are the roots of $x^n + 1 + (x+1)^n = 0$, then n must be
 (a) even integer (b) odd integer
 (c) rational but non-integer (d) None of these
- 143.** The quadratic equation whose roots are cubes of the roots of $x^2 + bx + c = 0$, is
 (a) $x^2 + b(b^2 - 3c)x - c^3 = 0$
 (b) $x^2 + b(b^2 - 3c)x + c^3 = 0$
 (c) $x^2 - b(b^2 - 3c)x + c^3 = 0$
 (d) None of the above
- 144.** If α and β are the roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are $\frac{1}{a\alpha + b}$ and $\frac{1}{a\beta + b}$, is
 (a) $cax^2 - bx + 1 = 0$ (b) $cax^2 + bx + 1 = 0$
 (c) $cax^2 + bx - 1 = 0$ (d) None of these
- 145.** If $x = 2 + 2^{1/3} + 2^{2/3}$, then the value of $x^3 - 6x^2 + 6x$ is
 (a) 3 (b) 2
 (c) 1 (d) None of these
- 146.** If roots of the equation $x^n - 1 = 0$ are $1, a_1, a_2, a_3, \dots, a_{n-1}$, then the value of $(1 - a_1)(1 - a_2)(1 - a_3) \dots (1 - a_{n-1})$ will be
 (a) n (b) n^2 (c) n^n (d) 0
- 147.** If α and β are the roots of the quadratic $x^2 - 2\cos\theta x + 1 = 0$, then equation whose roots are α^n, β^n , is
 (a) $x^2 - (2\cos n\theta)x + 1 = 0$
 (b) $2x^2 - (2\cos n\theta)x - 1 = 0$
 (c) $x^2 + (2\cos n\theta)x + 1 = 0$
 (d) $x^2 + (2\cos n\theta)x - 1 = 0$
- 148.** If $1, \omega, \omega^2, \dots, \omega^{n-1}$ are the n th roots of unity, then $(1 + \omega)(1 + \omega^2) \dots (1 + \omega^{n-1})$ equals
 (a) 0, n is even (b) 1, n is even
 (c) n , n is even (d) n^2 , n is even
- 149.** If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, then $f(-5 + 2\sqrt{-4})$ is equal to
 (a) -160 (b) 160
 (c) 0 (d) None of these
- 150.** If $p \in [-1, 1]$, then the value of x for which $4x^3 - 3x - p = 0$ has a root lies in
 (a) $[0, 1]$ (b) $[\frac{1}{2}, 1]$ (c) $[0, \frac{1}{2}]$ (d) $[-1, 1]$
- 151.** If α, β, γ and σ are the roots of the equation $x^4 + 4x^3 - 6x^2 + 7x - 9 = 0$, then the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$ is
 (a) 6 (b) 11
 (c) 13 (d) 5
- 152.** If $\tan \theta_1, \tan \theta_2$ and $\tan \theta_3$ are the real roots of the $x^3 - (a+1)x^2 + (b-a)x - b = 0$, where $\theta_1 + \theta_2 + \theta_3 \in (0, \pi)$, then $\theta_1 + \theta_2 + \theta_3$ is equal to
 (a) $\pi/2$ (b) $\pi/4$ (c) $3\pi/4$ (d) π
- 153.** If α, β and γ are the roots of $x^3 - x^2 - 1 = 0$, then $(1 + \alpha)/(1 - \alpha) + (1 + \beta)/(1 - \beta) + (1 + \gamma)/(1 - \gamma)$ is equal to
 (a) -5 (b) -6
 (c) -7 (d) -2
- 154.** The equation formed by decreasing each root of $ax^2 + bx + c = 0$ by 1 is $2x^2 + 8x + 2 = 0$, then
 (a) $a = -b$ (b) $b = -c$
 (c) $c = -a$ (d) $b = a + c$
- 155.** If the equations $ax^2 + bx + c = 0$ and $x^2 + x + 1 = 0$ have a common root, then
 (a) $a + b + c = 0$ (b) $a = b = c$
 (c) $a = b$ or $b = c$ or $c = a$ (d) $a - b + c = 0$
- 156.** If $a, b, c \in R$ and the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two roots in common, then
 (a) $a = b \neq c$ (b) $a = b = -c$
 (c) $a = b = c$ (d) None of these
- 157.** If the equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$, $a \neq c$ have a negative common root, then the value of $a - b + c$ is
 (a) 0 (b) 2
 (c) 1 (d) None of these
- 158.** If the equations $x^2 + ix + a = 0$ and $x^2 - 2x + ia = 0$, $a \neq 0$ have a common root, then
 (a) a is real
 (b) $a = \frac{1}{2} + i$
 (c) $a = \frac{1}{2} - i$
 (d) the other root is also common

- 159.** If the equations $x^2 + 2x + 3\lambda = 0$ and $2x^2 + 3x + 5\lambda = 0$ have a non-zero common root, then λ is equal to
 (a) 1 (b) -1
 (c) 3 (d) None of these
- 160.** If the equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 3 = 0$ have a common root, then $a : b : c$ is equal to
 (a) 2 : 4 : 5 (b) 1 : 3 : 4
 (c) 1 : 2 : 3 (d) None of these
- 161.** If the equations $x^2 - px + q = 0$ and $x^2 - ax + b = 0$ have a common root and the other root of the second equation is the reciprocal of the other root of the first, then $(q - b)^2$ is equal to
 (a) $aq(p - b)^2$ (b) $bq(p - a)^2$
 (c) $bq(p - b)^2$ (d) None of these
- 162.** Let $f(x) = 1 + 2x + 3x^2 + \dots + (n + 1)x^n$, where n is even. Then, the number of real roots of the equation $f(x) = 0$ is
 (a) 0 (b) 1
 (c) n (d) None of these
- 163.** The value of $x^2 + 2bx + c$ is positive, if
 (a) $b^2 - 4ac > 0$ (b) $b^2 - 4ac < 0$
 (c) $c^2 < b$ (d) $b^2 < c$
- 164.** If $ax^2 + bx + 10 = 0$ does not have two distinct real roots, then the least value of $5a + b$ is
 (a) -3 (b) -2
 (c) 3 (d) None of these
- 165.** If $-a^2x^2 + 2x + 3a^2 > 0, \forall x \in (2, 4)$, then the values of a lie in the interval
 (a) $\left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(-\frac{2}{\sqrt{7}}, \frac{2}{\sqrt{7}}\right)$
 (c) $\left(-\frac{4}{7}, \frac{4}{7}\right)$ (d) $\left(-\frac{2\sqrt{2}}{\sqrt{13}}, \frac{2\sqrt{2}}{\sqrt{13}}\right)$
- 166.** The values of a which make the expression $x^2 - ax + 1 - 2a^2$ always positive for real values of x , are
 (a) $-\frac{2}{3} < a < \frac{2}{3}$ (b) $-\frac{2}{3} \leq a \leq \frac{2}{3}$
 (c) $-\frac{2}{3} < a < 1$ (d) $0 < a < \frac{2}{3}$
- 167.** If x is integer satisfying $x^2 - 6x + 5 \leq 0$ and $x^2 - 2x > 0$, then the number of possible values of x is
 (a) 3 (b) 4
 (c) 2 (d) infinite
- 168.** If $\frac{6x^2 - 5x - 3}{x^2 - 2x + 6} < 4$, then the least and highest values of $4x^2$ are
 (a) 0, 81 (b) 0, 36 (c) -10, 3 (d) 10, -3
- 169.** The inequality $|2x - 3| < 1$ is valid when x lies in the interval
 (a) (3, 4) (b) (1, 2)
 (c) (-1, 2) (d) (-4, 3)
- 170.** The set of real values of x satisfying $|x - 1| - 1 \leq 1$ is
 (a) $[-1, 3]$ (b) $[0, 2]$
 (c) $[-1, 1]$ (d) None of these
- 171.** If $5^x + (2\sqrt{3})^{2x} \geq 13^x$, then the solution set for x is
 (a) $[2, +\infty]$ (b) $(2, +\infty)$
 (c) $(4, +\infty)$ (d) None of these
- 172.** If $\log_{10} x + \log_{10} y \geq 2$, then the smallest possible value of $x + y$ is
 (a) 10 (b) 30
 (c) 20 (d) None of these
- 173.** If $x \in R$, then $\frac{x^2 - x + 1}{x^2 + x + 1}$ takes values in the interval
 (a) $\left(\frac{1}{3}, 3\right)$ (b) $\left[\frac{1}{3}, 3\right]$
 (c) (0, 3) (d) None of these
- 174.** If $y = f(x) = \tan x \cot 3x$, then
 (a) $\frac{1}{3} < y < 1$
 (b) $-\infty < y < \frac{1}{3}$ or $3 < y < \infty$
 (c) $\frac{1}{3} < y < \infty$
 (d) $-\infty < y < 1$
- 175.** The real values of x for which $3^{72} \left(\frac{1}{3}\right)^x \left(\frac{1}{3}\right)^{\sqrt{x}} > 1$, are
 (a) $x \in [0, 64]$ (b) $x \in (0, 64)$
 (c) $x \in [0, 64)$ (d) None of these
- 176.** If α and β are the roots of $4x^2 - 16x + \lambda = 0, \lambda \in R$ such that $1 < \alpha < 2$ and $2 < \beta < 3$, then the number of integral solutions of λ is
 (a) 5 (b) 6
 (c) 2 (d) 3
- 177.** The least integral value of k for which $(k - 2)x^2 + 8x + k + 4 > 0, \forall x \in R$, is
 (a) 5 (b) 4
 (c) 3 (d) None of these
- 178.** The values of a for which exactly one root of the equation $e^a x^2 - e^{2a} x + e^a - 1 = 0$ lies between 1 and 2 are given by
 (a) $\log \frac{(5 - \sqrt{17})}{4} < a < \log \frac{5 + \sqrt{17}}{4}$
 (b) $0 < a < 100$
 (c) $\log \frac{5}{4} < a < \log \frac{10}{3}$
 (d) None of the above

- 179.** If the roots of the equation $x^2 + 2ax + b = 0$ are real and distinct and they differ by at most $2m$, then b lies in the interval
 (a) $(a^2, a^2 + m^2)$ (b) $(a^2 - m^2, a^2)$
 (c) $[a^2 - m^2, a^2]$ (d) None of these
- 180.** If a, b, c and d are four consecutive terms of an increasing AP, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are
 (a) non-real complex (b) real and equal
 (c) integers (d) real and distinct
- 181.** The interval of a for which the equation $\tan^2 x - (a - 4)\tan x + 4 - 2a = 0$ has at least one solution, $\forall x \in [0, \pi/4]$, is
 (a) $a \in (2, 3)$
 (b) $a \in [2, 3]$
 (c) $a \in (1, 4)$
 (d) $a \in [1, 4]$
- 182.** If the expression $[mx - 1 + (1/x)]$ is non-negative for all positive real x , then the minimum value of m must be
 (a) $-1/2$ (b) 0 (c) $1/4$ (d) $1/2$
- 183.** If a, b and c are distinct positive numbers, then the nature of roots of the equation $1/(x - a) + 1/(x - b) + 1/(x - c) = 1/x$ is
 (a) all real and distinct
 (b) all real and at least two are distinct
 (c) at least two real
 (d) all non-real
- 184.** If a, b, c and d are four consecutive terms of an increasing AP, then the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are
 (a) real and distinct
 (b) non-real complex
 (c) real and equal
 (d) integers
- 185.** If $\cos^4 x + \sin^2 x - p = 0$, $p \in R$ has real solutions, then
 (a) $p \leq 1$ (b) $\frac{3}{4} \leq p \leq 1$
 (c) $p \geq \frac{3}{4}$ (d) None of these
- 186.** If the roots of $x^2 + x + a = 0$ exceed a , then
 (a) $2 < a < 3$ (b) $a > 3$
 (c) $-3 < a < -2$ (d) $a < -2$
- 187.** The necessary and sufficient condition for the equation $(1 - a^2)x^2 + 2ax - 1 = 0$ to have roots lying in the interval $(0, 1)$ is
 (a) $a \neq 0$
 (b) $a > 0$
 (c) $a < 2$ or $a > 2$
 (d) None of the above
- 188.** The values of a for which the equation $2x^2 - 2(2a + 1)x + a(a - 1) = 0$ has roots α and β satisfying the condition $\alpha < a < \beta$, are
 (a) $(-3, 0)$ (b) $(0, \infty)$
 (c) $(-\infty, -3) \cup (0, \infty)$ (d) None of these
- 189.** Let a, b and c be real and $ax^2 + bx + c = 0$ has two real roots α and β , where $\alpha < -1$ and $\beta > 1$, then $1 + \frac{c}{a} + \left| \frac{b}{a} \right|$
 (a) < 2 (b) < 1
 (c) < 0 (d) None of these
- 190.** If $2a + 3b + 6c = 0$, $a, b, c \in R$, then the quadratic equation $ax^2 + bx + c = 0$ has
 (a) at least one root in $[0, 1]$ (b) at least one root in $[2, 3]$
 (c) at least one root in $(3, 4]$ (d) None of these
- 191.** Let $f(x) = ax^2 + bx + c$ and $b, c \in R$, $a \neq 0$, satisfying $f(1) + f(2) = 0$. Then, the quadratic equation $f(x) = 0$ has
 (a) no real root (b) 1 and 2 as real roots
 (c) two equal roots (d) two distinct real roots
- 192.** If $x + \lambda y - 2$ and $x - \mu y + 1$ are factors of the expression $6x^2 - xy - y^2 - 6x + 8y - 12$, then
 (a) $\lambda = \frac{1}{3}, \mu = \frac{1}{2}$ (b) $\lambda = 2, \mu = 3$
 (c) $\lambda = \frac{1}{3}, \mu = -\frac{1}{2}$ (d) None of these
- 193.** If the expression $y^2 + 2xy + 2x + my - 3$ can be resolved into two rational factors, then m must be
 (a) any positive real number
 (b) any negative real number
 (c) -2
 (d) 3
- 194.** If the expression $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ can be resolved into rational factors, then
 (a) $a + b + c = 0$
 (b) $a^3 + b^3 + c^3 = 3abc$
 (c) $ab + ac + bc = 0$
 (d) None of the above
- 195.** If the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y , then
 (a) $1 \leq x \leq 3$ (b) $2 \leq x \leq 3$
 (c) $-\frac{1}{3} < y < 1$ (d) $0 < y < \frac{2}{3}$
- 196.** If $(m_r, 1/m_r)$, $r = 1, 2, 3, 4$ are four pairs of values of x and y that satisfy the equation $x^2 + y^2 + 2gx + 2fy + c = 0$, then the value of $m_1 m_2 m_3 m_4$ is
 (a) 0
 (b) 1
 (c) -1
 (d) None of the above

- 197.** The number of values of k for which the equation $x^2 - 3x + k = 0$ has two distinct roots lying in the interval $(0, 1)$, is
- (a) 3
(b) 2
(c) infinitely many
(d) no value of k satisfies the requirements

- 198.** The number of real solutions of the equation $e^x = x$ is
- (a) 1
(b) 2
(c) 0
(d) infinite

- 199.** The number of real solutions of the equation $\log_{0.5} x = |x|$ is
- (a) 1
(b) 2
(c) 0
(d) None of these

- 200.** If $(\log_5 x)^2 + \log_5 x < 2$, then x belongs to the interval
- (a) $\left(\frac{1}{25}, 5\right)$
(b) $\left(\frac{1}{5}, \frac{1}{\sqrt{5}}\right)$
(c) $(1, \infty)$
(d) None of the above

Type 2. More than One Correct Option

- 201.** The roots of the equation $(a + \sqrt{b})^{x^2-15} + (a - \sqrt{b})^{x^2-15} = 2a$, where $a^2 - b = 1$, are
- (a) ± 3 (b) ± 4 (c) $\pm \sqrt{14}$ (d) $\pm \sqrt{5}$

- 202.** If $0 < a < b < c$ and the roots α, β of the equation $ax^2 + bx + c = 0$ are imaginary, then
- (a) $|\alpha| = |\beta|$
(b) $|\alpha| = 1$
(c) $|\beta| < 1$
(d) None of the above

- 203.** If $\tan \alpha$ and $\tan \beta$ are roots of the equation $x^2 + px + q = 0$ with $p \neq 0$, then
- (a) $\sin^2(\alpha + \beta) + p \sin(\alpha + \beta) \cos(\alpha + \beta) + q \cos^2(\alpha + \beta) = q$

(b) $\tan(\alpha + \beta) = \frac{p}{q-1}$

(c) $\cos(\alpha + \beta) = -p$
(d) $\sin(\alpha + \beta) = 1 - q$

- 204.** If $(ax^2 + c)y + (a'x^2 + c') = 0$ and x is a rational function of y , then

(a) $\frac{a}{a'} = \frac{c}{c'}$ (b) $\frac{a}{c} = \frac{a'}{c'}$
(c) $ac' - a'c = 0$ (d) None of these

- 205.** The equation $\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = a(a-1)$ has

- (a) four real roots, if $a > 2$
(b) two real roots, if $1 < a < 2$
(c) no real root, if $a < -1$
(d) four real roots, if $a < -1$

- 206.** If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then which of the following expression will be the symmetric function of roots

(a) $\left| \log \frac{\alpha}{\beta} \right|$ (b) $\alpha^2 \beta^5 + \beta^2 \alpha^5$

(c) $\tan(\alpha - \beta)$ (d) $\left(\log \frac{1}{\alpha}\right)^2 + (\log \beta)^2$

- 207.** If the quadratic equation $ax^2 + bx + c = 0$ ($a > 0$) has $\sec^2 \theta$ and $\operatorname{cosec}^2 \theta$ as its roots, then which of the following must hold good?

(a) $b + c = 0$ (b) $b^2 - 4ac \geq 0$
(c) $c \geq 4a$ (d) $4a + b \geq 0$

- 208.** If the roots of the equation $x^3 + px^2 + qx - 1 = 0$ form an increasing GP, where p and q are real, then
- (a) $p + q = 0$
(b) $p \in (-3, \infty)$
(c) one of the roots is unity
(d) one root is smaller than 1 and one root is greater than 1

- 209.** Given that α, γ are roots of the equation $Ax^2 - 4x + 1 = 0$ and β, δ are the roots of the equation $Bx^2 - 6x + 1 = 0$ such that α, β, γ and δ are in HP, then

(a) $A = 3$ (b) $A = 4$
(c) $B = 2$ (d) $B = 8$

- 210.** If $(1+k)\tan^2 x - 4\tan x - 1 + k = 0$ has real roots $\tan x_1$ and $\tan x_2$, then

(a) $k^2 \leq 5$
(b) $\tan(x_1 + x_2) = 2$
(c) for $k = 2, x_1 = \pi/4$
(d) for $k = 1, x_1 = 0$

- 211.** If the equation $x^2 + px + q = 0$ has roots u and v , where p and q are non-zero constants, then

(a) $qx^2 + px + 1 = 0$ has roots $1/u$ and $1/v$
(b) $(x+p)(x-q) = 0$ has roots $u+v$ and uv
(c) $x^2 + p^2x + q^2 = 0$ has roots u^2 and v^2
(d) $x^2 + px + p = 0$ has roots $\frac{u}{v}$ and $\frac{v}{u}$

- 212.** For real x , the function $\frac{(x-a)(x-b)}{x-c}$ will assume all

real values provided
(a) $a > b > c$

- (b) $b \leq c \leq a$
(c) $a > c > b$
(d) $a \leq c \leq b$

213. Let $P(x) = x^2 + bx + c$, where b and c are integers. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, then

- (a) $P(x) = 0$ has imaginary roots
(b) $P(x) = 0$ has roots of opposite sign
(c) $P(1) = 4$
(d) $P(1) = 6$

214. If the equations $4x^2 - x - 1 = 0$ and $3x^2 + (\lambda + \mu)x + \lambda - \mu = 0$ have a common root, then the rational values of λ and μ are

- (a) $\lambda = \frac{-3}{4}$
(b) $\lambda = 0$
(c) $\mu = \frac{3}{4}$

- (d) $\mu = 0$

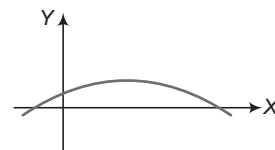
215. If $x, y \in R$ and $2x^2 + 6xy + 5y^2 = 1$, then

- (a) $|x| \leq \sqrt{5}$ (b) $|x| \geq \sqrt{5}$
(c) $y^2 \leq 2$ (d) $y^2 \leq 4$

216. If $\cos x - y^2 - \sqrt{y - x^2} - 1 \geq 0$, then

- (a) $y \geq 1$ (b) $x \in R$ (c) $y = 1$ (d) $x = 0$

217. If the following figure shows the graph of $f(x) = ax^2 + bx + c$, then



- (a) $ac < 0$ (b) $bc > 0$
(c) $ab > 0$ (d) $abc < 0$

Type 3. Assertion and Reason

Directions (Q. Nos. 218-219) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has four choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

218. Statement I If $\frac{a}{a_1}, \frac{b}{b_1}$ and $\frac{c}{c_1}$ are in AP, then a_1, b_1 and c_1 are in GP.

Statement II If $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ have a common root and $\frac{a}{a_1}, \frac{b}{b_1}$ and $\frac{c}{c_1}$ are in AP, then a_1, b_1 and c_1 are in GP.

219. Statement I If roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c = 1$.

Statement II If a, b and c are odd integers, then the roots of the equation $4abcx^2 + (b^2 - 4ac)x - b = 0$ are real and distinct.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 220-222) Let the roots of $f(x) = x^2 + bx + c$ be α and β , where $f(x)$ is quadratic polynomial $ax^2 + bx + c$. α and β are also the roots of $f(f(x)) = x$. Let the other two roots of $f(f(x)) = x$ be λ and δ .

220. The correct statements are

- I. If α and β are real and unequal, then λ and δ are also real.
II. If α and β are imaginary, then λ and δ are also imaginary.
(a) Only I (b) Only II
(c) both I and II (d) neither I nor II

221. If $\alpha + \beta = \lambda + \delta$, then the correct statements are

- I. α and β are real
II. λ and δ are real
III. α and β cannot be real
IV. λ and δ cannot be real

- (a) I and II (b) III and IV
(c) II and III (d) I and IV

222. If α and β are real and equal, then

- (a) λ and δ are imaginary
(b) λ and δ are real
(c) λ and δ may be real or imaginary
(d) Cannot say

Passage II (Q. Nos. 223-225) If $f(x)$ is a differentiable function wherever it is continuous and $f'(c_1) = f'(c_2) = 0$. $f''(c_1) \cdot f''(c_2) < 0$, $f(c_1) = 5$, $f(c_2) = 0$ and $(c_1 < c_2)$.

223. If $f(x)$ is continuous in $[c_1, c_2]$ and $f''(c_1) - f''(c_2) > 0$, then minimum number of roots of $f'(x) = 0$ in $[c_1 - 1, c_2 + 1]$ is

- (a) 2 (b) 3
(c) 4 (d) 5

- 224.** If $f(x)$ is continuous in $[c_1, c_2]$ and $f''(c_1) - f''(c_2) < 0$, then minimum number of roots of $f'(x) = 0$ in $[c_1 - 1, c_2 + 1]$ is
(a) 1 (b) 2 (c) 3 (d) 4

- 225.** If $f(x)$ is continuous in $[c_1, c_2]$ and $f''(c_1) - f''(c_2) > 0$, then minimum number of roots of $f'(x) = 0$ in $[c_1 - 1, c_2 + 1]$ is
(a) 2 (b) 3 (c) 4 (d) 5

Passage III (Q. Nos. 226-228) Let $f(x) = x^2 + b_1x + c_1$ and $g(x) = x^2 + b_2x + c_2$. If real roots of $f(x) = 0$ are α, β and real roots of $g(x) = 0$ are $\alpha + \delta, \beta + \delta$ and least value of $f(x)$ is $-\frac{1}{4}$. Then, least value of $g(x)$ occurs at $x = \frac{7}{2}$.

- 226.** The least value of $g(x)$ is

- (a) -1 (b) $-\frac{1}{2}$
(c) $-\frac{1}{4}$ (d) $-\frac{1}{3}$

- 227.** The value of b_2 is

- (a) 6 (b) -7

Type 5. Match the Columns

- 232.** Match the statements of Column I with the values of Column II.

Column I	Column II
A. The real values of a for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x + a^2 - 4a = 0$ possesses roots of opposite signs are given by	p. $a \geq 6$
B. If the equation $x^2 + 2(a+1)x + 9a - 5 = 0$ has only negative roots, then	q. $0 \leq a \leq 1$
C. The value of a for which the inequality $x^2 - 2(4a-1)x + 15a^2 - 2a - 7 > 0$ is valid for all $x \in R$, is	r. $0 < a < 4$
D. If $x \in R$, then $\frac{x^2 + 2x + a}{x^2 + 4x + 3a}$ can take all real values, if	s. $2 < a < 4$

- (c) 8 (d) 0

- 228.** The roots of $g(x) = 0$ are

- (a) 3, 4 (b) -3, 4
(c) 3, -4 (d) -3, -4

Passage IV (Q. Nos. 229-231) Let x_1, x_2, x_3 and x_4 be the roots (real or complex) of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$ and $x_1 + x_2 = x_3 + x_4$ and $a, b, c, d \in R$.

- 229.** If $a = 2$, then the value of $b - c$ is

- (a) -1 (b) 1 (c) -2 (d) 2

- 230.** If $b < 0$, then how many different values of a we may have?

- (a) 3 (b) 2 (c) 1 (d) 0

- 231.** If $b + c = 1$ and $a \neq -2$, then for real values of $a, c \in$

- (a) $\left(-\infty, \frac{1}{4}\right)$ (b) $(-\infty, 3)$
(c) $(-\infty, 1)$ (d) $(-\infty, 4)$

- 233.** Match the statements of Column I with the values of Column II.

Column I	Column II
A. If the equation $x^2 + 2(k+1)x + (9k-5) = 0$ has only negative roots, then	p. $2 < k < 4$
B. If the inequality $x^2 - 2(4k-1)x + 15k^2 - 2k - 7 > 0$ is valid for all x , then	q. $k \geq 6$
C. If $x^2 - 2(k-1)x + (2k+1) = 0$ has both roots positive, then	r. $k < -1$ or $k > 0$
D. If $2x^2 - 2(2k+1)x + k(k+1) = 0$ has one root less than k and other root greater than k , then	s. $k \geq 4$
E. The graph of the curve $x^2 = 3x - y - 2$ is strictly below the line y equal to	t. $\frac{1}{4}$

Type 6. Single Integer Answer Type Questions

- 234.** Let x_1, x_2, x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in GP, where $(x_1, x_2, x_3 > 0)$, then the maximum value of $[\beta] + [\gamma] + 2$ is _____, where $[\]$ denotes the greatest integer function.

- 235.** Consider the equation $x^3 - ax^2 + bx - c = 0$, where $a, b, c \in Q$ ($a \neq 1$). It is given that x_1, x_2 and x_1x_2 are

the real roots of the equation. If $b + c = 2(a + 1)$, then x_1x_2 is equal to _____.

- 236.** If α and β are the roots of $x^2 + px - q = 0$ and γ, δ are the roots of $x^2 + px + r = 0$, $q + r \neq 0$, then $\frac{(\alpha - \gamma)(\alpha - \delta)}{(\beta - \gamma)(\beta - \delta)}$ is equal to _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Let $a \in \mathbb{R}$ and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$, then [2014]
 - (a) $f(x)$ has three real roots, if $a > 4$
 - (b) $f(x)$ has only one real root, if $a > 4$
 - (c) $f(x)$ has three real roots, if $a < -4$
 - (d) $f(x)$ has three real roots, if $-4 < a < 4$
2. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then, the equation $p(p(x)) = 0$ has [2014]
 - (a) only purely imaginary roots
 - (b) all real roots
 - (c) two real and two purely imaginary roots
 - (d) neither real nor purely imaginary roots
3. The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$, is [2013]
 - (a) 6
 - (b) 4
 - (c) 2
 - (d) 0
4. If $3^x = 4^{x-1}$, then x is equal to [2013]
 - (a) $\frac{2 \log_3 2}{2 \log_3 2 - 1}$
 - (b) $\frac{2}{2 - \log_2 3}$
 - (c) $\frac{1}{1 - \log_4 3}$
 - (d) $\frac{2 \log_2 3}{2 \log_2 3 - 1}$
5. Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[5]{1+a} - 1) = 0$, where $a > -1$. Then, $\lim_{a \rightarrow 0^+} \alpha(a)$ and $\lim_{a \rightarrow 0^+} \beta(a)$ are [2012]
 - (a) $-\frac{5}{2}$ and 1
 - (b) $-\frac{1}{2}$ and -1
 - (c) $-\frac{7}{2}$ and 2
 - (d) $-\frac{9}{2}$ and 3
6. The value of $6 + \log_{3/2} \left(\frac{1}{3\sqrt{2}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \dots \right)$ is [2012]
 - (a) 4
 - (b) 3
 - (c) 1
 - (d) 0
7. Let (x_0, y_0) be solution of the following equations $(2x)^{\ln 2} = (3y)^{\ln 3}$ and $3^{\ln x} = 2^{\ln y}$. Then, x_0 is equal to [2011]
 - (a) $\frac{1}{6}$
 - (b) $\frac{1}{3}$
 - (c) $\frac{1}{2}$
 - (d) 6
8. Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is [2011]
 - (a) 1
 - (b) 2
 - (c) 3
 - (d) 4
9. A value of b for which the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$, have one root in common, is [2011]
 - (a) $-\sqrt{2}$
 - (b) $-i\sqrt{3}$
 - (c) $i\sqrt{5}$
 - (d) $\sqrt{2}$
10. Consider the polynomial $f(x) = 1 + 2x + 3x^2 + 4x^3$. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$. The real number s lies in the interval [2010]
 - (a) $\left(-\frac{1}{4}, 0\right)$
 - (b) $\left(-11, -\frac{3}{4}\right)$
 - (c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$
 - (d) $\left(0, \frac{1}{4}\right)$
11. Let p and q be real numbers such that $p \neq 0$, $p^3 \neq q$ and $p^3 \neq -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, then a quadratic equation having $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ as its roots, is [2010]
 - (a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
 - (b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
 - (c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
 - (d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$

JEE Main/AIEEE

12. Let α and β be the roots of equation $x^2 - 6x - 2 = 0$. If $a_n = \alpha^n - \beta^n$, for $n \geq 1$, then the value of $\frac{a_{10} - 2a_8}{2a_9}$ is equal to [2015]
 - (a) 6
 - (b) -6
 - (c) 3
 - (d) -3
13. If $a \in \mathbb{R}$ and the equation $-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$ (where, $[x]$ denotes the greatest integer $\leq x$) has no integral solution, then all possible values of a lie in the interval [2014]
 - (a) $(-1, 0) \cup (0, 1)$
 - (b) $(1, 2)$
 - (c) $(-2, -1)$
 - (d) $(-\infty, -2) \cup (2, \infty)$

- 14.** If the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$, $a, b, c \in R$, have a common root, then $a : b : c$ is [2013]
(a) 1 : 2 : 3 (b) 3 : 2 : 1 (c) 1 : 3 : 2 (d) 3 : 1 : 2
- 15.** The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has [2012]
(a) infinite number of real roots
(b) no real roots
(c) exactly one real root
(d) exactly four real roots
- 16.** Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line $\operatorname{Re} z = 1$, then it is necessary that [2011]
(a) $\beta \in (-1, 0)$ (b) $|\beta| = 1$
(c) $\beta \in (1, \infty)$ (d) $\beta \in (0, 1)$
- 17.** Sachin and Rahul attempted to solve a quadratic equation. Sachin made a mistake in writing down the constant term and ended up in roots (4, 3). Rahul made a mistake in writing down coefficient of x to get roots (3, 2). The correct roots of equation are [2011]
(a) -4, -3 (b) 6, 1 (c) 4, 3 (d) -6, -1
- 18.** If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009}$ is equal to [2010]
(a) -2 (b) -1
(c) 1 (d) 2
- 19.** If the roots of the equation $bx^2 + cx + a = 0$ are imaginary, then for all real values of x , the expression $3b^2x^2 + 6bcx + 2c^2$ is [2009]
(a) greater than $4ab$ (b) less than $4ab$
(c) greater than $-4ab$ (d) less than $-4ab$
- 20.** The quadratic equations $x^2 - 6x + a = 0$ and $x^2 - cx + 6 = 0$ have one root in common. The other roots of the first and second equations are integers in the ratio 4 : 3. Then, the common root is [2008]
(a) 2 (b) 1 (c) 4 (d) 3
- 21.** If the difference between the roots of the equation $x^2 + ax + 1 = 0$ is less than $\sqrt{5}$, then the set of possible values of a is [2007]
(a) $(-3, 3)$ (b) $(-3, \infty)$
(c) $(3, \infty)$ (d) $(-\infty, -3)$
- 22.** If the roots of the quadratic equation $x^2 + px + q = 0$ are $\tan 30^\circ$ and $\tan 15^\circ$ respectively, then the value of $2 + q - p$ is [2006]
(a) 3 (b) 0 (c) 1 (d) 2
- 23.** All the values of m for which both roots of the equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4 lie in the interval [2006]
(a) $m > 3$ (b) $-1 < m < 3$
(c) $1 < m < 4$ (d) $-2 < m < 0$
- 24.** The value of a for which the sum of the squares of the roots of the equation $x^2 - (a-2)x - a - 1 = 0$ assume the least value, is [2005]
(a) 2 (b) 3 (c) 0 (d) 1
- 25.** If the roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is equal to [2005]
(a) 1 (b) 2 (c) 3 (d) -2
- 26.** If both the roots of the quadratic equation $x^2 - 2kx + k^2 + k - 5 = 0$ are less than 5, then k lies in the interval [2005]
(a) $[4, 5]$ (b) $(-\infty, 4)$ (c) $(6, \infty)$ (d) $(5, 6]$
- 27.** If the equation $a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x = 0$, $a_1 \neq 0, n \geq 2$, has a positive root $x = \alpha$, then the equation $na_n x^{n-1} + (n-1)a_{n-1} x^{n-2} + \dots + a_1 = 0$ has a positive root, which [2005]
(a) is equal to α
(b) is greater than or equal to α
(c) belongs to $(0, \alpha)$
(d) is greater than α
- 28.** If $(1-p)$ is a root of quadratic equation $x^2 + px + (1-p) = 0$, then its roots are [2004]
(a) 0, 1 (b) -1, 1 (c) 0, -1 (d) -1, 2
- 29.** If one root of the equation $x^2 + px + 12 = 0$ is 4, while the equation $x^2 + px + q = 0$ has equal roots, then the value of q is [2004]
(a) $\frac{49}{4}$ (b) 12
(c) 3 (d) 4
- 30.** Let two numbers have arithmetic mean 9 and geometric mean 4. Then, these numbers are the roots of the quadratic equation [2004]
(a) $x^2 + 18x + 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$ (d) $x^2 - 18x - 16 = 0$
- 31.** Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further, assume that the origin, z_1 and z_2 form an equilateral triangle. Then, [2003]
(a) $a^2 = b$ (b) $a^2 = 2b$
(c) $a^2 = 3b$ (d) $a^2 = 4b$
- 32.** If the sum of the roots of the quadratic equation $ax^2 + bx + c = 0$ is equal to the sum of the squares of their reciprocals, then $\frac{a}{c}, \frac{b}{a}$ and $\frac{c}{b}$ are in [2003]
(a) AP (b) GP (c) HP (d) AGP

33. The number of the real solutions of the equation $x^2 - 3|x| + 2 = 0$ is [2003]
(a) 2 (b) 4 (c) 1 (d) 3
34. The value of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, is [2003]
(a) $2/3$ (b) $-2/3$ (c) $1/3$ (d) $-1/3$

Other Engineering Entrances

37. The number of digits in 20^{301} (given, $\log_{10} 2 = 0.3010$) is [WB JEE 2014]
(a) 602 (b) 301 (c) 392 (d) 391
38. The number of solutions of the equation $\log_{101} \log_7 (\sqrt{x+7} + \sqrt{x}) = 0$ is [WB JEE 2014]
(a) 3 (b) 7 (c) 9 (d) 49
39. The value of a , so that the sum of squares of the roots of the equation $x^2 - (a-2)x - a + 1 = 0$ assume the least value, is [BITSAT 2014]
(a) 2 (b) 0 (c) 3 (d) 1
40. If the roots of $x^2 - ax + b = 0$ are two consecutive odd integers, then $a^2 - 4b$ is equal to [Kerala CEE 2014]
(a) 3 (b) 4 (c) 5 (d) 6
41. If α and β are the roots of $x^2 - ax + b^2 = 0$, then $\alpha^2 + \beta^2$ is equal to [Kerala CEE 2014]
(a) $a^2 + 2b^2$ (b) $a^2 - 2b^2$ (c) $a^2 - 2b$ (d) $a^2 + 2b$ (e) $a^2 - b^2$
42. If α and β are the roots of the equation $x^2 + 3x - 4 = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta}$ is equal to [Kerala CEE 2014]
(a) $-\frac{3}{4}$ (b) $\frac{3}{4}$ (c) $-\frac{4}{3}$ (d) $\frac{4}{3}$ (e) $\frac{3}{2}$
43. If the roots of the equation $x^2 + 2bx + c = 0$ are α and β , then $b^2 - c$ is equal to [Kerala CEE 2014]
(a) $\frac{(\alpha - \beta)^2}{4}$ (b) $(\alpha + \beta)^2 - \alpha\beta$ (c) $(\alpha + \beta)^2 + \alpha\beta$ (d) $\frac{(\alpha - \beta)^2}{2} + \alpha\beta$ (e) $\frac{(\alpha + \beta)^2}{2} + \alpha\beta$

35. If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$, then the equation having α/β and β/α as its roots, is [2002]
(a) $3x^2 + 19x + 3 = 0$ (b) $3x^2 - 19x + 3 = 0$ (c) $3x^2 - 19x - 3 = 0$ (d) $x^2 - 16x + 1 = 0$
36. The number of real roots of $3^{2x^2 - 7x + 7} = 9$ is [2002]
(a) 0 (b) 2 (c) 1 (d) 4

44. If a , b and c are positive numbers in a GP, then the roots of the quadratic equation $(\log_e a)x^2 - (2\log_e b)x + (\log_e c) = 0$ are [WB JEE 2014]
(a) -1 and $\frac{\log_e c}{\log_e a}$ (b) 1 and $-\frac{\log_e c}{\log_e a}$ (c) 1 and $\log_a c$ (d) -1 and $\log_e a$
45. If α, β are the roots of $ax^2 + bx + c = 0$ ($a \neq 0$) and $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$ ($p \neq 0$), then the ratio of the squares of their discriminants is [WB JEE 2014]
(a) $a^2 : p^2$ (b) $a : p^2$ (c) $a^2 : p$ (d) $a : 2p$
46. Let $f(x) = 2x^2 + 5x + 1$. If we write $f(x)$ is $f(x) = [a(x+1)(x-2) + b(x-2)(x-1) + c(x-1)(x+1)]$ for real numbers a, b and c , then [WB JEE 2014]
(a) there are infinite number of choices for a, b and c
(b) only one choice for a but infinite number of choices for b and c
(c) exactly one choice for each of a, b and c
(d) more than one but finite number of choices for a, b and c
47. If α and β are the roots of the equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$, then $aS_{n+1} + bS_n + cS_{n-1}$ is equal to [AMU 2014]
(a) 0 (b) abc (c) $a + b + c$ (d) None of the above
48. If the roots of $ax^2 + bx + c = 0$ are $\sin \alpha$ and $\cos \alpha$ for some α , then which one of the following is correct? [RPET 2014]
(a) $a^2 + b^2 = 2ac$ (b) $b^2 - c^2 = 2ab$ (c) $b^2 - a^2 = 2ac$ (d) $b^2 + c^2 = 2ab$
49. The value of x such that $3^{2x} - 2(3^{x+2}) + 81 = 0$ is [Kerala CEE 2014]
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5

- 50.** The equation whose roots are the squares of the roots of the equation $2x^2 + 3x + 1 = 0$, is [Kerala CEE 2014]
 (a) $4x^2 + 5x + 1 = 0$ (b) $4x^2 - x + 1 = 0$
 (c) $4x^2 - 5x - 1 = 0$ (d) $4x^2 - 5x + 1 = 0$
 (e) $4x^2 + 5x - 1 = 0$
- 51.** In a $\triangle ABC$, $\tan A$ and $\tan B$ are the roots of $pq(x^2 + 1) = r^2x$. Then, $\triangle ABC$ is [WB JEE 2014]
 (a) a right angled triangle (b) an acute angled triangle
 (c) an obtuse angled triangle (d) an equilateral triangle
- 52.** If α and β are the roots of the quadratic equation $x^2 + px + q = 0$, then the values of $\alpha^3 + \beta^3$ and $\alpha^4 + \alpha^2\beta^2 + \beta^4$ are respectively [WB JEE 2014]
 (a) $3pq - p^3$ and $p^4 - 3p^2q + 3q^2$
 (b) $-p(3q - p^2)$ and $(p^2 - q)(p^2 + 3q)$
 (c) $pq - 4$ and $p^4 - q^4$
 (d) $3pq - p^3$ and $(p^2 - q)(p^2 - 3q)$
- 53.** The number of solution(s) of the equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ is [WB JEE 2014]
 (a) 2 (b) 0 (c) 3 (d) 1
- 54.** If $9\sqrt{x} = \sqrt{12} + \sqrt{147}$, then the value of x is [RPET 2014]
 (a) -2 (b) 2 (c) 3 (d) -3
- 55.** Solve the equation $x^2 - \sqrt{3}x + 1 = 0$. [J&K CET 2014]
 (a) $x = (-\sqrt{3} \pm 2i)/2$ (b) $x = (-\sqrt{3} \pm i)/2$
 (c) $x = (-\sqrt{3} \pm i)$ (d) $x = (\sqrt{3} \pm i)/2$
- 56.** If $a = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$, then the quadratic equation whose roots are $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$, is [Manipal 2014]
 (a) $x^2 - x + 2 = 0$ (b) $x^2 + 2x + 2 = 0$
 (c) $x^2 + x + 2 = 0$ (d) $x^2 + x - 2 = 0$
- 57.** Let p and q be real numbers. If α is the root of $x^2 + 3p^2x + 5q^2 = 0$, β is the root of $x^2 + 9p^2x + 15q^2 = 0$ and $0 < \alpha < \beta$, then the equation $x^2 + 6p^2x + 10q^2 = 0$ has a root γ that always satisfies [WB JEE 2014]
 (a) $\gamma = \frac{\alpha}{4} + \beta$ (b) $\beta < \gamma$
 (c) $\gamma = \frac{\alpha}{2} + \beta$ (d) $\alpha < \gamma < \beta$
- 58.** If the roots of the equation $\lambda^2 + 8\lambda + \mu^2 + 6\mu = 0$ are real, then μ lies between [RPET 2014]
 (a) -2 and 8 (b) -3 and 6
 (c) -8 and 2 (d) -6 and 3
- 59.** The number of solutions of the inequation $|x - 2| + |x + 2| < 4$ is [Kerala CEE 2014]
 (a) 1 (b) 2 (c) 4
 (d) 0 (e) 5
- 60.** Solve the inequation $2x - 5 \leq (4x - 7)/3$. [J&K CET 2014]
 (a) $x \in (-\infty, 4]$ (b) $x \in (-\infty, 4]$
 (c) $x \in (-\infty, 8]$ (d) $x \in (-\infty, -4]$
- 61.** Solve the inequation $3x + 2 > -16$ and $2x - 3 \leq 11$. [J&K CET 2014]
 (a) $(-6, 7]$ (b) $[-6, 7)$ (c) $(-6, 7)$ (d) $[-6, 7]$
- 62.** The number of real solutions of the equation $\sin e^x = 5^x + 5^{-x}$ is [Karnataka CET 2013]
 (a) 1 (b) 0
 (c) 2 (d) None of these
- 63.** Let a, b and c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is a root of $a^2x^2 - bx - c = 0$ and $0 < \alpha < \beta$. Then, the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies [UP SEE 2013]
 (a) $\gamma = \alpha$ (b) $\alpha < \beta < \gamma$
 (c) $\alpha < \gamma < \beta$ (d) $\gamma = \frac{\alpha + \beta}{2}$
- 64.** If the arithmetic mean of the roots of a quadratic equation is 8 and the geometric mean is 5, then the equation is [BITSAT 2013]
 (a) $x^2 - 16x - 25 = 0$ (b) $x^2 + 16x - 25 = 0$
 (c) $x^2 - 16x + 25 = 0$ (d) $x^2 - 8x + 5 = 0$
- 65.** If the roots of the equation $ax^2 + bx + c = 0$ are of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$, then $(a+b+c)^2$ is equal to [Manipal 2013]
 (a) $b^2 - 4ac$ (b) $b^2 - 2ac$
 (c) $2b^2 - ac$ (d) Σa^2
- 66.** If the equations $2ax^2 - 3bx + 4c = 0$ and $3x^2 - 4x + 5 = 0$ have a common root, then $(a+b)/(b+c)$ is equal to $(a, b, c \in R)$ [Kerala CEE 2013]
 (a) 1/2 (b) 3/35 (c) 34/31
 (d) 29/23 (e) None of these
- 67.** If $\log_2(9^{x-1} + 7) - \log_2(3^{x-1} + 1) = 2$, then the values of x are [Karnataka CET 2012]
 (a) 0, 2 (b) 0, 1
 (c) 1, 4 (d) 1, 2
- 68.** If a, b and c are in arithmetic progression, then the roots of the equation $ax^2 - 2bx + c = 0$ are [WB JEE 2012]
 (a) 1 and $\frac{c}{a}$ (b) $-\frac{1}{a}$ and $-c$
 (c) -1 and $-\frac{c}{a}$ (d) -2 and $-\frac{c}{2a}$

- 69.** The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + 8 + 2\sqrt{5} = 0$ is [Manipal 2012]
 (a) 2 (b) 4 (c) 6 (d) 8
- 70.** If α, β and γ are the roots of the equation $x^3 + 4x + 2 = 0$, then $\alpha^3 + \beta^3 + \gamma^3$ is equal to [Karnataka CET 2012]
 (a) 2 (b) 6 (c) -2 (d) -6
- 71.** If $(\alpha + \sqrt{\beta})$ and $(\alpha - \sqrt{\beta})$ are the roots of the equation $x^2 + px + q = 0$, where α, β, p and q are real, then the roots of the equation $(p^2 - 4q)(p^2x^2 + 4px) - 16q = 0$ are [WB JEE 2012]
 (a) $\left(\frac{1}{\alpha} + \frac{1}{\sqrt{\beta}}\right)$ and $\left(\frac{1}{\alpha} - \frac{1}{\sqrt{\beta}}\right)$
 (b) $\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\beta}\right)$ and $\left(\frac{1}{\sqrt{\alpha}} - \frac{1}{\beta}\right)$
 (c) $\left(\frac{1}{\sqrt{\alpha}} + \frac{1}{\sqrt{\beta}}\right)$ and $\left(\frac{1}{\sqrt{\alpha}} - \frac{1}{\sqrt{\beta}}\right)$
 (d) $(\sqrt{\alpha} + \sqrt{\beta})$ and $(\sqrt{\alpha} - \sqrt{\beta})$
- 72.** If the product of the roots of the equation $x^2 - 2\sqrt{2}kx + 2e^{2\log k} - 1 = 0$ is 31, then the roots of the equation are real for k , is equal to [AMU 2012]
 (a) -4 (b) 1
 (c) 4 (d) 0
- 73.** If α and β are the roots of the equation $x^2 - 2x + 4 = 0$, then the value of $\alpha^n + \beta^n$ will be [BITSAT 2011]
 (a) $i2^{n+1} \sin(n\pi/3)$
 (b) $2^{n+1} \cos(n\pi/3)$
 (c) $i2^{n-1} \sin(n\pi/3)$
 (d) $2^{n-1} \cos(n\pi/3)$
- 74.** If α, β and γ are the roots of $x^3 - 2x + 1 = 0$, then the value of $\Sigma \left(\frac{1}{\alpha + \beta - \gamma}\right)$ is [Karnataka CET 2011]
 (a) -1/2 (b) -1
 (c) 0 (d) 1/2
- 75.** The number of integral values of b , for which the equation $x^2 + bx - 16$ has integral roots, is [Kerala CEE 2011]
 (a) 2 (b) 3
 (c) 4 (d) 5
 (e) 6
- 76.** If a, b and c are three real numbers such that $a + 2b + 4c = 0$. Then, the equation $ax^2 + bx + c = 0$ [WB JEE 2011]
 (a) has both the roots complex
 (b) has its roots lying within $-1 < x < 0$
 (c) has one of roots equal to $\frac{1}{2}$
 (d) has its roots lying within $2 < x < 6$
- 77.** If one of the roots of $2x^2 - cx + 3 = 0$ is 3 and another equation $2x^2 - cx + d = 0$ has equal roots, where c and d are real numbers, then d is equal to [UP SEE 2011]
 (a) 3 (b) 49/8 (c) 8/49 (d) -3
- 78.** The value of $\frac{(\sqrt{\sqrt{3}+1} + \sqrt{\sqrt{3}-1})^2 (\sqrt{3} - \sqrt{2})}{(\sqrt{\sqrt{3}+1})^2 - (\sqrt{\sqrt{3}-1})^2}$ is [UP SEE 2011]
 (a) $\frac{\sqrt{3} + \sqrt{2}}{\sqrt{3}}$ (b) 1
 (c) $\sqrt{3}$ (d) $\frac{1}{\sqrt{3}}$
- 79.** If $|2x - 3| < |x + 5|$, then x lies in the interval [Kerala CEE 2011]
 (a) $(-3, 5)$ (b) $(5, 9)$
 (c) $\left(-\frac{2}{3}, 8\right)$ (d) $\left(-8, \frac{2}{3}\right)$
 (e) $\left(-5, \frac{2}{3}\right)$
- 80.** If α and β are the roots of the equation $ax^2 + bx + c = 0$, then the equation whose roots are $\frac{k}{\alpha}$ and $\frac{k}{\beta}$, is [MP PET 2011]
 (a) $cx^2 + kbx + k^2a = 0$ (b) $cx^2 + k^2bx + ka = 0$
 (c) $kcx^2 + bx + k^2a = 0$ (d) $k^2cx^2 + bx + ka = 0$
- 81.** If $3 \leq 3t - 18 \leq 18$, then which one of the following is correct? [Kerala CEE 2010]
 (a) $15 \leq 2t + 1 \leq 20$ (b) $8 \leq t < 12$
 (c) $8 \leq t + 1 \leq 13$ (d) $21 \leq 3t \leq 24$
 (e) $t \leq 7$ or $t \geq 12$
- 82.** The solution set of the inequation $\frac{x+11}{x-3} > 0$ is [Kerala CEE 2010]
 (a) $(-\infty, -11) \cup (3, \infty)$
 (b) $(-\infty, -10) \cup (2, \infty)$
 (c) $(-100, -11) \cup (1, \infty)$
 (d) $(-5, 0) \cup (3, 7)$
 (e) $(0, 5) \cup (-1, 0)$
- 83.** In a right angled triangle, the sides are a, b and c , with c as hypotenuse and $c - b \neq 1, c + b \neq 1$. Then, the value of $(\log_{c+b} a + \log_{c-b} a) / (2 \log_{c+b} a \times \log_{c-b} a)$ will be [WB JEE 2010]
 (a) 2 (b) -1 (c) 1/2 (d) 1
- 84.** If α and β are the roots of the equation $ax^2 + bx + c = 0, (c \neq 0)$, then the equation whose roots are $\frac{1}{a\alpha + b}$ and $\frac{1}{a\beta + b}$, is [BITSAT 2010]
 (a) $acx^2 - bx + 1 = 0$ (b) $x^2 - acx + bc + 1 = 0$
 (c) $acx^2 + bx - 1 = 0$ (d) $x^2 + acx - bc + 11 = 0$

- 85.** If one root of the equation $x^2 + px + q = 0$ is $2 + \sqrt{3}$, then the value of p and q are, respectively [VITEEE 2010]
 (a) $-4, 1$ (b) $4, -1$ (c) $2, \sqrt{3}$ (d) $-2, -\sqrt{3}$
- 86.** If $x^2 + px + q = 0$ has the roots α and β , then the value of $(\alpha - \beta)^2$ is [VITEEE 2010]
 (a) $p^2 - 4q$ (b) $(p^2 - 4q)^2$
 (c) $p^2 + 4q$ (d) $(p^2 + 4q)^2$
- 87.** If a and b are the roots of the equation $x^2 + ax + b = 0, a \neq 0, b \neq 0$, then the values of a and b are, respectively [Kerala CEE 2010]
 (a) 2 and -2 (b) 2 and -1 (c) 1 and -2
 (d) 1 and 2 (e) -1 and 2
- 88.** If the roots of the equation $x^2 - bx + c = 0$ are two consecutive integers, then $b^2 - 4c$ is equal to [Kerala CEE 2010]
 (a) -1 (b) 0 (c) 1
 (d) 2 (e) 3
- 89.** If α and β are the roots of the quadratic equation $x^2 + x + 1 = 0$, then the equation whose roots are α^{19} and β^7 , is [WB JEE 2010]
 (a) $x^2 - x + 1 = 0$ (b) $x^2 - x - 1 = 0$
 (c) $x^2 + x - 1 = 0$ (d) $x^2 + x + 1 = 0$
- 90.** The quadratic equation $x^2 + 15|x| + 14 = 0$ has [BITSAT 2010]
 (a) only positive solutions
 (b) only negative solutions
 (c) no solution
 (d) both positive and negative solutions
- 91.** The value of a for which the equation $2x^2 + 2\sqrt{6}x + a = 0$ has equal roots, is [VITEEE 2010]
 (a) 3
 (b) 4
 (c) 2
 (d) $\sqrt{3}$
- 92.** If $\frac{3}{2} + \frac{7}{2}i$ is a solution of the equation $ax^2 - 6x + b = 0$, where a and b are real numbers, then the value of $a + b$ is [Kerala CEE 2010]
 (a) 10 (b) 22 (c) 30
 (d) 29 (e) 31
- 93.** The roots of the quadratic equation $x^2 - 2\sqrt{3}x - 22 = 0$ are [WB JEE 2010]
 (a) imaginary
 (b) real, rational and equal
 (c) real, irrational and unequal
 (d) real, rational and unequal

Answers

Work Book Exercise 5.1

1. (a) 2. (c) 3. (d) 4. (b) 5. (a)

Work Book Exercise 5.2

1. (c) 2. (a) 3. (d) 4. (b) 5. (a) 6. (a) 7. (a) 8. (b) 9. (a) 10. (d)

Work Book Exercise 5.3

1. (a) 2. (a) 3. (c) 4. (d) 5. (a,b,c,d)

Work Book Exercise 5.4

1. (b) 2. (a) 3. (c) 4. (c) 5. (d) 6. (a) 7. (b) 8. (d) 9. (b) 10. (c)
11. (d) 12. (c) 13. (c) 14. (c)

Work Book Exercise 5.5

1. (a) 2. (b) 3. (a) 4. (c) 5. (d) 6. (b) 7. (c) 8. (a) 9. (d) 10. (c)
11. (c) 12. (c) 13. (c) 14. (c)

Target Exercises

1. (b) 2. (b) 3. (a) 4. (c) 5. (a) 6. (b) 7. (a) 8. (d) 9. (c) 10. (c)
11. (b) 12. (c) 13. (b) 14. (d) 15. (d) 16. (c) 17. (d) 18. (b) 19. (b) 20. (c)
21. (d) 22. (c) 23. (d) 24. (c) 25. (d) 26. (c) 27. (d) 28. (c) 29. (a) 30. (a)
31. (c) 32. (b) 33. (b) 34. (c) 35. (d) 36. (b) 37. (b) 38. (c) 39. (d) 40. (b)
41. (c) 42. (b) 43. (a) 44. (a) 45. (b) 46. (a) 47. (b) 48. (b) 49. (a) 50. (a)
51. (b) 52. (a) 53. (a) 54. (b) 55. (a) 56. (a) 57. (c) 58. (c) 59. (a) 60. (c)
61. (c) 62. (a) 63. (a) 64. (a) 65. (c) 66. (b) 67. (c) 68. (a) 69. (d) 70. (a)
71. (b) 72. (c) 73. (b) 74. (b) 75. (c) 76. (a) 77. (b) 78. (a) 79. (b) 80. (a)
81. (d) 82. (a) 83. (d) 84. (b) 85. (a) 86. (c) 87. (d) 88. (c) 89. (b) 90. (c)
91. (b) 92. (b) 93. (b) 94. (d) 95. (a) 96. (d) 97. (c) 98. (b) 99. (d) 100. (d)
101. (c) 102. (d) 103. (d) 104. (b) 105. (b) 106. (b) 107. (a) 108. (c) 109. (c) 110. (b)
111. (b) 112. (d) 113. (d) 114. (a) 115. (a) 116. (b) 117. (c) 118. (d) 119. (c) 120. (a)
121. (a) 122. (d) 123. (c) 124. (c) 125. (b) 126. (c) 127. (d) 128. (c) 129. (b) 130. (d)
131. (a) 132. (d) 133. (b) 134. (b) 135. (c) 136. (b) 137. (a) 138. (a) 139. (b) 140. (b)
141. (a) 142. (a) 143. (b) 144. (a) 145. (b) 146. (a) 147. (a) 148. (a) 149. (a) 150. (b)
151. (c) 152. (b) 153. (a) 154. (b) 155. (b) 156. (c) 157. (a) 158. (c) 159. (b) 160. (c)
161. (b) 162. (a) 163. (d) 164. (b) 165. (d) 166. (a) 167. (a) 168. (a) 169. (b) 170. (a)
171. (d) 172. (c) 173. (b) 174. (b) 175. (c) 176. (d) 177. (a) 178. (a) 179. (c) 180. (d)
181. (b) 182. (c) 183. (a) 184. (a) 185. (b) 186. (d) 187. (c) 188. (c) 189. (c) 190. (a)
191. (d) 192. (a) 193. (c) 194. (b) 195. (a) 196. (b) 197. (d) 198. (c) 199. (a) 200. (a)
201. (b,c) 202. (a,b) 203. (a,b) 204. (a,b,c) 205. (a,b,d) 206. (a,b,d) 207. (a,b,c) 208. (a,c,d) 209. (a,d) 210. (a,b,c,d)
211. (a,b) 212. (b,d) 213. (a,c) 214. (a,d) 215. (a,c) 216. (c,d) 217. (a,b,d) 218. (d) 219. (b) 220. (c)
221. (b) 222. (a) 223. (c) 224. (b) 225. (a) 226. (c) 227. (b) 228. (a) 229. (b) 230. (c)
231. (a) 232. (*) 233. (**) 234. (1) 235. (2) 236. (1)

* $A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$

** $A \rightarrow q; B \rightarrow p; C \rightarrow s; D \rightarrow r; E \rightarrow t$

Entrances Gallery

1. (b,d) 2. (d) 3. (c) 4. (a,b,c) 5. (b) 6. (a) 7. (c) 8. (c) 9. (b) 10. (c)
11. (b) 12. (c) 13. (a) 14. (a) 15. (d) 16. (c) 17. (b) 18. (c) 19. (c) 20. (a)
21. (a) 22. (a) 23. (b) 24. (d) 25. (a) 26. (b) 27. (c) 28. (c) 29. (a) 30. (b)
31. (c) 32. (c) 33. (b) 34. (a) 35. (b) 36. (b) 37. (c) 38. (c) 39. (d) 40. (b)
41. (b) 42. (b) 43. (a) 44. (c) 45. (a) 46. (c) 47. (a) 48. (c) 49. (b) 50. (d)
51. (a) 52. (d) 53. (b) 54. (c) 55. (d) 56. (c) 57. (d) 58. (c) 59. (d) 60. (a)
61. (a) 62. (d) 63. (c) 64. (c) 65. (a) 66. (c) 67. (d) 68. (a) 69. (b) 70. (d)
71. (a) 72. (c) 73. (b) 74. (b) 75. (a) 76. (c) 77. (b) 78. (b) 79. (c) 80. (a)
81. (c) 82. (a) 83. (d) 84. (a) 85. (a) 86. (a) 87. (c) 88. (c) 89. (d) 90. (c)
91. (a) 92. (e) 93. (c)

Explanations

Target Exercises

1. The given inequation reduces to

$$2 < \frac{8x}{10} \quad \text{or} \quad x > \frac{5}{2}$$

$\therefore$ Least integer is 3.

2. $a - b + c < 1, a + b + c > -1, 9a + 3b + c < -4$

$$\Rightarrow \left. \begin{array}{l} a - b + c < 1 \\ \text{and} \quad -a - b - c < 1 \end{array} \right\} \Rightarrow -2b < 2$$

$$\text{or} \quad 2b + 2 > 0$$

$$\Rightarrow b + 1 > 0$$

$$\begin{aligned} 3. (a^4 + b^4) - (a^3b + ab^3) &= (a^4 - a^3b) - (ab^3 - b^4) \\ &= a^3(a - b) - b^3(a - b) \\ &= (a - b)(a^3 - b^3) \\ &= (a - b)^2(a^2 + ab + b^2) \end{aligned}$$

which is positive as both factors are positive.

$$\text{Thus, } (a^4 + b^4) - (a^3b + ab^3) > 0$$

$$\Rightarrow a^4 + b^4 > a^3b + ab^3$$

4. Using number line rule,

$$x \in (-\infty, 0) \cup \left(\frac{2}{3}, \infty\right)$$

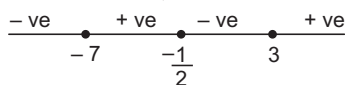
5. Using number line rule,

$$x \in (-\infty, 3) \cup (10, \infty)$$

6. Using number line rule,

$$x \in (-\infty, -4)$$

7. Using number line rule,



$$x \in (-\infty, -7) \cup \left(-\frac{1}{2}, 3\right)$$

$$8. 5x + 2 < 3x + 8 \quad \text{and} \quad \frac{x+2}{x-1} < 4$$

$$\Rightarrow x \in (-\infty, 1) \cup (2, 3)$$

$$9. |3x + 2| < 1 \Leftrightarrow -1 < 3x + 2 < 1$$

$$\Leftrightarrow -3 < 3x < -1$$

$$\Leftrightarrow -1 < x < -\frac{1}{3}$$

10. $|3 - x| = x - 3$ is true only when $x - 3 \geq 0$

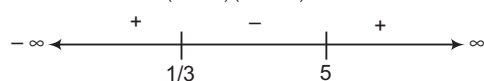
i.e. $x \geq 3$

11. $|2x - 3| < |x + 2|$

Squaring on both sides, we get

$$(2x - 3)^2 - (x + 2)^2 < 0$$

$$(x - 5)(3x - 1) < 0$$



$$\Rightarrow x \in \left(\frac{1}{3}, 5\right)$$

12. Let us first solve $|3x + 1| < \frac{1}{3}$

$$\Rightarrow -\frac{1}{3} < 3x + 1 < \frac{1}{3}$$

$$\Rightarrow -\frac{4}{3} < 3x < -\frac{2}{3}$$

$$\Rightarrow -\frac{4}{9} < x < -\frac{2}{9}$$

Also, $0 < |3x + 1|$ is satisfied by each x except when

$$3x + 1 = 0$$

$$\Rightarrow x = -\frac{1}{3}$$

$$\therefore \text{Solution is } \left(-\frac{4}{9}, -\frac{2}{9}\right) - \left\{-\frac{1}{3}\right\}.$$

13. $|x - 1|$ is the distance of x from 1

$|x - 3|$ is the distance of x from 3

The points $x = 2$ is equidistant from 1 and 3.

Hence, the solution consists of all $x \geq 2$.

14. $||x| - 1| < |1 - x| \quad \dots(i)$

Case I $x \geq 0$

$\therefore$ Eq. (i) becomes $|x - 1| < x - 1$ or $|1 - x| < 1 - x$

which is not satisfied by any x , because

$$|a| \geq a, \forall a \in R$$

Case II $-1 \leq x < 0$

$\therefore$ Eq. (i) becomes $|-1 - x| < 1 - x$

or $|x + 1| < 1 - x$

$$\Rightarrow x + 1 < 1 - x \text{ or } x < 0$$

Thus, Eq. (i) is satisfied for $-1 \leq x < 0$

Case III $x < -1$

Eq. (i) becomes $|-1 - x| < 1 - x$

$$\Rightarrow |1 + x| < 1 - x$$

$$\Rightarrow -(1 + x) < 1 - x$$

$$\Rightarrow -2 < 0, \text{ which is true}$$

So, solution set is $(-\infty, 0)$.

$$15. \left|\frac{1}{x} - 2\right| < 4 \Rightarrow -4 < \frac{1}{x} - 2 < 4$$

$$\Rightarrow -2 < \frac{1}{x} < 6$$

$$\text{Hence, } x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{1}{6}, \infty\right).$$

$$16. \left|1 + \frac{3}{x}\right| > 2$$

$$\text{Case I} \quad 1 + \frac{3}{x} > 2 \Rightarrow \frac{3}{x} > 1 \quad [\text{clearly } x > 0]$$

$$\Rightarrow 3 > x \text{ or } x < 3$$

$$\text{Case II} \quad 1 + \frac{3}{x} < -2 \Rightarrow \frac{3}{x} < -3 \quad [\text{hence } x < 0]$$

$$\Rightarrow 3 > -3x$$

$$\Rightarrow -1 < x \text{ or } x > -1$$

Hence, either $0 < x < 3$ or $-1 < x < 0$

17. $|x^2 - 10| \leq 6$

$$\begin{aligned} \Rightarrow -6 < x^2 - 10 \leq 6 \\ \Rightarrow 4 \leq x^2 \leq 16 \\ \Rightarrow 2 \leq |x| \leq 4 \\ \therefore \text{Either } 2 \leq x \leq 4 \\ \text{or} \quad -4 \leq x \leq -2 \\ \therefore x \in [-4, -2] \cup [2, 4] \end{aligned}$$

18. $\left|x + \frac{1}{x}\right| > 2$ [clearly $x \neq 0$]

$$\begin{aligned} \Rightarrow \left|\frac{x^2 + 1}{x}\right| > 2 \Rightarrow \frac{x^2 + 1}{|x|} > 2 \quad [\because x^2 + 1 > 0] \\ \Rightarrow x^2 + 1 > 2|x| \\ \Rightarrow |x|^2 - 2|x| + 1 > 0 \\ \Rightarrow (|x| - 1)^2 > 0 \\ \Rightarrow |x| \neq 1 \Rightarrow x \neq -1, 1 \\ \therefore x \in \mathbb{R} - \{-1, 0, 1\} \end{aligned}$$

19. $\frac{|x+1|}{|x|} + |x+1| = \frac{|x+1|^2}{|x|}$

$$\begin{aligned} \text{Since, } |a| + |b| = |a+b| \text{ iff } ab \geq 0 \\ \therefore \frac{(x+1)}{x} \cdot (x+1) \geq 0 \Rightarrow \frac{(x+1)^2}{x} \geq 0 \\ \Rightarrow x > 0 \text{ or } \{-1\} \end{aligned}$$

20. $(-3 \leq x-1 \leq 3) \text{ and } (x-1 \leq -1 \text{ or } x-1 \geq 1)$

$$\begin{aligned} \Rightarrow (-2 \leq x \leq 4) \text{ and } (x \leq 0 \text{ or } x \geq 2) \\ \Rightarrow -2 \leq x \leq 0 \text{ or } 2 \leq x \leq 4 \end{aligned}$$

21. Here, $x > \sqrt{1-x}$ i.e. greater than zero

$$\begin{aligned} \therefore x^2 > 1-x \text{ and } 1-x > 0, x > 0 \\ \Rightarrow x^2 + x - 1 > 0 \text{ and } 0 < x < 1 \\ \Rightarrow \left(x + \frac{1}{2}\right)^2 > \frac{5}{4} \text{ and } 0 < x < 1 \\ \Rightarrow x > \frac{\sqrt{5}-1}{2} \text{ and } 0 < x < 1 \\ \Rightarrow \frac{\sqrt{5}-1}{2} < x < 1 \end{aligned}$$

22. $|x-2|^2 - 3|x-2| + 2 = 0$

$$\begin{aligned} \Rightarrow (|x-2|-1)(|x-2|-2) = 0 \Rightarrow x = 3, 1, 4, 0 \\ \therefore \text{Product of roots} = 0 \end{aligned}$$

23. Here, $a > b \Rightarrow ax > bx$, only when $x > 0$
and $a > b > 0$

$$\begin{aligned} \text{Option (a) is false} \\ |x| > |y| \Rightarrow x > y > 0, \\ \text{Option (b) is false} \end{aligned}$$

$$a > b > 0 \Rightarrow \frac{1}{a} < \frac{1}{b}$$

$$\begin{aligned} \text{Option (c) is false} \\ a > b \Rightarrow a+c > b+c, \text{ for } c \in \mathbb{R} \\ \therefore \text{Option (d) is true.} \end{aligned}$$

24. For $x \leq 1$, the given inequation gives $x \leq 0$ and for $x \geq 3$, the given inequation gives $x \geq 4$.

For $1 < x \leq 2$ or $2 < x < 3$, the given inequation gives no solution.

$\therefore$ Solution is $(-\infty, 0] \cup [4, \infty)$.

25. Here, $5 = 4^a$ and $6 = 5^b$

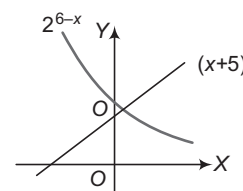
$$\begin{aligned} \text{Let } \log_3 2 &= x \\ \text{Then, } 2 &= 3^x \\ \text{Now, } 6 &= 5^b = (4^a)^b = 4^{ab} \\ \text{or } 3 &= 2^{2ab-1} \\ \therefore 2 &= (2^{2ab-1})^x = 2^{x(2ab-1)} \\ \Rightarrow x(2ab-1) &= 1 \end{aligned}$$

26. $\frac{\log a}{\log 5} \cdot \frac{\log x}{\log a} = 2$

$$\Rightarrow \log_5 x = 2 \Rightarrow x = 5^2 = 25$$

27. $x + 5 = 2^{6-x}$

Its graph is as follows



Clearly, there cannot be more than one solution and by trial, the solution is $x = 3$.

28. $|4 - 5x| > 2^2 = 4$

$$\begin{aligned} \Rightarrow \frac{5x}{4} - 1 > 1 \text{ or } \frac{5x}{4} - 1 < -1 \\ \therefore x > \frac{8}{5} \text{ or } x < 0 \end{aligned}$$

So, the solution set is $(-\infty, 0) \cup \left(\frac{8}{5}, +\infty\right)$

29. $\frac{x+2}{x} \geq (0.2)^1$ or $\frac{x+2}{x} \geq \frac{1}{5}$

$$\begin{aligned} 5x(x+2) \geq x^2 \text{ or } 4x^2 + 10x \geq 0 \\ \therefore x \geq 0 \text{ or } x \leq -\frac{5}{2} \end{aligned}$$

Also, $\frac{x+2}{x} > 0$

$$\Rightarrow x(x+2) > 0$$

$$\therefore x < -2 \text{ or } x > 0$$

$\therefore$ The solution set is $\left(-\infty, -\frac{5}{2}\right] \cup (0, +\infty)$.

30. $\log_{16} x = \frac{1 \pm \sqrt{1 - 4\log_{16} k}}{2}$

$$\begin{aligned} \text{For exactly one solution, } 4\log_{16} k &= 1 \\ \therefore k^4 &= 16 \end{aligned}$$

$$\begin{aligned} \text{Hence, } k &= 2, -2, 2i, -2i \\ \text{But } k &\text{ is positive and real.} \end{aligned}$$

31. $x^{\log_x a \cdot \log_a y \cdot \log_y z}$

$$\Rightarrow x^{\frac{\log a}{\log x} \cdot \frac{\log y}{\log a} \cdot \frac{\log z}{\log y}} \Rightarrow x^{\log_x z} = z$$

32. $\log_{10} x = 25\log_{10}(5) = 25\{1 - \log_{10} 2\}$

$$\begin{aligned} &= 25\{1 - 0.30103\} \\ &= 25\{0.69897\} \\ &= 17.469 \end{aligned}$$

$\therefore$ Number of zeroes coming immediately after the decimal place is 17.

$$33. \frac{\log x}{y-z} = \frac{\log y}{z-x} = \frac{\log z}{x-y}$$

$$= \frac{x \log x + y \log y + z \log z}{0}$$

$$\Rightarrow \log(x^x y^y z^z) = 0 \Rightarrow x^x y^y z^z = 1$$

$$34. \log_{1/3}(x^2 + x + 1) < -1 = \log_{1/3}\left(\frac{1}{3}\right)^{-1}$$

$$\Rightarrow x^2 + x + 1 < \left(\frac{1}{3}\right)^{-1}$$

$$\Rightarrow x^2 + x - 2 < 0$$

$$\Rightarrow x \in (-2, 1)$$

$$35. \log_{10} x^2 \geq 0 \Rightarrow \log_{10} x^2 \geq \log_{10} 1$$

$$\Rightarrow x^2 \geq 1 \Rightarrow x \geq 1 \Rightarrow x \leq -1$$

$$36. \text{Value} = \log_{x_1} \log_{x_2} \log_{x_3} \dots \log_{x_{n-1}} (x_{n-1}^{x_{n-2}^{x_{n-3}^{x_1}}} \log_{x_n} x_n)$$

$$= \log_{x_1} \log_{x_2} \log_{x_3} \dots \log_{x_{n-1}} x_{n-1}^{x_{n-2}^{x_{n-3}^{x_1}}}$$

$$= \dots = \log_{x_1} x_1 = 1$$

$$37. 4^{\sin^2 x} + \frac{4}{4^{\sin^2 x}} \geq 2 \sqrt{4^{\sin^2 x} \cdot \frac{4}{4^{\sin^2 x}}}$$

$$\Rightarrow 4^{\sin^2 x} + 4^{\cos^2 x} \geq 4$$

$$38. y = \frac{3^x}{3} + \frac{1}{3^x \cdot 3}, \text{ we know AM} \geq \text{GM}$$

$$\Rightarrow \frac{\frac{3^x}{3} + \frac{1}{3^x \cdot 3}}{2} \geq \sqrt{\frac{3^x}{3} \cdot \frac{1}{3^x \cdot 3}}$$

$$\Rightarrow 3^{x-1} + 3^{-x-1} \geq \frac{2}{3}$$

$$39. \text{Since, AM} \geq \text{GM}$$

$$\frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} \geq 6$$

$$\Rightarrow \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 6$$

Hence, the least value is 6.

$$40. \text{Since, AM} > \text{GM for different numbers.}$$

$$\therefore \frac{(b+c-a) + (c+a-b)}{2} > [(b+c-a)(c+a-b)]^{1/2}$$

$$\Rightarrow c > [(b+c-a)(c+a-b)]^{1/2}$$

$$\text{Similarly, } b > [(b+c-a)(a+b-c)]^{1/2}$$

$$\text{and } a > [(a+b-c)(c+a-b)]^{1/2}$$

On multiplying, we get

$$abc > (b+c-a)(c+a-b)(a+b-c)$$

$$\Rightarrow (b+c-a)(c+a-b)(a+b-c) - abc < 0$$

$$41. \text{Here, } x_1 \cdot x_2 \dots x_n = 1$$

$$\therefore \frac{x_1 + x_2 + \dots + x_n}{n} \geq (x_1 \cdot x_2 \dots x_n)^{1/n}$$

$$\Rightarrow x_1 + x_2 + \dots + x_n \geq n$$

$$42. \text{Since, AM} \geq \text{GM}$$

$$\therefore \frac{a^2 + b^2 + c^2}{3} \geq (a^2 b^2 c^2)^{1/3}$$

$$\text{and } \frac{1}{3} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \geq \left(\frac{1}{a^2 b^2 c^2} \right)^{1/3}$$

On multiplying, we get

$$\frac{1}{3} (a^2 + b^2 + c^2) \cdot \frac{1}{3} \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \geq (a^2 b^2 c^2)^{1/3} \left(\frac{1}{a^2 b^2 c^2} \right)^{1/3}$$

$$\Rightarrow (a^2 + b^2 + c^2) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right) \geq 9$$

$\therefore$ Least value is 9.

$$43. \text{Since, AM} \geq \text{GM}$$

$$\therefore \frac{a^2 + b^2}{2} \geq \sqrt{a^2 b^2} = ab, \frac{b^2 + c^2}{2} \geq bc$$

$$\text{and } \frac{c^2 + a^2}{2} \geq ca$$

On adding, we get $a^2 + b^2 + c^2 \geq ab + bc + ca$

$\Rightarrow$ Option (a) holds.

$$\text{Again, } \frac{a^3 + b^3 + c^3}{3} \geq (a^3 b^3 c^3)^{1/3}$$

$$\Rightarrow a^3 + b^3 + c^3 \geq 3abc$$

$\Rightarrow$ Option (d) does not hold.

$$\text{Next, } \frac{b+c}{2} \geq \sqrt{bc}, \frac{c+a}{2} \geq \sqrt{ca}, \frac{a+b}{2} \geq \sqrt{ab}$$

$$\Rightarrow \left(\frac{b+c}{2} \right) \left(\frac{c+a}{2} \right) \left(\frac{a+b}{2} \right) \geq \sqrt{a^2 b^2 c^2}$$

$$\Rightarrow (b+c)(c+a)(a+b) \geq 8abc$$

$\Rightarrow$ Option (b) does not hold.

$$\text{Again, } \frac{1}{3} \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \geq \left(\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} \right)^{1/3}$$

$$\Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3$$

$\Rightarrow$ Option (c) does not hold.

$$44. 3(a^2 + b^2 + c^2) - (a+b+c)^2$$

$$= 2(a^2 + b^2 + c^2 - bc - ca - ab)$$

$$= (b-c)^2 + (c-a)^2 + (a-b)^2 \geq 0$$

$$\Rightarrow 3(a^2 + b^2 + c^2) \geq (a+b+c)^2 > 9$$

$$\Rightarrow a^2 + b^2 + c^2 > 3$$

$\Rightarrow$ Option (a) holds.

$$\text{Now, } a^6 + b^6 \geq 12a^2 b^2 - 64$$

$$\text{If } a^6 + b^6 + 64 \geq 12a^2 b^2$$

$$\text{i.e. if } a^6 + b^6 + 2^6 \geq 3 \cdot 2^2 \cdot a^2 b^2$$

$$\text{i.e. if } \frac{a^6 + b^6 + 2^6}{3} \geq (2^6 a^2 b^2)^{1/3} \quad [\because \text{AM} \geq \text{GM}]$$

$\Rightarrow$ Option (b) does not hold.

Again, AM $\geq$ HM

$$\therefore \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

$$\Rightarrow \frac{\alpha}{3} \geq \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}} \Rightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{9}{\alpha}$$

$\Rightarrow$ Option (c) does not hold.

$$45. \text{Using AM} \geq \text{GM, one can show}$$

$$(b+c)(c+a)(a+b) \geq 8abc$$

$$\Rightarrow (p-a)(p-b)(p-c) \geq 8abc$$

$\Rightarrow$ Option (b) holds.

$$\begin{aligned} \text{Also, } \frac{(p-a)+(p-b)+(p-c)}{3} &\geq [(p-a)(p-b)(p-c)]^{1/3} \\ \Rightarrow \frac{3p-(a+b+c)}{3} &\geq [(p-a)(p-b)(p-c)]^{1/3} \\ \Rightarrow \frac{2p}{3} &\geq [(p-a)(p-b)(p-c)]^{1/3} \\ \Rightarrow (p-a)(p-b)(p-c) &\leq \frac{8p^3}{27} \end{aligned}$$

⇒ Option (a) does not hold.

$$\text{Again, } \frac{1}{2} \left(\frac{bc}{a} + \frac{ca}{b} \right) \geq \sqrt{\left(\frac{bc}{a} \cdot \frac{ca}{b} \right)}, \text{ etc.}$$

On adding the inequalities, we get

$$\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq a+b+c=p$$

⇒ Option (c) does not hold.

46. AM ≥ GM

$$\Rightarrow \frac{bcx + cay + abz}{3} \geq (a^2b^2c^2 \cdot xyz)^{1/3}$$

$$\text{or } \frac{bcx + cay + abz}{3} \geq 3xyz$$

$$47. \frac{2\left(\frac{a}{2}\right) + 3\left(\frac{b}{3}\right) + 4\left(\frac{c}{4}\right)}{9} \geq \left(\left(\frac{a}{2}\right)^2 \left(\frac{b}{3}\right)^3 \left(\frac{c}{4}\right)^4 \right)^{1/9}$$

$$\Rightarrow 2^9 \geq \left(\frac{a^2b^3c^4}{2^2 \cdot 3^3 \cdot 4^4} \right)$$

$$\text{or } a^2b^3c^4 \leq 2^9 \cdot 2^2 \cdot 3^3 \cdot 4^4$$

$$\text{or } a^2b^3c^4 \leq 4^2 \cdot 6^3 \cdot 8^4$$

⇒ Maximum value of P is $4^2 \cdot 6^3 \cdot 8^4$.

$$48. \text{ Here, } \frac{(x_1^2 + x_2^2 + \dots + x_n^2)}{n} \geq \left(\frac{x_1 + x_2 + \dots + x_n}{n} \right)^2$$

$$\Rightarrow n \sum_{i=1}^n x_i^2 \geq \left(\sum_{i=1}^n x_i \right)^2$$

49. $x + y + z = 2$

$$\text{Now, } (2-x)(2-y)(2-z) = (y+z)(z+x)(x+y)$$

$$\text{We know, } (y+z) \geq 2\sqrt{yz}, (x+z) \geq 2\sqrt{xz}, (x+y) \geq 2\sqrt{xy}$$

$$\therefore (y+z)(z+x)(x+y) \geq 8xyz$$

$$\Rightarrow (2-x)(2-y)(2-z) \geq 8xyz$$

$$\text{Also, } \left(\frac{x^{-1} + y^{-1} + z^{-1}}{3} \right) \geq \left(\frac{x+y+z}{3} \right)^{-1}$$

$$\Rightarrow x^{-1} + y^{-1} + z^{-1} \geq 3 \cdot \left(\frac{2}{3} \right)^{-1}$$

$$\Rightarrow x^{-1} + y^{-1} + z^{-1} \geq \frac{9}{2}$$

$$50. p^2 = (a_1 + a_2 + \dots + a_n)^2 = \sum_{i=1}^n a_i^2 + 2 \sum_{i < j} a_i a_j$$

$$\Rightarrow p^2 - 2q = \sum_{i=1}^n a_i^2 \geq 0 \Rightarrow q \leq \frac{1}{2} p^2$$

$$51. D = b^2 - 4ac, D' = d^2 + 4ac$$

$$\Rightarrow D + D' = b^2 + d^2 > 0$$

∴ Atleast one of D, D' is positive.

$$52. D = c^2(3a^2 + b^2)^2 - 4abc^2(-6a^2 - ab + 2b^2) = c^2(3a^2 - b^2 + 4ab)^2$$

53. The sum of the coefficients

$$= (a+2) + (a-3) - (2a-1) = 0$$

So, $x=1$ is a root.

$$\therefore \text{ Other root} = -\frac{2a-1}{a+2} = \text{Rational, } \forall a, a \neq -2$$

$$54. |\alpha - \beta| > \sqrt{3p} \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta > 3p$$

$$\Rightarrow p^2 - 3p - 4 > 0 \text{ and } p > 0$$

$$\Rightarrow (p-4)(p+1) > 0 \text{ and } p > 0$$

$$\Rightarrow p > 4$$

$$55. \alpha = \frac{1}{4-3i}, \beta = \frac{1}{4+3i} \Rightarrow \alpha\beta = \frac{1}{a}$$

$$\text{or } a = 25$$

$$\text{Also, } \alpha + \beta = \frac{-b}{a}$$

$$\Rightarrow \frac{8}{25} = -\frac{b}{25}$$

$$\Rightarrow b = -8$$

$$56. \text{ For } ax^2 - bxy - ay^2, D = b^2 + 4a^2 > 0$$

$$57. (16+9k)x^2 + 2(6-k)x + (39+11k)$$

$$\Rightarrow D = 0$$

∴ Only one possible integral solution for k.

$$58. \alpha + \alpha^2 = -\frac{p}{3}$$

$$\text{and } \alpha \cdot \alpha^2 = \frac{3}{3} \Rightarrow \alpha^3 = 1$$

$$\Rightarrow \alpha = 1, \omega, \omega^2$$

If $\alpha = 1$, then $p = -6$

If $\alpha = \omega$, then $p = 3$

$$\text{But } p > 0 \Rightarrow p \neq -6$$

$$\text{Hence, } p = 3$$

$$59. \alpha + \beta = -p, \alpha\beta = p^3 \text{ and } \beta^2 = \alpha$$

Hence, the roots are 4 and -2.

60. Since, a, b and c are positive, therefore

$$ax^2 + b|x| + c > 0$$

Hence, the equation $ax^2 + b|x| + c = 0$ has no real root.

$$61. \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

$$\Rightarrow \{(\alpha+k) + (\beta+k)\}^2 - 4(\alpha+k)(\beta+k) = (\alpha+\beta)^2 - 4\alpha\beta$$

$$\Rightarrow \frac{q^2}{p^2} - 4\frac{r}{p} = \frac{b^2}{a^2} - \frac{4c}{a}$$

$$\Rightarrow \frac{q^2 - 4pr}{p^2} = \frac{b^2 - 4ac}{a^2}$$

$$\Rightarrow \frac{b^2 - 4ac}{q^2 - 4pr} = \left(\frac{a}{p} \right)^2$$

$$62. \text{ Since, } \frac{\alpha}{\beta} = \frac{m}{n}$$

$$\Rightarrow \frac{\alpha + \beta}{\alpha - \beta} = \frac{m+n}{m-n}$$

$$\Rightarrow \frac{(\alpha + \beta)^2}{(\alpha + \beta)^2 - 4\alpha\beta} = \frac{(m+n)^2}{(m+n)^2 - 4mn}$$

$$\Rightarrow \frac{4\alpha\beta}{(\alpha + \beta)^2} = \frac{4mn}{(m+n)^2}$$

$$\Rightarrow \frac{c/a}{b^2/a^2} = \frac{mn}{(m+n)^2}$$

$$\Rightarrow ac(m+n)^2 = mnb^2$$

63. $\left(x^3 + \frac{1}{x^3}\right) - 5\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right)$

$$= \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) - 5\left[\left(x + \frac{1}{x}\right)^2 - 2\right] + \left(x + \frac{1}{x}\right)$$

$$= \left(x + \frac{1}{x}\right)^3 - 5\left(x + \frac{1}{x}\right)^2 - 2\left(x + \frac{1}{x}\right) + 10$$

$$= 125 - 125 - 2(5) + (10)$$

$$= 0$$

64. Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \dots}}} = \sqrt{6 + x}$

Squaring on both sides, we get

$$x^2 = 6 + x$$

$$\Rightarrow x = -2, 3$$

But x cannot be negative.

Hence, $x = 3$

65. $\alpha + \beta = -p, \alpha\beta = q$

Given, $\alpha + \beta = \alpha^2 + \beta^2$

$$= (\alpha + \beta)^2 - 2\alpha\beta$$

$$\Rightarrow -p = p^2 - 2q$$

$$\Rightarrow p^2 + p = 2q$$

66. Here, $q - r + r - p + p - q = 0$

$\therefore$ Its roots are $1, \frac{p-q}{q-r}$.

67. $2x^2 - 25x = 3(14)$

$$\Rightarrow 2x^2 - 25x - 42 = 0$$

$$\Rightarrow (2x + 3)(x - 14) = 0$$

$$\Rightarrow x = -\frac{3}{2}, 14$$

But x cannot be negative.

Hence, $x = 14$

68. Given, $\frac{x^2 - bx}{ax - c} = \frac{\lambda - 1}{\lambda + 1}$

$$\Rightarrow (\lambda + 1)(x^2 - bx) = (\lambda - 1)(ax - c)$$

$$\Rightarrow (\lambda + 1)x^2 - [\lambda b + b + \lambda a - a]x + (\lambda - 1)c = 0$$

Since, $\alpha + \beta = 0$

$$\Rightarrow \frac{\lambda b + b + \lambda a - a}{\lambda + 1} = 0$$

$$\Rightarrow \lambda(a + b) = a - b$$

$$\Rightarrow \lambda = \frac{a - b}{a + b}$$

69. Discriminant = 0

$$\Rightarrow 4(ac + bd)^2 - 4(a^2 + b^2)(c^2 + d^2) = 0$$

$$\Rightarrow 2abcd - a^2d^2 - b^2c^2 = 0$$

$$\Rightarrow (ad - bc)^2 = 0$$

$$\Rightarrow ad = bc$$

$$\Rightarrow \frac{a}{b} = \frac{c}{d}$$

70. Given, p, q and r are in AP $\Rightarrow 2q = p + r$

The roots of $px^2 + qx + r = 0$ are real, if

$$q^2 - 4pr \geq 0 \Rightarrow \left(\frac{p+r}{2}\right)^2 - 4pr \geq 0$$

$$\therefore \left(\frac{r}{p}\right)^2 - 14\left(\frac{r}{p}\right) + 1 \geq 0 \Rightarrow \left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$$

71. Given, $\frac{2x+1}{9} + \frac{x^2+7}{x^2-7} = \frac{2x+17}{9}$

$$\Rightarrow 7x^2 = 175$$

$$\Rightarrow x^2 = 25$$

$$\Rightarrow x = \pm 5$$

72. $x = 1$, satisfy $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$

$$\therefore \alpha = \beta = 1$$

$$\Rightarrow c(a-b) = a(b-c) \Rightarrow b = \frac{2ac}{a+c}$$

Hence, a, b and c are in HP.

73. Since, α and β are roots of equation $x^2 + x\sqrt{\alpha} + \beta = 0$.

$$\therefore \alpha + \beta = -\sqrt{\alpha}, \alpha\beta = \beta \Rightarrow \alpha = 1$$

[$\because \beta = 0$ does not satisfy the equation]

$$\therefore 1 + \beta = -1$$

$$\Rightarrow \beta = -2$$

74. α and β are roots of $8x^2 - 3x + 27 = 0$.

$$\Rightarrow \alpha + \beta = \frac{3}{8}, \alpha\beta = \frac{27}{8}$$

$$\therefore \left(\frac{\alpha^2}{\beta}\right)^{\frac{1}{3}} + \left(\frac{\beta^2}{\alpha}\right)^{\frac{1}{3}} = \frac{(\alpha^3)^{\frac{1}{3}} + (\beta^3)^{\frac{1}{3}}}{(\alpha\beta)^{\frac{1}{3}}}$$

$$= \frac{\alpha + \beta}{(\alpha\beta)^{\frac{1}{3}}} = \frac{3/8}{(27/8)^{\frac{1}{3}}}$$

$$= \frac{3/8}{3/2} = \frac{1}{4}$$

75. Since, $f(1) = 0$ and $f(x)$ is perfect square.

$$\therefore \alpha = \beta = 1$$

$$\Rightarrow \alpha\beta = 1$$

or $c^2(a^2 - b^2) = a^2(b^2 - c^2)$

$$\Rightarrow b^2(a^2 + c^2) = 2a^2c^2$$

$$\Rightarrow b^2 = \frac{2a^2c^2}{a^2 + c^2}$$

Hence, a^2, b^2 and c^2 are in HP.

76. We have, $(1+m)x^2 - 2(1+3m)x + (1+8m) = 0$... (i)

Let D be the discriminant of Eq. (i). Roots of Eq. (i) will be equal, if $D = 0$

$$\Rightarrow m^2 - 3m = 0 \Rightarrow m(m-3) = 0$$

$$\therefore m = 0, 3$$

77. Let the other root be β .

Then, $\alpha + \beta = -\frac{2}{4} = -\frac{1}{2}$

$$\Rightarrow \beta = -\frac{1}{2} - \alpha$$

and $4\alpha^2 + 2\alpha - 1 = 0$

Now, $4\alpha^3 - 3\alpha = \alpha(4\alpha^2 - 3) = \alpha(1 - 2\alpha - 3)$

[$\because 4\alpha^2 + 2\alpha - 1 = 0$]

$$\begin{aligned} &= -2\alpha^2 - 2\alpha = -\frac{1}{2}(4\alpha^2) - 2\alpha \\ &= -\frac{1}{2}(1 - 2\alpha) - 2\alpha \quad [\because 4\alpha^2 = 1 - 2\alpha] \\ &= -\frac{1}{2} - \alpha = \beta \quad [\text{from Eq. (i)}] \end{aligned}$$

Hence, $4\alpha^3 - 3\alpha$ is the other root.

78. Since, the roots are equal.

$$\begin{aligned} \therefore B^2 - 4AC &= 0 \\ \Rightarrow (c - a)^2 - 4(a - b)(b - c) &= 0 \\ \Rightarrow c^2 + a^2 - 2ca - 4ab + 4ac + 4b^2 - 4bc &= 0 \\ \Rightarrow c^2 + a^2 + 2ac + 4b^2 - 4b(c + a) &= 0 \\ \Rightarrow (c + a)^2 + (2b)^2 - 2 \cdot 2b(c + a) &= 0 \\ \Rightarrow [(c + a) - (2b)]^2 &= 0 \\ \Rightarrow c + a - 2b &= 0 \\ \Rightarrow 2b &= a + c \end{aligned}$$

Hence, a, b and c are in AP.

Aliter

As $x = 1$ satisfy given equation.

$\therefore \alpha = 1$ is the root

$$\begin{aligned} \Rightarrow \alpha &= \beta = 1 \\ \therefore \alpha\beta &= 1 = \frac{a - b}{b - c} \\ \Rightarrow 2b &= a + c \end{aligned}$$

79. Let the roots be α and β .

$$\begin{aligned} \text{Then, } \alpha + \beta &= \frac{-n}{l}, \alpha\beta = \frac{n}{l} \text{ and } \frac{\alpha}{\beta} = \frac{p}{q} \\ \text{Now, } \sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{n}{l}} &= \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\alpha\beta} \\ &= \frac{\alpha + \beta + \alpha\beta}{\sqrt{\alpha\beta}} = \frac{-\frac{n}{l} + \frac{n}{l}}{\sqrt{\frac{n}{l}}} = 0 \end{aligned}$$

80. The given equation can be written as

$$(ab - 1)x^2 + (a + b)x - ab = 0$$

$$\begin{aligned} \text{Here, } D &= (a + b)^2 + 4ab(ab - 1) \\ &= (a + b)^2 - 4ab + 4(ab)^2 \\ &= (a - b)^2 + (2ab)^2 \geq 0, \forall a, b \end{aligned}$$

$\therefore$ Roots are always real.

81. Put $\sqrt{x - 1} = t \Rightarrow x - 1 = t^2 \Rightarrow x = t^2 + 1$

The given equation reduces to

$$\sqrt{t^2 + 1 + 3 - 4t} + \sqrt{t^2 + 1 + 8 - 6t} = 1, \text{ where } t \geq 0$$

$$\Rightarrow |t - 2| + |t - 3| = 1, \text{ where } t \geq 0.$$

This equation will be satisfied, if $2 \leq t \leq 3$

$$\therefore 2 \leq \sqrt{x - 1} \leq 3 \text{ or } 5 \leq x \leq 10$$

Hence, the given equation is satisfied for all values of lying in $[5, 10]$.

82. Let α and β be the roots of the given equation.

$$\text{Then, } \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\text{Given, } \alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{(\alpha\beta)^2} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{(\alpha\beta)^2}$$

$$\Rightarrow -\frac{b}{a} = \frac{\frac{b^2}{a^2} - \frac{2c}{a}}{\frac{c^2}{a^2}} = \frac{b^2 - 2ca}{c^2}$$

$$\Rightarrow 2ca^2 = bc^2 + ab^2$$

Hence, bc^2, ca^2 and ab^2 are in AP.

83. Since, $\sin \theta$ and $\cos \theta$ are the roots of the equation $ax^2 + bx + c = 0$.

$$\therefore \sin \theta + \cos \theta = -\frac{b}{a} \text{ and } \sin \theta \cos \theta = \frac{c}{a}$$

$$\text{Now, } (\sin \theta + \cos \theta)^2 = 1 + 2 \sin \theta \cos \theta$$

$$\therefore \frac{b^2}{a^2} = 1 + \frac{2c}{a} = \frac{a + 2c}{a}$$

$$\Rightarrow b^2 = a(a + 2c) = a^2 + 2ac$$

$$\Rightarrow b^2 + c^2 = a^2 + 2ac + c^2 = (a + c)^2$$

$$\text{Hence, } (a + c)^2 = b^2 + c^2$$

84. Let the speed of the bus = x km/h, then the speed of the car = $(x + 25)$ km/h

$$\text{Time taken by the bus to cover 500 km} = \frac{500}{x} \text{ h}$$

$$\text{and time taken by the car to cover 500 km} = \frac{500}{(x + 25)} \text{ h}$$

$$\text{Given, } \frac{500}{x} = \frac{500}{(x + 25)} + 10 \Rightarrow \frac{50}{x} = \frac{50}{(x + 25)} + 1$$

$$\Rightarrow \frac{50}{x} - \frac{50}{(x + 25)} = 1$$

$$\Rightarrow \frac{50(x + 25 - x)}{x(x + 25)} = 1$$

$$\Rightarrow 1250 = x^2 + 25x$$

$$\Rightarrow x^2 + 25x - 1250 = 0$$

$$\Rightarrow x^2 + 50x - 25x - 1250 = 0$$

$$\Rightarrow (x + 50)(x - 25) = 0$$

$$\therefore x \neq -50 \text{ or } x = 25$$

Hence, the speed of the bus = 25 km/h and speed of the car = $(25 + 25) = 50$ km/h.

85. The given equation can be written as

$$(ay + a')x^2 + (by + b')x + (cy + c') = 0$$

The condition that x may be rational function of y is,

$(by + b')^2 - 4(ay + a')(cy + c')$ is a perfect square

$$\text{i.e. } (b^2 - 4ac)y^2 + (2bb' - 4ac' - 4a'c)y + b'^2$$

$$- 4a'c' \text{ is a perfect square } \Rightarrow D = 0$$

$$\text{i.e. } 4(bb' - 2ac' - 2a'c)^2 - 4(b^2 - 4ac)(b'^2 - 4a'c') = 0$$

$$\text{or } (ac' + a'c)^2 - 4aa'cc' = abb'c + a'bb'c$$

$$- a'bb'c - a'c'c'b^2 - acb'^2$$

$$\Rightarrow (ac' - a'c)^2 = (ab' - a'b)(bc' - b'c)$$

86. The corresponding quadratic equation of the given expression is

$$a(b - c)x^2 + b(c - a)xy + c(a - b)y^2 = 0$$

$$\Rightarrow a(b - c)\left(\frac{x}{y}\right)^2 + b(c - a)\frac{x}{y} + c(a - b) = 0$$

$$\text{Let } \frac{x}{y} = X$$

$$\text{Then, } a(b - c)X^2 + b(c - a)X + c(a - b) = 0$$

The given expression will be perfect square, if the discriminant of the corresponding equation of the given expression is zero.

$$\begin{aligned} \therefore b^2(c-a)^2 - 4a(b-c) \cdot c(a-b) &= 0 \\ \Rightarrow (bc + ab - 2ac)^2 &= 0 \\ \Rightarrow b(a+c) &= 2ac \\ \Rightarrow b &= \frac{2ac}{a+c} \end{aligned}$$

Hence, a , b and c are in HP.

87. $ax^2 - bx + c = 0$

$$\begin{aligned} \Rightarrow \alpha + \beta &= \frac{b}{a} \\ \text{and } \alpha\beta &= \frac{c}{a} \\ \text{Clearly, } y &= \frac{1}{x^2} \\ \Rightarrow a \cdot \frac{1}{y} - \frac{b}{\sqrt{y}} + c &= 0 \\ \Rightarrow a - b\sqrt{y} + cy &= 0 \\ \Rightarrow b^2y &= (a+cy)^2 \\ \Rightarrow (a+cy)^2 &= b^2y \text{ has roots } \frac{1}{\alpha^2}, \frac{1}{\beta^2}. \end{aligned}$$

88. $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$$\begin{aligned} \Rightarrow -\frac{b}{a} &= \frac{b^2 - 2ac}{c^2} \\ \Rightarrow -bc^2 &= ab^2 - 2a^2c \\ \Rightarrow \frac{b}{c} + \frac{c}{a} &= \frac{2a}{b} \\ \Rightarrow \frac{c}{a}, \frac{a}{b} \text{ and } \frac{b}{c} &\text{ are in AP.} \\ \text{i.e. } \frac{a}{c}, \frac{b}{a} \text{ and } \frac{c}{b} &\text{ are in HP.} \end{aligned}$$

89. $\alpha + \beta = \frac{4 + \sqrt{5}}{5 + \sqrt{2}}$ and $\alpha\beta = \frac{8 + 2\sqrt{5}}{5 + \sqrt{2}}$

$$\begin{aligned} \text{HM} &= \frac{2\alpha\beta}{\alpha + \beta} \\ &= \frac{16 + 4\sqrt{5}}{4 + \sqrt{5}} = 4 \end{aligned}$$

90. Since, $x^2 - 3x + 2$

$$\begin{aligned} \text{i.e. } (x-2)(x-1) &\text{ is a factor of } x^4 - px^2 + q. \\ \therefore x=2, x=1 &\text{ are roots of } x^4 - px^2 + q = 0 \\ \Rightarrow 16 - 4p + q &= 0 \text{ and } 1 - p + q = 0 \\ \text{On solving these equations, we get} \\ p &= 5, q = 4 \end{aligned}$$

91. $(3^x)^2 - 36 \cdot 3^x + 243 = 0$

$$\begin{aligned} \text{Put } 3^x &= y, \text{ the equation becomes} \\ y^2 - 36y + 243 &= 0 \\ \Rightarrow (y-9)(y-27) &= 0 \\ \Rightarrow y &= 9, 27 \\ \Rightarrow 3^x &= 9 \\ \text{or } 3^x &= 27 \\ \Rightarrow x &= 2 \\ \text{or } x &= 3 \end{aligned}$$

$\therefore$ Solution pair is $\{2, 3\}$.

92. We have, $\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$

$$\begin{aligned} \Rightarrow r(x+q+x+p) &= (x+p)(x+q) \\ \text{or } x^2 + x(p+q-2r) + pq - r(p+q) &= 0 \end{aligned}$$

Let its roots be α and $-\alpha$.

$$\text{Then, } \alpha + (-\alpha) = -(p+q-2r)$$

$$\Rightarrow p+q = 2r$$

$$\begin{aligned} \text{Also, product} &= \alpha(-\alpha) = pq - r(p+q) \\ &= pq - \frac{(p+q)^2}{2} = -\frac{1}{2}(p^2 + q^2) \end{aligned}$$

93. $\frac{|x+1|}{|x|} + |x+1| = \frac{|x+1|^2}{|x|}$

$$\Rightarrow |x+1| \left\{ \frac{1}{|x|} + 1 - \frac{(x+1)}{|x|} \right\} = 0$$

$$\begin{aligned} \therefore |x+1| &= 0 \Rightarrow 1+|x| - |x+1| = 0 \\ |x+1| &= 0 \Rightarrow x+1 > 0 \text{ and } x \neq 0 \\ \therefore x &= -1, x > 0 \end{aligned}$$

Aliter

$$\frac{(x+1)^2}{x} > 0$$

$$\Rightarrow x > 0 \cup \{-1\}$$

94. Given, $|x-2| = x^2$

On taking $x=1$ and $x=-2$, the equation is satisfied.
Hence, its solution is $\{-2, 1\}$.

95. Given equation is $|x|^2 - 3|x| + 2 = 0$

$$\begin{aligned} \Rightarrow (|x|-1)(|x|-2) &= 0 \\ \text{Either } |x| &= 1 \text{ or } |x| = 2 \\ \Rightarrow x &= \pm 1, x = \pm 2 \end{aligned}$$

Hence, the given equation has four solutions.

96. Put $2^x = y$, the equation becomes $16y^2 = 8y + 48$

$$\begin{aligned} \Rightarrow 2y^2 - y - 6 &= 0 \\ \Rightarrow (y-2)(2y+3) &= 0 \\ \Rightarrow y &= 2, -\frac{3}{2} \end{aligned}$$

But 2^x cannot be negative.

$$\text{Hence, } 2^x = 2 \Rightarrow x = 1$$

97. $5 - 2\sqrt{6} = \frac{1}{5 + 2\sqrt{6}}$

$\therefore$ Given equation

$$(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$$

$$\text{Becomes } y + \frac{1}{y} = 10, \text{ where } y = (5 + 2\sqrt{6})^{x^2-3}$$

$$\Rightarrow y^2 - 10y + 1 = 0$$

$$\Rightarrow y = 5 \pm 2\sqrt{6}$$

$$\Rightarrow (5 + 2\sqrt{6})^{x^2-3} = (5 + 2\sqrt{6})^1$$

$$\Rightarrow (5 + 2\sqrt{6})^{x^2-3} = (5 + 2\sqrt{6})^{-1}$$

$$\Rightarrow x^2 - 3 = 1$$

$$\text{or } x^2 - 3 = -1$$

$$\Rightarrow x^2 = 4 \text{ or } x^2 = 2$$

$$\begin{aligned} \text{Hence, } x &= \pm 2 \\ \text{or } x &= \pm \sqrt{2} \end{aligned}$$

98. The given equation can be written as

$$\left(\frac{5}{3}\right)^{3x} + \left(\frac{5}{3}\right)^x = 2$$

On putting $\left(\frac{5}{3}\right)^x = t$, the equation becomes

$$\begin{aligned} t^3 + t - 2 &= 0 \\ \Rightarrow t^3 - 1 + (t - 1) &= 0 \\ \Rightarrow (t - 1)(t^2 + t + 1) + (t - 1) &= 0 \\ \Rightarrow (t - 1)(t^2 + t + 2) &= 0 \\ \Rightarrow t = 1 \text{ or } t^2 + t + 2 &= 0 \end{aligned}$$

But $t^2 + t + 2 = 0$ does not have real solutions.

$$\text{Therefore, } t = 1 \Rightarrow \left(\frac{5}{3}\right)^x = 1 \Rightarrow x = 0$$

99. Since, α and β are the roots of the equation

$$x^2 - px + q = 0$$

$$\begin{aligned} \therefore \alpha + \beta &= p \\ \text{and } \alpha\beta &= q \end{aligned}$$

$$\begin{aligned} \text{Now, } (\alpha^{1/4} + \beta^{1/4})^4 &= (\alpha^{1/4} + \beta^{1/4})^2)^2 \\ &= [\alpha^{1/2} + \beta^{1/2} + 2(\alpha\beta)^{1/4}]^2 \\ &= [\sqrt{\alpha} + \sqrt{\beta} + 2\sqrt{\alpha\beta} + 2(\alpha\beta)^{1/4}]^2 \\ &= [\sqrt{p + 2\sqrt{q}} + 2(q)^{1/4}]^2 \\ &= p + 6\sqrt{q} + 4q^{1/4} \sqrt{p + 2\sqrt{q}} \\ \therefore \alpha^{1/4} + \beta^{1/4} &= [p + 6\sqrt{q} + 4q^{1/4} \sqrt{p + 2\sqrt{q}}]^{1/4} \end{aligned}$$

100. The function $y = \sqrt{x(x-3)}$ is defined for $x(x-3) \geq 0$
i.e. $x \leq 0$ or $x \geq 3$... (i)

The given equation can be written as

$$\begin{aligned} 9|x|^2 - 19|x| + 2 &= 0 \\ \Rightarrow (9|x| - 1)(|x| - 2) &= 0 \\ \Rightarrow |x| = 2 \text{ or } |x| = \frac{1}{9} \end{aligned}$$

$\therefore$ Solutions of the given equation are $\pm 2, \pm \frac{1}{9}$.

In the domain (i), the required solutions are $-2, -\frac{1}{9}$.

101. We have, $2^{x+2} \cdot 3^{3x/(x-1)} = 9 = 3^2$

$$\Rightarrow (x+2)\log 2 + \frac{3x}{x-1}\log 3 = 2\log 3$$

$$\Rightarrow (x+2)\log 2 + \left(\frac{3x}{x-1} - 2\right)\log 3 = 0$$

$$\Rightarrow (x+2)\left(\log 2 + \frac{1}{x-1}\log 3\right) = 0$$

$$\Rightarrow x = -2 \text{ or } x = 1 - \frac{\log 3}{\log 2}$$

102. We have, $f(x) + f\left(\frac{1}{x}\right) = 1$

$$\Rightarrow x - [x] + \frac{1}{x} - \left[\frac{1}{x}\right] = 1$$

$$\Rightarrow x + \frac{1}{x} - 1 = [x] + \left[\frac{1}{x}\right]$$

$$\Rightarrow \frac{x^2 + 1 - x}{x} = (\text{integer}) k \quad [\text{say}]$$

$$\Rightarrow x^2 - (k+1)x + 1 = 0$$

Since, x is real, so $(k+1)^2 - 4 \geq 0$

$$\Rightarrow k^2 + 2k - 3 \geq 0 \Rightarrow (k+3)(k-1) \geq 0$$

$$\Rightarrow k \leq -3 \text{ or } k \geq 1$$

Therefore, number of solutions is infinite.

103. The given equation can be written as

$$\left(\frac{3}{2}\right)^{3/x} + \left(\frac{3}{2}\right)^{1/x} = 2$$

Put $\left(\frac{3}{2}\right)^{1/x} = t$, the equation becomes $t^3 + t - 2 = 0$

$$\Rightarrow (t-1)(t^2 + t + 2) = 0$$

But $t^2 + t + 2 = 0$ has no real root. $[\because t = 1]$

$$\Rightarrow \left(\frac{3}{2}\right)^{1/x} = 1$$

$$\Rightarrow \frac{1}{x} = 0$$

which is not possible for any value of x .

104. Here, $\alpha + \beta = -b < 0$ $[\because b > 0]$

$$\alpha\beta = c < 0 \quad [\because c < 0]$$

Since, $\alpha\beta < 0$ and $\alpha < \beta \Rightarrow \alpha$ is a negative root.

$$\alpha + \beta < 0 \text{ and } \alpha < 0 \Rightarrow \beta < |\alpha|$$

Hence, $\alpha < 0 < \beta < |\alpha|$

105. Given, $x^2 - 2x + \sin^2 \alpha = 0$

$$\Rightarrow \sin^2 \alpha = 2x - x^2$$

$$\text{Since, } 0 \leq \sin^2 \alpha \leq 1$$

$$\Rightarrow 0 \leq 2x - x^2 \leq 1$$

$$\Rightarrow x^2 - 2x \leq 0 \text{ and } x^2 - 2x + 1 \geq 0$$

$$\Rightarrow x \in [0, 2] \text{ and } x \in R$$

$$\Rightarrow x \in [0, 2]$$

106. $(m+1)^2 - 4(m+4) \geq 0$

$$\Rightarrow (m-1)^2 \geq 4^2$$

$$\Rightarrow |m-1| \geq 4$$

$$\Rightarrow m-1 \leq -4 \text{ or } m-1 \geq 4$$

$$\Rightarrow m \leq -3 \text{ or } m \geq 5$$

$\therefore$ Their product > 0 , i.e. $m+4 > 0$

$$\Rightarrow m > -4$$

$$\text{Hence, } -4 < m \leq -3$$

107. If $x - a < 0$, $|x - a| = -(x - a)$

$\therefore$ Equation becomes $x^2 + 2a(x - a) - 3a^2 = 0$

$$\Rightarrow x^2 + 2ax - 5a^2 = 0$$

$$\Rightarrow x = -(1 + \sqrt{6})a, (-1 + \sqrt{6})a$$

$$\therefore x < a \leq 0$$

$$\therefore x = (-1 + \sqrt{6})a$$

$$\text{If } x - a \geq 0, |x - a| = x - a$$

$\therefore$ The equation becomes $x^2 - 2a(x - a) - 3a^2 = 0$

$$\Rightarrow x^2 - 2ax - a^2 = 0$$

$$\Rightarrow x = (1 + \sqrt{2})a, (1 - \sqrt{2})a$$

$$\Rightarrow x = (1 + \sqrt{2})a, (1 - \sqrt{2})a$$

$$\therefore x \geq a \text{ and } a \leq 0$$

$$\therefore x = (1 - \sqrt{2})a$$

108. For $x \geq a$, the given equation becomes

$$x^2 - 2a(x - a) - 3a^2 = 0$$

$$\Rightarrow x^2 - 2ax - a^2 = 0$$

$$\Rightarrow x = a(1 \pm \sqrt{2})$$

As $a \geq 0$ and $x \geq a$, the possible root is $a(1 + \sqrt{2})$.

For $x < a$, the given equation becomes

$$x^2 + 2a(x - a) - 3a^2 = 0$$

$$\Rightarrow x^2 + 2ax - 5a^2 = 0$$

$$\Rightarrow x = a(-1 \pm \sqrt{6})$$

As $a \geq 0$
and $x < a$

$\therefore$ Possible root is $a(-1 - \sqrt{6})$.

109. $\left(\frac{\sqrt{2}}{\sqrt{5+2\sqrt{2}}}\right)^x + \left(\frac{\sqrt{2}+1}{\sqrt{5+2\sqrt{2}}}\right)^x = 1$, which is of the form

$$\cos^x \alpha + \sin^x \alpha = 1$$

$$\therefore x = 2$$

110. $[x]^2 - [x] - 2 = 0$

$$\Rightarrow [x] = 2, -1$$

$$\Rightarrow x \in [-1, 0) \cup [2, 3)$$

111. Let $e^{\sin x} = t$

$$t - \frac{1}{t} - 4 = 0$$

$$\Rightarrow t = 2 \pm \sqrt{5}$$

$$\Rightarrow e^{\sin x} = 2 + \sqrt{5}, 2 - \sqrt{5}$$

But $e^{\sin x}$ cannot be negative.

$$\therefore e^{\sin x} = 2 + \sqrt{5}$$

$$\Rightarrow e^{\sin x} > e$$

$\Rightarrow \sin x > 1$, which is not possible.

So, the given equation has no real solution.

112. Given, $x^{\sqrt{x}} = \sqrt{x^x}$

$$\Rightarrow x^{\sqrt{x}} = (x^x)^{1/2}, x > 0$$

$$\Rightarrow x^{\sqrt{x}} = x^{x/2}, x > 0$$

$$\Rightarrow \text{Either } x = 1$$

or $\sqrt{x} = \frac{x}{2}, x > 0$

Hence, $x = 1$ and 4

113. $\sec \theta = \frac{6 \pm \sqrt{36 - 32}}{16} = \frac{6 \pm 2}{16} = \frac{1}{2}, \frac{1}{4}$

But $\sec \theta \leq -1$ or $\sec \theta \geq 1$

Hence, the given equation has no roots.

114. Given, $\log_2 x + \frac{1}{\log_2 x} = 3 + \frac{1}{3}$

$$= \log_2 y + \frac{1}{\log_2 y}$$

$$\Rightarrow \log_2 x = 3 \text{ and } \log_2 y = \frac{1}{3} \quad [\because x \neq y]$$

$$\Rightarrow x = 2^3 \text{ and } y = 2^{1/3}$$

Hence, $x + y^3 = 8 + 2 = 10$

115. $\sin(e^x) = 5^x + 5^{-x}$, let $5^x = t$, then

$$\sin(e^x) = t + \frac{1}{t} = \left(\sqrt{t} - \frac{1}{\sqrt{t}}\right)^2 + 2 \quad [\because 5^x > 0]$$

$\Rightarrow \sin(e^x) \geq 2$, which is not possible for any real x .

116. $D = a^2 + b^2 + c^2 = a^2 + (a+1)^2 + a^2(a+1)^2$

$$= a^2 \{1 + (a+1)^2\} + (a+1)^2$$

$$= a^2(a^2 + 2a + 2) + (a^2 + 2a + 1)$$

$$= a^4 + 2a^3 + 3a^2 + 2a + 1$$

$$= (a^2 + a + 1)^2$$

$\therefore \sqrt{D} = a(a+1) + 1 = \text{an odd integer}$

117. $\sum_{k=2}^n \left(x^2 - \left(\frac{1}{k-1} + \frac{1}{k}\right)x + \frac{1}{k(k-1)}\right)$

where, constant term is given by $\frac{1}{k \cdot (k-1)}$

$$\therefore \sum_{k=2}^{\infty} \frac{1}{k(k-1)} = \left(\frac{1}{1 \cdot 2}\right) + \left(\frac{1}{2 \cdot 3}\right) + \left(\frac{1}{3 \cdot 4}\right) + \dots$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \infty$$

$$= 1$$

118. $y^{x^2 + 7x + 12} = 1$

$$\Rightarrow x^2 + 7x + 12 = 0$$

$$\Rightarrow x = -3, -4$$

$$\Rightarrow y = 9, 10 \quad [\text{when } y \neq 1]$$

Again, when $y = 1$, $x = 5$

$\therefore$ Solutions are $(-3, 9)$, $(-4, 10)$ and $(5, 1)$.

119. As $(\lambda + 1)x^2 + 2 = \lambda x + 3$ has only one solution.

$$D = 0$$

$$\Rightarrow \lambda^2 - 4(\lambda + 1)(-1) = 0$$

$$\Rightarrow \lambda^2 + 4\lambda + 4 = 0$$

$$\Rightarrow (\lambda + 2)^2 = 0$$

$$\therefore \lambda = -2$$

120. Given quadratic expression is

$$x^2 + 2(a + b + c)x + 3(bc + ca + ab)$$

This quadratic expression will be a perfect square, if the discriminant of its corresponding equation is zero. Hence,

$$4(a + b + c)^2 - 4 \times 3(bc + ca + ab) = 0$$

$$\Rightarrow (a + b + c)^2 - 3(bc + ca + ab) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca - 3(bc + ca + ab) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 - ab - bc - ca = 0$$

$$\Rightarrow \frac{1}{2}[2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ca] = 0$$

$$\Rightarrow \frac{1}{2}[(a^2 + b^2 - 2ab) + (b^2 + c^2 - 2bc) + (c^2 + a^2 - 2ca)] = 0$$

$$\Rightarrow \frac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0$$

which is possible only when $(a - b)^2 = 0$, $(b - c)^2 = 0$ and $(c - a)^2 = 0$ i.e. $a = b = c$

121. $(x^2 + x - 2)(x^2 + x - 3) = 12$... (i)

Put $x^2 + x = y$, so that Eq. (i) becomes

$$\begin{aligned} & (y - 2)(y - 3) = 12 \\ \Rightarrow & y^2 - 5y - 6 = 0 \\ \Rightarrow & (y - 6)(y + 1) = 0 \Rightarrow y = 6, -1 \end{aligned}$$

When $y = 6$, we get

$$\begin{aligned} & x^2 + x - 6 = 0 \\ \Rightarrow & (x + 3)(x - 2) = 0 \Rightarrow x = -3, 2 \end{aligned}$$

When $y = -1$, we get

$$x^2 + x + 1 = 0$$

which has non-real roots and sum of roots is -1 .

122. Here, $x = 0$ is not a root. Divide both the numerator and denominator by x and put $x + 3/x = y$ to obtain

$$\frac{4}{y+1} + \frac{5}{y-5} = \frac{3}{2} \Rightarrow y = -5, 3$$

$x + 3/x = -5$ has two irrational roots and $x + 3/x = 3$ has imaginary roots.

123. Given α and β are roots of equation

$$\begin{aligned} & x^2 - 2x + 3 = 0 \\ \Rightarrow & \alpha^2 - 2\alpha + 3 = 0 \quad \dots (i) \\ \text{and} & \beta^2 - 2\beta + 3 = 0 \quad \dots (ii) \\ \Rightarrow & \alpha^2 = 2\alpha - 3 \\ \Rightarrow & \alpha^3 = 2\alpha^2 - 3\alpha \\ \Rightarrow & P = (2\alpha^2 - 3\alpha) - 3\alpha^2 + 5\alpha - 2 \\ & = -\alpha^2 + 2\alpha - 2 = 3 - 2 = 1 \quad [\text{using Eq. (i)}] \end{aligned}$$

Similarly, we have $Q = 2$

Now, sum of roots is 3 and product of roots is 2. Hence, the required equation is $x^2 - 3x + 2 = 0$.

124. Since, α and β are the roots of the equation

$$\begin{aligned} & 2x^2 - 35x + 2 = 0. \\ \therefore & 2\alpha^2 - 35\alpha = -2 \\ \Rightarrow & 2\alpha - 35 = -\frac{2}{\alpha} \\ \text{and} & 2\beta^2 - 35\beta = -2 \\ \Rightarrow & 2\beta - 35 = -\frac{2}{\beta} \\ \text{Now, } & (2\alpha - 35)(2\beta - 35) = \left(-\frac{2}{\alpha}\right)\left(-\frac{2}{\beta}\right) \\ & = \frac{8 \times 8}{\alpha^3 \beta^3} = \frac{64}{1} = 64 \quad [\because \alpha\beta = 1] \end{aligned}$$

125. $(31 + 8\sqrt{15})^{x^2-3} + 1 = (32 + 8\sqrt{15})^{x^2-3}$
 $\Rightarrow (31 + 8\sqrt{15})^{x^2-3} + 1^{x^2-3} = (32 + 8\sqrt{15})^{x^2-3}$
 $\Rightarrow x^2 - 3 = 1 \quad \text{or} \quad x = \pm 2 \quad [\because a^n + b^n = (a+b)^n]$

126. $a\alpha^2 + c = -b\alpha, a\alpha + b = -\frac{c}{\alpha}$

Hence, the given expression is

$$\frac{b}{c}(\alpha^2 + \beta^2) = \frac{b(b^2 - 2ac)}{a^2c}$$

127. Since, α and β are roots of $x^2 + px + q = 0$.

$$\begin{aligned} \therefore & \alpha + \beta = -p \quad \text{and} \quad \alpha\beta = q \\ \text{Now, } & \alpha^4 \text{ and } \beta^4 \text{ are roots of } x^2 - rx + q = 0. \\ \therefore & \alpha^4 + \beta^4 = r \quad \text{and} \quad \alpha^4\beta^4 = q \end{aligned}$$

Now, for equation $x^2 - 4qx + 2q^2 - r = 0$, product of roots is

$$\begin{aligned} 2q^2 - r &= 2(\alpha\beta)^2 - (\alpha^4 + \beta^4) \\ &= -(\alpha^2 - \beta^2)^2 < 0 \end{aligned}$$

As product of roots is negative, so the roots must be real.

128. $A = a(b - c)(a + b + c)$

$$B = b(c - a)(a + b + c)$$

$$C = c(a - b)(a + b + c)$$

$$\text{Now, } Ax^2 + Bx + C = 0$$

$$\Rightarrow (a + b + c)[a(b - c)x^2 + b(c - a)x + c(a - b)] = 0$$

Given that roots are equal.

$$\begin{aligned} \therefore & D = 0 \\ \Rightarrow & b^2(c - a)^2 - 4ac(b - c)(a - b) = 0 \\ \Rightarrow & b^2c^2 - 2ab^2c + b^2a^2 - 4a^2bc + 4acb^2 \\ & \quad + 4a^2c^2 - 4abc^2 = 0 \\ \Rightarrow & b^2c^2 + b^2a^2 + 4a^2c^2 + 2ab^2c - 4a^2bc - 4abc^2 = 0 \\ \Rightarrow & (bc + ab - 2ac)^2 = 0 \\ \Rightarrow & bc + ab = 2ac \\ \Rightarrow & \frac{1}{a} + \frac{1}{c} = \frac{2}{b} \end{aligned}$$

Hence, a, b and c are in HP.

129. $\because ax^2 - bx + c = 0$

$$\therefore \alpha + \beta = \frac{b}{a}$$

$$\text{and} \quad \alpha\beta = \frac{c}{a}$$

$$\text{Also, } (a + cy)^2 = b^2y$$

$$\Rightarrow c^2y^2 - (b^2 - 2ac)y + a^2 = 0$$

$$\Rightarrow \left(\frac{c}{a}\right)^2 y^2 - \left[\left(\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)\right]y + 1 = 0$$

$$\Rightarrow (\alpha\beta)^2 y^2 - (\alpha^2 + \beta^2)y + 1 = 0$$

$$\Rightarrow y^2 - (\alpha^{-2} + \beta^{-2})y + \alpha^{-2}\beta^{-2} = 0$$

$$\Rightarrow (y - \alpha^{-2})(y - \beta^{-2}) = 0$$

Hence, the roots are α^{-2} and β^{-2} .

130. Here, $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$
 $= [(\alpha + \beta)^2 - 2\alpha\beta]^2 - 2(\alpha\beta)^2$
 $= \left(p^2 + \frac{1}{p^2}\right)^2 - \frac{1}{p^4}$
 $= p^4 + \frac{1}{2p^4} + 2$
 $= \left(p^2 - \frac{1}{\sqrt{2}p^2}\right)^2 + 2 + \sqrt{2} \geq 2 + \sqrt{2}$

131. $x_1(x - x_2)^2 + x_2(x - x_1)^2 = 0$

$$\Rightarrow x^2(x_1 + x_2) - 4x_1x_2 + x_1x_2(x_1 + x_2) = 0$$

$$D = 16(x_1x_2)^2 - 4x_1x_2(x_1 + x_2)^2 > 0 \quad [\because x_1x_2 < 0]$$

The product of roots is $x_1x_2 < 0$.

Thus, the roots are real and of opposite signs.

132. $(x + a)(x + 1991) + 1 = 0$

$$\begin{aligned} \Rightarrow (x + a)(x + 1991) &= -1 \\ \Rightarrow (x + a) &= 1 \\ \text{and } x + 1991 &= -1 \\ \Rightarrow a &= 1993 \\ \text{or } x + a &= -1 \\ \text{and } x + 1991 &= 1 \\ \Rightarrow a &= 1989 \end{aligned}$$

133. Let $f(x) = -3 + x - x^2$. Then, $f(x) < 0$ for all x , because coefficient of x^2 is less than 0 and $D < 0$. Thus, LHS of the given equation is always positive, whereas the RHS is always less than zero. Hence, there is no solution.

134. $x = 3 \cos \theta$ and $y = 3 \sin \theta$

$$\begin{aligned} z &= 2 \cos \phi, t = 2 \sin \phi \\ \therefore 6 \cos \theta \sin \phi - 6 \sin \theta \cos \phi &= 6 \\ \Rightarrow \sin(\phi - \theta) &= 1 \\ \Rightarrow \phi - \theta &= 90^\circ + \theta \\ \Rightarrow P = xz &= -6 \sin \theta \cos \theta = -3 \sin 2\theta \\ \Rightarrow P_{\max} &= 3 \end{aligned}$$

135. $D > 0 \Rightarrow (a - 3)^2 + 4(a + 2) > 0$

$$\begin{aligned} \Rightarrow a^2 - 6a + 9 + 4a + 8 &> 0 \\ \Rightarrow a^2 - 2a + 17 &> 0 \\ \Rightarrow a &\in R \\ \therefore \frac{a^2 + 1}{a^2 + 2} &= 1 - \frac{1}{a^2 + 2} \geq \frac{1}{2} \end{aligned}$$

136. $x = 2 + \sqrt{3}$

$$\begin{aligned} \Rightarrow (x - 2)^2 &= 3 \\ \Rightarrow x^2 - 4x + 1 &= 0 \quad \dots(i) \\ \Rightarrow (x - 2)^4 &= 9 \\ \Rightarrow x^4 - 8x^3 + 24x^2 - 32x + 16 &= 9 \\ \Rightarrow x^4 - 8x^3 + 18x^2 - 8x + 2 + 6(x^2 - 4x + 1) - 1 &= 0 \\ \text{From Eq. (i), we get} \\ x^4 - 8x^3 + 18x^2 - 8x + 2 &= 1 \end{aligned}$$

137. $x^{3^n} + (-x)^{3^n} = 0$, if $n \geq 0$ and n is an integer.

138. At $x = 2$,

$$\begin{aligned} x^3 - 2(1 + \alpha)x^2 + (4\alpha + \alpha^2 + \beta^2)x - 2(\alpha^2 + \beta^2) \\ = 8 - 8(1 + \alpha) + 8\alpha + 2\alpha^2 + 2\beta^2 - 2\alpha^2 - 2\beta^2 = 0 \\ \therefore x = 2, z_0 \text{ and } \bar{z}_0 \text{ are three roots.} \end{aligned}$$

139. Let $\alpha = -4$, $\beta = -4\omega$ and $\gamma = -4\omega^2$

$$\therefore \left(\frac{\alpha}{\beta}\right)^2 = \frac{1}{\omega^2} = \omega \quad \text{and} \quad \left(\frac{\alpha}{\gamma}\right)^2 = \frac{1}{\omega^4} = \omega^2$$

$\therefore$ Required equation is

$$x^2 - (\omega + \omega^2)x + \omega^3 = 0 \Rightarrow x^2 + x + 1 = 0$$

140. Let $f(x) = x^4 + ax^3 + bx^2 + cx - 1$

Since, $(x - 1)^3$ is a factor.

$$\begin{aligned} \therefore f(1) = 0, f'(1) = 0, f''(1) = 0 \\ \Rightarrow a + b + c = 0, 3a + 2b + c = -4 \\ \text{and } 3a + b = -6 \\ \Rightarrow a = -2, b = 0, c = 2 \end{aligned}$$

So, other factor is $(x + 1)$.

141. $\alpha \pm i\beta$ are complex roots of equation $x^3 + qx + r = 0$ and γ is its real root.

$$\Rightarrow (x - \gamma)(x^2 - 2\alpha x + \alpha^2 + \beta^2) = x^3 + qx + r$$

On comparing coefficients, we get

$$\begin{aligned} -\gamma - 2\alpha &= 0 \\ \Rightarrow \gamma &= -2\alpha \\ \text{Since, } \gamma^3 + q\gamma + r &= 0 \\ \Rightarrow (-2\alpha)^3 + q(-2\alpha) + r &= 0 \\ \text{i.e. } 2\alpha \text{ is root of cubic equation } x^3 + qx - r &= 0. \end{aligned}$$

142. Since, α and β are the roots of $x^2 + px + q = 0$.

$$\begin{aligned} \therefore \alpha + \beta &= -p, \alpha\beta = q \quad \dots(i) \\ \alpha^{2n} + p^n \alpha^n + q^n &= 0 \\ \beta^{2n} + p^n \beta^n + q^n &= 0 \end{aligned}$$

On subtraction,

$$\alpha^n + \beta^n = -p^n$$

Now, $\frac{\alpha}{\beta}$ is a root of $x^n + 1 + (x + 1)^n = 0$.

$$\begin{aligned} \therefore \frac{\alpha^n}{\beta^n} + 1 + \left(\frac{\alpha}{\beta} + 1\right)^n &= 0 \\ \Rightarrow \frac{\alpha^n + \beta^n}{\beta^n} + \frac{(\alpha + \beta)^n}{\beta^n} &= 0 \\ \Rightarrow -p^n + (-p)^n &= 0 \end{aligned}$$

This is true only, if n is an even integer.

143. Let α and β be the roots of $x^2 + bx + c = 0$.

Then, $\alpha + \beta = -b, \alpha\beta = c$

Roots of the required equation are α^3 and β^3 . So, the equation is

$$\begin{aligned} x^2 - (\alpha^3 + \beta^3)x + \alpha^3\beta^3 &= 0 \\ \Rightarrow x^2 - \{(\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)\}x + (\alpha\beta)^3 &= 0 \\ \Rightarrow x^2 - (-b^3 + 3cb)x + c^3 &= 0 \\ \Rightarrow x^2 + b(b^2 - 3c)x + c^3 &= 0 \end{aligned}$$

144. Since, α and β are roots of the equation $ax^2 + bx + c = 0$.

$$\begin{aligned} \therefore \alpha + \beta &= -\frac{b}{a} \\ \text{and } \alpha\beta &= \frac{c}{a} \end{aligned}$$

The required equation is

$$\begin{aligned} x^2 - \left(\frac{1}{a\alpha + b} + \frac{1}{a\beta + b}\right)x + \frac{1}{a\alpha + b} \cdot \frac{1}{a\beta + b} &= 0 \\ \Rightarrow \{a^2\alpha\beta + ab(\alpha + \beta) + b^2\}x^2 \\ - \{a(\alpha + \beta) + 2b\}x + 1 &= 0 \\ \Rightarrow (ca - b^2 + b^2)x^2 - (2b - b)x + 1 &= 0 \\ \Rightarrow cax^2 - bx + 1 &= 0 \end{aligned}$$

145. $x = 2 + 2^{1/3} + 2^{2/3}$

$$\begin{aligned} \Rightarrow x - 2 &= 2^{1/3} + 2^{2/3} \\ \Rightarrow (x - 2)^3 &= (2^{1/3} + 2^{2/3})^3 \\ \Rightarrow x^3 - 8 - 6x(x - 2) \\ &= (2^{1/3})^3 + (2^{2/3})^3 + 3(2^{1/3}) \cdot (2^{2/3}) \cdot (2^{1/3} + 2^{2/3}) \end{aligned}$$

On putting $2^{1/3} + 2^{2/3} = x - 2$ in above equation, we get

$$\begin{aligned} x^3 - 6x^2 + 12x - 8 &= 6 + 6(x - 2) \\ \Rightarrow x^3 - 6x^2 + 6x &= 2 \end{aligned}$$

146. $1, a_1, a_2, \dots, a_{n-1}$ are roots of $x^n - 1 = 0$

$$\Rightarrow \frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \dots (x - a_{n-1})$$

$$\Rightarrow x^{n-1} + x^{n-2} + x^{n-3} + \dots + x + 1 = (x - a_1)(x - a_2) \dots (x - a_{n-1})$$

Put $x = 1$ on the both sides, we get
 $1 + 1 + 1 + \dots + 1 = (1 - a_1)(1 - a_2) \dots (1 - a_{n-1})$
 $\Rightarrow (1 - a_1)(1 - a_2) \dots (1 - a_{n-1}) = n$

147. $x = \frac{2 \cos \theta \pm \sqrt{4 \cos^2 \theta - 4}}{2} = \cos \theta \pm i \sin \theta$

Take $\alpha = \cos \theta + i \sin \theta$, then

$$\beta = \cos \theta - i \sin \theta$$

$$\Rightarrow \alpha^n = (\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

$$\text{Also, } \beta^n = (\cos \theta - i \sin \theta)^n = \cos n\theta - i \sin n\theta$$

$$\text{Now, } \alpha^n + \beta^n = 2 \cos n\theta \text{ and } \alpha^n \beta^n = 1$$

$\therefore$ Required equation is $x^2 - (\alpha^n + \beta^n)x + \alpha^n \beta^n = 0$

$$\Rightarrow x^2 - (2 \cos n\theta)x + 1 = 0$$

148. $\frac{x^n - 1}{x - 1} = (x - a_1)(x - a_2) \dots (x - a_{n-1})$

$$\Rightarrow x^{n-1} + x^{n-2} + \dots + x + 1 = (x - a_1)(x - a_2) \dots (x - a_{n-1})$$

Put $x = -1$, we get

$$\begin{aligned} (-1)^{n-1} (1 + a_1)(1 + a_2) \dots (1 + a_{n-1}) \\ = 1 - 1 + 1 - \dots + 1 \\ = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

$$\text{Thus, } (1 + \omega)(1 + \omega^2) \dots (1 + \omega^{n-1}) = n$$

149. Let $y = -5 + 4i \Rightarrow y^2 + 10y + 41 = 0$

$$\begin{aligned} \therefore f(x) &= x^4 + 9x^3 + 35x^2 - x + 4 \\ &= x^2(x^2 + 10x + 41) - x(x^2 + 10x + 41) \\ &\quad + 4(x^2 + 10x + 41) - 160 \\ &= -160 \\ \therefore f(-5 + 4i) &= -160 \end{aligned}$$

150. Let $x = \cos \theta$, we get $4 \cos^3 \theta - 3 \cos \theta = p$

$$\Rightarrow p = \cos 3\theta$$

$$\Rightarrow \theta = \frac{1}{3} \cos^{-1}(p) \text{ and } x = \cos \theta$$

$$\Rightarrow x = \cos \left(\frac{1}{3} \cos^{-1}(p) \right) \quad \dots (i)$$

$$\text{Since, } -1 \leq p \leq 1 \Rightarrow 0 \leq \cos^{-1} p \leq \pi$$

$$\text{or } 0 \leq \frac{1}{3} \cos^{-1} p \leq \frac{\pi}{3}, \text{ as we know } \cos x \text{ is decreasing.}$$

$$\therefore \cos 0 \geq \cos \left(\frac{1}{3} \cos^{-1} p \right) \geq \cos \frac{\pi}{3} \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\frac{1}{2} \leq x \leq 1 \Rightarrow x \in \left[\frac{1}{2}, 1 \right]$$

151. Since, α, β, γ and σ are the roots of the given equation, therefore

$$x^4 + 4x^3 - 6x^2 + 7x - 9 = (x - \alpha)(x - \beta)(x - \gamma)(x - \sigma)$$

On putting $x = i$ and then $x = -i$, we get

$$1 - 4i + 6 + 7i - 9 = (i - \alpha)(i - \beta)(i - \gamma)(i - \sigma)$$

$$\text{and } 1 + 4i + 6 - 7i - 9 = (-i - \alpha)(-i - \beta)(-i - \gamma)(-i - \sigma)$$

On multiplying these two equations, we get

$$(-2 + 3i)(-2 - 3i) = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$$

$$\Rightarrow 13 = (1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \sigma^2)$$

152. $\tan \theta_1 + \tan \theta_2 + \tan \theta_3 = (a + 1)$

$$\Sigma \tan \theta_1 \tan \theta_2 = (b - a)$$

$$\tan \theta_1 \tan \theta_2 \tan \theta_3 = b$$

$$\begin{aligned} \therefore \tan(\theta_1 + \theta_2 + \theta_3) &= \frac{\Sigma \tan \theta_1 - \Pi \tan \theta_1}{1 - \Sigma \tan \theta_1 \tan \theta_2} \\ &= \frac{(a + 1 - b)}{1 - (b - a)} = 1 \end{aligned}$$

$$\Rightarrow \theta_1 + \theta_2 + \theta_3 = \frac{\pi}{4}$$

153. $\Sigma \alpha = 1, \Sigma \alpha\beta = 0, \alpha\beta\gamma = 1$

$$\begin{aligned} \Sigma \frac{1 + \alpha}{1 - \alpha} &= -\Sigma \frac{-\alpha + 1 - 2}{1 - \alpha} = \Sigma \left(\frac{2}{1 - \alpha} - 1 \right) \\ &= 2 \Sigma \frac{1}{1 - \alpha} - 3 \end{aligned}$$

$$\text{Now, } \frac{1}{(x - \alpha)} + \frac{1}{(x - \beta)} + \frac{1}{(x - \gamma)} = \frac{3x^2 - 2x}{x^3 - x^2 - 1}$$

$$\Rightarrow \frac{1}{1 - \alpha} + \frac{1}{1 - \beta} + \frac{1}{1 - \gamma} = \frac{3 - 2}{1 - 1 - 1} = -1$$

$$\Rightarrow \frac{1 + \alpha}{1 - \alpha} = -5$$

154. Here, $x \rightarrow x + 1$

$$\Rightarrow ax^2 + (b + 2a)x + a + b + c = 0$$

$$\text{But equation given is } 2x^2 + 8x + 2 = 0$$

$$\Rightarrow a = 2, b + 2a = 8, a + b + c = 2$$

$$\Rightarrow b = 4, c = -4$$

$$\Rightarrow b = -c$$

155. $x^2 + x + 1 = 0$ has roots ω, ω^2 .

$$\therefore ax^2 + bx + c = 0$$

$\therefore$ Both roots common.

$$\Rightarrow \frac{a}{1} = \frac{b}{1} = \frac{c}{1}$$

156. We have, $x^3 + 3x^2 + 3x + 2 = 0$

$$\Rightarrow (x + 1)^3 + 1 = 0$$

$$\Rightarrow (x + 1 + 1) \{ (x + 1)^2 - (x + 1) + 1 \} = 0$$

$$\Rightarrow (x + 2)(x^2 + x + 1) = 0$$

$$\Rightarrow x = -2, \frac{-1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow x = -2, \omega, \omega^2$$

Since, $a, b, c \in R$, $ax^2 + bx + c = 0$ cannot have one real and one imaginary root. Therefore, two common roots of $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ are ω and ω^2 .

$$\text{Thus, } -\frac{b}{a} = \omega + \omega^2 = -1$$

$$\Rightarrow a = b \text{ and } \frac{c}{a} = \omega \cdot \omega^2 = 1 \Rightarrow c = a$$

$$\Rightarrow a = b = c$$

157. As the coefficients are in reverse order, the roots of $ax^2 + bx + c = 0$ are α and β , while the roots of

$$cx^2 + bx + a = 0 \text{ are } \frac{1}{\alpha} \text{ and } \frac{1}{\beta}.$$

One negative root is common.

$$\Rightarrow \alpha = \frac{1}{\alpha} < 0$$

$$\text{So, } \alpha = -1 \Rightarrow a - b + c = 0$$

158. Let α be the common root.

Then, $\alpha^2 + i\alpha + a = 0$,

$$\alpha^2 - 2\alpha + ia = 0$$

$$\Rightarrow \frac{\alpha^2}{a} = \frac{\alpha}{a(1-i)} = \frac{1}{-2-i}$$

$$\Rightarrow a^2(1-i)^2 = a(-2-i)$$

$$\Rightarrow a = \frac{1}{2} - i$$

159. $\alpha^2 + 2\alpha + 3\lambda = 0$ and $2\alpha^2 + 3\alpha + 5\lambda = 0$

$$\Rightarrow \frac{\alpha^2}{\lambda} = \frac{\alpha}{\lambda} = \frac{1}{-1}$$

$$\Rightarrow \alpha = 1 \text{ and } \alpha = -\lambda$$

$$\Rightarrow \lambda = -1$$

160. For the equation $x^2 + 2x + 3 = 0$,

$$\text{Discriminant} = (2)^2 - 4 \cdot 1 \cdot 3 < 0$$

$\therefore$ Roots of $x^2 + 2x + 3 = 0$ are imaginary. Since, the equations $x^2 + 2x + 3 = 0$ and $ax^2 + bx + c = 0$ are given to have a common root, therefore both roots will be common. Hence, both the equations are identical.

$$\Rightarrow a : b : c = 1 : 2 : 3$$

161. Let α and β be the roots of $x^2 - px + q = 0$ and α and $\frac{1}{\beta}$

be the roots of $x^2 - ax + b = 0$.

$$\text{Then, } \alpha + \beta = p \text{ and } \alpha\beta = q$$

$$\text{Also, } \alpha + \frac{1}{\beta} = a \text{ and } \frac{\alpha}{\beta} = b$$

$$\begin{aligned} \text{Now, } (q - b)^2 &= \left(\alpha\beta - \frac{\alpha}{\beta} \right)^2 = \alpha^2 \left(\beta - \frac{1}{\beta} \right)^2 \\ &= \frac{\alpha}{\beta} \cdot \beta \alpha \left[\left(\alpha + \beta \right) - \left(\alpha + \frac{1}{\beta} \right) \right]^2 \\ &= bq(p - a)^2 \end{aligned}$$

162. For $x > 0$, $f(x) > 0$, so there is no positive real root of $f(x) = 0$

Now, $f(0) = 1 \neq 0$, so $x = 0$ is not a root

$$f(x) = 1 + 2x + 3x^2 + \dots + (n+1)x^n \quad \dots(i)$$

Multiplying Eq. (i) by x , we get

$$xf(x) = x + 2x^2 + \dots + nx^n + (n+1)x^{n+1}$$

$$\Rightarrow (1-x)f(x) = (1+x+x^2+\dots+x^n) - (n+1)x^{n+1}$$

$$= \frac{1-x^{n+1}}{1-x} - (n+1)x^{n+1}$$

$$\Rightarrow f(x) = \frac{1-x^{n+1}}{(1-x)^2} - \frac{(n+1)x^{n+1}}{(1-x)}$$

$$= \frac{1 - (n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}$$

So, the equation $f(x) > 0$, $\forall x$.

Hence, no solution.

163. $x^2 + 2bx + c > 0$

So, the given equation has no roots.

$$\Rightarrow D < 0 \Rightarrow 4b^2 - 4c < 0$$

$$\Rightarrow b^2 < c$$

164. $f(0) = 10 > 0$, as $f(x) = 0$ does not have distinct real roots.

$$\Rightarrow f(5) \geq 0$$

$$\Rightarrow 25a + 5b + 10 \geq 0 \Rightarrow 5a + b \geq -2$$

165. $f(2) > 0$ and $f(4) > 0$

$$\Rightarrow a^2 < 4 \text{ and } 13a^2 < 8$$

$$\Rightarrow -2 < a < 2 \text{ and } \frac{-2\sqrt{2}}{\sqrt{13}} < a < \frac{2\sqrt{2}}{\sqrt{13}}$$

$$\Rightarrow -\frac{2\sqrt{2}}{\sqrt{13}} < a < \frac{2\sqrt{2}}{\sqrt{13}}$$

166. Since, the coefficient of x^2 is 1 which is positive.

$\therefore$ The given expression is positive for all real values of x , if $D < 0$.

$$\Rightarrow (-a)^2 - 4(1 - 2a^2) < 0$$

$$\Rightarrow 9a^2 - 4 < 0$$

$$\Rightarrow (3a+2)(3a-2) < 0$$

$$\Rightarrow -\frac{2}{3} < a < \frac{2}{3}$$

167. $x^2 - 6x + 5 \leq 0$ and $x^2 - 2x > 0$

$$\Rightarrow 1 \leq x \leq 5$$

$$\text{and } x < 0 \text{ or } x > 2$$

$$\Rightarrow 2 < x \leq 5$$

$$\Rightarrow x \in \{3, 4, 5\}$$

Hence, there are three solutions.

168. Denominator $= (x-1)^2 + 5 > 0$, $\forall x$

$$\Rightarrow 2x^2 + 3x - 27 = (2x+9)(x-3) \leq 0$$

$$\text{which gives } -\frac{9}{2} \leq x \leq 3 \text{ leading to } 0 \leq 4x^2 \leq 81$$

169. Given, $|2x-3| < 1 \Rightarrow -1 < 2x-3 < 1$

$$\Rightarrow 1 < x < 2$$

$$\Rightarrow x \in (1, 2)$$

170. $||x-1|-1| \leq 1$

$$\Rightarrow 0 \leq |x-1| \leq 2$$

$$\Rightarrow |x-1| \leq 2$$

$$\Rightarrow -1 \leq x \leq 3$$

171. $\left(\frac{5}{13}\right)^x + \left(\frac{12}{13}\right)^x \geq 1$

$$\therefore \cos^x \alpha + \sin^x \alpha \geq 1$$

$$\text{where, } \cos \alpha = \frac{5}{13}$$

$$\text{and } \sin \alpha = \frac{12}{13}$$

Equality holds for $x = 2$.

If $x < 2$, both $\cos \alpha$ and $\sin \alpha$ increase (being positive fractions).

$$\text{So, } \cos^x \alpha + \sin^x \alpha > 1$$

If $x < 2$. Thus, $x \leq 2$.

172. Here, $xy \geq 100$

$$\begin{aligned} \Rightarrow \frac{x+y}{2} &\geq \sqrt{xy} \\ \Rightarrow x+y &\geq 2 \cdot 10 \\ \Rightarrow x+y &\geq 20 \end{aligned}$$

173. Let $\frac{x^2 - x + 1}{x^2 + x + 1} = m$

$$\begin{aligned} \text{Discriminant} &\geq 0 \\ \Rightarrow (1+m)^2 - 4(1-m)^2 &\geq 0 \\ \Rightarrow (m-3)\left(m-\frac{1}{3}\right) &\leq 0 \\ \Rightarrow m &\in \left[\frac{1}{3}, 3\right] \end{aligned}$$

174. $y = \frac{\tan x}{\tan 3x} = \frac{\tan x (1 - 3 \tan^2 x)}{3 \tan x - \tan^3 x}$

$$\begin{aligned} \Rightarrow y &< \frac{1}{3} \quad \text{or} \quad y > 3 \\ \Rightarrow -\infty < y < \frac{1}{3} \quad \text{or} \quad 3 < y < \infty \end{aligned}$$

175. The given inequation is valid only when $x \geq 0$... (i)

The given inequation can be written in the form

$$\begin{aligned} 3^{72-x-\sqrt{x}} &> 1 \\ \Rightarrow 72-x-\sqrt{x} &> 0 \quad [\because 3 > 1] \\ \Rightarrow x + \sqrt{x} - 72 &< 0 \\ \Rightarrow (\sqrt{x} + 9)(\sqrt{x} - 8) &< 0 \\ \text{But } \sqrt{x} + 9 &> 0, \forall x \geq 0 \\ \therefore \sqrt{x} - 8 &< 0 \Rightarrow \sqrt{x} < 8 \\ \therefore 0 &\leq x < 64 \end{aligned}$$

176. $f(1) > 0, f(2) < 0, f(3) > 0$ and $D > 0$

$$\begin{aligned} \Rightarrow -12 + \lambda &> 0, \\ \lambda &< 16, \\ \lambda &> 12 \quad \text{and} \quad 16(16 - \lambda) > 0 \\ \Rightarrow 12 &< \lambda < 16 \\ \therefore \lambda &\in \{13, 14, 15\} \end{aligned}$$

Hence, three integral solutions.

177. $k - 2 > 0$ and

$$\begin{aligned} 64 - 4(k-2)(k+4) &< 0 \\ k > 2 \quad \text{and} \quad 16 - (k^2 + 2k - 8) &< 0 \\ k > 2 \quad \text{and} \quad k^2 + 2k - 24 &> 0 \\ k > 2 \quad \text{and} \quad (k-4)(k+6) &> 0 \\ \Rightarrow k &> 4 \end{aligned}$$

$\therefore$ Least integral value of $k = 5$

178. $(e^a - e^{2a} + e^a - 1)(4e^a - 2e^{2a} + e^a - 1) < 0$

$$\Rightarrow (e^{2a} - 2e^a + 1)(2e^{2a} - 5e^a + 1) < 0$$

Let

$$x = e^a$$

$$\begin{aligned} \Rightarrow (x-1)^2 (2x^2 - 5x + 1) &< 0 \\ \Rightarrow (x-1)^2 \left(x - \frac{5-\sqrt{17}}{4}\right) \left(x - \frac{5+\sqrt{17}}{4}\right) &< 0 \end{aligned}$$

$$\Rightarrow \frac{5-\sqrt{17}}{4} < x < \frac{5+\sqrt{17}}{4}$$

$$\Rightarrow \log \left(\frac{5-\sqrt{17}}{4} \right) < a < \log \left(\frac{5+\sqrt{17}}{4} \right)$$

179. Let the roots be α and β .

$$\therefore \alpha + \beta = -2a \quad \text{and} \quad \alpha\beta = b$$

$$\text{Given, } |\alpha - \beta| \leq 2m$$

$$\Rightarrow |\alpha - \beta|^2 \leq (2m)^2$$

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta \leq 4m^2$$

$$\Rightarrow 4a^2 - 4b \leq 4m^2$$

$$\Rightarrow a^2 - m^2 \leq b$$

$$\text{and discriminant } D > 0 \quad \text{or} \quad 4a^2 - 4b > 0$$

$$\Rightarrow a^2 - m^2 \leq b \quad \text{and} \quad b < a^2$$

$$\text{Hence, } b \in [a^2 - m^2, a^2]$$

180. Given, $a < b < c < d$. Let

$$f(x) = (x-a)(x-c) + 2(x-b)(x-d)$$

$$\Rightarrow f(b) = (b-a)(b-c) < 0$$

$$\text{and } f(d) = (d-a)(d-c) > 0$$

Hence, $f(x) = 0$ has one root in (b, d) . Also, $f(a)f(c) < 0$. So, the other root lies in (a, c) . Hence, roots of the equation are real and distinct.

$$\begin{aligned} 181. \tan x &= \frac{(a-4) - \sqrt{(a-4)^2 - 4(4-2a)}}{2} \\ &= \frac{a-4-a}{2} = a-2, -2 \end{aligned}$$

$$\therefore \tan x = a-2 \quad [\because \tan x \neq -2]$$

$$\therefore x \in \left[0, \frac{\pi}{4}\right]$$

$$\therefore 0 \leq a-2 \leq 1$$

$$\Rightarrow 2 \leq a \leq 3$$

182. We know that, $ax^2 + bx + c \geq 0, \forall x \in R$

If $a > 0$ and $b^2 - 4ac \leq 0$.

$$\therefore mx - 1 + \frac{1}{x} \geq 0$$

$$\Rightarrow \frac{mx^2 - x + 1}{x} \geq 0$$

$$\Rightarrow mx^2 - x + 1 \geq 0 \text{ as } x > 0$$

$$\text{Now, } mx^2 - x + 1 \geq 0, \text{ if } m > 0 \text{ and } 1 - 4m \leq 0$$

$$\Rightarrow m > 0 \quad \text{and} \quad m \geq \frac{1}{4}$$

Thus, the minimum value of m is $\frac{1}{4}$.

183. The equation on simplifying gives

$$\begin{aligned} x(x-b)(x-c) + x(x-c)(x-a) + x(x-a)(x-b) \\ - (x-a)(x-b)(x-c) = 0 \end{aligned} \quad \dots (i)$$

$$\text{Let } f(x) = x(x-b)(x-c) + x(x-c)(x-a) + x(x-a)(x-b) - (x-a)(x-b)(x-c)$$

We can assume without loss of generality that $a < b < c$. Now,

$$f(a) = a(a-b)(a-c) > 0$$

$$f(b) = b(b-c)(b-a) < 0$$

$$f(c) = c(c-a)(c-b) > 0$$

So, one root of Eq. (i) lies in (a, b) and one root in (b, c) . Obviously the third root must also be real.

184. As $a < b < c < d$ are in AP.

$$\therefore f(x) = (x-a)(x-c) + 2(x-b)(x-d)$$

$$\Rightarrow f(a) = +ve, f(b) = -ve, f(c) = -ve \text{ and } f(d) = +ve$$

$\Rightarrow$ One root lies in (a, b) and other in (c, d) .

185. $\cos^4 x - \cos^2 x + 1 - p = 0$; $0 \leq \cos^2 x \leq 1$,
The roots of $y^2 - y + 1 - p = 0$ lie between 0 and 1.

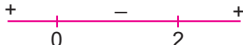
$$\begin{aligned} \therefore \quad & \alpha \geq 0, \beta \geq 0 \\ & \alpha - 1 \leq 0 \\ & \beta - 1 \leq 0 \\ \text{and} \quad & D \geq 0 \\ \Rightarrow \quad & \alpha + \beta \geq 0 \\ & \alpha\beta \geq 0, \alpha + \beta - 2 \leq 0, \\ & \alpha\beta - (\alpha + \beta) + 1 \geq 0 \\ \text{and} \quad & D \geq 0 \\ \therefore \quad & 1 \geq 0, 1 - p \geq 0, \\ & 1 - 2 \leq 0, 1 - p - 1 + 1 \geq 0 \\ \text{and} \quad & 1 - 4(1 - p) \geq 0 \\ \Rightarrow \quad & \frac{3}{4} \leq p \leq 1 \end{aligned}$$

186. Since, both roots of $f(x) = 0$ exceed a . Therefore,

$$\begin{aligned} \text{Discriminant} &> 0 && \dots(i) \\ f(a) &> 0 && \dots(ii) \\ -\frac{1}{2} &> a && \dots(iii) \end{aligned}$$

On solving Eqs. (i), (ii) and (iii), we get
 $a < -2$

187. $f(0) \cdot f(1) > 0$

$$\begin{aligned} & (-1)(1 - a^2 + 2a - 1) > 0 \\ \Rightarrow \quad & a^2 - 2a > 0 \\ \Rightarrow \quad & a(a - 2) > 0 \\ \Rightarrow \quad & a < 0 \\ \text{or} \quad & a > 2 \end{aligned}$$


188. Since, a lies between the roots of the given equation.

$$\begin{aligned} \therefore \quad & 2f(a) < 0 \\ \Rightarrow \quad & f(a) < 0 \\ \text{Here,} \quad & f(x) = 2x^2 - 2(2a + 1)x + a(a - 1) \\ \therefore \quad & f(a) = 2a^2 - 2(2a + 1)a + a(a - 1) < 0 \\ \Rightarrow \quad & -a^2 - 3a < 0 \\ \Rightarrow \quad & a(a + 3) > 0 \\ \Rightarrow \quad & a \in (-\infty, -3) \cup (0, \infty) \end{aligned}$$

189. Since, $\alpha < -1$ and $\beta > 1$

$$\begin{aligned} \therefore \quad & \alpha + \lambda = -1 \text{ and } \beta = 1 + \mu, \text{ where } \lambda, \mu > 0 \\ \text{Now, } 1 + \frac{c}{a} + \left| \frac{b}{a} \right| &= 1 + \alpha\beta + |\alpha + \beta| \\ &= 1 + (-1 - \lambda)(1 + \mu) + |-1 - \lambda + 1 + \mu| \\ &= 1 - 1 - \mu - \lambda - \lambda\mu + |\mu - \lambda| \\ &= \begin{cases} -\mu - \lambda - \lambda\mu + \mu - \lambda, & \text{if } \mu > \lambda \\ -\mu - \lambda - \lambda\mu + \lambda - \mu, & \text{if } \lambda > \mu \end{cases} \\ \therefore \quad 1 + \frac{c}{a} + \left| \frac{b}{a} \right| &= -2\lambda - \lambda\mu \text{ or } -2\mu - \lambda\mu \\ \text{In both cases, } 1 + \frac{c}{a} + \left| \frac{b}{a} \right| &< 0 \quad [\because \lambda, \mu > 0] \end{aligned}$$

190. Let $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx + d$
where, $f(0) = f(1) = d$
 $\therefore f'(x)$ has atleast one root $[0, 1]$
or $ax^2 + bx + c$ has atleast one root $[0, 1]$.

191. Since, $f(1) = -f(2)$,

i.e. $f(1) \cdot f(2) < 0$,
Hence, the equation must have a root $x \in (1, 2)$.
 $\therefore$ Two distinct real roots.

192. $6x^2 - xy - y^2 - 6x + 8y - 12 = 6(x + \lambda y - 2)(x - \mu y + 1)$
 $\Rightarrow \quad \lambda = \frac{1}{3}, \mu = \frac{1}{2}$

193. The given equation can be written as

$$\begin{aligned} y^2 + y(2x + m) + (2x - 3) &= 0 \\ \text{Here,} \quad D &= (2x + m)^2 - 4(2x - 3) \\ &= 4x^2 + 4xm - 8x + m^2 + 12 \\ &= 4x^2 + 4x(m - 2) + (m^2 + 12) \\ &= [4x^2 + 4x(m - 2) + (m - 2)^2] + (m^2 + 12) - (m - 2)^2 \\ &= [2x + (m - 2)]^2 + [12 + 4m - 4] \end{aligned}$$

For rational factors, D should be a perfect square.

$$\therefore \quad 8 + 4m = 0 \Rightarrow m = -2$$

Aliter

$$\begin{aligned} \text{Use } abc + 2fgh - af^2 - bg^2 - ch^2 &= 0 \\ \Rightarrow \quad m &= -2 \end{aligned}$$

194. The given expression

$$\begin{aligned} &= z^2 \left[a \left(\frac{x}{z} \right)^2 + b \left(\frac{y}{z} \right)^2 + c + 2a \left(\frac{y}{z} \right) + 2b \left(\frac{x}{z} \right) + 2c \left(\frac{x}{z} \right) \left(\frac{y}{z} \right) \right] \\ &= z^2 [aX^2 + bY^2 + c + 2aY + 2bX + 2cXY] \quad \dots(i) \end{aligned}$$

$$\text{where,} \quad \frac{x}{z} = X \quad \text{and} \quad \frac{y}{z} = Y$$

The given expression can be resolved into rational factors when the expression under the brackets given in Eq. (i) can be resolved into rational factors.

The condition for this is

$$\begin{aligned} abc + 2 \cdot a \cdot b \cdot c - a \cdot a^2 - b \cdot b^2 - c \cdot c^2 &= 0 \\ [\because \text{here, } f = a, g = b, h = c] \\ \Rightarrow \quad a^3 + b^3 + c^3 &= 3abc \end{aligned}$$

195. Given equation is

$$\begin{aligned} x^2 + 9y^2 - 4x + 3 &= 0 \\ \text{or} \quad x^2 - 4x + 9y^2 + 3 &= 0 \quad \dots(i) \end{aligned}$$

Since, x is real.

$$\begin{aligned} \therefore \quad (-4)^2 - 4(9y^2 + 3) &\geq 0 \\ \Rightarrow \quad 16 - 4(9y^2 + 3) &\geq 0 \\ \Rightarrow \quad 4 - 9y^2 - 3 &\geq 0 \\ \Rightarrow \quad 9y^2 - 1 &\leq 0 \\ \Rightarrow \quad (3y - 1)(3y + 1) &\leq 0 \\ \Rightarrow \quad -\frac{1}{3} &\leq y \leq \frac{1}{3} \end{aligned}$$

Eq. (i) can also be written as

$$9y^2 + 0y + x^2 - 4x + 3 = 0$$

Since, y is real.

$$\begin{aligned} \therefore \quad 0^2 - 4 \times 9(x^2 - 4x + 3) &\geq 0 \\ \Rightarrow \quad x^2 - 4x + 3 &\leq 0 \\ \Rightarrow \quad (x - 1)(x - 3) &\leq 0 \\ \Rightarrow \quad 1 &\leq x \leq 3 \end{aligned}$$

196. If $\left[m_r, \left(\frac{1}{m_r}\right)\right]$ satisfy the given equation

$x^2 + y^2 + 2gx + 2fy + c = 0$, then

$$m_r^2 + \frac{1}{m_r^2} + 2gm_r + \frac{2f}{m_r} + c = 0$$

$$\Rightarrow m_r^4 + 2gm_r^3 + cm_r^2 + 2fm_r + 1 = 0$$

Now, roots of given equation are m_1, m_2, m_3 and m_4 .

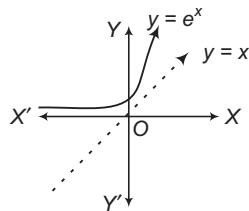
The product of roots,

$$m_1 m_2 m_3 m_4 = \frac{\text{Constant term}}{\text{Coefficient of } m_r^4} = \frac{1}{1} = 1$$

197. Let $f(x) = 0$, where $f(x) = x^2 - 3x + k$ has two roots α and β in $(0, 1)$ for some value of k , then by Rolle's theorem $f'(x) = 2x - 3 = 0$ has a root in $(0, 1)$ [$\because \alpha, \beta \in (0, 1)$] but $f'(x) = 0$ gives $x = \frac{3}{2} \notin (0, 1)$, which is contradiction.

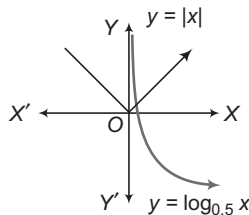
Hence, $f(x) = 0$ has no root in $(0, 1)$ for any value of k .

198.



$\therefore$ No solution.

199.



$\therefore$ Only one solution.

200. We have, $(\log_5 x)^2 + \log_5 x < 2$

Put $\log_5 x = a$, then $a^2 + a < 2$

$$\Rightarrow a^2 + a - 2 < 0$$

$$\Rightarrow (a + 2)(a - 1) < 0$$

$$\Rightarrow -2 < a < 1 \quad \text{or} \quad -2 < \log_5 x < 1$$

$$\therefore 5^{-2} < x < 5 \quad \text{i.e.} \quad \frac{1}{25} < x < 5$$

$$201. a - \sqrt{b} = \frac{(a - \sqrt{b})(a + \sqrt{b})}{a + \sqrt{b}} = \frac{a^2 - b}{a + \sqrt{b}} = \frac{1}{a + \sqrt{b}}$$

$\therefore$ Given equation reduces to

$$(a + \sqrt{b})^{x^2 - 15} + \frac{1}{(a + \sqrt{b})^{x^2 - 15}} = 2a$$

$$\Rightarrow y + \frac{1}{y} = 2a$$

$$\text{where,} \quad y = (a + \sqrt{b})^{x^2 - 15}$$

$$\Rightarrow y^2 - 2ay + 1 = 0$$

$$\Rightarrow y = \frac{2a \pm \sqrt{4a^2 - 4}}{2} = a \pm \sqrt{a^2 - 1}$$

$$\Rightarrow y = a \pm \sqrt{b} \quad [\because a^2 - b = 1]$$

$$\text{Case I } y = a + \sqrt{b} \Rightarrow (a + \sqrt{b})^{x^2 - 15} = a + \sqrt{b}$$

$$\Rightarrow x^2 - 15 = 1 \Rightarrow x = \pm 4$$

$$\text{Case II } y = a - \sqrt{b} \Rightarrow y = \frac{1}{a + \sqrt{b}}$$

$$\Rightarrow y = (a + \sqrt{b})^{-1}$$

$$\Rightarrow (a + \sqrt{b})^{x^2 - 15} = (a + \sqrt{b})^{-1}$$

$$\Rightarrow x^2 - 15 = -1$$

$$\Rightarrow x = \pm \sqrt{14}$$

202. The roots of $ax^2 + bx + c = 0$ are imaginary.

$$\therefore b^2 - 4ac < 0$$

$$\text{Roots are } \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i$$

$$\text{Let } \alpha = -\frac{b}{2a} + \frac{\sqrt{4ac - b^2}}{2a} i$$

$$\text{and } \beta = -\frac{b}{2a} - \frac{\sqrt{4ac - b^2}}{2a} i$$

$$\therefore |\alpha| = |\beta| \quad [\because \bar{\alpha} = \beta]$$

$$\text{Also, } |\alpha| = \sqrt{\frac{b^2}{4a^2} + \frac{4ac - b^2}{4a^2}} = \sqrt{\frac{c}{a}} > 1$$

$$[\because 0 < a < b < c]$$

$$|\alpha| = |\beta| > 1$$

$\therefore$ Option (c) is false.

203. We have, $\tan \alpha + \tan \beta = -p$ and $\tan \alpha \tan \beta = q$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{-p}{1 - q} = \frac{p}{q - 1}$$

$\therefore$ Option (b) is true.

It can be easily checked that options (c) and (d) are false. LHS of expression in option (a)

$$= \cos^2(\alpha + \beta) [\tan^2(\alpha + \beta) + p \tan(\alpha + \beta) + q]$$

$$= \frac{1}{1 + \left(\frac{p}{q-1}\right)^2} \left[\left(\frac{p}{q-1}\right)^2 + p \left(\frac{p}{q-1}\right) + q \right]$$

$$= \frac{1}{(q-1)^2 + p^2} [p^2 + p^2(q-1)^2 + q(q-1)^2] = q$$

204. We have, $(ax^2 + c)y + (a'x^2 + c') = 0$

$$\therefore (ay + a')x^2 + (cy + c') = 0$$

Since, x is a rational function, the discriminant must be a perfect square.

$$\text{Discriminant} = 0 - 4(ay + a')(cy + c')$$

$$= -4[acy^2 + (ac' + a'c)y + a'c']$$

$\therefore$ Roots of $acy^2 + (ac' + a'c)y + a'c' = 0$ must be equal.

$$\therefore \text{Discriminant} = 0, \text{ i.e. } (ac' + a'c)^2 - 4ac \cdot a'c' = 0$$

$$\Rightarrow (ac' - a'c)^2 = 0$$

$$\Rightarrow ac' - a'c = 0$$

$$\Rightarrow ac' = a'c$$

$$\Rightarrow \frac{a}{a'} = \frac{c}{c'} \Rightarrow \frac{a}{c} = \frac{a'}{c'}$$

205. We have, $\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = a(a-1)$

$$\Rightarrow \left(\frac{x}{x+1} + \frac{x}{x-1}\right)^2 - 2\left(\frac{x}{x+1}\right)\left(\frac{x}{x-1}\right) = a(a-1)$$

$$\Rightarrow \left(\frac{2x^2}{x^2-1}\right)^2 - \frac{2x^2}{x^2-1} = a(a-1)$$

$$\Rightarrow z^2 - z - a(a-1) = 0, \text{ where } z = \frac{2x^2}{x^2-1}$$

$$\Rightarrow z = a \text{ or } z = a-1$$

$$z = a \Rightarrow \frac{2x^2}{x^2-1} = a$$

$$\Rightarrow 2x^2 = ax^2 - a$$

$$\Rightarrow x = \pm \sqrt{\frac{a}{a-2}}$$

$$z = a-1 \Rightarrow \frac{2x^2}{x^2-1} = a-1$$

$$\Rightarrow 2x^2 = (1-a)x^2 - 1 + a$$

$$\Rightarrow x = \pm \sqrt{\frac{a-1}{a+1}}$$

$$\therefore x = \pm \sqrt{\frac{a}{a-2}}, \pm \sqrt{\frac{a-1}{a+1}}$$

$$\Rightarrow a < -1 \Rightarrow \text{All roots are real}$$

$$1 < a < 2 \Rightarrow x = \pm \sqrt{\frac{a}{2-a}}, \pm \sqrt{\frac{a-1}{a+1}}$$

$$\Rightarrow \text{Only two roots are real.}$$

$$\Rightarrow a > 2 \Rightarrow \text{All roots are real.}$$

206. Symmetric functions are those which do not change by interchanging α and β .

207. $\sec^2 \theta + \operatorname{cosec}^2 \theta = \sec^2 \theta \cdot \operatorname{cosec}^2 \theta$

Sum of the roots is equal to their product and the roots are real.

$$\therefore -\frac{b}{a} = \frac{c}{a} \Rightarrow b + c = 0$$

Also, $b^2 - 4ac \geq 0$

$$\Rightarrow c^2 - 4ac \geq 0$$

$$\Rightarrow c(c - 4a) \geq 0$$

$$\Rightarrow c - 4a \geq 0 \quad [\because c > 0]$$

Further $b^2 + 4ab \geq 0$

$$\Rightarrow b + 4a \leq 0 \quad [\because b < 0]$$

208. Let the roots be $a/r, a, ar$, where $a > 0, r > 1$.

Now, $a/r + a + ar = -p$... (i)

$$a(a/r) + a(ar)(a/r) = q$$
 ... (ii)

$$(a/r)(a)(ar) = 1$$
 ... (iii)

$$\Rightarrow a^3 = 1 \Rightarrow a = 1$$

Hence, option (c) is correct.

From Eq. (i), putting $a = 1$, we get

$$-p - 3 > 0 \quad \left[\because r + \frac{1}{r} > 2 \right]$$

$$\Rightarrow p < -3$$

Hence, option (b) is not correct.

Also, $1/r + 1 + r = -p$... (iv)

From Eq. (ii), putting $a = 1$, we get

$$1/r + r + 1 = q \quad \dots (v)$$

From Eqs. (iv) and (v), we have

$$-p = q \Rightarrow p + q = 0$$

Hence, option (a) is correct.

Now, as $r > 1$

$$a/r = 1/r < 1$$

and

$$ar = r > 1$$

Hence, option (d) is correct.

209. Since α, β, γ and δ are in HP. So, $1/\alpha, 1/\beta, 1/\gamma$ and $1/\delta$

are in AP and they may be taken as $a - 3d, a - d, a + d, a + 3d$. Replacing x by $1/x$, we get the equation whose roots are $1/\alpha, 1/\beta, 1/\gamma, 1/\delta$.

Therefore, equation $x^2 - 4x + A = 0$ has roots $a - 3d, a + d$ and equation $x^2 - 6x + B = 0$ has roots $a - d, a + 3d$. Sum of the roots is

$$2(a - d) = 4 \text{ and } 2(a + d) = 6$$

$$\therefore a = 5/2, d = 1/2$$

Product of the roots is

$$(a - 3d)(a + d) = A = 3$$

$$(a - d)(a + 3d) = B = 8$$

210. $(1 + k) \tan^2 x - 4 \tan x - 1 + k = 0$... (i)

Since, roots are real, we have

$$(-4)^2 - 4(1 + k)(-1 + k) \geq 0$$

$$\Rightarrow 16 - 4(k^2 - 1) \geq 0$$

$$\Rightarrow k^2 \leq 5$$

$$\text{We have, } \tan x_1 + \tan x_2 = -\left(\frac{-4}{1+k}\right) = \frac{4}{1+k}$$

$$\text{and } \tan x_1 \cdot \tan x_2 = \frac{-1+k}{1+k}$$

$$\therefore \tan(x_1 + x_2) = \frac{\frac{4}{1+k}}{1 - \left(\frac{-1+k}{1+k}\right)} = \frac{4}{2} = 2$$

For $k = 2$, from Eq. (i),

$$3 \tan^2 x - 4 \tan x + 1 = 0$$

$$\Rightarrow \tan x = 1, \frac{1}{3}$$

$$\therefore x_1 = \frac{\pi}{4}, x_2 = \tan^{-1} \frac{1}{3}$$

For $k = 1$, from Eq. (i), $2 \tan^2 x - 4 \tan x = 0$

$$\Rightarrow \tan x = 0, 2 \Rightarrow x_1 = 0, x_2 = \tan^{-1} 2$$

211. We have, $u + v = -p$ and $uv = q$

$$(a) \frac{1}{u} + \frac{1}{v} = \frac{u+v}{uv} = \frac{-p}{q} \text{ and } \frac{1}{u} \cdot \frac{1}{v} = \frac{1}{uv} = \frac{1}{q}$$

$\therefore$ The required equation is

$$x^2 - \left(-\frac{p}{q}\right)x + \frac{1}{q} = 0$$

$$\Rightarrow qx^2 + px + 1 = 0$$

(b) $(u + v) + uv = -p + q$ and $(u + v)uv = -pq$

$\therefore$ The required equation is

$$x^2 - (-p + q)x + (-pq) = 0$$

$$\Rightarrow x^2 + (p - q)x - pq = 0$$

$$\Rightarrow (x + p)(x - q) = 0$$

$$(c) u^2 + v^2 = (u + v)^2 - 2uv = p^2 - 2q$$

$$\text{and } u^2 v^2 = (uv)^2 = q^2$$

$$\therefore \text{The required equation is}$$

$$x^2 - (p^2 - 2q)x + q^2 = 0$$

$$(d) \frac{u}{v} + \frac{v}{u} = \frac{u^2 + v^2}{uv} = \frac{p^2 - 2q}{q} \quad \text{and} \quad \frac{u}{v} \cdot \frac{v}{u} = 1$$

$$\therefore \text{The required equation is}$$

$$x^2 - \left(\frac{p^2 - 2q}{q} \right) x + 1 = 0$$

$$\Rightarrow qx^2 - (p^2 - 2q)x + q = 0$$

$$212. \frac{(x-a)(x-b)}{x-c} = k$$

$$\Rightarrow (x-a)(x-b) - k(x-c) = 0$$

$$\Rightarrow x^2 - (a+b+k)x + ab + kc = 0$$

$$\Rightarrow \{-(a+b+k)\}^2 - 4 \cdot 1 \cdot (ab + kc) \geq 0 \quad [\because x \text{ is real}]$$

$$\Rightarrow k^2 + 2(a+b-2c) + (a-b)^2 \geq 0$$

This is true for all real k .

$$\therefore \text{Discriminant} = 4(a+b-2c)^2 - 4 \cdot 1 \cdot (a-b)^2 \leq 0$$

$$\Rightarrow (a+b-2c+a-b)(a+b-2c-a+b) \leq 0$$

$$\Rightarrow 2(a-c)2(b-c) \leq 0$$

$$\Rightarrow (a-c)(b-c) \leq 0$$

$$\Rightarrow (c-a)(c-b) \leq 0$$

$$\Rightarrow a \leq c \leq b \quad \text{or} \quad b \leq c \leq a$$

213. Since, $P(x)$ divides both of them. So, $P(x)$ also divides

$$(3x^4 + 4x^2 + 28x + 5) - 3(x^4 + 6x^2 + 25)$$

$$= -14x^2 + 28x - 70$$

$$= -14(x^2 - 2x + 5)$$

which is a quadratic.

$$\text{Hence, } P(x) = x^2 - 2x + 5$$

$$\Rightarrow P(1) = 4$$

214. Roots of $4x^2 - x - 1 = 0$ are irrational. So, one root common implies both roots are common.

$$\therefore \frac{4}{3} = \frac{-1}{\lambda + \mu} = \frac{-1}{\lambda - \mu} \Rightarrow \lambda = \frac{-3}{4}, \mu = 0$$

$$215. 2x^2 + 6xy + 5y^2 = 1 \quad \dots(i)$$

Eq. (i) can be rewritten as

$$2x^2 + (6y)x + 5y^2 - 1 = 0$$

Since, x is real.

$$36y^2 - 8(5y^2 - 1) \geq 0$$

$$\Rightarrow y^2 \leq 2$$

$$\Rightarrow -\sqrt{2} \leq y \leq \sqrt{2}$$

Eq. (i) can also be rewritten as

$$5y^2 + (6x)y + 2x^2 - 1 = 0$$

Since, y is real.

$$\therefore 36x^2 - 20(2x^2 - 1) \geq 0$$

$$\Rightarrow 36x^2 - 40x^2 + 20 \geq 0$$

$$\Rightarrow -4x^2 \geq -20$$

$$\Rightarrow x^2 \leq 5$$

$$\Rightarrow -\sqrt{5} \leq x \leq \sqrt{5}$$

$$216. \cos x - y^2 - \sqrt{y - x^2 - 1} \geq 0 \quad \dots(i)$$

Now, $\sqrt{y - x^2 - 1}$ is defined when $y - x^2 - 1 \geq 0$ or $y \geq x^2 + 1$. So, minimum value of y is 1.

From Eq. (i),

$$\cos x - y^2 \geq \sqrt{y - x^2 - 1}$$

where, $\cos x - y^2 \leq 0$ [as when $\cos x$ is maximum (= 1) and y^2 is minimum (= 1), so $\cos x - y^2$ is maximum].

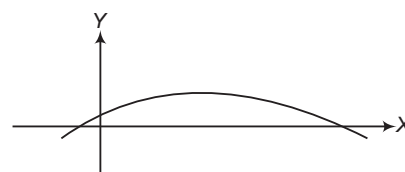
$$\text{Also, } \sqrt{y - x^2 - 1} \geq 0$$

$$\text{Hence, } \cos x - y^2 = \sqrt{y - x^2 - 1} = 0$$

$$\Rightarrow y = 1 \quad \text{and} \quad \cos x = 1, y = x^2 + 1$$

$$\Rightarrow x = 0, y = 1$$

217.



From the graph,

$$f(0) = c > 0 \quad \dots(i)$$

Also, the graph is concave downward.

$$\therefore a < 0 \quad \dots(ii)$$

$$\text{Further, abscissa of the vertex} = \frac{b}{2a} \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$ac < 0, ab < 0 \text{ and } bc > 0$$

218. For $ax^2 + bx + c = 0$ and $a_1x^2 + b_1x + c_1 = 0$ have a common root

$$(a_1c - ac_1)^2 = 4(ab_1 - ba_1)(bc_1 - b_1c) \quad \dots(i)$$

If $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in AP.

$$\frac{ba_1 - ab_1}{a_1b_1} = \frac{cb_1 - c_1b}{b_1c_1} = k$$

$$\text{and } \frac{cb_1 - ac_1}{a_1c_1} = 2k$$

On putting these values in Eq. (i), we have

$$c_1a_1 = b_1^2$$

219. **Statement I** Given equation is $x^2 - bx + c = 0$

Let α and β be two roots such that

$$|\alpha - \beta| = 1$$

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$$

$$\Rightarrow b^2 - 4c = 1$$

Statement II Given equation is

$$4abcx^2 + (b^2 - 4ac)x - b = 0.$$

$$\therefore D = (b^2 - 4ac)^2 + 16ab^2c$$

$$= (b^2 + 4ac)^2 > 0$$

Hence, roots are real and unequal.

Solutions (Q.Nos. 220-222)

Let the other roots of $f\{f(x)\} = x$ be λ and δ .

$$\Rightarrow f\{f(\lambda)\} = \lambda$$

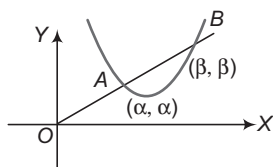
$$\text{Let } f(\lambda) = \gamma$$

$$\Rightarrow f(\gamma) = \gamma$$

Other roots γ and δ lie on the line

$$y = -x + c$$

There must be two points C and D on the parabola $y = ax^2 + bx + c$ which are image of each other in the line $y = x$



$\Rightarrow$ If α and β are real so are λ and δ .

If $\alpha + \beta = \lambda + \delta \Rightarrow$ middle points of AB and CD become same.

This is not possible $\Rightarrow \alpha, \beta$ and λ, δ cannot be real.

Also, if α and β are equal to then λ, δ cannot be real.

223. $f''(c_2)f''(c_1) < 0$ and $f'(c_1) = f'(c_2) = 0$

$$\Rightarrow f''(c_1) - f''(c_2) > 0$$

$$\Rightarrow f''(c_1) > 0$$

$$\text{and } f''(c_2) > 0$$

$\Rightarrow c_2$ is local maximum and c_1 is local minimum for $f(x)$

$\Rightarrow f'(x) = 0$ has atleast four times in $[c_1 - 1, c_2 + 1]$.

224. Here, c_1 is local maximum and c_2 is local minimum.

$\Rightarrow f'(x) = 0$ has atleast two roots in $[c_1 - 1, c_2 + 1]$.

225. As c_2 is local maximum and c_1 is local minimum.

$\Rightarrow f(x) = 0$ has atleast two solutions.

Solutions (Q. Nos. 226-228)

$$(\beta - \alpha) = (\beta + \delta) - (\alpha + \delta)$$

$$(\beta + \alpha)^2 - 4\alpha\beta = [(\beta + \delta) + (\alpha + \delta)]^2 - 4(\beta + \delta)(\alpha + \delta)$$

$$(-b_1)^2 - 4c_1 = (-b_2)^2 - 4c_2$$

$$D_1 = D_2$$

Least value of $f(x)$ is

$$-\frac{D_1}{4} = -\frac{1}{4} \Rightarrow D_1 = 1 \Rightarrow D_2 = 1$$

$\therefore$ Least value of $g(x)$ is

$$-\frac{D_2}{4} = -\frac{1}{4}$$

Least value of $g(x)$ occurs at

$$x = -\frac{b_2}{2} = \frac{7}{2} \Rightarrow b_2 = -7$$

$$b_2^2 - 4c_2 = D_2 \Rightarrow 49 - 4c_2 = 1$$

$$\Rightarrow \frac{48}{4} = c_2 \Rightarrow c_2 = 12$$

$$x^2 - 7x + 12 = 0 \Rightarrow x = 3, 4$$

Solutions (Q. Nos. 229-231)

$$\text{Let } x^4 + ax^3 + bx^2 + cx + d$$

$$= (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

$$\text{Let } (x - x_1)(x - x_2) = x^2 + px + q$$

$$\text{and } (x - x_3)(x - x_4) = x^2 + px + r$$

$$\therefore q = x_1x_2 \text{ and } r = x_3x_4$$

$$x^4 + ax^3 + bx^2 + cx + d$$

$$= x^4 + 2px^3 + (p^2 + q + r)x^2 + p(q + r)x + qr$$

$$\therefore a = 2p; b = p^2 + q + r; c = p(q + r); d = qr$$

$$\text{Clearly, } a^3 - 4ab + 8c = 0 \quad \dots(i)$$

229. If $a = 2$, then

$$b - c = 1$$

230. Investigating the nature of the cubic equation of a

$$\text{let } f(a) = a^3 - 4ab + 8c$$

$$f'(a) = 3a^2 - 4b$$

$$\text{If } b < 0, \text{ then } f'(a) > 0$$

$\therefore$ The equation $a^3 - 4ab + 8c = 0$ has only one real root.

231. On substituting $c = 1 - b$ in Eq. (i), we get

$$(a + 2)[(a - 1)^2 + 3 - 4b] = 0$$

$$\Rightarrow 4b - 3 > 0$$

$$\Rightarrow b > \frac{3}{4}$$

$$\Rightarrow c < \frac{1}{4}$$

232. A. Roots of the given equation will be opposite sign, if

$$D \geq 0 \text{ and product of roots } < 0.$$

$$(a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0$$

$$\text{and } \frac{a^2 - 4a}{2} < 0$$

$$\Rightarrow a^2 - 4a < 0$$

$$\left[\begin{array}{l} \therefore a^2 - 4a < 0 \\ \Rightarrow (a^3 + 8a - 1)^2 - 8(a^2 - 4a) \geq 0 \end{array} \right]$$

$$\Rightarrow 0 < a < 4$$

B. Equation has negative roots of discriminant ≥ 0 ,

$$\alpha < 0, \beta < 0$$

$$\text{i.e. } \alpha + \beta < 0 \text{ and } f(0) > 0$$

$$\therefore 4(a + 1)^2 - 36a + 20 > 0$$

$$\Rightarrow a^2 - 7a + 6 \geq 0$$

$$\Rightarrow a \leq 1 \text{ or } a \geq 6 \quad \dots(i)$$

$$\alpha + \beta < 0$$

$$\Rightarrow a + 1 > 0$$

$$\Rightarrow a > -1 \quad \dots(ii)$$

$$\text{and } f(0) > 0$$

$$9a - 5 > 0$$

$$\Rightarrow a > \frac{5}{9} \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$a \geq 6$$

C. Given, $x^2 - 2(4a - 1)x + 15a^2 - 2a - 7 > 0$

$$\therefore \text{Discriminant} < 0 \quad [\because \text{coefficient of } x^2 > 0]$$

$$\Rightarrow 4(4a - 1)^2 - 4(15a^2 - 2a - 7) < 0$$

$$\Rightarrow a^2 - 6a + 8 < 0$$

$$\Rightarrow 2 < a < 4$$

$$\text{D. Let } y = \frac{x^2 + 2x + a}{x^2 + 4x + 3a}$$

$$\Rightarrow x^2(y - 1) + 2(2y - 1)x + a(3y - 1) = 0$$

$$\text{As } x \in R, D \geq 0$$

$$\therefore 4(2y - 1)^2 - 4(y - 1)a(3y - 1) \geq 0$$

$$\Rightarrow (4 - 3a)y^2 - 4(1 - a)y + 1 - a \geq 0 \Rightarrow a < \frac{4}{3}$$

$$\text{and } a(a - 1) \leq 0$$

$$\therefore 0 \leq a \leq 1$$

233. A. Since, $f(x) = x^2 + 2(k+1)x + (9k-5)$ has both negative roots.

(i) Discriminant ≥ 0 (ii) $\alpha < 0, \beta < 0$ (iii) $f(0) > 0$

i.e. $\alpha + \beta < 0$

$\therefore$ Discriminant ≥ 0

$$\Rightarrow 4(k+1)^2 - 36k + 20 > 0$$

$$\Rightarrow k^2 - 7k + 6 \geq 0$$

$$\Rightarrow (k-1)(k-6) \geq 0$$

$$\Rightarrow k \leq 1 \text{ or } k \geq 6 \quad \dots(i)$$

$$\alpha + \beta < 0$$

$$\Rightarrow -2(k+1) < 0$$

$$\Rightarrow k+1 > 0$$

$$\Rightarrow k > -1 \quad \dots(ii)$$

and $f(0) > 0$

$$\Rightarrow 9k - 5 > 0$$

$$\Rightarrow k > \frac{5}{9} \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get
 $k \geq 6$

B. $f(x) > 0$

$\Rightarrow$ Discriminant < 0

$$\Rightarrow 4(4k-1)^2 - 4(15k^2 - 2k - 7) < 0$$

$$\Rightarrow k^2 - 6k + 8 < 0$$

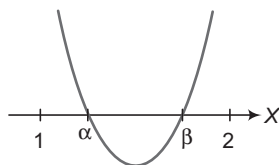
$$\Rightarrow 2 < k < 4$$

C. $f(x) = x^2 - 2(k-1)x + (2k+1)$ has both roots positive, if

(i) Discriminant > 0

(ii) Sum of the roots > 0

(iii) $f(0) > 0$



Now, discriminant ≥ 0

$$\Rightarrow 4(k-1)^2 - 4(2k+1) \geq 0$$

$$\Rightarrow k^2 - 4k \geq 0$$

$$\Rightarrow k \leq 0 \text{ and } k \geq 4 \quad \dots(i)$$

Sum of roots > 0

$$\Rightarrow 2(k-1) > 0$$

$$\Rightarrow k > 1$$

and $f(0) > 0$

$$\Rightarrow (2k+1) > 0$$

$$\Rightarrow k > -\frac{1}{2} \quad \dots(ii)$$

From Eqs. (i), (ii) and (iii), we get
 $k \geq 4$

D. Since, k lies between the roots of

$$f(x) = 2x^2 - 2(2k+1)x + k(k+1)$$

$\Rightarrow$ Discriminant > 0 and $f(k) < 0$

$$\Rightarrow 4(2k+1)^2 - 8k(k+1) > 0$$

$$\text{and } 2k^2 - 2k(2k+1) + k(k+1) < 0$$

$$\Rightarrow 8\left(k^2 + k + \frac{1}{2}\right) \geq 0$$

$$\text{and } -k^2 - k < 0$$

This is always true and $k > 0$ or $k < -1$.

$$E. y = -x^2 + 3x - 2$$

$$= \frac{9}{4} - \left(x^2 - 3x + \frac{9}{4}\right) - 2$$

$$y = \frac{1}{4} - \left(\frac{x-3}{2}\right)^2 \leq \frac{1}{4}$$

234. Let $f(x) = x^3 - x^2 + \beta x + \gamma \quad \dots(i)$

$f(x) = 0$ has three positive real roots in GP.

$\Rightarrow f'(x) = 0$ will have two distinct real roots.

$\Rightarrow 3x^2 - 2x + \beta = 0$ has two distinct roots.

$$\therefore D > 0$$

$$4 - 12\beta > 0 \Rightarrow \beta < \frac{1}{3} \quad \dots(ii)$$

Also, from Eq. (i),

$$x_1x_2 + x_2x_3 + x_3x_1 = \beta$$

$$x_1 + x_2 + x_3 = 1$$

$$x_1x_2x_3 = -\gamma$$

$$\Rightarrow x_2^3 = -r > 0 \quad [\because x_2^2 = x_1x_3]$$

$$\Rightarrow r > 0$$

$$\text{Also, } x_2(x_1 + x_3) + x_2^2 = \beta$$

$$\Rightarrow x_2(1 - x_2) + x_2^2 = \beta$$

$$\Rightarrow x_2 = \beta > 0$$

From Eqs. (ii) and (iv), we get $0 < \beta < \frac{1}{3}$ and $\gamma < 0$

$$[\beta]_{\max} = 0, [\gamma]_{\max} = -1$$

$$\Rightarrow [\beta] + [\gamma] + 2 = [0] + [-1] + 2 = 1$$

235. $x_1 + x_2 + x_1x_2 = a \quad \dots(i)$

$$x_1x_2 + x_1^2x_2 + x_1x_2^2 = b \quad \dots(ii)$$

$$x_1^2x_2^2 = c \quad \dots(iii)$$

From Eq. (ii), we get

$$x_1x_2(1 + x_1 + x_2) = b$$

$$x_1x_2(1 + a - x_1x_2) = b$$

$$x_1x_2(1 + a) - c = b$$

$$x_1x_2 = \frac{b+c}{a+1}, \text{ which is clearly a rational number.}$$

$$\therefore b + c = 2(a+1)$$

$$\therefore x_1x_2 = 2$$

236. Here, $\left. \begin{matrix} \alpha + \beta = -\rho \\ \gamma + \delta = -\rho \end{matrix} \right\} \Rightarrow \alpha + \beta = \gamma + \delta$

$$\text{Now, } (\alpha - \gamma)(\alpha - \delta) = \alpha^2 - \alpha(\gamma + \delta) + \gamma\delta$$

$$= \alpha^2 - \alpha(\alpha + \beta) + r$$

$$= -\alpha\beta + r$$

$$= -(-q) + r = q + r$$

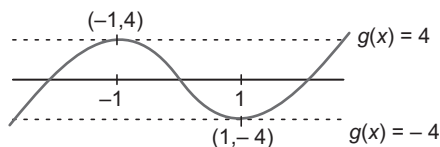
By symmetry of the results

$$(\beta - \gamma)(\beta - \delta) = q + r$$

Hence, the required ratio is 1.

Entrances Gallery

1. Let $y = x^5 - 5x$ and $g(x) = -a$. Then, it is clear from the graph that $f(x)$ has only real roots, if $a > 4$ or $a < -4$ and $f(x)$ has three real roots, if $-4 < a < 4$.



2. $P(x) = ax^2 + b$ with a, b of same sign.

$$P(P(x)) = a(ax^2 + b)^2 + b$$

If $x \in R$ or $ix \in R$

$$\Rightarrow x^2 \in R$$

$$\Rightarrow P(x) \in R$$

$$\Rightarrow P\{P(x)\} \neq 0$$

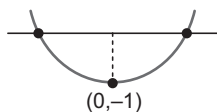
Hence, real or purely imaginary number cannot satisfy $P(P(x)) = 0$.

3. Let $f(x) = x^2 - x \sin x - \cos x \Rightarrow f'(x) = 2x - x \cos x$

$$\lim_{x \rightarrow \infty} f(x) \rightarrow \infty$$

$$\lim_{x \rightarrow -\infty} f(x) \rightarrow \infty$$

$$f(0) = -1$$



Hence, 2 solutions.

4. $\log_2 3^x = (x-1) \log_2 4 = 2(x-1)$

$$\Rightarrow x \log_2 3 = 2x - 2$$

$$\Rightarrow x = \frac{2}{2 - \log_2 3}$$

After rearranging, we get

$$x = \frac{2}{2 - \frac{1}{\log_3 2}} = \frac{2 \log_3 2}{2 \log_3 2 - 1}$$

Rearranging, again

$$x = \frac{\log_3 4}{\log_3 4 - 1} = \frac{\frac{1}{\log_4 3}}{\frac{1}{\log_4 3} - 1} = \frac{1}{1 - \log_4 3}$$

5. Let $1 + a = y$

$$\Rightarrow (y^{1/3} - 1)x^2 + (y^{1/2} - 1)x + y^{1/6} - 1 = 0$$

$$\Rightarrow \left(\frac{y^{1/3} - 1}{y - 1}\right)x^2 + \left(\frac{y^{1/2} - 1}{y - 1}\right)x + \frac{y^{1/6} - 1}{y - 1} = 0$$

Taking $\lim_{y \rightarrow 1}$ on both the sides, we get

$$\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{6} = 0$$

$$\Rightarrow 2x^2 + 3x + 1 = 0$$

$$\therefore x = -1, -\frac{1}{2}$$

6. Let $\sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \sqrt{4 - \frac{1}{3\sqrt{2}}} \dots = y$

$$\text{So, } 4 - \frac{1}{3\sqrt{2}} y = y^2 \quad [\because y > 0]$$

$$\Rightarrow y^2 + \frac{1}{3\sqrt{2}} y - 4 = 0 \Rightarrow y = \frac{8}{3\sqrt{2}}$$

$$\text{So, the required value is } 6 + \log_{3/2} \left(\frac{1}{3\sqrt{2}} \times \frac{8}{3\sqrt{2}} \right)$$

$$= 6 + \log_{3/2} \frac{4}{9} = 6 - 2 = 4$$

7. $(2x)^{\ln 2} = (3y)^{\ln 3}$... (i)

$$3^{\ln x} = 2^{\ln y} \quad \dots (ii)$$

$$\Rightarrow (\log x)(\log 3) = (\log y)(\log 2)$$

$$\Rightarrow \log y = \frac{(\log x)(\log 3)}{\log 2} \quad \dots (iii)$$

In Eq. (i), taking log on both sides, we get

$$(\log 2) \{ \log 2 + \log x \} = \log 3 \{ \log 3 + \log y \}$$

$$\Rightarrow (\log 2)^2 + (\log 2)(\log x) = (\log 3)^2 + \frac{(\log 3)^2 (\log x)}{\log 2}$$

[from Eq. (iii)]

$$\Rightarrow (\log 2)^2 - (\log 3)^2 = \frac{(\log 3)^2 - (\log 2)^2}{\log 2} (\log x)$$

$$\Rightarrow -\log 2 = \log x$$

$$\Rightarrow x = \frac{1}{2} \Rightarrow x_0 = \frac{1}{2}$$

8. $a_n = \alpha^n - \beta^n$

$$\alpha^2 - 6\alpha - 2 = 0$$

Multiply with α^8 on both sides, we get

$$\alpha^{10} - 6\alpha^9 - 2\alpha^8 = 0 \quad \dots (i)$$

$$\text{Similarly, } \beta^{10} - 6\beta^9 - 2\beta^8 = 0 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\alpha^{10} - \beta^{10} - 6(\alpha^9 - \beta^9) = 2(\alpha^8 - \beta^8)$$

$$\Rightarrow a_{10} - 6a_9 = 2a_8 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3$$

9. $x^2 + bx - 1 = 0$

$$\text{and } x^2 + x + b = 0 \quad \dots (i)$$

Common root is

$$(b-1)x - 1 - b = 0 \Rightarrow x = \frac{b+1}{b-1}$$

This value of x satisfies Eq. (i).

$$\therefore \frac{(b+1)^2}{(b-1)^2} + \frac{b+1}{b-1} + b = 0$$

$$\Rightarrow b = \sqrt{3}i, -\sqrt{3}i, 0$$

10. Since, $f\left(-\frac{1}{2}\right) \cdot f\left(-\frac{3}{4}\right) < 0$, so lies in $\left(-\frac{3}{4}, -\frac{1}{2}\right)$.

11. $\alpha^3 + \beta^3 = q$

$$\Rightarrow (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = q$$

$$\Rightarrow -p^3 + 3p\alpha\beta = q$$

$$\Rightarrow \alpha\beta = \frac{q + p^3}{3p}$$

$$\Rightarrow x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha} \right) x + \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 0$$

$$\Rightarrow x^2 - \frac{(\alpha^2 + \beta^2)}{\alpha\beta} x + 1 = 0$$

$$\Rightarrow x^2 - \left(\frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \right) x + 1 = 0$$

$$\Rightarrow x^2 - \left[\frac{p^2 - 2 \left(\frac{p^3 + q}{3p} \right)}{\frac{p^3 + q}{3p}} \right] x + 1 = 0$$

$$\Rightarrow (p^3 + q)x^2 - (3p^3 - 2p^3 - 2q)x + (p^3 + q) = 0$$

$$\Rightarrow (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$$

12. Given, α and β are the roots of the equation $x^2 - 6x - 2 = 0$.

$$\therefore a_n = \alpha^n - \beta^n \text{ for } n \geq 1$$

$$\therefore a_{10} = \alpha^{10} - \beta^{10}$$

$$a_8 = \alpha^8 - \beta^8 \Rightarrow a_9 = \alpha^9 - \beta^9$$

Now, consider

$$\frac{a_{10} - 2a_8}{2a_9} = \frac{\alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8)}{2(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)}{2(\alpha^9 - \beta^9)}$$

$$= \frac{\alpha^8 \cdot 6\alpha - \beta^8 \cdot 6\beta}{2(\alpha^9 - \beta^9)} \quad \left[\because \alpha \text{ and } \beta \text{ are the roots of } \begin{cases} x^2 - 6x - 2 = 0 \\ \text{So, } x^2 = 6x + 2 \\ \Rightarrow \alpha^2 = 6\alpha + 2 \\ \Rightarrow \alpha^2 - 2 = 6\alpha \\ \text{and } \beta^2 = 6\beta - 2 \\ \Rightarrow \beta^2 - 2 = 6\beta \end{cases} \right]$$

$$= \frac{6\alpha^9 - 6\beta^9}{2(\alpha^9 - \beta^9)} = \frac{6}{2} = 3$$

Aliter

Since, α and β are the roots of the equation

$$x^2 - 6x - 2 = 0$$

$$\Rightarrow x^2 = 6x + 2 \Rightarrow \alpha^2 = 6\alpha + 2$$

$$\Rightarrow \alpha^{10} = 6\alpha^9 + 2\alpha^8 \quad \dots (i)$$

$$\text{Similarly, } \beta^{10} = 6\beta^9 + 2\beta^8 \quad \dots (ii)$$

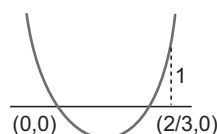
On subtracting Eq. (ii) from Eq. (i), we get

$$\alpha^{10} - \beta^{10} = 6(\alpha^9 - \beta^9) + 2(\alpha^8 - \beta^8)$$

$$\Rightarrow \alpha^{10} - \beta^{10} = 6a_9 + 2a_8 \quad [\because a_n = \alpha^n - \beta^n]$$

$$\Rightarrow a_{10} - 2a_8 = 6a_9 \Rightarrow \frac{a_{10} - 2a_8}{2a_9} = 3$$

13. $\therefore a^2 = 3t^2 - 2t$



For non-integral solution,

$$0 < a^2 < 1 \Rightarrow a \in (-1, 0) \cup (0, 1)$$

It is assumed that a real solution of given equation exists.

14. Given equations are

$$x^2 + 2x + 3 = 0 \quad \dots (i)$$

$$\text{and } ax^2 + bx + c = 0 \quad \dots (ii)$$

Since, Eq. (i) has imaginary roots.

So, Eq. (ii) will also have both roots same as Eq. (i).

$$\text{Thus, } \frac{a}{1} = \frac{b}{2} = \frac{c}{3}$$

Hence, $a : b : c$ is $1 : 2 : 3$

15. Given equation is

$$e^{\sin x} - e^{-\sin x} = 4$$

$$\Rightarrow e^{\sin x} - \frac{1}{e^{\sin x}} = 4$$

$$\text{Now, let } y = e^{\sin x}$$

$$\therefore y - \frac{1}{y} = 4$$

$$\Rightarrow y^2 - 4y - 1 = 0$$

$$\Rightarrow y = \frac{4 \pm \sqrt{16 + 4}}{2}$$

$$\Rightarrow y = 2 \pm \sqrt{5}$$

On substituting the value of y , we get

$$e^{\sin x} = 2 \pm \sqrt{5}$$

Now, sine is a bounded function,

i.e. $-1 \leq \sin x \leq 1$.

$$\therefore e^{-1} \leq e^{\sin x} \leq e$$

$$\Rightarrow e^{\sin x} \in \left[\frac{1}{e}, e \right]$$

16. Let $z = x + iy$, given $\text{Re}(z) = 1$

$$\therefore x = 1$$

$$\Rightarrow z = 1 + iy$$

Since, the complex roots are conjugate of each other.

$\therefore z = 1 + iy$ and $1 - iy$ are two roots of $z^2 + \alpha z + \beta = 0$

Product of roots = β

$$\Rightarrow (1 + iy)(1 - iy) = \beta$$

$$\therefore \beta = 1 + y^2 \geq 1$$

$$\Rightarrow \beta \in (1, \infty)$$

17. Let the quadratic equation be

$$ax^2 + bx + c = 0$$

Sachin made a mistake in writing down constant terms.

$\therefore$ Sum of roots is correct.

$$\text{i.e. } \alpha + \beta = 7$$

Rahul made mistake in writing down coefficient of x .

$\therefore$ Product of roots is correct.

$$\text{i.e. } \alpha\beta = 6$$

$\Rightarrow$ Correct quadratic equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\Rightarrow x^2 - 7x + 6 = 0 \text{ having roots } 1 \text{ and } 6.$$

18. Since, α and β are roots of the equation $x^2 - x + 1 = 0$.

$$\Rightarrow \alpha + \beta = 1, \alpha\beta = 1$$

$$\Rightarrow x = \frac{1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow x = \frac{1 + \sqrt{3}i}{2} \text{ or } \frac{1 - \sqrt{3}i}{2}$$

$$\Rightarrow x = -\omega \text{ or } -\omega^2$$

Thus, $\alpha = -\omega^2$, then $\beta = -\omega$

or $\alpha = -\omega$, then $\beta = -\omega^2$ [where, $\omega^3 = 1$]

$$\begin{aligned} \text{Hence, } \alpha^{2009} + \beta^{2009} &= (-\omega)^{2009} + (-\omega^2)^{2009} \\ &= -[(\omega^3)^{669} \cdot \omega^2 + (\omega^3)^{1339} \cdot \omega] \\ &= -[\omega^2 + \omega] = -(-1) = 1 \end{aligned}$$

19. Given $bx^2 + cx + a = 0$ has imaginary roots

$$\begin{aligned} \Rightarrow c^2 - 4ab < 0 &\Rightarrow c^2 < 4ab \\ \Rightarrow -c^2 > -4ab &\dots(i) \end{aligned}$$

Let $f(x) = 3b^2x^2 + 6bcx + 2c^2$

Here, $3b^2 > 0$

So, the given expression has a minimum value.

$$\begin{aligned} \therefore \text{Minimum value} &= \frac{-D}{4a} \\ &= \frac{4ac - b^2}{4a} = \frac{4(3b^2)(2c^2) - 36b^2c^2}{4(3b^2)} \\ &= -\frac{12b^2c^2}{12b^2} = -c^2 > -4ab \quad [\text{from Eq. (i)}] \end{aligned}$$

20. Let the roots of $x^2 - 6x + a = 0$ be $\alpha, 4\beta$ and that of $x^2 - cx + 6 = 0$ be $\alpha, 3\beta$.

$$\therefore \alpha + 4\beta = 6 \quad \text{and} \quad 4\alpha\beta = a$$

$$\text{and} \quad \alpha + 3\beta = c \quad \text{and} \quad 3\alpha\beta = 6$$

$$\Rightarrow \frac{a}{6} = \frac{4}{3} \Rightarrow a = 8$$

$$\therefore x^2 - 6x + 8 = 0$$

$$\Rightarrow (x-4)(x-2) = 0$$

$$\Rightarrow x = 2, 4$$

$$\text{and} \quad x^2 - cx + 6 = 0$$

$$\text{If } x = 2, \text{ then } 2^2 - 2c + 6 = 0 \Rightarrow c = 5$$

$$\therefore x^2 - 5x + 6 = 0$$

$$\Rightarrow x = 2, 3$$

Hence, the common root is 2.

21. Let α and β be the roots of equation $x^2 + ax + 1 = 0$, then

$$\alpha + \beta = -a \quad \text{and} \quad \alpha\beta = 1$$

$$\text{Now, } |\alpha - \beta| = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$$

$$\Rightarrow |\alpha - \beta| = \sqrt{a^2 - 4}$$

According to given condition,

$$\sqrt{a^2 - 4} < \sqrt{5}$$

$$\Rightarrow a^2 - 4 < 5$$

$$\Rightarrow a^2 < 9$$

$$\Rightarrow |a| < 3$$

$$\Rightarrow a \in (-3, 3)$$

22. Since, $\tan 30^\circ$ and $\tan 15^\circ$ are the roots of equation $x^2 + px + q = 0$.

$$\therefore \tan 30^\circ + \tan 15^\circ = -p$$

$$\text{and} \quad \tan 30^\circ \tan 15^\circ = q$$

$$\begin{aligned} \text{Now, } 2 + q - p &= 2 + \tan 30^\circ \tan 15^\circ \\ &\quad + (\tan 30^\circ + \tan 15^\circ) \\ &= 2 + \tan 30^\circ \tan 15^\circ + 1 - \tan 30^\circ \tan 15^\circ \end{aligned}$$

$$\left[\because \tan 45^\circ = \frac{\tan 30^\circ + \tan 15^\circ}{1 - \tan 30^\circ \tan 15^\circ} \right]$$

$$\Rightarrow 2 + q - p = 3$$

23. Since, both roots of equation $x^2 - 2mx + m^2 - 1 = 0$ are greater than -2 but less than 4 .

$$\therefore D \geq 0, -2 < -\frac{b}{2a} < 4,$$

$$f(4) > 0 \quad \text{and} \quad f(-2) > 0$$

$$\text{Now, } D \geq 0; 4m^2 - 4m^2 + 4 \geq 0$$

$$\Rightarrow 4 > 0, \forall m \in R \quad \dots(i)$$

$$-2 < -\frac{b}{2a} < 4$$

$$\Rightarrow -2 < \left(\frac{2m}{2 \cdot 1}\right) < 4$$

$$\Rightarrow -2 < m < 4 \quad \dots(ii)$$

$$f(4) > 0$$

$$\Rightarrow 16 - 8m + m^2 - 1 > 0$$

$$\Rightarrow m^2 - 8m + 15 > 0$$

$$\Rightarrow (m-3)(m-5) > 0$$

$$\Rightarrow -\infty < m < 3 \quad \text{and} \quad 5 < m < \infty \quad \dots(iii)$$

$$\text{and} \quad f(-2) > 0$$

$$\Rightarrow 4 + 4m + m^2 - 1 > 0$$

$$\Rightarrow m^2 + 4m + 3 > 0$$

$$\Rightarrow (m+3)(m+1) > 0$$

$$\Rightarrow -\infty < m < -3 \quad \text{and} \quad -1 < m < \infty \quad \dots(iv)$$

From Eqs. (i), (ii), (iii) and (iv), we get m lies between -1 and 3 .

24. Let α and β be the roots of equation

$$x^2 - (a-2)x - a - 1 = 0$$

$$\text{Then, } \alpha + \beta = a - 2 \quad \text{and} \quad \alpha\beta = -a - 1$$

$$\text{Now, } \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$\Rightarrow \alpha^2 + \beta^2 = (a-2)^2 + 2(a+1)$$

$$\Rightarrow \alpha^2 + \beta^2 = a^2 - 2a + 6$$

$$\Rightarrow \alpha^2 + \beta^2 = (a-1)^2 + 5$$

$$\text{The value of } \alpha^2 + \beta^2 \text{ will be least, if } a - 1 = 0$$

$$\Rightarrow a = 1$$

Aliter

$$\text{Since, } \alpha + \beta = (a-2) \quad \text{and} \quad \alpha\beta = -a-1$$

$$\begin{aligned} \text{Let } f(a) &= \alpha^2 + \beta^2 \\ &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a-2)^2 + 2(a+1) \\ &= a^2 - 2a + 6 \end{aligned}$$

$$\Rightarrow f'(a) = 2a - 2$$

For maxima or minima, put $f'(a) = 0$.

$$\therefore 2a - 2 = 0 \Rightarrow a = 1$$

$$\text{Now, } f''(a) = 2 \Rightarrow f''(1) = 2 > 0$$

$\therefore f(a)$ is minimum at $a = 1$.

25. Let n and $(n+1)$ be the two consecutive roots of

$$x^2 - bx + c = 0. \text{ Then, } n + (n+1) = b \text{ and } n(n+1) = c$$

$$\therefore b^2 - 4c = (2n+1)^2 - 4n(n+1)$$

$$= 4n^2 + 4n + 1 - 4n^2 - 4n = 1$$

26. Let $f(x) = x^2 - 2kx + k^2 + k - 5$

Since, both roots are less than 5 .

$$\text{Then, } D \geq 0, -\frac{b}{2a} < 5$$

$$\text{and} \quad f(5) > 0$$

Now, $D = 4k^2 - 4(k^2 + k - 5)$
 $= -4k + 20 \geq 0$
 $\Rightarrow k \leq 5 \quad \dots(i)$
 $-\frac{b}{2a} < 5 \Rightarrow k < 5 \quad \dots(ii)$
 and $f(5) > 0$
 $\Rightarrow 25 - 10k + k^2 + k - 5 > 0$
 $\Rightarrow k^2 - 9k + 20 > 0$
 $\Rightarrow (k - 5)(k - 4) > 0$
 $\Rightarrow k < 4 \text{ and } k > 5 \quad \dots(iii)$
 From Eqs. (i), (ii) and (iii), we get
 $k < 4$

27. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x = 0 \quad \dots(i)$

and $f(0) = 0$
 $f(\alpha) = 0$
 According to the Rolle's theorem,
 $f'(x) = 0$
 has atleast one root between $(0, \alpha)$.
 $\Rightarrow f'(x) = 0$ has a positive root less than α .

28. Since, $(1 - p)$ is a root of quadratic equation
 $x^2 + px + (1 - p) = 0 \quad \dots(i)$

So, $(1 - p)$ satisfied the above equation.
 $\therefore (1 - p)^2 + p(1 - p) + (1 - p) = 0$
 $\Rightarrow (1 - p)(1 - p + p + 1) = 0$
 $\Rightarrow (1 - p)2 = 0$
 $\Rightarrow p = 1$

On putting the value of p in Eq. (i), we get $x^2 + x = 0$
 $\Rightarrow x = 0, -1$

29. Since, one of the roots of equation $x^2 + px + 12 = 0$ is 4.

$\therefore 16 + 4p + 12 = 0$
 $\Rightarrow 4p = -28$
 $\Rightarrow p = -7$

So, the other equation is $x^2 - px + q = 0$ whose roots are equal. Let the roots be α and α .

$\therefore$ Sum of roots $= \alpha + \alpha = \frac{7}{1}$
 $\Rightarrow \alpha = \frac{7}{2}$

and product of roots $= \alpha \cdot \alpha = q$
 $\Rightarrow \left(\frac{7}{2}\right)^2 = q \Rightarrow q = \frac{49}{4}$

30. Let α and β be two numbers whose arithmetic mean is 9 and geometric mean is 4.

$\therefore \alpha + \beta = 18 \text{ and } \alpha\beta = 16$

$\therefore$ Required equation is
 $x^2 - (\alpha + \beta)x + (\alpha\beta) = 0$
 $\Rightarrow x^2 - 18x + 16 = 0$

31. Since, origin z_1 and z_2 are the vertices of an equilateral triangle, then

$z_1^2 + z_2^2 = z_1 z_2$
 $\Rightarrow (z_1 + z_2)^2 = 3z_1 z_2 \quad \dots(i)$

Again, z_1 and z_2 are the roots of the equation
 $z^2 + az + b = 0$

Then, $z_1 + z_2 = -a$
 and $z_1 z_2 = b$
 On putting these values in Eq. (i), we get
 $(-a)^2 = 3b$
 $\Rightarrow a^2 = 3b$

32. Given equation is $ax^2 + bx + c = 0$
 Let α and β be the roots of the equation.
 Then, $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$
 Also given, $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2} = \frac{\alpha^2 + \beta^2}{\alpha^2 \beta^2}$

$\Rightarrow \alpha + \beta = \left(\frac{\alpha + \beta}{\alpha\beta}\right)^2 - \frac{2}{\alpha\beta}$
 $\Rightarrow \left(-\frac{b}{a}\right) = \left(\frac{-b/a}{c/a}\right)^2 - \frac{2}{c/a}$
 $\Rightarrow -\frac{b}{a} = \left(\frac{b}{c}\right)^2 - \frac{2a}{c}$
 $\Rightarrow \frac{2a}{c} = \frac{b}{c} \left(\frac{b}{c} + \frac{c}{a}\right)$
 $\Rightarrow \frac{2a}{b} = \frac{b}{c} + \frac{c}{a}$
 $\Rightarrow \frac{c}{a}, \frac{a}{b} \text{ and } \frac{b}{c} \text{ are in AP.}$
 $\Rightarrow \frac{a}{c}, \frac{b}{a} \text{ and } \frac{c}{b} \text{ are in HP.}$

33. Given equation is $x^2 - 3|x| + 2 = 0$

Case I When $x > 0$, then $|x| = x$
 $\therefore x^2 - 3x + 2 = 0$
 $\Rightarrow (x - 1)(x - 2) = 0$
 $\Rightarrow x = 1, 2$

Case II When $x < 0$, then $|x| = -x$
 $\therefore x^2 + 3x + 2 = 0$
 $\Rightarrow (x + 1)(x + 2) = 0$
 $\Rightarrow x = -1, -2$
 Hence, four solutions are possible.

34. Since, one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as the other, then let their roots be α and 2α .

$\therefore \alpha + 2\alpha = -\frac{(3a - 1)}{(a^2 - 5a + 3)}$
 $\Rightarrow 3\alpha = -\frac{(3a - 1)}{(a^2 - 5a + 3)}$

and $\alpha \cdot 2\alpha = \frac{2}{(a^2 - 5a + 3)}$

$\Rightarrow 2\alpha^2 = \frac{2}{(a^2 - 5a + 3)}$

$\Rightarrow \frac{(3a - 1)^2}{9(a^2 - 5a + 3)^2} = \frac{1}{(a^2 - 5a + 3)}$

$\Rightarrow (3a - 1)^2 = 9(a^2 - 5a + 3)$

$\Rightarrow 9a^2 - 6a + 1 = 9a^2 - 45a + 27$

$\Rightarrow 45a - 6a = 27 - 1$

$\Rightarrow a = \frac{26}{39} = \frac{2}{3}$

35. Since, $\alpha^2 = 5\alpha - 3$

$$\Rightarrow \alpha^2 - 5\alpha + 3 = 0 \quad \text{and} \quad \beta^2 = 5\beta - 3$$

$$\Rightarrow \beta^2 - 5\beta + 3 = 0$$

These two equations show that α and β are the roots of the equation

$$x^2 - 5x + 3 = 0$$

$$\therefore \alpha + \beta = 5 \quad \text{and} \quad \alpha\beta = 3$$

$$\begin{aligned} \text{Now,} \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} &= \frac{\alpha^2 + \beta^2}{\alpha\beta} \\ &= \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta} \\ &= \frac{25 - 6}{3} = \frac{19}{3} \end{aligned}$$

$$\text{and} \quad \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} = 1$$

Thus, the equation having roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ is given by

$$\begin{aligned} x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \frac{\alpha}{\beta} \cdot \frac{\beta}{\alpha} &= 0 \\ \Rightarrow x^2 - \frac{19}{3}x + 1 &= 0 \\ \Rightarrow 3x^2 - 19x + 3 &= 0 \end{aligned}$$

36. Given, $3^{2x^2 - 7x + 7} = 3^2$

$$\Rightarrow 2x^2 - 7x + 7 = 2$$

$$\Rightarrow 2x^2 - 7x + 5 = 0$$

$$\begin{aligned} \text{Now,} \quad D &= b^2 - 4ac \\ &= (-7)^2 - 4 \times 2 \times 5 \\ &= 49 - 40 = 9 > 0 \end{aligned}$$

Hence, it has two real roots.

37. Let $y = 20^{301}$

$$\begin{aligned} \text{Number of digits} &= \text{Integral part of } (301 \log_{10} 20) + 1 \\ &= \text{Integral part of } [301(\log_{10} 10 + \log_{10} 2)] + 1 \\ &= \text{Integral part of } [301(1 + 0.3010)] + 1 \\ &= \text{Integral part of } [301 \times 1.3010] + 1 \\ &= \text{Integral part of } [391.601] + 1 \\ &= 391 + 1 = 392 \end{aligned}$$

38. Given, $\log_{101} \log_7 (\sqrt{x+7} + \sqrt{x}) = 0$

$$\therefore \log_7 (\sqrt{x+7} + \sqrt{x}) = (101)^0$$

$$\Rightarrow \log_7 (\sqrt{x+7} + \sqrt{x}) = 1$$

$$\Rightarrow (\sqrt{x+7} + \sqrt{x}) = 7^1$$

$$\Rightarrow \sqrt{x+7} + \sqrt{x} = 7$$

On squaring both sides, we get

$$\begin{aligned} (x+7) + x + 2\sqrt{x^2 + 7x} &= 49 \\ \Rightarrow 2x - 42 &= -2\sqrt{x^2 + 7x} \\ \Rightarrow x - 21 &= -\sqrt{x^2 + 7x} \end{aligned}$$

Again, squaring on both sides, we get

$$\begin{aligned} x^2 + 441 - 42x &= x^2 + 7x \\ \Rightarrow 49x &= 441 \\ \Rightarrow x &= \frac{441}{49} \\ \therefore x &= 9 \end{aligned}$$

39. Let α and β be the roots of the equation

$$x^2 - (a-2)x - a + 1 = 0$$

$$\therefore \alpha + \beta = a - 2$$

$$\text{and} \quad \alpha\beta = -(a-1)$$

$$\begin{aligned} \Rightarrow s &= \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a-2)^2 + 2(a-1) \\ &= a^2 - 4a + 4 + 2a - 2 \\ &= a^2 - 2a + 2 \end{aligned}$$

$$\text{Now,} \quad \frac{ds}{da} = 2a - 2$$

For maximum and minimum, put $ds / da = 0$

$$\Rightarrow 2a - 2 = 0 \Rightarrow a = 1$$

$$\text{Also,} \quad \frac{d^2s}{da^2} = 2 > 0$$

Hence, at $a = 1$, s will have minimum value.

40. Let two consecutive odd integers be $(\alpha + 1)$ and $(\alpha + 3)$.

$$\therefore \text{Sum of roots, } \alpha + 1 + \alpha + 3 = a$$

$$\Rightarrow 2\alpha = a - 4 \Rightarrow \alpha = \frac{a-4}{2}$$

and product of roots, $(\alpha + 1)(\alpha + 3) = b$

$$\Rightarrow \alpha^2 + 4\alpha + 3 = b$$

$$\Rightarrow \left(\frac{a-4}{2}\right)^2 + 4\left(\frac{a-4}{2}\right) + 3 = b$$

$$\Rightarrow \frac{a^2 + 16 - 8a}{4} + \frac{4a - 16}{2} + 3 = b$$

$$\Rightarrow a^2 + 16 - 8a + 8a - 32 + 12 = 4b$$

$$\therefore a^2 - 4b = 4$$

41. Since, α and β are the roots of $x^2 - ax + b^2 = 0$.

$$\text{Then,} \quad \alpha + \beta = a$$

$$\text{and} \quad \alpha\beta = b^2$$

$$\begin{aligned} \text{Now,} \quad \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (a)^2 - 2(b)^2 = a^2 - 2b^2 \end{aligned}$$

42. Since, α and β are the roots of the equation

$$x^2 + 3x - 4 = 0.$$

$$\text{Then,} \quad \alpha + \beta = -3 \quad \text{and} \quad \alpha\beta = -4$$

$$\text{Now,} \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{-3}{-4} = \frac{3}{4}$$

43. Since, α and β are the roots of $x^2 + 2bx + c = 0$.

$$\text{Then,} \quad \alpha + \beta = -2b$$

$$\text{and} \quad \alpha\beta = c$$

$$\begin{aligned} \text{Now,} \quad b^2 - c &= \left(\frac{\alpha + \beta}{-2}\right)^2 - \alpha\beta \\ &= \frac{\alpha^2 + \beta^2 + 2\alpha\beta - 4\alpha\beta}{4} \\ &= \frac{\alpha^2 + \beta^2 - 2\alpha\beta}{4} = \frac{(\alpha - \beta)^2}{4} \end{aligned}$$

44. Since, a, b and c are in GP.

$$\therefore b^2 = ac$$

Given equation is

$$(\log_e a) x^2 - (2 \log_e b) x + (\log_e c) = 0$$

Put $x = 1$, we get

$$\log_e a - 2 \log_e b + \log_e c = 0$$

$$\Rightarrow 2 \log_e b = \log_e a + \log_e c$$

$$\Rightarrow \log_e b^2 = \log_e ac$$

$\Rightarrow b^2 = ac$, which is true.

Hence, one of the root of given equation is 1.

Let another root be α .

$$\therefore \text{Sum of roots, } 1 + \alpha = \frac{2 \log_e b}{\log_e a} = \frac{\log_e b^2}{\log_e a}$$

$$\Rightarrow \alpha = \frac{\log_e ac}{\log_e a} - 1$$

$$= \frac{(\log_e a + \log_e c)}{\log_e a} - 1$$

$$= \frac{\log_e c}{\log_e a} = \log_a c$$

Hence, roots are 1 and $\log_a c$.

45. Given α, β are the roots of $ax^2 + bx + c = 0$ and $\alpha + h, \beta + h$ are the roots of $px^2 + qx + r = 0$.

$$\therefore \alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

$$\text{and } \alpha + h + \beta + h = -\frac{q}{p}, (\alpha + h)(\beta + h) = \frac{r}{p}$$

$$\text{Now, } (\alpha + h) - (\beta + h) = \alpha - \beta$$

$$\Rightarrow [(\alpha + h) - (\beta + h)]^2 = (\alpha - \beta)^2$$

$$\Rightarrow [(\alpha + h) + (\beta + h)]^2 - 4(\alpha + h)(\beta + h)$$

$$= (\alpha + \beta)^2 - 4\alpha\beta$$

$$\Rightarrow \frac{q^2}{p^2} - \frac{4r}{p} = \frac{b^2}{a^2} - \frac{4c}{a}$$

$$\Rightarrow \frac{q^2 - 4pr}{p^2} = \frac{b^2 - 4ac}{a^2}$$

$$\therefore \frac{b^2 - 4ac}{q^2 - 4pr} = \frac{a^2}{p^2}$$

Hence, the ratio of the square of their discriminants is $a^2 : p^2$.

46. Given, $f(x) = 2x^2 + 5x + 1$... (i)

$$\text{Also, } f(x) = a(x+1)(x-2) + b(x-2)(x-1) + c(x-1)(x+1)$$

$$= a(x^2 - x - 2) + b(x^2 - 3x + 2) + c(x^2 - 1)$$

$$\Rightarrow f(x) = (a+b+c)x^2 + (-a-3b)x + (-2a+2b-c) \dots (ii)$$

On equating the coefficients of x^2, x and constant terms in Eqs. (i) and (ii), we get

$$a + b + c = 2, -a - 3b = 5 \text{ and } -2a + 2b - c = 1$$

On solving above equations, we get

$$a = -4, b = -1/3 \text{ and } c = 19/3$$

Hence, exactly one choice for each of a, b and c .

47. Given, α and β are the roots of equation

$$ax^2 + bx + c = 0.$$

$$\therefore \alpha + \beta = -\frac{b}{a} \text{ and } \alpha\beta = \frac{c}{a}$$

$$\text{Now, } S_{n+1} = \alpha^{n+1} + \beta^{n+1}$$

$$= \alpha^{n+1} + \beta^{n+1} + \alpha^n\beta + \beta^n\alpha - \alpha^n\beta - \beta^n\alpha$$

$$= \alpha^n(\alpha + \beta) + \beta^n(\alpha + \beta) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$$

$$= (\alpha + \beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$$

$$= -\frac{b}{a}S_n - \frac{c}{a}S_{n-1}$$

$$\Rightarrow S_{n+1} = \frac{-bS_n - cS_{n-1}}{a}$$

$$\therefore aS_{n+1} + bS_n + cS_{n-1} = 0$$

48. Since, $\sin \alpha$ and $\cos \alpha$ are the roots of $ax^2 + bx + c = 0$.

$$\therefore \sin \alpha + \cos \alpha = -\frac{b}{a} \text{ and } \sin \alpha \cos \alpha = \frac{c}{a}$$

$$\Rightarrow (\sin \alpha + \cos \alpha)^2 = \left(-\frac{b}{a}\right)^2$$

$$\Rightarrow \sin^2 \alpha + \cos^2 \alpha + 2 \sin \alpha \cos \alpha = \frac{b^2}{a^2}$$

$$\Rightarrow 1 + 2 \cdot \frac{c}{a} = \frac{b^2}{a^2}$$

$$\Rightarrow a(a + 2c) = b^2$$

$$\Rightarrow b^2 - a^2 = 2ac$$

49. Given, $3^{2x} - 2(3^x + 2) + 81 = 0$

$$\therefore (3^x)^2 - 2(3^x) + 81 = 0$$

$$\text{Let } 3^x = y$$

$$\therefore y^2 - 18y + 81 = 0$$

$$\Rightarrow (y - 9)^2 = 0$$

$$\Rightarrow y = 9 \Rightarrow 3^x = 3^2 \quad [\because 3^x = y]$$

$$\therefore x = 2$$

50. Let the roots of the equation $2x^2 + 3x + 1 = 0$ be α and β .

$$\text{Then, } \alpha + \beta = -\frac{3}{2} \dots (i)$$

$$\text{and } \alpha\beta = \frac{1}{2} \dots (ii)$$

$$\therefore \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= \left(-\frac{3}{2}\right)^2 - 2\left(\frac{1}{2}\right) = \frac{9}{4} - 1 = \frac{5}{4}$$

$$\text{and } \alpha^2\beta^2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$\therefore$ Required equation is

$$x^2 - (\alpha^2 + \beta^2)x + \alpha^2\beta^2 = 0$$

$$\Rightarrow x^2 - \frac{5}{4}x + \frac{1}{4} = 0$$

$$\Rightarrow 4x^2 - 5x + 1 = 0$$

51. Given equation can be rewritten as

$$x^2 - \frac{r^2}{pq}x + 1 = 0$$

$$\therefore \tan A + \tan B = \frac{r^2}{pq} \text{ and } \tan A \tan B = 1$$

We know that, $A + B + C = 180^\circ$

$$\Rightarrow (A + B) = 180^\circ - C$$

$$\Rightarrow \tan(A + B) = \tan(180^\circ - C)$$

$$\Rightarrow \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\Rightarrow \frac{r^2/pq}{1 - 1} = -\tan C$$

$$\Rightarrow \infty = -\tan C$$

$$\therefore C = 90^\circ$$

Hence, ΔABC is a right angled triangle.

52. $\therefore$ Sum of roots, $\alpha + \beta = -p$ and $\alpha\beta = q$

$$\begin{aligned}\therefore (\alpha^3 + \beta^3) &= (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \\ &= (-p)^3 - 3q(-p) = -p^3 + 3pq \\ \text{and } \alpha^4 + \alpha^2\beta^2 + \beta^4 &= (\alpha^4 + \beta^4) + (\alpha\beta)^4 \\ &= (\alpha^2 + \beta^2)^2 - (\alpha\beta)^2 \\ &= [(\alpha + \beta)^2 - 2\alpha\beta]^2 - (\alpha\beta)^2 \\ &= [(-p)^2 - 2q]^2 - 3(q)^2 \\ &= [p^2 - 2q]^2 - 3q^2 \\ &= p^4 - 4p^2q + 4q^2 - q^2 \\ &= p^4 - 4p^2q + 3q^2 \\ &= p^4 - 3p^2q - p^2q + 3q^2 \\ &= p^2(p^2 - 3q) - q(p^2 - 3q) \\ &= (p^2 - q)(p^2 - 3q)\end{aligned}$$

53. Given equation is $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$... (i)

On squaring both sides, we get

$$\begin{aligned}(x+1) + (x-1) - 2\sqrt{x^2-1} &= 4x-1 \\ \Rightarrow 2x - 2\sqrt{x^2-1} &= 4x-1 \\ \Rightarrow -2\sqrt{x^2-1} &= 2x-1\end{aligned}$$

Again, squaring on both sides, we get

$$\begin{aligned}4(x^2-1) &= 4x^2 + 1 - 4x \\ \Rightarrow -4 &= 1 - 4x \\ \Rightarrow 4x &= 5 \Rightarrow x = 5/4\end{aligned}$$

But when we put $x = \frac{5}{4}$ in Eq. (i), we get

$$\begin{aligned}\sqrt{\frac{5}{4}+1} - \sqrt{\frac{5}{4}-1} &= \sqrt{4 \times \frac{5}{4} - 1} \\ \Rightarrow \sqrt{\frac{9}{4}} - \sqrt{\frac{1}{4}} &= \sqrt{5-1} \\ \Rightarrow \frac{3}{2} - \frac{1}{2} &= 2 \Rightarrow 1 = 2, \text{ which is not true.}\end{aligned}$$

Hence, no value of x satisfy the given equation.

54. Given, $9\sqrt{x} = \sqrt{12} + \sqrt{147}$

On squaring both sides, we get

$$\begin{aligned}81x &= 12 + 147 + 2\sqrt{12} \cdot \sqrt{147} \\ \Rightarrow 81x &= 159 + 84 \\ \Rightarrow 81x &= 243 \\ \therefore x &= 3\end{aligned}$$

Aliter

$$\begin{aligned}9\sqrt{x} &= \sqrt{12} + \sqrt{147} \\ \Rightarrow 9\sqrt{x} &= 2\sqrt{3} + 7\sqrt{3} \\ \Rightarrow 9\sqrt{x} &= 9\sqrt{3} \\ \Rightarrow \sqrt{x} &= \sqrt{3} \\ \therefore x &= 3\end{aligned}$$

55. $x^2 - \sqrt{3}x + 1 = 0$

$$\Rightarrow x = \frac{\sqrt{3} \pm \sqrt{3-4}}{2} = \frac{\sqrt{3} \pm i}{2}$$

56. Given, $a = \cos \frac{2\pi}{7} + i \sin \frac{2\pi}{7}$

$$\therefore a^7 = \cos 2\pi + i \sin 2\pi \quad [\because e^{i\theta} = \cos \theta + i \sin \theta]$$

$$= 1$$

Also, $\alpha = a + a^2 + a^4$ and $\beta = a^3 + a^5 + a^6$

Then, sum of the roots,

$$S = \alpha + \beta = a + a^2 + a^4 + a^3 + a^5 + a^6$$

$$\Rightarrow S = \frac{a(1-a^6)}{1-a} = \frac{a-a^7}{1-a} = \frac{a-1}{1-a} = -1 \quad [\because a^7 = 1]$$

Product of the roots,

$$\begin{aligned}P = \alpha\beta &= (a + a^2 + a^4)(a^3 + a^5 + a^6) \\ &= a^4 + a^5 + 1 + a^6 + 1 + a^2 + 1 + a + a^3 \quad [\because a^7 = 1] \\ &= 3 + (a + a^2 + a^3 + a^4 + a^5 + a^6) = 3 - 1 = 2\end{aligned}$$

Hence, the required quadratic equation is

$$x^2 + x + 2 = 0.$$

57. Since, α is a root of $x^2 + 3p^2x + 5q^2 = 0$, then

$$\alpha^2 + 3p^2\alpha + 5q^2 = 0 \quad \dots (i)$$

and β is a root of $x^2 + 9p^2x + 15q^2 = 0$, then

$$\therefore \beta^2 + 9p^2\beta + 15q^2 = 0 \quad \dots (ii)$$

Let $f(x) = x^2 + 6p^2x + 10q^2$

$$\begin{aligned}\text{Then, } f(\alpha) &= \alpha^2 + 6p^2\alpha + 10q^2 \\ &= (\alpha^2 + 3p^2\alpha + 5q^2) + 3p^2\alpha + 5q^2 \\ &= 0 + 3p^2\alpha + 5q^2 \quad [\text{from Eq. (i)}]\end{aligned}$$

$$\begin{aligned}\Rightarrow f(\alpha) &> 0 \quad \text{and } f(\beta) = \beta^2 + 6p^2\beta + 10q^2 \\ &= (\beta^2 + 9p^2\beta + 15q^2) - (3p^2\beta + 5q^2) \\ &= 0 - (3p^2\beta + 5q^2)\end{aligned}$$

$$\Rightarrow f(\beta) < 0$$

Thus, $f(x)$ is a polynomial such that $f(\alpha) > 0$ and $f(\beta) < 0$.

Therefore, there exists γ satisfying $\alpha < \gamma < \beta$ such that $f(\gamma) = 0$.

58. Given equation is $\lambda^2 + 8\lambda + \mu^2 + 6\mu = 0$.

Since, roots are real.

$$\begin{aligned}\therefore b^2 - 4ac &\geq 0 \\ \Rightarrow (8)^2 - 4(\mu^2 + 6\mu) &\geq 0 \\ \Rightarrow 64 - 4(\mu^2 + 6\mu) &\geq 0 \\ \Rightarrow 16 - (\mu^2 + 6\mu) &\geq 0 \\ \Rightarrow \mu^2 + 6\mu - 16 &\leq 0 \\ \Rightarrow \mu^2 + 8\mu - 2\mu - 16 &\leq 0 \\ \Rightarrow \mu(\mu + 8) - 2(\mu + 8) &\leq 0 \\ \Rightarrow (\mu + 8)(\mu - 2) &\leq 0 \\ \therefore -8 &\leq \mu \leq 2\end{aligned}$$

59. Given, $|x-2| + |x+2| < 4$

Case I When $x < -2$

$$\begin{aligned}\Rightarrow -(x-2) - (x+2) &< 4 \\ \Rightarrow -2x &< 4 \\ \Rightarrow -x < 2 \Rightarrow x > -2, \text{ which is not true.}\end{aligned}$$

Hence, no value of x exist.

Case II When $-2 < x < 2$

$$\begin{aligned}\Rightarrow -(x-2) + x + 2 &< 4 \\ \Rightarrow 4 &< 4, \text{ which is not true.}\end{aligned}$$

Hence, no value of x exist.

Case III When $x > 2$

$$\begin{aligned}\Rightarrow x - 2 + x + 2 &< 4 \\ \Rightarrow 2x &< 4 \Rightarrow x < 2, \text{ which is not true.}\end{aligned}$$

So, no value of x exist.

Hence, the number of solutions of the inequation is 0.

60. $2x - 5 \leq \frac{4x - 7}{3}$

$$\Rightarrow 6x - 15 \leq 4x - 7$$

$$\Rightarrow 2x \leq 8 \Rightarrow x \leq 4$$

61. $3x + 2 > -16$

$$\Rightarrow x > -6$$

$$\text{and } 2x - 3 \leq 11$$

$$\Rightarrow x \leq 7$$

$$\therefore x \in (-6, 7]$$

62. We can write, $5^x + 5^{-x} = 5^x + \frac{1}{5^x}$

$$= 5^x + \frac{1}{5^x} - 2 + 2$$

$$= \left(5^x - \frac{1}{5^x}\right)^2 + 2$$

This means $5^x + 5^{-x} > 1$

Thus, $\sin(e^x) > 1$, which is not possible.

63. By hypothesis, $a^2\alpha^2 + b\alpha + c = 0$... (i)

and $a^2\beta^2 - b\beta - c = 0$... (ii)

where, $0 < \alpha < \beta$

and $\alpha \neq 0$

Let $f(x) = a^2x^2 + 2bx + 2c$

Clearly, it is continuous on $[\alpha, \beta]$.

$$\therefore f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c = -\alpha^2a^2 < 0$$

$$\text{and } f(\beta) = a^2\beta^2 + 2b\beta + 2c = 3a^2\beta^2 > 0$$

Therefore, by the intermediate value theorem of continuous function, $f(x) = 0$ has a root γ between α and β .

64. Let α and β be the roots, then

$$\alpha + \beta = 16$$

and $\alpha\beta = 25$

$$\therefore x^2 - x(\alpha + \beta) + \alpha\beta = 0$$

$$\Rightarrow x^2 - 16x + 25 = 0$$

65. We have,

$$\frac{k+1}{k} + \frac{k+2}{k+1} = -\frac{b}{a} \quad \dots(i)$$

and $\frac{k+1}{k} \cdot \frac{k+2}{k+1} = \frac{c}{a} \quad \dots(ii)$

From Eq. (i), we have

$$1 + \frac{1}{k} + 1 + \frac{1}{k+1} = -\frac{b}{a}$$

$$\Rightarrow 2 + \frac{1}{k} + \frac{1}{k+1} = -\frac{b}{a} \quad \dots(iii)$$

From Eq. (ii), we have

$$\frac{k+2}{k} = \frac{c}{a}$$

$$\Rightarrow 1 + \frac{2}{k} = \frac{c}{a}$$

$$\Rightarrow \frac{2}{k} = \frac{c}{a} - 1$$

$$\therefore k = \frac{2a}{c-a}$$

Now, on substituting the value of k in Eq. (iii), we get

$$2 + \frac{c-a}{2a} + \frac{1}{\frac{2a}{c-a} + 1} = -\frac{b}{a}$$

$$\Rightarrow 2 + \frac{c-a}{2a} + \frac{c-a}{a+c} = -\frac{b}{a}$$

$$\Rightarrow \frac{2(2a)(a+c) + (c-a)(c+a) + 2a(c-a)}{2a(a+c)} = -\frac{b}{a}$$

$$\Rightarrow a^2 + c^2 + 6ac = -2ab - 2bc$$

$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = b^2 - 4ac$$

$$\therefore (a+b+c)^2 = b^2 - 4ac$$

66. Since, the second equation has imaginary roots.

$$\therefore \frac{2a}{3} = \frac{-3b}{-4} = \frac{4c}{5} = k$$

$$\Rightarrow a = \frac{3k}{2}, b = \frac{4k}{3}, c = \frac{5k}{4}$$

$$\therefore \frac{a+b}{b+c} = \frac{\frac{3k}{2} + \frac{4k}{3}}{\frac{4k}{3} + \frac{5k}{4}} = \frac{34}{31}$$

67. $\log_2 \left(\frac{9^{x-1} + 7}{3^{x-1} + 1} \right) = 2 \log_2 2$

$$\Rightarrow \frac{(3^2)^{x-1} + 7}{(3^{x-1} + 1)} = 4 \Rightarrow \frac{(3^{x-1})^2 + 7}{3^{x-1} + 1} = 4$$

Let $3^{x-1} = y$

$$\therefore \frac{y^2 + 7}{y + 1} = 4$$

$$\Rightarrow y^2 + 7 = 4y + 4$$

$$\Rightarrow y^2 - 4y + 3 = 0$$

$$\Rightarrow y = 1, 3$$

$$\therefore 3^{x-1} = 3^0 \text{ and } 3^{x-1} = 3^1$$

$$\Rightarrow x = 1, 2$$

68. Since, a, b and c are in AP.

$$\therefore 2b = a + c$$

Let α and β be the roots of the given equation.

$$\therefore \alpha + \beta = \frac{2b}{a}$$

$$= \frac{a+c}{a} = 1 + \frac{c}{a} \quad \dots(i)$$

and $\alpha\beta = \frac{c}{a}$

Now, $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$

$$= \left(1 + \frac{c}{a}\right)^2 - 4 \cdot \frac{c}{a} = \left(1 - \frac{c}{a}\right)^2$$

$$\Rightarrow \alpha - \beta = 1 - \frac{c}{a} \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$2\alpha = 2 \Rightarrow \alpha = 1$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\beta = \frac{2c}{a}$$

$$\therefore \beta = \frac{c}{a}$$

69. Let α and β be the roots of the given equation.

$$\therefore \alpha + \beta = \frac{4 + \sqrt{5}}{5 + \sqrt{2}}$$

$$\text{and } \alpha\beta = \frac{8 + 2\sqrt{5}}{5 + \sqrt{2}}$$

Now, harmonic mean of α and $\beta = \frac{2\alpha\beta}{\alpha + \beta}$

$$\begin{aligned} &= \frac{16 + 4\sqrt{5}}{5 + \sqrt{2}} \\ &= \frac{4 + \sqrt{5}}{5 + \sqrt{2}} \\ &= \frac{16 + 4\sqrt{5}}{4 + \sqrt{5}} \times \frac{(4 - \sqrt{5})}{(4 - \sqrt{5})} \\ &= \frac{64 - 16\sqrt{5} + 16\sqrt{5} - 4 \times 5}{16 - 5} \\ &= \frac{44}{11} = 4 \end{aligned}$$

70. $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$

$$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha)$$

$$\Rightarrow \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = 0$$

$$\Rightarrow \alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma = 3 \times (-2) = -6$$

71. $(\alpha + \sqrt{\beta}) + (\alpha - \sqrt{\beta}) = -p$

$$\Rightarrow 2\alpha = -p$$

$$\Rightarrow \alpha = -\frac{p}{2}$$

$$\text{and } [(\alpha + \sqrt{\beta})(\alpha - \sqrt{\beta})] = q$$

$$\Rightarrow \alpha^2 - \beta = q$$

$$\Rightarrow \beta = \alpha^2 - q$$

$$= \left(-\frac{p}{2}\right)^2 - q$$

$$= \frac{p^2}{4} - q$$

$$\Rightarrow p^2 - 4q = 4\beta$$

The given equation is

$$(p^2 - 4q)(p^2x^2 + 4px) - 16q = 0$$

$$\Rightarrow 4\beta(p^2x^2 + 4px) - 16(\alpha^2 - \beta) = 0 \quad [\because \alpha^2 - \beta = q]$$

$$\Rightarrow \beta(4\alpha^2x^2 - 8\alpha x) - 4(\alpha^2 - \beta) = 0 \quad [\because p = -2\alpha]$$

$$\Rightarrow \alpha^2\beta x^2 - 2\alpha\beta x + \beta = \alpha^2$$

$$\Rightarrow (\alpha x\sqrt{\beta} - \sqrt{\beta})^2 = \alpha^2$$

$$\Rightarrow \alpha x\sqrt{\beta} - \sqrt{\beta} = \pm \alpha$$

$$\therefore x = \frac{1}{\alpha} \pm \frac{1}{\sqrt{\beta}}$$

72. Given, $x^2 - 2\sqrt{2}kx + 2e^{2\log k} - 1 = 0$

$$\therefore \text{Product of roots} = 2e^{2\log k} - 1 = 31$$

$$\Rightarrow 2e^{2\log k} = 32$$

$$\Rightarrow e^{\log k^2} = 16$$

$$\Rightarrow k^2 = 16$$

$$\therefore k = \pm 4$$

But $k = -4$, $\log k$ is not defined.

Hence, required value of k is 4.

73. Here, $\alpha + \beta = 2$ and $\alpha\beta = 4$

$$\begin{aligned} \text{Now, } (\alpha - \beta) &= \sqrt{(\alpha + \beta)^2 - 4\alpha\beta} \\ &= \sqrt{4 - 16} = 2\sqrt{3}i \end{aligned}$$

$$\begin{aligned} \therefore \alpha &= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= 2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) \end{aligned}$$

$$\begin{aligned} \text{and } \beta &= \frac{2 - 2\sqrt{3}i}{2} \\ &= 2\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right) \end{aligned}$$

$$\begin{aligned} \therefore \alpha^n + \beta^n &= \left[2\left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)\right]^n \\ &\quad + \left[2\left(\cos \frac{\pi}{3} - i \sin \frac{\pi}{3}\right)\right]^n \\ &= 2^n \left[2 \cos \frac{n\pi}{3}\right] \\ &= 2^{n+1} \cos \frac{n\pi}{3} \end{aligned}$$

74. Since, α, β and γ are the roots of $x^3 - 2x + 1 = 0$.

$$\text{We have, } \Sigma\alpha = 0 \quad \dots(i)$$

$$\Sigma\alpha\beta = -2 \quad \dots(ii)$$

$$\alpha\beta\gamma = -1 \quad \dots(iii)$$

$$\text{Now, } \Sigma\left(\frac{1}{\alpha + \beta - \gamma}\right) = \Sigma\left(\frac{1}{-\gamma - \gamma}\right)$$

$$= \Sigma\left(\frac{-1}{2\gamma}\right) = \frac{-1}{2} \Sigma \frac{1}{\gamma}$$

$$= \frac{-1}{2} \left[\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right]$$

$$= \frac{-1}{2} \left[\frac{\Sigma\alpha\beta}{\alpha\beta\gamma}\right] = \frac{-1}{2} \left[\frac{-2}{-1}\right] = -1$$

75. (i) b must be even integer.

(ii) Its discriminant is a perfect square of an even number.

$$\begin{aligned} \text{So, } D &= b^2 + 4 \cdot 16 \\ &= b^2 + 64 \end{aligned}$$

Here, 0 and 6 only the even integers value of b , on which the discriminant of given quadratic equation have perfect square of an even number.

So, the number of integral values of b is 2.

76. On taking option (c), it is clear that,

$$a\left(\frac{1}{2}\right)^2 + b\left(\frac{1}{2}\right) + c = 0$$

$$\Rightarrow a + 2b + 4c = 0$$

77. $2(3)^2 - c(3) + 3 = 0$

$$\Rightarrow c = 7$$

$$\text{and } 2x^2 - 7x + d = 0$$

$$\text{Also, } B^2 = 4AC$$

$$\Rightarrow 49 = 4 \times 2 \times d$$

$$\therefore d = \frac{49}{8}$$

$$78. \frac{(\sqrt{\sqrt{3}+1} + \sqrt{\sqrt{3}-1})^2 (\sqrt{3}-\sqrt{2})}{(\sqrt{\sqrt{3}+1})^2 - (\sqrt{\sqrt{3}-1})^2}$$

$$= \frac{(\sqrt{3}+1 + \sqrt{3}-1 + 2\sqrt{2})(\sqrt{3}-\sqrt{2})}{(\sqrt{3}+1) - (\sqrt{3}-1)}$$

$$= \frac{2(\sqrt{3}+\sqrt{2})(\sqrt{3}-\sqrt{2})}{2} = 1$$

79. We have, $|2x-3| < |x+5|$

Case I If $-\infty < x < -5$

$$\begin{array}{c} \leftarrow -5 \quad \quad \quad 3/2 \rightarrow \end{array}$$

$$\Rightarrow -2x+3 < -x-5$$

$$\Rightarrow -x < -8$$

$$\therefore x > 8$$

...(i)

So, no solution exist.

Case II If $-5 < x < 3/2$

$$\begin{array}{l} -2x+3 < x+5 \\ \Rightarrow -3x < 2 \Rightarrow x > -\frac{2}{3} \end{array}$$

$$\therefore -\frac{2}{3} < x < \frac{3}{2}$$

...(ii)

Case III If $\frac{3}{2} < x < \infty$

$$\Rightarrow 2x-3 < x+5$$

$$\Rightarrow x < 8$$

$$\therefore \frac{3}{2} < x < 8$$

...(iii)

From Eqs. (i), (ii) and (iii), we get

$$x \in \left(-\frac{2}{3}, 8\right)$$

$$80. \text{ We have, } \frac{k}{\alpha} + \frac{k}{\beta} = k \left(\frac{\alpha+\beta}{\alpha\beta} \right) = k \left(\frac{-b/a}{c/a} \right)$$

$$= \frac{-kb}{c} \quad \text{and} \quad \frac{k}{\alpha} \cdot \frac{k}{\beta} = \frac{k^2}{\alpha\beta} = \frac{k^2 a}{c}$$

$$\therefore x^2 - \left(\frac{-kb}{c}\right)x + \frac{k^2 a}{c} = 0$$

$$\Rightarrow cx^2 + kbx + k^2 a = 0$$

$$81. 3 \leq 3t-18 \leq 18 \Rightarrow 21 \leq 3t \leq 36$$

$$\Rightarrow 7 \leq t \leq 12 \Rightarrow 8 \leq t+1 \leq 13$$

$$82. \frac{x+11}{x-3} > 0$$

$$\Rightarrow (x-3)(x+11) > 0$$

$$\Rightarrow x < -11, x > 3$$

$$\Rightarrow x \in (-\infty, -11) \cup (3, \infty)$$

$$83. \frac{\log_{c+b} a + \log_{c-b} a}{2 \log_{c+b} a \cdot \log_{c-b} a}$$

$$= \frac{\frac{\log a}{\log(c+b)} + \frac{\log a}{\log(c-b)}}{2 \cdot \frac{\log a}{\log(c+b)} \cdot \frac{\log a}{\log(c-b)}}$$

$$= \frac{\log a \{ \log(c-b) + \log(c+b) \}}{2(\log a)^2}$$

$$= \frac{\log(c^2 - b^2)}{2 \log a} = \frac{\log a^2}{\log a^2} \quad [\because a^2 + b^2 = c^2]$$

$$= 1$$

$$84. ax^2 + bx + c = 0$$

$[\because c \neq 0]$

Replacing x by $\frac{1-bx}{ax}$, we get the required equation

$$a \left(\frac{1-bx}{ax} \right)^2 + b \left(\frac{1-bx}{ax} \right) + c = 0$$

$$\Rightarrow a(1+b^2x^2-2bx) + ax(b-b^2x) + ca^2x^2 = 0$$

$$\Rightarrow a + ab^2x^2 - 2abx + abx - ab^2x^2 + a^2cx^2 = 0$$

$$\Rightarrow acx^2 - bx + 1 = 0$$

85. Since, one root of the equation $x^2 + px + q = 0$ is

$2 + \sqrt{3}$, then the other root will be $2 - \sqrt{3}$.

$\therefore$ Sum of roots, $2 + \sqrt{3} + 2 - \sqrt{3} = -p$

$$\Rightarrow p = -4$$

and product of roots, $(2 + \sqrt{3})(2 - \sqrt{3}) = q$

$$\therefore q = 1$$

$$86. \alpha + \beta = -p, \alpha\beta = q$$

$$\therefore \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= p^2 - 2q$$

$$\Rightarrow (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$$

$$= (p^2 - 2q) - 2q = p^2 - 4q$$

$$87. a + b = -a$$

...(i)

$$ab = b$$

...(ii)

$$\text{From Eq. (ii), } a = 1$$

$$\text{From Eq. (i), } b = -2$$

$[\because b \neq 0]$

88. Let α and β be the roots, then

$$\alpha + \beta = b, \alpha\beta = c$$

Given,

$$|\alpha - \beta| = 1$$

$$\Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = 1$$

$$\Rightarrow b^2 - 4c = 1$$

89. $\therefore \alpha = \omega, \beta = \omega^2$ will satisfy the given equation.

$$\text{Now, } \alpha^{19} = \omega^{19} = \omega, \beta^7 = \omega^{14} = \omega^2$$

$\therefore$ Required equation is

$$x^2 - (\omega + \omega^2)x + \omega^3 = 0$$

$$\Rightarrow x^2 + x + 1 = 0$$

$$90. \therefore x^2 + 15|x| + 14 = |x|^2 + 15|x| + 14 > 0$$

for all real x .

So, given equation has no solution.

91. Since, roots are equal.

$$\therefore (2\sqrt{6})^2 = 4 \cdot 2 \cdot a$$

$$\Rightarrow 24 = 8a \Rightarrow a = 3$$

$$92. \text{ Let } \alpha = \frac{3}{2} + \frac{7}{2}i \text{ and } \beta = \frac{3}{2} - \frac{7}{2}i$$

$$\therefore \alpha + \beta = 3, \alpha\beta = \frac{9}{4} + \frac{49}{4} = \frac{29}{2}$$

$$\Rightarrow \frac{6}{a} = 3, \frac{b}{a} = \frac{29}{2}$$

$$\Rightarrow a = 2, b = 29$$

$$\therefore a + b = 31$$

$$93. \text{ Discriminant } (D) = (-2\sqrt{3})^2 + 88$$

$$= 100 = 10^2$$

So, roots are real, irrational and unequal.

6

Permutation and Combination

Fundamental Principles of Counting (FPC)

Fundamental Principle of Multiplication

If there are two jobs such that one of them can be completed in m ways and when it has been completed in anyone of these m ways, second job can be completed in n ways, then the two jobs in succession can be completed in $m \times n$ ways.

➤ **Example 1.** In a class, there are 15 boys and 10 girls. The teacher wants to select a boy and a girl to represent the class in a function, then the total number of selection by teacher is

- (a) 15 (b) 10 (c) 25 (d) 150

Sol. (d) Here, the teacher is to perform two jobs:

- (i) Selecting a boy among 15 boys.
(ii) Selecting a girl among 10 girls.

The first job can be performed in 15 ways and the second in 10 ways. Therefore, by fundamental principle of multiplication, the total number of selection = $15 \times 10 = 150$ ways.

Fundamental Principle of Addition

If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.

➤ **Example 2.** There are 20 students in a class, in which 12 boys and 8 girls. The class teacher select either a boy or a girl for monitor of the class. In how many ways, the teacher can make this selection?

- (a) 120 (b) 12 (c) 96 (d) 20

Sol. (d) By fundamental principle of addition, the number of ways in which either a boy or a girl can be chosen as a monitor = $12 + 8 = 20$ ways.

➤ **Example 3.** Three dices are rolled. The number of possible outcomes in which atleast one dice shows 5, is

- (a) 36 (b) 18 (c) 90 (d) 91

Chapter Snapshot

- Fundamental Principles of Counting (FPC)
- Factorial
- Exponent of Prime p in Factorial n
- Representation of Symbols ${}^nP_r$ and ${}^nC_r$
- Some Basic Arrangements and Selections
- Summation of Numbers (3 different ways)
- Permutations under Certain Conditions
- Circular Permutations
- Geometrical Applications of ${}^nC_r$
- Selection of One or More Objects
- Number of Divisors and the Sum of the Divisors of a Given Natural Number
- Division of Objects into Groups
- Dearrangements
- Number of Integral Solutions of Linear Equations and Inequations

Sol. (d) When a die is rolled, there are six possible outcomes. So, by FPC, the total number of outcomes when three dice is rolled = $6 \times 6 \times 6 = 216$.

Now, the number of possible outcomes in which atleast one die shows 5 = Total number of possible outcomes – Number of possible outcomes in which 5 does not appear on any dice

$$= 6^3 - 5^3 = 216 - 125 = 91$$

➤ **Example 4.** How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5, if repetition of digits is not allowed?

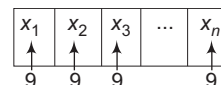
- (a) 120 (b) 1080 (c) 300 (d) 600

Sol. (c) In a four-digit numbers, 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways (i.e. 1, 2, 3, 4, 5). Since, repetition of digits is not allowed and 0 can be used at hundred's place. So, hundred's place can be filled in 5 ways. Now, anyone of the remaining 4 digits can be used to fill up ten's place. So, ten's place can be filled in 4 ways. One's place can be filled from the remaining 3 digits in 3 ways. Hence, by fundamental principle of multiplication, the required number of numbers = $5 \times 5 \times 4 \times 3 = 300$.

➤ **Example 5.** The total number of n -digit numbers ($n > 1$) having the property that no two consecutive digits are same, is

- (a) 9^n (b) 10^n
(c) $10^n - 9^n$ (d) None of these

Sol. (a)



The digit x_1 can be selected in 9 ways as '0' cannot be selected. The digit x_2 can be selected in 9 ways as 0 can be selected but digit in position x_1 cannot be selected. Similarly, each of the remaining digit can also be selected in 9 ways. Thus, the total number of such numbers is 9^n .

➤ **Example 6.** The number of polynomials of the form $x^3 + ax^2 + bx + c$ that are divisible by $x^2 + 1$ where $a, b, c \in \{1, 2, 3, \dots, 9, 10\}$, are

- (a) 7 (b) 8
(c) 9 (d) 10

$$\begin{array}{r} \text{Sol. (d) } x^2 + 1 \overline{) \begin{array}{r} x^3 + ax^2 + bx + c \\ \underline{x^3} \\ ax^2 + (b-1)x + c \end{array}} \\ \underline{ax^2} + a \\ (b-1)x + c - a \end{array}$$

Now, remainder $(b-1)x + c - a$ must be zero for any x , then

$$b-1=0 \quad \text{and} \quad c-a=0$$

$$\Rightarrow b=1 \quad \text{and} \quad c=a$$

Now, c or a can be selected in 10 ways. Hence, number of polynomials are 10.

Work Book Exercise 6.1

- 1 Twenty buses run between New Delhi to Meerut. If a man goes from Meerut to New Delhi by a bus and comes back to Meerut by another bus. The total possible ways is

20 40 39 380

- 2 Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Then, the total number of words which have atleast one letter repeated, is

30240 69760
99748 None of these

- 3 An examination paper contains 8 questions of which 4 have 3 possible answers, 3 have 2 possible answers and the remaining one question has 5 possible answers. The total number of possible answers to all the question is

2880 178
94 3240

- 4 There are stalls for 10 animals in a ship. The number of ways in which ship load can be made, if there are cows, calves and horses to be transported, animals of each kind being not less than 10, is

4^{10}
 13^{10}
130
None of the above

- 5 A letter lock consists of three rings, each marked with 10 different letters, the number of ways in which it is possible to make an unsuccessful attempt to open the lock is

999 899
479 None of these

- 6 How many two-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, if the digits can be repeated?

120 20 25 10

- 7 A five-digit numbers divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done, is

216 600 240 3125

- 8 If $p, q \in (1, 2, 3, 4)$, then the number of equations of the form $px^2 + qx + 1 = 0$ having real roots, is

4 5 6 7

- 9 Find the number ordered pairs (x, y) , if $x, y \in \{0, 1, 2, 3, \dots, 10\}$ and if $|x - y| > 5$.

30 20
15 None of these

- 10 Find the total number of ways in which two small squares can be selected on the normal chessboard, if they are not in same row or same column.

64 1296
1568 None of these

Factorial

For any natural number n , we define factorial of n ($n!$ or $\underline{n}$) as follows:

$$n! = n(n-1)(n-2)\dots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$\therefore 1! = 1, 2! = 2 \times 1 = 2, 3! = 3 \times 2 \times 1 = 6$ and so on.

✎ We can extend the definition of the factorial to zero and negative integers as follows:

(i) $0! = 1$

(ii) Factorial of negative integers and fractions are not defined.

(iii) $n! = n(n-1)! = n(n-1)(n-2)!$

(iv) $\frac{n!}{r!} = n(n-1)(n-2)\dots, n > r$

✎ **Example 7.** If $(n+1)! = 12 \times (n-1)!$, then n is equal to

- (a) 4 (b) 5
(c) 3 (d) 6

Sol. (c) We have, $(n+1)! = 12 \times (n-1)!$
 $\Rightarrow (n+1)(n)(n-1)! = 12 \times (n-1)!$
 $\Rightarrow n^2 + n = 12$
 $\Rightarrow n^2 + n - 12 = 0$
 $\Rightarrow (n+4)(n-3) = 0$
 $\therefore n = 3, n = -4$

✎ **Example 8.** If $\frac{1}{8!} + \frac{1}{9!} = \frac{x}{10!}$, then x is equal to

- (a) 100 (b) 10
(c) 90 (d) None of these

Sol. (a) We have,
 $\frac{1}{8!} + \frac{1}{9 \times 8!} = \frac{x}{10 \times 9 \times 8!}$
 $\Rightarrow 1 + \frac{1}{9} = \frac{x}{10 \times 9} \Rightarrow \frac{10}{9} = \frac{x}{10 \times 9}$
 $\therefore x = 100$

✎ **Example 9.** Find the remainder when $1! + 2! + 3! + \dots + n!$ is divided by 15, if $n \geq 5$.

- (a) 13 (b) 12 (c) 3 (d) 5

Sol. (c) Let $N = 1! + 2! + 3! + \dots + n!$
 $\Rightarrow \frac{N}{15} = \frac{1! + 2! + 3! + 4! + 5! + 6! + \dots + n!}{15}$
 $\Rightarrow \frac{N}{15} = \frac{1! + 2! + 3! + 4!}{15} + \frac{5! + 6! + 7! + \dots + n!}{15}$
 $\Rightarrow \frac{N}{15} = \frac{1 + 2 + 6 + 24}{15} + 8k$
 $\Rightarrow \frac{N}{15} = \frac{33}{15}$ [as $5!, 6!, 7!, 8!$ are divisible by 15]

Hence, remainder is 3.

✎ **Example 10.** The value of $\frac{(2n)!}{n!}$ is

- (a) $\{1 \cdot 3 \cdot 5 \dots (2n-1)\} 2^n$ (b) $\{1 \cdot 3 \cdot 5 \dots (n-1)\} 2^n$
(c) $\{1 \cdot 3 \cdot 5 \dots (2n+1)\} 2^n$ (d) None of these

Sol. (a) We have,

$$\begin{aligned} \frac{(2n)!}{n!} &= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \dots (2n-2)(2n-1)(2n)}{n!} \\ &= \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)\} \cdot \{2 \cdot 4 \cdot 6 \cdot 8 \dots (2n-2)(2n)\}}{n!} \\ &= \frac{\{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)\} \cdot 2^n \cdot n!}{n!} \\ &= \{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)\} \cdot 2^n \end{aligned}$$

✎ **Example 11.** The one's and ten's digits in $1! + 2! + 3! + \dots + 49!$ are respectively
(a) 4 and 1 (b) 3 and 1 (c) 4 and 2 (d) 5 and 1

Sol. (b) We know that $n!$ is divisible by 100 for $n \geq 10$ and $1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! = 409113$.

Therefore,

$$1! + 2! + 3! + \dots + 49! = 409113 + \text{a multiple of } 100$$

Hence, one's and ten's digits in $1! + 2! + \dots + 49!$ are 3 and 1, respectively.

✎ **Example 12.** If

$S = (1)(1!) + (2)(2!) + (3)(3!) + \dots + n(n!)$, then

- (a) $\frac{S+1}{n!} \in \text{integer}$
(b) $\frac{S+1}{n!} \notin \text{integer}$
(c) $\frac{S+1}{n!}$ cannot be discussed
(d) None of the above

Sol. (a) We have,

$$\begin{aligned} S &= \sum_{k=1}^n k(k!) = \sum_{k=1}^n \{(k+1)-1\}(k!) \\ &= \sum_{k=1}^n \{(k+1)k! - k!\} \\ &= (n+1)! - 1 \end{aligned}$$

$$\Rightarrow S + 1 = (n+1)!$$

Thus, $\frac{S+1}{n!} \in \text{Integer}$

✎ **Example 13.** $S_n = 1! + 2! + 3! + 4! + \dots + n!$ cannot be the square of natural number except for n , is

- (a) 1, 4 (b) 2, 3 (c) 1, 3 (d) 2, 4

Sol. (c) For $n = 1$, we have

$S_1 = 1! = 1$, which is a perfect square.

For $n = 2$, we have

$S_2 = 1! + 2! = 1 + 2 = 3$, which is not a perfect square.

For $n = 3$, we have

$$S_3 = 1! + 2! + 3! = 1 + 2 + 6 = 9$$

which is a perfect square.

For $n = 4$, we have

$$S_4 = 1! + 2! + 4! = 1 + 2 + 6 + 24 = 33$$

which is not a perfect square.

For $n \geq 5$, we find that the digits at unit's place in $n!$ is 0 and $S_4 = 1! + 2! + 3! + 4!$ has 3 as the digit at unit's place. Therefore, for $n \geq 5$, S_n has 3 at unit's place. But the square of any natural number does not have 3 at unit's place.

Therefore, S_n is not a perfect square for $n \geq 5$.

Hence, $S_n = 1! + 2! + 3! + \dots + n!$ is not a perfect square of a natural number except for $n = 1, 3$.

➤ **Example 14.** Let p be a prime number such that $p \geq 11$. Let $n = p! + 1$. The number of primes in the list $n+1, n+2, n+3, \dots, n+p-1$, is

- (a) $p-1$ (b) 2
(c) 1 (d) None of these

Sol. (d) For $1 \leq i \leq p-1$, $p!$ is divisible by $(i+1)$.

Thus, $n+i = p! + (i+1)$ is divisible by $(i+1)$

for $1 \leq i \leq p-1$

$\therefore$ None of $n+1, n+2, \dots, n+p-1$ is prime.

Exponent of Prime p in Factorial n

Let p be a prime number and n be a positive integer.

Let $E_p n!$ denotes the exponent of the prime p in the positive integer n . Then,

$$E_p(n!) = \left\lfloor \frac{n}{p} \right\rfloor + \left\lfloor \frac{n}{p^2} \right\rfloor + \left\lfloor \frac{n}{p^3} \right\rfloor + \dots + \left\lfloor \frac{n}{p^s} \right\rfloor$$

where, s is the largest positive integer such that $p^s \leq n < p^{s+1}$.

➤ **Example 15.** The exponent of 3 in $100!$ is

- (a) 33 (b) 48 (c) 45 (d) 50

Sol. (b) We have, $n = 100!$

$$E_3(100!) = \left\lfloor \frac{100}{3} \right\rfloor + \left\lfloor \frac{100}{3^2} \right\rfloor + \left\lfloor \frac{100}{3^3} \right\rfloor + \left\lfloor \frac{100}{3^4} \right\rfloor$$

$$= 33 + 11 + 3 + 1 = 48$$

Hence, the exponent of 3 in $100! = 48$.

➤ **Example 16.** The exponent of 12 in $50!$ is

- (a) 22 (b) 23 (c) 45 (d) 48

Sol. (a) We have,

$$12 = 2^2 \times 3 = 2 \times 2 \times 3 \text{ and } 50! = 2^a \times 3^b \times 5^c \times \dots$$

$$\text{Now, } a = E_2(50!) = \left\lfloor \frac{50}{2} \right\rfloor + \left\lfloor \frac{50}{2^2} \right\rfloor + \left\lfloor \frac{50}{2^3} \right\rfloor + \left\lfloor \frac{50}{2^4} \right\rfloor + \left\lfloor \frac{50}{2^5} \right\rfloor$$

$$= 25 + 12 + 6 + 3 + 1 = 47$$

$$\text{and } b = E_3(50!) = \left\lfloor \frac{50}{3} \right\rfloor + \left\lfloor \frac{50}{3^2} \right\rfloor + \left\lfloor \frac{50}{3^3} \right\rfloor$$

$$= 16 + 5 + 1 = 22$$

$$\therefore 50! = 2^{47} \times 3^{22} \times 5^c \times \dots$$

$$\Rightarrow 50! = (2^2 \times 3)^{22} \times 2^3 \times 5^c \times \dots$$

$$\Rightarrow 50! = (12)^{22} \times 2^3 \times 5^c \times \dots$$

$$\text{Hence, } E_{12}(50!) = 22$$

➤ **Example 17.** Let

$$E = \left\lfloor \frac{1}{3} + \frac{1}{50} \right\rfloor + \left\lfloor \frac{1}{3} + \frac{2}{50} \right\rfloor + \left\lfloor \frac{1}{3} + \frac{3}{50} \right\rfloor + \dots + 50 \text{ terms,}$$

then the exponent of 2 in $E!$ is

- (a) 17 (b) 25 (c) 15 (d) 20

Sol. (c) Let $E = \sum_{x=1}^{50} \left\lfloor \frac{1}{3} + \frac{x}{50} \right\rfloor$

$$\text{For } 1 \leq x \leq 33, \frac{1}{3} < \frac{1}{3} + \frac{x}{50} < 1$$

$$\therefore \left\lfloor \frac{1}{3} + \frac{x}{50} \right\rfloor = 0, \text{ for } 1 \leq x \leq 33$$

$$\text{For } 34 \leq x \leq 50, 1 < \frac{1}{3} + \frac{x}{50} < \frac{4}{3}$$

$$\Rightarrow \left\lfloor \frac{1}{3} + \frac{x}{50} \right\rfloor = 1, \text{ for } 34 \leq x \leq 50$$

$$\text{Thus, } E = 17 \text{ and } p = 2$$

$$\therefore 2^4 < 17 < 2^5$$

$$\therefore \text{Exponent of 2 in } 17! = \left\lfloor \frac{17}{2} \right\rfloor + \left\lfloor \frac{17}{4} \right\rfloor + \left\lfloor \frac{17}{8} \right\rfloor + \left\lfloor \frac{17}{16} \right\rfloor$$

$$= 8 + 4 + 2 + 1 = 15$$

Work Book Exercise 6.2

1 The exponent of 7 in $100!$ is

- 14 15
16 None of these

2 If $\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$, then x is equal to

- 49 36
64 None of these

3 If $n!$, $3 \times n!$ and $(n+1)!$ are in GP, then $n!$, $5 \times n!$ and $(n+1)!$ are in

- AP GP
HP None of these

4 The number of zero at the end of $70!$ is

- 16
5
7
70

5 Sum of the series $\sum_{r=1}^n (r^2 + 1)r!$ is

- $(n+1)!$
 $(n+2)! - 1$
 $n(n+1)!$
None of the above

Representation of Symbols ${}^nP_r$ and ${}^nC_r$

Symbol ${}^nP_r$ or $P(n, r)$ If n is a natural number and r is an integer satisfying $0 \leq r \leq n$, then the natural number $\frac{n!}{(n-r)!}$ is denoted by the symbol ${}^nP_r$ or $P(n, r)$.

i.e. ${}^nP_r = P(n, r) = \frac{n!}{(n-r)!}$

e.g. ${}^4P_2 = P(4, 2) = \frac{4!}{(4-2)!}$
 $= \frac{4!}{2!} = \frac{4 \times 3 \times 2!}{2!} = 12$

${}^7P_3 = P(7, 3) = \frac{7!}{(7-3)!} = \frac{7!}{4!}$
 $= \frac{7 \times 6 \times 5 \times 4!}{4!} = 210$

Symbol ${}^nC_r$ or $C(n, r)$ If n is a natural number and r is an integer satisfying $0 \leq r \leq n$, then the natural number $\frac{n!}{(n-r)! r!}$ is denoted by the symbol ${}^nC_r$ or $C(n, r)$.

i.e. ${}^nC_r = C(n, r) = \frac{n!}{(n-r)! r!}$

e.g. ${}^8C_2 = C(8, 2) = \frac{8!}{2! (8-2)!} = \frac{8 \times 7 \times 6!}{2 \times 1 \times 6!} = 28$

${}^{10}C_7 = C(10, 7) = \frac{10!}{7! (10-7)!}$
 $= \frac{10!}{7! 3!} = \frac{10 \times 9 \times 8 \times 7!}{7! \times 3 \times 2 \times 1} = 120$

Some Properties Related to ${}^nP_r$ and ${}^nC_r$

- i. ${}^nP_n = n!$
- ii. ${}^nP_r = {}^{n-1}P_r + r {}^{n-1}P_{r-1}$
- iii. $\sum_{r=1}^n r {}^r P_r = {}^{n+1}P_{n+1} - 1$
- iv. ${}^nC_r$ is a natural number and exists only if $n \geq r$; $n, r (\geq 0) \in \text{integers}$.
- v. ${}^nC_0 = 1 = {}^nC_n$
- vi. ${}^nC_r = {}^nC_{n-r}, 0 \leq r \leq n$
- vii. ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r, 1 \leq r \leq n$

- viii. ${}^nC_r = {}^nC_s$ implies $r = s$ or $r + s = n$
- ix. ${}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)} \cdot {}^{n-2}C_{r-2} = \dots$
- x. $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-(r-1)}{r}$
- xi. ${}^nC_r$ is greatest = $\begin{cases} r = \frac{n}{2}, & \text{if } n \text{ is even} \\ r = \frac{n \pm 1}{2}, & \text{if } n \text{ is odd} \end{cases}$
- xii. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$
- xiii. ${}^nC_0 + {}^nC_2 + \dots = {}^nC_1 + {}^nC_3 + \dots = 2^{n-1}$
- xiv. ${}^{2n+1}C_0 + {}^{2n+1}C_1 + \dots + {}^{2n+1}C_n = 2^{2n}$
- xv. ${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \dots + {}^nC_r = {}^{n+1}C_{r+1}$
- xvi. The number of combinations of n distinct objects taken $r (\leq n)$ at a time, when $k (0 \leq k \leq r)$ particular objects always occur, is ${}^{n-k}C_{r-k}$.
- xvii. The number of combinations of n distinct objects taken r at a time, when $k (1 \leq k \leq r)$ objects never occur, is ${}^{n-k}C_r$.
- xviii. The total number of selections of one or more objects from n different objects
 $= 2^n - 1 = ({}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n)$

➤ **Example 18.** If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, then r is equal to

- (a) 39
- (b) 40
- (c) 41
- (d) None of these

Sol. (c) We have, $\frac{56!}{(50-r)!} \cdot \frac{(51-r)!}{54!} = 30800$

$\Rightarrow 56 \times 55 (51-r) = 30800$

$\Rightarrow 51-r = \frac{30800}{280 \times 11} = \frac{30800}{3080} = 10$

$\therefore r = 51 - 10 = 41$

➤ **Example 19.** ${}^{4n}C_{2n} : {}^{2n}C_n$ is equal to

- (a) $\frac{\{1 \cdot 3 \cdot 5 \dots (4n-1)\}}{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}^2}$
- (b) $\frac{\{1 \cdot 3 \cdot 5 \dots (4n-1)\}^2}{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}}$
- (c) $\frac{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}^2}{\{1 \cdot 3 \cdot 5 \dots (4n-1)\}}$
- (d) None of these

Sol. (a) We have, $\frac{{}^{4n}C_{2n}}{{}^{2n}C_n} = \frac{4n!}{2n!2n!} \times \frac{n!n!}{2n!}$

$$= \frac{\{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \dots (4n-1) 4n!\} n! n!}{(1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \dots 2n)^2 2n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \dots (4n-1)\} \{2 \cdot 4 \cdot 6 \dots 4n\} n! n!}{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}^2 \{2 \cdot 4 \cdot 6 \dots 2n\}^2 2n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \dots (4n-1)\} 2^{2n} 2n! n! n!}{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}^2 2^{2n} \{1 \cdot 2 \cdot 3 \dots n\}^2 2n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \dots (4n-1)\} 2^{2n} 2n! n! n!}{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}^2 2^{2n} (n!)^2 2n!}$$

$$= \frac{\{1 \cdot 3 \cdot 5 \dots (4n-1)\}}{\{1 \cdot 3 \cdot 5 \dots (2n-1)\}^2}$$

➤ **Example 20.** The number of positive terms in the sequence $x_n = \frac{195}{4 \cdot {}^nP_n} - \frac{{}^{n+3}P_3}{{}^{n+1}P_{n+1}}$, $n \in N$ are

- (a) 2 (b) 3
(c) 4 (d) None of these

Sol. (c) We have, $x_n = \frac{195}{4 \cdot {}^nP_n} - \frac{{}^{n+3}P_3}{{}^{n+1}P_{n+1}}$

$$= \frac{195}{4 \cdot n!} - \frac{(n+3)(n+2)(n+1)}{(n+1)!} = \frac{195}{4 \cdot n!} - \frac{(n+3)(n+2)}{n!}$$

$$= \frac{195 - 4n^2 - 20n - 24}{4 \cdot n!} = \frac{171 - 4n^2 - 20n}{4 \cdot n!}$$

$\therefore x_n$ is positive.

$$\therefore \frac{171 - 4n^2 - 20n}{4 \cdot n!} > 0 \Rightarrow 4n^2 + 20n - 171 < 0$$

which is true for $n = 1, 2, 3, 4$.

Hence, the given sequence has 4 positive terms.

➤ **Example 21.** The value of $({}^7C_0 + {}^7C_1) + ({}^7C_1 + {}^7C_2) + \dots + ({}^7C_6 + {}^7C_7)$ is

(a) $2^8 - 2$ (b) $2^8 - 1$ (c) $2^8 + 1$ (d) 2^8

Sol. (a) $({}^7C_0 + {}^7C_1) + ({}^7C_1 + {}^7C_2) + \dots + ({}^7C_6 + {}^7C_7)$

$$= {}^8C_1 + {}^8C_2 + \dots + {}^8C_7$$

$$= {}^8C_0 + {}^8C_1 + {}^8C_2 + \dots + {}^8C_7 + {}^8C_8 - ({}^8C_0 + {}^8C_8)$$

$$= 2^8 - (1 + 1) = 2^8 - 2$$

➤ **Example 22.** The value of n for which

$${}^{n-1}C_4 - {}^{n-1}C_3 - \frac{5}{4} \cdot {}^{n-2}P_2 < 0, \text{ where } n \in N, \text{ is}$$

- (a) $\{5, 6, 7, 8, 9, 10\}$ (b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$
(c) $(0, 2) \cup (3, 11)$ (d) $(-\infty, 2) \cup (3, 11)$

Sol. (a) We have, ${}^{n-1}C_4 - {}^{n-1}C_3 - \frac{5}{4} \cdot {}^{n-2}P_2 < 0$

$$\Rightarrow \frac{(n-1)(n-2)(n-3)(n-4)}{4!} - \frac{(n-1)(n-2)(n-3)}{3!} - \frac{5}{4} \cdot \frac{(n-2)(n-3)}{2!} < 0$$

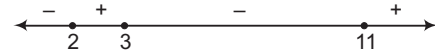
$$\Rightarrow \frac{(n-2)(n-3)}{24} \{(n-1)(n-4) - 4(n-1) - 30\} < 0$$

$$\Rightarrow \frac{(n-2)(n-3)}{24} \{n^2 - 9n - 22\} < 0$$

$$\Rightarrow \frac{(n-2)(n-3)(n-11)(n+2)}{24} < 0$$

$$\Rightarrow (n-2)(n-3)(n-11) < 0 \quad [\because n+2 > 0 \text{ for } n \in N]$$

$$\Rightarrow n \in (-\infty, 2) \cup (3, 11)$$



$\Rightarrow n \in (0, 2) \cup (3, 11) \Rightarrow n = 1, 4, 5, 6, 7, 8, 9, 10$
But ${}^{n-1}C_4$ and ${}^{n-2}P_2$ both are meaningful for $n \geq 5$.
Hence, $n = 5, 6, 7, 8, 9, 10$.

➤ **Example 23.** The value of $\frac{{}^{20}C_n}{{}^{25}C_m}$ when both numerator and denominator have their greatest values, is

- (a) $\frac{140}{403}$ (b) $\frac{140}{4025}$ (c) $\frac{143}{4025}$ (d) $\frac{140}{2045}$

Sol. (c) We know that,

$$\text{Greatest value of } {}^nC_r = \begin{cases} {}^nC_{n/2}, & \text{if } n \text{ is even} \\ {}^nC_{(n+1)/2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore {}^{20}C_n \text{ is greatest for } n = \frac{20}{2} = 10$$

$$\text{and } {}^{25}C_m \text{ is greatest for } m = \frac{25+1}{2} = 13$$

For $n = 10$ and $m = 13$, we have

$$\frac{{}^{20}C_n}{{}^{25}C_m} = \frac{{}^{20}C_{10}}{{}^{25}C_{13}} = \frac{20!}{10! \times 10!} \times \frac{12! \times 13!}{25!} = \frac{143}{4025}$$

➤ **Example 24.** $(n!)!$ is divisible by

- (a) $28(2n-1)^{n-1}$ (b) $(3n)!$
(c) $(n!)^{n!}$ (d) $(n!)^{(n-1)!}$

Sol. (d) Clearly, $(n!)!$ is the product of natural numbers from 1 to $n!$.

$$\therefore (n!)! = \{1 \times 2 \times 3 \times \dots \times n\} \times \{(n+1)(n+2) \dots (2n)\}$$

$$\times \{(2n+1)(2n+2) \dots (3n)\}$$

$$\vdots$$

$$\times \{(n! - n + 1)(n! - n + 2) \dots (n! - n + 3) \dots n!\} \dots (i)$$

We observe that,

Last term of first bracket on RHS of Eq. (i) is n .

Last term of second bracket on RHS of Eq. (i) is $2 \cdot n$.

Last term of third bracket on RHS of Eq. (i) is $3 \cdot n$ and so on.

Last term of the last bracket on RHS of Eq. (i) is $n!$ i.e.

$n(n-1)!$. It is clear from above pattern that there are $(n-1)!$

brackets on the RHS of Eq. (i) and in each bracket, there is product of n consecutive natural numbers.

From Eq. (i), we have

$$(n!)! = \prod_{r=1}^{(n-1)!} \{[r-1]n+1\} \{[r-1]n+2\} \dots \{(r-1)n+3\} \dots \{(r-1)n+n\}$$

We know that, the product of n consecutive natural numbers is divisible by $n!$.

So, let

$$\{(r-1)n+1\} \{[r-1]n+2\} \dots \{(r-1)n+n\} = n! I_r$$

where, $r = 1, 2, \dots, (n-1)!$

$$\therefore (n!)! = \prod_{r=1}^{(n-1)!} n! I_r = (n!)^{n-1!} \prod_{r=1}^{(n-1)!} I_r$$

$$= (n!)^{(n-1)!} \times \text{A natural number}$$

Thus, $(n!)!$ is divisible by $(n!)^{(n-1)!}$.

Work Book Exercise 6.3

6

Permutation and Combination

- If ${}^{15}P_r = {}^{14}P_8 + 8 \cdot {}^{14}P_7$, then r is equal to
a 5 b 6 c 7 d 8
- Which of the following is incorrect?
a ${}^nC_r = {}^nC_{n-r}$ b ${}^nC_r = {}^{n+1}C_r + {}^nC_{n-r}$
c ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$ d $r! \cdot {}^nC_r = {}^nP_r$
- If ${}^{n+2}C_8 : {}^{n-2}P_4 = 57 : 16$, then the value of n is
a 20 b 19 c 18 d 17
- The exponent of 7 in ${}^{100}C_{50}$ is
a 0 b 2
c 4 d None of these
- If ${}^{n-1}C_r = (k^2 - 3) {}^nC_{r+1}$, then k belongs to
a $[-\sqrt{3}, \sqrt{3}]$ b $(-\infty, -2)$
c $(2, \infty)$ d $(\sqrt{3}, 2)$
- If n is even and ${}^nC_0 < {}^nC_1 < {}^nC_2 < \dots < {}^nC_r$
 $> {}^nC_{r+1} > {}^nC_{r+2} > \dots > {}^nC_n$, then r is equal to
a $\frac{n}{2}$ b $\frac{n-1}{2}$ c $\frac{n-2}{2}$ d $\frac{n+2}{2}$
- The number of positive integers satisfying the inequality ${}^{n+1}C_{n-2} - {}^{n+1}C_{n-1} \leq 100$, are
a 9 b 8
c 5 d None of these

Some Basic Arrangements and Selections

Combinations

Each of the different selections made by taking some or all of a number of distinct objects or items, irrespective of their arrangements or order in which they are placed is called a combination.

Permutations

Each of the different arrangements which can be made by taking some or all of a number of distinct objects is called a permutation.

- Let r and n be positive integers such that $1 \leq r \leq n$. Then, the number of all permutations of n distinct items or objects taken r at a time is

$$n(n-1)(n-2)(n-3)\dots(n-r+1)$$

e.g. We have, $n(n-1)(n-2)\dots(n-r+1)$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)(n-r)!}{(n-r)!}$$

$$= \frac{n!}{(n-r)!} = {}^nP_r$$

So, the total number of arrangements (permutations) of n distinct items, taking r at a time is ${}^nP_r$ or $P(n, r)$.

- The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.
- The number of ways of selecting r items or objects from a group of n distinct items or objects is

$$\frac{n!}{(n-r)!r!} = {}^nC_r$$

e.g. Let there be n distinct objects which are to be arranged in a row by taking r at a time. The job of arranging n items by taking r at a time can be divided into two ordered sub-jobs J_1 and J_2 given by

J_1 : Selecting r items from n distinct items.

J_2 : Arranging r selected items.

We find that the sub-job J_1 can be completed in ${}^nC_r$ ways and sub-job J_2 can be completed in $r!$ ways. Therefore, number of ways of arranging n distinct items, taking r at a time

$$= {}^nC_r \times r!$$

Clearly, it is consistent with the result stated in point (1), because

$${}^nC_r \times r! = n(n-1)(n-2)\dots(n-r+1) = {}^nP_r$$

→ **Example 25.** Seven athletes are participating in a race. In how many ways can the first three prizes be won?

- (a) 420 (b) 210
(c) 35 (d) 105

Sol. (b) The total number of ways in which first three prizes can be won is equivalent to the number of arrangements of seven different things taken 3 at a time

$$= {}^7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210$$

→ **Example 26.** How many numbers between 400 and 1000 can be made with the digits 2, 3, 4, 5, 6 and 0?

- (a) 120
(b) 720
(c) 60
(d) None of the above

Sol. (c) Any number between 400 and 1000 will be three digits.

Since, the number should be greater than 400, therefore hundred's place can be filled up by any one of the three digits 4, 5 and 6 in 3 ways. Remaining two places can be filled up by remaining five digits in ${}^5P_2$ ways.

$$\therefore \text{Required number} = 3 \times {}^5P_2$$

$$= 3 \times \frac{5!}{3!} = 60$$

➤ **Example 27.** The number of numbers lying between 40000 and 70000 that can be made from the digits 0, 1, 2, 4, 5, 7, if digits can be repeated in the same number, is

- (a) 2591 (b) 932
(c) 766 (d) None of these

Sol. (a) The numbers are of five digits having 4 or 5 in ten thousand's place and the remaining digits are any of the given six digits.

$$\therefore \text{Number of ways to fill ten thousand's place} = {}^2P_1 = 2$$

$$\text{Number of ways to fill other four places} = 6^4$$

$$\therefore \text{Required number of numbers} = 2 \times 6^4 - 1 = 2591$$

➤ **Example 28.** In a network of railways, a small island has 15 stations. The number of different types of tickets to be printed for each class, if every station must have tickets for other station, are

- (a) 230
(b) 210
(c) 340
(d) None of the above

Sol. (b) For each pair of stations, two different types of tickets are required. Now, the number of selections of 2 stations from 15 stations = ${}^{15}C_2$.

$$\therefore \text{Required number of types of tickets} \\ = 2 \cdot {}^{15}C_2 = 2 \cdot \frac{15!}{2!13!} = 15 \times 14 = 210$$

➤ **Example 29.** In a certain test, a_i students gave wrong answers to atleast i questions, where $i = 1, 2, 3, \dots, k$. No student gave more than k wrong answers. The total number of wrong answers given is

- (a) $a_1 + a_2 + \dots + a_k$
(b) $a_1 + a_2 + \dots + a_{k-1}$
(c) $a_1 + a_2 + \dots + a_{k+1}$
(d) None of the above

Sol. (a) Total number of wrong answers
 $= 1 \cdot (a_1 - a_2) + 2 \cdot (a_2 - a_3) + \dots + (k-1) \cdot (a_{k-1} - a_k) + ka_k$
 $= a_1 + a_2 + a_3 + \dots + a_k$

➤ **Example 30.** In a class tournament, the participants were to play one game with another, two class players fell ill, having played 3 games each. If the total number of games played is 84, then the number of participants at the beginning, was

- (a) 22 (b) 15
(c) 17 (d) None of these

Sol. (b) Suppose, the two players did not play at all, so that the remaining $(n-2)$ players played ${}^{n-2}C_2$ matches.

Since, these two players played 3 matches each, hence the total number of matches

$$= {}^{n-2}C_2 + 3 + 3 = 84 \quad [\text{given}]$$

$$\begin{aligned} \Rightarrow \frac{(n-2)(n-3)}{1 \cdot 2} &= 78 \\ \Rightarrow n^2 - 5n + 6 &= 156 \\ \Rightarrow n^2 - 5n - 150 &= 0 \\ \Rightarrow (n-15)(n+10) &= 0 \\ \therefore n &= 15 \quad [\because n \neq -10] \end{aligned}$$

➤ **Example 31.** The maximum number of points of intersection of 8 circles are

- (a) 56 (b) 28
(c) 24 (d) 16

Sol. (a) Maximum number of points
 $= {}^8P_2 = 8 \times 7 = 56$

➤ **Example 32.** On a railway route, there are 15 stations. The number of tickets required in order that it may be possible to book a passenger from every station to every other, is

- (a) $\frac{15!}{13!2!}$ (b) $\frac{15!}{13!}$
(c) 15! (d) $\frac{15!}{2!}$

Sol. (a) Required number of tickets = ${}^{15}C_2 = \frac{15!}{2!13!}$

➤ **Example 33.** If a denotes the number of permutations of $(x+2)$ things taken all at a time, b denotes the number of permutations of x things taken 11 at a time and c denotes the number of permutations of $(x-11)$ things taken all at a time such that $a = 182bc$, then the value of x is

- (a) 15 (b) 12
(c) 10 (d) 19

Sol. (b) We have, ${}^{x+2}P_{x+2} = a \Rightarrow a = (x+2)!$

$$\text{Also, } {}^xP_{11} = b \Rightarrow b = \frac{x!}{(x-11)!}$$

$$\text{and } {}^{x-11}P_{x-11} = c$$

$$\Rightarrow c = (x-11)!$$

$$\therefore a = 182bc$$

$$\therefore (x+2)! = 182 \cdot \frac{x!}{(x-11)!} \cdot (x-11)!$$

$$\Rightarrow (x+2)(x+1) = 182 = 14 \times 13$$

$$\Rightarrow x+1 = 13$$

$$\text{and } x+2 = 14$$

$$\therefore x = 12$$

➤ **Example 34.** In a certain test, there are n questions. In this test, 2^{n-i} students gave wrong answers to atleast i questions, where $i = 1, 2, 3, \dots, n$. If the total number of wrong answers given is 2047, then n is equal to

- (a) 10 (b) 11
(c) 12 (d) 13

Sol. (b) The number of students answering exactly i ($1 \leq i \leq n-1$) questions wrongly is $2^{n-i} - 2^{n-i-1}$. The number of students answering all n questions wrongly is 2^0 . Hence, the total number of wrong answer is

$$\sum_{i=1}^{n-1} i(2^{n-i} - 2^{n-i-1}) + n(2^0) = 2047$$

$$\Rightarrow 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2^0 = 2047$$

$$\Rightarrow 2^n - 1 = 2047$$

$$\Rightarrow 2^n = 2048 \Rightarrow 2^n = 2^{11}$$

$$\therefore n = 11$$

➤ **Example 35.** The number of numbers of 9 different non-zero digits such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle, is

- (a) $2(4!)$ (b) $(4!)^2$
(c) $8!$ (d) None of these

Sol. (b) The required number of numbers = ${}^4P_4 \times {}^4P_4 = (4!)^2$

Summation of Numbers (3 different ways)

Sum of all the numbers greater than 10000 formed by the digits 1, 3, 5, 7, 9, no digit being repeated.

Method 1

All possible numbers = $5! = 120$

If one occupies the unit's place, then total number = 24

Hence, 1 enjoys unit's place 24 times.

				1

Similarly, 1 enjoys each place 24 times.

Sum due to 1 = $1 \times 24(1 + 10 + 10^2 + 10^3 + 10^4)$

Similarly, sum due to the digit

$$3 = 3 \times 24(1 + 10 + 10^2 + 10^3 + 10^4)$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots$$

Required total sum = $24(1 + 10 + 10^2 + 10^3 + 10^4)$
(1 + 3 + 5 + 7 + 9)

Method 2

In 1st column, there are twenty four 1's, twenty four 3's and so on and their sum = $24 \times 25 = 600$

Hence, add in vertical column normally, we get

	5th		2nd	1st	
	X	X	X	X	X
	X	X	X	X	X
	⋮	⋮	⋮	⋮	⋮
	X	X	X	X	X
120 Number	666	6	6	0	0 = 6666600

Method 3

Applicable only if the digits used are such that they have the same common difference. (Valid even, if the digits are repeating)

Writing all the numbers in ascending order of magnitude,

$$S = (13579 + 13597) + \dots + 97513 + 97531$$

$$S = (13579 + 99531) + (13597 + 97513) + \dots$$

$$= (111110) 60 \text{ times} = 6666600$$

$$S = \frac{n}{2}(l + L), \text{ where } n = \text{Number of numbers,}$$

$$l = \text{Smallest, } L = \text{Largest}$$

➤ **Example 36.** The sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time, is equal to

- (a) 93324
(b) 93328
(c) 92324
(d) None of the above

Sol. (a) The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time = Number of arrangements of 4 digits taken all at a time = ${}^4P_4 = 4! = 24$

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's place in all these numbers. Each of the digits 2, 3, 4, 5 occur in 3! i.e. 6 times in the unit's place.

So, total for the digits in the unit's place in all the numbers = $(2 + 3 + 4 + 5) \times 3! = 84$.

Since, each of the digits 2, 3, 4, 5 occurs 3! times in anyone of the remaining places. So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers = $(2 + 3 + 4 + 5) \times 3! = 84$

Hence, the sum of all the numbers
= $84(10^0 + 10^1 + 10^2 + 10^3) = 93324$

✎ Sum of all the numbers that can be formed with n (non-zero) digits, when repetition is not allowed and each digit being used

$$\text{once} = (n-1)! \frac{(10^n - 1)}{(10 - 1)} \text{ (sum of digits)}$$

$$= n-1! \text{ (sum of digits)} \times (111 \dots n \text{ times})$$

➤ **Example 37.** The sum of all the numbers greater than 10000 formed by using digits 0, 2, 4, 6, 8, no digit repeated in any number, is equal to

- (a) 5199960
(b) 5209960
(c) 5199980
(d) 5299960

Sol. (a) The total number of numbers formed by arranging the digits 0, 2, 4, 6, 8 taken all at a time is $5! = 120$. Some of these numbers are less than 10000 and some are greater than 10000. Infact all those numbers having 0 at ten thousand's place are four-digit numbers.

∴ Required sum = Sum of the numbers formed by using digits 0, 2, 4, 6, 8 – Sum of the numbers formed by using digits 2, 4, 6, 8

$$= (0 + 2 + 4 + 6 + 8)(5 - 1)! \left(\frac{10^5 - 1}{10 - 1} \right) - (2 + 4 + 6 + 8)(4 - 1)! \left(\frac{10^4 - 1}{10 - 1} \right)$$

$$= 480 \left(\frac{10^5 - 1}{10 - 1} \right) - 120 \left(\frac{10^4 - 1}{10 - 1} \right)$$

$$= 5333280 - 133320 = 5199960$$

Permutations under Certain Conditions

The number of all permutations (arrangements) of n different objects taken r at a time,

1. when a particular object is to be always included in each arrangement, is ${}^{n-1}C_{r-1} \times r!$.
2. when a particular object is never taken in each arrangement, is ${}^{n-1}C_r \times r!$.
3. when two specified objects always occur together is ${}^{n-2}C_{r-2} \times (r-1)! \times 2!$.

Proof

1. In order to arrange n distinct items by taking r at a time when a particular object is to be always included in each arrangement,

□ M □ M □ M □ ... □ M □
1st 2nd 3rd nth

Let us first put aside the specified item and select $(r-1)$ items from the remaining $(n-1)$ items. This can be done in ${}^{n-1}C_{r-1}$ ways. Now, we have r items, namely, one specified item and $(r-1)$ selected items. These r items can be arranged in $r!$ ways. Hence, the required number of arrangements is ${}^{n-1}C_{r-1} \times r!$.

2. If a particular objects is never taken in each arrangement, then we first select r objects from the remaining $(n-1)$ objects and then they are arranged. So, required number of arrangements is ${}^{n-1}C_r \times r!$.
3. In order to arrange n distinct items by taking r at a time when two specified objects always occur together, we first select $(r-2)$ objects from the remaining $(n-2)$ objects. This can be done in ${}^{n-2}C_{r-2}$ ways. Now, we consider two specified objects temporarily as a single object and mix it with $(r-2)$ selected objects. Now, these $(r-1)$ objects can be arranged in $(r-1)!$ ways. But two specified objects can be put together in $2!$ ways. Hence, required number of arrangements is

$${}^{n-2}C_{r-2} \times (r-1)! \times 2!$$

Example 38. There are 5 boys and 3 girls. In how many ways can they be seated in a row, so that all the three girls do not sit together?

- (a) $8!$ (b) $8!5!$
(c) $8! - 5!$ (d) $50 \cdot 6!$

Sol. (d) Total number of persons = $5 + 3 = 8$

When there is no restriction, then they can be seated in a row in $8!$ ways.

But when all the three girls sit together, consider the three girls as one person, we have only $(5 + 1) = 6$ persons. These 6 persons can be arranged in a row in $6!$ ways. But the three girls can be arranged among themselves in $3!$ ways.

∴ Number of ways when three girls are together = $6! \cdot 3!$

∴ Required number of ways in which all the three girls do not sit together

$$= 8! - 6! \cdot 3! = 6! (56 - 6) = 6! \cdot 50 = 50 \cdot 6!$$

Example 39. In how many ways can 10 examination paper be arranged, so that the best and the worst papers never come together?

- (a) $9 \cdot 8!$ (b) $8 \cdot 9!$
(c) $10 \cdot 9!$ (d) $9 \cdot 10!$

Sol. (b) The number of permutation of 10 papers when there is no restriction = $10!$

When the best and the worst papers come together, consider the two as one paper, we have only 9 papers. These 9 papers can be arranged in $9!$ ways.

But these two papers can be arranged among themselves in $2!$ ways.

∴ Number of arrangement when the best and the worst paper do not come together = $10! - 9! \cdot 2! = 8 \cdot 9!$

Example 40. The number of ways in which a committee of 5 can be chosen from 10 candidates so as to exclude the youngest, if it includes the oldest, is

- (a) 196
(b) 178
(c) 202
(d) None of the above

Sol. (a) There are two different ways of forming the committee:

- (i) oldest may be included
(ii) oldest may be excluded

- (i) If oldest is included, then youngest has to be excluded and we are to select 4 candidates out of 8.

This can be done in

$${}^8C_4 = \frac{8!}{4!4!} = \frac{8 \times 7 \times 6 \times 5}{4 \times 3 \times 2} = 70 \text{ ways}$$

- (ii) If oldest is excluded, then we are to select 5 candidates from 9 which can be done in

$${}^9C_5 = \frac{9!}{5!4!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 126 \text{ ways}$$

Hence, the total number of ways in which committee can be formed

$$= 126 + 70 = 196$$

➤ **Example 41.** The number of ways in which the letters of the word VOWEL can be arranged so that the letters O, E occupy only even places, is

- (a) 12 (b) 18
(c) 24 (d) None of these

Sol. (a) V O W E L
1 2 3 4 5
× ×

∴ O, E occupy only even places.

∴ O and E can be arranged in two × marked places in ${}^2P_2$ ways = $2! = 2$.

Also, the remaining 3 letters can be arranged in themselves in $3!$ ways = $3 \times 2 \times 1 = 6$

∴ Required number of words = $3! = 2 \times 6 = 12$

➤ **Example 42.** The number of ways in which the letters of word FRACTION be arranged so that no two vowels are together, is

- (a) 17330 (b) 14400
(c) 16440 (d) None of these

Sol. (b) The word 'FRACTION' consists of 8 different letters out of which A, I, O are 3 vowels and F, R, C, T, N are 5 consonants.

First of all, we arrange the consonants among themselves.

This can be done in $P(5, 5) = 5! = 120$ ways.

Let one such way be $\times F \times R \times C \times T \times N \times$

Now, no two vowels are together, if they are put at places marked (×). The 3 vowels can fill up these 6 places in $P(6, 3)$ ways.

Hence, the total number of words

$$= 120 \times P(6, 3) = 120 \times 6 \times 5 \times 4 = 120 \times 120 = 14400$$

➤ **Example 43.** A boat is to be managed by eight men of whom 2 can only row on bow side and 3 can only row on stroke side, the number of ways in which the crew can be arranged, is

- (a) 4360 (b) 5760
(c) 5930 (d) None of these

Sol. (b) First, we have to select 2 men for bow side and 3 for stroke side.

∴ The number of selections of the crew for two sides
 $= {}^5C_2 \times {}^3C_3$

For each selection, there are 4 persons, each on both sides who can be arranged in $4! \times 4!$ ways.

∴ Required number of arrangements

$$= {}^5C_2 \times {}^3C_3 \times 4! \times 4! = \frac{5 \times 4}{2} \times 1 \times 24 \times 24 = 5760$$

➤ **Example 44.** A teaparty is arranged for 16 people along two sides of a large table with 8 chairs on each side. Four men want to sit on one particular side and two on the other side. The number of ways in which they can be seated, is

- (a) $\frac{6!8!10!}{4!6!}$ (b) $\frac{8!8!10!}{4!6!}$
(c) $\frac{8!8!6!}{6!4!}$ (d) None of these

Sol. (b) There are 8 chairs on each side of the table. Let the sides be represented by A and B. Let four persons sit on side A, then number of ways of arranging 4 persons on 8 chairs on side A = ${}^8P_4$ and then two persons sit on side B. The number of ways of arranging 2 persons on 8 chairs on side B = ${}^8P_2$ and the remaining 10 persons can be arranged in remaining 10 chairs in $10!$ ways.

Hence, the total number of ways in which the person can be arranged

$$= {}^8P_4 \times {}^8P_2 \times 10! = \frac{8!8!10!}{4!6!}$$

➤ **Example 45.** The number of permutations of n different things taken r at a time, in which p ($r \leq n - p$) particular things will never occur, is

- (a) $P(n - p, r)$
(b) $P(n, r) - P(n, p)$
(c) $P(n, r) \times P(n, p)$
(d) $P(n - p, r) \times P(n, n - p)$

Sol. (a) As, p things are excluded.

∴ Number of things left = $(n - p)$, out of which r are to be arranged.

Now, the number of permutations = $P(n - p, r)$

➤ **Example 46.** The number of permutations of n different objects taken r (≥ 3) at a time which include 3 particular objects, is

- (a) ${}^nP_r \cdot 3!$ (b) ${}^nP_{r-3} \cdot 3!$
(c) ${}^{n-3}P_{r-3} \cdot 3!$ (d) ${}^rP_3 \cdot {}^{n-3}P_{r-3}$

Sol. (d) First, we arrange 3 particular things in r places.

This can be done in ${}^rP_3$ ways. Then, remaining $n - 3$ things can be arranged taken $r - 3$ at a time in ${}^{n-3}P_{r-3}$ ways.

∴ Total number of ways = ${}^rP_3 \cdot {}^{n-3}P_{r-3}$

➤ **Example 47.** We are required to form different words with the help of the letters of the word INTEGER. Let m_1 be the number of words in which I and N are never together and m_2 be the number of words which begin with I and end with R, then $\frac{m_1}{m_2}$ is given by

- (a) 42 (b) 30 (c) 6 (d) $\frac{1}{30}$

Sol. (b) The number of words in which I and N are together

$$= \frac{6!}{2!} \times 2! = 6! = 720$$

∴ The number of words in which I and N are never together,

$$m_1 = \frac{7!}{2!} - 720 = 1800$$

The number of words which begin with I and end with R,

$$m_2 = \frac{5!}{2!} = 60$$

$$\therefore \frac{m_1}{m_2} = \frac{1800}{60} = 30$$

➤ **Example 48.** The number of six-digit numbers that can be formed from the digits 1, 2, 3, 4, 5, 6 and 7, so that digits do not repeat and the terminal digits are even, is

- (a) 144 (b) 72 (c) 288 (d) 720

Sol. (d) Terminal digits are the first and last digits.

Since, terminal digits are even.

∴ 1st place can be filled in 3 ways and last place can be filled in 2 ways and remaining places can be filled in ${}^5P_4 = 120$ ways.

Hence, the number of six digit numbers, so that the terminal digits are even = $3 \times 120 \times 2 = 720$.

Permutations of Objects not all Distinct

1. The number of mutually distinguishable permutations of n things taken all at a time, of which p are alike of one kind, q alike of second kind such that $p + q = n$, is $\frac{n!}{p!q!}$.

2. The number of permutations of n things, of which p_1 are alike of one kind, p_2 are alike of second kind, p_3 are alike of third kind, ..., p_r are alike of r th kind such that

$$p_1 + p_2 + \dots + p_r = n, \text{ is } \frac{n!}{p_1! p_2! p_3! \dots p_r!}$$

3. The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\frac{n!}{p!q!}$.

4. Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \dots, p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then, the total number of permutations of these r objects to the above condition is

$$\frac{(p_1 + p_2 + \dots + p_r)!}{p_1! p_2! p_3! \dots p_r!}$$

➤ **Example 49.** The number of arrangements which can be made out of the letters of the word ALGEBRA, without changing the relative order (positions) of vowels and consonants, is

- (a) 72 (b) 54
(c) 36 (d) 62

Sol. (a) We have to arrange the 3 vowels in their places, i.e. in the 1st, 4th, 7th places and the 4 consonants in their places. The number of ways to arrange 3 vowels in 1st, 4th and 7th places = $\frac{3!}{2!}$ [∵ there are 2A's]

The number of ways to arrange 4 consonants in their places = $4!$

∴ Required number of arrangements

$$= \frac{3!}{2!} \times 4! = 3 \times 24 = 72$$

➤ **Example 50.** The number of ways in which two 10 paise, two 20 paise, three 25 paise and one 50 paise coins can be distributed among 8 children, so that each child gets only one coin, is

- (a) 1720 (b) 1680
(c) 1570 (d) None of these

Sol. (b) Total number of coins = $2 + 2 + 3 + 1 = 8$

∴ 2 coins are of 10 paise, 2 coins are of 20 paise, 3 coins are of 25 paise and 1 coin is of 50 paise.

∴ Required number of ways

$$= \frac{8!}{2! \times 2! \times 3! \times 1!} = \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 1 \times 2 \times 1 \times 3 \times 2 \times 1 \times 1} = 8 \times 7 \times 6 \times 5 = 1680$$

➤ **Example 51.** A boy has 3 library tickets and 9 books of his interest in the library. Out of these 8, he does not want to borrow Chemistry part II, unless Chemistry part I is also borrowed. The number of ways in which he can choose the three books to be borrowed, is

- (a) 41 (b) 32
(c) 51 (d) None of these

Sol. (a) The following are the different possibilities in which three books can be borrowed.

(i) When Chemistry part II is selected, then Chemistry part I is also borrowed and the third book is selected from the remaining 6 books = $C(6, 1) = 6$.

(ii) When Chemistry part II is not selected, in this case, he has to select the three books from the remaining 7 books. First choice can be made in

$$C(7, 3) = \frac{7 \times 6 \times 5}{1 \times 2 \times 3} = 35 \text{ ways.}$$

∴ Total number of ways in which he can choose the three books to be borrowed = $6 + 35 = 41$.

➤ **Example 52.** There are three copies, each of 4 different books. The number of ways in which they can be arranged on a shelf, is

- (a) $\frac{12!}{(3!)^4}$ (b) $\frac{11!}{(3!)^2}$
(c) $\frac{9!}{(3!)^2}$ (d) None of these

Sol. (a) Total number of books = $3 \times 4 = 12$ in which each of 4 different books is repeated 3 times.

Hence, the required number of arrangements

$$= \frac{12!}{3! \times 3! \times 3! \times 3!} = \frac{12!}{(3!)^4}$$

➤ **Example 53.** A library has a copies of one book, b copies of two books, c copies of each of three books and single copy of d books. The total number of ways in which these books can be distributed, is

- (a) $\frac{(a + 2b + 3c + d)!}{a! b! c!}$ (b) $\frac{(a + b + c + d)!}{a! b! c!}$
(c) $\frac{(a + 2b + 3c + d)!}{a! (b!)^2 (c!)^3}$ (d) None of these

Sol. (c) Total number of books = $a + 2b + 3c + d$
Since, there are b copies each of two books, c copies each of three books and single copy of d books.

$$\therefore \text{Total number of arrangements} = \frac{(a + 2b + 3c + d)!}{a! (b!)^2 (c!)^3}$$

➤ **Example 54.** How many different arrangements can be made out of the letters in the expansion $A^2 B^3 C^4$, when written in full?

$$(a) \frac{9!}{2! + 3! + 4!} \quad (b) \frac{9!}{2!3!4!}$$

$$(c) 2! + 3! + 4! (2!3!4!) \quad (d) 2!3! - 4$$

Sol. (b) $A^2 B^3 C^4$ written in full is AA BBB CCCC.

$$\therefore \text{Required number of ways} = \frac{9!}{2!3!4!}$$

➤ **Example 55.** The number of ways in which the six faces of a cube is painted with six different colours, is

- (a) 6
(b) $6!$
(c) ${}^6C_2$
(d) None of the above

Sol. (d) Number of ways = $\frac{6!}{6!} = 1$ [all faces are alike]

➤ **Example 56.** The number of numbers greater than a million that can be formed with the digits 2, 3, 0, 3, 4, 2 and 3, is

- (a) 360
(b) 340
(c) 370
(d) None of these

Sol. (a) Any number greater than a million must be of 7 or more than 7 digits. Here, number of given digits is seven, therefore we have to form numbers of seven digits only. Now, there are seven digits of which 3 occurs thrice and two occurs twice.

$$\therefore \text{Number of numbers formed} = \frac{7!}{2!3!} = 420$$

But this also includes those numbers of seven digits, whose first digit is zero and so in fact, they are only six digit numbers.

$$\text{Number of numbers of seven digits having zero in the first place} = 1 \times \frac{6!}{3!2!} = 60$$

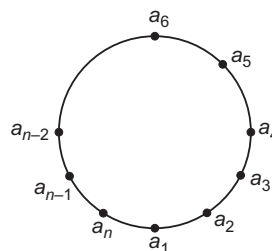
$$\text{Hence, required number} = 420 - 60 = 360$$

Circular Permutations

The number of circular permutations of n distinct objects is $(n-1)!$.

Proof Let $a_1, a_2, a_3, \dots, a_{n-1}, a_n$ be n distinct objects.

Let the total number of circular permutations be x . Consider one of these x permutations as shown in figure.



Clearly, this circular permutation provides n linear permutations as given below:

$$\begin{aligned} &a_1, a_2, a_3, \dots, a_{n-1}, a_n \\ &a_2, a_3, a_4, \dots, a_n, a_1 \\ &a_3, a_4, a_5, \dots, a_n, a_1, a_2 \\ &a_4, a_5, a_6, \dots, a_n, a_1, a_2, a_3 \\ &\dots \quad \dots \quad \dots \quad \dots \\ &\dots \quad \dots \quad \dots \quad \dots \\ &a_n, a_1, a_2, a_3, \dots, a_{n-1} \end{aligned}$$

Thus, each circular permutation gives n linear permutations. But there are x circular permutations, so that number of linear permutations is xn . But the number of linear permutations of n distinct objects is $n!$.

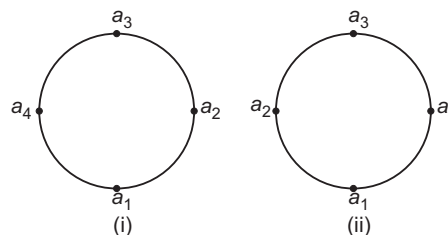
$$\therefore \quad xn = n! \quad \Rightarrow \quad x = \frac{n!}{n} = (n-1)!$$

✎ In the above theorem, anti-clockwise and clockwise orders of arrangements are considered as distinct permutations.

Difference between Clockwise and Anti-clockwise Arrangements

Consider the following circular permutations:

We observe that in both the order of the circular arrangements is a_1, a_2, a_3, a_4 . In Fig. (i), the order is anti-clockwise whereas in Fig. (ii), the order is clockwise.



Thus, the number of circular permutation of n things in which clockwise and anti-clockwise arrangements give rise to different permutations is $(n-1)!$. e.g. The number of permutations of 5 persons seated around a table is $(5-1)! = 4!$, because with respect to the table, the clockwise and anti-clockwise arrangements are distinct.

If anti-clockwise and clockwise orders of arrangements are not distincts, then the number of circular permutations of n distinct items is $\frac{1}{2} \{(n-1)!\}$.

e.g. Arrangements of beads in a necklace, arrangements of flowers in a garland.

➤ **Example 57.** The Chief Minister of 11 states of India meet to discuss the language problem. The number of ways they can seat themselves at a round table so that Punjab and Chennai Chief Ministers sit together, is

- (a) $9! \times 2!$
(b) $9!$
(c) $10!$
(d) None of the above

Sol. (a) Treat the Punjab and Chennai Chief Ministers as one $(P, C) + 9$ others.

So, we have to arrange 10 persons around round table. This can be done in $(10 - 1)! = 9!$ ways. Corresponding to each of these $9!$ ways, Punjab and Chennai Chief Ministers can interchange their places in $2!$ ways.
∴ Associating the two operations, total number of ways = $9! \times 2!$.

➤ **Example 58.** The number of ways in which n things of which r are alike, can be arranged in a circular order, is

- (a) $\frac{(n-1)!}{r!}$
(b) $\frac{n!}{r!}$
(c) $\frac{n!}{(r-1)!}$
(d) None of the above

Sol. (a) Let the required number of arrangements = x

Consider anyone of these x arrangements.

If in this arrangements, r alike things are replaced by r different things, then these r things can be arranged among themselves in $r!$ ways.

Since, one arrangement gives rise to $r!$ arrangements.

∴ x arrangements will give rise to $xr!$ arrangements.

But if all the n things are different, then the number of circular arrangements

$$= (n-1)!$$

Thus, $x \cdot r! = (n-1)!$

Hence, $x = \frac{(n-1)!}{r!}$

➤ **Example 59.** The number of ways in which 7 people can be arranged at a round table, so that 2 particular persons may be together, is

- (a) 132
(b) 148
(c) 240
(d) None of the above

Sol. (c) The two particular persons can be arranged among themselves in ${}^2P_2$ ways i.e. $2!$ ways.

Taking them as one person and keeping him fixed, we can arrange the remaining 5 persons among themselves in $5!$ ways.

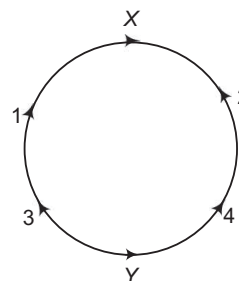
Hence, the required number of ways of which 2 particular persons come together

$$= 5! \times 2! = 120 \times 2 = 240$$

➤ **Example 60.** Three boys and three girls are to be seated around a table, in a circle. Among them, the boy X does not want any girl neighbour and the girl Y does not want any boy neighbour. The number of such arrangements possible is

- (a) 4
(b) 6
(c) 8
(d) None of the above

Sol. (a) As shown in figure, 1, 2 and X are the three boys and 3, 4 and Y are three girls. Boy X will have neighbours as boys 1 and 2 and the girl Y will have neighbours as girls 3 and 4.



1 and 2 can be arranged in

$$P(2, 2) = 2! = 2 \times 1 = 2 \text{ ways}$$

Also, 3 and 4 can be arranged in

$$P(2, 2) = 2! = 2 \times 1 = 2 \text{ ways}$$

Hence, required number of permutations

$$= 2 \times 2 = 4$$

➤ **Example 61.** There are n seats round a table numbered 1, 2, 3, ..., n . The number of ways in which m ($\leq n$) persons can take seats, is

- (a) ${}^nP_m$ (b) ${}^nC_m \cdot (m-1)!$
(c) $\frac{1}{2} \cdot {}^nP_m$ (d) $n-1 P_m$

Sol. (a) Since, the seats are numbered.

So, the arrangement is not circular.

Hence, the required number of ways = The number of arrangements of n things taken m at a time = ${}^nP_m$

Work Book Exercise 6.4

6

Permutation and Combination

- If the letters of the word 'VARUN' are written in all possible ways and then are arranged as in a dictionary, then the rank of the word 'VARUN' is
98 99 100 101
- 5 Indian and 5 American couples meet at a party and shake hands. If no wife shakes hands with her own husband and no Indian wife shakes hands with a male, then the number of hand shakes that takes place in the party, is
95 110 135 150
- If m denotes the number of 5-digit numbers, if each successive digits are in their descending order of magnitude and n is the corresponding figure, when the digits are in their ascending order of magnitude, then $(m - n)$ has the value
 ${}^{10}C_4$ ${}^9C_5$ ${}^{10}C_3$ ${}^9C_3$
- The number of ways in which 9 different prizes be given to 5 students, if one particular boy receives 4 prizes and the rest of the students can get any numbers of prizes, is
 ${}^9C_4 \cdot 2^{10}$ ${}^9C_5 \cdot 5^4$
 $4 \cdot 4^5$ None of these
- The number of ways in which 8 non-identical apples can be distributed among 3 boys such that every boy should get atleast 1 apple and atmost 4 apples is $k \cdot {}^7P_3$, where k has the value equal to
88 66 44 22
- A rack has 5 different pairs of shoes. The number of ways in which 4 shoes can be chosen from it so that there will be no complete pair, is
1920 200
110 80
- The number of ways in which 7 people can occupy six seats, 3 seats on each side in a first class railway compartment, if two specified persons are to be always included and occupy adjacent seats on the same side, is $(5!) \cdot k$, then k has the value equal to
2 4
8 None of these
- The number of different ways in which five 'dashes' and eight 'dots' can be arranged, using only seven of these 13 'dashes' and 'dots', is
1287 119
120 1235520
- There are n identical red balls and m identical green balls. The number of different linear arrangements consisting of n red balls but not necessarily all the green balls is ${}^xC_y$, then
 $x = m + n, y = m$
 $x = m + n + 1, y = m$
 $x = m + n + 1, y = m + 1$
 $x = m + n, y = n$
- In a unique hockey series between India and Pakistan, they decide to play on till a team wins 5 matches. The number of ways in which the series can be won by India, if no match ends in a draw, is
126 252
225 None of these
- The number of ways in which n things of which r alike and the rest different can be arranged in a circle distinguishing between clockwise and anti-clockwise arrangement, is
 $\frac{(n+r-1)!}{r!}$ $\frac{(n-1)!}{r}$
 $\frac{(n-1)!}{(r-1)!}$ $\frac{(n-1)!}{r!}$
- A gentleman invites a party of $m + n$ ($m \neq n$) friends to dinner and places m at one table T_1 and n at another table T_2 , the table being round. If not all people shall have the same neighbour in any two arrangement, then the number of ways in which he can arrange the guests, is
 $\frac{(m+n)!}{4mn}$ $\frac{1}{2} \cdot \frac{(m+n)!}{mn}$
 $2 \cdot \frac{(m+n)!}{mn}$ None of these
- There are 12 guests at a dinner party. Supposing that the master and mistress of the house have fixed seats opposite one another and that there are two specified guests who must always, be placed next to one another, the number of ways in which the company can be placed, is
 $20 \cdot 10!$
 $22 \cdot 10!$
 $44 \cdot 10!$
None of the above
- The number of signals that can be generated by using 6 differently coloured flags, when any number of them may be hoisted at a time, is
1956 1957
1958 1959
- The sum of all five-digit numbers that can be formed using the digits 1, 2, 3, 4, 5 when repetition of digits is not allowed, is
366000 660000
360000 3999960
- By using the digits 0, 1, 2, 3, 4 and 5 (repetitions not allowed), any number of digits being used, the number of non-zero numbers that can be formed, is
1030 1630 1200 1530
- The total number of numbers of not more than 20 digits that are formed by using the digits 0, 1, 2, 3 and 4, is
 5^{20}
 $5^{20} - 1$
 $5^{20} + 1$
None of the above

- 18 Six boys and six girls sit along a line alternately in x ways and along a circle (again alternately in y ways), then
 $x = y$ $y = 12x$ $x = 10y$ $x = 12y$
- 19 The number of times the digit 3 will be written when listing the integers from 1 to 1000, is
 269 300 271 302
- 20 The number of ways in which a mixed double game can be arranged from amongst 9 married couples, if no husband and wife play in the same game, is
 756 1512
 3024 None of these
- 21 The number of different seven digit-numbers that can be written using only the three digits 1, 2 and 3 with the condition that the digit 2 occurs twice in each number, is
 ${}^2P_2 \cdot 2^5$ ${}^7C_2 \cdot 2^5$
 ${}^7C_2 \cdot 5^2$ None of these
- 22 The number of ways one can put 5 different balls in 5 different boxes such that atmost three boxes is empty, is
 3000 3010 2990 3120
- 23 Given that n is odd, the number of ways in which three numbers in AP can be selected from $1, 2, 3, \dots, n$, is
 $\frac{(n-1)^2}{2}$ $\frac{(n+1)^2}{4}$ $\frac{(n+1)^2}{2}$ $\frac{(n-1)^2}{4}$
- 24 The number of ways in which we can select four numbers from 1 to 30, so as to exclude every selection of four consecutive numbers, is
 27378 27405
 27399 None of these
- 25 The number of ways in which 10 candidates $A_1, A_2, \dots, A_{10}$ can be ranked, so that A_1 is always above A_2 , is
 $10!$
 $\frac{10!}{2}$
 $9!$
 None of the above
- 26 There are n white and n black balls marked $1, 2, 3, \dots, n$. The number of ways in which we can arrange these balls in a row, so that neighbouring balls are of different colours, is
 $n!$ $(2n)!$
 $2(n!)^2$ $\frac{(2n)!}{(n!)^2}$
- 27 A man invites in a party of $(m+n)$ friends to dinner and places m at one round table and n at another. The number of ways of arranging the guests is
 $\frac{(m+n)!}{m!n!}$
 $\frac{(m+n)!}{(m-1)!(n-1)!}$
 $\frac{(m+n)!}{(m-1)!(n-1)!}$
 None of the above

Geometrical Applications of ${}^nC_r$

Some basic Geometrical Applications on ${}^nC_r$ are as follow:

- Out of n non-concurrent and non-parallel straight lines, points of intersection are ${}^nC_2$.
- Out of n points, the number of straight lines (when three points are not collinear) are ${}^nC_2$.
- If out of n points, m are collinear, then
 Number of straight lines = ${}^nC_2 - {}^mC_2 + 1$
- In a polygon, total number of diagonals out of n points (when three points are not collinear)

$$= \frac{n(n-3)}{2}$$
- Number of triangles formed from n points (when three points are not collinear) are ${}^nC_3$.
- Number of triangles out of n points in which m are collinear, are ${}^nC_3 - {}^mC_3$.
- Number of triangles that can be formed out of n points (when none of the side is common to the sides of polygon) are ${}^nC_3 - {}^nC_1 - {}^nC_1 \cdot {}^{n-4}C_1$.

8. Number of parallelogram formed by two system of parallel lines (when 1st set contains m parallel lines and 2nd set contains n parallel lines)

$$= {}^nC_2 \times {}^mC_2$$

9. Number of squares formed by two system of parallel lines in which 1st set is perpendicular 2nd sets of lines (when 1st set contains m parallel lines and 2nd set contains n parallel lines)

$$= \sum_{r=1}^{m-1} (m-r)(n-r); m < n$$

- ☞ **Example 62.** The number of parallelograms that can be formed from a set of four parallel lines intersecting another set of three parallel lines, is
 (a) 6 (b) 18
 (c) 12 (d) 9

Sol. (b) Required number of parallelograms

$$= {}^4C_2 \times {}^3C_2 = \frac{4!}{2!2!} \times \frac{3!}{2!1!}$$

$$= \frac{4 \times 3}{2 \times 1} \times \frac{3}{1} = 18$$

➤ **Example 63.** The number of diagonals that can be drawn by joining the vertices of an octagon, is

- (a) 28 (b) 48
(c) 20 (d) None of these

Sol. (c) The total number of lines obtained by joining 8 vertices of octagon

$$= {}^8C_2 = 28 \text{ ways}$$

Out of these 28 lines, 8 are sides and remaining diagonals.

Hence, the number of diagonals
 $= 28 - 8 = 20$

➤ **Example 64.** There are n concurrent lines and another line parallel to one of them. The number of different triangles that will be formed by the $(n+1)$ lines, is

- (a) $\frac{(n-1)n}{2}$ (b) $\frac{(n-1)(n-2)}{2}$
(c) $\frac{n(n+1)}{2}$ (d) $\frac{(n+1)(n+2)}{2}$

Sol. (b) The number of triangles = Number of selections of 2 lines from the $(n-1)$ concurrent lines which are cut by the another line

$$= {}^{n-1}C_2 = \frac{(n-1)!}{2!(n-3)!}$$

$$= \frac{(n-1)(n-2)}{2}$$

➤ **Example 65.** In a plane, there are two sets of parallel lines, one of m lines and the other of n lines. If the lines of one set cut those of the other, then the number of different parallelograms that can be formed is

- (a) $\frac{mn(m-1)(n-1)}{2}$
(b) $\frac{m(m-1)(n-1)}{6}$
(c) $\frac{nm(m-1)(n-1)}{4}$

(d) None of the above

Sol. (c) The number of parallelograms = Number of selections of 4 lines, two from each set

$$\left[\begin{array}{l} \because \text{parallelogram has two} \\ \text{sets of parallel sides} \end{array} \right]$$

$$= {}^nC_2 \times {}^mC_2 = \frac{n(n-1)}{2} \times \frac{m(m-1)}{2}$$

$$= \frac{nm(m-1)(n-1)}{4}$$

➤ **Example 66.** The maximum number of points into which 4 circles and 4 straight lines intersect, is

- (a) 26 (b) 50 (c) 56 (d) 72

Sol. (b) 4 lines intersect each other in ${}^4C_2 = 6$ points and 4 circles intersect in ${}^4P_2 = 12$ points.

Each line cuts 4 circles into 8 points.

$\therefore$ 4 lines cut 4 circles into 32 points.

$\therefore$ Required number = $6 + 12 + 32$
 $= 50$

➤ **Example 67.** The straight lines I_1, I_2, I_3 are parallel and lie in the same plane. A total number of m points are taken on I_1 , n points on I_2 , k points on I_3 . The maximum number of triangles formed with vertices at these points, are

- (a) ${}^{m+n+k}C_3$
(b) ${}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$
(c) ${}^mC_3 + {}^nC_3 + {}^kC_3$
(d) None of the above

Sol. (b) Total number of points are $m+n+k$. The triangles formed by these points = ${}^{m+n+k}C_3$.

Joining 3 points on the same line gives no triangle, such triangles are ${}^mC_3 + {}^nC_3 + {}^kC_3$.

$\therefore$ Required number of triangles
 $= {}^{m+n+k}C_3 - {}^mC_3 - {}^nC_3 - {}^kC_3$

➤ **Example 68.** Two straight lines intersect at a point O . Points $A_1, A_2, \dots, A_n$ are taken on one line and points $B_1, B_2, \dots, B_n$ on the other. If the point O is not to be used, then the number of triangles that can be drawn using these points as vertices, is

- (a) $n(n-1)$
(b) $n(n-1)^2$
(c) $n^2(n-1)$
(d) $n^2(n-1)^2$

Sol. (c) Number of triangles = ${}^{2n}C_3 - {}^nC_3 - {}^nC_3$
 $= \frac{2n(2n-1)(2n-2)}{6} - \frac{2n(n-1)(n-2)}{6}$
 $= \frac{1}{3}n(n-1)(3n) = n^2(n-1)$

➤ **Example 69.** There are 10 points in a plane, of which no 3 points are collinear and 4 points are concyclic. The number of different circles that can be drawn through at least 3 of these points, is

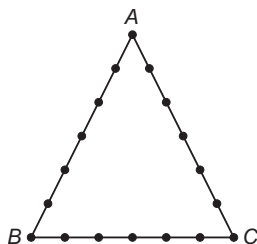
- (a) 116
(b) 120
(c) 117
(d) None of the above

Sol. (c) The required number of circles
 $= ({}^{10}C_3 - {}^4C_3) + 1$
 $= 117$

- 1 The interior angles of a regular polygon measure 150° each. The number of diagonals of the polygon is

35 44
54 78

- 2 18 points are indicated on the perimeter of a $\triangle ABC$ (see figure). How many triangles are there with vertices at these points?



331 408 710 711

- 3 There are m points on a straight line AB and n points on the line AC , none of them being the point A . Triangles are formed with these points as vertices, when

- (i) A is excluded.
(ii) A is included.

Then, the ratio of number of triangles in the two cases is

$$\frac{m+n-2}{m+n}$$

$$\frac{m+n-2}{m+n-1}$$

- 4 Let there be 9 fixed points on the circumference of a circle. Each of these points is joined to every one of the remaining 8 points by a straight line and the points are so positioned on the circumference that atmost 2 straight lines meet in any interior point of the circle. The number of such interior intersection points is

126
351
756
None of the above

- 5 The greatest possible number of points of intersection of 9 different straight lines and 9 different circles in a plane, is

117
153
270
None of the above

- 6 The number of triangles whose vertices are at the vertices of an octagon but none of whose sides happen to come from the sides of the octagon, is

24 52
48 16

Selection of One or More Objects

Selection from Different Items

The number of ways of selecting one or more items from a group of n distinct items is $2^n - 1$.

Out of n items, 1 item can be selected in ${}^nC_1$ ways, 2 items can be selected in ${}^nC_2$ ways, 3 items can be selected in ${}^nC_3$ ways and so on.

Hence, the required number of ways

$$= {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$$

$$= ({}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n) - {}^nC_0 = 2^n - 1$$

- **Example 70.** Nishi has 5 coins, each of the different denomination. The number of different sums of money, she can form, is

- (a) 32
(b) 25
(c) 31
(d) None of the above

Sol. (c) Required number of ways

$$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$$

$$= 2^5 - 1 = 31$$

- **Example 71.** A man has 6 friends. In how many ways he can invite one or more of them at dinner?

- (a) 64 (b) 65 (c) 63 (d) 61

Sol. (c) The man has to select some or all of his 6 friends.
So, required number of ways = $2^6 - 1 = 63$

- ✎ • The number of ways of selecting r items out of n identical items is 1.
• The total number of ways of selecting zero or more i.e. items from a group of n identical items is $(n+1)$.
• The total number of selections of one or more out of $p+q+r$ items, where p are alike of one kind, q are alike of second kind and rest are alike of third kind is

$$[(p+1)(q+1)(r+1)] - 1.$$

Selection from Identical and Distinct Objects

The total number of ways of selecting one or more items from p identical items of one kind, q identical items of second kind, r identical items of third kind and n different items is

$$(p+1)(q+1)(r+1)2^n - 1$$

Number of Divisors and the Sum of the Divisors of a Given Natural Number

Let $n = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \dots p_k^{n_k}$... (i)

where, $p_1, p_2, \dots, p_k$ are distinct prime numbers and $n_1, n_2, \dots, n_k$ are positive integers.

Clearly, any divisor of n is of the form

$$d = p_1^{m_1} \cdot p_2^{m_2} \cdot p_3^{m_3} \dots p_k^{m_k} \quad \dots (ii)$$

where, $m_1, m_2, \dots, m_k$ are natural numbers such that

$$0 \leq m_i \leq n_i, i = 1, 2, 3, \dots, k$$

Therefore, the total number of divisors of Eq. (i) will be equal to the number of ways of selecting atleast one from n_1 identical primes p_1 , n_2 identical primes p_2 and so on, finally n_k identical primes p_k . The number of such ways is

$$(n_1 + 1)(n_2 + 1) \dots (n_k + 1)$$

Hence, the total number of divisors of

$$n = p_1^{n_1} \cdot p_2^{n_2} \cdot p_3^{n_3} \dots p_k^{n_k} \text{ is}$$

$$(n_1 + 1)(n_2 + 1)(n_3 + 1) \dots (n_k + 1)$$

This includes 1 and n as divisors. Therefore, number of divisors other than 1 and n is

$$(n_1 + 1)(n_2 + 1)(n_3 + 1) \dots (n_k + 1) - 2$$

The sum of all divisors of Eq.(i) is given by

$$\begin{aligned} & \sum_{r_1=0}^{n_1} \sum_{r_2=0}^{n_2} \sum_{r_3=0}^{n_3} \dots \sum_{r_k=0}^{n_k} p_1^{r_1} \cdot p_2^{r_2} \cdot p_3^{r_3} \dots p_k^{r_k} \\ &= \left\{ \frac{p_1^{n_1+1} - 1}{p_1 - 1} \right\} \left\{ \frac{p_2^{n_2+1} - 1}{p_2 - 1} \right\} \left\{ \frac{p_3^{n_3+1} - 1}{p_3 - 1} \right\} \\ & \quad \dots \left\{ \frac{p_k^{n_k+1} - 1}{p_k - 1} \right\} \end{aligned}$$

The number of ways of expressing n as a product of two natural numbers

$$= \begin{cases} \frac{1}{2}(n_1 + 1)(n_2 + 1) \dots (n_k + 1), & \text{if } n \text{ is not perfect square} \\ \frac{1}{2}[(n_1 + 1)(n_2 + 1) \dots (n_k + 1) + 1], & \text{if } n \text{ is perfect square} \end{cases}$$

e.g. To find number of ways in which the number 94864 can be resolved as a product of two factors.

$$94864 = 2^4 \times 7^2 \times 11^2$$

So, 94864 is a perfect square.

$$\text{Hence, the number of ways} = \frac{1}{2}[(4 + 1)(2 + 1)(2 + 1) + 1] = 23$$

➤ **Example 72.** The total number of factors (excluding 1) of 2160 is

- (a) 40 (b) 39 (c) 41 (d) 38

Sol. (b) We have,

$$2160 = 2^4 \times 3^3 \times 5^1$$

The total number of divisors is same as the number of ways selecting same or all out of four 2's, three 3's and one 5's.

$$\begin{aligned} \text{The number of ways} \\ &= (4 + 1)(3 + 1)(1 + 1) - 1 = 39 \end{aligned}$$

➤ **Example 73.** The total number of proper factors of 7875 is

- (a) 23
(b) 24
(c) 22
(d) 21

Sol. (c) We have,

$$7875 = 3^2 \times 5^3 \times 7^1$$

The total number of factor

$$= (2 + 1)(3 + 1)(1 + 1) - 1 = 23$$

But this includes the given number itself.

$$\therefore \text{Number of proper factors} = 23 - 1 = 22$$

➤ **Example 74.** The number of factors (excluding 1 and the expression itself) of the product $a^7 b^4 c^3$ def, where a, b, c, d, e and f are all prime numbers, is

- (a) 1279
(b) 1278
(c) 1280
(d) 1277

Sol. (a) The total number of factors of the product $a^7 b^4 c^3$ def is equal to number of ways of selecting atleast one from seven a's, four b's, three c's, d, e, f are different.

$$\begin{aligned} \therefore \text{The number of ways} \\ &= (7 + 1)(4 + 1)(3 + 1)2^3 - 1 = 1279 \end{aligned}$$

➤ **Example 75.** The number of even divisors of 10800 is

- (a) 12 (b) 24 (c) 36 (d) 48

Sol. (d) We have,

$$10800 = 2^4 \times 3^3 \times 5^2$$

For a divisor to be even, we must have atleast one 2 out of four 2's any number of 3's and 5's.

$$\begin{aligned} \therefore \text{Total number of even divisors} \\ &= 4(3 + 1)(2 + 1) = 48 \end{aligned}$$

➤ **Example 76.** The sum of divisors of 2520 is

- (a) 9360 (b) 5040
(c) 9600 (d) None of these

Sol. (a) We have,

$$2520 = 2^3 \times 3^2 \times 5 \times 7$$

The sum of divisors of 2520

$$\begin{aligned} &= (1 + 2 + 2^2 + 2^3)(1 + 3 + 3^2)(1 + 5)(1 + 7) \\ &= 15 \times 13 \times 6 \times 8 \\ &= 9360 \end{aligned}$$

Division of Objects into Groups

Division of Items into Groups of Unequal Size

Number of ways in which $(m + n)$ distinct items can be divided into two unequal groups containing m and n items is $\frac{(m + n)!}{m! n!}$.

Proof

The number of ways in which $(m + n)$ items are divided into two groups containing m and n items is same as the number of combinations of $(m + n)$ things taken m at a time. Thus, the required number

$$= {}^{m+n}C_m = \frac{(m + n)!}{m! n!}.$$

- The number of ways in which $(m + n + p)$ items can be divided into unequal groups containing m, n, p items, is

$${}^{m+n+p}C_m \cdot {}^{n+p}C_n = \frac{(m + n + p)!}{m! n! p!}$$

- The number of ways to distribute $(m + n + p)$ items among 3 persons in the groups containing m, n and p items
 $= (\text{Number of ways to divide}) \times (\text{Number of groups})!$
 $= \frac{(m + n + p)!}{m! n! p!} \times 3!$

- **Example 77.** The number of ways to 16 different things to three persons A, B, C so that B gets one more than A and C gets two more than B, is

- (a) $\frac{16!}{4!5!7!}$ (b) $\frac{16!}{3!5!8!}$
 (c) $4!5!7!$ (d) None of these

Sol. (a) Suppose, A gets x things, then B gets $(x + 1)$ and C gets $(x + 3)$ things.

$$\therefore x + x + 1 + x + 3 = 16$$

$$\Rightarrow x = 4$$

Thus, we have to distribute 16 things to A, B and C in such that A gets 4, B gets 5 and C gets 7 things.

$$\therefore \text{Required number of ways} = \frac{(4 + 5 + 7)!}{4! 5! 7!}$$

$$= \frac{16!}{4! 5! 7!}$$

Division of Objects into Groups of Equal Size

The number of ways in which mn different objects can be divided equally into m groups, each containing n objects and the order of groups is not important, is

$$\left(\frac{(mn)!}{(n!)^m} \right) \frac{1}{m!}$$

The number of ways in which mn different items can be divided equally into m groups, each containing n objects and the order of groups is important, is

$$\left(\frac{(mn)!}{(n!)^m} \times \frac{1}{m!} \right) m! = \frac{(mn)!}{(n!)^m}$$

- **Example 78.** The number of ways in which a pack of 52 cards can be divided equally into four groups, is

- (a) $\frac{52!}{(13!)^4}$ (b) $\frac{52!}{(13!)^4 \times 4!}$
 (c) $\frac{52!}{(13!) \times 4!}$ (d) None of these

Sol. (b) Here, 52 cards are to be divided in four equal groups and order of the group is not important.

$$\text{So, required number of ways} = \frac{52!}{(13!)^4 \times 4!}$$

- **Example 79.** The number of ways 12 different books can be distributed equally among 4 persons, is

- (a) $\frac{12!}{(4!)^3}$ (b) $\frac{12!}{(3!)^4 \times 3}$ (c) $\frac{12!}{(3!)^4}$ (d) $\frac{12!}{(4!)^4}$

Sol. (c) Here, 12 different books can be divided equally among 4 persons that each person will get 3 books. Hence, required number of distribution

$$= {}^{12}C_3 \cdot {}^9C_3 \cdot {}^6C_3 \cdot {}^3C_3 = \frac{12!}{(3!)^4}$$

Division of Identical Objects into Groups

- i. The total number of ways of dividing n identical items among r persons, each one of whom, can receive 0, 1, 2 or more items ($\leq n$) is ${}^{n+r-1}C_{r-1}$.

Or

The total number of ways of dividing n identical objects into r groups, if blank groups are allowed, is ${}^{n+r-1}C_{r-1}$.

- **Example 80.** The total number of ways can 15 identical Mathematics books be distributed among six students, is

- (a) $\frac{15!}{6!9!}$ (b) $\frac{20!}{5!15!}$
 (c) ${}^{15}P_6$ (d) None of these

Sol. (b) We know that the total number of ways in which n things all alike can be distributed into r different boxes is ${}^{n+r-1}C_{r-1}$, when blank box is allowed. Hence, number of ways in which 15 identical books be distributed among six students

$$= {}^{15+6-1}C_{6-1} = {}^{20}C_5 = \frac{20!}{5!15!}$$

- ii. The total number of ways of dividing n identical items among r persons, each one of whom, receives atleast one item is ${}^{n-1}C_{r-1}$.

Or

The number of ways in which n identical items can be divided into r groups such that blank groups are not allowed is ${}^{n-1}C_{r-1}$.

➤ **Example 81.** The number of ways in which 10 identical balls can be distributed in 4 distinct boxes, so that none of the boxes remain empty, is

- (a) ${}^{10}C_4$ (b) ${}^{10}P_4$
(c) ${}^9P_3$ (d) None of these

Sol. (d) The required number of ways of distributing 10 identical items to three persons such that each person receives atleast one item.

So, required number of ways

$$= {}^{10-1}C_{4-1} = {}^9C_3 = \frac{9!}{3!6!} = 84$$

- iii. The number of ways in which n identical items can be divided into r groups so that no group contains less than m items and more than k ($m < k$) items is coefficient of x^n in the expansion of $(x^m + x^{m+1} + \dots + x^k)^r$.

➤ **Example 82.** Five distinct letters are to be transmitted through a communication channel. A total number of 15 blanks is to be inserted between the letters with atleast three between every two. The number of ways in which this can be done, is

- (a) 1200 (b) 1800
(c) 2400 (d) 3000

Sol. (c) For $1 \leq i \leq 4$, let $x_i (\geq 3)$ be the number of blanks between i th and $(i+1)$ th letters. Then,

$$x_1 + x_2 + x_3 + x_4 = 15 \quad \dots(i)$$

The number of solutions of Eq. (i)

$$= \text{Coefficient of } t^{15} \text{ in } (t^3 + t^4 + \dots)^4$$

$$= \text{Coefficient of } t^3 \text{ in } (1-t)^{-4}$$

$$= \text{Coefficient of } t^3 \text{ in } (1 + {}^4C_1 t + {}^5C_2 t^2 + {}^6C_3 t^3 + \dots)$$

But 5 letters can be permuted in $5! = 120$ ways.

Thus, the required number of arrangements

$$= (120)(20) = 2400$$

Dearrangements

If n distinct objects are arranged in a row, then the number of ways in which they can be dearranged, so that none of them occupies its original place is

$$n! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \dots + (-1)^n \frac{1}{n!} \right\}$$

and it is denoted by $D(n)$.

If r ($0 \leq r \leq n$) objects occupy the places assigned to them i.e. their original places and none of the remaining $(n-r)$ objects occupies its original places, then the number of such ways is $D(n-r) = {}^nC_r \cdot D(n-r)$

$$= {}^nC_r \cdot (n-r)! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^{n-r} \frac{1}{(n-r)!} \right\}$$

➤ **Example 83.** There are four balls of different colours and four boxes of colours, same as those of the balls. The number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour is

- (a) 9
(b) 13
(c) 8
(d) None of the above

Sol. (a) The required number of ways

$$= 4! \left[1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right]$$

$$= 12 - 4 + 1 = 9$$

➤ **Example 84.** The number of ways can 10 letters be placed in 10 marked envelopes, so that no letter is in the right envelope, are

- (a) $10! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{10!} \right\}$
(b) $10! \left\{ 1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{10!} \right\}$
(c) $\left\{ 1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{10!} \right\}$
(d) $9! \left\{ 1 + \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \dots - \frac{1}{10!} \right\}$

Sol. (a) The required number of ways is equal to the number of dearrangements of 10 objects.

Hence, required number of ways

$$= 10! \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{1}{10!} \right\}$$

➤ **Example 85.** Ajay writes letters to his five friends and addresses the corresponding. The number of ways can the letters be placed in the envelopes, so that atleast two of them are in the wrong envelopes, are

- (a) 120
(b) 125
(c) 119
(d) None of the above

Sol. (c) Required number of ways

$$= \sum_{r=2}^5 {}^5C_{5-r} D(r)$$

$$= \sum_{r=2}^5 \frac{5!}{r!(5-r)!} \left\{ 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^r}{r!} \right\}$$

$$\begin{aligned}
 &= \sum_{r=2}^5 \frac{5!}{(5-r)!} \left\{ \frac{1}{2!} - \frac{1}{3!} + \dots + \frac{(-1)^r}{r!} \right\} \\
 &= \frac{5!}{3!} \left(\frac{1}{2!} \right) + \frac{5!}{2!} \left(\frac{1}{2!} - \frac{1}{3!} \right) + \frac{5!}{1!} \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right) \\
 &\quad + \frac{5!}{0!} \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right) \\
 &= 10 + 20 + (60 - 20 + 5) + (60 - 20 + 5 - 1) \\
 &= 10 + 20 + 45 + 44 = 119
 \end{aligned}$$

Number of Integral Solutions of Linear Equations and Inequations

Consider the equation

$$x_1 + x_2 + x_3 + x_4 + \dots + x_r = n \quad \dots(i)$$

where, $x_1, x_2, x_3, x_4, \dots, x_r$ and n are non-negative integers.

This equation may be interpreted as that n identical objects are to be divided into r groups, where a group may contain any number of objects. Therefore,

$$\begin{aligned}
 &\text{Total number of solutions of Eq. (i)} \\
 &= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + \dots + x^n)^r \\
 &= {}^{n+r-1}C_n \text{ or } {}^{n+r-1}C_{r-1}
 \end{aligned}$$

- The total number of non-negative integral solutions of the equation $x_1 + x_2 + \dots + x_r = n$ is ${}^{n+r-1}C_{r-1}$ and the total number of solutions of the same equation in the set N of natural numbers is ${}^{n-1}C_{r-1}$.
- If the upper limit of a variable in solving an equation of the form $x_1 + x_2 + \dots + x_m = n$ subject to the conditions $a_i \leq x_i \leq b_i$; $i = 1, 2, \dots, m$ is more than or equal to the sum required and lower limit of all the variables are non-negative, then upper limit of that variable can be taken as infinite.
- In order to solve inequations of the form

$$x_1 + x_2 + \dots + x_m \leq n$$

We introduce a dummy (artificial) variable x_{m+1} such that

$$x_1 + x_2 + \dots + x_m + x_{m+1} = n, \text{ where } x_{m+1} \geq 0$$

The number of solutions of this equation are same as the number of solutions of Eq. (i).

- **Example 86.** In an examination, the maximum marks for each of three papers is n and that for fourth paper is $2n$. Then, the number of ways in which a candidate can get $3n$ marks, is

- (a) $\frac{1}{6}(n-1)(5n^2 + 10n + 6)$
 (b) $\frac{1}{6}(n+1)(5n^2 + 10n + 6)$
 (c) $\frac{1}{6}(n+1)(5n^2 + n + 6)$
 (d) None of the above

Sol. (b) The total number of ways of getting $3n$ marks
 = Coefficient of x^{3n} in
 $(x^0 + x^1 + x^2 + \dots + x^n)^3 \times (x^0 + x^1 + \dots + x^{2n})$

$$\begin{aligned}
 &= \text{Coefficient of } x^{3n} \text{ in } \left(\frac{1-x^{n+1}}{1-x} \right)^3 \left(\frac{1-x^{2n+1}}{1-x} \right) \\
 &= \text{Coefficient of } x^{3n} \text{ in } (1-x^{n+1})^3 (1-x^{2n+1})(1-x)^{-4} \\
 &= \text{Coefficient of } x^{3n} \text{ in} \\
 &\quad [(1-3x^{n+1}+3x^{2n+2}-x^{3n+3})(1-x^{2n+1})(1-x)^{-4}] \\
 &= \text{Coefficient of } x^{3n} \text{ in} \\
 &\quad [(1-3x^{n+1}-x^{2n+1}+3x^{2n+2}+\dots)(1-x)^{-4}] \\
 &= \text{Coefficient of } x^{3n} \text{ in } (1-x)^{-4} \\
 &\quad - 3 \text{ Coefficient of } x^{2n-1} \text{ in } (1-x)^{-4} \\
 &\quad - \text{Coefficient of } x^{n-1} \text{ in } (1-x)^{-4} \\
 &\quad + 3 \text{ Coefficient of } x^{n-2} \text{ in } (1-x)^{-4} \\
 &= {}^{3n+4-1}C_{4-1} - 3 \times {}^{2n-1+4-1}C_{4-1} - {}^{n-1+4-1}C_{4-1} \\
 &\quad + 3 \times {}^{n-2+4-1}C_{4-1} \\
 &= {}^{3n+3}C_3 - 3 \times {}^{2n+2}C_3 - {}^{n+2}C_3 + 3 \times {}^{n+1}C_3 \\
 &= \frac{(3n+3)(3n+2)(3n+1)}{6} - 3 \cdot \frac{(2n+2)(2n+1)(2n)}{6} \\
 &\quad - \frac{(n+2)(n+1)(n)}{6} + 3 \cdot \frac{(n+1)(n)(n-1)}{6} \\
 &= \frac{1}{6}(n+1)(5n^2 + 10n + 6)
 \end{aligned}$$

- **Example 87.** The number of ways of selecting n objects out of $3n$ objects, of which n are alike and rest are different, is

- (a) $2^{2n-1} + \frac{(2n-1)!}{n!(n-1)!}$ (b) $2^{2n-1} - \frac{(2n-1)!}{n!(n-1)!}$
 (c) $2^{2n+1} + \frac{(2n+1)!}{n!(n+1)!}$ (d) None of these

Sol. (a) The required number of ways

$$\begin{aligned}
 &= \text{Coefficient of } x^n \text{ in } (x^0 + x^1 + x^2 + \dots + x^n)(x^0 + x)^{2n} \\
 &= \text{Coefficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x} \right) (1+x)^{2n} \\
 &= \text{Coefficient of } x^n \text{ in } (1-x)^{-1} (1+x)^{2n} \\
 &= \text{Coefficient of } x^n \text{ in } \left(\sum_{r=0}^{\infty} x^r \right) (1+x)^{2n} \\
 &= \text{Coefficient of } x^n \text{ in } \left(\sum_{r=0}^n x^r \right) (1+x)^{2n} \\
 &= \text{Coefficient of } x^n \text{ in } \sum_{r=0}^n x^r (1+x)^{2n} \\
 &= \sum_{r=0}^n \text{Coefficient of } x^{n-r} \text{ in } (1+x)^{2n} \\
 &= \sum_{r=0}^n {}^{2n}C_{n-r} \\
 &= {}^{2n}C_n + {}^{2n}C_{n+1} + \dots + {}^{2n}C_1 + {}^{2n}C_0 \\
 &= \frac{1}{2} [{}^{2n}C_n + \{ {}^{2n}C_0 + {}^{2n}C_1 + \dots + {}^{2n}C_n + {}^{2n+1}C_{n+1} \\
 &\quad + \dots + {}^{2n}C_{2n} \}] \quad [\because {}^nC_r = {}^nC_{n-r}] \\
 &= \frac{1}{2} [{}^{2n}C_n + 2^{2n}] = \frac{1}{2} \left[\frac{2n!}{n!n!} + 2^{2n} \right] = 2^{2n-1} + \frac{(2n-1)!}{n!(n-1)!}
 \end{aligned}$$

- **Example 88.** The number of ways of selecting 4 letters from the word EXAMINATION, is
(a) 136 (b) 140 (c) 126 (d) 102

Sol. (a) There are 11 letters in the word EXAMINATION viz. 2 A's, 2 I's, 2 N's, E, X, M, T, O.

Therefore,

Number of ways of selecting 4 letters from these letters
= Coefficient of x^4 in $(x^0 + x^1 + x^2)^3 (1 + x)^5$

$$= \text{Coefficient of } x^4 \text{ in } \left(\frac{1-x^3}{1-x} \right)^3 (1+x)^5$$

$$= \text{Coefficient of } x^4 \text{ in } (1-x^3)^3 (1-x)^{-3} (1+x)^5$$

$$= \text{Coefficient of } x^4 \text{ in } (1-3x^3 + 3x^6 - x^9)(1+x)^5 (1-x)^{-3}$$

$$= \text{Coefficient of } x^4 \text{ in } (1-3x^3)(1 + {}^5C_1x + {}^5C_2x^2 + {}^5C_3x^3 + {}^5C_4x^4)(1-x)^{-3}$$

$$= \text{Coefficient of } x^4 \text{ in } (1+5x+10x^2+7x^3-10x^4)(1-x)^{-3}$$

$$= {}^{4+3-1}C_{3-1} + 5 \times {}^{3+3-1}C_{3-1} + 10 \times {}^{2+3-1}C_{3-1} + 7 \times {}^{1+3-1}C_{3-1} - 10$$

$$= {}^6C_2 + 5 \times {}^5C_2 + 10 \times {}^4C_2 + 7 \times {}^3C_2 - 10$$

$$= 15 + 50 + 60 + 21 - 10 = 136$$

- **Example 89.** The number of ways in which 16 identical things can be distributed among 4 persons, if each person gets atleast 3 things, is
(a) 33 (b) 35
(c) 38 (d) None of these

Sol. (b) Required number

$$= \text{Coefficient of } x^{16} \text{ in } (x^3 + x^4 + \dots + x^7)^4$$

$$= \text{Coefficient of } x^4 \text{ in } (1 + x + \dots + x^4)^4$$

$$= \text{Coefficient of } x^4 \text{ in } \left(\frac{1-x^5}{1-x} \right)^4$$

$$= \text{Coefficient of } x^4 \text{ in } [(1-x^5)^4(1-x)^{-4}]$$

$$= \text{Coefficient of } x^4 \text{ in } (1-x)^{-4} = {}^7C_3 = \frac{4 \cdot 5 \cdot 6 \cdot 7}{4!} = 35$$

Aliter

$$x_1 + x_2 + x_3 + x_4 = 16, \text{ where } x_1, x_2, x_3, x_4 \geq 3$$

$$\text{or } x_1 - 3 = a_1, x_2 - 3 = a_2, x_3 - 3 = a_3, x_4 - 3 = a_4$$

$$\therefore a_1, a_2, a_3, a_4 \geq 0 \text{ or } a_1 + a_2 + a_3 + a_4 = 8$$

$$\Rightarrow \text{Number of solutions} = {}^{4+4-1}C_{4-1} = {}^7C_3$$

$$= \frac{7 \times 6 \times 5}{3 \times 2} = 35$$

- **Example 90.** If N is the number of positive integral solution of $x_1 x_2 x_3 x_4 = 770$. Then,
(a) N is 250 (b) N is 252
(c) N is 254 (d) N is 256

Sol. (d) We have, $770 = 2 \cdot 5 \cdot 7 \cdot 11$

We can assign 2 to x_1 or x_2 or x_3 or x_4 i.e. 2 can be assigned in 4 ways.

Similarly, each of 5, 7 or 11 can be assigned in 4 ways.

Thus, the number of positive integral solution of

$$x_1 x_2 x_3 x_4 = 770 \text{ is } 4^4 = 2^8 = 256$$

Aliter

$$x_1 x_2 x_3 x_4 = 2^1 \cdot 5^1 \cdot 7^1 \cdot 11^1$$

$$\therefore a_1 + a_2 + a_3 + a_4 = 1$$

i.e. 2^1 can be assigned to a_1, a_2, a_3, a_4 .

$$\Rightarrow {}^{1+4-1}C_{4-1} = {}^4C_3 = 4$$

Similarly, for $5^1, 7^1, 11^1$

$$\therefore \text{Total number of ways} = 4 \cdot 4 \cdot 4 \cdot 4 = 256$$

Work Book Exercise 6.6

- 1 The total number of ways of dividing 15 different things into groups of 8, 4 and 3 respectively, is

$$\frac{15!}{7!3!2!}$$

$$\frac{15!}{9!5!4!}$$

None of these

- 2 The number of ways of distributing 50 identical things among 8 persons in such a way that three of them get 8 things each, two of them get 7 things each and remaining 3 get 4 things each, is

$$\frac{8!}{(3!)^2 (2!)}$$

$$\frac{8!}{(2!)^2 3!}$$

None of these

- 3 In how many ways can we put 12 different balls in three different boxes such that the first box contains exactly 5 balls?

$${}^{12}C_3 \cdot 5^7$$

$${}^{12}C_5 \cdot 2^7$$

c ${}^{12}C_5 \cdot 5^2$

None of these

- 4 The number of ways, 3 persons A, B and C having 6 one rupee coins, 7 one rupee coins and 8 one rupee coins respectively can donate 10 one rupee coins collectively, is

a ${}^7C_3$

${}^8C_3$

c ${}^{10}C_3$

None of these

- 5 In an examination, the maximum marks for each of the three papers are 50. Maximum marks for the fourth paper is 100. The number of ways in which a candidate can score 60% marks on the whole, is

a 100550

101551

c 110551

d None of these

- 6 The number of ways of dividing 20 persons into 10 couples, is

a $\frac{20!}{2^{10}}$

${}^{20}C_{10}$

c $\frac{20!}{(2!)^9}$

None of these

- 7** The number of integer solutions for the equation $x + y + z + t = 20$, where $x, y, z, t \geq -1$, is
a ${}^{20}C_4$ **b** ${}^{23}C_3$
c ${}^{27}C_4$ **d** ${}^{27}C_3$
- 8** The number of ways in which mn students can be distributed equally among m sections, is
a $\frac{(mn)!}{n!}$ **b** $\frac{(mn)!}{(n!)^m}$
c $\frac{(mn)!}{m!n!}$ **d** $(mn)^m$
- 9** The number of integers which lie between 1 and 10^6 and which have the sum of the digits equal to 12, is
a 8550 **b** 5382 **c** 6062 **d** 8055
- 10** The number of integral solutions of $x + y + z = 10$, with $x \geq -5, y \geq -5, z \geq -5$, is
a 135 **b** 136 **c** 455 **d** 105
- 11** The number of ways in which a score of 11 can be made from a throw by three persons, each throwing a single die once, is
a 45 **b** 18
c 27 **d** None of these
- 12** If $x, y, z, \dots$ are $(m+1)$ distinct prime numbers, then then number of factors of $x^n y z \dots$ is
a $m(n+1)$ **b** $2^n m$
c $(n+1)2^m$ **d** $n \cdot 2^m$
- 13** There are m copies of each of n different books in a university library. The number of ways in which one or more than one book can be selected, is
a $m^n + 1$ **b** $(m+1)^n - 1$
c $(n+1)^n - m$ **d** m
- 14** The number of ways in which one or more balls can be selected out of 10 white, 9 green and 7 blue balls, is
a 892 **b** 881
c 891 **d** 879
- 15** The number of all three element subsets of the set $\{a_1, a_2, a_3, \dots, a_n\}$ which contain a_3 , is
a ${}^nC_3$ **b** ${}^{n-1}C_3$
c ${}^{n-1}C_2$ **d** None of these
- 16** If one quarter of all three element subsets of the set $A = \{a_1, a_2, a_3, \dots, a_n\}$ contains the element a_3 , then n is equal to
a 10 **b** 12
c 14 **d** None of the above
- 17** The number of positive integral solutions of $abc = 30$ is
a 30 **b** 27
c 8 **d** None of these
- 18** The number of non-negative integral solutions of $x + y + z \leq n$, where $n \in N$ is
a ${}^{n+3}C_3$ **b** ${}^{n+4}C_4$
c ${}^{n+5}C_5$ **d** None of these
- 19** If n objects are arranged in a row, then the number of ways of selecting three of these objects, so that no two of them are next to each other, is
a ${}^{n-2}C_3$ **b** ${}^{n-3}C_2$
c ${}^{n-3}C_3$ **d** None of these
- 20** There are three piles of identical red, blue and green balls and each pile contains atleast 10 balls. The number of ways of selecting 10 balls, if twice as many red balls as green balls are to be selected, is
a 3 **b** 4
c 6 **d** 8
- 21** Two classrooms A and B having capacity of 25 and $(n-25)$ seats, respectively. Let A_n denotes the number of possible sitting arrangements of room A , when n students are to be seated in these rooms, starting from room A which is to be filled up full to its capacity. If $A_n - A_{n-1} = 25!({}^{49}C_{25})$, then n equals
a 49 **b** 48 **c** 50 **d** 51
- 22** Between two junction stations A and B , there are 12 intermediate stations. The number of ways in which a train can be made to stop at 4 of these stations, so that no two of these halting stations are consecutive, is
a ${}^8C_4$ **b** ${}^9C_4$
c ${}^{12}C_4 - 4$ **d** None of these
- 23** The sum of the divisors of $2^5 \cdot 3^4 \cdot 5^2$ is
a $3^2 \cdot 7^1 \cdot 11^2$
b $63 \cdot 121 \cdot 31$
c $3 \cdot 7 \cdot 11 \cdot 31$
d None of the above
- 24** The number of ways arranging m positive and $n < m+1$ negative signs in a row, so that no two negative signs are together, is
a ${}^{m+1}P_n$
b ${}^{n+1}P_m$
c ${}^{m+1}C_n$
d ${}^{n+1}C_m$
- 25** If $a_i, i = 1, 2, 3, 4$ is four real numbers of the same sign, then the minimum value of $\sum \frac{a_i}{a_j}$, $i, j \in \{1, 2, 3, 4\}, i \neq j$, is
a 8
b 6
c 12
d None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Number of positive integer n less than 17, for which $n! + (n+1)! + (n+2)!$ is an integral multiple of 49, is

- (a) 0 (b) 3
(c) 5 (d) 2

Sol. Here, $n! + (n+1)! + (n+2)!$
 $= n![1 + (n+1) + (n+2)(n+1)] = n!(n+2)^2$
 Either 7 divides $(n+2)$ or 49 divides $n!$.
 i.e. $n = 5, 12, 14, 15, 16$ $[\because n < 17]$
 Thus, number of solutions is five.
 Hence, (c) is the correct answer.

Ex 2. If $\frac{2}{9!} + \frac{2}{3!7!} + \frac{1}{5!5!} = \frac{2^a}{b!}$, where $a, b \in N$, then the ordered pair (a, b) is

- (a) (9, 10)
(b) (10, 9)
(c) (7, 10)
(d) (10, 7)

Sol. Here, $\frac{2}{9!} + \frac{2}{3!7!} + \frac{1}{5!5!}$
 $= \frac{1}{1!9!} + \frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{3!7!} + \frac{1}{9!1!}$
 $= \frac{1}{10!} \left\{ \frac{10!}{9!1!} + \frac{10!}{7!3!} + \frac{10!}{5!5!} + \frac{10!}{3!7!} + \frac{10!}{9!1!} \right\}$
 $= \frac{1}{10!} \{ {}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9 \}$
 $= \frac{1}{(10)!} \cdot (2^{10-1}) = \frac{2^9}{10!}$, but it is $\frac{2^a}{b!}$.
 $\Rightarrow a = 9 \text{ and } b = 10$
 Hence, (a) is the correct answer.

Ex 3. Number of ordered triplets (x, y, z) such that x, y, z are primes and $x^y + 1 = z$, is

- (a) 0
(b) 1
(c) 3
(d) None of the above

Sol. Here, $x^y + 1 = z$, where x, y, z are prime.
 Thus, y cannot be odd, if y is prime.
 $\Rightarrow x^y + 1$ is divisible by $(x+1)$.
 Now, z must be odd.
 $\Rightarrow x$ must be even. $[\because x^y = z - 1]$
 Thus, only even value (i.e. prime) is $x = 2$.
 $\Rightarrow x = 2, y = 2, z = 5$
 So, there is only one such triplet $(2, 2, 5)$.
 Hence, (b) is the correct answer.

Ex 4. $\sum_{0 \leq i < j < k < l \leq n} \sum$ is equal to

- (a) ${}^{n+1}C_4$ (b) $n^{n+1}C_4$
(c) ${}^{n+1}C_3$ (d) $n(n+1)$

Sol. As $\sum_{0 \leq i < j < k < l \leq n} \sum n = n \sum_{0 \leq i < j < k < l \leq n} \sum$ represents selection of four terms from $\{0, 1, 2, \dots, n\}$ i.e. $(n+1)$ terms.
 Thus, $\sum_{0 \leq i < j < k < l \leq n} \sum n = n \cdot {}^{n+1}C_4$
 Hence, (b) is the correct answer.

Ex 5. Let A denotes the property that two elements of $A = \{1, 5, 9, 13, \dots, 1093\}$ add upto 1094. Then, maximum number of elements in A can be

- (a) 273 (b) 136
(c) 137 (d) 138

Sol. Since, $A = \{1, 5, 9, 13, \dots, 1093\}$ is an arithmetic progression.
 $\therefore$ Number of terms $= \frac{1093 - 1}{4} + 1$
 $= 274$
 Since, sum of equidistant terms in AP is equal to sum of first and last terms $= 1 + 1093 = 1094$.
 $\therefore$ Maximum number of elements in A that add upto 1094
 $= \frac{274}{2} = 137$

Hence, (c) is the correct answer.

Ex 6. Let $n_1 = x_1x_2x_3$ and $n_2 = y_1y_2y_3$ be two 3-digit numbers, then the pairs of n_1 and n_2 can be formed, so that n_1 can be subtracted from n_2 without borrowing, is

- (a) $55 \cdot 54$ (b) $45 \cdot 55$
(c) $45 \cdot (55)^2$ (d) $55 \cdot (45)^2$

Sol. Here, $n_1 = x_1x_2x_3$ and $n_2 = y_1y_2y_3$
 $\Rightarrow n_1$ can be subtracted from n_2 without borrowing, if $y_i \geq x_i$ for $i = 1, 2, 3$.
 $\therefore$ Let $x_i = r \Rightarrow \begin{cases} r = 0, 1, 2, \dots, 9, \text{ for } x_2 \text{ and } x_3 \\ r = 1, 2, 3, \dots, 9, \text{ for } x_1 \end{cases}$
 $\therefore y_i = r, r+1, \dots, 9$
 Thus, for y_1, y_2 and y_3 , we have $(10-r)$ choices each.
 Now, total number of ways for choosing y_i and x_i
 $= \left\{ \sum_{r=1}^9 (10-r) \right\} \left\{ \sum_{r=0}^9 (10-r) \right\} \left\{ \sum_{r=0}^9 (10-r) \right\}$
 $= 45 \cdot 55 \cdot 55 = 45 \cdot (55)^2$
 Hence, (d) is the correct answer.

Ex 7. Let

$$E = (2n+1)(2n+3)(2n+5)\dots(4n-3)(4n-1);$$

$n > 1$, then $2^n E$ is divisible by
 (a) ${}^nC_{n/2}$ (b) ${}^{2n}C_n$ (c) ${}^{3n}C_n$ (d) ${}^{4n}C_{2n}$

Sol. Here, $E = (2n+1)(2n+3)(2n+5)\dots(4n-3)(4n-1)$

$$\text{or } E = \frac{(2n+1)(2n+2)(2n+3)(2n+4)\dots(4n-1)(4n)}{(2n+2)(2n+4)\dots(4n)} \cdot \frac{(2n)!}{(2n)!}$$

$$= \frac{(4n)!}{(2n)! \cdot 2^n \cdot (n+1)(n+2)\dots(2n)}$$

$$E = \frac{(4n)! \cdot n!}{(2n)! 2^n \cdot (2n)!}$$

$$\Rightarrow 2^n E = \frac{(4n)!}{(2n)! (2n)!} \cdot n! = {}^{4n}C_{2n} \cdot n!$$

$\therefore 2^n E$ is divisible by ${}^{4n}C_{2n}$.

Hence, (d) is the correct answer.

Ex 8. $\sum_{0 \leq i \leq j \leq 10} {}^{10}C_j {}^jC_i$ is equal to

- (a) $2^{10} - 1$ (b) 2^{10}
 (c) $3^{10} - 1$ (d) None of these

Sol. We have, $\sum_{0 \leq i \leq j \leq 10} {}^{10}C_j {}^jC_i$

$$\begin{aligned} &= {}^{10}C_0 \cdot {}^0C_0 + {}^{10}C_1 ({}^1C_0 + {}^1C_1) + {}^{10}C_2 ({}^2C_0 + {}^2C_1 + {}^2C_2) \\ &\quad + {}^{10}C_3 ({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3) + \dots + {}^{10}C_{10} \\ &\quad ({}^{10}C_0 + {}^{10}C_1 + {}^{10}C_2 + \dots + {}^{10}C_{10}) \\ &= {}^{10}C_0 \cdot 2^0 + {}^{10}C_1 \cdot 2^1 + {}^{10}C_2 \cdot 2^2 + {}^{10}C_3 \cdot 2^3 + \dots + {}^{10}C_{10} \cdot 2^{10} \\ &= (1+2)^{10} = 3^{10} \end{aligned}$$

Hence, (d) is the correct answer.

Ex 9. $\lim_{n \rightarrow \infty} \frac{\sum_{1 \leq i \leq j \leq n} (i+1)(j+1)}{n^4}$ is equal to

- (a) $\frac{1}{4}$ (b) $\frac{1}{6}$
 (c) $\frac{1}{8}$ (d) None of these

Sol. Here, $\lim_{n \rightarrow \infty} \frac{\sum_{1 \leq i \leq j \leq n} (i+1)(j+1)}{n^4}$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{\left(\sum_{i=1}^n (i+1) \right)^2 - \sum_{i=1}^n (i+1)^2}{2n^4} \\ &= \lim_{n \rightarrow \infty} \frac{\left\{ \frac{(n+2)(n+1)}{2} - 1 \right\}^2 - \left\{ \frac{(n+1)(n+2)(2n+3)}{6} - 1 \right\}}{2n^4} \\ &= \lim_{n \rightarrow \infty} \frac{n^4 \left(\frac{1}{4} \right) + n^3 \dots \text{and decreasing powers of } n}{2n^4} = \frac{1}{8} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 10. A seven-digit numbers divisible by 9 is to be formed by using 7 out of numbers $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of ways in which this can be done, is

- (a) $7!$ (b) $2 \cdot (7)!$
 (c) $3 \cdot (7)!$ (d) $4 \cdot 7!$

Sol. Sum of 7 digits = a multiple of 9

We know, sum of numbers 1, 2, 3, 4, 5, 6, 7, 8, 9 is 45.

So, two left numbers should also have sum as 9. The pairs to be left are (1, 8), (2, 7), (3, 6), (4, 5) with each pair left number of 7 digit numbers is $7!$.

So, with all 4 pairs = $4 \times 7!$

Hence, (d) is the correct answer.

Ex 11. Among $9!$ permutations of the digits 1, 2, 3, ..., 9. Consider those arrangements which have the property that if we take any five consecutive positions, the product of the digits in those positions is divisible by 7, the number of such arrangements is

- (a) $7!$ (b) $3 \cdot 7!$ (c) $8!$ (d) $4 \cdot 7!$

Sol. Since, amongst $9!$ arrangements of digits 1, 2, 3, 4, 5, 6, 7, 8, 9 i.e. the number formed is of the form $x_1 x_2 x_3 x_4 x_5 x_6 x_7 x_8 x_9$. Taking any five consecutive positions product is divisible by 7 only, if $x_5 = 7$.

So, the required arrangements = $8!$

Hence, (c) is the correct answer.

Ex 12. Let $S_n = \sum_{r=1}^n r!$; ($n > 6$), then $S_n - 7 \left\lceil \frac{S_n}{7} \right\rceil$

(where, $\lceil \cdot \rceil$ denotes the greatest integer function) is equal to

- (a) $\left\lceil \frac{n}{7} \right\rceil$ (b) $n! - 7 \left\lceil \frac{n}{7} \right\rceil$
 (c) 5 (d) 3

Sol. All the number $r!$; ($r \geq 7$) will be multiple of 7.

So, for remainder, consider

$$S_6 = 1 + 2 + 6 + 24 + 120 + 720 = 873$$

which leaves the remainder 5, when divided by 7.

Hence, (c) is the correct answer.

Ex 13. The number of ways of arranging m numbers out of 1, 2, 3, ..., n , so that maximum is $(n-2)$ and minimum is 2 (repetitions of numbers is allowed) such that maximum and minimum both occur exactly once ($n > 5, m > 3$), is

- (a) ${}^{n-3}C_{m-2}$ (b) ${}^mC_2 (n-3)^{m-2}$
 (c) $m(m-1)(n-5)^{m-2}$ (d) ${}^nC_2 \cdot {}^nC_m$

Sol. First, we take one number as 2 and one as $(n-2)$ and put them in $m(m-1)$ ways. Now, remaining $(m-2)$, numbers can be anyone from, 3, 4, ..., $(n-4)$, $(n-3)$. Which we can do in $(n-5)^{m-2}$.

$\therefore$ Total number of ways = $m(m-1)(n-5)^{m-2}$

Hence, (c) is the correct answer.

- Ex 14.** Total number of integer n such that $2 \leq n \leq 2000$ and HCF of n and 36 is one, is equal to
 (a) 666
 (b) 667
 (c) 665
 (d) None of the above

Sol. $\because 36 = 2^2 \cdot 3^2$

Total number of integers $\in [2, 2000]$ that are divisible by 2 is

$$\left\lfloor \frac{2000}{2} \right\rfloor = 1000$$

The integer $\in [2, 2000]$ that are divisible by 3 is

$$\left\lfloor \frac{2000}{3} \right\rfloor = 666$$

Integers that are divisible by 2 and 3 and contained in

$$[2, 2000] = \left\lfloor \frac{2000}{6} \right\rfloor = 333$$

Thus, total number of integers that are neither divisible by 2 nor by 3

$$= 1999 - (1000 + 666 - 333) = 666$$

Hence, (a) is the correct answer.

- Ex 15.** If $E = \frac{1}{4} \cdot \frac{2}{6} \cdot \frac{3}{8} \cdot \frac{4}{10} \cdots \frac{30}{62} \cdot \frac{31}{64} = 8^x$, then value of x is

- (a) -7
 (b) -9
 (c) -10
 (d) -12

Sol. We have,

$$E = \frac{31!}{2^{31}(32)!} = \frac{1}{2^{31}(32)} = \frac{1}{2^{36}} \\ = 2^{-36} = (2^3)^{-12} = 8^{-12}, \text{ but it is } 8^x.$$

Thus, $x = -12$

Hence, (d) is the correct answer.

- Ex 16.** The number of rational numbers lying in the interval (2002, 2003) all whose digits after the decimal points are non-zero and are in decreasing order, is

- (a) $\sum_{i=1}^9 {}^9P_i$
 (b) $\sum_{i=1}^{10} {}^9P_i$
 (c) $2^9 - 1$
 (d) $2^{10} - 1$

Sol. A rational number of the desired category is of the form 2002. $x_1x_2 \dots x_k$, where $1 \leq k \leq 9$ and $9 \geq x_1 > x_2 > \dots > x_k \geq 1$. We can choose k digits out of 9 in ${}^9C_k$ ways and arrange them in decreasing order in just one way. Thus, the desired number of rational numbers is

$${}^9C_1 + {}^9C_2 + \dots + {}^9C_9 = 2^9 - 1$$

Hence, (c) is the correct answer.

- Ex 17.** The number of ways of dividing 15 men and 15 women into 15 couples, each consisting of a man and a woman, is
 (a) 1240 (b) 1840 (c) 1820 (d) 2005

Sol. The number of ways of choosing first couple is $({}^{15}C_1)({}^{15}C_1) = 15^2$, the number of ways of choosing 2nd couple is $({}^{14}C_1)({}^{14}C_1) = 14^2$ and so on. Thus, the number of ways of choosing the couple is

$$15^2 + 14^2 + 13^2 + \dots + 2^2 + 1^2 \\ = \frac{15 \times (15 + 1)[2(15) + 1]}{6} = 1240$$

Hence, (a) is the correct answer.

- Ex 18.** The number of ordered pairs (m, n) , $m, n \in \{1, 2, \dots, 50\}$ such that $6^n + 9^m$ is a multiple of 5, is
 (a) 2500 (b) 1250 (c) 625 (d) 500

Sol. All the numbers of the form 6^n will end with 6 and 9^m will end with 9, if m is odd and will end with 1, if m is even, so $6^n + 9^m$ will end with 5, if n is any number and m is odd.

So, ordered pairs will be $50 \times 25 = 1250$.

Hence, (b) is the correct answer.

- Ex 19.** n is selected from the set $\{1, 2, 3, \dots, 100\}$ and the number $2^n + 3^n + 5^n$ is formed. Total number of ways of selecting n , so that the formed number is divisible by 4, is equal to

- (a) 50
 (b) 49
 (c) 48
 (d) None of these

Sol. If n is odd, $3^n = 4\lambda_1 - 1$, $5^n = 4\lambda_2 + 1$

$\Rightarrow 2^n + 3^n + 5^n$ is not divisible by 4, as $2^n + 3^n + 5^n$ will be in the form of $4\lambda + 2$.

Thus, total number of ways of selecting n is 49.

Hence, (b) is the correct answer.

- Ex 20.** The number of ordered pairs (m, n) , $m, n \in \{1, 2, \dots, 100\}$ such that $7^m + 7^n$ is divisible by 5, is
 (a) 1250 (b) 2000 (c) 2500 (d) 5000

Sol. Note that 7^r ($r \in N$) ends in 7, 9, 3 or 1 (corresponding to $r = 1, 2, 3$ and 4, respectively).

Thus, $7^m + 7^n$ cannot end in 5 for any values of $m, n \in N$.

In other words, for $7^m + 7^n$ to be divisible by 5, it should end in 0.

For $7^m + 7^n$ to end in 0, the forms of m and n should be as follows:

S.No.	m	n
1.	$4r$	$4s + 2$
2.	$4r + 1$	$4s + 3$
3.	$4r + 2$	$4s$
4.	$4r + 3$	$4s + 1$

Thus, for a given value of m , there are just 25 values of n for which $7^m + 7^n$ ends in 0.

[for instance, if $m = 4r$, then $n = 2, 6, 10, \dots, 98$]

$\therefore$ There are $100 \times 25 = 2500$ ordered pairs (m, n) for which $7^m + 7^n$ is divisible by 5.

Hence, (c) is the correct answer.

Ex 21. If the number of ways of selecting K coupons out of an unlimited number of coupons bearing the letters A, T, M, so that they cannot be used to spell the word MAT is 93, then K is equal to

- (a) 3 (b) 5
(c) 7 (d) None of these

Sol. Here, A, T, M bears unlimited number of coupons.

Thus, selecting K in which word MAT is not spelled.
If atleast one letter is not selected. i.e. ${}^3C_1$ (i)

From the remaining two we have to select K i.e. for each selection of K coupons we have 2 ways.

For K coupons, we have 2^K ways but to select atleast one, we have

$$\Rightarrow 2^K - 1 \quad \dots (ii)$$

Words cannot be used to spell MAT

$$\Rightarrow 3(2^K - 1) = 93 \Rightarrow K = 5$$

Hence, (b) is the correct answer.

Ex 22. In a shooting competition, a man can score 5, 4, 3, 2 or 0 points for each shot. Then, the number of different ways in which he can score 30 in seven shots, is

- (a) 419 (b) 418 (c) 420 (d) 421

Sol. Number of ways of making 30 in 7 shots

$$= \text{Coefficient of } x^{30} \text{ in } (x^0 + x^2 + x^3 + x^4 + x^5)^7$$

$$= \text{Coefficient of } x^{30} \text{ in } \{(x^0 + x^2 + x^3) + x^4(x + 1)\}^7$$

$$= \text{Coefficient of } x^{30} \text{ in}$$

$$\{x^{28}(x + 1)^7 + {}^7C_1 x^{24}(x + 1)^6(1 + x^2 + x^3) + {}^7C_2 x^{20}(x + 1)^5(x^3 + x + 1)^2 + \dots\}$$

$$= {}^7C_5 + {}^7C_1(6C_3 + 6C_2 + 6C_0) + {}^7C_2(5C_1 + 2)$$

$$= 21 + 252 + 147 = 420$$

Hence, (c) is the correct answer.

Ex 23. The number of ways in which the sum of upper faces of four distinct dices can be six, is

- (a) 10 (b) 4
(c) 6 (d) 7

Sol. The number of required ways will be equal to the number of solutions of

$$x_1 + x_2 + x_3 + x_4 = 6$$

where, $x_1, x_2, x_3, x_4 \geq 1$

$$\text{Let, } x_1 - 1 = a_1, x_2 - 1 = a_2, x_3 - 1 = a_3, x_4 - 1 = a_4 \geq 0$$

$$\Rightarrow (a_1 + 1) + (a_2 + 1) + (a_3 + 1) + (a_4 + 1) = 6$$

$$\text{where, } a_1, a_2, a_3, a_4 \geq 0$$

$$\Rightarrow a_1 + a_2 + a_3 + a_4 = 2$$

$$\therefore \text{Number of solutions} = {}^{2+4-1}C_{4-1} = {}^5C_3 = 10$$

Hence, (a) is the correct answer.

Ex 24. The number of ways can 14 identical toys be distributed among three boys, so that each one gets atleast one toy and no two boys get equal number of toys, is

- (a) 45 (b) 48
(c) 60 (d) None of these

Sol. Let the boys gets a, b and c toys, respectively.

$$\Rightarrow a + b + c = 14, a, b, c \geq 1 \text{ and } a, b \text{ and } c \text{ are distinct.}$$

$$\text{Let } a < b < c \text{ and } x_1 = a, x_2 = b - a, x_3 = c - b$$

$$\text{So, } 3x_1 + 2x_2 + x_3 = 14; x_1, x_2, x_3 \geq 1$$

$\therefore$ The number of solutions

$$= \text{Coefficient of } t^{14} \text{ in } \{(t^3 + t^6 + t^9 + \dots)(t^2 + t^4 + \dots)(t + t^2 + \dots)\}$$

$$= \text{Coefficient of } t^8 \text{ in } \{(1 + t^3 + t^6 + \dots)$$

$$(1 + t^2 + t^4 + \dots)(1 + t + t^2 + \dots)\}$$

$$= \text{Coefficient of } t^8 \text{ in } \{(1 + t^2 + t^3 + t^4 + t^5 + 2t^6$$

$$+ t^7 + 2t^8)(1 + t + t^2 + \dots + t^8)\}$$

$$= 1 + 1 + 1 + 1 + 1 + 2 + 1 + 2 = 10$$

Since, three distinct numbers can be assigned to three boys in $3!$ ways.

$$\text{So, total number of ways} = 10 \times 3! = 60$$

Hence, (c) is the correct answer.

Ex 25. Total number of positive integral solutions of $15 < x_1 + x_2 + x_3 \leq 20$ is equal to

- (a) 1125 (b) 1150
(c) 1245 (d) 685

Sol. $15 < x_1 + x_2 + x_3 \leq 20$

$$\Rightarrow x_1 + x_2 + x_3 = 16 + r; r = 0, 1, 2, 3, 4$$

Now, number of positive integral solutions of

$$x_1 + x_2 + x_3 = 16 + r \text{ is } {}^{13+r+3-1}C_{13+r}$$

$$\text{i.e. } {}^{15+r}C_{13+r} = {}^{15+r}C_2$$

Thus, total solutions

$$= \sum_{r=0}^4 {}^{15+r}C_2 = {}^{15}C_2 + {}^{16}C_2 + {}^{17}C_2 + {}^{18}C_2 + {}^{19}C_2$$

$$= {}^{20}C_3 - {}^{15}C_3 = 685$$

Hence, (d) is the correct answer.

Ex 26. Ten persons numbered 1, 2, ..., 10 play a chess tournament, each player playing against every other player exactly one game. It is known that no game ends in a draw. Let $w_1, w_2, \dots, w_{10}$ be the number of games won by players 1, 2, 3, ..., 10 respectively and $l_1, l_2, \dots, l_{10}$ be the number of games lost by the players 1, 2, ..., 10, respectively. Then,

$$(a) \sum w_i^2 = 81 - \sum l_i^2 \quad (b) \sum w_i^2 = 81 + \sum l_i^2$$

$$(c) \sum w_i^2 = \sum l_i^2 \quad (d) \text{None of these}$$

Sol. Clearly, each player will play 9 games and total number of games = ${}^{10}C_2 = 45$

$$\text{Clearly, } w_i + l_i = 9$$

$$\text{and } \sum w_i = \sum l_i = 45$$

$$\Rightarrow w_i = 9 - l_i$$

$$\Rightarrow w_i^2 = 81 + l_i^2 - 18l_i$$

$$\Rightarrow \sum w_i^2 = 81 \cdot 10 + \sum l_i^2 - 18 \sum l_i$$

$$= 810 + \sum l_i^2 - 18 \cdot 45 = \sum l_i^2$$

Hence, (c) is the correct answer.

Ex 27. The number of subsets of the set $A = \{a_1, a_2, \dots, a_n\}$ which contain even number of elements, is

- (a) 2^{n-1} (b) $2^n - 1$ (c) 2^{n-2} (d) 2^n

Sol. For each of the first $(n-1)$ elements $a_1, a_2, \dots, a_{n-1}$, we have two choices : either a_i ($1 \leq i \leq n-1$) lies in the subset or a_i does not lie in the subset. For the last element we have just one choice. If even number elements have already been taken, we do not include a_n , in the subset, otherwise (when odd number of elements have been added), we include it in the subset.

Thus, the number of subsets of $A = \{a_1, a_2, \dots, a_n\}$

which contain even number of elements is equal to 2^{n-1} .

Hence, (a) is the correct answer.

Ex 28. A is a set containing n elements. A subset P of A is chosen. The set A is reconstructed by replacing the elements of P . A subset Q of A is again chosen. The number of ways of choosing P and Q , so that $P \cap Q = \emptyset$, is

- (a) $2^{2n} - 2^n C_n$ (b) 2^n
(c) $2^n - 1$ (d) 3^n

Sol. Let $A = \{a_1, a_2, \dots, a_n\}$. For $a_i \in A$, we have the following choices:

- (i) $a_i \in P$ and $a_i \in Q$
(ii) $a_i \in P$ and $a_i \notin Q$
(iii) $a_i \notin P$ and $a_i \in Q$
(iv) $a_i \notin P$ and $a_i \notin Q$

Out of these, only (ii), (iii) and (iv), $a_i \notin P \cap Q$.

Therefore, the number of required subsets is 3^n .

Hence, (d) is the correct answer.

Ex 29. The number of integral points that lie exactly in the interior of the triangle with vertices $(0, 0)$, $(0, 21)$, $(21, 0)$, is

- (a) 133 (b) 190
(c) 233 (d) 105

Sol. The integral points lying on the line $x = 1$ which also lie in the interior of $\triangle OAB$ are

$(1, 1), (1, 2), \dots, (1, 19)$.

$\therefore$ There are 19 points lying on the line $x = 1$.

Similarly, the number of integral points $x = 2$ is 18 and on $x = 3$ is 17 and so on. The number of integral points on $x = 19$ is just 1.

Thus, number of integral points lying in the interior of the triangle is

$$19 + 18 + \dots + 1 = \frac{1}{2}(19)(19 + 1) = 190$$

Hence, (b) is the correct answer.

Ex 30. Let $S = \{1, 2, 3, \dots, n\}$. If X denotes the set of all subsets of S containing exactly two elements, then the value of $\sum_{A \in X} (\min A)$ is

- (a) ${}^{n+1}C_3$ (b) ${}^nC_3$
(c) ${}^nC_2$ (d) None of these

Sol. There are exactly $(n-1)$ subsets of S containing two elements having 1 as least element, exactly $(n-2)$ subsets of S having 2 as least element and so on.

Thus,

$$\sum_{A \in X} \min(A) = 1(n-1) + 2(n-2) + \dots + (n-1)(1)$$

$$\begin{aligned} &= \sum_{r=1}^{n-1} r(n-r) = n \sum_{r=1}^{n-1} r - \sum_{r=1}^{n-1} r^2 \\ &= n \left\{ \frac{1}{2}n(n-1) \right\} - \frac{1}{6}n(n-1)(2n-1) \\ &= \frac{1}{6}n(n-1)\{3n-2n+1\} \\ &= \frac{1}{6}(n+1)n(n-1) = {}^{n+1}C_3 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 31. The sum of factors of $8!$ which are odd and are of the form $3m+2$, where m is a natural number, is

- (a) 40
(b) 8
(c) 45
(d) 35

Sol. Here, $8! = 2^7 \cdot 3^2 \cdot 5^1 \cdot 7^1$

Obviously, the factors are not multiple of either 2 or 3. So, the factors may be 1, 5, 7, 35 of which 5 and 35 are of the form $3m+2$.

So, the sum = 40

Hence, (a) is the correct answer.

Ex 32. The number of positive integral solutions of the equation $x_1 x_2 x_3 x_4 x_5 = 420$ is

- (a) 1875
(b) 1600
(c) 1250
(d) None of the above

Sol. Since, $x_1 x_2 x_3 x_4 x_5 = 420$

$$\therefore x_1 x_2 x_3 x_4 x_5 = 2^2 \times 3 \times 5 \times 7$$

Each of 3, 5 or 7 can take 5 places and 2^2 can be disposed in 15 ways.

So, required number of solutions

$$= 5^3 ({}^5C_1 + {}^5C_2) = 5^3 \times 15 = 1875$$

Hence, (a) is the correct answer.

Aliter

$$x_1 x_2 x_3 x_4 x_5 = 2^2 \times 3 \times 5 \times 7$$

Here, 2^2 can be assigned to a_1, a_2, a_3, a_4, a_5

$$\Rightarrow a_1 + a_2 + a_3 + a_4 + a_5 = 2.$$

$$\Rightarrow {}^{2+5-1}C_{5-1} = {}^6C_4 = 15.$$

Also, 3^1 can be assigned to b_1, b_2, b_3, b_4, b_5

$$\Rightarrow b_1 + b_2 + b_3 + b_4 + b_5 = 1$$

$$\Rightarrow {}^{1+5-1}C_{5-1} = {}^5C_4 = 5$$

Similarly, for 5 and 7.

$\therefore$ Total number of positive integral solutions

$$= 15 \times 5 \times 5 \times 5 = 1875$$

Ex 33. The number of numbers lying in (2, 3), where all the digits after decimal are non-zero and in increasing order, is

(a) $\sum_{i=1}^9 {}^9P_i$ (b) 511 (c) $\sum_{i=1}^{10} {}^{10}P_i$ (d) 1023

Sol. Since, the digits required is non-zero.

$\therefore$ To select amongst {1, 2, 3, 4, 5, 6, 7, 8, 9}

Thus, any number is of the form

$2x_1 x_2 \dots x_k 3$, where $1 \leq x_1 < x_2 < x_3 < \dots < x_k \leq 9$

i.e. number of numbers lying between (2, 3)

$$= {}^9C_1 + {}^9C_2 + {}^9C_3 + \dots + {}^9C_9$$

$$= 2^9 - 1 = 1023$$

Hence, (d) is the correct answer.

Ex 34. The number of divisors of $n = 2^7 \cdot 3^5 \cdot 5^3$, which are of the form $4t + 1$ ($t \in N$), is

(a) 12 (b) 10
(c) 11 (d) 14

Sol. Any number will be of the form $4t + 1$, if it does not contain any power of 2 but contain even number of odds of the form $4k + 3$ and any number of odds of the form $4k + 1$, so total factors = $1 \times 3 \times 4 = 12$

But one will be 1, also so required number = $12 - 1 = 11$

Hence, (c) is the correct answer.

Aliter

Odd numbers of the type $4t + 1$ or $4t + 3$ and they are symmetrical.

Number of odd numbers = $(1 + 5)(1 + 3) = 24$

[any number of 5's and 3's and no 2's]

Required number = 12 but 1 is included.

$\therefore$ Required number of numbers = $12 - 1 = 11$

Ex 35. Total number of divisors of $n = 2^5 \cdot 3^4 \cdot 5^{10} \cdot 7^6$ that are of the form $4\lambda + 2$, $\lambda \geq 1$, is equal to

(a) 385 (b) 55 (c) 384 (d) 54

Sol. $\because 4\lambda + 2 = 2(2\lambda + 1)$ = odd multiple of 2

Thus, total divisors = $1 \cdot 5 \cdot 11 \cdot 7 - 1 = 384$

[one is subtracted because there will be case when selected powers of three, five and seven are zero each and this will make $\lambda = 0$]

Hence, (c) is the correct answer.

Ex 36. Total number of divisors of $n = 3^5 \cdot 5^7 \cdot 7^9$ that are of the form $4\lambda + 1$, $\lambda \geq 0$, is equal to

(a) 240 (b) 30 (c) 120 (d) 15

Sol. $\because 3^{n_1} = (4 - 1)^{n_1} = 4\lambda_1 + (-1)^{n_1}$,

$$5^{n_2} = (4 + 1)^{n_2} = 4\lambda_2 + 1$$

$$\text{and } 7^{n_3} = (8 - 1)^{n_3} = 4\lambda_3 + (-1)^{n_3}$$

Hence, any positive integer power of 5 will be in the form of $4\lambda_2 + 1$. Even powers of 3 and 7 will be in the form of $4\lambda + 1$ and odd powers of 3 and 7 will be in the form of $4\lambda - 1$. Thus, required number of divisors

$$= 8 \cdot (3 \cdot 5 + 3 \cdot 5) = 240$$

Hence, (a) is the correct answer.

Ex 37. Total number of positive integral solutions of $x_1 \cdot x_2 \cdot x_3 = 60$ is equal to

(a) 27 (b) 54
(c) 64 (d) None of these

Sol. $\because x_1 \cdot x_2 \cdot x_3 = 2^2 \cdot 3 \cdot 5$

Total number of positive integral solutions

$$= 3 \cdot 3 \cdot ({}^3C_1 + {}^3C_2) = 54$$

Hence, (b) is the correct answer.

Aliter

$$x_1 \cdot x_2 \cdot x_3 = 2^2 \cdot 3 \cdot 5$$

Here, 2^2 can be assigned to a_1, a_2, a_3 .

$$\Rightarrow a_1 + a_2 + a_3 = 2$$

$$\Rightarrow {}^{2+3-1}C_{3-1} = {}^4C_2 = 6$$

and 3^1 can be assigned to b_1, b_2, b_3 .

$$\Rightarrow b_1 + b_2 + b_3 = 1 \Rightarrow {}^{1+3-1}C_{3-1} = 3$$

$$\therefore x_1 \cdot x_2 \cdot x_3 = 2^2 \cdot 3^1 \cdot 5^1$$

Total number of positive integral solutions

$$= 6 \times 3 \times 3 = 54$$

Ex 38. The number of positive integral solutions of the equation $x_1 x_2 x_3 x_4 x_5 = 1050$, is

(a) 1800 (b) 1600
(c) 1400 (d) None of these

Sol. Using prime factorisation of 1050, we can write the given equation as

$$x_1 x_2 x_3 x_4 x_5 = 2 \times 3 \times 5^2 \times 7$$

We can assign 2, 3 or 7 to any of variables. We can assign entire 5^2 to just one variable in 5 ways or can assign $5^2 = 5 \times 5$ to two variables in ${}^5C_2$ ways. Thus, 5^2 can be assigned in

$${}^5C_1 + {}^5C_2 = 5 + 10 = 15 \text{ ways}$$

Hence, required number of solutions

$$= 5 \times 5 \times 5 \times 15 = 1875$$

Hence, (d) is the correct answer.

Aliter

$$x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 2 \times 3 \times 5^2 \times 7$$

For 5^2 , the number of ways = ${}^{5+2-1}C_{5-1} = {}^6C_4 = 15$

For 2^1 , the number of ways = ${}^{1+5-1}C_{5-1} = {}^5C_4 = 5$

$$\therefore \text{Positive integral solutions for } x_1 \cdot x_2 \cdot x_3 \cdot x_4 \cdot x_5 = 2 \times 3 \times 5^2 \times 7 \text{ is } 15 \times 5 \times 5 \times 5 = 1875$$

Ex 39. Let a be a factor of 120, then the number of positive integral solutions of $x_1 x_2 x_3 = a$ is

(a) 160 (b) 320 (c) 480 (d) 960

Sol. Let x_4 be such that $x_4 = \frac{120}{a}$, then the number of positive

integral solutions of $x_1 x_2 x_3 = a$ is same as that of number of positive integral solutions of $x_1 x_2 x_3 x_4$

$$= 120 = 2^3 \times 3 \times 5$$

We can assign 3 and 5 to unknown quantities in 4×4 ways.

We can assign all 2 to one unknown in ${}^4C_1$ ways, to two unknown in ${}^4C_2$ and to three unknown in ${}^4C_3$ ways.

Hence, the number of required solutions

$$= 4 \times 4 \times [{}^4C_1 + ({}^4C_2)(2) + {}^4C_3] = 4 \times 4 \times 20 = 320$$

Hence, (b) is the correct answer.

Aliter

$$x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 120$$

$$\Rightarrow x_1 \cdot x_2 \cdot x_3 \cdot x_4 = 2^3 \times 3 \times 5$$

$$\text{For } 2^3, \text{ the number of ways} = {}^{3+4-1}C_{4-1} = {}^6C_3 = 20$$

$$\text{For } 3^1, \text{ the number of ways} = {}^{1+4-1}C_{4-1} = {}^4C_3 = 4$$

$$\therefore \text{Total number of integral solutions} = 20 \times 4 \times 4 = 320$$

- Ex 40.** The number of ways of choosing triplets (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, \dots, n, n+1\}$, is
 (a) ${}^{n+1}C_3 + {}^{n+2}C_3$ (b) $n(n+1)(2n+1)$
 (c) $1^1 + 2^2 + \dots + (n-1)^2$ (d) None of these

Sol. Triplets with $x = y < z, x < y < z, y < z < x$ can be chosen in ${}^{n+1}C_2, {}^{n+1}C_3, {}^{n+1}C_3$ ways.

$$\therefore \text{There } {}^{n+1}C_2 + 2({}^{n+1}C_3) = {}^{n+2}C_3 + {}^{n+1}C_3 \text{ ways}$$

Hence, (a) is the correct answer.

- Ex 41.** The number of maps (functions) from the set $A = \{1, 2, 3\}$ into the set $B = \{1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \leq f(j)$ whenever $i < j$, is
 (a) 84 (b) 90
 (c) 88 (d) None of these

Sol. If the function is one-one, then select any three from the set B in ${}^7C_3$ ways i.e. 35 ways.

If the function is many-one, then there are two possibilities.

All three corresponds to same element number of such functions $= {}^7C_1 = 7$ ways.

Two corresponds to same element. Select any two from the set B . The larger one corresponds to the larger and the smaller one corresponds to the smaller the third may corresponds to any two.

$$\text{Number of such functions} = {}^7C_2 \times 2 = 42$$

$$\text{So, the required number of maps} = 35 + 7 + 42 = 84$$

Hence, (a) is the correct answer.

- Ex 42.** If $f: \{1, 2, 3, 4, 5\} \rightarrow \{x, y, z, t\}$, then the total number of onto functions is equal to
 (a) 242 (b) 245 (c) 102 (d) 240

Sol. Total functions $= 4^5$

$$\text{Total functions when any one element is the left out} = {}^4C_1 \cdot 3^5$$

$$\text{Total functions when any two elements are left out} = {}^4C_2 \cdot 2^5$$

$$\therefore \text{Total number of onto functions} = 4^5 - ({}^4C_1 \cdot 3^5 - {}^4C_2 \cdot 2^5 + {}^4C_3) = 240$$

Hence, (d) is the correct answer.

- Ex 43.** The function $f: \{1, 2, 3, 4, 5\} \rightarrow \{1, 2, 3, 4, 5\}$ that are onto and $f(i) \neq i$, is equal to

- (a) 9
 (b) 44
 (c) 16
 (d) None of the above

Sol. Total number of required functions

= Number of dearrangement of 5 objects

$$= 5! \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} \right) = 44$$

Hence, (b) is the correct answer.

- Ex 44.** The total number of functions f from the set $\{1, 2, 3\}$ into the set $\{1, 2, 3, 4, 5\}$ such that $f(i) \leq f(j), \forall i < j$, is equal to

- (a) 35 (b) 30
 (c) 50 (d) 60

Sol. Let 1 is associated with $r, r \in \{1, 2, 3, 4, 5\}$, then 2 can be associated with $r, r+1, \dots, 5$.

Let 2 is associated with j , then 3 can be associated with $j, j+1, \dots, 5$. Thus, required number of functions

$$\begin{aligned} &= \sum_{r=1}^5 \left(\sum_{j=r}^5 (6-j) \right) = \sum_{r=1}^5 \frac{(6-r)(7-r)}{2} \\ &= \frac{1}{2} \left(\sum_{r=1}^5 (42 - 13r + r^2) \right) \\ &= \frac{1}{2} \left(42 \cdot 5 - 13 \cdot \frac{6 \cdot 5}{2} + \frac{5 \cdot 6 \cdot 11}{6} \right) \\ &= 35 \end{aligned}$$

Hence, (a) is the correct answer.

- Ex 45.** The number of functions f from the set $A = \{0, 1, 2\}$ into the set $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$ such that $f(i) \leq f(j)$ for $i < j$ and $i, j \in A$, is

- (a) ${}^8C_3$
 (b) ${}^8C_3 + 2({}^8C_2)$
 (c) ${}^{10}C_3$
 (d) None of the above

Sol. A function $f: A \rightarrow B$ such that $f(0) \leq f(1) \leq f(2)$ falls in one of the following four categories.

Case I $f(0) < f(1) < f(2)$

There are ${}^8C_3$ functions in this category.

Case II $f(0) = f(1) < f(2)$

There are ${}^8C_2$ functions in this category.

Case III $f(0) < f(1) = f(2)$

There are again ${}^8C_2$ functions in this category.

Case IV $f(0) = f(1) = f(2)$

There are ${}^8C_1$ functions in this category.

Thus, the number of desired functions is ${}^8C_3 + {}^8C_2 + {}^8C_2 + {}^8C_1 = {}^9C_3 + {}^9C_2 = {}^{10}C_3$

Hence, (c) is the correct answer.

Ex 46. The integers from 1 to 1000 are written in order around a circle. Starting at 1, every fifteenth number is marked (i.e. 1, 16, 31 etc.) This process is continued until a number is reached which has already been marked, then unmarked numbers are

- (a) 200 (b) 400 (c) 600 (d) 800

Sol. In first round all the integers, which leaves the remainder 1 when divided by 15, will be marked.

Last number of this category is 991.

Next number to be marked is $(991 + 15 - 1000) = 6$

Again, second round of integers which leaves the remainder 6 when divided by 15, will be marked.

Last number of this category is 996.

Next number to be marked is $(996 + 15 - 1000) = 11$

Thus, third round of integers which leaves the remainder 11 when divided by 15, will be marked last number of this category is 986.

Next number to be marked is $986 + 15 - 1000 = 1$, which is already been marked.

Hence, the numbers of the form $15\lambda + 1, 15\lambda + 6, 15\lambda + 11$ are marked. All these numbers can also be put in the form of $5\lambda_1 + 1 (\lambda_1 \geq 0)$.

Hence, total number of marked numbers

$$= 1 + \left[\frac{999}{5} \right] = 200$$

$\therefore$ Unmarked numbers = 800

Hence, (d) is the correct answer.

Ex 47. Let $f(n)$ denotes the number of different ways the positive integer n can be expressed as the sum of 1's and 2's. For example $f(4) = 5$, since $4 = 1 + 1 + 1 + 1, 1 + 1 + 2, 1 + 2 + 1, 2 + 1 + 1, 2 + 2$ (**Note** Order of 1's and 2's is important). Then, $f\{f(6)\}$ is

- (a) $f(6)$ (b) $f(12)$
(c) $f(13)$ (d) None of these

Sol. As $f(4) = 5$ [given]

$\therefore f(6)$ can be written using 1's and 2's as

Number of 1's	Number of 2's	Number of arrangements
0	3	$\frac{3!}{3!} = 1$
2	2	$\frac{4!}{2!2!} = 6$
4	1	$\frac{5!}{4!} = 5$
6	0	$\frac{6!}{6!} = 1$
Total		13

$\therefore f(6) = 13$

$\therefore f\{f(6)\} = f(13)$

Hence, (c) is the correct answer.

Ex 48. The number of ways in which we can choose 3 squares on a chessboard such that one of the squares has its two sides common to other two squares, is

- (a) 290 (b) 292
(c) 294 (d) 296

Sol. Either we have to choose  or 

(i) Every square of (2 by 2)  will contribute four

 shaped figures by removing any one square out of it.

Number of ways to choose these squares = $7 \times 7 = 49$

So, total  shaped figures = $49 \times 4 = 196$

(ii) In every line it is possible to have 6  shaped

figures.

So, total number of such figures = $6 \times 8 \times 2 = 96$

$\therefore$ Total number of cases = $196 + 96 = 292$

Hence, (b) is the correct answer.

Ex 49. A man on the verge of dying. He has unlimited amount of money and intends to distributed some of it among his n relatives, so that total money that is to be distributed is a positive multiple of four and no relative gets more than ₹ $(4n - 1)$, then the number of ways in which the rich man can write his will, is

- (a) $4^{n-1} - 1$
(b) $n^n - 1$
(c) $4^{n-1} \cdot n^n$
(d) $4^{n-1} \cdot n^n - 1$

Sol. Let us assume the total money that is to be distributed is $4K$; ($K \in$ Natural numbers) and the relative x_i receives ₹ p_i .

$$\Rightarrow \sum_{i=1}^n p_i = 4K \quad \text{and} \quad 0 \leq p_i \leq 4n - 1$$

Thus, to obtain the number of integral solutions of

$p_1 + p_2 + p_3 + \dots + p_n = 4K, K = 1, 2, \dots$ is

Coefficient of x^{4K} in $(1 + x + x^2 + \dots + x^{4n-1})^n \dots$ (i)

$$\text{Let } (1 + x + x^2 + \dots + x^{4n-1})^n = \sum_{r=0}^{4n^2-n} a_r x^r \quad \dots$$
 (ii)

Now, required coefficients = $a_4 + a_8 + a_{12} + \dots$

Putting $x = 1, -1, i$ and $-i$ in Eq. (ii) and adding, we get

$$4(a_0 + a_4 + a_8 + \dots) = (4n)^n + 0 + 0 + 0$$

$$\Rightarrow a_0 + a_4 + a_8 + \dots = 4^{n-1} \cdot n^n \quad \dots$$
 (iii)

From Eqs. (i) and (iii), we get

Coefficient of x^{4K} in $(1 + x + x^2 + \dots + x^{4n-1})^n$ is $(4^{n-1} \cdot n^n - 1)$.

Hence, (d) is the correct answer.

Ex 50. A seven-digit number made up of all distinct digits 8, 7, 6, 4, 2, x and y is divisible by 3. Then, possible number of order pair (x, y) is

- (a) 4
(b) 8
(c) 2
(d) None of the above

Sol. We have, 8, 7, 6, 4, 2, x and y

Any number is divisible by 3, if sum of digits is divisible by 3.

i.e. $x + y + 27$ is divisible by 3.

x and y can take values from 0, 1, 3, 5, 9.

$\therefore$ Possible pairs are (5, 1), (3, 0), (9, 0), (9, 3), (1, 5), (0, 3), (0, 9) and (3, 9).

Hence, (b) is the correct answer.

Ex 51. 10 different toys are to be distributed among 10 children. Total number of ways of distributing these toys so that exactly 2 children do not get any toy, is equal to

- (a) $(10!)^2 \left(\frac{1}{3!2!7!} + \frac{1}{(2!)^5 6!} \right)$
(b) $(10!)^2 \left(\frac{1}{3!2!7!} + \frac{1}{(2!)^4 6!} \right)$
(c) $(10!)^2 \left(\frac{1}{3!7!} + \frac{1}{(2!)^5 6!} \right)$
(d) $(10!)^2 \left(\frac{1}{3!7!} + \frac{1}{(2!)^4 6!} \right)$

Sol. It is possible in two mutually exclusive cases.

Case I 2 children get none, one child gets three and all remaining children get one each.

Case II 2 children get none, 2 children get 2 each and all remaining children get one each.

In first case, number of ways = $\frac{10!}{3!2!7!} \cdot 10!$

In second case, number of ways = $\frac{10!}{(2!)^4 \cdot 6!} \cdot 10!$

Thus, total number of ways

$$= (10!)^2 \left(\frac{1}{3!2!7!} + \frac{1}{(2!)^4 6!} \right)$$

Hence, (b) is the correct answer.

Ex 52. A delegation of five students is to be formed from a group of 10 students. If three particular students want to remain together whereas two particular students do not want to remain together, then the number of selection is

- (a) 10 (b) 20
(c) 30 (d) None of these

Sol. Let A, B and C want to remain together and D, E do not want to remain together.

In	Out	Number of selections
A, B, C, D	E	${}^5C_1 = 5$
A, B, C, E	D	${}^5C_1 = 5$
D	A, B, C, E	${}^5C_4 = 5$
E	A, B, C, D	${}^5C_4 = 5$

$\therefore$ Required number of selections

$$= {}^5C_1 + {}^5C_1 + {}^5C_4 + {}^5C_4 = 20$$

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 53. If m and n are positive integers more than or equal to 2, $m > n$, then $(mn)!$ is divisible by

- (a) $(m!)^n$
(b) $(n!)^m$
(c) $(m+n)!$
(d) $(m-n)!$

Sol. $\frac{(mn)!}{(m!)^n}$ is the number of ways of distribution mn distinct

objects in n persons equally.

Hence, $\frac{(mn)!}{(m!)^n}$ is an integer $\Rightarrow (m!)^n / (mn)!$

Similarly, $(n!)^m / (mn)!$

Further $m + n < 2m \leq mn$

$\Rightarrow (m+n)! / (mn)!$

and $m - n < m < mn$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 54. Let n be a positive integer with $f(n) = 1! + 2! + 3! + \dots + n!$ and $P(x), Q(x)$ be polynomials in x such that $f(n+2) = P(n)$

$f(n+1) + Q(n)f(n)$ for all $n \geq 1$. Then,

- (a) $P(x) = x + 3$
(b) $Q(x) = -x - 2$
(c) $P(x) = -x - 2$
(d) $Q(x) = x + 3$

Sol. $\therefore f(n) = 1! + 2! + 3! + \dots + n!$

$$f(n+1) = 1! + 2! + 3! + \dots + (n+1)!$$

$$f(n+2) = 1! + 2! + 3! + \dots + (n+2)!$$

$$\therefore f(n+2) - f(n+1) = (n+2)! = (n+2)(n+1)!$$

$$= (n+2)[f(n+1) - f(n)]$$

$$\Rightarrow f(n+2) = (n+3)f(n+1) - (n+2)f(n)$$

$$\Rightarrow P(x) = x + 3, Q(x) = -x - 2$$

Hence, (a) and (b) are the correct answers.

Ex 55. The number of ways in which we can choose 2 distinct integers from 1 to 100, such that the difference between them is atmost 10, is

- (a) $^{100}C_2 - ^{90}C_2$
 (b) $^{100}C_{98} - ^{90}C_{88}$
 (c) $^{100}C_2 - ^{90}C_{88}$
 (d) None of the above

Sol. Let the chosen integers be x_1 and x_2 .

Let a be an integer before x_1 , b be integer between x_1 and x_2 and c be integer after x_2 .

$\therefore a + b + c = 98$, where $a \geq 0, b \geq 10, c \geq 0$.

Now, if we consider the choices, where difference is atleast 11, then the number of solutions is $^{88+3-1}C_{3-1} = ^{90}C_2$

$\therefore$ The number of ways in which b is less than 10 is $^{100}C_2 - ^{90}C_2$ which is equal to options (a), (b) and (c).

Hence, (a), (b) and (c) are the correct answers.

Ex 56. n locks and n corresponding keys are available. But the actual combination is not known. The maximum numbers of trials that are needed to assign the keys to their corresponding locks, are

- (a) $^nC_2$
 (b) $\sum_{k=2}^n (k-1)$
 (c) $n!$
 (d) $^{n+1}C_2$

Sol. First key will be tried for atmost $(n-1)$ locks. Second key will be tried for atmost $(n-2)$ locks and so on. Thus, the maximum number of trials needed

$$= (n-1) + (n-2) + \dots + 1 \\ = \frac{n(n-1)}{2} = ^nC_2$$

Hence, (a) and (b) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 57-61) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

Ex 57. Statement I The maximum number of points of intersection of 8 circles of unequal radii is 56.

Statement II The maximum number of points into which 4 circles of unequal radii and 4 non-coincident straight lines intersect is 50.

Sol. Statement I Two circles intersect in 2 points.

$\therefore$ Maximum number of points of intersection
 $= 2 \times \text{Number of selection of two circles from 8 circles}$
 $= 2 \times {}^8C_2 = 2 \times 28 = 56$

Statement II 4 lines intersect each other in ${}^4C_2 = 6$ points and 4 circles intersect each other in $2 \times {}^4C_2 = 12$ points.

Further, one line and one circle intersect in two points. So, 4 lines will intersect four circles in 32 points.

$\therefore$ Maximum number of points $= 6 + 12 + 32 = 50$

Hence, (b) is the correct answer.

Ex 58. Statement I If there are six letters, $L_1, L_2, L_3, L_4, L_5, L_6$ and their corresponding six envelopes are $E_1, E_2, E_3, E_4, E_5, E_6$. Letters having odd value can be put into odd value envelopes and even value letters can be put into even value envelopes, so that no letter go into the right envelopes, the number of arrangements will be equal to 4.

Statement II If P_n is the number of ways in which n letter can be put in n corresponding envelopes, such that no letters goes to correct envelopes, then

$$P_n = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \dots + \frac{(-1)^n}{n!} \right)$$

Sol. $\begin{matrix} L_1 & L_3 & L_5 & L_2 & L_4 & L_6 \\ E_1 & E_3 & E_5 & E_2 & E_4 & E_6 \end{matrix}$
 $\therefore$ Number of ways

$$= 3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \right) \cdot 3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \right) = 4$$

Hence, (a) is the correct answer.

Ex 59. Statement I The maximum value of k , such that $(50)^k$ divides 100! is 2.

Statement II If p is any prime number, then power of p in $n!$ is equal to $\left[\frac{n}{p} \right] + \left[\frac{n}{p^2} \right] + \left[\frac{n}{p^3} \right] + \dots$, where $[]$ represents the greatest integer function.

Sol. If power of p is l in $n!$, then $l = \left[\frac{n}{p} \right] + \left[\frac{n}{p^2} \right] + \dots$

Power of 2 in 100!

$$= \left[\frac{100}{2} \right] + \left[\frac{100}{2^2} \right] + \left[\frac{100}{2^3} \right] + \left[\frac{100}{2^4} \right] + \left[\frac{100}{2^5} \right] + \left[\frac{100}{2^6} \right]$$

$$= 50 + 25 + 12 + 6 + 3 + 1 = 97$$

So, power of 5 in 100! is 24.

$\therefore$ Exponent of 25 in $(100)! = 12$

$\Rightarrow$ Maximum value of k is 12.

Hence, (d) is the correct answer.

Ex 60. Statement I A five-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 with repetition. The total number formed are 216.

Statement II If sum of digits of any number is divisible by 3, then the number must be divisible by 3.

Sol. Number of numbers form by using 1, 2, 3, 4, 5 = $5! = 120$

Number of numbers formed by using 0, 1, 2, 4, 5

$$\boxed{4} \boxed{4} \boxed{3} \boxed{2} \boxed{1} = 4 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 96$$

Statement I is incorrect and Statement II is correct.

Hence, (d) is the correct answer.

Ex 61. Statement I The number of ordered pairs (m, n) ; $m, n \in \{1, 2, 3, \dots, 20\}$, such that $3^m + 7^n$ is a multiple of 10, is equal to 100.

Statement II $3^m + 7^n$ has last digit zero, when m is of $4k - 2$ type and n is of $4l$ type where $k, l \in W$.

Sol. 3^m gives 1 at unit place, if $m = 4k$

and these are 5 in numbers.

7^n gives 9 at unit place, if $n = 4l + 2$

and these are 5 in numbers.

$\therefore$ Number of ways = 25

3^m gives 3 at unit place, if $m = 4k + 1$

and these are 5 in numbers.

7^n gives 7 at unit place, if $n = 4l + 1$

and these are 5 in numbers.

$\therefore$ Number of ways = 25

3^m gives 7 at unit place, if $m = 4k + 3$

and these are 5 in numbers.

7^n gives 3 at unit place, if $n = 4l + 3$

and these are 5 in numbers.

$\therefore$ Number of ways = 25

3^m gives 9 at unit place, if $m = 3k + 2$

and these are 5 in numbers.

7^n gives 1 at unit place, if $n = 4l$

and these are 5 in numbers.

Hence, (c) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 62-64) One of the most important techniques of counting is the principle of exclusion and inclusion. Let $A_1, A_2, \dots, A_n$ be m sets and $n(A_i)$ represents the cardinality of the set A_i (the number of elements in the set A_i), then according to the principle of exclusion and inclusion

$$n(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^m n(A_i) - \sum_{i \neq j} n(A_i \cap A_j) + \sum_{i < j < k} n(A_i \cap A_j \cap A_k) - \dots + (-1)^{m+1} n(A_1 \cap A_2 \cap \dots \cap A_m)$$

In particular, if A, B, C are three sets, then

$$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

Principle of exclusion and inclusion must be applied whenever there is a chance of repeated counting of some of the samples.

Ex 62. The number of numbers from 1 to 100, which are neither divisible by 3 nor by 5 nor by 7, is
(a) 67 (b) 55 (c) 45 (d) 33

Sol. A : Set of numbers, which are divisible by 3.

B : Set of numbers, which are divisible by 5.

C : Set of numbers, which are divisible by 7.

Number of numbers, which are divisible by atleast one of 3 or 5 or 7

$$= n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(B \cap C) - n(C \cap A) + n(A \cap B \cap C)$$

$$= \left[\frac{100}{3} \right] + \left[\frac{100}{5} \right] + \left[\frac{100}{7} \right] - \left[\frac{100}{15} \right] - \left[\frac{100}{21} \right] - \left[\frac{100}{35} \right] + \left[\frac{100}{105} \right] = 55$$

$\therefore$ Desired number = $100 - 55 = 45$

Hence, (c) is the correct answer.

Ex 63. A six letters word is formed using the letters of the word 'ALMIGHTY' with or without repetition. The number of words that contain exactly three different letters, is

(a) 15600

(b) 30240

(c) ${}^8P_6 - {}^8P_3$

(d) None of the above

Sol. We can select three different letters in ${}^8C_3$ ways.

Suppose we select A, L, M.

Let X = Set of words in which A is absent,

Y = Set of words in which L is absent and

Z = Set of words in which M is absent.

$$\text{Then, } n(X \cup Y \cup Z) = (2^6 + 2^6 + 2^6) - (1^6 + 1^6 + 1^6) + 0 = 189$$

So, the number of six letter words formed by using A, L and M = $3^6 - 189 = 540$

$$\therefore \text{The desired number of words} = {}^8C_3 \times 540 = 30240$$

Hence, (b) is the correct answer.

Ex 64. The number of natural numbers less than or equal to 2985984, which are neither perfect squares nor perfect cubes, is

- (a) 2984124
(b) 2984244
(c) 2959595
(d) None of the above

Sol. Desired number
 $= 2985984 - \{\sqrt{2985984} + \sqrt[3]{2985984} - \sqrt[6]{2985984}\}$
 $= 2985984 - 1860 = 2984124$
 $[\because 2985984 = 2^{12} \cdot 3^6]$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 65-67) Consider the letters of the word 'MATHEMATICS'. There are eleven letters some of them are identical. Letters are classified as repeating and non-repeating letters. Set of repeating letters = {M, A, T} and set of non-repeating letters = {H, E, I, C, S}.

Ex 65. Possible number of words taking all letters at a time such that atleast one repeating letter is at odd position in each word, is

- (a) $\frac{9!}{2!2!2!}$ (b) $\frac{11!}{2!2!2!}$
(c) $\frac{11!}{2!2!2!} - \frac{9!}{2!2!}$ (d) $\frac{9!}{2!2!}$

Sol. Since, there are 5 even places and 3 pairs of repeated letters, therefore atleast one of these must be at an odd place.

$\therefore$ The number of ways = $\frac{11!}{2!2!2!}$

Hence, (b) is the correct answer.

Ex 66. Possible number of words taking all letters at a time such that in each word both M's are together and both T's are together but both A's are not together, is

- (a) $7! \cdot {}^8C_2$ (b) $\frac{11!}{2!2!2!} - \frac{10!}{2!2!}$
(c) $\frac{6!4!}{2!2!}$ (d) $\frac{9!}{2!2!2!}$

Sol. Make a bundle of both M's and another bundle of T's. Then, except A's we have 5 letters remaining, so M's, T's and the letters except A's can be arranged in 7! ways.

$\therefore$ Total number of arrangements = $7! \times {}^8C_2$

Hence, (a) is the correct answer.

Ex 67. Possible number of words in which no two vowels are together, is

- (a) $\frac{7!}{2!2!} \cdot {}^8C_4 \cdot \frac{4!}{2!}$
(b) $\frac{7!}{2!} \cdot {}^8C_4 \cdot \frac{4!}{2!}$
(c) $7! \cdot {}^8C_4 \cdot \frac{4!}{2!}$
(d) $\frac{7!}{2!2!2!} \cdot {}^8C_4 \cdot \frac{4!}{2!}$

Sol. Consonants can be placed in $\frac{7!}{2!2!}$ ways.

Then, there are 8 places and 4 vowels.

$\therefore$ Number of ways = $\frac{7!}{2!2!} \cdot {}^8C_4 \cdot \frac{4!}{2!}$

Hence, (a) is the correct answer.

Passage III (Ex. Nos. 68-70) Let set $S = \{1, 2, 3, \dots, n\}$ be a set of first n natural and $A \subseteq S$. Suppose, $n(A)$ represents cardinal number and $\min(A)$ represents least number among the elements of set A .

Ex 68. The greatest value of $\min(A)$, where $A \subseteq S$ and $n(A) = r, 1 \leq r \leq n$, is

- (a) r
(b) $(n - r)$
(c) $n - r + 1$
(d) $r + 1$

Sol. Consider, $A = \{n - r + 1, n - r + 2, \dots, n\} \subset S$

Clearly, $n(A) = r$ and $\min(A) = n - r + 1$

$\therefore$ Greatest value of $\min(A) = n - r + 1$

Hence, (c) is the correct answer.

Ex 69. The number of subsets A of S for which $n(A) = r$ and $\min(A) = k$, is

- (a) ${}^{n-k}C_{r-1}$
(b) ${}^nC_{r-1}$
(c) ${}^{n-k+1}C_{r-1}$
(d) ${}^{n-k-1}C_{r-1}$

Sol. k is the least member of A and rest of the $r - 1$ members of A are among the numbers $k + 1, k + 2, k + 3, \dots, n$.

$\therefore$ Number of ways = ${}^{n-k}C_{r-1}$

Hence, (a) is the correct answer.

Ex 70. The value of $\sum_{n(A)=r} \{\min(A)\} = k$ is

- (a) $n \cdot {}^{n-k}C_{r-1}$
(b) $(n + 1) \cdot {}^{n-k}C_{r-1} - r \cdot {}^{n-k+1}C_r$
(c) $k \cdot {}^{n-k}C_{r-1} + n \cdot {}^{n-k+1}C_r$
(d) ${}^nC_r$

Sol. Now, $\sum_{\substack{A \subseteq S \\ |A|=r}} [\min(A)] = \sum_{k=1}^{n-r+1} (k \cdot {}^{n-k}C_{r-1})$

Now, $k \cdot {}^{n-k}C_{r-1} = [(n + 1) - (n - k + 1)] \cdot {}^{n-k}C_{r-1}$
 $= (n + 1) \cdot {}^{n-k}C_{r-1} - (n - k + 1) \cdot {}^{n-k}C_{r-1}$

Also, $(n - k + 1) \cdot {}^{n-k}C_{r-1} = \frac{(n - k + 1) \cdot (n - k)!}{(r - 1)! [n - k - r + 1]!}$

$= \frac{r \cdot (n - k + 1)!}{r! (n - k + 1 - r)!} = r \cdot {}^{n-k+1}C_r$

$\therefore k \cdot {}^{n-k}C_{r-1} = (n + 1) \cdot {}^{n-k}C_{r-1} - r \cdot {}^{n-k+1}C_r$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 71. Match the statements of Column I with values of Column II.

Column I	Column II
A. The number of five-digit numbers having the product of digits 20 is	p. 77
B. A man took 5 space plays out of an engine to clean them. The number of ways in which he can place atleast two plays in the engine from where they came out, is	q. 31
C. The number of integers between 1 and 1000 inclusive in which atleast two consecutive digits are equal, is	r. 50
D. The value of $\frac{1}{15} \sum_{1 \leq i \leq j \leq 9} i \cdot j$ is	s. 181

Sol. A. There are two cases:

Case I 5, 4, 1, 1, 1, $\frac{5!}{3!} = 20$

Case II 5, 2, 2, 1, 1, $\frac{5!}{2!2!} = 30$

$\therefore$ Total = 20 + 30 = 50

B. $5! - D_5 = 5 \cdot D_4$
 $[D_5 \text{ stands for dearrangements of 5 things}]$
 $= 120 - 44 - 5 \times 9 = 31$

C. $1000 - 9^3 - 9^2 - 9 = 181$

D. $\frac{1}{15} \sum_{1 \leq i \leq j \leq 9} i \cdot j = \frac{1}{15} \left\{ \sum_{1 \leq i \leq j \leq 9} i \cdot j + (1^2 + 2^2 + \dots + 9^2) \right\}$
 $= \frac{1}{15} \left[\frac{(1+2+\dots+9)^2 - (1^2 + 2^2 + \dots + 9^2)}{2} + (1^2 + 2^2 + \dots + 9^2) \right]$
 $= \frac{1}{30} \left[\left(\frac{9 \times 10}{2} \right)^2 + \frac{9 \times 10 \times 19}{6} \right] = 77$

$\therefore A \rightarrow r; B \rightarrow q; C \rightarrow s, D \rightarrow p$

Ex 72. Match the statements of Column I with values of Column II.

Column I	Column II
A. The total number of selections of fruits which can be made from, 3 bananas, 4 apples and 2 oranges, is	p. greater than 50
B. If 7 points out of 12 are in the the same straight line, then the number of triangles formed is	q. greater than 100
C. The number of ways of selecting 10 balls from unlimited number of red, black, white and green balls, is	r. greater than 150
D. The total number of proper divisors of 38808 is	s. greater than 200

Sol. A. Required number of ways

$$= (2+1)(3+1)(4+1) - 1 = 59$$

B. The number of ways of selecting 3 points out of 12 points is ${}^7C_3$ ways.

Hence, the number of triangles formed is
 ${}^{12}C_3 - {}^7C_3 = 185$

C. Required number of ways

$$= \text{Coefficient of } x^{10} \text{ in } (1+x+x^2+\dots)^4$$

$$= \text{Coefficient of } x^{10} \text{ in } (1-x)^{-4}$$

$$= {}^{10+4-1}C_{4-1}$$

$$= {}^{13}C_3 = 286$$

D. Factorising the given number, we have

$$38808 = 2^3 \cdot 3^2 \cdot 7^2 \cdot 11$$

The total number of divisors of this number is same as the number of ways of selecting some or all of two 2's, two 3's, two 7's and one 11. Therefore, the total number of divisors

$$= (3+1)(2+1)(2+1)(1+1) = 72$$

Hence, the required number of divisors

$$= 72 - 2 = 70$$

$$A \rightarrow p; B \rightarrow p, q, r; C \rightarrow p, q, r, s; D \rightarrow p$$

Ex 73. Match the statements of Column I with values of Column II. Consider the word 'HONOLULU'.

Column I	Column II
A. Number of words that can be formed using the letters of given word in which consonants and vowels are alternate, is	p. 26
B. Number of words that can be formed without changing the order of vowels, is	q. 144
C. Number of ways in which 4 letters can be selected from the letters of the given word, is	r. 840
D. Number of words in which two O's are together but U's are separated, is	s. 900

Sol. A. Required number of ways = $2! \cdot \frac{4!}{2!} \cdot \frac{4!}{2!2!} = 144$

B. Required number of words = ${}^8C_4 \cdot \frac{4!}{2!} = 840$

C. Required number of ways of

Case I 2 alike + 2 alike = ${}^3C_2$

Case II 2 alike + 2 different = ${}^3C_1 {}^4C_2$

Case III All different = ${}^5C_4 = 3 + 18 + 5 = 26$

D. Required number of words = $\frac{5!}{2!} \cdot {}^6C_2 = 900$

$$A \rightarrow q; B \rightarrow r; C \rightarrow p; D \rightarrow s$$

Type 6. Single Integer Answer Type Questions

Ex 74. The possible number of ordered triads (m, n, p) , where $m, n, p \in N$ is $(6250)k$, such that $1 \leq m \leq 100$, $1 \leq n \leq 50$, $1 \leq p \leq 25$ and $2^m + 2^n + 2^p$ is divisible by 3, then k is _____.

Sol. (5) Here, $2^m + 2^n + 2^p = (3-1)^m + (3-1)^n + (3-1)^p$
 $= 3k + (-1)^m + (-1)^n + (-1)^p$ ($k \in I$)

So that $2^m + 2^n + 2^p$ is divisible by 3, if m, n, p all are odd or all are even.

$\therefore$ Number of possible ordered triads

$$= 50 \times 25 \times 12 + 50 \times 25 \times 13 = 31250 = 6250 \times 5$$

But it is $6250k$.

$$\therefore k = 5$$

Ex 75. If the total number of garlands that can be formed using 3 flowers of 1st kind and 12 flowers of the 2nd kind is 19λ , then λ is _____.

Sol. (1) This is a case equivalent to distributing 12 identical flowers into 3 identical boxes is same as $x + y + z = 12$ taking only different combinations of x, y and z .

Case I $x = y = z = 4$ only 1 way

Case II $x = y \neq z$

$$2x + z = 12$$

$$(a) \ x > z, \ x - z = a, \ a \geq 1$$

$$\Rightarrow x = z + a$$

$$\text{So, } 2x + z = 12$$

$$\Rightarrow 3z + 2a = 12, \ a \geq 1, \ z \geq 0$$

i.e. coefficient of x^{12} in

$$(1 + x^3 + x^6 + \dots)(x^2 + x^4 + \dots) = 2$$

$$(b) \ x < z, \ z - x = a$$

$$\Rightarrow z = x + a$$

$$\text{So, } 3x + a = 12, \ a \geq 1, \ x \geq 0$$

i.e. coefficient of x^{12} in

$$(1 + x^3 + x^6 + x^9 + \dots)(x + x^2 + \dots) = 4$$

Case III $x + y + z = 12, \ x \neq y \neq z$

Taking only different combinations of x, y and z .

Let $x > y > z$

$$\Rightarrow y - z = a$$

$$\Rightarrow y = z + a, \ a \geq 1$$

$$x - y = b$$

$$\Rightarrow x = z + a + b, \ b \geq 1$$

$$x + y + z = 12$$

$$\Rightarrow 3z + 2a + b = 12; \ a, b \geq 1, \ z > 0$$

Coefficient of x^{12} in

$$(1 + x^3 + x^6 + \dots)(x^2 + x^4 + x^6 + \dots)(x + x^2 + x^3 + \dots) = 12$$

$$\text{So, total ways} = 1 + 2 + 4 + 12 = 19$$

$$\text{Hence, } \lambda = 1$$

Ex 76. If the number of ordered triplet (a, b, c) , such that $\text{LCM}(a, b) = 1000$, $\text{LCM}(b, c) = 2000$ and $\text{LCM}(c, a) = 2000$ is $10k$, then k is _____.

$$\text{Sol. (7)} \because 1000 = 2^3 \cdot 5^3, 2000 = 2^4 \cdot 5^3$$

$$\text{So, } a = 2^A \cdot 5^R, b = 2^B \cdot 5^S, c = 2^C \cdot 5^T$$

$$\max(A, B) = 3, \max(A, C) = \max(B, C) = 4$$

$$\max(R, S) = \max(R, T) = \max(S, T) = 3$$

There are 10 ways of choosing R, S, T .

1 with all 3, 3 with one 2, 3 with one 1, 3 with one 0.

C must be 4. There are 7 ways of choosing A, B .

1 with both 3, 3 with $A = 3, B$ not, 3 with $B = 3$ and A not thus, 7 ways of choosing A, B, C , hence $7 \cdot 10 = 70$ ways

$$\Rightarrow k = 7$$

Ex 77. N is the number of ways in which a person can walk up a stairway which has 7 steps. If he can take 1 or 2 steps up the stairs at a time, then the value of $N/3$ is _____.

Sol. (7) If x denotes the number of times, he can take unit step and y denotes the number of times, he can take 2 steps, then $x + 2y = 7$.

We must have $x = 1, 3, 5$.

If $x = 1$, then steps will be 1, 2, 2, 2.

$$\Rightarrow \text{Number of ways} = \frac{4!}{3!} = 4$$

If $x = 3$, then steps will be 1, 1, 1, 2, 2.

$$\Rightarrow \text{Number of ways} = \frac{5!}{3!2!} = 10$$

If $x = 5$, then steps will be 1, 1, 1, 1, 1, 2.

$$\Rightarrow \text{Number of ways} = {}^6C_1 = 6$$

If $x = 7$, then steps will be 1, 1, 1, 1, 1, 1, 1.

$$\Rightarrow \text{Number of ways} = {}^7C_0 = 1$$

Hence, total number of ways $(N) = 4 + 10 + 6 + 1 = 21$

$$\Rightarrow \frac{N}{3} = \frac{21}{3} = 7$$

Target Exercises

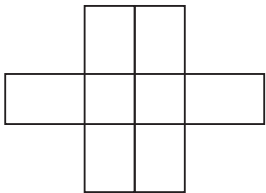
Type 1. Only One Correct Option

- If $\alpha = {}^m C_2$, then ${}^\alpha C_2$ is equal to
(a) ${}^{m+1} C_4$ (b) ${}^{m-1} C_4$ (c) $3^{m+2} C_4$ (d) $3^{m+1} C_4$
- If ${}^n C_3 + {}^n C_4 > {}^{n+1} C_3$, then
(a) $n > 6$ (b) $n > 7$
(c) $n < 6$ (d) None of these
- The value of $\sum_{r=0}^{n-1} {}^n C_r / ({}^n C_r + {}^n C_{r+1})$ equals
(a) $n+1$ (b) $n/2$
(c) $n+2$ (d) None of these
- The value of $\sum_{r=1}^n \frac{{}^n P_r}{r!}$ is
(a) 2^n (b) $2^n - 1$ (c) 2^{n-1} (d) $2^n + 1$
- If ${}^{2n+1} P_{n-1} : {}^{2n-1} P_n = 3 : 5$, then n is equal to
(a) 4 (b) 6 (c) 3 (d) 8
- The number less than 1000 that can be formed using the digits 0, 1, 2, 3, 4, 5 when repetition is not allowed is equal to
(a) 130 (b) 131
(c) 156 (d) 155
- The numbers greater than 1000 but not greater than 4000, which can be formed with the digits 0, 1, 2, 3, 4 (repetition of digits is allowed), are
(a) 350 (b) 375
(c) 450 (d) 576
- A variable name in certain computer language must be either an alphabet or an alphabet followed by a decimal digit. The total number of different variable names that can exist in that language is equal to
(a) 280 (b) 290
(c) 286 (d) 296
- The number of five-digit numbers that contain 7 exactly once is
(a) $(41)(9^3)$ (b) $(37)(9^3)$
(c) $(7)(9^4)$ (d) $(41)(9^4)$
- The total number of flags with three horizontal strips in order, which can be formed by using 2 identical red, 2 identical green and 2 identical white strips, is equal to
(a) $4!$
(b) $3 \times (4!)$
(c) $2 \times (4!)$
(d) None of these
- Let A be a set of $n (\geq 3)$ distinct elements. The number of triplets (x, y, z) of the elements of A in which atleast two coordinates same, is
(a) ${}^n P_3$ (b) $n^3 - {}^n P_3$ (c) $3n^2 - 2n$ (d) $3n^2 (n-1)$
- The total number of six-digit numbers that can be formed, having the property that every succeeding digit is greater than the preceding digit, is
(a) ${}^9 C_3$ (b) ${}^{10} C_3$ (c) ${}^9 P_3$ (d) ${}^{10} P_3$
- The number of four-digit numbers that can be made with the digits 1, 2, 3, 4 and 5 in which atleast two digits are identical, is
(a) $4^5 - 5!$ (b) 505
(c) 600 (d) None of these
- The total number of five-digit numbers of different digits in which the digit in the middle is the largest, is
(a) $\sum_{n=4}^9 {}^n P_4$ (b) $33(3!)$
(c) $30(3!)$ (d) None of these
- The number of nine non-zero digit numbers such that all the digits in the first four places are less than the digit in the middle and all the digits in the last four places are greater than that in the middle, is
(a) $2(4!)$ (b) $3(7!)/2$ (c) $2(7!)$ (d) ${}^4 P_4 \times {}^4 P_4$
- The total number of three-digit numbers, the sum of whose digits is even, is equal to
(a) 450 (b) 350 (c) 250 (d) 325
- The permutations of n objects taken
(i) atleast r objects at a time
(ii) atmost r objects at a time
(if repetition of the objects is allowed)
are respectively
(a) $\frac{n^{n-r+1}}{n-1}$ and $\frac{n^{r+1}-1}{n-1}$
(b) $\frac{n^{n-r+1}}{n-1}$ and $\frac{n^r(n^{r+1}-1)}{n-1}$
(c) $\frac{n^r(n^{n-r}-1)}{n-1}$ and $\frac{n^{r+1}-1}{n-1}$
(d) None of these
- How many ten-digit numbers can be written by using the digits 1 and 2?
(a) ${}^{10} C_1 + {}^9 C_2$ (b) 2^{10}
(c) ${}^{10} C_2$ (d) $10!$

19. A five-digit number divisible by 3 is to be formed using the numerals 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways this can be done is
(a) 216 (b) 240 (c) 600 (d) 3125
20. The total number of 9-digit numbers which have all different digits is
(a) $10!$ (b) $9!$ (c) $9 \cdot 9!$ (d) $10 \cdot 10!$
21. Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4 and then the men select the chairs from amongst the remaining. The number of possible arrangements is
(a) ${}^4C_3 \times {}^4C_2$ (b) ${}^4C_2 \times {}^4P_3$
(c) ${}^4P_2 \times {}^4P_3$ (d) None of these
22. From 4 Officers and 8 Jawans, a committee of 6 is to be chosen to include exactly one officer. The number of such committee is
(a) 160 (b) 200 (c) 224 (d) 300
23. Seven different lecturers are to deliver lectures in seven periods of a class on a particular day. A , B and C are three of the lecturers. The number of ways in which a routine for the day can be made such that A delivers his lecture before B and B before C , is
(a) 420 (b) 120
(c) 210 (d) None of these
24. The number of arrangements of letters of the word 'BANANA' in which the two N's do not appear together is
(a) 40 (b) 60 (c) 80 (d) 100
25. How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits, so that odd digits occupy even positions?
(a) 16 (b) 36 (c) 60 (d) 180
26. Let $A = \{x \mid x \text{ is a prime number and } x < 30\}$. The number of different rational numbers whose numerator and denominator belong to A , is
(a) 90 (b) 180
(c) 91 (d) None of these
27. Let S be the set of all functions from the set A to the set A . If $n(A) = k$, then $n(S)$ is
(a) $k!$ (b) k^k (c) $2^k - 1$ (d) 2^k
28. Let A be the set of 4-digit numbers $a_1a_2a_3a_4$, where $a_1 > a_2 > a_3 > a_4$, then $n(A)$ is equal to
(a) 126 (b) 84
(c) 210 (d) None of these
29. An n -digit number is a positive number with exactly n -digits. Nine hundred distinct n -digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is
(a) 6 (b) 7 (c) 8 (d) 9
30. The number of 5-digit telephone numbers having atleast one of their digits repeated is
(a) 90000 (b) 100000 (c) 30240 (d) 69760
31. If letters of the word 'KUBER' are written in all possible orders and arranged as in a dictionary, then rank of the word 'KUBER' will be
(a) 67 (b) 68 (c) 65 (d) 69
32. The total number of 4-digit numbers that are greater than 3000, that can be formed using the digits 1, 2, 3, 4, 5, 6 (no digit is being repeated in any number) is equal to
(a) 120 (b) 240 (c) 480 (d) 80
33. In a country, no two persons have identical set of teeth and there is no person without a tooth, also no person has more than 32 teeth. If shape and size of tooth is disregarded and only the position of tooth is considered, then maximum population of that country can be
(a) 2^{32} (b) $2^{32} - 1$
(c) Cannot be determined (d) None of these
34. n different toys have to be distributed among n children. Total number of ways in which these toys can be distributed, so that exactly one child gets no toy, is equal to
(a) $n!$ (b) $n! \cdot {}^nC_2$
(c) $(n-1)! \cdot {}^nC_2$ (d) $n! \cdot {}^{n-1}C_2$
35. The total number of permutations of k different things, in a row, taken not more than r at a time (each thing may be repeated any number of times) is equal to
(a) $k^r - 1$ (b) k^r (c) $\frac{k^r - 1}{k - 1}$ (d) $\frac{k(k^r - 1)}{(k - 1)}$
36. A person predicts the outcome of 20 cricket matches of his home team. Each match can result either in a win, loss or tie for the home team. Total number of ways in which he can make the predictions, so that exactly 10 predictions are correct, is equal to
(a) ${}^{20}C_{10} \cdot 2^{10}$ (b) ${}^{20}C_{10} \cdot 3^{20}$
(c) ${}^{20}C_{10} \cdot 3^{10}$ (d) ${}^{20}C_{10} \cdot 2^{20}$
37. A team of four students is to be selected from a total of 12 students. Total number of ways in which team can be selected such that two particular students refuse to be together and other two particular students wish to be together only, is equal to
(a) 220 (b) 182
(c) 226 (d) None of these
38. The total number of numbers that are less than $3 \cdot 10^8$ and can be formed using the digits 1, 2, 3, is equal to
(a) $\frac{1}{2}(3^9 + 4 \cdot 3^8)$ (b) $\frac{1}{2}(3^9 - 3)$
(c) $\frac{1}{2}(7 \cdot 3^8 - 3)$ (d) $\frac{1}{2}(3^9 - 3 + 3^8)$

39. Four couples (husband and wife) decide to form a committee of four members. The number of different committees that can be formed in which no couple finds a place, is
(a) 10 (b) 12 (c) 14 (d) 16
40. In an examination of 9 papers, a candidate has to pass in more papers than the number of papers in which he fails in order to be successful. The number of ways in which he can be unsuccessful, is
(a) 255 (b) 256 (c) 193 (d) 319
41. The number of 5-digit numbers that can be made using the digits 1 and 2 and in which atleast one digit is different, is
(a) 30 (b) 31
(c) 32 (d) None of these
42. In a club election, the number of contestants is one more than the number of maximum candidates for which a voter can vote. If the total number of ways in which a voter can vote be 62, then the number of candidates is
(a) 7 (b) 5
(c) 6 (d) None of these
43. Two players P_1 and P_2 play a series of ' $2n$ ' games. Each game can result in either a win or loss for P_1 . Total number of ways in which P_1 can win the series of these games, is equal to
(a) $\frac{1}{2} (2^{2n} - {}^{2n}C_n)$ (b) $\frac{1}{2} (2^{2n} - 2 \cdot {}^{2n}C_n)$
(c) $\frac{1}{2} (2^n - {}^{2n}C_n)$ (d) $\frac{1}{2} (2^n - 2 \cdot {}^{2n}C_n)$
44. In the decimal system of numeration, the number of 6-digit numbers in which the sum of digits is divisible by 5, is
(a) 180000 (b) 540000
(c) 5×10^5 (d) None of these
45. The number of possible outcomes in a throw of n ordinary dice in which atleast one of the dice shows an odd number, is
(a) $6^n - 1$ (b) $3^n - 1$
(c) $6^n - 3^n$ (d) None of these
46. The number of different matrices that can be formed with elements 0, 1, 2 or 3, each matrix having 4 elements, is
(a) 3×2^4 (b) 2×4^4
(c) 3×4^4 (d) None of these
47. There are 20 questions in a question paper. If no two students solve the same combination of questions but solve equal number of questions, then the maximum number of students who appeared in the examination, is
(a) ${}^{20}C_9$ (b) ${}^{20}C_{11}$
(c) ${}^{20}C_{10}$ (d) None of these
48. The number of different pairs of words ($\square\square\square\square$, $\square\square\square$) that can be made with the letters of the word 'STATICS' is
(a) 828 (b) 1260
(c) 396 (d) None of these
49. A shopkeeper sells three varieties of perfumes and he has a large number of bottles of the same size of each variety in his stock. There are 5 places in a row in his showcase. The number of different ways of displaying the three varieties of perfumes in the showcase is
(a) 6 (b) 50
(c) 150 (d) None of these
50. The number of ways in which 3 boys and 3 girls (all are of different heights) can be arranged in a line, so that boys as well as girls among themselves are in decreasing order of height from left to right, is
(a) 1 (b) 6!
(c) 20 (d) None of these
51. The number of words of four letters containing equal number of vowels and consonants, repetition allowed, is
(a) 105^2 (b) 210×243
(c) 105×243 (d) None of these
52. There are three teams each of a chairman, a supervisor and a worker three different companies. There are nine bonus of different denominations to be paid to these nine persons in all. In how many ways can this be done with due respect to superiority is given in every team?
(a) 865 (b) 129
(c) 1680 (d) None of these
53. If two numbers are selected from numbers 1 to 25, then the number of ways that their difference does not exceed 10 is
(a) 105 (b) 195
(c) ${}^{15}C_2$ (d) None of these
54. The number of ways in which a mixed double game can be arranged amongst 9 married couples, if no husband and wife play in the same game, is
(a) 756 (b) 1512
(c) 3024 (d) None of these
55. In a certain test, there are n questions. In this test 2^{n-i} students gave wrong answers to atleast i question, where $i = 1, 2, \dots, n$. If the total number of wrong answers given is 2047, then n is equal to
(a) 10 (b) 11 (c) 12 (d) 13
56. The number of natural numbers which are less than $2 \cdot 10^8$ and which can be written by means of the digit 1 and 2, is
(a) 772 (b) 870 (c) 900 (d) 766
57. The number of times digit 3 will be written when listing the integer from 1 to 1000, is
(a) 269 (b) 300 (c) 271 (d) 302

58. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. The number of ways in which we can place the balls in the boxes (order is not considered in the box) so that no box remains empty is
(a) 150 (b) 300
(c) 200 (d) None of these
59. The number of permutations of the letters of the word 'HINDUSTAN' such that neither the pattern 'HIN' nor 'DUS' nor 'TAN' appear, are
(a) 166674 (b) 168474 (c) 166680 (d) 181434
60. Two teams are to play a series of 5 matches between them. A match ends in a win or loss or draw for a team. A number of people forecast the result of each match and no two people make the same forecast for the series of matches. The smallest group of people in which one person forecasts correctly for all the matches will contain n people, where n is
(a) 81 (b) 243
(c) 486 (d) None of these
61. In a class of 20 students, every student had a hand shake with every other student. The total number of hand shakes were
(a) 180 (b) 190 (c) 200 (d) 210
62. There are 10 persons among whom two are brother. The total number of ways in which these persons can be seated around a round table so that exactly one person sit between the brothers, is equal to
(a) $(2!)(7!)$ (b) $(2!)(8!)$
(c) $(3!)(7!)$ (d) $(3!)(8!)$
63. The number of ways in which a couple can sit around a table with 6 guests, if the couple take consecutive seats, is
(a) 1440 (b) 720
(c) 5040 (d) None of these
64. The number of different garlands, that can be formed using 3 flowers of one kind and 3 flowers of other kind, is
(a) 60 (b) 20
(c) 4 (d) 5
65. In the next world cup of cricket, there will be 12 teams, divided equally in two groups. Teams of each group will play a match against each other. From each group, 3 top teams will qualify for the next round. In this round each team will play against others once. Four top teams of this round will qualify for the semifinal round, where each team will play against the other three. Two top teams of this round will go to the final round, where they will play the best of three matches. The minimum number of matches in the next world cup will be
(a) 54 (b) 53
(c) 38 (d) None of these
66. The value of ${}^{10}C_5 / {}^{11}C_6$, when numerator and denominator takes their greatest value, is
(a) $\frac{6}{11}$ (b) $\frac{5}{11}$ (c) $\frac{10}{6}$ (d) $\frac{10}{5}$
67. The number of subsets $\{1, 2, 3, \dots, n\}$ having least element m and greatest element k , $1 \leq m < k \leq n$, is
(a) $2^{n-(k-m)}$ (b) 2^{k-m-2} (c) 2^{k-m-1} (d) 2^{k-m+1}
68. A class has 21 students. The class teacher has been asked to make n groups of r students each and go to zoo taking one group at a time. The size of group zoo i.e. the value of r for which the teacher goes to the maximum number of times is (no group can go to the zoo twice)
(a) 9 or 10 (b) 10 or 11 (c) 11 or 12 (d) 12 or 13
69. A class has n students. We have to form a team of the students including atleast two students and also excluding atleast two students. The number of ways of forming the team is
(a) $2^n - 2n$ (b) $2^n - 2n - 2$
(c) $2^n - 2n - 4$ (d) None of these
70. From 4 gentlemen and 6 ladies, a committee of five is to be selected. The number of ways, in which the committee can be formed so that gentlemen are in majority, is
(a) 66 (b) 156
(c) 60 (d) None of these
71. If the total number of m elements subsets of the set $A = \{a_1, a_2, a_3, \dots, a_n\}$ is k times the number of m elements subsets containing a_4 , then n is
(a) $(m-1)k$ (b) mk
(c) $(m+1)k$ (d) None of these
72. An urn contains 3 red pens, 4 green pens and 6 yellow pens. The number of ways of drawing 4 pens from the urn, if atleast one red pen is to be included in the draw, is (all the pens are different from each other)
(a) 500 (b) 505
(c) 510 (d) None of these
73. Two numbers are chosen from 1, 3, 5, 7, ..., 147, 149 and 151 and multiplied together in all possible ways. The number of ways which will give us the product a multiple of 5, is
(a) 1710 (b) 2900
(c) 1700 (d) None of these
74. Let T_n denotes the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, then n equals
(a) 5 (b) 7 (c) 6 (d) 4
75. If a polygon has 44 diagonals, then the number of its sides are
(a) 11 (b) 7
(c) 8 (d) None of these

76. The sides AB, BC, CA of a $\triangle ABC$ have 3, 4, 5 interior points, respectively on them. Total number of triangles that can be formed using these points as vertices, is equal to
(a) 135 (b) 145 (c) 178 (d) 205
77. $ABCD$ is a convex quadrilateral 3, 4, 5 and 6 points are marked on the sides AB, BC, CD and DA , respectively. The number of triangles with vertices on different sides is
(a) 270 (b) 220 (c) 282 (d) None of these
78. In a polygon, no three diagonals are concurrent. If the total number of points of intersection of diagonals interior to the polygon is 70, then the number of diagonals of the polygon is
(a) 20 (b) 28 (c) 8 (d) None of these
79. n lines are drawn in a plane such that no two of them are parallel and no three of them are concurrent. The number of different points at which these lines will cut, is
(a) $\sum_{k=1}^n k$ (b) $n(n-1)$ (c) n^2 (d) None of these
80. If n objects are arranged in a row, then the number of ways of selecting three objects, so that no two of them are next to each other, is
(a) $\frac{(n-2)(n-3)(n-4)}{3}$ (b) $n^2 C_3$ (c) $n^2 C_3 + n^3 C_2$ (d) None of these
81. There are three coplanar parallel lines. If any p points are taken on each of the lines, the maximum number of triangles with vertices at these points is
(a) $3p^2(p-1) + 1$ (b) $3p^2(p-1)$ (c) $p^2(4p-3)$ (d) None of these
82. In a plane, there are two families of lines $y = x + r$, $y = -x + r$, where $r \in \{0, 1, 2, 3, 4\}$. The number of squares of diagonals of length 2 formed by the lines, is
(a) 9 (b) 16 (c) 25 (d) None of these
83. Line L_1 contains l_1 point and line L_2 contains l_2 point. If the points on L_1 are joined to the points on L_2 , then number of points of intersection of new lines, is
(a) ${}^l C_2 \times {}^l C_2$ (b) $4 \cdot {}^l C_2 \times {}^l C_2$ (c) $2 \cdot {}^l C_2 \times {}^l C_2$ (d) None of the above
84. The number of triangles whose vertices are at the vertices of an octagon but none of whose sides happen to come from the octagon, is
(a) 16 (b) 28 (c) 56 (d) 70
85. The sides AB, BC and CA of a $\triangle ABC$ have a, b and c interior points on them respectively, then the number of triangles that can be constructed using these interior points as vertices, is
(a) $\frac{ab+bc+ca}{2}$ (b) $\Sigma ab(a+b-2)$ (c) $\Sigma ab(a+b)$ (d) None of these
86. The number of points in the cartesian plane with integral coordinates satisfying the inequalities $|x| \leq k, |y| \leq k, |x-y| \leq k$, is
(a) $(k+1)^3 - k^3$ (b) $(k+2)^3 - (k+1)^3$ (c) (k^2+1) (d) None of these
87. Given 5 different green dyes, 4 different blue dyes and 3 different red dyes. The number of combinations of dyes, which can be chosen taking atleast one green and one blue dye, is
(a) 3600 (b) 3720 (c) 3800 (d) None of these
88. Six X 's have to be placed in the squares of the figure given below such that each row contains atleast one X '. The number of ways in which this can be done is
- 
- (a) 26 (b) 27 (c) 22 (d) None of these
89. The total number of six-digit numbers $x_1 x_2 x_3 x_4 x_5 x_6$ having the property $x_1 < x_2 \leq x_3 < x_4 < x_5 \leq x_6$, is
(a) ${}^{10}C_6$ (b) ${}^{12}C_6$ (c) ${}^{11}C_6$ (d) None of these
90. A class contains three girls and four boys. Every Saturday five students go on a picnic, a different group of students is being sent each week. During the picnic, each girl in the group is given doll by the accompanying teacher. All possible groups of five have gone once, the total number of dolls the girls have got, is
(a) 21 (b) 45 (c) 27 (d) 24
91. The total number of ways of selecting two numbers from the set $\{1, 2, 3, 4, \dots, 3n\}$, so that their sum of divisible by 3, is
(a) $\frac{2n^2-n}{2}$ (b) $\frac{3n^2-n}{2}$ (c) $2n^2-n$ (d) $3n^2-n$
92. If $r > p > q$, the number of different selections of $p+q$ things taking r at a time, where p things are identical and q other things are identical, is
(a) $p+q-r$ (b) $p+q-r+1$ (c) $r-p-q+1$ (d) None of these

93. The number of proper divisors of $2^p \cdot 6^q \cdot 15^r$ is

(a) $(p + q + 1)(q + r + 1)(r + 1)$
 (b) $(p + q + 1)(q + r + 1)(r + 1) - 2$
 (c) $(p + q)(q + r)r - 2$
 (d) None of the above

94. The number of even proper divisors of 1008 is

(a) 23 (b) 24
 (c) 22 (d) None of these

95. The number of ways to fill each of the four cells of the table with a distinct natural number such that the sum of the numbers is 10 and the sum of the numbers places diagonally are equal, is

(a) $2! \times 2!$ (b) $4!$
 (c) $2(4!)$ (d) None of these

96. In the figure, two 4-digit numbers are to be formed by filling the places with digits. The number of different ways in which the places can be filled by digits so that the sum of the numbers formed is also a 4-digit number and in no place the addition is with carrying, is

Th	H	T	U

(a) 55^4 (b) $36 \cdot (55)^3$
 (c) 45^4 (d) None of these

97. A bag contains 2 apples, 3 oranges and 4 bananas. The number of ways in which 3 fruits can be selected, if atleast one banana is always in the combination (assume fruit of same species to be alike) is

(a) 6 (b) 10 (c) 29 (d) 7

98. The number of divisors of the number 38808 (excluding 1 and the number itself), is

(a) 70 (b) 72
 (c) 71 (d) None of these

99. The number of integral solutions of $x_1 + x_2 + x_3 = 0$, with $x_i \geq -5$, is

(a) ${}^{15}C_2$ (b) ${}^{16}C_2$ (c) ${}^{17}C_2$ (d) ${}^{18}C_2$

100. If $33!$ is divisible by 2^n , then the maximum value of n is equal to

(a) 30 (b) 31 (c) 32 (d) 33

101. Let p be a prime number such that $p \geq 3$. Let $n = p! + 1$.

The number of primes in the list

$n + 1, n + 2, n + 3, \dots, n + p - 1$ is

(a) $p - 1$ (b) 2
 (c) 1 (d) None of these

102. A person writes letters to 6 friends and addresses the corresponding envelopes. The number of ways in which all 5 letters can be placed in wrong envelopes, is

(a) 264 (b) 210
 (c) 206 (d) None of these

103. The number of ways in which $m + n$ ($n \leq m + 1$) different things can be arranged in a row such that no two of the n things may be together, is

(a) $\frac{(m + n)!}{m! n!}$ (b) $\frac{m! (m + 1)!}{(m + n)!}$
 (c) $\frac{m! n!}{(m - n + 1)!}$ (d) None of these

104. Number of divisors of the form $4n + 2$ ($n \geq 0$) of the integer 240, is

(a) 4 (b) 8
 (c) 10 (d) 3

105. Total number of non-negative integral solutions of $x_1 + x_2 + x_3 = 10$ is equal to

(a) ${}^{12}C_3$ (b) ${}^{10}C_3$
 (c) ${}^{12}C_2$ (d) ${}^{10}C_2$

106. Total number of positive integral solutions of $x_1 + x_3 + 2x_2 = 15$ is equal to

(a) 42 (b) 70
 (c) 32 (d) None of these

107. Total number of ways in which 15 identical blankets can be distributed among 4 persons, so that each of them gets atleast two blankets, is

(a) ${}^{10}C_3$ (b) ${}^9C_3$
 (c) ${}^{11}C_3$ (d) None of these

108. Total number of 3 letter words that can be formed from the letter of the word 'AAAHNPPRRS' is equal to

(a) 210 (b) 237 (c) 247 (d) 227

109. 15 identical balls have to be put in 5 different boxes. Each box can contain any number of balls. Total number of ways of putting the balls into box, so that each box contains atleast 2 balls, is equal to

(a) ${}^9C_5$ (b) ${}^{10}C_5$
 (c) ${}^6C_5$ (d) ${}^{10}C_6$

110. The minimum marks required for clearing a certain screening paper is 210 out of 300. The screening paper consists of 3 sections each of Physics, Chemistry and Mathematics. Each section has 100 as maximum marks. Assuming, there is no negative marking and marks obtained in each section are integers, the number of ways in which a student can qualify the examination, assuming no cut-off limit, is

(a) ${}^{210}C_3 - {}^{90}C_3$
 (b) ${}^{93}C_3$
 (c) ${}^{213}C_3$
 (d) $(210)^3$

111. Number of ways of distributing 10 identical objects among 8 persons (one or many persons may not be getting any object), is
(a) 8^{10} (b) 10^8 (c) ${}^{17}C_7$ (d) ${}^{10}C_8$
112. A bag contains 3 black, 4 white and 2 red balls, all the balls being different. The number of selections of atmost 6 balls containing balls of all the colours, is
(a) $42(4!)$
(b) $2^6 \times 4!$
(c) $(2^6 - 1)(4!)$
(d) None of the above
113. The number of ways to give 16 different things to three persons, so that B gets 1 more than A and C gets 2 more than B, is
(a) $\frac{16!}{4!5!7!}$
(b) $4!5!7!$
(c) $\frac{16!}{3!5!8!}$
(d) None of the above
114. The number of positive integral solutions of $x + y + z = n, n \in N, n \geq 3$, is
(a) ${}^{n-1}C_2$
(b) ${}^{n-1}P_2$
(c) $n(n-1)$
(d) None of the above
115. The number of non-negative integral solutions of $a + b + c + d = n, n \in N$, is
(a) ${}^{n-1}P_2$
(b) $\frac{(n+1)(n+2)(n+3)}{6}$
(c) ${}^{n-1}C_{n-4}$
(d) None of these
116. The number of points (x, y, z) in space, whose each coordinate is a negative integer such that $x + y + z + 12 = 0$, is
(a) 385 (b) 55
(c) 110 (d) None of these
117. If a, b, c are three natural numbers in AP and $a + b + c = 21$, then the possible number of values of the ordered triplet (a, b, c) is
(a) 15
(b) 14
(c) 13
(d) None of the above
118. If a, b, c, d are odd natural numbers such that $a + b + c + d = 20$, then the number of values of the ordered 4-tuple (a, b, c, d) is
(a) 165
(b) 455
(c) 310
(d) None of the above

Type 2. More than One Correct Option

119. Number of ways in which three numbers in AP can be selected from $1, 2, 3, \dots, n$, is
(a) $\left(\frac{n-1}{2}\right)^2$, if n is even
(b) $\frac{n(n-2)}{4}$, if n is even
(c) $\frac{(n-1)^2}{4}$, if n is odd
(d) None of the above
120. Kanchan has 10 friends among whom two are married to each other. She wishes to invite five of them for a party. If the married couples refuse to attend separately, then the number of different ways in which she can invite five friends, is
(a) ${}^8C_5$ (b) $2 \times {}^8C_3$
(c) ${}^{10}C_5 - 2 \times {}^8C_4$ (d) None of these
121. A forecast is to be made of the results of five cricket matches, each of which can be a win or a draw or a loss for Indian team. Let
 p = number of forecasts with exactly 1 error
 q = number of forecasts with exactly 3 errors and
 r = number of forecasts with all five errors
Then, the correct statement(s) is/are
(a) $2q = 5r$ (b) $8p = q$
(c) $8p = 5r$ (d) $2(p+r) > q$
122. If x is the number of 5-digit numbers sum of whose digits is even and y is the number of 5-digit numbers sum of whose digits is odd, then
(a) $x = y$
(b) $x + y = 90000$
(c) $x = 45000$
(d) $x < y$
123. If ${}^nC_\alpha = {}^nC_\beta$, then it may be true that
(a) $\alpha = \beta$
(b) $\alpha + \beta = 2n$
(c) $\alpha + \beta = n$
(d) All of the above
124. There are n married couples at a party. Each person shakes hand with every person other than her or his spouse. The total number of hand shakes must be
(a) ${}^{2n}C_2 - n$
(b) ${}^{2n}C_2 - (n-1)$
(c) $2n(n-1)$
(d) ${}^{2n}C_2$
125. There are n lines in a plane, no two of which are parallel and no three of which are concurrent. If plane is divided in u_n parts, then
(a) $u_4 = 11$
(b) $u_3 = 7$
(c) $u_2 = 3$
(d) $u_n = u_{n-1} + 4$

Type 3. Assertion and Reason

Directions (Q. Nos. 126-133) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

126. Statement I The number of positive integral solutions of $abc = 30$ is 27.

Statement II Number of ways in which three prizes can be distributed among three persons is 3^3 .

127. Statement I Number of ways in which 10 identical toys can be distributed among three students, if each receives at least two toys, is ${}^9C_2$.

Statement II Number of positive integral solutions of $x + y + z + w = 7$ is ${}^6C_3$.

128. Statement I $(n^2)!/(n!)^n$ is a natural number, $\forall n \in N$.

Statement II Number of ways in which n^2 objects can be distributed among n persons equally is $(n^2)!/(n!)^n$.

129. Statement I The number of ways of writing 1400 as a product of two positive integers is 12.

Statement II 1400 is divisible by exactly three prime factors.

130. Statement I Let $E = \{1, 2, 3, 4\}$ and $F = \{a, b\}$. Then, the number of onto functions from E to F is 14.

Statement II Number of ways in which four distinct objects can be distributed into two different boxes, if no box remains empty is 14.

131. Statement I Number of ways in which India can win the series of 11 matches, if no match is drawn is 2^{10} .

Statement II For each match, there are two possibilities, either India wins or loses.

132. Statement I Number of ways in which Indian team (11 players) can bat, if Yuvraj wants to bat before Dhoni and Pathan wants to bat after Dhoni is $11!/3!$.

Statement II Yuvraj, Dhoni and Pathan can be arranged in batting order in $3!$ ways.

133. Statement I When number of ways of arranging 21 objects of which r objects are identical of one type and remaining are identical of second type is maximum, then maximum value of ${}^{13}C_r$ is 78.

Statement II ${}^{2n+1}C_r$ is maximum, when $r = n$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 134-136) We have to choose 11 players for cricket team from eight batsmen, six bowlers, four all rounders and two wicket keepers in the following questions.

134. The number of selections, when atmost one all rounder and one wicket keeper will play, is

- (a) ${}^4C_1 \times {}^{14}C_{10} + {}^2C_1 \times {}^{14}C_{10} + {}^4C_1 \times {}^2C_1 \times {}^{14}C_9 + {}^{14}C_{11}$
(b) ${}^4C_1 \times {}^{15}C_{11} + {}^{15}C_{11}$
(c) ${}^4C_1 \times {}^{15}C_{10} + {}^{15}C_{11}$
(d) None of the above

135. Number of selections when two particular batsmen do not want to play when a particular bowler will play, is

- (a) ${}^{17}C_{10} + {}^{19}C_{11}$ (b) ${}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$
(c) ${}^{17}C_{10} + {}^{20}C_{11}$ (d) ${}^{19}C_{10} + {}^{19}C_{11}$

136. Number of selections when a particular batsman and a particular wicket keeper do not want to play together, is

- (a) $2^{18}C_{10}$ (b) ${}^{19}C_{11} + {}^{18}C_{10}$
(c) ${}^{19}C_{10} + {}^{19}C_{11}$ (d) None of these

Passage II (Q. Nos. 137-139) 12 persons are to be arranged around two round tables such that one table can accommodate seven persons and another five persons only.

137. Number of ways in which these 12 persons can be arranged is

- (a) ${}^{12}C_5 6! 2!$ (b) $6! 4!$
(c) ${}^{12}C_5 6! 4!$ (d) None of these

138. Number of ways of arrangements, if two particular persons A and B do not want to be on the same table, is

- (a) ${}^{10}C_4 6! 4!$ (b) $2^{10}C_6 6! 4!$
(c) ${}^{11}C_6 6! 4!$ (d) None of these

139. Number of ways of arrangement, if two particular persons A and B want to be together and consecutive, is

- (a) ${}^{10}C_7 6! 3! 2! + {}^{10}C_5 4! 5! 2!$
(b) ${}^{10}C_5 6! 3! + {}^{10}C_7 4! 5!$
(c) ${}^{10}C_7 6! 2! + {}^{10}C_5 5! 2!$
(d) None of the above

Passage III (Q. Nos. 140-142) Number of ways of distributing n different things into r different groups is r^n when blank groups are taken into account and is $r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \dots + (-1)^{r-1} {}^rC_{r-1}$ when blank groups are not permitted.

- 140.** 4 candidates are competing for two managerial posts. In how many ways can the candidates be selected?
 (a) 4^2 (b) $4C_2$
 (c) 2^4 (d) None of these

- 141.** 8 different balls can be distributed among 3 children so that every child receives atleast 1 ball, is
 (a) 3^8 (b) $8C_3$
 (c) 8^3 (d) None of these
- 142.** 5 letters can be posted into 3 letter boxes in
 (a) 3^5 ways
 (b) 5^3 ways
 (c) $5C_3$ ways
 (d) None of the above

Type 5. Match the Columns

- 143.** Match the following.

Column I	Column II
A. Number of straight lines joining any two of 20 points of which four points are collinear, is	p. 56
B. Maximum number of points of intersection of 20 straight lines in the plane, is	q. 60
C. Maximum number of points of intersection of 8 circles in the plane, is	r. 185
D. Maximum number of points of intersection of six parabolas, is	s. 190

- 144.** A function is defined as $f: \{a_1, a_2, a_3, a_4, a_5, a_6\} \rightarrow \{b_1, b_2, b_3\}$. Then, match the following.

Column I	Column II
A. Number of surjective functions is	p. divisible by 6
B. Number of functions in which $f(a_i) \neq b_i$, is	q. divisible by 2
C. Number of invertible functions is	r. divisible by 4
D. Number of many-one functions is	s. divisible by 3
	t. not possible

- 145.** Match the statements of Column I with values of Column II.

Column I	Column II
A. The value of $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n!$ is	p. $(n+2)2^{n-1} - (n+1)$
B. The value of $n \cdot {}^nC_1 + (n-1) \cdot {}^nC_2 + (n-1) \cdot {}^nC_3 + \dots + 1 \cdot {}^nC_n$ is	q. $2^n C_n$
C. The value of ${}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^nC_2$ is	r. $(n+1)! - 1$
D. The value of $\sum_{r=0}^n ({}^nC_r)^2$ is	s. ${}^{n+1}C_3$
	t. ${}^{2n}C_{n-1} - 1$

Type 6. Single Integer Answer Type Questions

- 146.** A class has three teachers, Mr. P, Ms. Q and Mrs. R and six students A, B, C, D, E, F. Number of ways in which they can be seated in a line of 9 chairs, if between any two teachers, there are exactly two students, is $k!(18)$, then the value of k is _____.
- 147.** Consider, the five points comprising the vertices of a square and the intersection point of its diagonals. How many triangles can be formed using these points?
- 148.** If number of selections of 6 different letters that can be made from the words 'SUMAN' and 'DIVYA', so that each selection contains 3 letters from each word is N^2 , then the value of N is _____.
- 149.** There are n distinct white and n distinct black balls. If the number of ways of arranging them in a row, so that neighbouring balls are of different colours, is 1152, then the value of n is _____.
- 150.** There are 4 oranges, 5 apples and 6 mangoes in a fruit basket. If $(21k + 20)$ ways can a person make a selection of fruits from among the fruits in the basket, when all fruits of the same type, are identical, then k refers to _____.
- 151.** The number of ways of filling three boxes (named A, B and C) by 12 or less number of identical balls, if no box is empty, box B has atleast 3 balls and box C has atmost 5 balls, is 55λ , where λ , refers to _____.
- 152.** The number of non-negative integral solutions $2x + y + z = 21$ is $22(k)$, where k refers to _____.

Entrances Gallery

JEE Advanced/IIT JEE

- Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then, the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is _____. [2014]
- Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of n is _____. [2014]
- Six cards and six envelopes are numbered 1, 2, 3, 4, 5, 6 and cards are to be placed in envelopes so that each envelope contains exactly one card and no card is placed in the envelope bearing the same number and moreover the card numbered 1 is always placed in envelope numbered 2. Then, the number of ways it can be done is [2014]
(a) 264 (b) 265 (c) 53 (d) 67
- Consider the set of eight vectors $V = \{a\hat{i} + b\hat{j} + c\hat{k} ; a, b, c \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then, p is _____. [2013]

JEE Main/AIEEE

- The number of integers greater than 6000 that can be formed, using the digits 3, 5, 6, 7 and 8 without repetition, is [2015]
(a) 216 (b) 192
(c) 120 (d) 72
- Let A and B be two sets containing 2 elements and 4 elements, respectively. The number of subsets of $A \times B$ having 3 or more elements, is [2013]
(a) 256 (b) 220
(c) 219 (d) 211
- The total number of ways in which 5 balls of different colours can be distributed among 3 persons so that each person gets atleast one ball, is [2012]
(a) 75 (b) 150 (c) 210 (d) 243

Directions (Q. Nos. 8-9) Let a_n denotes the number of all n -digit positive integers formed by the digits 0, 1 or both, such that no consecutive digits in them are 0. Let b_n be the number of such n digit integers ending with digit 1 and c_n be the number of such n digit integers ending with digit 0. [2012]

- Which of the following is correct?
(a) $a_{17} = a_{16} + a_{15}$
(b) $c_{17} \neq c_{16} + c_{15}$
(c) $b_{17} \neq b_{16} + c_{16}$
(d) $a_{17} = c_{17} + b_{16}$
- The value of b_6 is
(a) 7 (b) 8 (c) 9 (d) 11
- Statement I** The number of ways distributing 10 identical balls in 4 distinct boxes such that no box is empty is ${}^9C_3$.
Statement II The number of ways of choosing any 3 places from 9 different places is ${}^9C_3$. [2011]
- (a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true, Statement II is false
(c) Statement I is false, Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- There are 10 points in a plane, out of these 6 are collinear. If N is the number of triangles formed by joining these points, then [2011]
(a) $N > 190$ (b) $N \leq 100$
(c) $100 < N \leq 140$ (d) $140 < N \leq 190$
- There are two urns. Urn A has 3 distinct red balls and urn B has 9 distinct blue balls. From each urn, two balls are taken out at random and then transferred to the other. The number of ways in which this can be done is [2010]
(a) 3 (b) 36
(c) 66 (d) 108
- From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on the shelf so that the dictionary is always in the middle. Then, the number of such arrangements is [2009]
(a) atleast 500 but less than 750
(b) atleast 750 but less than 1000
(c) atleast 1000
(d) less than 500
- In a shop, there are five types of ice-creams available. A child buys six ice-creams.
Statement I The number of different ways the child can buy the six ice-creams is ${}^{10}C_5$.
Statement II The number of different ways the child can buy the six ice-creams is equal to the number of different ways of arranging 6A's and 4B's in a row. [2008]

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true
- 15.** How many different words can be formed by jumbling the letters in the word 'MISSISSIPPI' in which no two S are adjacent? [2008]
(a) $8 \cdot {}^6C_4 \cdot {}^7C_4$
(b) $6 \cdot 7 \cdot {}^8C_4$
(c) $6 \cdot 8 \cdot {}^7C_4$
(d) $7 \cdot {}^6C_4 \cdot {}^8C_4$
- 16.** The set $S : \{1, 2, 3, \dots, 12\}$ is to be partitioned into three sets A, B, C of equal size.
Thus, $A \cup B \cup C = S$, $A \cap B = B \cap C = A \cap C = \phi$.
The number of ways to partition S is [2007]
(a) $12! / 3! (4!)^3$ (b) $12! / 3! (3!)^4$
(c) $12! / (4!)^3$ (d) $12! / (3!)^4$
- 17.** At an election, a voter may vote for any number of candidates not greater than the number to be elected. There are 10 candidates and 4 are to be elected. If a voter votes for atleast one candidate, then the number of ways in which he can vote, is [2006]
(a) 6210 (b) 385 (c) 1110 (d) 5040
- 18.** If the letters of the word 'SACHIN' are arranged in all possible ways and these words are written out as in dictionary, then the word 'SACHIN' appears at serial number is [2005]
(a) 602 (b) 603 (c) 600 (d) 601
- 19.** The range of the function $f(x) = {}^{7-x}P_{x-3}$ is [2004]
(a) $\{1, 2, 3\}$ (b) $\{1, 2, 3, 4, 5, 6\}$
(c) $\{1, 2, 3, 4\}$ (d) $\{1, 2, 3, 4, 5\}$
- 20.** How many ways are there to arrange the letters in the word 'GARDEN' with the vowels in alphabetical order? [2004]
(a) 120 (b) 240 (c) 360 (d) 480
- 21.** The number of ways of distributing 8 identical balls in 3 distinct boxes, so that none of the boxes is empty, is [2004]
(a) 5 (b) 21 (c) 3^8 (d) ${}^8C_3$
- 22.** The number of ways in which 6 men and 5 women can dine at a round table, if no two women are to sit together, is given by [2003]
(a) $6! \times 5!$ (b) 30 (c) $5! \times 4!$ (d) $7! \times 5!$
- 23.** If ${}^nC_r$ denotes the number of combinations of n things taken r at a time, then the expression ${}^nC_{r+1} + {}^nC_{r-1} + 2 \times {}^nC_r$ equals [2003]
(a) ${}^{n+2}C_r$ (b) ${}^{n+2}C_{r+1}$ (c) ${}^{n+1}C_r$ (d) ${}^{n+1}C_{r+1}$

Other Engineering Entrances

- 24.** How many 5-digit telephone numbers can be constructed using the digits 0 to 9, if each number starts with 67 and no digit appears more than once? [Karnataka CET 2014]
(a) 335 (b) 336 (c) 338 (d) 337
- 25.** A student is allowed to select atmost n books from a collection of $(2n + 1)$ books. If the number of ways in which he can do this, is 64, then the value of n is [BITSAT 2014]
(a) 6 (b) n
(c) 3 (d) None of these
- 26.** $S = \{1, 2, 3, \dots, 20\}$ is to be partitioned into four sets A, B, C and D of equal size. The number of ways it can be done, is [Manipal 2014]
(a) $\frac{20!}{4! \times 5!}$ (b) $\frac{20!}{4^5}$
(c) $\frac{20!}{(5!)^4}$ (d) $\frac{20!}{(4!)^5}$
- 27.** In how many ways can a student choose a program of 5 courses, if 9 courses are available and 2 specific courses are compulsory for every student? [Manipal 2014]
(a) 34 (b) 36
(c) 35 (d) 37
- 28.** 10 different toys are to be distributed among 10 children. Total number of ways of distributing these toys, so that exactly two children do not get any toy, is [Manipal 2014]
(a) $\frac{10!}{2! 3! 7!}$ (b) $\frac{10!}{(2!)^4 \times 6!}$
(c) $(10!)^2 \left[\frac{1}{(2!)^4 \times 6!} + \frac{1}{2! \times 3!} \right]$ (d) $\frac{10! \times 10!}{(2!)^2 \times 6!} \left[\frac{25}{84} \right]$
- 29.** Out of 7 consonants and 4 vowels, the number of words (not necessarily meaningful) that can be made, each consisting of 3 consonants and 2 vowels, is [WB JEE 2014]
(a) 24800 (b) 25100 (c) 25200 (d) 25400
- 30.** ${}^mC_{r+1} + \sum_{k=m}^n {}^kC_r$ is equal to [AMU 2014]
(a) ${}^nC_{r+1}$ (b) ${}^{n+1}C_{r+1}$
(c) ${}^nC_r$ (d) None of these
- 31.** $\frac{C_1}{C_0} + 2 \frac{C_2}{C_1} + 3 \frac{C_3}{C_2} + \dots + n \frac{C_n}{C_{n-1}}$ is equal to [AMU 2014]
(a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+1)}{2}$
(c) $\frac{(n+1)(n+2)}{2}$ (d) None of these

32. If n is an integer with $0 \leq n \leq 11$, then the minimum value of $n!(11-n)!$ is attained, when a value of n is [EAMCET 2014]
 (a) 11 (b) 5
 (c) 7 (d) 9
33. Find ${}^nC_{21}$, if ${}^nC_{10} = {}^nC_{11}$. [J&K CET 2014]
 (a) 1 (b) 0
 (c) 11 (d) 10
34. Determine n , if ${}^{2n}C_2 : {}^nC_2 = 9 : 2$. [J&K CET 2014]
 (a) 5 (b) 4
 (c) 3 (d) 2
35. Out of thirty points in a plane, eight of them are collinear. The number of straight lines that can be formed by joining these points, is [EAMCET 2014]
 (a) 296 (b) 540
 (c) 408 (d) 348
36. The number of positive integers which can be formed by using any number of digits from 0, 1, 2, 3, 4, 5 but using each digit not more than once in each number is [Karnataka CET 2013]
 (a) 1200 (b) 1500
 (c) 1600 (d) 1630
37. The total number of ways in which five '+' and three '-' signs can be arranged in a line such that no two '-' sign occur together, is [AMU 2013]
 (a) 10
 (b) 20
 (c) 15
 (d) None of the above
38. A student is to answer 10 out of 13 questions in an examination such that he must choose atleast 4 from the first five questions. The number of choices available to him is [Manipal 2013]
 (a) 140 (b) 196
 (c) 280 (d) 346
39. A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways can 3 balls be drawn from the box, if atleast one black ball is to be included in the draw? [AMU 2013]
 (a) 64 (b) 24 (c) 3 (d) 12
40. Four speakers will address a meeting, where speaker Q will always speak after speaker P . Then, the number of ways in which the order of speakers can be prepared is [WB JEE 2012]
 (a) 256
 (b) 128
 (c) 24
 (d) 12
41. Sum of digits in the unit's place formed by the digits 1, 2, 3 and 4 taken all at a time is [OJEE 2012]
 (a) 30
 (b) 60
 (c) 59
 (d) 61
42. A teacher takes 3 children from her class to the zoo at a time as often as she can, but she does not take the same three children to the zoo more than once. She finds that she goes to the zoo 84 times more than a particular child goes to the zoo. The number of children in her class is [GGSIPU 2011]
 (a) 12
 (b) 10
 (c) 60
 (d) None of the above
43. A student is allowed to select atmost n books from a collection of $(2n+1)$ books. If the total number of ways in which he can select atleast one book is 225, then the value of n is [J&K CET 2011]
 (a) 6 (b) 5
 (c) 4 (d) 3
44. In how many ways can 5 prizes be distributed among four students, when every student can take one or more prizes? [BITSAT 2011]
 (a) 1024 (b) 625
 (c) 120 (d) 600
45. The number of ways in which the digits 1, 2, 3, 4, 3, 2, 1 can be arranged so that the odd digits always occupy the odd places, is [MP PET 2011]
 (a) 6 (b) 12
 (c) 18 (d) 24
46. The number of integers greater than 6000 that can be formed with 3, 5, 6, 7 and 8, where no digit is repeated, is [Kerala CEE 2011]
 (a) 120 (b) 192
 (c) 216 (d) 72
 (e) 202
47. If ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30800 : 1$, then the value of r is [Kerala CEE 2010]
 (a) 40 (b) 51
 (c) 41 (d) 510
 (e) 101
48. The number of permutations by taking all letters and keeping the vowels of the word 'COMBINE' in the odd places, is [WB JEE 2010]
 (a) 96
 (b) 144
 (c) 512
 (d) 576
49. From 12 books, the difference between number of ways of selection of 5 books when one specified book is always excluded and one specified book is always included, is [Kerala CEE 2010]
 (a) 64 (b) 118
 (c) 132 (d) 330
 (e) 462
50. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$, then n is just greater than integer [WB JEE 2010]
 (a) 5 (b) 6
 (c) 4 (d) 7

Answers

Work Book Exercise 6.1

1. (d) 2. (b) 3. (d) 4. (b) 5. (a) 6. (d) 7. (a) 8. (d) 9. (a) 10. (c)

Work Book Exercise 6.2

1. (c) 2. (c) 3. (a) 4. (a) 5. (c)

Work Book Exercise 6.3

1. (d) 2. (b) 3. (b) 4. (a) 5. (d) 6. (a) 7. (b)

Work Book Exercise 6.4

1. (c) 2. (c) 3. (b) 4. (a) 5. (d) 6. (d) 7. (c) 8. (c) 9. (b) 10. (a)
11. (d) 12. (a) 13. (a) 14. (a) 15. (d) 16. (b) 17. (a) 18. (d) 19. (b) 20. (b)
21. (b) 22. (d) 23. (d) 24. (a) 25. (b) 26. (c) 27. (d)

Work Book Exercise 6.5

1. (c) 2. (d) 3. (a) 4. (a) 5. (c) 6. (d)

Work Book Exercise 6.6

1. (c) 2. (a) 3. (b) 4. (a) 5. (c) 6. (d) 7. (d) 8. (b) 9. (c) 10. (b)
11. (c) 12. (c) 13. (b) 14. (d) 15. (c) 16. (b) 17. (b) 18. (a) 19. (a) 20. (b)
21. (c) 22. (b) 23. (b) 24. (c) 25. (c)

Target Exercises

1. (d) 2. (a) 3. (b) 4. (b) 5. (a) 6. (b) 7. (b) 8. (c) 9. (a) 10. (a)
11. (b) 12. (a) 13. (b) 14. (d) 15. (d) 16. (a) 17. (d) 18. (b) 19. (a) 20. (c)
21. (d) 22. (c) 23. (d) 24. (a) 25. (c) 26. (c) 27. (b) 28. (c) 29. (b) 30. (d)
31. (a) 32. (b) 33. (b) 34. (b) 35. (d) 36. (a) 37. (c) 38. (c) 39. (d) 40. (b)
41. (a) 42. (c) 43. (a) 44. (a) 45. (c) 46. (c) 47. (c) 48. (b) 49. (c) 50. (c)
51. (d) 52. (d) 53. (d) 54. (b) 55. (b) 56. (d) 57. (b) 58. (a) 59. (b) 60. (b)
61. (b) 62. (b) 63. (a) 64. (d) 65. (b) 66. (a) 67. (d) 68. (b) 69. (b) 70. (a)
71. (b) 72. (b) 73. (d) 74. (b) 75. (a) 76. (d) 77. (d) 78. (a) 79. (d) 80. (b)
81. (c) 82. (a) 83. (c) 84. (a) 85. (d) 86. (a) 87. (b) 88. (a) 89. (c) 90. (b)
91. (b) 92. (b) 93. (b) 94. (a) 95. (a) 96. (b) 97. (a) 98. (a) 99. (c) 100. (b)
101. (d) 102. (a) 103. (d) 104. (a) 105. (c) 106. (a) 107. (a) 108. (c) 109. (a) 110. (b)
111. (c) 112. (a) 113. (a) 114. (a) 115. (b) 116. (b) 117. (c) 118. (a) 119. (b,c) 120. (b,c)
121. (a,b,d) 122. (a,b,c) 123. (a,c) 124. (a,c) 125. (a,b) 126. (a) 127. (d) 128. (a) 129. (b) 130. (a)
131. (b) 132. (a) 133. (d) 134. (a) 135. (b) 136. (b) 137. (c) 138. (b) 139. (a) 140. (b)
141. (d) 142. (a) 143. (*) 144. (**) 145. (***) 146. (6) 147. (8) 148. (8) 149. (4) 150. (9)
151. (2) 152. (6)

* $A \rightarrow r$; $B \rightarrow s$; $C \rightarrow p$; $D \rightarrow q$

** $A \rightarrow p,q,r,s$; $B \rightarrow p,q,r,s$; $C \rightarrow p,q,r,s,t$; $D \rightarrow s$

*** $A \rightarrow r$; $B \rightarrow p$; $C \rightarrow s$; $D \rightarrow q$

Entrances Gallery

1. (7) 2. (5) 3. (c) 4. (5) 5. (b) 6. (c) 7. (b) 8. (a) 9. (b) 10. (a)
11. (b) 12. (d) 13. (c) 14. (d) 15. (d) 16. (c) 17. (b) 18. (d) 19. (a) 20. (c)
21. (b) 22. (a) 23. (b) 24. (b) 25. (c) 26. (c) 27. (c) 28. (d) 29. (c) 30. (b)
31. (b) 32. (b) 33. (a) 34. (a) 35. (c) 36. (d) 37. (b) 38. (b) 39. (a) 40. (d)
41. (b) 42. (d) 43. (c) 44. (a) 45. (c) 46. (b) 47. (c) 48. (d) 49. (c) 50. (d)

Explanations

Target Exercises

- We have, $\alpha = {}^m C_2 \Rightarrow \alpha = \frac{m(m-1)}{2}$

$$\therefore {}^\alpha C_2 = \frac{\alpha(\alpha-1)}{2} = \frac{1}{2} \cdot \frac{m(m-1)}{2} \left\{ \frac{m(m-1)}{2} - 1 \right\}$$

$$= \frac{1}{8} m(m-1)(m-2)(m+1)$$

$$= \frac{1}{8} (m+1)m(m-1)(m-2) = 3 {}^{m+1} C_4$$
- Given, ${}^n C_3 + {}^n C_4 > {}^{n+1} C_3$

$$\Rightarrow {}^{n+1} C_4 > {}^{n+1} C_3 \quad [\because {}^n C_r + {}^n C_{r+1} = {}^{n+1} C_{r+1}]$$

$$\Rightarrow \frac{{}^{n+1} C_4}{{}^{n+1} C_3} > 1 \Rightarrow \frac{n-2}{4} > 1 \Rightarrow n > 6$$
- We have, $\sum_{r=0}^{n-1} \frac{{}^n C_r}{{}^n C_r + {}^n C_{r+1}} = \sum_{r=0}^{n-1} \frac{1}{1 + \frac{{}^n C_{r+1}}{{}^n C_r}}$

$$= \sum_{r=0}^{n-1} \frac{1}{1 + \frac{n-r}{r+1}}$$

$$= \sum_{r=0}^{n-1} \frac{r+1}{n+1} = \frac{1}{n+1} \sum_{r=0}^{n-1} (r+1)$$

$$= \frac{1}{(n+1)} [1+2+\dots+n] = \frac{n}{2}$$
- We know that, $\frac{{}^n P_r}{r!} = {}^n C_r$

$$\therefore \sum_{r=1}^n {}^n C_r = 2^n - 1 \quad [1 \text{ corresponding to } {}^n C_0]$$
- Given, $\frac{(2n+1)!(n-1)!}{(n+2)!(2n-1)!} = \frac{3}{5}$

$$\frac{(2n+1)(2n)}{(n+2)(n+1)n} = \frac{3}{5}$$

$$\Rightarrow 10(2n+1) = 3(n^2 + 3n + 2)$$

$$\Rightarrow 3n^2 - 11n - 4 = 0$$

$$\Rightarrow (n-4)(3n+1) = 0$$

$$\therefore n = 4$$
- The number of one-digit numbers = 0
 The number of two-digit numbers = $5 \times 5 = 25$.
 The number of three-digit numbers = $5 \times 5 \times 4 = 100$.
 Hence, the total numbers are 131.
- Numbers greater than 1000 and less than or equal to 4000 will be of 4-digits and will have either 1 or 2 or 3 in the 1st place. After fixing 1 at first place, then 2nd place can be filled by any of the 5-digits. Similarly, the 3rd place can be filled up in 5 ways and 4th place can be filled up in 5 ways. Thus, there will be $5 \times 5 \times 5 = 125$ ways in which 1 will be in first place but this also includes 1000. Hence, there will be 124 numbers having 1 in the first place. Similarly, 125 for each 2 or 3. One number will be there in which 4 will be in the first place i.e. 4000. Hence, the required number of numbers

$$= 124 + 125 + 125 + 1$$

$$= 375$$

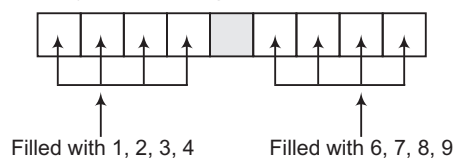
- Total number of variables, if only alphabet is used is 26.
 Total number of variables, if alphabets and digits both are used is 26×10 . Hence, the total number of variables is $26(1+10) = 286$.
- If 7 is used at first place, the number of numbers is 9^4 and for any other four places, it is 8×9^3 .
 Now, required number of numbers = $9^4 + 4 \times 8 \times 9^3$

$$= 9^3(9 + 32) = 41(9^3)$$
- If all strips are of different colours, then number of flags is $3! = 6$. When two strips are of same colour, then number of flags is ${}^3 C_1 \times (3!/2) \times {}^2 C_1 = 18$.
 $\therefore$ Total number of flags = $6 + 18 = 24 = 4!$
- Total number of triplets without restriction is $n \times n \times n$. The number of triplets with all different coordinates is ${}^n P_3$.
 Therefore, the required number of triplets is $n^3 - {}^n P_3$.
- $x_1 < x_2 < x_3 < x_4 < x_5 < x_6$, when the number is $x_1 x_2 x_3 x_4 x_5 x_6$. Clearly, no digit can be zero. Also, all the digits are distinct. So, let us first select six digits from the list of digits 1, 2, 3, 4, 5, 6, 7, 8, 9 which can be done in ${}^9 C_6$ ways.
 After selecting these digits, they can be put only in one order.
 Thus, total number of such numbers = ${}^9 C_6 \times 1 = {}^9 C_3$
- The number of numbers when repetition is allowed is 5^4 . The number of numbers when digits cannot be repeated is ${}^5 P_4$.
 Therefore, the required number of numbers is $5^4 - 5! = 505$.

Middle digit	Digits available for remaining four places	Pattern	Number of ways of filling remaining four places
4	0, 1, 2, 3		$3 \times {}^3 P_3$
5	0, 1, ..., 4		$4 \times {}^4 P_3$
6	0, 1, ..., 5	...	$5 \times {}^5 P_3$
7	0, 1, ..., 6	...	$6 \times {}^6 P_3$
8	0, 1, ..., 7	...	$7 \times {}^7 P_3$
9	0, 1, ..., 8	...	$8 \times {}^8 P_3$

Thus, number of required numbers = 5292

- According to the given conditions, numbers can be formed by the following format:



$$\therefore \text{Number of required numbers} = {}^4 P_4 \times {}^4 P_4$$

16. Total number of digits (sum of digits is even) = 450

17. (i) Required number of permutations = Number of permutations of n objects (taken r at a time + taken $(r+1)$ at a time + taken $(r+2)$ at a time + ... + taken n at a time]

$$= n^r + n^{r+1} + n^{r+2} + \dots + n^n$$

$$= \frac{n^r (n^{n-r+1} - 1)}{n - 1}$$

(ii) Required number of permutations = Number of permutations of n objects (taken 0 at a time + taken 1 at a time + ... + taken r at a time)

$$= n^0 + n^1 + n^2 + \dots + n^r = \frac{1 \times (n^{r+1} - 1)}{n - 1}$$

18. Total number of required numbers

$$= 2 \times 2 \times \dots 10 \text{ times} = 2^{10}$$

19. Number of numbers using 1, 2, 3, 4, 5 = $5! = 120$
 Number of numbers using 0, 1, 2, 4, 5

$$= 5! - 4! = 120 - 24 = 96$$
 [∵ there are $4!$ numbers beginning with '0']
 ∴ Total number of required numbers = $120 + 96 = 216$

20. Number of ways of filling first place = 9
 Number of ways of filling remaining 8 places

$$= 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 = 9!$$

 ∴ Total number of numbers = $9 \times 9!$

21. Number of ways for women = ${}^4P_2$
 Number of ways for men = ${}^6P_3$
 ∴ Total number of ways = ${}^4P_2 \times {}^6P_3$

22. Number of committees = ${}^4C_1 \times {}^8C_5 = 4 \times 56 = 224$

23. As the order of A, B and C is not to change, therefore number of ways of assigning three periods to them is ${}^7C_3$. Now, we left with 4 periods for 4 lecturers that can assign in $4!$ ways.
 ∴ Required number of ways = ${}^7C_3 \times 4!$

$$= \frac{7!}{3! \times 4!} \times 4! = \frac{7!}{3!} = 840$$

24. Total number of arrangements of the letters of the given word

$$= \frac{6!}{3!2!} = 60$$

 Number of arrangements in which two N's are together

$$= \frac{5!}{3!} = 20$$

 ∴ Number of arrangements in which two N's are not together

$$= 60 - 20 = 40$$

25. There are 5 odd places and that can be filled by using 2, 2, 8, 8, 8 in $\frac{5!}{2!3!} = 10$ ways and remaining 4 even places can be filled by using 3, 3, 5, 5 in $\frac{4!}{2!2!} = 6$ ways
 ∴ Total number of numbers = $10 \times 6 = 60$

26. $A = \{2, 3, 5, 7, 11, 13, 17, 19, 23, 29\}$. A rational number is made by taking any two in any order.
 So, the required number of rational numbers

$$= {}^{10}P_2 + 1 = 9$$
 [including 1]

27. Each element of the set A (domain) can be give the image in the set A (codomain) in k ways. So, the required number of functions.
 i.e. $n(S) = k \times k \times \dots (k \text{ times}) = k^k$

28. Any selection of four digits from the ten digits 0, 1, 2, 3, ..., 9 gives one such number. So, the required number of numbers = ${}^{10}C_4 = 210$

29. Number of distinct n digit numbers using 2, 5, 7 = 3^n .
 Now consider, $3^n > 900 \Rightarrow n = 7$ [for minimum]

30. Number of numbers with repetition

$$= 10 \times 10 \times 10 \times 10 \times 10 = 100000$$

 Number of numbers without repetition

$$= 10 \times 9 \times 8 \times 7 \times 6 = 30240$$

 ∴ Number of numbers with atleast one digit repeated

$$= 100000 - 30240 = 69760$$

31. Total words starting with B = $4! = 24$
 Total words starting with E = $4! = 24$
 Total words starting with KB = $3! = 6$
 Total words starting with KE = $3! = 6$
 Total words starting with KR = $3! = 6$
 The first word starting with KU will be KUBER.
 So, rank of the word KUBER

$$= 24 + 24 + 18 + 1 = 67$$

32. Total number of such numbers

$$= 4 \cdot 5 \cdot 4 \cdot 3 = 240$$

33. Maximum population = $2^{32} - 1$ [as each position of tooth has two options]

34. Total number of ways

$$= {}^nC_2 \times {}^nC_1 \times (n-1)! \cdot \frac{n!}{2!(n-2)!} \cdot n! = n! \cdot {}^nC_2$$

35. Total number of permutations = $0 + k + k^2 + k^3 + \dots + k^r$

$$= \frac{k(k^r - 1)}{(k - 1)}$$

36. Matches whose predictions are correct can be selected in ${}^{20}C_{10}$ ways. Now, each wrong prediction can be made in 2 ways. Thus, total number of ways = ${}^{20}C_{10} \cdot 2^{10}$.

37. Let S_1 and S_2 refuse to be together and S_3 and S_4 want to be together only. Total number of ways, when S_3 and S_4 are selected

$$= ({}^8C_2 + {}^2C_1 \cdot {}^8C_1) = 44$$

 Total number of ways when S_3 and S_4 are not selected

$$= ({}^8C_2 + {}^2C_1 \cdot {}^8C_3) = 182$$

 Thus, total number of ways = $44 + 182 = 226$

38. Formed number can be atmost of nine digits.
 Total number of such numbers

$$= 3 + 3^2 + 3^3 + \dots + 3^8 + 2 \cdot 3^8$$

$$= \frac{3(3^8 - 1)}{3 - 1} + 2 \cdot 3^8 = \frac{3^9 - 3 + 4 \cdot 3^8}{2} = \frac{7 \cdot 3^8 - 3}{2}$$

39. The number of committees of 4 gentlemen = ${}^4C_4 = 1$
 The number of committees of 3 gentlemen, 1 wife
 $= {}^4C_3 \times {}^1C_1$ [after selecting 3 gentlemen, only 1 wife is left, which can be included]
 The number of committees of 2 gentlemen, 2 wives
 $= {}^4C_2 \times {}^2C_2$
 The number of committees of 1 gentleman, 3 wives
 $= {}^4C_1 \times {}^3C_3$
 The number of committees of 4 wives = 1
 $\therefore$ The required number of committees
 $= 1 + 4 + 6 + 4 + 1 = 16$
40. The candidate is unsuccessful, if he fails in 9 or 8 or 7 or 6 or 5 papers.
 $\therefore$ The number of ways to be unsuccessful
 $= {}^9C_9 + {}^9C_8 + {}^9C_7 + {}^9C_6 + {}^9C_5$
 $= 1 + 9 + 36 + 84 + 126 = 256$
41. Total number of numbers of without restriction = 2^5
 Two numbers have all the digits equal. So, the required number of numbers = $2^5 - 2$
 $= 32 - 2 = 30$
42. $62 = {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-1}$
 $= 2^n - {}^nC_0 - {}^nC_n = 2^n - 2$
 $\Rightarrow 62 + 2 = 2^n$
 $\therefore 2^n = 64 = 2^6 \Rightarrow n = 6$
43. Total number of ways
 $= \sum_{r=1}^n {}^{2n}C_{n+r} = {}^{2n}C_{n+1} + \dots + {}^{2n}C_{2n}$
 $= \frac{1}{2} (2^{2n} - {}^{2n}C_n)$
44. Ways $\overset{\times}{9} \overset{\times}{10} \overset{\times}{10} \overset{\times}{10} \overset{\times}{10} \overset{\times}{2}$
 0 cannot be placed in the first place. In the other four places, any digit can go. After filling the first five places, the last place can be filled by 0 or 5.
 $\therefore$ Required number of numbers = $9 \times 10^4 \times 2 = 18000$
45. Total number of outcomes = $6 \times 6 \times \dots$ to n times = 6^n
 Total number of outcomes which show only even numbers
 $= 3 \times 3 \times 3 \dots$ to n times = 3^n
46. The matrix will be of the order 4×1 or 1×4 or 2×2 .
 For each order, the number of different matrices
 $=$ The number of ways to fill four places by 0, 1, 2, 3
 $= 4 \times 4 \times 4 \times 4$
47. If r questions are solved by each student, then the number of possible selections of questions is ${}^{20}C_r$.
 $\therefore$ The number of students = ${}^{20}C_r$
 [$\therefore$ each student has solved different combinations of questions]
 $\therefore$ The maximum number of students
 $=$ Maximum value of ${}^{20}C_r = {}^{20}C_{10}$
 because ${}^{20}C_{10}$ is the largest among ${}^{20}C_0, {}^{20}C_1, \dots, {}^{20}C_{20}$ being the middle one.
48. From every arrangement of 7 letters, we get a pair by putting a sign (,).
 $\therefore$ The required number of pairs = $\frac{7!}{2!2!} = 1260$

49. Possibilities Selections Arrangements
 One triplet, two different ${}^3C_1 \times {}^2C_2$ ${}^3C_1 \times {}^2C_2 \times \frac{5!}{3!} = 60$
 Two pairs, one different ${}^3C_2 \times {}^1C_1$ ${}^3C_2 \times {}^1C_1 \times \frac{5!}{2!2!} = 90$
 $\therefore$ Required number of ways = 150
50. Total number of ways = $\frac{6!}{3! \times 3!} = 20$
51. The number of selections of 1 pair of vowels and 1 pair of consonants = ${}^5C_1 \times {}^{21}C_1$
 $\therefore$ Required number of words
 $= {}^5C_1 \times {}^{21}C_1 \times \frac{4!}{2!2!} + {}^5C_2 \times {}^{21}C_2 \times 4!$
52. Required number of ways = $3! \cdot 3! \cdot 3! = 216$
53. If $x = 1, y \in \{1, 2, \dots, 11\}$
 $x = 2, y \in \{1, 2, \dots, 12\}$
 $\dots \dots \dots$
 $x = 11, y \in \{1, 2, \dots, 21\}$
 $\therefore$ Number of ways = $11 + 12 + \dots + 21$
54. We can choose two men out of 9 in ${}^9C_2$ ways. Since, no husband and wife are to play in the same game, two women out of the remaining 7 can be chosen in ${}^7C_2$ ways. If M_1, M_2, W_1 and W_2 are chosen, then a team may consist of M_1 and W_1 or M_1 and W_2 . Thus, the number of ways of arranging the game is
 ${}^9C_2 \cdot {}^7C_2 = \frac{9 \times 8}{2} \times \frac{7 \times 6}{2} \times 2 = 1512$
55. The number of students answering exactly i ($1 \leq i \leq n-1$) questions wrongly is $2^{n-i} - 2^{n-i-1}$. The number of students answering all n questions wrongly is 2^0 .
 Thus, the total number of wrong answers is
 $1(2^{n-1} - 2^{n-2}) + 2(2^{n-2} - 2^{n-3})$
 $+ \dots + (n-1)(2^1 - 2^0) + n(2^0)$
 $= 2^{n-1} + 2^{n-2} + 2^{n-3} + \dots + 2^0 = 2^n - 1$
 Now, $2^n - 1 = 2047$
 $\Rightarrow 2^n = 2048 = 2^{11}$
 $\therefore n = 11$
56. The required numbers are
 1, 2, 11, 12, 21, 22, ..., 122222222
 Let us calculate how many numbers are these.
 There are 2 one-digit such numbers. There are 2^2 two-digit such numbers and so on.
 There are 2^8 eight-digit such numbers. Also, the 9-digit numbers beginning with 1 and written by means of 1 and 2 are smaller than $2 \cdot 10^8$. Thus, there are 2^8 such nine-digit numbers.
 Hence, the required number of numbers
 $= 2 + 2^2 + 2^3 + \dots + 2^8 + 2^8$
 $= \frac{2(2^8 - 1)}{2 - 1} + 2^8$
 $= 2^9 - 2 + 2^8$
 $= 766$

57. Since, 3 does not occur in 1000, we have to count the number of times 3 occurs when we list the integers from 1 to 999. Any number between 1 and 999 is of the form xyz , where $0 \leq x, y, z \leq 9$. Let us first count the numbers in which 3 occurs exactly once. Since, 3 can occur at one place in ${}^3C_1$ ways, there are ${}^3C_1 (9 \times 9) = 3 \times 9^2$ such numbers. Next, 3 can occur exactly two places in ${}^3C_2 (9) = 3 \times 9$ such numbers.

Lastly, 3 can occur in all three-digits in one number only. Hence, the number of times 3 occurs is

$$1 \times (3 \times 9^2) + 2 \times (3 \times 9) + 3 \times 1 = 300$$

58. One possible arrangements is

$$\begin{matrix} 2 & 2 & 3 \end{matrix}$$

Three such arrangements are possible. Therefore, the number of ways is $({}^5C_2)({}^3C_2)({}^1C_1)(3) = 90$

The other possible arrangements is

$$\begin{matrix} 1 & 1 & 3 \end{matrix}$$

Three such arrangement are possible. In this case, the number of ways is $({}^5C_1)({}^4C_1)({}^3C_3)(3) = 60$

Hence, the total number of ways = $90 + 60 = 150$

59. Total number of permutations = $\frac{9!}{2!}$

Number of permutation of those containing 'HIN' = $7!$

Number of permutation of those containing 'DUS' = $\frac{7!}{2!}$

Number of permutation of those containing 'TAN' = $7!$

Number of permutation of those containing 'HIN' and 'DUS' = $5!$

Number of permutation of those containing 'HIN' and 'TAN' = $5!$

Number of permutation of those containing 'TAN' and 'DUS' = $5!$

Number of permutation of those containing 'HIN', 'DUS' and 'TAN' = $3!$

$$\therefore \text{Required number} = \frac{9!}{2!} - \left(7! + 7! + \frac{7!}{2} \right) - 3 \times 5! - 3! = 168474$$

60. The smallest number of people
= Total number of possible forecast
= Total number of possible results
= $3 \times 3 \times 3 \times 3 \times 3 = 243$

61. Since, every pair of the students gives us a hand shake

$\therefore$ Total number of hand shakes

= Total number of pairs of students

= Number of ways of choosing two students out of

$$= {}^{20}C_2 = \frac{20!}{2! \cdot 18!}$$

$$= \frac{20 \times 19 \times 18!}{2 \times 18!} = 190$$

62. Total number of ways = $8!2!$

63. A couple and 6 guests can be arranged in $(7-1)!$ ways.

But the two people forming the couple can be arranged among themselves in $2!$ ways.

$\therefore$ Required number of ways = $6! \times 2! = 1440$

64. Required number of garlands = $\frac{5!}{2 \cdot 2! \cdot 3!} = 5$

65. The number of matches in the first round = ${}^6C_2 + {}^6C_2$

The number of matches in the next round = ${}^6C_2$

The number of matches in the semifinal round = ${}^4C_2$

So, the required number of matches

$$= {}^6C_2 + {}^6C_2 + {}^6C_2 + {}^4C_2 + 2 = 53$$

66. $\frac{{}^{10}C_5}{{}^{11}C_6} = \frac{6}{11}$

67. Required set is $\{m, m+1, \dots, k\}$

$$\Rightarrow \text{Number of subsets} = 2^{k-(m-1)}$$

68. Number of groups each having r students = ${}^{21}C_r$

Let G = Number of times the teacher can go to the zoo

Z = Number of groups she can form = ${}^{21}C_r$

$\therefore G$ is maximum, when Z is maximum.

$$\text{i.e. } r = \frac{21-1}{2} \quad \text{or} \quad \frac{21+1}{2}$$

$$\text{i.e. } r = 10 \quad \text{or} \quad 11$$

69. Required number of ways

$$= {}^nC_2 + {}^nC_3 + \dots + {}^nC_{n-3} + {}^nC_{n-2}$$

$$= 2^n - ({}^nC_0 + {}^nC_1 + {}^nC_{n-1} + {}^nC_n)$$

$$= 2^n - (1 + n + n + 1)$$

$$= 2^n - 2(n+1)$$

70. In a committee of 5 persons, the gentlemen will be in majority, if the number of gentlemen ≥ 3 .

Total number of ways of forming committee

$$= {}^4C_3 \times {}^6C_2 + {}^4C_4 \times {}^6C_1$$

$$= 4 \times 15 + 1 \times 6$$

$$= 66$$

71. Total number of m elements subsets of $A = {}^nC_m$

Number of m elements subsets of A each containing the element a_4 = Number of ways of choosing remaining $(m-1)$ elements of the subset, out of remaining $(n-1)$ elements of A

$$= {}^{n-1}C_{m-1}$$

$$\text{Given that, } {}^nC_m = k \times {}^{n-1}C_{m-1}$$

$$\frac{n!}{m!(n-m)!} = k \cdot \frac{(n-1)!}{(m-1)!(n-m)!}$$

$$\Rightarrow n = km$$

72. Required number of ways

= Total number of ways of choosing 4 pens

– Number of ways of choosing 4 non-red pens

$$= {}^{3+4+6}C_4 - {}^4C_4 = {}^{13}C_4 - {}^{10}C_4$$

$$= \frac{13 \cdot 12 \cdot 11 \cdot 10}{4 \cdot 3 \cdot 2 \cdot 1} - \frac{10 \cdot 9 \cdot 8 \cdot 7}{4 \cdot 3 \cdot 2 \cdot 1}$$

$$= 715 - 210 = 505$$

73. Out of the given numbers, the numbers which are multiple of 5, are 5, 10, 15, 20, ..., 145.

This is an AP whose first term $a = 5$,

Common difference = 5

n th term, $t_n = 145$

$$\therefore 145 = 5 + (n-1) \times 5$$

$$\Rightarrow n = 29$$

Now, if total number of numbers is m , then

$$151 = 1 + (m-1) \times 2$$

$$\Rightarrow m-1 = \frac{151-1}{2} \Rightarrow m = 76$$

So, number of ways in which product is a multiple of 5

= Both two numbers from 5, 10, 15, 25, ..., 145

+ One number from 5, 10, 15, 25, ..., 145 and one from remaining numbers

$$= {}^{29}C_2 + {}^{29}C_1 \times {}^{76}C_1$$

$$= 29 \times 14 + 29 \times 76 = 29 \times 90 = 2610$$

$$74. \therefore {}^{n+1}C_3 - {}^nC_3 = 21$$

$$\therefore \frac{(n+1)(n-1)n}{6} - \frac{n(n-1)(n-2)}{6} = 21$$

$$\Rightarrow \frac{n(n-1)}{6} [(n+1) - (n-2)] = 21$$

$$\Rightarrow \frac{n(n-1)}{6} \times 3 = 21$$

$$\Rightarrow n(n-1) = 42$$

$$\Rightarrow n^2 - n - 42 = 0$$

$$\Rightarrow n = -6, 7$$

$$\therefore n = 7$$

$$75. \therefore {}^nC_2 - n = 44 \Rightarrow \frac{n(n-1)}{2} - n = 44$$

$$\therefore n^2 - 3n - 88 = 0$$

$$\Rightarrow n = -8, 11$$

$$\therefore n = 11$$

$$76. \text{Total number of triangles} = {}^{12}C_3 - {}^3C_3 - {}^4C_3 - {}^5C_3 = 205$$

77. The number of triangles with vertices on sides

$$AB, BC, CD = {}^3C_1 \times {}^4C_1 \times {}^5C_1$$

Similarly, for other cases, the total number of triangles

$$= {}^3C_1 \times {}^4C_1 \times {}^5C_1 + {}^3C_1 \times {}^4C_1 \times {}^6C_1 + {}^3C_1 \times {}^5C_1 \times {}^6C_1$$

$$+ {}^4C_1 \times {}^5C_1 \times {}^6C_1$$

$$= 342$$

78. A selection of four vertices of the polygon gives an interior intersection.

$$\therefore \text{The number of sides} = n$$

$$\Rightarrow {}^nC_4 = 70$$

$$\Rightarrow n(n-1)(n-2)(n-3) = 24 \times 70$$

$$= 8 \times 7 \times 6 \times 5$$

$$\therefore n = 8$$

$$\therefore \text{The number of diagonals} = {}^8C_2 - 8 = 20$$

79. ${}^nC_2$ is point of intersection.

80. Let x_0 be the number of objects to left of the first object chosen, x_1 be the number of objects between the first and the second, x_2 be the number of objects between the second and the third and x_3 be the number of objects to the right of the third object. We have,

$$x_0, x_3 \geq 0, x_1, x_2 \geq 1$$

$$\text{and } x_0 + x_1 + x_2 + x_3 = n - 3 \quad \dots (i)$$

The number of solutions of Eq. (i)

$$= \text{Coefficient of } y^{n-3} \text{ in } (1 + y + y^2 + \dots)(1 + y + y^2 + \dots)$$

$$(y + y^2 + y^3 + \dots)(y + y^2 + y^3 + \dots)$$

$$= \text{Coefficient of } y^{n-3} \text{ in } y^2(1 + y + y^2 + y^3 + \dots)^4$$

$$= \text{Coefficient of } y^{n-5} \text{ in } (1 - y)^{-4}$$

$$= \text{Coefficient of } y^{n-5} \text{ in } (1 + {}^4C_1 y + {}^5C_2 y^2 + {}^6C_3 y^3 + \dots)$$

$$= {}^{n-5+3}C_{n-5} = {}^{n-2}C_3 = \frac{(n-2)(n-3)(n-4)}{6}$$

$$81. \text{The number of triangles with vertices on different lines}$$

$$= {}^pC_1 \times {}^pC_1 \times {}^pC_1 = p^3$$

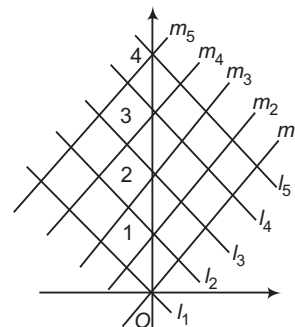
The number of triangles with two vertices on one line and the third vertex on any one of the other two lines

$$= {}^3C_1 \{ {}^pC_2 \times {}^{2p}C_1 \} = 6p \cdot \frac{p(p-1)}{2}$$

So, the required number of triangles

$$= p^3 + 3p^2(p-1) = p^2(4p-3)$$

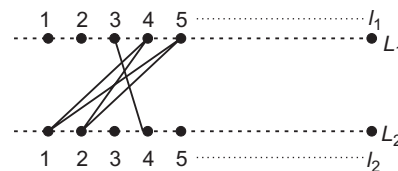
82. There are two sets of five parallel lines at equal distances.



Clearly, lines like l_1, l_3, m_1, m_3 form a square whose diagonals length is 2. So, the number of required squares = $3 \times 3 = 9$.

Since, choices are $(l_1, l_3), (l_2, l_4), (l_3, l_5)$ for one set.

83.



Required number of points on intersection of new lines

$$= 2 \times \text{Number of choosing 1 point on } L_1 \text{ and 2 point on } L_2$$

$$= 2 \cdot {}^1C_1 \times {}^2C_2$$

84. Required number of triangles

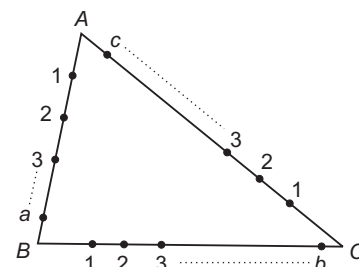
= Total number of triangles

– Number of triangles having two sides common

– Number of triangles having one side common

$$= {}^8C_3 - 8 - 8 \times 4 = 16$$

85.



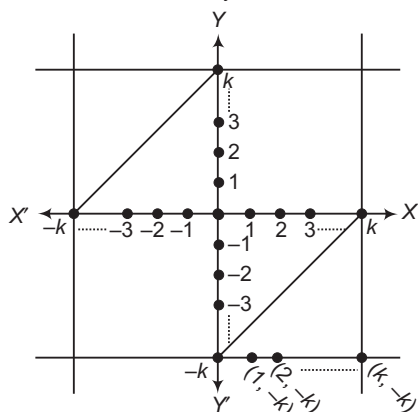
Required number of triangles

= Total number of ways of choosing 3 points

– Number of ways of choosing all the 3 points from AB or BC or CA

$$= {}^{a+b+c}C_3 - ({}^aC_3 + {}^bC_3 + {}^cC_3)$$

86. $|x| \leq k \Rightarrow -k \leq x \leq k$... (i)
 and $|y| \leq k \Rightarrow -k \leq y \leq k$... (ii)
 and $|x - y| \leq k \Rightarrow |y - x| \leq k$
 $\Rightarrow -k \leq y - x \leq k$
 $\Rightarrow x - k \leq y \leq x + k$... (iii)



$\therefore$ Number of points having integral coordinates
 $= (2k + 1)^2 - 2[k + (k - 1) + \dots + 2 + 1]$
 $= (3k^2 + 3k + 1)$

87. Total number of ways $= 31 \times 15 \times 8 = 3720$
88. Total number of squares $= 8$. We are to choose 6 squares for six X's and this can be done is ${}^8C_6 = 28$ ways. This also include the possibility that either the top horizontal row does not have any X or the bottom horizontal row have no X. These possibilities are not required.
 $\therefore$ Total number of ways $= 28 - 2 = 26$
89. Total number of such numbers
 $= {}^9C_6 + {}^9C_5 + {}^9C_5 + {}^9C_4 = {}^{10}C_6 + {}^{10}C_5 = {}^{11}C_6$
90. The number of times, a particular girl goes on picnic $= {}^6C_4$
 $\therefore$ Total number of dolls given to the girls
 $= {}^6C_4 \cdot 3 = \frac{6!}{4!2!} \cdot 3 = 45$
91. Case I 1 4 7 ... $3n - 2$
 Case II 2 5 8 ... $3n - 1$
 Case III 3 6 9 ... $3n$
 That means we must take 2 numbers from last row or one number each from first and second rows.
 Total number of ways $= {}^nC_2 + {}^nC_1 \cdot {}^nC_1$
 $= \frac{n(n-1)}{2} + n^2 = \frac{3n^2 - n}{2}$
92. The number of selections of p things from q identical things are $(r - p)$ things from q identical things $= 1 \times 1$
 $p \rightarrow p \quad p - 1 \dots r - q$
 $q \rightarrow r - p \quad r - p + 1 \dots q$
 Similarly, in all other cases, total number of ways
 $= p - (r - q) + 1$ or $q - (r - p) + 1 = p + q - r + 1$
93. $2^p \cdot 6^q \cdot 15^r = 2^{p+q} \cdot 3^{q+r} \cdot 5^r$
 $\therefore$ The number of proper divisors
 $= \{\text{Total number of selections from } (p + q) \text{ twos, } (q + r) \text{ threes and } r \text{ fives}\} - 2$
 $= (p + q + 1)(q + r + 1)(r + 1) - 2$

94. $1008 = 2^4 \times 3^2 \times 7$
 $\therefore$ The required number of even proper divisors
 $= \text{Total number of selections of atleast one 2 and any number of 3's or 7's}$
 $= 4 \times (2 + 1) \times (1 + 1) - 1 = 23$
95. The natural numbers are 1, 2, 3 and 4.
 Clearly, in one diagonal we have to place 1, 4 and in the other 2, 3.
 The number of ways $= 2! \times 2! = 4$
96. If 0 is placed in the unit's place of the upper number, then the unit's place of the lower number can be filled in 10 ways (filling by any one of 0, 1, 2, ..., 9).
 If 1 is placed in the units place of the upper number, then the unit's place of the lower number can be filled in 9 ways (filling by any one of 0, 1, 2, ..., 8) etc.
 $\therefore$ The unit's column can be filled in $10 + 9 + 8 + \dots + 1$, i.e. 55 ways. Similarly, for the second and the third columns. The number of ways for the fourth column
 $= 8 + 7 + \dots + 1 = 36$
 $\therefore$ The required number of ways
 $= 55 \times 55 \times 55 \times 36$
97. Coefficient of x^3 in
 $(x^0 + x^1 + x^2)(x^0 + x^1 + x^2 + x^3)(x^1 + x^2 + x^3 + x^4)$
 $= 6$
98. $38808 = 2^3 \times 3^2 \times 7^2 \times 11^1$
 $\therefore$ Divisors $= 4 \times 3 \times 3 \times 2 - 2 = 70$
99. ${}^{15+3-1}C_{3-1} = {}^{17}C_2$
100. $\left[\frac{33}{2}\right] + \left[\frac{33}{4}\right] + \left[\frac{33}{8}\right] + \left[\frac{33}{16}\right] + \left[\frac{33}{32}\right]$
 $= 16 + 8 + 4 + 2 + 1 = 31$
101. For $1 \leq k \leq p - 1$, $n + k = p! + k + 1$, is clearly divisible by $k + 1$. Therefore, there is no prime number in the given list.
102. The number of ways in which all 5 letters be placed in wrong envelopes
 $= {}^6C_5 \cdot 5! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$
 $= 720 \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right) = 264$
103. There is $(m + 1)$ space between them (including the beginning and the end) and n things can be placed in all possible spaces in ${}^{m+1}P_n$ ways.
 $\therefore$ Required number of ways
 $= m! \cdot {}^{m+1}P_n = \frac{m! \cdot (m + 1)!}{(m + 1 - n)!}$
104. We have, $240 = 2^4 \times 3^1 \times 5^1$
 $\therefore$ There are 4 divisors of 240 which of the form $4n + 2$, $n \geq 0$.
105. Number of non-negative integral solutions
 $= {}^{12}C_{10} = {}^{12}C_2$
106. $\therefore x_1 + x_2 + 2x_3 = 15 \Rightarrow x_1 + x_2 = 15 - 2x_3$
 where, x_3 can be 1, 2, ..., 7.
 Let $x_3 = r$
 $\Rightarrow x_1 + x_2 = 15 - 2r$

Number of positive integral solutions of this equation

$$= (14 - 2r)$$

∴ Total number of solutions

$$= \sum_{r=1}^7 (14 - 2r) = 14 \cdot 7 - 2 \cdot \left(\frac{7 \cdot 8}{2} \right) = 42$$

107. $x_1 + x_2 + x_3 + x_4 = 15$ and $x_i \geq 2$

$$\Rightarrow (x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) = 7$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 = 7, \text{ where } y_i = x_i - 2 \geq 0$$

$$\Rightarrow {}^{10}C_7 = {}^{10}C_3 \text{ are number of non-negative.}$$

108. Coefficient of x^3 in

$$\{(x^0 + x^1 + x^2 + x^3)(x^0 + x^1 + x^2) \cdot (x^0 + x^1)^4\}$$

$$= 247$$

109. Let the balls put in the box be x_1, x_2, x_3, x_4 and x_5 .

We have,

$$x_1 + x_2 + x_3 + x_4 + x_5 = 15, x_i \geq 2$$

$$\Rightarrow (x_1 - 2) + (x_2 - 2) + (x_3 - 2) + (x_4 - 2) + (x_5 - 2) = 5$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 + y_5 = 5, y_i = x_i - 2 \geq 0$$

$$= {}^{5+5-1}C_{5-1} = {}^9C_4$$

110. Coefficient of x^{210} in $\{(x^0 + x^1 + x^2 + x^3 + \dots + x^{100})^3\}$

$$= {}^{93}C_3$$

111. Required number of ways = Number of non-negative integral solution of $x_1 + x_2 + x_3 + x_4 + \dots + x_8 = 10$

$$= {}^{10+8-1}C_{8-1} = {}^{17}C_7$$

112. The required number of selections

$$= {}^3C_1 \times {}^4C_1 \times {}^2C_1 ({}^6C_3 + {}^6C_2 + {}^6C_1 + {}^6C_0)$$

113. $x + y + z = 16, y = x + 1$ and $z = y + 2$

$$\therefore x = 4, y = 5, z = 7$$

Now, number of ways to give 16 different things

$$= {}^{16}C_4 \times {}^{12}C_5 \times {}^7C_7$$

114. $x + y + z = n$

$$x, y, z \geq 1 \text{ or } x - 1 = a, y - 1 = b, z - 1 = c \geq 0$$

$$\Rightarrow a + b + c = n - 3$$

The required number of solutions

$$= {}^{n-3+3-1}C_{3-1} = {}^{n-1}C_2$$

115. The number of non-negative integral solutions of

$$a + b + c + d = n$$

$$\Rightarrow {}^{n+3}C_n = \frac{(n+3)!}{n!3!} = \frac{(n+3)(n+2)(n+1)}{6}$$

116. The required number of solutions

= The number of positive integral solution of

$$(a + b + c = 12)$$

$$= \frac{(12-1)(12-2)}{2} = 55$$

117. $a + a + d + a + 2d = 21$ or $a + d = 7$

$$\therefore a + c = 14$$

and

$$b = 7$$

The number of positive integral solutions of $(a + c = 14)$ is 13.

118. Let $a = 2p + 1, b = 2q + 1, c = 2r + 1, d = 2s + 1$, where p

q, r and s are non-negative integers.

$$\therefore 2p + 1 + 2q + 1 + 2r + 1 + 2s + 1 = 20$$

or

$$p + q + r + s = 8$$

The required number of solutions

= The number of non-negative integral solutions of $(p + q + r + s = 8)$

$$= \frac{(8+3)(8+2)(8+1)}{6} = 165$$

119. If a, b and c are in AP, then a and c both are odd or both are even.

Case I When n is even.

The number of ways of selection of two even numbers, a and c is ${}^{n/2}C_2$. Number of ways of selection of two odd numbers is ${}^{n/2}C_2$.

Hence, the number of AP's

$$2 \cdot {}^{n/2}C_2 = 2 \cdot \frac{\frac{n}{2} \left(\frac{n}{2} - 1 \right)}{2} = \frac{n(n-2)}{4}$$

Case II When n is odd.

The number of ways of selection of two odd numbers a and c is ${}^{(n+1)/2}C_2$. The number of ways of selection of two even numbers a and c is ${}^{(n-1)/2}C_2$.

Hence, the number of AP's = ${}^{(n+1)/2}C_2 + {}^{(n-1)/2}C_2$

$$= \frac{\left(\frac{n+1}{2} \right) \left(\frac{n+1}{2} - 1 \right)}{2} + \frac{\left(\frac{n-1}{2} \right) \left(\frac{n-1}{2} - 1 \right)}{2}$$

$$= \frac{1}{8} (n-1)[(n+1) + (n-3)] = \frac{(n-1)^2}{4}$$

120. The number of ways of inviting, when the couple not included is ${}^8C_5$. The number of ways of inviting with the couple included is ${}^8C_3$. Therefore, the required number of ways = ${}^8C_5 + {}^8C_3 = {}^8C_3 + {}^8C_3$ [$\because {}^8C_5 = {}^8C_3$]

$$\text{Also, } {}^{10}C_5 - 2 \times {}^8C_4 = \frac{10!}{5!5!} - 2 \times \frac{8!}{4!4!}$$

$$= \frac{10 \times 9 \times 8 \times 7 \times 6}{120} - 2 \times \frac{8 \times 7 \times 6 \times 5}{24}$$

$$= 9 \times 4 \times 7 - 140 = 112 = 2 \times \frac{8!}{3!5!}$$

121. $\therefore p = {}^5C_4 \times {}^2C_1 = 10$

$$q = {}^5C_2 ({}^2C_1)^3 = 80$$

$$\text{and } r = {}^5C_0 ({}^2C_1)^5 = 32$$

$$\therefore 2q = 5r, 8p = q$$

$$\text{and } 2(p+r) > q$$

122. By principle of homogeneity, $x = y$ is evident and since sum of digits is either odd or even.

$$\Rightarrow x + y = \text{Total 5-digit numbers}$$

$$= 9 \times 10 \times 10 \times 10 \times 10$$

$$\Rightarrow 2x = 90000$$

$$\therefore x = 45000$$

123. Since, ${}^nC_r = {}^nC_{n-r}$, hence options (a) and (c) follow.

124. Total hand shakes possible = ${}^{2n}C_2$. This will include n hand shakes in which a person shakes hand with her or his spouse.

∴ Required number of hand shakes

$$= {}^{2n}C_2 - n = 2n(n-1)$$

125. The n th line will rise to n additional parts, when u_{n-1} parts are already there.

$$\Rightarrow u_n = u_{n-1} + n$$

$\Rightarrow$ Option (d) is incorrect.

Again, $u_0 = 1, u_1 = 2$

We easily get $u_2 = 4, u_3 = 7, u_4 = 11$

Hence, options (a) and (b) are correct and option (c) is false.

126. We have, $30 = 2 \times 3 \times 5$. So, 2 can be assigned to either a or b or c i.e. 2 can be assigned in 3 ways. Similarly, each of 3 and 5 can be assigned in 3 ways. Thus, the number of solutions is $3 \times 3 \times 3 = 27$.

Also, the number of ways in which three prizes can be distributed among three persons

$$= 3 \times 3 \times 3 = 3^3$$

127. As each student receives atleast two toys. So, let us first give each student one toy. Now, we are left with 7 toys, which can be distributed among three students such that each receives atleast one toy, which is equivalent to number of positive integral solutions of the equation $x + y + z + w = 7$, which is given by ${}^{7-1}C_{4-1} = {}^6C_3$. Hence, Statement I is false and Statement II is correct.

128. Number of ways of dividing n^2 objects into n groups of same size is $\frac{(n^2)!}{(n!)^n n!}$.

Now, number of ways of distributing these n groups among n persons is

$$\left[\frac{(n^2)!}{(n!)^n n!} \right] n! = \frac{(n^2)!}{(n!)^n}, \text{ which is always an integer.}$$

Also, we know that product of r is divisible by $r!$.

Now, $(n^2)! = 1 \times 2 \times 3 \times 4 \times \dots \times n^2$

$$= 1 \times 2 \times 3 \times \dots \times n \times (n+1) \times (n+2) \times \dots \times 2n \\ \times (2n+1) \times (2n+2) \times \dots \times 3n \times \{n^2 - (n^2 - 1)\} \\ \{n^2 - (n^2 - 1)\} \dots n^2$$

Thus, in $n^2!$, there are n rows each consisting product of n integers. Each row is divisible by $n!$.

Hence, $(n^2)!$ is divisible by $(n!)^n$ or $\frac{(n^2)!}{(n!)^n}$ is a natural number.

Hence, both statements are correct and Statement II is the correct explanation of Statement I.

129. $\therefore 1400 = 2^3 \cdot 5^2 \cdot 7$

The number of ways in which 1400 can be expressed as a product of two positive integers is

$$\frac{(3+1)(2+1)(1+1)}{2} = 12$$

Statement II is correct but does not explain Statement I as it just gives the information of prime factor about which 1400 is divisible.

130. In onto functions, each image must be assigned to atleast one pre-image. Now, if we consider the images a and b as two different boxes, then four distinct objects 1, 2, 3 and 4 (pre-images) can be distributed, such that no box remains empty, in $2^4 - {}^2C_1(2-1)^4 = 14$ ways.

Hence, both statements are correct and Statement II is the correct explanation of Statement I.

131. India must win atleast 6 matches out of 11 matches. Then, number of ways in which India can win the series

$$= {}^{11}C_6 + {}^{11}C_7 + \dots + {}^{11}C_{11} = 2^{10}$$

Thus, both the statements are true, but Statement II is not a correct explanation of Statement I.

132. The batting order of 11 players can be decided in $11!$ ways. Now, Yuvraj, Dhoni and Pathan can be arranged in $3!$ ways. But the order of these three players is fixed i.e. Yuvraj-Dhoni-Pathan. Now, $11!$ answer is $3!$ times more, hence the required answer is $11!/3!$.

133. Number of ways of arranging 21 identical objects, when r is identical of one type and remaining are identical of second type, is $\frac{21!}{r!(21-r)!} = {}^{21}C_r$, which is maximum

when $r = 10$ or 11 .

Therefore, ${}^{13}C_r = {}^{13}C_{10}$ or ${}^{13}C_{11}$, hence maximum value of ${}^{13}C_r$ is ${}^{13}C_{10} = 286$.

Hence, Statement I is false. Obviously, Statement II is true.

134. When one all rounder and ten players from bowlers and batsman play, number of ways is ${}^4C_1 {}^{14}C_{10}$.

When one wicketkeeper and 10 players from bowlers and batsman play, number of ways is ${}^2C_1 {}^{14}C_{10}$.

When one all rounder, one wicketkeeper and nine from batsman and bowlers play, number of ways is ${}^4C_1 {}^2C_1 {}^{14}C_9$.

When all eleven players play from bowlers and batsmen then number of ways is ${}^{14}C_{11}$.

Total number of selections

$$= {}^4C_1 {}^{14}C_{10} + {}^2C_1 {}^{14}C_{10} + {}^4C_1 {}^2C_1 {}^{14}C_9 + {}^{14}C_{11}$$

135. If the particular bowler plays, then two batsmen will not play. So, rest of 10 players can be selected from 17 other players. Number of such selections is ${}^{17}C_{10}$.

If the particular bowler does not play, then number of selections = ${}^{19}C_{11}$

If all the three players do not play, then number of selections = ${}^{17}C_{11}$

$$\therefore \text{Total number of selections} = {}^{17}C_{10} + {}^{19}C_{11} + {}^{17}C_{11}$$

136. If the particular batsman is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If particular wicketkeeper is selected, then rest of 10 players can be selected in ${}^{18}C_{10}$ ways.

If both are not selected, then number of ways is ${}^{18}C_{11}$.

Hence, total number of ways

$$= 2 \cdot {}^{18}C_{10} + {}^{18}C_{11} = {}^{19}C_{11} + {}^{18}C_{10}$$

137. Seven persons can be selected for first table in ${}^{12}C_7$ ways. Now, these seven persons can be arranged in $6!$ ways. The remaining five persons can be arranged on the second table in $4!$ ways.

Hence, total number of ways = ${}^{12}C_5 \cdot 6! \cdot 4!$

138. Here, A can sit on first table and B on the second or A on second table and B on the first table.

If A is on the first table, then remaining six for first table can be selected in ${}^{10}C_6$ ways. Now, these seven persons can be arranged in $6!$ ways. Remaining five can be arranged on the other table in $4!$ ways.

Hence, total number of ways is $2 \cdot {}^{10}C_6 \cdot 6! \cdot 4!$.

139. If A, B are on the first table, then remaining five can be selected in ${}^{10}C_5$ ways.

Now, seven persons including A and B can be arranged on the first table in which A and B are together in $2! \cdot 5!$ ways. Remaining five can be arranged on the second table in $4!$ ways. Total number of ways is ${}^{10}C_5 \cdot 4! \cdot 5! \cdot 2!$.

If A and B are on the second table, then remaining three can be selected in ${}^{10}C_3$ ways.

Now, five persons including A and B can be arranged on the second table in which A and B are together in $2! \cdot 3!$ ways. Remaining seven can be arranged on the first table in $6!$ ways. Total number of ways for first table is ${}^{10}C_7 \cdot 6! \cdot 3! \cdot 2!$.

$\therefore$ Total number of ways = ${}^{10}C_7 \cdot 6! \cdot 3! \cdot 2! + {}^{10}C_5 \cdot 4! \cdot 5! \cdot 2!$

140. 2 candidates can be selected out of 4 in ${}^4C_2$ ways.

141. Required number of ways = $3^8 - {}^3C_1 \cdot 2^8 + {}^3C_2$

142. Each letter has 3 choices of letter boxes in 3^5 ways.

143. A. ${}^{20}C_2 - {}^4C_2 + 1 = 190 - 6 + 1 = 185$

B. $1 \times {}^{20}C_2 = 190$

C. $2 \times {}^8C_2 = 56$

D. ${}^6C_2 \times 4 = 60$

144. A. Number of surjective functions is

$$3^6 - {}^3C_1 (3-1)^6 + {}^3C_2 (3-2)^6 = 729 - 192 + 3 = 540$$

- B. If $f(a_i) \neq b_i$, then pre-image a_1, a_2, a_3 cannot be assigned to images b_1, b_2, b_3 , respectively.

Hence, each of a_1, a_2, a_3 can be assigned to images in 2 ways. a_4, a_5, a_6 can be assigned to images in 3 ways each.

Hence, number of functions is $2^3 \cdot 3^3 = 216$

- C. One-one functions are not possible, as pre-images are more than images.

- D. Number of many one functions

$$= \text{Total number of functions}$$

$$- \text{Number of one-one functions}$$

$$= 3^6 - 0 = 729$$

145. A. For $k \geq 1$, we have

$$k \cdot k! = [(k+1) - 1]k! = (k+1)! - k!$$

$$\text{Thus, } 1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n!$$

$$= (2! - 1!) + (3! - 2!) + (4! - 3!) + \dots + \{(n+1)! - n!\}$$

$$= (n+1)! - 1$$

- B. Let $S = n \cdot {}^nC_1 + (n-1) \cdot {}^nC_2 + (n-2) \cdot {}^nC_3 + \dots + 1 \cdot {}^nC_n$

$$= (n-1) \cdot {}^nC_1 + (n-2) \cdot {}^nC_2 + \dots + 1 \cdot {}^nC_{n-1}$$

$$+ ({}^nC_1 + {}^nC_2 + \dots + {}^nC_n)$$

$$\text{Let } C = (n-1) \cdot {}^nC_1 + (n-2) \cdot {}^nC_2 + \dots + 1 \cdot {}^nC_{n-1} \dots (i)$$

We use ${}^nC_r = {}^nC_{n-r}$ and rewrite the terms in the reverse order, to obtain

$$C = 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + \dots + (n-1) \cdot {}^nC_{n-1} \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2C = n({}^nC_1 + {}^nC_2 + \dots + {}^nC_{n-1})$$

$$= n(2^n - {}^nC_0 - {}^nC_n) = n(2^n - 2)$$

$$\Rightarrow C = n(2^{n-1} - 1)$$

$$\text{Thus, } S = n(2^{n-1} - 1) + (2^n - 1) = (n+2)2^{n-1} - (n+1)$$

C. We have,

$${}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^nC_2 = {}^{n+1}C_3$$

$$\text{D. We have, } \sum_{r=0}^n ({}^nC_r)^2 = \sum_{r=0}^n ({}^nC_r)({}^nC_{n-r})$$

$\therefore$ The number of ways of selecting n persons out of n men and n women = ${}^{2n}C_n$

146. Let T and S denote teacher and student, respectively. Then, we have following possible patterns according to questions

(i) T S S T S S T S S

(ii) S T S S T S S T S

(iii) S S T S S T S S T

Hence, total number of arrangements are

$$3 \cdot (3!)6! = 18 \times 6! \Rightarrow k = 6$$

147. To form a triangle, 3 points out of 5 can be chosen in ${}^5C_3 = 10$ ways.

But of these, the three points lying on the 2 diagonals will be collinear.

So, $10 - 2 = 8$ triangles can be formed.

148. Here, A is common letter in words SUMAN and DIVYA. Now, for selecting six different letters, we must select A either from word SUMAN or from word DIVYA.

Hence, for possible selections, we have

A excluded from SUMAN + A included in SUMAN

$$= {}^4C_3 \cdot {}^5C_3 + {}^4C_2 \cdot {}^4C_3$$

$$= 40 + 24 = 64$$

$$\text{Hence, } N^2 = 64 \Rightarrow N = 8$$

149. Number of arrangements are $2n!n!$.

$$\text{Given that, } 2n!n! = 1152$$

$$\Rightarrow (n!)^2 = 576$$

$$\therefore n! = 24 \Rightarrow n = 4$$

150. Zero or more oranges can be selected out of 4 identical oranges in $4 + 1 = 5$ ways.

Zero or more apples can be selected out of 5 identical apples in $5 + 1 = 6$ ways.

Zero or more mangoes can be selected out of 6 identical mangoes in $6 + 1 = 7$ ways.

$\therefore$ Total number of selections when all the three types of fruits are selected = $5 \times 6 \times 7 = 210$

But in one of these selections number of each type of fruit is zero and hence this selection must be excluded.

$\therefore$ Required number of ways = $210 - 1 = 209$

151. Suppose box A has x_1 balls, box B has x_2 balls and box C has x_3 balls. Then,

$$x_1 + x_2 + x_3 \leq 12, x_1 \geq 1, x_2 \geq 3, 1 \leq x_3 \leq 5$$

Let $x_4 = 12 - (x_1 + x_2 + x_3)$.

Then, $x_1 + x_2 + x_3 + x_4 = 12$

$$[1 \leq x_1 \leq 8, 3 \leq x_2 \leq 10, 1 \leq x_3 \leq 5 \text{ and } 0 \leq x_4 \leq 7]$$

$\therefore$ Required number of ways = Coefficient of x^{12} in

$$(x^1 + x^2 + \dots + x^8)(x^3 + x^4 + \dots + x^{10})$$

$$(x^1 + x^2 + \dots + x^5)(x^0 + x^1 + \dots + x^7)$$

= Coefficient of x^7 in $(1 + x + x^2 + \dots + x^7)$

$$(1 + x + x^2 + \dots + x^7)(1 + x + x^2 + x^3 + x^4)$$

$$(1 + x + x^2 + \dots + x^7)$$

= Coefficient of x^7 in $(1 - x)^{-4} (1 - x^5)(1 - x^8)$

= Coefficient of x^7 in $(1 - x)^{-4}(1 - x^5)$

= Coefficient of x^7 in

$$(1 - x^5)(1 + {}^4C_1x + {}^5C_2x^2 + {}^6C_3x^3 + \dots)$$

$$= {}^{10}C_7 - {}^5C_2 = 110$$

152. Clearly, $x = 0, 1, 2, 3, \dots, 10$. Let $x = k$, then $0 \leq k \leq 10$

When $x = k$, $y + z = 21 - 2k$

The number of non-negative integral solutions in above equation = Number of ways to distribute $(21 - 2k)$ identical things (each thing is number 1) among 2 persons

$$= {}^{21-2k+2-1}C_{2-1} = {}^{22-2k}C_1 = 22 - 2k$$

$$\therefore \text{Required number of solutions} = \sum_{k=0}^{10} (22 - 2k)$$

$$= 22 + 20 + 18 + \dots + 2$$

$$= 2(1 + 2 + 3 + \dots + 11)$$

$$= 2 \times \frac{11 \times 12}{2} = 132$$

Entrances Gallery

1. When n_5 takes value from 10 to 6 the carry forward moves from 0 to 4 which can be arranged in

$${}^4C_0 + \frac{{}^4C_1}{4} + \frac{{}^4C_2}{3} + \frac{{}^4C_3}{2} + \frac{{}^4C_4}{1} = 7$$

Aliter

Possible solutions are

$$1, 2, 3, 4, 10$$

$$1, 2, 3, 5, 9$$

$$1, 2, 3, 6, 8$$

$$1, 2, 4, 5, 8$$

$$1, 2, 4, 6, 7$$

$$1, 3, 4, 5, 7$$

$$2, 3, 4, 5, 6$$

Hence, there are 7 solutions.

2. Number of red lines = ${}^nC_2 - n$

Number of blue lines = n

$$\text{Hence, } {}^nC_2 - n = n$$

$$\Rightarrow {}^nC_2 = 2n$$

$$\Rightarrow \frac{n(n-1)}{2} = 2n$$

$$\Rightarrow n-1 = 4$$

$$\therefore n = 5$$

3. Required number of ways

$$= 5! - \{4 \cdot 4! - {}^4C_2 \cdot 3! + {}^4C_3 \cdot 2! - 1\} = 53$$

4. Let $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(-1, -1, 1)$ be vectors a, b, c, d and rest of the vectors be $-a, -b, -c, -d$ and let us find the number of ways of selecting coplanar vectors.

Observe that out of any three coplanar vectors, two will be collinear (anti-parallel).

Number of ways of selecting the anti-parallel pair = 4

Number of ways of selecting the third vector = 6

Total = 24

$\therefore$ Number of non-coplanar selections

$$= {}^8C_3 - 24 = 32 - 24 = 8, p = 5$$

Aliter

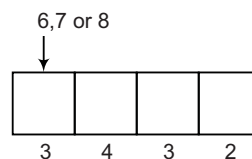
$$\text{Required value} = \frac{8 \times 6 \times 4}{3!}$$

$$\therefore p = 5$$

5. The integer greater than 6000 may be of 4 digit or 5 digit. So, here two cases arise.

Case I When number is of 4 digit.

Four-digit number can start from 6, 7 or 8.



Thus, total number of 4-digit number, which are greater than 600 = $3 \times 4 \times 3 \times 2 = 72$

Case II When number is of 5 digit.

Total number of five-digit number which are greater than 6000 = $5! = 120$

$\therefore$ Total number of integers = $72 + 120 = 192$

6. Given, $n(A) = 2, n(B) = 4$

$$\therefore n(A \times B) = 8$$

Now, the number of subsets of $A \times B$ having 3 or more elements

$$= {}^8C_3 + {}^8C_4 + \dots + {}^8C_8$$

$$= ({}^8C_0 + {}^8C_1 + \dots + {}^8C_8) - ({}^8C_0 + {}^8C_1 + {}^8C_2)$$

$$= 2^8 - {}^8C_0 - {}^8C_1 - {}^8C_2 [\because {}^nC_0 + {}^nC_1 + \dots + {}^nC_n = 2^n]$$

$$= 256 - 1 - 8 - 28 = 219$$

7. Number of ways = $3^5 - {}^3C_1 \cdot 2^5 + {}^3C_2 \cdot 1^5$

$$= 243 - 96 + 3 = 150$$

8. As, $a_n = b_n + c_n$

$$\Rightarrow a_n = 1 \dots (1 \text{ or } 0)$$

$$\Rightarrow a_n = \underbrace{1}_{(n-1) \text{ places}} = a_{n-1} + \underbrace{1}_{(n-2) \text{ places}} = a_{n-2}$$

$$a_n = a_{n-1} + a_{n-2}$$

$$\therefore a_{17} = a_{16} + a_{15}$$

9. b_6 = Six-digit numbers ending with 1.

$$\Rightarrow \boxed{1}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{1}$$

Now, the four places are to be filled.

Case I $\boxed{1}\dots\dots\dots\boxed{1}\boxed{1}$

For 3 places, all 1's are used in 1 way.

One zero is used = ${}^3C_1 = 3$

Two zeros are used = 1 way 0 1 0 1...

Total = 5 ways

Case II $\boxed{1}\dots\dots\dots\boxed{0}\boxed{1}$

For 3 places,

All 1's are used = 1 way

One zero is used = ${}^2C_1 = 2$ ways

$\therefore$ Total ways = 3 ways

Thus, $b_6 = 5 + 3 = 8$

10. The number of ways of distributing n identical objects among r persons such that each person gets atleast one object is same as the number of ways of selecting $(r - 1)$ places out of $(n - 1)$ different places, i.e. is ${}^{n-1}C_{r-1}$.

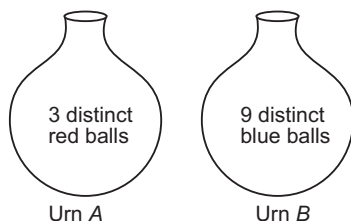
11. If out of n points, m are collinear.

Number of triangles = ${}^nC_3 - {}^mC_3$

$\therefore$ Number of triangles = ${}^{10}C_3 - {}^6C_3 = 120 - 20$

$\Rightarrow N = 100$

- 12.



The number of ways in which two balls from urn A and two balls from urn B can be selected

$$= {}^3C_2 \times {}^9C_2 = 3 \times 36 = 108$$

13. The number of ways in which 4 novels can be selected

$$= {}^6C_4 = 15$$

The number of ways in which 1 dictionary can be selected

$$= {}^3C_1 = 3$$

4 novels can be arranged in 4! ways.

[since, dictionary is fixed in the middle]

$\therefore$ Total number of ways = $15 \times 3 \times 4!$

$$= 15 \times 3 \times 24 = 1080$$

14. The number of different ways the child can buy the six ice-creams = Total number of non-negative integral solution of the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 6$

$$= {}^{6+5-1}C_{5-1} = {}^{10}C_4$$

and number of ways to arrange 6A's and 4B's in a row

$$= \frac{10!}{6!4!} = {}^{10}C_4$$

$\therefore$ Statement I is false and Statement II is true.

15. Given word is MISSISSIPPI.

Here, I = 4 times, S = 4 times, P = 2 times, M = 1 time

M I I I I P P

$$\begin{aligned} \therefore \text{Required number of words} &= {}^8C_4 \times \frac{7!}{4!2!} \\ &= {}^8C_4 \times \frac{7 \times 6!}{4!2!} \\ &= 7 \cdot {}^8C_4 \cdot {}^6C_4 \end{aligned}$$

16. Required number of ways = ${}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$$= \frac{12!}{8! \times 4!} \times \frac{8!}{4! \times 4!} \times 1 = \frac{12!}{(4!)^3}$$

17. Total number of ways

$$\begin{aligned} &= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + {}^{10}C_4 \\ &= 10 + 45 + 120 + 210 \\ &= 385 \end{aligned}$$

18. In the word SACHIN, order of alphabets is A, C, H, I, N, S. Number of words start with A = 5! so with C, H, I, N. Now, words start with S and after that ACHIN are in ascending orders of position, so $5 \cdot 5! = 600$ words are in dictionary before words with start S. Thus, position of given word is 601.

19. Given that, $f(x) = {}^{7-x}P_{x-3}$

The above function is defined, if

$$7 - x \geq 0 \quad \text{and} \quad x - 3 \geq 0$$

$$\text{and} \quad 7 - x \geq x - 3$$

$$\Rightarrow x \leq 7 \quad \text{and} \quad x \geq 3$$

$$\text{and} \quad x \leq 5$$

$$\therefore D_f = \{3, 4, 5\}$$

$$\text{Now,} \quad f(3) = {}^4P_0 = 1$$

$$f(4) = {}^3P_1 = 3$$

$$\text{and} \quad f(5) = {}^2P_2 = 2$$

$$\therefore R_f = \{1, 2, 3\}$$

20. Total number of ways in which all letters can be arranged = 6!

There are two vowels (A, E) in the word GARDEN. Total number of ways in which these two vowels can be arranged = 2!

$\therefore$ Total number of required ways

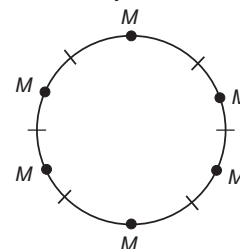
$$= \frac{6!}{2!} = 360$$

21. Required number of ways = ${}^{8-1}C_{3-1} = {}^7C_2 = \frac{7!}{2!5!}$

$$= \frac{7 \cdot 6}{2 \cdot 1} = 21$$

22. First, we fix the position of men, the number of ways in which men can sit = 5!

Now, the number of ways in which women can sit = ${}^6P_5$



$$\begin{aligned} \therefore \text{Total number of ways} &= 5! \times {}^6P_5 \\ &= 5! \times 6! \end{aligned}$$

23. Now, ${}^nC_{r+1} + {}^nC_{r-1} + 2 {}^nC_r$
 $= {}^nC_{r+1} + {}^nC_r + {}^nC_{r-1} + {}^nC_r$
 $= {}^{n+1}C_{r+1} + {}^{n+1}C_r = {}^{n+2}C_{r+1}$

24. Since, telephone number start with 67, so two digits is already fixed. Now, we have to arrange three digits from remaining eight digits i.e. 0, 1, 2, 3, 4, 5, 8, 9 in ${}^8P_3$ ways
 $= \frac{8!}{5!} = 8 \times 7 \times 6 = 336$ ways

Thus, required number of telephone numbers = 336

25. Number of ways
 $= {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \dots + {}^{2n+1}C_n$
 $= \frac{1}{2} (2^{2n+1}) = 2^{2n}$

Thus, $2^{2n} = 64 \Rightarrow 2^{2n} = 2^6$

On comparing, $2n = 6$

$\therefore n = 3$

26. Set $S = \{1, 2, 3, \dots, 20\}$ is to be partitioned into four sets of equal size i.e. having 5-5 numbers. The number of ways in which it can be done

$$= \frac{{}^{20}C_5 \times {}^{15}C_5 \times {}^{10}C_5 \times {}^5C_5}{15! \times 5! \times 10! \times 5! \times 5! \times 5!} \times 1$$

$$= \frac{(20!)}{(5!)^4}$$

27. Total number of available courses = 9

Out of these, 5 courses can be chosen. But it is given that 2 courses are compulsory for every student i.e. you have to choose only 3 courses instead of 5, out of 7 instead of 9.

It can be done in ${}^7C_3$ ways = $\frac{7 \times 6 \times 5}{6}$
 $= 35$ ways

28. There may be two cases.

Case I Two children get none and one get three and others get one each. Then, total number of ways

$$= \frac{10!}{2! \times 3! \times 7!} \times 10!$$

Case II Two get none and two get 2 each and the others get one each. Then, total number of ways

$$= \frac{10!}{(2!)^4 \times 6!} \times 10!$$

Hence, total number of ways

$$= \frac{(10!)^2}{2! \times 3! \times 7!} + \frac{(10!)^2}{(2!)^4 \times 6!}$$

$$= \frac{(10!)^2 \times 25}{(2!)^2 \times 6! \times 84}$$

29. 3 consonants can be selected from 7 consonants in ${}^7C_3$ ways.

2 vowels can be selected from 4 vowels in ${}^4C_2$ ways.

$\therefore$ Required number of words = ${}^7C_3 \times {}^4C_2 \times 5!$

$$\left[\begin{array}{l} \text{selected 5! letters can be arranged} \\ \text{in 5! ways, to get different words} \end{array} \right]$$

$$= 35 \times 6 \times 120 = 25200$$

30. We have, ${}^mC_{r+1} + \sum_{k=m}^n {}^kC_r = ({}^mC_{r+1} + {}^mC_r) + {}^{m+1}C_r$
 $+ {}^{m+2}C_r + \dots + {}^nC_r$
 $= ({}^{m+1}C_{r+1} + {}^{m+1}C_r) + {}^{m+2}C_r + \dots + {}^nC_r$
 $= {}^{m+2}C_{r+1} + {}^{m+2}C_r + \dots + {}^nC_r$
 $\vdots$
 $= {}^{n+1}C_{r+1}$

31. We have, $\frac{C_1}{C_0} + 2 \frac{C_2}{C_1} + 3 \frac{C_3}{C_2} + \dots + n \frac{C_n}{C_{n-1}}$
 $= \frac{{}^nC_1}{{}^nC_0} + 2 \frac{{}^nC_2}{{}^nC_1} + 3 \frac{{}^nC_3}{{}^nC_2} + \dots + n \frac{{}^nC_n}{{}^nC_{n-1}}$
 $= \frac{n(n-1)}{1} + 2 \times \frac{n(n-1)}{2} + 3 \times \frac{n(n-1)(n-2)}{2} + \dots + n \times \frac{1}{n}$
 $= n + (n-1) + (n-2) + \dots + 1 = \Sigma n = \frac{n(n+1)}{2}$

32. We know, ${}^{11}C_n$ is maximum when $n = 5$.

$$\therefore {}^{11}C_n = \frac{11!}{n!(11-n)!}$$

$\therefore n!(11-n)!$ is minimum, when $n = 5$.

33. Given, ${}^nC_{10} = {}^nC_{11}$

$$\Rightarrow \frac{n!}{10!(n-10)!} = \frac{n!}{11!(n-11)!}$$

$$\Rightarrow 11 \times 10!(n-11)! = 10!(n-10)(n-11)!$$

$$\Rightarrow 11 = n - 10 \Rightarrow n = 11 + 10 = 21$$

Now, ${}^nC_{21} = {}^{21}C_{21} = 1$

34. Given, ${}^{2n}C_2 : {}^nC_2 = 9 : 2 \Rightarrow \frac{{}^{2n}C_2}{{}^nC_2} = \frac{9}{2}$

$$\Rightarrow \frac{(2n)!}{2!(2n-2)!} \times \frac{2!(n-2)!}{n!} = \frac{9}{2}$$

$$\Rightarrow \frac{(2n)(2n-1)(2n-2)!}{2(2n-2)!} \times \frac{2(n-2)!}{n(n-1)(n-2)!} = \frac{9}{2}$$

$$\Rightarrow \frac{2(2n-1)}{n-1} = \frac{9}{2} \Rightarrow 4(2n-1) = 9(n-1)$$

$$\Rightarrow 8n - 4 = 9n - 9 \Rightarrow 9n - 8n = -4 + 9$$

$$\therefore n = 5$$

35. Total number of points in a plane is 30.

Out of them, 8 points are collinear.

$\therefore$ Total number of straight lines formed

$$= {}^{30}C_2 - {}^8C_2 + 1 = \frac{30 \times 29}{2} - \frac{8 \times 7}{2 \times 1} + 1$$

$$= 435 - 28 + 1 = 408$$

36. We have, 0, 1, 2, 3, 4, 5 i.e. six numbers, one zero and 5 positive numbers.

One-digit number = ${}^5P_1 = 5$

Two-digit numbers = ${}^6P_2 - {}^5P_1 = 25$

Three-digit numbers = ${}^6P_3 - {}^5P_2 = 100$

Four-digit numbers = ${}^6P_4 - {}^5P_3 = 300$

Five-digit numbers = ${}^6P_5 - {}^5P_4 = 600$

Six-digit numbers = ${}^6P_6 - {}^5P_5 = 600$

$\therefore$ Total possible numbers

$$= 5 + 25 + 100 + 300 + 600 + 600 = 1630$$

37. First, we fix the 5 plus '+' in five alternate positions.

$$\dots + \dots + \dots + \dots + \dots + \dots$$

Now, select the three positions for three 'minus' out of 6 blank positions.

∴ Required number of ways

$$= 1 \times \frac{{}^6P_3}{3!} = 1 \times {}^6C_3 = 1 \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1} = 1 \times 5 \times 4 = 20$$

38. **Case I** When he chooses 4 questions from first five questions.

$$\begin{aligned} \text{Now, the number of choices} &= {}^5C_4 \times {}^8C_6 \\ &= 5 \times \frac{8 \times 7}{2 \times 1} = 140 \end{aligned}$$

Case II When he chooses 5 questions from first five questions.

$$\begin{aligned} \text{Now, the number of choices} &= {}^5C_5 \times {}^8C_3 = 1 \times {}^8C_3 = \frac{8 \times 7 \times 6}{3 \times 2 \times 1} = 56 \end{aligned}$$

Now, the total number of choices available
= 140 + 56 = 196

39. **Case I** When 3 drawn balls contain 1 black ball

$$\begin{aligned} &= {}^3C_1 \cdot {}^2C_2 \cdot {}^4C_0 + {}^3C_1 \cdot {}^2C_1 \cdot {}^4C_1 + {}^3C_1 \cdot {}^4C_2 \cdot {}^2C_0 \\ &= 3 \cdot 1 + 3 \cdot 2 \cdot 4 + 3 \cdot 6 = 3 + 24 + 18 = 45 \end{aligned}$$

Case II When 3 drawn balls contains 2 black balls

$$\begin{aligned} &= {}^3C_2 \cdot {}^2C_1 \cdot {}^4C_0 + {}^3C_2 \cdot {}^4C_1 \cdot {}^2C_0 \\ &= 3 \cdot 2 \cdot 1 + 3 \cdot 4 \cdot 1 = 6 + 12 = 18 \end{aligned}$$

Case III When 3 drawn balls contain all black balls

$$= {}^3C_3 \cdot {}^2C_0 \cdot {}^4C_0 = 1$$

On adding all three cases, we get required ways
= 45 + 18 + 1 = 64

Aliter

Case I When 3 drawn balls contains 1 black ball

$$= {}^6C_2 \times {}^3C_1 = \frac{6 \times 5}{2 \times 1} \times 3 = 45$$

Case II When 3 drawn balls contain 2 black balls

$$= {}^6C_1 \times {}^3C_2 = 6 \times \frac{3 \times 2}{2 \times 1} = 18$$

Case III When 3 drawn balls contain all black balls

$$= {}^3C_3 = 1$$

On adding all three cases, we get

$$\text{Required ways} = 45 + 18 + 1 = 64$$

40. Four speakers will address the meeting in 4! ways = 24 different ways in which half number of cases will be such that P speaks before Q and half number of cases will be such that P speaks after Q.

$$\therefore \text{Required number of ways} = \frac{24}{2} = 12$$

41. Required sum = 3!(1 + 2 + 3 + 4) = 6(10) = 60

42. Number of ways of selecting 3 children from n children = ${}^nC_3$

Since, the number of times she will go to the garden = 84 × (Number of ways that a particular child goes to the zoo)

$$\Rightarrow {}^nC_3 = 84 \times (1 \times {}^{n-1}C_2) \Rightarrow {}^nC_3 = 84 \times \frac{(n-1)(n-2)}{2}$$

$$\Rightarrow \frac{n(n-1)(n-2)}{3 \times 2 \times 1} = 84 \times \frac{(n-1)(n-2)}{2}$$

$$\Rightarrow \frac{n}{3} = 84$$

$$\therefore n = 252$$

43. Since, the student is allowed to select atmost n books out of (2n + 1) books. Therefore, in order to select atleast one book, he has the choice to select one, two, three, ..., n books. Thus, if T is the total number of ways of selecting atleast one book, then

$$T = {}^{2n+1}C_1 + {}^{2n+1}C_2 + \dots + {}^{2n+1}C_n = 255 \quad \dots (i)$$

$$\begin{aligned} \text{Now, } {}^{2n+1}C_0 + {}^{2n+1}C_1 + \dots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} \\ + {}^{2n+1}C_{n+2} + \dots + {}^{2n+1}C_{2n+1} \end{aligned}$$

$$= (1+1)^{2n+1} = 2^{2n+1}$$

$$\therefore {}^{2n+1}C_0 + 2({}^{2n+1}C_1 + \dots + {}^{2n+1}C_n) + {}^{2n+1}C_{2n+1} = 2^{2n+1}$$

$$\Rightarrow 1 + 2T + 1 = 2^{2n+1}$$

$$\Rightarrow 1 + 2(255) + 1 = 2^{2n+1} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 255 + 1 = 2^{2n} \Rightarrow 256 = 2^{2n} \Rightarrow 4^4 = 4^n$$

$$\therefore n = 4$$

44. Clearly, first prize can be distributed among 4 students in ${}^4C_1 = 4$ ways

and second prize can be distributed among 4 students in ${}^4C_1 = 4$ ways.

Similarly, 3rd, 4th and 5th prizes can be distributed in 4 ways each.

Hence, by fundamental principle of counting, the total number of ways = $4 \times 4 \times 4 \times 4 \times 4 = 4^5 = 1024$

45. Required number of ways = $\frac{4!}{2!2!} \times \frac{3!}{2!} = 18$

46. The number of integers greater than 6000
= $5! + (3 \times 4 \times 3 \times 2) = 192$

$$47. \therefore \frac{{}^{56}P_{r+6}}{{}^{54}P_{r+3}} = 30800$$

$$\therefore \frac{56!}{54!} \times \frac{(51-r)!}{(50-r)!} = 30800$$

$$\Rightarrow 56 \times 55 \times (51-r) = 56 \times 55 \times 10$$

$$\Rightarrow 51-r = 10$$

$$\therefore r = 41$$

48. The vowels in the word COMBINE are O, I, E which can be arranged at 4 places in ${}^4P_3$ ways and other words can be arranged in 4! ways.

Hence, total number of ways

$$= {}^4P_3 \times 4! = 4! \times 4! = 576$$

49. Required number of ways

$$= {}^{11}C_5 - {}^{11}C_4 = \frac{11!}{5!6!} - \frac{11!}{4!7!} = 132$$

50. ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$

$$\Rightarrow {}^nC_4 > {}^nC_3 \quad [\because {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}]$$

$$\Rightarrow \frac{n!}{(n-4)!4!} > \frac{n!}{(n-3)!3!}$$

$$\Rightarrow (n-3)(n-4)! > (n-4)!4$$

$$\therefore n > 7$$

7

Mathematical Induction

Introduction

There are two basic processes of reasoning.

Deduction (result is deduced from general to particular) and **induction** (result is generalised from a particular result).

Deduction is the application of a general case to a particular case i.e. given a statement to be proven, often called a conjecture or a theorem in Mathematics, valid deductive steps are derived and a proof may or may not be established.

In contrast to deduction, induction depends on working with each case and developing a conjecture by observing incidences till each and every case is observed.

Induction is a key aspect of scientific reasoning, where collecting and analysing data is the norm. Thus, induction means the generalisation from particular cases or facts.

In algebra or in other disciplines of Mathematics, there are certain results or statements that are formulated in terms of n (where, n is a positive integer).

To prove such statements, the well suited principles that is used based on the specific technique, is the principle of mathematical induction.

Statement

A sentence or description which can be judged to be true or false, is called a statement.

e.g. (i) 3 divides 9.

(ii) Jaipur is the capital of Rajasthan.

A statement involving mathematical relations is known as the mathematical statement. e.g. 3 divides 9.

Principle of Mathematical Induction

Suppose there is a given statement $P(n)$ involving the natural number n , such that

- (i) statement is true for $n = 1$ i.e. $P(1)$ is true and
- (ii) if the statement is true for $n = k$ (where, k is some positive integer), then statement is also true for $n = k + 1$ i.e. truth of $P(k)$ implies the truth of $P(k + 1)$.

Chapter Snapshot

- Introduction
- Statement
- Principle of Mathematical Induction
- Algorithm for Mathematical Induction
- Types of Problems

Then, $P(n)$ is true for all natural numbers n .

- i. It is simply a statement of fact. There may be a situation when a statement is true for all $n \geq 4$. In this case, (i) will start from $n = 4$ and we shall verify the result for $n = 4$ i.e. $P(4)$.
- ii. It is a conditional property, it does not assert that the given statement is true for $n = k$ but only that, if it is true for $n = k$, then it is also true for $n = k + 1$. So, to prove that property holds, only prove that conditional proposition.

✎ The step 'if the statement is true for $n = k$, then it is also true for $n = k + 1$ ' is sometimes referred as the inductive step and the assumption in this step that statement is true for $n = k$ is called the inductive hypothesis.

Algorithm for Mathematical Induction

The steps used to prove that a statement is true for all natural numbers by using principle of mathematical induction are given below:

- Step I** Consider the given statement as $P(n)$.
- Step II** Put $n = 1$, then verify that it is a true statement.
- Step III** Suppose that the statement is true, for $n = k$ (any arbitrary number).
- Step IV** Show that the statement is true for $n = k + 1$.
Combining the results of steps II and IV conclude by principle of mathematical induction that $P(n)$ is true for all $n \in N$.

✎ This algorithm for principle of mathematical induction is used to solve problems which can be categorised into the following types.

Types of Problems

Identity Type Problems

In this type of problems, we use principle of mathematical induction to show LHS is equal to RHS.

✎ **Example 1.** $1 + 2 + 3 + \dots + n$ is equal to

$$(a) \frac{n(n-1)}{2} \quad (b) \frac{n(n+1)}{2} \quad (c) \frac{n(n+1)}{3} \quad (d) \frac{n(n-1)}{3}$$

Sol. (b) By the help of AP, we have

$$P(n) : 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Put $n = 1$, LHS = 1

$$\text{and} \quad \text{RHS} = \frac{1(1+1)}{2} = 1 = \text{LHS}$$

Therefore, $P(1)$ is true.

Suppose that the statement is true, for $n = k$, $k \in N$.

$$\therefore P(k) : 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2} \quad \dots (i)$$

Now, show that the result is true, for $n = k + 1$.

$$P(k+1) : 1 + 2 + 3 + \dots + k + (k+1) \\ = \frac{(k+1)(k+2)}{2}$$

$$\text{Then,} \quad \text{LHS} = 1 + 2 + 3 + \dots + k + (k+1) \\ = \frac{k(k+1)}{2} + (k+1) \quad [\text{from Eq. (i)}] \\ = \left(\frac{k+1}{2} \right) (k+2) = \text{RHS}$$

Therefore, $P(k+1)$ is true, whenever $P(k)$ is true.

Hence, from the principle of mathematical induction, the statement is true for all natural numbers.

✎ **Example 2.** For a natural number n , which one is the correct statement?

- (a) $1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$
- (b) $1^3 + 2^3 + 3^3 + \dots + n^3 > (1 + 2 + 3 + \dots + n)^2$
- (c) $1^3 + 2^3 + 3^3 + \dots + n^3 < (1 + 2 + 3 + \dots + n)^2$
- (d) $1^3 + 2^3 + 3^3 + \dots + n^3 \neq (1 + 2 + 3 + \dots + n)^2$

$$\text{Sol. (a) We have, } 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2 \\ = (1 + 2 + 3 + \dots + n)^2$$

Aliter

$$\text{Let } P(n) : 1^3 + 2^3 + 3^3 + \dots + n^3 = (1 + 2 + 3 + \dots + n)^2$$

For $n = 1$, we have

$$P(1) : 1^3 = 1^2 \Rightarrow 1 = 1, \text{ which is true and for } n = 2, \text{ we have}$$

$$P(2) : 1^3 + 2^3 = (1 + 2)^2 \Rightarrow 9 = 9, \text{ which is true.}$$

Similarly, we can prove that $P(n)$ is true for all $n \in N$.

Divisibility Type Problems

In this type of problems, we use principle of mathematical induction to show that given statement $P(n)$ is divisible by given number or $P(n)$ is a multiple of given number.

This process can also be explained with the following examples:

✎ **Example 3.** The greatest positive integer, which divides $(n+2)(n+3)(n+4)(n+5)(n+6)$ for all $n \in N$, is

- (a) 4
- (b) 120
- (c) 240
- (d) 24

$$\text{Sol. (b) Let } P(n) : (n+2)(n+3)(n+4)(n+5)(n+6)$$

$$\text{Put } n = 1, \quad P(1) = (1+2)(1+3)(1+4)(1+5)(1+6) \\ = 3 \times 4 \times 5 \times 6 \times 7 = 120 \times 21$$

$$\text{Put } n = 2, \quad P(2) = (2+2)(2+3)(2+4)(2+5)(2+6) \\ = 4 \times 5 \times 6 \times 7 \times 8 = 120 \times 56 \text{ and so}$$

on.

Hence, $P(n)$ is always divisible by 120.

Aliter

$\therefore n$ consecutive positive integer is divided by $n!$.

$$\therefore 5 \text{ consecutive positive integer is divided by } 5! \\ = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

➤ **Example 4.** If $n \in N$, then $11^{n+2} + 12^{2n+1}$ is divisible by

- (a) 113 (b) 123
(c) 133 (d) None of these

Sol. (c) Putting $n = 1$ in $11^{n+2} + 12^{2n+1}$, we get
 $11^3 + 12^3 = 3059$, which is divisible by 133.

Inequality Type Problems

In this type of problems, we have inequality symbols like $>$, $<$, $\geq$ or $\leq$ and use the principle of mathematical induction to show that LHS and RHS satisfies the given inequality.

➤ **Example 5.** For each $n \in N$, the correct statement is

- (a) $2^n < n$
(b) $n^2 > 2n$
(c) $n^4 < 10^n$
(d) $2^{3n} > 7n + 1$

Sol. (c) Let $n = 1$, then options (a), (b) and (d) do not satisfy.
Only option (c) satisfied.

➤ **Example 6.** For any natural number n , $2^n (n-1)! < n^n$, if

- (a) $n < 2$ (b) $n > 2$ (c) $n \geq 2$ (d) Never

Sol. (b) Check through option, the condition $2^n (n-1)! < n^n$ is satisfied for $n > 2$.

Recursion Type Problems

In this type of problems, we get a sequence in which later terms are deduced from earlier ones by using the principle of mathematical induction.

This process can be explained by following examples:

➤ **Example 7.** If $u_{n+1} = 3u_n - 2u_{n-1}$ and $u_0 = 2$, $u_1 = 3$, then u_n is equal to

- (a) $1 - 2^n$ (b) $2^n + 1$ (c) $2^n - 1$ (d) $2^n + 2$

Sol. (b) We have, $u_{n+1} = 3u_n - 2u_{n-1}$... (i)

Step I Given, $u_{1+1} = 3 = 2 + 1 = 2^1 + 1$

which is true, for $n = 1$.

Put $n = 1$ in Eq. (i), then $u_{1+1} = 3u_1 - 2u_{1-1}$

$$\Rightarrow u_2 = 3u_1 - 2u_0 = 3 \cdot 3 - 2 \cdot 2 = 5 = 2^2 + 1$$

which is true, for $n = 2$.

Therefore, the result are true, for $n = 1$ and $n = 2$.

Step II Assume it is true for $n = k$, then it is also true for $n = k - 1$.

Then, $u_k = 2^k + 1$... (ii)

and $u_{k-1} = 2^{k-1} + 1$... (iii)

Step III On putting $n = k$ in Eq. (i), we get

$$\begin{aligned} u_{k+1} &= 3u_k - 2u_{k-1} \\ &= 3(2^k + 1) - 2(2^{k-1} + 1) \quad [\text{from Eqs. (ii) and (iii)}] \\ &= 3 \cdot 2^k + 3 - 2 \cdot 2^{k-1} - 2 \\ &= 3 \cdot 2^k + 3 - 2^k - 2 = (3 - 1)2^k + 1 \\ &= 2 \cdot 2^k + 1 = 2^{k+1} + 1 \end{aligned}$$

This shows that the result is true for $n = k + 1$, hence by the principle of mathematical induction, the result is true for all $n \in N$.

➤ **Example 8.** If $a_1 = 2$, $a_n = 5a_{n-1}$, for all natural numbers $n \geq 2$, then the value of a_4 is equal to

- (a) 150 (b) 50
(c) 200 (d) None of these

Sol. (d) We have, $a_1 = 2$

Then, $a_2 = 5a_{2-1} = 5a_1 = 10$

$$a_3 = 5a_{3-1} = 5a_2 = 5 \times 10 = 50$$

$$\therefore a_4 = 5a_{4-1} = 5a_3 = 5 \times 50 = 250$$

Work Book Exercise

1 Let $P(n)$ be a statement and let $P(n) \rightarrow P(n+1)$ is true for all natural numbers n , then $P(n)$ is true

- a for all n
b for all $n > 1$
c for all $n > m$, m being a fixed integer
d Nothing can be said

2 The value of the natural numbers n such that the inequality $2^n > 2n + 1$ is valid, is

- a for $n \geq 3$ b for $n < 3$
c for $n \in N$ d for any n

3 If $n \in N$, then $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is always divisible by

- a 25 b 35
c 45 d None of these

4 For every positive integral value of n , $3^n > n^3$ when

- a $n > 2$ b $n \geq 3$ c $n \geq 4$ d $n < 4$

5 For every positive integer n , $2^n < n!$ when

- a $n < 4$ b $n \geq 4$
c $n < 3$ d None of these

6 For every natural number n , $n(n+1)$ is always

- a even b odd
c multiple of 3 d multiple of 4

7 If n is a natural number, then $\left(\frac{n+1}{2}\right)^n > n!$ is true,

when

- a $n < 1$ b $n \geq 1$
c $n \geq 2$ d $n < 3$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The value of sum in the n th bracket of
 $(1) + (2 + 3) + (4 + 5 + 6 + 7) + (8 + 9 + 10 + \dots + 15) + \dots$, is

- (a) $2^n (2^n + 2^{n-1} - 1)$
 (b) $2^{n-1} (2^n + 2^{n-1} - 1)$
 (c) $2^{n-2} (2^n + 2^{n-1} - 1)$
 (d) None of the above

Sol. If $n = 1$, then $2^n (2^n + 2^{n-1} - 1)$
 $= 2 (2 + 1 - 1) = 2(2) = 4$,
 $2^{n-1} (2^n + 2^{n-1} - 1) = 2^0 (2 + 2^0 - 1) = (2 + 1 - 1) = 2$
 and $2^{n-2} (2^n + 2^{n-1} - 1) = 2^{-1} (2^1 + 2^0 - 1)$
 $= 2^{-1} (2) = 1$

$\therefore$ Sum of n th bracket $= 2^{n-2} (2^n + 2^{n-1} - 1)$

Hence, (c) is the correct answer.

Ex 2. If $(1) + (2 + 3 + 4) + (5 + 6 + 7 + 8 + 9) + \dots$, then the sum of terms in the n th bracket is

- (a) $(n-1)^3 + n^3$
 (b) $(n+1)^3 + 8n^2$
 (c) $\frac{(n+1)(n+2)}{6n}$
 (d) $(n+1)^3 + n^3$

Sol. If $n = 1$, then $(n-1)^3 + n^3 = 0 + 1 = 1$,

$$(n+1)^3 + 8n^2 = 8 + 8 = 16,$$

$$\frac{(n+1)(n+2)}{6n} = \frac{6}{6} = 1,$$

and $(n+1)^3 + n^3 = 8 + 1 = 9$

If $n = 2$, then $(n-1)^3 + n^3 = 1 + 8 = 9$,

$$\frac{(n+1)(n+2)}{6n} = \frac{12}{12} = 1$$

$\therefore$ Sum of n th bracket $= (n-1)^3 + n^3$

Hence, (a) is the correct answer.

Ex 3. $(1) + (2 + 3) + (4 + 5 + 6) + \dots$ n brackets is equal to

- (a) $\frac{n(n+1)(n^2+n+2)}{8}$
 (b) $\frac{n(n+1)(n^2-n+2)}{8}$
 (c) $\frac{n(n-1)(n^2+n+2)}{8}$
 (d) $\frac{n(n-1)(n^2-n+2)}{8}$

Sol. If $n = 1$, then $\frac{n(n+1)(n^2+n+2)}{8} = \frac{2(4)}{8} = 1$

Hence, (a) is the correct answer.

Ex 4. Sum of n th bracket of

$(1) + \left(\frac{1}{3} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5}\right) + \dots$ is

- (a) $\frac{(3^n-1)^3}{2 \cdot 4^{n-1}}$ (b) $\frac{(3^n-1)}{2 \cdot 3^{(n-1)(n+2)/2}}$
 (c) $\frac{(3^n+1)}{3 \cdot 7^{n-1}}$ (d) None of these

Sol. If $n = 1$, then $\frac{(3^n-1)^3}{2 \cdot 4^{n-1}} = \frac{(2)^3}{2} = 4$;

$$\frac{(3^n-1)}{2 \cdot 3^{(n-1)(n+2)/2}} = \frac{2}{2 \cdot 3^0} = 1 \quad \text{and} \quad \frac{3^n+1}{3 \cdot 7^{n-1}} = \frac{4}{3}$$

$\therefore$ Sum of n th bracket $= \frac{(3^n-1)}{2 \cdot 3^{(n-1)(n+2)/2}}$

Aliter

n th bracket

$$= \frac{1}{3^{n(n-1)/2}} + \dots + \frac{1}{3^{(n+1)n/2-1}} = \frac{1}{3^{n(n-1)/2}} \left[\frac{1-1/3^n}{1-1/3} \right]$$

$$= \left[\frac{3(3^n-1)}{2 \cdot 3^n 3^{n(n-1)/2}} \right] = \frac{3^n-1}{2 \cdot 3^{(n-1)(n+2)/2}}$$

Hence, (b) is the correct answer.

Ex 5. The sum of $(1^2) + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + \dots$ n brackets is equal to

- (a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)^2(n+2)}{12}$
 (c) $n(n+1)(n+2)$ (d) $\frac{n(n+1)(2n+1)}{6}$

Sol. $\therefore t_n = 1^2 + 2^2 + \dots + n^2$

$$= \frac{n(n+1)(2n+1)}{6} = \frac{n(2n^2+3n+1)}{6}$$

$$= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}$$

$$\therefore S_n = \sum \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right)$$

$$= \frac{n^2(n+1)^2}{12} + \frac{n(n+1)(2n+1)}{12} + \frac{n(n+1)}{12}$$

$$= \left[\frac{n(n+1)}{12} \right] [n(n+1) + 2n + 1 + 1]$$

$$= \frac{n(n+1)(n^2+3n+2)}{12} = \frac{n(n+1)^2(n+2)}{12}$$

Hence, (b) is the correct answer.

Ex 6. $(1) + (1+3) + (1+3+5) + \dots$ n brackets is equal to

- (a) $\frac{n(n+1)(n+2)}{6}$
 (b) $\frac{n(n+1)(3n^2+23n+46)}{12}$
 (c) $\frac{n(27n^3+90n^2+45n-50)}{4}$
 (d) $\frac{n(n+1)(2n+1)}{6}$

Sol. n th bracket $= 1 + 3 + 5 + \dots + (2n-1) = n^2$

$$\therefore S_n = \Sigma n^2 = \frac{n(n+1)(2n+1)}{6}$$

Hence, (d) is the correct answer.

Ex 7. $(1) + (1+2) + (1+2+3) + \dots$ n brackets is equal to

- (a) $\frac{n(n+1)(n+2)}{6}$
 (b) $\frac{n(n+1)(3n^2+23n+46)}{12}$
 (c) $\frac{n(27n^3+90n^2+45n-50)}{4}$
 (d) $\frac{n(n+1)(2n+1)}{16}$

Sol. If $n = 1$, then the sum of the terms $= (1) = 1$,

$$\therefore \frac{n(n+1)(n+2)}{6} = \frac{1 \times 2 \times 3}{6} = 1$$

Hence, (a) is the correct answer.

Ex 8. $1 \cdot 3 \cdot 4 + 2 \cdot 4 \cdot 5 + 3 \cdot 5 \cdot 6 + \dots$ upto n terms is equal to

- (a) $\frac{n(n+1)(n+2)}{6}$
 (b) $\frac{n(n+1)(3n^2+23n+46)}{12}$
 (c) $\frac{n(27n^3+90n^2+45n-50)}{4}$
 (d) $\frac{n(n+1)(2n+1)}{6}$

Sol. Given expression

$$\begin{aligned} \Sigma n(n+2)(n+3) &= \Sigma (n^3 + 5n^2 + 6n) \\ &= \frac{n^2(n+1)^2}{4} + \frac{5n(n+1)(2n+1)}{6} + \frac{6n(n+1)}{2} \\ &= \frac{n(n+1)}{12} [3n(n+1) + 10(2n+1) + 36] \\ &= \frac{n(n+1)(3n^2+23n+46)}{12} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 9. $2 \cdot 1^2 + 3 \cdot 2^2 + 4 \cdot 3^2 + \dots + (n+1)n^2$ is equal to

- (a) $\frac{n(n+1)(n+2)(3n+5)}{12}$
 (b) $\frac{n(n+1)(n+2)(n+3)}{4}$
 (c) $\frac{2n(n+1)(n+2)(n+3)}{12}$
 (d) $\frac{n(n+1)(n+2)(3n+1)}{12}$

$$\begin{aligned} \text{Sol. } \therefore t_n &= (n+1)n^2 = n^3 + n^2 \\ &= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \\ &= \frac{n(n+1)}{12} [3n(n+1) + 2(2n+1)] \\ &= \frac{n(n+1)(3n^2+7n+2)}{12} \\ &= \frac{n(n+1)(n+2)(3n+1)}{12} \end{aligned}$$

Hence, (d) is the correct answer.

Ex 10. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots + n(n+1)^2$ is equal to

- (a) $\frac{n(n+1)(n+2)(3n+5)}{12}$
 (b) $\frac{n(n+1)(n+2)(n+3)}{4}$
 (c) $\frac{2n(n+1)(n+2)(n+3)}{12}$
 (d) $\frac{n(n+1)(n+2)(3n+1)}{12}$

$$\begin{aligned} \text{Sol. } \therefore t_n &= n(n+1)^2 = n(n^2+2n+1) = n^3+2n^2+n \\ \therefore S_n &= \Sigma n^3 + 2\Sigma n^2 + \Sigma n \\ &= \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \frac{n(n+1)}{12} [3n(n+1) + 4(2n+1) + 6] \\ &= \frac{n(n+1)(3n^2+11n+10)}{12} \\ &= \frac{n(n+1)(n+2)(3n+5)}{12} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 11. $\frac{1^2}{3} + \frac{1^2+2^2}{5} + \frac{1^2+2^2+3^2}{7} + \dots$ upto n terms is equal to

- (a) $\frac{n(n+1)(n+2)}{18}$ (b) $\frac{n(n+1)(n+2)}{6}$
 (c) $\frac{n(n+1)(n+2)}{3}$ (d) $\frac{2n(n+1)(n+2)}{3}$

$$\begin{aligned} \text{Sol. } \therefore t_n &= \frac{1^2+2^2+\dots+n^2}{2n+1} = \frac{n(n+1)}{6} \\ \therefore S_n &= \Sigma \frac{(n^2+n)}{6} = \frac{1}{6} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right] \\ &= \frac{n(n+1)(n+2)}{18} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 12. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots$ upto n terms is equal to

- (a) $\frac{n(n+1)(n+5)}{3}$ (b) $\frac{n(n+1)(n+2)}{3}$
 (c) $\frac{n(4n^2 + 6n - 1)}{3}$ (d) $n(n+1)(n+2)$

Sol. If $n = 1$, then the sum of the terms $= 1 \cdot 2 = 2$

$$\frac{n(n+1)(n+2)}{3} = \frac{1 \times 2 \times 3}{3} = 2$$

Hence, (b) is the correct answer.

Ex 13. $1 \cdot 3 + 2 \cdot 3^2 + 3 \cdot 3^3 + \dots + n \cdot 3^n$ is equal to

- (a) $\frac{(2n-1)3^{n+1} + 3}{4}$ (b) $\frac{(2n+1)3^{n+1} + 3}{4}$
 (c) $\frac{(2n+1)3^{n+1} - 3}{4}$ (d) $\frac{(2n-1)3^{n+1} - 3}{4}$

Sol. If $n = 1$, then

$$\frac{(2n-1)3^{n+1} + 3}{4} = 3, \frac{(2n+1)3^{n+1} + 3}{4} = \frac{15}{2},$$

$$\frac{(2n+1)3^{n+1} - 3}{4} = 6, \frac{(2n-1)3^{n+1} - 3}{4} = \frac{3}{2}$$

$$\therefore \text{Sum} = \frac{(2n-1)3^{n+1} + 3}{4}$$

Hence, (a) is the correct answer.

Ex 14. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n!$ is equal to

- (a) $(n+1)! - 1$
 (b) $(n-1)! + 1$
 (c) $(n+1)! + 1$
 (d) $(n-1)! - 1$

Sol. If $n = 1$, then $(n+1)! - 1 = 2 - 1 = 1$, $(n-1)! + 1 = 2$,
 $(n+1)! + 1 = 2 + 1 = 3$, $(n-1)! - 1 = 1 - 1 = 0$

$$\therefore \text{Sum} = (n+1)! - 1$$

Hence, (a) is the correct answer.

Ex 15. $1 \cdot 3 + 3 \cdot 5 + 5 \cdot 7 + \dots$ upto n terms is equal to

- (a) $\frac{n(n+1)(n+5)}{3}$
 (b) $\frac{n(n+1)(n+2)}{3}$
 (c) $\frac{n(4n^2 + 6n - 1)}{3}$
 (d) $n(n+1)(n+2)$

Sol. If $n = 1$, then $\frac{n(n+1)(n+5)}{3} = 4$, $\frac{n(n+1)(n+2)}{3} = 2$,

$$\frac{n(4n^2 + 6n - 1)}{3} = 3, n(n+1)(n+2) = 6$$

Hence, (c) is the correct answer.

Ex 16. $1 + 4 + 13 + 40 + \dots$ upto n terms is equal to

- (a) $\frac{3^{n+1} - 2^n}{2n}$ (b) $\frac{(3^{n+1} - 2n - 3)}{4}$
 (c) $\frac{3n - 1 + 3^n}{9}$ (d) $\frac{(3^{n-1} + 2n^2)}{8}$

Sol.

$$S_n = 1 + 4 + 13 + 40 + \dots + t_n$$

$$S_n = 1 + 4 + 13 + \dots + t_{n+1} + t_n$$

$$\Rightarrow 0 = 1 + 3 + 9 + 27 + \dots - t_n$$

$$\Rightarrow t_n = \frac{(3^n - 1)}{(3 - 1)} = \frac{(3^n - 1)}{2}$$

$$\therefore S_n = \sum \frac{(3^n - 1)}{2}$$

$$= \frac{3(3^n - 1)}{2(3 - 1)} - \frac{n}{2}$$

$$= \frac{(3^{n+1} - 3 - 2n)}{4}$$

Hence, (b) is the correct answer.

Ex 17. $1 + 4 + 10 + 19 + \dots + \frac{(3n^2 - 3n + 2)}{2}$ is equal to

- (a) $\frac{n(n+1)(2n+1)}{6}$
 (b) $\frac{n^2(n+1)^2}{4}$
 (c) $\frac{n(n+1)(n+2)}{6}$
 (d) $\frac{n(n^2+1)}{2}$

Sol. Sum $= \sum \frac{(3n^2 - 3n + 2)}{2}$

$$= \frac{3}{2} \sum n^2 - \frac{3}{2} \sum n + \sum 1$$

$$= \frac{3n(n+1)(2n+1)}{12} - \frac{3n(n+1)}{4} + n$$

$$= \frac{n}{4} [2n^2 + 3n + 1 - 3n - 3 + 4]$$

$$= \frac{n}{4} [2n^2 + 2] = \frac{n(n^2 + 1)}{2}$$

Hence, (d) is the correct answer.

Ex 18. $\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \dots \left(1 - \frac{1}{n+1}\right)$ is equal to

- (a) $\frac{1}{n+1}$
 (b) $\frac{n}{n+1}$
 (c) $\frac{n}{2n+1}$
 (d) $\frac{n}{3n+1}$

Sol. If $n = 1$, then $\frac{1}{n+1} = \frac{1}{2}$, $\frac{n}{n+1} = \frac{1}{2}$,

$$\frac{n}{2n+1} = \frac{1}{3}, \frac{n}{3n+1} = \frac{1}{4}$$

$$\text{If } n = 2, \text{ then } \frac{1}{n+1} = \frac{1}{3}, \frac{n}{n+1} = \frac{2}{3}$$

$$\therefore \text{Sum} = \frac{n}{n+1}$$

Hence, (b) is the correct answer.

Target Exercises

Type 1. Only One Correct Option

- $\frac{3}{4} + \frac{15}{16} + \frac{63}{64} + \dots$ n terms is equal to
 - $n - \frac{1}{3}4^n - \frac{1}{3}$
 - $n + \frac{1}{3}4^{-n} - \frac{1}{3}$
 - $n + \frac{1}{3}4^n - \frac{1}{3}$
 - $n - \frac{1}{3}4^{-n} + \frac{1}{3}$
- $\frac{1}{1^3} + \frac{2}{1^3 + 2^3} + \frac{2 \cdot 3}{1^3 + 2^3 + 3^3} + \frac{3 \cdot 4}{2 \cdot 2} + \dots$ n terms is equal to
 - $\frac{n^2}{(n+1)^2}$
 - $\frac{n^3}{(n+1)^3}$
 - $\frac{n}{n+1}$
 - $\frac{1}{n+1}$
- $\Sigma \frac{1^2 + 2^2 + 3^2 + \dots + n^2}{1 + 2 + 3 + \dots + n}$ is equal to
 - $\frac{(n^2 + 2n)}{3}$
 - $\frac{n^2 - 2n}{6}$
 - $\frac{n^2 + 11}{12n}$
 - None of these
- $\frac{1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + \dots + n(n+1)^2}{1^2 \cdot 2 + 2^2 \cdot 3 + 3^2 \cdot 4 + \dots + n^2(n+1)}$ is equal to
 - $\frac{3n+5}{3n+1}$
 - $\frac{3n+1}{3n+5}$
 - $(3n+1)(3n+5)$
 - None of these
- $\frac{1^4}{1 \cdot 3} + \frac{2^4}{3 \cdot 5} + \frac{3^4}{5 \cdot 7} + \dots + \frac{n^4}{(2n+1)(2n-1)}$ is equal to
 - $\frac{n(n+1)(n+2)}{6n}$
 - $\frac{n(n+1)(n^2+n+1)}{6(2n+1)}$
 - $n(n+2)(n+3)^2$
 - None of the above
- $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots$ n terms is equal to
 - $\frac{1}{n+1}$
 - $\frac{n}{n+1}$
 - $\frac{n}{2n+1}$
 - $\frac{n}{3n+1}$
- $\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots$ n terms is equal to
 - $\frac{1}{n+1}$
 - $\frac{n}{n+1}$
 - $\frac{n}{2n+1}$
 - $\frac{n}{3n+1}$
- $\cos \theta \cos 2\theta \cos 4\theta \dots \cos 2^{n-1}\theta$ is equal to
 - $\frac{\sin 2^n \theta}{2^n \sin \theta}$
 - $\frac{\sin 2^n \theta}{\sin \theta}$
 - $\frac{\cos 2^n \theta}{2^n \cos \theta}$
 - $\frac{\cos 2^n \theta}{2^n \sin \theta}$
- $1 + \frac{x}{a_1} + \frac{x(x+a_1)}{a_1 a_2} + \dots + \frac{x(x+a_1)(x+a_2)\dots(x+a_{n-1})}{a_1 a_2 \dots a_n}$ is equal to
 - $\frac{(x+a_1)(x+a_2)\dots(x+a_n)}{a_1 a_2 \dots a_n}$
 - $\frac{(x-a_1)(x-a_2)\dots(x-a_n)}{a_1 a_2 \dots a_n}$
 - $(x+a_1)(x+a_2)\dots(x+a_n)$
 - $(x-a_1)(x-a_2)\dots(x-a_n)$
- Using mathematical induction, the numbers a_n 's are defined by $a_0 = 1, a_{n+1} = 3n^2 + n + a_n, (n \geq 0)$. Then, a_n is equal to
 - $n^3 + n^2 + 1$
 - $n^3 - n^2 + 1$
 - $n^3 - n^2$
 - $n^3 + n^2$
- For $n \in N, \frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n$ is
 - an integer
 - a natural number
 - a positive fraction
 - None of these
- If $n \in N$, then $x^n - y^n$ is divisible by
 - $x - 1$
 - $x - y$
 - $x + y$
 - $y - 1$
- $x^n + a^n$ is divisible by $x + a$ for n is any
 - positive integer
 - integer
 - odd positive integer
 - None of these
- $x^{n+1} + (x+1)^{2n-1}$ is divisible by
 - x
 - $x+1$
 - $x^2 + x + 1$
 - $x^2 - x + 1$
- If $n \in N$, then 2^n
 - $= n$
 - $> n$
 - $< n$
 - None of these
- If $n \in N$, then $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \dots + \frac{1}{\sqrt{n}}$
 - $= \sqrt{n}$
 - $\geq \sqrt{n}$
 - $\leq \sqrt{n}$
 - None of these
- $\frac{(2n)!}{2^{2n}(n!)^2} \leq$
 - $\frac{1}{\sqrt{3n+1}}$
 - $\frac{1}{\sqrt{3n+2}}$
 - $\frac{1}{\sqrt{3n+4}}$
 - $\frac{1}{\sqrt{3n+5}}$
- If $n > 1$, then
 - $\frac{(2n)!}{(n!)^2} = \frac{4n}{2n+1}$
 - $\frac{(2n)!}{(n!)^2} < \frac{4n}{2n+1}$
 - $\frac{(2n)!}{(n!)^2} > \frac{4n}{2n+1}$
 - None of these

Type 2. More than One Correct Option

19. If $a_1 = 1$ and $a_n = na_{n-1} + 5$ for all positive integer $n \geq 2$, then a_5 is equal to
 (a) 550
 (b) 120
 (c) $5a_4 + 5$
 (d) 24
20. $x(x^{n-1} - na^{n-1}) + a^n(n-1)$ is divisible by $(x-a)^2$ for

- (a) $n \geq 1 (n \in \mathbb{Z})$
 (b) $n > 2$
 (c) all $n \in \mathbb{N}$
 (d) None of these

21. If p is prime number, then $n^p - n$ is divisible by p , when n is
 (a) natural number greater than 1
 (b) irrational number
 (c) complex number
 (d) positive integer greater than or equal to 2

Type 3. Assertion and Reason

Directions (Q. Nos. 22-24) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

22. **Statement I** Sum of 20th group of $\{1\}$, $\{1, 2\}$, $\{1, 2, 3\}$, ... is 110.

Statement II Sum of n natural numbers is $\frac{n(n+1)}{2}$.

23. **Statement I** If n is a positive integer, then $3^{2n} + 7$ is divisible by 8.

Statement II HCF of 16 and 88 is 8.

24. **Statement I** If $n \in \mathbb{N}$, product of $n(n+1)(n+2)$ is divisible by 6.

Statement II Product of 3 consecutive positive integers is divisible by $3!$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 25-26) If $a_n = \sqrt{7 + \sqrt{7 + \sqrt{7 + \dots}}}$ having n radical signs, then by methods of mathematical induction, answer the following questions.

25. Which of the following is true?

- (a) $a_n > 7, \forall n \geq 1$
 (b) $a_n > 3, \forall n \geq 1$
 (c) $a_n < 4, \forall n \geq 1$
 (d) $a_n < 3, \forall n \geq 1$

26. The value of $a_1 + a_2 + a_3$ is

- (a) < 6
 (b) < 4
 (c) < 12
 (d) None of the above

Passage II (Q. Nos. 27-28) Consider $S(k)$ is the sum of first k odd natural numbers expressed as

$$S(k): 1 + 3 + 5 + \dots + (2k - 1)$$

27. $S(k)$ can be expressed as

- (a) k^2
 (b) $2k^2$
 (c) $2 + k^2$
 (d) None of the above

28. If $S(k)$ expressed as $3 + k^2$, then

- (a) $S(1)$ is true
 (b) $S(k) \Rightarrow S(k+1)$
 (c) $S(k) \Rightarrow S(k+1)$
 (d) None of these

Passage III (Q. Nos. 29-30) Consider $a_n = 2^{2^n} + 1$, if $n > 1$.

29. The last digit of a_2 is

- (a) 1
 (b) 2
 (c) 3
 (d) None of these

30. The last digit of a_n is

- (a) 2
 (b) 3
 (c) 7
 (d) 4

Type 5. Match the Column

31. Match the following:

Column I	Column II
A. The smallest positive integer for which the statement $3^{n+1} < 4^n$ holds is	p. 1
B. If $x^n - 1$ is divisible by $x - k$. Then, the least positive integral value of k is	q. 4
C. If $49^n + 16n + k$ is divisible by 64 for $n \in N$. Then, the numerically least negative integral value of k is	r. 2
D. For all $n \in N$, $2^{4n} - 15n - 1$ is divisible by 15^k . Then, the maximum integer value of k is	s. 3

Type 6. Single Integer Answer Type Questions

32. If $\frac{(n+2)!}{6(n-1)!}$ divisible by n , $n \in N$ and $1 \leq n \leq 9$, then n is _____.

33. If $n \in N$ and $1 < n \leq 9$, then $n^3 + 2n$ is divisible by _____.

34. The minimum value of n for which $10^n + 3 \cdot 4^{n+2} + 5$ is an integral multiple of 97, is _____.

Entrances Gallery

JEE Main/AIEEE

1. $2^{3n} - 7n - 1$ is divisible by [2014]
 (a) 64 (b) 36
 (c) 49 (d) 25

2. **Statement I** The sum of the series
 $1 + (1 + 2 + 4) + (4 + 6 + 9) + (9 + 12 + 16) + \dots + (361 + 380 + 400)$ is 8000.

Statement II $\sum_{k=1}^n [k^3 - (k-1)^3] = n^3$, for any natural number n . [2012]

- (a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (b) Statement I is true, Statement II is false
 (c) Statement I false, Statement II is true
 (d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

3. **Statement I** For each natural number n , $(n+1)^7 - n^7 - 1$ is divisible by 7.

Statement II For each natural number n , $n^7 - n$ is divisible by 7. [2011]

- (a) Statement I is incorrect, Statement II is correct
 (b) Statement I is correct, Statement II is correct; Statement II is correct explanation for Statement I
 (c) Statement I is correct, Statement II is correct; Statement II is not a correct explanation for Statement I
 (d) Statement I is correct, Statement II is incorrect

4. **Statement I** For every natural number $n \geq 2$

$$\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Statement II For every natural number $n \geq 2$

$$\sqrt{n(n+1)} < n+1. \quad [2008]$$

- (a) Statement I is correct, Statement II is correct; Statement II is a correct explanation for Statement I
 (b) Statement I is correct, Statement II is correct; Statement II is not a correct explanation for Statement I
 (c) Statement I is correct, Statement II is incorrect
 (d) Statement I is incorrect, Statement II is correct

5. If $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, then which one of the

following holds, $\forall n \geq 1$ by the principle of mathematical induction? [2005]

- (a) $A^n = 2^{n-1}A + (n-1)I$ (b) $A^n = nA + (n-1)I$
 (c) $A^n = 2^{n-1}A - (n-1)I$ (d) $A^n = nA - (n-1)I$

6. Let $S(k) = 1 + 3 + 5 + \dots + (2k-1) = 3 + k^2$.

Then, which of the following is true? [2004]

- (a) $S(1)$ is correct
 (b) Principle of mathematical induction can be used to prove the formula
 (c) $S(k) \not\Rightarrow S(k+1)$
 (d) $S(k) \Rightarrow S(k+1)$

7. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \dots + n$ terms is equal to [2002]

- (a) $\frac{n(n+1)(n+2)(3n+5)}{12}$ (b) $\frac{n(n+1)(n+2)(n+3)}{4}$
 (c) $2n(n+1)(n+2)(n+3)$ (d) $\frac{n(n+1)(n+2)(3n+1)}{12}$

Other Engineering Entrances

8. If n is a positive integer, then $n^3 + 2n$ is divisible by
[BITSAT 2014]
(a) 2 (b) 6 (c) 15 (d) 3
9. $10^n \times 3(4^{n+2}) + 5$ is divisible by ($n \in N$)
[BITSAT 2014]
(a) 7 (b) 5 (c) 9 (d) 17
10. Let $n \geq 2$ be an integer,

$$A = \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
 and I is the identity matrix of order 3. Then,
[WB JEE 2014]
 (a) $A^n = I$ and $A^{n-1} \neq I$
 (b) $A^m \neq I$ for any positive integer m
 (c) A is not invertible
 (d) $A^m = 0$ for a positive integer m
11. If $\frac{1}{2 \times 4} + \frac{1}{4 \times 6} + \frac{1}{6 \times 8} + \dots n \text{ terms} = \frac{kn}{n+1}$, then
[EAMCET 2012]
 k is equal to
 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$
 (c) 1 (d) $\frac{1}{8}$
12. If a, b and n are natural numbers, then $a^{2n-1} + b^{2n-1}$ is divisible by
[EAMCET 2011]
 (a) $a + b$
 (b) $a - b$
 (c) $a^3 + b^3$
 (d) $a^2 + b^2$
13. Which of the following result is valid? [AMU 2011]
 (a) $(1+x)^n > (1+nx)$, for all natural numbers n
 (b) $(1+x)^n \geq (1+nx)$, for all natural numbers n , where $x > -1$
 (c) $(1+x)^n \leq (1+nx)$, for all natural numbers n
 (d) $(1+x)^n < (1+nx)$, for all natural numbers n
14. If n is a natural number, then [AMU 2011,13]
 (a) $1^2 + 2^2 + \dots + n^2 < \frac{n^3}{3}$
 (b) $1^2 + 2^2 + \dots + n^2 = \frac{n^3}{3}$
 (c) $1^2 + 2^2 + \dots + n^2 > n^3$
 (d) $1^2 + 2^2 + \dots + n^2 > \frac{n^3}{3}$
15. If $a_1 = 1$ and $a_n = na_{n-1}$ for all positive integers $n \geq 2$, then a_5 is equal to [Kerala CEE 2010]
 (a) 125 (b) 120
 (c) 100 (d) 115
 (e) None of these

Answers

Work Book Exercise

1. (d) 2. (a) 3. (a) 4. (c) 5. (b) 6. (a) 7. (c)

Target Exercises

1. (b) 2. (c) 3. (a) 4. (a) 5. (b) 6. (b) 7. (c) 8. (a) 9. (a) 10. (b)
 11. (b) 12. (b) 13. (c) 14. (c) 15. (b) 16. (b) 17. (a) 18. (c) 19. (a,c) 20. (a,c)
 21. (a,d) 22. (d) 23. (a) 24. (a) 25. (c) 26. (c) 27. (a) 28. (c) 29. (d) 30. (c)
 31. (*) 32. (1) 33. (3) 34. (2)

* $A \rightarrow q$; $B \rightarrow p$; $C \rightarrow p$; $D \rightarrow r$

Entrances Gallery

1. (c) 2. (d) 3. (b) 4. (b) 5. (d) 6. (d) 7. (b) 8. (d) 9. (c) 10. (a)
 11. (a) 12. (a) 13. (b) 14. (d) 15. (b)

Explanations

Target Exercises

1. If $n = 1$, then

$$\begin{aligned} n - \frac{1}{3} 4^n - \frac{1}{3} &= 1 - \frac{4}{3} - \frac{1}{3} = \frac{-2}{3}, \\ n + \frac{1}{3} 4^n - \frac{1}{3} &= 1 + \frac{4}{3} - \frac{1}{3} = \frac{9}{3} = 3, \\ n + \frac{1}{3} 4^n - \frac{1}{3} &= 1 + \frac{4}{3} - \frac{1}{3} = 2, \\ \text{and } n - \frac{1}{3} 4^n + \frac{1}{3} &= 1 - \frac{4}{3} + \frac{1}{3} = \frac{15}{3} = 5 \\ \therefore \text{Sum} &= n + \frac{1}{3} 4^n - \frac{1}{3} \end{aligned}$$

2. If $n = 1$, then

$$\begin{aligned} \frac{n^2}{(n+1)^2} &= \frac{1}{4}, \\ \frac{n^3}{(n+1)^3} &= \frac{1}{8}, \\ \frac{n}{n+1} &= \frac{1}{2}, \\ \text{and } \frac{1}{n+1} &= \frac{1}{2} \end{aligned}$$

If $n = 2$, then

$$\begin{aligned} \frac{n}{n+1} &= \frac{2}{3}, \\ \frac{1}{n+1} &= \frac{1}{3} \\ \therefore \text{Sum} &= \frac{n}{n+1} \end{aligned}$$

3. We have, $\frac{1^2 + 2^2 + 3^2 + \dots + n^2}{1 + 2 + 3 + \dots + n}$

$$\begin{aligned} &= \frac{n(n+1)(2n+1)2}{n(n+1)6} = \frac{(2n+1)}{3} \\ S_n &= 2 \frac{(\Sigma n)}{3} + \frac{n}{3} = \frac{n(n+1)}{3} + \frac{n}{3} \\ &= \frac{(n^2 + 2n)}{3} \end{aligned}$$

4. $\Sigma n(n+1)^2 = \Sigma (n^3 + 2n^2 + n)$

$$\begin{aligned} &= \frac{n^2(n+1)^2}{4} + \frac{2n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \\ &= \left[\frac{n(n+1)}{12} \right] [3n(n+1) + 4(2n+1) + 6] \\ &= \frac{n(n+1)(3n^2 + 11n + 10)}{12} \\ &= \frac{n(n+1)(n+2)(3n+5)}{12} \\ \Sigma n^2(n+1) &= \Sigma (n^3 + n^2) \\ &= \frac{n^2(n+1)^2}{4} + \frac{n(n+1)(2n+1)}{6} \\ &= \left[\frac{n(n+1)}{12} \right] [3n(n+1) + 2(2n+1)] \\ &= \frac{n(n+1)(3n^2 + 7n + 2)}{12} \end{aligned}$$

$$\begin{aligned} &= \frac{n(n+1)(n+2)(3n+1)}{12} \\ \therefore \frac{\Sigma n(n+1)^2}{\Sigma n^2(n+1)} &= \frac{\frac{n(n+1)(n+2)(3n+5)}{12}}{\frac{n(n+1)(n+2)(3n+1)}{12}} \\ &= \frac{3n+5}{3n+1} \end{aligned}$$

5. $S_n = \Sigma \frac{n^4}{(2n+1)(2n-1)}$

$$\begin{aligned} &= \Sigma \frac{n^4}{4n^2 - 1} \\ &= \Sigma \left(\frac{n^2}{4} + \frac{1}{16} + \frac{1}{16(2n+1)(2n-1)} \right) \\ &= \frac{n(n+1)(2n+1)}{24} + \frac{n}{16} + \Sigma \frac{(2n+1) - (2n-1)}{32(2n+1)(2n-1)} \\ &= \frac{n(n+1)(2n+1)}{24} + \frac{n}{16} + \frac{1}{32} \left(1 - \frac{1}{2n+1} \right) \\ &= \frac{n(n+1)(2n+1)}{24} + \frac{n}{16} + \frac{n}{16(2n+1)} \\ &= \frac{n(n+1)(2n+1)}{24} + \frac{n(2n+2)}{16(2n+1)} \\ &= \frac{n(n+1) \left[\frac{(2n+1)^2 + 3}{2n+1} \right]}{24} \\ &= \frac{n(n+1)(n^2 + n + 1)}{6(2n+1)} \end{aligned}$$

6. $t_n = \frac{1}{n(n+1)}$

$$\begin{aligned} \therefore S_n &= \frac{2-1}{1 \cdot 2} + \frac{3-2}{2 \cdot 3} + \frac{4-3}{3 \cdot 4} + \dots + \frac{(n+1)-n}{n(n+1)} \\ &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \dots + \frac{1}{n} - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} = \frac{n}{n+1} \end{aligned}$$

7. If $n = 1$, then

$$\begin{aligned} \frac{1}{n+1} &= \frac{1}{2}, \\ \frac{n}{n+1} &= \frac{1}{2}, \\ \frac{n}{2n+1} &= \frac{1}{3}, \\ \frac{n}{3n+1} &= \frac{1}{4} \\ \text{and } \therefore \text{Sum} &= \frac{n}{2n+1} \end{aligned}$$

8. If $n = 1$, then

$$\begin{aligned} \frac{\sin 2^n \theta}{2^n \sin \theta} &= \frac{\sin 2\theta}{2 \sin \theta} \\ &= \frac{2 \sin \theta \cos \theta}{2 \sin \theta} = \cos \theta \end{aligned}$$

9. If $n = 1$, then given sum $= 1 + \frac{x}{a_1} = \frac{x + a_1}{a_1}$

10. Given, $a_0 = 1, a_{n+1} = 3n^2 + n + a_n$

$$\Rightarrow a_1 = 3(0) + 0 + a_0 = 1$$

$$\Rightarrow a_2 = 3(1)^2 + 1 + a_1 = 3 + 1 + 1 = 5$$

From option (b),

Let $P(n) = n^3 - n^2 + 1$

$\therefore P(0) = 0 - 0 + 1 = 1 = a_0$

$P(1) = 1^3 - 1^2 + 1 = 1 = a_1$

and $P(2) = (2)^3 - (2)^2 + 1 = 5 = a_2$

$\therefore a_n = n^3 - n^2 + 1$

11. If $n = 1$, then

$$\begin{aligned} & \frac{1}{5}n^5 + \frac{1}{3}n^3 + \frac{7}{15}n \\ &= \frac{1}{5} + \frac{1}{3} + \frac{7}{15} \\ &= \frac{3 + 5 + 7}{15} \\ &= 1 \end{aligned}$$

12. If $n = 1$, then

$x^n - y^n = x - y$, divisible by $x - y$.

13. If $n = 1$, then

$x^n + a^n = x + a$

If $n = 3$, then

$x^n + a^n = x^3 + a^3 = (x + a)(x^2 + a^2 + ax)$

$\therefore x^n + a^n$ is divisible for odd positive integer n .

14. If $n = 1$, then

$x^{n+1} + (x+1)^{2n-1}$

$= x^2 + x + 1$, divisible by $x^2 + x + 1$.

15. If $n = 1$, then

$2^n = 2, > 1 = n$

$\Rightarrow 2^n > n$

16. If $n = 2$, then given sum $= 1 + \frac{1}{\sqrt{2}}$

$= \frac{\sqrt{2} + 1}{\sqrt{2}} = \frac{2 + \sqrt{2}}{2}$

$= \frac{3.414}{2}$

$= 1.707$

$\Rightarrow \sqrt{2} = 1.414$

$\therefore \text{Sum} \geq \sqrt{n}$

17. If $n = 1$, then

$\frac{(2n)!}{2^{2n}(n!)^2} = \frac{2}{4} = \frac{1}{2}$

Now, $\frac{1}{\sqrt{3n+1}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$,

$\frac{1}{\sqrt{3n+2}} = \frac{1}{\sqrt{5}}$,

$\frac{1}{\sqrt{3n+4}} = \frac{1}{\sqrt{7}}$,

and $\frac{1}{\sqrt{3n+5}} = \frac{1}{\sqrt{8}}$

$\therefore \frac{(2n)!}{2^{2n}(n!)^2} \leq \frac{1}{\sqrt{3n+1}}$

18. If $n = 2$, then

$\frac{(2n)!}{(n!)^2} = \frac{24}{4} = 6$

and $\frac{4n}{2n+1} = \frac{8}{3}$

$\Rightarrow \frac{(2n)!}{(n!)^2} > \frac{4n}{2n+1}$

19. Given, $a_n = na_{n-1} + 5$

For $n = 2$, then

$a_2 = 2a_1 + 5 = 7$ [$\because a_1 = 1$]

$a_3 = 3a_2 + 5 = 21 + 5 = 26$

$a_4 = 4a_3 + 5 = 104 + 5 = 109$

$\therefore a_5 = 5a_4 + 5 = 545 + 5 = 550$

20. Check through options, the condition is satisfied, $\forall n \in N$.

21. $n^p - n$ is divisible by p for any natural number greater than 1.

It is Fermet's theorem.

Aliter

Let $n = 4$ and $p = 2$

Then, $(4)^2 - 4 = 16 - 4 = 12$

It is divisible by 2. So, it is true for any natural number greater than 1.

22. Sum of 20th group $= \frac{20 \cdot 21}{2} = 210$

Sum of n natural numbers $= \frac{n(n+1)}{2}$

It is true, for $n = 1, \frac{1 \cdot 2}{2} = 1$

Assume, it is true, for $n = k$.

So, $1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}$

Now, for $k + 1, 1 + 2 + 3 + \dots + k + k + 1$
 $= \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$

So, Statement II is true, for all $n \in N$.

23. Let statement $P(n) = 3^{2n} + 7$

For $P(n = 1) = 16$

For $P(n = 2) = 88$

$\therefore$ HCF of 16 and 88 is 8.

24. Let $P(n) \equiv n(n+1)(n+2)$

$P(n = 1) \equiv 1 \cdot 2 \cdot 3 = 3!$

$P(n = 2) \equiv 2 \cdot 3 \cdot 4 = 2 \cdot (1 \cdot 2 \cdot 3) = 2 \cdot 3!$

$P(n = 3) \equiv 3 \cdot 4 \cdot 5 = 2 \cdot 5 \cdot (1 \cdot 2 \cdot 3) = 10 \cdot 3!$

So, Statement I is true, Statement II is also true and Statement II is a correct explanation of Statement I.

25. We have, $a_1 = \sqrt{7} < 4$. Assume $a_k < 4$ for some natural number k .

We have,

$a_{k+1} = \sqrt{7 + a_k} < \sqrt{7 + 4} < 4$

$\therefore a_n < 4, \forall n \geq 1$

26. $\because a_1 < 4, a_2 < 4, a_3 < 4$

$\therefore a_1 + a_2 + a_3 < 4 + 4 + 4 = 12$

27. Consider $1 + 3 + 5 + \dots + (2k - 1) = \frac{k}{2} [1 + (2k - 1)]$

$\left[\because \text{if } a \text{ and } l \text{ are the first term and last term of an AP, then } S_n = \frac{n}{2} (a + l) \right]$

$\Rightarrow 1 + 3 + 5 + \dots + (2k - 1) = k^2$

Thus, $S(k)$ can be expressed as k^2 .

i.e. $S(k): 1 + 3 + 5 + \dots + (2k - 1) = k^2$

28. Consider $S(k): 1 + 3 + 5 + \dots + (2k - 1) = 3 + k^2$

Then, $S(1): 1 = 3 + 1^2$, which is not true.

Suppose $S(k)$ is true, then

$1 + 3 + 5 + \dots + (2k - 1) = 3 + k^2$

On adding $(2k + 1)$ both sides, we get

$1 + 3 + 5 + \dots + (2k - 1) + (2k + 1) = 3 + k^2 + (2k + 1)$
 $= 3 + (k + 1)^2$

which is $S(k + 1)$.

Thus, $S(k) \Rightarrow S(k + 1)$

29. For $n = 2, a_2 = 2^{2^2} + 1 = 2^4 + 1$

$= 16 + 1 = 17$

Thus, the last digit of a_2 is 7.

30. Assume that $a_k = 2^{2^k} + 1 = 10m + 7$

where, $k > 1$ and m is some positive integer.

Now, $a_{k+1} = 2^{2^{k+1}} + 1 = (2^{2^k})^2 + 1$

$= (10m + 7)^2 + 1$

$= 10(10m^2 + 12m + 3) + 7$

Thus, last digit of a_n is 7, $\forall n > 1$.

31. A. If $n = 1$, then $3^2 \nless 4$

If $n = 2$, then $27 \nless 16$

If $n = 3$, then $81 \nless 64$

If $n = 4$, then $243 < 256$

B. $x^n - 1$ is divisible by $x - 1$, so $k = 1$

C. Value of $k = -1$, so numerically 1 is smallest.

D. If $n = 1 \Rightarrow 2^{4n} - 15n - 1 = 0; n = 2$

$\Rightarrow 256 - 30 - 1 = 225 = 15^2$

So, maximum value of k is 2.

32. If $n = 1$, then $\frac{(n+2)!}{6(n-1)!} = \frac{3!}{6 \times 0!} = \frac{6}{6} = 1$, divisible by 1.

33. If $n = 1$, then $n^3 + 2n = 1 + 2 = 3$, divisible by 3.

34. For $n = 1$, we have given expression $= 207 = 9 \times 23$

and for $n = 2$, we have given expression $= 873 = 9 \times 97$

Thus, for $n = 2$ given expression is an integral multiple of 97.

which is divisible by 7.

For $n = k, P_2(k) = k^7 - k$

Let $P_2(k)$ be divisible by 7.

$\therefore k^7 - k = 7\lambda \dots(i)$

For $n = k + 1, P_2(k + 1) = (k + 1)^7 - (k + 1)$

$= (k^7 - k) + 7(k^6 + 3k^5 + \dots + k)$

$= 7\lambda + 7(k^6 + 3k^5 + \dots + k)$ [from Eq. (i)]

$\Rightarrow$ Divisible by 7.

4. Let $P(n) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$

$\therefore P(2) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$
 $= 1.707 > \sqrt{2}$

Let us assume that

$P(k) = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} > \sqrt{k}$ is true, for $n = k + 1$

LHS $= \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}}$

$> \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k(k+1)} + 1}{\sqrt{(k+1)}}$

$> \frac{k+1}{\sqrt{k+1}} [\because \sqrt{k(k+1)} > k, \forall k \geq 0]$

$= \sqrt{k+1}$

$\therefore P(k+1) > \sqrt{k+1}$

Entrances Gallery

1. Let $P(n) = 2^{3n} - 7n - 1$

$\Rightarrow P(1) = 0, P(2) = 49$

$P(1)$ and $P(2)$ are divisible by 49.

Let $P(k) = 2^{3k} - 7k - 1 = 49\lambda$

$P(k+1) = 2^{3k+3} - 7k - 8$

$= 8(49\lambda + 7k + 1) - 7k - 8$

$= 49(8\lambda) + 49k = 49\lambda$

[where $\lambda = 8\lambda + k$, which is an integer]

2. We have, $\sum_{k=1}^n [k^3 - (k-1)^3]$

$= (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + \dots + [n^3 - (n-1)^3] = n^3$

$\therefore 1 + (1+2+4) + (4+6+9) + (9+12+16)$

$+ \dots + (361+380+400)$

$= (1^3 - 0^3) + (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3)$

$+ \dots + (20^3 - 19^3)$

$= (20)^3 = 8000$

3. Let $P_1(n) = (n+1)^7 - n^7 - 1$

$= ({}^7C_0 n^7 + {}^7C_1 n^6 + \dots + {}^7C_6 n + 1) - n^7 - 1$

$= (n^7 + {}^7C_1 n^6 + \dots + {}^7C_6 n + 1) - n^7 - 1$

$= {}^7C_1 n^6 + \dots + {}^7C_6 n$

$= 7n^6 + \frac{7(7-1)}{2} n^5 + \dots + 7n$

which is divisible by 7.

Now, let $P_2(n) = (n)^7 - n$

for $n = 1, P_2(1) = 0$

By mathematical induction, Statement I is true, $\forall n \geq 2$.

$$\begin{aligned} \text{Now, let } \alpha(n) &= \sqrt{n(n+1)} \\ \therefore \alpha(2) &= \sqrt{2(2+1)} \\ &= \sqrt{6} < 3 \end{aligned}$$

Let us assume that

$$\alpha(k) = \sqrt{k(k+1)} < (k+1) \text{ is true.}$$

For $n = k + 1$

$$\text{LHS} = \sqrt{(k+1)(k+2)} < (k+2) \quad [\because (k+1) < (k+2)]$$

$$\therefore \alpha(k+1) < (k+2)$$

By mathematical induction, Statement II is true but Statement II is not a correct explanation for Statement I.

$$5. A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}$$

$$\vdots \quad \vdots \quad \vdots$$

$$A^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \text{ can be verified by induction.}$$

Now, taking options

$$\begin{aligned} (b) nA + (n-1)I &= \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} + \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix} \\ &= \begin{bmatrix} 2n-1 & 0 \\ n & 2n-1 \end{bmatrix} \neq A^n \end{aligned}$$

$$\begin{aligned} (d) nA - (n-1)I &= \begin{bmatrix} n & 0 \\ n & n \end{bmatrix} - \begin{bmatrix} n-1 & 0 \\ 0 & n-1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} = A^n \end{aligned}$$

$$\begin{aligned} 6. 1 + 3 + 5 + \dots + (2k+1) &= [1 + 3 + 5 + \dots + (2k-1)] + (2k+1) \\ &= 3 + k^2 + 2k + 1 \quad [\text{given}] \\ &= 3 + (k+1)^2 \end{aligned}$$

$$\therefore S(k) \Rightarrow S(k+1)$$

$$\begin{aligned} 7. t_n &= n(n+1)(n+2) \\ &= n(n^2 + 3n + 2) \\ &= n^3 + 3n^2 + 2n \end{aligned}$$

$$\begin{aligned} \therefore S_n &= \Sigma n^3 + 3 \Sigma n^2 + 2 \Sigma n \\ &= \frac{n^2(n+1)^2}{4} + \frac{3n(n+1)(2n+1)}{6} + \frac{2n(n+1)}{2} \\ &= \left[\frac{n(n+1)}{4} \right] [n(n+1) + 4n + 2 + 4] \\ &= \frac{n(n+1)(n^2 + 5n + 6)}{4} \\ &= \frac{n(n+1)(n+2)(n+3)}{4} \end{aligned}$$

$$\begin{aligned} 8. \text{ Let } P(n) &= n^3 + 2n \\ \text{Now, } P(1) &= 1 + 2 = 3 \\ P(2) &= 8 + 4 = 12 \\ P(3) &= 27 + 6 = 33 \end{aligned}$$

Clearly, we see that all these numbers are divisible by 3.

$$\begin{aligned} 9. \text{ For } n=1, 10^n + 3 \cdot 4^{n+2} + 5 &= 10 + 3 \cdot 4^3 + 5 \\ &= 207, \text{ which is divisible by 9.} \end{aligned}$$

So, by induction, the result is divisible by 9.

$$10. \text{ Given, } A = \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now, } A \times A = \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} \cos\left(\frac{2\pi}{n}\right) & \sin\left(\frac{2\pi}{n}\right) & 0 \\ -\sin\left(\frac{2\pi}{n}\right) & \cos\left(\frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Arrange properly in a line

$$\begin{aligned} &= \begin{bmatrix} \cos^2\left(\frac{2\pi}{n}\right) - \sin^2\left(\frac{2\pi}{n}\right) & & \\ -\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) - \sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & & \\ 0 & \cos\left(\frac{2\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & 0 \\ & -\sin^2\left(\frac{2\pi}{n}\right) + \cos^2\left(\frac{2\pi}{n}\right) & 0 \\ & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \cos\left(2 \times \frac{2\pi}{n}\right) & 2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & 0 \\ -2\sin\left(\frac{2\pi}{n}\right)\cos\left(\frac{2\pi}{n}\right) & \cos\left(2 \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\left(2 \times \frac{2\pi}{n}\right) & \sin\left(2 \times \frac{2\pi}{n}\right) & 0 \\ -\sin\left(2 \times \frac{2\pi}{n}\right) & \cos\left(2 \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly,

$$\begin{aligned} A^n &= \begin{bmatrix} \cos\left(2^{n-1} \times \frac{2\pi}{n}\right) & \sin\left(2^{n-1} \times \frac{2\pi}{n}\right) & 0 \\ -\sin\left(2^{n-1} \times \frac{2\pi}{n}\right) & \cos\left(2^{n-1} \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I \end{aligned}$$

$$\text{and } A^{n-1} = \begin{bmatrix} \cos\left(2^{n-2} \times \frac{2\pi}{n}\right) & \sin\left(2^{n-2} \times \frac{2\pi}{n}\right) & 0 \\ -\sin\left(2^{n-2} \times \frac{2\pi}{n}\right) & \cos\left(2^{n-2} \times \frac{2\pi}{n}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \neq I$$

$$\begin{aligned} 11. \frac{kn}{n+1} &= \left[\frac{1}{2 \cdot 4} + \frac{1}{4 \cdot 6} + \frac{1}{6 \cdot 8} + \dots n \text{ terms} \right] \\ &= \frac{1}{2} \left[\frac{4-2}{2 \cdot 4} + \frac{6-4}{4 \cdot 6} + \frac{8-6}{6 \cdot 8} + \dots + \frac{2n+2-2n}{2n(2n+2)} \right] \\ &= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{4} + \frac{1}{4} - \frac{1}{6} + \frac{1}{6} - \frac{1}{8} + \dots + \frac{1}{2n} - \frac{1}{2n+2} \right] \\ &= \frac{1}{2} \left[\frac{1}{2} - \frac{1}{2(n+1)} \right] \\ &= \frac{n}{4(n+1)} \Rightarrow k = \frac{1}{4} \end{aligned}$$

12. $2n-1$ is an odd positive integer.
 $\Rightarrow a^{2n-1} + b^{2n-1}$ is divisible by $a+b$.

13. Let $P(n): (1+x)^n \geq (1+nx)$
 For $n=1$, $(1+x)^1 = 1+x = 1+1 \cdot x \geq 1+1 \cdot x$
 $\therefore (1+x)^1 \geq 1+1 \cdot x$

For $n=k$, let $P(k): (1+x)^k \geq (1+kx)$ is true.

For $n=k+1$, $P(k+1): (1+x)^{k+1} \geq \{1+(k+1)x\}$ is also true.

We will show $P(k+1)$ is true.

$$\begin{aligned} \text{Consider } (1+x)^{k+1} &= (1+x)^k \cdot (1+x) \geq (1+kx)(1+x) \\ &= 1+x+kx+kx^2 \geq 1+x+kx \quad [\text{if } x > -1] \\ &= 1+(k+1)x \quad [\because k > 0 \text{ and } x > -1] \end{aligned}$$

Thus, $(1+x)^{k+1} \geq 1+(k+1)x$, if $x > -1$

14. By taking option (d),

When $n=1$, then $1 > \frac{1}{3}$ [true]

When $n=2$, then $5 > \frac{8}{3}$, [true]

When $n=3$, then $14 > 9$, [true]

When $n=4$, then $30 > \frac{64}{3} = 21.33$, [true]

15. Given, $a_n = na_{n-1}$

$$\Rightarrow a_2 = 2a_1 = 2 \quad [\because a_1 = 1]$$

$$a_3 = 3a_2 = 3(2) = 6$$

$$a_4 = 4(a_3) = 4(6) = 24$$

$$\therefore a_5 = 5(a_4) = 5(24) = 120$$

8

Binomial Theorem

Binomial Theorem for Positive Integral Index

If x and a are real numbers, then for all $n \in \mathbb{N}$,

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_r x^{n-r} a^r + \dots + {}^nC_{n-1} x^1 a^{n-1} + {}^nC_n x^0 a^n$$

i.e.
$$(x+a)^n = \sum_{r=0}^n {}^nC_r x^{n-r} a^r$$

➤ **Example 1.** $x^{10} + 10x^8 a + 40x^6 a^2 + 80x^4 a^3 + 80x^2 a^4 + 32a^5$ is equal to

(a) $(x^2 + a)^5$ (b) $(x + a)^{10}$ (c) $(x^2 + 2a)^5$ (d) $(x^2 - 2a)^5$

Sol. (c) Using binomial theorem, we have

$$\begin{aligned} (x^2 + 2a)^5 &= {}^5C_0 (x^2)^5 (2a)^0 + {}^5C_1 (x^2)^4 (2a)^1 + {}^5C_2 (x^2)^3 (2a)^2 + {}^5C_3 (x^2)^2 (2a)^3 \\ &\quad + {}^5C_4 (x^2)(2a)^4 + {}^5C_5 (x^2)^0 (2a)^5 \\ &= x^{10} + 5(x^8)(2a) + 10(x^6)(4a^2) + 10(x^4)(8a^3) + 5(x^2)(16a^4) + 32a^5 \\ &= x^{10} + 10x^8 a + 40x^6 a^2 + 80x^4 a^3 + 80x^2 a^4 + 32a^5 \end{aligned}$$

Properties of Binomial Expansion

i. The above expansion is also valid when x and a are complex numbers.

ii. We have, $(x+a)^n = \sum_{r=0}^n {}^nC_r x^{n-r} a^r$. Since, r can have values from 0 to n , the total number of terms in the expansion is $(n+1)$.

iii. The sum of the indices of x and a in each term is n .

e.g. $(2x+3y)^9 = \sum_{r=0}^9 {}^9C_r (2x)^{9-r} (3y)^r$ and it has $(9+1)$ i.e. 10 terms.

Also, the sum of indices of $2x$ and $3y$ in each term is 9.

iv. Replacing a by $-a$, in the expansion of $(x+a)^n$, we get

$$(x-a)^n = {}^nC_0 x^n a^0 - {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + \dots + (-1)^r {}^nC_r x^{n-r} a^r + \dots + (-1)^n {}^nC_n x^0 a^n$$

i.e.
$$(x-a)^n = \sum_{r=0}^n (-1)^r {}^nC_r x^{n-r} a^r$$

Chapter Snapshot

- Binomial Theorem for Positive Integral Index
- Multinomial Theorem
- Greatest Coefficient
- Greatest Term
- R - f Factor Relation
- Divisibility Problems
- Properties of Binomial Coefficients
- Binomial Theorem for any Index
- Approximation
- Exponential Series
- Logarithmic Series

v. Putting $x=1$ and $a=x$ in the expansion of $(x+a)^n$, we get

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$$

$$\text{i.e. } (1+x)^n = \sum_{r=0}^n {}^nC_r x^r$$

known as the expansion of $(1+x)^n$ in ascending powers of x .

vi. Putting $a=1$ in the expansion of $(x+a)^n$, we get

$$(x+1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_r x^{n-r} + \dots + {}^nC_{n-1} x + {}^nC_n$$

$$\text{i.e. } (x+1)^n = \sum_{r=0}^n {}^nC_r x^{n-r}$$

known as the expansion of $(x+1)^n$ in descending powers of x .

vii. Putting $x=1$ and $a=-x$ in the expansion of $(x+a)^n$, we get

$$(1-x)^n = {}^nC_0 - {}^nC_1 x + {}^nC_2 x^2 - {}^nC_3 x^3 + \dots + (-1)^r {}^nC_r x^r + \dots + (-1)^n {}^nC_n x^n$$

$$\text{i.e. } (1-x)^n = \sum_{r=0}^n (-1)^r {}^nC_r x^r$$

➤ **Example 2.** Expand $(2x-3y)^4$ by binomial theorem.

Sol. Using binomial theorem, we have

$$\begin{aligned} (2x-3y)^4 &= \{2x + (-3y)\}^4 = \sum_{r=0}^4 (-1)^r {}^4C_r (2x)^{4-r} (3y)^r \\ &= {}^4C_0 (2x)^4 (3y)^0 - {}^4C_1 (2x)^3 (3y)^1 + {}^4C_2 (2x)^2 (3y)^2 \\ &\quad - {}^4C_3 (2x)(3y)^3 + {}^4C_4 (2x)^0 (3y)^4 \\ &= 16x^4 - 4(8x^3)(3y) + 6(4x^2)(9y^2) - 4(2x)(27y^3) + 81y^4 \\ &= 16x^4 - 96x^3y + 216x^2y^2 - 216xy^3 + 81y^4 \end{aligned}$$

viii.

$$\begin{aligned} \text{We have,} \\ (x+a)^n + (x-a)^n &= 2[{}^nC_0 x^n a^0 + {}^nC_2 x^{n-2} a^2 + \dots] \\ \text{and } (x+a)^n - (x-a)^n &= 2[{}^nC_1 x^{n-1} a^1 + {}^nC_3 x^{n-3} a^3 + \dots] \end{aligned}$$

ix. If n is odd, then $\{(x+a)^n + (x-a)^n\}$ and $\{(x+a)^n - (x-a)^n\}$ both have the same number of terms equal to $\left(\frac{n+1}{2}\right)$ whereas, if n is even, then $\{(x+a)^n + (x-a)^n\}$ has $\left(\frac{n}{2} + 1\right)$ terms and

$$\{(x+a)^n - (x-a)^n\} \text{ has } \left(\frac{n}{2}\right) \text{ terms.}$$

➤ **Example 3.** The number of terms in the expansion of $(1+5\sqrt{2}x)^9 + (1-5\sqrt{2}x)^9$ are

- (a) 3 (b) 6
(c) 5 (d) None of these

Sol. (c) Since, here $n=9$ is odd, therefore the expansion of $(1+5\sqrt{2}x)^9 + (1-5\sqrt{2}x)^9$ has $\left(\frac{9+1}{2}\right) = 5$ terms.

➤ **Example 4.** Using binomial theorem, expand $\{(x+y)^5 + (x-y)^5\}$ and hence find the value of $\{(\sqrt{2}+1)^5 + (\sqrt{2}-1)^5\}$.

Sol. We have,

$$\begin{aligned} (x+y)^5 + (x-y)^5 &= 2[{}^5C_0 x^5 + {}^5C_2 x^3 y^2 + {}^5C_4 x^1 y^4] \\ &= 2(x^5 + 10x^3 y^2 + 5xy^4) \end{aligned}$$

Putting $x=\sqrt{2}$ and $y=1$, we get

$$\begin{aligned} (\sqrt{2}+1)^5 + (\sqrt{2}-1)^5 &= 2\{(\sqrt{2})^5 + 10(\sqrt{2})^3 + 5\sqrt{2}\} \\ &= 2(4\sqrt{2} + 20\sqrt{2} + 5\sqrt{2}) = 58\sqrt{2} \end{aligned}$$

x. We have,

$$(x+a)^n = {}^nC_0 x^n a^0 + {}^nC_1 x^{n-1} a^1 + {}^nC_2 x^{n-2} a^2 + \dots + {}^nC_r x^{n-r} a^r + \dots + {}^nC_n x^0 a^n$$

Here, $(r+1)$ th term is given by ${}^nC_r x^{n-r} a^r$.

If T_{r+1} denotes the $(r+1)$ th term, then

$$T_{r+1} = {}^nC_r x^{n-r} a^r$$

This is called the general term, because by giving different values to r we can determine all terms of the expansion.

Also, the general term can be expressed as $\frac{n!}{r!s!} x^s a^r$, where $r+s=n$.

xi. In the binomial expansion of $(x-a)^n$, the general term is given by

$$T_{r+1} = (-1)^r {}^nC_r x^{n-r} a^r$$

xii. In the binomial expansion of $(1+x)^n$, we have

$$T_{r+1} = {}^nC_r x^r$$

xiii. In the binomial expansion of $(1-x)^n$, we have

$$T_{r+1} = (-1)^r {}^nC_r x^r$$

➤ To find the terms free from radicals or rational indicies in the expansion of $(a^{1/p} + b^{1/q})^n$, where a and b are prime numbers, first

find the general term $T_{r+1} = {}^nC_r (a^{1/p})^{n-r} (b^{1/q})^r = {}^nC_r a^{\frac{N-r}{p}} b^{\frac{r}{q}}$, then put the values of r , where $0 \leq r \leq N$, for which indicies of a and b become integers.

➤ **Example 5.** If the r th term in the expansion of

$$\left(\frac{x}{3} - \frac{2}{x^2}\right)^{10} \text{ contains } x^4, \text{ then } r \text{ is equal to}$$

- (a) 2 (b) 1 (c) 3 (d) 5

Sol. (c) Clearly, the general term in the binomial expansion of $\left(\frac{x}{3} - \frac{2}{x^2}\right)^{10}$ is given by

$$\begin{aligned} T_{r+1} &= {}^{10}C_r \left(\frac{x}{3}\right)^{10-r} \left(\frac{-2}{x^2}\right)^r \\ &= {}^{10}C_r \left(\frac{1}{3}\right)^{10-r} (-2)^r x^{10-r-2r} \\ &= {}^{10}C_r \left(\frac{1}{3}\right)^{10-r} (-2)^r x^{10-3r} \end{aligned}$$

Now, replace r by $r-1$, we get the r th term i.e.

$$T_r = {}^{10}C_{r-1} \left(\frac{1}{3}\right)^{11-r} (-2)^{r-1} x^{13-3r}$$

Since, the r th term contains the x^4 , therefore we have

$$\begin{aligned} 13 - 3r &= 4 \\ \Rightarrow 3r &= 9 \Rightarrow r = 3 \end{aligned}$$

➤ **Example 6.** The number of integral terms in

the expansion of $\left(5^{\frac{1}{2}} + 7^{\frac{1}{6}}\right)^{642}$ is

- (a) 106 (b) 108
(c) 103 (d) 109

Sol. (b) Clearly, the general term in binomial expansion of

$\left(5^{\frac{1}{2}} + 7^{\frac{1}{6}}\right)^{642}$ is given by

$$T_{r+1} = {}^{642}C_r \left(5^{\frac{1}{2}}\right)^{642-r} \left(7^{\frac{1}{6}}\right)^r$$

Obviously, r should be a multiple of 6.

Total numbers = $\frac{642}{6} = 107$, but first term for $r = 0$ is

also integral. Hence, total terms = $107 + 1 = 108$

xiv. The coefficient of x^k in the expansion of $(x+a)^n$ can be obtained by equating the power of x in the general term to k . This will give value of r and on substituting this value in general term we can find the coefficient of x^k .

➤ If r is not a positive integer, then there will be no term containing x^k and hence coefficient of x^k will be zero.

xv. The coefficient of $(r+1)$ th term in the expansion of $(1+x)^n$ is ${}^nC_r$.

xvi. The coefficient of x^r in the expansion of $(1+x)^n$ is ${}^nC_r$.

➤ **Example 7.** The coefficient of x^{11} in the

expansion of $\left(x^3 - \frac{2}{x^2}\right)^{12}$ is

- (a) -25344 (b) 25344
(c) 23544 (d) -23544

Sol. (a) Let the general term i.e. $(r+1)$ th contains x^{11} .

$$\begin{aligned} \text{We have, } T_{r+1} &= {}^{12}C_r (x^3)^{12-r} \left(\frac{-2}{x^2}\right)^r \\ &= {}^{12}C_r x^{36-3r-2r} (-1)^r 2^r \\ &= {}^{12}C_r (-1)^r 2^r x^{36-5r} \end{aligned} \quad \dots(i)$$

to find the coefficient of x^{11} , put $36 - 5r = 11$

$$\Rightarrow 5r = 25 \Rightarrow r = 5$$

On substituting the value of r in Eq. (i), we get

$$T_6 = {}^{12}C_5 (-1)^5 2^5 x^{11}$$

Thus, the coefficient of x^{11} is

$$\begin{aligned} {}^{12}C_5 (-1)^5 2^5 &= \frac{-12 \times 11 \times 10 \times 9 \times 8}{5 \times 4 \times 3 \times 2} \times 32 \\ &= -25344 \end{aligned}$$

➤ **Example 8.** The coefficient of x^p and x^q (p and q are positive integers) in the expansion of $(1+x)^{p+q}$ are

- (a) equal
(b) equal with opposite signs
(c) reciprocal of each other
(d) None of the above

Sol. (a) Coefficient of x^p and x^q in the expansion of $(1+x)^{p+q}$ are ${}^{p+q}C_p$ and ${}^{p+q}C_q$

$$\text{and } {}^{p+q}C_p = {}^{p+q}C_q = \frac{(p+q)!}{p!q!}$$

xvii. Independent term or constant term of a binomial expansion is the term in which exponent of the variable is zero. i.e. the coefficient of x^0 .

➤ **Example 9.** The ratio of the coefficient of x^{15}

to the term independent of x in $\left(x^2 + \frac{2}{x}\right)^{15}$ is

- (a) 12 : 32 (b) 1 : 32
(c) 32 : 12 (d) 32 : 1

Sol. (b) Clearly, the general term in the expansion of

$\left(x^2 + \frac{2}{x}\right)^{15}$ is given by

$$\begin{aligned} T_{r+1} &= {}^{15}C_r (x^2)^{15-r} \left(\frac{2}{x}\right)^r \\ &= {}^{15}C_r 2^r x^{30-3r} \end{aligned} \quad \dots(i)$$

Now, for the coefficient of term containing x^{15} , $30 - 3r = 15$, i.e. $r = 5$

Therefore, ${}^{15}C_5 2^5$ is the coefficient of x^{15} [from Eq.(i)]

Now, to find the term independent of x , put $30 - 3r = 0$

Thus, ${}^{15}C_{10} 2^{10}$ is the term independent of x [from Eq.(i)]

$$\text{Now, the ratio is } \frac{{}^{15}C_5 2^5}{{}^{15}C_{10} 2^{10}} = \frac{1}{2^5} = \frac{1}{32}$$

xviii. In the binomial expansion of $(x+a)^n$, the r th term from the end is $(n-r+2)$ th term from the beginning.

➤ **Example 10.** The 4th term from the end in the expansion of $\left(\frac{x^3}{2} - \frac{2}{x^2}\right)^9$ is

- (a) $\frac{670}{x^3}$ (b) $\frac{671}{x^3}$
(c) $\frac{672}{x^3}$ (d) None of these

Sol. (c) Clearly, the 4th term from the end is $9 - 4 + 2$ i.e. 7th term from the beginning, which is given by

$$T_7 = {}^9C_6 \left(\frac{x^3}{2}\right)^3 \left(\frac{-2}{x^2}\right)^6 = {}^9C_3 \frac{x^9}{8} \cdot \frac{64}{x^{12}} \\ = \frac{9 \times 8 \times 7}{3 \times 2 \times 1} \times \frac{64}{x^3} = \frac{672}{x^3}$$

ix. **Middle term in a binomial expansion** If n is an even natural number, then in the binomial expansion of $(x + a)^n$, $\left(\frac{n}{2} + 1\right)$ th term is the middle term.

xx. If n is odd natural number, then $\left(\frac{n+1}{2}\right)$ th and $\left(\frac{n+3}{2}\right)$ th are the middle terms in the binomial expansion of $(x + a)^n$.

➤ When there are two middle terms in the expansion, then their binomial coefficients are equal.

➤ **Example 11.** The middle term (terms) in the expansion of $\left(\frac{p}{x} + \frac{x}{p}\right)^9$ is/are

- (a) $\frac{126p}{x}$ (b) $\frac{126x}{p}$ (c) $\frac{125p}{x}$ (d) $\frac{125x}{p}$

Sol. (a, b) Since, the power of binomial is odd, therefore we have two middle terms which are 5th and 6th terms. These are given by

$$T_5 = {}^9C_4 \left(\frac{p}{x}\right)^5 \left(\frac{x}{p}\right)^4 = {}^9C_4 \frac{p}{x} = \frac{126p}{x} \\ \text{and } T_6 = {}^9C_5 \left(\frac{p}{x}\right)^4 \left(\frac{x}{p}\right)^5 = {}^9C_5 \frac{x}{p} = \frac{126x}{p}$$

Multinomial Theorem

i. We know,
$$(x + a)^n = \sum_{r=0}^n {}^nC_r x^{n-r} a^r, n \in \mathbb{N} \\ = \sum_{r=0}^n \frac{n!}{(n-r)! r!} x^{n-r} a^r \\ = \sum_{r+s=n} \frac{n!}{r! s!} x^s a^r, \text{ where } s = n - r$$

Some another form of binomial expansion

ii.
$$(x + y + z)^n = \sum_{r+s+t=n} \frac{n!}{r! s! t!} x^r y^s z^t$$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

iii.
$$(x + y + z + u)^n = \sum_{p+q+r+s=n} \frac{n!}{p! q! r! s!} x^p y^q z^r u^s$$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Above result can be generalized in the following form (known as multinomial theorem)

$$(x_1 + x_2 + \dots + x_k)^n = \sum_{r_1+r_2+\dots+r_k=n} \frac{n!}{r_1! r_2! \dots r_k!} x_1^{r_1} x_2^{r_2} \dots x_k^{r_k}$$

The general term in the above expansion is

$$\frac{n!}{r_1! r_2! r_3! \dots r_k!} x_1^{r_1} x_2^{r_2} x_3^{r_3} \dots x_k^{r_k}$$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + \dots + r_k = n$, because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$.

➤ The coefficient of $x_1^{n_1} \cdot x_2^{n_2} \dots x_m^{n_m}$ in the expansion of $(x_1 + x_2 + \dots + x_m)^n$ is $\frac{n!}{n_1! n_2! n_3! \dots n_m!}$

➤ **Example 12.** The number of distinct terms in the expansion of $(x + y - z)^{16}$ is

- (a) 136 (b) 153 (c) 16 (d) 17

Sol. (b) $(x + y - z)^{16} = {}^{16}C_0 x^{16} + {}^{16}C_1 x^{15} (y - z) + \dots \\ + {}^{16}C_r x^{16-r} (y - z)^r + \dots + {}^{16}C_{16} (y - z)^{16}$

Clearly, all the terms are distinct.

$\therefore$ The number of distinct terms
 $= 1 + 2 + 3 + \dots + 17 \\ = \frac{17 \times 18}{2} = 153$

Aliter

Number of terms in $(x_1 + x_2 + \dots + x_m)^n$
 $= {}^{n+m-1}C_{m-1}$

$\therefore$ Number of terms in $(x + y - z)^{16}$
 $= {}^{16+3-1}C_{3-1} = {}^{18}C_2 = \frac{18 \times 17}{2} = 153$

➤ **Example 13.** The coefficient of $a^8 b^6 c^4$ in the expansion of $(a + b + c)^{18}$ is

- (a) ${}^{18}C_{14} \cdot {}^{14}C_8$ (b) ${}^{18}C_{14}$
(c) ${}^{12}C_8$ (d) None of these

Sol. (a) In $(a + b + c)^{18}$, general term is given by

$$\frac{(18)!}{(r_1)!(r_2)!(r_3)!} (a)^{r_1} (b)^{r_2} (c)^{r_3} \quad \dots (i)$$

$\therefore$ Coefficient of $a^8 b^6 c^4$ is obtained by putting

$$r_1 = 8, r_2 = 6, r_3 = 4 \text{ in Eq. (i)}$$

$$\text{i.e.} \quad \frac{(18)!}{8! \cdot 6! \cdot 4!} a^8 b^6 c^4$$

$$\text{Thus, required coefficient} = \frac{(18)!}{8! \cdot 6! \cdot 4!} \times \frac{14!}{14!} = {}^{18}C_{14} \times {}^{14}C_8$$

➤ **Example 14.** The coefficient of $x^3 y^4 z^2$ in the expansion of $(2x - 3y + 4z)^9$ is

- (a) 12063600 (b) 13063680
(c) 11063600 (d) None of these

Sol. (b) The general term in the expansion of $(2x - 3y + 4z)^9$

$$\begin{aligned} &= \frac{9!}{\alpha_1! \alpha_2! \alpha_3!} \cdot (2x)^{\alpha_1} (-3y)^{\alpha_2} (4z)^{\alpha_3} \\ &= \frac{9!}{\alpha_1! \alpha_2! \alpha_3!} \cdot 2^{\alpha_1} (-3)^{\alpha_2} 4^{\alpha_3} \cdot x^{\alpha_1} y^{\alpha_2} z^{\alpha_3} \end{aligned}$$

$\therefore$ Coefficient of $x^3 y^4 z^2$

$$\begin{aligned} &= \frac{9!}{3!4!2!} \cdot 2^3 (-3)^4 \cdot 4^2 \quad [\text{taking } \alpha_1 = 3, \alpha_2 = 4, \alpha_3 = 2] \\ &= 1260 \times 10368 = 13063680 \end{aligned}$$

➤ **Example 15.** The coefficient of x^3 in the expansion of $(1 - x + x^2)^5$ is

- (a) 10 (b) 8 (c) -50 (d) -30

Sol. (d) $(1 - x + x^2)^5 = \{1 + x(x - 1)\}^5$
 $= {}^5C_0 + {}^5C_1 x(x - 1) + {}^5C_2 x^2(x - 1)^2 + {}^5C_3 x^3(x - 1)^3 + \dots$
 $\therefore$ The coefficient of $x^3 = -2 \cdot {}^5C_2 - {}^5C_3 = -30$

Aliter

General term of $(1 - x + x^2)^5$ is

$$\frac{5!}{(r_1)!(r_2)!(r_3)!} (1)^{r_1} (-x)^{r_2} (x^2)^{r_3} \quad \dots (i)$$

$\therefore$ For coefficient of x^3 , $r_2 + 2r_3 = 3$

where $r_1 + r_2 + r_3 = 5$

Let $r_3 = 0$, then $r_2 = 3$ and $r_1 = 2$

$r_3 = 1$, then $r_2 = 1$ and $r_1 = 3$

If $r_3 = 2, 3, 4, 5$, then r_2 is not possible as negative.

$\therefore$ Only two terms contains x^3 i.e.

$$(r_1 = 2, r_2 = 3, r_3 = 0) \text{ and } (r_1 = 3, r_2 = 1, r_3 = 1)$$

Thus, from Eq. (i),

$$= \frac{5!}{2!3!0!} (-1)^3 + \frac{5!}{3!1!1!} (-1)^1 = -10 - 20 = -30$$

➤ **Example 16.** The total number of terms in the expansion of $(a + b + c + d)^n$, $n \in N$ is

- (a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)(n+2)(n+3)}{6}$
(c) $\frac{(n+1)(n+2)(n+3)}{6}$ (d) None of these

Sol. (c) The number of terms in the expansion of $(a + b + c + d)^n$

$$= {}^{n+4-1}C_{4-1} = {}^{n+3}C_3 = \frac{(n+1)(n+2)(n+3)}{6}$$

Work Book Exercise 8.1

1 If the second, third and fourth terms in the expansion of $(x + y)^n$ are 135, 30 and $10/3$ respectively, then

- a $n = 7$ b $n = 5$
c $n = 6$ d None of these

2 The total number of dissimilar terms in the expansion of $(x_1 + x_2 + \dots + x_n)^3$ is

- a n^3 b $\frac{n^3 + 3n^2}{4}$
c $\frac{n(n+1)(n+2)}{6}$ d $\frac{n^2(n+1)^2}{4}$

3 If the coefficients of r th and $(r + 1)$ th term in the expansion of $(3 + 7x)^{29}$ are equal, then r equals

- a 15 b 21 c 14 d 25

4 If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \dots + a_{20}x^{20}$, then a_1 equals

- a 10 b 20 c 210 d 110

5 The number of terms which are free from radical signs in the expansion of $(y^{1/5} + x^{1/10})^{55}$ is

- a 5 b 6 c 7 d 4

6 The coefficient of x^6 in the expansion of $(1 + x^2 - x^3)^8$ is

- a 80 b 84 c 88 d 92

7 The coefficient of x^5 in the expansion of $(1 + x)^{21} + (1 + x)^{22} + \dots + (1 + x)^{30}$ is

- a ${}^{51}C_5$ b ${}^9C_5$
c ${}^{31}C_6 - {}^{21}C_6$ d ${}^{30}C_5 + {}^{20}C_5$

8 If the ninth term in the expansion of

$$[3^{\log_3 \sqrt{25^x - 1} + 7} + 3^{-1/8 \log_3 (5^x - 1) + 1}]^{10}$$

is equal to 180 and $x > 1$, then x equals

- a $\log_{10} 15$ b $\log_5 15$
c $\log_e 15$ d None of these

9 The expression

$$\frac{1}{\sqrt{4x+1}} \left\{ \left(\frac{1 + \sqrt{4x+1}}{2} \right)^7 - \left(\frac{1 - \sqrt{4x+1}}{2} \right)^7 \right\}$$

is a polynomial in x of degree

- a 7 b 5
c 4 d 3

10 If $\frac{1}{\sqrt{4x+1}} \left\{ \left(\frac{1 + \sqrt{4x+1}}{2} \right)^n - \left(\frac{1 - \sqrt{4x+1}}{2} \right)^n \right\}$

$= a_0 + a_1x + \dots + a_5x^5$, then n is equal to

- a 11 b 9
c 10 d 8

Greatest Coefficient

In the binomial expansion of $(x + a)^n$, the coefficient ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ are known as the binomial coefficients.

For given value of n ,

(a) If n is even, then greatest coefficient = ${}^nC_{n/2}$

(b) If n is odd, then greatest coefficient are

$$\frac{{}^nC_{n-1}}{2} \text{ and } \frac{{}^nC_{n+1}}{2}.$$

- Binomial coefficient of middle term is the greatest binomial coefficient.
- The greatest coefficient in the expansion of $(x_1 + x_2 + \dots + x_m)^n$ is $\frac{n!}{(q!)^{m-1}[(q+1)!]^r}$, where q and r are the quotient and remainder respectively when n is divided by m .

Example 17. Find the value of r for which ${}^{200}C_r$ is the greatest.

Sol. Coefficient of terms in the expansion of $(x + y)^{200}$ are ${}^{200}C_0, {}^{200}C_1, {}^{200}C_2, \dots, {}^{200}C_{200}$.

Since, middle term has greatest coefficient.

$\therefore$ Greatest coefficient = Coefficient of middle term

= Coefficient of 101st term = ${}^{200}C_{100}$

Hence, ${}^{200}C_r$ is the greatest, when $r = 100$

Example 18. The greatest coefficient in the expansion of $(x + y + z + w)^{15}$ is

(a) $\frac{15!}{3!(4!)^3}$

(b) $\frac{15!}{(3!)^3 4!}$

(c) $\frac{15!}{2!(4!)^2}$

(d) None of these

Sol. (a) The greatest coefficient is

$$\begin{aligned} &= \frac{n!}{(q!)^{k-r}[(q+1)!]^r} \quad [\text{here, } n = 15, q = 3, r = 3, k = 4] \\ &= \frac{15!}{(3!)^{4-3}[(3+1)!]^3} = \frac{15!}{3!(4!)^3} \end{aligned}$$

Greatest Term

Let T_{r+1} and T_r be $(r+1)$ th and r th terms, respectively in the expansion of $(x + a)^n$. Then,

$$T_{r+1} = {}^nC_r x^{n-r} a^r \text{ and } T_r = {}^nC_{r-1} x^{n-r+1} a^{r-1}$$

$$\begin{aligned} \therefore \frac{T_{r+1}}{T_r} &= \frac{{}^nC_r x^{n-r} a^r}{{}^nC_{r-1} x^{n-r+1} a^{r-1}} \\ &= \frac{n!}{(n-r)! r!} \times \frac{(r-1)!(n-r+1)!}{n!} \cdot \frac{a}{x} \\ &= \frac{n-r+1}{r} \cdot \frac{a}{x} \end{aligned}$$

$$\text{Now, } T_{r+1} \geq T_r \Rightarrow \frac{T_{r+1}}{T_r} \geq 1$$

$$\Rightarrow \frac{n-r+1}{r} \cdot \frac{a}{x} \geq 1$$

$$\Rightarrow \left\{ \left(\frac{n+1}{r} \right) - 1 \right\} \frac{a}{x} \geq 1$$

$$\Rightarrow \frac{n+1}{r} - 1 \geq \frac{x}{a} \Rightarrow \frac{n+1}{r} \geq \left(1 + \frac{x}{a} \right)$$

$$\Rightarrow \frac{n+1}{1 + \frac{x}{a}} \geq r \Rightarrow r \geq \left(\frac{n+1}{1 + \frac{x}{a}} \right)$$

Thus, $T_{r+1} \geq T_r$ according as

$$r \geq \left(\frac{n+1}{1 + \frac{x}{a}} \right) \quad \dots(i)$$

Now, two cases arise.

Case I When $\frac{n+1}{1 + \frac{x}{a}}$ is an integer.

Let $\frac{n+1}{1 + \frac{x}{a}} = m$. Then, from Eq. (i), we have

m th and $(m+1)$ th terms are greatest terms.

Case II When $\frac{n+1}{1 + \frac{x}{a}}$ is not an integer.

Let m be the integral part of $\frac{n+1}{1 + \frac{x}{a}}$. Then, from

Eq. (i), we have $(m+1)$ th term is the greatest term.

Above discussion suggests the following algorithm to find the greatest term in a binomial expansion.

Algorithm

Step I Write T_{r+1} and T_r from the given expansion.

Step II Find $\frac{T_{r+1}}{T_r}$

Step III Put $\frac{T_{r+1}}{T_r} > 1$

Step IV Solve the inequality in step III for r to get an inequality of the form $r < m$ or $r > m$.

If m is an integer, then m th and $(m+1)$ th terms are equal in magnitude and these two are the greatest terms. If m is not an integer, then obtain the integral part of m , say k . In this case, $(k+1)$ th term is the greatest term.

Shortcut Method

To find the greatest term (numerically) in the expansion of $(1+x)^n$.

- Calculate $\frac{|x|(n+1)}{|x|+1}$.
- If m is integer, then T_m and T_{m+1} are equal and both are greatest term.
- If m is not integer, then $T_{[m]+1}$ is the greatest term, where $[]$ denotes the greatest integral part.

➤ To find greatest term in $(x+y)^n$, express $(x+y)^n = x^n \left(1 + \frac{y}{x}\right)^n$.

Then, find the greatest term in $\left(1 + \frac{y}{x}\right)^n$.

➤ **Example 19.** The greatest term (numerically)

in the expansion of $(3-5x)^{15}$, when $x = \frac{1}{5}$ is

- ${}^{15}C_3 \times 3^{10}$
- ${}^{15}C_3 \times 3^{11}$
- ${}^{15}C_3 \times 3^{12}$
- None of these

Sol. (c) Let T_{r+1} and T_r denote the $(r+1)$ th and r th terms respectively. Then,

$$T_{r+1} = {}^{15}C_r 3^{15-r} (-5x)^r \text{ and } T_r = {}^{15}C_{r-1} 3^{15-r+1} (-5x)^{r-1}$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{{}^{15}C_r 3^{15-r} (-5x)^r}{{}^{15}C_{r-1} 3^{16-r} (-5x)^{r-1}}$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{15-r+1}{r} \left(\frac{-5x}{3} \right)$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{16-r}{r} \times \left(-\frac{5}{3} \times \frac{1}{5} \right), \text{ when } x = \frac{1}{5}$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{16-r}{r} \times \frac{1}{3}, \text{ numerically}$$

[neglecting minus sign]

Now,

$$\frac{T_{r+1}}{T_r} > 1 \text{ (numerically)}$$

$$\Rightarrow \frac{16-r}{r} \times \frac{1}{3} > 1$$

$$\Rightarrow 16 > 4r$$

$$\Rightarrow r < 4$$

Since, 4 is an integer. Therefore, 4th and 5th terms are numerically greatest terms.

$$\text{Now, } T_4 = T_{3+1} = {}^{15}C_3 3^{15-3} (-5x)^3$$

$$\Rightarrow T_4 = {}^{15}C_3 \times 3^{12} \left(-5 \times \frac{1}{5} \right)^3, \text{ when } x = \frac{1}{5}$$

$$\Rightarrow T_4 = {}^{15}C_3 \times 3^{12} \text{ (numerically)}$$

$$\text{and } T_5 = T_{4+1} = {}^{15}C_4 3^{15-4} (-5x)^4$$

$$= {}^{15}C_4 3^{11} \left(-5 \times \frac{1}{5} \right)^4, \text{ when } x = \frac{1}{5}$$

$$= {}^{15}C_4 \times 3^{11} \text{ (numerically)}$$

$$= {}^{15}C_3 \times 3^{12}$$

From above, we see that the values of both greatest terms are equal.

➤ **Example 20.** The greatest term (numerically) in the expansion of $(2+3x)^9$, when $x = \frac{3}{2}$, is

$$(a) \frac{5 \times 3^{11}}{2}$$

$$(b) \frac{5 \times 3^{13}}{2}$$

$$(c) \frac{7 \times 3^{13}}{2}$$

(d) None of these

Sol. (c) We have,

$$(2+3x)^9 = 2^9 \left(1 + \frac{3x}{2}\right)^9 = 2^9 \left(1 + \frac{9}{4}\right)^9 \quad \left[\because x = \frac{3}{2} \right]$$

$$\therefore r = \left\lfloor \frac{x(n+1)}{(x+1)} \right\rfloor \quad \left[\because -\frac{1}{3} < 0 \right]$$

$$= \left\lfloor \frac{\left(\frac{9}{4}\right)(9+1)}{\left(\frac{9}{4}\right)+1} \right\rfloor = \frac{90}{13} = 6\frac{12}{13} \neq \text{Integer}$$

The greatest term in the expansion is $T_{[r]+1} = T_{6+1} = T_7$
Hence, the greatest term $= 2^9 \cdot T_7$

$$= 2^9 \cdot T_{6+1} = 2^9 \cdot {}^9C_6 \left(\frac{9}{4}\right)^6 = 2^9 \cdot {}^9C_3 \left(\frac{9}{4}\right)^6$$

$$= 2^9 \cdot \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} \cdot \frac{3^{12}}{2^{12}} = \frac{7 \times 3^{13}}{2}$$

R-f Factor Relation

To find the integral and fractional parts of an irrational number of the form $(a+b\sqrt{c})^n$, where a, b, c and n are natural number, follow the following algorithm.

Algorithm

Step I Write the given expression equal to $I + f$, where I is its integral part and f is the fractional part.

Step II Define G by replacing '+' sign in the given expression by '-'. Note that G always lies between 0 and 1.

Step III Either add G to the expression in step I or subtract G from the expression in step I so that RHS is an integer.

Step IV If G is added to the expression in step I, then $G + f$ will always come out to be equal to 1 i.e. $G = 1 - f$. If G is subtracted from the expression in step I, then G will always come out to be equal to f .

Step V Obtain the value of the desired expression after getting f in terms of G .

➤ **Example 21.** Integral part of $(2+\sqrt{5})^{2n+1}$ is ($n \in \mathbb{N}$)

(a) an even number

(b) an odd number

(c) an even or an odd number depending upon the value of n

(d) None of the above

Sol. (a) Here, $\forall n \in \mathbb{N}, (2 + \sqrt{5})^{2n+1} \notin \mathbb{N}$

$\therefore$ We denote $(2 + \sqrt{5})^{2n+1}$ by $I + f$

where I is an integer and $f \in \mathbb{R}$ such that $0 < f < 1$

$\therefore 0 < (\sqrt{5} - 2) < 1$

$\therefore$ We denote $(\sqrt{5} - 2)^{2n+1}$ by G

where $G \in \mathbb{R}$ such that $0 < G < 1$

Now, $I + f = (\sqrt{5} + 2)^{2n+1}$

$$= (\sqrt{5})^{2n+1} + {}^{2n+1}C_1(\sqrt{5})^{2n} \cdot 2 + {}^{2n+1}C_2(\sqrt{5})^{2n-1} \cdot 2^2 + \dots \dots (i)$$

$$G = (\sqrt{5} - 2)^{2n+1} = (\sqrt{5})^{2n+1} - {}^{2n+1}C_1(\sqrt{5})^{2n} \cdot 2 + {}^{2n+1}C_2(\sqrt{5})^{2n-1} \cdot 2^2 - \dots \dots (ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$I + f - G = 2[{}^{2n+1}C_1(5)^n \cdot 2 + {}^{2n+1}C_3(5)^{n-1} \cdot 2^3 + \dots]$$

$$= 2k, \text{ where } k \text{ is an integer.}$$

$\therefore I$ is an integer.

$\therefore f - G = 2k - I$ is an integer. $\dots (iii)$

$$\left. \begin{array}{l} \text{Now, } 0 < f < 1 \\ \text{and } 0 < G < 1 \end{array} \right\} \Rightarrow \left. \begin{array}{l} 0 < f < 1 \\ -1 < -G < 0 \end{array} \right\} \Rightarrow -1 < f - G < 1 \dots (iv)$$

From Eqs. (iii) and (iv), $f - G = 0$

Now, from Eq. (iii), $I = 2k - (f - G) = 2k - 0 = 2k$

➤ **Example 22.** The integral part of $(\sqrt{2} + 1)^6$ is

- (a) 198 (b) 196
(c) 197 (d) 199

Sol. (c) Let $(\sqrt{2} + 1)^6 = I + f$, where $I \in \mathbb{N}$ and $0 < f < 1$.

Clearly, I is the greatest integer less than or equal to $(\sqrt{2} + 1)^6$.

Let $G = (\sqrt{2} - 1)^6$. Then,

$$I + f + G = (\sqrt{2} + 1)^6 + (\sqrt{2} - 1)^6$$

$$\Rightarrow I + f + G = 2\{{}^6C_0(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 + \dots\}$$

$$\Rightarrow I + f + G = \text{An even integer}$$

$$\Rightarrow f + G = 1$$

Again, $I + f + G$

$$= 2\{{}^6C_0(\sqrt{2})^6 + {}^6C_2(\sqrt{2})^4 + {}^6C_4(\sqrt{2})^2 + {}^6C_6(\sqrt{2})^0\}$$

$$\Rightarrow I + 1 = 2(8 + 15 \times 4 + 15 \times 2 + 1) \quad [\because f + G = 1]$$

$$\Rightarrow I = 197$$

Divisibility Problems

From the expansion,

$$(1 + \alpha)^n = 1 + {}^nC_1\alpha + {}^nC_2\alpha^2 + \dots + {}^nC_n\alpha^n$$

We can conclude that

- i. $(1 + \alpha)^n - 1 = {}^nC_1\alpha + {}^nC_2\alpha^2 + \dots + {}^nC_n\alpha^n$ is divisible by α i.e. it is a multiple of α .
In short, we can write that $(1 + \alpha)^n - 1 = M(\alpha)$
- ii. $(1 + \alpha)^n - 1 - n\alpha = {}^nC_2\alpha^2 + {}^nC_3\alpha^3 + \dots + {}^nC_n\alpha^n = M(\alpha^2)$
- iii. $(1 + \alpha)^n - 1 - n\alpha - \frac{n(n-1)}{2}\alpha^2 = {}^nC_3\alpha^3 + {}^nC_4\alpha^4 + \dots + {}^nC_n\alpha^n = M(\alpha^3)$

e.g. (i) $(1 + 7)^{51} - 1 = 8^{51} - 1$ is $M(7)$ i.e. multiple of 7

(ii) $(1 + 7)^{51} - 1 - 51 \times 7 = 8^{51} - 357$ is $M(7^2)$ i.e. multiple of 49.

(iii) $(1 + 7)^{51} - 1 - 51 \times 7 - \frac{51 \times 50}{2} \times 7^2$
 $= 8^{51} - 357 - 62475$
 $= 8^{51} - 62832$ is $M(7^3)$ i.e. multiple of 343.

➤ **Example 23.** Which of the following expression is divisible by 1225?

- (a) $6^{2n} - 35n - 1$ (b) $6^{2n} - 35n + 1$
(c) $6^{2n} - 35n$ (d) $6^{2n} - 35n + 2$

Sol. (a) Consider expression $= 6^{2n} - 35n - 1$

$$= (6^2)^n - 35n - 1 = 36^n - 35n - 1$$

$$\text{So, } 6^{2n} - 35n - 1 = (1 + 35)^n - 35n - 1$$

$$= 1 + 35n + {}^nC_2 \cdot 35^2 + \dots + 35^n - 35n - 1$$

$$= 35^2[{}^nC_2 + {}^nC_3 \cdot 35 + \dots + 35^{n-2}]$$

$$= 1225 \times \text{a positive integer, if } n \geq 2$$

If $n = 1$, then $6^{2n} - 35n - 1 = 0$, which is divisible by 1225.

So, options (b), (c) and (d) are not divisible by 1225.

➤ **Example 24.** If 7^{103} is divided by 25, then the remainder is

- (a) 20 (b) 16 (c) 18 (d) 15

Sol. (c) We have, $7^{103} = 7(49)^{51} = 7(50 - 1)^{51}$

$$= 7\{50^{51} - {}^{51}C_1 50^{50} + {}^{51}C_2 50^{49} - \dots - 1\}$$

$$= 7\{(50^{51} - {}^{51}C_1 50^{50} + {}^{51}C_2 50^{49} - \dots) - (7 + 18 - 18)\}$$

$$= k + 18 \text{ (say)} \quad [\because k \text{ is divisible by } 25]$$

$\therefore$ Remainder is 18.

➤ **Example 25.** The fractional part of $\frac{2^{4n}}{15}$ is

- (a) $\frac{1}{15}$ (b) $\frac{2}{15}$
(c) $\frac{4}{15}$ (d) None of these

$$\text{Sol. (a) } \frac{2^{4n}}{15} = \frac{16^n}{15} = \frac{(1 + 15)^n}{15}$$

$$= \frac{1 + {}^nC_1 15 + {}^nC_2 15^2 + \dots + {}^nC_n 15^n}{15}$$

$$= \frac{1 + 15k}{15}, \text{ where } k \in \mathbb{N} = \frac{1}{15} + k$$

$$\therefore \left\{ \frac{2^{4n}}{15} \right\} = \left\{ \frac{1}{15} + k \right\} = \frac{1}{15}$$

➤ **Example 26.** Larger of $99^{50} + 100^{50}$ and 101^{50} is

- (a) 101^{50} (b) $99^{50} + 100^{50}$
(c) both are equal (d) None of these

Sol. (a) We have, $101^{50} = (100 + 1)^{50}$

$$= 100^{50} + 50 \cdot 100^{49} + \frac{50 \cdot 49}{1 \cdot 2} \cdot 100^{48} + \dots \dots (i)$$

and $99^{50} = (100 - 1)^{50}$

$$= 100^{50} - 50 \cdot 100^{49} + \frac{50 \cdot 49}{1 \cdot 2} \cdot 100^{48} - \dots \dots (ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\begin{aligned} 101^{50} - 99^{50} &= 2 \left[50 \cdot 100^{49} + \frac{50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3} \times 100^{47} + \dots \right] \\ &= 100^{50} + 2 \cdot \frac{50 \cdot 49 \cdot 48}{1 \cdot 2 \cdot 3} \cdot 100^{47} + \dots > 100^{50} \end{aligned}$$

$$\Rightarrow 101^{50} > 99^{50} + 100^{50}$$

➤ **Example 27.** If $9^7 + 7^9$ is divisible by

- (a) 6 (b) 24 (c) 64 (d) 72

Sol. (c) We have, $9^7 + 7^9 = (1 + 8)^7 - (1 - 8)^9$
 $= (1 + {}^7C_1 \cdot 8^1 + {}^7C_2 \cdot 8^2 + \dots + {}^7C_7 \cdot 8^7)$
 $- (1 - {}^9C_1 \cdot 8^1 + {}^9C_2 \cdot 8^2 - \dots - {}^9C_9 \cdot 8^9)$
 $= 16 \times 8 + 64[({}^7C_2 + \dots + {}^7C_7 \cdot 8^5)$
 $- ({}^9C_2 - \dots - {}^9C_9 \cdot 8^7)]$
 $= 64$ (an integer)

➤ **Example 28.** The greatest integer which divides the number $101^{100} - 1$ is

- (a) 100
(b) 1000
(c) 10000
(d) 100000

Sol. (c) By binomial theorem,

$$(1 + x)^n = \left[1 + nx + \frac{n(n-1)}{2} \cdot x^2 + \dots + x^n \right]$$

$$\Rightarrow (1 + x)^n - 1 = nx + \frac{n(n-1)}{2} x^2 + \dots + x^n$$

$$\text{If } x = n, (1 + n)^n - 1 = n^2 + \frac{n(n-1)}{2} n^2 + \dots + n^n$$

$$(1 + n)^n - 1 = n^2 \left[1 + \frac{n(n-1)}{2} + \dots + n^{n-2} \right]$$

Put $n = 100$,

$$(1 + 100)^{100} - 1 = (100)^2 \left[1 + \frac{100(100-1)}{2} + \dots + 100^{98} \right]$$

Clearly, $(101)^{100} - 1$ is divisible by $(100)^2 = 10000$

Work Book Exercise 8.2

1 If n is even, then the greatest coefficient in the expansion of $(x + a)^n$ is

- a ${}^nC_{\frac{n}{2}+1}$ b ${}^nC_{\frac{n}{2}-1}$
c ${}^nC_{\frac{n}{2}}$ d None of these

2 If n is even positive integer, then the condition that the greatest term in the expansion of $(1 + x)^n$ may have the greatest coefficient also is

- a $\frac{n}{n+2} < x < \frac{n+2}{n}$
b $\frac{n+1}{n} < x < \frac{n}{n+1}$
c $\frac{n}{n+4} < x < \frac{n+4}{4}$
d None of the above

3 If $n > 1$, then $(1 + x)^n - nx - 1$ is divisible by

- a x^5 b x^2 c x^3
d x^4

4 The positive integer just greater than $(1 + 0.0001)^{10000}$ is

- a 3
b 4
c 5
d None of the above

5 If $[x]$ denotes the greatest integer less than or equal to x , then $[(1 + 0.0001)^{10000}]$ equals

- a 3
b 2
c 0
d None of the above

6 If $R = (6\sqrt{6} + 14)^{2n+1}$ and $f = R - [R]$, where $[]$ denotes the greatest integer function, then Rf equals

- a 20^n b 20^{2n}
c 20^{2n+1} d None of these

7 If $R = (\sqrt{2} + 1)^{2n+1}$ and $f = R - [R]$, where $[]$ denotes the greatest integer function, then $[R]$ equals

- a $f + \frac{1}{f}$
b $f - \frac{1}{f}$
c $\frac{1}{f} - f$
d None of the above

8 If $(5 + 2\sqrt{6})^n = I + f$, where $I \in N$, $n \in N$ and $0 \leq f < 1$, then I equals

- a $\frac{1}{f} - f$ b $\frac{1}{1+f} - f$
c $\frac{1}{1-f} - f$ d $\frac{1}{1-f} + f$

9 If $R = (7 + 4\sqrt{3})^{2n} = I + f$, where $I \in N$ and $0 < f < 1$, then $R(1 - f)$ equals

- a $(7 - 4\sqrt{3})^{2n}$ b $\frac{1}{(7 + 4\sqrt{3})^{2n}}$
c 1 d None of these

10 The digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$ is

- a 0 b 1
c 2 d 3

Properties of the Binomial Coefficients

8

Binomial Theorem

- i. In the expansion of $(1+x)^n$ the coefficients of terms equidistant from the beginning and the end are equal.

Proof

We have,

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_r x^r + \dots + C_{n-1}x^{n-1} + C_n x^n$$

In this expansion the coefficient of $(r+1)$ th term from the beginning is ${}^nC_r$. The $(r+1)$ th term from the end is $(n-r+1)$ th term from the beginning. Therefore, its coefficient is ${}^nC_{n-r}$. But, we have ${}^nC_r = {}^nC_{n-r}$.

Hence, the coefficients of terms equidistant from the beginning and the end are equal.

- ii. The sum of the binomial coefficients in the expansion of $(1+x)^n$ is 2^n .

i.e. $C_0 + C_1 + C_2 + \dots + C_n = 2^n$

or $\sum_{r=0}^n {}^nC_r = 2^n$

Proof

We have, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_{n-1}x^{n-1} + C_n x^n \dots (i)$

Putting $x=1$ in this expansion, we get

$$2^n = C_0 + C_1 + C_2 + C_3 + \dots + C_{n-1} + C_n$$

- It follows from the above result that
- $\sum_{r=0}^n {}^nC_r = 2^n$, $\sum_{r=1}^n {}^{n-1}C_{r-1} = 2^{n-1}$ etc.
- By putting $x=-1$ in Eq. (i), we get $\sum_{r=0}^n (-1)^r C_r = 0$ i.e. $C_0 - C_1 + C_2 - C_3 + \dots = 0$
- Sum of coefficients in the expansion of $(a+bx+cx^2)^n$ is $(a+b+c)^n$.

➤ **Example 29.** The sum of the last ten coefficients in the expansion of $(1+x)^{19}$ when expanded in ascending powers of x , is

- (a) 2^{18} (b) 2^{19}
(c) $2^{18} - {}^{19}C_{10}$ (d) None of these

Sol. (a) The required sum

$$= {}^{19}C_{10} + {}^{19}C_{11} + \dots + {}^{19}C_{19} \dots (i)$$

$$= {}^{19}C_0 + {}^{19}C_1 + {}^{19}C_2 + \dots + {}^{19}C_9 \dots (ii)$$

$$[\because {}^nC_r = {}^nC_{n-r}]$$

On adding Eqs. (i) and (ii), we get

$$2 \times (\text{Required sum}) = {}^{19}C_0 + {}^{19}C_1 + \dots + {}^{19}C_{19} = 2^{19}$$

$$\therefore \text{Required sum} = 2^{18}$$

➤ **Example 30.** If the sum of the coefficients in the expansion of $(a^2x^2 - 2ax + 1)^{51}$ vanishes, then the value of a is

- (a) 2 (b) -1 (c) 1 (d) -2

Sol. (c) The sum of the coefficients of the polynomial $(a^2x^2 - 2ax + 1)^{51}$ is obtained by putting $x=1$ in $(a^2x^2 - 2ax + 1)^{51}$.

By hypothesis, $(a^2 - 2a + 1)^{51} = 0$

$$\Rightarrow a = 1$$

- iii. The sum of the coefficients of the odd terms in the expansion of $(1+x)^n$ is equal to the sum of the coefficients of the even terms and each is equal to 2^{n-1} .

i.e.

$$C_0 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots = 2^{n-1}$$

Proof

We have, $(1+x)^n = C_0 + C_1x + C_2x^2 + C_3x^3 + \dots + C_{n-1}x^{n-1} + C_n x^n$

Putting $x=1$ and -1 respectively in the above expansion, we get

$$2^n = C_0 + C_1 + C_2 + C_3 + \dots + C_{n-1} + C_n$$

$$\text{and } 0 = C_0 - C_1 + C_2 - C_3 + \dots + (-1)^{n-1} C_{n-1} + (-1)^n C_n$$

On adding and subtracting these two expressions, we get

$$2^n = 2(C_0 + C_2 + C_4 + \dots)$$

and $2^n = 2(C_1 + C_3 + C_5 + \dots)$

$$\Rightarrow C_0 + C_2 + C_4 + \dots = 2^{n-1} = C_1 + C_3 + C_5 + \dots$$

iv. ${}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$ and

so on.

Proof

We have, ${}^nC_r = \frac{n!}{(n-r)!r!}$

$$= \frac{n(n-1)!}{r(r-1)! \{(n-1)-(r-1)\}!}$$

$$= \frac{n}{r} \cdot {}^{n-1}C_{r-1}$$

Similarly, we have ${}^{n-1}C_{r-1} = \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$

$$\therefore {}^nC_r = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$$

$$= \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$$

➤ **Example 31.** $\frac{1}{n!} + \frac{1}{2!(n-2)!} + \frac{1}{4!(n-4)!} + \dots$ is

equal to

(a) $\frac{2^{n-1}}{n!}$

(b) $\frac{2^n}{(n+1)!}$

(c) $\frac{2^n}{n!}$

(d) $\frac{2^{n-2}}{(n-1)!}$

Sol. (a) Given expression

$$= \frac{1}{n!} [{}^nC_0 + {}^nC_2 + {}^nC_4 + \dots] = \frac{2^{n-1}}{n!}$$

➤ **Example 32.** The sum

$$1 \cdot {}^{20}C_1 - 2 \cdot {}^{20}C_2 + 3 \cdot {}^{20}C_3 - \dots - 20 \cdot {}^{20}C_{20} \text{ is}$$

equal to

(a) 2^{19}

(b) 0

(c) $2^{20} - 1$

(d) None of these

Sol. (b) Using $r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1}$

$$\begin{aligned} \therefore \sum_{r=1}^{20} (-1)^{r-1} r \cdot {}^{20}C_r &= \sum_{r=1}^{20} (-1)^{r-1} \cdot 20 \cdot {}^{19}C_{r-1} \\ &= 20 \{ {}^{19}C_0 - {}^{19}C_1 + {}^{19}C_2 - \dots - {}^{19}C_{19} \} \\ &= 20 \times 0 = 0 \end{aligned}$$

$$\begin{aligned} \text{v. } C_0 C_r + C_1 C_{r+1} + \dots + C_{n-r} C_n &= {}^{2n}C_{n+r} \\ &= \frac{(2n)!}{(n+r)!(n-r)!} \end{aligned}$$

Proof Replacing x by $\frac{1}{x}$ in Eq. (i), we get

$$\left(1 + \frac{1}{x}\right)^n = C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} \quad \dots \text{(ii)}$$

On multiplying Eqs. (i) and (ii), we get

$$\begin{aligned} \frac{(1+x)^{2n}}{x^n} &= (C_0 + C_1 x + C_2 x^2 + \dots) \\ &\quad \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots \right) \end{aligned}$$

Now, comparing coefficient of x^r on both sides, we get

$$\begin{aligned} C_0 C_r + C_1 C_{r+1} + \dots + C_{n-r} C_n &= {}^{2n}C_{n+r} \\ &= \frac{(2n)!}{(n+r)!(n-r)!} \quad \dots \text{(iii)} \end{aligned}$$

vi. Sum of squares of coefficients is ${}^{2n}C_n$ i.e.

$$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n! n!}$$

Proof On putting $r=0$ in Eq. (iii), we get

$$C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2 = \frac{(2n)!}{n! n!}$$

➤ **Example 33.** If

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n, \text{ then}$$

$\sum_{0 \leq i < j \leq n} (C_i + C_j)^2$ is equal to

(a) $(n-1) \cdot {}^{2n}C_n + 2^{2n}$

(b) $n \cdot {}^{2n}C_n + 2^{2n}$

(c) $(n+1) \cdot {}^{2n}C_n + 2^{2n}$

(d) None of the above

Sol. (a) $\sum_{0 \leq i < j \leq n} (C_i + C_j)^2$

$$\begin{aligned} &= n(C_0^2 + C_1^2 + \dots + C_n^2) + 2 \sum_{0 \leq i < j \leq n} C_i C_j \\ &= n \cdot {}^{2n}C_n + [(C_0 + C_1 + \dots + C_n)^2 - (C_0^2 + C_1^2 + \dots + C_n^2)] \\ &= n \cdot {}^{2n}C_n + (2^n)^2 - {}^{2n}C_n \\ &= (n-1) \cdot {}^{2n}C_n + 2^{2n} \end{aligned}$$

vii.

$$C_0 C_1 + C_1 C_2 + C_2 C_3 + \dots + C_{n-1} C_n = {}^{2n}C_{n+1}$$

Proof On putting $r=1$ in Eq. (iii), we get

$$\begin{aligned} C_0 C_1 + C_1 C_2 + C_2 C_3 + \dots + C_{n-1} C_n &= {}^{2n}C_{n+1} \\ &= \frac{(2n)!}{(n+1)!(n-1)!} \end{aligned}$$

viii.

$$C_0 C_2 + C_1 C_3 + C_2 C_4 + \dots + C_{n-2} C_n = {}^{2n}C_{n+2}$$

Proof On putting $r=2$ in Eq. (iii), we get

$$\begin{aligned} C_0 C_2 + C_1 C_3 + C_2 C_4 + \dots + C_{n-2} C_n &= {}^{2n}C_{n+2} \\ &= \frac{(2n)!}{(n+2)!(n-2)!} \end{aligned}$$

ix.

$$C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n = 2^{n-1}(n+2)$$

Proof Consider,

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$$

Now, on multiplying both sides by x , we get

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + \dots + C_n x^{n+1}$$

On differentiating both sides w.r.t. x , we get

$$(1+x)^n + nx(1+x)^{n-1} = C_0 + 2C_1 x + 3C_2 x^2 + \dots + (n+1)C_n x^n$$

Now, putting $x=1$, we get

$$\begin{aligned} 2^n + n(2)^{n-1} &= C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n \\ \Rightarrow C_0 + 2C_1 + 3C_2 + \dots + (n+1)C_n &= 2^{n-1}(n+2) \end{aligned}$$

x. (i) $C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} = \frac{2^{n+1}-1}{n+1}$

(ii) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + \frac{(-1)^n C_n}{n+1} = \frac{1}{n+1}$

Proof (i) Integrating expansion of $(1+x)^n$ w.r.t. x , between the limits 0 to 1, we get

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 = \left[C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + C_n \frac{x^{n+1}}{n+1} \right]_0^1$$

$$\Rightarrow \frac{2^{n+1} - 1}{n+1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1}$$

(ii) Integrating expansion of $(1+x)^n$ between limits -1 to 0 , we get

$$\left[\frac{(1+x)^{n+1}}{n+1} \right]_{-1}^0 = \left[C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \dots + \frac{C_n x^{n+1}}{n+1} \right]_{-1}^0$$

$$\Rightarrow \frac{1}{n+1} = C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \frac{C_3}{4} + \dots + \frac{(-1)^n C_n}{n+1}$$

Some Other Important Properties

- xi.* ${}^n C_r = {}^n C_s$, then either $r = s$ or $r + s = n$
- xii.* $\frac{{}^n C_r}{{}^n C_{r-1}} = \frac{n-r+1}{r}$
- xiii.* ${}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$
- xiv.* $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots$
 $= \begin{cases} 0, & \text{if } n \text{ is odd} \\ (-1)^{n/2} {}^n C_{n/2}, & \text{if } n \text{ is even} \end{cases}$

➤ **Example 34.** The value of

$${}^{47} C_4 + \sum_{j=0}^3 {}^{50-j} C_3 + \sum_{k=0}^5 {}^{56-k} C_{53-k} \text{ is}$$

- (a) ${}^{57} C_4$ (b) ${}^{57} C_3$ (c) ${}^{57} C_5$ (d) ${}^{57} C_6$

Sol. (a) We have, ${}^{47} C_4 + \sum_{j=0}^3 {}^{50-j} C_3 + \sum_{k=0}^5 {}^{56-k} C_{53-k}$

$$= {}^{47} C_4 + ({}^{50} C_3 + {}^{49} C_3 + {}^{48} C_3 + {}^{47} C_3) +$$

$$({}^{56} C_{53} + {}^{55} C_{52} + {}^{54} C_{51} + {}^{53} C_{50} + {}^{52} C_{49} + {}^{51} C_{48})$$

$$= {}^{47} C_4 + ({}^{47} C_3 + {}^{48} C_3 + {}^{49} C_3 + {}^{50} C_3) +$$

$$({}^{56} C_3 + {}^{55} C_3 + {}^{54} C_3 + {}^{53} C_3 + {}^{52} C_3 + {}^{51} C_3)$$

$$[\because {}^n C_r = {}^n C_{n-r}]$$

$$= ({}^{47} C_4 + {}^{47} C_3) + ({}^{48} C_3 + {}^{49} C_3 + {}^{50} C_3) +$$

$$({}^{51} C_3 + {}^{52} C_3 + \dots + {}^{56} C_3)$$

$$[\because {}^n C_r + {}^n C_{r-1} = {}^{n+1} C_r]$$

$$= ({}^{48} C_4 + {}^{48} C_3) + {}^{49} C_3 + {}^{50} C_3 + \dots + {}^{56} C_3 = {}^{57} C_4$$

➤ **Example 35.** If n is a positive integer and

$C_k = {}^n C_k$, then the value of $\sum_{k=1}^n k^3 \left(\frac{C_k}{C_{k-1}} \right)^2$ is

- (a) $\frac{n(n+2)(n+1)^2}{12}$ (b) $\frac{n(n+1)(n+2)^2}{12}$
 (c) $\frac{n(n+1)(n+2)}{12}$ (d) None of these

Sol. (a) $\sum_{k=1}^n k^3 \left(\frac{C_k}{C_{k-1}} \right)^2 = \sum_{k=1}^n k^3 \left(\frac{n-k+1}{k} \right)^2$

$$\left[\because \frac{{}^n C_k}{{}^n C_{k-1}} = \frac{n-k+1}{k} \right]$$

$$= \sum_{k=1}^n k(n-k+1)^2 = \sum_{k=1}^n k[(n+1)^2 - 2k(n+1) + k^2]$$

$$= (n+1)^2 \sum_{k=1}^n k - 2(n+1) \sum_{k=1}^n k^2 + \sum_{k=1}^n k^3$$

$$= (n+1)^2 \cdot \frac{n(n+1)}{2} - 2(n+1) \cdot \frac{n(n+1)(2n+1)}{6} + \frac{n^2(n+1)^2}{4}$$

$$= \frac{n(n+1)^2}{12} [6(n+1) - 4(2n+1) + 3n]$$

$$= \frac{n(n+1)^2}{12} \cdot (n+2) = \frac{n(n+2)(n+1)^2}{12}$$

➤ **Example 36.** If

$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, then
 $(C_0 + C_1)(C_1 + C_2) \dots (C_{n-1} + C_n)$
 $= k \cdot C_1 C_2 C_3 \dots C_n$, where k is equal to

- (a) $\frac{(n+1)^n}{n!}$ (b) $\frac{n^n}{(n-1)!}$
 (c) $\frac{(n+1)^n}{(n-1)!}$ (d) None of these

Sol. (a) We have, $C_0 + C_1 = {}^{n+1} C_1$, $C_1 + C_2 = {}^{n+1} C_2, \dots$,
 $C_{n-1} + C_n = {}^{n+1} C_n$

$\therefore$ Given expression $= {}^{n+1} C_1 {}^{n+1} C_2 \dots {}^{n+1} C_n \dots$ (i)

Now, ${}^{n+1} C_r = \frac{(n+1)!}{r!(n-r+1)!} = \frac{(n+1)!}{r!(n-r+1)(n-r)!}$

or ${}^{n+1} C_r = \frac{n+1}{n-r+1} {}^n C_r \dots$ (ii)

Putting $n = 1, 2, 3, \dots, n$ in Eq. (ii), we get

$$(C_0 + C_1)(C_1 + C_2) \dots (C_{n-1} + C_n)$$

$$= \frac{n+1}{n} \cdot C_1 \cdot \frac{n+1}{n-1} C_2 \dots \frac{n+1}{1} C_n = \frac{(n+1)^n}{n!} C_1 C_2 C_3 \dots C_n$$

➤ **Example 37.** The value of

$${}^{15} C_0^2 - {}^{15} C_1^2 + {}^{15} C_2^2 - \dots - {}^{15} C_{15}^2 \text{ is}$$

- (a) 15 (b) -15 (c) 0 (d) 51

Sol. (c) As we know that,

$$C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots + (-1)^n C_n^2 = 0 \text{ (if } n \text{ is odd)}$$

Here, $n = 15$ (odd)

$$\therefore C_0^2 - C_1^2 + C_2^2 - \dots - C_{15}^2 = 0$$

➤ **Example 38.** If $(1+x-2x^2)^6 = 1 + a_1x + a_2x^2 + a_3x^3 + \dots$, then the value of $a_2 + a_4 + a_6 + \dots + a_{12}$ will be
(a) 32 (b) 31 (c) 64 (d) 1024

Sol. (b) We have, $(1+x-2x^2)^6 = 1 + a_1x + a_2x^2 + \dots$
Putting $x = 1$, we get
 $0 = 1 + a_1 + a_2 + a_3 + \dots + a_{12} \dots (i)$
Putting $x = -1$, we get
 $64 = 1 - a_1 + a_2 - a_3 + \dots + a_{12} \dots (ii)$
On adding Eqs. (i) and (ii), we get
 $64 = 2 + 2a_2 + 2a_4 + 2a_6 + \dots + 2a_{12}$
 $\Rightarrow a_2 + a_4 + a_6 + \dots + a_{12} = 31$

➤ **Example 39.** The sum ${}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{10}$ is equal to

- (a) $2^{20} + \frac{20!}{(10!)^2}$
(b) $2^{19} - \frac{1}{2} \cdot \frac{20!}{(10!)^2}$
(c) $2^{19} + {}^{20}C_{10}$
(d) None of the above

Sol. (d) Let $S = {}^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \dots + {}^{20}C_{10} \dots (i)$
 $S = {}^{20}C_{20} + {}^{20}C_{19} + {}^{20}C_{18} + \dots + {}^{20}C_{10}$
 $[\because {}^nC_r = {}^nC_{n-r}] \dots (ii)$
On adding Eqs. (i) and (ii), we get
 $2S = ({}^{20}C_0 + {}^{20}C_1 + \dots + {}^{20}C_{20}) + {}^{20}C_{10}$
 $= 2^{20} + {}^{20}C_{10}$
 $\therefore S = 2^{19} + \frac{{}^{20}C_{10}}{2}$

➤ **Example 40.** If $A = {}^{2n}C_0 \cdot {}^{2n}C_1 + {}^{2n}C_1 \cdot {}^{2n-1}C_1 + {}^{2n}C_2 \cdot {}^{2n-2}C_1 + \dots$, then A is
(a) 0 (b) 2^n (c) $n \cdot 2^{2n}$ (d) 1

Sol. (c) $A =$ Coefficient of x in $[{}^{2n}C_0(1+x)^{2n} + {}^{2n}C_1(1+x)^{2n-1} + \dots]$
 $=$ Coefficient of x in $(1+(1+x))^{2n}$
 $=$ Coefficient of x in $(2+x)^{2n}$
 $=$ Coefficient of x in $2^{2n} \left(1 + \frac{x}{2}\right)^{2n} = n \cdot 2^{2n}$

➤ **Example 41.** $\sum_{r=0}^n {}^nC_r \sin rx \cos (n-r)x$ is equal to

- (a) $2^{n-1} \sin (n-1)x$ (b) $2^n \sin nx$
(c) $2^{n-1} \sin nx$ (d) None of these

Sol. (c) We have, $\sum_{r=0}^n {}^nC_r \sin rx \cos (n-r)x$
 $= \frac{1}{2} [({}^nC_0 \sin 0x \cos nx + {}^nC_n \sin nx \cos 0x)$
 $+ ({}^nC_1 \sin x \cos (n-1)x + {}^nC_{n-1} \sin (n-1)x \cos x)$
 $+ \dots + ({}^nC_n \sin nx \cos 0x + {}^nC_0 \sin 0x \cos nx)]$
 $= \frac{1}{2} [{}^nC_0 \sin nx + {}^nC_1 \sin nx + \dots + {}^nC_n \sin nx]$
 $= \frac{1}{2} [({}^nC_0 + {}^nC_1 + \dots + {}^nC_n) \sin nx] = \frac{2^n \sin nx}{2}$
 $\therefore \sum_{r=0}^n {}^nC_r \sin rx \cos (n-r)x = 2^{n-1} \sin nx$

Work Book Exercise 8.3

- If the sum of the coefficients in the expansion of $(a+b)^n$ is 4096, then the greatest coefficient in the expansion is
a 924 b 792
c 1594 d None of these
- If the sum of the coefficients in the expansion of $(1+2x)^n$ is 6561, the greatest term in the expansion for $x = \frac{1}{2}$, is
a 4th b 5th
c 6th d None of these
- If the sum of the coefficients in the expansion of $(1+2x)^n$ is 6561, then the greatest coefficient in the expansion is
a 896 b 3594
c 1792 d None of the above

- If n is an odd natural number, then $\sum_{r=0}^n \frac{(-1)^r}{{}^nC_r}$ equals
a 0 b $\frac{1}{n}$
c $\frac{n}{2^n}$ d None of these
- If n is an even natural number, then $\sum_{r=0}^n \frac{(-1)^r}{{}^nC_r}$ equals
a 0 b $\frac{1}{n}$
c $\frac{(-1)^{n/2}}{{}^nC_{n/2}}$ d None of these
- If $a_n = \sum_{r=0}^n \frac{1}{{}^nC_r}$, then $\sum_{r=0}^n \frac{r}{{}^nC_r}$ equals
a $(n-1)a_n$ b na_n
c $\frac{n}{2}a_n$ d None of these

- 7 The value of $1 \cdot C_1 + 3 \cdot C_3 + 5 \cdot C_5 + 7 \cdot C_7 + \dots$, where $C_0, C_1, C_2, \dots, C_n$ are the binomial coefficients in the expansion of $(1+x)^n$, is
- a $n \cdot 2^{n-1}$ b $n \cdot 2^{n-2}$
c $n(n-1)2^{n-1}$ d None of these

- 8 If C_r is the coefficient of x^r in $(1+x)^n$, then the value of $\sum_{r=0}^n (r+1)^2 C_r$ is

- a $(n+1)(n+4)2^{n-2}$ b $(n+1)(n+4)2^{n-1}$
c $(n+1)^2 2^{n-2}$ d None of these

- 9 If C_r stands for ${}^nC_r$, then the sum of the series

$$\frac{2\left(\frac{n}{2}\right)! \left(\frac{n}{2}\right)!}{n!} [C_0^2 - 2C_1^2 + \dots + (-1)^n (n+1)C_n^2],$$

where n is an even positive integer, is equal to

- a 0 b $(-1)^{n/2}(n+1)$
c $(-1)^{n/2}(n+2)$ d $(-1)^n n$

- 10 In the expansion of $(1+x)^n(1+y)^n(1+z)^n$, the sum of the coefficients of the terms of degree r is

- a $({}^nC_r)^3$ b $3 \cdot {}^nC_r$
c ${}^{3n}C_r$ d ${}^nC_{3r}$

- 11 There are two bags each of which contains n balls. A man has to select an equal number of balls from both the bags. The number of ways in which a man can choose atleast one ball from each bag is

- a ${}^{2n}C_n$ b $({}^nC_n)^2$
c ${}^{2n}C_1$ d ${}^{2n}C_n - 1$

- 12 P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the elements of A . A subset B of P is chosen again. The number of ways of choosing A and B such that A and B have no common elements is

- a 2^n
b 3^n
c 4^n
d None of the above

- 13 P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the elements of A . A subset B of P is chosen again. The number of ways of choosing A and B such that $A = B$, is

- a 2^n b 3^n
c ${}^{2n}C_n$ d None of these

- 14 P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the elements of A . A subset B of P is chosen again. The number of ways of choosing A and B such that B is a subset of A , is

- a 2^n b 3^n
c ${}^{2n}C_n$ d None of these

- 15 P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the elements of A . A subset B of P is chosen again. The number of ways of choosing A and B such that B contains just one element more than A , is

- a ${}^{2n}C_{n-1}$ b 3^n c $(2^n)^2$ d ${}^{2n}C_n$

- 16 P is a set containing n elements. A subset A of P is chosen and the set P is reconstructed by replacing the elements of A . A subset B of P is chosen again. The number of ways of choosing A and B such that A and B have equal number of elements, is

- a 2^n b 3^n c $(2^n)^2$ d ${}^{2n}C_n$

- 17 If $n > 3$, then

$$xyC_0 - (x-1)(y-1)C_1 + (x-2)(y-2)C_2 - (x-3)(y-3)C_3 + \dots + (-1)^n(x-n)(y-n)C_n$$

- equals
- a $xy \times 2^n$ b nxy
c xy d None of these

- 18 If $n > 3$, then $xyzC_0 - (x-1)(y-1)(z-1)C_1$

$$+ (x-2)(y-2)(z-2)C_2 - (x-3)(y-3)(z-3)C_3 + \dots + (-1)^n(x-n)(y-n)(z-n)C_n$$

- equals
- a xyz b 0
c $-xyz$ d None of these

Binomial Theorem for any Index

Statement (Binomial theorem for any index)

Let n be a rational number and x be a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots + \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

- The condition $|x| < 1$ is unnecessary, if n is a whole number while the same condition is essential if n is a rational number other than a whole number.
- There are infinite number of terms in the expansion of $(1+x)^n$, when n is a negative integer or a fraction.

- If first term is not 1, then make first term unity in the following ways,

$$(x+a)^n = a^n \left(1 + \frac{x}{a}\right)^n, \text{ if } \left|\frac{x}{a}\right| < 1$$

- Expansion of $(x+a)^n$ for any index

Case I When $|x| > |a|$ i.e. $\left|\frac{a}{x}\right| < 1$

$$\begin{aligned} \text{In this case, } (x+a)^n &= \left\{x \left(1 + \frac{a}{x}\right)\right\}^n = x^n \left(1 + \frac{a}{x}\right)^n \\ &= x^n \left\{1 + n \cdot \frac{a}{x} + \frac{n(n-1)}{2!} \left(\frac{a}{x}\right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{a}{x}\right)^3 + \dots\right\} \end{aligned}$$

Case II When $|x| < |a|$ i.e. $\left|\frac{x}{a}\right| < 1$.

In this case, $(x+a)^n = \left\{ a \left(1 + \frac{x}{a} \right) \right\}^n = a^n \left(1 + \frac{x}{a} \right)^n$

$$= a^n \left\{ 1 + n \cdot \frac{x}{a} + \frac{n(n-1)}{2!} \left(\frac{x}{a} \right)^2 + \frac{n(n-1)(n-2)}{3!} \left(\frac{x}{a} \right)^3 + \dots \right\}$$

- If n is a positive integer, the expansion of $(1+x)^n$ contains $(n+1)$ terms and coincides with

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n,$$

because ${}^nC_0 = 1, {}^nC_1 = n,$

$${}^nC_2 = \frac{n(n-1)}{2!},$$

$${}^nC_3 = \frac{n(n-1)(n-2)}{3!}, \dots$$

- The general term in the expansion of $(1+x)^n$ is given by

$$T_{r+1} = \frac{n(n-1)(n-2)\dots\{n-(r-1)\}}{r!} x^r$$

Some Important Deductions

- i. Replacing n by $-n$ in the expansion for $(1+x)^n$, we get

$$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots + (-1)^r \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots$$

The general term in this expansion is

$$T_{r+1} = (-1)^r \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r$$

- ii. Replacing x by $-x$ and n by $-n$ in the expansion of $(1+x)^n$, we get

$$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots + \frac{n(n+1)(n+2)\dots(n+r-1)}{r!}x^r + \dots$$

The general term in this expansion is

$$T_{r+1} = \frac{n(n+1)(n+2)\dots(n+r-1)}{r!} x^r$$

- iii. Replacing x by $-x$ in the expansion of $(1+x)^n$, we get

$$(1-x)^n = 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + \dots + (-1)^r \frac{n(n-1)(n-2)\dots(n-r+1)}{r!}x^r + \dots$$

The general term is

$$T_{r+1} = (-1)^r \frac{n(n-1)(n-2)\dots(n-r+1)}{r!} x^r$$

- iv. Putting $n=1, 2, 3$ in the expansion of $(1+x)^{-n}$, we get

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \dots + (-1)^r x^r + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - \dots + (-1)^r (r+1)x^r + \dots$$

$$(1+x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \dots + (-1)^r \frac{(r+1)(r+2)}{2} x^r + \dots$$

Replacing x by $-x$ in the above expansions, we get

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots + x^r + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots + (r+1)x^r + \dots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + \dots + \frac{(r+1)(r+2)}{2} x^r + \dots$$

In all these expansion, it is assumed that $|x| < 1$. Students are advised to learn these expansions.

Example 42. The expansion of $\frac{1}{(4-3x)^{1/2}}$,

binomial theorem will be valid, if

(a) $x < 1$

(b) $|x| < 1$

(c) $-\frac{2}{\sqrt{3}} < x < \frac{2}{\sqrt{3}}$

(d) None of these

Sol. (d) The given expression can be written as

$$4^{-1/2} \left(1 - \frac{3}{4}x \right)^{-1/2} \text{ and it is valid only when } \left| \frac{3}{4}x \right| < 1 \Rightarrow -\frac{4}{3} < x < \frac{4}{3}$$

Example 43. Evaluate $\frac{1}{\sqrt[3]{6-3x}}$.

(a) $6^{1/3} \left[1 + \frac{x}{6} + \frac{2x^2}{6^2} + \dots \right]$ (b) $6^{-1/3} \left[1 + \frac{x}{6} + \frac{2x^2}{6^2} + \dots \right]$

(c) $6^{1/3} \left[1 - \frac{x}{6} + \frac{2x^2}{6^2} - \dots \right]$ (d) $6^{-1/3} \left[1 - \frac{x}{6} + \frac{2x^2}{6^2} - \dots \right]$

Sol. (b) $\frac{1}{(6-3x)^{1/3}} = (6-3x)^{-1/3} = 6^{-1/3} \left(1 - \frac{x}{2} \right)^{-1/3}$

$$= 6^{-1/3} \left[1 + \left(-\frac{1}{3} \right) \left(-\frac{x}{2} \right) + \frac{\left(-\frac{1}{3} \right) \left(-\frac{4}{3} \right)}{2 \cdot 1} \left(-\frac{x}{2} \right)^2 + \dots \right]$$

$$= 6^{-1/3} \left[1 + \frac{x}{6} + \frac{2x^2}{6^2} + \dots \right]$$

Example 44. If $\frac{1}{1-2x+x^2} = 1 + a_1x + a_2x^2 + \dots$,

then the value of a_r is

(a) $2r$

(b) $r+1$

(c) r

(d) $r-1$

Sol. (b) $\frac{1}{1-2x+x^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + \dots$
 $= 1 + a_1x + a_2x^2 + \dots \Rightarrow a_r = r + 1$

➤ **Example 45.** The coefficient of x^n in the expansion of $\frac{1}{(1-x)(1-2x)(1-3x)}$ is

(a) $\frac{1}{2}(2^{n+2} - 3^{n+3} + 1)$ (b) $\frac{1}{2}(3^{n+2} - 2^{n+3} + 1)$
 (c) $\frac{1}{2}(2^{n+3} - 3^{n+2} + 1)$ (d) None of these

Sol. (b) We have, $\frac{1}{(1-x)(1-2x)(1-3x)}$
 $= \frac{1}{2(1-x)} - \frac{4}{1-2x} + \frac{9}{2(1-3x)}$
 [by resolving into partial fractions]
 $= \frac{1}{2}(1-x)^{-1} - 4(1-2x)^{-1} + \frac{9}{2}(1-3x)^{-1}$
 $= \frac{1}{2}[1 + x + x^2 + \dots + x^n + \dots]$
 $- 4[1 + 2x + (2x)^2 + \dots + (2x)^n + \dots]$
 $+ \frac{9}{2}[1 + (3x) + (3x)^2 + \dots + (3x)^n + \dots]$
 $\therefore$ Coefficient of $x^n = \frac{1}{2}[1 - 8 \cdot 2^n + 9 \cdot 3^n]$
 $= \frac{1}{2}[1 - 2^{n+3} + 3^{n+2}]$

➤ **Example 46.** If x is nearly equal to one, then the approximate value of $\frac{mx^m - nx^n}{m-n}$ is

(a) x^m (b) x^n
 (c) x^{m+n} (d) None of these

Sol. (c) Since, x is very nearly equal to one, let $x = 1 + h$, where h is very nearly equal to zero, so that h^2 may be neglected.
 Now, $\frac{mx^m - nx^n}{m-n} = \frac{m(1+h)^m - n(1+h)^n}{m-n}$
 $= \frac{m(1+mh) - n(1+nh)}{m-n}$
 [neglecting h^2 and higher powers]
 $= \frac{(m-n)[1 + (m+n)h]}{m-n}$
 $= 1 + (m+n)h = (1+h)^{m+n} = x^{m+n}$

Approximation

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \dots$$

If $x < 1$, the terms of the above expansion go on decreasing and if x is very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion.

Thus, if x is so small that its squares and higher powers may be neglected, then $(1+x)^n = 1 + nx$, approximately. This is an approximate value of $(1+x)^n$.

➤ **Example 47.** If x is so small such that its square and higher powers may be neglected, then the value of $\frac{(8+3x)^{2/3}}{(2+3x)(4-5x)^{1/2}}$ is

(a) $1 - \frac{3}{2}x$ (b) $1 + \frac{5}{8}x$
 (c) $1 - \frac{5}{8}x$ (d) None of these

Sol. (c) We have, $\frac{(8+3x)^{2/3}}{(2+3x)\sqrt{4-5x}}$
 $= \frac{8^{2/3}\left(1 + \frac{3x}{8}\right)^{2/3}}{2\left(1 + \frac{3x}{2}\right)2\sqrt{1 - \frac{5x}{4}}}$
 $= \left(1 + \frac{3x}{8}\right)^{2/3} \left(1 + \frac{3x}{2}\right)^{-1} \left(1 - \frac{5x}{4}\right)^{-1/2}$
 $= \left(1 + \frac{2}{3} \cdot \frac{3x}{8} + \dots\right) \left(1 - \frac{3x}{2} + \dots\right) \left(1 + \frac{5}{8}x + \dots\right)$
 $= 1 + \left(\frac{1}{4} - \frac{3}{2} + \frac{5}{8}\right)x = 1 - \frac{5}{8}x$

➤ **Example 48.** If $\frac{(1-3x)^{1/2} + (1-x)^{5/3}}{\sqrt{4-x}}$ is

approximately equal to $a + bx$ for all small values of x , then

(a) $a = 1, b = -\frac{35}{24}$ (b) $a = -\frac{35}{24}, b = 1$
 (c) $a = 1, b = \frac{35}{24}$ (d) None of these

Sol. (a) Since, the given expression is equal to $a + bx$, so we have to neglect the terms x^2 and higher powers of x .

$\therefore \frac{(1-3x)^{1/2} + (1-x)^{5/3}}{\sqrt{4-x}}$
 $= [(1-3x)^{1/2} + (1-x)^{5/3}][4-x]^{-1/2}$
 $= \left[1 + \frac{1}{2}(-3x) + 1 + \frac{5}{3}(-x)\right]4^{-1/2}\left(1 - \frac{x}{4}\right)^{-1/2}$
 $= \frac{1}{2}\left(2 - \frac{3}{2}x - \frac{5x}{3}\right)\left[1 + \left(-\frac{1}{2}\right)\left(-\frac{x}{4}\right)\right]$
 $= \frac{1}{2}\left(2 - \frac{19}{6}x\right)\left(1 + \frac{x}{8}\right)$
 $= \frac{1}{2}\left(2 + \frac{x}{4} - \frac{19x}{6}\right) = \frac{1}{2}\left(2 - \frac{35x}{12}\right) = 1 - \frac{35x}{24}$
 $\therefore 1 - \frac{35x}{24} = a + bx \Rightarrow a = 1 \text{ and } b = -\frac{35}{24}$

➤ **Example 49.** Cube root of 217 is

(a) 6.01 (b) 6.04
 (c) 6.02 (d) None of these

Sol. (a) Consider, $(217)^{1/3} = (6^3 + 1)^{1/3} = 6\left(1 + \frac{1}{6^3}\right)^{1/3}$
On expanding by binomial theorem, we get
 $(217)^{1/3} = 6\left(1 + \frac{1}{3 \times 216} - \frac{1 \times 2}{3 \times 3 \times 2} \left(\frac{1}{216}\right)^2 + \dots\right) = 6.01$

➤ **Example 50.** The approximate value of $(7.995)^{1/3}$ correct to four decimal places is
(a) 1.9995 (b) 1.9996 (c) 1.9990 (d) 1.9991

Sol. (a) $(7.995)^{1/3} = (8 - 0.005)^{1/3} = (8)^{1/3} \left[1 - \frac{0.005}{8}\right]^{1/3}$
$$= 2 \left[1 - \frac{1}{3} \times \frac{0.005}{8} + \frac{1 \left(\frac{1}{3} - 1\right)}{2!} \left(\frac{0.005}{8}\right)^2 - \dots\right]$$
$$= 2 \left[1 - \frac{0.005}{24} - \frac{1}{3} \times \frac{1}{3} \times \left(\frac{0.005}{8}\right)^2 - \dots\right]$$
$$= 2(1 - 0.000208) = 2 \times 0.999792 = 1.9995$$

Exponential Series

(a) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$, where x may be any

real or complex and $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n$

(b) $a^x = 1 + \frac{x}{1!} \ln a + \frac{x^2}{2!} \ln^2 a + \frac{x^3}{3!} \ln^3 a + \dots$ where
 $a > 0$

- $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \dots$
- e is an irrational number lying between 2.7 and 2.8. Its value correct upto 10 places of decimal is 2.7182818284.
- $e + e^{-1} = 2 \left(1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \dots\right)$
- $e - e^{-1} = 2 \left(1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \dots\right)$
- Logarithms to the base e are known as the Napierian system, so named after Napier, their inventor. They are also called natural logarithm.

➤ **Example 51.** If $x < 1$, then find the coefficient of x^n in the expansion of $\frac{e^x}{1-x}$.

Sol. We have, $\frac{e^x}{1-x} = e^x (1-x)^{-1}$
$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots + \frac{x^{n-2}}{(n-2)!} + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} + \dots\right)$$
$$(1 + x + x^2 + \dots + x^{n-2} + x^{n-1} + x^n + \dots)$$

 $\therefore$ Coefficient of x^n in $\frac{e^x}{(1-x)}$
$$= 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-1)!} + \frac{1}{n!}$$

➤ **Example 52.** The value of $\frac{1 + \frac{2^2}{2!} + \frac{2^4}{3!} + \frac{2^6}{4!} + \dots}{1 + \frac{1}{2!} + \frac{2}{3!} + \frac{2^2}{4!} + \dots}$

is

(a) e^2 (b) $e^2 + 1$ (c) $e^2 - 1$ (d) $e + 1$

Sol. (c) Numerator $= 1 + \frac{2^2}{2!} + \frac{2^4}{3!} + \frac{2^6}{4!} + \dots$
$$= \frac{1}{2^2} \left\{ 2^2 + \frac{2^4}{2!} + \frac{2^6}{3!} + \frac{2^8}{4!} + \dots \right\}$$
$$= \frac{1}{2^2} \left\{ \frac{(2^2)^1}{1!} + \frac{(2^2)^2}{2!} + \frac{(2^2)^3}{3!} + \frac{(2^2)^4}{4!} + \dots \right\}$$
$$= \frac{1}{2^2} \{e^{2^2} - 1\}$$
$$= \frac{e^4 - 1}{4} \quad \dots(i)$$

and denominator

$$= 1 + \frac{1}{2!} + \frac{2}{3!} + \frac{2^2}{4!} + \dots = \frac{1}{2^2} \left\{ 2^2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \dots \right\}$$
$$= \frac{1}{4} \left\{ 1 + \left(1 + 2 + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \dots \right) \right\}$$
$$= \frac{1}{4} \{1 + e^2\} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{1 + \frac{2^2}{2!} + \frac{2^4}{3!} + \frac{2^6}{4!} + \dots}{1 + \frac{1}{2!} + \frac{2}{3!} + \frac{2^2}{4!} + \dots} = \frac{\frac{1}{4}(e^4 - 1)}{\frac{1}{4}(e^2 + 1)} = \frac{(e^2 + 1)(e^2 - 1)}{(e^2 + 1)}$$
$$= e^2 - 1$$

Logarithmic Series

(a) $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$

where, $-1 < x \leq 1$

(b) $\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots$

where, $-1 \leq x < 1$

(c) $\ln \frac{(1+x)}{(1-x)} = 2 \left(x + \frac{x^3}{3} + \frac{x^5}{5} + \dots \right), |x| < 1$

Remember

(a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \ln 2$ (b) $e^{\ln x} = x$

(c) $\ln 2 = 0.693$ (d) $\ln 10 = 2.303$

(e) $\sum_{n=1}^{\infty} \frac{n}{n!} = e$ (f) $\sum_{n=1}^{\infty} \frac{n^2}{n!} = 2e$

(g) $\sum_{n=1}^{\infty} \frac{n^3}{n!} = 5e$ (h) $\sum_{n=1}^{\infty} \frac{n^4}{n!} = 15e$

➤ **Example 53.** The value of

$$1 + \left(\frac{1}{2} + \frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6} + \frac{1}{7}\right)\frac{1}{4^3} + \dots \text{ is}$$

- (a) $\log_e \sqrt{12}$ (b) $\frac{1}{3} \log_e 12$
(c) $\frac{1}{4} \log_e 4$ (d) None of these

Sol. (a) Consider,

$$\begin{aligned} & 1 + \left(\frac{1}{2} + \frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4} + \frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6} + \frac{1}{7}\right)\frac{1}{4^3} + \dots \\ &= \left(\frac{1}{2} \cdot \frac{1}{4} + \frac{1}{4} \cdot \frac{1}{4^2} + \frac{1}{6} \cdot \frac{1}{4^3} + \dots\right) \\ &+ \left(1 + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{5} \cdot \frac{1}{4^2} + \frac{1}{7} \cdot \frac{1}{4^3} + \dots\right) \\ &= \frac{1}{2} \left[\frac{1}{4} + \frac{1}{2} \left(\frac{1}{4}\right)^2 + \frac{1}{3} \left(\frac{1}{4}\right)^3 + \dots \right] \\ &+ 2 \left[\frac{1}{2} + \frac{1}{3} \left(\frac{1}{2}\right)^3 + \frac{1}{5} \left(\frac{1}{2}\right)^5 + \dots \right] \\ &= \frac{1}{2} \left\{ -\log_e \left(1 - \frac{1}{4}\right) \right\} + \log_e \left(\frac{1 + \frac{1}{2}}{1 - \frac{1}{2}} \right) \end{aligned}$$

$$\begin{aligned} &= -\frac{1}{2} \log_e \left(\frac{3}{4}\right) + \log_e (3) = \log_e 3 - \log_e \sqrt{\frac{3}{4}} \\ &= \log_e 3 + \log_e \left(\frac{2}{\sqrt{3}}\right) = \log_e \left(3 \cdot \frac{2}{\sqrt{3}}\right) = \log_e (2\sqrt{3}) \\ &= \log_e (\sqrt{12}) \end{aligned}$$

➤ **Example 54.** In the expansion of

$2 \log_e x - \log_e (x+1) - \log_e (x-1)$, the coefficient of x^{-4} is

- (a) $\frac{1}{2}$
(b) -1
(c) 1
(d) None of the above

Sol. (a) $2 \log_e x - \log_e \left\{ \left(1 + \frac{1}{x}\right)x \right\} - \log_e \left\{ \left(1 - \frac{1}{x}\right)x \right\}$

$$\begin{aligned} &= 2 \log_e x - \left\{ \log_e \left(1 + \frac{1}{x}\right) + \log_e x \right\} \\ &\quad - \left\{ \log_e \left(1 - \frac{1}{x}\right) + \log_e x \right\} \\ &= - \left\{ \log_e \left(1 + \frac{1}{x}\right) + \log_e \left(1 - \frac{1}{x}\right) \right\} = 2 \left\{ \frac{1}{2x^2} + \frac{1}{4x^4} + \dots \right\} \end{aligned}$$

Thus, coefficient of $x^{-4} = 2 \cdot \frac{1}{4} = \frac{1}{2}$

Work Book Exercise 8.4

- $\frac{1}{\left(x^2 + \frac{1}{x}\right)^{4/3}}$ can be expanded by binomial theorem, if
 a $x < 1$ b $|x| < 1$
 c $x > 1$ d $|x| > 1$
- If $|x| < 1$, then $1 + n \left(\frac{2x}{1+x}\right) + \frac{n(n+1)}{2!} \left(\frac{2x}{1+x}\right)^2 + \dots$ is equal to
 a $\left(\frac{1-x}{1+x}\right)^n$ b $\left(\frac{1+x}{1-x}\right)^n$
 c $\left(\frac{1+x}{2x}\right)^n$ d $\left(\frac{2x}{1+x}\right)^n$
- The coefficient of x^3 in the expansion of $\frac{(1+3x)^2}{1-2x}$ will be
 a 8 b 32
 c 50 d None of these
- The coefficient of x^r in the expansion of $(1-4x)^{-1/2}$ is
 a $\frac{(2r)!}{(r!)^2}$
 b ${}^{2r}C_{r+1}$
 c $\frac{1 \cdot 3 \cdot 5 \dots (2r-1)}{2^r \cdot r!}$
 d None of the above

- If the binomial expansion of $(a+bx)^{-2}$ is $\frac{1}{4} - 3x + \dots$, then the values of a and b are
 a 2, 12 b 2, 10
 c 1, 12 d None of these
- In the expansion of $(e^x - 1)(e^{-x} + 1)$, the coefficient of x^3 is
 a 0 b $\frac{1}{3}$
 c $\frac{2}{3}$ d $\frac{1}{6}$
- In the expansion of $\frac{e^{7x} + e^{3x}}{e^{5x}}$, the constant term is
 a 0 b 1
 c 2 d None of these
- The value of $\frac{1}{1 \cdot 3} + \frac{1}{2} \cdot \frac{1}{3 \cdot 5} + \frac{1}{3} \cdot \frac{1}{5 \cdot 7} + \dots$ is
 a $2 \log_e 2 - 1$ b $\log_e 2 - 1$
 c $\log_e 2$ d None of these
- The coefficient of n^{-r} in the expansion of $\log_{10} \left(\frac{n}{n-1} \right)$ is
 a $\frac{1}{r \log_e 10}$ b $\frac{-1}{r \log_e 10}$
 c $\frac{-1}{r! \log_e 10}$ d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The coefficient of x^{50} in the expansion
 $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998}$
 $+ \dots + 1001x^{1000}$ is
 (a) $^{1002}C_{50}$ (b) $^{1002}C_{51}$ (c) $^{1005}C_{50}$ (d) $^{1005}C_{48}$

Sol. Let $S = (1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998}$
 $+ \dots + 1000(1+x)x^{999} + 1001x^{1000} \dots$ (i)
 $\Rightarrow \frac{xS}{1+x} = x(1+x)^{999} + 2x^2(1+x)^{998}$
 $+ \dots + 1000x^{1000} + 1001 \frac{x^{1001}}{1+x} \dots$ (ii)

On subtracting Eq.(ii) from Eq.(i), we get

$$S - \frac{xS}{1+x} = \{(1+x)^{1000} + x(1+x)^{999} + x^2(1+x)^{998} + \dots + x^{1000}\} - 1001 \frac{x^{1001}}{1+x}$$

$$\Rightarrow \frac{S}{1+x} = (1+x)^{1000} \left[\frac{1 - \left(\frac{x}{1+x}\right)^{1001}}{1 - \frac{x}{1+x}} \right] - 1001 \frac{x^{1001}}{1+x}$$

$$\Rightarrow \frac{S}{1+x} = (1+x)^{1001} - x^{1001} - 1001 \frac{x^{1001}}{1+x}$$

$$\Rightarrow S = (1+x)^{1002} - x^{1001}(1+x) - 1001x^{1001}$$

$$\Rightarrow S = (1+x)^{1002} - 1002x^{1001} - x^{1002}$$

$\therefore$ Coefficient of x^{50} in S = Coefficient of x^{50} in
 $(1+x)^{1002}$
 $= ^{1002}C_{50}$

Hence, (a) is the correct answer.

Ex 2. If $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$,
 where $a_0, a_1, a_2, \dots, a_{2n}$ are in AP, then a_n is
 equal to

(a) $\frac{1}{2n+1}$ (b) $\frac{1}{2n-1}$ (c) $\frac{2}{2n-1}$ (d) $\frac{1}{n+1}$

Sol. We have,

$$(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$$

Putting $x=1$ on both sides, we get

$$a_0 + a_1 + a_2 + \dots + a_n + \dots + a_{2n} = 1$$

$$\Rightarrow \frac{2n+1}{2} [a_0 + a_{2n}] = 1$$

[$\because a_0, a_1, \dots, a_{2n}$ are in AP]

$$\Rightarrow a_0 + a_{2n} = \frac{2}{2n+1}$$

$$\Rightarrow a_0 + [a_0 + (2n+1-1)d] = \frac{2}{2n+1}$$

where, d denotes the common difference of the AP.

$$\Rightarrow a_0 + nd = \frac{1}{2n+1}$$

$$\Rightarrow a_n = \frac{1}{2n+1}$$

$$[\because a_n = (n+1)\text{th term} = a_0 + (n+1-1)d]$$

Hence, (a) is the correct answer.

Ex 3. The coefficient of x^r $\{0 \leq r \leq (n-1)\}$ in the
 expansion of

$$(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2$$

$$(x+1)^2 + \dots + (x+1)^{n-1}$$
 is equal to

- (a) $(2^{n+r} + 1)^n C_r$
 (b) $(2^{n-r} + 1)^n C_r$
 (c) $(2^{n-r} - 1)^n C_r$
 (d) $(2^{n+r} - 1)^n C_r$

Sol. We have,

$$(x+2)^{n-1} + (x+2)^{n-2}(x+1) + (x+2)^{n-3}(x+1)^2$$

$$+ \dots + (x+1)^{n-1}$$

$$= (x+2)^{n-1} \left[\frac{1 - \left(\frac{x+1}{x+2}\right)^n}{1 - \frac{x+1}{x+2}} \right]$$

$$= (x+2)^n - (x+1)^n$$

$\therefore$ Coefficient of x^r in the given expression

$$= \text{Coefficient of } x^r \text{ in } (x+2)^n$$

$$- \text{Coefficient of } x^r \text{ in } (x+1)^n$$

$$= {}^nC_r 2^{n-r} - {}^nC_r$$

$$= (2^{n-r} - 1) {}^nC_r$$

Hence, (c) is the correct answer.

Ex 4. If k and n are positive integer and
 $s_k = 1^k + 2^k + \dots + n^k$, then $\sum_{r=1}^m {}^{m+1}C_r s_r$ is equal to

- (a) $(n+1)^{m+1} - (n+1)$
 (b) $(n+1)^{m+1} + (n+1)$
 (c) $(n-1)^{m+1} - (n-1)$
 (d) None of the above

Sol. We have, $\sum_{r=1}^m {}^{m+1}C_r s_r$

$$= \sum_{r=1}^m {}^{m+1}C_r (1^r + 2^r + \dots + n^r)$$

$$= \sum_{k=1}^n \left\{ \sum_{r=1}^m {}^{m+1}C_r k^r \right\}$$

$$\begin{aligned}
 &= \sum_{k=1}^n \left\{ \sum_{r=0}^{m+1} {}^{m+1}C_r k^r \right\} - {}^{m+1}C_0 - {}^{m+1}C_{m+1} k^{m+1} \\
 &= \sum_{k=1}^n \{ (1+k)^{m+1} - 1 - k^{m+1} \} \left[\because \sum_{r=0}^n {}^nC_r x^r = (1+x)^n \right] \\
 &= \sum_{k=1}^n \{ (1+k)^{m+1} - k^{m+1} \} - \sum_{k=1}^n 1 \\
 &= \sum_{k=1}^n \{ (1+k)^{m+1} - k^{m+1} \} - n \\
 &= [(2^{m+1} - 1^{m+1}) + (3^{m+1} - 2^{m+1}) \\
 &\quad + \dots + \{ (n+1)^{m+1} - n^{m+1} \}] - n \\
 &= \{ (n+1)^{m+1} - 1 \} - n = (n+1)^{m+1} - (n+1)
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 5. If $C_0, C_1, C_2, \dots, C_n$ denote the binomial coefficients in the expansion of $(1+x)^n$, then

$\sum_{r=0}^n (-1)^r {}^nC_r \frac{1+r \log_e 10}{(1+\log_e 10)^r}$ is equal to

- (a) 0
(b) 1
(c) 2
(d) Cannot be discussed

Sol. Let $\log_e 10 = x$. Then,

$$\begin{aligned}
 \sum_{r=0}^n (-1)^r {}^nC_r \frac{1+r \log_e 10}{(1+\log_e 10)^r} &= \sum_{r=0}^n (-1)^r {}^nC_r \frac{1+rx}{(1+nx)^r} \\
 &= \sum_{r=0}^n (-1)^r {}^nC_r \left(\frac{1}{1+nx} \right)^r + \sum_{r=0}^n (-1)^r \frac{n}{r} {}^{n-1}C_{r-1} \frac{rx}{(1+nx)^r} \\
 &= \sum_{r=0}^n (-1)^r {}^nC_r \left(\frac{1}{1+nx} \right)^r \\
 &\quad - \frac{nx}{1+nx} \cdot \sum_{r=1}^n (-1)^{r-1} {}^{n-1}C_{r-1} \left(\frac{1}{1+nx} \right)^{r-1} \\
 &= \left(1 - \frac{1}{1+nx} \right)^n - \frac{nx}{1+nx} \left(1 - \frac{1}{1+nx} \right)^{n-1} \\
 &= \left(\frac{nx}{1+nx} \right)^n - \left(\frac{nx}{1+nx} \right)^n = 0
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 6. The coefficient of $x^3 y^4 z^5$ in the expansion of $(xy + yz + zx)^6$ is

- (a) 70
(b) 60
(c) 50
(d) None of the above

Sol. We have,

$$\begin{aligned}
 (xy + yz + zx)^6 &= \sum_{r+s+t=6} \frac{6!}{r!s!t!} (xy)^r (yz)^s (zx)^t \\
 &= \sum_{r+s+t=6} \frac{6!}{r!s!t!} x^{r+t} y^{r+s} z^{s+t}
 \end{aligned}$$

If the general term in the above expansion contains $x^3 y^4 z^5$, then

$$r+t=3, r+s=4 \text{ and } s+t=5$$

Also, $r+s+t=6$

On solving these equations, we get

$$r=1, s=3, t=2$$

$$\therefore \text{Coefficient of } x^3 y^4 z^5 = \frac{6!}{1!3!2!} = \frac{6!}{2!3!} = 60$$

Hence, (b) is the correct answer.

Ex 7. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, $n \in N$, then for $2 \leq m \leq n$,

$C_0 - C_1 + C_2 - C_3 + \dots + (-1)^{m-1} C_{m-1}$ is equal to

- (a) $(-1)^{m-1} \cdot {}^{n-1}C_{m-1}$ (b) ${}^{n-1}C_{m-1}$
(c) $(-1)^m \cdot {}^{n-1}C_{m-1}$ (d) None of these

Sol. We have, $(1-x)^n (1-x)^{-1}$

$$\begin{aligned}
 &= \{C_0 - C_1 x + C_2 x^2 - C_3 x^3 + \dots + (-1)^{m-1} C_{m-1} x^{m-1} \\
 &\quad + \dots + (-1)^n C_n x^n\} \\
 &\quad \times \{1 + x + x^2 + x^3 + \dots + x^{m-1} + x^m + \dots\}
 \end{aligned}$$

Equating coefficients of x^{m-1} on both sides, we get

$$(-1)^{m-1} \cdot {}^{n-1}C_{m-1} = C_0 - C_1 + C_2 - C_3 + \dots + (-1)^{m-1} C_{m-1}$$

$$\begin{aligned}
 \Rightarrow C_0 - C_1 + C_2 - C_3 + \dots + (-1)^{m-1} C_{m-1} \\
 &= (-1)^{m-1} \frac{(n-1)!}{(n-m)! (m-1)!} \\
 &= (-1)^{m-1} \cdot {}^{n-1}C_{m-1}
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 8. For any positive integers m, n with $n \geq m$, let

$\binom{n}{m} = {}^nC_m$. Then,

$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m}$ is equal to

- (a) ${}^nC_m$ (b) ${}^nC_{m+1}$
(c) ${}^{n+1}C_{m+1}$ (d) None of these

Sol. We know that,

${}^nC_r = \text{Coefficient of } x^r \text{ in } (1+x)^n$

$$\therefore \binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \dots + \binom{m}{m}$$

= Coefficient of x^m in $(1+x)^n$ + Coefficient of x^m in $(1+x)^{n-1} + \dots$ + Coefficient of x^m in $(1+x)^m$

= Coefficient of x^m in $[(1+x)^n + (1+x)^{n-1} + \dots + (1+x)^m]$

$$= \text{Coefficient of } x^m \text{ in } (1+x)^m \left[\frac{(1+x)^{n-m+1} - 1}{(1+x) - 1} \right]$$

= Coefficient of x^{m+1} in $\{(1+x)^{n+1} - (1+x)^m\}$

= Coefficient of x^{m+1} in $(1+x)^{n+1} = {}^{n+1}C_{m+1}$

Hence, (c) is the correct answer.

Ex 9. Let n be an even integer and $k = \frac{3n}{2}$. Then, the

value of $\sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1}$ is

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. Since, n is an even integer. Therefore, $n = 2m, m \in \mathbb{N}$

$$\text{Also, } k = \frac{3n}{2} \Rightarrow k = 3m$$

$$\begin{aligned} \therefore \sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1} &= \sum_{r=1}^{3m} (-3)^{r-1} {}^{6m}C_{2r-1} \\ &= {}^{6m}C_1 - {}^{6m}C_3(3) + {}^{6m}C_5(3)^2 - {}^{6m}C_7(3)^3 \\ &\quad + \dots + (-3)^{3m-1} {}^{6m}C_{6m-1} \end{aligned}$$

$$\text{Now, } (1 + i\sqrt{3})^{6m} = \sum_{r=0}^{6m} {}^{6m}C_r (i\sqrt{3})^r$$

$$\begin{aligned} \Rightarrow 2^{6m} \left\{ \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right\}^{6m} &= {}^{6m}C_0 + {}^{6m}C_1(i\sqrt{3}) \\ &\quad + {}^{6m}C_2(i\sqrt{3})^2 + {}^{6m}C_3(i\sqrt{3})^3 \\ &\quad + \dots + {}^{6m}C_{6m}(i\sqrt{3})^{6m} \\ \Rightarrow 2^{6m}(\cos 2m\pi + i \sin 2m\pi) \end{aligned}$$

$$\begin{aligned} &= \{ {}^{6m}C_0 - {}^{6m}C_2 3 + {}^{6m}C_4 3^2 - \dots \} \\ &\quad + i\sqrt{3} \{ {}^{6m}C_1 - {}^{6m}C_3(3) + {}^{6m}C_5(3)^2 - \dots \} \\ &\quad [\text{using De-Moivre's theorem on LHS}] \end{aligned}$$

Equating imaginary parts on both sides, we get

$$\begin{aligned} 2^{6m} \sin 2m\pi &= \sqrt{3} \{ {}^{6m}C_1 - {}^{6m}C_3(3) + {}^{6m}C_5(3)^2 - \dots \} \\ \Rightarrow {}^{6m}C_1 - {}^{6m}C_3(3) + {}^{6m}C_5(3)^2 - \dots &= 0 [\because \sin 2m\pi = 0] \end{aligned}$$

$$\Rightarrow \sum_{r=1}^{3m} (-3)^{r-1} {}^{6m}C_{2r-1} = 0$$

$$\Rightarrow \sum_{r=1}^k (-3)^{r-1} {}^{3n}C_{2r-1} = 0,$$

$$\text{where } n = 2m \text{ and } k = \frac{3n}{2}$$

Hence, (a) is the correct answer.

Ex 10. The value of ${}^nC_3 + {}^nC_7 + {}^nC_{11} + \dots$ is

- (a) $\frac{1}{2} \left\{ 2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4} \right\}$
 (b) $\frac{1}{2} \left\{ 2^{n-1} + 2^{n/2} \sin \frac{n\pi}{4} \right\}$
 (c) $\frac{1}{4} \left\{ 2^{n+1} - 2^{n/2} \sin \frac{n\pi}{4} \right\}$
 (d) None of the above

Sol. We have, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \dots (i)$

Putting $x = 1$ and -1 respectively in Eq. (i), we get

$$C_0 + C_1 + C_2 + \dots + C_n = 2^n$$

$$C_0 - C_1 + C_2 - C_3 + \dots + (-1)^n C_n = 0$$

On adding and subtracting these two, we get

$$C_0 + C_2 + C_4 + \dots = 2^{n-1}$$

$$C_1 + C_3 + C_5 + \dots = 2^{n-1} \dots (ii)$$

Putting $x = i$ in Eq. (i), we get

$$\begin{aligned} (C_0 - C_2 + C_4 - C_6 + \dots) + i(C_1 - C_3 + C_5 - \dots) &= (1+i)^n \\ \Rightarrow (C_0 - C_2 + C_4 - C_6 + \dots) &\quad + i(C_1 - C_3 + C_5 - C_7 + \dots) \\ &= 2^{n/2} \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) \end{aligned}$$

On equating imaginary parts on both sides, we get

$$C_1 - C_3 + C_5 - C_7 + \dots = 2^{n/2} \sin \frac{n\pi}{4} \dots (iii)$$

On subtracting Eq. (iii) from Eq. (ii), we get

$$\begin{aligned} 2(C_3 + C_7 + C_{11} + \dots) &= \left[2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4} \right] \\ \Rightarrow C_3 + C_7 + C_{11} + \dots &= \frac{1}{2} \left[2^{n-1} - 2^{n/2} \sin \frac{n\pi}{4} \right] \end{aligned}$$

Hence, (a) is the correct answer.

Ex 11. The integer just greater than $(\sqrt{3} + 1)^{2m}$ contains

- (a) 2^{m+2} as a factor (b) 2^{m+1} as a factor
 (c) 2^{m+3} as a factor (d) None of these

Sol. Let $(\sqrt{3} + 1)^{2m} = I + F$, where $I \in \mathbb{N}$ and $0 < F < 1$

Let $G = (\sqrt{3} - 1)^{2m}$. Then,

$$\begin{aligned} I + F + G &= (\sqrt{3} + 1)^{2m} + (\sqrt{3} - 1)^{2m} \\ &= 2^m (2 + \sqrt{3})^m + 2^m (2 - \sqrt{3})^m \\ &= 2^{m+1} \times \text{an integer} \dots (i) \end{aligned}$$

$$\Rightarrow I + F + G = \text{an even integer}$$

$$\Rightarrow F + G = \text{an even integer} - I$$

$$\Rightarrow F + G = \text{an integer}$$

$$\Rightarrow F + G = 1 \quad [\because 0 < F < 1, 0 < G < 1]$$

Putting $F + G = 1$ in Eq. (i), we get

$$I + 1 = 2^{m+1} \times \text{an integer}$$

$$\Rightarrow 2^{m+1} \text{ is a factor of the integer just greater than } (\sqrt{3} + 1)^{2m}.$$

Hence, (b) is the correct answer.

Ex 12. If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ and $a_k = 1$ for

all $k \geq n$, then b_n is equal to

- (a) ${}^{n+1}C_{2n+1}$ (b) ${}^{2n+1}C_{n+1}$
 (c) ${}^{2n+1}C_{n+2}$ (d) None of these

Sol. Clearly, b_n is the coefficient of $(x-3)^n$ in the expression

$$\sum_{r=0}^{2n} b_r (x-3)^r. \text{ Therefore,}$$

$$b_n = \text{Coefficient of } (x-3)^n \text{ in } \left(\sum_{r=0}^{2n} a_r (x-2)^r \right) \dots (i)$$

$$= \text{Coefficient of } (x-3)^n \text{ in}$$

$$\left\{ \sum_{r=0}^{n-1} a_r (x-2)^r + \sum_{r=n}^{2n} a_r (x-2)^r \right\}$$

= Coefficient of $(x-3)^n$ in

$$\left\{ \sum_{r=0}^{n-1} a_r (x-2)^r + \sum_{r=n}^{2n} (x-2)^r \right\}$$

[$\because a_k = 1$ for all $k \geq n$]

= Coefficient of $(x-3)^n$ in $\sum_{r=0}^{2n} (x-2)^r$

= Coefficient of $(x-3)^n$ in $\left[(x-2)^n \left\{ \frac{(x-2)^{n+1}-1}{(x-2)-1} \right\} \right]$

= Coefficient of $(x-3)^n$ in $\left\{ \frac{(x-2)^{2n+1} - (x-2)^n}{x-3} \right\}$

= Coefficient of $(x-3)^{n+1}$ in $\{(x-2)^{2n+1} - (x-2)^n\}$

= Coefficient of $(x-3)^{n+1}$ in $(x-2)^{2n+1}$

= Coefficient of $(x-3)^{n+1}$ in $[(x-3)+1]^{2n+1}$

= Coefficient of $(x-3)^{n+1}$ in $\left\{ \sum_{r=0}^{2n+1} {}^{2n+1}C_r (x-3)^r \right\}$

= ${}^{2n+1}C_{n+1}$

Hence, (b) is the correct answer.

Ex 13. Let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$ and

$$g(x) = b_0 + b_1x + b_2x^2 + \dots + b_{n-1}x^{n-1} + x^n + x^{n+1} + \dots + x^{2n}$$

If $f(x) = g(x+1)$, then a_n in terms of n is equal to

(a) ${}^{2n+1}C_{n+1}$ (b) ${}^{2n-1}C_{n-1}$

(c) ${}^{2n-1}C_{n+1}$ (d) ${}^{n+1}C_{n+1}$

Sol. We have, $f(x) = g(x+1)$

$$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{n-1} b_r (x+1)^r + \sum_{r=n}^{2n} (x+1)^r$$

$$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{n-1} b_r (x+1)^r + (x+1)^n + (x+1)^{n+1} + \dots + (x+1)^{2n}$$

$$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{n-1} b_r (x+1)^r + (x+1)^n \left[\frac{(x+1)^{n+1}-1}{x+1-1} \right]$$

$$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{n-1} b_r (x+1)^r + \frac{(x+1)^{2n+1} - (x+1)^n}{x}$$

Now, a_n = Coefficient of x^n in $\sum_{r=0}^{2n} a_r x^r$

= Coefficient of x^n in

$$\left\{ \sum_{r=0}^{n-1} b_r (x+1)^r + \frac{(x+1)^{2n+1} - (x+1)^n}{x} \right\}$$

= Coefficient of x^n in $\left\{ \frac{(x+1)^{2n+1} - (x+1)^n}{x} \right\}$

= Coefficient of x^{n+1} in $\{(x+1)^{2n+1} - (x+1)^n\}$

= Coefficient of x^{n+1} in $(x+1)^{2n+1}$

= ${}^{2n+1}C_{n+1}$

Hence, (a) is the correct answer.

Ex 14. The greatest term in the expansion of $(1+x)^{10}$, when $x = \frac{2}{3}$, is

(a) $210 \left(\frac{2}{3} \right)^4$

(b) $\left(\frac{2}{3} \right)^4$

(c) $210 \left(\frac{1}{3} \right)^4$

(d) $210 \left(\frac{2}{3} \right)^6$

Sol. Let T_r and T_{r+1} denote the r th and $(r+1)$ th terms in the expansion of $(1+x)^{10}$. Then,

$$T_r = {}^{10}C_{r-1} x^{r-1} \quad \text{and} \quad T_{r+1} = {}^{10}C_r x^r$$

$$\therefore \frac{T_{r+1}}{T_r} = \frac{{}^{10}C_r x^r}{{}^{10}C_{r-1} x^{r-1}}$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{{}^{10}C_r}{{}^{10}C_{r-1}} \cdot x$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \frac{11-r}{r} \cdot x$$

$$\Rightarrow \frac{T_{r+1}}{T_r} = \left(\frac{11-r}{r} \right) \times \frac{2}{3} \quad \left[\because x = \frac{2}{3} \right]$$

Now, $\frac{T_{r+1}}{T_r} > 1$

$$\Rightarrow \left(\frac{11-r}{r} \right) \times \frac{2}{3} > 1$$

$$\Rightarrow 22 > 5r$$

$$\Rightarrow r < 4 \frac{2}{5}$$

$\therefore$ (4+1)th i.e. 5th term is the greatest term.

Putting $r = 4$ in T_{r+1} , we get

$$T_5 = {}^{10}C_4 x^4$$

$$\Rightarrow T_5 = {}^{10}C_4 \left(\frac{2}{3} \right)^4 \quad \left[\because x = \frac{2}{3} \right]$$

$$\Rightarrow T_5 = 210 \left(\frac{2}{3} \right)^4$$

Hence, (a) is the correct answer.

Ex 15. The greatest term in the expansion of $(1+x)^{2n}$ has also the greatest coefficient, then

(a) $x \in \left(\frac{n}{n-1}, \frac{n+1}{n} \right)$ (b) $x \in \left(\frac{n}{n-1}, \frac{n-1}{n} \right)$

(c) $x \in \left(\frac{n}{n+1}, \frac{n+1}{n} \right)$ (d) $x \in \left(\frac{n}{n-1}, \frac{n+1}{n} \right)$

Sol. Let $(r+1)$ th term be the greatest term in the expansion of $(1+x)^{2n}$. Then,

$$T_{r+1} = {}^{2n}C_r x^{2n-r}$$

The coefficient of T_{r+1} is ${}^{2n}C_r$. Clearly, ${}^{2n}C_r$ will be

greatest, if $r = \frac{2n}{2} = n$

Thus, $T_{n+1} = {}^{2n}C_n x^n$ has the greatest coefficient.

Now, T_{n+1} will be the greatest term, if

$$\Rightarrow T_n < T_{n+1} > T_{n+2}$$

$$T_{n+1} > T_n \quad \text{and} \quad T_{n+2} < T_{n+1}$$

$$\begin{aligned} \Rightarrow \frac{T_{n+1}}{T_n} &> 1 \quad \text{and} \quad \frac{T_{n+2}}{T_{n+1}} < 1 \\ \Rightarrow \frac{{}^{2n}C_n x^n}{{}^{2n}C_{n-1} x^{n-1}} &> 1 \quad \text{and} \quad \frac{{}^{2n}C_{n+1} x^{n+1}}{{}^{2n}C_n x^n} < 1 \\ \Rightarrow \left(\frac{n+1}{n}\right)x &> 1 \quad \text{and} \quad \left(\frac{n}{n+1}\right)x < 1 \\ \Rightarrow x &> \frac{n}{n+1} \quad \text{and} \quad x < \frac{n+1}{n} \\ \Rightarrow \frac{n}{n+1} &< x < \frac{n+1}{n} \\ \Rightarrow x &\in \left(\frac{n}{n+1}, \frac{n+1}{n}\right) \end{aligned}$$

Hence, (c) is the correct answer.

Ex 16. $\sum_{r=0}^n (-1)^r \frac{{}^nC_r}{{}^{r+3}C_r}$ is equal to

- (a) $\frac{3!}{2(n-3)}$
 (b) $\frac{3!}{2(n+3)}$
 (c) $\frac{3!}{(n+3)}$
 (d) None of the above

Sol. We have, $\sum_{r=0}^n (-1)^r \frac{{}^nC_r}{{}^{r+3}C_r}$

$$\begin{aligned} &= \sum_{r=0}^n (-1)^r \frac{n! \cdot 3!}{(n-r)! (r+3)!} \\ &= 3! \sum_{r=0}^n (-1)^r \frac{n!}{(n-r)! (r+3)!} \\ &= \frac{3!}{(n+1)(n+2)(n+3)} \sum_{r=0}^n \frac{(-1)^r (n+3)!}{\{(n+3)-(r+3)\}! (r+3)!} \\ &= \frac{3!}{(n+1)(n+2)(n+3)} \sum_{r=0}^n (-1)^r {}^{n+3}C_{r+3} \\ &= \frac{3!}{(n+1)(n+2)(n+3)} \sum_{s=3}^{n+3} (-1)^{s-3} {}^{n+3}C_s \\ &= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)} \sum_{s=3}^{n+3} (-1)^s {}^{n+3}C_s \\ &= \frac{-3!}{(n+1)(n+2)(n+3)} \left[\sum_{s=0}^{n+3} (-1)^s \cdot {}^{n+3}C_s \right. \\ &\quad \left. - {}^{n+3}C_0 + {}^{n+3}C_1 - {}^{n+3}C_2 \right] \\ &= \frac{3!}{2(n+3)} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 17. The value of $\sum_{r=0}^n (-2)^r \left(\frac{{}^nC_r}{{}^{r+2}C_r} \right)$ is

- (a) $\frac{1}{n+1}$, n is odd (b) $\frac{1}{n+2}$, n is even
 (c) $\frac{1}{n+1}$, n is even (d) None of these

Sol. For $r \geq 0$, we have $\frac{{}^nC_r}{{}^{r+2}C_r}$

$$\begin{aligned} &= \frac{n!}{(n-r)! r!} \times \frac{r! 2!}{(r+2)!} \\ &= \frac{n! \times 2!}{(n-r)! (r+2)!} \\ &= \frac{2}{(n+1)(n+2)} \times \frac{(n+1)(n+2)n!}{\{(n+2)-(r+2)\}! (r+2)!} \\ &= \frac{2}{(n+1)(n+2)} \times {}^{n+2}C_{r+2} \\ \therefore \sum_{r=0}^n (-2)^r \left(\frac{{}^nC_r}{{}^{r+2}C_r} \right) &= \frac{2}{(n+1)(n+2)} \sum_{r=0}^n {}^{n+2}C_{r+2} (-2)^r \\ &= \frac{1}{2(n+1)(n+2)} \sum_{r=0}^n {}^{n+2}C_{r+2} (-2)^{r+2} \\ &= \frac{1}{2(n+1)(n+2)} \sum_{s=2}^{n+2} {}^{n+2}C_s (-2)^s \\ &\quad \text{[putting } r+2=s\text{]} \\ &= \frac{1}{2(n+1)(n+2)} \\ &\quad \times \left[\sum_{s=0}^{n+2} {}^{n+2}C_s (-2)^s - {}^{n+2}C_0 (-2)^0 - {}^{n+2}C_1 (-2)^1 \right] \\ &= \frac{1}{2(n+1)(n+2)} [(1-2)^{n+2} - 1 + 2(n+2)] \\ &= \begin{cases} \frac{1}{2(n+1)(n+2)} (2n+4), & \text{if } n \text{ is even} \\ \frac{1}{2(n+1)(n+2)} (2n+3-1), & \text{if } n \text{ is odd} \end{cases} \\ &= \begin{cases} \frac{1}{n+1}, & \text{if } n \text{ is even} \\ \frac{1}{n+2}, & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 18. The sum of $\sum_{r=0}^n (-1)^r \frac{{}^nC_r}{{}^{r+2}C_r}$ is

- (a) $\frac{2}{n+1}$ (b) $\frac{1}{n+2}$
 (c) $\frac{2}{n-2}$ (d) $\frac{2}{n+2}$

Sol. We have, $\sum_{r=0}^n (-1)^r \frac{{}^nC_r}{r+2} C_r$

$$= \sum_{r=0}^n (-1)^r \frac{n!}{(n-r)! r!} \times \frac{2! r!}{(r+2)!}$$

$$= 2 \sum_{r=0}^n (-1)^r \frac{n!}{(n-r)! (r+2)!}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{r=0}^n (-1)^r \frac{(n+2)!}{\{(n+2)-(r+2)\}! (r+2)!}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{r=0}^n (-1)^{r+2} {}^{n+2}C_{r+2}$$

$$= \frac{2}{(n+1)(n+2)} \sum_{s=2}^{n+2} (-1)^s {}^{n+2}C_s$$

$$= \frac{2}{(n+1)(n+2)} \left[\left(\sum_{s=0}^{n+2} (-1)^s {}^{n+2}C_s \right) - ({}^{n+2}C_0 - {}^{n+2}C_1) \right]$$

$$= \frac{2}{n+2}$$

Hence, (d) is the correct answer.

Ex 19. If ${}^nC_0, {}^nC_1, {}^nC_2, \dots, {}^nC_n$ denote the binomial coefficients in the expansion of $(1+x)^n$ and $p+q=1$, then $\sum_{r=0}^n r^2 {}^nC_r p^r q^{n-r}$ is

- (a) np (b) npq
(c) $n^2 p^2 + npq$ (d) None of these

Sol. We have, $\sum_{r=0}^n r^2 {}^nC_r p^r q^{n-r}$

$$= \sum_{r=0}^n [r(r-1) + r] {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=0}^n r(r-1) {}^nC_r p^r q^{n-r} + \sum_{r=0}^n r {}^nC_r p^r q^{n-r}$$

$$= \sum_{r=2}^n r(r-1) \frac{n!}{r!} \cdot \frac{n-1}{r-1} {}^{n-2}C_{r-2} p^r q^{n-r}$$

$$+ \sum_{r=1}^n r \cdot \frac{n!}{r!} {}^{n-1}C_{r-1} p^r q^{n-r}$$

$$= n(n-1) p^2 (p+q)^{n-2} + np(p+q)^{n-1}$$

$$= n(n-1) p^2 + np \quad [\because p+q=1]$$

$$= n^2 p^2 - np^2 + np$$

$$= n^2 p^2 + npq \quad [\because p+q=1]$$

Hence, (c) is the correct answer.

Ex 20. Let $a_n = (\log 3)^n \sum_{r=1}^n \frac{r^2}{r! (n-r)!}$, $n \in N$. Then, the sum of the series $a_1 + a_2 + a_3 + a_4 + \dots$ is

(a) $(1+\log 3)(9\log 3)$ (b) $(1+\log 3)$
(c) $(1-\log 3)(9\log 3)$ (d) $(1+\log 3)(\log 3)$

Sol. We have,

$$a_n = (\log 3)^n \sum_{r=1}^n \frac{r^2}{r! (n-r)!}$$

$$= \frac{(\log 3)^n}{n!} \sum_{r=1}^n \frac{n!}{(n-r)! r!} \cdot r^2$$

$$= \frac{(\log 3)^n}{n!} \sum_{r=1}^n {}^nC_r \cdot r^2$$

$$= \frac{(\log 3)^n}{n!} [n(n-1) 2^{n-2} + n \cdot 2^{n-1}]$$

$$\left[\because \sum_{r=1}^n r^2 \cdot {}^nC_r = n(n-1) 2^{n-2} + n \cdot 2^{n-1} \right]$$

$$= \frac{(\log 3)^n}{n!} [n(n-1) + 2n] 2^{n-2}$$

$$= \frac{1}{(n-2)!} (\log 3)^n 2^{n-2} + \frac{2}{(n-1)!} (\log 3)^n \cdot 2^{n-2}$$

$$= (\log 3)^2 \frac{(2\log 3)^{n-2}}{(n-2)!} + (\log 3) \frac{(2\log 3)^{n-1}}{(n-1)!}$$

$\therefore$ Required sum

$$= \sum_{n=1}^{\infty} a_n$$

$$= (\log 3)^2 \sum_{n=2}^{\infty} \frac{(\log 9)^{n-2}}{(n-2)!} + (\log 3) \sum_{n=1}^{\infty} \frac{(\log 9)^{n-1}}{(n-1)!}$$

$$= (\log 3)^2 e^{\log 9} + (\log 3) e^{\log 9}$$

$$\left[\because \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=2}^{\infty} \frac{x^{n-2}}{(n-2)!} = e^x \right]$$

$$= (1+\log 3)(9\log 3)$$

Hence, (a) is the correct answer.

Ex 21. If $(1+x)^n = \sum_{r=0}^n {}^nC_r x^r$, then $\frac{C_0}{1 \cdot 2} 2^2 + \frac{C_1}{2 \cdot 3} 2^3 + \frac{C_2}{3 \cdot 4} 2^4 + \dots + \frac{C_n}{(n+1)(n+2)} 2^{n+2}$ is equal to

(a) $\frac{3^{n+2} + 2n - 5}{(n+1)(n+2)}$
(b) $\frac{3^{n+2} - 2n + 5}{(n+1)(n+2)}$
(c) $\frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}$
(d) None of the above

Sol. We have,

$$\frac{C_0}{1 \cdot 2} 2^2 + \frac{C_1}{2 \cdot 3} 2^3 + \frac{C_2}{3 \cdot 4} 2^4 + \dots + \frac{C_n}{(n+1)(n+2)} 2^{n+2}$$

$$= \sum_{r=0}^n \frac{1}{(r+1)(r+2)} {}^nC_r \cdot 2^{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^n \frac{n+2}{r+2} \cdot \frac{n+1}{r+1} {}^nC_r 2^{r+2}$$

$$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^n {}^{n+2}C_{r+2} 2^{n+2}$$

$$\begin{aligned}
 &= \frac{1}{(n+1)(n+2)} \left\{ \sum_{s=2}^{n+2} {}^{n+2}C_s 2^s \right\} \\
 &= \frac{1}{(n+1)(n+2)} \left[\left\{ \sum_{s=0}^{n+2} {}^{n+2}C_s 2^s \right\} - \{ {}^{n+2}C_0 2^0 + {}^{n+2}C_1 2^1 \} \right] \\
 &= \frac{1}{(n+1)(n+2)} [(1+2)^{n+2} - \{1+2(n+2)\}] \\
 &= \frac{3^{n+2} - 2n - 5}{(n+1)(n+2)}
 \end{aligned}$$

Hence, (c) is the correct answer.

Ex 22. The value of

$${}^{n-1}C_0 \cdot {}^nC_1 + {}^{n-1}C_1 \cdot {}^nC_2 + {}^{n-1}C_2 \cdot {}^nC_3 + \dots + {}^{n-1}C_{n-1} \cdot {}^nC_n \text{ is equal to}$$

- (a) $2^n C_n$
 (b) $2^{n-1} C_{n-1}$
 (c) $2^n C_{n+1}$
 (d) None of the above

Sol. We have,

$$\begin{aligned}
 &({}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_{n-1} x + {}^nC_n) \\
 &\times ({}^{n-1}C_0 + {}^{n-1}C_1 x + {}^{n-1}C_2 x^2 + \dots + {}^{n-1}C_{n-1} x^{n-1}) \\
 &= (1+x)^n (1+x)^{n-1} \\
 \Rightarrow &({}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_{n-1} x + {}^nC_n) \\
 &\times ({}^{n-1}C_0 + {}^{n-1}C_1 x + {}^{n-1}C_2 x^2 + \dots + {}^{n-1}C_{n-1} x^{n-1}) \\
 &= (1+x)^{2n-1}
 \end{aligned}$$

On equating the coefficient of x^{n-1} both sides, we get
 ${}^{n-1}C_0 {}^nC_1 + {}^{n-1}C_1 {}^nC_2 + \dots + {}^{n-1}C_{n-1} {}^nC_n = 2^{n-1} C_{n-1}$

Hence, (b) is the correct answer.

Ex 23. The value of

$${}^{n-1}C_0 \cdot {}^nC_2 + {}^{n-1}C_1 \cdot {}^nC_3 + {}^{n-1}C_2 \cdot {}^nC_4 + \dots + {}^{n-1}C_{n-2} \cdot {}^nC_n \text{ is}$$

- (a) $2^n C_{n-2}$
 (b) $2^{n-1} C_n$
 (c) $2^{n-1} C_{n-2}$
 (d) None of the above

Sol. We have,

$$\begin{aligned}
 &({}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_{n-1} x + {}^nC_n) \\
 &\times ({}^{n-1}C_0 + {}^{n-1}C_1 x + {}^{n-1}C_2 x^2 + \dots + {}^{n-1}C_{n-1} x^{n-1}) \\
 &= (1+x)^n (1+x)^{n-1} \\
 \Rightarrow &({}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_{n-1} x + {}^nC_n) \\
 &\times ({}^{n-1}C_0 + {}^{n-1}C_1 x + {}^{n-1}C_2 x^2 + \dots + {}^{n-1}C_{n-1} x^{n-1}) \\
 &= (1+x)^{2n-1}
 \end{aligned}$$

On equating the coefficient of x^{n-2} both sides, we get
 ${}^{n-1}C_0 {}^nC_2 + {}^{n-1}C_1 {}^nC_3 + \dots + {}^{n-1}C_{n-2} {}^nC_n = 2^{n-1} C_{n-2}$

Hence, (c) is the correct answer.

Ex 24. If $(1+x)^n = C_0 + C_1 x + \dots + C_n x^n$, then the value of $\sum_{r=0}^n \sum_{s=0}^n (C_r + C_s)$ is equal to

- (a) $(n+1) 2^{n+1}$
 (b) $(n-1) 2^{n+1}$
 (c) $(n+1) 2^n$
 (d) None of the above

Sol. $\sum_{r=0}^n \sum_{s=0}^n (C_r + C_s)$

$$\begin{aligned}
 &= \sum_{r=0}^n \sum_{s=0}^n C_r + \sum_{r=0}^n \sum_{s=0}^n C_s \\
 &= \sum_{s=0}^n \left(\sum_{r=0}^n C_r \right) + \sum_{r=0}^n \left(\sum_{s=0}^n C_s \right) \\
 &= \sum_{s=0}^n 2^n + \sum_{r=0}^n 2^n \\
 &= (n+1) 2^n + (n+1) 2^n = (n+1) 2^{n+1}
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 25. If $(1+x)^n = C_0 + C_1 x + \dots + C_n x^n$, then the value of $\sum_{r=0}^n \sum_{s=0}^n C_r C_s$ is equal to

- (a) 2^n (b) 2^{2n} (c) 2^{n+1} (d) 2^{3n}

Sol. We have, $\sum_{r=0}^n \sum_{s=0}^n C_r C_s = \sum_{r=0}^n \left(C_r \sum_{s=0}^n C_s \right)$

$$\begin{aligned}
 &= \sum_{r=0}^n 2^n \cdot C_r = 2^n \left(\sum_{r=0}^n C_r \right) \\
 &= 2^n \cdot 2^n = (2^n)^2 = 2^{2n}
 \end{aligned}$$

Aliter

$$\sum_{r=0}^n \sum_{s=0}^n C_r C_s = \left(\sum_{r=0}^n C_r \right) \left(\sum_{s=0}^n C_s \right) = 2^n \cdot 2^n = 2^{2n}$$

Hence, (b) is the correct answer.

Ex 26. If $(1+x)^n = C_0 + C_1 x + \dots + C_n x^n$, then the value of $\sum_{0 \leq r < s \leq n} C_r C_s$ is equal to

- (a) $\frac{1}{2} [2^{2n} - 2^n C_n]$ (b) $\frac{1}{4} [2^{2n} - 2^n C_n]$
 (c) $\frac{1}{2} [2^{2n} + 2^n C_n]$ (d) $\frac{1}{2} [2^{2n} - 2^n C_n]$

Sol. We have,

$$\begin{aligned}
 \sum_{r=0}^n \sum_{s=0}^n C_r C_s &= \left(\sum_{r=0}^n C_r^2 \right) + 2 \sum_{0 \leq r < s \leq n} C_r C_s \\
 \Rightarrow 2^{2n} &= 2^n C_n + 2 \sum_{0 \leq r < s \leq n} C_r C_s \\
 \Rightarrow \sum_{0 \leq r < s \leq n} C_r C_s &= \frac{1}{2} [2^{2n} - 2^n C_n]
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 27. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then the value of $\sum_{0 \leq r < s \leq n} (r \cdot s) C_r C_s$ is

- (a) $n^2 \left[2^{2n-3} + \frac{1}{2} 2^{n-2} C_{n-1} \right]$
 (b) $\left[2^{2n-3} - \frac{1}{2} 2^{n-2} C_{n-1} \right]$
 (c) $n^2 \left[2^{2n-3} - \frac{1}{2} 2^{n-2} C_{n-1} \right]$
 (d) None of the above

Sol. We have, $\sum_{0 \leq r < s \leq n} (r \cdot s) C_r C_s = \sum_{0 \leq r < s \leq n} (r \cdot {}^nC_r) (s \cdot {}^nC_s)$

$$= \sum_{0 \leq r < s \leq n} \left(r \cdot \frac{n}{r} {}^{n-1}C_{r-1} \right) \left(s \cdot \frac{n}{s} {}^{n-1}C_{s-1} \right)$$

$$= n^2 \cdot \frac{1}{2} [2^{2(n-1)} - 2^{(n-1)} C_{n-1}]$$

$$= n^2 \left[2^{2n-3} - \frac{1}{2} 2^{n-2} C_{n-1} \right]$$

Hence, (c) is the correct answer.

Ex 28. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then the value of $\sum_{0 \leq r < s \leq n} (C_r \pm C_s)^2$ is

- (a) $(n+1) {}^{2n}C_n$ (b) $(n \mp 1) {}^{2n}C_n \pm 2^n$
 (c) $(n \mp 1) {}^{2n}C_n$ (d) $(n \mp 1) {}^{2n}C_n \pm 2^{2n}$

Sol. We have, $\sum_{0 \leq r < s \leq n} (C_r \pm C_s)^2$

$$= (C_0 \pm C_1)^2 + (C_0 \pm C_2)^2 + \dots + (C_0 \pm C_n)^2$$

$$+ (C_1 \pm C_2)^2 + (C_1 \pm C_3)^2 + \dots + (C_1 \pm C_n)^2$$

$$+ (C_2 \pm C_3)^2 + \dots + (C_2 \pm C_n)^2$$

$$\dots + (C_{n-2} \pm C_{n-1})^2 + (C_{n-2} \pm C_n)^2 + (C_{n-1} \pm C_n)^2$$

$$= n(C_0^2 + C_1^2 + \dots + C_n^2) \pm 2 \sum_{0 \leq r < s \leq n} C_r C_s$$

$$= n \cdot {}^{2n}C_n \pm [2^{2n} - {}^{2n}C_n] = (n \mp 1) {}^{2n}C_n \pm 2^{2n}$$

Hence, (d) is the correct answer.

Ex 29. The value of $\sum_{r=0}^n \sum_{s=1}^n {}^nC_s {}^sC_r$ is

- (a) $3^n - 1$ (b) $3^n + 1$
 (c) 3^n (d) None of these

Sol. We have, $\sum_{r=0}^n \sum_{s=1}^n {}^nC_s {}^sC_r$

$$= \sum_{s=1}^n {}^nC_s ({}^sC_0 + {}^sC_1 + {}^sC_2 + \dots + {}^sC_s) \quad [\because r \leq s]$$

$$= \sum_{s=1}^n {}^nC_s (2)^s = \sum_{s=0}^n {}^nC_s 2^n - {}^nC_0 2^0$$

$$= (1+2)^n - 1 = 3^n - 1$$

Hence, (a) is the correct answer.

Ex 30. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then $\sum_{r=0}^n \sum_{s=0}^n (r+s) C_r C_s$ is equal to

- (a) 2^{2n} (b) $n \cdot 2^{2n}$
 (c) $n \cdot 2^n$ (d) None of these

Sol. We have, $\sum_{r=0}^n \sum_{s=0}^n (r+s) C_r C_s$

$$= \sum_{r=0}^n \sum_{s=0}^n (r \cdot {}^nC_r + s \cdot {}^nC_s)$$

$$= \sum_{r=0}^n \sum_{s=0}^n r \cdot {}^nC_r {}^nC_s + \sum_{r=0}^n \sum_{s=0}^n s \cdot {}^nC_r {}^nC_s$$

$$= \sum_{r=0}^n r \cdot {}^nC_r \left(\sum_{s=0}^n {}^nC_s \right) + \sum_{s=0}^n s \cdot {}^nC_s \left(\sum_{r=0}^n {}^nC_r \right)$$

$$= \sum_{r=0}^n r \cdot {}^nC_r \cdot 2^n + \sum_{s=0}^n s \cdot {}^nC_s \cdot 2^n$$

$$= 2^n \left(\sum_{r=0}^n r \cdot {}^nC_r \right) + 2^n \left(\sum_{s=0}^n s \cdot {}^nC_s \right)$$

$$= 2^n \left(\sum_{r=0}^n r \cdot \frac{n}{r} {}^{n-1}C_{r-1} \right) + 2^n \left(\sum_{s=0}^n s \cdot \frac{n}{s} {}^{n-1}C_{s-1} \right)$$

$$= 2^n (n \cdot 2^{n-1}) + 2^n (n \cdot 2^{n-1})$$

$$= n \cdot 2^{2n-1} + n \cdot 2^{2n-1} = n \cdot 2^{2n}$$

Hence, (b) is the correct answer.

Ex 31. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then $\sum_{0 \leq r < s \leq n} (r+s) C_r C_s$ is equal to

- (a) $n [2^{2n-1} - {}^{2n-1}C_{n-1}]$
 (b) $n [2^{2n-1} + {}^{2n-1}C_{n-1}]$
 (c) $2n [2^{2n-1} - {}^{2n-1}C_{n-1}]$
 (d) None of the above

Sol. We have, $\sum_{r=0}^n \sum_{s=0}^n (r+s) C_r C_s$

$$= \sum_{r=0}^n 2r C_r^2 + 2 \sum_{0 \leq r < s \leq n} (r+s) C_r C_s$$

$$\Rightarrow \sum_{r=0}^n \sum_{s=0}^n (r+s) C_r C_s = 2 \sum_{r=0}^n r \cdot C_r^2$$

$$+ 2 \sum_{0 \leq r < s \leq n} (r+s) C_r C_s$$

$$\Rightarrow n \cdot 2^{2n} = 2 \cdot \left(\frac{n}{2} {}^{2n}C_n \right) + 2 \sum_{0 \leq r < s \leq n} (r+s) C_r C_s$$

$$\Rightarrow \sum_{0 \leq r < s \leq n} (r+s) C_r C_s = \frac{1}{2} [n \cdot 2^{2n} - n \cdot {}^{2n}C_n]$$

$$= \frac{n}{2} \left[2^{2n} - \frac{2n}{n} {}^{2n-1}C_{n-1} \right]$$

$$= n [2^{2n-1} - {}^{2n-1}C_{n-1}]$$

Hence, (a) is the correct answer.

Ex 32. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then the value of $\sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s)$ is

- (a) $n^2 \cdot 2^n$ (b) $n \cdot 2^n$
(c) $n^2 \cdot 2^{2n}$ (d) None of these

Sol. We have, $\sum_{r=0}^n \sum_{s=0}^n (r+s) (C_r + C_s)$

$$= \sum_{r=0}^n \sum_{s=0}^n (rC_r + rC_s + sC_r + sC_s)$$

$$= \sum_{r=0}^n \left[\sum_{s=0}^n rC_r + r \sum_{s=0}^n C_s + C_r \sum_{s=0}^n s + \sum_{s=0}^n sC_s \right]$$

$$= \sum_{r=0}^n \left[(n+1)rC_r + r2^n + \frac{n(n+1)}{2}C_r + n \cdot 2^{n-1} \right]$$

$$= (n+1)(n \cdot 2^{n-1}) + 2^n \frac{n(n+1)}{2} + \frac{n(n+1)}{2} 2^n + n \cdot 2^{n-1}(n+1)$$

$$= n(n+1)2^n + n(n+1)2^n = 2n(n+1)2^n \quad \dots(i)$$

Also, $\sum_{r=0}^n \sum_{s=0}^n (r+s) (C_r + C_s)$

$$= \sum_{r=0}^n 4rC_r + 2 \sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s)$$

$$\therefore 2n(n+1)2^n = 4n \cdot 2^{n-1} + 2 \sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s)$$

$$\Rightarrow \sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s) = n^2 \cdot 2^n$$

Hence, (a) is the correct answer.

Ex 33. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then the value of $\sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s + C_rC_s)$ is

- (a) $n^2 \cdot 2^n - \frac{n}{2} [2^{2n} - 2^n C_n]$
(b) $n^2 \cdot 2^n + \frac{n}{2} [2^{2n} - 2^n C_n]$
(c) $n^2 \cdot 2^n + \frac{n}{2} [2^n + 2^n C_n]$
(d) None of the above

Sol. We have, $\sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s + C_rC_s)$

$$= \sum_{0 \leq r < s \leq n} (r+s) (C_r + C_s) + \sum_{0 \leq r < s \leq n} (r+s) C_rC_s$$

$$= n^2 \cdot 2^n + \frac{n}{2} [2^{2n} - 2^n C_n]$$

Hence, (b) is the correct answer.

Ex 34. Let n be a positive integer such that $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$,

then $\sum_{r=0}^{2n} a_r$ is

- (a) 3^n
(b) 3^{n-1}
(c) $\frac{3^n}{2}$
(d) None of the above

Sol. We have, $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$
Putting $x=1$ on both sides, we get

$$a_0 + a_1 + a_2 + \dots + a_{2n} = 3^n$$

Hence, (a) is the correct answer.

Ex 35. Let n be a positive integer such that $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then a_r is

- (a) $a_{2n}, 0 \leq r \leq 2n$ (b) $a_{2n-r}, 0 \leq r \leq 2n$
(c) $a_{n-r}, 0 \leq r \leq 2n$ (d) None of these

Sol. We have,

$$(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r \quad \dots(i)$$

$$\Rightarrow \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} \frac{a_r}{x^r} \quad \left[\text{replacing } x \text{ by } \frac{1}{x} \right]$$

$$\Rightarrow (1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^{2n-r} \quad \dots(ii)$$

[multiplying both sides by x^{2n}]

$$\Rightarrow \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \quad [\text{from Eqs. (i) and (ii)}]$$

On equating the coefficient of x^{2n-r} both sides, we get

$$a_{2n-r} = a_r, \text{ for } 0 \leq r \leq 2n$$

Hence, (b) is the correct answer.

Ex 36. Let n be a positive integer such that $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then $a_0 + a_1 + a_2 + \dots + a_{n-1}$ is

- (a) $\frac{1}{2} (3^n + a_n)$ (b) $\frac{1}{2} (3^{3n} - a_n)$
(c) $\frac{1}{2} (3^n - a_n)$ (d) None of these

Sol. We have, $a_r = a_{2n-r}$ for $0 \leq r \leq 2n$

$$\Rightarrow \sum_{r=0}^{n-1} a_r = \sum_{r=0}^{n-1} a_{2n-r}$$

$$\Rightarrow a_0 + a_1 + \dots + a_{n-1} = a_{2n} + a_{2n-1} + \dots + a_{n+1}$$

$$\Rightarrow 2(a_0 + a_1 + \dots + a_{n-1}) + a_n = 3^n \quad \left[\because \sum_{r=0}^{2n} a_r = 3^n \right]$$

$$\Rightarrow a_0 + a_1 + \dots + a_{n-1} = \frac{1}{2} (3^n - a_n)$$

Hence, (c) is the correct answer.

Ex 37. Let n be a positive integer such that $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then $(n-r)a_r + (2n-r+1)a_{r-1}, 0 < r < 2n$ is

- (a) $(r+1)a_{r+1}$ (b) $(r-1)a_{r+1}$
(c) $(2r+1)a_{r+1}$ (d) None of these

Sol. We have, $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$

Differentiating both sides w.r.t. x , we get

$$n(1+x+x^2)^{n-1}(1+2x) = \sum_{r=0}^{2n} r a_r x^{r-1}$$

$$\Rightarrow n(1+2x)(1+x+x^2)^n = (1+x+x^2) \left[\sum_{r=0}^{2n} r a_r x^{r-1} \right]$$

[multiplying both sides by $(1+x+x^2)$]

$$\Rightarrow n(1+2x) \left(\sum_{r=0}^{2n} a_r x^r \right) = (1+x+x^2) \left(\sum_{r=0}^{2n} r a_r x^{r-1} \right)$$

Equating the coefficients of x^r ($0 < r < 2n$) on both sides, we get

$$na_r + 2na_{r-1} = (r+1)a_{r+1} + ra_r + (r-1)a_{r-1}$$

$$\Rightarrow (r+1)a_{r+1} = (n-r)a_r + (2n-r+1)a_{r-1}$$

Hence, (a) is the correct answer.

Ex 38. The value of ${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{2n-2}C_n + {}^nC_2 \cdot {}^{2n-4}C_n - \dots$ is

- (a) 3^n
(b) 4^n
(c) 2^n
(d) None of the above

Sol. We have,

$${}^nC_0 \cdot {}^{2n}C_n - {}^nC_1 \cdot {}^{2n-2}C_n + {}^nC_2 \cdot {}^{2n-4}C_n - \dots + (-1)^n {}^nC_n \cdot {}^{2n-2n}C_n$$

$$= \text{Coefficient of } x^n \text{ in } [{}^nC_0 (1+x)^{2n} - {}^nC_1 (1+x)^{2n-2} + {}^nC_2 (1+x)^{2n-4} - \dots + (-1)^n {}^nC_n (1+x)^{2n-2n}]$$

$$= \text{Coefficient of } x^n \text{ in } [(1+x)^2 - 1]^n$$

$$= \text{Coefficient of } x^n \text{ in } (2+x^2)^n$$

$$= \text{Coefficient of } x^0 \text{ in } (2+x)^n = 2^n$$

Hence, (c) is the correct answer.

Ex 39. The value of ${}^nC_0 \cdot {}^{2n}C_r - {}^nC_1 \cdot {}^{2n-2}C_r + {}^nC_2 \cdot {}^{2n-4}C_r - \dots + (-1)^n {}^nC_n \cdot {}^{2n-2n}C_r$ is

- (a) $\begin{cases} 2^{2n-r} {}^nC_{r-n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$
(b) $\begin{cases} 3^{2n-r} {}^nC_{r-n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$
(c) $\begin{cases} 2^{2n+r} {}^nC_{r+n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$
(d) None of the above

Sol. We have, ${}^nC_0 \cdot {}^{2n}C_r - {}^nC_1 \cdot {}^{2n-2}C_r + {}^nC_2 \cdot {}^{2n-4}C_r - \dots + (-1)^n {}^nC_n \cdot {}^{2n-2n}C_r$

$$= \text{Coefficient of } x^r \text{ in } [{}^nC_0 (1+x)^{2n} - {}^nC_1 (1+x)^{2n-2} + {}^nC_2 (1+x)^{2n-4} - \dots + (-1)^n {}^nC_n (1+x)^{2n-2n}]$$

$$= \text{Coefficient of } x^r \text{ in } [(1+x)^2 - 1]^n$$

$$= \text{Coefficient of } x^r \text{ in } (2x+x^2)^n$$

$$= \text{Coefficient of } x^{r-n} \text{ in } (2+x)^n$$

$$= \begin{cases} 2^{2n-r} {}^nC_{r-n}, & \text{if } r > n \\ 0, & \text{if } r < n \end{cases}$$

Hence, (a) is the correct answer.

Ex 40. If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then

$$\sum_{r=0}^n (-1)^r a_r {}^nC_r = {}^nC_{n/3}, \text{ where } n \text{ is}$$

- (a) $3k+1$ (b) $3k+2$
(c) $3k$ (d) None of these

Sol. We have,

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n} \quad \dots(i)$$

$$\text{and } (x-1)^n = {}^nC_0x^n - {}^nC_1x^{n-1} + {}^nC_2x^{n-2} - \dots + (-1)^n {}^nC_n \quad \dots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x+x^2)^n (x-1)^n = (a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}) \times \{ {}^nC_0x^n - {}^nC_1x^{n-1} + {}^nC_2x^{n-2} - \dots + (-1)^n {}^nC_n \}$$

$$\Rightarrow (x^3-1)^n = (a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}) \times \{ {}^nC_0x^n - {}^nC_1x^{n-1} + \dots + (-1)^n {}^nC_n \} \quad \dots(iii)$$

Now, coefficient of x^n on RHS of Eq. (iii)

$$= a_0 {}^nC_0 - a_1 {}^nC_1 + a_2 {}^nC_2 - \dots + (-1)^n a_n {}^nC_n$$

LHS of Eq. (iii)

$$= (x^3-1)^n = (-1)^n (1-x^3)^n = (-1)^n \sum_{r=0}^n {}^nC_r (-x^3)^r$$

$$= (-1)^n \sum_{r=0}^n (-1)^r {}^nC_r x^{3r} \quad \dots(iv)$$

Clearly, if n is not a multiple of 3, then x^n does not occur in Eq. (iv).

$\therefore$ Coefficient of x^n in LHS = 0

When n is not a multiple of 3.

If n is a multiple of 3 i.e. if $n = 3m$, then

$$\text{LHS} = (-1)^{3m} (-1)^m {}^{3m}C_m$$

$$= {}^{3m}C_m = {}^nC_{n/3} \quad [\because n = 3m]$$

Thus, equating the coefficients of x^n on both sides, we get

$$a_0 {}^nC_0 - a_1 {}^nC_1 + a_2 {}^nC_2 - a_3 {}^nC_3 + \dots + (-1)^n a_n {}^nC_n = \begin{cases} 0, & \text{if } n \text{ is not a multiple of 3} \\ {}^nC_{n/3}, & \text{if } n \text{ is multiple of 3} \end{cases}$$

Hence, (c) is the correct answer.

Ex 41. If $n \in N$ and $(1+x+x^2+\dots+x^p)^n = \sum_{r=0}^{np} a_r x^r$,

if $0 \leq r \leq np$ and r is not a multiple of $p+1$, then ${}^nC_0 a_r - {}^nC_1 a_{r-1} + {}^nC_2 a_{r-2} - \dots + (-1)^r a_0 {}^nC_r$ is

- (a) 3^n (b) 2^n
(c) 0 (d) None of these

Sol. We have,

$$(1-x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \dots + (-1)^n {}^nC_n x^n$$

$$\text{and } (1+x+x^2+\dots+x^p)^n = (a_0 + a_1x + a_2x^2 + \dots + a_{np}x^{np})$$

$$\therefore (1-x)^n (1+x+x^2+\dots+x^p)^n = ({}^nC_0 - {}^nC_1x + {}^nC_2x^2 - {}^nC_3x^3 + \dots + (-1)^n {}^nC_n x^n) \times (a_0 + a_1x + a_2x^2 + \dots + a_{np}x^{np}) \quad \dots(i)$$

Now, coefficient of x^r on RHS of Eq. (i)
 $= a_r {}^nC_0 - a_{r-1} {}^nC_1 + a_{r-2} {}^nC_2 - \dots + a_0 (-1)^r {}^nC_r$

$$\begin{aligned} \text{LHS of Eq. (i)} &= (1-x)^n (1+x+x^2+\dots+x^p) \\ &= (1-x)^n \left\{ \frac{1-x^{p+1}}{1-x} \right\} \\ &= (1-x^{p+1})^n \end{aligned}$$

Since, r is not a multiple of $(p+1)$. Therefore, the expansion of $(1-x^{p+1})^n$ does not contain x^r in any term.

$\therefore$ Coefficient of x^r on LHS of Eq. (i) = 0

Hence,

$${}^nC_0 a_r - {}^nC_1 a_{r-1} + {}^nC_2 a_{r-2} - \dots + (-1)^r {}^nC_r a_0 = 0,$$

if r is not a multiple of $(p+1)$.

Hence, (c) is the correct answer.

Ex 42. When $32^{(32)^{(32)}}$ is divided by 7, then the remainder is

- (a) 2
 (b) 8
 (c) 4
 (d) None of the above

Sol. We have, $32^{32} = (2^5)^{32} = 2^{160} = (3-1)^{160}$
 $= {}^{160}C_0 3^{160} - {}^{160}C_1 3^{159} + \dots - {}^{160}C_{159} 3 + {}^{160}C_{160} 3^0$
 $= 3m + 1$, where $m \in N$ $32^{(32)^{(32)}} = (32)^{3m+1}$

Type 2. More than One Correct Option

Ex 44. The largest term in the expansion of $(3+2x)^{50}$,

where $x = \frac{1}{5}$ is

- (a) 5th (b) 51st (c) 7th (d) 6th

Sol. Consider, $(3+2x)^{50} = 3^{50} \left(1 + \frac{2x}{3} \right)^{50}$

Let T_{r+1} be the largest term in the expansion.

$$\therefore \frac{T_{r+1}}{T_r} \geq 1 \Rightarrow 102 - 2r \geq 15r \Rightarrow r \leq 6$$

Hence, (a) and (d) are the correct answers.

Ex 45. If $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$, $n \in N$, then

$$C_0 - C_1 + C_2 - \dots + (-1)^{m-1} C_{m-1}$$

is equal to $(m < n)$

- (a) $\frac{(n-1)(n-2)\dots(n-m+1)}{(m-1)!} (-1)^{m-1}$
 (b) ${}^{n-1}C_{m-1} (-1)^{m-1}$
 (c) $\frac{(n-1)(n-2)\dots(n-m)}{(m-1)!} (-1)^{m-1}$
 (d) ${}^{n-1}C_{n-m} (-1)^{m-1}$

$$\begin{aligned} &= (2^5)^{3m+1} = 2^{15m+5} \\ &= 2^3 (2^{5m+1}) \cdot 2^2 = (2^3)^{5m+1} \cdot 2^2 = (7+1)^{5m+1} \times 4 \\ &= \{ {}^{5m+1}C_0 7^{5m+1} + {}^{5m+1}C_1 7^{5m} \\ &\quad + \dots + {}^{5m+1}C_{5m} 7 + {}^{5m+1}C_{5m+1} 7^0 \} \times 4 \\ &= (7n+1) \times 4, \\ \text{where, } n &= {}^{5m+1}C_0 7^{5m+1} + \dots + {}^{5m+1}C_{5m} 7 \\ &= 28n+4 \end{aligned}$$

Thus, when $32^{(32)^{(32)}}$ is divided by 7, the remainder is 4.

Hence, (c) is the correct answer.

Ex 43. The coefficient of $x^n y^n$ in the expansion of

$$\{(1+x)(1+y)(x+y)\}^n \text{ is}$$

- (a) $\sum_{r=0}^n C_r^2$ (b) $\sum_{r=0}^n C_r^3$
 (c) $\sum_{r+s=0}^n {}^nC_r {}^nC_s^2$ (d) None of these

Sol. We have, $\{(1+x)(1+y)(x+y)\}^n$
 $= (C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n)$
 $\times (C_0 y^n + C_1 y^{n-1} + \dots + C_{n-1} y + C_n)$
 $\times (C_0 x^n + C_1 x^{n-1} y + \dots + C_{n-1} x y^{n-1} + C_n y^n)$

Clearly, coefficient of $x^n y^n$ on RHS

$$= C_0^3 + C_1^3 + C_2^3 + \dots + C_n^3 = \sum_{r=0}^n C_r^3$$

Hence, (b) is the correct answer.

$$\begin{aligned} \text{Sol. } &\frac{(n-1)(n-2)\dots(n-m+1)}{(m-1)!} \\ &= \frac{(n-1)(n-2)\dots(n-m+1)(n-m)\dots 2 \cdot 1}{(n-m)! (m-1)!} \\ &= {}^{n-1}C_{m-1} \end{aligned}$$

= Coefficient of x^{m-1} in $(1+x)^{n-1}$

= Coefficient of x^{m-1} in $(1+x)^n (1+x)^{-1}$

$$\text{Now, } (1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_{m-1} x^{m-1} + \dots + C_n x^n \dots (i)$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots + (-1)^{m-1} x^{m-1} + \dots \dots (ii)$$

Collecting the coefficients of x^{m-1} in the product of Eqs. (i) and (ii), we get

$$\begin{aligned} &(-1)^{m-1} C_0 + (-1)^{m-2} C_1 + \dots + C_{m-1} \\ &= \text{Coefficient of } x^{m-1} \text{ in } (1+x)^{n-1} = {}^{n-1}C_{m-1} \end{aligned}$$

$$\therefore C_0 - C_1 + C_2 - \dots + (-1)^{m-1} C_{m-1}$$

$$= {}^{n-1}C_{m-1} (-1)^{m-1}$$

$$= \frac{(n-1)(n-2)\dots(n-m+1)}{(m-1)!} (-1)^{m-1}$$

Hence, (a), (b) and (d) are the correct answers.

Ex 46. For which of the following values of x , 5th term is the numerically greatest term in the expansion of $(1 + x/3)^{10}$?

- (a) -2 (b) 1.8
(c) 2 (d) -1.9

Sol. Let T_5 be numerically the greatest term in the expansion of $(1 + x/3)^{10}$.

$$\text{Then, } \left| \frac{T_5}{T_4} \right| \geq 1$$

$$\text{and } \left| \frac{T_6}{T_5} \right| \leq 1$$

$$\text{Now, } \frac{T_{r+1}}{T_r} = \frac{10-r+1}{r} \cdot \frac{x}{3}$$

$$\Rightarrow \left| \frac{7}{4} \times \frac{x}{3} \right| \geq 1$$

$$\text{and } \left| \frac{6}{5} \cdot \frac{x}{3} \right| \leq 1$$

$$\Rightarrow |x| \geq \frac{12}{7}$$

$$\text{and } |x| \leq \frac{5}{2} \quad \dots(i)$$

$$\Rightarrow \frac{12}{7} \leq |x| \leq \frac{5}{2}$$

$$\Rightarrow x \in \left[-\frac{5}{2}, -\frac{12}{7} \right] \cup \left[\frac{12}{7}, \frac{5}{2} \right]$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 47. The coefficient of $a^8 b^6 c^4$ in the expansion of $(a + b + c)^{18}$ is

- (a) ${}^{18}C_{14} \cdot {}^{14}C_8$ (b) ${}^{18}C_{10} \cdot {}^{10}C_6$
(c) ${}^{18}C_6 \cdot {}^{12}C_8$ (d) ${}^{18}C_4 \cdot {}^{14}C_6$

Sol. Given expression

$$= \{a + (b + c)\}^{18} = {}^{18}C_0 a^{18} + {}^{18}C_1 a^{17} (b + c) + \dots + {}^{18}C_{10} a^8 (b + c)^{10} + \dots$$

$$\therefore \text{The coefficient of } a^8 b^6 c^4 = {}^{18}C_{10} \cdot {}^{10}C_6 = \frac{18!}{8!6!4!}$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 48. The term independent of x in the expansion of

$$(1 + x)^n \cdot \left(1 - \frac{1}{x}\right)^n \text{ is}$$

- (a) 0, if n is odd
(b) $(-1)^{\frac{n-1}{2}} \cdot {}^nC_{\frac{n-1}{2}}$, if n is odd
(c) $(-1)^{n/2} \cdot {}^nC_{n/2}$, if n is even
(d) None of the above

$$\text{Sol. Expression} = (-1)^n \cdot \frac{(1 - x^2)^n}{x^n}$$

$\therefore$ The term independent of x

= The coefficient of x^n in $(-1)^n \cdot (1 - x^2)^n$

= The coefficient of x^n in

$(-1)^n \{ {}^nC_0 + {}^nC_1(-x^2) + {}^nC_2(-x^2)^2 + \dots + {}^nC_n(-x^2)^n \}$,
and the expansion contains only even powers of x .

Hence, (a) and (c) are the correct answers.

Ex 49. Let $R = (8 + 3\sqrt{7})^{20}$ and $[R]$ = The greatest integer less than or equal to R . Then,

- (a) $[R]$ is even
(b) $[R]$ is odd
(c) $R - [R] = 1 - \frac{1}{(8 + 3\sqrt{7})^{20}}$
(d) None of the above

$$\text{Sol. } R = [R] + g = (8 + 3\sqrt{7})^{20}$$

$$= {}^{20}C_0 8^{20} + {}^{20}C_1 8^{19} (3\sqrt{7}) + \dots, \text{ where } 0 < g < 1$$

$$f = (8 - 3\sqrt{7})^{20} = {}^{20}C_0 8^{20} - {}^{20}C_1 8^{19} (3\sqrt{7}) + \dots,$$

where $0 < f < 1$

$$\therefore [R] + g + f = 2[{}^{20}C_0 8^{20} + {}^{20}C_2 8^{18} (3\sqrt{7})^2 + \dots] = \text{an even integer}$$

$$\therefore g + f = \text{an integer} = 1 \text{ as } 0 < g < 1, 0 < f < 1$$

$$\therefore [R] = \text{an even integer} - 1 = \text{an odd integer}$$

$$\text{Also, } R - [R] = g = 1 - f$$

$$= (8 - 3\sqrt{7})^{20} = 1 - \frac{1}{(8 + 3\sqrt{7})^{20}}$$

Hence, (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 50-54) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 50. Statement I Coefficient of $a^2 b^3 c^4$ in the expansion of $(a + b + c)^8$ is $\frac{8!}{2!3!4!}$.

Statement II Coefficient of $a^\alpha b^\beta c^\gamma$, where $\alpha + \beta + \gamma = n$, in the expansion of $(a + b + c)^n$ is $\frac{n!}{\alpha! \beta! \gamma!}$.

$$\text{Sol. } \therefore (a + b + c)^n = \sum \frac{n!}{p!q!r!} \cdot a^p b^q c^r, p + q + r = n$$

In Statement I, $p + q + r$ exceeds n .

Hence, (d) is the correct answer.

Ex 51. Statement I If $q = \frac{1}{3}$ and $p + q = 1$, then

$$\sum_{r=0}^{15} r {}^{15}C_r p^r q^{15-r} = 15 \times \frac{1}{3} = 5$$

Statement II If $p + q = 1, 0 < p < 1$, then

$$\sum_{r=0}^n r {}^nC_r p^r q^{n-r} = np$$

Sol. $\sum_{r=0}^n r {}^nC_r p^r q^{n-r}$

$$= np \sum_{r=1}^n {}^{n-1}C_{r-1} p^{r-1} q^{n-r} = np (q + p)^{n-1} = np$$

Hence, (d) is the correct answer.

Ex 52. Statement I Coefficient of x^{51} in the expansion of $(x-1)(x^2-2)(x^3-3)\dots(x^{10}-10)$ is -1 .

Statement II Coefficient of $x^{\frac{n(n+1)}{2}-4}$, $n \geq 4$, in the expansion of $(x-1)(x^2-2)(x^3-3)\dots(x^n-n)$ is $-4 + (-1)(-3) = -1$

Sol. Statement II is obviously true and it explains Statement I. Hence, (a) is the correct answer.

Ex 53. Statement I The greatest value of ${}^{40}C_0 \cdot {}^{60}C_r + {}^{40}C_1 \cdot {}^{60}C_{r-1} \dots {}^{40}C_{40} \cdot {}^{60}C_{r-40}$ is ${}^{100}C_{50}$.

Statement II The greatest value of ${}^{2n}C_r$ occurs at $r = n$.

Sol. The number of ways of selecting committee of r persons among 40 women and 60 men = ${}^{100}C_r$. This will assume greatest value at $r = 50$. Hence, (a) is the correct answer.

Ex 54. Statement I If $x = {}^nC_{n-1} + {}^{n+1}C_{n-1} + \dots + {}^{2n}C_{n-1}$, then $\frac{x+1}{2n+1}$ is an integer.

Statement II ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ and ${}^nC_r$ is divisible by n , if n and r are coprime.

Sol. $1 + x = {}^nC_n + {}^nC_{n-1} + {}^{n+1}C_{n-1} + \dots + {}^{2n}C_{n-1} = {}^{2n+1}C_n$

Since, $2n+1$ and n are coprime for every natural number n .

$\therefore {}^{2n+1}C_n$ is divisible by $2n+1$.

$\therefore \frac{x+1}{2n+1}$ is an integer.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 55-57) Consider the identity $(1+x)^{6m} = \sum_{r=0}^{6m} {}^{6m}C_r \cdot x^r$. By putting different values of x on

both sides, we can get summation of several series involving binomial coefficient. For example, By putting

$x = \frac{1}{2}$, we get $\sum_{r=0}^{6m} {}^{6m}C_r \cdot \frac{1}{2^r} = \left(\frac{3}{2}\right)^{6m}$.

Ex 55. The value of $\sum_{r=0}^{6m} {}^{6m}C_r \cdot 2^{r/2}$ is equal to

- (a) $\frac{3^{6m}}{2}$
- (b) $(1+\sqrt{2})^{3m}$
- (c) $(3+2\sqrt{2})^{3m}$
- (d) None of the above

Sol. $\sum_{r=0}^{6m} {}^{6m}C_r \cdot 2^{r/2}$ put $x = \sqrt{2} = (1+\sqrt{2})^{6m} = (3+2\sqrt{2})^{3m}$

Hence, (c) is the correct answer.

Ex 56. The value of $\sum_{r=0}^{3m} (-1)^r {}^{6m}C_{2r}$ is

- (a) 2^{3m}
- (b) 0, if m is odd
- (c) -2^{3m} , if m is even
- (d) None of the above

Sol. $\sum_{r=0}^{3m} (-1)^r {}^{6m}C_{2r} = \frac{1}{2} \left[(\sqrt{2})^{6m} \left(\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)^{6m} + (\sqrt{2})^{6m} \left(\frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \right)^{6m} \right]$

$$= 2^{3m} \cos \frac{3m\pi}{2} = \begin{cases} 0, & \text{if } m \text{ is odd} \\ (-1)^{m/2} 2^{3m}, & \text{if } m \text{ is even} \end{cases}$$

Hence, (b) is the correct answer.

Ex 57. The value of $\sum_{r=1}^{3m} (-3)^{r-1} {}^{6m}C_{2r-1}$ is

- (a) 0
- (b) 1
- (c) depends on m
- (d) None of the above

Sol. $\sum_{r=1}^{3m} (-3)^{r-1} {}^{6m}C_{2r-1}$

$$= \frac{1}{\sqrt{3}i} \{ \sqrt{3}i {}^{6m}C_1 + (\sqrt{3}i)^3 {}^{6m}C_3 + (\sqrt{3}i)^5 {}^{6m}C_5 + \dots + (\sqrt{3}i)^{6m-1} {}^{6m}C_{6m-1} \}$$

$$(1+\sqrt{3}i)^{6m} = {}^{6m}C_0 + \sqrt{3}i {}^{6m}C_1 + (\sqrt{3}i)^2 {}^{6m}C_2 + (\sqrt{3}i)^3 {}^{6m}C_3 + \dots$$

$$(1-\sqrt{3}i)^{6m} = {}^{6m}C_0 - \sqrt{3}i {}^{6m}C_1 + (\sqrt{3}i)^2 {}^{6m}C_2 - (\sqrt{3}i)^3 {}^{6m}C_3 + \dots$$

$$\begin{aligned} \therefore (1 + \sqrt{3}i)^{6m} - (1 - \sqrt{3}i)^{6m} \\ = 2[\sqrt{3}i^{6m}C_1 + (\sqrt{3}i)^3{}^{6m}C_3 + \dots] \\ \therefore \text{Given expression} \\ = \frac{1}{2\sqrt{3}i}[(1 + \sqrt{3}i)^{6m} - (1 - \sqrt{3}i)^{6m}] \\ = \frac{2^{6m}}{2\sqrt{3}i} \\ (\cos 2m\pi + i \sin 2m\pi - \cos 2m\pi + i \sin 2m\pi) \\ = 0 \end{aligned}$$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 58-60)

If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n \dots (i)$

Then, sum of the series $C_0 + C_k + C_{2k} + \dots$ can be obtained by putting all the roots of the equation $x^k - 1 = 0$ in Eq. (i) and then adding vertically.

For example, Sum of the series $C_0 + C_2 + C_4 + \dots$ can be obtained by putting roots of the equation $x^2 - 1 = 0$

$$\begin{aligned} \Rightarrow x = \pm 1 \text{ in Eq. (i)} \\ x = 1, C_0 + C_1 + C_2 + C_3 + \dots = 2^n \\ x = -1, C_0 - C_1 + C_2 - C_3 + \dots = 0 \\ 2(C_0 + C_2 + C_4 + \dots) = 2^n \\ C_0 + C_2 + C_4 + \dots = 2^{n-1} \end{aligned}$$

Ex 58. The values of x , we should substitute in Eq. (i) to get the sum of the series $C_0 + C_3 + C_6 + C_9 + \dots$, are

- (a) $1, -1, \omega$ (b) $\omega, \omega^2, \omega^3$
(c) $\omega, \omega^2, -1$ (d) None of these

Ex 59. If n is a multiple of 3, then $C_0 + C_3 + C_6 + \dots$ is equal to

- (a) $\frac{2^n + 2}{3}$ (b) $\frac{2^n - 2}{3}$
(c) $\frac{2^n + 2(-1)^n}{3}$ (d) $\frac{2^n - 2(-1)^n}{3}$

Ex 60. Sum of values of x , which we should substitute in Eq. (i) to give the sum of the series $C_0 + C_4 + C_8 + C_{12} + \dots$ is

- (a) 2
(b) $2(1 + i)$
(c) $2(1 - i)$
(d) 0

Sol. (Ex. Nos. 58-60) $x^3 - 1$

$$x = 1, \omega, \omega^2 \text{ or } x = \omega, \omega^2, \omega^3$$

$$x = 1, C_0 + C_1 + C_2 + C_3 + C_4 + \dots = 2^n$$

$$x = \omega, C_0 + C_1\omega + C_2\omega^2 + C_3\omega^3 + C_4\omega^4 + \dots = (1 + \omega)^n$$

$$x = \omega^2, C_0 + C_1\omega^2 + C_2\omega^4 + C_3\omega^6 + \dots = (1 + \omega^2)^n$$

$$\text{Now, } 3(C_0 + 0 + 0 + C_3 + 0 + 0 + C_6 + \dots) = 2^n + (-\omega^2)^n + (-\omega)^n = 2^n + (-1)^n + (-1)^n$$

$$\therefore C_0 + C_3 + C_6 + \dots = \frac{2^n + 2(-1)^n}{3}$$

$$x^4 - 1 = 0$$

$$\Rightarrow x = \pm 1, \pm i$$

$$\therefore \text{Sum of values of } x = 1 + (-1) + i + (-i) = 0$$

58. (b) **59.** (c) **60.** (d)

Type 5. Match the Columns

Ex 61. Match the statement of Column I with values of Column II.

Column I	Column II
A. ${}^mC_1 {}^nC_m - {}^mC_2 {}^{2n}C_m + {}^mC_3 {}^{3n}C_m \dots$ is	p. the coefficient of x^m in the expansion of $((1+x)^n - 1)^m$
B. ${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m$ is	q. the coefficient of x^m in $\frac{(1+x)^{n+1}}{x}$
C. $C_0C_n + C_1C_{n-1} + \dots + C_nC_0$ is	r. the coefficient of x^n in $(1+x)^{2n}$
D. $2^k {}^nC_0 - 2^{k-1} {}^nC_1 + 2^{k-2} {}^nC_2 - \dots + (-1)^k {}^nC_k + \dots + (-1)^{n-k} {}^nC_n$ is	s. the coefficient of x^k in the expansion of $(1+x)^n$

Sol. A. $({}^mC_1 {}^nC_m - {}^mC_2 {}^{2n}C_m + {}^mC_3 {}^{3n}C_m - \dots + (-1)^{m-1} {}^mC_m {}^{mn}C_m)$
= Coefficient of x^m in $[{}^mC_1(1+x)^n - {}^mC_2(1+x)^{2n} + {}^mC_3(1+x)^{3n} - \dots + (-1)^{m-1} {}^mC_m \cdot (1+x)^{mn}]$

$$\begin{aligned} &= \text{Coefficient of } x^m \text{ in } ({}^mC_0 - [{}^mC_0 - {}^mC_1(1+x)^n \\ &\quad + {}^mC_2(1+x)^{2n} - {}^mC_3(1+x)^{3n} \\ &\quad + \dots + (-1)^m {}^mC_m(1+x)^{mn}]) \end{aligned}$$

$$= \text{Coefficient of } x^m \text{ in } (1 - [1 - (1+x)^n])^m$$

$$= \text{Coefficient of } x^m \text{ in } (1 - (1+x)^n)^m$$

B. ${}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m$ is the coefficient of x^m in the series $(1+x)^n + (1+x)^{n-1} + (1+x)^{n-2} + \dots + (1+x)^m$
= $(1+x)^m [1 + (1+x) + (1+x)^2 + \dots + (1+x)^{n-m}]$

$$= (1+x)^m \left(\frac{1 - (1+x)^{n-m+1}}{1 - (1+x)} \right)$$

$$= \frac{(1+x)^{n+1} - (1+x)^m}{x}$$

$$\therefore \frac{(1+x)^m}{x} \text{ does not have } x^m,$$

$$\therefore {}^nC_m + {}^{n-1}C_m + {}^{n-2}C_m + \dots + {}^mC_m \text{ is the coefficient of } x^m \text{ in the series } \frac{(1+x)^{n+1}}{x}.$$

$$C. (1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_nx^n \dots (i)$$

$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \dots + {}^nC_nx^n \dots (ii)$$

On multiplying Eqs. (i) and (ii) and equate coefficient of x^n both sides, we get

$$\text{Coefficient of } x^n \text{ in } (1+x)^n(1+x)^n = {}^nC_0 {}^nC_n + {}^nC_1 {}^nC_{n-1} + {}^nC_2 {}^nC_{n-2} + \dots + {}^nC_n {}^nC_0$$

$$\therefore \text{Coefficient of } x^n \text{ in } (1+x)^{2n} = C_0C_n + C_1C_{n-1} + \dots + C_nC_0$$

$$D. 2^k {}^nC_k = \text{Coefficient of } x^k \text{ in } (1+2x)^n$$

$$2^{k-1} {}^{n-1}C_{k-1} = \text{Coefficient of } x^{k-1} \text{ in } (1+2x)^{n-1}$$

$$= \text{Coefficient of } x^k \text{ in } x(1+2x)^{n-1}$$

$$\therefore {}^nC_0(1+2x)^n - {}^nC_1x(1+2x)^{n-1} + {}^nC_2x^2(1+2x)^{n-2} - \dots = (1+2x-x)^n$$

Hence, given expression is coefficient of x^k in $(1+x)^n$.

$$A \rightarrow p; B \rightarrow q; C \rightarrow r; D \rightarrow s$$

Ex 62. Match the statement of Column I with values of Column II.

Column I	Column II
A. If $(r+1)$ th term is the first negative term in the expansion of $(1+x)^{7/2}$, then the value of r (where $ x < 1$) is	p. divisible by 2
B. The coefficient of y in the expansion of $(y^2 + 1/y)^5$ is	q. divisible by 5

Column I	Column II
C. If the second term in the expansion $\left(a^{1/3} + \frac{a}{\sqrt{a-1}}\right)^n$ is $14a^{5/2}$, then the value of n is	r. divisible by 10
D. The coefficient of x^4 in the expression $(1+2x+3x^2+4x^3+\dots)^{1/2}$ is	s. a prime number

$c, c \in N$, then $c+1$ (where $|x| < 1$) is

$$\text{Sol. A. } T_{r+1} = \frac{\frac{7}{2} \left(\frac{7}{2}-1\right) \left(\frac{7}{2}-2\right) \dots \left(\frac{7}{2}-r+1\right) x^r}{r!}$$

$$\text{First negative term, if } \frac{7}{2} - r + 1 < 0 \text{ i.e. } r > \frac{9}{2}$$

$$\text{Hence, } r = 5$$

$$B. T_{r+1} = {}^5C_r (y^2)^{5-r} \left(\frac{1}{y}\right)^r = {}^5C_r y^{10-3r}$$

$$\therefore 10 - 3r = 1$$

$$\Rightarrow r = 3$$

$$\text{So, coefficient of } y = {}^5C_3 = 10$$

$$C. T_2 = 14a^{5/2} \Rightarrow {}^nC_1 (a^{1/3})^{n-1} (a^{3/2})^1 = 14a^{5/2}$$

$$\Rightarrow n a^{\frac{n-1}{3} + \frac{3}{2}} = 14a^{5/2}$$

$$\Rightarrow n = 14$$

$$D. (1+2x+3x^2+4x^3+\dots)^{1/2} = [(1-x)^{-2}]^{1/2} = (1-x)^{-1}$$

$$= 1 + x + x^2 + \dots + x^n + \dots$$

$$\text{Hence, coefficient of } x^4 = 1$$

$$\therefore c = 1$$

$$\text{Hence, } c+1 = 2$$

$$A \rightarrow q, s; B \rightarrow p, q, r; C \rightarrow p; D \rightarrow p, s$$

Type 6. Single Integer Answer Type Questions

Ex 63. If $6^{83} + 8^{83}$ is divided by 49, then the sum of the digits of remainder is _____.

$$\text{Sol. } (8)(1+7)^{83} + (7-1)^{83} = (1+7)^{83} - (1-7)^{83}$$

$$= 2[{}^{83}C_1 \cdot 7 + {}^{83}C_3 \cdot 7^3 + \dots + {}^{83}C_{83} \cdot 7^{83}]$$

$$= (2 \cdot 7 \cdot 83) + 49I, \text{ where } I \text{ is an integer.}$$

$$\therefore 14 \times 83 = 1162$$

$$\Rightarrow \frac{1162}{49} = 23 \frac{35}{49}$$

$$\therefore \text{Remainder is } 35.$$

$$\Rightarrow \text{Sum of digits} = 8$$

Ex 64. The remainder, when $(15^{23} + 23^{23})$ is divided by 19, is _____.

$$\text{Sol. } (0) E = (19-4)^{23} + (19+4)^{23}$$

$$= 2[19^{23} + {}^{23}C_2 \cdot 19^{21} \cdot 4^2 + \dots + {}^{23}C_{22} \cdot 19 \cdot 4^{22}]$$

$$= 2 \cdot 19[19^{22} + {}^{23}C_2 \cdot 19^{20} \cdot 4^2 + \dots + {}^{23}C_{22} \cdot 4^{22}]$$

$$\Rightarrow E \text{ is divisible by } 19.$$

$$\Rightarrow \text{Remainder} = 0$$

Ex 65. The term independent of x in the expansion of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$, $x > 0$, is α times the corresponding binomial coefficient. Then, α is _____.

Sol. (1) $T_{r+1} = {}^{18}C_r \cdot (9x)^{18-r} \cdot (-1)^r \cdot \frac{(x)^{-r/2}}{3^r}$

$$T_{r+1} = {}^{18}C_r \cdot 9^{18-r} \cdot \frac{(-1)^r}{3^r} \cdot x^{18-\frac{3r}{2}}$$

$$\therefore 18 - \frac{3r}{2} = 0$$

$$\Rightarrow r = 120$$

$${}^{18}C_{12} \cdot \frac{9^6}{3^{12}} = \alpha \cdot {}^{18}C_{12}$$

$$\therefore \alpha = 1$$

Ex 66. $(1+x)(1+x+x^2)(1+x+x^2+x^3)\dots(1+x+x^2+\dots+x^{100})$ when written in the ascending power of x , then the highest exponent of x is λ (1010), where λ equals ____.

Sol. (5) Highest exponent in the product of first two is

$$3 = 1 + 2$$

Highest exponent in the product of first three is

$$6 = 1 + 2 + 3$$

Similarly, highest exponent in the product of first hundred

$$= 1 + 2 + \dots + 100 = 5050 = 5(1010)$$

$$\Rightarrow \lambda = 5$$

Ex 67. Given $(1-2x+5x^2-10x^3)(1+x)^n = 1+a_1x+a_2x^2+\dots$ and that $a_1^2 = 2a_2$, then the value of n is _____.

Sol. (6) $(1-2x+5x^2-10x^3)[C_0+C_1x+C_2x^2+\dots]$

$$= 1 + a_1x + a_2x^2 + \dots$$

$\Rightarrow$

$$a_1 = n - 2$$

and

$$a_2 = \frac{n(n-1)}{2} - 2n + 5$$

Now, $a_1^2 = 2a_2$. Therefore, we have

$$(n-2)^2 = n(n-1) - 4n + 10$$

$\Rightarrow$

$$n^2 - 4n + 4 = n^2 - 5n + 10$$

$\therefore$

$$n = 6$$

Ex 68. If $(1+x-3x^2)^{2145} = a_0 + a_1x + a_2x^2 + \dots$, then $a_0 - a_1 + a_2 - a_3 + \dots$ ends with _____.

Sol. (3) Put $x = -1$, $(-3)^{2145} = a_0 - a_1 + a_2 - a_3 + \dots$;

$$- (3)^{2145} = - (3^4)^{536} \cdot 3$$

which ends with 3.

Target Exercises

Type 1. Only One Correct Option

- The expansion of $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree
(a) 5 (b) 6
(c) 7 (d) 8
- The value of x , for which the 6th term in the expansion of $\left\{2^{\log_2 \sqrt{9^{x-1} + 7}} + \frac{1}{2^{(1/5) \log_2 (3^{x-1} + 1)}}\right\}^7$ is 84, is equal to
(a) 4 (b) 3
(c) -2 (d) 1
- The 10th term in the expansion of $(x-1)^{11}$ (in decreasing powers of x) is
(a) $-x$ (b) $-11x$
(c) $-x^2$ (d) $-^{11}C_9 x^2$
- If the third term in the expansion of $[x + x^{\log_{10} x}]^5$ is 10^6 , then x must be
(a) 1 (b) $\sqrt{10}$
(c) 10 (d) $10^{-3/5}$
- If the 4th term in the expansion of $(px + x^{-1})^m$ is 2.5 for all $x \in R$, then
(a) $p = \frac{5}{2}, m = 3$
(b) $p = \frac{1}{2}, m = 6$
(c) $p = -\frac{1}{2}, m = 6$
(d) None of the above
- If the coefficient of 4th term in the expansion of $\left(x + \frac{\alpha}{2x}\right)^n$ is 20, then the respective values of α and n are
(a) 2, 7 (b) 5, 8
(c) 3, 6 (d) 2, 6
- If coefficients of 2nd, 3rd and 4th terms in the expansion of $(1+x)^{2n}$ are in AP, then
(a) $2n^2 + 9n + 7 = 0$ (b) $2n^2 - 9n + 7 = 0$
(c) $2n^2 - 9n - 7 = 0$ (d) $2n^2 + 9n - 7 = 0$
- If the numerical coefficient of the p th term in the expansion of $(2x+3)^6$ is 4860, then the value(s) of p is/are
(a) 2 (b) 3
(c) 4 (d) 5
- The coefficient of the $(r+1)$ th term of $\left(x + \frac{1}{x}\right)^{20}$ when expanded in the descending powers of x , is equal to the coefficient of the 6th term of $\left(x^2 + 2 + \frac{1}{x^2}\right)^{10}$ when expanded in ascending powers of x . The value of r is
(a) 5
(b) 6
(c) 14
(d) None of the above
- Let $n \in N$. If $(1+x)^n = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ and $a_{n-3}, a_{n-2}, a_{n-1}$ are in AP, then
(a) a_1, a_2, a_3 are in AP
(b) a_1, a_2, a_3 are in HP
(c) $n = 17$
(d) $n = 14$
- The number of terms with integral coefficients in the expansion of $(7^{1/3} + 5^{1/2} \cdot x)^{600}$ is
(a) 100 (b) 50
(c) 101 (d) None of these
- The sum of the rational terms in the expansion of $(\sqrt{2} + \sqrt[3]{10})^{10}$ is
(a) 32 (b) 9
(c) 41 (d) None of these
- The coefficient of x^3 in $\left(\sqrt{x^5} + \frac{3}{\sqrt{x^3}}\right)^6$ is
(a) 0 (b) 120 (c) 420 (d) 540
- If the coefficients of x^7 and x^8 in $\left(2 + \frac{x}{3}\right)^n$ are equal, then n is equal to
(a) 56 (b) 55
(c) 45 (d) 15
- If $(1 + 2x + x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then a_r is equal to
(a) $(^nC_r)^2$ (b) $^nC_r \cdot ^nC_{r+1}$
(c) $^{2n}C_r$ (d) $^{2n}C_{r+1}$
- If in the expansion of $(1+x)^m(1-x)^n$, the coefficients of x and x^2 are 3 and -6 respectively, then m is
(a) 6 (b) 9
(c) 12 (d) 24

17. The coefficient of x^{53} in $\sum_{r=0}^{100} {}^{100}C_r (x-3)^{100-r} 2^r$ is
 (a) ${}^{100}C_{51}$ (b) ${}^{100}C_{52}$ (c) $-{}^{100}C_{53}$ (d) ${}^{100}C_{54}$
18. If the coefficients of x^2 and x^3 in the expansion of $(3+kx)^9$ are equal, then the value of k is
 (a) $-9/7$ (b) $9/7$
 (c) $7/9$ (d) None of these
19. The coefficient of x^6 in $\{(1+x)^6 + (1+x)^7 + \dots + (1+x)^{15}\}$ is
 (a) ${}^{16}C_9$
 (b) ${}^{16}C_5 - {}^6C_5$
 (c) ${}^{16}C_6 - 1$
 (d) None of the above
20. The coefficient of x^{13} in the expansion of $(1-x)^5 (1+x+x^2+x^3)^4$ is
 (a) 4 (b) -4
 (c) 0 (d) None of these
21. The coefficient of $x^6 \cdot y^{-2}$ in the expansion of $\left(\frac{x^2}{y} - \frac{y}{x}\right)^{12}$ is
 (a) ${}^{12}C_6$ (b) $-{}^{12}C_5$
 (c) 0 (d) None of these
22. Let $f(x) = \left(\sqrt{x^2+1} + \sqrt{x^2-1}\right)^6 + \left(\frac{2}{\sqrt{x^2+1} + \sqrt{x^2-1}}\right)^6$. Then,
 (a) $f(x)$ is a polynomial of the fourth degree in x
 (b) $f(x)$ has exactly two terms
 (c) $f(x)$ is not a polynomial in x
 (d) coefficient of x^6 is 46
23. The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^n$ is
 (a) ${}^nC_n$ (b) ${}^nC_4 + {}^nC_2$
 (c) ${}^nC_4 + {}^nC_1 + {}^nC_4 - {}^nC_2$ (d) ${}^nC_4 + {}^nC_2 + {}^nC_1 \cdot {}^nC_2$
24. The coefficient of $\lambda^n \mu^n$ in the expansion of $(1+\lambda)^n (1+\mu)^n (\lambda+\mu)^n$ is
 (a) $\sum_{r=0}^n ({}^nC_r)^2$ (b) $\sum_{r=0}^n ({}^nC_{r-2})^2$
 (c) $\sum_{r=0}^n ({}^nC_{r+3})^2$ (d) $\sum_{r=0}^n ({}^nC_r)^3$
25. If in the expansion of $(1+x)^n$, a, b and c are three consecutive coefficients, then n is equal to
 (a) $\frac{ac+ab+bc}{b^2+ac}$ (b) $\frac{2ac+ab+bc}{b^2-ac}$
 (c) $\frac{ab+ac}{b^2-ac}$ (d) None of these
26. If $f(x) = 1 - x + x^2 - x^3 + \dots - x^{15} + x^{16} - x^{17}$. Then, the coefficient of x^2 in $f(x-1)$ is
 (a) 826 (b) 816
 (c) 822 (d) None of these
27. The value of $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} + \dots + \binom{30}{20}\binom{30}{30}$ is equal to
 (a) ${}^{60}C_{20}$ (b) ${}^{30}C_{10}$
 (c) ${}^{60}C_{30}$ (d) ${}^{40}C_{30}$
28. If the coefficient of x^n in $(1+x)^{101} (1-x+x^2)^{100}$ is non-zero, then n cannot be of the form
 (a) $3r+1$ (b) $3r$
 (c) $3r+2$ (d) None of these
29. If the expansion of $\left(x^2 + \frac{2}{x}\right)^n$ for positive n has a term independent of x , then n is
 (a) 23 (b) 18
 (c) 16 (d) 0
30. The term independent of x in $\left[\sqrt{\frac{x}{3}} + \sqrt{\frac{3}{2x^2}}\right]^{10}$ is
 (a) 1
 (b) ${}^{10}C_1$
 (c) $5/12$
 (d) None of these
31. The term independent of x in the expansion of $[(t^{-1}-1)x + (t^{-1}+1)^{-1}x^{-1}]^8$ is
 (a) $56\left(\frac{1-t}{1+t}\right)^3$ (b) $56\left(\frac{1+t}{1-t}\right)^3$
 (c) $70\left(\frac{1-t}{1+t}\right)^4$ (d) $70\left(\frac{1+t}{1-t}\right)^4$
32. The term that is independent of x , in the expansion of $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$ is
 (a) ${}^9C_6\left(\frac{3}{2}\right)^5\left(-\frac{1}{3}\right)^4$
 (b) ${}^9C_6\left(\frac{1}{6}\right)^3$
 (c) ${}^9C_4\left(\frac{3}{2}\right)^4\left(-\frac{1}{3}\right)^5$
 (d) ${}^9C_6\left(\frac{3}{2}\right)^6\left(-\frac{1}{3}\right)^6$
33. Total number of terms, that are dependent on the value of x , in the expansion of $\left(x^2 - 2 + \frac{1}{x^2}\right)^n$ is equal to
 (a) $2n+1$ (b) $2n$
 (c) n (d) $n+1$

34. The range of the values of the term independent of x in the expansion of $\left(x \sin^{-1} \alpha + \frac{\cos^{-1} \alpha}{x}\right)^{10}$, $\alpha \in [-1, 1]$, is
- (a) $\left[-\frac{{}^{10}C_5 \pi^{10}}{2^5}, \frac{{}^{10}C_5 \pi^{10}}{2^{20}}\right]$ (b) $\left[-\frac{{}^{10}C_5 \pi^2}{2^{20}}, \frac{{}^{10}C_5 \pi^2}{2^5}\right]$
 (c) $[1, 2]$ (d) $(1, 2)$
35. The term independent of x in the expansion of $(1-x)^2 \left(x + \frac{1}{x}\right)^{10}$ is
- (a) ${}^{11}C_5$ (b) ${}^{10}C_5$
 (c) ${}^{10}C_4$ (d) None of these
36. The term independent of x in the expansion of $(1+x)^n \cdot \left(1 - \frac{1}{x}\right)^n$ is
- (a) 0, if n is odd
 (b) $(-1)^{\frac{n-1}{2}} \cdot {}^nC_{\frac{n-1}{2}}$, if n is odd
 (c) $(-1)^{n/2} \cdot {}^nC_{\frac{n}{2}-1}$, if n is even
 (d) None of the above
37. The term independent of x in the expansion of $(1+x)^n \left(1 + \frac{1}{x}\right)^n$ is
- (a) $C_0^2 + 2C_1^2 + 3C_2^2 + \dots + (n+1)C_n^2$
 (b) $(C_0 + C_1 + \dots + C_n)^2$
 (c) $C_0^2 + C_1^2 + \dots + C_n^2$
 (d) None of the above
38. $\left(\sqrt[3]{3}\sqrt{2} + \frac{1}{\sqrt[3]{3}}\right)^n$, t_7 from the 1st = $\frac{1}{6}$, t_7 from the last = $\frac{1}{6}$, n is equal to
- (a) 9 (b) 10
 (c) 8 (d) 7
39. If sum of middle terms is S in the expansion of $\left(2a - \frac{a^2}{4}\right)^9$, then the value(s) of S is/are
- (a) $\left(\frac{63}{32}\right)a^{14}(8+a)$ (b) $\left(\frac{63}{32}\right)a^{13}(8+a)$
 (c) $\left(\frac{63}{32}\right)a^{14}(8-a)$ (d) $\left(\frac{63}{32}\right)a^{13}(8-a)$
40. If the r th term is the middle term in the expansion of $\left(x^2 - \frac{1}{2x}\right)^{20}$, then the $(r+3)$ th term is
- (a) ${}^{20}C_{14} \cdot \frac{1}{2^{14}} \cdot x$ (b) ${}^{20}C_{12} \cdot \frac{1}{2^{12}} \cdot x^2$
 (c) $-\frac{1}{2^{13}} \cdot {}^{20}C_{13} \cdot x$ (d) None of these
41. In the expansion of $\left(\sqrt[3]{4} + \frac{1}{\sqrt[4]{6}}\right)^{20}$
- (a) the number of rational terms = 4
 (b) the number of irrational terms = 18
 (c) the middle term is irrational
 (d) the number of irrational terms = 17
42. If n is an integer between 0 and 21, then the minimum value of $n!(21-n)!$ is attained for n , is equal to
- (a) 1 (b) 10
 (c) 12 (d) 20
43. If the middle term in the expansion of $\left(x^2 + \frac{1}{x}\right)^n$ is $924x^6$, then n is equal to
- (a) 10 (b) 12
 (c) 14 (d) None of these
44. Middle term in the expansion of $(1+3x+3x^2+x^3)^6$ is
- (a) 4th (b) 3rd
 (c) 10th (d) None of these
45. The coefficient of $a^8b^4c^9d^9$ in $(abc+abd+acd+bcd)^{10}$ is
- (a) 10! (b) $\frac{10!}{8!4!9!9!}$
 (c) 2520 (d) None of these
46. The coefficient of x^2y^3 in the expansion of $(1-x+y)^{20}$ is
- (a) $\frac{20!}{2!3!}$ (b) $-\frac{20!}{2!3!}$
 (c) $\frac{20!}{5!2!3!}$ (d) None of these
47. The coefficient of x^2 in the expansion of $(1+x+x^2)^{10}$ is
- (a) ${}^{10}C_2$ (b) ${}^{10}C_1$
 (c) ${}^{11}C_2$ (d) None of these
48. The coefficient of x^4 in the expansion of $(1+x-2x^2)^7$ is
- (a) -81 (b) -91
 (c) 81 (d) 91
49. The total number of different terms in the expansion of $(x_1+x_2+x_3+\dots+x_n)^m$ is equal to
- (a) ${}^{n+m}C_m$ (b) ${}^{n+m-1}C_{n-1}$
 (c) ${}^{n+m+1}C_{m+1}$ (d) ${}^{n+m+1}C_{n+1}$
50. The coefficient of a^3b^4c in the expansion of $(1+a-b+c)^9$ is equal to
- (a) $\frac{9!}{3!6!}$ (b) $\frac{9!}{4!5!}$
 (c) $\frac{9!}{3!5!}$ (d) $\frac{9!}{3!4!}$

51. The number of distinct terms in the expansion of $(x + 2y - 3z + 5w - 7u)^n$ is
 (a) $n + 1$ (b) $n - 4C_4$
 (c) $n + 4C_{n+2}$ (d) None of these
52. If n is an even integer and a, b and c are distinct numbers, then the number of distinct terms in the expansion of $(a + b + c)^n + (a + b - c)^n$ is
 (a) $\left(\frac{n+2}{2}\right)^2$ (b) $n + 2$
 (c) $\frac{n+4}{2}$ (d) None of these
53. In the expansion of $(x + y + z)^{25}$
 (a) every term is of the form ${}^{25}C_r \cdot {}^rC_k \cdot x^{25-r} \cdot y^r \cdot z^k$
 (b) the coefficient of $x^8 y^9 z^9$ is ${}^{25}C_8 \cdot {}^8C_{17}$
 (c) the number of terms is 325
 (d) None of the above
54. The greatest coefficient in the expansion of $(1 + x)^{2n}$ is
 (a) $\frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{n!} \cdot 2^n$ (b) ${}^{2n}C_{n-1}$
 (c) ${}^{2n}C_{n+1}$ (d) None of these
55. The largest term in the expansion of $(3 + 2x)^{50}$, where $x = 1/5$, is
 (a) 15th (b) 51st (c) 7th (d) 6th
56. If n is positive integer and $(3\sqrt{3} + 5)^{2n+1} = \alpha + \beta$, where α is an integer and $0 < \beta < 1$, then
 (a) α is an even integer
 (b) $(\alpha + \beta)^2$ is divisible by 2^{2n+1}
 (c) α is an odd integer
 (d) None of the above
57. If $x = (2 + \sqrt{3})^n$, then the value of $x - x^2 + x[x]$, where $[]$ denotes the greatest integer function, is equal to
 (a) 1 (b) 2
 (c) 2^{2n} (d) 2^n
58. The last digit of $(2137)^{754}$ is
 (a) 2 (b) 3
 (c) 7 (d) 9
59. The remainder when 3^{37} is divided by 80 is
 (a) 78 (b) 3
 (c) 2 (d) 35
60. The remainder when 4^{101} is divided by 101 is
 (a) 4 (b) 64
 (c) 84 (d) 36
61. The remainder obtained when $1! + 2! + 3! + \dots + 95!$ is divided by 15, is
 (a) 3
 (b) 14
 (c) 1
 (d) None of the above
62. The value of $\left\{ \frac{3^{2003}}{28} \right\}$, where $\{ \}$ denotes the fractional part, is equal to
 (a) $\frac{15}{28}$ (b) $\frac{5}{28}$ (c) $\frac{19}{28}$ (d) $\frac{9}{28}$
63. p is prime number and $n < p < 2n$, if $N = {}^{2n}C_n$, then
 (a) p divides N (b) p^2 divides N
 (c) p cannot divide N (d) None of these
64. Let $f(n) = 10^n + 3 \cdot 4^{n+2} + 5, n \in N$. The greatest value of the integer which divides $f(n)$ for all n is
 (a) 27 (b) 9
 (c) 3 (d) None of these
65. 2^{60} when divided by 7 leave the remainder
 (a) 1 (b) 6 (c) 5 (d) 2
66. If sum of coefficients in the expression of $(\alpha x^2 - 2x + 1)^{35}$ is equal to sum of coefficients in the expansion of $(x - \alpha y)^{35}$, then α is equal to
 (a) 0 (b) 1
 (c) any real number (d) None of these
67. The sum of the coefficient of the binomial expansion of $\left(\frac{1}{x} + 2x \right)^n$ is equal to 6561. The constant term in the expansion is
 (a) ${}^8C_4$ (b) $16 \cdot {}^8C_4$
 (c) ${}^6C_4 \cdot 2^4$ (d) None of these
68. The sum of the numerical coefficients in the expansion of $\left(1 + \frac{x}{3} + \frac{2y}{3} \right)^{12}$ is
 (a) 1 (b) 2
 (c) 2^{12} (d) None of these
69. The sum of the coefficients $x^{2r}, r = 1, 2, 3, \dots$, in the expansion of $(1 + x)^n$ is
 (a) 2^n (b) $2^{n-1} - 1$
 (c) $2^n - 1$ (d) $2^{n-1} + 1$
70. The sum of the coefficients in the polynomial expansion of $(1 + x - 3x^2)^{2163}$ is
 (a) 1 (b) -1
 (c) 0 (d) None of these
71. If $(1 + x - 2x^2)^8 = a_0 + a_1x + a_2x^2 + \dots + a_{16}x^{16}$, then the sum $a_1 + a_3 + a_5 + \dots + a_{15}$ is equal to
 (a) -2^7 (b) 2^7
 (c) 28 (d) None of these
72. If $(1 + x)^{10} = a_0 + a_1x + a_2x^2 + \dots + a_{10}x^{10}$, then $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$ is equal to
 (a) 3^{10} (b) 2^{10}
 (c) 2^9 (d) None of these

73. In the expansion of $(1+x)^{50}$, let S be the sum of coefficients of odd power of x , then S is
(a) 0 (b) 2^{49} (c) 2^{50} (d) 2^{51}
74. If $(1+x-2x^2)^{20} = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_{40}x^{40}$, then $a_1 + a_3 + a_5 + \dots + a_{39}$ is equal to
(a) -2^{19} (b) -2^{20} (c) -2^{21} (d) -2^{18}
75. If $n \in N$ and n is not a multiple of three and $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then the value of $\sum_{r=0}^n (-1)^r \cdot a_r \cdot {}^nC_r$ is equal to
(a) 2 (b) -2 (c) 1 (d) None of these
76. If $(1+x)^{2n} = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then
(a) $a_0 + a_2 + a_4 + \dots = 2^n$
(b) $a_{n+1} > a_n$
(c) $a_{n-3} = a_{n+3}$
(d) None of the above
77. The value of $\sum_{r=0}^n {}^nC_r \sin rx \cdot \cos (n-r)x$ is equal to
(a) $2^{n-1} \cdot \sin (nx)$ (b) $2^{n+1} \cdot \sin (nx)$
(c) $2^n \cdot \sin (nx)$ (d) None of these
78. The value of $\sum_{r=0}^n {}^nC_r (\sin rx)$ is equal to
(a) $2^n \cdot \cos^n \frac{x}{2} \cdot \sin \frac{nx}{2}$ (b) $2^n \cdot \sin^n \frac{x}{2} \cdot \cos \frac{nx}{2}$
(c) $2^{n+1} \cdot \cos^n \frac{x}{2} \cdot \sin \frac{nx}{2}$ (d) $2^{n+1} \cdot \sin^n \frac{x}{2} \cdot \cos \frac{nx}{2}$
79. The value of $\sum_{r=1}^n \frac{r \cdot {}^nC_r}{{}^nC_{r-1}}$ is equal to
(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n-1)}{2}$
(c) $n(n+1)$ (d) $n(n-1)$
80. The value of $\sum_{r=0}^{2n} r \cdot ({}^{2n}C_r) \cdot \frac{1}{(r+2)}$ is equal to
(a) $\frac{2^{2n+1}(2n^2+2n-1)}{(2n+1)(2n+2)}$
(b) $\frac{2^{2n+1}(2n^2-n+1)-2}{(2n+1)(2n+2)}$
(c) $\frac{2^{2n+1}(2n^2+2n+1)}{(2n+1)(2n+2)}$
(d) None of the above
81. In $\triangle ABC$, the value of $\sum_{r=0}^n {}^nC_r a^r \cdot b^{n-r} \cdot \cos \{rB - (n-r)A\}$ is equal to
(a) c^n (b) zero
(c) a^n (d) b^n
82. The sum of the series $\sum_{r=1}^n (-1)^{r-1} \cdot {}^nC_r (a-r)$ is equal to
(a) $n \cdot 2^{n-1} + a$ (b) 0
(c) a (d) None of these
83. The sum $\sum_{r=1}^n r \cdot {}^{2n}C_r$ is equal to
(a) $n \cdot 2^{2n-1}$ (b) 2^{2n-1}
(c) $2^{n-1} + 1$ (d) None of these
84. If $(1-x+x^2)^n = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then $a_0 + a_2 + a_4 + \dots + a_{2n}$ is equal to
(a) $\frac{3^n+1}{2}$ (b) $\frac{3^n-1}{2}$
(c) $3^n - \frac{1}{2}$ (d) $3^n + \frac{1}{2}$
85. $aC_0 + (a+b)C_1 + (a+2b)C_2 + \dots + (a+nb)C_n$ is equal to
(a) $(2a+nb)2^n$ (b) $(2a+nb)2^{n-1}$
(c) $(na+2b)2^n$ (d) $(na+2b)2^{n-1}$
86. If $C_1, C_2, \dots, C_n$ are coefficients in the expansion of $(1+x)^n = 1 + C_1x + C_2x^2 + \dots + C_nx^n$, then the value of $C_1^2 - 2C_2^2 + 3C_3^2 - \dots - 2nC_{2n}^2$ is
(a) n^2 (b) $(-1)^{n-1}n$
(c) $2(-1)^{n-1}n \cdot 2^{n-1}C_n$ (d) $-n^2$
87. $({}^{10}C_0)^2 - ({}^{10}C_1)^2 + \dots - ({}^{10}C_9)^2 + ({}^{10}C_{10})^2$ equals
(a) ${}^{10}C_5$ (b) $-{}^{10}C_5$ (c) $({}^{10}C_5)^2$ (d) $(n!)^2$
88. The sum to $(n+1)$ terms of the series $\frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \dots$ is
(a) $\frac{1}{n+1}$ (b) $\frac{1}{n+2}$
(c) $\frac{1}{n(n+1)}$ (d) None of these
89. If $C_0, C_1, C_2, \dots, C_n$ denote the coefficients in the expansion of $(1+x)^n$, then the value of $C_1 + 2C_2 + 3C_3 + \dots + nC_n$ is
(a) $n2^{n-1}$ (b) $(n+2)2^n$
(c) $(n+1)2^{n-1}$ (d) $(n+2)2^{n-1}$
90. The value of $C_0 + 3C_1 + 5C_2 + 7C_3 + \dots + (2n+1)C_n$ is equal to
(a) 2^n (b) $2^n + n2^{n-1}$
(c) $2^n(n+1)$ (d) None of these
91. The value of $\sum_{r=0}^{50} ({}^{100}C_r \cdot {}^{200}C_{150+r})$ is equal to
(a) ${}^{300}C_{50}$ (b) ${}^{100}C_{50} \cdot {}^{200}C_{150}$
(c) ${}^{100}C_{50}$ (d) None of these

92. The value of $\sum_{r=0}^{2n} (-1)^r \cdot ({}^{2n}C_r)^2$ is equal to
 (a) ${}^{4n}C_{2n}$ (b) ${}^{2n}C_n$
 (c) $(-1)^n \cdot {}^{2n}C_n$ (d) None of these
93. The value of $\sum_{r=0}^{2n} r \cdot ({}^{2n}C_r)^2$ is equal to
 (a) $n \cdot {}^{4n}C_{2n}$ (b) $4n \cdot {}^{4n}C_{2n}$
 (c) $2n \cdot {}^{4n}C_{2n}$ (d) None of these
94. The value of $\sum_{i=0}^{n-r} {}^{n-i}C_r$, $r \in [1, n]$ is equal to
 (a) ${}^{n+1}C_r$ (b) ${}^nC_r$ (c) ${}^nC_{r+1}$ (d) ${}^{n+1}C_{r+1}$
95. If $C_0, C_1, C_2, \dots, C_n$ are the coefficients of the expansion of $(1+x)^n$, then the value of $\sum_{k=0}^n \frac{C_k}{k+1}$ is
 (a) 0 (b) $\frac{2^n - 1}{n}$
 (c) $\frac{2^{n+1} - 1}{n+1}$ (d) None of these
96. If $(1+x)^{2n} = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then $\left(1 + \frac{a_1}{a_0}\right)\left(1 + \frac{a_2}{a_1}\right)\left(1 + \frac{a_3}{a_2}\right) \dots \left(1 + \frac{a_{2n}}{a_{2n-1}}\right)$ is equal to
 (a) $\frac{n^n}{(2n)!}$ (b) $\frac{(2n+1)^{2n}}{(2n)!}$
 (c) $\frac{n^{n+1}}{(2n+1)!}$ (d) None of these
97. The value of $\sum_{0 \leq i < j \leq n} i \cdot {}^nC_j$ is equal to
 (a) $n(n+1) \cdot 2^{n-3}$ (b) $n^2 \cdot 2^{n-3}$
 (c) $n(n-1) \cdot 2^{n-3}$ (d) None of these
98. The value of $\sum_{0 \leq i \leq j \leq n} j \cdot {}^nC_j$ is equal to
 (a) $n \cdot 2^{n-3}$ (b) $n(n+3) \cdot 2^{n-3}$
 (c) $(n+3) \cdot 2^{n-3}$ (d) None of these
99. The value of $\sum_{r=0}^n (-1)^r \cdot r^2 \cdot {}^nC_r$ is equal to
 (a) n (b) $-n$
 (c) $(n+1)$ (d) None of these
100. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$, then $\sum_{0 \leq i < j \leq n} (C_i - C_j)^2$
 (a) $(n+1) \cdot {}^{2n}C_n - 2^{2n}$ (b) $(n+1)^2 \times {}^{2n}C_n - 2^n$
 (c) $(n+1) \times {}^{2n}C_n + 2^n$ (d) None of these
101. If $A = {}^{2n}C_0 {}^{2n}C_1 + {}^{2n}C_1 {}^{2n-1}C_1 + {}^{2n}C_2 {}^{2n-2}C_1 + \dots$, then A is
 (a) 0 (b) $2n$ (c) $n \cdot 2^{2n}$ (d) 1
102. $\sum_{0 \leq i < j \leq n} {}^nC_i$ is equal to
 (a) $n \cdot 2^{n-1}$ (b) $(n+1) \cdot 2^{n-1}$
 (c) $(n+1) \cdot 2^n$ (d) $n \cdot 2^n$
103. The value of $\sum_{0 \leq i < j < k < l \leq n} 2$ is equal to
 (a) $2(n+1)^3$ (b) $2^{n+1}C_4$
 (c) $2(n+1)^4$ (d) None of these
104. The value of $\sum_{1 \leq i < j \leq n-1} (i \cdot j) {}^nC_i \cdot {}^nC_j$ is equal to
 (a) $\frac{n^2}{2} (2^{2n-2} - 2^{n-2}C_{n-1})$
 (b) $\frac{n^2}{2} (2^{2n-2} + 2^{n-2}C_{n-1})$
 (c) $n^2 (2^{2n-2} + 2^{n-2}C_{n-1})$
 (d) $n^2 (2^{2n-2} - 2^{n-2}C_{n-1})$
105. If $S = \sum_{n=1}^{\infty} \left(\frac{{}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n}{{}^nP_n} \right)$, then S is equal to
 (a) $2e$ (b) $2e - 1$
 (c) $2e + 1$ (d) None of these
106. The value of $\sum_{r=0}^n r(n-r) \cdot ({}^nC_r)^2$ is equal to
 (a) $n^2 \cdot {}^{2n-1}C_n$ (b) $n^2 \cdot {}^{2n}C_{n-1}$
 (c) $n^2 \cdot {}^{2n-1}C_{n-1}$ (d) $n^2 \cdot {}^{2n-2}C_n$
107. If ${}^{n+1}C_{r+1} : {}^nC_r : {}^{n-1}C_{r-1} = 11:6:3$, then nr is equal to
 (a) 20 (b) 30 (c) 40 (d) 50
108. If $f(x) = x^n$, then the value of $f(1) + \frac{f'(1)}{1!} + \frac{f''(1)}{2!} + \dots + \frac{f^{(n)}(1)}{n!}$, where $f^{(r)}(x)$ denotes the r th order derivative of $f(x)$ with respect to x , is
 (a) n (b) 2^n
 (c) 2^{n-1} (d) None of these
109. The value of $\sum_{k=1}^{\infty} \sum_{r=0}^k \frac{1}{3^k} ({}^kC_r)$ is
 (a) $\frac{2}{3}$ (b) $\frac{4}{3}$ (c) 2 (d) 1
110. $1 + \frac{1}{3}x + \frac{1 \times 4}{3 \times 6}x^2 + \frac{1 \times 4 \times 7}{3 \times 6 \times 9}x^3 + \dots$ is equal to
 (a) x (b) $(1+x)^{1/3}$ (c) $(1-x)^{1/3}$ (d) $(1-x)^{-1/3}$
111. If $\alpha = \frac{5}{2!3} + \frac{5 \cdot 7}{3!3^2} + \frac{5 \cdot 7 \cdot 9}{4!3^3} + \dots$, then $\alpha^2 + 4\alpha$ is equal to
 (a) 21 (b) 23 (c) 25 (d) 27

Type 2. More than One Correct Option

112. For the expansion $(x \sin p + x^{-1} \cos p)^{10}$, ($p \in R$)
- the greatest value of the term independent of x is $10!/2^5(5!)^2$
 - the least value of sum of coefficient is zero
 - the greatest value of sum of coefficient is 32
 - the least value of the term independent of x occurs when $p = (2n+1)\frac{\pi}{4}$, $n \in Z$
113. In the expansion of $\left(x + \frac{a}{x^2}\right)^n$ ($a \neq 0$), if term independent of x does not exist, then n must be
- 20
 - 16
 - 15
 - 10
114. If the expansion of $\left(x + \frac{\alpha}{x}\right)^n$ and $\left(x + \frac{\beta}{x^2}\right)^n$ in powers of n have one term independent of x , then n is divisible by
- 2
 - 3
 - 1
 - None of these
115. If n is a positive integer, then in the trinomial expansion of $(x^2 + 2x + 2)^n$ coefficient of
- x is $2^n \cdot n$
 - x^2 is $n^2 \cdot 2^{n-1}$
 - x^3 is $2^n \cdot n+1C_3$
 - None of these
116. Which of the following will not be true?
- The last two digits of 3^{100} will be 73
 - The last two digits of 3^{50} will be 51
 - The last two digits of 3^{50} will be 49
 - The last three digits of 3^{50} will be 249
117. If $ac > b^2$, then the sum of the coefficients in the expansion of $(a\alpha^2x^2 + 2b\alpha x + c)^n$, where $a, b, c, \alpha \in R$ and $n \in N$,
- positive, if $a > 0$
 - positive, if $c > 0$
 - negative, if $a < 0$ and n is odd
 - positive, if $c < 0$ and n is even
118. If $f(n) = \sum_{r=1}^n [r(n^{n-1}C_{r-1} - r^n C_{r-1}) + (2r+1)^n C_r]$, then
- $f(n) = n^2 - 1$
 - $f(n) = (n+1)^2 - 1$
 - $\sum_{n=1}^{10} f(n) = 495$
 - $\sum_{n=1}^{10} f(n) = 374$
119. The value of ${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \dots + {}^{n+k}C_k$ is equal to
- ${}^{n+k+1}C_k$
 - ${}^{n+k+1}C_{n+1}$
 - ${}^{n+k}C_{n+1}$
 - None of these
120. Number of values of r satisfying the equation ${}^{69}C_{3r-1} - {}^{69}C_{r^2} = {}^{69}C_{r^2-1} - {}^{69}C_{3r}$ is
- 1
 - 2
 - 3
 - 7
121. Suppose $x_1, x_2, \dots, x_n$ ($n > 2$) are real numbers such that $x_i = -x_{n-i+1}$ for $1 \leq i \leq n$. Consider the sum $S = \sum \sum \sum x_i x_j x_k$ ($1 \leq i, j, k \leq n$) (i, j, k distinct), then which of the following is not true?
- $S = n! x_1, x_2, \dots, x_n$
 - $S = (n-3)(n-4)$
 - $S = (n-3)(n-4)(n-5)$
 - $S = 0$ for all n
122. If $f(m) = \sum_{i=0}^m \binom{30}{30-i} \binom{20}{m-i}$, where $\binom{p}{q} = {}^pC_q$, then
- maximum value of $f(m)$ is ${}^{50}C_{25}$
 - $f(0) + f(1) + \dots + f(50) = 2^{50}$
 - $f(m)$ is always divisible by 50 ($1 \leq m \leq 49$)
 - The value of $\sum_{m=0}^{50} (f(m))^2 = {}^{100}C_{50}$

Type 3. Assertion and Reason

Directions (Q. Nos.123-128) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a),(b),(c) and (d) out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- Statement I is true, Statement II is false
- Statement I is false, Statement II is true

123. **Statement I** Greatest term in the expansion of $(1+x)^{12}$, when $x = 11/10$ is 7th.

Statement II 7th term in the expansion of $(1+x)^{12}$ has the factor ${}^{12}C_6$ which is greatest value of ${}^{12}C_r$.

124. **Statement I** If n is an odd prime, then integral part of $(\sqrt{5} + 2)^n - 2^{n+1}$ is divisible by $20n$.

Statement II If n is prime, then ${}^nC_1, {}^nC_2, {}^nC_3, \dots, {}^nC_{n-1}$ must be divisible by n .

125. Statement I Any positive integral power of $(\sqrt{2} - 1)$ can be expressed as $\sqrt{N} - \sqrt{N-1}$ for some natural number $N > 1$.

Statement II Any positive integral power of $\sqrt{2} - 1$ can be expressed as $A + B\sqrt{2}$, where A and B are integers.

126. Two integers n and m are such that $m = n^2 - n$, where $n \geq 3$.

Statement I The number $m(m-2)$ is divisible by 24.

Statement II Product of r successive integers are divisible by $r!$.

127. Statement I If $S_n = {}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - \dots + \frac{(-1)^{n-1} {}^nC_n}{n}$ and $T_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$, then $S_n = T_n$ for all n .

Statement II $S_{n+1} - S_n = T_{n+1} - T_n$

128. Statement I $\sum_{0 \leq i < j \leq n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j} \right)$ is equal to $\frac{n^2}{2} a$, where $a = \sum_{r=0}^n \frac{1}{{}^nC_r}$

Statement II $\sum_{r=0}^n \frac{r}{{}^nC_r} = \sum_{r=0}^n \frac{n-r}{{}^nC_r}$

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 129-131) Consider following two infinite series in real r and θ

$$C = 1 + r \cos \theta + \frac{r^2 \cos 2\theta}{2!} + \frac{r^3 \cos 3\theta}{3!} + \dots$$

$$S = r \sin \theta + \frac{r^2 \sin 2\theta}{2!} + \frac{r^3 \sin 3\theta}{3!} + \dots$$

If θ remains constant and r varies, then

129. The expression $C \frac{dC}{dr} + S \frac{dS}{dr}$ is equal to

- (a) $C^2 + S^2$ (b) $(C^2 + S^2) \cos \theta$
(c) $(C^2 + S^2) \sin^2 \theta$ (d) 1

130. The expression $C \frac{dS}{dr} - S \frac{dC}{dr}$ is equal to

- (a) $C^2 + S^2$ (b) $(C^2 + S^2) \cos \theta$
(c) $(C^2 + S^2) \sin \theta$ (d) CS

131. $\left(\frac{dC}{dr} \right)^2 + \left(\frac{dS}{dr} \right)^2$ is equal to

- (a) $C^2 + S^2$ (b) $(C^2 + S^2) \cos^2 \theta$
(c) $(C^2 + S^2) \sin^2 \theta$ (d) 1

Passage II (Q. Nos. 132-135)

If $(x+y)^n = \sum_{r=0}^n C_r x^{n-r} y^r$, where $C_r = {}^nC_r$.

132. $\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \frac{C_7}{8} + \dots$ is equal to

- (a) $\frac{2^n - 1}{n+1}$ (b) $\frac{2^n + 1}{n+1}$
(c) $\frac{2^{n+1} - 1}{n+1}$ (d) None of these

133. If $\sum_{r=0}^n \left(\frac{r+2}{r+1} \right) C_r = \frac{2^8 - 1}{6}$, then n is equal to

- (a) 8 (b) 6 (c) 5 (d) 4

134. $C_0 - C_2 + C_4 - C_6 + \dots$ is equal to

- (a) $2^{n/2}$ (b) $2^{n/2} (1+i)$
(c) $2^{n/2} \cos \pi/4$ (d) None of these

135. If $(1+x)^n = \sum_{r=0}^n {}^nC_r x^r$ and $\sum_{r=0}^n \frac{1}{{}^nC_r} = a$, then the

value of $\sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j} \right)$ is equal to

- (a) $n^2 a$ (b) $\frac{n^2}{2a}$
(c) $\frac{n}{2} a$ (d) None of these

Passage III (Q. Nos. 136-138) If $(1+x)^n = \sum_{r=0}^n {}^nC_r x^r$,

then the sum of the binomial coefficients can be obtained by substituting $x=1$. But in case we have to find the sum of coefficients in some particular order, we can substitute x by $\pm ix$ or $\pm \omega x$ or $\pm \omega^2 x$ depending upon certain requirements.

136. The sum of binomial coefficients ${}^nC_0 + {}^nC_4 + {}^nC_8 + \dots$ must be

- (a) $2^{n/2} \cos \left(\frac{\pi}{8} \right)$ (b) $2^{n/2} \sin \left(\frac{n\pi}{8} \right)$
(c) $2^n + 2^{n/2} \cos \left(\frac{n\pi}{4} \right)$ (d) $2^{n-2} + 2^{\frac{n}{2}-1} \cos \left(\frac{n\pi}{4} \right)$

137. The sum of the binomial coefficients ${}^nC_0 + {}^nC_3 + {}^nC_6 + \dots$ must be

- (a) $\frac{2^n}{3}$
(b) $\frac{1}{3} \left[2^n + \cos \left(\frac{n\pi}{3} \right) \right]$
(c) $\frac{1}{3} \left[2^n + 2 \cos \left(\frac{n\pi}{3} \right) \right]$
(d) $\frac{1}{3} \left[2^n + 2 \sin \left(\frac{n\pi}{3} \right) \right]$

138. If n is an even positive integer and $k = \frac{3n}{2}$, then the

value of $\sum_{r=1}^n (-3)^{r-1} {}^{3n}C_{2r-1}$ is

- (a) 3^n (b) 3^{n-1} (c) 6^n (d) zero

Passage IV (Q. Nos. 139-141) Let n be positive integer such that $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$.

139. The value of $\sum_{r=0}^n \sum_{s=0}^n (C_r + C_s)$ is

- (a) $(n+1)2^{n+1}$ (b) $n2^n$
(c) $(n+1)2^n$ (d) $n2^{n-1}$

140. The value of $\sum_{r=0}^s \sum_{s=1}^n {}^nC_s \cdot {}^sC_r$ (when $r \leq s$) is

- (a) 3^n
(b) $n3^{n-1}$
(c) $n3^{n-1} - 1$
(d) $3^n - 1$

141. The value of $\sum_{r=0}^n \sum_{s=0}^n \sum_{t=0}^n \sum_{u=0}^n (1)$ is

- (a) ${}^nC_4$
(b) ${}^{n+1}C_4$
(c) ${}^{4n+1}C_4$
(d) $(n+1)^4$

Type 5. Match the Columns

142. Consider an expression $f(x) = x_n + x^{n+1}$, $n \in N$.

$f(x)$ is differentiated successively an arbitrary number of times, then multiplied by $(x+1)$ and again differentiated successively till it attains the form of $Ax + B$. It is found that $A - B$ is always divisible by a proper integer λ which depends on n . Now, in Column I different values of n are given and in Column II different values of λ are given, match the corresponding values of n and λ .

Column I		Column II	
A.	5	p.	15
B.	7	q.	81
C.	9	r.	49
D.	13	s.	91

143. Match the following.

Column I		Column II	
A.	If $x, y, z \in N$, then number of ordered triplet (x, y, z) satisfying $xyz = 243$ is	p.	19
B.	The number of terms in the expansion of $(x + y + z)^6$ is	q.	28
C.	If $x \in N$, then number of solutions of $x^2 + x - 400 \leq 0$ is	r.	21
D.	If $x, y, z \in N$, then number of solutions of $x + y + z = 10$ is	s.	36

144. Match the following.

Column I		Column II	
A.	The last non-zero digit in the sum $p. \quad 2 \sum_{0 \leq i \leq j \leq 10} i \cdot j {}^{10}C_i {}^{10}C_j$ is	p.	2
B.	The largest real value for x such that $q. \quad \sum_{k=0}^4 \left(\frac{3^{4-k}}{(4-k)!} \right) \left(\frac{x^k}{k!} \right) = \frac{32}{3}$	q.	4
C.	If the expansion of $(1 + x + x^2)^n$ be written as $a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n}$, then the value of $\frac{a_0 + a_1 + a_3 + a_4 + a_6 + a_7 + \dots}{a_2 + a_5 + a_8 + \dots}$ is	r.	1
D.	The value of $s. \quad \left[\frac{{}^{47}C_4}{{}^{57}C_4} + \sum_{j=0}^3 \frac{{}^{50-j}C_3}{{}^{57}C_{53}} + \sum_{j=0}^5 \frac{{}^{56-k}C_{53-k}}{{}^{57}C_4} \right]$ is	s.	3

145. Match the statements of Column I with values of Column II.

Column I		Column II	
A.	${}^{32}C_0^2 - {}^{32}C_1^2 + {}^{32}C_2^2 - \dots + {}^{32}C_{32}^2 =$	p.	${}^{63}C_{32}$
B.	${}^{32}C_0^2 + {}^{32}C_1^2 + {}^{32}C_2^2 + \dots + {}^{32}C_{32}^2 =$	q.	${}^{32}C_{16}$
C.	$\frac{1}{32} (1 \times {}^{32}C_1^2 + 2 \times {}^{32}C_2^2 + \dots + 32 \times {}^{32}C_{32}^2) =$	r.	0
D.	${}^{31}C_0^2 - {}^{31}C_1^2 + {}^{31}C_2^2 - \dots - {}^{31}C_{31}^2 =$	s.	${}^{64}C_{32}$

Type 6 Single Integer Answer Type Questions

146. Let $a = 3^{\frac{1}{223}} + 1$, for all $n \geq 3$ and let $f(x) = {}^nC_0 a^{n-1} - {}^nC_1 a^{n-2} + {}^nC_2 a^{n-3} - \dots + (-1)^{n-1} {}^nC_{n-1} a^0$.
If the value of $f(2007) + f(2008) = 3^k$, where $k \in N$, then the value of k is _____.
147. The sum of roots of x for which the sixth term of $\left(\sqrt{2^{\log(10-3^x)}} + 5\sqrt{2^{(x-2)\log 3}}\right)^m$ is equal to 21 and binomial coefficients of second, third and fourth terms are the first, third and fifth terms of an arithmetic progression, is _____.
148. The fractional part of the sum of all the rational terms in the expansion of $(3^{1/4} + 4^{1/3})^{12}$, is _____.
149. The coefficient of x^{83} in $(1+x+x^2+x^3+x^4)^n (x-1)^{n+3}$, is $-{}^nC_{2\lambda}$, then λ is _____.
150. The coefficient of x^4 in the expansion of $(1+x+x^2+x^3)^{11}$ is λ . Then, the number of divisors of λ of the form $9k$, is _____.
151. If the middle term in the expansion of $\left(\frac{x}{2} + 2\right)^8$ is 1120, then the sum of possible real values of x is _____.
152. The number formed by last two digits of the number $(17)^{256}$ is ab , then $(a+b)$ is _____.
153. The remainder, if $1+2+2^2+2^3+\dots+2^{1999}$ is divided by 5, is _____.
154. The largest real value for $x = 2\sqrt{2} - \lambda$ such that $\sum_{k=0}^4 \left(\frac{5^{4-k}}{(4-k)!}\right) \left(\frac{x^k}{k!}\right) = \frac{8}{3}$, then λ is _____.
155. The value of $C_0 - C_1 \frac{(1+x)}{(1+nx)} + C_2 \cdot \frac{(1+2x)}{(1+nx)^2} - C_3 \cdot \frac{(1+3x)}{(1+nx)^3} + \dots = 1$, then λ is

Entrances Gallery

JEE Advanced/IIT JEE

1. Coefficient of x^{11} in the expansion of $(1+x^2)^4 (1+x^3)^7 (1+x^4)^{12}$ is [2014]
(a) 1051 (b) 1106
(c) 1113 (d) 1120
2. The coefficients of three consecutive terms of $(1+x)^{n+5}$ are in the ratio 5:10:14. Then, n is... [2013]
3. For $r = 0, 1, \dots, 10$, let A_r, B_r and C_r denote, respectively the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Then, $\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$ is equal to [2010]
(a) $B_{10} - C_{10}$ (b) $A_{10} (B_{10}^2 - C_{10} A_{10})$
(c) 0 (d) $C_{10} - B_{10}$

JEE Main/AIEEE

4. The sum of coefficients of integral powers of x in the binomial expansion of $(1-2\sqrt{x})^{50}$ is [2015]
(a) $\frac{1}{2}(3^{50}+1)$ (b) $\frac{1}{2}(3^{50})$
(c) $\frac{1}{2}(3^{50}-1)$ (d) $\frac{1}{2}(2^{50}+1)$
5. If the coefficients of x^3 and x^4 in the expansion of $(1+ax+bx^2)(1-2x)^{18}$ in powers of x are both zero, then (a, b) is equal to [2014]
(a) $\left(16, \frac{251}{3}\right)$ (b) $\left(14, \frac{251}{3}\right)$
(c) $\left(14, \frac{272}{3}\right)$ (d) $\left(16, \frac{272}{3}\right)$
6. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to [2014]
(a) N
(b) $Y - X$
(c) X
(d) Y
7. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is [2014]
(a) 4 (b) 120
(c) 210 (d) 310

8. If n is a positive integer, then $(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n}$ is [2012]

(a) an irrational number
(b) an odd positive integer
(c) an even positive integer
(d) a rational number other than positive integers

9. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is [2011]

(a) -132 (b) -144
(c) 132 (d) 144

10. Let $S_1 = \sum_{j=1}^{10} j(j-1)^{10} C_j$, $S_2 = \sum_{j=1}^{10} j^{10} C_j$
and $S_3 = \sum_{j=1}^{10} j^2 {}^{10}C_j$

Statement I $S_3 = 55 \times 2^9$

Statement II $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^9$ [2010]

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

11. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9, is [2009]

(a) 0 (b) 2
(c) 7 (d) 8

12. **Statement I** $\sum_{r=0}^n (r+1) {}^nC_r = (n+2) 2^{n-1}$

Statement II $\sum_{r=0}^n (r+1) {}^nC_r x^r = (1+x)^n + nx(1+x)^{n-1}$ [2008]

(a) Statement I is false, Statement II is true
(b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(d) Statement I is true, Statement II is false

13. In the binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ equals [2007]

(a) $\frac{5}{n-4}$ (b) $\frac{6}{n-5}$
(c) $\frac{n-5}{6}$ (d) $\frac{n-4}{5}$

14. The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \dots + {}^{20}C_{10}$ is [2007]

(a) $-{}^{20}C_{10}$
(b) $\frac{1}{2} {}^{20}C_{10}$
(c) 0
(d) ${}^{20}C_{10}$

15. If the expansion in powers of x of the function $\frac{1}{(1-ax)(1-bx)}$ is $a_0 + a_1x + a_2x^2 + a_3x^3 + \dots$, then a_n is [2006]

(a) $\frac{a^n - b^n}{b-a}$ (b) $\frac{a^{n+1} - b^{n+1}}{b-a}$
(c) $\frac{b^{n+1} - a^{n+1}}{b-a}$ (d) $\frac{b^n - a^n}{b-a}$

16. For natural numbers m, n , if $(1-y)^m(1+y)^n = 1 + a_1y + a_2y^2 + \dots$ and $a_1 = a_2 = 10$, then (m, n) is [2006]

(a) (35, 20)
(b) (45, 35)
(c) (35, 45)
(d) (20, 45)

17. The value of ${}^{50}C_4 + \sum_{r=1}^6 {}^{56-r}C_3$ is [2005]

(a) ${}^{56}C_4$ (b) ${}^{56}C_3$
(c) ${}^{55}C_3$ (d) ${}^{55}C_4$

18. If the coefficients of r th, $(r+1)$ th and $(r+2)$ th terms in the binomial expansion of $(1+y)^m$ are in AP, then m and r satisfy the equation [2005]

(a) $m^2 - m(4r-1) + 4r^2 + 2 = 0$
(b) $m^2 - m(4r+1) + 4r^2 - 2 = 0$
(c) $m^2 - m(4r+1) + 4r^2 + 2 = 0$
(d) $m^2 - m(4r-1) + 4r^2 - 2 = 0$

19. If the coefficient of x^7 in $\left[ax^2 + \frac{1}{bx}\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax - \frac{1}{bx^2}\right]^{11}$, then a and b satisfy the relation [2005]

(a) $ab = 1$ (b) $\frac{a}{b} = 1$
(c) $a + b = 1$ (d) $a - b = 1$

20. If x is so small that x^3 and higher powers of x may be neglected, then $\frac{(1+x)^{3/2} - \left(1 + \frac{1}{2}x\right)^3}{(1-x)^{1/2}}$ may be approximated as [2005]

(a) $\frac{x}{2} - \frac{3}{8}x^2$ (b) $-\frac{3}{8}x^2$
(c) $3x + \frac{3}{8}x^2$ (d) $1 - \frac{3}{8}x^2$

21. The coefficient of the middle term in the binomial expansion in powers of x of $(1+\alpha x)^4$ and of $(1-\alpha x)^6$ is the same, if α equals [2004]

(a) $-\frac{5}{3}$ (b) $\frac{10}{3}$
(c) $-\frac{3}{10}$ (d) $\frac{3}{5}$

22. The coefficient of x^n in expansion of $(1+x)(1-x)^n$ is [2004]

- (a) $(n-1)$ (b) $(-1)^n(1-n)$
(c) $(-1)^{n-1}(n-1)^2$ (d) $(-1)^{n-1}n$

23. If $s_n = \sum_{r=0}^n \frac{1}{{}^nC_r}$ and $t_n = \sum_{r=0}^n \frac{r}{{}^nC_r}$, then $\frac{t_n}{s_n}$ is equal to [2004]

- (a) $\frac{n}{2}$ (b) $\frac{n}{2} - 1$ (c) $n - 1$ (d) $\frac{2n-1}{2}$

24. The number of integral terms in the expansion of $(\sqrt{3} + \sqrt[3]{5})^{256}$ is [2003]

- (a) 32 (b) 33 (c) 34 (d) 35

25. If x is positive, the first negative term in the expansion of $(1+x)^{27/5}$ is [2003]

- (a) 7th term (b) 5th term
(c) 8th term (d) 6th term

26. The coefficient of x^5 in $(1+2x+3x^2+\dots)^{-3/2}$ is [2002]

- (a) 21 (b) 25
(c) 26 (d) None of the above

27. If $|x| < 1$, then the coefficient of x^n in expansion of $(1+x+x^2+x^3+\dots)^2$ is [2002]

- (a) n (b) $n-1$ (c) $n+2$ (d) $n+1$

Other Engineering Entrances

28. The sum of the coefficients in the expansion of $(5x-4y)^n$, where n is a positive integer, is [BITSAT 2014]

- (a) 0 (b) n (c) 1 (d) -1

29. If 21st and 22nd terms in the expansion of $(1+x)^{44}$ are equal, then x is equal to [Karnataka CET 2014]

- (a) $8/7$ (b) $21/22$ (c) $7/8$ (d) $23/24$

30. If the coefficient of x^8 in $\left(ax^2 + \frac{1}{bx}\right)^{13}$ is equal to the coefficient of x^{-8} in $\left(ax - \frac{1}{bx^2}\right)^{13}$, then a and b will satisfy the relation [WB JEE 2014]

- (a) $ab+1=0$ (b) $ab=1$
(c) $a=1-b$ (d) $a+b=-1$

31. The coefficient of x^3 in the infinite series expansion of $\frac{2}{(1-x)(2-x)}$, for $|x| < 1$ is [WB JEE 2014]

- (a) $-\frac{1}{16}$ (b) $\frac{15}{8}$ (c) $-\frac{1}{8}$ (d) $\frac{15}{16}$

32. If $(a+bx)^{-3} = \frac{1}{27} + \frac{1}{3}x + \dots$, then the ordered pair (a, b) equals [EAMCET 2014]

- (a) $(3, -27)$ (b) $\left(1, \frac{1}{3}\right)$ (c) $(3, 9)$ (d) $(3, -9)$

33. If the term free from x in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is 405, then the value of k is [Manipal 2014]

- (a) ± 1 (b) ± 3
(c) ± 4 (d) ± 2

34. Middle term in the expansion of $\left(x^2 + \frac{1}{x^2} + 2\right)^n$ is [Manipal 2014]

- (a) $\frac{n!}{(n!)^2}$ (b) $\frac{(2n)!}{(n!)^2}$
(c) $\frac{(2n-1)!}{n!}$ (d) $\frac{(2n)!}{n!}$

35. The term independent of x in the expansion of $\left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)^{18}$ is [EAMCET 2014]

- (a) $-{}^{18}C_9 2^9$ (b) ${}^{18}C_9 2^{12}$
(c) ${}^{18}C_6 2^6$ (d) ${}^{18}C_9 2^8$

36. The value of the sum $({}^nC_1)^2 + ({}^nC_2)^2 + ({}^nC_3)^2 + \dots + ({}^nC_n)^2$ is [WB JEE 2014]

- (a) $({}^nC_n)^2$ (b) ${}^{2n}C_n$
(c) ${}^{2n}C_n + 1$ (d) ${}^{2n}C_n - 1$

37. Let $S = \frac{2}{1} {}^nC_0 + \frac{2^2}{2} {}^nC_1 + \frac{2^3}{3} {}^nC_2 + \dots + \frac{2^{n+1}}{n+1} {}^nC_n$.

Then, S equals [WB JEE 2014]

- (a) $\frac{2^{n+1}-1}{n+1}$ (b) $\frac{3^{n+1}-1}{n+1}$
(c) $\frac{3^n-1}{n}$ (d) $\frac{2^n-1}{n}$

38. ${}^mC_{r+1} + \sum_{k=m}^n {}^kC_r$ is equal to [AMU 2014]

- (a) ${}^nC_{r+1}$
(b) ${}^{n+1}C_{r+1}$
(c) ${}^nC_r$
(d) None of the above

39. $\frac{C_1}{C_0} + 2\frac{C_2}{C_1} + 3\frac{C_3}{C_2} + \dots + n\frac{C_n}{C_{n-1}}$ is equal to [AMU 2014]

- (a) $\frac{n(n-1)}{2}$ (b) $\frac{n(n+1)}{2}$
(c) $\frac{(n+1)(n+2)}{2}$ (d) None of these

40. If n is an integer with $0 \leq n \leq 11$, then the minimum value of $n!(11-n)!$ is attained when a value of n equals [EAMCET 2014]

- (a) 11 (b) 5
(c) 7 (d) 9

41. If the 6th term in the expansion of the binomial $[\sqrt{2}^{\log(10-3^x)} + \sqrt[5]{2}^{(x-2)\log 3}]^m$ is equal to 21 and it is known that the binomial coefficient of 2nd, 3rd and 4th terms in the expansion represent, respectively the first, third and fifth terms of an AP, then x is equal to [BITSAT 2013]
(a) 0 (b) 1 (c) -2 (d) 3
42. If the 4th term in the expansion of $(ax + \frac{1}{x})^n$ is $\frac{5}{2}$, $\forall x \in R$, then the values of a and n are [AMU 2013]
(a) $\frac{1}{2}, 6$ (b) $6, \frac{1}{2}$
(c) 2, 6 (d) None of these
43. $\sum_{0 \leq r < s \leq n} ({}^nC_r + {}^nC_s)$ is equal to [Manipal 2013]
(a) 2^{n+1} (b) $2^{n+1} - 1$
(c) $n2^n$ (d) $(n+1)2^n$
44. $(1 + \frac{C_1}{C_0})(1 + \frac{C_2}{C_1})(1 + \frac{C_3}{C_2}) \dots (1 + \frac{C_n}{C_{n-1}})$ is equal to [Manipal 2013]
(a) $\frac{n+1}{n!}$ (b) $\frac{(n+1)^n}{(n-1)!}$
(c) $\frac{(n+1)^n}{n!}$ (d) $\frac{(n-1)^n}{n!}$
45. If $\frac{e^x}{1-x} = B_0 + B_1x + B_2x^2 + \dots + B_nx^n + \dots$, then $B_n - B_{n-1}$ equals [Manipal 2012]
(a) $\frac{1}{n!}$ (b) $\frac{1}{(n-1)!}$
(c) $\frac{1}{n!} - \frac{1}{(n-1)!}$ (d) 1
46. The coefficient of x^{10} in the expansion of $1 + (1+x) + \dots + (1+x)^{20}$ is [WB JEE 2012]
(a) ${}^{19}C_9$ (b) ${}^{20}C_{10}$
(c) ${}^{21}C_{11}$ (d) ${}^{22}C_{12}$
47. If r th and $(r+1)$ th terms in the expansion of $(p+q)^n$ are equal, then $\frac{(n+1)q}{r(p+q)}$ is [Karnataka CET 2012]
(a) 0 (b) 1
(c) $\frac{1}{4}$ (d) $\frac{1}{2}$
48. If $(1+ax)^n = 1 + 6x + \frac{27}{2}x^2 + \dots + a^nx^n$, then the values of a and n are, respectively [Kerala CEE 2012]
(a) 2, 3 (b) 3, 2 (c) $\frac{3}{2}, 4$
(d) 1, 6 (e) $\frac{3}{2}, 6$
49. The 13th term in the expansion of $(x^2 + \frac{2}{x})^n$ is independent of x , then the sum of the divisors of n is [Karnataka CET 2012]
(a) 36 (b) 37 (c) 38 (d) 39
50. Let $(1+x)^{10} = \sum_{r=0}^{10} C_r x^r$ and $(1+x)^7 = \sum_{r=0}^7 d_r x^r$. If $P = \sum_{r=0}^5 C_{2r}$ and $Q = \sum_{r=0}^3 d_{2r+1}$, then $\frac{P}{Q}$ is equal to [WB JEE 2012]
(a) 4 (b) 8 (c) 16 (d) 32
51. If the binomial coefficients of $(3r)$ th and $(r+2)$ th terms in the binomial expansion of $(1+x)^{2n}$ are equal, then [AMU 2012]
(a) $n=r$ (b) $n=r+1$
(c) $n=2r$ (d) $n=2r-1$
52. If A and B are coefficients of x^n in the expansions of $(1+x)^{2n}$ and $(1+x)^{2n-1}$ respectively, then A/B is equal to [WB JEE 2011; AMU 2011]
(a) 4 (b) 2
(c) 9 (d) 6
53. In the expansion of $(x^3 - \frac{1}{x^2})^n$, $n \in N$, if the sum of the coefficients of x^5 and x^{10} is 0, then n is [GGSIPU 2011]
(a) 25 (b) 20
(c) 15 (d) None of these
54. 5th term from the end in the expansion of $(\frac{x^3}{2} - \frac{2}{x^2})^{12}$ is [GGSIPU 2011]
(a) $-7920x^{-4}$ (b) $7920x^4$
(c) $7920x^{-4}$ (d) $-7920x^4$
55. The coefficient of $p^n q^n$ in the expansion of $[(1+p)(1+q)(p+q)]^n$ is [J&K CET 2011]
(a) $\sum_{k=0}^n [C(n, k)]^2$ (b) $\sum_{k=0}^n [C(n, k+2)]^2$
(c) $\sum_{k=0}^n [C(n, k+3)]^2$ (d) $\sum_{k=0}^n [C(n, k)]^3$
56. In the expansion of $(1-3x+3x^2-x^3)^{2n}$, the middle term is [MP PET 2011]
(a) $(n+1)$ th term (b) $(2n+1)$ th term
(c) $(3n+1)$ th term (d) None of the above
57. If ${}^nC_4, {}^nC_5$ and ${}^nC_6$ are in AP, then n is [WB JEE 2011]
(a) 7 or 14 (b) 7
(c) 14 (d) 14 or 21
58. ${}^{15}C_3 + {}^{15}C_5 + \dots + {}^{15}C_{15}$ will be equal to [WB JEE 2011]
(a) 2^{14} (b) $2^{14} - 15$ (c) $2^{14} + 15$ (d) $2^{14} - 1$

59. Let $C_0, C_1, \dots, C_n$ denote the binomial coefficients in the expansion of $(1+x)^n$. The value of $C_1 - 2C_2 + 3C_3 - 4C_4 + \dots$ (upto n terms) is [UP SEE 2011]
- (a) 2^n (b) 2^{-n}
(c) 0 (d) 1
60. The expression ${}^nC_0 + 2 \cdot {}^nC_1 + 3 \cdot {}^nC_2 + \dots + (n+1) \cdot {}^nC_n$ is equal to [AMU 2011]
- (a) $(n+1) 2^n$ (b) $2^n (n+2)$
(c) $(n+2) 2^{n-1}$ (d) $(n+2) 2^{n+1}$
61. If in the expansion of $(a-2b)^n$, the sum of the 5th and 6th terms is zero, then the value of $\frac{a}{b}$ is [WB JEE 2010]
- (a) $\frac{n-4}{5}$ (b) $\frac{2(n-4)}{5}$
(c) $\frac{5}{n-4}$ (d) $\frac{5}{2(n-4)}$
62. The coefficient of the middle term in the expansion of $(x+2y)^6$ is [Kerala CEE 2010]
- (a) ${}^6C_3$
(b) $8 {}^6C_3$
(c) $8 {}^6C_5$
(d) ${}^6C_4$
(e) $8 {}^6C_4$
63. Let $(1+x)^n = 1 + a_1x + a_2x^2 + \dots + a_nx^n$. If a_1, a_2 and a_3 are in AP, then the value of n is [BITSAT 2010]
- (a) 4
(b) 5
(c) 6
(d) 7
64. If the sum of the coefficients in the expansion of $(a^2x^2 - 6ax + 11)^{10}$, where a is constant is 1024, then the value of a is [Kerala CEE 2010]
- (a) 5 (b) 1 (c) 2
(d) 3 (e) 4

Answers

Work Book Exercise 8.1

1. (b) 2. (c) 3. (b) 4. (b) 5. (b) 6. (b) 7. (c) 8. (b) 9. (d) 10. (a)

Work Book Exercise 8.2

1. (c) 2. (a) 3. (b) 4. (a) 5. (b) 6. (c) 7. (c) 8. (c) 9. (c) 10. (b)

Work Book Exercise 8.3

1. (a) 2. (b) 3. (c) 4. (a) 5. (d) 6. (c) 7. (b) 8. (a) 9. (c) 10. (c)
11. (d) 12. (b) 13. (a) 14. (b) 15. (a) 16. (d) 17. (d) 18. (b)

Work Book Exercise 8.4

1. (d) 2. (a) 3. (c) 4. (a) 5. (a) 6. (b) 7. (c) 8. (a) 9. (a)

Target Exercises

1. (c) 2. (d) 3. (d) 4. (c) 5. (b) 6. (d) 7. (b) 8. (d) 9. (a) 10. (a)
11. (c) 12. (c) 13. (d) 14. (b) 15. (c) 16. (c) 17. (c) 18. (b) 19. (a) 20. (a)
21. (c) 22. (b) 23. (d) 24. (d) 25. (b) 26. (b) 27. (b) 28. (c) 29. (b) 30. (d)
31. (c) 32. (b) 33. (c) 34. (a) 35. (a) 36. (a) 37. (c) 38. (b) 39. (d) 40. (c)
41. (b) 42. (b) 43. (b) 44. (c) 45. (c) 46. (d) 47. (c) 48. (b) 49. (b) 50. (d)
51. (d) 52. (d) 53. (a) 54. (a) 55. (c) 56. (a) 57. (a) 58. (d) 59. (b) 60. (a)
61. (a) 62. (c) 63. (a) 64. (b) 65. (a) 66. (b) 67. (b) 68. (c) 69. (b) 70. (b)
71. (a) 72. (b) 73. (b) 74. (a) 75. (d) 76. (c) 77. (a) 78. (a) 79. (a) 80. (b)
81. (a) 82. (c) 83. (a) 84. (a) 85. (b) 86. (c) 87. (b) 88. (d) 89. (a) 90. (c)
91. (a) 92. (c) 93. (a) 94. (d) 95. (c) 96. (b) 97. (c) 98. (b) 99. (d) 100. (a)
101. (c) 102. (a) 103. (b) 104. (a) 105. (d) 106. (d) 107. (d) 108. (b) 109. (c) 110. (d)
111. (b) 112. (a,b,c) 113. (a,b,d) 114. (a,b) 115. (a,b,c) 116. (a,b) 117. (a,b) 118. (b, c) 119. (a,b) 120. (c,d)
121. (a,b,c) 122. (a,b,d) 123. (b) 124. (a) 125. (b) 126. (a) 127. (a) 128. (a) 129. (b) 130. (c)
131. (a) 132. (a) 133. (c) 134. (d) 135. (d) 136. (d) 137. (c) 138. (d) 139. (a) 140. (d)
141. (d) 142. (*) 143. (**) 144. (***) 145. (****) 146. (9) 147. (2) 148. (0) 149. (8) 150. (8)
151. (0) 152. (9) 153. (0) 154. (5) 155. (0)

* $A \rightarrow p$; $B \rightarrow p, r$; $C \rightarrow p, q$; $D \rightarrow p, q, s$

** $A \rightarrow r$; $B \rightarrow q$; $C \rightarrow p$; $D \rightarrow s$

*** $A \rightarrow q$; $B \rightarrow r$; $C \rightarrow p$; $D \rightarrow p$

**** $A \rightarrow q$; $B \rightarrow s$; $C \rightarrow p$; $D \rightarrow r$

Entrances Gallery

1. (c) 2. (6) 3. (d) 4. (a) 5. (d) 6. (d) 7. (c) 8. (a) 9. (b) 10. (b)
11. (b) 12. (b) 13. (d) 14. (b) 15. (c) 16. (c) 17. (a) 18. (b) 19. (a) 20. (b)
21. (c) 22. (b) 23. (a) 24. (b) 25. (c) 26. (d) 27. (d) 28. (c) 29. (c) 30. (a)
31. (b) 32. (d) 33. (b) 34. (b) 35. (a) 36. (d) 37. (b) 38. (b) 39. (b) 40. (b)
41. (a) 42. (a) 43. (c) 44. (c) 45. (a) 46. (c) 47. (b) 48. (c) 49. (d) 50. (b)
51. (c) 52. (b) 53. (c) 54. (c) 55. (d) 56. (c) 57. (a) 58. (b) 59. (c) 60. (c)
61. (b) 62. (b) 63. (d) 64. (d)

Explanations

Target Exercises

$$1. (x + \sqrt{x^3 - 1})^5 + (x - \sqrt{x^3 - 1})^5$$

$$= 2 \left[{}^5C_0 x^5 + {}^5C_2 x^3 (\sqrt{x^3 - 1})^2 + {}^5C_4 x (\sqrt{x^3 - 1})^4 \right]$$

$$= 2 [x^5 + 10x^3(x^3 - 1) + 5x(x^3 - 1)^2]$$

This is a polynomial of degree 7.

$$2. \text{ Given expression } = \left[\sqrt{9^{x-1} + 7} + \frac{1}{(3^{x-1} + 1)^{1/5}} \right]^7$$

$$\therefore T_6 = T_{5+1} = {}^7C_5 (\sqrt{9^{x-1} + 7})^{7-5} \left(\frac{1}{(3^{x-1} + 1)^{1/5}} \right)^5$$

$$= 21(9^{x-1} + 7) \cdot \frac{1}{3^{x-1} + 1}$$

$$= \frac{21(3^{2x-2} + 7)}{3^{x-1} + 1} = 84 \quad [\text{given}]$$

$$\Rightarrow 3^{2x-2} + 7 = 4(3^{x-1} + 1)$$

$$\Rightarrow y^2 + 7 - 4y - 4 = 0, \text{ where } y = 3^{x-1}$$

$$\therefore y = 1 \Rightarrow 3^{x-1} = 1$$

$$\Rightarrow x - 1 = 0 \Rightarrow x = 1$$

$$\text{and } y = 3 \Rightarrow 3^{x-1} = 3$$

$$\Rightarrow x - 1 = 1 \Rightarrow x = 2$$

$$3. T_{10} = T_{9+1} = {}^{11}C_9 (x)^{11-9} (-1)^9 = -{}^{11}C_9 x^2$$

$$4. T_3 = T_{2+1} = {}^5C_2 x^{5-2} (x^{\log_{10} x})^2$$

$$= 10x^3 x^{2\log_{10} x} = 10x^{(3+2\log_{10} x)} = 10^6 \quad [\text{given}]$$

$$\Rightarrow x^{3+2\log_{10} x} = 10^5$$

$$\Rightarrow (3+2\log_{10} x) \log_{10} x = 5 \log_{10} 10 = 5$$

$$\Rightarrow \log_{10} x = -\frac{5}{2} \Rightarrow x = 10^{-5/2}$$

$$\text{and } \log_{10} x = 1 \Rightarrow x = 10^1 = 10$$

$$\therefore x = 10^{-5/2}, 10$$

$$5. \ln \left(px + \frac{1}{x} \right)^m, t_4 = 2.5$$

$$\therefore {}^mC_3 (px)^{m-3} \left(\frac{1}{x} \right)^3 = 2.5$$

$$\Rightarrow {}^mC_3 p^{m-3} x^{m-6} = 2.5, \text{ which is independent of } x.$$

$$\therefore m - 6 = 0 \Rightarrow m = 6$$

$$\therefore {}^6C_3 p^3 = \frac{5}{2} \Rightarrow 20p^3 = \frac{5}{2}$$

$$\text{or } p^3 = \frac{1}{8} \Rightarrow p = \frac{1}{2}$$

$$\text{Thus, } p = \frac{1}{2} \text{ and } m = 6$$

$$6. T_4 = {}^nC_3 x^{n-3} \left(\frac{\alpha}{2x} \right)^3 \Rightarrow {}^nC_3 \left(\frac{\alpha}{2} \right)^3 x^{n-6} = 20$$

Since, independent of x .

$$\Rightarrow n - 6 = 0 \text{ or } n = 6$$

$$\therefore {}^6C_3 \left(\frac{\alpha}{2} \right)^3 = 20 \Rightarrow 20 \cdot \frac{\alpha^3}{2^3} = 20$$

$$\Rightarrow \alpha = 2$$

$$7. \text{ Since, } {}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3 \text{ are in AP.}$$

$$\therefore 2^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$$

$$\Rightarrow 2n^2 - 9n + 7 = 0$$

$$8. T_p = T_{(p-1)+1} = {}^6C_{p-1} (2x)^{6-(p-1)} 3^{p-1}$$

$$= {}^6C_{p-1} 2^{7-p} 3^{p-1} x^{7-p}$$

$$\therefore \text{Coefficient of } T_p = {}^6C_{p-1} 2^{7-p} 3^{p-1} = 4860 \quad [\text{given}]$$

Clearly, $p = 5$ is the solution.

$$9. T_{r+1} = x^{20} \cdot {}^{20}C_r \left(\frac{1}{x^2} \right)^r \text{ in descending powers of } x.$$

$$T_6 = \frac{1}{x^{20}} \cdot {}^{20}C_5 (x^2)^{15} \text{ in ascending powers of } x$$

$$\Rightarrow {}^{20}C_r = {}^{20}C_5$$

$$\therefore r = 5$$

10. In an expansion, binomial coefficients equidistance from the beginning and the end are equal.

$$\therefore a_{n-1} = a_1, a_{n-2} = a_2, a_{n-3} = a_3$$

Thus, a_1, a_2, a_3 are in AP.

$$\Rightarrow 2 \cdot {}^nC_2 = {}^nC_1 + {}^nC_3$$

$$\Rightarrow 2 = \frac{{}^nC_1}{{}^nC_2} + \frac{{}^nC_3}{{}^nC_2}$$

$$\Rightarrow 2 = \frac{2}{n-1} + \frac{n-2}{3}$$

$$\Rightarrow n^2 - 9n + 14 = 0$$

$$\therefore n = 2 \text{ or } 7, \text{ neglecting } n = 2; \text{ we get } n = 7$$

$$11. T_{r+1} = {}^{600}C_r \cdot 7^{\frac{600-r}{3}} \cdot 5^{\frac{r}{2}} x^r$$

$$\text{Here, } 0 \leq r \leq 600 \text{ and } \frac{r}{2}, 200 - \frac{r}{3} \text{ are integers.}$$

$$\therefore r \text{ should be a multiple of 6.}$$

$$\therefore r = 0, 6, 12, \dots, 600 \text{ i.e. 101 terms}$$

$$12. T_{r+1} = {}^{10}C_r \cdot 2^{\frac{10-r}{2}} \cdot 3^{\frac{r}{5}}. \text{ This is rational, if } \frac{r}{5} \text{ and } 5 - \frac{r}{2} \text{ are}$$

integers. Also, $0 \leq r \leq 10$.

Now, r must be a multiple of 10 (LCM of 2 and 5).

$$\text{So, } r = 0, 10$$

$\therefore$ The rational terms are

$${}^{10}C_0 \cdot 2^5 \cdot 3^0, {}^{10}C_{10} \cdot 2^0 \cdot 3^2$$

$$\therefore \text{Sum of rational terms} = 32 + 9 = 41$$

$$13. T_{r+1} = {}^6C_r (x^{5/2})^{6-r} \left(\frac{3}{x^{3/2}} \right)^r$$

$$= {}^6C_r 3^r x^{15 - \frac{5r}{2} - \frac{3r}{2}}$$

$$= {}^6C_r 3^r x^{15-4r}$$

$$\therefore 15 - 4r = 3$$

$$\Rightarrow r = 3$$

$$\therefore T_{r+1} = T_{3+1} = {}^6C_3 (3)^3 x^{15-4(3)}$$

$$= 20 \times 27 \times x^3$$

$$= 540x^3$$

$$\therefore \text{Coefficient of } x^3 = 540$$

$$14. T_{r+1} \text{ in } \left(2 + \frac{x}{3}\right)^n = {}^nC_r 2^{n-r} \left(\frac{x}{3}\right)^r = {}^nC_r \frac{2^{n-r}}{3^r} x^r$$

$$\therefore \text{Coefficient of } x^r = {}^nC_r \frac{2^{n-r}}{3^r}$$

$$\therefore {}^nC_7 \cdot \frac{2^{n-7}}{3^7} = {}^nC_8 \cdot \frac{2^{n-8}}{3^8}$$

$$\Rightarrow \frac{2}{n-7} = \frac{1}{8 \times 3}$$

$$\Rightarrow n-7 = 48$$

$$\Rightarrow n = 55$$

$$15. \text{ We have, } (1+2x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$$

$$\Rightarrow (1+x)^{2n} = \sum_{r=0}^{2n} a_r x^r$$

Comparing the coefficients of x^r , we get
 ${}^{2n}C_r = a_r$

$$16. (1+x)^m(1-x)^n = \left(1+mx + \frac{m(m-1)}{2}x^2 + \dots\right) \times \left(1-nx + \frac{n(n-1)}{2}x^2 + \dots\right)$$

$$\therefore \text{Coefficient of } x = m-n=3 \quad [\text{given}]$$

$$\text{Coefficient of } x^2 = \frac{n(n-1)}{2} - mn + \frac{m(m-1)}{2} = -6$$

[given]

$$\Rightarrow n^2 - n - 2mn + m^2 - m = -12$$

$$\Rightarrow (m-n)^2 - (m+n) = -12$$

$$\Rightarrow (3)^2 - (m+n) = -12$$

$$\Rightarrow m+n=21$$

$\therefore$ Solving $m-n=3$, $m+n=21$, we get

$$m=12$$

$$n=9$$

$$17. \sum_{r=0}^{100} {}^{100}C_r (x-3)^{100-r} 2^r$$

$$= \{(x-3)+2\}^{100} = (x-1)^{100} = (1-x)^{100}$$

$$\therefore \text{Coefficient of } x^{53} = (-1)^{53} {}^{100}C_{53} = -{}^{100}C_{53}$$

$$18. \text{ Coefficient of } x^2 = \text{Coefficient of } x^3$$

$$\therefore {}^9C_2 3^{9-2} k^2 = {}^9C_3 3^{9-3} k^3$$

$$\Rightarrow 36 = 28k$$

$$\Rightarrow k = \frac{9}{7}$$

$$19. \text{ Expression} = \frac{(1+x)^6 \{1-(1+x)^{10}\}}{1-(1+x)}$$

[sum of 10 terms of GP]

$$= \frac{(1+x)^{16} - (1+x)^6}{x}$$

$\therefore$ Required coefficient

$$= \text{Coefficient of } x^7 \text{ in } \{(1+x)^{16} - (1+x)^6\}$$

$$= {}^{16}C_7 - {}^6C_7 = {}^{16}C_9$$

$$20. (1-x)^5(1+x)^4(1+x^2)^4$$

$$= (1-x^2)^4(1+x^2)^4(1-x)$$

$$= (1-x^4)^4(1-x)$$

$$\therefore \text{Coefficient of } x^{13} \text{ in } \{(1-x)(1-4x^4+6x^8-4x^{12})\} = 4$$

$$21. T_{r+1} = {}^{12}C_r \left(\frac{x^2}{y}\right)^{12-r} \left(-\frac{y}{x}\right)^r$$

$$= {}^{12}C_r x^{24-3r} \cdot y^{2r-12} (-1)^r$$

$\therefore$ Coefficient of $x^6 y^{-2}$ is obtained,

when $24-3r=6$ and $2r-12=-2$

$\Rightarrow r=6$ and $r=5$, which is not possible.

$$22. f(x) = (\sqrt{x^2+1} + \sqrt{x^2-1})^6 + (\sqrt{x^2+1} - \sqrt{x^2-1})^6$$

After expansion and simplification

$$f(x) = 64x^6 - 48x^2$$

$$23. (1+x+x^2+x^3)^n = [(1+x)(1+x^2)]^n = (1+x)^n(1+x^2)^n$$

$$= ({}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + {}^nC_4x^4 + \dots)$$

$$\times ({}^nC_0 + {}^nC_1x^2 + {}^nC_2x^4 + \dots)$$

$$\therefore \text{Coefficient of } x^4 = {}^nC_0 \cdot {}^nC_2 + {}^nC_2 \cdot {}^nC_1 + {}^nC_4 \cdot {}^nC_0$$

$$= {}^nC_2 + {}^nC_1 \cdot {}^nC_2 + {}^nC_4$$

$$24. \text{ Coefficient of } \lambda^n \mu^n \text{ in } (1+\lambda)^n(1+\mu)^n(\lambda+\mu)^n$$

$\Rightarrow$ Coefficient of $\lambda^n \mu^n$ in

$$\{({}^nC_0 + {}^nC_1\lambda + {}^nC_2\lambda^2 + \dots + {}^nC_n\lambda^n)$$

$$({}^nC_0\mu^n + {}^nC_1\mu^{n-1} + \dots + {}^nC_n\mu^n)\}$$

$$({}^nC_0\lambda^n + {}^nC_0\lambda^{n-1}\mu + \dots + {}^nC_n\mu^n)\}$$

Clearly, coefficient of $\lambda^n \mu^n$ is $(C_0^3 + C_1^3 + C_2^3 + \dots + C_n^3)$

$$25. \text{ Here, } a = {}^nC_r, b = {}^nC_{r+1} \text{ and } c = {}^nC_{r+2}$$

Put $n=2$, $r=0$, then option (b) holds the condition

$$\text{i.e. } n = \frac{2ac + ab + bc}{b^2 - ac}$$

$$26. f(x) = 1-x+x^2-x^3+\dots-x^{15}+x^{16}-x^{17}$$

$$= \frac{1-x^{18}}{1+x}$$

$$\Rightarrow f(x-1) = \frac{1-(x-1)^{18}}{x}$$

Therefore, required coefficient of x^2 is equal to coefficient of x^3 in $1-(x-1)^{18}$, which is given by ${}^{18}C_3 = 816$.

$$27. (1-x)^{30} = {}^{30}C_0 x^0 - {}^{30}C_1 x^1 + {}^{30}C_2 x^2$$

$$- \dots + (-1)^{30} {}^{30}C_{30} x^{30} \dots (i)$$

$$(x+1)^{30} = {}^{30}C_0 x^{30} + {}^{30}C_1 x^{29} + {}^{30}C_2 x^{28}$$

$$+ \dots + {}^{30}C_{10} x^{20} + \dots + {}^{30}C_{30} x^0 \dots (ii)$$

On multiplying Eqs. (i) and (ii) and equating the coefficient of x^{20} on both sides, we get required sum is equal to coefficient of x^{20} in $(1+x^2)^{30}$, which is given by ${}^{30}C_{10}$.

$$28. \text{ We have, } (1+x)^{101} (1-x+x^2)^{100} = (1+x) \{(1+x)(1-x+x^2)\}^{100}$$

$$= (1+x) (1+x^3)^{100} = (1+x) \{C_0 + C_1 x^3 + C_2 x^6$$

$$+ \dots + C_{100} x^{300}\}$$

$$= (1+x) \sum_{r=0}^n {}^nC_r x^{3r} = \sum_{r=0}^n {}^nC_r x^{3r} + \sum_{r=0}^n {}^nC_r x^{3r+1}$$

Hence, there will be no term containing $3r+2$.

$$29. {}^nC_r 2^r \cdot x^{2n-3r} \Rightarrow 2n-3r=0$$

$$\Rightarrow 2n=3r$$

n is multiple of $3/2$.

$$30. T_{r+1} = {}^{10}C_r \left(\sqrt{\frac{x}{3}} \right)^{10-r} \left(\sqrt{\frac{3}{2x^2}} \right)^r$$

$$= {}^{10}C_r \left(\frac{1}{\sqrt{3}} \right)^{10-r} \left(\sqrt{\frac{3}{2}} \right)^r x^{5-\frac{r}{2}-r}$$

Let T_{r+1} be the term independent of x .

$$\therefore 5 - \frac{r}{2} - r = 0 \Rightarrow \frac{3r}{2} = 5$$

$$\Rightarrow r = \frac{10}{3}$$

$\therefore r$ is not a whole number.

$\therefore$ Given expression does not have term independent of x .

$$31. [(t^{-1} - 1)x + (t^{-1} + 1)^{-1}x^{-1}]^8$$

$$= \left[\left(\frac{1}{t} - 1 \right)x + \left(\frac{1}{t} + 1 \right)^{-1} \frac{1}{x} \right]^8$$

$$T_{r+1} = {}^8C_r \left[\left(\frac{1}{t} - 1 \right)x \right]^{8-r} \left[\left(\frac{1}{t} + 1 \right)^{-1} \frac{1}{x} \right]^r$$

$$= {}^8C_r \left(\frac{1}{t} - 1 \right)^{8-r} \left(\frac{1}{t} + 1 \right)^{-r} x^{8-2r}$$

Let T_{r+1} be the term independent of x .

$$\therefore 8 - 2r = 0$$

$$\text{i.e. } r = 4$$

$$\therefore T_{r+1} = T_{4+1} = {}^8C_4 \left(\frac{1}{t} - 1 \right)^{8-4} \left(\frac{1}{t} + 1 \right)^{-4} x^{8-2(4)}$$

$$= 70 \left(\frac{1-t}{t} \right)^4 \left(\frac{t}{1+t} \right)^4 = 70 \left(\frac{1-t}{1+t} \right)^4$$

$$32. \left(\frac{3}{2}x^2 - \frac{1}{3x} \right)^9 = \sum_{r=0}^9 {}^9C_r \left(\frac{3}{2}x^2 \right)^{9-r} \left(-\frac{1}{3x} \right)^r$$

For the term that is independent of x , we must have

$$18 - 2r - r = 0$$

$$\Rightarrow r = 6$$

$$\text{Thus, required term} = {}^9C_6 \left(\frac{3}{2}x^2 \right)^3 \left(-\frac{1}{3x} \right)^6$$

$$\text{i.e. } {}^9C_6 \left(\frac{3}{2} \right)^3 \left(\frac{1}{3} \right)^6 = {}^9C_6 \left(\frac{1}{6} \right)^3$$

$$33. \left(x^2 - 2 + \frac{1}{x^2} \right)^n = \frac{1}{x^{2n}} (x^4 - 2x^2 + 1)^n$$

$$= \frac{(x^2 - 1)^{2n}}{x^{2n}}$$

Total number of terms that are dependent on x = number of terms in the expansion of $(x^2 - 1)^{2n}$ that have degree of x different from $2n$.

$$= (n+1) - 1 = n$$

$$34. \text{Term independent of } x = {}^{10}C_5 (\sin^{-1} \alpha)^5 (\cos^{-1} \alpha)^5$$

$$= {}^{10}C_5 (\sin^{-1} \alpha)^5 \left(\frac{\pi}{2} - \sin^{-1} \alpha \right)^5$$

$$= {}^{10}C_5 \left\{ \frac{\pi}{2} \sin^{-1} \alpha - (\sin^{-1} \alpha)^2 \right\}^5$$

$$- {}^{10}C_5 \left\{ \left(\sin^{-1} \alpha - \frac{\pi}{4} \right)^2 - \frac{\pi^2}{16} \right\}^5$$

$$\Rightarrow \text{Minimum value is } - {}^{10}C_5 \left(\frac{9\pi^2 - \pi^2}{16} \right)^5 = \frac{- {}^{10}C_5 \pi^{10}}{2^5}$$

$$\text{and maximum value is } - {}^{10}C_5 \left(\frac{-\pi^{10}}{16^5} \right) = \frac{{}^{10}C_5 \pi^{10}}{2^{20}}$$

$$35. \text{Expression} = \frac{1}{x^{10}} (1 - 2x + x^2)(1 + x^2)^{10}$$

$$= \frac{1}{x^{10}} (1 + x^2)^{10} - \frac{2}{x^9} (1 + x^2)^{10} + \frac{1}{x^8} (1 + x^2)^{10}$$

$\therefore$ The term independent of x

$$= {}^{10}C_5 - 2 \times 0 + {}^{10}C_4 = {}^{11}C_5$$

$$36. \text{Expression} = (-1)^n \cdot \frac{(1 - x^2)^n}{x^n}$$

$\therefore$ Term independent of x

$$= \text{Coefficient of } x^n \text{ in } (-1)^n \cdot (1 - x^2)^n$$

= Coefficient of x^n in

$$(-1)^n \{ {}^nC_0 + {}^nC_1(-x)^2 + {}^nC_2(-x^2)^2 + \dots + {}^nC_n(-x^2)^n \}$$

Since, the expansion contains only even powers of x .

Thus, if n is odd $\Rightarrow$ Coefficient is zero.

$$37. (1 + x)^n \left(1 + \frac{1}{x} \right)^n = (C_0 + C_1x + C_2x^2 + \dots + C_nx^n)$$

$$\times \left(C_0 + \frac{C_1}{x} + \frac{C_2}{x^2} + \dots + \frac{C_n}{x^n} \right)$$

Term independent of x on the RHS

$$= C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2$$

$$38. T_n \text{ in } \left[\sqrt[6]{3}\sqrt{2} + \frac{1}{\sqrt[3]{3}} \right]^n = {}^nC_6 (2^{1/2}3^{1/6})^{n-6} \left[\frac{1}{3^{1/3}} \right]^6$$

$$7\text{th term from the end in } \left[\sqrt[6]{3}\sqrt{2} + \frac{1}{\sqrt[3]{3}} \right]^n$$

$$= T_n \text{ in } \left[\frac{1}{\sqrt[3]{3}} + \sqrt[6]{3}\sqrt{2} \right]^n = {}^nC_6 \left(\frac{1}{3^{1/3}} \right)^{n-6} (3^{1/6}2^{1/2})^6$$

$$\therefore \frac{{}^nC_6 [2^{1/2}3^{1/6}]^{n-6} \left[\frac{1}{3^{1/3}} \right]^6}{{}^nC_6 \left[\frac{1}{3^{1/3}} \right]^{n-6} [2^{1/2}3^{1/6}]^6} = \frac{1}{6}$$

$$\Rightarrow \frac{\left[2^{\frac{n-6}{2}} 3^{\frac{n-6}{6}} \right] \times [3^{-2}]}{3^{-\left(\frac{n-6}{3}\right)} 2^3 \cdot 3^1} = \frac{1}{6}$$

$$\Rightarrow \frac{2^{\frac{n-6}{2}} 3^{\frac{n-6}{6}}}{3^{\frac{6-n}{3}} 2^3 \cdot 3^1} \cdot \frac{1}{3^2} = \frac{1}{6}$$

$$\Rightarrow \frac{2^{\frac{n-6}{2}}}{3^{\frac{6-n}{3}}} \cdot \frac{1}{3^{\left(\frac{n-6}{6}\right)}} \cdot \frac{1}{2^3 \cdot 3^3} = \frac{1}{6}$$

$$\Rightarrow \frac{2^{\frac{n-6}{2}}}{3^{\frac{6-n}{3}}} = \frac{2^3 \cdot 3^3}{6}$$

$$\Rightarrow \frac{2^{\frac{n-6}{2}}}{3^{\frac{n-6}{2}}} \cdot 3^{\frac{n-6}{2}} = 2^2 \times 3^2$$

Comparing the coefficients of 2 and 3, we get

$$\frac{n-6}{2} = 2 \Rightarrow n = 10$$

39. Middle terms are $T_{\frac{9+1}{2}} = T_5$ and $T_{\frac{9+3}{2}} = T_6$

$$\therefore S = {}^9C_4(2a)^{9-4} \left(-\frac{a^2}{4}\right)^4 + {}^9C_5(2a)^{9-5} \left(-\frac{a^2}{4}\right)^5$$

$$= 126 \times 32a^5 \times \frac{a^8}{256} - 126 \times 16a^4 \times \frac{a^{10}}{1024}$$

$$= \frac{63}{4}a^{13} - \frac{63}{32}a^{14} = \frac{63}{32}(8-a)a^{13}$$

40. As the index = 20, the 11th term is the middle term. So, we have to find the 14th term.

$$T_{14} = {}^{20}C_{13}(x^2)^{20-13} \left(-\frac{1}{2x}\right)^{13}$$

$$= -{}^{20}C_{13}x^{14} \cdot \frac{1}{2^{13}x^{13}}$$

$$= -\frac{1}{2^{13}} {}^{20}C_{13}x$$

41. $T_{r+1} = {}^{20}C_r \cdot 4^{\frac{20-r}{3}} \cdot 6^{-\frac{r}{4}} = {}^{20}C_r \cdot 2^{\frac{40-2r}{3}} \cdot 2^{\frac{r}{4}} \cdot 3^{-\frac{r}{4}}$

$$= {}^{20}C_r \cdot 2^{\frac{160-11r}{12}} \cdot 3^{-\frac{r}{4}}$$

For rational terms, $\frac{r}{4}$ and $\frac{160-11r}{12}$ must be integers and $0 \leq r \leq 20$.

So, $\frac{r}{4} = \text{integer} \Rightarrow r = 0, 4, 8, 12, 16, 20$

Clearly, for $r = 8, 16$ and 20 , $\frac{160-11r}{12}$ is also an integer.

$\therefore$ Only three terms are rational. So, $21-3$, i.e. 18 terms are irrational.

42. $n!(21-n)! = 21! \frac{n!(21-n)!}{21!} = \frac{21!}{{}^{21}C_n}$, which is minimum when ${}^{21}C_n$ is maximum, which occurs when $n = 10$.

43. Since, n is even, therefore $\left(\frac{n}{2} + 1\right)$ th term is middle term.

Hence, ${}^nC_{n/2}(x^2)^{n/2} \left(\frac{1}{x}\right)^{n/2} = 924x^6$

$$\Rightarrow x^{n/2} = x^6 \Rightarrow n = 12$$

44. $(1+3x+3x^2+x^3)^6 = \{(1+x)^3\}^6 = (1+x)^{18}$

Hence, the middle term is 10th.

45. $a^{10}b^{10}c^{10}d^{10} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{10}$

Therefore, the required coefficient is equal to the coefficient of $a^{-2}b^{-6}c^{-1}d^{-1}$ in $\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\right)^{10}$, which is given by $\frac{10!}{2!6!1!1!} = \frac{10 \times 9 \times 8 \times 7}{2} = 2520$

46. The general term in the expansion of $(1-x+y)^{20}$ is

$$\frac{20!}{r!s!t!} 1^r (-x)^s (y)^t, \text{ where } r+s+t=20$$

For x^2y^3 , we have the term

$$\frac{20!}{15!2!3!} 1^{15} (-x)^2 (y)^3$$

Hence, the coefficient of x^2y^3 is $\frac{20!}{15!2!3!}$

47. $(1+x+x^2)^{10} = \left(\frac{(1-x)(1+x+x^2)}{1-x}\right)^{10}$

$$= (1-x^3)^{10} (1-x)^{-10}$$

Thus, coefficient of x^2 in $(1+x+x^2)^{10}$ = coefficient of x^2 in $(1-x)^{-10}$ i.e. ${}^{10+2-1}C_2 = {}^{11}C_2$

48. $(1+x-2x^2)^7 = \sum_{r=0}^7 {}^7C_r (1+x)^{7-r} (-2x^2)^r$

Thus, required coefficient

$$= {}^7C_0 \cdot {}^7C_4 + {}^7C_1 (-2) \cdot {}^6C_2 + {}^7C_2 \cdot (-2)^2 \cdot {}^5C_0$$

$$= 1 \cdot 35 - 7 \cdot 2 \cdot 15 + 21 \cdot 4 \cdot 1 = -91$$

49. Total number of different terms in the expansion of $(x_1 + x_2 + \dots + x_n)^m$ is same as the number of non-negative integral solutions of $y_1 + y_2 + \dots + y_n = m$ i.e. ${}^{n+m-1}C_{n-1}$

50. $(1+a-b+c)^9$

$$= \sum \frac{9!}{x_1!x_2!x_3!x_4!} \cdot (1)^{x_1} (a)^{x_2} (-b)^{x_3} (c)^{x_4}$$

$$\Rightarrow \text{Coefficient of } a^3b^4c = \frac{9!}{1!3!4!1!} = \frac{9!}{3!4!}$$

51. Number of terms = ${}^{n+5-1}C_{5-1} = {}^{n+4}C_4$

52. $(a+b+c)^n + (a+b-c)^n = 2\{{}^nC_0 c^0 (a+b)^n + {}^nC_2 c^2 (a+b)^{n-2} + \dots + {}^nC_n c^n (a+b)^0\}$

Number of terms = $(n+1) + (n-1) + (n-3) + \dots + 1$

$$= \left(\frac{n+2}{2}\right)[1+(n+1)] = \frac{(n+2)^2}{2}$$

53. $(x+y+z)^{25} = \{x+(y+z)\}^{25}$

$$= {}^{25}C_0 x^{25} + {}^{25}C_1 x^{24} (y+z) + \dots + {}^{25}C_r x^{25-r} (y+z)^r + \dots$$

$$= \dots + {}^{25}C_r x^{25-r} (\dots + {}^rC_k y^r - k z^k + \dots) + \dots$$

$8+9+9 \neq 25$. So, there is no term like $x^8y^9z^9$.

The number of terms

$$= 1+2+3+\dots+26$$

$$= \frac{26 \times 27}{2}$$

$$= 351$$

54. The greatest coefficient

= Coefficient of the middle term

$$= {}^{2n}C_n = \frac{(2n)!}{n!n!}$$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3)(2n-4)\dots 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{n!n!}$$

$$= \frac{2^n \{n \cdot (n-1) \dots 2 \cdot 1\} \{(2n-1)(2n-3)(2n-5)\dots 5 \cdot 3 \cdot 1\}}{n!n!}$$

$$= \frac{2^n \cdot n! \{(2n-1)(2n-3)(2n-5)\dots 5 \cdot 3 \cdot 1\}}{n!n!}$$

$$= \frac{2^n \{1 \cdot 3 \cdot 5 \dots (2n-1)\}}{n!}$$

55. For largest term $\frac{n-r+1}{r} \cdot \left|\frac{y}{x}\right| \geq 1$

$$\Rightarrow \frac{51-r}{r} \cdot \frac{2x}{3} \geq 1; \text{ if } \frac{51-r}{r} \cdot \frac{2}{3} \left(\frac{1}{5}\right) \geq 1$$

if $102-2r \geq 15r$; if $17r \leq 102$

if $r \leq 6$

$\therefore$ We take $r = 6$

$\therefore$ Largest term = $T_{r+1} = T_{6+1} = T_7$

56. $\alpha + \beta = (3\sqrt{3} + 5)^{2n+1}$
 $\beta_1 = (3\sqrt{3} - 5)^{2n+1}$, where $\beta_1 < 1$
 $\Rightarrow \alpha + \beta + \beta_1 = \text{Even integer}$, $\beta - \beta_1 = 0$
 $\Rightarrow \alpha = \text{Even integer}$
57. $x - x^2 + x[x] = x - x(x - [x]) = x - x\{x\}$
 $= x(1 - \{x\})$
 $x = (2 + \sqrt{3})^n = {}^nC_0 2^n + {}^nC_1 2^{n-1} \sqrt{3}$
 $+ {}^nC_2 2^{n-2} (\sqrt{3})^2 + \dots$
 Let $x_1 = (2 - \sqrt{3})^n$
 $= {}^nC_0 2^n - {}^nC_1 2^{n-1} \sqrt{3} + {}^nC_2 2^{n-2} (\sqrt{3})^2 - \dots$
 $x + x_1 = 2({}^nC_0 2^n + {}^nC_2 2^{n-2} (\sqrt{3})^2 + \dots)$
 $= \text{Even integer}$
 Clearly, $x_1 \in (0, 1)$, $\forall n \in \mathbb{N}$
 $\Rightarrow [x] + \{x\} + x_1 = \text{Even integer}$
 $\Rightarrow \{x\} + x_1 = \text{Integer}$
 $\{x\} \in (0, 1)$, $x_1 \in (0, 1)$
 $\Rightarrow \{x\} + x_1 \in (0, 2)$
 $\Rightarrow \{x\} + x_1 = 1$
 $\Rightarrow x_1 = 1 - \{x\}$
 $\Rightarrow x(1 - \{x\}) = x \cdot x_1 = (2 + \sqrt{3})^n (2 - \sqrt{3})^n = 1$
58. End digit of 7^{4n+1} is 7 of 7^{4n+2} is 9,
 of $7^{4n+3} = 1$ and $7^{4n} = 7n \in \mathbb{N}$
 Now, $754 = 4 \cdot 188 + 2$. Thus, end digit of $(2137)^{754}$ will be
 same as that of 7^{4n+2} i.e. 9.
59. $3^{37} = 3^{4 \cdot 9} \cdot 3 = 3 \cdot (81)^9 = 3(80 + 1)^9$
 $= 3({}^9C_0 80^9 + {}^9C_1 80^8 + \dots + {}^9C_9)$
 Thus, required remainder is equal to 3.
60. $4^{101} = 4^{100} \cdot 4$, clearly $101 \div 4^{100}$
 Thus, required remainder is 4.
61. $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$
 $\Rightarrow \text{Remainder} = 3$
62. $3^{2003} = 3^{2001} \cdot 3^2 = 9(27)^{667} = 9(28 - 1)^{667}$
 $= 9[{}^{667}C_0 28^{667} + {}^{667}C_1 (28)^{666} + \dots + {}^{667}C_{667} (-1)^{667}]$
 When 3^{2003} is divided by 28, remainder is 19.
 Thus, $\left\{ \frac{3^{2003}}{28} \right\} = \frac{19}{28}$
63. $N = {}^{2n}C_n = \frac{(2n)!}{(n!)^2} = \frac{(n+1)(n+2) \dots (n+n)}{(n!)}$
 $\Rightarrow (n!)N = (n+1)(n+2) \dots (n+n)$
 Since, $n < p < 2n$
 $\Rightarrow p \mid (n+1)(n+2) \dots (n+n)$
 $\Rightarrow p \mid N$
64. Expression $= 10^n - 1 + 3 \cdot 4^{n+2} + 6$
 $= (10^n - 1) + 6(2^{2n+3} + 1)$
 $= (10 - 1)\{10^{n-1} + 10^{n-2} + \dots + 10 + 1\}$
 $+ 6(2 + 1)\{2^{2n+2} - 2^{2n+1} + \dots - 2 + 1\}$
 $10^{n-1} + 10^{n-2} + \dots + 10 + 1$ is an odd number which is
 not divisible by any odd number of all n .
 Again, $2^{2n+2} - 2^{2n+1} + \dots - 2 + 1$ is also an odd
 number which is not divisible by any odd number for
 all n .
 $\therefore$ The expression is divisible by 9.

65. $2^{60} = (1 + 7)^{20}$
 $= {}^{20}C_0 + {}^{20}C_1 \cdot 7 + {}^{20}C_2 \cdot 7^2 + \dots + {}^{20}C_{20} \cdot 7^{20}$
 $\therefore$ The remainder $= {}^{20}C_0 = 1$
66. $(\alpha - 2 + 1)^{35} = (1 - \alpha)^{35}$
 $\Rightarrow (\alpha - 1)^{35} = -(\alpha - 1)^{35} \Rightarrow \alpha = 1$
67. The sum of all the coefficients in an expansion is
 obtained by putting $x = 1$ in the expression.
 $\therefore \left(\frac{1}{1} + 2 \cdot 1\right)^n = 6561$
 $\therefore 3^n = 3^8 \Rightarrow n = 8$
 $\ln\left(\frac{1}{x} + 2x\right), T_{r+1} = {}^8C_r \cdot \left(\frac{1}{x}\right)^{8-r} \cdot (2x)^r$
 This is a constant, if $8 - 2r = 0$ i.e. $r = 4$
 $\therefore$ The constant term $= T_5 = {}^8C_4 \cdot 2^4$
68. The sum of all the numerical coefficients in the
 expansion is obtained by putting $x = 1, y = 1$
 i.e. $\left(1 + \frac{1}{3} + \frac{2}{3}\right)^{12} = (2)^{12}$
69. Sum of coefficient of $x^2, x^4, x^6, x^8, \dots$ is
 ${}^nC_2 + {}^nC_4 + {}^nC_6 + \dots = 2^{n-1} - 1$
70. The sum of coefficient is obtained by putting variable
 term as 1.
 $\therefore (1 + x - 3x^2)^{2163} = (1 + 1 - 3)^{2163} = -1$,
 as sum of coefficient.
71. Putting $x = 1, x = -1$ and adding, we get
 Sum $= \frac{1}{2} \{(a_0 + a_1 + a_2 + \dots + a_{16}) - (a_0 - a_1 + a_2 - \dots + a_{16})\}$
 $\therefore a_1 + a_3 + a_5 + \dots + a_{15}$
 $= \frac{1}{2} \{(1 + 1 - 2)^8 - (1 - 1 - 2)^8\} = \frac{-1}{2} (2^8) = -2^7$
72. Putting $x = i, -i$ and multiplying both the results, we get
 the value of the required expression as,
 $(a_0 - a_2 + a_4 - a_6 + a_8 - a_{10})^2 + (a_1 - a_3 + a_5 - a_7 + a_9)^2$
 $= (1 + i)^{10} (1 - i)^{10} = (1 + 1)^{10} = 2^{10}$
73. Sum of coefficients of odd powers of x .
 ${}^{50}C_1 + {}^{50}C_3 + \dots + {}^{50}C_{49} = 2^{50-1} = 2^{49}$
74. $(1 + x - 2x^2)^{20} = a_0 + a_1 x + a_2 x^2 + \dots + a_{40} x^{40}$
 $a_0 + a_1 + a_2 + a_3 + \dots + a_{40} = 0$
 $a_0 - a_1 + a_2 - a_3 + \dots + a_{40} = 2^{20}$
 $\therefore a_1 + a_3 + \dots + a_{39} = -2^{19}$
75. $(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r \dots (i)$
 We know that, $(1 - x)^n = \sum_{r=0}^n (-1)^r \cdot {}^nC_r x^r$
 $= \sum_{r=0}^n (-1)^{n-r} \cdot {}^nC_r x^{n-r} \dots (ii)$
 On multiplying Eqs. (i) and (ii), we get
 $\sum_{r=0}^n (-1)^{n-r} \cdot {}^nC_r a_r = \text{Coefficient of } x^n \text{ in } (1 - x^3)^n$
 Since, $n \neq M(3)$
 $\therefore \sum_{r=0}^n (-1)^{n-r} \cdot a_r \cdot {}^nC_r = 0 \Rightarrow \sum_{r=0}^n (-1)^r \cdot a_r \cdot {}^nC_r = 0$

76. $a_0 + a_1 + a_2 + \dots = 2^{2n}$ and $a_0 + a_2 + a_4 + \dots = 2^{2n-1}$
 $a_n = {}^{2n}C_n$ = The greatest coefficient, being the middle coefficients.

$$a_{n-3} = {}^{2n}C_{n-3} = {}^{2n}C_{2n-(n-3)} = {}^{2n}C_{n+3} = a_{n+3}$$

$$77. I = \sum_{r=0}^n {}^nC_r \sin rx \cdot \cos(n-r)x$$

$$= \sum_{r=0}^n {}^nC_{n-r} \sin(n-r)x \cdot \cos rx$$

$$\Rightarrow 2I = \sum_{r=0}^n {}^nC_r \cdot \sin nx = \sin nx \cdot 2^n$$

$$\Rightarrow I = 2^{n-1} \cdot \sin nx$$

$$78. \sum_{r=0}^n {}^nC_r \sin rx = \operatorname{Im} \left(\sum_{r=0}^n {}^nC_r e^{irx} \right)$$

$$= \operatorname{Im} \left(\sum_{r=0}^n {}^nC_r (e^{ix})^r \right) = \operatorname{Im}((1 + e^{ix})^n)$$

$$= \operatorname{Im}(1 + \cos x + i \sin x)^n$$

$$= \operatorname{Im} \left(2 \cos^2 \frac{x}{2} + 2i \sin \frac{x}{2} \cdot \cos \frac{x}{2} \right)^n$$

$$= \operatorname{Im} \left(2 \cos \frac{x}{2} \left(\cos \frac{x}{2} + i \sin \frac{x}{2} \right) \right)^n$$

$$= 2^n \cdot \cos^n \frac{x}{2} \cdot \sin \frac{nx}{2}$$

$$79. \sum_{r=1}^n \frac{r \cdot {}^nC_r}{{}^nC_{r-1}} = \sum_{r=1}^n r \cdot \frac{n-r+1}{r}$$

$$= (n+1) \sum_{r=1}^n 1 - \sum_{r=1}^n r$$

$$= n(n+1) - \frac{n(n+1)}{2} = \frac{n(n+1)}{2}$$

$$80. \sum_{r=0}^{2n} r \cdot {}^{2n}C_r \cdot \frac{1}{r+2} = \sum_{r=0}^{2n} (r+2-2) \cdot \frac{{}^{2n}C_r}{r+2}$$

$$= \sum_{r=0}^{2n} {}^{2n}C_r - 2 \cdot \sum_{r=0}^{2n} \frac{{}^{2n}C_r}{r+2}$$

$$= 2^{2n} - 2 \cdot \int_0^1 x(1+x)^{2n} dx$$

$$= 2^{2n} - 2 \left\{ \left[x \cdot \frac{(1+x)^{2n+1}}{2n+1} \right]_0^1 - \left[\frac{(1+x)^{2n+2}}{(2n+1)(2n+2)} \right]_0^1 \right\}$$

$$= \frac{2^{2n+1}(2n^2 - n + 1) - 2}{(2n+1)(2n+2)}$$

$$81. \sum_{r=0}^n {}^nC_r a^r \cdot b^{n-r} \cdot \cos\{rB - (n-r)A\}$$

$$= \operatorname{Re} \left(\sum_{r=0}^n {}^nC_r a^r \cdot b^{n-r} \cdot e^{i\{rB - (n-r)A\}} \right)$$

$$= \operatorname{Re} \left(\sum_{r=0}^n {}^nC_r (a \cdot e^{iB})^r \cdot (b \cdot e^{-iA})^{n-r} \right)$$

$$= \operatorname{Re}(ae^{iB} + be^{-iA})^n$$

$$= \operatorname{Re}(a \cos B + i a \sin B + b \cos A - i b \sin A)^n$$

$$= (a \cos B + b \cos A)^n = c^n$$

$$82. \text{Sum} = a \sum_{r=1}^n (-1)^{r-1} \cdot {}^nC_r - \sum_{r=1}^n (-1)^{r-1} \cdot r \cdot {}^nC_r$$

$$= a \{ {}^nC_1 - {}^nC_2 + {}^nC_3 - \dots \} - n \{ {}^{n-1}C_0 - {}^{n-1}C_1 + \dots \}$$

$$[\because r \cdot {}^nC_r = n \cdot {}^{n-1}C_{r-1} \text{ and } {}^nC_0 - {}^nC_1 + {}^nC_2 - \dots = 0]$$

$$= a \cdot {}^nC_0 = a$$

$$83. r \cdot {}^{2n}C_r = 2n \cdot {}^{2n-1}C_{r-1}$$

$$\therefore \sum_{r=1}^n r \cdot {}^{2n}C_r = 2n \{ {}^{2n-1}C_0 + {}^{2n-1}C_1 + \dots + {}^{2n-1}C_{n-1} \}$$

$$= 2n \cdot 2^{(2n-1)-1} = n \cdot 2^{2n-1}$$

84. Putting $x = 1$ and -1 , we get

$$1 = a_0 + a_1 + a_2 + a_3 + \dots + a_{2n}$$

$$\text{and } 3^n = a_0 - a_1 + a_2 - a_3 + \dots + a_{2n}$$

$$\text{Adding, we get } 1 + 3^n = 2(a_0 + a_2 + a_4 + \dots + a_{2n})$$

$$\therefore a_0 + a_2 + a_4 + \dots + a_{2n} = \frac{3^n + 1}{2}$$

$$85. \sum_{r=0}^n (a + rb) {}^nC_r = a(2^n) + b \sum_{r=0}^n r \cdot \frac{n}{r} \cdot {}^{n-1}C_{r-1}$$

$$= a \cdot 2^n + nb \cdot 2^{n-1} = (2a + nb) 2^{n-1}$$

$$86. (1+x)^{2n} = C_0 + C_1x + C_2x^2 + \dots + C_{2n}x^{2n}$$

On differentiating w.r.t. x , we get

$$2n(1+x)^{2n-1} = C_1 + 2C_2x + \dots + 2nC_{2n}x^{2n-1} \dots (i)$$

$$\text{Also, } (1-x)^{2n} = C_0 - C_1x + C_2x^2 - \dots + C_{2n}x^{2n}$$

Replacing x by $\frac{1}{x}$, we get

$$\therefore \left(1 - \frac{1}{x}\right)^{2n} = C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \dots + \frac{C_{2n}}{x^{2n}} \dots (ii)$$

On multiplying Eqs. (i) and (ii), we get

$$2n(1+x)^{2n-1} \left(1 - \frac{1}{x}\right)^{2n}$$

$$= (C_1 + 2C_2x + \dots + 2nC_{2n}x^{2n-1})$$

$$\times \left(C_0 - \frac{C_1}{x} + \frac{C_2}{x^2} - \dots + \frac{C_{2n}}{x^{2n}} \right)$$

Coefficient of $\frac{1}{x}$ on RHS

$$= -C_1^2 + 2C_2^2 - 3C_3^2 + \dots + 2nC_{2n}^2$$

$$= -(C_1^2 - 2C_2^2 + 3C_3^2 - \dots - 2nC_{2n}^2)$$

Also, coefficient of $\frac{1}{x}$ on LHS

$$= \text{Coefficient of } \frac{1}{x} \text{ in } \frac{2n(1+x)^{2n-1}(x-1)^{2n}}{x^{2n}}$$

$$= \text{Coefficient of } x^{2n-1} \text{ in } 2n(1-x^2)^{2n-1}(1-x)$$

$$= 2n \cdot {}^{2n-1}C_n (-1)^{n-1}$$

$$\therefore C_1^2 - 2C_2^2 + 3C_3^2 - \dots - 2nC_{2n}^2 = 2n \cdot {}^{2n-1}C_n (-1)^{n-1}$$

$$87. (1+x)^{10} = {}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + \dots + {}^{10}C_{10}x^{10} \dots (i)$$

Also,

$$(1-x)^{10} = {}^{10}C_0 - {}^{10}C_1x + {}^{10}C_2x^2 - \dots + {}^{10}C_{10}x^{10} \dots (ii)$$

On multiplying, we get

$$(1-x^2)^{10} = ({}^{10}C_0 + {}^{10}C_1x + {}^{10}C_2x^2 + \dots + {}^{10}C_{10}x^{10}) \\ \times ({}^{10}C_0 - {}^{10}C_1x + {}^{10}C_2x^2 - \dots + {}^{10}C_{10}x^{10})$$

Equating the coefficients of x^{10} , we get

$${}^{10}C_5(-1)^5 = {}^{10}C_0 {}^{10}C_{10} - {}^{10}C_1 {}^{10}C_9 + {}^{10}C_2 {}^{10}C_8 \\ - \dots + {}^{10}C_{10} {}^{10}C_0$$

$$\Rightarrow -{}^{10}C_5 = ({}^{10}C_0)^2 - ({}^{10}C_1)^2 + ({}^{10}C_2)^2 - \dots + ({}^{10}C_{10})^2$$

88. $(1-x)^n = C_0 - C_1x + C_2x^2 - C_3x^3 + \dots$ n terms

$$\therefore x(1-x)^n = C_0x - C_1x^2 + C_2x^3 - C_3x^4 + \dots$$

$$\therefore \int_0^1 x(1-x)^n dx$$

$$= \int_0^1 (C_0x - C_1x^2 + C_2x^3 - C_3x^4 + \dots) dx$$

$$\Rightarrow -\int_1^0 (1-t)t^n dt$$

$$= \left[C_0 \frac{x^2}{2} - C_1 \frac{x^3}{3} + C_2 \frac{x^4}{4} - C_3 \frac{x^5}{5} + \dots \right]_0^1$$

(where $t = 1-x$)

$$\Rightarrow \left(\frac{t^{n+1}}{n+1} - \frac{t^{n+2}}{n+2} \right)_1^0 = \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \dots$$

$$\therefore \frac{C_0}{2} - \frac{C_1}{3} + \frac{C_2}{4} - \frac{C_3}{5} + \dots$$

$$= \frac{1}{n+1} - \frac{1}{n+2}$$

$$= \frac{1}{(n+1)(n+2)}$$

89. $\ln(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$

On differentiating both sides w.r.t. x , we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + \dots + nC_nx^{n-1}$$

Put

$$x = 1$$

$$n \cdot 2^{n-1} = C_1 + 2C_2 + \dots + nC_n$$

90. $\ln(1+x^2)^n x = C_0x + C_1x^3 + C_2x^5 + \dots + C_nx^{2n+1}$

On differentiating both sides w.r.t. x and putting $x = 1$

$$(n+1) \cdot 2^n = C_0 + 3C_1 + 5C_2 + \dots + (2n+1)C_n$$

91. ${}^{100}C_r \cdot {}^{200}C_{150+r} = {}^{100}C_r \cdot {}^{200}C_{50-r}$

$$\text{Thus, } \sum_{r=0}^{50} {}^{100}C_r \cdot {}^{200}C_{50-r} = {}^{300}C_{50}$$

92. $\sum_{r=0}^{2n} (-1)^r \cdot ({}^{2n}C_r)^2 = \sum_{r=0}^{2n} (-1)^r \cdot {}^{2n}C_r \cdot {}^{2n}C_r$

$$= \sum_{r=0}^{2n} (-1)^r \cdot {}^{2n}C_r \cdot {}^{2n}C_{2n-r}$$

$$= \text{Coefficient of } x^{2n} \text{ in } (1-x)^{2n} \cdot (1+x)^{2n}$$

$$= \text{Coefficient of } x^{2n} \text{ in } (1-x^2)^{2n} = (-1)^n \cdot {}^{2n}C_n$$

93. $I = \sum_{r=0}^{2n} r \cdot ({}^{2n}C_r)^2 = \sum_{r=0}^{2n} r \cdot \frac{2n}{r} \cdot {}^{2n-1}C_{r-1} \cdot {}^{2n}C_r$

$$= 2n \sum_{r=0}^{2n} {}^{2n-1}C_{r-1} \cdot {}^{2n}C_r$$

$$= 2n \cdot {}^{4n-1}C_{2n}$$

$$\Rightarrow I = n \cdot {}^{4n}C_{2n}$$

94. $\sum_{i=0}^{n-r} {}^{n-i}C_r = {}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \dots + {}^{r+1}C_r + {}^rC_r$

$$= {}^{n+1}C_{r+1}$$

95. Here, $T_{r+1} = \frac{{}^nC_r}{r+1} = \frac{1}{r+1} \cdot {}^nC_r$

$$= \frac{1}{n+1} \cdot {}^{n+1}C_{r+1}$$

Putting $r = 0, 1, 2, \dots, n$ and adding, we get

$$\sum_{k=0}^n \frac{C_k}{k+1} = \sum_{k=0}^n \frac{1}{n+1} \cdot {}^{n+1}C_{k+1}$$

$$= \frac{1}{n+1} [{}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \dots + {}^{n+1}C_{n+1}]$$

$$= \frac{1}{n+1} [2^{n+1} - {}^{n+1}C_0] = \frac{2^{n+1} - 1}{n+1}$$

96. Clearly, $a_r = {}^{2n}C_r$

$$\Rightarrow \frac{a_r}{a_{r-1}} = \frac{{}^{2n}C_r}{{}^{2n}C_{r-1}} = \frac{(2n-r+1)}{r}$$

$$\Rightarrow 1 + \frac{a_r}{a_{r-1}} = \frac{2n+1}{r}$$

$$\Rightarrow \prod_{r=1}^{2n} \left(1 + \frac{a_r}{a_{r-1}} \right) = \prod_{r=1}^{2n} \frac{(2n+1)}{r} = \frac{(2n+1)^{2n}}{(2n)!}$$

97. $\sum_{0 \leq i < j \leq n} i \cdot {}^nC_j = \sum_{r=1}^n {}^nC_r (0+1+2+\dots+r-1)$

$$= \frac{1}{2} \{n(n-1) \cdot 2^{n-2} + n \cdot 2^{n-1} - n \cdot 2^{n-1}\}$$

$$= n(n-1) \cdot 2^{n-3}$$

98. $\sum_{0 \leq i < j \leq n} j \cdot {}^nC_i = \sum_{r=0}^{n-1} {}^nC_r [(r+1) + (r+2) + \dots + (n)]$

$$= \sum_{r=0}^{n-1} {}^nC_r \left[\frac{(n-r)}{2} (r+1+n) \right]$$

$$= \sum_{r=0}^{n-1} {}^nC_r \left[\frac{(n+1)}{2} (n-r) + \frac{r(n-r)}{2} \right]$$

$$= \frac{n+1}{2} \sum_{r=0}^n (n-r) \cdot {}^nC_r - \frac{n}{2} \sum_{r=0}^n r \cdot {}^nC_r + \frac{1}{2} \sum_{r=0}^n r^2 \cdot {}^nC_r$$

$$= \frac{n+1}{2} \sum_{r=0}^n r \cdot {}^nC_r - \frac{n}{2} \sum_{r=0}^n r \cdot {}^nC_r + \frac{1}{2} \sum_{r=0}^n r^2 \cdot {}^nC_r$$

$$= \frac{1}{2} \left(\sum_{r=0}^n r \cdot {}^nC_r + \sum_{r=0}^n r^2 \cdot {}^nC_r \right) = n(n+3) \cdot 2^{n-3}$$

99. $\sum_{r=0}^n (-1)^r \cdot r^2 \cdot {}^nC_r$

$$\text{Since, } (1-x)^n = \sum_{r=0}^n (-1)^r \cdot {}^nC_r \cdot x^r$$

On differentiating both sides w.r.t. x , we get

$$-n(1-x)^{n-1} = \sum_{r=0}^n (-1)^r \cdot {}^nC_r \cdot r \cdot x^{r-1}$$

$$\Rightarrow -nx(1-x)^{n-1} = \sum_{r=0}^n (-1)^r \cdot {}^nC_r \cdot r \cdot x^r$$

111. Given that,

$$\alpha = \frac{5}{2!3} + \frac{5 \cdot 7}{3!3^2} + \frac{5 \cdot 7 \cdot 9}{4!3^3} + \dots \quad \dots(i)$$

We know that,

$$(1+x)^n = 1 + \frac{nx}{1!} + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), with respect to factorial

$$n(n-1)x^2 = \frac{5}{3} \quad \dots(iii)$$

$$n(n-1)(n-2)x^3 = \frac{5 \cdot 7}{3^2} \quad \dots(iv)$$

$$\text{and } n(n-1)(n-2)(n-3)x^4 = \frac{5 \cdot 7 \cdot 9}{3^3} \quad \dots(v)$$

On dividing Eq. (iv) by Eq. (iii) and Eq. (v) by Eq. (iv),

$$\text{we get } (n-2)x = \frac{7}{3} \quad \dots(vi)$$

$$\text{and } (n-3)x = 3 \quad \dots(vii)$$

Again, dividing Eq. (vi) by Eq. (vii), we get

$$\frac{n-2}{n-3} = \frac{7}{9}$$

$$\Rightarrow 9n - 18 = 7n - 21$$

$$\Rightarrow 2n = -3 \Rightarrow n = -\frac{3}{2}$$

On putting the value of n in Eq. (vi), we get

$$\left(-\frac{3}{2} - 2\right)x = \frac{7}{3}$$

$$\Rightarrow x = -\frac{2}{3}$$

$\therefore$ From Eq. (ii), we get

$$\left(1 - \frac{2}{3}\right)^{-3/2} = 1 + 1 + \frac{5}{2!3} + \frac{5 \cdot 7}{3!3^2} + \dots$$

$$\Rightarrow 3^{3/2} - 2 = \frac{5}{2!3} + \frac{5 \cdot 7}{3!3^2} + \dots$$

$$\Rightarrow \alpha = 3^{3/2} - 2 \quad [\text{from Eq. (i)}]$$

$$\begin{aligned} \text{Now, } \alpha^2 + 4\alpha &= (3^{3/2} - 2)^2 + 4(3^{3/2} - 2) \\ &= 27 + 4 - 4 \cdot 3^{3/2} + 4 \cdot 3^{3/2} - 8 \\ &= 23 \end{aligned}$$

112. $(x \sin p + x^{-1} \cos p)^{10}$

The general term in the expansion is

$$T_{r+1} = {}^{10}C_r (x \sin p)^{10-r} (x^{-1} \cos p)^r$$

For the term independent of x , we have $10 - 2r = 0$ or $r = 5$.

Hence, the independent term is

$${}^{10}C_5 \sin^5 p \cos^5 p = {}^{10}C_5 \frac{\sin^5 2p}{32}$$

which is the greatest when $\sin 2p = 1$

$$\text{The least value of } {}^{10}C_5 \frac{\sin^5 2p}{32} \text{ is } -\frac{10!}{2^5(5!)^2},$$

$$\text{when } \sin 2p = -1 \text{ or } p = (4n-1)\frac{\pi}{4}, n \in \mathbb{Z}.$$

Sum of coefficients is $(\sin p + \cos p)^{10}$, when $x = 1$

or $(1 + \sin 2p)^5$, which is least, when $\sin 2p = -1$

Hence, least sum of coefficients is zero. Greatest sum of coefficients occurs, when $\sin 2p = 1$.

Hence, the greatest sum is $2^5 = 32$.

113. The general term of $\left(x + \frac{a}{x^2}\right)^n$ is

$$T_{r+1} = {}^nC_r \left(\frac{a}{x^2}\right)^r x^{n-r} = {}^nC_r a^r x^{n-3r}$$

If the term independent of x does not exist, then $n - 3r$ must not be zero for positive integers n and r .

$\Rightarrow r = \frac{n}{3}$ must not be an integer. Now, the integers

given in options (a), (b) and (d) are not divisible by 3.

Options (a), (b) and (d) are correct options.

Since, 15 is divisible by 3, so option (c) is not true.

114. $\ln \left(x + \frac{\alpha}{x}\right)^n$,

$$\begin{aligned} T_{r+1} &= {}^nC_r x^{n-r} \left(\frac{\alpha}{x}\right)^r \\ &= {}^nC_r x^{n-2r} \alpha^r \end{aligned}$$

T_{r+1} is independent of x , if $n - 2r = 0$

$\Rightarrow 2r \Rightarrow n$ is divisible by 2.

Again, in $\left(x + \frac{\beta}{x^2}\right)^n$

$$T_{r+1} = {}^nC_r x^{n-r} \left(\frac{\beta}{x^2}\right)^r = {}^nC_r x^{n-3r} \beta^r$$

T_{r+1} is independent of x , if $n - 3r = 0$ i.e. $n = 3r$ n is divisible by 3.

$$\begin{aligned} 115. (x^2 + 2x + 2)^n &= \{2 + x(2 + x)\}^n \\ &= 2^n + {}^nC_1 2^{n-1} x(2 + x) + {}^nC_2 2^{n-2} x^2 (2 + x)^2 \\ &\quad + {}^nC_3 2^{n-3} x^3 (2 + x)^3 + \dots \end{aligned}$$

It is easy to observe that coefficient of $x = n \cdot 2^n$ and the coefficient of x^2

$$= n \cdot 2^{n-1} + \frac{n(n-1)2^{n-2}}{2} = n^2 \cdot 2^{n-1}$$

$$\begin{aligned} \text{Coefficient of } x^3 &= {}^nC_2 \cdot 2^{n-2} \cdot 4 + {}^nC_3 \cdot 2^{n-3} \cdot 2^3 \\ &= 2^n \cdot {}^{n+1}C_3 \end{aligned}$$

116. (a) $3^{100} = 9^{50} = (10 - 1)^{50}$

$$= 10^{50} - {}^{50}C_1 10^{49} + \dots - {}^{50}C_{49} 10^1 + {}^{50}C_{50}$$

$\Rightarrow$ A multiple of $100 + 1$.

$\Rightarrow$ Last two digits of 3^{100} are 01.

$\Rightarrow$ Choice (a) is wrong.

$$\text{Again, } 3^{50} = 9^{25} = (10 - 1)^{25}$$

$$= 10^{25} - {}^{25}C_1 10^{24} + \dots + {}^{25}C_{24} 10 - {}^{25}C_{25}$$

$=$ A multiple of $100 + 249$

$\Rightarrow$ The remainder when 3^{50} divided by 100 is 49

$\Rightarrow$ Last two digits of 3^{50} are 49.

$\Rightarrow$ Option (b) is false and option (c) becomes true.

(d) From the same expansion, we get

$$3^{50} = \text{A multiple of } 1000 + 249$$

$\Rightarrow$ The remainder when 3^{50} is divided by 1000 is 249.

$\therefore$ The last three digits are 249.

117. Sum of coefficients in $(a\alpha^2 x^2 + 2b\alpha x + c)^n$ is

$$(a\alpha^2 + 2b\alpha + c)$$

$$\text{Let } f(\alpha) = a\alpha^2 + 2b\alpha + c$$

$$\text{Such that sum} = \{f(\alpha)\}^n$$

Now, $f(\alpha) > 0, \forall n$ and $\alpha \in \mathbb{R}$, then $a > 0$

Also, $f(\alpha) < 0, \forall n \in \text{odd}$ and $\alpha \in \mathbb{R}$, if $a < 0$

$$118. f(n) = \sum_{r=1}^n \{(r+1)^2 {}^nC_r - r^2 {}^nC_{r-1}\}$$

$$= (n+1)^2 {}^nC_{n-1} = (n+1)^2 - 1$$

Now, $\sum_{n=1}^{10} f(n) = (2^2 - 1) + (3^2 - 1) + (4^2 - 1) + (5^2 - 1)$

$$+ (6^2 - 1) + (7^2 - 1) + (8^2 - 1) + (9^2 - 1)$$

$$+ (10^2 - 1) + (11^2 - 1)$$

$$= \frac{11 \cdot (11+1)(22+1)}{6} - 11 = 495$$

119. The given expression is the coefficient of x^n in $(1+x)^n + (1+x)^{n+1} + \dots + (1+x)^{n+k}$

$$\Rightarrow \text{Coefficient of } x^n \text{ in } (1+x)^n \left\{ \frac{(1+x)^{k+1} - 1}{1+x-1} \right\}$$

$$= \frac{(1+x)^n (1+x)^{k+1} - (1+x)^n}{x}$$

$$\Rightarrow \text{Coefficient of } x^{n+1} \text{ in } [(1+x)^{n+k+1} - (1+x)^n]$$

i.e. ${}^{n+k+1}C_{n+1} \text{ or } {}^{n+k+1}C_k$

120. Given, ${}^{69}C_{3r-1} + {}^{69}C_{3r} = {}^{69}C_{r-1} + {}^{69}C_{r^2}$, ${}^{70}C_{3r} = {}^{70}C_{r^2}$

Either $3r = r^2$ or $r^2 + 3r = 70$

i.e. either $r = 0, 3$ or $r = 7, -10$

But the given equation is not defined for $r = 0, -10$

Hence, $r = 3$ or 7

121. Since, i, j and k are distinct, $n-i+1, n-j+1, n-k+1$ are also distinct and they all lie from 1 to n .

Now, $S = \sum \sum \sum (-x_{n-i+1})(-x_{n-j+1})(-x_{n-k+1})$

$$= -\sum \sum \sum x_i x_j x_k = -S \Rightarrow S = 0 \text{ for all } n$$

Options (a), (b) and (c) are correct since the statements given in options (a), (b) and (c) are not correct.

122. We have,

$$f(m) = \sum_{i=0}^m \binom{30}{30-i} \binom{20}{m-i} = \sum_{i=0}^m \binom{30}{i} \binom{20}{m-i} = {}^{50}C_m$$

Now, $f(m)$ is the greatest, when $m = 25$.

Also, $f(0) + f(1) + \dots + f(50)$

$$= {}^{50}C_0 + {}^{50}C_1 + \dots + {}^{50}C_{50} = 2^{50}$$

Also, ${}^{50}C_m$ is not divisible by 50 for any m as 50 is not a prime number.

$$\sum_{m=0}^{50} (f(m))^2 = ({}^{50}C_0)^2 + ({}^{50}C_1)^2 + \dots + ({}^{50}C_{50})^2 = {}^{100}C_{50}$$

123. $\frac{T_{r+1}}{T_r} = \frac{12-r+1}{r} \cdot \frac{11}{10}$

Let $T_{r+1} \geq T_r \Rightarrow 13-r \geq 11x$

$$\Rightarrow 13 \geq 2.1r \Rightarrow r \leq 6.19$$

Hence, the greatest term occurs for $r = 6$. Hence, 7th term is the greatest term. Also, the binomial coefficient of 7th term is ${}^{12}C_6$, which is the greatest binomial coefficient.

But this is not the reason for which T_7 is the greatest. Here, it is coincident that the greatest term has the greatest binomial coefficient.

Hence, Statement I is true, Statement II is true; but Statement II is not a correct explanation of Statement I.

124. Let $(\sqrt{5} + 2)^n = N + f$, where N is an integer and $0 < f < 1$

Let $(\sqrt{5} - 2)^n = f'$, then $0 < f' < 1$

Let $(\sqrt{5} + 2)^n - (\sqrt{5} - 2)^n = \text{integer}$ [$\because n$ is odd]

where $N = (\sqrt{5} + 2)^n - (\sqrt{5} - 2)^n$ [$\because f = f'$]

$$= 2 \left[{}^nC_1 \cdot 2 \cdot 5^{\frac{n-1}{2}} + {}^nC_3 \cdot 2^3 \cdot 5^{\frac{n-3}{2}} + \dots \right]$$

$\Rightarrow N$ is divisible by $20n$ on using Statement II.

[if n is prime and $r < n$, then there is no factor which will cut n , so ${}^nC_r$ will be divisible by n]

125. Statement I is true, since

$$(\sqrt{2} - 1)^2 = 3 - 2\sqrt{2} = \sqrt{9} - \sqrt{8}$$

$$(\sqrt{2} - 1)^3 = 5\sqrt{2} - 7 = \sqrt{50} - \sqrt{49}$$

[this could be proved by induction]

Statement II is also true, since any integral power of $(\sqrt{2} - 1)$ will have rational and irrational parts. The irrational part will have only one surd $\sqrt{2}$. Thus, $(\sqrt{2} - 1)^n = A + B\sqrt{2}$, where A and B are integers.

But, Statement II does not explain the Statement I.

126. $X = (n^2 - n)(n^2 - n - 2) = n(n-1)(n-2)(n+1)$

$$= (n-2)(n-1)n(n+1)$$

$$\Rightarrow \frac{X}{24} = {}^{n+1}C_4 \in \mathbb{I}$$

Hence, $m(m-2)$ is divisible by 24.

127. Since, $S_1 = {}^1C_1$ and $T_1 = 1 \Rightarrow S_1 = T_1$

Thus, if Statement II is true and if $S_n = T_n$ for some n , then $S_{n+1} = T_{n+1}$

$$\Rightarrow S_n = T_n \text{ for all } n.$$

Thus, Statement I is true, if Statement II is true. Let us prove Statement II.

Now, $T_{n+1} - T_n = \frac{1}{n+1}$

and $S_{n+1} - S_n = {}^{n+1}C_1 - \frac{{}^{n+1}C_2}{2} + \frac{{}^{n+1}C_3}{3} - \dots$

$$\dots + \frac{(-1)^{n-1} {}^{n+1}C_n}{n} + \frac{(-1)^n {}^{n+1}C_{n+1}}{n+1}$$

$$= \left({}^nC_1 - \frac{{}^nC_2}{2} + \frac{{}^nC_3}{3} - \dots + \frac{(-1)^{n-1} {}^nC_n}{n} \right)$$

$$= ({}^{n+1}C_1 - {}^nC_1) - \frac{1}{2} ({}^{n+1}C_2 - {}^nC_1) + \frac{1}{3} ({}^{n+1}C_3 - {}^nC_3)$$

$$- \dots + \frac{(-1)^{n-1}}{n} ({}^{n+1}C_n - {}^nC_n) + \frac{(-1)^n}{n+1}$$

$$= {}^nC_0 - \frac{{}^nC_1}{2} + \frac{{}^nC_2}{2} - \dots + \frac{(-1)^n {}^nC_n}{n+1}$$

$$= \int_0^1 (1-x)^n dx = \frac{1}{n+1}$$

Thus, Statement II is true.

128. $S = \sum_{0 \leq i < j \leq n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j} \right) = \sum_{0 \leq i < j \leq n} \left(\frac{n-i}{{}^nC_{n-i}} + \frac{n-j}{{}^nC_{n-j}} \right)$

$$= n \sum_{0 \leq i < j \leq n} \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j} \right) - S$$

$$\Rightarrow S = \frac{n}{2} \sum_{0 \leq i < j \leq n} \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j} \right)$$

$$= \frac{n}{2} \left(\sum_{r=0}^{n-1} \frac{n-r}{{}^nC_r} + \sum_{r=1}^n \frac{r}{{}^nC_r} \right) = \frac{n}{2} \left(\sum_{r=0}^n \frac{n}{{}^nC_r} \right) = \frac{n^2}{2} a$$

Solutions (Q. Nos. 129-131)

We have,

$$C + iS = 1 + re^{i\theta} + \frac{r^2 e^{2i\theta}}{2!} + \frac{r^3 e^{3i\theta}}{3!} + \dots = e^{re^{i\theta}} \quad \dots(i)$$

$$\text{and } C - iS = 1 + re^{-i\theta} + \frac{r^2 e^{-2i\theta}}{2!} + \frac{r^3 e^{-3i\theta}}{3!} + \dots = e^{re^{-i\theta}} \quad \dots(ii)$$

Clearly, $C^2 + S^2$

$$= e^{re^{i\theta}} \cdot e^{re^{-i\theta}} = e^{r(e^{i\theta} + e^{-i\theta})} = e^{2r \cos \theta} \quad \dots(iii)$$

On differentiating Eqs. (i) and (ii) w.r.t. r , we get

$$\frac{dC}{dr} + i \frac{dS}{dr} = e^{i\theta} + re^{2i\theta} + \frac{r^2 e^{3i\theta}}{2!} + \dots = e^{i\theta} e^{re^{i\theta}} \quad \dots(iv)$$

$$\begin{aligned} \frac{dC}{dr} - i \frac{dS}{dr} &= e^{-i\theta} + re^{-2i\theta} + \frac{r^2 e^{-3i\theta}}{2!} + \dots \\ &= e^{-i\theta} \left(1 + r \cdot e^{-i\theta} + \frac{r^2 e^{-2i\theta}}{2!} + \dots \right) \\ &= e^{-i\theta} \cdot e^{re^{-i\theta}} \quad \dots(v) \end{aligned}$$

On multiplying Eqs. (iv) and (v), we get

$$\begin{aligned} \left(\frac{dC}{dr} \right)^2 + \left(\frac{dS}{dr} \right)^2 &= e^{-i\theta} \cdot e^{re^{-i\theta}} \cdot e^{i\theta} \cdot e^{re^{i\theta}} \\ &= e^{(-i\theta + i\theta)} \cdot e^{r(e^{-i\theta} + e^{i\theta})} \\ &= e^{2r \cos \theta} = C^2 + S^2 \end{aligned}$$

Now, on multiplying Eqs. (ii) and (iv), we get

$$\begin{aligned} \left(\frac{dC}{dr} + i \frac{dS}{dr} \right) (C - iS) &= e^{i\theta} e^{r(e^{i\theta} + e^{-i\theta})} \\ &= e^{i\theta} e^{2r \cos \theta} \\ &= (\cos \theta + i \sin \theta) e^{2r \cos \theta} \quad \dots(vi) \end{aligned}$$

On equating real and imaginary parts from Eq. (vi)

$$C \frac{dC}{dr} + S \frac{dS}{dr} = (C^2 + S^2) \cos \theta$$

$$\text{and } C \frac{dS}{dr} - S \frac{dC}{dr} = (C^2 + S^2) \sin \theta$$

132. Given, $(x + y)^n = \sum_{r=0}^n {}^nC_r x^{n-r} y^r$, where $C_r = {}^nC_r$

$$\text{Using } (1 + x)^n = \sum_{r=0}^n {}^nC_r x^r \quad \dots(i)$$

On integrating between 0 to 1, we get

$$\begin{aligned} \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^1 &= C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \dots + \frac{C_n}{n+1} \\ \therefore C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \dots + \frac{C_n}{n+1} &= \frac{2^{n+1} - 1}{n+1} \quad \dots(ii) \end{aligned}$$

Also, integrating Eq. (i) between 0 to -1, we get

$$\begin{aligned} \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^{-1} &= -C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \frac{C_3}{4} - \dots \\ \therefore -C_0 + \frac{C_1}{2} - \frac{C_2}{3} + \frac{C_3}{4} - \dots &= -\frac{1}{n+1} \quad \dots(iii) \end{aligned}$$

On adding Eqs. (ii) and (iii), we get

$$\begin{aligned} 2 \left(\frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots \right) &= \frac{2^{n+1} - 2}{n+1} \\ \therefore \frac{C_1}{2} + \frac{C_3}{4} + \frac{C_5}{6} + \dots &= \frac{2^n - 1}{n+1} \end{aligned}$$

133. Now, $\sum_{r=0}^n \left(\frac{r+1+1}{r+1} \right) {}^nC_r = \sum_{r=0}^n {}^nC_r + \frac{1}{(n+1)} \sum_{r=0}^n {}^{n+1}C_{r+1}$

$$= 2^n + \frac{1}{(n+1)} (2^{n+1} - 1)$$

$$\text{Since, } \sum_{r=0}^n \left(\frac{r+2}{r+1} \right) {}^nC_r = \frac{2^8 - 1}{6} \quad [\text{given}]$$

$$\therefore 2^n + \frac{2^{n+1} - 1}{n+1} = \frac{2^8 - 1}{6}$$

$$\Rightarrow n = 5$$

134. Using, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \dots$

On putting $x = i$, we get

$$\begin{aligned} (1+i)^n &= (C_0 - C_2 + C_4 - C_6 + \dots) \\ &\quad + i(C_1 - C_3 + C_5 - C_7 + \dots) \\ (\sqrt{2})^n \left(\cos \frac{n\pi}{4} + i \sin \frac{n\pi}{4} \right) &= (C_0 - C_2 + C_4 - C_6 + \dots) \\ &\quad + i(C_1 - C_3 + C_5 - C_7 + \dots) \end{aligned}$$

On equating the real part, we get

$$C_0 - C_2 + C_4 - C_6 + \dots = 2^{\frac{n}{2}} \cdot \cos \frac{n\pi}{4}$$

135. Given, $a = \sum_{r=0}^n \frac{1}{{}^nC_r}$

$$\begin{aligned} \text{Let } y &= \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} \left(\frac{i}{{}^nC_i} + \frac{j}{{}^nC_j} \right) \\ &= \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} \left(\frac{n-i}{{}^nC_{n-i}} + \frac{n-j}{{}^nC_{n-j}} \right) \\ &= n \sum_{0 \leq i \leq n} \sum_{0 \leq j \leq n} \left(\frac{1}{{}^nC_{n-i}} + \frac{1}{{}^nC_{n-j}} \right) - y \\ \Rightarrow 2y &= n \{ a \cdot (n+1) + a \cdot (n+1) \} \Rightarrow y = na(n+1) \end{aligned}$$

136. Since, $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n$

Replace x by ix , we get

$$\begin{aligned} (1+ix)^n &= C_0 + iC_1 x - C_2 x^2 - iC_3 x^3 + C_4 x^4 + iC_5 x^5 - \dots \\ \Rightarrow (1+ix)^n &= (C_0 - C_2 x^2 + C_4 x^4 + \dots) \\ &\quad + i(C_1 x - C_3 x^3 + C_5 x^5 + \dots) \quad \dots(i) \end{aligned}$$

$$\begin{aligned} \text{Similarly, } (1-ix)^n &= (C_0 - C_2 x^2 + C_4 x^4 + \dots) \\ &\quad - i(C_1 x - C_3 x^3 + C_5 x^5 + \dots) \quad \dots(ii) \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$(1+ix)^n + (1-ix)^n = 2(C_0 - C_2 x^2 + C_4 x^4 - \dots)$$

On putting $x = 1$, we get

$$\begin{aligned} C_0 - C_2 + C_4 - C_6 + C_8 - \dots \\ = (1+i)^n + (1-i)^n = 2^{n/2} \cos \left(\frac{n\pi}{4} \right) \quad \dots(iii) \end{aligned}$$

$$\text{Also, } C_0 + C_2 + C_4 + C_6 + C_8 + \dots = 2^{n-1} \quad \dots(iv)$$

On adding Eqs. (iii) and (iv), we get

$$C_0 + C_4 + C_8 + C_{12} + \dots = 2^{n-2} + 2^{\frac{n}{2}-1} \cos \frac{n\pi}{4}$$

137. ${}^nC_0 + {}^nC_3 + {}^nC_6 + \dots = \frac{1}{3} [2^n + (-1)^n \omega^{2n} + (-1)^n \bar{\omega}^n]$

$$\begin{aligned} &= \frac{1}{3} [2^n + (-1)^n (\omega^n + \bar{\omega}^{2n})] \\ &= \frac{1}{3} [2^n + (-1)^n (\bar{\omega}^n + \omega^n)] \quad [\because \omega^2 = \bar{\omega}] \\ &= \frac{1}{3} \left(2^n + 2 \cos \left(\frac{n\pi}{3} \right) \right) \end{aligned}$$

$$\begin{aligned}
 138. \sum_{r=1}^n (-3)^{r-1} \cdot {}^{3n}C_{2r-1} \\
 = \sum_{r=1}^n (\sqrt{3}i)^{2r-2} \cdot {}^{3n}C_{2r-1} = \frac{1}{\sqrt{3}i} \sum_{r=1}^n (\sqrt{3}i)^{2r-1} \cdot {}^{3n}C_{2r-1} \\
 = \frac{1}{2i\sqrt{3}} \text{Imaginary part of } \{(1+i\sqrt{3})^{2n} - (1-i\sqrt{3})^{2n}\} \\
 = 0
 \end{aligned}$$

$$\begin{aligned}
 139. \sum_{r=0}^n \sum_{s=0}^n (C_r + C_s) &= \sum_{r=0}^n \sum_{s=0}^n C_r + \sum_{r=0}^n \sum_{s=0}^n C_s \\
 &= \sum_{s=0}^n \left(\sum_{r=0}^n C_r \right) + \sum_{r=0}^n \left(\sum_{s=0}^n C_s \right) \\
 &= (n+1)2^n + (n+1)2^n \\
 &= (n+1)2^{n+1}
 \end{aligned}$$

$$\begin{aligned}
 140. \sum_{r=0}^s \sum_{s=1}^n {}^nC_s {}^sC_r \\
 = \sum_{s=1}^n {}^nC_s ({}^sC_0 + {}^sC_1 + \dots + {}^sC_s) = \sum_{s=1}^n {}^nC_s (2^s) \\
 = \sum_{s=0}^n {}^nC_s 2^0 - {}^sC_0 2^0 = (1+2)^n - 1 \\
 = 3^n - 1
 \end{aligned}$$

$$\begin{aligned}
 141. \sum_{r=0}^n \sum_{s=0}^n \sum_{t=0}^n \sum_{u=0}^n (1) &= \sum_{r=0}^n (1) \sum_{s=0}^n (1) \sum_{t=0}^n (1) \sum_{u=0}^n (1) \\
 &= (n+1)(n+1)(n+1)(n+1) \\
 &= (n+1)^4
 \end{aligned}$$

$$\begin{aligned}
 142. \text{ Given, } f(x) &= x^n + x^{n+1} \\
 f^k(x) &= \frac{n!}{(n-k)!} x^{n-k} + \frac{(n+1)!}{(n+1-k)!} x^{n+1-k} \\
 \Rightarrow (1+x)f^k(x) &= \left(\frac{n!}{(n-k)!} x^{n-k} + \frac{(n+1)!}{(n+1-k)!} x^{n+1-k} \right) (1+x) = g(x) \\
 &\quad \text{[say]} \\
 g^{(n+1-k)}(x) &= \left(\frac{(n+1)!}{(n+1-k)!} (n+2-k)! \right) x \\
 &\quad + \left(\frac{n!}{(n-k)!} + \frac{(n+1)!}{(n+1-k)!} \right) (n+1-k)! \\
 \therefore A-B &= \frac{(n+1)!}{(n+1-k)!} (n+2-k)! \\
 &\quad - \left(\frac{n!}{(n-k)!} + \frac{(n+1)!}{(n+1-k)!} \right) (n+1-k)! \\
 &= (n!)n(n+1-k)
 \end{aligned}$$

143. A. Since, $xyz = 3^5$
 Number of solutions of $(x, y, z) = {}^{3+5-1}C_5 = 21$
 B. Number of terms = ${}^{6+3-1}C_{3-1} = 28$
 C. Since, $x^2 + x - 400 \leq 0$
 $\Rightarrow x(x+1) \leq 400$
 Solutions are 1, 2, 3, ..., 19.
 Number of solutions = 19
 D. Since, $x + y + z = 10$
 Number of solutions
 = Coefficient of t^{10} in $(t + t^2 + \dots + t^8)^3$

$$\begin{aligned}
 &= \text{Coefficient of } t^{10} \text{ in } t^3 \left(\frac{1-t^8}{1-t} \right)^3 \\
 &= \text{Coefficient of } t^7 \text{ in } (1 + {}^3C_1 t + {}^3C_2 t^2 + \dots) \\
 &= {}^9C_7 = 36
 \end{aligned}$$

$$\begin{aligned}
 144. \text{ A. Let } S &= \sum_{0 \leq i \leq j \leq 10} i \cdot j \cdot {}^nC_i \cdot {}^nC_j \\
 &= \sum_{i=0}^n \sum_{j=0}^n i \cdot j \cdot {}^nC_i \cdot {}^nC_j - S + \sum_{i=0}^n i^2 ({}^nC_i)^2 \\
 \Rightarrow 2S &= \sum_{i=0}^n \sum_{j=0}^n i \cdot j \cdot {}^nC_i \cdot {}^nC_j - \sum_{i=0}^n i^2 ({}^nC_i)^2 \\
 \Rightarrow 2S &= (n2^{n-1}) \cdot (n2^{n-1}) - \sum_{i=0}^n i^2 ({}^nC_i)^2
 \end{aligned}$$

where, $\sum_{i=0}^n i^2 ({}^nC_i)^2$ is the coefficient of x^0 in the product of $n(1+x)^{n-1} \left(1 + \frac{1}{x}\right)^{n-1}$.

$$\begin{aligned}
 \text{Coefficient of } x^0 \text{ in the product of } \frac{n^2(1+x)^{2n-2}}{x^{n-1}} \\
 = \text{Coefficient of } x^{n-1} \text{ in the product of } n^2(1+x)^{2n-1} \\
 = n^2 {}^{2n-2}C_{n-1} \\
 \therefore 2S = n^2 \{2^{2n-2} - {}^{2n-2}C_{n-1}\}
 \end{aligned}$$

For $n = 10$, we have

$$2S = 100 \{2^{18} - {}^{18}C_9\} \times 100$$

Last digit in 2^{18} is 4 and that the last digit in ${}^{18}C_9$ is zero (0).

Hence, last non-zero digit in $2^{18} - {}^{18}C_9$ is 4.

$$\begin{aligned}
 \text{B. } \sum_{k=0}^4 \left(\frac{3^{4-k}}{(4-k)!} \right) \left(\frac{x^k}{k!} \right) \\
 = \sum_{k=0}^4 \left(\frac{3^{4-k}}{(4-k)!} \right) \left(\frac{x^k}{k!} \right) \left(\frac{4!}{4!} \right) \\
 = \sum_{k=0}^4 \frac{{}^4C_k \cdot 3^{4-k} \cdot x^k}{4!} = \frac{(3+x)^4}{4!}
 \end{aligned}$$

According to the question,

$$\begin{aligned}
 \frac{(3+x)^4}{4!} &= \frac{32}{3} \\
 \Rightarrow (3+x)^4 &= 256 \\
 \Rightarrow x+3 &= 4 \\
 \Rightarrow x &= 1
 \end{aligned}$$

$$\begin{aligned}
 \text{C. Since, } (1+x+x^2)^n \\
 = a_0 + a_1x + a_2x^2 + \dots + a_{2n}x^{2n} \quad \dots (i)
 \end{aligned}$$

On substituting $x = \omega, \omega^2$ and 1 and then adding them together

$$a_0 + a_3 + a_6 + \dots = 3^{n-1}$$

On multiplying Eq. (i) by x^2 and then repeating the same process again, we get

$$a_1 + a_4 + a_7 + \dots = 3^{n-1}$$

$$\begin{aligned}
 \Rightarrow a_0 + a_3 + a_6 + \dots &= a_1 + a_4 + a_7 + \dots \\
 &= a_2 + a_5 + a_8 + \dots
 \end{aligned}$$

$$\therefore \text{The required ratio is } \frac{2 \cdot 3^{n-1}}{3^{n-1}} = 2$$

$$\begin{aligned}
 152. (17)^{256} &= (289)^{128} \\
 &= (300 - 11)^{128} \\
 &= {}^{128}C_0(-11)^{128} + 100m, \text{ for some integer } m \\
 &= 11^{128} + 100m \\
 &= (10 + 1)^{128} + 100m \\
 &= {}^{128}C_0 1^{128} + {}^{128}C_1 10 + 100m_1 + 100m \\
 &\quad \text{for some integer } m_1 \\
 &= 1 + 1280 + 100k, m + m_1 = k \\
 &= 1281 + 100k
 \end{aligned}$$

Hence, the required number is $81 = ab$
 $\Rightarrow a + b = 9$

$$\begin{aligned}
 153. \frac{1(2^{2000} - 1)}{1} &= 2^{2000} - 1 \\
 &= (5 - 1)^{1000} - 1 = (1 - 5)^{1000} - 1 \\
 &= 1 - {}^{1000}C_1 \cdot 5 + {}^{1000}C_2 \cdot 5^2 - \dots + {}^{1000}C_{1000} \cdot 5^{1000} - 1 \\
 &\text{which is divisible by 5.}
 \end{aligned}$$

$$\begin{aligned}
 154. \sum_{k=0}^4 \left(\frac{5^4 - k}{(4 - k)!} \right) \left(\frac{x^k}{k!} \right) &= \sum_{k=0}^4 \left(\frac{5^4 - k}{(4 - k)!} \right) \left(\frac{x^k}{k!} \right) \frac{4!}{4!} \\
 &= \sum_{k=0}^4 \frac{{}^4C_k \cdot 5^{4-k} \cdot x^k}{4!} = \frac{(5 + x)^4}{4!}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \frac{(5 + x)^4}{4!} &= \frac{8}{3} \\
 \Rightarrow (5 + x)^4 &= 64 \\
 \Rightarrow x + 5 &= 2^{3/2} = 2\sqrt{2} \\
 \Rightarrow x &= 2\sqrt{2} - 5 \\
 \therefore \lambda &= 5
 \end{aligned}$$

Entrances Gallery

$$\begin{aligned}
 1. 2x_1 + 3x_2 + 4x_3 &= 11 \\
 \text{Possibilities are } (0, 1, 2); (1, 3, 0); (2, 1, 1); (4, 1, 0). \\
 \therefore \text{ Required coefficients} \\
 &= ({}^4C_0 \times {}^7C_1 \times {}^{12}C_2) + ({}^4C_1 \times {}^7C_3 \times {}^{12}C_0) \\
 &\quad + ({}^4C_2 \times {}^7C_1 \times {}^{12}C_1) + ({}^4C_4 \times {}^7C_1 \times {}^{12}C_0) \\
 &= (1 \times 7 \times 66) + (4 \times 35 \times 1) + (6 \times 7 \times 12) + (1 \times 7) \\
 &= 462 + 140 + 504 + 7 \\
 &= 1113
 \end{aligned}$$

$$2. \text{ Let } T_{r-1}, T_r, T_{r+1} \text{ be three consecutive terms of } (1 + x)^{n+5}.$$

$$T_{r-1} = {}^{n+5}C_{r-2} (x)^{r-2}, T_r = {}^{n+5}C_{r-1} x^{r-1},$$

$$T_{r+1} = {}^{n+5}C_r x^r$$

$$\text{where, } {}^{n+5}C_{r-2} : {}^{n+5}C_{r-1} : {}^{n+5}C_r = 5 : 10 : 14$$

$$\therefore \frac{{}^{n+5}C_{r-2}}{5} = \frac{{}^{n+5}C_{r-1}}{10} = \frac{{}^{n+5}C_r}{14}$$

$$\text{Now, } \frac{{}^{n+5}C_{r-2}}{5} = \frac{{}^{n+5}C_{r-1}}{10}$$

$$\Rightarrow \frac{n-3r}{10} = -9 \quad \dots (i)$$

$$\text{and } \frac{{}^{n+5}C_{r-1}}{10} = \frac{{}^{n+5}C_r}{14}$$

$$\Rightarrow \frac{5n-12r}{14} = -30 \quad \dots (ii)$$

From Eqs. (i) and (ii), $n = 6$

$$\begin{aligned}
 155. C_0 - C_1 \frac{(1+x)}{(1+nx)} + C_2 \frac{(1+2x)}{(1+nx)^2} - C_3 \frac{(1+3x)}{(1+nx)^3} + \dots \\
 = \left[C_0 - \frac{C_1}{(1+nx)} + \frac{C_2}{(1+nx)^2} - \frac{C_3}{(1+nx)^3} + \dots \right] \\
 + \frac{x}{1+nx} \left[-C_1 + \frac{2C_2}{1+nx} - \frac{3C_3}{(1+nx)^2} + \dots \right] \quad \dots (i)
 \end{aligned}$$

$$\therefore (1-y)^n = C_0 - C_1 y + C_2 y^2 - C_3 y^3 + \dots + (-1)^n y^n \quad \dots (ii)$$

$$\text{On differentiating w.r.t. } y, \text{ we get} \\
 -(1-y)^{n-1} = -C_1 + 2C_2 y - 3C_3 y^2 + \dots \quad \dots (iii)$$

$$\text{Now, } I_1 = \left(1 - \frac{1}{1+nx} \right)^n = \left(\frac{nx}{1+nx} \right)^n \quad \dots (iv)$$

$$\begin{aligned}
 \text{and } I_2 &= -C_1 + 2C_2 \cdot \frac{1}{1+nx} - 3C_3 \cdot \frac{1}{(1+nx)^2} + \dots \\
 &= (-n) \left(1 - \frac{1}{1+nx} \right)^{n-1} = -n \left(\frac{nx}{1+nx} \right)^{n-1} \quad \dots (v)
 \end{aligned}$$

Putting I_1 and I_2 from Eqs. (iii) and (v) in Eq. (i), we get

$$\begin{aligned}
 C_0 - C_1 \frac{(1+x)}{(1+nx)} + C_2 \frac{(1+x)}{(1+nx)^2} - C_3 \frac{(1+3x)}{(1+nx)^3} \\
 + \dots + (-1)^n C_n \frac{1+nx}{(1+nx)^n} \\
 = \left(\frac{nx}{1+nx} \right)^n - \left(\frac{x}{1+nx} \right) \cdot n \cdot \left(\frac{nx}{1+nx} \right)^{n-1} \\
 = \left(\frac{nx}{1+nx} \right)^n - \left(\frac{nx}{1+nx} \right)^n = 0 \Rightarrow \lambda = 0
 \end{aligned}$$

$$3. \text{ Let } y = \sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$$

$$\sum_{r=1}^{10} A_r B_r = \text{Coefficient of } x^{20} \text{ in } \{(1+x)^{10} (x+1)^{20}\} - 1$$

$$= C_{20} - 1 = C_{10} - 1 \text{ and } \sum_{r=1}^{10} (A_r)^2 = \text{Coefficient of } x^{10} \text{ in}$$

$$\{(1+x)^{10} (x+1)^{10}\} - 1 = B_{10} - 1$$

$$\Rightarrow y = B_{10} (C_{10} - 1) - C_{10} (B_{10} - 1) = C_{10} - B_{10}$$

$$4. \text{ Let } T_{r+1} \text{ be the general term in the expansion of } (1 - 2\sqrt{x})^{50}.$$

$$\therefore T_{r+1} = {}^{50}C_r (1)^{50-r} (-2x^{1/2})^r = {}^{50}C_r 2^r x^{r/2} (-1)^r$$

For the integral power of x , r should be even integer.

$$\begin{aligned}
 \therefore \text{Sum of coefficients} &= \sum_{r=0}^{25} {}^{50}C_{2r} (2)^{2r} \\
 &= \frac{1}{2} [(1+2)^{50} + (1-2)^{50}] \\
 &= \frac{1}{2} [30^{50} + 1]
 \end{aligned}$$

Aliter

$$\text{We have, } (1 - 2\sqrt{x})^{50} = C_0 - C_1 2\sqrt{x} + C_2 (2\sqrt{x})^2 - \dots + C_{50} (2\sqrt{x})^{50} \quad \dots (i)$$

$$\text{and } (1 + 2\sqrt{x})^{50} = C_0 + C_1 2\sqrt{x} + C_2 (2\sqrt{x})^2 + \dots + C_{50} (2\sqrt{x})^{50} \quad \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$(1-2\sqrt{x})^{50} + (1+2\sqrt{x})^{50} = 2[C_0 + C_2(2\sqrt{x})^2 + \dots + C_{50}(2\sqrt{x})^{50}]$$

$$\Rightarrow \frac{(1-2\sqrt{x})^{50} + (1+2\sqrt{x})^{50}}{2} = C_0 + C_2(2\sqrt{x})^2 + \dots + C_{50}(2\sqrt{x})^{50}$$

On putting $x = 1$, we get

$$\frac{(1-2\sqrt{1})^{50} + (1+2\sqrt{1})^{50}}{2} = C_0 + C_2(2)^2 + \dots + C_{50}(2)^{50}$$

$$\Rightarrow \frac{(-1)^{50} + (3)^{50}}{2} = C_0 + C_2(2)^2 + \dots + C_{50}(2)^{50}$$

$$\Rightarrow \frac{1 + 3^{50}}{2} = C_0 + C_2(2)^2 + \dots + C_{50}(2)^{50}$$

5. Consider, $(1 + ax + bx^2)(1 - 2x)^{18}$

$$= 1(1 - 2x)^{18} + ax(1 - 2x)^{18} + bx^2(1 - 2x)^{18}$$

Coefficient of x^3

$$(-2)^3 {}^{18}C_3 + a(-2)^2 {}^{18}C_2 + b(-2) {}^{18}C_1 = 0$$

$$\frac{4 \times (17 \times 16)}{(3 \times 2)} - 2a \cdot \frac{17}{2} + b = 0 \quad \dots (i)$$

Coefficient of x^4

$$(-2)^4 {}^{18}C_4 + a(-2)^3 {}^{18}C_3 + b(-2)^2 {}^{18}C_2 = 0$$

$$(4 \times 20) - 2a \cdot \frac{16}{3} + b = 0 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$4\left(\frac{17 \times 8}{3} - 20\right) + 2a\left(\frac{16}{3} - \frac{17}{2}\right) = 0$$

$$\Rightarrow 4\left(\frac{17 \times 8 - 60}{3}\right) + \frac{2a(-19)}{6} = 0$$

$$\Rightarrow a = \frac{4 \times 76 \times 6}{3 \times 2 \times 19} \Rightarrow a = 16$$

$$\Rightarrow b = \frac{2 \times 16 \times 16}{3} - 80$$

$$= \frac{272}{3}$$

6. Set X contains elements of the form

$$4^n - 3n - 1 = (1 + 3)^n - 3n - 1$$

$$= 3^n + {}^nC_{n-1}3^{n-1} + \dots + {}^nC_23^2$$

$$= 9(3^{n-2} + {}^nC_{n-1}3^{n-3} + \dots + {}^nC_2)$$

Set X has natural numbers which are multiples of 9 (not all)

Set Y has all multiples of 9

$$X \cup Y = Y$$

7. Consider, $\left[\frac{x+1}{x^{2/3} - x^{1/3} + 1} - \frac{x-1}{x - x^{1/2}} \right]^{10}$

$$= \left[\frac{(x^{1/3})^3 + 1^3}{x^{2/3} - x^{1/3} + 1} - \frac{\{(\sqrt{x})^2 - 1\}}{\sqrt{x}(\sqrt{x} - 1)} \right]^{10}$$

$$= \left[(x^{1/3} + 1) - \frac{(\sqrt{x} + 1)}{\sqrt{x}} \right]^{10}$$

$$= (x^{1/3} - x^{-1/2})^{10}$$

$\therefore$ The general term is

$$T_{r+1} = {}^{10}C_r (x^{1/3})^{10-r} (-x^{-1/2})^r$$

$$= {}^{10}C_r (-1)^r x^{\frac{10-r}{3} - \frac{r}{2}}$$

For independent of x , put

$$\frac{10-r}{3} - \frac{r}{2} = 0 \Rightarrow 20 - 2r - 3r = 0$$

$$\Rightarrow 20 = 5r \Rightarrow r = 4$$

$$\therefore T_5 = {}^{10}C_4 = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

8. $(\sqrt{3} + 1)^{2n} = {}^{2n}C_0 (\sqrt{3})^{2n} + {}^{2n}C_1 (\sqrt{3})^{2n-1} + {}^{2n}C_2 (\sqrt{3})^{2n-2} + \dots + {}^{2n}C_{2n} (\sqrt{3})^{2n-2n}$

$$\Rightarrow (\sqrt{3} - 1)^{2n} = {}^{2n}C_0 (\sqrt{3})^{2n} (-1)^0 + {}^{2n}C_1 (\sqrt{3})^{2n-1} (-1)^1 + {}^{2n}C_2 (\sqrt{3})^{2n-2} (-1)^2 + \dots + {}^{2n}C_{2n} (\sqrt{3})^{2n-2n} (-1)^{2n}$$

On subtracting both the binomial expansions, we get

$$(\sqrt{3} + 1)^{2n} - (\sqrt{3} - 1)^{2n} = 2[{}^{2n}C_1 (\sqrt{3})^{2n-1} + {}^{2n}C_3 (\sqrt{3})^{2n-3} + {}^{2n}C_5 (\sqrt{3})^{2n-5} + \dots + {}^{2n}C_{2n-1} (\sqrt{3})^{2n-(2n-1)}]$$

which is most certainly an irrational number because of odd powers of $\sqrt{3}$ in each of the terms.

9. Here, $(1 - x - x^2 + x^3)^6 = \{(1 - x) - x^2(1 - x)\}^6$

$$= \{(1 - x)(1 - x^2)\}^6 = (1 - x)^6 \cdot (1 - x^2)^6$$

$$= \left\{ \sum_{r=0}^6 (-1)^r {}^6C_r \cdot x^r \right\} \left\{ \sum_{s=0}^6 (-1)^s {}^6C_s \cdot x^{2s} \right\}$$

$$= \sum_{r=0}^6 \sum_{s=0}^6 (-1)^{r+s} {}^6C_r \cdot {}^6C_s \cdot x^{r+2s}$$

Now, consider $r + 2s = 7$

i.e. $(s = 1, r = 5)$

or $(s = 2, r = 3)$ or $(s = 3, r = 1)$

$\therefore$ Coefficient of x^7 is

$$\{(-1)^{5+1} \cdot {}^6C_5 \cdot {}^6C_1\} + \{(-1)^{3+2} \cdot {}^6C_3 \cdot {}^6C_2\} + \{(-1)^{1+3} \cdot {}^6C_1 \cdot {}^6C_3\}$$

$$= (36) - (20)(15) + 6(20)$$

$$= 36 - 300 + 120 = -144$$

10. $\therefore S_1 = \sum_{j=1}^{10} j(j-1) \frac{10!}{j(j-1)(j-2)!(10-j)!}$

$$= 90 \sum_{j=2}^{10} \frac{8!}{(j-2)!\{8-(j-2)\}!} = 90 \cdot 2^8$$

and $S_2 = \sum_{j=1}^{10} j \frac{10!}{j(j-1)!\{9-(j-1)\}!}$

$$= 10 \sum_{j=1}^{10} \frac{9!}{(j-1)!\{9-(j-1)\}!} = 10 \cdot 2^9$$

Also, $S_3 = \sum_{j=1}^{10} [j(j-1) + j] \frac{10!}{j!(10-j)!}$

$$= \sum_{j=1}^{10} j(j-1) {}^{10}C_j + \sum_{j=1}^{10} j {}^{10}C_j$$

$$= 90 \cdot 2^8 + 10 \cdot 2^9$$

$$= 90 \cdot 2^8 + 20 \cdot 2^8 = 110 \cdot 2^8 = 55 \cdot 2^9$$

11. $8^{2n} - (62)^{2n+1} = (1 + 63)^n - (63 - 1)^{2n+1}$

$$= (1 + 63)^n + (1 - 63)^{2n+1}$$

$$= [1 + {}^nC_1 63 + {}^nC_2 (63)^2 + \dots + (63)^n]$$

$$+ [1 - {}^{(2n+1)}C_1 63 + {}^{(2n+1)}C_2 (63)^2 + \dots + (-1)^{(2n+1)} (63)^{(2n+1)}]$$

$$= 2 + 63 [{}^nC_1 + {}^nC_2 (63) + \dots + (63)^{n-1} - (2n+1){}^nC_1 + (2n+1){}^nC_2 (63) - \dots - (63)^{2n}]$$

∴ Remainder is 2.

12. Since, $\sum_{r=0}^n {}^nC_r \cdot x^r = (1+x)^n$

On multiplying by x, we get

$$\sum_{r=0}^n {}^nC_r \cdot x^{r+1} = x(1+x)^n$$

On differentiating w.r.t. x, we get

$$\sum_{r=0}^n (r+1) \cdot {}^nC_r \cdot x^r = (1+x)^n + nx(1+x)^{n-1}$$

Statement II is true.

If x = 1, then

$$\sum_{r=0}^n (r+1) \cdot {}^nC_r = 2^n + n(2)^{n-1} = (n+2)2^{n-1}$$

∴ Statement I is true and Statement II is a correct explanation of Statement I.

13. Since, in a binomial expansion of $(a-b)^n$, $n \geq 5$, the sum of 5th and 6th terms is equal to zero.

$$\therefore {}^nC_4 a^{n-4} (-b)^4 + {}^nC_5 a^{n-5} (-b)^5 = 0$$

$$\Rightarrow \frac{n!}{(n-4)! 4!} a^{n-4} \cdot b^4 - \frac{n!}{(n-5)! 5!} a^{n-5} \cdot b^5 = 0$$

$$\Rightarrow \frac{n!}{(n-5)! 4!} a^{n-5} \cdot b^4 \left(\frac{a}{n-4} - \frac{b}{5} \right) = 0$$

$$\Rightarrow \frac{a}{b} = \frac{n-4}{5}$$

14. We know that,

$$(1+x)^{20} = {}^{20}C_0 + {}^{20}C_1 x + \dots + {}^{20}C_{10} x^{10} + \dots + {}^{20}C_{20} x^{20}$$

On putting x = -1 in the above expansion, we get

$$0 = {}^{20}C_0 - {}^{20}C_1 + \dots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_{11} + \dots + {}^{20}C_{20}$$

$$\Rightarrow 0 = {}^{20}C_0 - {}^{20}C_1 + \dots - {}^{20}C_9 + {}^{20}C_{10} - {}^{20}C_{11} + \dots + {}^{20}C_{20}$$

$$\Rightarrow 0 = 2({}^{20}C_0 - {}^{20}C_1 + \dots - {}^{20}C_9) + {}^{20}C_{10}$$

$$\Rightarrow {}^{20}C_{10} = 2({}^{20}C_0 - {}^{20}C_1 + \dots + {}^{20}C_{10})$$

$$\Rightarrow {}^{20}C_0 - {}^{20}C_1 + \dots + {}^{20}C_{10} = \frac{1}{2} {}^{20}C_{10}$$

15. Now, $(1-ax)^{-1} (1-bx)^{-1}$

$$= (1+ax+a^2x^2+\dots)(1+bx+b^2x^2+\dots)$$

Hence, a_n = Coefficient of x^n in $(1-ax)^{-1} (1-bx)^{-1}$

$$= a^0 b^n + a b^{n-1} + \dots + a^n b^0$$

$$= a^0 b^n \left(1 + \frac{a}{b} + \left(\frac{a}{b} \right)^2 + \dots \right)$$

$$= a^0 b^n \left(\frac{\left(\frac{a}{b} \right)^{n+1} - 1}{\frac{a}{b} - 1} \right)$$

$$= \frac{b^n (a^{n+1} - b^{n+1})}{a-b} \cdot \frac{b}{b^{n+1}}$$

$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

16. $(1-y)^m (1+y)^n = 1 + a_1 y + a_2 y^2 + a_3 y^3 + \dots$

On differentiating w.r.t. y, we get

$$-m(1-y)^{m-1} (1+y)^n + (1-y)^m n(1+y)^{n-1} = a_1 + 2a_2 y + 3a_3 y^2 + \dots \quad \dots(i)$$

On putting y = 0 in Eq. (i), we get

$$-m + n = a_1 = 10 \quad \dots(ii)$$

[∵ $a_1 = 10$, given]

On again differentiating Eq. (i), we get

$$-m[-(m-1)(1-y)^{m-2}(1+y)^n + (1-y)^{m-1}n(1+y)^{n-1}] + n[-m(1-y)^{m-1}(1+y)^{n-1} + (1-y)^m(n-1)(1+y)^{n-2}] = 2a_2 + 6a_3 y + \dots \quad \dots(iii)$$

On putting y = 0 in Eq. (iii), we get

$$-m[-(m-1) + n] + n[-m + (n-1)] = 2a_2 = 20$$

$$\Rightarrow m(m-1) - mn - mn + n(n-1) = 20$$

$$\Rightarrow m^2 + n^2 - m - n - 2mn = 20$$

$$\Rightarrow (m-n)^2 - (m+n) = 20$$

$$\Rightarrow 100 - (m+n) = 20$$

$$\Rightarrow m+n = 80 \quad \dots(iv)$$

On solving Eqs. (ii) and (iv), we get

$$m = 35 \text{ and } n = 45$$

17. Now, ${}^{50}C_4 + {}^{55}C_3 + {}^{54}C_3 + {}^{53}C_3 + {}^{52}C_3 + {}^{51}C_3 + {}^{50}C_3$
 $= {}^{50}C_3 + {}^{50}C_4 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$
 $= {}^{51}C_4 + {}^{51}C_3 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$
 $[\because {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r]$

$$= {}^{52}C_4 + {}^{52}C_3 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$$

$$= {}^{53}C_4 + {}^{53}C_3 + {}^{54}C_3 + {}^{55}C_3$$

$$= {}^{54}C_4 + {}^{54}C_3 + {}^{55}C_3 = {}^{55}C_4 + {}^{55}C_3 = {}^{56}C_4$$

18. Since, the coefficient of given terms are ${}^mC_{r-1}$, ${}^mC_r$, ${}^mC_{r+1}$ respectively and they also in AP.

$$\therefore {}^mC_{r-1} + {}^mC_{r+1} = 2{}^mC_r$$

$$\Rightarrow \frac{m!}{(r-1)!(m-r+1)!} + \frac{m!}{(r+1)!(m-r-1)!} = 2 \cdot \frac{m!}{r!(m-r)!}$$

$$\Rightarrow \frac{1}{(m-r+1)(m-r)} + \frac{1}{(r+1)r} = \frac{2}{r(m-r)}$$

$$\Rightarrow \frac{r(r+1) + (m-r+1)(m-r)}{r(r+1)(m-r+1)(m-r)} = \frac{2}{r(m-r)}$$

$$\Rightarrow r^2 + r + m^2 + r^2 - 2mr + m - r = 2(mr - r^2 + r + m - r + 1)$$

$$\Rightarrow 4r^2 - 4mr - m - 2 + m^2 = 0$$

$$\Rightarrow m^2 - m(4r+1) + 4r^2 - 2 = 0$$

19. Let x^7 is contained in $(r+1)$ th term in the expansion of

$$\left(ax^2 + \frac{1}{bx} \right)^{11}$$

$$\therefore T_{r+1} = {}^{11}C_r (ax^2)^{11-r} \left(\frac{1}{bx} \right)^r = {}^{11}C_r \frac{a^{11-r}}{b^r} \cdot x^{22-3r}$$

For coefficient of x^7 , put

$$22 - 3r = 7 \Rightarrow r = 5$$

$$\therefore T_6 = {}^{11}C_5 \frac{a^6}{b^5} \cdot x^7$$

∴ Coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is

$${}^{11}C_5 \frac{a^6}{b^5}.$$

Similarly, coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6 \frac{a^5}{b^6}$.

According to the given condition,

$$\begin{aligned} {}^{11}C_5 \frac{a^6}{b^5} &= {}^{11}C_6 \frac{a^5}{b^6} \\ \Rightarrow \frac{a^6}{b^5} &= \frac{a^5}{b^6} \Rightarrow ab = 1 \end{aligned}$$

$$\begin{aligned} 20. \quad & \frac{(1+x)^{3/2} - \left(1 + \frac{1}{2}x\right)^3}{(1-x)^{1/2}} \\ &= \frac{\left(1 + \frac{3}{2}x + \frac{3 \cdot 2}{2}x^2\right) - \left(1 + \frac{3x}{2} + \frac{3 \cdot 2}{2} \cdot \frac{x^2}{4}\right)}{(1-x)^{1/2}} \\ &= \frac{3x^2}{(1-x)^{1/2}} \quad [\text{neglecting higher powers of } x] \\ &= -\frac{3x^2}{8} (1-x)^{-1/2} \\ &= -\frac{3x^2}{8} \left(1 + \frac{1}{2}x + \frac{1 \cdot 3}{2} \cdot \frac{x^2}{2}\right) = -\frac{3x^2}{8} \quad [\because \text{higher powers of } x^2 \text{ can be neglected}] \end{aligned}$$

21. The coefficient of the middle term in powers of x of $(1+\alpha x)^4 = {}^4C_2 \alpha^2$

The coefficient of the middle term in powers of x of $(1-\alpha x)^6 = {}^6C_3 (-\alpha)^3$

According to the given condition,

$$\begin{aligned} {}^4C_2 \alpha^2 &= {}^6C_3 (-\alpha)^3 \\ \Rightarrow \frac{4!}{2!2!} \alpha^2 &= -\frac{6!}{3!3!} \alpha^3 \Rightarrow 6\alpha^2 = -20\alpha^3 \\ \Rightarrow \alpha &= -\frac{6}{20} \Rightarrow \alpha = -\frac{3}{10} \end{aligned}$$

22. The coefficient of x^n in the expansion of $(1+x)(1-x)^n$ = Coefficient of x^n in $(1-x)^n$

$$\begin{aligned} &+ \text{Coefficient of } x^{n-1} \text{ in } (1-x)^n \\ &= (-1)^n {}^nC_n + (-1)^{n-1} {}^nC_{n-1} \\ &= (-1)^n \left(1 - \frac{n!}{1!(n-1)!}\right) = (-1)^n (1-n) \end{aligned}$$

$$\begin{aligned} 23. \quad & \text{Given that, } s_n = \sum_{r=0}^n \frac{1}{{}^nC_r} \\ & s_n = \sum_{r=0}^n \frac{1}{{}^nC_{n-r}} \quad [\because {}^nC_r = {}^nC_{n-r}] \\ \Rightarrow & ns_n = \sum_{r=0}^n \frac{n}{{}^nC_{n-r}} \\ \Rightarrow & ns_n = \sum_{r=0}^n \left(\frac{n-r}{{}^nC_{n-r}} + \frac{r}{{}^nC_{n-r}} \right) \end{aligned}$$

$$\begin{aligned} \Rightarrow ns_n &= \sum_{r=0}^n \frac{n-r}{{}^nC_{n-r}} + \sum_{r=0}^n \frac{r}{{}^nC_r} \\ \Rightarrow ns_n &= \left(\frac{n}{{}^nC_n} + \frac{n-1}{{}^nC_{n-1}} + \dots + \frac{1}{{}^nC_1} \right) + \sum_{r=0}^n \frac{r}{{}^nC_r} \\ \Rightarrow ns_n &= t_n + t_n \quad \left[\because t_n = \sum_{r=0}^n \frac{r}{{}^nC_r}, \text{ given} \right] \\ \Rightarrow ns_n &= 2t_n \Rightarrow \frac{t_n}{s_n} = \frac{n}{2} \end{aligned}$$

24. The general term of $(\sqrt{3} + \sqrt[8]{5})^{256}$ is

$$T_{r+1} = {}^{256}C_r (3)^{(256-r)/2} (5)^{r/8}$$

For integral terms, $\frac{256-r}{2}$ and $\frac{r}{8}$ are both positive integers.

i.e. $r = 0, 8, 16, 24, 32, \dots, 256$

Hence, total number of terms are 33.

25. Since, $(r+1)$ th term in the expansion of $(1+x)^{27/5}$

$$= \frac{27}{5} \left(\frac{27}{5} - 1 \right) \dots \left(\frac{27}{5} - r + 1 \right) \frac{x^r}{r!}$$

Now, this term will be negative, if the last factor in numerator is the only one negative factor.

$$\begin{aligned} \Rightarrow \frac{27}{5} - r + 1 &< 0 \\ \Rightarrow \frac{32}{5} &< r \\ \Rightarrow 6.4 < r &\Rightarrow \text{least value of } r \text{ is } 7. \end{aligned}$$

Thus, first negative term will be 8th.

$$\begin{aligned} 26. \quad & (1+2x+3x^2+\dots)^{-3/2} = [(1-x)^{-2}]^{-3/2} \\ &= (1-x)^3 \end{aligned}$$

∴ Coefficient of x^5 in $(1+2x+3x^2+\dots)^{-3/2}$

= Coefficient of x^5 in $(1-x)^3 = 0$

$$\begin{aligned} 27. \quad & (1+x+x^2+x^3+\dots)^2 = [(1-x)^{-1}]^2 \\ &= (1-x)^{-2} \end{aligned}$$

Coefficient of x^n in $(1+x+x^2+\dots)^2$

= Coefficient of x^n in $(1-x)^{-2}$

$$= {}^{n+2-1}C_{2-1} = {}^{n+1}C_1 = n+1$$

28. Using binomial theorem,

$$\begin{aligned} (5x-4y)^n &= {}^nC_0 (5x)^n + {}^nC_1 (5x)^{n-1} (-4y) \\ &+ {}^nC_2 (5x)^{n-2} (-4y)^2 + \dots + {}^nC_n (-4y)^n \end{aligned}$$

Sum of coefficients

$$\begin{aligned} &= {}^nC_0 5^n + {}^nC_1 5^{n-1} (-4) + {}^nC_2 5^{n-2} (-4)^2 \\ &+ \dots + {}^nC_n (-4)^n \\ &= (5-4)^n = 1^n = 1 \end{aligned}$$

29. Given, in the expansion of $(1+x)^{44}$, 21st and 22nd terms are equal.

$$\begin{aligned} \text{i.e. } T_{21} &= T_{22} \\ {}^{44}C_{20} x^{20} &= {}^{44}C_{21} x^{21} \Rightarrow x = \frac{{}^{44}C_{20}}{{}^{44}C_{21}} \\ &= \frac{44!}{20! \times 24!} = \frac{21! \times 23!}{20! \times 24!} = \frac{21}{24} = \frac{7}{8} \end{aligned}$$

30. The general term in $\left(ax^2 + \frac{1}{bx}\right)^{13}$ is

$$T_{r+1} = {}^{13}C_r (ax^2)^{13-r} \left(\frac{1}{bx}\right)^r$$

$$= {}^{13}C_r a^{13-r} \times b^{-r} (x)^{26-3r}$$

For coefficient of x^8 , put $26 - 3r = 8$

$$\Rightarrow 3r = 18 \Rightarrow r = 6$$

$$\therefore T_7 = {}^{13}C_6 a^{13-6} b^{-6} x^8 = {}^{13}C_6 a^7 b^{-6} x^8$$

Now, the general term in $\left(ax - \frac{1}{bx^2}\right)^{13}$ is

$$T'_{r+1} = {}^{13}C_r (ax)^{13-r} \left(-\frac{1}{bx^2}\right)^r$$

$$= {}^{13}C_r a^{13-r} \times b^{-r} \times (x)^{13-3r} (-1)^r$$

For coefficient of x^{-8} , put $13 - 3r = -8$

$$\Rightarrow 3r = 21 \Rightarrow r = 7$$

$$\therefore T'_8 = (-1)^7 {}^{13}C_7 a^{13-7} b^{-7} x^{-8}$$

$$= (-1)^7 {}^{13}C_7 a^6 b^{-7} x^{-8}$$

According to the given condition,

Coefficient of x^8 in $\left(ax^2 + \frac{1}{bx}\right)^{13}$

$$= \text{Coefficient of } x^{-8} \text{ in } \left(ax - \frac{1}{bx^2}\right)^{13}$$

$$\therefore {}^{13}C_6 a^7 b^{-6} = - {}^{13}C_7 a^6 b^{-7}$$

$$\Rightarrow {}^{13}C_7 \frac{a^7}{b^6} = - {}^{13}C_7 \frac{a^6}{b^7}$$

$$\Rightarrow \frac{a^7}{b^6} = - \frac{b^6}{b^7} \Rightarrow a = - \frac{1}{b} \Rightarrow ab + 1 = 0$$

31. Given expression can be rewritten as

$$2(1-x)^{-1} \times 2^{-1} \left(1 - \frac{x}{2}\right)^{-1}$$

$$= [1 + x + x^2 + x^3 + \dots]$$

$$\left[1 + \left(\frac{x}{2}\right) + \left(\frac{x}{2}\right)^2 + \left(\frac{x}{2}\right)^3 + \dots\right]$$

$\therefore$ Coefficient of x^3 in the above expression

$$= \left[\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right) + 1\right]$$

$$= \left(\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + 1\right) = \left(\frac{1+2+4+8}{8}\right) = \frac{15}{8}$$

32. Now, $(a + bx)^{-3} = a^{-3} \left(1 + \frac{bx}{a}\right)^{-3}$

$$= \frac{1}{a^3} \left(1 - {}^3C_1 \left(\frac{bx}{a}\right) + \dots\right)$$

$$= \frac{1}{a^3} - {}^3C_1 \frac{1}{a^3} \left(\frac{bx}{a}\right) + \dots$$

$$= \frac{1}{27} + \frac{1}{3} x + \dots \quad [\text{given}]$$

On equating constant terms and coefficients of x both sides, we get

$$\frac{1}{a^3} = \frac{1}{27}$$

$$\text{and } - {}^3C_1 \frac{1}{a^3} \left(\frac{b}{a}\right) = \frac{1}{3}$$

$$\Rightarrow a = 3$$

$$\text{and } - {}^3C_1 \left(\frac{1}{3^4}\right) \times b = \frac{1}{3}$$

$$\Rightarrow -3 \times \frac{1}{3^4} b = \frac{1}{3}$$

$$\Rightarrow -b = 3^2 \Rightarrow b = -9$$

$$\therefore (a, b) = (3, -9)$$

33. General term in the expansion of $\left(\sqrt{x} - \frac{k}{x^2}\right)^{10}$ is

$$T_{r+1} = {}^{10}C_r (\sqrt{x})^{10-r} \cdot \left(-\frac{k}{x^2}\right)^r$$

$$= {}^{10}C_r x^{\frac{10-r}{2}} \cdot (-k)^r \cdot x^{-2r}$$

$$= {}^{10}C_r (-k)^r x^{\left(\frac{10-5r}{2}\right)}$$

The term is free from x

$$\text{Put } \frac{10-5r}{2} = 0 \Rightarrow r = 2$$

$$\text{Now, } {}^{10}C_2 (-k)^2 = 405$$

$$\Rightarrow \frac{10 \times 9}{1 \times 2} \cdot k^2 = 405 \Rightarrow k^2 = \frac{405}{45} = 9$$

$$\therefore k = \pm 3$$

34. The given expression is

$$\left(x^2 + \frac{1}{x^2} + 2\right)^n = \left[\left(x + \frac{1}{x}\right)^2\right]^n = \left(x + \frac{1}{x}\right)^{2n}$$

The binomial contains $(2n + 1)$ terms in its expansion.

The middle term is T_{n+1} .

$$\therefore T_{n+1} = {}^{2n}C_n (x)^{2n-n} \left(\frac{1}{x}\right)^n$$

$$= {}^{2n}C_n = \frac{(2n)!}{(2n-n)! n!} = \frac{(2n)!}{(n!)^2}$$

35. General term in the expansion of $\left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)^{18}$ is

$$T_{r+1} = {}^{18}C_r (\sqrt{x})^{18-r} \left(-\frac{2}{\sqrt{x}}\right)^r$$

$$= {}^{18}C_r (x)^{\frac{18-r}{2}} (-2)^r (x)^{-\frac{r}{2}} = {}^{18}C_r (-2)^r x^{9-r}$$

For independent of x , put

$$9 - r = 0 \Rightarrow r = 9$$

$$\therefore T_{10} = {}^{18}C_9 (-2)^9 = - {}^{18}C_9 2^9$$

36. We know that,

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n \quad \dots (i)$$

$$\text{and } (x-1)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_n \quad \dots (ii)$$

On multiplying Eqs. (i) and (ii), we get

$$(1+x)^{2n} = ({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n) \times ({}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \dots + {}^nC_n)$$

Coefficient of x^n in RHS

$$= ({}^nC_0)^2 + ({}^nC_1)^2 + \dots + ({}^nC_n)^2$$

and coefficient of x^n in LHS $= {}^{2n}C_n$

$$\therefore ({}^nC_0)^2 + ({}^nC_1)^2 + \dots + ({}^nC_n)^2 = \frac{{}^{2n}C_n}{n! n!}$$

$$\Rightarrow ({}^nC_1)^2 + \dots + ({}^nC_n)^2 = \frac{(2n)!}{n! n!} - 1 = {}^{2n}C_n - 1$$

37. We know that,

$$(1-x)^n = {}^nC_0 + x {}^nC_1 + x^2 {}^nC_2 + \dots + x^n {}^nC_n$$

On integrating both sides from 0 to 2, we get

$$\begin{aligned} & \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^2 \\ &= \left[x {}^nC_0 + \frac{x^2}{2} {}^nC_1 + \frac{x^3}{3} {}^nC_2 + \dots + \frac{x^{n+1}}{n+1} {}^nC_n \right]_0^2 \\ &= \frac{(3)^{n+1}}{n+1} - \frac{1}{n+1} + 2^n {}^nC_0 = \frac{2^2}{2} {}^nC_1 + \frac{2^3}{3} {}^nC_2 \\ & \quad + \dots + \frac{2^{n+1}}{n+1} {}^nC_n - 0 \\ &= \frac{2}{1} {}^nC_0 + \frac{2^2}{2} {}^nC_1 + \frac{2^3}{3} {}^nC_2 + \dots + \frac{2^{n+1}}{n+1} {}^nC_n \\ &= \frac{3^{n+1} - 1}{n+1} \end{aligned}$$

$$\begin{aligned} 38. {}^mC_{r+1} + \sum_{k=m}^n {}^kC_r &= ({}^mC_{r+1} + {}^mC_r) + {}^{m+1}C_r \\ & \quad + {}^{m+2}C_r + \dots + {}^nC_r \\ &= ({}^{m+1}C_{r+1} + {}^{m+1}C_r) + {}^{m+2}C_r + \dots + {}^nC_r \\ &= ({}^{m+2}C_{r+1} + {}^{m+2}C_r) + \dots + {}^nC_r \\ & \quad \vdots \quad \quad \quad \vdots \\ &= {}^{n+1}C_{r+1} \end{aligned}$$

$$\begin{aligned} 39. \frac{C_1}{C_0} + 2 \frac{C_2}{C_1} + 3 \frac{C_3}{C_2} + \dots + n \frac{C_n}{C_{n-1}} \\ &= \frac{{}^nC_1}{{}^nC_0} + 2 \frac{{}^nC_2}{{}^nC_1} + 3 \frac{{}^nC_3}{{}^nC_2} + \dots + n \frac{{}^nC_n}{{}^nC_{n-1}} \\ &= \frac{n(n-1)}{1} + 2 \times \frac{n(n-1)}{2} + 3 \times \frac{n(n-1)(n-2)}{2} + \dots + n \times \frac{1}{n} \\ &= n + (n-1) + (n-2) + \dots + 1 = \Sigma n = \frac{n(n+1)}{2} \end{aligned}$$

40. We know that, ${}^{11}C_n$ is maximum when $n = 5$

$$\therefore {}^{11}C_n = \frac{11!}{n!(11-n)!}$$

$\therefore n!(11-n)!$ is minimum, when $n = 5$.

$$\begin{aligned} 41. \text{By given condition, } 2 \cdot {}^mC_2 &= {}^mC_1 + {}^mC_3 \\ \Rightarrow m(m-1) &= m + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} \end{aligned}$$

$$\Rightarrow m^2 - 9m + 14 = 0$$

$$\Rightarrow (m-2)(m-7) = 0$$

$$\Rightarrow m = 2, 7$$

Since, 6th term is 21, $m = 2$ is ruled out, so we have

$$T_6 = 21 = {}^7C_5 [2^{\log(10-3^x)}]^{7-5} \times [\sqrt[5]{2^{(x-2)\log 3}}]^5$$

$$\Rightarrow 21 = \frac{7 \times 6}{2 \times 1} 2^{\log(10-3^x)} \cdot 2^{(x-2)\log 3}$$

$$\Rightarrow 2^{\log(10-3^x) + (x-2)\log 3} = 1 = 2^0$$

$$\Rightarrow \log(10-3^x) = (2-x)\log 3$$

$$\Rightarrow \log(10-3^x) = \log 3^{2-x}$$

$$\Rightarrow 10-3^x = 3^{2-x} \Rightarrow 3^x \cdot 10 - 3^{2x} = 9$$

$$\Rightarrow 3^{2x} - 3^x \cdot 10 + 9 = 0$$

$$\text{Now, put } t = 3^x$$

$$\Rightarrow t^2 - 10t + 9 = 0$$

$$\Rightarrow (t-1)(t-9) = 0$$

$$\therefore t = 1 \text{ or } 9$$

$$\Rightarrow 3^x = 1 \text{ or } 3^x = 9$$

$$\Rightarrow x = 0 \text{ or } x = 2$$

42. Given expression is $\left(ax + \frac{1}{x}\right)^n$.

$$\text{Here, } T_4 = T_{(3+1)} = {}^nC_3 (ax)^{n-3} \left(\frac{1}{x}\right)^3 = \frac{5}{2} \text{ [given]}$$

$$\Rightarrow {}^nC_3 \cdot a^{n-3} \cdot x^{n-6} = \frac{5}{2} \quad \dots(i)$$

Since, $x \in R$

$$\therefore n-6=0 \Rightarrow n=6$$

Now, for a , on substituting the value of n in Eq. (i), we get

$${}^6C_3 a^3 = \frac{5}{2} \Rightarrow a^3 = \frac{5}{2} \times \frac{3 \times 2 \times 1}{6 \times 5 \times 4}$$

$$\Rightarrow a^3 = \frac{1}{8} = \left(\frac{1}{2}\right)^3 \Rightarrow a = \frac{1}{2}$$

43. We have,

$$\sum_{r=0}^n \sum_{s=0}^n (C_r + C_s) = \sum_{r=0}^n (C_r + C_r) + 2 \sum_{0 \leq r < s \leq n} (C_r + C_s)$$

$$\Rightarrow (n+1) \cdot 2^{n+1} = 2 \left(\sum_{r=0}^n C_r \right) + 2 \sum_{0 \leq r < s \leq n} (C_r + C_s)$$

$$(n+1) \cdot 2^{n+1} = 2 \cdot 2^n + 2 \sum_{0 \leq r < s \leq n} (C_r + C_s)$$

$$\Rightarrow \sum_{0 \leq r < s \leq n} (C_r + C_s) = n2^n$$

$$\begin{aligned} 44. \left(1 + \frac{C_1}{C_0}\right) \left(1 + \frac{C_2}{C_1}\right) \left(1 + \frac{C_3}{C_2}\right) \dots \left(1 + \frac{C_n}{C_{n-1}}\right) \\ &= \left(1 + \frac{n}{1}\right) \left(1 + \frac{n(n-1)}{2n}\right) \dots \left(1 + \frac{1}{n}\right) \\ &= \left(\frac{1+n}{1}\right) \left(\frac{1+n}{2}\right) \dots \left(\frac{1+n}{n}\right) = \frac{(1+n)^n}{n!} \end{aligned}$$

$$45. \text{We have, } \frac{e^x}{1-x} = B_0 + B_1x + B_2x^2 + \dots + B_nx^n + \dots$$

$$\Rightarrow e^x (1-x)^{-1} = B_0 + B_1x + B_2x^2 + \dots + B_nx^n + \dots$$

$$\Rightarrow \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots\right) (1 + x + x^2 + \dots)$$

$$= B_0 + B_1x + B_2x^2 + \dots$$

$$\Rightarrow (1 + x + x^2 + \dots) + \left(\frac{x}{1!} + \frac{x^2}{1!} + \dots\right)$$

$$+ \left(\frac{x^2}{2!} + \frac{x^3}{2!} + \dots\right) + \dots = B_0 + B_1x + B_2x^2 + \dots$$

$$\Rightarrow 1 + \left(1 + \frac{1}{1!}\right)x + \left(1 + \frac{1}{1!} + \frac{1}{2!}\right)x^2 + \dots$$

$$= B_0 + B_1x + B_2x^2 + \dots$$

On comparing, we get

$$B_0 = 1, B_1 = 1 + \frac{1}{1!}, B_2 = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots$$

$$\therefore B_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$$

$$\text{Now, } B_n - B_{n-1} = \left(1 + \frac{1}{1!} + \dots + \frac{1}{n!}\right) - \left(1 + \frac{1}{1!} + \dots + \frac{1}{(n-1)!}\right) = \frac{1}{n!}$$

46. The given series is in GP.

Hence, its sum

$$S = \frac{1\{(1+x)^{20+1} - 1\}}{(1+x) - 1} \quad [\because a = 1, r = 1+x]$$

$$= \frac{(1+x)^{21} - 1}{x}$$

Therefore, the required coefficient of x^{10} in the expansion of $\frac{(1+x)^{21} - 1}{x}$ = Coefficient of x^{11} in the expansion of $(1+x)^{21} - 1 = {}^{21}C_{11}$

47. We have, $T_r = T_{r+1}$

$$\Rightarrow {}^nC_{r-1} p^{n-r+1} \cdot q^{r-1} = {}^nC_r p^{n-r} q^r$$

$$\Rightarrow {}^nC_{r-1} p^{n-r} \cdot p q^{r-1} \cdot q^{-1} = {}^nC_r p^{n-r} q^r$$

$$\Rightarrow \frac{{}^nC_r}{{}^nC_{r-1}} = \frac{p}{q}$$

$$\Rightarrow \frac{(r-1)!(n-r+1)!}{r!(n-r)!} = \frac{p}{q}$$

$$\Rightarrow \frac{n-r+1}{r} = \frac{p}{q}$$

$$\Rightarrow nq + q - qr = pr$$

$$\Rightarrow (n+1)q = r(p+q)$$

$$\therefore \frac{(n+1)q}{r(p+q)} = 1$$

48. Here, $(1+ax)^n = 1 + nax + \frac{n(n-1)}{2!}(ax)^2 + \dots$

On comparing the coefficients of like powers in the given equation, we get

$$na = 6 \quad \text{and} \quad \frac{27}{2} = \frac{n(n-1)}{2} \cdot a^2$$

$$\Rightarrow (n-1)a = \frac{9}{2} \quad [\because na = 6]$$

$$\Rightarrow \frac{(n-1)6}{n} = \frac{9}{2}$$

$$\therefore n = 4 \quad \text{and} \quad a = \frac{3}{2}$$

49. 13th term in the expansion of $\left(x^2 + \frac{2}{x}\right)^n$ is given by

$$T_{13} = {}^nC_{12} (x^2)^{n-12} \left(\frac{2}{x}\right)^{12} = {}^nC_{12} x^{2n-24} \left(\frac{2^{12}}{x^{12}}\right)$$

$$= {}^nC_{12} x^{2n-24-12} (2^{12})$$

$$= {}^nC_{12} x^{2n-36} (2^{12})$$

If the 13th term will be independent of x , then x^{2n-36} must be 1.

$$\text{i.e. } 2n - 36 = 0$$

$$\Rightarrow 2n = 36$$

$$\Rightarrow n = 18$$

The divisors of $n = 18$ are 1, 2, 3, 6, 9, 18 and their sum is $1 + 2 + 3 + 6 + 9 + 18 = 39$.

$$50. P = \sum_{r=0}^5 C_{2r} = {}^{10}C_0 + {}^{10}C_2 + \dots + {}^{10}C_{10} = \frac{2^{10}}{2} = 2^9$$

$$Q = \sum_{r=0}^3 d_{2r+1} = d_1 + d_3 + d_5 + d_7$$

$$= {}^7C_1 + {}^7C_3 + {}^7C_5 + {}^7C_7 = \frac{2^7}{2} = 2^6$$

$$\therefore \frac{P}{Q} = \frac{2^9}{2^6} = 2^3 = 8$$

51. Given expansion is $(1+x)^{2n}$.

According to the given condition,

$${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$$

$$\Rightarrow (2n - 3r + 1) = (r + 1)$$

$$\Rightarrow 2n = 4r \Rightarrow n = 2r$$

52. Here, $A = {}^{2n}C_n$ and $B = {}^{2n-1}C_n$

$$\therefore \frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = \frac{2n}{n} = 2$$

53. $T_{r+1} = {}^nC_r (x^3)^{n-r} \left(\frac{-1}{x^2}\right)^r$

$$= {}^nC_r x^{3n-3r} (-1)^r x^{-2r} = {}^nC_r x^{3n-5r} (-1)^r$$

For coefficient of x^5 and x^{10} , substitute $3n - 5r = 5$ and 10 respectively, we get coefficient of

$$x^5 = {}^nC_{\frac{3n-5}{5}} (-1)^{\frac{3n-5}{5}} \quad \text{and} \quad \text{coefficient of}$$

$$x^{10} = {}^nC_{\frac{3n-10}{5}} (-1)^{\frac{3n-10}{5}}$$

Coefficient of x^5 + Coefficient of $x^{10} = 0$

$$\Rightarrow {}^nC_{\frac{3n-5}{5}} (-1)^{\frac{3n-5}{5}} = - {}^nC_{\frac{3n-10}{5}} (-1)^{\frac{3n-10}{5}}$$

$$\Rightarrow {}^nC_{\frac{3n-5}{5}} (-1)^{\frac{3n-5}{5}} = {}^nC_{\frac{3n-10}{5}} (-1)^{\frac{3n-5}{5}}$$

$$\Rightarrow {}^nC_{\frac{3n-5}{5}} = {}^nC_{\frac{3n-10}{5}}$$

$$\Rightarrow \frac{3n-5}{5} + \frac{3n-10}{5} = n$$

$$\Rightarrow \frac{6n-15}{5} = n$$

$$\Rightarrow 6n - 15 = 5n$$

$$\therefore n = 15$$

54. 5th term from end = $(12 - 5 + 2)$ th term from beginning = 9th term

$$= {}^{12}C_8 \left(\frac{x^3}{2}\right)^{12-8} \left(\frac{-2}{x^2}\right)^8 = {}^{12}C_8 x^{-4} (2)^4$$

$$= \frac{12 \times 11 \times 10 \times 9}{4 \times 3 \times 2 \times 1} \times 2^4 \times x^{-4} = 7920 x^{-4}$$

55. For $n = 1$, we have

$$= [(1+p)(1+q)(p+q)]^n$$

$$= (1+q+p+pq)(p+q)$$

$$= p+q+pq+q^2+p^2+pq+p^2q+pq^2$$

$$= p+q+2pq+q^2+p^2+p^2q+pq^2$$

$$\therefore \text{Coefficient of } p'q' = 2 = 1 + 1 = [C(1,0)]^3 + [C(1,1)]^3$$

For $n = 2$, we have

$$\begin{aligned} [(1+p)(1+q)(p+q)]^2 &= (1+p)^2(1+q)^2(p+q)^2 \\ &= (1+p^2+2p)(1+q^2+2q)(p^2+q^2+2pq) \\ &= (1+q^2+2q+q^2+p^2q^2+2p^2q \\ &\quad +2p+2pq^2+4pq)(p^2+q^2+2pq) \end{aligned}$$

$$\begin{aligned} \therefore \text{Coefficient of } p^2q^2 &= 1+1+8=10 \\ &= [C(2,0)]^3 [C(2,2)]^3 + [C(2,1)]^3 \\ &= \sum_{k=0}^2 [C(2,k)]^3 \end{aligned}$$

Thus, the coefficient of p^nq^n in $[(1+p)(1+q)(p+q)]^n$ is

$$\sum_{k=0}^n [(n,k)]^3.$$

56. Given expansion is $(1-3x+3x^2-x^3)^{2n} = [(1-x)^3]^{2n}$
 $= (1-x)^{6n}$
 $\therefore$ Middle term = $\frac{(6n+2)\text{th}}{2}$ term = $(3n+1)\text{th}$ term
 [$\because$ here, $6n$ is even]

57. Since, ${}^nC_4$, ${}^nC_5$ and ${}^nC_6$ are in AP.

$$\begin{aligned} \therefore \quad & 2 {}^nC_5 = {}^nC_4 + {}^nC_6 \\ \Rightarrow \quad & \frac{2}{5(n-5)} = \frac{1}{(n-4)(n-5)} + \frac{1}{30} \\ \Rightarrow \quad & \frac{2}{5(n-5)} = \frac{30+n^2-9n+20}{30(n-4)(n-5)} \\ \Rightarrow \quad & 12(n-4) = 30+n^2-9n+20 \\ \Rightarrow \quad & 12n-48 = n^2-9n+50 \\ \Rightarrow \quad & n^2-21n+98=0 \\ \Rightarrow \quad & (n-14)(n-7)=0 \\ \Rightarrow \quad & n=14 \text{ or } 7 \end{aligned}$$

58. Consider, ${}^{15}C_3 + {}^{15}C_5 + \dots + {}^{15}C_{15}$
 $= ({}^{15}C_1 + \dots + {}^{15}C_{15}) - {}^{15}C_1$
 $= \left(\frac{2^{15}}{2}\right) - 15 = 2^{14} - 15$

59. Since, $(1+x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$
 On differentiating w.r.t. x , we get
 $n(1+x)^{n-1} = C_1 + 2C_2x + \dots + nC_nx^{n-1}$
 On putting $x = -1$, we get
 $0 = C_1 - 2C_2 + \dots + n(-1)^{n-1}C_n$

60. We know, $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \dots + {}^nC_nx^n$
 $\Rightarrow x(1+x)^n = {}^nC_0x + {}^nC_1x^2 + {}^nC_2x^3 + \dots + {}^nC_nx^{n+1}$

On differentiating w.r.t. x , we get

$$(1+x)^n + nx(1+x)^{n-1} = {}^nC_0 + 2 {}^nC_1x + \dots + (n+1) {}^nC_nx^n$$

Put $x = 1$, we get

$$\begin{aligned} 2^n + n2^{n-1} &= {}^nC_0 + 2 {}^nC_1 + \dots + (n+1) {}^nC_n \\ \Rightarrow 2^{n-1}(n+2) &= {}^nC_0 + 2 {}^nC_1 + \dots + (n+1) {}^nC_n \end{aligned}$$

61. Here, $T_5 + T_6 = 0$

$$\begin{aligned} \Rightarrow {}^nC_4a^{n-4}(-2b)^4 + {}^nC_5a^{n-5}(-2b)^5 &= 0 \\ \Rightarrow 16 \cdot {}^nC_4a^{n-4}b^4 &= 32 {}^nC_5a^{n-5}b^5 \\ \Rightarrow \frac{{}^nC_5}{{}^nC_4} \cdot \frac{a^{n-5}b^5}{a^{n-4}b^4} &= \frac{1}{2} \Rightarrow \frac{b}{a} = \frac{1}{2} \cdot \frac{{}^nC_4}{{}^nC_5} \\ \Rightarrow \frac{a}{b} &= \frac{2 \cdot \frac{n!}{5!(n-5)!}}{\frac{n!}{4!(n-4)!}} = \frac{4!(n-4)!}{5!(n-5)!} \times 2 = \frac{2(n-4)}{5} \end{aligned}$$

62. In the expansion of $(x+2y)^6$, $\left(\frac{6}{2}+1\right)$ th term is the middle term.

$$\text{So, } T_4 = T_{3+1} = {}^6C_3 x^{6-3} (2y)^3 = 8 ({}^6C_3) (xy)^3$$

$$\therefore \text{Coefficient of middle term} = 8 ({}^6C_3)$$

63. $a_1 = {}^nC_1$, $a_2 = {}^nC_2$, $a_3 = {}^nC_3$

$\therefore a_1, a_2$ and a_3 are in AP.

$$\begin{aligned} \therefore \quad & 2a_2 = a_1 + a_3 \\ \Rightarrow \quad & 2 \cdot {}^nC_2 = {}^nC_1 + {}^nC_3 \\ \Rightarrow \quad & 2 \cdot \frac{n(n-1)}{2!} = n + \frac{n(n-1)(n-2)}{3!} \\ \Rightarrow \quad & n^2 - 9n + 14 = 0 \\ \Rightarrow \quad & (n-2)(n-7) = 0 \\ \therefore \quad & n = 7 \quad [\because n = 2 \text{ is not possible}] \end{aligned}$$

64. $(a^2 - 6a + 11)^{10} = 1024$

$$\begin{aligned} \Rightarrow (a^2 - 6a + 11)^{10} &= 2^{10} \\ \Rightarrow a^2 - 6a + 11 &= 2 \\ \Rightarrow a^2 - 6a + 9 &= 0 \\ \Rightarrow (a-3)^2 &= 0 \\ \therefore a &= 3 \end{aligned}$$

9

Trigonometric Functions and Equations

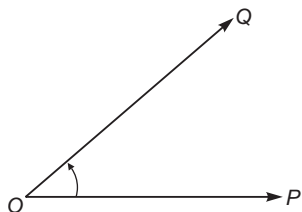
Introduction

Trigonometry is a branch of Mathematics in which we study triangles and relationships between the lengths of their sides and the angles between the sides.

The word 'Trigonometry' is derived from two Greek words '*trigonon*' and '*metron*'. The word '*trigonon*' means a triangle and the word '*metron*' means a measure. i.e. trigonometry means measuring the angles and sides of a triangle. Trigonometry defines the trigonometric functions, which describe the relationships and have applicability to physical phenomena such as waves.

Measure of Angles

The measure of angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final positions of the revolving rays are respectively called the initial and terminal sides. If initial and final positions of the revolving ray are OP and OQ , then the angle formed will be $\angle POQ$.



If the rotation is in clockwise direction, then the angle measured is negative and if the rotation is in anti-clockwise direction, then the angle measured is positive.

Chapter Snapshot

- Introduction
- Measure of Angles
- Systems of Measurement of Angles
- Trigonometric Ratios
- Trigonometric Function
- Graph of Trigonometric Functions
- Trigonometrical Identities
- Trigonometric Ratios of Allied Angles
- Trigonometrical Ratios of Compound Angles
- Trigonometric Ratios of Multiples of an Angle
- Maximum and Minimum Values of Trigonometrical Expressions
- Trigonometric Equations
- General Solution of Trigonometric Equations
- Solution of Trigonometric Inequality

Systems of Measurement of Angles

There are three systems for measuring angles:

- (i) **Sexagesimal system** In this system, a right angle is divided into 90 equal parts, called degrees. The symbol 1° is used to denote one degree. Thus, one degree is one-ninetieth $\left(\frac{1}{90}\text{th}\right)$ part of a right

angle. Each degree is divided into 60 equal parts, called minutes and one minute is divided into 60 equal parts, called seconds. The symbols $1'$ and $1''$ are used to denote one minute and one second respectively.

Thus, 1 right angle = 90 degree ($= 90^\circ$)

$$1^\circ = 60 \text{ minutes } (= 60')$$

$$1' = 60 \text{ seconds } (= 60'')$$

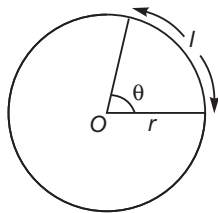
- (ii) **Centesimal system** In this system, a right angle is divided into 100 equal parts, called grades. Each grade is subdivided into 100 minutes and each minute is subdivided into 100 seconds. The symbols 1^g , $1'$ and $1''$ are used to denote a grade, a minute and a second, respectively.

Thus, 1 right angle = 100 grades ($= 100^g$)

$$1 \text{ grade} = 100 \text{ minutes } (= 100')$$

$$1 \text{ minute} = 100 \text{ seconds } (= 100'')$$

- (iii) **Circular system** If the angle subtended by an arc of length l to the centre of circle of radius r is θ , then $\theta = \frac{l}{r}$.



If the length of arc is equal to the radius of the circle, then the angle subtended at the centre of the circle will be one radian. One radian is denoted by 1^c .

The ratio of the circumference of the circle to the diameter of the circle is denoted by a Greek letter π and it is a constant quantity.

$$\therefore \frac{\text{Circumference of circle}}{\text{Diameter of circle}} = \pi$$

This constant quantity π is an irrational quantity and generally its approximate value is $\frac{22}{7}$ and its

general value upto six places of decimal is 3.142857.

- When an angle is expressed in radians, the word radian is generally omitted.

e.g. $1^c = 1,22^\circ = 22'$

- We have,

$$\pi \text{ radian} = 180^\circ$$

$$\therefore 1 \text{ radian} = \frac{180^\circ}{\pi} = \left(\frac{180}{22} \times 7\right)^\circ = 57^\circ 16' 45'' (\text{approx})$$

- We have, $180^\circ = \pi \text{ radian}$

$$\Rightarrow 1^\circ = \frac{\pi}{180} \text{ radian}$$

$$= \left(\frac{22}{7 \times 180}\right) \text{ radian} = 0.01746 \text{ radian}$$

Relation among Three Systems of Measurement of an Angle

Let D be the number of degree, R be the number of radian and G be the number of grade in an angle θ .

Now, $90^\circ = 1 \text{ right angle}$

$$\Rightarrow 1^\circ = \frac{1}{90} \text{ right angle}$$

$$\Rightarrow D^\circ = \frac{D}{90} \text{ right angle}$$

$$\Rightarrow \theta = \frac{D}{90} \text{ right angle} \quad \dots(i)$$

Again, $\pi \text{ radian} = 2 \text{ right angles}$

$$\Rightarrow 1 \text{ radian} = \frac{2}{\pi} \text{ right angles}$$

$$\Rightarrow R \text{ radian} = \frac{2R}{\pi} \text{ right angles} \quad \dots(ii)$$

and $100 \text{ grades} = 1 \text{ right angle}$

$$\Rightarrow 1 \text{ grade} = \frac{1}{100} \text{ right angle}$$

$$\Rightarrow G \text{ grades} = \frac{G}{100} \text{ right angles}$$

$$\Rightarrow \theta = \frac{G}{100} \text{ right angles} \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\frac{D}{90} = \frac{G}{100} = \frac{2R}{\pi}$$

This is the required relation between the three systems of measurement of an angle.

➤ **Example 1.** If the three angles of a quadrilateral are 60° , 60^g and $\frac{5\pi}{6}$ radian. Then,

the fourth angle is

(a) 60°

(b) 96°

(c) 96^g

(d) None of the above

Sol. (b) First angle = 60°

$$\text{Second angle} = 60^\circ = 60 \times \frac{90}{100} = 54^\circ$$

$$\begin{aligned} \text{Third angle} &= \frac{5\pi}{6} \text{ radian} \\ &= \frac{5 \times 180^\circ}{6} = 150^\circ \end{aligned}$$

$$\begin{aligned} \text{Fourth angle} &= 360^\circ - (60^\circ + 54^\circ + 150^\circ) \\ &= 360^\circ - 264^\circ \\ &= 96^\circ \end{aligned}$$

➤ **Example 2.** The angle between the hour hand and the minute hand of a clock at half past three, is

- (a) 75°
(b) 60°
(c) 30°
(d) 175°

Sol. (a) The angle traced by the hour hand in 12 h is of 360° .

∴ The angle traced by the hour hand in 3 h 30 min.

$$\text{i.e. } \frac{7}{2} \text{ h} = \left(\frac{360}{12} \times \frac{7}{2} \right)^\circ = 105^\circ$$

The angle traced by the minute hand in 60 min is 360° .

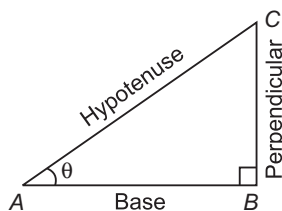
∴ The angle traced by the minute hand in 30 min

$$= \left(\frac{360}{60} \times 30 \right)^\circ = 180^\circ$$

Hence, the required angle between two hands
= $180^\circ - 105^\circ = 75^\circ$

Trigonometric Ratios

Let us take a right angled $\triangle ABC$ right angled at B .



Let $\angle CAB = \theta$, then

$$\sin \theta = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{BC}{AC}$$

$$\cos \theta = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC}$$

$$\tan \theta = \frac{\text{Perpendicular}}{\text{Base}} = \frac{BC}{AB}$$

$$\cot \theta = \frac{1}{\tan \theta} = \frac{\text{Base}}{\text{Perpendicular}} = \frac{AB}{BC}$$

$$\sec \theta = \frac{1}{\cos \theta} = \frac{\text{Hypotenuse}}{\text{Base}} = \frac{AC}{AB}$$

$$\operatorname{cosec} \theta = \frac{1}{\sin \theta} = \frac{\text{Hypotenuse}}{\text{Perpendicular}} = \frac{AC}{BC}$$

Except these six ratios, there are two other terms which are known as 'versed sine' and 'coverd sine', respectively.

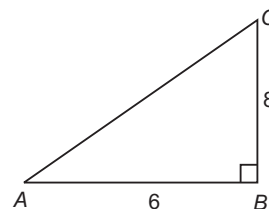
$$\begin{aligned} \text{i.e. } \quad \text{ver sin } \theta &= 1 - \cos \theta \\ \text{and } \quad \text{cover sin } \theta &= 1 - \sin \theta \end{aligned}$$

✎ Trigonometric ratios of any angle is always constant.

➤ **Example 3.** In a right angled $\triangle ABC$, if base is 6 and perpendicular height is 8, then the trigonometric ratios $\sin A$, $\cos A$ and $\tan A$, respectively are

- (a) $\frac{4}{5}, \frac{3}{5}, \frac{4}{3}$ (b) $\frac{2}{5}, \frac{3}{5}, \frac{4}{5}$
(c) $\frac{4}{5}, \frac{1}{5}, \frac{2}{3}$ (d) None of these

Sol. (a) By Pythagoras theorem,



$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ &= 6^2 + 8^2 = 36 + 64 = 100 \end{aligned}$$

$$\Rightarrow AC = 10$$

$$\therefore \sin A = \frac{\text{Perpendicular}}{\text{Hypotenuse}} = \frac{BC}{AC} = \frac{8}{10} = \frac{4}{5}$$

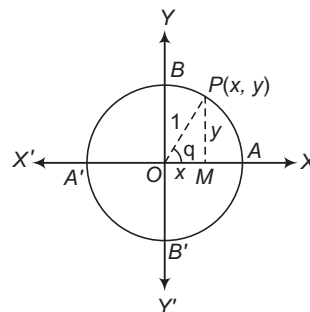
$$\cos A = \frac{\text{Base}}{\text{Hypotenuse}} = \frac{AB}{AC} = \frac{6}{10} = \frac{3}{5}$$

$$\tan A = \frac{\text{Perpendicular}}{\text{Base}} = \frac{BC}{AB} = \frac{8}{6} = \frac{4}{3}$$

Trigonometric Functions

Let $X'OX$ and YOY' be the coordinate axes.

Taking O as the centre and a unit radius, draw a circle, cutting the coordinate axes at A, B, A' and B' , as shown in the figure.



Let $\angle AOP = \theta$

$$\left[\because \angle AOP = \frac{\text{arc } AP}{\text{radius } OP} = \frac{\theta}{1} = \theta^c, \text{ using } \theta = \frac{l}{r} \right]$$

Now, the six trigonometric functions may be defined as under:

$$(i) \cos \theta = \frac{OM}{OP} = x$$

$$(ii) \sin \theta = \frac{PM}{OP} = y$$

$$(iii) \sec \theta = \frac{OP}{OM} = \frac{1}{x}, x \neq 0$$

$$(iv) \operatorname{cosec} \theta = \frac{OP}{PM} = \frac{1}{y}, y \neq 0$$

$$(v) \tan \theta = \frac{PM}{OM} = \frac{y}{x}, x \neq 0$$

$$(vi) \cot \theta = \frac{OM}{PM} = \frac{x}{y}, y \neq 0$$

Domain and Range of Trigonometrical Functions

The domain of a trigonometric ratio is the set of all values of angle θ for which it is meaningful and the range is the set of all values of the trigonometric ratio for different values of θ for which it is meaningful.

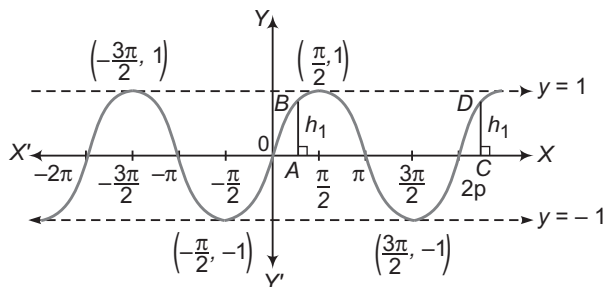
The following are domain and ranges of trigonometric ratios defined in the above section:

Trigonometric ratio	Domain	Range
$\sin \theta$	R	$[-1, 1]$
$\cos \theta$	R	$[-1, 1]$
$\tan \theta$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	R
$\operatorname{cosec} \theta$	$R - \{n\pi, n \in I\}$	$R - (-1, 1)$
$\sec \theta$	$R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$	$R - (-1, 1)$
$\cot \theta$	$R - \{n\pi, n \in I\}$	R

→ $|\sin \theta| \leq 1, |\cos \theta| \leq 1, |\sec \theta| \geq 1, |\operatorname{cosec} \theta| \geq 1$ for all values of θ for which these functions are defined.

Graph of Trigonometric Functions

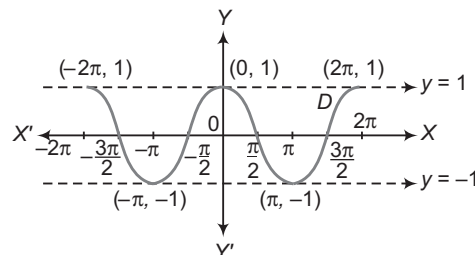
(i) Graph of $\sin x$



Facts related to $\sin x$

- (a) Period = 2π
- (b) Graph of $\sin x$ is continuous for all real values of x .

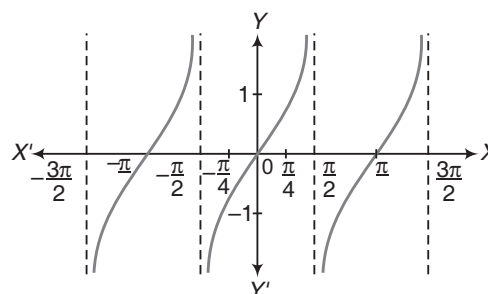
(ii) Graph of $\cos x$



Facts related to $\cos x$

- (a) Period = 2π
- (b) Graph of $\cos x$ is continuous for all real values of x .

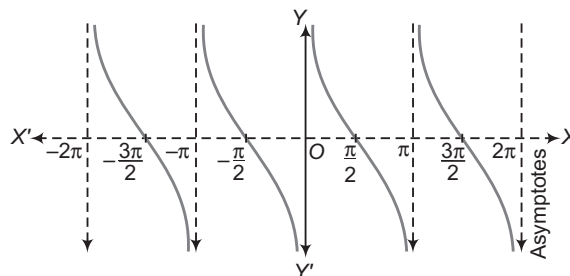
(iii) Graph of $\tan x$



Facts related to $\tan x$

- (a) Period = π
- (b) Graph of $\tan x$ is discontinuous at $x = \frac{m\pi}{2}$, where m is an odd integer.

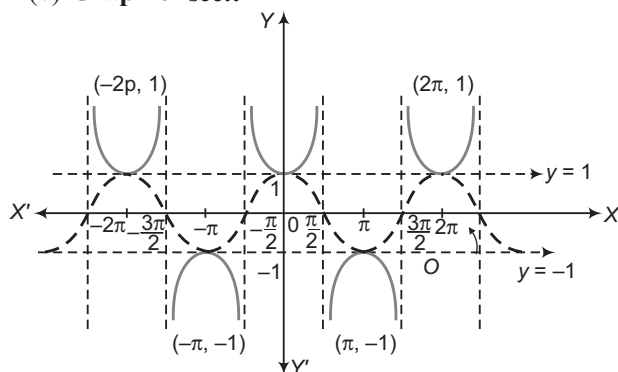
(iv) Graph of $\cot x$



Facts related to $\cot x$

- (a) Period = π
- (b) Graph of $\cot x$ is discontinuous at $x = m\pi$, where m is an integer.

(v) Graph of $\sec x$

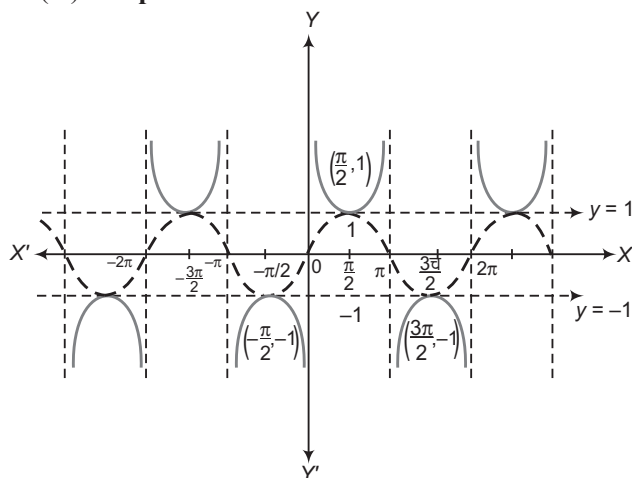


Facts related to $\sec x$

(a) Period = 2π

(b) Graph of $\sec x$ is discontinuous at $x = \frac{m\pi}{2}$, where m is an odd integer.

(vi) Graph of $\csc x$



Facts related to $\csc x$

(a) Period = 2π

(b) Graph of $\csc x$ is discontinuous at $x = m\pi$, where m is an integer.

Trigonometrical Identities

An equation involving trigonometric functions which is true for all those angles for which the functions are defined, is called a trigonometrical identity.

Following are some fundamental trigonometrical identities:

$$(i) \sin \theta = \frac{1}{\csc \theta} \text{ or } \csc \theta = \frac{1}{\sin \theta}$$

$$(ii) \cos \theta = \frac{1}{\sec \theta} \text{ or } \sec \theta = \frac{1}{\cos \theta}$$

$$(iii) \cot \theta = \frac{1}{\tan \theta} \text{ or } \tan \theta = \frac{1}{\cot \theta}$$

$$(iv) \tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$(v) \cot \theta = \frac{\cos \theta}{\sin \theta}$$

$$(vi) \cos^2 \theta + \sin^2 \theta = 1 \text{ or } 1 - \cos^2 \theta = \sin^2 \theta$$

$$\text{or } 1 - \sin^2 \theta = \cos^2 \theta$$

$$(vii) 1 + \tan^2 \theta = \sec^2 \theta$$

$$(viii) 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

➤ **Example 4.** $(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2$

is

(a) ≥ 9

(b) ≤ 9

(c) $= 9$

(d) None of the above

Sol. (a) $(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2$

$$= \sin^2 \theta + \operatorname{cosec}^2 \theta + 2 + \cos^2 \theta + \sec^2 \theta + 2$$

$$= 7 + \tan^2 \theta + \cot^2 \theta \quad \dots(i)$$

$$\text{Now, } \frac{\tan^2 \theta + \cot^2 \theta}{2} \geq \sqrt{\tan^2 \theta \cdot \cot^2 \theta} \quad [\because \text{AM} \geq \text{GM}]$$

$$\Rightarrow \tan^2 \theta + \cot^2 \theta \geq 2 \quad \dots(ii)$$

From Eq. (i), we get

$$(\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 \geq 7 + 2$$

$$\Rightarrow (\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2 \geq 9 \quad [\text{from Eq. (ii)}]$$

➤ **Example 5.** Prove that

$$\sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}} = \sec \theta + \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

$$\text{Sol. LHS} = \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta}}$$

$$= \sqrt{\frac{1 + \sin \theta}{1 - \sin \theta} \cdot \frac{1 + \sin \theta}{1 + \sin \theta}}$$

$$= \sqrt{\frac{(1 + \sin \theta)^2}{1 - \sin^2 \theta}}$$

$$= \sqrt{\frac{(1 + \sin \theta)^2}{\cos^2 \theta}}$$

$$= \frac{1 + \sin \theta}{\cos \theta}$$

$$= \frac{1}{\cos \theta} + \frac{\sin \theta}{\cos \theta}$$

$$= \sec \theta + \tan \theta = \text{RHS}$$

Hence proved.

Signs of Trigonometrical Ratios

We have introduced six trigonometric ratios. Signs of these ratios depend on the quadrant in which the terminal side of the angle lies. We always take the length $OP = r$ to be positive.

Thus, $\sin \theta = \frac{y}{r}$ has the sign of y , $\cos \theta = \frac{x}{r}$ has the sign of x . The sign of $\tan \theta$ depends on the signs of x and y and similarly the signs of other trigonometric ratios are decided by the signs of x and/or y . Thus, we have the following:

In first quadrant

We have, $x > 0$, $y > 0$

$$\therefore \sin \theta = \frac{y}{r} > 0,$$

$$\cos \theta = \frac{x}{r} > 0,$$

$$\tan \theta = \frac{y}{x} > 0,$$

$$\operatorname{cosec} \theta = \frac{r}{y} > 0,$$

$$\sec \theta = \frac{r}{x} > 0$$

$$\text{and} \quad \cot \theta = \frac{x}{y} > 0$$

Thus, in the first quadrant, all trigonometric functions are positive.

In second quadrant

We have, $x < 0$, $y > 0$

$$\therefore \sin \theta = \frac{y}{r} > 0,$$

$$\cos \theta = \frac{x}{r} < 0,$$

$$\tan \theta = \frac{y}{x} < 0,$$

$$\operatorname{cosec} \theta = \frac{r}{y} > 0,$$

$$\sec \theta = \frac{r}{x} < 0$$

$$\text{and} \quad \cot \theta = \frac{x}{y} < 0$$

Thus, in the second quadrant, sine and cosecant functions are positive and all others are negative.

In third quadrant

We have, $x < 0$, $y < 0$

$$\therefore \sin \theta = \frac{y}{r} < 0,$$

$$\cos \theta = \frac{x}{r} < 0,$$

$$\tan \theta = \frac{y}{x} > 0,$$

$$\operatorname{cosec} \theta = \frac{r}{y} < 0,$$

$$\sec \theta = \frac{r}{x} < 0$$

$$\text{and} \quad \cot \theta = \frac{x}{y} > 0$$

Thus, in the third quadrant, all trigonometric functions are negative except tangent and cotangent.

In fourth quadrant

We have, $x > 0$, $y < 0$

$$\therefore \sin \theta = \frac{y}{r} < 0,$$

$$\cos \theta = \frac{x}{r} > 0,$$

$$\tan \theta = \frac{y}{x} < 0,$$

$$\operatorname{cosec} \theta = \frac{r}{y} < 0,$$

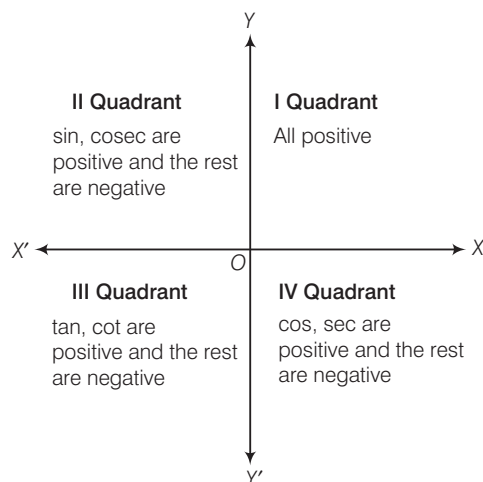
$$\sec \theta = \frac{r}{x} > 0$$

$$\text{and} \quad \cot \theta = \frac{x}{y} < 0$$

Thus, in the fourth quadrant, all trigonometric functions are negative except cosine and secant.

AID to Memory

The signs of trigonometrical ratios in different quadrants is the four word phrases 'AFTER SCHOOL TO COLLEGE'. The first letter of the first word in this phrase is 'A'. This may be taken to indicate that all trigonometric ratios are positive in the first quadrant. The first letter of the second word is 'S'. This indicates that sine and its reciprocal are positive in the second quadrant. The first letter of third word is 'T'. This may be taken as to indicate that tangent and its reciprocal are positive in the third quadrant. The first letter of the fourth word in the phrase is 'C' which may be taken as to indicate that only cosine and its reciprocal are positive in the fourth quadrant.



Variation of Values of Trigonometrical Ratios in Different Quadrants

		Y			
		II		I	
X'	sine	decreases from 1 to 0		sine	increases from 0 to 1
	cosine	decreases from 0 to -1		cosine	decreases from 1 to 0
	tangent	increases from $-\infty$ to 0		tangent	increases from 0 to ∞
	cotangent	decreases from 0 to $-\infty$		cotangent	decreases from ∞ to 0
	secant	increases from $-\infty$ to -1		secant	increases from 1 to ∞
		cosecant	increases from 1 to ∞	cosecant	decreases from ∞ to 1
			O		
		III		IV	
X	sine	decreases from 0 to -1		sine	increases from -1 to 0
	cosine	increases from -1 to 0		cosine	increases from 0 to 1
	tangent	increases from 0 to ∞		tangent	increases from $-\infty$ to 0
	cotangent	decreases from ∞ to 0		cotangent	decreases from 0 to $-\infty$
	secant	decreases from -1 to $-\infty$		secant	decreases from ∞ to 1
		cosecant	increases from $-\infty$ to -1	cosecant	decreases from -1 to $-\infty$
			Y'		

Trigonometric Ratios of Allied Angles

Two angles are said to be allied when their sum or difference is either zero or a multiple of 90° .

The angles $-\theta, 90^\circ \pm \theta, 180^\circ \pm \theta, 360^\circ \pm \theta$, etc., are angles allied to the angle θ , if θ is measured in degree. However, if θ is measured in radian, then the angles allied to θ are $-\theta, \frac{\pi}{2} \pm \theta, \pi \pm \theta, 2\pi \pm \theta$, etc.

Using trigonometric ratios of allied angles, we can find the trigonometric ratios of angles of any magnitude.

Following are the trigonometrical ratios of allied angles:

$$\begin{aligned} \sin(-\theta) &= -\sin \theta & \cos(-\theta) &= \cos \theta \\ \tan(-\theta) &= -\tan \theta & \cot(-\theta) &= -\cot \theta \\ \operatorname{cosec}(-\theta) &= -\operatorname{cosec} \theta & \sec(-\theta) &= \sec \theta \end{aligned}$$

$$\sin\left(\frac{\pi}{2} - \theta\right) = \cos \theta \quad \cos\left(\frac{\pi}{2} - \theta\right) = \sin \theta$$

$$\tan\left(\frac{\pi}{2} - \theta\right) = \cot \theta \quad \cot\left(\frac{\pi}{2} - \theta\right) = \tan \theta$$

$$\sec\left(\frac{\pi}{2} - \theta\right) = \operatorname{cosec} \theta \quad \operatorname{cosec}\left(\frac{\pi}{2} - \theta\right) = \sec \theta$$

$$\sin\left(\frac{\pi}{2} + \theta\right) = \cos \theta \quad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin \theta$$

$$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot \theta \quad \cot\left(\frac{\pi}{2} + \theta\right) = -\tan \theta$$

$$\sec\left(\frac{\pi}{2} + \theta\right) = -\operatorname{cosec} \theta \quad \operatorname{cosec}\left(\frac{\pi}{2} + \theta\right) = \sec \theta$$

$$\sin(\pi - \theta) = \sin \theta \quad \cos(\pi - \theta) = -\cos \theta$$

$$\tan(\pi - \theta) = -\tan \theta \quad \cot(\pi - \theta) = -\cot \theta$$

$$\sec(\pi - \theta) = -\sec \theta \quad \operatorname{cosec}(\pi - \theta) = \operatorname{cosec} \theta$$

$$\begin{aligned}\sin(\pi + \theta) &= -\sin \theta & \cos(\pi + \theta) &= -\cos \theta \\ \tan(\pi + \theta) &= \tan \theta & \cot(\pi + \theta) &= \cot \theta \\ \sec(\pi + \theta) &= -\sec \theta & \operatorname{cosec}(\pi + \theta) &= -\operatorname{cosec} \theta \\ \sin\left(\frac{3\pi}{2} - \theta\right) &= -\cos \theta & \cos\left(\frac{3\pi}{2} - \theta\right) &= -\sin \theta \\ \tan\left(\frac{3\pi}{2} - \theta\right) &= \cot \theta & \cot\left(\frac{3\pi}{2} - \theta\right) &= \tan \theta \\ \sec\left(\frac{3\pi}{2} - \theta\right) &= -\operatorname{cosec} \theta & \operatorname{cosec}\left(\frac{3\pi}{2} - \theta\right) &= -\sec \theta \\ \sin\left(\frac{3\pi}{2} + \theta\right) &= -\cos \theta & \cos\left(\frac{3\pi}{2} + \theta\right) &= \sin \theta \\ \tan\left(\frac{3\pi}{2} + \theta\right) &= -\cot \theta & \cot\left(\frac{3\pi}{2} + \theta\right) &= -\tan \theta \\ \sec\left(\frac{3\pi}{2} + \theta\right) &= \operatorname{cosec} \theta & \operatorname{cosec}\left(\frac{3\pi}{2} + \theta\right) &= -\sec \theta \\ \sin(2\pi - \theta) &= -\sin \theta & \cos(2\pi - \theta) &= \cos \theta \\ \tan(2\pi - \theta) &= -\tan \theta & \cot(2\pi - \theta) &= -\cot \theta \\ \sec(2\pi - \theta) &= \sec \theta & \operatorname{cosec}(2\pi - \theta) &= -\operatorname{cosec} \theta \\ \sin(2\pi + \theta) &= \sin \theta & \cos(2\pi + \theta) &= \cos \theta \\ \tan(2\pi + \theta) &= \tan \theta & \cot(2\pi + \theta) &= \cot \theta \\ \sec(2\pi + \theta) &= \sec \theta & \operatorname{cosec}(2\pi + \theta) &= \operatorname{cosec} \theta\end{aligned}$$

- $\sin\{n\pi + (-1)^n \theta\} = \sin \theta, n \in I$
- $\cos(2n\pi \pm \theta) = \cos \theta, n \in I$
- $\tan(n\pi + \theta) = \tan \theta, n \in I$

➤ **Example 6.** The value of $\operatorname{cosec}(-1410^\circ)$ is
(a) 3 (b) 2 (c) -3 (d) -1

Sol. (b) $\operatorname{cosec}(-1410^\circ)$
 $= -\operatorname{cosec}(1410^\circ)$ [$\because \operatorname{cosec}(-\theta) = -\operatorname{cosec} \theta$]
 $= -\operatorname{cosec}(360 \times 4 - 30^\circ)$
 $= -(-\operatorname{cosec} 30^\circ)$ [$\because \operatorname{cosec}(2n\pi - \theta) = -\operatorname{cosec} \theta$]
 $= \operatorname{cosec} 30^\circ = 2$

➤ **Example 7.** The value of $\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ$ is
(a) 0 (b) 1 (c) 2 (d) 3

Sol. (b) $\tan 1^\circ \tan 2^\circ \tan 3^\circ \dots \tan 89^\circ$
 $= (\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ)$
 $\dots (\tan 44^\circ \tan 46^\circ) \tan 45^\circ$
 $= (\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ) \dots \tan 45^\circ$
 $= 1 \cdot 1 \dots 1 = 1$

➤ **Example 8.** The value of

$$2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3} \text{ is}$$

(a) 10 (b) -10 (c) 12 (d) 14

Sol. (a) $2 \sin^2 \frac{3\pi}{4} + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3}$
 $= 2 \sin^2 \left(\pi - \frac{\pi}{4}\right) + 2 \cos^2 \frac{\pi}{4} + 2 \sec^2 \frac{\pi}{3}$
 $= 2 \sin^2 \frac{\pi}{4} + 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 + 2(2)^2$ [$\because \sin(\pi - \theta) = \sin \theta$]
 $= 2 \times \left(\frac{1}{\sqrt{2}}\right)^2 + 2 \times \frac{1}{2} + 2 \times 4 = 1 + 1 + 8 = 10$

Work Book Exercise 9.1

- The value of $\cos 10^\circ - \sin 10^\circ$ is
 a positive b negative
 c 0 d 1
- A circular wire of radius 7 cm is cut and bend again into an arc of a circle of radius 12 cm. The angle subtended by the arc at the centre is
 a 50° b 210°
 c 100° d 60°
- Which of the following is correct?
 a $\sin 1^\circ > \sin 1$ b $\sin 1^\circ < \sin 1$
 c $\sin 1^\circ = \sin 1$ d $\sin 1^\circ = \frac{\pi}{180} \sin 1$
- $\frac{\sec 8A - 1}{\sec 4A - 1}$ is equal to
 a $\frac{\tan 8A}{\tan 2A}$
 b $\frac{\cot 8A}{\cot 2A}$
 c $\frac{\tan 8A}{\tan 2A}$
 d None of the above
- In a right angled triangle, if the hypotenuse is four times as long as the perpendicular drawn to it from the opposite vertex. One of the acute angle is
 a 15° b 30°
 c 45° d None of these
- If the interior angles of a polygon are in AP with common difference 5° and the smallest angle 120° , then the number of sides of the polygon is
 a 9 or 16 b 9
 c 13 d 16
- The value of x for which $f(x) = \sqrt{\sin x - \cos x}$ defined, is
 a $\left(0, \frac{\pi}{4}\right)$ b $\left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$
 c $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ d None of these
- If A, B, C, D are angles of a cyclic quadrilateral, then $\cos A + \cos B + \cos C + \cos D$ equals
 a 0 b 1
 c -1 d None of these

- 10** If $1 + \sin x + \sin^2 x + \dots = 4 + 2\sqrt{3}$, $0 < x < \pi$, then x is equal to
- a $\frac{\pi}{6}$ b $\frac{\pi}{4}$
- c $\frac{\pi}{3}$ or $\frac{\pi}{6}$ d $\frac{\pi}{3}$ or $\frac{2\pi}{3}$

- 12** If $(\sec A - \tan A)(\sec B - \tan B)(\sec C - \tan C) = (\sec A + \tan A)(\sec B + \tan B)(\sec C + \tan C)$, then each side is equal to
- a 0 b 1
c -1 d ±1

426

$$(ix) \sin(A+B+C) = \sin A \cos B \cos C + \cos A \sin B \cos C + \cos A \cos B \sin C - \sin A \sin B \sin C$$

$$\text{Also, } \sin(A+B+C) = \cos A \cos B \cos C (\tan A + \tan B + \tan C - \tan A \tan B \tan C)$$

$$(x) \cos(A+B+C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \sin A \cos B \sin C - \cos A \sin B \sin C$$

$$\text{Also, } \cos(A+B+C) = \cos A \cos B \cos C (1 - \tan A \tan B - \tan B \tan C - \tan C \tan A)$$

$$(xi) \tan(A+B+C) = \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}$$

$$(xii) \sin(A_1 + A_2 + \dots + A_n) = \cos A_1 \cos A_2 \dots \cos A_n (S_1 - S_3 + S_5 - S_7 + \dots)$$

$$(xiii) \cos(A_1 + A_2 + \dots + A_n) = \cos A_1 \cos A_2 \dots \cos A_n (1 - S_2 + S_4 - S_6 + \dots)$$

$$(xiv) \tan(A_1 + A_2 + \dots + A_n) = \frac{S_1 - S_3 + S_5 - S_7 + \dots}{1 - S_2 + S_4 - S_6 + \dots}$$

where, $S_1 = \tan A_1 + \tan A_2 + \dots + \tan A_n$
= sum of the tangents of the separate angles

$S_2 = A_1 \tan A_2 + \tan A_1 \tan A_3 + \dots$
= sum of the tangents taken two at a time

$$S_3 = \tan A_1 \tan A_2 \tan A_3$$

+ $\tan A_2 \tan A_3 \tan A_4 + \dots$
= sum of the tangents taken three at a time and so on.

If $A_1 = A_2 = \dots = A_n = A$, we have

$$S_1 = n \tan A, S_2 = {}^nC_2 \tan^2 A,$$

$$S_3 = {}^nC_3 \tan^3 A, \dots$$

➤ **Example 9.** The value of $\cos(45^\circ - A) \cos(45^\circ - B) - \sin(45^\circ - A) \sin(45^\circ - B)$ is

$$(a) \sin(A+B)$$

$$(b) \cos(A+B)$$

$$(c) \sin(A-B)$$

$$(d) \cos(A-B)$$

Sol. (a) We have,
 $\cos(45^\circ - A) \cos(45^\circ - B) - \sin(45^\circ - A) \sin(45^\circ - B)$
= $\cos\{(45^\circ - A) + (45^\circ - B)\}$
= $\cos\{90^\circ - (A+B)\}$
= $\sin(A+B)$

➤ **Example 10.** If $\cos \theta + \sin \phi = m$ and $\sin \theta + \cos \phi = n$, then $\frac{1}{2}(m^2 + n^2 - 2)$ is equal to

$$(a) \sin(\theta - \phi)$$

$$(b) \sin(\theta + \phi)$$

$$(c) \cos(\theta - \phi)$$

$$(d) \cos(\theta + \phi)$$

Sol. (b) Given, $m = \cos \theta + \sin \phi$... (i)
and $n = \sin \theta + \cos \phi$... (ii)

On squaring and adding Eqs. (i) and (ii), we get

$$m^2 + n^2 = \cos^2 \theta + \sin^2 \phi + 2 \sin \phi \cos \theta + \sin^2 \theta + \cos^2 \phi + 2 \sin \theta \cos \phi$$

$$\Rightarrow m^2 + n^2 = 2 + 2(\sin \theta \cos \phi + \cos \theta \sin \phi)$$

$$\Rightarrow \sin(\theta + \phi) = \frac{m^2 + n^2 - 2}{2}$$

➤ **Example 11.** If $\tan A - \tan B = x$ and $\cot B - \cot A = y$, then $\cot(A-B)$ is equal to

$$(a) \frac{1}{y} - \frac{1}{x}$$

$$(b) \frac{1}{x} - \frac{1}{y}$$

$$(c) \frac{1}{x} + \frac{1}{y}$$

$$(d) \text{None of these}$$

Sol. (c) We have, $\tan A - \tan B = x$ and $\cot B - \cot A = y$

Now, $\cot B - \cot A = y$

$$\Rightarrow \frac{\tan A - \tan B}{\tan A \tan B} = y \Rightarrow \frac{x}{\tan A \tan B} = y$$

$$\Rightarrow \tan A \tan B = \frac{x}{y}$$

$$\text{Now, } \cot(A-B) = \frac{1}{\tan(A-B)} = \frac{1 + \tan A \tan B}{\tan A - \tan B}$$

$$= \frac{1 + \frac{x}{y}}{\frac{x}{y}} = \frac{y + x}{xy} = \frac{1}{x} + \frac{1}{y}$$

➤ **Example 12.** If $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$, then prove that $1 + \cot \alpha \tan \beta = 0$.

Sol. Given, $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$

$$\Rightarrow \cos \alpha \cos \beta - \sin \alpha \sin \beta = 1$$

$$\Rightarrow \cos(\alpha + \beta) = 1$$

$$\begin{aligned} \text{Now, } 1 + \cot \alpha \tan \beta &= 1 + \frac{\cos \alpha}{\sin \alpha} \times \frac{\sin \beta}{\cos \beta} \\ &= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \cos \beta} \\ &= \frac{\sin(\alpha + \beta)}{\sin \alpha \cos \beta} = \frac{0}{\sin \alpha \cos \beta} \\ &= 0 \end{aligned}$$

Hence proved.

Transformation Formulae

In the previous section, we have learnt about the following formulae:

$$\sin A \cos B + \cos A \sin B = \sin(A+B) \quad \dots(i)$$

$$\sin A \cos B - \cos A \sin B = \sin(A-B) \quad \dots(ii)$$

$$\cos A \cos B - \sin A \sin B = \cos(A+B) \quad \dots(iii)$$

$$\cos A \cos B + \sin A \sin B = \cos (A - B) \quad \dots (iv)$$

On adding Eqs. (i) and (ii), we obtain

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

On adding Eqs. (iii) and (iv), we get

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

On subtracting Eq. (iii) from Eq. (iv), we get

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

Thus, we obtain the following formulae:

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$$

$$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$$

$$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$$

$$2 \sin A \sin B = \cos (A - B) - \cos (A + B)$$

These four formulae convert the product of two sines or two cosines or one sine and one cosine into the sum or difference of two sines or two cosines.

In the above formulae, if we substitute $A + B = C$ and $A - B = D$, we obtain the following formulae to convert the sum into the product.

$$\sin C + \sin D = 2 \sin \left(\frac{C + D}{2} \right) \cos \left(\frac{C - D}{2} \right)$$

$$\sin C - \sin D = 2 \sin \left(\frac{C - D}{2} \right) \cos \left(\frac{C + D}{2} \right)$$

$$\cos C + \cos D = 2 \cos \left(\frac{C + D}{2} \right) \cos \left(\frac{C - D}{2} \right)$$

$$\cos C - \cos D = -2 \sin \left(\frac{C + D}{2} \right) \sin \left(\frac{C - D}{2} \right)$$

or

$$\cos C - \cos D = 2 \sin \left(\frac{C + D}{2} \right) \sin \left(\frac{D - C}{2} \right)$$

➤ **Example 13.** The value of $2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13}$ is

(a) 2

(b) 0

(c) 1

(d) 3

Sol. (b) We have,

$$\begin{aligned} & 2 \cos \frac{\pi}{13} \cos \frac{9\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= \cos \left(\frac{9\pi}{13} + \frac{\pi}{13} \right) + \cos \left(\frac{9\pi}{13} - \frac{\pi}{13} \right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= \cos \frac{10\pi}{13} + \cos \frac{8\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= \cos \left(\pi - \frac{3\pi}{13} \right) + \cos \left(\pi - \frac{5\pi}{13} \right) + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= -\cos \frac{3\pi}{13} - \cos \frac{5\pi}{13} + \cos \frac{3\pi}{13} + \cos \frac{5\pi}{13} \\ &= 0 \end{aligned}$$

➤ **Example 14.** Prove that $\cos 18^\circ - \sin 18^\circ = \sqrt{2} \sin 27^\circ$.

Sol. Consider,

$$\begin{aligned} \cos 18^\circ - \sin 18^\circ &= \cos 18^\circ - \sin (90^\circ - 72^\circ) \\ &= \cos 18^\circ - \cos 72^\circ \\ &= 2 \sin \left(\frac{18^\circ + 72^\circ}{2} \right) \sin \left(\frac{72^\circ - 18^\circ}{2} \right) \\ &= 2 \sin 45^\circ \sin 27^\circ \\ &= 2 \cdot \frac{1}{\sqrt{2}} \cdot \sin 27^\circ \\ &= \sqrt{2} \sin 27^\circ \end{aligned}$$

Hence proved.

➤ **Example 15.** If $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2} \right)$, then

$$\sin \alpha + \sin \beta + \sin \gamma$$

$$(a) > \sin (\alpha + \beta + \gamma) \quad (b) < \sin (\alpha + \beta + \gamma)$$

$$(c) = \sin (\alpha + \beta + \gamma) \quad (d) \leq \sin (\alpha + \beta + \gamma)$$

Sol. (a) We have,

$$\begin{aligned} & \sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) \\ &= 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} - 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha + \beta + 2\gamma}{2} \\ &= 2 \sin \frac{\alpha + \beta}{2} \left\{ \cos \frac{\alpha - \beta}{2} - \cos \frac{\alpha + \beta + 2\gamma}{2} \right\} \\ &= 2 \sin \frac{\alpha + \beta}{2} \left\{ 2 \sin \frac{\alpha + \gamma}{2} \sin \frac{\beta + \gamma}{2} \right\} \\ &= 4 \sin \frac{\alpha + \beta}{2} \sin \frac{\beta + \gamma}{2} \sin \frac{\gamma + \alpha}{2} \end{aligned}$$

We have, $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2} \right)$

$$\Rightarrow \frac{\alpha + \beta}{2}, \frac{\beta + \gamma}{2}, \frac{\gamma + \alpha}{2} \in \left(0, \frac{\pi}{2} \right)$$

$$\Rightarrow \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\beta + \gamma}{2} \right) \sin \left(\frac{\gamma + \alpha}{2} \right) > 0$$

$$\Rightarrow 4 \sin \left(\frac{\alpha + \beta}{2} \right) \sin \left(\frac{\beta + \gamma}{2} \right) \sin \left(\frac{\gamma + \alpha}{2} \right) > 0$$

$$\Rightarrow \sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma) > 0$$

$$\therefore \sin \alpha + \sin \beta + \sin \gamma > \sin (\alpha + \beta + \gamma)$$

Trigonometric Ratios of Multiples of an Angle

$$(i) \sin 2\theta = 2 \sin \theta \cos \theta = \frac{2 \tan \theta}{1 + \tan^2 \theta}$$

$$(ii) \cos 2\theta = \cos^2 \theta - \sin^2 \theta = 1 - 2 \sin^2 \theta$$

$$= 2 \cos^2 \theta - 1 = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta}$$

$$(iii) \cos^2 \theta = \frac{1}{2} (1 + \cos 2\theta), \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta)$$

$$(iv) \tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

$$(v) \cot 2\theta = \frac{\cot^2 \theta - 1}{2 \cot \theta}$$

$$\begin{aligned} \text{(vi)} \quad \sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\ \text{(vii)} \quad \cos 3\theta &= 4\cos^3\theta - 3\cos\theta \\ \text{(viii)} \quad \tan 3\theta &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \\ \text{(ix)} \quad \cot 3\theta &= \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1} = \frac{3\cot\theta - \cot^3\theta}{1 - 3\cot^2\theta} \end{aligned}$$

Trigonometric Ratios of Submultiple Angles

$$\begin{aligned} \text{(i)} \quad \sin \theta &= 2\sin \frac{1}{2}\theta \cos \frac{1}{2}\theta = \frac{2\tan \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta} \\ \text{(ii)} \quad \cos \theta &= \cos^2 \frac{1}{2}\theta - \sin^2 \frac{1}{2}\theta = 1 - 2\sin^2 \frac{1}{2}\theta \\ &= 2\cos^2 \frac{1}{2}\theta - 1 \\ &= \frac{1 - \tan^2 \frac{1}{2}\theta}{1 + \tan^2 \frac{1}{2}\theta} \\ \text{(iii)} \quad \tan \theta &= \frac{2\tan \frac{1}{2}\theta}{1 - \tan^2 \frac{1}{2}\theta} \\ \text{(iv)} \quad \cot \theta &= \frac{\cot^2 \frac{1}{2}\theta - 1}{2\cot \frac{1}{2}\theta} \\ \text{(v)} \quad \sin A &= 3\sin \left(\frac{A}{3}\right) - 4\sin^3 \left(\frac{A}{3}\right) \\ \text{(vi)} \quad \cos A &= 4\cos^3 \left(\frac{A}{3}\right) - 3\cos \left(\frac{A}{3}\right) \\ \text{(vii)} \quad \tan A &= \frac{3\tan \left(\frac{A}{3}\right) - \tan^3 \left(\frac{A}{3}\right)}{1 - 3\tan^2 \left(\frac{A}{3}\right)} \end{aligned}$$

$$\bullet \sin(\alpha) + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \dots + \sin[\alpha + (n-1)\beta]$$

$$= \frac{\sin \left\{ \alpha + (n-1) \left(\frac{\beta}{2} \right) \right\} \sin \left(\frac{n\beta}{2} \right)}{\sin \left(\frac{\beta}{2} \right)}$$

$$\bullet \cos(\alpha) + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \dots + \cos[\alpha + (n-1)\beta]$$

$$= \frac{\cos \left\{ \alpha + (n-1) \left(\frac{\beta}{2} \right) \right\} \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}$$

➤ **Example 16.** Prove that

$$\text{(i)} \quad \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \cot \theta.$$

$$\text{(ii)} \quad \frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \tan \frac{\theta}{2}.$$

$$\text{(iii)} \quad \frac{\cos \theta}{1 + \sin \theta} = \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right).$$

$$\text{Sol. (i) LHS} = \frac{1 + \sin 2\theta + \cos 2\theta}{1 + \sin 2\theta - \cos 2\theta} = \frac{(1 + \cos 2\theta) + \sin 2\theta}{(1 - \cos 2\theta) + \sin 2\theta}$$

$$= \frac{2\cos^2 \theta + 2\sin \theta \cos \theta}{2\sin^2 \theta + 2\sin \theta \cos \theta} = \frac{2\cos \theta (\cos \theta + \sin \theta)}{2\sin \theta (\cos \theta + \sin \theta)}$$

$$= \cot \theta = \text{RHS}$$

Hence proved.

$$\text{(ii) LHS} = \frac{1 + \sin \theta - \cos \theta}{1 + \sin \theta + \cos \theta} = \frac{(1 - \cos \theta) + \sin \theta}{(1 + \cos \theta) + \sin \theta}$$

$$= \frac{2\sin^2 \frac{\theta}{2} + 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2} + 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}$$

$$= \frac{2\sin \frac{\theta}{2} \left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)}{2\cos \frac{\theta}{2} \left(\sin \frac{\theta}{2} + \cos \frac{\theta}{2} \right)} = \tan \frac{\theta}{2} = \text{RHS}$$

Hence proved.

$$\text{(iii) LHS} = \frac{\cos \theta}{1 + \sin \theta}$$

$$= \frac{\sin \left(\frac{\pi}{2} - \theta \right)}{1 + \cos \left(\frac{\pi}{2} - \theta \right)}$$

$$= \frac{2\sin \left(\frac{\pi}{4} - \frac{\theta}{2} \right) \cos \left(\frac{\pi}{4} - \frac{\theta}{2} \right)}{2\cos^2 \left(\frac{\pi}{4} - \frac{\theta}{2} \right)}$$

$$= \tan \left(\frac{\pi}{4} - \frac{\theta}{2} \right) = \text{RHS}$$

Hence proved.

➤ **Example 17.** The value of

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8\theta}}}, \text{ where } 0 < \theta < \frac{\pi}{8}, \text{ is}$$

$$\text{(a) } 2\cos \theta$$

$$\text{(b) } \cos \theta$$

$$\text{(c) } 2\sin \theta$$

$$\text{(d) } -2\cos \theta$$

Sol. (a) We have,

$$\sqrt{2 + \sqrt{2 + \sqrt{2 + 2\cos 8\theta}}}$$

$$= \sqrt{2 + \sqrt{2 + \sqrt{4\cos^2 4\theta}}} \quad \left[\because 1 + \cos 8\theta = 2\cos^2 \frac{8\theta}{2} \right]$$

$$= \sqrt{2 + \sqrt{2 + 2\cos 4\theta}} = \sqrt{2 + \sqrt{2(1 + \cos 4\theta)}}$$

$$= \sqrt{2 + \sqrt{2(2\cos^2 2\theta)}} \quad \left[\because 1 + \cos 4\theta = 2\cos^2 2\theta \right]$$

$$= \sqrt{2 + 2\cos 2\theta} = \sqrt{2(1 + \cos 2\theta)}$$

$$= \sqrt{2(2\cos^2 \theta)} = 2\cos \theta$$

➤ **Example 18.** The value of $\frac{\sec 8\theta - 1}{\sec 4\theta - 1}$ is

- (a) $\frac{\tan 2\theta}{\tan 8\theta}$ (b) $\frac{\tan 8\theta}{\tan 4\theta}$ (c) $\frac{\tan 8\theta}{\tan 2\theta}$ (d) $\frac{\tan 6\theta}{\tan 2\theta}$

Sol. (c) We have, $\frac{\sec 8\theta - 1}{\sec 4\theta - 1}$

$$\begin{aligned} &= \frac{\frac{1}{\cos 8\theta} - 1}{\frac{1}{\cos 4\theta} - 1} \\ &= \frac{1 - \cos 8\theta}{\cos 8\theta} \cdot \frac{\cos 4\theta}{1 - \cos 4\theta} \\ &= \frac{2\sin^2 4\theta}{\cos 8\theta} \cdot \frac{\cos 4\theta}{1 - \cos 4\theta} \\ &= \frac{(2\sin 4\theta \cos 4\theta)}{\cos 8\theta} \times \frac{\sin 4\theta}{2\sin^2 2\theta} \\ &= \left(\frac{2\sin 4\theta \cos 4\theta}{\cos 8\theta} \right) \times \left(\frac{2\sin 2\theta \cos 2\theta}{2\sin^2 2\theta} \right) \\ &= \left(\frac{\sin 2(4\theta)}{\cos 8\theta} \right) \times \left(\frac{\cos 2\theta}{\sin 2\theta} \right) = \left(\frac{\sin 8\theta}{\cos 8\theta} \right) \times \left(\frac{\cos 2\theta}{\sin 2\theta} \right) \\ &= \tan 8\theta \cdot \cot 2\theta = \frac{\tan 8\theta}{\tan 2\theta} \end{aligned}$$

➤ **Example 19.** If $\tan \alpha = \frac{p}{q}$, where $\alpha = 6\beta$ and α

being an acute angle, then $\frac{1}{2} [p \operatorname{cosec} 2\beta - q \sec 2\beta]$

is equal to

- (a) $\sqrt{p^2 + q^2}$ (b) $\sqrt{p^2 - q^2}$
(c) $\sqrt{2p^2 + q^2}$ (d) None of these

Sol. (a) We have, $\tan \alpha = \frac{p}{q}$

$$\therefore \sin \alpha = \frac{p}{\sqrt{p^2 + q^2}} \text{ and } \cos \alpha = \frac{q}{\sqrt{p^2 + q^2}}$$

$$\text{Now, } \frac{1}{2} [p \operatorname{cosec} 2\beta - q \sec 2\beta]$$

$$\begin{aligned} &= \frac{\sqrt{p^2 + q^2}}{2} \left[\frac{p}{\sqrt{p^2 + q^2}} \operatorname{cosec} 2\beta - \frac{q}{\sqrt{p^2 + q^2}} \sec 2\beta \right] \\ &= \frac{\sqrt{p^2 + q^2}}{2} \left[\frac{\sin \alpha}{\sin 2\beta} - \frac{\cos \alpha}{\cos 2\beta} \right] \\ &= \sqrt{p^2 + q^2} \left[\frac{\sin(\alpha - 2\beta)}{2\sin 2\beta \cos 2\beta} \right] \\ &= \sqrt{p^2 + q^2} \left[\frac{\sin(6\beta - 2\beta)}{\sin 4\beta} \right] \quad [\because \alpha = 6\beta] \\ &= \sqrt{p^2 + q^2} \end{aligned}$$

➤ **Example 20.** If $\tan^2 \theta = 2 \tan^2 \phi + 1$, then

$\cos 2\theta + \sin^2 \phi$ is equal to

- (a) 1 (b) 2
(c) -1 (d) None of these

Sol. (d) We have, $\cos 2\theta$

$$\begin{aligned} &= \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \\ &= \frac{1 - (2 \tan^2 \phi + 1)}{1 + 2 \tan^2 \phi + 1} \quad [\because \tan^2 \theta = 2 \tan^2 \phi + 1] \\ &= \frac{-2 \tan^2 \phi}{2 + 2 \tan^2 \phi} = \frac{-\tan^2 \phi}{\sec^2 \phi} = -\sin^2 \phi \end{aligned}$$

$$\therefore \cos 2\theta + \sin^2 \phi = 0$$

➤ **Example 21.** The smallest positive value of x (in degree) for which

$$\tan(x + 100^\circ) = \tan(x + 50^\circ) \tan x \tan(x - 50^\circ), \text{ is}$$

- (a) 60° (b) 90°
(c) 45° (d) 30°

Sol. (d) We have,

$$\begin{aligned} \tan(x + 100^\circ) &= \tan(x + 50^\circ) \tan x \tan(x - 50^\circ) \\ \Rightarrow \frac{\tan(x + 100^\circ)}{\tan(x - 50^\circ)} &= \tan(x + 50^\circ) \tan x \\ \Rightarrow \frac{\sin(x + 100^\circ) \cos(x - 50^\circ)}{\cos(x + 100^\circ) \sin(x - 50^\circ)} &= \frac{\sin(x + 50^\circ) \sin x}{\cos(x + 50^\circ) \cos x} \\ \Rightarrow \frac{\sin(2x + 50^\circ)}{\sin 150^\circ} &= \frac{\cos 50^\circ}{-\cos(2x + 50^\circ)} \\ &\quad [\text{applying componendo and dividendo}] \\ \Rightarrow \sin(2x + 50^\circ) \cos(2x + 50^\circ) &= -\sin 150^\circ \cos 50^\circ \\ \Rightarrow 2 \sin(2x + 50^\circ) \cos(2x + 50^\circ) &= -\cos 50^\circ \\ \Rightarrow \sin(4x + 100^\circ) &= \sin(270^\circ - 50^\circ) \\ \Rightarrow \sin(4x + 100^\circ) &= \sin 220^\circ \\ \Rightarrow 4x + 100^\circ &= 220^\circ \Rightarrow x = 30^\circ \end{aligned}$$

➤ **Example 22.** The value of

$$\sum_{n=1}^{\infty} \frac{\tan\left(\frac{\theta}{2^n}\right)}{2^{n-1} \cos\left(\frac{\theta}{2^{n-1}}\right)} \text{ is}$$

- (a) $\frac{2}{\sin 2\theta} - \frac{1}{\theta}$ (b) $\frac{2}{\sin 2\theta} + \frac{1}{\theta}$
(c) $\frac{1}{\sin 2\theta} - \frac{1}{\theta}$ (d) $\frac{2}{\sin \theta} - \frac{1}{\theta}$

Sol. (a) We have,

$$\begin{aligned} &\sum_{r=1}^n \frac{\tan\left(\frac{\theta}{2^r}\right)}{2^{r-1} \cos\left(\frac{\theta}{2^{r-1}}\right)} \\ &= \sum_{r=1}^n \frac{\sin\left(\frac{\theta}{2^r}\right)}{2^{r-1} \cos\left(\frac{\theta}{2^r}\right) \cos\left(\frac{\theta}{2^{r-1}}\right)} \\ &= \sum_{r=1}^n \frac{2\sin^2\left(\frac{\theta}{2^r}\right)}{2^{r-1} \left\{ 2\sin\left(\frac{\theta}{2^r}\right) \cos\left(\frac{\theta}{2^r}\right) \right\} \cos\left(\frac{\theta}{2^{r-1}}\right)} \\ &= \sum_{r=1}^n \frac{1 - \cos\left(\frac{\theta}{2^{r-1}}\right)}{2^{r-1} \sin\left(\frac{\theta}{2^{r-1}}\right) \cos\left(\frac{\theta}{2^{r-1}}\right)} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{r=1}^n \left[\frac{1}{2^{r-2} \sin\left(\frac{\theta}{2^{r-2}}\right)} - \frac{1}{2^{r-1} \sin\left(\frac{\theta}{2^{r-1}}\right)} \right] \\
 &= \left[\left(\frac{2}{\sin 2\theta} - \frac{1}{\sin \theta} \right) + \left(\frac{1}{\sin \theta} - \frac{1}{2 \sin\left(\frac{\theta}{2}\right)} \right) \right. \\
 &\quad \left. + \left(\frac{1}{2 \sin\left(\frac{\theta}{2}\right)} - \frac{1}{2^2 \sin\left(\frac{\theta}{2^2}\right)} \right) + \dots + \left(\frac{1}{2^{n-2} \sin\left(\frac{\theta}{2^{n-2}}\right)} - \frac{1}{2^{n-1} \sin\left(\frac{\theta}{2^{n-1}}\right)} \right) \right] \\
 &= \frac{2}{\sin 2\theta} - \frac{1}{2^{n-1} \sin\left(\frac{\theta}{2^{n-1}}\right)} \\
 \therefore \sum_{n=1}^{\infty} \frac{\tan(\theta/2^n)}{2^{n-1} \cos\left(\frac{\theta}{2^{n-1}}\right)} &= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\tan(\theta/2^r)}{2^{r-1} \cos\left(\frac{\theta}{2^{r-1}}\right)} \\
 &= \lim_{n \rightarrow \infty} \left[\frac{2}{\sin 2\theta} - \frac{1}{2^{n-1} \sin\left(\frac{\theta}{2^{n-1}}\right)} \right] \\
 &= \frac{2}{\sin 2\theta} - \lim_{n \rightarrow \infty} \left\{ \frac{\theta/2^{n-1}}{\theta \cdot \sin\left(\frac{\theta}{2^{n-1}}\right)} \right\} = \frac{2}{\sin 2\theta} - \frac{1}{\theta}
 \end{aligned}$$

➤ **Example 23.** The value of $\cos A \cos 2A \cos 2^2 A \cos 2^3 A \dots \cos 2^{n-1} A$ is

- (a) $\frac{\cos 2^n A}{2^n \cos A}$ (b) $\frac{\sin 2^n A}{2^n \sin A}$
 (c) $\frac{\sin 2^n A}{2^n \cos A}$ (d) $\frac{\cos 2^n A}{2^n \sin A}$

Sol. (b) We have,

$$\begin{aligned}
 &\cos A \cos 2A \cos 2^2 A \cos 2^3 A \dots \cos 2^{n-1} A \\
 &= \frac{1}{2 \sin A} [(2 \sin A \cos A) \cos 2A \cos 2^2 A \cos 2^3 A \dots \cos 2^{n-1} A] \\
 &= \frac{1}{2 \sin A} [(\sin 2A \cos 2A) \cos 2^2 A \cos 2^3 A \dots \cos 2^{n-1} A] \\
 &= \frac{1}{2^2 \sin A} [(2 \sin 2A \cos 2A) \cos 2^2 A \cos 2^3 A \dots \cos 2^{n-1} A] \\
 &= \frac{1}{2^2 \sin A} [\sin 2(2A) \cdot \cos 2^2 A \cos 2^3 A \dots \cos 2^{n-1} A] \\
 &= \frac{1}{2^3 \sin A} [(2 \sin 2^2 A \cos 2^2 A) \cos 2^3 A \dots \cos 2^{n-1} A] \\
 &= \frac{1}{2^3 \sin A} [\sin(2 \cdot 2^2 A) \cos 2^3 A \dots \cos 2^{n-1} A] \\
 &= \frac{1}{2^3 \sin A} [(\sin 2^3 A \cos 2^3 A \cos 2^4 A \dots \cos 2^{n-1} A)] \\
 &\vdots \\
 &= \frac{1}{2^{n-1} \sin A} [\sin 2^{n-1} A \cos 2^{n-1} A]
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2^n \sin A} [2 \sin 2^{n-1} A \cos 2^{n-1} A] \\
 &= \frac{1}{2^n \sin A} \cdot \sin(2 \cdot 2^{n-1} A) \\
 &= \frac{1}{2^n \sin A} \cdot \sin 2^n A = \frac{\sin 2^n A}{2^n \sin A}
 \end{aligned}$$

Remark

Students are advised to learn above result as standard formula.

➤ **Example 24.** $\sin \frac{\pi}{n} + \sin \frac{3\pi}{n} + \sin \frac{5\pi}{n} + \dots$ to n terms is equal to

- (a) 1 (b) 2
 (c) 3 (d) 0

Sol. (d) $\sin \frac{\pi}{n} + \sin \frac{3\pi}{n} + \sin \frac{5\pi}{n} + \dots$ to n terms

$$\begin{aligned}
 &= \frac{\sin \frac{n \cdot 2\pi}{2n}}{\sin \frac{2\pi}{2n}} \cdot \sin \left(\frac{2 \cdot \frac{\pi}{n} + (n-1) \frac{2\pi}{n}}{2} \right) \\
 &= \frac{\sin \frac{\pi}{n}}{\sin \frac{\pi}{n}} \cdot \sin \left(\frac{2\pi + 2n\pi - 2\pi}{2n} \right) = \frac{\sin^2 \pi}{\sin \frac{\pi}{n}} = 0
 \end{aligned}$$

$$\therefore \sin \frac{\pi}{n} + \sin \frac{3\pi}{n} + \sin \frac{5\pi}{n} + \dots \text{ to } n \text{ terms} = 0$$

Computation of Trigonometrical Ratios of Some Useful Angles

By using the formulae introduced in the previous sections, we can now find the trigonometrical ratios of some important angles.

Value of $\sin 18^\circ$

Let $\theta = 18^\circ$. Then,

$$5\theta = 90^\circ$$

$$\Rightarrow 2\theta + 3\theta = 90^\circ$$

$$\Rightarrow 2\theta = 90^\circ - 3\theta$$

$$\Rightarrow \sin 2\theta = \sin (90^\circ - 3\theta)$$

$$\Rightarrow \sin 2\theta = \cos 3\theta$$

$$\Rightarrow 2 \sin \theta \cos \theta = 4 \cos^3 \theta - 3 \cos \theta$$

$$\Rightarrow \cos \theta (2 \sin \theta - 4 \cos^2 \theta + 3) = 0$$

$$\Rightarrow 2 \sin \theta - 4 \cos^2 \theta + 3 = 0 \quad [\because \cos \theta = \cos 18^\circ \neq 0]$$

$$\Rightarrow 2 \sin \theta - 4(1 - \sin^2 \theta) + 3 = 0$$

$$\Rightarrow 4 \sin^2 \theta + 2 \sin \theta - 1 = 0$$

$$\Rightarrow \sin \theta = \frac{-2 \pm \sqrt{4+16}}{8}$$

$$\Rightarrow \sin \theta = \frac{-1 \pm \sqrt{5}}{4}$$

$$\Rightarrow \sin \theta = \frac{-1 \pm \sqrt{5}}{4} = \frac{\sqrt{5}-1}{4} \quad [\because \sin 18^\circ > 0]$$

$$\text{Hence, } \sin 18^\circ = \frac{\sqrt{5}-1}{4}$$

Value of $\cos 36^\circ$

Since, $\cos 2\theta = 1 - 2\sin^2 \theta$

$$\therefore \cos 36^\circ = 1 - 2\sin^2 18^\circ [\text{putting } \theta = 18^\circ]$$

$$\begin{aligned} \Rightarrow \cos 36^\circ &= 1 - 2\left(\frac{\sqrt{5}-1}{4}\right)^2 \\ &= 1 - 2\left(\frac{6-2\sqrt{5}}{16}\right) = 1 - \left(\frac{3-\sqrt{5}}{4}\right) \\ &= \frac{\sqrt{5}+1}{4} \end{aligned}$$

Hence, $\cos 36^\circ = \frac{\sqrt{5}+1}{4}$

Value of $\sin 36^\circ$

Putting $\theta = 36^\circ$ in $\sin \theta = \sqrt{1 - \cos^2 \theta}$, we obtain

$$\begin{aligned} \sin 36^\circ &= \sqrt{1 - \cos^2 36^\circ} \\ &= \sqrt{1 - \left(\frac{\sqrt{5}+1}{4}\right)^2} \\ &= \sqrt{\frac{16 - (6+2\sqrt{5})}{16}} \\ &= \frac{\sqrt{10-2\sqrt{5}}}{4} \end{aligned}$$

Hence, $\sin 36^\circ = \frac{\sqrt{10-2\sqrt{5}}}{4}$

Trigonometrical Ratios of Some More Angles between 0° and 90°

Angle	$0^\circ/0$	$15^\circ/\frac{\pi}{12}$	$18^\circ/\frac{\pi}{10}$	$22.5^\circ/\frac{\pi}{8}$	$30^\circ/\frac{\pi}{6}$	$36^\circ/\frac{\pi}{5}$	$45^\circ/\frac{\pi}{4}$	$54^\circ/\frac{3\pi}{10}$	$60^\circ/\frac{\pi}{3}$	$67.5^\circ/\frac{3\pi}{8}$	$72^\circ/\frac{2\pi}{5}$	$75^\circ/\frac{5\pi}{12}$	$90^\circ/\frac{\pi}{2}$
$\sin \theta$	0	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{5}+1}{4}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}+\sqrt{2}}{2}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	1
$\cos \theta$	1	$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\sqrt{10+2\sqrt{5}}}{4}$	$\frac{\sqrt{2}+\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{5}+1}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{10-2\sqrt{5}}}{4}$	$\frac{1}{2}$	$\frac{\sqrt{2}-\sqrt{2}}{2}$	$\frac{\sqrt{5}-1}{4}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$	0
$\tan \theta$	0	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{2}-1$	$\frac{1}{\sqrt{3}}$	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	1	$\frac{\sqrt{10-2\sqrt{5}}}{\sqrt{5}+1}$	$\sqrt{3}$	$\sqrt{2}+1$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	∞
$\cot \theta$	∞	$\frac{\sqrt{3}+1}{\sqrt{3}-1}$	$\frac{\sqrt{10+2\sqrt{5}}}{\sqrt{5}-1}$	$\sqrt{2}+1$	$\sqrt{3}$	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	1	$\frac{\sqrt{5}+1}{\sqrt{10-2\sqrt{5}}}$	$\frac{1}{\sqrt{3}}$	$\sqrt{2}-1$	$\frac{\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}$	$\frac{\sqrt{3}-1}{\sqrt{3}+1}$	0
$\sec \theta$	1	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{2}{\sqrt{3}}$	$\sqrt{5}-1$	$\sqrt{2}$	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	2	$\sqrt{4+2\sqrt{2}}$	$\sqrt{5}+1$	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	∞
$\operatorname{cosec} \theta$	∞	$\frac{2\sqrt{2}}{\sqrt{3}-1}$	$\sqrt{5}+1$	$\sqrt{4+2\sqrt{2}}$	2	$\frac{4}{\sqrt{10-2\sqrt{5}}}$	$\sqrt{2}$	$\sqrt{5}-1$	$\frac{2}{\sqrt{3}}$	$\sqrt{4-2\sqrt{2}}$	$\frac{4}{\sqrt{10+2\sqrt{5}}}$	$\frac{2\sqrt{2}}{\sqrt{3}+1}$	1

Example 25. The value of $\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ$ is

- (a) 1 (b) $\frac{1}{2}$
(c) -1 (d) $-\frac{1}{2}$

Sol. (d) Consider,

$$\begin{aligned} &\cos 12^\circ + \cos 84^\circ + \cos 156^\circ + \cos 132^\circ \\ &= (\cos 12^\circ + \cos 132^\circ) + (\cos 84^\circ + \cos 156^\circ) \\ &= 2\cos\left(\frac{12^\circ + 132^\circ}{2}\right)\cos\left(\frac{132^\circ - 12^\circ}{2}\right) \\ &\quad + 2\cos\left(\frac{84^\circ + 156^\circ}{2}\right)\cos\left(\frac{156^\circ - 84^\circ}{2}\right) \\ &= 2\cos 72^\circ \cos 60^\circ + 2\cos 120^\circ \cos 36^\circ \\ &= 2\sin 18^\circ \cos 60^\circ + 2\cos 120^\circ \cos 36^\circ \\ &= 2\left(\frac{\sqrt{5}-1}{4}\right)\frac{1}{2} + 2\left(\frac{-1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right) \\ &= \frac{-1}{2} \end{aligned}$$

Example 26. Prove that

$$4\sin 27^\circ = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}}$$

Sol. Consider, $(\cos 27^\circ + \sin 27^\circ)^2 = \sin^2 27^\circ + \cos^2 27^\circ + 2\sin 27^\circ \cos 27^\circ$

$$= 1 + \sin 54^\circ = 1 + \frac{\sqrt{5}+1}{4} = \frac{1}{4}(5+\sqrt{5})$$

$$\therefore \cos 27^\circ + \sin 27^\circ = \frac{1}{2}\sqrt{5+\sqrt{5}} \quad \dots(i)$$

[$\because \sin 27^\circ > 0, \cos 27^\circ > 0$]

Similarly, we have

$$\begin{aligned} (\cos 27^\circ - \sin 27^\circ)^2 &= 1 - \sin 54^\circ \\ &= 1 - \frac{\sqrt{5}+1}{4} = \frac{3-\sqrt{5}}{4} \end{aligned}$$

$$\therefore \cos 27^\circ - \sin 27^\circ = \frac{1}{2}\sqrt{3-\sqrt{5}} \quad \dots(ii)$$

[$\because 0 < \theta < \frac{\pi}{4}, \cos \theta > \sin \theta$]

On subtracting Eq. (ii) from Eq. (i), we get

$$2\sin 27^\circ = \frac{1}{2}\sqrt{5+\sqrt{5}} - \frac{1}{2}\sqrt{3-\sqrt{5}}$$

$$\Rightarrow 4\sin 27^\circ = \sqrt{5+\sqrt{5}} - \sqrt{3-\sqrt{5}} \quad \text{Hence proved.}$$

➤ **Example 27.** If $\theta = \frac{-4\pi}{3}$, justify whether the

relation $2\sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} + \sqrt{1-\sin\theta}$ is true.

Sol. $\theta = -\frac{4\pi}{3} = -240^\circ \Rightarrow \frac{\theta}{2} = -120^\circ$

$$\therefore \sin\frac{\theta}{2} = \sin(-120^\circ) = -\sin 120^\circ = -\sin(180^\circ - 60^\circ)$$

$$= -\sin 60^\circ = -\frac{\sqrt{3}}{2}$$

$$\cos\frac{\theta}{2} = \cos(-120^\circ) = \cos 120^\circ = \cos(180^\circ - 60^\circ)$$

$$= -\cos 60^\circ = -\frac{1}{2}$$

$$\therefore \cos\frac{\theta}{2} + \sin\frac{\theta}{2} = -\frac{1}{2} - \frac{\sqrt{3}}{2} = -\frac{\sqrt{3}+1}{2} < 0$$

$$\text{and } \cos\frac{\theta}{2} - \sin\frac{\theta}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}-1}{2} > 0$$

$$\text{Thus, } \cos\frac{\theta}{2} + \sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} [\because \sqrt{1+\sin\theta} > 0] \quad \dots(i)$$

$$\text{and } \cos\frac{\theta}{2} - \sin\frac{\theta}{2} = \sqrt{1-\sin\theta} \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$2\sin\frac{\theta}{2} = -\sqrt{1+\sin\theta} - \sqrt{1-\sin\theta},$$

which shows that the given relation is not true.

➤ **Example 28.** Show that

$$\tan\frac{\pi}{24} = \sqrt{6} - \sqrt{3} + \sqrt{2} - 2.$$

Sol. Consider, $\sin 15^\circ = \sin(45^\circ - 30^\circ)$

$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\cos 15^\circ = \cos(45^\circ - 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\therefore \tan\frac{\pi}{24} = \frac{\sin\frac{\pi}{24}}{\cos\frac{\pi}{24}} = \frac{2\sin^2\frac{\pi}{24}}{2\sin\frac{\pi}{24}\cos\frac{\pi}{24}} = \frac{1-\cos\frac{\pi}{12}}{\sin\frac{\pi}{12}}$$

$$= \frac{1-\cos 15^\circ}{\sin 15^\circ} = \frac{1-\frac{\sqrt{3}+1}{2\sqrt{2}}}{\frac{\sqrt{3}-1}{2\sqrt{2}}}$$

$$= \frac{2\sqrt{2}-\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{(\sqrt{3}+1)(2\sqrt{2}-\sqrt{3}-1)}{(\sqrt{3}+1)(\sqrt{3}-1)}$$

$$= \frac{2\sqrt{6}-3-\sqrt{3}+2\sqrt{2}-\sqrt{3}-1}{2}$$

$$= \frac{2(\sqrt{6}-\sqrt{3}+\sqrt{2}-2)}{2}$$

$$= \sqrt{6}-\sqrt{3}+\sqrt{2}-2$$

Hence proved.

➤ **Example 29.** Prove that $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ = 1$.

Sol. $\tan 6^\circ \tan 42^\circ \tan 66^\circ \tan 78^\circ$

$$= \frac{\sin 6^\circ \sin 42^\circ \sin 66^\circ \sin 78^\circ}{\cos 6^\circ \cos 42^\circ \cos 66^\circ \cos 78^\circ}$$

$$= \frac{(2 \sin 66^\circ \sin 6^\circ)(2 \sin 78^\circ \sin 42^\circ)}{(2 \cos 66^\circ \cos 6^\circ)(2 \cos 78^\circ \cos 42^\circ)}$$

$$= \frac{(\cos 60^\circ - \cos 72^\circ)(\cos 36^\circ - \cos 120^\circ)}{(\cos 60^\circ + \cos 72^\circ)(\cos 120^\circ + \cos 36^\circ)}$$

$$= \frac{\left(\frac{1}{2} - \sin 18^\circ\right)\left(\cos 36^\circ + \frac{1}{2}\right)}{\left(\frac{1}{2} + \sin 18^\circ\right)\left(-\frac{1}{2} + \cos 36^\circ\right)}$$

$$= \frac{\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)}{\left(\frac{1}{2} + \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} - \frac{1}{2}\right)}$$

$$= \frac{(3-\sqrt{5})(3+\sqrt{5})}{(1+\sqrt{5})(\sqrt{5}-1)}$$

$$= \frac{9-5}{5-1} = \frac{4}{4} = 1$$

Hence proved.

➤ **Example 30.** Prove that

$$2\cos\frac{\pi}{32} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}}$$

Sol. Consider, $2 + \sqrt{2} = 2\left(1 + \frac{1}{\sqrt{2}}\right) = 2\left(1 + \cos\frac{\pi}{4}\right)$

$$= 2 \cdot 2\cos^2\frac{\pi}{8} = \left(2\cos\frac{\pi}{8}\right)^2$$

$$\therefore 2\cos\frac{\pi}{8} = \sqrt{2 + \sqrt{2}} \quad \left[\because \cos\frac{\pi}{8} > 0\right]$$

$$\text{Again, } 2 + \sqrt{2 + \sqrt{2}} = 2 + 2\cos\frac{\pi}{8} = 2\left(1 + \cos\frac{\pi}{8}\right)$$

$$= 2 \times 2\cos^2\frac{\pi}{16} = \left(2\cos\frac{\pi}{16}\right)^2$$

$$\therefore \sqrt{2 + \sqrt{2 + \sqrt{2}}} = 2\cos\frac{\pi}{16}$$

$$\text{Now, } 2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}} = 2 + 2\cos\frac{\pi}{16}$$

$$= 2\left(1 + \cos\frac{\pi}{16}\right)$$

$$= 2 \cdot 2\cos^2\frac{\pi}{32} = \left(2\cos\frac{\pi}{32}\right)^2$$

$$\therefore 2\cos\frac{\pi}{32} = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2}}}} \quad \left[\because \cos\frac{\pi}{32} > 0\right]$$

Hence proved.

Maximum and Minimum Values of Trigonometrical Expressions

In this section, we shall discuss problems on finding the maximum and minimum values of various trigonometrical expressions.

Consider the expression $a \cos \theta \pm b \sin \theta$, where θ is a variable.

Let $y = a \cos \theta \pm b \sin \theta$

Further, let $a = r \cos \alpha$ and $b = r \sin \alpha$. Then,

$$r = \sqrt{a^2 + b^2} \text{ and } \tan \alpha = \frac{b}{a}$$

$$\therefore y = r \cos \alpha \cos \theta \pm r \sin \alpha \sin \theta$$

$$\Rightarrow y = r \cos (\theta \mp \alpha)$$

We know that,

$$-1 \leq \cos (\theta \mp \alpha) \leq 1, \forall \theta$$

$$\Rightarrow -r \leq r \cos (\theta \mp \alpha) \leq r, \forall \theta$$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq y \leq \sqrt{a^2 + b^2}, \forall \theta$$

$$\Rightarrow -\sqrt{a^2 + b^2} \leq a \cos \theta \pm b \sin \theta \leq \sqrt{a^2 + b^2} \text{ for all } \theta.$$

If follows from the above discussion that

$-\sqrt{a^2 + b^2}$ and $\sqrt{a^2 + b^2}$ are minimum and maximum values of $a \cos \theta \pm b \sin \theta$ for varying values of θ .

✎ The maximum and minimum values of $a \cos \theta \pm b \sin \theta + c$ are $c + \sqrt{a^2 + b^2}$ and $c - \sqrt{a^2 + b^2}$ respectively i.e.

$$c - \sqrt{a^2 + b^2} \leq a \cos \theta \pm b \sin \theta + c \leq c + \sqrt{a^2 + b^2}.$$

✎ **Example 31.** The maximum and minimum values of $7 \cos \theta + 24 \sin \theta$ are

- (a) 25 and -25 (b) 24 and -24
(c) 5 and -5 (d) None of these

Sol. (a) We know that the maximum and minimum values of $a \cos \theta + b \sin \theta$ are $\sqrt{a^2 + b^2}$ and $-\sqrt{a^2 + b^2}$ respectively.

So, the maximum and minimum values of

$$7 \cos \theta + 24 \sin \theta \text{ are } \sqrt{7^2 + 24^2} = 25$$

$$\text{and } -\sqrt{7^2 + 24^2} = -25 \text{ respectively.}$$

✎ **Example 32.** Prove that

$$5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3} \right) + 3 \text{ lies between } -4 \text{ and } 10.$$

Sol. We have, $5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3} \right) + 3$

$$= 5 \cos \theta + 3 \left(\cos \theta \cos \frac{\pi}{3} - \sin \theta \sin \frac{\pi}{3} \right) + 3$$

$$= 5 \cos \theta + \frac{3}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3$$

$$= \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 \quad \dots (i)$$

$$\text{Now, } -\sqrt{\left(\frac{13}{2} \right)^2 + \left(\frac{3\sqrt{3}}{2} \right)^2} \leq \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta$$

$$\leq \sqrt{\left(\frac{13}{2} \right)^2 + \left(\frac{3\sqrt{3}}{2} \right)^2}$$

$$\Rightarrow -7 \leq \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta \leq 7, \forall \theta$$

$$\Rightarrow -7 + 3 \leq \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3 \leq 7 + 3, \forall \theta$$

$$\Rightarrow -4 \leq 5 \cos \theta + 3 \cos \left(\theta + \frac{\pi}{3} \right) + 3 \leq 10, \forall \theta \text{ [from Eq. (i)]}$$

Hence proved.

✎ **Example 33.** Find a and b such that the inequality holds good for all θ , $a \leq 3 \cos \theta + 5 \sin \left(\theta - \frac{\pi}{6} \right) \leq b$.

Sol. We have, $3 \cos \theta + 5 \sin \left(\theta - \frac{\pi}{6} \right)$

$$= 3 \cos \theta + 5 \left(\sin \theta \cos \frac{\pi}{6} - \cos \theta \sin \frac{\pi}{6} \right)$$

$$= 3 \cos \theta + 5 \left(\frac{\sqrt{3}}{2} \sin \theta - \frac{1}{2} \cos \theta \right)$$

$$= \frac{1}{2} \cos \theta + \frac{5\sqrt{3}}{2} \sin \theta$$

$$\text{We have, } \sqrt{\left(\frac{1}{2} \right)^2 + \left(\frac{5\sqrt{3}}{2} \right)^2} \leq \frac{1}{2} \cos \theta + \frac{5\sqrt{3}}{2} \sin \theta$$

$$\sin \theta \leq \sqrt{\left(\frac{1}{2} \right)^2 + \left(\frac{5\sqrt{3}}{2} \right)^2}, \forall \theta$$

$$\Rightarrow -\sqrt{19} \leq 3 \cos \theta + 5 \sin \left(\theta - \frac{\pi}{6} \right) \leq \sqrt{19}, \forall \theta$$

Hence, $a = -\sqrt{19}$ and $b = \sqrt{19}$.

✎ **Example 34.** Show that for all real value θ , the expression $a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta$ lies between $\frac{1}{2} \{ (a+c) - \sqrt{b^2 + (a-c)^2} \}$ and

$$\frac{1}{2} \{ (a+c) + \sqrt{b^2 + (a-c)^2} \}.$$

Sol. Let $y = a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta$. Then,

$$y = a \left(\frac{1 - \cos 2\theta}{2} \right) + \frac{b}{2} \sin 2\theta + c \left(\frac{1 + \cos 2\theta}{2} \right)$$

$$\Rightarrow y = \frac{a+c}{2} + \frac{1}{2} \{ (c-a) \cos 2\theta + b \sin 2\theta \}$$

$$\text{Now, } -\sqrt{(c-a)^2 + b^2} \leq (c-a) \cos 2\theta + b \sin 2\theta \leq \sqrt{(c-a)^2 + b^2}, \forall \theta \in R$$

$$\Rightarrow -\frac{1}{2} \sqrt{(c-a)^2 + b^2} \leq \frac{1}{2} \{ (c-a) \cos 2\theta + b \sin 2\theta \} \leq \frac{1}{2} \sqrt{(c-a)^2 + b^2}, \forall \theta \in R$$

$$\Rightarrow \frac{a+c}{2} - \frac{1}{2} \sqrt{(c-a)^2 + b^2} \leq \frac{a+c}{2} + \frac{1}{2} \{ (c-a) \cos 2\theta + b \sin 2\theta \} \leq \frac{a+c}{2} + \frac{1}{2} \sqrt{(c-a)^2 + b^2}, \forall \theta \in R$$

$$\Rightarrow \frac{1}{2} \{ (a+c) - \sqrt{b^2 + (c-a)^2} \}$$

$$\leq a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta$$

$$\leq \frac{1}{2} \{ (a+c) + \sqrt{b^2 + (c-a)^2} \}, \forall \theta \in R \text{ Hence proved.}$$

➤ **Example 35.** Show that for varying θ and fixed α , the expression $\cos \theta (\sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha})$ always lies between $-\sqrt{1 + \sin^2 \alpha}$ and $\sqrt{1 + \sin^2 \alpha}$.

Sol. Let $y = \cos \theta (\sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha})$.

$$\begin{aligned} \text{Then, } y &= \sin \theta \cos \theta + \cos \theta \sqrt{\sin^2 \theta + \sin^2 \alpha} \\ \Rightarrow (y - \sin \theta \cos \theta)^2 &= \cos^2 \theta (\sin^2 \theta + \sin^2 \alpha) \\ \Rightarrow y^2 - 2y \sin \theta \cos \theta &= \cos^2 \theta \sin^2 \alpha \\ \Rightarrow y^2 - 2y \sin \theta \cos \theta + \cos^2 \theta &= \cos^2 \theta + \cos^2 \theta \sin^2 \alpha \\ \Rightarrow y^2 \sec^2 \theta - 2y \tan \theta + 1 &= 1 + \sin^2 \alpha \\ &\quad [\text{dividing throughout by } \cos^2 \theta] \\ \Rightarrow y^2 \tan^2 \theta - 2y \tan \theta + 1 &= (1 + \sin^2 \alpha) - y^2 \\ \Rightarrow (y \tan \theta - 1)^2 &= (1 + \sin^2 \alpha) - y^2 \\ \Rightarrow (1 + \sin^2 \alpha) - y^2 &\geq 0 \\ \Rightarrow y^2 - (\sqrt{1 + \sin^2 \alpha})^2 &\leq 0 \\ \Rightarrow -\sqrt{1 + \sin^2 \alpha} \leq y &\leq \sqrt{1 + \sin^2 \alpha} \\ \text{Hence, } -\sqrt{1 + \sin^2 \alpha} &\leq \cos \theta (\sin \theta + \sqrt{\sin^2 \theta + \sin^2 \alpha}) \\ &\leq \sqrt{1 + \sin^2 \alpha}, \forall \theta \text{ Hence proved.} \end{aligned}$$

➤ **Example 36.** If in a ΔABC , $\tan \frac{A}{2}$, $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$ are in HP, then find the minimum value of $\cot \frac{B}{2}$.

Sol. In ΔABC , we have

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \quad \dots (i)$$

It is given that, $\tan \frac{A}{2}$, $\tan \frac{B}{2}$ and $\tan \frac{C}{2}$ are in HP.

$$\Rightarrow \cot \frac{A}{2}, \cot \frac{B}{2} \text{ and } \cot \frac{C}{2} \text{ are in AP.}$$

$$\Rightarrow 2 \cot \frac{B}{2} = \cot \frac{A}{2} + \cot \frac{C}{2} \quad \dots (ii)$$

From Eqs. (i) and (ii), we have

$$\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = 3 \cot \frac{B}{2} \Rightarrow \cot \frac{A}{2} \cot \frac{C}{2} = 3$$

Since, $AM \geq GM$

$$\begin{aligned} \Rightarrow \frac{\cot \frac{A}{2} + \cot \frac{C}{2}}{2} &\geq \sqrt{\cot \frac{A}{2} \cot \frac{C}{2}} \\ \Rightarrow \cot \frac{B}{2} &\geq \sqrt{3} \quad \left[\because \cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2} \right] \end{aligned}$$

Hence, the minimum value of $\cot \frac{B}{2}$ is $\sqrt{3}$.

Work Book Exercise 9.2

1 If $\sin A = \frac{1}{\sqrt{10}}$ and $\sin B = \frac{1}{\sqrt{5}}$, where A, B are acute angles, then $A + B$ equals

- a $\frac{\pi}{4}$ b $\frac{\pi}{2}$
c $\frac{\pi}{3}$ d π

2 If the angle α is in the third quadrant and $\tan \alpha = 2$, then $\sin \alpha$ is equal to

- a $\frac{2}{\sqrt{5}}$ b $\frac{2}{\sqrt{3}}$
c $\frac{\sqrt{5}}{2}$ d None of these

3 If θ lies in the first quadrant, then which of the following is not true?

- a $\frac{\theta}{2} < \tan \left(\frac{\theta}{2} \right)$ b $\frac{\theta}{2} < \sin \frac{\theta}{2}$
c $\theta \cos^2 \left(\frac{\theta}{2} \right) < \sin \theta$ d $\theta \sin \left(\frac{\theta}{2} \right) < 2 \sin \frac{\theta}{2}$

4 $\cos 2\theta + 2 \cos \theta$ is always

- a $> -\frac{3}{2}$ b $\geq \frac{3}{2}$
c $\leq -\frac{3}{2}$ d None of these

5 If $A = \tan 6^\circ \tan 42^\circ$ and $B = \cot 66^\circ \cot 78^\circ$, then

- a $A = 2B$ b $A = \frac{1}{3}B$
c $A = B$ d $3A = 2B$

6 The value of $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$ is

- a 2 b 1
c 4 d -4

7 The equation $\sin^2 \theta = \frac{x^2 + y^2}{2xy}$ is possible, if

- a $x = y$ b $x = -y$
c $2x = y$ d None of these

8 If $\frac{\sin(x+y)}{\sin(x-y)} = \frac{a+b}{a-b}$, then $\frac{\tan x}{\tan y}$ is equal to

- a $\frac{b}{a}$ b $\frac{a}{b}$
c ab d None of these

9 If $\tan^2 \theta = 2 \tan^2 \phi + 1$, then $\cos 2\theta + \sin^2 \phi$ equals

- a -1
b 0
c 1
d None of the above

10 The value of

$$\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} \sin \frac{7\pi}{14} \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$$

- a $\frac{1}{16}$
b $\frac{1}{64}$
c $\frac{1}{128}$
d None of the above

- 11** If $x \cos \alpha + y \sin \alpha = 2a$, $x \cos \beta + y \sin \beta = 2a$ and $2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} = 1$, then
- $\cos \alpha + \cos \beta = \frac{2ax}{x^2 + y^2}$
 - $\cos \alpha \cos \beta = \frac{2a^2 - y^2}{x^2 + y^2}$
 - $y^2 = 4a(\alpha - x)$
 - $\cos \alpha + \cos \beta = 2 \cos \alpha \cos \beta$
- 12** If $\sin \theta - \cos \theta < 0$, then θ lies between
- $n\pi - \frac{3\pi}{4}$ and $n\pi + \frac{\pi}{4}$, $n \in I$
 - $n\pi - \frac{\pi}{4}$ and $n\pi + \frac{\pi}{4}$, $n \in I$
 - $2n\pi - \frac{3\pi}{4}$ and $2n\pi - \frac{\pi}{4}$, $n \in I$
 - $2n\pi - \frac{3\pi}{4}$ and $2n\pi + \frac{\pi}{4}$, $n \in I$
- 13** If $2 \cos \frac{A}{2} = \sqrt{1 + \sin A} + \sqrt{1 - \sin A}$, then $\frac{A}{2}$ lies between ($n \in I$)
- $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$
 - $2n\pi - \frac{\pi}{4}$ and $2n\pi + \frac{\pi}{4}$
 - $2n\pi - \frac{3\pi}{4}$ and $2n\pi - \frac{\pi}{4}$
 - $-\infty$ and $+\infty$
- 14** The angle θ whose cosine equals to its tangent, is given by
- $\cos \theta = 2 \cos 18^\circ$
 - $\cos \theta = 2 \sin 18^\circ$
 - $\sin \theta = 2 \sin 18^\circ$
 - $\sin \theta = 2 \cos 18^\circ$
- 15** The value of $\tan 5\theta$ is
- $\frac{5 \tan \theta - 10 \tan^3 \theta + \tan^5 \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$
 - $\frac{5 \tan \theta + 10 \tan^3 \theta - \tan^5 \theta}{1 + 10 \tan^2 \theta - 5 \tan^4 \theta}$
 - $\frac{5 \tan^5 \theta - 10 \tan^3 \theta + \tan \theta}{1 - 10 \tan^2 \theta + 5 \tan^4 \theta}$
 - None of the above
- 16** If $\left| a \sin^2 \theta + b \sin \theta \cos \theta + c \cos^2 \theta - \frac{1}{2}(a+c) \right| \leq \frac{1}{2}k$, then k^2 is equal to
- $b^2 + (a-c)^2$
 - $a^2 + (b-c)^2$
 - $c^2 + (a-b)^2$
 - None of the above
- 17** If α, β, γ and δ are the smallest positive angles in ascending order of magnitude which have their sines equal to the positive quantity λ , then the value of $4 \sin \frac{\alpha}{2} + 3 \sin \frac{\beta}{2} + 2 \sin \frac{\gamma}{2} + \sin \frac{\delta}{2}$ is
- $\sqrt{1+\lambda}$
 - $2\sqrt{1+\lambda}$
 - $\sqrt{1-\lambda}$
 - $2\sqrt{1-\lambda}$
- 18** If $\tan \left(\frac{\theta}{2} \right) = \frac{5}{2}$ and $\tan \left(\frac{\phi}{2} \right) = \frac{3}{4}$, then the value of $\cos(\theta + \phi)$ is
- $-\frac{364}{725}$
 - $-\frac{627}{725}$
 - $-\frac{240}{339}$
 - None of the above
- 19** If $\alpha, \beta, \gamma \in \left(0, \frac{\pi}{2} \right)$, then the value of $\frac{\sin(\alpha + \beta + \gamma)}{\sin \alpha + \sin \beta + \sin \gamma}$ is
- < 1
 - > 1
 - $= 1$
 - None of these
- 20** If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$, then the value of $\sin x$ is
- $2 \cos 18^\circ$
 - $\cos 18^\circ$
 - $\sin 18^\circ$
 - $2 \sin 18^\circ$

Trigonometric Equations

An equation involving one or more trigonometrical ratios of unknown angle is called a trigonometric equation.

e.g. $\sin^2 x - 4 \sin x = 1$

It is to be noted that a trigonometrical identity is satisfied for every value of the unknown angle, whereas trigonometric equation is satisfied only for some values (finite or infinite) of unknown angle.

e.g. $\sin^2 x + \cos^2 x = 1$ is a trigonometrical identity as it is satisfied for every value of $x \in R$.

Solutions or Roots of a Trigonometric Equation

A value of the unknown angle which satisfies the given equation, is called a solution or root of the equation.

The trigonometric equation may have infinite number of solutions and can be classified as

- (i) Principal solution (ii) General solution

(i) **Principal solution** The value of unknown angle which satisfies the given equation and lies in the interval $[0, 2\pi)$, is called a principal solution of trigonometric equation.

(ii) **General solution** We know that trigonometric functions are periodic and solutions of trigonometric equations can be generalised with the help of the periodicity of the trigonometric functions.

The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

➤ A function $f(x)$ is said to be a periodic function, if a least positive real number T is such that $f(x+T) = f(x)$, then T is known as period of function $f(x)$.

General Solution of Trigonometric Equations

(i) $\sin \theta = 0$	$\theta = n\pi, n \in I$
(ii) $\cos \theta = 0$	$\theta = (2n+1)\frac{\pi}{2}, n \in I$
(iii) $\tan \theta = 0$	$\theta = n\pi, n \in I$
(iv) $\sin \theta = \sin \alpha$	$\theta = n\pi + (-1)^n \alpha, \alpha \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right], n \in I$
(v) $\cos \theta = \cos \alpha$	$\theta = 2n\pi \pm \alpha, \alpha \in [0, \pi], n \in I$
(vi) $\tan \theta = \tan \alpha$	$\theta = n\pi + \alpha, \alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), n \in I$
(vii) $\left. \begin{aligned} \sin^2 \theta &= \sin^2 \alpha \\ \cos^2 \theta &= \cos^2 \alpha \\ \tan^2 \theta &= \tan^2 \alpha \end{aligned} \right\}$	$\theta = n\pi \pm \alpha, n \in I$
(viii) $\sin \theta = 1$	$\theta = (4n+1)\frac{\pi}{2}, n \in I$
(ix) $\cos \theta = 1$	$\theta = 2n\pi, n \in I$
(x) $\cos \theta = -1$	$\theta = (2n+1)\pi, n \in I$
(xi) $\sin x = \sin \alpha$ and $\cos x = \cos \alpha$	$x = 2n\pi + \alpha$

- If θ is an odd multiple of $\frac{\pi}{2}$ i.e. when $\theta = (2n+1)\frac{\pi}{2}, n \in I$, then $\sec \theta$ and $\tan \theta$ are not defined.
- If θ is an even multiple of $\frac{\pi}{2}$ i.e. when $\theta = n\pi, n \in I$, then $\csc \theta$ and $\cot \theta$ are not defined.
- The general solution of equations containing $\csc \theta, \sec \theta$ and $\cot \theta$ is equivalent to that of the equation involving $\sin \theta, \cos \theta$ and $\tan \theta$.
- When $\cos \theta = 0$ then $\sin \theta = 1$ or -1

$$\therefore \theta = \left(n + \frac{1}{2}\right)\pi$$

If $\sin \theta = 1$, then n is even and if $\sin \theta = -1$, then n is odd.

Similarly, when $\sin \theta = 0$ then $\cos \theta = 1$ or -1

$$\therefore \theta = n\pi$$

If $\cos \theta = 1$, then n is even and if $\cos \theta = -1$, then n is odd.

The general solution should be given unless the solution is required in a specified interval or range.

➤ **Example 37.** If the equation $\sin^2 \theta - \cos \theta = \frac{1}{4}$, then the values of θ lying in the interval $0 \leq \theta \leq 2\pi$ are

(a) $\frac{\pi}{3}, \frac{5\pi}{3}$ (b) $\frac{\pi}{3}, \frac{2\pi}{3}$ (c) $\frac{4\pi}{3}, \frac{5\pi}{3}$ (d) $\frac{3\pi}{5}, \frac{\pi}{5}$

Sol. (a) The given equation can be written as

$$1 - \cos^2 \theta - \cos \theta = \frac{1}{4}$$

$$\Rightarrow \cos^2 \theta + \cos \theta - \frac{3}{4} = 0$$

$$\Rightarrow 4\cos^2 \theta + 4\cos \theta - 3 = 0$$

$$\Rightarrow \cos \theta = \frac{-4 \pm \sqrt{16+48}}{8} = \frac{1}{2}, -\frac{3}{2}$$

Since, $\cos \theta = -\frac{3}{2}$ is not possible.

$$\therefore \cos \theta = \frac{1}{2} = \cos \frac{\pi}{3}$$

$$\Rightarrow \theta = 2n\pi \pm \frac{\pi}{3}$$

For the given interval, $n = 0$ and $n = 1$

$$\Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

➤ **Example 38.** The solution of $\tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$ is

(a) $x = n\pi, n \in I$ (b) $x = \frac{n\pi}{2}, n \in I$

(c) $x = \frac{n\pi}{3}, n \in I$ (d) None of these

Sol. (c) We know that,

$$\tan 3x = \tan (x+2x) = \frac{\tan x + \tan 2x}{1 - \tan x \tan 2x}$$

$$\Rightarrow \tan 3x - \tan x \tan 2x \tan 3x = \tan x + \tan 2x$$

$$\Rightarrow \tan x \tan 2x \tan 3x = \tan 3x - \tan x - \tan 2x \quad \dots (i)$$

$$\text{Now, } \tan x + \tan 2x + \tan 3x = \tan x \tan 2x \tan 3x$$

$$\Rightarrow \tan x + \tan 2x + \tan 3x = \tan 3x - \tan x - \tan 2x$$

[from Eq. (i)]

$$\Rightarrow 2 \tan x = -2 \tan 2x$$

$$\Rightarrow \tan 2x = \tan(-x)$$

$$2x = n\pi + (-x) \Rightarrow 3x = n\pi$$

$$\therefore x = \frac{n\pi}{3}, \text{ where } n \in I$$

➤ **Example 39.** The number of distinct solution of $\sin 5\theta \cos 3\theta = \sin 9\theta \cos 7\theta$ in $\left[0, \frac{\pi}{2}\right]$, is

(a) 4 (b) 5
(c) 8 (d) 9

Sol. (d) $\sin 8\theta + \sin 2\theta = \sin 16\theta + \sin 2\theta$ or $\sin 16\theta = \sin 8\theta$

$$\therefore 16\theta = n\pi + (-1)^n 8\theta$$

$$\Rightarrow 8\theta = 2m\pi, \text{ when } n \text{ is even, } n = 2m$$

$$24\theta = (2m+1)\pi, \text{ when } n \text{ is odd, } n = (2m+1)$$

$$\therefore \theta = \frac{m\pi}{4}, \frac{(2m+1)\pi}{24}, \text{ when } m \in I$$

$$= 0, \frac{\pi}{2}, \frac{\pi}{4} \text{ and } \frac{\pi}{24}, \frac{\pi}{8}, \frac{5\pi}{24}, \frac{9\pi}{24}, \frac{7\pi}{24}, \frac{11\pi}{24}$$

➤ **Example 40.** If $0 \leq x \leq \frac{\pi}{2}$ and

$81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x is equal to

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) 0

Sol. (a) Given, $81^{\sin^2 x} + 81^{\cos^2 x} = 30$

$$\Rightarrow 81^{\sin^2 x} + 81^{1-\sin^2 x} = 30$$

$$\Rightarrow 81^{\sin^2 x} + \frac{81}{81^{\sin^2 x}} = 30$$

$$\text{Let } 81^{\sin^2 x} = y$$

$$\therefore y + \frac{81}{y} = 30$$

$$\Rightarrow y^2 - 30y + 81 = 0$$

$$\Rightarrow (y - 27)(y - 3) = 0$$

$$\Rightarrow y = 27 \text{ or } y = 3$$

$$\Rightarrow 81^{\sin^2 x} = 27 \text{ or } 81^{\sin^2 x} = 3$$

$$\Rightarrow 3^{4\sin^2 x} = 3^3 \text{ or } 3^{4\sin^2 x} = 3^1$$

$$\Rightarrow \sin^2 x = \frac{3}{4} \text{ or } 4\sin^2 x = 1$$

$$\Rightarrow \sin x = \frac{\sqrt{3}}{2} \text{ or } \sin x = \frac{1}{2} \left[\because x \in \left[0, \frac{\pi}{2} \right] \right]$$

$$\Rightarrow x = \frac{\pi}{3} \text{ or } x = \frac{\pi}{6}$$

➤ **Example 41.** If $\frac{\tan 3\theta - 1}{\tan 3\theta + 1} = \sqrt{3}$, then the general value of θ is

- (a) $\frac{n\pi}{3} - \frac{\pi}{12}$ (b) $n\pi + \frac{7\pi}{12}$ (c) $\frac{n\pi}{3} + \frac{7\pi}{36}$ (d) $n\pi + \frac{\pi}{12}$

Sol. (c) Given, $\frac{\tan 3\theta - 1}{\tan 3\theta + 1} = \sqrt{3}$

$$\Rightarrow \tan 3\theta - 1 = \sqrt{3} \tan 3\theta + \sqrt{3}$$

$$\Rightarrow \tan 3\theta - 1 - \sqrt{3} \tan 3\theta - \sqrt{3} = 0$$

$$\Rightarrow \tan 3\theta (1 - \sqrt{3}) - (1 + \sqrt{3}) = 0$$

$$\Rightarrow \tan 3\theta = \frac{1 + \sqrt{3}}{1 - \sqrt{3}}$$

$$\Rightarrow \tan 3\theta = \tan 105^\circ = \tan \frac{7\pi}{12}$$

$$\therefore 3\theta = n\pi + \frac{7\pi}{12} \Rightarrow \theta = \frac{n\pi}{3} + \frac{7\pi}{36}$$

Solution of Trigonometric Equation of the Form $a \cos \theta + b \sin \theta = c$

Let the equation be

$$a \cos \theta + b \sin \theta = c$$

On dividing both sides by $\sqrt{a^2 + b^2}$, we get

$$\frac{a}{\sqrt{a^2 + b^2}} \cos \theta + \frac{b}{\sqrt{a^2 + b^2}} \sin \theta = \frac{c}{\sqrt{a^2 + b^2}} \dots (i)$$

$$\text{Let } \tan \alpha = \frac{b}{a}$$

$$\therefore \sin \alpha = \frac{b}{\sqrt{a^2 + b^2}}$$

$$\text{and } \cos \alpha = \frac{a}{\sqrt{a^2 + b^2}}$$

From Eq. (i), we get

$$\cos \theta \cos \alpha + \sin \theta \sin \alpha = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \cos (\theta - \alpha) = \frac{c}{\sqrt{a^2 + b^2}}$$

$$\text{If } |c| > \sqrt{a^2 + b^2}$$

Then, equation $a \cos \theta + b \sin \theta = c$ has no solution and if

$$|c| \leq \sqrt{a^2 + b^2}, \text{ then let}$$

$$\cos \phi = \frac{|c|}{\sqrt{a^2 + b^2}}$$

$$\Rightarrow \cos (\theta - \alpha) = \cos \phi$$

$$\Rightarrow \theta - \alpha = 2n\pi \pm \phi$$

$$\Rightarrow \theta = 2n\pi \pm \phi + \alpha, n \in I$$

- While solving a trigonometric equation, squaring the equation at any step should be avoided as far as possible.
- If squaring is necessary, check the solution for extraneous values.
- Never cancel terms containing unknown terms on the two sides, which are in product. It may cause loss of genuine solution.
- The answer should not contain such values of angles which make any of the terms undefined or infinite.
- Domain should not change. If it changes, necessary corrections must be made.
- Check that denominator is not zero at any stage while solving equations.

➤ **Example 42.** If $\sin x + \sqrt{3} \cos x = \sqrt{2}$, then x is equal to

- (a) $2n\pi + \frac{\pi}{12}, 2n\pi - \frac{\pi}{3}$ (b) $2n\pi + \frac{5\pi}{12}, 2n\pi - \frac{\pi}{12}$
(c) $2n\pi - \frac{\pi}{12}, 2n\pi - \frac{\pi}{12}$ (d) None of the above

Sol. (b) Given, $\sqrt{3} \cos x + \sin x = \sqrt{2}$

$$\Rightarrow \frac{\sqrt{3}}{2} \cos x + \frac{1}{2} \sin x = \frac{\sqrt{2}}{2}$$

$$\Rightarrow \cos \left(x - \frac{\pi}{6} \right) = \cos \frac{\pi}{4}$$

$$\Rightarrow x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4}$$

$$\Rightarrow x = 2n\pi \pm \frac{\pi}{4} + \frac{\pi}{6}$$

$$\Rightarrow x = 2n\pi + \frac{5\pi}{12}, 2n\pi - \frac{\pi}{12}, \text{ where } n \in I$$

➤ **Example 43.** General solution of

$$\sin x + \cos x = \min_{a \in \mathbb{R}} \{1, a^2 - 4a + 6\} \text{ is}$$

- (a) $\frac{n\pi}{2} + (-1)^n \frac{\pi}{4}$ (b) $2n\pi + (-1)^n \frac{\pi}{4}$
 (c) $n\pi + (-1)^{n+1} \frac{\pi}{4}$ (d) $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$

Sol. (d) Given,

$$\sin x + \cos x = \min_{a \in \mathbb{R}} \{1, a^2 - 4a + 6\}$$

$$\text{Now, } a^2 - 4a + 6 = (a - 2)^2 + 2$$

$$\therefore \min_{a \in \mathbb{R}} \{1, a^2 - 4a + 6\} = \min\{1, 2\} = 1$$

$$\text{Now, } \sin x + \cos x = 1$$

$$\Rightarrow \sin\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x + \frac{\pi}{4} = n\pi + (-1)^n \cdot \frac{\pi}{4}$$

$$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{4}$$

➤ **Example 44.** The equation $\sqrt{3} \sin x + \cos x = 4$ has

- (a) only one solution
 (b) two solutions
 (c) infinitely many solutions
 (d) no solution

Sol. (d) We know that,

$$a \cos x + b \sin x = c$$

$$\text{has solutions, if } |c| \leq \sqrt{a^2 + b^2}$$

So, given equation is

$$\sqrt{3} \sin x + \cos x = 4$$

$$\therefore \sqrt{a^2 + b^2} = \sqrt{1^2 + (\sqrt{3})^2} = \sqrt{4} = 2$$

$$\text{and } |c| = 4 \Rightarrow |c| > \sqrt{a^2 + b^2}$$

Hence, given equation has no solution.

➤ **Example 45.** If the equation $\tan \theta + \sec \theta = \sqrt{3}$, then the value of $\theta \in (0, 2\pi)$ is

- (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{2}$
 (c) $\frac{\pi}{6}$ (d) None of these

Sol. (c) Given, $\frac{\sin \theta}{\cos \theta} + \frac{1}{\cos \theta} = \sqrt{3}$ [$\because \cos \theta \neq 0$]

$$\Rightarrow \sqrt{3} \cos \theta - \sin \theta = 1$$

On dividing both sides by 2, we have

$$\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = \frac{1}{2}$$

$$\Rightarrow \cos\left(\theta + \frac{\pi}{6}\right) = \cos \frac{\pi}{3}$$

$$\Rightarrow \theta + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$$

Taking positive sign,

$$\theta + \frac{\pi}{6} = 2n\pi + \frac{\pi}{3}$$

$$\Rightarrow \theta = 2n\pi + \frac{\pi}{6} \quad \dots (i)$$

Taking negative sign,

$$\theta + \frac{\pi}{6} = 2n\pi - \frac{\pi}{3}$$

$$\Rightarrow \theta = 2n\pi - \frac{\pi}{2}$$

$$\Rightarrow \theta = (4n - 1) \frac{\pi}{2} \quad \dots (ii)$$

From Eqs. (i) and (ii), values of θ between 0 and 2π are $\frac{\pi}{6}, \frac{3\pi}{2}$.

But when $\theta = \frac{3\pi}{2}$, then $\tan \theta$ and $\sec \theta$ does not exist, so it is rejected.

$$\text{Hence, } \theta = \frac{\pi}{6}$$

Work Book Exercise 9.3

- 1 In a $\triangle ABC$, $\angle A$ is greater than $\angle B$. If the values of the angles A and B satisfy the equation $3 \sin x - 4 \sin^3 x - k = 0$, $0 < k < 1$, then the measure of $\angle C$ is

- a $\frac{\pi}{3}$ b $\frac{\pi}{2}$
 c $\frac{2\pi}{3}$ d $\frac{5\pi}{6}$

- 2 The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$, where x is a variable, has real roots. Then, p lies in

- a $(0, 2\pi)$ b $(-\pi, 0)$ c $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ d $(0, \pi)$

- 3 The smallest positive root of the equation $\tan x - x = 0$, lies in

- a $\left(0, \frac{\pi}{2}\right)$ b $\left(\frac{\pi}{2}, \pi\right)$ c $\left(\pi, \frac{3\pi}{2}\right)$ d $\left(\frac{3\pi}{2}, 2\pi\right)$

- 4 General solution of the equation

$$(\sqrt{3} - 1) \sin \theta + (\sqrt{3} + 1) \cos \theta = 2 \text{ is}$$

- a $2n\pi \pm \frac{\pi}{4} + \frac{\pi}{12}$ b $n\pi + (-1)^n \frac{\pi}{4} + \frac{\pi}{12}$
 c $2n\pi \pm \frac{\pi}{4} - \frac{\pi}{12}$ d $n\pi + (-1)^n \frac{\pi}{4} - \frac{\pi}{12}$

- 5 In a right angled triangle, the hypotenuse is $2\sqrt{2}$ times the length of perpendicular drawn from the opposite vertex on the hypotenuse, then the other two angles are

- a $\frac{\pi}{3}, \frac{\pi}{6}$
 b $\frac{\pi}{4}, \frac{\pi}{4}$
 c $\frac{\pi}{8}, \frac{3\pi}{8}$
 d $\frac{\pi}{12}, \frac{5\pi}{12}$

- 6 The solution set of $(2 \cos x - 1)(3 + 2 \cos x) = 0$ in the interval $0 \leq x \leq 2\pi$, is

- a $\left\{\frac{\pi}{3}\right\}$
b $\left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$
c $\left\{\frac{\pi}{3}, \frac{5\pi}{3}, \cos^{-1}\left(-\frac{3}{2}\right)\right\}$
d None of the above

- 7 The most general value of θ which satisfies both the equations $\tan \theta = -1$ and $\cos \theta = \frac{1}{\sqrt{2}}$, will be

- a $n\pi + \frac{7\pi}{4}$
b $n\pi + (-1)^n \frac{7\pi}{4}$
c $2n\pi + \frac{7\pi}{4}$
d None of the above

- 8 General solution of $4 \cot 2\theta = \cot^2 \theta - \tan^2 \theta$ is

- a $n\pi, n \in I$ b $n\pi + (-1)^n \frac{\pi}{4}, n \in I$
c $n\pi \pm \frac{\pi}{4}, n \in I$ d None of these

- 9 The number of solutions of $2 \sin^2 x + \sin^2 2x = 2$, $x \in [0, 2\pi]$ is

- a 4 b 5
c 7 d 6

- 10 If the expression $\frac{\sin \frac{x}{2} + \cos \frac{x}{2} - i \tan x}{1 + 2i \sin \frac{x}{2}}$ is real,

then x is equal to

- a $2n\pi + 2 \tan^{-1} k, k \in R, n \in Z$
b $2n\pi + 2 \tan^{-1} k$, where $k \in (0, 1), n \in Z$
c $2n\pi + 2 \tan^{-1} k$, where $k \in (1, 2), n \in Z$
d $2n\pi + 2 \tan^{-1} k, k \in (2, 3), n \in Z$

Solving Simultaneous Equations

Here, we will discuss problems related to the solution of two equation satisfied simultaneously. We may divide the problems in two categories :

- (i) Two equations in one 'unknown' satisfied simultaneously.
(ii) Two equations in two 'unknowns' satisfied simultaneously.

➤ **Example 46.** If the equations $\sin \theta = -\frac{1}{2}$ and

$\tan \theta = \frac{1}{\sqrt{3}}$, then most common general value of θ is

- (a) $2n\pi \pm \frac{7\pi}{6}$
(b) $2n\pi - \frac{7\pi}{6}$
(c) $2n\pi + \frac{7\pi}{6}$
(d) None of the above

Sol. (c) First find the values of θ lying between 0 and 2π and satisfying the two given equations separately. Select the value of θ which satisfies both the equations, then generalise it.

$$\begin{aligned} \sin \theta &= -\frac{1}{2} \\ \Rightarrow \theta &= \frac{7\pi}{6} \text{ or } \frac{11\pi}{6} \\ \tan \theta &= \frac{1}{\sqrt{3}} \\ \Rightarrow \theta &= \frac{\pi}{6} \text{ or } \frac{5\pi}{6} \end{aligned}$$

[values between 0 and 2π]

Common value of $\theta = \frac{7\pi}{6}$

The required solution is

$$\theta = 2n\pi + \frac{7\pi}{6}$$

➤ **Example 47.** If $r > 0$, $-\pi \leq \theta \leq \pi$ and r, θ satisfy $r \sin \theta = 3$ and $r = 4(1 + \sin \theta)$, then the number of possible solutions of the pair (r, θ) is

- (a) 2 (b) 4 (c) 0 (d) infinite

Sol. (a) On eliminating θ from two given equations, we get

$$r = 4 \left(1 + \frac{3}{r}\right)$$

$$\Rightarrow (r - 6)(r + 2) = 0 \Rightarrow r = 6$$

Neglecting $r = -2$, as $r > 0$

$$\therefore \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\text{So, } (r, \theta) = \left(6, \frac{\pi}{6}\right), \left(6, \frac{5\pi}{6}\right)$$

➤ **Example 48.** If $3 \sin^2 \theta + 2 \sin^2 \phi = 1$ and

$3 \sin 2\theta = 2 \sin 2\phi$, $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$, then the

value of $\theta + 2\phi$ is

- (a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$
(c) 0 (d) None of these

Sol. (a) Here, $3 \sin^2 \theta = \cos 2\phi$ and $3 \sin \theta \cos \theta = \sin 2\phi$

On squaring and adding, we get

$$9 \sin^2 \theta (\sin^2 \theta + \cos^2 \theta) = 1$$

$$\text{i.e. } \sin \theta = \frac{1}{3} \text{ and } \cos \theta = \frac{2\sqrt{2}}{3}$$

$$\therefore \cos 2\phi = 3 \cdot \frac{1}{9} = \frac{1}{3} \text{ and } \sin 2\phi = \frac{2\sqrt{2}}{3}$$

Now, $\cos(\theta + 2\phi) = \cos \theta \cos 2\phi - \sin \theta \sin 2\phi$

$$= \frac{2\sqrt{2}}{3} \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{2\sqrt{2}}{3} = 0$$

$$\text{and } \theta + 2\phi < \frac{3\pi}{2}$$

$$\therefore \theta + 2\phi = \frac{\pi}{2}$$

Solution of Trigonometric Inequality

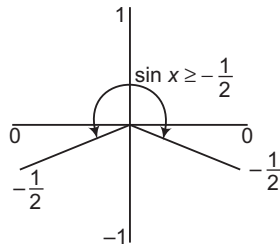
To solve trigonometric inequality, we must use graph of the trigonometric functions which would be clear from the following examples:

➤ **Example 49.** If $2\sin x + 1 \geq 0$ and $x \in [0, 2\pi]$, then the solution set for x is

- (a) $\left[0, \frac{7\pi}{6}\right]$ (b) $\left[0, \frac{7\pi}{6}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$
 (c) $\left[\frac{11\pi}{6}, 2\pi\right]$ (d) None of these

Sol. (b) $2\sin x + 1 \geq 0 \Rightarrow \sin x \geq -\frac{1}{2}$

The value scheme for this is shown below:



From the figure,

$$\frac{11\pi}{6} \leq x \leq 2\pi \text{ or } 0 \leq x \leq \frac{7\pi}{6}$$

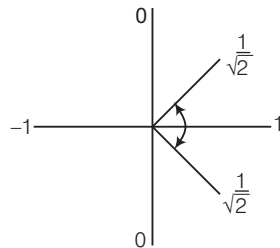
$$\therefore x \in \left[0, \frac{7\pi}{6}\right] \cup \left[\frac{11\pi}{6}, 2\pi\right]$$

➤ **Example 50.** If $\cos x - \sin x \geq 1$ and $0 \leq x \leq 2\pi$, then the solution set for x is

- (a) $\left[0, \frac{\pi}{4}\right] \cup \left[\frac{7\pi}{4}, 2\pi\right]$ (b) $\left[\frac{3\pi}{2}, \frac{7\pi}{4}\right] \cup \{0\}$
 (c) $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$ (d) None of these

Sol. (c) $\cos \left(x + \frac{\pi}{4}\right) \geq \frac{1}{\sqrt{2}}$

The value scheme for this is shown below:



From the figure, $-\frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{\pi}{4}$

and in general, $2n\pi - \frac{\pi}{4} \leq x + \frac{\pi}{4} \leq 2n\pi + \frac{\pi}{4}$

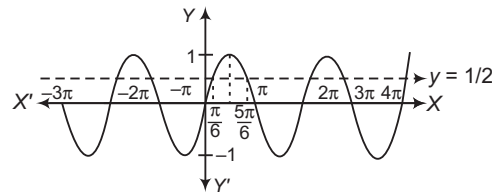
$$\therefore 2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi$$

For $n = 0$, $-\frac{\pi}{2} \leq x \leq 0$; for $n = 1$, $\frac{3\pi}{2} \leq x \leq 2\pi$

➤ **Example 51.** The solution set of inequality $\sin x > \frac{1}{2}$ is

- (a) $\bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$
 (b) $\bigcup_{n \in I} \left(2n\pi - \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$
 (c) $\bigcup_{n \in I} \left(n\pi + \frac{\pi}{6}, n\pi + \frac{5\pi}{6}\right)$
 (d) $\bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$

Sol. (d) From the graph of $y = \sin x$, it is obvious that between 0 and 2π ,



$$\sin x > \frac{1}{2} \text{ for } \frac{\pi}{6} < x < \frac{5\pi}{6}$$

$$\text{Hence, } \sin x > \frac{1}{2} \Rightarrow 2n\pi + \frac{\pi}{6} < x < 2n\pi + \frac{5\pi}{6}$$

$$\text{The required solution set is } \bigcup_{n \in I} \left(2n\pi + \frac{\pi}{6}, 2n\pi + \frac{5\pi}{6}\right)$$

Problems on Extremum Values

➤ **Example 52.** If equation $\sin^4 x = 1 + \tan^8 x$, then x is equal to

- (a) ϕ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

Sol. (a) $\sin^4 x = 1 + \tan^8 x$ only when $\sin^4 x = 1$ and $1 + \tan^8 x = 1$

$$\Rightarrow \sin^2 x = 1 \text{ and } \tan^8 x = 0$$

which is never possible, since $\sin x$ and $\tan x$ vanish simultaneously. Therefore, the given equation has no solution.

➤ **Example 53.** If $\sin^2 x + \cos^2 y = 2\sec^2 z$, then

- (a) $x = (2m+1)\frac{\pi}{2}$, $y = n\pi$ and $z = t\pi$
 (b) $x = (m+1)\frac{\pi}{2}$, $y = 2n\pi$ and $z = t\pi$
 (c) $x = (2m-1)\frac{\pi}{2}$, $y = n\pi$ and $z = 2t\pi$
 (d) None of the above

Sol. (a) $\sin^2 x + \cos^2 y = 2\sec^2 z$ only when $\sin^2 x = 1$, $\cos^2 y = 1$, $\sec^2 z = 1$

$$\Rightarrow \cos^2 x = 0, \sin^2 y = 0, \cos^2 z = 1$$

$$\Rightarrow \cos x = 0, \sin y = 0, \sin z = 0$$

$$\therefore x = (2m+1)\frac{\pi}{2}, y = n\pi \text{ and } z = t\pi$$

where, m , n and t are integers.

➤ **Example 54.** The number of solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \leq x \leq 2\pi$, is

- (a) 0 (b) 1
(c) 2 (d) 4

Sol. (b) Here, $x^3 + (x+2)^2 + 2\sin x = 4$

Clearly, $x = 0$ satisfies the equation.

If $0 < x \leq \pi$, $x^3 + (x+2)^2 + 2\sin x > 4$

If $\pi < x \leq 2\pi$, $x^3 + (x+2)^2 + 2\sin x > 27 + 25 - 2$

So, $x = 0$ is the only solution.

➤ **Example 55.** The number of real solutions of $\sin e^x \cdot \cos e^x = 2^{x-2} + 2^{-x-2}$ is

- (a) 0 (b) 1
(c) 2 (d) infinite

Sol. (a) Here, $\frac{1}{2} \sin(2e^x) = \frac{1}{4}(2^x + 2^{-x}) \geq \frac{1}{4} \cdot 2 = \frac{1}{2}$

So, $\sin(2e^x) \geq 1$

But $\sin(2e^x) \leq 1$

$\therefore \sin(2e^x) = 1$

But equality can hold when $2^x = 2^{-x} = 1$

i.e. $x = 0$

$\sin(2 \cdot e^0) = 1$

which is not true.

➤ **Example 56.** If $2\cos^2 \frac{x}{2} \sin^2 x = x^2 + x^{-2}$,

$0 < x \leq \frac{\pi}{2}$, then x is equal to

- (a) ϕ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}, \frac{\pi}{3}$

Sol. (a) LHS $= 2\cos^2 \frac{x}{2} \sin^2 x$

$$= (1 + \cos x) \sin^2 x < 2 \quad [\because 1 + \cos x < 2, \sin^2 x \leq 1]$$

$$\text{RHS} = x^2 + x^{-2} = x^2 + \frac{1}{x^2} \geq 2$$

Hence, the given equation has no solution.

➤ **Example 57.** If $\sin^6 x = 1 + \cos^4 3x$, then x is equal to

- (a) $(2n-1)\frac{\pi}{4}, n \in I$ (b) $(2n+1)\frac{3\pi}{4}, n \in I$
(c) $(2n+1)\frac{\pi}{2}, n \in I$ (d) None of these

Sol. (c) $\sin^6 x = 1 + \cos^4 3x \leq 1$

$\sin^6 x = 1 + \cos^4 3x \leq 1$ is possible only when

$$\sin^6 x = 1 + \cos^4 3x = 1$$

$$\Rightarrow \sin^6 x = 1 \quad \text{and} \quad 1 + \cos^4 3x = 1$$

$$\Rightarrow \sin^2 x = 1 \quad \text{and} \quad \cos^4 3x = 0$$

$$\Rightarrow \cos^2 x = 0 \quad \text{and} \quad \cos 3x = 0$$

$$\Rightarrow \cos x = 0 \quad \text{and} \quad \cos 3x = 0$$

$$\Rightarrow x = (2m+1)\frac{\pi}{2}$$

$$\text{and} \quad 3x = (2n+1)\frac{\pi}{2}, \text{ where } m, n \in I$$

$$\Rightarrow x = (2m+1)\frac{\pi}{2}$$

$$\text{and} \quad x = (2n+1)\frac{\pi}{6}$$

Common value of x is $(2n+1)\frac{\pi}{2}$, where $n \in I$.

The required solution is $x = (2n+1)\frac{\pi}{2}, n \in I$.

Work Book Exercise 9.4

1 If A and B are acute positive angles satisfying the equations $3\sin^2 A + 2\sin^2 B = 1$ and

$3\sin 2A - 2\sin 2B = 0$, then $A + 2B$ is equal to

- a 0 b $\frac{\pi}{2}$
c $\frac{\pi}{4}$ d $\frac{\pi}{3}$

2 If $\sin A = \sin B$, $\cos A = \cos B$, then the value of A in terms of B is

- a $n\pi + B$
b $n\pi + (-1)^n B$
c $2n\pi + B$
d $2n\pi - B$

3 The number of pairs (x, y) satisfying the equations $\sin x + \sin y = \sin(x+y)$ and $|x| + |y| = 1$ is

- a 2
b 4
c 6
d infinite

4 If $\cos(\alpha - \beta) = 1$ and $\cos(\alpha + \beta) = \frac{1}{e}$, then the number of ordered pairs $(\alpha, \beta) \in [-\pi, \pi]$ is

- a 0 b 1
c 2 d 4

5 The number of all possible ordered pairs (x, y) , $x, y \in R$ satisfying the system of equations

$$x + y = \frac{2\pi}{3}, \cos x + \cos y = \frac{3}{2}$$

- a 0
b 1
c infinite
d None of the above

6 The number of distinct roots of the equation $A\sin^3 x + B\cos^3 x + C = 0$ no two of which differ by 2π , is

- a 3
b 4
c infinite
d 6

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The maximum value of

$$4\sin^2 x + 3\cos^2 x + \sin \frac{x}{2} + \cos \frac{x}{2} \text{ is}$$

- (a) $4 + \sqrt{2}$ (b) $3 + \sqrt{2}$
(c) 9 (d) 4

Sol. Maximum value of $4\sin^2 x + 3\cos^2 x$ i.e. $\sin^2 x + 3$ is 4

and that of $\sin \frac{x}{2} + \cos \frac{x}{2}$ is $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \sqrt{2}$, both

attained at $x = \frac{\pi}{4}$. So, the given function has maximum value $4 + \sqrt{2}$.

Hence, (a) is the correct answer.

Ex 2. If in a $\triangle ABC$, $\angle C = 90^\circ$, then the maximum value of $\sin A \sin B$ is

- (a) $\frac{1}{2}$ (b) 1
(c) 2 (d) None of these

$$\begin{aligned} \text{Sol. } \sin A \sin B &= \frac{1}{2} \times 2 \sin A \sin B \\ &= \frac{1}{2} [\cos(A - B) - \cos(A + B)] \\ &= \frac{1}{2} [\cos(A - B) - \cos 90^\circ] \\ &= \frac{1}{2} \cos(A - B) \leq \frac{1}{2} \end{aligned}$$

$$\therefore \text{Maximum value of } \sin A \sin B = \frac{1}{2}$$

Hence, (a) is the correct answer.

Ex 3. The maximum value of $\sin(\cos x)$ is

- (a) $\sin 1$ (b) 1
(c) $\sin\left(\frac{1}{\sqrt{2}}\right)$ (d) $\sin\left(\frac{\sqrt{3}}{2}\right)$

Sol. $\cos x \in [-1, 1]$, $\forall x \in R$ and $\sin x$ is increasing in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\therefore \text{Maximum value of } \sin(\cos x) = \sin 1$$

Hence, (a) is the correct answer.

Ex 4. If $x \in (\pi, 2\pi)$ and

$$\frac{\sqrt{1+\cos x} + \sqrt{1-\cos x}}{\sqrt{1+\cos x} - \sqrt{1-\cos x}} = \cot\left(a + \frac{x}{2}\right), \text{ then } a$$

is equal to

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{3}$ (d) None of these

$$\begin{aligned} \text{Sol. } \therefore \sqrt{1+\cos x} &= \sqrt{2\cos^2 \frac{x}{2}} = \sqrt{2} \left| \cos \frac{x}{2} \right| \\ \text{and } \sqrt{1-\cos x} &= \sqrt{2\sin^2 \frac{x}{2}} = \sqrt{2} \left| \sin \frac{x}{2} \right| \\ \therefore \frac{\sqrt{1+\cos x} + \sqrt{1-\cos x}}{\sqrt{1+\cos x} - \sqrt{1-\cos x}} &= \frac{\left| \cos \frac{x}{2} \right| + \left| \sin \frac{x}{2} \right|}{\left| \cos \frac{x}{2} \right| - \left| \sin \frac{x}{2} \right|} = \frac{-\cos \frac{x}{2} + \sin \frac{x}{2}}{-\cos \frac{x}{2} - \sin \frac{x}{2}} \\ &= \frac{\cos \frac{x}{2} - \sin \frac{x}{2}}{\cos \frac{x}{2} + \sin \frac{x}{2}} \\ &= \frac{1 - \tan \frac{x}{2}}{1 + \tan \frac{x}{2}} = \tan\left(\frac{\pi}{4} - \frac{x}{2}\right) \\ &= \cot\left[\frac{\pi}{2} - \left(\frac{\pi}{4} + \frac{x}{2}\right)\right] = \cot\left(\frac{\pi}{4} + \frac{x}{2}\right) \end{aligned}$$

$$\therefore a = \frac{\pi}{4}$$

Hence, (a) is the correct answer.

Ex 5. Let $0 < A, B < \frac{\pi}{2}$ satisfying the equation

$$3\sin^2 A + 2\sin^2 B = 1 \text{ and } 3\sin 2A - 2\sin 2B = 0, \text{ then } A + 2B \text{ is equal to}$$

- (a) π (b) $\frac{\pi}{2}$
(c) $\frac{\pi}{4}$ (d) 2π

Sol. From the second equation, we have

$$\sin 2B = \frac{3}{2} \sin 2A \quad \dots(i)$$

and from the first equality,

$$3\sin^2 A = 1 - 2\sin^2 B = \cos 2B \quad \dots(ii)$$

$$\text{Now, } \cos(A + 2B) = \cos A \cos 2B - \sin A \sin 2B$$

$$= 3\cos A \sin^2 A - \frac{3}{2} \sin A \sin 2A$$

$$= 3\cos A \sin^2 A - 3\sin^2 A \cos A = 0$$

$$\Rightarrow A + 2B = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}$$

$$\text{Given that, } 0 < A < \frac{\pi}{2} \text{ and } 0 < B < \frac{\pi}{2}$$

$$\Rightarrow 0 < A + 2B < \frac{\pi}{2} + \pi$$

$$\text{So, } A + 2B = \frac{\pi}{2}, \text{ neglecting } \frac{3\pi}{2}$$

Hence, (b) is the correct answer.

Ex 6. If α, β, γ and δ are the solutions of the equation

$\tan\left(\theta + \frac{\pi}{4}\right) = 3 \tan 3\theta$, no two of which have equal tangents, then the value of $\tan \alpha + \tan \beta + \tan \gamma + \tan \delta$ is

- (a) 1 (b) -1
(c) 2 (d) 0

Sol. We have,

$$\tan\left(\theta + \frac{\pi}{4}\right) = 3 \tan 3\theta$$

$$\Rightarrow \frac{1 + \tan \theta}{1 - \tan \theta} = 3 \cdot \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$$

$$\Rightarrow \frac{1+t}{1-t} = 3 \left(\frac{3t-t^3}{1-3t^2} \right) \quad [\text{putting } t = \tan \theta]$$

$$\Rightarrow 3t^4 - 6t^2 + 8t - 1 = 0$$

$$\text{So, } t_1 + t_2 + t_3 + t_4 = \frac{0}{3} = 0$$

$$\therefore \tan \alpha + \tan \beta + \tan \gamma + \tan \delta = 0$$

Hence, (d) is the correct answer.

Ex 7. For all θ in $\left[0, \frac{\pi}{2}\right]$, $\cos(\sin \theta)$ is

- (a) $< \sin(\cos \theta)$ (b) $> \sin(\cos \theta)$
(c) $= \sin(\cos \theta)$ (d) None of these

Sol. We have,

$$\begin{aligned} \cos \theta + \sin \theta &= \sqrt{2} \left[\frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta \right] \\ &= \sqrt{2} \left[\cos \frac{\pi}{4} \cos \theta + \sin \frac{\pi}{4} \sin \theta \right] \\ &= \sqrt{2} \cos \left(\theta - \frac{\pi}{4} \right) \end{aligned}$$

$$\Rightarrow \cos \theta + \sin \theta \leq \sqrt{2} < \frac{\pi}{2}$$

$$\Rightarrow \cos \theta < \frac{\pi}{2} - \sin \theta$$

$$\Rightarrow \sin(\cos \theta) < \sin \left(\frac{\pi}{2} - \sin \theta \right)$$

$$\Rightarrow \sin(\cos \theta) < \cos(\sin \theta)$$

$$\Rightarrow \cos(\sin \theta) > \sin(\cos \theta)$$

Hence, (b) is the correct answer.

Ex 8. The smallest positive number p for which the equation $\cos(p \sin x) = \sin(p \cos x)$ has a solution $x \in [0, 2\pi]$, is equal to

- (a) π (b) $\frac{\pi}{\sqrt{2}}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2\sqrt{2}}$

Sol. We have, $\cos(p \sin x) = \sin(p \cos x)$

$$\Rightarrow \sin\left(\frac{\pi}{2} \pm p \sin x\right) = \sin(p \cos x)$$

$$\Rightarrow \frac{\pi}{2} \pm p \sin x = p \cos x$$

$$\Rightarrow \frac{\pi}{2} = p \cos x \pm p \sin x$$

$$\Rightarrow \frac{\pi}{2p} = \cos x \pm \sin x = \sqrt{2} \left(\frac{1}{\sqrt{2}} \cos x \pm \frac{1}{\sqrt{2}} \sin x \right)$$

$$\Rightarrow \frac{\pi}{2\sqrt{2}p} = \cos\left(x \pm \frac{\pi}{4}\right) \leq 1$$

$$\Rightarrow p \geq \frac{\pi}{2\sqrt{2}}$$

$$\text{Now, the smallest value of } p = \frac{\pi}{2\sqrt{2}}$$

Hence, (d) is the correct answer.

Ex 9. If α and β are solutions of $\sin^2 x + a \sin x + b = 0$ as well that of $\cos^2 x + c \cos x + d = 0$, then $(\alpha + \beta)$ is equal to

- (a) $\frac{2bd}{b^2 + d^2}$ (b) $\frac{a^2 + c^2}{2ac}$
(c) $\frac{b^2 + d^2}{2bd}$ (d) $\frac{2ac}{a^2 + c^2}$

Sol. According to the given condition,

$$\sin \alpha + \sin \beta = -a \text{ and } \cos \alpha + \cos \beta = -c$$

$$\Rightarrow 2 \sin \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} = -a$$

$$\text{and } 2 \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} = -c$$

$$\Rightarrow \tan \frac{\alpha + \beta}{2} = \frac{a}{c}$$

$$\Rightarrow \sin(\alpha + \beta) = \frac{2 \tan \frac{\alpha + \beta}{2}}{1 + \tan^2 \frac{\alpha + \beta}{2}} = \frac{2ac}{a^2 + c^2}$$

Hence, (d) is the correct answer.

Ex 10. If $S = \cos^2 \frac{\pi}{n} + \cos^2 \frac{2\pi}{n} + \dots + \cos^2 \frac{(n-1)\pi}{n}$, then S equals

- (a) $\frac{n}{2}(n+1)$ (b) $\frac{1}{2}(n-1)$
(c) $\frac{1}{2}(n-2)$ (d) $\frac{n}{2}$

$$\begin{aligned} \text{Sol. } S &= \cos^2 \frac{\pi}{n} + \cos^2 \frac{2\pi}{n} + \dots + \cos^2 \frac{(n-1)\pi}{n} \\ &= \frac{1}{2} \left[1 + \cos \frac{2\pi}{n} + 1 + \cos \frac{4\pi}{n} + 1 + \cos \frac{6\pi}{n} \right. \\ &\quad \left. + \dots + 1 + \cos 2(n-1) \frac{\pi}{n} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left[n - 1 + \sum_{k=1}^{n-1} \cos \frac{2k\pi}{n} \right] \\ &= \frac{1}{2} [n - 1 - 1] = \frac{1}{2}(n - 2) \end{aligned}$$

Hence, (c) is the correct answer.

Ex 11. If $\sin \theta = 3 \sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2 \tan \alpha$ is

- (a) 3 (b) 2 (c) 1 (d) 0

Sol. Given,

$$\sin \theta = 3 \sin(\theta + 2\alpha)$$

$$\Rightarrow \sin(\theta + \alpha - \alpha) = 3 \sin(\theta + \alpha + \alpha)$$

$$\Rightarrow \sin(\theta + \alpha) \cos \alpha - \cos(\theta + \alpha) \sin \alpha$$

$$= 3 \sin(\theta + \alpha) \cos \alpha + 3 \cos(\theta + \alpha) \sin \alpha$$

$$\Rightarrow -2 \sin(\theta + \alpha) \cos \alpha = 4 \cos(\theta + \alpha) \sin \alpha$$

$$\Rightarrow \frac{\sin(\theta + \alpha)}{\cos(\theta + \alpha)} = \frac{2\sin\alpha}{\cos\alpha}$$

$$\Rightarrow \tan(\theta + \alpha) + 2\tan\alpha = 0$$

Hence, (d) is the correct answer.

Ex 12. The numerical value of $\tan 20^\circ \tan 80^\circ \cot 50^\circ$ is

- (a) $\sqrt{3}$ (b) $\frac{1}{\sqrt{3}}$
(c) $2\sqrt{3}$ (d) $\frac{1}{2\sqrt{3}}$

Sol. $\tan 20^\circ \tan 80^\circ \cot 50^\circ$

$$= \tan(50^\circ - 30^\circ) \tan(50^\circ + 30^\circ) \frac{1}{\tan 50^\circ}$$

$$= \frac{\tan 50^\circ - 1/\sqrt{3}}{1 + \tan 50^\circ/\sqrt{3}} \cdot \frac{\tan 50^\circ + 1/\sqrt{3}}{1 - \tan 50^\circ/\sqrt{3}} \cdot \frac{1}{\tan 50^\circ}$$

$$= \frac{3\tan^2 50^\circ - 1}{3\tan 50^\circ - \tan^3 50^\circ} = -\cot(150^\circ) = \cot 30^\circ = \sqrt{3}$$

Hence, (a) is the correct answer.

Ex 13. If $A + B = \frac{\pi}{3}$, where $A, B \in R^+$, then the minimum value of $\sec A + \sec B$ is

- (a) $\frac{2}{\sqrt{3}}$ (b) $\frac{4}{\sqrt{3}}$
(c) $2\sqrt{3}$ (d) None of these

Sol. For $y = \sec x$, $x \in [0, \pi/2]$, tangent drawn to it any point lies completely below the graph of $y = \sec x$, thus

$$\frac{\sec A + \sec B}{2} \geq \sec\left(\frac{A+B}{2}\right)$$

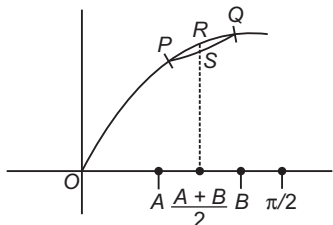
$$\Rightarrow \sec A + \sec B \geq 2 \cdot \sec\left(\frac{\pi}{6}\right) = \frac{4}{\sqrt{3}}$$

Hence, (b) is the correct answer.

Ex 14. If $A + B = \frac{\pi}{2}$, where $A, B \in R^+$, then the maximum value of $\sin A + \sin B$ is

- (a) 1
(b) $\sqrt{3}$
(c) $\sqrt{2}$
(d) None of the above

Sol. For $y = \sin x$; $x \in [0, \pi/2]$



From the above curve,

$$\frac{\sin A + \sin B}{2} \leq \sin\left(\frac{A+B}{2}\right) = \sin \frac{\pi}{4}$$

$$\Rightarrow \sin A + \sin B \leq \sqrt{2}$$

Hence, (c) is the correct answer.

Ex 15. If $\cos x = \frac{2\cos y - 1}{2 - \cos y}$, where $x, y \in (0, \pi)$, then

$\tan \frac{x}{2} \cot \frac{y}{2}$ is equal to

- (a) $\sqrt{2}$ (b) $\sqrt{3}$
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

Sol. Given, $\cos x = \frac{2\cos y - 1}{2 - \cos y}$

$$\Rightarrow \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{\frac{2(1 - \tan^2 y/2) - 1}{1 + \tan^2 y/2} - 1}{2 - \frac{1 - \tan^2 y/2}{1 + \tan^2 y/2}}$$

$$\Rightarrow 6\tan^2 \frac{y}{2} = 2\tan^2 \frac{x}{2}$$

$$\Rightarrow \tan \frac{x}{2} \cdot \cot \frac{y}{2} = \sqrt{3}$$

Hence, (b) is the correct answer.

Ex 16. If $\sin x + \sin^2 x = 1$, then the value of $\cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$ is

- (a) 1 (b) -1
(c) 2 (d) -2

Sol. Let $I = \cos^{12} x + 3\cos^{10} x + 3\cos^8 x + \cos^6 x$
 $= \cos^{12} x + 3 \cdot \cos^8 x (\cos^2 x + 1) + \cos^6 x$
 $= (\cos^4 x)^3 + 3\cos^6 x (\cos^4 x + \cos^2 x) + (\cos^2 x)^3$
 $= (\cos^4 x + \cos^2 x)^3 = [\cos^2 x (\cos^2 x + 1)]^3$

We have, $\sin x = 1 - \sin^2 x = \cos^2 x$

$$\Rightarrow \sin x = \cos^2 x = \frac{-1 + \sqrt{5}}{2}$$

$$\Rightarrow I = \left[\frac{\sqrt{5} - 1}{2} \left(\frac{\sqrt{5} + 1}{2} \right) \right]^3 = 1$$

Hence, (a) is the correct answer.

Ex 17. If $\frac{1 + \sin 2x}{1 - \sin 2x} = \cot^2(a + x)$,

$\forall x \in R \sim \left(n\pi + \frac{\pi}{4} \right)$, $n \in N$, then a is equal to

- (a) $\frac{\pi}{4}$
(b) $\frac{\pi}{2}$
(c) $\frac{3\pi}{4}$
(d) None of the above

Sol. $\frac{1 + \sin 2x}{1 - \sin 2x} = \frac{(\sin x + \cos x)^2}{(\sin x - \cos x)^2} = \left(\frac{1 + \cot x}{1 - \cot x} \right)^2$
 $\Rightarrow \left(\frac{1 + \tan x}{1 - \tan x} \right)^2 = \left[\tan\left(\frac{\pi}{4} + x\right) \right]^2 = \tan^2\left(\frac{\pi}{4} + x\right)$
 $= \cot^2\left(\frac{\pi}{2} + \frac{\pi}{4} + x\right) = \cot^2\left(\frac{3\pi}{4} + x\right)$

Hence, (c) is the correct answer.

Ex 18. If $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$, then the value of $\cot \alpha \tan \beta$ is

- (a) -1 (b) 0
(c) 1 (d) None of these

Sol. Given, $\sin \alpha \sin \beta - \cos \alpha \cos \beta + 1 = 0$
 $\Rightarrow \cos(\alpha + \beta) = 1 \Rightarrow \alpha + \beta = 2n\pi$
 $1 + \cot \alpha \tan \beta = 1 + \frac{\cos \alpha}{\sin \alpha} \cdot \frac{\sin \beta}{\cos \beta}$
 $= \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\sin \alpha \cos \beta} = \frac{\sin(\alpha + \beta)}{\sin \alpha \cos \beta}$
 $= \frac{\sin(2n\pi)}{\sin \alpha \cos \beta} = 0$
 $\Rightarrow \cot \alpha \tan \beta = -1$
Hence, (a) is the correct answer.

Ex 19. If $\tan x = n \tan y$, $n \in R^+$, then maximum value of $\sec^2(x - y)$ is

- (a) $\frac{(n+1)^2}{2n}$ (b) $\frac{(n+1)^2}{n}$ (c) $\frac{(n+1)^2}{2}$ (d) $\frac{(n+1)^2}{4n}$

Sol. Given, $\tan x = n \tan y$
We know that, $\cos(x - y) = \cos x \cos y + \sin x \sin y$
 $\Rightarrow \cos(x - y) = \cos x \cos y (1 + \tan x \tan y)$
 $= \cos x \cos y (1 + n \tan^2 y)$
 $\Rightarrow \sec^2(x - y) = \frac{\sec^2 x \sec^2 y}{(1 + n \tan^2 y)^2}$
 $= \frac{(1 + \tan^2 x)(1 + \tan^2 y)}{(1 + n \tan^2 y)^2}$
 $= \frac{(1 + n^2 \tan^2 y)(1 + \tan^2 y)}{(1 + n \tan^2 y)^2}$
 $= 1 + \frac{(n-1)^2 \tan^2 y}{(1 + n \tan^2 y)^2}$
Now, $\left(\frac{1 + n \tan^2 y}{2} \right)^2 \geq n \tan^2 y$
 $\Rightarrow \frac{\tan^2 y}{(1 + n \tan^2 y)^2} \leq \frac{1}{4n}$
 $\Rightarrow \sec^2(x - y) \leq 1 + \frac{(n-1)^2}{4n} = \frac{(n+1)^2}{4n}$
Hence, (d) is the correct answer.

Ex 20. If $f(x) = \cos x (\sin x + \sqrt{\sin^2 x + \sin^2 \theta})$, where θ is a given constant, then maximum value of $f(x)$ is

- (a) $\sqrt{1 + \cos^2 \theta}$ (b) $\sqrt{1 + \sin^2 \theta}$
(c) $|\cos \theta|$ (d) $|\sin \theta|$

Sol. $f(x) = \cos x (\sin x + \sqrt{\sin^2 x + \sin^2 \theta})$
 $\Rightarrow (f(x) \cdot \sec x - \sin x)^2 = \sin^2 x + \sin^2 \theta$
 $\Rightarrow f^2(x) \cdot \sec^2 x - 2f(x) \cdot \tan x + \sin^2 x = \sin^2 x + \sin^2 \theta$
 $\Rightarrow f^2(x) \cdot \tan^2 x - 2f(x) \cdot \tan x + f^2(x) - \sin^2 \theta = 0$
 $\Rightarrow 4f^2(x) \geq 4f^2(x)[f^2(x) - \sin^2 \theta]$
 $\Rightarrow f^2(x) \leq 1 + \sin^2 \theta \Rightarrow |f(x)| \leq \sqrt{1 + \sin^2 \theta}$
Hence, (b) is the correct answer.

Ex 21. If $a \cos 2\theta + b \sin 2\theta = c$ has α and β as its solutions, then $\tan \alpha + \tan \beta$ and $\tan \alpha \tan \beta$ are, respectively

- (a) $\frac{2b}{c+a}, \frac{c-a}{c+a}$ (b) $\frac{c+a}{2b}, \frac{c-a}{c+a}$
(c) $\frac{2b}{c-a}, \frac{c-a}{c+a}$ (d) $\frac{2b}{c+a}, \frac{c+a}{c-a}$

Sol. We have, $a \cos 2\theta + b \sin 2\theta = c$
 $\Rightarrow a(\cos^2 \theta - \sin^2 \theta) + 2b \sin \theta \cos \theta = c$
 $\Rightarrow a(1 - \tan^2 \theta) + 2b \tan \theta = c \sec^2 \theta = c(1 + \tan^2 \theta)$
 $\Rightarrow \tan^2 \theta(c+a) - 2b \tan \theta + c-a = 0$
This equation has $\tan \alpha$ and $\tan \beta$ as its roots.
 $\Rightarrow \tan \alpha + \tan \beta = \frac{2b}{(c+a)}$ and $\tan \alpha \tan \beta = \left(\frac{c-a}{c+a} \right)$
Hence, (a) is the correct answer.

Ex 22. If $\frac{\tan^3 A}{\tan A} = k$, then $\frac{\sin 3A}{\sin A}$ is equal to

- (a) $\frac{2k}{k-1}$ (b) $\frac{2k}{k+1}$
(c) $\frac{3k}{k+1}$ (d) None of these

Sol. $\frac{\tan 3A}{\tan A} = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A} \cdot \frac{1}{\tan A} = k$
 $\Rightarrow \frac{3 - \tan^2 A}{1 - 3 \tan^2 A} = k$
 $\Rightarrow 3 - \tan^2 A = k(1 - 3 \tan^2 A)$
 $\Rightarrow (3k - 1) \tan^2 A = k - 3$
 $\therefore \tan^2 A = \frac{k-3}{3k-1}$
Now, $\frac{\sin 3A}{\sin A} = \frac{3 \sin A - 4 \sin^3 A}{\sin A}$
 $= 3 - 4 \sin^2 A = 3 - \frac{4}{1 + \frac{1}{\tan^2 A}}$
 $= 3 - \frac{4}{1 + \frac{3k-1}{k-3}} = 3 - \frac{4(k-3)}{4(k-1)}$
 $= \frac{3k-3-k+3}{k-1} = \frac{2k}{k-1}$
Hence, (a) is the correct answer.

Ex 23. If $1 + \cot \theta \leq \frac{\pi}{2}$, for $0 < \theta < \pi$, then θ is equal to

- (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{8}$

Sol. $\therefore \tan \theta = \frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}}$
 $\Rightarrow \cot \theta = \frac{\cot^2 \frac{\theta}{2} - 1}{2 \cot \frac{\theta}{2}}$

$$\begin{aligned} \text{Now, } 1 + \cot\theta - \cot\frac{\theta}{2} &= 1 + \frac{\cot^2\frac{\theta}{2} - 1}{2\cot\frac{\theta}{2}} - \cot\frac{\theta}{2} \\ &= \frac{2\cot\frac{\theta}{2} + \cot^2\frac{\theta}{2} - 1 - 2\cot^2\frac{\theta}{2}}{2\cot\frac{\theta}{2}} \\ &= \frac{-\left(\cot\frac{\theta}{2} - 1\right)^2}{2\cot\frac{\theta}{2}} \leq 0, \text{ for } 0 < \theta < \pi \\ \Rightarrow 1 + \cot\theta - \cot\frac{\theta}{2} &\leq 0 \\ \text{Equality holds when } \cot\frac{\theta}{2} - 1 &= 0 \\ \therefore \theta &= \frac{\pi}{2} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 24. The value of $\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15}$ is

- (a) $\frac{1}{16}$ (b) $\frac{1}{32}$
(c) $\frac{1}{64}$ (d) $\frac{1}{8}$

$$\begin{aligned} \text{Sol. } \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} &= \frac{1}{2\sin\frac{2\pi}{15}} \cdot 2\sin\frac{2\pi}{15}\cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} \\ &= \frac{1}{2\sin\frac{2\pi}{15}} \cdot \sin\frac{4\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} \\ &= \frac{1}{4\sin\frac{2\pi}{15}} \cdot \sin\frac{8\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} \\ &= \frac{1}{8\sin\frac{2\pi}{15}} \cdot \sin\frac{16\pi}{15}\cos\frac{16\pi}{15} \\ &= \frac{1}{16\sin\frac{2\pi}{15}} \cdot \sin\frac{32\pi}{15} \\ &= \frac{1}{16\sin\frac{2\pi}{15}} \cdot \sin\left(2\pi + \frac{2\pi}{15}\right) \\ &= \frac{1}{16\sin\frac{2\pi}{15}} \cdot \sin\frac{2\pi}{15} = \frac{1}{16} \end{aligned}$$

Aliter

$$\begin{aligned} \text{Use } \cos A \cos 2A \cos 2^2 A \dots \cos 2^{n-1} A &= \frac{\sin 2^n A}{2^n \sin A} \\ \therefore \cos\frac{2\pi}{15}\cos\frac{4\pi}{15}\cos\frac{8\pi}{15}\cos\frac{16\pi}{15} &= \frac{\sin 2^4 \cdot \frac{\pi}{15}}{2^4 \sin \frac{\pi}{15}} = \frac{1}{16} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 25. The value of $\left(1 + \cos\frac{\pi}{8}\right)\left(1 + \cos\frac{3\pi}{8}\right)$
 $\left(1 + \cos\frac{5\pi}{8}\right)\left(1 + \cos\frac{7\pi}{8}\right)$ is
(a) $\frac{1}{4}$ (b) $\frac{1}{6}$ (c) $\frac{1}{8}$ (d) $\frac{1}{2}$

$$\begin{aligned} \text{Sol. Using } \cos\frac{5\pi}{8} &= \cos\left(\frac{\pi}{2} + \frac{\pi}{8}\right) = -\sin\frac{\pi}{8} \\ \Rightarrow \cos\frac{3\pi}{8} &= \cos\left(\frac{\pi}{2} - \frac{\pi}{8}\right) = \sin\frac{\pi}{8} \\ \Rightarrow \cos\frac{7\pi}{8} &= \cos\left(\pi - \frac{\pi}{8}\right) = -\cos\frac{\pi}{8} \\ \therefore \left(1 + \cos\frac{\pi}{8}\right)\left(1 - \cos\frac{\pi}{8}\right)\left(1 + \sin\frac{\pi}{8}\right)\left(1 - \sin\frac{\pi}{8}\right) \\ &= \left(1 - \cos^2\frac{\pi}{8}\right)\left(1 - \sin^2\frac{\pi}{8}\right) = \sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8} \\ &= \frac{1}{4}\left(\sin\frac{\pi}{4}\right)^2 = \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 26. Let $\cos A + \cos B + \cos C = \frac{3}{2}$ in a $\triangle ABC$. Then,

the triangle is

- (a) isosceles (b) right angled
(c) equilateral (d) None of these

$$\begin{aligned} \text{Sol. In } \triangle ABC, A + B + C &= \pi \\ \Rightarrow \cos A + \cos B + \cos C &= 2\cos\left(\frac{A+B}{2}\right) \\ &\quad \cos\left(\frac{A-B}{2}\right) + \cos C = \frac{3}{2} \\ \Rightarrow 2\cos\left(\frac{\pi}{2} - \frac{C}{2}\right)\cos\left(\frac{A-B}{2}\right) + 1 - 2\sin^2\frac{C}{2} &= \frac{3}{2} \\ \Rightarrow 4\sin^2\frac{C}{2} - 4\sin\frac{C}{2}\cos\left(\frac{A-B}{2}\right) + 1 &= 0 \quad \dots(i) \end{aligned}$$

Now, $\sin\frac{C}{2}$ is real.

$$\Rightarrow 16\cos^2\left(\frac{A-B}{2}\right) - 16 > 0$$

$$\Rightarrow \cos^2\left(\frac{A-B}{2}\right) - 1 \geq 0$$

$$\Rightarrow \cos^2\left(\frac{A-B}{2}\right) = 1 = \cos 0$$

$$\therefore A = B \quad \dots(ii)$$

Similarly, it can be shown that $B = C, C = A$. So, the triangle is equilateral.

Hence, (c) is the correct answer.

Ex 27. $\left(\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}\right)$
 $\left(\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7}\right)$ is equal to
(a) -105 (b) 105
(c) ± 105 (d) None of these

Sol. Let $\theta = \frac{n\pi}{7}$

$$\begin{aligned} \Rightarrow 7\theta &= n\pi \\ \Rightarrow 4\theta + 3\theta &= n\pi \\ \Rightarrow \tan 4\theta &= \tan(n\pi - 3\theta) \\ \Rightarrow \tan 4\theta &= -\tan 3\theta \\ \Rightarrow \frac{4\tan\theta - 4\tan^3\theta}{1 - 6\tan^2\theta + \tan^4\theta} &= -\frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta} \\ \Rightarrow \frac{4z - 4z^3}{1 - 6z^2 + z^4} &= -\frac{3z - z^3}{1 - 3z^2} \end{aligned}$$

[where, $\tan\theta = z$ (say)]

$$\begin{aligned} \Rightarrow (4 - 4z^2)(1 - 3z^2) &= -(3 - z^2)(1 - 6z^2 + z^4) \\ \Rightarrow z^6 - 21z^4 + 35z^2 - 7 &= 0 \quad \dots(i) \end{aligned}$$

This is a cubic equation in z^2 i.e. in $\tan^2\theta$. Therefore, the roots of this equation are $\tan^2\frac{\pi}{7}$, $\tan^2\frac{2\pi}{7}$ and

$$\tan^2\frac{3\pi}{7}.$$

From Eq. (i), sum of the roots = $\frac{-(-21)}{1} = 21$

$$\Rightarrow \tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7} = 21 \quad \dots(ii)$$

On putting $\frac{1}{y}$ in place of z in Eq. (i), we get

$$\begin{aligned} -7y^6 + 35y^4 - 21y^2 + 1 &= 0 \\ \Rightarrow 7y^6 - 35y^4 + 21y^2 - 1 &= 0 \quad \dots(iii) \end{aligned}$$

This is a cubic equation in y^2 i.e. in $\cot^2\theta$. Therefore, the roots of this equation are, $\cot^2\frac{\pi}{7}$, $\cot^2\frac{2\pi}{7}$ and

$$\cot^2\frac{3\pi}{7}.$$

From Eq. (iii), sum of the roots = $\frac{35}{7}$

$$\Rightarrow \cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7} = 5 \quad \dots(iv)$$

By multiplying Eqs. (ii) and (iv), we get

$$\begin{aligned} \left(\tan^2\frac{\pi}{7} + \tan^2\frac{2\pi}{7} + \tan^2\frac{3\pi}{7}\right) \\ \left(\cot^2\frac{\pi}{7} + \cot^2\frac{2\pi}{7} + \cot^2\frac{3\pi}{7}\right) &= 21 \times 5 = 105 \end{aligned}$$

Hence, (b) is the correct answer.

Ex 28. $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$ is equal to

- (a) $\frac{\sqrt{7}}{2}$
(b) $\sqrt{7}$
(c) 2
(d) $\frac{\sqrt{7}}{4}$

Sol. Let $x = \sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7}$

On squaring both sides, we get

$$\begin{aligned} x^2 &= \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7} + 2\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} \\ &\quad + 2\sin\frac{4\pi}{7}\sin\frac{8\pi}{7} + 2\sin\frac{8\pi}{7}\sin\frac{2\pi}{7} \end{aligned}$$

Now consider,

$$\begin{aligned} 2\sin\frac{2\pi}{7}\sin\frac{4\pi}{7} + 2\sin\frac{4\pi}{7}\sin\frac{8\pi}{7} + 2\sin\frac{8\pi}{7}\sin\frac{2\pi}{7} \\ = \left[\cos\frac{2\pi}{7} - \cos\frac{6\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{12\pi}{7} \right. \\ \left. + \cos\frac{6\pi}{7} - \cos\frac{10\pi}{7} \right] \\ = \left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\left(2\pi - \frac{2\pi}{7}\right) \right. \\ \left. - \cos\left(2\pi - \frac{4\pi}{7}\right) \right] \end{aligned}$$

$$= \left[\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} - \cos\frac{2\pi}{7} - \cos\frac{4\pi}{7} \right] = 0$$

$$\begin{aligned} \Rightarrow x^2 &= \sin^2\frac{2\pi}{7} + \sin^2\frac{4\pi}{7} + \sin^2\frac{8\pi}{7} \\ &= \frac{1 - \cos\frac{4\pi}{7}}{2} + \frac{1 - \cos\frac{8\pi}{7}}{2} + \frac{1 - \cos\frac{16\pi}{7}}{2} \\ &= \frac{1}{2} \left[3 - \left(\cos\frac{4\pi}{7} + \cos\frac{8\pi}{7} + \cos\frac{16\pi}{7} \right) \right] \\ &= \frac{1}{2} \left[3 - \frac{1}{2\sin\frac{\pi}{7}} \left\{ 2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{4\pi}{7} \right. \right. \\ &\quad \left. \left. + 2\sin\frac{\pi}{7}\cos\frac{6\pi}{7} \right\} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{1}{2} \left[3 - \frac{1}{2\sin\frac{\pi}{7}} \left\{ \left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7} \right) \right. \right. \\ &\quad \left. \left. + \left(\sin\frac{5\pi}{7} - \sin\frac{3\pi}{7} \right) + \left(\sin\frac{7\pi}{7} - \sin\frac{5\pi}{7} \right) \right\} \right] \\ &= \frac{1}{2} \left(3 + \frac{1}{2} \right) = \frac{7}{4} \Rightarrow x = \frac{\sqrt{7}}{2} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 29. If $3\sin\theta + 5\cos\theta = 5$, then the value of $5\sin\theta - 3\cos\theta$ is

- (a) 5 (b) 3
(c) 4 (d) None of these

Sol. $3\sin\theta = 5(1 - \cos\theta) = 5 \times 2\sin^2\frac{\theta}{2} \Rightarrow \tan\frac{\theta}{2} = \frac{3}{5}$

$$\begin{aligned} \therefore 5\sin\theta - 3\cos\theta &= 5 \times \frac{2\tan\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} - 3 \times \frac{\left(1 - \tan^2\frac{\theta}{2}\right)}{1 + \tan^2\frac{\theta}{2}} \\ &= 5 \times \frac{2 \times \frac{3}{5}}{1 + \frac{9}{25}} - \frac{3 \times \left(1 - \frac{9}{25}\right)}{1 + \frac{9}{25}} = 3 \end{aligned}$$

Hence, (b) is the correct answer.

Ex 30. In a $\triangle ABC$, the maximum value of

$$\frac{a\cos^2\frac{A}{2} + b\cos^2\frac{B}{2} + c\cos^2\frac{C}{2}}{a + b + c} \text{ is}$$

- (a) $\frac{1}{4}$ (b) 2 (c) $\frac{3}{4}$ (d) 1

$$\begin{aligned}
 \text{Sol. } & \frac{a \cos^2 \frac{A}{2} + b \cos^2 \frac{B}{2} + c \cos^2 \frac{C}{2}}{a + b + c} \\
 &= \frac{a(1 + \cos A) + b(1 + \cos B) + c(1 + \cos C)}{2(a + b + c)} \\
 &= \frac{a \cos A + b \cos B + c \cos C}{2(a + b + c)} + \frac{1}{2} \\
 &= \frac{R[\sin 2A + \sin 2B + \sin 2C]}{2(a + b + c)} + \frac{1}{2} \\
 &= \frac{4R \sin A \sin B \sin C}{2 \cdot 2s} + \frac{1}{2} \\
 &= \frac{abc}{8sR^2} + \frac{1}{2} = \frac{r}{2R} + \frac{1}{2} \leq \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \text{ as } R \geq 2r \\
 \Rightarrow & \frac{a \cos^2 \frac{A}{2} + b \cos^2 \frac{B}{2} + c \cos^2 \frac{C}{2}}{a + b + c} \leq \frac{3}{4}
 \end{aligned}$$

Thus, the maximum value is $\frac{3}{4}$.

Hence, (c) is the correct answer.

Ex 31. If $A + B + C = \pi$, then the value of $\cos^2 A + \cos^2 B + \cos^2 C$ is

- (a) $1 - \cos A \cos B \cos C$
- (b) $1 - 2 \sin A \sin B \sin C$
- (c) $1 - \sin A \sin B \sin C$
- (d) $1 - 2 \cos A \cos B \cos C$

$$\begin{aligned}
 \text{Sol. } & \text{We have, } \cos^2 A + \cos^2 B + \cos^2 C \\
 &= \cos^2 A + (1 - \sin^2 B) + \cos^2 C \\
 &= (\cos^2 A - \sin^2 B) + \cos^2 C + 1 \\
 &= \cos(A + B) \cos(A - B) + \cos^2 C + 1 \\
 &= \cos(\pi - C) \cos(A - B) + \cos^2 C + 1 \\
 &= -\cos C \cos(A - B) + \cos^2 C + 1 \\
 &= -\cos C [\cos(A - B) - \cos C] + 1 \\
 &= -\cos C [\cos(A - B) - \cos\{\pi - (A + B)\}] + 1 \\
 &= -\cos C [\cos(A - B) + \cos(A + B)] + 1 \\
 &= -2 \cos C \cos A \cos B + 1
 \end{aligned}$$

Hence, (d) is the correct answer.

Ex 32. Let A, B and C be three angles such that $A = \frac{\pi}{4}$

and $\tan B \tan C = p$. Then, all possible values of p such that A, B and C are the angles of a triangle, is

- (a) $\leq (\sqrt{2} + 1)^2$
- (b) $\geq (\sqrt{2} + 1)^2$
- (c) $> (\sqrt{2} + 1)^2$
- (d) $< (\sqrt{2} + 1)^2$

$$\begin{aligned}
 \text{Sol. } & \because A + B + C = \pi \\
 \Rightarrow & B + C = \frac{3\pi}{4} \\
 \Rightarrow & 0 < B, C < \frac{3\pi}{4} \\
 \text{Also, } & \tan B \tan C = p \\
 \Rightarrow & \frac{\sin B \sin C}{\cos B \cos C} = \frac{p}{1} \\
 \Rightarrow & \frac{\cos B \cos C - \sin B \sin C}{\cos B \cos C + \sin B \sin C} = \frac{1 - p}{1 + p}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow & \frac{\cos(B + C)}{\cos(B - C)} = \frac{1 - p}{1 + p} \\
 \Rightarrow & \frac{1 + p}{\sqrt{2}(p - 1)} = \cos(B - C) \quad \dots(i)
 \end{aligned}$$

Since, B or C can vary from 0 to $\frac{3\pi}{4}$

$$\begin{aligned}
 0 \leq B - C < \frac{3\pi}{4} \\
 \Rightarrow -\frac{1}{\sqrt{2}} < \cos(B - C) \leq 1
 \end{aligned}$$

Eq. (i) will now lead to

$$\begin{aligned}
 \Rightarrow & -\frac{1}{\sqrt{2}} < \frac{p + 1}{\sqrt{2}(p - 1)} \leq 1 \\
 \Rightarrow & \frac{2p}{p - 1} > 0 \\
 \Rightarrow & p < 0 \text{ or } p > 1 \quad \dots(ii)
 \end{aligned}$$

$$\text{Also, } \frac{p + 1 - \sqrt{2}(p - 1)}{\sqrt{2}(p - 1)} \leq 0$$

$$\begin{aligned}
 \Rightarrow & \frac{\{p - (\sqrt{2} + 1)\}^2}{(p - 1)} \geq 0 \\
 \Rightarrow & p < 1 \text{ or } p \geq (\sqrt{2} + 1)^2 \quad \dots(iii)
 \end{aligned}$$

Combining Eqs. (ii) and (iii), we get $p \geq (\sqrt{2} + 1)^2$

Hence, (b) is the correct answer.

Ex 33. If A, B and C are angles of any one of the triangle such that $\tan A + \tan B + \tan C = 100$. Then,

- (a) there exist exactly three non-similar isosceles triangles
- (b) there exist exactly two non-similar isosceles triangles
- (c) there exist exactly three similar isosceles triangles
- (d) None of the above

$$\begin{aligned}
 \text{Sol. } & \text{Let } A = B, \text{ then } 2A + C = 180^\circ \\
 \text{and } & 2 \tan A + \tan C = 100 \\
 \text{Now, } & 2A + C = 180^\circ
 \end{aligned}$$

$$\Rightarrow \tan 2A = -\tan C \quad \dots(i)$$

$$\text{Also, } 2 \tan A + \tan C = 100$$

$$\Rightarrow 2 \tan A - 100 = -\tan C \quad \dots(ii)$$

From Eqs. (i) and (ii), we have

$$2 \tan A - 100 = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\text{Let } \tan A = x, \text{ then } \frac{2x}{1 - x^2} = 2x - 100$$

$$\Rightarrow x^3 - 50x^2 + 50 = 0$$

$$\text{Let } f(x) = x^3 - 50x^2 + 50 = 0$$

$$\text{Then, } f'(x) = 3x^2 - 100x$$

$$\text{So, } f'(x) = 0 \text{ has roots } 0, \frac{100}{3}.$$

$$\text{Also, } f(0) \cdot f\left(\frac{100}{3}\right) < 0$$

Thus, $f(x) = 0$ has exactly three distinct real roots. Therefore, $\tan A$ and hence A has three distinct values.

So, there exist exactly three non-similar isosceles triangles.

Hence, (a) is the correct answer.

Ex 34. The sum of the series $\sin \theta \sec 3\theta + \sin 3\theta \sec 3^2\theta + \sin 3^2\theta \sec 3^3\theta + \dots n$ terms, is

- (a) $\frac{1}{2}[\tan 3^n\theta - \tan 3^{n-1}\theta]$
 (b) $[\tan 3^n\theta - \tan \theta]$
 (c) $\frac{1}{2}[\tan 3^n\theta - \tan \theta]$
 (d) None of the above

Sol. Here, $\sin \theta \sec 3\theta + \sin 3\theta \sec 3^2\theta + \sin 3^2\theta \sec 3^3\theta + \dots n$ terms

$$\begin{aligned} &= \sum_{r=1}^n \sin 3^{r-1}\theta \cdot \sec 3^r\theta = \sum_{r=1}^n \frac{\sin 3^{r-1}\theta}{\cos 3^r\theta} \\ &= \sum_{r=1}^n \frac{2\cos 3^{r-1}\theta \cdot \sin 3^{r-1}\theta}{2\cos 3^{r-1}\theta \cdot \cos 3^r\theta} \\ &= \frac{1}{2} \sum_{r=1}^n \frac{\sin(2 \cdot 3^{r-1}\theta)}{\cos 3^{r-1}\theta \cdot \cos 3^r\theta} \\ &= \frac{1}{2} \sum_{r=1}^n \frac{\sin(3^r\theta - 3^{r-1}\theta)}{\cos 3^{r-1}\theta \cdot \cos 3^r\theta} \\ &= \frac{1}{2} \sum_{r=1}^n \frac{\sin 3^r\theta \cdot \cos 3^{r-1}\theta - \cos 3^r\theta \cdot \sin 3^{r-1}\theta}{\cos 3^{r-1}\theta \cdot \cos 3^r\theta} \\ &= \frac{1}{2} \sum_{r=1}^n (\tan 3^r\theta - \tan 3^{r-1}\theta) \\ &= \frac{1}{2}[(\tan 3\theta - \tan \theta) + (\tan 3^2\theta - \tan 3\theta) \\ &\quad + \dots + (\tan 3^n\theta - \tan 3^{n-1}\theta)] \\ &= \frac{1}{2}[\tan 3^n\theta - \tan \theta] \end{aligned}$$

Hence, (c) is the correct answer.

Ex 35. The set of values of x for which

$$\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1, \text{ is}$$

- (a) ϕ
 (b) $\left\{\frac{\pi}{4}\right\}$
 (c) $\left\{n\pi + \frac{\pi}{4}, n = 1, 2, 3, \dots\right\}$
 (d) $\left\{2n\pi + \frac{\pi}{4}, n = 1, 2, 3, \dots\right\}$

Sol. We have, $\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1$

$$\begin{aligned} \Rightarrow \tan(3x - 2x) &= 1 \Rightarrow \tan x = 1 \\ \Rightarrow \tan x &= \tan \frac{\pi}{4} \\ \Rightarrow x &= n\pi + \frac{\pi}{4} \end{aligned}$$

But for this value of x , $\tan 2x = \tan\left(2n\pi + \frac{\pi}{2}\right) = \infty$

which does not satisfy the given equation as it reduces to indeterminate form.

Hence, (a) is the correct answer.

Ex 36. If $\exp^{[(\sin^2 x + \sin^4 x + \sin^6 x + \dots) \ln 2]}$ satisfies the equation $y^2 - 9y + 8 = 0$, then the value of

$$\frac{\cos x}{\cos x + \sin x}, 0 < x < \frac{\pi}{2}, \text{ is}$$

- (a) $\sqrt{3} + 1$ (b) $\frac{\sqrt{3} - 1}{2}$
 (c) $\sqrt{3} - 1$ (d) None of these

Sol. $\sin^2 x + \sin^4 x + \sin^6 x + \dots = \frac{\sin^2 x}{1 - \sin^2 x} = \tan^2 x$

$$\Rightarrow \exp^{[(\sin^2 x + \sin^4 x + \dots) \ln 2]} = e^{\tan^2 x \ln 2} = e^{\ln 2^{\tan^2 x}}$$

Given equation, $y^2 - 9y + 8 = 0$

$$\Rightarrow (y - 1)(y - 8) = 0$$

Either $y = 1$, then $2^{\tan^2 x} = 1 = 2^0$

$$\Rightarrow \tan^2 x = 0, \text{ but } x \in \left(0, \frac{\pi}{2}\right)$$

$\therefore$ Neglecting $x = 0$

$$\text{or } y = 8 \Rightarrow \tan^2 x = 3$$

$$\Rightarrow \tan x = \pm \sqrt{3}$$

$$\Rightarrow x = \frac{\pi}{3}, \text{ as } 0 < x < \frac{\pi}{2}$$

$$\Rightarrow \frac{\cos x}{\cos x + \sin x} = \frac{1/2}{1/2 + \sqrt{3}/2} = \frac{1}{\sqrt{3} + 1} = \frac{\sqrt{3} - 1}{2}$$

Hence, (b) is the correct answer.

Ex 37. If $\tan(\pi \cos \theta) = \cot(\pi \sin \theta)$, then the value(s)

of $\cos\left(\theta - \frac{\pi}{4}\right)$ is/are

- (a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$
 (c) $\pm \frac{1}{2\sqrt{2}}$ (d) None of these

Sol. We have, $\tan(\pi \cos \theta) = \cot(\pi \sin \theta)$

$$\Rightarrow \tan(\pi \cos \theta) = \tan\left(\frac{\pi}{2} - \pi \sin \theta\right)$$

$$\Rightarrow \pi \cos \theta = \left(\frac{\pi}{2} - \pi \sin \theta\right) + n\pi, n \in \mathbb{Z}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \cos \theta + \frac{1}{\sqrt{2}} \sin \theta = \frac{2n + 1}{2\sqrt{2}}, n \in \mathbb{Z}$$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \frac{2n + 1}{2\sqrt{2}}, n \in \mathbb{Z}$$

$$\Rightarrow \cos\left(\theta - \frac{\pi}{4}\right) = \pm \frac{1}{2\sqrt{2}}$$

[for $n = 0$ and $n = -1$]

Hence, (c) is the correct answer.

Ex 38. The set of all x in $(-\pi, \pi)$ satisfying

$$|4 \sin x - 1| < \sqrt{5} \text{ is given by}$$

- (a) $\left(-\frac{\pi}{10}, \frac{3\pi}{10}\right)$ (b) $\left(\frac{\pi}{10}, \frac{3\pi}{10}\right)$
 (c) $\left(\frac{\pi}{10}, -\frac{3\pi}{10}\right)$ (d) None of these

Sol. We have, $|4 \sin x - 1| < \sqrt{5}$

$$\begin{aligned} \Rightarrow -\sqrt{5} &< 4 \sin x - 1 < \sqrt{5} \\ \Rightarrow -\left(\frac{\sqrt{5}-1}{4}\right) &< \sin x < \left(\frac{\sqrt{5}+1}{4}\right) \\ \Rightarrow -\sin \frac{\pi}{10} &< \sin x < \cos \frac{\pi}{5} \\ \Rightarrow \sin \left(-\frac{\pi}{10}\right) &< \sin x < \sin \left(\frac{\pi}{2} - \frac{\pi}{5}\right) \\ \Rightarrow \sin \left(-\frac{\pi}{10}\right) &< \sin x < \sin \frac{3\pi}{10} \\ \Rightarrow x &\in \left(-\frac{\pi}{10}, \frac{3\pi}{10}\right) \end{aligned}$$

Hence, (a) is the correct answer.

Ex 39. Total number of solutions of $\cos x = \sqrt{1 - \sin 2x}$ in $[0, 2\pi]$, is
(a) 2 (b) 3
(c) 5 (d) None of these

Sol. $\cos x = \sqrt{1 - \sin 2x} = |\sin x - \cos x|$

Case I $\sin x \leq \cos x$
 $\Rightarrow \cos x = \cos x - \sin x$
 $\Rightarrow \sin x = 0$
 where, $x \in \left[0, \frac{\pi}{4}\right] \cup \left[\frac{5\pi}{4}, 2\pi\right]$

$\therefore \sin x = 0$
 $\Rightarrow x = 0, 2\pi$

Case II $\sin x > \cos x$
 $\Rightarrow \tan x = 2$
 where, $x \in \left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\therefore \tan x = 2$
 $\Rightarrow x = \tan^{-1}(2)$

Thus, the given equation has three solutions.
 Hence, (b) is the correct answer.

Ex 40. Total number of solutions of

$$|\cot x| = \cot x + \frac{1}{\sin x}, x \in [0, 3\pi], \text{ is}$$

- (a) 1 (b) 2
(c) 3 (d) 0

Sol. $|\cot x| = \cot x + \frac{1}{\sin x}$

Let $\cot x > 0 \Rightarrow \cot x = \cot x + \frac{1}{\sin x} = 0$

$\Rightarrow \frac{1}{\sin x} = 0$, which is not possible.

Let $\cot x \leq 0 \Rightarrow -\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow -2\cot x = \frac{1}{\sin x}$

$\Rightarrow \cos x = -\frac{1}{2}$

$\Rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$

So, the number of solutions is 2.
 Hence, (b) is the correct answer.

Ex 41. The equation $p \cos x - q \sin x = r$ admits of a solution for x only, if

- (a) $r < \max\{p, q\}$
 (b) $-\sqrt{p^2 + q^2} < r < \sqrt{p^2 + q^2}$
 (c) $r^2 = p^2 + q^2$
 (d) None of the above

Sol. Let $p = a \cos \theta$ and $-q = a \sin \theta$

$\therefore a(\cos \theta \cos x + \sin \theta \sin x) = r$
 $\Rightarrow a \cos(x - \theta) = r$

Thus, the condition for solution is

$$-a \leq r \leq a$$

where, $a = \sqrt{p^2 + q^2}$

Thus, equation $p \cos x - q \sin x = r$ admits solution for

$$-\sqrt{p^2 + q^2} \leq r \leq \sqrt{p^2 + q^2}$$

Hence, (b) is the correct answer.

Ex 42. If $4 \cos^2 x \sin x - 2 \sin^2 x = 3 \sin x$, then x is equal to

- (a) $n\pi + \left(\frac{3\pi}{10}\right)$ (b) $n\pi + (-1)^{n+1} \left(\frac{3\pi}{10}\right)$
 (c) $n\pi - \left(\frac{3\pi}{10}\right)$ (d) None of these

Sol. We have, $[4 \cos^2 x - 2 \sin x - 3] \sin x = 0$

Either $\sin x = 0 \Rightarrow x = n\pi$

or $4(1 - \sin^2 x) - 2 \sin x - 3 = 0$

$\Rightarrow 4 \sin^2 x + 2 \sin x - 1 = 0$

$\Rightarrow \sin x = \frac{-2 \pm \sqrt{4 + 16}}{8}$

$$= \frac{-1 \pm \sqrt{5}}{4} = \frac{\sqrt{5}-1}{4}, -\frac{1+\sqrt{5}}{4}$$

$\Rightarrow \sin x = \sin \frac{\pi}{10}$

$\Rightarrow x = n\pi + (-1)^n \left(\frac{\pi}{10}\right)$

or $\sin x = -\cos \frac{\pi}{5} = \sin \left(-\frac{\pi}{10}\right)$

$\Rightarrow x = n\pi + (-1)^{n+1} \left(\frac{3\pi}{10}\right)$

Hence, (b) is the correct answer.

Ex 43. If $\cos 3x + \cos 2x = \sin \frac{3x}{2} + \sin \frac{x}{2}$, $0 \leq x \leq 2\pi$,

then the number of values of x is

- (a) 6 (b) 7
(c) 4 (d) 5

Sol. We have, $2 \cos \frac{5x}{2} \cos \frac{x}{2} = 2 \sin x \cos \frac{x}{2}$

Either $\cos \frac{x}{2} = 0$

$\Rightarrow \frac{x}{2} = (2n+1) \frac{\pi}{2} \Rightarrow x = (2n+1)\pi$

or $\cos \frac{5x}{2} = \sin x = \cos \left(\frac{\pi}{2} - x\right)$

$$\Rightarrow \frac{5x}{2} = 2n\pi + \left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow \frac{7x}{2} = 2n\pi + \frac{\pi}{2} \Rightarrow x = \frac{4n\pi}{7} + \frac{\pi}{7}$$

$$\text{or } \frac{3x}{2} = 2n\pi - \frac{\pi}{2} \Rightarrow x = \frac{4n\pi}{3} - \frac{\pi}{3}$$

$$\text{For } 0 \leq x \leq 2\pi, x = \frac{\pi}{7}, \frac{5\pi}{7}, \frac{9\pi}{7}, \frac{13\pi}{7}, \pi$$

Hence, (d) is the correct answer.

Ex 44. The number of points of intersection of the curves $y = \cos x$, $y = \sin 3x$, if $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, is

- (a) 3 (b) 4
(c) 5 (d) 6

Sol. The point of intersection is given by

$$\sin 3x = \cos x = \sin\left(\frac{\pi}{2} - x\right)$$

$$\Rightarrow 3x = n\pi + (-1)^n\left(\frac{\pi}{2} - x\right)$$

Case I Let n be even i.e. $n = 2m \Rightarrow 3x = 2m\pi + \left(\frac{\pi}{2} - x\right)$

$$\Rightarrow x = \frac{m\pi}{2} + \frac{\pi}{8}$$

Case II Let n be odd i.e. $n = 2m + 1$

$$\Rightarrow 3x = (2m + 1)\pi - \left(\frac{\pi}{2} - x\right) = 2m\pi + \left(\frac{\pi}{2} + x\right)$$

$$\Rightarrow x = m\pi + \frac{\pi}{4}$$

$$\text{Now, } -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}$$

Points of intersection are

$$\left(\frac{\pi}{8}, \cos \frac{\pi}{8}\right), \left(\frac{\pi}{4}, \cos \frac{\pi}{4}\right), \left(\frac{3\pi}{8}, \cos \frac{3\pi}{8}\right).$$

Hence, (a) is the correct answer.

Ex 45. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, satisfying the equation

$$(1 - \tan \theta)(1 + \tan \theta) \sec^2 \theta + 2 \tan^2 \theta = 0 \text{ is}$$

- (a) 2 (b) 3
(c) 4 (d) None of these

Sol. $(1 - \tan \theta)(1 + \tan \theta) \sec^2 \theta + 2 \tan^2 \theta = 0$

$$\Rightarrow (1 - \tan^2 \theta)(1 + \tan^2 \theta) + 2 \tan^2 \theta = 0$$

$$\Rightarrow 1 + 2 \tan^2 \theta = \tan^4 \theta$$

By observation, we have $\tan^2 \theta = 3$

$$\Rightarrow \theta = n\pi \pm \left(\frac{\pi}{3}\right)$$

Moreover there will be values of θ , satisfying,

$3 < \tan^2 \theta < 4$ and satisfying the given equation as if $f(x) = x^2 - 2^x - 1$, then $f(3^+)f(4^-) < 0$.

So, the number of values of θ is 4.

Hence, (c) is the correct answer.

Ex 46. The general solution of the equation

$$\left(\cos \frac{x}{4} - 2 \sin x\right) \sin x + \left(1 + \sin \frac{x}{4} - 2 \cos x\right) \cos x = 0 \text{ is}$$

- (a) $2\pi + 8m\pi$ (b) $2\pi + m\pi$
(c) $2\pi + 4m\pi$ (d) None of these

Sol. The given equation can be simplified to

$$\sin \frac{5x}{4} + \cos x = 2$$

Since, the greatest value of $\sin\left(\frac{5x}{4}\right)$ and $\cos x$ is 1,

their sum is equal to 2 only, if $\sin\left(\frac{5x}{4}\right) = 1$ and

$\cos x = 1$ simultaneously i.e. $\frac{5x}{4} = 2n\pi + \frac{\pi}{2}$ and

$x = 2k\pi$; $n, k \in I$.

Since, we have to choose those values of x which satisfy both of these equations, we have

$$2k\pi = \frac{8n\pi}{5} + \frac{2\pi}{5} \Rightarrow k = \frac{4n+1}{5}$$

where, k and n are integers.

We write $k = n - \frac{n-1}{5}$

For $\frac{n-1}{5} = m$, we have $k = 1 + 4m$ ($m \in I$)

Therefore, $x = 2\pi + 8m\pi$ ($m \in I$)

Hence, (a) is the correct answer.

Ex 47. The values of x between 0 and 2π which satisfy the equation $\sin x \sqrt{8 \cos^2 x} = 1$ are in AP with common difference

- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{8}$ (c) $\frac{3\pi}{8}$ (d) $\frac{5\pi}{8}$

Sol. We have, $\sin x \sqrt{8 \cos^2 x} = 1$

$$\Rightarrow \sin x |\cos x| = \frac{1}{2\sqrt{2}}$$

Case I When $\cos x > 0$

$$\text{In this case, } \sin x \cos x = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \sin 2x = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

$$\Rightarrow x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

As x lies between 0 and 2π and $\cos x > 0$, $x = \frac{\pi}{8}, \frac{3\pi}{8}$.

Case II When $\cos x < 0$

$$\text{In this case, } \sin x |\cos x| = \frac{1}{2\sqrt{2}}$$

$$\Rightarrow \sin x \cos x = -\frac{1}{2\sqrt{2}} \text{ or } \sin 2x = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow x = \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}$$

$$\Rightarrow x = \frac{5\pi}{8}, \frac{7\pi}{8}$$

Thus, the values of x satisfying the given equation which lie between 0 and 2π are $\frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$.

These are in AP with common difference $\frac{\pi}{4}$.

Hence, (a) is the correct answer.

Ex 48. The values of θ , for which the system $(\sin 3\theta)x - y + z = 0$, $(\cos 2\theta)x + 4y + 3z = 0$ and $2x + 7y + 7z = 0$, has a non-trivial solution, are

- (a) $n\pi, n\pi + (-1)^n \frac{\pi}{6}$ (b) $n\pi, n\pi - (-1)^n \frac{\pi}{6}$
(c) $2n\pi, n\pi + (-1)^n \frac{\pi}{6}$ (d) $n\pi, n\pi + (-1)^n \frac{\pi}{3}$

Sol. The system has non-trivial solutions, if

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

On expanding, we get

$$\begin{aligned} 7\sin 3\theta + 14\cos 2\theta - 14 &= 0 \\ \Rightarrow \sin 3\theta + 2\cos 2\theta - 2 &= 0 \\ \Rightarrow \sin \theta (4\sin^2 \theta + 4\sin \theta - 3) &= 0 \\ \Rightarrow \sin \theta = 0 \Rightarrow \theta &= n\pi \\ \text{or } 4\sin^2 \theta + 4\sin \theta - 3 &= 0 \\ \Rightarrow (2\sin \theta - 1)(2\sin \theta + 3) &= 0 \\ \Rightarrow 2\sin \theta - 1 = 0 \\ \Rightarrow \sin \theta = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) \end{aligned}$$

or $4\sin^2 \theta + 4\sin \theta - 3 = 0$, which is not possible.

$$\begin{aligned} \Rightarrow \theta &= n\pi + (-1)^n \frac{\pi}{6} \\ \Rightarrow \theta &= n\pi \text{ or } n\pi + (-1)^n \frac{\pi}{6} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 49. A $\triangle ABC$ is such that $\sin(2A + B) = \frac{1}{2}$.

If A, B and C are in AP, then the values of A, B and C are

- (a) $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$ (b) $\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}$
(c) $\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}$ (d) None of these

Sol. $\sin(2A + B) = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right)$

$$\Rightarrow 2A + B = n\pi + (-1)^n \frac{\pi}{6} \quad \dots(i)$$

Also, $A + B + C = \pi$ and $2B = A + C$

$$\Rightarrow 3B = \pi \Rightarrow B = \frac{\pi}{3}$$

From Eq. (i), for $n = 1$, $2A + B = \frac{5\pi}{6}$

$$\Rightarrow A = \frac{\pi}{4}$$

$$\text{and } C = \frac{5\pi}{12}$$

Hence, (a) is the correct answer.

Ex 50. The equation

$$3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$$

is satisfied for the values of x given by

- (a) $\cos x = 0, \tan x = -1$ (b) $\tan x = 0$
(c) $\tan x = 1$ (d) None of these

Sol. The given equation is

$$3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28$$

$$\Rightarrow 3^{\sin 2x + 2\cos^2 x} + 3^{3 - (\sin 2x + 2\cos^2 x)} = 28$$

$$\Rightarrow y + \frac{27}{y} = 28$$

$$\text{where, } y = 3^{\sin 2x + 2\cos^2 x} \Rightarrow y^2 - 28y + 27 = 0$$

$$\Rightarrow y = 27 \text{ or } 1$$

$$\text{If } y = 27, \text{ then } 3^{\sin 2x + 2\cos^2 x} = 3^3$$

$$\Rightarrow \sin 2x + 2\cos^2 x = 3$$

$$\Rightarrow \sin 2x + \cos 2x = 2$$

$$\Rightarrow \sin 2x = \cos 2x = 1$$

which is not possible for any value of x and so $y \neq 27$.

Also, we have $y = 1$

$$\Rightarrow 3^{\sin 2x + 2\cos^2 x} = 1 = 3^0$$

$$\Rightarrow \sin 2x + 2\cos^2 x = 0$$

$$\Rightarrow 2\cos x (\sin x + \cos x) = 0$$

$$\text{Either } \cos x = 0 \text{ or } \tan x = -1$$

Hence, (a) is the correct answer.

Ex 51. If $\cos^4 x + a \cos^2 x + 1 = 0$ has atleast one solution, then

- (a) $a \in [2, \infty)$ (b) $a \in [-2, 2]$
(c) $a \in (-\infty, -2]$ (d) $a \in \mathbb{R} \sim (-2, 2)$

Sol. $\cos^4 x + a \cos^2 x + 1 = 0$

$$\text{Let } t = \cos^2 x, \text{ then } t \in [0, 1]$$

$$\text{Thus, we have } t^2 + at + 1 = 0$$

This quadratic must have atleast one root in $[0, 1]$.

$$\Rightarrow a^2 - 4 \geq 0, 1 + a + 1 \leq 0$$

$$\Rightarrow a \in (-\infty, -2] \cup [2, \infty), a \leq -2$$

$$\Rightarrow a \in (-\infty, -2]$$

Hence, (c) is the correct answer.

Ex 52. Total number of solutions of $\cos x \cos 2x \cos 3x = \frac{1}{4}$ in $[0, \pi]$ is

- (a) 4 (b) 6
(c) 8 (d) None of these

Sol. Given, $\cos x \cos 2x \cos 3x = \frac{1}{4}$

$$\Rightarrow 4 \cos 3x \cos x \cos 2x = 1$$

$$\Rightarrow 2(\cos 4x + \cos 2x) \cos 2x = 1$$

$$\Rightarrow 2(2\cos^2 2x - 1 + \cos 2x) \cos 2x = 1$$

$$\Rightarrow 4\cos^3 2x + 2\cos^2 2x - 2\cos 2x - 1 = 0$$

$$\Rightarrow 2\cos^2 2x(2\cos 2x + 1) - (2\cos 2x + 1) = 0$$

$$\Rightarrow (2\cos 2x + 1)(2\cos^2 2x - 1) = 0$$

$$\Rightarrow \cos 4x(2\cos 2x + 1) = 0$$

$$\Rightarrow \cos 4x = 0 \Rightarrow 4x = (2n_1 + 1)\frac{\pi}{2}$$

$$\Rightarrow x = (2n_1 + 1)\frac{\pi}{8} = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$$

$$\text{or } \cos 2x = -\frac{1}{2} \Rightarrow 2x = 2n_2\pi \pm \frac{2\pi}{3}$$

$$\Rightarrow x = n_2\pi \pm \frac{\pi}{3} \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}$$

Thus, there are six solutions.

Hence, (b) is the correct answer.

- Ex 53.** Total number of solutions of $\sin^4 x + \cos^4 x = \sin x \cos x$ in $[0, 2\pi]$ is
(a) 2 (b) 4 (c) 6 (d) 8

Sol. $\sin^4 x + \cos^4 x = \sin x \cos x$
 $\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cos^2 x = \sin x \cos x$
 $\Rightarrow 1 - \frac{\sin^2 2x}{2} = \frac{\sin 2x}{2} \Rightarrow \sin^2 2x + \sin 2x - 2 = 0$
 $\Rightarrow (\sin 2x + 2)(\sin 2x - 1) = 0 \Rightarrow \sin 2x = 1$
 $\Rightarrow 2x = (4n_1 + 1)\frac{\pi}{2} \Rightarrow x = (4n_1 + 1)\frac{\pi}{4}$
 $\Rightarrow x = \frac{\pi}{4}, \frac{5\pi}{4}$

Thus, there are two solutions.

Hence, (a) is the correct answer.

- Ex 54.** Total number of integral values of n so that $\sin x(\sin x + \cos x) = n$ has atleast one solution, is
(a) 2 (b) 1
(c) 3 (d) 0

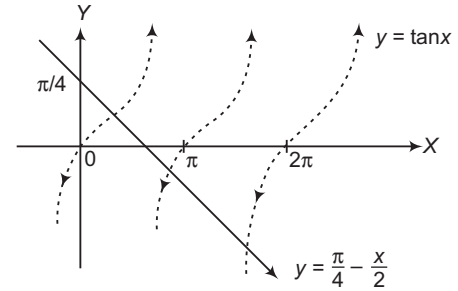
Sol. $\sin x(\sin x + \cos x) = n$
 $\Rightarrow \sin^2 x + \sin x \cos x = n$
 $\Rightarrow \frac{1 - \cos 2x}{2} + \frac{\sin 2x}{2} = n$
 $\Rightarrow \sin 2x - \cos 2x = 2n - 1$
 $\Rightarrow -\sqrt{2} \leq 2n - 1 \leq \sqrt{2}$
 $\Rightarrow \frac{1 - \sqrt{2}}{2} \leq n \leq \frac{1 + \sqrt{2}}{2}$
 $\Rightarrow n = 0, 1$
Hence, (a) is the correct answer.

- Ex 55.** The equation $\tan^4 x - 2 \sec^2 x + a^2 = 0$ will have atleast one solution, if
(a) $|a| \leq 4$ (b) $|a| \leq 2$
(c) $|a| \leq \sqrt{3}$ (d) None of these

Sol. $\tan^4 x - 2 \sec^2 x + a^2 = 0$
 $\Rightarrow \tan^4 x - 2(1 + \tan^2 x) + a^2 = 0$
 $\Rightarrow \tan^4 x - 2\tan^2 x + 1 = 3 - a^2$
 $\Rightarrow (\tan^2 x - 1)^2 = 3 - a^2 \Rightarrow 3 - a^2 \geq 0$
 $\Rightarrow a^2 \leq 3 \Rightarrow |a| \leq \sqrt{3}$
Hence, (c) is the correct answer.

- Ex 56.** The number of roots of the equation $x + 2 \tan x = \frac{\pi}{2}$ in the interval $[0, 2\pi]$ is
(a) 1 (b) 2
(c) 3 (d) infinite

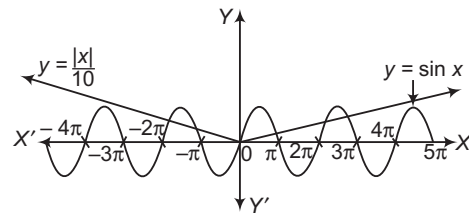
Sol. We have, $x + 2 \tan x = \frac{\pi}{2} \Rightarrow \tan x = \frac{\pi}{4} - \frac{x}{2}$



Now, the graph of the curve $y = \tan x$ and $y = \frac{\pi}{4} - \frac{x}{2}$ in the interval $[0, 2\pi]$ intersect at three points. The abscissa of these three points are the roots of the equation.
Hence, (c) is the correct answer.

- Ex 57.** Total number of solutions of $\sin x = \frac{|x|}{10}$ is
(a) 4 (b) 6
(c) 7 (d) None of these

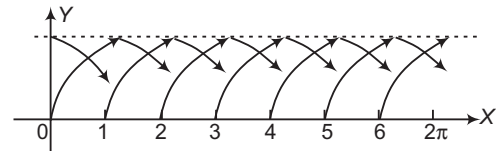
Sol. Graph of $y = \sin x$ and $y = \frac{|x|}{10}$ meet exactly six times.
So, there are six solutions.



Hence, (b) is the correct answer.

- Ex 58.** Total number of solutions of $\sin \{x\} = \cos \{x\}$, where $\{ \}$ denotes the fractional part, in $[0, 2\pi]$ is
(a) 5 (b) 7
(c) 8 (d) None of these

Sol. $\sin \{x\} = \cos \{x\}$
The graph of $y = \sin \{x\}$ and $y = \cos \{x\}$ meet exactly 7 times in $[0, 2\pi]$.



Hence, (b) is the correct answer.

- Ex 59.** The number of all possible 5-tuples $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$ holds for all x , is
(a) zero (b) 1
(c) 2 (d) infinite

Sol. Since, the equation $a_1 + a_2 \sin x + a_3 \cos x + a_4 \sin 2x + a_5 \cos 2x = 0$ holds for all values of x .
 $a_1 + a_3 + a_5 = 0$ [on putting $x = 0$]

$$\Rightarrow \begin{matrix} a_1 - a_3 + a_5 = 0 \\ a_3 = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} a_1 + a_5 = 0 \end{matrix} \quad \dots(i)$$

Putting $x = \frac{\pi}{2}$ and $\frac{3\pi}{2}$, we get

$$\Rightarrow \begin{matrix} a_1 + a_2 - a_5 = 0 \\ a_1 = 0 \end{matrix} \quad \text{and} \quad \begin{matrix} a_1 - a_2 + a_5 = 0 \\ a_2 - a_5 = 0 \end{matrix} \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$a_1 = a_2 = a_3 = a_5 = 0$$

The given equation reduces to $a_4 \sin 2x = 0$. This is true for all values of x , therefore $a_4 = 0$.

So, $a_1 = a_2 = a_3 = a_4 = a_5 = 0$

Thus, the number of 5-tuples is one.

Hence, (b) is the correct answer.

Ex 60. The equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has

- (a) no solution (b) two solutions
(c) three solutions (d) None of these

Sol. The given equation can be written as $e^{2 \sin x} - 4e^{\sin x} - 1 = 0$

$$\Rightarrow e^{\sin x} = \frac{4 \pm \sqrt{16 + 4}}{2} = 2 + \sqrt{5}$$

$$\Rightarrow \sin x = \ln(2 + \sqrt{5})$$

$[\because \ln(2 - \sqrt{5})$ is not defined as $2 - \sqrt{5}$ is $-ve$]

Now, $2 + \sqrt{5} > e \Rightarrow \ln(2 + \sqrt{5}) > 1 \Rightarrow \sin x > 1$

which is not possible.

$\therefore$ No real solution exists.

Hence, (a) is the correct answer.

Ex 61. If $0 \leq x \leq \frac{\pi}{2}$ and $81^{\sin^2 x} + 81^{\cos^2 x} = 30$, then x

is equal to

- (a) $\frac{\pi}{6}, \frac{\pi}{3}$ (b) $\frac{\pi}{3}, \frac{2\pi}{3}$
(c) $\frac{5\pi}{6}, \frac{\pi}{6}$ (d) None of these

Sol. Let $81^{\sin^2 x} = y$

$$\text{Then, } 81^{\cos^2 x} = 81^{1 - \sin^2 x} = 81y^{-1}$$

So that, the given equation can be written as

$$y^2 - 30y + 81 = 0$$

$$\Rightarrow y = 3 \quad \text{or} \quad y = 27$$

$$\Rightarrow 81^{\sin^2 x} = 3 \quad \text{or} \quad 27$$

$$\Rightarrow 3^{4 \sin^2 x} = 3^1 \quad \text{or} \quad 3^3$$

$$\Rightarrow \sin^2 x = \frac{1}{4} \quad \text{or} \quad \frac{3}{4}$$

$$\Rightarrow \sin x = \pm \frac{1}{2} \quad \text{or} \quad \pm \frac{\sqrt{3}}{2}$$

$$\Rightarrow \sin x = \frac{1}{2}, \frac{\sqrt{3}}{2} \quad \left[\text{neglecting } -\frac{1}{2}, -\frac{\sqrt{3}}{2} \text{ as } 0 \leq x \leq \frac{\pi}{2} \right]$$

$$\Rightarrow x = \frac{\pi}{6} \quad \text{or} \quad \frac{\pi}{3}$$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 62. If $x = (\sec \phi - \tan \phi)$ and $y = \operatorname{cosec} \phi + \cot \phi$, then

- (a) $x = \frac{y+1}{y-1}$ (b) $x = \frac{y-1}{y+1}$
(c) $y = \frac{1+x}{1-x}$ (d) $xy + x - y + 1 = 0$

Sol. We have, $x = \frac{1 - \sin \phi}{\cos \phi}$, $y = \frac{1 + \cos \phi}{\sin \phi}$

On multiplying, we get

$$xy = \frac{(1 - \sin \phi)(1 + \cos \phi)}{\cos \phi \sin \phi}$$

$$\Rightarrow xy + 1 = \frac{1 - \sin \phi + \cos \phi - \sin \phi \cos \phi + \sin \phi \cos \phi}{\cos \phi \sin \phi}$$

$$= \frac{1 - \sin \phi + \cos \phi}{\cos \phi \sin \phi}$$

$$\text{and } x - y = \frac{(1 - \sin \phi)\sin \phi - \cos \phi(1 + \cos \phi)}{\cos \phi \sin \phi}$$

$$= \frac{\sin \phi - \sin^2 \phi - \cos \phi - \cos^2 \phi}{\cos \phi \sin \phi}$$

$$= \frac{\sin \phi - \cos \phi - 1}{\cos \phi \sin \phi} = -(xy + 1)$$

$$\text{Thus, } xy + x - y + 1 = 0$$

$$\Rightarrow x = \frac{y-1}{y+1}$$

$$\text{and } y = \frac{1+x}{1-x}$$

Hence, (b), (c) and (d) are the correct answers.

Ex 63. If $\tan x = \frac{2b}{a-c}$ ($a \neq c$),

$$y = a \cos^2 x + 2b \sin x \cos x + c \sin^2 x \quad \text{and}$$

$$z = a \sin^2 x - 2b \sin x \cos x + c \cos^2 x, \text{ then}$$

- (a) $y = z$
(b) $y + z = a + c$
(c) $y - z = a - c$
(d) $y - z = (a - c)^2 + 4b^2$

Sol. Adding the expression for y and z , we get

$$y + z = a(\cos^2 x + \sin^2 x) + c(\sin^2 x + \cos^2 x) = a + c$$

and subtracting them,

$$y - z = a(\cos^2 x - \sin^2 x) + 4b \sin x \cos x - c(\cos^2 x - \sin^2 x)$$

$$= a \cos 2x + 2b \sin 2x - c \cos 2x$$

$$= (a - c) \cos 2x + 2b \sin 2x$$

$$\text{Now, } \cos 2x = \frac{1 - \tan^2 x}{1 + \tan^2 x} = \frac{(a - c)^2 - 4b^2}{(a - c)^2 + 4b^2}$$

$$\text{and } \sin 2x = \frac{2 \tan x}{1 + \tan^2 x} = \frac{4b(a - c)}{(a - c)^2 + 4b^2}$$

$$\Rightarrow y - z = \frac{(a - c)[(a - c)^2 - 4b^2] + 8b^2(a - c)}{(a - c)^2 + 4b^2}$$

$$= \frac{(a - c)[(a - c)^2 + 4b^2]}{(a - c)^2 + 4b^2} = a - c$$

As $a \neq c$, we get $y \neq z$

Hence, (b) and (c) are the correct answers.

Ex 64. If $x = \sin(\alpha - \beta)\sin(\gamma - \delta)$,

$y = \sin(\beta - \gamma)\sin(\alpha - \delta)$ and

$z = \sin(\gamma - \alpha)\sin(\beta - \delta)$, then

- (a) $x + y + z = 0$ (b) $x + y - z = 0$
(c) $y + z - x = 0$ (d) $x^3 + y^3 + z^3 = 3xyz$

Sol. $2x = \cos(\alpha - \beta - \gamma + \delta) - \cos(\alpha - \beta + \gamma - \delta)$

$2y = \cos(\beta - \gamma - \alpha + \delta) - \cos(\beta - \gamma + \alpha - \delta)$

and similarly for $2z$,

On adding, we get $2x + 2y + 2z = 0$

$\Rightarrow x + y + z = 0$

Hence, (a) and (d) are the correct answers.

Ex 65. $\sin \frac{15\pi}{32} \sin \frac{7\pi}{16} \sin \frac{3\pi}{8}$ is equal to

- (a) $\frac{1}{8\sqrt{2} \cos \frac{15\pi}{32}}$ (b) $\frac{1}{8 \sin \frac{\pi}{32}}$
(c) $\frac{1}{4\sqrt{2}} \operatorname{cosec} \frac{\pi}{16}$ (d) $\frac{1}{8\sqrt{2}} \operatorname{cosec} \frac{\pi}{32}$

Sol. $\sin \frac{15\pi}{32} \sin \frac{7\pi}{16} \sin \frac{3\pi}{8}$
 $= \cos \left(\frac{\pi}{2} - \frac{15\pi}{32} \right) \cos \left(\frac{\pi}{2} - \frac{7\pi}{16} \right) \cos \left(\frac{\pi}{2} - \frac{3\pi}{8} \right)$
 $= \cos \frac{\pi}{32} \cos \frac{\pi}{16} \cos \frac{\pi}{8}$
 $= \frac{\left(2 \sin \frac{\pi}{32} \cos \frac{\pi}{32} \right) \cos \frac{\pi}{16} \cos \frac{\pi}{8}}{2 \sin \frac{\pi}{32}}$
 $= \frac{2 \sin \frac{\pi}{16} \cos \frac{\pi}{16} \cos \frac{\pi}{8}}{4 \sin \frac{\pi}{32}} = \frac{2 \sin \frac{\pi}{8} \cos \frac{\pi}{8}}{8 \sin \frac{\pi}{32}}$
 $= \frac{\sin \frac{\pi}{4}}{8 \sin \frac{\pi}{32}} = \frac{1}{8\sqrt{2} \sin \frac{\pi}{32}} = \frac{1}{8\sqrt{2} \cos \frac{15\pi}{32}}$

Hence, (a) and (d) are the correct answers.

Ex 66. If $a = \frac{1}{5 \cos x + 12 \sin x}$, then for all real x

- (a) the least positive value of a is $\frac{1}{13}$
(b) the greatest negative value of a is $-\frac{1}{13}$
(c) $a \leq \frac{1}{13}$
(d) $-\frac{1}{13} \leq a \leq \frac{1}{13}$

Sol. We know that,

$$-\sqrt{5^2 + 12^2} \leq 5 \cos x + 12 \sin x \leq \sqrt{5^2 + 12^2}$$

$$\therefore -13 \leq 5 \cos x + 12 \sin x < 0$$

$$\Rightarrow -13 \leq \frac{1}{a} < 0 \quad \text{and} \quad 0 < 5 \cos x + 12 \sin x \leq 13$$

$$\Rightarrow 0 < \frac{1}{a} \leq 13$$

$$\therefore a \leq -\frac{1}{13} \quad \text{and} \quad a \geq \frac{1}{13}$$

Hence, (a) and (b) are the correct answers.

Ex 67. Let $[x]$ be the greatest integer less than or equal to x . The equation $\sin x = [1 + \sin x] + [1 - \cos x]$ has

- (a) no solution in $\left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$
(b) no solution in $\left[\frac{\pi}{2}, \pi \right]$
(c) no solution in $\left[\pi, \frac{3\pi}{2} \right]$
(d) no solution for $x \in R$

Sol. At $x = -\frac{\pi}{2}, \frac{3\pi}{2}$; $[1 + \sin x] = 0, [1 - \cos x] = 1$

$$\therefore \sin x = 0 + 1 \Rightarrow -1 = 1 \quad [\text{absurd}]$$

$$\text{At } x = 0, [1 + \sin x] = 1, [1 - \cos x] = 0$$

$$\therefore \sin x = 1 + 0 \Rightarrow 0 = 1 \quad [\text{absurd}]$$

$$\text{At } x = \frac{\pi}{2}, [1 + \sin x] = 2, [1 - \cos x] = 1$$

$$\therefore \sin x = 2 + 1 = 3 \quad [\text{absurd}]$$

$$\text{At } x = \pi, [1 + \sin x] = 1, [1 - \cos x] = 2$$

$$\therefore \sin x = 1 + 2 = 3 \quad [\text{absurd}]$$

$$\text{In } \left(-\frac{\pi}{2}, 0 \right), [1 + \sin x] = 0, [1 - \cos x] = 0$$

$$\therefore \sin x = 0 + 0 = 0 \quad [\text{absurd}]$$

$$\text{In } \left(0, \frac{\pi}{2} \right), [1 + \sin x] = 1, [1 - \cos x] = 0$$

$$\therefore \sin x = 1 + 0 = 1 \quad [\text{absurd}]$$

$$\text{In } \left(\frac{\pi}{2}, \pi \right), [1 + \sin x] = 1, [1 - \cos x] = 1$$

$$\therefore \sin x = 1 + 1 = 2 \quad [\text{absurd}]$$

$$\text{In } \left(\pi, \frac{3\pi}{2} \right), [1 + \sin x] = 0, [1 - \cos x] = 1$$

$$\therefore \sin x = 0 + 1 = 1 \quad [\text{absurd}]$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 68. If $\sin \theta = a$ for exactly one value of $\theta \in \left[0, \frac{7\pi}{3} \right]$,

then the value of a is

- (a) $\frac{\sqrt{3}}{2}$ (b) 1
(c) 0 (d) -1

Sol. Clearly, $-1 \leq a \leq 1$

For any value of a other than 1, -1 we know $\sin \theta$ has two values (in quadrants I, II or III, IV).

Hence, (b) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 69-77) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

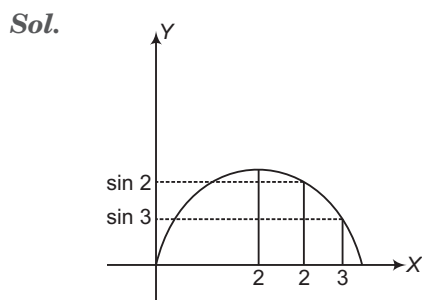
Ex 69. Statement I The number of integral values of λ , for which the equation $7\cos x + 5\sin x = 2\lambda + 1$ has a solution, is 8.

Statement II $a\cos\theta + b\sin\theta = c$ has at least one solution, if $|c| > \sqrt{a^2 + b^2}$

Sol. $7\cos x + 5\sin x = 2\lambda + 1$
 $|2\lambda + 1| \leq \sqrt{49 + 25}$
 $\Rightarrow |2\lambda + 1| \leq \sqrt{74}$
 $-\sqrt{74} \leq 2\lambda + 1 \leq \sqrt{74}$
 $-9.6 \leq 2\lambda \leq 7.6, -4.8 \leq \lambda \leq 3.8$
 $\lambda = -4, -3, -2, -1, 0, 1, 2, 3$
 Also, $a\cos\theta + b\sin\theta = c$
 has no solution, if $|c| > \sqrt{a^2 + b^2}$
 Hence, (c) is the correct answer.

Ex 70. Statement I $\sin 2 > \sin 3$

Statement II If $x, y \in \left(\frac{\pi}{2}, \pi\right), x < y$, then $\sin x > \sin y$.



Hence, (a) is the correct answer.

Ex 71. Let $\alpha, \beta, \gamma > 0$ and $\alpha + \beta + \gamma = \frac{\pi}{2}$

Statement I If

$\left| \tan \alpha \tan \beta - \frac{a!}{6} \right| + \left| \tan \beta \tan \gamma - \frac{b!}{2} \right| + \left| \tan \gamma \tan \alpha - \frac{c!}{3} \right| \leq 0$, where $n! = 1 \cdot 2 \cdot \dots \cdot n$, then $\tan \alpha \tan \beta, \tan \beta \tan \gamma, \tan \gamma \tan \alpha$ are in AP.

Statement II

$$\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 1$$

Sol. Statement II $\alpha + \beta = \frac{\pi}{2} - \gamma$

$$\frac{\tan \alpha - \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{1}{\tan \gamma}$$

$$\Rightarrow \Sigma \tan \alpha \tan \beta = 1$$

$\therefore$ Statement II is true.

Statement I $\tan \alpha \tan \beta = \frac{a!}{6}, \tan \beta \tan \gamma = \frac{b!}{2}$

$$\text{and } \tan \alpha \tan \gamma = \frac{c!}{3}$$

$$\frac{a!}{6} + \frac{b!}{2} + \frac{c!}{3} = 1$$

$$\Rightarrow a! = 1, b! = 1, c! = 1$$

$\Rightarrow \tan \alpha \tan \beta, \tan \gamma \tan \alpha$ and $\tan \beta \tan \gamma$ are in AP.

$\therefore$ Statement I is false.

Hence, (d) is the correct answer.

Ex 72. The angles of a triangle are given by the three values of x obtained from the equation $\tan^3 x - 3\sqrt{3} \tan^2 x + 3\sqrt{3} \tan x - 3 = 0$.

Statement I The triangle so obtained is an equilateral triangle.

Statement II If roots of the equation are $\tan A, \tan B$ and $\tan C$, then $\tan A + \tan B + \tan C = 3\sqrt{3}$

Sol. $\tan A + \tan B + \tan C = 3\sqrt{3}$ and $\tan A \tan B \tan C = 3$
 $\therefore \tan A + \tan B + \tan C \neq \tan A \tan B \tan C$
 So, triangle does not exist.
 Hence, (b) is the correct answer.

Ex 73. Statement I $x = n\pi + \frac{\pi}{3}, n \in I$ satisfies the trigonometric equation

$$\tan x \sin x + \sqrt{3} = \sqrt{3} \sin x + \tan x$$

Statement II $\tan x \sin x + \sqrt{3} = \sqrt{3} \sin x + \tan x$

$$\Rightarrow (\tan x - \sqrt{3})(\sin x - 1) = 0$$

$\therefore$ General solution is given by

$$x = n\pi + \frac{\pi}{6}, n \in I \cup x = r\pi + \frac{\pi}{2}, r \in I.$$

Sol. $\tan x \sin x + \sqrt{3} = \sqrt{3} \sin x + \tan x$... (i)

$$\Rightarrow \tan x(\sin x - 1) + \sqrt{3}(1 - \sin x) = 0$$

$$\Rightarrow (\sin x - 1)(\tan x - \sqrt{3}) = 0$$

$\therefore \sin x = 1, \tan x = \sqrt{3}$, but $\sin x = 1$ is not possible as $\tan x$ is not defined at $x = (2n+1)\frac{\pi}{2}$.

i.e. domain of Eq. (i) is $R - \left\{ (2n+1)\frac{\pi}{2} \right\}$.

$$\therefore \tan x = \sqrt{3} = \tan\left(\frac{\pi}{3}\right)$$

$$\Rightarrow x = n\pi + \frac{\pi}{3}, n \in \mathbb{I}$$

$\therefore$ Statement I is true and Statement II is false.

Hence, (c) is the correct answer.

Ex 74. Statement I $x = \frac{k\pi}{13}, k \in \mathbb{I}$ does not represent the general solution of trigonometric equation $\sin 13x - \sin 13x \cos 2x = 0$.

Statement II Both $x = r\pi, r \in \mathbb{I}$ and $x = \frac{k\pi}{13}, k \in \mathbb{I}$ satisfies the trigonometric equation $\sin 13x - \sin 13x \cos 2x = 0$.

$$\begin{aligned} \text{Sol. } \sin 13x(1 - \cos 2x) &= 0 \\ \Rightarrow \sin 13x &= 0 \text{ or } \cos 2x = 1 \\ \therefore 13x &= k\pi \text{ or } \cos 2x = 1 \\ x &= \frac{k\pi}{13}, k \in \mathbb{I} \text{ or } 2x = 2n\pi \end{aligned}$$

$$\Rightarrow x = n\pi, n \in \mathbb{I}$$

So, Statement I is false and Statement II is true.

Hence, (d) is the correct answer.

Ex 75. Statement I Common value(s) of x satisfying the equations

$$\log_{\sin x}(\sec x + 8) > 0 \text{ and}$$

$$\log_{\sin x} \cos x + \log_{\cos x} \sin x = 2 \text{ in } (0, 4\pi) \text{ does not exist.}$$

Statement II On solving above trigonometric equations we have to take intersection of trigonometric chains given by $\sec x > 1$ and $x = n\pi + \frac{\pi}{4}, n \in \mathbb{I}$.

$$\begin{aligned} \text{Sol. } \log_{\sin x} \cos x + \log_{\cos x} \sin x &= 2 \text{ only when} \\ \sin x &= \cos x \\ \Rightarrow x &= n\pi + \frac{\pi}{4} \end{aligned} \quad \dots(i)$$

$$\begin{aligned} \text{Also, } \log_{\sin x}(\sec x + 8) &> 0 \\ \Rightarrow \sec x + 8 &< 1 \\ \Rightarrow \sec x &< -7 \end{aligned} \quad \dots(ii)$$

Clearly, $x = \frac{\pi}{4} + n\pi$ satisfy Eq. (ii).

$\therefore$ Statement I is true and Statement II is false.

Hence, (c) is the correct answer.

Ex 76. Statement I If $f(x) = \cos x + 2 \cos kx, k \in \theta, k \neq 0$, then $f(x) = 3$ has infinite solutions.

Statement II The function is periodic.

Sol. If $f(x) = \cos x + 2 \cos kx, k \in \theta, k \neq 0$, then $f(x) = 3$ has only one solution when $x = 0$. It has infinite solutions, if $k \in \mathbb{I}$ but it is given as $k \in \theta$.

$\therefore$ Statement I is false.

Hence, (d) is the correct answer.

Ex 77. Statement I The maximum value of $\sin \sqrt{2}x + \sin ax$ cannot be 2 (where, a is a positive rational number).

Statement II $\frac{\sqrt{2}}{a}$ is irrational.

$$\text{Sol. } f(x) = \sin \sqrt{2}x + \sin ax$$

$$\text{The period of } \sin \sqrt{2}x = \frac{2\pi}{\sqrt{2}}$$

$$\text{and period of } \sin ax = \frac{2\pi}{a}$$

$f(x)$ has period as

$$\text{LCM of } \left\{ \frac{2\pi}{\sqrt{2}}, \frac{2\pi}{a} \right\} \text{ does not exist.}$$

As $\sqrt{2}$ is irrational and a is rational.

$\therefore$ Maximum value of $f(x)$ cannot be 2 as both cannot be 1.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 78-80) If θ is an angle, one measured in radian and $\theta \in [0, 2\pi]$, then $r\theta$ is length of arc AB, of circle of radius r , subtending angle θ at the centre O, of the circle. Area of sector OAB is $\frac{1}{2}r^2\theta$.

Ex 78. The angle between minute hand and hour hand of a clock at 'half past 4' equals

- (a) 42°
- (b) 43°
- (c) 44°
- (d) None of the above

Sol. Angle subtended by two consecutive marks at centre = 30°

So, at 'half past 4', the angle is 45° .

Hence, (d) is the correct answer.

Ex 79. The wheel of a train is 1 m in diameter and it makes 5 revolutions per second. Then, the speed of the train is (approximately)

- (a) 57 km/h
- (b) 66 km/h
- (c) 68 km/h
- (d) 42.6 km/h

$$\text{Sol. Distance covered in 1s} = 5 \left(2\pi \cdot \frac{1}{2} \right) = 5\pi \text{ m}$$

$$\text{Distance covered in 1h} = \frac{5\pi}{1000} \times 60 \times 60 = 56.52 \text{ km}$$

So, speed of the train is 57 km/h.

Hence, (a) is the correct answer.

Ex 80. Two lines drawn through a point on the circumference of a circle divided the circle into three regions of equal area. Then, the angle θ between the lines is given by

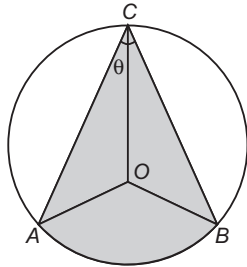
- (a) $3\theta + 3\sin \theta = \pi$
- (b) $6\theta + 3\sin \theta = \pi$
- (c) $2\theta + \sin \theta = \pi$
- (d) $\theta + \sin \theta = \pi/2$

Sol. Area of region $ABC = \frac{\pi r^2}{3}$

$$\text{Area of } OAB = \frac{1}{2} r^2 2\theta = r^2 \theta$$

$$\text{Area of } \triangle OAC = \frac{1}{2} r^2 \sin \theta = \text{Area of } \triangle OBC$$

$$\Rightarrow \frac{1}{2} r^2 \sin \theta + \frac{1}{2} r^2 \sin \theta + r^2 \theta = \frac{\pi r^2}{3}$$



$$\Rightarrow 3 \sin \theta + 3\theta = \pi$$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 81-83) Given,

$$\cos 2^m \theta \cos 2^{m+1} \theta \dots \cos 2^n \theta = \frac{\sin 2^{n+1} \theta}{2^{n-m+1} \sin 2^m \theta}, \quad \text{where}$$

$$2^m \theta \neq k\pi, n, m, k \in \mathbb{I}.$$

Ex 81. $\sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14}$ is equal to

- (a) $\frac{1}{64}$ (b) $-\frac{1}{64}$
(c) $\frac{1}{8}$ (d) $-\frac{1}{8}$

$$\begin{aligned} \text{Sol. } \sin \frac{9\pi}{14} \sin \frac{11\pi}{14} \sin \frac{13\pi}{14} &= \sin \frac{5\pi}{14} \sin \frac{3\pi}{14} \sin \frac{\pi}{14} \\ &= \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{3\pi}{7} \\ &= -\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \\ &= -\frac{\sin \frac{8\pi}{7}}{8 \sin \frac{\pi}{7}} = \frac{1}{8} \end{aligned}$$

Hence, (c) is the correct answer.

Ex 82. $\cos 2^3 \frac{\pi}{10} \cos 2^4 \frac{\pi}{10} \cos 2^5 \frac{\pi}{10} \dots \cos 2^{10} \frac{\pi}{10}$ is

equal to

- (a) $\frac{1}{128}$ (b) $\frac{1}{256}$
(c) $\frac{1}{512} \sin \frac{\pi}{10}$ (d) $\frac{\sqrt{5}-1}{512} \sin \frac{3\pi}{10}$

$$\begin{aligned} \text{Sol. } \cos 2^3 \frac{\pi}{10} \cos 2^4 \frac{\pi}{10} \cos 2^5 \frac{\pi}{10} \dots \cos 2^{10} \frac{\pi}{10} \\ = \frac{\sin 2^{11} \frac{\pi}{10}}{256 \sin 2^3 \frac{\pi}{10}} = \frac{1}{256} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 83. $\cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \dots \cos \frac{11\pi}{11}$ is equal to

- (a) $-\frac{1}{32}$ (b) $\frac{1}{512}$ (c) $\frac{1}{1024}$ (d) $-\frac{1}{2048}$

$$\begin{aligned} \text{Sol. } \cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \dots \cos \frac{11\pi}{11} \\ = \left(\cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{3\pi}{11} \cos \frac{4\pi}{11} \cos \frac{5\pi}{11} \right)^2 \\ = \left(\cos \frac{\pi}{11} \cos \frac{2\pi}{11} \cos \frac{4\pi}{11} \cos \frac{8\pi}{11} \cos \frac{5\pi}{11} \right)^2 \\ = \left(\frac{\sin \frac{16\pi}{11}}{16 \sin \frac{\pi}{11}} \cdot \cos \frac{5\pi}{11} \right)^2 = \left(\frac{2 \sin \frac{5\pi}{11} \cos \frac{5\pi}{11}}{32 \sin \frac{\pi}{11}} \right)^2 = \frac{1}{1024} \end{aligned}$$

Hence, (c) is the correct answer.

Passage III (Ex. Nos. 84-88) $y = ax^2 + bx + c = 0$ is a quadratic equation which has real roots if and only if $b^2 - 4ac \geq 0$. If $(x, y) = 0$ is a second degree equation, then using above fact we can get the range of x and y by treating it as quadratic equation in y or x . Similarly, $ax^2 + bx + c \leq 0, \forall x \in \mathbb{R}$, if $a > 0$ and $b^2 - 4ac \leq 0$.

Ex 84. If $0 < \alpha, \beta < 2\pi$, then the number of ordered pairs (α, β) satisfying

$$\sin^2(\alpha + \beta) - 2 \sin \alpha \sin(\alpha + \beta) + \sin^2 \alpha + \cos^2 \beta = 0, \text{ is}$$

- (a) 2 (b) 0
(c) 4 (d) None of these

Sol. On solving it, we get

$$\sin(\alpha + \beta) = \sin \alpha \pm \sqrt{-\cos^2 \beta}$$

$$\Rightarrow \cos \beta = 0 \Rightarrow \beta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{If } \beta = \frac{\pi}{2}, \text{ then } \tan \alpha = 1, \alpha \in \left\{ \frac{\pi}{4}, \frac{5\pi}{4} \right\}$$

$$\text{If } \beta = \frac{3\pi}{2}, \text{ then } \tan \alpha = -1, \alpha \in \left\{ \frac{3\pi}{4}, \frac{7\pi}{4} \right\}$$

Hence, (c) is the correct answer.

Ex 85. If $A + B + C = \pi$, then the maximum value of $\cos A + \cos B + k \cos C$ (where, $k > 1/2$) is

- (a) $\frac{1}{k} + \frac{k}{2}$ (b) $\frac{2k^2 + 1}{3}$
(c) $\frac{k^2 + 2}{3}$ (d) $\frac{1}{2k} + k$

Sol. Let $\cos A + \cos B + k \cos C = y$

$$\Rightarrow 2k \sin^2 \frac{C}{2} - 2 \cos \left(\frac{A-B}{2} \right) \sin \frac{C}{2} + y - k = 0$$

$$\text{As } D \geq 0$$

$$\Rightarrow k(y - k) \leq \frac{1}{2} \cos^2 \left(\frac{A-B}{2} \right) \leq \frac{1}{2}$$

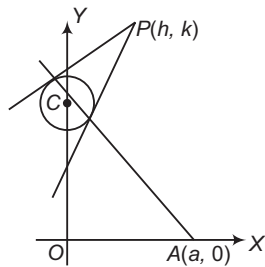
$$\Rightarrow y \leq \frac{1}{2k} + k$$

Hence, (d) is the correct answer.

Ex 86. A circle with radius $|a|$ and centre on Y -axis slides along it and a variable line through $(a, 0)$ cuts the circle at points P and Q . The region in which the point of intersection of tangents to the circle at points P and Q lies is represented by

- (a) $y^2 \geq 4(ax - a^2)$
 (b) $y^2 \leq 4(ax - a^2)$
 (c) $y \geq 4(ax - a^2)$
 (d) $y \leq (ax - a^2)$

Sol. Let the centre be $(0, \alpha)$ and equation of circle be $x^2 + (y - \alpha)^2 = |a|^2$



Equation of chord of contact for $P(h, k)$ is $xh + yk - \alpha(y + k) + \alpha^2 - a^2 = 0$

It passes through $(a, 0)$.

$$\therefore \alpha^2 - \alpha k + ah - a^2 = 0$$

$$\text{As } a \text{ is real } \Rightarrow k^2 - 4(ah - a^2) \geq 0$$

Thus, required equation is

$$y^2 \geq 4(ax - a^2)$$

Hence, (a) is the correct answer.

Ex 87. In above question let A_a represents the region asked in the problem (obviously $|a|$ is radius of circle), then the region represented by $\bigcap_{a \in R} A_a$ will be

- (a) $h^2 - k^2 \leq 0$ (b) $h^2 - k^2 \geq 0$
 (c) $h + 2k < 0$ (d) $h + 2k > 0$

Sol. Intersection of all region is nothing but the region which satisfy $k^2 - 4ah + 4a^2 \geq 0$ for all values of $a \in R$.

$$\Rightarrow 16h^2 - 16k^2 \leq 0$$

$$\Rightarrow h^2 - k^2 \leq 0$$

Hence, (a) is the correct answer.

Ex 88. If $3^{2x+2} + (a^2 - 4a - 2)3^x + 1 > 0, \forall x \in R$,

then

- (a) $a \in R$ (b) $a \in R^+$
 (c) $a \in [1, \infty)$ (d) $a \in R - \{2\}$

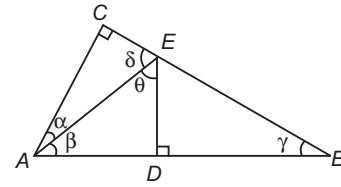
$$\text{Sol. } 3^x \left(9 \cdot 3^x + \frac{1}{3^x} + (a^2 - 4a - 2) \right) > 0$$

$$\Rightarrow 3^x \left[\left(3 \cdot 3^{x/2} - \frac{1}{3^{x/2}} \right)^2 (a - 2)^2 \right] > 0$$

$$\Rightarrow a \neq 2$$

Hence, (d) is the correct answer.

Passage IV (Ex. Nos. 89-93) In the figure, it is given that $\angle C = 90^\circ$, $AD = DB$, ED is perpendicular to AB , $AB = 20$ and $AC = 12$.



Ex 89. Area of $\triangle AEC$ is

- (a) 24 sq units (b) 21 sq units
 (c) 42 sq units (d) $\frac{21}{2}$ sq units

Sol. Given, $AD = BD$

$$\text{Now, } BC^2 = AB^2 - AC^2 = 16^2$$

$$\text{i.e. } BC = 16$$

$$\triangle AED \cong \triangle BED \quad [\text{by SAS property}]$$

$$\tan \gamma = \frac{AC}{BC} = \frac{12}{16} = \frac{3}{4} \quad \dots(i)$$

$$\gamma = \beta \quad \dots(ii)$$

$$\cos \gamma = \frac{BD}{BE} \Rightarrow BE = \frac{10}{4/5} = \frac{25}{2} \quad \dots(iii)$$

$$\tan \gamma = \frac{DE}{BD} \Rightarrow DE = 10 \left(\frac{3}{4} \right) = \frac{15}{2} \quad \dots(iv)$$

$$CE = BC - BE = 16 - \frac{25}{2} = \frac{7}{2}$$

$$\text{Area of } \triangle AEC = \frac{1}{2} (AC)(CE) = \frac{1}{2} \times 12 \times \frac{7}{2} = 21$$

Hence, (b) is the correct answer.

Ex 90. The value of $\tan \gamma \tan (60^\circ - \gamma) \tan (60^\circ + \gamma)$ is

- (a) $-\frac{117}{44}$ (b) $\frac{17}{4}$ (c) $\frac{3}{4}$ (d) $\frac{5}{4}$

Sol. $\tan \gamma \tan (60^\circ - \gamma) \tan (60^\circ + \gamma)$

$$= \tan \gamma \left(\frac{\sqrt{3} - \tan \gamma}{1 + \sqrt{3} \tan \gamma} \right) \left(\frac{\sqrt{3} + \tan \gamma}{1 - \sqrt{3} \tan \gamma} \right)$$

Put $\tan \gamma = \frac{3}{4}$, we get

$$\tan \gamma \tan (60^\circ - \gamma) \tan (60^\circ + \gamma) = -\frac{117}{44}$$

Hence, (a) is the correct answer.

Ex 91. The value of $\cos (\alpha + \theta)$ is

- (a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{117}{125}$ (d) $\frac{44}{125}$

Sol. We have, $\cos (\alpha + \theta) = \cos \alpha \cos \theta - \sin \alpha \sin \theta$

$$\Rightarrow \tan \alpha = \frac{CE}{AC} = \frac{7}{2 \times 12} = \frac{7}{24}$$

$$\Rightarrow \cos \alpha = \frac{24}{25}; \sin \alpha = \frac{7}{25}$$

$$\Rightarrow \tan \theta = \frac{10 \times 2}{15} = \frac{4}{3}$$

$$\Rightarrow \cos \theta = \frac{3}{5}; \sin \theta = \frac{4}{5}$$

$$\Rightarrow \cos (\alpha + \theta) = \frac{24}{25} \cdot \frac{3}{5} - \frac{7}{25} \cdot \frac{4}{5} = \frac{72 - 28}{125} = \frac{44}{125}$$

Hence, (d) is the correct answer.

Ex 92. The value of

$$\sin\left(\frac{\alpha + \theta + \gamma + \delta}{2}\right) \sin\left(\frac{\alpha + \theta - \gamma - \delta}{2}\right) \text{ is}$$

- (a) $\frac{4}{5}$ (b) $\frac{3}{5}$
(c) $\frac{117}{125}$ (d) $-\frac{44}{125}$

Sol.
$$\sin\left(\frac{\alpha + \theta + \gamma + \delta}{2}\right) \sin\left(\frac{\alpha + \theta - \gamma - \delta}{2}\right)$$

$$= \frac{1}{2} [\cos(\gamma + \delta) - \cos(\alpha + \theta)]$$

$$= \frac{1}{2} [-\cos(\alpha + \theta) - \cos(\alpha + \theta)]$$

$$[-\because \gamma + \delta = \pi - (\alpha + \theta)]$$

$$= -\cos(\alpha + \theta) = -\frac{44}{125}$$

Hence, (d) is the correct answer.

Ex 93. Which of the following is true?

- (a) $\gamma + \delta = \theta + \alpha$ (b) $\theta + \delta > \gamma + \alpha$
(c) $\alpha + \theta > \gamma + \delta$ (d) $\alpha + \beta > \gamma + \delta$

Sol. Since, $AC > CE$

$$\delta > \alpha \quad \dots(i)$$

Since, $AD > DE$

$$\theta > \beta$$

Since, $\beta = \gamma$

$$\theta > \gamma \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

$$\delta + \theta > \alpha + \gamma$$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 94. Match the statements of Column I with values of Column II.

Column I	Column II
A. The number of real roots of the equation $\cos^7 x + \sin^4 x = 1$ in $(-\pi, \pi)$ is	p. 1
B. The value of $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$ is	q. 4
C. $4 \cos 36^\circ - 4 \cos 72^\circ + 4 \sin 18^\circ \cos 36^\circ$ equals	r. 3
D. The number of values of x , where $x \in [-2\pi, 2\pi]$, which satisfy $\operatorname{cosec} x = 1 + \cot x$, is	s. 2

Sol. A. $\cos^7 x + \sin^4 x = 1$

$$\cos^7 x = (1 + \sin^2 x) \cos^2 x$$

$$\Rightarrow \cos x = 0$$

$$\text{or } \cos^5 x = 1 + \sin^2 x$$

$$\cos x = 0$$

$$\Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$$

$$\text{or } \cos^5 x = 1 + \sin^2 x$$

$$\Rightarrow x = 0 \quad [\because \text{LHS} \leq 1 \text{ and RHS} \geq 1]$$

$$\therefore x = -\frac{\pi}{2}, 0, \frac{\pi}{2}$$

B. $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$

$$= \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sqrt{3} \cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{2 \left(\frac{\sqrt{3}}{2} \cos 20^\circ - \frac{1}{2} \sin 20^\circ \right)}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4 \cos 50^\circ}{\sin 40^\circ} = 4$$

C. $4 \cos 36^\circ - 4 \cos 72^\circ + 4 \sin 18^\circ \cos 36^\circ$

$$= 4 \left(\frac{\sqrt{5} + 1}{4} \right) - 4 \left(\frac{\sqrt{5} - 1}{4} \right) + 4 \left(\frac{\sqrt{5} - 1}{4} \right) \left(\frac{\sqrt{5} + 1}{4} \right)$$

$$= \sqrt{5} + 1 - \sqrt{5} + 1 + 1 = 3$$

D. $\operatorname{cosec} x = 1 + \cot x$

$$\Rightarrow \frac{1}{\sin x} = \frac{\sin x + \cos x}{\sin x}$$

$$\Rightarrow \sin x + \cos x = 1 \text{ and } \sin x \neq 0$$

$$\cos \left(x - \frac{\pi}{4} \right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x - \frac{\pi}{4} = -2\pi + \frac{\pi}{4}, \frac{\pi}{4}$$

$$\left[\because x - \frac{\pi}{4} \in \left[-2\pi - \frac{\pi}{4}, 2\pi - \frac{\pi}{4} \right] \right]$$

$$\Rightarrow x = -\frac{3\pi}{2}, \frac{\pi}{2}$$

Hence, $A \rightarrow r$; $B \rightarrow q$; $C \rightarrow r$; $D \rightarrow s$

Ex 95. Match the statements of Column I with values of Column II.

Column I	Column II
A. The number of solutions of the equation $ \cot x = \cot x + \frac{1}{\sin x}$ ($0 < x < \pi$) is	p. No solution
B. If $\sin \theta + \sin \phi = \frac{1}{2}$ and $\cos \theta + \cos \phi = 2$, then $\cot \left(\frac{\theta + \phi}{2} \right)$ is equal to	q. $\frac{1}{3}$
C. $\sin^2 \alpha + \sin \left(\frac{\pi}{3} - \alpha \right) \sin \left(\frac{\pi}{3} + \alpha \right)$ is equal to	r. 1
D. If $\tan \theta = 3 \tan \phi$, then maximum value of $\frac{\tan^2(\theta - \phi)}{\tan^2 \theta}$ is	s. $\frac{3}{4}$

Sol. A. $|\cot x| = \cot x + \frac{1}{\sin x}$

$$\text{If } 0 < x < \frac{\pi}{2}, \text{ then } \cot x > 0$$

$$\text{So, } \cot x = \cot x + \frac{1}{\sin x}$$

$$\Rightarrow \frac{1}{\sin x} = 0, \text{ no solution}$$

$$\text{If } \frac{\pi}{2} < \cot x < \pi, \text{ then } -\cot x = \cot x + \frac{1}{\sin x}$$

$$\frac{2 \cos x}{\sin x} + \frac{1}{\sin x} = 0$$

$$\Rightarrow 1 + 2 \cos x = 0 \text{ and } \sin x \neq 0 \Rightarrow x = \frac{2\pi}{3}$$

B. Since, $\sin \phi + \sin \theta = \frac{1}{2}$ and $\cos \theta + \cos \phi = 2$ has no solution.

$$\begin{aligned} \text{C. } \sin^2 \alpha + \sin \left(\frac{\pi}{3} - \alpha \right) \sin \left(\frac{\pi}{3} + \alpha \right) \\ = \sin^2 \alpha + \sin^2 \frac{\pi}{3} - \sin^2 \alpha = \frac{3}{4} \end{aligned}$$

$$\begin{aligned} \text{D. } \tan \theta = 3 \tan \phi \\ \tan (\theta - \phi) = \frac{\tan \theta - \tan \phi}{1 + \tan \theta \tan \phi} = \frac{2 \tan \phi}{1 + 3 \tan^2 \phi} \\ = \frac{2}{\cot \phi + 3 \tan \phi} \quad [\text{maximum, if } \tan \theta > 0] \\ \frac{\cot \phi + 3 \tan \phi}{2} \geq \sqrt{3} \quad [\text{using AM} \geq \text{GM}] \\ \Rightarrow (\cot \phi + 3 \tan \phi)^2 \geq 12 \Rightarrow \tan^2 (\theta - \phi) \leq \frac{1}{3} \end{aligned}$$

$$A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$$

Ex 96. Match the statements of Column I with values of Column II.

Column I	Column II
A. The value of $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 170^\circ$ is	p. 4
B. The value of $\sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ$ is	q. 1
C. Maximum value of $3 \sin x + 4 \cos x$ is	r. 0
D. $\tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ$ is	s. 5

Sol. A. $\cos 1^\circ \cos 2^\circ \cos 3^\circ \dots \cos 170^\circ = 0$

$$\begin{aligned} \text{B. } \sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ \\ = \frac{\sqrt{3}}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sqrt{3} \cos 20^\circ - \sin 20^\circ}{\cos 20^\circ \sin 20^\circ} \\ = \frac{4[\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ]}{2 \cos 20^\circ \sin 20^\circ} = \frac{4 \sin 40^\circ}{\sin 40^\circ} = 4 \end{aligned}$$

C. $3 \sin x + 4 \cos x$

Maximum value = 5

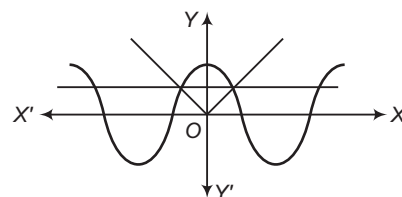
$$\begin{aligned} \text{D. } \tan 15^\circ + \tan 30^\circ + \tan 15^\circ \tan 30^\circ \\ = \frac{\sqrt{3}-1}{\sqrt{3}+1} + \frac{1}{\sqrt{3}} + \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{1}{\sqrt{3}} \\ = \frac{\sqrt{3}-1}{\sqrt{3}+1} \left(\frac{1}{\sqrt{3}} + 1 \right) + \frac{1}{\sqrt{3}} \\ = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{\sqrt{3}+1}{\sqrt{3}} + \frac{1}{\sqrt{3}} = 1 \end{aligned}$$

$$A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$$

Ex 97. Match the statements of Column I with values of Column II.

Column I	Column II
A. Maximum value of $-x^2 - 2x + 2$ is	p. 3
B. Number of real solutions of $\cos(e^x) = 2^x + 2^{-x}$ is	q. 2
C. Number of solutions of equation $ x = \cos x$ is	r. 1
D. Minimum value of $x^2 - 2x + 4, \forall x \in R$, is	s. 0

Sol. A. Maximum value = $-\frac{D}{4a} = 3$



B. LHS ≤ 1 , whereas RHS ≥ 2

So, no solution.

C. Number of solutions = 2

D. Minimum value = $-\frac{D}{4a} = -\left[-\frac{12}{4}\right] = 3$

$$A \rightarrow p; B \rightarrow s; C \rightarrow q; D \rightarrow p$$

Type 6. Single Integer Answer Type Questions

Ex 98. In a $\triangle ABC$, $2 \left(\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \right)$ has the maximum value λ , then λ is equal to _____.

Sol. (3) Let $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} = k$
or $2 \sin \frac{A+B}{4} \cos \frac{A-B}{4} + \cos \frac{A+B}{2} = k$
 $\Rightarrow 2 \sin^2 \frac{A+B}{4} - 2 \cos \frac{A-B}{4} \sin \frac{A+B}{4} + k - 1 = 0$

Since, $\sin \frac{A+B}{4}$ is real, $4 \cos^2 \frac{A-B}{4} - 8(k-1) \geq 0$

$$\Rightarrow 2(k-1) \leq \cos^2 \frac{A-B}{4} \leq 1$$

$$\Rightarrow k \leq \frac{3}{2}$$

So, $2 \cos \frac{A-B}{4} \sin \frac{A+B}{4} - 2 \sin^2 \frac{A+B}{4} + 1 \leq \frac{3}{2}$

Hence, $\lambda = 3$

Ex 99. If $\sqrt{a+b} \tan \frac{\theta}{2} = \sqrt{a-b} \tan \frac{\phi}{2}$ and $(b - a \sec \theta) (b + a \cos \phi)$ has maximum value λ , then λ is equal to _____.

Sol. (0) Given that,
 $\sqrt{a+b} \tan \frac{\theta}{2} = \sqrt{a-b} \tan \frac{\phi}{2} \Rightarrow a \geq b \quad \dots(i)$

$$\Rightarrow \cos \theta = \frac{1 - \tan^2 \frac{\theta}{2}}{1 + \tan^2 \frac{\theta}{2}} = \frac{a \cos \phi + b}{a + b \cos \phi}$$

$$\Rightarrow (b - a \sec \theta) (b + a \cos \phi) = b^2 - a^2$$

$$\Rightarrow (b - a \sec \theta) (b + a \cos \phi) \leq 0 \quad [\text{from Eq. (i)}]$$

 So, the maximum value is 0.
 $\therefore \lambda = 0$

Ex 100. If $\log_{|\sin x|} |\cos x| + \log_{|\cos x|} |\sin x| = 2$, then $|\tan x|$ is equal to _____.

Sol. (1) Here, $\log_{|\sin x|} |\cos x| + \frac{1}{\log_{|\sin x|} |\cos x|} = 2$

$$\Rightarrow y + \frac{1}{y} = 2 \text{ where } y = \log_{|\sin x|} |\cos x|$$

$$\Rightarrow y = 1$$

$$\Rightarrow \log_{|\sin x|} |\cos x| = 1$$

$$\Rightarrow |\sin x| = |\cos x|$$

$$\Rightarrow |\tan x| = 1$$

Ex 101. If $16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta - \cos \frac{5\pi}{8} \right) \left(\cos \theta - \cos \frac{7\pi}{8} \right) = \lambda \cos 4\theta$, then the value of λ is _____.

Sol. (2) LHS = $16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta - \cos \frac{5\pi}{8} \right) \left(\cos \theta - \cos \frac{7\pi}{8} \right)$

$$= 16 \left(\cos \theta - \cos \frac{\pi}{8} \right) \left(\cos \theta - \cos \frac{3\pi}{8} \right) \left(\cos \theta + \cos \frac{3\pi}{8} \right) \left(\cos \theta + \cos \frac{\pi}{8} \right)$$

$$= 16 \left(\cos^2 \theta - \cos^2 \frac{\pi}{8} \right) \left(\cos^2 \theta - \cos^2 \frac{3\pi}{8} \right)$$

$$= 16 \left(\cos^4 \theta - \cos^2 \theta + \sin^2 \frac{\pi}{8} \cos^2 \frac{\pi}{8} \right)$$

$$= 16 \left(\cos^4 \theta - \cos^2 \theta + \frac{1}{4} \sin^2 \frac{\pi}{4} \right)$$

$$= 16 \left(\cos^4 \theta - \cos^2 \theta + \frac{1}{8} \right)$$

$$= 16 \left(-\cos^2 \theta \sin^2 \theta + \frac{1}{8} \right) = 16 \left(\frac{-\sin^2 2\theta}{4} + \frac{1}{8} \right)$$

$$= 16 \left(\frac{1 - 2 \sin^2 2\theta}{8} \right) = 16 \left(\frac{\cos^2 2\theta - \sin^2 2\theta}{8} \right)$$

$$= \frac{16 \cos 4\theta}{8} = 2 \cos 4\theta$$

 Hence, the value of λ is 2.

Ex 102. If $x \in [0, 2\pi]$ for which

$2 \cos x \leq |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}| \leq \sqrt{2}$
 has solution set $x \in \left[\frac{\lambda\pi}{4}, \frac{\mu\pi}{4} \right]$, then $\mu - \lambda$ is equal to _____.

Sol. (6) Let $y = |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}|$

$$\Rightarrow y^2 = 2 - 2 |\cos 2x|$$

 If $x \in \left[0, \frac{\pi}{4} \right]$ or $\left[\frac{3\pi}{4}, \frac{5\pi}{4} \right]$ or $\left[\frac{7\pi}{4}, 2\pi \right]$
 $\cos 2x$ is non-negative.
 So, $y^2 = 2 - 2 \cos 2x = 4 \sin^2 x$

$$y = 2 |\sin x|$$

$$\Rightarrow \cos x \leq |\sin x|$$

 Except for $x \in \left[0, \frac{\pi}{4} \right]$ and $\left[\frac{7\pi}{4}, 2\pi \right]$
 so that leaves $\left[\frac{3\pi}{4}, \frac{5\pi}{4} \right]$

In which we certainly have, $\sin x \leq \frac{1}{\sqrt{2}}$

If $x \in \left(\frac{\pi}{4}, \frac{3\pi}{4} \right)$ or $\left(\frac{5\pi}{4}, \frac{7\pi}{4} \right)$, then
 $\cos 2x$ is negative, so $y^2 = 2 + 2 \cos 2x = 4 \cos^2 x$

$$\Rightarrow y = 2 |\cos x|$$

 So the first inequality certainly holds, the second also holds.
 Thus, solution set is $x \in \left[\frac{\pi}{4}, \frac{7\pi}{4} \right]$.
 $\Rightarrow \mu - \lambda = 6$

Ex 103. If $2^7 \cos^3 \theta \sin^5 \theta = a \sin 8\theta - b \sin 6\theta + c \sin 4\theta + d \sin 2\theta$ and θ is real, then the value of $a + b + c + d$ must be _____.

Sol. (7) Let $z = e^{i\theta}$
 $\therefore 2 \cos \theta = z + \frac{1}{z}$ and $2i \sin \theta = z - \frac{1}{z}$
 Now, $(2 \cos \theta)^3 \cdot (2i \sin \theta)^5 = \left(z + \frac{1}{z} \right)^3 \left(z - \frac{1}{z} \right)^5$

$$= \left(z^8 - \frac{1}{z^8} \right) - 2 \left(z^6 - \frac{1}{z^6} \right) - 2 \left(z^4 - \frac{1}{z^4} \right) + 6 \left(z^2 - \frac{1}{z^2} \right)$$

$$= 2i \sin 8\theta - 4i \sin 6\theta - 4i \sin 4\theta + 12i \sin 2\theta$$

$$\therefore 2^7 \cos^3 \theta \sin^5 \theta = \sin 8\theta - 2 \sin 6\theta - 2 \sin 4\theta + 6 \sin 2\theta$$

 On comparing, $a = 1, b = 2, c = -2, d = 6$
 $\Rightarrow a + b + c + d = 7$

Ex 104. If $\sin x + \sin y \geq \cos \alpha \cos x, \forall x \in R$, then $\sin y + \cos \alpha$ is equal to _____.

Sol. (1) $\sin x + \sin y \geq \cos \alpha \cos x, \forall x \in R$
 Let $x = -\frac{\pi}{2}$
 $\therefore -1 + \sin y \geq 0$
 $\Rightarrow \sin y \geq 1 \Rightarrow \sin y = 1$

If $\sin y = 1$, then
 $\sin x + 1 \geq \cos \alpha \cos x$
 $\Rightarrow \cos \alpha \cos x - \sin x \leq 1 \quad \dots(i)$
 Using $a \sin \theta + b \cos \theta \leq \sqrt{a^2 + b^2}, \forall \theta \in R$
 $\cos \alpha \cos x - \sin x \leq \sqrt{1 + \cos^2 \alpha}$
 $\therefore \sqrt{1 + \cos^2 \alpha} \leq 1$
 $\Rightarrow \cos \alpha = 0$
 $\Rightarrow \sin y + \cos \alpha = 1$

Ex 105. If $0 \leq A, B, C \leq \pi$ and $A + B + C = \pi$, then the minimum value of $\sin 3A + \sin 3B + \sin 3C$ is _____.

Sol. (–2) Let $y = \sin 3A + \sin 3B + \sin 3C$ for $\sin 3A$ to be non-positive.
 We have, $2\pi \leq 3A \leq 3\pi$
 $\Rightarrow \frac{2\pi}{3} < A < \pi$
 Since, $A + B + C = \pi$
 All of $\sin 3A, \sin 3B, \sin 3C$ can not be negative.
 Let us take $\sin 3A = -1$
 $\Rightarrow A = \frac{\pi}{2} \Rightarrow \sin 3B = -1 \Rightarrow B = \frac{\pi}{2}$
 $\therefore C = 0$
 i.e. $\sin 3A = -1, \sin 3B = -1, \sin 3C = 0$
 Hence, minimum value of
 $\sin 3A + \sin 3B + \sin 3C = -2$

Ex 106. If θ_1, θ_2 and θ_3 are three values lying in $[0, 2\pi]$ for which $\tan \theta = \lambda$, then

$\left| \tan \frac{\theta_1}{3} \tan \frac{\theta_2}{3} + \tan \frac{\theta_2}{3} \tan \frac{\theta_3}{3} + \tan \frac{\theta_1}{3} \tan \frac{\theta_3}{3} \right|$
 is equal to _____.

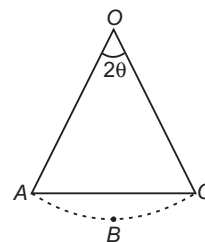
Sol. (3) $\tan \theta = \frac{3 \tan \frac{\theta}{3} - \tan^3 \frac{\theta}{3}}{1 - 3 \tan^2 \frac{\theta}{3}}$

$\Rightarrow \tan^3 \frac{\theta}{3} - 3\lambda \tan \frac{\theta}{3} - 3 \tan \frac{\theta}{3} + \lambda = 0$
 $\therefore \tan \frac{\theta_1}{3} \tan \frac{\theta_2}{3} + \tan \frac{\theta_2}{3} \tan \frac{\theta_3}{3} + \tan \frac{\theta_1}{3} \tan \frac{\theta_3}{3} = -3$
 $\Rightarrow \left| \tan \frac{\theta_1}{3} \tan \frac{\theta_2}{3} + \tan \frac{\theta_2}{3} \tan \frac{\theta_3}{3} + \tan \frac{\theta_1}{3} \tan \frac{\theta_3}{3} \right| = 3$

Ex 107. The number of solutions of the equation $\log_{|x-1|} |x^2 - 1| = [\sin x] + [\cos x]$ where, $[\]$ denotes greatest integer function, is _____.

Sol. (1) As $1 \leq |\sin x| + |\cos x| \leq \sqrt{2}$
 $\therefore [|\sin x| + |\cos x|] = 1$
 $\Rightarrow \log_{|x-1|} |x^2 - 1| = 1$
 $\Rightarrow |x^2 - 1| = |x - 1|$
 $\Rightarrow x = 1, 0, -2$, but $x \neq 0, 1$
 $\therefore$ Number of solutions is one.

Ex 108. For a given sector $OABC$, if $\theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right)$, then minimum area of segment $ABCA$ is $r^2 \left(\frac{\pi}{6} - \frac{\sqrt{3}}{\lambda}\right)$, then λ is _____.



Sol. (4) Area of segment $ABCA$
 $= \text{Area of sector } AOC - \text{Area of } \Delta AOC$
 $= \frac{2\theta}{360^\circ} \times \pi r^2 - \frac{r^2}{2} \sin 2\theta$
 $= r^2 \left\{ \theta - \frac{1}{2} \sin 2\theta \right\}, \theta \in \left(\frac{\pi}{6}, \frac{\pi}{3}\right) \quad \dots(i)$
 Let $f(\theta) = \theta - \frac{1}{2} \sin 2\theta$
 $\Rightarrow f'(\theta) = 1 - \cos 2\theta > 0$ [increasing]
 Hence, area of segment $ABCA = r^2 \left\{ \frac{\pi}{6} - \frac{1}{2} \sin \frac{2\pi}{3} \right\}$
 $= r^2 \left\{ \frac{\pi}{6} - \frac{\sqrt{3}}{4} \right\} = r^2 \left\{ \frac{\pi}{6} - \frac{\sqrt{3}}{\lambda} \right\} \Rightarrow \lambda = 4$

Ex 109. The number of solutions for $2 \sin^2 x + \sin^2 2x = 2$ when $-\pi < x < \pi$, is _____.

Sol. (6) We have, $2 \sin^2 x + \sin^2 2x = 2$
 $\Rightarrow 1 - \cos 2x + 1 - \cos^2 2x = 2$
 $\Rightarrow \cos 2x + \cos^2 2x = 0$
 $\Rightarrow \cos 2x(1 + \cos 2x) = 0$
 $\Rightarrow \cos 2x = 0, \cos 2x = -1$
 $\Rightarrow 2x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, 2x = \pm \pi$
 $\Rightarrow x = \pm \frac{\pi}{4}, \pm \frac{3\pi}{4}, \pm \frac{\pi}{2}$
 Hence, the number of solutions is six.

Target Exercises

Type 1. Only One Correct Option

- The numerical value of $\tan \frac{\pi}{3} + 2 \tan \frac{2\pi}{3} + 4 \tan \frac{4\pi}{3} + 8 \tan \frac{8\pi}{3}$ is
(a) $-5\sqrt{3}$ (b) $-\frac{5}{\sqrt{3}}$
(c) $5\sqrt{3}$ (d) $\frac{5}{\sqrt{3}}$
- A quadratic equation whose roots are $\csc^2 \theta$ and $\sec^2 \theta$, can be
(a) $x^2 - 2x + 2 = 0$ (b) $x^2 - 3x + 3 = 0$
(c) $x^2 - 5x + 5 = 0$ (d) None of these
- $\cos^2 \frac{3\pi}{5} + \cos^2 \frac{4\pi}{5}$ is equal to
(a) $\frac{4}{5}$ (b) $\frac{5}{2}$ (c) $\frac{5}{4}$ (d) $\frac{3}{4}$
- The value of $\sin 600^\circ \cos 330^\circ + \cos 120^\circ \sin 150^\circ$ is
(a) -1 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{3}{2}$
- The value of expression $\frac{1 - 4 \sin 10^\circ \sin 70^\circ}{2 \sin 10^\circ}$ is
(a) $\frac{1}{2}$ (b) 1
(c) 2 (d) None of these
- The value of the expression $(\sqrt{3} \sin 75^\circ - \cos 75^\circ)$ is
(a) $2 \sin 15^\circ$ (b) $1 + \sqrt{3}$ (c) $2 \sin 105^\circ$ (d) $\sqrt{2}$
- If $\sin \alpha = \frac{1}{\sqrt{5}}$ and $\sin \beta = \frac{3}{5}$, then $\beta - \alpha$ lies in the interval
(a) $\left(0, \frac{\pi}{4}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$ (b) $\left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$
(c) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ (d) $\left(\pi, \frac{5\pi}{4}\right]$
- The numerical value of $\sqrt{3} \csc 20^\circ - \sec 20^\circ$ is
(a) 2 (b) 4
(c) 6 (d) None of these
- Which of the following numbers is rational?
(a) $\sin 15^\circ$ (b) $\cos 15^\circ$
(c) $\sin 15^\circ \cos 15^\circ$ (d) $\sin 15^\circ \cos 75^\circ$
- The value of $\cos \frac{\pi}{7} \cos \frac{4\pi}{7} \cos \frac{5\pi}{7}$ is
(a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $-\frac{1}{8}$ (d) $\frac{1}{8}$
- The value of $6(\sin^6 \theta + \cos^6 \theta) - 9(\sin^4 \theta + \cos^4 \theta) + 4$ is
(a) -3 (b) 0 (c) 1 (d) 3
- If $\frac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha} = x$, then $\frac{1 - \cos \alpha - \sin \alpha}{\cos \alpha}$ is equal to
(a) $\frac{1}{x}$ (b) x
(c) $1 - x$ (d) None of these
- Which of the following is not correct?
(a) $\sin \theta = -\frac{1}{5}$ (b) $\cos \theta = 1$
(c) $\sec \theta = \frac{1}{2}$ (d) $\tan \theta = 20$
- $\log(\sin 1^\circ) \log(\sin 2^\circ) \log(\sin 3^\circ) \dots \log(\sin 179^\circ)$ is equal to
(a) 1 (b) 0 (c) 2 (d) -1
- The value of $3 \left[\sin^4 \left(\frac{3\pi}{2} - \alpha \right) + \sin^4 (3\pi + \alpha) \right] - 2 \left[\sin^6 \left(\frac{\pi}{2} + \alpha \right) + \sin^6 (5\pi - \alpha) \right]$ is
(a) 0 (b) 1
(c) 3 (d) $\sin 4\alpha + \sin 6\alpha$
- If $\tan x + \cot x = 2$, then $\sin^{2n} x + \cos^{2n} x$ is equal to
(a) 2^n (b) $-\frac{1}{2}$
(c) $\frac{1}{2}$ (d) None of these
- If $\frac{\pi}{2} < \alpha < \pi$, $\pi < \beta < \frac{3\pi}{2}$; $\sin \alpha = \frac{15}{17}$ and $\tan \beta = \frac{12}{5}$, then the value of $\sin(\beta - \alpha)$ is
(a) $-\frac{171}{221}$ (b) $-\frac{21}{221}$ (c) $\frac{21}{221}$ (d) $\frac{171}{221}$
- If $\cos(\theta - \alpha) = a$ and $\sin(\theta - \beta) = b$, then $\cos^2(\alpha - \beta) + 2ab \sin(\alpha - \beta)$ is equal to
(a) $4a^2b^2$ (b) $a^2 - b^2$
(c) $a^2 + b^2$ (d) $-a^2b^2$
- The expression $\frac{\cos 6x + 6 \cos 4x + 15 \cos 2x + 10}{\cos 5x + 5 \cos 3x + 10 \cos x}$ is equal to
(a) $\cos 2x$ (b) $2 \cos x$
(c) $\cos^2 x$ (d) $1 + \cos x$

- 20.** $\sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ$ is equal to
 (a) $\sin 36^\circ$ (b) $\cos 36^\circ$
 (c) $\sin 7^\circ$ (d) $\cos 7^\circ$
- 21.** The value of $\cos y \cos \left(\frac{\pi}{2} - x\right) - \cos \left(\frac{\pi}{2} - y\right) \cos x$
 $+ \sin y \cos \left(\frac{\pi}{2} - x\right) + \cos x \sin \left(\frac{\pi}{2} - y\right)$ is zero, if
 (a) $x = 0$ (b) $y = 0$
 (c) $x = y$ (d) $x = n\pi - \frac{\pi}{4} + y$
- 22.** If $\operatorname{cosec} A + \cot A = \frac{11}{2}$, then $\tan A$ is equal to
 (a) $\frac{21}{22}$ (b) $\frac{15}{16}$ (c) $\frac{44}{117}$ (d) $\frac{117}{43}$
- 23.** The value of the expression $\frac{\sin^3 x}{1 + \cos x} + \frac{\cos^3 x}{1 - \sin x}$ is
 (a) $\sqrt{2} \cos \left[\frac{\pi}{4} - x\right]$ (b) $\sqrt{2} \cos \left[\frac{\pi}{4} + x\right]$
 (c) $\sqrt{2} \sin \left[\frac{\pi}{4} - x\right]$ (d) None of these
- 24.** In ΔABC , the value of the expression
 $\operatorname{cosec} A (\sin B \cos C + \cos B \sin C)$ is
 (a) 1 (b) $\frac{c}{a}$
 (c) $\frac{a}{c}$ (d) None of these
- 25.** If $\cos(A - B) = \frac{3}{5}$ and $\tan A \tan B = 2$, then
 (a) $\cos A \cos B = \frac{1}{5}$ (b) $\sin A \sin B = -\frac{2}{5}$
 (c) $\cos(A + B) = \frac{1}{5}$ (d) None of these
- 26.** $f(\theta) = \sin^2 \theta + \sin^2 \left(\theta + \frac{2\pi}{3}\right) + \sin^2 \left(\theta + \frac{4\pi}{3}\right)$, then
 $f\left(\frac{\pi}{15}\right)$ is equal to
 (a) $\frac{2}{3}$ (b) $\frac{3}{2}$
 (c) $\frac{1}{3}$ (d) None of these
- 27.** If $\cos(x - y)$, $\cos x$ and $\cos(x + y)$ are in HP, then
 $\cos x \cdot \sec\left(\frac{y}{2}\right)$ is equal to
 (a) ± 1 (b) $\pm \sqrt{2}$
 (c) $\pm \sqrt{3}$ (d) None of these
- 28.** If $\sin^3 x \sin 3x = \sum_{r=0}^n a_r \cos(rx)$, $\forall x \in R$, where
 $a_0, a_1, \dots, a_n$ are constants and $a_n \neq 0$, then
 (a) $n = 5$
 (b) $n = 6$
 (c) $n = 7$
 (d) None of the above
- 29.** If $A + B = \frac{\pi}{4}$, where $A, B \in R^+$, then the minimum
 value of $(1 + \tan A)(1 + \tan B)$ is
 (a) 2 (b) 4
 (c) 1 (d) None of these
- 30.** If $\frac{\sin x}{\sin y} = \frac{1}{2}$, $\frac{\cos x}{\cos y} = \frac{3}{2}$, where $x, y \in \left(0, \frac{\pi}{2}\right)$, then the
 value of $\tan(x + y)$ is
 (a) $\sqrt{13}$ (b) $\sqrt{14}$ (c) $\sqrt{17}$ (d) $\sqrt{15}$
- 31.** In ΔABC , if $\angle C = \frac{2\pi}{3}$, then the value of
 $\cos^2 A + \cos^2 B - \cos A \cos B$ is
 (a) $\frac{3}{4}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
- 32.** The value of $\sum_{r=0}^{10} \cos^3 \frac{\pi r}{3}$ is
 (a) $-\frac{9}{2}$ (b) $-\frac{7}{2}$ (c) $-\frac{9}{8}$ (d) $-\frac{1}{8}$
- 33.** If in any ΔABC , the value of $\angle A$ is obtained from the
 equation $3\cos A + 2 = 0$, then the equation whose
 roots are $\sin A$ and $\tan A$, is
 (a) $6x^2 + \sqrt{5}x - 5 = 0$ (b) $6x^2 + 5x - 5 = 0$
 (c) $x^2 + \sqrt{5}x - 5 = 0$ (d) None of these
- 34.** If $\cot^2 x = \cot(x - y)\cot(x - z)$, then $\cot 2x$ is equal
 to (where, $x \neq \pm \pi/4$)
 (a) $\frac{1}{2}(\tan y + \tan z)$ (b) $\frac{1}{2}(\cot y + \cot z)$
 (c) $\frac{1}{2}(\sin y + \sin z)$ (d) None of these
- 35.** If $\alpha + \beta = \frac{\pi}{2}$ and $\beta + \gamma = \alpha$, then $\tan \alpha$ is equal to
 (a) $2(\tan \beta + \tan \gamma)$ (b) $\tan \beta + \tan \gamma$
 (c) $\tan \beta + 2\tan \gamma$ (d) $2\tan \beta + \tan \gamma$
- 36.** If $\sin \theta = n \sin(\theta + 2\alpha)$, then $\tan(\theta + \alpha)$ is equal to
 (a) $\left(\frac{2+n}{2-n}\right) \tan \alpha$ (b) $\left(\frac{1+n}{1-n}\right) \tan \alpha$
 (c) $\tan \alpha$ (d) $\left(\frac{1-n}{1+n}\right) \tan \alpha$
- 37.** If $\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$, $\frac{x}{a} \cos \beta + \frac{y}{b} \sin \beta = 1$ and
 $\frac{\cos \alpha \cos \beta}{a^2} + \frac{\sin \alpha \sin \beta}{b^2} = 0$, then
 (a) $\tan \alpha \tan \beta = \frac{b^2(x^2 + a^2)}{a^2(y^2 + b^2)}$ (b) $x^2 + y^2 = a^2 + b^2$
 (c) $\tan \alpha \tan \beta = \frac{a^2}{b^2}$ (d) None of these
- 38.** If $x = \sin(\alpha - \beta) \sin(\gamma - \delta)$, $y = \sin(\beta - \gamma) \sin(\alpha - \delta)$
 and $z = \sin(\gamma - \alpha) \sin(\beta - \delta)$, then
 (a) $x + y + z = 0$ (b) $x + y - z = 0$
 (c) $y + z - x = 0$ (d) None of these

39. $\cos\left(\frac{\pi}{4} - x\right)\cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right)\sin\left(\frac{\pi}{4} - y\right)$ is equal to
 (a) $\sin(x - y)$ (b) $\sin(x + y)$
 (c) $\cos(x + y)$ (d) $\cos(x - y)$
40. If $x \in (0, \pi)$ and $\sin x \cos^3 x > \cos x \sin^3 x$, then complete set of values of x is
 (a) $x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$
 (b) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \cup \left(\frac{3\pi}{4}, \pi\right)$
 (c) $x \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
 (d) None of the above
41. For $0 < \phi < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n} \phi$, $y = \sum_{n=0}^{\infty} \sin^{2n} \phi$ and $z = \sum_{n=0}^{\infty} \cos^{2n} \phi \sin^{2n} \phi$, then
 (a) $xyz = xz + y$ (b) $xyz = xy + y$
 (c) $xyz = x + y + z$ (d) $xyz = yz + x$
42. If $\frac{x}{\cos \theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$, then $x + y + z$ is equal to
 (a) 1 (b) 0
 (c) -1 (d) None of these
43. If $\cos \theta = \frac{1}{2}\left(x + \frac{1}{x}\right)$, then $\frac{1}{2}\left(x^2 + \frac{1}{x^2}\right)$ is equal to
 (a) $\sin 2\theta$ (b) $\cos 2\theta$
 (c) $\tan 2\theta$ (d) None of these
44. If $-\pi \leq x \leq \pi$, $-\pi \leq y \leq \pi$ and $\cos x + \cos y = 2$, then the value of $\cos(x - y)$ is
 (a) -1 (b) 0
 (c) 1 (d) None of these
45. If $\cos x + \cos y + \cos \alpha = 0$ and $\sin x + \sin y + \sin \alpha = 0$, then $\cot\left(\frac{x + y}{2}\right)$ is equal to
 (a) $\sin \alpha$ (b) $\cos \alpha$
 (c) $\cot \alpha$ (d) $\sin\left(\frac{x + y}{2}\right)$
46. If $\alpha + \beta - \gamma = \pi$, then $\sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma$ is equal to
 (a) $2 \sin \alpha \sin \beta \cos \gamma$ (b) $2 \cos \alpha \cos \beta \cos \gamma$
 (c) $2 \sin \alpha \sin \beta \sin \gamma$ (d) None of these
47. If $\alpha + \beta + \gamma = 2\pi$, then
 (a) $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$
 (b) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} + \tan \frac{\gamma}{2} \tan \frac{\alpha}{2} = 1$
 (c) $\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = -\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$
 (d) None of the above
48. If $\sin x + \cos x = \frac{\sqrt{7}}{2}$, where $x \in \left[0, \frac{\pi}{4}\right]$, then $\tan \frac{x}{2}$ is equal to
 (a) $\frac{3 - \sqrt{7}}{3}$ (b) $\frac{\sqrt{7} - 2}{3}$
 (c) $\frac{7 - \sqrt{7}}{4}$ (d) None of these
49. If $\sin B = \frac{1}{5} \sin(2A + B)$, then $\frac{\tan(A + B)}{\tan A}$ is equal to
 (a) $\frac{5}{3}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) $\frac{3}{5}$
50. If $\tan \alpha = \frac{1}{7}$ and $\sin \beta = \frac{1}{\sqrt{10}}$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then 2β is equal to
 (a) $\frac{\pi}{4} - \alpha$ (b) $\frac{3\pi}{4} - \alpha$ (c) $\frac{\pi}{8} - \alpha$ (d) $\frac{3\pi}{8} - \frac{\alpha}{2}$
51. $\frac{3 \cos \theta + \cos 3\theta}{3 \sin \theta - \sin 3\theta}$ is equal to
 (a) $1 + \cot^2 \theta$ (b) $\cot^4 \theta$ (c) $\cot^3 \theta$ (d) $2 \cot \theta$
52. If $3 \sin \alpha = 5 \sin \beta$, then $\frac{\tan \frac{\alpha + \beta}{2}}{\tan \frac{\alpha - \beta}{2}}$ is equal to
 (a) 1 (b) 2 (c) 3 (d) 4
53. If $\cos 2B = \frac{\cos(A + C)}{\cos(A - C)}$, then
 (a) $\tan A, \tan B$ and $\tan C$ are in AP
 (b) $\tan A, \tan B$ and $\tan C$ are in GP
 (c) $\tan A, \tan B$ and $\tan C$ are in HP
 (d) None of the above
54. If $\tan x = \frac{b}{a}$, then the value of $a \cos 2x + b \sin 2x$ is
 (a) a (b) $a - b$
 (c) $a + b$ (d) b
55. The graph of the function $\cos x \cos(x + 2) - \cos^2(x + 1)$ is
 (a) a straight line passing through $(0, -\sin^2 \theta)$ with slope 2
 (b) a straight line passing through $(0, 0)$
 (c) a parabola with vertex $(1, -\sin^2 1)$
 (d) a straight line passing through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$ and parallel to the X -axis
56. If $\alpha \cos^2 3\theta + \beta \cos^4 \theta = 16 \cos^6 \theta + 9 \cos^2 \theta$ is an identity, then
 (a) $\alpha = 1, \beta = 18$ (b) $\alpha = 1, \beta = 24$
 (c) $\alpha = 3, \beta = 24$ (d) $\alpha = 4, \beta = 2$
57. Let n be an odd integer. If $\sin n\theta = \sum_{r=0}^n b_r \sin^r \theta$ for every value of θ , then
 (a) $b_0 = 1, b_1 = 3$ (b) $b_0 = 0, b_1 = n$
 (c) $b_0 = -1, b_1 = n$ (d) $b_0 = 0, b_1 = n^2 - n + 3$

- 58.** If $0 \leq \theta \leq \frac{\pi}{2}$ and $x = X \cos \theta + Y \sin \theta$,
 $y = X \sin \theta - Y \cos \theta$ such that
 $x^2 + 2xy + y^2 = aX^2 + bY^2$, where a and b are
 constants, then
 (a) $a = -1, b = -3$ (b) $\theta = \frac{\pi}{2}$
 (c) $a = 3, b = -1$ (d) $\theta = \frac{\pi}{3}$
- 59.** If θ_1 and θ_2 are two values lying in $[0, 2\pi]$ for which
 $\tan \theta = \lambda$, then $\tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2}$ is equal to
 (a) 0 (b) -1 (c) 2 (d) 1
- 60.** If $\alpha \in (0, \pi/2)$, then the expression
 $\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}$ is always greater than or equal to
 (a) $2 \tan \alpha$ (b) $2 \cos \alpha$
 (c) 1 (d) $\sec^2 \alpha$
- 61.** In $\triangle ABC$, $\tan \frac{A}{2}, \tan \frac{B}{2}$ and $\tan \frac{C}{2}$ are in HP, then the
 value of $\cot \frac{A}{2} \cot \frac{C}{2}$ is
 (a) 1 (b) 2
 (c) 3 (d) 4
- 62.** The expression
 $\cos^2 \phi + \cos^2 (a + \phi) - 2 \cos a \cos \phi \cos (a + \phi)$ is
 independent of
 (a) ϕ (b) a
 (c) both a and ϕ (d) None of these
- 63.** If $\cos x - \sin \alpha \cot \beta \sin x = \cos \alpha$, then $\tan \frac{x}{2}$ is equal
 to
 (a) $\cot \frac{\alpha}{2} \tan \frac{\beta}{2}$ (b) $\cot \frac{\beta}{2} \tan \frac{\alpha}{2}$
 (c) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2}$ (d) None of these
- 64.** If $A + B = \frac{\pi}{3}$ and $\cos A + \cos B = 1$, then
 (a) $\cos (A - B) = \frac{1}{3}$ (b) $|\cos A - \cos B| = \sqrt{\frac{2}{3}}$
 (c) $\cos (A - B) = \frac{2}{3}$ (d) None of these
- 65.** Let $I(n) = 2 \cos nx, \forall n \in N$, then $I(1) \cdot I(n+1) - I(n)$
 is equal to
 (a) $I(n+3)$ (b) $I(n+2)$
 (c) $I(n+1) \cdot I(2)$ (d) $I(n+2) \cdot I(2)$
- 66.** The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ in
 x has real roots. Then, the set of values of p is
 (a) $[0, 2\pi]$ (b) $[-\pi, 0]$
 (c) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) $[0, \pi]$
- 67.** If $x = \alpha, \beta$ satisfies both the equation
 $\cos^2 x + a \cos x + b = 0$ and $\sin^2 x + p \sin x + q = 0$,
 then the relation between a, b, p and q is
 (a) $1 + b + a^2 = p^2 - q$ (b) $a^2 + b^2 = p^2 + q^2$
 (c) $b + q = a^2 + p^2 - 2$ (d) None of these
- 68.** If $f(\theta) = |\sin \theta| + |\cos \theta|, \theta \in R$, then
 (a) $f(\theta) \in [0, 2]$ (b) $f(\theta) \in [0, \sqrt{2}]$
 (c) $f(\theta) \in [0, 1]$ (d) $f(\theta) \in [1, \sqrt{2}]$
- 69.** If $f(\theta) = (\sin \theta + \operatorname{cosec} \theta)^2 + (\cos \theta + \sec \theta)^2$, then
 minimum value of $f(\theta)$ is
 (a) 7 (b) 8
 (c) 9 (d) None of these
- 70.** If $f(\theta) = \sin^4 \theta + \cos^4 \theta + 1$, then range of $f(\theta)$ is
 (a) $\left[\frac{3}{2}, 2\right]$ (b) $\left[1, \frac{3}{2}\right]$
 (c) $[1, 2]$ (d) None of these
- 71.** If $f(x) = \sin^6 x + \cos^6 x + 1$, then range of $f(\theta)$ is
 (a) $\left[\frac{1}{4}, 1\right]$ (b) $\left[\frac{1}{4}, \frac{3}{4}\right]$
 (c) $\left[\frac{3}{4}, 1\right]$ (d) None of these
- 72.** The maximum value of $\sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$ in
 the interval $\left(0, \frac{\pi}{2}\right)$ is attained at
 (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
- 73.** The number of integral value of k for which the
 equation $7 \cos x + 5 \sin x = 2k + 1$ has a solution, is
 (a) 4 (b) 8 (c) 10 (d) 12
- 74.** If $f(x) = \begin{vmatrix} \sin^2 \theta & \cos^2 \theta & x \\ \cos^2 \theta & x & \sin^2 \theta \\ x & \sin^2 \theta & \cos^2 \theta \end{vmatrix}, \theta \in (0, \pi/2)$, then
 roots of $f(x) = 0$ are
 (a) $1/2, -1, 0$ (b) $1/2, -1$
 (c) $-1/2, 1, -1$ (d) $-1/2, -1, 1$
- 75.** If $f(\theta) = \frac{1}{2 - 3 \cos \theta}$, then range of $f(\theta)$ is
 (a) $\left(-\infty, \frac{1}{5}\right] \cup [1, \infty)$ (b) $\left[-1, \frac{1}{5}\right]$
 (c) $(-\infty, -1] \cup \left[\frac{1}{5}, \infty\right)$ (d) None of these
- 76.** If $f(\theta) = 5 \cos \theta + 3 \cos\left(\theta + \frac{\pi}{3}\right) + 3$, then range of
 $f(\theta)$ is
 (a) $[-5, 11]$ (b) $[-3, 9]$
 (c) $[-2, 10]$ (d) $[-4, 10]$

- 77.** If $a \sin x + b \cos(x + \theta) + b \cos(x - \theta) = d$, then the minimum value of $|\cos \theta|$ is
 (a) $\frac{1}{2|b|} \sqrt{d^2 - a^2}$ (b) $\frac{1}{2|a|} \sqrt{d^2 - a^2}$
 (c) $\frac{1}{2|d|} \sqrt{d^2 - a^2}$ (d) None of these
- 78.** The minimum value of the expression $\sin \alpha + \sin \beta + \sin \gamma$, where α, β and γ are real numbers satisfying $\alpha + \beta + \gamma = \pi$, is
 (a) positive (b) zero
 (c) negative (d) None of these
- 79.** If $A \geq 0, B \geq 0, A + B = \frac{\pi}{3}$ and $y = \tan A \tan B$, then
 (a) $y \in \left(-\infty, \frac{1}{3}\right] \cup [3, \infty)$ (b) $0 \leq y \leq \frac{1}{3}$
 (c) $y > \frac{1}{3}$ (d) None of these
- 80.** If $(a \sin^4 \theta + b)(a + b) = ab \cos^4 \theta$, then
 (a) $|a| \geq |b|$
 (b) $|b| = |a|$
 (c) $|a| \leq |b|$
 (d) None of the above
- 81.** If $\sin \theta = x + \frac{a}{x}, \forall x \in R \sim \{0\}$, then
 (a) $a \geq \frac{1}{4}$ (b) $a \geq \frac{1}{2}$
 (c) $a \leq \frac{1}{4}$ (d) $a \leq \frac{1}{2}$
- 82.** If $\sin^2 x + a \sin x + 1 = 0$ has no real number solution, then
 (a) $|a| \geq 2$ (b) $|a| \geq 1$
 (c) $|a| < 2$ (d) None of these
- 83.** If $\sin \theta_1 - \sin \theta_2 = a$ and $\cos \theta_1 + \cos \theta_2 = b$, then
 (a) $a^2 + b^2 \geq 4$ (b) $a^2 + b^2 \leq 4$
 (c) $a^2 + b^2 \geq 3$ (d) $a^2 + b^2 \leq 2$
- 84.** If $0 \leq a \leq 3, 0 \leq b \leq 3$ and the equation $x^2 + 4 + 3 \cos(ax + b) = 2x$ has atleast one solution, then the value of $a + b$ is
 (a) 0 (b) $\frac{\pi}{2}$
 (c) π (d) None of these
- 85.** If $\tan^2 x + \sec x - a = 0$ has atleast one solution, then the complete set of values of a is
 (a) $(-\infty, 1]$ (b) $[1, 1]$
 (c) $[-1, 1]$ (d) $[-1, \infty)$
- 86.** $\sec^2 \theta = \frac{4xy}{(x+y)^2}$ is true if and only if
 (a) $x + y \neq 0$
 (b) $x = y, x \neq 0$
 (c) $x = y$
 (d) $x \neq 0, y \neq 0$
- 87.** If $\sin \theta = \frac{x^2 + y^2}{x^2 - y^2}$, then
 (a) $x \neq y$ (b) $x \neq y$ and $x, y \in R$
 (c) $x = 0$ or $y = 0$ (d) $x \neq y$ and $x \neq 0, y \neq 0$
- 88.** The most general values of θ satisfying the equation $(1 + 2 \sin \theta)^2 + (\sqrt{3} \tan \theta - 1)^2 = 0$ is/are given by
 (a) $n\pi \pm \frac{\pi}{6}$ (b) $n\pi + (-1)^n \frac{7\pi}{6}$
 (c) $2n\pi + \frac{7\pi}{6}$ (d) $2n\pi + \frac{11\pi}{6}$
- 89.** The number of solutions of the equation $\tan^2 x - \sec^{10} x + 1 = 0$ in $(0, 10)$, is
 (a) 3 (b) 8
 (c) 10 (d) None of these
- 90.** Total number of solutions of $\tan x + \cot x = 2 \operatorname{cosec} x$ in $[-2\pi, 2\pi]$ is
 (a) 2 (b) 4 (c) 6 (d) 8
- 91.** Total number of solutions of the equation $3x + 2 \tan x = \frac{5\pi}{2}$ in $x \in [0, 2\pi]$, is
 (a) 1 (b) 2 (c) 3 (d) 4
- 92.** If $a, b \in [0, \pi]$ and the equation $x^2 + 4 + 3 \sin(ax + b) - 2x = 0$ has atleast one solution, then the value of $(a + b)$ can be
 (a) $\frac{7\pi}{2}$ (b) $\frac{3\pi}{2}$
 (c) $\frac{9\pi}{2}$ (d) None of these
- 93.** If $\frac{1}{6} \sin \theta, \cos \theta$ and $\tan \theta$ are in GP, then the general value(s) of θ is/are
 (a) $2n\pi \pm \frac{\pi}{3}, n \in I$ (b) $2n\pi \pm \frac{\pi}{6}, n \in I$
 (c) $2n\pi + (-1)^n \frac{\pi}{3}, n \in I$ (d) $n\pi + \frac{\pi}{3}, n \in I$
- 94.** The general values of x for which $\cos 2x, \frac{1}{2}$ and $\sin 2x$ are in AP, are given by
 (a) $n\pi, n\pi + \frac{\pi}{2}$ (b) $n\pi, n\pi + \frac{\pi}{4}$
 (c) $n\pi + \frac{\pi}{4}$ (d) $n\pi$
- 95.** The most general values of θ satisfying $\tan \theta + \tan\left(\frac{3\pi}{4} + \theta\right) = 2$ are
 (a) $n\pi \pm \frac{\pi}{3}, n \in I$
 (b) $2n\pi \pm \frac{\pi}{6}, n \in I$
 (c) $2n\pi \pm \frac{\pi}{3}, n \in I$
 (d) $2n\pi + (-1)^n \frac{\pi}{3}, n \in I$

- 96.** The most general solutions of the equation $\sec x - 1 = (\sqrt{2} - 1) \tan x$ are given by
 (a) $n\pi + \frac{\pi}{8}, n\pi$ (b) $2n\pi, 2n\pi + \frac{\pi}{4}$
 (c) $2n\pi, n\pi$ (d) None of these
- 97.** If $n \tan y = \tan x, n \in R^+$, then the maximum value of $\sec^2(x - y)$ is equal to
 (a) $\frac{(n+1)^2}{2n}$ (b) $\frac{(n+1)^2}{n}$
 (c) $\frac{(n+1)^2}{2}$ (d) $\frac{(n+1)^2}{4n}$
- 98.** Let α, β be any two positive values of x for which $2 \cos x, |\cos x|$ and $1 - 3\cos^2 x$ are in GP. The minimum value of $|\alpha - \beta|$ is
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$
 (c) $\frac{\pi}{2}$ (d) None of these
- 99.** The value(s) of $x \in [-2\pi, 2\pi]$ such that $\frac{\sin x + i \cos x}{1+i}, i = \sqrt{-1}$ is purely imaginary, is/are given by
 (a) $n\pi - \frac{\pi}{4}$ (b) $n\pi + \frac{\pi}{4}$
 (c) $n\pi$ (d) None of these
- 100.** The number of solutions of the equation $1 + \sin x \sin^2 \frac{x}{2} = 0$ in $[-\pi, \pi]$ is
 (a) 0 (b) 1
 (c) 2 (d) 3
- 101.** The number of solutions of $\cos x = |1 + \sin x|$, $0 \leq x \leq 3\pi$ is
 (a) 3 (b) 2
 (c) 4 (d) None of these
- 102.** The most general solutions of $2^1 + |\cos x| + \cos^2 x + |\cos x|^3 + \dots = 4$ are given by
 (a) $n\pi \pm \frac{\pi}{3}, n \in I$ (b) $2n\pi \pm \frac{\pi}{3}, n \in I$
 (c) $2n\pi \pm \frac{2\pi}{3}, n \in I$ (d) None of these
- 103.** If $x \neq \frac{n\pi}{2}$ and $(\cos x)^{\sin^2 x - 3 \sin x + 2} = 1$, then all solutions of x is/are given by
 (a) $2n\pi + \frac{\pi}{2}$ (b) $(2n+1)\pi - \frac{\pi}{2}$
 (c) $2n\pi + (-1)^n \frac{\pi}{2}$ (d) None of these
- 104.** If $0 \leq x \leq 3\pi, 0 \leq y \leq 3\pi$ and $\cos x \sin y = 1$, then the possible number of values of the ordered pair (x, y) is
 (a) 6 (b) 12
 (c) 8 (d) 15
- 105.** If $x, y \in [0, 2\pi]$, then total number of ordered pairs (x, y) satisfying $\sin x \cos y = 1$ is
 (a) 1 (b) 3 (c) 5 (d) 7
- 106.** The number of values of x for which $\sin 2x + \cos 4x = 2$ is
 (a) 0 (b) 1 (c) 2 (d) infinite
- 107.** If $|\tan x| \leq 1$ and $x \in [-\pi, \pi]$, then the solution set for x is
 (a) $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
 (b) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
 (c) $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$
 (d) None of the above
- 108.** If $4\sin^2 x - 8\sin x + 3 \leq 0, 0 \leq x \leq 2\pi$, then the solution set for x is
 (a) $\left[0, \frac{\pi}{6}\right]$ (b) $\left[0, \frac{5\pi}{6}\right]$ (c) $\left[\frac{5\pi}{6}, 2\pi\right]$ (d) $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$
- 109.** The set of values of x for which $\sin x \cos^3 x > \cos x \sin^3 x, 0 \leq x \leq 2\pi$, is
 (a) $(0, \pi)$ (b) $\left(0, \frac{\pi}{4}\right)$
 (c) $\left(\frac{\pi}{4}, \pi\right)$ (d) None of these
- 110.** If x_1 and x_2 are two positive values of x for which $2\cos x, |\cos x|$ and $3\sin^2 x - 2$ are in GP. The minimum value of $|x_1 - x_2|$ is
 (a) $\frac{4\pi}{3}$ (b) $\frac{\pi}{3}$
 (c) $2\cos^{-1}\left(\frac{2}{3}\right)$ (d) $\cos^{-1}\left(\frac{2}{3}\right)$
- 111.** If $3\sin x + 4\cos ax = 7$ has atleast one solution, then a has to be necessarily a/an
 (a) odd integer
 (b) even integer
 (c) rational number
 (d) irrational number
- 112.** If the equation $a_1 + a_2 \cos 2x + a_3 \sin^2 x = 1$ is satisfied by every real value of x , then the number of possible values of the triplet (a_1, a_2, a_3) is
 (a) 0 (b) 1 (c) 3 (d) infinite
- 113.** If $\sin^2 \theta - 2\sin \theta - 1 = 0$ is to be satisfied for exactly 4 distinct values of $\theta \in [0, n\pi], n \in N$, then the least value of n is
 (a) 2 (b) 6 (c) 4 (d) 15
- 114.** If $2\tan^2 x - 5\sec x$ is equal to 1 for exactly 7 distinct values of $x \in \left[0, \frac{n\pi}{2}\right], n \in N$, then the greatest value of n is
 (a) 6 (b) 12 (c) 13 (d) 15

- 115.** If $\sin \alpha$, 1 and $\cos 2\alpha$ are in GP, then α is equal to
 (a) $n\pi + (-1)^n \frac{\pi}{2}, n \in I$ (b) $n\pi + (-1)^{n-1} \frac{\pi}{2}, n \in I$
 (c) $2n\pi, n \in I$ (d) None of these
- 116.** If $\max_{\theta \in R} \{5\sin \theta + 3\sin (\theta - \alpha)\} = 7$, then set of possible values of α is
 (a) $\left\{x \mid x = 2n\pi \pm \frac{\pi}{3}; n \in I\right\}$
 (b) $\left\{x \mid x = 2n\pi \pm \frac{2\pi}{3}; n \in I\right\}$
 (c) $\left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$
 (d) None of the above
- 117.** The number of solutions of $|\cos x| = \sin x$, $0 \leq x \leq 4\pi$ is
 (a) 8 (b) 4
 (c) 2 (d) None of these
- 118.** The values of a for which the equation $\cos^4 x - (a+2)\cos^2 x - (a+3) = 0$ possesses a solution, lies in
 (a) $[-3, -2]$ (b) $(-3, -2)$ (c) $(0, 2)$ (d) $(0, 3)$
- 119.** The least positive non-integral solution of $\sin \pi(x^2 + x) - \sin \pi x^2 = 0$ is
 (a) rational
 (b) irrational of the form $\sqrt{p}$
 (c) irrational of the form $\frac{\sqrt{p}-1}{4}$, where p is an odd integer
 (d) irrational of the form $\frac{\sqrt{p}+1}{4}$, where p is an even integer
- 120.** If $\theta \in [0, 5\pi]$ and $r \in R$ such that $2\sin \theta = r^4 - 2r^2 + 3$, then the maximum number of values of the pair (r, θ) is
 (a) 8
 (b) 10
 (c) 6
 (d) None of the above
- 121.** Total number of solutions of $[\sin x] = \cos x$, where $[]$ denotes the greatest integer function in $[0, 3\pi]$, is
 (a) 2
 (b) 6
 (c) 5
 (d) None of the above
- 122.** If $1 - \sin x = \frac{\sqrt{3}}{2} \left|x - \frac{\pi}{2}\right| + a$ has no solution, where $a \in R^+$, then
 (a) $a \in R^+$
 (b) $a > \frac{3}{2} - \frac{\pi}{\sqrt{3}}$
 (c) $a \in \left(0, \frac{3}{2} + \frac{\pi}{\sqrt{3}}\right)$
 (d) None of the above
- 123.** If $\sin^2 (\theta - \alpha) \cos \alpha = \cos^2 (\theta - \alpha) \sin \alpha = m \sin \alpha \cos \alpha$, then
 (a) $|m| \leq \frac{1}{\sqrt{2}}$
 (b) $|m| \geq \frac{1}{\sqrt{2}}$
 (c) $|m| \geq 1$
 (d) $|m| \leq 1$
- 124.** If $\cos x - \frac{\cot \beta \sin x}{2} = \frac{\sqrt{3}}{2}$, then the value of $\tan \frac{x}{2}$ is
 (a) $\tan \frac{\beta}{2} \tan 15^\circ$
 (b) $\tan \frac{\beta}{2}$
 (c) $\tan 15^\circ$
 (d) None of the above
- 125.** If the expression $n \sin^2 \theta + 2n \cos (\theta + \alpha) \sin \alpha \sin \theta + \cos 2(\alpha + \theta)$ is independent of θ , then the value of n is
 (a) 1
 (b) 2
 (c) 3
 (d) 4
- 126.** For any real value of $\theta \neq \pi$, the value of the expression $y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta}$ is
 (a) $1 \leq y \leq 2$
 (b) $y < 0$ and $y > 2$
 (c) $-1 \leq y \leq 1$
 (d) $y \geq 1$
- 127.** Solution of equation $[\sin x] = [1 + \sin x] + [1 - \cos x]$, $0 \leq x \leq 2\pi$ is
 (a) $x = \frac{3\pi}{2}$
 (b) $x = \frac{3\pi}{4}$
 (c) no real solution
 (d) None of the above
- 128.** The general solution of the equation $\sin^{50} x - \cos^{50} x = 1$ is
 (a) $2n\pi + \frac{\pi}{2}$
 (b) $2n\pi + \frac{\pi}{3}$
 (c) $n\pi + \frac{\pi}{2}$
 (d) $n\pi + \frac{\pi}{3}$
- 129.** Number of ordered pairs (a, x) satisfying the equation $\sec^2 (a+2)x + a^2 - 1 = 0$, $-\pi < x < \pi$ is
 (a) 2
 (b) 1
 (c) 3
 (d) infinite

Type 2. More than One Correct Option

- 130.** If $f_n(\theta) = \tan \frac{\theta}{2} (1 + \sec \theta) (1 + \sec 2\theta) (1 + \sec 4\theta) \dots (1 + \sec 2^n \theta)$, then
- (a) $f_2\left(\frac{\pi}{16}\right) = 1$ (b) $f_3\left(\frac{\pi}{32}\right) = 1$
 (c) $f_4\left(\frac{\pi}{64}\right) = 1$ (d) $f_5\left(\frac{\pi}{128}\right) = 1$
- 131.** Let $0 \leq \theta \leq \frac{\pi}{2}$ and $x = X \cos \theta + Y \sin \theta$,
 $y = X \sin \theta - Y \cos \theta$ such that
 $x^2 + 4xy + y^2 = aX^2 + bY^2$, where a and b are constants. Then,
- (a) $a = -1, b = 3$ (b) $\theta = \frac{\pi}{4}$
 (c) $a = 3, b = -1$ (d) $\theta = \frac{\pi}{3}$
- 132.** $\frac{\sin^2 x + \alpha^2 - 2}{\cos 2x} = \frac{\alpha^2}{1 - \tan^2 x}$ has a solution, if
- (a) $\alpha \leq -1$ (b) $\alpha \geq 1$
 (c) $\alpha = 1/2$ (d) α is any real number
- 133.** If $\cos^3 x \sin 2x = \sum_{m=1}^n a_m \sin mx$ is identity in x . Then,
- (a) $a_3 = \frac{3}{8}, a_2 = 0$ (b) $n = 6, a_1 = \frac{1}{2}$
 (c) $n = 5, a_1 = \frac{1}{4}$ (d) $\sum a_n = \frac{3}{4}$
- 134.** If $0 \leq x, y \leq 180^\circ$ and $\sin(x - y) = \cos(x + y) = 1/2$, then x and y are given by
- (a) $x = 45^\circ, y = 15^\circ$ (b) $x = 45^\circ, y = 135^\circ$
 (c) $x = 165^\circ, y = 15^\circ$ (d) $x = 165^\circ, y = 135^\circ$
- 135.** If $\tan \frac{x}{2} = \operatorname{cosec} x - \sin x$, then $\tan^2 \frac{x}{2}$ is equal to
- (a) $2 - \sqrt{5}$ (b) $\sqrt{5} - 2$
 (c) $(9 - 4\sqrt{5})(2 + \sqrt{5})$ (d) $(9 + 4\sqrt{5})(2 - \sqrt{5})$
- 136.** If $y = \frac{\sqrt{1 - \sin 4A} + 1}{\sqrt{1 + \sin 4A} - 1}$, then one of the values of y is
- (a) $-\tan A$ (b) $\cot A$
 (c) $\tan\left(\frac{\pi}{4} + A\right)$ (d) $-\cot\left(\frac{\pi}{4} + A\right)$
- 137.** If $x = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} + \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$, then $x^2 = a^2 + b^2 + 2\sqrt{p(a^2 + b^2) - p^2}$, where p is equal to
- (a) $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$
 (b) $a^2 \sin^2 \alpha + b^2 \cos^2 \alpha$
 (c) $\frac{1}{2}[a^2 + b^2 + (a^2 - b^2) \cos 2\alpha]$
 (d) $\frac{1}{2}[a^2 + b^2 - (a^2 - b^2) \cos 2\alpha]$
- 138.** If $\tan \alpha$ and $\tan \beta$ are the roots of the equation $x^2 + px + q = 0$ with $p \neq 0$, then
- (a) $\sin^2(\alpha + \beta) + p \sin(\alpha + \beta) \cos(\alpha + \beta) + q \cos^2(\alpha + \beta) = q$
 (b) $\tan(\alpha + \beta) = \frac{p}{q - 1}$
 (c) $\cos(\alpha + \beta) = 1 - q$
 (d) $\sin(\alpha + \beta) = -p$
- 139.** If $y = \sin^2 x + \cos^4 x$, then for all real x
- (a) the maximum value of y is $\frac{2}{3}$
 (b) the minimum value of y is $\frac{3}{4}$
 (c) $y \leq 1$
 (d) $y \geq \frac{1}{4}$
- 140.** If $k_1 = \sin x \cos^3 x$ and $k_2 = \cos x \sin^3 x$, then
- (a) $k_1 - k_2 > 0, x \in \left(0, \frac{\pi}{4}\right)$
 (b) $k_1 - k_2 < 0, x \in \left(0, \frac{\pi}{4}\right)$
 (c) $k_1 + k_2 > 0, x \in \left(0, \frac{\pi}{2}\right)$
 (d) $k_1 + k_2 < 0, x \in \left(0, \frac{\pi}{4}\right)$
- 141.** If $\alpha \in [-2\pi, 2\pi]$ and $\cos \frac{\alpha}{2} + \sin \frac{\alpha}{2} = \sqrt{2} (\cos 36^\circ - \sin 18^\circ)$, then a value of α is
- (a) $\frac{7\pi}{6}$ (b) $\frac{\pi}{6}$
 (c) $\frac{5\pi}{6}$ (d) $-\frac{\pi}{6}$
- 142.** If $\cos x = \sqrt{1 - \sin 2x}$, where $0 \leq x \leq \pi$, then a value of x is
- (a) π
 (b) 0
 (c) $\tan^{-1} 2$
 (d) None of the above
- 143.** If $\sin \theta + \sqrt{3} \cos \theta = 6x - x^2 - 11$, $0 \leq \theta \leq 4\pi$, $x \in R$, then holds for
- (a) no value of x and θ
 (b) one value of x and two values of θ
 (c) two values of x and two values of θ
 (d) two pairs of values of (x, θ)
- 144.** If $0 \leq x \leq 2\pi$ and $|\cos x| \leq \sin x$, then
- (a) the set of values of x is $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$
 (b) the number of solutions that are integral multiples of $\frac{\pi}{4}$ is three
 (c) the sum of the largest and the smallest solution is $\frac{3\pi}{4}$
 (d) $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4}\right]$

Type 3. Assertion and Reason

Directions (Q. Nos. 145-151) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

145. Statement I If A, B and C are the angles of a triangle such that $\angle A$ is obtuse, then $\tan B \tan C > 1$.

Statement II In any triangle, $\tan A = \frac{\tan B + \tan C}{\tan B \tan C - 1}$

146. Statement I If $2\sin \frac{\theta}{2} = \sqrt{1 + \sin \theta} + \sqrt{1 - \sin \theta}$, then $\frac{\theta}{2}$ lies between $2n\pi + \frac{\pi}{4}$ and $2n\pi + \frac{3\pi}{4}$.

Statement II If $\frac{\theta}{2}$ runs from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$, then $\sin \frac{\theta}{2} > 0$.

147. Statement I The numbers $\sin 18^\circ$ and $-\sin 54^\circ$ are solutions of such quadratic equations with integer coefficients.

Statement II If $x = 18^\circ$, then $5x = 90^\circ$ and if $y = -54^\circ$, then $5y = -270^\circ$.

148. Statement I $\cos 36^\circ > \tan 36^\circ$

Statement II $\cos 36^\circ > \sin 36^\circ$

149. Statement I The minimum value of the expression $\sin \alpha + \sin \beta + \sin \gamma$, where α, β and γ are real numbers such that $\alpha + \beta + \gamma = \pi$, is negative.

Statement II α, β and γ are angles of a triangle.

150. Statement I The equation $\sin x + \operatorname{cosec} x = 2$ possesses unique solution, if $x \in \left[0, \frac{\pi}{2}\right] \cup \left[2\pi, \frac{5\pi}{2}\right]$.

Statement II $x + \frac{1}{x} \geq 2, x > 0$ equality holds, only when $x = 1$.

151. Statement I If $(\sin x)^2 + (\tan x)^2 = 0$, then $x = n\pi$,

Statement II If $a^2 + b^2 = 0$, then $a, b = 0$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 152-154) For each positive real number k , let C_k denotes the circle with centre at origin and radius k units. On a circle C_k , particle a moves k units in the counter-clockwise direction. After completing its motion on C_k , the particle moves onto the circle $C_{k+\gamma}$ in some well defined manner, where $\gamma > 0$. The motion of the particle continues in this manner.

152. Let $k \in I^+$ and $\gamma = 1$, particle moves in the radial direction from circle C_k to C_{k+1} . If particle starts from the point $P(-1, 0)$, then

- (a) it will cross the X -axis again at $(3, 0)$
(b) it will cross the X -axis again at $(4, 0)$
(c) it will cross the positive Y -axis at $(0, 4)$
(d) it will cross the positive Y -axis at $(0, 6)$

153. Let $k \in I^+$ and $\gamma \in R$, particle moves tangentially from the circle C_k to $C_{k+\gamma}$, such that length of tangent is equal to k unit itself. If particle starts from the point $Q(1, 0)$, then

- (a) the particle will cross positive X -axis again for $2\sqrt{2} < x < 4$
(b) the particle will cross positive X -axis again at $x = 2\sqrt{2}$
(c) the particle will cross X -axis again at $x = 4$
(d) the particle will cross X -axis again at $x = 3$

154. Let the particle starts from the point $A(2, 0)$ and moves $\frac{\pi}{2}$ units on circle C_2 in the counter-clockwise

direction, then moves on circle C_3 along the tangential path. Let this straight line (tangential path traced by particle) intersect the circle C_3 at points A and B and tangents drawn at A and B intersect at

- (a) $(\sqrt{2}, \sqrt{2})$ (b) $(9, 9)$
(c) $\left(\frac{9}{2\sqrt{2}}, \frac{9}{2\sqrt{2}}\right)$ (d) $(9\sqrt{2}, 9\sqrt{2})$

Passage II (Q. Nos. 155-157) Let $f(\theta, \alpha) = 2 \sin^2 \theta + 4 \cos(\theta + \alpha) \sin \theta \sin \alpha + \cos 2(\theta + \alpha)$

General solution of $\cos x = 0$ is $x = (2n+1)\frac{\pi}{2}, n \in I$

General solution of $\sin x = 0$ is $x = n\pi, n \in I$.

155. The value of $f\left(\frac{\pi}{3}, \frac{\pi}{4}\right)$ is

- (a) 0 (b) 1 (c) 2 (d) 3

156. The general solution of $f\left(\frac{\pi}{3}, \alpha\right) = 0$ is

- (a) $\alpha = (2n+1)\frac{\pi}{2}, n \in I$ (b) $\alpha = (2n+1)\frac{\pi}{4}, n \in I$
(c) $\alpha = (2n+1)\pi, n \in I$ (d) $\alpha = (2n+1)\frac{\pi}{3}, n \in I$

157. If $f\left(\theta, \frac{\pi}{6}\right) = \frac{1}{2}$, then

- (a) $\theta = n\pi : n \in I$
 (b) $\theta = \frac{n\pi}{2} : n \in I$
 (c) $\theta = 2n\pi : n \in I$
 (d) θ is any real

Passage III (Q. Nos. 158-160) If A, B and C are angles of a triangle such that $\tan A + \tan B + \tan C = k$, then find the interval in which k should lie.

158. There exists exactly one isosceles $\triangle ABC$, if

- (a) $k < 0$ or $k = 3\sqrt{3}$
 (b) $k > 0$
 (c) $-1 < k < 0$
 (d) $0 < k < 3$

159. There exist exactly two isosceles $\triangle ABC$, if

- (a) $k < 3$
 (b) $k > 3\sqrt{3}$
 (c) $k < 3\sqrt{3}$
 (d) $0 < k < 3\sqrt{3}$

160. There exist three or more non-similar isosceles triangles for

- (a) no value of k
 (b) $0 < k < 3\sqrt{3}$
 (c) $k > \sqrt{3}$
 (d) $k > 0$

Passage IV (Q. Nos. 161-163) Use boundedness for solving trigonometric equations. Consider an equation

$$\sin x + \sin y = 2 \quad \dots(i)$$

We know that, $\sin x \leq 1, \sin y \leq 1$ and $\sin x + \sin y \leq 2, \forall x$ and y .

Therefore, $\sin x + \sin y = 2$ if and only if $\sin x = 1$ and $\sin y = 1 \Leftrightarrow x = 2m\pi + \frac{\pi}{2}$ and $y = 2n\pi + \frac{\pi}{2}$

which is the required solution of equation. To solve the Eq. (i), we have used the boundedness of $\sin x$ rather than using conventional methods of solving equations.

In general, we employ one or more of the following extreme value conditions

- (i) $-1 \leq \sin x \leq 1 \Rightarrow |\sin x| \leq 1$ and $\sin^2 x \leq 1$
 (ii) $-1 \leq \cos x \leq 1 \Rightarrow |\cos x| \leq 1$ and $\cos^2 x \leq 1$
 (iii) $-\sqrt{a^2 + b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2}$
 $\Rightarrow |a \sin x + b \cos x| \leq \sqrt{a^2 + b^2}$

161. The minimum value of $27^{\cos 2x} \cdot 81^{\sin 2x}$ is

- (a) 1 (b) $\frac{1}{9}$ (c) $\frac{1}{81}$ (d) $\frac{1}{243}$

162. Number of roots of the equation $\cos^7 x + \sin^4 x = 1$ in the interval $[0, 2\pi]$ is

- (a) 0 (b) 1 (c) 2 (d) 4

163. The smallest positive number p for which the equation $\cos(p \sin x) = \sin(p \cos x)$ has a solution in $[a, 2\pi]$, is

- (a) $\frac{\pi}{4}$
 (b) $\frac{\pi}{3}$
 (c) $\frac{\pi}{4\sqrt{2}}$
 (d) $\frac{\pi}{2\sqrt{2}}$

Type 5. Match the Columns

164. Match the items of Column I with that of Column II.

Column I	Column II
A. The number of pairs (x, y) satisfying the equation $\sin x + \sin y = \sin(x + y)$ $ x + y = 1$, is	p. 3
B. The number of values of x for which $f(x)$ is valid, where $f(x) = \sqrt{\sec^{-1}\left(\frac{1- x }{2}\right)}$, is	q. 8
C. If $x, y \in [0, 2\pi]$, then total number of ordered pairs (x, y) satisfying $\sin x \cos y = 1$, is	r. ∞
D. $f(x) = \sin x - \cos x - kx + b$ decreases for all values of real values of x when $4\sqrt{2}k$ is always greater than	s. 6

165. Match the items of Column I with that of Column II.

Column I	Column II
A. If in a $\triangle ABC$, $\sin^2 A + \sin^2 B = \sin(A + B)$, then the triangle must be	p. right angled
B. If in a $\triangle ABC$, $\frac{bc}{2\cos A} = b^2 + c^2 - 2bc \cos A$, then the triangle must be	q. equilateral
C. If in a $\triangle ABC$, $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \sqrt{3}$, then the triangle must be	r. isosceles
D. If in a triangle, the sides and the altitudes are in AP, then the triangle must be	s. obtuse angles

166. Match the items of Column I with that of Column II.

Column I	Column II
A. The number of solutions of $\frac{x}{2} + \frac{\sin x}{\cos x} = \frac{\pi}{4}$ in $[-\pi, \pi]$, is	p. 1
B. The number of solutions of equation $\sin^{-1}(x^2 - 1) + \cos^{-1}(2x^2 - 5) = \frac{\pi}{2}$ is	q. 0
C. The number of solutions of $x^4 - 2x^2 \sin^2 \frac{\pi}{2} x + 1 = 0$ is	r. 3
D. The number of solutions of $x^2 + 2x + 2 \sec^2 \pi x + \tan^2 \pi x = 0$ is	s. 2

167. Match the items of Column I with that of Column II.

Column I	Column II
A. $\sin x \cos^3 x > \cos x$ $\sin^3 x, 0 \leq x \leq 2\pi$, is	p. $\left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$
B. $4 \sin^2 x - 8 \sin x + 3 \leq 0$, $0 \leq x \leq 2\pi$, is	q. $\left[\frac{3\pi}{2}, 2\pi\right] \cup \{0\}$
C. $ \tan x \leq 1$ and $x \in [-\pi, \pi]$, is	r. $\left(0, \frac{\pi}{4}\right)$
D. $\cos x - \sin x \geq 1$ $0 \leq x \leq 2\pi$, is	s. $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$

170. Match the items of Column I with that of Column II.

Column I	Column II
A. The number of solutions of the equation $ \tan 2x = \sin x, x \in [0, \pi]$ is	p. 1
B. The value of $4 \tan \frac{\pi}{16} - 4 \tan^3 \frac{\pi}{16} + 6 \tan^2 \frac{\pi}{16} - \tan^4 \frac{\pi}{16} - 1$ is	q. 4
C. If the equation $\tan(p \cot x) = \cot(p \tan x)$ has a solution in $(0, \pi) - \left\{\frac{\pi}{2}\right\}$, then $\frac{4}{\pi} \rho_{\max}$ is equal to	r. 3
D. The value of $\frac{2x}{\pi}$ in $[0, 2\pi]$, if $5^{\cos^2 2x} + 2 \sin^2 x + 5^{2 \cos^2 x + \sin^2 2x} = 126$ has a solution, is	s. 2

168. Match the items of Column I with that of Column II.

Column I	Column II
A. The ratio of greatest value of $2 - \cos x + \sin^2 x$ to its least value is $\frac{k}{4}$, then k equals	p. 0
B. The equation $\cos 2x + p \sin x = 2p - 7$ possesses a solution, then number of integral values of p satisfying equation is	q. 12
C. If $\cos \alpha = \frac{3}{4}$, then value of $16 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2}$ is	r. 13
D. If a is a real constant and A, B, C are variable angles and $\sqrt{a^2 - 4 \tan A} + a \tan B + \sqrt{a^2 - 4 \tan C} = 6a$, then least value of $\tan^2 A + \tan^2 B + \tan^2 C$ is	s. 5

169. Match the items of Column I with that of Column II.

Column I	Column II
A. If $\cos A = \frac{3}{4}$, then the value of $\sin \frac{A}{2} \sin \frac{5A}{2}$ is	p. 0
B. The value of $\sin \frac{\pi}{7} + \sin \frac{2\pi}{7} + \sin \frac{3\pi}{7}$ is	q. $\frac{11}{32}$
C. If $\tan^2 \theta = 2 \tan^2 \phi + 1$, then $\cos 2\theta + \sin^2 \phi$ equals	r. $\frac{1}{2} \cot \frac{\pi}{14}$
D. If $\sin B = \frac{1}{5} \sin(2A + B)$, then $\frac{\tan(A + B)}{\tan A}$ is equal to	s. $\frac{3}{2}$

Type 6. Single Integer Answer Type Questions

171. The number of real solutions of

$[x]^2 - 5[x] + 6 - \sin x = 0$, where $[]$ denotes the greatest integer function, is _____.

172. If $f(n\theta) = \frac{2 \sin 2\theta}{\cos 2\theta - \cos 4n\theta}$ and

$$f(\theta) + f(2\theta) + f(3\theta) + \dots + f(n\theta) = \frac{\sin \lambda \theta}{\sin \theta \sin \mu \theta},$$

then the value of $\mu - \lambda$ is _____.

173. If $\frac{3 - \tan^2 \frac{\pi}{7}}{1 - \tan^2 \frac{\pi}{7}} = \lambda \cos \frac{\pi}{7}$, then λ is _____.

174. The number of integral values of α such that $\sin x \cos 3x - \alpha \cos x \sin 3x = 0$ does not have any real root other than $(2n+1)\frac{\pi}{2}, n \in I$ for any real value of x , is _____.

- 175.** The expression $n \sin^2 \theta + 2n \cos(\theta + \alpha) \sin \alpha \sin \theta + \cos 2(\alpha + \theta)$ is independent of θ , the value of n is _____.
- 176.** For any real value of $\theta \neq \pi$, the value of the expression $y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta}$, $y \in R - [a, b]$, then the value of $(b - a)$ is _____.
- 177.** The number of solutions of equation $[\sin x] = [1 + \sin x] + [1 - \cos x]$, $0 \leq x \leq 2\pi$, is _____.
- 178.** In a $\triangle ABC$, $\angle B = \frac{\pi}{3}$ and $\sin A \sin C = \lambda$, then the set of all possible values of $\lambda \in [a, b]$, then $(b - a)$ is _____.
- 179.** In a $\triangle ABC$, $\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \geq \lambda \sqrt{3}$, then the value of λ is _____.
- 180.** If in a $\triangle ABC$, $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$ are in HP and $\cot \frac{B}{2} \geq \sqrt{\lambda}$, then the value of λ is _____.
- 181.** If α and β satisfy the equation $12 \sin \alpha + 5 \cos \alpha = 2\beta^2 - 8\beta + 21$, then the value of $\beta^2 - 2\beta$ is _____.
- 182.** If the values of x and y satisfy the equation $\tan^2(x + y) + \cot^2(x + y) = 1 - 2x - x^2$, then the value of $x^2 - 3x + 2$ is _____.
- 183.** If the smallest positive values of x and y satisfying $x - y = \frac{\pi}{4}$ and $\cot x + \cot y = 2$ are $x = \frac{\lambda\pi}{12}$ and $y = \frac{\mu\pi}{6}$, then $\lambda + \mu$ is equal to _____.
- 184.** The system of simultaneous equations for x and y , where $x, y \in \left(0, \frac{\pi}{2}\right)$, are $4^{\sin x} + 3^{\frac{1}{\cos y}} = 11$ and $5 \cdot 16^{\sin x} - 2 \cdot 3^{\frac{1}{\cos y}} = 2$. If $x + y = \frac{\pi}{\lambda}$, then λ is equal to _____.
- 185.** The number of solutions for $\left. \begin{aligned} \sin\left(x - \frac{\pi}{4}\right) - \cos\left(x + \frac{3\pi}{4}\right) &= 1 \\ \frac{2 \cos 7x}{\cos 3 + \sin 3} &> 2^{\cos 2x} \end{aligned} \right\}$ in $(0, 2\pi)$ is _____.
- 186.** The integral value of p for which $\sqrt{p} \cos x - 2 \sin x = \sqrt{2} + \sqrt{2 - p}$ has a solution, is _____.

Entrances Gallery

JEE Advanced/IIT JEE

- 1.** For $x \in (0, \pi)$, the equation $\sin x + 2 \sin 2x - \sin 3x = 3$ has _____ [2014]
(a) infinitely many solutions
(b) three solutions
(c) one solution
(d) no solution
- 2.** The number of points in $(-\infty, \infty)$, for which $x^2 - x \sin x - \cos x = 0$, is _____ [2013]
(a) 6 (b) 4
(c) 2 (d) 0
- 3.** Let $\theta, \phi \in [0, 2\pi]$ be such that $2 \cos \theta (1 - \sin \phi) = \sin^2 \theta \left(\tan \frac{\theta}{2} + \cot \frac{\theta}{2} \right) \cos \phi - 1$, $\tan(2\pi - \theta) > 0$ and $-1 < \sin \theta < -\frac{\sqrt{3}}{2}$. Then, ϕ cannot satisfy _____ [2012]
(a) $0 < \phi < \frac{\pi}{2}$ (b) $\frac{\pi}{2} < \phi < \frac{4\pi}{3}$
(c) $\frac{4\pi}{3} < \phi < \frac{3\pi}{2}$ (d) $\frac{3\pi}{2} < \phi < 2\pi$
- 4.** Let $f: (-1, 1) \rightarrow \mathbb{R}$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2 \theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then, the value(s) of $f\left(\frac{1}{3}\right)$ is/are _____ [2012]
(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$ (c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$
- 5.** The positive integer value of $n > 3$ satisfying the equation $\frac{1}{\sin\left(\frac{\pi}{n}\right)} = \frac{1}{\sin\left(\frac{2\pi}{n}\right)} + \frac{1}{\sin\left(\frac{3\pi}{n}\right)}$ is _____ [2011]
- 6.** Let $f(x) = x^2$ and $g(x) = \sin x$, for all $x \in \mathbb{R}$. Then, the set of all x satisfying $(f \circ g \circ g \circ f)(x) = (g \circ g \circ f)(x)$, where $(f \circ g)(x) = f[g(x)]$, is _____ [2011]
(a) $\pm \sqrt{n\pi}, n \in \{0, 1, 2, \dots\}$
(b) $\pm \sqrt{n\pi}, n \in \{1, 2, \dots\}$
(c) $\frac{\pi}{2} + 2n\pi, n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$
(d) $2n\pi, n \in \{\dots, -2, -1, 0, 1, 2, \dots\}$

7. The number of all possible values of θ , where $0 < \theta < \pi$, for which the system of equations

$$(y+z)\cos 3\theta = (xyz)\sin 3\theta$$

$$x\sin 3\theta = \frac{2\cos 3\theta}{y} + \frac{2\sin 3\theta}{z}$$

$$(xyz)\sin 3\theta = (y+2z)\cos 3\theta + y\sin 3\theta$$

have a solution (x_0, y_0, z_0) with $y_0 z_0 \neq 0$, is _____. [2010]

8. The number of values of θ in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

such that $\theta \neq \frac{n\pi}{5}$ for $n = 0, \pm 1, \pm 2$ and $\tan \theta = \cot 5\theta$ as well as $\sin 2\theta = \cos 4\theta$, is _____. [2010]

9. The maximum value of the expression $\frac{1}{\sin^2 \theta + 3\sin \theta \cos \theta + 5\cos^2 \theta}$ is _____. [2010]

JEE Main/AIEEE

10. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$, where, $x \in R$ and $k \geq 1$. Then, $f_4(x) - f_6(x)$ equals [2014]

- (a) $\frac{1}{6}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{12}$

11. The expression $\frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A}$ can be written as [2013]

- (a) $\sin A \cos A + 1$ (b) $\sec A \operatorname{cosec} A + 1$
(c) $\tan A + \cot A$ (d) $\sec A + \operatorname{cosec} A$

12. In a ΔPQR , if $3\sin P + 4\cos Q = 6$ and $4\sin Q + 3\cos P = 1$, then $\angle R$ is equal to [2012]

- (a) $\frac{5\pi}{6}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{3\pi}{4}$

13. If $A = \sin^2 x + \cos^4 x$, then for all real x [2011]

- (a) $\frac{13}{16} \leq A \leq 1$ (b) $1 \leq A \leq 2$
(c) $\frac{3}{4} \leq A \leq \frac{13}{16}$ (d) $\frac{3}{4} \leq A \leq 1$

14. The possible values of $\theta \in (0, \pi)$ such that $\sin \theta + \sin 4\theta + \sin 7\theta = 0$ are [2011]

- (a) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{4\pi}{9}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{8\pi}{9}$
(b) $\frac{\pi}{4}, \frac{5\pi}{12}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$
(c) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{35\pi}{36}$
(d) $\frac{2\pi}{9}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{8\pi}{9}$

15. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and $\sin(\alpha - \beta) = \frac{5}{13}$, where

$0 \leq \alpha, \beta \leq \frac{\pi}{4}$. Then, $\tan 2\alpha$ is equal to [2010]

- (a) $\frac{25}{16}$ (b) $\frac{56}{33}$ (c) $\frac{19}{12}$ (d) $\frac{20}{7}$

16. Let A and B denote the statements
 $A : \cos \alpha + \cos \beta + \cos \gamma = 0$
 $B : \sin \alpha + \sin \beta + \sin \gamma = 0$

If $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$, then [2009]

- (a) A is true and B is false (b) A is false and B is true
(c) both A and B are true (d) both A and B are false

17. The number of values of x in the interval $[0, 3\pi]$ satisfying the equation $2\sin^2 x + 5\sin x - 3 = 0$ is [2006]

- (a) 6 (b) 1 (c) 2 (d) 4

18. A triangular park is enclosed on two sides by a fence and on the third side by a straight river bank. The two sides having fence are of same length x . The maximum area enclosed by the park is [2006]

- (a) $\sqrt{\frac{x^3}{8}}$ (b) $\frac{1}{2}x^2$ (c) πx^2 (d) $\frac{3}{2}x^2$

19. If $0 < x < \pi$ and $\cos x + \sin x = \frac{1}{2}$, then $\tan x$ is equal to [2006]

- (a) $\frac{(4 - \sqrt{7})}{3}$ (b) $-\frac{(4 + \sqrt{7})}{3}$
(c) $\frac{(1 + \sqrt{7})}{4}$ (d) $\frac{(1 - \sqrt{7})}{4}$

20. Let α and β be such that $\pi < \alpha - \beta < 3\pi$. If $\sin \alpha + \sin \beta = -\frac{21}{65}$ and $\cos \alpha + \cos \beta = -\frac{27}{65}$, then the value of $\cos\left(\frac{\alpha - \beta}{2}\right)$ is [2004]

- (a) $-\frac{3}{\sqrt{130}}$ (b) $\frac{3}{\sqrt{130}}$
(c) $\frac{6}{65}$ (d) $-\frac{6}{65}$

21. If $f : R \rightarrow S$, defined by $f(x) = \sin x - \sqrt{3} \cos x + 1$, is onto, then the interval of S is [2004]

- (a) $[0, 3]$ (b) $[-1, 1]$
(c) $[0, 1]$ (d) $[-1, 3]$

22. $\sin^2 \theta = \frac{4xy}{(x+y)^2}$ is true if and only if [2002]

- (a) $x - y \neq 0$
(b) $x = -y$
(c) $x + y \neq 0$
(d) $x \neq 0, y \neq 0$

23. The value of $\frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ}$ is [2002]

- (a) 1 (b) $\sqrt{3}$ (c) $\sqrt{3}/2$ (d) 2

- 24.** If $\tan \theta = -\frac{4}{3}$, then $\sin \theta$ is [2002]
 (a) $-\frac{4}{5}$ but not $\frac{4}{5}$
 (b) $-\frac{4}{5}$ or $\frac{4}{5}$
 (c) $\frac{4}{5}$ but not $-\frac{4}{5}$
 (d) None of the above
- 25.** If $\sin(\alpha + \beta) = 1$ and $\sin(\alpha - \beta) = \frac{1}{2}$, then $\tan(\alpha + 2\beta) \tan(2\alpha + \beta)$ is equal to [2002]
 (a) 1
 (b) -1
 (c) 0
 (d) None of the above
- 26.** If $y = \sin^2 \theta + \operatorname{cosec}^2 \theta$, $\theta \neq 0$, then [2002]
 (a) $y = 0$
 (b) $y \leq 2$
 (c) $y \geq -2$
 (d) $y \geq 2$
- 27.** The equation $a \sin x + b \cos x = c$, where $|c| > \sqrt{a^2 + b^2}$ has [2002]
 (a) a unique solution
 (b) infinite number of solutions
 (c) no solution
 (d) None of the above
- 28.** If α is a root of $25 \cos^2 \theta + 5 \cos \theta - 12 = 0$, $\frac{\pi}{2} < \alpha < \pi$, then $\sin 2\alpha$ is equal to [2002]
 (a) $\frac{24}{25}$ (b) $-\frac{24}{25}$ (c) $\frac{13}{18}$ (d) $-\frac{13}{18}$

Other Engineering Entrances

- 29.** The sum of the series $\sum_{n=1}^{\infty} \sin\left(\frac{n! \pi}{720}\right)$ is [WB JEE 2014]
 (a) $\sin\left(\frac{\pi}{180}\right) + \sin\left(\frac{\pi}{360}\right) + \sin\left(\frac{\pi}{540}\right)$
 (b) $\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{30}\right) + \sin\left(\frac{\pi}{120}\right) + \sin\left(\frac{\pi}{360}\right)$
 (c) $\sin\left(\frac{\pi}{6}\right) + \sin\left(\frac{\pi}{30}\right) + \sin\left(\frac{\pi}{120}\right) + \sin\left(\frac{\pi}{360}\right) + \sin\left(\frac{\pi}{720}\right)$
 (d) $\sin\left(\frac{\pi}{180}\right) + \sin\left(\frac{\pi}{360}\right)$
- 30.** If $\cos x = \tan y$, $\cot y = \tan z$ and $\cot z = \tan x$, then $\sin x$ is equal to [EAMCET 2014]
 (a) $\frac{\sqrt{5}+1}{4}$ (b) $\frac{\sqrt{5}-1}{4}$ (c) $\frac{\sqrt{5}+1}{2}$ (d) $\frac{\sqrt{5}-1}{2}$
- 31.** The value of $2 \cot^2(\pi/6) + 4 \tan^2(\pi/6) - 3 \operatorname{cosec}(\pi/6)$ is [J&K CET 2014]
 (a) 2 (b) 4 (c) 4/3 (d) 3/4
- 32.** Find the value of $\cos(29\pi/3)$. [J&K CET 2014]
 (a) 1 (b) 0 (c) $\sqrt{3}/2$ (d) $1/2$
- 33.** If $\sin \theta = \sin \alpha$, then [Karnataka CET 2014]
 (a) $\frac{\theta + \alpha}{2}$ is any odd multiple of $\frac{\pi}{2}$ and $\frac{\theta - \alpha}{2}$ is any multiple of π
 (b) $\frac{\theta + \alpha}{2}$ is any odd multiple of π and $\frac{\theta - \alpha}{2}$ is any multiple of π
 (c) $\frac{\theta + \alpha}{2}$ is any multiple of $\frac{\pi}{2}$ and $\frac{\theta - \alpha}{2}$ is any even multiple of π
 (d) $\frac{\theta + \alpha}{2}$ is any even multiple of $\frac{\pi}{2}$ and $\frac{\theta - \alpha}{2}$ is any odd multiple of π
- 34.** $\tan 81^\circ - \tan 63^\circ - \tan 27^\circ + \tan 9^\circ$ equals [EAMCET 2014]
 (a) 6 (b) 0 (c) 2 (d) 4
- 35.** If $\sin A + \cos A = \sqrt{2}$, then the value of $\cos^2 A$ is [RPET 2014]
 (a) $\sqrt{2}$ (b) $1/2$ (c) 4 (d) -1
- 36.** If $\tan \frac{\alpha}{2}$ and $\tan \frac{\beta}{2}$ are the roots of $8x^2 - 26x + 15 = 0$, then $\cos(\alpha + \beta)$ is equal to [Manipal 2014]
 (a) $\frac{627}{726}$ (b) $-\frac{627}{725}$
 (c) -1 (d) None of these
- 37.** If $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2}$, then the value of $\cos \frac{x}{2}$ is [Karnataka CET 2014]
 (a) $-\frac{1}{\sqrt{10}}$ (b) $\frac{3}{\sqrt{10}}$ (c) $\frac{1}{\sqrt{10}}$ (d) $-\frac{3}{\sqrt{10}}$
- 38.** $\cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7}$ [WB JEE 2014]
 (a) is equal to zero (b) lies between 0 and 3
 (c) is a negative number (d) lies between 3 and 6
- 39.** The value of $\tan \frac{\pi}{5} + 2 \tan \frac{2\pi}{5} + 4 \cot \frac{4\pi}{5}$ is [WB JEE 2014]
 (a) $\cot \frac{\pi}{5}$ (b) $\cot \frac{2\pi}{5}$ (c) $\cot \frac{4\pi}{5}$ (d) $\cot \frac{3\pi}{5}$
- 40.** If x and y are acute angles such that $\cos x + \cos y = \frac{3}{2}$ and $\sin x + \sin y = \frac{3}{4}$, then $\sin(x + y)$ equals [Karnataka CET 2014]
 (a) $\frac{2}{5}$ (b) $\frac{3}{4}$
 (c) $\frac{3}{5}$ (d) $\frac{4}{5}$

41. Find the value of $\cos(x/2)$, if $\tan x = 5/12$ and x lies in third quadrant. [J&K CET 2014]
 (a) $5\sqrt{13}$ (b) $5/\sqrt{26}$ (c) $5/13$ (d) $\sqrt{\frac{1}{26}}$
42. Let n be a positive integer such that $\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{n}}{2}$, then [AMU 2014]
 (a) $6 \leq n \leq 8$ (b) $4 < n \leq 8$
 (c) $4 \leq n < 8$ (d) $4 < n < 8$
43. The sum of the solutions in $(0, 2\pi)$ of the equation $\cos x \cos\left(\frac{\pi}{3} - x\right) \cos\left(\frac{\pi}{3} + x\right) = \frac{1}{4}$ is [EAMCET 2014]
 (a) 4π (b) π (c) 2π (d) 3π
44. Which one of the following is not correct? [AMU 2013]
 (a) $|\sin x| \leq 1$
 (b) $-1 \leq \cos x \leq 1$
 (c) $|\sec x| < 1$
 (d) $\operatorname{cosec} x \geq 1$ or $\operatorname{cosec} x \leq -1$
45. If $0 < \beta < \alpha < \frac{\pi}{4}$, $\cos(\alpha + \beta) = \frac{3}{5}$ and $\cos(\alpha - \beta) = \frac{4}{5}$, then $\sin 2\alpha$ is equal to [Karnataka CET 2013]
 (a) 0 (b) 1
 (c) 2 (d) None of these
46. The most general solutions of the equation $\sec^2 x = \sqrt{2}(1 - \tan^2 x)$ are given by [BITSAT 2013]
 (a) $n\pi \pm \frac{\pi}{4}$ (b) $2n\pi + \frac{\pi}{4}$
 (c) $n\pi \pm \frac{\pi}{8}$ (d) None of these
47. The number of solutions of the equation $(1 - \cos 2x)(\cos 2x + \cos^2 x) = 0$, $0 \leq x \leq 2\pi$ is [Manipal 2013]
 (a) 3 (b) 2 (c) 1 (d) 0
48. If $\log_{\cos x} \tan x + \log_{\sin x} \cot x = 0$, then the most general solution of x is [UP SEE 2013]
 (a) $2n\pi - \frac{3\pi}{4}$, $n \in \mathbb{Z}$ (b) $2n\pi + \frac{\pi}{4}$, $n \in \mathbb{Z}$
 (c) $n\pi - \frac{\pi}{4}$, $n \in \mathbb{Z}$ (d) None of these
49. The number of real solutions of the equation $x^3 + x^2 + 4x + 2\sin x = 0$ in $0 \leq x \leq 2\pi$ is [OJEE 2013]
 (a) 4 (b) 2 (c) 1 (d) 0
50. The general solution of $\sin 3x + \sin x - 3\sin 2x = \cos 3x + \cos x - 3\cos 2x$ is [AMU 2013]
 (a) $\frac{n\pi}{2} + \frac{\pi}{8}$ for n integer (b) $\frac{n\pi}{2} - \frac{\pi}{8}$ for n integer
 (c) $n\pi + \frac{\pi}{6}$ for n integer (d) $n\pi - \frac{\pi}{6}$ for n integer
51. If $\sin A + \sin B + \sin C = 3$, then $\cos A + \cos B + \cos C$ is equal to [MP PET 2012]
 (a) 3 (b) 2 (c) 1 (d) 0
52. If $\cos \alpha + 2\cos \beta + 3\cos \gamma = 0$, $\sin \alpha + 2\sin \beta + 3\sin \gamma = 0$ and $\alpha + \beta + \gamma = \pi$, then $\sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma$ is equal to [Karnataka CET 2012]
 (a) -18 (b) 0 (c) 3 (d) 9
53. If $\cos A = m\cos B$ and $\cot\left(\frac{A+B}{2}\right) = \lambda/\tan\left(\frac{B-A}{2}\right)$, then λ is equal to [Manipal 2012]
 (a) $\frac{m}{m-1}$ (b) $\frac{m+1}{m}$
 (c) $\frac{m+1}{m-1}$ (d) None of these
54. The least value of $3\sin^2 \theta + 4\cos^2 \theta$ is [OJEE 2012]
 (a) 2 (b) 3 (c) 0 (d) 1
55. If $\sin 2x = 4\cos x$, then x is equal to [Karnataka CET 2012]
 (a) $\frac{n\pi}{2} \pm \frac{\pi}{4}$, $n \in \mathbb{Z}$ (b) no value
 (c) $n\pi + (-1)^n \frac{\pi}{4}$, $n \in \mathbb{Z}$ (d) $2n\pi \pm \frac{\pi}{2}$, $n \in \mathbb{Z}$
56. Number of solutions of the equation $\tan x + \sec x = 2\cos x$, $x \in [0, \pi]$ is [WB JEE 2012]
 (a) 0 (b) 1
 (c) 2 (d) 3
57. The value of θ satisfying $\cos \theta + \sqrt{3}\sin \theta = 2$ is [OJEE 2012]
 (a) 30° (b) 60°
 (c) 45° (d) 90°
58. If $\sin \theta = \frac{2t}{1+t^2}$ and θ lies in second quadrant, then $\cos \theta$ is equal to [WB JEE 2011]
 (a) $\frac{1-t^2}{1+t^2}$ (b) $\frac{t^2-1}{1+t^2}$
 (c) $\frac{-|1-t^2|}{1+t^2}$ (d) $\frac{1+t^2}{|1-t^2|}$
59. If $\sin \theta = 3\sin(\theta + 2\alpha)$, then the value of $\tan(\theta + \alpha) + 2\tan \alpha$ is [Kerala CEE 2011]
 (a) 3 (b) 2 (c) -1
 (d) 0 (e) 1
60. If $\alpha, \beta, \gamma \in [0, \pi]$ and α, β, γ are in AP, then $\frac{\sin \alpha - \sin \gamma}{\cos \gamma - \cos \alpha}$ is equal to [Kerala CEE 2011]
 (a) $\sin \beta$ (b) $\cos \beta$ (c) $\cot \beta$
 (d) $\operatorname{cosec} \beta$ (e) $2\cos \beta$
61. If A, B and C are the angles of a triangle such that $\sec(A-B)$, $\sec A$ and $\sec(A+B)$ are in AP, then [UP SEE 2011]
 (a) $\operatorname{cosec}^2 A = 2\operatorname{cosec}^2 \frac{B}{2}$ (b) $2\sec^2 A = \sec^2 \frac{B}{2}$
 (c) $2\operatorname{cosec}^2 A = \operatorname{cosec}^2 \frac{B}{2}$ (d) $2\sec^2 B = \sec^2 \frac{A}{2}$

- 62.** If $\sin \theta = \frac{1}{2}$ and $\tan \theta = \frac{1}{\sqrt{3}}$, $\forall n \in I$, then most general value of θ is [BITSAT 2011]
 (a) $2n\pi + \frac{\pi}{6}, \forall n \in I$ (b) $2n\pi + \frac{\pi}{4}, \forall n \in I$
 (c) $2n\pi + \frac{\pi}{3}, \forall n \in I$ (d) $2n\pi + \frac{\pi}{5}, \forall n \in I$
- 63.** A value of θ satisfying $\sin 5\theta - \sin 3\theta + \sin \theta = 0$ such that $0 < \theta < \frac{\pi}{2}$ is [Karnataka CET 2011]
 (a) $\frac{\pi}{12}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{2}$
- 64.** The number of solutions of $2\sin x + \cos x = 3$ is [WB JEE 2011]
 (a) 1 (b) 2
 (c) infinite (d) no solution
- 65.** If $\sin \theta + \cos \theta = 0$ and $0 < \theta < \pi$, then θ is equal to [WB JEE 2011]
 (a) 0 (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{3\pi}{4}$
- 66.** The solution of trigonometric equation $\cos^4 x + \sin^4 x = 2\cos(2x + \pi)\cos(2x - \pi)$ is [UP SEE 2011]
 (a) $x = \frac{n\pi}{2} \pm \sin^{-1}\left(\frac{1}{5}\right)$
 (b) $x = \frac{n\pi}{4} + \frac{(-1)^n}{4} \sin^{-1}\left(\frac{\pm 2\sqrt{2}}{3}\right)$
 (c) $x = \frac{n\pi}{2} \pm \cos^{-1}\frac{1}{5}$
 (d) $x = \frac{n\pi}{4} - \frac{(-1)^n}{4} \cos^{-1}\left(\frac{1}{5}\right)$
- 67.** If $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$, then the number of distinct real roots of this equation in the interval $-\pi/2 < x < \pi/2$ is [AMU 2011]
 (a) 1 (b) 3 (c) 2 (d) 4
- 68.** If $\frac{\cos A}{3} = \frac{\cos B}{4} = \frac{1}{5} - \frac{\pi}{2} < A < 0$ and $-\frac{\pi}{2} < B < 0$, then the value of $2\sin A + 4\sin B$ is [BITSAT 2010]
 (a) 4 (b) -2 (c) -4 (d) 0
- 69.** The value of $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ}$ is [VITEEE 2010]
 (a) 0 (b) 2
 (c) 3 (d) 1
- 70.** The value of $\tan 40^\circ + \tan 20^\circ + \sqrt{3} \tan 20^\circ \tan 40^\circ$ is [BITSAT 2010]
 (a) $\sqrt{12}$ (b) $\frac{1}{\sqrt{3}}$
 (c) $\sqrt{2}$ (d) $\sqrt{3}$
- 71.** The value of $\frac{\sin 55^\circ - \cos 55^\circ}{\sin 10^\circ}$ is [VITEEE 2010]
 (a) $\frac{1}{\sqrt{2}}$ (b) 2
 (c) 1 (d) $\sqrt{2}$
- 72.** If $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$, $\sin \alpha = \frac{4}{5}$ and $\cos(\alpha + \beta) = -\frac{12}{13}$, then $\sin \beta$ is equal to [Kerala CEE 2010]
 (a) $\frac{63}{65}$ (b) $\frac{61}{65}$
 (c) $\frac{3}{5}$ (d) $\frac{5}{13}$
 (e) $\frac{8}{65}$
- 73.** If $\tan \alpha = \frac{b}{a}$, $a > b > 0$ and $0 < \alpha < \frac{\pi}{4}$, then $\sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a-b}{a+b}}$ is equal to [Kerala CEE 2010]
 (a) $\frac{2\sin \alpha}{\sqrt{\cos 2\alpha}}$ (b) $\frac{2\cos \alpha}{\sqrt{\cos 2\alpha}}$
 (c) $\frac{2\sin \alpha}{\sqrt{\sin 2\alpha}}$ (d) $\frac{2\cos \alpha}{\sqrt{\sin 2\alpha}}$
 (e) $\frac{2\tan \alpha}{\sqrt{\cos 2\alpha}}$
- 74.** The value of $\frac{\cot x - \tan x}{\cot 2x}$ is [WB JEE 2010]
 (a) 1 (b) 2 (c) -1 (d) 4
- 75.** The number of points of intersection of $2y = 1$ and $y = \sin x$ in $-2\pi \leq x \leq 2\pi$ is [VITEEE 2010]
 (a) 1 (b) 2 (c) 3 (d) 4
- 76.** The value of x in $\left(0, \frac{\pi}{2}\right)$ satisfying the equation $\sin x \cos x = \frac{1}{4}$ is [Kerala CEE 2010]
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{8}$ (d) $\frac{\pi}{4}$
 (e) $\frac{\pi}{12}$
- 77.** The number of solutions of $\cos 2\theta = \sin \theta$ in $(0, 2\pi)$ is [Kerala CEE 2010]
 (a) 1 (b) 2 (c) 3 (d) 4
 (e) 0
- 78.** If $\sin 6\theta + \sin 4\theta + \sin 2\theta = 0$, then the general values of θ are [WB JEE 2010]
 (a) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{3}$
 (b) $\frac{n\pi}{4}, n\pi \pm \frac{\pi}{6}$
 (c) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{3}$
 (d) $\frac{n\pi}{4}, 2n\pi \pm \frac{\pi}{6}$

Answers

Work Book Exercise 9.1

1. (a) 2. (b) 3. (b) 4. (a) 5. (a) 6. (b) 7. (b) 8. (a) 9. (a) 10. (d)
11. (d) 12. (d)

Work Book Exercise 9.2

1. (a) 2. (d) 3. (b) 4. (a) 5. (c) 6. (c) 7. (a) 8. (b) 9. (b) 10. (b)
11. (c) 12. (d) 13. (a) 14. (c) 15. (a) 16. (a) 17. (b) 18. (b) 19. (a) 20. (d)

Work Book Exercise 9.3

1. (c) 2. (d) 3. (c) 4. (a) 5. (c) 6. (b) 7. (c) 8. (c) 9. (d) 10. (c)

Work Book Exercise 9.4

1. (b) 2. (c) 3. (c) 4. (d) 5. (a) 6. (c)

Target Exercises

1. (a) 2. (c) 3. (d) 4. (a) 5. (b) 6. (d) 7. (a) 8. (b) 9. (c) 10. (d)
11. (c) 12. (d) 13. (c) 14. (b) 15. (b) 16. (d) 17. (d) 18. (c) 19. (b) 20. (d)
21. (d) 22. (c) 23. (a) 24. (a) 25. (a) 26. (b) 27. (b) 28. (b) 29. (a) 30. (d)
31. (a) 32. (d) 33. (a) 34. (b) 35. (c) 36. (b) 37. (b) 38. (c) 39. (b) 40. (a)
41. (c) 42. (b) 43. (b) 44. (c) 45. (c) 46. (a) 47. (a) 48. (b) 49. (c) 50. (a)
51. (c) 52. (d) 53. (b) 54. (a) 55. (d) 56. (b) 57. (b) 58. (c) 59. (b) 60. (a)
61. (c) 62. (a) 63. (c) 64. (b) 65. (b) 66. (d) 67. (c) 68. (d) 69. (c) 70. (a)
71. (a) 72. (a) 73. (b) 74. (b) 75. (c) 76. (d) 77. (a) 78. (c) 79. (a) 80. (a)
81. (c) 82. (c) 83. (b) 84. (c) 85. (d) 86. (b) 87. (c) 88. (c) 89. (a) 90. (c)
91. (c) 92. (b) 93. (a) 94. (b) 95. (a) 96. (b) 97. (d) 98. (d) 99. (a) 100. (a)
101. (a) 102. (a) 103. (d) 104. (a) 105. (b) 106. (a) 107. (a) 108. (d) 109. (b) 110. (c)
111. (c) 112. (d) 113. (c) 114. (d) 115. (b) 116. (a) 117. (b) 118. (a) 119. (c) 120. (c)
121. (d) 122. (b) 123. (b) 124. (a) 125. (b) 126. (b) 127. (c) 128. (c) 129. (c) 130. (all)
131. (b,c) 132. (a,b) 133. (a,c,d) 134. (a,c,d) 135. (b,c) 136. (all) 137. (all) 138. (a,b) 139. (b,c) 140. (a,c)
141. (a,d) 142. (b,c) 143. (b,d) 144. (b,d) 145. (d) 146. (b) 147. (a) 148. (b) 149. (c) 150. (d)
151. (c) 152. (b) 153. (a) 154. (c) 155. (a) 156. (b) 157. (d) 158. (a) 159. (b) 160. (a)
161. (d) 162. (d) 163. (d) 164. (*) 165. (*) 166. (*) 167. (*) 168. (*) 169. (*) 170. (*)
171. (1) 172. (1) 173. (4) 174. (3) 175. (2) 176. (2) 177. (0) 178. (1) 179. (3) 180. (3)
181. (0) 182. (6) 183. (6) 184. (2) 185. (1) 186. (2)

* 164. $A \rightarrow s; B \rightarrow r; C \rightarrow p; D \rightarrow q$

167. $A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$

170. $A \rightarrow q; B \rightarrow s; C \rightarrow p; D \rightarrow p, r$

165. $A \rightarrow p; B \rightarrow r; C \rightarrow q; D \rightarrow q$

168. $A \rightarrow r; B \rightarrow s; C \rightarrow s; D \rightarrow q$

166. $A \rightarrow r; B \rightarrow s; C \rightarrow s; D \rightarrow q$

169. $A \rightarrow q; B \rightarrow r; C \rightarrow p; D \rightarrow s$

Entrances Gallery

1. (d) 2. (c) 3. (a,c,d) 4. (a,b) 5. (7) 6. (a) 7. (3) 8. (3) 9. (2) 10. (d)
11. (b) 12. (b) 13. (d) 14. (a) 15. (b) 16. (c) 17. (d) 18. (b) 19. (b) 20. (a)
21. (d) 22. (c) 23. (c) 24. (b) 25. (a) 26. (d) 27. (c) 28. (b) 29. (c) 30. (d)
31. (c) 32. (d) 33. (a) 34. (d) 35. (b) 36. (b) 37. (a) 38. (c) 39. (a) 40. (d)
41. (d) 42. (d) 43. (c) 44. (c) 45. (b) 46. (c) 47. (d) 48. (b) 49. (c) 50. (a)
51. (d) 52. (b) 53. (c) 54. (b) 55. (d) 56. (c) 57. (b) 58. (c) 59. (d) 60. (c)
61. (b) 62. (a) 63. (b) 64. (d) 65. (d) 66. (b) 67. (c) 68. (c) 69. (b) 70. (d)
71. (d) 72. (a) 73. (a) 74. (b) 75. (d) 76. (e) 77. (c) 78. (a)

Explanations

Target Exercises

$$1. \tan \frac{\pi}{3} + 2 \tan \frac{2\pi}{3} + 4 \tan \frac{4\pi}{3} + 8 \tan \frac{8\pi}{3}$$

$$= \tan \frac{\pi}{3} + 2 \tan \left(\pi - \frac{\pi}{3} \right) + 4 \tan \left(\pi + \frac{\pi}{3} \right)$$

$$+ 8 \tan \left(3\pi - \frac{\pi}{3} \right)$$

$$= \tan \frac{\pi}{3} - 2 \tan \frac{\pi}{3} + 4 \tan \frac{\pi}{3} - 8 \tan \frac{\pi}{3}$$

$$= -5 \tan \frac{\pi}{3} = -5\sqrt{3}$$

$$2. \sec^2 \theta + \operatorname{cosec}^2 \theta = \frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta} = \frac{4}{\sin^2 2\theta} \geq 4$$

$$\text{Also, } \sec^2 \theta \operatorname{cosec}^2 \theta = \frac{4}{\sin^2 2\theta} \geq 4$$

Thus, required quadratic equation will be $x^2 - tx + t = 0$, where $t \geq 4$

Hence, $x^2 - 5x + 5 = 0$ can be the required equation.

$$3. \cos^2 \frac{3\pi}{5} + \cos^2 \frac{4\pi}{5} = \cos^2 108^\circ + \cos^2 144^\circ.$$

$$= \sin^2 18^\circ + \cos^2 36^\circ = \left(\frac{\sqrt{5}-1}{4} \right)^2 + \left(\frac{\sqrt{5}+1}{4} \right)^2$$

$$= \frac{2(5+1)}{16} = \frac{3}{4}$$

$$4. \sin(2 \times 360^\circ - 120^\circ) \cos(360^\circ - 30^\circ)$$

$$+ \cos 120^\circ \sin(180^\circ - 30^\circ)$$

$$= -\sin 120^\circ \cos 30^\circ + \cos 120^\circ \sin 30^\circ$$

$$= -\sin(90^\circ + 30^\circ) \cos 30^\circ + \cos(90^\circ + 30^\circ) \sin 30^\circ$$

$$= -\cos^2 30^\circ - \sin^2 30^\circ$$

$$= -1$$

5. Given expression

$$= \frac{1 - 2(\cos 60^\circ - \cos 80^\circ)}{2 \sin 10^\circ} = \frac{1 - 2\left(\frac{1}{2} - \cos 80^\circ\right)}{2 \sin 10^\circ}$$

$$= \frac{2 \cos 80^\circ}{2 \sin 10^\circ} = \frac{\cos(90^\circ - 10^\circ)}{\sin 10^\circ} = \frac{\sin 10^\circ}{\sin 10^\circ} = 1$$

6. Given expression

$$= 2 \left(\frac{\sqrt{3}}{2} \sin 75^\circ - \frac{1}{2} \cos 75^\circ \right)$$

$$= 2(\cos 30^\circ \sin 75^\circ - \sin 30^\circ \cos 75^\circ)$$

$$= 2 \sin(75^\circ - 30^\circ) = 2 \sin 45^\circ = 2 \times \frac{1}{\sqrt{2}} = \sqrt{2}$$

$$7. \text{ Given, } \sin \alpha = \frac{1}{\sqrt{5}} \text{ and } \sin \beta = \frac{3}{5}$$

$$\Rightarrow \cos \alpha = \frac{2}{\sqrt{5}} \text{ and } \cos \beta = \frac{4}{5}$$

$$\therefore \sin(\beta - \alpha) = \sin \beta \cos \alpha - \cos \beta \sin \alpha$$

$$= \left(\frac{3}{5} \right) \left(\frac{2}{\sqrt{5}} \right) - \left(\frac{4}{5} \right) \left(\frac{1}{\sqrt{5}} \right) = \frac{2}{5\sqrt{5}}$$

$$\text{Clearly, } 0 < \sin(\beta - \alpha) < \frac{1}{\sqrt{2}}$$

$$\Rightarrow 0 < (\beta - \alpha) < \frac{\pi}{4} \Rightarrow \frac{3\pi}{4} < (\beta - \alpha) < \pi$$

$$\Rightarrow (\beta - \alpha) \in \left(0, \frac{\pi}{4} \right) \Rightarrow (\beta - \alpha) \in \left(\frac{3\pi}{4}, \pi \right)$$

$$8. \sqrt{3} \operatorname{cosec} 20^\circ - \sec 20^\circ = \frac{\sqrt{3} \cos 20^\circ - \sin 20^\circ}{\sin 20^\circ \cos 20^\circ}$$

$$= \frac{4(\sqrt{3}/2 \cos 20^\circ - 1/2 \sin 20^\circ)}{2 \cdot \sin 20^\circ \cos 20^\circ} = \frac{4 \cdot \sin 40^\circ}{\sin 40^\circ} = 4$$

$$9. \text{ We have, } \sin 15^\circ = \sqrt{\frac{1}{2}(1 - \cos 30^\circ)}$$

$$= \sqrt{\frac{1}{2} \left(1 - \frac{\sqrt{3}}{2} \right)} = \frac{1}{2} \sqrt{2 - \sqrt{3}}$$

which is an irrational number.

$\cos 15^\circ = \frac{1}{2} \sqrt{2 + \sqrt{3}}$ is also an irrational number.

$$\sin 15^\circ \cos 15^\circ = \frac{1}{2} \sin 30^\circ = \frac{1}{4}$$

which is a rational number.

Further, $\sin 15^\circ \cos 75^\circ = \sin 15^\circ \cos(90^\circ - 15^\circ)$

$$= \sin 15^\circ \sin 15^\circ = \frac{1}{4}(2 - \sqrt{3})$$

which is an irrational number.

10. We have,

$$\cos \frac{\pi}{7} \cos \frac{4\pi}{7} \cos \frac{5\pi}{7} = -\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7}$$

$$\text{As } \cos \frac{5\pi}{7} = \cos \left(\pi - \frac{2\pi}{7} \right) = -\cos \frac{2\pi}{7}$$

$$= -\frac{1}{2 \sin \frac{\pi}{7}} \left[2 \sin \frac{\pi}{7} \cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} \right]$$

$$= -\frac{1}{2^2 \sin \frac{\pi}{7}} \left[\sin \frac{4\pi}{7} \cos \frac{4\pi}{7} \right] = -\frac{1}{8 \sin \frac{\pi}{7}} \left[\sin \frac{8\pi}{7} \right]$$

$$= -\frac{1}{8 \sin \frac{\pi}{7}} \left[\sin \left(\pi + \frac{\pi}{7} \right) \right] = -\frac{1}{8 \sin \frac{\pi}{7}} \left[-\sin \frac{\pi}{7} \right] = \frac{1}{8}$$

$$11. 6(\sin^6 \theta + \cos^6 \theta) - 9(\sin^4 \theta + \cos^4 \theta) + 4$$

$$= 6[(\sin^2 \theta + \cos^2 \theta)^3 - 3\sin^2 \theta \cos^2 \theta (\sin^2 \theta + \cos^2 \theta)]$$

$$- 9[(\sin^2 \theta + \cos^2 \theta)^2 - 2\sin^2 \theta \cos^2 \theta] + 4$$

$$= 6[1 - 3\sin^2 \theta \cos^2 \theta] - 9[1 - 2\sin^2 \theta \cos^2 \theta] + 4$$

$$= 6 - 9 + 4 = 1$$

$$12. \text{ Given, } \frac{2 \sin \alpha}{1 + \cos \alpha + \sin \alpha} = x$$

$$\Rightarrow \frac{2 \sin \alpha (1 - \cos \alpha - \sin \alpha)}{(1 + \cos \alpha + \sin \alpha)(1 - \cos \alpha - \sin \alpha)} = x$$

$$\Rightarrow \frac{2 \sin \alpha (1 - \cos \alpha - \sin \alpha)}{1 - \sin^2 \alpha - \cos^2 \alpha - 2 \sin \alpha \cos \alpha} = x$$

$$\Rightarrow \frac{1 - \cos \alpha - \sin \alpha}{\cos \alpha} = -x$$

13. Since, $\sec \theta \notin (-1, 1)$

$$\therefore \sec \theta \neq \frac{1}{2}$$

14. Since, $\log(\sin 90^\circ)$ is a factor in the given expression and the value of which is $\log(1) = 0$. Hence, the value of the given expression is 0.

$$\begin{aligned} 15. \text{ We have, } & 3 \left[\sin^4 \left(\frac{3\pi}{2} - \alpha \right) + \sin^4(3\pi + \alpha) \right] \\ & - 2 \left[\sin^6 \left(\frac{\pi}{2} + \alpha \right) + \sin^6(5\pi - \alpha) \right] \\ & = 3 [(-\cos \alpha)^4 + (-\sin \alpha)^4] - 2 [\cos^6 \alpha + \sin^6 \alpha] \\ & = 3 [(\cos^2 \alpha + \sin^2 \alpha)^2 - 2 \sin^2 \alpha \cos^2 \alpha] \\ & \quad - 2 [(\cos^2 \alpha + \sin^2 \alpha)^3 - 3 \cos^2 \alpha \sin^2 \alpha] \\ & \quad (\cos^2 \alpha + \sin^2 \alpha) \\ & = 3 - 6 \sin^2 \alpha \cos^2 \alpha - 2 + 6 \sin^2 \alpha \cos^2 \alpha = 3 - 2 = 1 \end{aligned}$$

16. Since, $\tan x + \frac{1}{\tan x} = 2$

$$\Rightarrow \tan x = 1 \Rightarrow x = \frac{\pi}{4}$$

$$\therefore \sin x = \frac{1}{\sqrt{2}} \text{ and } \cos x = \frac{1}{\sqrt{2}}$$

$$\text{Hence, } \sin^{2n} x + \cos^{2n} x = \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

17. Given, $\sin \alpha = \frac{15}{17}$, $\tan \beta = \frac{12}{5}$

$$\text{Since, } \frac{\pi}{2} < \alpha < \pi, \pi < \beta < \frac{3\pi}{2}$$

$$\therefore \cos \alpha = -\frac{8}{17}, \sin \beta = -\frac{12}{13}$$

$$\text{and } \cos \beta = -\frac{5}{13}$$

$$\begin{aligned} \text{Now, } \sin(\beta - \alpha) &= \sin \beta \cos \alpha - \cos \beta \sin \alpha \\ &= -\frac{12}{13} \left(-\frac{8}{17} \right) - \left(-\frac{5}{13} \right) \left(\frac{15}{17} \right) \\ &= \frac{96}{221} + \frac{75}{221} = \frac{171}{221} \end{aligned}$$

$$\begin{aligned} 18. \cos^2(\alpha - \beta) + 2ab \sin(\alpha - \beta) \\ &= \cos^2(\alpha - \beta) + 2 \cos(\theta - \alpha) \sin(\theta - \beta) \sin(\alpha - \beta) \\ &= \cos^2(\alpha - \beta) + [\sin\{2\theta - (\alpha + \beta)\} \\ & \quad + \sin(\alpha - \beta)] \sin(\alpha - \beta) \\ &= 1 + \sin\{2\theta - (\alpha + \beta)\} \sin(\alpha - \beta) \\ &= 1 + \sin\{(\theta - \alpha) + (\theta - \beta)\} \sin\{(\theta - \beta) - (\theta - \alpha)\} \\ &= 1 + \sin^2(\theta - \beta) - \sin^2(\theta - \alpha) \\ &= \cos^2(\theta - \alpha) + \sin^2(\theta - \beta) = a^2 + b^2 \end{aligned}$$

19. Put $x = 60^\circ$, then expression reduces to 1, therefore (b) is the appropriate choice.

$$\begin{aligned} 20. \sin 47^\circ + \sin 61^\circ - \sin 11^\circ - \sin 25^\circ \\ &= 2 \sin 54^\circ \cos 7^\circ - 2 \sin 18^\circ \cos 7^\circ \\ &= 2 \cos 7^\circ (\sin 54^\circ - \sin 18^\circ) \\ &= 2 \cos 7^\circ (\cos 36^\circ - \sin 18^\circ) \\ &= 2 \cos 7^\circ \left[\left(\frac{\sqrt{5} + 1}{4} \right) - \left(\frac{\sqrt{5} - 1}{4} \right) \right] \\ &= \cos 7^\circ \end{aligned}$$

$$\begin{aligned} 21. \cos y \cos \left(\frac{\pi}{2} - x \right) - \cos \left(\frac{\pi}{2} - y \right) \cos x \\ + \sin y \cos \left(\frac{\pi}{2} - x \right) + \cos x \sin \left(\frac{\pi}{2} - y \right) = 0 \end{aligned}$$

$$\Rightarrow \cos y \sin x - \cos x \sin y + \sin y \sin x + \cos x \cos y = 0$$

$$\Rightarrow \sin(x - y) + \cos(x - y) = 0$$

$$\Rightarrow \tan(x - y) = -1$$

$$\Rightarrow x = n\pi - \frac{\pi}{4} + y$$

$$22. \operatorname{cosec} A + \cot A = \frac{11}{2}$$

$$\Rightarrow \operatorname{cosec}^2 A - \cot^2 A = 1$$

$$\Rightarrow \operatorname{cosec} A - \cot A = \frac{2}{11}$$

$$\Rightarrow 2 \cot A = \frac{117}{22} \Rightarrow \tan A = \frac{44}{117}$$

23. Let $\frac{\sin^3 x}{1 + \cos x} + \frac{\cos^3 x}{1 - \sin x} = A$, then

$$A = \frac{(\sin^3 x + \cos^3 x) + (\cos^4 x - \sin^4 x)}{(1 + \cos x)(1 - \sin x)}$$

$$\{(\sin^3 x + \cos^3 x)\} \{(\cos x + \sin x)\}$$

$$\Rightarrow A = \frac{(\cos x - \sin x)(\cos^2 x + \sin^2 x)}{(1 + \cos x)(1 - \sin x)}$$

$$(\sin x + \cos x) \{ (1 - \sin x \cos x) \}$$

$$\Rightarrow A = \frac{+ (\cos x - \sin x)}{1 + \cos x - \sin x - \sin x \cos x}$$

$$\Rightarrow A = \sin x + \cos x$$

$$\Rightarrow A = \sqrt{2} \left[\frac{1}{\sqrt{2}} \sin x + \frac{1}{\sqrt{2}} \cos x \right] \quad \dots(i)$$

$$\begin{aligned} \Rightarrow A &= \sqrt{2} \left[\cos \frac{\pi}{4} \sin x + \sin \frac{\pi}{4} \cos x \right] \\ &= \sqrt{2} \sin \left[\frac{\pi}{4} + x \right] \end{aligned}$$

Again, by Eq. (i), we get

$$A = \sqrt{2} \left[\sin \frac{\pi}{4} \sin x + \cos \frac{\pi}{4} \cos x \right] = \sqrt{2} \cos \left[\frac{\pi}{4} - x \right]$$

24. Given expression = $\operatorname{cosec} A \sin(B + C)$

$$= \operatorname{cosec} A \cdot \sin(\pi - A) \quad [\because A + B + C = \pi]$$

$$= \operatorname{cosec} A \cdot \sin A = 1$$

$$25. \cos(A - B) = \frac{3}{5} \Rightarrow \tan(A - B) = \frac{4}{3}$$

$$\Rightarrow \frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{4}{3}$$

$$\Rightarrow \tan A - \tan B = 4$$

$$\text{Now, } (\tan A + \tan B)^2 = (\tan A - \tan B)^2 + 4 \tan A \tan B = 24$$

$$\Rightarrow \tan A + \tan B = 2\sqrt{6}$$

$$\therefore \tan A = 2 + \sqrt{6} \text{ and } \tan B = \sqrt{6} - 2$$

$$\cos A = \frac{1}{\sqrt{11 + 4\sqrt{6}}}$$

$$\cos B = \frac{1}{\sqrt{11 - 4\sqrt{6}}}$$

$$\therefore \cos A \cos B = \frac{1}{\sqrt{121 - 96}} = \frac{1}{5}$$

$$\sin A \sin B = \frac{(2 + \sqrt{6})(\sqrt{6} - 2)}{5} = \frac{2}{5}$$

$$\therefore \cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$= \frac{1}{5} - \frac{2}{5} = -\frac{1}{5}$$

$$26. f(\theta) = \sin^2 \theta + \sin^2 \left(\theta + \frac{2\pi}{3} \right) + \sin^2 \left(\theta + \frac{4\pi}{3} \right)$$

$$= \frac{1}{2} \left[3 - \cos 2\theta - \cos \left(\frac{4\pi}{3} + 2\theta \right) - \cos \left(\frac{8\pi}{3} + 2\theta \right) \right]$$

$$= \frac{1}{2} \left[3 - \cos 2\theta - 2 \cos(2\pi + 2\theta) \cos \left(\frac{2\pi}{3} \right) \right]$$

$$= \frac{1}{2} (3 - \cos 2\theta + \cos 2\theta) = \frac{3}{2}$$

$$\Rightarrow f\left(\frac{\pi}{15}\right) = \frac{3}{2}$$

$$27. \text{ We have, } \frac{2}{\cos x} = \frac{1}{\cos(x-y)} + \frac{1}{\cos(x+y)}$$

$$= \frac{2 \cos x \cos y}{\cos^2 x - \sin^2 y}$$

$$\Rightarrow 2 \cos^2 x - 2 \sin^2 y = 2 \cos^2 x \cos y$$

$$\Rightarrow 2 \cos^2 x (1 - \cos y) = 2 \sin^2 y$$

$$\Rightarrow 2 \cos^2 x \times 2 \sin^2 \frac{y}{2} = 8 \sin^2 \frac{y}{2} \cos^2 \frac{y}{2}$$

$$\Rightarrow \cos^2 x = 2 \cos^2 \frac{y}{2}$$

$$\Rightarrow \cos^2 x \sec^2 \frac{y}{2} = 2$$

$$\Rightarrow \cos x \sec \frac{y}{2} = \pm \sqrt{2}$$

$$28. \sin^3 x \sin 3x = \frac{1}{2} \sin^2 x (\cos 2x - \cos 4x)$$

$$= \frac{1}{4} (1 - \cos 2x) (\cos 2x - \cos 4x)$$

$$= \frac{1}{4} (\cos 2x - \cos 4x - \cos^2 2x + \cos 2x \cos 4x)$$

$$= \frac{1}{4} \left(\cos 2x - \cos 4x - \frac{1 + \cos 4x}{2} + \frac{1}{2} (\cos 6x + \cos 2x) \right)$$

$$= \frac{1}{4} \left(\frac{3}{2} \cos 2x - \frac{3}{2} \cos 4x + \frac{1}{2} \cos 6x - \frac{1}{2} \right)$$

$$= -\frac{1}{8} + \frac{3}{8} \cos 2x - \frac{3}{8} \cos 4x + \frac{1}{8} \cos 6x \Rightarrow n = 6$$

$$29. A + B = \frac{\pi}{4} \Rightarrow \tan(A + B) = 1$$

$$\Rightarrow \tan A + \tan B = 1 - \tan A \tan B$$

$$\Rightarrow \tan A + \tan B + \tan A \tan B + 1 = 2$$

$$\Rightarrow (1 + \tan A)(1 + \tan B) = 2$$

$$30. \frac{\sin x}{\sin y} = \frac{1}{2}, \frac{\cos x}{\cos y} = \frac{3}{2} \Rightarrow \frac{\tan x}{\tan y} = \frac{1}{3}$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{4 \tan x}{1 - 3 \tan^2 x}$$

Also, $\sin y = 2 \sin x$ and $\cos y = \frac{2}{3} \cos x$

$$\Rightarrow \sin^2 y + \cos^2 y = 4 \sin^2 x + \frac{4 \cos^2 x}{9} = 1$$

$$\Rightarrow 27 \tan^2 x = 5 \Rightarrow \tan x = \frac{\sqrt{5}}{3\sqrt{3}}$$

$$\Rightarrow \tan(x + y) = \sqrt{15}$$

$$31. \text{ Given, } \angle C = \frac{2\pi}{3} \Rightarrow A + B = \frac{\pi}{3}$$

Now, $\cos^2 A + \cos^2 B - \cos A \cos B$

$$= \frac{1}{2} [2 + \cos 2A + \cos 2B - \cos(A + B) - \cos(A - B)]$$

$$= \frac{1}{2} \left[2 + 2 \cos(A + B) \cos(A - B) - \frac{1}{2} - \cos(A - B) \right]$$

$$= \frac{1}{2} \left[\frac{3}{2} + \cos(A - B) - \cos(A - B) \right] = \frac{3}{4}$$

$$32. \text{ Let } I = \sum_{r=0}^{10} \frac{1}{4} \left(\cos 3 \frac{\pi r}{3} + 3 \cos \frac{\pi r}{3} \right)$$

$$= \sum_{r=0}^{10} \frac{1}{4} \left(\cos \pi r + 3 \cos \frac{\pi r}{3} \right) = \frac{1}{4} (I_1 + I_2)$$

$$\therefore I_1 = \sum_{r=0}^{10} \cos \pi r = 1 - 1 + 1 - 1 + \dots - 1 + 1 = 1$$

$$\text{and } I_2 = \sum_{r=0}^{10} \cos \frac{\pi r}{3} = \frac{\cos \left(\frac{10}{2} \cdot \frac{\pi}{3} \right) \cdot \sin \frac{11\pi}{6}}{\sin \frac{\pi}{6}} = -\frac{1}{2}$$

$$\therefore I = \frac{1}{4} \left(1 - \frac{3}{2} \right) = -\frac{1}{8}$$

33. The given equation is

$$3 \cos A + 2 = 0 \Rightarrow \cos A = -\frac{2}{3}$$

Since, A is angle of triangle.

$\therefore A < \pi \Rightarrow \sin A$ is positive.

$$\therefore \sin A = \sqrt{1 - \cos^2 A} = \frac{\sqrt{5}}{3}$$

$$\Rightarrow \tan A = \frac{\sin A}{\cos A} = \frac{-\sqrt{5}}{2}$$

$$\sin A + \tan A = \frac{-\sqrt{5}}{6}$$

$$\sin A \tan A = \frac{-5}{6}$$

Hence, the equation whose roots are $\sin A$ and $\tan A$, is

$$x^2 - (\sin A + \tan A)x + \sin A \tan A = 0$$

$$\Rightarrow x^2 + \frac{\sqrt{5}}{6}x + \left(\frac{-5}{6} \right) = 0$$

$$\Rightarrow 6x^2 + \sqrt{5}x - 5 = 0$$

$$34. \cot^2 x = \cot(x - y) \cdot \cot(x - z)$$

$$\Rightarrow \cot^2 x = \left(\frac{\cot x \cot y + 1}{\cot y - \cot x} \right) \left(\frac{\cot x \cot z + 1}{\cot z - \cot x} \right)$$

$$\Rightarrow \cot^2 x \cot y \cot z - \cot^3 x \cot y - \cot^3 x \cot z + \cot^4 x = 0$$

$$\Rightarrow \cot^2 x \cot y \cot z + \cot x \cot y + \cot x \cot z + 1 = 0$$

$$\Rightarrow \cot x \cot y (1 + \cot^2 x) + \cot x \cot z (1 + \cot^2 x) + 1 - \cot^4 x = 0$$

$$\Rightarrow \cot x (\cot y + \cot z) (1 + \cot^2 x) + (1 - \cot^2 x) (1 + \cot^2 x) = 0$$

$$\Rightarrow \cot x (\cot y + \cot z) + (1 - \cot^2 x) = 0$$

$$\Rightarrow \frac{\cot^2 x - 1}{2 \cot x} = \frac{1}{2} (\cot y + \cot z)$$

$$\Rightarrow \frac{1}{2} (\cot y + \cot z) = \cot 2x$$

35. Since, $\alpha + \beta = \frac{\pi}{2}$, then

$$\tan \beta = \tan\left(\frac{\pi}{2} - \alpha\right) = \cot \alpha$$

and $\beta + \gamma = \alpha \Rightarrow \gamma = \alpha - \beta$

$$\begin{aligned} \therefore \tan \gamma &= \tan(\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \\ &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \cot \alpha} = \frac{\tan \alpha - \tan \beta}{2} \end{aligned}$$

Hence, $\tan \alpha = \tan \beta + 2 \tan \gamma$

36. We have, $\sin \theta = n \sin(\theta + 2\alpha)$

$$\Rightarrow \frac{\sin(\theta + 2\alpha)}{\sin \theta} = \frac{1}{n}$$

$$\Rightarrow \frac{\sin(\theta + 2\alpha) + \sin \theta}{\sin(\theta + 2\alpha) - \sin \theta} = \frac{1+n}{1-n}$$

$$\Rightarrow \frac{2 \sin(\theta + \alpha) \cos \alpha}{2 \sin \alpha \cos(\theta + \alpha)} = \frac{1+n}{1-n}$$

$$\Rightarrow \tan(\theta + \alpha) = \left(\frac{1+n}{1-n}\right) \tan \alpha$$

37. Given, $\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$... (i)

$$\frac{x}{a} \cos \beta + \frac{y}{b} \sin \beta = 1 \quad \dots (ii)$$

and $\frac{\cos \alpha \cos \beta}{a^2} + \frac{\sin \alpha \sin \beta}{b^2} = 0 \quad \dots (iii)$

From Eqs. (i) and (ii), we get

$$x = a \frac{\sin \alpha - \sin \beta}{\sin(\alpha - \beta)} \text{ and } y = b \frac{\cos \beta - \cos \alpha}{\sin(\alpha - \beta)}$$

$$\text{Now, } x^2 + y^2 = \frac{a^2(\sin \alpha - \sin \beta)^2 + b^2(\cos \beta - \cos \alpha)^2}{\sin^2(\alpha - \beta)}$$

$$= \frac{a^2(\sin^2 \alpha + \sin^2 \beta) + b^2(\cos^2 \alpha + \cos^2 \beta)}{\sin^2(\alpha - \beta)}$$

$$= \frac{-2(a^2 \sin \alpha \sin \beta + b^2 \cos \alpha \cos \beta)}{\sin^2(\alpha - \beta)}$$

$$= \frac{a^2(\sin^2 \alpha + \sin^2 \beta) + b^2(\cos^2 \alpha + \cos^2 \beta)}{\sin^2(\alpha - \beta)}$$

[from Eq. (iii)]

$$= a^2 + b^2 + \frac{\left[a^2\{\sin^2 \alpha + \sin^2 \beta - \sin^2(\alpha - \beta)\} + b^2\{\cos^2 \alpha + \cos^2 \beta - \sin^2(\alpha - \beta)\} \right]}{\sin^2(\alpha - \beta)}$$

$$= a^2 + b^2 + \frac{1}{\sin^2(\alpha - \beta)}$$

$$\begin{aligned} &[a^2(\sin^2 \alpha + \sin^2 \beta - \sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta \\ &+ 2 \sin \alpha \sin \beta \cos \alpha \cos \beta) + b^2\{\cos^2 \alpha(1 - \sin^2 \beta) \\ &+ \cos^2 \beta(1 - \sin^2 \alpha) + 2 \sin \alpha \sin \beta \sin \alpha \sin \beta \cos \alpha \cos \beta\}] \end{aligned}$$

$$= a^2 + b^2 + \frac{\left[2a^2 \sin \alpha \sin \beta \cos(\alpha - \beta) + 2b^2 \cos \alpha \cos \beta \cos(\alpha - \beta) \right]}{\sin^2(\alpha - \beta)}$$

$$= a^2 + b^2 + \frac{2a^2 b^2 \cos(\alpha - \beta)}{\sin^2(\alpha - \beta)} \left[\frac{\sin \alpha \sin \beta}{b^2} + \frac{\cos \alpha \cos \beta}{a^2} \right]$$

[from Eq. (iii)]

$$= a^2 + b^2 + 0 = a^2 + b^2$$

Thus, $x^2 + y^2 = a^2 + b^2$

Now, $x^2 + y^2 = a^2 + b^2$

$$\Rightarrow x^2 - a^2 = -(y^2 - b^2)$$

$$\Rightarrow \frac{x^2 - a^2}{y^2 - b^2} = -1$$

$$\Rightarrow \frac{b^2(x^2 - a^2)}{a^2(y^2 - b^2)} = -\frac{b^2}{a^2} = \tan \alpha \tan \beta \text{ [from Eq. (iii)]}$$

38. $2x = \cos(\alpha - \beta - \gamma + \delta) - \cos(\alpha - \beta + \gamma - \delta)$

$$2y = \cos(\beta - \gamma - \alpha + \delta) - \cos(\beta - \gamma + \alpha - \delta)$$

and similarly for $2z$,

On adding, we get $2x + 2y + 2z = 0$

$$\Rightarrow x + y + z = 0$$

39. $\cos\left(\frac{\pi}{4} - x\right) \cos\left(\frac{\pi}{4} - y\right) - \sin\left(\frac{\pi}{4} - x\right) \sin\left(\frac{\pi}{4} - y\right)$

Let $\frac{\pi}{4} - x = A$ and $\frac{\pi}{4} - y = B$

Then, $\cos A \cos B - \sin A \sin B = \cos(A + B)$

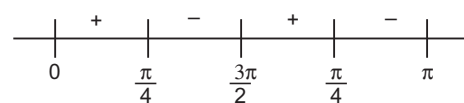
$$= \left(\cos \frac{\pi}{4} - x + \frac{\pi}{4} - y \right)$$

$$= \cos\left(\frac{\pi}{2} - x - y\right) = \cos\left[\frac{\pi}{2} - (x + y)\right] = \sin(x + y)$$

40. $\sin x \cos^3 x > \cos x \sin^3 x$

$$\Rightarrow \sin x \cos x (\cos^2 x - \sin^2 x) > 0$$

$$\Rightarrow \sin x \cos x \cdot \cos 2x > 0 \Rightarrow \cos x \cos 2x > 0$$



$$\therefore x \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{2}, \frac{3\pi}{4}\right)$$

41. We have,

$$x = \sum_{n=0}^{\infty} \cos^{2n} \phi = 1 + \cos^2 \phi + \cos^4 \phi + \dots$$

$$= \frac{1}{1 - \cos^2 \phi} = \frac{1}{\sin^2 \phi}$$

Similarly, $y = \frac{1}{1 - \sin^2 \phi} = \frac{1}{\cos^2 \phi}$

and $z = \frac{1}{1 - \sin^2 \phi \cos^2 \phi}$

$$= \frac{1}{1 - \frac{1}{x} \cdot \frac{1}{y}} = \frac{xy}{xy - 1} \Rightarrow xyz = xy + z$$

Also, $x + y = \frac{1}{\sin^2 \phi} + \frac{1}{\cos^2 \phi} = \frac{1}{\sin^2 \phi \cos^2 \phi} = xy$

Thus, $xyz = xy + z = x + y + z$

42. We have, $\frac{x}{\cos \theta} = \frac{y}{\cos\left(\theta - \frac{2\pi}{3}\right)} = \frac{z}{\cos\left(\theta + \frac{2\pi}{3}\right)}$

Therefore, each ratio

$$= \frac{x + y + z}{\cos \theta + \cos\left(\theta - \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{2\pi}{3}\right)}$$

$$= \frac{x + y + z}{\cos \theta + 2 \cos \theta \cos \frac{2\pi}{3}} = \frac{x + y + z}{0} = 0$$

43. Given that, $\cos \theta = \frac{1}{2} \left(x + \frac{1}{x} \right)$

$$\Rightarrow x + \frac{1}{x} = 2 \cos \theta \quad \dots(i)$$

We know that,

$$\begin{aligned} x^2 + \frac{1}{x^2} &= \left(x + \frac{1}{x} \right)^2 - 2 \\ &= (2 \cos \theta)^2 - 2 = 4 \cos^2 \theta - 2 \\ &= 2 \cos 2\theta \quad [\text{from Eq. (i)}] \end{aligned}$$

$$\therefore \frac{1}{2} \left(x^2 + \frac{1}{x^2} \right) = \frac{1}{2} \times 2 \cos 2\theta = \cos 2\theta$$

44. Maximum value of $\cos \theta = 1$

So, the equation can have solution only when

$$\cos x = 1, \cos y = 1$$

$$\Rightarrow x = 0, y = 0$$

$$\therefore \cos(x - y) = \cos 0 = 1$$

45. Given equations may be written as

$$\cos x + \cos y = -\cos \alpha$$

$$\text{and } \sin x + \sin y = -\sin \alpha$$

$$\Rightarrow 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) = -\cos \alpha \quad \dots(i)$$

$$\text{and } 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) = -\sin \alpha \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\frac{2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)}{2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)} = \frac{\cos \alpha}{\sin \alpha}$$

$$\Rightarrow \cot \left(\frac{x+y}{2} \right) = \cot \alpha$$

46. We have, $\alpha + \beta - \gamma = \pi$

$$\text{Now, } \sin^2 \alpha + \sin^2 \beta - \sin^2 \gamma$$

$$= \sin^2 \alpha + \sin(\beta - \gamma) \sin(\beta + \gamma)$$

$$= \sin^2 \alpha + \sin(\pi - \alpha) \sin(\beta + \gamma) [\because \alpha + \beta - \gamma = \pi]$$

$$= \sin^2 \alpha + \sin \alpha \sin(\beta + \gamma)$$

$$= \sin \alpha [\sin \alpha + \sin(\beta + \gamma)]$$

$$= \sin \alpha [\sin \{ \pi - (\beta - \gamma) \} + \sin(\beta + \gamma)]$$

$$= \sin \alpha [\sin(\beta - \gamma) + \sin(\beta + \gamma)]$$

$$= \sin \alpha [2 \sin \beta \cos \gamma]$$

$$= 2 \sin \alpha \sin \beta \cos \gamma$$

47. We have, $\alpha + \beta + \gamma = 2\pi$

$$\Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} + \frac{\gamma}{2} = \pi$$

$$\Rightarrow \frac{\alpha}{2} + \frac{\beta}{2} = \pi - \frac{\gamma}{2}$$

$$\Rightarrow \tan \left(\frac{\alpha}{2} + \frac{\beta}{2} \right) = \tan \left(\pi - \frac{\gamma}{2} \right)$$

$$\Rightarrow \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = -\tan \frac{\gamma}{2}$$

$$\Rightarrow \tan \frac{\alpha}{2} + \tan \frac{\beta}{2} + \tan \frac{\gamma}{2} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$$

48. $\sin x + \cos x = \frac{\sqrt{7}}{2}$

$$\Rightarrow \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{\sqrt{7}}{2}$$

$$\Rightarrow (\sqrt{7} + 2) \tan^2 \frac{x}{2} - 4 \tan \frac{x}{2} + (\sqrt{7} - 2) = 0$$

$$\Rightarrow \tan \frac{x}{2} = \frac{4 \pm \sqrt{16 - 4(7-4)}}{2(\sqrt{7} + 2)} = \frac{1}{\sqrt{7} + 2} \text{ as } \frac{x}{2} < \frac{\pi}{8}$$

$$\Rightarrow \tan \frac{x}{2} = \frac{\sqrt{7} - 2}{3}$$

49. $\sin B = \frac{1}{5} \sin(2A + B) \Rightarrow \frac{\sin(2A + B)}{\sin B} = 5$

$$\Rightarrow \frac{\sin(2A + B) + \sin B}{\cos(2A + B) - \sin B} = \frac{6}{4}$$

$$\Rightarrow \frac{\sin(A + B) \cos A}{\cos(A + B) \sin A} = \frac{3}{2}$$

$$\Rightarrow \frac{\tan(A + B)}{\tan A} = \frac{3}{2}$$

50. If $\tan \alpha = \frac{1}{7}$ and $\sin \beta = \frac{1}{\sqrt{10}}$

$$\sin 2\beta = 2 \sin \beta \cos \beta = 2 \cdot \frac{1}{\sqrt{10}} \cdot \frac{3}{\sqrt{10}} = \frac{3}{5}$$

$$\tan(\alpha + 2\beta) = \frac{\tan \alpha + \tan 2\beta}{1 - \tan \alpha \tan 2\beta} = \frac{\frac{1}{7} + \frac{3}{4}}{1 - \frac{1}{7} \cdot \frac{3}{4}} = \frac{25}{25} = 1$$

$$\therefore 2\beta = \frac{\pi}{4} - \alpha$$

51. $\frac{3 \cos \theta + \cos 3\theta}{3 \sin \theta - \sin 3\theta} = \frac{4 \cos^3 \theta}{4 \sin^3 \theta} = \cot^3 \theta$

52. $3 \sin \alpha = 5 \sin \beta \Rightarrow \frac{\sin \alpha}{\sin \beta} = \frac{5}{3}$

$$\Rightarrow \frac{\sin \alpha + \sin \beta}{\sin \alpha - \sin \beta} = \frac{8}{2} \Rightarrow \frac{\tan \left(\frac{\alpha + \beta}{2} \right)}{\tan \left(\frac{\alpha - \beta}{2} \right)} = 4$$

53. $\cos 2B = \frac{\cos(A + C)}{\cos(A - C)}$

$$\Rightarrow \frac{1 - \cos 2B}{1 + \cos 2B} = \frac{\cos(A - C) - \cos(A + C)}{\cos(A - C) + \cos(A + C)}$$

$$\Rightarrow \frac{2 \sin^2 B}{2 \cos^2 B} = \frac{2 \sin A \sin C}{2 \cos A \cos C}$$

$$\Rightarrow \tan A \tan C = \tan^2 B$$

Hence, $\tan A, \tan B, \tan C$ are in GP.

54. $a \cos 2x + b \sin 2x = a \frac{1 - \tan^2 x}{1 + \tan^2 x} + b \frac{2 \tan x}{1 + \tan^2 x}$

$$= a \cdot \frac{1 - \frac{b^2}{a^2}}{1 + \frac{b^2}{a^2}} + b \cdot \frac{\frac{2b}{a}}{1 + \frac{b^2}{a^2}} = a \left(\frac{a^2 - b^2}{a^2 + b^2} \right) + \frac{2ab^2}{a^2 + b^2}$$

$$= \left(\frac{a}{a^2 + b^2} \right) (a^2 - b^2 + 2b^2) = a$$

55. Let $y = \cos x \cos(x+2) - \cos^2(x+1)$

$$\begin{aligned} &= \frac{1}{2} [\cos(2x+2) + \cos 2] - \cos^2(x+1) \\ &= \frac{1}{2} [2\cos^2(x+1) - 1 + \cos 2] - \cos^2(x+1) \\ &= -\frac{1}{2} (1 - \cos 2) = -\frac{1}{2} (2\sin^2 1) = -\sin^2 1 \end{aligned}$$

This shows that $y = -\sin^2 1$ is a straight line which is parallel to X-axis and clearly passes through the point $\left(\frac{\pi}{2}, -\sin^2 1\right)$.

56. From the given identity, we have

$$\begin{aligned} &\alpha[4\cos^3 \theta - 3\cos \theta]^2 + \beta \cos^4 \theta \\ &\quad = 16\cos^6 \theta + 9\cos^2 \theta \\ \Rightarrow &16\alpha \cos^6 \theta + (\beta - 24\alpha) \cos^4 \theta + 9\alpha \cos^2 \theta \\ &\quad = 16\cos^6 \theta + 9\cos^2 \theta \\ \Rightarrow &\alpha = 1 \text{ and } \beta - 24\alpha = 0 \Rightarrow \alpha = 1 \text{ and } \beta = 24 \end{aligned}$$

57. Given that,

$$\sin n\theta = \sum_{r=0}^n b_r \sin^r \theta = b_0 + b_1 \sin \theta + b_2 \sin^2 \theta + \dots + b_n \sin^n \theta \quad \dots (i)$$

Putting $\theta = 0$ in Eq. (i), we get $0 = b_0$

Again, Eq. (i) can be written as $\sin n\theta = \sum_{r=0}^n b_r \sin^r \theta$

$$\Rightarrow \frac{\sin n\theta}{\sin \theta} = \sum_{r=1}^n b_r \sin^{r-1} \theta$$

Taking limit as $\theta \rightarrow 0$, we get

$$\begin{aligned} &\lim_{\theta \rightarrow 0} \frac{\sin n\theta}{\sin \theta} = b_1 \\ \Rightarrow &\lim_{\theta \rightarrow 0} n \left(\frac{\sin \theta}{\theta} \right) \left(\frac{\theta}{\sin \theta} \right) = b_1 \\ \Rightarrow &n = b_1 \\ \Rightarrow &b_0 = 0, b_1 = n \end{aligned}$$

58. $x^2 + y^2 = X^2 + Y^2$,

$$\begin{aligned} xy &= (X^2 - Y^2) \sin \theta \cdot \cos \theta - XY(\cos^2 \theta - \sin^2 \theta) \\ x^2 + 4xy + y^2 &= X^2 + Y^2 + 2(X^2 - Y^2) \sin 2\theta \\ &\quad - 2XY \cos 2\theta \\ &= (1 + 2 \sin 2\theta) X^2 + (1 - 2 \sin 2\theta) Y^2 - 2 \cos 2\theta \cdot XY \end{aligned}$$

According to the question,

$$\begin{aligned} a &= 1 + 2 \sin 2\theta, b = 1 - 2 \sin 2\theta, \cos 2\theta = 0 \\ \cos 2\theta &= 0 \end{aligned}$$

$$\Rightarrow \theta = \frac{\pi}{4}, \text{ then}$$

$$a = 1 + 2 \sin \frac{\pi}{2}, b = 1 - 2 \sin \frac{\pi}{2}$$

$$\therefore a = 3, b = -1$$

59. $\tan \theta = \lambda \Rightarrow \frac{2 \tan \theta / 2}{1 - \tan^2 \theta / 2} = \lambda$

$$\Rightarrow \lambda \tan^2 \frac{\theta}{2} + 2 \tan \frac{\theta}{2} - \lambda = 0$$

$$\Rightarrow \tan \frac{\theta_1}{2} \cdot \tan \frac{\theta_2}{2} = -1$$

60. Since, $AM \geq GM$

$$\frac{1}{2} \left[\sqrt{x^2 + x} + \frac{\tan^2 \alpha}{\sqrt{x^2 + x}} \right] \geq \sqrt{\sqrt{x^2 + x} \times \frac{\tan^2 \alpha}{\sqrt{x^2 + x}}} = |\tan \alpha|$$

$$61. \frac{2}{\tan \frac{B}{2}} = \frac{1}{\tan \frac{A}{2}} + \frac{1}{\tan \frac{C}{2}}$$

$$\Rightarrow 2 \tan \frac{A}{2} \tan \frac{C}{2} = \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{B}{2} \tan \frac{A}{2}$$

$$\Rightarrow 2 \tan \frac{A}{2} \tan \frac{C}{2} = \tan \frac{B}{2} \left(\tan \frac{A}{2} + \tan \frac{C}{2} \right)$$

$$\Rightarrow \frac{2 \tan \frac{A}{2} \tan \frac{C}{2}}{1 - \tan \frac{A}{2} \tan \frac{C}{2}} = \tan \frac{B}{2} \left(\frac{\tan \frac{A}{2} + \tan \frac{C}{2}}{1 - \tan \frac{A}{2} \tan \frac{C}{2}} \right)$$

$$= \tan \frac{B}{2} \tan \left(\frac{A}{2} + \frac{C}{2} \right)$$

$$= \tan \frac{B}{2} \tan \left(\frac{\pi}{2} - \frac{B}{2} \right)$$

$$\Rightarrow \frac{2 \tan \frac{A}{2} \tan \frac{C}{2}}{1 - \tan \frac{A}{2} \tan \frac{C}{2}} = 1$$

$$\Rightarrow 2 \tan \frac{A}{2} \tan \frac{C}{2} = 1 - \tan \frac{A}{2} \tan \frac{C}{2}$$

$$\Rightarrow \tan \frac{A}{2} \tan \frac{C}{2} = \frac{1}{3}$$

$$\Rightarrow \cot \frac{A}{2} \cot \frac{C}{2} = 3$$

62. The given expression

$$\begin{aligned} &= \cos^2 \phi + \cos^2(a + \phi) - [\cos(a + \phi) \\ &\quad + \cos(a - \phi)] \cos(a + \phi) \\ &= \cos^2 \phi - \cos(a + \phi) \cos(a - \phi) \\ &= \cos^2 \phi - (\cos^2 \phi - \sin^2 a) = \sin^2 a \end{aligned}$$

which is independent of ϕ .

63. $\cos x - \sin \alpha \cot \beta \sin x = \cos \alpha$

$$\Rightarrow \sin \beta \cos x - \sin \alpha \cos \beta \sin x = \cos \alpha \sin \beta$$

$$\begin{aligned} \Rightarrow &\sin \beta \left(1 - \tan^2 \frac{x}{2} \right) - \sin \alpha \cos \beta \cdot 2 \tan \frac{x}{2} \\ &= \cos \alpha \sin \beta \left(1 + \tan^2 \frac{x}{2} \right) \end{aligned}$$

$$\Rightarrow \tan^2 \frac{x}{2} (-\sin \beta - \cos \alpha \sin \beta) - \sin \alpha \cos \beta \cdot 2 \tan \frac{x}{2} + \sin \beta (1 - \cos \alpha) = 0$$

$$\Rightarrow \tan \frac{x}{2} = \frac{-2 \sin \alpha \cos \beta \pm \sqrt{4 [\sin^2 \alpha \cos^2 \beta + \sin^2 \beta]}}{2 \sin \beta (1 + \cos \alpha)}$$

$$= \frac{-\sin \alpha \cos \beta \pm \sqrt{\sin^2 \alpha (\sin^2 \beta + \cos^2 \beta)}}{\sin \beta (1 + \cos \alpha)}$$

$$= \frac{-\sin \alpha \cos \beta \pm \sin \alpha}{\sin \beta (1 + \cos \alpha)}$$

$$= \frac{\sin \alpha (-\cos \beta \pm 1)}{\sin \beta (1 + \cos \alpha)}$$

$$= \tan \frac{\beta}{2} \tan \frac{\alpha}{2} \text{ or } -\tan \frac{\alpha}{2} \cot \frac{\beta}{2}$$

$$64. 2 \cos\left(\frac{A+B}{2}\right) \cdot \cos\left(\frac{A-B}{2}\right) = 1 \text{ or } 2 \cdot \frac{\sqrt{3}}{2} \cdot \cos\left(\frac{A-B}{2}\right) = 1$$

$$\therefore \cos\left(\frac{A-B}{2}\right) = \frac{1}{\sqrt{3}}$$

$$\therefore \cos(A-B) = 2 \cos^2\left(\frac{A-B}{2}\right) - 1 = \frac{2}{3} - 1 = -\frac{1}{3}$$

$$|\cos A - \cos B| = \left| 2 \sin\left(\frac{A+B}{2}\right) \cdot \sin\left(\frac{A-B}{2}\right) \right|$$

$$= 2 \cdot \frac{1}{2} \sqrt{1 - \frac{1}{3}} = \sqrt{\frac{2}{3}}$$

$$65. I(n) = 2 \cos nx \Rightarrow I(1) \cdot I(n+1) - I(n)$$

$$= 4 \cos x \cos(n+1)x - 2 \cos nx$$

$$= 2[2 \cos(n+3)x \cdot \cos x - \cos nx]$$

$$= 2[\cos(n+2)x + \cos nx - \cos nx]$$

$$= 2 \cos(n+2)x = I(n+2)$$

$$66. D \geq 0 \Rightarrow \cos^2 p - 4 \sin p \cdot (\cos p - 1) \geq 0$$

$$\text{or } \cos^2 p + 4 \sin p(1 - \cos p) \geq 0 \quad \dots(i)$$

$$\text{As, } \cos^2 p \geq 0 \text{ and } 1 - \cos p \geq 0$$

$$\text{Eq. (i) will hold for all } \sin p \geq 0. \text{ So, } p \in [0, \pi]$$

$$67. \cos \alpha + \cos \beta = -a, \cos \alpha \cdot \cos \beta = b$$

$$\text{and } \sin \alpha + \sin \beta = -p, \sin \alpha \cdot \sin \beta = q$$

$$\therefore a^2 + p^2 = 2 + \cos(\alpha - \beta)$$

$$\text{and } b + q = \cos(\alpha - \beta)$$

$$\therefore a^2 + p^2 = 2 + b + q$$

$$68. f(\theta) = |\sin \theta| + |\cos \theta|, \forall \theta \in R. \text{ Clearly, } f(\theta) > 0.$$

$$\text{Also, } f^2(\theta) = \sin^2 \theta + \cos^2 \theta + 2 \sin \theta \cdot \cos \theta$$

$$= 1 + |\sin 2\theta|$$

$$0 \leq |\sin 2\theta| \leq 1$$

$$\Rightarrow 1 \leq f^2(\theta) \leq 2$$

$$\Rightarrow 1 \leq f(\theta) \leq \sqrt{2}$$

$$69. f(\theta) = (\sin \theta + \csc \theta)^2 + (\cos \theta + \sec \theta)^2$$

$$= \sin^2 \theta + \cos^2 \theta + \csc^2 \theta + \sec^2 \theta + 4$$

$$= 5 + \frac{1}{\sin^2 \theta} + \frac{1}{\cos^2 \theta}$$

$$= 5 + \frac{4}{4 \sin^2 \theta \cos^2 \theta}$$

$$= 5 + \frac{4}{\sin^2 2\theta} \geq 9$$

$$70. f(\theta) = \sin^4 \theta + \cos^4 \theta + 1$$

$$= (\sin^2 \theta + \cos^2 \theta)^2 - 2 \sin^2 \theta \cdot \cos^2 \theta + 1$$

$$= 2 - \frac{1}{2} \sin^2 2\theta$$

$$\Rightarrow f(\theta) \in \left[\frac{3}{2}, 2 \right]$$

$$71. f(x) = \cos^6 x + \sin^6 x$$

$$= (\cos^2 x + \sin^2 x)(\sin^4 x + \cos^4 x - \cos^2 x \cdot \sin^2 x)$$

$$= [(\sin^2 x + \cos^2 x)^2 - 3 \sin^2 x \cdot \cos^2 x]$$

$$= 1 - \frac{3}{4} \sin^2 2x$$

$$\Rightarrow f(x) \in \left[\frac{1}{4}, 1 \right]$$

$$72. \sin\left(x + \frac{\pi}{6}\right) + \cos\left(x + \frac{\pi}{6}\right)$$

$$= \sqrt{2} \left[\sin\left(x + \frac{\pi}{6} + \frac{\pi}{4}\right) \right] = \sqrt{2} \sin\left(x + \frac{5\pi}{12}\right) \leq \sqrt{2}$$

$$\text{Equality holds, when } x + \frac{5\pi}{12} = \frac{\pi}{2} \text{ i.e. } x = \frac{\pi}{12}$$

Therefore, maximum value of given expression is attained at $x = \frac{\pi}{12}$.

$$73. \text{ Since, } -\sqrt{a^2 + b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2}$$

$$\therefore -\sqrt{74} \leq 7 \cos x + 5 \sin x \leq \sqrt{74}$$

$$\text{So, } -\sqrt{74} < 2k + 1 < \sqrt{74}$$

$$\text{Therefore, } 2k + 1 = \pm 8, \pm 7 \pm 6, \dots, \pm 1, 0$$

$$\text{So, } k = -4, \pm 3, \pm 2, \pm 1, 0 \text{ i.e. 8 values of } k.$$

$$74. f(x) = (\sin^2 \theta + \cos^2 \theta + x) \begin{vmatrix} 1 & \cos^2 \theta & x \\ 1 & x & \sin^2 \theta \\ 1 & \sin^2 \theta & \cos^2 \theta \end{vmatrix}$$

$$= (x+1) \begin{vmatrix} 1 & \cos^2 \theta & x \\ 0 & x - \cos^2 \theta & \sin^2 \theta - x \\ 0 & \sin^2 \theta - \cos^2 \theta & \cos^2 \theta - x \end{vmatrix}$$

$$= (x+1)[(x - \cos^2 \theta)(\cos^2 \theta - x) - (\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta - x)]$$

$$= (x+1)[-x^2 - \cos^4 \theta + 2x \cos^2 \theta - x \cos^2 \theta + x \sin^2 \theta - \sin^4 \theta + \sin^2 \theta \cos^2 \theta]$$

$$= (-1/2)(x+1)[(x - \sin^2 \theta)^2 + (x - \cos^2 \theta)^2 + (\sin^2 \theta - \cos^2 \theta)^2]$$

$$\text{So } f(x) = 0, \text{ if } x = -1$$

$$\text{or } x = \sin^2 \theta = \cos^2 \theta$$

$$\text{If } \sin^2 \theta = \cos^2 \theta, \text{ then } \theta = \frac{\pi}{4} \Rightarrow x = 1/2$$

$$\text{Hence } x = -1, 1/2$$

$$75. 3 \geq 3 \cos \theta \geq -3 \Rightarrow -3 \leq -3 \cos \theta \leq 3$$

$$\text{For } f(\theta) = \frac{1}{2 - 3 \cos \theta}, -1 \leq 2 - 3 \cos \theta \leq 5 \text{ and } \neq 0$$

$$\text{For } -1 \leq 2 - 3 \cos \theta < 0, -1 \geq f(\theta) > -\infty$$

$$\text{For } 0 < 2 - 3 \cos \theta \leq 5, \infty > f(\theta) \geq \frac{1}{5}$$

$$\text{Thus, the range of } f(\theta) \text{ is } (-\infty, -1] \cup \left[\frac{1}{5}, \infty \right).$$

$$76. \therefore f(\theta) = 5 \cos \theta + 3 \cos\left(\theta + \frac{\pi}{3}\right) + 3$$

$$= 5 \cos \theta + \frac{3}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3$$

$$= \frac{13}{2} \cos \theta - \frac{3\sqrt{3}}{2} \sin \theta + 3$$

$$\text{As, } -\sqrt{a^2 + b^2} \leq a \sin \theta + b \cos \theta \leq \sqrt{a^2 + b^2}$$

Thus, range of $f(\theta)$ is $[-4, 10]$.

$$77. a \sin x + b \cos(x + \theta) + b \cos(x - \theta) = d$$

$$\Rightarrow a \sin x + 2b \cdot \cos x \cdot \cos \theta = d$$

$$\Rightarrow |d| \leq \sqrt{a^2 + 4b^2 \cdot \cos^2 \theta}$$

$$\Rightarrow \frac{d^2 - a^2}{4b^2} \leq \cos^2 \theta$$

$$\Rightarrow |\cos \theta| \geq \frac{\sqrt{d^2 - a^2}}{2|b|}$$

78. Given that, $\alpha + \beta + \gamma = \pi$

Taking $\alpha = -\frac{\pi}{2}$, $\beta = -\frac{\pi}{2}$ and $\gamma = 2\pi$

$\therefore \sin \alpha + \sin \beta + \sin \gamma = -1 - 1 + 0 = -2$

But $\sin \alpha + \sin \beta + \sin \gamma \geq -3$ for any α, β, γ .

Hence, the minimum value of $\sin \alpha + \sin \beta + \sin \gamma$ is negative.

79. $y = \tan A \cdot \tan\left(\frac{\pi}{3} - A\right)$

$\Rightarrow \tan^2 A + \sqrt{3}(y-1)\tan A + y = 0$

As $\tan A$ is real, $D \geq 0 \Rightarrow 3(y-1)^2 - 4y \geq 0$

$\therefore 3y^2 - 10y + 3 \geq 0 \Rightarrow y \leq \frac{1}{3}$ or $y \geq 3$

80. We have,

$(a \sin^4 \theta + b)(a + b) = ab \cos^4 \theta$

$\Rightarrow a^2 \sin^4 \theta + ab \sin^4 \theta + ba + b^2 = ab \cos^4 \theta$

$\Rightarrow a^2 \sin^4 \theta + ab + b^2 = ab (\cos^4 \theta - \sin^4 \theta)$

$\Rightarrow a^2 \sin^4 \theta + ab + b^2 = ab (\cos^2 \theta + \sin^2 \theta) \times (\cos^2 \theta - \sin^2 \theta)$

$[\because a^2 - b^2 = (a+b)(a-b)]$

$\Rightarrow a^2 \sin^4 \theta + ab + b^2 = ab \cos 2\theta$

$[\because \cos 2\theta = \cos^2 \theta - \sin^2 \theta]$

$\Rightarrow a^2 \sin^4 \theta + ab + b^2 = ab (1 - 2\sin^2 \theta)$

$\Rightarrow a^2 \sin^4 \theta + ab + b^2 = ab - 2ab \sin^2 \theta$

$\Rightarrow a^2 \sin^4 \theta + 2ab \sin^2 \theta + b^2 = 0$

$\Rightarrow (a \sin^2 \theta + b)^2 = 0$

$\Rightarrow a \sin^2 \theta + b = 0 \Rightarrow \sin^2 \theta = \frac{-b}{a}$

$\Rightarrow \left| \frac{-b}{a} \right| \leq 1 \Rightarrow \left| \frac{b}{a} \right| \leq 1 \Rightarrow |b| \leq |a|$

81. $x^2 - x \sin \theta + a = 0$

As, $D \geq 0$ for real roots.

$\Rightarrow \sin^2 \theta - 4a \geq 0 \Rightarrow 4a \leq \sin^2 \theta \leq 1 \Rightarrow a \leq \frac{1}{4}$

82. $\sin^2 x + a \sin x + 1 = 0$

Let $t = \sin x$, then we have

$\Rightarrow t^2 + at + 1 = 0$

Clearly, $t = 0$ does not satisfy this equation.

Dividing throughout by t , we get $t + \frac{1}{t} = -a$

$\Rightarrow |a| = \left| t + \frac{1}{t} \right| \geq 2 \Rightarrow |a| \geq 2$

Thus, for no real solution, $|a| < 2$.

83. $\sin \theta_1 - \sin \theta_2 = a$, $\cos \theta_1 + \cos \theta_2 = b$

$\Rightarrow a^2 + b^2 = 2 + 2 \cos(\theta_1 + \theta_2)$

$\Rightarrow 0 \leq a^2 + b^2 \leq 4$

84. $x^2 - 2x + 4 = -3 \cos(ax + b)$

$\Rightarrow (x-1)^2 + 3 = -3 \cos(ax + b)$

As, $-1 \leq \cos(ax + b) \leq 1$

and $(x-1)^2 \geq 0$, the above is possible only, if

$\cos(ax + b) = -1$, when $x = 1$

So, $a + b = \pi, 3\pi, 5\pi$, etc., and $3\pi > 6$

85. $\tan^2 x + \sec x - a = 0$

$\Rightarrow \sec^2 x + \sec x - 1 - a = 0$

$\Rightarrow \left(\sec x + \frac{1}{2} \right)^2 = 1 + a + \frac{1}{4} = \frac{5}{4} + a$

Now, $\sec x \geq 1$ or ≤ -1

$\Rightarrow \sec x + \frac{1}{2} \geq \frac{3}{2}$ or $\leq -\frac{1}{2}$

$\Rightarrow \left(\sec x + \frac{1}{2} \right)^2 \geq \frac{1}{4} \Rightarrow \frac{5}{4} + a \geq \frac{1}{4}$

$\Rightarrow a \geq -1, a \in [-1, \infty)$

86. Since, the value of $\sec^2 \theta$ is always ≥ 1 .

$\therefore \frac{4xy}{(x+y)^2} \geq 1 \Rightarrow (x+y)^2 - 4xy \leq 0$

$\Rightarrow (x-y)^2 \leq 0 \Rightarrow (x-y)^2 = 0$

As, $(x-y)^2$ can never be negative.

$\Rightarrow x = y$

Here, the fraction $\frac{4xy}{(x+y)^2}$ is not defined for

$x + y = 0$

$\therefore x + y \neq 0 \Rightarrow x + x \neq 0$

$\Rightarrow 2x \neq 0$ or $x \neq 0$

Hence, $x = y, x \neq 0$

87. $\sin \theta = \frac{x^2 + y^2}{x^2 - y^2} \Rightarrow \left| \frac{x^2 + y^2}{x^2 - y^2} \right| \leq 1$

$\Rightarrow -1 \leq \frac{x^2 + y^2}{x^2 - y^2} \leq 1$

From $\frac{x^2 + y^2}{x^2 - y^2} \leq 1$, we get $\frac{2y^2}{x^2 - y^2} \leq 0$

$\Rightarrow y^2 = 0$ or $x^2 - y^2 \leq 0$... (i)

Also, $-1 \leq \frac{x^2 + y^2}{x^2 - y^2} \Rightarrow \frac{2x^2}{x^2 - y^2} \geq 0$

$\Rightarrow x^2 = 0$ or $x^2 - y^2 > 0$... (ii)

From Eqs. (i) and (ii), we get $x^2 = 0$ and $y^2 = 0$

Thus, $x = 0$ or $y = 0$ is one only possibility.

88. Here, $\sin \theta = -\frac{1}{2}$ and $\tan \theta = \frac{1}{\sqrt{3}}$

As, $a^2 + b^2 = 0 \Rightarrow a = b = 0$

$\sin \theta = -\frac{1}{2} \Rightarrow \theta = m\pi + (-1)^n \left(-\frac{\pi}{6} \right)$

and $\tan \theta = \frac{1}{\sqrt{3}} \Rightarrow \theta = m\pi + \frac{\pi}{6}$

For common values, m must be odd.

i.e. $m = 2n + 1 \Rightarrow \theta = 2n\pi + \frac{7\pi}{6}$

89. $\sec^2 x - \sec^{10} x = 0$

$\Rightarrow \sec^2 x (1 - \sec^8 x) = 0$

$\Rightarrow 1 - \sec^8 x = 0$ as $\sec^2 x \geq 1$

$\Rightarrow \sec^8 x = 1 \Rightarrow \cos^8 x = 1 \Rightarrow \cos x = \pm 1$

$\Rightarrow x = n\pi, n \in \mathbb{I}$

$\Rightarrow x = \{\pi, 2\pi, 3\pi\} \in (0, 10)$

So, the required number is 3.

$$90. \tan x + \cot x = 2 \operatorname{cosec} x \Rightarrow \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = \frac{2}{\sin x}$$

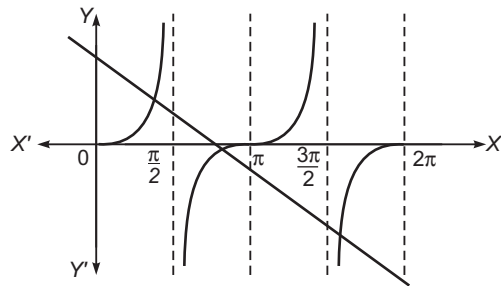
$$\Rightarrow \frac{1}{\sin x \cdot \cos x} = \frac{2}{\sin x}$$

$$\Rightarrow \cos x = \frac{1}{2} \Rightarrow x = \pm \frac{\pi}{3}, \pm \frac{4\pi}{3}, \pm \frac{5\pi}{3}$$

Thus, there are six solutions.

$$91. 3x + 2 \tan x = \frac{5\pi}{2} \Rightarrow \tan x = \frac{5\pi}{4} - \frac{3x}{2}$$

Graphs of $y = \frac{5\pi}{4} - \frac{3x}{2}$ and $y = \tan x$, meet exactly three times in $[0, 2\pi]$.



Thus, there are 3 solutions.

$$92. x^2 + 4 - 2x + 3 \sin(ax + b) = 0$$

$$(x - 1)^2 + 3 + 3 \sin(ax + b) = 0$$

$$\Rightarrow x = 1 \text{ and } \sin(ax + b) = -1$$

$$\Rightarrow \sin(a + b) = -1 \Rightarrow a + b = \frac{3\pi}{2}$$

$$93. \cos^2 \theta = \frac{1}{6} \sin \theta \cdot \tan \theta \text{ or } 6 \cos^3 \theta + \cos^2 \theta - 1 = 0$$

As, $\cos \theta = \frac{1}{2}$ satisfies the equation (by trial)

$$(2 \cos \theta - 1)(3 \cos^2 \theta + 2 \cos \theta + 1) = 0$$

$$\Rightarrow \cos \theta = \frac{1}{2} \text{ [other values of } \cos \theta \text{ are imaginary]}$$

$$\text{So, } \theta = 2n\pi \pm \frac{\pi}{3}, n \in I$$

$$94. 1 = \cos 2x + \sin 2x$$

$$\Rightarrow \sin\left(2x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2x + \frac{\pi}{4} = n\pi + (-1)^n \frac{\pi}{4} \Rightarrow x = n\pi, n\pi + \frac{\pi}{4}$$

$$95. \text{ Here, } \tan \theta + \frac{-1 + \tan \theta}{1 + \tan \theta} = 2$$

$$\Rightarrow \tan \theta + \tan^2 \theta - 1 + \tan \theta = 2 + 2 \tan \theta$$

$$\Rightarrow \tan^2 \theta = 3 \Rightarrow \theta = n\pi \pm \frac{\pi}{3}, n \in I$$

$$96. 1 - \cos x = (\sqrt{2} - 1) \sin x$$

$$\Rightarrow \sin \frac{x}{2} \left\{ \sin \frac{x}{2} - (\sqrt{2} - 1) \cos \frac{x}{2} \right\} = 0$$

$$\therefore \sin \frac{x}{2} = 0 \text{ or } \tan \frac{x}{2} = \sqrt{2} - 1 = \tan \frac{45^\circ}{2}$$

$$\Rightarrow \frac{x}{2} = n\pi \text{ or } \frac{x}{2} = n\pi + \frac{\pi}{8}$$

$$\therefore x = 2n\pi, 2n\pi + \frac{\pi}{4}$$

$$97. \tan x = n \tan y, \cos(x - y) = \cos x \cos y + \sin x \sin y$$

$$\Rightarrow \cos(x - y) = \cos x \cos y (1 + \tan x \cdot \tan y)$$

$$= \cos x \cos y (1 + n \tan^2 y)$$

$$\Rightarrow \sec^2(x - y) = \frac{\sec^2 x \sec^2 y}{(1 + n \tan^2 y)^2} = \frac{(1 + \tan^2 x)(1 + \tan^2 y)}{(1 + n \tan^2 y)^2}$$

$$= \frac{(1 + n^2 \tan^2 y)(1 + \tan^2 y)}{(1 + n \tan^2 y)^2}$$

$$= 1 + \frac{(n - 1)^2 \tan^2 y}{(1 + n \tan^2 y)^2}$$

$$\text{Now, } \left(\frac{1 + n \tan^2 y}{2} \right)^2 \geq n \tan^2 y \quad [\because \text{AM} \geq \text{GM}]$$

$$\Rightarrow \frac{\tan^2 y}{(1 + n \tan^2 y)^2} \leq \frac{1}{4n}$$

$$\Rightarrow \sec^2(x - y) \leq 1 + \frac{(n - 1)^2}{4n} = \frac{(n + 1)^2}{4n}$$

$$98. \cos^2 x = 2 \cos x \cdot (1 - 3 \cos^2 x)$$

$$\Rightarrow \cos x \{6 \cos^2 x + \cos x - 2\} = 0$$

$$\therefore \cos x = 0, \frac{1}{2}, -\frac{2}{3}$$

The two smallest positive values of x are $\frac{\pi}{3}$ and $\frac{\pi}{2}$.

$$\Rightarrow |\alpha - \beta| = \frac{\pi}{6}$$

$$99. \frac{\sin x + i \cos x}{1 + i} = \frac{(1 - i)(\sin x + i \cos x)}{(1 + i)(1 - i)}$$

$$= \frac{\sin x + \cos x + i(\cos x - \sin x)}{2}$$

which is purely imaginary.

$$\Rightarrow \sin x + \cos x = 0 \Rightarrow \tan x = -1$$

$$\therefore x = n\pi - \frac{\pi}{4}$$

$$100. \text{ Since, } 1 + \sin x \sin^2 \frac{x}{2} = 0$$

$$\therefore 1 + \sin x \left(\frac{1 - \cos x}{2} \right) = 0$$

$$\Rightarrow 2 + \sin x - \sin x \cos x = 0$$

$$\Rightarrow \sin 2x - 2 \sin x = 4$$

which is not possible for any x in $[-\pi, \pi]$.

$$101. \text{ Clearly, } 1 + \sin x \geq 0$$

So, the equation is

$$\cos x - \sin x = 1 \Rightarrow \cos\left(x + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4} \Rightarrow x = 0, \frac{3\pi}{2}, 2\pi, \frac{7\pi}{2}, \dots$$

$$\text{As } 0 \leq x \leq 3\pi \Rightarrow x = 0, \frac{3\pi}{2}, 2\pi$$

Thus, there are 3 solutions.

$$102. 2^{\frac{1}{1 - |\cos x|}} = 2^2 \Rightarrow \frac{1}{1 - |\cos x|} = 2 \Rightarrow |\cos x| = \frac{1}{2}$$

$$\Rightarrow \cos x = \pm \frac{1}{2}$$

$$\therefore x = 2n\pi \pm \frac{\pi}{3}, 2n\pi \pm \left(\pi - \frac{\pi}{3}\right)$$

$$= 2n\pi \pm \frac{\pi}{3}, (2n \pm 1)\pi \mp \frac{\pi}{3} = n\pi \pm \frac{\pi}{3}$$

103. As, $x \neq \frac{n\pi}{2}$, we get $\cos x \neq 0, 1, -1$

$$\text{So, } \sin^2 x - 3 \sin x + 2 = 0$$

$$\Rightarrow \sin x = 1, 2$$

which is not possible as $x \neq \frac{n\pi}{2}$.

$\therefore$ No solution.

104. Max $\cos \theta = 1$. So, $\cos x \cdot \sin y = 1$

$$\Rightarrow \cos x = 1, \sin y = 1$$

$$\text{or } \cos x = -1, \sin y = -1$$

$$\cos x = 1, \sin y = 1$$

$$\Rightarrow x = 0, 2\pi$$

$$\text{and } y = \frac{\pi}{2}, \frac{5\pi}{2} \quad [\text{from the equation}]$$

$$\cos x = -1, \sin y = -1 \Rightarrow x = \pi, 3\pi$$

$$\text{and } y = \frac{3\pi}{2} \quad [\text{from the equation}]$$

$\therefore$ Required number of ordered pair
 $= 2 \times 2 + 2 \times 1 = 6$

105. $\sin x \cdot \cos y = 1 \Rightarrow \sin x = 1, \cos y = 1$

$$\text{or } \sin x = -1, \cos y = -1$$

If $\sin x = 1$, then $\cos y = 1$

$$\Rightarrow x = \frac{\pi}{2}, y = 0, 2\pi$$

If $\sin x = -1$, then $\cos y = -1$

$$\Rightarrow x = \frac{3\pi}{2}, y = \pi$$

Thus, possible ordered pairs are

$$\left(\frac{\pi}{2}, 0\right), \left(\frac{\pi}{2}, 2\pi\right), \left(\frac{3\pi}{2}, \pi\right).$$

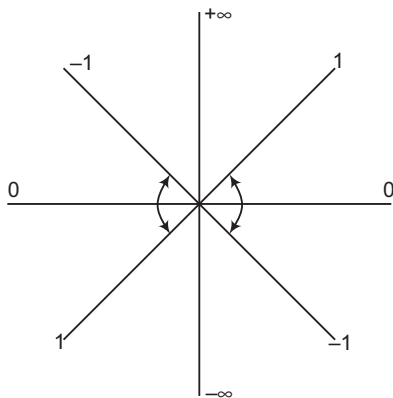
106. $\sin 2x + \cos 4x = 2$

$$\Rightarrow \sin 2x = 1, \cos 4x = 1$$

$$\therefore 1 - 2 \sin^2 2x = 1 \quad \text{or} \quad 1 - 2 = 1 \quad [\text{absurd}]$$

Thus, no solution.

107. $-1 \leq \tan \theta \leq 1$ is shown below:



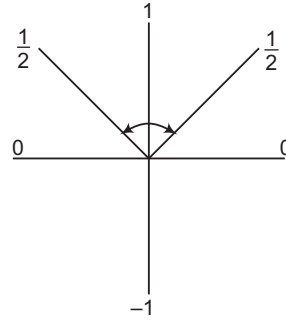
From the figure, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

$$\text{or } -\pi \leq x \leq -\frac{3\pi}{4}$$

$$\text{or } \frac{3\pi}{4} \leq x \leq \pi$$

$$\therefore x \in \left[-\pi, -\frac{3\pi}{4}\right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right]$$

108. Here, $(2 \sin x - 1)(2 \sin x - 3) \leq 0$. But $2 \sin x - 3$ is always negative.



$$\therefore 2 \sin x - 1 \geq 0 \quad \text{i.e. } \sin x \geq \frac{1}{2}$$

$$\text{From the figure, } \frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$$

109. $\sin x \cdot \cos x (\cos^2 x - \sin^2 x) > 0$

$$\Rightarrow \frac{1}{2} \sin 2x \cdot \cos 2x > 0$$

$$\Rightarrow \sin 4x > 0$$

$$\therefore 0 < 4x < \pi$$

$$\Rightarrow 0 < x < \frac{\pi}{4}$$

110. $\cos^2 x = 2 \cos x (3 \sin^2 x - 2)$

$$\Rightarrow \cos x [\cos x - 2 \{3(1 - \cos^2 x) - 2\}] = 0$$

$$\Rightarrow \cos x (6 \cos^2 x - 2 + \cos x) = 0$$

$$\Rightarrow \cos x = 0, \text{ which is not possible.}$$

$$\text{or } 6 \cos^2 x + \cos x - 2 = 0$$

$$\Rightarrow \cos x = \frac{1}{2}, -\frac{2}{3}$$

$$\Rightarrow x = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$\text{or } x = \pi - \cos^{-1}\left(\frac{2}{3}\right), \pi + \cos^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow |x_1 - x_3| = \frac{4\pi}{3} \quad \text{or} \quad |x_1 - x_2| = 2 \cos^{-1}\left(\frac{2}{3}\right)$$

$$\Rightarrow |x_1 - x_2|_{\min} = 2 \cos^{-1}\left(\frac{2}{3}\right)$$

111. $3 \sin x + 4 \cos ax = 7$

$$\Rightarrow \sin x = 1, \cos ax = 1$$

$$\Rightarrow x = (4n_1 + 1)\frac{\pi}{2}, ax = 2n_2\pi$$

$$\Rightarrow (4n_1 + 1)\frac{\pi}{2} = \frac{2n_2\pi}{a}$$

$$\Rightarrow a = \frac{4n_2}{4n_1 + 1} = \text{Rational}$$

112. $a_1 + a_2 (2 \cos^2 x - 1) + a_3 (1 - \cos^2 x) = 1$

$$\text{or } (2a_2 - a_3) \cos^2 x + (a_1 - a_2 + a_3 - 1) = 0$$

$$\text{This can hold for all } x, \text{ if } 2a_2 - a_3 = 0$$

$$\text{and } a_1 - a_2 + a_3 - 1 = 0$$

As there are two equations in three unknowns, so the number of solutions is infinite.

113. Here, $\sin \theta = 1 \pm \sqrt{2}$, but $|\sin \theta| \leq 1$. So, $\sin x = 1 - \sqrt{2}$, which gives two values of θ in each of $[0, 2\pi], (2\pi, 4\pi], (4\pi, 6\pi], \text{ etc.}$

Thus, the least value of n is 4.

114. Here, $(2\sec x + 1)(\sec x - 3) = 0$; but $|\sec x| \geq 1$. So, $\sec x = 3$, which gives two values of θ in each of $[0, 2\pi]$, $(2\pi, 4\pi]$, $(4\pi, 6\pi]$ and one value in $\left(6\pi, 6\pi + \frac{3\pi}{2}\right]$.

Thus, $n = 15$

115. Here, $1 = \sin \alpha \cdot \cos 2\alpha$

$\Rightarrow \sec 2\alpha = \sin \alpha$
which is only possible, if both are 1 or -1.

$$\Rightarrow \alpha = n\pi + (-1)^{n-1} \frac{\pi}{2}$$

116. $5\sin \theta + 3(\sin \theta \cdot \cos \alpha - \cos \theta \cdot \sin \alpha)$

$$= (5 + 3\cos \alpha) \sin \theta - 3\sin \alpha \cdot \cos \theta$$

$$\therefore \theta \in R \min \{5\sin \theta + 3\sin(\theta - \alpha)\}$$

$$= \sqrt{(5 + 3\cos \alpha)^2 + 9\sin^2 \alpha}$$

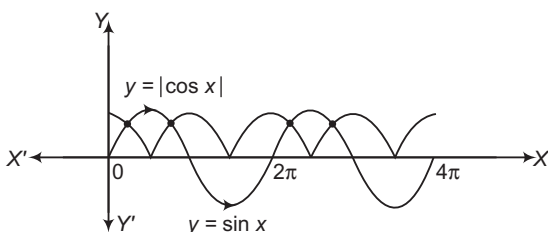
$$= \sqrt{34 + 30\cos \alpha}$$

$\therefore$ The given equation is $34 + 30\cos \alpha = 49$

$$\Rightarrow \cos \alpha = \frac{1}{2}$$

$$\Rightarrow \alpha = 2n\pi \pm \frac{\pi}{3}$$

117.



Clearly, from the graph, there are four solutions.

118. We have,

$$\cos^4 x - (a+2)\cos^2 x - (a+3) = 0$$

$$\Rightarrow \cos^2 x = \frac{(a+2) \pm \sqrt{(a+2)^2 + 4(a+3)}}{2}$$

$$\Rightarrow \cos^2 x = \frac{(a+2) \pm (a+4)}{2} [\because \cos^2 x \neq -1]$$

$$\Rightarrow \cos^2 x = a+3$$

We know, $0 \leq \cos^2 x \leq 1$

$$\Rightarrow 0 \leq a+3 \leq 1$$

$$\Rightarrow -3 \leq a \leq -2$$

$$\text{or } a \in [-3, -2]$$

119. $\sin \pi(x^2 + x) = \sin \pi x^2$

$$\Rightarrow \pi(x^2 + x) = n\pi + (-1)^n \pi x^2$$

$$\therefore x^2 + x = 2m + x^2, \text{ where } n = 2m$$

$$\Rightarrow x = 2m \in \mathbb{Z}$$

$$\text{or } x^2 + x = p' - x^2$$

where, p' is an odd integer.

$$\Rightarrow x^2 + x = p' - x^2$$

For non-integral solution,

$$x = \frac{-1 \pm \sqrt{1+8p'}}{4} = \frac{-1 + \sqrt{1+8p'}}{4}$$

[taking the positive value]

$$= \frac{\sqrt{p}-1}{4}, \text{ where } p = (8p' + 1) \quad [\text{an odd integer}]$$

120. $2\sin \theta = (r^2 - 1)^2 + 2 \geq 2$. But $\max \sin \theta = 1$

$\therefore$ The equation implies $\sin \theta = 1, r^2 = 1$

$$\sin \theta = 1 \Rightarrow \theta = \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2} \quad [\text{from the equation}]$$

$$r^2 = 1 \Rightarrow r = 1, -1$$

$\therefore$ Number of values of pair $(r, \theta) = 2 \times 3 = 6$

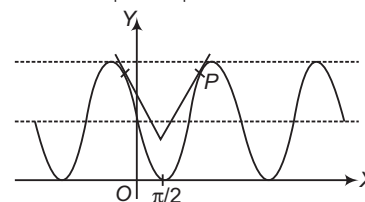
121. $[\sin x] = \cos x \Rightarrow \cos x = -1, 0, 1$

$$x = -\pi, \frac{\pi}{2}, 2\pi \Rightarrow [\sin x] = 0, 1, 0$$

Thus, $[\sin x] = \cos x$ has no solution.

122. Slope of $y = 1 - \sin x$ is $-\cos x$.

$$\text{Slope of } y = \frac{\sqrt{3}}{2} \left| x - \frac{\pi}{2} \right| + a \text{ for } x > \frac{\pi}{2} \text{ is } \frac{\sqrt{3}}{2}.$$



$$\Rightarrow \cos x = -\frac{\sqrt{3}}{2} \Rightarrow x = \frac{7\pi}{6} \Rightarrow P \equiv \left(\frac{7\pi}{6}, \frac{3}{2}\right)$$

If $y = \frac{\sqrt{3}}{2} \left| x - \frac{\pi}{2} \right| + a$ passes through P , then

$$a = \frac{3}{2} - \frac{\pi}{\sqrt{3}} \Rightarrow a > \frac{3}{2} - \frac{\pi}{\sqrt{3}}$$

123. $\sin^2(\theta - \alpha) \cos \alpha = \cos^2(\theta - \alpha) \sin \alpha = m \sin \alpha \cos \alpha$

$$\Rightarrow \frac{\sin^2(\theta - \alpha)}{\sin \alpha} = \frac{\cos^2(\theta - \alpha)}{\cos \alpha} = m$$

$$= \frac{\sin^2(\theta - \alpha) + \cos^2(\theta - \alpha)}{\sin \alpha + \cos \alpha} = \frac{1}{\sin \alpha + \cos \alpha}$$

$$\Rightarrow \sin \alpha + \cos \alpha = \frac{1}{m} \Rightarrow \sin \left(\alpha + \frac{\pi}{4}\right) = \frac{1}{\sqrt{2}m}$$

$$\Rightarrow \left| \frac{1}{\sqrt{2}m} \right| \leq 1 \Rightarrow |m| \geq \frac{1}{\sqrt{2}}$$

124. $\cos x - \frac{\cot \beta \sin x}{2} = \frac{\sqrt{3}}{2}$

$$\Rightarrow \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} - \frac{\cot \beta \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow 1 - \tan^2 \frac{x}{2} - \cot \beta \tan \frac{x}{2} = \frac{\sqrt{3}}{2} \left(1 + \tan^2 \frac{x}{2}\right)$$

$$\Rightarrow (2 + \sqrt{3}) \tan^2 \frac{x}{2} + 2 \cot \beta \tan \frac{x}{2} + (\sqrt{3} - 2) = 0$$

$$\Rightarrow \tan \frac{x}{2} = \frac{-2 \cot \beta \pm \sqrt{4 \cot^2 \beta + 4}}{2(2 + \sqrt{3})}$$

$$\Rightarrow \tan \frac{x}{2} = \frac{-2 \cot \beta \pm 2 \operatorname{cosec} \beta}{2(2 + \sqrt{3})}$$

$$\Rightarrow \tan \frac{x}{2} = \frac{-\cot \beta + \operatorname{cosec} \beta}{(2 + \sqrt{3})}$$

$$\text{or } \tan \frac{x}{2} = \frac{-\cot \beta - \operatorname{cosec} \beta}{(2 + \sqrt{3})}$$

$$\Rightarrow \tan \frac{x}{2} = \tan \frac{\beta}{2} \tan 15^\circ$$

$$\begin{aligned}
 125. n \sin^2 \theta + 2n \cos(\theta + \alpha) \sin \alpha \sin \theta + \cos 2(\alpha + \theta) \\
 = n \sin^2 \theta + n \cos(\theta + \alpha) \{ \cos(\theta - \alpha) - \cos(\theta + \alpha) \} \\
 \quad + 2 \cos^2(\theta + \alpha) - 1 \\
 = n \sin^2 \theta + n (\cos^2 \theta - \sin^2 \alpha) - n \cos^2(\theta + \alpha) \\
 \quad + 2 \cos^2(\alpha + \theta) - 1 \\
 = n \sin^2 \theta + n \cos^2 \theta - n \sin^2 \alpha + (2 - n) \cos^2(\theta + \alpha) - 1 \\
 = (n - 1) - n \sin^2 \alpha + (2 - n) \cos^2(\theta + \alpha) \Rightarrow n = 2
 \end{aligned}$$

$$\begin{aligned}
 126. \text{ Given that, } y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta} \\
 \Rightarrow (y - 1) \cos^2 \theta + y \cos \theta + 1 = 0 \\
 \Rightarrow \cos \theta = -1 \\
 \text{and } \cos \theta = \frac{1}{1 - y} \\
 \Rightarrow -1 < \frac{1}{1 - y} < 1 \\
 \Rightarrow y < 0 \text{ and } y > 2
 \end{aligned}$$

$$\begin{aligned}
 127. [\sin x] = [1 + \sin x] + [1 - \cos x] \\
 \Rightarrow 0 = 2 + [-\cos x] \\
 \Rightarrow [-\cos x] = -2 \\
 \text{Since, } -1 \leq \cos x \leq 1 \\
 \Rightarrow -1 \leq -\cos x \leq 1 \\
 \Rightarrow [-\cos x] \neq -2 \\
 \text{Hence, } [-\cos x] = -2 \text{ has no real solution.}
 \end{aligned}$$

$$\begin{aligned}
 128. \text{ We have, } \sin^{50} x - \cos^{50} x = 1 \\
 \Rightarrow \sin^{50} x = 1 + \cos^{50} x \\
 \text{Since, } \sin^{50} x \leq 1 \text{ and } \cos^{50} x \geq 1, \text{ therefore the two} \\
 \text{sides are equal only, if} \\
 \sin^{50} x = 1 = 1 + \cos^{50} x \text{ i.e. } \sin^{50} x = 1 \\
 \text{and } \cos^{50} x = 0 \\
 \therefore x = n\pi + \frac{\pi}{2}, n \in \mathbb{I}
 \end{aligned}$$

$$\begin{aligned}
 129. \text{ Given equation is } \sec^2(a + 2)x + a^2 - 1 = 0 \\
 \Rightarrow \tan^2(a + 2)x + a^2 = 0 \\
 \Rightarrow \tan^2(a + 2)x = 0 \\
 \text{and } a = 0 \\
 \Rightarrow \tan^2 2x = 0 \\
 \Rightarrow x = 0, \frac{\pi}{2}, -\frac{\pi}{2} \\
 \text{Hence, } (0, 0), \left(0, \frac{\pi}{2}\right), \left(0, -\frac{\pi}{2}\right) \text{ are ordered pair} \\
 \text{satisfying the equation.}
 \end{aligned}$$

$$\begin{aligned}
 130. 1 + \sec \theta = \frac{1 + \cos \theta}{\cos \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{\cos \theta} \\
 \text{Similarly, for others} \\
 f_n(\theta) = \tan \frac{\theta}{2} \cdot \frac{2 \cos^2 \frac{\theta}{2}}{\cos \theta} \cdot \frac{2 \cos^2 \theta}{\cos 2\theta} \dots \frac{2 \cos^2 2^{n-1} \theta}{\cos 2^n \theta} \\
 = \tan \frac{\theta}{2} \cdot 2^{n+1} \cdot (\cos \theta \cos 2\theta \dots \cos 2^{n-1} \theta) \cdot \frac{\cos^2 \frac{\theta}{2}}{\cos 2^n \theta}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{\sin \theta}{\cos 2^n \theta} \cdot 2^n \left\{ \frac{2 \sin \theta \cdot \cos \theta \dots \cos 2^{n-1} \theta}{2 \sin \theta} \right\} = \dots \\
 &= \frac{\sin \theta}{\cos 2^n \theta} \cdot 2^n \cdot \frac{\sin 2^n \theta}{2^n \sin \theta} = \tan 2^n \theta
 \end{aligned}$$

$$\therefore f_2\left(\frac{\pi}{16}\right) = \tan\left(4 \cdot \frac{\pi}{16}\right) = 1, f_3\left(\frac{\pi}{32}\right) = \tan\left(8 \cdot \frac{\pi}{32}\right) = 1$$

and so on.

$$\begin{aligned}
 131. x^2 + y^2 = X^2 + Y^2, xy = (X^2 - Y^2) \sin \theta \cdot \cos \theta \\
 \quad - XY (\cos^2 \theta - \sin^2 \theta) \\
 x^2 + 4xy + y^2 = X^2 + Y^2 + 2(X^2 - Y^2) \sin 2\theta \\
 \quad - 4XY \cos 2\theta \\
 = (1 + 2 \sin 2\theta) X^2 + (1 - 2 \sin 2\theta) Y^2 - 4 \cos 2\theta \cdot XY \\
 \text{According to the question,} \\
 a = 1 + 2 \sin 2\theta, b = 1 - 2 \sin 2\theta, \cos 2\theta = 0 \\
 \cos 2\theta = 0 \Rightarrow \theta = \frac{\pi}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{Then, } a = 1 + 2 \sin \frac{\pi}{2}, b = 1 - 2 \sin \frac{\pi}{2} \\
 \Rightarrow a = 3, b = -1
 \end{aligned}$$

$$\begin{aligned}
 132. \frac{\alpha^2}{1 - \tan^2 x} &= \frac{(\sin^2 x + \alpha^2 - 2)(1 + \tan^2 x)}{1 - \tan^2 x} \\
 \Rightarrow \alpha^2 \cos^2 x &= \sin^2 x + \alpha^2 - 2 \Rightarrow 2 = \sin^2 x (1 + \alpha^2) \\
 \Rightarrow \sin^2 x &= \frac{2}{1 + \alpha^2} \Rightarrow 0 \leq \frac{2}{1 + \alpha^2} \leq 1 \\
 \Rightarrow \alpha^2 &\geq 1 \Rightarrow \alpha \leq -1 \text{ or } \alpha \geq 1
 \end{aligned}$$

$$\begin{aligned}
 133. \cos^3 x \sin 2x &= \frac{\cos 3x + 3 \cos x}{4} \cdot \sin 2x \\
 &= \frac{1}{8} (\sin 5x - \sin x) + \frac{3}{8} (\sin 3x + \sin x) \\
 &= \frac{1}{4} \sin x + \frac{3}{8} \sin 3x + \frac{1}{8} \sin 5x \\
 \therefore n = 5, a_1 &= \frac{1}{4}, a_2 = 0, a_3 = \frac{3}{8}, a_4 = 0, a_5 = \frac{1}{8}
 \end{aligned}$$

$$\begin{aligned}
 134. \sin(x - y) = 1/2 \Rightarrow x - y = 30^\circ \text{ or } 150^\circ \dots (i) \\
 \text{and } \cos(x + y) = 1/2 \Rightarrow x + y = 60^\circ \text{ or } 300^\circ \dots (ii) \\
 \text{Since, } x \text{ and } y \text{ lie between } 0^\circ \text{ and } 180^\circ, \text{ Eqs. (i) and (ii)} \\
 \text{are simultaneously true when } x = 45^\circ, y = 15^\circ \text{ or } \\
 x = 165^\circ, y = 135^\circ. \text{ But for the values given by (b) or (c),} \\
 \text{Eqs. (i) and (ii) do not hold simultaneously.}
 \end{aligned}$$

$$\begin{aligned}
 135. \text{ The given relation can be written as} \\
 \tan \frac{x}{2} = \frac{1}{\sin x} - \sin x \\
 \Rightarrow \tan \frac{x}{2} = \frac{1 + \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}} - \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} \\
 \Rightarrow 2 \tan^2 \frac{x}{2} \left(1 + \tan^2 \frac{x}{2}\right) = 1 \left(1 + \tan^2 \frac{x}{2}\right)^2 - 4 \tan^2 \frac{x}{2} \\
 \Rightarrow 2y(1 + y) = (1 + y)^2 - 4y \quad \left[\text{where, } y = \tan^2 \frac{x}{2}\right] \\
 \Rightarrow y^2 + 4y - 1 = 0 \\
 \Rightarrow y = \frac{-4 \pm \sqrt{16 + 4}}{2} = -2 \pm \sqrt{5}
 \end{aligned}$$

$$\begin{aligned}
 \text{Since, } y > 0, \text{ we get} \\
 y = \sqrt{5} - 2 = \frac{(\sqrt{5} - 2)^2}{\sqrt{5} - 2} \cdot \frac{2 + \sqrt{5}}{2 + \sqrt{5}} = (9 - 4\sqrt{5})(2 + \sqrt{5})
 \end{aligned}$$

$$136. y = \frac{\sqrt{(\cos 2A - \sin 2A)^2 + 1}}{\sqrt{(\cos 2A + \sin 2A)^2 - 1}}$$

$$\Rightarrow y = \frac{\pm (\cos 2A - \sin 2A) + 1}{\pm (\cos 2A + \sin 2A) - 1}$$

which gives us four values of y , say y_1, y_2, y_3 and y_4 .
We have,

$$\begin{aligned} y_1 &= \frac{\cos 2A - \sin 2A + 1}{\cos 2A + \sin 2A - 1} \\ &= \frac{(1 + \cos 2A) - \sin 2A}{(\cos 2A - 1) + \sin 2A} \\ &= \frac{2\cos^2 A - 2\sin A \cos A}{-2\sin^2 A + 2\sin A \cos A} \\ &= \frac{\cos A (\cos A - \sin A)}{\sin A (\cos A - \sin A)} = \cot A \end{aligned}$$

$$\begin{aligned} y_2 &= \frac{-(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1} \\ &= \frac{(1 - \cos 2A) + \sin 2A}{-(1 + \cos 2A) - \sin 2A} \\ &= \frac{2\sin^2 A + 2\sin A \cos A}{-2\cos^2 A - 2\sin A \cos A} = -\tan A \end{aligned}$$

$$\begin{aligned} y_3 &= \frac{(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1} \\ &= \frac{(1 + \cos 2A) - \sin 2A}{-(1 + \cos 2A) - \sin 2A} \\ &= \frac{2\cos^2 A - 2\sin A \cos A}{-2\cos^2 A - 2\sin A \cos A} \\ &= -\frac{\cos A - \sin A}{\cos A + \sin A} \\ &= -\frac{1 - \tan A}{1 + \tan A} = -\tan\left(\frac{\pi}{4} - A\right) = -\cot\left(\frac{\pi}{4} + A\right) \end{aligned}$$

$$\begin{aligned} y_4 &= \frac{-(\cos 2A - \sin 2A) + 1}{(\cos 2A + \sin 2A) - 1} \\ &= \frac{(1 - \cos 2A) + \sin 2A}{(1 - \cos 2A) + \sin 2A} \\ &= \frac{2\sin^2 A - 2\sin A \cos A}{-2\sin^2 A + 2\sin A \cos A} \\ &= \frac{\cos A + \sin A}{\cos A - \sin A} \\ &= \frac{1 + \tan A}{1 - \tan A} = \tan\left(\frac{\pi}{4} + A\right) \end{aligned}$$

$$137. x = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} + \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$$

$$\Rightarrow x^2 = a^2 + b^2 + 2\sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)(a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)}$$

$$= a^2 + b^2 + k$$

$$\text{where, } k^2 = \sqrt{[(a^2 + b^2) - (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)] \times (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha)}$$

$$\therefore x = a^2 + b^2 + 2\sqrt{(a^2 + b^2)p - p^2}$$

$$\text{where, } p = a^2 \sin^2 \alpha + b^2 \cos^2 \alpha$$

$$\begin{aligned} \Rightarrow p &= \frac{a^2}{2}(1 - \cos 2\alpha) + \frac{b^2}{2}(1 + \cos 2\alpha) \\ &= \frac{1}{2}[a^2 + b^2 - (a^2 - b^2)\cos 2\alpha] \end{aligned}$$

$$\text{Also, } x^2 = a^2 + b^2 + 2\sqrt{(a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)(a^2 + b^2) - (a^2 \cos^2 \alpha + b^2 \sin^2 \alpha)}$$

$$= a^2 + b^2 + 2\sqrt{(a^2 + b^2)p - p^2}$$

$$\text{where, } p = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$$

$$\Rightarrow p = \frac{1}{2}[a^2 + b^2 + (a^2 - b^2)\cos 2\alpha]$$

138. We have, $\tan \alpha + \tan \beta = -p$ and $\tan \alpha \tan \beta = q$, so that

$$\tan(\alpha + \beta) = \frac{-p}{1-q} = \frac{p}{q-1}$$

$$\begin{aligned} \Rightarrow \cos^2(\alpha + \beta) &= \frac{1}{1 + \frac{p^2}{(q-1)^2}} \\ &= \frac{(q-1)^2}{(q-1)^2 + p^2} \neq (q-1)^2 \end{aligned}$$

Similarly, $\sin^2(\alpha + \beta) \neq p$. Thus, option (b) is correct while options (c) and (d) are not.

Finally, the LHS of option (a) can be written as

$$\cos^2(\alpha + \beta) [\tan^2(\alpha + \beta) + p \tan(\alpha + \beta) + q]$$

$$= \frac{(q-1)^2}{p^2 + (q-1)^2} \left[\frac{p^2}{(q-1)^2} + p \left(\frac{p}{q-1} \right) \right] = q$$

So, option (a) is also true.

$$139. y = \cos^4 x - \cos^2 x + 1 = \left(\cos^2 x - \frac{1}{2} \right)^2 + \frac{3}{4}$$

$\therefore y_{\min} = \frac{3}{4}$ and y is maximum when $\left(\cos^2 x - \frac{1}{2} \right)^2$ is the maximum.

$$\therefore y_{\max} = \frac{1}{4} + \frac{3}{4} = 1$$

$$140. k_1 - k_2 = \sin x \cos x (\cos^2 x - \sin^2 x)$$

$$= \frac{1}{2} \sin 2x \cos 2x = \frac{1}{4} \sin 4x$$

$$\Rightarrow k_1 - k_2 > 0, 0 < 4x < \pi$$

$$\Rightarrow 0 < x < \frac{\pi}{4}$$

$$\Rightarrow k_1 + k_2 = \frac{\sin 2x}{2}$$

$$\Rightarrow k_1 + k_2 > 0$$

$$\Rightarrow 0 < 2x < \pi$$

$$\Rightarrow 0 < x < \frac{\pi}{2}$$

$$141. \text{ Here, } \cos\left(\frac{\alpha}{2} - \frac{\pi}{4}\right) = \frac{\sqrt{5}+1}{4} - \frac{\sqrt{5}-1}{4} = \frac{1}{2}$$

$$\Rightarrow \frac{\frac{\pi}{2} - \frac{\pi}{4}}{2} = 2n\pi \pm \frac{\pi}{3}$$

$$\therefore \alpha = 4n\pi + \frac{\pi}{2} \pm \frac{2\pi}{3} = 4n\pi + \frac{7\pi}{6}, 4n\pi - \frac{\pi}{6}$$

$$\text{For } n = 0, \alpha = \frac{7\pi}{6}, -\frac{\pi}{6}$$

$$142. \text{ Here, } \cos x = |\cos x - \sin x|$$

$$\text{In } \left[0, \frac{\pi}{4}\right] \cos x \geq \sin x \text{ and in } \left(\frac{\pi}{4}, \pi\right), \cos x < \sin x$$

$$\therefore \cos x = \cos x - \sin x, 0 \leq x \leq \frac{\pi}{4} \quad \dots (i)$$

and $\cos x = \sin x - \cos x$, $\frac{\pi}{4} < x \leq \pi$... (ii)

From Eq. (i), we get $\sin x = 0$

$\therefore x = 0$

From Eq. (ii), we get $2 \cos x = \sin x$

$\therefore \tan x = 2$

$\therefore x = \tan^{-1} 2$

143. $\sin \theta + \sqrt{3} \cos \theta = -2 - (x^2 - 6x + 9) = -2 - (x - 3)^2$

$\therefore \sin x + \sqrt{3} \cos \theta \leq -2$

But the minimum value of $\sin x + \sqrt{3} \cos \theta$

So, we have $\sin \theta + \sqrt{3} \cos \theta = -2$ and then $x = 3$

$\therefore x = 3$ and $\cos \left(\theta - \frac{\pi}{6} \right) = -1$

i.e. $\theta - \frac{\pi}{6} = \pi, 3\pi$

144. $|\cos x| \geq |0|$. So, $|\cos x| \leq \sin x \Rightarrow \sin x \geq 0$

So, there is no x in $(\pi, 2\pi)$.

Now, if $x = 2\pi$, $|\cos 2\pi| = 1$, $\sin 2\pi$, which is not true.

So, $0 \leq x \leq \pi$

If $0 \leq x \leq \frac{\pi}{2}$, then $|\cos x| \leq \sin x$

$\cos x \leq \sin x \Rightarrow \tan x \geq 1$

$\therefore \frac{x}{4} \leq x \leq \frac{\pi}{2}$

If $\frac{x}{2} < x \leq \pi$, then $|\cos x| \leq \sin x$

$-\cos x \leq \pi \Rightarrow \tan x \leq -1$ [$\because \cos x < 0$]

$\therefore \frac{x}{2} < x \leq \frac{3\pi}{4}$

$\therefore x \in \left[\frac{\pi}{4}, \frac{\pi}{2} \right] \cup \left[\frac{\pi}{2}, \frac{3\pi}{4} \right]$

145. In any triangle,

$$\begin{aligned} \tan A &= -\tan(B+C) \\ &= \frac{-\tan B - \tan C}{1 - \tan B \tan C} = \frac{\tan B + \tan C}{\tan B \tan C - 1} \end{aligned}$$

Statement II is true.

Since, $\tan A < 0$ as $\angle A$ is obtuse.

So, $\tan B \tan C - 1$ must be negative.

$\Rightarrow \tan B, \tan C > 0$

$\therefore$ Statement I is false.

146. The RHS of the expression given in Statement I

$$\begin{aligned} &= \sqrt{\left(\cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right)^2} + \sqrt{\left(\cos \frac{\theta}{2} - \sin \frac{\pi}{2} \right)^2} \\ &= \left| \cos \frac{\theta}{2} + \sin \frac{\theta}{2} \right| + \left| \cos \frac{\theta}{2} - \sin \frac{\theta}{2} \right| \quad \dots (i) \\ &= \left| \sqrt{2} \sin \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \right| + \left| \sqrt{2} \cos \left(\frac{\theta}{2} + \frac{\pi}{4} \right) \right| \end{aligned}$$

From Eq. (i), it is clear that this expression will simplify to $2 \sin \frac{\theta}{2}$, if

$$\cos \frac{\theta}{2} + \sin \frac{\theta}{2} > 0 \quad \text{and} \quad \cos \frac{\theta}{2} - \sin \frac{\theta}{2} < 0$$

[$\because |x| = x$, if $x > 0$, $|x| = -x$, if $x < 0$]

This will lead to

$$\sin \left(\frac{\theta}{2} + \frac{\pi}{4} \right) > 0, \cos \left(\frac{\theta}{2} + \frac{\pi}{4} \right) < 0$$

$$\Rightarrow 2n\pi + \frac{\pi}{2} < \frac{\theta}{2} + \frac{\pi}{4} < 2n\pi + \pi$$

$$\Rightarrow 2n\pi + \frac{\pi}{4} < \frac{\theta}{2} < 2n\pi + \frac{3\pi}{4}$$

Thus, Statement I is true but does not follow from Statement II.

147. Observe the following steps:

$$x = 18^\circ, y = -54^\circ$$

$$\Rightarrow 5x = 90^\circ, 5y = -270^\circ$$

$$\Rightarrow 3x = 90^\circ - 2x, 3y = -270^\circ - 2y$$

$$\Rightarrow \cos 3x = \sin 2x, \cos 3y = \sin 2y$$

$$\Rightarrow 4 \cos^3 x - 3 \cos x = 2 \sin x \cos x$$

$$4 \cos^3 y - 3 \cos y = 2 \sin y \cos y$$

$$\Rightarrow 4 \cos^2 x - 3 = 2 \sin x, 4 \cos^2 y - 3 = 2 \sin y$$

$$\Rightarrow 4 \sin^2 x + 2 \sin x - 1 = 0, 4 \sin^2 y + 2 \sin y - 1 = 0$$

$\Rightarrow \sin x$ (i.e. $\sin 18^\circ$) and $\sin y$ ($-\sin 54^\circ$) are roots of the same quadratic equation.

148. The Statement I is true but does not follow from Statement II.

Indeed $\cos 36^\circ > \tan 36^\circ$, if $\cos^2 36^\circ > \sin 36^\circ$... (i)

At this stage, we may think when $\cos 36^\circ$ is greater than $\sin 36^\circ$, $\cos^2 36^\circ$ should also be greater than $\sin 36^\circ$ but this is not true, since $\cos^2 36^\circ < \cos 36^\circ$.

Eq. (i) is proved as follows:

$$1 + \cos 72^\circ > 2 \sin 36^\circ$$

$$\Rightarrow 1 + \sin 18^\circ \geq 2 \sin (30^\circ + 6^\circ)$$

$$\Rightarrow 1 + \sin 18^\circ \geq 2 (\sin 30^\circ \cos 6^\circ + \cos 30^\circ \sin 6^\circ)$$

$$1 + 2 \sin 9^\circ \cos 9^\circ \geq \cos 6^\circ + 2 \cos 30^\circ \sin 6^\circ$$

The last inequality is true in the light of the facts

$$1 > \cos 6^\circ, \sin 9^\circ > \sin 6^\circ, \cos 9^\circ > \cos 30^\circ$$

149. The minimum value of sum of three sines can be -3 provided it is attainable through some angle. Indeed value -3 is possible, if

$$\sin \alpha = -1, \sin \beta = -1, \sin \gamma = -1$$

$$\Rightarrow \alpha = (4l - 1) \frac{\pi}{2}, \beta = (4m - 1) \frac{\pi}{2}, \gamma = (4n - 1) \frac{\pi}{2}$$

Now, $\alpha + \beta + \gamma = \pi$

$$\Rightarrow \frac{\pi}{2} [4(l + m + n) - 3] = \pi$$

$$\Rightarrow 4(l + m + n) = 5 \quad \dots (i)$$

Since, l, m and n are integers, so Eq. (i) is impossible.

$\Rightarrow$ Minimum cannot be -3 .

But for $\alpha = \frac{3\pi}{2}, \beta = \frac{2\pi}{2}, \gamma = -2\pi, \alpha + \beta + \gamma = \pi$

and $\sin \alpha + \sin \beta + \sin \gamma = -2$

Thus, $\sin \alpha + \sin \beta + \sin \gamma$ can attain negative values, hence the Statement I is true. The Statement II is false, since α, β and γ are arbitrary real numbers.

Though their sum is π but they can be of the type $\frac{\pi}{2}, \frac{\pi}{2}, 0$.

150. Statement I $\sin x + \operatorname{cosec} x = 2 \Rightarrow \sin x = \operatorname{cosec} x = 1$

$$\therefore x = \frac{\pi}{2}, \frac{5\pi}{2} \Rightarrow \text{Two solutions}$$

$\Rightarrow$ Statement I is false.

Statement II $x + \frac{1}{x} \geq 2$, only when $x \geq 0$.

$\therefore$ Statement II is false.

151. $\sin^2 x + \tan^2 x = 0$ iff $\sin x = \tan x = 0$

$$\Rightarrow x = n\pi$$

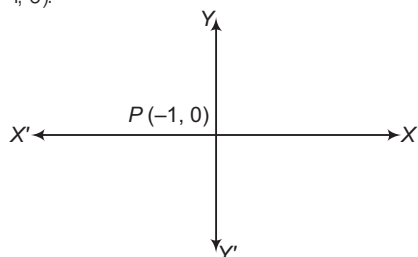
So, Statement I is true.

But Statement II is not always true.

i.e. if $a, b \in$ complex numbers, say $a = 2$ and $b = -2i$

$$\Rightarrow a^2 + b^2 = 0$$

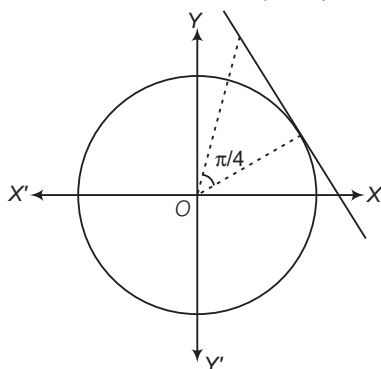
152. Particle moves 1 radian on each circle starts from $P(-1, 0)$.



To cross X-axis, it should cover π radian, hence will be on 4th circle.

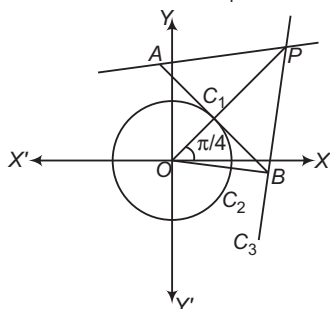
So, to cross positive Y-axis, it should cover a distance of $\frac{3\pi}{2}$ radian i.e. 5th circle.

153. Angular distance travelled from starting of motion on one circle to reach to other is $\left(1 + \frac{\pi}{4}\right)$.



$\Rightarrow$ On 4th circle, particle will cross the positive X-axis again $\Rightarrow x \in (2\sqrt{2}, 4)$

154. Angular distance travelled is $\frac{\pi}{4}$.



$\Rightarrow$ Coordinates of point C are $(\sqrt{2}, \sqrt{2})$.

$\Rightarrow$ Equation of tangent AB is $x + y = \sqrt{2}$.

It intersect the circle $x^2 + y^2 = 9$ at A and B.

$$\Rightarrow P = \left(\frac{9}{2\sqrt{2}}, \frac{9}{2\sqrt{2}}\right)$$

155. $f(\theta, \alpha) = 2 \sin^2 \theta + 4 \cos \theta (\theta + \alpha) \sin \theta \sin \alpha$
 $+ 2 \cos^2 (\theta + \alpha) - 1$

$$= 2 \sin^2 \theta + 2 \cos (\theta + \alpha) [2 \sin \theta \sin \alpha + \cos (\theta + \alpha)] - 1$$

$$= 2 \sin^2 \theta + 2 \cos (\theta + \alpha) [\sin \theta \sin \alpha + \cos \theta \cos \alpha] - 1$$

$$= 2 \sin^2 \theta + 2 \cos (\theta + \alpha) \cos (\theta - \alpha) - 1$$

$$= 2 \sin^2 \theta + 2 \cos^2 \theta - 2 \sin^2 \alpha - 1$$

$$= \cos 2\alpha$$

$$\therefore f\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \cos\left(2 \times \frac{\pi}{4}\right) = 0$$

156. $f(\theta, \alpha) = 0$

$$\Rightarrow \cos 2\alpha = 0$$

$$\Rightarrow 2\alpha = (2n + 1) \frac{\pi}{2}$$

$$\therefore \alpha = (2n + 1) \frac{\pi}{4}, n \in I$$

157. $f\left(\theta, \frac{\pi}{6}\right) = \frac{1}{2}$

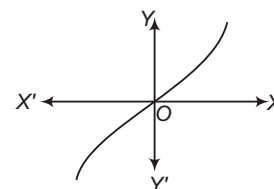
$$\cos\left(\frac{2\pi}{6}\right) = \frac{1}{2}$$

$$\Rightarrow \frac{1}{2} = \frac{1}{2}$$

$$\Rightarrow \theta \text{ is any real.}$$

Solutions (Q. Nos. 158-160)

$k = 0$, graph will be shown as given below:



So, no such triangle is possible.

Let $A = B$, then $2A + C = 180^\circ$

$$\text{and } 2 \tan A + \tan C = k \quad \dots(i)$$

$$\text{Now, } 2A + C = 180^\circ$$

$$\tan 2A = -\tan C$$

$$\text{Also, } 2 \tan A + \tan C = k$$

$$\Rightarrow 2 \tan A + \tan (180^\circ - 2A) = k$$

$$\Rightarrow 2 \tan A + \tan (180^\circ - 2A) = k$$

$$\Rightarrow 2 \tan A - \tan 2A = k$$

$$\Rightarrow -2 \tan^3 A = k (1 - \tan^2 A)$$

$$\Rightarrow 2 \tan^3 A - k \tan^2 A + k = 0$$

$$\text{Let } \tan A = x, x > 0$$

$$\text{Consider, } f(x) = 2x^3 - kx^2 + k \quad \dots(ii)$$

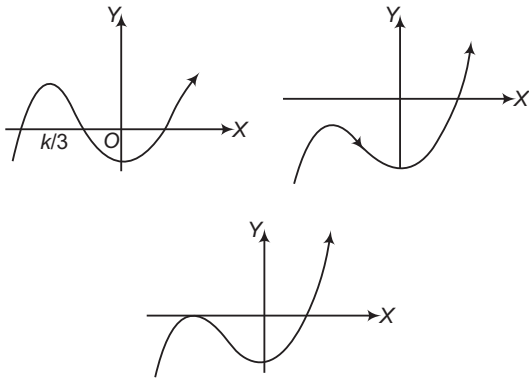
$$f'(x) = 6x^2 - 2kx = 0$$

$$x = \frac{k}{3}, 0$$

Following cases are arise:

Case I If $k < 0$, then $f(0) < 0$ and the equation has only one positive solution.

∴ There is only one triangle.

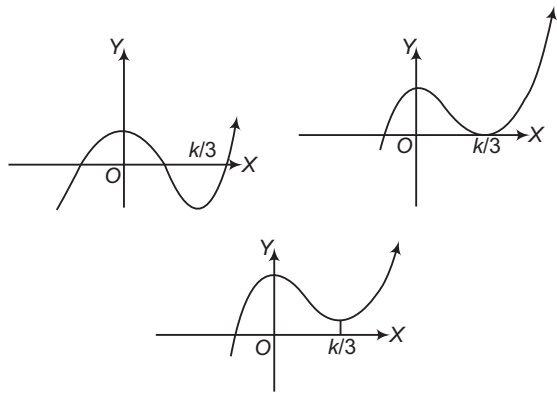


Case II If $k = 0$, then $f(x) = 2x^3$

Thus, $f(x) = 0$ iff $x = 0$ which is not possible.

Case III If $k > 0$, then $0, \frac{k}{3}$ are critical points.

$$f(0) = k > 0 \text{ and } f\left(\frac{k}{3}\right) = \frac{27k - k^3}{27}$$



There is one isosceles triangle, if $f\left(\frac{k}{3}\right) = 0$ i.e. $k = 3\sqrt{3}$

There are two isosceles triangle, if $f\left(\frac{k}{3}\right) < 0$ i.e. $k > 3\sqrt{3}$

There can never be three isosceles triangles.

161. Let $y = 27^{\cos 2x} \cdot 81^{\sin 2x} = 3^{3 \cos 2x + 4 \sin 2x}$

$$\text{Now, } -\sqrt{3^2 + 4^2} \leq 3 \cos 2x + 4 \sin 2x \leq \sqrt{3^2 + 4^2}$$

$$\Rightarrow -5 \leq 3 \cos 2x + 4 \sin 2x \leq 5$$

$$\therefore 3^{-5} \leq 3^{3 \cos 2x + 4 \sin 2x} \leq 3^5$$

162. $\cos^7 x \leq \cos^2 x$ and $\sin^4 x = \sin^2 x$

$$\Rightarrow \cos^7 x + \sin^4 x \leq \cos^2 x + \sin^2 x = 1$$

So, the given equation is satisfied if and only if $\cos^7 x = 0$ and $\sin^2 x = 1$

$$\text{or } \cos^7 x = 1 \text{ and } \sin^2 x = 0$$

$$\Rightarrow x = (2n+1)\frac{\pi}{2} \text{ or } x = 2m\pi$$

$$\therefore 0 \leq x \leq 2\pi, \text{ so } x = 0, \frac{\pi}{2}, \frac{3\pi}{2}, 2\pi$$

163. $\cos(p \sin x) = \sin(p \cos x) = \cos\left(\frac{\pi}{2} - p \cos x\right)$

$$\Rightarrow p \sin x = 2n\pi \pm \left(\frac{\pi}{2} - p \cos x\right), n \in \mathbb{I}$$

Consider the positive sign, we have

$$p(\sin x + \cos x) = (4n+1)\frac{\pi}{2}$$

$$\Rightarrow \sin x + \cos x = \frac{(4n+1)\pi}{2p}$$

$$\therefore |\cos x + \sin x| \leq \sqrt{2}$$

$$\Rightarrow \left|\frac{(4n+1)\pi}{2p}\right| \leq \sqrt{2} \text{ or } p \geq \frac{|(4n+1)|\pi}{2\sqrt{2}}$$

So, p is smallest positive, if $n = 0$, giving again $p = \frac{\pi}{2\sqrt{2}}$.

164. A. $2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x-y)$

$$-2 \sin \frac{1}{2}(x+y) \cos \frac{1}{2}(x+y) = 0$$

$$\text{or } 2 \sin \frac{1}{2}(x+y) \cdot 2 \sin \frac{1}{2}x \sin \frac{1}{2}y = 0$$

$$\therefore \text{Either } \sin \frac{1}{2}(x+y) = 0, \sin \frac{1}{2}x = 0$$

$$\text{or } \sin \frac{1}{2}y = 0$$

$$\therefore x+y=0, x=0, y=0$$

$$|x| + |y| = 1$$

$$\text{i.e. } x+y=1, x-y=1, x+y=-1, x-y=-1$$

$$\text{When } x+y=0, \text{ reject } x+y=1$$

$$\text{and } x+y=-1$$

$$\text{Solve with } x-y=1 \text{ or } x-y=-1$$

$$\text{which gives } \left(\frac{1}{2}, -\frac{1}{2}\right) \text{ or } \left(-\frac{1}{2}, -\frac{1}{2}\right)$$

Again, solving with $x=0$, we get $(0, \pm 1)$ and solving with $y=0$, we get $(\pm 1, 0)$

∴ There are six set of solution.

B. For $f(x)$ to be obtained,

$$\frac{1-|x|}{2} \leq -1 \Rightarrow 1-|x| \leq -2 \Rightarrow |x| \geq 3$$

$$\Rightarrow x \leq -3 \text{ or } x \geq 3 \quad \dots(i)$$

$$\text{or } \frac{1-|x|}{2} \geq 1 \Rightarrow 1-|x| \geq 2 \text{ or } |x| \leq -1$$

which is not possible.

$$\text{Also, } \sec^{-1}\left(\frac{1-|x|}{2}\right) \geq 0 \text{ is always true as}$$

$$0 \leq \sec^{-1} x \leq \pi \Rightarrow \sec^{-1} x \neq \frac{\pi}{2}$$

Hence, value of x lies between $(-\infty, -3] \cup [3, \infty)$.

∴ Infinite values of x .

C. $\sin x \cos y = 1$

$$\Rightarrow \sin x = 1, \cos y = 1$$

$$\sin x = -1, \cos y = -1 \Rightarrow x = \frac{3\pi}{2}, y = \pi$$

Thus, ordered pairs are three.

D. $f(x) = \sin x - \cos x - kx + b$

$$\Rightarrow f'(x) = \cos x + \sin x - k$$

$$\text{Maximum value of } \cos x + \sin x = \sqrt{2}$$

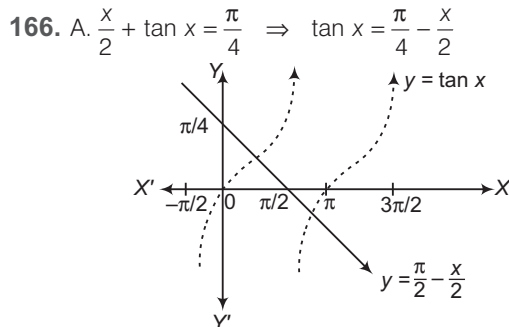
$$\text{If } k > \sqrt{2}, \text{ then } f'(x) < 0, \forall x \in \mathbb{R}$$

and hence $f(x)$ will decreasing function, $\forall x \in \mathbb{R}$.

$$\therefore k > \sqrt{2}$$

$$\Rightarrow 4\sqrt{2}k > 8$$

165. A. $\sin^2 A + \sin^2 B = \sin(A + B)$
 $\Rightarrow 1 - \frac{\cos 2A + \cos 2B}{2} = \sin C$
 $\Rightarrow 1 - \cos(A + B) \cos(A - B) = \sin(A + B)$
 $\Rightarrow \cos^2 C \cos^2(A - B) + 2 \cos C \cos(A - B) = \cos^2 C$
 $\Rightarrow \cos C = 0$
Hence, triangle is right angled.
- B. $\frac{bc}{2 \cos A} = b^2 + c^2 - 2bc \cos A$
 $\Rightarrow \cos A = \frac{bc}{2a^2} \Rightarrow b^2 c^2 = a^2 (b^2 + c^2 - a^2)$
 $\Rightarrow (a^2 - b^2)(a^2 - c^2) = 0$
 $\Rightarrow a = b \text{ or } a = c$
Hence, triangle is isosceles.
- C. $\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \sqrt{3}$
 $\Rightarrow \left(\tan \frac{A}{2} - \tan \frac{B}{2} \right)^2 + \left(\tan \frac{B}{2} - \tan \frac{C}{2} \right)^2 + \left(\tan \frac{C}{2} - \tan \frac{A}{2} \right)^2 = 0$
Hence, triangle is equilateral.
- D. If a, b and c are in AP and h_a, h_b, h_c are in AP, where h_a, h_b, h_c are the altitudes, then $a = b = c$
Hence, the triangle is equilateral.

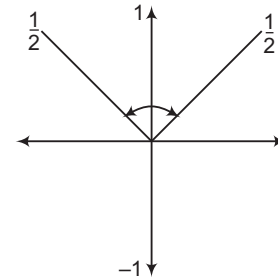


From the graph, the equation has 3 solutions in $[-\pi, \pi]$.

- B. $\sin^{-1} |x^2 - 1| + \cos^{-1} |2x^2 - 5| = \frac{\pi}{2}$
 $\Rightarrow |x^2 - 1| = |2x^2 - 5|$
 $\Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2}$ (two solutions)
- C. $x^4 - 2x^2 \sin^2 \frac{\pi x}{2} + 1 = 0$
 $\Rightarrow \left(x^2 - \sin^2 \frac{\pi x}{2} \right)^2 + 1 - \sin^4 \frac{\pi x}{2} = 0$
 $\Rightarrow x = (2n + 1), n \in \mathbb{I} \text{ and } x^2 = \sin^2 \frac{\pi x}{2} = 1$
 $\Rightarrow x = \pm 1$ is the solution
- D. Let $y = x^2 + 2x + 2 \sec^2 \pi x + \tan^2 \pi x$
 $\Rightarrow y = (x + 1)^2 + (2 \sec^2 \pi x - 1) + \tan^2 \pi x > 0$
 $\Rightarrow$ No solution.

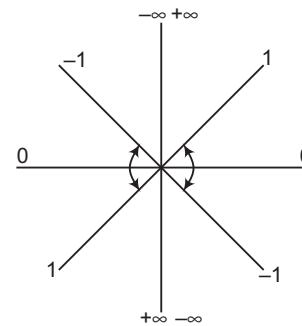
167. A. $\sin x \cdot \cos x (\cos^2 x - \sin^2 x) > 0$
 $\Rightarrow \frac{1}{2} \sin 2x \cdot \cos 2x > 0 \Rightarrow \sin 4x > 0$
 $\therefore 0 < 4x < \pi$

- B. Here, $(2 \sin x - 1)(2 \sin x - 3) \leq 0$
But $2 \sin x - 3$ is always negative.
 $\therefore 2 \sin x - 1 \geq 0$
 $\Rightarrow \sin x \geq \frac{1}{2}$



From the figure, $\frac{\pi}{6} \leq x \leq \frac{5\pi}{6}$

- C. $-1 \leq \tan \theta \leq 1$
The value of scheme for this is shown below



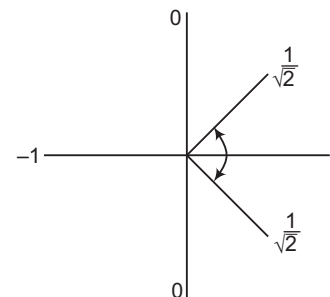
From the figure, $-\frac{\pi}{4} \leq x \leq \frac{\pi}{4}$

or $-\pi \leq x \leq -\frac{3\pi}{4}$ or $\frac{3\pi}{4} \leq x \leq \pi$

$\therefore x \in \left[-\pi, -\frac{3\pi}{4} \right] \cup \left[-\frac{\pi}{4}, \frac{\pi}{4} \right] \cup \left[\frac{3\pi}{4}, \pi \right]$

- D. $\cos \left(x + \frac{\pi}{4} \right) \geq \frac{1}{\sqrt{2}}$

The value for this is shown below:



From the figure, $-\frac{\pi}{4} \leq x + \frac{\pi}{4} \leq \frac{\pi}{4}$

and in general, $2n\pi - \frac{\pi}{2} \leq x + \frac{\pi}{4} \leq 2n\pi + \frac{\pi}{4}$

$\therefore 2n\pi - \frac{\pi}{2} \leq x \leq 2n\pi$

For $n = 0$, $-\frac{\pi}{2} \leq x \leq 0$; for $n = 1$, $\frac{3\pi}{2} \leq x \leq 2\pi$

168. A. $f(x) = 2 - \cos x + \sin^2 x = 3 - \cos x - \cos^2 x$

$$= -\left(\cos x + \frac{1}{2}\right)^2 + \frac{13}{4}$$

$\therefore$ Greatest value $= \frac{13}{4}$ and least value $= 1$

$\therefore$ Ratio $= \frac{13}{4}$

$\Rightarrow k = 13$

B. $1 - 2 \sin^2 x + p \sin x = 2p - 7$

i.e. $2 \sin^2 x - p \sin x + 2p - 8 = 0$

It has solution, if

(i) $p^2 - 8(2p - 8) \geq 0$ i.e. $(p - 8)^2 \geq 0$ which is always true.

(ii) $\sin x = \frac{p \pm \sqrt{p^2 - 8p + 64}}{4} = 2, \frac{p - 4}{2}$

iff $-1 \leq \frac{p - 4}{2} \leq 1$, iff $2 \leq p \leq 6$

$\therefore p$ has 5 integral values.

C. $\cos \alpha = \frac{3}{4}$

$$16 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2} = 8 \left(2 \sin \frac{\alpha}{2} \sin \frac{3\alpha}{2} \right)$$

$$= 8 (\cos \alpha - \cos 2\alpha)$$

$$= 8 (\cos \alpha - 2 \cos^2 \alpha + 1)$$

$$= 8 \left(\frac{3}{4} - \frac{18}{16} + 1 \right) = 5$$

D. Let $A = \sqrt{a^2 - 4} \hat{i} + a \hat{j} + \sqrt{a^2 - 4} \hat{k}$

$$B = \tan A \hat{i} + \tan B \hat{j} + \tan C \hat{k}$$

$$A \cdot B = |A| |B| \cos \theta$$

$$\text{or } \sqrt{a^2 - 4} \tan A + a \tan B + \sqrt{a^2 - 4} \tan C$$

$$= \sqrt{a^2 - 4 + a^2 + a^2 + 4}$$

$$\sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \cos \theta$$

$$= \sqrt{3} a \sqrt{\tan^2 A + \tan^2 B + \tan^2 C} \cos \theta$$

$$\text{or } \cos \theta = \frac{\sqrt{a^2 - 4} \tan A + a \tan B + \sqrt{a^2 - 4} \tan C}{\sqrt{3} a \sqrt{\tan^2 A + \tan^2 B + \tan^2 C}}$$

Since, $\cos \theta \leq 1$

$$\therefore \frac{6a}{\sqrt{3} a \sqrt{\tan^2 A + \tan^2 B + \tan^2 C}} \leq 1$$

or $\tan^2 A + \tan^2 B + \tan^2 C \geq 12$

169. A. $\sin \frac{A}{2} \sin \frac{5A}{2} = \frac{1}{2} (\cos 2A - \cos 3A)$

$$= \frac{1}{2} (2 \cos^2 A - 1 - 4 \cos^3 A + 3 \cos A)$$

$$= \frac{1}{2} \left(2 \times \frac{9}{16} - 1 - 4 \times \frac{27}{64} + 3 \times \frac{3}{4} \right) = \frac{11}{32}$$

B. $\sin \frac{\pi}{7} + \sin \frac{2\pi}{7} + \sin \frac{3\pi}{7} = \frac{1}{2 \sin \frac{\pi}{7}}$

$$\left\{ \left(1 - \cos \frac{2\pi}{7} \right) + 2 \sin \frac{\pi}{7} \sin \frac{2\pi}{7} + 2 \sin \frac{3\pi}{7} \sin \frac{\pi}{7} \right\}$$

$$= \frac{1}{2 \sin \frac{\pi}{7}} \left\{ 1 - \cos \frac{2\pi}{7} + \cos \frac{\pi}{7} - \cos \frac{3\pi}{7} + \cos \frac{2\pi}{7} - \cos \frac{4\pi}{7} \right\}$$

$$= \frac{1}{2 \sin \frac{\pi}{7}} \left\{ 1 + \cos \frac{\pi}{7} \right\} = \frac{1}{2} \cot \frac{\pi}{14}$$

C. $\tan^2 \theta = 2 \tan^2 \phi + 1$

$$1 + \tan^2 \theta = 2(1 + \tan^2 \phi)$$

$$\Rightarrow \sec^2 \theta = 2 \cos^2 \theta = 1 + \cos 2\theta$$

$$\Rightarrow \cos^2 \theta = -\sin^2 \theta$$

$$\cos 2\theta + \sin^2 \phi = 0$$

D. $\frac{\sin (2A + B)}{\sin B} = 5$

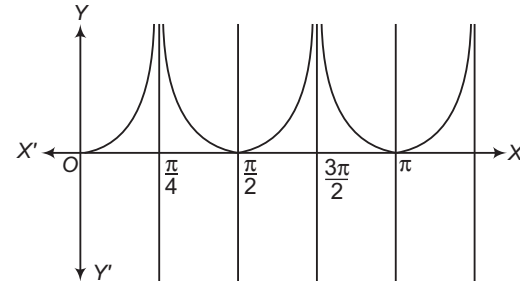
Using componendo and dividendo,

$$\frac{\sin (2A + B) + \sin B}{\sin (2A + B) - \sin B} = \frac{6}{4}$$

$$\Rightarrow \frac{2 \sin (2A + B) \cos A}{2 \cos (2A + B) \sin A} = \frac{3}{2}$$

$$\therefore \frac{\tan (A + B)}{\tan A} = \frac{3}{2}$$

170. A. Clearly, number of solutions of $|\tan 2x| = \sin x$ in $[0, \pi]$ are 4.



B. $\tan 4A = \frac{2 \tan 2A}{1 - \tan^2 2A} = \frac{4 \tan A}{1 - \tan^2 A}$

$$1 - \left(\frac{2 \tan A}{1 - \tan^2 A} \right)^2$$

$$\tan 4A = \frac{4 \tan A (1 - \tan^2 A)}{1 + \tan^4 A - 6 \tan^2 A}$$

$$4 \tan A - 4 \tan^3 A + (6 \tan^2 A - \tan^4 A - 1) \tan 4A = 0$$

If $A = \frac{\pi}{16}$

$$4 \tan \frac{\pi}{16} - 4 \tan^3 \frac{\pi}{16} + 6 \tan^2 \frac{\pi}{16} - \tan^4 \frac{\pi}{16} - 1 = 0$$

$\therefore$ Required value is 2.

C. $\tan (p \cot x) = \cot (p \tan x)$

$$\tan (p \cot x) = \tan \left(\frac{\pi}{2} - p \tan x \right)$$

$$p \cot x = n\pi + \frac{\pi}{2} - p \tan x$$

$$p = \frac{n\pi + \frac{\pi}{2}}{\tan x + \cot x} = \frac{\pi}{2} \sin x \cos x [\because x \in [0, \pi]]$$

$$\Rightarrow P_{\max} = \frac{\pi}{4} \Rightarrow \frac{4P_{\max}}{\pi} = 1$$

$$\begin{aligned}
 \text{D. } 5^{\cos^2 2x + 2\sin^2 x} + 5^{2\cos^2 x + \sin^2 2x} &= 126 \\
 \Rightarrow 5^y + 5^{3-y} &= 126 \\
 \Rightarrow 5^y + \frac{125}{5^y} &= 126 \\
 \Rightarrow 5^y &= 125, 1 \\
 \Rightarrow y &= 3, 0 \\
 \Rightarrow \cos^2 2x + 2\sin^2 x &= 3 \\
 \Rightarrow x &= \frac{\pi}{2}, \frac{3\pi}{2} \\
 \therefore \cos^2 2x + 2\sin^2 x &\neq 0 \\
 \Rightarrow \frac{2x}{\pi} &= 1, 3
 \end{aligned}$$

171. $[x]^2 - 5[x] + 6 = \sin x$

$$\Rightarrow ([x] - 1) = \sin x$$

LHS is always represents an integer.

RHS must be an integer.

Now, integer values of $\sin x$ are 0, +1 and -1.

Case I Let $\sin x = 1$

$$\Rightarrow [x]^2 - 5[x] + 5 = 0$$

$$\Rightarrow [x] = \frac{5 \pm \sqrt{5}}{2} \quad [\text{not possible}]$$

Case II Let $\sin x = -1$

$$\Rightarrow \sin x = -1$$

$$\Rightarrow [x]^2 - 5[x] + 7 = 0 \quad [\text{this equation has no real roots}]$$

Case III Let $\sin x = 0$

$$\Rightarrow [x]^2 - 5[x] + 6 = 0$$

$$\Rightarrow [x] = 2, [x] = 3$$

$$\Rightarrow x \in [2, 4) \text{ and } \sin x = 0 \text{ for } x \in [2, 4) \text{ at } x = \pi$$

$\Rightarrow x = \pi$ is the only solution.

172. Given that, $f(n\theta) = \frac{2\sin 2\theta}{\cos 2\theta - \cos 4n\theta}$

$$= \frac{2\sin 2\theta}{2\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\sin\{(2n+1)\theta - (2n-1)\theta\}}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \frac{\begin{bmatrix} \sin(2n+1)\theta \cos(2n-1)\theta \\ -\cos(2n+1)\theta \sin(2n-1)\theta \end{bmatrix}}{\sin(2n+1)\theta \sin(2n-1)\theta}$$

$$= \cot(2n-1)\theta - \cot(2n+1)\theta$$

$$\Rightarrow f(\theta) + f(2\theta) + f(3\theta) + \dots + f(n\theta)$$

$$= (\cot \theta - \cot 3\theta) + (\cot 3\theta - \cot 5\theta)$$

$$+ (\cot 5\theta - \cot 7\theta) + \dots + \cot(2n-1)\theta - \cot(2n+1)\theta$$

$$= \cot \theta - \cot(2n+1)\theta = \frac{\cos \theta}{\sin \theta} - \frac{\cos(2n+1)\theta}{\sin(2n+1)\theta}$$

$$= \frac{\sin 2n\theta}{\sin \theta \sin(2n+1)\theta}$$

$$\Rightarrow \mu = 2n + 1$$

$$\Rightarrow \lambda = 2n$$

$$\Rightarrow \mu - \lambda = 1$$

173. Let $\theta = \frac{\pi}{7} \Rightarrow 3\theta = \pi - 4\theta$

$$\Rightarrow \sin 3\theta = \sin 4\theta$$

$$\Rightarrow 3\sin \theta - 4\sin^3 \theta = 4\sin \theta \cos \theta \cos 2\theta$$

$$\Rightarrow 3 - 4\sin^2 \theta = 4\cos \theta (2\cos^2 \theta - 1)$$

$$\Rightarrow 8\cos^3 \theta - 4\cos \theta = 4\cos^2 \theta - 1$$

$$\Rightarrow 4\cos \theta (2\cos^2 \theta - 1) = 4\cos^2 \theta - 1$$

$$\Rightarrow 4\cos \theta = \frac{4\cos^2 \theta - 1}{2\cos^2 \theta - 1} = \frac{3 - \tan^2 \theta}{1 - \tan^2 \theta}$$

$$\Rightarrow 4\cos \frac{\pi}{7} = \frac{3 - \tan^2 \frac{\pi}{7}}{1 - \tan^2 \frac{\pi}{7}}$$

Hence, $\lambda = 4$

174. $\sin x \cos 3x - \alpha \cos x \sin 3x = 0 \quad \dots(i)$

$$\Rightarrow \alpha = \frac{\sin x \cos 3x}{\cos x \sin 3x}$$

$$\Rightarrow \alpha = \frac{\tan x}{\tan 3x}$$

$$\Rightarrow \alpha = \frac{\tan x (1 - 3\tan^2 x)}{3\tan x - \tan^3 x}$$

$$\Rightarrow \alpha = \frac{1 - 3\tan^2 x}{3 - \tan^2 x} \quad \dots(ii)$$

For real value of x , RHS of 2nd never lies between $\left(\frac{1}{3}, 3\right)$.

So, the number of integral values is $\{0, 1, 2\}$ i.e. 3

175. $n \sin^2 \theta + 2n \cos(\theta + \alpha) \sin \alpha \sin \theta + \cos 2(\alpha + \theta)$
 $= n \sin^2 \theta + n \cos(\theta + \alpha) [\cos(\theta - \alpha) - \cos(\theta + \alpha)]$
 $+ 2\cos^2(\theta + \alpha) - 1$
 $= n \sin^2 \theta + n (\cos^2 \theta - \sin^2 \alpha) - n \cos^2(\theta + \alpha)$
 $+ 2\cos^2(\alpha + \theta) - 1$
 $= n \sin^2 \theta + n \cos^2 \theta - n \sin^2 \alpha + (2 - n) \cos^2(\theta + \alpha) - 1$
 $= (n - 1) - n \sin^2 \alpha + (2 - n) \cos^2(\theta + \alpha)$
 $\therefore n = 2$

176. Given that, $y = \frac{\cos^2 \theta - 1}{\cos^2 \theta + \cos \theta}$

$$\Rightarrow (y - 1) \cos^2 \theta + y \cos \theta + 1 = 0$$

$$\Rightarrow \cos \theta = -1 \text{ and } \cos \theta = \frac{1}{1 - y}$$

$$\Rightarrow -1 < \frac{1}{1 - y} < 1$$

$$\Rightarrow y < 0 \text{ and } y > 2 \Rightarrow y \in R - [0, 2]$$

$$\therefore b - a = 2$$

177. $[\sin x] = [1 + \sin x] + [1 - \cos x]$

$$\Rightarrow 0 = 2 + [-\cos x]$$

$$\Rightarrow [-\cos x] = -2$$

Since, $-1 \leq \cos x \leq 1$

$$\Rightarrow -1 \leq -\cos x \leq 1$$

$$\Rightarrow [-\cos x] \neq -2$$

Hence, $[-\cos x] = -2$ has no real solution.

178. We have, $\angle B = \frac{\pi}{3}$

$$\begin{aligned} \therefore A + B + C &= \pi \\ \Rightarrow A + C &= \frac{2\pi}{3} \\ \Rightarrow C &= \frac{2\pi}{3} - A \end{aligned}$$

Now, $\sin A \sin C = \lambda$

$$\Rightarrow \sin A \sin \left(\frac{2\pi}{3} - A \right) = \lambda$$

$$\Rightarrow \sin A \left(\frac{\sqrt{3}}{2} \cos A + \frac{1}{2} \sin A \right) = \lambda$$

$$\Rightarrow \frac{\sqrt{3}}{4} \sin 2A + \frac{1}{4} \sin^2 A = \lambda$$

$$\Rightarrow \frac{\sqrt{3}}{4} \sin 2A + \frac{1}{4} (1 - \cos 2A) = \lambda$$

$$\Rightarrow \frac{\sqrt{3}}{4} \sin 2A - \frac{1}{4} \cos 2A = \lambda - \frac{1}{4}$$

$$\text{Now, } -\sqrt{\frac{3}{16} + \frac{1}{16}} \leq \frac{\sqrt{3}}{4} \sin 2A - \frac{1}{4} \cos 2A \leq \sqrt{\frac{3}{16} + \frac{1}{16}}$$

$$\Rightarrow -\frac{1}{2} \leq \frac{\sqrt{3}}{4} \sin 2A - \frac{1}{4} \cos 2A \leq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{2} \leq \lambda - \frac{1}{4} \leq \frac{1}{2}$$

$$\Rightarrow -\frac{1}{4} \leq \lambda - \frac{3}{4}$$

$$\Rightarrow \lambda \in \left[-\frac{1}{4}, \frac{3}{4} \right]$$

$$\Rightarrow b = \frac{3}{4}, a = -\frac{1}{4}$$

$$\therefore b - a = 1$$

179. In $\triangle ABC$, we have

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$$

$$\Rightarrow \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$$

Now, $AM \geq GM$

$$\Rightarrow \left\{ \frac{\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}}{3} \right\} \geq \left\{ \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \right\}^{\frac{1}{3}}$$

$$\Rightarrow \left\{ \frac{\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}}{3} \right\} \geq \left\{ \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \right\}^{\frac{1}{3}}$$

$$\Rightarrow \left(\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \right)^{\frac{2}{3}} \geq 3$$

$$\Rightarrow \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \geq 3\sqrt{3}$$

$$\Rightarrow \lambda = 3$$

180. In $\triangle ABC$, we have

$$\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \quad \dots (i)$$

It is given that, $\tan \frac{A}{2}, \tan \frac{B}{2}, \tan \frac{C}{2}$ are in HP.

$$\Rightarrow \cot \frac{A}{2}, \cot \frac{B}{2}, \cot \frac{C}{2} \text{ are in HP.}$$

$$\Rightarrow 2 \cot \frac{B}{2} = \cot \frac{A}{2} + \cot \frac{C}{2} \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = 3 \cot \frac{B}{2}$$

$$\Rightarrow \cot \frac{A}{2} \cot \frac{C}{2} = 3$$

Now, $AM \geq GM$

$$\Rightarrow \frac{\cot \frac{A}{2} + \cot \frac{C}{2}}{2} \geq \sqrt{\cot \frac{A}{2} \cot \frac{C}{2}}$$

$$\Rightarrow \cot \frac{B}{2} \geq \sqrt{3} \quad \left[\because \cot \frac{A}{2} + \cot \frac{C}{2} = 2 \cot \frac{B}{2} \right]$$

$$\Rightarrow \lambda = 3$$

181. We know that, $|\sin x + b \cos x| \leq \sqrt{a^2 + b^2}$ for all x .

$$\therefore |12 \sin \alpha + 5 \cos \alpha| \leq 13$$

$$\text{Also, } 2\beta^2 - 8\beta + 21 = 2(\beta^2 - 4\beta + 4) + 13 \\ = 2(\beta - 2)^2 + 13 \geq 13$$

$$\therefore 12 \sin \alpha = 5 \cos \alpha = 2\beta^2 - 8\beta + 21$$

$$\Rightarrow 12 \sin \alpha + 5 \cos \alpha = 13$$

$$\text{and } 2\beta^2 - 8\beta + 21 = 13$$

$$\Rightarrow \frac{12}{13} \sin \alpha + \frac{5}{13} \cos \alpha = 1 \quad \text{and } \beta = 2$$

$$\Rightarrow \cos(\alpha - \theta) = 1 \quad \text{and } \beta = 2$$

$$\text{where, } \tan \theta = \frac{12}{5}$$

$$\Rightarrow \alpha - \theta = 2n\pi$$

$$\text{and } \beta = 2, n \in \mathbb{Z}$$

$$\Rightarrow \beta^2 - 2\beta = 0$$

182. We observe that,

$$LHS \geq 2 \quad \text{and} \quad RHS = 2 - (x - 1)^2 \leq 2$$

So, the given equation is valid, if

$$LHS = 2 \quad \text{and} \quad RHS = 2$$

$$\Rightarrow \tan^2(x + y) + \cot^2(x + y) = 2$$

$$\text{and } 1 - 2x - x^2 = 2$$

$$\Rightarrow x = -1$$

$$\Rightarrow x^2 - 3x + 2 = 6$$

183. We have, $x - y = \frac{\pi}{4}$ and $\cot x + \cot y = 2$

Now, $\cot x + \cot y = 2$

$$\Rightarrow \sin(x + y) = 2 \sin x \sin y$$

$$\Rightarrow \sin(x + y) = \cos(x - y) - \cos(x + y)$$

$$\Rightarrow \sin(x + y) + \cos(x + y) = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}} \sin(x + y) + \frac{1}{\sqrt{2}} \cos(x + y) = \frac{1}{2}$$

$$\Rightarrow \sin(x + y) \cos \frac{\pi}{4} + \cos(x + y) \sin \frac{\pi}{4} = \frac{1}{2}$$

$$\Rightarrow \sin \left(x + y + \frac{\pi}{4} \right) = \sin \frac{\pi}{6} \quad \text{or} \quad \sin \left(x + y + \frac{\pi}{4} \right) = \sin \frac{5\pi}{2}$$

$$\Rightarrow x + y + \frac{\pi}{4} = \frac{\pi}{6}$$

$$\text{or } x + y + \frac{\pi}{4} = \frac{5\pi}{6}$$

$$\Rightarrow x + y = \frac{7\pi}{12} \quad [\because x, y \geq 0]$$

On solving this equation with $x - y = \frac{\pi}{4}$, we get

$$x = \frac{5\pi}{12} \text{ and } y = \frac{\pi}{6}$$

$$\Rightarrow \lambda = 5, \mu = 1$$

$$\Rightarrow \lambda + \mu = 5 + 1 = 6$$

184. Let $4^{\sin x} = u$ and $3^{\cos y} = v$. Then, the given system of equations reduces to

$$u + v = 11 \quad \dots(i)$$

$$5u^2 - 2v = 2 \quad \dots(ii)$$

Eliminating v between Eqs. (i) and (ii), we get

$$5u^2 - 22 + 2u = 2$$

$$\Rightarrow 5u^2 + 2u - 24 = 0$$

$$\Rightarrow (5u + 12)(u - 2) = 0$$

$$\Rightarrow u - 2 = 0$$

$$[\because u = 4^{\sin x} > 0 \text{ for all } x, \text{ so } 5u + 12 > 0]$$

$$\Rightarrow u = 2 \Rightarrow 4^{\sin x} = 2$$

$$\Rightarrow 2^{2\sin x} = 2^1 \Rightarrow 2\sin x = 1$$

$$\Rightarrow \sin x = \frac{1}{2}$$

$$\Rightarrow \sin x = \sin \frac{\pi}{6} \Rightarrow x = \frac{\pi}{6}$$

Putting the value of u in Eq. (i), we get

$$v = 9$$

$$\Rightarrow \frac{1}{3^{\cos y}} = 3^2$$

$$\Rightarrow \frac{1}{\cos y} = 2 \Rightarrow \cos y = \frac{1}{2}$$

$$\Rightarrow \cos y = \cos \frac{\pi}{3}$$

$$\Rightarrow y = \frac{\pi}{3}$$

$$\therefore x + y = \frac{\pi}{2} \Rightarrow \lambda = 2$$

185. We have, $\sin \left(x - \frac{\pi}{4} \right) - \cos \left(x + \frac{3\pi}{4} \right) = 1$

$$\Rightarrow \sin \left(x - \frac{\pi}{4} \right) + \sin \left(x + \frac{\pi}{4} \right) = 1 \quad [\because \sin \theta = \sin (\pi - \theta)]$$

$$\Rightarrow 2 \sin x \cos \frac{\pi}{4} = 1$$

$$\Rightarrow \sin x = \frac{1}{\sqrt{2}} \Rightarrow x = \frac{\pi}{4}, \frac{3\pi}{4}$$

We know that, 3 radians $\cong 171^\circ 22'$. Therefore, $\sin 3 > 0$, $\cos 3 < 0$ and $|\cos 3| > \sin 3$

$$\therefore \cos 3 + \sin 3 < 0$$

$$\text{Now, } \frac{2 \cos 7x}{\cos 3 + \sin 3} > 2^{\cos 2x}$$

$$\Rightarrow 2 \cos 7x < 2^{\cos 2x} (\cos 3 + \sin 3)$$

$$[\because \cos 3 + \sin 3 < 0]$$

$$\Rightarrow 2 \cos 7x < 0 \quad [\because 2^{\cos 2x} > 0]$$

$$\Rightarrow \cos 7x < 0$$

Clearly, $x = \frac{3\pi}{4}$ satisfies this equation.

Hence, $x = \frac{3\pi}{4}$ is the required solution.

So, only one solution.

186. The given equation will have a solution, if

$$p \geq 0, 2 - p \geq 0$$

$$\text{and } |\sqrt{p} \cos x - 2 \sin x| \leq \sqrt{p + 4}$$

$$\Rightarrow p \geq 0, 2 - p \geq 0$$

$$\text{and } |\sqrt{2} + \sqrt{2 - p}| \leq \sqrt{p + 4}$$

$$\Rightarrow p \geq 0, p \leq 2$$

$$\text{and } (\sqrt{2} + \sqrt{2 - p})^2 \leq p + 4$$

$$\Rightarrow p \geq 0, p \leq 2$$

$$\text{and } p^2 + 2p - 4 \geq 0$$

$$\Rightarrow p \geq 0, p \leq 2$$

$$\text{and } p \in (-\infty, -\sqrt{5} - 1) \cup [\sqrt{5} - 1, \infty)$$

$$\Rightarrow p \in (\sqrt{5} - 1, 2]$$

So, the integral value of p is 2.

Entrances Gallery

1. $\sin x + 2 \sin 2x - \sin 3x = 3$

$$\Rightarrow \sin x + 4 \sin x \cos x - 3 \sin x + 4 \sin^3 x = 3$$

$$\Rightarrow \sin x [-2 + 4 \cos x + 4(1 - \cos^2 x)] = 3$$

$$\Rightarrow \sin x [2 - (4 \cos^2 x - 4 \cos x + 1) + 1] = 3$$

$$\Rightarrow \sin x [3 - (2 \cos x - 1)^2] = 3$$

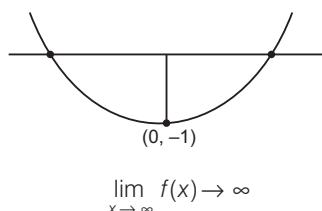
$$\Rightarrow \sin x = 1 \text{ and } 2 \cos x - 1 = 0$$

$$\Rightarrow x = \frac{\pi}{2} \text{ and } x = \frac{\pi}{3}$$

which is not possible at same time. Hence, no solution.

2. Let $f(x) = x^2 - x \sin x - \cos x$

$$\Rightarrow f'(x) = 2x - x \cos x$$



$$\lim_{x \rightarrow -\infty} f(x) \rightarrow \infty$$

$$f(0) = -1$$

Hence, there are two solutions.

3. $2 \cos \theta (1 - \sin \phi) = \frac{2 \sin^2 \theta}{\sin \theta} \cos \phi - 1$

$$= 2 \sin \theta \cos \phi - 1$$

$$\Rightarrow 2 \cos \theta - 2 \cos \theta \sin \phi = 2 \sin \theta \cos \phi - 1$$

$$\Rightarrow 2 \cos \theta + 1 = 2 \sin (\theta + \phi) \quad \dots(i)$$

$$\tan(2\pi - \theta) > 0 \Rightarrow \tan \theta < 0 \quad \dots(ii)$$

$$\text{and } -1 < \sin \theta < -\frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta \in \left(\frac{3\pi}{2}, \frac{5\pi}{3} \right) \quad \dots(iii)$$

$$\therefore \frac{1}{2} < \sin (\theta + \phi) < 1 \quad [\text{from Eqs. (i) and (iii)}]$$

$$\Rightarrow 2\pi + \frac{\pi}{6} - \theta_{\max} < \phi < 2\pi + \frac{5\pi}{6} - \theta_{\min}$$

$$\Rightarrow \frac{\pi}{2} < \phi < \frac{4\pi}{3}$$

4. For $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

Let $\cos 4\theta = \frac{1}{3}$,

$$\cos 2\theta = \pm \sqrt{\frac{1 + \cos 4\theta}{2}} = \pm \sqrt{\frac{2}{3}}$$

$$f\left(\frac{1}{3}\right) = \frac{2}{2 - \sec^2 \theta} = \frac{2 \cos^2 \theta}{2 \cos^2 \theta - 1} = 1 + \frac{1}{\cos 2\theta}$$

$$\Rightarrow f\left(\frac{1}{3}\right) = 1 - \sqrt{\frac{3}{2}} \text{ or } 1 + \sqrt{\frac{3}{2}}$$

5. $\frac{1}{\sin \frac{\pi}{n}} - \frac{1}{\sin \frac{3\pi}{n}} = \frac{1}{\sin \frac{2\pi}{n}}$

$$\Rightarrow \frac{\sin \frac{3\pi}{n} - \sin \frac{\pi}{n}}{\sin \frac{\pi}{n} \sin \frac{3\pi}{n}} = \frac{1}{\sin \frac{2\pi}{n}}$$

$$\Rightarrow \frac{\left(2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} - \sin \frac{\pi}{n}\right) \sin \frac{2\pi}{n}}{\sin \frac{\pi}{n} \sin \frac{3\pi}{n}} = 1$$

$$\Rightarrow \sin \frac{4\pi}{n} = \sin \frac{3\pi}{n}$$

$$\Rightarrow \frac{4\pi}{n} + \frac{3\pi}{n} = \pi \Rightarrow n = 7$$

6. $(f \circ g \circ f)(x) = \sin^2(\sin x^2)$

$$(g \circ f)(x) = \sin(\sin x^2)$$

$$\therefore \sin^2(\sin x^2) = \sin(\sin x^2)$$

$$\Rightarrow \sin(\sin x^2) [\sin(\sin x^2) - 1] = 0$$

$$\Rightarrow \sin(\sin x^2) = 0 \text{ or } 1$$

$$\Rightarrow \sin x^2 = n\pi \text{ or } 2m\pi + \frac{\pi}{2}, \text{ where } m, n \in \mathbb{I}$$

$$\Rightarrow \sin x^2 = 0 \Rightarrow x^2 = n\pi$$

$$\Rightarrow x = \pm \sqrt{n\pi}, n \in \{0, 1, 2, \dots\}$$

7. Given, $(y + z) \cos 3\theta - (xyz) \sin 3\theta = 0$... (i)

and $xyz \sin 3\theta = (2 \cos 3\theta) z + (2 \sin 3\theta) y$... (ii)

$$\therefore (y + z) \cos 3\theta = (2 \cos 3\theta) z + (2 \sin 3\theta) y$$

$$= (y + 2z) \cos 3\theta + y \sin 3\theta$$

$$\Rightarrow y (\cos 3\theta - 2 \sin 3\theta) = z \cos 3\theta$$

and $y (\sin 3\theta - \cos 3\theta) = 0$

$$\Rightarrow \sin 3\theta - \cos 3\theta = 0$$

$$\Rightarrow \sin 3\theta = \cos 3\theta$$

$$\therefore 3\theta = n\pi + \frac{\pi}{4}$$

8. $\tan \theta = \cot 5\theta$

$$\Rightarrow \cos 6\theta = 0$$

$$4 \cos^3 2\theta - 3 \cos 2\theta = 0$$

$$\Rightarrow \cos 2\theta = 0 \text{ or } \pm \frac{\sqrt{3}}{2}$$

Also, we have $\sin 2\theta = \cos 4\theta$

$$\Rightarrow 2 \sin^2 2\theta + \sin 2\theta - 1 = 0$$

$$\Rightarrow 2 \sin^2 2\theta + 2 \sin 2\theta - \sin 2\theta - 1 = 0$$

$$\sin 2\theta = -1 \text{ or } \sin 2\theta = \frac{1}{2}$$

$$\cos 2\theta = 0 \text{ and } \sin 2\theta = -1$$

$$\Rightarrow 2\theta = -\frac{\pi}{2} \Rightarrow \theta = -\frac{\pi}{4}$$

$$\cos 2\theta = \pm \frac{\sqrt{3}}{2}, \sin 2\theta = \frac{1}{2}$$

$$\Rightarrow 2\theta = \frac{\pi}{6}, \frac{5\pi}{6} \Rightarrow \theta = \frac{\pi}{12}, \frac{5\pi}{12}$$

$$\therefore \theta = -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}$$

9. $\frac{1}{4 \cos^2 \theta + 1 + \frac{3}{2} \sin 2\theta}$

$$= \frac{1}{2[1 + \cos 2\theta] + 1 + \frac{3}{2} \sin 2\theta}$$

lies between $\frac{1}{2}$ to $\frac{11}{2}$.

$\therefore$ Maximum value is 2.

Minimum value of $1 + 4 \cos^2 \theta + 3 \sin \theta \cos \theta$

$$= 1 + \frac{4(1 + \cos 2\theta)}{2} + \frac{3}{2} \sin 2\theta$$

$$= 1 + 2 + 2 \cos 2\theta + \frac{3}{2} \sin 2\theta$$

$$= 3 + 2 \cos 2\theta + \frac{3}{2} \sin 2\theta$$

$$= 3 - \sqrt{4 + \frac{9}{4}} = 3 - \frac{5}{2} = \frac{1}{2}$$

So, maximum value of $\frac{1}{4 \cos^2 \theta + 1 + \frac{3}{2} \sin 2\theta}$ is 2.

10. $\frac{1}{4} (\sin^4 x + \cos^4 x) - \frac{1}{6} (\sin^6 x + \cos^6 x)$

$$= \frac{3(\sin^4 x + \cos^4 x) - 2(\sin^6 x + \cos^6 x)}{12}$$

$$= \frac{3(1 - 2 \sin^2 x \cos^2 x) - 2(1 - 3 \sin^2 x \cos^2 x)}{12}$$

$$= \frac{1}{12}$$

11. Given expression

$$= \frac{\tan A}{1 - \cot A} + \frac{\cot A}{1 - \tan A} = \frac{\sin A}{\cos A} \times \frac{\sin A}{\sin A - \cos A} + \frac{\cos A}{\sin A} \times \frac{\cos A}{\cos A - \sin A}$$

$$= \frac{1}{\sin A - \cos A} \left\{ \frac{\sin^3 A - \cos^3 A}{\cos A \sin A} \right\}$$

$$= \frac{\sin^2 A + \sin A \cos A + \cos^2 A}{\sin A \cos A}$$

$$= \frac{1 + \sin A \cos A}{\sin A \cos A} = 1 + \sec A \operatorname{cosec} A$$

12. Given, a ΔPQR such that

$$3 \sin P + 4 \cos Q = 6 \quad \dots (i)$$

and $4 \sin Q + 3 \cos P = 1 \quad \dots (ii)$

Squaring and adding Eqs. (i) and (ii), we get

$$(3 \sin P + 4 \cos Q)^2 + (4 \sin Q + 3 \cos P)^2 = 36 + 1$$

$$\Rightarrow 9(\sin^2 P + \cos^2 P) + 16(\sin^2 Q + \cos^2 Q)$$

$$+ 2 \times 3 \times 4(\sin P + \cos Q + \sin Q \cos P) = 37$$

$$\Rightarrow 24[\sin(P + Q)] = 37 - 25$$

$$\Rightarrow \sin(P+Q) = \frac{1}{2}$$

Since, P and Q are angles of ΔPQR , hence

$$\Rightarrow 0^\circ < P+Q < \pi$$

$$\Rightarrow P+Q = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

$$\Rightarrow R = \frac{5\pi}{6} \text{ or } \frac{\pi}{6}$$

Hence, 2 cases arise here.

Case I $R = \frac{5\pi}{6}$

$$\Rightarrow P+Q = \frac{\pi}{6}$$

$$\Rightarrow 0^\circ < P, Q < \frac{\pi}{6}$$

$$\Rightarrow \sin P < \frac{1}{2}, \cos Q < 1$$

$$\Rightarrow 3\sin P + 4\cos Q < \frac{3}{2} + 4$$

$$\Rightarrow 3\sin P + 4\cos Q < \frac{11}{2} < 6$$

$$\Rightarrow 3\sin P + 4\cos Q = 6 \text{ is not possible.}$$

Case II $R = \frac{\pi}{6}$

Hence, $R = \frac{\pi}{6}$ is the only possibility.

13. $A = \sin^2 x + \cos^4 x$

$$\begin{aligned} \Rightarrow A &= 1 - \cos^2 x + \cos^4 x \\ &= \cos^4 x - \cos^2 x + \frac{1}{4} + \frac{3}{4} \\ &= \left(\cos^2 x - \frac{1}{2}\right)^2 + \frac{3}{4} \end{aligned}$$

...(i)

where, $0 \leq \left(\cos^2 x - \frac{1}{2}\right)^2 \leq \frac{1}{4}$

...(ii)

$$\therefore \frac{3}{4} \leq A \leq 1$$

14. $\sin \theta + \sin 4\theta + \sin 7\theta = 0$

$$\Rightarrow \sin 4\theta + (\sin \theta + \sin 7\theta) = 0$$

$$\Rightarrow \sin 4\theta + 2\sin 4\theta \cdot \cos 3\theta = 0$$

$$\Rightarrow \sin 4\theta \{1 + 2\cos 3\theta\} = 0$$

$$\Rightarrow \sin 4\theta = 0 \text{ and } \cos 3\theta = -\frac{1}{2}$$

As $0 < \theta < \pi$

$$\therefore 0 < 4\theta < 4\pi$$

$$\therefore 4\theta = \pi, 2\pi, 3\pi$$

$$\cos 3\theta = -\frac{1}{2}$$

$$0 < 3\theta < 3\pi$$

$$\Rightarrow 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}$$

$$\Rightarrow \theta = \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$$

15. $\cos(\alpha + \beta) = \frac{4}{5} \Rightarrow \alpha + \beta \in 1^{\text{st}} \text{ quadrant}$

and $\sin(\alpha - \beta) = \frac{5}{13} \Rightarrow \alpha - \beta \in 1^{\text{st}} \text{ quadrant}$

$$\Rightarrow 2\alpha = (\alpha + \beta) + (\alpha - \beta)$$

$$\therefore \tan 2\alpha = \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta)\tan(\alpha - \beta)}$$

$$= \frac{\frac{3}{4} + \frac{5}{12}}{1 - \frac{3}{4} \cdot \frac{5}{12}} = \frac{56}{33}$$

16. $\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta) = -\frac{3}{2}$

$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + 3 = 0$$

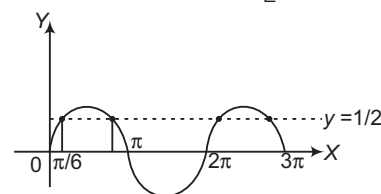
$$\Rightarrow 2[\cos(\beta - \gamma) + \cos(\gamma - \alpha) + \cos(\alpha - \beta)] + \sin^2 \alpha + \cos^2 \alpha + \sin^2 \beta + \cos^2 \beta + \sin^2 \gamma + \cos^2 \gamma = 0$$

$$\Rightarrow (\sin \alpha + \sin \beta + \sin \gamma)^2 + (\cos \alpha + \cos \beta + \cos \gamma)^2 = 0$$

17. Given equation is $2\sin^2 x + 5\sin x - 3 = 0$

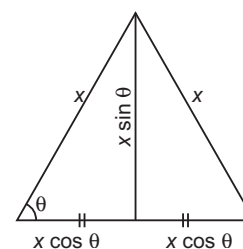
$$\Rightarrow (2\sin x - 1)(\sin x + 3) = 0$$

$$\Rightarrow \sin x = \frac{1}{2} \quad [\because \sin x \neq -3]$$



It is clear from figure that the curve intersect the line at four points in the given interval. Hence, number of solutions are 4.

18. Area = $\frac{1}{2} \times \text{Base} \times \text{Altitude}$



$$= \frac{1}{2} \times (2x \cos \theta) \times (x \sin \theta) = \frac{1}{2} x^2 \sin 2\theta$$

[since, maximum value of $\sin 2\theta$ is 1]

$$\therefore \text{Maximum area} = \frac{1}{2} x^2$$

19. Given, $\cos x + \sin x = \frac{1}{2}$

$$\therefore \frac{1 - \tan^2 \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} + \frac{2 \tan \frac{x}{2}}{1 + \tan^2 \frac{x}{2}} = \frac{1}{2}$$

Let $\tan \frac{x}{2} = t$

$$\Rightarrow \frac{1 - t^2}{1 + t^2} + \frac{2t}{1 + t^2} = \frac{1}{2}$$

$$\Rightarrow 3t^2 - 4t - 1 = 0$$

$$\Rightarrow t = \frac{2 \pm \sqrt{7}}{3}$$

As $0 < x < \pi \Rightarrow 0 < \frac{x}{2} < \frac{\pi}{2}$

Since, $\tan \frac{x}{2}$ is positive.

$$\therefore t = \tan \frac{x}{2} = \frac{2 + \sqrt{7}}{3}$$

Now,
$$\tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}} = \frac{2t}{1 - t^2}$$

$$\Rightarrow \tan x = \frac{2 \left(\frac{2 + \sqrt{7}}{3} \right)}{1 - \left(\frac{2 + \sqrt{7}}{3} \right)^2}$$

$$\Rightarrow \tan x = -\frac{2 + \sqrt{7}}{1 + 2\sqrt{7}} \times \frac{1 - 2\sqrt{7}}{1 - 2\sqrt{7}}$$

$$\Rightarrow \tan x = -\left(\frac{4 + \sqrt{7}}{3} \right)$$

20. Given, $\sin \alpha + \sin \beta = -\frac{21}{65}$... (i)

and $\cos \alpha + \cos \beta = -\frac{27}{65}$... (ii)

On squaring and adding Eqs. (i) and (ii), we get
 $\sin^2 \alpha + \sin^2 \beta + 2 \sin \alpha \sin \beta + \cos^2 \alpha + \cos^2 \beta + 2 \cos \alpha \cos \beta$

$$= \left(-\frac{21}{65} \right)^2 + \left(-\frac{27}{65} \right)^2$$

$$\Rightarrow 2 + 2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) = \frac{441}{4225} + \frac{729}{4225}$$

$$\Rightarrow 2[1 + \cos(\alpha - \beta)] = \frac{1170}{4225}$$

$$\Rightarrow \cos^2 \left(\frac{\alpha - \beta}{2} \right) = \frac{1170}{4 \times 4225}$$

$$\Rightarrow \cos^2 \left(\frac{\alpha - \beta}{2} \right) = \frac{9}{130}$$

$$\Rightarrow \cos \left(\frac{\alpha - \beta}{2} \right) = -\frac{3}{\sqrt{130}}$$

$$\left[\because \pi < \alpha - \beta < 3\pi \Rightarrow \frac{\pi}{2} < \frac{\alpha - \beta}{2} < \frac{3\pi}{2} \right]$$

21. Given, $f(x) = \sin x - \sqrt{3} \cos x + 1$

Since, $-2 \leq \sin x - \sqrt{3} \cos x \leq 2$

$$\Rightarrow -1 \leq \sin x - \sqrt{3} \cos x + 1 \leq 3$$

$\therefore$ Range of $f(x) = [-1, 3]$

22. Since, $\sin^2 \theta \leq 1$

$$\Rightarrow \frac{4xy}{(x+y)^2} \leq 1 \left[\because \sin^2 \theta = \frac{4xy}{(x+y)^2}, \text{ given} \right]$$

$$\Rightarrow x^2 + y^2 + 2xy - 4xy \geq 0$$

$$\Rightarrow (x - y)^2 \geq 0$$

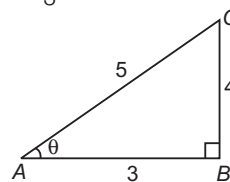
which is true for all real values of x and y provided

$x + y \neq 0$, otherwise $\frac{4xy}{(x+y)^2}$ will be meaningless.

23. We know that,

$$\frac{1 - \tan^2 15^\circ}{1 + \tan^2 15^\circ} = \cos 30^\circ = \frac{\sqrt{3}}{2}$$

24. Since, $\tan \theta = -\frac{4}{3}$



$$\therefore \sin \theta = \frac{BC}{AC} = \frac{4}{5}$$

But $\tan \theta$ is negative which is possible only, if θ lies in second and fourth quadrants.

$$\therefore \sin \theta \text{ may be } \frac{4}{5} \text{ or } -\frac{4}{5}$$

25. Since, $\sin(\alpha + \beta) = 1$

$$\Rightarrow \alpha + \beta = \frac{\pi}{2} \quad \dots (i)$$

and $\sin(\alpha - \beta) = \frac{1}{2}$

$$\Rightarrow \alpha - \beta = \frac{\pi}{6} \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$\alpha = \frac{\pi}{3} \text{ and } \beta = \frac{\pi}{6}$$

$$\therefore \tan(\alpha + 2\beta) \tan(2\alpha + \beta)$$

$$= \tan \left(\frac{2\pi}{3} \right) \tan \left(\frac{5\pi}{6} \right)$$

$$= \tan \left(\pi - \frac{\pi}{3} \right) \tan \left(\pi - \frac{\pi}{6} \right)$$

$$= \left(-\tan \frac{\pi}{3} \right) \left(-\tan \frac{\pi}{6} \right) = \frac{1}{\sqrt{3}} \times \sqrt{3} = 1$$

26. Given, $y = \sin^2 \theta + \operatorname{cosec}^2 \theta$

$$\therefore y = (\sin \theta - \operatorname{cosec} \theta)^2 + 2$$

$$\Rightarrow y \geq 2, \theta \neq 0$$

27. Since, $|c| > \sqrt{a^2 + b^2}$

$$\Rightarrow c < -\sqrt{a^2 + b^2} \text{ and } c > \sqrt{a^2 + b^2}$$

But $-\sqrt{a^2 + b^2} \leq a \sin x + b \cos x \leq \sqrt{a^2 + b^2} \quad \dots (i)$

and $a \sin x + b \cos x = c \quad \dots (ii)$

From Eqs. (i) and (ii), we see that no solution exist.

28. Since, α is a root of $25 \cos^2 \theta + 5 \cos \theta - 12 = 0$

$$\therefore 25 \cos^2 \alpha + 5 \cos \alpha - 12 = 0$$

$$\Rightarrow (5 \cos \alpha - 3)(5 \cos \alpha + 4) = 0$$

$$\Rightarrow \cos \alpha = -\frac{4}{5}, \frac{3}{5}$$

But $\frac{\pi}{2} < \alpha < \pi$ i.e. in IInd quadrant.

$$\therefore \cos \alpha = -\frac{4}{5} \text{ and } \sin \alpha = \frac{3}{5}$$

Now, $\sin 2\alpha = 2 \sin \alpha \cos \alpha$

$$= 2 \times \frac{3}{5} \times \left(-\frac{4}{5} \right) = -\frac{24}{25}$$

29. We know that, $\sin n\pi = 0, \forall n \in \mathbb{N}$

$$\begin{aligned} \therefore E &= \sum_{n=1}^{\infty} \sin\left(\frac{n! \pi}{720}\right) = \sin\left(\frac{\pi}{720}\right) + \sin\left(\frac{2! \pi}{720}\right) + \\ &\quad \sin\left(\frac{3! \pi}{720}\right) + \sin\left(\frac{4! \pi}{720}\right) + \sin\left(\frac{5! \pi}{720}\right) \\ &\quad + \sin\left(\frac{6! \pi}{720}\right) + \dots + \sin\left(\frac{720! \pi}{720}\right) \\ &= \sin\left(\frac{\pi}{720}\right) + \sin\left(\frac{\pi}{360}\right) + \sin\left(\frac{\pi}{120}\right) + \sin\left(\frac{\pi}{30}\right) \\ &\quad + \sin\left(\frac{\pi}{6}\right) + \sin \pi + \dots + \sin\left(\frac{720! \pi}{720}\right) \end{aligned}$$

Here, we see that after five terms of the above series, all angles are multiples of π i.e. $\sin n\pi$, so all further values are zero.

$$\begin{aligned} \therefore E &= \sin\left(\frac{\pi}{720}\right) + \sin\left(\frac{\pi}{360}\right) + \sin\left(\frac{\pi}{120}\right) \\ &\quad + \sin\left(\frac{\pi}{30}\right) + \sin\left(\frac{\pi}{6}\right) \end{aligned}$$

30. Given, $\cos x = \tan y$, $\cot y = \tan z$ and $\cot z = \tan x$

$$\begin{aligned} \therefore \cos x = \tan y &\Rightarrow \cos x = \frac{1}{\tan z} \\ \Rightarrow \cos x = \cot z &\Rightarrow \cos x = \tan x \\ \Rightarrow \cos x = \sin x / \cos x &\Rightarrow \cos^2 x = \sin x \\ \Rightarrow 1 - \sin^2 x = \sin x &\Rightarrow \sin^2 x + \sin x - 1 = 0 \\ \therefore \sin x &= \frac{-1 \pm \sqrt{1 - 4 \times (-1)}}{2 \times 1} \\ &= \frac{-1 \pm \sqrt{5}}{2} \\ \Rightarrow \sin x &= \frac{\sqrt{5} - 1}{2} \quad \left[\because \frac{-1 - \sqrt{5}}{2} < -1 \right] \end{aligned}$$

$$\begin{aligned} 31. 2 \cot^2\left(\frac{\pi}{6}\right) + 4 \tan^2\left(\frac{\pi}{6}\right) - 3 \operatorname{cosec} \frac{\pi}{6} \\ &= 2(\sqrt{3})^2 + 4\left(\frac{1}{\sqrt{3}}\right)^2 - 3(2) \\ &= 2 \times 3 + \frac{4}{3} - 6 = \frac{4}{3} \end{aligned}$$

$$\begin{aligned} 32. \cos\left(\frac{29\pi}{3}\right) &= \cos\left[\frac{(27+2)\pi}{3}\right] = \cos\left(9\pi + \frac{2\pi}{3}\right) \\ &= -\cos\frac{2\pi}{3} = -\cos\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \sin\frac{\pi}{6} = \frac{1}{2} \end{aligned}$$

33. Given, $\sin \theta = \sin \alpha$

$$\begin{aligned} \therefore \sin \theta - \sin \alpha &= 0 \\ \Rightarrow 2 \cos\left(\frac{\theta + \alpha}{2}\right) \sin\left(\frac{\theta - \alpha}{2}\right) &= 0 \end{aligned}$$

It is not necessary that $\frac{\theta + \alpha}{2}$ is odd multiple of $\frac{\pi}{2}$ and

$\frac{\theta - \alpha}{2}$ is any multiple of π should be simultaneously

hold good for the above equation to be true. Hence, correct answer should be $\frac{\theta + \alpha}{2}$ is odd multiple of $\frac{\pi}{2}$ or

$\frac{\theta - \alpha}{2}$ is any multiple of π .

34. $\tan 81^\circ - \tan 63^\circ - \tan 27^\circ + \tan 9^\circ$

$$\begin{aligned} &= (\tan 81^\circ + \tan 9^\circ) - [\tan(63^\circ) + \tan(27^\circ)] \\ &= \left[\frac{\sin 81^\circ}{\cos 81^\circ} + \frac{\sin 9^\circ}{\cos 9^\circ}\right] - \left[\frac{\sin 63^\circ}{\cos 63^\circ} + \frac{\sin 27^\circ}{\cos 27^\circ}\right] \\ &= \left[\frac{\sin 81^\circ \cos 9^\circ + \sin 9^\circ \cos 81^\circ}{\cos 9^\circ \cos 81^\circ}\right] \\ &\quad - \left[\frac{\sin 63^\circ \cos 27^\circ + \sin 27^\circ \cos 63^\circ}{\cos 27^\circ \cos 63^\circ}\right] \\ &= \frac{[\sin(81^\circ + 9^\circ)]}{\cos 9^\circ \cos(90^\circ - 9^\circ)} - \frac{[\sin(63^\circ + 27^\circ)]}{\cos 27^\circ \cos(90^\circ - 27^\circ)} \\ &= \frac{\sin 90^\circ}{\cos 9^\circ \sin 9^\circ} - \frac{\sin 90^\circ}{\cos 27^\circ \sin 27^\circ} \\ &= \frac{1}{\sin 9^\circ \cos 9^\circ} \times \frac{2}{2} - \frac{1}{\sin 27^\circ \cos 27^\circ} \times \frac{2}{2} \\ &= \frac{2}{\sin 18^\circ} - \frac{2}{\sin 54^\circ} \\ &= 2 \left[\frac{\sin 54^\circ - \sin 18^\circ}{\sin 18^\circ \sin 54^\circ} \right] = 2 \left[\frac{2 \cos 36^\circ \sin 18^\circ}{\sin 18^\circ \cos 36^\circ} \right] = 4 \end{aligned}$$

35. Given, $\sin A + \cos A = \sqrt{2}$

$$\begin{aligned} \therefore \frac{1}{\sqrt{2}} \sin A + \frac{1}{\sqrt{2}} \cos A &= 1 \\ \Rightarrow \sin \frac{\pi}{4} \sin A + \cos A \cos \frac{\pi}{4} &= 1 \\ \Rightarrow \cos\left(A - \frac{\pi}{4}\right) &= 1 \\ \Rightarrow A - \frac{\pi}{4} = 0 &\Rightarrow A = \frac{\pi}{4} \\ \text{Now, } \cos^2 A &= \cos^2 \frac{\pi}{4} = \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \end{aligned}$$

36. The given equation is $8x^2 - 26x + 15 = 0$.

The sum of roots,

$$\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = \frac{26}{8} = \frac{13}{4}$$

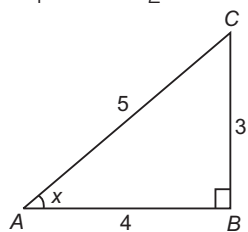
and product of roots,

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{15}{8}$$

$$\therefore \tan\left(\frac{\alpha + \beta}{2}\right) = \frac{\tan \frac{\alpha}{2} + \tan \frac{\beta}{2}}{1 - \tan \frac{\alpha}{2} \tan \frac{\beta}{2}} = \frac{\frac{13}{4}}{1 - \frac{15}{8}} = -\frac{26}{7}$$

$$\begin{aligned} \text{Now, } \cos(\alpha + \beta) &= \frac{1 - \tan^2\left(\frac{\alpha + \beta}{2}\right)}{1 + \tan^2\left(\frac{\alpha + \beta}{2}\right)} \\ &\quad \left[\because \cos 2\theta = \frac{1 - \tan^2 \theta}{1 + \tan^2 \theta} \right] \\ &= \frac{1 - \left(-\frac{26}{7}\right)^2}{1 + \left(-\frac{26}{7}\right)^2} \\ &= \frac{49 - 676}{49 + 676} = -\frac{627}{725} \end{aligned}$$

37. Given, $\tan x = \frac{3}{4}$, $\pi < x < \frac{3\pi}{2} \Rightarrow \cos x = -\frac{4}{5}$



$$\begin{aligned} \therefore 1 + \cos x &= 2 \cos^2 \left(\frac{x}{2} \right) \\ \therefore 1 - \frac{4}{5} &= 2 \cos^2 \left(\frac{x}{2} \right) \\ \Rightarrow \frac{1}{5} &= 2 \cos^2 \left(\frac{x}{2} \right) \\ \Rightarrow \cos^2 \left(\frac{x}{2} \right) &= \frac{1}{10} \\ \Rightarrow \cos \left(\frac{x}{2} \right) &= \pm \frac{1}{\sqrt{10}} \\ \Rightarrow \cos \left(\frac{x}{2} \right) &= -\frac{1}{\sqrt{10}} \quad \left[\because \frac{\pi}{2} < \frac{x}{2} < \frac{3\pi}{4} \right] \end{aligned}$$

38. Let $\frac{2\pi r}{7} = \theta \Rightarrow 2\pi r = 3\theta + 4\theta$

$$\begin{aligned} \Rightarrow 4\theta &= 2\pi r - 3\theta \\ \Rightarrow \sin 4\theta &= \sin (2\pi r - 3\theta) \\ \Rightarrow \sin 4\theta &= -\sin 3\theta \\ \Rightarrow 2 \sin 2\theta \cos 2\theta &= -[3 \sin \theta - 4 \sin^3 \theta] \\ \Rightarrow 2 \times 2 \sin \theta \cos \theta (2 \cos^2 \theta - 1) &= -3 \sin \theta + 4 \sin^3 \theta \\ \Rightarrow \sin \theta [8 \cos^3 \theta - 4 \cos \theta + 3 - 4(1 - \cos^2 \theta)] &= 0 \\ \Rightarrow 8 \cos^3 \theta + 4 \cos^2 \theta - 4 \cos \theta - 1 &= 0 \dots (i) \end{aligned}$$

Thus, $\cos \frac{2\pi}{7}$, $\cos \frac{4\pi}{7}$

and $\cos \frac{6\pi}{7}$ are the roots of Eq. (i).

$$\therefore \cos \frac{2\pi}{7} + \cos \frac{4\pi}{7} + \cos \frac{6\pi}{7} = -\frac{1}{2}$$

39. $\tan \frac{\pi}{5} + 2 \tan \frac{2\pi}{5} + 4 \cot \frac{4\pi}{5}$

$$\begin{aligned} &= \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} + 2 \frac{\sin \frac{2\pi}{5}}{\cos \frac{2\pi}{5}} + 4 \frac{\cos \frac{4\pi}{5}}{\sin \frac{4\pi}{5}} \\ &= \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} + 2 \frac{\sin \frac{2\pi}{5}}{\cos \frac{2\pi}{5}} + \frac{4 \left(\cos^2 \frac{2\pi}{5} - \sin^2 \frac{2\pi}{5} \right)}{2 \sin \frac{2\pi}{5} \cos \frac{2\pi}{5}} \\ &= \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} + 2 \frac{\sin \frac{2\pi}{5}}{\cos \frac{2\pi}{5}} + \frac{2 \left(\cos^2 \frac{2\pi}{5} - \sin^2 \frac{2\pi}{5} \right)}{\sin \frac{2\pi}{5} \cos \frac{2\pi}{5}} \\ &= \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} + 2 \left[\frac{\sin^2 \frac{2\pi}{5} + \cos^2 \frac{2\pi}{5} - \sin^2 \frac{2\pi}{5}}{\cos \frac{2\pi}{5} \sin \frac{2\pi}{5}} \right] \end{aligned}$$

$$\begin{aligned} &= \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} + \frac{2 \cos \frac{2\pi}{5}}{\sin \frac{2\pi}{5}} = \frac{\sin \frac{\pi}{5}}{\cos \frac{\pi}{5}} + \frac{2 \left(\cos^2 \frac{\pi}{5} - \sin^2 \frac{\pi}{5} \right)}{2 \sin \frac{\pi}{5} \cos \frac{\pi}{5}} \\ &= \frac{\sin^2 \frac{\pi}{5} + \cos^2 \frac{\pi}{5} - \sin^2 \frac{\pi}{5}}{\sin \frac{\pi}{5} \cos \frac{\pi}{5}} = \frac{\cos^2 \frac{\pi}{5}}{\sin \frac{\pi}{5} \cos \frac{\pi}{5}} = \cot \frac{\pi}{5} \end{aligned}$$

40. Given, $\cos x + \cos y = \frac{3}{2}$

$$\Rightarrow 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) = \frac{3}{2} \dots (i)$$

and $\sin x + \sin y = \frac{3}{4}$

$$\Rightarrow 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right) = \frac{3}{4} \dots (ii)$$

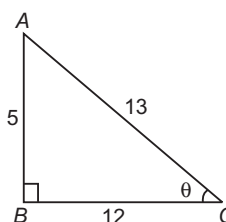
On dividing Eq. (ii) by Eq. (i), we get

$$\begin{aligned} \frac{2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)}{2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)} &= \frac{3/4}{3/2} \\ \Rightarrow \tan \left(\frac{x+y}{2} \right) &= \frac{1}{2} \end{aligned}$$

$$\begin{aligned} \therefore \sin (x+y) &= \frac{2 \tan \left(\frac{x+y}{2} \right)}{1 + \tan^2 \left(\frac{x+y}{2} \right)} = \frac{2 \times \frac{1}{2}}{1 + \left(\frac{1}{2} \right)^2} \\ &= \frac{4}{4+1} = \frac{4}{5} \end{aligned}$$

41. Given, $\tan x = \frac{5}{12}$ and x is in IIIrd quadrant.

$$\therefore \sin x = \frac{-5}{13} \text{ and } \cos x = \frac{-12}{13}$$



Now, $\cos x = 2 \cos^2 \frac{x}{2} - 1$

$$\begin{aligned} \Rightarrow \cos^2 \frac{x}{2} &= \frac{1}{2} (\cos x + 1) \\ &= \frac{1}{2} \left(\frac{-12}{13} + 1 \right) = \frac{1}{2} \left(\frac{1}{13} \right) = \frac{1}{26} \end{aligned}$$

$$\therefore \cos \frac{x}{2} = \sqrt{\frac{1}{26}}$$

42. $\sin \frac{\pi}{2n} + \cos \frac{\pi}{2n} = \frac{\sqrt{n}}{2}$

On squaring both sides, we get

$$\begin{aligned} \sin^2 \frac{\pi}{2n} + \cos^2 \frac{\pi}{2n} + 2 \sin \frac{\pi}{2n} \cos \frac{\pi}{2n} &= \frac{n}{4} \\ \Rightarrow 1 + \sin \frac{2\pi}{2n} &= \frac{n}{4} \Rightarrow 1 + \sin \frac{\pi}{n} = \frac{n}{4} \end{aligned}$$

Consider $n = 6$

$$\begin{aligned} \therefore 1 + \sin \frac{\pi}{6} &= \frac{6}{4} \Rightarrow 1 + \frac{1}{2} = \frac{6}{4} \\ \Rightarrow \frac{3}{2} &= \frac{3}{2} \quad [\text{satisfy}] \\ \therefore 4 &< n < 8 \end{aligned}$$

43. We know that,

$$\cos x \cos \left(\frac{\pi}{3} - x \right) \cos \left(\frac{\pi}{3} + x \right) = \frac{1}{4} \cos 3\theta$$

But it is given that

$$\cos x \cos \left(\frac{\pi}{3} - x \right) \cos \left(\frac{\pi}{3} + x \right) = \frac{1}{4}$$

$$\begin{aligned} \therefore \cos 3\theta &= 1 \\ \Rightarrow 3\theta &= 2\pi, 4\pi \\ \Rightarrow \theta &= \frac{2\pi}{3}, \frac{4\pi}{3} \\ \therefore \text{Sum of solutions} &= \frac{2\pi}{3} + \frac{4\pi}{3} = \frac{6\pi}{3} = 2\pi \end{aligned}$$

44. We know that,

Range of $\sin x \in [-1, 1]$
Range of $\cos x \in [-1, 1]$
Range of $\sec x \in (-\infty, -1] \cup [1, \infty)$
and range of $\operatorname{cosec} x \in (-\infty, -1] \cup [1, \infty)$

45. $\sin 2\alpha = \sin\{(\alpha + \beta) + (\alpha - \beta)\}$

$$= \sin(\alpha + \beta) \cos(\alpha - \beta) + \cos(\alpha + \beta) \sin(\alpha - \beta) \quad \dots (i)$$

$$\begin{aligned} \text{Now, } \sin(\alpha + \beta) &= \sqrt{1 - \cos^2(\alpha + \beta)} \\ &= \sqrt{1 - \frac{9}{25}} = \frac{4}{5} \quad \left[\because 0 < \alpha + \beta < \frac{\pi}{2} \right] \end{aligned}$$

$$\begin{aligned} \text{and } \sin(\alpha - \beta) &= \sqrt{1 - \cos^2(\alpha - \beta)} \\ &= \sqrt{1 - \frac{16}{25}} = \frac{3}{5} \quad \left[\because 0 < \alpha - \beta < \frac{\pi}{4} \right] \end{aligned}$$

From Eq. (i), we have

$$\sin 2\alpha = \frac{4}{5} \cdot \frac{4}{5} + \frac{3}{5} \cdot \frac{3}{5} = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

$$46. \tan^2 x = \frac{\sqrt{2} - 1}{\sqrt{2} + 1} = (\sqrt{2} - 1)^2 = \tan^2 \frac{\pi}{8}$$

$$\Rightarrow \tan x = \tan \left(\pm \frac{\pi}{8} \right) \Rightarrow x = n\pi \pm \frac{\pi}{8}$$

$$47. (1 - \cos 2x)(\cos 2x + \cot^2 x) = 0 \Rightarrow \cos 2x = 1$$

$$\Rightarrow x = 0, \pi, 2\pi$$

But for all values of x , $\cot x$ is not defined.
Hence, no solution exist.

$$48. \log_{\cos x} \sin x - 1 + \log_{\sin x} \cos x - 1 = 0$$

$$\text{or } y + \frac{1}{y} - 2 = 0, \text{ where } y = \log_{\cos x} \sin x$$

$$\therefore (y - 1)^2 = 0 \Rightarrow \log_{\cos x} \sin x = 1$$

$$\Rightarrow \sin x = \cos x \Rightarrow \tan x = 1$$

$$\Rightarrow x = n\pi + \frac{\pi}{4}$$

Also, $\sin x, \cos x, \tan x, \cot x$ must be positive and $\sin x \neq 1, \cos x \neq 0$.

$$\therefore x = 2n\pi + \frac{\pi}{4}, n \in \mathbb{Z}$$

$$49. \text{ Given, } x^3 + x^2 + 4x + 2\sin x = 0$$

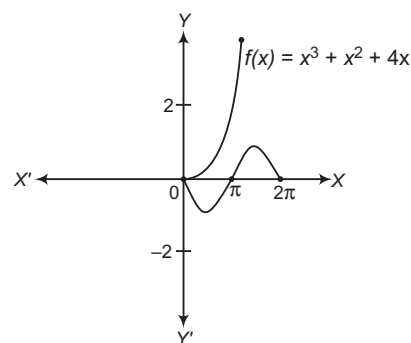
$$\Rightarrow x^3 + x^2 + 4x = -2\sin x$$

$$\text{As } x \in [0, 2\pi], \sin x \in [-1, 1]$$

$$\Rightarrow -2\sin x \in [-2, 2]$$

$$\text{Now, consider } f(x) = x^3 + x^2 + 4x$$

$$\Rightarrow f'(x) = 3x^2 + 2x + 4$$



$$\text{Here, } a > 0, D = 4 - 4(3)(4) < 0$$

$$\therefore f'(x) > 0, \forall x$$

So, $f(x)$ is a strictly increasing function.

$$\text{Now, } f(0) = 0$$

As evident from graph, $-2\sin x$ and $x^3 + x^2 + 4x$ will intersect only at $x = 0$.

$\therefore$ Number of solutions = 1

50. Given,

$$(\sin 3x + \sin x) - 3\sin 2x = (\cos 3x + \cos x) - 3\cos 2x$$

$$\Rightarrow 2\sin 2x \cos x - 3\sin 2x = 2\cos 2x \cos x - 3\cos 2x$$

$$\Rightarrow \sin 2x (2\cos x - 3) = \cos 2x (2\cos x - 3)$$

$$\Rightarrow (2\cos x - 3)(\sin 2x - \cos 2x) = 0$$

$$\therefore \cos x = \frac{3}{2}, \text{ which is not possible. } [\because -1 \leq \cos x \leq 1]$$

$$\therefore \sin 2x = \cos 2x$$

$$\Rightarrow \tan 2x = 1 = \tan \frac{\pi}{4}$$

$$\Rightarrow 2x = n\pi + \frac{\pi}{4}$$

$$\therefore x = \frac{n\pi}{2} + \frac{\pi}{8} \text{ for } n \text{ integers}$$

51. Given, $\sin A + \sin B + \sin C = 3$

$$\therefore \sin A = \sin B = \sin C = 1 \quad [\because -1 \leq \sin x \leq 1]$$

$$\Rightarrow A = B = C = \frac{\pi}{2}$$

$$\Rightarrow \cos A + \cos B + \cos C = 0$$

52. Let $a = \cos \alpha + i \sin \alpha, b = \cos \beta + i \sin \beta$

$$\text{and } c = \cos \gamma + i \sin \gamma$$

$$\text{Then, } a + 2b + 3c = (\cos \alpha + 2\cos \beta + 3\cos \gamma)$$

$$+ i(\sin \alpha + 2\sin \beta + 3\sin \gamma) = 0$$

$$\text{We know that, } a^3 + 8b^3 + 27c^3 = 18abc$$

$$\therefore \cos 3\alpha + 8\cos 3\beta + 27\cos 3\gamma = 18\cos(\alpha + \beta + \gamma)$$

$$\text{and } \sin 3\alpha + 8\sin 3\beta + 27\sin 3\gamma$$

$$= 18\sin(\alpha + \beta + \gamma) = 18\sin \pi = 0$$

53. Given, $\frac{\cos A}{\cos B} = m$

On applying componendo and dividendo rule, we get

$$\begin{aligned} \frac{\cos A + \cos B}{\cos A - \cos B} &= \frac{m+1}{m-1} \\ \Rightarrow \frac{2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right)}{2 \sin \left(\frac{A+B}{2} \right) \sin \left(\frac{B-A}{2} \right)} &= \frac{m+1}{m-1} \\ \Rightarrow \lambda = \frac{m+1}{m-1} &\Rightarrow \frac{\cot \left(\frac{A+B}{2} \right)}{\tan \left(\frac{B-A}{2} \right)} = \frac{m+1}{m-1} \end{aligned}$$

54. Consider, $3 \sin^2 \theta + 4 \cos^2 \theta$
 $= 3 \sin^2 \theta + 3 \cos^2 \theta + \cos^2 \theta = 3 + \cos^2 \theta$
 We know that, least value of $\cos^2 \theta$ is 0.
 Hence, least value of $3 \sin^2 \theta + 4 \cos^2 \theta$ is 3.

55. We have, $\sin 2x = 4 \cos x \Rightarrow 2 \sin x \cos x = 4 \cos x$
 $\Rightarrow \cos x (\sin x - 2) = 0$
 $\Rightarrow \cos x = 0$ [$\because \sin x \neq 2$]
 $\Rightarrow \cos x = 0 = \cos \pi/2$
 $\Rightarrow x = 2n\pi \pm \frac{\pi}{2}, n \in \mathbb{Z}$

56. We have, $\tan x + \sec x = 2 \cos x$
 $\Rightarrow \frac{\sin x}{\cos x} + \frac{1}{\cos x} = 2 \cos x$
 $\Rightarrow \sin x + 1 = 2 \cos^2 x = 2(1 - \sin^2 x) = 2 - 2 \sin^2 x$
 $\Rightarrow 2 \sin^2 x + \sin x - 1 = 0$
 $\Rightarrow (\sin x + 1)(2 \sin x - 1) = 0$
 $\Rightarrow \sin^2 x = -1$, which is not possible. [$\because 0 \leq x \leq \pi$]
 or $\sin x = \frac{1}{2} \Rightarrow x = \frac{\pi}{6}, \frac{5\pi}{6}$
 $\therefore x \in [0, \pi]$
 Hence, the number of solutions is 2.

57. Consider, $\cos \theta + \sqrt{3} \sin \theta = 2$
 $\Rightarrow \frac{1}{2} \cos \theta + \frac{\sqrt{3}}{2} \sin \theta = 1$
 $\Rightarrow \cos 60^\circ \cos \theta + \sin 60^\circ \sin \theta = 1$
 $\Rightarrow \cos(60^\circ - \theta) = 1 = \cos 0^\circ \Rightarrow 60^\circ - \theta = 0^\circ$
 $\therefore \theta = 60^\circ$

58. Given, $\sin \theta = 2t/1+t^2$
 $\therefore \cos \theta < 0$
 $\Rightarrow \cos \theta = -\sqrt{1 - \sin^2 \theta}$
 $= -\sqrt{1 - \frac{4t^2}{(1+t^2)^2}} = -\frac{|1-t^2|}{1+t^2}$

59. $\therefore \frac{\sin \theta}{\sin(\theta + 2\alpha)} = \frac{3}{1}$
 On applying componendo and dividendo rule, we get
 $\frac{\sin \theta + \sin(\theta + 2\alpha)}{\sin \theta - \sin(\theta + 2\alpha)} = \frac{3+1}{3-1}$
 $\Rightarrow \frac{2 \sin(\theta + \alpha) \cos \alpha}{-2 \cos(\theta + \alpha) \sin \alpha} = 2$
 $\Rightarrow -\tan(\theta + \alpha) \cot \alpha = 2$
 $\Rightarrow \tan(\theta + \alpha) + 2 \tan \alpha = 0$

60. Since, α, β and γ are in AP.

$$\begin{aligned} \therefore 2\beta &= \alpha + \gamma \\ \text{Now, } \frac{\sin \alpha - \sin \gamma}{\cos \gamma - \cos \alpha} &= \frac{2 \cos \left(\frac{\alpha + \gamma}{2} \right) \sin \left(\frac{\alpha - \gamma}{2} \right)}{2 \sin \left(\frac{\alpha + \gamma}{2} \right) \sin \left(\frac{\alpha - \gamma}{2} \right)} \\ &= \cot \left(\frac{\alpha + \gamma}{2} \right) = \cot \left(\frac{2\beta}{2} \right) \quad [\because 2\beta = \alpha + \gamma] \\ &= \cot \beta \end{aligned}$$

61. $\therefore \sec A = \frac{\sec(A-B) + \sec(A+B)}{2}$
 $= \frac{\cos(A+B) + \cos(A-B)}{2 \cos(A+B) \cos(A-B)}$
 $= \frac{\cos A \cos B}{[\cos^2 A \cos^2 B - \sin^2 A \sin^2 B]}$
 $\Rightarrow \sec A = \frac{\cos A \cos B}{[\cos^2 A \cos^2 B - (1 - \cos^2 A)(1 - \cos^2 B)]}$

$$\begin{aligned} \Rightarrow \cos^2 A + \cos^2 B - 1 &= \cos^2 A \cos B \\ \Rightarrow \cos^2 A &= 1 + \cos B \Rightarrow \cos^2 A = 2 \cos^2 \frac{B}{2} \\ \Rightarrow \sec^2 A &= \frac{1}{2} \sec^2 \frac{B}{2} \Rightarrow 2 \sec^2 A = \sec^2 \frac{B}{2} \end{aligned}$$

62. $\therefore \sin \theta = \frac{1}{2} = \sin \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}, \pi - \frac{\pi}{6}$
 and $\tan \theta = \frac{1}{\sqrt{3}} = \tan \frac{\pi}{6} \Rightarrow \theta = \frac{\pi}{6}, \pi + \frac{\pi}{6}$
 $\therefore$ Common value of θ is $\frac{\pi}{6}$.
 $\therefore$ General value of θ is $2n\pi + \frac{\pi}{6}, \forall n \in \mathbb{I}$.

63. Consider, $\sin 5\theta - \sin 3\theta + \sin \theta = 0$
 $\Rightarrow 2 \cos 4\theta \sin \theta + \sin \theta = 0$
 $\Rightarrow \sin \theta (2 \cos 4\theta + 1) = 0$
 $\Rightarrow \cos 4\theta = -\frac{1}{2}$ [$\because \sin \theta \neq 0$, for $0 < \theta < \frac{\pi}{2}$]
 $\Rightarrow \cos 4\theta = \cos \frac{2\pi}{3} = \cos \frac{4\pi}{3}$
 $\therefore \theta = \frac{\pi}{6} \text{ or } \frac{\pi}{3}$

64. Note that maximum value of $2 \sin x + \cos x$
 $= \sqrt{2^2 + 1^2} = \sqrt{5} = 2.236$
 Hence, $2 \sin x + \cos x$ has no solution.

65. Given, $\sin \theta + \cos \theta = 0 \Rightarrow \tan \theta = -1$
 $\therefore \theta = \frac{3\pi}{4} \in (0, \pi)$

66. $\cos^4 x + \sin^4 x = 2 \cos(2x + \pi) \cos(2x - \pi)$
 $\Rightarrow (\cos^2 x + \sin^2 x)^2 - 2 \sin^2 x \cos^2 x = \cos 4x + \cos 2\pi$
 $\Rightarrow 1 - \frac{1}{2} \sin^2 2x = \cos 4x + 1$
 $\Rightarrow -\frac{1}{2} \left(\frac{1 - \cos 4x}{2} \right) = \cos 4x \Rightarrow -\frac{1}{4} = \frac{3}{4} \cos 4x$
 $\Rightarrow \cos 4x = -\frac{1}{3} \Rightarrow \sin 4x = \pm \frac{2\sqrt{2}}{3}$
 $\Rightarrow x = \frac{2n\pi}{4} \pm \frac{1}{4} \cos^{-1} \left(-\frac{1}{3} \right)$
 $\Rightarrow x = \frac{n\pi}{4} + \frac{(-1)^n}{4} \sin^{-1} \left(\pm \frac{2\sqrt{2}}{3} \right)$

67. Let $\Delta = \begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$ and taking common from R_1 , we get

$$\Delta = (\sin x + \cos x + \cos x) \begin{vmatrix} 1 & 1 & 1 \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}$$

On applying $C_2 \rightarrow C_2 - C_1, C_3 \rightarrow C_3 - C_1$, we get

$$\Delta = (\sin x + 2 \cos x) \begin{vmatrix} 1 & 0 & 1 \\ \cos x & \sin x - \cos x & \cos x \\ \cos x & 0 & \sin x - \cos x \end{vmatrix}$$

$$\Rightarrow 0 = (\sin x + 2 \cos x) 1 [(\sin x - \cos x)^2]$$

$$\Rightarrow \tan x = -2, \tan x = 1$$

$$\text{or } x = \frac{\pi}{4}, \tan^{-1}(-2)$$

Hence, two values of x exist in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

68. Given, $\cos A = \frac{3}{5}$ and $\cos B = \frac{4}{5}$

$\therefore \angle A$ and $\angle B$ lie on IVth quadrant.

$$\therefore \sin A = -\sqrt{1 - \frac{9}{25}}$$

$$\text{and } \sin B = -\sqrt{1 - \frac{16}{25}}$$

$$\Rightarrow \sin A = -\frac{4}{5}$$

$$\text{and } \sin B = -\frac{3}{5}$$

$$\text{Now, } 2 \sin A + 4 \sin B = 2 \left(-\frac{4}{5}\right) + 4 \left(-\frac{3}{5}\right) = -\frac{8}{5} - \frac{12}{5} = -\frac{20}{5} = -4$$

69. $\frac{\cot 54^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot 70^\circ} = \frac{\cot (90^\circ - 36^\circ)}{\tan 36^\circ} + \frac{\tan 20^\circ}{\cot (90^\circ - 20^\circ)}$
 $= \frac{\tan 36^\circ}{\tan 36^\circ} + \frac{\tan 20^\circ}{\tan 20^\circ}$
 $= 1 + 1 = 2$

70. $\therefore \tan 60^\circ = \tan (40^\circ + 20^\circ)$

$$\Rightarrow \sqrt{3} = \frac{\tan 40^\circ + \tan 20^\circ}{1 - \tan 40^\circ \tan 20^\circ}$$

$$\Rightarrow \sqrt{3} - \sqrt{3} \tan 40^\circ \tan 20^\circ = \tan 40^\circ + \tan 20^\circ$$

$$\therefore \tan 40^\circ + \tan 20^\circ + \sqrt{3} \tan 40^\circ \tan 20^\circ = \sqrt{3}$$

71. $\frac{\sin 55^\circ - \cos 55^\circ}{\sin 10^\circ} = \frac{\sin 55^\circ - \sin 35^\circ}{\sin 10^\circ}$
 $= \frac{2 \cos 45^\circ \sin 10^\circ}{\sin 10^\circ} = \sqrt{2}$

72. Given, $\cos(\alpha + \beta) = -\frac{12}{13}$

Here, $0 < (\alpha + \beta) < \pi$

$$\therefore \sin(\alpha + \beta) = \sqrt{1 - \cos^2(\alpha + \beta)}$$

$$= \sqrt{1 - \frac{144}{169}} = \frac{5}{13}$$

Now, $\sin \beta = \sin[(\alpha + \beta) - \alpha]$

$$= \sin(\alpha + \beta) \cos \alpha - \cos(\alpha + \beta) \sin \alpha$$

$$= \frac{5}{13} \cdot \frac{3}{5} - \left(-\frac{12}{13}\right) \cdot \frac{4}{5} = \frac{15}{65} + \frac{48}{65} = \frac{63}{65}$$

73. Given, $\tan \alpha = \frac{b}{a}, \sqrt{\frac{a+b}{a-b}} - \sqrt{\frac{a-b}{a+b}} = \sqrt{\frac{1+\frac{b}{a}}{1-\frac{b}{a}}} - \sqrt{\frac{1-\frac{b}{a}}{1+\frac{b}{a}}}$
 $= \frac{\sqrt{1+\tan \alpha}}{\sqrt{1-\tan \alpha}} - \frac{\sqrt{1-\tan \alpha}}{\sqrt{1+\tan \alpha}} = \frac{(1+\tan \alpha) - (1-\tan \alpha)}{\sqrt{1-\tan^2 \alpha}}$
 $= \frac{2 \tan \alpha}{\sqrt{1-\tan^2 \alpha}} = \frac{2 \sin \alpha}{\sqrt{\cos 2\alpha}}$

74. $\frac{\cot x - \tan x}{\cot 2x} = \tan 2x \left(\cot x - \frac{1}{\cot x} \right)$
 $= \tan 2x \left(\frac{\cot^2 x - 1}{\cot x} \right)$
 $= \tan 2x \left(\frac{\cot^2 x - 1}{2 \cot x} \right) \cdot 2$
 $= \tan 2x \cot 2x \cdot 2 = 2$

75. $\therefore \sin x = \frac{1}{2}$

$$\Rightarrow \sin x = \sin \frac{\pi}{6}$$

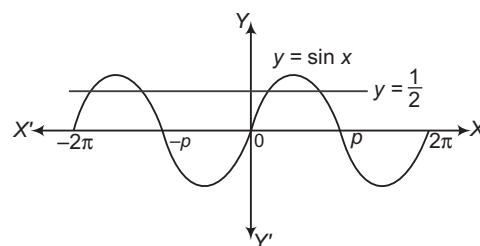
$$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$$

For $-2\pi \leq x \leq 2\pi$,

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{11\pi}{6}$$

$\therefore$ Number of points of intersection of two given curves
 $= 4$

Aliter



So, the number of points of intersection is 4.

76. $\therefore 2 \sin x \cos x = \frac{1}{2}$

$$\Rightarrow \sin 2x = \frac{1}{2} = \sin \frac{\pi}{6}$$

$$\Rightarrow 2x = n\pi + (-1)^n \frac{\pi}{6}$$

$$\Rightarrow x = \frac{n\pi}{2} + (-1)^n \frac{\pi}{12}$$

For $x \in \left(0, \frac{\pi}{2}\right), x = \frac{\pi}{12}$ [for $n = 0$]

77. $\therefore \cos 2\theta = \sin \theta$

$$\Rightarrow 1 - 2 \sin^2 \theta = \sin \theta$$

$$\Rightarrow 2 \sin^2 \theta + \sin \theta - 1 = 0$$

$$\Rightarrow 2 \sin^2 \theta + 2 \sin \theta - \sin \theta - 1 = 0$$

$$\begin{aligned} \Rightarrow 2 \sin \theta (\sin \theta + 1) - (\sin \theta + 1) &= 0 \\ \Rightarrow (\sin \theta + 1)(2 \sin \theta - 1) &= 0 \\ \Rightarrow \sin \theta = -1 \quad \text{and} \quad \sin \theta &= \frac{1}{2} \\ \Rightarrow \sin \theta &= \sin \frac{3\pi}{2} \\ \text{and} \quad \sin \theta &= \sin \frac{\pi}{6} \\ \Rightarrow \theta &= n\pi + (-1)^n \frac{3\pi}{2} \\ \text{and} \quad \theta &= m\pi + (-1)^m \frac{\pi}{6} \\ \text{For } \theta \in (0, 2\pi), \\ \theta &= \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6} \\ \text{Hence, the number of solutions is 3.} \end{aligned}$$

$$\begin{aligned} 78. \therefore \sin 6\theta + \sin 4\theta + \sin 2\theta &= 0 \\ \Rightarrow (\sin 6\theta + \sin 2\theta) + \sin 4\theta &= 0 \\ \Rightarrow 2 \sin 4\theta \cos 2\theta + \sin 4\theta &= 0 \\ \Rightarrow \sin 4\theta (2 \cos 2\theta + 1) &= 0 \\ \text{Either} \quad \sin 4\theta &= 0 \\ \text{or} \quad \cos 2\theta &= -\frac{1}{2} \\ \text{When } \sin 4\theta = 0, \text{ then} \\ 4\theta &= n\pi \\ \Rightarrow \theta &= \frac{n\pi}{4} \\ \text{When } \cos 2\theta = -\frac{1}{2} = \cos \frac{2\pi}{3}, \text{ then} \\ 2\theta &= 2n\pi \pm \frac{2\pi}{3} \\ \Rightarrow \theta &= n\pi \pm \frac{\pi}{3} \end{aligned}$$

10

Properties of Triangles, Heights and Distances

Introduction

Properties of triangle is a part of trigonometry, which is a branch of Mathematics. In which we study about triangles and relationships between their sides and angles.

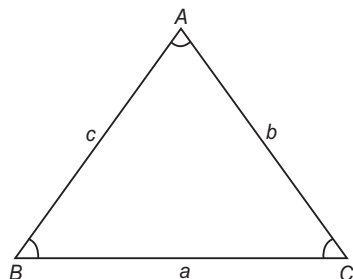
Height and distance is an application of trigonometry. Here, with the help of trigonometry, we find the height of any object or distance of any object from any point/object which cannot be measured directly.

Relation between the Sides and Angles of Triangle

In a $\triangle ABC$, the angles are denoted by capital letters A, B and C and the lengths of the sides opposite to these angles are denoted by small letters a, b and c , respectively.

Semi-perimeter of the triangle is written as

$$s = \frac{a + b + c}{2} \text{ and its area denoted by } s \text{ or } \Delta.$$



- The longest side of a triangle have corresponding largest angle and vice-versa.
- The smallest side of a triangle have corresponding smallest angle and vice-versa.

Chapter Snapshot

- Introduction
- Relation between the Sides and Angles of Triangle
- Trigonometric Ratios of Half Angles of a Triangle
- Area of a Triangle
- Conditional Identities
- Solution of Triangles
- Circles Connected with Triangle
- The Orthocentre and the Pedal Triangle
- Cyclic Quadrilateral
- Regular Polygon
- Heights and Distances
- Some Important Properties of Triangles
- Some Properties Related to Circle

Sine rule In any $\triangle ABC$, the sines of the angles are proportional to the lengths of the opposite sides i.e.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \quad [\text{say}]$$

Then, $a = k \sin A$, $b = k \sin B$, $c = k \sin C$

Cosine rule In any $\triangle ABC$, cosine of an angle can be expressed in terms of sides.

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

Projection formula In any $\triangle ABC$,

$$a = b \cos C + c \cos B$$

$$b = a \cos C + c \cos A$$

$$c = a \cos B + b \cos A$$

Napier's analogy In any $\triangle ABC$,

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b} \cot \frac{C}{2}$$

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot \frac{A}{2}$$

$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a} \cot \frac{B}{2}$$

➤ **Example 1.** The sides of a triangle are 8 cm, 10 cm and 12 cm. Then, the greatest angle is

- (a) triple of the smallest angle
(b) double of the smallest angle
(c) equal to the smallest angle
(d) None of the above

Sol. (b) Let $a = 8$ cm, $b = 10$ cm and $c = 12$ cm. Then, greatest angle is C and the smallest angle is A . Applying cosine rule, we get

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 100 - 144}{2 \cdot 8 \cdot 10} = \frac{1}{8}$$

and $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{100 + 144 - 64}{2 \cdot 10 \cdot 12} = \frac{3}{4}$

Now, $\cos 2A = 2\cos^2 A - 1 = 2 \cdot \frac{9}{16} - 1 = \frac{1}{8}$

$$\Rightarrow \cos C = \cos 2A$$

$$\Rightarrow C = 2A$$

Thus, the greatest angle is double of the smallest angle.

➤ **Example 2.** If k is the perimeter of $\triangle ABC$, then

$$b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} \text{ is equal to}$$

- (a) k
(b) $2k$
(c) $\frac{k}{2}$
(d) None of the above

Sol. (c) $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2}$

$$= \frac{b}{2} (1 + \cos C) + \frac{c}{2} (1 + \cos B)$$

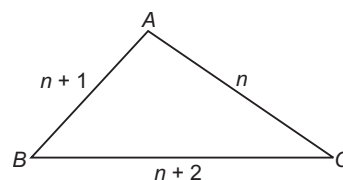
$$= \frac{b}{2} + \frac{c}{2} + \frac{1}{2} (b \cos C + c \cos B)$$

$$= \frac{a + b + c}{2} = \frac{k}{2}$$

➤ **Example 3.** The sides of a triangle are three consecutive natural numbers and its largest angles is twice the smallest one. Then, the sides of the triangle are

- (a) 3, 4, 5
(b) 4, 5, 6
(c) 2, 5, 3
(d) 4, 3, 7

Sol. (b) Let the sides be $AC = n$, $AB = n + 1$, $BC = n + 2$. Smallest angle is B and the largest one is A .



$$\Rightarrow A = 2B$$

Also, $A + B + C = 180^\circ$

$$\Rightarrow 3B + C = 180^\circ$$

$$\Rightarrow C = 180^\circ - 3B$$

We have, $\frac{\sin A}{n+2} = \frac{\sin B}{n} = \frac{\sin C}{n+1}$

$$\Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin(180^\circ - 3B)}{n+1}$$

$$\Rightarrow \frac{\sin 2B}{n+2} = \frac{\sin B}{n} = \frac{\sin 3B}{n+1}$$

$$\Rightarrow \cos B = \frac{n+2}{2n}$$

and $\frac{\sin 2B}{n+2} = \frac{3 \sin B - 4 \sin^3 B}{n+1}$

$$\Rightarrow \frac{2 \cos B}{n+2} = \frac{4 \cos^2 B - 1}{n+1}$$

$$\Rightarrow \frac{1}{n} = \frac{(n+2)^2 - 1}{n^2}$$

$$\Rightarrow n = 4$$

Hence, the sides are 4, 5 and 6.

➤ **Example 4.** In a $\triangle ABC$,

$$\Sigma (b+c) \tan \frac{A}{2} \tan \left(\frac{B-C}{2} \right) \text{ is equal to}$$

- (a) a
(b) b
(c) c
(d) 0

Sol. (d) Now, $(b+c) \tan \frac{A}{2} \tan \left(\frac{B-C}{2} \right)$

$$= (b+c) \tan \frac{A}{2} \times \frac{b-c}{b+c} \cot \frac{A}{2} = b - c$$

$$\therefore \Sigma (b+c) \tan \left(\frac{B-C}{2} \right) \tan \frac{A}{2} = b - c + c - a + a - b = 0$$

Trigonometric Ratios of Half Angles of a Triangle

In a $\triangle ABC$, if the sides of the triangle corresponding to angles A, B and C , are a, b, c respectively and s is the semi-perimeter, then

$$\begin{aligned} \sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}, \sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}, \\ \sin \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}, \\ \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}, \cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}, \\ \cos \frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}, \\ \tan \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \tan \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \\ \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} \end{aligned}$$

Area of a Triangle

In a $\triangle ABC$, if the sides of the triangle are a, b, c and corresponding angles are A, B and C , respectively, then area of triangle is given by

- When two sides and angle between them is known, then

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B$$
- When one side and corresponding angles are known, then

$$\Delta = \frac{c^2 \sin A \sin B}{2 \sin C} = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin C \sin A}{2 \sin B}$$
- When all the three sides are known, then

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}$$

It is known as Heron's formula.

➤ **Example 5.** In a $\triangle ABC$, $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm

and $\text{ar}(\triangle ABC) = \frac{9\sqrt{3}}{2} \text{ cm}^2$. Then, a is

- (a) $6\sqrt{3}$ cm (b) 9 cm
(c) 18 cm (d) None of these

Sol. (b) $\frac{1}{2} bc \sin \frac{2\pi}{3} = \frac{9\sqrt{3}}{2}$

$$\Rightarrow \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot bc = \frac{9\sqrt{3}}{2} \Rightarrow bc = 18$$

$$\text{Also, } \cos \frac{2\pi}{3} = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow -\frac{1}{2} = \frac{(b-c)^2 + 2bc - a^2}{2bc}$$

$$\Rightarrow (b-c)^2 + 3bc - a^2 = 0$$

$$\Rightarrow 27 + 54 = a^2 \Rightarrow a^2 = 81 \Rightarrow a = 9 \text{ cm}$$

➤ **Example 6.** The area of a $\triangle ABC$ is $a^2 - (b-c)^2$. Then, $\tan A$ is equal to

- (a) $\frac{4}{3}$ (b) $\frac{3}{4}$
(c) $\frac{8}{15}$ (d) None of these

Sol. (c) Here, $\Delta = a^2 - (b-c)^2$

$$\Rightarrow \Delta = (2s-2c)(2s-2b) = 4(s-b)(s-c)$$

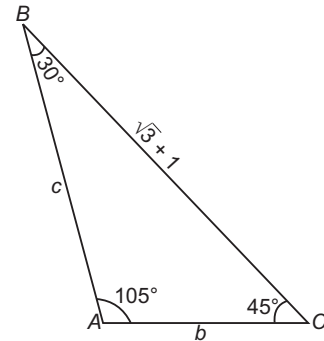
$$\therefore \frac{1}{4} = \frac{(s-b)(s-c)}{\Delta} = \tan^2 \frac{A}{2}$$

$$\Rightarrow \tan A = \frac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}} = \frac{2 \times \frac{1}{4}}{1 - \left(\frac{1}{4}\right)^2} = \frac{8}{15}$$

➤ **Example 7.** If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3} + 1)$ cm, then the area of the triangle (in cm^2) is

- (a) $\frac{1}{2}(\sqrt{3} + 1)$ (b) $\frac{1}{4}(\sqrt{3} + 1)$
(c) $\frac{1}{2}(\sqrt{3} - 1)$ (d) $\frac{1}{4}(\sqrt{3} - 1)$

Sol. (a) We have, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$



$$\Rightarrow \frac{\sin 105^\circ}{\sqrt{3} + 1} = \frac{\sin 30^\circ}{b} = \frac{\sin 45^\circ}{c}$$

$$\Rightarrow b = \frac{\sqrt{3} + 1}{2 \sin 105^\circ}, c = \frac{\sqrt{3} + 1}{\sqrt{2} \sin 105^\circ}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} bc \sin A = \frac{1}{2} bc \sin 105^\circ$$

$$= \frac{1}{2} \cdot \frac{(\sqrt{3} + 1)^2}{2 \sqrt{2} \sin(60^\circ + 45^\circ)}$$

$$= \frac{(\sqrt{3} + 1)^2}{4\sqrt{2} \left[\frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \right]} = \frac{(\sqrt{3} + 1)^2}{2(\sqrt{3} + 1)} = \frac{1}{2}(\sqrt{3} + 1) \text{ cm}^2$$

Conditional Identities

Some standard identities in triangle are given below :

- i. $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
- ii. $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{C}{2} \tan \frac{B}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$
- iii. $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$
- iv. $\cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C$
- v. $\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
- vi. $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$

➤ **Example 8.** If A, B, C are vertices of triangle, then $\cos^2 A + \cos^2 B + \cos^2 C + 2 \cos A \cos B \cos C$ is equal to

- (a) 1 (b) -1
(c) 0 (d) None of these

Sol. (a) Consider, $\cos^2 A + \cos^2 B + \cos^2 C$

$$= \frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} + \frac{1 + \cos 2C}{2}$$

$$= \frac{1}{2}(\cos 2A + \cos 2B + \cos 2C) + \frac{3}{2}$$

$$= \frac{1}{2}(-1 - 4 \cos A \cos B \cos C) + \frac{3}{2}$$

$$= 1 - 2 \cos A \cos B \cos C$$

Solution of Triangles

The three sides a, b, c and the three angles A, B, C are called the elements of the ΔABC . When any three of these six elements (except all the three angles) of a triangle are given, the triangle is known completely; that is the other three elements can be expressed in terms of the given elements and can be evaluated. This process is called the solution of triangles.

- i. If the three sides a, b and c are given, then angle A is obtained from

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \text{ or } \cos A = \frac{b^2 + c^2 - a^2}{2bc}$$
 B and C can be obtained in a similar way.
- ii. If two sides b and c and the included angle A are given, then $\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c} \cot \frac{A}{2}$ gives $\frac{B-C}{2}$.

Also, $\frac{B+C}{2} = 90^\circ - \frac{A}{2}$, so B and C can be evaluated.

The third side is given by

$$a = b \frac{\sin A}{\sin B}$$

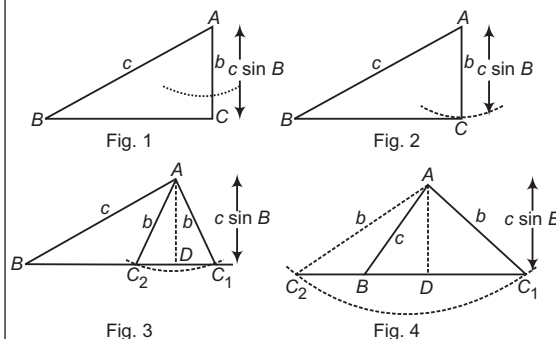
or $a^2 = b^2 + c^2 - 2bc \cos A$

- iii. If two sides b and c and the angle B (opposite to side b) are given, then

$$\sin C = \frac{c}{b} \sin B, A = 180^\circ - (B + C) \text{ and}$$

$$a = \frac{b \sin A}{\sin B} \text{ give the remaining elements. If}$$

$b < c \sin B$, then no triangle is possible (Fig. 1). If $b = c \sin B$ and B is an acute angle, then there is only one triangle possible (Fig. 2). If $c \sin B < b < c$ and B is an acute angle, then there are two values of $\angle C$ (Fig. 3). If $c \sin B < b, c < b$ and B is an acute angle, then there is only one triangle (Fig. 4).



This is, sometimes called an ambiguous case.

Aliter

By applying cosine rule, we have

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\Rightarrow a^2 - (2c \cos B)a + (c^2 - b^2) = 0$$

$$\Rightarrow a = c \cos B \pm \sqrt{(c \cos B)^2 - (c^2 - b^2)}$$

$$\Rightarrow a = c \cos B \pm \sqrt{b^2 - (c \sin B)^2}$$

This equation leads to following cases:

Case I If $b < c \sin B$, no such triangle is possible.

Case II Let $b = c \sin B$.

There are further following cases :

- (a) B is an obtuse angle $\Rightarrow \cos B$ is negative.
There exists no such triangle.
- (b) B is an acute angle $\Rightarrow \cos B$ is positive.
There exists only one such triangle.

Case III Let $b > c \sin B$.

There are further following cases:

- (a) B is an acute angle $\Rightarrow \cos B$ is positive. In this case two values of a will exist if and only if $c \cos B > \sqrt{b^2 - (c \sin B)^2}$ or $c > b$
 $\Rightarrow$ Two such triangle is possible. If $c < b$, only one such triangle is possible.
- (b) B is an obtuse angle $\Rightarrow \cos B$ is negative. In this case triangle will exist if and only if $\sqrt{b^2 - (c \sin B)^2} > |c \cos B| \Rightarrow b > c$.

So, in this case only one such triangle is possible. If $b < c$, there exists no such triangle.

- If one side a and angles B and C are given, then

$$A = 180^\circ - (B + C), b = \frac{a \sin B}{\sin A} \text{ and } c = \frac{a \sin C}{\sin A}$$

- If the three angles A, B and C are given, we can only find the ratios of the sides a, b and c by using sine rule (since, there are infinite similar triangles possible).

Example 9. In any $\triangle ABC$, the sides are 6 cm, 10 cm and 14 cm. Then, the triangle is obtuse angled with the obtuse angle equals

- (a) 180° (b) 90°
 (c) 120° (d) 160°

Sol. (c) Let $a = 14, b = 10, c = 6 \Rightarrow s = \frac{14 + 10 + 6}{2} = 15$

The largest angle is opposite to the longest side. Then,

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \\ = \sqrt{\frac{5 \times 9}{15}} = \sqrt{3}$$

$$\Rightarrow \frac{A}{2} = 60^\circ \\ \therefore A = 120^\circ$$

Aliter

By cosine rule, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$= \frac{100 + 36 - 196}{120} \\ = -\frac{1}{2}$$

$$\Rightarrow A = 120^\circ$$

Example 10. Two sides of a triangle are $\sqrt{3} - 1$ and $\sqrt{3} + 1$ units and their included angle is 60° .

Then, the third side of the triangle is

- (a) $\sqrt{6}$ (b) $\sqrt{8}$
 (c) $1 + \sqrt{6}$ (d) $\sqrt{5}$

Sol. (a) Let $A = 60^\circ \Rightarrow B + C = 120^\circ$

$$\text{Then, } \tan \left(\frac{B-C}{2} \right) = \frac{b-c}{b+c} \cot \frac{A}{2} \\ = \frac{\sqrt{3} + 1 - \sqrt{3} + 1}{\sqrt{3} + 1 + \sqrt{3} - 1} \cot 30^\circ = 1$$

$$\Rightarrow \frac{B-C}{2} = 45^\circ$$

$$\Rightarrow B - C = 90^\circ$$

$$\Rightarrow B = 105^\circ, C = 15^\circ$$

Also, $a = b \frac{\sin A}{\sin B} = \frac{(\sqrt{3} + 1) \sin 60^\circ}{\sin 105^\circ} = \frac{(\sqrt{3} + 1) \sin 60^\circ}{\sin(60^\circ + 45^\circ)}$

$$= \frac{(\sqrt{3} + 1) \sin 60^\circ}{\sin 60^\circ \cos 45^\circ + \cos 60^\circ \sin 45^\circ} \\ = \frac{\sqrt{2}(\sqrt{3} + 1)\sqrt{3}}{\sqrt{3} + 1} = \sqrt{6}$$

Aliter

By cosine rule, $a^2 = b^2 + c^2 - 2bc \cos A$

$$= (\sqrt{3} + 1)^2 + (\sqrt{3} - 1)^2 - 2(\sqrt{3} + 1)(\sqrt{3} - 1) \cos 60^\circ \\ = 8 - 4 \cdot \frac{1}{2} = 6 \\ \Rightarrow a = \sqrt{6}$$

Example 11. If a, b and A are given in a triangle and c_1, c_2 are the possible values of the third side, then $c_1^2 + c_2^2 - 2c_1c_2 \cos 2A$ is equal to

- (a) $a^2 \cos^2 A$ (b) $4b^2 \cos^2 A$
 (c) $4a^2 \cos^2 A$ (d) None of these

Sol. (c) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0$$

$$\Rightarrow c_1 + c_2 = 2b \cos A \text{ and } c_1 c_2 = b^2 - a^2$$

$$\Rightarrow c_1^2 + c_2^2 - 2c_1 c_2 \cos 2A = (c_1 + c_2)^2 - 2c_1 c_2 (1 + \cos 2A) \\ = 4b^2 \cos^2 A - 2(b^2 - a^2) 2 \cos^2 A \\ = 4a^2 \cos^2 A$$

Work Book Exercise 10.1

1 In $\triangle ABC$, $\frac{\sin B}{\sin(A+B)}$ is equal to

- a $\frac{b}{a+b}$
 b $\frac{b}{c}$
 c $\frac{c}{b}$
 d None of the above

2 In $\triangle ABC$, $c \cos(A - \alpha) + a \cos(C + \alpha)$ is equal to
 a $a \cos \alpha$ b $b \cos \alpha$ c $c \cos \alpha$ d $2b \cos \alpha$

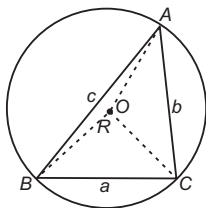
3 If in $\triangle ABC$, $a = 6$ cm, $b = 8$ cm, $c = 10$ cm, then the value of $\sin 2A$ is

- a $\frac{6}{25}$ b $\frac{8}{25}$
 c $\frac{10}{25}$ d $\frac{24}{25}$

- 4 If $A = 45^\circ$, $B = 75^\circ$, then $a + c\sqrt{2}$ is equal to
 a $2b$ b $3b$ c $\sqrt{2}b$ d b
- 5 In a $\triangle ABC$, AD is the altitude from A . Given $b > c$, $C = 23^\circ$ and $AD = \frac{abc}{b^2 - c^2}$, then $\angle B$ is equal to
 a 23° b 113°
 c 67° d 90°
- 6 If $4\sin A = 4\sin B = 3\sin C$ in a $\triangle ABC$, then $\cos A$ is equal to
 a $\frac{1}{3}$ b $\frac{1}{9}$
 c $\frac{1}{27}$ d $\frac{1}{18}$
- 7 In an acute angled $\triangle ABC$, the least value of $\sec A + \sec B + \sec C$ is
 a 6 b 3 c 9 d 4
- 8 In a $\triangle ABC$, if $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ and the sides $a = 2$, then area of triangle is
 a 1 b 2 c $3\sqrt{2}$ d $\sqrt{3}$
- 9 In $\triangle ABC$, $a^2 + b^2 + c^2 = ac + ab\sqrt{3}$, then triangle is
 a equilateral b isosceles
 c Right angled d None of these
- 10 If the line segment joining the points $A(a, b)$ and $B(c, d)$ subtends an angle θ at the origin, then $\cos \theta$ is equal to
 a $\frac{ab + cd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$
 b $\frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$
 c $\frac{ac - bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$
 d None of the above

Circles Connected with Triangle

- i. **Circumcircle of a triangle** The circle passing through the vertices of a $\triangle ABC$ is called circumcircle. Its radius R is called the circumradius and its centre is known as circumcentre. Circumcentre is the point of intersection of perpendicular bisectors of the sides.

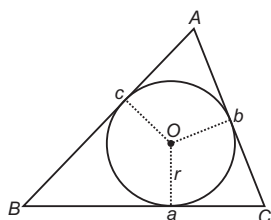


In $\triangle ABC$,

$$(a) R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C}$$

$$(b) R = \frac{abc}{4\Delta}$$

- ii. **Incircle of a triangle** The circle touching all the sides of a triangle internally is known as an incircle of the triangle. Its centre is called incentre and its radius r is called the inradius of the circle. Incentre is the point of intersection of bisectors of the angles of the triangle.



In $\triangle ABC$,

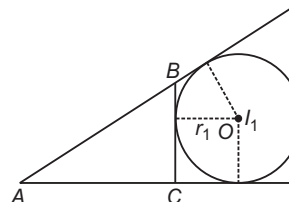
$$(a) r = \frac{\Delta}{s}$$

$$(b) r = (s - a) \tan \frac{A}{2} = (s - b) \tan \frac{B}{2} = (s - c) \tan \frac{C}{2}$$

$$(c) r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = \frac{b \sin \frac{A}{2} \sin \frac{C}{2}}{\cos \frac{B}{2}} = \frac{c \sin \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{C}{2}}$$

$$(d) r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

- iii. **Escribed circles of a triangle** The circle touching BC and the two sides AB and AC produced of $\triangle ABC$ externally is called the escribed circle opposite to A . Its radius is denoted by r_1 . The centre of this circle is known as excentre and denoted by I_1 . It is the point, where the external bisectors of the $\angle B$ and $\angle C$ and the internal bisector of the $\angle A$ meet.



Similarly, r_2 and r_3 denote the radii of the escribed circles opposite to the angles B and C , respectively and excentres are denoted by I_2 and I_3 . r_1, r_2 and r_3 are called the exradii of $\triangle ABC$. Here,

$$\begin{aligned} \text{(a) } r_1 &= \frac{\Delta}{s-a} = s \tan \frac{A}{2} \\ &= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\ &= \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} \end{aligned}$$

$$\begin{aligned} \text{(b) } r_2 &= \frac{\Delta}{s-b} = s \tan \frac{B}{2} \\ &= 4R \sin \frac{B}{2} \cos \frac{C}{2} \cos \frac{A}{2} \\ &= \frac{b \cos \frac{A}{2} \cos \frac{C}{2}}{\cos \frac{B}{2}} \end{aligned}$$

$$\begin{aligned} \text{(c) } r_3 &= \frac{\Delta}{s-c} = s \tan \frac{C}{2} \\ &= 4R \sin \frac{C}{2} \cos \frac{A}{2} \cos \frac{B}{2} = \frac{c \cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{C}{2}} \end{aligned}$$

- $r_1 + r_2 + r_3 = 4R + r$
 • $r_1 r_2 + r_2 r_3 + r_3 r_1 = s^2 = \frac{r_1 r_2 r_3}{r}$
 • $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$ and $r_1 r_2 r_3 = r^2 \left(\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} \right)^2$

- **Example 12.** In any $\triangle ABC$,
 $a \cos A + b \cos B + c \cos C$ is equal to
 (a) $4R$ (b) $\sin A \sin B \sin C$
 (c) $R \sin A \sin B \sin C$ (d) $4R \sin A \sin B \sin C$

Sol. (d) $a \cos A + b \cos B + c \cos C$
 $= 2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C$
 $[\because a = 2R \sin A, b = 2R \sin B \text{ and } c = 2R \sin C]$
 $= R(\sin 2A + \sin 2B + \sin 2C)$
 $= R[2 \sin(A+B) \cos(A-B) + 2 \sin C \cos C]$
 $= 2R[\sin(\pi - C) \cos(A-B) + \sin C \cos \{\pi - (A+B)\}]$
 $= 2R[\sin C \{\cos(A-B) - \cos(A+B)\}]$
 $= 4R \sin A \sin B \sin C$

- **Example 13.** In a $\triangle ABC$, $\cos A + \cos B + \cos C$ is equal to

- (a) $\frac{r}{R}$ (b) $1 - \frac{r}{R}$ (c) $\frac{R}{r}$ (d) $1 + \frac{r}{R}$

Sol. (d) $\cos A + \cos B + \cos C$
 $= 2 \cos \left(\frac{A+B}{2} \right) \cos \left(\frac{A-B}{2} \right) + \cos C$
 $= 2 \cos \left(\frac{\pi}{2} - \frac{C}{2} \right) \cos \left(\frac{A-B}{2} \right) + 1 - 2 \sin^2 \frac{C}{2}$
 $= 1 + 2 \sin \frac{C}{2} \cos \left(\frac{A-B}{2} \right) - 2 \sin^2 \frac{C}{2}$
 $= 1 + 2 \sin \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \sin \frac{C}{2} \right]$
 $= 1 + 2 \sin \frac{C}{2} \left[\cos \left(\frac{A-B}{2} \right) - \cos \left(\frac{A+B}{2} \right) \right]$
 $= 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$
 $= \left(1 + \frac{r}{R} \right) \left[\because r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right]$

- **Example 14.** If $r_1 = r_2 + r_3 + r$, then triangle is
 (a) right angled
 (b) obtuse angled
 (c) equilateral
 (d) None of the above

Sol. (a) $\because r_1 - r = r_2 + r_3$
 $\Rightarrow \frac{\Delta}{s-a} - \frac{\Delta}{s} = \frac{\Delta}{s-b} + \frac{\Delta}{s-c}$
 $\Rightarrow \frac{s-s+a}{s(s-a)} = \frac{s-c+s-b}{(s-b)(s-c)}$
 $\Rightarrow \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$
 $\Rightarrow s^2 - (b+c)s + bc = s^2 - as$
 $\Rightarrow \frac{s(-a+b+c)}{(b+c+a)(b+c-a)} = bc$
 $\Rightarrow \frac{2}{(b+c)^2 - a^2} = bc$
 $\Rightarrow b^2 + c^2 + 2bc - a^2 = 2bc$
 $\Rightarrow b^2 + c^2 = a^2$
 $\therefore \angle A = 90^\circ$

Thus, the triangle is right angled triangle.

- **Example 15.** If in a triangle, R and r are the circumradius and inradius respectively, then the HM of the exradii of the triangle is
 (a) $3r$ (b) $2R$
 (c) $R + r$ (d) None of these

Sol. (a) $\Sigma \frac{1}{r_i} = \Sigma \frac{s-a}{\Delta} = \frac{3s - (a+b+c)}{\Delta} = \frac{s}{\Delta}$
 $\therefore \text{HM of exradii} = \frac{1}{\Sigma 1/r_i} = \frac{3}{s/\Delta} = \frac{3\Delta}{s} = 3r$

- **Example 16.** If the exradii of a triangle are in HP, then the corresponding sides are in
 (a) AP
 (b) GP
 (c) HP
 (d) None of the above

Sol. (a) $\frac{1}{r_1}, \frac{1}{r_2}$ and $\frac{1}{r_3}$ are in AP.
 $\Rightarrow \frac{s-a}{\Delta}, \frac{s-b}{\Delta}, \frac{s-c}{\Delta}$ are in AP.
 $\Rightarrow s-a, s-b, s-c$ are in AP.
 $\Rightarrow -a, -b, -c$ are in AP.
 $\Rightarrow a, b, c$ are in AP.

- **Example 17.** In a $\triangle ABC$, the inradius and three exradii are r, r_1, r_2 and r_3 , respectively. In usual notations, the value of $r \cdot r_1 \cdot r_2 \cdot r_3$ is equal to
- (a) 2Δ (b) Δ^2
 (c) $\frac{abc}{4R}$ (d) None of these

Sol. (b) $r \cdot r_1 \cdot r_2 \cdot r_3 = \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}$
 $= \frac{\Delta^4}{\Delta^2} = \Delta^2$

- **Example 18.** In a right angled triangle, R is equal to
- (a) $\frac{s+r}{2}$ (b) $\frac{s-r}{2}$
 (c) $s-r$ (d) $\frac{s+r}{a}$

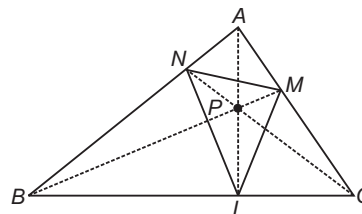
Sol. (b) In a right angled triangle, the circumcentre lies on the hypotenuse.
 $\Rightarrow R = \frac{a}{2}$, where $A = 90^\circ$
 Also, $r = (s-a)\tan \frac{A}{2}$
 $= (s-a)\tan 45^\circ$
 $= (s-a) = s-2R$
 $\Rightarrow r+2R = s$
 $\Rightarrow R = \frac{s-r}{2}$

- **Example 19.** In a $\triangle ABC$, $a=1$ and the perimeter is six times the AM of the sines of the angles. The measure of $\angle A$ is
- (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

Sol. (c) $1+b+c = 6 \left(\frac{\sin A + \sin B + \sin C}{3} \right)$
 $= 2 \left(\frac{1}{2R} + \frac{b}{2R} + \frac{c}{2R} \right)$
 $= \frac{1}{R}(1+b+c)$
 $\Rightarrow R = 1$ [$\because 1+b+c \neq 0$]
 $\therefore \frac{1}{\sin A} = 2R = 2$
 $\Rightarrow \sin A = \frac{1}{2}$
 $\Rightarrow A = \frac{\pi}{6}$

The Orthocentre and the Pedal Triangle

Orthocentre is a point P at which all the three altitudes, drawn from the vertices to the opposite sides of the triangle, meet.

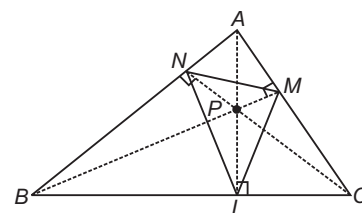


The $\triangle LMN$ which is formed by joining the feet of these three altitudes is called the pedal triangle.

- Orthocentre of the triangle is the incentre of the pedal triangle.
- If I_1, I_2 and I_3 are the centres of escribed circles which are opposite to A, B and C , respectively and I is the centre of incircle, then $\triangle ABC$ is the pedal triangle of $\triangle I_1 I_2 I_3$ and I is the orthocentre of $\triangle I_1 I_2 I_3$.
- The centroid of the triangle lies on the line joining the circumcentre to the orthocentre and divides it in the ratio 1:2.
- Circle circumscribing the pedal triangle of a given triangle bisects the sides of the given triangle and also the lines joining the vertices of the given triangle to the orthocentre of the given triangle. This circle is known as nine point circle.
- Circumcentre of the pedal triangle of a given triangle bisects the line joining the circumcentre of the triangle to the orthocentre.

- **Example 20.** The distance of the orthocentre from the vertices of the triangle are
- (a) $2R \cos A, 2R \cos B, 2R \cos C$
 (b) $2R \sin A, 2R \sin B, 2R \sin C$
 (c) $2R \cos A, R \cos B, R \cos C$
 (d) $2R \operatorname{cosec} A, 2R \operatorname{cosec} B, 2R \operatorname{cosec} C$

Sol. (a) Let L, M and N be the feet of the altitudes from the vertices A, B and C of the triangle, respectively.



In $\triangle ABL$, $BL = c \cos B$
 In $\triangle BPL$,
 $BP = BL \sec \angle PBL = c \cos B \sec (90^\circ - C)$
 $= c \cos B \operatorname{cosec} C = 2R \cos B$
 Similarly, $PC = 2R \cos C$ and $PA = 2R \cos A$

➤ **Example 21.** $\sin A, \sin B$ and $\sin C$ are in AP for the ΔABC . Then,

- (a) the altitudes are in AP
(b) the altitudes are in HP
(c) the medians are in GP
(d) the medians are in AP

Sol. (b) $\sin A, \sin B, \sin C$ are in AP.

$$\Rightarrow \frac{a}{2R}, \frac{b}{2R}, \frac{c}{2R} \text{ are in AP.}$$

$$\Rightarrow \frac{2\Delta}{h_1}, \frac{2\Delta}{h_2} \text{ and } \frac{2\Delta}{h_3} \text{ are in AP.}$$

$$\Rightarrow \frac{1}{h_1}, \frac{1}{h_2} \text{ and } \frac{1}{h_3} \text{ are in AP.}$$

$$\Rightarrow h_1, h_2 \text{ and } h_3 \text{ are in HP.}$$

Now, medians are

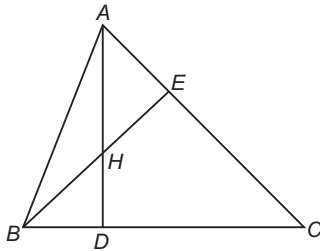
$$\sqrt{\frac{c^2 + b^2}{2} - \frac{a^2}{4}}, \sqrt{\frac{a^2 + c^2}{2} - \frac{b^2}{4}}, \sqrt{\frac{a^2 + b^2}{2} - \frac{c^2}{4}}$$

These are neither in AP nor GP.

➤ **Example 22.** The ratio of the distances of the orthocentre of an acute angled ΔABC from the sides BC, AC and AB is

- (a) $\cos A : \cos B : \cos C$
(b) $\sin A : \sin B : \sin C$
(c) $\sec A : \sec B : \sec C$
(d) None of the above

Sol. (c) $HD = BD \cdot \tan \angle EBC = c \cos B \cdot \tan (90^\circ - C)$



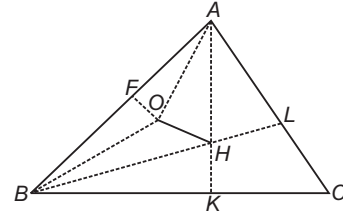
$$\begin{aligned} &= \frac{c \cos B \cdot \cos C}{\sin C} \\ &= \frac{2R \cos B \cos C}{\frac{2R \cos A \cos B \cos C}{\cos A}} \end{aligned}$$

Similarly, for others. So, the ratio of the distances of the orthocentre from the sides = $\frac{1}{\cos A} : \frac{1}{\cos B} : \frac{1}{\cos C}$.

➤ **Example 23.** The distance between the circumcentre and the orthocentre of a ΔABC is

- (a) $R \sqrt{1 - 8 \cos A \cos B \cos C}$
(b) $R \sqrt{1 + 8 \cos A \cos B \cos C}$
(c) $R \sqrt{1 - 4 \cos A \cos B \cos C}$
(d) None of the above

Sol. (a) Let O and H be the circumcentre and the orthocentre, respectively. If OF is the perpendicular to AB .



We have,

$$\begin{aligned} \angle OAF &= 90^\circ - \angle AOF \\ &= 90^\circ - C \end{aligned}$$

Also,

$$\angle HAL = 90^\circ - C$$

Hence,

$$\begin{aligned} \angle OAH &= A - \angle OAF - \angle HAL \\ &= A - 2(90^\circ - C) \\ &= A + 2C - (A + B + C) \\ &= C - B \end{aligned}$$

Also, $OA = R$ and $PA = 2R \cos A$

Now, in ΔAOH ,

$$\begin{aligned} OH^2 &= OA^2 + PA^2 - 2OA \cdot HA \cos OAH \\ &= R^2 + 4R^2 \cos^2 A - 4R^2 \cos A \cos (C - B) \\ &= R^2 + 4R^2 \cos A [\cos A - \cos (C - B)] \\ &= R^2 - 4R^2 \cos A [\cos (B + C) + \cos (C - B)] \\ &= R^2 - 8R^2 \cos A \cos B \cos C \end{aligned}$$

$$\text{Hence, } OH = R \sqrt{1 - 8 \cos A \cos B \cos C}$$

Cyclic Quadrilateral

A cyclic quadrilateral is a quadrilateral which can be circumscribed by a circle.

- Sum of the opposite angles of a cyclic quadrilateral is 180° .
- In a cyclic quadrilateral, sum of the products of the opposite sides is equal to the product of the diagonals. This is known as Ptolemy's theorem.
- If sum of the opposite sides of a quadrilateral is equal, then a circle can be inscribed in the quadrilateral.
- The area of the cyclic quadrilateral with sides a, b, c and d is $\sqrt{(s-a)(s-b)(s-c)(s-d)}$.

➤ **Example 24.** The area of a cyclic

quadrilateral $ABCD$ is $\frac{(3\sqrt{3})}{4}$. The radius of the

circle circumscribing cyclic quadrilateral is 1. If $AB = 1$ and $BD = \sqrt{3}$, then $BC \cdot CD$ is equal to

- (a) 2 (b) $3 - \frac{1}{\sqrt{3}}$
(c) $2\sqrt{3} + 1$ (d) None of these

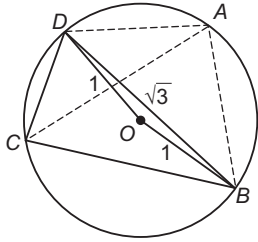
Sol. (a) Clearly, $\angle BOD = 2C$

$$\therefore \cos 2C = \frac{1^2 + 1^2 - (\sqrt{3})^2}{2 \cdot 1 \cdot 1}$$

$$\Rightarrow C = 60^\circ$$

$$\therefore A = 180^\circ - C = 120^\circ$$

So, $\cos 120^\circ = \frac{AD^2 + 1^2 - (\sqrt{3})^2}{2 \cdot AD \cdot 1}$

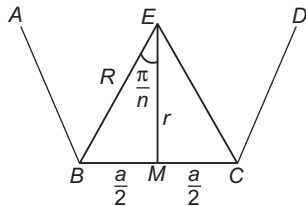


$$\begin{aligned} \therefore -\frac{1}{2} &= \frac{AD^2 - 2}{2AD} \\ \Rightarrow AD^2 + AD - 2 &= 0 \\ \Rightarrow AD &= 1 \\ \text{Now, } \ar(ABCD) &= \ar(\triangle ADB) + \ar(\triangle BCD) \\ \Rightarrow \frac{3\sqrt{3}}{4} &= \frac{1}{2} \cdot 1 \cdot 1 \cdot \sin 120^\circ + \frac{1}{2} \cdot BC \cdot CD \cdot \sin 60^\circ \\ \Rightarrow BC \cdot CD &= 2 \end{aligned}$$

Regular Polygon

A regular polygon is a polygon which has all its sides as well as all its angles equal. If the polygon has n sides, sum of its internal angles is $(n-2)\pi$ and each angle is $\frac{(n-2)\pi}{n}$. The circle passing through all the vertices of a regular polygon is called circumscribed circle and the circle which touches all sides of a regular polygon is called its inscribed circle.

Radius of circumcircle, $R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$ and radius of incircle, $r = \frac{a}{2} \cot \frac{\pi}{n}$, where a is the length of the side of a regular polygon.



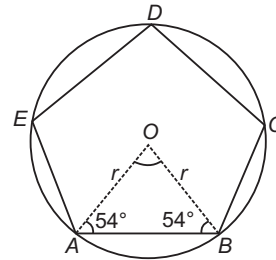
$$\begin{aligned} \text{Area of a regular polygon} &= \frac{1}{4} n a^2 \cot \frac{\pi}{n} \\ &= n r^2 \tan \frac{\pi}{n} \\ &= \frac{n}{2} R^2 \sin \frac{2\pi}{n} \end{aligned}$$

- Sum of the exterior angles of a polygon taken in one direction (clockwise or anti-clockwise) remains constant and it is equal to 360° .
- In the regular polygon, the centroid, the circumcentre and the incentre are same.

➤ **Example 25.** If the area of circle is A_1 and area of regular pentagon inscribed in the circle is A_2 . Then, $\frac{A_1}{A_2}$ is equal to

- (a) $\frac{2\pi}{5}$
 (b) $\frac{2\pi}{5} \sec \frac{\pi}{10}$
 (c) $\sec \frac{\pi}{10}$
 (d) None of the above

Sol. (b) In $\triangle OAB$, $OA = OB = r$ and $\angle AOB = 72^\circ$



$$\begin{aligned} \therefore \text{Area of } \triangle AOB &= \frac{1}{2} r \cdot r \sin 72^\circ \\ &= \frac{1}{2} r^2 \cos 18^\circ \end{aligned}$$

Area of pentagon = 5 (Area of $\triangle AOB$)

$$\Rightarrow A_2 = \frac{5}{2} r^2 \cos 18^\circ \quad \dots (i)$$

Also, we know that,

$$\text{Area of circle} = \pi r^2$$

$$\Rightarrow A_1 = \pi r^2 \quad \dots (ii)$$

Thus,
$$\frac{A_1}{A_2} = \frac{\pi r^2}{\frac{5}{2} r^2 \cos 18^\circ} = \frac{2\pi}{5} \sec \frac{\pi}{10}$$

➤ **Example 26.** Let A_0, A_1, A_2, A_3, A_4 and A_5 be the consecutive vertices of a regular hexagon inscribed in a unit circle. The product of the lengths of A_0A_1, A_0A_2 and A_0A_4 is

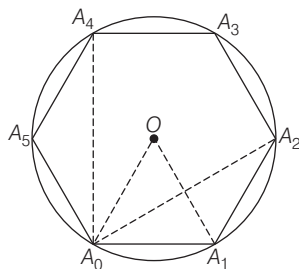
- (a) $\frac{3}{4}$ (b) $3\sqrt{3}$
 (c) 3 (d) $\frac{3\sqrt{3}}{2}$

Sol. (c) $\cos 60^\circ = \frac{OA_0^2 + OA_1^2 - A_0A_1^2}{2 \cdot OA_0 \cdot OA_1}$

$$\Rightarrow \frac{1}{2} \times 2 \times 1 \times 1 = 2 - A_0A_1^2$$

$$\Rightarrow A_0A_1^2 = 2 - 1 = 1$$

$$\Rightarrow A_0A_1 = A_1A_2$$



$$\begin{aligned}\cos 120^\circ &= \frac{A_0A_1^2 + A_1A_2^2 - A_0A_2^2}{2 \cdot A_0A_1 \cdot A_1A_2} \\ &= \frac{1 + 1 - A_0A_2^2}{2 \cdot 1 \cdot 1}\end{aligned}$$

$$\begin{aligned}\therefore A_0A_2 &= \sqrt{3} = A_0A_4 \\ \Rightarrow A_0A_1 \times A_0A_2 \times A_0A_4 &= 1 \times \sqrt{3} \times \sqrt{3} = 3\end{aligned}$$

➤ **Example 27.** If regular pentagon and a regular decagon have the same perimeter, then the ratio of their areas is

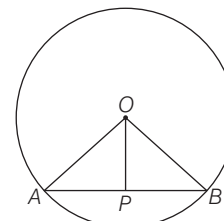
- (a) $1:\sqrt{5}$
- (b) $2:\sqrt{5}$
- (c) $\sqrt{5}:2$
- (d) $\sqrt{5}:1$

Sol. (b) Let the perimeter of the pentagon and decagon be $10x$.

Then, each side of the pentagon is $2x$ and its area

$$= 5x^2 \cot \frac{\pi}{5} \quad \dots(i)$$

$$[\because n = 5 \text{ and } a = 2x]$$



Also, as each side of decagon is x , so its area

$$= \frac{5}{2} x^2 \cot \frac{\pi}{10} \quad [\because n = 10 \text{ and } a = x]$$

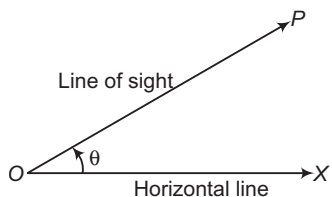
$$\begin{aligned}\Rightarrow \frac{\text{Area of pentagon}}{\text{Area of decagon}} &= \frac{2 \cot 36^\circ}{\cot 18^\circ} = \frac{2 \cos 36^\circ \sin 18^\circ}{\sin 36^\circ \cos 18^\circ} \\ &= \frac{2 \cos 36^\circ \sin 18^\circ}{2 \sin 18^\circ \cos 18^\circ \cos 18^\circ} = \frac{\cos 36^\circ}{\cos^2 18^\circ} \\ &= \frac{2 \cos 36^\circ}{2 \cos^2 18^\circ} = \frac{2 \cos 36^\circ}{1 + \cos 36^\circ} \\ &= \frac{2(\sqrt{5} + 1)}{4 \left\{ 1 + \frac{\sqrt{5} + 1}{4} \right\}} = \frac{2(\sqrt{5} + 1)}{5 + \sqrt{5}} = \frac{2}{\sqrt{5}}\end{aligned}$$

Work Book Exercise 10.2

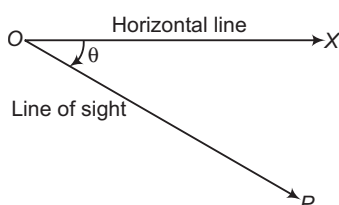
- If the radius of the circumcircle of an isosceles $\triangle ABC$ is equal to $AB = AC$, then the angle A is
 - a $\frac{\pi}{6}$
 - b $\frac{\pi}{3}$
 - c $\frac{\pi}{2}$
 - d $\frac{2\pi}{3}$
- If a^2, b^2, c^2 are in AP, then $\cot A, \cot B$ and $\cot C$ are in
 - a AP
 - b GP
 - c HP
 - d None of the above
- Given an isosceles triangle, whose one angle is 120° and radius of its incircle is $\sqrt{3}$. Then, the area of the triangle (in sq units) is
 - a $7 + 12\sqrt{3}$
 - b $12 - 7\sqrt{3}$
 - c $12 + 7\sqrt{3}$
 - d 4π
- A regular polygon of nine sides, each of length 2, is inscribed in a circle. The radius of the circle is
 - a $\sec \frac{\pi}{9}$
 - b $\sin \frac{\pi}{9}$
 - c $\operatorname{cosec} \frac{\pi}{9}$
 - d $\tan \frac{\pi}{9}$
- In a $\triangle ABC$, $\sin A + \sin B + \sin C = 1 + \sqrt{2}$, $\cos A + \cos B + \cos C = \sqrt{2}$, if the triangle is
 - a equilateral
 - b isosceles
 - c right angled
 - d right angle isosceles
- A circle is inscribed in an equilateral triangle of side a . The area of any square inscribed in the circle is
 - a $\frac{a^2}{4}$
 - b $\frac{a^2}{6}$
 - c $\frac{a^2}{9}$
 - d $\frac{2a^2}{3}$
- In a $\triangle ABC$ the line joining the circumcentre to the incentre is parallel to BC , then $\cos B + \cos C$ is equal to
 - a $\frac{3}{2}$
 - b 1
 - c $\frac{3}{4}$
 - d $\frac{1}{2}$
- In a triangle $a = 13, b = 14, c = 15$, then r is equal to
 - a 4
 - b 8
 - c 2
 - d 6
- The exradii of a triangle r_1, r_2 and r_3 are in harmonic progression, then the sides a, b and c are in
 - a AP
 - b GP
 - c HP
 - d None of these
- In an equilateral triangle, the inradius and the circumradius are connected by
 - a $r = 4R$
 - b $r = \frac{R}{2}$
 - c $r = \frac{R}{3}$
 - d None of these

Heights and Distances

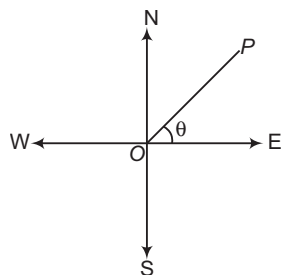
Angle of elevation If O is the observer's eye and OX is the horizontal line through O . If the object P is at a higher level than eye, then $\angle POX$ is called the angle of elevation.



Angle of depression If the object P is at a lower level than O , then $\angle POX$ is called the angle of depression.

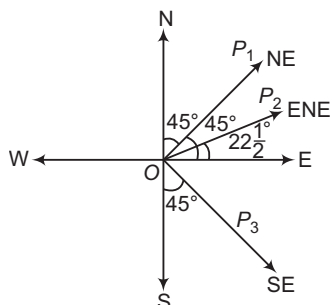


Bearing If the observer and the object i.e. O and P are on the same level, then bearing is defined. To measure the bearing the four standard directions East, West, North and South are taken as the cardinal directions. Angle between the line of observation i.e. OP and any one standard direction East, West, North or South is measured.



Thus, $\angle POE$ is called the bearing of the point P with respect to O measured from East to North.

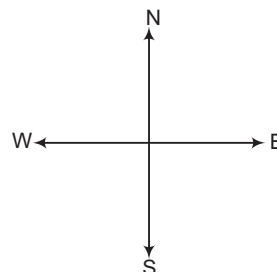
In other words, the bearing of P as seen from O is the direction in which P is seen from O .



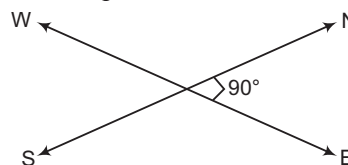
North-East means equally inclined to North and East. South-East means equally inclined to South and East. ENE means equally inclined to East and North-East.

Dimensions

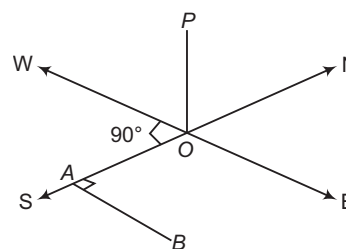
- (a) If the total actual figure is located in one plane, the problem is of two dimensions. For directions in two dimensional figures, cross vertically as in given figure.



- (b) If the total actual figure is located in more than one plane, the problem will be of three dimensions. For directions in three dimensional figures, cross obliquely as shown. Clearly, this oblique cross represents the horizontal plane.



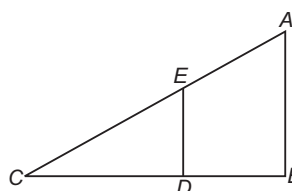
- (c) If OP is a vertical tower perpendicular to the plane, then it will be represented as shown in the figure. Clearly, $\angle POA = 90^\circ$.



If the observer at A moves towards East, we draw a line AB parallel to OE to represent this movement. Clearly, $\angle OAB = 90^\circ$ (angle between North and East).

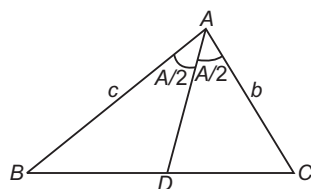
Some Important Properties of Triangles

- i. In a $\triangle ABC$, if $DE \parallel AB$, then $\frac{AB}{DE} = \frac{BC}{DC} = \frac{AC}{EC}$

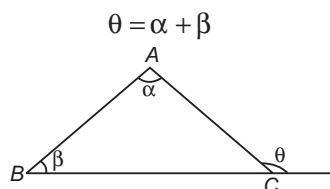


- ii. In any $\triangle ABC$, if AD is the angle bisector of $\angle BAC$, then

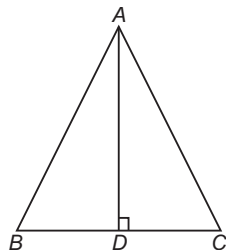
$$\frac{BD}{DC} = \frac{AB}{AC} = \frac{c}{b}$$



- iii. In any $\triangle ABC$, the exterior angle is equal to the sum of interior opposite angles.

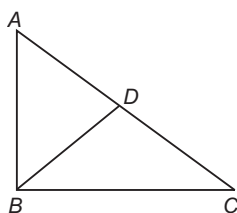


- iv. In an isosceles triangle, the median is perpendicular to the base. $\triangle ABC$ is isosceles and $AD \perp BC$.



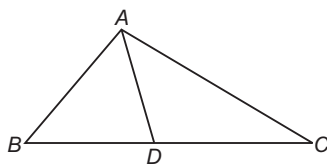
- v. In similar triangle, the corresponding sides are proportional.

- vi. In a right angled $\triangle ABC$, the mid-point D of hypotenuse AC is equidistant from its vertices A, B and C .



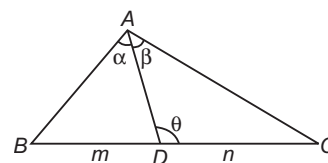
i.e. $AD = BD = CD$

- vii. **Apollonius theorem** If AD is median of the $\triangle ABC$, then



$$AB^2 + AC^2 = 2(AD^2 + BD^2)$$

- viii. **m-n theorem** If in a $\triangle ABC$, D is a point on the line BC such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$, $\angle DAC = \beta$, then

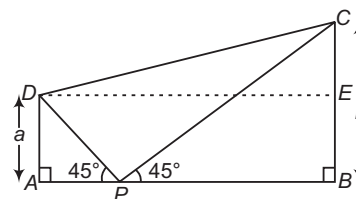


- (a) $(m+n)\cot\theta = m\cot\alpha - n\cot\beta$
(b) $(m+n)\cot\theta = n\cot B - m\cot C$

- Example 28. P is a point on the segment joining the feet of two vertical poles of heights a and b . The angles of elevation of the tops of the poles from P are 45° each. Then, the square of the distance between the tops of the poles is

- (a) $\frac{a^2 + b^2}{2}$
(b) $a^2 + b^2$
(c) $2(a^2 + b^2)$
(d) $4(a^2 + b^2)$

Sol. (c) In $\triangle APD$, $\tan 45^\circ = \frac{a}{AP} \Rightarrow AP = a$



and in $\triangle BPC$, $\tan 45^\circ = \frac{b}{PB}$

$\Rightarrow PB = b$

$\therefore DE = a + b$

and $CE = b - a$

In $\triangle DEC$, $DC^2 = DE^2 + EC^2$
 $= (a+b)^2 + (b-a)^2 = 2(a^2 + b^2)$

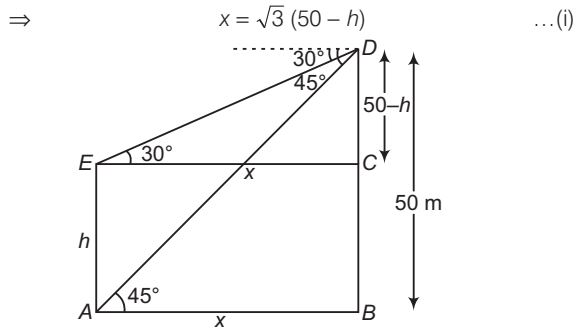
- Example 29. From the top of a cliff 50 m high, the angles of depression of the top and bottom of a tower are observed to be 30° and 45° . The height of tower is

- (a) 50 m
(b) $50\sqrt{3}$ m
(c) $50(\sqrt{3} - 1)$ m
(d) $50\left(1 - \frac{\sqrt{3}}{3}\right)$ m

Sol. (d) Let the height of the cliff be $BD = 50$ m and height of the tower be $AE = h$ m.

In $\triangle DEC$, $\tan 30^\circ = \frac{50-h}{x}$

$\Rightarrow x = \frac{50-h}{\frac{1}{\sqrt{3}}}$



and in $\triangle BAD$,

$$\tan 45^\circ = \frac{50}{x}$$

$$\Rightarrow x = 50 \text{ m}$$

$$\text{From Eq. (i), } 50 = \sqrt{3}(50 - h)$$

$$\Rightarrow \sqrt{3}h = 50\sqrt{3} - 50$$

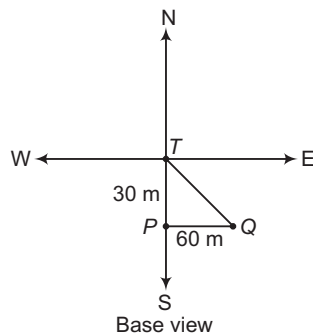
$$\Rightarrow h = \frac{50(\sqrt{3} - 1)}{\sqrt{3}} \text{ m}$$

$$\Rightarrow h = 50 \left(1 - \frac{\sqrt{3}}{3} \right) \text{ m}$$

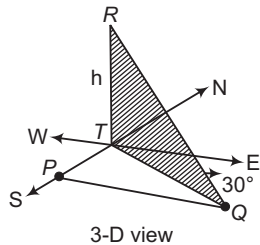
➤ **Example 30.** A man standing 30 m South of the tower of height h , walks 60 m to the East and finds the angle of elevation of the top of the tower to be 30° . Then, the height of tower is equal to

- (a) $15\sqrt{10} \text{ m}$ (b) $15\sqrt{3} \text{ m}$
(c) $10\sqrt{15} \text{ m}$ (d) None of these

Sol. (c) Let the lines East-West and North-South meet at the foot of the tower T .



Let the man be standing at P , 30 m South of the tower. The man walks 60 m to Q parallel to the line WE . The angle of elevation of the tower at Q is 30° .



$$\text{Hence, } TQ = h \cot 30^\circ = \sqrt{3}h$$

$$\text{From the } \triangle PQT, QT^2 = PT^2 + PQ^2$$

$$\Rightarrow 3h^2 = (30)^2 + (60)^2$$

$$= 4500$$

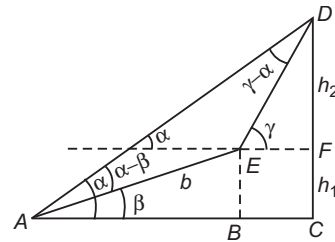
$$\Rightarrow h = 10\sqrt{15} \text{ m}$$

➤ **Example 31.** The angle of elevation of the top of a hill from a point is α . After walking $b \text{ m}$ towards the top up, a slope inclined at an angle β to the horizon, the angle of elevation of the top becomes γ . Then, the height of the hill is

- (a) $\frac{b \sin \alpha \sin (\gamma - \beta)}{\sin (\gamma - \alpha)}$ (b) $\frac{b \sin \alpha \sin (\gamma - \alpha)}{\sin (\gamma - \beta)}$
(c) $\frac{b \sin (\gamma - \beta)}{\sin (\gamma - \alpha)}$ (d) $\frac{\sin (\gamma - \beta)}{b \sin \alpha \sin (\gamma - \alpha)}$

Sol. (a) In $\triangle ABE$, $\sin \beta = \frac{BE}{b}$

$$\Rightarrow BE = h_1 = b \sin \beta \quad \dots (i)$$



Using sine rule in $\triangle AED$,

$$\frac{\sin (\alpha - \beta)}{ED} = \frac{\sin (\gamma - \alpha)}{b} \Rightarrow ED = \frac{b \sin (\alpha - \beta)}{\sin (\gamma - \alpha)}$$

$$\text{Now, in } \triangle FED, \sin \gamma = \frac{h_2}{ED}$$

$$\Rightarrow h_2 = \frac{b \sin (\alpha - \beta) \sin \gamma}{\sin (\gamma - \alpha)} \quad \dots (ii)$$

$$\therefore \text{Total height, } CD = h_1 + h_2 = b \sin \beta + \frac{b \sin (\alpha - \beta) \sin \gamma}{\sin (\gamma - \alpha)} \quad [\text{from Eqs. (i) and (ii)}]$$

$$= \frac{b [\sin \beta \sin (\gamma - \alpha) + \sin (\alpha - \beta) \sin \gamma]}{\sin (\gamma - \alpha)}$$

$$= \frac{b \{ \sin \beta (\sin \gamma \cos \alpha - \cos \gamma \sin \alpha) + \sin \gamma (\sin \alpha \cos \beta - \sin \beta \cos \alpha) \}}{\sin (\gamma - \alpha)}$$

$$= \frac{b (\sin \beta \sin \gamma \cos \alpha - \sin \beta \cos \gamma \sin \alpha + \sin \gamma \sin \alpha \cos \beta - \sin \gamma \sin \beta \cos \alpha)}{\sin (\gamma - \alpha)}$$

$$= \frac{b \sin \alpha \sin (\gamma - \beta)}{\sin (\gamma - \alpha)}$$

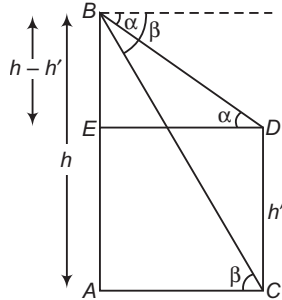
➤ **Example 32.** From the top of a hill $h \text{ m}$ high, the angle of depressions of the top and the bottom of the pillar are α and β , respectively. The height (in metre) of the pillar is

- (a) $\frac{h(\tan \beta - \tan \alpha)}{\tan \beta}$
(b) $\frac{h(\tan \alpha - \tan \beta)}{\tan \alpha}$
(c) $\frac{h(\tan \beta + \tan \alpha)}{\tan \beta}$
(d) $\frac{h(\tan \beta + \tan \alpha)}{\tan \alpha}$

Sol. (a) Let AB be a hill whose height is h m and CD be a pillar of height h' m.

$$\text{In } \triangle EDB, \quad \tan \alpha = \frac{h - h'}{ED} \quad \dots (i)$$

$$\text{and in } \triangle ACB, \quad \tan \beta = \frac{h}{AC} = \frac{h}{ED} \quad \dots (ii)$$



$$\text{From Eqs. (i) and (ii),} \quad \frac{\tan \alpha}{\tan \beta} = \frac{h - h'}{h}$$

$$\Rightarrow h \cdot \frac{\tan \alpha}{\tan \beta} = h - h'$$

$$\therefore h' = \frac{h(\tan \beta - \tan \alpha)}{\tan \beta}$$

➤ **Example 33.** $ABCD$ is a rectangular field. A vertical lamp post of height 12 m stands at the corner A . If the angle of elevation of its top from B is 60° and from C is 45° , then the area of the field is

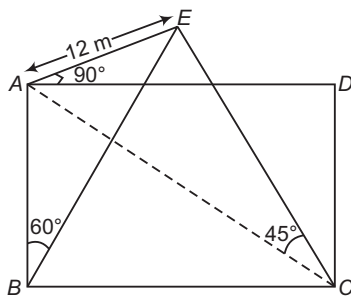
(a) $48\sqrt{2}$ sq m

(b) $48\sqrt{3}$ sq m

(c) 48 sq m

(d) $12\sqrt{2}$ sq m

Sol. (a) In $\triangle ABE$, $\tan 60^\circ = \frac{12}{AB}$
 $\Rightarrow AB = 4\sqrt{3}$ m



and in $\triangle ACE$,

$$\tan 45^\circ = \frac{12}{AC}$$

$$\Rightarrow AC = 12 \text{ m}$$

In $\triangle ABC$,

$$BC = \sqrt{AC^2 - AB^2} = \sqrt{144 - 48} = 4\sqrt{6} \text{ m}$$

$\therefore$ Area of rectangular field

$$= AB \times BC$$

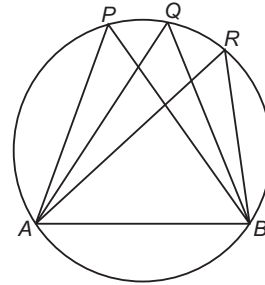
$$= 4\sqrt{3} \times 4\sqrt{6}$$

$$= 48\sqrt{2} \text{ sq m}$$

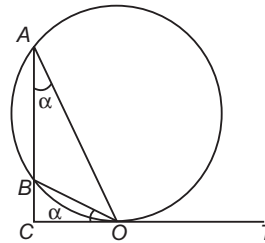
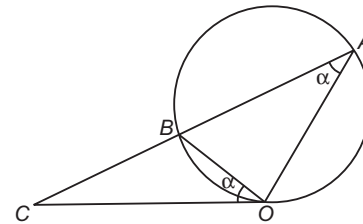
Some Properties Related to Circle

i. Angles in the same segment of a circle are equal.

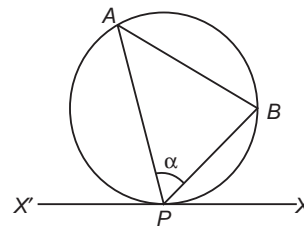
$$\text{i.e. } \angle APB = \angle AQB = \angle ARB$$



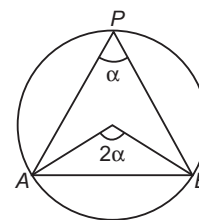
ii. Angles in the alternate segments of a circle are equal.



iii. If the line joining two points A and B subtends the greatest angle α at a point P , then the circle will touch the straight line XX' at the point P .

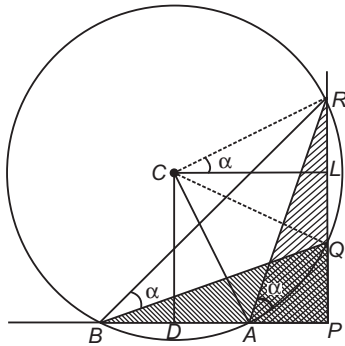


iv. The angle subtended by any chord at the centre is twice the angle subtended by the same on any point on the circumference of the circle.



- **Example 34.** A statue on the top of a pillar subtends the same angle α at distances 9 m and 11 m from the pillar in the horizontal plane through the foot of the pillar. If $\tan \alpha = \frac{1}{10}$, then the height of the pillar and height of the statue are
- (a) 9 m, 2 m
(b) 1 m, 9 m
(c) 10 m, 9 m
(d) 1 m, 10 m

Sol. (a) Let PQ be the pillar and QR the statue. Let A be a point 9 m from P and B be the point 11 m from P . Since, $\angle RAQ = \angle RBQ$, the four points A, B, R and Q are concyclic.



Let D be the mid-point of AB and L be the mid-point of RQ . Draw perpendiculars from D on AB and from L on RQ respectively. Let them meet at C . Then, C is the centre of the circle passing through A, B, R and Q .

$$\begin{aligned} \Rightarrow \angle RCQ &= 2\angle RAQ = 2\alpha \\ \Rightarrow \angle RCL &= \alpha. \text{ Also, } CL = DP = 10 \text{ m} \\ \Rightarrow RL &= LC \tan \alpha = 10 \cdot \frac{1}{10} = 1 \text{ m} \end{aligned}$$

Hence, the height of the statue is 2 m.

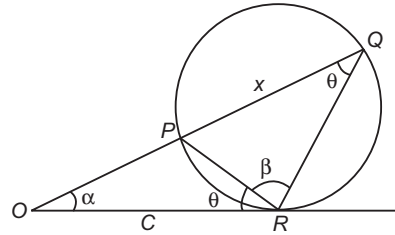
Also, $AB = 2 \text{ m}$

$$\begin{aligned} \Rightarrow LP &= CD \text{ and } CR = CA \quad [\text{radii of the circle}] \\ \Rightarrow \triangle ABC &\text{ and } \triangle RQC \text{ are congruent.} \\ \Rightarrow LP &= CD = CL = 10 \text{ m} \\ \therefore PQ &= LP - LQ = 10 - 1 = 9 \text{ m} \end{aligned}$$

- **Example 35.** A person walking along a canal observes that two objects are in the same line which is inclined at an angle α to the canal. He walks a distance c further and observes that the objects subtend their greatest angle β . Then, the distance between the object is

- (a) $\frac{2c \sin \alpha \sin \beta}{\cos \beta - \cos \alpha}$
(b) $\frac{2c \sin \alpha \sin \beta}{\cos \alpha - \cos \beta}$
(c) $\frac{2c \sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$
(d) None of the above

Sol. (c) Let O be the initial position of the man and P and Q be the positions of the objects. Since, PQ subtends the greatest angle at R , a circle will pass through P, Q, R and OR will be tangential to this circle at R .



Let $\angle PQR = \angle PRO = \theta$ [from the property of circle]

We have, to find PQ i.e. x

Clearly, $\angle OPR = \theta + \beta$

Applying sine rule in $\triangle PRQ$, we get

$$\begin{aligned} \frac{x}{\sin \beta} &= \frac{PR}{\sin \theta} \\ \Rightarrow x &= \frac{PR \sin \beta}{\sin \theta} \quad \dots(i) \end{aligned}$$

Applying sine rule in $\triangle POR$, we get

$$\begin{aligned} \frac{PR}{\sin \alpha} &= \frac{c}{\sin (\theta + \beta)} \\ \Rightarrow PR &= \frac{c \sin \alpha}{\sin (\theta + \beta)} \quad \dots(ii) \end{aligned}$$

From Eqs. (i) and (ii), we have

$$\begin{aligned} x &= \frac{c \sin \alpha \sin \beta}{\sin (\theta + \beta) \sin \theta} = \frac{2c \sin \alpha \sin \beta}{2 \sin (\theta + \beta) \sin \theta} \\ &= \frac{2c \sin \alpha \sin \beta}{\cos \beta - \cos (2\theta + \beta)} \quad \dots(iii) \end{aligned}$$

From $\triangle ORQ$,

$$\alpha + \theta + \beta + \theta = 180^\circ$$

$$\Rightarrow 2\theta + \beta = 180^\circ - \alpha$$

$$\Rightarrow \cos (2\theta + \beta) = \cos (180^\circ - \alpha) = -\cos \alpha \quad \dots(iv)$$

From Eqs. (iii) and (iv), we get

$$x = \frac{2c \sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$$

- 528

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. If the data given to construct a $\triangle ABC$ is $a = 5, b = 7, \sin A = \frac{3}{4}$, then it is possible to construct

- (a) only one triangle
- (b) two triangle
- (c) infinitely many triangles
- (d) no triangle

Sol. We have, $\frac{\sin A}{a} = \frac{\sin B}{b} \Rightarrow \frac{3/4}{5} = \frac{\sin B}{7}$
 $\Rightarrow \sin B = \frac{21}{20} > 1$, which is not possible

Thus, no triangle is possible.
Hence, (d) is the correct answer.

Ex 2. In $\triangle ABC$, $a = 5, b = 4$ and $\cos(A - B) = \frac{31}{32}$, then side c is

- (a) 6
- (b) 7
- (c) 9
- (d) None of the above

Sol. We have, $\tan \frac{A - B}{2} = \sqrt{\frac{1 - \cos(A - B)}{1 + \cos(A - B)}}$
 $= \sqrt{\frac{1 - \frac{31}{32}}{1 + \frac{31}{32}}} = \frac{1}{\sqrt{63}}$
 $\Rightarrow \frac{a - b}{a + b} \cot \frac{C}{2} = \frac{1}{\sqrt{63}}$
 $\left[\because \tan \frac{A - B}{2} = \frac{a - b}{a + b} \cot \frac{C}{2} \right]$

$$\Rightarrow \frac{1}{9} \cot \frac{C}{2} = \frac{1}{\sqrt{63}}$$

$$\Rightarrow \tan \frac{C}{2} = \frac{\sqrt{7}}{3}$$

$$\text{Now, } \cos C = \frac{1 - \tan^2 \frac{C}{2}}{1 + \tan^2 \frac{C}{2}}$$

$$\Rightarrow \cos C = \frac{1 - \frac{7}{9}}{1 + \frac{7}{9}} = \frac{1}{8}$$

Since, $c^2 = a^2 + b^2 - 2ab \cos C$
 $\Rightarrow c^2 = 25 + 16 - 40 \times \frac{1}{8} = 36$
 $\Rightarrow c = 6$

Hence, (a) is the correct answer.

Ex 3. In an acute angled $\triangle ABC$, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then

- (a) $b + 2c < 2a < 2b + 2c$
- (b) $b + 4c < 4a < 2b + 4c$
- (c) $b + 4c < 4a < 4b + 4c$
- (d) $b + 3c < 3a < 3b + 3c$

Sol. $r - r_2 = r_3 - r_1$
 $\Rightarrow \frac{\Delta}{s} - \frac{\Delta}{s - b} = \frac{\Delta}{s - c} - \frac{\Delta}{s - a}$
 $\Rightarrow \frac{-b}{s(s - b)} = \frac{c - a}{(s - a)(s - c)}$
 $\Rightarrow \frac{(s - a)(s - c)}{s(s - b)} = \frac{a - c}{b}$
 $\Rightarrow \tan^2 \frac{B}{2} = \frac{a - c}{b}$

But $\frac{B}{2} \in \left(\frac{\pi}{6}, \frac{\pi}{4} \right)$

$$\Rightarrow \tan^2 \frac{B}{2} \in \left(\frac{1}{3}, 1 \right)$$

$$\Rightarrow \frac{1}{3} < \frac{a - c}{b} < 1$$

$$\Rightarrow b < 3a - 3c < 3b$$

$$\Rightarrow b + 3c < 3a < 3b + 3c$$

Hence, (d) is the correct answer.

Ex 4. The sides of a triangle are $3x + 4y, 4x + 3y$ and $5x + 5y$ units, where $x, y > 0$. The triangle is

- (a) right angled
- (b) equilateral
- (c) obtuse angled
- (d) None of these

Sol. Let $a = 3x + 4y, b = 4x + 3y$ and $c = 5x + 5y$
Clearly, c is the largest side and then the largest angle C is given by

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{-2xy}{2(12x^2 + 25xy + 12y^2)} < 0$$

$\Rightarrow C$ is an obtuse angle.

Hence, (c) is the correct answer.

Ex 5. In a $\triangle ABC$, a, b, c, A are given and c_1, c_2 are two values of the third side c . The sum of the areas of two triangles with sides a, b, c_1 and a, b, c_2 is

- (a) $\frac{1}{2} b^2 \sin 2A$
- (b) $\frac{1}{2} a^2 \sin 2A$
- (c) $b^2 \sin 2A$
- (d) None of these

Sol. We have, $\cos A = \frac{c^2 + b^2 - a^2}{2bc}$

$$\Rightarrow c^2 - 2bc \cos A + b^2 - a^2 = 0$$

It is given that c_1 and c_2 are roots of this equation.

Therefore, $c_1 + c_2 = 2b \cos A$ and $c_1 c_2 = b^2 - a^2$

$$\Rightarrow k(\sin C_1 + \sin C_2) = 2k \sin B \cos A$$

$$\Rightarrow \sin C_1 + \sin C_2 = 2 \sin B \cos A$$

Now, sum of the areas of two triangles

$$\begin{aligned} &= \frac{1}{2} ab \sin C_1 + \frac{1}{2} ab \sin C_2 \\ &= \frac{1}{2} ab (\sin C_1 + \sin C_2) = \frac{1}{2} ab (2 \sin B \cos A) \\ &= b \cdot b \sin A \cos A = \frac{1}{2} b^2 \sin 2A \end{aligned}$$

Hence, (a) is the correct answer.

Ex 6. In a $\triangle ABC$, a, b, A are given and b_1, b_2 are two values of the third side b such that $b_2 = 2b_1$. Then, $\sin A$ is equal to

(a) $\sqrt{\frac{9a^2 - c^2}{8a^2}}$

(b) $\sqrt{\frac{9a^2 - c^2}{8c^2}}$

(c) $\sqrt{\frac{9a^2 - c^2}{8b^2}}$

(d) None of the above

Sol. We have, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$

$$\Rightarrow b^2 - 2bc \cos A + (c^2 - a^2) = 0$$

It is given that b_1 and b_2 are roots of this equation.

Therefore, $b_1 + b_2 = 2c \cos A$ and $b_1 b_2 = c^2 - a^2$

$$\Rightarrow 3b_1 = 2c \cos A$$

$$\text{and } 2b_1^2 = c^2 - a^2 \quad [\because b_2 = 2b_1, \text{ given}]$$

$$\Rightarrow 2 \left(\frac{2c}{3} \cos A \right)^2 = c^2 - a^2$$

$$\Rightarrow 8c^2 (1 - \sin^2 A) = 9c^2 - 9a^2$$

$$\Rightarrow \sin A = \sqrt{\frac{9a^2 - c^2}{8c^2}}$$

Hence, (b) is the correct answer.

Ex 7. If in $\triangle ABC$, $\angle A = 90^\circ$ and $c, \sin B, \cos B$ are rational numbers, then

- (a) a and b are rational (b) a is irrational
(c) only b is rational (d) None of these

Sol. Let AD be perpendicular from A on BC . Then,

$$\cos B = \frac{BD}{c} \Rightarrow BD = c \cos B$$

$\Rightarrow BD$ is rational.

Similarly, $AD = c \sin B \Rightarrow AD$ is rational

$$\text{Now, } \sin C = \cos B = \frac{AD}{b}$$

$\Rightarrow b$ is rational

$$\text{Since, } \cos C = \sin B = \frac{DC}{b}$$

$\Rightarrow DC$ is rational.

Now, $a = BD + DC$ is rational.

Hence, (a) is the correct answer.

Ex 8. If P is a point on the altitude AD of the $\triangle ABC$ such that $\angle CDP = \frac{B}{3}$, then AP is equal to

(a) $2a \sin \frac{C}{3}$
(c) $2c \sin \frac{B}{3}$

(b) $2b \sin \frac{C}{3}$
(d) $2c \sin \frac{C}{3}$

Sol. $\angle BPA = 90^\circ + \frac{B}{3}$, $\angle ABP = \frac{2B}{3}$

In $\triangle ABP$, by sine rule,

$$\frac{AP}{\sin \frac{2B}{3}} = \frac{c}{\sin \left(90^\circ + \frac{B}{3} \right)} = \frac{c}{\cos \left(\frac{B}{3} \right)}$$

$$\Rightarrow AP = \frac{c \sin \left(\frac{2B}{3} \right)}{\cos \left(\frac{B}{3} \right)} = \frac{2c \sin \left(\frac{B}{3} \right) \cos \left(\frac{B}{3} \right)}{\cos \left(\frac{B}{3} \right)}$$

$$\Rightarrow AP = 2c \sin \left(\frac{B}{3} \right)$$

Hence, (c) is the correct answer.

Ex 9. If in a $\triangle ABC$, CD is the angle bisector of the $\angle ABC$, then CD is equal to

(a) $\frac{a+b}{2ab} \cos \frac{C}{2}$
(c) $\frac{2ab}{a+b} \cos \frac{C}{2}$

(b) $\frac{a+b}{ab} \cos \frac{C}{2}$
(d) None of these

Sol. $\ar(\triangle CAB) = \ar(\triangle CAD) + \ar(\triangle CDB)$

$$\Rightarrow \frac{1}{2} ab \sin C = \frac{1}{2} b \cdot CD \sin \left(\frac{C}{2} \right) + \frac{1}{2} a \cdot CD \sin \left(\frac{C}{2} \right)$$

$$\Rightarrow CD(a+b) \sin \left(\frac{C}{2} \right) = ab \left[2 \sin \left(\frac{C}{2} \right) \cos \left(\frac{C}{2} \right) \right]$$

$$\therefore CD = \frac{2ab \cos \left(\frac{C}{2} \right)}{(a+b)}$$

and in $\triangle CAD$,

$$\frac{CD}{\sin A} = \frac{b}{\sin \angle CDA} \quad [\text{by sine rule}]$$

$$\Rightarrow CD = \frac{b \sin A}{\sin \left(B + \frac{C}{2} \right)}$$

Hence, (c) is the correct answer.

Ex 10. In any $\triangle ABC$, the least value of $\left(\frac{\sin^2 A + \sin A + 1}{\sin A} \right)$ is

(a) 3
(c) 9

(b) $\sqrt{3}$
(d) None of these

Sol. $\frac{\sin^2 A + \sin A + 1}{\sin A} = \sin A + \frac{1}{\sin A} + 1$

$$= \left[\left(\sqrt{\sin A} - \frac{1}{\sqrt{\sin A}} \right)^2 + 3 \right] > 3$$

Hence, (a) is the correct answer.

Ex 11. If A is the area and $2s$ is the sum of the sides of a triangle, then

- (a) $A \leq \frac{s^2}{3\sqrt{3}}$ (b) $A \geq \frac{s^2}{3\sqrt{3}}$
 (c) $A > \frac{s^2}{\sqrt{3}}$ (d) None of these

Sol. We have, $2s = a + b + c$, $A^2 = s(s-a)(s-b)(s-c)$

Now, $\frac{AM \geq GM}{s + (s-a) + (s-b) + (s-c)} \Rightarrow \frac{4s-2s}{4} \geq [s(s-a)(s-b)(s-c)]^{1/4}$
 $\Rightarrow \frac{4s-2s}{4} \geq [A^2]^{1/4}$
 $\Rightarrow \frac{s}{2} \geq A^{1/2} \Rightarrow A \leq \frac{s^2}{4}$
 Also, $\frac{(s-a) + (s-b) + (s-c)}{3} \geq [(s-a)(s-b)(s-c)]^{1/3}$
 or $\frac{s}{3} \geq \left[\frac{A^2}{s}\right]^{1/3}$ or $\frac{A^2}{s} \leq \frac{s^3}{27}$
 $\Rightarrow A \leq \frac{s^2}{3\sqrt{3}}$

Hence, (a) is the correct answer.

Ex 12. If in a $\triangle ABC$, $\angle B = 90^\circ$, then $\tan^2\left(\frac{A}{2}\right)$ is equal

- to
 (a) $\frac{b-c}{b+c}$
 (b) $\frac{b+c}{b-c}$
 (c) $\frac{b-2c}{b+c}$
 (d) None of the above

Sol. $\cos A = \frac{c}{b}$

$\Rightarrow \frac{1 - \tan^2 \frac{A}{2}}{1 + \tan^2 \frac{A}{2}} = \frac{c}{b}$
 $\Rightarrow \frac{\left(1 + \tan^2 \frac{A}{2}\right) - \left(1 - \tan^2 \frac{A}{2}\right)}{\left(1 + \tan^2 \frac{A}{2}\right) + \left(1 - \tan^2 \frac{A}{2}\right)} = \frac{(b-c)}{(b+c)}$
 [by componendo and dividendo rule]
 $\therefore \tan^2 \frac{A}{2} = \frac{b-c}{b+c}$

Hence, (a) is the correct answer.

Ex 13. In a $\triangle ABC$, $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab}$ is equal to

- (a) $\frac{1}{2R} - \frac{1}{r}$ (b) $2R - r$
 (c) $r - 2R$ (d) $\frac{1}{r} - \frac{1}{2R}$

Sol. $\frac{r_1}{bc} = \frac{4R \sin\left(\frac{A}{2}\right) \cos\left(\frac{B}{2}\right) \cos\left(\frac{C}{2}\right)}{2R \sin B \cdot 2R \sin C}$
 $= \frac{\sin\left(\frac{A}{2}\right)}{4R \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)}$
 $= \frac{\sin^2\left(\frac{A}{2}\right)}{4R \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right)} = \frac{\sin^2\left(\frac{A}{2}\right)}{r}$

So,

$\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} \left[\sin^2\left(\frac{A}{2}\right) + \sin^2\left(\frac{B}{2}\right) + \sin^2\left(\frac{C}{2}\right) \right]$
 $= \frac{1}{2r} (1 - \cos A + 1 - \cos B + 1 - \cos C)$
 $= \frac{1}{2r} [3 - (\cos A + \cos B + \cos C)]$
 $= \frac{1}{2r} \left[3 - \left\{ 1 + 4 \sin\left(\frac{A}{2}\right) \sin\left(\frac{B}{2}\right) \sin\left(\frac{C}{2}\right) \right\} \right]$
 $= \frac{1}{2r} \left(2 - \frac{r}{R} \right) = \frac{1}{r} - \frac{1}{2R}$

Hence, (d) is the correct answer.

Ex 14. If A, B and C are angles of triangle such that $\angle A$ is obtuse, then $\tan B \tan C$ will be less than

- (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{\sqrt{3}}{2}$
 (c) 1 (d) None of these

Sol. Since, A is obtuse angle. Then, $90^\circ < A < 180^\circ$

$\Rightarrow 90^\circ > B + C > 0$
 $\Rightarrow B + C < 90^\circ \Rightarrow B < 90^\circ - C$
 $\Rightarrow \tan B < \tan (90^\circ - C)$
 $\Rightarrow \tan B < \cot C$
 $\Rightarrow \tan B \tan C < 1$

Hence, (c) is the correct answer.

Ex 15. The sides of two angles of a triangle are equal to $\frac{5}{13}$ and $\frac{99}{101}$. The cosine of the third angle is

- (a) $\frac{255}{1315}$ (b) $\frac{251}{1313}$
 (c) $\frac{255}{1313}$ (d) None of these

Sol. Let $\sin B = \frac{5}{13}$ and $\sin C = \frac{99}{101}$
 $\cos B = \frac{12}{13}$ and $\cos C = \frac{20}{101}$
 $\cos A = \cos \{180^\circ - (B + C)\}$
 $= -\cos (B + C)$
 $= -(\cos B \cos C - \sin B \sin C)$
 $= \sin B \sin C - \cos B \cos C$
 $= \frac{5}{13} \times \frac{99}{101} - \frac{12}{13} \times \frac{20}{101} = \frac{255}{1313}$

Hence, (c) is the correct answer.

Ex 16. If a, b, c and d are the sides of a quadrilateral, then the minimum value of $\frac{a^2 + b^2 + c^2}{d^2}$ is

- (a) 1 (b) $\frac{1}{2}$
(c) $\frac{1}{3}$ (d) $\frac{1}{4}$

Sol. Since, $(a-b)^2 + (b-c)^2 + (c-a)^2 \geq 0$
 $\Rightarrow 2(a^2 + b^2 + c^2) \geq 2ab + 2bc + 2ca$
 $\Rightarrow 3(a^2 + b^2 + c^2) \geq a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$
 $\Rightarrow 3(a^2 + b^2 + c^2) \geq (a+b+c)^2 > d^2$
 $\Rightarrow 3(a^2 + b^2 + c^2) > d^2$
 $\Rightarrow \frac{a^2 + b^2 + c^2}{d^2} > \frac{1}{3}$
 So, minimum value of $\frac{a^2 + b^2 + c^2}{d^2}$ is $\frac{1}{3}$.
 Hence, (c) is the correct answer.

Ex 17. If $b=2, c=\sqrt{3}$ and $\angle A=30^\circ$, then inradius of $\triangle ABC$ is

(a) $\frac{\sqrt{3}-1}{2}$ (b) $\frac{\sqrt{3}+1}{2}$
 (c) $\frac{\sqrt{3}-1}{4}$ (d) None of these

Sol. $b=2, c=\sqrt{3}$ and $\angle A=30^\circ$
 $a = \sqrt{b^2 + c^2 - 2bc \cos A}$
 $= \sqrt{4 + 3 - 2 \cdot 2 \cdot \sqrt{3} \cdot \frac{\sqrt{3}}{2}} = \sqrt{1} = 1$
 $r = (s-a) \tan \left(\frac{A}{2} \right) = \frac{b+c-a}{2} \tan \frac{A}{2}$
 $= \frac{\sqrt{3}+1}{2} \tan 15^\circ$
 $= \frac{\sqrt{3}+1}{2} \cdot \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{\sqrt{3}-1}{2}$

Hence, (a) is the correct answer.

Ex 18. If in a $\triangle ABC$,
 $\cos A \cos B + \sin A \sin B \sin C = 1$,
 then $a : b : c$ is equal to

(a) $1 : 1 : \sqrt{2}$ (b) $1 : 1 : \sqrt{3}$
 (c) $1 : \sqrt{2} : 1$ (d) $1 : \sqrt{3} : 1$

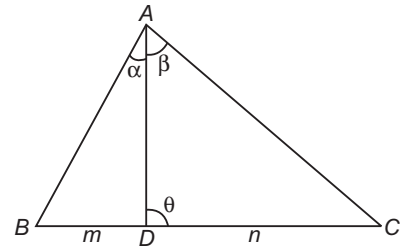
Sol. Here, $\sin C = \frac{1 - \cos A \cos B}{\sin A \sin B} \leq 1$
 $\Rightarrow \frac{1 - \cos A \cos B}{\sin A \sin B} - 1 \leq 0$
 $\Rightarrow 1 - \cos A \cos B - \sin A \sin B \leq 0$
 $\Rightarrow 1 - \cos(A-B) \leq 0$
 $\Rightarrow \cos(A-B) \geq 1$
 But $\cos(A-B)$ cannot be greater than 1.
 $\Rightarrow \cos(A-B) = 1$
 $\Rightarrow A-B=0 \Rightarrow A=B$

$\Rightarrow \sin C = \frac{1 - \cos^2 A}{\sin^2 A} = \frac{\sin^2 A}{\sin^2 A} = 1$
 $\Rightarrow C = 90^\circ$
 $\Rightarrow A = B = 45^\circ$
 Now, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$
 $\Rightarrow \frac{1}{\sqrt{2}a} = \frac{1}{\sqrt{2}b} = \frac{1}{c}$
 $\Rightarrow a : b : c = 1 : 1 : \sqrt{2}$
 Hence, (a) is the correct answer.

Ex 19. If D is a point on the side BC of a $\triangle ABC$ such that $BD : DC = m : n$ and $\angle ADC = \theta$, $\angle BAD = \alpha$ and $\angle DAC = \beta$. Then, $(m+n) \cot \theta$ is equal to

(a) $m \cot \alpha + n \cot \beta$ (b) $n \cot \beta - m \cot \alpha$
 (c) $m \cot \alpha - n \cot \beta$ (d) $m \cot \beta - n \cot \alpha$

Sol. Given, $\frac{BD}{DC} = \frac{m}{n}$ and $\angle ADC = \theta$



Since, $\angle ADB = (180^\circ - \theta)$, $\angle BAD = \alpha$ and $\angle DAC = \beta$,
 $\angle ABD = 180^\circ - (\alpha + 180^\circ - \theta) = \theta - \alpha$ and
 $\angle ACD = 180^\circ - (\theta + \beta)$

From $\triangle ABD$, $\frac{BD}{\sin \alpha} = \frac{AD}{\sin(\theta - \alpha)}$... (i)

From $\triangle ADC$, $\frac{DC}{\sin \beta} = \frac{AD}{\sin[180^\circ - (\theta + \beta)]}$

or $\frac{DC}{\sin \beta} = \frac{AD}{\sin(\theta + \beta)}$... (ii)

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{BD \sin \beta}{DC \sin \alpha} = \frac{\sin(\theta + \beta)}{\sin(\theta - \alpha)}$$

$$\Rightarrow \frac{m \sin \beta}{n \sin \alpha} = \frac{\sin \theta \cos \beta + \cos \theta \sin \beta}{\sin \theta \cos \alpha - \cos \theta \sin \alpha}$$

$$\Rightarrow m \sin \beta (\sin \theta \cos \alpha - \cos \theta \sin \alpha) = n \sin \alpha (\sin \theta \cos \beta + \cos \theta \sin \beta)$$

$$\Rightarrow m \cot \alpha - m \cot \theta = n \cot \beta + n \cot \theta$$

[dividing both sides by $\sin \alpha \sin \beta \sin \theta$]

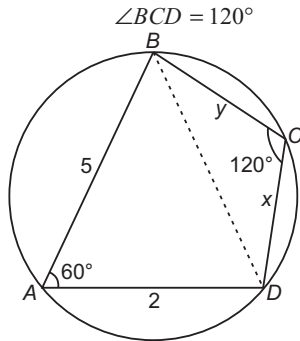
$$\Rightarrow (m+n) \cot \theta = m \cot \alpha - n \cot \beta$$

Hence, (c) is the correct answer.

Ex 20. The two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60° . If the area of the quadrilateral is $4\sqrt{3}$, then the remaining two sides are

- (a) 2, 3 (b) 4, 6
(c) $\sqrt{3}+1, \sqrt{3}$ (d) 3, 3

Sol. Let $AD = 2$, $AB = 5$, $\angle DAB = 60^\circ$. Since, the quadrilateral is cyclic.



$$\text{Area of } \triangle ABD = \frac{1}{2} \cdot 2 \cdot 5 \sin 60^\circ = 5 \cdot \frac{\sqrt{3}}{2}$$

$$\begin{aligned} \text{Area of } \triangle BCD &= \text{Area of quadrilateral } ABCD \\ &\quad - \text{Area of } \triangle ABD \\ &= 4\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{2} \end{aligned}$$

Let $CD = x$ and $BC = y$

$$\text{Now, area of } \triangle BCD = \frac{1}{2} \cdot x \cdot y \cdot \sin 120^\circ$$

$$\text{or } \frac{3\sqrt{3}}{2} = \frac{1}{2} \cdot xy \cdot \frac{\sqrt{3}}{2}$$

$$\Rightarrow xy = 6 \quad \dots(i)$$

On applying cosine rule in $\triangle BAD$, we get

$$\begin{aligned} \cos 60^\circ &= \frac{AD^2 + AB^2 - BD^2}{2AD \cdot AB} \\ \Rightarrow \frac{1}{2} &= \frac{2^2 + 5^2 - BD^2}{2 \cdot 2 \cdot 5} \Rightarrow BD^2 = 19 \end{aligned}$$

On applying cosine rule in $\triangle BCD$, we get

$$\begin{aligned} \cos 120^\circ &= \frac{x^2 + y^2 - 19}{2xy} \\ \Rightarrow -\frac{1}{2} &= \frac{x^2 + y^2 - 19}{2xy} \end{aligned}$$

$$\Rightarrow x^2 + y^2 + xy = 19$$

$$\text{Now, } x^2 + y^2 + 2xy = 13 + 12 = 25$$

$$\begin{aligned} \Rightarrow (x + y)^2 &= 25 \\ \Rightarrow x + y &= 5 \quad \dots(ii) \end{aligned}$$

$$\text{and } x^2 + y^2 - 2xy = 13 - 12$$

$$\Rightarrow x - y = \pm 1 \quad \dots(iii)$$

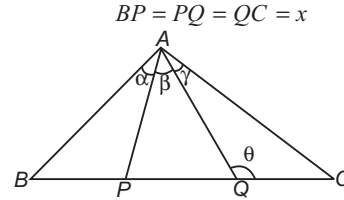
On solving, we get $x = 3$, $y = 2$ or $x = 2$, $y = 3$

Hence, (a) is the correct answer.

Ex 21. The base of a triangle is divided into three equal parts. If t_1 , t_2 and t_3 are the tangents of the angles subtended by these parts at the opposite vertex, then $\left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right)$ is equal to

- (a) $4\left(1 - \frac{1}{t_2^2}\right)$ (b) $4\left(1 + \frac{1}{t_2^2}\right)$
(c) $4\left(1 + \frac{1}{t_1^2}\right)$ (d) $4\left(1 + \frac{1}{t_2^2}\right)$

Sol. Let the points P and Q divide the side BC in three equal parts.



Also, let $\angle BAP = \alpha$, $\angle PAQ = \beta$,
 $\angle QAC = \gamma$ and $\angle AQC = \theta$

From equation,

$$\begin{aligned} \tan \alpha &= t_1, \tan \beta = t_2, \tan \gamma = t_3 \\ \text{On applying } m:n \text{ rule in } \triangle ABC, \text{ we get} \\ (2x + x) \cot \theta &= 2x \cot(\alpha + \beta) - x \cot \gamma \quad \dots(i) \end{aligned}$$

From $\triangle APC$, we get

$$\begin{aligned} (x + x) \cot \theta &= x \cot \beta - x \cot \gamma \quad \dots(ii) \\ \text{On dividing Eq. (i) by Eq. (ii), we get} \\ \frac{3}{2} &= \frac{2 \cot(\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma} \end{aligned}$$

$$\begin{aligned} \Rightarrow 3 \cot \beta - 3 \cot \gamma &= 4 \cot(\alpha + \beta) - 2 \cot \gamma \\ \Rightarrow 3 \cot \beta - \cot \gamma &= \frac{4(\cot \alpha \cot \beta - 1)}{\cot \beta + \cot \alpha} \end{aligned}$$

$$\begin{aligned} \Rightarrow 3 \cot^2 \beta - \cot \beta \cot \gamma + 3 \cot \alpha \cot \beta - \cot \alpha \cot \gamma \\ = 4 \cot \alpha \cot \beta - 4 \end{aligned}$$

$$\begin{aligned} \Rightarrow 4 + 4 \cot^2 \beta &= \cot^2 \beta + \cot \alpha \cot \beta \\ &\quad + \cot \beta \cot \gamma + \cot \gamma \cot \alpha \end{aligned}$$

$$\Rightarrow 4(1 + \cot^2 \beta) = (\cot \beta + \cot \alpha)(\cot \beta + \cot \gamma)$$

$$\Rightarrow 4\left(1 + \frac{1}{t_2^2}\right) = \left(\frac{1}{t_1} + \frac{1}{t_2}\right)\left(\frac{1}{t_2} + \frac{1}{t_3}\right)$$

Hence, (d) is the correct answer.

Ex 22. In the $\triangle ABC$, if $(a^2 + b^2) \sin(A - B) = (a^2 - b^2) \sin(A + B)$, then the triangle is

- (a) either isosceles or right angled
(b) only right angled
(c) only isosceles triangle
(d) None of the above

Sol. Let $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$ [say]

$$\Rightarrow a = k \sin A, b = k \sin B, c = k \sin C$$

Now, the given relation is

$$\begin{aligned} (a^2 + b^2) \sin(A - B) &= (a^2 - b^2) \sin(A + B) \\ \Rightarrow k^2 [\sin^2 A + \sin^2 B] \sin(A - B) \\ &= k^2 [\sin^2 A - \sin^2 B] \sin(A + B) \\ \Rightarrow [\sin^2 A + \sin^2 B] \sin(A - B) \\ &= \sin^2(A + B) \sin(A - B) \end{aligned}$$

$$\Rightarrow \sin(A - B) [\sin^2 A + \sin^2 B - \sin^2 C] = 0$$

Hence, either the first factor = 0 or the second factor = 0

$$\text{If } \sin(A - B) = 0 \Rightarrow A - B = 0 \Rightarrow A = B$$

$\Rightarrow$ The triangle is isosceles.

$$\text{If } \sin^2 A + \sin^2 B - \sin^2 C = 0$$

$$\Rightarrow \frac{a^2}{k^2} + \frac{b^2}{k^2} - \frac{c^2}{k^2} = 0$$

$$\Rightarrow a^2 + b^2 - c^2 = 0 \Rightarrow a^2 + b^2 = c^2$$

So, the triangle is right angled.

Hence, (a) is the correct answer.

Ex 23. The sides of a $\triangle ABC$ are in AP, if the angles A and C are the greatest and the smallest angles respectively, then

- $4(1 - \cos A)(1 - \cos C)$ is equal to
- (a) $\cos^2 \frac{A+C}{2}$ (b) $4 \cos^2 \frac{A+C}{2}$
- (c) $4 \cos^2 \frac{A-C}{2}$ (d) $4 \cos^2 \frac{A-C}{4}$

Sol. Since, A is the greatest and C is the smallest angle, a is the greatest and c is the smallest side.

$\Rightarrow a, b$ and c are in AP.

$$\Rightarrow 2b = a + c$$

$$\Rightarrow 4R \sin B = 2R (\sin A + \sin C)$$

$$\Rightarrow 2 \cdot 2 \sin \frac{B}{2} \cos \frac{B}{2} = 2 \sin \frac{A+C}{2} \cos \frac{A-C}{2}$$

$$\Rightarrow 2 \sin \frac{B}{2} = \cos \frac{A-C}{2} \left[\because \frac{A+C}{2} = 90^\circ - \frac{B}{2} \right]$$

$$\Rightarrow 2 \cos \frac{A+C}{2} = \cos \frac{A-C}{2} \quad \dots(i)$$

$$\text{Now, } 4(1 - \cos A)(1 - \cos C) = 4 \cdot 2 \sin^2 \frac{A}{2} \cdot 2 \sin^2 \frac{C}{2}$$

$$\Rightarrow 4 \left[2 \sin \frac{A}{2} \cdot \sin \frac{C}{2} \right]^2 = 4 \left[\cos \frac{A-C}{2} - \cos \frac{A+C}{2} \right]^2$$

$$\Rightarrow 4 \left[2 \cos \frac{A+C}{2} - \cos \frac{A+C}{2} \right]^2 = 4 \cos^2 \frac{A+C}{2}$$

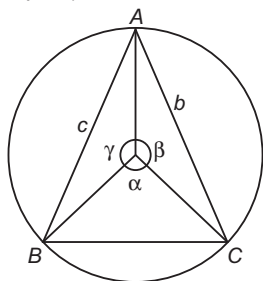
[from Eq. (i)]

Hence, (b) is the correct answer.

Ex 24. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the centre. Then, the minimum value of the arithmetic mean of

- $\cos \left(\alpha + \frac{\pi}{2} \right), \cos \left(\beta + \frac{\pi}{2} \right), \cos \left(\gamma + \frac{\pi}{2} \right)$ is
- (a) $\frac{\sqrt{3}}{2}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\sqrt{2}}{3}$ (d) $-\frac{\sqrt{3}}{2}$

Sol. Here, $\alpha + \beta + \gamma = 2\pi$



Now, arithmetic mean

$$\begin{aligned} & \frac{\cos \left(\alpha + \frac{\pi}{2} \right) + \cos \left(\beta + \frac{\pi}{2} \right) + \cos \left(\gamma + \frac{\pi}{2} \right)}{3} \\ &= \frac{-\sin \alpha - \sin \beta - \sin \gamma}{3} \\ &= -\left(\frac{\sin \alpha + \sin \beta + \sin \gamma}{3} \right) \end{aligned}$$

For this arithmetic mean to be minimum, $\sin \alpha + \sin \beta + \sin \gamma$ must be maximum.

Clearly, $\sin \alpha + \sin \beta + \sin \gamma$ will be maximum when $\alpha = \beta = \gamma$. Also, $\alpha + \beta + \gamma = 2\pi$

$$\Rightarrow \alpha = \beta = \gamma = \frac{2\pi}{3}$$

Now, $(\sin \alpha + \sin \beta + \sin \gamma)_{\max}$

$$= \sin \frac{2\pi}{3} + \sin \frac{2\pi}{3} + \sin \frac{2\pi}{3} = \frac{3\sqrt{3}}{2}$$

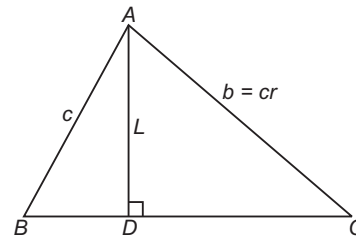
$$\therefore (AM)_{\min} = -\frac{3\sqrt{3}}{3 \cdot 2} = -\frac{\sqrt{3}}{2}$$

Hence, (d) is the correct answer.

Ex 25. In a triangle of base a , the ratio of the other sides is $r (< 1)$. The altitude of the triangle is less than or equal to

- (a) $\frac{ar}{1+r^2}$
- (b) $\frac{ar}{1-r^2}$
- (c) $\frac{ar}{1 \pm r^2}$
- (d) None of the above

Sol. Let D be the foot of the altitude from A and $BC = a$, $AB = c$, $AC = b$, $AD = L = c \sin B$



$$\begin{aligned} \text{Now, } \frac{ar}{1-r^2} &= \frac{ac^2 r}{c^2 - r^2 c^2} = \frac{abc}{c^2 - b^2} \\ &= \frac{a \sin B \sin C}{\sin^2 C - \sin^2 B} = \frac{a \sin B \sin C}{\sin(C-B) \sin(C+B)} \\ &= \frac{a \sin B \sin C}{\sin A \sin(C-B)} \\ &= \frac{c \sin B}{\sin(C-B)} = \frac{L}{\sin(C-B)} \end{aligned}$$

$$\Rightarrow L = \frac{ar}{1-r^2} \cdot \sin(C-B)$$

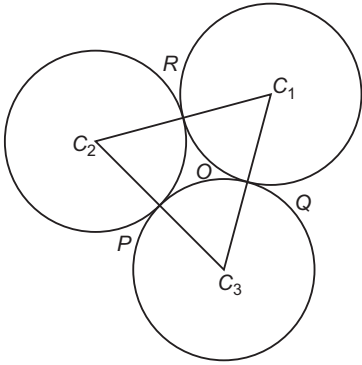
$$\Rightarrow L \leq \frac{ar}{1-r^2}$$

Hence, (b) is the correct answer.

Ex 26. Three circles touch one another externally. The tangents at their point of contact meet at a point whose distance from a point of contact is 4. Then, the ratio of the product of the radii to the sum of the radii of circles is

- (a) 16:1
- (b) 1:16
- (c) 8:1
- (d) None of the above

Sol. Let r_1, r_2 and r_3 be the radii of the three circles with centres at C_1, C_2 and C_3 . Let the circles touch at P, Q and R .



Also, $C_1C_2 = r_1 + r_2, C_2C_3 = r_2 + r_3, C_3C_1 = r_3 + r_1$
Let O be the point whose distance from the points of contact is 4.

Then, O is the incentre of the $\Delta C_1C_2C_3$ with $OP = OQ = OR = 4$, being the radius of the incircle

$$\therefore 4 = \frac{\Delta C_1C_2C_3}{\frac{1}{2}[C_1C_2 + C_2C_3 + C_3C_1]} = \frac{S}{s} \quad \dots(i)$$

where, $s = r_1 + r_2 + r_3$

$$\therefore \Delta^2 = s(s - C_1C_2)(s - C_2C_3)(s - C_3C_1) = s(r_1)(r_2)(r_3)$$

$$\text{Eq. (i) gives } 16 = \frac{\Delta^2}{s^2} = \frac{s(r_1)(r_2)(r_3)}{s^2} = \frac{r_1r_2r_3}{r_1 + r_2 + r_3}$$

Now, the ratio of the product of the radii to the sum of the radii = 16 : 1.

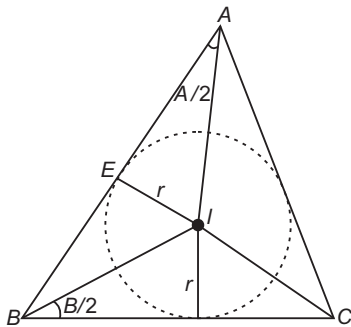
Hence, (a) is the correct answer.

Ex 27. In any ΔABC , if I is the incentre, then $AI + BI + CI$ is

- (a) $\geq 3r$ (b) $\geq 2r$
(c) $\geq 6r$ (d) None of these

Sol. In ΔAEI , $AI = EI \operatorname{cosec} \frac{A}{2}$

$$\Rightarrow AI = r \operatorname{cosec} \frac{A}{2}$$



$$\text{Similarly, } BI = r \operatorname{cosec} \frac{B}{2} \text{ and } CI = r \operatorname{cosec} \frac{C}{2}$$

Now,

$$AI + BI + CI = r \left(\operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2} \right)$$

On applying AM $\geq$ GM, we get

$$\begin{aligned} \operatorname{cosec} \frac{A}{2} + \operatorname{cosec} \frac{B}{2} + \operatorname{cosec} \frac{C}{2} &\geq 3 \left(\operatorname{cosec} \frac{A}{2} \operatorname{cosec} \frac{B}{2} \operatorname{cosec} \frac{C}{2} \right)^{1/3} \\ &\geq 3 \left(\frac{1}{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \right)^{1/3} \\ &\geq 3(8)^{1/3} \left[\text{as } \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \geq \frac{1}{8} \right] \end{aligned}$$

$$\therefore AI + BI + CI \geq r \cdot 3 \cdot (8)^{1/3}$$

$$\Rightarrow AI + BI + CI \geq 6r$$

Hence, (c) is the correct answer.

Ex 28. In any ΔABC , if $\sin A, \sin B$ and $\sin C$ are in AP, then the maximum value of $\tan \frac{B}{2}$ is

- (a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{3}$ (d) None of these

Sol. $2\sin B = \sin A + \sin C = \sin A + \sin(A + B)$
 $= \sin A + \sin A \cos B + \cos A \sin B$

$$\Rightarrow \sin B (2 - \cos A) = \sin A (1 + \cos B)$$

$$\Rightarrow \frac{\sin B}{1 + \cos B} = \frac{\sin A}{2 - \cos A}$$

$$\Rightarrow \tan \frac{B}{2} = \frac{\sin A}{2 - \cos A} = k \quad [\text{say}]$$

$$\Rightarrow \sin A = 2k - k \cos A$$

$$\Rightarrow \sin A + k \cos A = 2k$$

$$\Rightarrow \sin(A + \alpha) = \frac{2k}{\sqrt{1 + k^2}}, \text{ where } \tan \alpha = \frac{k}{1}$$

$$\Rightarrow \frac{2k}{\sqrt{1 + k^2}} \leq 1 \Rightarrow 2k \leq \sqrt{1 + k^2}$$

$$\Rightarrow 4k^2 \leq 1 + k^2$$

$$\Rightarrow 3k^2 - 1 \leq 0 \Rightarrow |k| \leq \frac{1}{\sqrt{3}}$$

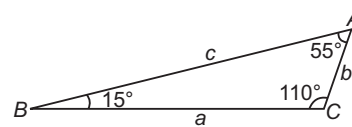
$$\therefore \text{Maximum value of } \tan \frac{B}{2} \text{ is } \frac{1}{\sqrt{3}}.$$

Hence, (b) is the correct answer.

Ex 29. In a triangle, if $\angle A = 55^\circ, \angle B = 15^\circ$ and $\angle C = 110^\circ$, then $c^2 - a^2$ is equal to

- (a) ab (b) $2ab$
(c) $-ab$ (d) None of these

$$\text{Sol. } \frac{c}{\sin 110^\circ} = \frac{b}{\sin 15^\circ} = \frac{a}{\sin 55^\circ} = k \quad [\text{say}]$$



$$\begin{aligned} \text{Now, } c^2 - a^2 &= k^2 \sin^2 110^\circ - k^2 \sin^2 55^\circ \\ &= k^2 (\sin 110^\circ + \sin 55^\circ) (\sin 110^\circ - \sin 55^\circ) \end{aligned}$$

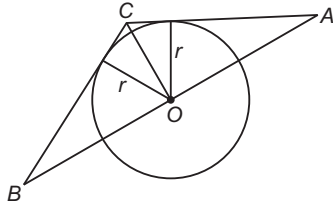
$$\begin{aligned}
 &= k^2 \left(2 \sin \frac{165^\circ}{2} \cos \frac{55^\circ}{2} \right) \left(2 \cos \frac{165^\circ}{2} \sin \frac{55^\circ}{2} \right) \\
 &= k^2 \sin 165^\circ \sin 55^\circ \\
 &= (k \sin 15^\circ) (k \sin 55^\circ) = ab
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 30. A circle touches two of the smaller sides of a $\triangle ABC$ ($a < b < c$) and has its centre on the greatest side. Then, the radius of the circle is

- (a) $\frac{a-b-c}{2}$
 (b) $\frac{abc}{2}$
 (c) $\frac{2\Delta}{a+b}$
 (d) None of the above

Sol. Let O be the centre and r be the radius of the circle.



$$\text{Then, } ar(\triangle BOC) + ar(\triangle AOC) = ar(\triangle ABC)$$

$$\Rightarrow (a+b) \frac{r}{2} = \sqrt{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow r = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{a+b} = \frac{2\Delta}{a+b}$$

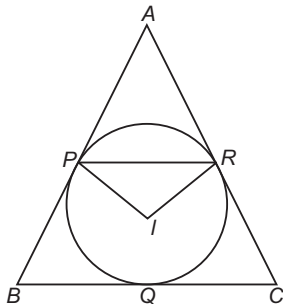
$$\text{where, } s = \frac{a+b+c}{2}$$

Hence, (c) is the correct answer.

Ex 31. Incircle of a $\triangle ABC$ touches sides AB , BC and CA respectively at P , Q and R , then the maximum value of $\frac{PR}{AI} + \frac{PQ}{BI} + \frac{RQ}{CI}$ is (where, I is incentre of the triangle)

- (a) $3\sqrt{3}$ (b) $\frac{3\sqrt{3}}{2}$
 (c) 2 (d) None of these

Sol. Given that, $\angle ARI = \angle API = 90^\circ$
 So, points A, R, I and P are concyclic.
 $\Rightarrow \angle I = \pi - \angle A$



On applying sine rule in $\triangle PRA$, we get

$$\frac{PR}{\sin A} = \frac{AI}{1}$$

$$\Rightarrow \frac{PR}{AI} = \sin A$$

$$\Rightarrow \frac{PR}{AI} + \frac{PQ}{BI} + \frac{RQ}{CI} = \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

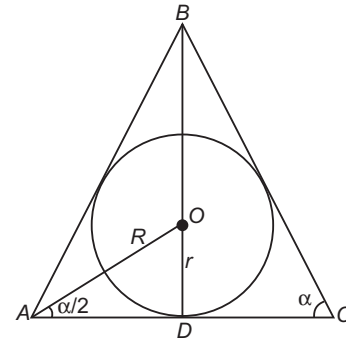
$$\Rightarrow \text{Maximum value is } \frac{3\sqrt{3}}{2}.$$

Hence, (b) is the correct answer.

Ex 32. The ratio of the radii of the inscribed and circumscribed circle for an isosceles triangle with the base angle α is

- (a) $\sin 2\alpha \tan \frac{\alpha}{2}$ (b) $\sin \alpha \tan \frac{\alpha}{2}$
 (c) $\sin 2\alpha \tan \alpha$ (d) None of these

Sol. Let ABC be an isosceles triangle and $\angle A = \angle C = \alpha$ denoting $|AC| = b$, by the theorem of sines, we have
 $\frac{b}{\sin B} = 2R$



where, R is the radius of the circumscribed circle. Since, the base angle is equal to α

$$B = 180^\circ - 2\alpha, \sin B = \sin(180^\circ - 2\alpha) = \sin 2\alpha$$

$$\text{Hence, } R = \frac{b}{2 \sin 2\alpha}$$

Let O be the centre of the inscribed circle, then $\angle OAD = \frac{\alpha}{2}$

$$\text{Therefore, } r = |OD| = |AD| \tan \frac{\alpha}{2} = \frac{b}{2} \tan \frac{\alpha}{2}$$

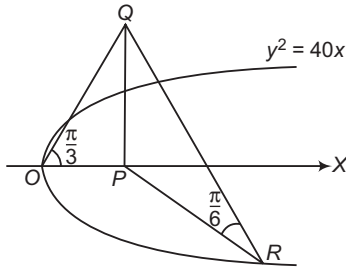
$$\therefore \frac{r}{R} = \sin 2\alpha \tan \frac{\alpha}{2}$$

Hence, (a) is the correct answer.

Ex 33. A vertical tower erected at the focus of the parabola $y^2 = 40x$ subtends an angle $\frac{\pi}{3}$ at the vertex of the parabola. If the tower subtends an angle $\frac{\pi}{6}$ at point R lying on the parabola, then possible coordinates of point R are

- (a) $\left(\frac{7}{3}, 20\sqrt{\frac{7}{30}}\right)$ (b) $\left(\frac{1}{40}, 1\right)$
 (c) $(20, 20\sqrt{2})$ (d) $\left(\frac{7}{20}, \sqrt{14}\right)$

Sol. Let PQ be the tower of height h .
We have,



$$OP = 10 = h \cot \frac{\pi}{3} \Rightarrow h = 10\sqrt{3}$$

$$PR = h \cdot \cot \frac{\pi}{6} = 10\sqrt{3} \cdot \sqrt{3} = 30$$

If $R \equiv (10t^2, 20t)$, then $30 = 10 + 10t^2$

$$\Rightarrow t^2 = 2$$

$$\Rightarrow R \equiv (20, 20\sqrt{2})$$

Hence, (c) is the correct answer.

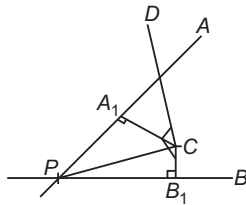
Ex 34. Two straight roads PA and PB intersect at P . A vertical tower CD , C being its foot is situated between the angle formed by these roads and subtends the angle $\frac{\pi}{4}$ and $\frac{\pi}{6}$ at the points A_1 and

B_1 , respectively. It is given that A_1 lies on PA and is nearest to point C , similarly B_1 lies on PB and is also nearest to point C . If $PB_1 = 6$ m, $PA_1 = 9$ m, then height of tower is

- (a) $3\sqrt{5}$ m
- (b) $2\sqrt{5}$ m
- (c) $4\sqrt{5}$ m
- (d) None of the above

Sol. We have, $\angle DA_1C = \frac{\pi}{4}$, $\angle DB_1C = \frac{\pi}{6}$, $PB_1 = 6$, $PA_1 = 9$.

Let $CD = h$



$$CB_1 = h \cot \frac{\pi}{6} = h\sqrt{3}$$

$$CA_1 = h \cot \frac{\pi}{4} = h$$

$$\text{We have, } PC^2 = PA_1^2 + A_1C^2 = PB_1^2 + B_1C^2$$

$$\Rightarrow 81 + h^2 = 36 + 3h^2$$

$$\Rightarrow 2h^2 = 81 - 36$$

$$\Rightarrow h^2 = 45$$

$$\Rightarrow h = 3\sqrt{5} \text{ m}$$

Hence, (a) is the correct answer.

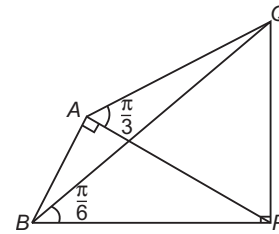
Ex 35. A man finds that at a point due South of a verticle tower the angle of elevation of the tower is $\frac{\pi}{3}$. He, then walks due West $10\sqrt{6}$ m on the horizontal plane and the angle of elevation of the tower to be $\frac{\pi}{6}$. Original distance of the man from the tower is

- (a) $5\sqrt{3}$ m
- (b) $15\sqrt{3}$ m
- (c) $10\sqrt{3}$ m
- (d) None of these

Sol. Let PQ be the tower and A, B be points of observations.
We have,

$$\angle PAB = \frac{\pi}{2}, \angle QAP = \frac{\pi}{3}, \angle QBP = \frac{\pi}{6}$$

$$\text{Now, } AP = PQ \cdot \cot \frac{\pi}{3} = \frac{PQ}{\sqrt{3}}$$



$$BP = PQ \cdot \cot \frac{\pi}{6} = PQ\sqrt{3}$$

In right angled $\triangle PAB$,

$$BP^2 = AP^2 + AB^2 \Rightarrow 3PQ^2 = \frac{PQ^2}{3} + 600$$

$$\Rightarrow \frac{8}{3}PQ^2 = 600 \Rightarrow PQ^2 = \frac{600 \cdot 3}{8}$$

$$\Rightarrow PQ = 15 \text{ m}$$

$$\text{Now, } AP = \frac{PQ}{\sqrt{3}} = \frac{15}{\sqrt{3}} \text{ m} = 5\sqrt{3} \text{ m}$$

Hence, (a) is the correct answer.

Ex 36. At the foot of a mountain, the elevation of its peak is found to be $\pi/4$, after ascending h m toward the mountain up a slope of $\pi/6$ inclination, the elevation is found to be $\pi/3$. Height of the mountain is

- (a) $\frac{h}{2}(\sqrt{3} + 1)$ m
- (b) $h(\sqrt{3} + 1)$ m
- (c) $\frac{h}{2}(\sqrt{3} - 1)$ m
- (d) $h(\sqrt{3} - 1)$ m

Sol. Let A be the top of hill and P be a point on its foot.

We have, $\angle APB = \frac{\pi}{4}$, $\angle QPB = \frac{\pi}{6}$, $\angle AQR = \frac{\pi}{3}$,

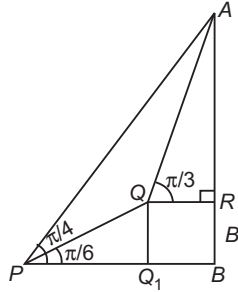
$$PQ = h \text{ m}$$

$$\text{Now, } PB = AB \cdot \cot \frac{\pi}{4} = AB$$

$$QQ_1 = PQ \cdot \sin \frac{\pi}{6} = \frac{h}{2}$$

$$\text{and } PQ_1 = PQ \cdot \cos \frac{\pi}{6} = \frac{h\sqrt{3}}{2}$$

$$\text{In } \Delta AQR, \tan \frac{\pi}{3} = \frac{AR}{QR} = \frac{AB - QQ_1}{PB - PQ_1}$$



$$\Rightarrow \frac{AB - \frac{h}{2}}{AB - \frac{h\sqrt{3}}{2}} = \sqrt{3}$$

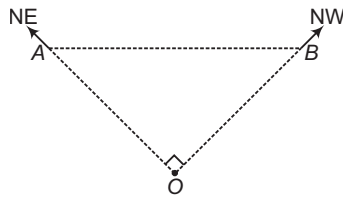
$$\Rightarrow AB = \frac{h}{(\sqrt{3} - 1)} = \frac{h(\sqrt{3} + 1)}{2} \text{ m}$$

Hence, (a) is the correct answer.

Ex 37. A light house, facing North, sends out a fan shaped beam of light extending from North-East to North-West. An observer on a steamer, sailing due West at a uniform speed, first sees the light when he is 5 km away from the light house and continues to see it for $30\sqrt{2}$ min. Speed of the steamer is

- (a) $\frac{5}{30}$ km/min
 (b) $\frac{3}{30}$ km/min
 (c) $\frac{5}{25}$ km/min
 (d) None of the above

Sol. Let A be the point, where observer first see the light and B be the point when observer finally sees the light.



We have, $OA = 5 \text{ km} = OB$

$$\angle AOB = \frac{\pi}{2}$$

$$\Rightarrow AB = 5\sqrt{2} \text{ km}$$

If speed of steamer is v km/min, then

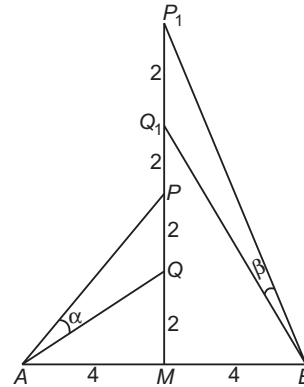
$$v = \frac{AB}{30\sqrt{2}} = \frac{5\sqrt{2}}{30\sqrt{2}} = \frac{5}{30} \text{ km/min}$$

Hence, (a) is the correct answer.

Ex 38. A two metre long object is fired vertically upwards from the mid-point of two locations A and B, 8 m apart. The speed of the object after t second is given by $\frac{ds}{dt} = 2t + 1$ m/s. Let α and β be the angles subtended by the object at A and B, respectively after one and two seconds. Then, the value of $\cos(\alpha - \beta)$ is

- (a) $\frac{5}{\sqrt{26}}$
 (b) $\frac{5}{26}$
 (c) $\frac{5}{2\sqrt{26}}$
 (d) None of the above

Sol. The speed of the object is $\frac{ds}{dt} = 2t + 1$



$$\Rightarrow s = t^2 + t + c, \text{ where } c \text{ is a constant.}$$

At $t = 1, s = 2$ and at $t = 2, s = 6$

From ΔAMQ ,

$$AQ = \sqrt{16 + 4} = \sqrt{20}$$

From ΔAMP ,

$$AP = \sqrt{16 + 16} = \sqrt{32}$$

Also,

$$PQ = 2$$

$$\Rightarrow \cos \alpha = \frac{32 + 20 - 4}{2\sqrt{32}\sqrt{20}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \sin \alpha = \frac{1}{10}$$

From ΔBQ_1M ,

$$BQ_1 = \sqrt{36 + 16} = \sqrt{52}$$

From ΔBP_1M ,

$$BP_1 = \sqrt{64 + 16} = \sqrt{80}$$

Also,

$$P_1Q_1 = 2$$

$$\Rightarrow \cos \beta = \frac{52 + 80 - 4}{2\sqrt{52}\sqrt{80}} = \frac{8}{\sqrt{65}}$$

$$\Rightarrow \sin \beta = \frac{1}{\sqrt{65}}$$

$$\Rightarrow \cos(\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta = \frac{24}{\sqrt{650}} + \frac{1}{\sqrt{650}} = \frac{25}{\sqrt{650}} = \frac{5}{\sqrt{26}}$$

Hence, (a) is the correct answer.

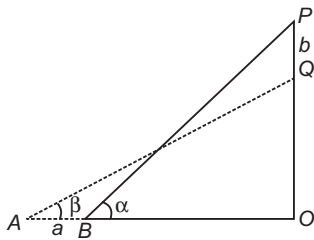
Ex 39. A ladder rests against a wall at an angle α to the horizontal. Its foot is pulled away from the wall through a distance a , so that it slides a distance b down the wall making an angle β with the horizontal. Then, a is equal to

- (a) $b \tan \frac{1}{2}(\alpha + \beta)$
 (b) $b \tan \frac{1}{2}(\alpha - \beta)$
 (c) $a \tan \frac{1}{2}(\alpha - \beta)$
 (d) None of the above

Sol. Let l be the length of the ladder.

$$a = OA - OB = l \cos \beta - l \cos \alpha$$

$$b = OP - OQ = l \sin \alpha - l \sin \beta$$



$$\Rightarrow \frac{a}{b} = \frac{\cos \beta - \cos \alpha}{\sin \alpha - \sin \beta}$$

$$\Rightarrow \frac{a}{b} = \frac{2 \sin \frac{\alpha + \beta}{2} \cdot \sin \frac{\alpha - \beta}{2}}{2 \sin \frac{\alpha - \beta}{2} \cdot \cos \frac{\alpha + \beta}{2}}$$

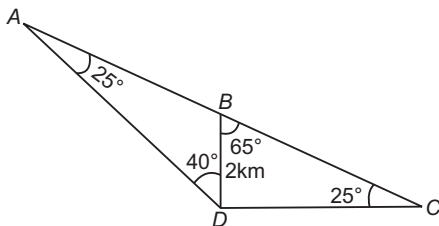
$$\Rightarrow a = b \tan \frac{\alpha + \beta}{2}$$

Hence, (a) is the correct answer.

Ex 40. Four ships A, B, C and D are at sea in the following relative positions B is on the straight line segment AC , B is due North of D and D is due West of C . The distance between B and D is 2 km and $\angle BDA = 40^\circ$, $\angle BCD = 25^\circ$. Then, the distance between A and D is equal to (given, $\sin 25^\circ = 0.423$)

- (a) 4 km
 (b) 6 km
 (c) 4.28 km
 (d) 5 km

Sol. $\angle DBC = 65^\circ$



$$\Rightarrow \angle DAB = 25^\circ$$

$$\Rightarrow AD = DC = 2 \cot 25^\circ$$

$$= 2 \frac{\cos 25^\circ}{\sin 25^\circ} = \frac{2 \times 0.906}{0.423} = \frac{1.812}{0.423}$$

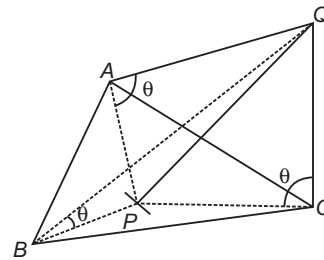
$$= 4.28 \text{ km}$$

Hence, (c) is the correct answer.

Ex 41. PQ is a vertical tower, P is the foot, Q is the top of the tower. A, B and C are three points in the horizontal plane through P . The angle of elevation of Q from A, B and C are equal and each is equal to θ . The sides of the $\triangle ABC$ are a, b, c and the area of the $\triangle ABC$ is Δ . Then, the height of the tower is

- (a) $\frac{abc \tan \theta}{3\Delta}$
 (b) $\frac{abc \tan \theta}{\Delta}$
 (c) $\frac{abc \tan \theta}{4}$
 (d) $\frac{abc \tan \theta}{4\Delta}$

Sol. Let h be the height of the tower PQ .



$$\Rightarrow \tan \theta = \frac{h}{AP} = \frac{h}{BP} = \frac{h}{CP}$$

$$\Rightarrow AP = BP = CP = h \cot \theta \text{ and that } P \text{ is the circumcentre of the } \triangle ABC. \text{ If } R \text{ is the radius of the circumcircle, then}$$

$$\Delta = \frac{abc}{4R} = \frac{abc}{4AP} = \frac{abc}{4h \cot \theta} \quad [\because R = AP]$$

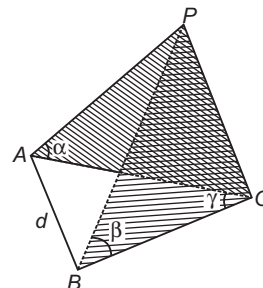
$$\Rightarrow h = \frac{abc \tan \theta}{4\Delta}$$

Hence, (d) is the correct answer.

Ex 42. A vertical pole stands at a point O on a horizontal ground. A and B are points on the ground d m apart. The pole subtends angles α and β at A and B , respectively. AB subtends $\angle \gamma$ at O . Then, the height of the pole is equal to

- (a) $\frac{d}{[\cot^2 \alpha + \cot^2 \beta + 2 \cot \alpha \cot \beta \cos \gamma]^{1/2}}$
 (b) $\frac{d}{[\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma]^2}$
 (c) $\frac{d}{[\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma]^{1/2}}$
 (d) None of the above

Sol. Let the height of the pole be OP .



From $\triangle OAP$ and $\triangle OBP$,

$$\tan \alpha = \frac{h}{OA}, \quad \tan \beta = \frac{h}{OB}$$

$$\Rightarrow OA = h \cot \alpha, OB = h \cot \beta$$

On applying cosine rule to $\triangle OAB$,

$$\Rightarrow \cos \gamma = \frac{OA^2 + OB^2 - d^2}{2OA \cdot OB}$$

$$\Rightarrow 2h^2 \cot \alpha \cot \beta \cos \gamma = h^2 \cot^2 \alpha + h^2 \cot^2 \beta - d^2$$

$$\Rightarrow h = \frac{d}{[\cot^2 \alpha + \cot^2 \beta - 2 \cot \alpha \cot \beta \cos \gamma]^{1/2}}$$

Hence, (c) is the correct answer.

Ex 43. A sign-post in the form of an isosceles $\triangle ABC$ is mounted on a pole of height h fixed to the ground. The base BC of the triangle is parallel to the ground. A man standing on the ground at a distance d from the sign-post finds that the top vertex A of the triangle subtends an $\angle \beta$ and either of the other two vertices subtends the same angle α at his feet. Then, the area of the triangle is

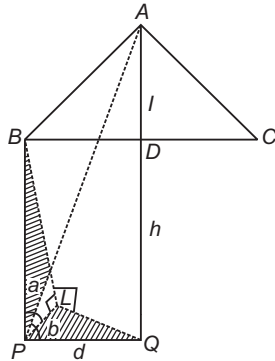
(a) $(d \tan \beta + h) \sqrt{h^2 \cot^2 \alpha + d^2}$

(b) $(d \tan \beta - h) \sqrt{h^2 \cot^2 \alpha - d^2}$

(c) $(d \tan \beta - h) \sqrt{h^2 \tan^2 \alpha - d^2}$

(d) None of the above

Sol. Let the base $BC = 2a$. Let l be the altitude of the $\triangle ABC$. Let Q be the foot of the pole and D be the mid-point of BC .



Also, we assume that L is the foot of the perpendicular from B on the ground, so that PQL is a right angled triangle with $LQ = a$.

$$\Rightarrow PL = \sqrt{a^2 + d^2}$$

Also, $AQ = l + h$

$$\text{In } \triangle APQ, \frac{l+h}{d} = \tan \beta$$

$$\Rightarrow l = d \tan \beta - h \quad \dots(i)$$

$$\text{In } \triangle BLP, \frac{BL}{PL} = \tan \alpha$$

$$\Rightarrow h^2 \cot^2 \alpha = a^2 + d^2 \Rightarrow a = \sqrt{h^2 \cot^2 \alpha - d^2}$$

$$\text{Area of } \triangle ABC = \frac{1}{2} BC \cdot l$$

$$= al = (d \tan \beta - h) \sqrt{h^2 \cot^2 \alpha - d^2}$$

Hence, (b) is the correct answer.

Ex 44. ABC is a triangular park with $AB = AC = 100$ m. A television tower stands at the mid-point of BC . The angles of elevation of the top of the tower at A, B and C are $45^\circ, 60^\circ$ and 60° , respectively. Then, the height of the tower is equal to

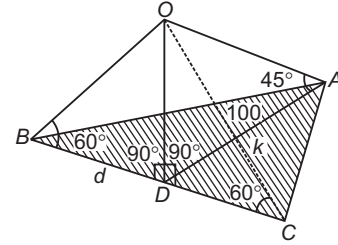
(a) 50 m

(b) $\frac{50}{3}$ m

(c) $50\sqrt{3}$ m

(d) None of these

Sol. Let $OD = h, D$ be the mid-point of $BC, BD = d$ and $AD = k$



$$\text{In } \triangle OAD, \frac{h}{k} = \tan 45^\circ = 1 \Rightarrow h = k$$

$$\text{In } \triangle OBD, \frac{h}{d} = \tan 60^\circ = \sqrt{3} \Rightarrow d = \frac{h}{\sqrt{3}}$$

Now, in $\triangle ABD, AB = 100$ m.

$$\therefore d^2 + k^2 = (100)^2$$

$$\Rightarrow \frac{h^2}{3} + h^2 = 100 \times 100 \Rightarrow \frac{4h^2}{3} = 100 \times 100$$

$$\Rightarrow h^2 = 7500 \Rightarrow h = 50\sqrt{3} \text{ m}$$

Hence, (c) is the correct answer.

Ex 45. A straight pillar PQ stands at a point P . The points A and B are situated due South and East of P , respectively. M is the mid-point of AB . PAM is an equilateral triangle and N is the foot of the perpendicular from P on AB . Suppose, $AN = 20$ m and the angle of elevation of the top of the pillar at N is $\tan^{-1} 2$. Then, the height of the pillar and the angles of elevation of its top at A and B are

(a) $40\sqrt{3}$ m, $60^\circ, 45^\circ$

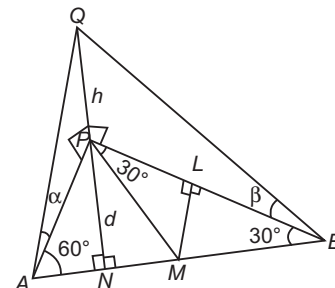
(b) $40\sqrt{3}$ m, $45^\circ, 30^\circ$

(c) $20\sqrt{3}$ m, $65^\circ, 50^\circ$

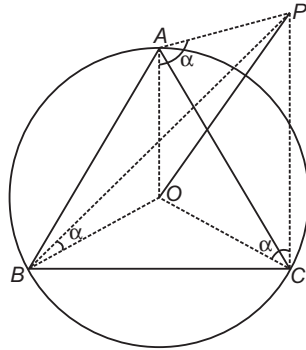
(d) None of the above

Sol. Since, PAM is an equilateral triangle.

Therefore, PN is perpendicular to AM and N is the mid-point of AM . It is given that $AN = 20$ m.



Sol. Since, the tower makes equal angles at the vertices of the triangle, therefore foot of the tower is at the circumcentre.



From $\triangle OAP$, we have

$$\tan \alpha = \frac{OP}{OA}$$

$$\Rightarrow OP = OA \tan \alpha$$

$$\Rightarrow OP = R \tan \alpha$$

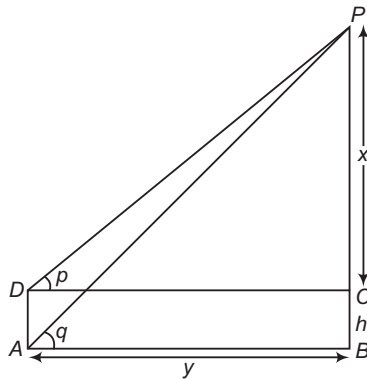
Hence, (d) is the correct answer.

Ex 49. The top of a hill observed from the top and bottom of a building of height h is at angles of elevation p and q , respectively. The height of the hill is

- (a) $\frac{h \cot q}{\cot q - \cot p}$ (b) $\frac{h \cot p}{\cot p - \cot q}$
 (c) $\frac{h \tan p}{\tan p - \tan q}$ (d) None of these

Sol. Let AD be the building of height h and BP be the hill. Then,

$$\tan q = \frac{h+x}{y} \text{ and } \tan p = \frac{x}{y}$$



$$\begin{aligned} \Rightarrow \tan q &= \frac{h+x}{y} = \frac{h+x}{x \cot p} \\ \Rightarrow x \cot p &= (h+x) \cot q \\ \Rightarrow x &= \frac{h \cot q}{\cot p - \cot q} \\ \Rightarrow h+x &= \frac{h \cot q}{\cot p - \cot q} + h \\ &= \frac{h \cot p}{\cot p - \cot q} \end{aligned}$$

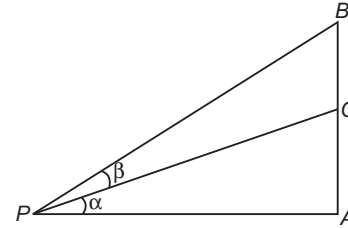
Hence, (b) is the correct answer.

Ex 50. AB is a vertical pole. The end A is on the level ground. C is the middle point of AB , P is a point on the level ground. The portion BC subtends an angle β at P . If $AP = nAB$, then the $\tan \beta$ is equal to

- (a) $\frac{n}{2n^2 + 1}$ (b) $\frac{n}{n^2 - 1}$
 (c) $\frac{n}{n^2 + 1}$ (d) None of these

Sol. Let $\angle APC = \alpha$, then

$$\tan \alpha = \frac{AC}{AP} = \frac{AC}{nAB} = \frac{AB}{nAB} = \frac{1}{n}$$



In $\triangle APB$,

$$\tan(\alpha + \beta) = \frac{AB}{AP} = \frac{AB}{nAB} = \frac{1}{n}$$

Now,

$$\beta = \alpha + \beta - \alpha$$

$$\begin{aligned} \Rightarrow \tan \beta &= \frac{\tan(\alpha + \beta) - \tan \alpha}{1 + \tan(\alpha + \beta) \tan \alpha} \\ &= \frac{\frac{1}{n} - \frac{1}{2n}}{1 + \frac{1}{n} \cdot \frac{1}{2n}} = \frac{\frac{n}{2n^2 + 1}}{1 + \frac{1}{2n^2}} = \frac{n}{2n^2 + 1} \end{aligned}$$

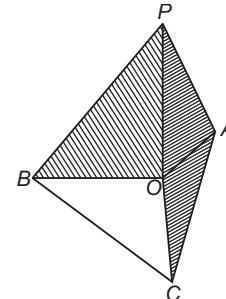
Hence, (a) is the correct answer.

Ex 51. A tripod has three legs each being 5 m long and the feet are planted in the ground at points 3 m apart. The height of the top of the tripod above the ground is

- (a) $\sqrt{26}$ m (b) $\sqrt{28}$ m
 (c) $\sqrt{24}$ m (d) None of these

Sol. Let the legs of tripod be planted in the ground at A, B, C and P be the top of tripod. We have,

$$PA = PB = PC = 5 \text{ m}, AB = BC = CA = 3 \text{ m}$$



$$\text{We have, } OA = \frac{AB}{2 \cdot \sin \frac{\pi}{3}} = \sqrt{3} \text{ m}$$

$$OP^2 = PA^2 - OA^2 = 25 - 3 = 22$$

$$\Rightarrow OP = \sqrt{22} \text{ m}$$

Hence, (d) is the correct answer.

Ex 52. A and B are two points on one bank of a straight river and C, D are two other points on the other bank of river. If direction from A to B is same as that from C to D and $AB = a$, $\angle CAD = \alpha$, $\angle DAB = \beta$, $\angle CBA = \gamma$, then CD is equal to

- (a) $\frac{a \sin \beta \cdot \sin \gamma}{\sin \alpha \cdot \sin(\alpha + \beta + \gamma)}$ (b) $\frac{a \sin \alpha \cdot \sin \gamma}{\sin \beta \cdot \sin(\alpha + \beta + \gamma)}$
 (c) $\frac{a \sin \alpha \cdot \sin \beta}{\sin \gamma \cdot \sin(\alpha + \beta + \gamma)}$ (d) None of these

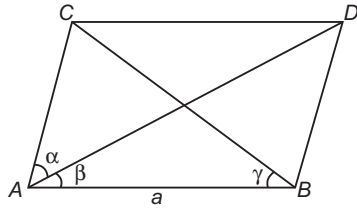
Sol. Clearly,

$$\angle CDA = \beta, \angle DCB = \gamma, \angle ACB = \pi - (\alpha + \beta + \gamma)$$

Using sine rule in $\triangle CAB$, we get

$$\frac{AB}{\sin(\pi - (\alpha + \beta + \gamma))} = \frac{CA}{\sin \gamma}$$

$$\Rightarrow CA = \frac{a \cdot \sin \gamma}{\sin(\alpha + \beta + \gamma)}$$



Now, using sine rule in $\triangle CAD$, we get

$$\frac{CA}{\sin \beta} = \frac{CD}{\sin \alpha}$$

$$\Rightarrow CD = \frac{CA \cdot \sin \alpha}{\sin \beta} = \frac{a \sin \alpha \cdot \sin \gamma}{\sin \beta \cdot \sin(\alpha + \beta + \gamma)}$$

Hence, (b) is the correct answer.

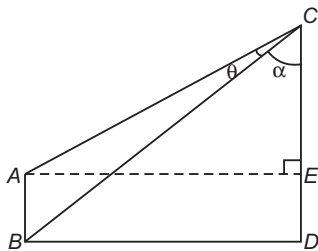
Ex 53. The vertical poles, AB of the length 2 m and CD of length 20 m are erected with base B and D respectively. It is given that distance between the poles is more than 20 m and $\tan(\angle ACB) = \frac{2}{77}$, then distance between the poles is

- (a) 68 m
 (b) 24 m
 (c) 72 m
 (d) None of the above

Sol. We have, $\angle ACB = \theta$

$$\text{where, } \tan \theta = \frac{2}{77}, AB = 2, CD = 20.$$

Let $\angle BCD = \alpha$



$$\Rightarrow \tan \alpha = \frac{BD}{CD} = \frac{BD}{20}$$

$$\text{and } \tan(\theta + \alpha) = \frac{AE}{CE} = \frac{BD}{18}$$

$$\Rightarrow \frac{BD}{18} = \frac{\frac{2}{77} + \frac{BD}{20}}{1 - \frac{2BD}{77 \cdot 20}}$$

$$\Rightarrow BD^2 - 77BD + 360 = 0$$

$$\Rightarrow BD = 5, 72 \text{ but } BD > 20$$

$$\Rightarrow BD = 72 \text{ m}$$

Hence, (c) is the correct answer.

Ex 54. Two persons who are 500 m apart, observe the direction and the angle of elevation of a balloon at the same instant. One find the elevation to be $\frac{\pi}{3}$ and direction being

South-West, while the other find the elevation to be $\frac{\pi}{4}$ and direction being West. Height of the balloon is

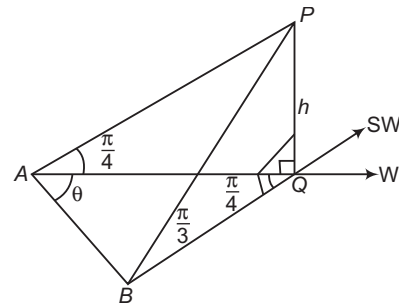
- (a) $500 \sqrt{\frac{3}{4 + \sqrt{6}}}$ m (b) $500 \sqrt{\frac{3}{4 - \sqrt{6}}}$ m
 (c) $250 \sqrt{\frac{3}{4 + \sqrt{6}}}$ m (d) $250 \sqrt{\frac{3}{4 - \sqrt{6}}}$ m

Sol. Let P represents the position of balloon at the time of observations and A and B are the points of observations.

$$\text{We have, } \angle PAQ = \frac{\pi}{4},$$

$$\angle PBQ = \frac{\pi}{3}, \angle AQB = \frac{\pi}{4} \text{ and } AB = 500 \text{ m}$$

$$AQ = PQ \cot \frac{\pi}{4} = h, BQ = PQ \cdot \cot \frac{\pi}{3} = \frac{h}{\sqrt{3}}$$



Using cosine rule in $\triangle AQB$, we get

$$\cos \frac{\pi}{4} = \frac{AQ^2 + BQ^2 - AB^2}{2AQ \cdot BQ}$$

$$\Rightarrow \frac{1}{\sqrt{2}} = \frac{h^2 + \frac{h^2}{3} - 250000}{\frac{2h^2}{\sqrt{3}}}$$

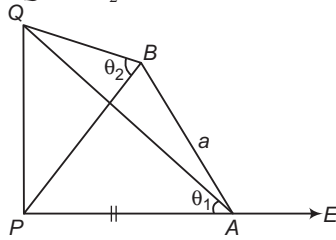
$$\Rightarrow h = 500 \sqrt{\frac{3}{4 - \sqrt{6}}} \text{ m}$$

Hence, (b) is the correct answer.

Ex 55. A man stands due East of the top of the tower to be θ_1 . He, then walks a m North-West and finds the angle of elevation to be θ_2 . Height of the tower is the root of the equation

- (a) $t^2(\cot^2 \theta_2 - \cot^2 \theta_1) + t\sqrt{2}a \cot \theta_1 + a^2 = 0$
 (b) $t^2(\cot^2 \theta_1 - \cot^2 \theta_2) - t\sqrt{2}a \cot \theta_1 + a^2 = 0$
 (c) $t^2(\cot^2 \theta_1 - \cot^2 \theta_2) - t\sqrt{2} \cot \theta_2 + a^2 = 0$
 (d) $t^2(\cot^2 \theta_2 - \cot^2 \theta_1) - t\sqrt{2} \cot \theta_2 + a^2 = 0$

Sol. Let PQ be the tower of height h and A, B be the points of observation. We have, $AB = a$, $\angle PAB = \frac{\pi}{4}$, $\angle QAP = \theta_1$, and $\angle QBP = \theta_2$



$$PA = PQ \cdot \cot \theta_1 = h \cot \theta_1$$

$$PB = PQ \cdot \cot \theta_2 = h \cot \theta_2$$

Using cosine rule in $\triangle PAB$, we get

$$\cos \frac{\pi}{4} = \frac{PA^2 + AB^2 - PB^2}{2PA \cdot AB}$$

$$\Rightarrow \sqrt{2}h a \cot \theta_1 = h^2 \cot^2 \theta_1 + a^2 - h^2 \cot^2 \theta_2$$

$$\Rightarrow h^2(\cot^2 \theta_1 - \cot^2 \theta_2) - h\sqrt{2}a \cot \theta_1 + a^2 = 0$$

$$\Rightarrow h \text{ is the root of } t^2(\cot^2 \theta_1 - \cot^2 \theta_2) - t\sqrt{2}a \cot \theta_1 + a^2 = 0$$

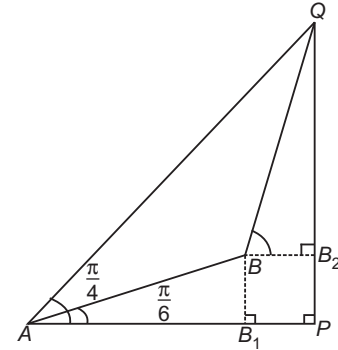
Hence, (b) is the correct answer.

Ex 56. The angle of elevation of the top of vertical tower from a point A on the horizontal ground is found to be $\frac{\pi}{4}$. From A , a man walks 10 m up a path sloping at an angle $\frac{\pi}{6}$. After this, the slope becomes steeper and after walking up another 10 m, the man reaches the top of the tower. The distance of A from the foot of the tower is

- (a) $5(1 + \sqrt{3})$ m
 (b) $\frac{5}{2}(1 + \sqrt{3})$ m
 (c) $5(\sqrt{3} - 1)$ m
 (d) $\frac{5}{2}(\sqrt{3} - 1)$ m

Sol. Let PQ be the tower of height h and A, B be the points of observations. Then,

$$\angle QAP = \frac{\pi}{4}, \angle BAP = \frac{\pi}{6}, AB = 10 \text{ m and } BQ = 10 \text{ m}$$



$$\angle QAB = \frac{\pi}{12} = \angle AQB$$

$$\Rightarrow ABQ = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

Applying cosine rule in $\triangle ABQ$, we get

$$AQ^2 = AB^2 + BQ^2 - 2AB \cdot BQ \cos \frac{5\pi}{6}$$

$$= 100 + 100 + 200 \cdot \frac{\sqrt{3}}{2}$$

$$= 200 + 100\sqrt{3}$$

$$= 100(2 + \sqrt{3})$$

$$\Rightarrow AQ = 10\sqrt{2 + \sqrt{3}}$$

$$AP = AQ \cdot \cos \frac{\pi}{4} = \frac{10\sqrt{2 + \sqrt{3}}}{\sqrt{2}}$$

$$= 5\sqrt{4 + 2\sqrt{3}}$$

$$= 5\sqrt{(\sqrt{3} + 1)^2}$$

$$\Rightarrow AP = 5(1 + \sqrt{3}) \text{ m}$$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 57. If in $\triangle ABC$, $a^4 + b^4 + c^4 = 2a^2(b^2 + c^2)$, then $\angle A$ is equal to

- (a) 45° (b) 60° (c) 90° (d) 135°

Sol. We have, $a^4 + b^4 + c^4 = 2a^2(b^2 + c^2)$

$$\Rightarrow a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 + 2b^2c^2 = 2b^2c^2$$

$$\Rightarrow (b^2 + c^2 - a^2)^2 = 2b^2c^2$$

$$\Rightarrow b^2 + c^2 - a^2 = \sqrt{2}bc$$

$$\text{or } b^2 + c^2 - a^2 = -\sqrt{2}bc$$

$$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{1}{\sqrt{2}}$$

$$\text{or } \frac{b^2 + c^2 - a^2}{2bc} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos A = \frac{1}{\sqrt{2}} = \cos 45^\circ$$

$$\text{or } \cos A = -\frac{1}{\sqrt{2}} = \cos \left(\pi - \frac{\pi}{4} \right)$$

$$= \cos \frac{3\pi}{4}$$

$$\Rightarrow A = 45^\circ$$

$$\text{or } A = 135^\circ$$

Hence, (a) and (d) are the correct answers.

Ex 58. If $\cos(\theta - \alpha)$, $\cos \theta$ and $\cos(\theta + \alpha)$ are in HP, then $\cos \theta \sec\left(\frac{\alpha}{2}\right)$ is equal to

- (a) -1 (b) $-\sqrt{2}$ (c) $\sqrt{2}$ (d) 2

Sol. We are given,

$$\cos \theta = \frac{2 \cos(\theta - \alpha) \cos(\theta + \alpha)}{\cos(\theta - \alpha) + \cos(\theta + \alpha)} = \frac{2(\cos^2 \theta - \sin^2 \alpha)}{2 \cos \theta \cos \alpha}$$

$$\Rightarrow \cos^2 \theta \cos \alpha = \cos^2 \theta - \sin^2 \alpha$$

$$\Rightarrow \cos^2 \theta = \frac{\sin^2 \alpha}{1 - \cos \alpha} \Rightarrow \cos \theta = \frac{4 \sin^2 \frac{\alpha}{2} \cos^2 \frac{\alpha}{2}}{2 \sin^2 \frac{\alpha}{2}}$$

$$\Rightarrow \cos^2 \theta \sec^2 \frac{\alpha}{2} = 2 \Rightarrow \cos \theta \sec \frac{\alpha}{2} = \pm \sqrt{2}$$

Hence, (b) and (c) are the correct answers.

Ex 59. In $\triangle ABC$, which of the following statements are true?

- (a) Maximum value of $\sin 2A + \sin 2B + \sin 2C$ is same as the maximum value of $\sin A + \sin B + \sin C$
 (b) $R \geq 2r$, where R is circumradius and r is inradius
 (c) $R^2 \geq \frac{abc}{a+b+c}$
 (d) $\triangle ABC$ is right angled, if $r + 2R = s$, where s is semi-perimeter

Sol. (a) $\therefore \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$

Maximum value of

$$\sin 2A + \sin 2B + \sin 2C = 4 \cdot 1 \cdot 1 \cdot 1 = 4$$

$$\text{and } \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$$

$\therefore$ Maximum value of

$$\sin A + \sin B + \sin C = 4 \cdot 1 \cdot 1 \cdot 1 = 4$$

(b) We know that

$$\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq \frac{1}{8}$$

$$\Rightarrow 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \leq 1$$

$$\Rightarrow 2 \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) \leq R$$

$$\Rightarrow 2r \leq R$$

$$\therefore R \geq 2r$$

(c) From the above relation $R \geq 2r$,

$$R^2 \geq 2rR \Rightarrow R^2 \geq \frac{2\Delta R}{s}$$

$$\Rightarrow R^2 \geq \frac{2abc}{4 \left(\frac{a+b+c}{2} \right)} \Rightarrow R^2 \geq \frac{abc}{(a+b+c)}$$

(d) $\therefore \angle ABC = 90^\circ$

$$\therefore \angle B = 90^\circ \Rightarrow \sin B = 1$$

$$\Rightarrow 2R \sin B = 2R \Rightarrow b = 2R$$

$$\Rightarrow (s-b) + 2R = s \Rightarrow r \tan \frac{B}{2} + 2R = s$$

$$\Rightarrow r \cdot 1 + 2R = s \Rightarrow r + 2R = s$$

Hence, (a), (b), (c) and (d) are the correct answers.

Ex 60. There exists a $\triangle ABC$ satisfying the conditions

- (a) $b \sin A = a$, $A < \frac{\pi}{2}$
 (b) $b \sin A > a$, $A > \frac{\pi}{2}$
 (c) $b \sin A > a$, $A < \frac{\pi}{2}$
 (d) $b \sin A < a$, $A < \frac{\pi}{2}$, $b > a$

Sol. The sine formula is $\frac{a}{\sin A} = \frac{b}{\sin B}$
 $\Rightarrow a \sin B = b \sin A$

$$(a) b \sin A = a \Rightarrow a \sin B = a \Rightarrow B = \pi/2$$

Since, $\angle A < \pi/2$. Therefore, the triangle is possible.

(b) and (c)

$$b \sin A > a \Rightarrow a \sin B > a \Rightarrow \sin B > 1$$

Therefore, $\triangle ABC$ is not possible.

$$(d) b \sin A < a \Rightarrow a \sin B < a \Rightarrow \sin B < 1$$

$\Rightarrow \angle B$ exists

$$\text{Now, } b > a \Rightarrow B > A$$

$$\text{Since, } A < \frac{\pi}{2}$$

Therefore, the triangle is possible.

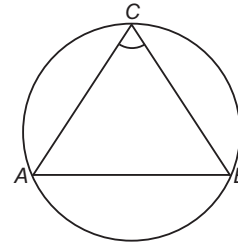
Hence, (a) and (d) are the correct answers.

Ex 61. If a right angled $\triangle ABC$ of maximum area is inscribed within a circle of radius R , then

- (a) $\Delta = 2R^2$ (b) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{\sqrt{2}+1}{R}$
 (c) $r = (\sqrt{2}-1)R$ (d) $s = (1+\sqrt{2})R$

Sol. For a right angled inscribed in a circle of radius R , the length of the hypotenuse is $2R$.

$\therefore$ The area is maximum when the triangle is isosceles with each side $= \sqrt{2}R$.



$$\Rightarrow s = \frac{1}{2} (2\sqrt{2} + 2)R = (\sqrt{2} + 1)R$$

$$\Rightarrow \Delta = \frac{1}{2} \sqrt{2}R \cdot \sqrt{2}R = R^2$$

$$\Rightarrow \frac{1}{r} = \frac{\sqrt{2}+1}{R}$$

$$\text{Now, } \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta}$$

$$= \frac{s}{\Delta} = \frac{1}{R} = \frac{\sqrt{2}+1}{R}$$

Hence, (b) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 62-66) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 62. Statement I In a ΔABC , if $a < b < c$ and r is in radius and r_1, r_2, r_3 are the exradii opposite to angles A, B, C respectively, then $r < r_1 < r_2 < r_3$.

Statement II For ΔABC ,

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{r_1 r_2 r_3}{r}.$$

Sol. Statement I $a < b < c$

$$\begin{aligned} s - a &> s - b > s - c \\ s &> s - a > s - b > s - c \\ \frac{\Delta}{s} &< \frac{\Delta}{s - a} < \frac{\Delta}{s - b} < \frac{\Delta}{s - c} \end{aligned}$$

$$r < r_1 < r_2 < r_3$$

Statement II $r_1 = \frac{\Delta}{s - a}, r_2 = \frac{\Delta}{s - b}, r_3 = \frac{\Delta}{s - c}, r = \frac{\Delta}{s}$

$$\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{s}{\Delta} = \frac{1}{r}$$

Hence, (b) is the correct answer.

Ex 63. Statement I If the sides of a triangle are 13, 14 and 15, then the radius of incircle is 4.

Statement II In a ΔABC ,

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}, \text{ where } s = \frac{a+b+c}{2}$$

$$\text{and } r = \frac{\Delta}{s}.$$

$$\text{Sol. } s = 21, \Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6} = \sqrt{3 \cdot 7 \cdot 2^4 \cdot 7 \cdot 3} = 3 \cdot 7 \cdot 4 = 84$$

$$r = \frac{\Delta}{s} = \frac{84}{21} = 4$$

Hence, (a) is the correct answer.

Ex 64. Statement I In a ΔABC , $\sum \frac{\cos^2 \frac{A}{2}}{a}$ has the

value equal to $\frac{s^2}{abc}$.

Statement II In a ΔABC ,

$$\cos \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \cos \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{ac}},$$

$$\cos \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}.$$

$$\text{Sol. } \therefore \frac{\cos^2 \frac{A}{2}}{a} = \frac{s(s-a)}{abc}$$

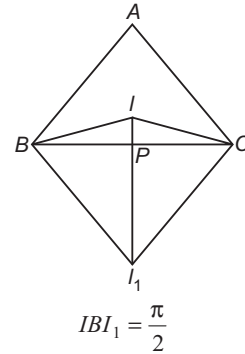
$$\therefore \sum \frac{\cos^2 \frac{A}{2}}{a} = \frac{s^2}{abc}$$

Hence, (c) is the correct answer.

Ex 65. Statement I If I is incentre of ΔABC and I_1 is excentre opposite to A and P is the intersection of II_1 and BC , then $IP \cdot I_1 P = BP \cdot PC$.

Statement II In a ΔABC , I is incentre and I_1 is excentre opposite to A , then $IBI_1 C$ must be square.

$$\text{Sol. } \therefore \angle ICI_1 = \frac{\pi}{2}$$



$\therefore BICI_1$ is cyclic quadrilateral

$$BP \cdot PC = IP \cdot I_1 P$$

Hence, (c) is the correct answer.

Ex 66. All the notations used in Statement I and Statement II are usual.

Statement I In ΔABC , if

$$\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}, \text{ then value of}$$

$$\frac{r_1 + r_2 + r_3}{r} \text{ is equal to 9.}$$

Statement II In ΔABC ,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R, \text{ where } R \text{ is circumradius.}$$

Sol. Statement II is true.

Statement I $\tan A = \tan B = \tan C$

$$A = B = C \text{ i.e. } a = b = c$$

$$r_1 = r_2 = r_3$$

$$\therefore \frac{r_1 + r_2 + r_3}{3} = 3 \cdot \frac{r_1}{r} = 3 \cdot \frac{\frac{\Delta}{s-a}}{\frac{\Delta}{s}}$$

$$= 3 \cdot \frac{a+b+c}{b+c-a} = 9$$

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 67-69) Let a, b and c be the sides opposite to angles A, B and C respectively in a $\triangle ABC$
 $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$ and $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$. If $a = 6, b = 3$ and $\cos(A-B) = \frac{4}{5}$.

Ex 67. $\angle C$ is equal to

- (a) $\frac{\pi}{4}$
- (b) $\frac{\pi}{2}$
- (c) $\frac{3\pi}{4}$
- (d) $\frac{2\pi}{3}$

Ex 68. Area of the triangle is equal to

- (a) 8
- (b) 9
- (c) 10
- (d) 11

Ex 69. The value of $\sin A$ is

- (a) $\frac{1}{\sqrt{5}}$
- (b) $\frac{2}{\sqrt{5}}$
- (c) $\frac{1}{2\sqrt{5}}$
- (d) $\frac{1}{\sqrt{3}}$

Sol. (Ex. Nos. 67-69)

$$\begin{aligned} \cos(A-B) &= \frac{4}{5} \\ \Rightarrow \frac{1 - \tan^2 \frac{A-B}{2}}{1 + \tan^2 \frac{A-B}{2}} &= \frac{4}{5} \\ \Rightarrow \frac{2 \tan^2 \frac{A-B}{2}}{2} &= \frac{1}{9} \Rightarrow \tan \frac{A-B}{2} = \frac{1}{3} \\ \Rightarrow \frac{1}{3} &= \frac{6-3}{6+3} \cot \frac{C}{2} \\ \Rightarrow \cos \frac{C}{2} &= 1 \Rightarrow C = 90^\circ \end{aligned}$$

$$\begin{aligned} \text{Area of triangle} &= \frac{1}{2} ab \sin C \\ &= \frac{1}{2} \times 6 \times 3 \times 1 = 9 \\ \therefore \frac{a}{\sin A} &= \frac{\sqrt{a^2 + b^2}}{1} \\ \Rightarrow \frac{6}{\sin A} &= \sqrt{45} \\ \Rightarrow \sin A &= \frac{2}{\sqrt{5}} \end{aligned}$$

67. (b)

68. (b)

69. (b)

Passage II (Ex. Nos. 70-72) Consider a $\triangle ABC$, where x, y and z are the length of perpendicular drawn from the vertices of the triangle to the opposite sides a, b and c respectively, let the letters R, r, S and Δ denote the circumradius, inradius, semi-perimeter and area of the triangle, respectively.

Ex 70. If $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{a^2 + b^2 + c^2}{k}$, then the value of k is

- (a) R
- (b) S
- (c) $2R$
- (d) $\frac{3}{2}R$

$$\begin{aligned} \text{Sol. } \frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} &= b \sin B + c \sin C + a \sin A \\ &= \frac{b^2 + c^2 + a^2}{2R} \end{aligned}$$

$$\therefore k = 2R$$

Hence, (c) is the correct answer.

Ex 71. If $\cot A + \cot B + \cot C = k \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right)$,

then the value of k is

- (a) R^2
- (b) rR
- (c) Δ
- (d) $a^2 + b^2 + c^2$

$$\begin{aligned} \text{Sol. } \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} + \frac{a^2}{4\Delta^2} + \frac{c^2}{4\Delta^2} &= \frac{a^2 + b^2 + c^2}{4\Delta^2} \\ \cot A + \cot B + \cot C &= \frac{R}{abc} \\ (b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2) \\ &= \frac{R}{abc} (b^2 + c^2 + a^2) = \frac{R}{abc} \left(\frac{4\Delta^2}{x^2} + \frac{4\Delta^2}{y^2} + \frac{4\Delta^2}{z^2} \right) \\ &= \frac{4\Delta^2 R}{abc} \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) \\ &= \frac{4\Delta R}{abc} \cdot \Delta \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) = \Delta \left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \right) \\ \therefore k &= \Delta \end{aligned}$$

Hence, (c) is the correct answer.

Ex 72. The value of

$$\frac{c \sin B + b \sin C}{x} + \frac{a \sin C + c \sin A}{y} + \frac{b \sin A + a \sin B}{z}$$

- (a) $\frac{R}{r}$
- (b) $\frac{S}{R}$
- (c) 2
- (d) 6

$$\text{Sol. } \Sigma \frac{c \sin B + b \sin C}{x}, \Sigma \frac{x+x}{x} = 6$$

Hence, (d) is the correct answer.

Type 5. Match the Columns

Ex 73. Match the statements of Column I with values of Column II.

Column I	Column II
A. If in $\triangle ABC$, $\angle B = 60^\circ$, then $a^2 + b^2 + c^2$ is equal to	p. $a + b + c$
B. If l, m and n are perpendicular drawn from the vertices of triangle having sides a, b and c , then $\sqrt{2R \left(\frac{bl}{c} + \frac{cm}{a} + \frac{an}{b} \right) + 2ab + 2bc + 2ca}$ is equal to	q. $a - b$
C. In a $\triangle ABC$, $R (b^2 \sin 2C + c^2 \sin 2B)$ equals	r. $2b^2 + ac$
D. In a right angled $\triangle ABC$, $\angle C = \frac{\pi}{2}$, then $4R \sin \frac{A+B}{2} \cdot \sin \frac{A-B}{2}$	s. abc

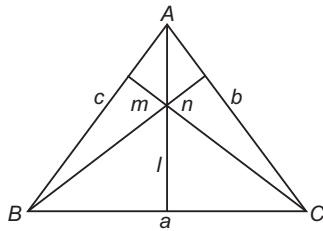
Sol. A. $\frac{1}{2} = \cos B = \frac{c^2 + a^2 - b^2}{2ca}$
 $\Rightarrow c^2 + a^2 = b^2 + ca$
 $\Rightarrow a^2 + b^2 + c^2 = 2b^2 + ca$

B. $\sin C = \frac{l}{b}$... (i)

$\sin B = \frac{n}{a}$... (ii)

$\sin A = \frac{m}{c}$... (iii)

$\therefore 2R \left(\frac{bl}{c} + \frac{cm}{a} + \frac{an}{b} \right) + 2ab + 2bc + 2ca$
 $= \frac{c}{\sin c} \left(\frac{bl}{c} \right) + \frac{a}{\sin A} \left(\frac{cm}{a} \right) + \frac{b}{\sin B} \left(\frac{an}{b} \right)$



From Eqs. (i), (ii) and (iii), we get

$2R \left(\frac{bl}{c} + \frac{cm}{a} + \frac{an}{b} \right) = a^2 + b^2 + c^2$

$\therefore \sqrt{2R \left(\frac{bl}{c} + \frac{cm}{a} + \frac{an}{b} \right) + 2ab + 2bc + 2ca}$
 $= a + b + c$

C. $R = \frac{b}{2 \sin B} = \frac{c}{2 \sin C}$

$\therefore Rb^2 \sin 2C + Rc^2 \sin 2B$
 $= b^2 c \cos C + c^2 b \cos B$
 $= bc(b \cos C + c \cos B)$
 $= abc$

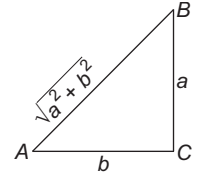
D. $2R(-\cos A + \cos B)$

$= 2R \left(-\frac{b}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \right)$

$\therefore R = \frac{\sqrt{a^2 + b^2}}{2}$

$\therefore 2R \left(-\frac{b}{\sqrt{a^2 + b^2}} + \frac{a}{\sqrt{a^2 + b^2}} \right)$

$= \sqrt{a^2 + b^2} \left(\frac{a - b}{\sqrt{a^2 + b^2}} \right) = a - b$



$A \rightarrow r; B \rightarrow p; C \rightarrow s; D \rightarrow q$

Ex 74. Match the statements of Column I with values of Column II.

Column I	Column II
A. In a $\triangle ABC$, $(a + b + c)(b + c - a) = \lambda bc$, where $\lambda \in I$, then greatest value of λ is	p. 3
B. In a $\triangle ABC$, $\tan A + \tan B + \tan C = 9$ and if $\tan^2 A + \tan^2 B + \tan^2 C = k$, then least value of k satisfying is	q. $9(3)^{1/3}$
C. In a $\triangle ABC$, if line joining the circumcentre to the incentre is parallel to BC , then value of $\cos B + \cos C$ is	r. 1
D. If the sides of $\triangle ABC$ are in AP (order being a, b, c) and satisfy $\frac{2!}{1!9!} + \frac{2}{3!7!} + \frac{1}{5!5!} = \frac{8^a}{(2b)!}$, then the value of $\cos A + \cos B$ is	s. $12/7$

Sol. A. $(b + c)^2 - a^2 = \lambda bc$

$\Rightarrow b^2 + c^2 - a^2 = (\lambda - 2)bc$

$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$

$\cos A = \frac{\lambda - 2}{2} < 1$

or $\lambda - 2 < 2$

$\Rightarrow \lambda < 4$

or $\lambda = 3$

B. $\tan A + \tan B + \tan C = 9$ in any triangle

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$

$\frac{\tan^2 A + \tan^2 B + \tan^2 C}{3} \geq (\tan A \tan B \tan C)^{2/3}$

$\Rightarrow k \geq 3(9)^{2/3}$

or $k \geq 9 \cdot 3^{1/3}$

C. Since, the line joining the circumcentre to the incentre is parallel to BC .

$\therefore r = R \cos A$

$\therefore 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = R \cos A$

$\Rightarrow -1 + \cos A + \cos B + \cos C = \cos A$

$\Rightarrow \cos B + \cos C = 1$

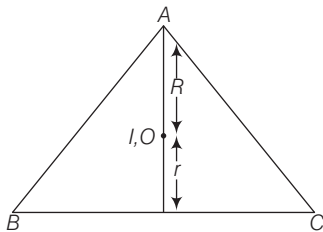
$$\begin{aligned} \text{D. } \frac{2!}{1!9!} + \frac{2}{3!7!} + \frac{1}{5!5!} &= \frac{2}{10!}({}^{10}C_1) + \frac{2}{10!}({}^{10}C_3) \\ &\quad + \frac{1}{10!}({}^{10}C_5) \\ &= \frac{1}{10!}({}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9) \\ \frac{1}{10!}(2^{10-1}) &= \frac{2^9}{10!} = \frac{8^a}{(2b)!} \quad [\text{given}] \\ \Rightarrow 3a &= 9, 2b = 10 \\ \Rightarrow a &= 3, b = 5 \\ \text{Also, } a, b, c &\text{ are in AP.} \\ \Rightarrow c &= 7 \\ \therefore \cos A &= \frac{b^2 + c^2 - a^2}{2bc} \end{aligned}$$

$$\begin{aligned} &= \frac{25 + 49 - 9}{2 \times 5 \times 7} \\ &= \frac{65}{2 \times 5 \times 7} = \frac{13}{14} \\ \text{and } \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{9 + 49 - 25}{2 \times 3 \times 7} = \frac{11}{14} \\ \therefore \cos A + \cos B &= \frac{13 + 11}{14} \\ &= \frac{24}{14} = \frac{12}{7} \\ A \rightarrow p; B \rightarrow q; C \rightarrow r; D \rightarrow s \end{aligned}$$

Type 6. Single Integer Answer Type Questions

Ex 75. ABC is an equilateral triangle of side 4 cm. If R , r and h are the circumradius, inradius and altitude respectively, then $\frac{R+r}{h}$ is equal to _____.

Sol. (1) In equilateral triangle, circumcentre (O) and incentre (I) coincide.



Also, from the diagram, $R + r = h$
 $\Rightarrow \frac{R+r}{h} = 1$

Ex 76. If in a $\triangle ABC$, if $a = 6$, $b = 3$ and $\cos(A - B) = \frac{4}{5}$, then find the area of the triangle.

Sol. (9) Here, $\cos(A - B) = \frac{4}{5}$

$$\therefore \frac{1 - \tan^2 \frac{A-B}{2}}{1 + \tan^2 \frac{A-B}{2}} = \frac{4}{5}$$

By componendo and dividendo, $\frac{2 \tan^2 \frac{A-B}{2}}{2} = \frac{5-4}{5+4}$

$$\Rightarrow \tan^2 \frac{A-B}{2} = \frac{1}{9} \Rightarrow \tan \frac{A-B}{2} = \frac{1}{3} \quad [\because A > B]$$

But $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$

$$\therefore \frac{1}{3} = \frac{6-3}{6+3} \cot \frac{C}{2} \Rightarrow \cot \frac{C}{2} = 1 \Rightarrow C = \frac{\pi}{2}$$

$\therefore$ The area of the triangle
 $= \frac{1}{2} ab \sin C = \frac{1}{2} \cdot 6 \cdot 3 \cdot \sin \frac{\pi}{2} = 9 \text{ sq units}$

Ex 77. In a $\triangle ABC$, $\cos A \cos C = \frac{\lambda(c^2 - a^2)}{3ca}$, where AD is the median through A and $AD \perp AC$, then the value of λ is _____.

Sol. (2) From the $\triangle ABC$, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$... (i)

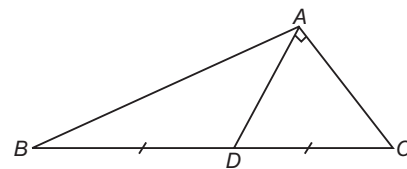
From $\triangle CAD$, $\cos C = \frac{AC}{CD} = \frac{b}{\frac{a}{2}} = \frac{2b}{a}$... (ii)

From $\triangle ABD$, $\frac{BD}{\sin(A - 90^\circ)} = \frac{AB}{\sin \angle ADB}$

$$\Rightarrow \frac{\frac{a}{2}}{-\cos A} = \frac{c}{\sin(90^\circ + C)}$$

$$\Rightarrow \frac{a}{-2\cos A} = \frac{c}{\cos C}$$

$$\therefore \cos A = \frac{a \cos C}{-2c} = \frac{a}{-2c} \cdot \frac{2b}{a} = -\frac{b}{c} \quad [\text{from Eq. (ii)}]$$



From Eq. (i), $\frac{b^2 + c^2 - a^2}{2bc} = -\frac{b}{c}$

$$\Rightarrow b^2 + c^2 - a^2 = -2b^2$$

$$\Rightarrow c^2 - a^2 = -3b^2 \quad \dots (iii)$$

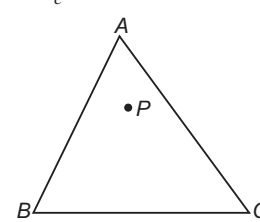
Now, $\cos A \cdot \cos C = \frac{b^2 + c^2 - a^2}{2bc} \cdot \frac{2b}{a} = \frac{b^2 + c^2 - a^2}{ca}$
 $= \frac{3b^2 + 3(c^2 - a^2)}{3ca}$
 $= \frac{a^2 - c^2 + 3(c^2 - a^2)}{3ca} \quad [\text{from Eq. (iii)}]$
 $= \frac{2(c^2 - a^2)}{3ca}$

$$\Rightarrow \lambda = 2$$

Target Exercises

Type 1. Only One Correct Option

- If the angles of a triangle are in the ratio $4 : 1 : 1$, then the ratio of the longest side to the perimeter is
(a) $\sqrt{3} : (2 + \sqrt{3})$ (b) $1 : 6$
(c) $1 : (2 + \sqrt{3})$ (d) $2 : 3$
- If two adjacent sides of a cyclic quadrilateral are 2 and 5 and the angle between them is 60° . If the third side is 3, then the remaining fourth side is
(a) 2 (b) 3 (c) 4 (d) 5
- In $\triangle ABC$, $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2}$ is equal to
(a) a^2 (b) b^2
(c) c^2 (d) None of these
- In a $\triangle ABC$, $\angle C = 60^\circ$, then $\frac{1}{a + c} + \frac{1}{b + c}$ is equal to
(a) $\frac{1}{a + b + c}$ (b) $\frac{2}{a + b + c}$
(c) $\frac{3}{a + b + c}$ (d) None of these
- In any $\triangle ABC$, $4 \Delta (\cot A + \cot B + \cot C)$ is equal to
(a) $3(a^2 + b^2 + c^2)$ (b) $2(a^2 + b^2 + c^2)$
(c) $a^2 + b^2 + c^2$ (d) None of these
- If in $\triangle ABC$, $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is
(a) right angled (b) obtuse angled
(c) equilateral (d) isosceles
- If $\sin(A + B + C) = 1$, $\tan(A - B) = \frac{1}{\sqrt{3}}$ and $\sec(A + C) = 2$, then
(a) $A = 90^\circ, B = 60^\circ, C = 30^\circ$
(b) $A = 120^\circ, B = 60^\circ, C = 0^\circ$
(c) $A = 60^\circ, B = 30^\circ, C = 0^\circ$
(d) None of the above
- The value of $\frac{3 + \cot 76^\circ \cot 16^\circ}{\cot 76^\circ + \cot 16^\circ}$ is
(a) $\cot 60^\circ$ (b) $\tan 2^\circ$ (c) $\tan 44^\circ$ (d) $\cot 44^\circ$
- Sides of a $\triangle ABC$ are in AP, if $a < \text{minimum } \{b, c\}$, then $\cos A$ is equal to
(a) $\frac{3c - 4b}{2b}$ (b) $\frac{3c - 4b}{2c}$
(c) $\frac{4c - 3b}{2b}$ (d) $\frac{4c - 3b}{2c}$
- If a, b and c are the sides of a $\triangle ABC$, then $\sqrt{a} + \sqrt{b} - \sqrt{c}$ is always
(a) negative (b) positive
(c) non-negative (d) non-positive
- If sides of $\triangle ABC$ are a, b and c such that $2b = a + c$, then exhaustive range of $\frac{b}{c}$ is
(a) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{3}, 2\right)$
(c) $\left(\frac{2}{3}, 2\right)$ (d) $\left(\frac{3}{2}, 2\right)$
- Two sides of a triangle are given by the roots of the equation $x^2 - 2\sqrt{3}x + 2 = 0$. The angle between the sides is $\frac{\pi}{3}$. The perimeter of the triangle is
(a) $6 + \sqrt{3}$ (b) $2\sqrt{3} + \sqrt{6}$
(c) $2\sqrt{3} + \sqrt{10}$ (d) None of these
- The sides of $\triangle ABC$ are $AB = \sqrt{13}$ cm, $BC = 4\sqrt{3}$ cm, $CA = 7$ cm. Then, $\sin \theta$, where θ is the smallest angle of the triangle, is equal to
(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$
(c) $\frac{\sqrt{3} - 1}{2\sqrt{2}}$ (d) None of these
- In a $\triangle ABC$, the sides a, b and c are such that they are the roots of $x^3 - 11x^2 + 38x - 40 = 0$. Then, $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ is equal to
(a) $\frac{3}{4}$ (b) 1
(c) $\frac{9}{16}$ (d) None of these
- If in a $\triangle ABC$, $2\cos A \sin C = \sin B$, then the triangle is
(a) equilateral
(b) isosceles
(c) right angled
(d) None of the above
- If in a $\triangle ABC$, $\frac{2\cos A}{a} + \frac{\cos B}{b} + \frac{2\cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$, then $\angle A$ is equal to
(a) 90° (b) 45°
(c) 60° (d) None of these

- 35.** In $\triangle ABC$, $1 - \tan \frac{B}{2} \cdot \tan \frac{C}{2}$ is equal to
- (a) $\frac{2a}{a+b+c}$ (b) $\frac{2b}{a+b+c}$
 (c) $\frac{2c}{a+b+c}$ (d) None of these
- 36.** In $\triangle ABC$, $a : b : c = (1+x) : 1 : (1-x)$, where $x \in (0, 1)$. If $\angle A = \frac{\pi}{2} + \angle C$, then x is equal to
- (a) $\frac{1}{\sqrt{6}}$ (b) $\frac{1}{2\sqrt{3}}$
 (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{2\sqrt{7}}$
- 37.** If a, b and c are the sides of a triangle such that $b \cdot c = \lambda^2$, for some positive λ , then
- (a) $c \geq 2\lambda \sin \frac{C}{2}$ (b) $b \geq 2\lambda \sin \frac{B}{2}$
 (c) $a \geq 2\lambda \sin \frac{A}{2}$ (d) None of these
- 38.** ABC is a triangle, $B = 60^\circ, C = 30^\circ$. BC is produced to D so that $\angle ADB$, then given $\sqrt{3} = 1 \frac{8}{11}$ and $BC \times CD = 2^3 \times 3^2 \times 19 \times 11$, the square of the altitude from A to BC is
- (a) 3250 (b) 3248
 (c) 3249 (d) None of these
- 39.** If $c^2 = a^2 + b^2$, then $4s(s-a)(s-b)(s-c)$ is equal to
- (a) a^2b^2 (b) c^2a^2
 (c) b^2c^2 (d) s^4
- 40.** In $\triangle ABC$, $(a+b-c)(b+c-a)(c+a-b) - abc$ is always
- (a) non-negative (b) non-positive
 (c) negative (d) positive
- 41.** If the area of a $\triangle ABC$ be λ , then $a^2 \sin 2B + b^2 \sin 2A$ is equal to
- (a) 2λ (b) λ
 (c) 4λ (d) None of these
- 42.** In a $\triangle ABC$, R = circumradius and r = inradius. The value of $\frac{a \cos A + b \cos B + c \cos C}{a+b+c}$ is equal to
- (a) $\frac{R}{r}$ (b) $\frac{R}{2r}$
 (c) $\frac{r}{R}$ (d) $\frac{2r}{R}$
- 43.** In a $\triangle ABC$, $2s$ = perimeter and R = circumradius. Then, $\frac{s}{R}$ is equal to
- (a) $\sin A + \sin B + \sin C$ (b) $\cos A + \cos B + \cos C$
 (c) $\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2}$ (d) None of these
- 44.** In a $\triangle ABC$, if $\frac{R}{r} \leq 2$, then the triangle is
- (a) scalene
 (b) isosceles
 (c) right angled
 (d) equilateral
- 45.** In the given figure, P is any arbitrary interior point of the $\triangle ABC$, H_a, H_b and H_c are the length of altitudes drawn from vertices A, B and C respectively. If x_a, x_b and x_c represent the distance of P from sides BC, AC and AB respectively, then $\frac{x_a}{H_a} + \frac{x_b}{H_b} + \frac{x_c}{H_c}$ is always equal to
- 
- (a) 3 (b) 2
 (c) 1 (d) None of these
- 46.** In a $\triangle ABC$, $a = 2b$ and $|A - B| = \frac{\pi}{3}$. The measure of $\angle C$ is
- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{3}$
 (c) $\frac{\pi}{6}$ (d) None of these
- 47.** If the sides of a right angled triangle are in AP, then tangents of the acute angled triangle are
- (a) $\sqrt{3} + \frac{1}{2}, \sqrt{3} - \frac{1}{2}$ (b) $\sqrt{5} - \frac{1}{2}, \sqrt{5} + \frac{1}{2}$
 (c) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (d) $\frac{3}{4}, \frac{4}{3}$
- 48.** In a $\triangle ABC$, the angles A and B are two values of θ satisfying $\sqrt{3} \cos \theta + \sin \theta = k, |k| < 2$. The triangle
- (a) is an acute angled (b) is a right angled
 (c) is an obtuse angled (d) has one angle $\frac{\pi}{3}$
- 49.** If in an obtuse angled triangle the obtuse angle is $\frac{3\pi}{4}$ and the other two angles are equal to two values of θ satisfying $a \tan \theta + b \sec \theta = c$, where $|b| \leq \sqrt{a^2 + c^2}$, then $a^2 - c^2$ is equal to
- (a) ac (b) $2ac$
 (c) $\frac{a}{c}$ (d) None of these
- 50.** The length of the sides of a triangle are x, y and $\sqrt{x^2 + y^2 + xy}$. The measure of the greatest angle is
- (a) 120°
 (b) 150°
 (c) 135°
 (d) None of the above

51. The greatest angle of a triangle whose sides are $x^2 + x + 1$, $2x + 1$ and $x^2 - 1$, is

- (a) 60° (b) 90°
(c) 120° (d) None of these

52. In $\triangle ABC$, internal angle bisector of $\angle A$ makes an angle θ with side BC . Value of $\sin \theta$ is equal to

- (a) $\left| \sin \left(\frac{B-C}{2} \right) \right|$ (b) $\left| \sin \left(\frac{B}{2} - C \right) \right|$
(c) $\cos \left(\frac{B-C}{2} \right)$ (d) $\cos \left(\frac{B}{2} - C \right)$

53. In a $\triangle ABC$, $a : b : c = 4 : 5 : 6$. The ratio of the radius of the circumcircle to that of the incircle is

- (a) $\frac{16}{9}$ (b) $\frac{16}{7}$ (c) $\frac{11}{7}$ (d) $\frac{7}{16}$

54. If r_1 , r_2 and r_3 are exradii of any triangle, then $r_1 r_2 + r_2 r_3 + r_3 r_1$ is equal to

- (a) $\frac{\Delta}{r}$ (b) $\frac{\Delta^2}{r^2}$
(c) $\frac{r}{\Delta}$ (d) $\frac{r^2}{\Delta^2}$

55. In an equilateral triangle, the ratio of the inradius, circumradius and exradius are

- (a) $1 : 3 : 2$ (b) $1 : 1 : 1$
(c) $2 : 3 : 4$ (d) $1 : 2 : 3$

56. In a triangle, if $r_1 > r_2 > r_3$, then

- (a) $a > b > c$ (b) $a < b < c$
(c) $a > b$ and $b < c$ (d) $a < b$ and $b > c$

57. If x, y and z are perpendicular drawn from the vertices of triangle having sides a, b and c , then the value of $\frac{bx}{c} + \frac{cy}{a} + \frac{az}{b}$ will be

- (a) $\frac{a^2 + b^2 + c^2}{2R}$ (b) $\frac{a^2 + b^2 + c^2}{R}$
(c) $\frac{a^2 + b^2 + c^2}{4R}$ (d) $\frac{2(a^2 + b^2 + c^2)}{R}$

58. If the sides of a triangle are in the ratio $3 : 7 : 8$, then $R : r$ is equal to

- (a) $2 : 7$ (b) $7 : 2$ (c) $3 : 7$ (d) $7 : 3$

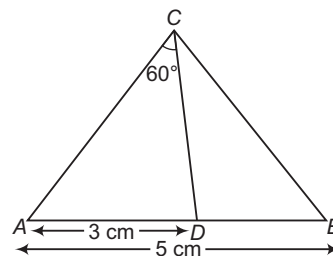
59. If the length of median AA_1 , BB_1 and CC_1 of $\triangle ABC$ are m_a , m_b and m_c , respectively. Then,

- (a) $\Sigma m_a > \frac{3}{2}s$ (b) $\Sigma m_a > 3s$
(c) $\Sigma m_a > \frac{5}{2}s$ (d) $\Sigma m_a > 2s$

60. In a $\triangle ABC$, $B = 90^\circ$, $AC = h$ and the length of the perpendicular from B to AC is p such that $h = 4p$. If $AB < BC$, then $\angle C$ has the measure

- (a) $\frac{5\pi}{12}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{12}$ (d) None of these

61. In the given figure, ABC is a triangle in which $C = 90^\circ$ and $AB = 5$ cm. D is a point on AB such that $AD = 3$ cm and $\angle ACD = 60^\circ$. Then, the length of AC is



- (a) $5\sqrt{\frac{3}{7}}$ cm (b) $\sqrt{\frac{7}{3}}$ cm
(c) $\frac{3}{\sqrt{7}}$ cm (d) None of these

62. If α , β and γ are the altitudes of a $\triangle ABC$ and $2s$ denotes its perimeter, then $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ is equal to

- (a) $\frac{\Delta}{s}$ (b) $\frac{s}{\Delta}$
(c) $s \cdot \Delta$ (d) None of these

63. If in a $\triangle ABC$, $a^2 + b^2 + c^2 = 8R^2$, where R = circumradius, then the triangle is

- (a) equilateral
(b) isosceles
(c) right angled
(d) None of the above

64. A $\triangle ABC$ is right angled at B . Then, the diameter of the incircle of the triangle is

- (a) $2(c + a - b)$ (b) $c + a - 2b$
(c) $c + a - b$ (d) None of these

65. In any $\triangle ABC$, $4 \left(\frac{s}{a} - 1 \right) \left(\frac{s}{b} - 1 \right) \left(\frac{s}{c} - 1 \right)$ is equal to

- (a) $\frac{r}{R}$ (b) $\frac{2r}{R}$
(c) $\frac{3r}{R}$ (d) None of these

66. In any $\triangle ABC$, $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$ is equal to

- (a) $\frac{a^2 + b^2 + c^2}{2\Delta^2}$ (b) $\frac{a^2 + b^2 + c^2}{\Delta^2}$
(c) $\frac{a^2 + b^2 + c^2}{3\Delta^2}$ (d) None of these

67. In any $\triangle ABC$, $rr_1 + r_2 r_3$ is equal to

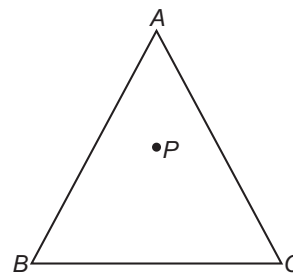
- (a) ab
(b) ac
(c) bc
(d) None of the above

68. In an equilateral triangle,

- (a) $r_1 = r_2 = r_3 = 2r$
(b) $r_1 = r_2 = r_3 = r$
(c) $r_1 = r_2 = r_3 = 3r$
(d) None of the above

69. If in a $\triangle ABC$, AD , BE and CF are the altitudes and R is the circumradius, then the radius of the circle DEF is
 (a) $\frac{R}{2}$ (b) $2R$
 (c) R (d) None of these
70. If R is the radius of the circumcircle of the $\triangle ABC$ and its area is Δ , then
 (a) $R = \frac{a+b+c}{\Delta}$ (b) $R = \frac{a+b+c}{4\Delta}$
 (c) $R = \frac{abc}{4\Delta}$ (d) $R = \frac{abc}{\Delta}$
71. The radii r_1, r_2, r_3 of escribed circles of the $\triangle ABC$ are in HP. If its area is 24 sq cm and its perimeter is 24 cm, then the lengths of its sides are
 (a) 4, 6, 8 (b) 3, 9, 11
 (c) 6, 8, 10 (d) None of these
72. In $\triangle ABC$, BB_1 is the bisector of $\angle B$. Altitude drawn from A to BB_1 meets side BC in D_1 and BB_1 in D_2 . Value of $\frac{AD_1}{AD_2}$ is
 (a) $\frac{4}{3}$ (b) $\frac{6}{5}$
 (c) 2 (d) $\frac{5}{2}$
73. In $\triangle ABC$, line joining the circumcentre and orthocentre is parallel to side AC , then the value of $\tan A \cdot \tan C$ is equal to
 (a) $\sqrt{3}$
 (b) 3
 (c) $3\sqrt{3}$
 (d) None of the above
74. In $\triangle ABC$, the value of $\frac{r_1 + r_2}{1 + \cos C}$ is always equal to
 (a) $2r$ (b) $2R$ (c) $\frac{2r^2}{R}$ (d) $\frac{2R^2}{r}$
75. If A, A_1, A_2, A_3 are the area of the incircle and excircles, then $\frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$ is equal to
 (a) $\frac{1}{\sqrt{A}}$ (b) $\frac{2}{\sqrt{A}}$
 (c) $\frac{3}{\sqrt{A}}$ (d) None of these
76. If H is the orthocentre of $\triangle ABC$, then AH is equal to
 (a) $c \cot A$ (b) $b \cot A$ (c) $a \cot B$ (d) $a \cot A$
77. ABC is an isosceles triangle inscribed in a circle of radius r . If $AB = AC$ and h is the altitude from A to BC , then the triangle has perimeter $P = 2(\sqrt{2hr - h^2} + \sqrt{2hr})$ and area A is equal to
 (a) $h\sqrt{3rh - h^2}$ (b) $h\sqrt{2rh - h^2}$
 (c) $2h\sqrt{2rh - h^2}$ (d) None of these

78. In the given figure, P is any interior point of the equilateral $\triangle ABC$ of side length 2 units.



If x_a, x_b and x_c represent the distance of P from the sides BC, CA and AB respectively, then $x_a + x_b + x_c$ is equal to

- (a) 6 (b) $\sqrt{3}$ (c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{3}$
79. In $\triangle ABC$, base BC and area of triangle ' Δ ' are fixed. Locus of the centroid of $\triangle ABC$ is a straight line that is
 (a) parallel to side BC
 (b) right bisector of side BC
 (c) right angle of BC
 (d) inclined at an angle $\sin^{-1}\left(\frac{\sqrt{\Delta}}{BC}\right)$ to side BC
80. In $\triangle ABC$, $\angle A = \frac{\pi}{2}$, AA_1 and AA_2 are the median and altitude respectively. If $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC$, then $\sin^3 \frac{A}{3} \cdot \cos \frac{A}{3}$ is equal to
 (a) $\frac{3a^3}{16b^2c}$ (b) $\frac{3a^3}{64b^2c}$
 (c) $\frac{3a^2}{4b^2c}$ (d) $\frac{3a^3}{128b^2c}$
81. At a point A , the angle of elevation of a tower is such that its tangent is $\frac{5}{12}$, on walking 120 m nearer the tower the tangent of the angle of elevation is $\frac{3}{4}$. The height of the tower is
 (a) 225 m
 (b) 200 m
 (c) 230 m
 (d) None of the above
82. A observer on the top of a cliff 200 m above the sea level, observes the angles of depression of two ships on opposite sides of the cliff to be 45° and 30° , respectively. The distance between the ships if the line joining them points to the base of cliff, is
 (a) $100(\sqrt{3} + 1)$ m
 (b) $200(\sqrt{3} + 1)$ m
 (c) $150(\sqrt{3} + 1)$ m
 (d) None of the above

- 83.** A tower is observed from two stations A and B , where B is East of A at a distance of 200 m. The tower is due North of A and due West of B . The angles of elevation of the tower from A and B are complementary. The height of the tower is
(a) $100(2)^{1/4}$ m (b) $200\sqrt{2}$ m
(c) $200(2)^{1/4}$ m (d) None of these
- 84.** The angle of elevation of the top of an incomplete vertical pillar at a horizontal distance of 100 m from its base is 45° . If the angle of elevation of the top of the complete pillar at the same point is to be 60° , then the height of the incomplete pillar is to be increased by
(a) $50\sqrt{2}$ m (b) 100 m
(c) $100(\sqrt{3} - 1)$ m (d) $100(\sqrt{3} + 1)$ m
- 85.** A tree is broken by wind, its upper part touches the ground at a point 10 m from the foot of the tree and makes an angle of 45° with the ground. The entire length of the tree is
(a) 15 m (b) 20 m
(c) $10(1 + \sqrt{2})$ m (d) $10\left(1 + \frac{\sqrt{3}}{2}\right)$ m
- 86.** If a flagstaff 6 m high placed on the top of a tower throws a shadow of $2\sqrt{3}$ m along the ground, then the angle (in degree) that the Sun makes with the ground, is
(a) 30° (b) 60°
(c) 45° (d) None of these
- 87.** The angle of elevation of the top of a tower from the top and bottom of a building of height a are 30° and 45° , respectively. If the tower and the building stand at the same level, the height of the tower is
(a) $\frac{a(3 + \sqrt{3})}{2}$ (b) $a(3 + \sqrt{3})$
(c) $a\sqrt{3}$ (d) $a(\sqrt{3} - 1)$
- 88.** At a point 15 m away from the base of a 15 m high house, the angle of elevation of the top is
(a) 90° (b) 60°
(c) 30° (d) 45°
- 89.** From the top of a light house 60 m high with its base at sea level, the angle of depression of a boat is 15° . The distance of the boat from the light house is
(a) $30\left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right)$ m (b) $30\left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right)$ m
(c) $60\left(\frac{\sqrt{3} + 1}{\sqrt{3} - 1}\right)$ m (d) $60\left(\frac{\sqrt{3} - 1}{\sqrt{3} + 1}\right)$ m
- 90.** A tower subtends an angle of 30° at a point distant d from the foot of the tower and on the same level as the foot of the tower. At a second point, h vertically above the first, the angle of depression of the foot of the tower is 60° . The height of the tower is
(a) $\frac{h}{3}$ (b) $\frac{h}{3d}$ (c) $3h$ (d) $\frac{3h}{d}$
- 91.** The angle of depression of a point situated at a distance of 70 m from the base of a tower is 45° . The height of the tower is
(a) 70 m (b) $70\sqrt{2}$ m (c) $\frac{70}{\sqrt{2}}$ m (d) 35 m
- 92.** The horizontal distance between two tower is 60 m. The angular elevation of the top of the taller tower as seen from the top of the shorter one is 30° . If the height of the taller tower is 150 m, the height of the shorter one is
(a) $(150 - 20\sqrt{3})$ m (b) 200 m
(c) 216 m (d) None of these
- 93.** If the elevation of the Sun is 30° , then the length of the shadow cast by a tower of 150 ft height, is
(a) $75\sqrt{3}$ ft (b) $200\sqrt{3}$ ft
(c) $150\sqrt{3}$ ft (d) None of these
- 94.** An aeroplane flying at a height of 300 m above the ground passes vertically above another plane at an instant when the angle of elevation of the two planes from the same point on the ground are 60° and 45° , respectively. Then, the height of the lower plane from the ground (in metre) is
(a) $100\sqrt{3}$ (b) $\frac{100}{\sqrt{3}}$
(c) 50 (d) $150(\sqrt{3} + 1)$
- 95.** A person walking alone a straight road observes that at two points 1000 m apart, the angles of elevation of a vertical tower in front of him are $\frac{\pi}{6}$ and $\frac{5\pi}{12}$. The height of the tower is equal to
(a) $250(\sqrt{3} + 1)$ m (b) $250(\sqrt{3} - 1)$ m
(c) $250(\sqrt{2} + 1)$ m (d) $250(\sqrt{2} - 1)$ m
- 96.** A chimney of 20 m high, standing vertically on the top of a building, subtends an angle $\tan^{-1}\left(\frac{1}{6}\right)$ at a distance of 70 m from the foot of the building. The height of the building is
(a) 25 m (b) 90 m (c) 50 m (d) 30 m
- 97.** A and B are two points 30 m apart in a line on the horizontal plane through the foot of a tower lying on opposite sides of the tower. The distances of the top of the tower from A and B are 20 m and 15 m respectively. If the angle of elevation of the top of the tower at A is θ and the height of the tower is h , then
(a) $\cos \theta = 28/36$ (b) $\cos \theta = 43/48$
(c) $h = 5\sqrt{5}/\sqrt{3}$ (d) $h = 5\sqrt{455}/12$
- 98.** The longer side of a parallelogram is 10 cm and the shorter is 6 cm. If the longer diagonal makes an angle 30° with the longer side, then length of the longer diagonal is
(a) $(5\sqrt{3} + \sqrt{11})$ cm (b) $(4\sqrt{3} + \sqrt{11})$ cm
(c) $(5\sqrt{3} + \sqrt{13})$ cm (d) None of these

- 99.** PQ is a post of height a and AB is a tower at some distance, α and β are the angles of elevation of B , the top of the tower, at P and Q , respectively. The height of the tower is
- (a) $\frac{a \sin \alpha \cos \beta}{\sin (\alpha - \beta)}$ (b) $\frac{a \cos \alpha \sin \beta}{\sin (\alpha - \beta)}$
 (c) $\frac{a \sin \alpha \sin \beta}{\sin (\alpha - \beta)}$ (d) None of these
- 100.** A man observes that when he move up a distance c m on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30° and when he moves up further a distance c m the angle of depression of that point is 45° . The angle of inclination of the slope with the horizontal is
- (a) 60° (b) 45° (c) 75° (d) 30°
- 101.** A tower of height b subtends an angle at a point O on the level of the foot of the tower and at a distance a from the foot of the tower. If the pole mounted on the tower also subtends on equal angle at O , then height of the pole is
- (a) $b \left(\frac{a^2 - b^2}{a^2 + b^2} \right)$ (b) $b \left(\frac{a^2 + b^2}{a^2 - b^2} \right)$
 (c) $a \left(\frac{a^2 - b^2}{a^2 + b^2} \right)$ (d) $a \left(\frac{a^2 + b^2}{a^2 - b^2} \right)$
- 102.** AB is a vertical pole and C is its middle point. The end A is on the level ground and P is any point on the level ground other than A . The portion CB subtends an angle β at P . If $AP : AB = 2 : 1$, then β is equal to
- (a) $\tan^{-1} \frac{4}{9}$ (b) $\tan^{-1} \frac{1}{9}$
 (c) $\tan^{-1} \frac{5}{9}$ (d) $\tan^{-1} \frac{2}{9}$
- 103.** Each side of an equilateral triangle subtends an angle of $\pi/3$ at the top of a tower of height h , located at the centre of the triangle. If a is the length of the side of the triangle, then
- (a) $3a^2 = h^2$ (b) $2a^2 = 3h^2$
 (c) $2a^2 = h^2$ (d) None of these
- 104.** A house of height 100 m subtends a right angle at the window of an opposite house. If the height of the window is 64 m, then the distance between the two houses is
- (a) 48 m (b) 36 m
 (c) 54 m (d) 72 m
- 105.** The angle of elevation of a cloud from a point h m above a lake is θ_1 and the angle of depression of its reflection is θ_2 . Then, height of the cloud above the lake is
- (a) $h \frac{\tan (\theta_1 + \theta_2)}{\tan (\theta_2 - \theta_1)}$ (b) $h \frac{\cos (\theta_1 + \theta_2)}{\cos (\theta_1 - \theta_2)}$
 (c) $h \frac{\sin (\theta_1 + \theta_2)}{\sin (\theta_2 - \theta_1)}$ (d) $h \frac{\cot (\theta_1 + \theta_2)}{\cot (\theta_1 - \theta_2)}$
- 106.** The angle of elevation of a tower from a point A due South of it is x and from a point B due East of A is y . If $AB = l$, then the height h of the tower is given by
- (a) $\frac{l}{\sqrt{\cot^2 y - \cot^2 x}}$ (b) $\frac{l}{\sqrt{\tan^2 y - \tan^2 x}}$
 (c) $\frac{2l}{\sqrt{\cot^2 y - \cot^2 x}}$ (d) None of these
- 107.** A pole stands vertically on the centre of a square. When α is the elevation of the Sun, its shadow just reaches the side of the square and is at a distance x and y from the ends of that side. The height of the pole is
- (a) $\sqrt{\frac{x^2 + y^2}{2}} \tan \alpha$ (b) $\sqrt{\frac{x^2 + y^2}{2}} \cot \alpha$
 (c) $\sqrt{\frac{x^2 - y^2}{2}} \tan \alpha$ (d) None of these
- 108.** If a flagstaff subtends equal angles at the points A, B, C and D on the horizontal ground through the foot of flagstaff, then the points A, B, C and D necessarily form a
- (a) rectangle (b) parallelogram
 (c) square (d) None of these
- 109.** The angles of elevation of the top of a tower from two points (collinear with foot of tower) on the ground at a distance a and b ($b > a$) from its foot are found to be α and β . If h is the height of tower, then
- (a) $(b - a) = h (\tan \alpha - \tan \beta)$
 (b) $(b - a) = h (\cot \alpha - \cot \beta)$
 (c) $(b - a) = h (\tan \beta - \tan \alpha)$
 (d) $(b - a) = h (\cot \beta - \cot \alpha)$
- 110.** A flagstaff standing vertically at the vertex A of an isosceles $\triangle ABC$, subtends angle $\frac{\pi}{3}$ at the mid-point of the base BC and an angle $\frac{\pi}{6}$ at the vertex B . If $BC = a$, then height of flagstaff is
- (a) $\frac{a}{2} \sqrt{\frac{3}{2}}$ (b) $\frac{a}{3} \sqrt{\frac{3}{2}}$
 (c) $a \sqrt{\frac{3}{2}}$ (d) $\frac{a}{4} \sqrt{\frac{3}{2}}$
- 111.** A tower of height h standing vertically at the centre of a square of side length a subtends the same angle θ at all the corner points of the square. Then,
- (a) $2h^2 = a^2 \tan^2 \theta$ (b) $a^2 = 2h^2 \tan^2 \theta$
 (c) $a^2 = 2h^2 \cot^2 \theta$ (d) $2a^2 = h^2 \cot^2 \theta$
- 112.** The angle of elevation of the top of a vertical tower PQ , Q being its foot standing on the horizontal ground, from two points A and B on ground are found to be θ_A and θ_B , respectively. If Q, A and B are collinear and $QA = 4$ m, $QB = 9$ m and θ_A, θ_B are complementary angles, then height of tower is
- (a) 3 m (b) 6 m (c) 4 m (d) 16 m

- 113.** A monument $ABCD$ stands at a point A on a level ground. At a point P on the ground, the portions AB, AC, AD subtend angles α, β, γ respectively. If $AB = a, AC = b, AD = c, AP = x$ and $\alpha + \beta + \gamma = 180^\circ$, then
- (a) $x^2 = (a + b + c) / abc$
 (b) $x^2 = 1 / (a + b + c)$
 (c) $\tan \alpha + \tan \beta + \tan \gamma = (a + b + c)^{1/2} / \sqrt{abc}$
 (d) $\tan \alpha \tan \beta \tan \gamma = (a + b + c)^{3/2} / \sqrt{abc}$
- 114.** Three vertical tower standing at A, B, C subtends the angle θ_A, θ_B and θ_C , respectively at the circumcentre of $\triangle ABC$, then $\tan \theta_A, \tan \theta_B$ and $\tan \theta_C$ are in
- (a) AP
 (b) GP
 (c) HP
 (d) None of the above
- 115.** On the top of a hemispherical dome of radius r , there stands a flag of height h . From a point on the ground the elevation of the top of the flag is 30° . After moving a distance d towards the dome, when the flag is just visible, then elevation is 45° , then r and h are

- (a) $\frac{d}{\sqrt{2}}, \frac{d(\sqrt{3} + 1)}{2\sqrt{2}}$
 (b) $\frac{d(\sqrt{3} + 1)}{2\sqrt{2}}, \frac{d(\sqrt{3} + 1)(\sqrt{2} - 1)}{2\sqrt{2}}$
 (c) $\frac{d(\sqrt{3} + 1)}{2\sqrt{2}}, \frac{d(\sqrt{2} - 1)}{2\sqrt{2}}$
 (d) None of the above

- 116.** A spherical ball of radius r subtends an angle θ_1 at a point P on the ground. If the angle of elevation of centre of the ball at P is θ_2 , then height of the centre of the ball from the ground is
- (a) $r \sin \theta_2 \cdot \operatorname{cosec} \frac{\theta_1}{2}$ (b) $r \sin \frac{\theta_2}{2} \cdot \operatorname{cosec} \theta_1$
 (c) $r \sin \frac{\theta_1}{2} \cdot \operatorname{cosec} \theta_2$ (d) $r \sin \theta_1 \cdot \operatorname{cosec} \frac{\theta_2}{2}$
- 117.** A circular ring of radius r m is suspended from a point h m vertically above the centre of ring by four strings of equal lengths attached at equal intervals to the circumference of the ring. If θ is the angle between two consecutive strings, then $\cos \theta$ is equal to
- (a) $\frac{h^2}{h^2 + r^2}$ (b) $\frac{r^2}{h^2 + r^2}$ (c) $\frac{2h^2}{h^2 + r^2}$ (d) $\frac{2r^2}{h^2 + r^2}$

Type 2. More than One Correct Option

- 118.** If in a $\triangle ABC$,
 $\sin C + \cos C + \sin (2B + C) - \cos (2B + C) = 2\sqrt{2}$,
 then $\triangle ABC$ is
- (a) equilateral
 (b) isosceles
 (c) right angled
 (d) obtuse angled
- 119.** If r_1, r_2 and r_3 are the radii of the escribed circles of a $\triangle ABC$ and r is the radius of its incircle, then the root(s) of the equation $x^2 - r(r_1 r_2 + r_2 r_3 + r_3 r_1) + r_1 r_2 r_3 - 1 = 0$ is/are
- (a) 1 (b) $r_1 + r_2 + r_3$
 (c) r (d) $r_1 r_2 r_3 - 1$
- 120.** Let ABC be an isosceles triangle with base BC . If r is the radius of the circle inscribed in the $\triangle ABC$ and r_1 be the radius of the circle escribed opposite to the angle A , then the product $r_1 r$ can be equal to
- (a) $R^2 \sin^2 A$
 (b) $R^2 \sin^2 2B$
 (c) $\frac{1}{2} a^2$
 (d) $\frac{a^2}{4}$
- 121.** If $\cos B \cos C + \sin B \sin C \sin^2 A = 1$, then ABC is
- (a) isosceles
 (b) right angled
 (c) equilateral
 (d) None of the above
- 122.** A pole of length h stands inside a triangular plot ABC and subtends equal angles α at its vertices, then
- (a) $2 h \cos \alpha \sin A = a \sin \alpha$
 (b) $2 h \cos \alpha \sin C = c \sin \alpha$
 (c) $2 h \cos \alpha \sin B = b \sin \alpha$
 (d) $2 h \cos \alpha = R$
- 123.** The area of a regular polygon of n sides is (where, r is inradius, R is circumradius and a is side of the triangle)
- (a) $\frac{nR^2}{2} \sin\left(\frac{2\pi}{n}\right)$
 (b) $nr^2 \tan\left(\frac{\pi}{n}\right)$
 (c) $\frac{na^2}{4} \cot \frac{\pi}{n}$
 (d) $nR^2 \tan\left(\frac{\pi}{n}\right)$

Type 3. Assertion and Reason

Directions (Q. Nos. 124-128) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

124. In acute angled $\triangle ABC$, $a > b > c$

Statement I $r_1 > r_2 > r_3$

Statement II $\cos A < \cos B < \cos C$

125. Statement I In $\triangle ABC$, if $\tan A : \tan B : \tan C = 1 : 2 : 3$, then $A = 45^\circ$.

Statement II If $p : q : r = 1 : 2 : 3$, then $p = 1$.

126. Statement I If in a triangle, orthocentre, circumcentre and centroid are rational points, then its vertices must also be rational points.

Statement II If the vertices of a triangle are rational points, then the centroid, circumcentre and orthocentre are also rational points.

127. Let the equations of the sides BC, CA and AB of the $\triangle ABC$ be $L_i : a_i x + b_i y + c_i = 0$ for $i = 1, 2, 3$ respectively such that $a_i^2 + b_i^2 = k$ for $i = 1, 2, 3$. Let the equations of bisectors of internal angles B and C be $L_1 + L_3 = 0$ and $L_1 + L_2 = 0$.

Statement I The bisector of internal angle A is $L_2 - L_3 = 0$.

Statement II Any line passing through the points of intersection of the lines L_1 and L_2 can be written as $L_1 + \lambda L_2 = 0$.

128. In the $\triangle ABC$, AD is the angle bisector of $\angle BAC$ meeting BC at D . Circumcircles S_1 and S_2 are drawn to circumscribe $\triangle ABD$ and $\triangle ACD$, respectively. If S_1 cuts AC at E and S_2 cuts AB at F , then

Statement I $BF = EC$

Statement II If from an exterior point P , two secants are drawn to cut a circle at A, B and C, D , then $PA \times PB = PC \times PD$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 129-131) Given that

$$\Delta = 6, r_1 = 2, r_2 = 3, r_3 = 6$$

129. Circumradius R is equal to

- (a) 2.5
(b) 3.5
(c) 1.5
(d) 4.2

130. Inradius r is equal to

- (a) 2
(b) 1
(c) 1.5
(d) 2.5

131. Difference between the greatest and the least angles is

- (a) $\cos^{-1} \frac{4}{5}$
(b) $\tan^{-1} \frac{3}{4}$
(c) $\cos^{-1} \frac{3}{5}$
(d) None of the above

Passage II (Q. Nos. 132-134) Let ABC be a triangle, from vertices A, B and C altitudes AD, BE and CF are drawn to opposite sides BC, CA and AB respectively, which meet at a point O . Now, $\triangle DEF$ is completed, also length OA, OB and OC , respectively are p, q and r .

132. Ratio of area of $\triangle DEF$ to $\triangle ABC$ is

- (a) $2 \cos A \cos B \cos C$
(b) $2 \sin A \sin B \sin C$
(c) $2 \cos A \cos B \sin C$
(d) None of the above

133. Radius of the circumcircle of $\triangle DEF$ is

- (a) $\frac{R}{2}$
(b) R
(c) $\frac{R}{4}$
(d) None of the above

134. Radius of the incircle of $\triangle DEF$ is

- (a) $2R \cos A \cos B \cos C$
(b) $R \cos A \cos B \cos C$
(c) $\frac{abcc \cos A \cos B \cos C}{\Delta ABC}$
(d) None of the above

Passage III (Q. Nos. 135-137) O is the circumcentre of $\triangle ABC$. R_1, R_2, R_3 are the radii of the circumcircles of $\triangle OBC, \triangle OCA$ and $\triangle OAB$, respectively. r_1, r_2, r_3 are the radii of the circles drawn on the altitudes OD, OE and OF of these triangles respectively. $\Delta_1, \Delta_2, \Delta_3$ being the respective areas of these triangles. R, r are respectively the circumradius and inradius of $\triangle ABC$, Δ being its area and a, b, c are the lengths of the sides BC, CA and AB , respectively.

135. $\frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3}$ equals

- (a) $\frac{4\Delta}{R^2}$ (b) $\frac{4\Delta}{R^2}$
(c) $\frac{4}{R^2}(a+b+c)$ (d) None of these

136. $\frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3}$ equals

- (a) $4 \tan A \tan B \tan C$ (b) $4 \sin A \sin B \sin C$
(c) $4 \cot A \cot B \cot C$ (d) $4 \cos A \cos B \cos C$

137. $\frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3}$ equals

- (a) $R^2 \left[\frac{\sin A}{\Delta_1} + \frac{\sin B}{\Delta_2} + \frac{\sin C}{\Delta_3} \right]$
(b) $R^2 \left[\frac{\cos A}{\Delta_1} + \frac{\cos B}{\Delta_2} + \frac{\cos C}{\Delta_3} \right]$
(c) $R^2 \left[\frac{\tan A}{\Delta_1} + \frac{\tan B}{\Delta_2} + \frac{\tan C}{\Delta_3} \right]$
(d) $R^2 \left[\frac{\cot A}{\Delta_1} + \frac{\cot B}{\Delta_2} + \frac{\cot C}{\Delta_3} \right]$

Passage IV (Q. Nos. 138-140) The two adjacent sides AB and BC of a cyclic quadrilateral $ABCD$ are 2 and 5 units, respectively and the angle between them is 60° . If the area of quadrilateral is $4\sqrt{3}$ sq units.

138. The area of $\triangle ADC$ is

- (a) $3\sqrt{3}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{3\sqrt{3}}{2}$ (d) $\frac{3}{2}$

139. The area of circle circumscribing the quadrilateral $ABCD$ is

- (a) 48π
(b) $\frac{19\pi}{3}$
(c) 12π
(d) $\frac{19\pi}{12}$

140. For given quadrilateral AD may be equal to

- (a) AB
(b) BC
(c) DC
(d) None of the above

Type 5. Match the Columns

141. Match the statements of Column I with the values of Column II.

Column I	Column II
A. In a $\triangle ABC$, if $a^4 - 2(b^2 + c^2)a^2 + b^4 + b^2c^2 + c^4 = 0$, then $\angle A$ is	p. 30°
B. In a $\triangle ABC$, if $a^4 + b^4 + c^4 = a^2b^2 + 2b^2c^2 + 2c^2a^2$, then $\angle C$ is	q. 60°
C. In a $\triangle ABC$, if $a^4 + b^4 + c^4 + 2a^2c^2 = 2a^2b^2 + 2b^2c^2$, then $\angle B$ is	r. 90°
	s. 120°
	t. 150°

142. If r_1, r_2, r_3 represent the radii of the escribed circles opposite to angles A, B and C , respectively of a $\triangle ABC$ and r and R represent the inradius and circumradius of the same triangle respectively, then match the following:

Column I	Column II
A. In a $\triangle ABC$, if $R = 40$, then r can be equal to	p. $\frac{1}{2009}$
B. If $\frac{r_1 + r_2 + r_3}{r} = \lambda$, then λ can be equal to	q. 25
C. If $\frac{r_1 r_2 r_3}{r^3} = \mu$, then μ can be equal to	r. 10
D. In $\triangle ABC$, angle A is obtuse, then $\tan B \cdot \tan C$ can be equal to	s. 2009

Type 6. Single Integer Answer Type Questions

143. In a $\triangle ABC$, if a is the arithmetic mean and b, c are two geometric means between any two positive number.

Then, $\frac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C}$ is equal to _____.

144. In a $\triangle ABC$, if $A = 60^\circ$, then $\frac{b}{c+a} + \frac{c}{a+b}$ is equal to _____.

145. In a triangle of the base a the ratio of the other sides is $r(<1)$ and the altitude of the triangle is less than or equal to $\frac{ar}{\lambda - \mu r^2}$, then $\lambda - \mu$ is _____.

146. In $\triangle ABC$, $\sin A \sin B + \sin B \sin C + \sin C \sin A = \frac{9}{4}$ and $a = 2$, then the value of $\sqrt{3}\Delta$, where Δ is area of triangle, is _____.

147. In any $\triangle ABC$, if $\sin A, \sin B$, and $\sin C$ are in AP and the maximum value of $\tan \frac{B}{2} = \sqrt{\frac{\lambda}{\mu}}$, then $\lambda + \mu$ is _____.

148. The incircle of $\triangle ABC$ touches BC at P , AC at Q and AB at R . If G is foot of perpendicular from P to QR , then $\frac{RG}{QG} \cdot \frac{CQ}{BR}$ is equal to _____.

Entrances Gallery

JEE Advanced/IIT JEE

- In a triangle, the sum of two sides is x and the product of the same two sides is y . If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the inradius to the circumradius of the triangle is [2014]
 (a) $\frac{3y}{2x(x+c)}$ (b) $\frac{3y}{2c(x+c)}$ (c) $\frac{3y}{4x(x+c)}$ (d) $\frac{3y}{4c(x+c)}$
- In a $\triangle PQR$, P is the largest angle and $\cos P = \frac{1}{3}$. Further, the incircle of the triangle touches the sides PQ , QR and RP at N , L and M respectively, such that the lengths of PN , QL and RM are consecutive even integers. Then, possible length(s) of the side(s) of the triangle is/are [2013]
 (a) 16 (b) 18
 (c) 24 (d) 22
- Let $\triangle PQR$ be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a , b and c are the lengths of the sides of the triangle opposite to the angles at P , Q and R respectively. Then, $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals [2012]
 (a) $\frac{3}{4\Delta}$ (b) $\frac{45}{4\Delta}$ (c) $\left(\frac{3}{4\Delta}\right)^2$ (d) $\left(\frac{45}{4\Delta}\right)^2$
- If the angles A , B and C of a triangle are in an arithmetic progression and if a , b and c denote the lengths of the sides opposite to A , B and C , respectively, then the value of the expression $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ is [2010]
 (a) $\frac{1}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) 1 (d) $\sqrt{3}$
- Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and a , b and c denote the lengths of the sides opposite to A , B and C , respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is(are) [2010]
 (a) $-(2 + \sqrt{3})$
 (b) $1 + \sqrt{3}$
 (c) $2 + \sqrt{3}$
 (d) $4\sqrt{3}$
- Consider a $\triangle ABC$ and let a , b and c denote the lengths of the sides opposite to vertices A , B and C , respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to _____. [2010]

JEE Main/AIEEE

- If the angles of elevation of the top of a tower from three collinear points A , B and C on a line leading to the foot of the tower are 30° , 45° and 60° respectively, then the ratio $AB : BC$ is [2015]
 (a) $\sqrt{3} : 1$ (b) $\sqrt{3} : \sqrt{2}$
 (c) $1 : \sqrt{3}$ (d) $2 : 3$
- A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45° . It flies off horizontally straight away from the point O . After one second, the elevation of the bird from O is reduced to 30° . Then, the speed (in m/s) of the bird is [2014]
 (a) $40(\sqrt{2} - 1)$
 (b) $40(\sqrt{3} - \sqrt{2})$
 (c) $20\sqrt{2}$
 (d) $20(\sqrt{3} - 1)$
- The x -coordinates of the incentre of the triangle that has the coordinates of mid-points of its sides as $(0, 1)$, $(1, 1)$ and $(1, 0)$ is [2013]
 (a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$ (c) $1 + \sqrt{2}$ (d) $1 - \sqrt{2}$
- $ABCD$ is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to [2013]
 (a) $\frac{(p^2 + q^2)\sin\theta}{p\cos\theta + q\sin\theta}$
 (b) $\frac{p^2 + q^2\cos\theta}{p\cos\theta + q\sin\theta}$
 (c) $\frac{p^2 + q^2}{p^2\cos\theta + q^2\sin\theta}$
 (d) $\frac{(p^2 + q^2)\sin\theta}{(p\cos\theta + q\sin\theta)^2}$
- For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is [2010]
 (a) there is a regular polygon with $\frac{r}{R} = \frac{1}{2}$
 (b) there is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$
 (c) there is a regular polygon with $\frac{r}{R} = \frac{2}{3}$
 (d) there is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$

- ## Other Engineering Entrances

- 561

- 29.** In any $\triangle ABC$,

$$\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2}$$
 equals [EAMCET 2014]
 (a) $\sin^2 B$ (b) $\cos^2 A$ (c) $\cos^2 B$ (d) $\sin^2 A$
- 30.** If the angles of a triangle are in the ratio 1:1:4, then the ratio of the perimeter of the triangle to its largest side is [EAMCET 2014]
 (a) $\sqrt{2} + 2 : \sqrt{3}$ (b) 3 : 2
 (c) $\sqrt{3} + 2 : \sqrt{2}$ (d) $\sqrt{3} + 2 : \sqrt{3}$
- 31.** If angles A, B and C of a $\triangle ABC$ are in AP and $b : c = \sqrt{3} : \sqrt{2}$, then $\angle A$ is given by [EAMCET 2014]
 (a) 45° (b) 60° (c) 75° (d) 90°
- 32.** If in a $\triangle ABC$, $r_1 = 2$, $r_2 = 3$ and $r_3 = 6$, then a equals [EAMCET 2014]
 (a) 4 (b) 1
 (c) 2 (d) 3
- 33.** If in a $\triangle ABC$, $a \tan A + b \tan B = (a+b) \tan \frac{(A+B)}{2}$, then [BITSAT 2013]
 (a) $A = B = C$ (b) $C = A$
 (c) $A = B$ (d) $B = C$
- 34.** $a^3 \cos(B-C) + b^3 \cos(C-A) + c^3 \cos(A-B)$ is equal to [Manipal 2013]
 (a) $3abc$ (b) $3(a+b+c)$
 (c) $abc(a+b+c)$ (d) 0
- 35.** If in a $\triangle ABC$, $\frac{a}{\cos A} = \frac{b}{\cos B}$, then [Karnataka CET 2013]
 (a) $\sin^2 A + \sin^2 B = \sin^2 C$
 (b) $2 \sin A \cos B = \sin C$
 (c) $2 \sin A \sin B \sin C = 1$
 (d) None of the above
- 36.** The area of a circle and the area of a regular polygon of n sides and of perimeter equal to that of the circle are in the ratio [OJEE 2013]
 (a) $\tan \frac{\pi}{n} : \frac{\pi}{n}$ (b) $\cos \frac{\pi}{n} : \frac{\pi}{n}$
 (c) $\sin \frac{\pi}{n} : \frac{\pi}{n}$ (d) $\cot \frac{\pi}{n} : \frac{\pi}{n}$
- 37.** $ABCD$ is a square plot. The angle of elevation of the top of a pole standing at D from A or C is 30° and that from B is θ , then $\tan \theta$ is equal to [Karnataka CET 2013]
 (a) $\sqrt{6}$ (b) $\frac{1}{\sqrt{6}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{2}}{3}$
- 38.** A verticle pole PO is standing at the centre O of a square $ABCD$. If AC subtends an $\angle 90^\circ$ at the top P of the pole, then the angle subtended by a side of the square at P is [UP SEE 2013]
 (a) 30° (b) 45°
 (c) 60° (d) None of these
- 39.** If $A + B + C = 180^\circ$, then $\frac{\cot A + \cot B + \cot C}{\cot A \cot B \cot C}$ is equal to [BITSAT 2012]
 (a) 1 (b) $\cot A \cot B \cot C$
 (c) -1 (d) 0
- 40.** The base BC of a $\triangle ABC$ is 6 cm and $\angle B = 112.5^\circ$, $\angle C = 22.5^\circ$, then its altitude is [MP PET 2012]
 (a) 12 cm (b) 6 cm
 (c) 1.5 cm (d) 3 cm
- 41.** Let p, q and r be the sides opposite to the angles P, Q and R , respectively in a $\triangle PQR$. If $r^2 \sin P \sin Q = pq$, then the triangle is [WB JEE 2012]
 (a) equilateral
 (b) acute angled but not equilateral
 (c) obtuse angle
 (d) right angled
- 42.** In any $\triangle ABC$, the simplified form of $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2}$ is [Karnataka CET 2011]
 (a) $a^2 - b^2$ (b) $\frac{1}{a^2 - b^2}$ (c) $\frac{1}{a^2} - \frac{1}{b^2}$ (d) $a^2 + b^2$
- 43.** If in a $\triangle ABC$, $\sin A, \sin B$ and $\sin C$ are in AP, then [WB JEE 2011]
 (a) the altitudes are in AP (b) the altitudes are in HP
 (c) the angles are in AP (d) the angles are in HP
- 44.** Let C be right angle of a $\triangle ABC$, then $\frac{\sin^2 A}{\sin^2 B} - \frac{\cos^2 A}{\cos^2 B}$ is equal to [J&K CET 2011]
 (a) $\frac{a^2 - b^2}{ab}$ (b) $\frac{a^4 - b^4}{a^2 b^2}$ (c) $\frac{a^4 + b^4}{a^2 b^2}$ (d) $\frac{a^2 + b^2}{ab}$
- 45.** In a $\triangle ABC$, $a = 8$ cm, $b = 10$ cm, $c = 12$ cm. Then, relation between angles of the triangle is [J&K CET 2011]
 (a) $C = A + B$ (b) $C = 2B$
 (c) $C = 2A$ (d) $C = 3A$
- 46.** A ladder rests against a wall, so that its top touches the roof of the house. If the ladder makes an angle of 60° with the horizontal and height of the house be $6\sqrt{3}$ m, then the length of the ladder is [BITSAT 2011]
 (a) $12\sqrt{3}$ m (b) 12 m
 (c) $\frac{12}{\sqrt{3}}$ m (d) None of these
- 47.** A man from the top of a 100 m high tower sees a car moving towards the tower at an angle of depression of 30° . After sometime, the angle of depression becomes 60° . The distance travelled by the car during this is [MP PET 2011]
 (a) $100\sqrt{3}$ m (b) $\frac{200\sqrt{3}}{3}$ m
 (c) $\frac{100\sqrt{3}}{3}$ m (d) $200\sqrt{3}$ m

48. The angles of elevation of the top of a tower at two points, which are at distances a and b from the foot in the same horizontal line and on the same sides of the tower, are complementary. The height of the tower is

[MP PET 2011]

- (a) ab (b) $\sqrt{ab}$ (c) $\sqrt{\frac{a}{b}}$ (d) $\sqrt{\frac{b}{a}}$

49. Angles of elevation of the top of a tower from three points (collinear) A, B and C on a road leading to the foot of the tower are $30^\circ, 45^\circ$ and 60° , respectively. The ratio of AB to BC is

[Karnataka CET 2011]

- (a) $\sqrt{3} : 1$ (b) $\sqrt{3} : 2$ (c) $1 : 2$ (d) $2 : \sqrt{3}$

50. If angles A, B and C are in AP, then $\frac{a+c}{b}$ is equal to

[VITEEE 2010]

- (a) $2 \sin \left(\frac{A-C}{2} \right)$
 (b) $2 \cos \left(\frac{A-C}{2} \right)$
 (c) $\cos \left(\frac{A-C}{2} \right)$
 (d) $\sin \left(\frac{A-C}{2} \right)$

Answers

Work Book Exercise 10.1

1. (b) 2. (b) 3. (d) 4. (a) 5. (b) 6. (b) 7. (a) 8. (d) 9. (c) 10. (b)

Work Book Exercise 10.2

1. (d) 2. (a) 3. (c) 4. (c) 5. (d) 6. (b) 7. (b) 8. (a) 9. (a) 10. (b)

Work Book Exercise 10.3

1. (c) 2. (c) 3. (a) 4. (a) 5. (d) 6. (a) 7. (d) 8. (c) 9. (b) 10. (d)

Target Exercises

1. (a) 2. (a) 3. (c) 4. (c) 5. (c) 6. (c) 7. (c) 8. (d) 9. (d) 10. (b)
 11. (c) 12. (b) 13. (b) 14. (c) 15. (b) 16. (a) 17. (d) 18. (b) 19. (b) 20. (c)
 21. (c) 22. (c) 23. (b) 24. (a) 25. (a) 26. (b) 27. (a) 28. (b) 29. (a) 30. (a)
 31. (c) 32. (a) 33. (c) 34. (d) 35. (a) 36. (c) 37. (c) 38. (c) 39. (a) 40. (b)
 41. (c) 42. (c) 43. (a) 44. (d) 45. (c) 46. (b) 47. (d) 48. (c) 49. (b) 50. (a)
 51. (c) 52. (c) 53. (b) 54. (b) 55. (d) 56. (a) 57. (a) 58. (b) 59. (a) 60. (c)
 61. (a) 62. (b) 63. (c) 64. (c) 65. (a) 66. (b) 67. (c) 68. (c) 69. (a) 70. (c)
 71. (c) 72. (c) 73. (b) 74. (b) 75. (a) 76. (d) 77. (b) 78. (b) 79. (a) 80. (d)
 81. (a) 82. (b) 83. (c) 84. (c) 85. (c) 86. (b) 87. (a) 88. (d) 89. (c) 90. (a)
 91. (a) 92. (a) 93. (c) 94. (a) 95. (a) 96. (c) 97. (d) 98. (a) 99. (a) 100. (c)
 101. (b) 102. (d) 103. (b) 104. (a) 105. (c) 106. (a) 107. (a) 108. (d) 109. (d) 110. (d)
 111. (c) 112. (b) 113. (d) 114. (d) 115. (b) 116. (a) 117. (a) 118. (b,c) 119. (a,d) 120. (a,b)
 121. (a,b) 122. (a,b,c) 123. (a,b,c) 124. (b) 125. (c) 126. (d) 127. (b) 128. (a) 129. (a) 130. (b)
 131. (c) 132. (a) 133. (a) 134. (a) 135. (b) 136. (a) 137. (c) 138. (c) 139. (b) 140. (a)
 141. (*) 142. (**) 143. (2) 144. (1) 145. (0) 146. (3) 147. (4) 148. (1)

* $A \rightarrow q, s; B \rightarrow p, t; C \rightarrow r$

** $A \rightarrow p, r; B \rightarrow q, r, s; C \rightarrow s; D \rightarrow p$

Entrances Gallery

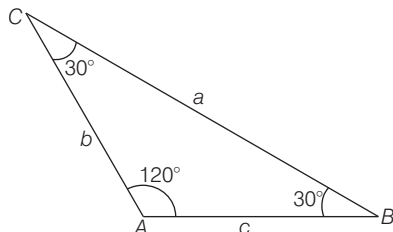
1. (b) 2. (b,d) 3. (c) 4. (d) 5. (b) 6. (3) 7. (a) 8. (d) 9. (b) 10. (a)
 11. (c) 12. (c) 13. (c) 14. (c) 15. (c) 16. (c) 17. (c) 18. (a) 19. (b) 20. (a)
 21. (c) 22. (b) 23. (b) 24. (a) 25. (b) 26. (d) 27. (c) 28. (a) 29. (d) 30. (d)
 31. (c) 32. (d) 33. (c) 34. (a) 35. (b) 36. (a) 37. (b) 38. (c) 39. (a) 40. (d)
 41. (d) 42. (c) 43. (b) 44. (b) 45. (c) 46. (b) 47. (b) 48. (b) 49. (a) 50. (b)

Explanations

Target Exercises

1. Here, ratio of angles are 4 : 1 : 1.

$$\begin{aligned} \Rightarrow 4x + x + x &= 180^\circ \\ \Rightarrow x &= 30^\circ \\ \therefore \angle A &= 120^\circ, \angle B = \angle C = 30^\circ \end{aligned}$$



Thus, the ratio of longest side to the perimeter

$$= \frac{a}{a+b+c}$$

$$\begin{aligned} \text{Let } b=c &= x \\ \therefore a^2 &= b^2 + c^2 - 2bc \cos A \\ \Rightarrow a^2 &= 2x^2 - 2x^2 \cos A = 2x^2(1 - \cos A) \\ \Rightarrow a^2 &= 4x^2 \sin^2 \frac{A}{2} \\ \Rightarrow a &= 2x \sin \frac{A}{2} \\ \Rightarrow a &= 2x \sin 60^\circ = \sqrt{3}x \end{aligned}$$

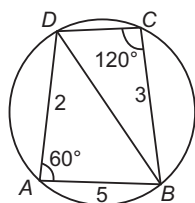
Thus, required ratio is

$$\frac{a}{a+b+c} = \frac{\sqrt{3}x}{x+x+\sqrt{3}x} = \frac{\sqrt{3}}{2+\sqrt{3}}$$

2. In $\triangle ABD$,

$$\cos 60^\circ = \frac{2^2 + 5^2 - BD^2}{2(2)(5)}$$

$$\Rightarrow BD^2 = 19$$



Now, in $\triangle BCD$,

$$\cos 120^\circ = \frac{CD^2 + 9 - 19}{(2)(3)(CD)}$$

$$\Rightarrow CD^2 + 3CD - 10 = 0$$

$$\Rightarrow CD = -5, 2$$

$$\Rightarrow CD = 2 \quad [\because CD \neq -5]$$

$$\begin{aligned} 3. (a-b)^2 \cos^2 \frac{C}{2} + (a+b)^2 \sin^2 \frac{C}{2} \\ &= (a^2 + b^2 - 2ab) \cos^2 \frac{C}{2} + (a^2 + b^2 + 2ab) \sin^2 \frac{C}{2} \\ &= a^2 + b^2 + 2ab \left(\sin^2 \frac{C}{2} - \cos^2 \frac{C}{2} \right) \\ &= a^2 + b^2 - 2ab \cos C \end{aligned}$$

$$\begin{aligned} &= a^2 + b^2 - (a^2 + b^2 - c^2) \\ &= c^2 \quad \left[\because \cos C = \frac{a^2 + b^2 - c^2}{2ab} \right] \end{aligned}$$

$$4. \text{ We have, } \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow \cos 60^\circ = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow a^2 + b^2 - c^2 = ab$$

$$\Rightarrow b^2 + bc + a^2 + ac = ab + ac + bc + c^2$$

$$\Rightarrow b(b+c) + a(a+c) = (a+c)(b+c)$$

On dividing by $(a+c)(b+c)$ and add 2 on both sides, we get

$$1 + \frac{b}{a+c} + 1 + \frac{a}{b+c} = 3$$

$$\Rightarrow \frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$$

5. We have, $4\Delta (\cot A + \cot B + \cot C)$

$$\begin{aligned} &= 4 \cdot \frac{1}{2} bc \sin A \cdot \frac{\cos A}{\sin A} + 4 \cdot \frac{1}{2} ca \sin B \cdot \frac{\cos B}{\sin B} \\ &\quad + 4 \cdot \frac{1}{2} ab \sin C \cdot \frac{\cos C}{\sin C} \end{aligned}$$

$$= 2bc \cos A + 2ca \cos B + 2ab \cos C$$

$$\begin{aligned} &= 2bc \cdot \frac{b^2 + c^2 - a^2}{2bc} + 2ca \cdot \frac{c^2 + a^2 - b^2}{2ca} \\ &\quad + 2ab \cdot \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

$$\begin{aligned} &= b^2 + c^2 - a^2 + c^2 + a^2 - b^2 + a^2 + b^2 - c^2 \\ &= a^2 + b^2 + c^2 \end{aligned}$$

$$6. \text{ We have, } \frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$$

$$\Rightarrow \frac{\cos A}{k \sin A} = \frac{\cos B}{k \sin B} = \frac{\cos C}{k \sin C}$$

$$\Rightarrow \cot A = \cot B = \cot C \Rightarrow A = B = C = 60^\circ$$

$\therefore$ The triangle is equilateral.

7. We have,

$$A + B + C = 90^\circ, A - B = 30^\circ \text{ and } A + C = 60^\circ$$

$$\Rightarrow B = 30^\circ, A = 60^\circ, C = 0^\circ$$

$$\begin{aligned} 8. \frac{3 + \cot 76^\circ \cot 16^\circ}{\cot 76^\circ + \cot 16^\circ} &= \frac{3 + \frac{\cos 76^\circ \cos 16^\circ}{\sin 76^\circ \sin 16^\circ}}{\frac{\cos 76^\circ}{\sin 76^\circ} + \frac{\cos 16^\circ}{\sin 16^\circ}} \\ &= \frac{3 \sin 76^\circ \sin 16^\circ + \cos 76^\circ \cos 16^\circ}{\cos 76^\circ \sin 16^\circ + \sin 76^\circ \cos 16^\circ} \\ &= \frac{2 \sin 76^\circ \sin 16^\circ + \cos (76^\circ - 16^\circ)}{\sin (76^\circ + 16^\circ)} \\ &= \frac{2 \sin 76^\circ \sin 16^\circ + \frac{1}{2} (\cos 60^\circ - \cos 92^\circ) + \frac{1}{2}}{\sin 92^\circ} \\ &= \frac{1 - \cos 92^\circ}{\sin 92^\circ} = \frac{2 \sin^2 46^\circ}{2 \sin 46^\circ \cos 46^\circ} = \tan 46^\circ = \cot 44^\circ \end{aligned}$$

9. Sides are in AP and $a < \min\{b, c\}$.

$\Rightarrow$ Order of AP can be b, c, a or c, b, a .

Case I When $2c = a + b$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2c - b)^2}{2bc}$$

$$= \frac{4bc - 3c^2}{2bc} = \frac{4b - 3c}{2b}$$

Case II $2b = a + c$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + c^2 - (2b - c)^2}{2bc}$$

$$= \frac{4bc - 3b^2}{2bc} = \frac{4c - 3b}{2c}$$

10. Clearly, $a + b > c$

$$\Rightarrow (\sqrt{a} + \sqrt{b})^2 - 2\sqrt{ab} > (\sqrt{c})^2$$

$$\Rightarrow (\sqrt{a} + \sqrt{b})^2 > (\sqrt{c})^2$$

$$\Rightarrow \sqrt{a} + \sqrt{b} > \sqrt{c}$$

$$\Rightarrow \sqrt{a} + \sqrt{b} - \sqrt{c} > 0$$

11. $a + b > c \Rightarrow 2(b - c) + b > 0 \Rightarrow 3b > 2c$

$$\Rightarrow \frac{b}{c} > \frac{2}{3}$$

$$\text{Also, } b + c > a \Rightarrow b + c > 2b - c \Rightarrow 2c > b$$

$$\Rightarrow \frac{b}{c} < 2$$

$$\text{and } c + a > b \Rightarrow c = 2a - c > b \Rightarrow 2b > b$$

$$\text{Thus, } \frac{b}{c} \in \left(\frac{2}{3}, 2\right)$$

12. Here, $a + b = 2\sqrt{3}$, $ab = 2$ and $C = \frac{\pi}{3}$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab} \Rightarrow \frac{1}{2} = \frac{a^2 + b^2 - c^2}{2ab}$$

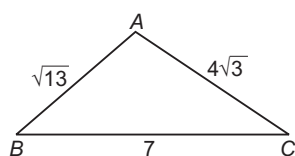
$$\Rightarrow a^2 + b^2 - c^2 = ab$$

$$\Rightarrow (a + b)^2 - 2ab - c^2 = ab$$

$$\Rightarrow 12 - 4 - c^2 = 2 \text{ or } c = \sqrt{6}$$

$$\therefore \text{The perimeter of triangle} = a + b + c = 2\sqrt{3} + \sqrt{6}$$

13. $\cos \theta = \frac{49 + 48 - 13}{2 \cdot 28\sqrt{3}} = \frac{\sqrt{3}}{2}$



$$\Rightarrow \theta = 30^\circ$$

$$\Rightarrow \sin \theta = \frac{1}{2}$$

14. $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$
- $$= \frac{b^2 + c^2 - a^2}{2abc} + \frac{c^2 + a^2 - b^2}{2abc} + \frac{a^2 + b^2 - c^2}{2abc}$$
- $$= \frac{a^2 + b^2 + c^2}{2abc}$$
- $$= \frac{(a + b + c)^2 - 2(ab + bc + ca)}{2abc}$$
- $$= \frac{11^2 - 2 \cdot 38}{2 \cdot 40} = \frac{9}{16}$$

15. $2 \cos A \sin C = \sin(C + A)$

$$= \sin C \cdot \cos A + \cos C \cdot \sin A$$

$$\Rightarrow \cos A \sin C - \sin A \cos C = 0$$

$$\Rightarrow \sin(C - A) = 0$$

$$\Rightarrow C = A$$

16. We have, $\frac{2 \cos A}{a} + \frac{\cos B}{b} + \frac{2 \cos C}{c} = \frac{a}{bc} + \frac{b}{ca}$
- $$\Rightarrow \frac{2(b^2 + c^2 - a^2)}{2bca} + \frac{c^2 + a^2 - b^2}{2bac} + \frac{2(a^2 + b^2 - c^2)}{2abc}$$
- $$= \frac{a^2 + b^2}{abc}$$
- $$\Rightarrow 2b^2 + 2c^2 - 2a^2 + c^2 + a^2 - b^2 + 2a^2 + 2b^2 - 2c^2 = 2a^2 + 2b^2$$

$$\Rightarrow 3b^2 + c^2 + a^2 = 2a^2 + 2b^2$$

$$\Rightarrow b^2 + c^2 = a^2$$

$$\text{Hence, } \angle A = 90^\circ$$

17. Given equation becomes $(b + c)^2 - a^2 = \lambda bc$

$$\Rightarrow b^2 + c^2 - a^2 = (\lambda - 2)bc$$

$$\Rightarrow \frac{b^2 + c^2 - a^2}{2bc} = \frac{\lambda - 2}{2}$$

$$\Rightarrow \frac{\lambda - 2}{2} = \cos A$$

$$\text{But } -1 < \cos A < 1$$

$$\therefore -1 < \frac{\lambda - 2}{2} < 1$$

$$\Rightarrow 0 < \lambda < 4$$

18. $\cos^2 B + \cos^2 C = \cos^2 B + \cos^2\left(\frac{\pi}{2} - B\right)$
- $$= \cos^2 B + \sin^2 B = 1$$

19. We have,

$$\frac{b^2 - c^2}{a} \cdot \cos A + \frac{c^2 - a^2}{b} \cdot \cos B + \frac{a^2 - b^2}{c} \cdot \cos C$$

$$= \frac{b^2 - c^2}{a} \cdot \frac{b^2 + c^2 - a^2}{2bc} + \frac{c^2 - a^2}{b} \cdot \frac{c^2 + a^2 - b^2}{2ac}$$

$$+ \frac{a^2 - b^2}{c} \cdot \frac{a^2 + b^2 - c^2}{2ab}$$

$$= 0$$

20. We have, $a^2 = b^2 + c^2 - 2bc \cos A$

$$\Rightarrow c^2 - 2bc \cos A + (b^2 - a^2) = 0$$

Clearly, c_1, c_2 are the roots of this equation such that $c_1 + c_2 = 2ab \cos A$ and $c_1 c_2 = b^2 - a^2$

$$\therefore c_1 - c_2 = \sqrt{(c_1 + c_2)^2 - 4c_1 c_2}$$

$$= \sqrt{4b^2 \cos^2 A - 4(b^2 - a^2)}$$

$$= 2\sqrt{a^2 - b^2(1 - \cos^2 A)}$$

$$= 2\sqrt{a^2 - b^2 \sin^2 A}$$

21. $\cos^2 A + \cos^2 B - \cos^2 C = 1$

$$\Rightarrow 1 - \sin^2 A + 1 - \sin^2 B - 1 + \sin^2 C = 1$$

$$\Rightarrow \sin^2 A + \sin^2 B = \sin^2 C$$

$$\Rightarrow a^2 + b^2 = c^2$$

Thus, triangle is right angled at C.

34. $2b = a + c$

$$\begin{aligned} \Rightarrow 2 \sin B &= \sin A + \sin C \\ \Rightarrow 2 \sin \frac{B}{2} &= \cos \left(\frac{A-C}{2} \right) \\ \Rightarrow \cos \frac{A}{2} \cdot \cos \frac{C}{2} &= 3 \sin \frac{A}{2} \cdot \sin \frac{C}{2} \\ \Rightarrow \tan \frac{A}{2} \cdot \tan \frac{C}{2} &= \frac{1}{3} \\ \therefore \frac{\tan \frac{A}{2} + \tan \frac{C}{2}}{\cot \frac{B}{2}} &= \frac{\cos \frac{B}{2}}{\cos \frac{A}{2} \cdot \cos \frac{C}{2}} \cdot \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} \\ &= \frac{\cos \left(\frac{A+C}{2} \right)}{\cos \frac{A}{2} \cdot \cos \frac{C}{2}} \\ &= 1 - \tan \frac{A}{2} \cdot \tan \frac{C}{2} = 1 - \frac{1}{3} = \frac{2}{3} \end{aligned}$$

35. $1 - \tan \frac{B}{2} \cdot \tan \frac{C}{2}$

$$\begin{aligned} &= 1 - \sqrt{\frac{(s-a)(s-c)}{s(s-b)} \cdot \frac{(s-a)(s-b)}{s(s-c)}} \\ &= 1 - \frac{s-a}{s} = \frac{a}{s} = \frac{2a}{a+b+c} \end{aligned}$$

36. $a = (1+x)\lambda, b = \lambda, c = (1-x)\lambda, \frac{A}{2} - \frac{C}{2} = \frac{\pi}{4}$

$$\begin{aligned} \Rightarrow \cos \frac{A}{2} \cdot \cos \frac{C}{2} + \sin \frac{A}{2} \cdot \sin \frac{C}{2} &= \frac{1}{\sqrt{2}} \\ \Rightarrow \sqrt{\frac{s^2(s-a)(s-c)}{bc \cdot ab}} &+ \sqrt{\frac{(s-b)(s-c)}{bc} \cdot \frac{(s-a)(s-b)}{ab}} = \frac{1}{\sqrt{2}} \\ \Rightarrow \frac{s}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} + \frac{(s-b)}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} &= \frac{1}{\sqrt{2}} \\ \Rightarrow \left(\frac{2s-b}{b} \right) \sqrt{\frac{(s-a)(s-c)}{ac}} &= \frac{1}{\sqrt{2}} \\ \Rightarrow \frac{a+c}{b} \sqrt{\frac{(s-a)(s-c)}{ac}} &= \frac{1}{\sqrt{2}} \\ \Rightarrow 2 \left(\frac{a+c}{b} \right)^2 &= \frac{ac}{(s-a)(s-c)} \\ a+c &= 2\lambda, b = \lambda \\ \Rightarrow \frac{a+c}{b} = 2, s &= \frac{a+b+c}{2} = \frac{3\lambda}{2} \\ s-a &= \frac{(1-2x)\lambda}{2}, (s-c) = \frac{(1+2x)\lambda}{2} \\ \Rightarrow 8 &= \frac{(1-x^2)4}{(1-4x^2)} \Rightarrow x = \frac{1}{\sqrt{7}} \end{aligned}$$

37. Since, $AM \geq GM$

$$\therefore \frac{b+c}{2} \geq \sqrt{bc} \Rightarrow b+c \geq 2\lambda \quad \dots(i)$$

Also, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \frac{b+c}{\sin B + \sin C}$

$$\Rightarrow \frac{a}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{b+c}{2 \sin \left(\frac{B+C}{2} \right) \cos \left(\frac{B-C}{2} \right)}$$

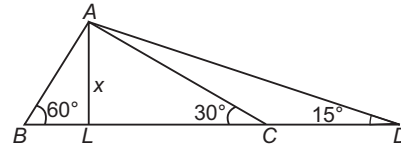
$$\Rightarrow a = \frac{(b+c) \sin A/2}{\cos \left(\frac{B-C}{2} \right)} \quad \dots(ii)$$

Since, $\cos \left(\frac{B-C}{2} \right)$ is always positive.

From Eqs. (i) and (ii), we have

$$a \geq 2\lambda \sin \frac{A}{2}$$

38. $BC = x (\cot 60^\circ + \cot 30^\circ)$
 $= (\sqrt{3} + 1/\sqrt{3})x = (4/\sqrt{3})x$
 $CD = x (\cot 15^\circ - \cot 30^\circ)$
 $= x (2 + \sqrt{3} - \sqrt{3}) = 2x$



$$BC \times CD = (4/\sqrt{3}) \cdot 2x^2 = 2^3 \times 3^2 \times 19 \times 11 \quad [\text{given}]$$

$$\Rightarrow x^2 = \frac{19}{11} \times 3^2 \times 19 \times 11 = (19 \times 3)^2 = 3249$$

39. Since, $c^2 = a^2 + b^2$
 $\therefore \triangle ABC$ is right angled with $\angle C = 90^\circ$.
 $\therefore$ Area of triangle $= \frac{1}{2} ab$
 $\Rightarrow \sqrt{s(s-a)(s-b)(s-c)} = \frac{1}{2} ab$
 $\Rightarrow 4s(s-a)(s-b)(s-c) = a^2 b^2$

40. $(a+b-c)(b+c-a)(c+a-b) - abc$
 $= 8(s-c)(s-a)(s-b) - abc$
 $= 8 \cdot \frac{\Delta^2}{s} - R \cdot 4\Delta = 4\Delta \left(\frac{2\Delta}{s} - R \right)$
 $= 4\Delta (2r - R) \leq 0$

41. $a^2 \sin 2B + b^2 \sin 2A$
 $= 2a^2 \sin B \cdot \cos B + 2b^2 \sin A \cdot \cos A$
 $= \frac{a^2 b}{R} \cos B + \frac{b^2 a}{R} \cos A$
 $= \frac{ab}{R} (a \cos B + b \cos A) = \frac{abc}{R} = 2bc \sin A$
 $= 4 \left(\frac{1}{2} bc \sin A \right) = 4\lambda$

42. $\frac{a \cos A + b \cos B + c \cos C}{a+b+c}$
 $= \frac{2R \sin A \cos A + 2R \sin B \cos B + 2R \sin C \cos C}{2s}$
 $= \frac{R}{2s} (\sin 2A + \sin 2B + \sin 2C)$
 $= \frac{R}{2s} \cdot 4 \sin A \sin B \sin C$
 $= \frac{2R}{s} \cdot \frac{abc}{8R^3} = \frac{abc}{4sR^2}$

But $R = \frac{abc}{4\Delta}, r = \frac{\Delta}{s}$

So, the value $= \frac{4\Delta R}{4 \cdot \frac{\Delta}{r} \cdot R^2} = \frac{r}{R}$

$$43. \frac{s}{R} = \frac{(a+b+c)}{2R} = \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R} \\ = \sin A + \sin B + \sin C$$

$$44. \text{As, } \frac{r}{R} = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ \Rightarrow \frac{r}{R} = 2 \left\{ \cos \left(\frac{A+B}{2} \right) - \cos \left(\frac{A+B}{2} \right) \right\} \cdot \sin \frac{C}{2} \\ \Rightarrow \frac{r}{R} = 2 \left[\cos \left(\frac{A+B}{2} \right) - \sin \left(\frac{C}{2} \right) \right] \cdot \sin \frac{C}{2} \\ \Rightarrow \frac{r}{R} \leq 2 \left(1 - \sin \frac{C}{2} \right) \sin \frac{C}{2} \\ \Rightarrow \frac{r}{R} \leq 2 \left(\sin \frac{C}{2} - \sin^2 \frac{C}{2} \right) \\ \Rightarrow \frac{r}{R} \leq 2 \left\{ \frac{1}{4} - \left(\frac{1}{2} - \sin \frac{C}{2} \right)^2 \right\} \\ \Rightarrow \frac{r}{R} \leq \frac{2}{4} \text{ or } \frac{R}{r} \geq 2 \text{ but } \frac{R}{r} \leq 2 \\ \Rightarrow \frac{R}{r} = 2$$

$$\text{i.e. if } \frac{A-B}{2} = 0 \text{ and } \frac{1}{2} - \sin \frac{C}{2} = 0 \\ \Rightarrow A = B \text{ and } C = 60^\circ \\ \Rightarrow A = B = C = 60^\circ$$

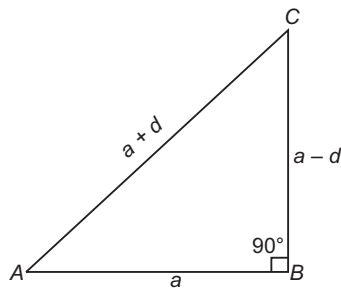
So, triangle is equilateral.

$$45. \text{We have, } \Delta = \Delta_{BPC} + \Delta_{APC} + \Delta_{APB} \\ \Rightarrow \Delta = \frac{1}{2} ax_a + \frac{1}{2} bx_b + \frac{1}{2} cx_c \\ \text{Also, } \Delta = \frac{1}{2} aH_a = \frac{1}{2} bH_b = \frac{1}{2} cH_c \\ \Rightarrow \Delta = \Delta \left(\frac{x_a}{H_a} + \frac{x_b}{H_b} + \frac{x_c}{H_c} \right) \Rightarrow \frac{x_a}{H_a} + \frac{x_b}{H_b} + \frac{x_c}{H_c} = 1$$

$$46. \text{Clearly, } A > B \quad [\because a > b]$$

$$\text{Now, } \tan \left(\frac{A-B}{2} \right) = \frac{a-b}{a+b} \cot \frac{C}{2} \\ \Rightarrow \tan 30^\circ = \frac{1}{3} \cot \frac{C}{2} \\ \therefore \sqrt{3} = \cot \frac{C}{2} \Rightarrow \frac{C}{2} = \frac{\pi}{6} \Rightarrow C = \frac{\pi}{3}$$

$$47. \text{Let sides of the given triangle be } a-d, a, a+d. \\ \text{Now, } (a+d)^2 = (a-d)^2 + a^2$$



$$\text{or } 4ad = a^2 \\ \Rightarrow a = 4d \\ \text{Now, } \tan A = \frac{a-d}{a} = \frac{4d-d}{4d} = \frac{3}{4} \\ \tan C = \frac{a}{a-d} = \frac{4d}{4d-d} = \frac{4}{3}$$

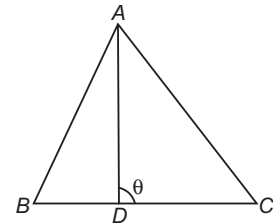
$$48. \sqrt{3} \cos A + \sin A = \sqrt{3} \cos B + \sin B \\ \Rightarrow \frac{\sin A - \sin B}{\cos B - \cos A} = \sqrt{3} \\ \therefore \sqrt{3} = \frac{2 \cos \left(\frac{A+B}{2} \right) \cdot \sin \left(\frac{A-B}{2} \right)}{2 \sin \left(\frac{A+B}{2} \right) \cdot \sin \left(\frac{A-B}{2} \right)} = \cot \frac{A+B}{2} \\ \Rightarrow \frac{A+B}{2} = \frac{\pi}{6} \Rightarrow A+B = \frac{\pi}{3} \Rightarrow C = \pi - \frac{\pi}{3} = \frac{2\pi}{3} \\ \text{So, triangle is an obtuse angled.}$$

$$49. c \cos \theta - a \sin \theta = b \\ \text{Therefore, } c \cos \alpha - a \sin \alpha = c \cos \beta - a \sin \beta, \text{ where } \alpha \text{ and } \beta \text{ are the other two angles of the triangle.} \\ \therefore c (\cos \alpha - \cos \beta) = a (\sin \alpha - \sin \beta) \\ \Rightarrow \frac{c}{a} = \frac{2 \cos \left(\frac{\alpha+\beta}{2} \right) \sin \left(\frac{\alpha-\beta}{2} \right)}{2 \sin \left(\frac{\alpha+\beta}{2} \right) \sin \left(\frac{\beta-\alpha}{2} \right)} = -\cot \left(\frac{\alpha+\beta}{2} \right) \\ \Rightarrow \tan \left(\frac{\alpha+\beta}{2} \right) = \frac{-a}{c} \\ \Rightarrow \tan (\alpha+\beta) = \frac{2 \times (-a/c)}{1 - a^2/c^2} = \frac{2ac}{a^2 - c^2} \\ \Rightarrow \tan \left(\pi - \frac{3\pi}{4} \right) = \frac{2ac}{a^2 - c^2} \\ \therefore a^2 - c^2 = 2ac$$

$$50. \text{Let } a = x, b = y \text{ and } c = \sqrt{x^2 + y^2 + xy} \\ \text{Then, } a^2 = x^2, b^2 = y^2 \text{ and } c^2 = x^2 + y^2 + xy \\ \therefore c \text{ is the length of the greatest side and hence } \angle C \text{ is the greatest angle} \\ \therefore \text{By cosine rule,} \\ \cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{x^2 + y^2 - x^2 - y^2 - xy}{2xy} = -\frac{1}{2} \\ \therefore \angle C = 120^\circ$$

$$51. \text{Since, the sides are positive.} \\ \therefore x > 1 \text{ or else } c \text{ will be negative.} \\ a = x^2 + x + 1 > x + x + 1 = 2x + 1 = b \\ \therefore a > b \\ \text{Also, } a > c. \\ \text{Hence, } a \text{ is the greatest side, so } A \text{ is the greatest angle.} \\ \therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{1}{2} \Rightarrow A = 120^\circ$$

$$52. \theta = B + \frac{A}{2} = \pi - \left(C + \frac{A}{2} \right) \\ \Rightarrow \sin \theta = \sin \left(B + \frac{A}{2} \right) = \sin \left(C + \frac{A}{2} \right) \\ = \sin \left(B + \frac{\pi}{2} - \frac{B+C}{2} \right) \\ = \sin \left(C + \frac{\pi}{2} - \frac{B+C}{2} \right) = \cos \left(\frac{B-C}{2} \right)$$



53. We have, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$
 $\therefore \frac{R}{r} = \frac{abc}{4\Delta} \cdot \frac{s}{\Delta} = \frac{abc}{4(s-a)(s-b)(s-c)}$
 Since, $a : b : c = 4 : 5 : 6$
 $\Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{6} = k$ [say]
 Thus, $\frac{R}{r} = \frac{(4k)(5k)(6k)}{4\left(\frac{15k}{2} - 4k\right)\left(\frac{15k}{2} - 5k\right)\left(\frac{15k}{2} - 6k\right)}$
 $= \frac{120k^3 \cdot 2}{k^2 \cdot 7 \cdot 5 \cdot 3} = \frac{16}{7}$

54. $\therefore r_1 r_2 + r_2 r_3 + r_3 r_1$
 $= \frac{\Delta}{(s-a)} \times \frac{\Delta}{(s-b)} + \frac{\Delta}{s-b} \times \frac{\Delta}{s-c} + \frac{\Delta}{s-c} \times \frac{\Delta}{s-a}$
 $= \frac{\Delta^2}{(s-b)(s-a)} + \frac{\Delta^2}{(s-b)(s-c)} + \frac{\Delta^2}{(s-a)(s-c)}$
 $= \frac{\Delta^2[(s-c) + (s-a) + (s-b)]}{(s-a)(s-b)(s-c)}$
 $= \frac{\Delta^2[3s - (a+b+c)]}{(s-a)(s-b)(s-c)} \times \frac{s}{s} = \frac{\Delta^2 s^2}{\Delta^2} = s^2 = \frac{\Delta^2}{r^2}$

55. The area of an equilateral triangle $= \frac{\sqrt{3}}{4} a^2$, where a is side.

Also, $s = \frac{a+a+a}{2} = \frac{3a}{2}$
 $\therefore$ Inradius, $r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4} a^2 \times 2}{4 \times 3a} = \frac{a}{2\sqrt{3}}$
 Circumradius, $R = \frac{abc}{4\Delta} = \frac{a^3}{\sqrt{3} a^2} = \frac{a}{\sqrt{3}}$
 and exradius, $r_1 = \frac{\Delta}{s-a} = \frac{(\sqrt{3}/4)a^2}{a/2} = \frac{\sqrt{3}}{2} a$
 Hence, $r : R : r_1 = \frac{a}{2\sqrt{3}} : \frac{a}{\sqrt{3}} : \frac{\sqrt{3}}{2} a = 1 : 2 : 3$

56. We know that, $r_1 = \frac{\Delta}{s-a}$, $r_2 = \frac{\Delta}{s-b}$, $r_3 = \frac{\Delta}{s-c}$

Given that, $r_1 > r_2 > r_3$
 $\Rightarrow \frac{\Delta}{s-a} > \frac{\Delta}{s-b} > \frac{\Delta}{s-c}$
 $\Rightarrow \frac{1}{s-a} > \frac{1}{s-b} > \frac{1}{s-c}$
 $\Rightarrow s-a < s-b < s-c$
 [$\because s-a, s-b, s-c$ are positive]
 $\Rightarrow -a < -b < -c \Rightarrow a > b > c$

57. Let area of triangle be Δ , then according to the question,
 $\Delta = \frac{1}{2} ax = \frac{1}{2} by = \frac{1}{2} cz$
 $\therefore \frac{bx}{c} + \frac{cy}{a} + \frac{az}{b} = \frac{b}{c} \left(\frac{2\Delta}{a} \right) + \frac{c}{a} \left(\frac{2\Delta}{b} \right) + \frac{a}{b} \left(\frac{2\Delta}{c} \right)$
 $= \frac{2\Delta(b^2 + c^2 + a^2)}{abc}$
 $= \frac{2(a^2 + b^2 + c^2)}{abc} \cdot \frac{abc}{4R} \left[\because \Delta = \frac{abc}{4R} \right]$
 $= \frac{a^2 + b^2 + c^2}{2R}$

58. Let the sides be $a = 3x, b = 7x, c = 8x$.

Then, $2s = a + b + c \Rightarrow s = 9x$
 $\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$
 $= \sqrt{9x \cdot 6x \cdot 2x \cdot x} = 6\sqrt{3}x^2$

Now, $R = \frac{abc}{4\Delta}$ and $r = \frac{\Delta}{s}$
 $\therefore \frac{R}{r} = \frac{sabc}{4\Delta^2} = \frac{9x \cdot 3x \cdot 7x \cdot 8x}{4 \cdot 108 \cdot x^4} = \frac{7}{2}$

59. If G is the centroid, then we have the following

$BG = CG > a \Rightarrow \frac{2}{3}m_b + \frac{2}{3}m_c > a$

Similarly, $\frac{2}{3}m_a + \frac{2}{3}m_b > c, \frac{2}{3}m_a + \frac{2}{3}m_c > b$

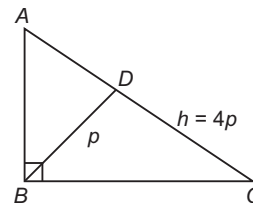
$\Rightarrow \frac{4}{3}(m_a + m_b + m_c) > a + b + c$

$\Rightarrow m_a + m_b + m_c > \frac{3}{4}(a + b + c)$

$\Rightarrow m_a + m_b + m_c > \frac{3}{2}s$

60. Here, $CD = p \cot C$ and $AD = p \cot A = p \tan C$

Adding, $4p = p \cot C + p \tan C$



$\Rightarrow \tan C + \cot C = 4$

$\Rightarrow \frac{1}{\sin C \cdot \cos C} = 4$

$\Rightarrow \sin 2C = \frac{1}{2}$

$\Rightarrow C = 15^\circ$

61. Using $(m+n)\cot\theta = n\cos\beta - m\cot\alpha$, we get

$(3+2)\cot\angle CDA = 2\cot 30^\circ - 3\cot 60^\circ$

$\Rightarrow \cot\angle CDA = \frac{\sqrt{3}}{5} \Rightarrow \sin\angle CDA = \frac{5}{2\sqrt{7}}$

$\therefore \frac{AC}{\sin\angle CDA} = \frac{3}{\sin 60^\circ}$

$\Rightarrow AC = \frac{3 \cdot 2}{\sqrt{3}} \cdot \frac{5}{2\sqrt{7}} = 5\sqrt{\frac{3}{7}}$

62. $\frac{1}{2} \cdot \alpha \cdot a = \Delta$

$\therefore \frac{1}{\alpha} = \frac{a}{2\Delta}$

Similarly, $\frac{1}{\beta} = \frac{b}{2\Delta}, \frac{1}{\gamma} = \frac{c}{2\Delta}$

$\therefore \alpha^{-1} + \beta^{-1} + \gamma^{-1} = \frac{1}{2\Delta}(a+b+c) = \frac{s}{\Delta}$

63. $a^2 + b^2 + c^2 = 8R^2$

$\Rightarrow \sin^2 A + \sin^2 B + \sin^2 C = 2$

$\Rightarrow \cos 2A + \cos 2B + \cos 2C + 1 = 0$

$\therefore -4\cos A \cos B \cos C = 0$

$\Rightarrow A \text{ or } B \text{ or } C = \frac{\pi}{2}$

$$64. r = \frac{\Delta}{s} = \frac{(1/2) \cdot ac}{(1/2) \cdot (a+b+c)} = \frac{ac}{a+b+c}$$

$$= \frac{ac(c+a-b)}{(c+a)^2 - b^2} = \frac{ac(c+a-b)}{c^2 + 2ca + a^2 - b^2}$$

$$= \frac{ac(c+a-b)}{2ca + b^2 - b^2} = \frac{c+a-b}{2} \quad [\because a^2 + c^2 = b^2]$$

$$65. \text{ We have, } 4 \left(\frac{s}{a} - 1 \right) \left(\frac{s}{b} - 1 \right) \left(\frac{s}{c} - 1 \right)$$

$$= 4 \cdot \frac{(s-a)(s-b)(s-c)}{abc}$$

$$= 4 \cdot \frac{s(s-a)(s-b)(s-c)}{s \cdot abc}$$

$$= 4 \cdot \frac{\Delta^2}{s \cdot abc}$$

$$= \frac{4\Delta}{abc} \cdot \frac{\Delta}{s} = \frac{r}{R}$$

$$66. \text{ We have, } \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$$

$$= \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2} + \frac{s^2}{\Delta^2}$$

$$= \frac{1}{\Delta^2} [4s^2 - 2s(a+b+c) + a^2 + b^2 + c^2]$$

$$= \frac{1}{\Delta^2} [4s^2 - 2s \cdot 2s + a^2 + b^2 + c^2]$$

$$= \frac{a^2 + b^2 + c^2}{\Delta^2}$$

$$67. \text{ We have, } r_1 + r_2 + r_3$$

$$= \frac{\Delta}{s} \cdot \frac{\Delta}{s-a} + \frac{\Delta}{s} \cdot \frac{\Delta}{s-b} + \frac{\Delta}{s} \cdot \frac{\Delta}{s-c} = \Delta^2 \left[\frac{1}{s(s-a)} + \frac{1}{(s-b)(s-c)} \right]$$

$$= \Delta^2 \left[\frac{(s-b)(s-c) + s(s-a)}{s(s-a)(s-b)(s-c)} \right]$$

$$= \Delta^2 \left[\frac{2s^2 - s(a+b+c) + bc}{\Delta^2} \right]$$

$$= 2s^2 - s \cdot 2s + bc = bc$$

$$68. \text{ In an equilateral triangle, } a = b = c$$

$$\therefore s = \frac{a+a+a}{2} = \frac{3a}{2}$$

$$r = \frac{\Delta}{s} = \frac{\Delta}{\frac{3a}{2}} = \frac{2\Delta}{3a}$$

$$r_1 = \frac{\Delta}{s-a} = \frac{\Delta}{a/2} = \frac{2\Delta}{a} = 3 \cdot \frac{2\Delta}{3a} = 3r$$

$$r_2 = \frac{\Delta}{s-b} = \frac{\Delta}{s-a} = 3r$$

$$r_3 = \frac{\Delta}{s-c} = \frac{\Delta}{s-a} = 3r$$

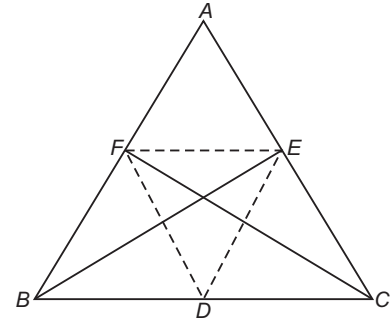
$$\therefore r_1 = r_2 = r_3 = 3r$$

$$69. \text{ Circle on } BC \text{ as diameter passes through } E, F.$$

[since, $\angle BEC = \angle CFB = 90^\circ$]

and radius of this circle = $\frac{a}{2}$

Also, $\angle FBE = 90^\circ - A$
 $FE = 2R = \sin A = a \cos A$
 and $\angle FDE = 180^\circ - 2A$



$\therefore$ Radius of circle DEF

$$= \frac{FE}{2 \sin \angle FDE} = \frac{a \cos A}{2 \sin (180^\circ - 2A)}$$

$$= \frac{a \cos A}{2 \sin 2A} = \frac{2R \sin A \cos A}{2 \cdot 2 \sin A \cos A}$$

$$= \frac{R}{2} \quad \left[\because \frac{a}{2 \sin A} = R \right]$$

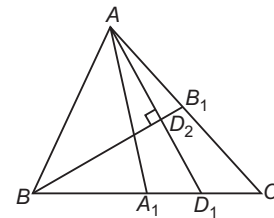
$$70. \text{ We have, } a = 2R \sin A = 2R \cdot \frac{\Delta \cdot 2}{bc}$$

$$\therefore a = \frac{4R\Delta}{bc} \Rightarrow R = \frac{abc}{4\Delta}$$

$$71. r_1, r_2 \text{ and } r_3 \text{ are in HP or } \frac{s-a}{\Delta}, \frac{s-b}{\Delta} \text{ and } \frac{s-c}{\Delta} \text{ are in AP.}$$

or $s-a, s-b, s-c$ are in AP or a, b, c are in AP.
 Now, $a+b+c=24 \Rightarrow b=8, s=12, c=4-a$
 or $a^2 - 16a + 60 = 0$ or $a=10, 6$
 If $a=10, b=6$ and if $a=6, b=10$
 $\therefore$ Sides of the triangle are 6, 8, 10 cm.

$$72. \text{ In } \triangle ABD_1, \angle ABD_2 = \angle D_2BD_1 = \frac{B}{2}$$



$\Rightarrow \triangle ABD_2$ and $\triangle D_1D_2B$ are congruent.

$$\Rightarrow \frac{AD_1}{AD_2} = \frac{2}{1} = 2$$

$$73. \text{ Distance of circumcentre from side } AC = R \cos B \text{ and distance of orthocentre from side } AC = 2R \cos A \cdot \cos C$$

$$\Rightarrow R \cos B = 2R \cos A \cdot \cos C$$

$$\Rightarrow -\cos(A+C) = 2 \cos A \cdot \cos C$$

$$\Rightarrow \sin A \cdot \sin C = 3 \cos A \cdot \cos C \Rightarrow \tan A \cdot \tan C = 3$$

$$74. r_1 + r_2 = 4R \left(\sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2} + \sin \frac{B}{2} \cdot \cos \frac{A}{2} \cdot \cos \frac{C}{2} \right)$$

$$= 4R \left(\cos \frac{C}{2} \left(\sin \frac{A}{2} \cdot \cos \frac{B}{2} + \sin \frac{B}{2} \cdot \cos \frac{A}{2} \right) \right)$$

$$= 4R \left(\cos^2 \frac{C}{2} \right) = 2R (1 + \cos C)$$

$$\Rightarrow \frac{r_1 + r_2}{1 + \cos C} = 2R$$

75. Area of circle = $\pi \times (\text{Radius})^2$

$$\therefore A = \pi r^2, A_1 = \pi r_1^2, A_2 = \pi r_2^2, A_3 = \pi r_3^2$$

$$\begin{aligned} \therefore \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}} &= \frac{1}{r_1 \sqrt{\pi}} + \frac{1}{r_2 \sqrt{\pi}} + \frac{1}{r_3 \sqrt{\pi}} \\ &= \frac{1}{\sqrt{\pi}} \left[\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right] = \frac{1}{\sqrt{\pi}} \left[\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} \right] \\ &= \frac{1}{\sqrt{\pi}} \left[\frac{3s - (a+b+c)}{\Delta} \right] = \frac{1}{\sqrt{\pi}} \cdot \frac{3s-2s}{\Delta} \\ &= \frac{1}{\sqrt{\pi}} \cdot \frac{s}{\Delta} = \frac{1}{r \sqrt{\pi}} = \frac{1}{\sqrt{\pi r^2}} = \frac{1}{\sqrt{A}} \end{aligned}$$

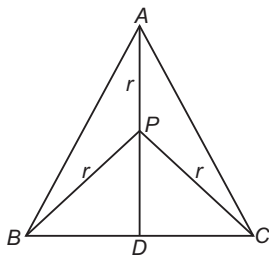
76. From $\triangle AHB$, $\frac{AH}{\sin(90^\circ - A)} = \frac{c}{\sin(A+B)}$

$$\Rightarrow AH = \frac{c \cos A}{\sin(180^\circ - C)} = \frac{c \cos A}{\sin C} = \frac{a}{\sin A} \cos A$$

$$\left[\because \frac{a}{\sin A} = \frac{c}{\sin C} \right]$$

$$= a \cot A$$

77. In $\triangle ABC$, $AB = BC$ and altitude $AD = h$
Ler r be the radius of circumscribed circle.



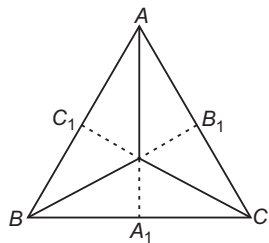
$$\begin{aligned} \Rightarrow PA &= PB = PC = r \\ \Rightarrow PD &= h - r \\ \text{Now, } BD &= \sqrt{BP^2 - PD^2} \\ &= \sqrt{r^2 - (h-r)^2} = \sqrt{2rh - h^2} \end{aligned}$$

$$\therefore BC = 2BD = 2\sqrt{2rh - h^2}$$

Thus, area of $\triangle ABC = \frac{1}{2} \times BC \times DA$

$$= \frac{1}{2} \times 2\sqrt{2rh - h^2} \times h = h\sqrt{2rh - h^2}$$

78. Let A_1, B_1 and C_1 be the foot of altitudes drawn from P to sides BC, CA and AB , respectively. We have,
 $PA_1 = x_a, PB_1 = x_b$ and $PC_1 = x_c$
ar ($\triangle ABC$) = ar ($\triangle PBC$) + ar ($\triangle CPA$) + ar ($\triangle APB$)



$$\begin{aligned} \Rightarrow \frac{\sqrt{3}}{4} \cdot 4 &= \frac{1}{2} \cdot 2(x_a + x_b + x_c) \\ \Rightarrow x_a + x_b + x_c &= \sqrt{3} \end{aligned}$$

79. $\Delta = \frac{1}{2}(BC)h$

where, h is the distance of vertex A from side BC .

$$\therefore \Delta_{GBC} = \frac{\Delta}{3} = \frac{(BC)h}{6}, \text{ where } G \text{ is centroid.}$$

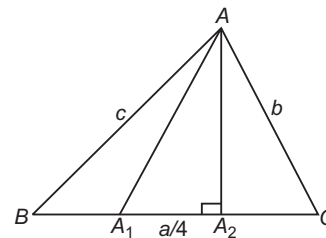
$$\Rightarrow h = \frac{2\Delta}{BC} = \text{constant}$$

Thus, distance of vertex A from side is fixed. This in turn implies that distance of centroid from side BC will be fixed, hence locus of G will be a line parallel to BC .

80. $\angle BAA_1 = \angle A_1AA_2 = \angle A_2AC = \frac{A}{3}$

Clearly, $\triangle AA_1C$ is isosceles.

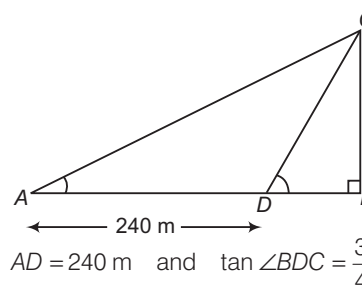
$$\Rightarrow AA_1 = AC = b$$



$$A_1A_2 = A_2C = \frac{a}{4}$$

$$\begin{aligned} \text{Now, } \sin^3 \frac{A}{3} \cdot \cos \frac{A}{3} &= \frac{1}{2} \sin^2 \frac{A}{3} \cdot 2 \sin \frac{A}{3} \cdot \cos \frac{A}{3} \\ &= \frac{1}{2} \sin^2 \frac{A}{3} \cdot \sin \frac{2A}{3} \\ &= \frac{1}{2} \cdot \left(\frac{A_2C}{b} \right)^2 \cdot \frac{BA_2}{AB} = \frac{1}{2} \cdot \frac{a^2}{16b^2} \cdot \frac{3a}{4} \cdot \frac{1}{c} \\ &= \frac{3a^3}{128b^2c} \end{aligned}$$

81. Let BC be the tower and $\tan \angle BAC = \frac{5}{12}$.



$$\begin{aligned} \text{Now, } \tan \angle BAC &= \frac{BC}{AB} = \frac{5}{12} \\ \Rightarrow \frac{BC}{240 + BD} &= \frac{5}{12} \\ \Rightarrow BC &= (240 + BD) \frac{5}{12} \quad \dots (i) \end{aligned}$$

$$\text{Again, } \tan \angle BDC = \frac{BC}{BD} = \frac{3}{4} \Rightarrow BC = \frac{3}{4} BD \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\begin{aligned} \frac{3}{4} BD &= (240 + BD) \frac{5}{12} \\ \Rightarrow 9BC &= 1200 + 5BD \\ \Rightarrow 4BD &= 1200 \end{aligned}$$

$$\Rightarrow BD = \frac{1200}{4} = 300 \text{ m}$$

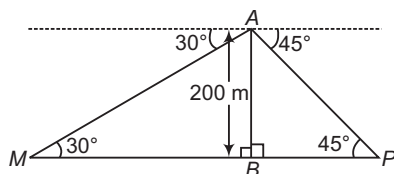
$$\therefore BC = \frac{3}{4} BD$$

$$= \frac{3}{4} \times 300$$

$$= 225 \text{ m}$$

Hence, the height of the tower is 225 m.

82. Let $AB = 200 \text{ m}$ be the cliff and P, M be the two ships on opposite sides of the cliff such that PBM is a straight line.



Distance between this ships = PM

In right angled $\triangle ABP$,

$$\frac{BP}{200} = \cot 45^\circ$$

$$\Rightarrow BP = 200 \times \cot 45^\circ$$

$$= 200 \times 1 = 200 \text{ m}$$

In right angled $\triangle ABM$,

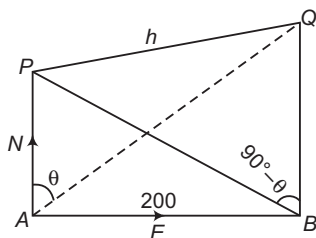
$$\frac{BM}{200} = \cot 30^\circ$$

$$\therefore BM = 200 \times \cot 30^\circ = 200 \times \sqrt{3}$$

Hence, $PM = PB + BM = 200 + 200\sqrt{3}$

$$= 200(\sqrt{3} + 1) \text{ m}$$

83. Let PQ be the tower. Let θ be the angle of elevation of Q as seen from A . Then, the angle of elevation of Q as seen from B is $90^\circ - \theta$.



From the figure, we have

$$\angle QAP = \theta$$

and

$$\angle QBP = 90^\circ - \theta$$

Let h be the height of tower PQ .

Now,

$$\angle APQ = 90^\circ$$

$\therefore$

$$h = AP = \tan \theta$$

...(i)

Also, $\angle BPQ = 90^\circ$

$\therefore$

$$h = BP \tan (90^\circ - \theta) = BP \cot \theta$$

...(ii)

From Eqs. (i) and (ii), we get

$$h^2 = AP \times BP$$

...(iii)

In right angled $\triangle BAP$,

$$AB = AP = 200 \text{ m}$$

$\therefore$

$$BP = 200\sqrt{2}$$

...(iv)

From Eqs. (iii) and (iv), we get

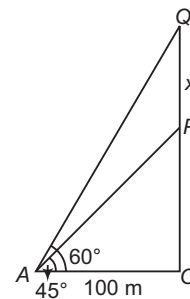
$$h^2 = 200 \times 200\sqrt{2}$$

$\Rightarrow$

$$h = 200(2)^{1/4} \text{ m}$$

84. Let OP be the incomplete and OQ be the complete pillar.

If A is a point on the horizontal plane, such that $OA = 100 \text{ m}$.



Then,

and

In $\triangle OAP$,

$$\angle OAP = 45^\circ$$

$$\angle OAQ = 60^\circ$$

$$\tan 45^\circ = \frac{OP}{OA}$$

$\therefore$

$$OP = OA = 100 \text{ m}$$

In $\triangle OAQ$,

$$\tan 60^\circ = \frac{OQ}{OA}$$

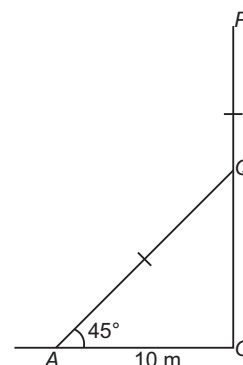
$\Rightarrow$

$$\sqrt{3} = \frac{(100 + x)}{100}$$

$\Rightarrow$

$$x = 100(\sqrt{3} - 1) \text{ m}$$

85. Let $AQ (= PQ)$ be the broken part of the tree OP , whose upper part touches the ground at A such that



$$\angle QAO = 45^\circ$$

$\therefore$

$$OA = 10 \text{ m}$$

$$[\because OQ = 10 \text{ m}]$$

Now, in $\triangle OAQ$,

$$AQ = \sqrt{(OA)^2 + (OQ)^2}$$

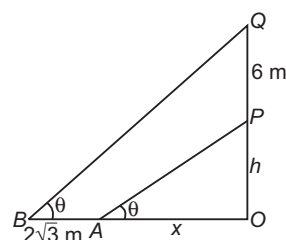
$$= 10\sqrt{2} \text{ m}$$

Hence, the entire length of the tree

$$= OP = OQ + PQ = OQ + AQ$$

$$= 10(1 + \sqrt{2}) \text{ m}$$

86. Let OP be the tower of height, say h , m and PQ be the flagstaff of height 6 m.



Let the Sun makes an angle θ with the ground and OA and $AB = 2\sqrt{3}$ m be the shadows of the tower and the flagstaff, respectively.

Let $OA = x$

$$\text{In } \triangle OAP, \tan \theta = \frac{h}{x} \quad \dots (i)$$

$$\text{and in } \triangle OBQ, \tan \theta = \frac{(h+6)}{(x+2\sqrt{3})}$$

$$\therefore \frac{h}{x} = \frac{(h+6)}{(x+2\sqrt{3})} \quad \dots (ii)$$

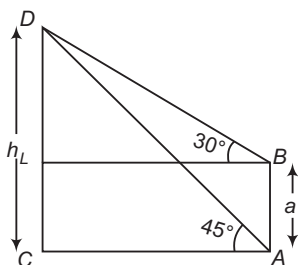
From Eqs. (i) and (ii), we get

$$hx + 2\sqrt{3}h = xh + 6x \Rightarrow 2\sqrt{3}h = 6x$$

$$\Rightarrow \tan \theta = \frac{h}{x} = \frac{6}{2\sqrt{3}} = \sqrt{3} \Rightarrow \theta = 60^\circ$$

87. CD is tower of height h , AB is building of height a .

$$\text{In } \triangle BLD, \tan 30^\circ = \frac{h-a}{LB}$$



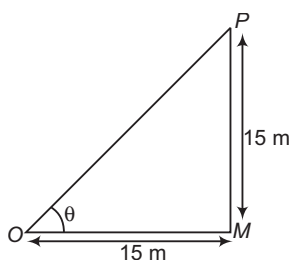
$$\therefore LB = \frac{(h-a)}{\tan 30^\circ} = \sqrt{3}(h-a)$$

$$\text{In } \triangle ACD, \tan 45^\circ = \frac{h}{LB} \Rightarrow h(\sqrt{3}-1) = \sqrt{3}a$$

$$\therefore h = \frac{\sqrt{3}a}{\sqrt{3}-1} = \frac{\sqrt{3}(\sqrt{3}+1)a}{2}$$

$$\Rightarrow h = \left(\frac{3+\sqrt{3}}{2} \right) a$$

88. Let MP denotes the house. Let θ be the angle of elevation.

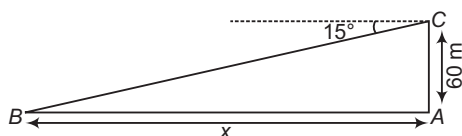


$$\therefore \tan \theta = \frac{MP}{OM} = \frac{15}{15} = 1$$

$$\Rightarrow \theta = 45^\circ$$

89. Here, B is the position of boat and AC is light house.

$$\text{Now, } \frac{AC}{x} = \tan 15^\circ = \tan(45^\circ - 30^\circ)$$

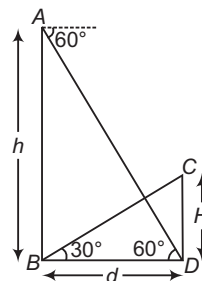


$$= \frac{1 - \tan 30^\circ}{1 + \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}}$$

$$\Rightarrow \frac{AC}{x} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\therefore x = 60 \left(\frac{\sqrt{3}+1}{\sqrt{3}-1} \right) \text{ m}$$

90. Let CD be the tower.



$$\text{From } \triangle BCD, \frac{H}{d} = \tan 30^\circ \quad \dots (i)$$

$$\text{and from } \triangle ABD, \frac{h}{d} = \tan 60^\circ \quad \dots (ii)$$

$$\therefore \frac{H}{h} = \frac{\tan 30^\circ}{\tan 60^\circ}$$

$$\Rightarrow H = \frac{h}{3}$$

91. AB is tower of height h .

$$\text{In } \triangle ABC, \frac{AB}{BC} = \tan 45^\circ$$

$$\Rightarrow h = 70 \tan 45^\circ$$

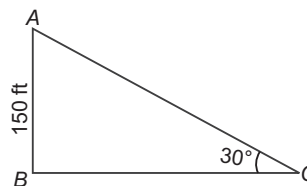
$$\Rightarrow h = 70 \text{ m}$$

92. From figure,

$$\text{In } \triangle ADE, \frac{150-h}{60} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$

$$\therefore h = 150 - \frac{60}{\sqrt{3}} = (150 - 20\sqrt{3}) \text{ m}$$

93. Given that height of the tower $AB = h = 150$ ft



Angle of elevation of the Sun = 30°

Let BC be the length of the shadow cast by the tower.

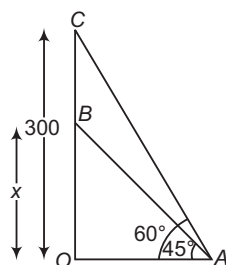
Then, from right angled $\triangle ABC$, we have

$$\tan 30^\circ = \frac{AB}{BC} = \frac{h}{BC}$$

$$\Rightarrow \frac{150}{BC} = \frac{1}{\sqrt{3}}$$

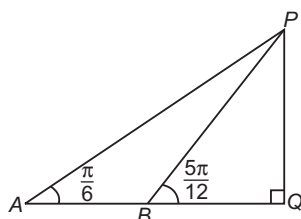
$$\Rightarrow BC = 150\sqrt{3} \text{ ft}$$

94. In $\triangle OAB$, $\tan 45^\circ = \frac{OB}{OA}$



$$\begin{aligned} \Rightarrow \quad \frac{x}{OA} &= 1 \\ \therefore \quad OA &= x \\ \text{In } \triangle OAC, \quad \tan 60^\circ &= \frac{OC}{OA} \\ \Rightarrow \quad \frac{300}{OA} &= \sqrt{3} \\ \therefore \quad OA &= \frac{300}{\sqrt{3}} \\ &= 100\sqrt{3} = x \end{aligned}$$

95. Let PQ be the vertical tower of height h and A and B be the points of observation.
We have, $AB = 1000$ m

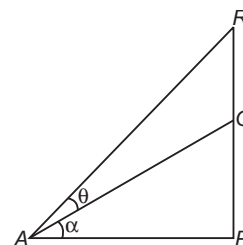


$$\begin{aligned} \text{In } \triangle AQP, \quad AQ &= h \cot \frac{\pi}{6} = h\sqrt{3} \\ \text{In } \triangle BQP, \quad BQ &= h \cot \frac{5\pi}{12} \\ &= h \cot \left(\frac{\pi}{6} + \frac{\pi}{4} \right) \\ \Rightarrow \quad BQ &= h \frac{\cot \frac{\pi}{6} \cdot \cot \frac{\pi}{4} - 1}{\cot \frac{\pi}{6} + \cot \frac{\pi}{4}} \\ &= h \frac{(\sqrt{3} - 1)}{(\sqrt{3} + 1)} \\ &= \frac{h(\sqrt{3} - 1)^2}{2} \\ \text{Now,} \quad AB &= AQ - BQ \\ &= h \left[\sqrt{3} - \left(\frac{4 - 2\sqrt{3}}{2} \right) \right] \\ &= h(\sqrt{3} - 2 + \sqrt{3}) \\ \Rightarrow \quad 1000 &= 2h(\sqrt{3} - 1) \\ \Rightarrow \quad h &= \frac{500}{(\sqrt{3} - 1)} \\ &= 250(\sqrt{3} + 1) \text{ m} \end{aligned}$$

96. Let RQ be the chimney, PQ be the vertical tower and A be the point of observation.

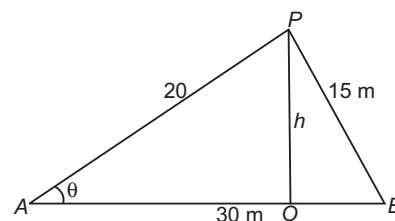
We have,

$$AP = 70 \text{ m}, RQ = 20 \text{ m} \quad \text{and} \quad \tan \theta = \frac{1}{6}$$



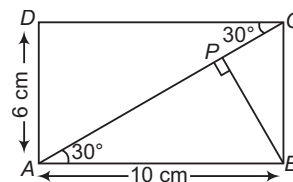
$$\begin{aligned} \text{If } \angle QAP &= \alpha \quad \text{and} \quad PQ = H \\ \text{and} \quad \tan(\theta + \alpha) &= \frac{H + 20}{70}, \quad \tan \alpha = \frac{H}{70} \\ \Rightarrow \quad \frac{H + 20}{70} &= \frac{\frac{H}{70} + \frac{1}{6}}{1 - \frac{H}{6 \cdot 70}} = \frac{6H + 70}{6 \cdot 70 - H} \\ \Rightarrow \quad 6 \cdot 70H - H^2 + 120 \cdot 70 - 20H &= 70 \cdot 6H + 70^2 \\ \Rightarrow \quad H^2 + 20H - 70 \cdot 50 &= 0 \\ \Rightarrow \quad H^2 + 20H + 100 &= 3500 + 100 = 3600 \\ \Rightarrow \quad (H + 10)^2 &= (60)^2 \\ \Rightarrow \quad H &= 50 \text{ m} \end{aligned}$$

97. Let OP be the tower of height h , then $AB = 30$, $AP = 20$ and $BP = 15$.



$$\begin{aligned} \text{From } \triangle ABP, \\ \cos \theta &= \frac{20^2 + 30^2 - 15^2}{2 \times 20 \times 30} \\ &= \frac{1075}{1200} = \frac{43}{48} \\ \Rightarrow \quad \sin \theta &= \sqrt{1 - \left(\frac{43}{48} \right)^2} \\ &= \frac{\sqrt{2304 - 1849}}{48} = \frac{\sqrt{455}}{48} \\ \text{and} \quad h &= 20 \sin \theta = 20 \times \frac{\sqrt{455}}{48} = \frac{5\sqrt{455}}{12} \end{aligned}$$

98. Draw $BP \perp AC$, the longer diagonal. From right angled $\triangle APB$, we have



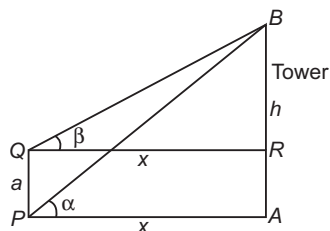
$$\begin{aligned} \frac{AP}{AB} &= \cos 30^\circ \\ \Rightarrow AP &= AB \cos 30^\circ \\ \therefore AP &= 10 \times \frac{\sqrt{3}}{2} \\ &= 5\sqrt{3} \text{ cm} \end{aligned}$$

$$\begin{aligned} \text{Also, } \frac{BP}{AB} &= \sin 30^\circ \\ \therefore BP &= AB \sin 30^\circ \\ &= 10 \times \frac{1}{2} \\ &= 5 \text{ cm} \end{aligned}$$

From right angled ΔBPC , we have

$$\begin{aligned} BP^2 + PC^2 &= BC^2 \\ \Rightarrow 5^2 + PC^2 &= 6^2 \\ \Rightarrow 6^2 - 5^2 &= 11 \\ \Rightarrow PC &= \sqrt{11} \text{ cm} \\ \therefore \text{Longer diagonal, } AC &= AP + PC \\ &= (5\sqrt{3} + \sqrt{11}) \text{ cm} \end{aligned}$$

99. Let $AB = h$ be the height of tower at a distance x from the post $PQ = a$.



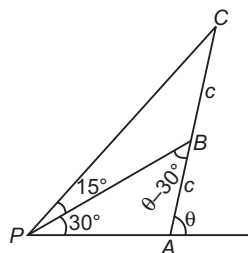
AB subtends angles α and β at P and Q , respectively. We have, to determine h and x in terms of known quantities α , β and a .

$$\begin{aligned} AB &= x \tan \alpha, BR = x \tan \beta \\ PQ &= AR = AB - BR = x (\tan \alpha - \tan \beta) \\ \therefore x &= \frac{PQ}{\tan \alpha - \tan \beta} = \frac{a}{\tan \alpha - \tan \beta} = \frac{a \cos \alpha \cos \beta}{\sin(\alpha - \beta)} \\ \therefore h &= AB = x \tan \alpha = \frac{a \cos \alpha \cos \beta}{\sin(\alpha - \beta)} \cdot \frac{\sin \alpha}{\cos \alpha} \\ &= \frac{a \sin \alpha \cos \beta}{\sin(\alpha - \beta)} \end{aligned}$$

100. Applying $m-n$ theorem of trigonometry, we get
 $(c + c) \cot(\theta - 30^\circ) = c \cot 15^\circ - c \cot 30^\circ$

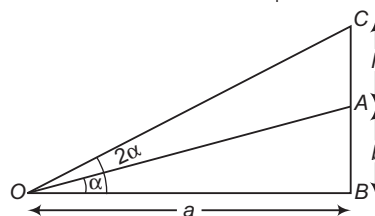
$$\Rightarrow \cot(\theta - 30^\circ) = \frac{1}{2} \cdot \frac{\sin(30^\circ - 15^\circ)}{\sin 15^\circ \sin 30^\circ}$$

$$\Rightarrow \cot(\theta - 30^\circ) = \frac{1}{2} \cdot \frac{1}{\sin 30^\circ}$$



$$\begin{aligned} \Rightarrow \theta - 30^\circ &= 45^\circ \\ \therefore \theta &= 75^\circ \end{aligned}$$

101. Let AB be the tower and AC be pole of height h .

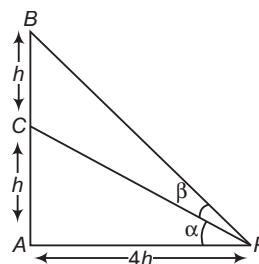


$$\text{From } \Delta ABO, \quad \frac{b}{a} = \tan \alpha \quad \dots(i)$$

$$\begin{aligned} \text{From } \Delta CBO, \quad \frac{b+h}{a} &= \tan 2\alpha \\ \Rightarrow \frac{b+h}{a} &= \frac{2 \tan \alpha}{1 - \tan^2 \alpha} \\ \Rightarrow b+h &= \frac{2a \times (b/a)}{1 - (b^2/a^2)} \\ \Rightarrow h &= b \left(\frac{a^2 + b^2}{a^2 - b^2} \right) \end{aligned}$$

102. Given that, $AC = CB = h$

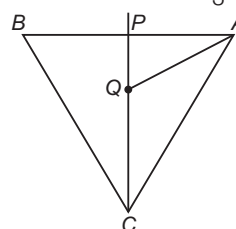
[say]



$$\begin{aligned} \angle CPA &= \alpha, \angle BPC = \beta \\ \text{and } \frac{AP}{AB} &= \frac{2}{1} \Rightarrow AP = 2AB = 4h \\ \text{In } \Delta APC, \quad \tan \alpha &= \frac{h}{4h} \Rightarrow \tan \alpha = \frac{1}{4} \\ \text{In } \Delta ABP, \quad \tan(\alpha + \beta) &= \frac{2h}{4h} = \frac{1}{2} \\ \Rightarrow \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta} &= \frac{1}{2} \Rightarrow \frac{\frac{1}{4} + \tan \beta}{1 - \frac{1}{4} \tan \beta} = \frac{1}{2} \\ \Rightarrow \frac{1 + 4 \tan \beta}{4 - \tan \beta} &= \frac{1}{2} \Rightarrow 2 + 8 \tan \beta = 4 - \tan \beta \\ \Rightarrow 9 \tan \beta &= 2 \Rightarrow \beta = \tan^{-1} \frac{2}{9} \end{aligned}$$

103. Let ABC be the equilateral triangle having its centre at Q and PQ be the vertical tower. We have,

$$AQ = QC = \frac{a}{2 \cdot \sin \frac{\pi}{3}} = \frac{a}{\sqrt{3}}$$



$$\text{Now, } PA^2 = PC^2 = PQ^2 + AQ^2 = h^2 + \frac{a^2}{3}$$

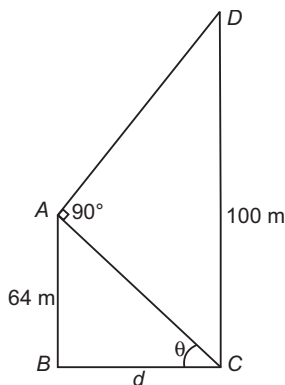
Since, $\angle APC = \frac{\pi}{3}$

$$\Rightarrow \cos \frac{\pi}{3} = \frac{PA^2 + PC^2 - AC^2}{2PA \cdot PC} = \frac{1}{2}$$

$$\Rightarrow 2 \left(h^2 + \frac{a^2}{3} \right) - a^2 = h^2 + \frac{a^2}{3}$$

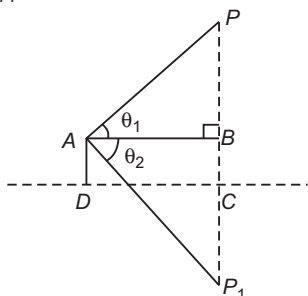
$$\Rightarrow 3h^2 = 2a^2$$

104. $64 \cot \theta = d$



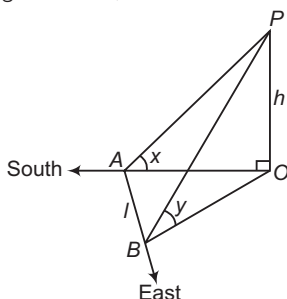
Also, $(100 - 64) \tan \theta = d$
 or $(64)(36) = d^2$
 $\therefore d = 8 \times 6 = 48 \text{ m}$

105. Let CD be the surface of lake, P be the cloud and P_1 be the reflection of cloud.
 We have, $AD = h$, $\angle PAB = \theta_1$, $\angle P_1AB = \theta_2$
 Let $PC = H$



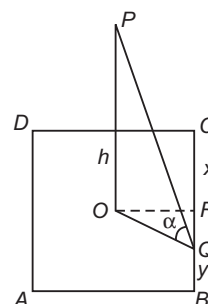
Now, $AB = PB \cdot \cot \theta_1 = (H - h) \cot \theta_1$
 Also, $AB = P_1B \cdot \cot \theta_2 = (H + h) \cot \theta_2$
 $\Rightarrow (H - h) \cot \theta_1 = (H + h) \cot \theta_2$
 $\Rightarrow H = \frac{h(\cot \theta_1 + \cot \theta_2)}{(\cot \theta_1 - \cot \theta_2)}$
 $= \frac{h \sin(\theta_1 + \theta_2)}{\sin(\theta_2 - \theta_1)}$

106. Let OP be the tower of height h .
 In right angled $\triangle OAP$, $\angle OAP = x$



$\frac{OA}{h} = \cot x$
 $\Rightarrow OA = h \cot x \quad \dots (i)$
 In right angled $\triangle OBP$, $\angle OBP = y$
 $\therefore \frac{OB}{h} = \cot y$
 $\Rightarrow OB = h \cot y \quad \dots (ii)$
 In right angled $\triangle OAB$, $AB^2 + OA^2 = OB^2$
 $\therefore l^2 + h^2 \cot^2 x = h^2 \cot^2 y$
 $\Rightarrow h^2(\cot^2 y - \cot^2 x) = l^2$
 $\Rightarrow h = \frac{l}{\sqrt{\cot^2 y - \cot^2 x}}$

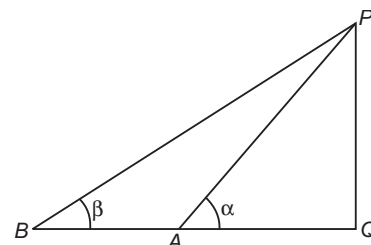
107. Let O be the centre of the square, OP the pole. Shadow of the pole OP is OQ . From question, $BQ = y$ and $CQ = x$. Then, $BC = x + y$



Let $OR \perp BC$
 $\therefore OR = \frac{x+y}{2}$ and $BR = \frac{x+y}{2}$
 $RQ = \frac{x+y}{2} - y = \frac{x-y}{2}$

Let h be the height of the pole.
 From right angled $\triangle POQ$,
 $\cot \alpha = \frac{OQ}{h} \Rightarrow OQ = h \cot \alpha$
 Now, from right angled $\triangle ORQ$, $OQ^2 = OR^2 + RQ^2$
 $\Rightarrow h^2 \cot^2 \alpha = \left(\frac{x+y}{2} \right)^2 + \left(\frac{x-y}{2} \right)^2$
 $\Rightarrow h^2 \cot^2 \alpha = \frac{2(x^2 + y^2)}{4} \Rightarrow h = \sqrt{\frac{x^2 + y^2}{2}} \tan \alpha$

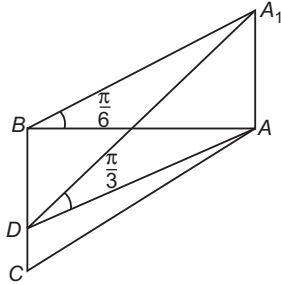
108. Let PQ be the vertical tower, having Q as its foot. We have, $AQ = BQ = CQ = DQ$, which implies that the points A, B, C and D are concyclic having Q as its centre.
 109. Let PQ be the tower and A, B be the points of observations. We have, $PQ = h$, $AQ = a$, $BQ = b$



Now, $AQ = PQ \cdot \cot \alpha \Rightarrow a = h \cot \alpha$
 and $BQ = PQ \cdot \cot \beta \Rightarrow b = h \cot \beta$
 $\Rightarrow (b - a) = h(\cot \beta - \cot \alpha)$

110. Let AA_1 be the flagstaff of height h .

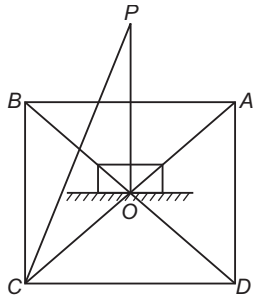
We have, $BC = a$, $\angle A_1DA = \frac{\pi}{3}$, $\angle A_1BA = \frac{\pi}{6}$



Now, $AD = AA_1 \cdot \cot \frac{\pi}{3} = \frac{h}{\sqrt{3}}$
 and $AB = AA_1 \cdot \cot \frac{\pi}{6} = h\sqrt{3}$
 In $\triangle ABD$, $AB^2 = AD^2 + BD^2$
 $\Rightarrow 3h^2 = \frac{h^2}{3} + \frac{a^2}{4} \Rightarrow \frac{8h^2}{3} = \frac{a^2}{4}$
 $\Rightarrow h = \frac{a\sqrt{3}}{4\sqrt{2}} = \frac{a}{4\sqrt{2}}$

111. Let $ABCD$ be the square of side length a and OP be the vertical tower.

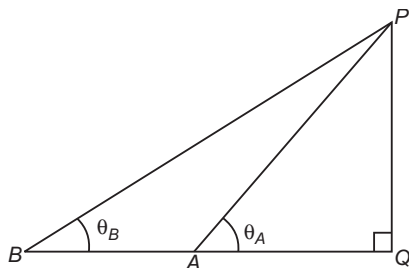
We have, $OP = h$, $BC = AB = CD = AD = a$, $\angle PCO = \theta$



Now, $OC = h \cot \theta$
 $\Rightarrow \frac{a}{\sqrt{2}} = h \cot \theta \Rightarrow a^2 = 2h^2 \cot^2 \theta$

112. Let $PQ = h$.

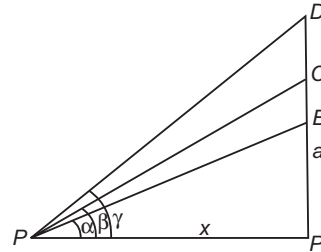
We have, $QA = 4$, $QB = 9$, $\theta_A + \theta_B = \frac{\pi}{2}$



Now, $QA = PQ \cdot \cot \theta_A$
 $QB = PQ \cdot \cot \theta_B$
 $QA \cdot QB = PQ^2 \cdot \cot \theta_A \cdot \cot \theta_B$
 $= PQ^2 \cdot \cot \theta_A \cdot \tan \theta_A = PQ^2$
 $\Rightarrow PQ = \sqrt{QA \cdot QB} = 6\text{m}$

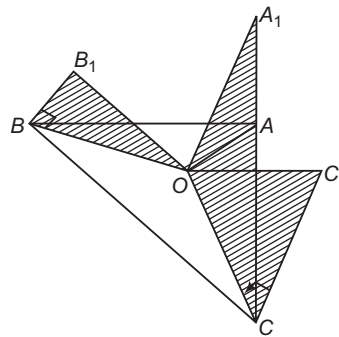
113. We have,

$a = AB = AP \tan \alpha = x \tan \alpha$
 $b = AC = x \tan \beta$
 and $c = AD = x \tan \gamma$,
 so that $a + b + c = x (\tan \alpha + \tan \beta + \tan \gamma)$
 $= x \tan \alpha \tan \beta \tan \gamma$ [$\because \alpha + \beta + \gamma = 180^\circ$]
 and $abc = x^3 \tan \alpha \tan \beta \tan \gamma$



$\Rightarrow x^2 = \frac{abc}{a + b + c}$,
 so that $\tan \alpha + \tan \beta + \tan \gamma = \tan \alpha \tan \beta \tan \gamma$
 $= (a + b + c)/x$
 $= (a + b + c)^{3/2}/\sqrt{abc}$

114. Let AA_1 , BB_1 and CC_1 be the towers and O be the circumcentre of $\triangle ABC$. We have,
 $\angle A_1OA = \theta_A$, $\angle B_1OB = \theta_B$, $\angle C_1OC = \theta_C$

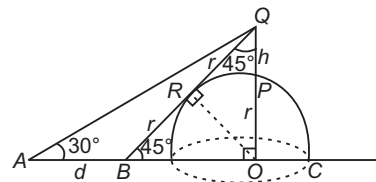


Now, $OA = AA_1 \cdot \cot \theta_A$, $OB = BB_1 \cdot \cot \theta_B$
 $OC = CC_1 \cdot \cot \theta_C$
 Since, $OA = OB = OC$

$\Rightarrow AA_1 \cot \theta_A = BB_1 \cot \theta_B = CC_1 \cot \theta_C$
 $\Rightarrow \frac{\tan \theta_A}{AA_1} = \frac{\tan \theta_B}{BB_1} = \frac{\tan \theta_C}{CC_1}$

Any other relationship between $\tan \theta_A$, $\tan \theta_B$, $\tan \theta_C$ cannot be established.

115. Let $PQ = h$ be a pole standing on a hemispherical dome, of radius r . The elevation of top Q as seen from A is 30° .



$\therefore r + h = OA \tan 30^\circ$... (i)
 On walking $AB = d$, the top is just visible from B which means that BQ is a tangent touching at R .
 OR is normal.

$$\begin{aligned}\angle OBQ &= 45^\circ = \angle OQB \\ \therefore OB &= OQ = r + h \\ \text{and } OA &= r + h + d \\ \therefore r &= OR = RQ = RB \\ \text{and } BQ &= BR + RQ = 2r\end{aligned}$$

Hence, from right angled $\triangle OBQ$, we have

$$\begin{aligned}(2r)^2 &= (r + h)^2 + (r + h)^2 \\ \Rightarrow h &= r(\sqrt{2} - 1) \quad \dots (ii)\end{aligned}$$

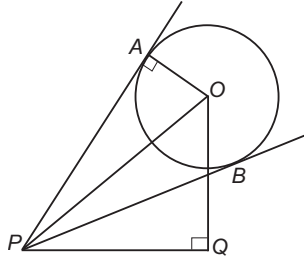
$$\text{From Eq. (i), } (r + h) = (r + h + d) \cdot \frac{1}{\sqrt{3}}$$

$$\begin{aligned}\Rightarrow (\sqrt{3} - 1)(r + h) &= d \\ \Rightarrow (\sqrt{3} - 1) \cdot \sqrt{2}r &= d \quad [\text{from Eq. (ii)}]\end{aligned}$$

$$\Rightarrow r = \frac{d}{\sqrt{2}(\sqrt{3} - 1)} = \frac{d(\sqrt{3} + 1)}{2\sqrt{2}}$$

$$\text{Also, } h = r(\sqrt{2} - 1) = \frac{d(\sqrt{3} + 1)(\sqrt{2} - 1)}{2\sqrt{2}}$$

116. Let O be the centre of spherical ball. We have, $\angle APB = \theta_1$, $\angle OPQ = \theta_2$, $OA = OB = r$

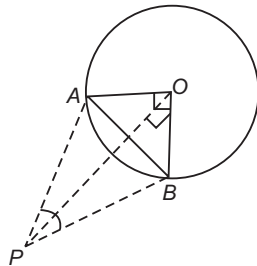


$$\text{Clearly, } \angle APO = \angle OPB = \frac{\theta_1}{2}$$

$$\text{Now, } OP = OA \cdot \operatorname{cosec}(\angle APO) = r \operatorname{cosec} \frac{\theta_1}{2}$$

$$\text{and } OQ = OP \cdot \sin \theta_2 = r \sin \theta_2 \cdot \operatorname{cosec} \frac{\theta_1}{2}$$

117. Let O be the centre of ring and P be the point of suspension. If PA and PB are two consecutive string, then $\angle AOB = \frac{\pi}{2}$



We have, $OA = OB = r$, $OP = h$

In $\triangle AOB$,

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow AB = r\sqrt{2}$$

In $\triangle OPB$,

$$PB^2 = OP^2 + OB^2 = h^2 + r^2$$

$$\Rightarrow PA = PB = \sqrt{h^2 + r^2}$$

If $\angle APB = \theta$

$$\begin{aligned}\Rightarrow \cos \theta &= \frac{PA^2 + PB^2 - AB^2}{2PA \cdot PB} \\ &= \frac{2(h^2 + r^2) - 2r^2}{2(h^2 + 4r^2)} = \frac{h^2}{h^2 + r^2}\end{aligned}$$

118. We have,

$$\begin{aligned}\sin C + \cos C + \sin(2B + C) - \cos(2B + C) &= 2\sqrt{2} \\ \Rightarrow [\sin(2B + C) + \sin C] + [\cos C - \cos(2B + C)] &= 2\sqrt{2} \\ \Rightarrow 2 \sin(B + C) \cos B + 2 \sin(B + C) \sin B &= 2\sqrt{2} \\ \Rightarrow \sin(180^\circ - A) \cos B + \sin(180^\circ - A) \sin B &= \sqrt{2} \\ \Rightarrow \sin A(\cos B + \sin B) &= \sqrt{2}\end{aligned}$$

$$\Rightarrow \sin A \left(\frac{1}{\sqrt{2}} \cos B + \frac{1}{\sqrt{2}} \sin B \right) = 1$$

$$\Rightarrow \sin A \sin \left(\frac{\pi}{4} + B \right) = 1, \text{ is possible only when}$$

$$\sin A = 1 \text{ and } \sin \left(\frac{\pi}{4} + B \right) = 1$$

$$\therefore \angle A = 90^\circ \text{ and } \angle B = 45^\circ, \text{ then } \angle C = 45^\circ$$

119. We have,

$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

$$\text{and } r_1 r_2 r_3 = \frac{\Delta^3}{(s-a)(s-b)(s-c)}$$

$$\therefore x^2 - r_1 r_2 + r_2 r_3 + r_3 r_1 x + r_1 r_2 r_3 - 1 = 0$$

It satisfied by $x = 1$.

So, one root is $x = 1$ and therefore other root is $r_1 r_2 r_3 - 1$.

$$120. r = \frac{\Delta}{s}, r_1 = \frac{\Delta}{s-a}$$

$$r_1 r = \frac{\Delta^2}{s(s-a)} = \frac{s(s-a)(s-b)(s-c)}{s(s-a)}$$

$$= (s-b)(s-c) = (s-b)^2 \quad [\because b=c]$$

$$= \frac{(2s-2b)^2}{4} = \frac{(a+b+c-2b)^2}{4} \quad [\because b=c]$$

$$= \frac{a^2}{4} = \frac{4R^2 \sin^2 A}{4} = R^2 \sin^2 A$$

Also, if $\angle B = \theta$

$$\Rightarrow \angle A = \pi - 2\theta$$

$$\therefore r_1 r = R^2 \sin^2(\pi - 2\theta)$$

$$= R^2 \sin^2 2\theta = R^2 \sin^2 2B$$

121. We are given that

$$\cos B \cos C + \sin B \sin C \sin^2 A = 1 \quad \dots (i)$$

$$\text{and } \sin^2 A \leq 1 \quad \dots (ii)$$

Now, $\sin B, \sin C$ are positive in a triangle.

$$\therefore \sin B \sin C \sin^2 A \leq \sin B \sin C \quad \dots (iii)$$

$$\Rightarrow 1 - \cos B \cos C \leq \sin B \sin C \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 1 \leq \cos(B - C) \text{ or } \cos(B - C) \geq 1$$

But $\cos \theta$ cannot be > 1 , hence we must have sign of equality throughout.

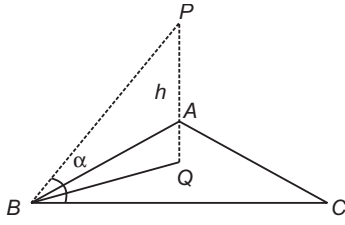
$$\Rightarrow \cos(B - C) = 1$$

$$\Rightarrow B - C = 0$$

$$\therefore B = C$$

Also, $\sin^2 A = 1$, i.e. $A = \frac{\pi}{2}$, hence the triangle both right angled and isosceles.

122. From the figure (if PQ is the pole of height h),



$$BQ = h \cot \alpha = CQ = AQ$$

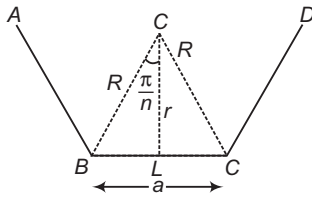
$\therefore Q$ is the circumcentre of $\triangle ABC$.

$$\text{Hence, } \frac{a}{\sin A} = 2R = 2h \cot \alpha$$

$$\Rightarrow a \sin \alpha = 2h \cos \alpha \sin A$$

123. Area of polygon,

$$\Delta = n \text{ times the area of } \triangle OBC = \frac{na^2}{4} \cot \frac{\pi}{n} \quad \dots (i)$$



$$\begin{aligned} \text{Now, } a &= 2r \tan \frac{\pi}{n} \\ \Rightarrow \Delta &= nr^2 \tan \left(\frac{\pi}{n} \right) \quad [\text{from Eq. (i)}] \end{aligned}$$

$$\begin{aligned} \text{Also, } a &= 2R \sin \frac{\pi}{n} \\ \Rightarrow \Delta &= \frac{nR^2}{2} \sin \left(\frac{2\pi}{n} \right) \quad [\text{from Eq. (i)}] \end{aligned}$$

124. $a > b > c \Rightarrow s - a > s - b > s - c$
 $\Rightarrow \frac{\Delta}{s-a} > \frac{\Delta}{s-b} > \frac{\Delta}{s-c} \Rightarrow r_1 > r_2 > r_3$

Also, cosine is a decreasing function for $x \in [0, \pi/2]$
 For $a > b > c \Rightarrow A > B > C \Rightarrow \cos A < \cos B < \cos C$
 Hence, both statements are true but Statement II does not explain Statement I.

125. Statement II is false. Because if $p = 2, q = 4, r = 6$,
 then $p : q : r = 1 : 2 : 3$ but $p \neq 1$.
 For Statement I, let $\tan A = k, \tan B = 2k, \tan C = 3k$,
 then from $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
 (in a triangle), we get

$$\begin{aligned} 6k &= 6k^3 \\ \Rightarrow k &= 0, 1, -1, \text{ but } k = 0, -1 \text{ is not possible, so } k = 1. \\ \Rightarrow \tan A &= 1 \\ \Rightarrow A &= 45^\circ \end{aligned}$$

So, Statement I is correct.

126. Statement II is obviously correct.

For Statement I, let the circumcentre be at $(0, 0)$ and the vertices of the triangle be $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) , then orthocentre of the triangle becomes

$$(x_1 + x_2 + x_3, y_1 + y_2 + y_3)$$

This implies that, if the centroid is rational, then orthocentre is also rational and $x_1 + x_2 + x_3$ can be rational even, if x_1, x_2, x_3 are not all rational.

127. Since, $(L_1 + L_3) - (L_1 + L_2) = 0$ passes through the point of intersection of the two lines $L_1 + L_3 = 0$ and $L_1 + L_2 = 0$ and also passes through the point A, hence it has to be the bisector of internal $\angle A$. Statement II is also correct but is not a correct explanation of Statement I.

128. We have, $BA \times BF = BD \times BC$ and $CE \times CA = CD \times CB$

$$\text{Dividing, we have } \frac{BA \times BF}{CE \times CA} = \frac{BD}{CD}$$

$$\begin{aligned} \text{But } \frac{AB}{AC} &= \frac{BD}{DC} \Rightarrow \frac{BD}{DC} \cdot \frac{BF}{CE} = \frac{BD}{CD} \\ \Rightarrow BF &= CE \end{aligned}$$

$$129. R = \frac{abc}{4\Delta} = \frac{60}{24} = 2.5$$

$$130. r = \frac{\Delta}{s} = \frac{6}{6} = 1$$

131. Since, the triangle is right angled, greatest angle is 90° . Also, the least angle is opposite to side a, which is $\sin^{-1} \frac{3}{5}$.

$$\Rightarrow 90^\circ - \sin^{-1} \frac{3}{5} = \cos^{-1} \frac{3}{5}$$

$$\begin{aligned} 132. \Delta_{DEF} &= \frac{1}{2} a \cos A b \cos B \sin(\pi - 2C) a \\ &= \frac{ab \cos A \cos B \sin 2C}{2} \\ &= 2 \cos A \cos B \cos C \cdot \frac{ab \sin C}{2} \\ \Rightarrow \frac{\Delta_{DEF}}{\Delta_{ABC}} &= 2 \cos A \cos B \cos C \quad \left[\because \Delta_{ABC} = ab \frac{\sin C}{2} \right] \end{aligned}$$

133. Let R be the radius of the circumcircle of $\triangle DEF$, then

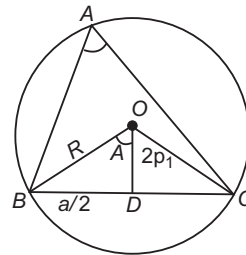
$$R' = \frac{(a \cos A)(b \cos B)(c \cos C)}{4\Delta_{DEF}} = \frac{(abc)(\cos A \cos B \cos C)}{4 \cdot 2(\cos A \cos B \cos C) \Delta_{ABC}} = \frac{1}{2} \cdot \frac{abc}{4\Delta_{ABC}} = \frac{1}{2} R$$
 [here, R is circumradius of $\triangle ABC$]

$$\begin{aligned} 134. \text{Radius of } \triangle DEF &= \frac{\Delta_{DEF}}{S_{DEF}} \\ &= \frac{(2 \cos A \cos B \cos C) \frac{1}{2} ab \sin C}{\left(\frac{a \cos A + b \cos B + c \cos C}{2} \right)} \\ &= 2R \cos A \cos B \cos C \end{aligned}$$

Solutions (Q. Nos. 135-137)

We know that, $\Delta = abc / 4R$

$$\therefore \Delta_1 = \frac{OB \times OC \times BC}{4R_1} = \frac{aR^2}{4R_1}$$



$$\Rightarrow \frac{a}{R_1} = \frac{4\Delta_1}{R^2}, \text{ similarly } \frac{b}{R^2} = \frac{4\Delta_2}{R^2}, \frac{c}{R_3} = \frac{4\Delta_3}{R^2}$$

$$135. \frac{a}{R_1} + \frac{b}{R_2} + \frac{c}{R_3} = \frac{4}{R^2} (\Delta_1 + \Delta_2 + \Delta_3) = \frac{4\Delta}{R^2}$$

$$\angle BOD = \frac{1}{2} \angle BOC = \angle A$$

$$\text{From } \triangle OBD, \frac{a/2}{2r_1} = \tan A$$

$$\Rightarrow \frac{a}{r_1} = 4 \tan A, \text{ similarly } \frac{b}{r_2} = 4 \tan B, \frac{c}{r_3} = 4 \tan C$$

$$136. \frac{a}{r_1} + \frac{b}{r_2} + \frac{c}{r_3} = 4 (\tan A + \tan B + \tan C)$$

$$= 4 \tan A \tan B \tan C \quad [\because A + B + C = 180^\circ]$$

$$a = 4r_1 \tan A = \frac{4\Delta_1 R_1}{R^2}$$

$$\Rightarrow \frac{R_1}{r_1} = \frac{R^2 \tan A}{\Delta_1}$$

$$\text{Similarly, } \frac{R_2}{r_2} = \frac{R^2 \tan B}{\Delta_2}, \frac{R_3}{r_3} = \frac{R^2 \tan C}{\Delta_3}$$

$$137. \frac{R_1}{r_1} + \frac{R_2}{r_2} + \frac{R_3}{r_3} = R^2 \left[\frac{\tan A}{\Delta_1} + \frac{\tan B}{\Delta_2} + \frac{\tan C}{\Delta_3} \right]$$

$$138. \text{Area of } \triangle ABC = \frac{1}{2} \cdot 2 \cdot 5 \cdot \frac{\sqrt{3}}{2} = \frac{5\sqrt{3}}{2}$$

$$\text{Area of } \triangle ADC = 4\sqrt{3} - \frac{5\sqrt{3}}{2} = \frac{3\sqrt{3}}{2}$$

$$139. \cos 60^\circ = \frac{29 - AC^2}{20} = \frac{1}{2}$$

$$\Rightarrow AC^2 = 19$$

$$\Rightarrow R = \sqrt{\frac{19}{3}}$$

$$\therefore \text{Area of circle} = \frac{19\pi}{3}$$

$$140. \text{Area of } \triangle ADC = \frac{3\sqrt{3}}{2}, \text{ let } AD = x \text{ and } DC = y$$

$$\Rightarrow \frac{1}{2} xy \sin 120^\circ = \frac{3\sqrt{3}}{2} \Rightarrow xy = 6$$

$$AC^2 = 19$$

$$\Rightarrow x^2 + y^2 + xy = 19$$

$$\Rightarrow x^2 + y^2 = 13$$

$$\Rightarrow (x, y) = (2, 3) \text{ or } (3, 2)$$

Hence, AD may be equal to AB = 2

$$141. A. \because a^4 - 2(b^2 + c^2)a^2 + b^4 + b^2c^2 + c^4 = 0$$

$$\Rightarrow (b^2 + c^2 - a^2)^2 - (b^2 + c^2)^2 + b^4 + b^2c^2 + c^4 = 0$$

$$\Rightarrow (b^2 + c^2 - a^2)^2 = b^2c^2$$

$$\Rightarrow \left(\frac{b^2 + c^2 - a^2}{2bc} \right)^2 = \frac{1}{4}$$

$$\Rightarrow \cos^2 A = \frac{1}{4} \Rightarrow \cos A = \pm \frac{1}{2}$$

$$\therefore A = 60^\circ \text{ or } 120^\circ$$

$$B. \because \cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\Rightarrow a^2 + b^2 - c^2 = 2ab \cos C$$

$$\text{On squaring both sides, then}$$

$$a^4 + b^4 + c^4 + 2a^2b^2 - 2a^2c^2 - 2b^2c^2$$

$$= 4a^2b^2 \cos^2 C$$

$$\Rightarrow 3a^2b^2 = 4a^2b^2 \cos^2 C$$

$$[\because a^4 + b^4 + c^4 = a^2b^2 + 2b^2c^2 + 2c^2a^2]$$

$$\Rightarrow \cos^2 C = \frac{3}{4}$$

$$\Rightarrow \cos C = \pm \frac{\sqrt{3}}{2}$$

$$\therefore C = 30^\circ \text{ or } 150^\circ$$

$$C. \because a^4 + b^4 + c^4 + 2a^2c^2 = 2a^2b^2 + 2b^2c^2$$

$$\Rightarrow b^4 - 2(a^2 + c^2)b^2 + (a^2 + c^2)^2 = 0$$

$$\Rightarrow \{b^2 - (a^2 + c^2)\}^2 = 0$$

$$\Rightarrow a^2 + c^2 = b^2 \Rightarrow \cos B = 0$$

$$\therefore B = 90^\circ$$

$$142. A. R \geq 2r, \text{ if } R = 40, r \leq 20$$

$$B. \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}, AM \geq HM \frac{r_1 + r_2 + r_3}{r} \geq 9$$

$$C. \because GM \geq HM$$

$$\therefore (r_1 r_2 r_3)^{1/3} \geq \frac{3}{\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3}} \Rightarrow (r_1 r_2 r_3)^{1/3} \geq \frac{3\Delta}{s} = 3r$$

$$\Rightarrow \frac{r_1 r_2 r_3}{r^3} \geq 27$$

$$D. \tan B \cdot \tan C < 1$$

$$143. \text{Let } a = \frac{x+y}{2} \text{ and } b = xr, c = xr^2$$

$$\text{Then, } y = xr^3$$

$$\text{Now, } \frac{\sin^3 B + \sin^3 C}{\sin A \sin B \sin C} = \frac{b^3 + c^3}{\frac{abc}{x^3 r^3 + x^3 r^6}} = \frac{2r^3(1+r^3)}{\frac{x + xr^3}{2} \cdot xr \cdot xr^2} = \frac{2r^3(1+r^3)}{r^3(1+r^3)} = 2$$

$$144. \cos 60^\circ = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\Rightarrow \frac{b^2 + c^2}{2bc} = \frac{a^2 + bc}{2bc} \quad \dots(i)$$

$$\therefore \frac{b}{c+a} + \frac{c}{a+b} = \frac{ab + b^2 + c^2 + ac}{(a+b)(c+a)}$$

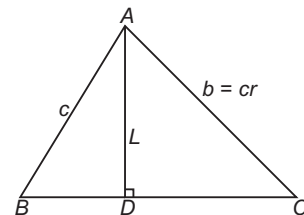
$$= \frac{ab + a^2 + bc + ac}{a^2 + ac + ab + bc} \quad [\text{from Eq. (i)}]$$

$$= 1$$

$$145. \text{Let } D \text{ be the foot of the altitude from } A \text{ and } BC = a,$$

$$AB = c, AC = cr, AD = L = c \sin B$$

$$\text{Now, } \frac{ar}{1-r^2} = \frac{ac^2 r}{c^2 - r^2 c^2} = \frac{abc}{c^2 - b^2}$$



$$= \frac{a \sin B \sin C}{\sin^2 C - \sin^2 B} = \frac{a \sin B \sin C}{\sin(C-B) \sin(C+B)}$$

$$= \frac{a \sin B \sin C}{\sin A \sin(C-B)} = \frac{c \sin B}{\sin(C-B)} = \frac{L}{\sin(C-B)}$$

$$\Rightarrow L \leq \frac{ar}{1-r^2} = \frac{ar}{\lambda - \mu^2} \Rightarrow \lambda - \mu = 0$$

146. The given equation is

$$\begin{aligned} 4 \sin A \sin B + 4 \sin B \sin C + 4 \sin C \sin A &= 9 \\ \Rightarrow 2 \cos(A-B) - 2 \cos(A+B) + 2 \cos(B-C) \\ &\quad - 2 \cos(B+C) + 2 \cos(C-A) - 2 \cos(C+A) = 9 \\ \Rightarrow 2 [\cos(A-B) + \cos(B-C) + \cos(C-A)] \\ &= 9 - 2 [\cos A + \cos B + \cos C] \geq 9 - 2 \times \frac{3}{2} = 6 \end{aligned}$$

$$\begin{aligned} \Rightarrow \cos(A-B) + \cos(B-C) + \cos(C-A) &\geq 3 \\ \text{But } \cos(A-B) &\leq 1, \cos(B-C) \leq 1, \cos(C-A) \leq 1 \\ \Rightarrow \cos(A-B) &= 1, \cos(B-C) = 1, \cos(C-A) = 1 \\ \Rightarrow A=B=C, &\text{ therefore } \triangle ABC \text{ is equilateral.} \\ \text{Hence, } \Delta &= \sqrt{3} \end{aligned}$$

147. $2 \sin B = \sin A + \sin C = \sin A + \sin(A+B)$

$$\begin{aligned} &= \sin A + \sin A \cos B + \cos A \sin B \\ \Rightarrow \sin B(2 - \cos A) &= \sin A(1 + \cos B) \\ \Rightarrow \frac{\sin B}{1 + \cos B} &= \frac{\sin A}{2 - \cos A} = k \quad [\text{say}] \\ \Rightarrow \tan \frac{B}{2} &= \frac{\sin A}{2 - \cos A} = k \\ \Rightarrow \sin A &= 2k - k \cos A \\ \Rightarrow \sin A + k \cos A &= 2k \\ \Rightarrow \sin(A + \alpha) &= \frac{2k}{\sqrt{1+k^2}} \end{aligned}$$

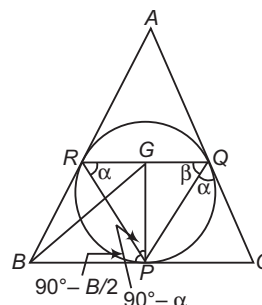
$$\begin{aligned} \text{where, } \tan \alpha &= \frac{k}{1} \\ \Rightarrow \frac{2k}{\sqrt{1+k^2}} &\leq 1 \\ \Rightarrow 2k &\leq \sqrt{1+k^2} \Rightarrow 4k^2 \leq 1+k^2 \\ \Rightarrow 3k^2 - 1 &\leq 0 \Rightarrow k \leq \frac{1}{\sqrt{3}} \end{aligned}$$

$\Rightarrow$ Maximum value of $\tan \frac{B}{2}$ is

$$\frac{1}{\sqrt{3}} = \sqrt{\frac{\lambda}{\mu}}$$

$$\therefore \lambda + \mu = 4$$

148. Since, $BP = BR \Rightarrow \angle BPR = \angle BRP = 90^\circ - \frac{B}{2}$



$$\Rightarrow PR = 2BP \sin \frac{B}{2} \Rightarrow RG = 2BP \sin \frac{B}{2} \cos \alpha$$

Similarly, $QG = 2PC \sin \frac{C}{2} \cos \beta$

$$\Rightarrow \frac{RG}{QG} = \frac{BP}{PC} \cdot \frac{\sin \frac{B}{2} \cos \alpha}{\sin \frac{C}{2} \cos \beta}$$

$$\Rightarrow \frac{RG}{QG} = \frac{BP}{CQ} \cdot \frac{\sin \frac{B}{2} \cos \left(90^\circ - \frac{C}{2}\right)}{\sin \frac{C}{2} \cos \left(90^\circ - \frac{B}{2}\right)}$$

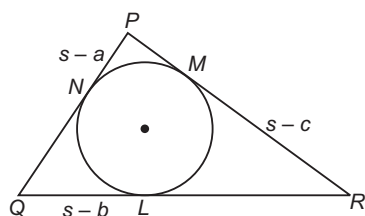
$$\Rightarrow \frac{RG}{QG} = \frac{BR}{CQ} \Rightarrow \frac{RG}{QG} \cdot \frac{CQ}{BR} = 1$$

Entrances Gallery

1. $x = a + b$

$$\begin{aligned} &\frac{y}{x^2 - c^2} = \frac{ab}{y} \\ \Rightarrow \frac{a^2 + b^2 - c^2}{2ab} &= -\frac{1}{2} = \cos 120^\circ \\ \Rightarrow \angle C = \frac{2\pi}{3} \Rightarrow R &= \frac{abc}{4\Delta}, r = \frac{\Delta}{s} \\ \Rightarrow \frac{r}{R} = \frac{4\Delta^2}{s(abc)} &= \frac{4 \left[\frac{1}{2} ab \sin \left(\frac{2\pi}{3} \right) \right]^2}{\frac{x+c}{2} \cdot y \cdot c} \Rightarrow \frac{r}{R} = \frac{3y}{2c(x+c)} \end{aligned}$$

2. Let $s - a = 2k - 2, s - b = 2k, s - c = 2k + 2, k \in I, k > 1$
Adding, we get $s = 6k$



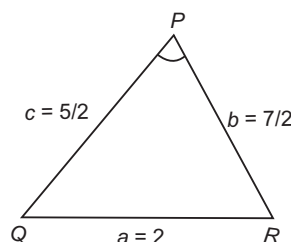
$$\text{So, } a = 4k + 2, b = 4k, c = 4k - 2$$

Now,

$$\begin{aligned} \cos P &= \frac{1}{3} \\ \Rightarrow \frac{b^2 + c^2 - a^2}{2bc} &= \frac{1}{3} \\ \Rightarrow 3[(4k)^2 + (4k-2)^2 - (4k+2)^2] &= 2 \times 4k(4k-2) \\ \Rightarrow 3[16k^2 - 4(4k) \times 2] &= 8k(4k-2) \\ \Rightarrow 48k^2 - 96k &= 32k^2 - 16k \\ \Rightarrow 16k^2 &= 80k \\ \Rightarrow k &= 5 \end{aligned}$$

So, sides are 22, 20, 18.

$$3. \frac{2 \sin P - 2 \sin P \cos P}{2 \sin P + 2 \sin P \cos P} = \frac{1 - \cos P}{1 + \cos P} = \frac{2 \sin^2 \frac{P}{2}}{2 \cos^2 \frac{P}{2}}$$



$$\begin{aligned}
 &= \tan^2 \frac{P}{2} \\
 &= \frac{(s-b)(s-c)}{s(s-a)} \\
 &= \frac{[(s-b)(s-c)]^2}{\Delta^2} \\
 &= \frac{\left[\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\right]^2}{\Delta^2} = \left(\frac{3}{4\Delta}\right)^2
 \end{aligned}$$

4. $B = 60^\circ$

$$\begin{aligned}
 \therefore \frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A &= 2 \sin A \cos C + 2 \sin C \cos A \\
 &= 2 \sin (A+C) = 2 \sin B \\
 &= 2 \times \frac{\sqrt{3}}{2} = \sqrt{3}
 \end{aligned}$$

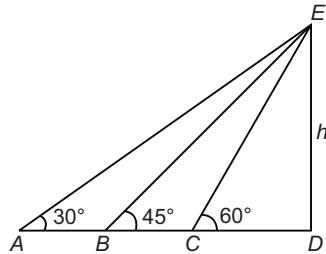
5. Using cosine rule for $\angle C$,

$$\begin{aligned}
 \frac{\sqrt{3}}{2} &= \frac{(x^2 + x + 1)^2 + (x^2 - 1)^2 - (2x + 1)^2}{2(x^2 + x + 1)(x^2 - 1)} \\
 \Rightarrow \sqrt{3} &= \frac{2x^2 + 2x - 1}{x^2 + x + 1} \\
 \Rightarrow (\sqrt{3} - 2)x^2 + (\sqrt{3} - 2)x + (\sqrt{3} + 1) &= 0 \\
 \Rightarrow x &= \frac{(2 - \sqrt{3}) \pm \sqrt{3}}{2(\sqrt{3} - 2)} \\
 \Rightarrow x &= -(2 + \sqrt{3}), 1 + \sqrt{3} \\
 \Rightarrow x &= 1 + \sqrt{3} \quad [\text{as } x > 0]
 \end{aligned}$$

6. $\Delta = \frac{1}{2} ab \sin C$

$$\begin{aligned}
 \Rightarrow \sin C &= \frac{2\Delta}{ab} = \frac{2 \times 15\sqrt{3}}{6 \times 10} = \frac{\sqrt{3}}{2} \\
 \Rightarrow C &= 120^\circ \\
 \Rightarrow c &= \sqrt{a^2 + b^2 - 2ab \cos C} \\
 &= \sqrt{6^2 + 10^2 - 2 \times 6 \times 10 \times \cos 120^\circ} = 14 \\
 \therefore r &= \frac{\Delta}{s} \Rightarrow r^2 = \frac{225 \times 3}{\left(\frac{6+10+4}{2}\right)^2} = 3
 \end{aligned}$$

7. According to the given information, the figure should be as follows.



Let the height of tower be h .

In $\triangle EDA$,

$$\begin{aligned}
 \tan 30^\circ &= \frac{ED}{AD} \\
 \Rightarrow \frac{1}{\sqrt{3}} &= \frac{ED}{AD} = \frac{h}{AD} \\
 \Rightarrow AD &= h\sqrt{3}
 \end{aligned}$$

In $\triangle EDB$,

$$\begin{aligned}
 \tan 45^\circ &= \frac{h}{BD} \\
 \Rightarrow BD &= h
 \end{aligned}$$

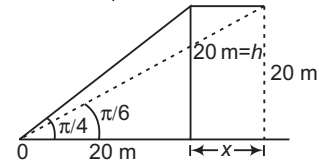
In $\triangle EDC$,

$$\begin{aligned}
 \tan 60^\circ &= \frac{h}{CD} \\
 \Rightarrow CD &= \frac{h}{\sqrt{3}}
 \end{aligned}$$

Now,

$$\begin{aligned}
 \frac{AB}{BC} &= \frac{AD - BD}{BD - CD} \\
 \Rightarrow \frac{AB}{BC} &= \frac{h\sqrt{3} - h}{h - \frac{h}{\sqrt{3}}} \\
 \Rightarrow \frac{AB}{BC} &= \frac{h(\sqrt{3} - 1)}{h(\sqrt{3} - 1)} \\
 \Rightarrow \frac{AB}{BC} &= \frac{\sqrt{3} - 1}{(\sqrt{3} - 1)} \times \sqrt{3} \\
 \Rightarrow \frac{AB}{BC} &= \frac{\sqrt{3}}{1} \\
 \therefore AB : BC &= \sqrt{3} : 1
 \end{aligned}$$

$$8. \tan 30^\circ = \frac{20}{20 + x} = \frac{1}{\sqrt{3}}$$

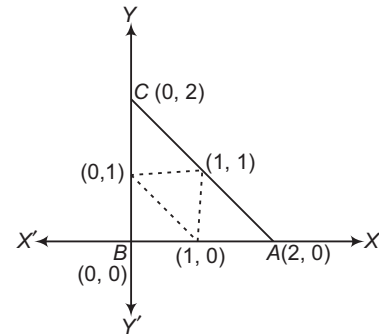


$$\begin{aligned}
 20 + x &= 20\sqrt{3} \\
 x &= 20(\sqrt{3} - 1)
 \end{aligned}$$

$\Rightarrow$ Speed is $20(\sqrt{3} - 1)$ m/s.

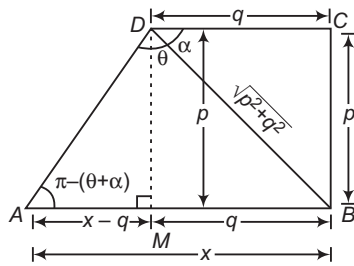
9. Given, mid-points of a triangle are $(0, 1)$, $(1, 1)$ and $(1, 0)$. Plotting these points on a graph paper and make a triangle.

So, the sides of a triangle will be 2, 2 and $\sqrt{2^2 + 2^2}$ i.e. $2\sqrt{2}$.



$$\begin{aligned}
 \text{x-coordinate of incentre} &= \frac{2 \cdot 0 + 2\sqrt{2} \cdot 0 + 2 \cdot 2}{2 + 2 + 2\sqrt{2}} \\
 &= \frac{2}{2 + \sqrt{2}} \times \frac{2 - \sqrt{2}}{2 - \sqrt{2}} \\
 &= 2 - \sqrt{2}
 \end{aligned}$$

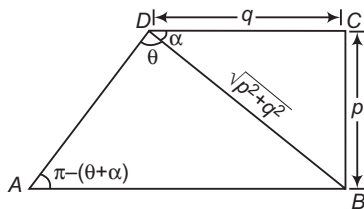
10. Let $AB = x$



$$\begin{aligned} \text{In } \triangle DAM, \tan(\pi - \theta - \alpha) &= \frac{p}{x-q} \\ \Rightarrow \tan(\theta + \alpha) &= \frac{p}{q-x} \\ \Rightarrow q-x &= p \cot(\theta + \alpha) \\ \Rightarrow x &= q - p \cot(\theta + \alpha) \\ &= q - p \left(\frac{\cot \theta \cot \alpha - 1}{\cot \alpha + \cot \theta} \right) \quad \left[\because \cot \alpha = \frac{q}{p} \right] \\ &= q - p \left(\frac{\frac{q}{p} \cot \theta - 1}{\frac{q}{p} + \cot \theta} \right) \\ &= q - p \left(\frac{q \cot \theta - p}{q + p \cot \theta} \right) \\ &= q - p \left(\frac{q \cos \theta - p \sin \theta}{q \sin \theta + p \cos \theta} \right) \\ \Rightarrow x &= \frac{q^2 \sin \theta + pq \cos \theta - pq \cos \theta + p^2 \sin \theta}{p \cos \theta + q \sin \theta} \\ \Rightarrow AB &= \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta} \end{aligned}$$

Aliter

Applying sine rule in $\triangle ABD$,



$$\begin{aligned} \frac{AB}{\sin \theta} &= \frac{\sqrt{p^2 + q^2}}{\sin \{\pi - (\theta + \alpha)\}} \\ \Rightarrow \frac{AB}{\sin \theta} &= \frac{\sqrt{p^2 + q^2}}{\sin(\theta + \alpha)} \\ \Rightarrow AB &= \frac{\sqrt{p^2 + q^2} \sin \theta}{\sin \theta \cos \alpha + \cos \theta \sin \alpha} \\ &= \frac{(p^2 + q^2) \sin \theta}{q \sin \theta + p \cos \theta} \\ &= \frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta} \end{aligned}$$

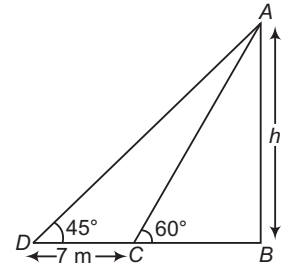
$$\left[\begin{aligned} \because \cos \alpha &= \frac{q}{\sqrt{p^2 + q^2}} \\ \text{and } \sin \alpha &= \frac{p}{\sqrt{p^2 + q^2}} \end{aligned} \right]$$

11. $\therefore \frac{a}{2R} = \sin \frac{\pi}{n}$ and $\frac{a}{2r} = \tan \frac{\pi}{n}$

$$\therefore \frac{r}{R} = \cos \frac{\pi}{n}$$

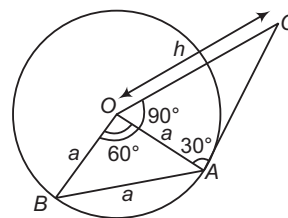
$$\begin{aligned} n=3 \text{ gives } \frac{r}{R} &= \frac{1}{2} \\ n=4 \text{ gives } \frac{r}{R} &= \frac{1}{\sqrt{2}} \\ n=6 \text{ gives } \frac{r}{R} &= \frac{\sqrt{3}}{2} \end{aligned}$$

12. In $\triangle ABC$, $BC = h \cot 60^\circ$
and in $\triangle ABD$, $BD = h \cot 45^\circ$
Since, $BD - BC = DC$
 $\Rightarrow h \cot 45^\circ - h \cot 60^\circ = 7$



$$\begin{aligned} \Rightarrow h &= \frac{7}{\cot 45^\circ - \cot 60^\circ} = \frac{7}{\left(1 - \frac{1}{\sqrt{3}}\right)} \\ &= \frac{7\sqrt{3}}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1} = \frac{7\sqrt{3}}{2} (\sqrt{3}+1) \text{ m} \end{aligned}$$

13. Let h be the height of a tower.
Since, $\angle AOB = 60^\circ$
So, $\triangle OAB$ is an equilateral.
 $\therefore OA = OB = AB = a$



$$\begin{aligned} \text{In } \triangle OAC, \tan 30^\circ &= \frac{h}{a} \Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{a} \\ \Rightarrow h &= \frac{a}{\sqrt{3}} \end{aligned}$$

14. Since, $\tan \frac{P}{2}$ and $\tan \frac{Q}{2}$ are the roots of equation $ax^2 + bx + c = 0$.

$$\therefore \left. \begin{aligned} \tan \frac{P}{2} + \tan \frac{Q}{2} &= -\frac{b}{a} \\ \text{and } \tan \frac{P}{2} \tan \frac{Q}{2} &= \frac{c}{a} \end{aligned} \right\} \dots(i)$$

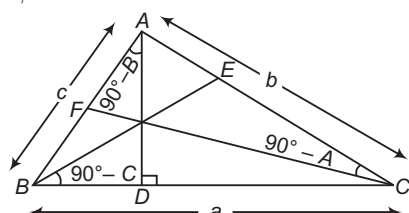
Also, $\frac{P}{2} + \frac{Q}{2} + \frac{R}{2} = \frac{\pi}{2}$ [$\because P + Q + R = \pi$]
 $\Rightarrow \frac{P+Q}{2} = \frac{\pi}{2} - \frac{R}{2}$
 $\Rightarrow \frac{P+Q}{2} = \frac{\pi}{4}$ [$\because \angle R = \frac{\pi}{2}$, given]
 $\Rightarrow \tan\left(\frac{P}{2} + \frac{Q}{2}\right) = 1$
 $\Rightarrow \frac{\tan \frac{P}{2} + \tan \frac{Q}{2}}{1 - \tan \frac{P}{2} \tan \frac{Q}{2}} = 1$
 $\Rightarrow \frac{-\frac{b}{a}}{1 - \frac{b}{a}} = 1$
 $\Rightarrow -\frac{b}{a} = 1 - \frac{b}{a}$ [from Eq. (i)]
 $\Rightarrow -b = a - b$
 $\Rightarrow c = a + b$

Aliter

Since, $\angle R = \frac{\pi}{2} \Rightarrow \angle P + \angle Q = \frac{\pi}{2}$
 $\Rightarrow \frac{\angle P}{2} = \frac{\pi}{4} - \frac{\angle Q}{2}$
 $\therefore \tan \frac{P}{2} = \tan\left(\frac{\pi}{4} - \frac{Q}{2}\right)$
 $= \frac{\tan \frac{\pi}{4} - \tan \frac{Q}{2}}{1 + \tan \frac{\pi}{4} \tan \frac{Q}{2}}$
 $\Rightarrow \tan \frac{P}{2} + \tan \frac{P}{2} \tan \frac{Q}{2} = 1 - \tan \frac{Q}{2}$
 $\Rightarrow \tan \frac{P}{2} + \tan \frac{Q}{2} = 1 - \tan \frac{P}{2} \tan \frac{Q}{2}$
 $\Rightarrow -\frac{b}{a} = 1 - \frac{b}{a} \Rightarrow -b = a - c$
 $\Rightarrow c = a + b$

15. We know that, $\frac{c}{\sin C} = 2R$
 $\Rightarrow c = 2R$ [$\because C = 90^\circ$]... (i)
 and $\tan \frac{C}{2} = \frac{r}{s-c} \Rightarrow \tan \frac{\pi}{4} = \frac{r}{s-c}$
 $\therefore r = s - c \Rightarrow r = \frac{a+b+c}{2} - c$
 $\Rightarrow a + b - c = 2r$... (ii)
 On adding Eqs. (i) and (ii), we get
 $2(r + R) = a + b$

16. In $\triangle BAD$,



$\cos(90^\circ - B) = \frac{AD}{c}$
 $\Rightarrow AD = c \sin B$

Similarly,

$BE = a \sin C$
 and $CF = b \sin A$
 Since, AD, BE and CF are in HP.
 $\therefore \frac{1}{\sin C \sin B}, \frac{1}{\sin A \sin C}, \frac{1}{\sin B \sin A}$ are in AP.
 $\Rightarrow \sin A, \sin B$ and $\sin C$ are in AP.

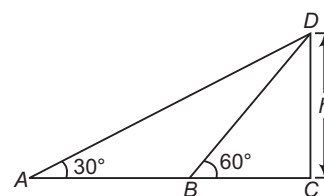
Aliter

$\text{ar}(\triangle ABC) = \frac{1}{2} \times BC \times AD$
 $\Rightarrow \Delta = \frac{1}{2} \times a \times AD$
 $\Rightarrow AD = \frac{2\Delta}{a}$
 Similarly, $BE = \frac{2\Delta}{b}$ and $CF = \frac{2\Delta}{c}$
 $\therefore AD, BE$ and CF are in HP.
 $\therefore \frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in HP.
 $\Rightarrow a, b$ and c are in AP.
 $\Rightarrow \sin A, \sin B$ and $\sin C$ are in AP.

17. Let $a = \sin \alpha, b = \cos \alpha, c = \sqrt{1 + \sin \alpha \cos \alpha}$
 Here, we see that the greatest side is c .

$\therefore \cos C = \frac{a^2 + b^2 - c^2}{2ab}$
 $\Rightarrow \cos C = \frac{\sin^2 \alpha + \cos^2 \alpha - 1 - \sin \alpha \cos \alpha}{2 \sin \alpha \cos \alpha}$
 $\Rightarrow \cos C = -\frac{\sin \alpha \cos \alpha}{2 \sin \alpha \cos \alpha}$
 $\Rightarrow \cos C = -\frac{1}{2} = \cos 120^\circ$
 $\Rightarrow \angle C = 120^\circ$

18. Let $CD (=h)$ be the height of the tree and $BC (=x)$ be the width of the river.



In $\triangle CBD$, $\tan 60^\circ = \frac{CD}{BC}$
 $\Rightarrow \sqrt{3} = \frac{h}{x}$
 $\Rightarrow h = x\sqrt{3}$... (i)
 and in $\triangle CAD$,
 $\tan 30^\circ = \frac{CD}{AC}$
 $\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{40+x}$
 $\Rightarrow h\sqrt{3} = 40+x$
 $\Rightarrow 3x = 40+x$ [from Eq. (i)]
 $\Rightarrow 2x = 40$
 $\Rightarrow x = 20 \text{ m}$

19. Let $AB = a$,

and $ON \perp AB$

$AN = BN$

In $\triangle AON$, $\tan \frac{\pi}{n} = \frac{AN}{ON}$

$$\Rightarrow ON = AN \cot \frac{\pi}{n} = \frac{a}{2} \cot \frac{\pi}{2n} \quad \dots(i)$$

and $\sin \frac{\pi}{n} = \frac{AN}{OA}$

$$\Rightarrow OA = AN \operatorname{cosec} \frac{\pi}{n} = \frac{a}{2} \operatorname{cosec} \frac{\pi}{2n} \quad \dots(ii)$$

Now, sum of the radii = $ON + OA$

$$= \frac{a}{2} \cot \frac{\pi}{2n} + \frac{a}{2} \operatorname{cosec} \frac{\pi}{2n} \quad [\text{from Eqs. (i) and (ii)}]$$

$$= \frac{a}{2} \left[\frac{\cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} + \frac{1}{\sin \frac{\pi}{2n}} \right] = \frac{a}{2} \left[\frac{1 + \cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} \right] = \frac{a}{2} \left[\frac{2 \cos^2 \frac{\pi}{4n}}{2 \sin \frac{\pi}{4n} \cos \frac{\pi}{4n}} \right] = \frac{a}{2} \cot \frac{\pi}{4n}$$

20. Given, $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = \frac{3b}{2}$

$$\Rightarrow a \left[\frac{s(s-c)}{ab} \right] + c \left[\frac{s(s-a)}{bc} \right] = \frac{3b}{2}$$

$$\Rightarrow \frac{s(s-c + s-a)}{b} = \frac{3b}{2}$$

$$\Rightarrow 2s(2s - c - a) = 3b^2$$

$$\Rightarrow 2s(a + b + c - c - a) = 3b^2$$

$$\Rightarrow (a + b + c)b = 3b^2$$

$$\Rightarrow a + b + c = 3b$$

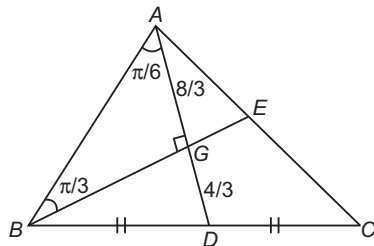
$$\Rightarrow 2b = a + c$$

Hence, a, b and c are in AP.

21. Given, $AD = 4$ and $BD = DC$

Since, the centroid G divides the line AD in the ratio $2 : 1$.

$$\therefore AG = \frac{8}{3} \text{ and } DG = \frac{4}{3}$$



In $\triangle ABG$,

$$\tan \frac{\pi}{3} = \frac{AG}{BG}$$

$$\Rightarrow BG = AG \cot \frac{\pi}{3}$$

$$\Rightarrow BG = \frac{8}{3} \times \frac{1}{\sqrt{3}} = \frac{8}{3\sqrt{3}}$$

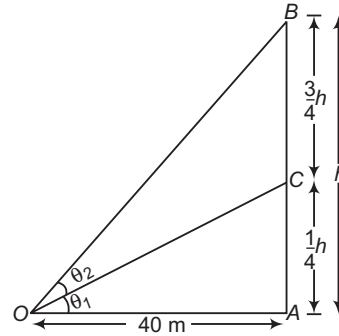
$$\text{Area of } \triangle ADB = \frac{1}{2} \times AD \times BG = \frac{1}{2} \times 4 \times \frac{8}{3\sqrt{3}} = \frac{16}{3\sqrt{3}}$$

Since, median divides a triangle into two triangles of equal area. Therefore,

$$\text{Area of } \triangle ABC = 2 \times \text{Area of } \triangle ADB$$

$$= 2 \times \frac{16}{3\sqrt{3}} = \frac{32}{3\sqrt{3}} \text{ sq units}$$

22. Given, $\theta_2 = \tan^{-1} \frac{3}{5}$



$$\Rightarrow \tan \theta_2 = \frac{3}{5} \quad \dots(i)$$

$$\text{In } \triangle AOC, \tan \theta_1 = \frac{AC}{AO}$$

$$\Rightarrow \tan \theta_1 = \frac{\frac{1}{4}h}{40} = \frac{h}{160} \quad \dots(ii)$$

and in $\triangle AOB$,

$$\tan(\theta_1 + \theta_2) = \frac{AB}{AO} = \frac{h}{40}$$

$$\Rightarrow \frac{\tan \theta_1 + \tan \theta_2}{1 - \tan \theta_1 \tan \theta_2} = \frac{h}{40}$$

$$\Rightarrow \frac{\frac{h}{160} + \frac{3}{5}}{1 - \frac{h}{160} \times \frac{3}{5}} = \frac{h}{40} \quad [\text{from Eqs. (i) and (ii)}]$$

$$\Rightarrow \frac{5[h + 96]}{800 - 3h} = \frac{h}{40}$$

$$\Rightarrow 200[h + 96] = 800h - 3h^2$$

$$\Rightarrow 3h^2 - 600h + 19200 = 0$$

$$\Rightarrow h^2 - 200h + 6400 = 0$$

$$\Rightarrow (h - 160)(h - 40) = 0$$

$$\Rightarrow h = 160 \text{ or } h = 40$$

Hence, height of the vertical pole is 40 m.

23. Since, $A + B + C = \pi$

$$\Rightarrow \frac{A + C}{2} = \frac{\pi - B}{2}$$

$$\Rightarrow \frac{A - B + C}{2} = \frac{\pi - B}{2}$$

$$\therefore 2ca \sin \left(\frac{A - B + C}{2} \right) = 2ca \sin \left(\frac{\pi - B}{2} \right)$$

$$= 2ac \cos B$$

$$= 2ac \left(\frac{a^2 + c^2 - b^2}{2ac} \right)$$

$$= a^2 + c^2 - b^2$$

24. Given that,

$$\begin{aligned} a &= 4, b = 3 \\ \angle A &= 60^\circ \\ \text{Now, } \cos 60^\circ &= \frac{c^2 + 9 - 16}{2 \times 3 \times c} \\ \Rightarrow \frac{1}{2} &= \frac{c^2 - 7}{2 \times 3c} \\ \Rightarrow c^2 - 3c - 7 &= 0 \end{aligned}$$

25. Given that,

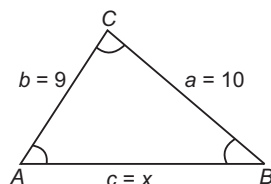
$$\begin{aligned} \tan \frac{A}{2} &= \frac{5}{6} \\ \text{and } \tan \frac{C}{2} &= \frac{2}{5} \\ \text{Now, } \tan \frac{A}{2} \tan \frac{C}{2} &= \frac{5}{6} \times \frac{2}{5} \\ \Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \cdot \sqrt{\frac{(s-a)(s-b)}{s(s-c)}} &= \frac{1}{3} \\ \Rightarrow \frac{s-b}{s} &= \frac{1}{3} \\ \Rightarrow 2s &= 3b \\ \Rightarrow a + c &= 2b \end{aligned}$$

$$[\because 2s = a + b + c]$$

$\Rightarrow a, b$ and c are in AP.

26. Let in the figure, $\angle A > \angle B > \angle C$

Since, angles are in AP.



$$\therefore \angle C = \theta - d, \angle B = \theta$$

$$\text{and } \angle A = \theta + d$$

$$\text{As, } \angle A + \angle B + \angle C = 180^\circ$$

$$\theta + d + \theta + \theta - d = 180^\circ$$

$$\Rightarrow 3\theta = 180^\circ$$

$$\Rightarrow \theta = 60^\circ$$

Now, in $\triangle ABC$,

$$\begin{aligned} \cos B &= \frac{a^2 + c^2 - b^2}{2ac} \\ \Rightarrow \cos 60^\circ &= \frac{(10)^2 + x^2 - 9^2}{2 \times 10 \times x} \\ \Rightarrow \frac{1}{2} &= \frac{100 + x^2 - 9^2}{20x} \\ \Rightarrow x^2 - 10x + 19 &= 0 \\ \therefore x &= \frac{10 \pm \sqrt{100 - 76}}{2} \\ &= \frac{10 \pm \sqrt{24}}{2} = 5 \pm \sqrt{6} \end{aligned}$$

27. Consider, $a [b \cos C - c \cos B]$

$$\begin{aligned} &= ab \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - ac \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \\ &= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} = b^2 - c^2 \end{aligned}$$

28. $\Sigma a^3 \sin(B - C) = \Sigma k^3 \sin^3 A \sin(B - C)$

$$\begin{aligned} &\left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k \right] \\ &= \Sigma k^3 [\sin^2 A \sin(B + C) \sin(B - C)] \\ &= k^3 \left[\left\{ \sin^2 A \times \frac{1}{2} (\cos 2C - \cos 2B) \right\} \right. \\ &\quad \left. + \{ \sin^2 B \times \frac{1}{2} (\cos 2A - \cos 2C) \} \right. \\ &\quad \left. + \{ \sin^2 C \times \frac{1}{2} (\cos 2B - \cos 2A) \} \right] \\ &= \frac{k^3}{2} [\sin^2 A (1 - 2 \sin^2 C - 1 + 2 \sin^2 B) \\ &\quad + \sin^2 B (1 - 2 \sin^2 A - 1 + 2 \sin^2 C) \\ &\quad + \sin^2 C (1 - 2 \sin^2 B - 1 + 2 \sin^2 A)] \\ &= \frac{k^3}{2} [-2 \sin^2 A \sin^2 C + 2 \sin^2 A \sin^2 B \\ &\quad - 2 \sin^2 B \sin^2 A + 2 \sin^2 B \sin^2 C \\ &\quad - 2 \sin^2 C \sin^2 B + 2 \sin^2 C \sin^2 A] \\ &= \frac{k^3}{2} [0] = 0 \end{aligned}$$

$$\begin{aligned} 29. \text{ Consider, } &\frac{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}{4b^2c^2} \\ &= \frac{2s(2s-2a)(2s-2b)(2s-2c)}{4b^2c^2} \quad [\because a+b+c=2s] \\ &= \frac{16[s(s-a)(s-b)(s-c)]}{4b^2c^2} \\ &= \frac{4\Delta^2}{b^2c^2} = \left(\frac{2\Delta}{bc} \right)^2 = \sin^2 A \quad \left[\because \sin A = \frac{2\Delta}{bc} \right] \end{aligned}$$

30. Given the ratio of angles of a triangle is 1 : 1 : 4.

Let angles of a triangle be A, B and C .

$$\therefore A : B : C = 1 : 1 : 4$$

$$\text{Let } A = x, B = x \text{ and } C = 4x$$

$$\therefore A + B + C = 180^\circ$$

$$\therefore x + x + 4x = 180^\circ$$

$$\Rightarrow 6x = 180^\circ \Rightarrow x = 30^\circ$$

$$\therefore A = 30^\circ, B = 30^\circ \text{ and } C = 120^\circ$$

Hence, the largest angle is 120° . So, the largest side of triangle is c .

$\therefore$ Perimeter of triangle : Largest side of a triangle

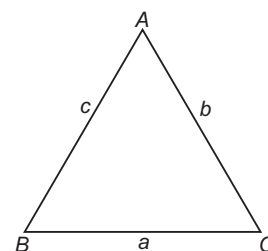
$$\begin{aligned} &= (a + b + c) : c \\ &= (\sin 30^\circ + \sin 30^\circ + \sin 120^\circ) : \sin 120^\circ \\ &= \left(\frac{1}{2} + \frac{1}{2} + \frac{\sqrt{3}}{2} \right) : \frac{\sqrt{3}}{2} = \left(1 + \frac{\sqrt{3}}{2} \right) : \frac{\sqrt{3}}{2} = 2 + \sqrt{3} : \sqrt{3} \end{aligned}$$

31. Since, A, B and C are in AP.

$$\text{We have, } 2B = A + C$$

$$\text{Also, } A + B + C = 180^\circ \quad \dots (i)$$

$$\Rightarrow 3B = 180^\circ \Rightarrow B = 60^\circ \quad \dots (ii)$$



Now, by using sine rule, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \sin C = \frac{c}{b} \sin B$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \sin 60^\circ$$

$$= \frac{\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{2} = \frac{1}{\sqrt{2}} = \sin 45^\circ$$

$$\Rightarrow C = 45^\circ \quad \dots(\text{iii})$$

$$\Rightarrow A = 75^\circ \quad [\text{from Eqs. (i), (ii) and (iii)}]$$

32. Given, $r_1 = 2$, $r_2 = 3$ and $r_3 = 6$

$$\therefore r_1 = \frac{\Delta}{s-a} \Rightarrow 2 = \frac{\Delta}{s-a}$$

$$\Rightarrow \frac{s-a}{\Delta} = \frac{1}{2} \quad \dots(\text{i})$$

$$r_2 = \frac{\Delta}{s-b} \Rightarrow 3 = \frac{\Delta}{s-b}$$

$$\Rightarrow \frac{s-b}{\Delta} = \frac{1}{3} \quad \dots(\text{ii})$$

$$\text{and } r_3 = \frac{\Delta}{s-c} \Rightarrow 6 = \frac{\Delta}{s-c} \Rightarrow \frac{s-c}{\Delta} = \frac{1}{6} \quad \dots(\text{iii})$$

On adding Eqs. (i), (ii) and (iii), we get

$$\frac{s-a}{\Delta} + \frac{s-b}{\Delta} + \frac{s-c}{\Delta} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$$

$$\Rightarrow \frac{3s - (a+b+c)}{\Delta} = \frac{3+2+1}{6}$$

$$\Rightarrow \frac{3s - 2s}{\Delta} = \frac{6}{6} = 1$$

$$\Rightarrow \frac{s}{\Delta} = 1 \quad \dots(\text{iv})$$

$$\therefore s^2 = r_1 r_2 + r_2 r_3 + r_3 r_1 = 2 \times 3 + 3 \times 6 + 6 \times 2$$

$$= 6 + 18 + 12 = 36$$

$$\Rightarrow s^2 = 36$$

$$\Rightarrow s = 6$$

$$\text{From Eq. (iv), } \frac{6}{\Delta} = 1, \Delta = 6$$

$$\text{Now, } r_1 = \frac{\Delta}{s-a}, 2 = \frac{6}{6-a}$$

$$\Rightarrow 6-a = 3 \Rightarrow a = 3$$

33. Rewrite the given relation as

$$a \left[\tan A - \tan \left(\frac{A+B}{2} \right) \right] = b \left[\tan \left(\frac{A+B}{2} \right) - \tan B \right]$$

$$\Rightarrow \frac{a \sin \left[A - \frac{A+B}{2} \right]}{\cos A \cos \left(\frac{A+B}{2} \right)} = \frac{b \sin \left[\frac{A+B}{2} - B \right]}{\cos B \cos \left(\frac{A+B}{2} \right)}$$

$$\Rightarrow \frac{a \sin \left(\frac{A-B}{2} \right)}{\cos A} = \frac{b \sin \left(\frac{A-B}{2} \right)}{\cos B}$$

$$\Rightarrow \frac{a}{\cos A} = \frac{b}{\cos B}$$

$$\Rightarrow \frac{\sin A}{\cos A} = \frac{\sin B}{\cos B}$$

$$\Rightarrow \tan A = \tan B$$

$$\Rightarrow A = B$$

34. By sine rule, $T_1 = k^3 \sin^3 A \cos (B-C)$

$$= k^3 \sin^2 A \sin (B+C) \cos (B-C)$$

$$= \frac{1}{2} k^3 [\sin^2 A (\sin 2B + \sin 2C)]$$

$$= \frac{1}{2} k^3 [2 \sin^2 A \sin B \cos B + 2 \sin^2 A \sin C \cos C]$$

$$\Rightarrow \Sigma a^3 \cos (B-C) = \Sigma k^3 \sin A \sin B (\sin A \cos B + \cos A \sin B)$$

$$= \Sigma k^3 \sin A \sin B \sin (A+B)$$

$$= \Sigma k^3 \sin A \sin B \sin C = \Sigma abc = 3abc$$

35. We have, $\frac{a}{\cos A} = \frac{b}{\cos B}$

$$\Rightarrow \frac{\sin A}{\cos A} = \frac{\sin B}{\cos B} \quad \left[\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \right]$$

$$\Rightarrow \sin A \cos B = \sin B \cos A \quad \dots(\text{i})$$

$$\Rightarrow 2 \sin A \cos B = \cos A \sin B + \cos A \sin B$$

$$= \cos A \sin B + \sin A \cos B \quad [\text{from Eq. (i)}]$$

$$= \sin (A+B) = \sin (\pi - C) = \sin C$$

36. Area of circle, $A_1 = \pi r^2$

Perimeter of a circle, $P = 2\pi r$

If a is side of regular polygon of n sides, then

$$na = 2\pi r$$

$$\Rightarrow a = \frac{2\pi r}{n} \quad \dots(\text{i})$$

Now, if A_2 = Area of regular polygon of n sides

Then, $A_2 = \frac{1}{4} na^2 \cot \left(\frac{\pi}{n} \right)$

$$= \frac{1}{4} n \left(\frac{2\pi r}{n} \right)^2 \cot \left(\frac{\pi}{n} \right) \quad [\text{from Eq. (i)}]$$

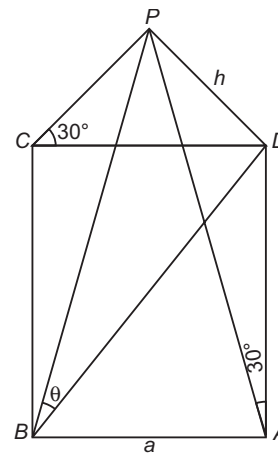
$$= \frac{1}{4} n \left(\frac{4\pi^2 r^2}{n^2} \right) \cot \left(\frac{\pi}{n} \right) = \frac{\pi^2 r^2}{n} \cot \left(\frac{\pi}{n} \right)$$

$$\therefore \frac{\text{Area of circle}}{\text{Area of regular polygon}} = \frac{\pi r^2}{\frac{\pi^2 r^2}{n} \cot \left(\frac{\pi}{n} \right)}$$

$$= \frac{\tan \left(\frac{\pi}{n} \right)}{\frac{\pi}{n}} = \tan \left(\frac{\pi}{n} \right) : \frac{\pi}{n}$$

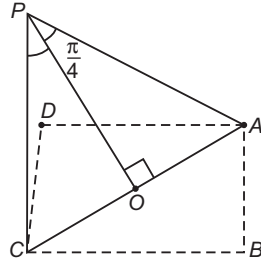
37. Let PD be a pole of height h .

From $\triangle PAD$ or $\triangle PCD$, $h = a \tan 30^\circ = \frac{a}{\sqrt{3}}$



From $\triangle PBD$, diagonal $BD = a\sqrt{2}$
 $\therefore \frac{a}{\sqrt{3}} = a\sqrt{2} \tan \theta$
 $\Rightarrow \tan \theta = \frac{1}{\sqrt{6}}$

38. Let $\angle OPA = \angle OPC = \frac{\pi}{4}$ and $OP = n$

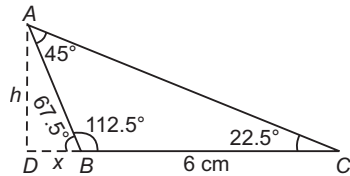


So, $\frac{OA}{OP} = \tan 45^\circ = 1$
 $\therefore OA = n, AC = 2n$
 Also, $\frac{OP}{PA} = \cos \frac{\pi}{4}$
 $\Rightarrow PA = \sqrt{2}n$
 Also, $AC = \sqrt{2} AB$
 $\Rightarrow 2n = \sqrt{2} \cdot AB$
 $\Rightarrow AB = \sqrt{2} n$
 $\therefore PA = AB = PB$

So, $\triangle BPA$ is equilateral.
 Hence, the required angle is 60° .

39. Since, $A + B + C = 180^\circ$
 $\Rightarrow \cot(A + B + C) = \frac{\Sigma \cot A \cot B - 1}{\cot A \cot B \cot C - \Sigma \cot A} = \frac{1}{0}$
 $\Rightarrow \cot A \cot B \cot C - \Sigma \cot A = 0$
 $\Rightarrow \frac{\cot A + \cot B + \cot C}{\cot A \cot B \cot C} = 1$

40. Applying sine rule in $\triangle ABC$,



$\frac{\sin 45^\circ}{6} = \frac{\sin 22.5^\circ}{AB}$
 $\Rightarrow AB = \frac{\sqrt{2 - \sqrt{2}} \times 6}{2 \times \frac{1}{\sqrt{2}}} = 3\sqrt{2} \sqrt{2 - \sqrt{2}}$
 In $\triangle ABD$, $\sin 67.5^\circ = \frac{h}{AB} \cdot \frac{\sqrt{2 + \sqrt{2}}}{2} = \frac{h}{3\sqrt{2} \sqrt{2 - \sqrt{2}}}$
 $\Rightarrow h = \frac{\sqrt{4 - 2} \times 3\sqrt{2}}{2} = \frac{3 \times 2}{2} = 3 \text{ cm}$

41. We have, $r^2 \sin P \sin Q = pq$
 $\Rightarrow \sin^2 R \sin P \sin Q = \sin P \sin Q$
 $\left[\because \frac{p}{\sin P} = \frac{q}{\sin Q} = \frac{r}{\sin R} \right]$

$\Rightarrow \sin^2 R = 1 \Rightarrow \sin R = 1 \quad [\because 0 < R \leq 90^\circ]$
 $\Rightarrow R = 90^\circ$
 $\Rightarrow \triangle PQR$ is a right angled triangle.

42. We have, $\frac{\cos 2A}{a^2} - \frac{\cos 2B}{b^2}$
 $= \frac{1 - 2\sin^2 A}{a^2} - \frac{(1 - 2\sin^2 B)}{b^2}$
 $= \frac{1}{a^2} - \frac{1}{b^2} - \frac{2\sin^2 A}{a^2} + \frac{2\sin^2 B}{b^2}$
 $= \frac{1}{a^2} - \frac{1}{b^2} + 2 \left[\frac{\sin^2 B}{b^2} - \frac{\sin^2 A}{a^2} \right]$
 $= \frac{1}{a^2} - \frac{1}{b^2} \quad \left[\because \frac{\sin A}{a} = \frac{\sin B}{b} \right]$

43. Since, $a = \frac{2\Delta}{p_1}, b = \frac{2\Delta}{p_2}, c = \frac{2\Delta}{p_3}$ and $\sin B = \frac{\sin A + \sin C}{2}$
 $[\because \sin A, \sin B \text{ and } \sin C \text{ are in AP}]$

$\Rightarrow b = \frac{a + c}{2}$
 $\Rightarrow \frac{2\Delta}{p_2} = \frac{2\Delta/p_1 + 2\Delta/p_3}{2}$
 $\Rightarrow p_2 = \frac{2p_1 p_3}{p_1 + p_3}$

Hence, altitudes are in HP.

44. $\therefore \angle C = 90^\circ$

$\therefore \angle A + \angle B = 90^\circ$
 Consider, $\frac{\sin^2 A}{\sin^2 B} - \frac{\cos^2 A}{\cos^2 B} = \frac{\sin^2 A}{\sin^2 B} - \frac{\cos^2 (90^\circ - B)}{\cos^2 (90^\circ - A)}$
 $= \frac{\sin^2 A}{\sin^2 B} - \frac{\sin^2 B}{\sin^2 A} = \frac{a^2}{b^2} - \frac{b^2}{a^2} = \frac{a^4 - b^4}{a^2 b^2}$

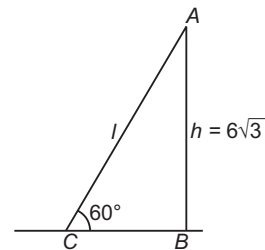
45. Clearly,

$\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{64 + 100 - 144}{2 \times 8 \times 10} = \frac{1}{8} \dots (i)$
 and $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{100 + 144 - 64}{2 \times 10 \times 12} = \frac{3}{4}$
 Also, $\cos 2A = 2\cos^2 A - 1 = 2 \times \frac{9}{16} - 1 = \frac{1}{8} \dots (ii)$

Now, from Eqs. (i) and (ii), $\cos 2A = \cos C \Rightarrow C = 2A$

46. Given, height of house is $6\sqrt{3}$ and the angle of elevation is 60° .

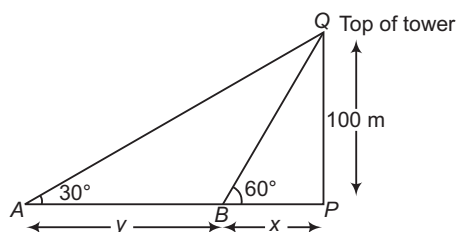
In $\triangle ABC$, $\sin 60^\circ = \frac{h}{l}$
 $\Rightarrow \frac{\sqrt{3}}{2} = \frac{h}{l}$



$\Rightarrow \sqrt{3}l = 6\sqrt{3} \times 2$
 $\therefore l = \frac{6\sqrt{3} \times 2}{\sqrt{3}} = 12 \text{ m}$

$$47. \therefore \tan 60^\circ = \frac{100}{x}$$

$$\Rightarrow x = \frac{100}{\tan 60^\circ} = \frac{100}{\sqrt{3}}$$



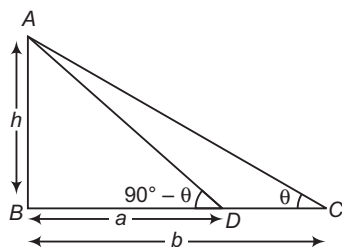
and $\tan 30^\circ = \frac{100}{x+y}$

$$\Rightarrow x+y = 100\sqrt{3}$$

$\therefore$ The required distance $= 100\sqrt{3} - \frac{100}{\sqrt{3}}$

$$= \frac{200\sqrt{3}}{3} \text{ m}$$

48. In $\triangle ABC$, $\tan \theta = \frac{h}{b}$... (i)



In $\triangle ABD$, $\tan(90^\circ - \theta) = \frac{h}{a}$

[$\because$ angles of elevation at C and D are complement]

$$\Rightarrow \cot \theta = \frac{h}{a}$$

$$\Rightarrow \tan \theta = \frac{a}{h} \quad \dots (ii)$$

On comparing Eqs. (i) and (ii), we get

$$\frac{h}{b} = \frac{a}{h}$$

$$\Rightarrow h^2 = ab$$

$$\therefore h = \sqrt{ab}$$

49. In $\triangle PDA$,

$$\tan 30^\circ = \frac{h}{AD}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{AD}$$

$$AD = \sqrt{3}h \quad \dots (i)$$

$\Rightarrow$

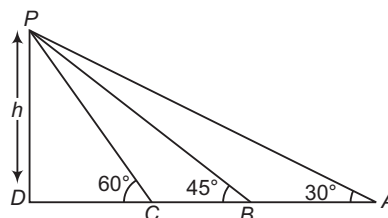
In $\triangle PDB$,

$$\tan 45^\circ = \frac{h}{DB} \Rightarrow 1 = \frac{h}{DB}$$

$$DB = h \quad \dots (ii)$$

$\Rightarrow$ In $\triangle PDC$,

$$\tan 60^\circ = \frac{h}{DC}$$



$$\Rightarrow \sqrt{3} = \frac{h}{DC} \Rightarrow DC = \frac{h}{\sqrt{3}} \quad \dots (iii)$$

Now, $\frac{AB}{BC} = \frac{AD - DB}{DB - DC} = \frac{\sqrt{3}h - h}{h - \frac{h}{\sqrt{3}}}$

[from Eqs. (i), (ii) and (iii)]

$$= \frac{h(\sqrt{3} - 1)\sqrt{3}}{h(\sqrt{3} - 1)} = \sqrt{3} : 1$$

50. We have, $A + B + C = \pi$ and $2B = A + C$

$$\Rightarrow 3B = \pi \Rightarrow B = \frac{\pi}{3}$$

$$\therefore \frac{a+c}{b} = \frac{\sin A + \sin C}{\sin B}$$

$$= \frac{2 \sin \left(\frac{A+C}{2} \right) \cos \left(\frac{A-C}{2} \right)}{\sin \frac{\pi}{3}}$$

$$= \frac{2 \sin \frac{\pi}{3} \cos \left(\frac{A-C}{2} \right)}{\sin \frac{\pi}{3}}$$

$$= 2 \cos \left(\frac{A-C}{2} \right)$$

11

Cartesian System of Rectangular Coordinates

Introduction

Coordinate geometry is the branch of Mathematics which includes the study of different curves and figures by representing points in a plane by ordered pairs of real number called cartesian coordinates. Also, it involves the methods of algebra to solve geometrical problems which is known as analytical geometry. Since, to study geometrical figure, we deal with coordinates that's why it is called coordinate geometry.

Coordinate System

i. **Cartesian coordinate system** Let XOX' and YOY' be two perpendicular straight lines drawn through any fix point O on the plane of the paper. Then,

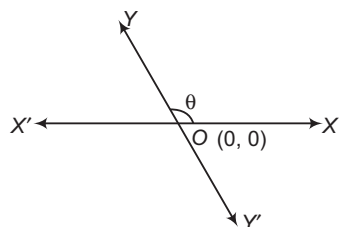
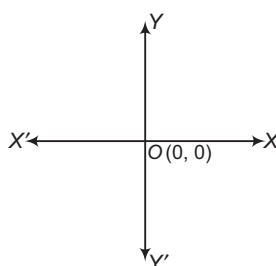
X-axis The horizontal line XOX' is called X -axis.

Y-axis The vertical line YOY' is called Y -axis.

Coordinate axes X -axis and Y -axis together are called axes of coordinate or axes of reference.

Origin The point O is called the origin of coordinates or the origin.

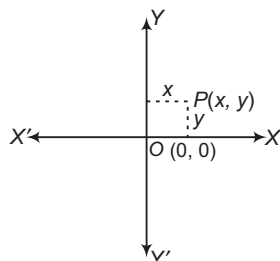
Oblique axes If both the axes are not perpendicular, then they are called as oblique axes as shown in the figure:



Chapter Snapshot

- Introduction
- Coordinate System
- Distance Formulae
- Applications of Distance Formula
- Section Formulae
- Area of a Triangle
- Area of a Quadrilateral
- Some Standard Points of a Triangle
- Locus
- Transformation of Axes

Cartesian coordinates The ordered pair of perpendicular distances from both axes of a point P lying in the plane is called cartesian coordinates of P . If the cartesian coordinates of a point P are (x, y) , then x is called the abscissa or x -coordinate of point P and y is called the ordinate or y -coordinate of point P .

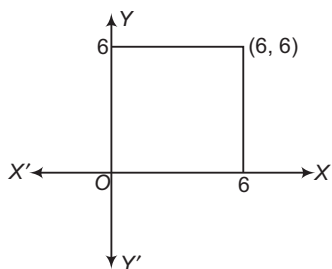


- Coordinates of the origin are $(0, 0)$.
- y -coordinate of a point on X -axis is zero.
- x -coordinate of a point on Y -axis is zero.

➤ **Example 1.** One of the vertices of a square is origin and adjacent sides of the square are coincident with positive axes. If length of a side is 6 units, then which will not be its vertex?

- (a) $(0, 6)$ (b) $(6, 0)$ (c) $(-6, -6)$ (d) $(0, 0)$

Sol. (c) Obviously, $(-6, -6)$ will not be the vertex of the square. Since, the square lies in the first quadrant as shown below:

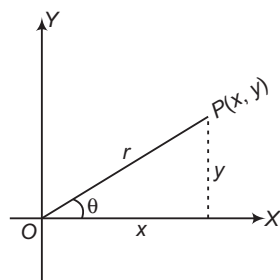


ii. **Polar coordinates system** Let P be a point whose distance be r from origin and makes an angle θ with X -axis.

$$\therefore \frac{x}{r} = \cos \theta \quad \text{and} \quad \frac{y}{r} = \sin \theta$$

$$\text{or} \quad x = r \cos \theta \quad \text{and} \quad y = r \sin \theta$$

Thus, to represent (x, y) in (r, θ) is called polar representation as shown in the figure:



➤ **Example 2.** If polar coordinates of any point are $\left(2, \frac{\pi}{3}\right)$, then its cartesian coordinates are

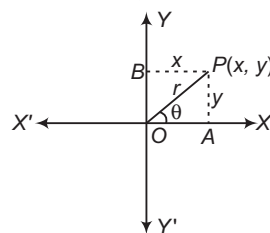
- (a) $(\sqrt{3}, 1)$ (b) $(1, \sqrt{3})$
(c) $(1, -\sqrt{3})$ (d) None of these

Sol. (b) Clearly, $x = 2 \cos \frac{\pi}{3}$ and $y = 2 \sin \frac{\pi}{3}$
 $\Rightarrow x = 1$ and $y = \sqrt{3}$
 $\therefore$ Cartesian coordinates of P are $(1, \sqrt{3})$.

iii. **Relation between polar and cartesian coordinates system** Let P be any point in a plane whose cartesian coordinates are $P(x, y)$ and let the same point when polar system is used have coordinates (r, θ) .

$$\text{Then,} \quad OA = x = r \cos \theta \quad \dots(i)$$

$$\text{and} \quad OB = y = r \sin \theta \quad \dots(ii)$$



On squaring and adding Eqs. (i) and (ii), we get

$$OP = r = \sqrt{x^2 + y^2}$$

Again, dividing Eq. (ii) by Eq. (i), we get

$$\theta = \tan^{-1} \left(\frac{y}{x} \right)$$

- On adding 360° (or any multiple of 360°) to the vectorial angle does not alter the final position of revolving lines so (r, θ) is always the same point as $(r, \theta + 2n\pi)$, where $n \in \mathbb{I}$.
- On adding 180° (or any odd multiple of 180°) to the vectorial angle and changing the sign of radius vector gives the same point as before.

Thus, the point $[-r, \theta + (2n+1)\pi]$ is same as $[-r, \theta + \pi]$.

➤ **Example 3.** The polar coordinates of a point having cartesian coordinates $(-1, -1)$ are

- (a) $\left(\sqrt{2}, \frac{\pi}{4}\right)$ (b) $\left(\sqrt{2}, \frac{3\pi}{4}\right)$
(c) $\left(\sqrt{2}, \frac{5\pi}{4}\right)$ (d) $\left(\sqrt{2}, -\frac{\pi}{4}\right)$

Sol. (c) The cartesian coordinates of P are $(-1, -1)$.

$$\text{So, } x = -1 = r \cos \theta \quad \text{and} \quad y = -1 = r \sin \theta$$

$$\Rightarrow r = \sqrt{1+1} = \sqrt{2}$$

$$\text{and} \quad \theta = \tan^{-1} \left(\frac{-1}{-1} \right) = \tan^{-1} (1)$$

Since, the point lies in the third quadrant.

$$\therefore \theta = \frac{5\pi}{4} \quad \text{or} \quad -\frac{3\pi}{4}$$

Hence, the polar coordinates of $P(-1, -1)$ are $P\left(\sqrt{2}, \frac{5\pi}{4}\right)$.



➤ **Example 4.** Point $P(5, 60^\circ)$ having polar coordinates is equivalent to
(a) $(-5, 60^\circ)$ (b) $(5, 240^\circ)$ (c) $(5, 420^\circ)$ (d) $(5, 300^\circ)$

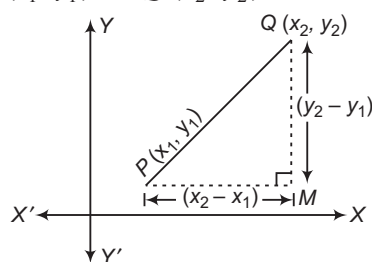
Sol. (c) Since, point having coordinates (r, θ) is equivalent to $(r, 2n\pi + \theta)$, where $n \in \mathbb{I}$.

Hence, $(5, 60^\circ) = (5, 2n\pi + 60^\circ) = (5, 420^\circ)$ [take $n = 1$]

Distance Formulae

i. **When coordinates of two points are given in cartesian form**

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be the two points.



Then, the distance between P and Q will be given by

$$PQ = \sqrt{(\text{Difference of abscissae})^2 + (\text{Difference of ordinates})^2}$$

$$\therefore d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

➤ **Example 5.** If the distance between the points $(x, 2)$ and $(3, 4)$ is 2, then the value of x is

- (a) 2 (b) 1 (c) 3 (d) 4

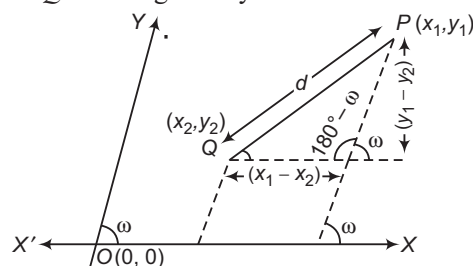
Sol. (c) Given, $2 = \sqrt{(x-3)^2 + (2-4)^2}$

$$\Rightarrow 4 = (x-3)^2 + 4 \Rightarrow x-3 = 0$$

$$\therefore x = 3$$

ii. **When coordinate axes are inclined at an angle ω**

If the coordinate system is oblique, i.e. if the coordinate axes are inclined at an angle ω . In this case, the distance between two points P and Q will be given by



$$PQ = d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2)\cos\omega}$$

➤ **Example 6.** The distance between points $P(-3, -2)$ and $Q(-6, 7)$, the axes being inclined at 60° , is

- (a) $3\sqrt{7}$ (b) $\sqrt{7}$
(c) $2\sqrt{7}$ (d) None of these

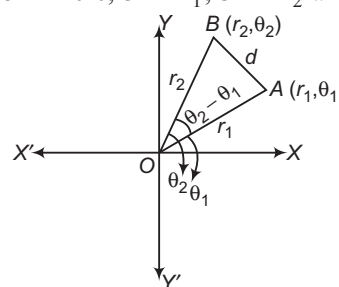
Sol. (a) $PQ = \sqrt{(3)^2 + (9)^2 + 2 \times 3 \times (-9) \cos 60^\circ}$
 $\Rightarrow \sqrt{9 + 81 - 54 \times \frac{1}{2}} = \sqrt{63} = 3\sqrt{7}$

iii. **When coordinates are in polar form**

If (r_1, θ_1) and (r_2, θ_2) are given points, then distance between them is

$$\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

Proof Here, $OA = r_1$, $OB = r_2$ and $AB = d$



By cosine law,

$$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)$$

$$\therefore |AB| = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}$$

➤ **Example 7.** The distance between the points $P(2, 60^\circ)$ and $Q(5, 270^\circ)$ is

- (a) $\sqrt{29 + 10\sqrt{3}}$
(b) $\sqrt{29 - 10\sqrt{3}}$
(c) $\sqrt{25 - 10\sqrt{3}}$
(d) $\sqrt{25 + 10\sqrt{3}}$

Sol. (a) $PQ = \sqrt{(2)^2 + (5)^2 - 2(2)(5)\cos(60^\circ - 270^\circ)}$
 $= \sqrt{4 + 25 - 20\cos(-210^\circ)}$
 $= \sqrt{29 - 20\cos(180^\circ + 30^\circ)}$
 $= \sqrt{29 + 20\cos 30^\circ} = \sqrt{29 + 10\sqrt{3}}$

Applications of Distance Formula

i. **Collinearity of three given points** The three given points A, B and C are collinear i.e. lie on the same straight line, if any of the three points (say B) lie on the straight line joining the other two points.

i.e. $AB + BC = AC$



➤ **Example 8.** The points $(1, 1)$, $(-2, 7)$ and

$(3, -3)$

(a) form a right angled triangle

(b) form an isosceles triangle

(c) are collinear

(d) None of the above

Sol. (c) Let $A(1, 1)$, $B(-2, 7)$ and $C(3, -3)$ be the given points, then

$$AB = \sqrt{(-2-1)^2 + (7-1)^2}$$

$$= \sqrt{9+36} = 3\sqrt{5}$$

$$BC = \sqrt{(3+2)^2 + (-3-7)^2}$$

$$= \sqrt{25+100} = 5\sqrt{5}$$

and

$$AC = \sqrt{(3-1)^2 + (-3-1)^2}$$

$$= \sqrt{4+16} = 2\sqrt{5}$$

Clearly,

$$BC = AB + AC$$

Hence, A , B and C are collinear.

ii. **Types of triangles** If A, B and C are vertices of triangle, then it would be

(a) equilateral triangle, when $AB = BC = CA$.

(b) isosceles triangle, when any two sides are equal.

(c) right angled triangle, when sum of square of any two sides is equal to square of the third side.

➤ **Example 9.** The points $A(2a, 4a)$, $B(2a, 6a)$ and

$C(2a + \sqrt{3}a, 5a)$ (when $a > 0$) are vertices of

(a) an obtuse angled triangle

(b) an equilateral triangle

(c) an isosceles obtuse angled triangle

(d) a right angled triangle

Sol. (b) $AB = \sqrt{(2a-2a)^2 + (4a-6a)^2} = 2a$

$$BC = \sqrt{(2a + \sqrt{3}a - 2a)^2 + (5a - 6a)^2}$$

$$= \sqrt{(\sqrt{3}a)^2 + a^2} = 2a$$

$$\text{and } CA = \sqrt{(2a - 2a - \sqrt{3}a)^2 + (4a - 5a)^2}$$

$$= \sqrt{(\sqrt{3}a)^2 + a^2} = 2a$$

Since, $AB = BC = CA$, A , B and C form an equilateral triangle.

➤ **Example 10.** The triangle formed by

$P(-2, 3)$, $Q(3, -2)$ and $R(0, 0)$ is

(a) an equilateral

(b) an isosceles

(c) a right angled

(d) None of these

Sol. (b) $PQ = \sqrt{25+25} = 5\sqrt{2}$

$$RQ = \sqrt{9+4} = \sqrt{13}$$

$$PR = \sqrt{4+9} = \sqrt{13}$$

$$RQ = PR$$

Hence, ΔPQR is isosceles.

➤ **Example 11.** The triangle joining the points

$P(-2, 7)$, $Q(-4, -1)$ and $R(2, 6)$ is

(a) an equilateral triangle

(b) a right angled triangle

(c) an isosceles triangle

(d) a scalene triangle

Sol. (b) Since, $PQ = \sqrt{68}$, $PR = \sqrt{17}$ and $QR = \sqrt{85}$

$$\therefore PQ^2 + PR^2 = QR^2$$

i.e. right angled triangle.

iii. **Position of four points** Let A, B, C and D be the four given points in a plane. By joining these points following figures are formed :

(a) Parallelogram, if

$$AB = DC, BC = AD, AC \neq BD$$

(b) Rectangle, if $AB = CD, BC = DA, AC = BD$

(c) Rhombus, if $AB = BC = CD = DA$

$$\text{and } AC \neq BD$$

(d) Square, if $AB = BC = CD = DA$

$$\text{and } AC = BD$$

Quadrilateral	Diagonals	Angles between diagonals
Parallelogram	Not equal	$\theta \neq \frac{\pi}{2}$
Rectangle	Equal	$\theta \neq \frac{\pi}{2}$
Rhombus	Not equal	$\theta = \frac{\pi}{2}$
Square	Equal	$\theta = \frac{\pi}{2}$

- Diagonals of square, rhombus, rectangle and parallelogram always bisect each other.
- Diagonals of rhombus and square bisect each other at right angle.
- Every parallelogram is not a rectangle but every rectangle is a parallelogram.
- Every rhombus is not a square but every square is a rhombus.
- Every parallelogram is not a rhombus but every rhombus is a parallelogram.
- Every rectangle is not a square but every square is a rectangle.

➤ **Example 12.** The figure formed by four points

$(1, -2)$, $(3, 6)$, $(5, 10)$ and $(3, 2)$ is a

(a) square

(b) rectangle

(c) rhombus

(d) parallelogram

Sol. (d) Let $A \equiv (1, -2)$, $B \equiv (3, 6)$, $C \equiv (5, 10)$

and $D \equiv (3, 2)$

$$\text{Then, } AB = \sqrt{(1-3)^2 + (-2-6)^2} = 2\sqrt{17}$$

$$BC = \sqrt{(3-5)^2 + (6-10)^2} = 2\sqrt{5}$$

$$CD = \sqrt{(5-3)^2 + (10-2)^2} = 2\sqrt{17}$$

$$AD = \sqrt{(1-3)^2 + (-2-2)^2} = 2\sqrt{5}$$



$$AC = \sqrt{(1-5)^2 + (-2-10)^2} = 4\sqrt{10}$$

and $BD = \sqrt{(3-3)^2 + (6-2)^2} = 4$

Clearly, $AB = CD$, $BC = AD$ and $AC \neq BD$

Hence, $ABCD$ is a parallelogram.

➤ **Example 13.** If vertices of a quadrilateral are $A(0, 0)$, $B(3, 4)$, $C(7, 7)$ and $D(4, 3)$, then quadrilateral $ABCD$ is

- parallelogram
- rectangle
- square
- rhombus

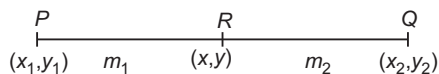
Sol. (d) Obviously, $AB = BC = CD = AD = 5$ and also $AC \neq BD$.

Therefore, it is a rhombus.

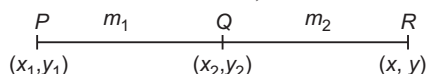
Section Formulae

Coordinates of a point which divides the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $m_1 : m_2$ are

i. $\left(\frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2} \right)$ (internal division)



ii. $\left(\frac{m_1 x_2 - m_2 x_1}{m_1 - m_2}, \frac{m_1 y_2 - m_2 y_1}{m_1 - m_2} \right)$ (external division)



When m_1 and m_2 are of opposite sign, then the division is external.

- Coordinates of any point which divide the line segment joining two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the ratio $\lambda : 1$ is given by

$$\left(\frac{x_1 + \lambda x_2}{\lambda + 1}, \frac{y_1 + \lambda y_2}{\lambda + 1} \right), (\lambda \neq -1)$$

- Line formed by joining (x_1, y_1) and (x_2, y_2) is divided by

> X-axis in the ratio $-\frac{y_1}{y_2}$

> Y-axis in the ratio $-\frac{x_1}{x_2}$

If ratio is positive, then the axis divides it internally and if ratio is negative, then the axis divides externally.

- Line $Ax + By + C = 0$ divides the line joining the points (x_1, y_1) and (x_2, y_2) in the ratio $\lambda : 1$, then

$$\lambda = -\left(\frac{Ax_1 + By_1 + C}{Ax_2 + By_2 + C} \right)$$

If λ is positive, then it divides internally and if λ is negative, then it divides externally.

- The mid-point of (x_1, y_1) and (x_2, y_2) is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right)$.

- Another conditions to prove that A, B, C and D are vertices of

Parallelogram Show that diagonals AC and BD bisect each other.

Rhombus Show that diagonals AC and BD bisect each other and adjacent sides AB and BC are equal.

Square Show that diagonals AC and BD are equal and bisect each other and adjacent sides AB and BC are equal.

Rectangle Show that diagonals AC and BD are equal and bisect each other.

- The coordinates of the points of trisection P and Q of the line segment joining A (x_1, y_1) and B (x_2, y_2) are $\left(\frac{2x_1 + x_2}{3}, \frac{2y_1 + y_2}{3} \right)$ and $\left(\frac{x_1 + 2x_2}{3}, \frac{y_1 + 2y_2}{3} \right)$ respectively.

➤ **Example 14.** The coordinates of point which divides the line segment joining points A $(0, 0)$ and B $(9, 12)$ in the ratio 1 : 2, are

- $(-3, 4)$
- $(3, 4)$
- $(3, -4)$
- None of these

Sol. (b) Let P divides the line segment joining points A $(0, 0)$ and B $(9, 12)$ internally in the ratio 1 : 2

$$\therefore x = \frac{9 \times 1 + 0 \times 2}{1 + 2} = \frac{9}{3} = 3$$

and $y = \frac{12 \times 1 + 0 \times 2}{1 + 2} = \frac{12}{3} = 4$

Hence, the coordinates of point P are $(3, 4)$.

➤ **Example 15.** The ratio in which the line $3x + y - 9 = 0$ divides the segment joining the points $(1, 3)$ and $(2, 7)$, is

- 1 : 2
- 4 : 3
- 3 : 4
- None of these

Sol. (c) Let the line $3x + y - 9 = 0$ divides the line segment joining A $(1, 3)$ and B $(2, 7)$ in the ratio $k : 1$ at point C, then the coordinates of C are $\left(\frac{2k + 1}{k + 1}, \frac{7k + 3}{k + 1} \right)$.

$\therefore$ C lies on line $3x + y - 9 = 0$, so it satisfies the equation of line.

$$\therefore 3 \left(\frac{2k + 1}{k + 1} \right) + \left(\frac{7k + 3}{k + 1} \right) - 9 = 0 \Rightarrow k = \frac{3}{4}$$

Hence, the required ratio is 3 : 4 (internally).

Aliter

$$\begin{aligned} \text{Required ratio} &= -\frac{(Ax_1 + By_1 + C)}{(Ax_2 + By_2 + C)} \\ &= -\frac{(3 + 3 - 9)}{(6 + 7 - 9)} = \frac{3}{4} \end{aligned}$$

➤ **Example 16.** The ratio in which the line segment joining the points $(3, -4)$ and $(-5, 6)$ is divided by the X-axis, is

- 2 : 3
- 3 : 2
- 6 : 4
- None of these

Sol. (a) Let P be the point on X-axis, which divides the line segment joining points in the ratio $\lambda : 1$.

Then, $x = \frac{-5\lambda + 3}{\lambda + 1}$ and $y = \frac{6\lambda - 4}{\lambda + 1}$



But as it is known for a point on X-axis, its y-coordinate = 0

$$\Rightarrow \frac{6\lambda - 4}{\lambda + 1} = 0 \Rightarrow \lambda = \frac{2}{3}$$

Aliter

$$\text{Required ratio} = -\frac{y_1}{y_2} = -\frac{(-4)}{6} = \frac{2}{3}$$

➤ **Example 17.** Prove that the points $(-2, -1)$, $(1, 0)$, $(4, 3)$ and $(1, 2)$ are the vertices of a parallelogram. Is it a rectangle?

Sol. Let the given points be A, B, C and D, respectively.

Then, the coordinates of the mid-point of AC are

$$\left(\frac{-2+4}{2}, \frac{-1+3}{2}\right) = (1, 1)$$

and coordinates of the mid-point of BD are

$$\left(\frac{1+1}{2}, \frac{0+2}{2}\right) = (1, 1)$$

Thus, AC and BD have the same mid-point.

Hence, ABCD is a parallelogram.

Now, we shall see whether ABCD is a rectangle or not.

We have,

$$AC = \sqrt{\{4 - (-2)\}^2 + \{3 - (-1)\}^2} = 2\sqrt{13}$$

$$\text{and } BD = \sqrt{(1-1)^2 + (0-2)^2} = 2$$

Clearly, $AC \neq BD$

So, ABCD is not a rectangle.

➤ **Example 18.** The points of trisection of line segment joining the points A(2, 1) and B(5, 3), are

- (a) $\left(3, \frac{7}{3}\right), \left(\frac{5}{3}, 4\right)$ (b) $\left(4, \frac{5}{3}\right), \left(3, \frac{7}{3}\right)$
 (c) $\left(3, \frac{5}{3}\right), \left(4, \frac{7}{3}\right)$ (d) $\left(4, \frac{7}{3}\right), \left(3, \frac{7}{3}\right)$

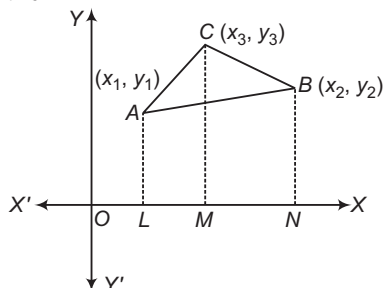
Sol. (c) Let P_1 and P_2 be the points, which trisect the line segment joining the points A(2, 1) and B(5, 3).

$$P_1(x, y) = \left(\frac{1 \times 5 + 2 \times 2}{1+2}, \frac{1 \times 3 + 2 \times 1}{1+2}\right) = \left(3, \frac{5}{3}\right)$$

$$P_2(x, y) = \left(\frac{2 \times 5 + 1 \times 2}{1+2}, \frac{2 \times 3 + 1 \times 1}{1+2}\right) = \left(4, \frac{7}{3}\right)$$

Area of a Triangle

Let A, B and C be the vertices of ΔABC , whose coordinates are given by A(x_1, y_1), B(x_2, y_2) and C(x_3, y_3), then



Area of ΔABC

$$= \frac{1}{2} [(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))]$$

This can also be written as

$$\Delta = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

➤ If area of triangle is zero. Then, the points will be collinear and if the points are collinear, then

$$\text{Area of triangle} = 0 \text{ or } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

➤ **Example 19.** If the vertices of a triangle are $(1, 2)$, $(4, -6)$ and $(3, 5)$, then its area is

(a) $\frac{23}{2}$ sq units

(b) $\frac{25}{2}$ sq units

(c) 12 sq units

(d) None of the above

Sol. (b) Area of triangle, whose vertices are given,

$$\begin{aligned} \Delta &= \frac{1}{2} [x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)] \\ &= \frac{1}{2} [1(-6 - 5) + 4(5 - 2) + 3(2 + 6)] \\ &= \frac{25}{2} \text{ sq units} \end{aligned}$$

➤ **Example 20.** The points $(k, 2 - 2k)$, $(-k + 1, 2k)$ and $(-4 - k, 6 - 2k)$ are collinear for

(a) all values of k

(b) $k = -1$ or $\frac{1}{2}$

(c) $k = -1$

(d) no values of k

Sol. (c) The given points are collinear, if area of $\Delta = 0$

$$\Rightarrow \begin{vmatrix} k & 2-2k & 1 \\ -k+1 & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix} = 0$$

On applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$, we get

$$\begin{vmatrix} k & 2-2k & 1 \\ -2k+1 & 4k-2 & 0 \\ -4-2k & 4 & 0 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -2k+1 & 4k-2 \\ -4-2k & 4 \end{vmatrix} = 0$$

$$\Rightarrow (1-2k)(k+1) = 0$$

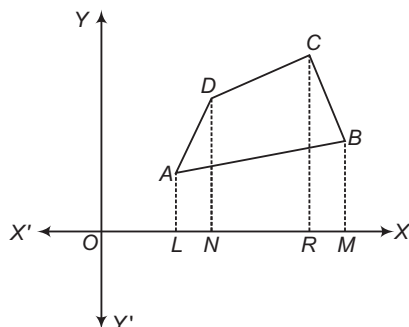
$\Rightarrow k = -1$ or $k = \frac{1}{2}$, neglecting $\frac{1}{2}$ as when $k = \frac{1}{2}$, points are same.



Area of a Quadrilateral

Objective Mathematics Vol. 1

If (x_1, y_1) , (x_2, y_2) , (x_3, y_3) and (x_4, y_4) are vertices of a quadrilateral, then its area will be given by



Area of quadrilateral = Area of trapezium $ALND$
+ Area of trapezium $DNRC$ + Area of trapezium $CRMB$ - Area of trapezium $ALMB$

On substituting the value of area of trapezium

$$= \frac{1}{2} (\text{Sum of parallel sides}) \times \text{Distance between them}$$

$\therefore$ Area of quadrilateral

$$= \frac{1}{2} [(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_4 - x_4 y_3) + (x_4 y_1 - x_1 y_4)]$$

• Area of polygon can be calculated as

> **Row Method** When vertices of polygon are given, then area of polygon will be

$$\frac{1}{2} \left| \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} + \dots + \begin{vmatrix} x_n & x_1 \\ y_n & y_1 \end{vmatrix} \right|$$

> **Stair Method** Again, when vertices of polygon are given say $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$. Then, its area is

$$\frac{1}{2} \left| \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ \vdots & \vdots \\ x_n & y_n \\ x_1 & y_1 \end{vmatrix} \right|$$

$$= \frac{1}{2} [(x_1 y_2 + x_2 y_3 + \dots + x_n y_1) - (y_1 x_2 + y_2 x_3 + \dots + y_n x_1)]$$

• If the two opposite vertices of a square are $A(x_1, y_1)$ and $C(x_2, y_2)$, then its area

$$= \frac{1}{2} AC^2 = \frac{1}{2} [(x_2 - x_1)^2 + (y_2 - y_1)^2]$$

➤ **Example 21.** The area of quadrilateral, whose vertices are $(1, 1)$, $(3, 4)$, $(5, -2)$ and $(4, -7)$, is

(a) 41 sq units

(b) $\frac{41}{2}$ sq units

(c) $\frac{31}{2}$ sq units

(d) 7 sq units

Sol. (b) Area of quadrilateral

$$= \frac{1}{2} [(1 \times 4) - (3 \times 1) + \{3 \times (-2)\} - (5 \times 4) + \{5 \times (-7)\} - \{4 \times (-2)\} + (4 \times 1) - \{1 \times (-7)\}]$$

$$= \left| -\frac{41}{2} \right| = \frac{41}{2} \text{ sq units}$$

➤ **Example 22.** The values of t , if the area of the pentagon $ABCDE$ is $\frac{45}{2}$ sq units, where

$A = (1, 3)$, $B = (-2, 5)$, $C = (-3, -1)$, $D = (0, -2)$ and $E = (2, t)$ and here the points are given in order, are
(a) -1, 89 (b) -2, 89 (c) 1, 89 (d) 2, 89

Sol. (a) Area of the pentagon $ABCDE$

$$= \frac{1}{2} \left| \begin{vmatrix} 1 & 3 \\ -2 & 5 \end{vmatrix} + \begin{vmatrix} -2 & 5 \\ -3 & -1 \end{vmatrix} + \begin{vmatrix} -3 & -1 \\ 0 & -2 \end{vmatrix} + \begin{vmatrix} 0 & -2 \\ 2 & t \end{vmatrix} + \begin{vmatrix} 2 & t \\ 1 & 3 \end{vmatrix} \right|$$

$$\therefore \frac{45}{2} = \frac{1}{2} [(5 + 6) + (2 + 15) + (6 - 0) + (0 + 4) + (6 - t)]$$

$$\Rightarrow \frac{45}{2} = \frac{1}{2} (11 + 17 + 6 + 4 + 6 - t) = \frac{1}{2} |44 - t|$$

$$\Rightarrow 44 - t = \pm 45$$

$$\Rightarrow t = 44 \mp 45 = -1, 89$$

➤ **Example 23.** The area of the pentagon, whose vertices are $A(1, 1)$, $B(7, 21)$, $C(7, -3)$, $D(12, 2)$ and $E(0, -3)$, is

(a) $\frac{137}{2}$ sq units

(b) $\frac{137}{3}$ sq units

(c) $\frac{137}{4}$ sq units

(d) None of these

$$\text{Sol. (a) Required area} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_1 & y_1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 1 \\ 7 & 21 \\ 7 & -3 \\ 12 & 2 \\ 0 & -3 \\ 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [(21 - 21 + 14 - 36 + 0) - (7 + 147 - 36 + 0 - 3)]$$

$$= \frac{137}{2} \text{ sq units}$$

➤ **Example 24.** If the coordinates of two opposite vertices of a square are $(6, -3)$ and $(-2, 3)$, then area of square is

(a) 50 sq units

(b) 25 sq units

(c) 35 sq units

(d) None of these

Sol. (a) Area of square = $\frac{1}{2} d^2$

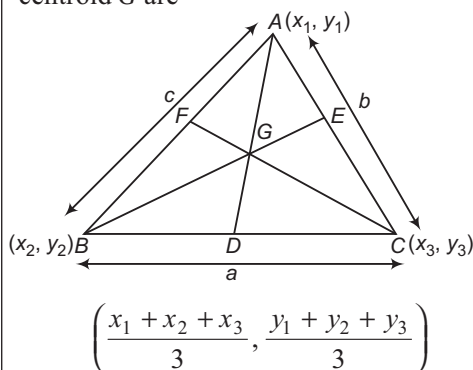
$$= \frac{1}{2} \{(6 + 2)^2 + (-3 - 3)^2\}$$

$$= \frac{1}{2} (64 + 36) = 50 \text{ sq units}$$

Some Standard Points of a Triangle

In this section, we will discuss problems on finding centroid, circumcentre, incentre and orthocentre of a triangle.

- i. **Centroid** The centroid of a triangle is the point of intersection of its medians. It divides the medians in the ratio 2 : 1. If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a $\triangle ABC$, then the coordinates of its centroid G are

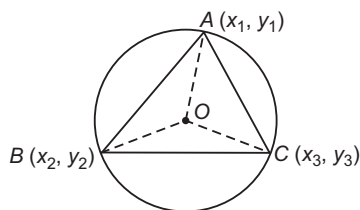


- ii. **Incentre** The point of intersection of the internal bisectors of the angles of a triangle is called its incentre.

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a $\triangle ABC$ such that $BC = a$, $CA = b$ and $AB = c$, then the coordinates of the incentre are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)$$

- iii. **Circumcentre** The circumcentre of a triangle is the point of intersection of the perpendicular bisectors of its sides.



It is the centre of the circle passing through the vertices of a triangle and so it is equidistant from the vertices of the triangle.

Thus, if O is the circumcentre of a $\triangle ABC$, then

$$OA = OB = OC$$

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$ and let $O(x, y)$ be its circumcentre. Then, the coordinates of O are obtained by solving

$$OA^2 = OB^2 = OC^2$$

$$\begin{aligned} \text{i.e. } (x-x_1)^2 + (y-y_1)^2 &= (x-x_2)^2 + (y-y_2)^2 \\ &= (x-x_3)^2 + (y-y_3)^2 \end{aligned}$$

The coordinates of the circumcentre are also given by

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C} \right)$$

- iv. **Orthocentre** The orthocentre of a triangle is the point of intersection of its altitudes.

In order to find the coordinates of the orthocentre of a triangle, we first find the equations of its altitudes and then we find the coordinates of the point of intersection of any two of them.

If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices of a $\triangle ABC$, then the coordinates of its orthocentre are

$$\left(\frac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \frac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C} \right)$$

- The orthocentre of a right angled triangle is at the vertex forming the right angle.
- The orthocentre O' , circumcentre O and centroid G of a triangle are collinear and G divides $O'O$ in the ratio 2:1 i.e. $O'G:OG = 2:1$.
- The circumcentre of a right angled triangle is the mid-point of its hypotenuse.

- v. **Nine point centre** Let ABC be a triangle such that AD , BE and CF are its altitudes drawn from A , B and C on the opposite sides; H , I , J are the mid-points of the sides BC , CA and AB , respectively and K , L , M are the mid-points of the line segments joining the orthocentre O' to the angular points A , B , C .

These nine points ($D, E, F, H, I, J, K, L, M$) are concyclic and the circle passing through these nine points is called the nine point circle. Its centre is known as the nine point centre.

The nine point centre of a triangle is collinear with the circumcentre O and the orthocentre O' and bisects the segment joining them. The radius of nine point circle of a triangle is half the radius of the circumcircle.

- The orthocentre, nine point centre, centroid and circumcentre are collinear.

vi.

Excentre of a triangle Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a $\triangle ABC$.

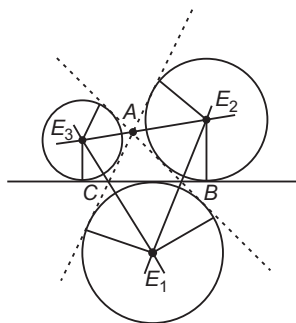
Let a, b and c be the lengths of the sides BC, CA and AB or a, b and c be the lengths of the sides opposite to $\angle A, \angle B$ and $\angle C$ respectively, then the circle which touches the side BC and two sides AB and AC produced is called the escribed circle opposite to the $\angle A$. The bisector of the external angles B and C meet at E_1 which is the centre of the escribed circle such that

$$E_1 \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c} \right)$$

Similarly,

$$E_2 \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c} \right)$$

$$\text{and } E_3 \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c} \right)$$



➤ **Example 25.** Centroid of the triangle, whose vertices are $(0, 0)$, $(2, 5)$ and $(7, 4)$, is

- (a) $(3, 3)$ (b) $(2, 3)$ (c) $(3, 2)$ (d) $(2, 2)$

Sol. (a) Centroid of the triangle

$$= \left(\frac{0 + 2 + 7}{3}, \frac{0 + 5 + 4}{3} \right) = (3, 3)$$

➤ **Example 26.** Incentre of triangle, whose vertices are $A(-36, 7)$, $B(20, 7)$ and $C(0, -8)$, is

- (a) $(0, -1)$
(b) $(0, 0)$
(c) $(-1, 0)$
(d) None of the above

Sol. (c) $\because a = BC = \sqrt{(20 - 0)^2 + (7 + 8)^2} = 25$

$$b = CA = \sqrt{(-36 - 0)^2 + (7 - 8)^2} = 39$$

$$\text{and } c = AB = \sqrt{(36 + 20)^2 + (7 - 7)^2} = 56$$

$$I = \left(\frac{\frac{25(-36) + 39(20) + 56(0)}{25 + 39 + 56}, \frac{25(7) + 39(7) + 56(-8)}{25 + 39 + 56} \right) = (-1, 0)$$

➤ **Example 27.** If the vertices A, B and C of a triangle are $(2, 1)$, $(5, 2)$ and $(3, 4)$, respectively. Then, the circumcentre is

(a) $\left(\frac{13}{4}, \frac{-9}{4} \right)$

(b) $\left(\frac{-13}{4}, \frac{9}{4} \right)$

(c) $\left(\frac{13}{4}, \frac{9}{4} \right)$

(d) $\left(\frac{13}{2}, \frac{9}{4} \right)$

Sol. (c) Let $P(x, y)$ be the coordinates of the centre of circumcircle.

$$\therefore PA^2 = PB^2 \quad \text{and} \quad PB^2 = PC^2$$

$$\therefore 3x + y = 12 \quad \dots(i)$$

$$\text{and } x - y = 1 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get the centre at $\left(\frac{13}{4}, \frac{9}{4} \right)$.

➤ **Example 28.** The orthocentre of the triangle formed by the line $2x^2 - xy - 6y^2 = 0$ and the line $x + y + 1 = 0$, is

(a) $\left(-\frac{4}{3}, -\frac{4}{3} \right)$

(b) $(4, 4)$

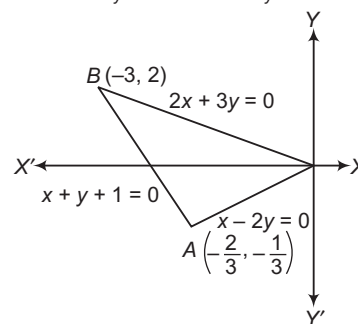
(c) $(-4, -4)$

(d) None of these

Sol. (a) We have, $2x^2 - xy - 6y^2 = 0$

$$\Rightarrow (2x + 3y)(x - 2y) = 0$$

$$\Rightarrow 2x + 3y = 0 \quad \text{and} \quad x - 2y = 0$$



Thus, the equations of sides of the triangle are

$$2x + 3y = 0, x - 2y = 0 \quad \text{and} \quad x + y + 1 = 0.$$

The equation of the altitude through the vertex $O(0, 0)$ is

$$y - 0 = 1(x - 0)$$

$$\Rightarrow y = x \quad \dots(i)$$

The equation of altitude through the vertex $B(-3, 2)$ is

$$y - 2 = -2(x + 3)$$

$$\Rightarrow 2x + y + 4 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = -\frac{4}{3} \quad \text{and} \quad y = -\frac{4}{3}$$

Hence, the coordinates of the orthocentre are

$$\left(-\frac{4}{3}, -\frac{4}{3} \right)$$

Locus

It is the path or curve traced by a moving point satisfying the given condition.

- i. **Equation of the locus of a point** The equation of the locus of a point is the algebraic relation which is satisfied by the coordinates of every point on the locus of the point.

Steps taken to find the equation of locus of a point

- Assumes the coordinates of the point say (h, k) , whose locus is to be find.
- Write the given condition involving (h, k) .
- Eliminate the variable (s), if any.
- Replace $h \rightarrow x$ and $k \rightarrow y$.

The equation so obtained is the locus of the point which moves under some definite conditions.

➤ **Example 29.** The sum of squares of the distances of a moving point from two fixed points $(a, 0)$ and $(-a, 0)$ is equal to a constant. Then, the equation of its locus is

- $x^2 - y^2 = C + a^2$
- $x^2 + y^2 = C - a^2$
- $x^2 + y^2 = C^2 + a^2$
- $x^2 - y^2 = C - a^2$

Sol. (b) Let $P(h, k)$ be any position of the moving point and let $A(a, 0)$ and $B(-a, 0)$ be the given points. Then,

$$PA^2 + PB^2 = \lambda \quad [\text{given}]$$

$$\Rightarrow (h-a)^2 + (k-0)^2 + (h+a)^2 + (k-0)^2 = \lambda$$

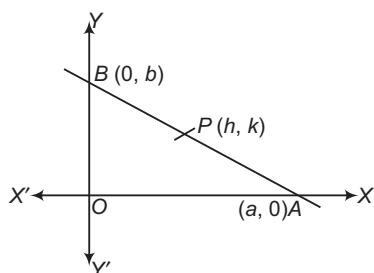
$$\Rightarrow h^2 - 2ah + a^2 + k^2 + h^2 + 2ah + a^2 + k^2 = \lambda$$

$$\therefore \text{Locus of } (h, k) \text{ is } x^2 + y^2 = C - a^2, \text{ where } C = \frac{\lambda}{2}$$

➤ **Example 30.** A rod of length l slides with its ends on two perpendicular lines. Then, the locus of its mid-point is

- $x^2 + y^2 = \frac{l^2}{4}$
- $x^2 + y^2 = \frac{l^2}{2}$
- $x^2 - y^2 = \frac{l^2}{4}$
- None of these

Sol. (a) Let the two perpendicular lines be the coordinate axes. Let AB be rod of length l and the coordinates of A and B be $(a, 0)$ and $(0, b)$, respectively.



Let $P(h, k)$ be the mid-point of rod AB in one of the infinite position it attains, then

$$h = \frac{a+0}{2} \quad \text{and} \quad k = \frac{0+b}{2}$$

$$\Rightarrow h = \frac{a}{2} \quad \text{and} \quad k = \frac{b}{2} \quad \dots(i)$$

From $\triangle OAB$, we have

$$AB^2 = OA^2 + OB^2$$

$$\Rightarrow a^2 + b^2 = l^2$$

$$\Rightarrow (2h)^2 + (2k)^2 = l^2$$

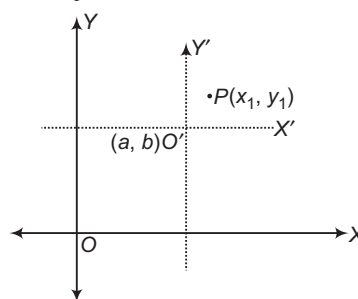
$$\Rightarrow 4h^2 + 4k^2 = l^2$$

$$\Rightarrow h^2 + k^2 = \frac{l^2}{4}$$

So, the equation of locus is $x^2 + y^2 = \frac{l^2}{4}$.

Transformation of Axes

- i. **To alter the origin of coordinates without altering the direction of the axes** Let origin $O(0, 0)$ be shifted to a point (a, b) by moving the X -axis and Y -axis parallel to themselves. If the coordinates of point P with reference to old axis are (x_1, y_1) , then coordinates of this point with respect to new axis will be $(x_1 - a, y_1 - b)$.



➤ **Example 31.** If the axes are transformed from origin to the point $(3, 4)$, then new coordinates of $(7, 8)$ are

- $(4, 3)$
- $(4, 2)$
- $(4, 4)$
- None of these

Sol. (c) We know that, if the origin is shifted to (h, k) , then new coordinates of (x, y) becomes $(x - h, y - k)$. Therefore, the new coordinates of $(7, 8)$ with respect to new origin are $(4, 4)$.

➤ **Example 32.** If the axes are shifted to the point $(1, -2)$ without rotation. What do the following equations become?

- $2x^2 + y^2 - 4x + 4y = 0$
- $y^2 - 4x + 4y + 8 = 0$

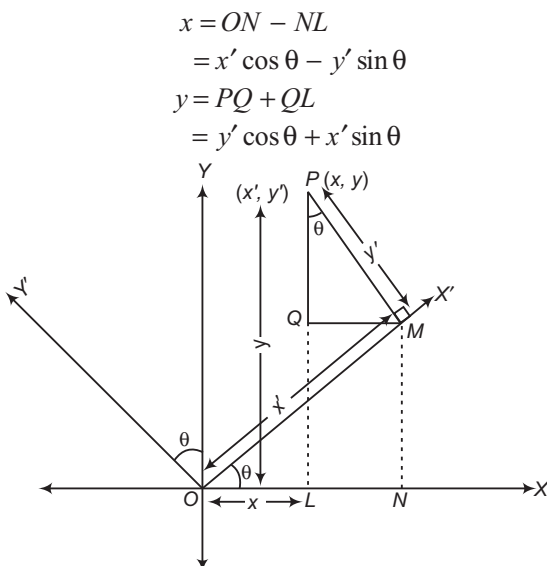
Sol. (i) Substituting $x = X + 1$, $y = Y + (-2) = Y - 2$ in the equation $2x^2 + y^2 - 4x + 4y = 0$, we get

$$2(X+1)^2 + (Y-2)^2 - 4(X+1) + 4(Y-2) = 0$$

$$\Rightarrow 2X^2 + Y^2 = 6, \text{ where } (X, Y) \text{ are new coordinates and } (x, y) \text{ are old coordinates.}$$


- (ii) Substituting $x = X + 1$, $y = Y - 2$ in the equation $y^2 - 4x + 4y + 8 = 0$, we get
 $(Y - 2)^2 - 4(X + 1) + 4(Y - 2) + 8 = 0$
 $\Rightarrow Y^2 = 4X$

ii. To change the direction of the axes of coordinates without changing origin Let OX and OY be the old axes and OX' and OY' be the new axes obtained by rotating the old OX and OY through an angle θ , then the coordinates of $P(x, y)$ with respect to new coordinate axes will be given by



Thus, if the axes are rotated through an angle θ . Then, the coordinates of a point with respect to new axis will be

$x' = x \cos \theta + y \sin \theta$
 and $y' = -x \sin \theta + y \cos \theta$
 The above relation between (x, y) and (x', y') can also be obtained from the table.

	$x \downarrow$	$y \downarrow$
$x' \rightarrow$	$\cos \theta$	$\sin \theta$
$y' \rightarrow$	$-\sin \theta$	$\cos \theta$

- x and y are old coordinates, x' and y' are new coordinates.
- The axes rotation in anti-clockwise is positive and clockwise rotation of axes is negative.

Example 33. Keeping the origin constant axes are rotated at an angle 30° in negative direction, then coordinates of $(2, 1)$ with respect to new axes are

- (a) $\left(\frac{2\sqrt{3}+1}{2}, \frac{2\sqrt{3}}{2}\right)$ (b) $\left(\frac{2\sqrt{3}+1}{2}, \frac{-2+\sqrt{3}}{2}\right)$
 (c) $\left(\frac{2+\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$ (d) None of these

Sol. (d) $X' = 2\cos(-30^\circ) + 1\sin(-30^\circ)$
 $= \frac{2\sqrt{3}-1}{2}$
 $Y' = -2\sin 30^\circ + \cos 30^\circ$
 $= \frac{-2+\sqrt{3}}{2}$

Example 34. If the axes be turned through an angle $\tan^{-1} 2$. What does the equation $4xy - 3x^2 = a^2$ become?

- (a) $X^2 - 4Y^2 = a^2$
 (b) $X^2 + 4Y^2 = a^2$
 (c) $X^2 + 4Y^2 = -a^2$
 (d) None of the above

Sol. (a) Here, $\tan \theta = 2$

So, $\cos \theta = \frac{1}{\sqrt{5}}$
 and $\sin \theta = \frac{2}{\sqrt{5}}$

For x and y , we have

$$x = X \cos \theta - Y \sin \theta = \frac{X - 2Y}{\sqrt{5}}$$

and $y = X \sin \theta + Y \cos \theta = \frac{2X + Y}{\sqrt{5}}$

The equation $4xy - 3x^2 = a^2$ reduces to

$$\frac{4(X - 2Y)}{\sqrt{5}} \cdot \frac{(2X + Y)}{\sqrt{5}} - 3\left(\frac{X - 2Y}{\sqrt{5}}\right)^2 = a^2$$

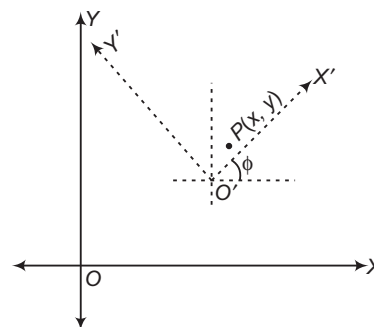
$$\Rightarrow 4(2X^2 - 2Y^2 - 3XY) - 3(X^2 - 4XY + 4Y^2) = 5a^2$$

$$\Rightarrow 5X^2 - 20Y^2 = 5a^2$$

$$\Rightarrow X^2 - 4Y^2 = a^2$$

iii.

To change the direction of the axes of coordinates with changing origin If $P(x, y)$ and the axes are shifted parallel to the original axes so that new origin is $O'(\alpha, \beta)$ and then the axes are rotated about the new origin (α, β) by angle ϕ in the anti-clockwise direction, then the coordinates of P will be given by

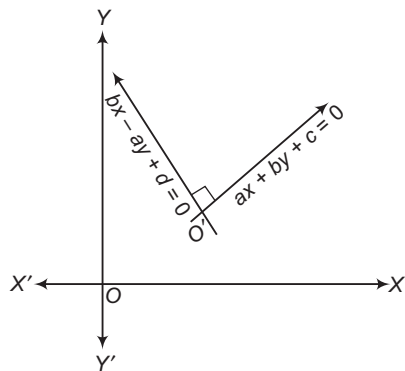


$$x = \alpha + x' \cos \phi - y' \sin \phi$$

$$y = \beta + x' \sin \phi + y' \cos \phi$$

where, x and y are old coordinates, x' and y' are new coordinates.

➤ If $P(x, y)$ and two mutually perpendicular lines $ax + by + c = 0$ and $bx - ay + d = 0$ are taken as new axes such that new coordinates of P are (x', y') , then



$$x' = \frac{bx - ay + d}{\sqrt{b^2 + a^2}} \text{ and } y' = \frac{ax + by + c}{\sqrt{a^2 + b^2}}$$

➤ **Example 35.** What does the equation $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$ become when referred to rectangular axes through the point $(-2, -3)$, the new axes being inclined at an angle of 45° with the old axes?

Sol. Let O' be $(-2, -3)$. Since, the axes are rotated about O' by an angle 45° in anti-clockwise direction, let (x', y') be the new coordinates with respect to new axes and (x, y) be the coordinates with respect to old axes. Then, we have

$$x = -2 + x' \cos 45^\circ - y' \sin 45^\circ \\ = -2 + \left(\frac{x' - y'}{\sqrt{2}} \right)$$

$$y = -3 + x' \sin 45^\circ + y' \cos 45^\circ = -3 + \left(\frac{x' + y'}{\sqrt{2}} \right)$$

The new equation is

$$2 \left\{ -2 + \left(\frac{x' - y'}{\sqrt{2}} \right) \right\}^2 + 4 \left\{ -2 + \left(\frac{x' - y'}{\sqrt{2}} \right) \right\} \left\{ -3 + \left(\frac{x' + y'}{\sqrt{2}} \right) \right\} \\ - 5 \left\{ -3 + \left(\frac{x' + y'}{\sqrt{2}} \right) \right\}^2 + 20 \left\{ -2 + \left(\frac{x' - y'}{\sqrt{2}} \right) \right\} \\ - 22 \left\{ -3 + \left(\frac{x' + y'}{\sqrt{2}} \right) \right\} - 14 = 0$$

$$\Rightarrow x'^2 - 14x'y' - 7y'^2 - 2 = 0$$

Hence, new equation of curve is

$$x^2 - 14xy - 7y^2 - 2 = 0$$

Points to be Remembered

1. Reflection (or image) of a point

Let (x, y) be any point, then its image with respect to

- (i) X -axis is $(x, -y)$
- (ii) Y -axis is $(-x, y)$
- (iii) origin is $(-x, -y)$
- (iv) line $y = x$ is (y, x)

2. A triangle is isosceles, if any two of its median are equal.
3. Triangle having integral coordinates can never be equilateral.
4. If $a_r x + b_r y + c_r = 0$ ($r = 1, 2, 3$) are the sides of a triangle, then the area of the triangle is given by

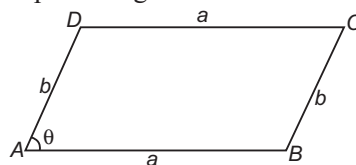
$$\frac{1}{2 C_1 C_2 C_3} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$

where, C_1, C_2 and C_3 are the cofactors of c_1, c_2 and c_3 in the determinant.

5. Area of parallelogram

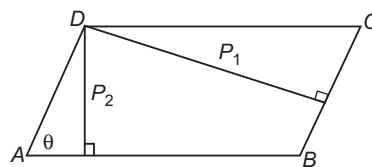
- (i) Whose sides are a and b and angle between them is θ , is given by

area of parallelogram $ABCD = ab \sin \theta$



- (ii) Whose length of perpendicular from one vertex to the opposite sides are P_1 and P_2 and angle between sides is θ , is given by

area of parallelogram $ABCD = \frac{P_1 P_2}{\sin \theta}$



6. A triangle having vertices $(at_1^2, 2at_1)$, $(at_2^2, 2at_2)$ and $(at_3^2, 2at_3)$, then area of triangle

$$= a^2 [(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)]$$

7. Area of triangle formed by coordinate axes and the line $ax + by + c$ is $\frac{c^2}{2|ab|}$.

8. Area of rhombus formed by $|ax| + |by| + |c| = 0$ is $\frac{2c^2}{|ab|}$.

9. Three points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are collinear, if $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$.

10. To remove the term of xy in the equation $ax^2 + 2hxy + by^2 = 0$. The angle θ through which the axes must be turned, is

$$\theta = \frac{1}{2} \tan^{-1} \left(\frac{2h}{a - b} \right)$$





Work Book Exercise

Objective Mathematics Vol. 1

- 1 If the line segment joining $(2, 3)$ and $(-1, 2)$ is divided internally in the ratio $3 : 4$ by the line $x + 2y = k$, then k is
 - a $\frac{41}{7}$
 - b $\frac{5}{7}$
 - c $\frac{36}{7}$
 - d $\frac{31}{7}$
- 2 The polar coordinates of the vertices of a triangle are $(0, 0)$, $(3, \frac{\pi}{2})$ and $(3, \frac{\pi}{3})$. Then, the triangle is
 - a right angled
 - b isosceles
 - c equilateral
 - d None of these
- 3 If the points $(-2, 0)$, $(-1, \frac{1}{\sqrt{3}})$ and $(\cos \theta, \sin \theta)$ are collinear, then the number of values of $\theta \in [0, 2\pi]$ is
 - a 0
 - b 1
 - c 2
 - d infinite
- 4 The incentre of the triangle formed by the axes and the line $\frac{x}{a} + \frac{y}{b} = 1$ is
 - a $(\frac{a}{2}, \frac{b}{2})$
 - b $(\frac{ab}{a+b+\sqrt{ab}}, \frac{ab}{a+b+\sqrt{ab}})$
 - c $(\frac{a}{3}, \frac{b}{3})$
 - d $(\frac{ab}{a+b+\sqrt{a^2+b^2}}, \frac{ab}{a+b+\sqrt{a^2+b^2}})$
- 5 The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then, $PQRS$ must be a
 - a rectangle
 - b square
 - c cyclic quadrilateral
 - d rhombus
- 6 The coordinates of three consecutive vertices of a parallelogram are $(1, 3)$, $(-1, 2)$ and $(2, 5)$. The coordinates of the fourth vertex are
 - a $(6, 4)$
 - b $(4, 6)$
 - c $(-2, 0)$
 - d None of the above
- 7 In a $\triangle ABC$, the coordinates of B are $(0, 0)$, $AB = 2$, $\angle ABC = \frac{\pi}{3}$ and the middle point of BC has the coordinates $(2, 0)$. The centroid of the triangle is
 - a $(\frac{1}{2}, \frac{\sqrt{3}}{2})$
 - b $(\frac{5}{3}, \frac{1}{\sqrt{3}})$
 - c $(\frac{4+\sqrt{3}}{3}, \frac{1}{3})$
 - d None of these
- 8 The points (α, β) , (δ, γ) , (α, δ) and (β, γ) taken in order, where $\alpha, \beta, \gamma, \delta$ are different real numbers, are
 - a collinear
 - b vertices of a square
 - c vertices of a rhombus
 - d concyclic
- 9 The limiting position of the point of intersection of the lines $3x + 4y = 1$ and $(1+c)x + 3c^2y = 2$ as c tends to 1, is
 - a $(-5, 4)$
 - b $(5, -4)$
 - c $(4, -5)$
 - d None of the above
- 10 A point moves in the xy -plane such that the sum of its distances from two mutually perpendicular lines is always equal to 3. The area enclosed by the locus of the point is
 - a 18 sq units
 - b $\frac{9}{2}$ sq units
 - c 7 sq units
 - d None of these
11. Three vertices of a quadrilateral in order are $(6, 1)$, $(7, 2)$ and $(-1, 0)$. If the area of the quadrilateral is 4 sq units, then the locus of fourth vertex has the equation
 - a $x - 7y = 1$
 - b $x - 7y + 15 = 0$
 - c $(x - 7y)^2 + 14(x - 7y) - 15 = 0$
 - d None of the above
12. If a vertex of an equilateral triangle is the origin and the side opposite to it has the equation $x + y = 1$, then the orthocentre of triangle is
 - a $(\frac{1}{3}, \frac{1}{3})$
 - b $(\frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$
 - c $(\frac{2}{3}, \frac{2}{3})$
 - d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The cartesian form of the equation $r^{1/2} = a^{1/2} \cos \frac{\theta}{2}$ is

- (a) $(2x^2 + 2y^2 - ax)^2 = a^2(x^2 + y^2)$
- (b) $(x^2 + y^2 - ax)^2 = 2a^2(x^2 - y^2)$
- (c) $(x^2 + y^2 - 2ax)^2 = a^2(x^2 - y^2)^2$
- (d) None of the above

Sol. Given, $r^{1/2} = a^{1/2} \cos \frac{\theta}{2}$... (i)

On squaring Eq. (i) both sides, we get

$$r = a \cos^2 \frac{\theta}{2} = \frac{a}{2}(1 + \cos \theta)$$

Now, by using the relation between cartesian and polar coordinates, the equation becomes

$$2r^2 = ar + ar \cos \theta$$

$$\text{i.e. } 2(x^2 + y^2) = a\sqrt{x^2 + y^2} + ax$$

$$\text{i.e. } (2x^2 + 2y^2 - ax)^2 = a^2(x^2 + y^2)$$

Hence, (a) is the correct answer.

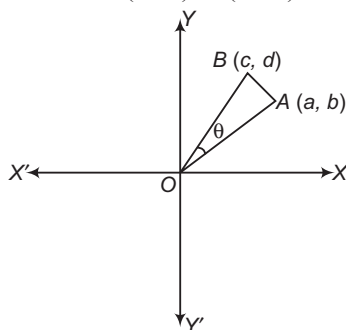
Ex 2. If the segments joining the points $A(a, b)$ and $B(c, d)$ subtend an angle θ at the origin, then

- (a) $\cos \theta = \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$
- (b) $\sin \theta = \frac{ac + bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$
- (c) $\cos \theta = \frac{ac - bd}{\sqrt{(a^2 + b^2)(c^2 + d^2)}}$
- (d) None of the above

Sol. Let O be the origin. Then,

$$OA^2 = a^2 + b^2, OB^2 = c^2 + d^2$$

$$\text{and } AB^2 = (c - a)^2 + (d - b)^2$$



Using cosine formula in $\triangle OAB$, we have

$$AB^2 = OA^2 + OB^2 - 2 \cdot OA \cdot OB \cos \theta$$

$$\Rightarrow (c - a)^2 + (d - b)^2 = a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \cos \theta$$

$$\begin{aligned} &\Rightarrow c^2 + a^2 - 2ac + d^2 + b^2 - 2bd \\ &= a^2 + b^2 + c^2 + d^2 - 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \cos \theta \\ &\Rightarrow 2(ac + bd) = 2\sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \cos \theta \\ \therefore \cos \theta &= \frac{ac + bd}{\sqrt{a^2 + b^2}\sqrt{c^2 + d^2}} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 3. If the point $P(x, y)$ is equidistant from the points $A(a + b, b - a)$ and $B(a - b, a + b)$, then

- (a) $ax = by$
- (b) $bx = ay$ and P can be (a, b)
- (c) $x^2 - y^2 = 2(ax + by)$
- (d) None of the above

Sol. We have, $PA = PB \Rightarrow (PA)^2 = (PB)^2$

$$\begin{aligned} &\Rightarrow [x - (a + b)]^2 + [y - (b - a)]^2 \\ &= [x - (a - b)]^2 + [y - (a + b)]^2 \\ &\Rightarrow [(x - a) - b]^2 + [(y - b) + a]^2 \\ &= [(x - a) + b]^2 + [(y - b) - a]^2 \\ &\Rightarrow [(x - a) + b]^2 - [(x - a) - b]^2 \\ &= [(y - b) + a]^2 - [(y - b) - a]^2 \\ &\Rightarrow 4b(x - a) = 4a(y - b) \\ &\Rightarrow bx = ay \end{aligned} \quad \dots (i)$$

Also, $P(a, b)$ satisfies Eq. (i), so that P can be (a, b) .

Hence, (b) is the correct answer.

Ex 4. If t_1, t_2 and t_3 are distinct and the points $(t_1, 2at_1 + at_1^3)$, $(t_2, 2at_2 + at_2^3)$, $(t_3, 2at_3 + at_3^3)$ are collinear, then

- (a) $t_1 t_2 t_3 = 1$
- (b) $t_1 + t_2 + t_3 = t_1 t_2 t_3$
- (c) $t_1 + t_2 + t_3 = 0$
- (d) $t_1 + t_2 + t_3 = -1$

Sol. The given points are collinear, if

$$\begin{vmatrix} t_1 & 2at_1 + at_1^3 & 1 \\ t_2 & 2at_2 + at_2^3 & 1 \\ t_3 & 2at_3 + at_3^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} t_1 & 2t_1 + t_1^3 & 1 \\ t_2 & 2t_2 + t_2^3 & 1 \\ t_3 & 2t_3 + t_3^3 & 1 \end{vmatrix} = 0$$

On applying $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$, we get

$$\begin{vmatrix} t_1 & 2t_1 + t_1^3 & 1 \\ t_2 - t_1 & 2(t_2 - t_1) + (t_2^3 - t_1^3) & 0 \\ t_3 - t_1 & 2(t_3 - t_1) + (t_3^3 - t_1^3) & 0 \end{vmatrix} = 0$$

$$\Rightarrow (t_2 - t_1)(t_3 - t_1) \begin{vmatrix} t_1 & 2t_1 + t_1^3 & 1 \\ 1 & 2 + t_2^2 + t_1^2 + t_2t_1 & 0 \\ 1 & 2 + t_3^2 + t_1^2 + t_3t_1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow (t_2 - t_1)(t_3 - t_1)(t_3 - t_2)(t_3 + t_2 + t_1) = 0$$

$$\Rightarrow t_1 + t_2 + t_3 = 0 \quad [\because t_1 \neq t_2 \neq t_3]$$

Hence, (c) is the correct answer.

Ex 5. If the vertices of a triangle have integral coordinates, then the triangle cannot be
 (a) equilateral (b) isosceles
 (c) scalene (d) None of these

Sol. Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$ be the vertices of a triangle and $x_1, x_2, x_3, y_1, y_2, y_3$ be integers.

$$\therefore BC^2 = (x_2 - x_3)^2 + (y_2 - y_3)^2, \text{ it is a positive integer.}$$

If the triangle is equilateral, then $AB = BC = CA = a$ [say]

$$\text{and} \quad \angle A = \angle B = \angle C = 60^\circ$$

$$\therefore \text{Area of the triangle} = \left(\frac{1}{2}\right)bc \sin A = \left(\frac{1}{2}\right)a^2 \sin 60^\circ$$

$$= \left(\frac{a^2}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \left(\frac{\sqrt{3}}{4}\right)a^2$$

which is irrational. $[\because a^2 \text{ is a positive integer}]$

Now, area of the triangle in terms of the coordinates

$$= \frac{1}{2} |x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)|$$

which is a rational number.

This contradicts that the area is an irrational number, if the triangle is equilateral.

Hence, (a) is the correct answer.

Ex 6. If the coordinates of two points A and B are $(3, 4)$ and $(5, -2)$ respectively. Then, the coordinates of any point P , if $PA = PB$ and area of $\triangle PAB = 10$, are

- (a) $(7, 5)$, $(1, 0)$ (b) $(7, 2)$, $(1, 0)$
 (c) $(7, 2)$, $(-1, 0)$ (d) None of these

Sol. Let the coordinates of P be (x, y) . Then,

$$PA = PB \Rightarrow PA^2 = PB^2$$

$$\Rightarrow (x - 3)^2 + (y - 4)^2 = (x - 5)^2 + (y + 2)^2$$

$$\Rightarrow x - 3y - 1 = 0 \quad \dots(i)$$

Now, area of $\triangle PAB = 10$

$$\Rightarrow \frac{1}{2} \begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ 5 & -2 & 1 \end{vmatrix} = \pm 10$$

$$\Rightarrow 6x + 2y - 26 = \pm 20$$

$$\Rightarrow 6x + 2y - 46 = 0 \text{ or } 6x + 2y - 6 = 0$$

$$\Rightarrow 3x + y - 23 = 0 \text{ or } 3x + y - 3 = 0 \quad \dots(ii)$$

On solving $x - 3y - 1 = 0$ and $3x + y - 23 = 0$, we get
 $x = 7$ and $y = 2$

On solving $x - 3y - 1 = 0$ and $3x + y - 3 = 0$, we get
 $x = 1$ and $y = 0$

Thus, the coordinates of P are $(7, 2)$ or $(1, 0)$.

Hence, (b) is the correct answer.

Ex 7. The area of the triangle with vertices

$[(a+1)(a+2), (a+2)]$ $[(a+2)(a+3), (a+3)]$
 and $[(a+3)(a+4), (a+4)]$ is

- (a) independent of a (b) dependent on a
 (c) Cannot be discussed (d) None of these

Sol. Area of the given triangle is given by

$$\Delta = \frac{1}{2} \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ (a+2)(a+3) & a+3 & 1 \\ (a+3)(a+4) & a+4 & 1 \end{vmatrix}$$

On applying $R_3 \rightarrow R_3 - R_2$ and $R_2 \rightarrow R_2 - R_1$, we get

$$\Delta = \frac{1}{2} \begin{vmatrix} (a+1)(a+2) & a+2 & 1 \\ 2(a+2) & 1 & 0 \\ 2(a+3) & 1 & 0 \end{vmatrix} = |-1| = 1$$

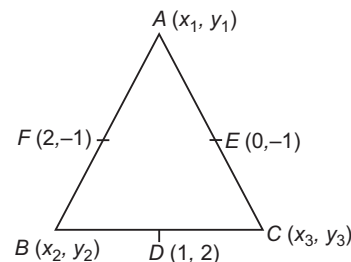
which is independent of a .

Hence, (a) is the correct answer.

Ex 8. If the coordinates of mid-points of the sides of a triangle are $(1, 2)$, $(0, -1)$ and $(2, -1)$. Then, the coordinates of its vertices are

- (a) $(1, -4)$, $(3, 2)$, $(-1, 2)$
 (b) $(1, 4)$, $(3, -2)$, $(1, 2)$
 (c) $(1, 4)$, $(3, 2)$, $(1, 2)$
 (d) None of the above

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$. Let $D(1, 2)$, $E(0, -1)$ and $F(2, -1)$ be the mid-points of BC , CA and AB , respectively.



$$\therefore \frac{x_2 + x_3}{2} = 1 \text{ and } \frac{y_2 + y_3}{2} = 2$$

$$\Rightarrow x_2 + x_3 = 2 \text{ and } y_2 + y_3 = 4 \quad \dots(i)$$

Similarly, E and F are the mid-points of CA and AB , respectively.

$$\frac{x_1 + x_3}{2} = 0 \text{ and } \frac{y_1 + y_3}{2} = -1$$

$$\Rightarrow x_1 + x_3 = 0 \text{ and } y_1 + y_3 = -2 \quad \dots(ii)$$

$$\text{and } \frac{x_1 + x_2}{2} = 2 \text{ and } \frac{y_1 + y_2}{2} = -1$$

$$\Rightarrow x_1 + x_2 = 4 \text{ and } y_1 + y_2 = -2 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$(x_2 + x_3) + (x_1 + x_3) + (x_1 + x_2) = 2 + 0 + 4$$

$$\text{and } (y_2 + y_3) + (y_1 + y_3) + (y_1 + y_2) = 4 - 2 - 2$$

$$\Rightarrow x_1 + x_2 + x_3 = 3 \text{ and } y_1 + y_2 + y_3 = 0 \quad \dots(iv)$$

From Eqs. (i) and (iv), we get

$$x_1 + 2 = 3 \text{ and } y_1 + 4 = 0$$

$$\therefore x_1 = 1$$

$$\text{and } y_1 = -4$$



So, the coordinates of A are $(1, -4)$.

From Eqs. (ii) and (iv), we get

$$x_2 + 0 = 3 \quad \text{and} \quad y_2 - 2 = 0$$

$$\therefore x_2 = 3 \quad \text{and} \quad y_2 = 2$$

So, the coordinates of B are $(3, 2)$.

From Eqs. (iii) and (iv), we get

$$x_3 + 4 = 3 \quad \text{and} \quad y_3 - 2 = 0$$

$$\therefore x_3 = -1 \quad \text{and} \quad y_3 = 2$$

So, coordinates of C are $(-1, 2)$.

Hence, the vertices of the ΔABC are $A(1, -4)$, $B(3, 2)$ and $C(-1, 2)$.

Hence, (a) is the correct answer.

- Ex 9.** If Δ_1 is the area of the triangle with vertices $(0, 0)$, $(a \tan \alpha, b \cot \alpha)$, $(a \sin \alpha, b \cos \alpha)$, Δ_2 is the area of the triangle with vertices (a, b) , $(a \sec^2 \alpha, b \operatorname{cosec}^2 \alpha)$, $(a + a \sin^2 \alpha, b + b \cos^2 \alpha)$ and Δ_3 is the area of the triangle with vertices $(0, 0)$, $(a \tan \alpha, -b \cot \alpha)$, $(a \sin \alpha, b \cos \alpha)$. Then,
- Δ_1, Δ_2 and Δ_3 are in GP
 - Δ_1, Δ_2 and Δ_3 are not in GP
 - Cannot be discussed
 - None of the above

Sol. We have,

$$\Delta_1 = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a \tan \alpha & b \cot \alpha & 1 \\ a \sin \alpha & b \cos \alpha & 1 \end{vmatrix} = \frac{1}{2} ab |\sin \alpha - \cos \alpha|$$

$$\text{and } \Delta_2 = \frac{1}{2} \begin{vmatrix} a & b & 1 \\ a \sec^2 \alpha & b \operatorname{cosec}^2 \alpha & 1 \\ a + a \sin^2 \alpha & b + b \cos^2 \alpha & 1 \end{vmatrix}$$

On applying $C_1 \rightarrow C_1 - aC_3$ and $C_2 \rightarrow C_2 - bC_3$, we get

$$\Delta_2 = \frac{1}{2} ab \begin{vmatrix} 0 & 0 & 1 \\ \tan^2 \alpha & \cot^2 \alpha & 1 \\ \sin^2 \alpha & \cos^2 \alpha & 1 \end{vmatrix}$$

$$= \frac{1}{2} ab |\sin^2 \alpha - \cos^2 \alpha|$$

$$\text{and } \Delta_3 = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a \tan \alpha & -b \cot \alpha & 1 \\ a \sin \alpha & b \cos \alpha & 1 \end{vmatrix}$$

$$= \frac{1}{2} ab |\sin \alpha + \cos \alpha|$$

$$\text{Clearly, } \Delta_1 \Delta_3 = \frac{1}{2} ab \Delta_2.$$

Now, Δ_1, Δ_2 and Δ_3 are in GP, if

$$\Delta_1 \Delta_3 = \Delta_2^2 \Rightarrow \frac{1}{2} ab \Delta_2 = \Delta_2^2 \Rightarrow \Delta_2 = \frac{1}{2} ab$$

$$\Rightarrow \frac{1}{2} ab (\sin^2 \alpha - \cos^2 \alpha) = \frac{1}{2} ab$$

$$\Rightarrow \sin^2 \alpha - \cos^2 \alpha = 1$$

Then, $\alpha = (2m+1)\frac{\pi}{2}$, $m \in I$. But for this value of α , the

vertices of the given triangles are not defined. Δ_1, Δ_2 and Δ_3 cannot be in GP for any value of α .

Hence, (b) is the correct answer.

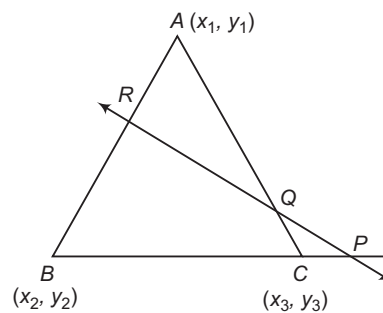
Ex 10. A line L intersects the three sides BC , CA and AB of a ΔABC at P , Q and R respectively.

Then, $\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB}$ is equal to

- 1
- 0
- 1
- None of the above

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of ΔABC and let $lx + my + n = 0$ be the equation of line, if P divides BC in the ratio $\lambda : 1$, then the coordinates of P are

$$\left(\frac{\lambda x_3 + x_2}{\lambda + 1}, \frac{\lambda y_3 + y_2}{\lambda + 1} \right)$$



Also, as P lies on L , we have

$$l \left(\frac{\lambda x_3 + x_2}{\lambda + 1} \right) + m \left(\frac{\lambda y_3 + y_2}{\lambda + 1} \right) + n = 0$$

$$\Rightarrow -\frac{lx_2 + my_2 + n}{lx_3 + my_3 + n} = \lambda = \frac{BP}{PC} \quad \dots(i)$$

Similarly, we obtain

$$\frac{CQ}{QA} = -\frac{lx_3 + my_3 + n}{lx_1 + my_1 + n} \quad \dots(ii)$$

$$\text{and } \frac{AR}{RB} = -\frac{lx_1 + my_1 + n}{lx_2 + my_2 + n} \quad \dots(iii)$$

On multiplying Eqs. (i), (ii) and (iii), we get

$$\frac{BP}{PC} \cdot \frac{CQ}{QA} \cdot \frac{AR}{RB} = -1$$

Hence, (c) is the correct answer.

Ex 11. Vertices of a triangle are $A(x_1, x_1 \tan \alpha_1)$, $B(x_2, x_2 \tan \alpha_2)$ and $C(x_3, x_3 \tan \alpha_3)$. If the circumcentre coincides with the origin and the orthocentre $H = (\bar{x}, \bar{y})$, then

$\bar{y}(\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3)$ is equal to

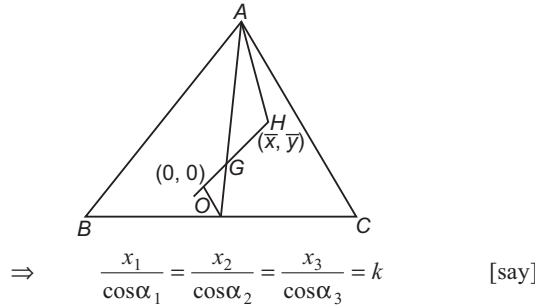
- $\bar{x}(\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3)$
- $\bar{x}(\sin \alpha_1 - \sin \alpha_2 - \sin \alpha_3)$
- $\bar{x}(\sin \alpha_3 - \sin \alpha_2 + \sin \alpha_1)$
- None of the above

Sol. Here, the circumcentre $O = (0, 0)$

So, $OA = OB = OC$

$$\therefore x_1^2 + x_1^2 \tan^2 \alpha_1 = x_2^2 + x_2^2 \tan^2 \alpha_2 = x_3^2 + x_3^2 \tan^2 \alpha_3$$

$$\Rightarrow x_1^2 \sec^2 \alpha_1 = x_2^2 \sec^2 \alpha_2 = x_3^2 \sec^2 \alpha_3$$



$\therefore$ The vertices of the triangle become

$$\begin{aligned} A &= (k \cos \alpha_1, k \sin \alpha_1) \\ B &= (k \cos \alpha_2, k \sin \alpha_2) \\ C &= (k \cos \alpha_3, k \sin \alpha_3) \end{aligned}$$

$\therefore$ The centroid G

$$= \left(k \frac{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3}{3}, k \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{3} \right)$$

We know that the orthocentre H , centroid G and circumcentre O are collinear.

So,

$$m \text{ of } HO = m \text{ of } GO$$

$$\therefore m \text{ of } HO = \frac{\bar{y} - 0}{\bar{x} - 0} = \frac{\bar{y}}{\bar{x}}$$

$$\text{and } m \text{ of } GO = \frac{k \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{3} - 0}{k \frac{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3}{3} - 0}$$

$$\therefore \frac{\bar{y}}{\bar{x}} = \frac{\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3}{\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3}$$

$$\Rightarrow \bar{y}(\cos \alpha_1 + \cos \alpha_2 + \cos \alpha_3) = \bar{x}(\sin \alpha_1 + \sin \alpha_2 + \sin \alpha_3)$$

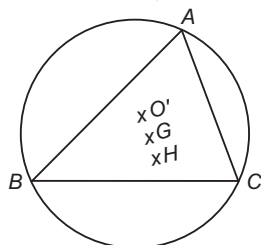
Hence, (a) is the correct answer.

Ex 12. The circumcentre of a triangle having vertices $A(a, a \tan \alpha)$, $B(b, b \tan \beta)$, $C(c, c \tan \gamma)$ is at the origin, where $\alpha + \beta + \gamma = \pi$. If the orthocentre lies on the line $\left(4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2}\right)x - \lambda y = 0$, then λ is

- (a) $4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$
- (b) $2 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$
- (c) $1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$
- (d) None of the above

Sol. Since, circumcentre O' is the origin and therefore equation of the circumcircle may be

$$x^2 + y^2 = r^2 \quad \dots(i)$$



Since, vertex $A(a, a \tan \alpha)$ lies on Eq. (i), therefore

$$a^2(1 + \tan^2 \alpha) = r^2$$

$$\Rightarrow a = r \cos \alpha$$

$$\therefore A \equiv (r \cos \alpha, r \sin \alpha)$$

Similarly, $B \equiv (r \cos \beta, r \sin \beta)$, $C \equiv (r \cos \gamma, r \sin \gamma)$

$\therefore$ Centroid G is

$$\left[\frac{r(\cos \alpha + \cos \beta + \cos \gamma)}{3}, \frac{r(\sin \alpha + \sin \beta + \sin \gamma)}{3} \right]$$

Circumcentre $O'(0, 0)$ and orthocentre $H(h, k)$ (let)

Since, we know that O', G and H are collinear, therefore

Slope of $O'G$ = Slope of $O'H$

$$\Rightarrow \frac{\sin \alpha + \sin \beta + \sin \gamma}{\cos \alpha + \cos \beta + \cos \gamma} = \frac{k}{h}$$

$$\Rightarrow x(\sin \alpha + \sin \beta + \sin \gamma) - y(\cos \alpha + \cos \beta + \cos \gamma) = 0$$

$$\Rightarrow x \left(4 \cos \frac{\alpha}{2} \cos \frac{\beta}{2} \cos \frac{\gamma}{2} \right) - y \left(1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2} \right) = 0$$

and hence the result becomes $\alpha + \beta + \gamma = \pi$

$$\text{Hence, } \lambda = 1 + 4 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}$$

Hence, (c) is the correct answer.

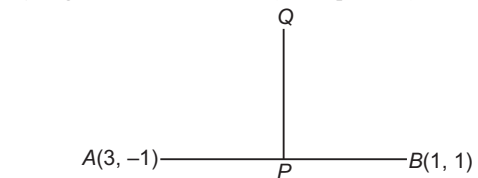
Ex 13. The mid-point of line segment joining $(3, -1)$ and $(1, 1)$ is shifted by two units (in the sense of increasing y) perpendicular to the line segment. Then, the coordinates of the point in the new position are

- (a) $(2 - \sqrt{2}, 2)$
- (b) $(2, 2 - \sqrt{3})$
- (c) $(2 + \sqrt{2}, \sqrt{2})$
- (d) None of these

Sol. Let P be the mid-point of the line segment joining $A(3, -1)$ and $B(1, 1)$.

$$\text{Then, } P \equiv \left(\frac{3+1}{2}, \frac{-1+1}{2} \right) = (2, 0)$$

Let P be shifted to Q , where $PQ = 2$ and y -coordinate of Q is greater than that of P (from question).



$$\text{Now, } m \text{ of } AB = \frac{1 - (-1)}{1 - 3} = \frac{2}{-2} = -1$$

$$\therefore m \text{ of } PQ = -\frac{1}{(-1)} = 1$$

Coordinates of Q by distance formula are

$$(2 \pm 2 \cos \theta, 0 \pm 2 \sin \theta), \text{ where } \tan \theta = 1$$

$$= \left(2 \pm 2 \cdot \frac{1}{\sqrt{2}}, 0 \pm 2 \cdot \frac{1}{\sqrt{2}} \right) = (2 \pm \sqrt{2}, \pm \sqrt{2})$$

As y -coordinate of Q is greater than that of P .

$$\therefore Q = (2 + \sqrt{2}, \sqrt{2})$$

This is the required point.

Hence, (c) is the correct answer.



Ex 14. ABC is a variable triangle with the fixed vertex $C(1, 2)$ and A, B having the coordinates $(\cos t, \sin t)$, $(\sin t, -\cos t)$ respectively, where t is a parameter. The locus of centroid of the ΔABC is

- (a) $3(x^2 + y^2) - 2x - 4y - 1 = 0$
 (b) $3(x^2 + y^2) - 2x - 4y + 1 = 0$
 (c) $3(x^2 + y^2) + 2x + 4y - 1 = 0$
 (d) $3(x^2 + y^2) + 2x + 4y + 1 = 0$

Sol. Let $G(\alpha, \beta)$ be the centroid in any position.

$$\text{Then, } (\alpha, \beta) = \left(\frac{1 + \cos t + \sin t}{3}, \frac{2 + \sin t - \cos t}{3} \right)$$

$$\therefore \alpha = \frac{1 + \cos t + \sin t}{3}, \beta = \frac{2 + \sin t - \cos t}{3}$$

or $3\alpha - 1 = \cos t + \sin t$... (i)
 and $3\beta - 2 = \sin t - \cos t$... (ii)

On squaring and adding Eqs. (i) and (ii), we get
 $(3\alpha - 1)^2 + (3\beta - 2)^2 = (\cos t + \sin t)^2 + (\sin t - \cos t)^2$
 $= 2(\cos^2 t + \sin^2 t) = 2$

$\therefore$ The equation of locus of the centroid is

$$(3x - 1)^2 + (3y - 2)^2 = 2$$

$$\Rightarrow 9(x^2 + y^2) - 6x - 12y + 3 = 0$$

$$\Rightarrow 3(x^2 + y^2) - 2x - 4y + 1 = 0$$

Hence, (b) is the correct answer.

Ex 15. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Then, the locus of the point which divides the line segment between these two points in the ratio 1 : 2, is

- (a) $16x^2 - 10xy + y^2 - 2 = 0$
 (b) $16x^2 + 10xy + y^2 + 2 = 0$
 (c) $16x^2 + 10xy - y^2 + 2 = 0$
 (d) $16x^2 + 10xy + y^2 - 2 = 0$

Sol. Let equation of the line be $y = 4x + c$, where c is parameter. It intersects the hyperbola $xy = 1$ at two points, for which $x(4x + c) = 1$.

$$\Rightarrow 4x^2 + cx - 1 = 0$$

Let x_1 and x_2 be the roots of this equation. Then,

$$x_1 + x_2 = -\frac{c}{4} \quad \text{and} \quad x_1 x_2 = -\frac{1}{4}$$

If A and B are the points of intersection of the lines and the hyperbola, then the coordinates A are $\left(x_1, \frac{1}{x_1}\right)$ and

that of B are $\left(x_2, \frac{1}{x_2}\right)$.

Let $R(h, k)$ be the point which divides AB in the ratio 1 : 2, then

$$h = \frac{2x_1 + x_2}{3} \quad \text{and} \quad k = \frac{\frac{2}{x_1} + \frac{1}{x_2}}{3} = \frac{2x_2 + x_1}{3x_1 x_2}$$

$$\Rightarrow 2x_1 + x_2 = 3h \quad \dots (i)$$

and $x_1 + 2x_2 = 3\left(-\frac{1}{4}\right)k = \left(-\frac{3}{4}\right)k \quad \dots (ii)$

On adding Eqs. (i) and (ii), we get

$$3(x_1 + x_2) = 3\left(h - \frac{k}{4}\right)$$

$$\Rightarrow 3\left(-\frac{c}{4}\right) = 3\left(h - \frac{k}{4}\right) \Rightarrow h - \frac{k}{4} = -\frac{c}{4} \quad \dots (iii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$x_1 - x_2 = 3\left(h + \frac{k}{4}\right)$$

$$\Rightarrow (x_1 - x_2)^2 = 9\left(h + \frac{k}{4}\right)^2$$

$$\Rightarrow \frac{c^2}{16} + 1 = 9\left(h + \frac{k}{4}\right)^2$$

$$\Rightarrow \left(h - \frac{k}{4}\right)^2 + 1 = 9\left(h + \frac{k}{4}\right)^2$$

$$\Rightarrow h^2 - \frac{1}{2}hk + \frac{k^2}{16} + 1 = 9\left(h^2 + \frac{1}{2}hk + \frac{k^2}{16}\right)$$

$$\Rightarrow 16h^2 + 10hk + k^2 - 2 = 0$$

So, the locus of $R(h, k)$ is $16x^2 + 10xy + y^2 - 2 = 0$.

Hence, (d) is the correct answer.

Ex 16. The equation of a pair of straight lines is $ax^2 + 2hxy + by^2 = 0$. By what angle must the axes be rotated so that the term containing xy in the equation may be removed?

- (a) $\frac{1}{2} \tan^{-1} \left(\frac{2h}{a+b} \right)$ (b) $\frac{1}{2} \tan^{-1} \left(\frac{2h}{a-b} \right)$
 (c) $\frac{1}{2} \tan^{-1} \left(\frac{2h}{ab} \right)$ (d) None of these

Sol. Here, the origin remains fixed. Let the axes be turned about the fixed origin through an angle ϕ in the anti-clockwise sense and new coordinates of the point (x, y) become (x', y') .

Then, the equations of transformation will be
 $x = x' \cos \phi - y' \sin \phi$ and $y = x' \sin \phi + y' \cos \phi$

$\therefore$ The changed equation will be

$$a(x' \cos \phi - y' \sin \phi)^2 + 2h(x' \cos \phi - y' \sin \phi)(x' \sin \phi + y' \cos \phi) + b(x' \sin \phi + y' \cos \phi)^2 = 0$$

$$\Rightarrow (a \cos^2 \phi + 2h \sin \phi \cos \phi + b \sin^2 \phi)x'^2$$

$$+ (-a \sin 2\phi + 2h \cos 2\phi + b \sin 2\phi)x' y'$$

$$+ (a \sin^2 \phi - 2h \sin \phi \cos \phi + b \cos^2 \phi)y'^2 = 0$$

$\therefore$ The transformed equation of the pair of lines for the new axes will be

$$(a \cos^2 \phi + h \sin 2\phi + b \sin^2 \phi)x'^2 + \{(b - a) \sin 2\phi$$

$$+ 2h \cos 2\phi\}x' y' + (a \sin^2 \phi - h \sin 2\phi + b \cos^2 \phi)y'^2 = 0$$

This equation will not contain xy , if

$$(b - a) \sin 2\phi + 2h \cos 2\phi = 0$$

$$\Rightarrow \tan 2\phi = \frac{2h}{a - b}$$

$$\Rightarrow \phi = \frac{1}{2} \tan^{-1} \frac{2h}{a - b}$$

which is the required angle.

Hence, (b) is the correct answer.



Type 2. More than One Correct Option

Ex 17. If the coordinates of the vertices of a triangle are rational numbers, then which of the following points of the triangle have rational coordinates?

- (a) centroid (b) incentre
(c) circumcentre (d) orthocentre

Sol. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$ such that $x_1, x_2, x_3, y_1, y_2, y_3$ are rational numbers. Then, $\frac{x_1 + x_2 + x_3}{3}$ and $\frac{y_1 + y_2 + y_3}{3}$ are also rational numbers.

So, the coordinates of the centroid are always rational numbers. Let P be the circumcentre of $\triangle ABC$. Then,

$$\begin{aligned} PA &= PB = PC \\ \Rightarrow PA^2 &= PB^2 = PC^2 \\ \Rightarrow PA^2 &= PB^2 \\ \text{and } PB^2 &= PC^2 \end{aligned}$$

These two relations provide two linear equations in terms of the coordinates of point P such that the coefficients are rational numbers. So, the coordinates of P are also rational numbers.

Since, the centroid G divides the line segment joining circumcentre P and orthocentre H in the ratio 1 : 2. Therefore, coordinates of H will also have rational values.

The coordinates of the incentre are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c} \right)$$

where, $a = BC$, $b = CA$ and $c = AB$

Clearly, $a = \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2}$

$$b = \sqrt{(x_3 - x_1)^2 + (y_3 - y_1)^2}$$

$$\text{and } c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

may not be rational number. Therefore, coordinates of the incentre may not be rational numbers.

Hence, (a), (c) and (d) are the correct answers.

Ex 18. If $a > 0$, $b > 0$, then the maximum area of the triangle formed by the points $O(0, 0)$, $A(a \cos \theta, b \sin \theta)$ and $B(a \cos \theta, -b \sin \theta)$ is (in sq units)

- (a) $\frac{3ab}{4}$, when $\theta = \frac{\pi}{4}$
(b) $\frac{ab}{2}$, when $\theta = \frac{\pi}{4}$
(c) $\frac{ab}{2}$, when $\theta = -\frac{\pi}{4}$
(d) $a^2 b^2$

Sol. Let Δ be the area of $\triangle ABC$. Then,

$$\Delta = \text{Absolute value of } \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a \cos \theta & b \sin \theta & 1 \\ a \cos \theta & -b \sin \theta & 1 \end{vmatrix}$$

$$= \text{Absolute value of } (-ab \sin \theta \cos \theta)$$

$$= ab |\sin \theta \cos \theta|$$

$$\therefore \Delta = \frac{ab}{2} |\sin 2\theta|$$

Clearly, Δ is maximum when $\sin 2\theta = \pm 1$ i.e. $\theta = \pm \frac{\pi}{4}$

and the maximum value of Δ is $\frac{ab}{2}$.

Hence, (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 19-20) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 19. Statement I If point of intersection of the lines $4x + 3y = \lambda$ and $3x - 4y = \mu$, $\forall \lambda, \mu \in R$ is (x_1, y_1) , then the locus of (λ, μ) is $x + 7y = 0$, $\forall x_1 = y_1$.

Statement II If $4\lambda + 3\mu > 0$ and $3\lambda - 4\mu > 0$, then (x_1, y_1) is in first quadrant.

Sol. On solving $4x + 3y = \lambda$ and $3x - 4y = \mu$, we get $x_1 = \frac{4\lambda + 3\mu}{25}$ and $y_1 = \frac{3\lambda - 4\mu}{25}$

$$\therefore \frac{4\lambda + 3\mu}{25} = \frac{3\lambda - 4\mu}{25}$$

$$\Rightarrow \lambda + 7\mu = 0$$

$\therefore$ Locus of (λ, μ) is $x + 7y = 0$

For first quadrant, $x_1 > 0$ and $y_1 > 0$

$$\text{i.e. } \frac{4\lambda + 3\mu}{25} > 0 \quad \text{and} \quad \frac{3\lambda - 4\mu}{25} > 0$$

$$\text{or } 4\lambda + 3\mu > 0 \quad \text{and} \quad 3\lambda - 4\mu > 0$$

Hence, (b) is the correct answer.

Ex 20. Statement I If centroid and circumcentre of a triangle are known, then its orthocentre and nine point centre can be found.

Statement II Orthocentre, nine point centre, centroid and circumcentre are collinear.

Sol. Orthocentre (O), nine point centre (N), centroid (G) and circumcentre (C) are collinear and G divides O and C in the ratio 2 : 1 (internally) and N is the mid-point of O and C .

Hence, (a) is the correct answer.



Type 4. Linked Comprehension Based Questions

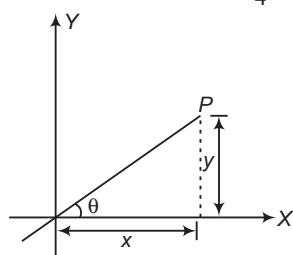
Passage (Ex. Nos. 21-23) Suppose we define the distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ as $d(P, Q) = \max\{|x_2 - x_1|, |y_2 - y_1|\}$.

Ex 21. For a point P along the line $y = \sqrt{3}x$, $d(0, P)$ is equal to

- (a) x (b) y
(c) x , if $x < 0$ (d) y , if $y \geq 0$

Sol. Clearly, $|y| < |x|$, if $\theta < \frac{\pi}{4}$,

$$|y| > |x|, \text{ if } \theta < \frac{\pi}{4}$$



and $|y| = |x|$, if $\theta = \frac{\pi}{4}$

$$\Rightarrow d(0, P) = |y| \text{ as } \theta > \frac{\pi}{4}$$

$$\Rightarrow d(0, P) = y, y \geq 0$$

Hence, (d) is the correct answer.

Ex 22. The area of the region bounded by the locus of a point P satisfying $d(P, A) = 4$, where A is $(1, 2)$, is

- (a) 64 sq units (b) 54 sq units
(c) 16π sq units (d) None of these

Sol. We have, $\max\{|x - 1|, |y - 2|\} = 4$

If $\{|x - 1| \geq |y - 2|\}$, then

$$|x - 1| = 4$$

i.e. if $(x + y - 3)(x - y + 1) \geq 0$, then

$$x = -3 \text{ or } 5$$

If $|y - 2| \geq |x - 1|$, then

$$|y - 2| = 4$$

i.e. $(x + y - 3)(x - y + 1) \leq 0$

Then, $y = -2 \text{ or } 6$

So, the locus of P bounds a square, whose sides are

$$x = -3, x = 5, y = -2, y = 6$$

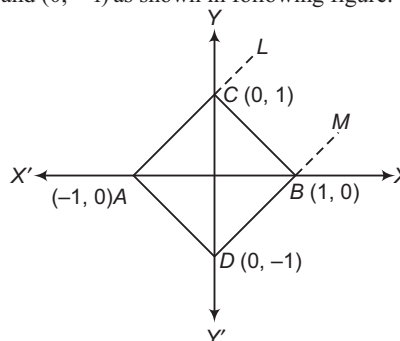
Thus, the area is $(8)^2 = 64$

Hence, (a) is the correct answer.

Ex 23. Suppose that points A and B have coordinates $(1, 0)$ and $(-1, 0)$ respectively, then for a variable point P on this plane the equation $d(P, A) + d(P, B) = 2$ represents

- (a) a line segment joining A and B
(b) an ellipse with foci at A and B
(c) region lying inside a square of area 2
(d) region inside a semi-circle with AB as diameter

Sol. Consider a square with vertices at $(-1, 0)$, $(1, 0)$, $(0, 1)$ and $(0, -1)$ as shown in following figure:



If we select a point P_1 in the region $BCLM$, then

$$d(A, P) = 1 + x$$

$$d(B, P) = y$$

$$\therefore d(A, P) + d(B, P) = 1 + x + y > 2$$

However, for points P_2 and P_3 lying respectively on the line BC and below the line BC ,

$$d(P, A) + d(P, B) = 2$$

The same argument holds for other quadrant also.

Hence, $d(P, A) + d(P, B) = 2$ represents the regions lying inside the square $ABCD$.

Hence, (c) is the correct answer.

Type 5. Match the Column

Ex 24. Match the following:

Column I	Column II
A. If the point $\{x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)\}$, divides the join of (x_1, y_1) and (x_2, y_2) internally, then	p. $-\frac{4\sqrt{2}}{3} < t < \frac{5\sqrt{2}}{6}$
B. Set of values of t for which point $P(t, t^2 - 2)$ lies inside the triangle formed by the lines $x + y = 1$, $y = x + 1$ and $y = -1$, is	q. $1 < t < \frac{\sqrt{13} - 1}{2}$
C. If $P\left(1 + \frac{t}{\sqrt{2}}, 2 + \frac{t}{\sqrt{2}}\right)$ is any point on a line, then the value of t for which the point P lies between parallel lines $x + 2y = 1$ and $2x + 4y = 15$, is	r. $0 < t < 1$
D. If the point $P(1, t)$ always remains in the interior of triangle formed by the lines $y = x$, $y = 0$ and $x + y = 4$, then	s. $0 \leq t \leq 1$

Sol. A. $\{x_1 + t(x_2 - x_1), y_1 + t(y_2 - y_1)\}$
 $= \left\{ \frac{(1-t)x_1 + tx_2}{(1-t) + t}, \frac{(1-t)y_1 + ty_2}{(1-t) + t} \right\}$

Clearly, these are coordinates of the point dividing the join of (x_1, y_1) and (x_2, y_2) in the ratio of $t : (1-t)$. For the internal division, we must have $t > 0$ and $1-t > 0$.

$$\Rightarrow 0 < t < 1$$

B. If the point $P(t, t^2 - 2)$ lies inside the $\triangle ABC$, then there are three cases arise.

Case I A and P must be on same side of BC whose equation is $y + 1 = 0$.

$$\therefore (1+1)(t^2 - 2 + 1) > 0 \Rightarrow |t| > 1 \quad \dots(i)$$

Case II B and P must lie on the same side of AC .

$$\begin{aligned} \therefore (-2 - 1 - 1)(t + t^2 - 2 - 1) &> 0 \\ \Rightarrow t^2 + t - 3 &< 0 \\ \Rightarrow \frac{-1 - \sqrt{13}}{2} < t < \frac{-1 + \sqrt{13}}{2} \quad \dots(ii) \end{aligned}$$

Case III C and P must lie on same side of AB .

$$\begin{aligned} \therefore (2 + 1 + 1)(t - t^2 + 2 + 1) &> 0 \\ \Rightarrow t^2 - t - 3 &< 0 \\ \Rightarrow \frac{1 - \sqrt{13}}{2} < t < \frac{1 + \sqrt{13}}{2} \quad \dots(iii) \end{aligned}$$

From Eqs. (i), (ii) and (iii), we get

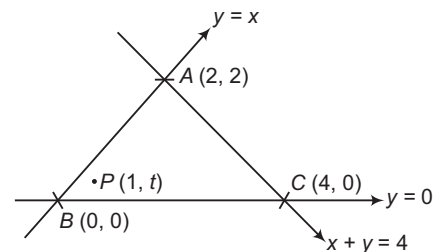
$$t \in \left(\frac{1 - \sqrt{13}}{2}, -1 \right) \cup \left(1, \frac{-1 + \sqrt{13}}{2} \right)$$

$$\text{or } 1 < |t| < \frac{-1 + \sqrt{13}}{2}$$

C. Since, P lies between the parallel lines

$$\begin{aligned} \Rightarrow \left(1 + \frac{t}{\sqrt{2}} + 4 + \frac{2t}{\sqrt{2}} - 1 \right) \\ \left(2 + \frac{2t}{\sqrt{2}} + 8 + \frac{4t}{\sqrt{2}} - 15 \right) < 0 \\ \Rightarrow \left(4 + \frac{3t}{\sqrt{2}} \right) \left(-5 + \frac{6t}{\sqrt{2}} \right) < 0 \\ \Rightarrow -\frac{4\sqrt{2}}{3} < t < \frac{5\sqrt{2}}{6} \end{aligned}$$

D. Since, $P(1, t)$ lies inside the $\triangle ABC$.



Case I A and P lies on same side of BC .

$$\Rightarrow 2t > 0 \text{ or } t > 0 \quad \dots(i)$$

Case II B and P lies on same side of AC .

$$\begin{aligned} \Rightarrow (0 + 0 - 4)(1 + t - 4) &> 0 \\ \Rightarrow (t - 3) < 0 \Rightarrow t < 3 \quad \dots(ii) \end{aligned}$$

Case III C and P lies on same side of AB .

$$\Rightarrow (4 - 0)(1 - t) > 0 \Rightarrow t < 1 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we get

$$0 < t < 1$$

$$A \rightarrow r, s; B \rightarrow q; C \rightarrow p; D \rightarrow r, s$$

Type 6. Single Integer Answer Type Questions

Ex 25. If the line segment joining $(2, 3)$ and $(-1, 2)$ is divided internally in the ratio $3 : 4$ by the line $x + 2y = \frac{\lambda}{\mu}$, then $\lambda - 5\mu$ is _____.

Sol. (6) Since, the required ratio is $3 : 4$.

$$\Rightarrow \frac{3}{4} = -\frac{\left(2 + 6 - \frac{\lambda}{\mu} \right)}{\left(-1 + 4 - \frac{\lambda}{\mu} \right)}$$

$$\Rightarrow 9 - 3\left(\frac{\lambda}{\mu} \right) = 4\left(\frac{\lambda}{\mu} \right) - 32$$

$$\Rightarrow \frac{\lambda}{\mu} = \frac{41}{7}$$

$$\therefore \lambda - 5\mu = 6$$

Ex 26. Number of values of b for which in an acute $\triangle ABC$, the coordinates of orthocentre H are $(4, b)$, centroid G are $(b, 2b - 8)$ and circumcentre S are $(-4, 8)$, is _____.

Sol. (0) As we know that centroid (G) divides the line segment joining orthocentre (H) and circumcentre (S) in the ratio $2 : 1$.

$$\begin{aligned} &\begin{array}{ccc} & 2 & 1 \\ H & G & S \\ (4, b) & (b, 2b - 8) & (-4, 8) \end{array} \\ \therefore \frac{-8 + 4}{3} = b &\Rightarrow b = \frac{-4}{3} \end{aligned}$$

$$\text{and } \frac{16 + b}{3} = 2b - 8 \Rightarrow b = 8$$

But no common value of b is possible.
Hence, the number of values of b is zero.

Target Exercises

Type 1. Only One Correct Option

- The vertices of a triangle are $A(1, 1)$, $B(4, 5)$ and $C(6, 13)$. Then, the value of $\cos A$ is
 (a) $\frac{63}{65}$ (b) $\frac{60}{65}$
 (c) $\frac{61}{63}$ (d) None of these
- Let α, β be the roots of the equation $a_1x^2 + 2b_1x + c_1 = 0$ and γ, δ be the roots of $a_2x^2 + 2b_2x + c_2 = 0$. If the four points $A(\alpha, 0)$, $B(\beta, 0)$, $C(\gamma, 0)$ and $D(\delta, 0)$ are such that the sum of the ratios in which C and D divide AB is zero, then $a_1c_2 + a_2c_1$ is
 (a) $2b_1b_2$
 (b) b_1b_2
 (c) $3b_1b_2$
 (d) None of the above
- The three vertices of a parallelogram taken in order are $(-1, 0)$, $(3, 1)$ and $(2, 2)$, respectively. Then, the coordinates of the fourth vertex are
 (a) $(0, 0)$ (b) $(2, 1)$
 (c) $(1, 2)$ (d) $(-2, 1)$
- The area of a triangle with vertices $(a \cos \theta_1, b \sin \theta_1)$ and $(a \cos \theta_2, b \sin \theta_2)$ and $(a \cos \theta_3, b \sin \theta_3)$, (where, $\theta_1 \neq \theta_2 \neq \theta_3$) is equal to
 (a) $2ab \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_2 - \theta_3}{2} \right) \sin \left(\frac{\theta_3 - \theta_1}{2} \right)$
 (b) $ab \sin \left(\frac{\theta_1 - \theta_2}{2} \right) \sin \left(\frac{\theta_2 - \theta_3}{3} \right) \sin \left(\frac{\theta_3 - \theta_1}{1} \right)$
 (c) $4ab \sin \left(\frac{\theta_1 - \theta_2}{1} \right) \sin \left(\frac{\theta_2 - \theta_3}{2} \right) \sin \left(\frac{\theta_3 - \theta_1}{3} \right)$
 (d) None of the above
- Two vertices of a triangle are $(3, -5)$ and $(-7, 4)$. If its centroid is $(2, -1)$. Then, the third vertex is
 (a) $(10, 2)$ (b) $(10, -2)$
 (c) $(2, 2)$ (d) None of these
- If α, β and γ are the roots of the equation $x^3 - 3px^2 + 3qx - 1 = 0$, then the centroid of the triangle, whose vertices are $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$ and $C\left(\gamma, \frac{1}{\gamma}\right)$, is
 (a) $(p, -q)$ (b) (p, q) (c) $(-p, q)$ (d) $(-p, -q)$
- Let $A\left(\alpha, \frac{1}{\alpha}\right)$, $B\left(\beta, \frac{1}{\beta}\right)$ and $C\left(\gamma, \frac{1}{\gamma}\right)$ be the vertices of a $\triangle ABC$, where α, β are the roots of the equation $x^2 - 6p_1x + 2 = 0$ and β, γ are the roots of the equation $x^2 - 6p_2x + 3 = 0$ and γ, α are the roots of the equation $x^2 - 6p_3x + 6 = 0$, p_1, p_2, p_3 being positive. Then, the coordinates of the centroid of $\triangle ABC$ are
 (a) $\left(1, \frac{11}{18}\right)$ (b) $\left(0, \frac{11}{8}\right)$
 (c) $\left(2, \frac{11}{18}\right)$ (d) None of these
- If $\tan \alpha, \tan \beta$ and $\tan \gamma$ are the roots of the equation $x^3 - 3ax^2 + 3bx - 1 = 0$, then the centroid of the $\triangle ABC$, whose vertices are $A(\tan \alpha, \cot \alpha)$, $B(\tan \beta, \cot \beta)$ and $C(\tan \gamma, \cot \gamma)$, is
 (a) (a, b)
 (b) $(-a, b)$
 (c) $(a, -b)$
 (d) None of the above
- If the coordinates of the mid-points of the sides of a triangle are $(1, 1)$, $(2, -3)$ and $(3, 4)$. Then, its incentre is
 (a) $\left(\frac{2\sqrt{13} + 20\sqrt{2}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}}, \frac{2\sqrt{13} - 6\sqrt{17}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}}\right)$
 (b) $\left(\frac{2\sqrt{13} + 20\sqrt{2}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}}, \frac{2\sqrt{13} - 6\sqrt{17}}{\sqrt{13} - \sqrt{17} - 5\sqrt{2}}\right)$
 (c) $\left(\frac{2\sqrt{13} - 20\sqrt{2}}{\sqrt{13} - \sqrt{17} - 5\sqrt{2}}, \frac{2\sqrt{13} - 6\sqrt{17}}{\sqrt{13} - \sqrt{17} - 5\sqrt{2}}\right)$
 (d) None of the above
- If (α, β) , $(\bar{x}, \bar{y})$ and (u, v) are respectively coordinates of the circumcentre, centroid and orthocentre of a triangle. Then,
 (a) $3\bar{x} = 2\alpha + u$ and $3\bar{y} = 2\beta + v$
 (b) $3\bar{x} = 2\alpha - u$ and $3\bar{y} = 2\beta - v$
 (c) $3\bar{x} = 2\alpha - u$ and $3\bar{y} = 2\beta + v$
 (d) $3\bar{x} = 2\alpha + u$ and $3\bar{y} = 2\beta - v$
- If in a $\triangle ABC$ (whose circumcentre is origin), $a \leq \sin A$, then for any point (x, y) inside the circumcircle of $\triangle ABC$
 (a) $|xy| < \frac{1}{8}$ (b) $|xy| > \frac{1}{8}$
 (c) $\frac{1}{8} < xy < \frac{1}{2}$ (d) None of these



12. If the coordinates of a point $(4, 5)$ become $(-3, 9)$, then the point of origin be shifted at
 (a) $(7, -4)$ (b) $(7, 4)$
 (c) $(-7, 4)$ (d) None of these
13. If the axes are rotated through an angle of 45° in the clockwise direction, the coordinates of a point in the new system are $(0, -2)$, then its original coordinates are
 (a) $(\sqrt{2}, \sqrt{2})$ (b) $(-\sqrt{2}, \sqrt{2})$
 (c) $(\sqrt{2}, -\sqrt{2})$ (d) $(-\sqrt{2}, -\sqrt{2})$
14. Shift the origin to a suitable point so that the equation $y^2 + 4y + 8x - 2 = 0$ will not contain term in y and the constant term, then the origin is
 (a) $\left(\frac{3}{4}, 2\right)$ (b) $\left(\frac{3}{4}, -2\right)$ (c) $\left(-\frac{3}{4}, -2\right)$ (d) $\left(-\frac{3}{4}, 2\right)$
15. If (x, y) and (X, Y) are the coordinates of the same point referred to two sets of rectangular axes with the same origin and if $ax + by$ becomes $pX + qY$, where a, b are independent of x, y , then
 (a) $a^2 - b^2 = p^2 + q^2$ (b) $a^2 - b^2 = p^2 - q^2$
 (c) $a^2 + b^2 = p^2 - q^2$ (d) $a^2 + b^2 = p^2 + q^2$
16. The angle through which the coordinates axes be rotated, so that xy -term in the equation $5x^2 + 4\sqrt{3}xy + 9y^2 = 0$ may be missing, is
 (a) $-\frac{\pi}{6}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$
17. If by rotating the coordinate axes without translating the origin, the expression $a_1x^2 + 2h_1xy + b_1y^2$ becomes $a_2X^2 + 2h_2XY + b_2Y^2$, then
 (a) $a_1 - b_1 = a_2 - b_2$ (b) $a_1 + b_1 = a_2 + b_2$
 (c) $a_1 + b_2 = a_2 + b_1$ (d) $a_1 - b_1 = a_2 + b_2$
18. If by rotating the coordinate axes without translating the origin, the expression $a_1x^2 + 2h_1xy + b_1y^2$ becomes $a_2X^2 + 2h_2XY + b_2Y^2$, then
 (a) $(a_1 - b_1)^2 + 4h_1^2 = (a_2 - b_2)^2 + 4h_2^2$
 (b) $(a_1 - b_1)^2 - 4h_1^2 = (a_2 - b_2)^2 + 4h_2^2$
 (c) $(a_1 - b_1)^2 + 4h_1^2 = (a_2 - b_2)^2 - 4h_2^2$
 (d) None of the above
19. If by rotating the coordinate axes without translating the origin, the expression $a_1x^2 + 2h_1xy + b_1y^2$ becomes $a_2X^2 + 2h_2XY + b_2Y^2$, then
 (a) $a_1b_1 + h_1^2 = a_2b_2 + h_2^2$
 (b) $a_1b_1 + h_1^2 = a_2b_2 - h_2^2$
 (c) $a_1b_1 - h_1^2 = a_2b_2 - h_2^2$
 (d) None of the above
20. The axes be shifted without rotation, so that the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ does not contain terms in x, y and constant term, then the point is
 (a) $\left(-\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$ (b) $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$
 (c) $\left(-\frac{hf - bg}{ab - h^2}, -\frac{hg - af}{ab - h^2}\right)$ (d) $\left(\frac{hf - bg}{ab - h^2}, -\frac{hg - af}{ab - h^2}\right)$
21. The angle through which the axes may be turned, so that the equation $Ax + By + C = 0$ may be reduced to the form $x = \text{constant}$, is
 (a) $\tan^{-1}\left(\frac{B}{A}\right)$ (b) $\tan^{-1}\left(\frac{A}{B}\right)$
 (c) $\frac{1}{2} \tan^{-1}\left(\frac{A}{B}\right)$ (d) $\frac{1}{2} \tan^{-1}\left(\frac{B}{A}\right)$

Type 2. More than One Correct Option

22. All points lying inside the triangle formed by the points $(1, 3)$, $(5, 0)$ and $(-1, 2)$ satisfy
 (a) $3x + 2y \geq 0$ (b) $2x + y - 13 \geq 0$
 (c) $2x - 3y - 12 \leq 0$ (d) $-2x + y \geq 0$
23. If $(-6, -4)$, $(3, 5)$ and $(-2, 1)$ are the vertices of a parallelogram, then remaining vertex can be
 (a) $(0, -1)$ (b) $(7, 9)$
 (c) $(-1, 0)$ (d) $(-11, -8)$
24. If $(-4, 0)$ and $(1, -1)$ are two vertices of a triangle of area 4 sq units, then its third vertex lies on
 (a) $y = x$ (b) $5x + y + 12 = 0$
 (c) $x + 5y - 4 = 0$ (d) $x + 5y + 12 = 0$

Type 3. Assertion and Reason

Directions (Q. Nos. 25-29) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

25. **Statement I** If the point $(2a - 5, a^2)$ is on the same side of the line $x + y - 3 = 0$ as that of the origin, then $a \in (2, 4)$.

Statement II The points (x_1, y_1) and (x_2, y_2) lie on the same or opposite sides of the line $ax + by + c = 0$, as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same or opposite signs.



26. Statement I The point (α, α^2) lies inside the triangle formed by the lines $2x + 3y - 1 = 0$, $x + 2y - 3 = 0$ and $5x - 6y - 1 = 0$ for every $\alpha \in \left(-\frac{3}{2}, -1\right) \cup \left(\frac{1}{2}, 1\right)$.

Statement II Two points (x_1, y_1) and (x_2, y_2) lie on the same side of straight line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of opposite signs.

27. Statement I Area of the triangle formed by $4x + y + 1 = 0$ with the coordinate axes is $1/8$ sq unit.

Statement II Area of the triangle made by the line $ax + by + c = 0$ with the coordinate axes is $\frac{c^2}{2|ab|}$.

28. Statement I The incentre of a triangle formed by the lines $x \cos \frac{\pi}{9} + y \sin \frac{\pi}{9} = \pi$, $x \cos \frac{8\pi}{9} + y \sin \frac{8\pi}{9} = \pi$ and $x \cos \frac{13\pi}{9} + y \sin \frac{13\pi}{9} = \pi$ is $(0, 0)$.

Statement II The point $(0, 0)$ is equidistant from the lines $x \cos \frac{\pi}{9} + y \sin \frac{\pi}{9} = \pi$, $x \cos \frac{8\pi}{9} + y \sin \frac{8\pi}{9} = \pi$ and $x \cos \frac{13\pi}{9} + y \sin \frac{13\pi}{9} = \pi$.

29. Statement I The area of triangle formed by the points $A(1000, 1002)$, $B(1001, 1004)$ and $C(1002, 1003)$ is same as the area formed by $A'(0, 0)$, $B'(1, 2)$ and $C'(2, 1)$.

Statement II The area of the triangle is constant with respect to translation of axes.

Type 4. Linked Comprehension Based Questions

Passage (Q. Nos. 30-32) Let $ABCD$ be a square with sides of unit length. Points E and F are taken on sides AB and AD respectively, so that $AE = AF$. Let P be a point inside the square $ABCD$.

30. The maximum possible area of quadrilateral $CDFE$ is

- (a) $1/8$
(b) $1/4$
(c) $5/8$
(d) $3/8$

31. The value of $(PA)^2 - (PB)^2 + (PC)^2 - (PD)^2$ is equal to

- (a) 3 (b) 2
(c) 1 (d) 0

32. Let a line passing through point A divides the square $ABCD$ into two parts so that area of one portion is double the other, then the length of portion of line inside the square is

- (a) $\sqrt{10}/3$ (b) $\sqrt{13}/3$ (c) $\sqrt{11}/3$ (d) $2/\sqrt{3}$

Type 5. Match the Column

33. Consider the lines represented by equation $(x^2 + xy - x)(x - y) = 0$, forming a triangle. Then, match the following:

Column I	Column II
A. Orthocentre of triangle	p. $(1/6, 1/2)$
B. Circumcentre	q. $\{1/2 + 2\sqrt{2}, 1/2\}$
C. Centroid	r. $(0, 1/2)$
D. Incentre	s. $(1/2, 1/2)$

Type 6. Single Integer Answer Type Questions

34. If the area of triangle formed by the points $(2a, b)$, $(a + b, 2b + a)$ and $(2b, 2a)$ is 2 sq units, then the area of the triangle whose vertices are $(a + b, a - b)$, $(3b - a, b + 3a)$ and $(3a - b, 3b - a)$ will be _____.

35. The equations of three sides of a triangle are $x = 2$, $y + 1 = 0$ and $x + 2y = 4$. The coordinates of the

circumcentre of the triangle are (λ, μ) , then $\lambda + \mu$ is _____.

36. The distance between the circumcentre and orthocentre of the triangle, whose vertices are $(0, 0)$, $(6, 8)$ and $(-4, 3)$, is L , then the value of $\frac{2}{\sqrt{5}}L$ is _____.

Entrances Gallery

JEE Main/AIEEE

- The number of points having both coordinates as integers that lie in the interior of the triangle with vertices $(0, 0)$, $(0, 41)$ and $(41, 0)$, is [2015]
 (a) 901 (b) 861
 (c) 820 (d) 780
- Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then [2014]
 (a) $2bc - 3ad = 0$ (b) $2bc + 3ad = 0$
 (c) $3bc - 2ad = 0$ (d) $3bc + 2ad = 0$
- The x -coordinate of the incentre of the triangle that has the coordinates of mid-points of its sides as $(0, 1)$, $(1, 1)$ and $(1, 0)$, is [2013]
 (a) $2 + \sqrt{2}$ (b) $2 - \sqrt{2}$
 (c) $1 + \sqrt{2}$ (d) $1 - \sqrt{2}$
- If the line $2x + y = k$ passes through the point which divides the line segment joining the points $(1, 1)$ and $(2, 4)$ in the ratio $3 : 2$, then k equals [2012]
 (a) $\frac{29}{5}$ (b) 5 (c) 6 (d) $\frac{11}{5}$
- If $A(2, -3)$ and $B(-2, 1)$ are two vertices of a triangle and third vertex moves on the line $2x + 3y = 9$, then the locus of the centroid of the triangle is [2011]
 (a) $2x - 3y = 1$ (b) $x - y = 1$
 (c) $2x + 3y = 1$ (d) $2x + 3y = 3$
- Three distinct points A, B and C are given in the 2-dimensional coordinate plane such that the ratio of the distance of anyone of them from the point $(1, 0)$ to the distance from the point $(-1, 0)$ is equal to $1/3$. Then, the circumcentre of the $\triangle ABC$ is at the point [2009]
 (a) $(0, 0)$ (b) $\left(\frac{5}{4}, 0\right)$ (c) $\left(\frac{5}{2}, 0\right)$ (d) $\left(\frac{5}{3}, 0\right)$
- Let $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ be the vertices of a right angled triangle with AC as its hypotenuse. If the area of the triangle is 1, then the set of values which k can take is given by [2007]
 (a) $\{1, 3\}$ (b) $\{0, 2\}$
 (c) $\{-1, 3\}$ (d) $\{-3, -2\}$
- If (a, a^2) falls inside the angle made by the lines $y = \frac{x}{2}$, $x > 0$ and $y = 3x$, $x > 0$, then a belongs to [2006]
 (a) $(3, \infty)$ (b) $\left(\frac{1}{2}, 3\right)$
 (c) $\left(-3, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{2}\right)$
- If a vertex of a triangle is $(1, 1)$ and the mid-points of two sides through this vertex are $(-1, 2)$ and $(3, 2)$, then the centroid of the triangle is [2005]
 (a) $\left(\frac{1}{3}, \frac{7}{3}\right)$ (b) $\left(1, \frac{7}{3}\right)$
 (c) $\left(-\frac{1}{3}, \frac{7}{3}\right)$ (d) $\left(-1, \frac{7}{3}\right)$
- Let $A(2, -3)$ and $B(-2, 1)$ be the vertices of a $\triangle ABC$. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line [2004]
 (a) $2x + 3y = 9$ (b) $2x - 3y = 7$
 (c) $3x + 2y = 5$ (d) $3x - 2y = 3$
- If the equation of the locus of a point equidistant from the points (a_1, b_1) and (a_2, b_2) is $(a_1 - a_2)x + (b_1 - b_2)y + c = 0$, then the value of c is [2003]
 (a) $\frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$ (b) $a_1^2 - a_2^2 + b_1^2 - b_2^2$
 (c) $\frac{1}{2}(a_1^2 + a_2^2 + b_1^2 + b_2^2)$ (d) $\sqrt{a_1^2 + b_1^2 - a_2^2 - b_2^2}$
- Locus of centroid of the triangle whose vertices are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$ and $(1, 0)$, where t is a parameter, is [2003]
 (a) $(3x - 1)^2 + (3y)^2 = a^2 - b^2$
 (b) $(3x - 1)^2 + (3y)^2 = a^2 + b^2$
 (c) $(3x + 1)^2 + (3y)^2 = a^2 + b^2$
 (d) $(3x + 1)^2 + (3y)^2 = a^2 - b^2$
- A triangle with vertices $(4, 0)$, $(-1, -1)$ and $(3, 5)$ is [2002]
 (a) isosceles and right angled
 (b) isosceles but not right angled
 (c) right angled but not isosceles
 (d) neither right angled nor isosceles
- The incentre of the triangle with vertices $(1, \sqrt{3})$, $(0, 0)$ and $(2, 0)$ is [2002]
 (a) $\left(1, \frac{\sqrt{3}}{2}\right)$ (b) $\left(\frac{2}{3}, \frac{1}{\sqrt{3}}\right)$
 (c) $\left(\frac{2}{3}, \frac{\sqrt{3}}{2}\right)$ (d) $\left(1, \frac{1}{\sqrt{3}}\right)$
- The locus of the mid-point of the distance between the axes of the variable line $x \cos \alpha + y \sin \alpha = p$, where p is constant, is [2002]
 (a) $x^2 + y^2 = 4p^2$ (b) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$
 (c) $x^2 + y^2 = \frac{4}{p^2}$ (d) $\frac{1}{x^2} + \frac{1}{y^2} = \frac{2}{p^2}$

Other Engineering Entrances



- 16.** The circumcentre of the triangle with vertices $(8, 6)$, $(8, -2)$ and $(2, -2)$ is at the point [Kerala CEE 2014]
 (a) $(2, -1)$ (b) $(1, -2)$ (c) $(5, 2)$
 (d) $(2, 5)$ (e) $(4, 5)$
- 17.** The triangle joining the points $P(2, 7)$, $Q(4, -1)$ and $R(-2, 6)$ is [UP SEE 2013]
 (a) isosceles triangle (b) equilateral triangle
 (c) right angled triangle (d) None of these
- 18.** ABC is a triangle with vertices $A(-1, 4)$, $B(6, -2)$ and $C(-2, 4)$. D , E and F are the points which divide each AB , BC and CA , respectively in the ratio $3:1$ internally. Then, the centroid of the $\triangle DEF$ is [Kerala CEE 2013]
 (a) $(3, 6)$ (b) $(1, 2)$ (c) $(4, 8)$
 (d) $(-3, 6)$ (e) None of these
- 19.** A point moves such that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always $2a$. Then, equation of its locus is [WB JEE 2013]
 (a) $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$ (b) $\frac{x^2}{a^2} - \frac{y^2}{a^2(1-e^2)} = 1$
 (c) $\frac{x^2}{a^2(1-e^2)} + \frac{y^2}{a^2} = 1$ (d) None of these
- 20.** The line joining $A(b \cos \alpha, b \sin \alpha)$ and $B(a \cos \beta, a \sin \beta)$, where $a \neq b$, is produced to the point $M(x, y)$, so that $AM:MB = b:a$. Then, $x \cos \left(\frac{\alpha+\beta}{2}\right) + y \sin \left(\frac{\alpha+\beta}{2}\right)$ is equal to [WB JEE 2012]
 (a) 0 (b) 1 (c) -1 (d) $a^2 + b^2$
- 21.** If the mid-points of the sides of a triangle are $(-2, 3)$, $(4, -3)$ and $(4, 5)$, then the centroid of the triangle is [Karnataka CET 2012]
 (a) $(5/3, 2)$ (b) $(5/6, 1)$ (c) $(2, 5/3)$ (d) $(1, 5/6)$
- 22.** The vertices of $\triangle ABC$ are $A(2, 2)$, $B(-4, -4)$ and $C(5, -8)$. Find the length of a median of a triangle, which is passing through the point C . [Guj CET 2011]
 (a) $\sqrt{65}$ (b) $\sqrt{117}$
 (c) $\sqrt{85}$ (d) $\sqrt{116}$
- 23.** The line $x + y = 4$ divides the line joining the points $(-1, 1)$ and $(5, 7)$ in the ratio [BITSAT 2011]
 (a) $2:1$ (b) $1:2$ internally
 (c) $1:2$ externally (d) None of these
- 24.** The points $A(1, 2)$, $B(2, 4)$ and $C(4, 8)$ form a/an [Karnataka CET 2011]
 (a) isosceles triangle
 (b) equilateral triangle
 (c) straight line
 (d) right angled triangle
- 25.** The vertices of a triangle are $A(0, 0)$, $B(0, 2)$ and $C(2, 0)$, then find the distance between its orthocentre and circumcentre. [Guj CET 2011]
 (a) 0 (b) $\sqrt{2}$
 (c) $\frac{1}{\sqrt{2}}$ (d) None of these
- 26.** Orthocentre of the triangle formed by the lines $x - y = 0$, $x + y = 0$ and $x = 3$ is [Guj CET 2011]
 (a) $(0, 0)$ (b) $(3, 0)$
 (c) $(0, 3)$ (d) Cannot be determined
- 27.** If $A = (-3, 4)$, $B = (-1, -2)$, $C = (5, 6)$ and $D = (x, -4)$ are vertices of a quadrilateral such that $\triangle ABD = 2\triangle ACD$. Then, x is equal to [GGSIPU 2011]
 (a) 6 (b) 9
 (c) 69 (d) 96
- 28.** The equation of the locus of the point of intersection of the straight lines $x \sin \theta + (1 - \cos \theta)y = a \sin \theta$ and $x \sin \theta - (1 + \cos \theta)y + a \sin \theta = 0$ is [WB JEE 2011]
 (a) $y = \pm ax$ (b) $x = \pm ay$
 (c) $y^2 = 4x$ (d) $x^2 + y^2 = a^2$
- 29.** A line segment of 8 units in length moves so, that its end points are always on the coordinate axes. Then, the equation of locus of its mid-point is [J&K CET 2011]
 (a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 16$
 (c) $x^2 + y^2 = 8$ (d) $|x| + |y| = 8$
- 30.** The distance between the points $(a \cos \alpha, a \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$ is [Kerala CEE 2011]
 (a) $a \cos \left(\frac{\alpha - \beta}{2}\right)$ (b) $2a \cos \left(\frac{\alpha - \beta}{2}\right)$
 (c) $a \sin \left(\frac{\alpha - \beta}{2}\right)$ (d) $2a \sin \left(\frac{\alpha - \beta}{2}\right)$
 (e) None of these
- 31.** The vertices of a rectangle $ABCD$ are $A(-1, 0)$, $B(2, 0)$, $C(a, b)$ and $D(-1, 4)$. Then, the length of the diagonal AC is [Kerala CEE 2011]
 (a) 2 (b) 3 (c) 4 (d) 5
 (e) 6
- 32.** The points $(0, 8/3)$, $(1, 3)$ and $(82, 30)$ are the vertices of [Karnataka CET 2011]
 (a) an equilateral triangle
 (b) an isosceles triangle
 (c) a right angled triangle
 (d) None of the above
- 33.** The locus of a point which moves so that its distance from X -axis is double of its distance from Y -axis, is [Karnataka CET 2011]
 (a) $x = 2y$ (b) $y = 2x$
 (c) $x = 5y + 1$ (d) $y = 2x + 3$



- 34.** The three distinct points $A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$ and $C(0, a)$ (where, a is a real number) are collinear, if [J&K CET 2011]
 (a) $t_1 t_2 = -1$ (b) $t_1 t_2 = 1$
 (c) $2t_1 t_2 = t_1 + t_2$ (d) $t_1 + t_2 = a$
- 35.** If the distance between $(2, 3)$ and $(-5, 2)$ is equal to the distance between $(x, 2)$ and $(1, 3)$, then the values of x are [BITSAT 2010]
 (a) $-6, 8$ (b) $6, 8$
 (c) $-8, 6$ (d) $-7, 7$
- 36.** If the three points $(0, 1)$, $(0, -1)$ and $(x, 0)$ are vertices of an equilateral triangle, then the values of x are [Kerala CEE 2010]
 (a) $\sqrt{3}, \sqrt{2}$ (b) $\sqrt{3}, -\sqrt{3}$
 (c) $-\sqrt{5}, \sqrt{3}$ (d) $\sqrt{2}, -\sqrt{2}$
 (e) $\sqrt{5}, -\sqrt{5}$
- 37.** If the three points $(3q, 0)$, $(0, 3p)$ and $(1, 1)$ are collinear, then which one is true? [WB JEE 2010]
 (a) $\frac{1}{p} + \frac{1}{q} = 0$ (b) $\frac{1}{p} + \frac{1}{q} = 1$
 (c) $\frac{1}{p} + \frac{1}{q} = 3$ (d) $\frac{1}{p} + \frac{3}{q} = 1$
- 38.** The area (in sq units) of the triangle formed by the points $(2, 2)$, $(5, 5)$ and $(6, 7)$ is [Kerala CEE 2010]
 (a) $9/2$ (b) 5 (c) 10 (d) $3/2$
 (e) 14
- 39.** Q, R and S are points on the line joining the points $P(a, x)$ and $T(b, y)$ such that $PQ = RS = ST$, then $\left(\frac{5a+3b}{8}, \frac{5x+3y}{8}\right)$ is the mid-point of the line segment [MP PET 2010]
 (a) RS (b) ST (c) PQ (d) QR

Answers

Work Book Exercise

1. (a) 2. (c) 3. (b) 4. (d) 5. (d) 6. (b) 7. (b) 8. (b) 9. (a) 10. (a)
 11. (c) 12. (a)

Target Exercises

1. (a) 2. (a) 3. (d) 4. (a) 5. (b) 6. (b) 7. (c) 8. (a) 9. (a) 10. (a)
 11. (a) 12. (a) 13. (d) 14. (b) 15. (d) 16. (a) 17. (b) 18. (a) 19. (c) 20. (d)
 21. (a) 22. (a,c) 23. (c,d) 24. (c,d) 25. (d) 26. (c) 27. (a) 28. (b) 29. (a) 30. (c)
 31. (d) 32. (b) 33. (*) 34. (8) 35. (4) 36. (5)

* $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow p$; $D \rightarrow q$

Entrances Gallery

1. (d) 2. (c) 3. (b) 4. (c) 5. (c) 6. (b) 7. (c) 8. (b) 9. (b) 10. (a)
 11. (a) 12. (b) 13. (a) 14. (d) 15. (b) 16. (c) 17. (c) 18. (b) 19. (a) 20. (a)
 21. (c) 22. (c) 23. (b) 24. (c) 25. (b) 26. (a) 27. (c) 28. (d) 29. (b) 30. (d)

Explanations

Target Exercises

1. We know that, $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$, where $a = BC$,
 $b = CA$ and $c = AB$ are the lengths of sides of the $\triangle ABC$.
 Now, $a = BC = \sqrt{(4-6)^2 + (5-13)^2} = \sqrt{68}$
 $b = CA = \sqrt{(6-1)^2 + (13-1)^2} = \sqrt{169} = 13$
 and $c = AB = \sqrt{(4-1)^2 + (5-1)^2} = 5$
 $\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = \frac{169 + 25 - 68}{2 \times 13 \times 5} = \frac{63}{65}$

2. Since, α, β are the roots of $a_1x^2 + 2b_1x + c_1 = 0$ and γ, δ are the roots of $a_2x^2 + 2b_2x + c_2 = 0$, therefore
 $\alpha + \beta = -\frac{2b_1}{a_1}, \alpha\beta = \frac{c_1}{a_1} \quad \dots(i)$

Suppose C divides AB in the ratio $\lambda : 1$ and D divides AB in the ratio $\mu : 1$. Then,

$$\gamma = \frac{\lambda\beta + \alpha}{\lambda + 1} \quad \text{and} \quad \delta = \frac{\mu\beta + \alpha}{\mu + 1}$$

$$\Rightarrow \lambda = \frac{\gamma - \alpha}{\beta - \gamma} \quad \text{and} \quad \mu = \frac{\delta - \alpha}{\beta - \delta} \quad \dots(ii)$$

$$\Rightarrow \lambda + \mu = \frac{\gamma - \alpha}{\beta - \gamma} + \frac{\delta - \alpha}{\beta - \delta}$$

$$\Rightarrow 0 = \frac{(\gamma - \alpha)(\beta - \delta) + (\delta - \alpha)(\beta - \gamma)}{(\beta - \gamma)(\beta - \delta)} \quad [\because \lambda + \mu = 0]$$

$$\Rightarrow (\gamma - \alpha)(\beta - \delta) + (\delta - \alpha)(\beta - \gamma) = 0$$

$$\Rightarrow \left(-\frac{2b_1}{a_1}\right) \times \left(-\frac{2b_2}{a_2}\right) - \frac{2c_1}{a_1} - \frac{2c_2}{a_2} = 0$$

$$\Rightarrow \frac{4b_1b_2}{a_1a_2} = \frac{2(c_1a_2 + c_2a_1)}{a_1a_2} \quad [\text{using Eqs. (i) and (ii)}]$$

$$\Rightarrow a_1c_2 + a_2c_1 = 2b_1b_2$$

3. Let $A(-1, 0), B(3, 1), C(2, 2)$ and $D(x, y)$ be the vertices of a parallelogram $ABCD$ taken in order. Since, the diagonals of a parallelogram bisect each other.

$\therefore$ Coordinates of the mid-point of AC
 = Coordinates of the mid-point of BD

$$\Rightarrow \left(\frac{-1+2}{2}, \frac{0+2}{2}\right) = \left(\frac{3+x}{2}, \frac{1+y}{2}\right)$$

$$\Rightarrow \left(\frac{1}{2}, 1\right) = \left(\frac{3+x}{2}, \frac{1+y}{2}\right)$$

$$\Rightarrow \frac{3+x}{2} = \frac{1}{2} \quad \text{and} \quad \frac{1+y}{2} = 1$$

$$\Rightarrow x = -2 \quad \text{and} \quad y = 1$$

Hence, the fourth vertex of the parallelogram is $(-2, 1)$.

4. Let Δ be the area of the triangle. Then,

$$\Delta = \frac{1}{2} \begin{vmatrix} a \cos \theta_1 & b \sin \theta_1 & 1 \\ a \cos \theta_2 & b \sin \theta_2 & 1 \\ a \cos \theta_3 & b \sin \theta_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2} ab \begin{vmatrix} \cos \theta_1 - \cos \theta_2 & \sin \theta_1 - \sin \theta_2 & 0 \\ \cos \theta_2 - \cos \theta_3 & \sin \theta_2 - \sin \theta_3 & 0 \\ \cos \theta_3 & \sin \theta_3 & 1 \end{vmatrix}$$

[applying $R_1 \rightarrow R_1 - R_2, R_2 \rightarrow R_2 - R_3$]

$$= \frac{1}{2} ab \begin{vmatrix} -2 \sin \left(\frac{\theta_1 + \theta_2}{2}\right) \sin \left(\frac{\theta_1 - \theta_2}{2}\right) & & \\ -2 \sin \left(\frac{\theta_2 + \theta_3}{2}\right) \sin \left(\frac{\theta_2 - \theta_3}{2}\right) & & \\ \cos \theta_3 & \sin \theta_3 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 2 \sin \left(\frac{\theta_1 - \theta_2}{2}\right) \cos \left(\frac{\theta_1 + \theta_2}{2}\right) & 0 & \\ 2 \sin \left(\frac{\theta_2 - \theta_3}{2}\right) \cos \left(\frac{\theta_2 + \theta_3}{2}\right) & 0 & \\ \sin \theta_3 & & 1 \end{vmatrix}$$

$$= 2ab \sin \left(\frac{\theta_1 - \theta_2}{2}\right) \sin \left(\frac{\theta_2 - \theta_3}{2}\right) \times \left[-\cos \left(\frac{\theta_2 + \theta_3}{2}\right) \sin \left(\frac{\theta_1 + \theta_2}{2}\right) + \cos \left(\frac{\theta_1 + \theta_2}{2}\right) \sin \left(\frac{\theta_2 + \theta_3}{2}\right)\right]$$

$$= 2ab \sin \left(\frac{\theta_1 - \theta_2}{2}\right) \sin \left(\frac{\theta_2 - \theta_3}{2}\right) \sin \left(\frac{\theta_2 + \theta_3}{2} - \frac{\theta_1 + \theta_2}{2}\right)$$

$$= 2ab \sin \left(\frac{\theta_1 - \theta_2}{2}\right) \sin \left(\frac{\theta_2 - \theta_3}{2}\right) \sin \left(\frac{\theta_3 - \theta_1}{2}\right)$$

5. Let the coordinates of the third vertex be (x, y) .

$$\text{Then, } \frac{x+3-7}{3} = 2$$

$$\text{and } \frac{y-5+4}{3} = -1$$

$$\Rightarrow x - 4 = 6 \quad \text{and} \quad y - 1 = -3$$

$$\Rightarrow x = 10 \quad \text{and} \quad y = -2$$

Thus, the coordinates of the third vertex are $(10, -2)$.

6. Since, α, β and γ are the roots of the equation

$$x^3 - 3px^2 + 3qx - 1 = 0$$

$$\text{Therefore, } \alpha + \beta + \gamma = 3p, \alpha\beta + \beta\gamma + \gamma\alpha = 3q \quad \dots(i)$$

$$\text{and } \alpha\beta\gamma = 1$$

Let $G(x, y)$ be the centroid of $\triangle ABC$. Then,

$$x = \frac{\alpha + \beta + \gamma}{3} \quad \text{and} \quad y = \frac{1}{3} \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right)$$

$$\Rightarrow x = \frac{\alpha + \beta + \gamma}{3} \quad \text{and} \quad y = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{3\alpha\beta\gamma}$$

$$\Rightarrow x = \frac{3p}{3} \quad \text{and} \quad y = \frac{3q}{3} \quad [\text{using Eq. (i)}]$$

$$\Rightarrow x = p \quad \text{and} \quad y = q$$

Hence, the coordinates of G are (p, q) .

7. It is given that α and β are the roots of the equation $x^2 - 6p_1x + 2 = 0$.

$$\therefore \alpha + \beta = 6p_1, \alpha\beta = 2 \quad \dots(ii)$$



β and γ are the roots of the equation $x^2 - 6\rho_2x + 3 = 0$.

$$\therefore \beta + \gamma = 6\rho_2, \beta\gamma = 3 \quad \dots (ii)$$

γ and α are the roots of the equation $x^2 - 6\rho_3x + 6 = 0$.

$$\therefore \gamma + \alpha = 6\rho_3, \gamma\alpha = 6 \quad \dots (iii)$$

From Eqs. (i), (ii) and (iii), we get

$$\alpha\beta\gamma = 6 \quad [\because \alpha, \beta, \gamma > 0]$$

$$\text{Now, } \alpha\beta = 2 \text{ and } \alpha\beta\gamma = 6 \Rightarrow \gamma = 3$$

$$\beta\gamma = 3 \text{ and } \alpha\beta\gamma = 6 \Rightarrow \alpha = 2$$

$$\gamma\alpha = 6 \text{ and } \alpha\beta\gamma = 6 \Rightarrow \beta = 1$$

$$\therefore \alpha + \beta = 6\rho_1 \Rightarrow 3 = 6\rho_1 \Rightarrow \rho_1 = \frac{1}{2}$$

$$\beta + \gamma = 6\rho_2 \Rightarrow 4 = 6\rho_2 \Rightarrow \rho_2 = \frac{2}{3}$$

$$\text{and } \gamma + \alpha = 6\rho_3 \Rightarrow 5 = 6\rho_3 \Rightarrow \rho_3 = \frac{5}{6}$$

The coordinates of the centroid of triangle are

$$\left[\frac{\alpha + \beta + \gamma}{3}, \frac{1}{3} \left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \right) \right]$$

$$= \left[\frac{6}{3}, \frac{1}{3} \left(\frac{1}{2} + 1 + \frac{1}{3} \right) \right] = \left(2, \frac{11}{18} \right)$$

8. Since, $\tan \alpha$, $\tan \beta$ and $\tan \gamma$ are the roots of the equation

$$x^3 - 3ax^2 + 3bx - 1 = 0$$

$$\text{Therefore, } \tan \alpha + \tan \beta + \tan \gamma = 3a \quad \dots (i)$$

$$\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha = 3b \quad \dots (ii)$$

$$\text{and } \tan \alpha \tan \beta \tan \gamma = 1 \quad \dots (iii)$$

Let $(\bar{x}, \bar{y})$ be the coordinates of the centroid G of $\triangle ABC$.

$$\text{Then, } \bar{x} = \frac{\tan \alpha + \tan \beta + \tan \gamma}{3} = a \quad [\text{using Eq. (i)}]$$

$$\bar{y} = \frac{\cot \alpha + \cot \beta + \cot \gamma}{3}$$

$$= \frac{\tan \alpha \tan \beta + \tan \beta \tan \gamma + \tan \gamma \tan \alpha}{3 \tan \alpha \tan \beta \tan \gamma}$$

$$= \frac{3b}{3} = b \quad [\text{using Eqs. (ii) and (iii)}]$$

Hence, the coordinates of the centroid G are (a, b) .

9. Let $P(1, 1)$, $Q(2, -3)$ and $R(3, 4)$ be the mid-points of sides AB , BC and CA , respectively of $\triangle ABC$. Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of $\triangle ABC$.

Then, P is the mid-point of AB .

$$\Rightarrow \frac{x_1 + x_2}{2} = 1$$

$$\text{and } \frac{y_1 + y_2}{2} = 1$$

$$\Rightarrow x_1 + x_2 = 2$$

$$\text{and } y_1 + y_2 = 2 \quad \dots (i)$$

Q is the mid-point of BC .

$$\Rightarrow \frac{x_2 + x_3}{2} = 2$$

$$\text{and } \frac{y_2 + y_3}{2} = -3$$

$$\Rightarrow x_2 + x_3 = 4 \text{ and } y_2 + y_3 = -6 \quad \dots (ii)$$

R is the mid-point of AC .

$$\Rightarrow \frac{x_1 + x_3}{2} = 3$$

$$\text{and } \frac{y_1 + y_3}{2} = 4$$

$$\Rightarrow x_1 + x_3 = 6 \text{ and } y_1 + y_3 = 8 \quad \dots (iii)$$

From Eqs. (i), (ii) and (iii), we get

$$x_1 + x_2 + x_2 + x_3 + x_1 + x_3 = 2 + 4 + 6$$

$$\text{and } y_1 + y_2 + y_2 + y_3 + y_1 + y_3 = 2 - 6 + 8$$

$$\Rightarrow x_1 + x_2 + x_3 = 6 \text{ and } y_1 + y_2 + y_3 = 2 \quad \dots (iv)$$

From Eqs. (i) and (iv), we get

$$x_3 + 2 = 6$$

$$\text{and } 2 + y_3 = 2$$

$$\therefore x_3 = 4$$

$$\text{and } y_3 = 0$$

So, the coordinates of C are $(4, 0)$.

From Eqs. (ii) and (iv), we get

$$x_1 + 4 = 6 \text{ and } y_1 - 6 = 2$$

$$\therefore x_1 = 2 \text{ and } y_1 = 8$$

So, the coordinates of A are $(2, 8)$.

From Eqs. (iii) and (iv), we get

$$x_2 + 6 = 6 \text{ and } y_2 + 8 = 2$$

$$\therefore x_2 = 0 \text{ and } y_2 = -6$$

So, the coordinates of B are $(0, -6)$.

$$\text{Now, } a = BC = \sqrt{(4-0)^2 + (0+6)^2} = \sqrt{52} = 2\sqrt{13}$$

$$b = AC = \sqrt{(4-2)^2 + (0-8)^2} = \sqrt{68} = 2\sqrt{17}$$

$$\text{and } c = AB = \sqrt{(2-0)^2 + (8+6)^2} = \sqrt{200} = 10\sqrt{2}$$

The coordinates of the incentre of the $\triangle ABC$ are

$$\left(\frac{ax_1 + bx_2 + cx_3}{a+b+c}, \frac{ay_1 + by_2 + cy_3}{a+b+c} \right)$$

$$\text{or } \left(\frac{2\sqrt{13} + 20\sqrt{2}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}}, \frac{2\sqrt{13} - 6\sqrt{17}}{\sqrt{13} + \sqrt{17} + 5\sqrt{2}} \right)$$

10. We know that the centroid of a triangle divides the segment joining the orthocentre and circumcentre internally in the ratio $2:1$. Therefore,

$$\bar{x} = \frac{2\alpha + u}{2+1}$$

$$\text{and } \bar{y} = \frac{2\beta + v}{2+1}$$

$$\Rightarrow 3\bar{x} = 2\alpha + u$$

$$\text{and } 3\bar{y} = 2\beta + v$$

11. Let $a \leq \sin A \Rightarrow \frac{a}{\sin A} \leq 1$

$$\Rightarrow 2R \leq 1 \Rightarrow R \leq \frac{1}{2}$$

So, for any point (x, y) inside the circumcircle,

$$x^2 + y^2 < \frac{1}{4}$$

$$\text{Using AM} \geq \text{GM}, \left(\frac{x^2 + y^2}{2} \geq |xy| \right) \Rightarrow |xy| < \frac{1}{8}$$

12. Let (h, k) be the point to which the origin shifted.

$$\text{Then, } x = 4, y = 5, X = -3, Y = 9$$

$$\therefore x = X + h \text{ and } y = Y + k$$

$$\Rightarrow 4 = -3 + h \text{ and } 5 = 9 + k$$

$$\Rightarrow h = 7 \text{ and } k = -4$$

Hence, the origin must be shifted to $(7, -4)$.

13. Here, $X = 0$, $Y = -2$ and $\theta = -45^\circ$

Let (x, y) be the original coordinates of the point. Then,

$$x = X \cos \theta - Y \sin \theta \text{ and } y = X \sin \theta + Y \cos \theta$$

$$\Rightarrow x = 2 \sin(-45^\circ) \text{ and } y = -2 \cos(-45^\circ)$$

$$\Rightarrow x = -\sqrt{2} \text{ and } y = -\sqrt{2}$$

Hence, the original coordinates are $(-\sqrt{2}, -\sqrt{2})$.



14. Let the origin be shifted to (h, k) . Then,
 $x = X + h$ and $y = Y + k$
 On substituting $x = X + h, y = Y + k$ in the equation
 $y^2 + 4y + 8x - 2 = 0$, we get

$$(Y + k)^2 + 4(Y + k) + 8(X + h) - 2 = 0$$

$$\Rightarrow Y^2 + (4 + 2k)Y + 8X + (k^2 + 4k + 8h - 2) = 0$$

For this equation to be free from the term containing Y and the constant term, we must have

$$4 + 2k = 0 \quad \text{and} \quad k^2 + 4k + 8h - 2 = 0$$

$$\Rightarrow k = -2 \quad \text{and} \quad h = \frac{3}{4}$$

Hence, the origin is shifted at the point $\left(\frac{3}{4}, -2\right)$.

15. Suppose one set of rectangular axes is obtained by rotating the coordinate axes of the other set by an angle θ in anti-clockwise sense. Let (x, y) be the coordinates of the point with respect to old axes and (X, Y) be the coordinates with respect to the new axes. Then,

$$x = X \cos \theta - Y \sin \theta, \quad y = X \sin \theta + Y \cos \theta$$

$$\therefore ax + by = a(X \cos \theta - Y \sin \theta) + b(X \sin \theta + Y \cos \theta)$$

$$= (a \cos \theta + b \sin \theta)X + (b \cos \theta - a \sin \theta)Y$$

It is given that $ax + by$ becomes $pX + qY$. Therefore,

$$p = a \cos \theta + b \sin \theta \quad \text{and} \quad q = b \cos \theta - a \sin \theta$$

$$\Rightarrow p^2 + q^2 = (a \cos \theta + b \sin \theta)^2 + (b \cos \theta - a \sin \theta)^2$$

$$\Rightarrow p^2 + q^2 = a^2 + b^2$$

16. In order to remove xy -term from the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, the axes are rotated through an angle θ given by

$$\tan 2\theta = \frac{2h}{a - b}$$

Here, $a = 5, b = 9$
 and $h = 2\sqrt{3}$
 $\therefore \tan 2\theta = \frac{2h}{a - b}$

$$\Rightarrow \tan 2\theta = \frac{4\sqrt{3}}{5 - 9} = -\sqrt{3} = \tan\left(-\frac{\pi}{3}\right)$$

$$\Rightarrow 2\theta = n\pi - \frac{\pi}{3}, n \in I$$

$$\Rightarrow \theta = \frac{n\pi}{2} - \frac{\pi}{6}, n \in I$$

17. Let the axes be rotated through an angle θ in anti-clockwise sense. Then,

$$x = X \cos \theta - Y \sin \theta \quad \text{and} \quad y = X \sin \theta + Y \cos \theta$$

$$\therefore a_1x^2 + 2h_1xy + b_1y^2 = a_1(X \cos \theta - Y \sin \theta)^2$$

$$+ 2h_1(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta)$$

$$+ b_1(X \sin \theta + Y \cos \theta)^2$$

$$= (a_1 \cos^2 \theta + b_1 \sin^2 \theta + h_1 \sin 2\theta)X^2$$

$$+ XY(2h_1 \cos \theta - a_1 \sin 2\theta + b_1 \sin 2\theta)$$

$$+ (a_1 \sin^2 \theta - h_1 \sin 2\theta + b_1 \cos^2 \theta)Y^2 \dots (i)$$

It is given that the expression $a_1x^2 + 2h_1xy + b_1y^2$ transforms to $a_2X^2 + 2h_2XY + b_2Y^2$... (ii)

by rotating the axes. Therefore, Eqs. (i) and (ii) are the same.

$$\therefore a_2 = a_1 \cos^2 \theta + b_1 \sin^2 \theta + h_1 \sin 2\theta \dots (iii)$$

$$2h_2 = 2h_1 \cos 2\theta - a_1 \sin 2\theta + b_1 \sin 2\theta \dots (iv)$$

$$\text{and} \quad b_2 = a_1 \sin^2 \theta - h_1 \sin 2\theta + b_1 \cos^2 \theta \dots (v)$$

On adding Eqs. (iii) and (v), we get

$$a_2 + b_2 = a_1 + b_1$$

18. We have, $a_2 - b_2 = a_1 \cos 2\theta - b_1 \cos 2\theta + 2h_1 \sin 2\theta$

$$\Rightarrow a_2 - b_2 = (a_1 - b_1) \cos 2\theta + 2h_1 \sin 2\theta$$

$$\Rightarrow (a_2 - b_2)^2 = (a_1 - b_1)^2 \cos^2 2\theta + 4h_1^2 \sin^2 2\theta$$

$$+ 4(a_1 - b_1)h_1 \sin 2\theta \cos 2\theta \dots (vi)$$

From Eq. (iv), we have

$$2h_2 = 2h_1 \cos 2\theta - (a_1 - b_1) \sin 2\theta$$

$$\Rightarrow 4h_2^2 = 4h_1^2 \cos^2 2\theta + (a_1 - b_1)^2 \sin^2 2\theta$$

$$- 4(a_1 - b_1)h_1 \sin 2\theta \cos 2\theta \dots (vii)$$

On adding Eqs. (vi) and (vii), we get

$$(a_2 - b_2)^2 + 4h_2^2$$

$$= (a_1 - b_1)^2 (\cos^2 2\theta + \sin^2 2\theta) + 4h_1^2 (\cos^2 2\theta + \sin^2 2\theta)$$

$$= (a_1 - b_1)^2 + 4h_1^2$$

Hence, $(a_2 - b_2)^2 + 4h_2^2 = (a_1 - b_1)^2 + 4h_1^2$

19. We have, $(a_1 - b_1)^2 + 4h_1^2 = (a_2 - b_2)^2 + 4h_2^2$

$$\text{and} \quad a_1 + b_1 = a_2 + b_2$$

$$\therefore (a_1 - b_1)^2 + 4h_1^2 - (a_1 + b_1)^2 = (a_2 - b_2)^2 + 4h_2^2$$

$$- (a_2 + b_2)^2$$

$$\Rightarrow -4a_1b_1 + 4h_1^2 = -4a_2b_2 + 4h_2^2$$

$$\Rightarrow h_1^2 - a_1b_1 = h_2^2 - a_2b_2$$

$$\Rightarrow a_1b_1 - h_1^2 = a_2b_2 - h_2^2$$

20. Let the origin be shifted to (x_1, y_1) . Then,

$$x = X + x_1 \quad \text{and} \quad y = Y + y_1$$

On substituting $x = X + x_1$ and $y = Y + y_1$ in
 $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2$$

$$+ 2g(X + x_1) + 2f(Y + y_1) + c = 0$$

$$\Rightarrow aX^2 + 2hXY + bY^2 + 2X(ax_1 + hy_1 + g)$$

$$+ 2Y(hx_1 + by_1 + f) + ax_1^2 + 2hx_1y_1$$

$$+ by_1^2 + 2gx_1 + 2fy_1 + c = 0$$

This equation will be free from the terms containing X, Y and constant term, if

$$ax_1 + hy_1 + g = 0 \dots (i)$$

$$hx_1 + by_1 + f = 0 \dots (ii)$$

$$\text{and} \quad ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0 \dots (iii)$$

$$\text{Now, } ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$\Rightarrow x_1(ax_1 + hy_1 + g) + y_1(hx_1 + by_1 + f) + (gx_1 + fy_1 + c) = 0$$

$$\Rightarrow x_1 \times 0 + y_1 \times 0 + gx_1 + fy_1 + c = 0$$

[using Eqs. (i) and (ii)]

$$\Rightarrow gx_1 + fy_1 + c = 0 \dots (iv)$$

On solving Eqs. (i) and (ii) by cross-multiplication, we get

$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{hg - af}{ab - h^2}$$

Hence, the origin must be shifted at $\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2}\right)$.

21. Let the axes be turned through an angle α . If the axes are turned through an angle α , then

$x = X \cos \alpha - Y \sin \alpha$ and $y = X \sin \alpha + Y \cos \alpha$
On substituting these values in $Ax + By + C = 0$, we get
 $A(X \cos \alpha - Y \sin \alpha) + B(X \sin \alpha + Y \cos \alpha) + C = 0$
 $\Rightarrow X(A \cos \alpha + B \sin \alpha) + Y(B \cos \alpha - A \sin \alpha) + C = 0$
...(i)

If this equation is of the form $X = \text{constant}$, then

$$B \cos \alpha - A \sin \alpha = 0 \quad [\text{coefficient of } Y = 0]$$

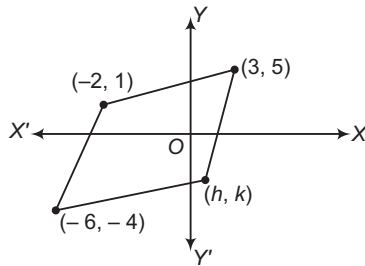
$$\Rightarrow \tan \alpha = \frac{B}{A} \Rightarrow \alpha = \tan^{-1}\left(\frac{B}{A}\right)$$

22. Substituting the coordinates of points $(1, 3)$, $(5, 0)$ and $(-1, 2)$ in $3x + 2y$, we obtain the values 9, 15 and 1 which are all positive. Therefore, all the points lying inside the triangle formed by given points satisfy $3x + 2y \geq 0$. Substituting the coordinates of the given points in $2x + 3y - 13$, we find the values -2 , -3 and -9 which are all negative. So, (b) is not correct.

Again, substituting the given points in $2x - 3y - 12$, we get -19 , -2 , -20 which are all negative. It follows that all points lying inside the Δ formed by given point satisfy $2x - 3y - 12 \leq 0$. So, (c) is the correct answer.

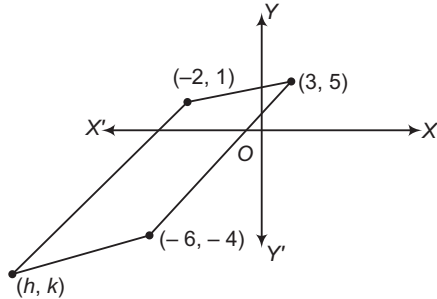
Finally, substituting the coordinates of the given point in $-2x + y$, we get 1 , -10 and 4 which are not all positive. So, (d) is not correct.

23. If the remaining vertex is (h, k) , then



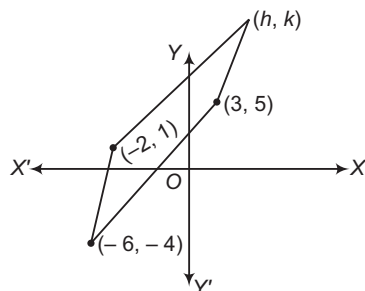
$$h - 2 = -6 + 3, k + 1 = 5 - 4$$

$$\Rightarrow h = -1, k = 0$$



$$h + 3 = -6 - 2, k + 5 = -4 + 1$$

$$\Rightarrow h = -11, k = -8$$



$$h - 6 = 3 - 2, k - 4 = 5 + 1$$

$$\Rightarrow h = 7, k = 10$$

24. Let the third vertex be (x, y) . Then,

$$\frac{1}{2} \begin{vmatrix} x & y & 1 \\ -4 & 0 & 1 \\ 1 & -1 & 1 \end{vmatrix} = \pm 4$$

$$\Rightarrow x + 5y + 4 = \pm 8$$

$$\Rightarrow x + 5y + 12 = 0$$

$$\text{or } x + 5y - 4 = 0$$

25. According to given data,

$$2a - 5 + a^2 - 3 < 0 \Rightarrow a^2 + 2a - 8 < 0 \Rightarrow (a - 2)(a + 4) < 0$$

$$\Rightarrow a \in (-4, 2)$$

26. A and P lie on same side of BC $\Rightarrow \alpha \in \left(-\frac{3}{2}, -1\right)$ on same side of CA

$$\alpha \in \left(-\infty, \frac{1}{3}\right) \cup \left(\frac{1}{2}, \infty\right) \text{ on same side of AB, if}$$

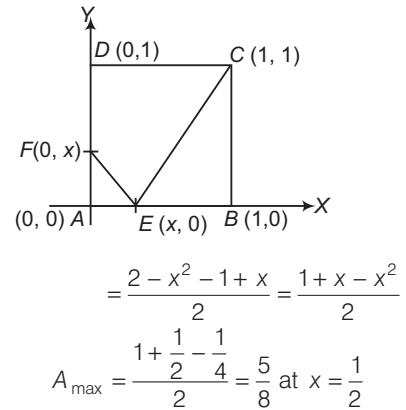
$$\alpha \in \left(-\infty, -1\right) \cup \left(\frac{1}{3}, \infty\right) \text{ taking intersection, we get result.}$$

27. Put in formula $\frac{c^2}{2|ab|} = \frac{1}{2|4 \times 1|} = \frac{1}{8}$ sq unit.

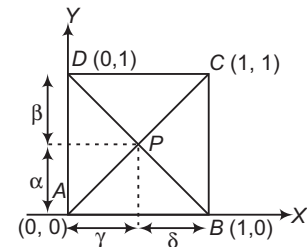
28. Since, the point $(0, 0)$ is equidistant from the lines, therefore the circumcentre is $(0, 0)$ and in equilateral triangle, circumcentre and incentre coincides.

29. Area of triangle is unaltered by shifting origin to any point. If origin is shifted to $(1000, 1002)$. A, B, C become $A'(0, 0)$, $B'(1, 2)$ and $C'(2, 1)$. Both are true.

30. Area of CDFE $A = 1 - \frac{1}{2}x^2 - \frac{1}{2}(1 - x)$



- 31.

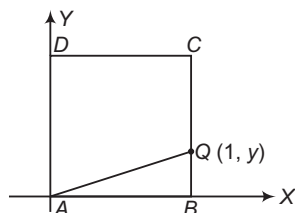


$$\Rightarrow (PA)^2 - (PB)^2 + (PC)^2 - (PD)^2$$

$$= (\alpha^2 + \gamma^2) - (\alpha^2 + \delta^2) + (\delta^2 + \beta^2) - (\gamma^2 + \beta^2) = 0$$



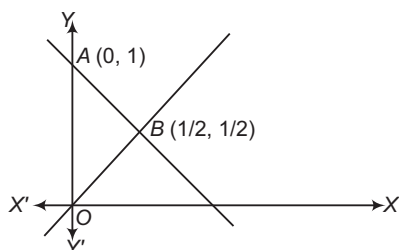
32.



$$\frac{1}{2}y(1) = \frac{1}{3}(1) \Rightarrow y = \frac{2}{3}$$

$$\therefore \text{Length of } AQ = \sqrt{(1)^2 + \left(\frac{2}{3}\right)^2} = \frac{\sqrt{13}}{3}$$

33.



The given lines are $x(x+y-1)(x-y)=0$. In other words, lines $x=0$, $x+y-1=0$ and $x-y=0$ form $\triangle OAB$ as shown in the above diagram.

The triangle is right angled at point B, hence orthocentre is $(1/2, 1/2)$. Also, circumcentre is mid-point of OA which is $(0, 1/2)$. The centroid is

$$\left(\frac{0 + \frac{1}{2} + 0}{3}, \frac{0 + \frac{1}{2} + 1}{3} \right) = \left(\frac{1}{6}, \frac{1}{2} \right)$$

Also, $OA = 1, OB = OC = 1/\sqrt{2}$. Hence, the incentre is

$$\left(\frac{0\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(1) + 0\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}}, \frac{0\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}(1) + 1\left(\frac{1}{\sqrt{2}}\right)}{\frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}}} \right)$$

$$= \left(\frac{1}{2 + 2\sqrt{2}}, \frac{1}{2} \right)$$

34. We know that the area of the triangle formed by joining the mid-points of any triangle is one-fourth of area of that triangle. Therefore, required area is 8.

35. One side of the triangle is parallel to the Y-axis and another side is parallel to the X-axis. So, the triangle is a right angled triangle. Hence, the mid-point of the hypotenuse is the circumcentre. Solving, $x=2$, $x+2y=4$, we get one end of the hypotenuse and solving $y+1=0$, $x+2y=4$, we get the other end. Their coordinates are $(2, 1)$ and $(6, -1)$.

$$\therefore \text{Circumcentre} = \left(\frac{2+6}{2}, \frac{1-1}{2} \right)$$

$$\Rightarrow \lambda + \mu = 4$$

36. Given, vertices of triangle are $O(0, 0)$, $B(6, 8)$ and $C(-4, 3)$.

$$\text{Clearly, } BC^2 = OB^2 + OC^2$$

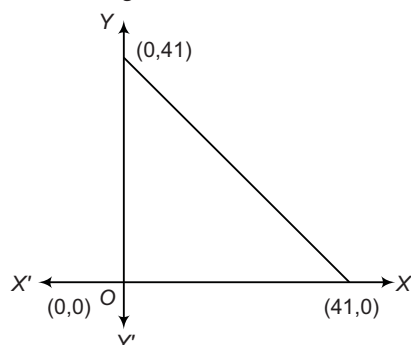
$\therefore \triangle OBC$ is right angled at O.

Now, circumcentre = mid-point of hypotenuse $BC = \left(1, \frac{11}{2} \right)$ and orthocentre = vertex $O(0, 0)$

$$\therefore \text{Required distance} = \sqrt{\left(1 + \frac{121}{4} \right)} = \frac{5\sqrt{5}}{2} \text{ units}$$

Entrances Gallery

1. Required points (x, y) are such that it satisfy $x + y < 41$ and $x > 0, y > 0$
Number of positive integral solution of the equation $x + y + k = 41$ will be number of integral coordinates in the bounded region.

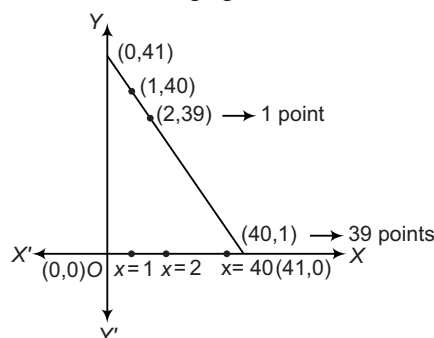


$$\therefore \text{Total number of integral coordinates} = {}^{41-1}C_{3-1} = {}^{40}C_2$$

$$= \frac{40!}{2!38!} = 780$$

Aliter

Consider the following figure:



Clearly, the number of required points

$$= 1 + 2 + 3 + \dots + 39 = \frac{39}{2}(39+1) = 780$$

2. Let point of intersection be $(h, -h)$.

$$\Rightarrow \begin{cases} 4ah - 2ah + c = 0 \\ 5bh - 2bh + d = 0 \end{cases}$$

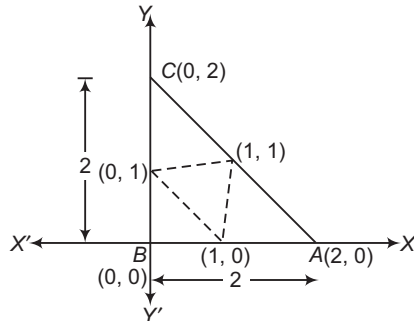
$$\text{So, } -\frac{c}{2a} = -\frac{d}{3b} \Rightarrow 3bc - 2ad = 0$$



3. Given, mid-points of sides of a triangle are $(0, 1)$, $(1, 1)$ and $(1, 0)$. Plotting these points on a graph paper and making a triangle.

So, the sides of a triangle will be 2, 2

$$\text{and } \sqrt{2^2 + 2^2} = 2\sqrt{2}$$



x-coordinate of incentre

$$\begin{aligned} &= \frac{2 \times 0 + 2 \cdot \sqrt{2} \cdot 0 + 2 \cdot 2}{2 + 2 + 2\sqrt{2}} \\ &= \frac{2}{2 + \sqrt{2}} \times \frac{2 - \sqrt{2}}{2 - \sqrt{2}} = 2 - \sqrt{2} \end{aligned}$$

4. Given line $L : 2x + y = k$ passes through point (say P) which divides a line segment (say AB) in the ratio 3 : 2, where $A(1, 1)$ and $B(2, 4)$.

Using section formula, the coordinates of the point P , which divides AB internally in the ratio 3 : 2, are

$$P\left(\frac{3 \times 2 + 2 \times 1}{3 + 2}, \frac{3 \times 4 + 2 \times 1}{3 + 2}\right) \equiv P\left(\frac{8}{5}, \frac{14}{5}\right)$$

Also, since the line L passes through P , hence substituting the coordinates of $P\left(\frac{8}{5}, \frac{14}{5}\right)$ in the

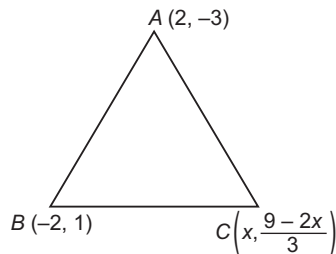
equation of $L : 2x + y = k$, we get

$$2\left(\frac{8}{5}\right) + \left(\frac{14}{5}\right) = k$$

$$\Rightarrow k = 6$$

5. The third vertex lies on $2x + 3y = 9$

$$\text{i.e. } \left(x, \frac{9 - 2x}{3}\right)$$



$\therefore$ Locus of centroid is

$$\left[\frac{2 - 2 + x}{3}, \frac{-3 + 1 + \frac{9 - 2x}{3}}{3}\right] = (h, k)$$

$$\therefore h = \frac{x}{3}, k = \frac{3 - 2x}{9}$$

$$\Rightarrow 9k = 3 - 2(3h)$$

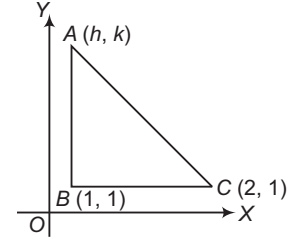
$$\begin{aligned} \Rightarrow 9k &= 3 - 6h \\ \Rightarrow 2h + 3k &= 1 \\ \text{or } 2x + 3y &= 1 \end{aligned}$$

6. According to the question, $\frac{\sqrt{(x-1)^2 + y^2}}{\sqrt{(x+1)^2 + y^2}} = \frac{1}{3}$

$$\Rightarrow x = \frac{1}{2}, 2$$

$$\therefore \text{Circumcentre} = \left(\frac{\frac{1}{2} + 2}{2}, 0\right) = \left(\frac{5}{4}, 0\right)$$

7. Since, $A(h, k)$, $B(1, 1)$ and $C(2, 1)$ are the vertices of a right angled $\triangle ABC$.



$$\begin{aligned} \text{Now, } AB &= \sqrt{(1-h)^2 + (1-k)^2} \\ BC &= \sqrt{(2-1)^2 + (1-1)^2} = 1 \\ \text{and } CA &= \sqrt{(h-2)^2 + (k-1)^2} \end{aligned}$$

From Pythagoras theorem,

$$\begin{aligned} AC^2 &= AB^2 + BC^2 \\ \Rightarrow 4 + h^2 - 4h + k^2 + 1 - 2k &= h^2 + 1 - 2h + k^2 + 1 - 2k + 1 \\ \Rightarrow 5 - 4h &= 3 - 2h \Rightarrow h = 1 \end{aligned} \quad \dots (i)$$

Now, given that area of triangle is 1.

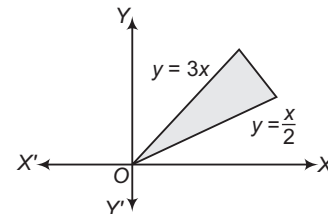
$$\begin{aligned} \text{Then, area of } \triangle ABC &= \frac{1}{2} \times AB \times BC \\ \Rightarrow 1 &= \frac{1}{2} \times \sqrt{(1-h)^2 + (1-k)^2} \times 1 \\ \Rightarrow 2 &= \sqrt{(1-h)^2 + (1-k)^2} \quad \dots (ii) \\ \Rightarrow 2 &= \sqrt{(k-1)^2} \quad [\text{from Eq. (i)}] \\ \Rightarrow 4 &= k^2 + 1 - 2k \\ \Rightarrow k^2 - 2k - 3 &= 0 \Rightarrow (k-3)(k+1) = 0 \end{aligned}$$

$$\therefore k = -1, 3$$

Thus, the set of values of k is $\{-1, 3\}$.

8. The graph of equations $x - 2y = 0$ and $3x - y = 0$ is shown in the figure. Since, given point (a, a^2) lies in the shaded region.

$$\text{Then, } a^2 - \frac{a}{2} > 0 \quad \text{and} \quad a^2 - 3a < 0$$



$$\Rightarrow a \in (-\infty, 0) \cup \left(\frac{1}{2}, \infty\right)$$

$$\text{and } a \in (0, 3) \Rightarrow a \in \left(\frac{1}{2}, 3\right)$$



9. Let D and E be the mid-points of AB and AC .

The coordinates of B are

$$-1 = \frac{1+x_2}{2} \quad \text{and} \quad 2 = \frac{1+y_2}{2}$$

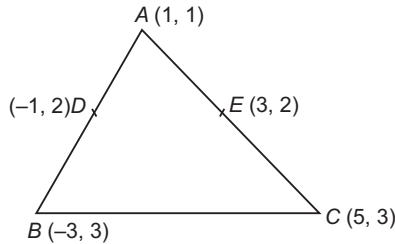
i.e. $x_2 = -3$ and $y_2 = 3$

and coordinates of C are

$$3 = \frac{1+x_3}{2}$$

and $2 = \frac{1+y_3}{2}$

i.e. $x_3 = 5$ and $y_3 = 3$



$\therefore$ Centroid of triangle

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right)$$

$$= \left(\frac{1 - 3 + 5}{3}, \frac{1 + 3 + 3}{3} \right) = \left(1, \frac{7}{3} \right)$$

10. Let (x, y) be the coordinates of vertex C and (x_1, y_1) be the coordinates of centroid of the triangle.

$\therefore x_1 = \frac{x+2-2}{3}$ and $y_1 = \frac{y-3+1}{3}$

$\Rightarrow x_1 = \frac{x}{3}$ and $y_1 = \frac{y-2}{3}$... (i)

Since, the centroid lies on the line $2x + 3y = 1$.

So, point (x_1, y_1) satisfies the equation of line.

$\therefore 2x_1 + 3y_1 = 1$

$\Rightarrow \frac{2x}{3} + \frac{3(y-2)}{3} = 1$ [from Eq. (i)]

$\Rightarrow 2x + 3y - 6 = 3$

$\Rightarrow 2x + 3y = 9$

Hence, this is the required equation of locus of the vertex C .

11. Let $P(\alpha, \beta)$ be the point which is equidistant to $A(a_1, b_1)$ and $B(a_2, b_2)$.

$\therefore PA = PB$

$\Rightarrow (\alpha - a_1)^2 + (\beta - b_1)^2 = (\alpha - a_2)^2 + (\beta - b_2)^2$

$\Rightarrow \alpha^2 + a_1^2 - 2\alpha a_1 + \beta^2 + b_1^2 - 2\beta b_1$

$= \alpha^2 + a_2^2 - 2\alpha a_2 + \beta^2 + b_2^2 - 2\beta b_2$

$\Rightarrow 2(a_2 - a_1)\alpha + 2(b_2 - b_1)\beta + (a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$

Thus, the equation of locus (α, β) is

$$(a_2 - a_1)x + (b_2 - b_1)y + \frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2) = 0$$

But the given equation is

$$(a_2 - a_1)x + (b_2 - b_1)y - c = 0$$

$\therefore c = -\frac{1}{2}(a_1^2 + b_1^2 - a_2^2 - b_2^2)$

$$= \frac{1}{2}(a_2^2 + b_2^2 - a_1^2 - b_1^2)$$

12. Since, the vertices of the triangle are $(a \cos t, a \sin t)$, $(b \sin t, -b \cos t)$ and $(1, 0)$.

Let the coordinate of centroid be

$$x = \frac{a \cos t + b \sin t + 1}{3}$$

$\Rightarrow 3x - 1 = a \cos t + b \sin t$... (i)

and $y = \frac{a \sin t - b \cos t + 0}{3}$

$\Rightarrow 3y = a \sin t - b \cos t$... (ii)

On squaring both sides and adding Eqs. (i) and (ii), we get

$$(3x - 1)^2 + (3y)^2 = a^2(\cos^2 t + \sin^2 t) + b^2(\sin^2 t + \cos^2 t)$$

$\Rightarrow (3x - 1)^2 + (3y)^2 = a^2 + b^2$

13. Let the vertices of $\triangle ABC$ be $A(4, 0)$, $B(-1, -1)$ and $C(3, 5)$.

Now, $AB = \sqrt{(-1-4)^2 + (-1-0)^2}$

$$= \sqrt{25+1} = \sqrt{26}$$

$$BC = \sqrt{(3+1)^2 + (5+1)^2} = \sqrt{4^2 + 6^2}$$

$$= \sqrt{16+36} = \sqrt{52}$$

and $CA = \sqrt{(4-3)^2 + (0-5)^2} = \sqrt{1+25} = \sqrt{26}$

$\therefore AC^2 + AB^2 = (\sqrt{26})^2 + (\sqrt{26})^2$

$$= 26 + 26 = 52 = BC^2$$

Hence, $CA^2 + AB^2 = BC^2$

Thus, the triangle is isosceles and right angled triangle.

14. Let the vertices of $\triangle ABC$ be $A(1, \sqrt{3})$, $B(0, 0)$ and $C(2, 0)$.

Again, let

$$a = BC = \sqrt{(2-0)^2 + (0-0)^2} = 2$$

$$b = AC = \sqrt{(2-1)^2 + (0-\sqrt{3})^2} = 2$$

and $c = AB = \sqrt{(0-1)^2 + (0-\sqrt{3})^2} = 2$

Since, all sides of a triangle are equal, therefore the triangle is an equilateral triangle.

Also, incentre is same as centroid of the triangle.

Hence, coordinates of incentre are

$$\left(\frac{1+0+2}{3}, \frac{\sqrt{3}+0+0}{3} \right) \text{ i.e. } \left(1, \frac{1}{\sqrt{3}} \right)$$

15. The straight line $x \cos \alpha + y \sin \alpha = p$ meets the X -axis

at the point $A\left(\frac{p}{\cos \alpha}, 0\right)$ and the Y -axis at the point

$B\left(0, \frac{p}{\sin \alpha}\right)$. Let (h, k) be the coordinates of the

mid-point of the line segment AB .

Then, $h = \frac{p}{2 \cos \alpha}$

and $k = \frac{p}{2 \sin \alpha}$

$\Rightarrow \cos \alpha = \frac{p}{2h}$

and $\sin \alpha = \frac{p}{2k}$

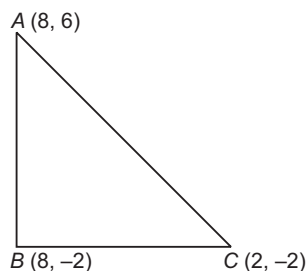
$\Rightarrow \sin^2 \alpha + \cos^2 \alpha = \frac{p^2}{4h^2} + \frac{p^2}{4k^2} = 1$

Hence, locus of the point (h, k) is

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{4}{p^2}$$



16. Let the vertices of a triangle be $A(8, 6)$, $B(8, -2)$ and $C(2, -2)$.



$$\begin{aligned} \text{Now, } AB &= \sqrt{(8-8)^2 + (6+2)^2} \\ &= \sqrt{0+8^2} = 8 \\ BC &= \sqrt{(2-8)^2 + (-2+2)^2} \\ &= \sqrt{36+0} = 6 \\ \text{and } CA &= \sqrt{(8-2)^2 + (6+2)^2} \\ &= \sqrt{6^2+8^2} \\ &= \sqrt{36+64} \\ &= \sqrt{100} = 10 \\ \text{Now, } AB^2 + BC^2 &= (8)^2 + (6)^2 \\ &= 64 + 36 \\ &= 100 = AC^2 \end{aligned}$$

So, ABC is a right angled triangle and right angled at B . We know that, in a right angled triangle, the circumcentre is the mid-point of hypotenuse.

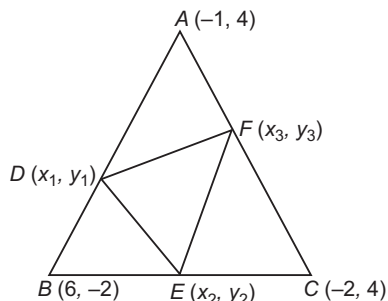
$$\therefore \text{Mid-point of } AC = \left(\frac{8+2}{2}, \frac{6-2}{2} \right) = (5, 2)$$

Hence, the required circumcentre is $(5, 2)$.

$$\begin{aligned} 17. \therefore PQ &= \sqrt{(4-2)^2 + (-1-7)^2} = 2\sqrt{17} \\ QR &= \sqrt{(-2-4)^2 + (6+1)^2} = \sqrt{85} \\ \text{and } RP &= \sqrt{(-2-2)^2 + (6-7)^2} = \sqrt{17} \\ \therefore PQ^2 + RP^2 &= QR^2 \end{aligned}$$

So, the triangle formed by joining the given points is right angled triangle, right angled at P .

18. Let (x_1, y_1) , (x_2, y_2) and (x_3, y_3) be coordinates of the points D, E and F which divide each AB, BC and CA , respectively in the ratio $3 : 1$ internally.



$$\begin{aligned} \therefore x_1 &= \frac{3 \times 6 - 1 \times 1}{4} = \frac{17}{4} \\ \text{and } y_1 &= \frac{-2 \times 3 + 4 \times 1}{4} \\ &= -\frac{2}{4} = -\frac{1}{2} \end{aligned}$$

$$\text{Similarly, } x_2 = 0, y_2 = \frac{5}{2}$$

$$\text{and } x_3 = -\frac{5}{4}, y_3 = 4$$

$$\text{For centroid, } x = \frac{x_1 + x_2 + x_3}{3} = \frac{\frac{17}{4} - \frac{5}{4} + 0}{3} = \frac{12}{12} = 1$$

$$\text{and } y = \frac{y_1 + y_2 + y_3}{3} = \frac{-\frac{1}{2} + \frac{5}{2} + 4}{3} = \frac{6}{3} = 2$$

Hence, $(x, y) = (1, 2)$

19. Let $A(ae, 0)$ and $B(-ae, 0)$ be two given points and (h, k) be the coordinates of the moving point P .

$$\text{Now, } PA + PB = 2a$$

$$\Rightarrow \sqrt{(h-ae)^2 + k^2} + \sqrt{(h+ae)^2 + k^2} = 2a \quad \dots(i)$$

But we know that,

$$[(h-ae)^2 + k^2] - [(h+ae)^2 + k^2] = -4aeh \quad \dots(ii)$$

On dividing Eq. (ii) by Eq. (i), we get

$$\sqrt{[(h-ae)^2 + k^2]} - \sqrt{[(h+ae)^2 + k^2]} = -2eh \quad \dots(iii)$$

On adding Eqs. (i) and (iii), we get

$$2\sqrt{[(h-ae)^2 + k^2]} = 2(a-eh)$$

On squaring both sides, we get

$$(h-ae)^2 + k^2 = (a-eh)^2$$

$$\Rightarrow \frac{h^2}{a^2} + \frac{k^2}{a^2(1-e^2)} = 1$$

Hence, locus of P is $\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1$.

$$20. \therefore \frac{AM}{MB} = \frac{b}{a}$$

$$\frac{(b \cos \alpha, b \sin \alpha) - (a \cos \beta, a \sin \beta)}{(a \cos \beta, a \sin \beta) - (x, y)} = \frac{b}{a}$$

So, using external division,

$$x = \frac{ba \cos \beta - ab \cos \alpha}{b-a}$$

$$= \frac{ab}{b-a} (\cos \beta - \cos \alpha)$$

$$\text{and } y = \frac{ba \sin \beta - ab \sin \alpha}{b-a} = \frac{ab}{b-a} (\sin \beta - \sin \alpha)$$

$$\therefore x \cos \left(\frac{\alpha + \beta}{2} \right) = \frac{ab}{b-a} \cos \left(\frac{\alpha + \beta}{2} \right) (\cos \beta - \cos \alpha)$$

$$\text{and } y \sin \left(\frac{\alpha + \beta}{2} \right) = \frac{ab}{b-a} \sin \left(\frac{\alpha + \beta}{2} \right) (\sin \beta - \sin \alpha)$$

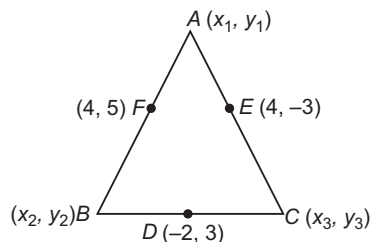
$$\therefore x \cos \left(\frac{\alpha + \beta}{2} \right) + y \sin \left(\frac{\alpha + \beta}{2} \right) = \frac{ab}{b-a}$$

$$\left[\cos \left(\frac{\alpha + \beta}{2} \right) \cdot \left\{ -2 \sin \left(\frac{\beta + \alpha}{2} \right) \sin \left(\frac{\beta - \alpha}{2} \right) \right\} \right. \\ \left. + \sin \left(\frac{\alpha + \beta}{2} \right) \cdot \left\{ 2 \cos \left(\frac{\beta + \alpha}{2} \right) \sin \left(\frac{\beta - \alpha}{2} \right) \right\} \right]$$

$$= \frac{ab}{b-a} \left[-\sin(\alpha + \beta) \sin \left(\frac{\beta - \alpha}{2} \right) \right. \\ \left. + \sin(\alpha + \beta) \sin \left(\frac{\beta - \alpha}{2} \right) \right] = 0$$



21. Let the vertices of the triangle are $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$, then



$$\begin{aligned}x_1 + x_2 &= 8 & \dots(i) \\y_1 + y_2 &= 10 & \dots(ii) \\x_2 + x_3 &= -4 & \dots(iii) \\y_2 + y_3 &= 6 & \dots(iv) \\x_3 + x_1 &= 8 & \dots(v) \\y_3 + y_1 &= -6 & \dots(vi)\end{aligned}$$

and

On solving these equations, we get

$$x_1 = 10, x_2 = -2, x_3 = -2, y_1 = -1, y_2 = 11 \text{ and } y_3 = -5$$

Hence, the centroid is $\left(2, \frac{5}{3}\right)$.

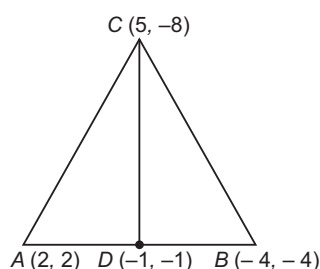
Aliter

As we know that the centroid of the $\triangle ABC$ and that of the triangle formed by joining the mid-points of the sides of $\triangle ABC$ is same.

Therefore, required centroid

$$= \left(\frac{4+4-2}{3}, \frac{5-3+3}{3}\right) = \left(2, \frac{5}{3}\right)$$

22. Let D be the mid-point of AB , then coordinates of D are $(-1, -1)$.



$$\begin{aligned}\therefore CD &= \sqrt{(-1-5)^2 + (-1+8)^2} \\&= \sqrt{36 + 49} = \sqrt{85}\end{aligned}$$

23. Let the line divides the line joining the points $(-1, 1)$ and $(5, 7)$ in the ratio $k : 1$ and at (x_1, y_1) .

$$\therefore x_1 = \frac{5k-1}{k+1}$$

$$\text{and } y_1 = \frac{7k+1}{k+1}$$

Since, (x_1, y_1) satisfies the equation $x + y = 4$.

$$\therefore \frac{5k-1}{k+1} + \frac{7k+1}{k+1} = 4$$

$$\Rightarrow 5k - 1 + 7k + 1 = 4k + 4$$

$$\Rightarrow 8k = 4$$

$$\Rightarrow k = 1/2$$

$\therefore$ Required ratio $= 1 : 2$, internally

$$24. AB = \sqrt{(2-1)^2 + (4-2)^2} = \sqrt{5}$$

$$BC = \sqrt{(4-2)^2 + (8-4)^2} = \sqrt{20} = 2\sqrt{5}$$

and

$$CA = \sqrt{(4-1)^2 + (8-2)^2} = \sqrt{45} = 3\sqrt{5}$$

Now,

$$AB + BC = CA \quad [\because \sqrt{5} + 2\sqrt{5} = 3\sqrt{5}]$$

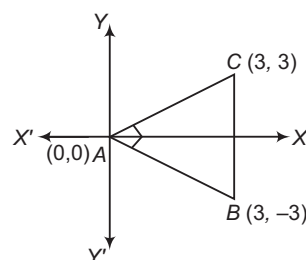
Hence, A, B and C are forming a straight line.

25. Given, vertices of $\triangle ABC$ are the vertices of right angled triangle, right angled at A .

In a right angled triangle, A is orthocentre and mid-point of BC is $D(1, 1)$ which is the circumcentre.

$$\therefore \text{Required distance} = AD = \sqrt{(1-0)^2 + (1-0)^2} = \sqrt{2}$$

26. From the intersection of given lines, we get the vertices of $\triangle ABC$ as $A(0, 0)$, $B(3, -3)$ and $C(3, 3)$.



Now,

$$AB = \sqrt{9+9} = \sqrt{18},$$

$$BC = \sqrt{36} = 6$$

and

$$CA = \sqrt{9+9} = \sqrt{18}$$

Now,

$$CA^2 + AB^2 = BC^2$$

$\therefore \triangle ABC$ is right angled at A .

Hence, orthocentre is $(0, 0)$.

$$27. \text{Area of } \triangle ACD = \frac{1}{2} \begin{vmatrix} -3 & 4 & 1 \\ 5 & 6 & 1 \\ x & -4 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [-3(6+4) - 4(5-x) + 1(-20-6x)]$$

$$= \frac{1}{2} [-30 - 20 + 4x - 20 - 6x]$$

$$= \frac{1}{2} [-2x - 70] = -x - 35$$

Similarly, area of $\triangle ABD = |3x + 1|$ (given)

$$\therefore \frac{\text{Area of } \triangle ABD}{\text{Area of } \triangle ACD} = \frac{2}{1}$$

$$\Rightarrow \frac{3x+1}{-x-35} = \pm \frac{2}{1}$$

$$\Rightarrow x = 69 \text{ or } x = -\frac{71}{5}$$

28. The equations of the locus of the point of intersection of given straight lines are

$$x \sin \theta + (1 - \cos \theta) y - a \sin \theta = 0 \quad \dots(i)$$

$$x \sin \theta - (1 + \cos \theta) y + a \sin \theta = 0 \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$y(1 - \cos \theta + 1 + \cos \theta) = 2a \sin \theta$$

$$\Rightarrow 2y = 2a \sin \theta$$

$$\Rightarrow y = a \sin \theta$$

$$\therefore x \sin \theta + a \sin \theta (1 - \cos \theta - 1) = 0$$

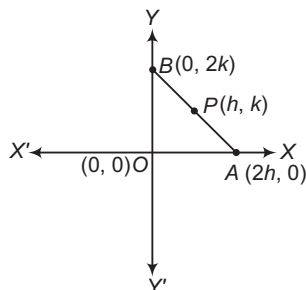
$$\Rightarrow x \sin \theta = a \sin \theta \cos \theta$$

$$\Rightarrow x = a \cos \theta$$



$$\begin{aligned}\therefore \sin \theta &= \frac{y}{a} \text{ and } \cos \theta = \frac{x}{a} \\ \Rightarrow \frac{y^2}{a^2} + \frac{x^2}{a^2} &= 1 \\ \Rightarrow x^2 + y^2 &= a^2\end{aligned}$$

29. Let $P(h, k)$ be the mid-point of AB , whose length is 8 units.



$$\begin{aligned}\therefore (2h - 0)^2 + (0 - 2k)^2 &= 8^2 \\ \Rightarrow 4h^2 + 4k^2 &= 64 \\ \Rightarrow h^2 + k^2 &= 16\end{aligned}$$

So, locus of the required point is $x^2 + y^2 = 16$.

30. Distance

$$\begin{aligned}&= \sqrt{a^2 (\cos \alpha - \cos \beta)^2 + a^2 (\sin \alpha - \sin \beta)^2} \\ &= a \sqrt{[\sin^2 \alpha + \cos^2 \alpha + \cos^2 \beta + \sin^2 \beta - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta]} \\ &= a \sqrt{2\{1 - \cos(\alpha - \beta)\}} = 2a \sin \left(\frac{\alpha - \beta}{2} \right)\end{aligned}$$

Aliter

Put $a = 1$, $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{6}$, then the points will be $(0, 1)$ and $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. Obviously, the distance between these two points is 1 which is given by option (d).

$$\left[\because 2a \sin \frac{\alpha - \beta}{2} = 2 \times 1 \times \sin \frac{(\pi/2) - (\pi/6)}{2} = 2 \times \frac{1}{2} = 1 \right]$$

31. In a rectangle $ABCD$, the diagonals are equal. So, $AC = BD$

$$\begin{aligned}\text{Now, length of } BD &= \sqrt{3^2 + 4^2} = 5 \\ \Rightarrow AC &= 5\end{aligned}$$

32. Since, the area of triangle formed by these points is zero, therefore the points are collinear.

33. Clearly, $x = 2y$ satisfy the given condition.

34. The given points are collinear, if

$$\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ 0 & a & 1 \end{vmatrix} = 0 \Rightarrow a^2 \begin{vmatrix} t_1^2 & 2t_1 & 1 \\ t_2^2 & 2t_2 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0$$

$$\begin{aligned}\Rightarrow t_1^2 (2t_2 - 1) - 2t_1 (t_2^2) + 1(t_2^2) &= 0 \\ \Rightarrow 2t_1^2 t_2 - t_1^2 - 2t_1 t_2^2 + t_2^2 &= 0 \\ \Rightarrow 2t_1 t_2 (t_1 - t_2) + (t_2 - t_1)(t_2 + t_1) &= 0 \\ \Rightarrow (t_1 - t_2)(2t_1 t_2 - t_1 - t_2) &= 0 \\ \Rightarrow t_1 &= t_2 \\ \text{or } t_1 + t_2 &= 2t_1 t_2\end{aligned}$$

35. $\sqrt{(2+5)^2 + (3-2)^2} = \sqrt{(x-1)^2 + (2-3)^2}$

$$\begin{aligned}\Rightarrow 49 + 1 &= (x-1)^2 + 1 \\ \Rightarrow x-1 &= \pm 7 \\ \Rightarrow x &= -6, 8\end{aligned}$$

36. Since, $A(0, 1)$, $B(0, -1)$ and $C(x, 0)$ are the vertices of an equilateral $\triangle ABC$.

$$\begin{aligned}\therefore AB &= BC \\ \Rightarrow \sqrt{0^2 + 4} &= \sqrt{x^2 + 1} \\ \Rightarrow x^2 &= 3 \\ \Rightarrow x &= \pm \sqrt{3}\end{aligned}$$

37. $\begin{vmatrix} 3q & 0 & 1 \\ 0 & 3p & 1 \\ 1 & 1 & 1 \end{vmatrix} = 0 \Rightarrow 3q(3p-1) + 1(0-3p) = 0$

$$\Rightarrow 9pq = 3p + 3q \Rightarrow \frac{1}{p} + \frac{1}{q} = 3$$

38. Area of triangle = $\frac{1}{2} \begin{vmatrix} 2 & 2 & 1 \\ 5 & 5 & 1 \\ 6 & 7 & 1 \end{vmatrix}$

$$\begin{aligned}&= \frac{1}{2} [2(5-7) - 2(5-6) + 1(35-30)] \\ &= \frac{1}{2} [-4 + 2 + 5] = \frac{3}{2} \text{ sq units}\end{aligned}$$

39. Given, $PQ = RS = ST$

$$P(a, x) \quad Q \quad R \quad S \quad T(b, y)$$

$$\text{Mid-point of } PT \text{ is } R \left(\frac{a+b}{2}, \frac{x+y}{2} \right).$$

$$\therefore \text{Point } Q \text{ is mid-point of } PR \text{ i.e. } Q \left(\frac{3a+b}{4}, \frac{3x+y}{4} \right).$$

$$\text{Hence, mid-point of } QR \text{ is } \left(\frac{5a+3b}{8}, \frac{5x+3y}{8} \right).$$

12

Straight Line and Pair of Straight Lines

Straight Line

When we say that a first degree equation in x, y i.e. $ax + by + c = 0$ represents a line it means that all points (x, y) satisfying $ax + by + c = 0$ lie along a line. Thus, a line is also defined as the locus of a point satisfying the condition $ax + by + c = 0$, where a, b and c are constants.

It follows from the above discussion that $ax + by + c = 0$ is the general equation of a line.

It should be noted that, in general equation of a line, there are only two unknowns, because equation of every straight line can be put in the form $ax + by + 1 = 0$, where a and b are two unknowns.

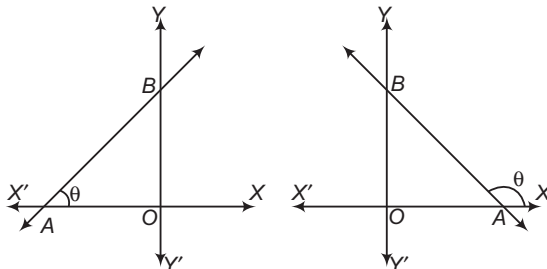
Note that x and y are not unknowns. Infact, these are the coordinates of any point on the line and are known as the current coordinates. Thus, to determine a line, we will need two conditions to determine the two unknowns. In the further discussion on straight line, you will find that whenever it will be asked to find a straight line, there will always be two conditions connecting the two unknowns.

Slope (Gradient) of a Line

The trigonometrical tangent of the angle that a line makes with the positive direction of the X -axis in anti-clockwise sense is called the slope or gradient of the line.

The slope of a line is generally denoted by m . Thus,

$$m = \tan \theta$$



Chapter Snapshot

- Straight Line
- Angle between Two Lines
- Point of Intersection of Two Lines
- Image of a Point with Respect to a Line
- Family of Lines through the Intersection of Two Given Lines
- Locus and its Equation
- Combined Equation of a Pair of Straight Lines
- Bisectors of the Angle between the Lines Given by a Homogeneous Equation
- General Equation of Second Degree
- Equations of the Angle Bisectors
- Distance between the Pair of Parallel Lines

Since, a line parallel to X -axis makes an angle of 0° with X -axis. Therefore, its slope is $\tan 0^\circ = 0$. A line parallel to Y -axis i.e. perpendicular to X -axis makes an angle of 90° with X -axis, so its slope is $\tan \frac{\pi}{2} = \infty$.

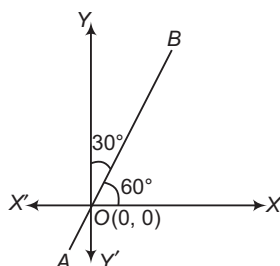
Also, the slope of a line equally inclined with axes is 1 or -1 as it makes an angle of 45° or 135° with X -axis.

✎ The angle of inclination of a line with the positive direction of X -axis in anti-clockwise sense always lies between 0° and 180° .

✎ **Example 1.** Slope of a line making an angle of 30° with Y -axis and lie in 1st quadrant and passing through origin, is

- (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\sqrt{3}$

Sol. (d) Since, line AB makes an angle of 30° with Y -axis. Hence, the inclination of line with X -axis is 60° .



$\therefore$ Slope $= m = \tan 60^\circ = \sqrt{3}$

Slope of a Line in Terms of Coordinates of any Two Points on it

Let (x_1, y_1) and (x_2, y_2) be coordinates of any two points on a line, then its slope m is given by

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\text{Difference of ordinates}}{\text{Difference of abscissae}}$$

✎ **Example 2.** The value of x , so that 2 is the slope of the line through $(2, 5)$ and $(x, 3)$, is

- (a) 1 (b) 2 (c) -1 (d) -2

Sol. (a) Slope of the line through $(2, 5)$ and $(x, 3)$,

$$m = \frac{3-5}{x-2} \Rightarrow 2 = \frac{3-5}{x-2}$$

$\Rightarrow x = 1$

Angle between Two Lines

The angle θ between the lines having slopes m_1 and m_2 is given by

$$\tan \theta = \pm \left(\frac{m_2 - m_1}{1 + m_1 m_2} \right)$$

and the acute angle between the lines is given by

$$\tan \theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right|$$

✎ **Example 3.** Find the angle between the lines joining the points $(0, 0)$, $(2, 3)$ and the points $(2, -2)$, $(3, 5)$.

Sol. Let θ be the angle between the given lines.

$$\begin{aligned} \text{Then, } m_1 &= \text{Slope of the line joining } (0, 0) \text{ and } (2, 3) \\ &= \frac{3-0}{2-0} = \frac{3}{2} \end{aligned}$$

$$\begin{aligned} \text{and } m_2 &= \text{Slope of the line joining } (2, -2) \text{ and } (3, 5) \\ &= \frac{5+2}{3-2} = 7 \end{aligned}$$

$$\begin{aligned} \therefore \tan \theta &= \pm \left(\frac{m_2 - m_1}{1 + m_1 m_2} \right) \\ &= \pm \left(\frac{7 - \frac{3}{2}}{1 + 7 \left(\frac{3}{2} \right)} \right) \\ &= \pm \frac{\frac{11}{2}}{\frac{23}{2}} = \pm \frac{11}{23} \end{aligned}$$

$$\Rightarrow \theta = \tan^{-1} \left(\pm \frac{11}{23} \right)$$

Condition of Parallelism and Perpendicularity of Two Lines

i. **Condition of parallelism of lines** If two lines of slopes m_1 and m_2 are parallel, then the angle θ between them is of 0° .

$$\therefore \tan \theta = \tan 0^\circ = 0$$

$$\Rightarrow \frac{m_2 - m_1}{1 + m_1 m_2} = 0$$

$$\Rightarrow m_2 = m_1$$

Thus, when two lines are parallel, their slopes are equal.

✎ Three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are collinear, if slope of

$$AB = \text{slope of } BC, \text{ i.e. } \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_3 - y_2}{x_3 - x_2}$$

✎ **Example 4.** The value of y , so that the line through $(3, y)$ and $(2, 7)$ is parallel to the line through $(-1, 4)$ and $(0, 6)$, is

- (a) 5 (b) -5
(c) 9 (d) -9

Sol. (c) Let $P(3, y)$, $Q(2, 7)$, $R(-1, 4)$ and $S(0, 6)$ be the given points. Then,

$$m_1 = \text{Slope of the line } PQ = \frac{7-y}{2-3} = y-7$$

$$m_2 = \text{Slope of the line } RS = \frac{6-4}{0-(-1)} = 2$$

Since, PQ and RS are parallel, therefore

$$m_1 = m_2 \Rightarrow y-7 = 2$$

$$\Rightarrow y = 9$$

➤ **Example 5.** If three points $A(h, 0)$, $P(a, b)$ and $B(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.

Sol. Since, the points $A(h, 0)$, $P(a, b)$ and $B(0, k)$ lie on a line.

∴ The points $A(h, 0)$, $P(a, b)$ and $B(0, k)$ are collinear.

⇒ Slope of PA = Slope of PB

$$\Rightarrow \frac{b - 0}{a - h} = \frac{k - b}{0 - a}$$

$$\Rightarrow ab = (a - h)(b - k)$$

$$\Rightarrow ab = ab - ak - bh + hk$$

$$\Rightarrow ak + bh = hk$$

$$\therefore \frac{a}{h} + \frac{b}{k} = 1$$

ii. **Condition of perpendicularity of lines** If two lines of slopes m_1 and m_2 are perpendicular, then the angle θ between them is of 90° .

$$\therefore \cot \theta = 0$$

$$\Rightarrow \frac{1 + m_1 m_2}{m_1 - m_2} = 0$$

$$\Rightarrow m_1 m_2 = -1$$

Thus, when two lines are perpendicular, the product of their slopes is -1 . If m is the slope of a line, then the slope of a line

perpendicular to it is $\left(-\frac{1}{m}\right)$.

➤ **Example 6.** $A(4, 4)$, $B(3, 5)$ and $C(-1, -1)$ are the vertices of

(a) an equilateral triangle

(b) a right angled triangle

(c) an isosceles triangle

(d) None of the above

Sol. (b) In $\triangle ABC$,

$$m_1 = \text{Slope of } AB = \frac{5 - 4}{3 - 4} = -1$$

$$m_2 = \text{Slope of } AC = \frac{-1 - 4}{-1 - 4} = 1$$

Clearly, $m_1 m_2 = -1$, this shows that $AB \perp AC$

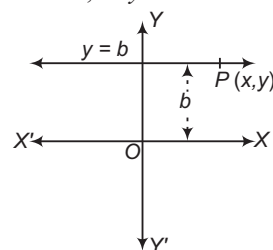
$$\text{i.e. } \angle CAB = \frac{\pi}{2}$$

Hence, the given points are the vertices of a right angled triangle.

Lines Parallel to the Coordinate Axes

i. **Line parallel to X-axis** Let AB be a straight line parallel to X -axis at a distance b from it. Then, the ordinate of each point on AB is b . Thus, AB can be considered as the locus of a point at a distance b from X -axis and if $P(x, y)$ is any point on AB , then $y = b$.

The equation of a line parallel to X -axis at a distance b from it, is $y = b$.



Since, X -axis is parallel to itself at a distance 0 from it. Therefore, the equation of X -axis is $y = 0$.

If a line is parallel to X -axis at a distance b and below X -axis, then its equation is $y = -b$.

➤ **Example 7.** Determine the equation of line through the point $(-4, -3)$ and parallel to X -axis.

Sol. We know, the equation of line which is parallel to X -axis is of the form $y = k$.

Here, line is passing through $(-4, -3)$.

Therefore, we have $-3 = k$

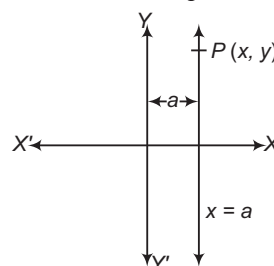
Thus, the required equation of line is

$$y = -3 \text{ or } y + 3 = 0$$

ii. **Line parallel to Y-axis** Let AB be a line parallel to Y -axis and at a distance a from it. Then, the abscissa of every point on AB is a . So, it can be treated as the locus of a point at a distance a from Y -axis. Thus, if $P(x, y)$ is any point on AB , then $x = a$.

The equation of a line parallel to Y -axis at a distance a from it, is $x = a$.

Since, Y -axis is parallel to itself at a distance 0 from it, therefore the equation of Y -axis is $x = 0$.



If a line is parallel to Y -axis at a distance a and to the left of Y -axis, then its equation is $x = -a$.

➤ **Example 8.** Find the equation of line, which is parallel to Y -axis and at a distance 3 units from left of the origin.

(a) $x = 3$ (b) $y = -3$ (c) $x = -3$ (d) $y = 3$

Sol. (c) We know, the equation of line which is parallel to Y -axis is of the form $x = k$.

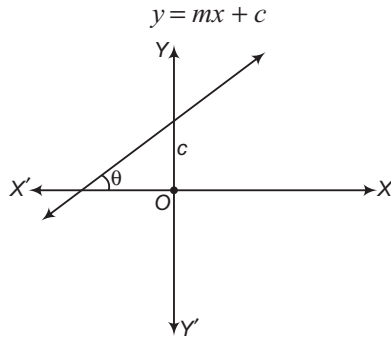
Here, we take $k = -3$, because it is left side of the origin.

$$\therefore x = -3$$

Different Forms of the Equation of a Straight Line

The equation of a line in the general form can also be written as $ax + by = c$. In this section, we shall discuss some standard forms of a line. These standard forms will also be compared with the general form.

- i. **The slope intercept form of a line** The equation of a line with slope m and making an intercept c on Y -axis is



and the equation of a line with slope m and x -intercept d is $y = m(x - d)$.

- If the line passes through the origin, then $0 = m \cdot 0 + c \Rightarrow c = 0$
Therefore, the equation of a line passing through the origin is $y = mx$, where m is the slope of the line.
- If the line is parallel to X -axis, then $m = 0$, therefore the equation of a line parallel to X -axis is $y = c$.

Example 9. The equation of a straight line which cut off an intercept of 5 units on negative direction of Y -axis and make an angle of 120° with positive direction of X -axis, is

- (a) $y + \sqrt{3}x - 5 = 0$
(b) $y + \sqrt{3}x + 5 = 0$
(c) $y - \sqrt{3}x + 5 = 0$
(d) $y - \sqrt{3}x - 5 = 0$

Sol. (b) Here, given that $m = \tan 120^\circ = -\sqrt{3}$ and $c = -5$
So, the equation of the line is given by
 $y = -\sqrt{3}x - 5 \Rightarrow y + \sqrt{3}x + 5 = 0$.

- ii. **Reduction of general form to slope intercept form** The general form of the equation of a line is

$$\begin{aligned} Ax + By + C &= 0 \\ \Rightarrow By &= -Ax - C \\ \Rightarrow y &= \left(-\frac{A}{B}\right)x + \left(-\frac{C}{B}\right) \end{aligned}$$

This is of the form $y = mx + c$,
where, $m = -\frac{A}{B}$ and $c = -\frac{C}{B}$.

Thus, for the straight line $Ax + By + C = 0$,

$$\text{Slope } (m) = -\frac{A}{B} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y}$$

and intercept on Y -axis

$$= -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

Example 10. Find the value of k for which the line $(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0$ is parallel to X -axis.

Sol. Let m be the slope of the line

$$(k - 3)x - (4 - k^2)y + k^2 - 7k + 6 = 0$$

$$\text{Then, } m = -\frac{\text{Coefficient of } x}{\text{Coefficient of } y} = \frac{-(k - 3)}{-(4 - k^2)} = \frac{k - 3}{4 - k^2}$$

As the line is parallel to X -axis, therefore we have

$$\frac{k - 3}{4 - k^2} = 0$$

$$\Rightarrow k - 3 = 0 \Rightarrow k = 3$$

- iii. **The point-slope form of a line** The equation of a line which passes through the point (x_1, y_1) and has the slope m is

$$y - y_1 = m(x - x_1).$$

Example 11. Find the equation of the perpendicular bisector of the line segment joining the points $A(2, 3)$ and $B(4, -5)$.

Sol. The slope of AB is given by

$$m = \frac{-5 - 3}{4 - 2} = -4$$

$$\therefore \text{Slope of a line perpendicular to } AB = \frac{-1}{m} = \frac{1}{4}$$

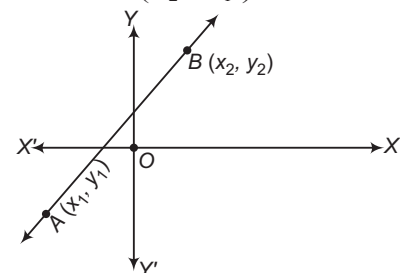
Let P be the mid-point of AB . Then, the coordinates of P are $\left(\frac{2 + 4}{2}, \frac{3 - 5}{2}\right)$ i.e. $(3, -1)$.

Thus, the required line passes through $P(3, -1)$ and has slope $\frac{1}{4}$. So, its equation is

$$y + 1 = \frac{1}{4}(x - 3) \Rightarrow 4y + 4 = x - 3 \Rightarrow x - 4y - 7 = 0$$

- iv. **The two point form of a line** The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1).$$



➤ **Example 12.** Find the equation of the line joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$.

Sol. Here, $x_1 = at_1^2$, $y_1 = 2at_1$, $x_2 = at_2^2$, $y_2 = 2at_2$

So, the equation of the required line is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2}(x - at_1^2)$$

$$\Rightarrow (y - 2at_1) = \frac{2}{t_1 + t_2}(x - at_1^2)$$

$$\Rightarrow y(t_1 + t_2) - 2at_1^2 - 2at_1t_2 = 2x - 2at_1^2$$

$$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$$

v. The equation of a line passing through two points (x_1, y_1) and (x_2, y_2) can also be written in the determinant form as

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

➤ **Example 13.** The equation of the line passing through the points $P(2, 3)$ and $Q(4, 5)$ is

- (a) $2x + y - 1 = 0$ (b) $2x - y - 1 = 0$
(c) $x + y + 1 = 0$ (d) $x - y + 1 = 0$

Sol. (d) The equation of the line through $P(2, 3)$ and $Q(4, 5)$ is

$$\begin{vmatrix} x & y & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 0$$

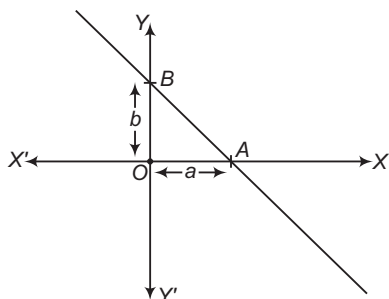
$$\Rightarrow x(3 - 5) - y(2 - 4) + 1(10 - 12) = 0$$

$$\Rightarrow -2x + 2y - 2 = 0$$

$$\Rightarrow -x + y - 1 = 0$$

$$\Rightarrow x - y + 1 = 0$$

vii. **The intercept form of a line** The equation of a line which cuts off intercepts a and b respectively on the X and Y -axes is $\frac{x}{a} + \frac{y}{b} = 1$.



➤ **Example 14.** The intercept made by a line on Y -axis is double to the intercept made by it on X -axis, if it passes through $(1, 2)$, then its equation is

- (a) $2x + y = 4$
(b) $2x + y + 4 = 0$
(c) $2x - y = 4$
(d) $2x - y + 4 = 0$

Sol. (a) Let the equation of the line be

$$\frac{x}{a} + \frac{y}{2a} = 1$$

Since, it passes through the point $(1, 2)$.

$$\text{So, } \frac{1}{a} + \frac{2}{2a} = 1$$

$$\Rightarrow \frac{1}{a} + \frac{1}{a} = 1$$

$\therefore$ The required equation is $2x + y = 4$.

➤ **Example 15.** The equation of the line through $(2, 3)$, so that the segment of the line intercepted between the axes is bisected at this point, is

- (a) $3x - 2y = 12$
(b) $3x + 2y = 12$
(c) $x - 2y = 12$
(d) $3x - y = 12$

Sol. (b) Let the equation of the line be $\frac{x}{a} + \frac{y}{b} = 1$, which meets

the X and Y -axes at $A(a, 0)$ and $B(0, b)$. The coordinates of the mid-point of AB are $(\frac{a}{2}, \frac{b}{2})$. Since, the point $(2, 3)$ bisects

AB . Therefore,

$$\frac{a}{2} = 2 \quad \text{and} \quad \frac{b}{2} = 3$$

$$\Rightarrow a = 4 \quad \text{and} \quad b = 6$$

Hence, the equation of the line is $\frac{x}{4} + \frac{y}{6} = 1$

$$\text{or } 3x + 2y = 12$$

vii. **Reduction of general equation of a line to intercept form**

The general equation of a line is $Ax + By + C = 0$.

$$\Rightarrow \frac{x}{-\frac{C}{A}} + \frac{y}{-\frac{C}{B}} = 1$$

Proof

$$Ax + By = -C$$

$$\Rightarrow \frac{Ax}{-C} + \frac{By}{-C} = 1$$

$$\Rightarrow \frac{x}{(-\frac{C}{A})} + \frac{y}{(-\frac{C}{B})} = 1$$

This is of the form $\frac{x}{a} + \frac{y}{b} = 1$.

Thus, for the straight line $Ax + By + C = 0$, we have

$$\text{Intercept on } X\text{-axis} = -\frac{C}{A} = -\frac{\text{Constant term}}{\text{Coefficient of } x}$$

$$\text{Intercept on } Y\text{-axis} = -\frac{C}{B} = -\frac{\text{Constant term}}{\text{Coefficient of } y}$$

➤ The intercept made by a line on X -axis can also be obtained by putting $y = 0$ in its equation. Similarly, y -intercept is the value of y obtained from the line when x is replaced by zero.

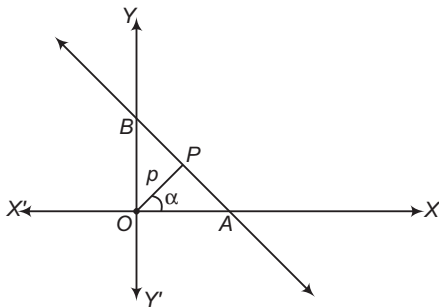
➤ **Example 16.** Transform the equation of the line $\sqrt{3}x + y - 8 = 0$ to intercept form and find intercepts on the coordinate axes.

Sol. We have, $\sqrt{3}x + y - 8 = 0$
 $\Rightarrow \sqrt{3}x + y = 8$
 $\Rightarrow \frac{\sqrt{3}}{8}x + \frac{y}{8} = 1$
 $\Rightarrow \frac{x}{\frac{8}{\sqrt{3}}} + \frac{y}{8} = 1$

This is the intercept form of the given line.

So, x-intercept = $\frac{8}{\sqrt{3}}$ and y-intercept = 8

viii. **The normal or perpendicular form of a line**
 The equation of the straight line upon which the length of the perpendicular from the origin is p and this perpendicular makes an angle α with X-axis is $x \cos \alpha + y \sin \alpha = p$.

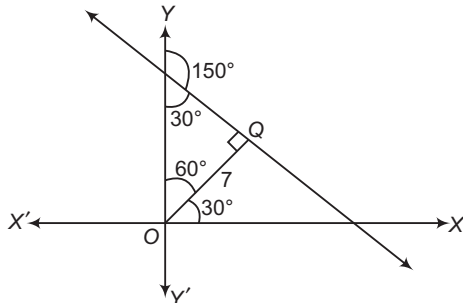


➤ **Example 17.** The length of the perpendicular from the origin to a line is 7 and the line makes an angle of 150° with the positive direction of Y-axis. Then, the equation of the line is

- (a) $\sqrt{3}x + y = 14$
 (b) $\sqrt{3}x - y = 14$
 (c) $3x - y = 14$
 (d) None of the above

Sol. (a) Given that, $p = 7, \alpha = 30^\circ$

$\therefore$ Equation of the required line is $x \cos 30^\circ + y \sin 30^\circ = 7$



$\Rightarrow \frac{x\sqrt{3}}{2} + \frac{y}{2} = 7$
 $\Rightarrow \sqrt{3}x + y = 14$

ix. **Reduction of the general equation of a line to the normal form**

The general equation of a line is

$$Ax + By + C = 0$$

Let its normal form be $x \cos \alpha + y \sin \alpha = p$.

Therefore, $\frac{A}{\cos \alpha} = \frac{B}{\sin \alpha} = \frac{C}{-p}$

$\Rightarrow \cos \alpha = -\frac{Ap}{C}$ and $\sin \alpha = -\frac{Bp}{C}$

$\therefore \cos^2 \alpha + \sin^2 \alpha = 1$

$\Rightarrow \frac{A^2 p^2}{C^2} + \frac{B^2 p^2}{C^2} = 1$

$\Rightarrow p = \frac{|C|}{\sqrt{A^2 + B^2}}$

$\therefore \cos \alpha = -\frac{|C|}{C} \cdot \frac{A}{\sqrt{A^2 + B^2}}$

and $\sin \alpha = -\frac{|C|}{C} \cdot \frac{B}{\sqrt{A^2 + B^2}}$

On putting the values of $\cos \alpha, \sin \alpha$ and p in $x \cos \alpha + y \sin \alpha = p$, we get

$$\left(-\frac{Ax}{\sqrt{A^2 + B^2}} - \frac{By}{\sqrt{A^2 + B^2}} \right) \frac{|C|}{C} = \frac{|C|}{\sqrt{A^2 + B^2}}$$

$$\Rightarrow \frac{Ax}{\sqrt{A^2 + B^2}} + \frac{By}{\sqrt{A^2 + B^2}} = \frac{-C}{\sqrt{A^2 + B^2}}$$

$$\Rightarrow \left(\frac{A}{\sqrt{A^2 + B^2}} \right) x + \left(\frac{B}{\sqrt{A^2 + B^2}} \right) y = \left(\frac{-C}{\sqrt{A^2 + B^2}} \right)$$

This is the normal form of the line

$$Ax + By + C = 0$$

It follows from the above discussion that to reduce the general equation of a line to normal form, we first shift the constant term on the RHS and make it positive if it is not, so and then divide both sides by

$$\sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2}$$

➤ **Example 18.** The normal form of the equation $x - \sqrt{3}y + 8 = 0$ is

- (a) $x \cos 120^\circ - y \sin 120^\circ = 4$
 (b) $x \cos 120^\circ + y \sin 120^\circ = 4$
 (c) $x \cos 120^\circ + y \sin 120^\circ = 8$
 (d) None of the above

Sol. (b) Given equation of line is

$$\begin{aligned} x - \sqrt{3}y + 8 &= 0 \\ \Rightarrow x - \sqrt{3}y &= -8 \\ \Rightarrow -x + \sqrt{3}y &= 8 \end{aligned}$$

Now, divide by

$$\begin{aligned} \sqrt{(\text{Coefficient of } x)^2 + (\text{Coefficient of } y)^2} \\ = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2, \text{ we get} \\ -\frac{1}{2}x + \frac{\sqrt{3}}{2}y = \frac{8}{2} \end{aligned}$$

$$\Rightarrow -\cos 60^\circ x + \sin 60^\circ y = 4$$

[$\because$ convert in form of $x \cos \alpha + y \sin \alpha = p$]

$$\Rightarrow x \cos (180^\circ - 60^\circ) + y \sin (180^\circ - 60^\circ) = 4$$

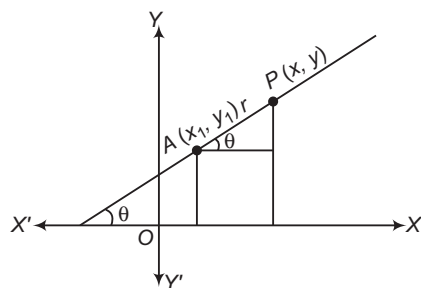
[$\because \cos x$ is negative and $\sin x$ is positive, which is possible in second quadrant]

$$\Rightarrow x \cos 120^\circ + y \sin 120^\circ = 4$$

x. The distance form of a line The equation of the straight line passing through (x_1, y_1) and making an angle θ with the positive direction of X -axis is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

where, r is the distance of the point (x, y) on the line from the point (x_1, y_1) .



• If the equation of the line is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$

Then,

$$x - x_1 = r \cos \theta$$

and

$$y - y_1 = r \sin \theta$$

$\Rightarrow$

$$x = x_1 + r \cos \theta$$

and

$$y = y_1 + r \sin \theta$$

Thus, the coordinates of any point on the line at a distance r from the given point (x_1, y_1) are $(x_1 + r \cos \theta, y_1 + r \sin \theta)$. If P is on the right side of (x_1, y_1) then r is positive and if P is on the left side of (x_1, y_1) then r is negative. Since, different values of r determine different points on the line, therefore the above form of the line is also called parametric form or symmetric form of a line.

• In the above form, we can determine the coordinates of any point on the line at a given distance from the given point through which it passes. At a given distance r from the point

(x_1, y_1) on the line $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta}$, there are two points viz.

$$(x_1 + r \cos \theta, y_1 + r \sin \theta) \text{ and } (x_1 - r \cos \theta, y_1 - r \sin \theta)$$

Example 19. A line is drawn from the point $P(\alpha, \beta)$ making an angle θ with the positive direction of X -axis to meet the line $ax + by + c = 0$ at Q . The length of PQ is

$$\begin{aligned} (a) & -\frac{a\alpha + b\beta + c}{a \cos \theta + b \sin \theta} & (b) & \left| \frac{a\alpha + b\beta + c}{\sqrt{a^2 + b^2}} \right| \\ (c) & \frac{a\alpha + b\beta + c}{a \cos \theta + b \sin \theta} & (d) & \text{None of these} \end{aligned}$$

Sol. (a) Equation of a straight line passing through the point $P(\alpha, \beta)$ and making an angle θ with positive direction of X -axis is

$$\frac{x - \alpha}{\cos \theta} = \frac{y - \beta}{\sin \theta} = r \quad [\text{say}]$$

Now, coordinates of any point at a distance r from $P(\alpha, \beta)$ on this line are $(\alpha + r \cos \theta, \beta + r \sin \theta)$.

If it lies on the line $ax + by + c = 0$, then

$$a(\alpha + r \cos \theta) + b(\beta + r \sin \theta) + c = 0$$

$$\Rightarrow r = -\frac{a\alpha + b\beta + c}{a \cos \theta + b \sin \theta}$$

$$\text{Thus, } PQ = r = -\frac{a\alpha + b\beta + c}{a \cos \theta + b \sin \theta}$$

Example 20. The equation of the line is passing through $P(4, 5)$ and making 30° with X -axis. Then, coordinates of point which is at a distance 4 units on either side of P , are

$$\begin{aligned} (a) & (4 - 2\sqrt{3}, 7) \\ (b) & (4 \pm 2\sqrt{3}, 7) \\ (c) & (4 + 2\sqrt{3}, 7), (4 - 2\sqrt{3}, 3) \\ (d) & (4 - 2\sqrt{3}, 7), (4 - 2\sqrt{3}, 7) \end{aligned}$$

Sol. (c) Since, the equation of the line passing through $P(4, 5)$ and making an angle 30° .

$\therefore$ Point on the line at a distance 4 from $(4, 5)$ is given by

$$\frac{x - 4}{\cos 30^\circ} = \frac{y - 5}{\sin 30^\circ} = \pm 4$$

$$x = 4 \pm 4 \cos 30^\circ$$

$$\text{and } y = 5 \pm 4 \sin 30^\circ$$

$$\Rightarrow x = 4 \pm 2\sqrt{3}$$

$$\text{and } y = 5 \pm 2$$

The coordinates of point are $(4 + 2\sqrt{3}, 7)$ and $(4 - 2\sqrt{3}, 3)$.

Position of Two Points Relative to a Given Line

In this section, we shall see how to check whether two given points are on the same side or opposite sides of a given line.

The two points (x_1, y_1) and (x_2, y_2) are on the same (or opposite) sides of the straight line $ax + by + c = 0$ according as the quantities $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same (or opposite) signs.

- A point (x_1, y_1) will lie on the side of the origin relative to a line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the same sign.
- A point (x_1, y_1) will lie on the opposite side of the origin relative to the line $ax + by + c = 0$, if $ax_1 + by_1 + c$ and c have the opposite sign.

➤ **Example 21.** The given line $3x - 4y = 8$, the points $(2, 3)$, $(-4, 5)$ are on the

(a) same sides
(b) opposite sides
(c) lie on the line
(d) None of the above

Sol. (a) The essential condition for the points on the same side or opposite side of the line is $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ are of same sign or opposite sign, respectively.

Let $c = 3x - 4y - 8$, then value of c_1 at $(2, 3)$ is given by

$$c_1 = 3 \times 2 - 4 \times 3 - 8$$

$$\Rightarrow c_1 = 6 - 12 - 8$$

$$\Rightarrow c_1 = -14$$

and c_2 at $(-4, 5)$ is given by

$$c_2 = 3 \times (-4) - 4 \times 5 - 8 = -40$$

Since, c_1 and c_2 are of same sign. Therefore, the two points are on the same sides of the given line.

Work Book Exercise 12.1

- Consider the equation $y - y_1 = m(x - x_1)$. In this equation, if m and x_1 are fixed and different lines are drawn for different values of y_1 , then
 - the line will pass through a single point
 - there will be one possible line only
 - there will be a set of parallel lines
 - None of the above
- If the coordinates of the points A and B are $(3, 3)$ and $(7, 6)$, then the length of the portion of the line AB intercepted between the axes is
 - $\frac{5}{4}$
 - $\frac{\sqrt{10}}{4}$
 - $\frac{\sqrt{13}}{3}$
 - None of these
- If $A(1, 1)$, $B(\sqrt{3} + 1, 2)$ and $C(\sqrt{3}, \sqrt{3} + 2)$ are three vertices of a square, then the diagonal through B is
 - $y = (\sqrt{3} - 2)x + (3 - \sqrt{3})$
 - $y = 0$
 - $y = x$
 - None of the above
- If each of the points $(x_1, 4)$ and $(-2, y_1)$ lies on the line joining the points $(2, -1)$ and $(5, -3)$, then the point $P(x_1, y_1)$ lies on the line
 - $x = 3y$
 - $x = -3y$
 - $y = 2x + 1$
 - $2x + 6y + 1 = 0$
- All points lying inside the triangle formed by the points $(1, 3)$, $(5, 0)$ and $(-1, 2)$ satisfy
 - $3x + 2y \geq 0$
 - $2x + y - 13 \geq 0$
 - $2x - 3y + 12 \leq 0$
 - $-2x + y \geq 0$
- A square is constructed on the portion of the line $x + y = 5$, which is intercepted between the axes on the side of the line away from origin. The equations to the diagonals of the square are
 - $x - 5, y = -5$
 - $x = 5, y = 5$
 - $x = -5, y = 5$
 - $x - y = 5, x - y = -5$
- A ray of light coming from the point $(1, 2)$ is reflected at a point A on the X -axis and then passes through the point $(5, 3)$. The coordinates of the point A are
 - $(\frac{13}{5}, 0)$
 - $(\frac{5}{13}, 0)$
 - $(-7, 0)$
 - None of these
- The larger of the two angles made with X -axis of a straight line drawn through $(1, 2)$, so that it intersects $x + y = 4$ at a distance $\frac{\sqrt{6}}{3}$ from $(1, 2)$, is
 - 105°
 - 75°
 - 60°
 - 15°
- If two vertices of a triangle are $(5, -1)$ and $(-2, 3)$ and its orthocentre lies at the origin, then the coordinates of the third vertex are
 - $(4, 7)$
 - $(-4, -7)$
 - $(2, -3)$
 - $(5, -1)$
- The equations of the line on which the perpendiculars from the origin make 30° angle with X -axis and which form a triangle of area $\frac{50}{\sqrt{3}}$ with axes, are
 - $x + \sqrt{3}y \pm 10 = 0$
 - $\sqrt{3}x + y \pm 10 = 0$
 - $x \pm \sqrt{3}y - 10 = 0$
 - None of these

Point of Intersection of Two Lines

Let the equations of two lines be

$$a_1x + b_1y + c_1 = 0 \quad \dots(i)$$

and

$$a_2x + b_2y + c_2 = 0 \quad \dots(ii)$$

Then, the coordinates of the point of intersection of Eqs. (i) and (ii) are

$$\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \right).$$

➤ To find the coordinates of the point of intersection of two non-parallel lines, we solve the given equations simultaneously and the values of x and y , so obtained determine the coordinates of the point of intersection.

➤ **Example 22.** Find the value(s) of m for which the lines $mx + (2m + 3)y + m + 6 = 0$ and $(2m + 1)x + (m - 1)y + m - 9 = 0$ intersect at a point on Y -axis.

Sol. The equations of the lines are

$$mx + (2m + 3)y + m + 6 = 0 \quad \dots(i)$$

$$\text{and } (2m + 1)x + (m - 1)y + m - 9 = 0 \quad \dots(ii)$$

Solving these two by cross-multiplication method, we have

$$\frac{x}{(2m + 3)(m - 9) - (m - 1)(m + 6)} = \frac{y}{(2m + 1)(m + 6) - m(m - 9)} = \frac{1}{m(m - 1) - (2m + 1)(2m + 3)}$$

$$\Rightarrow \frac{x}{m^2 - 20m - 21} = \frac{y}{m^2 + 22m + 6} = \frac{1}{-3(m^2 + 3m + 1)}$$

$$\Rightarrow x = \frac{m^2 - 20m - 21}{-3(m^2 + 3m + 1)}$$

$$\text{and } y = \frac{m^2 + 22m + 6}{-3(m^2 + 3m + 1)}$$

So, given lines intersect at point

$$\left(\frac{m^2 - 20m - 21}{-3(m^2 + 3m + 1)}, \frac{m^2 + 22m + 6}{-3(m^2 + 3m + 1)} \right)$$

It lies on Y-axis.

$$\therefore \frac{m^2 - 20m - 21}{-3(m^2 + 3m + 1)} = 0$$

$$\Rightarrow m^2 - 20m - 21 = 0$$

$$\Rightarrow (m - 21)(m + 1) = 0$$

$$\Rightarrow m = -1, 21$$

➤ **Example 23.** The number of integer values of m for which the x -coordinate of the point of intersection of the lines $3x + 4y = 9$ and $y = mx + 1$ is also an integer, is

- (a) 2 (b) 0
(c) 4 (d) 1

Sol. (a) On solving equation $3x + 4y = 9$ and $y = mx + 1$, we get

$$x = \frac{5}{3 + 4m}$$

Now, for x to be an integer,

$$3 + 4m = \pm 5 \text{ or } \pm 1$$

The integral values of m satisfying these conditions are -2 and -1 .

Condition of Concurrency of Three Lines

Three lines are said to be concurrent, if they pass through a common point i.e. they meet at a point. Thus, if three lines are concurrent the point of intersection of two lines lies on the third line.

Now, the lines $a_1x + b_1y + c_1 = 0 \quad \dots(i)$

$a_2x + b_2y + c_2 = 0 \quad \dots(ii)$

are concurrent and $a_3x + b_3y + c_3 = 0 \quad \dots(iii)$

$$\Leftrightarrow \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

This is the required condition of concurrency of three lines.

Another condition Three lines

$$L_1 \equiv a_1x + b_1y + c_1 = 0$$

$$L_2 \equiv a_2x + b_2y + c_2 = 0$$

and $L_3 \equiv a_3x + b_3y + c_3 = 0$

are concurrent iff there exist constants λ_1, λ_2 and λ_3 not all zero such that

$$\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$$

$$\text{i.e. } \lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) + \lambda_3(a_3x + b_3y + c_3) = 0$$

➤ **Example 24.** Given the four lines with the equations

$$x + 2y - 3 = 0, 3x + 4y - 7 = 0,$$

$$2x + 3y - 4 = 0, 4x + 5y - 6 = 0, \text{ then}$$

- (a) they are all concurrent
(b) they are the sides of a quadrilateral
(c) three lines are concurrent
(d) None of the above

Sol. (c) Given lines $x + 2y - 3 = 0$ and $3x + 4y - 7 = 0$ intersect at $(1, 1)$, which does not satisfy $2x + 3y - 4 = 0$ and $4x + 5y - 6 = 0$. Also, $3x + 4y - 7 = 0$ and $2x + 3y - 4 = 0$ intersect at $(5, -2)$, which does not satisfy $x + 2y - 3 = 0$ and $4x + 5y - 6 = 0$.

Lastly, intersection point of $x + 2y - 3 = 0$ and

$2x + 3y - 4 = 0$ is $(-1, 2)$ which satisfy $4x + 5y - 6 = 0$.

Hence, three lines are concurrent.

➤ **Example 25.** The least value of $|k|$, so that the lines $x = k + m$, $y = -2$ and $y = mx$ are concurrent, is

- (a) $2\sqrt{2}$ (b) $\sqrt{2}$
(c) $4\sqrt{2}$ (d) None of these

Sol. (a) Since, the given lines are concurrent, therefore

$$\begin{vmatrix} 1 & 0 & -k - m \\ 0 & 1 & 2 \\ m & -1 & 0 \end{vmatrix} = 0$$

$$\Rightarrow m^2 + km + 2 = 0 \quad \dots(i)$$

Since, m is real, the discriminant of Eq. (i) is greater than or equal to zero (0).

$$\Rightarrow k^2 \geq 8$$

$$\text{or } |k| \geq 2\sqrt{2}$$

Thus, the least value of $|k|$ is $2\sqrt{2}$.

➤ **Example 26.** Show that the following lines are concurrent

$$L_1 \equiv (a - b)x + (b - c)y + (c - a) = 0,$$

$$L_2 \equiv (b - c)x + (c - a)y + (a - b) = 0$$

and $L_3 \equiv (c - a)x + (a - b)y + (b - c) = 0.$

Sol. Clearly, $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 = 0$

$$\text{where, } \lambda_1 = \lambda_2 = \lambda_3 = 1$$

Hence, the given lines are concurrent.

Lines Parallel and Perpendicular to a Given Line

i. Equation of a line parallel to a given line
The equation of a line parallel to a given line $ax + by + c = 0$ is $ax + by + \lambda = 0$, where λ is a constant.

➤ To write a line parallel to a given line, we keep the expression containing x and y same and simply replace the given constant by a new constant λ . The value of λ can be determined by some given conditions.

➤ **Example 27.** The equation of a straight line parallel to $2x + 3y + 11 = 0$ and which is such that the sum of its intercept on the axes is 15, is

- (a) $2x + 3y - 18 = 0$
(b) $2x - 3y - 18 = 0$
(c) $2x + 3y + 18 = 0$
(d) $2x - 3y + 18 = 0$

Sol. (a) The equation of the line parallel to the line $2x + 3y + 11 = 0$ is

$$2x + 3y + \lambda = 0 \quad \dots(i)$$

where, λ is a constant.

The intercept by the line on X-axis is given by $\left(-\frac{\lambda}{2}\right)$ and

on the Y-axis is given by $\left(-\frac{\lambda}{3}\right)$.

It is given that the sum of the intercepts on the axes is 15.

$$\therefore \left(-\frac{\lambda}{2}\right) + \left(-\frac{\lambda}{3}\right) = 15$$

$$\Rightarrow -\frac{5\lambda}{6} = 15 \Rightarrow \lambda = -18$$

On putting $\lambda = -18$ in Eq. (i), we get $2x + 3y - 18 = 0$

Hence, the equation of the required line is $2x + 3y - 18 = 0$.

ii. Equation of a line perpendicular to a given line The equation of a line perpendicular to a given line $ax + by + c = 0$ is $bx - ay + \lambda = 0$, where λ is a constant and its value can be determined by some given condition.

➤ **Example 28.** The equation of the line perpendicular to the line $2x + 3y + 5 = 0$ and passing through $(1, 1)$, is

- (a) $3x + 2y + 1 = 0$
(b) $3x - 2y - 2 = 0$
(c) $3x - 2y - 1 = 0$
(d) $3x - y - 1 = 0$

Sol. (c) Equation of line perpendicular to the given line is

$$3x - 2y + \lambda = 0 \quad \dots(i)$$

Since, it passes through point $(1, 1)$.

$$\therefore 3 - 2 + \lambda = 0$$

$$\Rightarrow \lambda = -1$$

On putting the value of λ in Eq. (i), we get

$$3x - 2y - 1 = 0$$

Distance of a Point from a Line

The length of the perpendicular from a point (x_1, y_1) to a line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

➤ The length of the perpendicular from the origin to the line

$$ax + by + c = 0 \text{ is } \frac{|c|}{\sqrt{a^2 + b^2}}$$

• Distance between parallel lines $ax + by + c_1 = 0$ and

$$ax + by + c_2 = 0 \text{ is } \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}}$$

➤ **Example 29.** The distance of point $(-1, 1)$ from the line $12x - 5y + 9 = 0$ is

- (a) $\frac{13}{8}$ (b) $\frac{8}{13}$
(c) $\frac{5}{13}$ (d) $-\frac{8}{13}$

Sol. (b) Required distance = $\frac{|-12 - 5 + 9|}{\sqrt{(12)^2 + (5)^2}} = \frac{8}{13}$

➤ **Example 30.** The distance between the parallel lines $3x - 4y + 9 = 0$ and $6x - 8y - 15 = 0$, is

- (a) $-\frac{33}{10}$ (b) $\frac{10}{33}$ (c) $\frac{33}{10}$ (d) $\frac{33}{20}$

Sol. (c) Given equations of lines are

$$3x - 4y + 9 = 0 \quad \dots(i)$$

$$\text{and } 6x - 8y - 15 = 0 \quad \dots(ii)$$

Eq. (ii) can be rewritten as

$$3x - 4y - \frac{15}{2} = 0 \quad \dots(iii)$$

On comparing Eqs. (i) and (iii) with $ax + by + c_1 = 0$ and $ax + by + c_2 = 0$, we get

$$a = 3, b = -4, c_1 = 9, c_2 = -\frac{15}{2}$$

Now, distance between parallel lines

$$= \frac{|c_2 - c_1|}{\sqrt{a^2 + b^2}} = \frac{\left|-\frac{15}{2} - 9\right|}{\sqrt{(3)^2 + (-4)^2}} = \frac{\left|-\frac{33}{2}\right|}{\sqrt{25}} = \frac{33}{10}$$

Area of a Parallelogram

Let $ABCD$ be a parallelogram. The equations whose sides are AB, BC, CD and DA are $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$, $a_1x + b_1y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$. Then,

$$\text{Area} = \frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

➤ **Example 31.** Area of the parallelogram formed by the lines $y = mx$, $y = mx + 1$, $y = nx$ and $y = nx + 1$ equals

(a) $\frac{|m+n|}{(m-n)^2}$

(b) $\frac{2}{|m+n|}$

(c) $\frac{1}{|m+n|}$

(d) $\frac{1}{|m-n|}$

Sol. (d) The given equation of lines can be rewritten as $mx - y = 0$, $nx - y = 0$, $mx - y + 1 = 0$ and $nx - y + 1 = 0$

Now, using the formula $\frac{(c_1 - d_1)(c_2 - d_2)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$, we get

$$\text{Required area} = \frac{(0-1)(0-1)}{\begin{vmatrix} m & -1 \\ n & -1 \end{vmatrix}} = \frac{1}{|-m+n|} = \frac{1}{|m-n|}$$

➤ **Example 32.** Prove that the four straight lines $\frac{x}{a} + \frac{y}{b} = 1$, $\frac{x}{b} + \frac{y}{a} = 1$, $\frac{x}{a} + \frac{y}{b} = 2$ and $\frac{x}{b} + \frac{y}{a} = 2$ form a rhombus. Find its area.

Sol. The equations of the four sides are

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(i)$$

$$\frac{x}{b} + \frac{y}{a} = 1 \quad \dots(ii)$$

$$\frac{x}{a} + \frac{y}{b} = 2 \quad \dots(iii)$$

and $\frac{x}{b} + \frac{y}{a} = 2 \quad \dots(iv)$

Clearly, Eqs. (i), (iii) and Eqs. (ii), (iv) form two sets of parallel lines. So, the four lines form a parallelogram.

Let p_1 be the distance between parallel Eqs. (i) and (iii) and p_2 be the distance between parallel Eqs. (ii) and (iv). Then,

$$p_1 = \frac{2-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2 + b^2}} \left[\because d = \frac{c_1 - c_2}{\sqrt{a^2 + b^2}} \right]$$

and $p_2 = \frac{2-1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{ab}{\sqrt{a^2 + b^2}}$

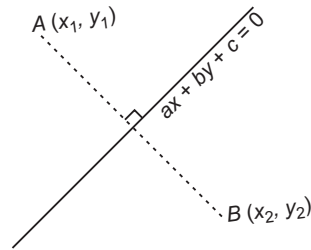
Clearly, $p_1 = p_2$. So, the given lines form a rhombus.

Area of the rhombus

$$= \frac{(2-1)(2-1)}{\begin{vmatrix} \frac{1}{a} & \frac{1}{b} \\ \frac{1}{b} & \frac{1}{a} \end{vmatrix}} = \frac{1}{\left(\frac{1}{a^2} - \frac{1}{b^2}\right)} = \frac{a^2 b^2}{|b^2 - a^2|}$$

Image of a Point with Respect to a Line

The image of (x_1, y_1) on $ax + by + c = 0$ be (x_2, y_2) , given by $\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -\frac{2(ax_1 + by_1 + c)}{a^2 + b^2}$.



Particular Cases

- i. The image of the point $P(x_1, y_1)$ with respect to X -axis is $(x_1, -y_1)$.
- ii. The image of the point $P(x_1, y_1)$ with respect to Y -axis is $(-x_1, y_1)$.
- iii. The image of the point $P(x_1, y_1)$ with respect to the line mirror $y = x$ is $Q(y_1, x_1)$.
- iv. The image of the point $P(x_1, y_1)$ with respect to the line mirror $y = x \tan \theta$ is given by $x = x_1 \cos 2\theta + y_1 \sin 2\theta$, $y = x_1 \sin 2\theta - y_1 \cos 2\theta$.
- v. The image of a point $P(x_1, y_1)$ with respect to the origin is the point $(-x_1, -y_1)$.

➤ **Example 33.** Find the image of the point $(3, 8)$ with respect of the line $x + 3y = 7$ assuming the line to be plane mirror.

- (a) $(1, 4)$
 (b) $(-1, -4)$
 (c) $(-1, 4)$
 (d) $(1, -4)$

Sol. (b) Given point (x_1, y_1) is $(3, 8)$ and line $x + 3y - 7$ is 0. By using the relation

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{-2(ax_1 + by_1 + c)}{a^2 + b^2}, \text{ we get}$$

$$\frac{x-3}{1} = \frac{y-8}{3} = \frac{-2(1 \times 3 + 3 \times 8 - 7)}{1^2 + 3^2}$$

$$\Rightarrow \frac{x-3}{1} = \frac{y-8}{3} = \frac{-2(20)}{10}$$

$$\Rightarrow \frac{x-3}{1} = \frac{y-8}{3} = -4$$

$$\Rightarrow x = -4 + 3, y = -12 + 8$$

$$\Rightarrow x = -1, y = -4$$

Equation of Straight Lines Passing through a Given Point and Making a Given Angle with a Given Line

The equations of the straight lines which pass through a given point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$$

➤ **Example 34.** The equations of the two straight lines through $(7, 9)$ and making an angle of 60° with the line $x - \sqrt{3}y - 2\sqrt{3} = 0$, are

(a) $x = 7$ and $x + \sqrt{3}y = 7 + 9\sqrt{3}$

(b) $x = -7$ and $x + \sqrt{3}y = 7 + 9\sqrt{3}$

(c) $x = 7$ and $x - \sqrt{3}y = 7 + 9\sqrt{3}$

(d) $x = 7$ and $x + \sqrt{3}y = 7 + \sqrt{3}$

Sol. (a) We know that the equations of two straight lines which pass through a point (x_1, y_1) and make a given angle α with the given straight line $y = mx + c$ are

$$y - y_1 = \frac{m \pm \tan \alpha}{1 \mp m \tan \alpha} (x - x_1)$$

Here, $x_1 = 7, y_1 = 9, \alpha = 60^\circ$

and $m =$ Slope of the line $x - \sqrt{3}y - 2\sqrt{3} = 0$

So, $m = \frac{1}{\sqrt{3}}$

Now, the equations of the required lines are

$$y - 9 = \frac{\frac{1}{\sqrt{3}} + \tan 60^\circ}{1 - \frac{1}{\sqrt{3}} \tan 60^\circ} (x - 7)$$

and $y - 9 = \frac{\frac{1}{\sqrt{3}} - \tan 60^\circ}{1 + \frac{1}{\sqrt{3}} \tan 60^\circ} (x - 7)$

$$\Rightarrow (y - 9) \left(1 - \frac{1}{\sqrt{3}} \tan 60^\circ\right) = \left(\frac{1}{\sqrt{3}} + \tan 60^\circ\right) (x - 7)$$

and $(y - 9) \left(1 + \frac{1}{\sqrt{3}} \tan 60^\circ\right) = \left(\frac{1}{\sqrt{3}} - \tan 60^\circ\right) (x - 7)$

$$\Rightarrow 0 = \left(\frac{1}{\sqrt{3}} + \sqrt{3}\right) (x - 7)$$

$$\Rightarrow x - 7 = 0 \text{ and } (y - 9)(2) = \left(\frac{1}{\sqrt{3}} - \sqrt{3}\right) (x - 7)$$

$$\Rightarrow x + \sqrt{3}y = 7 + 9\sqrt{3}$$

Hence, the required lines are

$$x = 7 \text{ and } x + \sqrt{3}y = 7 + 9\sqrt{3}$$

➤ **Example 35.** If $(1, 1)$ and $(-3, 5)$ are vertices of a diagonal of a square, then the equations of its sides through $(1, 1)$ are

(a) $2x - y = 1, y - 1 = 0$

(b) $3x + y = 4, x - 1 = 0$

(c) $x = 1, y = 1$

(d) None of these

Sol. (c) Since, each side of a square makes 45° angle with its diagonals, so equation of the sides through $(1, 1)$ are given by

$$y - 1 = \frac{-1 \pm \tan 45^\circ}{1 \mp (-1) \tan 45^\circ} (x - 1)$$

[$\because$ slope of diagonal, $m = -1$]

$$\Rightarrow y - 1 = \frac{-1 \pm 1}{1 \pm 1} (x - 1) \Rightarrow x - 1 = 0, y - 1 = 0$$

or $x = 1, y = 1$

Slope of a Line Equally Inclined to Two Given Lines

The slope m of a line which is equally inclined with two lines of slopes m_1 and m_2 , is given by $\frac{m - m_1}{1 + mm_1} = \frac{m_2 - m}{1 + mm_2}$.

➤ The above equation in m always gives two values of m which are the slopes of the lines parallel to the bisectors of the angles between the two given lines.

➤ **Example 36.** A line $4x + y = 1$ through the point $A(2, -7)$ meets the line BC whose equation is $3x - 4y + 1 = 0$ at the point B . If $AB = AC$, then the equation of the line AC is

(a) $52x - 89y + 519 = 0$

(b) $52x + 89y - 519 = 0$

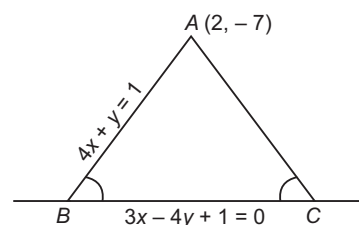
(c) $52x + 89y + 519 = 0$

(d) None of the above

Sol. (c) Since, $AB = AC$

$$\therefore \angle ABC = \angle ACB = \theta$$

[say]



Now, slope of $BC = \frac{3}{4}$ and slope of $AB = -4$

Let slope of AC be m .

Since, BC is equally inclined with AB and AC .

$$\frac{\frac{3}{4} + 4}{1 - 4 \cdot \frac{3}{4}} = \frac{m - \frac{3}{4}}{1 + \frac{3}{4}m}$$

$$\Rightarrow \frac{19}{8} = \frac{3 - 4m}{4 + 3m}$$

$$\Rightarrow 76 + 57m = 24 - 32m$$

$$\Rightarrow m = -\frac{52}{89}$$

$$\therefore \text{Equation of } AC \text{ is } (y + 7) = -\frac{52}{89}(x - 2)$$

$$\Rightarrow 52x + 89y + 519 = 0$$

Work Book Exercise 12.2

12

Straight Line and Pair of Straight Lines

- The image of the point $(-8, 12)$ with respect to the line mirror $4x + 7y + 13 = 0$ is
a $(16, -2)$ b $(-16, 2)$ c $(16, 2)$ d $(-16, -2)$
- The straight line $x + 2y - 9 = 0$, $3x + 5y - 5 = 0$ and $ax + by - 1 = 0$ are concurrent, if the straight line $35x - 22y + 1 = 0$ passes through the point
a (a, b) b (b, a) c $(a, -b)$ d $(-a, b)$
- The equations of the lines through $(-1, -1)$ and making angle 45° with the line $x + y = 0$ are given by
a $x + 1 = 0, y + 2 = 0$ b $x - 1 = 0, y - 2 = 0$
c $x - 1 = 0, y - 1 = 0$ d $x + 1 = 0, y + 1 = 0$
- The area enclosed $2|x| + 3|y| \leq 6$ is
a 3 sq units b 12 sq units
c 9 sq units d 24 sq units
- The equation of the line with gradient $-\frac{3}{2}$, which is concurrent with the lines $4x + 3y - 7 = 0$ and $8x + 5y - 1 = 0$, is
a $3x + 2y - 2 = 0$ b $3x + 2y - 63 = 0$
c $2y - 3x - 2 = 0$ d None of these
- The algebraic sum of the perpendicular distances from $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ to a variable line is zero, then the line passes through
a the orthocentre of $\triangle ABC$
b the centroid of $\triangle ABC$
c the circumcentre of $\triangle ABC$
d None of these
- If the algebraic sum of the perpendicular distances from the points $(2, 0)$, $(0, 2)$ and $(1, 1)$ to a variable straight lines be zero, then the line passes through the point
a $(-1, 1)$ b $(1, 1)$ c $(1, -1)$ d $(-1, -1)$
- The equation of the line passing through the intersection of $x - \sqrt{3}y + \sqrt{3} - 1 = 0$ and $x + y - 2 = 0$ and making an angle of 15° with the first line, is
a $x - y = 0$
b $x - y + 1 = 0$
c $y = 1$
d $\sqrt{3}x - y + 1 - \sqrt{3} = 0$
- The distance of the point $(3, 5)$ from the line $2x + 3y - 14 = 0$ measured parallel to the line $x - 2y = 1$ is
a $\frac{7}{\sqrt{5}}$ b $\frac{7}{\sqrt{13}}$
c $\sqrt{5}$ d $\sqrt{13}$
- A line is drawn from $P(x_1, y_1)$ in the direction α with the X-axis, to meet $Ax + By + C = 0$ at Q. Then, the length PQ is equal to
a $\left| \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}} \right|$ b $-\frac{Ax_1 + By_1 + C}{a \cos \alpha + B \sin \alpha}$
c $\frac{Ax_1 + By_1 + C}{A \cos \alpha + B \sin \alpha}$ d $-\frac{Ax_1 + By_1 + C}{A \sin \alpha + B \cos \alpha}$
- The image of the point $P(\alpha, \beta)$ with respect to the line $y = -x$ is the point Q and the image of Q with respect to the line $y = x$ is R. Then, the mid-point of R is
a $(\alpha + \beta, \beta + \alpha)$ b $(0, 0)$
c $(\alpha - \beta, \beta - \alpha)$ d None of these
- The equation of a straight line passing through $(1, 2)$ and having intercept of length 3 between the straight lines $3x + 4y = 24$ and $3x + 4y = 12$ is
a $7x + 24y - 55 = 0$ b $24x + 7y - 38 = 0$
c $24x - 7y - 10 = 0$ d $7x - 24y + 41 = 0$

Family of Lines through the Intersection of Two Given Lines

The equation of the family of lines passing through the intersection of the lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is

$$(a_1x + b_1y + c_1) + \lambda(a_2x + b_2y + c_2) = 0$$

where, λ is a parameter.

✎ The equation $L_1 + \lambda L_2 = 0$ represents a line passing through the intersecting of the line $L_1 = 0$ and $L_2 = 0$, which is a fixed point. Hence, $L_1 + \lambda L_2 = 0$ represents a family of straight lines for different values of λ , which pass through a fixed point.

➤ **Example 37.** The equation of the straight line through the intersection of line $2x + y = 1$ and $3x + 2y = 5$ and passing through the origin, is

- (a) $7x + 3y = 0$ (b) $7x - y = 0$
(c) $3x + 2y = 0$ (d) $x + y = 0$

Sol. (a) The family of line through the intersection of L_1 and L_2 is given by $L_1 + \lambda L_2 = 0$, where λ is constant.

∴ The equation of required line is given by

$$(2x + y - 1) + \lambda(3x + 2y - 5) = 0 \quad \dots(i)$$

Since, it passes through the origin $O(0, 0)$.

$$\therefore -1 - 5\lambda = 0$$

$$\Rightarrow \lambda = -\frac{1}{5}$$

On putting value of λ in Eq. (i), we get

$$x\left(2 - \frac{3}{5}\right) + y\left(1 - \frac{2}{5}\right) + (-1 + 1) = 0$$

$$\Rightarrow \frac{7}{5}x + \frac{3}{5}y = 0$$

$$\Rightarrow 7x + 3y = 0$$

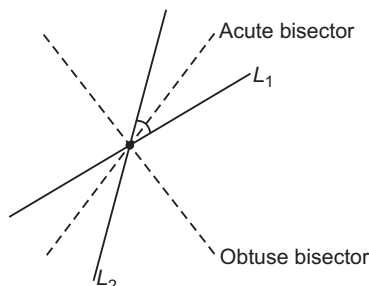
Equation of the Bisectors

The equation of the bisectors of the angles between the lines

$$a_1x + b_1y + c_1 = 0 \quad \dots(i)$$

and $a_2x + b_2y + c_2 = 0 \quad \dots(ii)$

are given by $\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$



where, if

- i. $c_1, c_2 > 0$ and $a_1a_2 + b_1b_2 > 0$, then the bisector corresponding to + sign gives the obtuse angle bisector and the bisector corresponding to '-' sign gives the acute angle bisector.
- ii. $c_1, c_2 > 0$ and $a_1a_2 + b_1b_2 < 0$, then the bisector corresponding to '+' and '-' signs gives the acute and obtuse angle bisector, respectively.

➤ A line which is equally inclined to given two lines is parallel to the angle bisectors of the given lines.

- **Example 38.** The equation of the bisector of the angle between the straight lines $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$, is
- (a) $21x + 27y + 131 = 0$
 - (b) $x + 27y - 131 = 0$
 - (c) $21x - 27y + 131 = 0$
 - (d) $21x + 27y - 131 = 0$

Sol. (d) The equation of the bisectors of the angles between $3x - 4y + 7 = 0$ and $12x - 5y - 8 = 0$ is

$$\frac{3x - 4y + 7}{\sqrt{3^2 + (-4)^2}} = \pm \frac{12x - 5y - 8}{\sqrt{(12)^2 + (-5)^2}}$$

$$\Rightarrow 39x - 52y + 91 = \pm (60x - 25y - 40)$$

Taking the positive sign, $21x + 27y - 131 = 0$ as one bisector.

- **Example 39.** The equation of the obtuse angle bisector of lines $12x - 5y + 7 = 0$ and $3y - 4x - 1 = 0$ is
- (a) $4x + 7y + 11 = 0$
 - (b) $4x + 7y - 11 = 0$
 - (c) $7x + 4y + 11 = 0$
 - (d) $4x + y + 11 = 0$

Sol. (a) First make the constant terms (c_1 and c_2) positive in the two equation i.e. write equation as

$$12x - 5y + 7 = 0$$

and $4x - 3y + 1 = 0$

Now, check $a_1a_2 + b_1b_2 = 12 \times 4 + (-5) \times (-3) = 63$, which is positive.

Now, $\frac{12x - 5y + 7}{\sqrt{(12)^2 + (-5)^2}} = + \frac{4x - 3y + 1}{\sqrt{(4)^2 + (-3)^2}}$

$$\Rightarrow 5(12x - 5y + 7) = 13(4x - 3y + 1)$$

So, $4x + 7y + 11 = 0$ is the required obtuse angle bisector.

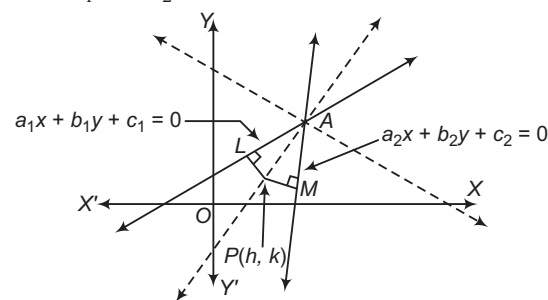
Bisector of the Angle Containing the Origin

Let the equations of two lines be

$$a_1x + b_1y + c_1 = 0 \quad \dots(i)$$

and $a_2x + b_2y + c_2 = 0 \quad \dots(ii)$

where, c_1 and c_2 are positive.



The equation of the bisector containing the origin is given by

$$\frac{|a_1x + b_1y + c_1|}{\sqrt{a_1^2 + b_1^2}} = \frac{|a_2x + b_2y + c_2|}{\sqrt{a_2^2 + b_2^2}}$$

$$\Rightarrow \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

[since $a_1x + b_1y + c_1$ and $a_2x + b_2y + c_2$ are either both positive or both negative]

Thus, we have the following algorithm to find the bisector of the angle containing the origin :

Algorithm

Step I Obtain the two lines, let the lines be $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$.

Step II Rewrite the equation of the two lines, so that their constant terms are positive.

Step III The bisector corresponding to the positive sign i.e.

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = + \frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the angle that contains the origin.

➤ The bisector corresponding to the negative sign

$$\text{i.e. } \frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = -\frac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}}$$

is the bisector of the angle not containing the origin.

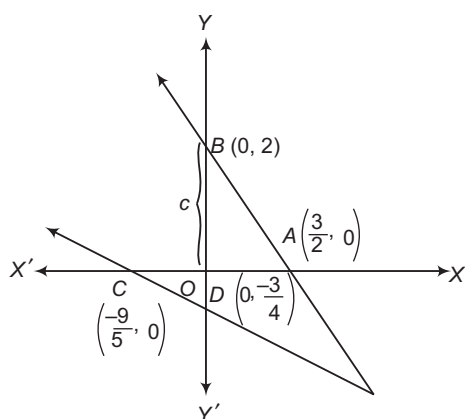
➤ **Example 40.** For the straight lines $4x + 3y - 6 = 0$ and $5x + 12y + 9 = 0$, find the equation of the bisector of the angle which contains the origin.

Sol. When constants have been made positive, then equation of lines become

$$-4x - 3y + 6 = 0$$

$$\text{and } 5x + 12y + 9 = 0$$

Equation of bisector of the angle containing the origin will be



$$\frac{-4x - 3y + 6}{5} = \frac{5x + 12y + 9}{13}$$

$$\Rightarrow 7x + 9y - 3 = 0$$

➤ On drawing the graphs of the given straight lines it will be found that the origin lies in the acute angle. Hence, the equation of the bisector of the acute angle is the bisector of the angle between the given lines containing the origin, which is $7x + 9y - 3 = 0$.

➤ **Example 41.** The equation of the bisector of that angle between the lines $x + 2y - 11 = 0$, $3x - 6y - 5 = 0$ which contains the point $(1, -3)$ is

(a) $3x = 19$

(b) $3y = 7$

(c) $3x = 19$ and $3y = 7$

(d) None of the above

Sol. (a) Since, the origin and the point $(1, -3)$ lie on the same side of $x + 2y - 11 = 0$ and on the opposite side of $3x - 6y - 5 = 0$. Therefore, the bisector of the angle containing $(1, -3)$ is the bisector of that angle which does not contain the origin and is given by

$$\frac{-x - 2y + 11}{\sqrt{5}} = -\left(\frac{-3x + 6y + 5}{\sqrt{45}}\right)$$

i.e. $3x = 19$

Locus and its Equation

It is the path or curve traced by a moving point satisfying the given condition.

i. **Equation of the locus of a point** The equation of the locus of a point is the algebraic relation which is satisfied by the coordinates of every point on the locus of the point.

Step taken to find the equation of locus of a point

(a) Assumes the coordinates of the point say (h, k) whose locus is to be find.

(b) Write the given condition involving (h, k) .

(c) Eliminate the variable(s) (if any).

(d) Replace $h \rightarrow x$ and $k \rightarrow y$

The equation, so obtained is the locus of the point which moves under some definite condition.

➤ **Example 42.** If $P = (1, 0)$, $Q = (-1, 0)$ and $R = (2, 0)$ are three given points, then locus of the point S satisfying the relation $SQ^2 + SR^2 = 2SP^2$, is

(a) a straight line parallel to X -axis

(b) a circle passing through the origin

(c) a circle with the centre at the origin

(d) a straight line parallel to Y -axis

Sol. (d) Let the coordinates of S be (h, k) .

$$\therefore SQ^2 + SR^2 = 2SP^2$$

$$\Rightarrow (h+1)^2 + k^2 + (h-2)^2 + k^2 = 2[(h-1)^2 + k^2]$$

$$\Rightarrow h^2 + 2h + 1 + k^2 + h^2 - 4h + 4 + k^2 = 2(h^2 - 2h + 1 + k^2)$$

$$\Rightarrow 2h + 3 = 0$$

$$\Rightarrow h = -\frac{3}{2} \Rightarrow x = -\frac{3}{2}$$

Hence, it is a straight line parallel to Y -axis.

➤ **Example 43.** If sum of distances of a point from the origin and lines $x = 2$ is 4, then its locus is

(a) $x^2 - 12y = 36$

(b) $y^2 + 12x = 36$

(c) $y^2 - 12x = 36$

(d) $x^2 + 12y = 36$

Sol. (b) Let point be $P(h, k)$. So, distance from the origin

$$= \sqrt{h^2 + k^2} \text{ and distance from the line } = (h - 2)$$

$$\therefore \sqrt{h^2 + k^2} + (h - 2) = 4$$

$$\Rightarrow \sqrt{h^2 + k^2} = (-h + 6)$$

$$\Rightarrow h^2 + k^2 = h^2 + 36 - 12h$$

$$\Rightarrow k^2 + 12h = 36$$

$$\Rightarrow y^2 + 12x = 36 \text{ is the required locus of point.}$$

Work Book Exercise 12.3

- The equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ is
 - $21x + 77y - 101 = 0$
 - $11x - 3y + 9 = 0$
 - $31x + 77y + 101 = 0$
 - $11x - 3y - 9 = 0$
- If the family of lines $x(a + 2b) + y(a + 3b) = a + b$ passes through the point for all values of a and b , then the coordinates of the point are
 - $(2, 1)$
 - $(2, -1)$
 - $(-2, 1)$
 - None of the above
- The straight lines through the point of intersection of $ax + by + c = 0$ and $a'x + b'y + c' = 0$ are parallel to the Y-axis has the equation
 - $x(ab' - a'b) + (cb' - c'b) = 0$
 - $x(ab' + a'b) + (cb' + c'b) = 0$
 - $y(a'b - ab') + (a'c - ac') = 0$
 - None of the above
- The equation of line bisecting the obtuse angle between $\frac{y - x - 2}{\sqrt{2}} = \frac{x + 2y - 5}{n}$ and $2y + x = 5$ is where n is equal to
 - 5
 - $\sqrt{2}$
 - $\sqrt{5}$
 - None of the above
- If $u = a_1x + b_1y + c_1 = 0$ and $v = a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} = \frac{c_1}{c_2}$, then $u + kv = 0$ represents
 - $u = 0$
 - a family of concurrent lines
 - a family of parallel lines
 - None of the above
- If $u \equiv a_1x + b_1y + c_1 = 0$, $v \equiv a_2x + b_2y + c_2 = 0$ and $\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$, then $u + kv = 0$ represents
 - a family of concurrent lines
 - a family of parallel lines
 - $u = 0$ or $v = 0$
 - None of the above
- The locus of a point which moves, so the difference of its distance from two fixed straight lines which are at right angles is equal to the distance from another fixed straight line is
 - a straight line
 - a circle
 - a parabola
 - an ellipse
- A and B are fixed points. The vertex C of $\triangle ABC$ moves such that $\cot A + \cot B = \text{constant}$. The locus of C is a straight line
 - perpendicular to AB
 - parallel to AB
 - inclined at an angle $(A - B)$ to AB
 - None of the above

Combined Equation of a Pair of Straight Lines

The combined equation of the straight line

$$a_1x + b_1y + c_1 = 0 \text{ and } a_2x + b_2y + c_2 = 0 \text{ is}$$

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0.$$

e.g. The combined equation of lines $2x + y + 3 = 0$ and $x - y + 4 = 0$ is $(2x + y + 3)(x - y + 4) = 0$ or $2x^2 - xy - y^2 + 11x + y + 12 = 0$.

- The equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is known as the general equation of second degree. Thus, the joint equation of a pair of straight lines is general equation of second degree.
- In order to find the separate equation of a pair of lines when their joint equation is given, write down the equation of pair of lines as a quadratic equation in x (or y) and from this write down the value of x in terms of y (or value of y in terms of x). Do cross-multiplication and bring all the terms on LHS making RHS equal to zero. The two equations, thus obtained will be the equations of the two lines.

Example 44. Find the separate equation of the lines represented by

$$2x^2 - xy - y^2 + 9x - 3y + 10 = 0.$$

Sol. Given equation is

$$2x^2 - xy - y^2 + 9x - 3y + 10 = 0 \quad \dots(i)$$

It can be written as

$$2x^2 - (y - 9)x - (y^2 + 3y - 10) = 0 \quad \dots(ii)$$

$$\therefore x = \frac{(y - 9) \pm \sqrt{(y - 9)^2 + 4 \cdot 2 \cdot (y^2 + 3y - 10)}}{4}$$

$$\Rightarrow 4x = (y - 9) \pm \sqrt{9y^2 + 6y + 1}$$

$$\Rightarrow 4x = y - 9 \pm (3y + 1)$$

$$\Rightarrow 4x = 4y - 8 \text{ and } 4x = -2y - 10$$

$$\Rightarrow x - y + 2 = 0 \text{ and } 2x + y + 5 = 0$$

Hence, the separate equations of the lines represented by Eq. (i) are

$$x - y + 2 = 0$$

$$\text{and } 2x + y + 5 = 0$$

Pair of Straight Lines through the Origin

Homogeneous equation of second degree A rational, integral, algebraic equation in two variables x and y is said to be a homogeneous equation of the second degree, if the sum of the indices (exponents) of x and y in each term is equal to 2.

The general form of the homogeneous equation of the second degree in x and y is $ax^2 + 2hxy + by^2 = 0$, where a , b and h are constants.

e.g. the equation $2x^2 - xy + 5y^2 = 0$ is a homogeneous equation of second degree, but the equation $2x^2 + 5xy + y^2 - 2x = 0$ is not homogeneous equation of second degree.

- The homogeneous equation of second degree $ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines passing through the origin. These lines are real and distinct, if $h^2 > ab$ and coincident, if $h^2 = ab$ and the lines are imaginary i.e. they do not exist, if $h^2 < ab$.
- To find the separate equation of lines proceed as follows :
 - > Divide by x^2 to obtain quadratic equation in $\frac{y}{x}$.
 - > Solve the quadratic equation to get the required lines.
- If $y = m_1x$ and $y = m_2x$ are lines represented by the equation $ax^2 + 2hxy + by^2 = 0$, then

$$m_1 + m_2 = -\frac{2h}{b} = -\frac{\text{Coefficient of } xy}{\text{Coefficient of } y^2}$$

$$\text{and } m_1 m_2 = \frac{a}{b} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } y^2}$$

➤ **Example 45.** If the equation $6x^2 - 5xy + y^2 = 0$ represents a pair of distinct straight lines each passing through the origin. Then, the separate equation of these lines are

- (a) $y - 3x = 0$ and $y - 2x = 0$
 (b) $y + 3x = 0$ and $y + 2x = 0$
 (c) $y - 3x = 4$ and $y - 2x = 0$
 (d) $y - 3x = 0$ and $y - 2x = 4$

Sol. (a) The given equation is a homogeneous equation of second degree. So, it represents a pair of straight lines passing through the origin.

On comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get $a = 6$, $b = 1$ and $2h = -5$

$$\therefore h^2 - ab = \frac{25}{4} - 6 = \frac{1}{4} > 0$$

$$\Rightarrow h^2 > ab$$

Hence, the given equation represents a pair of distinct lines passing through the origin.

Now consider, $6x^2 - 5xy + y^2 = 0$

$$\Rightarrow \left(\frac{y}{x}\right)^2 - 5\left(\frac{y}{x}\right) + 6 = 0 \quad [\text{dividing by } x^2]$$

$$\Rightarrow \left(\frac{y}{x}\right)^2 - 3\left(\frac{y}{x}\right) - 2\left(\frac{y}{x}\right) + 6 = 0$$

$$\Rightarrow \left(\frac{y}{x} - 3\right)\left(\frac{y}{x} - 2\right) = 0$$

$$\Rightarrow \frac{y}{x} - 3 = 0$$

$$\text{or } \frac{y}{x} - 2 = 0$$

$$\Rightarrow y - 3x = 0$$

$$\text{or } y - 2x = 0$$

So, the given equation represents the straight lines $y - 3x = 0$ and $y - 2x = 0$.

➤ **Example 46.** If the slope of one of the line represented by $ax^2 + 2hxy + by^2 = 0$ is the square of the other, then

$$(a) \frac{a-b}{h} + \frac{8h^2}{ab} = 6 \quad (b) \frac{a+b}{h} + \frac{8h^2}{ab} = 6$$

$$(c) \frac{a-b}{h} - \frac{8h^2}{ab} = 6 \quad (d) \text{None of these}$$

Sol. (b) Let m and m^2 be the slopes of the lines represented by $ax^2 + 2hxy + by^2 = 0$. Then,

$$m + m^2 = -\frac{2h}{b}$$

and

$$m \cdot m^2 = \frac{a}{b}$$

$$\left[\because m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1 m_2 = \frac{a}{b} \right]$$

$$\Rightarrow m + m^2 = -\frac{2h}{b} \text{ and } m = \left(\frac{a}{b}\right)^{1/3}$$

$$\Rightarrow \left(\frac{a}{b}\right)^{1/3} + \left(\frac{a}{b}\right)^{2/3} = -\frac{2h}{b}$$

$$\Rightarrow \left[\left(\frac{a}{b}\right)^{1/3} + \left(\frac{a}{b}\right)^{2/3}\right]^3 = \left(-\frac{2h}{b}\right)^3$$

$$\Rightarrow \frac{a}{b} + \left(\frac{a}{b}\right)^2 + 3\left(\frac{a}{b}\right)^{1/3}\left(\frac{a}{b}\right)^{2/3} \left\{\left(\frac{a}{b}\right)^{1/3} + \left(\frac{a}{b}\right)^{2/3}\right\} = -\frac{8h^3}{b^3}$$

$$\Rightarrow \frac{a}{b} + \frac{a^2}{b^2} + 3\left(\frac{a}{b}\right)\left(-\frac{2h}{b}\right) = -\frac{8h^3}{b^3}$$

$$\left[\because \left(\frac{a}{b}\right)^{1/3} + \left(\frac{a}{b}\right)^{2/3} = -\frac{2h}{b} \right]$$

$$\Rightarrow \frac{a}{b} + \frac{a^2}{b^2} + \frac{8h^3}{b^3} = \frac{6ah}{b^2}$$

$$\Rightarrow \frac{a(a+b)}{b^2} + \frac{8h^3}{b^3} = \frac{6ah}{b^2}$$

$$\Rightarrow \frac{a+b}{h} + \frac{8h^2}{ab} = 6$$

$$\left[\text{multiplying both sides by } \frac{b^2}{ah} \right]$$

Angle between the Pair of Lines Given by

$$ax^2 + 2hxy + by^2 = 0$$

The angle θ between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ is given by

$$\tan \theta = \pm \frac{2\sqrt{h^2 - ab}}{a + b}$$

and acute angle between the lines is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

➤ **Example 47.** The angle between the lines whose joint equation is $2x^2 - 3xy - 6y^2 = 0$, is

$$(a) \theta = \tan^{-1} \left(\frac{\sqrt{53}}{4} \right)$$

$$(b) \theta = \tan^{-1} \left(-\frac{\sqrt{57}}{4} \right)$$

$$(c) \theta = \tan^{-1} \left(\frac{\sqrt{3}}{4} \right)$$

$$(d) \theta = \frac{1}{2} \tan^{-1} \left(\frac{\sqrt{57}}{4} \right)$$

Sol. (b) On comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get $a = 2$, $b = -6$ and $h = -\frac{3}{2}$

Let θ be the angle between the given lines.

$$\text{Then, } \tan \theta = \pm \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\Rightarrow \tan \theta = \pm \frac{2\sqrt{\frac{9}{4} + 12}}{2 - 6} = \mp \frac{\sqrt{57}}{4}$$

$$\Rightarrow \theta = \tan^{-1} \left(-\frac{\sqrt{57}}{4} \right)$$

Condition for the Lines to be Coincident

The lines are coincident, if the angle between them is zero.

Thus, lines are coincident.

$$\Leftrightarrow \theta = 0$$

$$\Leftrightarrow \tan \theta = 0$$

$$\Leftrightarrow \frac{2\sqrt{h^2 - ab}}{a + b} = 0$$

$$\Leftrightarrow h^2 - ab = 0$$

$$\Leftrightarrow h^2 = ab$$

The lines represented by $ax^2 + 2hxy + by^2 = 0$ are coincident iff $h^2 = ab$.

➤ **Example 48.** If the lines represented by $2x^2 + 4xy + by^2 = 0$ are coincident, then find the value of b .

Sol. Since, the lines represented by $2x^2 + 4xy + by^2 = 0$ are coincident, therefore we have

$$2^2 = 2 \cdot b$$

$$\Rightarrow b = 2$$

Condition of the Lines to be Perpendicular

The lines are perpendicular, if the angle between them is

$\frac{\pi}{2}$. Thus, lines are perpendicular.

$$\Leftrightarrow \theta = \frac{\pi}{2}$$

$$\Leftrightarrow \cot \theta = \cot \frac{\pi}{2}$$

$$\Leftrightarrow \frac{a + b}{2\sqrt{h^2 - ab}} = 0$$

$$\Leftrightarrow a + b = 0$$

$$\Leftrightarrow \text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

The lines represented by $ax^2 + 2hxy + by^2 = 0$ are perpendicular iff $a + b = 0$ i.e. coefficient of x^2 + coefficient of $y^2 = 0$

➤ **Example 49.** Find the value of a for which the lines represented by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular.

Sol. The lines given by $ax^2 + 5xy + 2y^2 = 0$ are mutually perpendicular, if $a + 2 = 0$, i.e. $a = -2$.

Bisectors of the Angle between the Lines Given by a Homogeneous Equation

The joint equation of the bisectors of the angles between the lines represented by the equation $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$$

$$\Rightarrow hx^2 - hy^2 = (a - b)xy$$

$$\Rightarrow hx^2 - (a - b)xy - hy^2 = 0$$

➤ **Example 50.** The equation of the lines bisecting the angles between the pair of lines $3x^2 + xy - 2y^2 = 0$ is

$$(a) x^2 - 10xy + y^2 = 0$$

$$(b) x^2 - xy - y^2 = 0$$

$$(c) x^2 + 10xy - 5y^2 = 0$$

$$(d) x^2 - 10xy - y^2 = 0$$

Sol. (d) On comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we get

$$a = 3, h = \frac{1}{2} \text{ and } b = -2$$

Using the formula $\frac{x^2 - y^2}{a - b} = \frac{xy}{h}$, we obtain the equation of the bisectors as

$$\frac{x^2 - y^2}{3 - (-2)} = \frac{xy}{1/2}$$

$$\Rightarrow x^2 - 10xy - y^2 = 0$$

General Equation of Second Degree

The necessary and sufficient condition for $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ to represent a pair of straight lines is

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\text{or } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

➤ **Example 51.** The equation

$$3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0 \text{ represents}$$

- (a) a pair of straight lines
(b) a circle
(c) an ellipse
(d) a hyperbola

Sol. (a) The given equation is of the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

On comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we obtain}$$

$$a = 3, h = \frac{7}{2}, b = 2, g = \frac{5}{2}, f = \frac{5}{2} \text{ and } c = 2$$

$$\text{Now, } abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= 12 + 2 \times \frac{5}{2} \times \frac{5}{2} \times \frac{7}{2} - 3 \left(\frac{5}{2}\right)^2 - 2 \left(\frac{5}{2}\right)^2 - 2 \left(\frac{7}{2}\right)^2 = 0$$

Hence, the given equation represents a pair of straight lines.

➤ **Example 52.** The value of h , so that the equation $6x^2 + 2hxy + 12y^2 + 22x + 31y + 20 = 0$ represent two straight lines, is

- (a) $\frac{171}{20}$
(b) $\frac{171}{40}$
(c) $\frac{170}{21}$
(d) $\frac{17}{20}$

Sol. (a) Here, $a = 6, b = 12, g = 11, f = \frac{31}{2}$ and $c = 20$

Since, the given equation represents a pair of straight line, therefore

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0 \quad \dots(i)$$

$$\Rightarrow 20h^2 - 341h + \frac{2907}{2} = 0$$

$$\Rightarrow \left(h - \frac{17}{2}\right)(20h - 171) = 0$$

$$\text{Hence, } h = \frac{17}{2} \text{ or } \frac{171}{20}$$

Angle between the Pair of Lines Given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

The angle θ between pair of lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ can be obtained by using the formula

$$\tan \theta = \pm \frac{2\sqrt{h^2 - ab}}{a + b}$$

and the acute angle between the lines is given by

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right|$$

- The lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are parallel iff $h^2 = ab$ and $af^2 = bg^2$ or $\frac{a}{h} = \frac{h}{b} = \frac{g}{f}$.
- The lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are perpendicular iff $a + b = 0$.
i.e. coefficient of x^2 + coefficient of $y^2 = 0$
- The lines are coincident, if $g^2 = ac$ or $f^2 = bc$.

➤ **Example 53.** Find the angle between the pair of straight lines $x^2 - 5xy + 4y^2 + 3x - 4 = 0$.

Sol. Given equation is

$$x^2 - 5xy + 4y^2 + 3x - 4 = 0 \quad \dots(i)$$

Here,

$$a = \text{Coefficient of } x^2 = 1$$

$$b = \text{Coefficient of } y^2 = 4$$

$$2h = \text{Coefficient of } xy = -5$$

$$\therefore h = -\frac{5}{2}$$

If θ is the acute angle between the pair of line (i), then

$$\tan \theta = \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \left| \frac{2\sqrt{\frac{25}{4} - 4}}{1 + 4} \right| = \frac{3}{5}$$

$$\therefore \theta = \tan^{-1}\left(\frac{3}{5}\right)$$

Thus, acute angle between pair of lines (i) is $\tan^{-1}\left(\frac{3}{5}\right)$

and obtuse angle is $\pi - \tan^{-1}\left(\frac{3}{5}\right)$.

➤ **Example 54.** If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents parallel straight lines, then

- (a) $hf = bg$
(b) $h^2 = bc$
(c) $a^2 f = b^2 g$
(d) None of the above

Sol. (a) The given equation represents a pair of parallel straight lines, if $h^2 = ab$ and $bg^2 = af^2$, which on simplification gives $hf = bg$.

Equations of the Angle Bisectors

The equation of the bisectors of the angles between the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are given by

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h}$$

where, (x_1, y_1) is the point of intersection of the lines represented by the given equation.

Two pairs of straight lines are said to be equally inclined, if both have the same pair of bisectors.

Example 55. To the lines $ax^2 + 2hxy + by^2 = 0$ the lines $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ are

- (a) equally inclined
- (b) perpendicular
- (c) bisector of the angle
- (d) None of the above

Sol. (a) If the two pairs of straight lines have the same bisectors, then the two pairs are equally inclined.

The equation of the bisectors of the angle between the lines given by $ax^2 + 2hxy + by^2 = 0$ is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \quad \dots(i)$$

The equation of the bisectors of the angle between the lines given by $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ is

$$\frac{x^2 - y^2}{a^2 - b^2} = \frac{xy}{h(a+b)} \quad \dots(ii)$$

Clearly, Eqs. (i) and (ii) are the same. The two pairs of straight lines are equally inclined.

Distance between the Pair of Parallel Lines

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then the distance between

them is given by $2\sqrt{\frac{g^2 - ac}{a(a+b)}}$ or $2\sqrt{\frac{f^2 - bc}{b(a+b)}}$.

The distance between parallel lines can be found, by finding their separate equations and then find the distance between them.

Example 56. Distance between two lines represented by the pair of straight lines $x^2 - 6xy + 9y^2 + 3x - 9y - 4 = 0$ is

- (a) $\frac{1}{2}$
- (b) $\frac{5}{\sqrt{2}}$
- (c) $\sqrt{10}$
- (d) $\sqrt{\frac{5}{2}}$

Sol. (d) First check for parallel lines

$$\text{i.e. } \frac{a}{h} = \frac{h}{b} = \frac{g}{f}$$

$$\Rightarrow \frac{1}{-3} = \frac{-3}{9} = \frac{3/2}{-9/2}$$

which is true, hence lines are parallel.

$\therefore$ Distance between the pair of parallel straight lines

$$= 2\sqrt{\frac{g^2 - ac}{a(a+b)}}$$

$$= 2\sqrt{\frac{9}{4} + 4} = \sqrt{\frac{5}{2}}$$

Example 57. Find the distance between the pair of parallel line $x^2 + 4xy + 4y^2 + 3x + 6y - 4 = 0$.

Sol. Given equation of lines can be rewritten as

$$x^2 + (4y + 3)x + (4y^2 + 6y - 4) = 0$$

$$\Rightarrow x = \frac{-(4y + 3) \pm \sqrt{(4y + 3)^2 - 4(4y^2 + 6y - 4)}}{2}$$

$$\Rightarrow x = \frac{-(4y + 3) \pm \sqrt{16y^2 + 9 + 24y - 16y^2 - 24y + 16}}{2}$$

$$\Rightarrow x = \frac{-(4y + 3) \pm \sqrt{25}}{2} \Rightarrow x = \frac{-4y - 3 \pm 5}{2}$$

$$\Rightarrow x = \frac{-4y - 3 + 5}{2} \text{ or } x = \frac{-4y - 3 - 5}{2}$$

$$\Rightarrow x = -2y + 1 \text{ or } x = -2y - 4$$

$$\Rightarrow x + 2y - 1 = 0 \text{ or } x + 2y + 4 = 0$$

$$\therefore \text{ Required distance} = \frac{|4 - (-1)|}{\sqrt{1 + 4}} = \frac{5}{\sqrt{5}} = \sqrt{5}$$

Point of Intersection of Straight Lines

If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then they intersect at the point $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

The point of intersection of the two non-parallel lines can be found, by finding their separate equations and solve them.

Example 58. Find the point of intersection of lines represented by $2x^2 - xy - y^2 + 9x - 3y + 10 = 0$.

Sol. On comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get}$$

$$a = 2, h = -\frac{1}{2}, b = -1, g = \frac{9}{2}, f = -\frac{3}{2} \text{ and } c = 10$$

Now, by using the formula $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$, we get

$$\text{Point of intersection} = \left(\frac{\frac{3}{4} + \frac{9}{2}, \frac{-9}{4} + 3}{-2 - \frac{1}{4}, -2 - \frac{1}{4}}\right)$$

$$= \left(\frac{\frac{21}{4}, \frac{3}{4}}{-\frac{9}{4}, -\frac{9}{4}}\right) = \left(\frac{-7}{3}, \frac{-1}{3}\right)$$

➤ **Example 59.** Prove that the equation $2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0$ represents a pair of straight lines. Find the coordinates of their point of intersection.

Sol. Given equation is

$$2x^2 + 5xy + 3y^2 + 6x + 7y + 4 = 0 \quad \dots(i)$$

Writing the Eq. (i) as a quadratic equation in x , we have

$$2x^2 + (5y + 6)x + 3y^2 + 7y + 4 = 0$$

$$\begin{aligned} x &= \frac{-(5y + 6) \pm \sqrt{(5y + 6)^2 - 4 \cdot 2 \cdot (3y^2 + 7y + 4)}}{4} \\ &= \frac{-(5y + 6) \pm \sqrt{25y^2 + 60y + 36 - 24y^2 - 56y - 32}}{4} \\ &= \frac{-(5y + 6) \pm \sqrt{y^2 + 4y + 4}}{4} \\ &= \frac{-(5y + 6) \pm (y + 2)}{4} \end{aligned}$$

$$\therefore x = \frac{-5y - 6 + y + 2}{4}, \frac{-5y - 6 - y - 2}{4}$$

$$\Rightarrow 4x + 4y + 4 = 0 \quad \text{and} \quad 4x + 6y + 8 = 0$$

$$\Rightarrow x + y + 1 = 0 \quad \text{and} \quad 2x + 3y + 4 = 0$$

Hence, Eq. (i) represents a pair of straight lines whose equations are

$$x + y + 1 = 0 \quad \dots(ii)$$

$$\text{and} \quad 2x + 3y + 4 = 0 \quad \dots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$x = 1 \quad \text{and} \quad y = -2$$

Hence, coordinates of the point of intersection of

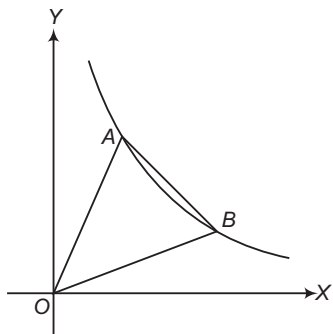
Eqs. (ii) and (iii) is $(1, -2)$.

Lines Joining the Origin to the Points of Intersection of a Line and a Curve

The combined equation of the straight lines joining origin to the points of intersection of a second degree curve

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and a straight line $lx + my + n = 0$ is



$$\begin{aligned} ax^2 + 2hxy + by^2 - 2g\left(\frac{lx + my}{n}\right)x - 2f\left(\frac{lx + my}{n}\right)y \\ + c\left(\frac{lx + my}{n}\right)^2 = 0 \end{aligned}$$

➤ **Example 60.** The angle between the lines joining the origin to the points of intersection of the line $y = 3x + 2$ and the curve $x^2 + 2xy + 3y^2 + 4x + 8y + 11 = 0$ is

$$(a) \frac{1}{2} \tan^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

$$(b) \tan^{-1} \left(\frac{\sqrt{2}}{3} \right)$$

$$(c) \tan^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

$$(d) \tan^{-1} \left(\frac{\sqrt{3}}{2\sqrt{2}} \right)$$

Sol. (c) The equation of the given straight line is

$$y = 3x + 2 \Rightarrow \frac{y - 3x}{2} = 1 \quad \dots(i)$$

The equation of the given curve is

$$x^2 + 2xy + 3y^2 + 4x + 8y + 11 = 0 \quad \dots(ii)$$

The combined equation of the straight lines joining the origin to the points of intersection of Eqs. (i) and (ii) is a homogeneous equation of second degree obtained with the help of Eqs. (i) and (ii).

Making Eq. (ii) homogeneous of the second degree in x and y with the help of Eq. (i), we get

$$\begin{aligned} x^2 + 2xy + 3y^2 + 4x\left(\frac{y - 3x}{2}\right) + 8y\left(\frac{y - 3x}{2}\right) \\ - 11\left(\frac{y - 3x}{2}\right)^2 = 0 \end{aligned}$$

$$\Rightarrow 4x^2 + 8xy + 12y^2 + 2(8y^2 - 12x^2 - 20xy) - 11(y^2 - 6xy + 9x^2) = 0$$

$$\Rightarrow 7x^2 - 2xy - y^2 = 0$$

This is the required equation. On comparing the equation with $ax^2 + 2hxy + by^2 = 0$, we obtain $a = 7$, $b = -1$ and $h = -1$.

Let θ be the required angle. Then,

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\Rightarrow \tan \theta = \frac{2\sqrt{1 + 7}}{7 - 1} = \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \theta = \tan^{-1} \left(\frac{2\sqrt{2}}{3} \right)$$

Work Book Exercise 12.4

- The equation $3x^2 + 2hxy + 3y^2 = 0$ represents a pair of straight lines passing through the origin. The two lines are
 - real and distinct, if $h^2 > 3$
 - real and distinct, if $h^2 > 9$
 - real and coincident, if $h^2 = 3$
 - real and coincident, if $h^2 > 3$
- The equation $ax^2 + by^2 + cx + cy = 0$, $c \neq 0$ represents a pair of straight lines, if
 - $a + b = 0$
 - $a - b = 0$
 - $b + c = 0$
 - None of the above
- The lines represented by the equation $9x^2 + 24xy + 16y^2 + 21x + 28y + 6 = 0$ are
 - parallel
 - coincident
 - perpendicular
 - None of the above
- If the angle between the pair of straight lines represented by the equation $x^2 - 3xy + \lambda y^2 + 3x - 5y + 2 = 0$ is $\tan^{-1}\left(\frac{1}{3}\right)$, where λ is a non-negative real number. Then, λ is
 - 2
 - 0
 - 3
 - 1
- The equation $y^2 - x^2 + 2x - 1 = 0$, represents
 - a pair of straight lines
 - a circle
 - a parabola
 - an ellipse
- The equation $8x^2 + 8xy + 2y^2 + 26x + 13y + 15 = 0$ represents a pair of straight lines. The distance between them is
 - $\frac{7}{\sqrt{15}}$
 - $\frac{7}{2\sqrt{5}}$
 - $\sqrt{\frac{7}{5}}$
 - None of these
- One bisectors of the angle between the lines given by $a(x-1)^2 + 2h(x-1)y + by^2 = 0$ is $2x + y - 2 = 0$. The other bisector is
 - $x - 2y - 1 = 0$
 - $x + 2y + 1 = 0$
 - $2x - y - 1 = 0$
 - None of these
- Two lines represented by the equation $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ will bisect the angles between the other two, if
 - $c + 6a = 0$ and $ab + d = 0$
 - $b + d = 0$ and $a + c = 0$
 - $c + 6a = 0$ and $b + d = 0$
 - None of the above
- The coordinates of the orthocentre of the triangle formed by the lines $2x^2 + 3xy + y^2 = 0$ and $x + y = 1$ are
 - (1, 1)
 - $\left(\frac{1}{2}, \frac{1}{2}\right)$
 - $\left(\frac{1}{3}, \frac{1}{3}\right)$
 - $\left(\frac{1}{4}, \frac{1}{4}\right)$
- The image of the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$ by the line mirror $y = 0$ is
 - $ax^2 + 2hxy + by^2 = 0$
 - $ax^2 - 2hxy - by^2 = 0$
 - $bx^2 + 2hxy + ay^2 = 0$
 - $ax^2 - 2hxy + by^2 = 0$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Given, a straight lines and a fixed point O , a straight line is drawn through O meeting lines in the points $R_1, R_2, R_3, \dots, R_n$ and on it a point R is taken such that $\frac{n}{OR} = \frac{1}{OR_1} + \frac{1}{OR_2} + \dots + \frac{1}{OR_n}$,

then the locus of R is

- (a) a straight line (b) a circle
(c) a parabola (d) None of these

Sol. Let equations of the given lines be $a_i x + b_i y + c_i = 0$, $i = 1, 2, \dots, n$ and the point O be the origin $(0, 0)$. Then, equation of the line through O can be written as

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta} = r, \text{ where } \theta \text{ is the angle made by the line}$$

with the positive direction of X -axis and r is the distance of any point on the line from the origin O .

Let $r, r_1, r_2, \dots, r_n$ be distance of the points $R, R_1, R_2, \dots, R_n$ from $O \Rightarrow OR = r$ and $OR_1 = r_1 \dots$

Then, coordinates of R are $(r \cos \theta, r \sin \theta)$ and of R_i are $(r_i \cos \theta, r_i \sin \theta)$, where $i = 1, 2, \dots, n$.

Since, R_i lies on $a_i x + b_i y + c_i = 0$

$$\Rightarrow a_i r_i \cos \theta + b_i r_i \sin \theta + c_i = 0, \text{ for } i = 1, 2, \dots, n$$

$$\Rightarrow -\frac{a_i}{c_i} \cos \theta - \frac{b_i}{c_i} \sin \theta = \frac{1}{r_i}, i = 1, 2, \dots, n$$

$$\Rightarrow \sum_{i=1}^n \frac{1}{r_i} = -\left[\sum_{i=1}^n \frac{a_i}{c_i} \right] \cos \theta - \left[\sum_{i=1}^n \frac{b_i}{c_i} \right] \sin \theta$$

$$\Rightarrow \frac{n}{r} = -\left[\sum_{i=1}^n \frac{a_i}{c_i} \right] \cos \theta - \left[\sum_{i=1}^n \frac{b_i}{c_i} \right] \sin \theta$$

Hence, the locus of R is

$$\left(\sum_{i=1}^n \frac{a_i}{c_i} \right) x + \left(\sum_{i=1}^n \frac{b_i}{c_i} \right) y + n = 0$$

which is a straight line.

Hence, (a) is the correct answer.

Ex 2. A variable straight line drawn through the point of intersection of lines $\frac{x}{a} + \frac{y}{b} = 1$ and

$\frac{x}{b} + \frac{y}{a} = 1$, meets the coordinate axes in A and

B . Then, the locus of the mid-point of AB is

(a) $xy(a+b) = ab(x+y)$

(b) $2xy(a-b) = ab(x+y)$

(c) $2xy(a-b) = ab(x-y)$

(d) $2xy(a+b) = ab(x+y)$

Sol. The equation of the variable line through the point of intersection of the given lines is of the type

$$\left(\frac{x}{a} + \frac{y}{b} - 1 \right) + k \left(\frac{x}{b} + \frac{y}{a} - 1 \right) = 0$$

$$\Rightarrow \left(\frac{1}{a} + \frac{k}{b} \right) x + \left(\frac{1}{b} + \frac{k}{a} \right) y = (1+k)$$

$$\Rightarrow (ak+b)x + (bk+a)y = ab(1+k)$$

$$\therefore \text{Points } A \text{ and } B \text{ are } \left(\frac{ab(1+k)}{ak+b}, 0 \right) \text{ and}$$

$$\left(0, \frac{ab(1+k)}{bk+a} \right).$$

Let $P(x_1, y_1)$ be the mid-point of AB .

$$\text{Then, } x_1 = \frac{ab(1+k)}{2(ak+b)} \text{ and } y_1 = \frac{ab(1+k)}{2(bk+a)} \dots (i)$$

Therefore, we write Eq. (i) in the following form

$$\frac{1}{x_1} = \frac{2(ak+b)}{ab(1+k)}, \frac{1}{y_1} = \frac{2(bk+a)}{ab(1+k)}$$

$$\Rightarrow \frac{1}{x_1} + \frac{1}{y_1} = \frac{2(a+b)(1+k)}{ab(1+k)}$$

$$\Rightarrow \frac{x_1 + y_1}{x_1 y_1} = \frac{2(a+b)}{ab}$$

Hence, the locus of $P(x_1, y_1)$ is $2xy(a+b) = ab(x+y)$.

Hence, (d) is the correct answer.

Ex 3. The equation of the line passing through the point $P(1, 2)$ and cutting the lines $x + y - 5 = 0$ and $2x - y = 7$ at A and B respectively such that the harmonic mean of PA and PB is 10, is

(a) $y - 2 = \tan \left(\cos^{-1} \frac{11}{\sqrt{146}} - \cos^{-1} \frac{14}{5\sqrt{146}} \right) (x - 1)$

(b) $y + 2 = \tan \left(\cos^{-1} \frac{11}{\sqrt{146}} + \cos^{-1} \frac{14}{5\sqrt{146}} \right) (x + 1)$

(c) $y - 2 = \tan \left(\cos^{-1} \frac{11}{\sqrt{146}} + \cos^{-1} \frac{14}{5\sqrt{146}} \right) (x - 1)$

(d) None of the above

Sol. Let the equation of the line passing through $P(1, 2)$ be

$$y - 2 = m(x - 1) \dots (i)$$

Let $PA = r_1, PB = r_2$

Then, $A = (1 + r_1 \cos \theta, 2 + r_1 \sin \theta)$

and $B = (1 + r_2 \cos \theta, 2 + r_2 \sin \theta)$,

where, $\tan \theta = m$

As A is on the line $x + y - 5 = 0$.

$$\Rightarrow 1 + r_1 \cos \theta + 2 + r_1 \sin \theta - 5 = 0$$

$$\Rightarrow r_1 (\cos \theta + \sin \theta) = 2$$

$$\therefore \frac{1}{r_1} = \frac{\cos \theta + \sin \theta}{2} \dots (ii)$$

As B is on the line $2x - y - 7 = 0$.

$$\Rightarrow 2(1 + r_2 \cos \theta) - (2 + r_2 \sin \theta) - 7 = 0$$

$$\Rightarrow r_2 (2 \cos \theta - \sin \theta) = 7$$

$$\therefore \frac{1}{r_2} = \frac{2 \cos \theta - \sin \theta}{7} \dots (iii)$$

Now, HM of PA and PB is 10.

$$\therefore 10 = \frac{2PA \cdot PB}{PA + PB}$$

$$\Rightarrow 10 = \frac{2r_1 r_2}{r_1 + r_2}$$

$$\Rightarrow \frac{r_1 + r_2}{r_1 r_2} = \frac{1}{5} \Rightarrow \frac{1}{r_1} + \frac{1}{r_2} = \frac{1}{5}$$

$$\Rightarrow \frac{2 \cos \theta - \sin \theta}{7} + \frac{\cos \theta + \sin \theta}{2} = \frac{1}{5}$$

[from Eqs. (i) and (iii)]

$$\Rightarrow 10(2 \cos \theta - \sin \theta) + 35(\cos \theta + \sin \theta) = 14$$

$$\Rightarrow 55 \cos \theta + 25 \sin \theta = 14$$

$$\text{where } \cos \alpha = \frac{55}{\sqrt{55^2 + 25^2}}$$

$$\therefore \cos(\theta - \alpha) = \frac{14}{5\sqrt{146}}$$

$$\Rightarrow \theta = \alpha + \cos^{-1} \frac{14}{5\sqrt{146}}$$

$$= \cos^{-1} \frac{11}{\sqrt{146}} + \cos^{-1} \frac{14}{5\sqrt{146}}$$

From Eq. (i), the equation of the line is

$$y - 2 = \tan \left(\cos^{-1} \frac{11}{\sqrt{146}} + \cos^{-1} \frac{14}{5\sqrt{146}} \right) (x - 1)$$

Hence, (c) is the correct answer.

Ex 4. A triangle has the line $y = m_1 x$ and $y = m_2 x$ for two of its sides with m_1 and m_2 being roots of the equation $bx^2 + 2hx + a = 0$. If $H(a, b)$ is the orthocentre of the triangle, then the equation of the third side is

(a) $(a + b)(ax + by) = ab(a + b - h)$

(b) $(a - b)(ax - by) = ab(a - b - 2h)$

(c) $(a - 2b)(ax + by) = ab(a + b - h)$

(d) $(a + b)(ax + by) = ab(a + b - 2h)$

Sol. The given lines $y = m_1 x$ and $y = m_2 x$ intersect at the origin $O(0, 0)$. Thus, one vertex of the triangle is at the origin O . Therefore, let OAB be the triangle and OA, OB be the lines respectively

$$y = m_1 x \quad \dots(i)$$

$$\text{and } y = m_2 x \quad \dots(ii)$$

Let the equation of the third side AB be

$$y = mx + c \quad \dots(iii)$$

Given that, $H(a, b)$ is the orthocentre of the ΔOAB .

$\therefore OH$ is perpendicular to AB .

$$\Rightarrow \left(\frac{b}{a} \right) \times m = -1$$

$$\Rightarrow m = -\frac{a}{b} \quad \dots(iv)$$

On solving Eq. (iii) with Eqs. (i) and (ii), the coordinates of A are

$$\left(\frac{c}{m_1 - m}, \frac{cm_1}{m_1 - m} \right)$$

$$\text{and those of } B \text{ are } \left(\frac{c}{m_2 - m}, \frac{cm_2}{m_2 - m} \right)$$

Now, equation of the line through A and perpendicular to OB is

$$y - \frac{cm_1}{m_1 - m} = -\frac{1}{m_2} \left(x - \frac{c}{m_1 - m} \right)$$

$$\Rightarrow y = -\frac{x}{m_2} + \frac{c(m_1 m_2 + 1)}{m_2(m_2 - m)} \quad \dots(v)$$

Similarly, equation of line through B and perpendicular to OA is

$$y = -\frac{x}{m_1} + \frac{c(m_1 m_2 + 1)}{m_1(m_2 - m)} \quad \dots(vi)$$

The point of intersection of Eqs. (v) and (vi) is the orthocentre $H(a, b)$.

On subtracting Eq. (vi) from Eq. (v), we get

$$x = a = \frac{-cm(m_1 m_2 + 1)}{(m_1 - m)(m_2 - m)}$$

$$\Rightarrow c = \frac{-[m_1 m_2 - m(m_1 + m_2) + m^2]a}{m(m_1 m_2 + 1)} \quad \dots(vii)$$

Since, m_1 and m_2 are the roots of the equation $bx^2 + 2hx + a = 0$.

$$\therefore m_1 + m_2 = -\frac{2h}{b} \quad \text{and} \quad m_1 m_2 = \frac{a}{b}$$

$$c = \frac{-\left[\frac{a}{b} + \frac{2hm}{b} + m^2 \right]a}{m\left(\frac{a}{b} + 1 \right)} = \frac{-[a + 2hm + bm^2]a}{m(a + b)}$$

From Eq. (iii), the equation of third side AB is

$$y = mx - \frac{(a + 2hm + bm^2)a}{m(a + b)}$$

$$\Rightarrow y = \left(-\frac{a}{b} \right) x - \frac{(a - 2ha/b + ba^2/b^2)a}{\left(-\frac{a}{b} \right)(a + b)}$$

$$\Rightarrow (ax + by)(a + b) = ab(a + b - 2h)$$

Hence, (d) is the correct answer.

Ex 5. The centroid $(\bar{x}, \bar{y})$ of the triangle with sides $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$, is

(a) $\frac{\bar{x}}{bl - hm} = \frac{\bar{y}}{am - hl} = \frac{2}{3(am^2 - 2hlm + bl^2)}$

(b) $\frac{\bar{x}}{bl + hm} = \frac{\bar{y}}{am - hl} = \frac{2}{3(am^2 - 2hlm - bl^2)}$

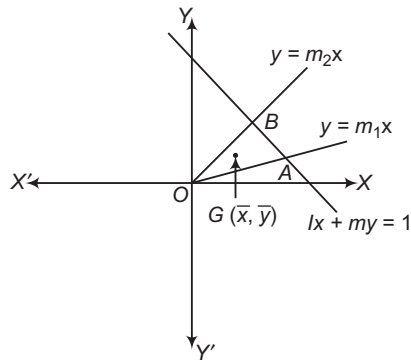
(c) $\frac{\bar{x}}{bl + hm} = \frac{\bar{y}}{am + hl} = \frac{2}{3(am^2 + 2hlm - bl^2)}$

(d) $\frac{\bar{x}}{bl - hm} = \frac{\bar{y}}{am + hl} = \frac{2}{3(am^2 - 2hlm - bl^2)}$

Sol. Let $y = m_1 x$ and $y = m_2 x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$. Then,

$$m_1 + m_2 = -\frac{2h}{b}$$

$$\text{and } m_1 m_2 = \frac{a}{b}$$



On solving $y = m_1x$ and $y = m_2x$ respectively with $lx + my = 1$, we obtain that the coordinates of A and B which are

$$A\left(\frac{1}{l + mm_1}, \frac{m_1}{l + mm_1}\right) \text{ and } B\left(\frac{1}{l + mm_2}, \frac{m_2}{l + mm_2}\right)$$

Since, $C(\bar{x}, \bar{y})$ is the centroid of ΔOAB . Therefore,

$$\bar{x} = \frac{1}{3} \left(\frac{1}{l + mm_1} + \frac{1}{l + mm_2} \right)$$

$$\text{and } \bar{y} = \frac{1}{3} \left(\frac{m_1}{l + mm_1} + \frac{m_2}{l + mm_2} \right)$$

$$\Rightarrow \bar{x} = \frac{1}{3} \left\{ \frac{2l + m(m_1 + m_2)}{l^2 + lm(m_1 + m_2) + m^2m_1m_2} \right\}$$

$$\text{and } \bar{y} = \frac{1}{3} \left\{ \frac{l(m_1 + m_2) + 2mm_1m_2}{l^2 + lm(m_1 + m_2) + m^2m_1m_2} \right\}$$

$$\Rightarrow \bar{x} = \frac{1}{3} \left\{ \frac{2l - \frac{2hm}{b}}{l^2 - \frac{2hlm}{b} + \frac{am^2}{b}} \right\}$$

$$\text{and } \bar{y} = \frac{1}{3} \left\{ \frac{-\frac{2hl}{b} + 2m\frac{a}{b}}{l^2 - \frac{2hlm}{b} + m^2\frac{a}{b}} \right\}$$

$$\Rightarrow \bar{x} = \frac{2}{3} \cdot \frac{bl - hm}{am^2 - 2hlm + bl^2}$$

$$\text{and } \bar{y} = \frac{2}{3} \cdot \frac{am - hl}{am^2 - 2hlm + bl^2}$$

$$\Rightarrow \frac{\bar{x}}{bl - hm} = \frac{\bar{y}}{am - hl} = \frac{2}{3(am^2 - 2hlm + bl^2)}$$

Hence, (a) is the correct answer.

Ex 6. The orthocentre of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is

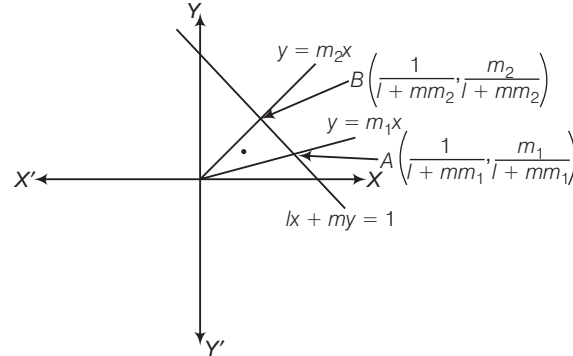
$$(a) \frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 + 2hlm + bl^2}$$

$$(b) \frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 + 2hlm - bl^2}$$

$$(c) \frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 - 2hlm - bl^2}$$

$$(d) \frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}$$

Sol. Let $y = m_1x$ and $y = m_2x$ be the lines represented by $ax^2 + 2hxy + by^2 = 0$, then



$$m_1 + m_2 = -\frac{2h}{b} \text{ and } m_1m_2 = \frac{a}{b} \quad \dots(i)$$

Let OAB be the triangle formed by the lines

$$y = m_1x, y = m_2x \text{ and } lx + my = 1$$

The coordinates of the vertices of ΔOAB are $O(0, 0)$,

$$A\left(\frac{1}{l + mm_1}, \frac{m_1}{l + mm_1}\right) \text{ and } B\left(\frac{1}{l + mm_2}, \frac{m_2}{l + mm_2}\right)$$

The equation of the altitude through A is

$$y - \frac{m_1}{l + mm_1} = -\frac{1}{m_2} \left(x - \frac{1}{l + mm_1} \right)$$

$$\Rightarrow x + m_2y = \frac{1 + m_1m_2}{1 + mm_1} \quad \dots(ii)$$

Similarly, the equation of the altitude through B is

$$x + m_1y = \frac{1 + m_1m_2}{l + mm_2} \quad \dots(iii)$$

From Eqs. (ii) and (iii), we get

$$\frac{\frac{x}{l(m_1 - m_2)(1 + m_1m_2)}}{(l + mm_1)(l + mm_2)} = \frac{\frac{y}{m(m_1 - m_2)(1 + m_1m_2)}}{(l + mm_1)(l + mm_2)} = \frac{1}{m_1 - m_2}$$

$$\Rightarrow \frac{x}{l} = \frac{y}{m} = \frac{1 + m_1m_2}{l^2 + lm(m_1 + m_2) + m^2m_1m_2}$$

$$\Rightarrow \frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 - 2hlm + bl^2} \quad [\text{using Eq. (i)}]$$

Hence, the coordinates (x, y) of the orthocentre of ΔOAB are given by

$$\frac{x}{l} = \frac{y}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}$$

Hence, (d) is the correct answer.

Ex 7. The straight lines represented by

$$(y - mx)^2 = a^2(1 + m^2) \text{ and }$$

$$(y - nx)^2 = a^2(1 + n^2), \text{ is}$$

- (a) square (b) rhombus
(c) rectangle (d) None of these

Sol. We have,

$$(y - mx)^2 = a^2(1 + m^2) \text{ and } (y - nx)^2 = a^2(1 + n^2)$$

$$\Rightarrow y - mx = \pm a\sqrt{1 + m^2} \text{ and } y - nx = \pm a\sqrt{1 + n^2}$$

$$\Rightarrow y = mx \pm a\sqrt{1+m^2} \text{ and } y = nx \pm a\sqrt{1+n^2}$$

Thus, the equation of the line represented by the given equation are

$$y = mx + a\sqrt{1+m^2} \quad \dots(i)$$

$$y = mx - a\sqrt{1+m^2} \quad \dots(ii)$$

$$y = nx + a\sqrt{1+n^2} \quad \dots(iii)$$

$$y = nx - a\sqrt{1+n^2} \quad \dots(iv)$$

Clearly, lines (iii) and (iv) are parallel and the distance between them is given by

$$d_1 = \left| \frac{a\sqrt{1+m^2} + a\sqrt{1+m^2}}{\sqrt{1+m^2}} \right| = |2a|$$

Similarly, lines (iii) and (iv) are parallel and the distance between them is given by

$$d_2 = \left| \frac{a\sqrt{1+n^2} + a\sqrt{1+n^2}}{\sqrt{1+n^2}} \right| = |2a|$$

Clearly, $d_1 = d_2$

The lines (i), (ii), (iii) and (iv) form a rhombus.

Hence, (b) is the correct answer.

Ex 8. The equation $\lambda(x^3 - 3xy^2) + y^3 - 3x^2y = 0$ represents

- (a) three lines equally inclined to the other
- (b) two lines equally inclined to each other
- (c) no line equally inclined to each other
- (d) None of the above

Sol. The given equation is a homogeneous cubic equation in x, y . So, it represents three lines passing through the origin. Let one of the three lines be $y = mx$, where $m = \tan\theta$, θ is the angle made by the line with X -axis. Then, $\lambda(x^3 - 3m^2x^3) + m^3x^3 - 3mx^3 = 0$

$$\Rightarrow \lambda(1 - 3m^2) + m^3 - 3m = 0$$

$$\Rightarrow \lambda = \frac{3m - m^3}{1 - 3m^2} \quad [\because m = \tan\theta]$$

$$\Rightarrow \lambda = \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$$

$$\Rightarrow \lambda = \tan 3\theta$$

$$\Rightarrow \tan 3\theta = \tan \phi,$$

$$\text{where, } \lambda = \tan \phi \text{ or } \phi = \tan^{-1} \lambda$$

$$\Rightarrow 3\theta = \phi, \pi + \phi, 2\pi + \phi$$

$$\Rightarrow \theta = \frac{\phi}{3}, \frac{\pi}{3} + \frac{\phi}{3}, \frac{2\pi}{3} + \frac{\phi}{3}$$

Thus, the given equation represents three lines passing through the origin and the lines make angles $\frac{\phi}{3}, \frac{\pi}{3} + \frac{\phi}{3}$

and $\frac{2\pi}{3} + \frac{\phi}{3}$ with X -axis.

Clearly, they are equally inclined with each other.

Hence, (a) is the correct answer.

Ex 9. A point moves in such a way that the distance between the feet of the perpendiculars drawn from it to the lines $ax^2 + 2hxy + by^2 = 0$ is a constant 2λ . Then, the locus of the point is

$$(a) (x^2 + y^2)(h^2 + ab) = \lambda^2[(a+b)^2 - 4h^2]$$

$$(b) (x^2 + y^2)(h^2 - ab) = \lambda^2[(a-b)^2 + 4h^2]$$

$$(c) (x^2 - y^2)(h^2 - ab) = \lambda^2[(a-b)^2 - 4h^2]$$

(d) None of the above

Sol. Let the point be $P(\alpha, \beta)$. Then,

$$2\lambda = 2\sqrt{\frac{(h^2 - ab)(\alpha^2 + \beta^2)}{(a-b)^2 + 4h^2}}$$

$$\Rightarrow \lambda^2[(a-b)^2 + 4h^2] = (\alpha^2 + \beta^2)(h^2 - ab)$$

Hence, the locus of (α, β) is

$$(x^2 + y^2)(h^2 - ab) = \lambda^2[(a-b)^2 + 4h^2]$$

Hence, (b) is the correct answer.

Ex 10. The condition that the pair of straight line joining the origin to the intersections of the line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ is at the right angles, is

$$(a) 2c^2 = a^2(1 - m^2) \quad (b) 2c^2 = a^2(1 + m^2)$$

$$(c) c^2 = a^2(1 + m^2) \quad (d) \text{None of these}$$

Sol. The equations of the given straight line and the given curve are

$$y = mx + c \text{ or } \frac{y - mx}{c} = 0 \quad \dots(i)$$

$$\text{and } x^2 + y^2 = a^2 \quad \dots(ii)$$

The combined equation of the straight lines joining the origin to the points of intersection of Eqs. (i) and (ii) is

$$x^2 + y^2 = \left(\frac{y - mx}{c} \right)^2 a^2$$

$$\Rightarrow x^2(c^2 - a^2m^2) + 2a^2mxy + y^2(c^2 - a^2) = 0 \quad \dots(iii)$$

The lines given by Eq. (iii) are at right angles, if Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow (c^2 - a^2m^2) + (c^2 - a^2) = 0$$

$$\Rightarrow 2c^2 = a^2(1 + m^2), \text{ which is the required condition.}$$

Hence, (b) is the correct answer.

Ex 11. All chords of the curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin pass through a fixed point. Then, the point is

$$(a) (-1, 2)$$

$$(b) (-1, -2)$$

$$(c) (1, -2)$$

$$(d) (1, 2)$$

Sol. Let $Ax + By = 1$ be a chord of the curve $3x^2 - y^2 - 2x + 4y = 0$. The combined equation of the straight lines joining the origin to the points of intersection of the lines $Ax + By = 1$ and the curve $3x^2 - y^2 - 2x + 4y = 0$ is

$$3x^2 - y^2 - 2x(Ax + By) + 4y(Ax + By) = 0$$

$$\Rightarrow x^2(3 - 2A) + 2xy(2A - B) + y^2(4B - 1) = 0 \quad \dots(i)$$

Since, the lines represented by Eq. (i) are at right angles.

$$\therefore \text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow 3 - 2A + 4B - 1 = 0$$

$$\Rightarrow 2A - 4B = 2$$

$$\Rightarrow A - 2B = 1$$

$$\Rightarrow A = 2B + 1$$

On putting $A = 2B + 1$ in $Ax + By = 1$, we get

$$\begin{aligned} x(2B + 1) + By &= 1 \\ \Rightarrow (x - 1) + B(2x + y) &= 0 \end{aligned}$$

Clearly, it is a line passing through the intersection of the lines $x - 1 = 0$ and $2x + y = 0$ i.e. $(1, -2)$.

Thus, the required point is $(1, -2)$.

Hence, (c) is the correct answer.

Ex 12. The lines joining the origin to the other two points of intersection of the curves $ax^2 + 2hxy + by^2 + 2gx = 0$ and $a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$ will be at right angles to one another, if

- (a) $g(a' - b') = g'(a + b)$
- (b) $g(a' + b') = g'(a - b)$
- (c) $g(a' - b') = g'(a - b)$
- (d) $g(a' + b') = g'(a + b)$

Sol. The given curves are

$$ax^2 + 2hxy + by^2 = -2gx \quad \dots(i)$$

$$\text{and } a'x^2 + 2h'xy + b'y^2 = -2g'x \quad \dots(ii)$$

The combined equation of the straight lines joining the origin to the points of intersection of Eqs. (i) and (ii) is a homogeneous equation of second degree. To obtain the desired equation we proceed as follows.

On multiplying Eq. (i) by g' and Eq. (ii) by g and subtracting, we get

$$(ag' - a'g)x^2 + 2(hg' - h'g)xy + (bg' - b'g)y^2 = 0 \quad \dots(iii)$$

The lines given by Eq. (iii) are at right angles, if

$$g(a' + b') = g'(a + b)$$

Hence, (d) is the correct answer.

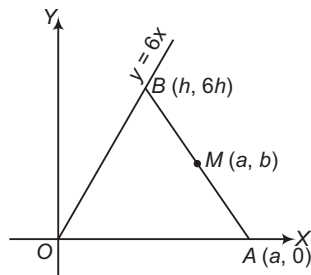
Ex 13. If a line AB of length $2l$ moves with the end A always on the X -axis and the end B always on the line $y = 6x$, then the equation of the locus of the mid-point of AB is

- (a) $9x^2 - 6xy - 10y^2 = 9l^2$
- (b) $9x^2 - 6xy + 10y^2 = 9l^2$
- (c) $9x^2 + 6xy + 10y^2 = 9l^2$
- (d) $9x^2 + 6xy - 10y^2 = 9l^2$

Sol. Let the coordinates of A be $(a, 0)$ and those of B be $(h, 6h)$.

$$\text{Given, } AB^2 = 4l^2$$

$$\Rightarrow (h - a)^2 + (6h)^2 = 4l^2 \quad \dots(i)$$



Let the coordinates of mid-point M of AB be (α, β) . Then,

$$\alpha = \frac{h + a}{2}, \beta = \frac{6h}{2}$$

$$\Rightarrow 2\alpha = h + a \quad \text{and} \quad \beta = 3h \quad \dots(ii)$$

We have to eliminate a and h from Eqs. (i) and (ii).

$$\text{From Eq. (ii), } h = \frac{\beta}{3} \text{ and } 2\alpha = \frac{\beta}{3} + a = \frac{\beta + 3a}{3}$$

$$\Rightarrow 6\alpha = \beta + 3a \Rightarrow a = \frac{6\alpha - \beta}{3}$$

Now, from Eq. (i),

$$\begin{aligned} 4l^2 &= \left(\frac{\beta}{3} - \frac{6\alpha - \beta}{3} \right)^2 + 36 \cdot \frac{\beta^2}{9} \\ \Rightarrow \frac{(2\beta - 6\alpha)^2}{9} + 4\beta^2 &= \frac{(2\beta - 6\alpha)^2 + 36\beta^2}{9} \\ \Rightarrow 36l^2 &= 4\beta^2 - 24\alpha\beta + 36\alpha^2 + 36\beta^2 + 36\alpha^2 \\ \Rightarrow 9l^2 &= 9\alpha^2 - 6\alpha\beta + 10\beta^2 \end{aligned}$$

Hence, the locus of (α, β) is

$$9x^2 - 6xy + 10y^2 = 9l^2$$

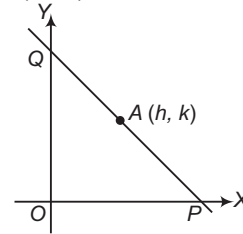
Hence, (b) is the correct answer.

Ex 14. Let (h, k) be a fixed point, where $h > 0, k > 0$.

A straight line passing through this point cuts the positive direction of the coordinate axes at the points P and Q . Then, the minimum area of the ΔOPQ , O being the origin, is

- (a) $4hk$ sq units
- (b) $2hk$ sq units
- (c) $3hk$ sq units
- (d) None of these

Sol. Let the equation of any line passing through $a(h, k)$ be $y - k = m(x - h)$.



Let this line cut the X -axis and Y -axis at P and Q .

$$\text{Then, } P \equiv \left(h - \frac{k}{m}, 0 \right) \text{ and } Q \equiv (0, k - mh)$$

Let S be the area of ΔOPQ , then

$$\begin{aligned} S &= \frac{1}{2} OP \times OQ = \frac{1}{2} \left(h - \frac{k}{m} \right) (k - mh) \\ &= \frac{1}{2} \frac{(mh - k)(k - mh)}{m} \end{aligned}$$

$$\Rightarrow 2mS = hkm - k^2 - h^2m^2 + khm$$

$$\Rightarrow h^2m^2 - 2(hk - S)m + k^2 = 0$$

Since, m is real.

$$\therefore \text{Discriminant, } D \geq 0$$

$$\Rightarrow 4(hk - S)^2 - 4h^2k^2 \geq 0$$

$$\Rightarrow S - 2hk \geq 0$$

$$\Rightarrow S \geq 2hk$$

Hence, minimum value of S is $2hk$ sq units.

Hence, (b) is the correct answer.

Ex 15. A straight line passes through a fixed point (h, k) . Then, the locus of the foot of the perpendicular on it from the origin is

- (a) $x^2 + y^2 - hx - ky = 0$
 (b) $x^2 - y^2 - hx + ky = 0$
 (c) $x^2 - y^2 + hx + ky = 0$
 (d) $x^2 + y^2 + hx + ky = 0$

Sol. Let the equation of the straight line be

$$x \cos \alpha + y \sin \alpha - p = 0 \quad \dots(i)$$

Since, the Eq. (i) passes through the point (h, k) , therefore

$$h \cos \alpha + k \sin \alpha - p = 0 \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$(x - h) \cos \alpha + (y - k) \sin \alpha = 0 \quad \dots(iii)$$

Now, the equation of the straight line which is perpendicular to Eq. (i) and passes through the origin, is

$$x \sin \alpha - y \cos \alpha = 0 \quad \dots(iv)$$

The required locus will be obtained by eliminating α from the Eqs. (iii) and (iv).

From Eq. (iv), $x \sin \alpha = y \cos \alpha$

$$\Rightarrow \frac{x}{\cos \alpha} = \frac{y}{\sin \alpha}$$

$$\Rightarrow \frac{x}{\cos \alpha} = \frac{y}{\sin \alpha} = \frac{\sqrt{x^2 + y^2}}{\sqrt{\cos^2 \alpha + \sin^2 \alpha}} = \sqrt{x^2 + y^2}$$

$$\Rightarrow x(x - h) + y(y - k) = 0$$

$$\Rightarrow x^2 - hx + y^2 - ky = 0$$

$$\therefore x^2 + y^2 - hx - ky = 0$$

which is the required locus.

Hence, (a) is the correct answer.

Ex 16. Abscissae and ordinates of n given points are in AP with first term a and common difference 1 and 2, respectively. If algebraic sum of perpendiculars from these given points on a variable line which always passes through the point $\left(\frac{13}{2}, 11\right)$ is zero, then the values of a and

n are

- (a) 2, 10
 (b) 4, 20
 (c) 3, 9
 (d) 0, 1

Sol. Let the n given points be

$(a, a), (a + 1, a + 2), (a + 2, a + 4), \dots$

i.e. $[a + i - 1, a + 2(i - 1)]; i = 1, 2, 3, \dots, n$

Let the variable line be

$$px + qy + r = 0 \quad \dots(i)$$

Since, the algebraic sum of perpendiculars drawn from these n points on the variable line (i) is always zero.

$$\therefore \sum \frac{p(a + i - 1) + q(a + 2i - 2) + r}{\sqrt{p^2 + q^2}} = 0$$

$$\Rightarrow \sum [p(a + i - 1) + q(a + 2i - 2) + r] = 0$$

$$\Rightarrow p \sum (a + i - 1) + q \sum (a + 2i - 2) + rn = 0$$

$$\Rightarrow p \sum \frac{a + i - 1}{n} + q \sum \frac{a + 2i - 2}{n} + r = 0$$

Hence, the line (i) always passes through a fixed point $\left(\sum \frac{a + i - 1}{n}, \sum \frac{a + 2i - 2}{n}\right)$.

But it is given that this passes through the point $\left(\frac{13}{2}, 11\right)$.

$$\therefore \sum \frac{a + i - 1}{n} = \frac{13}{2}$$

$$\text{and} \quad \sum \frac{a + 2i - 2}{n} = 11$$

On solving the above equations, we get

$$a = 2 \quad \text{and} \quad n = 10$$

Hence, (a) is the correct answer.

Ex 17. A variable line is drawn through the origin O . Two points A and B are taken on the line such that $OA = 1$ unit and $OB = 2$ units. Through points A and B two lines are drawn making equal angle ' α ' with the line AB . Then, the locus of the point of intersection of the lines is

$$(a) x^2 + y^2 = \frac{9 + \tan^2 \alpha}{4}$$

$$(b) x^2 - y^2 = \frac{9 - \tan^2 \alpha}{4}$$

$$(c) x^2 + y^2 = \frac{9 + \tan^2 \alpha}{2}$$

$$(d) x^2 + y^2 = \frac{9 + 2 \tan^2 \alpha}{4}$$

Sol. Let C be the mid-point of AB .

Given that, $AB = 1$

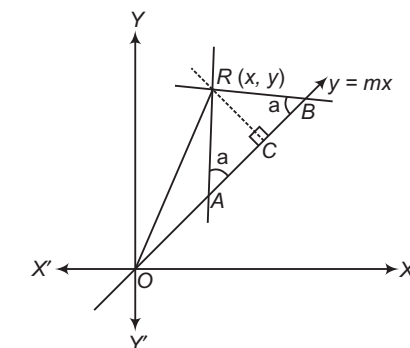
$$\Rightarrow AC = CB = \frac{1}{2}$$

$$OC = \frac{3}{2}$$

$$\text{and} \quad RC = \frac{1}{2} \tan \alpha$$

$$\Rightarrow OR^2 = OC^2 + RC^2$$

$$\Rightarrow x^2 + y^2 = \left(\frac{3}{2}\right)^2 + \frac{1}{4} \tan^2 \alpha$$



$$\Rightarrow x^2 + y^2 = \frac{9 + \tan^2 \alpha}{4} \text{ is the required locus of the}$$

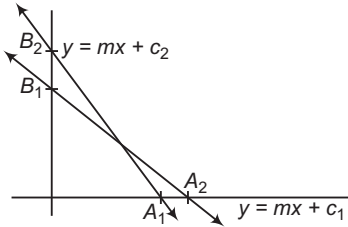
point of intersection of the lines.

Hence, (a) is the correct answer.

Ex 18. Straight lines $y = mx + c_1$ and $y = mx + c_2$, where $m \in R^+$, meet the X -axis at A_1 and A_2 respectively and Y -axis at B_1 and B_2 respectively. It is given that points A_1, A_2, B_1 and B_2 are concyclic. Locus of intersection of lines A_1B_2 and A_2B_1 is

- (a) $y = x$ (b) $y + x = 0$
(c) $xy = 1$ (d) $xy + 1 = 0$

Sol. If A_1, A_2, B_1, B_2 are concyclic.
 $\Rightarrow m^2 - 1 = 0 \Rightarrow m = 1$
 $A_1 \equiv (-c_1, 0), A_2 \equiv (-c_2, 0)$
 $B_1 \equiv (0, c_1), B_2 \equiv (0, c_2)$



Equation of A_1B_2 is $-\frac{x}{c_1} + \frac{y}{c_2} = 1$... (i)

Equation of A_2B_1 is $-\frac{x}{c_2} + \frac{y}{c_1} = 1$... (ii)

The locus of point of intersection of Eqs. (i) and (ii), is

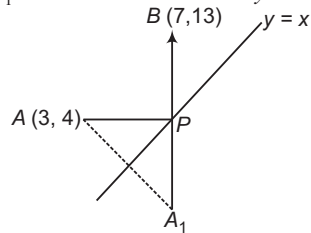
$$y \left(\frac{1}{c_2} - \frac{1}{c_1} \right) + x \left(\frac{1}{c_2} - \frac{1}{c_1} \right) = 0 \Rightarrow x + y = 0$$

Hence, (b) is the correct answer.

Ex 19. Consider the point $A \equiv (3, 4)$ and $B \equiv (7, 13)$. If P is a point of the line $y = x$ such that $PA + PB$ is minimum, then coordinates of P are

- (a) $\left(\frac{13}{7}, \frac{13}{7}\right)$ (b) $\left(\frac{23}{7}, \frac{23}{7}\right)$
(c) $\left(\frac{31}{7}, \frac{31}{7}\right)$ (d) $\left(\frac{33}{7}, \frac{33}{7}\right)$

Sol. Let A_1 be the reflection of A in $y = x$.



$\Rightarrow A_1 = (4, 3)$

Now, $PA + PB = A_1P + PB$, which is minimum, if A_1, P and B are collinear. Equation of A_1B is

$$(y - 3) = \frac{13 - 3}{7 - 4} (x - 4)$$

$\Rightarrow 3y = 10x - 31$

Solving it with $y = x$, we get

$$P \equiv \left(\frac{31}{7}, \frac{31}{7}\right)$$

Hence, (c) is the correct answer.

Ex 20. Consider the point $A \equiv (0, 1)$ and $B \equiv (2, 0)$ and P is a point on the line $4x + 3y + 9 = 0$. Coordinates of the point P such that $|PA - PB|$ is maximum, are

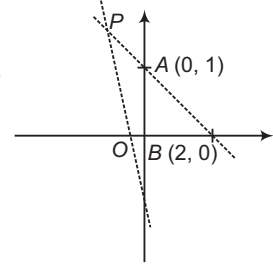
- (a) $\left(-\frac{12}{5}, \frac{17}{5}\right)$ (b) $\left(-\frac{84}{5}, \frac{13}{5}\right)$
(c) $\left(-\frac{6}{5}, \frac{17}{5}\right)$ (d) None of these

Sol. We have, $|PA - PB| \leq AB$.

Thus, for $|PA - PB|$ to be maximum point A, B and P must be collinear equation of AB is $x + 2y = 2$.

Solving it with given line, we get $P \equiv \left(-\frac{84}{5}, \frac{13}{5}\right)$

Hence, (b) is the correct answer.



Ex 21. In question number 20, coordinates of the point P such that $|PA - PB|$ is minimum, are

- (a) $\left(\frac{3}{20}, -\frac{14}{5}\right)$ (b) $\left(\frac{3}{20}, \frac{14}{5}\right)$
(c) $\left(-\frac{3}{30}, \frac{14}{5}\right)$ (d) $\left(-\frac{9}{20}, -\frac{12}{5}\right)$

Sol. Minimum value of $|PA - PB|$ is zero. It can be obtained, if $PA = PB$. That means P must lie on the right bisector of AB .

Equation of right bisector of AB is

$$y - \frac{1}{2} = 2(x - 1) \quad \text{i.e. } y = 2x - \frac{3}{2}$$

Solving with given line, we get

$$P \equiv \left(-\frac{9}{20}, -\frac{12}{5}\right)$$

Hence, (d) is the correct answer.

Ex 22. In the adjacent figure, ΔABC is right angled at B . If $AB = 4$ and $BC = 3$ and side AC slides along the coordinate axes in such a way that B always remains in the first quadrant, then B always lie on the straight line

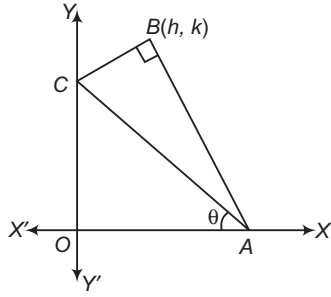
(a) $y = x$ (b) $3y = 3x$
(c) $4y = 3x$ (d) $x + y = 0$

Sol. Given, $AB = 4, BC = 3 \Rightarrow AC = 5$

$$A \equiv (5 \cos \theta, 0), C \equiv (0, 5 \sin \theta)$$

If BC and AB make the angle θ_1 and θ_2 with X -axis, then

$$\begin{aligned} \theta_1 &= C - \theta, \theta_2 = \pi - (A + \theta) \\ &= \pi - \left(\frac{\pi}{2} - C + \theta\right) = \frac{\pi}{2} + (C - \theta) \end{aligned}$$



Using parametric equation of line for BC, we get

$$\frac{h}{\cos(C-\theta)} = \frac{k-5\sin\theta}{\sin(C-\theta)} = 3 \quad \dots(i)$$

Similarly, using parametric equation of line for AB, we get

$$\frac{h}{-\sin(C-\theta)} = \frac{k}{\cos(C-\theta)} = 4 \quad \dots(ii)$$

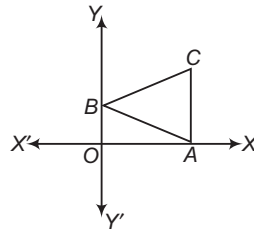
From Eqs. (i) and (ii), we get

$$\cos(C-\theta) = \frac{h}{3} = \frac{k}{4}$$

$\Rightarrow$ Locus of B is $3x = 3y$.

Hence, (b) is the correct answer.

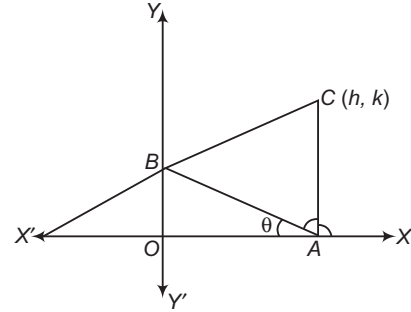
Ex 23. Adjacent figure represents an equilateral $\triangle ABC$ of side length 2 units. Locus of vertex C as the side AB slides along the coordinate axes, is



- (a) $x^2 + y^2 - xy + 1 = 0$
 (b) $x^2 + y^2 + xy\sqrt{3} = 1$
 (c) $x^2 + y^2 = 1 + xy\sqrt{3}$
 (d) $x^2 + y^2 - xy\sqrt{3} + 1 = 0$

Sol. $AB = BC = CA = 2$

Let $\angle BAO = \theta$
 $\Rightarrow A \equiv (2\cos\theta, 0), B \equiv (0, 2\sin\theta)$



BC makes an angle $\left(\frac{\pi}{3} - \theta\right)$ with X-axis and

AC makes an angle $\left[\pi - \left(\frac{\pi}{3} + \theta\right)\right]$ with X-axis.

If $C \equiv (h, k)$, then

$$\frac{h}{\cos\left(\frac{\pi}{3} - \theta\right)} = \frac{k - 2\sin\theta}{\sin\left(\frac{\pi}{3} - \theta\right)} = 2$$

$$\text{and } \frac{h - 2\cos\theta}{-\cos\left(\frac{\pi}{3} + \theta\right)} = \frac{k}{\sin\left(\frac{\pi}{3} + \theta\right)} = 2$$

$$\Rightarrow 2\cos\left(\frac{\pi}{3} - \theta\right) = h, 2\sin\left(\frac{\pi}{3} + \theta\right) = k$$

$$\Rightarrow \cos\theta + \sqrt{3}\sin\theta = h, \sqrt{3}\cos\theta + \sin\theta = k$$

$$\Rightarrow 2\cos\theta = (\sqrt{3}k - h), 2\sin\theta = (h\sqrt{3} - k)$$

Thus, locus of (h, k) is

$$1 = x^2 + y^2 - xy\sqrt{3} \quad \text{or} \quad x^2 + y^2 = 1 + xy\sqrt{3}$$

Hence, (c) is the correct answer.

Type 2. More than One Correct Option

Ex 24. If the equation $ax^2 - 6xy + y^2 + bx + cy + d = 0$ represents pair of lines whose slopes are m and m^2 , then value of a is/are

- (a) $a = -8$ (b) $a = 8$
 (c) $a = 27$ (d) $a = -27$

Sol. m and m^2 are roots of equation

$$\left(\frac{y}{x}\right)^2 - 6\left(\frac{y}{x}\right) + a = 0$$

$$\Rightarrow m + m^2 = 6$$

$$\Rightarrow m^3 = a$$

$$\Rightarrow m^3 + m^6 + 3m^3(m + m^2) = 216$$

$$\Rightarrow a + a^2 + 3a \cdot 6 = 216$$

$$\Rightarrow a^2 + 19a - 216 = 0$$

$$\therefore a = 8, -27$$

Hence, (b) and (d) are the correct answers.

Ex 25. Two sides of a triangle are the lines $(a+b)x + (a-b)y - 2ab = 0$ and $(a-b)x + (a+b)y - 2ab = 0$. If the triangle is isosceles and the third side passes through point $(b-a, a-b)$, then the equation of third side can be

- (a) $x + y = 0$ (b) $x = y + 2(b-a)$
 (c) $x - b + a = 0$ (d) $y - a + b = 0$

Sol. We have, $(a+b)x + (a-b)y - 2ab = 0$

$$\text{and } (a-b)x + (a+b)y - 2ab = 0$$

Equation of the angle bisectors are

$$(a+b)x + (a-b)y - 2ab$$

$$= \pm \{(a-b)x + (a+b)y - 2ab\}$$

$$\Rightarrow 2bx - 2by = 0$$

$$\text{i.e. } x = y$$

and $2ax + 2ay - 4ab = 0$

i.e. $x + y = 2b$

∴ Equation of third side is given by

(ii) $x - y = k$ satisfying the point $(b - a, a - b)$

∴ $k = 2b - 2a$

So, the line is $x - y = 2(b - a)$.

(ii) $x + y - 2b = k$ passing through the point $(b - a, a - b)$

∴ $k = -2b$

So, the line is $x + y = 0$.

Hence, (a) and (b) are the correct answers.

Ex 26. Line $\frac{x}{a} + \frac{y}{b} = 1$ cuts the coordinate axes at

$A(a, 0)$ and $B(0, b)$ and the line $\frac{x}{a'} + \frac{y}{b'} = -1$

at $A'(-a', 0)$ and $B'(0, -b')$. If the point A, B, A', B' are concyclic, then the orthocentre of the $\triangle ABA'$ is

- (a) $(0, 0)$ (b) $(0, b)$
(c) $\left(0, \frac{aa'}{b}\right)$ (d) $\left(0, \frac{bb'}{a}\right)$

Sol. $\frac{x}{a} + \frac{y}{b} = 1$

$-aa' = -bb'$

i.e.

$aa' = bb'$

...(i)

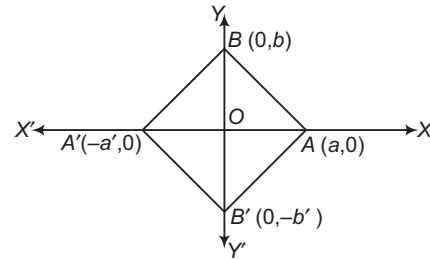
Equation of the altitude through A' is

$y - 0 = \frac{a}{b}(x + a')$

If it intersects the altitude $x = 0$ at

$y = \frac{aa'}{b}$

∴ Orthocentre is $\left(0, \frac{aa'}{b}\right)$.



which is the same as $(0, b')$ by Eq. (i).

Hence, (b) and (c) are the correct answers.

Type 3. Assertion and Reason

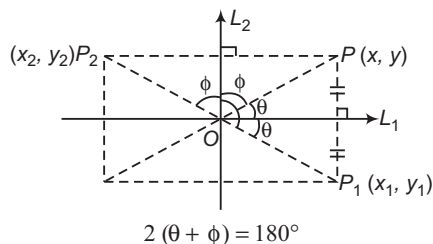
Directions (Ex. Nos. 27-30) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 27. Statement I If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are the images of point $P(x, y)$ about lines $L_1 \equiv ax + by + c = 0$ and $L_2 \equiv bx - ay + c' = 0$ respectively, then the line joining points P_1 and P_2 always passes through point of intersection of lines L_1 and L_2 .

Statement II Lines L_1 and L_2 are perpendicular.

Sol. Let the line joining P_1 and P_2 passes through intersection of L_1 and L_2 , then from the figure,



$\Rightarrow \theta + \phi = 90^\circ$, which is true as L_1 and L_2 are perpendicular.

∴ Both the statements are true and Statement II is correct explanation of Statement I.

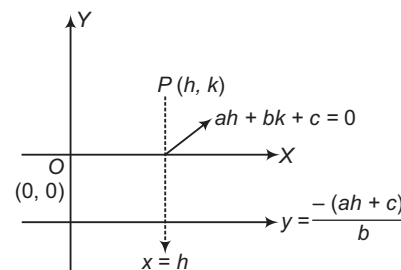
Hence, (a) is the correct answer.

Ex 28. Statement I If $ah + bk + c = 0$, then the point (h, k) lies above the line $ax + by + c = 0$ where a, b, c are non-zero real numbers.

Statement II If $x = h$, $y = \frac{-(ah + c)}{b}$ which is less than k (for $b > 0$) i.e. $\frac{-(ah + c)}{b} < k$, hence (h, k) lies above the line.

Sol. Statement I is false, as Statement II is true from figure $O(0, 0)$ and $P(h, k)$ lie on same side for

$ax + by + c = 0$



$\Rightarrow ah + bk + L > 0$ or $k > \frac{-(ah + c)}{b}$

Hence, (d) is the correct answer.

Ex 29. Statement I A homogeneous equation in x and y of degree n represents family of line (not more than n), all intersecting at origin.

Statement II Substituting $\frac{y}{x} = m$, gives a polynomial in m of degree n , which can have maximum n real roots of the form $y - m_1x = 0$, $y - m_2x = 0$, ...

Sol. Statement II is true and correct explanation of Statement I.

Hence, (a) is the correct answer.

Ex 30. Statement I Two mutually perpendicular lines intersect with coordinate axes, then the angle which one makes with positive X -axis anti-clockwise is equal to angle that other one makes with positive Y -axis anti-clockwise.

Statement II All quadrilaterals that can be formed by points of intersection of pair of lines with coordinate axes are concyclic.

Sol. Statement II is false as the quadrilateral formed by point of intersection of pair of lines with coordinate axes are concyclic only when they are perpendicular.

Hence, (c) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 31-33) The line $6x + 8y = 48$ intersects the coordinate axes at A and B , respectively. A line L bisects the area and the perimeter of the ΔOAB , where O is the origin.

Ex 31. The number of such lines possible is

- (a) 1
- (b) 2
- (c) 3
- (d) more than 3

Ex 32. The slope of the line L can be

- (a) $\frac{10 + 5\sqrt{3}}{10}$
- (b) $\frac{10 - 5\sqrt{6}}{10}$
- (c) $\frac{8 + 3\sqrt{6}}{10}$
- (d) None of the above

Ex 33. The line L does not intersect the side ... of the ΔOAB .

- (a) AB
- (b) OB
- (c) OA
- (d) can intersect all the sides

Sol. (Ex. Nos. 31-33)

Case I Let the line L cuts AO and AB at distances X and Y from A .

$\therefore$ Area of the triangle with sides x and y ,

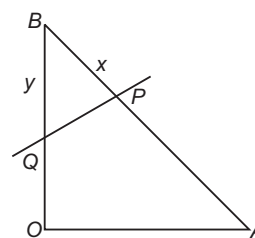
$$\frac{3xy}{10} = 12 \Rightarrow xy = 40$$

Also, $x + y = 12$ [using perimeter bisection] which is not possible.

Case II If the line L cuts OB and BA at distances y and x from B , then we have $xy = 30$ and $x + y = 12$.

$$\Rightarrow x = 6 + \sqrt{6} \text{ and } y = 6 - \sqrt{6}$$

Case III If the line L cuts the sides OA and OB at distances x and y from O , then $x + y = 12$ and $xy = 24$.



$$x, y = 6 \pm 2\sqrt{3} \quad [\text{not possible}]$$

So, there is a unique line possible.

Let point P be (α, β) .

Using parametric equation AB ,

$$\beta = 6 - \frac{3}{5}(6 + \sqrt{6})$$

$$\text{and } \alpha = \frac{4}{5}(6 + \sqrt{6})$$

$$\therefore \text{Slope of } PQ = \frac{\beta - 0}{\alpha - 0} = \frac{10 - 5\sqrt{6}}{10}$$

31. (a) 32. (b) 33. (c)

Passage II (Ex. Nos. 34-36) Given the equation of two sides of a square as $5x + 12y - 10 = 0$,

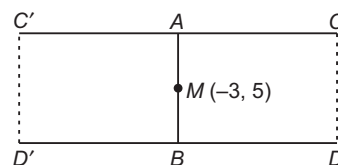
$5x + 12y + 29 = 0$. Also, given a point $M(-3, 5)$ lying on one of its side.

Ex 34. Number of possible squares are

- (a) 0
- (b) 1
- (c) 2
- (d) 3

Sol. Given, $5x + 12y - 10 = 0$

and $5x + 12y + 29 = 0$



Square can be $ABCD$ or $ABC'D'$.

Hence, (c) is the correct answer.

Ex 35. Area of square is

- (a) 3
(b) 6
(c) 9
(d) 12

Sol. $\therefore A = d^2$ [d = distance between parallel sides]

$$d = \frac{39}{\sqrt{5^2 + 12^2}} = 3$$

$$\Rightarrow A = 9$$

Hence, (c) is the correct answer.

Ex 36. If the possible equation of the remaining sides is $12x - 5y + \lambda = 0$, then λ cannot be

- (a) 61 (b) 22
(c) 100 (d) 36

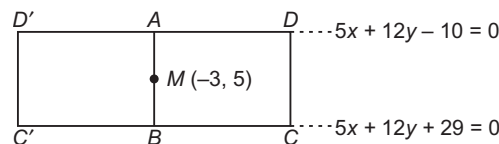
Sol. $AB (y - 5) = (x + 3) \left(\frac{12}{5} \right)$

$$\Rightarrow 5y - 25 = 12x + 36$$

$$\Rightarrow 12x - 5y + 61 = 0$$

Now, CD and CD'

$$\Rightarrow 12x - 5y + \lambda = 0$$



$$\therefore d = 3 = \frac{|61 - \lambda|}{13}$$

$$\Rightarrow 61 - \lambda = \pm 39$$

$$\lambda = 100, 22$$

$$\text{So, } \lambda = 100, 22, 61$$

Hence, (d) is the correct answer.

Type 5. Match the Column

Ex 37. Match the following :

Column I	Column II
A. $ax + by + c = 0$ is a variable straight line, where a, b, c are 1st, 4th, 6th terms of an increasing AP. Then, variable straight line always passes through a fixed point	p. $(x + y - 3)^2 + (x - y + 1)^2 = 20$
B. A variable line L is drawn through $P(2, 3)$ to meet lines $y - x - 10 = 0$ and $y - x - 20 = 0$ at points A and B respectively. A point Q is taken on L such that $PQ^2 = PA \cdot PB$. Locus of Q is	q. $(0, 3)$
C. Vertices of a triangle are $(1, 2)$, $(\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$ and $(\sqrt{5}\sin\theta, -\sqrt{5}\cos\theta)$. Locus of its orthocentre is	r. $\left(\frac{2}{3}, -\frac{5}{3}\right)$
D. A ray of light emerging from the point source placed at $(1, 2)$ is reflected at a point Q on the Y -axis and then passes through the point $(6, 9)$. Coordinates of Q are	s. $(y - x - 1)^2 = 171$

Sol. A. $b = a + 3d, c = a + 5d$

$$b - a = 3d, c - a = 5d$$

$$\Rightarrow \frac{b - a}{3} = \frac{c - a}{5}$$

$$\Rightarrow 5b - 5a = 3c - 3a \Rightarrow 2a - 5b + 3c = 0$$

$$\Rightarrow \frac{2}{3}a - \frac{5}{3}b + c = 0$$

So, straight line passes through $\left(\frac{2}{3}, -\frac{5}{3}\right)$.

B. $\frac{x - 2}{\cos\theta} = \frac{y - 3}{\sin\theta} = r$

$$\Rightarrow x = 2 + r\cos\theta, y = 3 + r\sin\theta$$

$$\Rightarrow 3 + r\sin\theta = 2 + r\cos\theta + 10$$

$$\Rightarrow r(\sin\theta - \cos\theta) = 12 - 3 = 9$$

$$PA = \frac{9}{\sin\theta - \cos\theta} \quad \dots(i)$$

Similarly, $PB = \frac{19}{\sin\theta - \cos\theta} \quad \dots(ii)$

and $PQ = r$
 $h = 2 + r\cos\theta, k = 3 + r\sin\theta$
 $r^2 = \left(\frac{9}{\sin\theta - \cos\theta}\right) \left(\frac{19}{\sin\theta - \cos\theta}\right)$

$$\Rightarrow (r\sin\theta - r\cos\theta)^2 = 19 \times 9$$

$$\Rightarrow (k - 3 - h + 2)^2 = 171$$

$$\Rightarrow (y - x - 1)^2 = 171$$

C. Let $A \equiv (1, 2), B \equiv (\sqrt{5}\cos\theta, \sqrt{5}\sin\theta)$,

$$C \equiv (\sqrt{5}\sin\theta, -\sqrt{5}\cos\theta)$$

Distance of A, B, C from $(0, 0)$ is $\sqrt{5}$ units.

$\Rightarrow (0, 0)$ is circumcentre C of $\triangle ABC$.

Let $C(h, k)$ be centroid of triangle.

$$3h = 1 + \sqrt{5}(\cos\theta + \sin\theta)$$

$$3k = 2 + \sqrt{5}(\sin\theta - \cos\theta)$$

If O is orthocentre of triangle having coordinates (α, β) .

$$OG : GC = 2 : 1$$

$$\Rightarrow \alpha = 3h, \beta = 3k$$

$$\cos\theta + \sin\theta = \frac{\alpha - 1}{\sqrt{5}} \quad \dots(i)$$

$$\sin\theta - \cos\theta = \frac{\beta - 2}{\sqrt{5}} \quad \dots(ii)$$

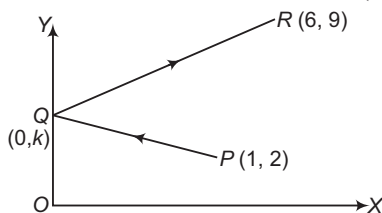
$$\Rightarrow \sin\theta = \frac{\alpha + \beta - 3}{2\sqrt{5}}$$

$$\Rightarrow \cos\theta = \frac{\alpha - \beta + 1}{2\sqrt{5}}$$

$$\left(\frac{\alpha + \beta - 3}{2\sqrt{5}}\right)^2 + \left(\frac{\alpha - \beta + 1}{2\sqrt{5}}\right)^2 = 1$$

$$\therefore \text{Required locus is } (x + y - 3)^2 + (x - y + 1)^2 = 20.$$

D. P' is reflection of P in Y -axis i.e. $P'(-1, 2)$.



Equation of $P'R$ is

$$y - 2 = \frac{9 - 2}{6 - 1}(x + 1)$$

$$\Rightarrow y - 2 = x + 1$$

$$\therefore y = x + 3$$

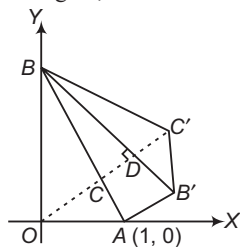
It meets Y -axis at $Q(0, 3)$.

$A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$

Type 6. Single Integer Answer Type Questions

Ex 38. A straight line cuts the X -axis at point $A(1, 0)$ and Y -axis at point B , such that $\angle OAB = \alpha$ ($\alpha > \pi/4$). C is a middle point of AB , if B' is a mirror image of point B with respect to line OC and C' is a mirror image of point C with respect to line BB' , then the ratio of the areas of $\triangle ABB'$ and $BB'C$ is _____.

Sol. (2) From the figure, $OB = \tan \alpha$



Coordinates of point C

$$= \frac{1}{2} \begin{vmatrix} k & 0 & 1 \\ k & k\lambda & 1 \\ \lambda + 1 & \lambda + 1 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 1 & 0 & 1 \\ 1 & \lambda & \lambda \\ \lambda + 1 & \lambda + 1 & 1 \end{vmatrix}$$

So, equation of line OC is $y = x \tan \alpha$ i.e. $\angle COA = \alpha$.

Now,

$$AB = \frac{1}{2} k^2 \left[\left(\frac{\lambda}{\lambda + 1} - \frac{\lambda - 1}{\lambda + 1} \right) + 1 \left(\frac{\lambda - 1}{\lambda + 1} \cdot \frac{1}{\lambda + 1} - 0 \right) \right]$$

$$\therefore AB = \sec \alpha \quad \dots(i)$$

$$\text{So, } AC = BC = \frac{k^2}{2} \left[\left(\frac{1}{\lambda + 1} + \frac{\lambda - 1}{(\lambda + 1)^2} \right) \right]$$

$$= \frac{k^2}{2} \frac{2\lambda}{(\lambda + 1)^2} \sec \alpha$$

$$\text{In } \triangle BDC, \angle BCD = (180^\circ - 2\alpha) \left(\frac{k^2 \lambda}{(\lambda + 1)} \angle COA = \alpha \right)$$

$$\Rightarrow BD = BC \sin (180^\circ - 2\alpha)$$

$$= \frac{3}{8} \times \sin 2\alpha = \frac{1}{3} \times \sec \alpha \times 2 \sin \alpha \times \cos \alpha = \sin \alpha$$

$$\Rightarrow BD = B'D' = \sin \alpha$$

$$\left[\left(0, \frac{k}{2} \right) B' \text{ is a mirror image of } B \text{ in line } OC \right]$$

Clearly, $\triangle BDC$ and $\triangle B'D'C'$ are congruent.

$$BC' = BC = \frac{1}{2} \sec \alpha \text{ and } \angle ABB' = \angle B'BC' \quad \dots(ii)$$

$$\therefore \frac{\text{Area of } \triangle ABB'}{\text{Area of } \triangle BB'C'} = \frac{AB \times BB' \times \sin \angle ABB'}{BB' \times BC' \times \sin \angle B'BC'} \quad [\because \angle ABB' = \angle B'BC']$$

$$= \frac{AB}{BC'} = \frac{\sec \alpha}{\frac{1}{2} \sec \alpha} = 2 \quad [\text{from Eqs. (i) and (ii)}]$$

Ex 39. Sides of a rhombus are parallel to the lines $x + y - 1 = 0$ and $7x - y - 5 = 0$. It is given that diagonal of the rhombus intersect at $(1, 3)$ and one vertex A of the rhombus lies on the line $y = 2x$. Then, the number of all possible coordinates of the vertex A is _____.

Sol. (2) It is clear that diagonals on the rhombus will be parallel to the bisectors of the given lines and will pass through $(1, 3)$.

Equations of bisectors of the given lines are

$$\frac{x + y - 1}{\sqrt{2}} = \pm \left(\frac{7x - y - 5}{5\sqrt{2}} \right)$$

$$\text{or } 2x - 6y = 0, 6x + 2y = 5$$

The equations of diagonal are $x - 3y + 8 = 0$ and $3x + y - 6 = 0$.

Thus, the required vertex will be the point where these lines meet the line $y = 2x$.

Solving these lines, we get possible coordinates as

$$\left(\frac{8}{5}, \frac{16}{5} \right) \text{ or } \left(\frac{6}{5}, \frac{12}{5} \right).$$

Hence, the number of possible coordinates are 2.

Ex 40. If the lines $ax + y + 1 = 0$, $x + by + 1 = 0$ and $x + y + c = 0$ (a, b and c being distinct and different from 1) are concurrent, then $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is equal to _____.

Sol. (1) Since, the given lines are concurrent

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

Operating $C_2 \rightarrow C_2 - C_1$ and $C_3 \rightarrow C_3 - C_1$, we get

$$\begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

$$\Rightarrow a(b-1)(c-1) - (b-1)(1-a) - (c-1)(1-a) = 0$$

$$\Rightarrow \frac{a}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 0$$

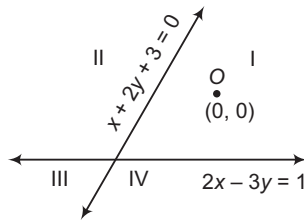
$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0 \Rightarrow \frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$$

Target Exercises

Type 1. Only One Correct Option

- If a , b and c are any three terms of an AP, then the $ax + by + c = 0$
 - has a fixed direction
 - always passes through a fixed point
 - always cuts intercepts on the axes such that their sum is zero
 - forms a triangle with the axes whose area is constant
- Equations $(b - c)x + (c - a)y + (a - b) = 0$ and $(b^3 - c^3)x + (c^3 - a^3)y + (a^3 - b^3) = 0$ (where, a , b and c are not all equal) will represent the same line, if
 - $b = c$
 - $c = a$
 - $c + a = 0$
 - $a + b + c = 0$
- A rectangle $ABCD$, $A \equiv (0, 0)$, $B \equiv (4, 0)$, $C \equiv (4, 2)$ and $D \equiv (0, 2)$ undergoes the following transformations successively:
 - $f_1(x, y) \rightarrow (y, x)$
 - $f_2(x, y) \rightarrow (x + 3y, y)$
 - $f_3(x, y) \rightarrow \left(\frac{x - y}{2}, \frac{x + y}{2}\right)$
 The final figure will be
 - a square
 - a rhombus
 - a rectangle
 - a parallelogram
- The points $A(0, -1)$, $B(2, 1)$, $C(0, 3)$ and $D(-2, 1)$ are the vertices of a
 - square
 - rectangle
 - parallelogram
 - None of these
- If the points $(-2, 0)$, $\left(-1, \frac{1}{\sqrt{3}}\right)$ and $(\cos \theta, \sin \theta)$ are collinear, then the number of values of $\theta \in [0, 2\pi]$ is
 - 0
 - 1
 - 2
 - infinite
- If a , b , c are in AP and α , β , γ are in GP, then the points (a, α) , (b, β) and (c, γ) are collinear, if
 - $\alpha = \beta = \gamma$
 - $\alpha = \beta^2$
 - $\beta = \gamma^2$
 - None of these
- The points (α, β) , (γ, δ) , (α, δ) and (γ, β) taken in order, where α , β , γ and δ are different real numbers, are
 - collinear
 - vertices of a square
 - vertices of a rhombus
 - concylic
- The diagonals of a parallelogram $PQRS$ are along the lines $x + 3y = 4$ and $6x - 2y = 7$, then $PQRS$ must be a
 - rectangle
 - square
 - cyclic quadrilateral
 - rhombus
- A line passes through $(2, 2)$ and is perpendicular to the line $3x + y = 3$, its y -intercept is
 - $\frac{1}{3}$
 - $\frac{2}{3}$
 - 1
 - $\frac{4}{3}$
- If one of the diagonals of a square is along the line $x = 2y$ and one of its vertices is $(3, 0)$, then its sides through this vertex are given by the equations
 - $y - 3x + 9 = 0$, $3y + x - 3 = 0$
 - $y + 3x + 9 = 0$, $3y + x - 3 = 0$
 - $y - 3x + 9 = 0$, $3y - x + 3 = 0$
 - $y - 3x + 3 = 0$, $3y + x + 9 = 0$
- The equation of a line in parametric form is given by
 - $(x - x_1)r \cos \theta = (y - y_1)r \sin \theta$
 - $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$
 - $(x - x_1) \cos \theta = (y - y_1) \sin \theta = r$
 - None of the above
- A line is drawn perpendicular to the line $y = 5x$, meeting the coordinate axes at A and B . If the area of $\triangle OAB$ is 10 sq units, where O is the origin, then equation of drawn line is
 - $x + 5y = 10$
 - $x - 5y = 10$
 - $x + 4y = 10$
 - $x - 4y = 10$
- The vertices of a $\triangle ABC$ are $(1, 1)$, $(4, -2)$ and $(5, 5)$ respectively. The equation of perpendicular dropped from C to the internal bisector of angle A is
 - $y - 5 = 0$
 - $x - 5 = 0$
 - $2x + 3y - 7 = 0$
 - None of these
- In an isosceles $\triangle ABC$, the coordinates of the points B and C on the base BC are respectively $(2, 1)$ and $(1, 2)$. If the equation of the line AB is $y = \frac{1}{2}x$, then the equation of the line AC is
 - $2y = x + 3$
 - $y = 2x$
 - $y = \frac{1}{2}(x - 1)$
 - $y = x - 1$
- The equation of the lines through the points $(2, 3)$ and making an intercept of length 2 units between the lines $y + 2x = 3$ and $y + 2x = 5$, are
 - $x + 3 = 0$, $3x + 4y = 12$
 - $y - 2 = 0$, $4x - 3y = 6$
 - $x - 2 = 0$, $3x + 4y = 18$
 - None of the above

16. A line through $A(-5, -4)$ meets the lines $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at B , C and D respectively. If $\left(\frac{15}{AB}\right)^2 + \left(\frac{10}{AC}\right)^2 = \left(\frac{6}{AD}\right)^2$, then the equation of the line is
- (a) $2x + 3y + 22 = 0$ (b) $5x - 4y + 7 = 0$
(c) $3x - 2y + 3 = 0$ (d) None of these
17. Two lines $2x - 3y = 1$ and $x + 2y + 3 = 0$ divide the XY -plane in four compartments which are named as shown in the figure. Consider the locations of the points $(2, -1)$, $(3, 2)$ and $(-1, -2)$, we get



- (a) $(2, -1) \in \text{IV}$ (b) $(3, 2) \in \text{III}$
(c) $(-2, -1) \in \text{II}$ (d) None of these
18. The pair of points which lie on the same side the straight line $3x - 8y - 7 = 0$, is
- (a) $(0, -1)$, $(0, 0)$ (b) $(24, -3)$, $(1, 1)$
(c) $(-1, -1)$, $(3, 7)$ (d) $(0, 1)$, $(3, 0)$
19. If $A\left(\sin \alpha, \frac{1}{\sqrt{2}}\right)$ and $B\left(\frac{1}{\sqrt{2}}, \cos \alpha\right)$, $-\pi \leq \alpha \leq \pi$, are two points on the same side of the line $x - y = 0$, then α belongs to the interval
- (a) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (b) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$
(c) $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$ (d) None of these
20. If the point $(1 + \cos \theta, \sin \theta)$ lies between the region corresponding to the acute angle between the lines $3y = x$ and $6y = x$, then
- (a) $\theta \in R$ (b) $\theta \in R \sim n\pi, n \in I$
(c) $\theta \in R \sim (2n + 1)\frac{\pi}{2}, n \in I$ (d) None of these
21. If $P\left(\frac{1+t}{\sqrt{2}}, \frac{2+t}{\sqrt{2}}\right)$ is any point on a line, then the range of values of t for which the point P lies between the parallel lines $x + 2y = 1$ and $2x + 4y = 15$, is
- (a) $-\frac{4\sqrt{2}}{3} < t < \frac{5\sqrt{2}}{6}$ (b) $0 < t < \frac{5\sqrt{2}}{6}$
(c) $-\frac{4\sqrt{2}}{3} < t < 0$ (d) None of these
22. If $A(n, n^2)$ (where, $n \in N$) is any point in the interior of the quadrilateral formed by the lines $x = 0$, $y = 0$, $3x + y - 4 = 0$ and $4x + y - 21 = 0$, then the possible number of positions of the point A is
- (a) 0 (b) 1 (c) 2 (d) 3

23. $P(3, 1)$, $Q(6, 5)$ and $R(x, y)$ are three points such that the $\angle PRQ$ is a right angle and the area of $\Delta PRQ = 7$, then the number of such points R is
- (a) 0 (b) 1
(c) 2 (d) 4
24. The slopes of the lines joining the origin to the points of trisection of the portion of the line $3x + y = 12$ intercepted between the axis are the roots of the equation
- (a) $2x^2 - 15x + 18 = 0$ (b) $3x^2 - 20x + 12 = 0$
(c) $12x^2 - 20x + 3 = 0$ (d) $18x^2 - 15x + 2 = 0$
25. $ABCD$ is square $A \equiv (1, 2)$, $B \equiv (3, -4)$. If line CD passes through $(3, 8)$, then mid-point of CD is
- (a) $(2, 6)$ (b) $(6, 2)$ (c) $(2, 5)$ (d) $\left(\frac{28}{5}, \frac{1}{5}\right)$
26. The area of triangle formed by the lines $Y = ax$, $x + y - a = 0$ and the Y -axis is equal to
- (a) $\frac{1}{2|1+a|}$ (b) $\frac{a^2}{|1+a|}$ (c) $\frac{1}{2} \left| \frac{a}{1+a} \right|$ (d) $\frac{a^2}{2|1+a|}$
27. The straight lines $7x - 2y + 10 = 0$ and $7x + 2y - 10 = 0$ forms and isosceles triangle with the line $y = 2$, area of this triangle is equal to
- (a) $\frac{15}{7}$ sq units (b) $\frac{10}{7}$ sq units
(c) $\frac{18}{7}$ sq units (d) None of these
28. If $x - 2y + 4 = 0$ and $2x + y - 5 = 0$ are the sides of a isosceles triangle having area 10 sq units. Equation of third side is
- (a) $x + 3y = 19$ (b) $3x - y - 11 = 0$
(c) $x - 3y = 19$ (d) $3x - y + 15 = 0$
29. The equation of a straight line equally inclined to the axes and equidistant from the points $(1, -2)$ and $(3, 4)$ is
- (a) $x + y + 1 = 0$ (b) $x - y + 1 = 0$
(c) $x - y - 1 = 0$ (d) $x + y - 1 = 0$
30. A pair of straight line drawn through the origin form with the line $2x + 3y = 6$ an isosceles right angled triangle, then the lines and the area of the triangle, thus formed is
- (a) $x - 5y = 0$, $5x + y = 0$; $\Delta = \frac{36}{13}$
(b) $3x - y = 0$, $x + 3y = 0$; $\Delta = \frac{12}{17}$
(c) $5x - y = 0$, $x + 5y = 0$; $\Delta = \frac{13}{5}$
(d) None of the above
31. The area enclosed within the curve $|x| + |y| = 1$, is
- (a) 2 sq units (b) 4 sq units
(c) 6 sq units (d) None of these

32. The sides of a rectangle are $x = 0$, $y = 0$, $x = 4$ and $y = 3$. The equation of the straight line having slope $\frac{1}{2}$ that divides the rectangle into two equal halves, is
(a) $2y = x + 1$ (b) $2x = y + 1$ (c) $2y + x = 1$ (d) $2x + y = 1$
33. The range of values of the ordinate of a point moving on the line $x = 1$ and always remaining in the interior of the triangle formed by the lines $y = x$, the X -axis and $x + y = 4$, is
(a) $(0, 1)$ (b) $[0, 1]$
(c) $[0, 4]$ (d) None of these
34. The lines $x + 2y + 3 = 0$, $x + 2y - 7 = 0$ and $2x - y - 4 = 0$ are the sides of a square. Equation of the remaining side of the square can be
(a) $2x - y - 14 = 0$ (b) $2x - y + 8 = 0$
(c) $2x - y - 10 = 0$ (d) $2x - y - 6 = 0$
35. The line, which is parallel to X -axis and crosses the curve $y = \sqrt{x}$ at an angle of 45° , is
(a) $x = \frac{1}{2}$ (b) $y = \frac{1}{4}$ (c) $y = \frac{1}{2}$ (d) $y = 1$
36. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$, is
(a) 1 sq unit (b) 2 sq units
(c) $2\sqrt{2}$ sq units (d) 4 sq units
37. The area bounded by the curves $x + 2|y| = 1$ and $x = 0$, is
(a) $\frac{1}{3}$ (b) $\frac{1}{2}$ (c) 2 (d) 3
38. A ray of light is sent along the line $x + y = 1$, after being reflected from the line $y - x = 1$ it is again reflected from the line $y = 0$, then the equation of the line representing the ray after second reflection may be given as
(a) $x + y = 1$ (b) $x - y = 1$
(c) $y - x = 1$ (d) None of these
39. A straight line L with negative slope passes through the point $(8, 2)$ and intersects the positive coordinate axes at point P and Q . As L varies the absolute minimum value of $OP + OQ$ is (where, O is origin)
(a) 12 (b) 14 (c) 18 (d) 20
40. The lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$ are concurrent only when
(a) $a^2 + b^2 + c^2 = 2abc$ (b) $a^3 + b^3 + c^3 = 3abc$
(c) $a + b + c = 3abc$ (d) None of these
41. If $3a - 2b + 5c = 0$, family of the lines $ax + by + c = 0$ are always concurrent at a point whose coordinates are
(a) $\left(-\frac{3}{5}, \frac{2}{5}\right)$ (b) $\left(\frac{3}{5}, -\frac{2}{5}\right)$
(c) $\left(\frac{3}{5}, \frac{2}{5}\right)$ (d) None of these
42. The equation of the base of an equilateral triangle is $x + y = 2$ and the vertex is $(2, -1)$. Length of its side is
(a) $\sqrt{\frac{1}{2}}$ (b) $\sqrt{\frac{3}{2}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\sqrt{2}$
43. The line L has intercepts a and b on the coordinate axes. When keeping the origin fixed, the coordinate axes are rotated through a fixed angle, then the same line has intercepts p and q on the rotated axes. Then,
(a) $a^2 + b^2 = p^2 + q^2$ (b) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2} + \frac{1}{q^2}$
(c) $a^2 + p^2 = b^2 + q^2$ (d) $\frac{1}{a^2} + \frac{1}{p^2} = \frac{1}{b^2} + \frac{1}{q^2}$
44. The coordinates of foot of the perpendicular drawn from the point $(2, 4)$ on the line $x + y = 1$, are
(a) $\left(\frac{1}{2}, \frac{3}{2}\right)$ (b) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (c) $\left(\frac{3}{2}, -\frac{1}{2}\right)$ (d) $\left(-\frac{1}{2}, -\frac{3}{2}\right)$
45. The two points on the line $x + y = 4$ that lie at a unit distance from the line $4x + 3y = 10$, are
(a) $(-3, 1)$, $(7, 11)$ (b) $(3, 1)$, $(-7, 11)$
(c) $(3, 1)$, $(7, 11)$ (d) None of these
46. The centroid of an equilateral triangle is $(0, 0)$. If two vertices of the triangle lie on $x + y = 2\sqrt{2}$, then one of them will have its coordinates as
(a) $(\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6})$ (b) $(\sqrt{2} + \sqrt{3}, \sqrt{2} - \sqrt{3})$
(c) $(\sqrt{2} + \sqrt{5}, \sqrt{2} - \sqrt{5})$ (d) None of these
47. ABC is an equilateral triangle such that the vertices B and C lie on two parallel lines at a distance 6. If A lies between the parallel lines at a distance 4 from one of them, then the length of a side of the equilateral triangle is
(a) 8 (b) $\sqrt{\frac{88}{3}}$
(c) $\frac{4\sqrt{7}}{\sqrt{3}}$ (d) None of these
48. If the straight lines $a_1x + b_1y + c_1 = 0$, $a_1x + b_1y + c_2 = 0$, $a_2x + b_2y + d_1 = 0$ and $a_2x + b_2y + d_2 = 0$ are the sides of rhombus, then
(a) $(a_1^2 + b_1^2)(c_1 - c_2)^2 = (a_1^2 + b_1^2)(d_1 - d_2)^2$
(b) $(a_1^2 + b_1^2)|d_1 - d_2| = (a_2^2 + b_2^2)|c_1 - c_2|$
(c) $(a_2^2 + b_2^2)(d_1 - d_2)^2 = (a_1^2 + b_1^2)(c_1 - c_2)^2$
(d) $(a_1^2 + b_1^2)|c_1 - c_2| = (a_2^2 + b_2^2)|d_1 - d_2|$
49. A straight line L_1 makes angle $\tan^{-1} \frac{4}{3}$ with the parallel lines $3x - 4y - 24 = 0$ and $3x - 4y - 12 = 0$. If L_1 makes an intercept of 3 units with these parallel lines, then the equation L_1 may be given as
(a) $x = 1$
(b) $y = 1$
(c) $x = 2y + 3$
(d) None of the above

50. The vertices of a $\triangle OBC$ are $O(0, 0)$, $B(-3, -1)$ and $C(-1, -3)$. The equation of a line parallel to BC and intersecting sides OB and OC whose distance from the origin is $\frac{1}{2}$, is
- (a) $x + y + \frac{1}{\sqrt{2}} = 0$ (b) $x + y - \frac{1}{\sqrt{2}} = 0$
 (c) $x + y - \frac{1}{2} = 0$ (d) $x + y + \frac{1}{2} = 0$
51. The image of the point $P(3, 5)$ with respect to the line $y = x$ is the point Q is the image of Q with respect to the line $y = 0$ is the point $R(a, b)$, then (a, b) is equal to
- (a) $(5, 3)$ (b) $(5, -3)$
 (c) $(-5, 3)$ (d) $(-5, -3)$
52. The equation of the diagonal through origin of the quadrilateral formed by the line $x = 0$, $y = 0$, $x + y - 1 = 0$ and $6x + y - 3 = 0$, is
- (a) $4x - 3y = 0$ (b) $3x - 2y = 0$
 (c) $x = y$ (d) $x + y = 0$
53. Consider a family of straight lines $(x + y) + \lambda(2x - y + 1) = 0$. Equation of the straight line belonging to this family that is farthest from $(1, -3)$ is
- (a) $13y + 6x = 7$ (b) $15y + 6x = 7$
 (c) $13y - 6x = 7$ (d) $15y - 6x = 7$
54. A family of lines is given by $(1 + 2\lambda)x + (1 - \lambda)y + \lambda = 0$, λ being the parameter. The line belonging to this family at the maximum distance from the point $(1, 4)$ is
- (a) $4x - y + 1 = 0$ (b) $33x + 12y + 7 = 0$
 (c) $12x + 33y = 7$ (d) None of these
55. Consider the family of lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ and $x - y + 1 + \lambda_2(2x - y - 2) = 0$. Equation of a straight line that belong to both families is
- (a) $25x - 62y + 86 = 0$ (b) $62x - 25y + 86 = 0$
 (c) $25x - 62y = 86$ (d) $5x - 2y - 7 = 0$
56. Family of a lines $x \sec^2 \theta + y \tan^2 \theta - 2 = 0$, for different real θ , is
- (a) not concurrent
 (b) concurrent at $(1, 1)$
 (c) concurrent at $(2, -2)$
 (d) concurrent at $(-2, 2)$
57. Consider the family of lines $y - y_1 = m(x - x_1)$, in which m is a constant and $x_1^2 + y_1^2 = 1$, then all the lines of the family are
- (a) concurrent at origin
 (b) normal lines to the circle $x^2 + y^2 = 2$
 (c) tangent lines to the circle $x^2 + y^2 = 1$
 (d) None of the above
58. A line is such that its segments between the straight lines $5x - y = 4$ and $3x + 4y - 4 = 0$ is bisected at the points $(1, 5)$. Its equation is
- (a) $23x - 7y + 6 = 0$
 (b) $7x + 4y + 3 = 0$
 (c) $83x - 35y + 92 = 0$
 (d) None of the above
59. The equation of the line which bisects the obtuse angle between the lines $x - 2y + 4 = 0$ and $4x - 3y + 2 = 0$, is
- (a) $(4 - \sqrt{5})x - (3 - 2\sqrt{5})y + (2 - 4\sqrt{5}) = 0$
 (b) $(3 - 2\sqrt{5})x - (4 - \sqrt{5})y + (2 + 4\sqrt{5}) = 0$
 (c) $(4 + \sqrt{5})x - (3 + 2\sqrt{5})y + (2 + 4\sqrt{5}) = 0$
 (d) None of the above
60. Equation of the straight line making equal angles with the straight lines $x + y = 1$ and $2x + 3y = 1$ and passing through the point $(1, 2)$ may be given as
- (a) $\sqrt{13}(x + y - 3) = \pm \sqrt{2}(2x + 3y - 8)$
 (b) $\sqrt{13}(x + y + 1) = \pm \sqrt{2}(2x + 3y + 6)$
 (c) $\sqrt{13}(x + y + 5) = \pm \sqrt{2}(2x + 3y + 9)$
 (d) None of the above
61. A light ray coming along the line $3x + 4y = 5$, gets reflected from the line $ax + by = 1$ and goes along the line $5x - 12y = 10$, then
- (a) $a = \frac{64}{115}, b = \frac{112}{15}$
 (b) $a = \frac{14}{15}, b = -\frac{8}{115}$
 (c) $a = \frac{64}{115}, b = -\frac{8}{115}$
 (d) $a = \frac{64}{15}, b = \frac{14}{15}$
62. Equations of the bisectors of the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$ are given by
- (a) $21x + 77y - 101 = 0, 11x - 3y + 9 = 0$
 (b) $11x - 6y + 111 = 0, 22x - 13y + 104 = 0$
 (c) $15x - 9y + 67 = 0, 15x + 4y + 33 = 0$
 (d) None of the above
63. θ_1 and θ_2 are the inclination of lines L_1 and L_2 with X -axis. If L_1 and L_2 pass through $P(x_1, y_1)$, then equation of one of the angle bisector of these lines is
- (a) $\frac{x - x_1}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)} = \frac{y - y_1}{\sin\left(\frac{\theta_1 - \theta_2}{2}\right)}$
 (b) $\frac{x - x_1}{-\sin\left(\frac{\theta_1 - \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 - \theta_2}{2}\right)}$
 (c) $\frac{x - x_1}{\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$
 (d) $\frac{x - x_1}{-\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$

64. A line $L_1 = 0$ passing through $(1, 2)$ has equation of its two bisectors with respect to other line $L_2 = 0$ as $B_1 \equiv 3x - 4y - 7 = 0$ and $B_2 \equiv 4x + 3y - 2 = 0$. Then, which of the following is true?

- (a) B_1 is acute angle bisector
(b) B_2 is acute angle bisector
(c) both B_1 and B_2 are equally inclined to the line
(d) nothing can be said

65. A variable line drawn through the point $(1, 3)$ meets the X -axis at A and Y -axis at B . If the rectangle $OAPB$ is completed, where O is the origin, then locus of P is

- (a) $\frac{1}{y} + \frac{3}{x} = 1$ (b) $x + 3y = 1$
(c) $\frac{1}{x} + \frac{3}{y} = 1$ (d) $3x + y = 1$

66. A rectangle $ABCD$ has its side AB parallel to line $y = x$ and vertices A, B and D lie on $y = 1$, $x = 2$ and $x = -2$ respectively. Locus of vertex C is

- (a) $x = 5$ (b) $x - y = 5$ (c) $y = 5$ (d) $x + y = 5$

67. The equation $\sqrt{(x^2 + 4y^2 - 4xy + 4)} + x - 2y = 1$ represents a/an

- (a) straight line (b) ellipse
(c) circle (d) parabola

68. $A \equiv (-4, 0)$, $B \equiv (4, 0)$, M and N are variable points on Y -axis such that M lies below N and $MN = 4$. If the line joining AM and BN intersect at P , then locus of P is

- (a) $2xy + 16 + x^2 = 0$ (b) $2xy - 16 + x^2 = 0$
(c) $2xy - 16 - x^2 = 0$ (d) $2xy + 16 - x^2 = 0$

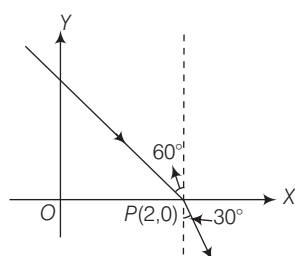
69. Family of lines $\lambda x + 3y - 6 = 0$ (λ is a real parameter) intersect the lines $x - 2y + 3 = 0$ and $x - y + 1 = 0$ in P and Q , then locus of the middle point of PQ is

- (a) $4x + 2y = 1$ (b) $x + y = 2$
(c) $2x - 2y = 0$ (d) $4x + 3y = 4$

70. ABC is a right angled isosceles triangle, right angled at $A(2, 1)$. If the equation of side BC is $-2x + y = 3$, then the combined equation of lines AB and AC is

- (a) $3x^2 - 3y^2 + 8xy + 20x + 10y + 25 = 0$
(b) $3x^2 - 3y^2 + 8xy - 20x - 10y + 25 = 0$
(c) $3x^2 - 3y^2 + 8xy + 10x + 15y + 20 = 0$
(d) None of the above

71. In the following figure combined equation of the incident and the refracted ray is



(a) $(x - 1)^2 - y^2 + \frac{4}{\sqrt{3}}(x - 2)y = 0$

(b) $(x - 2)^2 + y^2 - \frac{4}{\sqrt{3}}(x - 2)y = 0$

(c) $(x - 2)^2 + y^2 + \frac{4}{\sqrt{3}}(x - 2)y = 0$

(d) None of the above

72. From a common point on the lines $xy - 3y - 4x + 12 = 0$ mutually perpendicular lines L_1 and L_2 are drawn such that they intercept triangles of equal area with coordinate axes. If θ_1 and θ_2 are inclination of L_1 and L_2 with coordinate axes, then $|\theta_1 + \theta_2|$ is equal to

(a) $\tan^{-1}\left(\frac{48}{7}\right)$ (b) $\tan^{-1}\left(\frac{24}{7}\right)$ (c) $\tan^{-1}\left(\frac{1}{7}\right)$ (d) $\tan^{-1}\left(\frac{2}{7}\right)$

73. The value of λ for which the equation $6x^2 + 11xy - 10y^2 + x + 31y + \lambda = 0$ represents a pair of straight lines, is

- (a) -15 (b) 0
(c) 2 (d) None of these

74. If a and b are positive number ($a < b$), then the range of values of k for which a real λ is found such that the equation $ax^2 + 2\lambda xy + by^2 + 2k(x + y + 1) = 0$ represents a pair of straight lines, is

- (a) $a < k^2 < b$ (b) $a \leq k^2 \leq b$
(c) $k^2 \leq a, k^2 \geq b$ (d) None of these

75. The equation $x^3 + y^3 = 0$ represents

- (a) three real straight lines
(b) one point
(c) one real and two imaginary lines
(d) None of the above

76. If the line $y = \tan \theta x$ cuts the curve $x^3 + xy^2 + 2x^2 + 2y^2 + 3x + 1 = 0$ at the points A, B and C . If OA, OB and OC are in HP, then $\tan \theta$ is equal to

- (a) ± 1 (b) 0 (c) 2 (d) -2

77. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines equidistant from the origin, then

- (a) $f^4 = c(bf - ag)$ (b) $f^4 - g^4 = c(bf^2 - ag^2)$
(c) $f^4 + g^4 = c(bf^2 + ag^2)$ (d) None of these

78. Which of the following pairs of straight lines intersects at right angles?

- (a) $2x^2 = y(x + 2y)$ (b) $(x + y)^2 = x(y + 3x)$
(c) $2y(x + y) = xy$ (d) $y = \pm 2x$

79. If one of the lines of the pair $ax^2 + 2hxy + by^2 = 0$ bisects the angles between positive directions of the axes, then a, b and h satisfy the relation

- (a) $a + b = 2|h|$ (b) $a + b = -2h$
(c) $a - b = 2|h|$ (d) $(a - b)^2 = 4h^2$

80. If the straight lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ intersect on the X -axis, then
 (a) $ag = fh$ (b) $ah = fg$
 (c) $af = gh$ (d) None of these
81. The four straight lines given by the equations $12x^2 + 7xy - 12y^2 = 0$ and $12x^2 + 7xy - 12y^2 - x + 7y - 1 = 0$ lie along the sides of a
 (a) square (b) parallelogram
 (c) rectangle (d) rhombus
82. Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 9$, is
 (a) $\frac{27}{4}$ sq units (b) 0 sq unit
 (c) $\frac{9}{3}$ sq units (d) 27 sq units
83. The area of the triangle formed by the lines $4x^2 - 9xy - 9y^2 = 0$ and $x = 2$ is equal to
 (a) $\frac{20}{3}$ sq units (b) 3 sq units (c) $\frac{10}{3}$ sq units (d) 2 sq units
84. The value of λ for which the lines joining the point of intersection of curves C_1 and C_2 to the origin are equally inclined to the X -axis, where $C_1 : \lambda x^2 + 3y^2 - 2\lambda xy + 9x = 0$ and $C_2 : 3x^2 - 4y^2 + 8xy - 3x = 0$, is
 (a) $\frac{4}{3}$ (b) 12
 (c) 1 (d) None of these
85. The area of the triangle formed by joining the origin to the points of intersection of the line $\sqrt{5}x + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$, is
 (a) 6 sq units (b) 5 sq units (c) 4 sq units (d) 3 sq units
86. If the lines joining origin to the points of intersection of the line $fx - gy = \lambda$ and the curve $x^2 + hxy - y^2 + gx + fy = 0$ are mutually perpendicular, then
 (a) $\lambda = h$ (b) $\lambda = g$
 (c) $\lambda = fg$ (d) λ may have any value

Type 2. More than One Correct Option

87. Two particles start from the same point $(2, -1)$ one moving 2 units along the line $x + y = 1$ and the other 5 units along $x - 2y = 4$. If the particles move towards increasing y , then their new position will be
 (a) $(2 - \sqrt{2}, \sqrt{2} - 1)$
 (b) $(2\sqrt{5} + 2, \sqrt{5} - 1)$
 (c) $(2 + \sqrt{2}, \sqrt{2} + 1)$
 (d) $(2\sqrt{5} - 2, \sqrt{5} - 1)$
88. If one diagonal of a square is the portion of the line $\frac{x}{a} + \frac{y}{b} = 1$ intercepted by the axes, then the extremities of the other diagonal of the square are
 (a) $\left(\frac{a+b}{2}, \frac{a+b}{2}\right)$ (b) $\left(\frac{a-b}{2}, \frac{a+b}{2}\right)$
 (c) $\left(\frac{a-b}{2}, \frac{b-a}{2}\right)$ (d) $\left(\frac{a+b}{2}, \frac{b-a}{2}\right)$
89. A line which makes an acute angle θ with the positive direction of X -axis is drawn through the point $P(3, 4)$ to meet the line $x = 6$ at R and $y = 8$ at S , then
 (a) $PR = 3 \sec \theta$
 (b) $PS = 4 \operatorname{cosec} \theta$
 (c) $PR + PS = \frac{2(3 \sin \theta + 4 \cos \theta)}{\sin 2\theta}$
 (d) $\frac{9}{(PR)^2} + \frac{16}{(PS)^2} = 1$
90. The base of a triangle lies along the line $x = a$ and is of length a . The area of the triangle is a^2 , if the vertex lies on the line
 (a) $x = 0$ (b) $x = -a$ (c) $x = 3a$ (d) $x = -3a$
91. The equation of straight lines passing through ordered pairs (a, b) satisfying equation $\sec^2(a+2)b + a^2 - 1 = 0$, $-\pi < b < \pi$ and having slope $\frac{1}{2}$, is
 (a) $x - 2y = 0$ (b) $x + 2y = 1$
 (c) $x - 2y = \pi$ (d) $x - 2y + \pi = 0$
92. Two straight lines $u = 0$ and $v = 0$ passes through the origin forming an angle of $\tan^{-1}\left(\frac{7}{9}\right)$ with each other. If the ratio of the slopes of $v = 0$ and $u = 0$ is $\frac{9}{2}$, then their equations are
 (a) $y = 3x$ and $3y = 2x$
 (b) $2y = 3x$ and $3y = x$
 (c) $y + 3x = 0$ and $3y + 2x = 0$
 (d) $2y + 3x = 0$ and $3y + x = 0$
93. If pair of straight lines, $(\tan^2 \theta + \cos^2 \theta)x^2 - (2 \tan \theta)xy + (\sin^2 \theta)y^2 = 0$ are inclined at angles α and β with coordinate axes, then
 (a) $\tan \alpha + \tan \beta = \operatorname{cosec} 2\theta$
 (b) $\tan \alpha \cdot \tan \beta = \sec^2 \theta + \cot^2 \theta$
 (c) $\tan \alpha - \tan \beta = 2$
 (d) $\frac{\tan \alpha}{\tan \beta} = \frac{2 + \sin 2\theta}{2 - \sin 2\theta}$
94. A and B are two fixed points whose coordinates are $(3, 2)$ and $(5, 4)$, respectively. The coordinates of a point P , if ABP is an equilateral triangle, are
 (a) $(3 - \sqrt{3}, 3 + \sqrt{3})$ (b) $(4 + \sqrt{3}, 3 - \sqrt{3})$
 (c) $(3 - \sqrt{3}, 4 + \sqrt{3})$ (d) $(3 + \sqrt{3}, 4 - \sqrt{3})$

Type 3. Assertion and Reason

Directions (Q. Nos. 95-107) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

95. Statement I Perpendicular from point $A(1, 1)$ to the line joining the points $B(c \cos \alpha, c \sin \alpha)$ and $C(c \cos \beta, c \sin \beta)$ bisects BC for all values of α and β .

Statement II Perpendicular drawn from the vertex to the base of an isosceles triangle bisects the base.

96. Statement I Let the lines $2x + 3y + 19 = 0$ and $9x + 6y - 17 = 0$ cut the X -axis in A, B and Y -axis in C, D . Then, points A, B, C and D are concyclic.

Statement II Since, $OA \cdot OB = OC \cdot OD$, where O is origin, therefore A, B, C and D points are concyclic.

97. Statement I If the sides of the $\triangle ABC$ are along the lines L_1, L_2 and L_3 , then there is only one point in the plane of $\triangle ABC$ which is equidistant from the lines L_1, L_2 and L_3 .

Statement II Incentre of the $\triangle ABC$ is equidistance from the lines L_1, L_2 and L_3 .

98. Statement I Let $L_1 : a_1x + b_1y + c_1 = 0$, $L_2 : a_2x + b_2y + c_2 = 0$ and $L_3 : a_3x + b_3y + c_3 = 0$ be three concurrent lines, then

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$

Statement II If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the lines L_1, L_2 and L_3 must be concurrent.

99. Statement I The internal angle bisector of angle C of a $\triangle ABC$ with sides AB, AC and BC are $y = 0$, $3x + 2y = 0$ and $2x + 3y + 6 = 0$ respectively, is $5x + 5y + 6 = 0$.

Statement II Image of point A with respect to $5x + 5y + 6 = 0$ lies on side BC of the triangle.

100. Statement I The lines $(a + b)x + (a - 2b)y = a$ are concurrent at the point $\left(\frac{2}{3}, \frac{1}{3}\right)$.

Statement II The lines $x + y - 1 = 0$ and $x - 2y = 0$ intersect at the point $\left(\frac{2}{3}, \frac{1}{3}\right)$.

101. Statement I If a, b and c are variables such that $3a + 2b + 4c = 0$, then the family of lines given by $ax + by + c = 0$ pass through a fixed point $(3, 2)$.

Statement II The equation $ax + by + c = 0$ will represent a family of straight lines passing through a fixed point iff there exists a linear relation between a, b and c .

102. Statement I The joint equation of the lines $2x - y = 5$ and $x - 2y = 3$ is $2x^2 + 3xy - 2y^2 - 11x - 7y + 15 = 0$.

Statement II Every second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ always represent pair of straight lines.

103. Statement I The joint equation of lines $y = x$ and $y = -x$ is $y^2 = -x^2$, i.e. $x^2 + y^2 = 0$.

Statement II The joint equation of lines $ax + by = 0$ and $cx + dy = 0$ is $(ax + by)(cx + dy) = 0$, where a, b and c are constant.

104. Statement I If the lines represented by $x^2 - 2pxy - y^2 = 0$ are rotated about the origin through an angle θ , one in clockwise direction and other in anti-clockwise direction, then the equation of the bisectors of the angle between the lines in the new position is $px^2 + 2xy + py^2 = 0$.

Statement II The bisectors of the angles between the lines in new position are same as the bisectors of the angles between their old positions.

105. Statement I If $-2h = a + b$, then one line of the pair of lines $ax^2 + 2hxy + by^2 = 0$ bisects the angle between coordinate axes in positive quadrant.

Statement II If $ax + y(2h + a) = 0$ is a factor of $ax^2 + 2hxy + by^2 = 0$, then $b + 2h + a = 0$.

106. Statement I Two of the straight lines represented by the equation $ax^3 + bx^2 + y + cxy^2 + dy^3 = 0$ will be right angled, if $a^2 + ac + bc + d^2 = 0$.

Statement II Product of the slopes of two perpendicular lines is -1 .

107. Statement I If $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents two parallel straight lines, then the distance between them is $\frac{6}{\sqrt{10}}$.

Statement II If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of parallel straight lines, then the distance between them is $\sqrt{\frac{g^2 - ac}{a(a + b)}}$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 108-110) A $\triangle ABC$ is given where vertex A is $(1, 1)$ and the orthocentre is $(2, 4)$. Also, sides AB and BC are members of the family of lines $ax + by + c = 0$, where a, b and c are in AP.

108. The vertex B is

- (a) $(2, 1)$
- (b) $(1, -2)$
- (c) $(-1, 2)$
- (d) None of the above

109. The vertex C is

- (a) $(4, 16)$
- (b) $(17, -4)$
- (c) $(4, -17)$
- (d) $(-17, 4)$

110. $\triangle ABC$ is a/an

- (a) obtuse angled triangle
- (b) right angled triangle
- (c) acute angled triangle
- (d) equilateral triangle

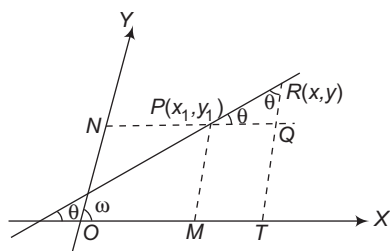
Passage II (Q. Nos. 111-113) Let us consider the situation when axes are inclined at an angle ω . If coordinates of a point P are (x_1, y_1) , then $PN = x_1$, $PM = y_1$, where PM is parallel to Y -axis and PN is parallel to X -axis.

Equation of straight line through line through P and makes an angle θ with X -axis is $RQ = y - y_1$, $PQ = x - x_1$

From $\triangle PQR$, we have $\frac{PQ}{\sin(\omega - \theta)} = \frac{RQ}{\sin \theta}$

$y - y_1 = \frac{\sin \theta}{\sin(\omega - \theta)} (x - x_1)$ written in the form of

$y - y_1 = m(x - x_1)$, where $m = \frac{\sin \theta}{\sin(\omega - \theta)}$



$\therefore$ If slope of line is m , then angle of inclination of line with X -axis is given by $\tan \theta = \left(\frac{m \sin \omega}{1 + m \cos \omega} \right)$.

111. The axes being inclined at an angle of 60° , then the inclination of the straight line $y = 2x + 5$ with the X -axis is

- (a) 30°
- (b) $\tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$
- (c) $\tan^{-1} 2$
- (d) 60°

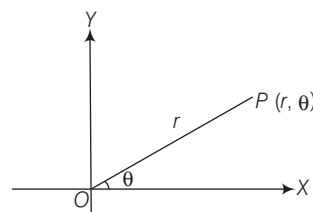
112. The axes being inclined at an angle of 60° , then angle between the two straight lines $y = 2x + 5$ and $2y + x + 7 = 0$ is

- (a) 90°
- (b) $\tan^{-1} \left(\frac{5}{3} \right)$
- (c) $\tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$
- (d) $\tan^{-1} \left(\frac{5}{\sqrt{3}} \right)$

113. The axes being inclined at an angle of 30° , the equation of straight line which makes an angle of 60° with the positive direction of X -axis and x -intercept equal to 2, is

- (a) $y - \sqrt{3}x = 0$
- (b) $\sqrt{3}y = x$
- (c) $y + \sqrt{3}x = 2\sqrt{3}$
- (d) $y + 2x = 0$

Passage III (Q. Nos. 114-116) Let $OP = r$ and $\angle XOP = \theta$, then the position of P in polar coordinate system is represented by (r, θ) . OX and OY are called the initial line and pole, respectively.



If p is the length of perpendicular from origin to line and if α is the angle which the perpendicular to the line makes with the initial line. The equation of line is given by $r \cos(\theta - \alpha) = p$.

114. The polar coordinates of the foot of perpendicular from the pole on the line joining the two points (r_1, θ_1) and (r_2, θ_2) are

- (a) $\left[\frac{r_1 r_2 \sin(\theta_2 - \theta_1)}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}}, \tan^{-1} \left(\frac{r_1 \cos \theta_1 - r_2 \cos \theta_2}{r_2 \sin \theta_2 - r_1 \sin \theta_1} \right) \right]$
- (b) $\left[\frac{r_1 r_2 \sin(\theta_2 + \theta_1)}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\theta_1 - \theta_2)}}, \tan^{-1} \left(\frac{r_1 \cos \theta_1 - r_2 \cos \theta_2}{r_2 \sin \theta_2 - r_1 \sin \theta_1} \right) \right]$
- (c) $\left[\frac{-r_1 r_2 \sin(\theta_2 - \theta_1)}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_1 - \theta_2)}}, \tan^{-1} \left(\frac{r_1 \cos \theta_1 - r_2 \cos \theta_2}{r_2 \sin \theta_2 - r_1 \sin \theta_1} \right) \right]$
- (d) None of the above

115. The equation of straight line $x + \sqrt{3}y = 1$ in polar coordinate is

- (a) $r = \frac{1}{2} \cos\left(\theta + \frac{\pi}{6}\right)$ (b) $2r = \operatorname{cosec}\left(\theta + \frac{\pi}{6}\right)$
(c) $r = \sin\left(\theta + \frac{\pi}{6}\right)$ (d) $r = 2 \cos\left(\theta + \frac{\pi}{6}\right)$

116. Perpendicular distance from pole to the line $\frac{l}{r} = 4 \cos \theta + 2 \sin \theta$ is

- (a) $2\sqrt{5}$ (b) $\frac{l}{2\sqrt{5}}$
(c) $\frac{2\sqrt{5}}{l}$ (d) None of these

Passage IV (Q. Nos. 117-119) $A(1, 3)$ and

$C\left(-\frac{2}{5}, -\frac{2}{5}\right)$ are the vertices of a $\triangle ABC$ and the equation of the angle bisector of $\angle ABC$ is $x + y = 2$.

117. Equation of side BC is

- (a) $7x + 3y - 4 = 0$
(b) $7x + 3y + 4 = 0$
(c) $7x - 3y + 4 = 0$
(d) $7x - 3y - 4 = 0$

118. Coordinates of vertex B are

- (a) $\left(\frac{3}{10}, \frac{7}{10}\right)$ (b) $\left(\frac{17}{10}, \frac{3}{10}\right)$
(c) $\left(-\frac{5}{2}, \frac{9}{2}\right)$ (d) $(1, 1)$

119. Equation of side AB is

- (a) $3x + 7y = 24$
(b) $3x + 7y + 24 = 0$
(c) $13x + 7y + 8 = 0$
(d) $13x - 7y + 8 = 0$

Passage V (Q. Nos. 120-122) A variable line L is drawn through $O(0, 0)$ to meet the lines L_1 and L_2 given by $y - x - 10 = 0$ and $y - x - 20 = 0$ at the points A and B , respectively.

120. A point P is taken on L such that $\frac{2}{OP} = \frac{1}{OA} + \frac{1}{OB}$.

Then, the locus of P is

- (a) $3x + 3y = 40$
(b) $3x + 3y + 40 = 0$
(c) $3x - 3y = 40$
(d) $3y - 3x = 40$

121. If $OP^2 = OA \times OB$, then locus of P is

- (a) $(y - x)^2 = 100$
(b) $(y + x)^2 = 50$
(c) $(y - x)^2 = 200$
(d) None of the above

122. If $\left(\frac{1}{OP^2}\right) = \left(\frac{1}{OB^2}\right) + \left(\frac{1}{OA^2}\right)$, then locus of P is

- (a) $(y - x)^2 = 80$
(b) $(y - x)^2 = 100$
(c) $(y - x)^2 = 64$
(d) None of the above

Type 5. Match the Columns

123. Match the following:

Column I	Column II
A. Two vertices of a triangle are $(5, -1)$ and $(-2, 3)$. If orthocentre is the origin, then coordinates of the third vertex are	p. $(-4, -7)$
B. A point on the line $x + y = 4$ which lies at a unit distance from the line $4x + 3y = 10$, is	q. $(-7, 11)$
C. Orthocentre of the triangle made by the lines $x + y - 1 = 0$, $x - y + 3 = 0$, $2x + y = 7$, is	r. $(1, -2)$
D. If a , b and c are in AP, then lines $ax + by = c$ are concurrent at	s. $(-1, 2)$

124. Match the following:

Column I	Column II
A. Lines $x - 2y - 6 = 0$, $3x + y - 4 = 0$ and $\lambda x + 4y + \lambda^2 = 0$ are concurrent, then the value of λ is	p. 2
B. The points $(\lambda + 1, 1)$, $(2\lambda + 1, 3)$ and $(2\lambda + 2, 2\lambda)$ are collinear, then the value of λ is	q. 4

Column I	Column II
C. If line $x + y - 1 - \lambda = 0$, passing through the intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ is perpendicular to one of them, then the value of λ is	r. $-\frac{1}{2}$
D. If line $y - x - 1 + \lambda = 0$ is equally inclined to axes and equidistant from the points $(1, -2)$ and $(3, 4)$, then λ is	s. -4

125. Match the following:

Column I	Column II
A. The distance between the lines $(x + 7y)^2 + 4\sqrt{2}(x + 7y) - 42 = 0$ is	p. 2
B. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is $ x + y = k$, where k is equal to	q. 7
C. If $6x + 6y + m = 0$ is acute angle bisector of lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$, then m is equal to	r. 3
D. Area of the triangle formed by the lines $y^2 - 9xy + 18x^2 = 0$ and $y = 6$ is	s. 1

Type 6. Single Integer Answer Type Questions

126. The straight lines $7x - 2y + 10 = 0$ and $7x + 2y - 10 = 0$ form an isosceles triangle with the line $y = 2$. Area of this triangle is equal to $\frac{2\lambda}{\mu}$ sq units. Then, the value of $\lambda - \mu$ is _____.
127. Each side of a square is of length 6 units and the centre of the square is $(-1, 2)$. One of its diagonals is parallel to $x + y = 0$. Then, the number of all coordinates of the vertices of the square, is _____.
128. If the points $P(a^2, a)$ lies in the region corresponding to the acute angle between the lines $2y = x$ and $4y = x$, then $a \in (\lambda, \mu)$. Then, the value of $\lambda + \mu$ is _____.
129. The number of values of non-negative real numbers h_1, h_2, h_3, k_1, k_2 and k_3 such that the algebraic sum of the perpendiculars drawn from points $(2, k_1), (3, k_2), (7, k_3), (h_1, 4), (h_2, 5)$ and $(h_3, -3)$ on a variable line passing through $(2, 1)$ is zero, is _____.
130. Let ABC be a triangle. Let A be the point $(1, 2)$, $y = x$ is the perpendicular bisector of AB and $x - 2y + 1 = 0$ is the angle bisector of $\angle C$. If equation of BC is given by $ax + by - 5 = 0$, then the value of $a + b$ is _____.
131. The side of a triangle have the combined equation $x^2 - 3y^2 - 2xy + 8y - 4 = 0$. The third side, which is variable always passes through the point $(-5, -1)$. If the range of values of the slope of the third line is such that the origin is an interior point of the triangle is (a, b) , then the value of $\left(a + \frac{1}{b}\right)$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distances of the point P from the lines $x - y = 0$ and $x + y = 0$, respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $2 \leq d_1(P) + d_2(P) \leq 4$, is _____. [2014]
2. For $a > b > c > 0$, the distance between $(1, 1)$ and the point of intersection of the lines $ax + by + c = 0$ and $bx + ay + c = 0$ is less than $2\sqrt{2}$, then [2013]
- (a) $a + b - c > 0$ (b) $a - b + c < 0$
(c) $a - b + c > 0$ (d) $a + b - c < 0$
3. A straight line L through the point $(3, -2)$ is inclined at an angle 60° to the line $\sqrt{3}x + y = 1$. If L also intersects the X -axis, then the equation of L is [2011]
- (a) $y + \sqrt{3}x + 2 - 3\sqrt{3} = 0$
(b) $y - \sqrt{3}x + 2 + 3\sqrt{3} = 0$
(c) $\sqrt{3}y - x + 3 + 2\sqrt{3} = 0$
(d) $\sqrt{3}y + x - 3 + 2\sqrt{3} = 0$

JEE Main/AIEEE

4. Locus of the image of the point $(2, 3)$ in the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$, $k \in R$, is a [2015]
- (a) straight line parallel to X -axis
(b) straight line parallel to Y -axis
(c) circle of radius $\sqrt{2}$
(d) circle of radius $\sqrt{3}$
5. Let PS be the median of the triangle with vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is [2014]
- (a) $4x - 7y - 11 = 0$ (b) $2x + 9y + 7 = 0$
(c) $4x + 7y + 3 = 0$ (d) $2x - 9y - 11 = 0$
6. A ray of light along $x + \sqrt{3}y = \sqrt{3}$ gets reflected upon reaching X -axis, the equation of the reflected ray is [2013]
- (a) $y = x + \sqrt{3}$ (b) $\sqrt{3}y = x - \sqrt{3}$
(c) $y = \sqrt{3}x - \sqrt{3}$ (d) $\sqrt{3}y = x - 1$
7. The lines $L_1: y - x = 0$ and $L_2: 2x + y = 0$ intersect the line $L_3: y + 2 = 0$ at P and Q , respectively. The bisector of the acute angle between L_1 and L_2 intersects L_3 at R .
- Statement I** The ratio $PR : RQ$ equals $2\sqrt{2} : \sqrt{5}$.
- Statement II** In any triangle, bisector of an angle divides the triangle into two similar triangles. [2011]
- (a) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(b) Statement I is true, Statement II is false
(c) Statement I is false, Statement II is true
(d) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I

8. The line $x + y = |a|$ and $ax - y = 1$ intersect each other in the first quadrant. Then, the set of all possible values of a in the interval [2011]
(a) $(-1, 1]$ (b) $(0, \infty)$ (c) $[1, \infty)$ (d) $(-1, \infty)$
9. The line L given by $\frac{x}{5} + \frac{y}{b} = 1$ passes through the point $(13, 32)$. The line K is parallel to L and has the equation $\frac{x}{c} + \frac{y}{3} = 1$. Then, the distance between L and K is [2010]
(a) $\frac{23}{\sqrt{15}}$ (b) $\sqrt{17}$ (c) $\frac{17}{\sqrt{15}}$ (d) $\frac{23}{\sqrt{17}}$
10. The lines $p(p^2 + 1)x - y + q = 0$ and $(p^2 + 1)^2x + (p^2 + 1)y + 2q = 0$ are perpendicular to a common line for [2009]
(a) exactly one value of p (b) exactly two values of p
(c) more than two values of p (d) no value of p
11. The perpendicular bisector of the line segment joining $P(1, 4)$ and $Q(k, 3)$ has y -intercept -4 . Then, a possible value of k is [2008]
(a) -4 (b) 1
(c) 2 (d) -2
12. If one of the lines $my^2 + (1 - m^2)xy - mx^2 = 0$ is a bisector of the angle between the lines $xy = 0$, then m is [2007]
(a) $-\frac{1}{2}$ (b) -2
(c) ± 1 (d) 2
13. A straight line through the point $A(3, 4)$ is such that its intercept between the axes is bisected at A . Its equation is [2006]
(a) $3x - 4y + 7 = 0$ (b) $4x + 3y = 24$
(c) $3x + 4y = 25$ (d) $x + y = 7$
14. If non-zero numbers a, b and c are in HP, then the straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ always passes through a fixed point. That point is [2005]
(a) $\left(1, -\frac{1}{2}\right)$ (b) $(1, -2)$
(c) $(-1, -2)$ (d) $(-1, 2)$
15. The equation of the straight line passing through the point $(4, 3)$ and making intercepts on the coordinate axes whose sum is -1 , is [2004]
(a) $\frac{x}{2} + \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$
(b) $\frac{x}{2} - \frac{y}{3} = -1$ and $\frac{x}{-2} + \frac{y}{1} = -1$
(c) $\frac{x}{2} + \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$
(d) $\frac{x}{2} - \frac{y}{3} = 1$ and $\frac{x}{-2} + \frac{y}{1} = 1$
16. If the sum of the slopes of the lines given by $x^2 - 2cxy - 7y^2 = 0$ is four times their product, then c has the value [2004]
(a) 1 (b) -1 (c) 2 (d) -2
17. If one of the lines given by $6x^2 - xy + 4cy^2 = 0$ is $3x + 4y = 0$, then c equals [2004]
(a) 1 (b) -1 (c) 3 (d) -3
18. If x_1, x_2, x_3 and y_1, y_2, y_3 are both in GP with the same common ratio, then the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) [2003]
(a) lie on a straight line (b) lie on an ellipse
(c) lie on a circle (d) are vertices of a triangle
19. A square of side a lies above the X -axis and has one vertex at the origin. The side passing through the origin makes an angle α ($0 < \alpha < \frac{\pi}{4}$) with the positive direction of X -axis. The equation of its diagonal not passing through the origin is [2003]
(a) $y(\cos \alpha - \sin \alpha) - x(\sin \alpha - \cos \alpha) = a$
(b) $y(\cos \alpha + \sin \alpha) + x(\sin \alpha - \cos \alpha) = a$
(c) $y(\cos \alpha + \sin \alpha) + x(\sin \alpha + \cos \alpha) = a$
(d) $y(\cos \alpha + \sin \alpha) + x(\cos \alpha - \sin \alpha) = a$
20. Three straight lines $2x + 11y - 5 = 0$, $24x + 7y - 20 = 0$ and $4x - 3y - 2 = 0$ [2002]
(a) form a triangle
(b) are only concurrent
(c) are concurrent with one line bisecting the angle between the other two
(d) None of these
21. A straight line through the point $(2, 2)$ intersects the lines $\sqrt{3}x + y = 0$ and $\sqrt{3}x - y = 0$ at the points A and B . The equation to the line AB , so that the $\triangle OAB$ is equilateral, is [2002]
(a) $x - 2 = 0$ (b) $y - 2 = 0$
(c) $x + y - 4 = 0$ (d) None of these

Other Engineering Entrances

22. The value of k such that the lines $2x - 3y + k = 0$, $3x - 4y - 13 = 0$ and $8x - 11y - 33 = 0$ are concurrent, is [BITSAT 2014]
(a) 20 (b) -7
(c) 7 (d) -20
23. The points $(2, 5)$ and $(5, 1)$ are the two opposite vertices of a rectangle. If other two vertices are points on the straight line $y = 2x + k$, then the value of k is [Kerala CEE 2014]
(a) 4 (b) 3 (c) -4
(d) -3 (e) 1

- 24.** A straight line perpendicular to the line $2x + y = 3$ is passing through $(1, 1)$. Its y -intercept is [Kerala CEE 2014]
 (a) 1 (b) 2 (c) 3
 (d) $\frac{1}{2}$ (e) $\frac{1}{3}$
- 25.** The ratio by which the line $2x + 5y - 7 = 0$ divides the straight line joining the points $(-4, 7)$ and $(6, -5)$ is [Kerala CEE 2014]
 (a) 1 : 4 (b) 1 : 2 (c) 1 : 1
 (d) 2 : 3 (e) 1 : 3
- 26.** The straight lines $x + y = 0$, $5x + y = 4$ and $x + 5y = 4$ form [WB JEE 2014]
 (a) an isosceles triangle (b) an equilateral triangle
 (c) a scalene triangle (d) a right angled triangle
- 27.** The slopes of the lines represented by $x^2 + 2hxy + 2y^2 = 0$ are in the ratio 1 : 2, then h equals [UP SEE 2014]
 (a) $\pm \frac{1}{2}$ (b) $\pm \frac{3}{2}$
 (c) ± 1 (d) ± 3
- 28.** Let $A(2, -3)$ and $B(-2, 1)$ be vertices of a $\triangle ABC$. If the centroid of this triangle moves on the line $2x + 3y = 1$, then the locus of the vertex C is the line [J&K CET 2014]
 (a) $3x - 2y = 3$ (b) $2x + 3y = 9$
 (c) $2x - 3y = 7$ (d) $3x + 2y = 5$
- 29.** A straight line passes through the points $(5, 0)$ and $(0, 3)$. The length of perpendicular from the point $(4, 4)$ on the line is [Karnataka CET 2014]
 (a) $\frac{15}{\sqrt{34}}$ (b) $\frac{\sqrt{17}}{2}$
 (c) $\frac{17}{2}$ (d) $\sqrt{\frac{17}{2}}$
- 30.** If p is the length of the perpendicular from the origin to the line, whose intercepts with the coordinate axes are $\frac{1}{3}$ and $\frac{1}{4}$, then the value of p is [Kerala CEE 2014]
 (a) $\frac{3}{4}$ (b) $\frac{1}{12}$ (c) 5
 (d) 12 (e) $\frac{1}{5}$
- 31.** If p and q are respectively the perpendiculars from the origin upon the straight lines, whose equations are $x \sec \theta + y \operatorname{cosec} \theta = a$ and $x \cos \theta - y \sin \theta = a \cos 2\theta$, then $4p^2 + q^2$ is equal to [Kerala CEE 2014]
 (a) $5a^2$ (b) $4a^2$ (c) $3a^2$
 (d) $2a^2$ (e) a^2
- 32.** If any point P is at the equal distances from points $A(a + b, a - b)$ and $B(a - b, a + b)$, then locus of a point is [RPET 2014]
 (a) $x - y = 0$ (b) $ax + by = 0$
 (c) $bx + ay = 0$ (d) $x + y = 0$
- 33.** The equation of the perpendicular bisector of the line segment joining $A(-2, 3)$ and $B(6, -5)$ is [Kerala CEE 2013]
 (a) $x - y = 1$ (b) $x - y = 3$ (c) $x + y = 3$
 (d) $x + y = 1$ (e) $x + y = -1$
- 34.** The equation of second degree $x^2 + 2\sqrt{2}xy + 2y^2 + 4x + 4\sqrt{2}y + 1 = 0$ represents a pair of straight lines. The distance between them is [UP SEE 2013]
 (a) $2\sqrt{3}$ (b) $2\sqrt{5}$ (c) 2 (d) 0
- 35.** Let $P = (-1, 0)$, $O = (0, 0)$ and $Q = (3, 3\sqrt{3})$ be three points. Then, the equation of the bisector of the $\angle POQ$ is [BITSAT 2013]
 (a) $y = \sqrt{3}x$ (b) $\sqrt{3}y = x$ (c) $y = -\sqrt{3}x$ (d) $\sqrt{3}y = -x$
- 36.** The point of intersection of line represented by the equation $3x^2 + 8xy - 3y^2 + 29x - 3y + 18 = 0$ is [MP PET 2012]
 (a) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (b) $\left(\frac{-3}{2}, \frac{-5}{2}\right)$ (c) $(-3, -5)$ (d) $(3, 5)$
- 37.** If lines $(\tan^2 \theta + \cos^2 \theta)x^2 - 2 \tan \theta \cdot xy + \sin^2 \theta \cdot y^2 = 0$ make with X -axis angles α and β , then $\tan \alpha - \tan \beta$ is equal to [MP PET 2012]
 (a) 2 (b) 4
 (c) $\tan \theta$ (d) $2 \tan \theta$
- 38.** If two pairs of lines $x^2 - 2mxy - y^2 = 0$ and $x^2 - 2nxy - y^2 = 0$ are such that one of them represents the bisector of the angles between the other, then [AMU 2012]
 (a) $mn = 1$ (b) $m + n = mn$
 (c) $mn = -1$ (d) $m - n = mn$
- 39.** If a straight line passes through the points $\left(-\frac{1}{2}, 1\right)$ and $(1, 2)$, then its x -intercept is [Kerala CEE 2011]
 (a) -2 (b) -1 (c) 2
 (d) 1 (e) 0
- 40.** The equation of the line passing through $(0, 0)$ and intersection $3x - 4y = 2$ and $x + 2y = -4$ is [J&K CET 2011]
 (a) $7x = 6y$ (b) $6x = 7y$
 (c) $5x = 8y$ (d) $x = 0$
- 41.** The line parallel to the X -axis and passing through the point of intersection of the lines $ax + 2by + 3b = 0$ and $bx - 2ay - 3a = 0$, where $(a, b) \neq (0, 0)$ is [Kerala CEE 2011]
 (a) above the X -axis at a distance of $3/2$
 (b) above the X -axis at a distance of $2/3$
 (c) below the X -axis at a distance of $2/3$
 (d) below the X -axis at a distance of $3/2$
 (e) below the X -axis at a distance of 3

- 42.** Equation of the bisector of the acute angle between lines $3x + 4y + 5 = 0$ and $12x - 5y - 7 = 0$ is
[BITSAT 2011]
(a) $21x + 77y + 100 = 0$ (b) $99x - 27y + 30 = 0$
(c) $99x + 27y + 30 = 0$ (d) $21x - 77y - 100 = 0$
- 43.** A variable line passes through a fixed point (a, b) and meets the coordinate axes in A and B . The locus of the point of intersection of lines through A, B parallel to coordinate axes is
[MP PET 2011]
(a) $\frac{x}{a} + \frac{y}{b} = 1$ (b) $\frac{a}{x} + \frac{b}{y} = 1$
(c) $\frac{x}{a} + \frac{y}{b} = 2$ (d) $\frac{x}{a} + \frac{y}{b} = 3$
- 44.** Locus of a point which moves such that its distance from the X -axis is twice its distance from the line $x - y = 0$ is
[Karnataka CET 2011]
(a) $x^2 + 4xy - y^2 = 0$ (b) $2x^2 - 4xy + y^2 = 0$
(c) $x^2 - 4xy + y^2 = 0$ (d) $x^2 - 4xy - y^2 = 0$
- 45.** The equations $y = \pm\sqrt{3}x$, $y = 1$ are the sides of
[BITSAT 2010]
(a) an equilateral triangle (b) a right angled triangle
(c) an isosceles triangle (d) a scalene triangle
- 46.** The equation of a straight line which passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to $x\sec\theta + y\csc\theta = a$ is
[Kerala CEE 2010]
(a) $\frac{x}{a} + \frac{y}{a} = a\cos\theta$
(b) $x\cos\theta - y\sin\theta = a\cos 2\theta$
(c) $x\cos\theta + y\sin\theta = a\cos 2\theta$
(d) $x\cos\theta + y\sin\theta - a\cos 2\theta = 1$
(e) $x\cos\theta - y\sin\theta + a\cos 2\theta = -1$
- 47.** If the line $px - qy = r$ intersects the coordinate axes at $(a, 0)$ and $(0, b)$, then the value of $a + b$ is equal to
[Kerala CEE 2010]
(a) $r\left(\frac{q+p}{pq}\right)$ (b) $r\left(\frac{q-p}{pq}\right)$ (c) $r\left(\frac{p-q}{pq}\right)$
(d) $r\left(\frac{p+q}{p-q}\right)$ (e) $r\left(\frac{p-q}{p+q}\right)$
- 48.** The equations of the line through $(1, 1)$ and making angles of 45° with the line $x + y = 0$ are [WB JEE 2010]
(a) $x - 1 = 0$, $x - y = 0$
(b) $x - y = 0$, $y - 1 = 0$
(c) $x + y - 2 = 0$, $y - 1 = 0$
(d) $x - 1 = 0$, $y - 1 = 0$
- 49.** If the sum of distances from a point P on two mutually perpendicular straight lines is 1 unit, then the locus of P is
[WB JEE 2010]
(a) a parabola
(b) a circle
(c) an ellipse
(d) a straight line
- 50.** The equation of one of the lines parallel to $4x - 3y = 5$ and at a unit distance from the point $(-1, -4)$ is
[Kerala CEE 2010]
(a) $3x + 4y - 3 = 0$
(b) $3x + 4y + 3 = 0$
(c) $4x - 3y + 3 = 0$
(d) $4x - 3y - 3 = 0$
(e) $4x - 3y - 4 = 0$
- 51.** A line has slope m and y -intercept 4. The distance between the origin and the line is equal to
[Kerala CEE 2010]
(a) $\frac{4}{\sqrt{1-m^2}}$ (b) $\frac{4}{\sqrt{m^2-1}}$
(c) $\frac{4}{\sqrt{m^2+1}}$ (d) $\frac{4m}{\sqrt{1+m^2}}$
(e) $\frac{4m}{\sqrt{m-1}}$
- 52.** The distance of the point $(1, 2)$ from the line $x + y + 5 = 0$ measured along the line parallel to $3x - y = 7$ is equal to
[Kerala CEE 2010]
(a) $4\sqrt{10}$ (b) 40
(c) $\sqrt{40}$ (d) $10\sqrt{2}$
(e) $2\sqrt{20}$
- 53.** A line through the point $A(2, 0)$ which makes an angle of 30° with the positive direction of X -axis is rotated about A in clockwise direction through an angle of 15° . Then, the equation of the straight line in the new position is
[WB JEE 2010]
(a) $(2 - \sqrt{3})x + y - 4 + 2\sqrt{3} = 0$
(b) $(2 - \sqrt{3})x - y - 4 + 2\sqrt{3} = 0$
(c) $(2 - \sqrt{3})x - y + 4 + 2\sqrt{3} = 0$
(d) $(2 - \sqrt{3})x + y + 4 + 2\sqrt{3} = 0$
- 54.** The number of points on the line $x + y = 4$ which are unit distance apart from the line $2x + 2y = 5$ is
[WB JEE 2010]
(a) 0 (b) 1
(c) 2 (d) ∞
- 55.** The slopes of the lines which makes an angle 45° with the line $3x - y = -5$ are
[Kerala CEE 2010]
(a) $(1, -1)$ (b) $\left(\frac{1}{2}, -1\right)$ (c) $\left(1, \frac{1}{2}\right)$
(d) $\left(2, -\frac{1}{2}\right)$ (e) $\left(-2, \frac{1}{2}\right)$
- 56.** The vertices of a triangle are $A(3, 7)$, $B(3, 4)$ and $C(5, 4)$. The equation of the bisector of the $\angle ABC$ is
[Kerala CEE 2010]
(a) $y = x + 1$ (b) $y = x - 1$
(c) $y = 3x - 5$ (d) $y = x$
(e) $y = -x$

Answers

Work Book Exercise 12.1

1. (c) 2. (a) 3. (d) 4. (d) 5. (a) 6. (b) 7. (a) 8. (b) 9. (b) 10. (b)

Work Book Exercise 12.2

1. (d) 2. (a) 3. (d) 4. (b) 5. (a) 6. (b) 7. (b) 8. (a) 9. (c) 10. (b)
11. (b) 12. (d)

Work Book Exercise 12.3

1. (a) 2. (b) 3. (a) 4. (c) 5. (b) 6. (b) 7. (a) 8. (b)

Work Book Exercise 12.4

1. (b) 2. (a) 3. (a) 4. (a) 5. (a) 6. (b) 7. (a) 8. (c) 9. (b) 10. (d)

Target Exercises

1. (b) 2. (d) 3. (d) 4. (a) 5. (b) 6. (a) 7. (b) 8. (d) 9. (d) 10. (a)
11. (b) 12. (a) 13. (b) 14. (b) 15. (c) 16. (a) 17. (a) 18. (c) 19. (a) 20. (d)
21. (a) 22. (b) 23. (a) 24. (a) 25. (d) 26. (d) 27. (c) 28. (b) 29. (c) 30. (a)
31. (a) 32. (a) 33. (a) 34. (a) 35. (c) 36. (b) 37. (b) 38. (b) 39. (c) 40. (b)
41. (b) 42. (c) 43. (b) 44. (b) 45. (b) 46. (a) 47. (c) 48. (a) 49. (a) 50. (a)
51. (b) 52. (b) 53. (d) 54. (c) 55. (d) 56. (c) 57. (d) 58. (c) 59. (a) 60. (a)
61. (c) 62. (a) 63. (d) 64. (b) 65. (c) 66. (c) 67. (a) 68. (b) 69. (d) 70. (d)
71. (c) 72. (b) 73. (a) 74. (d) 75. (c) 76. (a) 77. (b) 78. (a) 79. (b) 80. (c)
81. (a) 82. (a) 83. (c) 84. (b) 85. (b) 86. (d) 87. (a,b) 88. (a,c) 89. (all) 90. (b,c)
91. (a,c,d) 92. (all) 93. (b,c,d) 94. (a,b) 95. (d) 96. (a) 97. (d) 98. (c) 99. (b) 100. (a)
101. (d) 102. (c) 103. (d) 104. (d) 105. (b) 106. (b) 107. (c) 108. (b) 109. (d) 110. (a)
111. (b) 112. (d) 113. (c) 114. (a) 115. (b) 116. (b) 117. (b) 118. (c) 119. (a) 120. (d)
121. (c) 122. (a) 123. (*) 124. (**) 125. (***) 126. (2) 127. (4) 128. (6) 129. (1) 130. (2)
131. (4)

* $A \rightarrow p$; $B \rightarrow q$; $C \rightarrow s$; $D \rightarrow s$

** $A \rightarrow p$, s ; $B \rightarrow p$, r ; $C \rightarrow p$; $D \rightarrow p$

*** $A \rightarrow p$; $B \rightarrow s$; $C \rightarrow q$; $D \rightarrow r$

Entrances Gallery

1. (6) 2. (a) 3. (b) 4. (c) 5. (b) 6. (b) 7. (b) 8. (c) 9. (d) 10. (a)
11. (a) 12. (c) 13. (b) 14. (b) 15. (d) 16. (c) 17. (d) 18. (a) 19. (d) 20. (c)
21. (b) 22. (b) 23. (c) 24. (d) 25. (c) 26. (a) 27. (b) 28. (b) 29. (d) 30. (e)
31. (e) 32. (a) 33. (b) 34. (c) 35. (c) 36. (b) 37. (a) 38. (c) 39. (a) 40. (a)
41. (d) 42. (c) 43. (b) 44. (b) 45. (a) 46. (b) 47. (b) 48. (d) 49. (d) 50. (d)
51. (c) 52. (c) 53. (b) 54. (a) 55. (e) 56. (a)

Explanations

Target Exercises

1. As, $2b = a + c$

$$\therefore a - 2b + c = 0 \text{ and } ax + by + c = 0$$

Hence, it always passes through the point $(1, -2)$.

2. If $\frac{b^3 - c^3}{b - c} = \frac{c^3 - a^3}{c - a} = \frac{a^3 - b^3}{a - b}$

$$\Rightarrow (b^2 + c^2 + bc) = (c^2 + a^2 + ac) = (a^2 + b^2 + ab)$$

$$\Rightarrow b^2 - a^2 = -c(b - a)$$

$$\Rightarrow a + b + c = 0$$

3. On applying Eqs. (i), (ii) and (iii) transformation,

we finally get $A \equiv (0, 0)$, $B \equiv (4, 8)$, $C \equiv (5, 9)$,

$$D \equiv (1, 1), m_{AB} = \frac{8}{4}, m_{BC} = \frac{9-8}{5-4} = 1, m_{CD} = \frac{9-1}{5-1} = \frac{8}{4},$$

$$m_{AD} = 1, m_{AC} = \frac{9}{5}, m_{BD} = \frac{8-1}{4-1} = \frac{7}{3}$$

Hence, final figure will be a parallelogram.

4. All sides are equal and angles are 90° by Pythagoras theorem or $m_1 m_2 = -1$.

5. $\frac{1}{2} \begin{vmatrix} -2 & 0 & 1 \\ -1 & 1 & 1 \\ \cos \theta & \sin \theta & 1 \end{vmatrix} = 0$

$$\Rightarrow \sqrt{3} \sin \theta - \cos \theta = 2$$

$$\Rightarrow \sin \left(\theta - \frac{\pi}{6} \right) = 1 \Rightarrow \theta - \frac{\pi}{6} = \frac{\pi}{2}$$

6. The points $A(a, \alpha)$, $B(b, \beta)$ and $C(c, \gamma)$ are collinear, if

$$\text{Slope of } AB = \text{Slope of } BC$$

$$\Rightarrow \frac{\beta - \alpha}{b - a} = \frac{\gamma - \beta}{c - b}$$

$$\text{i.e. } (b - a)(\gamma - \beta) - (c - b)(\beta - \alpha) = 0$$

$$\text{i.e. } \gamma - \beta = \beta - \alpha$$

$[\because a, b \text{ and } c \text{ are in AP, so } b - a = c - b]$

$$\Rightarrow 2\beta = \gamma + \alpha$$

$$\Rightarrow \alpha = \beta = \gamma \quad [\because \alpha, \beta \text{ and } \gamma \text{ are in GP}]$$

7. Let the points be A, B, C and D .

Clearly, $m \text{ of } AD = 0 = m \text{ of } BC$

$$\therefore AD \parallel BC \parallel X\text{-axis. Also, } AD^2 = BC^2$$

$$m \text{ of } AC = \infty = m \text{ of } BD$$

$$\therefore AC \parallel BD \parallel Y\text{-axis. Also, } AC^2 = BD^2$$

Hence, the given points are vertices of a square.

8. Clearly, the lines $x + 3y = 4$ and $6x - 2y = 7$ are at right angles. So, diagonals of the parallelogram $PQRS$ intersect at right angles. Hence, $PQRS$ must be a rhombus.

9. Equation of the line passing through $(2, 2)$ and perpendicular to $3x + y = 3$ is

$$(x - 2) - 3(y - 2) = 0$$

$$\Rightarrow x - 3y = -4$$

$$\therefore y\text{-intercept of this line is } \frac{4}{3}$$

10. $\tan(\pm 45^\circ) = \frac{m - \frac{1}{2}}{1 + m \cdot \frac{1}{2}}$

$$\therefore m = 3, -\frac{1}{3}$$

$\therefore$ Sides are $y - 3x + 9 = 0$ and $3y + x - 3 = 0$.

11. $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$ [by definition]

12. Let the equation of line be $\frac{x}{a} + \frac{y}{b} = 1$

$$\Rightarrow -\frac{b}{a} \cdot 5 = -1 \Rightarrow 5b = a$$

$$\text{Area of } \Delta OAB = \frac{1}{2} |ab|$$

$$\Rightarrow 10 = \frac{1}{2} |5b^2| \Rightarrow b^2 = 4$$

$$\Rightarrow b = \pm 2, a = \pm 10$$

$$\text{The line can be } \frac{x}{10} + \frac{y}{2} = 1 \text{ or } \frac{x}{10} + \frac{y}{2} = -1$$

13. The bisector AD of ΔABC will divide the opposite side BC in the ratio of arms of the angle i.e. $AB : AC$ or $3\sqrt{2} : 4\sqrt{2}$ or $3 : 4$. Hence, the point D on BC by ratio formula is $\left(\frac{31}{4}, 1\right)$ and A is $(1, 1)$.

$\therefore$ Slope of $AD = 0$. Hence, the slope of CL which is perpendicular from C on AD is ∞ .

$$\therefore \text{Equation of } CL \text{ is } x - 5 = 0.$$

14. Here, ΔABC is isosceles.

Let (h, k) be the coordinates of the vertex A of the ΔABC .

$$\text{Since, } AB = AC$$

$$\Rightarrow (2 - h)^2 + (1 - k)^2 = (1 - h)^2 + (2 - k)^2$$

$$\Rightarrow h^2 + k^2 - 4h - 2k + 5 = h^2 + k^2 - 2h - 4k + 5$$

$$\Rightarrow h = k \quad \dots(i)$$

Again, $A(h, k)$ lies on the line AB whose equation is

$$y = \frac{1}{2}x$$

$$\text{Then, } k = \frac{1}{2}h \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$h = 0, k = 0$$

Now, equation of the line joining the points $A(0, 0)$ and $C(1, 2)$ is

$$y - 0 = \frac{2 - 0}{1 - 0}(x - 0) \Rightarrow y = 2x$$

15. $\frac{x - 2}{\cos \theta} = \frac{y - 3}{\sin \theta} = r \text{ or } (r + 2)$

$$\therefore [r \sin \theta + 3] + 2[r \cos \theta + 2] = 3 \text{ for } y + 2x = 3$$

$$\text{and } [(r + 2)\sin \theta + 3] + 2[(r + 2)\cos \theta + 2] = 5$$

For $y + 2x = 5$,

On subtracting, we get

$$\begin{aligned} 2\sin\theta + 4\cos\theta &= 2 \\ \therefore 2\cos\theta &= 1 - \sin\theta \\ \Rightarrow 4(1 - \sin^2\theta) &= (1 - \sin\theta)^2 \\ \Rightarrow \sin\theta &= 1 \text{ or } -\frac{3}{5} \end{aligned}$$

$$\begin{aligned} \text{From Eq. (i), } \cos\theta &= 0, \frac{4}{5} \\ \therefore \tan\theta &= \infty \text{ or } -\frac{3}{4} \\ \therefore x - 2 &= 0 \text{ and } 3x + 4y = 18 \end{aligned}$$

$$16. \frac{x+5}{\cos\theta} = \frac{y+4}{\sin\theta} = \frac{r_1}{AB} = \frac{r_2}{AC} = \frac{r_3}{AD}$$

Since, $(r_1 \cos\theta - 5, r_1 \sin\theta - 4)$ lies on $x + 3y + 2 = 0$.

$$\therefore r_1 = \frac{15}{\cos\theta + 3\sin\theta}$$

$$\text{Similarly, } \frac{10}{AC} = 2\cos\theta + \sin\theta$$

$$\text{and } \frac{6}{AD} = \cos\theta - \sin\theta$$

On putting in the given relation, we get
 $(2\cos\theta + 3\sin\theta)^2 = 0$

$$\begin{aligned} \therefore \tan\theta &= -\frac{2}{3} \Rightarrow y + 4 = -\frac{2}{3}(x + 5) \\ \Rightarrow 2x + 3y + 22 &= 0 \end{aligned}$$

17. Clearly, (a) is the only solution.

$$18. (-3 + 8 - 7)(9 - 56 - 7) \\ \Rightarrow (-1, -1) \text{ and } (3, 7) \text{ are required points.}$$

$$19. \sin\alpha - \frac{1}{\sqrt{2}} > 0 \text{ and } \frac{1}{\sqrt{2}} - \cos\alpha > 0 \\ \text{or } \sin\alpha - \frac{1}{\sqrt{2}} < 0 \text{ and } \frac{1}{\sqrt{2}} - \cos\alpha < 0 \\ \Rightarrow \alpha \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$$

20. If P lies between the acute region of the given lines, then
 $(6\sin\theta - 1 - \cos\theta)(3\sin\theta - 1 - \cos\theta) < 0$

$$\Rightarrow 72\cos^4\frac{\theta}{2} \left(\tan\frac{\theta}{2} - \frac{1}{6}\right) \left(\tan\frac{\theta}{2} - \frac{1}{3}\right) < 0$$

$$\Rightarrow \frac{1}{6} < \tan\left(\frac{\theta}{2}\right) < \frac{1}{3}$$

$$\Rightarrow n\pi + \tan^{-1}\left(\frac{1}{6}\right) < \frac{\theta}{2} < \tan^{-1}\left(\frac{1}{3}\right) + n\pi$$

$$\Rightarrow 2n\pi + 2\tan^{-1}\left(\frac{1}{6}\right) < \theta < 2n\pi + 2\tan^{-1}\left(\frac{1}{3}\right)$$

21. Let P be on $x + 2y = 1$, then

$$1 + \frac{t}{\sqrt{2}} + 2\left(2 + \frac{t}{\sqrt{2}}\right) = 1 \\ \Rightarrow t = -\frac{4\sqrt{2}}{3}$$

Let P be on $2x + 3y = 15$, then

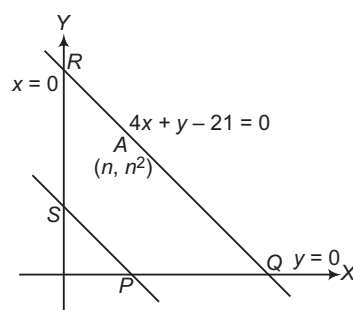
$$2\left(1 + \frac{t}{\sqrt{2}}\right) + 4\left(2 + \frac{t}{\sqrt{2}}\right) = 15$$

$$\Rightarrow t = \frac{5\sqrt{2}}{6}$$

$$\therefore \frac{-4\sqrt{2}}{3} < t < \frac{5\sqrt{2}}{6}$$

22. Origin is on the left of PS ,

$$\begin{aligned} 3x + y - 4, \text{ then } 0 + 0 - 4 < 0 \\ \Rightarrow A(n, n^2) \quad 3n + n^2 - 4 > 0 \\ \Rightarrow n^2 + 3n - 4 > 0 \\ \Rightarrow (n + 4)(n - 1) > 0 \Rightarrow n > 1 \\ \therefore n + 4 > 0 \end{aligned} \quad \dots(i)$$



Now, A and O lie on the same side of QR .

$$\begin{aligned} \therefore 4x + y - 21 = 0 + 0 - 21 < 0 \\ \Rightarrow A(n, n^2) \quad 4n + n^2 - 21 < 0 \\ \Rightarrow n^2 + 4n - 21 < 0 \Rightarrow (n + 7)(n - 3) < 0 \\ \Rightarrow 0 < n < 3 \\ \therefore n \in N \end{aligned} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$1 < n < 3 \Rightarrow n = 2$$

$A \equiv (2, 4)$ only one point.

23. $PR \perp QR \Rightarrow m_1 m_2 = -1$

$$\Rightarrow 3x + 4y = 13$$

Area of $\Delta PRQ = 7$

$$\Rightarrow \frac{1}{2}(9 - 4x + 3y) = \pm 7$$

$$\therefore 4x - 3y + 5 = 0 \text{ and } 4x - 3y - 23 = 0$$

$\therefore$ No solution.

24. The given line meets X -axis at $A(4, 0)$ and Y -axis at $B(0, 12)$. Let P divides AB in the ratio $1:2$ and Q divides AB in the ratio $2:1$, then coordinates of P and Q are $(8/3, 4)$ and $(4/3, 8)$, respectively. So, the equations of OP and OQ are respectively

$$y = \frac{4}{8/3}x \text{ and } y = \frac{8}{4/3}x$$

$$\text{or } 2y = 3x \text{ and } y = 6x$$

whose slopes $3/2$ and 6 are the roots of the equation

$$x^2 - \left(\frac{3}{2} + 6\right)x + \frac{3}{2} \times 6 = 0 \Rightarrow 2x^2 - 15x + 18 = 0$$

$$25. m_{AB} = \frac{-4-2}{3-1} = -3$$

Thus, equation of CD is $y - 8 = -3(x - 3)$

i.e. $y + 3x = 17$.

Equation of right bisector of AB is $y + 1 = \frac{1}{3}(x - 2)$

i.e. $3y = x - 5$.

Solving it with line CD , we get

$$x = \frac{28}{5}, y = \frac{1}{5}$$

Thus, mid-point of CD is $\left(\frac{28}{5}, \frac{1}{5}\right)$.

26. $B \equiv (0, a), A_1 \equiv \left(\frac{a}{1+a}, \frac{a^2}{1+a} \right)$

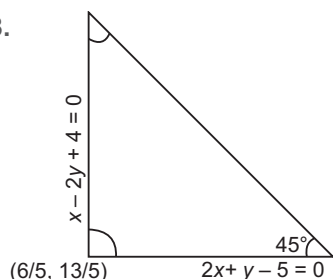
$$\Delta O A_1 B = \frac{1}{2} (OB) \left| \frac{a}{1+a} \right| = \frac{1}{2} \frac{a^2}{1+a}$$

27. We have, $A \equiv (0, 5), B \equiv \left(\frac{6}{7}, 2 \right), C \equiv \left(-\frac{6}{7}, 2 \right)$

$$\Rightarrow BC = \frac{12}{7}, AD = 3$$

$$\text{Area of } \Delta ABC = \frac{1}{2} \cdot \frac{12}{7} \cdot 3 = \frac{18}{7} \text{ sq units}$$

28.



$\therefore$ Area of triangle = 10

$\therefore$ Equation of third side is $3x - y - 11 = 0$.

29. Equation of straight line equally inclined to axes is $y = x + c$, equidistant from $(1, -2)$ and $(3, 4)$.

$$\Rightarrow \frac{|1+2+c|}{\sqrt{2}} = \frac{|3-4+c|}{\sqrt{2}}$$

$$\Rightarrow c+3 = -c+1 \Rightarrow c = -1$$

Hence, required straight line is $y = x - 1$.

30. $y = mx$

It makes an angle of $\pm 45^\circ$ with $2x + 3y = 6$.

$$\therefore \tan(\pm 45^\circ) = \frac{m - (-2/3)}{1 + m(-2/3)} = \pm 1$$

$$\Rightarrow 3m + 2 = \pm (3 - 2m)$$

$$\therefore m = \frac{1}{5}, -5$$

Hence, the sides are $x - 5y = 0, 5x + y = 0$

and $2x + 3y = 6$

Solving in pairs, the vertices are

$$(0, 0), \left(-\frac{6}{13}, \frac{30}{13} \right), \left(\frac{30}{13}, \frac{6}{13} \right)$$

$$\therefore \Delta = \frac{1}{2} (x_1 y_2 - x_2 y_1) = \frac{1}{2} \times \frac{936}{169} = \frac{36}{13}$$

31. $\therefore A(-1, 0), B(0, -1), C(1, 0)$ and $D(0, 1)$

Clearly, $ABCD$ is a square.

$\Rightarrow$ Area of square = 2 sq units

32. Let the required line be $2y = x + c$.

$$\text{We have, } A_1 \equiv \left(4, \frac{4+c}{2} \right), B_1 \equiv \left(0, \frac{c}{2} \right)$$

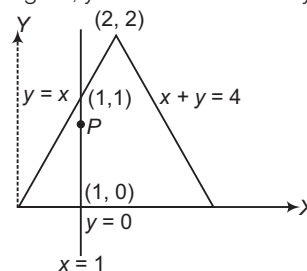
Area of rectangle $ABCD$ is 12 sq units.

Thus, area of trapezium OAA_1B_1 is

$$6 = \frac{1}{2} (4) \left(\frac{c}{2} + \frac{4+c}{2} \right) \Rightarrow c = 1$$

Thus, required line is $2y = x + 1$.

33. From the figure, y -coordinate of P vary from 0 to 1.



So, to be an interior point, $0 < \text{ordinate of } P < 1$.

34. Distance between $x + 2y + 3 = 0$ and $x + 2y - 7 = 0$ is $\frac{10}{\sqrt{5}}$.

Let the remaining side be $2x - y + \lambda = 0$.

We have, $\frac{|\lambda + 4|}{\sqrt{5}} = \frac{10}{\sqrt{5}} \Rightarrow \lambda = 6, -14$, thus remaining side is $2x - y + 6 = 0$ or $2x - y - 14 = 0$.

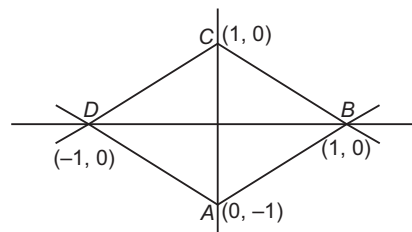
35. Any line parallel to X -axis is $y = k$ and it meets $y = \sqrt{x}$ in the point $P(k^2, k)$. Slope of the 1st line is 0 and

$$\left(\frac{dy}{dx} \right)_P = \frac{1}{2\sqrt{x}} = \frac{1}{2k} \text{ for the given curve.}$$

Since, angle of intersection is 45° .

$$\Rightarrow k = \frac{1}{2} \text{ or } x = \frac{1}{4} \text{ and } y = \frac{1}{2}$$

36. Required area of $ABCD = (AB)^2 = (\sqrt{2})^2 = 2$ sq units



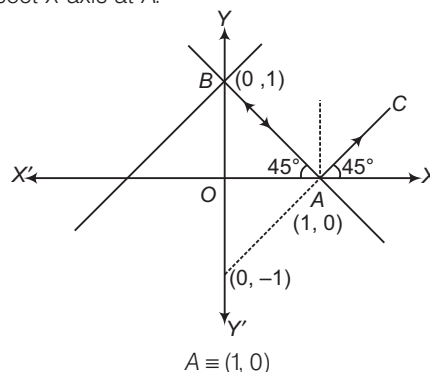
37. The given lines are $x + 2y = 1, x - 2y = 1$ and $x = 0$,

where vertices of curve are $A(1, 0), B\left(0, \frac{1}{2}\right)$ and

$$C\left(0, -\frac{1}{2}\right).$$

$$\therefore \text{Required area} = \frac{1}{2}$$

38. Since, incident ray is perpendicular to the line mirror $y - x = 1$, so after reflection, reflected ray goes back and intersect X -axis at A .



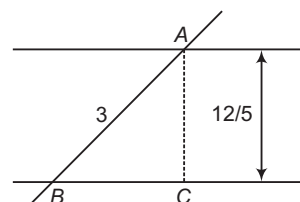
whose equation is $x - y = 1$.

39. The equation of the line is
 $(y - 2) = m(x - 8), m < 0$
 The coordinates of P and Q are $P\left(8 - \frac{2}{m}, 0\right)$ and $Q(0, 2 - 8m)$.
 Therefore, $OP + OQ = 8 - \frac{2}{m} + 2 - 8m$
 $= 10 + \frac{2}{-m} + 8(-m)$
 Using AM $\geq$ GM,
 $\frac{\frac{2}{(-m)} + 8(-m)}{2} \geq \sqrt{\frac{2}{(-m)} \times 8(-m)}$
 $\therefore OP + OQ \geq 10 + 2\sqrt{\frac{2}{-m} \times 8(-m)} = 18$
 Thus, absolute minimum value of $OP + OQ = 18$.
40. Since, the lines are concurrent, then
 $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$
 $\Rightarrow a(cb - a^2) - b(b^2 - ac) + c(ba - c^2) = 0$
 $\Rightarrow a^3 + b^3 + c^3 = 3abc$
41. We have, $3a - 2b + 5c = 0$
 $\Rightarrow (3/5)a + (-2/5)b + c = 0$
 Therefore, $ax + by + c = 0$ always passes through the point $(3/5, -2/5)$.
42. $\rho = a \sin 60^\circ$ or $\frac{1}{\sqrt{2}} = \frac{a\sqrt{3}}{2} \Rightarrow \sqrt{\frac{2}{3}} = a$
43. $\frac{|0 + 0 - 1|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = \frac{|0 + 0 - 1|}{\sqrt{\frac{1}{p^2} + \frac{1}{q^2}}} \Rightarrow \frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{a^2} + \frac{1}{b^2}$
44. Let the coordinates of foot of perpendicular be (α, β) .
 $\therefore \frac{\alpha - 2}{1} = \frac{\beta - 4}{1} = -\frac{(2 + 4 - 1)}{1^2 + 1^2} \Rightarrow \alpha - 2 = \beta - 4 = -\frac{5}{2}$
 $\therefore \alpha = 2 - \frac{5}{2}$ and $\beta = 4 - \frac{5}{2}$
 Thus, foot of perpendicular is $\left(-\frac{1}{2}, \frac{3}{2}\right)$.
45. $\frac{4k + 3k - 10}{\sqrt{4^2 + 3^2}} = \pm 1 \Rightarrow 4h + 3k = 10 \pm 5$
 and $h + k = 4 \Rightarrow h = 3, -7 \Rightarrow k = 1, 11$
 $\Rightarrow (3, 1)$ and $(-7, 11)$ are required points.
46. Let the vertices B and C lie on the given line.
 $OD = \frac{2\sqrt{2}}{\sqrt{2}} = 2$
 Equation of OD is
 $y = x \Rightarrow x = y = \sqrt{2}$ [for point D]
 Also, $BD = OD \cdot \tan 60^\circ = 2\sqrt{3}$ for the coordinates of B and C , using parametric equation of line, we get
 $\frac{x - \sqrt{2}}{-\frac{1}{\sqrt{2}}} = \frac{y - \sqrt{2}}{\frac{1}{\sqrt{2}}} = \pm 2\sqrt{3}$
 $\Rightarrow C \equiv (\sqrt{2} + \sqrt{6}, \sqrt{2} - \sqrt{6})$ and $D \equiv (\sqrt{2} - \sqrt{6}, \sqrt{2} + \sqrt{6})$

47. $A = (0, 0), B = (2 \cot \theta, 2), C = (4 \cot 60^\circ - \theta, -4)$
 $\therefore$ (Side of an equilateral triangle)²
 $= 4 \cot^2 \theta + 4 = 16 \cot^2 (60^\circ - \theta) + 16$
 $4 \operatorname{cosec}^2 \theta = 16 \operatorname{cosec}^2 (60^\circ - \theta)$
 $\therefore \tan \theta = \frac{\sqrt{3}}{5}$
 Hence, required length $= 2 \operatorname{cosec} \theta$
 $= 2 \times \frac{\sqrt{25 + 3}}{\sqrt{3}} = \frac{2\sqrt{28}}{\sqrt{3}} = \frac{4\sqrt{7}}{\sqrt{3}}$

48. Distance between parallel lines must be equal.
 $\Rightarrow \frac{|c_1 - c_2|}{\sqrt{a_1^2 + b_1^2}} = \frac{|d_1 - d_2|}{\sqrt{a_2^2 + b_2^2}}$
 $\Rightarrow (a_2^2 + b_2^2)(c_1 - c_2)^2 = (a_1^2 + b_1^2)(d_1 - d_2)^2$

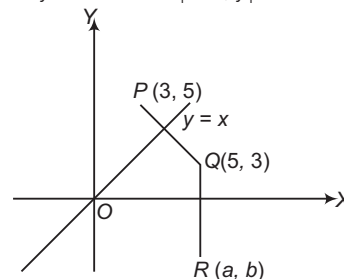
49. Slope of $L_1 = m = \tan(\theta \pm \alpha)$
 $= \frac{\tan \theta \pm \tan \alpha}{1 \mp \tan \theta \cdot \tan \alpha}$



m_1 is not defined and $m_2 = -\frac{7}{24}$
 $x = 1$ for which m is not defined.

50. The equation of line BC is $x + y + 4 = 0$.
 $\therefore$ Equation of a line parallel to BC is $x + y + k = 0$.
 which is at a distance $\frac{1}{2}$ from the origin.
 $\therefore \frac{|k|}{\sqrt{2}} = \frac{1}{2}$
 $\Rightarrow k = \pm \frac{1}{\sqrt{2}}$
 Since, BC and the required line are on the same side of the origin.
 $\therefore k = \frac{1}{\sqrt{2}}$
 Hence, the required line is $x + y + \frac{1}{\sqrt{2}} = 0$.

51. Let (x_1, y_1) be the image of the point $P(3, 5)$ with respect to the line $y = x$. Then, $x_1 = 5, y_1 = 3$



$\therefore Q = (5, 3)$
 Since, the image of the point $Q(5, 3)$ w.r.t. the line $y = 0$ is (a, b) .
 $\therefore a = 5$ and $b = -3$
 Hence, $(a, b) = (5, -3)$

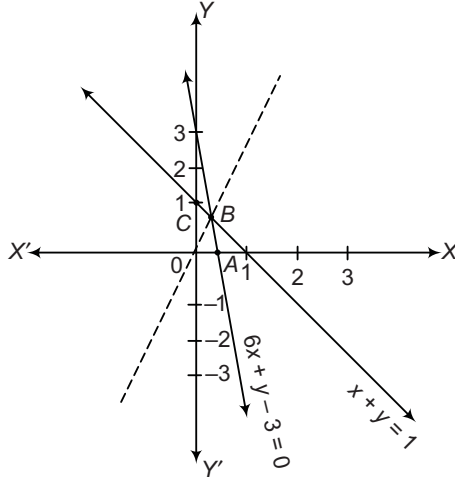
52. Clearly, the diagonal passing through $(0, 0)$ will also pass through vertex B . Now, equation of lines passing through B , are given by

$$(x + y - 1) + \lambda(6x + y - 3) = 0 \quad \dots(i)$$

Since, required line is also

passes through $(0, 0)$, therefore we have

$$-1 + \lambda(-3) = 0 \Rightarrow \lambda = -1/3$$



From Eq. (i), we get

$$(x + y - 1) + \left(-\frac{1}{3}\right)(6x + y - 3) = 0$$

$$\Rightarrow 3x - 2y = 0$$

which is the required line.

53. Given family is concurrent at $P\left(-\frac{1}{3}, \frac{1}{3}\right)$. If $Q \equiv (1, -3)$,

$$\text{then } m_{PQ} = \frac{-3 - \frac{1}{3}}{1 + \frac{1}{3}} = -\frac{5}{2}. \text{ Now, member of the family}$$

that is farthest from Q will have its slope as $\frac{2}{5}$.

$$\Rightarrow \frac{2}{5} = -\frac{(1 + 2\lambda)}{(1 - \lambda)} \Rightarrow \lambda = -\frac{7}{8}$$

Thus, equation of required line is

$$(x + y) - \frac{7}{8}(2x - y + 1) = 0 \Rightarrow 15y - 6x - 7 = 0$$

54. Here, $x + y + \lambda(2x - y + 1) = 0$

Each line passes through the point of intersection of the lines $x + y = 0$ and $2x - y + 1 = 0$, which is $\left(-\frac{1}{3}, \frac{1}{3}\right)$.

$$\therefore \text{Slope } (m) \text{ of the required line} = \frac{-1}{\frac{4 - 1/3}{1 + 1/3}} = -\frac{4}{11}$$

$$\therefore \text{The required line is } y - \frac{1}{3} = -\frac{4}{11}\left(x + \frac{1}{3}\right)$$

$$\Rightarrow 12x + 33y = 7$$

55. Lines $5x + 3y - 2 + \lambda_1(3x - y - 4) = 0$ are concurrent at $(1, -1)$ and lines $x - y + 1 + \lambda_2(2x - y - 2) = 0$ are concurrent at $(3, 4)$. Thus, equation of line common to both family is

$$y - 4 = \frac{-1 - 4}{1 - 3}(x - 3)$$

$$\Rightarrow 5x - 2y - 7 = 0$$

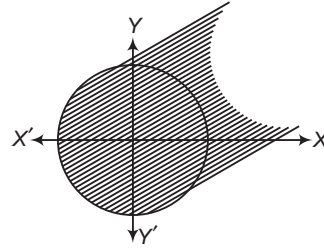
56. $x(1 + \tan^2 \theta) + y \tan^2 \theta - 2 = 0$

$$\Rightarrow (x - 2) + \tan^2 \theta (x + y) = 0$$

$\Rightarrow (x - 2) + \lambda(x + y) = 0$ passes through point of intersection of the lines $x - 2 = 0$ and $x + y = 0$.

$$\therefore \text{Point} \equiv (2, -2)$$

57. Since, $y - y_1 = m(x - x_1)$ represents family of parallel lines.



So, obvious family of parallel lines.

58. $(1, 5)$ is the mid-point of AB , where A and B are on 1st and 2nd line but in opposite directions.

$$\therefore \frac{x - 1}{\cos \theta} = \frac{y - 5}{\sin \theta} = \frac{r}{\text{for } A} = \frac{-r}{\text{for } B} \quad \dots(i)$$

$$A \text{ lies on 1st, } 5(r \cos \theta + 1) - (r \sin \theta + 5) = 4$$

$$B \text{ lies on 2nd,}$$

$$3(-r \cos \theta + 1) + 4(-r \sin \theta + 5) = 4$$

$$\therefore r(5 \cos \theta - \sin \theta) = 4$$

$$\text{and } r(3 \cos \theta + 4 \sin \theta) = 19$$

$$\therefore \tan \theta = \frac{83}{35}$$

Hence, equation of bisector is $83x - 35y + 92 = 0$.

59. Here, $a_1 = 1, b_1 = -2, c_1 = 4$

$$a_2 = 4, b_2 = -3, c_2 = 2$$

$$\therefore a_1 a_2 + b_1 b_2 = 4 + 6 = 10 > 0$$

$\therefore$ Equation of the bisector of obtuse angle between the given lines, is

$$\frac{x - 2y + 4}{\sqrt{1^2 + (-2)^2}} = \frac{4x - 3y + 2}{\sqrt{4^2 + (-3)^2}}$$

$$\Rightarrow \frac{x - 2y + 4}{\sqrt{5}} = \frac{4x - 3y + 2}{5}$$

$$\Rightarrow 5x - 10y + 20 = 4\sqrt{5}x - 3\sqrt{5}y + 2\sqrt{5}$$

$$\Rightarrow (4\sqrt{5} - 5)x + (10 - 3\sqrt{5})y + 2\sqrt{5} - 20 = 0$$

$$\Rightarrow (4 - \sqrt{5})x + (2\sqrt{5} - 3)y + 2 - 4\sqrt{5} = 0$$

60. Equation of angle bisectors are

$$\frac{x + y - 1}{\sqrt{2}} = \pm \frac{2x + 3y - 1}{\sqrt{13}}$$

$$\text{Slope, } m = \frac{\sqrt{13} - 2\sqrt{2}}{3\sqrt{2} - \sqrt{13}} \text{ and } -\left(\frac{\sqrt{13} + 2\sqrt{2}}{3\sqrt{2} + \sqrt{13}}\right)$$

Equation of required line is

$$(y - 2) = \frac{\sqrt{13} - 2\sqrt{2}}{3\sqrt{2} - \sqrt{13}}(x - 1)$$

$$\Rightarrow (3\sqrt{2} - \sqrt{13})y - 6\sqrt{2} + 2\sqrt{13} = (\sqrt{13} - 2\sqrt{2})x - \sqrt{13} + 2\sqrt{2}$$

$$\Rightarrow \sqrt{13}(x + y - 3) = \sqrt{2}(2x + 3y - 8)$$

$$\text{and } y - 2 = -\left(\frac{\sqrt{13} + 2\sqrt{2}}{3\sqrt{2} + \sqrt{13}}\right)(x - 1)$$

$$\Rightarrow (3\sqrt{2} + \sqrt{13})y - 6\sqrt{2} - 2\sqrt{13} = -(\sqrt{13} + 2\sqrt{2})x + \sqrt{13} + 2\sqrt{2}$$

$$\Rightarrow \sqrt{13}(x + y - 3) = -\sqrt{2}(2x + 3y - 8)$$

61. $ax + by = 1$ will be one of the bisectors of the given line.

Equations of bisectors of the given lines are,

$$\frac{3x + 4y - 5}{5} = \pm \left(\frac{5x - 12y - 10}{13} \right)$$

 $\Rightarrow 64x - 8y = 115$
 or $14x + 112y = 15$
 $\Rightarrow \left(a = \frac{64}{115}, b = -\frac{8}{115} \right)$ or $\left(a = \frac{14}{15}, b = \frac{112}{15} \right)$

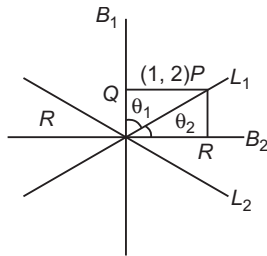
62. $\frac{3x - 4y + 7}{\sqrt{3^2 + 4^2}} = \pm \frac{12x + 5y - 2}{\sqrt{(12)^2 + 5^2}}$

$\Rightarrow 21x + 77y - 101 = 0$
 $\Rightarrow 11x - 3y + 9 = 0$

63. Angle bisectors will make the angles $\frac{\theta_1 + \theta_2}{2}$ and $\left(\frac{\pi}{2} + \frac{\theta_1 + \theta_2}{2} \right)$ with the X-axis. Hence, their equations are

$$\frac{x - x_1}{-\sin\left(\frac{\theta_1 + \theta_2}{2}\right)} = \frac{y - y_1}{\cos\left(\frac{\theta_1 + \theta_2}{2}\right)}$$

64. $PQ = \frac{|3 - 8 - 7|}{5} = \frac{12}{5}$, $PR = \frac{|4 + 6 - 2|}{5} = \frac{8}{5}$



$\therefore PQ > PR$
 So, B_1 is an obtuse angle bisector and B_2 is an acute angle bisector.

$\therefore \theta_1 > \theta_2$

65. Let the line be $\frac{x}{a} + \frac{y}{b} = 1$, then $\frac{1}{a} + \frac{3}{b} = 1$

$A \equiv (a, 0), B \equiv (0, b)$

$\Rightarrow P \equiv (a, b)$

Thus, locus of P is $\frac{1}{x} + \frac{3}{y} = 1$.

66. Let the equation of side BC be $y = x + a$.

$\Rightarrow A \equiv (1 - a, 1), B \equiv (2, 2 + a)$

Equation of side AD is

$y - 1 = -\{x - (1 - a)\}$

$\Rightarrow D \equiv (-2, 4 - a)$

Let $C \equiv (h, k) \Rightarrow h + 1 - a = 2 - 2$

$\Rightarrow h = a - 1$

and $k + 1 = 2 + a + 4 - a$

$\therefore k = 5$

Thus, locus of C is $y = 5$.

67. $\sqrt{x^2 + 4y^2 - 4xy + 4} = 1 - x + 2y$

Squaring both sides, we get

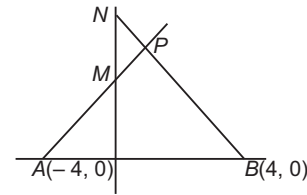
$x^2 + 4y^2 - 4xy + 4 = 1 + x^2 + 4y^2 - 2x + 4y - 4xy$

$\Rightarrow 2x - 4y + 3 = 0$

68. Let $M \equiv (0, h), N \equiv (0, h + 4)$

Equation of AM is

$\frac{x}{-4} + \frac{y}{h} = 1 \quad \dots (i)$



Equation of BN is

$\frac{x}{4} + \frac{y}{h+4} = 1 \quad \dots (ii)$

Elimination h from Eqs. (i) and (ii), we get
 $x^2 + 2xy - 16 = 0$

69. Family of lines passes through $(0, 2)$ pair of given lines

$S : (x - 2y + 3)(x - y + 1) = 0$

$\Rightarrow x^2 + 2y^2 + 4x - 3xy - 5y + 3 = 0$

Chord with middle point (h, k) is $T = S_1$

$\Rightarrow xh + 2yk + 2(x + h) - \frac{3(xh + yk)}{2} - \frac{5}{2}(y + k)$

$= h^2 + 2k^2 + 4h - 4hk - 5k + 3$

It passes through $(0, 2)$.

$\therefore 4k + 2h - \frac{3}{2}(2h) - \frac{5}{2}(2 + k) + 3 = 1$

$\Rightarrow 8y + 4x - 10 - 5y + 6 = 0$

$\Rightarrow 4x + 3y = 4$

70. We have, $\angle B = \angle C = \frac{\pi}{4}$. Let m be the slope of side

AB , then $1 = \left| \frac{m - 2}{1 + 2m} \right| \Rightarrow m = \frac{1}{3}$ and $m = -3$

Thus, equations of equal sides are $x - 3y + 1 = 0$

and $3x + y - 7 = 0$

Their combined equation is

$(x - 3y + 1)(3x + y - 7) = 0$

$\Rightarrow 3x^2 - 3y^2 - 8xy - 4x + 22y - 7 = 0$

71. Equation of incident ray is $y = -\frac{1}{\sqrt{3}}(x - 2)$

Equation of refracted ray is $y = -\sqrt{3}(x - 2)$

Combined equation is

$(x - 2 + y\sqrt{3})\left(x - 2 + \frac{y}{\sqrt{3}}\right) = 0$

$\Rightarrow (x - 2)^2 + y^2 + \frac{y}{\sqrt{3}}(x - 2) = 0$

72. Common point is $(3, 4)$.

Let L_1 and L_2 be $y - 4 = m(x - 3)$ and $y - 4 = -\frac{1}{m}(x - 3)$

Now,

$A_1 = A_2$

$\Rightarrow \frac{1}{2} \frac{(4m + 3)^2}{m} = \frac{(4 - 3m)^2}{m} \times \frac{1}{2}$

$\Rightarrow 7m^2 + 48m - 7 = 0$

$m_1 + m_2 = -\frac{48}{7}, m_1 m_2 = -1$

$\Rightarrow |\theta_1 + \theta_2| = \tan^{-1}\left(\frac{24}{7}\right)$

73. Given equation will represent a pair of straight lines, if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

Here, $a = 6, h = \frac{11}{2}, b = -10, g = \frac{1}{2}, f = \frac{31}{2}, c = \lambda$

$$\therefore \Delta = 0$$

$$\text{On comparing, } \lambda = \frac{-5415}{361} = -15$$

Also, $h^2 - ab = \frac{121}{4} + 60 = \frac{361}{4} > 0$

$$\therefore \lambda = -15 \text{ is valid.}$$

74. We have,

$$ax^2 + 2\lambda xy + by^2 + 2kx + 2ky + 2k = 0$$

On comparing, $h = \lambda, g = k, c = 2k, f = k$

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow ab \cdot (2k) + 2\lambda k^2 - ak^2 - bk^2 - 2\lambda^2 k = 0$$

$$\Rightarrow 2k\lambda^2 - 2k^2\lambda + (a+b)k^2 - 2abk = 0$$

For real $\lambda, B^2 - 4AC \geq 0$

$$\Rightarrow 4k^4 - 4 \cdot 2k [(a+b)k^2 - 2abk] \geq 0$$

$$\Rightarrow k^2 - 2(a+b)k + 4ab \geq 0$$

$$\Rightarrow (k-2a)(k-2b) \geq 0$$

$$\Rightarrow k \text{ does not lie between } 2a \text{ and } 2b.$$

$$\Rightarrow k \leq 2a, k \geq 2b$$

75. $x^3 + y^3 = 0$

$$\Rightarrow (x+y)(x^2 - xy + y^2) = 0$$

$$x+y=0 \quad [\text{real line}]$$

$$x^2 - xy + y^2 = 0 \quad [\text{imaginary line}]$$

76. Let $x = r \cos \theta$ and $y = r \sin \theta$

$$r^3 \cos^3 \theta + r^3 \cos \theta \sin^2 \theta + 2r^2 \cos^2 \theta + 2r^2 \sin^2 \theta + 3r \cos \theta + 1 = 0$$

$$r^3 \cos \theta + 2r^2 + 3r \cos \theta + 1 = 0$$

Let r_1, r_2, r_3 be the three roots.

$$r_1 + r_2 + r_3 = -\frac{2}{\cos \theta} \quad \dots (i)$$

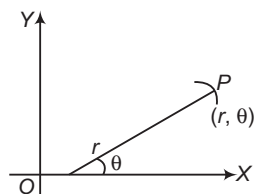
$$r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{3 \cos \theta}{\cos \theta} = 3 \quad \dots (ii)$$

$$r_1 r_2 r_3 = -\frac{1}{\cos \theta} \quad \dots (iii)$$

$\therefore$ OA, OB and OC are in HP.

$$\therefore r_2 = \frac{2r_1 r_3}{r_1 + r_3} \quad \dots (iv)$$

$$\Rightarrow r_2 r_1 + r_2 r_3 = 2r_1 r_3$$



From Eq. (ii), $3r_1 r_3 = 3$

$$\Rightarrow r_1 r_3 = 1$$

From Eq. (iii), $r_2 = -\frac{1}{\cos \theta}$

From Eq. (iv), $r_1 + r_3 = \frac{2r_1 r_3}{r_2}$

$$\Rightarrow r_1 + r_3 = -2 \cos \theta$$

From Eq. (i),

$$-2 \cos \theta - \frac{1}{\cos \theta} = -\frac{2}{\cos \theta}$$

$$\Rightarrow 2 \cos \theta = \frac{2}{\cos \theta} - \frac{1}{\cos \theta} = \frac{1}{\cos \theta}$$

$$\cos^2 \theta = \frac{1}{2}$$

$$\Rightarrow \sec^2 \theta = 2$$

$$\Rightarrow \tan^2 \theta = 1$$

$$\therefore \tan \theta = 1 \text{ or } -1$$

77. We have,

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$\equiv (l_1 x + m_1 y + n_1)(l_2 x + m_2 y + n_2)$$

Comparing the coefficient of similar term, we get

$$l_1 l_2 = a, m_1 m_2 = b, n_1 n_2 = c$$

$$l_1 m_2 + l_2 m_1 = 2h, l_1 n_2 + l_2 n_1 = 2g,$$

$$m_1 n_2 + m_2 n_1 = 2f$$

Now, the two lines are equidistant from origin.

$$\therefore \frac{0 \cdot l_1 + 0 \cdot m_1 + n_1}{\sqrt{l_1^2 + m_1^2}} = \frac{0 \cdot l_2 + 0 \cdot m_2 + n_2}{\sqrt{l_2^2 + m_2^2}}$$

$$\Rightarrow n_1^2(l_2^2 + m_2^2) = n_2^2(l_1^2 + m_1^2)$$

$$\Rightarrow n_1^2 l_2^2 - n_2^2 l_1^2 = n_2^2 m_1^2 - n_1^2 m_2^2$$

$$\Rightarrow (n_1 l_2 + n_2 l_1)^2 [(n_1 l_2 - n_2 l_1)^2 - 4n_1 n_2 l_1 l_2]$$

$$= (m_1 n_2 + m_2 n_1)^2 [(m_1 n_2 - m_2 n_1)^2 - 4m_1 m_2 n_1 n_2]$$

$$\therefore 4g^2[4g^2 - 4ac] = 4f^2[4f^2 - 4bc]$$

$$\Rightarrow f^4 - g^4 = c(bf^2 - ag^2)$$

78. Clearly, coefficient of x^2 + coefficient of $y^2 = 0$ in option (a).

79. $y = x$ should satisfy $ax^2 + 2hxy + by^2 = 0$

$$\Rightarrow a + b = -2h$$

80. If the lines intersect $(x_1, 0)$, then

$$ax^2 + h \cdot 0 + g = 0,$$

$$hx_1 + 0 \cdot b + f = 0$$

$$\Rightarrow x_1 = -\frac{g}{a} = -\frac{f}{h}$$

$$\therefore af = gh$$

81. The lines of first pair are $3x + 4y = 0$ and $4x - 3y = 0$.

The lines of second pair are $3x + 4y - 1 = 0$ and $4x - 3y + 1 = 0$. They may enclose a square or a rectangle.

But the distance between each pair of parallel lines is same i.e. $\frac{1}{5}$. Hence, it is a square.

82. Since, area of the triangle formed by

$$ax^2 + 2hxy + by^2 = 0 \text{ and } lx + my + n = 0, \text{ is}$$

$$\frac{n^2 \sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2} = \frac{81 \sqrt{\frac{81}{4} - 18}}{18 \cdot 1 + 2 \cdot \frac{9}{2} \cdot 1 \cdot 0 + 1 \cdot 0^2}$$

$$= \frac{81 \times 3}{2 \times 18} = \frac{27}{4} \text{ sq units}$$

83. Given, $4x^2 - 9xy - 9y^2 = 0$ and $x = 2$

$$\Rightarrow 4(2)^2 - 9(2)y - 9y^2 = 0$$

$$\Rightarrow 9y^2 + 18y - 16 = 0$$

$$\Rightarrow (3y + 8)(3y - 2) = 0$$

$$\Rightarrow y = -\frac{8}{3}, \frac{2}{3}$$

$\therefore$ Points of intersection are $\left(2, -\frac{8}{3}\right)$ and $\left(2, \frac{2}{3}\right)$.

$$\therefore \text{Required area} = \frac{10}{3} \text{ sq units}$$

84. Equation of a curve which passes through the point of intersection of two curves C_1 and C_2 , is

$$C_1 + 3C_2 = 0 \Rightarrow (\lambda + 9)x^2 - 9y^2 + (24 - 2\lambda)xy = 0 \dots (i)$$

which represents a pair of straight lines passing through origin.

Let $y = m_1x$ and $y = m_2x$ be two lines.

$$\therefore m_1 m_2 x^2 - y^2 - (m_1 + m_2)xy = 0 \dots (ii)$$

$$\Rightarrow \frac{m_1 m_2}{\lambda + 9} = -\frac{1}{9} = \frac{-(m_1 + m_2)}{24 - 2\lambda} \text{ [from Eqs. (i) and (ii)]}$$

$$\Rightarrow m_1 + m_2 = \frac{2\lambda - 24}{9}$$

Since, both lines are equally inclined with X-axis.

$$\therefore m_1 + m_2 = 0$$

$$\Rightarrow \lambda = 12$$

85. $x = \frac{5 \pm 2\sqrt{5}}{3}$, $y = \frac{2\sqrt{5} \mp 5}{3}$ are the point of intersection.

$\therefore$ Required area of the triangle, thus formed

$$= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ \frac{5+2\sqrt{5}}{3} & \frac{2\sqrt{5}-5}{3} & 1 \\ \frac{5-2\sqrt{5}}{3} & \frac{2\sqrt{5}+5}{3} & 1 \end{vmatrix} = 5 \text{ sq units}$$

86. Making the equation of curve homogeneous with the help of equation of line $\frac{fx - gy}{\lambda} = 1$ and to be

perpendicular both the lines represented by this homogeneous equation

$$a + b = 0 \Rightarrow \lambda + gf - \lambda - gf = 0 \Rightarrow 0 = 0$$

Hence, λ may have any value.

87. Let $P(2, -1)$ goes 2 units along $x + y = 1$ upto A and 5 units along $x - 2y = 4$ upto B .

$$\text{Slope of } PA = -1 = \tan 135^\circ, \text{ slope of } PB = \frac{1}{2} = \tan \theta$$

$$\therefore \sin \theta = \frac{1}{\sqrt{5}}, \cos \theta = \frac{2}{\sqrt{5}}$$

$$\therefore A \equiv (x_1 + r \cos 135^\circ, y_1 + r \sin 135^\circ)$$

$$= \left(2 + 2 \times -\frac{1}{\sqrt{2}}, -1 + 2 \times \frac{1}{\sqrt{2}}\right)$$

$$= (2 - \sqrt{2}, \sqrt{2} - 1)$$

$$B \equiv (x_1 + r \cos \theta, y_1 + r \sin \theta)$$

$$= \left(2 + 5 \times \frac{2}{\sqrt{5}}, -1 + \frac{5}{\sqrt{5}}\right)$$

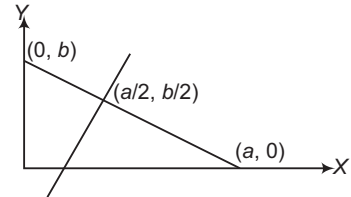
$$= (2\sqrt{5} + 2, \sqrt{5} - 1)$$

$$88. \frac{x - \frac{a}{2}}{\frac{a}{\sqrt{a^2 + b^2}}} = \frac{y - \frac{b}{2}}{\frac{b}{\sqrt{a^2 + b^2}}} = \pm \frac{\sqrt{a^2 + b^2}}{2}$$

$$x = \frac{a}{2} + \frac{b}{2}, y = \frac{b}{2} + \frac{a}{2}$$

and

$$x = \frac{a}{2} - \frac{b}{2}, y = \frac{b}{2} - \frac{a}{2}$$



$\therefore$ The required points are

$$\left(\frac{a+b}{2}, \frac{a+b}{2}\right) \text{ and } \left(\frac{a-b}{2}, \frac{b-a}{2}\right).$$

89. The equation of the line in parametric form is

$$\frac{x-3}{\cos \theta} = \frac{y-4}{\sin \theta} = r$$

Any point on this line is $(3 + r \cos \theta, 4 + r \sin \theta)$.

It lies on $x = 6$, if $3 + r \cos \theta = 6 \Rightarrow r = 3 \sec \theta$

$$\therefore PR = 3 \sec \theta$$

Again, the point lies on $y = 8$, if $4 + r \sin \theta = 8$

$$\therefore r = 4 \operatorname{cosec} \theta \text{ or } PS = 4 \operatorname{cosec} \theta$$

90. Let h be the height of the triangle.

Since, the area of the triangle is a^2 .

$$\therefore \frac{1}{2} \times a \times h = a^2$$

$$\Rightarrow h = 2a$$

Since, the base lies along the line $x = a$, the vertex lies on the line parallel to the base at a distance $2a$ from it.

So, the required lines are

$$x = a \pm 2a \text{ i.e. } x = -a \text{ or } x = 3a$$

91. $\tan^2(a+2)b + a^2 = 0$

$$a = 0, \tan^2(a+2)b = 0$$

$$\Rightarrow \tan^2 2b = 0$$

$$\Rightarrow b = 0, \frac{\pi}{2}, -\frac{\pi}{2}$$

$$(a, b) = (0, 0), \left(0, \frac{\pi}{2}\right), \left(0, -\frac{\pi}{2}\right)$$

$$\therefore y - 0 = \frac{1}{2}(x - 0), y - \frac{\pi}{2} = \frac{1}{2}(x - 0)$$

$$\text{and } y + \frac{\pi}{2} = \frac{1}{2}(x - 0)$$

$$\Rightarrow 2y = x, 2y - \pi = x$$

$$\text{and } 2y + \pi = x$$

92. Let the slope of $u = 0$ be m , then the slope of $v = 0$ is $\frac{9m}{2}$.

$$\therefore \frac{7}{9} = \left| \frac{m - \frac{9m}{2}}{1 + m \cdot \frac{9m}{2}} \right| = \left| \frac{-7m}{2 + 9m^2} \right|$$

$$\Rightarrow \frac{-7m}{2 + 9m^2} = \pm \frac{7}{9}$$

$$\Rightarrow 2 + 9m^2 = \pm 9m$$

$$\Rightarrow 9m^2 - 9m + 2 = 0$$

$$\text{or } 9m^2 + 9m + 2 = 0$$

$$m = \frac{9 \pm \sqrt{81 - 72}}{18}$$

$$= \frac{9 \pm 3}{18} = \frac{2}{3}, \frac{1}{3}$$

$$\text{or } m = \frac{-9 \pm 3}{18} = -\frac{2}{3}, -\frac{1}{3}$$

$\therefore$ Equations of the lines are

$$(i) 3y = x \text{ and } 2y = 3x$$

$$(ii) 3y = 2x \text{ and } y = 3x$$

$$(iii) x + 3y = 0 \text{ and } 3x + 2y = 0$$

$$(iv) 2x + 3y = 0 \text{ and } 3x + y = 0$$

$$93. \therefore m_1 + m_2 = -\frac{2h}{b}$$

$$\text{and } m_1 m_2 = \frac{a}{b}$$

$$\therefore \tan \alpha + \tan \beta = \frac{2 \tan \theta}{\sin^2 \theta} = 4 \operatorname{cosec} 2\theta$$

$$\text{Also, } \tan \alpha \tan \beta = \frac{\tan^2 \theta + \cos^2 \theta}{\sin^2 \theta} = \sec^2 \theta + \cot^2 \theta$$

$$\text{i.e. } \tan \alpha - \tan \beta = 2$$

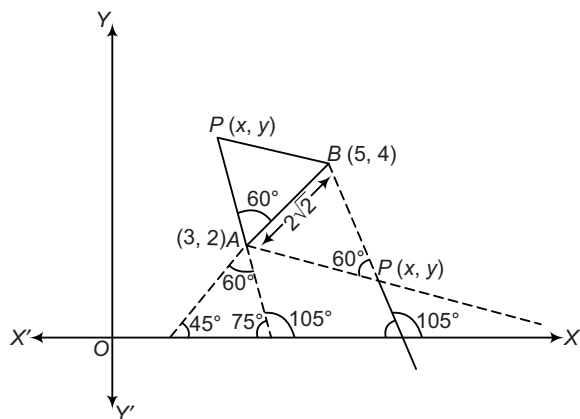
$$\text{Also, } \frac{\tan \alpha}{\tan \beta} = \frac{2 + \sin 2\theta}{2 - \sin 2\theta}$$

$$94. \text{ Clearly, the slope of line, } AB = \frac{4-2}{5-3} = 1 = \tan 45^\circ$$

Case I When P is above AB.

The line AP makes 105° angle with positive direction of X-axis.

$\therefore$ Equation of line AP is



$$\frac{x-3}{\cos 105^\circ} = \frac{y-2}{\sin 105^\circ} = 2\sqrt{2}$$

$$\Rightarrow x = 3 + 2\sqrt{2} \cos 105^\circ$$

$$\text{and } y = 2 + 2\sqrt{2} \sin 105^\circ$$

$$\Rightarrow x = 3 + 2\sqrt{2} \left(\frac{1-\sqrt{3}}{2\sqrt{2}} \right)$$

$$\text{and } y = 2 + 2\sqrt{2} \left(\frac{\sqrt{3}+1}{2\sqrt{2}} \right)$$

$$\Rightarrow x = 3 - \sqrt{3} \text{ and } y = 3 + \sqrt{3}$$

Case II When P lies below AB.

The line BP makes angle 105° with positive direction of X-axis.

$\therefore$ Equation of line BP is

$$\frac{x-5}{\cos 105^\circ} = \frac{y-4}{\sin 105^\circ} = -2\sqrt{2} \quad [\because P \text{ lies below of } (5, 4)]$$

$$\Rightarrow x = 5 - \cos 105^\circ (2\sqrt{2}) \text{ and } y = 4 - 2\sqrt{2} \sin 105^\circ$$

$$\Rightarrow x = 5 - (1 - \sqrt{3}) \text{ and } y = 4 - (1 + \sqrt{3}) \Rightarrow x = 4 + \sqrt{3} \text{ and } y = 3 - \sqrt{3}$$

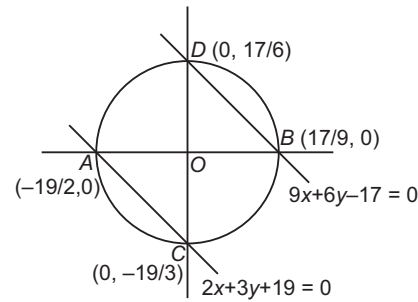
95. Statement II is true.

Statement I Since, AB may not be equal to AC.

$\therefore$ Perpendicular drawn from A to BC may not bisect BC.

Hence, Statement I is false.

96. Figure explains the both statements,



97. Statement II is true and Statement I is false as excentres of the triangle are also equidistant from the sides.

98. Statement I is true and Statement II is false as if L_1, L_2 and L_3 are parallel, then also $\Delta = 0$.

$$99. \text{ Bisector at C, } \frac{|3x + 2y|}{\sqrt{13}} = \frac{|2x + 3y + 6|}{\sqrt{13}}$$

$$\Rightarrow x - y - 6 = 0 \text{ and } 5x + 5y + 6 = 0$$

According to given equations of sides internal angle bisector at C will have negative slope, image of A will lie on BC with respect to both bisectors.

100. Statement I is true and follows from statement II as the family of lines can be written as $a(x + y - 1) + b(x - 2y) = 0$.

101. First, let the equation $ax + by + c = 0$ represents a family of straight lines passing through (a, b) for different values of a, b and c .

Then, we have to show that there is a linear relation between a, b and c . Then, we have to prove that the equation $ax + by + c = 0$ represents a family of lines passing through a fixed point. Let the linear relation be

$$la + mb + nc = 0 \Rightarrow \left(\frac{l}{n} \right) a + \left(\frac{m}{n} \right) b + c = 0$$

$\Rightarrow ax + by + c = 0$ always passes through a fixed point $\left(\frac{l}{n}, \frac{m}{n} \right)$.

$$\therefore 3a + 2b + 4c = 0 \Rightarrow \left(\frac{3}{4} \right) a + \left(\frac{2}{4} \right) b + c = 0$$

and $ax + by + c = 0$ represents system of concurrent lines passing through $\left(\frac{3}{4}, \frac{1}{2} \right)$.

Thus, Statement I is false and Statement II is true.

102. As $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents the general equation of second degree. But it represents straight line, if
- $$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$
- Thus, Statement II is false.
- And to the pair of straight lines formed by $2x - y = 5$ and $x + 2y = 3$ is given by
- $$(2x - y - 5)(x + 2y - 3) = 0$$
- $$\Rightarrow 2x^2 + 3xy - 2y^2 - 11x - 7y + 15 = 0$$
- Thus, Statement I is true.
- Hence, Statement I is true and Statement II is false.

103. The joint equation of $y = x$ and $y = -x$ is
- $$(x - y)(x + y) = 0, \text{ i.e. } x^2 - y^2 = 0.$$

104. The bisectors of the angles between the lines in new position are same as the bisectors of angles between their old positions, is always true.

$\therefore$ The equation of angle bisectors of the angle between the lines in the new position is

$$\frac{x^2 - y^2}{1 - (-1)} = \frac{xy}{-p}$$

$$\Rightarrow px^2 + 2xy - py^2 = 0$$

$\therefore$ Statement I is false and Statement II is true.

105. Put $2h = -(a + b)$ in $ax^2 + 2hxy + by^2 = 0$
- $$\Rightarrow ax^2 - (a + b)xy + by^2 = 0$$
- $$\Rightarrow (x - y)(ax - by) = 0$$
- $\Rightarrow$ One of the line bisects the angle between coordinate axes in positive quadrant.
- Also, put $b = -2h - a$ in $ax - by$, we have
- $$ax - by = ax - (-2h - a)y = ax + (2h + a)y$$
- Hence, $ax + (2h + a)y$ is a factor of
- $$ax^2 + 2hxy + by^2 = 0.$$

106. $ax^3 + bx^2 + y + cxy^2 + dy^3 = 0$
- $$\Rightarrow d\left(\frac{y}{x}\right)^3 + c\left(\frac{y}{x}\right)^2 + b\left(\frac{y}{x}\right) + a = 0$$
- $$\Rightarrow dm^3 + cm^2 + bm + a = 0$$
- $$m_1 m_2 m_3 = -a/d \quad \dots(i)$$
- $\Rightarrow m_3 = a/d$ as two lines are perpendicular, put
- $$m_3 = a/d \quad \dots(ii)$$
- $$\Rightarrow a^2 + ac + bd + d^2 = 0$$

107. Let $lx + my + n_1 = 0$ and $lx + my + n_2 = 0$ be two parallel lines represented by the given equation.
- $$\therefore ax^2 + 2hxy + by^2 + 2gx + 2fy + c = (lx + my + n_1)(lx + my + n_2)$$

Comparing coefficients of like terms, we get

$$l^2 = a, m^2 = b, 2lm = 2h, l(n_1 + n_2) = 2g,$$

$$m(n_1 + n_2) = 2f$$

$$\text{and } n_1 n_2 = c$$

Now, the distance between the parallel lines is

$$d = \frac{|n_1 - n_2|}{\sqrt{l^2 + m^2}} = \sqrt{\frac{(n_1 + n_2)^2 - 4n_1 n_2}{l^2 + m^2}}$$

$$= \sqrt{\frac{\left(\frac{2g}{l}\right)^2 - 4c}{a + b}} = \sqrt{\frac{4g^2 - 4cl^2}{a(a + b)}} = 2\sqrt{\frac{g^2 - ac}{a(a + b)}}$$

Hence, Statement II is false.

Also, $x^2 + 6xy + 9y^2 + 4x + 12y - 5 = 0$ represents two parallel straight lines.

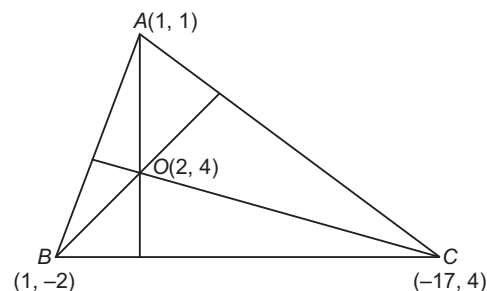
$$\therefore \text{Distance between them} = 2\sqrt{\frac{g^2 - ac}{a(a + b)}} = 2\sqrt{\frac{4 + 5}{1 + 9}} = \frac{6}{\sqrt{10}}$$

Hence, Statement I is true and Statement II is false.

108. Slope of AO = $\frac{4 - 1}{2 - 1} = 3$

$$\left(-\frac{a}{b}\right)3 = -1 \Rightarrow 3a = b$$

$$\Rightarrow b - 3a = 0$$



$$\therefore a + c = 2b$$

$$\Rightarrow c = 2b - a$$

$$\text{Now, consider } a(1) + b(-2) + c = a - 2b + c = a - 2b + 2b - a = 0$$

109. Consider, $-17a + 4b + c = -17a + 4b + 2b - a = -18a + 6b = 6(b - 3a) = 0$

110. Slope of AC (m_1) = $\frac{1 - 4}{1 + 17} = -\frac{3}{18} = -\frac{1}{6}$

$$\text{Slope of BC } (m_2) = \frac{-2 - 4}{1 + 17} = -\frac{6}{18} = -\frac{1}{3}$$

$$\therefore \tan C = \frac{m_2 - m_1}{1 + m_1 m_2} = \frac{-1/3 + 1/6}{1 + \left(-\frac{1}{6}\right)\left(-\frac{1}{3}\right)} < 0$$

Hence, $\triangle ACB$ is obtuse.

111. $\therefore \omega = 60^\circ, m = 2$

$$\therefore \tan \theta = \frac{m \sin \omega}{1 + m \cos \omega} = \frac{2 \sin 60^\circ}{1 + 2 \cos 60^\circ} = \frac{2 \times \sqrt{3}/2}{1 + 2 \times 1/2} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

112. $\therefore \omega = 60^\circ, m_1 = 2, m_2 = -\frac{1}{2}$

$$\therefore \tan \theta_1 = \frac{m_1 \sin \omega}{1 + m_1 \cos \omega} = \frac{2 \sin 60^\circ}{1 + 2 \cos 60^\circ} = \frac{2 \times \sqrt{3}/2}{1 + 2 \times 1/2} = \frac{\sqrt{3}}{2}$$

$$\text{and } \tan \theta_2 = \frac{-1/2 \times \sqrt{3}/2}{1 - 1/2 \times 1/2} = \frac{-\sqrt{3}/4}{3/4} = -\frac{1}{\sqrt{3}}$$

Let angle between the lines be ϕ , then

$$\tan \phi = \left| \frac{\tan \theta_1 - \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} \right| = \left| \frac{\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{3}}}{1 - \frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{3}}} \right|$$

$$\Rightarrow \phi = \tan^{-1} \left(\frac{5}{\sqrt{3}} \right)$$

$$113. m = \frac{\sin 60^\circ}{\sin (30^\circ - 60^\circ)} = -\sqrt{3}$$

$$\therefore \text{Equation of the line is } y - 0 = -\sqrt{3}(x - 2)$$

$$\Rightarrow \sqrt{3}x + y = 2\sqrt{3}$$

$$114. \text{ Let the straight line be } r \cos (\theta - \alpha) = p.$$

Since, it passes through two points.

$$\text{Then, } r_1 \cos \theta_1 \cos \alpha + r_1 \sin \theta_1 \sin \alpha - p = 0 \quad \dots(i)$$

$$\text{and } r_2 \cos \theta_2 \cos \alpha + r_2 \sin \theta_2 \sin \alpha - p = 0 \quad \dots(ii)$$

$$\Rightarrow \frac{\cos \alpha}{r_2 \sin \theta_2 - r_1 \sin \theta_1} = \frac{\sin \alpha}{r_1 \cos \theta_1 - r_2 \cos \theta_2}$$

$$= \frac{p}{r_1 r_2 \sin (\theta_2 - \theta_1)}$$

$$\Rightarrow p = \frac{r_1 r_2 \sin (\theta_2 - \theta_1)}{\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos (\theta_2 - \theta_1)}}$$

$$\text{and } \tan \alpha = \frac{r_1 \cos \theta_1 - r_2 \cos \theta_2}{r_2 \sin \theta_2 - r_1 \sin \theta_1}$$

$$115. \text{ Substituting } x = r \cos \theta \text{ and } y = r \sin \theta, \text{ we get}$$

$$2r = \operatorname{cosec} \left(\theta + \frac{\pi}{6} \right)$$

$$116. \text{ Equation in cartesian form is } l = 4x + 2y$$

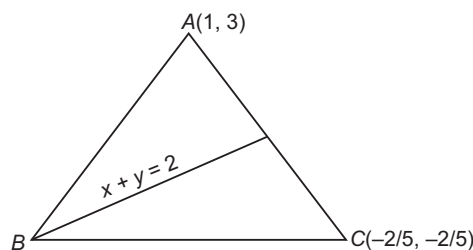
[replacing $x = r \cos \theta$ and $y = r \sin \theta$]

$$\Rightarrow \text{Perpendicular distance from } (0, 0) \text{ is } \frac{l}{2\sqrt{5}}.$$

$$117. \text{ Image of } A(1, 3) \text{ in line } x + y = 2 \text{ is}$$

$$\left(1 - \frac{2(2)}{2}, 3 - \frac{2(2)}{2} \right) \equiv (-1, 1)$$

$$\text{So, line } BC \text{ passes through } (-1, 1) \text{ and } \left(-\frac{2}{5}, -\frac{2}{5} \right).$$



$$\text{Equation of line } BC \text{ is } y - 1 = \frac{-2/5 - 1}{-2/5 + 1} (x + 1)$$

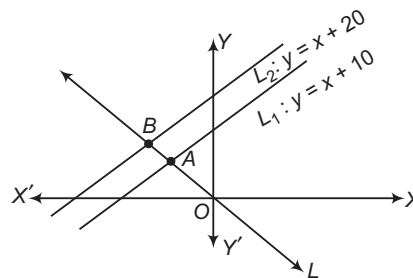
$$\Rightarrow 7x + 3y + 4 = 0$$

$$118. \text{ Vertex } B \text{ is point of intersection of } 7x + 3y + 4 = 0 \text{ and } x + y = 2 \text{ i.e. } B = (-5/2, 9/2)$$

$$119. \text{ Line } AB \text{ is } y - 3 = \frac{3 - 9/2}{1 + 5/2} (x - 1)$$

$$\Rightarrow 3x + 7y = 24$$

120.



Let the parametric equation of variable line be

$$\frac{x}{\cos \theta} = \frac{y}{\sin \theta} = r$$

$$\Rightarrow x = r \cos \theta, y = r \sin \theta$$

Putting it in L_1 , we get

$$r \sin \theta = r \cos \theta + 10$$

$$\Rightarrow \frac{1}{OA} = \frac{\sin \theta - \cos \theta}{10}$$

Similarly, putting the general point of variable line in the equation of L_2 , we get

$$\frac{1}{OB} = \frac{\sin \theta - \cos \theta}{20}$$

Let $P = (h, k)$ and $OP = r \Rightarrow r \cos \theta = h, r \sin \theta = k$, we have

$$\frac{2}{r} = \frac{\sin \theta - \cos \theta}{10} + \frac{\sin \theta - \cos \theta}{20}$$

$$\Rightarrow 40 = 3r \sin \theta - 3r \cos \theta$$

$$\Rightarrow 3y - 3x = 40$$

$$121. r^2 = \frac{10 \times 20}{(\sin \theta - \cos \theta)^2}$$

$$\Rightarrow (r \sin \theta - r \cos \theta)^2 = 200$$

Hence, locus is $(y - x)^2 = 200$.

$$122. \frac{1}{r^2} = \frac{(\sin \theta - \cos \theta)^2}{100} + \frac{(\sin \theta - \cos \theta)^2}{400}$$

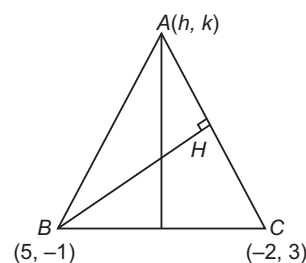
$$\Rightarrow 400 = 5(r \sin \theta - r \cos \theta)^2$$

Hence, the locus is $400 = 5(y - x)^2$, i.e. $(y - x)^2 = 80$.

$$123. A. AH \perp BC \Rightarrow \left(\frac{k}{h} \right) \left(\frac{3+1}{-2-5} \right) = -1$$

$$4k = 7h \quad \dots(i)$$

$$BH \perp AC \Rightarrow \left(\frac{0+1}{0-5} \right) \left(\frac{k-3}{h+2} \right) = -1$$



$$\Rightarrow k - 3 = 5(h + 2) \quad \dots(ii)$$

$$7h - 12 = 20h + 40$$

$$13h = -52$$

$$h = -4$$

$$\therefore k = -7$$

$$\therefore A = (-4, -7)$$

B. $x + y - 4 = 0$... (i)
 $4x + 3y - 10 = 0$... (ii)

Let $(h, 4 - h)$ be the point on Eq. (ii), then

$$\left| \frac{4h + 3(4 - h) - 10}{5} \right| = 1$$

$$\Rightarrow h + 2 = \pm 5$$

$$\Rightarrow h = 3 \text{ or } h = -7$$

$\therefore$ Required point is either $(3, 1)$ or $(-7, 11)$.

C. Orthocentre of the triangle is the point of intersection of the lines

$$x + y - 1 = 0$$

$$\text{and } x - y + 3 = 0$$

i.e. $(-1, 2)$

D. Since, a, b and c are in AP.

$$\therefore b = \frac{a + c}{2}$$

Now, the family of lines is $ax + \left(\frac{a + c}{2}\right)y = c$

$$\text{i.e. } a\left(x + \frac{y}{2}\right) + c\left(\frac{y}{2} - 1\right) = 0$$

$\therefore$ Point of concurrency is $(-1, 2)$.

124. A. $\therefore$ Lines are concurrent.

$$\therefore \begin{vmatrix} 1 & -2 & -6 \\ 3 & 1 & -4 \\ \lambda & 4 & \lambda^2 \end{vmatrix} = 0$$

$$\Rightarrow \lambda^2 + 2\lambda - 8 = 0 \Rightarrow \lambda = 2, -4$$

B. $\therefore$ Points are collinear.

$$\therefore \begin{vmatrix} \lambda + 1 & 1 & 1 \\ 2\lambda + 1 & 3 & 1 \\ 2\lambda + 2 & 2\lambda & 1 \end{vmatrix} = 0$$

$$\Rightarrow 2\lambda^2 - 3\lambda - 2 = 0 \Rightarrow \lambda = 2, -1/2$$

C. Point of intersection of $x - y + 1 = 0$ and $3x + y - 5 = 0$ is $(1, 2)$, it will satisfy $x + y - 1 - \lambda = 0$

$$\therefore \lambda = 2$$

D. Mid-point of $(1, -2)$ and $(3, 4)$ will satisfy

$$y - x - 1 + \lambda = 0$$

$$\Rightarrow \lambda = 2$$

125. A. $(x + 7y)^2 + 4\sqrt{2}(x + 7y) - 42 = 0$

$$\Rightarrow (x + 7y)^2 + 7\sqrt{2}(x + 7y) - 3\sqrt{2}(x + 7y) - 42 = 0$$

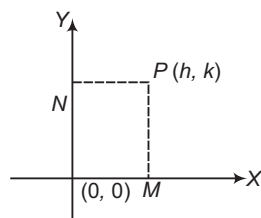
$$\Rightarrow (x + 7y)[x + 7y + 7\sqrt{2}] - 3\sqrt{2}(x + 7y + 7\sqrt{2}) = 0$$

$$\Rightarrow (x + 7y + 7\sqrt{2})(x + 7y - 3\sqrt{2}) = 0$$

$$\Rightarrow x + 7y + 7\sqrt{2} = 0 \text{ and } x + 7y - 3\sqrt{2} = 0$$

$$\therefore d = \frac{|7\sqrt{2} + 3\sqrt{2}|}{\sqrt{1 + 49}} = \frac{10\sqrt{2}}{\sqrt{50}} = 2$$

B. Let two perpendicular lines be coordinate axes.



$$\text{Then, } PM + PN = 1$$

$$\Rightarrow h + k = 1$$

Hence, the locus is $x + y = 1$.

But, if the point lies in other quadrants also, then $|x| + |y| = 1$. Hence, value of k is 1.

C. Angle bisector between the lines $x + 2y + 4 = 0$ and $4x + 2y - 1 = 0$ is

$$\frac{x + 2y + 4}{\sqrt{1 + 4}} = \pm \frac{(-4x - 2y + 1)}{\sqrt{16 + 4}}$$

$$\Rightarrow x + 2y + 4 = \pm \frac{(-4x - 2y + 1)}{2}$$

$$\Rightarrow 2(x + 2y + 4) = \pm (-4x - 2y + 1)$$

Since, $aa' + bb' < 0$, so positive sign + gives acute angle bisector. Hence, required bisector is

$$2x + 4y + 8 = -4x - 2y + 1$$

$$\Rightarrow 6x + 6y + 7 = 0 \therefore m = 7$$

D. We have, $y^2 - 9xy + 18x^2 = 0$

$$\text{or } y^2 - 6xy - 3xy + 18x^2 = 0$$

$$\Rightarrow y(y - 6x) - 3x(y - 6x) = 0$$

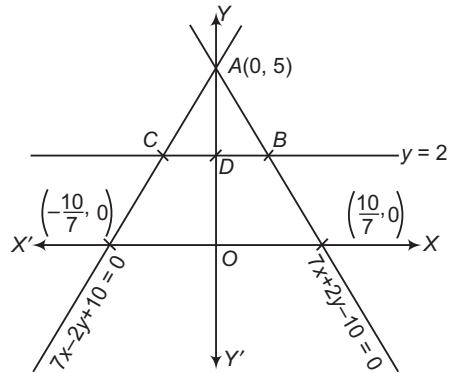
$$\Rightarrow (y - 3x) = 0 \text{ and } y - 6x = 0$$

The third line is $y = 6$. Therefore, area of the triangle formed by these lines

$$= \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 1 & 6 & 1 \\ 2 & 6 & 1 \end{vmatrix} = \frac{1}{2} |6 - 12| = 3 \text{ sq units}$$

126. We have, $B \equiv \left(\frac{6}{7}, 2\right), C \equiv \left(-\frac{6}{7}, 2\right)$

$$\Rightarrow BC = \frac{12}{7}, AD = 3$$

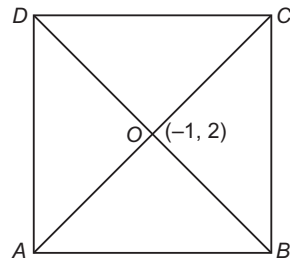


$$\therefore \text{Area of } \triangle ABC = \frac{1}{2} \cdot \frac{12}{7} \cdot 3 = \frac{18}{7} = \frac{2 \times 9}{7} \text{ sq units}$$

$$\Rightarrow \lambda = 9, \mu = 7 \Rightarrow \lambda - \mu = 9 - 7 = 2$$

127. Let ABCD be the given square with centre $(-1, 2)$ and side of length 6. BD is parallel to $x + y = 0$.

Equation of BD is $x + y = 1$. So, the equation to AC is $x - y + 3 = 0$ (note that $AC \perp BD$).



We have,

$$|OC| = |OB| = |OA| = |OD| = 3\sqrt{2} \text{ units}$$

We may write AC as $\frac{x - (-1)}{\cos 45^\circ} = \frac{y - 2}{\sin 45^\circ} = r$, where r is the algebraic distance of (x, y) from $(-1, 2)$. Therefore, A and C are given by

$$\frac{x+1}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm 3\sqrt{2} \text{ or } A \text{ is } (2, 5) \text{ and } C \text{ is } (-4, -1).$$

$$\text{Again, we may write } BD \text{ as } \frac{x+1}{1/\sqrt{2}} + \frac{y-2}{-1/\sqrt{2}} = r$$

B and D are given by

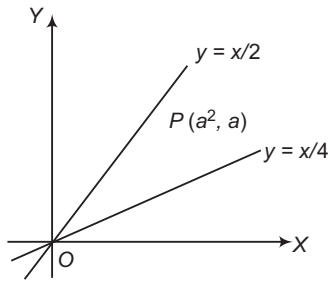
$$\frac{x+1}{1/\sqrt{2}} = \frac{y-2}{-1/\sqrt{2}} = \pm 3\sqrt{2}$$

B is $(-4, 5)$ and D is $(2, -1)$.

The vertices of the square are $(2, 5)$, $(-4, 5)$, $(-4, -1)$ and $(2, -1)$.

Hence, the number of all possible coordinates are 4.

128. We have, $a - \frac{a^2}{4} > 0$ and $a - \frac{a^2}{2} < 0$



$$\Rightarrow \left(a - \frac{a^2}{4}\right) \left(a - \frac{a^2}{2}\right) < 0$$

$$\Rightarrow a \in (2, 4)$$

$$\Rightarrow \lambda = 2 \text{ and } \mu = 4$$

$$\Rightarrow \lambda + \mu = 4 + 2 = 6$$

129. Let the equation of the variable line be $ax + by + c = 0$.

It is given that

$$\sum_{i=1}^6 \frac{ax_i + by_i + c}{\sqrt{a^2 + b^2}} = 0$$

$$\Rightarrow \sum_{i=1}^6 (ax_i + by_i + c) = 0$$

$$\Rightarrow \sum_{i=1}^6 ax_i + \sum_{i=1}^6 by_i + 6c = 0$$

$$\Rightarrow a \left(\frac{\sum_{i=1}^6 x_i}{6} \right) + b \left(\frac{\sum_{i=1}^6 y_i}{6} \right) + c = 0$$

$$\text{So, the fixed point must be } \left(\frac{\sum_{i=1}^6 x_i}{6}, \frac{\sum_{i=1}^6 y_i}{6} \right).$$

But the fixed point is $(2, 1)$, so that

$$\frac{2 + 3 + 7 + h_1 + h_2 + h_3}{6} = 2$$

$$\Rightarrow h_1 + h_2 + h_3 = 0$$

$$\Rightarrow h_1 = 0, h_2 = 0, h_3 = 0 \quad [\text{as } h_1, h_2, h_3 \geq 0]$$

Aliter

Let the variable line be $y - 1 = m(x - 2)$ or $mx - y + 1 - 2m = 0$.

Since, the algebraic sum of the perpendicular from the given points is zero i.e. $\frac{k_1 + k_2 + k_3 + 4 + 5 - 3}{6} = 1$

$$\Rightarrow k_1 = k_2 = k_3 = 0 \quad [\text{as } k_1, k_2, k_3 \geq 0]$$

$$\frac{m(2 + 3 + 7 + h_1 + h_2 + h_3 - 12) + (6 - k_1 - k_2 - k_3 - 4 - 5 + 3)}{\sqrt{1 + m^2}} = 0, \text{ for all } m$$

$$\Rightarrow m(h_1 + h_2 + h_3) - (k_1 + k_2 + k_3) = 0, \text{ for all } m$$

$$\Rightarrow h_1 + h_2 + h_3 = 0$$

$$\text{and } k_1 + k_2 + k_3 = 0$$

$$\Rightarrow h_1 = 0 = h_2 = h_3$$

$$\text{and } k_1 = 0 = k_2 = k_3$$

$\Rightarrow$ Only one solution.

130. The point B is $(2, 1)$. Image of A $(1, 2)$ in the line $x - 2y + 1 = 0$ is given by

$$\frac{x-1}{1} = \frac{y-2}{-2} = \frac{4}{5}$$

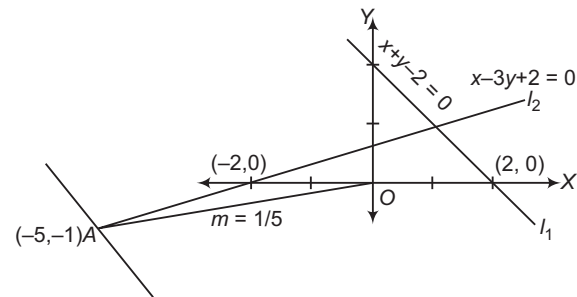
$\therefore$ Coordinates of the point are $\left(\frac{9}{5}, \frac{2}{5}\right)$.

Since, this point lies on BC.

$\therefore$ Equation of BC is $3x - y - 5 = 0$.

$$\therefore a + b = 2$$

131. $x^2 - 3y^2 - 2xy + 8y - 4 \equiv (x - 3y + 2)(x + y - 2)$



Now, $(-5, -1)$ lies on $x - 3y + 2 = 0$.

In limiting case line passing through $(-5, -1)$ can be parallel to $x + y - 2 = 0$

$$\text{i.e. } m > -1$$

and maximum slope can occur, if it passes through $(0, 0)$.

$$\text{i.e. } m < \frac{1}{5}$$

$$\Rightarrow m \in \left(-1, \frac{1}{5}\right)$$

$$\Rightarrow a = -1$$

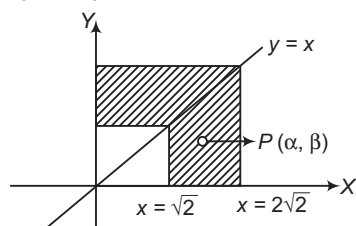
$$\text{and } b = \frac{1}{5}$$

$$\Rightarrow \left(a + \frac{1}{b}\right) = -1 + 5 = 4$$

Entrances Gallery

1. $2 \leq d_1(p) + d_2(p) \leq 4$

For $P(\alpha, \beta)$, $\alpha > \beta$



$$\Rightarrow 2\sqrt{2} \leq 2\alpha \leq 4\sqrt{2}$$

$$\Rightarrow \sqrt{2} \leq \alpha \leq 2\sqrt{2}$$

$$\therefore \text{Area of region} = (2\sqrt{2})^2 - (\sqrt{2})^2 = 8 - 2 = 6 \text{ sq units}$$

2. For point of intersection,

$$(a-b)x_1 = (a-b)y_1$$

$\Rightarrow$ Point lies on line $y = x$

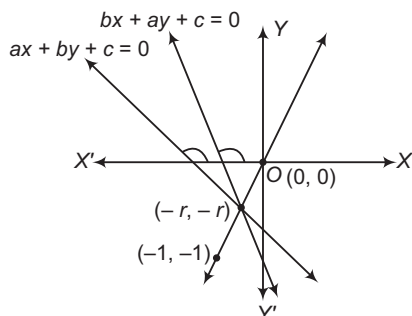
Let point be (r, r) .

$$\text{Then, } \sqrt{(r-1)^2 + (r-1)^2} < 2\sqrt{2}$$

$$\Rightarrow \sqrt{2}|r-1| < 2\sqrt{2}$$

$$\Rightarrow |r-1| < 2$$

$$\Rightarrow -1 < r < 3$$



$\Rightarrow (-1, 1)$ lies on the opposite side of origin for both lines.

$$\Rightarrow -a - b + c < 0$$

$$\Rightarrow a + b - c > 0$$

3. $\frac{m + \sqrt{3}}{1 - \sqrt{3}m} = \sqrt{3}$

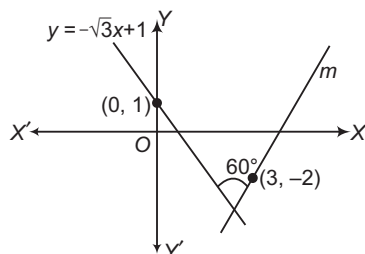
$$\Rightarrow m + \sqrt{3} = \pm(\sqrt{3} - 3m)$$

$$\Rightarrow 4m = 0$$

$$\Rightarrow m = 0$$

$$\text{or } 2m = 2\sqrt{3}$$

$$\Rightarrow m = \sqrt{3}$$



$$\therefore \text{Equation is } y + 2 = \sqrt{3}(x - 3)$$

$$\Rightarrow \sqrt{3}x - y - (2 + 3\sqrt{3}) = 0$$

4. First of all find the point of intersection of the lines $2x - 3y + 4 = 0$ and $x - 2y + 3 = 0$ (say A). Now, the line $(2x - 3y + 4) + k(x - 2y + 3) = 0$ is the perpendicular bisector of the line joining points $P(2, 3)$ and image $P'(h, k)$. Now, $AP = AP'$ and simplify.

Given line is

$$(2x - 3y + 4) + k(x - 2y + 3) = 0, k \in R \quad \dots(i)$$

This line will pass through the point of intersection of the lines

$$2x - 3y + 4 = 0 \quad \dots(ii)$$

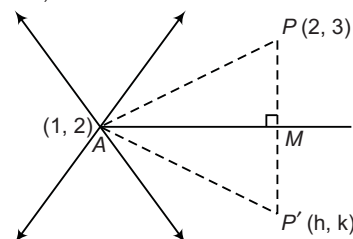
$$\text{and } x - 2y + 3 = 0 \quad \dots(iii)$$

On solving Eqs. (ii) and (iii), we get

$$x = 1, y = 2$$

$\therefore$ Point of intersection of lines (ii) and (iii) is $(1, 2)$.

Let M be the mid-point of PP' , then AM is perpendicular bisector of PP' (where, A is the point of intersection of given lines).



$\therefore$

$$AP = AP'$$

$$\Rightarrow \sqrt{(2-1)^2 + (3-2)^2} = \sqrt{(h-1)^2 + (k-2)^2}$$

$$\Rightarrow \sqrt{2} = \sqrt{h^2 + k^2 - 2h - 4k + 1 + 4}$$

$$\Rightarrow \sqrt{2} = \sqrt{h^2 + k^2 - 2h - 4k + 5}$$

$$\Rightarrow h^2 + k^2 - 2h - 4k + 5 = 2$$

$$\Rightarrow h^2 + k^2 - 2h - 4k + 3 = 0$$

Thus, the required locus is

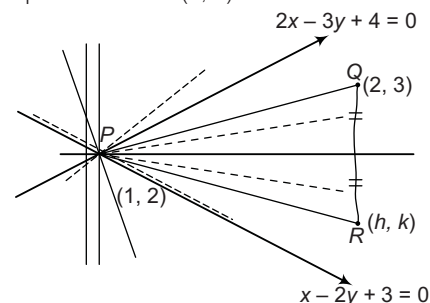
$$x^2 + y^2 - 2x - 4y + 3 = 0$$

which is a equation of circle with

$$\text{radius} = \sqrt{1 + 4 - 3} = \sqrt{2}$$

Aliter

$(2x - 3y + 4) + k(x - 2y + 3) = 0$ is family of lines passing through $(1, 2)$. By congruency of triangles, we can prove that mirror image (h, k) and the point $(2, 3)$ will be equidistant from $(1, 2)$.



$\therefore$ Locus of (h, k) is $PR = PQ$

$$\Rightarrow (h-1)^2 + (k-2)^2 = (2-1)^2 + (3-2)^2$$

$$\text{or } (x-1)^2 + (y-2)^2 = 2$$

Hence, locus is a circle of radius $\sqrt{2}$.

5. $\therefore S = \left(\frac{13}{2}, 1\right), P = (2, 2)$

Slope $= -\frac{2}{9}$

Equation will be $\frac{y+1}{x-1} = -\frac{2}{9}$

$\Rightarrow 9y + 9 + 2x - 2 = 0$

$\Rightarrow 2x + 9y + 7 = 0$

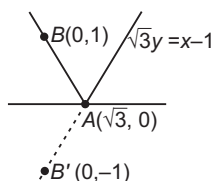
6. Take any point $B(0, 1)$ on given line.

Equation of AB' is

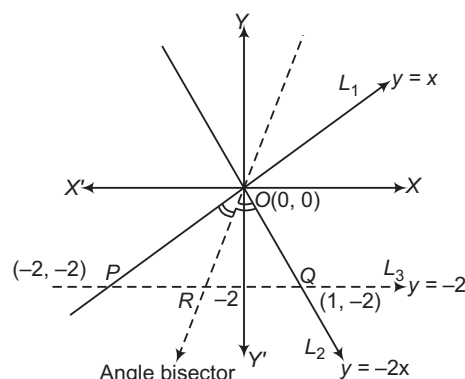
$y - 0 = \frac{-1-0}{0-\sqrt{3}}(x-\sqrt{3})$

$\Rightarrow -\sqrt{3}y = -x + \sqrt{3}$

$\Rightarrow x - \sqrt{3}y = \sqrt{3} \Rightarrow \sqrt{3}y = x - \sqrt{3}$



7. Here, $L_1 : y - x = 0$ and $L_2 : 2x + y = 0$ and $L_3 : y + 2 = 0$ shown as



$|PO| = \sqrt{4+4} = 2\sqrt{2}$

$|OQ| = \sqrt{1+4} = \sqrt{5}$

Since, OR is angle bisector.

$\therefore \frac{OP}{OQ} = \frac{PR}{RQ} \Rightarrow \frac{PR}{RQ} = \frac{2\sqrt{2}}{\sqrt{5}}$

$\therefore$ Statement I is true.

But it does not divide that Δ in two similar triangles.

$\therefore$ Statement II is false.

8. As $x + y = |a|$ and $ax - y = 1$ intersect in I quadrant.

$\therefore x$ and y -coordinates are positive.

$\therefore x = \frac{1+|a|}{1+a} \geq 0$ and $y = \frac{a|a|-1}{a+1} \geq 0$

$\Rightarrow 1+a \geq 0$ and $a|a|-1 \geq 0$

$\Rightarrow a \geq -1$ and $a|a| \geq 1$... (i)

If $-1 \leq a < 0$, then

$-a^2 \geq 1$ [not possible]

If $a \geq 0$, then

$a^2 \geq 1 \Rightarrow a \geq 1$... (ii)

$\therefore a \geq 1$ or $a \in [1, \infty)$

9. Since, the line L is passing through the point $(13, 32)$.

$\therefore \frac{13}{5} + \frac{32}{b} = 1 \Rightarrow \frac{32}{b} = -\frac{8}{5} \Rightarrow b = -20$

The line K is parallel to the line L , its equation must be

$\frac{x}{5} - \frac{y}{20} = a$ or $\frac{x}{5a} - \frac{y}{20a} = 1$

On comparing with $\frac{x}{c} + \frac{y}{3} = 1$, we get

$20a = -3$

$\therefore c = 5a = -\frac{3}{4}$

Hence, the distance between lines

$= \frac{|a-1|}{\sqrt{\frac{1}{25} + \frac{1}{400}}} = \frac{\left|-\frac{3}{20}-1\right|}{\sqrt{\frac{17}{400}}} = \frac{23}{\sqrt{17}}$

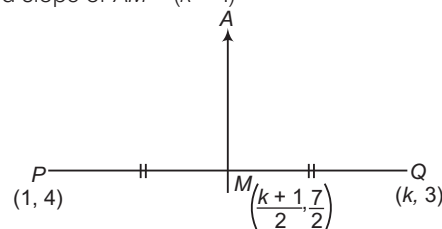
10. Lines perpendicular to same line are parallel to each other.

$\therefore -p(p^2+1) = p^2+1 \Rightarrow p = -1$

Hence, there is exactly one value of p .

11. Since, slope of $PQ = \frac{4-3}{1-k} = \frac{1}{1-k}$

and slope of $AM = (k-1)$



$\therefore$ Equation of AM is

$y - \frac{7}{2} = (k-1) \left[x - \left(\frac{k+1}{2} \right) \right]$

For y -intercept, $x = 0, y = -4$

$\Rightarrow -4 - \frac{7}{2} = -(k-1) \left(\frac{k+1}{2} \right) \Rightarrow \frac{15}{2} = \frac{k^2-1}{2}$

$\Rightarrow k^2 - 1 = 15$

$\Rightarrow k^2 = 16 \Rightarrow k = \pm 4$

12. Equations of bisectors of lines $xy = 0$ are $y = \pm x$.

Put $y = \pm x$ in $my^2 + (1-m^2)xy - mx^2 = 0$, we get

$mx^2 \pm (1-m^2)x^2 - mx^2 = 0$

$\Rightarrow (1-m^2)x^2 = 0 \Rightarrow m = \pm 1$

13. Since, A is the mid-point of line PQ .

$\therefore 3 = \frac{a+0}{2}$

$\Rightarrow a = 6$

and $4 = \frac{0+b}{2}$

$\Rightarrow b = 8$

Hence, the equation of line is

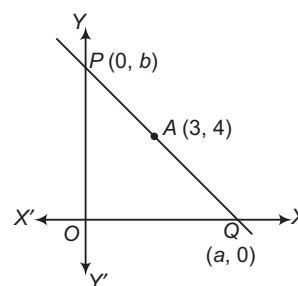
$\frac{x}{6} + \frac{y}{8} = 1$

$\Rightarrow 4x + 3y = 24$

14. Since, a, b and c are in HP. Then $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ are in AP.

$\therefore \frac{2}{b} = \frac{1}{a} + \frac{1}{c} \Rightarrow \frac{1}{a} - \frac{2}{b} + \frac{1}{c} = 0$

Hence, straight line $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ is always passes through a fixed point $(1, -2)$.



15. Let the intercepts on the coordinate axes be a and b .

According to the given condition,

$$a + b = -1$$

$$\Rightarrow b = -a - 1 = -(a + 1)$$

Let equation of line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$\Rightarrow \frac{x}{a} - \frac{y}{a+1} = 1 \quad \dots (i)$$

Since, this line passes through a point $(4, 3)$.

$$\therefore \frac{4}{a} - \frac{3}{a+1} = 1$$

$$\Rightarrow \frac{4a + 4 - 3a}{a(a+1)} = 1$$

$$\Rightarrow a + 4 = a^2 + a$$

$$\Rightarrow a^2 = 4$$

$$\Rightarrow a = \pm 2$$

On putting the values of a in Eq. (i), we get

$$\frac{x}{2} - \frac{y}{3} = 1$$

$$\text{and } \frac{x}{-2} + \frac{y}{1} = 1$$

16. The given pair of lines is $x^2 - 2cxy - 7y^2 = 0$.

On comparing the standard equation $ax^2 + 2hxy + by^2 = 0$, we get

$$a = 1, 2h = -2c, b = -7$$

$$\text{Now, } m_1 + m_2 = -\frac{2h}{b} = -\frac{2c}{-7}$$

$$\text{and } m_1 m_2 = \frac{a}{b} = -\frac{1}{7}$$

According to the given condition,

$$m_1 + m_2 = 4m_1 m_2$$

$$\therefore -\frac{2c}{7} = -\frac{4}{7}$$

$$\Rightarrow c = 2$$

17. Since, one of the two lines is $3x + 4y = 0$. Then, $y = -\frac{3x}{4}$

will satisfy the equation

$$6x^2 - xy + 4cy^2 = 0$$

$$\therefore 6x^2 - x\left(-\frac{3x}{4}\right) + 4c\left(-\frac{3x}{4}\right)^2 = 0$$

$$\Rightarrow 6x^2 + \frac{3x^2}{4} + 4c \cdot \frac{9x^2}{16} = 0$$

$$\Rightarrow x^2(27 + 9c) = 0$$

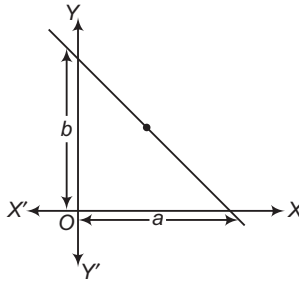
$$\Rightarrow c = -3$$

18. Since, x_1, x_2, x_3 and y_1, y_2, y_3 are in GP.

Then, $x_2 = rx_1, x_3 = r^2x_1$ and $y_2 = ry_1, y_3 = r^2y_1$

where, r is a common ratio.

The points become $(x_1, y_1), (rx_1, ry_1)$ and (r^2x_1, r^2y_1) .



$$\text{Now, } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & 1 \\ rx_1 & ry_1 & 1 \\ r^2x_1 & r^2y_1 & 1 \end{vmatrix}$$

Taking x_1 common from C_1 and y_1 from C_2 ,

$$= x_1 y_1 \begin{vmatrix} 1 & 1 & 1 \\ r & r & 1 \\ r^2 & r^2 & 1 \end{vmatrix} = x_1 y_1 (0) = 0$$

[$\because$ two columns are identical]

Hence, these points lie on a straight line.

Aliter

Let $x_1 = a \Rightarrow x_2 = ar$ and $x_3 = ar^2$

and $y_1 = b \Rightarrow y_2 = br$ and $y_3 = br^2$

where, r is a common ratio.

The given points will be $A(a, b), B(ar, br)$ and $C(ar^2, br^2)$.

$$\text{Now, slope of } AB = \frac{b(r-1)}{a(r-1)} = \frac{b}{a}$$

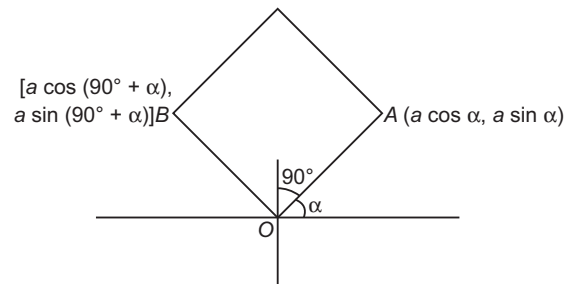
$$\text{and slope of } BC = \frac{b(r^2-r)}{a(r^2-r)} = \frac{b}{a}$$

$\therefore$ Slope of AB = Slope of BC

But B is a common point.

$\therefore A, B$ and C are collinear i.e. the points $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) lie on a straight line.

19. Since, line OA makes an angle α with X -axis and given $OA = a$, then coordinates of A are $(a \cos \alpha, a \sin \alpha)$. Also, $OB \perp OA$, then OB makes an angle $(90^\circ + \alpha)$ with X -axis, then coordinates of B are $[a \cos (90^\circ + \alpha), a \sin (90^\circ + \alpha)]$ i.e. $(-a \sin \alpha, a \cos \alpha)$.



Equation of the diagonal AB not passing through the origin is

$$(y - a \sin \alpha) = \frac{a \cos \alpha - a \sin \alpha}{-a \sin \alpha - a \cos \alpha} (x - a \cos \alpha)$$

$$\Rightarrow (\sin \alpha + \cos \alpha)(y - a \sin \alpha) = (\sin \alpha - \cos \alpha)(x - a \cos \alpha)$$

$$\Rightarrow y(\sin \alpha + \cos \alpha) + x(\cos \alpha - \sin \alpha) = a \sin \alpha(\sin \alpha + \cos \alpha) - a \cos \alpha(\sin \alpha - \cos \alpha)$$

$$= a(\sin^2 \alpha + \sin \alpha \cos \alpha - \cos \alpha \sin \alpha + \cos^2 \alpha) = a$$

20. The angle bisector for the given two lines

$$24x + 7y - 20 = 0 \quad \text{and} \quad 4x - 3y - 2 = 0, \text{ are}$$

$$\frac{24x + 7y - 20}{25} = \pm \frac{4x - 3y - 2}{5}$$

Taking positive sign, we get $2x + 11y - 5 = 0$

This equation of line is already given.

Therefore, the given three lines are concurrent with one line bisecting the angle between the other two.

21. Given lines $\sqrt{3}x + y = 0$ makes an angle of 120° with OX and $\sqrt{3}x - y = 0$ makes an angle of 60° with OX . So, the required line is $y - 2 = 0$.

22. Given lines are concurrent.

$$\text{So, } \begin{vmatrix} 2 & -3 & k \\ 3 & -4 & -13 \\ 8 & -11 & -33 \end{vmatrix} = 0$$

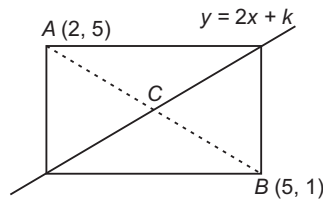
$$\Rightarrow 2(132 - 143) + 3(-99 + 104) + k(-33 + 32) = 0$$

$$\Rightarrow -22 + 15 - k = 0$$

$$\Rightarrow k = -7$$

23. Now, mid-point of $A(2, 5)$ and $B(5, 1)$ is

$$C\left(\frac{2+5}{2}, \frac{5+1}{2}\right) \text{ or } C\left(\frac{7}{2}, 3\right).$$



We know that, the diagonals of a rectangle bisect each other.

So, the mid-point $C\left(\frac{7}{2}, 3\right)$ lies on a given line.

$$\therefore 3 = 2 \times \frac{7}{2} + k \Rightarrow 3 = 7 + k \Rightarrow k = -4$$

24. Any straight line perpendicular to $2x + y = 3$ is

$$2y - x + c = 0 \quad \dots(i)$$

Since, it passes through $(1, 1)$.

$$\therefore 2(1) - 1 + c = 0$$

$$\Rightarrow c = -1$$

On putting $c = -1$ in Eq. (i), we get

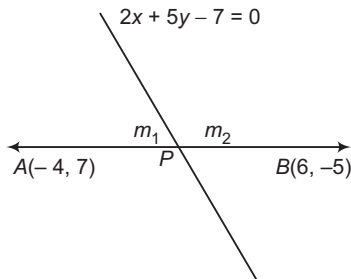
$$2y - x - 1 = 0$$

$$\Rightarrow 2y - x = 1$$

$$\Rightarrow \frac{y}{1/2} - \frac{x}{1} = 1$$

So, the y -intercept is $\frac{1}{2}$.

25. Let the given line divide line joining two points in the ratio $m_1 : m_2$.



$$\therefore \text{Coordinates of } P = \left(\frac{6m_1 - 4m_2}{m_1 + m_2}, \frac{-5m_1 + 7m_2}{m_1 + m_2} \right)$$

Since, the point P lies on the line

$$2x + 5y - 7 = 0$$

$$\therefore 2\left(\frac{6m_1 - 4m_2}{m_1 + m_2}\right) + 5\left(\frac{-5m_1 + 7m_2}{m_1 + m_2}\right) - 7 = 0$$

$$\Rightarrow 12m_1 - 8m_2 - 25m_1 + 35m_2 - 7m_1 - 7m_2 = 0$$

$$\Rightarrow -20m_1 + 20m_2 = 0$$

$$\Rightarrow m_1 = m_2 \Rightarrow \frac{m_1}{m_2} = 1$$

$$\therefore m_1 : m_2 = 1 : 1$$

26. Given lines are

$$x + y = 0 \quad \dots(i)$$

$$5x + y = 4 \quad \dots(ii)$$

$$\text{and } x + 5y = 4 \quad \dots(iii)$$

On solving above equations pairwise, we get

$$A(1, -1), B\left(\frac{2}{3}, \frac{2}{3}\right), C(-1, 1)$$

$$\text{Now, } AB = \sqrt{\left(\frac{2}{3} - 1\right)^2 + \left(\frac{2}{3} + 1\right)^2}$$

$$= \sqrt{\frac{1}{9} + \frac{25}{9}} = \sqrt{\frac{26}{9}} = \frac{\sqrt{26}}{3}$$

$$BC = \sqrt{\left(-1 - \frac{2}{3}\right)^2 + \left(1 - \frac{2}{3}\right)^2} = \sqrt{\frac{25}{9} + \frac{1}{9}}$$

$$= \sqrt{\frac{26}{9}} = \frac{\sqrt{26}}{3}$$

$$\text{and } CA = \sqrt{(1+1)^2 + (-1-1)^2} = \sqrt{4+4} = 2\sqrt{2}$$

Here, we see that $AB = BC$

Hence, given lines form an isosceles triangle.

27. $\therefore m + 2m = -\frac{2h}{2}$ and $m \times 2m = \frac{1}{2}$

$$\Rightarrow 3m = -h \text{ and } 2m^2 = \frac{1}{2} \Rightarrow h = -3m \text{ and } m = \pm \frac{1}{2}$$

$$\therefore h = -3\left(\pm \frac{1}{2}\right) \Rightarrow h = \pm \frac{3}{2}$$

28. Let the coordinates of C be (x_1, y_1) .

The coordinates of centroid of the triangle are

$$\left(\frac{2-2+x_1}{3}, \frac{-3+1+y_1}{3} \right)$$

$$\text{i.e. } \left(\frac{x_1}{3}, \frac{y_1-2}{3} \right)$$

So, this point will satisfy the given equation

$$2x + 3y = 1$$

$$\therefore \frac{2x_1}{3} + 3\left(\frac{y_1-2}{3}\right) = 1$$

$$\Rightarrow 2x_1 + 3y_1 = 9$$

Hence, the locus of the vertex C is the line

$$2x + 3y = 9$$

29. The equation of line passing through points $(5, 0)$ and $(0, 3)$ is

$$(y - 0) = \frac{(3-0)}{(0-5)}(x - 5) \Rightarrow y = \frac{3}{-5}(x - 5)$$

$$\Rightarrow -5y = 3x - 15 \Rightarrow 3x + 5y - 15 = 0$$

$\therefore$ The length of perpendicular from the point $(4, 4)$ to the above line is

$$d = \frac{(3 \times 4 + 5 \times 4 - 15)}{\sqrt{3^2 + 5^2}} = \frac{12 + 20 - 15}{\sqrt{9 + 25}}$$

$$= \frac{17}{\sqrt{34}} = \sqrt{\frac{17}{2}}$$

30. Equation of line in intercept form is

$$\frac{x}{a} + \frac{y}{b} = 1$$

Here, $a = \frac{1}{3}$ and $b = \frac{1}{4}$

$$\therefore \frac{x}{1/3} + \frac{y}{1/4} = 1$$

$$\Rightarrow 3x + 4y - 1 = 0$$

$\therefore$ Perpendicular distance from $(0, 0)$ to the line is

$$\frac{|3(0) + 4(0) - 1|}{\sqrt{3^2 + 4^2}} = p$$

$$\Rightarrow \frac{1}{\sqrt{9 + 16}} = p$$

$$\Rightarrow p = \frac{1}{5}$$

31. Since, the perpendicular distance from $(0, 0)$ to the line $x \sec \theta + y \csc \theta - a = 0$ is p .

$$\begin{aligned} \therefore p &= \frac{|0 + 0 - a|}{\sqrt{\sec^2 \theta + \csc^2 \theta}} = \frac{a}{\sqrt{\frac{1}{\cos^2 \theta} + \frac{1}{\sin^2 \theta}}} \\ &= \frac{a \sin \theta \cos \theta}{\sqrt{\sin^2 \theta + \cos^2 \theta}} = \frac{a \sin \theta \cos \theta}{1} \cdot \frac{2}{2} = \frac{a \sin 2\theta}{2} \end{aligned}$$

Also, the perpendicular distance from $(0, 0)$ to the line $x \cos \theta - y \sin \theta - a \cos 2\theta = 0$ is q .

$$\therefore q = \frac{|0 - 0 - a \cos 2\theta|}{\sqrt{\cos^2 \theta + \sin^2 \theta}} = a \cos 2\theta$$

$$\begin{aligned} \therefore 4p^2 + q^2 &= 4 \left(\frac{a \sin 2\theta}{2} \right)^2 + (a \cos 2\theta)^2 \\ &= a^2 \sin^2 2\theta + a^2 \cos^2 2\theta = a^2 \end{aligned}$$

32. Let coordinate of point P be (x, y) .

Given, $PA = PB \Rightarrow (PA)^2 = (PB)^2$

$$\Rightarrow \{x - (a + b)\}^2 + \{y - (a - b)\}^2 = \{x - (a - b)\}^2 + \{y - (a + b)\}^2$$

$$\begin{aligned} \Rightarrow x^2 + (a + b)^2 - 2x(a + b) + y^2 + (a - b)^2 - 2y(a - b) \\ = x^2 + (a - b)^2 - 2x(a - b) + y^2 + (a + b)^2 - 2y(a + b) \end{aligned}$$

$$\Rightarrow 2x(-a - b + a - b) + 2y(-a + b + a + b) = 0$$

$$\Rightarrow -2bx + 2by = 0$$

$$\Rightarrow -x + y = 0$$

$$\Rightarrow x - y = 0$$

33. Mid-point of AB is $\left(\frac{-2+6}{2}, \frac{3-5}{2}\right)$ i.e. $(2, -1)$.

Slope of the line $AB = \frac{-5-3}{6+2} = -1$

So, slope of the required line is 1.

$\therefore$ Required equation of the line is given by

$$y + 1 = 1(x - 2)$$

$$\Rightarrow y - x + 3 = 0 \Rightarrow x - y = 3$$

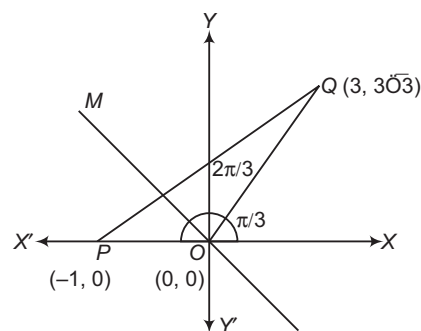
34. We have, $x^2 + 2\sqrt{2}xy + 2y^2 + 4x + 4\sqrt{2}y + 1 = 0$

Here, $a = 1, b = 2, h = \sqrt{2}, g = 2, f = 2\sqrt{2}, c = 1$

$\therefore$ Required distance

$$= 2 \sqrt{\frac{g^2 - ac}{a(a+b)}} = 2 \sqrt{\frac{2 \cdot 2 - 1 \cdot 1}{1(1+2)}} = 2 \sqrt{\frac{3}{3}} = 2 \text{ units}$$

35. Slope of $OQ = \frac{3\sqrt{3}-0}{3-0} = \sqrt{3}$



$\therefore \angle POQ$ is $\frac{2\pi}{3}$ and the line OM makes an angle $\frac{2\pi}{3}$ with positive direction of X -axis.

Thus, slope of the line $OM = \tan \frac{2\pi}{3} = -\sqrt{3}$

Hence, equation of line OM is $y = -\sqrt{3}x$

$$\Rightarrow \sqrt{3}x + y = 0$$

36. Here, $a = 3, h = 4, b = -3, g = \frac{29}{2}, f = -\frac{3}{2}, c = 18$

$\therefore$ The point of intersection

$$\begin{aligned} &= \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right) \\ &= \left[\frac{\left(-3 \times \frac{29}{2}\right) + \left(\frac{3}{2} \times 4\right)}{16 + 9}, \frac{\left(3 \times \frac{-3}{2} - \frac{29}{2} \times 4\right)}{16 + 9} \right] \\ &= \left[\frac{-87 + 12}{25}, \frac{-9 - 116}{25} \right] = \left[\frac{-3}{2}, \frac{-5}{2} \right] \end{aligned}$$

37. Here, $m_1 + m_2 = -\frac{2h}{b} = \frac{2 \tan \theta}{\sin^2 \theta} = \frac{2}{\sin \theta \cos \theta}$

and $m_1 m_2 = \frac{a}{b} = \frac{(\tan^2 \theta + \cot^2 \theta)}{\sin^2 \theta} = \sec^2 \theta + \cot^2 \theta$

Now, $m_1 - m_2 = \sqrt{(m_1 + m_2)^2 - 4m_1 m_2}$

$$= \sqrt{\frac{4}{\sin^2 \theta \cos^2 \theta} - 4(\sec^2 \theta + \cot^2 \theta)}$$

$$= \sqrt{\frac{4}{\sin^2 \theta \cos^2 \theta} - 4 \left[\frac{1}{\cos^2 \theta} + \frac{\cos^2 \theta}{\sin^2 \theta} \right]}$$

$$= \sqrt{\frac{4 - 4\sin^2 \theta - 4\cos^4 \theta}{\sin^2 \theta \cos^2 \theta}}$$

$$= \sqrt{\frac{4 - 4\sin^2 \theta - 4(1 - \sin^2 \theta)\cos^2 \theta}{\sin^2 \theta \cos^2 \theta}}$$

$$= \sqrt{\frac{4 - 4\sin^2 \theta - 4\cos^2 \theta + 4\sin^2 \theta \cos^2 \theta}{\sin^2 \theta \cos^2 \theta}}$$

$$= \sqrt{\frac{4\sin^2 \theta \cos^2 \theta}{\sin^2 \theta \cos^2 \theta}} = 2$$

38. Bisector of the angles between the lines $x^2 - 2mxy - y^2 = 0$ is

$$\frac{x^2 - y^2}{2} = \frac{xy}{-m}$$

$$\Rightarrow mx^2 + 2xy - my^2 = 0$$

But it represents by $x^2 - 2nxy - y^2 = 0$.

$$\therefore \frac{m}{1} = \frac{2}{-2n} \Rightarrow mn = -1$$

39. $\therefore (y-1) = \frac{2-1}{1+\frac{1}{2}} \left(x + \frac{1}{2}\right)$

$$\Rightarrow (y-1) = \frac{2}{3} \left(x + \frac{1}{2}\right) \Rightarrow 2x - 3y = -4$$

$$\Rightarrow \frac{x}{-2} + \frac{y}{\left(\frac{4}{3}\right)} = 1$$

So, x-intercept is -2.

40. Intersection point of $3x - 4y = 2$ and $x + 2y = -4$ is $\left(\frac{-6}{5}, \frac{-7}{5}\right)$.

$$\therefore \text{Slope of the required line} = \frac{\frac{-7}{5} - 0}{\frac{-6}{5} - 0} = \frac{7}{6}$$

$\therefore$ Required equation of line is

$$y - 0 = \frac{7}{6}(x - 0) \Rightarrow 6y = 7x$$

41. The line is parallel to the X-axis i.e. $m = 0$

Also, point of intersection of the lines

$$ax + 2by + 3b = 0$$

and $bx - 2ay - 3a = 0$ is $\left(0, -\frac{3}{2}\right)$.

$\therefore$ Required equation of the line is

$$y + \frac{3}{2} = 0(x - 0) \Rightarrow y = -\frac{3}{2}$$

So, the line is below the X-axis at a distance of $3/2$.

42. Here, $a_1 a_2 + b_1 b_2 = 3 \times 12 + 4 \times (-5) = 16 > 0$

For acute angle bisector,

$$\frac{3x + 4y + 5}{\sqrt{9 + 16}} = -\frac{(12x - 5y - 7)}{\sqrt{12^2 + (-5)^2}}$$

$$\Rightarrow \frac{3x + 4y + 5}{5} = -\frac{(12x - 5y - 7)}{13}$$

$$\Rightarrow 39x + 52y + 65 = -60x + 25y + 35$$

$$\Rightarrow 99x + 27y + 30 = 0$$

43. Let (h, k) be the point of the intersection, then coordinates of A and B are $(h, 0)$ and $(0, k)$, respectively. Therefore, equation of variables line is

$$y - 0 = \frac{k - 0}{0 - h}(x - h)$$

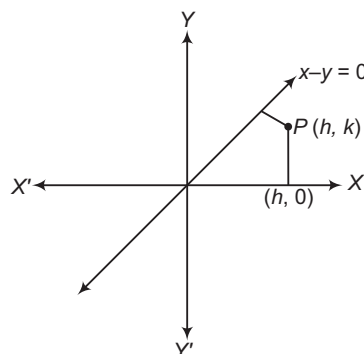
$$\Rightarrow y = -\frac{k}{h}(x - h)$$

Since, the line passes through (a, b) , we get

$$b = \frac{k}{h}(a - h) \Rightarrow \frac{b}{k} + \frac{a}{h} = 1$$

So, locus is $\frac{a}{x} + \frac{b}{y} = 1$.

44. Let $P(h, k)$ be the point from which its distance from X-axis i.e. is twice its distance from the line $x - y = 0$



$$\therefore k = 2 \left[\frac{|h - k|}{\sqrt{2}} \right]$$

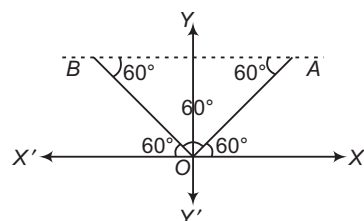
$$\Rightarrow k^2 = 2(h^2 + k^2 - 2hk)$$

$$\Rightarrow 2h^2 + k^2 - 4hk = 0$$

So, the locus of the point P is

$$2x^2 + y^2 - 4xy = 0$$

45. Equation of OA is $y = \sqrt{3}x$. Equation of OB is $y = -\sqrt{3}x$ and equation of AB is $y = 1$.



Clearly, from figure, $\triangle OAB$ is an equilateral triangle.

46. We have, $y = \frac{-x \sec \theta}{\operatorname{cosec} \theta} + \frac{a}{\operatorname{cosec} \theta}$

$$\Rightarrow \text{Slope} = -\frac{\sec \theta}{\operatorname{cosec} \theta}$$

$\therefore$ Required equation of line is

$$y - a \sin^3 \theta = \frac{\operatorname{cosec} \theta}{\sec \theta} (x - a \cos^3 \theta)$$

$$\Rightarrow y - a \sin^3 \theta = \frac{\cos \theta}{\sin \theta} (x - a \cos^3 \theta)$$

$$\Rightarrow y \sin \theta - a \sin^4 \theta = x \cos \theta - a \cos^4 \theta$$

$$\Rightarrow x \cos \theta - y \sin \theta + a \sin^4 \theta - a \cos^4 \theta = 0$$

$$\Rightarrow x \cos \theta - y \sin \theta - a \cos 2\theta = 0$$

47. Points $(a, 0)$ and $(0, b)$ will satisfy the equation of line $px - qy = r$.

$$\Rightarrow ap = r, -bq = r$$

$$\therefore a + b = \frac{r}{p} - \frac{r}{q} = r \left(\frac{q - p}{pq} \right)$$

48. Let m be the slope of required line.

$$\therefore \left| \frac{m - (-1)}{1 + m(-1)} \right| = 1 \Rightarrow \frac{m + 1}{1 - m} = \pm 1$$

$$\Rightarrow m + 1 = 1 - m, m + 1 = -1 + m$$

$$\Rightarrow m = 0, m = \infty$$

So, equation of line through $(1, 1)$ is

$$y - 1 = 0, x - 1 = 0$$

49. Locus is $|x| + |y| = 1$, which separately represents equation of straight lines.

50. Required equation can be $4x - 3y - k = 0$

$$\therefore \left| \frac{4 \times (-1) - 3 \times (-4) - k}{\sqrt{4^2 + (-3)^2}} \right| = 1$$

$$\Rightarrow \frac{-4 + 12 - k}{5} = \pm 1$$

$$\Rightarrow 8 - k = \pm 5$$

$$\Rightarrow k = 3 \text{ or } k = 13$$

$\therefore$ Equations of lines are $4x - 3y - 3 = 0$

and $4x - 3y - 13 = 0$.

51. Equation of line is $y = mx + 4$.

$$\therefore \text{Required distance} = \frac{4}{\sqrt{1 + m^2}}$$

52. Let equation of line parallel to $3x - y = 7$ be $3x - y = \lambda$.

Since, it passes through $(1, 2)$.

$$\therefore 3 - 2 = \lambda$$

$$\Rightarrow \lambda = 1$$

So, line is $3x - y = 1$

The point of intersection of $x + y + 5 = 0$

and $3x - y = 1$ is $(-1, -4)$.

So, distance between $(1, 2)$ and $(-1, -4)$

$$\begin{aligned} &= \sqrt{\{1 - (-1)\}^2 + \{2 - (-4)\}^2} \\ &= \sqrt{(2)^2 + (6)^2} = \sqrt{4 + 36} = \sqrt{40} \end{aligned}$$

53. The equation of line in new position is

$$y - 0 = \tan 15^\circ (x - 2)$$

$$\Rightarrow y = (2 - \sqrt{3})(x - 2)$$

$$\Rightarrow (2 - \sqrt{3})x - y - 4 + 2\sqrt{3} = 0$$

54. Given lines are $x + y = 4$

$$\text{and } 2x + 2y = 5$$

$$\Rightarrow x + y = \frac{5}{2}$$

The distance between two parallel lines.

$$\therefore d = \frac{4 - \frac{5}{2}}{\sqrt{1^2 + 1^2}} = \frac{3}{2\sqrt{2}} = \frac{3\sqrt{2}}{4} > 1$$

Hence, no point lies in it.

55. Let m be the required slope.

$$\therefore \left| \frac{m - 3}{1 + 3m} \right| = 1$$

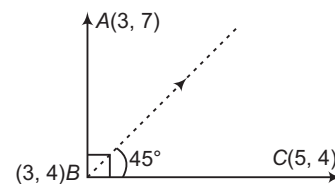
$$\Rightarrow \frac{m - 3}{1 + 3m} = \pm 1$$

$$\Rightarrow m - 3 = 1 + 3m$$

$$\text{and } m - 3 = -1 - 3m$$

$$\Rightarrow m = -2, m = \frac{1}{2}$$

56. Required line is passing through $(3, 4)$ and having slope 1.



$\therefore$ Equation of required line is

$$y - 4 = 1(x - 3)$$

$$\Rightarrow x - y + 1 = 0$$

$$\Rightarrow y = x + 1$$

13

Circle

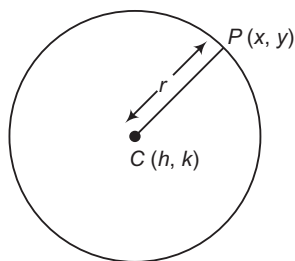
Introduction

A circle is the locus of a point which moves in a plane such that its distance from a fixed point in plane is always a constant. The fixed point is called the centre and the constant distance is called the radius of the circle.

Standard Equation of a Circle

The equation of a circle whose centre is at (h, k) and radius r , is given by

$$(x - h)^2 + (y - k)^2 = r^2$$



Proof Let C be the centre of the circle and its coordinates be (h, k) . Let the radius of the circle be r and let $P(x, y)$ be any point on the circumference. Then,

$$CP = r \Rightarrow CP^2 = r^2$$

$$\Rightarrow (x - h)^2 + (y - k)^2 = r^2$$

It is the required equation of the circle having centre at (h, k) and radius equal to r .

🔗 The above equation is known as the central form of the equation of a circle.

🔗 **Example 1.** The equation of the circle whose centre is $(2, -3)$ and radius 5, is

(a) $x^2 + y^2 + 4x - 6y - 12 = 0$

(b) $x^2 + y^2 - 4x + 6y - 12 = 0$

(c) $x^2 + y^2 - 6x + 4y - 12 = 0$

(d) None of the above

Sol. (b) The equation of the required circle is $(x - 2)^2 + (y + 3)^2 = 5^2$

$$\Rightarrow x^2 + y^2 - 4x + 6y - 12 = 0$$

Chapter Snapshot

- Introduction
- Standard Equation of a Circle
- Circle Passing through Three Points
- Position of a Point with Respect to a Circle
- Intersection of a Straight Line and a Circle
- Equation of Tangent
- Normal to a Circle
- Pair of Tangents
- Director Circle
- Pole and Polar
- Diameter of a Circle
- Angle of Intersection of Two Circles
- Family of Circles
- Coaxial System of Circles
- Limiting Points

➤ **Example 2.** The equation of the circle whose centre is $(1, 2)$ and which passes through the point $(4, 6)$, is

(a) $x^2 + y^2 - 8x - 12y + 20 = 0$

(b) $x^2 + y^2 - 2x - 4y - 25 = 0$

(c) $x^2 + y^2 - 2x - 4y - 20 = 0$

(d) $x^2 + y^2 - 8x - 12y - 20 = 0$

Sol. (c) We have, centre of the circle $(1, 2)$, since point $P(4, 6)$ lies on the circle, therefore radius of circle,

$$r = \sqrt{(4-1)^2 + (6-2)^2} = 5$$

Now, the equation of the required circle whose centre is $(1, 2)$ and radius 5 is $(x-1)^2 + (y-2)^2 = 5^2$

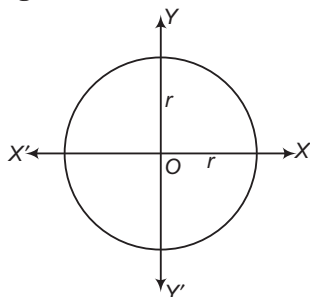
$$\Rightarrow x^2 + y^2 - 2x - 4y - 20 = 0$$

Some Particular Cases of the Central Form

The equation of a circle with centre at (h, k) and radius equal to r , is

$$(x-h)^2 + (y-k)^2 = r^2 \quad \dots(i)$$

i. **When the centre of the circle coincides with the origin i.e. $h = k = 0$**



Then, Eq. (i) reduces to

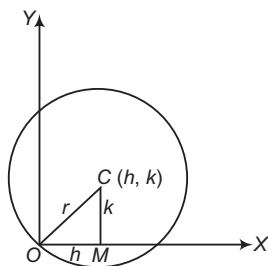
$$x^2 + y^2 = r^2$$

ii. **When the circle passes through the origin** Let O be the origin and $C(h, k)$ be the centre of the circle. Draw $CM \perp OX$.

In $\triangle OCM$, we have

$$OC^2 = OM^2 + CM^2$$

i.e.
$$r^2 = h^2 + k^2$$



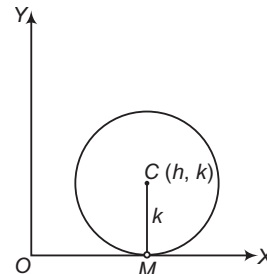
The equation of the circle (i) is

$$(x-h)^2 + (y-k)^2 = h^2 + k^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky = 0$$

iii. **When the circle touches X-axis** Let $C(h, k)$ be the centre of the circle. Since, the circle touches the X-axis.

$$\therefore r = k$$



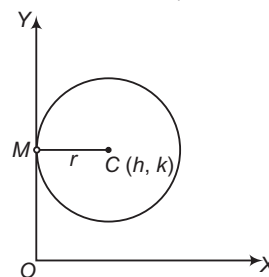
Hence, the equation of the circle is

$$(x-h)^2 + (y-r)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ry + h^2 = 0$$

iv. **When the circle touches Y-axis** Let $C(h, k)$ be the centre of the circle. Since, the circle touches the Y-axis.

$$\therefore h = r$$



Hence, the equation of the circle is

$$(x-r)^2 + (y-k)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2rx - 2ky + k^2 = 0$$

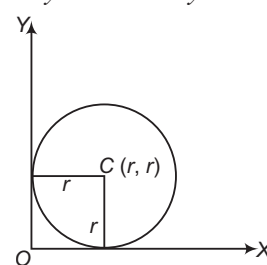
v. **When the circle touches both the axes** In this case,

$$h = k = r$$

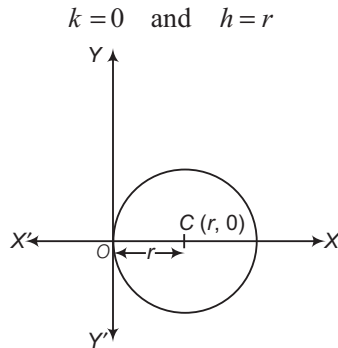
Hence, the equation of the circle is

$$(x-r)^2 + (y-r)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2rx - 2ry + r^2 = 0$$



vi. When the circle passes through the origin and centre lies on X-axis In this case, we have

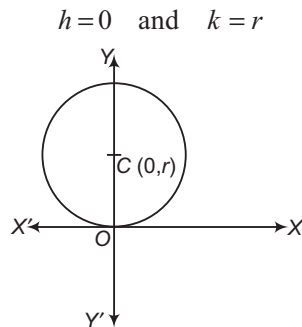


Hence, the equation of the circle is

$$(x-r)^2 + (y-0)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2rx = 0$$

vii. When the circle passes through the origin and centre lies on Y-axis In this case, we have



The equation of the circle is

$$(x-0)^2 + (y-r)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2ry = 0$$

Example 3. The equation of the circle which touches the Y-axis at the origin and passes through (3, 4) is

- (a) $2(x^2 + y^2) - 3x = 0$ (b) $3(x^2 + y^2) - 25x = 0$
 (c) $4(x^2 + y^2) - 25x = 0$ (d) None of these

Sol. (b) The circle touches Y-axis at origin i.e. centre of circle lies on X-axis.

$\therefore$ The equation of circle is

$$x^2 + y^2 - 2rx = 0$$

The circle passes through (3, 4).

$$\therefore 9 + 16 - 6r = 0$$

$$\Rightarrow r = \frac{25}{6}$$

Required equation of circle is

$$x^2 + y^2 - 2 \times \frac{25}{6} x = 0$$

$$\Rightarrow 3(x^2 + y^2) - 25x = 0$$

Example 4. The equation of the incircle of the triangle formed by the axes and the line $4x + 3y = 6$, is

(a) $x^2 + y^2 - 6x - 6y + 9 = 0$

(b) $4(x^2 + y^2 - x - y) + 1 = 0$

(c) $4(x^2 + y^2 + x + y) + 1 = 0$

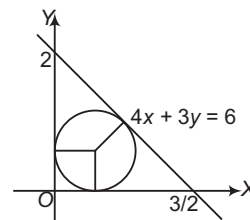
(d) None of the above

Sol. (b) If the inradius is r , then the centre is (r, r) and its distance from the line $4x + 3y = 6$ is r .

$$\text{So, } \frac{|4r + 3r - 6|}{\sqrt{4^2 + 3^2}} = r$$

$$\Rightarrow |7r - 6| = 5r$$

$$7r - 6 = \pm 5r \Rightarrow r = 3, \frac{1}{2}$$



Clearly, from the figure, $r \neq 3$

$\therefore$ The equation of the incircle is

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow x^2 + y^2 - x - y + \frac{1}{4} = 0$$

$$\Rightarrow 4(x^2 + y^2 - x - y) + 1 = 0$$

General Equation of a Circle

The equation $x^2 + y^2 + 2gx + 2fy + c = 0$ always represents a circle whose centre is $(-g, -f)$ and radius is $\sqrt{g^2 + f^2 - c}$.

Proof The given equation is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

$$\Rightarrow (x^2 + 2gx + g^2) + (y^2 + 2fy + f^2) = g^2 + f^2 - c$$

$$\Rightarrow (x+g)^2 + (y+f)^2 = (\sqrt{g^2 + f^2 - c})^2$$

$$\Rightarrow \{x - (-g)\}^2 + \{y - (-f)\}^2 = \{\sqrt{g^2 + f^2 - c}\}^2$$

This is of the form $(x-h)^2 + (y-k)^2 = r^2$, which represents a circle having centre at (h, k) and radius equal to r .

Hence, the given Eq. (i) represents a circle whose centre is $(-g, -f)$ i.e.

$$\left(-\frac{1}{2} \text{ coefficient of } x, -\frac{1}{2} \text{ coefficient of } y\right)$$

and radius $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(\frac{1}{2} \text{ coefficient of } x\right)^2 + \left(\frac{1}{2} \text{ coefficient of } y\right)^2 - \text{constant term}}$$

- If $g^2 + f^2 - c > 0$, then the radius of the circle is real and hence the circle is also real.
- If $g^2 + f^2 - c = 0$, then the radius of the circle is zero. Such a circle is known as point circle.
- If $g^2 + f^2 - c < 0$, then the radius $\sqrt{g^2 + f^2 - c}$ of circle is imaginary but the centre is real. Such a circle is called an imaginary circle as it is not possible to draw such a circle.
- Special features of the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of the circle are
 - > It is quadratic in both x and y .
 - > Coefficient of x^2 = Coefficient of y^2

In solving problems, it is advisable to keep the coefficient of x^2 and y^2 unity.

 - > There is no term containing xy i.e. the coefficient of xy is zero.
 - > It contains three arbitrary constants viz. g, f and c .- On comparing the general equation $x^2 + y^2 + 2gx + 2fy + c = 0$ of a circle with the general equation of second degree $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ We find that it represents a circle, if $a = b$ i.e. coefficient of x^2 = coefficient of y^2 and $h = 0$ i.e. coefficient of $xy = 0$.

➤ **Example 5.** The centre and radius of the circle $3x^2 + 3y^2 - 8x - 10y + 3 = 0$ are

- (a) $(4, 5), 1$
 (b) $(8, 10), 3$
 (c) $\left(\frac{-4}{3}, \frac{-5}{3}\right), \frac{4}{3}\sqrt{2}$
 (d) $\left(\frac{4}{3}, \frac{5}{3}\right), \frac{4}{3}\sqrt{2}$

Sol. (d) We have,

$$3x^2 + 3y^2 - 8x - 10y + 3 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{8}{3}x - \frac{10}{3}y + 1 = 0$$

Here, $g = \frac{-4}{3}$ and $f = \frac{-5}{3}$

and $r = \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(\frac{-4}{3}\right)^2 + \left(\frac{-5}{3}\right)^2 - 1}$$

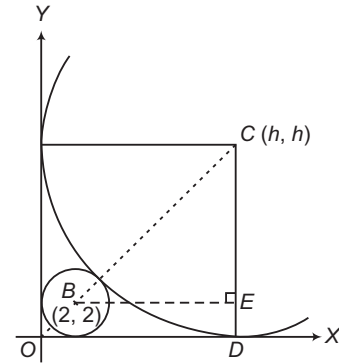
$$= \sqrt{\frac{32}{9}} = \frac{4}{3}\sqrt{2}$$

Hence, centre is $\left(\frac{4}{3}, \frac{5}{3}\right)$ and radius is $\frac{4}{3}\sqrt{2}$.

➤ **Example 6.** Radius of bigger circle touching the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ and both the coordinate axes, is

- (a) $(3 + 2\sqrt{2})$
 (b) $2(3 + 2\sqrt{2})$
 (c) $(6 + 2\sqrt{2})$
 (d) $2(6 + 2\sqrt{2})$

Sol. (b) Let the centre of required circle be (h, h) . We have,



$$\Rightarrow \angle COD = \angle CBE = \frac{\pi}{4}, CB = h + 2, BE = h - 2$$

$$\Rightarrow \frac{h - 2}{h + 2} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow h = \frac{2(\sqrt{2} + 1)}{(\sqrt{2} - 1)} = 2(3 + 2\sqrt{2})$$

➤ **Example 7.** If the equation $px^2 + (2 - q)xy + 3y^2 - 6qx + 30y + 6q = 0$ represents a circle, then the values of p and q are

- (a) $p = 2, q = 3$ (b) $p = 3, q = 2$
 (c) $p = 1, q = 2$ (d) $p = 2, q = 1$

Sol. (b) Second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, iff $a = b$ and $h = 0$
 In equation $px^2 + (2 - q)xy + 3y^2 - 6qx + 30y + 6q = 0$, it represents circle iff $p = 3$ and $2 - q = 0 \Rightarrow q = 2$
 $\therefore p = 3$ and $q = 2$

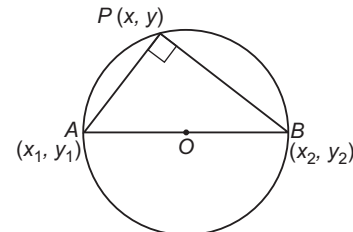
Equation of Circle when End Points of Diameter are Given

If A and B are end points of a diameter of a circle whose coordinates are (x_1, y_1) and (x_2, y_2) , respectively.

Then, the equation of circle is

$$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

Proof Let $P(x, y)$ be any point on the circle.



$\therefore$ Slope of $PA \cdot$ Slope of $PB = -1$

$$\Rightarrow \frac{y - y_2}{x - x_2} \times \frac{(y - y_1)}{(x - x_1)} = -1$$

$$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$$

- **Example 8.** If the abscissa and the ordinates of two points A and B are the roots of the equations $x^2 + 2ax - b^2 = 0$ and $y^2 + 2py - q^2 = 0$ respectively. Then, the equation of circle with AB as diameter is
- (a) $x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$
 (b) $x^2 + y^2 - 2ax + 2py + b^2 + q^2 = 0$
 (c) $x^2 + y^2 - 2ax - 2py + b^2 + q^2 = 0$
 (d) $x^2 + y^2 + 2ax + 2py + b^2 + q^2 = 0$

Sol. (a) Let (x_1, x_2) and (y_1, y_2) be the roots of the equation $x^2 + 2ax - b^2 = 0$ and $y^2 + 2py - q^2 = 0$ respectively. Then, $x_1 + x_2 = -2a$, $x_1 x_2 = -b^2$, $y_1 + y_2 = -2p$ and $y_1 y_2 = -q^2$. The equation of circle with $A(x_1, y_1)$ and $B(x_2, y_2)$ as diameter is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
 $\Rightarrow x^2 + y^2 - x(x_1 + x_2) - y(y_1 + y_2) + x_1 x_2 + y_1 y_2 = 0$
 $\Rightarrow x^2 + y^2 + 2ax + 2py - b^2 - q^2 = 0$

Circle Passing through Three Points

Let $x^2 + y^2 + 2gx + 2fy + c = 0$... (i)
 be the circle passing through three non-collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$. Then,
 $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$... (ii)
 $x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$... (iii)
 $x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$... (iv)

On solving Eqs. (ii), (iii) and (iv) as simultaneous linear equations in g, f and c , we can obtain the values of g, f and c . Substituting these values in Eq. (i), we obtain the equation of the desired circle.

Alternately, we eliminate g, f and c from Eqs. (i), (ii), (iii) and (iv) to obtain the equation of the required circle as follows :

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Remember If P is a point and C is the centre of a circle of radius r , then the maximum and minimum distances of P from the circle are $CP + r$ and $CP - r$ respectively.

- **Example 9.** The circle passing through the distinct points $(1, t)$, $(t, 1)$ and (t, t) for all values of t , passes through the point
- (a) $(1, 1)$ (b) $(-1, -1)$
 (c) $(1, -1)$ (d) $(-1, 1)$

Sol. (a) Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots (i)$$

It is passing through $(1, t)$, $(t, 1)$ and (t, t) , then

$$1 + t^2 + 2g + 2ft + c = 0 \quad \dots (ii)$$

$$t^2 + 1 + 2gt + 2f + c = 0 \quad \dots (iii)$$

$$\text{and } 2t^2 + 2gt + 2ft + c = 0 \quad \dots (iv)$$

On solving Eqs. (ii), (iii) and (iv), we get

$$2g(t - 1) + 2f(1 - t) = 0$$

$$\Rightarrow g - f = 0 \text{ and } t^2 - 1 + 2f(t - 1) = 0$$

$$f = -\frac{(t + 1)}{2} = g$$

$$\text{From Eq. (iv), } 2t^2 - t(t + 1) - t(t + 1) + c = 0$$

$$\Rightarrow c = 2t$$

From Eq. (i),

$$x^2 + y^2 - (t + 1)x - (t + 1)y + 2t = 0$$

$$\Rightarrow (x^2 + y^2 - x - y) - t(x + y - 2) = 0$$

$$\therefore P + tQ = 0$$

$$\therefore P = 0$$

$$\text{and } Q = 0$$

$$\text{Then, } x^2 + y^2 - x - y = 0 \quad \dots (v)$$

$$\text{and } x + y - 2 = 0 \quad \dots (vi)$$

On solving Eqs. (v) and (vi), we get $x = 1$ and $y = 1$.

Intercepts on the Axes

The lengths of intercepts made by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ with X and Y -axes are

$$2\sqrt{g^2 - c} \quad \text{and} \quad 2\sqrt{f^2 - c}$$

Proof Intercepts on X -axis:

Suppose the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

meets the X -axis at two points A_1 and A_2 .

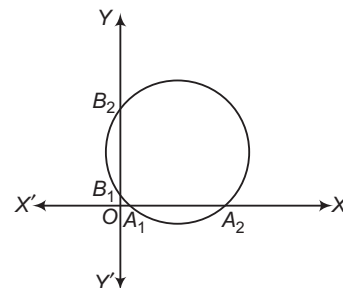
Since, on X -axis, $y = 0$. Therefore, x -coordinates (abscissa) of points A_1 and A_2 are roots of the equation

$$x^2 + 2gx + c = 0 \quad \dots (i)$$

Let x_1 and x_2 be the abscissa of A_1 and A_2 , respectively. Then, x_1 and x_2 are roots of Eq. (i).

$$\therefore x_1 + x_2 = -2g \quad \text{and} \quad x_1 x_2 = c$$

Now, intercept on X -axis $= A_1 A_2 = x_2 - x_1$



$$\begin{aligned} &= \sqrt{(x_2 + x_1)^2 - 4x_1 x_2} = \sqrt{4g^2 - 4c} \\ &= 2\sqrt{g^2 - c} \end{aligned}$$

Similarly, we have intercept on Y -axis $= 2\sqrt{f^2 - c}$

- If $g^2 > c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and distinct, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ meets the X-axis at two real and distinct points and the length of the intercept on X-axis is $2\sqrt{g^2 - c}$.
- If $g^2 = c$, then the roots of the equation $x^2 + 2gx + c = 0$ are real and equal, so the circle touches X-axis and the intercept on X-axis is zero.
- If $g^2 < c$, then the roots of the equation $x^2 + 2gx + c = 0$ are imaginary, so the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not meet X-axis at real points.
- Similarly, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the Y-axis at real and distinct points, touches or does not meet at real points according as $f^2 >, =$ or $< c$.

➤ **Example 10.** The equation of the circle touching Y-axis at (0, 3) and making intercept of 8 units on the axis

- (a) $x^2 + y^2 - 10x - 6y - 9 = 0$
 (b) $x^2 + y^2 - 10x - 6y + 9 = 0$
 (c) $x^2 + y^2 + 10x - 6y + 9 = 0$
 (d) $x^2 + y^2 + 10x + 6y + 9 = 0$

Sol. (b, c) Let the equation of the circle be
 $x^2 + y^2 + 2gx + 2fy + c = 0$

Since, circle touches Y-axis at (0, 3).

$$\therefore 9 + 6f + c = 0 \text{ and } f^2 = c$$

$$\Rightarrow f^2 + 6f + 9 = 0$$

$$\Rightarrow (f + 3)^2 = 0$$

$$\text{and } f = -3$$

$$\therefore c = f^2 = 9$$

Since, length of the intercept of circle on Y-axis is 8 units.

$$\therefore 2\sqrt{g^2 - c} = 8$$

$$\Rightarrow g^2 - c = 16$$

$$\Rightarrow g^2 = 16 + c$$

$$\Rightarrow g^2 = 16 + 9$$

$$\Rightarrow g = \pm 5$$

The centre of the required circle are $(-g, -f)$ i.e. $(-5, 3)$ and $(5, 3)$ and radius $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{25 + 9 - 9} = 5$$

Hence, equation of required circle are

$$(x + 5)^2 + (y - 3)^2 = 25$$

$$\text{or } (x - 5)^2 + (y - 3)^2 = 25$$

$$\Rightarrow x^2 + y^2 + 10x - 6y + 9 = 0$$

$$\text{or } x^2 + y^2 - 10x - 6y + 9 = 0$$

➤ **Example 11.** A circle touches a given straight line and cuts of a constant length $2d$ from another straight line perpendicular to the first straight line. The locus of the centre of the circle is

- (a) $y^2 - x^2 = d^2$ (b) $x^2 + y^2 = d^2$
 (c) $xy = d^2$ (d) None of these

Sol. (a) Let the two given straight lines be X and Y-axes and the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. It touches X-axis at $c = g^2$.

The circle cuts an intercept of length $2d$ on Y-axis.

$$\therefore 2d = 2\sqrt{f^2 - c}$$

$$\Rightarrow d^2 = f^2 - c$$

$$\Rightarrow d^2 = f^2 - g^2$$

Hence, the locus of $(-g, -f)$ is

$$d^2 = (-y)^2 - (-x)^2$$

$$\Rightarrow y^2 - x^2 = d^2$$

Position of a Point with Respect to a Circle

A point (x_1, y_1) lies outside, on or inside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ according as $S_1 >, =$ or < 0 , where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Proof Let $P(x_1, y_1)$ be the given point and let C be the centre of the circle. Then, the coordinates of C are $(-g, -f)$.

$$\text{Now, } CP = \sqrt{(x_1 + g)^2 + (y_1 + f)^2}$$

The point P lies outside, on or inside the circle according as, $CP >, =$ or $<$ radius of the circle

$$\text{i.e. } \sqrt{(x_1 + g)^2 + (y_1 + f)^2} >, = \text{ or } < \sqrt{g^2 + f^2 - c}$$

➤ **Example 12.** If the point $(a, -a)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then a lies in the interval

- (a) $(-1, 4)$
 (b) $(-\infty, -1)$
 (c) $(4, \infty)$
 (d) $[-1, 4]$

Sol. (a) Since, the point $(a, -a)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$.

$$\therefore a^2 + a^2 - 4a - 2a - 8 < 0$$

$$\Rightarrow 2a^2 - 6a - 8 < 0$$

$$\Rightarrow a^2 - 3a - 4 < 0$$

$$\Rightarrow (a - 4)(a + 1) < 0$$

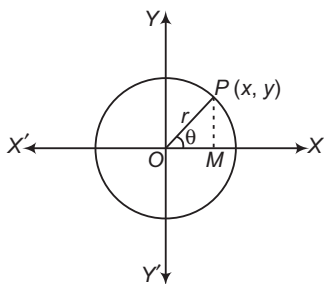
$$\Rightarrow -1 < a < 4$$

$$\Rightarrow a \in (-1, 4)$$

Equation of a Circle in Parametric Form

(i) Circle with Centre (0,0) and Radius r

Let $P(x, y)$ be any point on the circle $x^2 + y^2 = r^2$ and OP make an angle θ with the positive direction of X-axis. Let PM be perpendicular from P on X-axis. Then, $\angle MOP = \theta$.

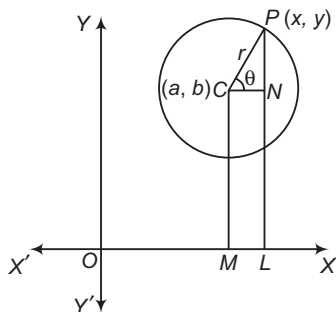


Thus, $x = r \cos \theta$, $y = r \sin \theta$ are the required parametric form of the equation $x^2 + y^2 = r^2$, where θ is parameter.

(ii) Circle with Centre (a, b) and Radius r

Let $P(x, y)$ be any point on the circle
 $(x - a)^2 + (y - b)^2 = r^2$

Let PL and CM be perpendiculars from P and C , respectively on X -axis and let CN be perpendicular from C on PL .



Let $\angle NCP = \theta$

Then, $x = OL = OM + ML$

$$= OM + CN$$

$$= a + r \cos \theta \quad [\because CN = CP \cos \theta = r \cos \theta]$$

and $y = PL$

$$= PN + NL = CM + PN$$

$$= b + r \sin \theta \quad [\because PN = CP \sin \theta = r \sin \theta]$$

Hence, $x = a + r \cos \theta$, $y = b + r \sin \theta$ is the required parametric equation of the circle

$$(x - a)^2 + (y - b)^2 = r^2$$

Example 13. α, β and γ are parametric angles of three points P, Q and R respectively on the circle $x^2 + y^2 = 1$ and A is the point $(-1, 0)$. If the lengths of the chords AP, AQ and AR are in GP, then $\cos \frac{\alpha}{2}, \cos \frac{\beta}{2}$ and $\cos \frac{\gamma}{2}$ are in

(a) AP

(b) GP

(c) HP

(d) None of these

Sol. (b) Coordinates of P, Q, R are $(\cos \alpha, \sin \alpha), (\cos \beta, \sin \beta)$ and $(\cos \gamma, \sin \gamma)$ respectively and $A \equiv (-1, 0)$.

$$\therefore AP = \sqrt{(1 + \cos \alpha)^2 + \sin^2 \alpha} = 2 \cos \frac{\alpha}{2}$$

$$AQ = \sqrt{(1 + \cos \beta)^2 + \sin^2 \beta} = 2 \cos \frac{\beta}{2}$$

$$AR = \sqrt{(1 + \cos \gamma)^2 + \sin^2 \gamma} = 2 \cos \frac{\gamma}{2}$$

Hence, AP, AQ and AR are in GP, then $\cos \frac{\alpha}{2}, \cos \frac{\beta}{2}, \cos \frac{\gamma}{2}$

are also in GP.

Example 14. Orthocentre of the ΔABC , where $A \equiv (a \cos \theta_1, a \sin \theta_1), B \equiv (a \cos \theta_2, a \sin \theta_2)$ and $C \equiv (a \cos \theta_3, a \sin \theta_3)$ is

(a) $\left(\frac{2a}{3} \sum \cos \theta_i, \frac{2a}{3} \sum \sin \theta_i \right)$

(b) $\left(\frac{a}{2} \sum \cos \theta_i, \frac{a}{2} \sum \sin \theta_i \right)$

(c) $(a \sum \cos \theta_i, a \sum \sin \theta_i)$

(d) $\left(\frac{3a}{2} \sum \cos \theta_i, \frac{3a}{2} \sum \sin \theta_i \right)$

Sol. (c) A, B and C lie on the circle $x^2 + y^2 = a^2$. That means circumcentre of ΔABC is $O \equiv (0, 0)$ and its centroid is

$$G \equiv \left(\frac{a}{3} \sum \cos \theta_i, \frac{a}{3} \sum \sin \theta_i \right)$$

If orthocentre of triangle is $H(x_p, y_p)$, then

$$OG : GH = 1 : 2$$

$$\Rightarrow x_p = a \sum \cos \theta_i, y_p = a \sum \sin \theta_i$$

Work Book Exercise 13.1

1 If $2(x^2 + y^2) + 4\lambda x + \lambda^2 = 0$ represents a circle of meaningful radius, then the range of real values of λ is

- a R
- b $(0, \infty)$
- c $(-\infty, 0)$
- d None of the above

2 If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is $(3, 4)$, then find the coordinates of the other end of the diameter.

- a $(2, 1)$
- b $(1, 2)$
- c $(1, 1)$
- d None of the above

3 The equation of circle with centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$ is

- a $x^2 + y^2 - 2ax - 2by - 2b^2 = 0$
- b $x^2 + y^2 + 2ax + by + 2b^2 = 0$
- c $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
- d None of the above

4 Equation of the circle with centre on the Y -axis and passing through the origin and the point $(2, 3)$ is

- a $x^2 + y^2 + 13y = 0$
- b $3x^2 + 3y^2 + 13x + 3 = 0$
- c $6x^2 + 6y^2 - 26y = 0$
- d $x^2 + y^2 + 13x + 3 = 0$

- 5 Circle $x^2 + y^2 - 2x - \lambda x - 1 = 0$ passes through two fixed points, coordinates of the points are
 a $(0, \pm 1)$ b $(\pm 1, 0)$
 c $(0, 1)$ and $(0, 2)$ d $(0, -1)$ and $(0, -2)$
- 6 Find the equation of the circle which passes through the points $(2, 3)$ and $(4, 5)$ and the centre lies on the straight line $y - 4x + 3 = 0$.
 a $x^2 + y^2 - 4x - 10y + 25 = 0$
 b $x^2 + y^2 - 4x - 10y - 25 = 0$
 c $x^2 + y^2 - 4x + 10y - 25 = 0$
 d None of the above
- 7 The circle $x^2 + y^2 - 10x - 14y + 24 = 0$ cuts an intercept on Y-axis of length
 a 5 b 10
 c 1 d None of these
- 8 Let $P \equiv (3, 6)$, $Q \equiv (3\cos \theta, 3\sin \theta)$ and $R \equiv (3\sin \theta, -3\cos \theta)$, then the locus of centroid to ΔPQR is (where, θ is real parameter)
 a $x^2 + y^2 - 2x + 4y + 3 = 0$
 b $x^2 + y^2 - 2x - 4y + 3 = 0$
 c $x^2 + y^2 - 2x + 3y = 0$
 d $x^2 + y^2 + 2x + 4y - 3 = 0$
- 9 If the circle $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$ (where, c and d are greater than zero). Then, maximum value of cd is
 a 25 b 125 c 425 d 625
- 10 From the point $A(0, 3)$ on the circle $x^2 + 4x + (y - 3)^2 = 0$, a chord AB is drawn and extended to a point P , such that $AP = 2AB$. The locus of P is
 a $x^2 + (y - 3)^2 = 0$
 b $x^2 + 4x + (y + 3)^2 = 0$
 c $x^2 + 8x + (y - 3)^2 = 16$
 d $(x + 4)^2 + (y - 3)^2 = 16$

Intersection of a Straight Line and a Circle

Let the equation of the circle be

$$x^2 + y^2 = r^2 \quad \dots(i)$$

and the equation of the line be

$$y = mx + c \quad \dots(ii)$$

Since, the points of intersection of Eqs. (i) and (ii) satisfy each of them. Therefore, to find the points of intersection of Eqs. (i) and (ii), we have to solve them simultaneously.

On substituting the value of y from Eq. (ii) in Eq. (i), we get

$$x^2 + (mx + c)^2 = r^2$$

$$\Rightarrow x^2(1 + m^2) + 2mcx + (c^2 - r^2) = 0 \quad \dots(iii)$$

i. When points of intersection are real and distinct

In this case, Eq. (iii) has two distinct roots.

$\therefore$ Discriminant > 0

$$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - r^2) > 0$$

$$\Rightarrow 4r^2(1 + m^2) - 4c^2 > 0$$

$$\Rightarrow r^2(1 + m^2) > c^2 \Rightarrow r^2 > \frac{c^2}{1 + m^2}$$

$$\Rightarrow r > \left| \frac{c}{\sqrt{1 + m^2}} \right|$$

ii. When the points of intersection are coincident

In this case, Eq. (iii) has two equal two roots.

$\therefore$ Discriminant $= 0$

$$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - r^2) = 0$$

$$\Rightarrow c^2 = r^2(1 + m^2) \Rightarrow r = \left| \frac{c}{\sqrt{1 + m^2}} \right|$$

iii. When the points of intersection are imaginary

In this case, Eq. (iii) has imaginary roots.

$\therefore$ Discriminant < 0

$$\Rightarrow 4m^2c^2 - 4(1 + m^2)(c^2 - r^2) < 0$$

$$\Rightarrow r^2(1 + m^2) - c^2 < 0$$

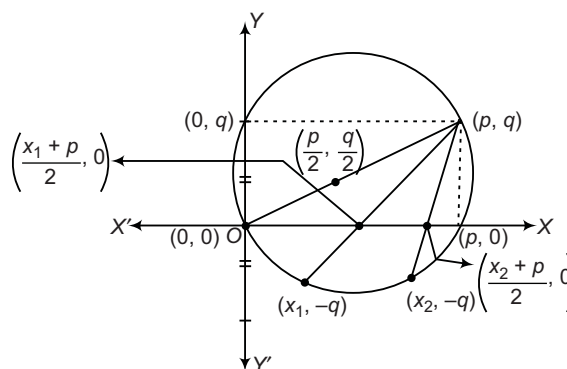
$$\Rightarrow r^2(1 + m^2) < c^2$$

$$\Rightarrow r < \left| \frac{c}{\sqrt{1 + m^2}} \right|$$

Example 15. Two distinct chords drawn from point (p, q) on the circle $x^2 + y^2 = px + qy$, where $pq \neq 0$ are bisected by the X-axis. Then,

(a) $|p| = |q|$ (b) $p^2 = 8q^2$ (c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$

Sol. (d) The equation of circle is $x^2 + y^2 - px - qy = 0$



On putting $y = -q$, we get
 $x^2 - px + 2q^2 = 0$

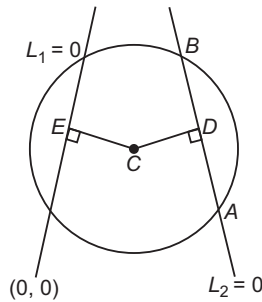
For real roots,

$$p^2 - 8q^2 > 0$$

$$\Rightarrow p^2 > 8q^2$$

- **Example 16.** L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by $x^2 + y^2 - x + 3y = 0$ on L_1 and L_2 are equal, then one possible equation of line L is
- (a) $x + 6y = 0$ (b) $x - 7y = 0$
 (c) $x + 7y = 0$ (d) $x - 6y = 0$

Sol. (c) Centre of circle is $\left(\frac{1}{2}, -\frac{3}{2}\right)$ and its radius is $\frac{\sqrt{5}}{2}$.



where, $L_1 : y = mx$ and $L_2 : x + y - 1 = 0$
 Since, length of intercept made by L_1 and L_2 are equal.

$\Rightarrow$ Distance from centre should also be equal.

$$\Rightarrow \frac{\left|\frac{m}{2} + \frac{3}{2}\right|}{\sqrt{m^2 + 1}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{2}}$$

$$\Rightarrow \left(\frac{m+3}{2}\right)^2 = 2(m^2 + 1)$$

$$\Rightarrow m^2 + 6m + 9 = 8m^2 + 8$$

$$\Rightarrow 7m^2 - 6m - 1 = 0$$

$$\Rightarrow (7m + 1)(m - 1) = 0$$

$$\Rightarrow m = 1, -\frac{1}{7}$$

$\therefore$ Required straight lines are

$$y = x \quad \text{and} \quad y = -\frac{1}{7}x$$

$$\Rightarrow y - x = 0 \quad \text{and} \quad x + 7y = 0$$

Length of the Intercept Cut off from a Line by a Circle

The length of the intercept cut off from the line $y = mx + c$ by the circle $x^2 + y^2 = r^2$ is

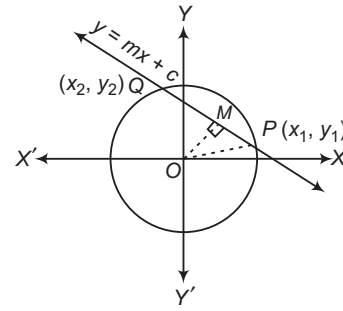
$$2\sqrt{\frac{r^2(1+m^2) - c^2}{(1+m^2)}}$$

Proof Draw $OM \perp PQ$. Join OP .

OM = Length of the perpendicular from $O(0, 0)$ to
 $y = mx + c$

$$= \frac{c}{\sqrt{1+m^2}}$$

and OP = radius = r



$$\therefore PQ = 2PM$$

$$\Rightarrow PQ = 2\sqrt{OP^2 - OM^2}$$

$$\Rightarrow PQ = 2\sqrt{r^2 - \frac{c^2}{1+m^2}}$$

$$\Rightarrow PQ = 2\sqrt{\frac{r^2(1+m^2) - c^2}{1+m^2}}$$

- **Example 17.** Length of the chord cut off by $y = 2x + 1$ from the circle $x^2 + y^2 = 4$ is equal to

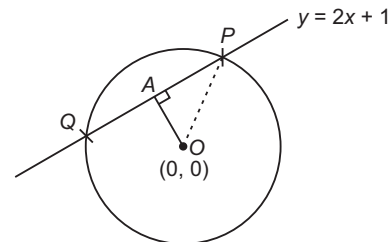
(a) $2\sqrt{\frac{13}{5}}$

(b) $2\sqrt{\frac{17}{5}}$

(c) $2\sqrt{\frac{11}{5}}$

(d) $2\sqrt{\frac{19}{5}}$

Sol. (d) Here, $OA = \frac{1}{\sqrt{5}}$



$$\text{Now, } PA^2 = OP^2 - OA^2 = 4 - \frac{1}{5} = \frac{19}{5}$$

$$\Rightarrow PQ = 2PA = 2\sqrt{\frac{19}{5}}$$

Condition of Tangency

The line $y = mx + c$ touches the circle $x^2 + y^2 = r^2$, if the length of the intercept is zero.

i.e. $PQ = 0$

$$\Rightarrow 2\sqrt{\frac{r^2(1+m^2) - c^2}{1+m^2}} = 0$$

$$\Rightarrow r^2(1+m^2) - c^2 = 0$$

$$\Rightarrow c = \pm r\sqrt{1+m^2}$$

which is the required condition.

➤ **Example 18.** The line $3x - 2y = k$ meets the circle $x^2 + y^2 = 4r^2$ at only one point, if k^2 is equal to

- (a) $20r^2$ (b) $52r^2$ (c) $\frac{52r^2}{9}$ (d) $\frac{20r^2}{9}$

Sol. (b) The line $3x - 2y = k$ meets the circle $x^2 + y^2 = 4r^2$ at only one point, so it is tangent to the circle.

$$\therefore \frac{k}{2} = \pm 2r \sqrt{1 + \left(\frac{3}{2}\right)^2}$$

$$\Rightarrow \frac{k^2}{4} = 4r^2 \left(1 + \frac{9}{4}\right) \Rightarrow k^2 = 52r^2$$

➤ **Example 19.** The condition that the straight line $cx - by + b^2 = 0$ may touch the circle $x^2 + y^2 = ax + by$, is

- (a) $abc = 1$ (b) $a = c$
(c) $b = ac$ (d) None of these

Sol. (b) Circle is given as $x^2 + y^2 = ax + by$

$$\Rightarrow x^2 + y^2 - ax - by = 0 \quad \dots(i)$$

The line is given as $cx - by + b^2 = 0$

$$\Rightarrow y = \left(\frac{c}{b}x + b\right) \quad \dots(ii)$$

On substituting the value of y from Eq. (ii) in Eq. (i), we get

$$x^2 + \left(\frac{c}{b}x + b\right)^2 - ax - b\left(\frac{c}{b}x + b\right) = 0$$

$$\Rightarrow x^2 + \frac{c^2}{b^2}x^2 + 2cx + b^2 - ax - cx - b^2 = 0$$

$$\Rightarrow x^2 \left(1 + \frac{c^2}{b^2}\right) + x(c - a) = 0 \quad \dots(iii)$$

If it is a perfect square, then $c - a = 0$ or $c = a$.

Equation of Tangent

i. **Slope form** The equation of a tangent of slope m to the circle $x^2 + y^2 = r^2$ is

$$y = mx \pm r\sqrt{1 + m^2}$$

The coordinates of the point of contact are

$$\left(\pm \frac{rm}{\sqrt{1 + m^2}}, \mp \frac{r}{\sqrt{1 + m^2}}\right)$$

- The equation of tangent of slope m to the circle $(x - a)^2 + (y - b)^2 = r^2$ is given by

$$y - b = m(x - a) \pm r\sqrt{1 + m^2}$$

and the coordinates of the points of contact are

$$\left(a \pm \frac{mr}{\sqrt{1 + m^2}}, b \mp \frac{r}{\sqrt{1 + m^2}}\right)$$

- The equation of the tangent of slope m to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\text{is } y + f = m(x + g) \pm \sqrt{(g^2 + f^2 - c)(1 + m^2)}$$

➤ **Example 20.** If $a > 2b > 0$, then the positive value of m for which $y = mx - b\sqrt{1 + m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x - a)^2 + y^2 = b^2$ is

- (a) $\frac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\frac{\sqrt{a^2 - 4b^2}}{2b}$
(c) $\frac{2b}{a - 2b}$ (d) $\frac{b}{a - 2b}$

Sol. (a) We have, $y = mx - b\sqrt{1 + m^2}$ is a tangent to the circle $x^2 + y^2 = b^2$ for all values of m . As it touches the circle $(x - a)^2 + y^2 = b^2$ and the equation of tangent of slope m to the circle $(x - a)^2 + y^2 = b^2$ is

$$y = m(x - a) \pm b\sqrt{1 + m^2}$$

$$\therefore y = mx - b\sqrt{1 + m^2}$$

and $y = m(x - a) \pm b\sqrt{1 + m^2}$ are same.

$$\therefore -ma + b\sqrt{1 + m^2} = -b\sqrt{1 + m^2}$$

$$\Rightarrow ma = 2b\sqrt{1 + m^2}$$

$$\Rightarrow m^2 a^2 = 4b^2(1 + m^2)$$

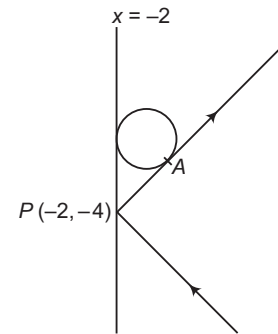
$$\Rightarrow m^2(a^2 - 4b^2) = 4b^2$$

$$\Rightarrow m = \frac{2b}{\sqrt{a^2 - 4b^2}}$$

➤ **Example 21.** A light ray gets reflected from the line $x = -2$. If the reflected ray touches the circle $x^2 + y^2 = 4$ and point of incident is $(-2, -4)$, then equation of incident ray is

- (a) $4y + 3x + 22 = 0$ (b) $3y + 4x + 20 = 0$
(c) $4y + 2x + 20 = 0$ (d) $x + y + 6 = 0$

Sol. (a) Any tangent of $x^2 + y^2 = 4$ is $y = mx \pm 2\sqrt{1 + m^2}$. If it passes through $(-2, -4)$, then $(2m - 4)^2 = 4(1 + m^2)$.



$$\Rightarrow 4m^2 + 16 - 16m = 4 + 4m^2$$

$$\Rightarrow m = \infty, m = \frac{3}{4}$$

Also, $m = \infty$ as $x = -2$ is reflected ray. Hence, slope of reflected ray is $\frac{3}{4}$. Thus, equation of incident ray is

$$(y + 4) = -\frac{3}{4}(x + 2)$$

$$\Rightarrow 4y + 3x + 22 = 0$$

ii. **Point form** The equation of the tangent at the point $P(x_1, y_1)$ to a circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \text{ is}$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

iii. **Parametric form** The equation of the tangent to the circle $(x - a)^2 + (y - b)^2 = r^2$ at the point $(a + r \cos \theta, b + r \sin \theta)$ is

$$(x - a) \cos \theta + (y - b) \sin \theta = r$$

Equation of tangent for $x^2 + y^2 = r^2$ at (x_1, y_1) is $xx_1 + yy_1 = r^2$.

➤ **Example 22.** The equation of tangent to the circle $x^2 + y^2 - 2ax = 0$ at the point $[a(1 + \cos \alpha), a \sin \alpha]$ is

(a) $x \cos \alpha + y \sin \alpha = a$

(b) $x \cos \alpha + y \sin \alpha = 1 + \cos \alpha$

(c) $x \cos \alpha + y \sin \alpha = a(1 + \cos \alpha)$

(d) None of the above

Sol. (c) The equation of tangent of $x^2 + y^2 - 2ax = 0$ at $[a(1 + \cos \alpha), a \sin \alpha]$ is

$$xa(1 + \cos \alpha) + ya \sin \alpha - a[x + a(1 + \cos \alpha)] = 0$$

$$ax \cos \alpha + ay \sin \alpha - a^2(1 + \cos \alpha) = 0$$

$$\Rightarrow x \cos \alpha + y \sin \alpha = a(1 + \cos \alpha)$$

➤ **Example 23.** Tangent to circle $x^2 + y^2 = 5$ at $(1, -2)$, also touches the circle

$x^2 + y^2 - 8x + 6y + 20 = 0$. The coordinates of the corresponding point of contact is

(a) $(2, 0)$

(b) $(3, -1)$

(c) $(-1, 3)$

(d) $(1, -2)$

Sol. (b) Equation of tangent to $x^2 + y^2 = 5$ at $(1, -2)$ is $x - 2y = 5$.

Putting $x = 2y + 5$ in second circle, we get

$$(2y + 5)^2 + y^2 - 8(2y + 5) + 6y + 20 = 0$$

$$\Rightarrow 5y^2 + 10y + 5 = 0$$

$$\Rightarrow (y + 1)^2 = 0$$

$$\Rightarrow y = -1$$

$$\Rightarrow x = -2 + 5 = 3$$

Thus, point of contact is $(3, -1)$.

Normal to a Circle

The normal at any point on a curve is a straight line which is perpendicular to the tangent to the curve at that point.

➤ **Example 24.** The normal to the circle $x^2 + y^2 - 3x - 6y - 10 = 0$ at the point $(-3, 4)$ is

(a) $2x + 9y - 30 = 0$

(b) $9x - 2y + 35 = 0$

(c) $2x - 9y + 30 = 0$

(d) $2x - 9y - 30 = 0$

Sol. (a) The equation of normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at any point (x_1, y_1) is

$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$

or $\frac{x - x_1}{x_1 + g} = \frac{y - y_1}{y_1 + f}$

$$\Rightarrow \frac{x + 3}{-3 - 3/2} = \frac{y - 4}{4 - 3}$$

$$\Rightarrow 2x + 9y - 30 = 0$$

Work Book Exercise 13.2

1 The equation of the circle which is touched by $y = x$, has its centre on the positive direction of the X-axis and cuts off a chord of length 2 units along the line $\sqrt{3}y - x = 0$, is

a $x^2 + y^2 - 4x + 2 = 0$

b $x^2 + y^2 - 4x + 1 = 0$

c $x^2 + y^2 - 8x + 8 = 0$

d $x^2 + y^2 - 4y + 2 = 0$

2 The locus of the point of intersection of perpendicular tangents to the circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ is

a $x^2 + y^2 = a^2 - b^2$

b $x^2 + y^2 = a^2 + b^2$

c $x^2 + y^2 = (a + b)^2$

d None of these

3 If the line $y = mx$ does not intersect the circle $(x + 10)^2 + (y + 10)^2 = 180$, then

a $m \in (-2, \infty)$

b $m \in \left(-\infty, -\frac{1}{2}\right)$

c $m \in \left(-2, -\frac{1}{2}\right)$

d None of these

4 If the circle $x^2 + y^2 + 4x + 22y + c = 0$ bisects the circumference of the circle $x^2 + y^2 - 2x + 8y - d = 0$, then $c + d$ is equal to

a 60

b 50

c 40

d 56

5 If AB is the intercept of the tangent of the circle $x^2 + y^2 = r^2$ between the coordinate axes, then the locus of the vertex P of the rectangle OAPB is

a $x^2 + y^2 = r^2$

b $\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{r^2}$

c $\frac{1}{x^2} + \frac{1}{y^2} = r^2$

d None of these

6 The area of the triangle formed by the tangent at the point (a, b) to the circle $x^2 + y^2 = r^2$ and the coordinate axes is

a $\frac{r^4}{2ab}$

b $\frac{r^4}{2|ab|}$

c $\frac{r^4}{ab}$

d $\frac{r^4}{|ab|}$

- 7 If two circles $a(x^2 + y^2) + bx + cy = 0$ and $A(x^2 + y^2) + Bx + Cy = 0$ touch each other, then
 a $aC = cA$ b $bC = cB$
 c $aB = bA$ d $aA = bB = cC$
- 8 The circles which can be drawn, to pass through $(1, 0)$ and $(3, 0)$ and touch the Y-axis intersect at an angle θ , then $\cos \theta$ is equal to
 a $\frac{1}{2}$ b $-\frac{1}{2}$
 c $\frac{1}{4}$ d $-\frac{1}{4}$
- 9 The locus of the point of intersection of the tangents to the circle $x^2 + y^2 = a^2$ at points whose parametric angles differ by $\frac{\pi}{3}$, is

- a $x^2 + y^2 = 4a^2$ b $3(x^2 + y^2) = a^2$
 c $3(x^2 + y^2) = 4a^2$ d $4(x^2 + y^2) = 3a^2$

- 10 If α, β are the roots of the equation $35x^2 - 24x - 35 = 0$, then the parametric equations $x = \alpha + a \cos \theta, y = \beta + a \sin \theta$ can represent a family of
 a straight lines passing through $\left(-\frac{5}{7}, -\frac{7}{5}\right)$
 b straight lines passing through $\left(\frac{5}{7}, -\frac{7}{5}\right)$
 c circles passing through $\left(-\frac{5}{7}, \frac{7}{5}\right)$ and $\left(\frac{7}{5}, \frac{5}{7}\right)$
 d concentric circles whose centres are at $\left(\frac{7}{5}, -\frac{5}{7}\right)$

Pair of Tangents

Theorem 1

From a given point, two tangents can be drawn to a circle which are real and distinct, coincident or imaginary according as the given point lies outside, on or inside the circle.

Proof Let (x_1, y_1) be a given point and the equation of the circle be $x^2 + y^2 = a^2$. Then, the equation of any tangent to the circle is of the form $y = mx \pm a\sqrt{1+m^2}$, where m is the slope of the tangent.

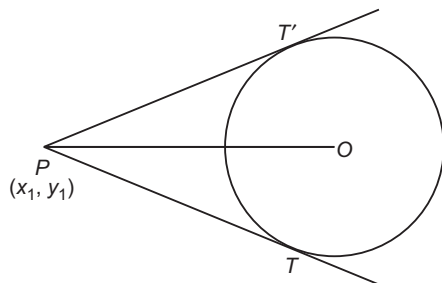
If this tangent passes through the given point (x_1, y_1) , we must have

$$y_1 = mx_1 \pm a\sqrt{1+m^2}$$

$$\Rightarrow (y_1 - mx_1)^2 = a^2(1+m^2)$$

$$\Rightarrow m^2(x_1^2 - a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0 \quad \dots(i)$$

This equation being quadratic in m , gives two values of m (real, coincident or imaginary), corresponding to any given values of x_1 and y_1 . For each value of m , we have a corresponding tangent.



Hence, in general, two tangents can be drawn from the point (x_1, y_1) to a circle.

The tangents are real, coincident or imaginary according as the values of m obtained from Eq. (i) are real, coincident or imaginary.

Or according as $4x_1^2y_1^2 - 4(x_1^2 - a^2)(y_1^2 - a^2) \geq$ or < 0

Or according as $x_1^2 + y_1^2 - a^2 >, =$ or < 0 .

Or according as the point $P(x_1, y_1)$ lies outside, on or inside the circle $x^2 + y^2 = a^2$.

Theorem 2

The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$ to the circle $x^2 + y^2 = a^2$ is $(x^2 + y^2 - a^2)(x_1^2 + y_1^2 - a^2) = (xx_1 + yy_1 - a^2)^2$ or $SS' = T^2$ where, $S = x^2 + y^2 - a^2, S' = x_1^2 + y_1^2 - a^2$ and $T = xx_1 + yy_1 - a^2$

Example 25. The angle between the pair of tangents from the point $\left(1, \frac{1}{2}\right)$ to the circle

$$x^2 + y^2 + 4x + 2y - 4 = 0, \text{ is}$$

- (a) $\cos^{-1}\left(\frac{4}{5}\right)$ (b) $\sin^{-1}\left(\frac{4}{5}\right)$
 (c) $\sin^{-1}\left(\frac{3}{5}\right)$ (d) None of these

Sol. (b) The equation of the pair of tangents is $SS_1 = T^2$.

$$\Rightarrow (x^2 + y^2 + 4x + 2y - 4) \left\{ 1^2 + \left(\frac{1}{2}\right)^2 + 4 \cdot 1 + 2 \cdot \frac{1}{2} - 4 \right\}$$

$$= \left\{ x \cdot 1 + y \cdot \frac{1}{2} + 2(x+1) + \left(y + \frac{1}{2}\right) - 4 \right\}^2$$

$$\Rightarrow \frac{9}{4}(x^2 + y^2 + 4x + 2y - 4) = \frac{9}{4}(2x + y - 1)^2$$

$$\Rightarrow 3x^2 + 4xy - 8x - 4y + 5 = 0$$

$$\therefore \text{Required angle} = \tan^{-1} \left| \frac{2\sqrt{(2)^2 - 3 \times 0}}{3 + 0} \right|$$

$$= \tan^{-1}\left(\frac{4}{3}\right) = \sin^{-1}\left(\frac{4}{5}\right)$$

➤ **Example 26.** The number of real tangents that can be drawn from (2, 2) to the circle $x^2 + y^2 - 6x - 4y + 3 = 0$ is

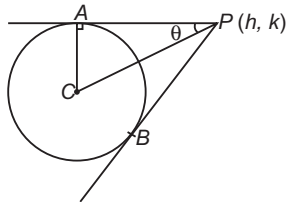
- (a) 0 (b) 1 (c) 2 (d) 3

Sol. (a) The point (2, 2) lies inside the circle. Hence, there cannot be a tangent drawn from (2, 2) to the given circle.

➤ **Example 27.** The angle between tangents drawn from point P to the circle $x^2 + y^2 + 4x - 6y + 9 \sin^2 \theta + 13 \cos^2 \theta = 0$ is 2θ. Locus of point P is

- (a) $x^2 + y^2 + 4x - 6y - 9 = 0$
(b) $x^2 + y^2 + 4x - 6y - 4 = 0$
(c) $x^2 + y^2 + 4x - 6y + 9 = 0$
(d) $x^2 + y^2 + 4x - 6y + 4 = 0$

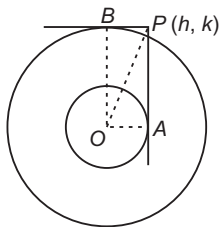
Sol. (c) Centre is (-2, 3) and radius is $2|\sin \theta|$.
we have $CP = \frac{CA}{\sin \theta} = 2$
Thus, $(h+2)^2 + (k-3)^2 = 4$
Hence, locus is $x^2 + y^2 + 4x - 6y + 9 = 0$



➤ **Example 28.** PA is tangent to $x^2 + y^2 = a^2$ and PB is tangent to $x^2 + y^2 = b^2$ ($b > a$). If $\angle APB = \frac{\pi}{2}$, then locus of point P is

- (a) $x^2 - y^2 = a^2 - b^2$ (b) $x^2 + y^2 = b^2 - a^2$
(c) $x^2 + y^2 = a^2 + b^2$ (d) $x^2 - y^2 = b^2 - a^2$

Sol. (c) If $\angle APB = \frac{\pi}{2}$, then OAPB will be a rectangle.



Thus, $OP^2 = OA^2 + OB^2$
 $\Rightarrow h^2 + k^2 = a^2 + b^2$
Thus, required locus is $x^2 + y^2 = a^2 + b^2$.

➤ **Example 29.** If the tangent from a point P to the circle $x^2 + y^2 = 1$ is perpendicular to the tangent from P to the circle $x^2 + y^2 = 3$, then the locus of P is a

- (a) circle of radius 2
(b) circle of radius 4
(c) circle of radius 3
(d) None of the above

Sol. (a) The equation of a tangent to $x^2 + y^2 = 1$ is

$$x \cos \alpha + y \sin \alpha = 1 \quad \dots(i)$$

The equation of tangent to $x^2 + y^2 = 3$, perpendicular to Eq. (i), is

$$x \sin \alpha - y \cos \alpha = \sqrt{3} \quad \dots(ii)$$

Let the coordinates of P be (h, k). Then,

$$h \cos \alpha + k \sin \alpha = 1$$

and

$$h \sin \alpha - k \cos \alpha = \sqrt{3}$$

Eliminating (h, k) from these two equations, we get

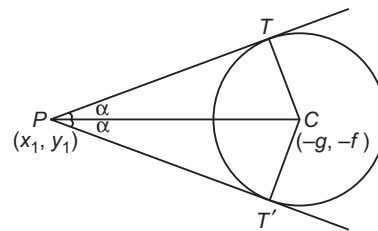
$$h^2 + k^2 = 4$$

So, locus of (h, k) is $x^2 + y^2 = 4$, which is a circle of radius 2.

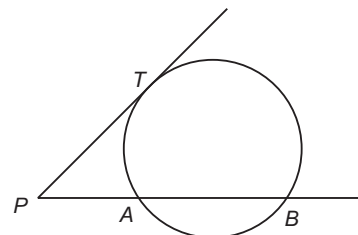
Length of the Tangents

The length of the tangent from the point $P(x_1, y_1)$ to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is equal to

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c} = \sqrt{S_1}$$

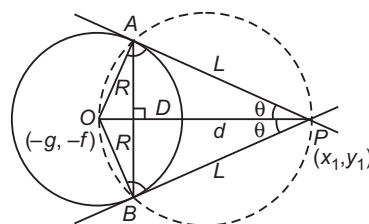


- If PT is the length of the tangent from a point P to a given circle, then PT^2 is called the power of the point with respect to the given circle.



$$\Rightarrow PT^2 = PA \cdot PB$$

All these formulae are applicable when coefficients of x^2 and y^2 is unity.



- Area of quadrilateral PAOB = $2 \Delta POA$
 $= 2 \cdot \frac{1}{2} RL = RL$

- Find AB i.e. length of chord of contact

$$AB = 2L \sin \theta, \text{ where } \tan \theta = \frac{R}{L}$$

$$= \frac{2RL}{\sqrt{R^2 + L^2}}$$

- Area of ΔPAB (triangle formed by pair of tangents and corresponding chord of contact)

$$\begin{aligned} &= \frac{1}{2} AB \times PD \\ &= \frac{1}{2} (2L \sin \theta) (L \cos \theta) \\ &= L^2 \sin \theta \cos \theta \\ &= \frac{RL^3}{R^2 + L^2} \end{aligned}$$

- Angle 2θ between the pair of tangents

$$\begin{aligned} \tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2 \theta} = \frac{2RL}{L^2 - R^2} \\ 2\theta &= \tan^{-1} \left(\frac{2RL}{L^2 - R^2} \right) \end{aligned}$$

- Equation of the circle circumscribing the ΔPAB . One such circle described on OP as diameter
i.e. $(x - x_1)(x + g) + (y - y_1)(y + f) = 0$

Example 30. The length of the tangent drawn from any point on the circle $x^2 + y^2 + 2gx + 2fy + \alpha = 0$ to the circle $x^2 + y^2 + 2gx + 2fy + \beta = 0$, is

(a) $\sqrt{\beta - \alpha}$ (b) $\sqrt{\alpha - \beta}$ (c) $\sqrt{\alpha\beta}$ (d) $\sqrt{\frac{\alpha}{\beta}}$

Sol. (a) Let (h, k) be any point on the first circle, then

$$h^2 + k^2 + 2gh + 2fk + \alpha = 0$$

Length of the tangent from (h, k) to the second circle

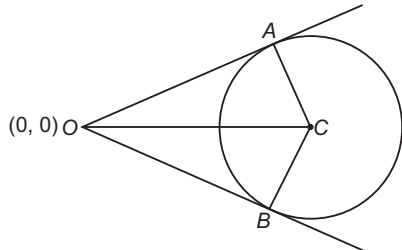
$$\begin{aligned} &= \sqrt{h^2 + k^2 + 2hg + 2fk + \beta} \\ &= \sqrt{\beta - \alpha} \end{aligned}$$

Example 31. If OA and OB are tangents from the origin to the circle, $x^2 + y^2 + 2gx + 2fy + c = 0$ and C is the centre of circle, then area of the quadrilateral $OACB$ is

(a) $\frac{1}{2} \sqrt{(g^2 + f^2 - c)}$ (b) $\sqrt{c(g^2 + f^2 - c)}$
(c) $c \sqrt{(g^2 + f^2 - c)}$ (d) $\sqrt{\frac{(g^2 + f^2 - c^2)}{c}}$

Sol. (b) Since, $OA = OB = \sqrt{c}$ (length of the tangent from the origin to the circle)

and $CA = CB = \sqrt{g^2 + f^2 - c}$



$$\begin{aligned} \text{Area of the quadrilateral } OACB &= 2 \text{ Area of } \Delta OAC \\ &= 2 \times \left(\frac{1}{2} \right) OA \times CA = \sqrt{c} \sqrt{g^2 + f^2 - c} \end{aligned}$$

Director Circle

The locus of the point of intersection of two perpendicular tangents to a given conic is known as its director circle.

The equation of the director circle of the circle $x^2 + y^2 = a^2$ is

$$x^2 + y^2 = 2a^2$$

Example 32. The locus of the point of intersection of tangents to the circle $x = a \cos \theta$, $y = a \sin \theta$ at the points, whose parametric angles differ by $\frac{\pi}{2}$ is a

- (a) straight line (b) circle
(c) pair of straight line (d) None of these

Sol. (b) Tangents at θ and $\left(\theta + \frac{\pi}{2}\right)$ are

$$\begin{aligned} &x \cos \theta + y \sin \theta = a \\ \text{and } &x \cos \left(\theta + \frac{\pi}{2}\right) + y \sin \left(\theta + \frac{\pi}{2}\right) = a \end{aligned}$$

$$\Rightarrow x \cos \theta + y \sin \theta = a \text{ and } -x \sin \theta + y \cos \theta = a$$

Squaring on both sides and adding, we get $x^2 + y^2 = 2a^2$.

Example 33. If the angle between tangents drawn to $x^2 + y^2 + 2gx + 2fy + c = 0$ from $(0, 0)$ is $\frac{\pi}{2}$, then

- (a) $g^2 + f^2 = 3c$ (b) $g^2 + f^2 = 2c$
(c) $g^2 + c^2 = 5c$ (d) $g^2 + f^2 = 4c$

Sol. (b) Clearly, $(0, 0)$ lies on director circle of the given circle. Now, equation of director circle is

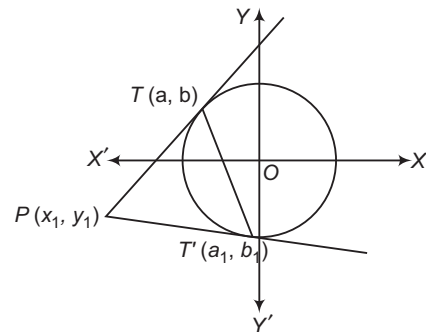
$$(x + g)^2 + (y + f)^2 = 2(g^2 + f^2 - c)$$

If $(0, 0)$ lies on it, then $g^2 + f^2 = 2(g^2 + f^2 - c)$

$$\Rightarrow g^2 + f^2 = 2c$$

Chord of Contact

The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the circle $x^2 + y^2 = a^2$ is



$$xx_1 + yy_1 = a^2 \text{ or } T = 0$$

Proof Let $T(\alpha, \beta)$ and $T'(\alpha_1, \beta_1)$ be the points of contact of tangents drawn from $P(x_1, y_1)$ to $x^2 + y^2 = a^2$. The equations of tangents at $T(\alpha, \beta)$ and $T'(\alpha_1, \beta_1)$ to the circle $x^2 + y^2 = a^2$ are

$$\alpha x + \beta y = a^2 \quad \dots(i)$$

$$\text{and} \quad \alpha_1 x + \beta_1 y = a^2 \quad \dots(ii)$$

Since, Eqs. (i) and (ii) passes through (x_1, y_1) . Therefore,

$$\alpha x_1 + \beta y_1 = a^2 \text{ and } \alpha_1 x_1 + \beta_1 y_1 = a^2 \\ \Rightarrow (\alpha, \beta) \text{ and } (\alpha_1, \beta_1) \text{ lies on } xx_1 + yy_1 = a^2.$$

Hence, the chord of contact is $xx_1 + yy_1 = a^2$ i.e. $T = 0$.

➤ **Example 34.** Tangents are drawn to the circle $x^2 + y^2 = 12$ at the points where it is met by the circle $x^2 + y^2 - 5x + 3y - 2 = 0$, the point of intersection of these tangents is

- (a) $\left(6, -\frac{18}{5}\right)$ (b) $\left(6, \frac{18}{5}\right)$
(c) $\left(\frac{18}{5}, 6\right)$ (d) None of these

Sol. (a) The circles are given as

$$x^2 + y^2 = 12 \quad \dots(i)$$

$$\text{and} \quad x^2 + y^2 - 5x + 3y - 2 = 0 \quad \dots(ii)$$

If A and B are the points of intersection of Eqs. (i) and (ii), clearly AB will be the common chord whose equation will be

$$(x^2 + y^2 - 12) - (x^2 + y^2 - 5x + 3y - 2) = 0 \\ \Rightarrow 5x - 3y - 10 = 0 \quad \dots(iii)$$

If P is the point, where the tangents at A and B with respect to Eq. (i), meet each other, AB will be the chord of contact of P. Let the coordinates of P be (α, β) .

Equation of the chord of contact of (α, β) with respect to Eq. (i) is

$$x\alpha + y\beta - 12 = 0 \quad \dots(iv)$$

As Eqs. (iii) and (iv) represent the same equation, on comparing the coefficients, we get

$$\frac{\alpha}{5} = \frac{\beta}{-3} = \frac{-12}{-10}$$

$$\Rightarrow \alpha = 6 \text{ and } \beta = -\frac{18}{5}$$

➤ **Example 35.** Tangents are drawn to $x^2 + y^2 = 1$ from any arbitrary point P on the line $2x + y - 4 = 0$. The corresponding chord of contact passes through a fixed point whose coordinates are

- (a) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, 1\right)$ (c) $\left(\frac{1}{2}, \frac{1}{4}\right)$ (d) $\left(1, \frac{1}{2}\right)$

Sol. (c) Let $P \equiv (a, 4 - 2a)$. Equation of chord of contact is

$$x \cdot a + y \cdot (4 - 2a) = 1$$

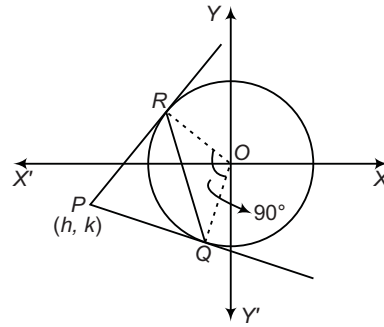
$$\Rightarrow (4y - 1) + a(x - 2y) = 0$$

It will always pass through a fixed point whose coordinates are $y = \frac{1}{4}$ and $x = 2y = \frac{1}{2}$.

➤ **Example 36.** If the chord of contact of tangents drawn from the point (h, k) to the circle $x^2 + y^2 = a^2$ subtends a right angle at the centre, then $h^2 + k^2$ is equal to

- (a) $\frac{a^2}{2}$ (b) a^2
(c) $2a^2$ (d) None of these

Sol. (c)



As shown in diagram, $\angle ROQ = \pi/2$

Also, $\angle PRO = \angle PQO = \pi/2$.

Then, quadrilateral PROQ is square and hence, $PR = RO$.

$$\Rightarrow \sqrt{h^2 + k^2 - a^2} = a \\ \Rightarrow h^2 + k^2 = 2a^2$$

➤ **Example 37.** If the chord of contact of tangents drawn from a point on the circle $x^2 + y^2 = a^2$ to the circle $x^2 + y^2 = b^2$ touches the circle $x^2 + y^2 = c^2$, then, a, b and c are in

- (a) AP
(b) GP
(c) HP
(d) None of the above

Sol. (b) Let any point on the circle $x^2 + y^2 = a^2$ be $(a \cos \theta, a \sin \theta)$. The chord of contact of this point with respect to the circle $x^2 + y^2 = b^2$ is $ax \cos \theta + ay \sin \theta = b^2$. If this chord touches the circle $x^2 + y^2 = c^2$, then

$$\left| \frac{0 + 0 - b^2}{\sqrt{a^2 \cos^2 \theta + a^2 \sin^2 \theta}} \right| = c \\ \Rightarrow \frac{b^2}{a} = c \\ \Rightarrow b^2 = ac$$

Chord Bisected at a Given Point

The equation of the chord of the circle $x^2 + y^2 = a^2$ bisected at the point (x_1, y_1) is given by

$$xx_1 + yy_1 - a^2 = x_1^2 + y_1^2 - a^2$$

or

$$T = S'$$

Proof Let AB be the chord of the circle $x^2 + y^2 = a^2$ which is bisected at $P(x_1, y_1)$. Then, $OP \perp AB$.

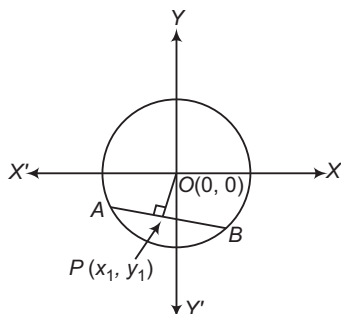
We have,

$$\text{Slope of } OP = \frac{y_1 - 0}{x_1 - 0} = \frac{y_1}{x_1}$$

$$\therefore \text{Slope of } AB = \frac{-x_1}{y_1}$$

So, the equation of AB is

$$y - y_1 = -\frac{x_1}{y_1}(x - x_1)$$



$$\begin{aligned} \Rightarrow y y_1 - y_1^2 &= -x x_1 + x_1^2 \\ \Rightarrow x x_1 + y y_1 &= x_1^2 + y_1^2 \\ \Rightarrow x x_1 + y y_1 - a^2 &= x_1^2 + y_1^2 - a^2 \\ \Rightarrow T &= S' \end{aligned}$$

➤ **Example 38.** Locus of mid-point of chords of $x^2 + y^2 + 2gx + 2fy + c = 0$ that pass through the origin, is

(a) $x^2 + y^2 + 2gx + 2fy = 0$

(b) $x^2 + y^2 + gx + fy + c = 0$

(c) $x^2 + y^2 + gx + fy = 0$

(d) $2(x^2 + y^2 + gx + fy) + c = 0$

Sol. (c) Let the mid-point of chords of circle be $P(h, k)$.

Equation of this chord is $T = S_1$

i.e. $xh + yk + g(x + h) + f(y + k) + c = h^2 + k^2 + 2gh + 2fk + c$.

It passes through $(0, 0)$, thus

$$h^2 + k^2 + gh + fk = 0$$

i.e. required locus is $x^2 + y^2 + gx + fy = 0$

➤ **Example 39.** Tangents PA and PB drawn to $x^2 + y^2 = 9$ from any arbitrary point P on the line $x + y = 25$. Locus of mid-point of chord AB is

(a) $25(x^2 + y^2) = 9(x + y)$ (b) $25(x^2 + y^2) = 3(x + y)$

(c) $5(x^2 + y^2) = 3(x + y)$ (d) None of these

Sol. (a) Let $P \equiv (a, 25 - a)$. Equation of chord AB is $T = O$ i.e. $xa + y(25 - a) = 9$. If mid-point of chord AB is $C(h, k)$, then equation of chord AB is $T = S_1$ i.e. $xh + yk = h^2 + k^2$.

Comparing the coefficients of like powers, we get

$$\begin{aligned} \frac{a}{h} &= \frac{25 - a}{k} = \frac{9}{h^2 + k^2} \\ &= \frac{a + 25 - a}{h + k} = \frac{25}{h + k} \end{aligned}$$

$$\Rightarrow 9(h + k) = 25(h^2 + k^2)$$

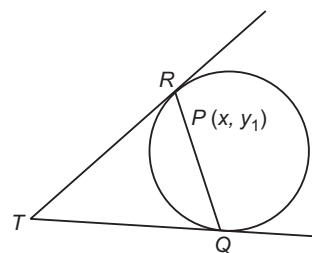
$$\Rightarrow 25(x^2 + y^2) = 9(x + y)$$

Pole and Polar

Polar of point with respect to a circle If through a point $P(x_1, y_1)$ (inside or outside a circle), there any straight line is drawn to meet the given circle at Q and R , then the locus of the point of intersection of the tangents at Q and R is called the polar of P and P is called the pole of the polar.

Theorem

The polar of a point $P(x_1, y_1)$ with respect to the circle $x^2 + y^2 = a^2$ is $xx_1 + yy_1 = a^2$ or $T = 0$.



Proof Suppose a straight line through the given point $P(x_1, y_1)$ intersects the given circle at Q and R . Let the tangents to the circle at Q and R meet at T whose coordinates are (h, k) (say).

Then, QR is the chord of contact of tangents drawn from $T(h, k)$ to the circle $x^2 + y^2 = a^2$. Therefore, the equation of QR is $hx + ky = a^2$.

Since, it passes through $P(x_1, y_1)$, therefore

$$hx_1 + ky_1 = a^2$$

$\therefore$ Locus of $T(h, k)$, i.e. polar of P is

$$xx_1 + yy_1 = a^2$$

Conjugate points Two points A and B are conjugate points with respect to a given circle, if each lies on the polar of the other with respect to the circle.

Conjugate lines If two lines be such that the pole of one line lies on the other, then they are called conjugate lines with respect to the given circle.

• The distance of any two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ from the centre of a circle are proportional to the distance of each from the polar of the other.

• If the polar of a point $P(x_1, y_1)$ with respect to a circle whose centre is O meets the straight line joining P to the centre of the circle at the point Q , then

> OP is perpendicular to the polar of P .

> P and Q are inverse points with respect to the circle i.e. $OP \cdot OQ = (\text{radius})^2$

$$OQ = (\text{radius})^2$$

• The polar of (x_1, y_1) with respect to the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$

which is same as the equation of the tangent at (x_1, y_1) if (x_1, y_1) lies on the circle.

➤ **Example 40.** The pole of the line $3x + 5y + 17 = 0$ with respect to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is

- (a) (1, 2) (b) (2, 1)
(c) (1, -4) (d) None of these

Sol. (a) Given circle is

$$x^2 + y^2 + 4x + 6y + 9 = 0 \quad \dots(i)$$

$$\text{and given line } 3x + 5y + 17 = 0 \quad \dots(ii)$$

Let (λ, β) be the pole of line with respect to circle.

Now, equation of polar of point (α, β) with respect to the circle is

$$\Rightarrow x\alpha + y\beta + 2(x + \alpha) + 3(y + \beta) + 9 = 0$$

$$\Rightarrow (\alpha + 2)x + (\beta + 3)y + 2\alpha + 3\beta + 9 = 0 \quad \dots(iii)$$

Now, lines (ii) and (iii) are same.

$$\therefore \frac{\alpha + 2}{3} = \frac{\beta + 3}{5} = \frac{2\alpha + 3\beta + 9}{17}$$

Solving these equations, we get

$$\alpha = 1 \text{ and } \beta = 2$$

Hence, required pole is (1, 2).

Diameter of a Circle

The locus of the middle points of a system of parallel chords of a circle is called a diameter of the circle.

Theorem

The equation of the diameter bisecting parallel chord $y = mx + c$ (c is a parameter) of the circle $x^2 + y^2 = a^2$ is $x + my = 0$.

Proof Let $P(h, k)$ be the mid-point of a chord $y = mx + c$. Then, x -coordinate of points A and B are the roots of the equation

$$x^2 + (mx + c)^2 = a^2$$

$$\Rightarrow x^2(1 + m^2) + 2mcx + c^2 - a^2 = 0 \quad \dots(i)$$

Let (x_1, y_1) and (x_2, y_2) be the coordinates of A and B respectively. Then, x_1, x_2 are the roots of Eq. (i).

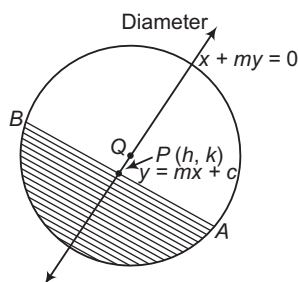
$$\therefore x_1 + x_2 = -\frac{2mc}{1 + m^2} \Rightarrow \frac{x_1 + x_2}{2} = \frac{-mc}{1 + m^2}$$

$$\Rightarrow h = -\frac{mc}{1 + m^2}$$

$$[\because P(h, k) \text{ is the mid-point of } AB]$$

$$\Rightarrow \frac{-h}{m}(1 + m^2) = c \quad \dots(ii)$$

$$\text{Since, } (h, k) \text{ lies on } y = mx + c. \text{ Therefore, } k = mh + c \quad \dots(iii)$$



Eliminating c from Eqs. (ii) and (iii), we get

$$-\frac{h}{m}(1 + m^2) = (k - mh)$$

$$\Rightarrow -h - hm^2 = mk - m^2h$$

$$\Rightarrow -h = mk$$

So, locus of (h, k) is

$$-x = my$$

$$\Rightarrow x + my = 0$$

Common Tangents to Two Circles

Let the equations of the two circles be

$$(x - h_1)^2 + (y - k_1)^2 = a_1^2 \quad \dots(i)$$

$$\text{and } (x - h_2)^2 + (y - k_2)^2 = a_2^2 \quad \dots(ii)$$

with centres $C_1(h_1, k_1)$ and $C_2(h_2, k_2)$ and radii a_1 and a_2 respectively. Then, the following cases of intersection of these two circles may arise :

Case I Circles neither intersect nor touch each other,

$$\text{when } C_1 C_2 > a_1 + a_2$$

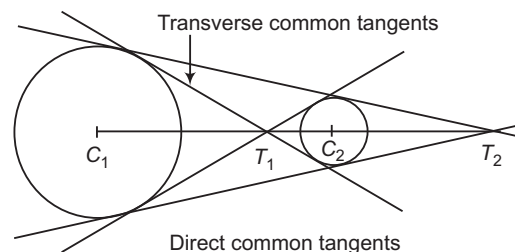
i.e. the distance between the centres is greater than the sum of the radii.

In this case, the two circles do not intersect with each other and four common tangents can be drawn to two circles. Two of them are direct common tangents and the other two are transverse common tangents.

The point of intersection of direct and transverse common tangents always lie on the line joining the centres C_1 and C_2 and divide it internally and externally respectively in the ratio $a_1 : a_2$ i.e.

$$\frac{C_1 T_2}{C_2 T_2} = \frac{a_1}{a_2} \quad [\text{externally}]$$

$$\text{and } \frac{C_1 T_1}{C_2 T_1} = \frac{a_1}{a_2} \quad [\text{internally}]$$

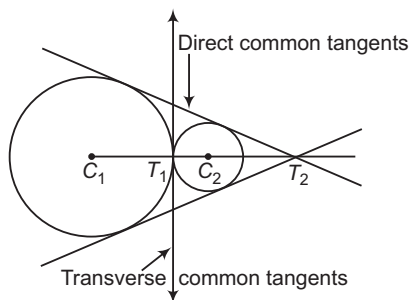


Case II Circles touch externally,

$$\text{when } C_1 C_2 = a_1 + a_2$$

i.e. the distance between the centres is equal to the sum of the radii.

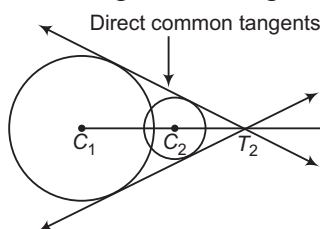
In this case, two direct tangents are real and distinct while the transverse tangents are coincident.



Case III Circles intersect each other when $|a_1 - a_2| < C_1C_2 < a_1 + a_2$

i.e. the distance between the centres is less than the sum of the radii.

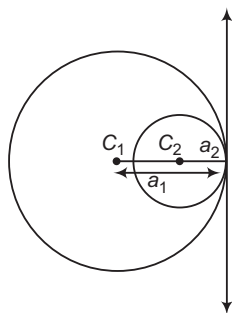
In this case, the two direct common tangents are real while the transverse tangents are imaginary.



Case IV Circles touch internally when $C_1C_2 = a_1 - a_2$

i.e. the distance between the centres is equal to the difference of the radii.

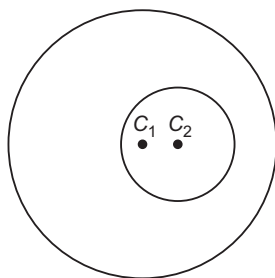
In this case, two tangents are real and coincident while the other two tangents are imaginary.



Case V Circles neither touch nor intersect when $C_1C_2 < a_1 - a_2$

i.e. the distance between the centres is less than the difference of the radii.

In this case, all four common tangents are imaginary.



Example 41. How many common tangents can be drawn to the following circles $x^2 + y^2 = 6x$ and $x^2 + y^2 + 6x + 2y + 1 = 0$?

- (a) 1 (b) 2 (c) 3 (d) 4

Sol. (d) The coordinates of the centre of the given circles are $C_1(3, 0)$, $C_2(-3, -1)$ and corresponding radii are $r_1 = 3$ and $r_2 = 1$ respectively.

$$\text{Now, } C_1C_2 = \sqrt{(3+3)^2 + (0+1)^2} = \sqrt{37}$$

$$\text{and } r_1 + r_2 = 4$$

$$\therefore C_1C_2 > r_1 + r_2$$

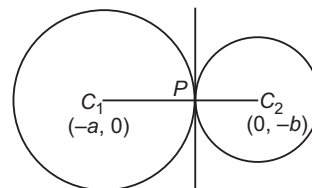
So, four common tangents can be drawn to the given circles.

Example 42. The two circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touching each other, then

- (a) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c^2}$ (b) $a^2 + b^2 = c^2$
(c) $\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{c}$ (d) None of these

Sol. (c) We have,

$$\text{and } \begin{aligned} x^2 + y^2 + 2ax + c &= 0 & \dots(i) \\ x^2 + y^2 + 2by + c &= 0 & \dots(ii) \end{aligned}$$



$$\begin{aligned} \therefore C_1C_2 &= r_1 + r_2 \\ \Rightarrow \sqrt{a^2 + b^2} &= \sqrt{a^2 - c} + \sqrt{b^2 - c} \\ \text{where } r_1 &= C_1P = \sqrt{a^2 - c}, r_2 = C_2P = \sqrt{b^2 - c} \\ \Rightarrow a^2 + b^2 &= a^2 - c + b^2 - c + 2\sqrt{(a^2 - c)(b^2 - c)} \\ \Rightarrow c &= \pm \sqrt{(a^2 - c)(b^2 - c)} \\ \Rightarrow c^2 &= a^2b^2 - a^2c - b^2c + c^2 \\ \Rightarrow a^2b^2 &= a^2c + b^2c \Rightarrow \frac{1}{c} = \frac{1}{a^2} + \frac{1}{b^2} \end{aligned}$$

Example 43. The centre of a set of circles, each of radius 3, lies on the circles $x^2 + y^2 = 25$. The locus of any point in the set is

- (a) $4 \leq x^2 + y^2 \leq 64$ (b) $x^2 + y^2 \leq 25$
(c) $x^2 + y^2 \geq 25$ (d) $3 \leq x^2 + y^2 \leq 9$

Sol. (a) Let (h, k) be any point in the set, then equation of circle is $(x - h)^2 + (y - k)^2 = 9$.

$$\text{But } (h, k) \text{ lie on } x^2 + y^2 = 25, \text{ then } h^2 + k^2 = 25$$

$$\therefore 2 \leq \text{Distance between the centres of two circles} \leq 8$$

$$\Rightarrow 2 \leq \sqrt{h^2 + k^2} \leq 8$$

$$4 \leq h^2 + k^2 \leq 64$$

$$\therefore \text{Locus of } (h, k) \text{ is } 4 \leq x^2 + y^2 \leq 64.$$

➤ **Example 44.** The equation of a circle with centre $(4, 3)$ touching the circle $x^2 + y^2 = 1$, is

- (a) $(x - 4)^2 + (y - 3)^2 = 15$
 (b) $(x - 4)^2 + (y - 3)^2 = 16$
 (c) $(x - 4)^2 + (y - 3)^2 = 36$
 (d) None of the above

Sol. (b, c) The given circle is $x^2 + y^2 = 1$, which has centre $C(0, 0)$ and radius $r_1 = 1$.

The required circle has centre $C_2(4, 3)$ and radius r_2 .

If the circles are touching externally, then $r_1 + r_2 = C_1C_2$

$$\Rightarrow r_2 = 5 - 1 = 4$$

If the circles are touching internally, then $r_2 - r_1 = C_1C_2$

$$\Rightarrow r_2 = 6$$

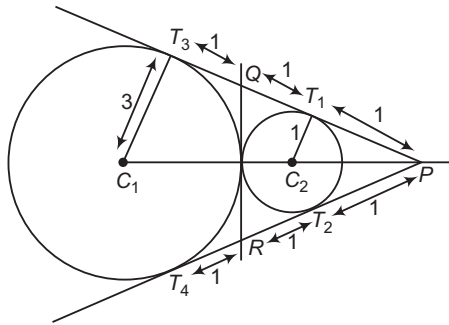
Thus, the required circle are $(x - 4)^2 + (y - 3)^2 = 16$

or $(x - 4)^2 + (y - 3)^2 = 36$.

➤ **Example 45.** The common tangents to the circles $x^2 + y^2 + 2x = 0$ and $x^2 + y^2 - 6x = 0$ form a triangle which is

- (a) equilateral (b) isosceles
 (c) right angled (d) None of these

Sol. (a) Clearly, the distance between the centres of the given circles is equal to the sum of their radii. So, two circles touch each other externally.



We have, $PT_1 = PT_2$ and $PT_3 = PT_4$

$$\Rightarrow T_1T_3 = T_2T_4$$

$$\Rightarrow T_1Q = T_2R$$

$$\Rightarrow PT_1 + T_1Q = PT_2 + T_2R$$

$$\Rightarrow PT_1 + T_1Q = PT_2 + T_2R$$

$$\Rightarrow PQ = PR = QR = 2$$

So, ΔPQR is equilateral.

Common Chord

The chord joining the points of intersection of two given circles is called their common chord.

The equation of the common chord of two circles

$$S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \dots(i)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \dots(ii)$$

$$\text{is } 2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$$

$$\text{i.e. } S_1 - S_2 = 0$$

➤ **Example 46.** The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ bisects the circumference of the circle

$$x^2 + y^2 + 2g'x + 2f'y + c' = 0, \text{ if}$$

$$(a) 2g'(g - g') + 2f'(f - f') = c - c'$$

$$(b) 2g'(g - g') - 2f'(f - f') = c - c'$$

$$(c) g'(g - g') + f'(f - f') = c + c'$$

$$(d) \text{ None of the above}$$

Sol. (a) Common chord $S_1 - S_2 = 0$ passes through the centre of $S_2 = 0$,

$$\text{where } S_1 \equiv x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0$$

Common chord is

$$2(g - g')x + 2(f - f')y + (c - c') = 0$$

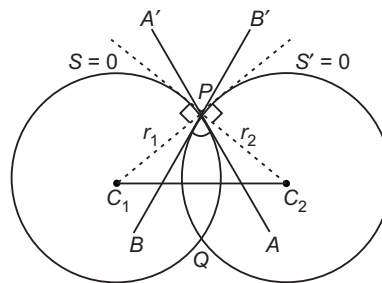
which passes through $(-g', -f')$.

$$2g'(g - g') + 2f'(f - f') = (c - c')$$

Angle of Intersection of Two Circles

The angle of intersection of two circles is defined as the angle between the tangents or angle between the normals to the two circles at their point of intersection is given by

$$\cos \theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$



Proof

$$\text{Let } x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \dots(i)$$

$$\text{and } x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \dots(ii)$$

be two circles having their centres at $C_1(-g_1, -f_1)$ and $C_2(-g_2, -f_2)$, respectively. Let P and Q be the points of intersection of these two circles. Since, the tangents at P are perpendicular to the radii of the corresponding circles and angle between two lines is same as the angle between their perpendiculars. Therefore, the angle of intersection of two circles is equal to the angle between their radii at the point of intersection. Thus, we have,

$$\cos \angle C_1PC_2 = \frac{C_1P^2 + C_2P^2 - C_1C_2^2}{2C_1P \cdot C_2P}$$

$$\therefore \cos \theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

where, $C_1P = r_1$, $C_2P = r_2$ and $C_1C_2 = d$

➤ **Example 47.** If θ is the angle of intersection of two circles $x^2 + y^2 = a^2$ and $(x - c)^2 + y^2 = b^2$, then the length of common chord of two circles is

- (a) $\frac{ab}{\sqrt{a^2 + b^2 - 2ab\cos\theta}}$ (b) $\frac{2ab}{\sqrt{a^2 + b^2 - 2ab\cos\theta}}$
 (c) $\frac{2ab\sin\theta}{\sqrt{a^2 + b^2 - 2ab\cos\theta}}$ (d) None of these

Sol. (c) The equation of two circles are

$$x^2 + y^2 = a^2 \quad \dots(i)$$

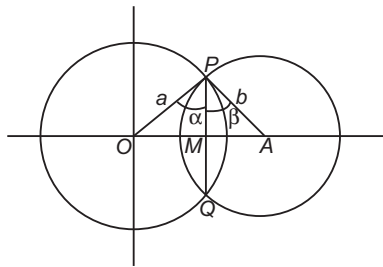
$$\text{and } (x - c)^2 + y^2 = b^2 \quad \dots(ii)$$

The radius of first and second circle are a and b respectively.

Let $\angle OPM = \alpha$ and $\angle APM = \beta$

$$\therefore \angle OPA = \alpha + \beta = \theta$$

Let $PQ = x$



$$\Rightarrow \cos\alpha = \frac{PM}{a} = \frac{x}{2a} \text{ and } \cos\beta = \frac{x}{2b}$$

$$\text{Now, } \cos\theta = \cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

$$\therefore \sin\alpha \sin\beta = \cos\alpha \cos\beta - \cos\theta$$

On squaring both sides, we get

$$\sin^2\alpha \sin^2\beta = \cos^2\alpha \cos^2\beta + \cos^2\theta - 2\cos\theta \cos\alpha \cos\beta$$

$$1 - \cos^2\alpha - \cos^2\beta + \cos^2\alpha + \cos^2\beta = \cos^2\alpha \cos^2\beta + \cos^2\theta - 2\cos\theta \cos\alpha \cos\beta$$

$$\therefore \sin^2\theta = \cos^2\alpha + \cos^2\beta - 2\cos\theta \cos\alpha \cos\beta$$

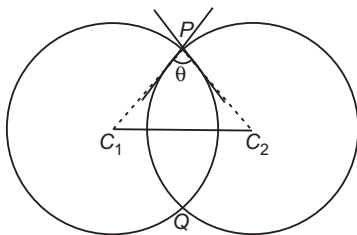
$$\Rightarrow \sin^2\theta = \frac{x^2}{4a^2} + \frac{x^2}{4b^2} - \frac{2x^2}{4ab} \cdot \cos\theta$$

$$\Rightarrow 4a^2b^2 \sin^2\theta = x^2(b^2 + a^2 - 2ab\cos\theta)$$

$$\therefore x = \frac{2ab\sin\theta}{\sqrt{a^2 + b^2 - 2ab\cos\theta}}$$

Orthogonal Circles

Two circles are said to intersect orthogonally, if their angle of intersection is a right angle.



If two circles intersect orthogonally, then we say that the circles are orthogonal circles.

If circles (i) and (ii) intersect orthogonally, then

$$\angle C_1PC_2 = 90^\circ$$

$$\Rightarrow \cos\angle C_1PC_2 = \cos 90^\circ$$

$$\Rightarrow C_1P^2 + C_2P^2 - C_1C_2^2 = 0$$

$$\Rightarrow C_1P^2 + C_2P^2 = C_1C_2^2$$

$$\Rightarrow g_1^2 + f_1^2 - c_1 + g_2^2 + f_2^2 - c_2 = (-g_1 - g_2)^2 + (-f_1 + f_2)^2$$

$$\Rightarrow 2(g_1g_2 + f_1f_2) = c_1 + c_2$$

This is the condition of orthogonality of two circles.

➤ **Example 48.** If the circles $x^2 + y^2 + 2x + 2ay + 6 = 0$ and $x^2 + y^2 + 4ay + a = 0$ intersect orthogonally, then value of a is equal to

- (a) $\frac{1 \pm 4\sqrt{6}}{8}$ (b) $\frac{1 \pm 4\sqrt{7}}{8}$
 (c) $\frac{1 \pm \sqrt{97}}{8}$ (d) $\frac{1 \pm 2\sqrt{6}}{8}$

Sol. (c) We have, $2g_1g_2 + 2f_1f_2 = c_1 + c_2$

$$\Rightarrow 2(1)(0) + 2(a)(2a) = 6 + a$$

$$\Rightarrow 4a^2 - a - 6 = 0$$

$$\Rightarrow a = \frac{1 \pm \sqrt{1 + 96}}{8} = \frac{1 \pm \sqrt{97}}{8}$$

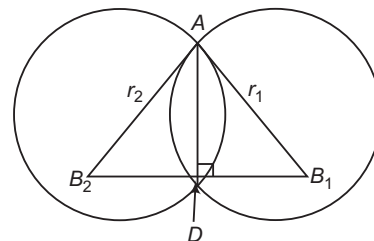
➤ **Example 49.** The circles having radii r_1 and r_2 intersect orthogonally. Length of their common chord is

- (a) $\frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$ (b) $\frac{2r_1^2r_2}{\sqrt{r_1^2 + r_2^2}}$
 (c) $\frac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$ (d) $\frac{2r_1^2r_2^2}{\sqrt{r_1^2 + r_2^2}}$

Sol. (a) Let $\angle AB_1B_2 = \theta$

$$\Rightarrow AD = r_1 \sin\theta \text{ and } AD = r_2 \cos\theta$$

$$\Rightarrow AD^2 \left(\frac{1}{r_1^2} + \frac{1}{r_2^2} \right) = 1$$



$$\Rightarrow AD = \frac{r_1r_2}{\sqrt{r_1^2 + r_2^2}}$$

$$\text{Thus, length of common chord} = \frac{2r_1r_2}{\sqrt{r_1^2 + r_2^2}}$$

➤ **Example 50.** The centre of circle S lies on $2x - 2y + 9 = 0$ and it cuts orthogonally the circle $x^2 + y^2 = 4$. Then, the circle passes through two fixed points

- (a) (1, 1) and (3, 3) (b) $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(-4, 4)$
(c) (0, 0) and (5, 5) (d) None of these

Sol. (b) Let $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore$ It cuts orthogonally.

$$\Rightarrow c = 4$$

Moreover, $-2g + 2f + 9 = 0$

$[\because (-g, -f)$ satisfy the given equation]

$$\therefore S \equiv x^2 + y^2 + 2gx + 2fy + 4 = 0$$

$$\Rightarrow x^2 + y^2 + (2f + 9)x + 2fy + 4 = 0$$

$$\Rightarrow (x^2 + y^2 + 9x + 4) + 2f(x + y) = 0$$

It is of the form $S + \lambda P = 0$ and hence passes through the intersection of $S = 0$ and $P = 0$ which when solved give $\left(-\frac{1}{2}, \frac{1}{2}\right)$ and $(-4, 4)$.

➤ **Example 51.** Locus of the centre of a circle that passes through (a, b) and cuts the circle $x^2 + y^2 = a^2$ orthogonally, is

- (a) $2ax + 2by = a^2 + b^2$ (b) $2ax + by = b^2 + 2a^2$
(c) $ax + 2by = 2b^2 + a^2$ (d) $2ax + 2by = b^2 + 2a^2$

Sol. (d) Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$. It passes through (a, b) , then

$$a^2 + b^2 + 2ag + 2fb + c = 0$$

It also cuts $x^2 + y^2 = a^2$ orthogonally, then

$$2g \cdot 0 + 2f \cdot 0 = c - a^2 \Rightarrow c = a^2$$

$$\Rightarrow 2ag + 2fb + b^2 + 2a^2 = 0$$

Thus, locus of centre is $2ax + 2by = b^2 + 2a^2$.

Radical Axis

The radical axis of two circles is the locus of a point which moves in such a way that the lengths of the tangents drawn from it to the two circles are equal.

The radical axis of two circles $S_1 = 0$ and $S_2 = 0$ is given by

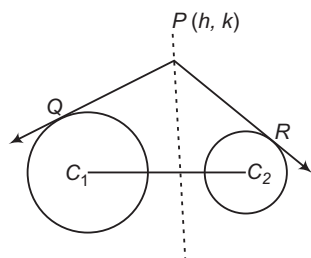
$$S_1 - S_2 = 0$$

Proof

$$\text{Let } S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0 \quad \dots(i)$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0 \quad \dots(ii)$$

be two circles.



Let $P(h, k)$ be a point on the radical axis of Eqs. (i) and (ii). Then,

$$\begin{aligned} PQ = PR &\Rightarrow PQ^2 = PR^2 \\ \Rightarrow h^2 + k^2 + 2g_1h + 2f_1k + c_1 &= h^2 + k^2 + 2g_2h + 2f_2k + c_2 \\ \Rightarrow 2h(g_1 - g_2) + 2k(f_1 - f_2) + c_1 - c_2 &= 0 \end{aligned}$$

Hence, the locus of (h, k) is

$$2x(g_1 - g_2) + 2y(f_1 - f_2) + c_1 - c_2 = 0$$

$$\text{or } S_1 - S_2 = 0$$

This is the equation of the radical axis of circles (i) and (ii).

As it is a first degree equation in x and y . So, it represents a straight line.

- The equations of radical axis and the common chord of two circles are identical.
- The radical axis of two circles is always perpendicular to the line joining the centres of the circles.
- The radical axis of three circles, whose centres are non-collinear, taken in pairs are concurrent.
- The centre of the circle cutting two given circles orthogonally, lies on their radical axis.

or

The locus of the centres of the circles cutting two given circles orthogonally, is their radical axis.

- **Radical centre** The point of intersection of radical axes of three circles whose centres are non-collinear, taken in pairs is called their radical centre.
- The circle with centre at the radical centre and radius equal to the length of the tangents from it to one of the circles intersects all the three circles orthogonally.

➤ **Example 52.** The point from which the lengths of the tangents to the following three circles $x^2 + y^2 + 4x + 6y - 19 = 0$, $x^2 + y^2 - 2x - 2y - 5 = 0$ and $x^2 + y^2 = 9$ are equal, is

- (a) (2, -1) (b) (2, -2) (c) (1, 1) (d) (1, -2)

Sol. (c) The required point will be the radical centre.

$\therefore$ The common chord R_{12} of the first two circles is

$$S_1 - S_2 = 0$$

$$\text{i.e. } 6x + 8y - 14 = 0$$

$$\Rightarrow 3x + 4y = 7 \quad \dots(i)$$

The common chord R_{23} of the second and third circles is

$$S_2 - S_3 = 0$$

$$\text{i.e. } -2x - 2y + 4 = 0$$

$$\Rightarrow x + y = 2 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get the point of intersection of R_{12} and R_{23} is (1, 1).

➤ **Example 53.** Equation of the circle that cuts the circle $x^2 + y^2 = a^2$, $(x - b)^2 + y^2 = a^2$ and $x^2 + (y - c)^2 = a^2$ orthogonally, is

- (a) $x^2 + y^2 - bx - cy - a^2 = 0$
(b) $x^2 + y^2 + bx + cy - a^2 = 0$
(c) $x^2 + y^2 + bx + cy + a^2 = 0$
(d) $x^2 + y^2 - bx - cy + a^2 = 0$

Sol. (d) $S_1 \equiv x^2 + y^2 - a^2 = 0$
 $S_2 \equiv x^2 + y^2 - 2bx + b^2 - a^2 = 0$
 $S_3 \equiv x^2 + y^2 - 2cy + c^2 - a^2 = 0$

Clearly, the radical centre of these circles is $P\left(\frac{b}{2}, \frac{c}{2}\right)$.

Length of tangent from P to S_1 is $\sqrt{\frac{b^2}{4} + \frac{c^2}{4} - a^2}$.

Thus, equation of required circle is

$$\left(x - \frac{b}{2}\right)^2 + \left(y - \frac{c}{2}\right)^2 = \frac{b^2}{4} + \frac{c^2}{4} - a^2$$

$$\Rightarrow x^2 + y^2 - bx - cy + a^2 = 0$$

Family of Circles

- i. The equation of a family of circles passing through the intersection of a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and line $L = lx + my + n = 0$ is $x^2 + y^2 + 2gx + 2fy + c + \lambda(lx + my + n) = 0$ or $S + \lambda L = 0$ where, λ is any real number.
- ii. The equation of the family of circles passing through the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$ or $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda L = 0$ where, $L = 0$ represents the line passing through $A(x_1, y_1)$ and $B(x_2, y_2)$ and $\lambda \in R$.
- iii. The equation of the family of circles touching the circle $S_3 \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at point $P(x_1, y_1)$ is $x^2 + y^2 + 2gx + 2fy + c + \lambda\{xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c\} = 0$ or $S + \lambda L = 0$ where, $L = 0$ is the equation of the tangent to $S = 0$ at (x_1, y_1) and $\lambda \in R$.
- iv. The equation of a family of circles passing through the intersection of the circles, $S_1 \equiv x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_2 \equiv x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ is $S_1 + \lambda S_2 = 0$ i.e. $(x^2 + y^2 + 2g_1x + 2f_1y + c_1) + \lambda(x^2 + y^2 + 2g_2x + 2f_2y + c_2) = 0$ where, $\lambda (\neq -1)$ is an arbitrary real number.

Example 54. Let S_1 be a circle passing through $A(0, 1)$, $B(-2, 2)$ and S_2 be a circle of radius $\sqrt{10}$ units such that AB is common chord of S_1 and S_2 . Then, the equation of S_2 is

- (a) $x^2 + y^2 + 2x - 3y + 2 = 0$
- (b) $x^2 + y^2 + 2x - 3y \pm \sqrt{7}(x + 2y - 2) = 0$
- (c) $x^2 + y^2 + 2x - 3y \pm \sqrt{5}(x + 2y - 2) = 0$
- (d) None of the above

Sol. (b) Equation of line AB is

$$y - 2 = \frac{2 - 1}{-2 - 0}(x + 2) = -\frac{1}{2}(x + 2)$$

$$\Rightarrow x + 2y - 2 = 0 \quad \dots(i)$$

Now, equation of circle whose diagonally opposite points are A and B .

$$(x - 0)(x + 2) + (y - 1)(y - 2) = 0$$

$$\Rightarrow x^2 + y^2 + 2x - 3y + 2 = 0 \quad \dots(ii)$$

and family of circles passing through the points of intersection of Eqs. (i) and (ii)

$$x^2 + y^2 + 2x - 3y + 2 + \lambda(x + 2y - 2) = 0$$

$$\Rightarrow x^2 + y^2 + (2 + \lambda)x + (2\lambda - 3)y + 2 - 2\lambda = 0 \quad \dots(iii)$$

Eq. (iii) represents a circle of radius $\sqrt{10}$ units.

$$\therefore \sqrt{\left(-\frac{2 + \lambda}{2}\right)^2 + \left(-\frac{2\lambda - 3}{2}\right)^2} - 2 + 2\lambda = \sqrt{10}$$

$$\Rightarrow (4 + 4\lambda + \lambda^2) + (4\lambda^2 + 9 - 12\lambda) + 8\lambda - 8 = 40$$

$$\Rightarrow \lambda = \pm \sqrt{7}$$

Hence, required circles are

$$x^2 + y^2 + 2x - 3y + 2 \pm \sqrt{7}(x + 2y - 2) = 0$$

There are two such circles possible.

Example 55. Radius of the largest circle that can be drawn to pass through the point $(0, 1)$, $(0, 6)$ and touching X -axis, is

- (a) $\frac{5}{2}$
- (b) $\frac{3}{2}$
- (c) $\frac{7}{2}$
- (d) $\frac{9}{2}$

Sol. (c) Equation of any circles through $(0, 1)$ and $(0, 6)$ is

$$x^2 + (y - 1)(y - 6) + \lambda x = 0$$

$$\Rightarrow x^2 + y^2 - 7y + 6 + \lambda x = 0$$

If it touches X -axis, then $x^2 + \lambda x + 6 = 0$ should have equal roots.

$$\Rightarrow \lambda^2 = 24 \Rightarrow \lambda = \pm \sqrt{24}$$

Radius of these circles = $\sqrt{6 + \frac{49}{4} - 6} = \frac{7}{2}$ units. Thus, we

can draw two circles but radius of both circles is $\frac{7}{2}$.

Example 56. A circle passes through the points $A(1, 0)$, $B(5, 0)$ and touches Y -axis at $C(0, h)$. If $\angle ACB$ is maximum, then

- (a) $h = \sqrt{5}$
- (b) $h = 2\sqrt{5}$
- (c) $h = \sqrt{10}$
- (d) $h = 2\sqrt{10}$

Sol. (a) Equation of any circle passing through (1, 0) and (5, 0) is

$$y^2 + (x-1)(x-5) + \lambda y = 0$$

$$\Rightarrow x^2 + y^2 + \lambda y - 6x + 5 = 0$$

If $\angle ACB$ is maximum, then this circle must touch the Y-axis at (0, h).

On putting $x = 0$ in the equation of circle, we get

$$y^2 + \lambda y + 5 = 0$$

where $y = h$ is its repeated root.

$$\Rightarrow h^2 = 5$$

$$\text{and } \lambda = -2h$$

$$\Rightarrow |h| = \sqrt{5}$$

Coaxial System of Circles

A system of circles is said to be coaxial system of circles, if every pair of the circles in the system has the same radical axis.

The following results are direct consequences from the above definition and concepts learnt in earlier sections.

- i. Since, the lines joining the centres of two circles is perpendicular to their radical axis. Therefore, the centres of all circles of a coaxial system lie on a straight line which is perpendicular to the common radical axis.
- ii. Circles passing through two fixed points P and Q form a coaxial system because every pair of circles has the same common chord PQ and therefore, the same radical axis which is perpendicular bisector of PQ .
- iii. If the equation of a member of a system of coaxial circles is $S = 0$ and the equation of the common radical axis is $L = 0$, then the equation representing the coaxial system of circle is

$$S + \lambda L = 0, \text{ where } \lambda \in R$$

In other words, $S + \lambda L = 0, \lambda \in R$ represents a family of coaxial circles having $L = 0$ as the common radical axis.
- iv. The equation of a coaxial system of circles of which two members are $S_1 = 0$ and $S_2 = 0$ is

$$S_1 + \lambda S_2 = 0 \text{ or } S_1 + \lambda(S_1 - S_2) = 0$$

$$\text{or } S_2 + \lambda(S_1 - S_2) = 0, \text{ where } \lambda \in R.$$

In other words, if $S_1 = 0$ and $S_2 = 0$ are two circles, then

$$S_1 + \lambda S_2 = 0$$

or $S + \lambda(S_1 - S_2) = 0$

or $S_2 + \lambda(S_1 - S_2) = 0, \text{ where } \lambda \in R$

represents a family of coaxial circles having $S_1 - S_2 = 0$ as the common radical axis.

Example 57. The equation of a circle which is coaxial with the circles $2x^2 + 2y^2 - 2x + 6y - 3 = 0$ and $x^2 + y^2 + 4x + 2y + 1 = 0$. It is given that the centre of the circle to be determined lies on the radical axis of the circle

$$(a) 3(x^2 + y^2) + 2x + 8y - 2 = 0$$

$$(b) 4(x^2 + y^2) + 6x + 10y - 1 = 0$$

$$(c) 2(x^2 + y^2) + 2x + 8y - 3 = 0$$

$$(d) \text{None of the above}$$

Sol. (b) Equation of the given circles are

$$S_1 = x^2 + y^2 - x + 3y - \frac{3}{2} = 0 \quad \dots(i)$$

$$\text{and } S_2 = x^2 + y^2 + 4x + 2y + 1 = 0 \quad \dots(ii)$$

The radical axis of Eqs. (i) and (ii) is

$$S_1 - S_2 = 0 \Rightarrow 10x - 2y + 5 = 0 \quad \dots(iii)$$

$\therefore$ Required circle will have the equation of the form

$$x^2 + y^2 + 4x + 2y + 1 + k(10x - 2y + 5) = 0$$

$$\Rightarrow x^2 + y^2 + 2(2 + 5k)x + 2(1 - k)y + (1 + 5k) = 0 \quad \dots(iv)$$

Its centre is $[-2 - 5k, k - 1]$.

From question, it lies on Eq. (iii).

$$\therefore 10(-2 - 5k) - 2(k - 1) + 5 = 0$$

$$\Rightarrow k = -\frac{1}{4}$$

On putting the value of k in Eq. (iv), we get

$$x^2 + y^2 + \frac{3}{2}x + \frac{5}{2}y - \frac{1}{4} = 0$$

$$\Rightarrow 4(x^2 + y^2) + 6x + 10y - 1 = 0$$

Limiting Points

Limiting points of a coaxial system of circles are the members of the system which are of zero radius.

It follows from the above definition that limiting points of a coaxial system of circles are the centres of point circles belonging to the system.

In the previous section, we have seen that for a fixed value of c and varying g , the equation

$$x^2 + y^2 + 2gx + c = 0 \quad \dots(i)$$

represents a family of coaxial circles having X -axis as the common radical axis.

The coordinates of the centre and radius of Eq.(i) are $(-g, 0)$ and $\sqrt{g^2 - c}$, respectively.

For the circle (i) to be a point circle, we must have radius = 0

$$\Rightarrow \sqrt{g^2 - c} = 0 \Rightarrow c = g^2$$

$$\Rightarrow g = \pm \sqrt{c}$$

Hence, the limiting points of the coaxial system of circles given by Eq. (i) are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$.

Clearly, these points are real and distinct, if $c > 0$, coincident for $c = 0$ and imaginary, if $c < 0$.

System of Coaxial Circles when Limiting Points are Given

Let (a, b) and (α, β) be two limiting points of a coaxial system of circles. Then, the corresponding point of circles are

$$S_1 \equiv (x - a)^2 + (y - b)^2 = 0$$

and

$$S_2 \equiv (x - \alpha)^2 + (y - \beta)^2 = 0$$

So, the coaxial system of circles is given by

$$S_1 + \lambda S_2 = 0, \lambda \neq -1$$

$$\Rightarrow \{(x - a)^2 + (y - b)^2\} + \lambda \{(x - \alpha)^2 + (y - \beta)^2\} = 0, \lambda \neq -1$$

The value of λ is determined from the given condition.

➤ **Example 58.** If one circle of a coaxial system is $x^2 + y^2 + 2gx + 2fy + c = 0$ and one limiting point is (a, b) , then equation of the radical axis will be

(a) $(g + a)x + (f + b)y + c - a^2 - b^2 = 0$

(b) $2(g + a)x + 2(f + b)y + c - a^2 - b^2 = 0$

(c) $2g + 2fy + c - a^2 - b^2 = 0$

(d) None of the above

Sol. (b) Given circle is

$$S_1 \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

∴ Equation of the second circle is

$$(x - a)^2 + (y - b)^2 = 0$$

$$S_2 \equiv x^2 + y^2 - 2ax - 2by + a^2 + b^2 = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), equation of radical axis is

$$S_1 - S_2 = 0$$

$$\Rightarrow 2(g + a)x + 2(f + b)y + c - a^2 - b^2 = 0$$

➤ **Example 59.** Two circles, each of radius 5 units touch each other at $(1, 2)$. If the equation of their common tangent is $4x + 3y - 10 = 0$, then equation of one such circle is

(a) $x^2 + y^2 - 6x + 2y + 15 = 0$

(b) $x^2 + y^2 - 10x - 10y + 25 = 0$

(c) $x^2 + y^2 + 6x - 2y - 15 = 0$

(d) $x^2 + y^2 - 10x - 10y - 25 = 0$

Sol. (b) Equation of circles will be

$$(x - 1)^2 + (y - 2)^2 + \lambda(4x + 3y - 10) = 0$$

$$\Rightarrow x^2 + y^2 + 2x(2\lambda - 1) + y(3\lambda - 4) + 5 - 10\lambda = 0$$

Since, its radius is 5 units.

$$\therefore (2\lambda - 1)^2 + \frac{1}{4}(3\lambda - 4)^2 - 5 + 10\lambda = 25$$

$$\Rightarrow \lambda = \pm 2$$

Thus, equation of circles are

$$x^2 + y^2 + 6x + 2y - 15 = 0$$

or

$$x^2 + y^2 - 10x - 10y + 25 = 0$$

➤ **Example 60.** The coordinates of the limiting points of a system of coaxial circles determined by the circles $S_1 \equiv x^2 + y^2 + 4x + 2y + 5 = 0$ and $S_2 \equiv x^2 + y^2 + 2x - 4y + 7 = 0$ are

(a) $(-2, -1)$ and $(0, -3)$

(b) $(2, -1)$ and $(1, -3)$

(c) $(2, 1)$ and $(0, 3)$

(d) $(-2, 1)$ and $(3, 0)$

Sol. (a) The equation of a coaxial system of circles determined by the given circles is

$$S_1 + \lambda(S_1 - S_2) = 0, \lambda \in R$$

$$\Rightarrow x^2 + y^2 + 4x + 2y + 5 + \lambda(2x - 2y - 2) = 0$$

$$\Rightarrow x^2 + y^2 + 2(2 + \lambda)x + 2(1 - \lambda)y + 5 - 2\lambda = 0 \quad \dots(i)$$

The coordinates of the centre and radius are $[-(2 + \lambda), \lambda - 1]$ and $\sqrt{(2 + \lambda)^2 + (1 - \lambda)^2 - 5 + 2\lambda}$.

Now, $R = 0$

$$\Rightarrow (2 + \lambda)^2 + (1 - \lambda)^2 - 5 + 2\lambda = 0$$

$$\Rightarrow 2\lambda^2 + 4\lambda = 0$$

$$\therefore \lambda = 0, -2$$

Hence, the limiting points are $(-2, -1)$ and $(0, -3)$.

➤ **Example 61.** The equation of the radical of a coaxial system of circle whose limiting points are $(2, -1)$ and $(-3, 2)$, is

(a) $5x + 3y - 4 = 0$

(b) $5x + 3y + 4 = 0$

(c) $5x - 3y + 4 = 0$

(d) $3x - 5y + 4 = 0$

Sol. (c) The equations of circles having $(2, -1)$ and $(-3, 2)$ as limiting points, are

$$S_1 \equiv (x - 2)^2 + (y + 1)^2 = 0 \quad \dots(i)$$

and $S_2 \equiv (x + 3)^2 + (y - 2)^2 = 0 \quad \dots(ii)$

Clearly, Eqs. (i) and (ii) are members of the coaxial system of circles whose limiting points are $(2, -1)$ and $(-3, 2)$.

So, the equation of the radical axis is

$$S_2 - S_1 = 0$$

$$\Rightarrow 10x - 6y + 8 = 0$$

$$\Rightarrow 5x - 3y + 4 = 0$$

Work Book Exercise 13.3

- The locus of middle points of the chords of the circle $x^2 + y^2 = 49$ passing through the point $(1, -1)$ is
 - $x^2 + y^2 - x - y = 0$
 - $x^2 + y^2 - x + y = 0$
 - $x^2 + y^2 + x - y = 0$
 - $x^2 + y^2 + x + y = 0$
- The point from which the tangents to the circles $x^2 + y^2 - 8x + 40 = 0$, $5x^2 + 5y^2 - 25x + 80 = 0$, and $x^2 + y^2 - 8x + 16y + 160 = 0$ are equal in length, is
 - $\left(8, \frac{15}{2}\right)$
 - $\left(-8, \frac{15}{2}\right)$
 - $\left(8, -\frac{15}{2}\right)$
 - None of the above
- For the two circles $x^2 + y^2 = 16$ and $x^2 + y^2 - 2y = 0$, there is/are
 - one pair of common tangents
 - two pairs of common tangents
 - three common tangents
 - no common tangents
- If the chord of contact of tangents from three points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will
 - be concyclic
 - be collinear
 - form the vertices of a triangle
 - None of the above

- 5** If a line is drawn through a fixed point $P(\alpha, \beta)$ to cut the circle $x^2 + y^2 = a^2$ at A and B , then $PA \cdot PB$ is equal to
- $\alpha^2 + \beta^2$
 - $\alpha^2 + \beta^2 - a^2$
 - a^2
 - $\alpha^2 + \beta^2 + a^2$
- 6** If the two conics $a_1x^2 + 2h_1xy + b_1y^2 = c_1$, $a_2x^2 + 2h_2xy + b_2y^2 = c_2$ intersect in four concyclic points, then
- $(a_1 - b_1)h_2 = (a_2 - b_2)h_1$
 - $(a_1 - b_1)h_1 = (a_2 - b_2)h_2$
 - $(a_1 + b_1)h_2 = (a_2 + b_2)h_1$
 - $(a_1 + b_1)h_1 = (a_2 + b_2)h_2$
- 7** If the polar of a point (p, q) with respect to the circle $x^2 + y^2 = a^2$ touches the circle $(x - c)^2 + (y - d)^2 = b^2$, then
- $b^2(p^2 + q^2) = (a^2 - cp - dq)^2$
 - $b^2(p^2 + q^2) = (a^2 - cp - dp)^2$
 - $a^2(p^2 + q^2) = (b^2 - cp - dq)^2$
 - None of the above
- 8** If the chord of contact of tangents from a point P to a given circle passes through Q , then the circle on PQ as diameter
- cuts the given circle orthogonally
 - touches the given circle externally
 - touches the given circle internally
 - None of the above
- 9** If the chord of contact of tangents from a point P to a circle passes through Q . If l_1 and l_2 are the lengths of the tangents from P and Q to the circle, then PQ is equal to
- $\frac{l_1 + l_2}{2}$
 - $\frac{l_1 - l_2}{2}$
 - $\sqrt{l_1^2 + l_2^2}$
 - $\sqrt{l_1^2 - l_2^2}$
- 10** The circle S_1 with centre $C_1(a_1, b_1)$ and radius r_1 touches externally the circle S_2 with centre $C_2(a_2, b_2)$ and radius r_2 . If the tangent at their common point passes through the origin, then
- $(a_1^2 + a_2^2) + (b_1^2 + b_2^2) = r_1^2 + r_2^2$
 - $(a_1^2 - a_2^2) + (b_1^2 - b_2^2) = r_1^2 - r_2^2$
 - $(a_1^2 - b_2^2) + (a_2^2 + b_2^2) = r_1^2 + r_2^2$
 - $(a_1^2 - b_1^2) + (a_2^2 + b_2^2) = r_1^2 + r_2^2$

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. Let C be any circle with centre $(0, \sqrt{2})$. Then, the number of rational points can be there on C is atmost (A rational point is a point both of whose coordinates are rational numbers)

- (a) four
- (b) three
- (c) two
- (d) one

Sol. The equation of any circle C with centre $(0, \sqrt{2})$ is given by

$$(x - 0)^2 + (y - \sqrt{2})^2 = r^2,$$

where r is any positive real number.

$$\Rightarrow x^2 + y^2 - 2\sqrt{2}y = r^2 - 2 \quad \dots(i)$$

If possible, let $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ be three distinct rational points on circle C , then

$$x_1^2 + y_1^2 - 2\sqrt{2}y_1 = r^2 - 2 \quad \dots(ii)$$

$$x_2^2 + y_2^2 - 2\sqrt{2}y_2 = r^2 - 2 \quad \dots(iii)$$

$$\text{and } x_3^2 + y_3^2 - 2\sqrt{2}y_3 = r^2 - 2 \quad \dots(iv)$$

We claim atleast two of y_1, y_2 and y_3 are distinct. Here, if $y_1 = y_2 = y_3$, then P, Q and R lie on a line parallel to X -axis and a line parallel to X -axis does not cross the circle in more than two points. Thus, we have either $y_1 \neq y_2$ or $y_1 \neq y_3$ or $y_2 \neq y_3$.

On subtracting Eq. (ii) from Eqs. (iii) and (iv), we get

$$(x_2^2 + y_2^2) - (x_1^2 + y_1^2) - 2\sqrt{2}(y_2 - y_1) = 0$$

$$\text{and } (x_3^2 + y_3^2) - (x_1^2 + y_1^2) - 2\sqrt{2}(y_3 - y_1) = 0$$

$$\Rightarrow a_1 - \sqrt{2}b_1 = 0 \text{ and } a_2 - \sqrt{2}b_2 = 0 \quad \dots(v)$$

$$\text{where, } a_1 = (x_2^2 + y_2^2) - (x_1^2 + y_1^2), b_1 = 2(y_2 - y_1)$$

$$a_2 = (x_3^2 + y_3^2) - (x_1^2 + y_1^2), b_2 = 2(y_3 - y_1)$$

Clearly, a_1, a_2 and b_1, b_2 are rational numbers as x_1, x_2 and x_3, y_1, y_2, y_3 are rational numbers.

Since, either $y_1 \neq y_2$ or $y_1 \neq y_3$

$$\therefore \text{ Either } b_1 \neq 0$$

$$\text{or } b_2 \neq 0$$

$$\text{If } b_1 \neq 0,$$

$$\text{then } a_1 - \sqrt{2}b_1 = 0 \quad [\text{from Eq. (v)}]$$

$$\Rightarrow \frac{a_1}{b_1} = \sqrt{2}$$

which is not possible because $\frac{a_1}{b_1}$ is a rational number

and $\sqrt{2}$ is an irrational number.

If $b_2 \neq 0$, then

$$a_2 - \sqrt{2}b_2 = 0 \Rightarrow \frac{a_2}{b_2} = \sqrt{2}$$

which is not possible because $\frac{a_2}{b_2}$ is a rational number

and $\sqrt{2}$ is an irrational number.

Thus, in both the cases we arrive at a contradiction. This means that our supposition is wrong. Hence, there can be atmost two rational points on circle C .

Aliter

Let there be three points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ with rational coordinates on circle C having its equation

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

Since, P, Q and R lie on circle Eq. (i), therefore

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0$$

$$\text{and } x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0$$

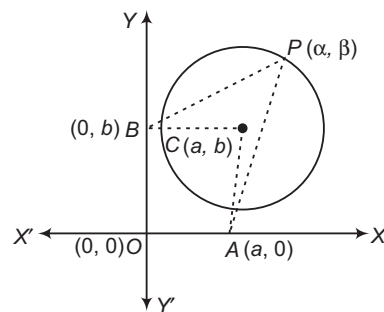
These are three linear equations in g, f and c with rational coefficients. So, we get rational values of g, f and c but $f = \sqrt{2}$. Thus, we arrive at a contradiction. Hence, there can be atmost two rational points on circle C .

Hence, (c) is the correct answer.

Ex 2. P is a variable point on a circle with centre at C . CA and CB are perpendicular from C to the X and Y -axes, respectively. Then, the locus of the centroid of ΔPAB is

- (a) a parabola
- (b) a circle
- (c) an ellipse
- (d) None of the above

Sol. Let $C(a, b)$ be the centre and r be the radius of the given circle. Let $P(\alpha, \beta)$ be a variable point on the circle. Then, coordinates of A and B are $(a, 0)$ and $(0, b)$, respectively. Let $Q(h, k)$ be the centroid of ΔPAB . Then,



$$h = \frac{a + \alpha}{3}$$

$$\text{and } k = \frac{\beta + b}{3} \Rightarrow \alpha = 3h - a$$

$$\text{and } \beta = 3k - b \quad \dots(i)$$

The equation of the circle is $(x - a)^2 + (y - b)^2 = r^2$, where r is the radius of the circle. Since, $P(\alpha, \beta)$ lies on the circle.

Therefore, $(\alpha - a)^2 + (\beta - b)^2 = r^2$
 $\Rightarrow (3h - 2a)^2 + (3k - 2b)^2 = r^2$ [from Eq. (i)]

Hence, the locus of (h, k) is

$$(3x - 2a)^2 + (3y - 2b)^2 = r^2$$

$$\Rightarrow \left(x - \frac{2a}{3}\right)^2 + \left(y - \frac{2b}{3}\right)^2 = \left(\frac{r}{3}\right)^2$$

Clearly, it represents a circle with centre $\left(\frac{2a}{3}, \frac{2b}{3}\right)$.

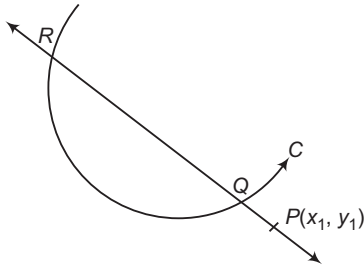
Hence, (b) is the correct answer.

Ex 3. Consider a curve $ax^2 + 2hxy + by^2 = 1$ and a point P not on the curve. A line drawn from the point P intersects the curve at points Q and R . If the product $PQ \cdot PR$ is independent of the slope of the line, then the curve is

- (a) a circle
- (b) a parabola
- (c) an ellipse
- (d) a hyperbola

Sol. The equation of a line passing through $P(x_1, y_1)$ is

$$\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$$



The coordinates of any point on this line are

$$(x_1 + r \cos \theta, y_1 + r \sin \theta)$$

If it lies on $ax^2 + 2hxy + by^2 = 1$, then

$$a(x_1 + r \cos \theta)^2 + 2h(x_1 + r \cos \theta)(y_1 + r \sin \theta) + b(y_1 + r \sin \theta)^2 = 1$$

$$\Rightarrow r^2(a \cos^2 \theta + h \sin 2\theta + b \sin^2 \theta) + 2r(ax_1 \cos \theta + hx_1 \sin \theta + hy_1 \cos \theta + by_1 \sin \theta) + ax_1^2 + 2hx_1y_1 + by_1^2 - 1 = 0 \dots (i)$$

This is a quadratic in r . So, it gives two values of r which represent the distances of two points of intersection of the line and the circle from the point P . Let $PQ = r_1$ and $PR = r_2$, then r_1 and r_2 are the roots of the Eq. (i).

$$\therefore r_1 r_2 = \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a \cos^2 \theta + h \sin 2\theta + b \sin^2 \theta}$$

$$\Rightarrow PQ \cdot PR = \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{a \cos^2 \theta + h \sin 2\theta + b \sin^2 \theta}$$

$$= \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{\left(\frac{a+b}{2}\right) + \left(\frac{a-b}{2}\right) \cos 2\theta + \sin 2\theta}$$

$$= \frac{ax_1^2 + 2hx_1y_1 + by_1^2 - 1}{\left(\frac{a+b}{2}\right) + \left(\sqrt{\left(\frac{a-b}{2}\right)^2 + h^2}\right) \sin(2\theta + \phi)}$$

Clearly, this will be independent of θ , if

$$\left(\frac{a-b}{2}\right)^2 + h^2 = 0$$

$$\Rightarrow \frac{a-b}{2} = 0$$

$$\text{and } h = 0$$

$$\Rightarrow a = b$$

$$\text{and } h = 0$$

On substituting $a = b$ and $h = 0$ in $ax^2 + 2hxy + by^2 = 1$, we get

$$ax^2 + ay^2 = 1$$

which represents a circle.

Hence, (a) is the correct answer.

Ex 4. If two curves whose equations are

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

and $a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$ intersect in four concyclic points, then

- (a) $\frac{a-b}{h} = \frac{a'+b'}{h'}$
- (b) $\frac{a+b}{h} = \frac{a'+b'}{h'}$
- (c) $\frac{a-b}{h'} = \frac{a'-b'}{h}$
- (d) $\frac{a-b}{h} = \frac{a'-b'}{h'}$

Sol. The equation of the curve passing through the intersection of given curves is

$$(ax^2 + 2hxy + by^2 + 2gx + 2fy + c) + \lambda(a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c') = 0 \dots (i)$$

$$\Rightarrow x^2(a + \lambda a') + y^2(b + \lambda b') + 2xy(h + \lambda h') + 2x(g + \lambda g') + 2y(f + \lambda f') + c + \lambda c' = 0$$

This will represent a circle, if

$$\text{Coefficient of } x^2 = \text{Coefficient of } y^2$$

and coefficient of $xy = 0$

$$\Rightarrow a + \lambda a' = b + \lambda b'$$

$$\text{and } h + \lambda h' = 0$$

$$\Rightarrow \lambda = -\frac{a-b}{a'-b'}$$

$$\text{and } \lambda = -\frac{h}{h'}$$

$$\Rightarrow \frac{a-b}{a'-b'} = \frac{h}{h'}$$

$$\Rightarrow \frac{a-b}{h} = \frac{a'-b'}{h'}$$

Hence, (d) is the correct answer.

Ex 5. If the curves

$ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$
and $lx^2 - 2hxy + (a + l - b)^2 - 2mx - 2ny + k = 0$
intersect at four concyclic points A, B, C and D
and P is the point $\left(\frac{g+m}{a+l}, \frac{f+n}{a+l}\right)$, then

- (a) $PA^2 + PB^2 + PC^2 + PD^2 = 4PA^2$
(b) $PA^2 + PB^2 + PC^2 + PD^2 = 2PA^2$
(c) $PA^2 + PB^2 + PC^2 + PD^2 = 3PA^2$
(d) $PA^2 + PB^2 + PC^2 + PD^2 = 2(PA + PB)^2$

Sol. The equation of a conic passing through the point of intersection of the given curves is

$$ax^2 + 2hxy + by^2 - 2gx - 2fy + c + \lambda \{lx^2 - 2hxy + (a + l - b)^2 - 2mx - 2ny + k\} = 0$$

$$\Rightarrow (a + \lambda l)x^2 + by^2 + 2h(1 - \lambda)xy - 2x(g + \lambda m) - 2y(f + \lambda n) + c + \lambda(a + l - b)^2 + k\lambda = 0 \dots (i)$$

This will represent a circle, if

$$a + \lambda l = b$$

$$\text{and } 2h(1 - \lambda) = 0$$

$$\Rightarrow \lambda = 1$$

$$\text{and } a + l = b$$

On substituting these values in Eq. (i), it reduces to

$$(a + l)x^2 + (a + l)y^2 - 2x(g + m) - 2y(f + n) + c + k = 0$$

$$\Rightarrow x^2 + y^2 - 2\left(\frac{g+m}{a+l}\right)x - 2\left(\frac{f+n}{a+l}\right)y + \frac{c+k}{a+l} = 0$$

Clearly, its centre is at $P\left(\frac{g+m}{a+l}, \frac{f+n}{a+l}\right)$

Thus, if A, B, C and D are four points on it. Then,

$$PA^2 = PB^2 = PC^2 = PD^2 = (\text{Radius})^2$$

$$\Rightarrow PA^2 + PB^2 + PC^2 + PD^2 = 4PA^2$$

Hence, (a) is the correct answer.

Ex 6. The equation of the circle of minimum radius which contains the three circles

$$x^2 + y^2 - 4y - 5 = 0 \dots (i)$$

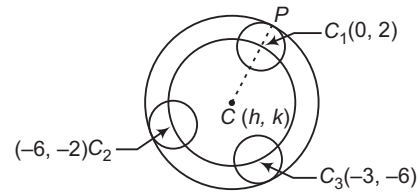
$$x^2 + y^2 + 12x + 4y + 31 = 0 \dots (ii)$$

$$\text{and } x^2 + y^2 + 6x + 12y + 36 = 0 \dots (iii)$$

is

- (a) $\left(x - \frac{31}{18}\right)^2 + \left(y - \frac{23}{12}\right)^2 = \left(3 - \frac{5}{36}\sqrt{949}\right)^2$
(b) $\left(x + \frac{23}{12}\right)^2 + \left(y + \frac{31}{18}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$
(c) $\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$
(d) None of the above

Sol. The coordinates of the centres and radii of three given circles are as given below :



	Centre	Radius
Circle (i)	$C_1(0, 2)$	$r_1 = 3$
Circle (ii)	$C_2(-6, -2)$	$r_2 = 3$
Circle (iii)	$C_3(-3, -6)$	$r_3 = 3$

Let $C(h, k)$ be the centre of the circle passing through the centres of the circles (i), (ii) and (iii). Then,

$$CC_1 = CC_2 = CC_3$$

$$\Rightarrow CC_1^2 = CC_2^2 = CC_3^2$$

$$\Rightarrow (h - 0)^2 + (k - 2)^2 = (h + 6)^2 + (k + 2)^2$$

$$= (h + 3)^2 + (k + 6)^2$$

$$\Rightarrow -4k + 4 = 12h + 4k + 40$$

$$= 6h + 12k + 45$$

$$\Rightarrow 12h + 8k + 36 = 0$$

$$\text{and } 6h - 8k - 5 = 0$$

$$\Rightarrow 3h + 2k + 9 = 0$$

$$\text{and } 6h - 8k - 5 = 0$$

$$\Rightarrow h = \frac{-31}{18}$$

$$k = \frac{-23}{12}$$

$$\therefore CC_1 = \sqrt{\left(0 + \frac{31}{18}\right)^2 + \left(2 + \frac{23}{12}\right)^2}$$

$$= \frac{5}{36}\sqrt{949}$$

$$\text{Now, } CP = CC_1 + C_1P$$

$$\Rightarrow CP = \left(\frac{5}{36}\sqrt{949} + 3\right)$$

Thus, required circle has its centre at $\left(-\frac{31}{18}, -\frac{23}{12}\right)$ and

$$\text{radius} = CP = \left(\frac{5}{36}\sqrt{949} + 3\right)$$

Hence, its equation is

$$\left(x + \frac{31}{18}\right)^2 + \left(y + \frac{23}{12}\right)^2 = \left(3 + \frac{5}{36}\sqrt{949}\right)^2$$

Hence, (c) is the correct answer.

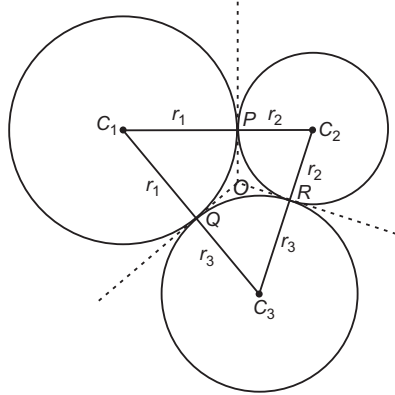
✎ In the above example, radii of all the circles were same. In case three circles are of distinct radii, then the radius of the required circle will be equal to $CC_1 + \text{radius of largest circle}$.

Ex 7. Three circles touch one another externally. The tangents at their points of contact meet at a point whose distance from a point of contact is 4. Then, the ratio of the product of the radii to the sum of the radii of the circles is

- (a) 16 (b) 4 (c) 8 (d) $\frac{1}{16}$

Sol. Let C_1, C_2 and C_3 be the centres of three circles of radii r_1, r_2 and r_3 respectively, then the lengths of the sides of $\Delta C_1C_2C_3$ are

$$C_1C_3 = r_1 + r_3, C_2C_3 = r_2 + r_3 \text{ and } C_1C_2 = r_1 + r_2$$



Let O be the point of intersection of common tangents in three circles taken in pairs. Then, $OP = OQ = OR$. Also, OP, OQ and OR are perpendicular to the sides C_1C_2, C_2C_3 and C_3C_1 respectively.

$$\begin{aligned} \text{Therefore, } OP = OQ = OR &= r [\text{inradius of } \Delta C_1C_2C_3] \\ \Rightarrow r &= 4 \\ \Rightarrow \frac{\text{Area of } \Delta C_1C_2C_3}{\text{Semi-perimeter}} &= 4 \quad \left[\because r = \frac{\Delta}{s} \right] \end{aligned}$$

$$\begin{aligned} \text{But area of } \Delta C_1C_2C_3 &= \sqrt{(r_1 + r_2 + r_3)r_1r_2r_3} \\ [\because s &= r_1 + r_2 + r_3] \end{aligned}$$

$$\begin{aligned} \therefore \frac{\text{Area of } \Delta C_1C_2C_3}{\text{Semi-perimeter}} &= 4 \\ \Rightarrow \frac{\sqrt{(r_1 + r_2 + r_3)r_1r_2r_3}}{r_1 + r_2 + r_3} &= 4 \\ \Rightarrow \frac{r_1r_2r_3}{r_1 + r_2 + r_3} &= 16 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 8. The equation of the circle circumscribing the triangle formed by the line $x + y = 6, 2x + y = 4$ and $x + 2y = 5$, is

- (a) $x^2 + y^2 - 17x - 19y + 50 = 0$
 (b) $x^2 + y^2 + 17x - 19y + 50 = 0$
 (c) $x^2 + y^2 - 19x - 17y + 50 = 0$
 (d) None of the above

Sol. Let the equation of sides AB, BC and CA of ΔABC are respectively

$$x + y = 6 \quad \dots(i)$$

$$2x + y = 4 \quad \dots(ii)$$

$$\text{and } x + 2y = 5 \quad \dots(iii)$$

The coordinates A, B and C are $(7, -1), (-2, 8)$ and $(1, 2)$, respectively.

Let the equation of the circumcircle of ΔABC be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(iv)$$

It passes through $A(7, -1), B(-2, 8)$ and $C(1, 2)$.

$$\text{Therefore, } 50 + 14g - 2f + c = 0 \quad \dots(v)$$

$$68 - 4g + 16f + c = 0 \quad \dots(vi)$$

$$\text{and } 5 + 2g + 4f + c = 0 \quad \dots(vii)$$

On solving these equations, we get

$$g = -\frac{17}{2}, f = -\frac{19}{2}$$

and

$$c = 50$$

On substituting the values of g, f and c in Eq. (iv), the equation of the required circumcircle is

$$x^2 + y^2 - 17x - 19y + 50 = 0$$

Aliter

Consider the equation

$$\begin{aligned} (x + y - 6)(2x + y - 4) + \lambda(2x + y - 4)(x + 2y - 5) \\ + \mu(x + 2y - 5)(x + y - 6) = 0 \quad \dots(i) \end{aligned}$$

where, λ and μ are constants.

This equation represents a curve passing through the vertices of the triangle formed by the given lines.

We have to determine the values of λ and μ , so that the curve given in Eq. (i) is a circle.

The curve given in Eq. (i) will be a circle, if

$$\begin{aligned} \text{Coefficient of } x^2 &= \text{Coefficient of } y^2 \text{ and Coefficient} \\ &\text{of } xy = 0 \end{aligned}$$

$$\Rightarrow 2 + 2\lambda + \mu = 1 + 2\lambda + 2\mu$$

$$\text{and } 3 + 5\lambda + 3\mu = 0$$

$$\Rightarrow \mu = 1$$

$$\text{and } 3\mu + 5\lambda + 3 = 0$$

$$\Rightarrow \mu = 1$$

$$\text{and } \lambda = -\frac{6}{5}$$

On substituting the values of λ and μ in Eq. (i), we get

$$(x + y - 6)(2x + y - 4) - \left(\frac{6}{5}\right)(2x + y - 4)$$

$$(x + 2y - 5) + (x + 2y - 5)(x + y - 6) = 0$$

$$\Rightarrow x^2 + y^2 - 17x - 19y + 50 = 0$$

This is the equation of the required circle.

Hence, (a) is the correct answer.

Ex 9. If $L_r = a_r x + b_r y + c_r = 0, r = 1, 2, 3$ be the sides of a ΔABC , then the equation of its circumcircle is

$$(a) \begin{vmatrix} \frac{1}{L_1} & \frac{1}{L_2} & \frac{1}{L_3} \\ a_2a_3 - b_2b_3 & a_3a_1 - b_3b_1 & a_1a_2 - b_1b_2 \\ a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 & a_1b_2 + a_2b_1 \end{vmatrix} = 0$$

$$(b) \begin{vmatrix} L_1 & L_2 & L_3 \\ a_2a_3 - b_2b_3 & a_3a_1 - b_3b_1 & a_1a_2 - b_1b_2 \\ a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 & a_1b_2 + a_2b_1 \end{vmatrix} = 0$$

$$(c) \begin{vmatrix} a_2a_3 + b_2b_3 & a_3a_1 + b_3b_1 & a_1a_2 - b_1b_2 \\ \frac{1}{L_1} & \frac{1}{L_2} & \frac{1}{L_3} \\ a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 & a_1b_2 + a_2b_1 \end{vmatrix} = 0$$

(d) None of the above

Sol. The equation of a curve passing through the intersection of the lines

$$L_1 = a_1x + b_1y + c_1 = 0$$

$$L_2 = a_2x + b_2y + c_2 = 0$$

$$\text{and } L_3 = a_3x + b_3y + c_3 = 0$$

$$\text{is } \lambda L_1 L_2 + \mu L_2 L_3 + \nu L_3 L_1 = 0 \quad \dots(i)$$

$$\Rightarrow \lambda(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + \mu(a_2x + b_2y + c_2)(a_3x + b_3y + c_3) + \nu(a_3x + b_3y + c_3)(a_1x + b_1y + c_1) = 0$$

This equation will represent a circle, if

$$\text{Coefficient of } x^2 = \text{Coefficient of } y^2$$

and coefficient of $xy = 0$

$$\therefore \lambda(a_1a_2 - b_1b_2) + \mu(a_2a_3 - b_2b_3) + \nu(a_3a_1 - b_3b_1) = 0 \dots(ii)$$

$$\text{and } \lambda(a_1b_2 + a_2b_1) + \mu(a_2b_3 + a_3b_2) + \nu(a_3b_1 + a_1b_3) = 0 \dots(iii)$$

Eliminating λ, μ and ν from Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} L_1L_2 & L_2L_3 & L_3L_1 \\ a_1a_2 - b_1b_2 & a_2a_3 - b_2b_3 & a_3a_1 - b_3b_1 \\ a_1b_2 + a_2b_1 & a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \frac{1}{L_1} & \frac{1}{L_2} & \frac{1}{L_3} \\ a_2a_3 - b_2b_3 & a_3a_1 - b_3b_1 & a_1a_2 - b_1b_2 \\ a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 & a_1b_2 + a_2b_1 \end{vmatrix} = 0$$

Hence, (a) is the correct answer.

Ex 10. The equation of the circumcircle of the triangle formed by the lines $x \cos \alpha_i + y \sin \alpha_i = p_i$, ($i = 1, 2, 3$), is

$$(a) \begin{vmatrix} L_1L_2 & L_2L_3 & L_3L_1 \\ \cos(\alpha_1 - \alpha_2) & \cos(\alpha_2 - \alpha_3) & \cos(\alpha_3 - \alpha_1) \\ \sin(\alpha_1 + \alpha_2) & \sin(\alpha_2 + \alpha_3) & \sin(\alpha_3 + \alpha_1) \end{vmatrix}$$

$$(b) \begin{vmatrix} L_3L_2 & L_1L_3 & L_2L_1 \\ \cos(\alpha_1 + \alpha_2) & \cos(\alpha_2 + \alpha_3) & \cos(\alpha_3 + \alpha_1) \\ \sin(\alpha_1 + \alpha_2) & \sin(\alpha_2 + \alpha_3) & \sin(\alpha_3 + \alpha_1) \end{vmatrix} = 0$$

$$(c) \begin{vmatrix} L_1L_2 & L_2L_3 & L_3L_1 \\ \cos(\alpha_1 + \alpha_2) & \cos(\alpha_2 + \alpha_3) & \cos(\alpha_3 + \alpha_1) \\ \sin(\alpha_1 + \alpha_2) & \sin(\alpha_2 + \alpha_3) & \sin(\alpha_3 + \alpha_1) \end{vmatrix} = 0$$

(d) None of the above

Sol. The equation of three lines are

$$L_i \equiv x \cos \alpha_i + y \sin \alpha_i - p_i = 0, (i = 1, 2, 3).$$

The equation of any curve passing through the points of intersection of the given straight lines is

$$\lambda L_1 L_2 + \mu L_2 L_3 + \nu L_3 L_1 = 0 \quad \dots(i)$$

$$\Rightarrow \lambda(x \cos \alpha_1 + y \sin \alpha_1 - p_1)(x \cos \alpha_2 + y \sin \alpha_2 - p_2) + \mu(x \cos \alpha_2 + y \sin \alpha_2 - p_2)(x \cos \alpha_3 + y \sin \alpha_3 - p_3) + \nu(x \cos \alpha_3 + y \sin \alpha_3 - p_3)(x \cos \alpha_1 + y \sin \alpha_1 - p_1) = 0$$

This equation will represent a circle, if

$$\text{Coefficient of } x^2 = \text{Coefficient of } y^2$$

and coefficient of $xy = 0$

$$\therefore \lambda(\cos \alpha_1 \cos \alpha_2 - \sin \alpha_1 \sin \alpha_2) + \mu(\cos \alpha_2 \cos \alpha_3 - \sin \alpha_2 \sin \alpha_3) + \nu(\cos \alpha_3 \cos \alpha_1 - \sin \alpha_3 \sin \alpha_1) = 0$$

$$\text{and } \lambda(\sin \alpha_1 \cos \alpha_2 + \cos \alpha_1 \sin \alpha_2) + \mu(\sin \alpha_2 \cos \alpha_3 + \cos \alpha_2 \sin \alpha_3) + \nu(\sin \alpha_3 \cos \alpha_1 + \cos \alpha_3 \sin \alpha_1) = 0$$

$$\Rightarrow \lambda \cos(\alpha_1 + \alpha_2) + \mu \cos(\alpha_2 + \alpha_3) + \nu \cos(\alpha_3 + \alpha_1) = 0 \dots(ii)$$

$$\text{and } \lambda \sin(\alpha_1 + \alpha_2) + \mu \sin(\alpha_2 + \alpha_3) + \nu \sin(\alpha_3 + \alpha_1) = 0 \dots(iii)$$

Eliminating λ, μ, ν from Eqs. (i), (ii) and (iii), we get

$$\begin{vmatrix} L_1L_2 & L_2L_3 & L_3L_1 \\ \cos(\alpha_1 + \alpha_2) & \cos(\alpha_2 + \alpha_3) & \cos(\alpha_3 + \alpha_1) \\ \sin(\alpha_1 + \alpha_2) & \sin(\alpha_2 + \alpha_3) & \sin(\alpha_3 + \alpha_1) \end{vmatrix} = 0$$

This is the equation of the required circle.

Hence, (c) is the correct answer.

Ex 11. The general equation of the circle passing through two given points $A(x_1, y_1)$ and $B(x_2, y_2)$ may be written as

$$(a) (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$(b) (x - x_2)(y - y_1) + (y - y_2)(x - x_1) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$(c) (x - x_1)(y - y_1) + (x - x_2)(y - y_2) + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

(d) None of the above

Sol. Let $P(h, k)$ be any point on the circle passing through points $A(x_1, y_1)$ and $B(x_2, y_2)$. Since, the angle in the same segment of a circle is always same. Therefore, $\angle APB = \theta$ or $\pi - \theta$, where θ is a constant.

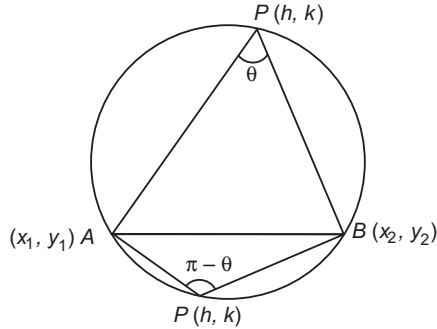
$$\text{Now, } m_1 = \text{Slope of } AP = \frac{k - y_1}{h - x_1},$$

$$\text{and } m_2 = \text{Slope of } BP = \frac{k - y_2}{h - x_2}$$

$$\tan \theta = \pm \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow \tan \theta = \pm \frac{\frac{k - y_1}{h - x_1} - \frac{k - y_2}{h - x_2}}{1 + \frac{k - y_1}{h - x_1} \times \frac{k - y_2}{h - x_2}}$$

$$\Rightarrow \tan \theta = \pm \frac{(h-x_2)(k-y_1) - (h-x_1)(k-y_2)}{(h-x_1)(h-x_2) + (k-y_1)(k-y_2)}$$



$$\Rightarrow \begin{aligned} & (h-x_1)(h-x_2) + (k-y_1)(k-y_2) \\ & = \pm \cot \theta \{ (h-x_2)(k-y_1) - (h-x_1)(k-y_2) \} \end{aligned}$$

Hence, the locus of (h, k) is

$$\begin{aligned} & (x-x_1)(x-x_2) + (y-y_1)(y-y_2) \\ & = \pm \cot \theta \{ (x-x_2)(y-y_1) - (x-x_1)(y-y_2) \} \\ \Rightarrow & (x-x_1)(x-x_2) + (y-y_1)(y-y_2) \\ & \pm \cot \theta \{ (x(y_1-y_2) + y(x_2-x_1) + x_1y_2 - x_2y_1) \} = 0 \\ \Rightarrow & (x-x_1)(x-x_2) + (y-y_1)(y-y_2) \end{aligned}$$

$$\pm \cot \theta \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{aligned} & (x-x_1)(x-x_2) + (y-y_1)(y-y_2) \\ & + \lambda \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 12. The equation of the circle circumscribing the quadrilateral formed by the lines in order are $5x + 3y - 9 = 0$, $x - 3y = 0$, $2x - y = 0$, $x + 4y - 2 = 0$ without finding the vertices of the quadrilateral, is

(a) $2x^2 + 2y^2 - \frac{16}{9}x + \frac{2}{3}y = 0$

(b) $2x^2 + y^2 + \frac{16}{9}x + \frac{2}{3}y = 0$

(c) $x^2 + y^2 + \frac{16}{9}x + \frac{1}{3}y = 0$

(d) $x^2 + y^2 - \frac{16}{9}x + \frac{1}{3}y = 0$

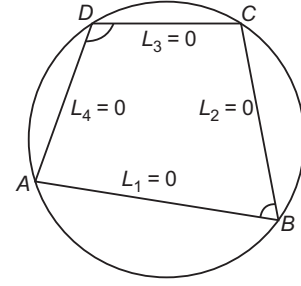
Sol. The equation of the sides of the quadrilateral are

$$L_1 \equiv 5x + 3y - 9 = 0$$

$$L_2 \equiv x - 3y = 0$$

$$L_3 \equiv 2x - y = 0$$

and $L_4 \equiv x + 4y - 2 = 0$



The equation of the curve passing through the vertices of the quadrilateral is

$$\begin{aligned} & = L_1L_3 + \lambda L_2L_4 = 0 \\ \Rightarrow & (5x + 3y - 9)(2x - y) + \lambda(x - 3y)(x + 4y - 2) = 0 \end{aligned} \quad \dots(i)$$

This will represent a circle, if

$$\text{Coefficient of } x^2 = \text{Coefficient of } y^2$$

and coefficient of $xy = 0$

$$\Rightarrow 10 + \lambda = -3 - 12\lambda$$

$$\Rightarrow 1 + \lambda = 0$$

$$\Rightarrow \lambda = -1$$

On substituting the value of λ in Eq. (i), we get

$$(5x + 3y - 9)(2x - y) - (x - 3y)(x + 4y - 2) = 0$$

$$\Rightarrow x^2 + y^2 - \frac{16}{9}x + \frac{1}{3}y = 0$$

which is the equation of the required circle.

Hence, (d) is the correct answer.

Ex 13. The equation of the circles passing through $(-4, 3)$ and touching the lines $x + y = 2$ and $x - y = 2$, is

(a) $x^2 + y^2 \pm 2(10 + 3\sqrt{6})x + (55 \pm 24\sqrt{6}) = 0$

(b) $x^2 + y^2 \pm 2(10 + \sqrt{6})x + (55 \pm \sqrt{6}) = 0$

(c) $x^2 - y^2 \pm 2(10 + 3\sqrt{6})y - (55 \pm 24\sqrt{6}) = 0$

(d) None of the above

Sol. Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

It passes through $(-4, 3)$. Therefore,

$$25 - 8g + 6f + c = 0 \quad \dots(ii)$$

Since, circle (i) touches the lines $x + y - 2 = 0$ and $x - y - 2 = 0$. Therefore,

$$\begin{aligned} \left| \frac{-g - f - 2}{\sqrt{2}} \right| &= \left| \frac{-g + f - 2}{\sqrt{2}} \right| \\ &= \sqrt{g^2 + f^2 - c} \quad \dots(iii) \end{aligned}$$

Now, $\left| \frac{-g - f - 2}{\sqrt{2}} \right| = \left| \frac{-g + f - 2}{\sqrt{2}} \right|$

$$\Rightarrow -g - f - 2 = \pm(-g + f - 2)$$

$$\Rightarrow -g - f - 2 = -g + f - 2$$

$$\text{or } -g - f - 2 = g - f + 2$$

$$\Rightarrow f = 0 \quad \text{or } g = -2$$

Case I When $f = 0$

From Eq. (iii), we have

$$\left| \frac{-g + f - 2}{\sqrt{2}} \right| = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow \left| \frac{-g - 2}{\sqrt{2}} \right| = \sqrt{g^2 - c} \quad [\because f = 0]$$

$$\Rightarrow (g + 2)^2 = 2(g^2 - c)$$

$$\Rightarrow g^2 - 4g - 4 - 2c = 0 \quad \dots(\text{iv})$$

Putting $f = 0$ in Eq. (ii), we get

$$25 - 8g + c = 0 \quad \dots(\text{v})$$

Eliminating c from Eqs. (iv) and (v), we get

$$g^2 - 20g + 46 = 0 \Rightarrow g = 10 \pm 3\sqrt{6}$$

On substituting the values of g , f and c in Eq. (i), we get

$$x^2 + y^2 \pm 2(10 \pm 3\sqrt{6})x + (55 \pm 24\sqrt{6}) = 0$$

This is the equation of the required circle.

Case II When $g = -2$

From Eq. (iii), we have

$$\left| \frac{-g + f - 2}{\sqrt{2}} \right| = \sqrt{g^2 + f^2 - c}$$

$$\Rightarrow f^2 = 2(4 + f^2 - c) \quad [\because g = -2]$$

$$\Rightarrow f^2 - 2c + 8 = 0 \quad \dots(\text{vi})$$

On putting $g = -2$ in Eq. (ii), we get

$$c = -6f - 41$$

On substituting the value of c in Eq. (vi), we get

$$f^2 + 12f + 82 + 8 = 0$$

$$\Rightarrow f^2 + 12f + 90 = 0$$

This equation gives imaginary values of f .

Hence, the equations of the required circles are

$$x^2 + y^2 \pm 2(10 + 3\sqrt{6})x + (55 \pm 24\sqrt{6}) = 0$$

Hence, (a) is the correct answer.

Ex 14. If $\left(m_i, \frac{1}{m_i}\right)$ are four distinct points on a circle,

then

(a) $m_1 m_2 m_3 m_4 = 1$

(b) $m_1 m_2 m_3 m_4 = -1$

(c) $m_1 m_2 m_3 m_4 = \frac{1}{2}$

(d) $\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} + \frac{1}{m_4} = \frac{1}{4}$

Sol. Let the points $\left(m_i, \frac{1}{m_i}\right)$, $i = 1, 2, 3, 4$ lie on the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Then, $m_i^2 + \frac{1}{m_i^2} + 2gm_i + \frac{2f}{m_i} + c = 0$, $i = 1, 2, 3, 4$

$$\Rightarrow m_i^4 + 2gm_i^3 + cm_i^2 + 2fm_i + 1 = 0, \quad i = 1, 2, 3, 4$$

$\Rightarrow m_1, m_2, m_3$ and m_4 are the roots of the equation

$$m^4 + 2gm^3 + cm^2 + 2fm + 1 = 0$$

$$\Rightarrow m_1 m_2 m_3 m_4 = \frac{1}{1} = 1$$

Hence, (a) is the correct answer.

Ex 15. Circles are drawn through the points (a, b) and $(b, -a)$, such that the chord joining the two points subtends an angle of 45° at any point of the circumference. Then, the distance between the centres is

(a) $\sqrt{3}$ times the radius of either circle

(b) 2 times the radius of either circle

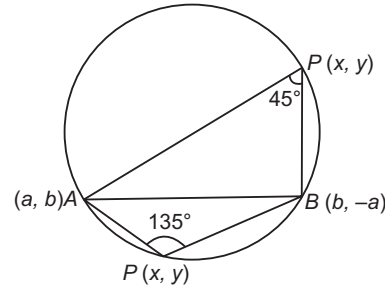
(c) $\frac{1}{\sqrt{2}}$ times the radius of either circle

(d) $\sqrt{2}$ times the radius of either circle

Sol. Let $P(x, y)$ be any point on the circumference of the circle.

Then, $m_1 = \text{Slope of } PA = \frac{b - y}{a - x}$

and $m_2 = \text{Slope of } PB = \frac{-a - y}{b - x}$



We have, $\angle APB = 45^\circ$ or 135°

$$\Rightarrow \frac{m_1 - m_2}{1 + m_1 m_2} = \tan 45^\circ \text{ or } \tan 135^\circ$$

$$\Rightarrow \frac{\frac{b - y}{a - x} - \frac{-a - y}{b - x}}{1 + \frac{b - y}{a - x} \times \frac{-a - y}{b - x}} = \pm 1$$

$$\Rightarrow \frac{(b - y)(b - x) + (a + y)(a - x)}{(a - x)(b - x) - (b - y)(a + y)} = \pm 1$$

$$\Rightarrow x^2 + y^2 = a^2 + b^2$$

$$\Rightarrow \{x - (a + b)\}^2 + \{y - (b - a)\}^2 = a^2 + b^2$$

Hence, the equations of the circles are

$$x^2 + y^2 = a^2 + b^2$$

and $\{x - (a + b)\}^2 + \{y - (b - a)\}^2 = a^2 + b^2$

The centres of these circles are $O(0, 0)$ and $C(a + b, b - a)$.

$$\therefore \text{Distance between the centre} = \sqrt{(a + b)^2 - (a - b)^2}$$

$$= \sqrt{2}\sqrt{a^2 + b^2} = \sqrt{2} \text{ (radius of either circle)}$$

Hence, (d) is the correct answer.

Ex 16. Let a_n , $n = 1, 2, 3, 4$ represent four distinct positive real numbers other than unit such that each pair of the logarithm of a_n and the reciprocal of logarithm denotes a point on a circle, whose centre lies on Y -axis. Then, the product of these four numbers is

(a) 0 (b) 1 (c) 2 (d) 3

Sol. Let $(0, b)$ be the centre and r be the radius of the given circle, then its equation is

$$(x - 0)^2 + (y - b)^2 = r^2$$

$$\Rightarrow x^2 + y^2 - 2yb + b^2 - r^2 = 0 \quad \dots(i)$$

It is given that the point $P_n \left(\log a_n, \frac{1}{\log a_n} \right); n = 1, 2, 3, 4$

lie on the circle given by Eq. (i). Therefore,

$$(\log a_n)^2 + \frac{1}{(\log a_n)^2} - \frac{2b}{\log a_n} + b^2 - r^2 = 0$$

where, $n = 1, 2, 3, 4$

$\Rightarrow \log a_1, \log a_2, \log a_3$ and $\log a_4$ are the roots of the equation

$$\lambda^4 + (b^2 - r^2)\lambda^2 - 2b\lambda + 1 = 0$$

$\therefore$ Sum of the roots = 0

$$\log a_1 + \log a_2 + \log a_3 + \log a_4 = 0$$

$$\Rightarrow \log(a_1 a_2 a_3 a_4) = 0$$

$$\Rightarrow a_1 a_2 a_3 a_4 = 1$$

Hence, (b) is the correct answer.

Ex 17. If $A(a, 0)$ and $B(b, 0)$ are two point on X -axis and a point $C(0, c)$ is on Y -axis. Then, the locus of a point P such that A, B, P and C are concyclic and the coordinates of a point D on the locus of P such that AB is parallel to CD , are

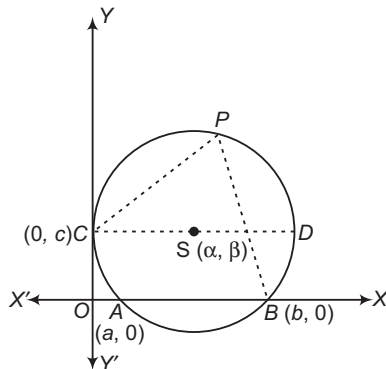
$$(a) \left(x - \frac{a+b}{2} \right)^2 + \left(y - \frac{c^2 + ab}{2c} \right)^2 = \frac{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2}{4c^2}, (a+b, c)$$

$$(b) \left(x - \frac{a+b}{2} \right)^2 + \left(y - \frac{c^2 + ab}{2c} \right)^2 = \frac{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2}{c^2}, (a-b, c)$$

$$(c) \left(x - \frac{a+b}{2} \right)^2 + \left(y - \frac{c^2 + ab}{2c} \right)^2 = \frac{c^2 + a^2 b^2 + b^2 c^2 + c^2 a^2}{4c^2}, (a+c, b)$$

(d) None of the above

Sol.



It is given that A, B, P and C are concyclic points. Let $S(\alpha, \beta)$ be the centre and r be the radius of the circle.

Then, the equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2 \quad \dots(i)$$

Let the coordinates of A, B and C be $(a, 0), (b, 0)$ and $(0, c)$ respectively. As A, B and C lie on Eq.(i).

Therefore, $(a - \alpha)^2 + \beta^2 = r^2$

$$(b - \alpha)^2 + \beta^2 = r^2$$

$$\text{and } \alpha^2 + (c - \beta)^2 = r^2$$

On solving these equations, we get

$$\alpha = \frac{(a+b)}{2}, \beta = \frac{c^2 + ab}{2c}$$

$$\text{and } r = \frac{\sqrt{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2}}{2c}$$

Let (h, k) be the coordinates of P . Then,

$$\Rightarrow (h - \alpha)^2 + (k - \beta)^2 = r^2$$

$$\Rightarrow \left(h - \frac{a+b}{2} \right)^2 + \left(k - \frac{c^2 + ab}{2c} \right)^2 = \frac{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2}{4c^2}$$

Hence, the locus of P is

$$\left(x - \frac{a+b}{2} \right)^2 + \left(y - \frac{c^2 + ab}{2c} \right)^2 = \frac{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2}{4c^2}$$

It is given that CD is parallel to AB . So, the equation of CD is $y = c$.

On putting $y = c$ in Eq. (ii), we get

$$\left(x - \frac{a+b}{2} \right)^2 + \left(c - \frac{c^2 + ab}{2c} \right)^2 = \frac{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2}{4c^2}$$

$$\left(x - \frac{a+b}{2} \right)^2 = \frac{c^4 + a^2 b^2 + b^2 c^2 + c^2 a^2 - (c^2 - ab)^2}{4c^2}$$

$$\Rightarrow \left(x - \frac{a+b}{2} \right)^2 = \left(\frac{a+b}{2} \right)^2$$

$$\Rightarrow x = a + b$$

$\therefore$ The coordinates of D are $(a + b, c)$.

Hence, (a) is the correct answer.

Ex 18. The set of value of a for which the point $(2a, a + 1)$ is an interior point of the larger segment of the circle $x^2 + y^2 - 2x - 2y - 8 = 0$ made by the chord $x - y + 1 = 0$, is

$$(a) \left(\frac{5}{9}, \frac{9}{5} \right) \quad (b) \left(0, \frac{5}{9} \right) \quad (c) \left(0, \frac{9}{5} \right) \quad (d) \left(1, \frac{9}{5} \right)$$

Sol. The point $(2a, a + 1)$ will be an interior point of the larger segment of the circle

$$x^2 + y^2 - 2x - 2y - 8 = 0$$

Since, the point $(2a, a + 1)$ is an interior point and the point $(2a, a + 1)$ and the centre $(1, 1)$ are on the same side of the chord $x - y + 1 = 0$.

$$\therefore (2a)^2 + (a + 1)^2 - 2(2a) - 2(a + 1) - 8 < 0$$

$$\text{and } (2a - a - 1 + 1)(1 - 1 + 1) > 0$$

$$\Rightarrow 5a^2 - 4a - 9 < 0 \quad \text{and} \quad a > 0$$

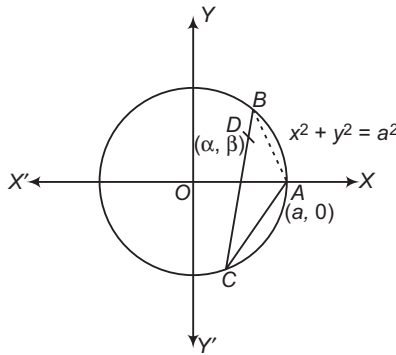
$$\begin{aligned} \Rightarrow (5a-9)(a+1) &< 0 \quad \text{and} \quad a > 0 \\ \Rightarrow -1 < a < \frac{9}{5} \quad \text{and} \quad a > 0 \\ \Rightarrow a &\in \left(0, \frac{9}{5}\right) \end{aligned}$$

Hence, (c) is the correct answer.

Ex 19. Consider the circle $x^2 + y^2 = a^2$. Let $A(a, 0)$ and D be given interior point of the circle. If BC is an arbitrary chord of the circle through point D , then the locus of the centroid of $\triangle ABC$ is

- (a) a circle whose radius is less than $2a/3$
(b) a circle whose radius is greater than $2a/3$
(c) a circle whose radius is equal to $2a/3$
(d) None of the above

Sol. Let $D(\alpha, \beta)$ be an interior point of circle.



Then, $\alpha^2 + \beta^2 < a^2$... (i)

The equation of BC is

$$\frac{x-\alpha}{\cos\theta} = \frac{y-\beta}{\sin\theta}$$

Let $DB = r$. Then, the coordinates of B are

$$(\alpha + r\cos\theta, \beta + r\sin\theta)$$

It lies on $x^2 + y^2 = a^2$

$$\text{Therefore, } (\alpha + r\cos\theta)^2 + (\beta + r\sin\theta)^2 = a^2$$

$$\Rightarrow r^2 + 2r(\alpha\cos\theta + \beta\sin\theta) + \alpha^2 + \beta^2 - a^2 = 0 \quad \dots (ii)$$

This is quadratic in r . So, it gives two values of r one each for point B and C .

Let $DB = r_1$ and $DC = r_2$

Then, r_1, r_2 are the roots of Eq. (ii).

$$\therefore r_1 + r_2 = -2(\alpha\cos\theta + \beta\sin\theta)$$

$$\text{and } r_1 r_2 = \alpha^2 + \beta^2 - a^2 \quad \dots (iii)$$

The coordinates of B and C are $(r_1\cos\theta, r_1\sin\theta)$ and $(r_2\cos\theta, r_2\sin\theta)$ respectively.

Let $C(h, k)$ be the centroid of $\triangle ABC$. Then,

$$3h = a + r_1\cos\theta + r_2\cos\theta$$

$$\text{and } 3k = r_1\sin\theta + r_2\sin\theta$$

$$\Rightarrow (3h-a)^2 + (3k)^2 = (r_1 + r_2)^2$$

Hence, the locus of (h, k) is

$$(3x-a)^2 + (3y)^2 = (r_1 + r_2)^2$$

$$\Rightarrow \left(x - \frac{a}{3}\right)^2 + y^2 = \left(\frac{r_1 + r_2}{3}\right)^2$$

which represents a circle of radius r given by

$$r = \left|\frac{r_1 + r_2}{3}\right|$$

Now,

$$r = \left|\frac{r_1 + r_2}{3}\right|$$

$$\Rightarrow r = \frac{2}{3}|\alpha\cos\theta + \beta\sin\theta| \Rightarrow r \leq \frac{2}{3}\sqrt{\alpha^2 + \beta^2}$$

$$\Rightarrow r < \frac{2}{3}a \quad [\text{using Eq. (i)}]$$

Hence, the locus of the centroid of $\triangle ABC$ is a circle having its centre at $\left(\frac{a}{3}, 0\right)$ and radius less than $\frac{2a}{3}$.

Hence, (a) is the correct answer.

Ex 20. The angular points of a triangle are $A(a\cos\alpha, a\sin\alpha)$, $B(a\cos\beta, a\sin\beta)$ and $C(a\cos\gamma, a\sin\gamma)$. The coordinates of the orthocentre of $\triangle ABC$ are

$$[a(\cos\alpha + \cos\beta + \cos\gamma), a(\sin\alpha + \sin\beta + \sin\gamma)]$$

If A, B, C and D are four points on a circle, then the orthocentre of the $\triangle ABC$, $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$ are

- (a) collinear
(b) concyclic
(c) not concyclic
(d) None of the above

Sol. Clearly, points A, B and C lie on the circle $x^2 + y^2 = a^2$.

Therefore, circumcentre of $\triangle ABC$ is at the origin. The coordinates of the centroid C are

$$\left(\frac{a}{3}(\cos\alpha + \cos\beta + \cos\gamma), \frac{a}{3}(\sin\alpha + \sin\beta + \sin\gamma)\right)$$

Let $O_1(\bar{x}, \bar{y})$ be orthocentre of $\triangle ABC$. Then, $O(0, 0)$,

$$C\left(\frac{a}{3}(\cos\alpha + \cos\beta + \cos\gamma), \frac{a}{3}(\sin\alpha + \sin\beta + \sin\gamma)\right)$$

and $O_1(\bar{x}, \bar{y})$ are collinear and C divides OO_1 in the ratio 1 : 2.

$$\therefore \frac{a}{3}(\cos\alpha + \cos\beta + \cos\gamma) = \frac{1 \times \bar{x} + 2 \times 0}{1+2}$$

$$\text{and } \frac{a}{3}(\sin\alpha + \sin\beta + \sin\gamma) = \frac{1 \times \bar{y} + 2 \times 0}{1+2}$$

$$\Rightarrow \bar{x} = a(\cos\alpha + \cos\beta + \cos\gamma)$$

$$\text{and } \bar{y} = a(\sin\alpha + \sin\beta + \sin\gamma)$$

Thus, the coordinates of the orthocentre are $[a(\cos\alpha + \cos\beta + \cos\gamma), a(\sin\alpha + \sin\beta + \sin\gamma)]$

Let $D(a\cos\delta, a\sin\delta)$ be a point on the circle $x^2 + y^2 = a^2$.

Then, the coordinates of the orthocentre of $\triangle BCD$, $\triangle CDA$ and $\triangle DAB$ are

$$O_2[a(\cos\beta + \cos\gamma + \cos\delta), a(\sin\beta + \sin\gamma + \sin\delta)]$$

$$O_3[a(\cos\gamma + \cos\delta + \cos\alpha), a(\sin\gamma + \sin\delta + \sin\alpha)]$$

$$O_4[a(\cos\delta + \cos\alpha + \cos\beta), a(\sin\delta + \sin\alpha + \sin\beta)]$$

Consider the circle

$$\{x - a(\cos\alpha + \cos\beta + \cos\gamma + \cos\delta)\}^2 + \{y - a(\sin\alpha + \sin\beta + \sin\gamma + \sin\delta)\}^2 = a^2 \quad \dots (i)$$

We observe that this circle passes through O_1, O_2, O_3 and O_4 . Hence, the four orthocentre are concyclic.

Hence, (b) is the correct answer.

Ex 21. Three concentric circles of which biggest is $x^2 + y^2 = 1$, have their radii in AP. If the line $y = x + 1$ cuts all the circles at real and distinct points, then the interval in which the common difference in an AP will lie, is

- (a) $\left[0, \left(1 - \frac{1}{\sqrt{2}}\right)\right]$ (b) $\left(0, \frac{1}{2}\right)$
(c) $(1, 1)$ (d) $\left[0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right]$

Sol. The equation of the biggest circle is $x^2 + y^2 = 1^2$

Clearly, it is centred at $O(0, 0)$ and has radius 1.

Let the radii of the other two circles be $1 - r, 1 - 2r$, where $r > 0$.

Thus, the equations of the concentric circles are

$$x^2 + y^2 = 1 \quad \dots(i)$$

$$x^2 + y^2 = (1 - r)^2 \quad \dots(ii)$$

$$x^2 + y^2 = (1 - 2r)^2 \quad \dots(iii)$$

Clearly, $y = x + 1$ cuts the circle (i) at $(1, 0)$ and $(0, 1)$. This line will cut circles (ii) and (iii) at real and distinct points, if

$$\left|\frac{1}{\sqrt{2}}\right| < 1 - r \quad \text{and} \quad \left|\frac{1}{\sqrt{2}}\right| < 1 - 2r$$

$$\Rightarrow \frac{1}{\sqrt{2}} < 1 - r$$

$$\text{and} \quad \frac{1}{\sqrt{2}} < 1 - 2r$$

$$\Rightarrow r < 1 - \frac{1}{\sqrt{2}}$$

$$\text{and} \quad r < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow r < \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow r \in \left[0, \frac{1}{2}\left(1 - \frac{1}{\sqrt{2}}\right)\right] \quad [\because r > 0]$$

Hence, (d) is the correct answer.

Ex 22. A line meets the coordinate axes at A and B . A circle is circumscribed about the $\triangle OAB$. If the distance of the point A and B from the tangent at the origin to the circle are m and n respectively, then the equation of the circle is

- (a) $x^2 - y^2 + \sqrt{m+n}(\sqrt{m}x + \sqrt{n}y) = 0$
(b) $x^2 + y^2 - \sqrt{m+n}(\sqrt{m}x + \sqrt{n}y) = 0$
(c) $x^2 + y^2 - \sqrt{mn}(\sqrt{m}x + \sqrt{n}y) = 0$
(d) None of the above

Sol. Let $OA = a$ and $OB = b$. Since, $\angle AOB = \frac{\pi}{2}$.

Therefore, AB is a diameter of the circle.

Thus, the equation of the circle is

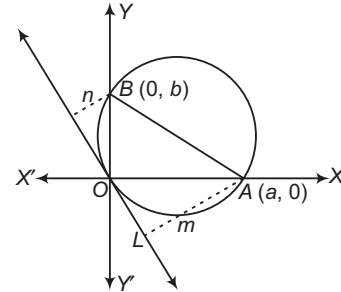
$$x(x - a) + y(y - b) = 0$$

$$\Rightarrow x^2 + y^2 - ax - by = 0 \quad \dots(i)$$

The equation of the tangent at $O(0, 0)$ is

$$0x + 0y - \frac{a}{2}(x + 0) - \frac{b}{2}(y + 0) = 0$$

$$\Rightarrow ax + by = 0 \quad \dots(ii)$$



$\therefore m =$ Length of the perpendicular from $A(a, 0)$ on Eq. (ii)

$$= \frac{a^2}{\sqrt{a^2 + b^2}} \quad \dots(iii)$$

and $n =$ Length of the perpendicular from $B(0, b)$ on Eq. (ii)

$$= \frac{b^2}{\sqrt{a^2 + b^2}} \quad \dots(iv)$$

$$\therefore m + n = \frac{a^2 + b^2}{\sqrt{a^2 + b^2}} = \sqrt{a^2 + b^2} \quad \dots(v)$$

From Eqs. (iii) and (v), we get

$$a = \sqrt{m(m + n)}$$

From Eqs. (iv) and (v) we, get

$$b = \sqrt{n(m + n)}$$

On substituting the values of a and b in Eq. (i), we get that the equation of the required circle is

$$x^2 + y^2 - \sqrt{m(m + n)}x - \sqrt{n(m + n)}y = 0$$

$$\Rightarrow x^2 + y^2 - \sqrt{m + n}(\sqrt{m}x + \sqrt{n}y) = 0$$

Hence, (b) is the correct answer.

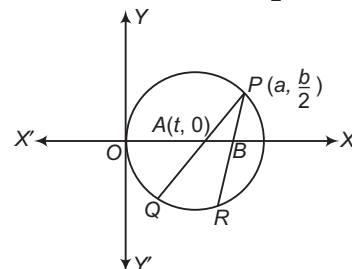
Ex 23. Let a circle be $2x(x - a) + y(2y - b) = 0$, $a \neq 0$, $b \neq 0$. Then, the condition on a and b if two chords each bisected by the X -axis, can be drawn to the circle from $\left(a, \frac{b}{2}\right)$, is

- (a) $a^2 > 2b^2$ (b) $a^2 < 2b^2$
(c) $a^2 = 2b^2$ (d) None of these

Sol. The equation of the circle is

$$2x(x - a) + y(2y - b) = 0$$

$$\Rightarrow x^2 + y^2 - ax - \frac{b}{2}y = 0$$



Let PQ and PR be two chords drawn from $P\left(a, \frac{b}{2}\right)$

such that they are bisected by X -axis.

Let $A(t, 0)$ be the mid-point of PQ . Then, its equation is

$$tx + 0y - \frac{a}{2}(x+t) - \frac{b}{4}(y+0) = t^2 - at \quad [\because T = S]$$

$$\Rightarrow \left(t - \frac{a}{2}\right)x - \frac{b}{4}y = t^2 - \frac{a}{2}t$$

This passes through $P\left(a, \frac{b}{2}\right)$.

$$\text{Therefore, } \left(t - \frac{a}{2}\right)a - \frac{b^2}{8} = t^2 - \frac{a}{2}t$$

$$\Rightarrow t^2 - \frac{3}{2}at + \left(\frac{a^2}{2} + \frac{b^2}{8}\right) = 0$$

This should give two distinct values of t for points A and B .

$$\therefore \frac{9}{4}a^2 - 4\left(\frac{a^2}{2} + \frac{b^2}{8}\right) > 0$$

$$\Rightarrow \frac{a^2}{4} - \frac{b^2}{2} > 0 \Rightarrow a^2 > 2b^2$$

Hence, (a) is the correct answer.

Ex 24. The length of the common chord of the two circles $(x-a)^2 + (y-b)^2 = c^2$ and $(x-b)^2 + (y-a)^2 = c^2$ is

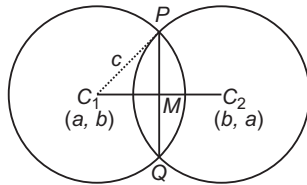
- (a) $\sqrt{4c^2 + 2(a-b)^2}$ (b) $\sqrt{4c^2 - (a-b)^2}$
(c) $\sqrt{4c^2 - 2(a-b)^2}$ (d) $\sqrt{2c^2 - 2(a-b)^2}$

Sol. The equation of two circles are

$$S_1 \equiv (x-a)^2 + (y-b)^2 = c^2 \quad \dots(i)$$

$$\text{and } S_2 \equiv (x-b)^2 + (y-a)^2 = c^2 \quad \dots(ii)$$

The equation of the common chord of these circle is



$$\Rightarrow S_1 - S_2 = 0$$

$$\Rightarrow (x-a)^2 - (x-b)^2 + (y-b)^2 - (y-a)^2 = 0$$

$$\Rightarrow (2x-a-b)(b-a) + (2y-b-a)(a-b) = 0$$

$$\Rightarrow 2x-a-b-2y+b+a = 0$$

$$\Rightarrow x-y = 0$$

Let C_1 and C_2 be centres of circles (i) and (ii) respectively. Then, the coordinates of C_1 and C_2 are (a, b) and (b, a) respectively.

$$\text{Now, } C_1M = \frac{|a-b|}{\sqrt{1+1}} = \frac{|a-b|}{\sqrt{2}}$$

In right angled ΔC_1PM , we have

$$PM = \sqrt{C_1P^2 - C_1M^2} = \sqrt{c^2 - \frac{(a-b)^2}{2}}$$

$$\therefore PQ = 2PM = 2\sqrt{c^2 - \frac{(a-b)^2}{2}} = \sqrt{4c^2 - 2(a-b)^2}$$

Hence, (c) is the correct answer.

Ex 25. P, Q, R and S are the centres as the four circles each of which is cut by a fixed circle orthogonally. If l_1^2, l_2^2, l_3^2 and l_4^2 are the squares of the lengths of tangents to the four circles from a point in their plane, then

$$(a) l_1^2 \Delta QRS - l_2^2 \Delta RSP + l_3^2 \Delta SPQ - l_4^2 \Delta PQR = -1$$

$$(b) l_1^2 \Delta QRS + l_2^2 \Delta RSP - l_3^2 \Delta SPQ + l_4^2 \Delta PQR = 0$$

$$(c) l_1^2 \Delta QRS - l_2^2 \Delta RSP + l_3^2 \Delta SPQ - l_4^2 \Delta PQR = 0$$

(d) None of the above

Sol. Let $x^2 + y^2 + 2g_ix + 2f_iy + c_i = 0, i = 1, 2, 3, 4$ be four circles and $x^2 + y^2 + 2gx + 2fy + c = 0$ be a fixed circle intersecting the four circles orthogonally. Then,

$$2(gg_i + ff_i) = c + c_i, i = 1, 2, 3, 4 \quad \dots(i)$$

Let $O(0, 0)$ be the fixed point in the plane of the circles.

Then, l_i^2 = Square of the length of the tangent from $O(0, 0)$ to

$$x^2 + y^2 + 2g_ix + 2f_iy = 0 \\ = c_i, i = 1, 2, 3, 4 \quad \dots(ii)$$

From Eqs. (i) and (ii), we have

$$2(gg_i + ff_i) = c + l_i^2; i = 1, 2, 3, 4$$

$$\Rightarrow 2gg_1 + 2ff_1 - c - l_1^2 = 0$$

$$2gg_2 + 2ff_2 - c - l_2^2 = 0$$

$$2gg_3 + 2ff_3 - c - l_3^2 = 0$$

$$\text{and } 2gg_4 + 2ff_4 - c - l_4^2 = 0$$

Eliminating g, f and c from these four equations, we get

$$\begin{vmatrix} g_1 & f_1 & -1 & -l_1^2 \\ g_2 & f_2 & -1 & -l_2^2 \\ g_3 & f_3 & -1 & -l_3^2 \\ g_4 & f_4 & -1 & -l_4^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} l_1^2 & g_1 & f_1 & 1 \\ l_2^2 & g_2 & f_2 & 1 \\ l_3^2 & g_3 & f_3 & 1 \\ l_4^2 & g_4 & f_4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow l_1^2 \begin{vmatrix} g_2 & f_2 & 1 \\ g_3 & f_3 & 1 \\ g_4 & f_4 & 1 \end{vmatrix} - l_2^2 \begin{vmatrix} g_1 & f_1 & 1 \\ g_3 & f_3 & 1 \\ g_4 & f_4 & 1 \end{vmatrix} + l_3^2 \begin{vmatrix} g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \\ g_4 & f_4 & 1 \end{vmatrix} - l_4^2 \begin{vmatrix} g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \\ g_3 & f_3 & 1 \end{vmatrix} = 0 \quad \dots(iii)$$

The coordinates of the centre of the four circles are $P(-2g_1-2f_1), Q(-2g_2-2f_2), R(-2g_3-2f_3)$ and $S(-2g_4-2f_4)$.

$\therefore$ Area of ΔPQR

$$= \frac{1}{2} \begin{vmatrix} -2g_1 & -2f_1 & 1 \\ -2g_2 & -2f_2 & 1 \\ -2g_3 & -2f_3 & 1 \end{vmatrix} = 2 \begin{vmatrix} g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \\ g_3 & f_3 & 1 \end{vmatrix}$$

$$\Rightarrow \begin{vmatrix} g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \\ g_3 & f_3 & 1 \end{vmatrix} = 2\Delta PQR$$

Similarly, we have

$$\begin{vmatrix} g_2 & f_2 & 1 \\ g_3 & f_3 & 1 \\ g_4 & f_4 & 1 \end{vmatrix} = 2\Delta QRS, \quad \begin{vmatrix} g_1 & f_1 & 1 \\ g_3 & f_3 & 1 \\ g_4 & f_4 & 1 \end{vmatrix} = 2\Delta RSP$$

and $\begin{vmatrix} g_1 & f_1 & 1 \\ g_2 & f_2 & 1 \\ g_4 & f_4 & 1 \end{vmatrix} = 2\Delta PQS$

On substituting these values in Eq. (iii), we get

$$l_1^2 \Delta QRS - l_2^2 \Delta RSP + l_3^2 \Delta SPQ - l_4^2 \Delta PQR = 0$$

Hence, (c) is the correct answer.

Ex 26. The line $Ax + By + C = 0$ cuts the circle $x^2 + y^2 + ax + by + c = 0$ at P and Q . The line $A'x + B'y + C' = 0$ cuts the circle $x^2 + y^2 + a'x + b'y + c' = 0$ at R and S . If P, Q, R and S are concyclic, then

(a) $\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

(b) $\begin{vmatrix} a+a' & b+b' & c+c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

(c) $\begin{vmatrix} a & b & c \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$

(d) None of the above

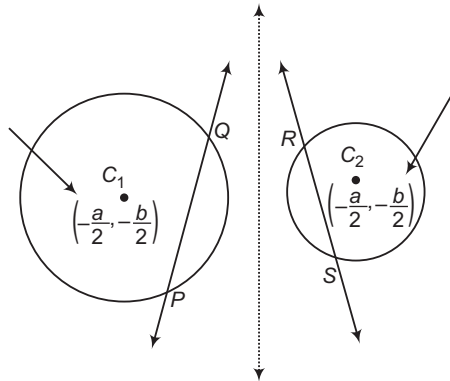
Sol. Let the given circles be $S_1 = 0$ and $S_2 = 0$. Let P, Q, R and S lie on the circle $S_3 = 0$.

Since, $Ax + By + C = 0$ cuts the circle $S_1 = 0$ at P and Q .

Therefore, $Ax + By + C = 0$ is the radical axis of $S_1 = 0$ and $S_3 = 0$

Similarly, $A'x + B'y + C' = 0$ is the radical axis of $S_2 = 0$ and $S_3 = 0$.

The radical axis of $S_1 = 0$ and $S_2 = 0$ is $S_1 - S_2 = 0$.



i.e. $x(a-a') + y(b-b') + c-c' = 0$

Since, the radical axes of three circles, taken in pairs, are concurrent. Therefore,

$$Ax + By + C = 0$$

$$A'x + B'y + C' = 0$$

and $x(a-a') + y(b-b') + c-c' = 0$

are concurrent. Consequently, we have

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Aliter

The equation of a family of circles passing through P and Q is

$$x^2 + y^2 + ax + by + c + \lambda(Ax + By + C) = 0 \quad \dots(i)$$

Similarly, the equation of a family of circles passing through R and S is

$$x^2 + y^2 + a'x + b'y + c' + \mu(A'x + B'y + C') = 0 \quad \dots(ii)$$

If points P, Q, R and S are concyclic points, then Eqs. (i) and (ii) must represent the same circle for some values of λ and μ .

$\therefore$

$$a + \lambda A = a' + \mu A'$$

$$b + \lambda B = b' + \mu B'$$

$$c + \lambda C = c' + \mu C'$$

$$\Rightarrow a - a' + \lambda A - \mu A' = 0 \quad \dots(iii)$$

$$b - b' + \lambda B - \mu B' = 0 \quad \dots(iv)$$

$$c - c' + \lambda C - \mu C' = 0 \quad \dots(v)$$

Eliminating λ and μ from Eqs. (iii), (iv) and (v), we get

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0$$

Hence, (a) is the correct answer.

Ex 27. A circle touches the hypotenuse of a right angled triangle at its middle point and passes through the mid-point of the shorter side. If a and b ($a < b$) are the length of the sides, then the radius is

(a) $\frac{b}{a} \sqrt{a^2 + b^2}$

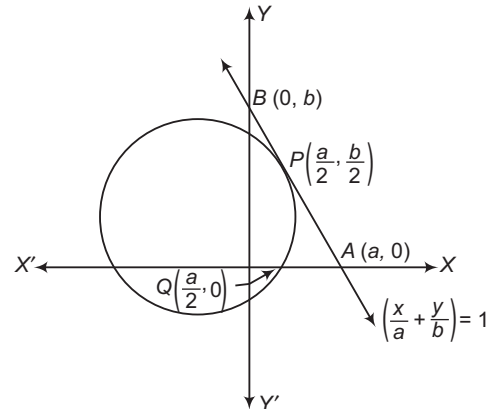
(b) $\frac{b}{2a} \sqrt{a^2 - b^2}$

(c) $\frac{b}{4a} \sqrt{a^2 + b^2}$

(d) None of these

Sol. The equation of the circle touching $\frac{x}{a} + \frac{y}{b} = 1$ at

$$P\left(\frac{a}{2}, \frac{a}{2}\right) \text{ is } \left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \lambda\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0.$$



It passes through $Q\left(\frac{a}{2}, 0\right)$.

Therefore, $\lambda = \frac{b^2}{2}$

On putting the value of λ in Eq. (i), we get

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b}{2}\right)^2 + \frac{b^2}{2}\left(\frac{x}{a} + \frac{y}{b} - 1\right) = 0$$

$$\Rightarrow x^2 + y^2 - \left(a - \frac{b^2}{2a}\right)x - \frac{by}{2} + \frac{a^2 - b^2}{4} = 0$$

Let r be the radius of this circle. Then,

$$r^2 = \frac{1}{4}\left(a - \frac{b^2}{2a}\right)^2 + \frac{b^2}{16} - \left(\frac{a^2 - b^2}{4}\right) = \frac{b^2(a^2 + b^2)}{16a^2}$$

$$\Rightarrow r = \frac{b}{4a}\sqrt{a^2 + b^2}$$

Hence, (c) is the correct answer.

Ex 28. Let P, Q and R be the centres and r_1, r_2, r_3 be the corresponding radii of three circles from a system of coaxial circle, then

$r_1^2 \cdot QR + r_2^2 \cdot RP + r_3^2 \cdot PQ$ is equal to

- (a) $PQ \cdot QR \cdot RP$
 (b) $-PQ \cdot QR \cdot RP$
 (c) $PQ + QR + RP$
 (d) $\frac{PQ}{QR} \times RP$

Sol. Let $x^2 + y^2 + 2gx + c = 0$,

where g is a variable and c is a constant, be a coaxial system of circle having common radical axis as X -axis.

Let $x^2 + y^2 + 2g_i x + c = 0, i = 1, 2, 3$

be three members of the given coaxial system of circles.

Then, the coordinates of their centres and radii are

$P(-g_1, 0), Q(-g_2, 0), R(-g_3, 0)$

$r_1^2 = g_1^2 - c, r_2^2 = g_2^2 - c$

and $r_3^2 = g_3^2 - c$

Now, $r_1^2 \cdot QR + r_2^2 \cdot RP + r_3^2 \cdot PQ$

$$= (g_1^2 - c)(g_2 - g_3) + (g_2^2 - c)(g_3 - g_1) + (g_3^2 - c)(g_1 - g_2)$$

$$= g_1^2(g_2 - g_3) + g_2^2(g_3 - g_1) + g_3^2(g_1 - g_2) - c\{(g_2 - g_3) + (g_3 - g_1) + (g_1 - g_2)\}$$

$$= -(g_1 - g_2)(g_2 - g_3)(g_3 - g_1)$$

$$= -PQ \cdot QR \cdot RP$$

Hence, (b) is the correct answer.

Ex 29. The limiting points of the coaxial system of circles given by $x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$ subtend a right angle at the origin, if

- (a) $-\frac{c}{g^2} + \frac{k}{f^2} = 2$ (b) $\frac{c}{g^2} + \frac{k}{f^2} = -2$
 (c) $\frac{c}{g^2} - \frac{k}{f^2} = 2$ (d) $\frac{c}{g^2} + \frac{k}{f^2} = 2$

Sol. The equation representing the coaxial system of circles is

$$x^2 + y^2 + 2gx + c + \lambda(x^2 + y^2 + 2fy + k) = 0$$

$$\Rightarrow x^2 + y^2 + \frac{2g}{1+\lambda}x + \frac{2f\lambda}{1+\lambda}y + \frac{c+k\lambda}{1+\lambda} = 0 \quad \dots(i)$$

The coordinates of the centre of this circle are

$$\left(-\frac{g}{1+\lambda}, -\frac{f\lambda}{1+\lambda}\right) \quad \dots(ii)$$

$$\text{and radius} = \sqrt{\frac{g^2 + f^2\lambda^2 - (c+k\lambda)(1+\lambda)}{(1+\lambda)^2}}$$

For the limiting points, we have

Radius = 0

$$\Rightarrow g^2 + f^2\lambda^2 - (c+k\lambda)(1+\lambda) = 0$$

$$\Rightarrow \lambda^2(f^2 - k) - \lambda(c+k) + (g^2 - c) = 0 \quad \dots(iii)$$

Let λ_1 and λ_2 be the roots of this equation. Then,

$$\lambda_1 + \lambda_2 = \frac{c+k}{f^2 - k} \quad \text{and} \quad \lambda_1\lambda_2 = \frac{g^2 - c}{f^2 - k} \quad \dots(iv)$$

Thus, the coordinates of limiting points L_1 and L_2 are

$$L_1\left(-\frac{g}{1+\lambda_1}, -\frac{f\lambda_1}{1+\lambda_1}\right)$$

$$\text{and} \quad L_2\left(-\frac{g}{1+\lambda_2}, -\frac{f\lambda_2}{1+\lambda_2}\right) \quad [\text{from Eq. (iv)}]$$

Now, L_1L_2 will subtend a right angle at the origin, if

Slope of $OL_1 \times$ Slope of $OL_2 = -1$

$$\text{Then,} \quad \frac{f\lambda_1}{g} \times \frac{f\lambda_2}{g} = -1$$

$$\Rightarrow f^2\lambda_1\lambda_2 = -g^2$$

$$\Rightarrow f^2\left(\frac{g^2 - c}{f^2 - k}\right) = -g^2$$

$$\Rightarrow f^2(g^2 - c) + g^2(f^2 - k) = 0$$

$$\Rightarrow \frac{c}{g^2} + \frac{k}{f^2} = 2$$

Hence, (d) is the correct answer.

Type 2. More than One Correct Option

Ex 30. The coordinates of the point(s) on the line $x + y = 5$, which is/are equidistant from the lines $|x| = |y|$, is/are

- (a) (5, 0)
 (b) (0, 5)
 (c) (-5, 0)
 (d) (0, -5)

Sol. $|x| = |y| \Rightarrow x + y = 0$ and $x - y = 0$

Bisector of the angles between these are

$$\frac{x - y}{\sqrt{2}} = \pm \frac{x + y}{\sqrt{2}} \Rightarrow y = 0 \quad \text{and} \quad x = 0$$

By solving $y = 0, x + y = 5$, we get (5, 0)

and $x = 0, x + y = 5$, we get (0, 5)

Hence, (a) and (b) are the correct answers.

Ex 31. The tangents drawn from the origin to the circle $x^2 + y^2 + 2gx + 2fy + f^2 = 0$ are perpendicular, if

- (a) $g = f$ (b) $g = -f$
(c) $g = 2f$ (d) $2g = f$

Sol. Radius of the given circle $= |g|$. For perpendicular tangents, origin should lie on the director circle, i.e. distance of origin from the centre of the given circle $= |g| = \sqrt{2}$

$$\Rightarrow g^2 + f^2 = 2g^2 \Rightarrow g = \pm f$$

Hence, (a) and (b) are the correct answers.

Ex 32. If $(a, 0)$ is a point on a diameter of the circle $x^2 + y^2 = 4$, then $x^2 - 4x - a^2 = 0$ has

- (a) exactly one real root in $(-1, 0]$
(b) exactly one real root in $[2, 5]$
(c) distinct roots greater than -1
(d) distinct roots less than 5

Sol. Since, $(a, 0)$ is a point on the diameter of the circle $x^2 + y^2 = 4$.

Maximum value of a^2 is 4.

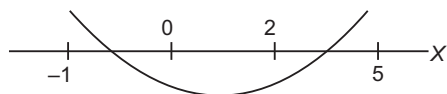
Let $f(x) = x^2 - 4x - a^2$

Clearly, $f(-1) = 5 - a^2 > 0$,

$$f(2) = -(a^2 + 4) < 0$$

$$f(0) = -a^2 < 0 \text{ and } f(5) = 5 - a^2 > 0$$

Graph of $f(x)$ will be as shown as

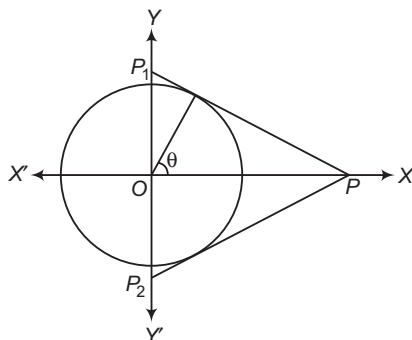


Hence, (a), (b), (c) and (d) are the correct answers.

Ex 33. Tangents are drawn to the circle $x^2 + y^2 = 50$ from a point P lying on the X -axis. These tangents meet the Y -axis at points P_1 and P_2 . Possible coordinates of P so that area of ΔPP_1P_2 is minimum, are

- (a) $(10, 0)$ (b) $(10\sqrt{2}, 0)$
(c) $(-10, 0)$ (d) $(-10\sqrt{2}, 0)$

Sol.



$$OP = 5\sqrt{2} \sec \theta$$

$$OP_1 = 5\sqrt{2} \csc \theta$$

$$\Delta PP_1P_2 = \frac{100}{\sin 2\theta}$$

$$(\Delta PP_1P_2)_{\min} = 100 \Rightarrow \theta = \frac{\pi}{4}$$

$$\Rightarrow OP = 10$$

$$\Rightarrow P = (10, 0), (-10, 0)$$

Hence, (a) and (c) are the correct answers.

Ex 34. Consider the circle

$$x^2 + y^2 - 10x - 6y + 30 = 0.$$

Let O be the centre of the circle and tangent at $A(7, 3)$ and $B(5, 1)$ meet at C . Let $S = 0$ represents family of circles passing through A and B , then

- (a) area of quadrilateral $OACB = 4$
(b) the radical axis for the family of circles $S = 0$ is $x + y = 10$
(c) the smallest possible circle of the family $S = 0$ is $x^2 + y^2 - 12x - 4y + 38 = 0$
(d) the coordinates of point C are $(7, 1)$

Sol. Coordinates of O are $(5, 3)$ and radius $= 2$

Equation of tangent at $A(7, 3)$ is

$$(7x + 3y - 59x + 7) - 3(y + 3) + 30 = 0$$

$$\text{i.e. } 2x - 15 = 0 \Rightarrow x = 7$$

Equation of tangent at $B(5, 1)$ is

$$5x + y - 5(x + 5) - 3(y + 1) + 30 = 0$$

$$\text{i.e. } -2y + 2 = 0 \text{ i.e. } y = 1$$

$\therefore$ Coordinates of C are $(7, 1)$.

$\therefore$ Area of $OACB = 4$

Equation of AB is $x - y = 4$ (radical axis)

Equation of the smallest circle is

$$(x - 7)(x - 5) + (y - 3)(y - 1) = 0$$

$$\text{i.e. } x^2 + y^2 - 12x - 4y + 38 = 0$$

Hence, (a), (c) and (d) are the correct answers.

Ex 35. The range of values of a such that the angle θ between the pair of tangents drawn from $(a, 0)$ to the circle $x^2 + y^2 = 1$ satisfies $\frac{\pi}{2} < \theta < \pi$, lies in

- (a) $(1, 2)$ (b) $(1, \sqrt{2})$
(c) $(-\sqrt{2}, -1)$ (d) $(-\sqrt{2}, -1) \cup (1, \sqrt{2})$

Sol. Equation of pair of tangent by $SS' = T^2$ is

$$(a^2 - 1)y^2 - x^2 + 2ax - a^2 = 0$$

If θ is the angle between the tangents, then

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b} = \frac{2\sqrt{-(a^2 - 1)(-1)}}{a^2 - 2} = \frac{2\sqrt{(a^2 - 1)}}{a^2 - 2}$$

If θ lies in II quadrant, then $\tan \theta < 0$

$$\therefore \frac{2\sqrt{(a^2 - 1)}}{a^2 - 2} < 0$$

$$\Rightarrow a^2 - 1 > 0 \text{ and } a^2 - 2 < 0$$

$$\Rightarrow |a| > 1 \text{ and } |a| < \sqrt{2}$$

$$\Rightarrow a \in (-\sqrt{2}, -1) \cup (1, \sqrt{2})$$

Hence, (c) and (d) are the correct answers.

- Ex 36.** If the line $3x - 4y - k = 0$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b) , then $k + a + b$ can be equal to
 (a) 20 (b) 22
 (c) -30 (d) -28

Sol. Since, the given line touches the given circle, then length of the perpendicular from the centre $(2, 4)$ of the circle from the line $3x - 4y - k = 0$ is equal to the radius $\sqrt{4 + 16 + 5} = 5$ of the circle.

$$\Rightarrow \frac{3 \times 2 - 4 \times 4 - k}{\sqrt{9 + 16}} = \pm 5$$

$$\Rightarrow k = 15 \text{ or } -35$$

Now, equation of the tangent at (a, b) to the given circle is

$$xa + yb - 2(x + a) - 4(y + b) - 5 = 0$$

$$\Rightarrow (a - 2)x + (b - 4)y - (2a + 4b + 5) = 0$$

If it represents the given line $3x - 4y - k = 0$

$$\text{Then, } \frac{a-2}{3} = \frac{b-4}{-4} = \frac{2a+4b+5}{k} = \lambda \quad [\text{say}] \dots (i)$$

$$\text{Then, } a = 3\lambda + 2, b = 4 - 4\lambda$$

$$\text{and } 2a + 4b + 5 = k\lambda$$

$$\Rightarrow 2(3\lambda + 2) + 4(4 - 4\lambda) + 5 = 15\lambda \quad [\text{if } k = 15]$$

$$\Rightarrow \lambda = 1$$

$$\Rightarrow a = 5, b = 0$$

$$\text{and } k + a + b = 20$$

Again, if $k = -35$, from Eq. (i),

$$25 - 10\lambda = -35\lambda \Rightarrow \lambda = -1 \Rightarrow a = -1$$

$$b = 8 \text{ and } k + a + b = -35 - 1 + 8 = -28$$

Hence, (a) and (d) are the correct answers.

- Ex 37.** If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts each of the circles $x^2 + y^2 - 4 = 0$
 $x^2 + y^2 - 6x - 8y + 10 = 0$
 and $x^2 + y^2 + 2x - 4y - 2 = 0$
 at the extremities of a diameter, then
 (a) $c = -4$
 (b) $g + f = c - 1$
 (c) $g^2 + f^2 - c = 17$
 (d) $gf = 6$

Sol. Let $S_1 = x^2 + y^2 + 2gx + 2fy + c = 0$

$$S_2 = x^2 + y^2 - 4 = 0$$

$$S_3 = x^2 + y^2 - 6x - 8y + 10 = 0$$

$$S_4 = x^2 + y^2 + 2x - 4y - 2 = 0$$

Equation of the common chords

$$S_1 - S_2 = 0, S_1 - S_3 = 0, S_1 - S_4 = 0.$$

Satisfy these equations by the coordinates of the centre of corresponding circles.

Hence, (a), (b), (c) and (d) are the correct answers.

- Ex 38.** If the area of the quadrilateral formed by the tangents from the origin to the circle $x^2 + y^2 + 6x - 10y + c = 0$ and the radii corresponding to the points of contact is 15, then a value of c is
 (a) 9
 (b) 4
 (c) 5
 (d) 25

Sol. Area of the quadrilateral $= \sqrt{c} \times \sqrt{9 + 25 - c} = 15$

$$\therefore c = 9, 25$$

Hence, (a) and (d) are the correct answers

- Ex 39.** The set of real values of a for which atleast one tangent to the parabola $y^2 = 4ax$ becomes normal to the circle $x^2 + y^2 - 2ax - 4ay + 3a^2 = 0$, is
 (a) $[1, 2]$
 (b) $[\sqrt{2}, 3]$
 (c) R
 (d) ϕ (null set)

Sol. Any tangent of parabola will be of the form $ty = at^2$ at the point $(at^2, 2at)$. If this is normal to the circle, then this will pass through centre of the circle which is $(a, 2a)$.

$$2at = a + at^2$$

$$\Rightarrow t^2 - 2t + 1 = 0$$

$$\Rightarrow t = 1, 1 \text{ for any value of } a.$$

So, the condition satisfies for all real values of a .

Hence, (a), (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 40-49) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The correct choices are

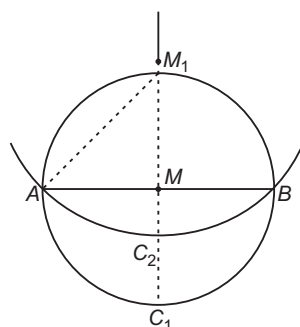
- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

- Ex 40. Statement I** The circle of smallest radius passing through two given points A and B must be of radius $\frac{1}{2} AB$.

Statement II A straight line is a shortest distance between two points.

Sol. Let C_1 be a circle which passes through A, B and C_2 be another circle which passes through A and B , then centres of C_1 and C_2 must lie on perpendicular bisector of AB . Indeed, centre of

C_1 is mid-point M of AB and centre of any other circle lies somewhere else on bisector.



Then, $M_1A > AM$
 [hypotenuse of right angled ΔAMM_1]
 $\Rightarrow$ Radius of $C_2 > 1/2 AB$
 $\Rightarrow C_1$ is the circle whose radius is least.
 Thus, Statement I is true but does not actually follow from Statement II which is certainly true.
 Hence, (b) is the correct answer.

Ex 41. Statement I The equation of chord of the circle $x^2 + y^2 - 6x + 10y - 9 = 0$ which is bisected at $(-2, 4)$ must be $x + y - 2 = 0$.

Statement II In notations, the equation of the chord of the circle $S = 0$ bisected at (x_1, y_1) must be $T = S_1$.

Sol. The Statement II is well known result but if applied to the data given in Statement I will yield
 $5x - 9y + 46 = 0$
 $\Rightarrow$ Statement I is false, Statement II is true.
 Hence, (d) is the correct answer.

Ex 42. Statement I If two circles $x^2 + y^2 + 2g'x + 2f'y = 0$ and $x^2 + y^2 + 2g''x + 2f''y = 0$ touch each other, then $f'g' = fg'$.

Statement II Two circles touch each other, if line joining their centres is perpendicular to all possible common tangents.

Sol. The Statement II is false, since line joining centres may not be parallel to common tangents. The assertion can be proved easily by using distance between centres = sum of radii.
 Hence, (c) is the correct answer.

Ex 43. Statement I The equations of the straight lines joining origin to the points of intersection of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$ is $(y - x)^2 = 0$.

Statement II $y + x = 0$ is a common chord of $x^2 + y^2 - 4x - 2y = 4$ and $x^2 + y^2 - 2x - 4y - 4 = 0$.

Sol. Statement I is true because common chord itself passes through origin.
 Statement II is false (common chord is $x - y = 0$).
 Hence, (c) is the correct answer.

Ex 44. Statement I Two orthogonal circles intersect to generate a common chord which subtends complementary angles at their circumferences.

Statement II Two orthogonal circles intersect to generate a common chord which subtends supplementary angles at their centres.

Sol. Common chord of two orthogonal circles subtends supplementary angles at the centres and so complementary angles on the circumferences of the two circles.
 $\therefore$ Both the statements are correct and Statement II is correct explanation for Statement I.
 Hence, (a) is the correct answer.

Ex 45. Statement I For two non-intersecting circles, direct common tangents subtends a right angle at either of point of intersection of circles with line segment joining the centres of circles.

Statement II If distance between the centres is more than sum of radii, then circles are non-intersecting.

Sol. Draw circle C with direct common tangent as diameter, then both the points of intersection of line joining the centres lie in the interior of the circle C . Therefore, the angles are obtuse angles.
 So, Statement I is false and Statement II is true.
 Hence, (d) is the correct answer.

Ex 46. Tangents are drawn from the point $(17, 7)$ to the circle $x^2 + y^2 = 169$.

Statement I The tangents are mutually perpendicular.

Statement II The locus of the points from which mutually perpendicular tangents can be drawn to the given circle is $x^2 + y^2 = 338$.

Sol. Since, the tangents are perpendicular.
 $\Rightarrow$ Locus of perpendicular tangents to circle $x^2 + y^2 = 169$ is a director circle having equation $x^2 + y^2 = 338$
 Hence, (a) is the correct answer.

Ex 47. Statement I The equation $x^2 + y^2 - 4x + 8y - 5 = 0$ represents a circle.

Statement II The general equation of degree two $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a circle, if $a = b$ and $h = 0$ circle will be real, if

$$g^2 + f^2 - c > 0.$$

Sol. Statement I is true and Statement II is true. Also, Statement II is the correct explanation of Statement I.
 Hence, (a) is the correct answer.

Ex 48. Statement I The least and greatest distances of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are 5 and 15 units respectively.

Statement II A point (x_1, y_1) lies outside a circle $S = x^2 + y^2 + 2gx + 2fy + c = 0$, if $S_1 > 0$, where $S_1 = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

Sol. Centre of the circle = $(2, 1)$

and $r = \sqrt{25} = 5$ units

Distance of $(10, 7)$ from $(2, 1)$ is 10 units.

Thus, required distances are 5 and 15 units, respectively.

Hence, (b) is the correct answer.

Ex 49. Statement I The chord of contact of tangent from three points A, B and C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B and C will be collinear.

Statement II A, B and C always lie on the normal to the circle $x^2 + y^2 = a^2$.

Sol. Equations of chord of contact from $A(x_1, y_1)$ are

$$xx_1 + yy_1 - a^2 = 0$$

$$xx_2 + yy_2 - a^2 = 0$$

$$xx_3 + yy_3 - a^2 = 0$$

i.e. A, B and C are collinear.

Hence, (c) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 50-54) Consider the two circles $C_1 : x^2 + y^2 = r_1^2$ and $C_2 : x^2 + y^2 = r_2^2$ ($r_2 < r_1$). Let A be a fixed point on the circle C_1 , say $A(r_1, 0)$ and B be a variable point on the circle C_2 . The line BA meets the circle C_2 again at C .

Ex 50. The maximum value of BC^2 is

- (a) $4r_1^2$ (b) $4r_2^2$
(c) $4r_2^2 - 4r_1^2$ (d) None of these

Ex 51. The minimum value of BC^2 is

- (a) $4r_1^2$ (b) $4r_2^2$
(c) $4r_2^2 - 4r_1^2$ (d) None of these

Ex 52. The set of values of $OA^2 + OB^2 + BC^2$ is

- (a) $[5r_2^2 - 3r_1^2, 5r_2^2 + r_1^2]$
(b) $[4r_2^2 - 4r_1^2, -4r_1^2]$
(c) $[4r_1^2, 4r_2^2]$
(d) $[5r_2^2 - 3r_1^2, 5r_2^2 + 3r_1^2]$

Ex 53. The locus of the mid-point of AB , O being the origin is

- (a) $\left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{2}$ (b) $\left(x - \frac{r_1}{2}\right)^2 + y^2 = \frac{r_2^2}{4}$
(c) $\left(x - \frac{r_2}{2}\right)^2 + y^2 = \frac{r_1^2}{2}$ (d) $\left(x - \frac{r_2}{2}\right)^2 + y^2 = \frac{r_1^2}{4}$

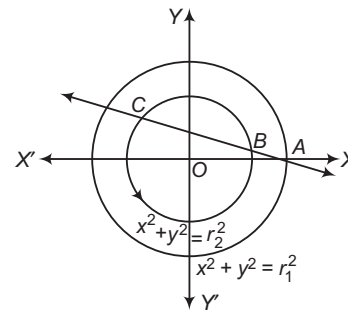
Ex 54. The locus of the mid-point of AB , when BC^2 is maximum, is

- (a) $x^2 + y^2 = r_2^2$ (b) $x^2 + y^2 = (r_1 + r_2)^2$
(c) $x^2 + y^2 = (r_1 - r_2)^2$ (d) None of these

Sol. (Ex. Nos. 50-54)

Let the equation of AB be

$$\frac{x - r_1}{\cos \theta} = \frac{y - 0}{\sin \theta} = r$$



$\therefore$ Coordinates of any point on this line are,

$$(r_1 + r \cos \theta, r \sin \theta)$$

If it lies on $x^2 + y^2 = r_2^2$. Then, we have

$$(r_1 + r \cos \theta)^2 + (r \sin \theta)^2 = r_2^2$$

$$\Rightarrow r^2 + 2rr_1 \cos \theta + r_1^2 - r_2^2 = 0 \quad \dots(i)$$

Let $AB = r_B$ and $AC = r_C$. Then, r_B and r_C are the roots of Eq. (i).

$$\therefore r_B + r_C = -2r_1 \cos \theta \quad \text{and} \quad r_B r_C = r_1^2 - r_2^2$$

$$(BC)^2 = (r_C - r_B)^2 = (r_C + r_B)^2 - 4r_B r_C \\ = 4r_1^2 \cos^2 \theta - 4r_1^2 + 4r_2^2$$

which is maximum, when $\cos^2 \theta = 1$.

$$\therefore (BC)^2_{\max} = 4r_2^2$$

Again, $(BC)^2$ is minimum when $\cos^2 \theta = 0$.

$$\therefore (BC)^2_{\min} = 4r_2^2 - 4r_1^2$$

Now, $OA^2 + OB^2 + BC^2$

$$\Rightarrow r_1^2 + r_2^2 + 4r_1^2 \cos^2 \theta - 4r_1^2 + 4r_2^2$$

$$\Rightarrow 5r_2^2 - 3r_1^2 + 4r_1^2 \cos^2 \theta$$

$$\text{Now, } 0 \leq \cos^2 \theta \leq 1 \Rightarrow 0 \leq 4r_1^2 \cos^2 \theta \leq 4r_1^2$$

$$\Rightarrow 5r_2^2 - 3r_1^2 \leq 5r_2^2 - 3r_1^2 + 4r_1^2 \cos^2 \theta \leq 5r_2^2 + r_1^2$$

$$\Rightarrow OA^2 + OB^2 + BC^2 \in [5r_2^2 - 3r_1^2, 5r_2^2 + r_1^2]$$

Let (h, k) be the mid-point of AB and let (α, β) be the coordinates of B . Then,

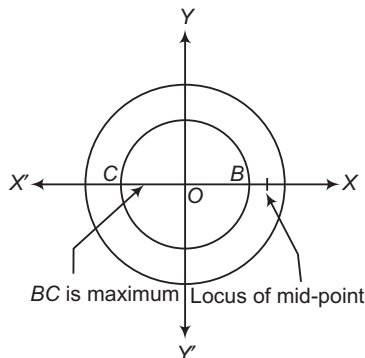
$$\frac{\alpha + r_1}{2} = h \quad \text{and} \quad \frac{\beta}{2} = k$$

$$\Rightarrow \alpha = 2h - r_1 \quad \text{and} \quad \beta = 2k$$

where, (α, β) lies on $x^2 + y^2 = r_2^2$

$$\begin{aligned} \therefore \quad \alpha^2 + \beta^2 &= r_2^2 \\ \Rightarrow \quad (2h - r_1)^2 + (2k)^2 &= r_2^2 \\ \therefore \quad \text{Locus of mid-point of } AB &\text{ is} \\ \left(x - \frac{r_1}{2}\right)^2 + y^2 &= \frac{r_2^2}{4} \end{aligned}$$

Again, the locus of mid-point of AB when BC is maximum is a fixed point M on X -axis shown as



50. (b) 51. (c) 52. (a) 53. (b) 54. (d)

Passage II (Ex. Nos. 55-57) Limiting points of a coaxial system of circles are the members of the system which are of zero radius.

Let (a, b) and (α, β) be two limiting points of a coaxial system of circles. Then, the corresponding point of circles are

$$S_1 \equiv (x - a)^2 + (y - b)^2 = 0$$

$$\text{and } S_2 \equiv (x - \alpha)^2 + (y - \beta)^2 = 0$$

So, the coaxial system of circles is given by

$$S_1 + \lambda S_2 = 0, \lambda \neq -1$$

$$\text{or } \{(x - a)^2 + (y - b)^2\} + \lambda \{(x - \alpha)^2 + (y - \beta)^2\} = 0$$

$$\lambda \neq -1$$

The value of λ is determined from the given condition.

Ex 55. The coordinates of the limiting points of a system of coaxial circles determined by the circles

$$S_1 \equiv x^2 + y^2 + 4x + 2y + 5 = 0$$

$$\text{and } S_2 \equiv x^2 + y^2 + 2x + 4y + 7 = 0 \text{ are}$$

- (a) (2, 1) and (0, 3)
(b) (-2, -1) and (0, -3)
(c) (-2, 1) and (0, -3)
(d) None of the above

Sol. Using, $S_1 + \lambda (S_1 - S_2) = 0$

Coordinates of centre are $(-(2 + \lambda), 1 - \lambda)$

$$\text{and } R = \sqrt{(2 + \lambda)^2 + (1 - \lambda)^2 - 5 + 2\lambda}$$

Now, $R = 0$

$$\Rightarrow (2 + \lambda)^2 + (1 - \lambda)^2 - 5 + 2\lambda = 0$$

$$\Rightarrow 2\lambda^2 + 4\lambda = 0$$

$$\Rightarrow \lambda = 0, -2$$

$\therefore$ Points are $(-2, -1)$ and $(0, -3)$.

Hence, (b) is the correct answer.

Ex 56. The equation of the circle which passes through the origin and belongs to the coaxial system of which the limiting points are (1, 2) and (4, 3), is given by

- (a) $2(x^2 + y^2) - x - 7y = 0$
(b) $x^2 + y^2 - x - 7y = 0$
(c) $3(x^2 + y^2) - x - 7y = 0$
(d) None of the above

Sol. The coaxial circle is

$$(x - 1)^2 + (y - 2)^2 + \lambda \{(x - 4)^2 + (y - 3)^2\} = 0 \quad \dots(i)$$

which will pass through origin.

$$\Rightarrow 1 + 4 + \lambda (16 + 9) = 0$$

$$\Rightarrow \lambda = -\frac{1}{5}$$

Putting $\lambda = -\frac{1}{5}$ in Eq. (i), we get

$$2(x^2 + y^2) - x - 7y = 0$$

Hence, (a) is the correct answer.

Ex 57. The equation of the radical axis of a coaxial system of circles whose limiting points are (2, -1) and (-3, 2) is given by

- (a) $x - 3y + 4 = 0$ (b) $x + 3y - 4 = 0$
(c) $5x - 3y + 4 = 0$ (d) None of these

Sol. Here, $S_1 \equiv (x - 2)^2 + (y + 1)^2 = 0$

$$S_2 \equiv (x + 3)^2 + (y - 2)^2 = 0$$

$\therefore$ Radical axis is $S_2 - S_1 = 0$.

$$\Rightarrow 10x - 6y + 8 = 0$$

$$\text{or } 5x - 3y + 4 = 0$$

Hence, (c) is the correct answer.

Passage III (Ex. Nos. 58-60) Let $y = f(x)$ and $y = g(x)$ be the pair of curves such that the tangents at point with equal abscissae intersect on Y -axis, the normals drawn at points with equal abscissae intersect on X -axis. One curve passes through (1, 1) and the other passes through (2, 3).

Ex 58. The curve $f(x)$ is given by

- (a) $\frac{2}{x} - x$ (b) $2x^2 - \frac{1}{x}$
(c) $\frac{2}{x^2} - x$ (d) None of these

Ex 59. The curve $g(x)$ is given by

- (a) $x - \frac{1}{x}$ (b) $x + \frac{2}{x}$
(c) $x^2 - \frac{1}{x^2}$ (d) None of these

Ex 60. The number of positive integral solutions for $f(x) = g(x)$ is

- (a) 4
(b) 5
(c) 6
(d) None of the above

Sol. (Ex. Nos. 58-60) Let $y = f(x)$ and $y = g(x)$ be the required curves. The equations of the tangents to these two curves at points with equal abscissae x are

$$Y - f(x) = f'(x)(X - x)$$

$$\text{and } Y - g(x) = g'(x)(X - x)$$

These two lines intersect on Y -axis.

$$\therefore Y - f(x) = -xf'(x)$$

$$\text{and } Y - g(x) = -xg'(x)$$

$$\Rightarrow Y = f(x) - xf'(x) = g(x) - xg'(x)$$

$$\Rightarrow f(x) - g(x) = x\{f'(x) - g'(x)\}$$

$$\Rightarrow f(x) - g(x) = x \frac{d}{dx} \{f(x) - g(x)\}$$

$$\Rightarrow \frac{d\{f(x) - g(x)\}}{f(x) - g(x)} = \frac{dx}{x}$$

On integrating both sides we get

$$\log\{f(x) - g(x)\} = \log x + \log c$$

$$f(x) - g(x) = cx \quad \dots(i)$$

The equation of the normals at points with equal abscissae x to the two curves are

$$Y - f(x) = -\frac{1}{f'(x)}(X - x)$$

$$\text{and } Y - g(x) = -\frac{1}{g'(x)}(X - x)$$

These intersect on X -axis.

$$\therefore 0 - f(x) = \frac{1}{f'(x)}(X - x)$$

$$\text{and } 0 - g(x) = -\frac{1}{g'(x)}(X - x)$$

$$\Rightarrow X = x + f(x)f'(x)$$

$$\text{and } X = x + g(x)g'(x)$$

$$\Rightarrow x + f(x)f'(x) = x + g(x)g'(x)$$

$$\Rightarrow f(x)f'(x) = g(x)g'(x)$$

On integrating, we get $\{f(x)\}^2 = \{g(x)\}^2 + C_1$

$$\Rightarrow \{f(x)\}^2 - \{g(x)\}^2 = C_1$$

$$\Rightarrow \{f(x) + g(x)\} Cx = C_1$$

$$\Rightarrow f(x) + g(x) = \frac{C_1}{Cx} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$f(x) = \frac{1}{2} \left(Cx + \frac{C_1}{x} \right) \quad \dots(iii)$$

$$\text{and } g(x) = \frac{1}{2} \left(\frac{C_1}{Cx} + Cx \right) \quad \dots(iv)$$

Since, Eq. (iii) passes through (1, 1) and Eq. (iv) passes through (2, 3).

$$\therefore 2 = C + \frac{C_1}{C} \quad \text{and} \quad 6 = \frac{C_1}{2C} - 2C$$

$$\Rightarrow C = -2 \quad \text{and} \quad C_1 = -8$$

$$\therefore f(x) = -x + \frac{2}{x} \quad \text{and} \quad g(x) = \frac{2}{x} + x$$

For number of positive integral solutions,

$$f(x) = g(x)$$

$$\Rightarrow x + \frac{2}{x} = \frac{2}{x} + x$$

$\therefore x = 0$, no solution.

58. (a)

59. (b)

60. (d)

Passage IV (Ex. Nos. 61-63) A system of circles is said to be coaxial when every pair of the circles has the same radical axis.

Ex 61. The equation of the circle which belong to the coaxial system of circles for which the limiting points are (1, -1), (2, 0) and which passes through the origin, is

$$(a) x^2 + y^2 - 4x = 0 \quad (b) x^2 + y^2 + 4x = 0$$

$$(c) x^2 + y^2 - 4y = 0 \quad (d) x^2 + y^2 + 4y = 0$$

$$\text{Sol. } (x-1)^2 + (y+1)^2 + \lambda [(x-2)^2 + y^2] = 0 \quad \dots(i)$$

Put (0, 0) in Eq. (i) to get λ .

$$\therefore \lambda = -\frac{1}{2}$$

The equation of circle is $x^2 + y^2 + 4y = 0$.

Hence, (d) is the correct answer.

Ex 62. If origin is a limiting point of a coaxial system one of whose member is

$$x^2 + y^2 - 2\alpha x - 2\beta y + c = 0, \text{ then the other limiting point is}$$

$$(a) \left(\frac{c\alpha}{\alpha^2 + \beta^2}, -\frac{c\beta}{\alpha^2 + \beta^2} \right)$$

$$(b) \left(\frac{c\alpha}{\alpha^2 + \beta^2}, \frac{c\beta}{\alpha^2 + \beta^2} \right)$$

$$(c) \left(\frac{c\beta}{\alpha^2 + \beta^2}, -\frac{c\alpha}{\alpha^2 + \beta^2} \right)$$

$$(d) \left(-\frac{c\beta}{\alpha^2 + \beta^2}, -\frac{c\alpha}{\alpha^2 + \beta^2} \right)$$

Sol. Family of circle,

$$x^2 + y^2 - 2\alpha x - 2\beta y + c + \lambda (x^2 + y^2) = 0$$

$$x^2 + y^2 - \frac{2\alpha}{\lambda + 1}x - \frac{2\beta}{\lambda + 1}y + \frac{c}{\lambda + 1} = 0$$

Radius of this circle = 0

$$\Rightarrow \left(\frac{\alpha}{\lambda + 1} \right)^2 + \left(\frac{\beta}{\lambda + 1} \right)^2 - \frac{c}{\lambda + 1} = 0$$

$$\alpha^2 + \beta^2 - c(\lambda + 1) = 0$$

$$\lambda + 1 = \frac{\alpha^2 + \beta^2}{c}$$

$$\text{Put } \left(\frac{\alpha}{\lambda + 1}, \frac{\beta}{\lambda + 1} \right) = \left(\frac{c\alpha}{\alpha^2 + \beta^2}, \frac{c\beta}{\alpha^2 + \beta^2} \right)$$

Hence, (b) is the correct answer.

Ex 63. The equation of the radical axis of the system of coaxial circles

$$x^2 + y^2 - 2ax + 2by + c + 2\lambda(ax - by + 1) = 0 \text{ is}$$

$$(a) ax - by + 1 = 0 \quad (b) bx + ay - 1 = 0$$

$$(c) 2(ax + by) + 1 = 0 \quad (d) 2(bx - ay) + 1 = 0$$

Sol. $S + \lambda L = 0$, $L = 0$ represents the radical axis

$$\therefore ax - by + 1 = 0$$

Hence, (a) is the correct answer.

Type 5. Match the Columns

Ex 64. Match the statement of Column I with values of Column II.

	Column I	Column II
A.	P, Q lie on the X -axis and R, S lie on the Y -axis. Maximum number of parabolas that can be drawn through these points is	p. 3
B.	From a point on the radical axis of circles $S_1 \equiv x^2 + y^2 + 4x - 6y - 12 = 0$, $S_2 \equiv x^2 + y^2 - 6y - 16 = 0$, tangents are drawn to S_2 . Chord of contact passes through $(-25, \alpha^2)$, then $[\alpha]$ can be	q. -3
C.	If (α, β) is the foot of normal drawn from the point $(6, 2)$ on the parabola $y^2 = 4x$, then roots of equation $x^2 = \beta(\alpha - 4)$ is/are	r. 2
D.	A variable circle whose centre lies on $y^2 - 26 = 0$ cuts rectangular hyperbola $xy = 16$ at $\left(4t_i, \frac{4}{t_i}\right), i = 1, 2, 3, 4$, then $\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}$ can be	s. -2

Sol. A. $(ax + by + c)(dx + by + c') + \lambda xy = 0$ represents parabola for maximum two values of λ .

B. Equation of radical axis is $x = -1$.

Let $P \equiv (-1, \beta)$

Equation of chord of contact is

$$(x + 3y + 16) - \beta(y - 3) = 0$$

$$\Rightarrow \alpha^2 = 3$$

$$\Rightarrow [\alpha] = 1 \text{ or } -2$$

C. Equation of normal at (α, β) is

$$2(y - \beta) + \beta(x - \alpha) = 0 \Rightarrow \beta(\alpha - 4) = 4$$

$$D. \frac{4}{4} \left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} \right) = \frac{0 \pm 6}{2} = \pm 3$$

$A \rightarrow r; B \rightarrow s; C \rightarrow s, r; D \rightarrow p, q$

Ex 65. $A(-2, 0)$ and $B(2, 0)$ are the two fixed points and P is a point such that $PA - PB = 2$. Let S be the circle $x^2 + y^2 = r^2$, then match the following :

	Column I	Column II
A.	If $r = 2$, then the number of points P satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	p. 2
B.	If $r = 1$, then the number of points satisfying $PA - PB = 2$ and lying on $x^2 + y^2 = r^2$ is	q. 4
C.	If $r = 2$, then number of common tangents is	r. 0
D.	If $r = \frac{1}{2}$, then number of common tangents is	s. 1

Sol. Locus of point P satisfying $PA - PB = 2$ is a branch of the hyperbola $x^2 - \frac{y^2}{3} = 1$.

For $r = 2$, the circle and the branch of the hyperbola intersect at two points.

For $r = 1$, there is no point of intersection.

If m is the slope of the common tangent, then

$$m^2 - 3 = r^2(1 + m^2) \Rightarrow m^2 = \frac{r^2 + 3}{1 - r^2}$$

Hence, no common tangents for $r > 1$ and two common tangents for $r < 1$.

$A \rightarrow p; B \rightarrow r; C \rightarrow r; D \rightarrow p$

Ex 66. Match the statements of Column I with values of Column II.

	Column I	Column II
A.	The radical axis of two circles	p. is the square of the distance between their centres
B.	The common tangent to two intersecting circles of equal radii	q. is perpendicular to the line joining the centres
C.	The common chord of two intersecting circles	r. is parallel to the line joining the centre
D.	The sum of the squares of the radii of two circles intersecting orthogonally	s. is bisected by the line joining the centres

Sol. Let the equations of two circles be

$$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$$

$$\text{and } x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$$

A. Equation of the radical axis is

$$2(g_1 - g_2)x + 2(f_1 - f_2)y + c_1 - c_2 = 0$$

$$\text{Slope of the radical axis} = -\frac{g_1 - g_2}{f_1 - f_2}$$

and slope of the line joining the centres

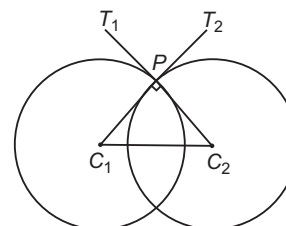
$$(-g_1, -f_1) \text{ and } (-g_2, -f_2) \text{ is } \frac{f_1 - f_2}{g_1 - g_2}$$

So that radical axis is perpendicular to the line joining the centre.

B. Common tangent to the intersecting circles of equal radii will be at the same distance from the centres of the two circles and hence will be parallel to the line joining the centres.

C. Since, the line joining the centres to the middle point of the common chord is perpendicular to the chord, the line joining the centres bisects the chord.

D. Since, $\angle C_1PC_2 = 90^\circ$ [see in the figure]



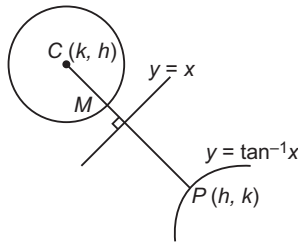
$$(C_1C_2)^2 = (C_1P)^2 + (C_2P)^2$$

$A \rightarrow q; B \rightarrow r; C \rightarrow s; D \rightarrow p$

Type 6. Single Integer Answer Type Questions

Ex 67. Find the number of points on $y = \tan^{-1} x$, $\forall x (0, \pi)$ whose image in $y = x$ is the centre of the circle with radius $\frac{\pi}{2\sqrt{2}}$ unit and which is at a minimum distance of $\frac{\pi}{2\sqrt{2}}$ unit from the circle.

Sol. (1) Let (h, k) be the point on the curve $y = \tan^{-1} x$. Image of (h, k) in $y = x$ is (k, h) which is the centre of a circle of radius $\frac{\pi}{2\sqrt{2}}$.



Given, $PM = \frac{\pi}{2\sqrt{2}}$ [shortest distance]

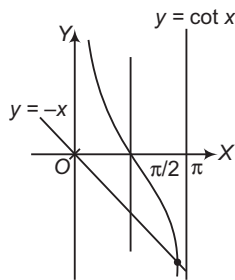
and $CM = \frac{\pi}{2\sqrt{2}}$ [radius of circle]

$$\Rightarrow CP = \left(\frac{\pi}{2\sqrt{2}}\right)^2 = \frac{\pi}{\sqrt{2}}$$

Now, $CP = \sqrt{(h-k)^2 + (k-h)^2} = \frac{\pi}{\sqrt{2}}$

$$\Rightarrow \sqrt{2}|h-k| = \frac{\pi}{\sqrt{2}}$$

$$\Rightarrow |h-k| = \frac{\pi}{2}$$



$$\Rightarrow h-k = \pm \frac{\pi}{2}$$

$$\Rightarrow k = h \pm \frac{\pi}{2}$$

Since, (h, k) lies on $y = \tan^{-1} x$

$$\Rightarrow k = h - \frac{\pi}{2}$$

Now, $h - \frac{\pi}{2} = \tan^{-1} h$

Since, $0 < h < \pi$

$$\Rightarrow -\frac{\pi}{2} < h - \frac{\pi}{2} < \frac{\pi}{2}$$

$$h = \tan\left(h - \frac{\pi}{2}\right) = -\cot h$$

$$\Rightarrow -h = \cot h$$

which gives only one solution for $h \in (0, \pi)$.

So, there is only one point on the curve.

Ex 68. $P(a, b)$ is a point in the first quadrant, circles are drawn through P touching the coordinate axes, such that the length of common chord of these circles is maximum, if possible values of a/b is k_1 and k_2 , then $k_1 + k_2$ is equal to ____.

Sol. (6) Let r be the radius of the circle.

Its equation is $x^2 + y^2 - 2r(x + y) + r^2 = 0$

Since, it passes through $P(a, b)$.

$$a^2 + b^2 - 2r(a + b) + r^2 = 0$$

Solving $r_1 = a + b + \sqrt{2ab}$... (i)

$$r_2 = a + b - \sqrt{2ab}$$

the common chord is

$$S - S' = 0 = 2(x + y) = r_1 + r_2$$

For maximum length of the common chord, it must pass through the centre (r_1, r_2) of the small circle $\frac{a}{b} = 3 \pm 2\sqrt{2}$.

$$\Rightarrow k_1 + k_2 = 6$$

Ex 69. The number of points (x, y) having integral coordinates satisfying the condition $x^2 + y^2 < 25$ is n , then integral part of $\frac{n}{9}$ is equal to ____.

Sol. (7) Since, $x^2 + y^2 < 25$ and x and y are integers, the possible values of x and $y \in (0, \pm 1, \pm 2, \pm 3, \pm 4)$. Thus, x and y can be chosen in 9 ways each and (x, y) can be chosen in 81 ways. However, we have to exclude cases $(\pm 3, \pm 4)$, $(\pm 4, \pm 3)$ and $(\pm 4, \pm 4)$ i.e. 12 cases.

$\therefore$ The number of permissible values

$$= 81 - 12 = 69 = n$$

$$\therefore \left\lceil \frac{n}{9} \right\rceil = 7$$

Ex 70. Let $A(2, -2)$ and $B(2, -2)$ be two points and AB subtends an angle of 45° at any point P in the plane in such a way that area of $\triangle APB$ is 8 sq units, then number of possible position(s) of P is ____.

Sol. (4) $\because AB = 4$ and area of $\triangle APB = 8$

$$\therefore \frac{1}{2} \times 4 \times h = 8$$

$$\Rightarrow h = 4$$

It is clear that, P lies on circle of radius $2\sqrt{2}$ units with AB as its chord, so there are four possible positions of P .

Ex 71. The sum of the square of the length of the chord intercepted by the line $x + y = n$, $n \in N$ on the circle $x^2 + y^2 = 4$ is p , then $\frac{p}{11}$ is equal to _____.

Sol. (2) $AB^2 = 4 AM^2$

$$4 \left(4 - \frac{n^2}{2} \right) = 2(8 - n^2), n \in N$$

$$\Rightarrow n = 1 \text{ or } 2$$

$$\text{Hence, } p = 2(8 - 1^2 + 8 - 2^2) = 22$$

$$\therefore \frac{p}{11} = 2$$

Ex 72. If (α, β) is a point on the circle whose centre is on the X -axis and which touches the line $x + y = 0$ at $(2, -2)$, then $[\alpha]$ is equal to _____.

Sol. (6) If $(a, 0)$ is the centre C and P is $(2, -2)$, then $\angle COP = 45^\circ$

Since, the equation of OP is $x + y = 0$.

$$\therefore OP = 2\sqrt{2} = CP$$

$$\text{Hence, } OC = 4$$

$$\therefore \alpha = OB = OC + CB = 4 + 2\sqrt{2}$$

$$\Rightarrow [\alpha] = 6$$

Ex 73. The number of such points $(a + 1, \sqrt{3}a)$, where a is any integer, lying inside the region bounded by the circles $x^2 + y^2 - 2x - 3 = 0$ and $x^2 + y^2 - 2x - 15 = 0$ is _____.

Sol. (0) The given circles are $(x - 1)^2 + y^2 = 4$ and $(x - 1)^2 + y^2 = 16$. Then, point $(a + 1, \sqrt{3}a)$ lies on the line $y = \sqrt{3}(x - 1)$ whose slope $= \sqrt{3}$ and makes an angle of 60° with X -axis.

$$A = (1 + 2 \cos 60^\circ, 2 \sin 60^\circ) = (2, \sqrt{3})$$

$$B = (1 + 4 \cos 60^\circ, 4 \sin 60^\circ) = (3, 2\sqrt{3})$$

Hence, there is no point on the line segment AB .

Ex 74. If the line $3x - 4y - k = 0$ touches the circle $x^2 + y^2 - 4x - 8y - 5 = 0$ at (a, b) , then the positive integral value of $\frac{k + a + b}{5}$ is equal to _____.

$$\text{Sol. (4) } r = 5, \pm 5 = \frac{(3 \times 2) + (-4 \times 4) - k}{\sqrt{9 + 16}}$$

$$\Rightarrow k = 15 \text{ or } -35$$

$$\text{Take } k = 15$$

Now, equation of tangent at (a, b) is

$$xa + yb - 2(x + a) - 4(y + b) - 5 = 0$$

If it represents the given line $3x - 4y - k = 0$, then

$$\frac{a - 2}{3} = \frac{b - 4}{-4} = \frac{2a + 4b + 5}{k} = \lambda$$

On simplification, $\lambda = 1$

$$\Rightarrow a = 5, b = 0$$

$$\text{and } k + a + b = 20$$

$$\Rightarrow \frac{k + a + b}{5} = \frac{20}{5} = 4$$

Target Exercises

Type 1. Only One Correct Option

- The equation $2x^2 + 2y^2 + 4x + 8y + 15 = 0$ represents
 - a pair of straight line
 - a circle
 - an ellipse
 - None of these
- Let $\phi(x, y) = 0$ be the equation of a circle. If $\phi(0, \lambda) = 0$ has equal roots 2, 2 and $\phi(\lambda, 0) = 0$ has roots $\frac{4}{5}, 5$, then the centre of the circle is
 - $\left(2, \frac{29}{10}\right)$
 - $\left(\frac{29}{10}, 2\right)$
 - $\left(-2, \frac{29}{10}\right)$
 - None of these
- The equation of the image of the circle $(x - 3)^2 + (y - 2)^2 = 1$ by the mirror $x + y = 19$ is
 - $(x - 14)^2 + (y - 13)^2 = 1$
 - $(x - 15)^2 + (y - 14)^2 = 1$
 - $(x - 16)^2 + (y - 15)^2 = 1$
 - $(x - 17)^2 + (y - 16)^2 = 1$
- The number of integral values of λ for which $x^2 + y^2 + \lambda x + (1 - \lambda)y + 5 = 0$ is the equation of a circle whose radius cannot exceed 5, is
 - 14
 - 18
 - 16
 - None of these
- The point $([p + 1], [p])$, where $[]$ denotes the greatest integer function lying inside the region bounded by the circle $x^2 + y^2 - 2x - 15 = 0$ and $x^2 + y^2 - 2x - 7 = 0$, then
 - $p \in [-1, 0) \cup [0, 1) \cup [1, 2)$
 - $p \in [-1, 2) - \{0, 1\}$
 - $p \in (-1, 2)$
 - None of the above
- If $(x, 3)$ and $(3, 5)$ are the extremities of a diameter of a circle with centre at $(2, y)$, then the values of x and y are
 - $x = 1, y = 4$
 - $x = 4, y = 1$
 - $x = 8, y = 2$
 - None of the above
- The line joining $(5, 0)$ to $(10\cos\theta, 12\sin\theta)$ is divided internally in the ratio 2 : 3 at P . If θ varies, then the locus of P is
 - a pair of straight line
 - a circle
 - a straight line
 - None of the above
- Circles are drawn through the point $(2, 0)$ to cut intercept of length 5 units on X -axis. If their centres lie in the first quadrant, then their equation is
 - $x^2 + y^2 - 9x + 2fy + 14 = 0$
 - $3x^2 + 3y^2 + 27x - 2fy + 42 = 0$
 - $x^2 + y^2 - 9x - 2fy + 14 = 0$
 - $x^2 + y^2 - 2fx - 9y + 14 = 0$
- The abscissae of two points A and B are the roots of the equation $x^2 + 2ax - b^2 = 0$ and their ordinates are the roots of the equation $x^2 + 2px - q^2 = 0$. The radius of the circle with AB as diameter, is
 - $\sqrt{a^2 + b^2 + p^2 + q^2}$
 - $\sqrt{a^2 + p^2}$
 - $\sqrt{b^2 + q^2}$
 - None of these
- The equation of a line inclined at an angle $\frac{\pi}{4}$ to the X -axis, such that the two circles $x^2 + y^2 = 4$, $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal lengths on it, is
 - $2x - 2y - 3 = 0$
 - $2x - 2y + 3 = 0$
 - $x - y + 6 = 0$
 - $x - y - 6 = 0$
- In $\triangle ABC$, right angled at A , on the leg AC as diameter, a semi-circle is described. If a chord joins A with the point of intersection D of the hypotenuse and the semi-circle, then the length of AC is equal to
 - $\frac{AB \times AD}{\sqrt{AB^2 + AD^2}}$
 - $\frac{AB \times AD}{AB + AD}$
 - $\sqrt{AB \times AD}$
 - $\frac{AB \times AD}{\sqrt{AB^2 - AD^2}}$
- If OA and OB are equal perpendicular chords of the circle $x^2 + y^2 - 2x + 4y = 0$, then equations of OA and OB , where O is origin, are
 - $3x + y = 0$ and $3x - y = 0$
 - $3x + y = 0$ and $3y - x = 0$
 - $x + 3y = 0$ and $y - 3x = 0$
 - $x + y = 0$ and $x - y = 0$
- If $S \equiv x^2 + y^2 + 2x + 3y + 1 = 0$ and $S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0$ are two circles, then the point $(-3, -2)$ lies
 - inside S' only
 - inside S only
 - inside S and S'
 - outside S and S'

14. The set of values of c , so that the equations $y = |x| + c$ and $x^2 + y^2 - 8|x| - 9 = 0$ have no solution, is
 (a) $(-\infty, -3) \cup (3, \infty)$ (b) $(-3, 3)$
 (c) $(-\infty, -5\sqrt{2}) \cup (5\sqrt{2}, \infty)$ (d) $(5\sqrt{2} - 4, \infty)$
15. If $(1 + ax)^n = 1 + 8x + 24x^2 + \dots$ and a line through $P(\alpha, n)$ cuts the circle $x^2 + y^2 = 4$ in A and B , then $PA \cdot PB$ is equal to
 (a) 4 (b) 8 (c) 16 (d) 32
16. A rhombus is inscribed in the region common to the two circles $x^2 + y^2 - 4x - 12 = 0$ and $x^2 + y^2 + 4x - 12 = 0$ with two of its vertices on the line joining the centres of the circles. The area of the rhombus is
 (a) $8\sqrt{3}$ sq units (b) $4\sqrt{3}$ sq units
 (c) $6\sqrt{3}$ sq units (d) None of these
17. A straight line with slope 2 and y -intercept 5, touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at a point Q . Then, the coordinates of Q are
 (a) $(-6, 11)$ (b) $(-9, -13)$
 (c) $(-10, -15)$ (d) $(-6, -7)$
18. Equation of the smaller circle that touches the circle $x^2 + y^2 = 1$ and passes through the point $(4, 3)$, is
 (a) $5(x^2 + y^2) - 24x - 18y + 25 = 0$
 (b) $x^2 + y^2 - 24x - 18y + 5 = 0$
 (c) $5(x^2 + y^2) - 24x + 18y + 25 = 0$
 (d) $5(x^2 + y^2) + 24x - 18y + 25 = 0$
19. If the tangent at the point P on the circle $x^2 + y^2 + 6x + 6y = 2$ meets the straight line $5x - 2y + 6 = 0$ at a point Q on Y -axis, then the length of PQ is
 (a) 4 units (b) $2\sqrt{5}$ units (c) 5 units (d) $3\sqrt{5}$ units
20. The locus of the centre of a circle which touches externally the circle $x^2 + y^2 - 6x - 6y + 14 = 0$ and also touches Y -axis, is given by the equation
 (a) $x^2 - 6x - 10y + 14 = 0$ (b) $x^2 - 10x - 6y + 14 = 0$
 (c) $y^2 - 6x - 10y + 14 = 0$ (d) $y^2 - 10x - 6y + 14 = 0$
21. If two circles with radii a and b touch each other externally such that θ is the angle between the direct common tangents ($a > b \geq 2$), then
 (a) $\theta = 2\cos^{-1}\left(\frac{a-b}{a+b}\right)$ (b) $\theta = 2\tan^{-1}\left(\frac{a+b}{a-b}\right)$
 (c) $\theta = 2\sin^{-1}\left(\frac{a+b}{a-b}\right)$ (d) $\theta = 2\sin^{-1}\left(\frac{a-b}{a+b}\right)$
22. Equation of the circumcircle of equilateral $\triangle ABC$ is $x^2 + y^2 + 2ax + 2by = 0$. If one vertex of the triangle coincides with origin, then equation of incircle of $\triangle ABC$ is
 (a) $4x^2 + 4y^2 + 8ax + 8by + 2a^2 + 3b^2 = 0$
 (b) $4x^2 + 4y^2 + 8ax + 8by + 3a^2 + 2b^2 = 0$
 (c) $4x^2 + 4y^2 + 8ax + 8by + 2a^2 + 2b^2 = 0$
 (d) $4x^2 + 4y^2 + 8ax + 8by + 3a^2 + 3b^2 = 0$
23. Let PQ and RS be tangents at the extremities of diameter PR of a circle of radius r . If PS and RQ intersect at a point X on the circumference of the circle, then $2r$ equals
 (a) $\sqrt{PQ \cdot RS}$ (b) $\frac{PQ + RS}{2}$
 (c) $\frac{2PQ + RS}{PQ + RS}$ (d) $\frac{\sqrt{PQ^2 + RS^2}}{2}$
24. If the angle between tangents drawn to $x^2 + y^2 + 2gx + 2fy + c = 0$ from $(0, 0)$ is $\frac{\pi}{2}$, then
 (a) $g^2 + f^2 = 3c$ (b) $g^2 + f^2 = 2c$
 (c) $g^2 + f^2 = 5c$ (d) $g^2 + f^2 = 4c$
25. Coordinates of the mid-point of the segment cut by the circle $x^2 + y^2 - 6x + 2y - 54 = 0$ on the line $2x - 5y + 18 = 0$, is
 (a) $(4, 1)$ (b) $(1, 3)$
 (c) $(1, 4)$ (d) $(3, 1)$
26. If AB is a chord of the circle $x^2 + y^2 = r^2$ subtending a right angle at the centre. Then, the locus of the centroid of $\triangle PAB$ as P moves on the circle, is
 (a) a parabola
 (b) a circle
 (c) an ellipse
 (d) a pair of straight line
27. The number of common tangents to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 6x - 8y = 24$, is
 (a) 0 (b) 1 (c) 3 (d) 4
28. Combined equation of tangents drawn to $x^2 + y^2 = 1$ from $(-1, -1)$, is
 (a) $x + y - xy - 1 = 0$
 (b) $x - y + xy + 1 = 0$
 (c) $x - y - xy - 1 = 0$
 (d) $x + y + xy + 1 = 0$
29. The radius of the circle touching the straight lines $x - 2y - 1 = 0$ and $3x - 6y + 7 = 0$, is
 (a) $\frac{3}{\sqrt{5}}$ units (b) $\frac{\sqrt{5}}{3}$ unit
 (c) $\sqrt{5}$ units (d) $\frac{1}{\sqrt{2}}$ unit
30. If a circle having the point $(-1, 1)$ as its centre touches the straight line $x + 2y + 9 = 0$, then the coordinates of the point(s) of contact are
 (a) $(-3, 3)$ (b) $(-3, -3)$
 (c) $(0, 0)$ (d) $\left(\frac{7}{3}, \frac{-17}{3}\right)$
31. If the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 8x + 12 = 0$ touch each other, then the equation of their common tangent at the point, where they touch, is given by
 (a) $x = -2$ (b) $y = 2$
 (c) $x = 2$ (d) $y = -2$

32. The equation of a circle which has a tangent $3x + 4y = 6$ and two normals given by $(x - 1)(y - 2) = 0$ is
 (a) $(x - 3)^2 + (y - 4)^2 = 5^2$
 (b) $x^2 + y^2 - 4x - 2y + 4 = 0$
 (c) $x^2 + y^2 - 2x - 4y + 4 = 0$
 (d) $x^2 + y^2 - 2x - 4y + 5 = 0$
33. If the circles $x^2 + y^2 + 2ax + b = 0$ and $x^2 + y^2 + 2cx + b = 0$ touch each other, then
 (a) $b > 0$ (b) $b < 0$
 (c) $b = 0$ (d) None of these
34. If the straight line $ax + by = 2$; $a, b \neq 0$ touches the circle $x^2 + y^2 - 2x = 3$ and is normal to the circle $x^2 + y^2 - 4y = 6$, then the values of a and b are
 (a) $a = 1, b = 2$ (b) $a = 1, b = -1$
 (c) $a = \frac{-4}{3}, b = 1$ (d) None of these
35. If tangents PA and PB are drawn to $x^2 + y^2 = 4$ from the point $P(3, 0)$, then area of $\triangle PAB$ is equal to
 (a) $\frac{5}{9}\sqrt{5}$ sq units (b) $\frac{1}{3}\sqrt{5}$ sq units
 (c) $\frac{10}{9}\sqrt{5}$ sq units (d) $\frac{20}{3}\sqrt{5}$ sq units
36. If tangents PA and PB are drawn to $x^2 + y^2 = 9$ from the point $P(4, 4)$, then area of $\triangle OAB$, O being the origin, is equal to
 (a) $\frac{9}{32}\sqrt{207}$ sq units (b) $\frac{9}{32}\sqrt{23}$ sq units
 (c) $\frac{9}{38}\sqrt{207}$ sq units (d) $\frac{9}{38}\sqrt{23}$ sq units
37. A circle touches the line $y = x$ at $(4, 4)$ and makes an intercept of length 6 units on the line $x + y = 0$. One possible coordinates of its centre are
 (a) $\left(4 - \frac{5}{\sqrt{2}}, 4 - \frac{\sqrt{5}}{2}\right)$
 (b) $\left(4 + \frac{5}{\sqrt{2}}, 4 + \frac{\sqrt{5}}{2}\right)$
 (c) $\left(4 - \frac{5}{\sqrt{2}}, 4 + \frac{5}{\sqrt{2}}\right)$
 (d) None of the above
38. Consider four circles $(x \pm 1)^2 + (y \pm 1)^2 = 1$. Equation of smaller circle touching these four circles is
 (a) $x^2 + y^2 = 3 - \sqrt{2}$
 (b) $x^2 + y^2 = 6 - 3\sqrt{2}$
 (c) $x^2 + y^2 = 5 - 2\sqrt{2}$
 (d) $x^2 + y^2 = 3 - 2\sqrt{2}$
39. Radius of bigger circle touching the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ and both the coordinate axes, is
 (a) $(3 + 2\sqrt{2})$ units (b) $2(3 + 2\sqrt{2})$ units
 (c) $(6 + 2\sqrt{2})$ units (d) $2(6 + 2\sqrt{2})$ units
40. The two circles $x^2 + y^2 - 2x - 4y = 0$ and $x^2 + y^2 - 8y - 4 = 0$
 (a) intersect each other
 (b) touch each other internally
 (c) touch each other externally
 (d) None of the above
41. Which of the following statement(s) is true regarding the following circles?
 $x^2 + y^2 - 4x - 8y = 0$, $x^2 + y^2 - 8x - 6y + 20 = 0$
 (a) These circles do not touch each other
 (b) These circles touch each other internally
 (c) These circles touch each other externally
 (d) None of the above
42. Circles are drawn having the sides of $\triangle ABC$ as the diameters. Radical centre of these circle is the
 (a) circumcentre of $\triangle ABC$ (b) incentre of $\triangle ABC$
 (c) orthocentre of $\triangle ABC$ (d) centroid of $\triangle ABC$
43. The equation of a circle is $x^2 + y^2 = 25$. The equation of its chord whose middle point is $(1, -2)$, is given by
 (a) $x + 2y + 5 = 0$ (b) $2x - y - 5 = 0$
 (c) $x - 2y - 5 = 0$ (d) $2x + y - 5 = 0$
44. A triangle is formed by the lines whose combined equation is given by $(x + y - 4)(xy - 2x - y + 2) = 0$. The equation of its circumcircle is
 (a) $x^2 + y^2 - 5x - 3y + 8 = 0$
 (b) $x^2 + y^2 - 3x - 5y + 8 = 0$
 (c) $x^2 + y^2 + 2x + 2y - 3 = 0$
 (d) None of the above
45. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB . The equation of the circle with AB as a diameter, is
 (a) $x^2 + y^2 + x + y = 0$
 (b) $x^2 + y^2 = x + y$
 (c) $x^2 + y^2 - 3x + y = 0$
 (d) None of the above
46. If p and q are the longest distance and the shortest distance, respectively of the point $(-7, 2)$ from any point (α, β) on the curve whose equation is $x^2 + y^2 - 10x - 14y - 51 = 0$, then GM of p and q is equal to
 (a) $2\sqrt{11}$ (b) $5\sqrt{5}$
 (c) 13 (d) None of these
47. The number of points on the circle $2x^2 + 2y^2 - 3x = 0$ which are at a distance of 2 units from the point $(-2, 1)$, is
 (a) 2 (b) 0
 (c) 1 (d) None of these
48. If (a, b) is a point on the chord AB of the circle, where the ends of the chord are $A = (2, -3)$ and $B = (3, 2)$, then
 (a) $a \in (-3, 2), b \in (2, 3)$ (b) $a \in (2, 3), b \in (-3, 2)$
 (c) $a \in (-2, 2), b \in (-3, 3)$ (d) None of these

49. A region in XY -plane is bounded by the curve $y = \sqrt{25 - x^2}$ and the line $y = 0$. If the point $(a, a + 1)$ lies in the interior of the region, then
 (a) $a \in (-4, 3)$ (b) $a \in (-\infty, -1) \cup (3, \infty)$
 (c) $a \in (-1, 3)$ (d) None of these
50. The range of values of r for which the point $\left(-5 + \frac{r}{\sqrt{2}}, -3 + \frac{r}{\sqrt{2}}\right)$ is an interior point of the major segment of the circle $x^2 + y^2 = 16$, cut off by the line $x + y = 2$, is
 (a) $(-\infty, 5\sqrt{2})$
 (b) $(4\sqrt{2} - \sqrt{14}, 5\sqrt{2})$
 (c) $(4\sqrt{2} - \sqrt{14}, 4\sqrt{2} + \sqrt{14})$
 (d) None of the above
51. There are two circles whose equations are $x^2 + y^2 = 9$ and $x^2 + y^2 - 8x - 6y + n^2 = 0$, $n \in I$. If the two circles have exactly two common tangents, then the number of possible values of n is
 (a) 2 (b) 8
 (c) 9 (d) None of these
52. The range of λ for which the circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 4\lambda x + 9 = 0$ have two common tangents, is
 (a) $\lambda \in \left(-\frac{13}{8}, \frac{13}{8}\right)$ (b) $\lambda > \frac{13}{8}$ or $\lambda < -\frac{13}{8}$
 (c) $1 < \lambda < \frac{13}{8}$ (d) None of these
53. The range of values of m for which the line $y = mx + 2$ cuts the circle $x^2 + y^2 = 1$ at distinct or coincident points, is
 (a) $(-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$ (b) $[-\sqrt{3}, \sqrt{3}]$
 (c) $[\sqrt{3}, \infty]$ (d) None of these
54. A foot of the normal from the point $(4, 3)$ to a circle is $(2, 1)$ and a diameter of circle has the equation $2x - y = 2$. Then, the equation of the circle is
 (a) $x^2 + y^2 + 2x - 1 = 0$ (b) $x^2 + y^2 - 2x - 1 = 0$
 (c) $x^2 + y^2 - 2y - 1 = 0$ (d) None of these
55. Lines are drawn through the point $P(-2, -3)$ to meet the circle $x^2 + y^2 - 2x - 10y + 1 = 0$. The length of the line segment PA , A being the point on the circle, where the line meets the circle at coincident points, is
 (a) 16 units (b) $4\sqrt{3}$ units
 (c) 48 units (d) None of these
56. A ray of light incident at the point $(-2, -1)$ gets reflected from the tangent at $(0, -1)$ to the circle $x^2 + y^2 = 1$. The reflected ray touches the circle. The equation of the line along which the incident ray moved, is
 (a) $4x - 3y + 11 = 0$ (b) $4x + 3y + 11 = 0$
 (c) $3x + 4y + 11 = 0$ (d) None of these
57. The point P moves in the plane of a regular hexagon such that the sum of the squares of its distances from the vertices of the hexagon is $6a^2$. If the radius of the circumcircle of the hexagon is $r (< a)$, then the locus of P is
 (a) a pair of straight lines
 (b) an ellipse
 (c) a circle of radius $\sqrt{a^2 - r^2}$
 (d) an ellipse of major axis a and minor axis r
58. If the circumcircle of the triangle formed by the line $ax + by + c = 0$, $bx + cy + a = 0$, $cx + ay + b = 0$ passes through the origin, then
 (a) $\begin{vmatrix} b^2 & c^2 & a^2 \\ ab & bc & ca \\ ac & ab & bc \end{vmatrix} = \begin{vmatrix} b^2 & c^2 & a^2 \\ bc & ca & ab \\ ac & ab & bc \end{vmatrix}$
 (b) $\begin{vmatrix} b^2 & c^2 & a^2 \\ ab & bc & ca \\ ac & ab & bc \end{vmatrix} = -\begin{vmatrix} b^2 & c^2 & a^2 \\ bc & ca & ab \\ ac & ab & bc \end{vmatrix}$
 (c) $-\begin{vmatrix} b^2 & c^2 & a^2 \\ ab & bc & ca \\ ac & bc & ab \end{vmatrix} = \begin{vmatrix} b^2 & c^2 & a^2 \\ bc & ca & ab \\ ac & ab & bc \end{vmatrix}$
 (d) None of the above
59. The equation of the sides of a quadrilateral are given by $L_r = a_r x + b_r y + c_r = 0$; $r = 1, 2, 3, 4$. If the quadrilateral is cyclic, then
 (a) $\frac{a_1 a_3 - b_1 b_3}{a_2 a_4 - b_2 b_4} = -\frac{a_1 b_3 + a_3 b_1}{a_2 b_4 + a_4 b_2}$
 (b) $\frac{a_1 a_3 + b_1 b_3}{a_2 a_4 + b_2 b_4} = \frac{a_1 b_3 - a_3 b_1}{a_2 b_4 - a_4 b_2}$
 (c) $\frac{a_1 a_3 - b_1 b_3}{a_2 a_4 - b_2 b_4} = \frac{a_1 b_3 + a_3 b_1}{a_2 b_4 + a_4 b_2}$
 (d) None of the above
60. A point moves in such a manner that the sum of the squares of its distances from the vertices of a triangle is constant. Then, the locus of the point is
 (a) a hyperbola (b) a parabola
 (c) an ellipse (d) a circle
61. A line L_1 intersects X and Y -axes at P and Q respectively and a line L_2 , perpendicular to L_1 cuts X and Y -axes at R and S , respectively. Then, the locus of point of intersection of the lines PS and QR is
 (a) $2x^2 + 2y^2 - ax - by = 0$ (b) $x^2 + y^2 - ax - by = 0$
 (c) $x^2 - y^2 - ax - by = 0$ (d) None of these
62. A circle of constant radius r passes through the origin O and cuts the axes at A and B . Then, the locus of the perpendicular from O to AB is
 (a) $(x^2 + y^2) = 4r^2 x^2 y^2$ (b) $(x^2 - y^2)^3 = 4r^2 x^2 y^2$
 (c) $(x^2 + y^2)^3 = r^2 x^2 y^2$ (d) $(x^2 + y^2)^3 = 4r^2 x^2 y^2$
63. O is a fixed point and a point R moves along a fixed line L not passing through O . If S is a point on OR such that $OR \cdot OS = \lambda^2$, then the locus of S is
 (a) an ellipse (b) a parabola (c) a circle (d) a polygon

64. The circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed in a triangle which has two of its sides along the coordinate axes. If the locus of the circumcentre of the triangle is $x + y - xy + k\sqrt{x^2 + y^2} = 0$, then the value of k is equal to
(a) 2 (b) 1 (c) 3 (d) -2
65. Consider the two circles $C_1 : x^2 + y^2 = r_1^2$ and $C_2 : x^2 + y^2 = r_2^2$ ($r_2 < r_1$). Let A be a fixed point on the circle C_1 , say $A(r_1, 0)$ and B be a variable point on the circle C_2 . Then, line BA meets the circle C_2 again at C . Then, the set of values of $OB^2 + OA^2 + BC^2$ is
(a) $[5r_2^2 - 3r_1^2, 5r_2^2 + r_1^2]$ (b) $[3r_2^2 - 5r_1^2, 5r_2^2 + r_1^2]$
(c) $[5r_2^2 - 3r_1^2, r_2^2 + 5r_1^2]$ (d) None of these
66. Two rods of length a and b slide along the coordinate axes, which are rectangular, in such a way that their ends are always concyclic, then the locus of the centre of the circle passing through these ends, is the curve
(a) $4(x^2 + y^2) = a^2 + b^2$ (b) $(x^2 - y^2) = a^2 - b^2$
(c) $4(x^2 - y^2) = a^2 - b^2$ (d) None of these
67. Through the point of intersection P , which has integral coordinates, of the circles $x^2 + y^2 = 1$ and $x^2 + y^2 + 2x + 4y + 1 = 0$, a common chord APB is drawn terminating on the two circles such that chords AP and BP of the given circles subtend equal angles at the centre. Then, the equation of this chord is
(a) $y = x + 1$ (b) $y = x - 1$
(c) $x^2 + y^2 = -1$ (d) $\frac{1}{x} + \frac{1}{y} = 0$
68. The circle $x^2 + y^2 - 6x - 10y + \lambda = 0$ does not touch or intersect the coordinate axes and the point $(1, 4)$ is inside the circle. Then, the set of values of λ is
(a) $(20, 25)$ (b) $(25, 30)$
(c) $(5, 9)$ (d) $(25, 29)$
69. If $4l^2 - 5m^2 + 6l + 1 = 0$ and the line $lx + my + 1 = 0$ touches a fixed circle. Then, the equation of circle is
(a) $(x + 3)^2 + (y - 0)^2 = 5$ (b) $(x - 3)^2 + (y - 0)^2 = 5$
(c) $(x - 3)^2 + (y - 0)^2 = 0$ (d) None of these
70. If $al^2 - bm^2 + 2dl + 1 = 0$, where a, b, d are fixed real numbers such that $a + b = d^2$ and $lx + my + 1 = 0$ touches a fixed circle. Then, the equation of fixed circle is
(a) $x^2 + y^2 - dx + a = 0$ (b) $2x^2 + y^2 - 2dx + 2a = 0$
(c) $x^2 + y^2 - 2dx + a = 5$ (d) $x^2 + y^2 - 2dx + a = 0$
71. The range of parameter a for which the variable line $y = 2x + a$ lies between the circle $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$ without intersecting or touching either circle, is
(a) $(-15 + 2\sqrt{5}, -\sqrt{5} - 1)$ (b) $(15 + 2\sqrt{5}, \sqrt{5} - 1)$
(c) $(-15 - 2\sqrt{5}, -\sqrt{5} + 1)$ (d) $(-15 + 2\sqrt{5}, \sqrt{5} - 1)$
72. If $P(a, b)$ and $Q(b, a)$ are two points such that $a \neq b$, then the equation of the circle touching OP and OQ and Q , where O is the origin, is
(a) $x^2 + y^2 - 2(x + y)\left(\frac{a^2 - b^2}{a + b}\right) + (a^2 + b^2) = 0$
(b) $x^2 + y^2 - 2(x + y) + (a^2 + b^2) = 0$
(c) $x^2 + y^2 - 2(x + y)\left(\frac{a^2 + b^2}{a + b}\right) + (a^2 + b^2) = 0$
(d) $x^2 + y^2 - 2(x + y) + (a^2 - b^2) = 0$
73. A circle touches the line $y = x$ at a point P such that $OP = 4\sqrt{2}$ units where O is the origin. The circle contains the point $(-10, 2)$ in its interior and the length of its chord on the line $x + y = 0$ is $6\sqrt{2}$ units. Then, the equation of the circle is
(a) $\frac{1}{x^2} + \frac{1}{y^2} + 18x - 2y + 32 = 0$
(b) $\frac{1}{x^2} + \frac{1}{y^2} + 18x - 16y + 32 = 0$
(c) $x^2 + y^2 + 18x - 16y + 32 = 0$
(d) $x^2 + y^2 + 18x - 2y + 32 = 0$
74. A tangent drawn from the point $(4, 0)$ to the circle $x^2 + y^2 = 8$ touches it at a point A in the first quadrant. Then, the coordinates of another point B on the circle such that $AB = 4$, is
(a) $(2, -2)$ and $(2, 2)$ (b) $(2, -2)$ and $(-2, 2)$
(c) $(-2, -2)$ and $(2, -2)$ (d) $(-2, 2)$ and $(-2, -2)$
75. The locus of the point of intersection of tangents to the circle $x^2 + y^2 = a^2$, which are inclined at an angle α with each other, is
(a) $(x^2 + y^2 - 2a^2)\tan^2 \alpha = 4a^2(x^2 + y^2 - a^2)$
(b) $(x^2 + y^2 - a^2)\tan^2 \alpha = 4a^2(x^2 + y^2 - 2a^2)$
(c) $(x^2 + y^2 - 2a^2)^2 \tan^2 \alpha = 4a^2(x^2 + y^2 - 2a^2)$
(d) $(x^2 + y^2 - 2a^2)^2 \tan^2 \alpha = 4a^2(x^2 + y^2 - a^2)$
76. If four points P, Q, R and S in the plane are taken and the square of the length of the tangent from P to the circle on QR as diameter is denoted by $\{P, QR\}$, then
(a) $\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 0$
(b) $\{P, RS\} + \{P, QS\} + \{Q, PR\} + \{Q, RS\} = 0$
(c) $\{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} = 1$
(d) $\{P, RS\} - \{Q, RS\} - \{Q, PR\} - \{P, QS\} = 0$
77. If the chord of contact of tangents from the point (α, β) to the circle $x^2 + y^2 = r_1^2$ is a tangent to the circle $(x - a)^2 + (y - b)^2 = r_2^2$, then
(a) $r_2^2(\alpha^2 - \beta^2) = (r_1^2 - a\alpha - b\beta)^2$
(b) $r_1^2(\alpha^2 - \beta^2) = (r_2^2 - a\alpha - b\beta)^2$
(c) $r_2^2(\alpha^2 + \beta^2) = (r_1^2 - a\alpha - b\beta)^2$
(d) $r_1^2(\alpha^2 + \beta^2) = (r_2^2 - a\alpha - b\beta)^2$
78. The locus of the middle points of the chords of the circle $x^2 + y^2 = a^2$ which subtend a right angle at the centre, is
(a) $(x^2 + y^2) - a^2 = 0$ (b) $2(x^2 + y^2) - 2a^2 = 0$
(c) $2(x^2 - y^2) + a^2 = 0$ (d) $2(x^2 + y^2) - a^2 = 0$

79. The locus of the middle of chords of the circle $x^2 + y^2 = a^2$ which subtend a right angle at $(c, 0)$, is
 (a) $2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$
 (b) $(x^2 + y^2) - 2cx + c^2 - a^2 = 0$
 (c) $2(x^2 + y^2) - cx + c^2 - a^2 = 0$
 (d) $2(x^2 + y^2) - 2cx + 2c^2 - 2a^2 = 0$
80. The locus of the mid-points of the chords of the circle $x^2 + y^2 = 4$ such that the segment intercepted by the chord on the curve $x^2 = 2(x + y)$ subtends a right angle at the origin, is
 (a) $x^2 + y^2 + 2x + 2y = 0$
 (b) $x^2 + y^2 - 2x - 2y = 0$
 (c) $x^2 + y^2 - x - y = 0$
 (d) None of the above
81. Through a fixed point (x_1, y_1) , secants are drawn to the circle $x^2 + y^2 = a^2$. Then, the locus of mid-points of the secants intercepted by the given circle is
 (a) $x^2 + y^2 = xx_1 + yy_1$
 (b) $x^2 + y^2 = xx_1 - yy_1$
 (c) $x^2 + y^2 = yy_1 - xx_1$
 (d) $x^2 - y^2 = yy_1 + xx_1$
82. The intervals of values of a for which the line $y + x = 0$ bisects two chords drawn from a point $\left(\frac{1+\sqrt{2}a}{2}, \frac{1-\sqrt{2}a}{2}\right)$ to the circle $2x^2 + 2y^2 - (1+\sqrt{2}a)x - (1-\sqrt{2}a)y = 0$, is
 (a) $(2, \infty)$
 (b) $(-\infty, -2) \cup (2, \infty)$
 (c) $(-\infty, 2)$
 (d) None of the above
83. The polar of a given point with respect to any one of the circles $x^2 + y^2 - 2kx + c^2 = 0$, (k being a variable) always passes through a fixed point whatever be the value of k . Then, the point of intersection is
 (a) $\left(-x_1, \frac{x_1^2 - c^2}{y_1}\right)$ (b) $\left(x_1, \frac{x_1^2 - c^2}{y_1}\right)$
 (c) $\left(-x_1, \frac{x_1^2 + c^2}{y_1}\right)$ (d) None of these
84. Let C_1 and C_2 be two circles with C_2 lying inside C_1 . If a circle C lying inside C_1 touches C_1 internally and C_2 externally. Then, the locus of the centre of C is
 (a) a circle (b) a parabola
 (c) a hyperbola (d) an ellipse
85. Two circles, each of radius 5 units, touch each other at $(1, 2)$. If the equation of their common tangent is $4x + 3y = 10$, then the equations of the circles are
 (a) $(x + 5)^2 - (y + 5)^2 = 5^2, (x + 3)^2 - (y + 1)^2 = 5^2$
 (b) $(x + 5)^2 + (y + 5)^2 = 5^2, (x + 3)^2 - (y + 1)^2 = 5^2$
 (c) $(x - 5)^2 + (y - 5)^2 = 5^2, (x + 3)^2 + (y + 1)^2 = 5^2$
 (d) None of the above
86. Two circles have centres $(a, 0)$, $(-a, 0)$ and radii b, c respectively such that $a > b > c$. Then, the points of contact of the common tangents to the two circles lie on
 (a) $x^2 + y^2 = a^2 \pm bc$ (b) $x^2 - y^2 = a^2 \pm bc$
 (c) $2x^2 + y^2 = a^2 \pm bc$ (d) None of these
87. The equation of the circle which touches the straight lines $x + y = 2$, $x - y = 2$ and also touches the circle $x^2 + y^2 = 1$, is
 (a) $(x - \sqrt{2})^2 + y^2 = (\sqrt{2} + 1)^2$
 (b) $(x - \sqrt{2})^2 + y^2 = (\sqrt{2} - 1)^2$
 (c) $(x - \sqrt{2})^2 - y^2 = (\sqrt{2} - 1)^2$
 (d) $(x + \sqrt{2})^2 + y^2 = (\sqrt{2} - 1)^2$
88. The equation to the circle cutting orthogonally the three circles $x^2 + y^2 - 2x + 3y - 7 = 0$, $x^2 + y^2 + 5x - 5y + 9 = 0$ and $x^2 + y^2 + 7x - 9y + 29 = 0$, is
 (a) $x^2 + y^2 - 18x - 16y - 4 = 0$
 (b) $x^2 + y^2 - 16x - 18y - 16 = 0$
 (c) $x^2 + y^2 + 16x + 18y - 4 = 0$
 (d) $x^2 + y^2 - 16x - 18y - 4 = 0$
89. The equation of a circle which is coaxial with the circles $x^2 + y^2 + 4x + 2y + 1 = 0$ and $x^2 + y^2 - x + 3y - \frac{3}{2} = 0$ and having its centre on the radical axis of these circles, is
 (a) $4x^2 + 4y^2 + 6x + 10y - 1 = 0$
 (b) $4x^2 + y^2 + 6x + 10y - 1 = 0$
 (c) $x^2 + y^2 + 6x + 10y - 1 = 0$
 (d) $4x^2 + 4y^2 - 6x - 10y - 1 = 0$
90. As λ varies, the circles $x^2 + y^2 + 2ax + 2by + 2\lambda(ax - by) = 0$ form a coaxial system. Then, the equation of the circle which are orthogonal of the circles of the above system, is
 (a) $x^2 + y^2 + c\left(\frac{x}{a} + \frac{y}{b}\right) + c = 0$
 (b) $x^2 + y^2 + \frac{c}{2}\left(\frac{x}{a} - \frac{y}{b}\right) + c = 0$
 (c) $x^2 + y^2 + c\left(\frac{x}{a} - \frac{y}{b}\right) + c = 0$
 (d) $x^2 + y^2 + \frac{c}{2}\left(\frac{x}{a} + \frac{y}{b}\right) + c = 0$
91. The polar of any point with respect to a system of coaxial circles pass through
 (a) a fixed point
 (b) a variable point
 (c) radical axis
 (d) None of the above
92. If P, Q and R are the centres of three circles from a coaxial system of vectors and l_1, l_2, l_3 are the lengths of the tangents to them from a fixed point. Then, $l_1^2 \cdot QR + l_2^2 \cdot RP + l_3^2 \cdot PQ$ is equal to
 (a) 1 (b) 2
 (c) 0 (d) None of these

93. One possible equation of the chord of $x^2 + y^2 = 100$ that passes through $(1, 7)$ and subtends an angle $\frac{2\pi}{3}$ at the origin, is
 (a) $3y + 4x - 25 = 0$ (b) $x + y - 8 = 0$
 (c) $3x + 4y - 31 = 0$ (d) None of these
94. Tangents are drawn to $x^2 + y^2 = 1$ from any arbitrary point P on the circle $C_1 : x^2 + y^2 - 4 = 0$. These tangent meets the circle C_1 again in A and B . Locus of point of intersection of tangents drawn to C_1 at A and B is
 (a) $x^2 + y^2 = 10$ (b) $x^2 + y^2 = 16$
 (c) $x^2 + y^2 = 5$ (d) $x^2 + y^2 = 25$
95. If the circle $x^2 + y^2 + 2a_1x + c = 0$ lies completely inside the circle $x^2 + y^2 + 2a_2x + c = 0$, then
 (a) $a_1a_2 > 0, c < 0$ (b) $a_1a_2 > 0, c > 0$
 (c) $a_1a_2 < 0, c < 0$ (d) $a_1a_2 < 0, c > 0$
96. If the angle between tangents drawn to $x^2 + y^2 - 2x - 4y + 1 = 0$ at the points, where it is cut by the line $y = 2x + c$, is $\frac{\pi}{2}$. Then,
 (a) $|c| = \sqrt{5}$ (b) $|c| = 2\sqrt{5}$ (c) $|c| = \sqrt{10}$ (d) $|c| = 2\sqrt{10}$
97. A circle touches the lines $y = \frac{x}{\sqrt{3}}$, $y = x\sqrt{3}$ and has unit radius. If the centre of this circle lies in the first quadrant, then possible equation of this circle is
 (a) $x^2 + y^2 - 2x(\sqrt{3} + 1) - 2y(\sqrt{3} + 1) + 8 + 4\sqrt{3} = 0$
 (b) $x^2 + y^2 - 2x(1 + \sqrt{3}) - 2y(1 + \sqrt{3}) + 5 + 4\sqrt{3} = 0$
 (c) $x^2 + y^2 - 2x(1 + \sqrt{3}) - 2y(1 + \sqrt{3}) + 7 + 4\sqrt{3} = 0$
 (d) $x^2 + y^2 - 2x(1 + \sqrt{3}) - 2y(1 + \sqrt{3}) + 6 + 4\sqrt{3} = 0$
98. The equation of the circle which passes through the point $(2a, 0)$ and whose radical axis with respect to the circle $x^2 + y^2 = a^2$ is the line $x = \frac{a}{2}$, is
 (a) $x^2 - y^2 + 2ax = 0$ (b) $x^2 + y^2 - 2ax = 0$
 (c) $x^2 + y^2 + 2ax = 0$ (d) None of these
99. The equation of the system of coaxial circles that are tangent at $(\sqrt{2}, 4)$ to the locus of the point of intersection of two mutually perpendicular tangents to the circle $x^2 + y^2 = 9$, is
 (a) $x^2 + y^2 + 9 - \lambda(\sqrt{2}x + 4y - 18) = 0$
 (b) $x^2 + y^2 + 9 + \lambda(\sqrt{2}x + 4y - 18) = 0$
 (c) $x^2 + y^2 + \lambda(\sqrt{2}x + 4y - 18) = 0$
 (d) $x^2 + y^2 - 9 + \lambda(\sqrt{2}x + 4y - 18) = 0$
100. The equation of the circle passing through the origin and cutting the circles $x^2 + y^2 - 4x + 6y + 10 = 0$ and $x^2 + y^2 + 12y + 6 = 0$ at right angles, is
 (a) $2(x^2 + y^2) - 7x + 2y = 0$ (b) $(x^2 + y^2) - 7x + 2y = 0$
 (c) $2(x^2 + y^2) + 7x - 2y = 0$ (d) $(x^2 + y^2) + 7x - 2y = 0$
101. The equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 6x + 4y - 3 = 0$, passes through $(3, 0)$ and touches Y -axis, is
 (a) $2x^2 + 2y^2 - 6x - 6y + 9 = 0$
 (b) $x^2 + 2y^2 - 6x - 6y + 9 = 0$
 (c) $x^2 + y^2 - 6x - 6y + 9 = 0$
 (d) $x^2 + y^2 - x - y + 9 = 0$
102. The equation of the radical axis of the circle $2x^2 + 2y^2 + 14x - 18y + 15 = 0$ and $4x^2 + 4y^2 - 3x - y + 5 = 0$, is
 (a) $31x + 35y - 25 = 0$ (b) $31x - 35y + 25 = 0$
 (c) $35x + 31y - 25 = 0$ (d) $35x - 31y + 25 = 0$
103. The equation of the circle whose diameter is the common chord of the circles $(x - a)^2 + y^2 = a^2$ and $x^2 + (y - b)^2 = b^2$, is
 (a) $(x^2 + y^2)(a^2 + b^2) = ab(bx + ay)$
 (b) $(x^2 + y^2)(a^2 + b^2) = 2ab(bx + ay)$
 (c) $(2x^2 + 2y^2)(a^2 + b^2) = ab(bx + ay)$
 (d) $x^2 + y^2 + a^2 + b^2 - bx - ay = 0$
104. The equation of the circle which touches the line $x - y = 0$ at the origin and bisects the circumference of the circle $x^2 + y^2 + 2y - 3 = 0$, is
 (a) $x^2 + y^2 - 5x - 5y = 0$ (b) $2x^2 + 2y^2 - 5x + 5y = 0$
 (c) $2x^2 + y^2 - 5x + 5y = 0$ (d) $x^2 + y^2 - 5x + 5y = 0$
105. The equation of the circle which passes through the origin and belong to the coaxial system of which the limiting points are $(1, 2)$ and $(4, 3)$, is
 (a) $x^2 + y^2 - x - 7y = 0$ (b) $3(x^2 + y^2) - x - 7y = 0$
 (c) $2(x^2 + y^2) - x - 7y = 0$ (d) None of these
106. The equation of the circle through the intersection of the circles $x^2 + y^2 - 8x - 2y + 7 = 0$ and $x^2 + y^2 - 4x + 10y + 8 = 0$ and that passes through the point $(-1, -2)$, is
 (a) $x^2 + 9y^2 - 40x + 78y + 71 = 0$
 (b) $9x^2 + y^2 - 40x + 78y + 71 = 0$
 (c) $x^2 + y^2 - 40x + 78y + 71 = 0$
 (d) $9x^2 + 9y^2 - 40x + 78y + 71 = 0$
107. The lengths of the tangents from any point of a fixed circle of coaxial system to another fixed circles of the system are in the ratio
 (a) $\sqrt{\frac{g_2 - g_1}{g_3 - g_1}}$ (b) $\sqrt{\frac{g_2 + g_1}{g_3 + g_1}}$ (c) $\sqrt{\frac{g_3 - g_1}{g_2 + g_1}}$ (d) $\sqrt{\frac{g_3 - g_2}{g_2 - g_1}}$
108. The radius of the smallest circle which touches the straight line $3x - y = 6$ at $(1, -3)$ and also touches the line $y = x$, is
 (a) 1 (b) 1.6 (c) 1.49 (d) $\frac{1}{2}$
109. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is touched by the line $y = x$ at point P such that $OP = 6\sqrt{2}$ units, where O is the origin, then the value of c is
 (a) 74 (b) 62 (c) 64 (d) 72

Type 2. More than One Correct Option

- 110.** The equation(s) of circle passing through $(3, -6)$ and touching both the axes, is/are
 (a) $x^2 + y^2 - 6x + 6y + 9 = 0$
 (b) $x^2 + y^2 + 6x - 6y + 9 = 0$
 (c) $x^2 + y^2 + 30x - 30y + 225 = 0$
 (d) $x^2 + y^2 - 30x + 30y + 225 = 0$
- 111.** The equation of the tangents drawn from the origin to the circle $x^2 + y^2 - 2rx - 2hy + h^2 = 0$, is/are
 (a) $x = 0$ (b) $y = 0$
 (c) $(h^2 - r^2)x - 2rhy = 0$ (d) $(h^2 - r^2)x + 2rhy = 0$
- 112.** Equation of a circle with centre $(4, 3)$ touching the circle $x^2 + y^2 = 1$ is
 (a) $x^2 + y^2 - 8x - 6y - 9 = 0$
 (b) $x^2 + y^2 - 8x - 6y + 11 = 0$
 (c) $x^2 + y^2 - 8x - 6y - 11 = 0$
 (d) $x^2 + y^2 - 8x - 6y + 9 = 0$
- 113.** The equation(s) of a tangent to the circle $x^2 + y^2 = 25$ passing through $(-2, 11)$, is/are
 (a) $4x + 3y = 25$ (b) $3x + 4y = 38$
 (c) $24x - 7y + 125 = 0$ (d) $7x + 24y = 230$
- 114.** The centre(s) of the circle(s) passing through the points $(0, 0)$, $(1, 0)$ and touching the circle $x^2 + y^2 = 9$ is/are
 (a) $\left(\frac{3}{2}, \frac{1}{2}\right)$ (b) $\left(\frac{1}{2}, \frac{3}{2}\right)$
 (c) $\left(\frac{1}{2}, 2^{1/2}\right)$ (d) $\left(\frac{1}{2}, -2^{1/2}\right)$
- 115.** Let x, y be real variable satisfying the equation of circle $x^2 + y^2 + 8x - 10y - 40 = 0$.
 If $a = \max\{(x+2)^2 + (y-3)^2\}$
 and $b = \min\{(x+2)^2 + (y-3)^2\}$, then
 (a) $a + b = 18$
 (b) $a + b = 4\sqrt{2}$
 (c) $a - b = 4\sqrt{2}$
 (d) $a \cdot b = 73$
- 116.** Coordinates of the centre of a circle, whose radius is 2 units and which touches the line pair $x^2 - y^2 - 2x + 1 = 0$, are
 (a) $(4, 0)$
 (b) $(1 + 2\sqrt{2}, 0)$
 (c) $(4, 1)$
 (d) $(1, 2\sqrt{2})$
- 117.** Point M moved on the circle $(x-4)^2 + (y-8)^2 = 20$. Then, it broke away from it and moving along a tangent to the circle, cuts the X -axis at the point $(-2, 0)$. The coordinates of a point on the circle at which the moving point broke away, are
 (a) $\left(-\frac{3}{5}, \frac{46}{5}\right)$ (b) $\left(-\frac{2}{5}, \frac{44}{5}\right)$
 (c) $(6, 4)$ (d) $(3, 5)$
- 118.** If PQR is the triangle formed by the common tangents to the circles $x^2 + y^2 + 6x = 0$ and $x^2 + y^2 - 2x = 0$, then
 (a) centroid of ΔPQR is $(1, 0)$
 (b) incentre of ΔPQR is $(1, 0)$
 (c) circumcentre of ΔPQR is $(1, 0)$
 (d) orthocentre of ΔPQR is $(2, 0)$

Type 3. Assertion and Reason

Directions (Q. Nos. 119-127) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

119. Statement I The number of circles that pass through the points $(1, -7)$ and $(-5, 1)$ and of radius 4, is two.

Statement II The centre of any circle that pass through the points A and B lies on the perpendicular bisector of AB .

120. Statement I If the chord of contact of tangent from three points A, B, C to the circle $x^2 + y^2 = a^2$ are concurrent, then A, B, C will be collinear.

Statement II Lines

$(a_1x + b_1y + c_1) + k(a_2x + b_2y + c_2) = 0$ always pass through a fixed point for $k \in R$.

121. Statement I Circles $x^2 + y^2 = 144$ and $x^2 + y^2 - 6x - 8y = 0$ do not have any common tangent.

Statement II If two circles are concentric, then they do not have common tangents.

122. Statement I The least and greatest distances of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are 5 and 15 units, respectively.

Statement II A point (x_1, y_1) lies outside a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$, if $S_1 > 0$, where $S_1 \equiv x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$.

123. Statement I Number of circles passing through $(1, 2)$, $(4, 8)$ and $(0, 0)$ is one.

Statement II Every triangle has one circumcircle.

- 124. Statement I** Let $S_1 : x^2 + y^2 - 10x - 12y - 39 = 0$, $S_2 : x^2 + y^2 - 2x - 4y + 1 = 0$ and $S_3 : 2x^2 + 2y^2 - 20x - 24y + 78 = 0$. The radical centre of these circles taken pairwise is $(-2, -3)$.

Statement II Point of intersection of three radical axes of three circles taken in pairs is known as radical centre.

- 125. Statement I** The equation $x^2 + y^2 - 2x - 2ay - 8 = 0$ represents for different values of a . A system of circle passing through two fixed points lying on X -axis.

Statement II If $S = 0$ is a circle and $L = 0$ is a straight line, then $S + lL = 0$ represents the family of circles passing through the point of intersection of circle and straight line, where l is an arbitrary parameter.

- 126. Statement I** The circles $x^2 + y^2 + 2px + r = 0$, $x^2 + y^2 + 2qy + r = 0$ touch,

$$\text{if } \frac{1}{p^2} + \frac{1}{q^2} = \frac{1}{r}.$$

Statement II Two circles with centres C_1, C_2 and radii r_1, r_2 touch each other, if $r_1 \pm r_2 = C_1C_2$.

- 127. Statement I** The equation of chord of the circle $x^2 + y^2 - 6x + 10y - 9 = 0$ which is bisected at $(-2, 4)$, must be $x + y - 2 = 0$.

Statement II In notations, the equation of the chord of the circle $S = 0$ bisected at (x_1, y_1) must be $T = S_1$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 128 -130) Each side of a square has length 4 units and its centre is at $(3, 4)$. If one of the diagonals is parallel to the line $y = x$.

- 128.** Which of the following is not the vertex of the square?
(a) $(1, 6)$ (b) $(5, 2)$ (c) $(1, 2)$ (d) $(4, 6)$

- 129.** The radius of the circle inscribed in the triangle formed by any three vertices is
(a) $2\sqrt{2}(\sqrt{2} + 1)$
(b) $2\sqrt{2}(\sqrt{2} - 1)$
(c) $2(\sqrt{2} + 1)$
(d) None of the above

- 130.** The radius of the circle inscribed in the triangle formed by any two vertices of square and the centre is
(a) $2(\sqrt{2} - 1)$ (b) $2(\sqrt{2} + 1)$
(c) $\sqrt{2}(\sqrt{2} - 1)$ (d) None of these

Passage II (Q. Nos. 131-133) Tangents PA and PB are drawn to the circle $(x - 4)^2 + (y - 5)^2 = 4$ from the point P on the curve $y = \sin x$, where A and B lie on the circle. Consider the function $y = f(x)$ represented by the locus of the centre of the circumcircle of $\triangle PAB$, then answer the following questions:

- 131.** Range of $y = f(x)$ is
(a) $[-2, 1]$ (b) $[-1, 4]$ (c) $[0, 2]$ (d) $[2, 3]$

- 132.** Period of $y = f(x)$ is
(a) 2π (b) 3π
(c) π (d) None of these

- 133.** Which of the following is true?
(a) $f(x) = 4$ has real roots
(b) $f(x) = 1$ has real roots
(c) Range of $y = f^{-1}(x)$ is $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$
(d) None of the above

Passage III (Q. Nos. 134-136) In the problems of coordinate geometry, use of pure geometry may be helpful rather than developing the equations especially in the case of circles. Simple facts like angles in the same segments of a circle are equal and the angle at the centre of a circle is double of the angle at the circumference standing on the same arc.

- 134.** $A(2, 0)$, $B(0, 2)$ and C are the vertices of a triangle inscribed in the circle $x^2 + y^2 = 4$. The bisector of $\angle C$ belongs to either of two families of concurrent lines, whose points of concurrency are
(a) $(\sqrt{2}, \sqrt{2})$
(b) $(2, 2)$
(c) $(\sqrt{2}, -\sqrt{2})$
(d) $(-\sqrt{2}, \sqrt{2})$

- 135.** $A(2, 0)$ and $B(0, 2)$ are two points on the circle $x^2 + y^2 - 2x - 2y = 0$. C is any point on the circle and I is the incentre of $\triangle ABC$. If CI intersects the circle at O , then the circumcentre of $\triangle AIB$ can be
(a) $(3, 3)$ (b) $(2, 2)$
(c) $(1, 1)$ (d) $(2, -2)$

- 136.** A and B are two points on a circle and P is any point on arc AB . If the bisectors of $\angle PBA$ and $\angle PAB$ intersect at O , then the locus of the point O is
(a) minor arc of a circle, if AB is a minor arc
(b) major arc of a circle, if AB is a major arc
(c) always on a major arc of a circle
(d) always on a minor arc of a circle

Passage IV (Q. Nos. 137-139) A variable circle of radius 2 units roots outside the circle $x^2 + y^2 + 4x = 0$. If C and C_1 denote the centres of respectively circles.

- 137.** Locus of centre of variable circle i.e. c is
(a) $x^2 + y^2 + 4y = 0$
(b) $x^2 + y^2 + 4x - 12 = 0$
(c) $x^2 + y^2 + 4y - 12 = 0$
(d) None of the above

138. If the line joining c and c_1 makes an angle of 60° with X -axis, then equation of common tangents of the circle can be

- (a) $\sqrt{3}x - y \pm 2 = 0, \sqrt{3}y + x - 2 = 0$
 (b) $\sqrt{3}x + y \pm 2 = 0, \sqrt{3}y + x - 2 = 0$
 (c) $\sqrt{3}x - y \pm 2 = 0, \sqrt{3}y - x - 2 = 0$
 (d) None of the above

139. The area of the region by the common tangents and the line $x + \sqrt{3}y + 2 = 0$, is

- (a) 2 sq units (b) 4 sq units (c) 8 sq units (d) 16 sq units

Passage V (Q. Nos. 140-142) Let α chord of a circle be that chord of the circle which subtends an angle α at the centre.

140. If $x + y = 1$ is α chord of $x^2 + y^2 = 1$, then α is equal to

- (a) $\pi/6$ (b) $\pi/6$
 (c) $\pi/6$ (d) $x + y = 1$ is not a chord

141. If slope of $\frac{\pi}{3}$ chord of $x^2 + y^2 = 4$ is 1, then its equation is

- (a) $x - y + \sqrt{6} = 0$
 (b) $x - y = 2\sqrt{3}$
 (c) $x - y = \sqrt{3}$
 (d) $x - y + \sqrt{3} = 0$

142. Distance of $\frac{2\pi}{3}$ chord of $x^2 + y^2 + 2x + 4y + 1 = 0$

from the centre, is

- (a) 1 unit
 (b) 2 units
 (c) $\sqrt{2}$ units
 (d) $\frac{1}{\sqrt{2}}$ unit

Type 5. Match the Columns

143. Match the statements of Column I with values of Column II.

Column I	Column II
A. Number of circles touching given three non-concurrent lines is	p. 1
B. Number of circles touching $y = x$ at $(2, 2)$ and also touching line $x + 2y - 4 = 0$, is	q. 2
C. Number of circles touching lines $x \pm y = 2$ and passing through the point $(4, 3)$ is	r. 4
D. Number of circles intersecting given three circles orthogonally	s. Infinite

144. Let $x^2 + y^2 + 2gx + 2fy + c = 0$ be an equation of circle.

Column I	Column II
A. If circle lie in first quadrant, then	p. $g < 0$
B. If circle lie above X -axis, then	q. $g > 0$
C. If circle lie on the left of Y -axis, then	r. $g^2 - c < 0$
D. If circle touches positive X -axis and does not intersect Y -axis, then	s. $c > 0$

145. Match the statements of Column I with values of Column II.

Column I	Column II
A. If the length of the common chord of two circles of radii 3 and 4 units which intersect orthogonally is $\frac{k}{5}$, then k is equal to	p. 1
B. If the circumference of the circle $x^2 + y^2 + 4x + 12y + p = 0$ is bisected by the circle $x^2 + y^2 - 2x + 8y - q = 0$, then $p + q$ is equal to	q. 24

Column I	Column II
C. Number of distinct chords of the circle $2x(x - \sqrt{2}) + y(2y - 1) = 0$; chords are passing through the point $(\sqrt{2}, \frac{1}{2})$ and are bisects on X -axis, is	r. 32
D. One of the diameters of the circle s . circumscribing the rectangle $ABCD$ is $4y = x + 7$. If A and B are the points $(-3, 4)$ and $(5, 4)$ respectively, then the area of the rectangle is	36

146. Match the statements of Column I with values of Column II.

Column I	Column II
A. If one of the diameter of the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ is a chord to the circle with centre $(2, 1)$, then the radius of the circle is equal to	p. $2 - \sqrt{3}$
B. If the straight line $y = mx$ touches or lies outside the circle $x^2 + y^2 - 20y + 90 = 0$, then $ m $ can be equal to	q. $2 + \sqrt{3}$
C. AB is a chord of the circle $x^2 + y^2 = 25$ and the tangents at A and B intersect at C . If $(2, 2\sqrt{3})$ is the mid-point of AB , then the area of quadrilateral $OACB$ is equal to (O being origin)	r. 2
D. A circle C of radius unity touches both the coordinate axes and lies in the first quadrant. If the circle C_1 , which also touches both the axes and lies in the first quadrant, intersect C orthogonally, then the radius C_1 can be equal to	s. 3
	t. 75/4

Type 6. Single Integer Answer Type Questions

147. The length of a common internal tangent to two circles is 7 and a common external tangent is 11. If the product of the radii of the two circles is p , then the value of $p/2$ is _____.
148. Line segments AC and BD are diameters of circle of radius one. If $\angle BDC = 60^\circ$, then the length of line segment AB is _____.
149. Let the lines $(y - 2) = m_1(x - 5)$ and $(y + 4) = m_2(x - 3)$ intersect at right angles at P , where m_1 and m_2 are parameters. If locus of P is $x^2 + y^2 + gx + fy + 7 = 0$, then the value of $|f + g|$ is _____.
150. Consider the family of circles $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$ passing through two fixed points A and B . Then, the distance between the points A and B is _____.
151. A circle touches the hypotenuse of a right angled triangle at its middle point and passes through the middle point of shorter side. If 3 and 4 units are the length of the sides and r is the radius of the circle, then find the value of $3r$.
152. Let $2x^2 + y^2 - 3xy = 0$ be the equations of a pair of tangents drawn from the origin O to a circle of radius 3 units with centre in the first quadrant. If A is one of the points of contact, and the length of OA is $3(3 + \sqrt{2\lambda})$, then find the numerical quantity λ .
153. If line L touch a circle C_1 of diameter 6. If the centre of C_1 , lies in the first quadrant, then the radius of the circle C_2 which is concentric with C_1 and cuts intercepts of length 8 on these lines is λ , then find the numerical quantity λ .
154. Suppose the line $y = 2x + a$ does not intersect or touch the circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 16x - 2y + 61 = 0$. If the circles lie on opposite side of the line, then a must lie in the interval $(2\sqrt{5} - 5k, -\sqrt{5} - 1)$. Find the numerical quantity k .

Entrances Gallery

IIT JEE/JEE Advanced

1. A circle S passes through the point $(0, 1)$ and is orthogonal to the circles $(x - 1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then, [2014]

- (a) radius of S is 8 (b) radius of S is 7
(c) centre of S is $(-7, 1)$ (d) centre of S is $(-8, 1)$

2. Let complex numbers α and $\frac{1}{\alpha}$ lie on circles

$$(x - x_0)^2 + (y - y_0)^2 = r^2 \text{ and}$$

$$(x - x_0)^2 + (y - y_0)^2 = 4r^2, \text{ respectively. If}$$

$$z_0 = x_0 + iy_0 \text{ satisfies the equation } 2|z_0|^2 = r^2 + 2,$$

then $|\alpha|$ is equal to [2013]

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{7}}$ (d) $\frac{1}{3}$

3. Circle(s) touching X -axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on Y -axis is/are [2013]

- (a) $x^2 + y^2 - 6x + 8y + 9 = 0$
(b) $x^2 + y^2 - 6x + 7y + 9 = 0$
(c) $x^2 + y^2 - 6x - 8y + 9 = 0$
(d) $x^2 + y^2 - 6x - 7y + 9 = 0$

4. The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is [2012]

- (a) $20(x^2 + y^2) - 36x + 45y = 0$
(b) $20(x^2 + y^2) + 36x - 45y = 0$
(c) $36(x^2 + y^2) - 20x + 45y = 0$
(d) $36(x^2 + y^2) + 20x - 45y = 0$

Directions (Q.Nos. 5-6) A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L , perpendicular to PT is a tangent to the circle $(x - 3)^2 + y^2 = 1$. [2012]

5. A possible equation of L is

- (a) $x - \sqrt{3}y = 1$ (b) $x + \sqrt{3}y = 1$
(c) $x - \sqrt{3}y = -1$ (d) $x + \sqrt{3}y = 5$

6. A common tangent of the two circles is

- (a) $x = 4$ (b) $y = 2$
(c) $x + \sqrt{3}y = 4$ (d) $x + 2\sqrt{2}y = 6$

7. The circle passing through the point $(-1, 0)$ and touching Y -axis at $(0, 2)$ also passes through the point [2011]

- (a) $\left(-\frac{3}{2}, 0\right)$ (b) $\left(-\frac{5}{2}, 2\right)$ (c) $\left(-\frac{3}{2}, \frac{5}{2}\right)$ (d) $(-4, 0)$

8. Two parallel chords of a circle of radius 2 are at a distance $\sqrt{3} + 1$ apart. If the chords subtend at the centre, angles of $\frac{\pi}{k}$ and $\frac{2\pi}{k}$, ($k > 0$), then the value of $[k]$, where $[k]$ denotes the largest integer $\leq k$, is [2010]

JEE Main/AIEEE

9. The number of common tangents to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ is [2015]
(a) 1 (b) 2 (c) 3 (d) 4
10. Let C be the circle with centre at $(1, 1)$ and radius 1. If T is the circle centred at $(0, y)$, passing through origin and touching the circle C externally, then the radius of T is equal to [2014]
(a) $\frac{\sqrt{3}}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{1}{2}$ (d) $\frac{1}{4}$
11. The circle passing through $(1, -2)$ and touching the X -axis at $(3, 0)$ also passes through the point [2013]
(a) $(-5, 2)$ (b) $(2, -5)$ (c) $(5, -2)$ (d) $(-2, 5)$
12. The length of the diameter of the circle which touches the X -axis at the point $(1, 0)$ and passes through the point $(2, 3)$, is [2012]
(a) $\frac{10}{3}$ (b) $\frac{3}{5}$
(c) $\frac{6}{5}$ (d) $\frac{5}{3}$
13. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ ($c > 0$) touch each other, if [2011]
(a) $|a| = c$ (b) $a = 2c$ (c) $|a| = 2c$ (d) $2|a| = c$
14. The equation of the circle passing through the point $(1, 0)$ and $(0, 1)$ and having the smallest radius, is [2011]
(a) $x^2 + y^2 + x + y - 2 = 0$
(b) $x^2 + y^2 - 2x - 2y + 1 = 0$
(c) $x^2 + y^2 - x - y = 0$
(d) $x^2 + y^2 + 2x + 2y - 7 = 0$
15. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points, if [2010]
(a) $-85 < m < -35$ (b) $-35 < m < 15$
(c) $15 < m < 65$ (d) $35 < m < 85$
16. If P and Q are the points of intersection of the circles $x^2 + y^2 + 3x + 7y + 2p - 5 = 0$ and $x^2 + y^2 + 2x + 2y - p^2 = 0$, then there is a circle passing through P, Q and $(1, 1)$ for [2009]
(a) all values of p
(b) all except one value of p
(c) all except two values of p
(d) exactly one value of p
17. The point diametrically opposite to the point $P(1, 0)$ on the circle $x^2 + y^2 + 2x + 4y - 3 = 0$ is [2008]
(a) $(3, 4)$ (b) $(3, -4)$
(c) $(-3, 4)$ (d) $(-3, -4)$
18. Consider a family of circles which are passing through the point $(-1, 1)$ and are tangent to X -axis. If (h, k) are the coordinates of the centre of the circles, then the set of values of k is given by the interval [2007]
(a) $0 < k < \frac{1}{2}$ (b) $k \geq \frac{1}{2}$
(c) $-\frac{1}{2} \leq k \leq \frac{1}{2}$ (d) $k \leq \frac{1}{2}$
19. If the lines $3x - 4y - 7 = 0$ and $2x - 3y - 5 = 0$ are two diameters of a circle of area 49π sq units, then equation of the circle is [2006]
(a) $x^2 + y^2 + 2x - 2y - 62 = 0$
(b) $x^2 + y^2 - 2x + 2y - 62 = 0$
(c) $x^2 + y^2 - 2x + 2y - 47 = 0$
(d) $x^2 + y^2 + 2x - 2y - 47 = 0$
20. Let C be the circle with centre $(0, 0)$ and radius 3 units. The equation of the locus of the mid-points of the chords of the circle C that subtend an angle of $\frac{2\pi}{3}$ at its centre, is [2006]
(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = \frac{27}{4}$
(c) $x^2 + y^2 = \frac{9}{4}$ (d) $x^2 + y^2 = \frac{3}{2}$
21. If the circles $x^2 + y^2 + 2ax + cy + a = 0$ and $x^2 + y^2 - 3ax + dy - 1 = 0$ intersect at two distinct points P and Q , then the line $5x + by - a = 0$ passes through P and Q for [2005]
(a) exactly two values of a
(b) infinitely many values of a
(c) no value of a
(d) exactly one value of a
22. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = p^2$ orthogonally, then the equation of the locus of its centre is [2005]
(a) $2ax + 2by - (a^2 + b^2 + p^2) = 0$
(b) $x^2 + y^2 - 2ax - 3by + (a^2 - b^2 - p^2) = 0$
(c) $2ax + 2by - (a^2 - b^2 + p^2) = 0$
(d) $x^2 + y^2 - 3ax - 4by + (a^2 + b^2 - p^2) = 0$
23. If a circle passes through the point (a, b) and cuts the circle $x^2 + y^2 = 4$ orthogonally, then the locus of its centre is [2004]
(a) $2ax + 2by + (a^2 + b^2 + 4) = 0$
(b) $2ax + 2by - (a^2 + b^2 + 4) = 0$
(c) $2ax - 2by + (a^2 + b^2 + 4) = 0$
(d) $2ax - 2by - (a^2 + b^2 + 4) = 0$
24. A variable circle passes through the fixed point $A(p, q)$ and touches X -axis. The locus of the other end of the diameter through A is [2004]
(a) $(x - p)^2 = 4qy$ (b) $(x - q)^2 = 4py$
(c) $(y - p)^2 = 4qx$ (d) $(y - q)^2 = 4px$

25. If the lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ lie along diameters of a circle of circumference 10π , then the equation of the circle is [2004]
 (a) $x^2 + y^2 - 2x + 2y - 23 = 0$
 (b) $x^2 + y^2 - 2x - 2y - 23 = 0$
 (c) $x^2 + y^2 + 2x + 2y - 23 = 0$
 (d) $x^2 + y^2 + 2x - 2y - 23 = 0$
26. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB . Equation of the circle on AB as a diameter is [2004]
 (a) $x^2 + y^2 - x - y = 0$ (b) $x^2 + y^2 - x + y = 0$
 (c) $x^2 + y^2 + x + y = 0$ (d) $x^2 + y^2 + x - y = 0$
27. If the two circles $(x-1)^2 + (y-3)^2 = r^2$ and $x^2 + y^2 - 8x + 2y + 8 = 0$ intersect in two distinct points, then [2003]
 (a) $2 < r < 8$ (b) $r < 2$ (c) $r = 2$ (d) $r > 2$
28. The greatest distance of the point $P(10, 7)$ from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is [2002]
 (a) 10 units (b) 15 units
 (c) 5 units (d) None of these
29. The equation of the tangent to the circle $x^2 + y^2 + 4x - 4y + 4 = 0$ which make equal intercepts on the positive coordinate axes, is [2002]
 (a) $x + y = 2$ (b) $x + y = 2\sqrt{2}$
 (c) $x + y = 4$ (d) $x + y = 8$

Other Engineering Entrances

30. Equation of circle with centre $(-a, -b)$ and radius $\sqrt{a^2 - b^2}$ is [Karnataka CET 2014]
 (a) $x^2 + y^2 + 2ax + 2by + 2b^2 = 0$
 (b) $x^2 + y^2 - 2ax - 2by - 2b^2 = 0$
 (c) $x^2 + y^2 - 2ax - 2by + 2b^2 = 0$
 (d) $x^2 + y^2 - 2ax + 2by + 2a^2 = 0$
31. A circle of radius $\sqrt{8}$ is passing through origin and the point $(4, 0)$. If the centre lies on the line $y = x$, then the equation of circle is [Kerala CEE 2014]
 (a) $(x-2)^2 + (y-2)^2 = 8$ (b) $(x+2)^2 + (y+2)^2 = 8$
 (c) $(x-3)^2 + (y-3)^2 = 8$ (d) $(x+3)^2 + (y+3)^2 = 8$
 (e) $(x-4)^2 + (y-4)^2 = 8$
32. Compute the shortest distance between the circle $x^2 + y^2 - 10x - 14y - 151 = 0$ and the point $(-7, 2)$. [GGSIPU 2014]
 (a) 0 (b) 1 (c) 2 (d) 4
33. If one end of a diameter of the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is $(3, 4)$, then find the coordinate of the other end of the diameter. [RPET 2014]
 (a) $(2, 1)$ (b) $(1, 2)$
 (c) $(1, 1)$ (d) None of these
34. The measure of the chord intercepted by circle $x^2 + y^2 = 9$ and the line $x - y + 2 = 0$ is [AMU 2014]
 (a) $\sqrt{28}$ (b) $2\sqrt{5}$ (c) 7 (d) 5
35. A circle with centre at $(2, 4)$ is such that the line $x + y + 2 = 0$ cuts a chord of length 6. The radius of the circle is [EAMCET 2014]
 (a) $\sqrt{41}$ cm (b) $\sqrt{11}$ cm
 (c) $\sqrt{21}$ cm (d) $\sqrt{31}$ cm
36. The condition for the lines $lx + my + n = 0$ and $l_1x + m_1y + n_1 = 0$ to be conjugate with respect to the circle $x^2 + y^2 = r^2$, is [EAMCET 2014]
 (a) $r^2(ll_1 + mm_1) = nn_1$ (b) $r^2(ll_1 - mm_1) = nn_1$
 (c) $r^2(ll_1 + mm_1) + nn_1 = 0$ (d) $r^2(ll_1 + l_1m) = nn_1$
37. The length of the common chord of the two circles $x^2 + y^2 - 4y = 0$ and $x^2 + y^2 - 8x - 4y + 11 = 0$ is [EAMCET 2014]
 (a) $\frac{\sqrt{145}}{4}$ cm (b) $\frac{\sqrt{11}}{2}$ cm (c) $\sqrt{135}$ cm (d) $\frac{\sqrt{135}}{4}$ cm
38. The shortest distance between the circles $(x-1)^2 + (y+2)^2 = 1$ and $(x+2)^2 + (y-2)^2 = 4$ is [Kerala CEE 2014]
 (a) 1 (b) 2 (c) 3
 (d) 4 (e) 5
39. The centre of the circle, whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at $(5, 5)$ is [Kerala CEE 2014]
 (a) $(10, 5)$ (b) $(5, 8)$ (c) $(5, 10)$
 (d) $(8, 9)$ (e) $(9, 8)$
40. A circle passes through the points $(0, 0)$ and $(0, 1)$ and also touches the circle $x^2 + y^2 = 16$. The radius of the circle is [Kerala CEE 2014]
 (a) 1 (b) 2 (c) 3
 (d) 4 (e) 5
41. If the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the three circles $x^2 + y^2 - 5 = 0$, $x^2 + y^2 - 8x - 6y + 10 = 0$ and $x^2 + y^2 - 4x + 2y - 2 = 0$ at the extremities of their diameters, then [WB JEE 2014]
 (a) $c = -5$ (b) $fg = \frac{147}{25}$
 (c) $g + 2f = c + 2$ (d) $4f = 3g$
42. If the squares of the tangents from a point P to the circles $x^2 + y^2 = a$, $x^2 + y^2 = b^2$ and $x^2 + y^2 = c^2$ are in AP, then [AMU 2014]
 (a) a, b, c are in AP (b) a, b, c are in GP
 (c) a^2, b^2, c^2 are in AP (d) a^2, b^2, c^2 are in GP
43. The point at which the circles $x^2 + y^2 - 4x - 4y + 7 = 0$ and $x^2 + y^2 - 12x - 10y + 45 = 0$ touch each other, is [EAMCET 2014]
 (a) $\left(\frac{13}{5}, \frac{14}{5}\right)$ (b) $\left(\frac{2}{5}, \frac{5}{6}\right)$
 (c) $\left(\frac{14}{5}, \frac{13}{5}\right)$ (d) $\left(\frac{12}{5}, 2 + \frac{\sqrt{21}}{5}\right)$

44. The locus of the centre of the circle, which cuts the circle $x^2 + y^2 - 20x + 4 = 0$ orthogonally and touches the line $x = 2$, is [EAMCET 2014]
 (a) $x^2 = 16y$ (b) $y^2 = 4x$
 (c) $y^2 = 16x$ (d) $x^2 = 4y$
45. If the equation of the tangent to the circle $x^2 + y^2 - 2x + 6y - 6 = 0$ parallel to $3x - 4y + 7 = 0$ is $3x - 4y + k = 0$, then the values of k are [Kerala CEE 2013]
 (a) 5, -35 (b) -5, 35
 (c) 7, -32 (d) -7, 32
 (e) None of these
46. Circles are drawn through the point (2, 0) to cut intercept of length 5 units on X -axis. If their centres lies in the first quadrant, then their equation is [UP SEE 2013]
 (a) $x^2 + y^2 - 9x + 2fy + 14 = 0$
 (b) $3x^2 + 3y^2 + 27x - 2fy + 42 = 0$
 (c) $x^2 + y^2 - 9x - 2fy + 14 = 0$
 (d) $x^2 + y^2 - 2fx - 9y + 14 = 0$
47. Two vertices of an equilateral triangle are (-1, 0) and (1, 0) and its circumcircle is [OJEE 2013]
 (a) $x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$ (b) $x^2 - \left(y + \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$
 (c) $x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = -\frac{4}{3}$ (d) None of these
48. The least and the greatest distances of the point (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ are [Karnataka CET 2012]
 (a) 10, 5 (b) 15, 20 (c) 12, 16 (d) 5, 15
49. Equation of unit circle concentric with circle $x^2 + y^2 + 8x + 4y - 8 = 0$ is [OJEE 2012]
 (a) $x^2 + y^2 + 8x + 4y + 19 = 0$
 (b) $x^2 + y^2 - 8x + 4y + 19 = 0$
 (c) $x^2 + y^2 - 8x - 4y + 19 = 0$
 (d) None of the above
50. If $a > 2b > 0$, then positive value of m for which $y = mx - b\sqrt{1+m^2}$ is a common tangent to $x^2 + y^2 = b^2$ and $(x-a)^2 + y^2 = b^2$ is [Manipal 2012]
 (a) $\frac{2b}{\sqrt{a^2 - 4b^2}}$ (b) $\frac{\sqrt{a^2 - 4b^2}}{2b}$
 (c) $\frac{2b}{a - 2b}$ (d) $\frac{b}{a - 2b}$
51. The equation of the two tangents from (-5, -4) to the circle $x^2 + y^2 + 4x + 6y + 8 = 0$ are [Karnataka CET 2012]
 (a) $x + 2y + 13 = 0$, $2x - y + 6 = 0$
 (b) $2x + y + 13 = 0$, $x - 2y = 6$
 (c) $3x + 2y + 23 = 0$, $2x - 3y + 4 = 0$
 (d) $x - 7y = 23$, $6x + 13y = 4$
52. The locus of the mid-points of the chord of the circle $x^2 + y^2 = 4$ which subtends a right angle at the origin is [Manipal 2012]
 (a) $x + y = 2$ (b) $x^2 + y^2 = 1$
 (c) $x^2 + y^2 = 2$ (d) $x + y = 1$
53. The circles $x^2 + y^2 - 6x - 8y = 0$ and $x^2 + y^2 - 6x + 8 = 0$ are [MP PET 2012]
 (a) intersecting in two points (b) non-intersecting
 (c) touching externally (d) touching internally
54. If the circles $x^2 + y^2 + 2x + 2ky + 6 = 0$ and $x^2 + y^2 + 2ky + k = 0$ intersect orthogonally, then k is equal to [WB JEE 2012]
 (a) 2 or $-\frac{3}{2}$ (b) -2 or $-\frac{3}{2}$
 (c) 2 or $\frac{3}{2}$ (d) -2 or $\frac{3}{2}$
55. The sum of the minimum distance and the maximum distance from the point (4, -3) to the circle $x^2 + y^2 + 4x - 10y - 7 = 0$ is [Kerala CEE 2011]
 (a) 20 (b) 12 (c) 10
 (d) 16 (e) 22
56. If the straight line $y = mx$ lies outside the circle $x^2 + y^2 - 20y + 90 = 0$, then the value of m will satisfy [WB JEE 2011; Guj CET 2011]
 (a) $m < 3$ (b) $|m| < 3$
 (c) $m > 3$ (d) $|m| > 3$
57. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB . Equation of the circle with AB as diameter is [WB JEE 2011]
 (a) $x^2 + y^2 = 1$
 (b) $x(x-1) + y(y-1) = 0$
 (c) $x^2 + y^2 = 2$
 (d) $(x-1)(x-2) + (y-1)(y-2) = 0$
58. The length (in units) of tangent from point (5, 1) to the circle $x^2 + y^2 + 6x - 4y - 3 = 0$ is [BITSAT 2011]
 (a) 81 (b) 29 (c) 7 (d) 21
59. Circle $ax^2 + ay^2 + 2gx + 2fy + c = 0$ touches X -axis, if [MP PET 2011]
 (a) $f^2 > ac$ (b) $g^2 > ac$ (c) $f^2 = bc$ (d) $g^2 = ac$
60. The locus of the point of intersection of the tangents at the extremities of a chord of the circle $x^2 + y^2 = a^2$, which touches the circle $x^2 + y^2 - 2ax = 0$ passes through the point, is [AMU 2011]
 (a) $\left(\frac{a}{2}, 0\right)$ (b) $\left(0, \frac{a}{2}\right)$
 (c) $(a, 0)$ (d) $(0, 0)$
61. If the lines joining the origin to the intersection of the line $y = mx + 2$ and the circle $x^2 + y^2 = 1$ are at right angles, then [AMU 2011]
 (a) $m = \sqrt{3}$ (b) $m = \pm\sqrt{7}$
 (c) $m = 1$ (d) $m = \sqrt{5}$

62. The total number of common tangents of $x^2 + y^2 - 6x - 8y + 9 = 0$ and $x^2 + y^2 = 1$ is
[Karnataka CET 2011]
(a) 4 (b) 2 (c) 3 (d) 1
63. The centre of a circle which cuts $x^2 + y^2 + 6x - 1 = 0$, $x^2 + y^2 - 3y + 2 = 0$ and $x^2 + y^2 + x + y - 3 = 0$ orthogonally is
[Karnataka CET 2011]
(a) $\left(\frac{1}{7}, \frac{9}{7}\right)$ (b) $\left(-\frac{1}{7}, -\frac{9}{7}\right)$
(c) $\left(\frac{1}{7}, -\frac{9}{7}\right)$ (d) $\left(-\frac{1}{7}, \frac{9}{7}\right)$
64. The line segment joining the points (4, 7) and (-2, -1) is a diameter of a circle. If the circle intersects X-axis at A and B, then AB is equal to [BITSAT 2010]
(a) 4 (b) 5
(c) 6 (d) 8
65. The distance of the mid-point of line joining two points (4, 0) and (0, 4) from the centre of the circle $x^2 + y^2 = 16$ is [VITEEE 2010]
(a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $2\sqrt{3}$
66. The equation of family of circles with centre at (h, k) touching X-axis, is given by [Kerala CEE 2010]
(a) $x^2 + y^2 - 2hx + h^2 = 0$
(b) $x^2 + y^2 - 2hx - 2ky + h^2 = 0$
(c) $x^2 + y^2 - 2hx - 2ky - h^2 = 0$
(d) $x^2 + y^2 - 2hx - 2ky = 0$
(e) $x^2 + y^2 + 2hx + 2ky = 0$
67. The straight line $x + y - 1 = 0$ meets the circle $x^2 + y^2 - 6x - 8y = 0$ at A and B. Then, the equation of the circle of which AB is a diameter, is [WB JEE 2010]
(a) $x^2 + y^2 - 2y - 6 = 0$
(b) $x^2 + y^2 + 2y - 6 = 0$
(c) $2(x^2 + y^2) + 2y - 6 = 0$
(d) $3(x^2 + y^2) + 2y - 6 = 0$
68. The equation of normal of $x^2 + y^2 - 2x + 4y - 5 = 0$ at (2, 1) is [WB JEE 2010]
(a) $y = 3x - 5$ (b) $2y = 3x - 4$
(c) $y = 3x + 4$ (d) $y = x + 1$
69. The equation of the chord of the circle $x^2 + y^2 = 81$, which is bisected at the point (-2, 3), is [Kerala CEE 2010]
(a) $3x - y = 13$
(b) $3x - 4y = 13$
(c) $2x - 3y = 13$
(d) $3x - 3y = 13$
(e) $2x - 3y = -13$
70. If the two circles $(x + 7)^2 + (y - 3)^2 = 36$ and $(x - 5)^2 + (y + 2)^2 = 49$ touch each other externally, then the point of contact is [Kerala CEE 2010]
(a) $\left(\frac{-19}{13}, \frac{19}{13}\right)$ (b) $\left(\frac{-19}{13}, \frac{9}{13}\right)$
(c) $\left(\frac{17}{13}, \frac{9}{13}\right)$ (d) $\left(\frac{-17}{13}, \frac{9}{13}\right)$
(e) $\left(\frac{19}{13}, \frac{19}{13}\right)$

Answers

Work Book Exercise 13.1

1. (a) 2. (b) 3. (c) 4. (c) 5. (a) 6. (a) 7. (b) 8. (b) 9. (d) 10. (d)

Work Book Exercise 13.2

1. (a) 2. (b) 3. (c) 4. (b) 5. (b) 6. (b) 7. (b) 8. (b) 9. (c) 10. (d)

Work Book Exercise 13.3

1. (b) 2. (c) 3. (d) 4. (b) 5. (b) 6. (a) 7. (a) 8. (a) 9. (c) 10. (b)

Target Exercises

1. (d) 2. (b) 3. (d) 4. (c) 5. (d) 6. (a) 7. (b) 8. (a) 9. (a) 10. (a)
 11. (d) 12. (c) 13. (a) 14. (d) 15. (c) 16. (a) 17. (d) 18. (a) 19. (c) 20. (d)
 21. (d) 22. (d) 23. (a) 24. (b) 25. (c) 26. (b) 27. (b) 28. (d) 29. (b) 30. (b)
 31. (c) 32. (c) 33. (c) 34. (c) 35. (c) 36. (a) 37. (c) 38. (d) 39. (b) 40. (b)
 41. (b) 42. (c) 43. (c) 44. (b) 45. (b) 46. (a) 47. (a) 48. (b) 49. (c) 50. (b)
 51. (c) 52. (b) 53. (a) 54. (b) 55. (b) 56. (b) 57. (c) 58. (a) 59. (c) 60. (d)
 61. (b) 62. (d) 63. (c) 64. (b) 65. (a) 66. (c) 67. (a) 68. (d) 69. (b) 70. (d)
 71. (a) 72. (c) 73. (d) 74. (b) 75. (d) 76. (a) 77. (c) 78. (d) 79. (a) 80. (b)
 81. (a) 82. (b) 83. (a) 84. (d) 85. (c) 86. (a) 87. (b) 88. (d) 89. (a) 90. (d)
 91. (a) 92. (c) 93. (a) 94. (b) 95. (b) 96. (c) 97. (c) 98. (b) 99. (d) 100. (a)
 101. (c) 102. (b) 103. (b) 104. (d) 105. (c) 106. (d) 107. (a) 108. (c) 109. (d) 110. (a,d)
 111. (a,c) 112. (c,d) 113. (a,c) 114. (c,d) 115. (a,c,d) 116. (b,d) 117. (b,c) 118. (a,b,c) 119. (d) 120. (a)
 121. (b) 122. (b) 123. (d) 124. (d) 125. (a) 126. (a) 127. (d) 128. (d) 129. (b) 130. (a)
 131. (d) 132. (c) 133. (c) 134. (a) 135. (b) 136. (d) 137. (b) 138. (a) 139. (c) 140. (b)
 141. (a) 142. (a) 143. (*) 144. (**) 145. (***) 146. (****) 147. (9) 148. (1) 149. (6) 150. (6)
 151. (5) 152. (5) 153. (5) 154. (3)

* $A \rightarrow r$; $B \rightarrow q$; $C \rightarrow q$; $D \rightarrow p$
 ** $A \rightarrow p, r, s$; $B \rightarrow r, s$; $C \rightarrow q, s$; $D \rightarrow p, s$
 *** $A \rightarrow q$; $B \rightarrow s$; $C \rightarrow q$; $D \rightarrow r$
 **** $A \rightarrow s$; $B \rightarrow s$; $C \rightarrow t$; $D \rightarrow p, q$

Entrances Gallery

1. (b,c) 2. (c) 3. (a,c) 4. (a) 5. (a) 6. (d) 7. (d) 8. (3) 9. (c) 10. (d)
 11. (c) 12. (a) 13. (a) 14. (c) 15. (b) 16. (c) 17. (d) 18. (b) 19. (c) 20. (c)
 21. (c) 22. (a) 23. (b) 24. (a) 25. (a) 26. (a) 27. (a) 28. (b) 29. (b) 30. (a)
 31. (a) 32. (c) 33. (b) 34. (a) 35. (a) 36. (a) 37. (d) 38. (b) 39. (e) 40. (b)
 41. (d) 42. (c) 43. (c) 44. (c) 45. (a) 46. (a) 47. (a) 48. (d) 49. (a) 50. (a)
 51. (a) 52. (c) 53. (d) 54. (a) 55. (a) 56. (b) 57. (b) 58. (c) 59. (d) 60. (a)
 61. (b) 62. (c) 63. (d) 64. (d) 65. (b) 66. (b) 67. (a) 68. (a) 69. (e) 70. (b)

Perpendicular from (0, 0) on line (i) is $\frac{|-c|}{\sqrt{2}} = \frac{c}{\sqrt{2}}$.

In ΔAON , $\sqrt{2^2 - \left(\frac{c}{\sqrt{2}}\right)^2} = AN$

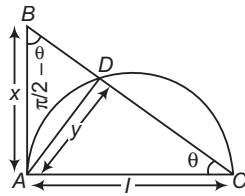
and in ΔCPM , $\sqrt{3^2 - \left(\frac{2-c}{\sqrt{2}}\right)^2} = CM$

Given, $AN = CM$
 $\Rightarrow 4 - \frac{c^2}{2} = 9 - \frac{(2-c)^2}{2}$
 $\Rightarrow c = \frac{-3}{2}$

Therefore, equation of the line is $y = x - \frac{3}{2}$

or $2x - 2y - 3 = 0$

11. ΔABC and ΔDBA are similar.



$\therefore lx = y\sqrt{l^2 + x^2}$
 $\Rightarrow l^2x^2 = y^2(l^2 + x^2)$
 $\Rightarrow l^2(x^2 - y^2) = x^2y^2$
 $\Rightarrow l = \frac{xy}{\sqrt{x^2 - y^2}} = \frac{AB \times AD}{\sqrt{AB^2 - AD^2}}$

12. Let the equation of the chord OA of the circle $x^2 + y^2 - 2x + 4y = 0$

be $y = mx$

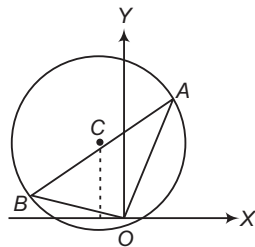
On solving Eqs. (i) and (ii), we get

$x^2 + m^2x^2 - 2x + 4mx = 0$

$\Rightarrow (1 + m^2)x^2 - (2 - 4m)x = 0$

$\Rightarrow x = 0$

and $x = \frac{2 - 4m}{1 + m^2}$



Hence, the points of intersection are

$(0, 0)$ and $A\left(\frac{2 - 4m}{1 + m^2}, \frac{m(2 - 4m)}{1 + m^2}\right)$

$\Rightarrow OA^2 = \left(\frac{2 - 4m}{1 + m^2}\right)^2 (1 + m^2) = \frac{(2 - 4m)^2}{1 + m^2}$

Since, OAB is an isosceles right angled triangle.

$\therefore OA^2 = \frac{1}{2} AB^2$

where, AB is a diameter of the given circle.

So, $OA^2 = 10$

$\Rightarrow \frac{(2 - 4m)^2}{1 + m^2} = 10$

$\Rightarrow 4 - 16m + 16m^2 = 10(1 + m^2)$

$\Rightarrow 3m^2 - 8m - 3 = 0$

$\Rightarrow m = 3 \text{ or } -\frac{1}{3}$

Hence, the required equation is $y = 3x$
 or $x + 3y = 0$

13. $S(-3, -2) = 9 + 4 - 6 - 6 + 1 = 2 > 0$

So, $(-3, -2)$ lies outside of S
 and $S'(-3, -2)$

$= 9 + 4 - 12 - 6 + 2 = -3 < 0$

This implies, $(-3, -2)$ lies inside of S' .

14. Since, $y = |x| + c$ and $x^2 + y^2 - 8|x| - 9 = 0$ both are symmetrical about Y -axis for $x > 0$, $y = x + c$.

Equation of tangent to circle $x^2 + y^2 - 8x - 9 = 0$

Parallel to $y = x + c$ is

$y = (x - 4) + 5\sqrt{1 + 1}$

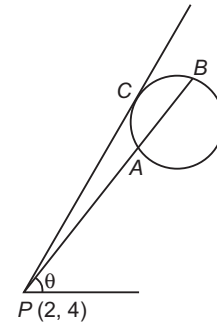
$\Rightarrow y = x + (5\sqrt{2} - 4)$

For no solution, $c > 5\sqrt{2} - 4$.

$\therefore c \in (5\sqrt{2} - 4, \infty)$

15. Given, $(1 + \alpha x)^n = 1 + 8x + 24x^2 + \dots$

$\Rightarrow 1 + n(\alpha x) + \frac{n(n-1)}{1 \cdot 2}(\alpha x)^2 + \dots$
 $= 1 + 8x + 24x^2 + \dots$



On equating the coefficients of x and x^2 , we get

$n\alpha = 8 \quad \frac{n\alpha(n\alpha - \alpha)}{1 \cdot 2} = 24$

or $\frac{8(8 - \alpha)}{2} = 24$

$\Rightarrow 8 - \alpha = 6$

$\therefore \alpha = 2 \text{ and } n = 4$

Equation of the line is $\frac{x-2}{\cos \theta} = \frac{y-4}{\sin \theta} = r$, then point

$(2 + r \cos \theta, 4 + r \sin \theta)$ lies on the circle $x^2 + y^2 = 4$.

$\Rightarrow (2 + r \cos \theta)^2 + (4 + r \sin \theta)^2 = 4$

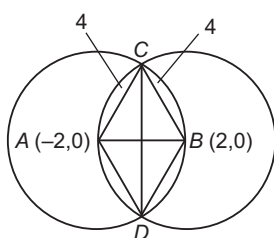
or $r^2 + 4r(\cos \theta + 2 \sin \theta) + 16 = 0$

$\therefore PA \cdot PB = r_1 r_2$
 $= \frac{16}{1} = 16$

Aliter

$PA \cdot PB = (PC)^2$
 $= 2^2 + 4^2 - 4 = 16$

16. Circles with centres $(2, 0)$ and $(-2, 0)$ each with radius 4.

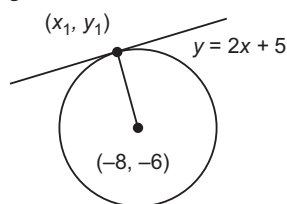


Thus, Y-axis is their common chord.

So, $\triangle ABC$ is equilateral.

Hence, area of $ABCD$ is $\frac{2\sqrt{3}}{4}(4)^2$ i.e. $8\sqrt{3}$ sq units.

17. From the figure, we have



$$y_1 = 2x_1 + 5 \text{ and } \frac{y_1 + 6}{x_1 + 8} \times 2 = -1$$

$$\Rightarrow x_1 = -6 \text{ and } y_1 = -7$$

18. For smallest circle, OA will become common normal.

$$\therefore OA = 5 \Rightarrow AB = 4$$

So, equation of line OA is $y = \frac{3}{4}x$.

Putting this value of y in $x^2 + y^2 = 1$, we get

$$x^2 + \frac{9x^2}{16} = 1 \Rightarrow x = \pm \frac{4}{5}$$

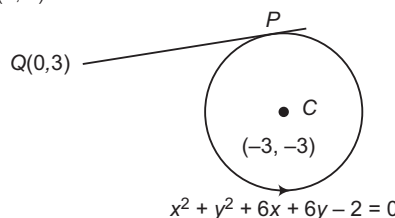
$$B = \left(\frac{4}{5}, \frac{3}{5}\right)$$

Thus, equation of required circle is

$$x^2 + y^2 - \frac{24}{5}x - \frac{18}{5}y + 5 = 0$$

19. Here, $5x - 2y + 6 = 0$ meets Y-axis at $(0, 3)$.

$$\therefore Q(0, 3)$$



Here, length of tangent $= \sqrt{S_1}$

$$\Rightarrow PQ = \sqrt{0 + 9 + 0 + 18 - 2} = 5 \text{ units}$$

20. $\therefore c = f^2$

$$\therefore (g - 3)^2 + (f - 3)^2 = \left[\sqrt{9 + 9 - 14} + \sqrt{g^2 + f^2 - c} \right]^2$$

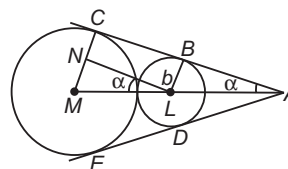
$$\Rightarrow g^2 + f^2 - 6g - 6f + 18 = (2 + g)^2$$

$$\Rightarrow f^2 - 10g - 6f + 14 = 0$$

Hence, locus of centre (g, f) is

$$y^2 - 10x - 6y + 14 = 0$$

- 21.



In $\triangle MLN$,

$$\sin \alpha = \frac{a - b}{a + b}$$

$$\alpha = \sin^{-1} \left(\frac{a - b}{a + b} \right)$$

$\therefore$ Angle between AB and AD

$$= 2\alpha = 2 \sin^{-1} \left(\frac{a - b}{a + b} \right)$$

22. Centre is $(-a, -b)$ and radius is $\sqrt{a^2 + b^2}$. Thus, equation of incircle is $(x + a)^2 + (y + b)^2 = \frac{a^2 + b^2}{4}$.

$$\Rightarrow 4x^2 + 4y^2 + 8ax + 8by + 3a^2 + 3b^2 = 0$$

23. Here,

$$PQ = 2r \cot \theta$$

$$RS = 2r \tan \theta$$

$$\Rightarrow$$

$$2r = \sqrt{PQ \cdot RS}$$

24. Clearly, $(0, 0)$ lies on director circle of the given circle.

Now, equation of director circle is

$$(x + g)^2 + (y + f)^2 = 2(g^2 + f^2 - c)$$

If $(0, 0)$ lies on it, then

$$g^2 + f^2 = 2(g^2 + f^2 - c)$$

$$\Rightarrow$$

$$g^2 + f^2 = 2c$$

25. Centre of circle is $(3, -1)$. Equation of any line passing through $(3, -1)$ and perpendicular to $2x - 5y + 18 = 0$ is

$$(y + 1) = -\frac{5}{2}(x - 3)$$

$$\Rightarrow$$

$$2y + 5x - 13 = 0$$

On solving it with the given chord, we get $x = 1, y = 4$.

Thus, required mid-point is $(1, 4)$.

26. If (h, k) is the centroid of the $\triangle PAB$, then

$$h = \frac{r(1 + \cos \theta)}{3}$$

and

$$k = \frac{r(1 - \sin \theta)}{3}$$

$$\Rightarrow \left(h - \frac{r}{3}\right)^2 + \left(k - \frac{r}{3}\right)^2 = \left(\frac{r}{3}\right)^2$$

Hence, locus of (h, k) is

$$\left(x - \frac{r}{3}\right)^2 + \left(y - \frac{r}{3}\right)^2 = \left(\frac{r}{3}\right)^2$$

27. $\therefore C_1 C_2 = |r_1 - r_2|$

circles touch internally.

Hence, the number of common tangents is one.

28. Tangents are clearly $x = -1, y = -1$.

Their combined equation is

$$(x + 1)(y + 1) = 0$$

$$\Rightarrow$$

$$x + y + xy + 1 = 0$$

29. Diameter of circle = Distance of the point (1, 0) from $3x - 6y + 7 = 0$

$$\frac{3(1) - 6(0) + 7}{\sqrt{3^2 + (-6)^2}} = \frac{10}{\sqrt{45}} = \frac{2}{3}\sqrt{5} \text{ units}$$

$$\therefore \text{Radius of circle} = \frac{1}{2} \left(\frac{2}{3}\sqrt{5} \right) = \frac{\sqrt{5}}{3} \text{ unit}$$

30. Equation of normal is $y - 1 = 2(x + 1)$
 $\Rightarrow 2x - y + 3 = 0$
 $x = -3$ and $y = -3$, solving tangent and normals.

31. $\because C_1C_2 = r_1 + r_2 = 4$
 So, circles touch each other externally.
 $\therefore$ Common tangent is
 $(x^2 + y^2 - 8x + 12) - (x^2 + y^2 - 4) = 0$
 $\Rightarrow -8x + 16 = 0 \Rightarrow x = 2$

32. $(x - 1)(y - 2) = 0 \Rightarrow x - 1 = 0$ and $y - 2 = 0$
 $\therefore$ Radius of circle = $\frac{3(1) + 4(2) - 6}{\sqrt{9 + 16}} = \frac{5}{5} = 1$ unit

Now, equation of the circle is
 $(x - 1)^2 + (y - 2)^2 = 1$
 $\Rightarrow x^2 + y^2 - 2x - 4y + 4 = 0$

33. Equation of common tangent is $S_1 - S_2 = 0$ i.e. $x = 0$
 On putting $x = 0$ in the equation of circles, we get
 $y^2 + b = 0$

It should have equal roots, hence $b = 0$.

34. Here, $\frac{a(1) + b(0) - 2}{\sqrt{a^2 + b^2}} = 2$

or $a - 2 = 2\sqrt{a^2 + b^2}$

Also, normal to the circle will pass through the centre (0, 2).

$$\begin{aligned} \Rightarrow a - 2 &= 2\sqrt{a^2 + 1} \\ \Rightarrow (a - 2)^2 &= 4(a^2 + 1) \\ \Rightarrow 3a^2 + 4a &= 0 \Rightarrow a = 0, -\frac{4}{3} \end{aligned}$$

35. Equation of AB is $T = 0$ i.e. $3x + y \cdot 0 = 4$

$$\begin{aligned} \Rightarrow x &= \frac{4}{3}, OD = \frac{4}{3} \\ \Rightarrow AD^2 &= OA^2 - OD^2 \\ \Rightarrow AD^2 &= 4 - \frac{16}{9} = \frac{20}{9} \\ \Rightarrow AD &= \frac{2\sqrt{5}}{3} \Rightarrow AB = \frac{4\sqrt{5}}{3} \\ \therefore \text{Area of } \triangle PAB &= \frac{1}{2} \cdot \frac{4\sqrt{5}}{3} \cdot \left(3 - \frac{4}{3} \right) = \frac{10\sqrt{5}}{9} \text{ sq units} \end{aligned}$$

36. Equation of AB is $x \cdot 4 + y \cdot 4 = 9$

$$\Rightarrow OD = \frac{9}{4\sqrt{2}}$$

Also, $AD^2 = OA^2 - OD^2 = 9 - \frac{81}{32} = \frac{207}{32}$

$$\Rightarrow AB = 2 \cdot AD = 2 \cdot \sqrt{\frac{207}{32}}$$

$$\therefore \text{Area of triangle} = \frac{1}{2} \cdot \frac{9}{4\sqrt{2}} \cdot \frac{\sqrt{207}}{2\sqrt{2}} = \frac{9\sqrt{207}}{32} \text{ sq units}$$

37. Let C be centre.

We have, $CD = 4$ and $AD = 3$

$$\Rightarrow CA^2 = 3^2 + 4^2$$

$$\Rightarrow CP = CA = 5$$

If C is (h, k), then $\frac{h - 4}{1} = \frac{k - 4}{1} = \pm 5$

$$\Rightarrow h = 4 - \frac{5}{\sqrt{2}}, k = 4 + \frac{5}{\sqrt{2}}$$

$$\text{or } h = 4 + \frac{5}{\sqrt{2}}, k = 4 - \frac{5}{\sqrt{2}}$$

38. $A_1B_1 = \sqrt{4 + 4} = 2\sqrt{2}$

$$\Rightarrow AB = 2\sqrt{2} - 2 = 2(\sqrt{2} - 1)$$

Thus, equation of required circle is
 $x^2 + y^2 = (\sqrt{2} - 1)^2$
 $= 3 - 2\sqrt{2}$

39. Let the centre of required circle be (h, h).

We have, $\angle COD = \angle CBE = \frac{\pi}{4}$

$$CB = h + 2$$

$$\text{and } BE = h - 2$$

$$\therefore \frac{h - 2}{h + 2} = \cos \frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow h = \frac{2(\sqrt{2} + 1)}{(\sqrt{2} - 1)} = 2(3 + 2\sqrt{2}) \text{ units}$$

40. $C_1C_2 = \sqrt{(0 - 1)^2 + (4 - 2)^2} = \sqrt{1 + 4} = \sqrt{5} = r_2 - r_1$

Thus, two circles touch each other internally.

41. $C_1C_2 = \sqrt{(4 - 2)^2 + (3 - 4)^2}$
 $= \sqrt{4 + 1} = \sqrt{5} = r_1 - r_2$

This show that these circles touch each other internally.

42. Radical axes of circles are indeed the altitudes of the triangle. Thus, radical centres of $S_1 = 0, S_2 = 0, S_3 = 0$ will be the orthocentre of the triangle.

43. $S_1 = T$ at (h, k)

$$\Rightarrow hx + ky = h^2 + k^2$$

So, (1, -2) is mid-point of chord whose equation is

$$1 \cdot x + (-2)y = 1^2 + (-2)^2$$

$$\Rightarrow x - 2y - 5 = 0$$

44. The sides are $x + y - 4 = 0, x - 1 = 0, y - 2 = 0$. So, the triangle is right angled at (1, 2). The hypotenuse is $x + y - 4 = 0$, whose ends are (1, 3) and (2, 2). This, circumcentre = $\left(\frac{1+2}{2}, \frac{3+2}{2} \right)$

$$\text{and circumradius} = \frac{1}{2} \sqrt{(1-2)^2 + (3-2)^2} = \frac{1}{\sqrt{2}}$$

Hence, the equation of circumcircle is $x^2 + y^2 - 3x - 5y + 8 = 0$.

45. On solving $y = x, x^2 + y^2 - 2x = 0$, we get

$$(x, y) = (0, 0), (1, 1)$$

So, the ends of the diameter are (0, 0), (1, 1).

Hence, the equation of the required circle is

$$(x - 0)(x - 1) + (y - 0)(y - 1) = 0$$

$$\text{or } x^2 + y^2 = x + y$$

46. The centre C of the circle is $(5, 7)$

and radius $= \sqrt{5^2 + 7^2 + 51} = 5\sqrt{5}$

$PC = \sqrt{12^2 + 5^2} = 13$

$\therefore PA = 13 - 5\sqrt{5}$

and $PB = 13 + 5\sqrt{5}$

$GM = \sqrt{PA \cdot PB}$
 $= \sqrt{(13 - 5\sqrt{5})(13 + 5\sqrt{5})} = 2\sqrt{11}$

47. Here, centre of circle $= \left(\frac{3}{4}, 0\right)$

and radius of circle $= \frac{3}{4}$

$\therefore PC = \sqrt{\left(\frac{3}{4} + 2\right)^2 + (0 - 1)^2}$
 $= \sqrt{\frac{121}{16} + 1} = \frac{\sqrt{137}}{4}$

and $PA = \frac{\sqrt{137}}{4} - \frac{3}{4} = \frac{\sqrt{137} - 3}{4} > 2$

Hence, there are two tangents.

48. Let (a, b) should be an internal or end point of the line segment joining A, B . So, the x -coordinate a will vary from 2 to 3 i.e. from the x -coordinate of A to the x -coordinate of B .

Similarly, $b \in (-3, 2)$

49. $y = \sqrt{25 - x^2}$, $y = 0$ bound the semi-circle above the X -axis.

$\therefore a + 1 > 0$ and $a^2 + (a + 1)^2 - 25 < 0$
 $\Rightarrow -1 < a < 3$

50. The given point is an interior point

$\therefore \left(-5 + \frac{r}{\sqrt{2}}\right)^2 + \left(-3 + \frac{r}{\sqrt{2}}\right)^2 - 16 < 0$

$\Rightarrow r^2 - 8\sqrt{2}r + 18 < 0$

$\Rightarrow 4\sqrt{2} - \sqrt{14} < r < 4\sqrt{2} + \sqrt{14}$

So, the point is on the major segment.

This, the centre and the point are on the same side of the line $x + y = 2$.

$\therefore -5 + \frac{r}{\sqrt{2}} - 3 + \frac{r}{\sqrt{2} - 2} < 0$

$\Rightarrow r < 5\sqrt{2}$

Hence, $4\sqrt{2} - \sqrt{14} < r < 5\sqrt{2}$

51. For $x^2 + y^2 = 9$, Centre $= (0, 0)$ and radius $= 3$

For $x^2 + y^2 - 8x - 6y + n^2 = 0$,

Centre $= (4, 3)$ and radius $= \sqrt{4^2 + 3^2 - n^2}$

$\therefore 4^2 + 3^2 - n^2 > 0$ or $n^2 < 5^2$ or $-5 < n < 5$

Since, circle should cut to have exactly two common tangents.

So, $r_1 + r_2 > d$

$\therefore 3 + \sqrt{25 - n^2} > \sqrt{4^2 + 3^2}$

$\Rightarrow \sqrt{25 - n^2} > 2 \Rightarrow 25 - n^2 > 4$

$\therefore n^2 < 21 \Rightarrow -\sqrt{21} < n < \sqrt{21}$

Therefore, common values of n should satisfy

$-\sqrt{21} < n < \sqrt{21}$

But $n \in I$. So, $n = -4, -3, \dots, 4$.

52. On solving the equation,

$x^2 + \left(\frac{13}{4\lambda}\right)^2 = 4$ or $x = \sqrt{4 - \left(\frac{13}{4\lambda}\right)^2}$

It should have two real and distinct values.

So, $4 - \left(\frac{13}{4\lambda}\right)^2 > 0 \Rightarrow \lambda < -\frac{13}{8}$ or $\lambda > \frac{13}{8}$

53. The length of the perpendicular from the centre $(0, 0)$ to the line is $\frac{2}{\sqrt{1 + m^2}}$.

The radius of the circle $= 1$. For the line to cut the circle at distinct or coincident points,

$\frac{2}{\sqrt{1 + m^2}} \leq 1$

$\Rightarrow 1 + m^2 \geq 4$

$\Rightarrow m^2 \geq 3$

54. The line joining $(4, 3)$ and $(2, 1)$ is also along a diameter. So, the centre is the intersection of the diameters.

$2x - y = 2$ and $y - 3 = \frac{3 - 1}{4 - 2} = (x - 4)$

Solving these, we get centre $= (1, 0)$

Also, radius $=$ Distances between $(1, 0)$ and $(2, 1)$
 $= \sqrt{2}$ units

55. $PA =$ Length of the tangent from P to the circle

$= \sqrt{(-2)^2 + (-3)^2 - 2(-2) - 10(-3) + 1}$
 $= \sqrt{48} = 4\sqrt{3}$ units

56. Any line through $P(-2, -1)$ is $y + 1 = m(x + 2)$.

It touches the circle, if $\left| \frac{2m - 1}{\sqrt{1 + m^2}} \right| = 1$.

$\Rightarrow m = 0, \frac{4}{3}$

$\therefore$ The equation of PB is

$y + 1 = \frac{4(x + 2)}{3}$

$\Rightarrow 4x - 3y + 5 = 0$

A point on PB is $(-5, -5)$. Its image by the line $y = -1$ is $P(-5, 3)$.

So, the equation of the incident ray is

$y - 3 = \frac{3 + 1}{-5 + 2}(x + 5)$

$\Rightarrow 4x + 3y + 11 = 0$

57. The vertices are $A(r, 0)$, $B(r \cos 60^\circ, r \sin 60^\circ)$, $C(-r \cos 60^\circ, r \sin 60^\circ)$, $D(-r, 0)$, $E(-r \cos 60^\circ, -r \sin 60^\circ)$ and $F(r \cos 60^\circ, -r \sin 60^\circ)$.

If $P = (x, y)$, then $\Sigma PA^2 = 6a^2$

$\Rightarrow x^2 + y^2 + r^2 = a^2$

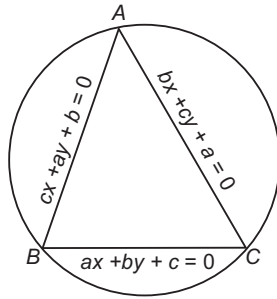
$\Rightarrow x^2 + y^2 = (\sqrt{a^2 - r^2})^2$

58. Let ABC be the triangle formed by the lines $ax + by + c = 0$, $bx + cy + a = 0$ and $cx + ay + b = 0$. The equation of a second degree curve passing through the vertices A, B and C is

$\lambda(ax + by + c)(bx + cy + a) + \mu(bx + cy + a)(cx + ay + b) + \gamma(cx + ay + b)(ax + by + c) = 0 \dots (i)$

This will represent a circle, if
Coefficient of x^2 = Coefficient of y^2 and coefficient of $xy = 0$

$$\Rightarrow \lambda(ab - bc) + \mu(bc - ca) + \gamma(ca - ab) = 0 \quad \dots(ii)$$

$$\text{and } \lambda(b^2 + ac) + \mu(c^2 + ab) + \gamma(a^2 + bc) = 0 \quad \dots(iii)$$


The circle given in Eq.(i) will pass through the origin $(0, 0)$, if

$$\lambda ac + \mu ab + \gamma bc = 0 \quad \dots(iv)$$

Eliminating λ, μ and γ from Eqs. (ii), (iii) and (iv), we get

$$\begin{vmatrix} b^2 + ac & c^2 + ab & a^2 + bc \\ ab - bc & bc - ca & ca - ab \\ ac & ab & bc \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} b^2 & c^2 & a^2 \\ ab - bc & bc - ca & ca - ab \\ ac & ab & bc \end{vmatrix} = 0$$

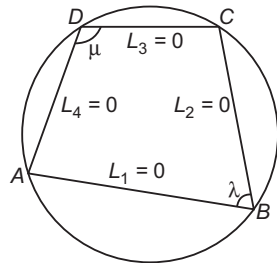
[applying $R_1 \rightarrow R_1 - R_3$]

$$\Rightarrow \begin{vmatrix} b^2 & c^2 & a^2 \\ ab & bc & ca \\ ac & ab & bc \end{vmatrix} + \begin{vmatrix} -bc & -ca & -ab \\ -bc & -ca & -ab \\ -bc & -ca & -ab \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} b^2 & c^2 & a^2 \\ ab & bc & ca \\ ac & ab & bc \end{vmatrix} - \begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ bc & ca & ab \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} b^2 & c^2 & a^2 \\ ab & bc & ca \\ ac & ab & bc \end{vmatrix} = \begin{vmatrix} bc & ca & ab \\ bc & ca & ab \\ bc & ca & ab \end{vmatrix}$$

59. The equation of a second degree curve passing through the vertices of the quadrilateral ABCD is



$$\lambda L_1 L_3 + \mu L_2 L_4 = 0 \quad \dots(i)$$

$$\text{or } \lambda(a_1x + b_1y + c_1)(a_3x + b_3y + c_3) + \mu(a_2x + b_2y + c_2)(a_4x + b_4y + c_4) = 0 \quad \dots(ii)$$

This will represent a circle, if
Coefficient of x^2 = Coefficient of y^2
and coefficient of $xy = 0$

$$\Rightarrow \lambda(a_1a_3 - b_1b_3) + \mu(a_2a_4 - b_2b_4) = 0$$

$$\text{and } \lambda(a_1b_3 + b_1a_3) + \mu(a_2b_4 + b_2a_4) = 0$$

$$\Rightarrow \frac{\lambda}{\mu} = - \left(\frac{a_2a_4 - b_2b_4}{a_1a_3 - b_1b_3} \right) = - \left(\frac{a_2b_4 + a_4b_2}{a_1b_3 + a_3b_1} \right)$$

$$\Rightarrow \frac{a_2a_4 - b_2b_4}{a_1a_3 - b_1b_3} = \frac{a_2b_4 + a_4b_2}{a_1b_3 + a_3b_1}$$

$$\Rightarrow \frac{a_1a_3 - b_1b_3}{a_2a_4 - b_2b_4} = \frac{a_1b_3 + a_3b_1}{a_2b_4 + a_4b_2}$$

Aliter

The quadrilateral ABCD will be a cyclic quadrilateral, if $\theta + \phi + \pi \Rightarrow \theta = \pi - \phi$

$$\Rightarrow \tan \theta = -\tan \phi$$

$$\Rightarrow \frac{-\frac{a_2}{b_2} + \frac{a_1}{b_1}}{1 + \frac{a_2}{b_2} \times \frac{a_1}{b_1}} = \frac{-\frac{a_4}{b_4} + \frac{a_3}{b_3}}{1 + \frac{a_4}{b_4} \times \frac{a_3}{b_3}}$$

$$\Rightarrow \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2} = \frac{a_3b_4 - a_4b_3}{a_3a_4 + b_3b_4}$$

$$\Rightarrow (a_1b_2 - a_2b_1)(a_3a_4 + b_3b_4) = -(a_1a_2 + b_1b_2)(a_3b_4 - a_4b_3)$$

$$\Rightarrow (a_1a_3 - b_1b_3)(a_2b_4 + a_4b_2) = (a_2a_4 - b_2b_4)(a_1b_3 + a_3b_1)$$

$$\Rightarrow \frac{a_1a_3 - b_1b_3}{a_2a_4 - b_2b_4} = \frac{a_1b_3 + a_3b_1}{a_2b_4 + a_4b_2}$$

60. Let $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ be the vertices of ΔABC . Again, let $P(h, k)$ be a variable point such that

$$PA^2 + PB^2 + PC^2 = \lambda^2 \quad [\text{constant}]$$

$$\Rightarrow (x_1 - h)^2 + (y_1 - k)^2 + (x_2 - h)^2 + (y_2 - k)^2 + (x_3 - h)^2 + (y_3 - k)^2 = \lambda^2$$

$$\Rightarrow 3h^2 + 3k^2 - 2h(x_1 + x_2 + x_3) - 2k(y_1 + y_2 + y_3) + x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 - \lambda^2 = 0$$

Hence, the locus of (h, k) is

$$3x^2 + 3y^2 - 2(x_1 + x_2 + x_3)x - 2(y_1 + y_2 + y_3)y + x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 - \lambda^2 = 0$$

$$\Rightarrow x^2 + y^2 - 2\left(\frac{x_1 + x_2 + x_3}{3}\right)x - 2\left(\frac{y_1 + y_2 + y_3}{3}\right)y + \frac{1}{3}(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2 - \lambda^2) = 0$$

which represents a circle.

61. Let the equation of the line L_1 be

$$\frac{x}{a} + \frac{y}{b} = 1 \quad \dots(i)$$

It cuts the coordinate axes at $P(a, 0)$ and $Q(0, b)$. The equation of the line L_2 is $\frac{x}{b} - \frac{y}{a} = \lambda$, where λ is parameter.

$$\Rightarrow \frac{x}{b\lambda} - \frac{y}{a\lambda} = 1$$

This cuts the coordinate axes at $R(b\lambda, 0)$ and $S(0, -a\lambda)$.

Equation of lines PS and QR are

$$\frac{x}{a} - \frac{y}{a\lambda} = 1 \quad \text{and} \quad \frac{x}{b\lambda} + \frac{y}{b} = 1$$

Let $T(h, k)$ be the point of intersection of PS and QR.

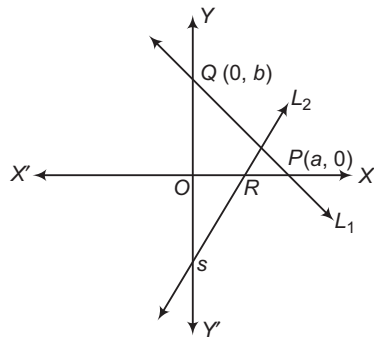
$$\text{Then, } \frac{h}{a} - \frac{k}{a\lambda} = 1 \quad \text{and} \quad \frac{h}{b\lambda} + \frac{k}{b} = 1$$

$$\Rightarrow \frac{h}{a} - 1 = \frac{k}{a\lambda} \quad \text{and} \quad \frac{h}{b\lambda} = 1 - \frac{k}{b}$$

$$\Rightarrow \frac{h-a}{k} = \frac{1}{\lambda} \quad \text{and} \quad \frac{b-k}{h} = \frac{1}{\lambda}$$

$$\Rightarrow \frac{h-a}{k} = \frac{b-k}{h}$$

$$\Rightarrow h^2 + k^2 - ah - bk = 0$$



Hence, the locus of $T(h, k)$ is

$$x^2 + y^2 - ax - by = 0$$

which represents a circle passing through the origin.

- 62.** Let the coordinates of A and B be $(a, 0)$ and $(0, b)$, respectively.

Then, the equation of the circle passing through $O(0, 0)$, $A(a, 0)$ and $B(0, b)$ is

$$x^2 + y^2 - ax - by = 0 \quad \dots(i)$$

It is given that the radius of circle (i) is r .

$$\therefore r^2 = \frac{a^2}{4} + \frac{b^2}{4}$$

$$\Rightarrow a^2 + b^2 = 4r^2 \quad \dots(ii)$$

The equation of sides AB is $\frac{x}{a} + \frac{y}{b} = 1$

Let $P(h, k)$ be the foot of the perpendicular from O to AB . Then,

$$\frac{k-0}{h-0} \times \frac{b-0}{0-a} = -1$$

$$\Rightarrow \frac{h}{b} = \frac{k}{a} = \lambda \quad [\text{say}]$$

$$\Rightarrow a = \frac{k}{\lambda} \quad \text{and} \quad b = \frac{h}{\lambda}$$

Since, (h, k) lies on $\frac{x}{a} + \frac{y}{b} = 1$.

$$\therefore \frac{h}{a} + \frac{k}{b} = 1 \Rightarrow \lambda \left(\frac{h}{k} + \frac{k}{h} \right) = 1$$

$$\Rightarrow \lambda = \frac{hk}{h^2 + k^2}$$

$$\therefore a = \frac{h^2 + k^2}{h} \quad \text{and} \quad b = \frac{h^2 + k^2}{k}$$

On substituting the values of a and b in Eq. (ii), we get

$$\frac{(h^2 + k^2)^2}{h^2} + \frac{(h^2 + k^2)^2}{k^2} = 4r^2$$

$$\Rightarrow (h^2 + k^2)^3 = 4h^2k^2r^2$$

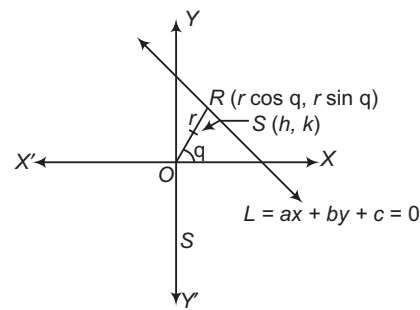
Hence, the locus of (h, k) is

$$(x^2 + y^2)^3 = 4x^2y^2r^2$$

- 63.** Let the fixed point O be at the origin $(0, 0)$ and the fixed line L be $ax + by + c = 0$ such that $c \neq 0$. Again, let R be a variable point on line L such that $OR = r$ and $\angle XOR = \theta$. Then, the coordinates of R are $(r \cos \theta, r \sin \theta)$.

Let $S(h, k)$ be a variable point on OR such that $OS = r_1$.

Then, $h = r_1 \cos \theta$, $k = r_1 \sin \theta$



It is given that

$$OR \cdot OS = \lambda^2$$

$$\Rightarrow r r_1 = \lambda^2 \Rightarrow r_1 = \frac{\lambda^2}{r}$$

$$\Rightarrow h = \frac{\lambda^2 \cos \theta}{r} \quad \text{and} \quad k = \frac{\lambda^2 \sin \theta}{r} \quad \dots(i)$$

$$\Rightarrow h^2 + k^2 = \frac{\lambda^4}{r^2} (\cos^2 \theta + \sin^2 \theta)$$

$$h^2 + k^2 = \frac{\lambda^4}{r^2} \quad \dots(ii)$$

Since, $R(r \cos \theta, r \sin \theta)$ lies on the line $ax + by + c = 0$.

$$\therefore (a \cos \theta + b \sin \theta) r + c = 0$$

$$\Rightarrow \left(\frac{ahr}{\lambda^2} + \frac{bkr}{\lambda^2} \right) r + c = 0 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow \left(\frac{ah + bk}{\lambda^2} \right) r^2 + c = 0 \quad \dots(iii)$$

On eliminating r from Eqs. (ii) and (iii), we get

$$(ah + bk) \frac{\lambda^2}{h^2 + k^2} + c = 0$$

$$\Rightarrow h^2 + k^2 + \frac{\lambda^2}{c} (ah + bk) = 0$$

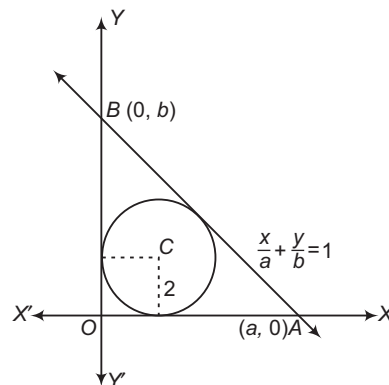
Thus, the locus of (h, k) is

$$x^2 + y^2 + \frac{\lambda^2}{c} (ax + by) = 0$$

which represents a circle.

- 64.** Let OAB be the triangle in which the circle $x^2 + y^2 - 4x - 4y + 4 = 0$ is inscribed.

Let equation of AB be $\frac{x}{a} + \frac{y}{b} = 1$.



Since, AB touches the circle

$$x^2 + y^2 - 4x - 4y + 4 = 0$$

Therefore,

$$\therefore \frac{\left| \frac{2}{a} + \frac{2}{b} - 1 \right|}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = 2$$

$$\Rightarrow \frac{\left(\frac{2}{a} + \frac{2}{b} - 1 \right)}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}} = 2$$

$[\because O(0, 0) \text{ and } C(2, 2) \text{ lies on the same side of } AB, \text{ therefore } \frac{2}{a} + \frac{2}{b} - 1 < 0]$

$$\Rightarrow \frac{-(2b + 2a - ab)}{\sqrt{a^2 + b^2}} = 2$$

$$\Rightarrow 2a + 2b - ab + 2\sqrt{a^2 + b^2} = 0 \quad \dots(i)$$

Let $P(h, k)$ be the circumcentre of ΔOAB .

Since, ΔOAB is a right angled triangle. So, its circumcentre is the mid-point of AB .

$$\therefore h = \frac{a}{2}$$

$$\text{and } k = \frac{b}{2}$$

$$\Rightarrow a = 2h \text{ and } b = 2k \quad \dots(ii)$$

Form Eqs. (i) and (ii), we get

$$4k + 4k - 4hk + 2\sqrt{4h^2 + 4k^2} = 0$$

$$\Rightarrow h + k - hk + \sqrt{h^2 + k^2} = 0$$

So, the locus of $P(h, k)$ is

$$x + y - xy + \sqrt{x^2 + y^2} = 0$$

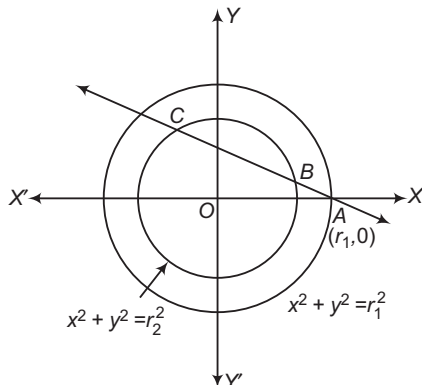
But the locus of the circumcentre is given to be

$$x + y - xy + k\sqrt{x^2 + y^2} = 0$$

Thus, the value of k is 1.

65. Let the equation of line AB be

$$\frac{x - r_1}{\cos \theta} = \frac{y - 0}{\sin \theta} = r$$



The coordinates of any point on this line are $(r_1 + r \cos \theta, r \sin \theta)$

If it lies on $x^2 + y^2 = r_2^2$, then

$$(r_1 + r \cos \theta)^2 + (r \sin \theta)^2 = r_2^2$$

$$\Rightarrow r^2 + 2r_1 r \cos \theta + r_1^2 - r_2^2 = 0 \quad \dots(i)$$

Let $AB = r_B$ and $AC = r_C$. Then, r_B and r_C are the roots of Eq. (i). Therefore,

$$r_B + r_C = -2r_1 \cos \theta \text{ and } r_B r_C = r_1^2 - r_2^2$$

$$\therefore BC^2 = (r_C - r_B)^2$$

$$= (r_C + r_B)^2 - 4r_B r_C$$

$$= 4r_1^2 \cos^2 \theta - 4r_1^2 + 4r_2^2$$

Now, $OA^2 + OB^2 + BC^2$

$$= r_1^2 + r_2^2 + 4r_1^2 \cos^2 \theta - 4r_1^2 + 4r_2^2$$

$$= 5r_2^2 - 3r_1^2 + 4r_1^2 \cos^2 \theta$$

Now, $0 \leq \cos^2 \theta \leq 1$

$$\Rightarrow 0 \leq 4r_1^2 \cos^2 \theta \leq 4r_1^2$$

$$\Rightarrow 5r_2^2 - 3r_1^2 \leq 5r_2^2 - 3r_1^2 + 4r_1^2 \cos^2 \theta$$

$$\leq 5r_2^2 - 3r_1^2 + 4r_1^2$$

$$\Rightarrow 5r_2^2 - 3r_1^2 \leq OA^2 + OB^2 + BC^2$$

$$\leq 5r_2^2 + r_1^2 \quad [\text{from Eq. (ii)}]$$

$$\Rightarrow OA^2 + OB^2 + BC^2 \in [5r_2^2 - 3r_1^2, 5r_2^2 + r_1^2]$$

66. Let A_1A_2 and B_1B_2 be two rods of lengths a and b which slide along OX and OY respectively and

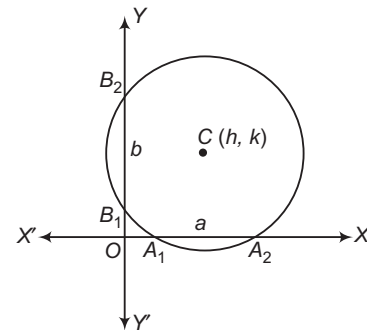
$$x^2 + y^2 + 2gx + 2fy + c = 0$$

be the circle passing through A_1, A_2, B_1 and B_2 . Then, A_1A_2 = Intercept on X-axis

$$\Rightarrow A_1A_2 = 2\sqrt{g^2 - c}$$

$$\Rightarrow a = 2\sqrt{g^2 - c} \quad \dots(i)$$

and B_1B_2 = Intercept on Y-axis



$$\Rightarrow B_1B_2 = 2\sqrt{f^2 - c} \quad \dots(ii)$$

$$\Rightarrow b = 2\sqrt{f^2 - c}$$

Here, c is the variable which is to be eliminated to find the locus of the centre $(-g, -f)$ of the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

On eliminating c from Eqs. (i) and (ii), we get

$$a^2 - b^2 = 4(g^2 - f^2)$$

Thus, the locus of $(-g, -f)$ is

$$a^2 - b^2 = 4(x^2 - y^2)$$

Hence, the locus of the centre of the circle is

$$a^2 - b^2 = 4(x^2 - y^2)$$

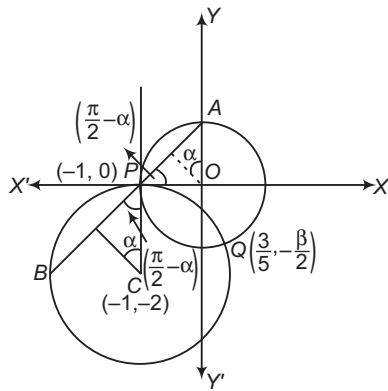
67. On solving the equations $x^2 + y^2 = 1$

$$\text{and } x^2 + y^2 + 2x + 4y + 1 = 0$$

We find that the two circles intersect at $P(-1, 0)$ and

$$Q\left(\frac{3}{5}, -\frac{4}{5}\right).$$

Clearly, point P has integral coordinates.



Let $y - 0 = m(x + 1)$ or $y = m(x + 1)$ be a chord AB passing through P meeting the two circles at A and B . It is given that chords AP and BP make the same angle 2α at the centres $O(0, 0)$ and $C(-1, -2)$ respectively of the given circles.

We have,

$$\begin{aligned} \text{Slope of } OP &= 0 \\ \text{and slope of } AP &= m \\ \Rightarrow m &= \tan\left(\frac{\pi}{2} - \alpha\right) \\ \Rightarrow m &= \cot \alpha \end{aligned} \quad \dots(i)$$

$$\text{Also, slope of } CP = \frac{2}{m'}$$

where,

$$m' = 0$$

We have,

$$\text{Slope of } BP = m \text{ and angle between } CP \text{ and } BP \text{ is } \left(\frac{\pi}{2} - \alpha\right).$$

$$\begin{aligned} \text{Therefore, } \tan\left(\frac{\pi}{2} - \alpha\right) &= \left| \frac{m - \frac{2}{m'}}{1 + m \times \frac{2}{m'}} \right| \\ &= \left| \frac{mm' - 2}{m' + 2m} \right| = \left| \frac{1}{m} \right| \\ \Rightarrow \cot \alpha &= \left| \frac{1}{m} \right| \end{aligned} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\begin{aligned} m &= \left| \frac{1}{m} \right| \\ \Rightarrow m|m| &= 1 \Rightarrow m^2 = 1 \\ \Rightarrow m &\pm 1 \end{aligned}$$

Clearly, for $m = -1$, we obtain chord PAB when A and B lie on the same side of P .

So, we take $m = 1$.

On putting $m = 1$ in $y = m(x + 1)$, we obtain $y = x + 1$

which is the equation of the chord APB .

68. We know that, the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ does not intersect or touch with the coordinate axes, if $g^2 - c < 0$ and $f^2 - c < 0$

Therefore, the circle $x^2 + y^2 - 6x - 10y + \lambda = 0$ will not intersect or touch the coordinate axes, if

$$\begin{aligned} 9 - \lambda < 0 \quad \text{and} \quad 25 - \lambda < 0 \\ \Rightarrow \lambda > 9 \quad \text{and} \quad \lambda > 25 \\ \Rightarrow \lambda > 25 \end{aligned} \quad \dots(i)$$

It is given that the point $(1, 4)$ lies inside the circle $x^2 + y^2 - 6x - 10y + \lambda = 0$

Therefore,

$$\begin{aligned} 17 - 6 - 40 + \lambda < 0 \\ \Rightarrow \lambda < 29 \end{aligned} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get $\lambda \in (25, 29)$

69. Let the line $lx + my + 1 = 0$ touch the circle

$$(x - a)^2 + (y - b)^2 = r^2$$

$$\begin{aligned} \text{Then, } \left| \frac{la + mb + 1}{\sqrt{l^2 + m^2}} \right| &= r \\ \Rightarrow (la + mb + 1)^2 &= r^2(l^2 + m^2) \\ \Rightarrow l^2(a^2 - r^2) + m^2(b^2 - r^2) + 2ablm + 2al &+ 2bm + 1 = 0 \end{aligned}$$

On comparing this with the equation $4l^2 - 5m^2 + 6l + 1 = 0$, we have

$$\begin{aligned} a^2 - r^2 &= 4 \\ b^2 - r^2 &= -5 \\ 2ab &= 0 \\ 2a &= 6 \\ 2b &= 0 \\ \Rightarrow a &= 3, b = 0 \\ \text{and } r^2 &= 5 \end{aligned}$$

Clearly, above equations are consistent. Thus, the line touches the fixed circle

$$(x - 3)^2 + (y - 0)^2 = 5$$

70. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

If $lx + my + 1 = 0$ touches this circle, then

$$\begin{aligned} \left| \frac{-lg - mf + 1}{\sqrt{l^2 + m^2}} \right| &= \sqrt{g^2 + f^2 - c} \\ \Rightarrow (f^2 - c)y^2 + (g^2 - c)m^2 + 2l(g - mfg) + 2mf - 1 &= 0 \\ \Rightarrow (c - f^2)y^2 + (c - g^2)m^2 + 2l(mfg - g) - 2mf + 1 &= 0 \end{aligned}$$

On comparing this equation with

$$al^2 - bm^2 + 2dl + 1 = 0, \text{ we get } c - f^2 = a, c - g^2 = -b$$

$$mfg - g = d \text{ and } f = 0$$

$$\Rightarrow c = a, g = -d$$

$$\text{and } c - d^2 = -b$$

$$\Rightarrow a - d^2 = -b$$

$$\Rightarrow a + b = d^2, \text{ which is the required condition.}$$

Hence, the fixed circle is $x^2 + y^2 - 2dx + a = 0$.

71. The equations of the given circles are

$$x^2 + y^2 - 2x - 2y + 1 = 0 \quad \dots(i)$$

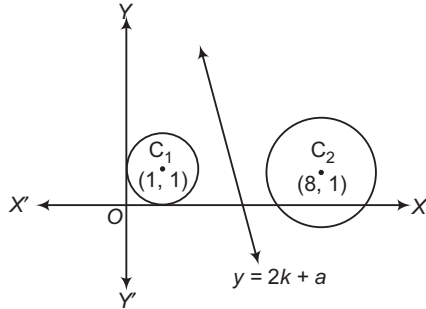
$$\text{and } x^2 + y^2 - 16x - 2y + 61 = 0 \quad \dots(ii)$$

The coordinates of the centres and radii of these two circles are $C_1(1, 1)$, $r_1 = 1$ and $C_2(8, 1)$, $r_2 = 2$ respectively.

For the line $y = 2x + a$ not to touch or intersect circle (i), we must have

$$\left| \frac{1+a}{\sqrt{5}} \right| > 1$$

[$\because$ length of perpendicular from centre C_1 > radius r_1]



$$\Rightarrow \left| \frac{1+a}{\sqrt{5}} \right| > \sqrt{5}$$

$$\Rightarrow a \in (-\infty, -\sqrt{5}-1) \cup (\sqrt{5}-1, \infty) \quad \dots(iii)$$

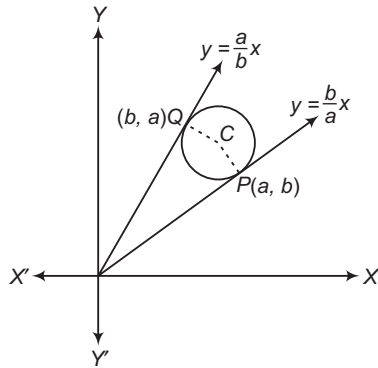
Similarly, for the line $y = 2x + a$ not to touch or intersect circle (ii), we must have

$$\left| \frac{15+a}{\sqrt{5}} \right| > 2$$

$$\Rightarrow |15+a| > 2\sqrt{5}$$

$$\Rightarrow a \in (-\infty, -15-2\sqrt{5}, -15+2\sqrt{5}, \infty) \quad \dots(iv)$$

The line $y = 2x + a$ will be between the circles, if their centres C_1 and C_2 are on the opposite side of it.



$$\therefore (2-1+a)(16-1+a) < 0$$

$$\Rightarrow (a+1)(a+15) < 0$$

$$\Rightarrow a \in (-15, -1) \quad \dots(v)$$

From Eqs. (iii), (iv) and (v), we get

$$a \in (-15+2\sqrt{5}, -\sqrt{5}-1)$$

72. The equations of OP and OQ are $y = \frac{b}{a}x$ and $y = \frac{a}{b}x$, respectively.

Clearly, centre C of the required circle is the point of intersection of CP and CQ .

The equation of a line passing through P and perpendicular to OP is

$$y - b = -\frac{a}{b}(x - a)$$

$$\Rightarrow ax + by = a^2 + b^2$$

Similarly, the equation of a line through Q and perpendicular to OQ is

$$y - a = -\frac{b}{a}(x - b)$$

$$\Rightarrow bx + ay = a^2 + b^2$$

On solving Eqs. (i) and (ii), we obtain the coordinates of C as

$$\left(\frac{a^2 + b^2}{a + b}, \frac{a^2 + b^2}{a + b} \right)$$

Now, CP = Length of perpendicular from C on

$$bx - ay = 0$$

$$= \frac{\left| b \left(\frac{a^2 + b^2}{a + b} \right) - a \left(\frac{a^2 + b^2}{a + b} \right) \right|}{\sqrt{b^2 + (-a)^2}}$$

$$= \frac{|b - a|}{a + b} \sqrt{a^2 + b^2}$$

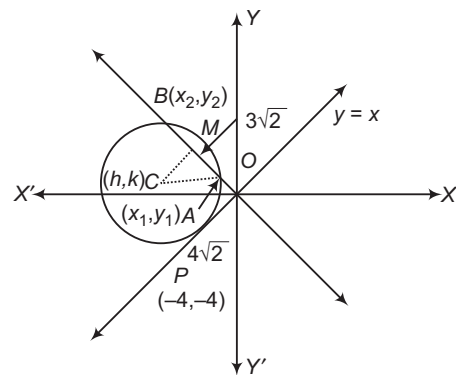
Thus, the equation of the required circle is

$$\left(x - \frac{a^2 + b^2}{a + b} \right)^2 + \left(y - \frac{a^2 + b^2}{a + b} \right)^2 = \left\{ \frac{|b - a|}{a + b} \sqrt{a^2 + b^2} \right\}^2$$

$$\Rightarrow x^2 + y^2 - 2(x + y) \left(\frac{a^2 + b^2}{a + b} \right) + (a^2 + b^2) = 0$$

73. The parametric form of OP is

$$\frac{x - 0}{\cos \pi/4} = \frac{y - 0}{\sin \pi/4}$$



Since, $OP = 4\sqrt{2}$ units

So, the coordinates of P are given by

$$\frac{x - 0}{\cos \pi/4} = \frac{y - 0}{\sin \pi/4} = -4\sqrt{2}$$

$$\Rightarrow x = -4, y = -4$$

Thus, the coordinates of P are $(-4, -4)$.

Let $C(h, k)$ be the centre of the circle and r be its radius.

Now, $CP \perp OP$

$$\Rightarrow \frac{k + 4}{h + 4} \times 1 = -1$$

$$\Rightarrow k + 4 = -h - 4$$

$$\Rightarrow h + k = -8 \quad \dots(i)$$

$$\text{Also, } CP^2 = (h + 4)^2 + (k + 4)^2$$

$$\Rightarrow (h + 4)^2 + (k + 4)^2 = r^2 \quad \dots(ii)$$

In $\triangle ACM$, we have

$$AC^2 = AM^2 + CM^2$$

$$\Rightarrow r^2 = (3\sqrt{2})^2 + \left(\frac{h+k}{\sqrt{2}}\right)^2$$

$$\Rightarrow r^2 = 18 + \left(\frac{-8}{\sqrt{2}}\right)^2$$

$$\Rightarrow r^2 = 18 + 32$$

$$\Rightarrow r = 5\sqrt{2} \text{ units}$$

Also, $CP = r$

$$\Rightarrow \left|\frac{h-k}{\sqrt{2}}\right| = r \Rightarrow \left|\frac{h-k}{\sqrt{2}}\right| = 5\sqrt{2}$$

$$\Rightarrow |h-k| = 10 \Rightarrow h-k = \pm 10 \quad \dots(iii)$$

From Eqs. (i) and (iv), we get

$$h = -9, k = 1 \text{ or } h = 1, k = -9$$

Thus, the equations of the circles are

$$(x+9)^2 + (y-1)^2 = (5\sqrt{2})^2$$

$$\text{and } (x-1)^2 + (y+9)^2 = (5\sqrt{2})^2$$

$$\text{or } x^2 + y^2 + 18x - 2y + 32 = 0$$

$$\text{and } x^2 + y^2 - 2x + 18y + 32 = 0$$

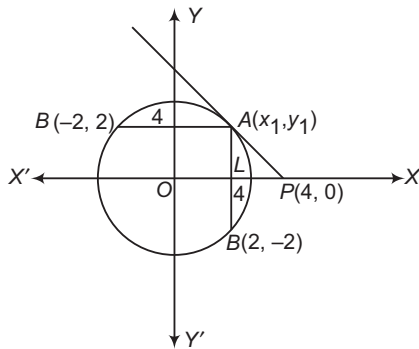
Clearly, $(-10, 2)$ lies in the interior of

$$x^2 + y^2 + 18x - 2y + 32 = 0$$

Hence, the equation of the required circle is

$$x^2 + y^2 + 18x - 2y + 32 = 0$$

74. Let $A(x_1, y_1)$ be a point on the circle $x^2 + y^2 = 8$ such that the tangent at A passes through $P(4, 0)$.



The equation of the tangent at $A(x_1, y_1)$ to the circle $x^2 + y^2 = 8$ is

$$xx_1 + yy_1 = 8$$

It passes through, $P(4, 0)$.

$$\therefore 4x_1 = 8 \Rightarrow x_1 = 2$$

Since, (x_1, y_1) lies on $x^2 + y^2 = 8$.

$$x_1^2 + y_1^2 = 8$$

$$\Rightarrow 4 + y_1^2 = 8$$

$$\Rightarrow y_1^2 = 4$$

$$\Rightarrow y_1 = \pm 2$$

$$\Rightarrow y_1 = 2$$

$\therefore A(x_1, y_1)$ lies in first quadrant]

Thus, the coordinates of A are $(2, 2)$.

Let $B(x_2, y_2)$ be a point on the given circle such that $AB = 4$. Then,

$$AB = 4$$

$$\Rightarrow (x_2 - 2)^2 + (y_2 - 2)^2 = 4^2$$

$$\Rightarrow x_2^2 + y_2^2 - 4(x_2 + y_2) = 8$$

$$\Rightarrow 8 - 4(x_2 + y_2) = 8$$

$$[\because B(x_2, y_2) \text{ lies on } x^2 + y^2 = 8, x_2^2 + y_2^2 = 8]$$

$$\Rightarrow x_2 + y_2 = 0$$

$$\Rightarrow y_2 = -x_2$$

$$\text{Now, } x_2^2 + y_2^2 = 8$$

$$\Rightarrow 2x_2^2 = 8 \Rightarrow x_2^2 = 4 \Rightarrow x_2 = \pm 2$$

$$\therefore y_2 = -x_2 \Rightarrow y_2 = \mp 2$$

Hence, the possible coordinates of B are

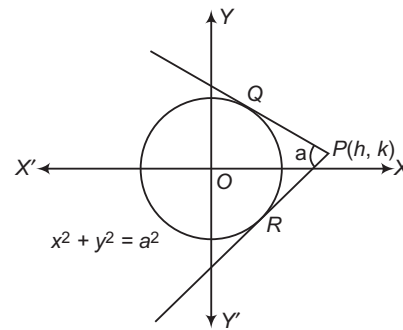
$$(2, -2) \text{ and } (-2, 2).$$

75. Let $P(h, k)$ be the point of intersection of tangents drawn from P to the circle

$$x^2 + y^2 = a^2$$

The equation of any tangent to the circle

$$x^2 + y^2 = a^2 \text{ is } y = mx + a\sqrt{1+m^2}$$



If it passes through $P(h, k)$ then

$$k = mh + a\sqrt{1+m^2}$$

$$\Rightarrow (k - mh)^2 = a^2(1+m^2)$$

$$\Rightarrow m^2(h^2 - a^2) - 2mhk + k^2 - a^2 = 0$$

Let m_1 and m_2 be the roots of this equation. Then,

$$m_1 + m_2 = \frac{2hk}{h^2 - a^2}$$

and

$$m_1 m_2 = \frac{k^2 - a^2}{h^2 - a^2}$$

Clearly, m_1 and m_2 represent slopes of two tangents drawn from P which are inclined at an angle α with each other.

$$\tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$$\Rightarrow (1 + m_1 m_2) \tan \alpha = \sqrt{(m_1 + m_2)^2 - 4m_1 m_2}$$

$$\Rightarrow \left(1 + \frac{k^2 - a^2}{h^2 - a^2}\right) \tan^2 \alpha = \left[\frac{4h^2 k^2}{(h^2 - a^2)^2} - 4\left(\frac{k^2 - a^2}{h^2 - a^2}\right)\right]$$

$$\Rightarrow (h^2 - k^2 - 2a^2)^2 \tan^2 \alpha = 4a^2(k^2 + h^2 - a^2)$$

Hence, the locus of $P(h, k)$ is

$$(x^2 + y^2 - 2a^2)^2 \tan^2 \alpha = 4a^2(x^2 + y^2 - a^2)$$

76. Let $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ and

$S(x_4, y_4)$ be four points in a plane. Then,

the equation of a circle with RS as diameter is

$$(x - x_3)(x - x_4) + (y - y_3)(y - y_4) = 0$$

$$\therefore \{P, RS\} = (x_1 - x_3)(x_1 - x_4) + (y_1 - y_3)(y_1 - y_4)$$

Similarly, we have

$$\begin{aligned}\{P, QS\} &= (x_1 - x_2)(x_1 - x_4) + (y_1 - y_2)(y_1 - y_4) \\ \{Q, PR\} &= (x_2 - x_1)(x_2 - x_3) + (y_2 - y_1)(y_2 - y_3) \\ \text{and } \{Q, RS\} &= (x_2 - x_3)(x_2 - x_4) + (y_2 - y_3)(y_2 - y_4) \\ \text{Hence, } \{P, RS\} - \{P, QS\} + \{Q, PR\} - \{Q, RS\} &= 0\end{aligned}$$

77. The equation of the chord of contact of tangent drawn from a point $\{\alpha, \beta\}$ to the circle

$$x^2 + y^2 = r_1^2 \text{ is } \alpha x + \beta y = r_1^2.$$

This line touches the circle $(x - a)^2 + (y - b)^2 = r_2^2$

$$\therefore \left| \frac{a\alpha + b\beta - r_1^2}{\sqrt{a^2 + b^2}} \right| = r_2^2$$

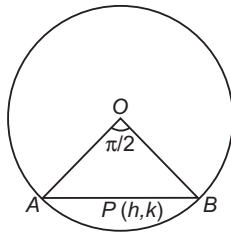
$$\Rightarrow r_2^2 (\alpha^2 + \beta^2) = (a\alpha + b\beta - r_1^2)^2$$

$$\Rightarrow r_2^2 (\alpha^2 + \beta^2) = (r_1^2 - a\alpha - b\beta)^2$$

78. Let (h, k) be the mid-point of a chord AB of the circle $x^2 + y^2 = a^2$. Then, the equation of AB is

$$hx + ky - a^2 = h^2 + k^2 - a^2 \quad [\because T = S]$$

$$\Rightarrow hx + ky = h^2 + k^2 \quad \dots(i)$$



The combined equation of OA and OB is

$$x^2 + y^2 = a^2 \left(\frac{hx + ky}{h^2 + k^2} \right)^2$$

$$\Rightarrow (h^2 + k^2)^2 (x^2 + y^2) - a^2 (hx + ky)^2 = 0$$

Since, OA and OB will be perpendicular, if Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow (h^2 + k^2)^2 - a^2 h^2 + (h^2 + k^2)^2 - a^2 k^2 = 0$$

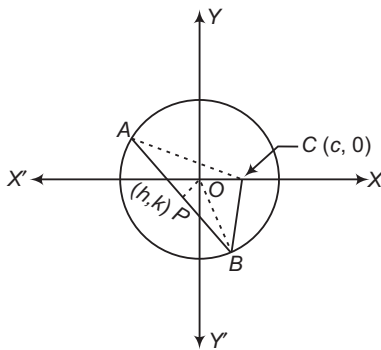
$$\Rightarrow 2(h^2 + k^2)^2 - a^2 (h^2 + k^2) = 0$$

$$\Rightarrow 2(h^2 + k^2) - a^2 = 0$$

So, locus of (h, k) is $2(x^2 + y^2) - a^2 = 0$

79. Let $P(h, k)$ be the mid-point of a chord AB of the circle $x^2 + y^2 = a^2$ which subtends a right angle at the point $C(c, 0)$. Since, mid-point of the hypotenuse of a right triangle is equidistant from the vertices.

$$\therefore PA = PB = PC$$



$$\Rightarrow PA = PB = \sqrt{(h - c)^2 + k^2}$$

Also, in $\triangle OPB$, we have

$$OB^2 = OP^2 + PB^2$$

$$\Rightarrow a^2 = h^2 + k^2 + PB^2$$

$$\Rightarrow a^2 = h^2 + k^2 + (h - c)^2 + k^2 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 2(h^2 + k^2) - 2hc + c^2 - a^2 = 0$$

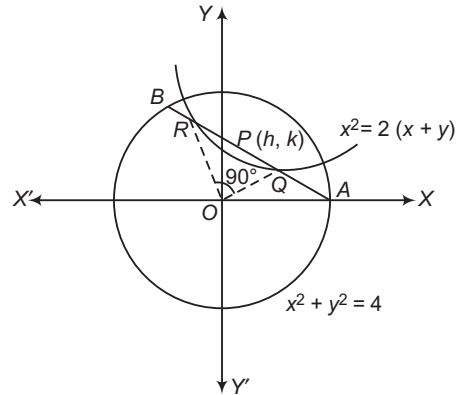
Hence, the locus of $P(h, k)$ is

$$2(x^2 + y^2) - 2cx + c^2 - a^2 = 0$$

80. Let $P(h, k)$ be the mid-point of a chord AB of the circle $x^2 + y^2 = 4$.

Then, the equation of the chord AB is

$$hx + ky = h^2 + k^2$$



Let QR be the segment intercepted by the chord AB on the curve $x^2 = 2(x + y)$ such that $\angle QOR = 90^\circ$.

The combined equation of OQ and OR is

$$x^2 = 2(x + y) \left(\frac{hx + ky}{h^2 + k^2} \right)$$

$$\Rightarrow x^2(h^2 + k^2) = (x + y)(hx + ky)$$

which represents a pair of perpendicular lines.

$\therefore$ Coefficient of x^2 + Coefficient of $y^2 = 0$

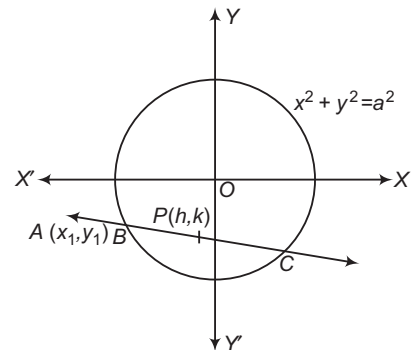
$$\Rightarrow h^2 + k^2 - 2h - 2k = 0$$

Hence, the locus of $P(h, k)$ is

$$x^2 + y^2 - 2x - 2y = 0$$

81. Let $P(h, k)$ be the mid-point of chord BC intercepted by the circle $x^2 + y^2 = a^2$ on the secant drawn through $A(x_1, y_1)$. Then, the equation of chord BC is

$$hx + ky = h^2 + k^2$$



Since, it passes through $A(x_1, y_1)$.

$$\therefore hx_1 + ky_1 = h^2 + k^2$$

Hence, the locus of (h, k) is

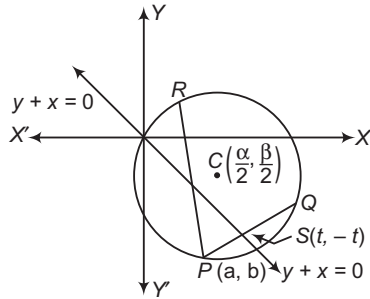
$$x^2 + y^2 = xx_1 + yy_1$$

82. Let $\alpha = \frac{1 + \sqrt{2}a}{2}$ and $\beta = \frac{1 - \sqrt{2}a}{2}$.

Then, the equation of the circle is

$$2x^2 + 2y^2 - 2\alpha x - 2\beta y = 0$$

$$\Rightarrow x^2 + y^2 - \alpha x - \beta y = 0 \quad \dots(i)$$



The coordinates of the given point are (α, β) . Clearly, it lies on the circle (i).

Let the chord PQ be bisected at $S(t, -t)$.

Then, its equation is

$$tx - ty - \frac{\alpha}{2}(x+t) - \frac{\beta}{2}(y-t) = 2t^2 - \alpha t + \beta t$$

$$\Rightarrow x\left(t - \frac{\alpha}{2}\right) - y\left(t - \frac{\beta}{2}\right) = \frac{4t^2 - \alpha t + \beta t}{2}$$

$$\Rightarrow 2\alpha t - \alpha^2 + 2\beta t - \beta^2 = 4t^2 - \alpha t + \beta t$$

$$\Rightarrow 4t^2 - 3\alpha t + 3\beta t + (\alpha^2 + \beta^2) = 0$$

$$\Rightarrow 4t^2 - 3(\alpha - \beta)t + (\alpha^2 + \beta^2) = 0$$

Since, t is real. Therefore, roots of the above equation are real.

Hence, $9(\alpha - \beta)^2 - 16(\alpha^2 + \beta^2) = 0$

$$\Rightarrow 9 \times 2a^2 - 8(1 + 2a^2) > 0$$

$$\Rightarrow 2a^2 - 8 > 0$$

$$\Rightarrow a^2 - 4 > 0$$

$$\Rightarrow a \in (-\infty, -2) \cup (2, \infty)$$

83. Let (x_1, y_1) be the given point. Then, the equation of the polar of (x_1, y_1) with respect to the circles $x^2 + y^2 - 2kx + c^2 = 0$ is

$$xx_1 + yy_1 - k(x + x_1) + c^2 = 0$$

$$\Rightarrow (xx_1 + yy_1 + c^2) - k(x + x_1) = 0$$

This is a straight line passing through a fixed point which is the point of intersection of the lines

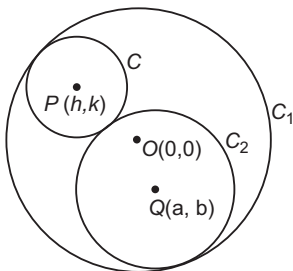
$$xx_1 + yy_1 + c^2 = 0 \quad \text{and} \quad x + x_1 = 0$$

Hence, the point of intersection is $\left(-x_1, \frac{x_1^2 - c^2}{y_1}\right)$.

84. Let the equations of the circles C_1 and C_2 be

$$x^2 + y^2 = r_1^2 \quad \dots(i)$$

and $(x - \alpha)^2 + (y - \beta)^2 = r_2^2 \quad \dots(ii)$



Let $P(h, k)$ be the centre of circle C and r be its radius. Since, circle C touches circle C_1 internally and the circle C_2 externally.

$$\therefore OP = r_1 - r \quad \text{and} \quad PQ = r + r_2$$

$$\Rightarrow \sqrt{h^2 + k^2} = r_1 - r$$

and $\sqrt{(h - \alpha)^2 + (k - \beta)^2} = r + r_2$

$$\Rightarrow \sqrt{h^2 + k^2} + \sqrt{(h - \alpha)^2 + (k - \beta)^2} = r + r_2$$

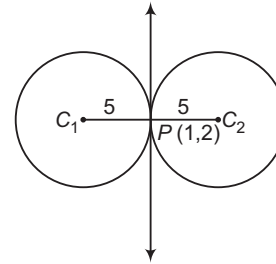
[on eliminating r]

Thus, the locus of (h, k) is

$$\sqrt{x^2 + y^2} + \sqrt{(x - \alpha)^2 + (y - \beta)^2} = r_1 + r_2$$

Clearly, it represents an ellipse having its foci at $O(0, 0)$ and $Q(\alpha, \beta)$ and the length of the major axis is $r_1 + r_2$.

85. We have,



Slope of the tangent $= -\frac{4}{3}$

$\therefore$ Slope of $C_1C_2 = \frac{3}{4}$

If C_1C_2 makes an angle θ with X -axis, then

$$\cos \theta = \frac{4}{5} \quad \text{and} \quad \sin \theta = \frac{3}{5}$$

So, the equation of C_1C_2 in parametric form is

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} \quad \dots(i)$$

Since, C_1 and C_2 are points on Eq. (i) at a distance of 5 units from P . So, the coordinates of C_1 and C_2 are given by

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} = \pm 5$$

$$\Rightarrow x = 1 \pm 4 \quad \text{and} \quad y = 2 \pm 3$$

Thus, the coordinates of C_1 and C_2 are $(5, 5)$ and $(-3, -1)$, respectively.

Hence, the equations of the two circles are

$$(x - 5)^2 + (y - 5)^2 = 5^2$$

and $(x + 3)^2 + (y + 1)^2 = 5^2$

86. The equations of the given circles are

$$(x - a)^2 + y^2 = b^2$$

$$\Rightarrow x^2 + y^2 - 2ax + a^2 - b^2 = 0 \quad \dots(i)$$

and $(x + a)^2 + y^2 = c^2$

$$\Rightarrow x^2 + y^2 + 2ax + a^2 - c^2 = 0 \quad \dots(ii)$$

Let $P(h, k)$ be a point of contact of common tangents to these circles such that it lies on Eq. (i).

Then, $h^2 + k^2 - 2ah + a^2 - b^2 = 0 \quad \dots(iii)$

$\therefore$ Equation of the tangent at $P(h, k)$ to

$$x^2 + y^2 - 2ax + a^2 - b^2 = 0$$

$$hx + ky - a(x + h) + a^2 - b^2 = 0$$

$$\Rightarrow x(h - a) + ky - ah + a^2 - b^2 = 0$$

This will touch circle (ii), if

$$\left| \frac{-a(h-a) + k \times 0 - ah + a^2 - b^2}{\sqrt{(h-a)^2 + k^2}} \right| = c$$

$$\Rightarrow \left| \frac{-2ah + 2a^2 - b^2}{b} \right| = c$$

$$\Rightarrow 2ah - (2a^2 - b^2) = \pm bc$$

$$\Rightarrow 2ah - (a^2 - b^2) = a^2 \pm bc$$

$$\Rightarrow h^2 + k^2 = a^2 \pm bc$$

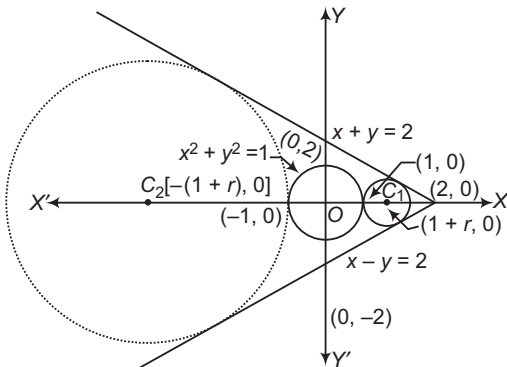
$$\text{Since, } 2ah - (a^2 - b^2) = h^2 + k^2$$

[from Eq. (iii)]

Hence, $P(h, k)$ lies on $x^2 + y^2 = a^2 \pm bc$.

87. Clearly, lines $x + y = 2$ and $x - y = 2$ intersect at $(2, 0)$.

So, the centre of the required circle lies on the bisector of the angle containing the origin. Let r be the radius of the required circle. Then, the coordinates of the centre are $C_1(1+r, 0)$ or $C_2[-(1+r), 0]$. As it touches $x + y = 2$.



$$\therefore \frac{|\pm(1+r) + 0 - 2|}{\sqrt{2}} = r$$

$$\Rightarrow \pm(1+r) - 2 = \pm\sqrt{2}r$$

$$\Rightarrow 1+r-2 = -\sqrt{2}r$$

$$\text{or } -1-r-2 = -\sqrt{2}r$$

$$\Rightarrow r = \frac{1}{\sqrt{2}+1}$$

$$\text{or } r = \frac{3}{\sqrt{2}-1}$$

$$\Rightarrow r = \sqrt{2}-1$$

$$\text{or } r = 3(\sqrt{2}+1)$$

Thus, coordinates of the centre and radius of the required circle are $(\sqrt{2}, 0)$ and $(\sqrt{2}-1)$ respectively.

Hence, the equation of the required circle is

$$(x - \sqrt{2})^2 + y^2 = (\sqrt{2}-1)^2$$

88. Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

Since, it is orthogonal to three given circles.

$$\therefore 2\left(-g + \frac{3}{2}f\right) = c - 7 \Rightarrow -2g + 3f = c - 7$$

$$\Rightarrow 2\left(\frac{5}{2}g - \frac{5}{2}f\right) = c + 9 \Rightarrow 5g - 5f = c + 9$$

$$\Rightarrow 2\left(\frac{7}{2}g - \frac{9}{2}f\right) = c + 29 \Rightarrow 7g - 9f = c + 29$$

On solving these equations, we get

$$g = -8, f = -9 \text{ and } c = -4$$

On putting the values of g, f and c in Eq. (i), we obtain the required circle as $x^2 + y^2 - 16x - 18y - 4 = 0$.

89. The given circles are

$$S_1 \equiv x^2 + y^2 + 4x + 2y + 1 = 0 \quad \dots(i)$$

$$\text{and } S_2 \equiv x^2 + y^2 - x + 3y - \frac{3}{2} = 0 \quad \dots(ii)$$

The radical axis of these two circles is

$$S_1 - S_2 = 0$$

$$\Rightarrow 5x - y + \frac{5}{2} = 0$$

$$\Rightarrow 10x - 2y + 5 = 0 \quad \dots(iii)$$

The equation of the family of circles coaxial with the circles (i) and (ii) is

$$S_1 + \lambda(S_1 - S_2) = 0$$

$$\Rightarrow (x^2 + y^2 + 4x + 2y + 1) + \lambda(10x - 2y + 5) = 0 \quad \dots(iv)$$

$$\Rightarrow x^2 + y^2 + 2x(2 + 5\lambda) + 2(1 - \lambda)y + 1 + 5\lambda = 0$$

This coordinates of the centre are

$$[-(2 + 5\lambda), -(1 - \lambda)]$$

Thus circle will have its centre on the radical axis of the given circles i.e. on Eq. (iii), if

$$-10(2 + 5\lambda) + 2(1 - \lambda) + 5 = 0$$

$$\Rightarrow \lambda = -\frac{1}{4}$$

On putting $\lambda = -\frac{1}{4}$ in Eq. (iv), we get

$$4x^2 + 4y^2 + 6x + 10y - 1 = 0$$

as the equation of the required circle.

90. We know that, $S + \lambda L = 0, \lambda \in R$ represents a family of coaxial circles having common radical axis $L = 0$.

So, the family of circles

$$x^2 + y^2 + 2ax + 2by + 2\lambda(ax - by) = 0$$

represents a family of coaxial circles having common radical axis

$$ax - by = 0.$$

$$\text{Let } x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

be the circle orthogonal to the above system of coaxial circles.

$$\text{Then, } 2\{ga(l + \lambda) + fb(1 - \lambda)\} = c + 0$$

[applying the condition of orthogonality]

$$\Rightarrow 2(ag + bf) - c + 2\lambda(ag - bf) = 0 \text{ for all } \lambda \in R$$

This is true for all real values of λ .

$$\therefore 2(ag - bf) = 0 \text{ and } 2(ag + bf) - c = 0$$

$$\Rightarrow ag = bf \text{ and } ag + bf = \frac{c}{2}$$

$$\Rightarrow ag = bf = \frac{c}{4}$$

$$\Rightarrow g = \frac{c}{4a} \text{ and } f = \frac{c}{4b}$$

On substituting the values of g, f and c in Eq. (i), we get

$$x^2 + y^2 + \frac{c}{2}\left(\frac{x}{a} + \frac{y}{b}\right) + c = 0$$

This is the equation of the circle orthogonal to the given coaxial system of circles.

91. Let $x^2 + y^2 + 2gx + c = 0$... (i)

where $g \in R$ and c is a constant, be a family of coaxial circles having Y-axis as the common radical axis.

Let $P(\alpha, \beta)$ be any point in the same plane. Then, the polar of $P(\alpha, \beta)$ with respect to Eq. (i) is

$$\alpha x + \beta y + g(x + \alpha) + c = 0$$

$$\Rightarrow (\alpha x + \beta y + c) + g(x + \alpha) = 0$$

This represents a family of straight lines passing through the point of intersection of the lines

$$\alpha x + \beta y + c = 0$$

$$\text{and } x + \alpha = 0$$

On solving these two equations, we get

$$x = -\alpha, y = \frac{\alpha^2 - c}{\beta}$$

Hence, the polar of $P(\alpha, \beta)$ with respect to Eq. (i) passes through the fixed point $\left(-\alpha, \frac{\alpha^2 - c}{\beta}\right)$.

92. Let $x^2 + y^2 + 2gx + c = 0$, where g is a variable and c is a constant, be a coaxial system of circles having X-axis as the common radical axis.

$$\text{Let } x^2 + y^2 + 2g_i x + c = 0, i = 1, 2, 3$$

be three members of the given coaxial system of circles

Then, the coordinates of their centres are

$$P(-g_1, 0), Q(-g_2, 0)$$

$$\text{and } R(-g_3, 0).$$

Let $A(\alpha, \beta)$ be any point and let l_1, l_2, l_3 be the lengths of the tangents drawn from A to the three circles. Then,

$$l_i^2 = \alpha^2 + \beta^2 + 2g_i \alpha + c, i = 1, 2, 3$$

$$\text{We have, } PQ = g_1 - g_2, QR = g_2 - g_3$$

$$\text{and } RP = g_3 - g_1$$

$$\therefore l_1^2 \cdot QR + l_2^2 \cdot RP + l_3^2 \cdot PQ$$

$$= l_1^2(g_2 - g_3) + l_2^2(g_3 - g_1) + l_3^2(g_1 - g_2)$$

$$= (\alpha^2 + \beta^2 + c)(g_2 - g_3 + g_3 - g_1 - g_1 + g_2)$$

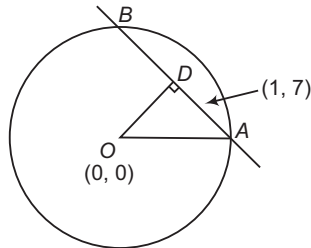
$$+ 2\alpha \{g_1(g_2 - g_3) + g_2(g_3 - g_1) + g_3(g_1 - g_2)\}$$

$$= 0$$

93. Let D be the mid-point of chord AB , drawn through the point $P(1, 7)$. We have,

$$\angle DOA = \frac{\pi}{3}$$

$$\Rightarrow OD = OA \cdot \cos \frac{\pi}{3} = 5$$



Let equation of chord AB be

$$(y - 7) = m(x - 1)$$

$$\Rightarrow y - mx + m - 7 = 0$$

$$\Rightarrow OD = \frac{|m - 7|}{\sqrt{1 + m^2}} = 5$$

$$\Rightarrow m^2 + 49 - 14m = 25 + 25m^2$$

$$\Rightarrow 24m^2 + 14m - 24 = 0$$

$$\Rightarrow m = \frac{3}{4}, -\frac{4}{3}$$

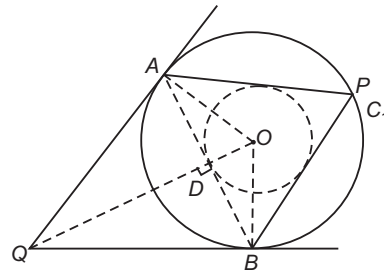
$\Rightarrow$ Equation of chord AB is

$$4y - 3x - 25 = 0$$

$$\Rightarrow 3y + 4x - 25 = 0$$

94. We have, $OD = 1, AO = 2$

$$\text{If } \angle AOD = \theta \Rightarrow \cos \theta = \frac{1}{2}$$



$$\Rightarrow \theta = \frac{\pi}{3}$$

$$\Rightarrow OQ = OA \cdot \sec \frac{\pi}{3}$$

$$= 2OA = 4$$

Thus, locus of Q is $x^2 + y^2 = 16$.

95. Equation of radical axis of the given circles is $x = 0$. If one circle lies completely inside the other, then the centre of both circles should lie on the same side of radical axis and radical axis should not intersect the circles.

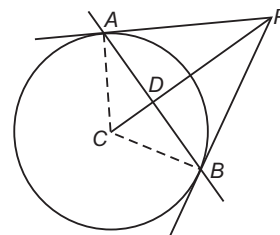
$$\Rightarrow (-a_1)(-a_2) > 0$$

$$\Rightarrow a_1 a_2 > 0$$

$$\text{and } y^2 + c = 0$$

should have imaginary roots $\Rightarrow c > 0$

96. Since, $\angle APB = \frac{\pi}{2}$



$$\Rightarrow \angle ACB = \frac{\pi}{2}$$

$$\Rightarrow \angle DCB = \frac{\pi}{4}$$

$$\Rightarrow CD = CB \cdot \cos \frac{\pi}{4}$$

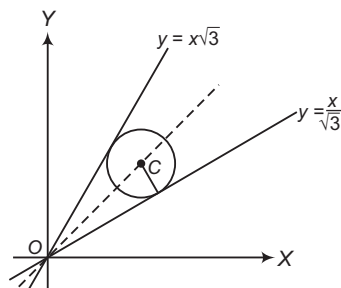
$$= \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\Rightarrow |c| = \sqrt{10}$$

97. Angle between lines is $60^\circ - 30^\circ = 30^\circ$. Thus, equation of their acute angle bisector is $y = \tan(30^\circ + 15^\circ) \cdot x$ i.e. $y = x$. Let $C \equiv (h, h)$, then

$$\frac{|h - h\sqrt{3}|}{2} = 1$$

$$\Rightarrow (\sqrt{3} - 1)h = 2$$



$$\Rightarrow h = \frac{2}{(\sqrt{3} - 1)} = \frac{2(\sqrt{3} + 1)}{2} = (\sqrt{3} + 1)$$

Thus, equation of circle is

$$(x - (\sqrt{3} + 1))^2 + (y - (\sqrt{3} + 1))^2 = 1$$

$$\Rightarrow x^2 + y^2 - 2x(\sqrt{3} + 1) - 2y(\sqrt{3} + 1) + 7 + 4\sqrt{3} = 0$$

98. The equation of the family of coaxial circles having $x = \frac{a}{2}$ as radical axis with respect to $x^2 + y^2 = a^2$, is

$$x^2 + y^2 - a^2 + \lambda \left(x - \frac{a}{2} \right) = 0 \quad \dots(i)$$

If it passes through $(2a, 0)$, then

$$4a^2 - a^2 + \lambda \left(2a - \frac{a}{2} \right) = 0$$

$$\Rightarrow 3a^2 + \frac{3\lambda a}{2} = 0$$

$$\Rightarrow \lambda = -2a$$

On putting $\lambda = -2a$ in Eq. (i), we get

$$x^2 + y^2 - 2ax = 0$$

This is the equation of the required circle.

99. The locus of the point of perpendicular tangents to $x^2 + y^2 = 9$ is its director circle given by

$$x^2 + y^2 = 18$$

The equation of the tangent to this circle at $(\sqrt{2}, 4)$ is

$$\sqrt{2}x + 4y = 18$$

Hence, the required equation of the system of coaxial circle is

$$x^2 + y^2 - 9 + \lambda(\sqrt{2}x + 4y - 18) = 0$$

100. Let the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

It passes through the origin. So, $(0, 0)$ satisfies Eq. (i) consequently, we have

$$c = 0$$

The Eq. (i) intersects the circles

$$x^2 + y^2 - 4x + 6y + 10 = 0$$

$$\text{and } x^2 + y^2 + 12y + 6 = 0$$

orthogonally. Therefore,

$$2(-2g + 3f) = c + 10$$

$$\text{and } 2(0g + 6f) = c + 6$$

$$\Rightarrow -2g + 3f = 5 \quad \text{and } f = \frac{1}{2}$$

$$\Rightarrow g = -\frac{7}{4} \quad \text{and } f = \frac{1}{2}$$

On substituting the values of g, f and c in Eq. (i), we get

$$x^2 + y^2 - \frac{7}{2}x + y = 0$$

$$\Rightarrow 2(x^2 + y^2) - 7x + 2y = 0$$

This is the equation of the required circle.

101. Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

It passes through $(3, 0)$ and touches Y -axis.

$$\therefore 9 + 0 + 6g + 0f + c = 0$$

$$\Rightarrow 9 + 6g + c = 0 \quad \dots(ii)$$

$$\text{and } c = f^2 \quad \dots(iii)$$

It is given that Eq. (i) is orthogonal to

$$x^2 + y^2 - 6x + 4y - 3 = 0$$

$$\therefore 2(-3g + 2f) = c + (-3)$$

$$\Rightarrow -6g + 4f = c - 3 \quad \dots(iv)$$

From Eqs. (ii) and (iii), we get

$$c = f^2 \quad \text{and } g = -\left(\frac{9 + f^2}{6}\right)$$

On substituting these values in Eq. (iv), we get

$$9 + f^2 + 4f = f^2 - 3 \Rightarrow f = -3$$

$$\therefore c = f^2 \Rightarrow c = 9$$

$$\text{and } g = -\left(\frac{9 + f^2}{6}\right) \Rightarrow g = -3$$

On substituting the values of g, f and c in Eq. (i), we get

$$x^2 + y^2 - 6x - 6y + 9 = 0$$

This is the equation of required circle.

102. The equation of the circles are

$$S_1 \equiv x^2 + y^2 + 7x - 9y + \frac{15}{2} = 0 \quad \dots(i)$$

$$\text{and } S_2 \equiv x^2 + y^2 - \frac{3}{4}x - \frac{1}{4}y + \frac{5}{4} = 0 \quad \dots(ii)$$

The radical axis of these two circles is given by

$$S_1 - S_2 = 0$$

$$\Rightarrow \frac{31x}{3} - \frac{35y}{4} + \frac{25}{4} = 0$$

$$\Rightarrow 31x - 35y + 25 = 0$$

103. The common chord of the given circles is

$$ax - by = 0 \quad \dots(i)$$

The equation of the family of circles passing through the intersection of the given circles is

$$(x - a)^2 + y^2 - a^2 + \lambda(ax - by) = 0$$

$$\Rightarrow x^2 + y^2 + ax(\lambda - 2) - \lambda by = 0 \quad \dots(ii)$$

The coordinates of the centre of this circle are

$$\left(\frac{-a(\lambda - 2)}{2}, \frac{\lambda b}{2} \right)$$

If the common chord i.e. Eq. (i) is a diameter of circle (ii), then

$$-\frac{a^2}{2}(\lambda - 2) - \frac{\lambda b^2}{2} = 0$$

$$\Rightarrow a^2(\lambda - 2) + \lambda b^2 = 0$$

$$\Rightarrow \lambda = \frac{2a^2}{a^2 + b^2}$$

On substituting the value of λ in Eq. (ii), we obtain the equation of the required circle which is

$$(x^2 + y^2)(a^2 + b^2) = 2ab(bx + ay)$$

- 104.** The equation of the family of circles touching the line $x - y = 0$ at the origin is

$$(x - 0)^2 + (y - 0)^2 + \lambda(x - y) = 0$$

$$\Rightarrow x^2 + y^2 + \lambda(x - y) = 0 \quad \dots(i)$$

This circle bisects the circumference of the circle

$$x^2 + y^2 + 2y - 3 = 0 \quad \dots(ii)$$

Therefore, the common chord of Eqs. (i) and (ii) is a diameter of circle (ii).

The equation of the common chord of Eqs. (i) and (ii) is

$$\lambda x - (\lambda + 2)y + 3 = 0 \quad \dots(iii)$$

This will be a diameter of Eq. (ii). If the centre $(0, -1)$ of circle (ii) satisfies the Eq. (iii).

$$\therefore \lambda + 2 + 3 = 0$$

$$\Rightarrow \lambda = -5$$

On putting $\lambda = -5$ in Eq. (i), we get

$$x^2 + y^2 - 5x + 5y = 0$$

- 105.** Equations of circles with limiting points $(1, 2)$ and $(4, 3)$ are

$$(x - 1)^2 + (y - 2)^2 = 0$$

$$\text{and } (x - 4)^2 + (y - 3)^2 = 0$$

Any circle coaxial with these circles is

$$(x - 1)^2 + (y - 2)^2 + \lambda \{(x - 4)^2 + (y - 3)^2\} = 0 \quad \dots(i)$$

This will pass through the origin, if

$$1 + 4 + \lambda(16 + 9) = 0$$

$$\Rightarrow \lambda = -\frac{1}{5}$$

On putting $\lambda = -\frac{1}{5}$ in Eq. (i), we get

$$2(x^2 + y^2) - x - 7y = 0$$

This is the equation of the required circle.

- 106.** The equation of any circle passing through the points of intersection of the circles $S_1 = 0$ and $S_2 = 0$ is $S_1 + \lambda S_2 = 0$

Hence, the equation of the required circle is

$$(x^2 + y^2 - 8x - 2y + 7) + \lambda(x^2 + y^2 - 4x + 10y + 8) = 0$$

$$\Rightarrow x^2(1 + \lambda) + y^2(1 + \lambda) - 4x(2 + \lambda) - 2y(1 - 5\lambda) + 7 + 8\lambda = 0 \quad \dots(i)$$

This circle passes through the point $(-1, -2)$.

$$\therefore (1 + \lambda) + 4(1 + \lambda) + 4(2 + \lambda) + 4(1 - 5\lambda) + 7 + 8\lambda = 0$$

$$\Rightarrow 24 - 3\lambda = 0$$

$$\Rightarrow \lambda = 8$$

On putting $\lambda = 8$ in Eq. (i), we get the required circle

$$9x^2 + 9y^2 - 40x + 78y + 71 = 0$$

- 107.** Let $x^2 + y^2 + 2g_i x + c = 0$; $i = 1, 2, 3$

be three fixed circles from a system of coaxial circles having Y-axis as the common radical axis.

Let $P(\alpha, \beta)$ be any point on the circle

$$x^2 + y^2 + 2g_1 x + c = 0$$

$$\text{Then, } \alpha^2 + \beta^2 + 2g_1 \alpha + c = 0 \quad \dots(i)$$

Now, l_1 = Length of the tangent from $P(\alpha, \beta)$ to

$$x^2 + y^2 + 2g_2 x + c = 0$$

$$= \sqrt{\alpha^2 + \beta^2 + 2g_2 \alpha + c}$$

$$= \sqrt{2(g_2 - g_1)\alpha} \quad [\text{using Eq. (i)}]$$

and l_2 = Length of the tangent from $P(a, b)$ to

$$x^2 + y^2 + 2g_3 x + c = 0$$

$$= \sqrt{\alpha^2 + \beta^2 + 2g_3 \alpha + c}$$

$$= \sqrt{2(g_3 - g_1)\alpha} \quad [\text{using Eq. (ii)}]$$

$$\therefore \frac{l_1}{l_2} = \frac{\sqrt{g_2 - g_1}}{\sqrt{g_3 - g_1}} = \text{Constant}$$

- 108.** The equation of the family of circles touching the straight line $3x - y = 6$ at $(1, -3)$ is

$$(x - 1)^2 + (y + 3)^2 + \lambda(3x - y - 6) = 0 \quad \dots(i)$$

The coordinates of the centre of the circle given in Eq. (i). Then,

$$r = \sqrt{\left(\frac{2 - 3\lambda}{2}\right)^2 + \left(\frac{\lambda - 6}{2}\right)^2 - 10 + 6\lambda} = \sqrt{\frac{5}{2}} |\lambda|$$

If the circle in Eq. (i) touches $y = x$, then

$$\left| \frac{\frac{2 - 3\lambda}{2} - \frac{\lambda - 6}{2}}{\sqrt{1 + 1}} \right| = \sqrt{\frac{5}{2}} |\lambda|$$

$$\Rightarrow \left| \frac{4 - 2\lambda}{\sqrt{2}} \right| = \sqrt{\frac{5}{2}} |\lambda|$$

$$\Rightarrow |4 - 2\lambda| = \sqrt{5} |\lambda|$$

$$\Rightarrow 4 - 2\lambda = \pm \sqrt{5} \lambda$$

$$\Rightarrow \lambda = \frac{4}{2 - \sqrt{5}}$$

$$\Rightarrow \lambda = \frac{4}{2 + \sqrt{5}}$$

$$\text{We have, } r = \sqrt{\frac{5}{4}} |\lambda|$$

Clearly, r will be least, if λ is least. The smaller of the two values of λ is $\frac{4}{2 + \sqrt{5}}$.

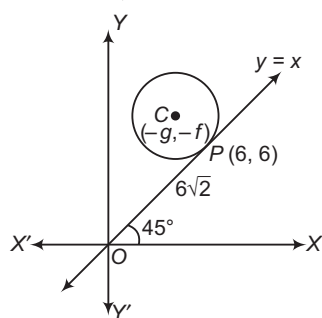
For this value of λ , we have

$$\begin{aligned} r &= \sqrt{\frac{5}{4}} \times \frac{4}{2 + \sqrt{5}} = 10\sqrt{2} - 4\sqrt{10} \\ &= 10 \times 1.414 - 4 \times 3.162 \\ &= 1.49 \end{aligned}$$

- 109.** The equation of the line $y = x$ in parametric form is

$$\frac{x}{\cos \pi/4} = \frac{y}{\sin \pi/4}$$

Since, $OP = 6\sqrt{2}$. So, the coordinates of P are given by



$$\frac{x}{\cos \pi/4} = \frac{y}{\sin \pi/4} = 6\sqrt{2}$$

$$\Rightarrow x = y = 6$$

So, the coordinates of P are $(6, 6)$.

The equation of the circle touching $y = x$ at $(6, 6)$ is

$$(x-6)^2 + (y-6)^2 + \lambda(x-y) = 0$$

$$x^2 + y^2 + (\lambda-12)x - y(\lambda+12) + 72 = 0$$

This will be same as $x^2 + y^2 + 2gx + 2fy + c = 0$, if

$$\lambda - 12 = 2g, (\lambda + 12) = -2f \quad \text{and} \quad c = 72$$

Hence, the required value of c is 72.

110. Equation of circle is

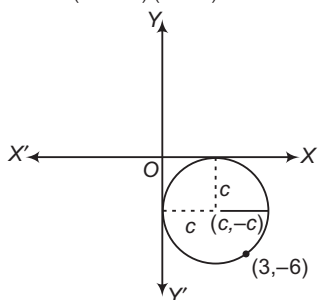
$$(x-c)^2 + (y+c)^2 = c^2$$

It passes through $(3, -6)$.

$$\Rightarrow (3-c)^2 + (c-6)^2 = c^2$$

$$\Rightarrow c^2 - 18c + 45 = 0$$

$$(c-15)(c-3) = 0$$



$$\therefore c = 3, 15$$

From Eq. (i), we get equation of circles are

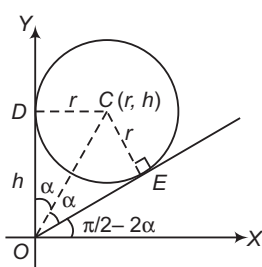
$$x^2 + y^2 - 6x + 6y + 9 = 0$$

$$\text{and} \quad x^2 + y^2 - 30x + 30y + 225 = 0$$

111. The given equation is

$$(x-r)^2 + (y-h)^2 = r^2$$

Tangents are $x = 0$



and

$$y = x \tan \left(\frac{\pi}{2} - 2\alpha \right)$$

$$y = x \cot 2\alpha$$

$$y = \frac{x(1 - \tan^2 \alpha)}{2 \tan \alpha}$$

$$y = \frac{x \left(1 - \frac{r^2}{h^2} \right)}{2 \left(\frac{r}{h} \right)} \quad \left[\because \text{in } \triangle ODC, \tan \frac{r}{h} \right]$$

$$\Rightarrow (h^2 - r^2)x - 2rhy = 0$$

112. Let radius of required circle be r .

$\therefore$ Now, to touch the line pair

Distance between centres = Radius of I circle

$\pm$ Radius of II circle

$$5 = r \pm 1$$

$$\therefore r = 5 \pm 1$$

$$= 6, 4$$

$\therefore$ Equation of circles are

$$(x-4)^2 + (y-3)^2 = 6^2$$

$$\text{and} \quad (x-4)^2 + (y-3)^2 = 4^2$$

$$\Rightarrow x^2 + y^2 - 8x - 6y - 11 = 0$$

$$\text{and} \quad x^2 + y^2 - 8x - 6y + 9 = 0$$

...(i)

113. The equation of tangent in terms of slope of $x^2 + y^2 = 25$ is

$$y = mx \pm 5\sqrt{1+m^2} \quad \dots(i)$$

Eq. (i) pass through $(-2, 11)$, then

$$11 = -2m \pm 5\sqrt{1+m^2}$$

On squaring both sides, we get

$$21m^2 - 44m - 96 = 0$$

$$\Rightarrow (7m-24)(3m+4) = 0$$

$$\therefore m = -4/3, 24/7$$

From Eq. (i), we get required tangents as

$$24x - 7y \pm 125 = 0$$

$$\text{and} \quad 4x + 3y = \pm 25$$

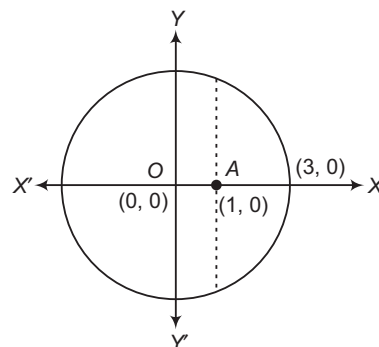
Hence, tangents are

$$24x - 7y + 125 = 0$$

$$\text{and} \quad 4x + 3y = 25$$

114. Consider family of circles through $(0, 0)$ and $(1, 0)$

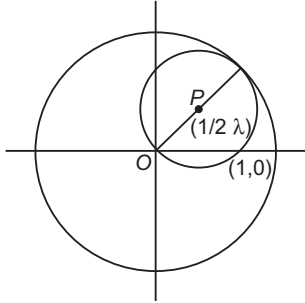
$$x(x-1) + y^2 + \lambda y = 0 \text{ touches } x^2 + y^2 = 9.$$



$$\therefore \text{Common chord} = -x + \lambda y + 9 = 0 \quad \dots(i)$$

∴ Perpendicular from (0, 0) on Eq. (i) is equal to 3.

$$\begin{aligned} \left| \frac{9}{\sqrt{1+\lambda^2}} \right| &= 3 \\ \Rightarrow \lambda^2 &= 8 \\ \Rightarrow \lambda &= \pm 2\sqrt{2} \\ \text{Circle } x(x-1) + y^2 + 2\sqrt{2}y &= 0 \\ \therefore \text{Centre } \left(\frac{1}{2}, -2^{-1/2} \right) \end{aligned}$$

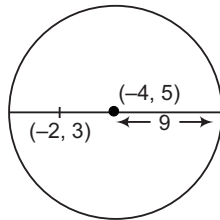


Aliter

$$\begin{aligned} OP &= \frac{3}{2} \\ \Rightarrow \sqrt{\frac{1}{4} + \lambda^2} &= \frac{3}{2} \Rightarrow \frac{1}{4} + \lambda^2 = \frac{9}{4} \\ \Rightarrow \lambda^2 &= 2 \Rightarrow \lambda = \pm \sqrt{2} \end{aligned}$$

115. $x^2 + y^2 + 8x - 10y - 40 = 0$

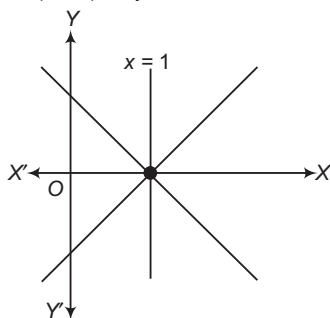
Centre of the circle is (-4, 5).
Its radius = 9 units



Distance of the centre (-4, 5) from the point (-2, 3) is

$$\begin{aligned} \sqrt{4+4} &= 2\sqrt{2} \\ \therefore a &= 2\sqrt{2} + 9 \\ \text{and } b &= -2\sqrt{2} + 9 \\ \therefore a+b &= 18 \\ a-b &= 4\sqrt{2} \\ a \cdot b &= 81 - 8 = 73 \end{aligned}$$

116. Line pair is $(x-1)^2 - y^2 = 0$



i.e. $x + y - 1 = 0$
 $x - y - 1 = 0$

Let the centre be $(\alpha, 0)$, then its distance from $x + y - 1 = 0$ is $\left| \frac{\alpha - 1}{\sqrt{2}} \right| = 2$ (radius)

i.e. $\alpha = 1 \pm 2\sqrt{2}$

∴ Centre may be $(1 + 2\sqrt{2}, 0), (1 - 2\sqrt{2}, 0)$

Now, let the centre be $(1, \beta)$, then $\left| \frac{1 + \beta - 1}{\sqrt{2}} \right| = 2$

$\Rightarrow \beta = \pm 2\sqrt{2}$

∴ Centre may be $(1, 2\sqrt{2}), (1, -2\sqrt{2})$.

117. $x^2 + y^2 - 8x - 16y + 60 = 0$... (i)

Equation of chord of contact from $(-2, 0)$ is

$$-2x - 4(x-2) - 8y + 60 = 0$$

$$3x + 4y - 34 = 0 \quad \dots (ii)$$

$$x^2 + \left(\frac{34 - 3x}{4} \right)^2 - 8x - 16 \left(\frac{34 - 3x}{4} \right) + 60 = 0$$

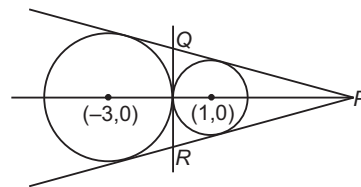
$$16x^2 + 1156 - 204x + 9x^2 - 128x - 2176 + 192x + 960 = 0$$

$$5x^2 - 28x - 12 = 0$$

$$\begin{aligned} \Rightarrow (x-6)(5x+2) &= 0 \\ x &= 6, -\frac{2}{5} \end{aligned}$$

∴ Points are $(6, 4)$ and $\left(-\frac{2}{5}, \frac{44}{5} \right)$.

118. Centre of the first circle is $(-3, 0)$ and the radius is 3 and radius of the second circle is $(1, 0)$ and the radius is 1 unit. Since, the distance between the centres is equal to the sum of the radii, the two circles touch each other externally at the origin, the common tangent at the origin is Y-axis.



Let $y = mx + c$ be a direct common tangent to the two circles, then

$$\frac{-3m + c}{\sqrt{1+m^2}} = \pm 3 \quad \text{and} \quad \frac{m + c}{\sqrt{1+m^2}} = \pm 1$$

$$\Rightarrow -6cm + c^2 = 9$$

$$\text{and } 2cm + c^2 = 1$$

$$\Rightarrow cm = -1 \quad \text{and} \quad c^2 = 3$$

$$\Rightarrow c = \pm \sqrt{3} \quad \text{and} \quad m = \pm \frac{1}{\sqrt{3}}$$

⇒ Equation of the common tangents are

$$y = -\frac{1}{\sqrt{3}}x + \sqrt{3}, y = \frac{1}{\sqrt{3}}x - \sqrt{3}, x = 0$$

Since, the lines $y = -\frac{1}{\sqrt{3}}x + \sqrt{3}$ and $y = \frac{1}{\sqrt{3}}x - \sqrt{3}$ make angle of 60° with $x = 0$.

The ΔPQR formed by these tangents is equilateral, so that the centroid, circumcentre and orthocentre of the triangle coincide with its incentre $(1, 0)$, the centre of the circle of smaller radius inscribed in the ΔPQR .

119. Statement II is true as the centre is equidistant from A and B, hence lies on the perpendicular bisector of AB. Statement I is false as the distance between the given points is 10 and hence any circle through A and B has radius more than or equal to 5 and hence there is no circle of radius 4 through A and B is possible.
120. We know that, chords of contact of given circle generated by any point on given line passes through the fixed point, as they form family of straight lines, hence both the statements are true and Statement II is the correct explanation of Statement I.
121. For circle $x^2 + y^2 = 144$, centre $C_1(0, 0)$ and radius $r_1 = 12$.
For circle $x^2 + y^2 - 6x - 8y = 0$, centre $C_2 = (3, 4)$ and radius $r_2 = 5$.
Now, $C_1C_2 = 5$ and $r_1 - r_2 = 7$, thus $C_1C_2 < r_1 - r_2$, hence one circle is completely lying inside other without touching it, hence there is no common tangent. Therefore, Statement I is true. Therefore, both the statements are true but Statement II is not correct explanation of Statement I.
122. Centre of the circle $C(2, 1)$ and radius $r = 5$.
Distance of $P(10, 7)$ from $C(2, 1)$ is 10 units, hence required distances are 5, 15, respectively. Therefore, Statement I is true. Statement II is true but not the correct explanation of Statement I, as the information is not sufficient to get distance said in Statement I.
123. Given points are collinear, therefore circle is not possible. Hence, Statement I is false, however Statement II is true.
124. Since, $S_1 = 0$ and $S_3 = 0$ has no radical axis.
 $\therefore$ Radical centre does not exist.
125. $x^2 + y^2 - 2x - 2ay - 8 = 0$
 $\Rightarrow (x^2 + y^2 - 2x - 8) - 2a(y) = 0$
 $S + lL = 0$
solving the two equations
$$\begin{aligned} x^2 + y^2 - 2x - 8 &= 0 \\ y &= 0 \\ x^2 - 4x + 2x - 8 &= 0 \\ x(x - 4) + 2(x - 4) &= 0 \\ x &= 4, x = -2 \end{aligned}$$

So, $(4, 0)$ and $(-2, 0)$ are the points of intersection which lie on X-axis.
126. Two circles touch each other $C_1C_2 = r_1 \pm r_2$
$$\sqrt{p^2 + q^2} = \sqrt{p^2 - r} + \sqrt{q^2 - r}$$

$$\Rightarrow p^2 + q^2 = p^2 - r + q^2 - r + 2\sqrt{(p^2 - r)(q^2 - r)}$$

$$\Rightarrow \frac{1}{r} = \frac{1}{p^2} + \frac{1}{q^2}$$
127. The Statement II is well known result but if applied to the data in Statement I will yield
 $5x - 9y + 46 = 0$
 $\Rightarrow$ Statement I is false, Statement II is true.

Solutions (Q. Nos. 128-130)

It is given that one of the diagonals of the square is parallel to the line $y = x$.

Also, the length of the diagonal of the square is $4\sqrt{2}$ unit.

Now, equation of one of diagonals is

$$\frac{x-3}{\frac{1}{\sqrt{2}}} = \frac{y-4}{\frac{1}{\sqrt{2}}} = r = \pm 2\sqrt{2}$$

$$\therefore x - 3 = y - 4 = \pm 2$$

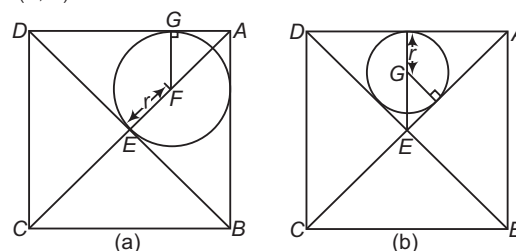
$$\Rightarrow x = 5, 1 \text{ and } y = 6, 2$$

So, two of the vertices are $(1, 2)$ and $(5, 6)$.

The other diagonal is parallel to the line $y = -x$, so that its equation is

$$\frac{x-3}{\frac{1}{\sqrt{2}}} = \frac{y-4}{-\frac{1}{\sqrt{2}}} = r = \pm 2\sqrt{2}$$

Hence, the two vertices on this diagonal are $(1, 6)$ and $(5, 2)$.



$$AB = 4 \text{ units, } AC = 4\sqrt{2} \text{ units}$$

$$\Rightarrow AE = 2\sqrt{2} \text{ units}$$

$$\text{In Fig. (a), } EF + FA = AE$$

$$\Rightarrow \sqrt{2}r + r = 2\sqrt{2}$$

$$\Rightarrow r = \frac{2\sqrt{2}}{\sqrt{2} + 1} = 2\sqrt{2}(\sqrt{2} - 1) \text{ units}$$

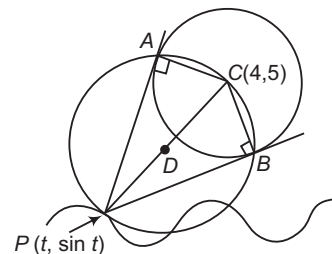
$$\text{In Fig. (b), } EG + GF = EF$$

$$\Rightarrow \sqrt{2}r + r = 2$$

$$\Rightarrow r = \frac{2}{\sqrt{2} + 1} = 2(\sqrt{2} - 1) \text{ units}$$

Solutions (Q. Nos. 131-133)

Centre of the given circle is $C(4, 5)$. Points P, A, C and B are concyclic such that PC is diameter of the circle. Hence, centre D of the circumcircle of $\triangle ABC$ is mid-point of PC, then we have



$$h = \frac{t+4}{2} \text{ and } k = \frac{\sin t - 5}{2}$$

$$\text{Eliminating } t, \text{ we have } k = \frac{\sin(2h-4)+5}{2}$$

$$\text{or } y = \frac{\sin(2x-4)+5}{2}$$

$$\Rightarrow f^{-1}(x) = \frac{\sin^{-1}(2x-5)+4}{2}$$

Thus, range of $y = \frac{\sin(2x-4)+5}{2}$ is $[2, 3]$ and period is π .

Also, $f(x) = 4$
 $\Rightarrow \sin(2x - 4) = 3$, which has no real solutions.
 For $f(x) = 1$
 $\Rightarrow \sin(2x - 4) = -3$, which has no real solutions.

But range of $y = \frac{\sin^{-1}(2x - 5) + 4}{2}$ is
 $\left[-\frac{\pi}{4} + 2, \frac{\pi}{4} + 2\right]$

134. The points of concurrency lie on the intersection of perpendicular bisectors of AB with the circle $x^2 + y^2 = 4$.

Hence, the points are $(\sqrt{2}, \sqrt{2})$ or $(-\sqrt{2}, -\sqrt{2})$.

135. Circumcentre lies on the point of intersection of perpendicular bisector of AB with the circle.

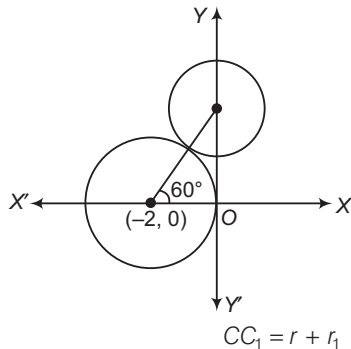
Hence, circumcentre is $(0, 0)$ or $(2, 2)$.

136. $\angle AOB = \pi - \left(\frac{A+B}{2}\right)$

$\Rightarrow \angle AOB$ is fixed and is rather than $\frac{\pi}{2}$.

$\Rightarrow O$ lies always on a minor arc of a circle.

- 137.



$$\Rightarrow \sqrt{(h+2)^2 + k^2} = 2 + 2 = 4$$

$\therefore$ Locus of c is $x^2 + y^2 + 4x - 12 = 0$.

138. If (h, t) denotes the centre of variable circle, then

$$h = -2 + 4 \cdot \cos 60^\circ = -2 + 4 \cdot \frac{1}{2} = 0$$

$$k = 0 + 4 \sin 60^\circ = 2\sqrt{3}$$

$\therefore$ Common tangent at the point of contact is

$$S_1 - S_2 = 0$$

$$\text{i.e. } x^2 + y^2 + 4x - (x^2 + y^2 - 2\sqrt{3}y - 4) = 0$$

$$4x + 4\sqrt{3}y - 8 = 0$$

$$\text{i.e. } x + \sqrt{3}y - 2 = 0$$

Equation of tangent to given circle is

$$y = m(x + 2) \pm 2\sqrt{1 + m^2}$$

$$mx - y + 2m \pm 2\sqrt{1 + m^2} = 0$$

$$\left| \frac{m \times 0 - 2\sqrt{3} + 2m \pm 2\sqrt{1 + m^2}}{\sqrt{1 + m^2}} \right| = 2$$

$$m - \sqrt{3} \pm \sqrt{1 + m^2} = \pm \sqrt{1 + m^2}$$

$$\Rightarrow m - 3 + \sqrt{1 + m^2} = \pm \sqrt{1 + m^2}$$

$$\text{i.e. } m = \sqrt{3}, -\frac{1}{\sqrt{3}}$$

$$\text{and } m - \sqrt{3} - \sqrt{1 + m^2} = \pm \sqrt{1 + m^2}$$

$$\Rightarrow m - \sqrt{3} = 0$$

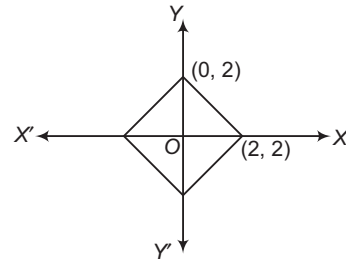
$$\Rightarrow m - \sqrt{3} = 2\sqrt{1 + m^2}$$

Equation of tangent is

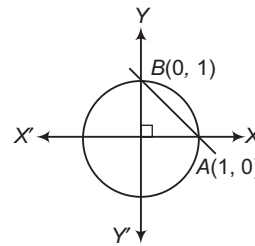
$$y = \sqrt{3}x \pm 2 \quad \text{and} \quad y = (-1/\sqrt{3})x + 2\sqrt{\frac{4}{3}}$$

$$\Rightarrow \sqrt{3}y + x = 2$$

139. Here, Area = $4 \left(\frac{1}{2} \times 2 \times 2 \right) = 8$ sq units



- 140.



Here, $x + y = 1$ subtends an angle $\frac{\pi}{2}$ at centre.

141. Slope of chord = 1

Since, the chord is $\frac{\pi}{3}$.

$\therefore$ Distance from the origin is $\sqrt{3}$ units.

Let the equation of the chord be

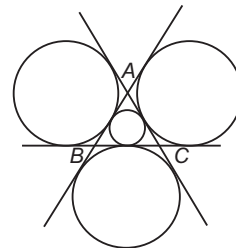
$$x - y + k = 0$$

$$\therefore \left| \frac{k}{\sqrt{2}} \right| = \sqrt{3} \quad \text{i.e. } k = \pm \sqrt{6}$$

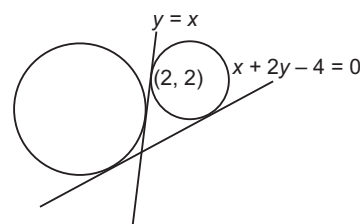
142. Radius of the circle = 2 units

$$\text{Distance from the origin} = 2 \cos \frac{\pi}{3} = 1$$

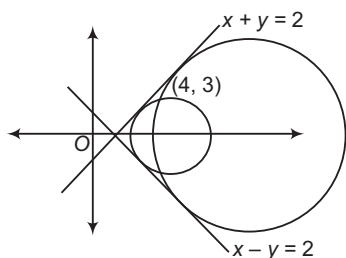
143. A.



- B.



C.



D. Obviously one (see theory).

144. A. $(-g, -f)$ lies in first quadrant, then $g < 0$ and $f < 0$; also X and Y -axes must not cut the circle.

Solving circle and X -axis, we have $x^2 + 2gx + c = 0$, which must have imaginary roots, then $g^2 - c < 0$, then c must be positive.

$$\text{Also, } f^2 - c < 0$$

- B. If circle lies above X -axis, then $x^2 + 2gx + c = 0$ must have imaginary roots, then $g^2 - c < 0$ and $c > 0$.

- C. $(-g, -f)$ lies in third or fourth quadrant, then $g > 0$. Also, Y -axis must not cut the circle.

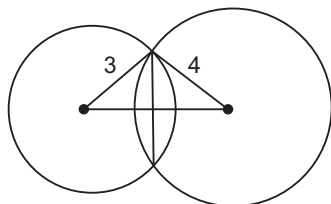
Solving circle and Y -axis, we have $y^2 + 2gy + c = 0$, which must have imaginary roots, then $f^2 - c < 0$, then c must be positive.

- D. $x^2 + 2gx + c = 0$ must have equal roots, then $g^2 = c$, hence $c > 0$

$$\text{Also, } -g > 0 \Rightarrow g < 0$$

145. A. Let length of common chord be $2a$, then

$$\begin{aligned} \sqrt{9-a^2} + \sqrt{16-a^2} &= 5 \\ \Rightarrow \sqrt{16-a^2} &= 5 - \sqrt{9-a^2} \\ \Rightarrow 16-a^2 &= 25 + 9-a^2 - 10\sqrt{9-a^2} \end{aligned}$$



$$\begin{aligned} \Rightarrow 10\sqrt{9-a^2} &= 18 \\ \Rightarrow 100(9-a^2) &= 324 \\ \Rightarrow 100a^2 &= 576 \\ \Rightarrow a &= \frac{\sqrt{576}}{\sqrt{100}} = \frac{24}{10} \\ \therefore 2a &= \frac{24}{5} = \frac{k}{5} \\ \Rightarrow k &= 24 \end{aligned}$$

- B. Equation of common chord is $6x + 4y + p + q = 0$ common chord pass through centre $(-2, -6)$ of circle $x^2 + y^2 + 4x + 12y + p = 0$

$$\therefore p + q = 36$$

- C. Equation of the circle is $2x^2 + 2y^2 - 2\sqrt{2}x - y = 0$. Let $(\alpha, 0)$ be mid-point of a chord. Then, equation of the chord is

$$2\alpha x - \sqrt{2}(x + \alpha) - \frac{1}{2}(y + 0) = 2\alpha^2 - 2\sqrt{2}\alpha$$

Since, it passes through the point $(\sqrt{2}, \frac{1}{2})$.

$$\therefore 2\sqrt{2}\alpha - \sqrt{2}(\sqrt{2} + \alpha) - \frac{1}{4} = 2\alpha^2 - 2\sqrt{2}\alpha$$

$$\Rightarrow 8\alpha^2 - 12\sqrt{2}\alpha + 9 = 0$$

$$\Rightarrow (2\sqrt{2}\alpha - 3)^2 = 0$$

$$\text{i.e. } \alpha = \frac{3}{2\sqrt{2}}, \frac{3}{2\sqrt{2}}$$

$\therefore$ Number of chords is 1.

- D. Mid-point of $AB = (1, 4)$

$\therefore$ Equation perpendicular bisector of AB is $x = 1$.

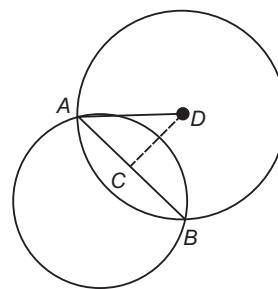
A diameter is $4y = x + 7$

$\Rightarrow$ Centre of the circle is $(1, 2)$.

$\Rightarrow$ Sides of the rectangle are 8 and 4.

$\therefore$ Area = 32 sq units

146. A. $CD = \sqrt{5}$ and $AC = 2$



$$AD = \sqrt{5+4} = 3$$

- B. Distance from the centre $(0, 10)$ of the line $y = mx$

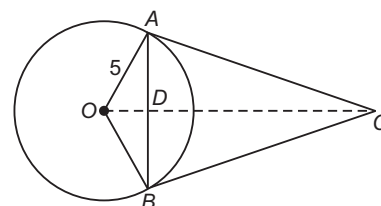
$$= \frac{10}{\sqrt{1+m^2}} \geq \text{Radius}$$

$$\Rightarrow |m| \leq 3$$

- C. $OD = 4$

$$\Rightarrow \sin \theta = \frac{4}{5}$$

$$\begin{aligned} \text{So, } AC &= OA \cot \theta \\ &= 5 \times \frac{3}{4} \\ &= \frac{15}{4} \end{aligned}$$



$$\therefore \text{Area of quadrilateral } OACB = OA \times AC = \frac{75}{4} \text{ sq units}$$

- D. The circle C is $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 2rx - 2ry + r^2 = 0$

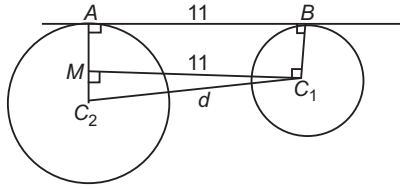
They intersect orthogonally.

$$2(r+r) = 1 + r^2$$

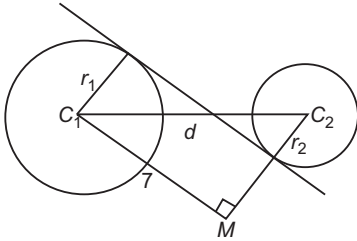
$$r^2 - 4r + 1 = 0$$

$$\text{So, } r = 2 \pm \sqrt{3}$$

147.



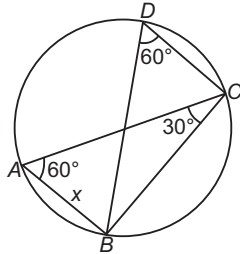
From the figure, $d^2 = 11^2 + (r_1 - r_2)^2$... (i)



From the figure, $d^2 = 7^2 + (r_1 + r_2)^2$... (ii)

From Eqs. (i) and (ii), $4r_1r_2 = 72$
 $\Rightarrow r_1r_2 = 18$

148.



$\angle A = 60^\circ = \angle D$
 $AC = 2$
 $\angle ABC = 90^\circ$ [given]
 $x = 1$ unit

149. Clearly, locus of point of intersection of lines is $(x - 5)$.

$(x - 3) + (y - 2)(y + 4) = 0$
 $\Rightarrow x^2 + y^2 - 8x + 2y + 7 = 0$
Hence, $|f + g| = |2 + (-8)| = 6$

150. $x^2 + y^2 - 2x - 2\lambda y - 8 = 0$

$\Rightarrow (x^2 + y^2 - 2x - 8) - 2\lambda y = 0$, which is of the form of $S + \lambda L = 0$

All the circles passing through the point of intersection of circle $x^2 + y^2 - 2x - 8 = 0$ and $y = 0$.

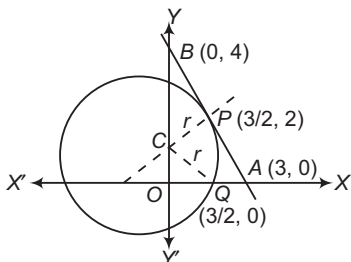
Solving, we get $x^2 - 2x + 8 = 0$

$\Rightarrow (x - 4)(x + 2) = 0$

$\Rightarrow A \equiv (4, 0)$ and $B \equiv (-2, 0)$

$\Rightarrow$ Distance between A and B is 6 units.

151. Let centre be $C(h, k)$



$\therefore CP \perp AB \Rightarrow \frac{2-k}{\frac{3}{2}-h} = \frac{3}{4}$
 $\Rightarrow 6h - 8k = -7$... (i)
 $\therefore CP = CQ$

$\Rightarrow \left(h - \frac{3}{2}\right)^2 + (k - 2)^2 = \left(h - \frac{3}{2}\right)^2 + k^2$

$\Rightarrow k = 1$

Put $k = 1$ in Eq. (i), we get

$6h = 1$
 $\Rightarrow h = \frac{1}{6}$

$\therefore$ Radius $(r) = CQ = \sqrt{\left(\frac{1}{6} - \frac{3}{2}\right)^2 + 1}$
 $= \sqrt{\left(\frac{1-9}{6}\right)^2 + 1} = \frac{5}{3}$

Now, $3r = 5$

152. Let (h, k) be the centre of circle with radius 3 units.

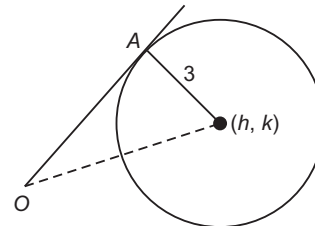
$2x^2 + y^2 - 3xy - 0 = (2x - y)(x - y)$ is an equation of pair of tangents to this circle from O.

Distance of (h, k) from each of $x - y = 0$ and $2x - y = 0$ is equal to radius.

$\Rightarrow |h - k| = 3\sqrt{2}$
and $|2h - k| = 3\sqrt{5}$

$\therefore$ Both h and k are > 0 , we must have, $h < k < 2h$

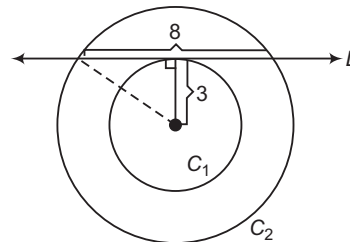
$\Rightarrow h = 3(\sqrt{5} + \sqrt{2})$
and $k = 3(\sqrt{5} + 2\sqrt{2})$



If A is point of contact, then

$OA^2 = (h^2 + k^2) = 9$
 $\Rightarrow OA^2 = 9(3 + \sqrt{10})^2$
 $\Rightarrow OA = 3(3 + \sqrt{10})$
 $\Rightarrow \lambda = 5$

153. Radius of $C_1 = 3$. If a tangent to C_1 cuts and intercept of length 8 on C_2 , then C_1 and C_2 being concentric. Radius of C_1 on the point of contact = Perpendicular drawn from center of C_2 on its chord.



$\Rightarrow$ Chord is bisected at the point of contact of C_1 .

$\therefore$ Radius of $C_2 = \sqrt{4^2 + 3^2} = 5$ [see the figure]

154. The circles are

$$(x-1)^2 + (y-1)^2 = 1$$

and

$$(x-8)^2 + (y-1)^2 = 4$$

If circles lie on opposite sides, then centre of circles must be on opposite sides.

$$\Rightarrow (1-2-a)(1-16-a) < 0$$

$$\Rightarrow a \in (-15, -1) \quad \dots (i)$$

Again, if line does not cut any circle, then distance from centre $>$ radius

$$\Rightarrow \frac{|1-2-a|}{\sqrt{5}} > 1$$

$$\frac{|1-16-a|}{\sqrt{5}} > 2$$

$$|1+a| > \sqrt{5}, |15+a| > 2\sqrt{5} \quad \dots (ii)$$

On taking the intersection of intervals, i.e. Eqs. (i) and (ii), we get

$$a \in (2\sqrt{5} - 15, -\sqrt{15} - 1)$$

$$\Rightarrow k = 3$$

Entrances Gallery

1. Given circle

$$x^2 + y^2 - 2x - 15 = 0$$

$$x^2 + y^2 - 1 = 0$$

$$\text{Radical axis } x + 7 = 0$$

... (i)

Centre of circle lies on Eq. (i).

Let the centre be $(-7, k)$.

Let equation be

$$x^2 + y^2 + 14x - 2ky + c = 0$$

Orthogonality gives

$$-14 = c - 15 \Rightarrow c = 1$$

$$(0, 1) \rightarrow 1 - 2k + 1 = 0 \Rightarrow k = 1$$

... (ii)

$$\text{Hence, radius} = \sqrt{7^2 + k^2 - c}$$

$$= \sqrt{49 + 1 - 1} = 7 \text{ units}$$

Aliter

$$\text{Given circles } x^2 + y^2 - 2x - 15 = 0$$

$$x^2 + y^2 - 1 = 0$$

Let equation of circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

Circle passes through $(0, 1)$.

$$\Rightarrow 1 + 2f + c = 0$$

Applying condition of orthogonality,

$$-2g = c - 15, 0 = c - 1 \Rightarrow c = 1, g = 7, f = -1$$

$$r = \sqrt{49 + 1 - 1} = 7 \text{ units; centre } (-7, 1)$$

2. $OB = |\alpha|$

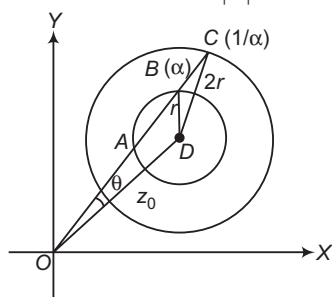
$$OC = \frac{1}{|\alpha|} = \frac{1}{|\alpha|}$$

In $\triangle OBD$,

$$\cos \theta = \frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|}$$

In $\triangle OCD$,

$$\cos \theta = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$$



$$\frac{|z_0|^2 + |\alpha|^2 - r^2}{2|z_0||\alpha|} = \frac{|z_0|^2 + \frac{1}{|\alpha|^2} - 4r^2}{2|z_0|\frac{1}{|\alpha|}}$$

$$\Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$$

3. Equation of circle can be written as

$$(x-3)^2 + y^2 + \lambda(y) = 0$$

$$\Rightarrow x^2 + y^2 - 6x + \lambda y + 9 = 0$$

$$\text{Now, (Radius)}^2 = 7 + 9 = 16$$

$$\Rightarrow 9 + \frac{\lambda^2}{4} - 9 = 16$$

$$\Rightarrow \lambda^2 = 64 \Rightarrow \lambda = \pm 8$$

$$\therefore \text{Equation is } x^2 + y^2 - 6x \pm 8y + 9 = 0.$$

4. Equation of the chord bisected at $P(h, k)$ is

$$hx + ky = h^2 + k^2 \quad \dots (i)$$

$$\text{Let any point on line be } \left(\alpha, \frac{4}{5}\alpha - 4\right).$$

Equation of the chord of contact is

$$\Rightarrow \alpha x + \left(\frac{4}{5}\alpha - 4\right)y = 9 \quad \dots (ii)$$

Comparing Eqs. (i) and (ii), we get

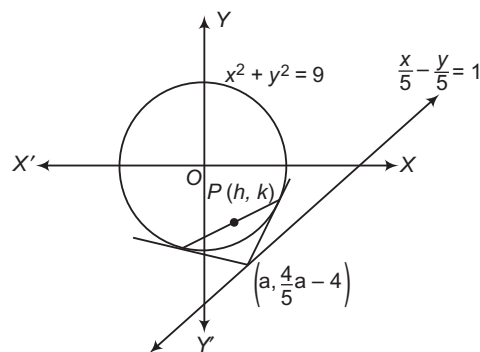
$$\frac{h}{\alpha} = \frac{k}{\frac{4}{5}\alpha - 4} = \frac{h^2 + k^2}{9}$$

$$\Rightarrow \alpha = \frac{20h}{4h - 5k}$$

$$\text{Now, } \frac{h(4h - 5k)}{20h} = \frac{h^2 + k^2}{9}$$

$$\Rightarrow 20(h^2 + k^2) = 9(4h - 5k)$$

$$\Rightarrow 20(x^2 + y^2) - 36x + 45y = 0$$



5. Equation of tangent at $P(\sqrt{3}, 1)$ is

$$\sqrt{3}x + y = 4$$

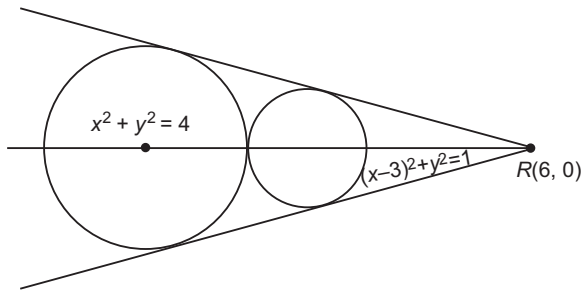
Slope of line perpendicular to above tangent is $1/\sqrt{3}$.
So, equation of tangents with slope $1/\sqrt{3}$ to $(x-3)^2 + y^2 = 1$ will be

$$y = \frac{1}{\sqrt{3}}(x-3) \pm 1\sqrt{1 + \frac{1}{3}}$$

$$\sqrt{3}y = x - 3 \pm 2$$

$$\sqrt{3}y = x - 1 \Rightarrow \sqrt{3}y = x - 5.$$

6. Point of intersection of direct common tangent is (6, 0)



So, let the equation of common tangent be

$$y - 0 = m(x - 6)$$

as it touches $x^2 + y^2 = 4$

$$\Rightarrow \frac{|0 - 0 + 6m|}{\sqrt{1 + m^2}} = 2$$

$$9m^2 = 1 + m^2$$

$$m = \pm \frac{1}{2\sqrt{2}}$$

So, equation of common tangent

$$y = \frac{1}{2\sqrt{2}}(x - 6), y = -\frac{1}{2\sqrt{2}}(x - 6) \text{ and also } x = 2$$

7. Circle touching Y-axis at (0, 2) is

$$(x - 0)^2 + (y - 2)^2 + \lambda x = 0$$

passes through (-1, 0).

$$\therefore 1 + 4 - \lambda = 0 \Rightarrow \lambda = 5$$

$$x^2 + y^2 + 5x - 4y + 4 = 0$$

Put $y = 0 \Rightarrow x = -1, -4$

$\therefore$ Circle passes through (-4, 0).

8. $2\cos \frac{\pi}{2k} + 2\cos \frac{\pi}{k} = \sqrt{3} + 1$

$$\cos \frac{\pi}{2k} + \cos \frac{\pi}{k} = \frac{\sqrt{3} + 1}{2}$$

$$\text{Let } \frac{\pi}{k} = \theta, \cos \theta + \cos \frac{\theta}{2} = \frac{\sqrt{3} + 1}{2}$$

$$\Rightarrow 2\cos^2 \frac{\theta}{2} - 1 + \cos \frac{\theta}{2} = \frac{\sqrt{3} + 1}{2}$$

$$\text{Now, } \cos \frac{\theta}{2} = t, 2t^2 + t - \frac{\sqrt{3} + 3}{2} = 0$$

$$\therefore t = \frac{-1 \pm \sqrt{1 + 4(3 + \sqrt{3})}}{4} = \frac{-1 \pm (2\sqrt{3} + 1)}{4}$$

$$= \frac{-2 - 2\sqrt{3}}{4}, \frac{\sqrt{3}}{2}$$

$$\therefore t \in [-1, 1], \cos \frac{\theta}{2} = \frac{\sqrt{3}}{2}$$

$$\frac{\theta}{2} = \frac{\pi}{6} \Rightarrow k = 3$$

9. Number of common tangents depend on the position of the circles with respect to each other.

(i) If circles touch externally $\Rightarrow C_1C_2 = r_1 + r_2$, 3 common tangents

(ii) If circles touch internally $\Rightarrow C_1C_2 = r_2 - r_1$, 1 common tangent

(iii) If circles do not touch each other, 4 common tangents

Given equations of circles are

$$x^2 + y^2 - 4x - 6y - 12 = 0 \quad \dots(i)$$

$$x^2 + y^2 + 6x + 18y + 26 = 0 \quad \dots(ii)$$

Centre of circle (i) is $C_1(2, 3)$ and radius $= \sqrt{4 + 9 + 12}$
 $= 5 = r_1$ [say]

Centre of circle (ii) is $C_2(-3, -9)$ and
radius $= \sqrt{9 + 81 - 26}$
 $= 8 = r_2$ [say]

$$\text{Now, } C_1C_2 = \sqrt{(2 + 3)^2 + (3 + 9)^2}$$

$$\Rightarrow C_1C_2 = \sqrt{5^2 + 12^2}$$

$$\Rightarrow C_1C_2 = \sqrt{25 + 144} = 13$$

$$\therefore r_1 + r_2 = 5 + 8 = 13$$

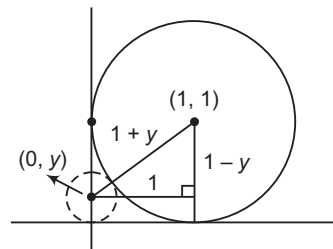
$$\text{Also, } C_1C_2 = r_1 + r_2$$

Thus, both circles touch each other externally. Hence, there are three common tangents.

10. According to the figure,

$$(1 + y)^2 = (1 - y)^2 + 1 \quad [\because y > 0]$$

$$\Rightarrow y = \frac{1}{4}$$



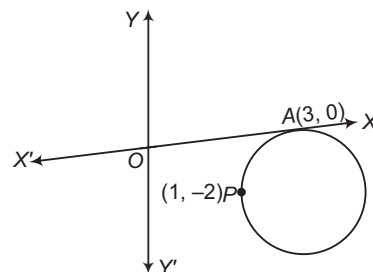
11. Let the equation of circle be

$$(x - 3)^2 + (y - 0)^2 + \lambda y = 0$$

As it passes through (1, -2)

$$\therefore (1 - 3)^2 + (-2)^2 + \lambda(-2) = 0$$

$$\Rightarrow 4 + 4 - 2\lambda = 0 \Rightarrow \lambda = 4$$

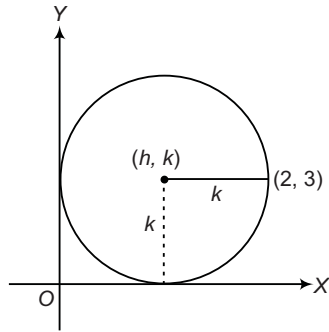


$\therefore$ Equation of circle is

$$(x - 3)^2 + y^2 + 4y = 0$$

By hit and trial method, we see that point (5, -2) satisfies equation of circle.

12. **Given** (i) A circle which touches X-axis at the point (1, 0).
(ii) The circle also passes through the point (2, 3).
To find The length of the diameter of the circle.



Let us assume that the coordinates of the centre of the circle are $C(h, k)$ and its radius be r .

Now, since the circle touches X-axis at (1, 0), hence its radius should be equal to ordinate of centre.

$$\Rightarrow r = k$$

Hence, the equation of the circle is

$$(x - h)^2 + (y - k)^2 = k^2$$

Also, given that the circle passes through points (1, 0) and (2, 3). Hence, substituting them in the equation of the circle, we get

$$(1 - h)^2 + (0 - k)^2 = k^2 \quad \dots(i)$$

$$(2 - h)^2 + (3 - k)^2 = k^2 \quad \dots(ii)$$

From Eq. (i), we get $h = 1$

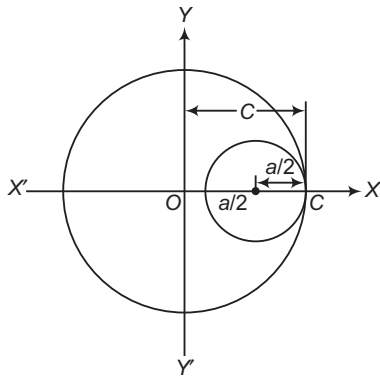
On substituting $h = 1$ in Eq. (ii), we get

$$(2 - 1)^2 + (3 - k)^2 = k^2$$

$$\Rightarrow k = \frac{5}{3}$$

Hence, the diameter of the circle is $2k = \frac{10}{3}$.

13. $x^2 + y^2 - ax = 0$ and $x^2 + y^2 = c^2$ touch each other.



- (i) If circles touch internally, then

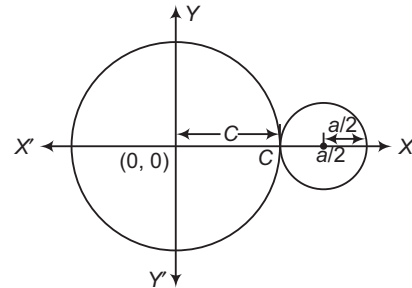
$$\left| c - \frac{a}{2} \right| = \frac{a}{2}$$

$$\Rightarrow c - \frac{a}{2} = \frac{a}{2}$$

$$\Rightarrow c = a, c > 0$$

$$\therefore |a| = c$$

- (ii) If circles touch externally, then



$$\left| c + \frac{a}{2} \right| = \frac{a}{2} \Rightarrow c + \frac{a}{2} = \frac{a}{2}$$

$$c = 0 \text{ i.e. not possible as } c > 0$$

$\therefore$ The circles should touch internally and

$$|a| = c$$

14. Circle whose diametric end points are (1, 0) and (0, 1) will be of smallest radius.

$$\Rightarrow (x - 1)(x - 0) + (y - 0)(y - 1) = 0$$

$$\Rightarrow x^2 + y^2 - x - y = 0$$

15. Since, the coordinates of the centre of the circle are (2, 4).

$$\text{Also, } r^2 = 4 + 16 + 5 = 25$$

The line will intersect the circle at two distinct points, if the distance of (2, 4) from $3x - 4y = m$ is less than radius of the circle.

$$\therefore \frac{|6 - 16 - m|}{5} < 5$$

$$\Rightarrow -25 < 10 + m < 25$$

$$\therefore -35 < m < 15$$

16. Let $S \equiv x^2 + y^2 + 3x + 7y + 2p - 5 = 0$

$$\text{and } S' \equiv x^2 + y^2 + 2x + 2y - p^2 = 0$$

Equation of the required circle is $S + \lambda S' = 0$

As it passes through (1, 1), the value of

$$\lambda = \frac{-(7 + 2p)}{(6 - p^2)}$$

Here, λ is not defined at $p = \pm \sqrt{6}$

Hence, it is true for all except two values of p .

17. Given equation can be rewritten as

$$(x + 1)^2 + (y + 2)^2 = (2\sqrt{2})^2$$

Let required point be $Q(\alpha, \beta)$.

Then, mid-point of $P(1, 0)$ and $Q(\alpha, \beta)$ is the centre of the circle

$$\text{i.e. } \frac{\alpha + 1}{2} = -1 \quad \text{and} \quad \frac{\beta + 0}{2} = -2$$

$$\Rightarrow \alpha = -3 \quad \text{and} \quad \beta = -4$$

Hence, required point is $(-3, -4)$.

18. Equation of circle which touches X-axis and coordinates of centre are (h, k) , is

$$(x - h)^2 + (y - k)^2 = k^2$$

Since, it is passing through $(-1, 1)$, then

$$(-1 - h)^2 + (1 - k)^2 = k^2$$

$$\Rightarrow h^2 + 2h - 2k + 2 = 0$$

For real circles, $D \geq 0$

$$\Rightarrow (2)^2 - 4(-2k + 2) \geq 0$$

$$\Rightarrow k \geq \frac{1}{2}$$

19. The given equations of diameters are

$$3x - 4y - 7 = 0 \quad \dots(i)$$

$$\text{and} \quad 2x - 3y - 5 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 1 \quad \text{and} \quad y = -1$$

$\therefore$ The intersection of two diameters is the centre of circle, i.e. $(1, -1)$.

Let r be the radius of circle, then

$$\text{Area of circle} = 49\pi \text{ sq units}$$

$$\Rightarrow \pi r^2 = 49\pi$$

$$\Rightarrow r = 7 \text{ units}$$

$\therefore$ Equation of required circle is

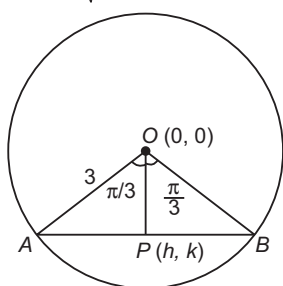
$$(x - 1)^2 + (y + 1)^2 = 49$$

$$\Rightarrow x^2 - 2x + 1 + y^2 + 2y + 1 = 49$$

$$\Rightarrow x^2 + y^2 - 2x + 2y - 47 = 0$$

20. Let the coordinates of a point P be (h, k) which is mid-point of the chord AB .

$$\text{Now, } OP = \sqrt{(h - 0)^2 + (k - 0)^2} = \sqrt{h^2 + k^2}$$



$$\text{In } \triangle AOP, \quad \cos \frac{\pi}{3} = \frac{OP}{OA}$$

$$\Rightarrow \frac{1}{2} = \frac{\sqrt{h^2 + k^2}}{3}$$

$$\Rightarrow h^2 + k^2 = \frac{9}{4}$$

Hence, the required locus is

$$x^2 + y^2 = \frac{9}{4}$$

21. Let equation of circles be

$$S_1 \equiv x^2 + y^2 + 2ax + cy + a = 0$$

$$\text{and} \quad S_2 \equiv x^2 + y^2 - 3ax + dy - 1 = 0$$

Chord through intersection points P and Q of the given circles is $S_1 - S_2 = 0$.

$$\therefore (x^2 + y^2 + 2ax + cy + a) - (x^2 + y^2 - 3ax + dy - 1) = 0$$

$$\Rightarrow 5ax + (c - d)y + a + 1 = 0$$

On comparing it with $5x + by - a = 0$, we get

$$\frac{5a}{5} = \frac{c - d}{b} = \frac{a + 1}{-a}$$

$$\Rightarrow a(-a) = a + 1$$

$$\Rightarrow a^2 + a + 1 = 0$$

which gives no real value of a .

Hence, the line will pass through P and Q for no value of a .

22. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

It cuts the circle $x^2 + y^2 = p^2$ orthogonally.

$$\therefore 2g(0) + 2f(0) = c - p^2$$

$$\Rightarrow c = p^2$$

Also, it passes through (a, b) .

$$\therefore a^2 + b^2 + 2ga + 2fb + p^2 = 0$$

So, locus of $(-g, -f)$ is

$$a^2 + b^2 - 2ax - 2by + p^2 = 0$$

$$\Rightarrow 2ax + 2by - (a^2 + b^2 + p^2) = 0$$

23. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0.$$

It cuts the circle $x^2 + y^2 = 4$ orthogonally, if

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$\therefore 2g \cdot 0 + 2f \cdot 0 = c - 4$$

$$\Rightarrow c = 4$$

Equation of circle is

$$x^2 + y^2 + 2gx + 2fy + 4 = 0$$

Since, it passes through the point (a, b) .

$$\therefore a^2 + b^2 + 2ag + 2fb + 4 = 0$$

Locus of centre $(-g, -f)$ will be

$$a^2 + b^2 - 2xa - 2yb + 4 = 0$$

$$\Rightarrow 2ax + 2by - (a^2 + b^2 + 4) = 0$$

Aliter

Let the centre of required circle be $(-g, -f)$. This circle cuts the circle $x^2 + y^2 = 4$ orthogonally. The centre and radius of circle $x^2 + y^2 = 4$ are $(0, 0)$ and 2 , respectively.

$$\therefore g^2 + f^2 = 4 + (a + g)^2 + (b + f)^2$$

$$\Rightarrow g^2 + f^2 = 4 + a^2 + g^2 + 2ag + 2bf + b^2 + f^2$$

$$\Rightarrow 4 + a^2 + b^2 + 2ag + 2bf = 0$$

Hence, the locus of centre is

$$2ax + 2by - (a^2 + b^2 + 4) = 0.$$

24. Since, the coordinates of one end of a diameter of a circle A are (p, q) and let the coordinates of other end B be (x_1, y_1) .

Equation of circle in diameter form is

$$(x - p)(x - x_1) + (y - q)(y - y_1) = 0$$

$$\Rightarrow x^2 - (p + x_1)x + px_1 + y^2 - (q + y_1)y + qy_1 = 0$$

$$\Rightarrow x^2 - (p + x_1)x + y^2 - (q + y_1)y + px_1 + qy_1 = 0$$

Since, this circle touches X -axis.

$$\text{i.e. } y = 0$$

$$\therefore x^2 - (p + x_1)x + px_1 + qy_1 = 0$$

Also, the discriminant of above equation will be

$$(p + x_1)^2 = 4(px_1 + qy_1)$$

$$\Rightarrow p^2 + x_1^2 + 2px_1 = 4px_1 + 4qy_1$$

$$\Rightarrow x_1^2 - 2px_1 + p^2 = 4qy_1$$

Hence, locus of point B is $(x - p)^2 = 4qy$.

25. Given lines $2x + 3y + 1 = 0$ and $3x - y - 4 = 0$ are the diameters of circle.

The intersection of two lines is the centre of circle $(1, -1)$. Circumference of circle $= 10\pi$ [given]

$$\Rightarrow 2\pi r = 10\pi$$

$$\Rightarrow r = 5$$

$\therefore$ Equation of circle having centre $(1, -1)$ and radius 5 is

$$(x-1)^2 + (y+1)^2 = 5^2$$

$$\Rightarrow x^2 + 1 - 2x + y^2 + 2y + 1 = 25$$

$$\Rightarrow x^2 + y^2 - 2x + 2y - 23 = 0$$

26. Given equation of line is $y = x$... (i)
and equation of circle is

$$x^2 + y^2 - 2x = 0 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$x^2 + x^2 - 2x = 0$$

$$\Rightarrow 2x(x-1) = 0$$

$$\Rightarrow x = 0, y = 1$$

On putting the value of x in Eq. (i) respectively, we get $y = 0, x = 1$.

Let coordinates of A be $(0, 0)$ and coordinates of B be $(1, 1)$.

$\therefore$ Equation of circle when AB as a diameter, is

$$(x-0)(x-1) + (y-0)(y-1) = 0$$

$$\Rightarrow x^2 - x + y^2 - y = 0$$

$$\Rightarrow x^2 + y^2 - x - y = 0$$

27. The equation of first circle is

$$(x-1)^2 + (y-3)^2 = r^2$$

whose centre is $C_1(1, 3)$ and radius, $r_1 = r$.

and equation of second circle is

$$x^2 + y^2 - 8x + 2y + 8 = 0$$

whose centre is $C_2(4, -1)$ and radius,

$$r_2 = \sqrt{4^2 + 1^2 - 8} = \sqrt{17-8} = 3 \text{ units}$$

If two circles intersect at two distinct points, then

$$r_1 - r_2 < C_1C_2 < r_1 + r_2$$

$$\Rightarrow r - 3 < \sqrt{(4-1)^2 + (-1-3)^2} < r + 3$$

$$\Rightarrow r - 3 < \sqrt{9+16} < r + 3$$

$$\Rightarrow r - 3 < 5 < r + 3$$

$$\Rightarrow r - 3 < 5 \text{ and } 5 < r + 3$$

$$\Rightarrow r < 8 \text{ and } 2 < r$$

$$\Rightarrow 2 < r < 8$$

28. Given equation of circle is

$$x^2 + y^2 - 4x - 2y - 20 = 0$$

whose centre is $C(2, 1)$ and radius is 5 units.

At point $(10, 7)$,

$$S_1 = 10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 = 75 > 0$$

So, P lies outside the circle

Now, $PC = \sqrt{(2-10)^2 + (1-7)^2}$

$$= \sqrt{8^2 + 6^2}$$

$$= \sqrt{10^2} = 10 \text{ units}$$

$\therefore$ Greatest distance between circle and the point is $P = 10 + 5 = 15$ units.

29. Given equation of circle is $x^2 + y^2 + 4x - 4y + 4 = 0$ whose centre is $(-2, 2)$ and radius $= 2$ units.

Let the equation of required tangent be

$$x + y = a$$

The perpendicular distance from centre to the circle is equal to radius of circle.

$$\therefore \left| \frac{-2+2-a}{\sqrt{2}} \right| = 2$$

$$\Rightarrow a = 2\sqrt{2}$$

Hence, equation of the tangent is

$$x + y = 2\sqrt{2}$$

30. We know that, equation of circle having centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2.$$

Here, centre $= (-a, -b)$ and radius $= \sqrt{a^2 - b^2}$

$\therefore$ Required equation of circle is

$$(x+a)^2 + (y+b)^2 = (\sqrt{a^2 - b^2})^2$$

$$\Rightarrow x^2 + a^2 + 2ax + y^2 + b^2 + 2by = a^2 - b^2$$

$$\Rightarrow x^2 + 2ax + y^2 + 2by + 2b^2 = 0$$

31. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0, \text{ whose centre is } (-g, -f).$$

Since, the circle passing through $(0, 0)$ and $(4, 0)$.

$$\therefore 0^2 + 0^2 + 2g(0) + 2f(0) + c = 0$$

$$\Rightarrow c = 0$$

$$\text{and } 4^2 + 0^2 + 2g(4) + 2f(0) + 0 = 0$$

$$\Rightarrow 16 + 8g = 0$$

$$\therefore g = -2$$

Also, the centre $(-g, -f)$ lies on the line $y = x$.

$$\therefore -f = -g$$

$$\Rightarrow f = g$$

$$\therefore f = -2$$

Now, the equation of circle having centre $(2, 2)$ and radius $\sqrt{8}$ is

$$(x-2)^2 + (y-2)^2 = (\sqrt{8})^2$$

$$\Rightarrow (x-2)^2 + (y-2)^2 = 8$$

32. We have an equation of circle

$$x^2 + y^2 - 10x - 14y - 151 = 0$$

Let us compare this equation with

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$$\text{Clearly, } g = -5, f = -7$$

$$\text{and } c = -151$$

$$\therefore \text{Centre of circle} = (-g, -f) = (5, 7)$$

$$\text{and radius} = \sqrt{g^2 + f^2 - c}$$

$$= \sqrt{25 + 49 + 151}$$

$$= \sqrt{225} = 15 \text{ units}$$

Now, distance between the centre and given point $(-7, 2)$

$$= \sqrt{(5+7)^2 + (5-2)^2}$$

$$= \sqrt{(12)^2 + 25}$$

$$= \sqrt{169} = 13 \text{ units}$$

Clearly, the point $(-7, 2)$ lies inside the circle.

$\therefore$ Shortest distance $=$ Radius $-$ Distance between point and centre $= 15 - 13 = 2$ units

33. The centre of the given circle $x^2 + y^2 - 4x - 6y + 11 = 0$ is (2, 3).

Let the other end of diameter be (x_1, y_1) .

Since, centre = $\left(\frac{x_1 + 3}{2}, \frac{y_1 + 4}{2}\right)$

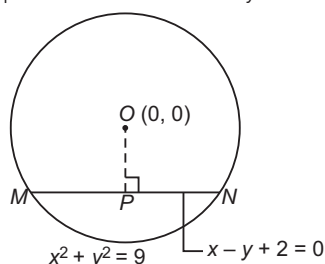
$\therefore (2, 3) = \left(\frac{x_1 + 3}{2}, \frac{y_1 + 4}{2}\right)$

$\Rightarrow 2 = \frac{x_1 + 3}{2}, 3 = \frac{y_1 + 4}{2}$

$\therefore x_1 = 1, y_1 = 2$

Hence, coordinates of other end of the diameter are (1, 2).

34. Given equation of circle is $x^2 + y^2 = 9$.



$\therefore$ Centre = (0, 0), radius = 3 units

As chord is intercepted by circle and line

$x - y + 2 = 0$

Draw $OP \perp MN$.

$$\text{Length of } OP = \frac{|0 - 0 + 2|}{\sqrt{(1)^2 + (-1)^2}} = \frac{2}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2} \text{ units}$$

Now, $\triangle OPN$ is a right angled triangle.

So, $ON^2 = OP^2 + PN^2$

[by Pythagoras theorem]

$\Rightarrow (3)^2 = (\sqrt{2})^2 + PN^2$
 $\left[\begin{array}{l} \because \text{radius of circle is 3 units} \\ \text{and } OP = \sqrt{2} \text{ units} \end{array} \right]$

$\Rightarrow 9 - 2 = PN^2$

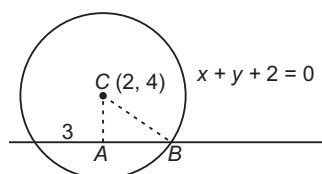
$\Rightarrow PN^2 = 7$

$\Rightarrow PN = \sqrt{7} \text{ units}$

$\therefore \triangle OPN \cong \triangle OPM$

$\therefore MN = 2PN$ [$\because MP = PN$]
 $= 2\sqrt{7}$
 $= \sqrt{28} \text{ units}$

35. Let r be the radius of the circle.



Now, perpendicular distance

$$AC = \frac{|2 + 4 + 2|}{\sqrt{1^2 + 1^2}} = \frac{8}{\sqrt{2}} = 4\sqrt{2}$$

In right angled $\triangle CAB$,

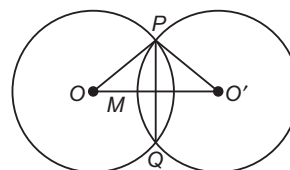
$$r^2 = (AC)^2 + (AB)^2 = (4\sqrt{2})^2 + (3)^2 = 32 + 9$$

$\Rightarrow r^2 = 41$

$\therefore r = \sqrt{41} \text{ cm}$

36. Condition for conjugate line is

$$r^2(l_1 + mm_1) = nn_1$$



37. Given equations of circles are

$$x^2 + y^2 - 4y = 0$$

and $x^2 + y^2 - 8x - 4y + 11 = 0$

$\therefore$ Equation of chord is

$$x^2 + y^2 - 4y - (x^2 + y^2 - 8x - 4y + 11) = 0$$

$\Rightarrow 8x - 11 = 0$

Centre and radius of first circle are O (0, 2)

and $OP = r = 2$.

Now, perpendicular distance from O (0, 2) to the line $8x - 11 = 0$ is

$$d = OM = \frac{|8 \times 0 - 11|}{\sqrt{8^2}} = \frac{11}{8} \text{ cm}$$

$$\begin{aligned} \text{In } \triangle OMP, PM &= \sqrt{OP^2 - OM^2} = \sqrt{2^2 - \left(\frac{11}{8}\right)^2} \\ &= \sqrt{4 - \frac{121}{64}} = \sqrt{\frac{256 - 121}{64}} = \frac{\sqrt{135}}{8} \text{ cm} \end{aligned}$$

$\therefore$ Length of chord, $PQ = 2PM = 2 \times \frac{\sqrt{135}}{8} = \frac{\sqrt{135}}{4} \text{ cm}$

38. Since, the centres of given circles are

$$C_1(1, -2) \text{ and } C_2(-2, 2)$$

and radii of given circles are

$$r_1 = 1, r_2 = 2$$

Now, $C_1C_2 = \sqrt{(-2 - 1)^2 + (2 + 2)^2} = \sqrt{9 + 16} = 5 \text{ units}$

$\therefore$ Shortest distance between two circles

= Distance between two centres - $(r_1 + r_2)$

$$= 5 - (1 + 2) = 2 \text{ units}$$

39. The centre and radius of given circle

$$x^2 + y^2 - 2x - 4y - 20 = 0$$

are $C_1(1, 2)$ and $r = \sqrt{1^2 + 2^2 + 20} = 5 \text{ units}$

Since, the radii of the two circles are equal, therefore the two circles will touch externally and the point of contact will be the mid-point between two centres.

Let coordinates of the center of required circle be (h, k) .

$\therefore \frac{h+1}{2} = 5 \text{ and } \frac{k+2}{2} = 5$

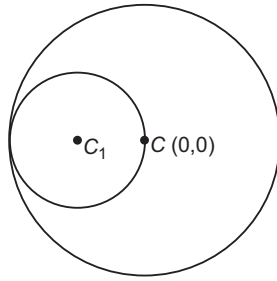
$\Rightarrow h = 9 \text{ and } k = 8$

Hence, the centre is (9, 8).

40. Let $S \equiv x^2 + y^2 = 16$

At point $(0, 0)$,

$$S \equiv 0 + 0 - 16 = -16 < 0$$



At point $(0, 1)$,

$$S \equiv 0 + 1 - 16 = -15 < 0$$

Thus, both points lie inside the circle.

It means required circle touch inside the given circle.

So, the centre and radius of given circle are $C(0, 0)$ and $r = 4$.

Since, the required circle is passing through centre $(0, 0)$.

$\therefore$ Diameter of required circle = 4 units

and radius of required circle = $\frac{4}{2} = 2$ units

41. Centres and constant terms in the circles $x^2 + y^2 - 5 = 0$, $x^2 + y^2 - 8x - 6y + 10 = 0$ and $x^2 + y^2 - 4x + 2y - 2 = 0$ are $C_1(0, 0)$, $C_2(4, 3)$, $C_3(2, -1)$, $c_1 = -5$, $c_2 = 10$ and $c_3 = -2$.

Also, centre and constant of circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

are

$$C_4(-g, -f) \text{ and } c_4 = c$$

Since, the first circle intersect all the three at the extremities of diameter, therefore they are orthogonal to each other.

$$\therefore 2(g_1g_2 + f_1f_2) = c_1 + c$$

$$\therefore 2[-g(0) + (-f)(0)] = c - 5$$

$$\Rightarrow c = 5$$

$$2[-g(4) + (-f)(3)] = c + 10$$

$$\Rightarrow -2(4g + 3f) = 5 + 10$$

$$\Rightarrow 4g + 3f = -\frac{15}{2} \quad \dots(i)$$

$$\text{and } 2[-g(2) + (-f)(-1)] = c - 2$$

$$\Rightarrow 2[-2g + f] = 5 - 2$$

$$\Rightarrow -2g + f = \frac{3}{2} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$f = -\frac{9}{10}$$

and

$$g = -\frac{12}{10}$$

$$\therefore fg = \frac{-9}{10} \times -\frac{12}{10} = \frac{27}{25}$$

and

$$4f = 4 \times -\frac{9}{10}$$

$$\Rightarrow 4f = -\frac{36}{10}$$

$$\therefore 4f = 3g$$

42. We know that, the length of the tangent from a point $P(x_1, y_1)$ is $\sqrt{S_1}$, where S_1 is the equation of circle at point P .

Given equations of circles are

$$P \equiv x^2 + y^2 - a^2,$$

$$Q \equiv x^2 + y^2 - b^2$$

and

$$R \equiv x^2 + y^2 - c^2$$

$\therefore$ Length of tangent at point $A(x_1, y_2)$ are

$$T_1 = \sqrt{P_1} = \sqrt{x_1^2 + y_1^2 - a^2}$$

$$T_2 = \sqrt{Q_1} = \sqrt{x_1^2 + y_1^2 - b^2}$$

and

$$T_3 = \sqrt{R_1} = \sqrt{x_1^2 + y_1^2 - c^2}$$

Since, it is given that that length of tangent are in an AP.

$$\begin{aligned} \therefore T_2^2 &= \frac{T_1^2 + T_3^2}{2} \\ \Rightarrow &= x_1^2 + y_1^2 - b^2 \\ &= \frac{(x_1^2 + y_1^2 - a^2) + (x_1^2 + y_1^2 - c^2)}{2} \end{aligned}$$

$$\Rightarrow x_1^2 + y_1^2 - b^2 = x_1^2 + y_1^2 - \frac{(a^2 + c^2)}{2}$$

$$\Rightarrow -b^2 = -\frac{(a^2 + c^2)}{2}$$

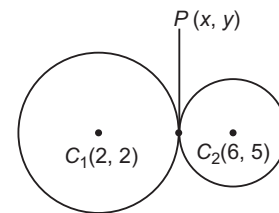
$$\Rightarrow b^2 = \frac{a^2 + c^2}{2}$$

Hence, a^2, b^2 and c^2 are in an AP.

43. Centres and radii of given circles are

$$C_1(2, 2), r_1 = \sqrt{2^2 + 2^2 - 7} = 1 \text{ unit}$$

and $C_2(6, 5)$,



$$\begin{aligned} r_2 &= \sqrt{(6)^2 + (5)^2 - 45} \\ &= \sqrt{36 + 25 - 45} = 4 \text{ units} \end{aligned}$$

Let P be the point at which the circle touch.

Using internal ratio formula,

$$\begin{aligned} P(x, y) &= \left(\frac{1 \times 6 + 4 \times 2}{1 + 4}, \frac{1 \times 5 + 4 \times 2}{1 + 4} \right) \\ &= \left(\frac{6 + 8}{5}, \frac{5 + 8}{5} \right) = \left(\frac{14}{5}, \frac{13}{5} \right) \end{aligned}$$

44. Let the equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

where, centre = $(-g, -f)$

The centre of given circle

$$x^2 + y^2 - 20x + 4 = 0 \text{ is } (10, 0).$$

Condition for cut the two circles,

$$\therefore 2(g_1g_2 + f_1f_2) = c_1 + c_2$$

$$2(-g) \times 10 + 0 \times (-f) = c + 4$$

$$2(-10g) = c + 4$$

Also, circle touches the line $x = 2$.

So, the perpendicular distance from centre to the circle is equal to radius of the circle.

$$\begin{aligned} \therefore \frac{|-g-2|}{\sqrt{1}} &= \sqrt{g^2 + f^2 - c} \\ \Rightarrow (g+2) &= \sqrt{g^2 + f^2 - c} \\ \Rightarrow g^2 + 4 + 4g &= g^2 + f^2 - c \\ \Rightarrow f^2 - 4g - c - 4 &= 0 \\ \Rightarrow f^2 - 4g + 4 + 20g - 4 &= 0 \\ \Rightarrow f^2 + 16g &= 0 \\ \text{Hence, the locus of } (-g, -f) &\text{ is} \\ y^2 - 16x &= 0 \\ \Rightarrow y^2 &= 16x \end{aligned}$$

45. We have equation of circle

$$x^2 + y^2 - 2x + 6y - 6 = 0$$

$$\begin{aligned} \Rightarrow g = -1, f = 3, c = -6 \\ \text{Radius } (r) &= \sqrt{1 + 9 + 6} = 4 \\ \text{and centre} &= (1, -3) \end{aligned}$$

Now, distance of tangent $3x - 4y + k = 0$ from the centre is

$$4 = \frac{|3 + 12 + k|}{\sqrt{25}} \Rightarrow 4 = \frac{k + 15}{\pm 5}$$

$$\begin{aligned} \Rightarrow \pm 20 &= k + 15 \\ \therefore k &= 5, -35 \end{aligned}$$

46. The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ passes through $(2, 0)$.

$$\therefore 4 + 4g + c = 0 \quad \dots(i)$$

Its intercept on X-axis is $2 \times \sqrt{g^2 - c} = 5$ [given]

$$\Rightarrow g^2 - c = \frac{25}{4} \Rightarrow c = \frac{4g^2 - 25}{4}$$

$$\therefore 4 + 4g + \frac{4g^2 - 25}{4} = 0 \quad [\text{using Eq. (i)}]$$

$$\Rightarrow 16 + 16g + 4g^2 - 25 = 0$$

$$\Rightarrow 4g^2 + 16g - 9 = 0$$

$$\Rightarrow g = \frac{-9}{2}, \frac{1}{2}$$

So, $g = -\frac{9}{2}$, since centre lies in first quadrant, also $c = 14$.

Hence, $x^2 + y^2 - 9x + 2fy + 14 = 0$ is the required equation.

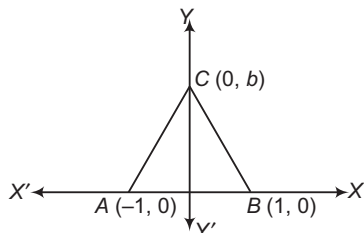
47. The third vertex of the equilateral triangle must lie on Y-axis. Let it be $C(0, b)$.

Now, $AC = BC = AB$

$$\therefore \sqrt{1 + b^2} = \sqrt{1 + b^2} = 2$$

$$\therefore b^2 = 4 - 1$$

$$\Rightarrow b = \sqrt{3}$$



As triangle is an equilateral, so circumcentre is centroid of $\triangle ABC$.

Now, centroid = Circumcentre

$$= \left(0, \frac{b}{3}\right) = \left(0, \frac{1}{\sqrt{3}}\right)$$

$$\therefore \text{Required circle is } (x - 0)^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = r^2$$

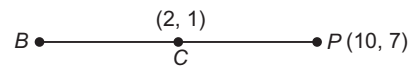
$$\Rightarrow (1 - 0)^2 = \left(0 - \frac{1}{\sqrt{3}}\right)^2 = r^2$$

$$\Rightarrow x^2 + \left(y - \frac{1}{\sqrt{3}}\right)^2 = \frac{4}{3}$$

48. We have,

$$10^2 + 7^2 - 4 \times 10 - 2 \times 7 - 20 > 0$$

So, the point $P(10, 7)$ lies outside the circle.



Also, centre of given circle is $C(2, 1)$

and

$$r = 5$$

$$\therefore PC = \sqrt{(10 - 2)^2 + (7 - 1)^2} = 10 \text{ units}$$

and $BC = 5$ [$\because B$ is a point on circle and line PC]

$\therefore$ Greatest distance = $10 + 5 = 15$ units

Least distance = $10 - 5 = 5$ units

49. $\therefore g = 4, f = 2, c = -8$

$\therefore$ Centre = $(-4, -2)$ = Centre of concentric circle

Also, radius of unit circle = 1 unit

$\therefore$ Required equation of circle is

$$(x + 4)^2 + (y + 2)^2 = 1$$

$$\Rightarrow x^2 + y^2 + 8x + 4y + 19 = 0$$

50. Given, $y = mx - b\sqrt{1 + m^2}$ touches both the circles.

$\therefore$ Distance from centre = Radius of both the circles

$$\frac{|ma - 0 - b\sqrt{1 + m^2}|}{\sqrt{m^2 + 1}} = b \quad \text{and} \quad \frac{|-b\sqrt{1 + m^2}|}{\sqrt{m^2 + 1}} = b$$

$$\Rightarrow |ma - b\sqrt{1 + m^2}| = |b\sqrt{1 + m^2}|$$

$$\begin{aligned} \Rightarrow m^2 a^2 - 2abm\sqrt{1 + m^2} + b^2(1 + m^2) \\ = b^2(1 + m^2) \end{aligned}$$

$$\Rightarrow ma - 2b\sqrt{1 + m^2} = 0$$

$$\Rightarrow m^2 a^2 = 4b^2(1 + m^2)$$

$$\Rightarrow m = \frac{2b}{\sqrt{a^2 - 4b^2}}$$

51. Any line through the point $(-5, -4)$ is

$$y + 4 = m(x + 5)$$

$$mx - y + (5m - 4) = 0 \quad \dots(i)$$

If it is a tangent, then perpendicular from centre $(-2, -3)$

is equal to radius = $\sqrt{(2)^2 + (3)^2} = \sqrt{13}$

$$= \sqrt{4 + 9 - 8} = \sqrt{5}$$

$$\Rightarrow \frac{m(-2) - (-3) + (5m - 4)}{\sqrt{m^2 + 1}} = \sqrt{5}$$

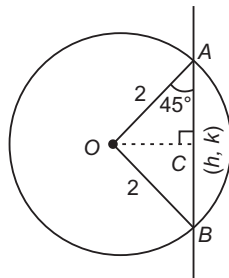
$$\Rightarrow -2m + 3 + 5m - 4 = \sqrt{5}\sqrt{m^2 + 1}$$

$$\Rightarrow (3m - 1)^2 = 5(m^2 + 1)$$

$$\begin{aligned} \Rightarrow 9m^2 + 1 - 6m &= 5m^2 + 5 \\ \Rightarrow 4m^2 - 6m - 4 &= 0 \\ \Rightarrow 2m^2 - 3m - 2 &= 0 \\ \Rightarrow m &= -\frac{1}{2}, 2 \end{aligned}$$

Hence, required equations are
 $2x - y + 6 = 0$ and $x + 2y + 13 = 0$

52. As we have to find the locus of mid-point of chord and we know perpendicular bisect the chord.



$$\therefore \angle OAC = 45^\circ \text{ or } \frac{OC}{OA} = \sin 45^\circ$$

$$\Rightarrow OC = \frac{2}{\sqrt{2}}$$

$$\therefore \sqrt{h^2 + k^2} = OC^2$$

or $x^2 + y^2 = 2$ is required equation of locus of mid-point of chord subtending right angle at centre.

53. Given, $C_1 = (3, 4)$ and $C_2 = (3, 0)$

$$\therefore |C_1C_2| = \sqrt{(3-3)^2 + (4-0)^2} = 4$$

$$r_1 = \sqrt{9+16-0} = 5$$

$$r_2 = \sqrt{9-8} = 1$$

$$\Rightarrow |r_1 - r_2| = 4$$

$$\therefore |r_1 - r_2| > |C_1C_2|$$

One circle lies inside the other and both the circles are touching internally.

54. Since, circles intersect orthogonally.

$$\therefore 2(g_1g_2 + f_1f_2) = C_1 + C_2$$

$$\Rightarrow 2(1 \cdot 0 + k \cdot k) = 6 + k$$

$$\Rightarrow 2k^2 - k - 6 = 0$$

$$\Rightarrow 2k^2 - 4k + 3k - 6 = 0$$

$$\Rightarrow 2k(k-2) + 3(k-2) = 0$$

$$\Rightarrow (2k+3)(k-2) = 0$$

$$\Rightarrow k = -\frac{3}{2} \text{ or } 2$$

55. Centre of the circle

$$x^2 + y^2 + 4x - 10y - 7 = 0$$

is $(-2, 5)$ and radius $(r) = \sqrt{4+25+7} = 6$ units

Distance of point $(4, -3)$ from centre $(-2, 5)$ is

$$\sqrt{(-2-4)^2 + (5+3)^2} = \sqrt{100} = 10 \text{ units}$$

Also, we check whether the point lies inside the circle or not.

$$\therefore (4)^2 + (-3)^2 + 4 \cdot 4 - 10 \cdot (-3) - 7 = 80 > 0$$

So, minimum distance $= 10 - 6 = 4$ units

and maximum distance $= 10 + 6 = 16$ units

$\therefore$ Sum of the minimum and maximum distances

$$= 4 + 16 = 20 \text{ units}$$

56. The intersection of line and circle is

$$x^2 + m^2x^2 - 20mx + 90 = 0$$

$$\Rightarrow x^2(1+m^2) - 20mx + 90 = 0$$

$$\text{Let } D < 0$$

$$\Rightarrow 400m^2 - 4 \times 90(1+m^2) < 0$$

$$\Rightarrow 40m^2 < 360$$

$$\therefore |m| < 3$$

57. On putting $y = x$ in the equation of given circle, we get

$$2x^2 - 2x = 0$$

$$\Rightarrow 2x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

$$\Rightarrow y = 0, 1$$

$\therefore$ Coordinates of diameter AB are

$$A(0, 0) \text{ and } B(1, 1)$$

Hence, required equation of circle is

$$(x-0)(x-1) + (y-0)(y-1) = 0$$

$$\Rightarrow x(x-1) + y(y-1) = 0$$

58. Length of tangent from $(5, 1)$ to the circle is $\sqrt{S_1}$

$$= \sqrt{5^2 + 1^2 + 6 \cdot 5 - 4 \cdot 1 - 3}$$

$$= \sqrt{49} = 7 \text{ units}$$

59. Since, $ax^2 + ay^2 + 2gx + 2fy + c = 0$ touches X-axis,

i.e. $y = 0$ satisfy the equation

$$\therefore ax^2 + 2gx + c = 0$$

$$\Rightarrow (2g)^2 = 4 \times ac \quad [\text{since, } x \text{ is a real}]$$

$$\Rightarrow 4g^2 = 4ac$$

$$\Rightarrow g^2 = ac$$

60. We have, $x^2 + y^2 = a^2$

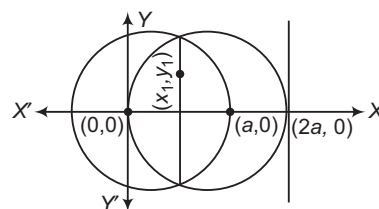
i.e. centre $= (0, 0)$ and radius $= a$

Also, we have

$$x^2 + y^2 - 2ax = 0$$

Centre $= (a, 0)$ and radius $= a$

So, the circles are intersecting each other.



Locus of the tangent to the extremities of the chord passes through (x_1, y_1) to circle $x^2 + y^2 = a^2$ is

$$T = 0$$

$$\Rightarrow xx_1 + yy_1 - a^2 = 0 \quad \dots(i)$$

Also, above equation is tangent to circle

$$x^2 + y^2 - 2ax = 0$$

Equation of tangent line is

$$x = 2a$$

$$\Rightarrow 1 \cdot x + 0 \cdot y - 2a = 0 \quad \dots(ii)$$

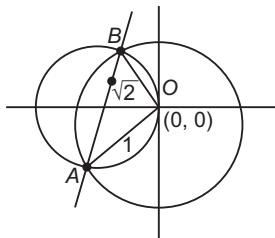
On comparing Eqs. (i) and (ii), we get

$$\frac{x_1}{1} = \frac{y_1}{0} = \frac{a^2}{2a}$$

$$\Rightarrow x_1 = \frac{a}{2}, y_1 = 0$$

$\therefore$ Line will pass through the fixed point $\left(\frac{a}{2}, 0\right)$.

61. We have, $x^2 + y^2 = 1$
Centre = (0, 0) and radius = 1



Since, lines are intersecting at right angles.
i.e. AB is the diameter of another circle which passes through (0, 0) and have radius

$$= \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \text{ unit}$$

$\therefore$ Distance of line $mx - y + 2 = 0$ from (0, 0) is

$$\frac{1}{\sqrt{2}} = \frac{|m \cdot 0 - 0 + 2|}{\sqrt{m^2 + 1}}$$

$$\Rightarrow (m^2 + 1) = 2 \cdot 2$$

$$\Rightarrow m^2 = 7$$

$$\therefore m = \pm \sqrt{7} \text{ unit}$$

62. $C_1 = (3, 4)$ and $C_2 = (0, 0)$,

$$r_1 = \sqrt{3^2 + 4^2 - 9} = 4 \text{ and } r_2 = 1$$

$$\therefore |C_1 C_2| = \sqrt{(0-3)^2 + (0-4)^2} = 5$$

$$r_1 + r_2 = 5$$

$$\text{So, } |C_1 C_2| = r_1 + r_2$$

Hence, there are three common tangents.

63. Let equation of required circle be
 $x^2 + y^2 + 2gx + 2fy + c = 0$

$$\therefore 2(g \cdot 3 + f \cdot 0) = c - 1$$

$$\Rightarrow 6g = c - 1 \quad \dots(i)$$

$$\text{Also, } 2\left(g \cdot 0 - f \cdot \frac{3}{2}\right) = c + 2$$

$$\Rightarrow -3f = c + 2 \quad \dots(ii)$$

$$\text{and } 2\left(g \cdot \frac{1}{2} + f \cdot \frac{1}{2}\right) = c - 3$$

$$\Rightarrow g + f = c - 3 \quad \dots(iii)$$

From Eqs. (i) and (ii), we get

$$6g + 3f = -3 \quad \dots(iv)$$

$$\text{Also, } g + f = 6g + 1 - 3$$

[using value of c from Eqs. (i) and (iii)]

$$\Rightarrow 5g - f = 2$$

$$\Rightarrow 15g - 3f = 6 \quad \dots(v)$$

From Eqs. (iv) and (v), we get

$$21g = 3$$

$$\Rightarrow g = \frac{1}{7} \text{ and } c = \frac{6}{7} + 1 = \frac{13}{7}$$

$$\text{and } f = \frac{-1}{3} \left(\frac{13}{7} + 2 \right) = \frac{-9}{7}$$

So, the centre of the required circle is $\left(\frac{-1}{7}, \frac{9}{7} \right)$.

64. Equation of circle is

$$(x-4)(x+2) + (y-7)(y+1) = 0$$

$$\Rightarrow x^2 - 2x - 8 + y^2 + y - 7y - 7 = 0$$

$$\Rightarrow x^2 + y^2 - 2x - 6y - 15 = 0$$

$$\text{Here, } g = -1, c = -15$$

$$\Rightarrow AB = 2\sqrt{g^2 - c} \Rightarrow AB = 2\sqrt{1+15}$$

$$\therefore AB = 8 \text{ units}$$

65. Mid-point of (4, 0) and (0, 4) is (2, 2).

$$\therefore \text{Required distance} = \sqrt{(2-0)^2 + (2-0)^2}$$

$$= \sqrt{8}$$

$$= 2\sqrt{2} \text{ units}$$

66. Required equation is

$$(x-h)^2 + (y-k)^2 = k^2$$

$$\Rightarrow x^2 + y^2 - 2hx - 2ky + h^2 = 0$$

67. Required equation of circle is

$$x^2 + y^2 - 6x - 8y + \lambda(x+y-1) = 0$$

$$\Rightarrow x^2 + y^2 - (6-\lambda)x - (8-\lambda)y - \lambda = 0$$

$$\text{whose centre is } \left(3 - \frac{\lambda}{2}, 4 - \frac{\lambda}{2} \right)$$

and which lies on the line $x + y - 1 = 0$.

$$\Rightarrow 3 - \frac{\lambda}{2} + 4 - \frac{\lambda}{2} - 1 = 0$$

$$\Rightarrow \lambda = 6$$

Hence, required equation is

$$x^2 + y^2 - 6x - 8y + 6x + 6y - 6 = 0$$

$$\Rightarrow x^2 + y^2 - 2y - 6 = 0$$

68. Centre of circle is (1, -2).

$\therefore$ Required equation of normal

Equation of straight line passing through (1, -2) and (2, 1)

$$\text{i.e. } y + 2 = \frac{-2-1}{1-2}(x-1)$$

$$\Rightarrow y + 2 = 3x - 3$$

$$\Rightarrow 3x - y - 5 = 0$$

69. Required equation of chord is

$$T = S_1$$

$$\Rightarrow -2x + 3y - 81 = 4 + 9 - 81$$

$$\Rightarrow 2x - 3y = -81$$

70. Here, $C_1(-7, 3)$, $r_1 = 6$ and $C_2(5, -2)$, $r_2 = 7$

$\therefore$ Required point of contact

$$= \left(\frac{r_1 x_2 + r_2 x_1}{r_1 + r_2}, \frac{r_1 y_2 + r_2 y_1}{r_1 + r_2} \right)$$

$$= \left(\frac{6 \times 5 + 7 \times (-7)}{6+7}, \frac{6 \times (-2) + 7 \times 3}{6+7} \right) = \left(-\frac{19}{13}, \frac{9}{13} \right)$$

14

Parabola

Conic Sections

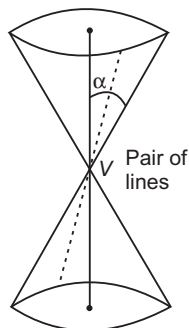
A conic section, as the name implies, is a section cut off from a circular (not necessary a right circular) cone by a plane in various ways. The shape of the section depends upon the position of the cutting plane.

Consider a double napped right circular cone of semi-vertical angle α and let it be cut by a plane inclined at an angle θ to the axis of the cone. We will get different sections (curves) as follows :

Case I If the plane passes through the vertex V

The curve of intersection is a pair of straight lines passing through the vertex which are

- (i) real and distinct, for $\theta < \alpha$.
- (ii) coincident, for $\theta = \alpha$ i.e. the plane touches the cone.
- (iii) imaginary, for $\theta > \alpha$.



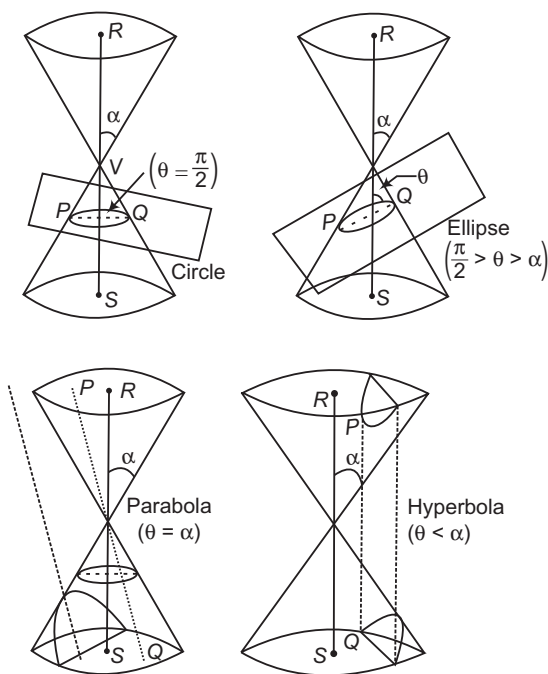
Case II If the plane does not pass through the vertex V

The curve of intersection is called

- (i) a circle, if $\theta = \frac{\pi}{2}$.
- (ii) a parabola, for $\theta = \alpha$ i.e. if the plane is parallel to the generator PQ .
- (iii) an ellipse, for $\theta > \alpha$ $\left(\theta \neq \frac{\pi}{2} \right)$ i.e. if the plane cuts both the generating lines PQ and RS .
- (iv) a hyperbola, for $\theta < \alpha$ i.e. if the plane cuts both the cones.

Chapter Snapshot

- Conic Sections
- Parabola
- Other Standard Forms of Parabola
- Position of a Point with Respect to a Parabola
- Intersection of a Line and a Parabola
- Equation of Tangent
- Angle of Intersection of Two Parabolas
- Equation of Normal to Parabola
- Number of Normals and Conormal Points
- Combined Equation of Pair of Tangents
- Director Circle
- Diameter of a Parabola
- Pole and Polar of a Parabola
- Lengths of Tangent, Subtangent, Normal and Subnormal



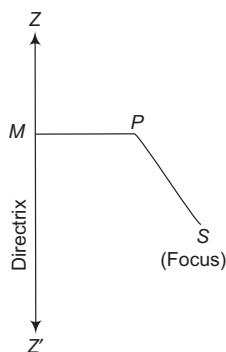
Analytical Definition of Conic Section

A conic section or a conic is the locus of a point which moves in such a way that its distance from a fixed point always bears a constant ratio to its distance from a fixed line, all being in the same plane.

Thus, if S is a fixed point in the plane of the paper and ZZ' is a fixed line in the same plane, then the locus of a point P which moves in the same plane in such a way that

$$\frac{SP}{PM} = \text{Constant} = e \quad [\text{say}]$$

is called a conic section or simply a conic.



The fixed point is called the focus of the conic section and the fixed line is known as its directrix. The constant ratio e is called the eccentricity of the conic section.

If $e = 1$, the conic is called a parabola.

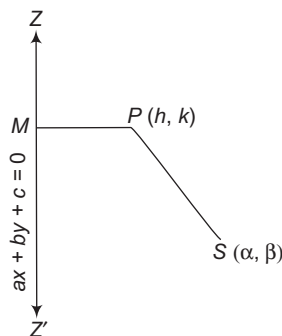
If $e < 1$, the conic is called an ellipse.

If $e > 1$, the conic is called a hyperbola.

General Equation of a Conic

If the focus, directrix and eccentricity of a conic are given, we can find its equation as given below.

Let $S(\alpha, \beta)$ be the focus, $ax + by + c = 0$ be the directrix and e be the eccentricity of a conic. Let $P(h, k)$ be any point on the conic such that PM is the perpendicular from P on the directrix. Then, by definition, we have



$$SP = e \cdot PM \Rightarrow SP^2 = e^2 \cdot PM^2$$

$$\Rightarrow (h - \alpha)^2 + (k - \beta)^2 = e^2 \left[\frac{ah + bk + c}{\sqrt{a^2 + b^2}} \right]^2$$

So, the locus of (h, k) is

$$(x - \alpha)^2 + (y - \beta)^2 = e^2 \left[\frac{ax + by + c}{\sqrt{a^2 + b^2}} \right]^2$$

This is the equation of the required conic.

This equation when simplified can be written in the form

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

which is the general equation of second degree.

➤ **Example 1.** The equation of a conic section whose focus is at $(-1, 0)$, directrix is the line $4x - 3y + 2 = 0$ and eccentricity is $\frac{1}{\sqrt{2}}$, is

(a) $34x^2 + 24xy + 41y^2 + 84x + 12y + 46 = 0$

(b) $41x^2 + 24xy + 34y^2 + 84x + 12y + 46 = 0$

(c) $34x^2 + 24xy + 41y^2 + 12x + 84y + 46 = 0$

(d) $34x^2 + 24xy + 41y^2 + 84x + 12y + 81 = 0$

Sol. (a) Let $P(h, k)$ be a point on the conic. Then, by definition

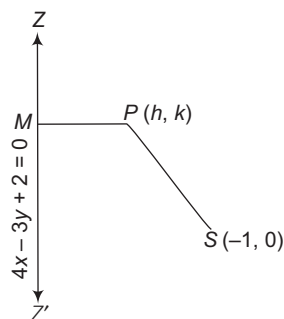
$$\begin{aligned} SP &= e \cdot PM \\ \Rightarrow \sqrt{(h+1)^2 + (k-0)^2} &= \frac{1}{\sqrt{2}} \left| \frac{4h-3k+2}{\sqrt{4^2 + (-3)^2}} \right| \\ \Rightarrow (h+1)^2 + k^2 &= \frac{1}{50} (4h-3k+2)^2 \end{aligned}$$

So, the locus of (h, k) is

$$(x+1)^2 + y^2 = \frac{1}{50}(4x-3y+2)^2$$

$$\Rightarrow 50\{(x+1)^2 + y^2\} = (4x-3y+2)^2$$

$$\Rightarrow 34x^2 + 41y^2 + 24xy + 84x + 12y + 46 = 0$$



Hence, the equation of the conic is

$$34x^2 + 24xy + 41y^2 + 84x + 12y + 46 = 0$$

Recognition of Conic Section

The general equation of second degree viz.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
 represents

(i) a pair of straight lines, if

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

(ii) a pair of parallel (or coincident) straight lines, if

$$\Delta = 0 \text{ and } h^2 = ab$$

(iii) a pair of perpendicular straight lines, if

$$\Delta = 0 \text{ and } a + b = 0$$

(iv) a point, if $\Delta = 0$ and $h^2 < ab$

The general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
, will represent

(i) a circle, if $\Delta \neq 0$, $a = b$ and $h = 0$.

(ii) a parabola, if $\Delta \neq 0$ and $h^2 = ab$.

(iii) an ellipse, if $\Delta \neq 0$ and $h^2 < ab$.

(iv) a hyperbola, if $\Delta \neq 0$ and $h^2 > ab$.

(v) a rectangular hyperbola, if $\Delta \neq 0$ and $h^2 > ab$ and $a + b = 0$.

➤ **Example 2.** The name of the conic represented by the equation $x^2 + y^2 - 2xy + 20x + 10 = 0$ is

- (a) a hyperbola (b) an ellipse
(c) a parabola (d) a circle

Sol. (c) On comparing the given equation with the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
, we have

$$a = 1, b = 1, h = -1, c = 10, g = 10 \text{ and } f = 0$$

$$\therefore abc + 2fgh - af^2 - bg^2 - ch^2 = 10 - 100 - 10$$

$$= -100 \neq 0 \text{ and } h^2 - ab = (-1)^2 - 1 \times 1 = 0$$

$$\Rightarrow h^2 = ab$$

Thus, we have

$$\Delta \neq 0 \text{ and } h^2 = ab$$

So, the given equation represents a parabola.

➤ **Example 3.** The name of the conic represented by the equation $2xy + 4x - 6y + 17 = 0$ is

- (a) an ellipse
(b) a circle
(c) a parabola
(d) a rectangular hyperbola

Sol. (d) On comparing the given equation with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
, we get

$$a = 0, b = 0, h = 1, c = 17, g = 2, f = -3$$

$$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= -12 - 17$$

$$= -29 \neq 0$$

$$\text{and } h^2 - ab = 1 - 0 > 0$$

Thus, we have $\Delta \neq 0$

$$\text{and } h^2 > ab, a + b = 0$$

So, the given equation represents a rectangular hyperbola.

Some Useful Terms Used in Conic Sections

(i) **Axis** The straight line passing through the focus and perpendicular to the directrix is called the axis of the conic section.

(ii) **Vertex** The point(s) of intersection of the conic section and the axis is (are) called the vertex (vertices) of the conic section.

(iii) **Centre** The point which bisects every chord of the conic passing through it, is called the centre of the conic section.

(iv) **Latusrectum** The latusrectum of a conic is the chord passing through the focus and perpendicular to the axis.

(v) **Focal chord** Any chord passing through the focus is called a focal chord of the conic section.

(vi) **Double ordinate** A chord perpendicular to the axis of a conic is known as its double ordinate.

(vii) If $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is the equation of a conic, then the coordinates of its centre are

$$\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2} \right)$$

In order to obtain the coordinates of the centre, we may solve the equations

$$\frac{\partial S}{\partial x} = 0$$

$$\text{and } \frac{\partial S}{\partial y} = 0$$

$$\Rightarrow 2ax + 2hy + 2g = 0$$

$$\text{and } 2hx + 2by + 2f = 0$$

➤ **Example 4.** The centre of the conic

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0 \text{ is}$$

- (a) (3, 2) (b) (2, 4)
(c) (2, 3) (d) (4, 2)

Sol. (c) Let $S \equiv 14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$ be the given conic.

$$\text{Then, } \frac{\partial S}{\partial x} = 0 \text{ and } \frac{\partial S}{\partial y} = 0$$

$$\Rightarrow 28x - 4y - 44 = 0 \text{ and } 22y - 4x - 58 = 0$$

$$\Rightarrow 7x - y - 11 = 0 \text{ and } 2x - 11y + 29 = 0$$

On solving these two equations, we get

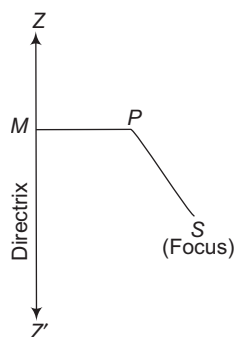
$$x = 2 \text{ and } y = 3$$

Hence, the coordinates of the centre of given conic is (2, 3).

Parabola

A parabola is the locus of a point which moves in a plane such that its distance from a fixed point in the plane is always equal to its distance from a fixed straight line in the same plane.

It is evident from this definition of a conic, that a parabola is a conic section with eccentricity $e = 1$.



The fixed point is called the focus and the fixed straight line is called the directrix of the parabola. The line through the focus and perpendicular to the directrix is the axis of the parabola. The point on the axis mid-way between the focus and directrix is called the vertex of the parabola.

Let S be the focus, ZZ' be the directrix and P be any point on the parabola. Then, by definition

$$SP = PM$$

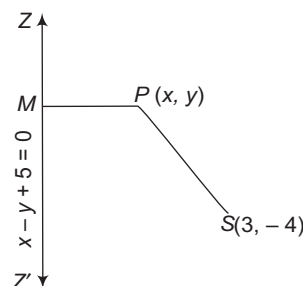
where, PM is the length of the perpendicular from P on the directrix ZZ' .

➤ **Example 5.** The equation of parabola whose focus is (3, -4) and directrix is $x - y + 5 = 0$, is

- (a) $x^2 + y^2 + 2xy - 22x + 26y + 25 = 0$
(b) $x^2 + y^2 + 2xy + 22x - 26y + 25 = 0$
(c) $x^2 + y^2 + 2xy + 22x + 26y - 25 = 0$
(d) None of the above

Sol. (a) Let $P(x, y)$ be any point on the parabola.

By definition of parabola,



$$SP = PM$$

$$\therefore \sqrt{(x-3)^2 + (y+4)^2} = \left| \frac{x-y+5}{\sqrt{1^2+1^2}} \right|$$

On squaring both sides, we get

$$(x-3)^2 + (y+4)^2 = \frac{(x-y+5)^2}{(\sqrt{2})^2}$$

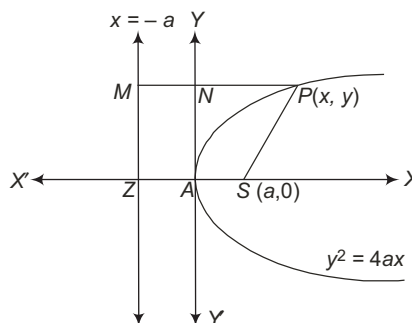
$$\Rightarrow 2(x^2 + y^2 - 6x + 8y + 25) = x^2 + y^2 - 2xy + 10x - 10y + 25$$

$$\Rightarrow x^2 + y^2 + 2xy - 22x + 26y + 25 = 0$$

which is the required equation of parabola.

Standard Equation of Parabola

Consider the focus of the parabola as $S(a, 0)$ and directrix is $x + a = 0$ and axis as X -axis.



Now, according to the definition of the parabola, for any point on the parabola, we must have

$$SP = PM$$

$$\Rightarrow \sqrt{(x-a)^2 + (y-0)^2} = PN + NM = x + a$$

$$\Rightarrow (x-a)^2 + y^2 = (x+a)^2$$

$$\Rightarrow y^2 = (x+a)^2 - (x-a)^2$$

$$\Rightarrow y^2 = 4ax$$

Vertex = (0, 0)

Tangent at vertex, $x = 0$

Equation of latusrectum, $x = a$

Extremities of latusrectum are $(a, 2a)$, $(a, -2a)$

Length of latusrectum = $4a$

Focal distance, $SP = PM = x + a$

Parametric forms are $x = at^2$ and $y = 2at$,

where t is a parameter.

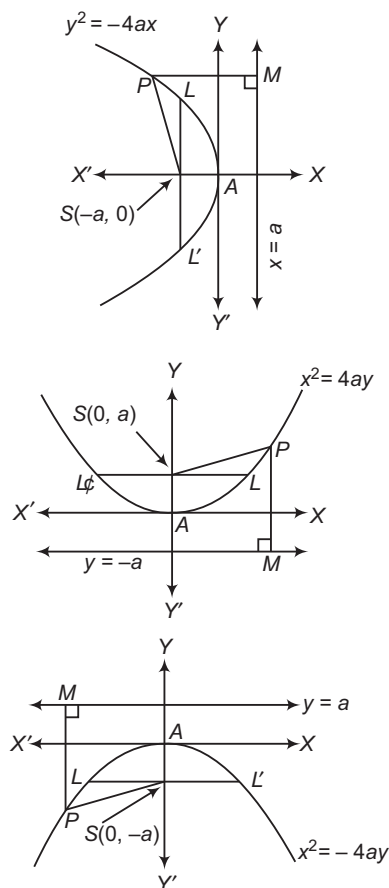
Some Terms Related to Parabola

- (i) **Axis** The axis of a parabola is a straight line through the focus and perpendicular to the directrix.
- (ii) **Vertex** The point of intersection of the parabola and its axis is called its vertex.
- (iii) **Double ordinate** A straight line that is drawn perpendicular to the axis and terminates at both ends of the parabola, is a double ordinate of the parabola.
- (iv) **Latusrectum** The double ordinate passes through the focus is called latusrectum of parabola.
- (v) **Chord** A chord of a parabola is the line segment joining any two points on the parabola.
- (vi) **Focal chord** A chord of a parabola passing through the focus is called focal chord.
- (vii) **Focal distance** The focal distance of any point $P(x, y)$ on the parabola is its distance from the focus.

	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Coordinates of vertex	(0, 0)	(0, 0)	(0, 0)	(0, 0)
Coordinates of focus	(a, 0)	(-a, 0)	(0, a)	(0, -a)
Equation of the directrix	$x = -a$	$x = a$	$y = -a$	$y = a$
Equation of the axis	$y = 0$	$y = 0$	$x = 0$	$x = 0$
Length of the latusrectum	4a	4a	4a	4a
Focal distance of a point $P(x, y)$	$x + a$	$a - x$	$y + a$	$a - y$

- If the vertex of the parabola is at the point $A(h, k)$ and its latusrectum is of length $4a$, then its equation is
 - $(y - k)^2 = 4a(x - h)$, if its axis is parallel to OX i.e. parabola opens rightward.
 - $(y - k)^2 = -4a(x - h)$, if its axis is parallel to OX' i.e. parabola opens leftward.
 - $(x - h)^2 = 4a(y - k)$, if its axis is parallel to OY i.e. parabola opens upward.
 - $(x - h)^2 = -4a(y - k)$, if its axis is parallel to OY' i.e. parabola opens downward.
- The general equation of a parabola whose axis is parallel to X -axis is $x = ay^2 + by + c$ and the general equation of a parabola whose axis is parallel to Y -axis is $y = ax^2 + bx + c$.

Other Standard Forms of Parabola



➤ **Example 6.** If the equation of parabola is $x^2 = -8y$, then

- (a) focus is $(0, -2)$
- (b) directrix is $y = 2$
- (c) length of latusrectum is 8
- (d) All of the above

Sol. (d) The given equation is of the form $x^2 = -4ay$, where a is positive.

∴ The focus is on Y -axis in the negative direction and parabola opens downward.

On comparing the given equation with standard form, we get $a = 2$.

Therefore, the coordinates of the focus are $(0, -2)$, then the equation of the directrix is $y = 2$ and the latusrectum is $4a$ i.e. 8.

Hence, all of these are correct.

➤ **Example 7.** The focus of the parabola $y^2 - x - 2y + 2 = 0$, is

- (a) $\left(\frac{5}{4}, 1\right)$
- (b) $\left(\frac{1}{4}, 0\right)$
- (c) $(1, 1)$
- (d) None of these

Sol. (a) The equation of parabola can be rewritten as $y^2 - 2y + 1 = x - 1 \Rightarrow (y - 1)^2 = x - 1$

On putting $x - 1 = X$ and $y - 1 = Y$, then the equation becomes

$$Y^2 = X \Rightarrow Y^2 = 4 \cdot \frac{1}{4} \cdot X$$

$$\therefore \text{Focus} = \left(\frac{1}{4}, 0\right)_{(X, Y)} = \left(\frac{5}{4}, 1\right)$$

➤ **Example 8.** The vertex of the parabola $x^2 + 2y = 8x - 7$ is

(a) $\left(4, \frac{7}{2}\right)$

(b) $\left(4, \frac{9}{2}\right)$

(c) $\left(\frac{9}{2}, 4\right)$

(d) (1, 0)

Sol. (b) We have, $x^2 - 8x = -2y - 7$

$$\Rightarrow (x - 4)^2 = -2\left(y - \frac{9}{2}\right)$$

Clearly, the vertex is at $\left(4, \frac{9}{2}\right)$.

➤ **Example 9.** The coordinates of an end point of the latusrectum of the parabola $(y - 1)^2 = 4(x + 1)$ are

(a) (0, -3)

(b) (0, -1)

(c) (0, 1)

(d) (1, 3)

Sol. (b) On shifting the origin at $(-1, 1)$, we have

$$\begin{cases} x = X - 1 \\ y = Y + 1 \end{cases} \quad \dots(i)$$

From Eq. (i), the given parabola becomes

$$Y^2 = 4X$$

The coordinates of the end points of latusrectum are $(X = 1, Y = 2)$ and $(X = 1, Y = -2)$.

From Eq. (i), the coordinates of the end points of the latusrectum are (0, 3) and (0, -1).

➤ **Example 10.** If the two ends of the latusrectum are given, then the maximum number of parabolas that can be drawn, is

(a) 1

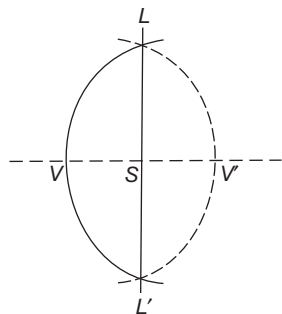
(b) 2

(c) 0

(d) infinite

Sol. (b) Given, L, L' are the ends of the latusrectum, S is mid-point of LL' and VSV' is the perpendicular bisector of LL', where $VS = \left(\frac{1}{4}\right)LL' = V'S$.

Clearly, two parabolas are possible.



➤ **Example 11.** If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B, then PA · PB is equal to [where, $P = (\sqrt{3}, 0)$]

(a) $\frac{4(\sqrt{3} + 2)}{3}$

(b) $\frac{4(2 - \sqrt{2})}{3}$

(c) $\frac{4\sqrt{3}}{2}$

(d) $\frac{2(\sqrt{3} + 2)}{3}$

Sol. (a) Given $y - \sqrt{3}x + 3 = 0$ can be rewritten as

$$\frac{y - 0}{\sqrt{3}/2} = \frac{x - \sqrt{3}}{1/2} = r \quad \dots(i)$$

On solving Eq.(i) with the parabola $y^2 = x + 2$, we get

$$\frac{3r^2}{4} = \frac{r}{2} + \sqrt{3} + 2$$

$$\Rightarrow 3r^2 - 2r - (4\sqrt{3} + 8) = 0$$

$$\therefore PA \cdot PB = |r_1 r_2| = \frac{4(\sqrt{3} + 2)}{3}$$

Parametric Equations of a Parabola

The simplest and the best form of representing the coordinates of a point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$ because these coordinates satisfy the equation $y^2 = 4ax$ for all values of t .

The equations $x = at^2$ and $y = 2at$ taken together are called the parametric equations of the parabola $y^2 = 4ax$, t being the parameter.

Parabola	$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$
Parametric coordinates	$(at^2, 2at)$	$(-at^2, 2at)$	$(2at, at^2)$	$(2at, -at^2)$
Parametric equations	$x = at^2, y = 2at$	$x = -at^2, y = 2at$	$x = 2at, y = at^2$	$x = 2at, y = -at^2$

- The parametric equations of $(y - k)^2 = 4a(x - h)$ are $x = h + at^2$ and $y = k + 2at$.

➤ **Example 12.** The point on $y^2 = 4ax$ nearest to the focus has its abscissa

(a) $x = -a$

(b) $x = a$

(c) $x = \frac{3}{2}$

(d) $x = 0$

Sol. (d) The focus is S (a, 0). If $P(at^2, 2at)$ is any point on the parabola, then

$$SP^2 = a^2(t^2 - 1)^2 + 4a^2t^2 = a^2(t^2 + 1)^2$$

This is least for $t = 0$. Hence, the abscissa of P is $x = 0$ for SP to be least.

14

Work Book Exercise 14.1

- The name of the curve described parametrically by the equations $x = t^2 + t + 1$, $y = t^2 - t + 1$, is
 - a circle
 - an ellipse
 - a hyperbola
 - a parabola
- The axis of parabola $9y^2 - 16x - 12y - 57 = 0$ is
 - $3y = 2$
 - $x + 3y = 3$
 - $2x = 3$
 - $y = 3$
- The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is
 - $x = -\frac{1}{2}$
 - $x = \frac{1}{2}$
 - $x = -\frac{3}{2}$
 - $x = \frac{3}{2}$
- The equation of the directrix of the parabola $x = -2at$, $y = -at^2$, $t \in \mathbb{R}$, is
 - $x - a = 0$
 - $y - a = 0$
 - $x + a = 0$
 - $y + a = 0$
- If the parabola $y^2 = 4ax$ passes through $(3, 2)$, then the length of its latusrectum is
 - $\frac{2}{3}$
 - $\frac{4}{3}$
 - $\frac{1}{3}$
 - 4
- The focal distance of a point on the parabola $y^2 = 4x$ and above its axis, is 10 units. Its coordinates are
 - $(9, 6)$
 - $(25, 10)$
 - $(25, -10)$
 - None of these
- The end points of the latusrectum of a parabola are $(7, 5)$ and $(7, 3)$, then possible coordinates of its vertex are
 - $\left(\frac{13}{2}, 4\right)$
 - $\left(4, \frac{13}{2}\right)$
 - $\left(4, \frac{15}{2}\right)$
 - $\left(\frac{15}{2}, 4\right)$

Position of a Point with Respect to a Parabola

The point (x_1, y_1) lies outside, on or inside the parabola $y^2 = 4ax$ according as $y_1^2 - 4ax_1 >, =$ or < 0 .

Let $P(x_1, y_1)$ be a point in the plane. From P , draw $PM \perp AX$, meeting the parabola $y^2 = 4ax$ at Q . Let the coordinates of Q be (x_2, y_2) . Since, Q lies on the parabola $y^2 = 4ax$.

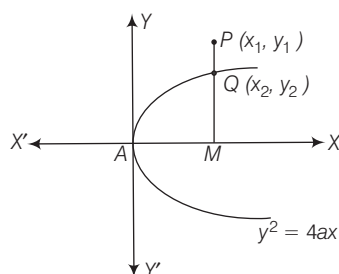
$$\therefore y_2^2 = 4ax_2$$

Now, point (x_2, y_2) will be outside, on or inside the parabola $y^2 = 4ax$ according as

$$PM >, = \text{ or } < QM$$

$$\Rightarrow PM^2 >, = \text{ or } < QM^2$$

$$\Rightarrow y_1^2 >, = \text{ or } < y_2^2$$



$$\Rightarrow y_1^2 >, = \text{ or } < 4ax_2$$

Example 13. The ends of a line segment are $P(1, 3)$ and $Q(1, 1)$. R is a point on the line segment PQ such that $PR : QR = 1 : \lambda$. If R is an interior point of a parabola $y^2 = 4ax$, then

- $\lambda \in (0, 1)$
- $\lambda \in \left(-\frac{3}{5}, 1\right)$
- $\lambda \in \left(\frac{1}{2}, \frac{3}{5}\right)$
- None of these

Sol. (a) The coordinates of R are $\left(1, \frac{1+3\lambda}{1+\lambda}\right)$.

It is an interior point of $y^2 - 4x = 0$, if $\left(\frac{1+3\lambda}{1+\lambda}\right)^2 - 4 < 0$.

$$\text{Therefore, } -\frac{3}{5} < \lambda < 1$$

$$\begin{aligned} \text{But } & \lambda > 0 \\ \Rightarrow & 0 < \lambda < 1 \\ \therefore & \lambda \in (0, 1) \end{aligned}$$

Example 14. The number of points with non-negative integral coordinates that lie in the interior of the region common to the circle $x^2 + y^2 = 16$ and the parabola $y^2 = 4x$, is

- 8
- 10
- 16
- None of these

Sol. (a) (λ, μ) is interior to both the curve, if $\lambda^2 + \mu^2 - 16 < 0$ and $\mu^2 - 4\lambda < 0$

$$\text{Now, } \mu^2 - 4\lambda < 0 \Rightarrow \lambda > \left(\frac{\mu}{2}\right)^2$$

If $\mu = 0$, $\lambda = 1, 2, 3$; if $\mu = 1$, $\lambda = 1, 2, 3$; if $\mu = 2$, $\lambda = 2, 3$ and if $\mu = 3$, $\lambda = 3, 4$.

$$\text{Also, } \lambda^2 + \mu^2 - 16 < 0$$

$$\Rightarrow \lambda^2 < 16 - \mu^2$$

If $\mu = 0$, $\lambda = 1, 2, 3$; if $\mu = 1$, $\lambda = 1, 2, 3$; if $\mu = 2$, $\lambda = 2, 3$ and if $\mu = 3$, $\lambda = 1, 2$.

Hence, $(1, 0)$, $(2, 0)$, $(3, 0)$, $(1, 1)$, $(2, 1)$, $(3, 1)$, $(2, 2)$, $(3, 2)$ are the possible points.

Equation of a Chord

- (i) **Parametric equation of a chord** Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be any two points on the parabola $y^2 = 4ax$.

Then, the equation of the chord PQ is

$$y - 2at_1 = \frac{2at_2 - 2at_1}{at_2^2 - at_1^2} (x - at_1^2)$$

$$\Rightarrow y - 2at_1 = \frac{2}{t_2 + t_1} (x - at_1^2)$$

$$\Rightarrow y(t_1 + t_2) = 2x + 2at_1t_2$$

- (ii) The equation of the chord joining points t_1 and t_2 on the parabola $y^2 = 4ax$ is

$$y(t_1 + t_2) = 2x + 2at_1t_2$$

If the chord PQ is a focal chord of the parabola, then $(a, 0)$ must satisfy the above equation.

$$0 = 2a + 2at_1t_2 \Rightarrow t_1t_2 = -1$$

It follows from this that t is the parameter for one end of a focal chord of the parabola $y^2 = 4ax$, then the parameter for the other end is $-1/t$ and the coordinates of the end points of a focal chord PQ of the parabola $y^2 = 4ax$ can be

taken as $P(at^2, 2at)$ and $Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

- (iii) **Length of a focal chord** Let $P(at^2, 2at)$ be one end of a focal chord PQ of the parabola $y^2 = 4ax$. Then, the coordinates of the other end

Q are $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

The length of the focal chord of the parabola $y^2 = 4ax$ is $a\left(t + \frac{1}{t}\right)^2$, where t is the parameter for one end of the chord.

We know that,

$$\left|t + \frac{1}{t}\right| \geq 2, \forall t \neq 0$$

$$\therefore a\left(t + \frac{1}{t}\right)^2 \geq 4a \Rightarrow PQ \geq 4a$$

Thus, the length of the smallest focal chord of the parabola is $4a$, which is the length of its latusrectum. Hence, the latusrectum of a parabola is the smallest focal chord.

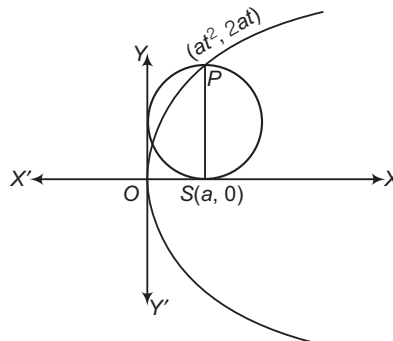
- (iv) If l_1 and l_2 are length of focal segments, then

$$4a = \frac{4l_1l_2}{l_1 + l_2}$$

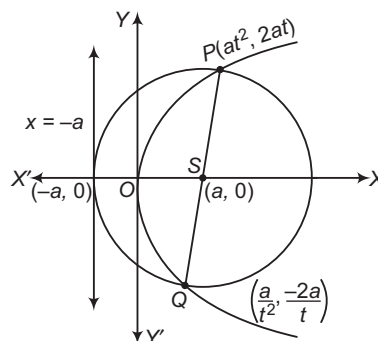
i.e. Length of latusrectum = 2 (Harmonic mean of focal segment)

- (v) The length of the focal chord which makes an angle θ with positive direction of X -axis is $4a \operatorname{cosec}^2 \theta$.

- (vi) Circle described on the focal length as diameter touches the tangent at vertex.



- (vii) Circle described on the focal chord as diameter touches directrix.



- **Example 15.** If $(2, -8)$ is at an end of a focal chord of the parabola $y^2 = 32x$, then the coordinate of the other end of chord is

- (a) $(8, -2)$
(b) $(16, 32)$
(c) $(32, 32)$
(d) None of the above

Sol. (c) We have, $y^2 = 32x$

Here, $a = 8$

Then, any point on this parabola is $(8t^2, 16t)$.

On comparing this with point $(2, -8)$, we get $t = -\frac{1}{2}$

Now, the other end of focal chord is $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$.

Hence, the coordinates of the other end of the focal chord is $(32, 32)$.

- **Example 16.** Circles are drawn with diameter being any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line.

The equation of line is

- (a) $16x + 17 = 0$
(b) $16x + 33 = 0$
(c) $4y + 17 = 0$
(d) $4y + 33 = 0$

Sol. (b) We have, $y^2 - 4x - y - 4 = 0$

$$\Rightarrow y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$$

$$\Rightarrow \left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$$

Circle drawn with diameter being any focal chord of the parabola always touches the directrix of the parabola.

Thus, circle will touch the line at $x + \frac{17}{16} = -1$

$$\text{i.e. } 16x + 33 = 0$$

➤ **Example 17.** The circles on focal radii of a parabola as diameter touch

(a) the tangent at the vertex

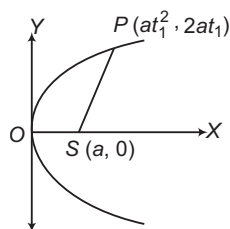
(b) the axis

(c) the directrix

(d) None of the above

Sol. (a) The circle on SP as diameter is

$$(x - at_1^2)(x - a) + (y - 2at_1)y = 0$$



It meets Y-axis when $x = 0$.

$$\Rightarrow y^2 - 2at_1y + a^2t_1^2 = 0$$

$$\Rightarrow (y - at_1)^2 = 0$$

Clearly, Y-axis meets the circle in two coincident point. Hence, the circle touches the tangent at the vertex.

➤ **Example 18.** The length of the chord of the parabola $x^2 = 4ay$ passing through the vertex and having slope $\tan \alpha$, is

(a) $4a \operatorname{cosec} \alpha \cot \alpha$

(b) $4a \tan \alpha \sec \alpha$

(c) $4a \cos \alpha \cot \alpha$

(d) $4a \sin \alpha \tan \alpha$

Sol. (b) Let A (0, 0) be the vertex and AP be a chord of $x^2 = 4ay$ such that slope of AP is $\tan \alpha$.

Again, let the coordinates of P be $(2at, at^2)$. Then,

$$\text{Slope of AP} = \frac{at^2}{2at} = \frac{t}{2}$$

$$\Rightarrow \tan \alpha = \frac{t}{2}$$

$$\Rightarrow t = 2 \tan \alpha$$

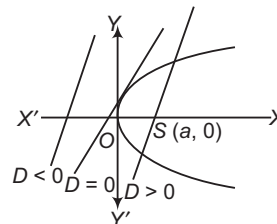
$$\text{Now, } AP = \sqrt{\{(2at - 0)^2 + (at^2 - 0)^2\}}$$

$$= at\sqrt{4 + t^2}$$

$$= 2a \tan \alpha \sqrt{4 + 4 \tan^2 \alpha}$$

$$= 4a \tan \alpha \sec \alpha$$

Intersection of a Line and a Parabola



Let the parabola be $y^2 = 4ax$... (i)

and the given line be $y = mx + c$... (ii)

Eliminating y from Eqs. (i) and (ii), we get

$$(mx + c)^2 = 4ax$$

$$\text{or } m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad \dots \text{(iii)}$$

This equation, being quadratic in x , gives two values of x and shows that every straight line will cut the parabola in two points may be real, coincident or imaginary, according as discriminant of Eq. (iii) is greater, equal or less than 0.

$$\text{i.e. } 4(mc - 2a)^2 - 4m^2c^2 >, = \text{ or } < 0$$

$$\text{or } 4a^2 - 4amc >, = \text{ or } < 0$$

$$\text{or } a >, = \text{ or } < mc \quad \dots \text{(iv)}$$

Condition of Tangency

The line $y = mx + c$ touches the parabola $y^2 = 4ax$,

if $c = \frac{a}{m}$ and the coordinates of the point of contact are $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.

The conditions of tangency of a line and a parabola in different forms are listed below for ready reference:

Parabola	Line	Point of contact	Condition of tangency
$y^2 = 4ax$	$y = mx + c$	$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$	$c = \frac{a}{m}$
$y^2 = -4ax$	$y = mx + c$	$\left(-\frac{a}{m^2}, -\frac{2a}{m}\right)$	$c = -\frac{a}{m}$
$x^2 = 4ay$	$x = my + c$	$\left(\frac{2a}{m}, \frac{a}{m^2}\right)$	$c = \frac{a}{m}$
$x^2 = -4ay$	$x = my + c$	$\left(-\frac{2a}{m}, -\frac{a}{m^2}\right)$	$c = -\frac{a}{m}$
$(y - k)^2 = 4a(x - h)$	$y = mx + c$	$\left(h + \frac{a}{m^2}, k + \frac{2a}{m}\right)$	$mh + c = k + \frac{a}{m}$
$(y - k)^2 = -4a(x - h)$	$y = mx + c$	$\left(h - \frac{a}{m^2}, k - \frac{2a}{m}\right)$	$mh + c = k - \frac{a}{m}$

Parabola	Line	Point of contact	Condition of tangency
$(x-h)^2 = 4a(y-k)$	$x = my + c$	$\left(h + \frac{2a}{m}, k + \frac{a}{m^2}\right)$	$mk + c = h + \frac{a}{m}$
$(x-h)^2 = -4a(y-k)$	$x = my + c$	$\left(h - \frac{2a}{m}, k - \frac{a}{m^2}\right)$	$mk + c = h - \frac{a}{m}$

➤ **Example 19.** If the straight line $y = mx + c$ meets the parabola $y^2 = 8x$ in real and distinct points, then

(a) $c < \frac{2}{m}$ (b) $c > \frac{2}{m}$ (c) $|c| < \frac{2}{m}$ (d) $|c| > \frac{2}{m}$

Sol. (a) Given, equation of parabola is $y^2 = 8x$ and equation of line is $y = mx + c$.

From $y^2 = 8x$
and $y = mx + c$
 $(mx + c)^2 = 8x$

$\Rightarrow m^2x^2 + 2mxc + c^2 - 8x = 0$

$\Rightarrow m^2x^2 + (2mc - 8)x + c^2 = 0$

For real and distinct points,

$(2mc - 8)^2 - 4m^2c^2 > 0$

$\Rightarrow 4m^2c^2 - 32mc + 64 - 4m^2c^2 > 0$

$\Rightarrow 32mc < 64 \Rightarrow c < \frac{2}{m}$

➤ **Example 20.** If the straight line $x \cos \alpha + y \sin \alpha = p$ touches the parabola $y^2 = 4ax$, then the point of contact is

(a) $a \cos \alpha, a \sin \alpha$ (b) $-p \sec \alpha, 2p \operatorname{cosec} \alpha$
(c) $a \cot^2 \alpha, -2a \cot \alpha$ (d) None of these

Sol. (b) The given line is $x \cos \alpha + y \sin \alpha = p$
or $y = -x \cot \alpha + p \operatorname{cosec} \alpha$
Since, the given line touches the parabola.

$\therefore c = \frac{a}{m^2}$ or $a = cm$

$\Rightarrow (p \operatorname{cosec} \alpha)(-\cot \alpha) = a$
and the point of contact

$= \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$= \left(\frac{a}{\cot^2 \alpha}, \frac{-2a}{\cot \alpha}\right)$

$= -p \sec \alpha, 2p \operatorname{cosec} \alpha$

Equation of Tangent

In this section, we will obtain the equation of tangents to various forms of the parabola in terms of slope of the tangent or in terms of the coordinates of the point of contact of the tangent line and the parabola or in terms of value of the parameter of the point of contact.

(i) **Point form** The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by
 $yy_1 = 2a(x + x_1)$

- The equation of the tangent at (x_1, y_1) to any second degree curve can also be obtained by replacing x^2 by xx_1 , y^2 by yy_1 , x by $\frac{x+x_1}{2}$, y by $\frac{y+y_1}{2}$ and xy by $\frac{xy_1+x_1y}{2}$ and without changing the constant (if any) in the equation of the curve.
- The equation of tangents to all standard forms of parabola at point (x_1, y_1) are given below for ready reference.

Equation of parabola	Equation of tangent
$y^2 = 4ax$	$yy_1 = 2a(x + x_1)$
$y^2 = -4ax$	$yy_1 = -2a(x + x_1)$
$x^2 = 4ay$	$xx_1 = 2a(y + y_1)$
$x^2 = -4ay$	$xx_1 = -2a(y + y_1)$

(ii) **Parametric form** The equation of the tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is given by

$ty = x + at^2$

- The parametric equations of tangents to all standard forms of parabola are as given below:

Equation of parabola	Point of contact	Equation of the tangent
$y^2 = 4ax$	$(at^2, 2at)$	$ty = x + at^2$
$y^2 = -4ax$	$(-at^2, 2at)$	$ty = -x + at^2$
$x^2 = 4ay$	$(2at, at^2)$	$tx = y + at^2$
$x^2 = -4ay$	$(2at, -at^2)$	$tx = -y + at^2$

(iii) **Slope form** The equation of the tangent of slope m to the parabola $y^2 = 4ax$ is

$y = mx + \frac{a}{m} \text{ at } \left(\frac{a}{m^2}, \frac{2a}{m}\right)$

- The equation of a tangent of slope m to the parabola $(y-k)^2 = 4a(x-h)$ is given by $y-k = m(x-h) + \frac{a}{m}$. The coordinates of the point of contact are $\left(h + \frac{a}{m^2}, k + \frac{2a}{m}\right)$.
- It should be noted that to obtain tangent and the coordinates of the point of contact of tangent to $x^2 = 4ay$, we can interchange x and y in the equation of the tangent to $y^2 = 4ax$ and also in the coordinates of the point of contact. Similarly, relationship exists between $x^2 = -4ay$ and $y^2 = -4ax$.

(iv) **Point of intersection of tangents** Let tangent at $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ intersect at R .

Then, their point of intersection is $R[at_1t_2, a(t_1 + t_2)]$ i.e. (GM of abscissa, AM of ordinate).

➤ **Example 21.** If the line $y = 3x + c$ touches the parabola $y^2 = 12x$ at point P , then the equation of the tangent at point Q , where PQ is a focal chord, is

- (a) $x + 3y + 27 = 0$
 (b) $3x - y - 27 = 0$
 (c) $x + 3y - 27 = 0$
 (d) None of the above

Sol. (a) Line $y = 3x + c$ touches $y^2 = 12x$, then we must have

$$c = \frac{a}{m} \quad \text{i.e.} \quad c = \frac{3}{3} = 1$$

The point of contact P is $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{1}{3}, 2\right)$

On comparing this point to $(at^2, 2at)$, we get

$$2at = 2 \quad \text{or} \quad t = \frac{1}{3}$$

Hence, point P has parameter $\frac{1}{3}$, then point Q has parameter -3 .

Now, tangent at point Q is $(-3)y = x + 3(-3)^2$
 or $x + 3y + 27 = 0$

➤ **Example 22.** The equation of tangent to the parabola $y^2 = 8x$ having slope 2, is

- (a) $y = 2x + 1$ (b) $y = 2x + 4$
 (c) $y = 2x + 3$ (d) None of these

Sol. (a) Equation of the tangent to $y^2 = 4ax$ having slope m is $y = mx + \frac{a}{m}$.

For the given parabola $y^2 = 8x$, we have $a = 2$ and $m = 2$

$$\therefore \text{Equation of tangent is } y = 2x + \frac{2}{2}$$

$$\Rightarrow y = 2x + 1$$

➤ **Example 23.** If two parabolas $y^2 = 4a(x - \lambda_1)$ and $x^2 = 4a(y - \lambda_2)$ always touch each other, where λ_1 and λ_2 being variable parameters. Then, their points of contact lie on a

- (a) straight line
 (b) circle
 (c) parabola
 (d) hyperbola

Sol. (d) Let $P(x_1, y_1)$ be the point of contact of the two parabolas. Tangents at P to the two parabolas are

$$yy_1 = 2a(x + x_1) - 4a\lambda_1 \quad \dots (i)$$

$$\Rightarrow 2ax - yy_1 = 2a(2\lambda_1 - x_1)$$

$$\text{and} \quad xx_1 = 2a(y + y_1) - 4a\lambda_2 \quad \dots (ii)$$

$$\Rightarrow xx_1 - 2ay = 2a(y_1 - 2\lambda_2) \quad \dots (iii)$$

Clearly, Eqs. (i) and (ii) represent the same line.

$$\therefore \frac{2a}{x_1} = \frac{y_1}{2a}$$

$$\Rightarrow x_1 y_1 = 4a^2$$

Hence, the locus of (x_1, y_1) is $xy = 4a^2$, which is a hyperbola.

➤ **Example 24.** The locus of the point of intersection of tangents to the parabola $y^2 = 4(x + 1)$ and $y^2 = 8(x + 2)$ which are perpendicular to each other, is

- (a) $x + 7 = 0$
 (b) $x - y = 4$
 (c) $x + 3 = 0$
 (d) $y - x = 12$

Sol. (c) The equation of any tangent to $y^2 = 4(x + 1)$ is

$$y = m(x + 1) + \frac{1}{m} \quad \dots (i)$$

and tangent to $y^2 = 8(x + 2)$ is

$$y = m'(x + 2) + \frac{2}{m'} \quad \dots (ii)$$

It is given that Eqs. (i) and (ii) are perpendicular.

On putting $m' = -\frac{1}{m}$ in Eq. (ii), we get

$$y = \left(-\frac{1}{m}\right)(x + 2) - 2m \quad \dots (iii)$$

On subtracting Eq. (iii) from Eq. (i), we get

$$0 = \left(m + \frac{1}{m}\right)x + 3\left(m + \frac{1}{m}\right)$$

$$\Rightarrow x + 3 = 0 \quad \left[\because m + \frac{1}{m} \neq 0 \right]$$

➤ **Example 25.** From an external point P , tangents are drawn to the parabola $y^2 = 4ax$, then the equation to the locus of P when these tangents makes angles θ_1 and θ_2 with the axis, such that $\tan \theta_1 + \tan \theta_2$ is constant ($= b$), is

- (a) $y = \frac{x}{b}$ (b) $y = bx$
 (c) $y = b^2x$ (d) None of these

Sol. (b) Let the coordinates of $P(h, k)$ and the given equation of the parabola be $y^2 = 4ax$. Any tangent on the parabola is given by $y = mx + \frac{a}{m}$.

$$\text{If this passes through } (h, k), \text{ then}$$

$$k = mh + \frac{a}{m}$$

$$\Rightarrow m^2h - mk + a = 0 \quad \dots (i)$$

which is a quadratic in m .

Let its roots be m_1 and m_2 , then $m_1 + m_2 = \frac{k}{h}$ and

$m_1 m_2 = \frac{a}{h}$. Now, if the two tangents through P make

angles θ_1 and θ_2 with X -axis and $m_1 = \tan \theta_1$ and $m_2 = \tan \theta_2$.

$$\therefore \tan \theta_1 + \tan \theta_2 = \frac{k}{h} \quad \dots (ii)$$

$$\text{and} \quad m_1 m_2 = \frac{a}{h} \quad \dots (iii)$$

$$\Rightarrow \tan \theta_1 \tan \theta_2 = \frac{a}{h} \quad \dots (iv)$$

By hypothesis, $\tan \theta_1 + \tan \theta_2 = b$

From Eq. (ii), we get

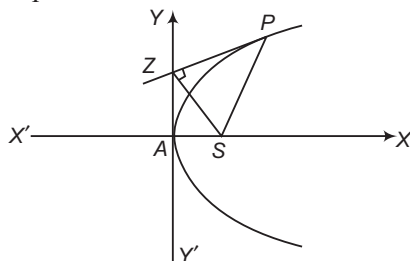
$$\frac{k}{h} = b \Rightarrow k = bh$$

Generalising, the locus of (h, k) is $y = bx$.

Some Useful Results on Tangents

In this section, we will be discussing some useful properties of tangents to a parabola.

- The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular on the directrix from the point. (It is known as reflection property of parabola).
- The tangent at the extremities of a focal chord of a parabola intersect at right angles on the directrix.
- The portion of the tangent to a parabola cut off between the directrix and the curve subtends a right angle at the focus.
- The perpendicular drawn from the focus on any tangent to a parabola intersect it at the point where it cuts the tangent at the vertex.
- If SZ is perpendicular to the tangent at a point P of a parabola, then Z lies on the tangent at the vertex and $SZ^2 = AS \cdot SP$, where A is the vertex of the parabola.



- The orthocentre of any triangle formed by three tangents to a parabola lies on the directrix.
- The circumcircle formed by the intersection points of tangents at any three points on a parabola passes through the focus of the parabola.
- The tangent at any point of a parabola is equally inclined to the focal distance of the point and the axis of the parabola.
- The length of the subtangent at any point on a parabola is equal to twice the abscissae of the point.

- **Example 26.** A ray of light moving parallel to the X -axis gets reflected from a parabolic mirror whose equation is $(y-2)^2 = 4(x+1)$. After reflection, the ray must pass through the point
- (0, 2)
 - (2, 0)
 - (0, -2)
 - (-1, 2)

Sol. (a) The equation of the axis of the parabola is $y - 2 = 0$, which is parallel to the X -axis. So, a ray parallel to the axis of the parabola. We know that any ray parallel to the axis of a parabola passes through the focus after reflection. Here, (0, 2) is the focus.

- **Example 27.** If y_1, y_2 are the ordinates of two points P and Q on the parabola and y_3 is the ordinate of the point of intersection of tangents at P and Q , then

- y_1, y_2, y_3 are in AP
- y_1, y_3, y_2 are in AP
- y_1, y_2, y_3 are in GP
- y_1, y_3, y_2 are in GP

Sol. (b) Let the coordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively. Then, $y_1 = 2at_1$ and $y_2 = 2at_2$. The coordinates of the point of intersection of the tangents at P and Q are $[at_1 t_2, a(t_1 + t_2)]$.

$$\therefore y_3 = a(t_1 + t_2)$$

$$\Rightarrow y_3 = \frac{y_1 + y_2}{2}$$

So, y_1, y_3, y_2 are in AP.

- **Example 28.** If parabola of latusrectum l , touches a fixed equal parabola, the axes of the two curves being parallel, then the locus of the vertex of the moving curve is

- a parabola of latusrectum $2l$
- a parabola of latusrectum l
- an ellipse whose major axis is $2l$
- an ellipse whose minor axis is $2l$

Sol. (a) Let (h, k) be the coordinates of the vertex of the moving parabola and its equation be

$$(y - k)^2 + l(x - h) = 0$$

The equation of the fixed curve $y^2 - lx = 0$; then the tangents at the point (lt^2, lt) to the two curves are

$$-2yt + x + lt^2 = 0$$

$$\text{and } y(lt - k) + \frac{l}{2}x + l^2t^2 + 2k^2 - 2lh - 2klt = 0$$

Since, they are coincident.

$$\therefore \frac{k - lt}{2t} = \frac{l/2}{1} = \frac{l^2t^2 + 2k^2 - 2lh - 2lt}{2lt^2}$$

$$\text{from which } t = \frac{k}{2l} \text{ and } k^2 = lh = klt$$

Eliminating t , we get $k^2 = 2lh$.

Hence, the locus of the vertex is $y^2 = 2lx$, which is a parabola having latusrectum as $2l$.

Common Tangents

In this section, we shall discuss some problems on finding the common tangents to two conics.

- **Example 29.** If two equal parabolas having the same vertex and their axes are at right angles. Then, the length of their common tangent is

- $3a$
- $3\sqrt{2}a$
- $\frac{3}{\sqrt{2}}a$
- $\sqrt{2}a$

Sol. (b) Let the two equal parabolas be $y^2 = 4ax$ and $x^2 = 4ay$.

The equation of any tangent to $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \text{ at the point } \left(\frac{a}{m^2}, \frac{2a}{m} \right) \quad \dots(i)$$

Now, $y = mx + \frac{a}{m}$
 $\Rightarrow x = \frac{y}{m} - \frac{a}{m^2}$
 It also touches the parabola $x^2 = 4ay$.
 Then, $\frac{-a}{m^2} = \frac{a}{1/m} \quad \left[\because \text{using } c = \frac{a}{m} \right]$
 $\Rightarrow m^3 = -1$
 $\Rightarrow m = -1$

The common tangent touches $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m} \right)$.

So, the coordinates of the point of contact are $P(a, -2a)$.

The common tangent of Eq. (i) touches $x^2 = 4ay$ at

$$\left(\frac{2a}{1/m}, \frac{a}{1/m^2} \right) \equiv Q(2am, am^2) \quad \left[\because x = my + \frac{a}{m} \text{ touches } x^2 = 4ay \text{ at } \left(\frac{2a}{m}, \frac{a}{m^2} \right) \right]$$

$$\equiv Q(-2a, a)$$

Hence, length of the common tangent PQ
 $= \sqrt{(a+2a)^2 + (-2a-a)^2} = 3\sqrt{2}a$

➤ **Example 30** If two lines are drawn at right angles, one being a tangent to $y^2 = 4ax$ and the other to $x^2 = 4by$. Then, the locus of their point of intersection is the curve

(a) $(ax + by)(x^2 + y^2) + (bx - ay)^2 = 0$

(b) $(ax + by)^2(x^2 + y^2) + (bx - ay) = 0$

(c) $(bx + ay)(x^2 + y^2) + (ax - by)^2 = 0$

(d) None of the above

Sol. (a) Let $y = mx + \frac{a}{m}$... (i)

be a tangent to $y^2 = 4ax$

and $x = m_1y + \frac{b}{m_1}$... (ii)

be a tangent to $x^2 = 4by$.

Lines (i) and (ii) will be perpendicular, if

$$m \times \frac{1}{m_1} = -1$$

$$\Rightarrow m_1 = -m$$

Let (h, k) be the point of intersection of Eqs. (i) and (ii).

Then, $k = mh + \frac{a}{m}$

and $h = m_1k + \frac{b}{m_1}$

$$\Rightarrow k = mh + \frac{a}{m}$$

$$\text{and } h = -mk - \frac{b}{m}$$

$$\Rightarrow m^2h - mk + a = 0$$

$$\text{and } m^2k + mh + b = 0$$

$$\therefore \frac{m^2}{-kb - ah} = \frac{m}{ak - bh} = \frac{1}{h^2 + k^2}$$

[by cross-multiplication]

$$\Rightarrow m = \frac{kb + ah}{bh - ak} \text{ and } \frac{ak - bh}{h^2 + k^2}$$

$$\Rightarrow \frac{kb + ah}{bh - ak} = \frac{ak - bh}{h^2 + k^2}$$

$$\Rightarrow (h^2 + k^2)(kb + ah) = -(bh - ak)^2$$

Hence, the locus of (h, k) is

$$(x^2 + y^2)(ax + by) = -(bx - ay)^2$$

$$\Rightarrow (x^2 + y^2)(ax + by) + (bx - ay)^2 = 0$$

Angle of Intersection of Two Parabolas

The angle θ between the tangents at point of intersection of two parabolas C_1 and C_2 is defined as the angle of intersection of two parabolas and is given by

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

where, m_1 and m_2 are slopes of tangent at point of intersection of two parabolas C_1 and C_2 , respectively.

If $\theta = \frac{\pi}{2}$, then two parabolas are said to be intersect orthogonally and condition for the same is

$$m_1 m_2 = -1$$

If $\theta = 0$ or π , then the two parabolas touch each other and the condition for same is $m_1 = m_2$.

➤ **Example 31.** If two parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect at an angle $\frac{\pi}{3}$ at a point other

than the origin, then the value of $\left(\frac{a}{b} \right)^{1/3} + \left(\frac{b}{a} \right)^{1/3}$

is

(a) $\sqrt{3}$

(b) $\frac{\sqrt{3}}{2}$

(c) $-\frac{\sqrt{3}}{2}$

(d) None of these

Sol. (b) By solving $y^2 = 4ax$ and $x^2 = 4by$, we get the points of intersection are $(0, 0)$ and $(4a^{1/3}b^{1/3}, 4a^{2/3}b^{1/3})$.

The tangent to $y^2 = 4ax$ at $(4a^{1/3}b^{1/3}, 4a^{2/3}b^{1/3})$ is

$$y \cdot 4a^{2/3}b^{1/3} = 2a(x + 4a^{1/3}b^{2/3}) \quad \dots (i)$$

$$\text{Similarly, } x^2 = 4by \text{ is } x \cdot 4a^{1/3}b^{1/3} = 2a(y + 4a^{2/3}b^{1/3}) \quad \dots (ii)$$

Slope of tangents are $\frac{a^{1/3}}{2b^{1/3}}$ and $\frac{2a^{1/3}}{b^{1/3}}$, respectively.

$$\tan \frac{\pi}{3} = \left| \frac{\frac{a^{1/3}}{2b^{1/3}} - \frac{2a^{1/3}}{b^{1/3}}}{1 + \frac{a^{1/3}}{2b^{1/3}} \cdot \frac{2a^{1/3}}{b^{1/3}}} \right|$$

$$\Rightarrow \sqrt{3} = \frac{3}{2} \left(\frac{a^{1/3}b^{1/3}}{a^{2/3} + b^{2/3}} \right)$$

$$\Rightarrow \left(\frac{a}{b} \right)^{1/3} + \left(\frac{b}{a} \right)^{1/3} = \frac{\sqrt{3}}{2}$$

Work Book Exercise 14.2

14

Parabola

- A tangent to the parabola $y^2 = 4ax$ is inclined at $\pi/3$ with the axis of the parabola. The point of contact is
 - $\left(\frac{a}{3}, \frac{-2a}{\sqrt{3}}\right)$
 - $(3a, -2\sqrt{3}a)$
 - $(3a, 2\sqrt{3}a)$
 - None of these
- The equation of tangent to the parabola $y^2 = 9x$ which goes through the point $(4, 10)$, is
 - $x + 4y + 1 = 0$
 - $9x + 4y + 4 = 0$
 - $x - 4y + 36 = 0$
 - $9x + 4y + 9 = 0$
- Let A, B and C be three points taken on the parabola $y^2 = 4ax$ with coordinates $(at_i^2, 2at_i)$; $i = 1, 2, 3$, where t_1, t_2 and t_3 are in AP. If AA', BB' and CC' are focal chords and coordinates of A', B', C' are $(at_i'^2, 2at_i')$; $i = 1, 2, 3$, then t_1', t_2' and t_3' are in
 - AP
 - GP
 - HP
 - None of the above
- If a focal chord of $y^2 = 16x$ is a tangent to $(x - 6)^2 + y^2 = 2$, then the values of the slope of this chord will be
 - $\{-1, 1\}$
 - $\{-2, 2\}$
 - $\{-2, 1/2\}$
 - $\{2, -1/2\}$
- The circle $x^2 + y^2 - 2x - 6y + 2 = 0$ intersects the parabola $y^2 = 8x$ orthogonally at the point P . The equation of the tangent to the parabola at P can be
 - $x - y - 4 = 0$
 - $2x + y - 2 = 0$
 - $x + y - 4 = 0$
 - $2x - y + 1 = 0$
- From a point T , a tangent is drawn at the point $P(16, 16)$ of the parabola $y^2 = 16x$. If S is the focus of the parabola, then $\angle TPS$ can be equal to
 - $\tan^{-1}(3/4)$
 - $\frac{1}{2} \tan^{-1}(1/2)$
 - $\tan^{-1}(1/2)$
 - $\frac{\pi}{4}$
- The angle between the tangents drawn from the point $(1, 4)$ to the parabola $y^2 = 4x$ is
 - $\pi/6$
 - $\pi/4$
 - $\pi/3$
 - $\pi/2$

Equation of Normal to Parabola

In this section, we will discuss equation of normal to a parabola in different forms.

- (i) **Point form** The equation of normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by

$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

- (ii) **Parametric form** The equation of normal to the parabola $y^2 = 4ax$ at point $(at^2, 2at)$ is given by

$$y + tx = 2at + at^3$$

- (iii) **Slope form** The equation of normal to the parabola $y^2 = 4ax$ in terms of its slope m , is given by

$$y = mx - 2am - am^3 \text{ at the point } (am^2, -2am).$$

- The equation of normals to various standard forms of the parabola in terms of the slope of the normal are as given below:

Equation of the parabola	Equation of the normal	Slope of the normal	Point of contact (Foot of the normal)
$y^2 = 4ax$	$y = mx - 2am - am^3$	m	$(am^2, -2am)$
$y^2 = -4ax$	$y = mx + 2am + am^3$	m	$(-am^2, 2am)$
$x^2 = 4ay$	$x = my - 2am - am^3$	$1/m$	$(-2am, am^2)$

$$x^2 = -4ay \quad x = my + 2am + am^3 \quad 1/m \quad (2am, -am^2)$$

- The equation of normals to the parabolas reducible to one of the standard forms are given below for ready reference :

Parabola	Equation of the normal
$(y - k)^2 = 4a(x - h)$	$y - k = m(x - h) - 2am - am^3$
$(y - k)^2 = -4a(x - h)$	$y - k = m(x - h) + 2am + am^3$
$(x - h)^2 = 4a(y - k)$	$x - h = m(y - k) - 2am - am^3$
$(x - h)^2 = -4a(y - k)$	$x - h = m(y - k) + 2am + am^3$

➤ **Example 32.** The equation of the normals at the ends of the latusrectum of the parabola $y^2 = 4ax$ are given by

- $x^2 - y^2 - 6ax + 9a^2 = 0$
- $x^2 - y^2 - 6ax - 6ay + 9a^2 = 0$
- $x^2 - y^2 + 6ay + 9a^2 = 0$
- None of the above

Sol. (a) The coordinates of the ends of the latusrectum of the parabola $y^2 = 4ax$ are $(a, 2a)$ and $(a, -2a)$ respectively. The equation of the normal at $(a, 2a)$ to $y^2 = 4ax$ is

$$y - 2a = -\frac{2a}{2a}(x - a)$$

$$\Rightarrow \quad x + y - 3a = 0 \quad \dots(i)$$

Similarly, the equation of the normal $(a, -2a)$ is

$$x - y - 3a = 0 \quad \dots(ii)$$

On combining Eqs. (i) and (ii), we get

$$x^2 - y^2 - 6ax + 9a^2 = 0$$

➤ **Example 33.** If the tangents and normals at the extremities of a focal chord of a parabola intersect at (x_1, y_1) and (x_2, y_2) respectively, then

- (a) $x_1 = y_2$ (b) $x_1 = y_1$
(c) $y_1 = y_2$ (d) $x_2 = y_1$

Sol. (c) Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola $y^2 = 4ax$.

The tangents at P and Q intersect at $[at_1t_2, a(t_1 + t_2)]$.

$$\therefore x_1 = at_1t_2 \text{ and } y_1 = a(t_1 + t_2)$$

$$\Rightarrow x_1 = -a \text{ and } y_1 = a(t_1 + t_2)$$

$$[\because PQ \text{ is a focal chord. } \therefore t_1t_2 = -1]$$

The normals at P and Q intersect at

$$[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$$

$$\therefore x_2 = 2a + a(t_1^2 + t_2^2 + t_1t_2)$$

$$\text{and } y_2 = -at_1t_2(t_1 + t_2)$$

$$\Rightarrow x_2 = 2a + a(t_1^2 + t_2^2 - 1) = a + a(t_1^2 + t_2^2)$$

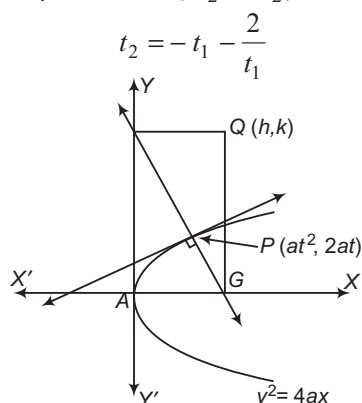
$$\text{and } y_2 = a(t_1 + t_2)$$

$$\text{Clearly, } y_1 = y_2$$

Some Useful Results on Normals to a Parabola

In this section, we will be discussing some useful results on normals to a parabola. These results are very useful to solve problems of parabola.

- (i) If the normal at the point $P(at_1^2, 2at_1)$ meets the parabola $y^2 = 4ax$ at $(at_2^2, 2at_2)$, then



- (ii) The tangent at one extremity of the focal chord of a parabola is parallel to the normal at the other extremity.
(iii) The normals at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ to the parabola $y^2 = 4ax$ intersect at the point $[2a + a(t_1^2 + t_2^2 + t_1t_2), -at_1t_2(t_1 + t_2)]$.
(iv) If the normals at points $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ on the parabola $y^2 = 4ax$ meet on the parabola, then $t_1t_2 = 2$.

- (v) If the normals at two points P and Q of a parabola $y^2 = 4ax$ intersect at a third point R on the curve, then the product of the ordinates of P and Q is $8a^2$.

- (vi) If the normal chord at a point $P(at^2, 2at)$ to the parabola $y^2 = 4ax$ subtends a right angle at the vertex of the parabola, then $t^2 = 2$.

- (vii) The normal chord of a parabola at a point whose ordinate is equal to the abscissae, subtends a right angle at the focus.

- (viii) The normal at any point of a parabola is equally inclined to the focal distance of the point and the axis of the parabola.

➤ **Example 34.** The normal at the point $P(ap^2, 2ap)$ meets the parabola at $Q(aq^2, 2aq)$ such that the lines joining the origin to P and Q are at right angle. Then,

- (a) $p^2 = 2$ (b) $q^2 = 2$ (c) $p = 2q$ (d) $q = 2p$

Sol. (a) Since, the normal at $(ap^2, 2ap)$ to $y^2 = 4ax$ meets the parabola at $(aq^2, 2aq)$.

$$q = -p - \frac{2}{p} \quad \dots(i)$$

Since, $OP \perp OQ$.

$$\frac{2ap - 0}{ap^2 - 0} \times \frac{2aq - 0}{aq^2 - 0} = -1$$

$$\Rightarrow pq = -4$$

$$\Rightarrow p \left(-p - \frac{2}{p} \right) = -4 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow p^2 = 2$$

➤ **Example 35.** The normal chord of a parabola $y^2 = 4ax$ at a point whose ordinate is equal to the abscissa, subtends a right angle at the

- (a) focus (b) vertex
(c) end of the latusrectum (d) None of these

Sol. (a) Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$.

$$\text{Then, } at^2 = 2at \Rightarrow t = 2$$

So, the coordinates of P are $(4a, 4a)$.

Let PQ be a normal chord at P to the parabola.

Let the coordinates of Q be $(at_1^2, 2at_1)$.

$$\text{Then, } t_1 = t - \frac{2}{t} = 2 - \frac{2}{2} = -3$$

So, the coordinates of Q are $(9a, -6a)$.

$$\therefore \text{Slope of } SP \times \text{Slope of } SQ = \frac{4a}{3a} \times \frac{-6a}{8a} = -1$$

Thus, the normal chord makes a right angle at the focus.

➤ **Example 36.** P is the point t on the parabola $y^2 = 4ax$ and PQ is a focal chord. PT is the tangent at P and QN is the normal at Q . If the angle between PT and QN is α and the distance between PT and QN is d , then

- (a) $0 < \alpha < 90^\circ$
(b) $\alpha = 90^\circ$

(c) $d = 0$

(d) $d = a \left(\sqrt{1+t^2} + \frac{1}{t^2 \sqrt{1+t^2}} \right)$

Sol. (d) Since, PQ is the focal chord.

So, coordinates of P and Q are $(at^2, 2at)$ and $\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$ respectively.

$\therefore$ Equation of the tangent at P is $ty = x + at^2$.

$\therefore$ Slope of $PT = \frac{1}{t}$

Equation of the normal at Q is $xt^2 - yt^3 - a - 2at^2 = 0$

Slope of $QN = \frac{1}{t}$

$\therefore PT$ and QN are parallel.

$\therefore \alpha = 0$

Given, distance between PT and QN is d .

So, $d = \left[\frac{-at^4 - a - 2at^2}{\sqrt{(t^4 + t^6)}} \right] = a \left(\sqrt{1+t^2} + \frac{1}{t^2 \sqrt{1+t^2}} \right)$

➤ **Example 37.** If the normal to the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ cuts the parabola again at $(aT^2, 2aT)$, then

(a) $-2 \leq T \leq 2$

(b) $T \in (-\infty, -8) \cup (8, \infty)$

(c) $T^2 < 8$

(d) $T^2 \geq 8$

Sol. (d) Equation of the normal of the parabola $y^2 = 4ax$ at the point $(at^2, 2at)$ is

$y + tx = 2at + at^3 \quad \dots(i)$

Since, Eq. (i) cuts the parabola again at $(aT^2, 2aT)$.

$\therefore 2aT + taT^2 = 2at + at^3$

$\Rightarrow 2a(T - t) = -at(T^2 - t^2)$

$\Rightarrow 2 = -t(T + t) \quad [\because t \neq T]$

$\Rightarrow t^2 + tT + 2 = 0$

Since, t is a real.

$\therefore T^2 - 4 \cdot 1 \cdot 2 \geq 0$

$\Rightarrow T^2 \geq 8$

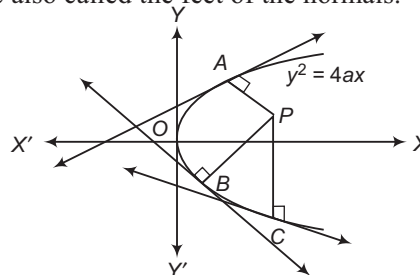
Number of Normals and Conormal Points

In this section, we will see that in general, three normals can be drawn from a point to a parabola. We shall also study about the relations between their slopes and conditions, so that three normals are distinct.

(i) Three normals can be drawn from a point to a parabola.

(ii) **Conormal points** The point on the parabola at which the normals pass through a common point

are called conormal points. The conormal points are also called the feet of the normals.



In figure A, B, C are conormal points with respect to point P .

Some Useful Results on Conormal Points

Let $P(h, k)$ be a point and $y^2 = 4ax$ be a parabola.

The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$

If it passes through the point $P(h, k)$, then

$k = mh - 2am - am^3$

$\Rightarrow am^3 + m(2a - h) + k = 0 \quad \dots(i)$

This is cubic equation in m . So, it gives three values of m , say m_1, m_2 and m_3 . Corresponding to each value of m , there is a normal passing through the point $P(h, k)$.

Let A, B, C be the feet of the normals. Then, AP, BP and CP are three normals passing through point P . Let m_1, m_2 and m_3 respectively be their slopes. Then, their equations are

$y = m_1x - 2am_1 - am_1^3$

$y = m_2x - 2am_2 - am_2^3$

$y = m_3x - 2am_3 - am_3^3$

The coordinates of A, B and C are

$(am_1^2, -2am_1), (am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$ respectively.

Since, m_1, m_2, m_3 are the roots of Eq. (i).

$m_1 + m_2 + m_3 = 0 \quad \dots(ii)$

$m_1m_2 + m_2m_3 + m_3m_1 = \frac{2a - h}{a} \quad \dots(iii)$

and $m_1m_2m_3 = \frac{-k}{a} \quad \dots(iv)$

We have the following results related to conormal points and the slopes of the normals at conormal points :

- The algebraic sum of the slopes of the normals at conormal points is zero.
- The sum of the ordinates of the conormal points is zero.
- The centroid of the triangle formed by the conormal points on a parabola lies on its axis.

➤ **Example 38.** If the normals drawn from any point to the parabola $y^2 = 4ax$ cut the line $x = 2a$ in points whose ordinates are in arithmetic progression, then tangents of the angles which the normals makes with the axis, are

- (a) in AP (b) in GP
(c) in HP (d) None of these

Sol. (b) Let the equation of the parabola be $y^2 = 4ax$. Then, the feet of the normals passing through a point are $(am_1^2, -2am_1)$, $(am_2^2, -2am_2)$ and $(am_3^2, -2am_3)$.

Equation of normal at $(am_1^2, -2am_1)$ is

$$y = m_1x - 2am_1 - am_1^3.$$

On solving with the line $x = 2a$, we get the point of intersection as $(2a, -am_1^3)$.

Similarly, on solving with the other normals, the ordinates of the points of intersection with $x = 2a$ are

$$-am_1^3, -am_2^3 \text{ and } -am_3^3.$$

These ordinates are in AP.

$$\text{So, } -am_1^3 - am_3^3 = -2am_2^3$$

$$\text{But } m_1^3 + m_3^3 = 2m_2^3 \quad \dots(i)$$

$$\text{as } \Sigma m_i = 0 \quad \dots(ii)$$

$$m_1^3 + m_2^3 + m_3^3 = 3m_1m_2m_3 \quad \dots(iii)$$

From Eqs. (i) and (iii), we get

$$3m_2^3 = 3m_1m_3$$

$$\Rightarrow m_2^3 = m_1m_3$$

Hence, m_1, m_2 and m_3 are in GP.

➤ **Example 39.** The number of distinct normals that can be drawn from $(-2, 1)$ to the parabola $y^2 - 4x - 2y - 3 = 0$, is

- (a) 1 (b) 2
(c) 3 (d) 0

Sol. (a) Here, $y^2 - 2y + 1 = 4(x + 1)$

$$\Rightarrow (y - 1)^2 = 4(x + 1)$$

So, the axis is $y - 1 = 0$. Also, $(-2, 1)$ lies on the axis and it is exterior to the parabola because

$$1^2 - 4(-2) - 2(1) - 3 = 4 > 0$$

Hence, only one normal is possible.

Combined Equation of Pair of Tangents

The combined equation of the pair of tangents drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$(y^2 - 4ax)(y_1^2 - 4ax_1) = \{yy_1 - 2a(x + x_1)\}^2$$

$$\text{or } SS_1 = T^2$$

$$\text{where, } S \equiv y^2 - 4ax, S_1 \equiv y_1^2 - 4ax_1$$

$$\text{and } T \equiv yy_1 - 2a(x + x_1)$$

➤ **Example 40.** The normal chord at a point t on the parabola $y^2 = 4ax$ subtends a right angle at the vertex. Then, t^2 is equal to

- (a) 4 (b) 2
(c) 1 (d) 3

Sol. (b) The equation of any normal to $y^2 = 4ax$ at $(at^2, 2at)$ is

$$y + tx = 2at + at^3 \quad \dots(i)$$

The combined equation of the lines joining the vertex (origin) to the points of intersection of the parabola and Eq. (i) is

$$y^2 = 4ax \left(\frac{y + tx}{2at + at^3} \right)$$

$$\Rightarrow (2t + t^3)y^2 = 4x(y + tx)$$

If Eq. (i) makes a right angle at the vertex, then

$$\text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$4t - 2t - t^3 = 0$$

$$\Rightarrow t^2 = 2$$

Director Circle

The locus of the point of intersection of perpendicular tangents to a conic is known as its director circle.

The director circle of a parabola is its directrix.

Chord of Contact

The chord joining the points of contact of two tangents drawn from an external point P to a parabola is known as the chord of contact of tangents drawn from P .

(i) The chord of contact of tangents drawn from a point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1)$$

(ii) The length of the chord of contact of tangents drawn from (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$\frac{1}{a} \sqrt{(y_1^2 - 4ax_1)(y_1^2 + 4a^2)}$$

(iii) The area of the triangle formed by the tangents drawn from (x_1, y_1) to $y^2 = 4ax$ and their chord

$$\text{of contact is } \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}.$$

(iv) The locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle α is

$$(x + a)^2 \tan^2 \alpha = y^2 - 4ax$$

(v) The angle between the pair of tangents drawn from (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$\tan^{-1} \left(\sqrt{\frac{S_1}{x_1 + a}} \right)$$

➤ **Example 41.** If the chord of contact of tangents from a point P to the parabola $y^2 = 4ax$, touches the parabola $x^2 = 4by$, then the locus of P is a/an

- (a) circle
(b) parabola
(c) ellipse
(d) hyperbola

Sol. (d) Let $P(h, k)$ be a point. Then, the chord of contact of tangents from P to $y^2 = 4ax$ is

$$ky = 2a(x + h) \quad \dots(i)$$

This touches the parabola $x^2 = 4by$. So, it should be of the form

$$x = my + \frac{b}{m} \quad \dots(ii)$$

Eq. (i) can be rewritten as

$$x = \left(\frac{k}{2a}\right)y - h \quad \dots(iii)$$

Since, Eqs. (ii) and (iii) represent the same line.

$$\therefore m = \frac{k}{2a}$$

$$\text{and } \frac{b}{m} = -h$$

Eliminating m from these two equations, we get

$$2ab = -hk$$

Hence, the locus of $P(h, k)$ is

$$xy = -2ab, \text{ which is a hyperbola.}$$

➤ **Example 42.** Tangents are drawn at the points, where the line $lx + my + n = 0$ intersected by the parabola $y^2 = 4ax$. The coordinates of the point of intersection of tangents are

$$(a) \left(\frac{n}{l}, \frac{-2am}{l}\right)$$

$$(b) \left(\frac{l}{n}, \frac{-2am}{n}\right)$$

$$(c) \left(\frac{n}{m}, \frac{-2al}{m}\right)$$

(d) None of the above

Sol. (a) Let the point of intersection of tangents be (x_1, y_1) , then the equation of the chord of contact of tangents drawn from P to the parabola $y^2 = 4ax$ is $yy_1 = 2a(x + x_1)$.

Clearly, $lx + my + n = 0$ is also the chord of contact of tangents.

Therefore, $yy_1 = 2a(x + x_1)$ and $lx + my + n = 0$ are same line.

$$\therefore \frac{2a}{l} = \frac{-y_1}{m} = \frac{2ax_1}{n}$$

$$\Rightarrow x_1 = \frac{n}{l}$$

$$\text{and } y_1 = \frac{-2am}{l}$$

$$\text{Hence, required point is } \left(\frac{n}{l}, \frac{-2am}{l}\right).$$

Equation of the Chord Bisected at a Given Point

Theorem

The equation of chord of the parabola $y^2 = 4ax$ which is bisected at (x_1, y_1) is

$$yy_1 - 2a(x + x_1) = y_1^2 - 4ax_1$$

or

$$T = S'$$

where, $S' = y_1^2 - 4ax_1$ and $T = yy_1 - 2a(x + x_1)$.

➤ If the tangent at the point $P(x_1, y_1)$ to the parabola $y^2 = 4ax$ meets the parabola $y^2 = 4ax + b$ at Q and R , then the mid-point of QR is also (x_1, y_1) .

➤ **Example 43.** The mid-point of the chord intercepted on the line $4x - 3y + 4 = 0$ by the parabola $y^2 = 8x$ is

$$(a) (5/4, 3)$$

$$(b) (2, 4)$$

$$(c) (5/2, 14/3)$$

$$(d) (5, 8)$$

Sol. (a) Let (α, β) be the mid-points of chord of the parabola $y^2 = 8x$.

$\therefore$ Equation of chord whose mid-point is (α, β) is

$$y\beta - 4(x + \alpha) = \beta^2 - 8\alpha$$

or

$$4x - y\beta + \beta^2 - 4\alpha = 0$$

Clearly, $4x - 3y + 4 = 0$ and $4x - y\beta + \beta^2 - 4\alpha = 0$

are same line.

$$\therefore \frac{4}{4} = \frac{3}{\beta} = \frac{4}{\beta^2 - 4\alpha}$$

$$\Rightarrow \beta = 3$$

$$\text{and } \alpha = 5/4$$

Hence, required coordinates is $(5/4, 3)$.

➤ **Example 44.** The locus of the mid-points of chords of the parabola $y^2 = 4ax$ which passes through the focus is

$$(a) y^2 + 2ax + 2a^2 = 0$$

$$(b) y^2 - ax + 2a^2 = 0$$

$$(c) y^2 - 2ax + 2a^2 = 0$$

$$(d) y^2 - 2ax + a^2 = 0$$

Sol. (c) Let (h, k) be the coordinates of the mid-points of one of the chord which passes through the focus $(a, 0)$.

Then, the equation of the chord, where mid-point is (h, k) , is

$$ky - 2a(x + h) = k^2 - 4ah$$

This line passes through the focus $(a, 0)$.

$$\therefore -2a^2 + 2ah - k^2 = 0$$

$$\Rightarrow k^2 - 2ah + 2a^2 = 0$$

Hence, the locus of (h, k) is

$$y^2 - 2ax + 2a^2 = 0$$

14

Diameter of a Parabola

Diameter of a parabola is the locus of the mid-point of a series of its parallel chords.

Equation of Diameter of a Parabola

The equation of the diameter bisecting chords of slope m of the parabola $y^2 = 4ax$ is $y = \frac{2a}{m}$.

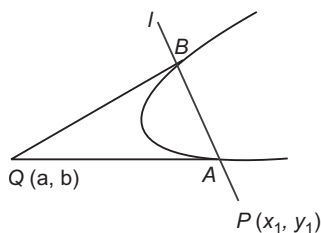
➤ **Example 45.** The locus of the middle points of parallel chords of a parabola $x^2 = 4ay$ is a

- (a) straight line parallel to X -axis
- (b) straight line parallel to Y -axis
- (c) circle
- (d) straight line parallel to a bisector of the angles between the axes

Sol. (b) The locus of the middle points of parallel chords is a line parallel to the axes of the parabola.

Pole and Polar of a Parabola

If a line through the point P , cuts the parabola at points A and B , then the locus of the point of intersection of tangents at A and B is a straight line. This line is called the polar of point P with respect to the parabola. If line l is the polar of point P with respect to a parabola, then point P is called the pole of line l with respect to the parabola.



- (i) The equation of polar of point $P(x_1, y_1)$ with respect to the parabola $y^2 = 4ax$ is $T = 0$

i.e. $yy_1 - 2a(x + x_1) = 0$

(ii) Properties of pole and polar

- (a) If polar of a point P with respect to a parabola passes through point Q , then polar of Q with respect to the parabola will pass through P . Such points P and Q are called conjugate points.
- (b) If the pole of line l_1 lies on line l_2 , then the pole of line l_2 lies on line l_1 . Such lines l_1 and l_2 are called the conjugate lines.

➤ **Example 46.** The locus of the poles of tangents to the parabola $y^2 = 4ax$ with respect to the parabola $y^2 = 4bx$, is

- (a) a circle
- (b) a parabola
- (c) a straight line
- (d) an ellipse

Sol. (b) The equation of any tangent to $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \quad \dots(i)$$

Let (h, k) be the pole of Eq. (i) with respect to the parabola $y^2 = 4bx$.

Then, the equation of polar is

$$ky = 2b(x + h) \quad \dots(ii)$$

$$\text{or } 2bx - ky + 2bh = 0 \quad \dots(iii)$$

Clearly, Eqs. (i) and (ii) represent the same straight line.

$$\therefore \frac{2b}{m} = \frac{-k}{-1} = \frac{2bh}{a/m}$$

$$\Rightarrow k = \frac{2b}{m}$$

$$\text{and } h = \frac{a}{m^2}$$

$$\Rightarrow m = \frac{2b}{k}$$

$$\text{and } m^2 = \frac{a}{h}$$

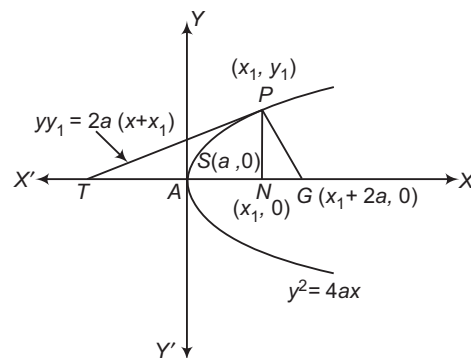
$$\Rightarrow \left(\frac{2b}{k}\right)^2 = \frac{a}{h}$$

$$\Rightarrow k^2 = \frac{4b^2h}{a}$$

Hence, the locus of (h, k) is $y^2 = \frac{4b^2x}{a}$, which is a parabola.

Lengths of Tangent, Subtangent, Normal and Subnormal

Subtangent and subnormal Let the tangent and normal at any point $P(x_1, y_1)$ on a parabola $y^2 = 4ax$ meet the axis in T and G , respectively. Then, PT is called the length of the tangent at P , PG is called the length of the normal at P , NT is called the subtangent at P and NG is called the subnormal at P .



Subtangent = NT = Twice the abscissa of P
and subnormal = $NG = 2a$ = Semi-latusrectum

It can be easily seen, with the help of the right angled $\triangle NPT$ and $\triangle PNG$, that

(i) The length of the tangent = $PT = PN \operatorname{cosec} \psi$

$$= y_1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$$

(ii) The length of the normal = $PG = PN \sec \psi$

$$= y_1 \sqrt{1 + \left(\frac{dx}{dy}\right)^2_{(x_1, y_1)}}$$

(iii) The length of the subtangent = $NT = PN \cot \psi$

$$= y_1 \left(\frac{dx}{dy}\right)_{(x_1, y_1)}$$

(iv) The length of the subnormal = $NG = PN \tan \psi$

$$= y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)}$$

$$\text{where, } \tan \psi = \frac{2a}{y_1} = m$$

➤ **Example 47.** The subtangent, ordinate and subnormal to the parabola $y^2 = 4ax$ at a point, different from the origin, are in

- (a) AP
(b) GP
(c) HP
(d) None of the above

Sol. (b) The equation of parabola is $y^2 = 4ax$.

$$\Rightarrow 2y \frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y}$$

At (x_1, y_1) , we have

$$\begin{aligned} \text{Subtangent} &= \frac{y_1}{(dy/dx)} = \frac{y_1}{2a/y_1} \\ &= \frac{y_1^2}{2a} \end{aligned}$$

$$\text{Subnormal} = y_1 \cdot \frac{dy}{dx} = 2a$$

Clearly, subtangent = $\frac{y_1^2}{2a}$, ordinate = y_1

and subnormal = $2a$ are in GP .

Work Book Exercise 14.3

- The chord of contact of the pair of tangents drawn from each point on the line $2x + y = 4$ to the parabola $y^2 = -4x$ pass through the point
 - $(2, -1)$
 - $(1/2, 1/4)$
 - $(-1/2, -1/4)$
 - $(-2, 1)$
- The inclination of the normal with positive direction of X -axis, drawn at another end of a normal to the parabola $y^2 = 4x$ is always
 - 60°
 - less than 60°
 - more than 60°
 - less than 45°
- Two parabolas with the same axis, focus of each being exterior to the other and the latusrectum being $4a$ and $4b$. The locus of middle points of the intercepts between the parabolas made on the lines parallel to the common axis is a/an
 - straight line, if $a > b$
 - parabola, if $a \neq b$
 - parabola, $\forall a, b$
 - ellipse, if $b > a$
- The set of real values of a for which atleast one tangent to $y^2 = 4ax$ becomes normal to the circle $x^2 + y^2 - 2ax - 4ay + 3a^2 = 0$, is
 - $[1, 2]$
 - $[\sqrt{2}, 3]$
 - R
 - None of these
- If P, Q and R are three points on parabola $y^2 = 4x$ and normal at P and R meet at Q , then the locus of mid-point of the chord PR is a parabola whose vertex is at
 - $(2, 0)$
 - $(0, -2)$
 - $(-2, 0)$
 - None of these
- Tangent and normal at any point P of the parabola $y^2 = 4ax$ ($a > 0$) meet the X -axis at T and N , respectively. If the lengths of subtangent and subnormal at this point are equal, then the area of $\triangle PTN$ is given by
 - $4a^2$
 - $6\sqrt{2}a^2$
 - $4\sqrt{2}a^2$
 - None of the above
- If $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) are the feet of the three normals drawn from a point to the parabola $y^2 = 4ax$, then

$$\frac{x_1 - x_2}{y_3} + \frac{x_2 - x_3}{y_1} + \frac{x_3 - x_1}{y_2}$$
 is equal to
 - $4a$
 - $2a$
 - a
 - 0

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The condition for all the three normals drawn from a given point (h, k) to the parabola $y^2 = 4ax$ to be real and distinct, is

- (a) $4(h - 2a)^3 > 27ak^2$
 (b) $(h - 2a)^3 > 27ak^2$
 (c) $4(h - 2a)^3 < 27ak^2$
 (d) $(h - 2a)^3 < 27ak^2$

Sol. Equation of the normal to the parabola $y^2 = 4ax$ at any point $(at^2, 2at)$ is

$$y = -tx + 2at + at^3 \quad \dots(i)$$

If this normal passes through the point (h, k) , then

$$k + th = 2at + at^3 \quad \dots(ii)$$

Let $f(t) = at^3 + (2a - h)t - k$

So,

$$f'(t) = 3at^2 + (2a - h) = 0 \text{ for critical points}$$

i.e. $t = \pm \sqrt{\frac{h - 2a}{3a}} = \pm \alpha$ [say] $\Rightarrow h > 2a$

All the roots of Eq. (ii) are real, if $f(\alpha) f(-\alpha) < 0$

$$\Rightarrow [a\alpha^3 + (2a - h)\alpha - k][-a\alpha^3 - (2a - h)\alpha - k] < 0$$

$$\Rightarrow [a\alpha^3 + (2a - h)\alpha]^2 - k^2 > 0$$

$$\Rightarrow \alpha^2[a\alpha^2 + (2a - h)]^2 - k^2 > 0$$

$$\Rightarrow \frac{h - 2a}{3a} \left[a \left(\frac{h - 2a}{3a} \right) + 2a - h \right]^2 - k^2 > 0$$

$$\Rightarrow 4(h - 2a)^3 > 27ak^2$$

Hence, (a) is the correct answer.

Ex 2. The locus of centres of a family of circles passing through the vertex of the parabola $y^2 = 4ax$ and cutting the parabola orthogonally at the other point of intersection, is

- (a) $y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$
 (b) $2y^2(2y^2 + x^2 - 12ax) = ax(3x + 4a)^2$
 (c) $2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$
 (d) None of the above

Sol. Let $P(at^2, 2at)$ be any point on $y^2 = 4ax$.

$$\text{Vertex } A = (0, 0)$$

The equation of tangent at P is

$$ty = x + at^2 \quad \dots(i)$$

Tangent at P will be normal to the circle as circle and parabola intersect orthogonally at P .

AP is a chord, whose mid-point is $\left(\frac{at^2}{2}, at\right)$ and slope

is $(2/t)$.

$\therefore$ Line passing through the mid-point of AP and perpendicular to AP will pass through the centre of the circle.

$$\therefore y - at = -\frac{t}{2} \left(x - \frac{at^2}{2} \right)$$

$$\Rightarrow tx + 2y = \frac{at^3}{2} + 2at \quad \dots(ii)$$

Eqs. (i) and (ii) both pass through (h, k) , which is the centre of the circle.

$$\therefore tk = h + at^2 \quad \dots(iii)$$

$$\Rightarrow 2th + 4k = at^3 + 4at \quad \dots(iv)$$

On multiplying Eq. (iii) by t and subtracting from Eq. (iv), we get

$$t^2k + t(4a - 3h) - 4k = 0 \quad \dots(v)$$

$$\text{Also, from Eq. (iii), } at^2 - tk + h = 0 \quad \dots(vi)$$

Eliminating t from Eqs. (v) and (vi), we have

$$\frac{t^2}{h(4a - 3h) - 4k^2} = \frac{t}{-4ak - hk} = \frac{1}{-k^2 - a(4a - 3h)}$$

$$\Rightarrow 2k^2(2k^2 + h^2 - 12ah) = ah(3h - 4a)^2$$

Hence, the required locus is

$$2y^2(2y^2 + x^2 - 12ax) = ax(3x - 4a)^2$$

Hence, (c) is the correct answer.

Ex 3. Normals are drawn from the point P with slopes $m_1 \cdot m_2 = \alpha$ and it is a part of the parabola itself, then α is equal to

- (a) 1 (b) 2 (c) 3 (d) -2

Sol. Let the point P be (h, k) .

$$\therefore k = mh - 2m - m^3$$

$$\Rightarrow m^3 + m(2 - h) + k = 0$$

$$\Rightarrow m_1 m_2 m_3 = -k$$

$$\Rightarrow m_3 = -\frac{k}{\alpha}$$

$$\Rightarrow \left(-\frac{k}{\alpha}\right)^3 - \frac{k}{\alpha}(2 - h) + k = 0$$

$$\Rightarrow k^2 = \alpha^2 h - 2\alpha^2 + \alpha^3$$

Thus, the locus of (h, k) is

$$\Rightarrow y^2 = \alpha^2 x - 2\alpha^2 + \alpha^3$$

On comparing it with $y^2 = 4x$, we get

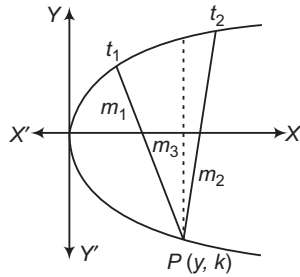
$$\alpha^2 = 4$$

$$\text{and } -2\alpha^2 + \alpha^3 = 0$$

$$\Rightarrow \alpha = 2$$

Aliter

Since, locus of P is a part of the parabola.



So, normals at any two points t_1 and t_2 meet at P .

$$\Rightarrow t_1 t_2 = 2$$

$$\Rightarrow (-m_1)(-m_2) = 2$$

$$\Rightarrow \alpha = 2$$

Hence, (b) is the correct answer.

Ex 4. Consider two concentric circles

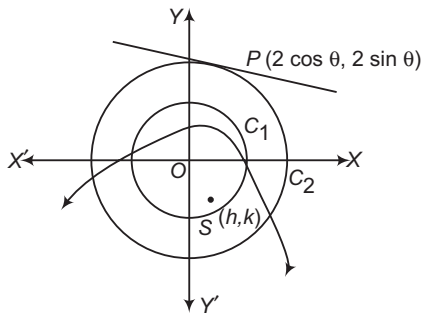
$C_1 : x^2 + y^2 - 1 = 0$ and $C_2 : x^2 + y^2 - 4 = 0$. A parabola is drawn through the points, where C_1 meets the X -axis and having arbitrary tangent of C_2 as its directrix. Then, the locus of the focus of drawn parabola is

(a) $\frac{4}{3}x^2 - y^2 = 3$ (b) $\frac{3}{4}x^2 - y^2 = 3$

(c) $\frac{4}{3}x^2 + y^2 = 3$ (d) $\frac{3}{4}x^2 + y^2 = 3$

Sol. Clearly, the parabola should pass through $(1, 0)$ and $(-1, 0)$. Let directrix of this parabola be

$$x \cos \theta + y \sin \theta = 2$$



If $S(h, k)$ is the focus of this parabola, then distance of $(\pm 1, 0)$ from S and from the directrix should be same.

$$\Rightarrow (h-1)^2 + k^2 = (\cos \theta - 2)^2 \quad \dots(i)$$

$$\text{and } (h+1)^2 + k^2 = (\cos \theta + 2)^2 \quad \dots(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$$4h = 8 \cos \theta \Rightarrow \cos \theta = \frac{h}{2}$$

On adding Eqs. (i) and (ii), we get

$$2(h^2 + k^2 + 1) = 2(\cos^2 \theta + 4)$$

$$\Rightarrow h^2 + k^2 + 1 = 4 + \frac{h^2}{4}$$

$$\Rightarrow \frac{3}{4}h^2 + k^2 = 3$$

Hence, locus of focus is

$$\frac{3}{4}x^2 + y^2 = 3$$

Hence, (d) is the correct answer.

Ex 5. A movable parabola touches the X and Y -axes at $(1, 0)$ and $(0, 1)$. Then, the locus of the focus of the parabola is

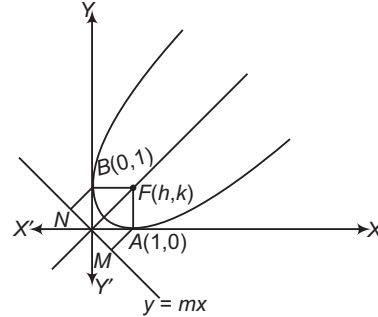
(a) $2x^2 - 2x + 2y^2 - 2y + 1 = 0$

(b) $x^2 - 2x + 2y^2 - 2y + 1 = 0$

(c) $2x^2 - 2x + 2y^2 + 2y + 2 = 0$

(d) $2x^2 + 2x - 2y^2 - 2y - 2 = 0$

Sol. Since, X -axis and Y -axis are two perpendicular tangents to the parabola and both meet at the origin, the directrix passes through the origin.



Let $y = mx$ be the directrix and (h, k) be the focus.

$$FA = AM$$

$$\Rightarrow \sqrt{(h-1)^2 + k^2} = \left| \frac{m}{\sqrt{1+m^2}} \right| \quad \dots(i)$$

and

$$FB = BN$$

$$\Rightarrow \sqrt{h^2 + (k-1)^2} = \left| \frac{1}{\sqrt{1+m^2}} \right| \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$(h-1)^2 + h^2 + k^2 + (k-1)^2 = 1$$

$$\Rightarrow 2x^2 - 2x + 2y^2 - 2y + 1 = 0$$

which is the required locus.

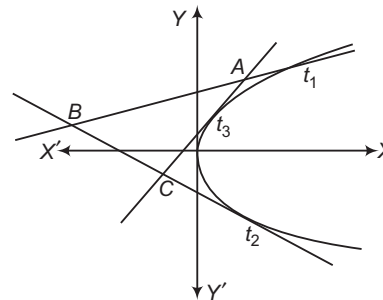
Hence, (a) is the correct answer.

Ex 6. The locus of vertices of the equilateral $\triangle ABC$ such that its sides touch the parabola $y^2 = 4ax$, is

(a) $y^2 = (x+a)^2 + 4ax$ (b) $y^2 = 3(x+a)^2 + 4ax$

(c) $y^2 = 3(x+a)^2 - ax$ (d) None of these

Sol. Any point on the parabola $y^2 = 4ax$ is $(at^2, 2at)$ and point of intersection of tangents at t_1, t_2 is $[at_1 t_2, a(t_1 + t_2)]$.



$$\Rightarrow A \equiv [at_1 t_3, a(t_1 + t_3)], B \equiv [at_1 t_2, a(t_1 + t_2)]$$

$$\text{and } C \equiv [at_2 t_3, a(t_2 + t_3)]$$

Given, $\triangle ABC$ is an equilateral triangle.

$$\Rightarrow \tan 60^\circ = \sqrt{3} = \left| \frac{m_{AB} - m_{AC}}{1 - m_{AB} \cdot m_{AC}} \right| = \left| \frac{\frac{1}{t_1} - \frac{1}{t_3}}{1 + \frac{1}{t_1} \cdot \frac{1}{t_3}} \right|$$

$$\Rightarrow 3(1 + t_1 t_3)^2 = (t_1 - t_3)^2$$

$$\Rightarrow 3(1 + t_1 t_3)^2 = (t_1 + t_3)^2 - 4t_1 t_3$$

If A is (h, k) , then $h = at_1 t_3$, $k = a(t_1 + t_3)$

$$\Rightarrow 3 \left(1 + \frac{h}{a} \right)^2 = \left(\frac{k}{a} \right)^2 - \frac{4h}{a}$$

So, the required locus is $y^2 = 3(x + a)^2 + 4ax$.

Hence, (b) is the correct answer.

Ex 7. A line L passing through the focus of the parabola $y^2 = 4(x - 1)$ intersects the parabola in two distinct points. If m is the slope of the line L , then

- (a) $-1 < m < 1$ (b) $m < -1$ or $m > 1$
(c) $m \in \mathbb{R}$ (d) None of these

Sol. Let $y = Y$, $x - 1 = X$. Then, the equation is $Y^2 = 4X$.

So, the focus = $(1, 0)$

$$X, Y = (2, 0)$$

Any line through the focus is $y = m(x - 2)$.

Solving this with $y^2 = 4(x - 1)$, we get

$$m^2(x - 2)^2 = 4(x - 1)$$

$$\Rightarrow m^2 x^2 - 4(m^2 + 1)x + 4(m^2 + 1) = 0$$

If $m \neq 0$, then $D = 16(m^2 + 1)^2 - 16m^2(m^2 + 1)$

$$= 16(m^2 + 1) > 0, \text{ for all } m$$

But, if $m = 0$, then x does not have two real distinct values.

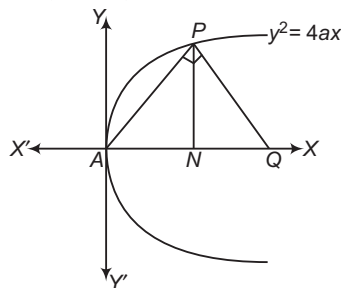
So, $m \in \mathbb{R}$, except $m = 0$.

Hence, (d) is the correct answer.

Ex 8. Vertex A of a parabola $y^2 = 4ax$ is joined to any point P on it and line PQ is drawn at right angle to AP to meet the axis at Q . Then, the projection of PQ on the axis is always equal to

- (a) $3a$ (b) $2a$
(c) $3a$ (d) $4a$

Sol. Let $P \equiv (at^2, 2at)$



Equation of the line PQ is

$$y - 2at = -\frac{t}{2}(x - at^2)$$

On putting $y = 0$, we get

$$x = 4a + at^2$$

So, coordinates of Q and N are $(4a + at^2, 0)$ and $(at^2, 0)$, respectively.

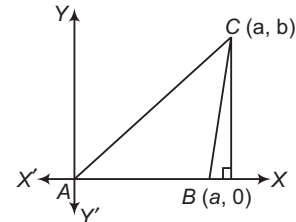
$\therefore$ Length of projection = $4a + at^2 - at^2 = 4a$

Hence, (d) is the correct answer.

Ex 9. If on a given base, triangle is described such that the sum of tangents of the base angles is constant (k), then the locus of third vertex is

- (a) a hyperbola (b) an ellipse
(c) a circle (d) a parabola

Sol. Let AB be the given base. Then,
 $\tan A + \tan B = k$



$$\Rightarrow \tan A = \frac{\beta}{\alpha}, \tan B = \frac{\beta}{\alpha - a}$$

$$\text{Now, } \frac{\beta}{\alpha} - \frac{\beta}{\alpha - a} = k$$

$$\Rightarrow -a\beta = k(\alpha^2 - a\alpha)$$

$$\Rightarrow x^2 = -\frac{a}{k}y + ax$$

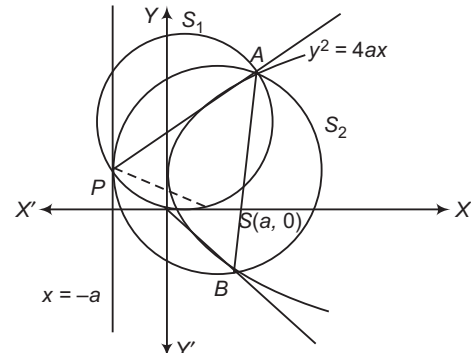
which is the equation of a parabola.

Hence, (d) is the correct answer.

Ex 10. From a point $A(t)$ on the parabola $y^2 = 4ax$, a focal chord and a tangent are drawn. Two circles are drawn in which one circle is drawn taking focal chord AB as diameter and other is drawn by taking intercept of tangent between point A and point P on the directrix as diameter. Then, the common chord of the circles is

- (a) line joining focus and P
(b) line joining focus and A
(c) tangent to the parabola at point A
(d) None of the above

Sol. Circle S_2 , taking focal chord AB as diameter will touch directrix at point P and circle S_1 , taking AP as diameter will pass through focus S (since, AP subtends angle 90° at focus of parabola).



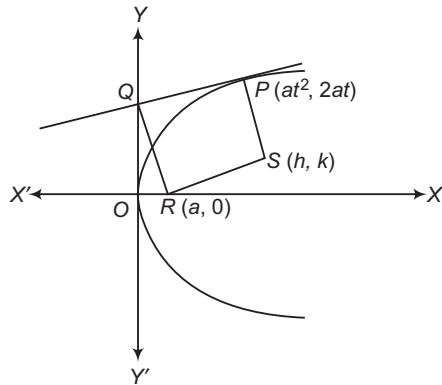
So, common chord of given circles is line AP (which is intercept of tangent at point A between point A and directrix).

Hence, (c) is the correct answer.

Ex 11. A tangent is drawn to the parabola $y^2 = 4ax$ at P such that it cuts Y -axis at Q . A line perpendicular to this tangent is drawn through Q which cuts the axis of the parabola at R . If the rectangle $PQRS$ is completed, then the locus of S is

- (a) $y^4 = (x+a)[a(x-a)^2 - y^2(x^2 - 2a)]$
 (b) $y^4 = (x-a)[a(x-a)^2 + y^2(x^2 - 2a)]$
 (c) $y^4 = (x-a)[a(x-a)^2 - y^2(x^2 + 2a)]$
 (d) None of the above

Sol. From the property of parabola, R is focus.



$$\frac{2at - k}{at^2 - h} \times \frac{k - 0}{h - a} = -1$$

and $\frac{k}{h - a} = \frac{1}{t}$
 $\Rightarrow t = \frac{h - a}{k}$

So, required locus is

$$\left[2a \left(\frac{h - a}{k} \right) - k \right] k = (a - h) \left[a \left(\frac{h - a}{k} \right)^2 - h \right]$$

$$\Rightarrow y^4 = (x - a)[a(x - a)^2 - y^2(x + 2a)]$$

Hence, (c) is the correct answer.

Ex 12. A point $P(t^2, 2t)$ lies on the parabola $y^2 = 4x$, where FP is produced to B , where F is focus. If $PB = f(t)$ and point B always lies on the line $y - x = 2$, then $f(t)$ is equal to

- (a) $\frac{(t^2 - 2t + 2)(1 + t^2)}{-t^2 + 2t + 1}$
 (b) $\frac{(t^2 - 2t + 2)(1 + t^2)}{t^2 + 2t + 1}$
 (c) $\frac{(t^2 - 2t + 2)(1 - t^2)}{-t^2 + 2t + 1}$
 (d) None of the above

Sol. Let θ be the angle in which the line FP makes with positive direction of X -axis, then

$$\tan \theta = \frac{2t}{t^2 - 1}$$

$$\sin \theta = \frac{2t}{1 + t^2} \text{ and } \cos \theta = \frac{t^2 - 1}{1 + t^2}$$

$$B \equiv \left\{ 1 + [1 + t^2 + f(t)] \frac{t^2 - 1}{1 + t^2}, [1 + t^2 + f(t)] \frac{2t}{1 + t^2} \right\}$$

$\therefore B$ lies on $y - x = 2$.

$$\Rightarrow [1 + t^2 + f(t)] \frac{2t}{1 + t^2} = 1 + [1 + t^2 + f(t)] \frac{t^2 - 1}{1 + t^2} + 2$$

$$\Rightarrow f(t) \left(\frac{2t - t^2 + 1}{1 + t^2} \right) = 3 + t^2 - 1 - 2t$$

$$\Rightarrow f(t) = \frac{(t^2 - 2t + 2)(1 + t^2)}{-t^2 + 2t + 1}$$

Hence, (a) is the correct answer.

Ex 13. Two equal parabolas have the same focus and their axes are at right angles, a normal to one is perpendicular to a normal to the other. Then, the locus of the point of intersection of these normal is

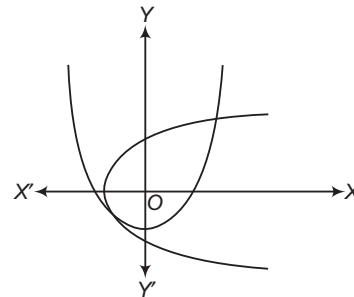
- (a) an ellipse (b) a circle
 (c) another parabola (d) a hyperbola

Sol. Taking the common focus as origin and axis of the parabola as axis of the coordinate, the equation of the parabola may be written as

$$y^2 = 4a(x + a) \quad \dots(i)$$

and $x^2 = 4a(y + a) \quad \dots(ii)$

where, $4a$ is the length of latusrectum of each parabola.



Equation of any normal to parabola (i) is

$$y = m(x + a) - 2am - am^3 \quad \dots(iii)$$

Equation of any normal to parabola (ii) is

$$x = m'(y + a) - 2am' - am'^3 \quad \dots(iv)$$

But the two normals are at right angles, then

$$m \times \frac{1}{m'} = -1$$

$$\Rightarrow m' = -m$$

On substituting $m' = -m$ in Eq. (iv), we get

$$x = -m(y + a) + 2am + am^3 \quad \dots(v)$$

The locus of the point of intersection of the normals can be obtained by the elimination of m from Eqs. (iii) and (v). So, on adding Eqs. (iii) and (v) after taking the terms to RHS in both cases, we get

$$0 = mx - y - x - my$$

$$\Rightarrow x + y = m(x - y) \Rightarrow m = \frac{x + y}{x - y}$$

On putting the value of m in Eq. (iii), we get

$$\begin{aligned} y &= \frac{x + y}{x - y}(x + a) - 2a \frac{x + y}{x - y} - a \left(\frac{x + y}{x - y} \right)^3 \\ \Rightarrow y(x - y)^3 &= (x + y) [(x + a)(x - y)^2 - 2a(x - y)^2 - a(x + y)^2] \\ &= (x + y) [(x - y)^2(x + a - 2a) - a(x + y)^2] \\ &= (x + y) [x(x - y)^2 - 2a(x^2 + y^2)] \\ \Rightarrow 2a(x^2 + y^2)(x + y) &= (x + y)x(x - y)^2 - y(x - y)^3 \\ \Rightarrow 2a(x^2 + y^2)(x + y) &= (x - y)^2 [x^2 + xy - xy + y^2] \\ \Rightarrow 2a(x + y) &= (x - y)^2 \end{aligned}$$

which is the required locus of a parabola.

Hence, (c) is the correct answer.

Ex 14. PN is ordinate of the parabola, a straight line is drawn parallel to the axis which bisect PN and meets the curve at Q , if NQ meets the tangent at the vertex at a point T , then AT is equal to (where, A is the vertex)

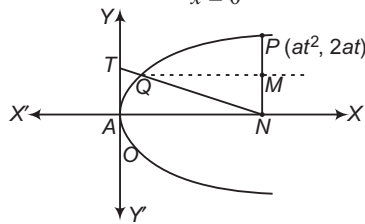
- (a) $\frac{PN}{3\sqrt{2}}$ (b) $\frac{2\sqrt{2}}{3}PN$ (c) $\frac{3}{2}NP$ (d) $\frac{2}{3}PN$

Sol. Let the equation of the parabola be

$$y^2 = 4ax \quad \dots(i)$$

Tangent at the vertex is Y -axis or

$$x = 0 \quad \dots(ii)$$



Again, let P be any point $(at^2, 2at)$ on the parabola.

So, $PN = 2at$ and coordinates of N are $(at^2, 0)$.

If M is the mid-point of PN , then $NM = \left(\frac{1}{2}\right)PN = at$.

As MQ is parallel to X -axis, its equation is

$$y = at \quad \dots(iii)$$

On putting the value of y from Eq. (iii) in Eq. (i), we get

$$a^2t^2 = 4ax \Rightarrow x = \frac{1}{4}at^2$$

So, the coordinates of Q are $\left(\frac{1}{4}at^2, at\right)$.

Equation of NQ will be

$$(y - 0) = \frac{at - 0}{\frac{1}{4}at^2 - at^2}(x - at^2)$$

$$\Rightarrow y = \frac{-4}{3t}(x - at^2) \quad \dots(iv)$$

If NQ meets the tangent at the vertex i.e. $x = 0$ at T , then on putting $x = 0$ in Eq. (iv), we get

$$y = \frac{4at}{3}$$

So, the coordinates of T are $\left(0, \frac{4at}{3}\right)$.

$$\text{or } AT = \frac{4at}{3}$$

$$\text{Clearly, } AT = \frac{2}{3} \times PN$$

Hence, (d) is the correct answer.

Ex 15. The sides of a triangle touch $y^2 - 4ax = 0$ and two of its angular points are on $y^2 - 4b(x + c) = 0$. Then, the locus of the third angular point is equal to

$$(a) y^2 = \left(\frac{2b}{a} + 1\right)^2(ax + 4bc)$$

$$(b) y^2 = 4\left(\frac{2b}{a} - 1\right)^2(ax + 4bc)$$

$$(c) y^2 = 4\left(\frac{2b}{a} + 1\right)(ax + 4bc)$$

(d) None of the above

Sol. Let the three tangents be

$$yt_1 - x - at_1^2 = 0 \quad \dots(i)$$

$$yt_2 - x - at_2^2 = 0 \quad \dots(ii)$$

$$\text{and } yt_3 - x - at_3^2 = 0 \quad \dots(iii)$$

Then, the three angular points are

$$[at_2t_3, a(t_2 + t_3)], [at_3t_1, a(t_3 + t_1)], [at_1t_2, a(t_1 + t_2)]$$

Let the last two be on the second parabola, then

$$a^2(t_3 + t_1)^2 - 4bat_3t_1 - 4bc = 0 \quad \dots(iv)$$

$$\text{and } a^2(t_1 + t_2)^2 - 4bat_1t_2 - 4bc = 0 \quad \dots(v)$$

On subtracting Eq. (v) from Eq. (iv), we get

$$a(t_2 + t_3) = (4b - 2a)t_1 \quad \dots(vi)$$

From Eqs. (v) and (vi),

$$a^2t_2t_3 = a^2t_1^2 - 4bc = \frac{a^4}{4(2b - a)^2}(t_2 + t_3)^2 - 4bc \dots(vii)$$

But for the third angular point, we have $x = at_2t_3$

$$\text{and } y = a(t_2 + t_3) \quad \dots(viii)$$

Hence, the required locus of the parabola is

$$ax = \frac{a^2}{4(2b - a)^2}y^2 - 4bc$$

From Eqs. (vii) and (viii), we get

$$y^2 = 4\left(\frac{2b}{a} - 1\right)^2(ax + 4bc)$$

Hence, (b) is the correct answer.

Ex 16. The parabola is $y^2 = \lambda x$ and $25[(x - 3)^2 + (y + 2)^2] = (3x - 4y - 2)^2$ are equal, if λ is equal to

(a) 1

(b) 2

(c) 3

(d) 6

Sol. Let us recall that two parabolas are equal, if the length of their latusrectum are equal.

Length of the latusrectum of $y^2 = \lambda x$ is λ .

$\therefore$ Equation of the second parabola is

$$25 \{(x-3)^2 + (y+2)^2\} = (3x-4y-2)^2$$

$$\Rightarrow \sqrt{(x-3)^2 + (y+2)^2} = \frac{|3x-4y-2|}{\sqrt{3^2+4^2}}$$

Clearly, it represents a parabola having focus at $(3, -2)$ and equation of the directrix as $3x - 4y - 2 = 0$

$\therefore$ Length of latusrectum

$$= 2(\text{Distance between focus and directrix})$$

$$= 2 \left| \frac{3 \times 3 - 4 \times (-2) - 2}{\sqrt{3^2 + (-4)^2}} \right| = 6$$

Thus, the two parabolas are equal, if $\lambda = 6$.

Hence, (d) is the correct answer.

Ex 17. Set of values of h for which the number of distinct common normals of $(x-2)^2 = 4(y-3)$ and $x^2 + y^2 - 2x - hy - c = 0$ ($c > 0$) is 3, is

- (a) $(2, \infty)$ (b) $(4, \infty)$
(c) $(2, 4)$ (d) $(10, \infty)$

Sol. The equation of any normal to the parabola $(x-2)^2 = 4(y-3)$ is

$$x - 2 = m(y - 3) - 2m - m^3$$

If it passes through $(1, h/2)$, then

$$1 - 2 = m \left(\frac{h}{2} - 3 \right) - 2m - m^3$$

$$\Rightarrow 2m^3 + m(10 - h) - 2 = 0$$

This equation will give three distinct values of m , if $f'(m) = 0$ has two distinct roots, where

$$f(m) = 2m^3 + m(10 - h) - 2$$

$$\text{Now, } f'(m) = 6m^2 + (10 - h)$$

$$\Rightarrow f'(m) = 0$$

$$\Rightarrow m = \pm \sqrt{\frac{h-10}{6}}$$

The values of m are real and distinct, if $h > 10$
i.e. $h \in (10, \infty)$.

Hence, (d) is the correct answer.

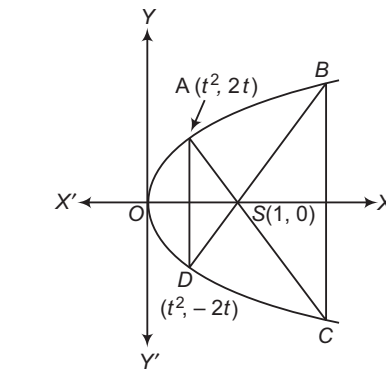
Ex 18. Area of the trapezium whose vertices lie on the parabola $y^2 = 4x$ and its diagonals pass through $(1, 0)$ and having length $\frac{25}{4}$ units each, is

- (a) $\frac{75}{4}$ sq units
(b) $\frac{625}{16}$ sq units
(c) $\frac{25}{4}$ sq units
(d) $\frac{25}{8}$ sq units

Sol. We have,

$AS = \text{Focal distance of } A$

$$AS = 1 + t^2$$



$$\therefore CS = AC - AS = \frac{25}{4} - (1 + t^2)$$

If PQ is a focal chord of parabola $y^2 = 4ax$, then

$$\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{a}$$

$$\therefore \frac{1}{AS} + \frac{1}{CS} = \frac{1}{a}$$

$$\Rightarrow \frac{1}{1+t^2} + \frac{1}{\frac{25}{4} - (1+t^2)} = 1$$

$$\Rightarrow \frac{1}{1+t^2} + \frac{4}{25-4(1+t^2)} = 1$$

$$\Rightarrow 25(1+t^2) - 4(1+t^2)^2 = 25$$

$$\Rightarrow 4(1+t^2)^2 - 25(1+t^2) + 25 = 0$$

$$\Rightarrow \{(1+t^2) - 5\} \{4(1+t^2) - 5\} = 0$$

$$\Rightarrow 1+t^2 = 5, \frac{5}{4} \Rightarrow t^2 = 4, \frac{1}{4}$$

$$\Rightarrow t = \pm 2, \pm \frac{1}{2}$$

Thus, the coordinates of A, B, C and D are

$A \equiv (1/4, 1), B \equiv (4, 4), C \equiv (4, -4)$ and $D \equiv (1/4, -1)$

$$\therefore AD = 2, BC = 8$$

$$\text{and distance between } AD \text{ and } BC = \frac{15}{4}$$

$$\therefore \text{Area of trapezium } ABCD = \frac{1}{2}(2+8) \times \frac{15}{4}$$

$$= \frac{75}{4} \text{ sq units}$$

Hence, (a) is the correct answer.

Ex 19. If (h, k) is a point on the axis of the parabola $2(x-1)^2 + 2(y-1)^2 = (x+y+2)^2$ from where three distinct normals may be drawn, then
(a) $h > 2$ (b) $h < 4$ (c) $h > 8$ (d) $h = 8$

Sol. We have,

$$2(x-1)^2 + 2(y-1)^2 = (x+y+2)^2$$

$$\Rightarrow \sqrt{(x-1)^2 + (y-1)^2} = \frac{|x+y+2|}{\sqrt{1+1}}$$

Clearly, it represents a parabola having its focus at $(1, 1)$ and directrix is $x + y + 2 = 0$.

The equation of the axis is $y - 1 = 1(x - 1)$ i.e. $y = x$.
Semi-latusrectum = Length of perpendicular from (1, 1) on the directrix

$$= \left| \frac{1 + 1 + 2}{\sqrt{1 + 1}} \right| = 2\sqrt{2}$$

The coordinates of the vertex are (0, 0).

So, the equation of the axis in parametric form is

$$\frac{x - 0}{\cos \pi/4} = \frac{y - 0}{\sin \pi/4} \quad \dots(i)$$

We know that three distinct normals can be drawn from a point $(h, 0)$ on the axis of the parabola $y^2 = 4ax$, if $h > 2a$ (= Semi-latusrectum).

The coordinates of a point on Eq. (i) at a distance $2\sqrt{2}$ from the vertex are given by

$$\frac{x}{\cos \pi/4} = \frac{y}{\sin \pi/4} = 2\sqrt{2}$$

$$\Rightarrow x = 2, y = 2$$

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 20. Consider a circle with its centre lying on the focus of the parabola $y^2 = 2px$ such that it touches the directrix of the parabola. Then, a point of intersection of the circle and the parabola is

- (a) $\left(\frac{p}{2}, p\right)$ (b) $\left(\frac{p}{2}, -p\right)$
(c) $\left(-\frac{p}{2}, p\right)$ (d) $\left(-\frac{p}{2}, -p\right)$

Sol. The focus of the parabola $y^2 = 2px$ is at $\left(\frac{p}{2}, 0\right)$ and its directrix is $x = -\frac{p}{2}$. Since, the circle touches the directrix.

$\therefore$ Radius = Distance between the point $\left(\frac{p}{2}, 0\right)$ and the

$$\text{line } x = -\frac{p}{2} = p$$

So, the equation of the circle is

$$\left(x - \frac{p}{2}\right)^2 + (y - 0)^2 = p^2$$

$$\Rightarrow x^2 + y^2 - px - \frac{3p^2}{4} = 0$$

On solving this equation with $y^2 = 2px$, we obtain

$$x^2 + px - \frac{3p^2}{4} = 0 \Rightarrow (2x - p)(2x + 3p) = 0$$

$$\Rightarrow x = \frac{p}{2} \text{ or } x = -\frac{3p}{2}$$

On putting $x = \frac{p}{2}$ in $y^2 = 2px$, we obtain $y = \pm p$

For $x = -\frac{3p}{2}$, y is imaginary. Hence, the circle and the parabola intersect at $\left(\frac{p}{2}, p\right)$ and $\left(\frac{p}{2}, -p\right)$.

Hence, (a) and (b) are the correct answers.

Ex 21. A variable circle is described to pass through the point (1, 0) and tangent to the curve $y = \tan(\tan^{-1} x)$. The locus of the centre of the circle is a parabola whose

- (a) length of the latusrectum is $2\sqrt{2}$
(b) axis of symmetry has the equation $x + y = 1$
(c) vertex has the coordinates $(3/4, 1/4)$
(d) None of the above

Sol. Let (h, k) be the centre, then

$$\left| \frac{h - k}{\sqrt{2}} \right| = \sqrt{(h - 1)^2 + k^2}$$

$$\therefore (h - k)^2 = 2(h^2 + k^2 - 2h + 1)$$

$$\Rightarrow h^2 + k^2 + 2hk - 4h + 2 = 0$$

$$\Rightarrow (h + k)^2 = 4h - 2$$

$$\Rightarrow (h + k)^2 = 2h + 2k + 2h - 2k - 2$$

$$\Rightarrow (h + k)^2 - 2(h + k) + 1 = 2\left(h - k - \frac{1}{2}\right)$$

$$\therefore \text{Locus is } (x + y - 1)^2 = 2(x - y - 1/2)$$

$$\left(\frac{x + y - 1}{\sqrt{2}}\right)^2 = \frac{2}{\sqrt{2}} \left(\frac{x - y - \frac{1}{2}}{\sqrt{2}}\right)$$

$\therefore$ Axis of symmetry is $x + y = 1$

Length of latusrectum = $\sqrt{2}$

Vertex is the point of intersection of $x + y = 1$ and $x - y = 1/2$ i.e. $(3/4, 1/4)$

Hence, (b) and (c) are the correct answers.

Ex 22. The locus of the mid-point of the focal radii of a variable point moving on the parabola $y^2 = 4ax$ is a parabola whose

(a) latusrectum is half the latusrectum of the original parabola

(b) vertex is $\left(\frac{a}{2}, 0\right)$

(c) directrix is Y-axis

(d) focus has the coordinates $(a, 0)$

Sol. Any point on the parabola is $(at^2, 2at)$.

$$\therefore \text{Mid-point of } (a, 0) \text{ and } (at^2, 2at) \text{ is } \left(\frac{a + at^2}{2}, at\right)$$

$$\therefore \text{Locus is given by } x = \frac{a + at^2}{2}$$

$$\Rightarrow 2x = a \left(1 + \frac{y^2}{a^2}\right) = a + \frac{y^2}{a} \quad \left[\because y = at \Rightarrow t = \frac{y}{a}\right]$$

$$\Rightarrow 2ax = a^2 + y^2 \Rightarrow y^2 = 2a(x - a/2)$$

Its a parabola with vertex at $\left(\frac{a}{2}, 0\right)$, latusrectum = $2a$

Directrix is $x - a/2 = a/2$

$$\Rightarrow x = a \text{ i.e. } (a, 0)$$

Hence, (a), (b), (c) and (d) are the correct answers.

- Ex 23.** The straight line $kx + y = 4$ touches the parabola $y = x - x^2$, if
 (a) $k = -5$
 (b) $k = 0$
 (c) $k = 3$
 (d) k takes any real value

Sol. $x^2 - x + y = 0$
 $\Rightarrow x^2 - x + 4 - kx = 0$
 $\Rightarrow x^2 - (1+k)x + 4 = 0$
 $\therefore (1+k)^2 - 16 = 0$
 $\Rightarrow k+1 = \pm 4 \Rightarrow k = 3, -5$
 Hence, (a) and (c) are the correct answers.

Type 3. Assertion and Reason

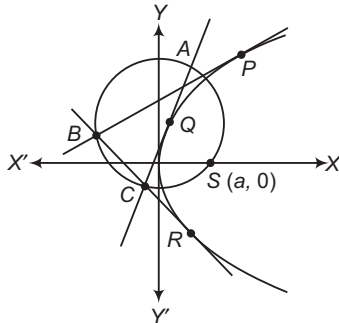
Directions (Ex. Nos. 24-27) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

Ex 24. Statement I Circumcircle of a triangle formed by the lines $x=0$, $x+y+1=0$ and $x-y+1=0$ also passes through the point $(1, 0)$.

Statement II Circumcircle of a triangle formed by three tangents of parabola passes through its focus.

Sol. Statement I Consider a parabola $y^2 = 4ax$
 Clearly, $x = 0$ is tangent to the parabola at $(0, 0)$.
 $y^2 = 4(-y-1) \Rightarrow (y+2)^2 = 0$
 $\therefore x+y+1=0$ is a tangent to the parabola
 Again, $y^2 = 4(y-1) \Rightarrow y^2 - 4y + 4 = 0$
 $\Rightarrow (y-2)^2 = 0$
 So, $x-y+1=0$ is a tangent to the parabola and $(1, 0)$ is the focus.
 $\therefore$ Statement I is true.
 Consider a parabola $y^2 = 4ax$.
 Let $P(t_1)$, $Q(t_2)$ and $R(t_3)$ be three points on it.



Tangents are drawn at these points which intersect

$$\begin{aligned} A &\equiv [at_1t_2, a(t_1+t_2)] \\ B &\equiv [at_1t_3, a(t_1+t_3)] \\ C &\equiv [at_2t_3, a(t_2+t_3)] \end{aligned}$$

Let $\angle SAC = \alpha$ and $\angle SBC = \beta$
 $\Rightarrow \tan \alpha = \frac{\frac{1}{t_2} - \frac{t_1+t_2}{t_1t_2-1}}{1 + \frac{1}{t_2} \left(\frac{t_1+t_2}{t_1t_2-1} \right)} = \left| \frac{1}{t_1} \right|$
 Similarly, $\tan \beta = \left| \frac{1}{t_1} \right| \Rightarrow \alpha = \beta$ or $\alpha + \beta = \pi$
 So, A, B, C and S are concyclic.
 Hence, (a) is the correct answer.

Ex 25. Statement I Length of focal chord of a parabola $y^2 = 8x$ making an angle of 60° with X -axis is 32.

Statement II Length of focal chord of a parabola $y^2 = 4ax$ making an angle α with X -axis is $4a \operatorname{cosec}^2 \alpha$.

Sol. Let AB be a focal chord.
 $\therefore$ Slope of $AB = \frac{2t}{t^2-1} = \tan \alpha$

 $\Rightarrow \tan \frac{\alpha}{2} = \frac{1}{t} \Rightarrow t = \cot \frac{\alpha}{2}$

Length of $AB = a \left(t + \frac{1}{t} \right)^2 = 4a \operatorname{cosec}^2 \alpha$

So, Statement II is true but Statement I is false.
 Hence, (d) is the correct answer.

Ex 26. Statement I Area of triangle formed by pair of tangents drawn from a point $(12, 8)$ to the parabola $y^2 = 4x$ and their corresponding chord of contact is 32 sq units.

Statement II If from a point $P(x_1, y_1)$, tangents are drawn to a parabola $y^2 = 4ax$, then area of triangle formed by these tangents and their corresponding chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{4|a|}$ sq units.

Sol. Statement II Area of triangle formed by these tangents and their corresponding chord of contact is $\frac{(y_1^2 - 4ax_1)^{3/2}}{2|a|}$.

$\therefore$ Statement II is false.

Statement I $x_1 = 12, y_1 = 8$

$$\begin{aligned} \text{Area} &= \frac{(y_1^2 - 4ax_1)^{3/2}}{2} \\ &= \frac{(64 - 48)^{3/2}}{2} \\ &= 32 \end{aligned}$$

Statement I is true.

Hence, (c) is the correct answer.

Ex 27. Statement I The perpendicular bisector of the line segment joining the points $(-a, 2at)$ and $(a, 0)$ is tangent to the parabola $y^2 = 4ax$, where $t \in \mathbb{R}$.

Statement II Number of parabolas with a given point as vertex and length of latusrectum equal to 4, is 2.

Sol. Image of $(a, 0)$ with respect to tangent $yt = x + at^2$ is $(-a, 2at)$.

$\therefore$ Perpendicular bisector of $(a, 0)$ and $(-a, 2at)$ is the tangent line $yt = x + at^2$ to the parabola.

$\therefore$ Statement I is true.

Statement II Infinitely many parabolas are possible.

$\therefore$ Statement II is false.

Hence, (c) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 28-30)

$$\text{If } a = \lim_{x \rightarrow \pi/2} \sqrt{\frac{\tan x - \sin[\tan^{-1}(\tan x)]}{\tan x + \cos^2(\tan x)}} \text{ and } r = \frac{1}{\lambda}$$

where, λ is the least integral value for which three real normals can be drawn to the parabola $y^2 = 8x$ from $(\lambda, 0)$.

Ex 28. Sum of infinite GP whose first term is a and common ratio is r , is

- (a) $\frac{2}{3}$ (b) $\frac{1}{5}$
(c) $\frac{5}{4}$ (d) $\frac{4}{3}$

Sol. $\therefore \lambda = 4$, so $r = \frac{1}{4}$

$$\therefore a = \lim_{x \rightarrow \frac{\pi^-}{2}} \sqrt{\frac{\tan x - \sin[\tan^{-1}(\tan x)]}{\tan x + \cos^2(\tan x)}}$$

$$\Rightarrow a = \lim_{x \rightarrow \frac{\pi^-}{2}} \sqrt{\frac{1 - \cos x}{\cos^2(\tan x)}} = 1$$

$$\Rightarrow a = 1$$

$$\therefore \text{Sum} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

Hence, (d) is the correct answer.

Ex 29. Radius of the circle whose centre is (a, λ) which touches the line $3x + 4y = 4$, is

- (a) 5
(b) 3
(c) 4
(d) 2

Sol. The point (a, λ) is $(1, 4)$.

Its distance from the line $3x + 4y - 4 = 0$ is

$$\left| \frac{3 + 16 - 4}{5} \right| = 3$$

Hence, (b) is the correct answer.

Ex 30. The value of $\lim_{x \rightarrow a\lambda} [[x - 5] + [4 - x]]$, where $[]$

denotes greatest integer function, is

- (a) -2 (b) -1
(c) 2 (d) Does not exist

Sol. $\lim_{x \rightarrow 4^+} [[x - 5] + [4 - x]] = -2$

When $x \rightarrow 4^+$, then $[x - 5] = -1$ and $[4 - x] = -1$

$$\lim_{x \rightarrow 4^-} [[x - 5] + [4 - x]] = -2$$

When $x \rightarrow 4^-$ then $[x - 5] = -2$ and $[4 - x] = 0$

Hence, (a) is the correct answer.

Passage II (Ex. Nos. 31-33) The straight line $y + tx = 2at + at^3$ is a normal to the parabola $y^2 = 4ax$ for all real values of t and foot of the normal is given by $(at^2, 2at)$.

Ex 31. Number of real normals that can be drawn from the point $(4, 0)$ to the parabola $y^2 = 16x$, is

- (a) 1 (b) 2
(c) 3 (d) None of these

Sol. $y + tx = 2at + at^3$

Since, $a = 4$ and normal passes through $(4, 0)$.

$$\therefore 0 + 4t = 8t + 4t^3$$

$$\Rightarrow 4t^3 + 4t = 0 \Rightarrow 4t(t^2 + 1) = 0$$

$\Rightarrow$ Only one real value of t i.e. $t = 0$

So, number of real normals is 1.

Hence, (a) is the correct answer.

Ex 32. If the ordinates of points P and Q on the parabola $y^2 = 12x$ are in the ratio 1:2, then the locus of the point of intersection of normals to the parabola at P and Q , is

- (a) $343y^2 = 12(x - 6)^3$
(b) $343y^2 = 12(x + 6)^3$
(c) $343y^2 = -12(x - 6)^3$
(d) None of the above

Sol. Let (h, k) be the point of intersection of the normals.

$$\therefore k + th = 2at + at^3 \quad \dots(i)$$

$$\Rightarrow at^3 + (2a - h)t - k = 0$$

$$\Rightarrow t_1 t_2 t_3 = \frac{k}{a}, t_1 + t_2 + t_3 = 0$$

$$\text{Also, } \frac{2at_1}{2at_2} = \frac{1}{2}$$

$$\Rightarrow 2t_1 = t_2, t_3 = -3t_1$$

$$\therefore (t_1)(2t_1)(-3t_1) = \frac{k}{a}$$

$$\Rightarrow t_1^3 = -\frac{k}{6a}$$

Since, t_1 satisfies Eq. (i).

$$\therefore (2a - h)t_1 = (k - at_1^3)$$

$$\Rightarrow -(2a - h)t_1^3 = \left(k - \frac{k}{6}\right)^3$$

$$\therefore -(6 - x)^3 = \frac{343}{12}y^2$$

$$\Rightarrow 343y^2 = -12(6 - x)^3$$

$$= 12(x - 6)^3$$

Hence, (a) is the correct answer.

Ex 33. A point through which three normals of parabola $y^2 = 4ax$ are passing, two of which are making angles α and β with X -axis, where $\tan \alpha \tan \beta = 2$, lies on the curve

(a) $y(y^2 - 2ax) = 0$

(b) $y(y^2 + 2ax) = 0$

(c) $y(y^2 - 4ax) = 0$

(d) $y(y^2 - ax) = 0$

Sol. $t_1 t_2 t_3 = \frac{k}{a}$

$$\Rightarrow t_1 t_2 = (-t_1)(-t_2) = \tan \alpha \tan \beta = 2$$

$$\Rightarrow t_3 = \frac{k}{2a}$$

Since, t_3 satisfy the equation $at^2 + (2a - h)t - k = 0$

$$a\left(\frac{k^3}{8a^3}\right) + (2a - h)\frac{k}{2a} - k = 0$$

$$\Rightarrow y[y^2 + 4a(2a - x) - 8a^2] = 0$$

$$\Rightarrow y(y^2 - 4ax) = 0$$

Hence, (c) is the correct answer.

Passage III (Ex. Nos. 34-36) $y = f(x)$ is parabola of the form $y = x^2 + ax + 1$, its tangent at the point of intersection of Y -axis and parabola also touches the circle $x^2 + y^2 = r^2$. It is known that no point of the parabola is below X -axis.

Ex 34. The radius of circle when a attains its maximum value, is

(a) $\frac{1}{\sqrt{10}}$ (b) $\frac{1}{\sqrt{5}}$ (c) 1 (d) $\sqrt{5}$

Sol. Since, no point of the parabola is below X -axis.

$$\therefore a^2 - 4 \leq 0 \quad [\because D < 0]$$

So, maximum value of a is 2.

Now, equation of the parabola, when $a = 2$ is

$$y = x^2 + 2x + 1$$

It intersects Y -axis at $(0, 1)$.

Equation of the tangent at $(0, 1)$ is $y = 2x + 1$.

Since, $y = 2x + 1$ touches the circle $x^2 + y^2 = r^2$.

$$\therefore r = \frac{1}{\sqrt{5}}$$

Hence, (b) is the correct answer.

Ex 35. The slope of the tangent when radius of the circle is maximum, is

(a) 0 (b) 1

(c) -1 (d) not defined

Sol. Equation of the tangent at $(0, 1)$ to the parabola

$$y = x^2 + ax + 1 \text{ is } \frac{y+1}{2} = \frac{a}{2}(x+0) + 1$$

i.e. $ax - y + 1 = 0$

$$\therefore r = \frac{1}{\sqrt{a^2 + 1}}$$

Radius is maximum, when $a = 0$.

$\therefore$ Equation of the tangent is $y = 1$ and slope of the tangent is 0.

Hence, (a) is the correct answer.

Ex 36. The minimum area bounded by the tangent and the coordinate axes is

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) 1

Sol. Equation of tangent is $y = ax + 1$ whose intercepts are $-\frac{1}{a}$ and 1.

$\therefore$ Area of the triangle bounded by tangent and the axes

$$= \frac{1}{2} \left| -\frac{1}{a} \cdot 1 \right|$$

$$= \frac{1}{2|a|}$$

It is minimum, when $a = 2$.

$$\therefore \text{Minimum area} = \frac{1}{4}$$

Hence, (a) is the correct answer.

14

Type 5. Match the Columns

Ex 37. Match the statements of Column I with values of Column II.

Column I	Column II
A. Area of a triangle formed by the tangents drawn from a point $(-2, 2)$ to the parabola $y^2 = 4(x + y)$ and their corresponding chord of contact is	p. 8
B. Length of the latusrectum of the conic $25\{(x-2)^2 + (y-3)^2\} = (3x + 4y - 6)^2$ is	q. $4\sqrt{3}$
C. If focal distance of a point on the parabola $y = x^2 - 4$ is $25/4$ and points are of the form $(\pm \sqrt{a}, b)$, then the value of $a + b$ is	r. 4
D. Length of side of an equilateral triangle inscribed in a parabola $y^2 - 2x - 2y - 3 = 0$ whose one angular point is vertex of the parabola is	s. $24/5$

Sol. A. Point of contact of tangent drawn from $(-2, 2)$ on $y^2 = 4(x + y)$ are $(0, 4)$ and $(0, 0)$.

$$\therefore \text{Area} = 4$$

B. The conic is a parabola having focus $(2, 3)$ and directrix $3x + 4y - 6 = 0$.

$\therefore$ Latusrectum $= 2 \times$ (Perpendicular distance of focus from the directrix)

$$= 2 \left(\frac{6 + 12 - 6}{5} \right) = \frac{24}{5}$$

$$C. y + 4 = x^2 \Rightarrow x^2 = 4 \cdot \frac{1}{4}(y + 4)$$

$$\text{Focal distance} = \frac{25}{4}$$

$$\therefore \text{Distance from directrix } y = \frac{-17}{4}$$

Now, ordinate of points of the parabola whose focal distance is $\frac{25}{4}$

$$= \frac{-17}{4} + \frac{25}{4} = 2$$

So, points are $(\pm \sqrt{6}, 2)$.

$$\Rightarrow a + b = 8$$

$$D. \text{Length of side} = 8\sqrt{3} \quad a = 8\sqrt{3} \cdot \frac{1}{2} = 4\sqrt{3}$$

$$A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$$

Ex 38. Match the statements of Column I with values of Column II.

Column I	Column II
A. If the parabola $y^2 = 4x$ and the circle having its centre at $(6, 5)$ intersects at right angle, at the point (a, a) , then one value of a is	p. 13
B. If the angle between the tangents drawn to $(y-2)^2 = 4(x+3)$ at the points, where it is intersected by the line $3x - y + 8 = 0$ is $\frac{4\pi}{p}$, then p has the value	q. 8
C. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the value of k is	r. $10\sqrt{5}$
D. Length of the normal chord of the parabola $y^2 = 8x$ at the point, where abscissa and ordinate are equal, is	s. 4

Sol. A. Since, point (a, a) lies on $y^2 = 4x$.

$$\therefore a^2 = 4a \Rightarrow a = 0, 4 \Rightarrow a = 4$$

B. The line $3x - y + 8 = 0$ passes through the focus $(-2, 2)$, so the tangents at the end points on the chord is $\frac{\pi}{2}$.

$$\therefore \frac{4\pi}{p} = \frac{\pi}{2} \Rightarrow p = 8$$

$$C. y^2 = k(x - 8/k)$$

$$\text{Equation of directrix is } x - \frac{8}{k} = -\frac{k}{4}$$

$$\Rightarrow x = \frac{8}{k} - \frac{k}{4}$$

But it is given that directrix is $x = 1$.

$$\Rightarrow \frac{8}{k} - \frac{k}{4} = 1 \Rightarrow k^2 + 4k - 32 = 0$$

$$\therefore k = 4, -8$$

D. End points of the normal chord will be $(8, 8)$ and

$$\left[2 \left(\frac{-5}{2} \right)^2, 2 \cdot 2 \left(\frac{-5}{2} \right) \right]$$

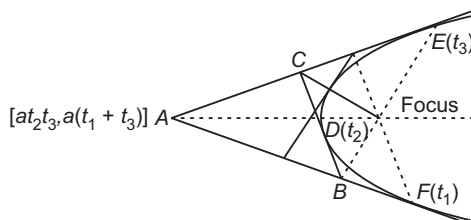
So, length of the chord will be $10\sqrt{5}$.

$$A \rightarrow s; B \rightarrow q; C \rightarrow s; D \rightarrow r$$

Type 6. Integer Answer Type Questions

Ex 39. The sides of $\triangle ABC$ are tangents to the parabola $y^2 = 4ax$. Let D, E, F be the point of contact of sides BC, CA and AB , respectively. If lines AD, BE and CF are concurrent at focus of the parabola and $\angle ABC$ of $\triangle ABC$ is $\frac{\pi}{k}$, then k equals ____.

Sol. (3) As AD passes through focus.



$$\begin{aligned} \therefore \frac{2at_2}{a(t_2^2 - 1)} &= \frac{a(t_1 + t_3)}{a(t_1 t_3 - 1)} \\ \Rightarrow 2t_1 t_2 t_3 - 2t_2 &= (t_1 + t_3)t_2^2 - (t_1 + t_3) \\ \Rightarrow t_2^3 - (t_1 + t_2 + t_3)t_2^2 - 3t_2 + 2t_1 t_2 t_3 \\ &\quad + (t_1 + t_2 + t_3) = 0 \\ \Rightarrow t_1, t_2, t_3 &\text{ are the roots of the equation} \\ t^3 - (t_1 + t_2 + t_3)t^2 - 3t + 2t_1 t_2 t_3 + (t_1 + t_2 + t_3) &= 0. \\ \Rightarrow t_1 t_2 t_3 &= -[2t_1 t_2 t_3 + (t_1 + t_2 + t_3)] \quad \dots(i) \\ \Rightarrow t^2 - (t_1 + t_2 + t_3)t^2 - 3t + \frac{(t_1 + t_2 + t_3)}{3} &= 0 \\ \Rightarrow \frac{t_1 + t_2 + t_3}{3} &= \left(\frac{3t - t^3}{1 - 3t^2} \right) \end{aligned}$$

As, $\frac{1}{t_1}, \frac{1}{t_2}, \frac{1}{t_3}$ are the slopes of the sides taking

$$\frac{1}{t_1} = \tan \theta_1, \frac{1}{t_2} = \tan \theta_2 \text{ and } \frac{1}{t_3} = \tan \theta_3$$

$$\Rightarrow \frac{3 \cot \theta - \cot^3 \theta}{1 - 3 \cot^2 \theta} = \cot \alpha$$

where, $\frac{t_1 + t_2 + t_3}{3} = \cot \alpha$ and the roots of the equations are $\theta_1, \theta_2, \theta_3$.

$$\Rightarrow \cot 3\theta = \cot \alpha$$

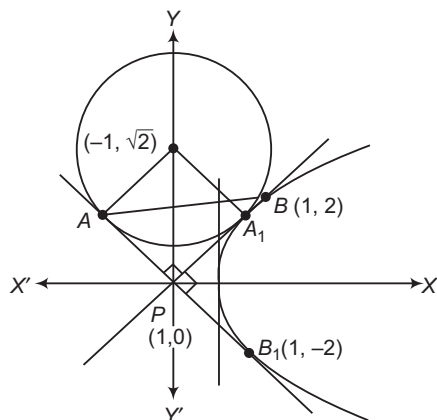
$$\Rightarrow \theta = \frac{n\pi + \alpha}{3}$$

$$\Rightarrow \theta_1 = \frac{\alpha}{3}, \theta_2 = \frac{\pi + \alpha}{3}, \theta_3 = \frac{2\pi + \alpha}{3}$$

$$\therefore \theta_2 - \theta_1 = \theta_3 - \theta_2 = \frac{\pi}{3}$$

Ex 40. If a point P on the axis of the parabola $y^2 = 4x$ is taken such that the point is at shortest distance from the circle $x^2 + y^2 + 2x - 2\sqrt{2}y + 2 = 0$ tangent are drawn to the circle and the parabola, then the area of ΔPAB is $\sqrt{a}$, where A and B are the points of contact on two different lines on circle and parabola respectively, find a .

Sol. (2) Axis of the parabola $y^2 = 4x$ is $y = 0$ and the point which is at shortest distance from the circle.



Since, $(-1, 0)$ lies on the directrix of the parabola, hence tangents to the parabola are these lines

$y = x + 1$ and $y = -x - 1$, also the tangents to the given circle. Here, ΔPAB is possible in two ways, one shown by using PAB and other by using PA_1B_1 . Length of $PA = PA_1 = 1$, also $B(1, 2)$ and $B_1(1, -2)$ as B and B_1 lies on the latusrectum of the parabola.

Here, ΔPAB and ΔPA_1B_1 are right angled triangles.

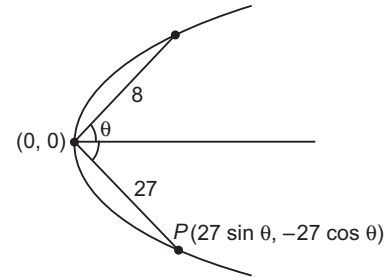
$$\begin{aligned} \text{So, area of } \Delta PAB &= \frac{1}{2} \times PA \times PB = \frac{1}{2} \times 1 \times 2\sqrt{2} \\ &= \sqrt{2} \text{ sq units} \end{aligned}$$

$$\begin{aligned} \text{Also, area of } \Delta PA_1B_1 &= \frac{1}{2} \times PA_1 \times PB \\ &= \frac{1}{2} \times 1 \times 2\sqrt{2} \\ &= \sqrt{2} \text{ sq units} \end{aligned}$$

$$\therefore \text{Area of } \Delta PAB = \sqrt{2} \text{ sq units}$$

Ex 41. 64, 27 are the lengths of any two perpendicular chords of parabola passing through the vertex of the parabola. If l is the length of latusrectum of the parabola, then write the value of $5l/16$ is _____.

Sol. (9) Let the equation of parabola be $y^2 = lx$ or $x^2 = ly$



By substituting P, Q in $y^2 = lx$ and eliminating θ , we get

$$l = \frac{144}{5} \Rightarrow 5l = 144$$

$$\therefore \frac{5l}{16} = \frac{144}{16} = 9$$

Ex 42. If tangent to the parabola $y^2 = 4x$ is normal to $x^2 = 4by$ and $|b| < \frac{1}{\sqrt{k}}$, then the numerical quantity of k should be _____.

Sol. (8) Any tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$ and any

normal to $x^2 = 4by$ is $y = m'x + 2b + \frac{b}{m'^2}$.

If these are same lines, then $m = m'$ and $\frac{1}{m} = 2b + \frac{b}{m^2}$,

$$2bm^2 - m + b = 0$$

This will give real roots, if $1 - 8b^2 > 0$

$$\Rightarrow b^2 < \frac{1}{8} \Rightarrow |b| < \frac{1}{\sqrt{8}}$$

$$\therefore k = 8$$

Target Exercises

Type 1. Only One Correct Option

- Equation of parabola having the extremities of its latusrectum as $(3, 4)$ and $(4, 3)$ is
 - $\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \left(\frac{x + y + 6}{2}\right)^2$
 - $\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{(x + y - 8)^2}{2}$
 - $\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{(x + y - 4)^2}{2}$
 - None of the above
- The length of the latusrectum of the parabola $2\{(x - a)^2 + (y - a)^2\} = (x + y)^2$ is
 - $2a$
 - $2\sqrt{2}a$
 - $4a$
 - $\sqrt{2}a$
- Double ordinate AB of the parabola $y^2 = 4ax$ subtends an angle $\frac{\pi}{2}$ at the vertex of the parabola, then tangents drawn to parabola at A and B will intersect at
 - $(-4a, 0)$
 - $(-2a, 0)$
 - $(-3a, 0)$
 - None of these
- AB is a focal chord of $x^2 - 2x + y - 2 = 0$, whose focus is S . If $AS = l_1$, then BS is equal to
 - $\frac{4l_1}{4l_1 - 1}$
 - $\frac{l_1}{4l_1 - 1}$
 - $\frac{2l_1}{4l_1 - 1}$
 - None of these
- OA and OB are two mutually perpendicular chords of $y^2 = 4ax$, O being the origin. Line AB will always pass through the point
 - $(2a, 0)$
 - $(6a, 0)$
 - $(8a, 0)$
 - $(4a, 0)$
- The length of latusrectum of the parabola, whose focus is $(a \sin 2\theta, a \cos 2\theta)$ and directrix is the line $y = a$, is
 - $|4a \cos^2 \theta|$
 - $|4a \sin^2 \theta|$
 - $|4a \cos 2\theta|$
 - $|a \cos 2\theta|$
- The point $(a, 2a)$ is an interior point of the region bounded by the parabola $y^2 = 16x$ and the double ordinate through the focus. Then, a belongs to the open interval
 - $a < 4$
 - $0 < a < 4$
 - $0 < a < 2$
 - $a > 4$
- The vertex of parabola $y^2 = 8x$ is at the centre of a circle and parabola cuts the circle at ends of its latusrectum. Then, equation of the circle is
 - $x^2 + y^2 = 4$
 - $x^2 + y^2 = 20$
 - $x^2 + y^2 = 80$
 - None of these
- AB is a chord of the parabola $y^2 = 4ax$ with vertex A . BC is drawn perpendicular to AB meeting the axis at C . The projection of BC on the axis of the parabola is
 - a
 - $2a$
 - $4a$
 - $8a$
- The circle $x^2 + y^2 + 2\lambda x = 0, \lambda \in R$, touches the parabola $y^2 = 4x$ externally. Then,
 - $\lambda > 0$
 - $\lambda < 0$
 - $\lambda > 1$
 - None of these
- Vertex of the parabola whose parametric equation is $x = t^2 - t + 1, y = t^2 + t + 1, t \in R$, is
 - $(1, 1)$
 - $(2, 2)$
 - $\left(\frac{1}{2}, \frac{1}{2}\right)$
 - $(3, 3)$
- The circle $x^2 + y^2 + 2gx + 2fy + c = 0$ cuts the parabola $x^2 = 4ay$ at point $A_i(x_i, y_i), i = 1, 2, 3, 4$, then
 - $\sum y_i = 0$
 - $\sum y_i = -4(f + 2a)$
 - $\sum x_i = -4(g + 2a)$
 - $\sum x_i = -2(g + 2a)$
- A circle is drawn to pass through the extremities of latusrectum of the parabola $y^2 = 8x$. It is given that this circle also touches the directrix of the parabola. Radius of this circle is equal to
 - 4
 - $\sqrt{21}$
 - 3
 - $\sqrt{26}$
- A circle having its centre at $(2, 3)$ is cut orthogonally by the parabola $y^2 = 4x$. The possible intersection point of these curves can be
 - $(1, 2)$ or $(3, 2\sqrt{3})$
 - $(9, 6)$ or $(2, 2\sqrt{2})$
 - $(1, 2)$ or $(4, 4)$
 - None of the above
- If the length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c , then
 - $2a^2 = bc$
 - $a^3 = b^2c$
 - $ac = b^2$
 - $b^2c = 4a^3$

16. The angle of intersection between the curves $x^2 = 4(y+1)$ and $x^2 = -4(y+1)$ is
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
 (c) 0 (d) $\frac{\pi}{2}$
17. The equation of a parabola is $y^2 = 4x$. If $P(1, 3)$ and $Q(1, 1)$ are two points in the XY -plane. Then, for the parabola,
 (a) P and Q are exterior points
 (b) P is an interior point while Q is an exterior point
 (c) P and Q are interior points
 (d) P is an exterior point while Q is an interior point
18. A double ordinate parabola $y^2 = 8px$ is of length $16p$. The angle subtended by it at the vertex of the parabola is
 (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$
 (c) $\frac{\pi}{3}$ (d) None of these
19. The parabola $y^2 = kx$ makes an intercept of length 4 on the line $x - 2y = 1$. Then, k is equal to
 (a) $\frac{\sqrt{105} - 5}{10}$ (b) $\frac{5 - \sqrt{105}}{10}$
 (c) $\frac{5 + \sqrt{105}}{10}$ (d) None of these
20. Radius of the largest circle which passes through the focus of the parabola $y^2 = 4x$ and contained in it, is
 (a) 8 (b) 4 (c) 2 (d) 5
21. The condition that the straight line $lx + my + n = 0$ touches the parabola $x^2 = 4ay$ is
 (a) $bn = am^2$ (b) $al^2 - mn = 0$
 (c) $ln = am^2$ (d) $am = ln^2$
22. The point $(-2m, m+1)$ is an interior point of the smaller region bounded by the circle $x^2 + y^2 = 4$ and parabola $y^2 = 4x$. Then, m belongs to the interval
 (a) $-5 - 2\sqrt{6} < m < 1$
 (b) $0 < m < 4$
 (c) $-1 < m < \frac{3}{5}$
 (d) $-1 < m < -5 + 2\sqrt{6}$
23. The locus of the points of trisection of the double ordinates of parabola $y^2 = 4ax$, is
 (a) $y^2 = ax$ (b) $9y^2 = 4ax$
 (c) $9y^2 = ax$ (d) $y^2 = 9ax$
24. If the tangents at P and Q on a parabola meet in T , then SP , ST and SQ are in
 (a) AP
 (b) GP
 (c) HP
 (d) None of the above
25. Let us define a region R in XY -plane as a set of points (x, y) satisfying $[x^2] = [y]$, where $[x]$ denotes greatest integer $\leq x$, then the region R defines
 (a) a parabola whose axis is horizontal
 (b) a parabola whose axis is vertical
 (c) integer point on the parabola $y = x^2$
 (d) None of the above
26. If the tangent at P on $y^2 = 4ax$ meets the tangent at the vertex in Q and S is the focus of the parabola, then $\angle SQP$ is equal to
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\frac{2\pi}{3}$
27. A water jet from a fountain reaches its maximum height of 4 m at a distance 0.5 m from the vertical passing through the point O of water outlet. The height of the jet above the horizontal OX at a distance of 0.75 m from the point O , is
 (a) 5 m (b) 6 m
 (c) 3 m (d) 7 m
28. If the normals at three points P, Q and R of the parabola $y^2 = 4ax$ meet in a point O and S is its focus, then $|SP| \cdot |SQ| \cdot |SR|$ is equal to
 (a) a^2 (b) $a(SO)^3$
 (c) $a(SO)^2$ (d) None of these
29. If the 4th term in the expansion of $\left(px + \frac{1}{x}\right)^n, n \in N$ is $\frac{5}{2}$ and three normals to the parabola $y^2 = x$ are drawn through a point $(q, 0)$, then
 (a) $q = p$ (b) $q > p$
 (c) $q < p$ (d) $pq = 1$
30. Tangent at the vertex divides the distance between directrix and latusrectum in the ratio
 (a) 1 : 1
 (b) 1 : 2
 (c) depends on directrix and focus
 (d) None of the above
31. The chord AB of the parabola $y^2 = 4ax$ cuts the axis of the parabola at C . If $A = (at_1^2, 2at_1)$, $B = (at_2^2, 2at_2)$ and $AC : AB = 1 : 3$, then
 (a) $t_2 = 2t_1$ (b) $t_2 + 2t_1 = 0$
 (c) $t_1 + 2t_2 = 0$ (d) None of these
32. AB is a chord of the parabola $y^2 = 4ax$. If its equation is $y = mx + c$ and it subtends a right angle at the vertex of the parabola, then
 (a) $c = 4am$ (b) $a = 4mc$
 (c) $c = -4am$ (d) $a + 4mc = 0$
33. Locus of trisection point of any double ordinate of $y^2 = 4ax$ is
 (a) $3y^2 = 4ax$ (b) $y^2 = 6ax$
 (c) $9y^2 = 4ax$ (d) None of these



34. Circle drawn having its diameter equal to focal distance of any point lying on the parabola $x^2 - 4x + 6y + 10 = 0$, will touch a fixed line whose equation is
 (a) $y = 2$ (b) $y = -1$
 (c) $x + y = 2$ (d) $x - y = 2$
35. An equilateral $\triangle SAB$ is inscribed in the parabola $y^2 = 4ax$ having its focus at S . If chord AB lies towards the left of S , then side length of this triangle is
 (a) $2a(2 - \sqrt{3})$
 (b) $4a(2 - \sqrt{3})$
 (c) $a(2 - \sqrt{3})$
 (d) $8a(2 - \sqrt{3})$
36. Normals AO, AA_1, AA_2 are drawn to parabola $y^2 = 8x$ from the point $A(h, 0)$. If $\triangle OA_1A_2$ is equilateral, then possible values of h is
 (a) 26 (b) 24
 (c) 28 (d) None of these
37. The locus of centre of a circle that passes through $P(a, b)$ and touch the line $y = mx + c$ (it is given that $b \neq ma + c$), is
 (a) a straight line (b) a circle
 (c) a parabola (d) a hyperbola
38. The parabola $y^2 = 4x$ and the circle $(x - 6)^2 + y^2 = r^2$ will have no common tangent, if
 (a) $r > \sqrt{20}$ (b) $r < \sqrt{20}$
 (c) $r > \sqrt{18}$ (d) $r \in (\sqrt{20}, \sqrt{28})$
39. Parabola $y^2 = 4x$ and the circle having its centre at $(6, 5)$ intersect at right angle. Possible point of intersection of these curves can be
 (a) $(9, 6)$ (b) $(2, \sqrt{8})$
 (c) $(1, 2)$ (d) $(3, 2\sqrt{3})$
40. $y = 2x + c'$, c being variable, is a chord of the parabola $y^2 = 4x$, meeting the parabola at A and B . Locus of point dividing the segment AB internally in the ratio 1:1, is
 (a) $y = 1$ (b) $x = 1$
 (c) $y = 2$ (d) $x = 2$
41. Radius of the circle that passes through origin and touches the parabola $y^2 = 4ax$ at the point $(a, 2a)$, is
 (a) $\frac{5}{\sqrt{2}}a$ (b) $2\sqrt{2}a$
 (c) $5\sqrt{2}a$ (d) $3\sqrt{2}a$
42. A line L passing through the focus of the parabola $(y - 2)^2 = 4(x + 1)$ intersects the parabola in two distinct points. If m is the slope of the line L , then m can take values in the interval
 (a) $(-1, 1)$ (b) $(-\infty, -1) \cup (1, \infty)$
 (c) $(-\infty, 0) \cup (0, \infty)$ (d) $(-\infty, \infty)$
43. A variable parabola having length of its latusrectum equal to $4a$ and having its axis parallel to X -axis drawn in such a way that it always touches the parabola $y^2 = 4ax$. Locus of the vertex of the variable parabola is
 (a) $y^2 = 8ax$ (b) $y^2 + 8ax = 0$
 (c) $y^2 = 6ax$ (d) $y^2 + 6ax = 0$
44. Set of values of m for which a chord of slope m of the circle $x^2 + y^2 = 4$ touches the parabola $y^2 = 4x$, is
 (a) $\left(-\infty, -\sqrt{\frac{\sqrt{2}-1}{2}}\right) \cup \left(\sqrt{\frac{\sqrt{2}-1}{2}}, \infty\right)$
 (b) $(-\infty, 1) \cup (1, \infty)$
 (c) $(-1, 1)$
 (d) R
45. If two tangents drawn from the point (α, β) to the parabola $y^2 = 4x$ is such that the slope of one tangent is double of the other, then
 (a) $\beta = \frac{2}{9}\alpha^2$ (b) $\alpha = \frac{2}{9}\beta^2$
 (c) $2\alpha = 9\beta^2$ (d) None of these
46. The tangents from the origin to the parabola $y^2 + 4 = 4x$ are inclined at
 (a) $\frac{\pi}{6}$ (b) $\frac{\pi}{4}$
 (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$
47. The minimum distance between the parabolas $y^2 - 4x - 8y + 40 = 0$ and $x^2 - 8x - 4y + 40 = 0$ is
 (a) 0 (b) $\sqrt{3}$ (c) $2\sqrt{2}$ (d) $\sqrt{2}$
48. The equation of common tangent to the equal parabolas $y^2 = 4ax$ and $x^2 = 4ay$, is
 (a) $x + y + a = 0$ (b) $x + y = a$
 (c) $x - y = a$ (d) None of these
49. The length of common chord of the parabola $2y^2 = 3(x + 1)$ and the circle $x^2 + y^2 + 2x = 0$, is
 (a) $\sqrt{3}$ (b) $2\sqrt{3}$
 (c) $\frac{\sqrt{3}}{2}$ (d) None of these
50. The locus of the middle points of chords of a parabola which subtend a right angle at the vertex of the parabola, is
 (a) a circle
 (b) an ellipse
 (c) a parabola
 (d) None of the above
51. The locus of a point from which tangents to a parabola are at right angles, is
 (a) a straight line
 (b) a pair of straight lines
 (c) a circle
 (d) a parabola

52. P is a point. Two tangents are drawn from it to the parabola $y^2 = 4x$, such that the slope of one tangent is three times the slope of the other. The locus of P is
(a) a straight line (b) a circle
(c) a parabola (d) an ellipse
53. The locus of the middle points of chords of the parabola $y^2 = 8x$ drawn through the vertex is a parabola whose
(a) focus is $(2, 0)$ (b) latusrectum is 8
(c) focus is $(0, 2)$ (d) latusrectum is 4
54. Circle drawn with diameter being any focal chord of the parabola $y^2 - 4x - y - 4 = 0$ will always touch a fixed line, whose equation is
(a) $16 + 33x = 0$ (b) $32x + 13 = 0$
(c) $13x + 32 = 0$ (d) $16x + 33 = 0$
55. Tangents are drawn to $y^2 = 4ax$ at point, where the line $lx + my + n = 0$ meets this parabola. Intersection point of these tangents is
(a) $\left(\frac{n}{l}, \frac{2am}{l}\right)$ (b) $\left(\frac{n}{l}, \frac{am}{l}\right)$
(c) $\left(\frac{n}{l}, \frac{-2am}{l}\right)$ (d) $\left(\frac{n}{l}, \frac{-am}{l}\right)$
56. Locus of mid-point of chords of the parabola $y^2 = 4ax$ that pass through the point $(3a, a)$, is
(a) $y^2 + 2ax + ay - 6a^2 = 0$
(b) $y^2 + 2ax - ay + 6a^2 = 0$
(c) $y^2 - 2ax + ay + 6a^2 = 0$
(d) $y^2 - 2ax - ay + 6a^2 = 0$
57. If straight lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents of $y^2 = 4ax$, then
(a) $m_1 + m_2 = 0$ (b) $m_1 m_2 = 1$
(c) $m_1 m_2 = -1$ (d) $m_1 + m_2 = 1$
58. Tangents drawn to parabola $y^2 = 4ax$ at the points A and B intersect at C . If S is the focus of the parabola, then SA , SC and SB forms
(a) an AP (b) a GP
(c) an HP (d) None of these
59. If the line $ax + by + c = 0$ is a tangent to the parabola $y^2 - 4y - 8x + 32 = 0$, then
(a) $4b^2 = a(7a + 2c + 4b)$ (b) $4b^2 = a(7a + c - 4b)$
(c) $4b^2 = a(7a + 2c + b)$ (d) $4b^2 = a(7a + 2c - b)$
60. If the tangent drawn from a point to the parabola $y^2 = 4ax$ are normals to the parabola $x^2 = 4by$, then
(a) $a^2 \geq b^2$ (b) $a^2 \geq 4b^2$
(c) $a^2 \geq 8b^2$ (d) $8a^2 \geq b^2$
61. If mutually perpendicular tangents TA and TB are drawn to $y^2 = 4ax$, then minimum length of AB is equal to
(a) $4a$ (b) $6a$ (c) $8a$ (d) $2a$
62. PA and PB are tangents drawn to $y^2 = 4ax$ from arbitrary point P . If the angle between tangents is $\frac{\pi}{4}$, then locus of point P is
(a) $y^2 = x^2 + a^2 + 6ax$ (b) $y^2 = x^2 - a^2 + 6ax$
(c) $y^2 = x^2 + a^2 - 6ax$ (d) $y^2 = x^2 - a^2 - 6ax$
63. PA and PB are the tangents drawn to $y^2 = 4ax$ from point P . These tangents meet Y -axis at the points A_1 and B_1 , respectively. If the area of $\triangle PA_1 B_1$ is 2 sq units, then locus of P is
(a) $a^2(y^2 + 2x)x^2 = 8$ (b) $a^2(y^2 + 4x)x^2 = 16$
(c) $a^2(y^2 - 4x)x^2 = 8$ (d) $a^2(y^2 - 4x)x^2 = 16$
64. Locus of the mid-point of any focal chord of $y^2 = 4ax$ is
(a) $y^2 = a(x - 2a)$ (b) $y^2 = 2a(x - 2a)$
(c) $y^2 = 2a(x - a)$ (d) None of these
65. Chord AB of the parabola $y^2 = 4ax$ subtends a right angle at the origin. Point of intersection of tangents drawn to parabola at A and B lie on the line
(a) $x + 2a = 0$ (b) $y + 2a = 0$
(c) $x + 4a = 0$ (d) $y + 4a = 0$
66. A normal is drawn to the parabola $y^2 = 4ax$ at a point P other than the vertex. If it cuts the parabola at point Q , then the least value of OQ , where O is the vertex of the parabola, is
(a) $4\sqrt{6}a$ (b) $2\sqrt{6}a$
(c) $3\sqrt{6}a$ (d) None of these
67. Tangents PA and PB are drawn to $x^2 = 4ay$. If m_{PA} and m_{PB} are the slope of these tangents and $m_{PA}^2 + m_{PB}^2 = 4$, then locus of P is
(a) $y^2 = 2x(x - a)$ (b) $y^2 = x(x - a)$
(c) $y^2 = 2x(2x - a)$ (d) $y^2 = 2x(x - 2a)$
68. If a line is drawn from $A(-2, 0)$ to intersect the curve $y^2 = 4x$ in P and Q in the first quadrant such that $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$, then slope of the line is always
(a) $> \sqrt{3}$ (b) $< \frac{1}{\sqrt{3}}$
(c) $> \sqrt{2}$ (d) $> \frac{1}{\sqrt{3}}$
69. The set of points on the axis of the parabola $y^2 = 4x + 8$ from which the three normals to the parabola are all real and different, is
(a) $\{(k, 0) | k \leq -2\}$ (b) $\{(k, 0) | k > -2\}$
(c) $\{(0, k) | k > -2\}$ (d) None of these
70. If three distinct and real normals can be drawn to $y^2 = 8x$ from the point $(a, 0)$, then
(a) $a > 2$ (b) $a > 4$
(c) $a \in (2, 4)$ (d) None of these



- 71.** Length of the shortest normal chord of the parabola $y^2 = 4x$ is
 (a) $a\sqrt{27}$ units (b) $3a\sqrt{3}$ units
 (c) $2a\sqrt{27}$ units (d) None of these
- 72.** Tangent and normal drawn to parabola at $A(at^2, 2at)$, $t \neq 0$ meet X -axis at points B and D , respectively. If the rectangle $ABCD$ is completed, then the locus of C is
 (a) $y = 2a$ (b) $y + 2a = 0$
 (c) $x = 2a$ (d) $x + 2a = 0$
- 73.** Normals PO , PA and PB , O being the origin, are drawn to $y^2 = 4x$ from $P(h, 0)$. If $\angle AOB = \frac{\pi}{2}$, then area of quadrilateral $OAPB$ is equal to
 (a) 12 sq units (b) 24 sq units
 (c) 6 sq units (d) 18 sq units
- 74.** Locus of the mid-point of any normal chords of $y^2 = 4ax$ is
 (a) $x = a \left(\frac{4a^2}{y^2} - 2 + \frac{y^2}{2a^2} \right)$
 (b) $x = a \left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2} \right)$
 (c) $x = a \left(\frac{4a^2}{y^2} - 2 - \frac{y^2}{2a^2} \right)$
 (d) $x = a \left(\frac{4a^2}{y^2} + 2 - \frac{y^2}{2a^2} \right)$
- 75.** The radius of the circle whose centre is $(-4, 0)$ and which cuts the parabola $y^2 = 8x$ at A and B such that its common chord AB subtends a right angle at the vertex of the parabola, is equal to
 (a) 4 (b) 3 (c) $\sqrt{18}$ (d) 5
- 76.** If the line $y = mx + c$ is a tangent to $y^2 = 4mx$, then distance of this tangent from the parallel normal is
 (a) $\sqrt{1 + m^2}$
 (b) $\sqrt{2 + m^2}$
 (c) $(1 + m^2)^{3/2}$
 (d) $(2 + m^2)^{3/2}$
- 77.** AB is a double ordinate of the parabola $y^2 = 4ax$. Tangents drawn to parabola at A and B meet Y -axis at A_1 and B_1 , respectively. If the area of trapezium AA_1B_1B is equal to $12a^2$, then angle subtended by A_1B_1 at the focus of the parabola is equal to
 (a) $2 \tan^{-1}(3)$
 (b) $\tan^{-1}(3)$
 (c) $2 \tan^{-1}(2)$
 (d) $\tan^{-1}(2)$
- 78.** Maximum number of common normals of $y^2 = 4ax$ and $x^2 = 4by$ can be equal to
 (a) 3 (b) 4 (c) 6 (d) 5
- 79.** If two different tangents of $y^2 = 4x$ are the normals to $x^2 = 4by$, then
 (a) $|b| > \frac{1}{2\sqrt{2}}$ (b) $|b| < \frac{1}{2\sqrt{2}}$
 (c) $|b| > \frac{1}{\sqrt{2}}$ (d) $|b| < \frac{1}{\sqrt{2}}$
- 80.** Two parabolas with the same axis, focus of each being exterior to the other and the latusrectum being $4a$ and $4b$. The locus of the middle points of the intercepts between the parabolas made on the lines parallel to the common axis, is a/an
 (a) straight line, if $a > b$
 (b) parabola, if $a \neq b$
 (c) parabola for all a, b
 (d) ellipse, if $b > a$

Type 2. More than One Correct Option

- 81.** The coordinates of an end point of the latusrectum of the parabola $(y - 1)^2 = 2(x + 2)$ are
 (a) $(-2, 1)$ (b) $(-3/2, 1)$ (c) $(-3/2, 2)$ (d) $(-3/2, 0)$
- 82.** Which of the following is/are true about the parabola $y^2 = 4ax$ ($a > 0$)?
 (a) If t_1, t_2 are end points of a focal chord, then $t_1 t_2 = -1$
 (b) Tangent at the end of a focal chord cuts at right angle at directrix
 (c) Distance of any point on the parabola from directrix is equal to the sum of a abscissa of the point
 (d) End points of latusrectum are $(a, 2a)$, $(-a, 2a)$
- 83.** The equation of the tangent of the parabola $y^2 = 9x$ which goes through the point $(4, 10)$, is
 (a) $x + 4y + 1 = 0$ (b) $9x + 4y + 4 = 0$
 (c) $x - 4y + 36 = 0$ (d) $9x - 4y + 4 = 0$
- 84.** The line(s) tangent to the curve $y = x^2 - x$ is/are
 (a) $x - y = 0$ (b) $x + y = 0$
 (c) $x - y = 1$ (d) $x + y = 1$
- 85.** A straight line touches the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$. The equation of the line is
 (a) $y = x + 2a$
 (b) $y = -x - 2a$
 (c) $y = x - 2a$
 (d) $y = x - a$
- 86.** If a normal chord of $y^2 = 4ax$ subtends an angle $\frac{\pi}{2}$ at the vertex of the parabola, then its slope is equal to
 (a) $\sqrt{2}$ (b) $-\sqrt{2}$
 (c) 1 (d) -1

Type 3. Assertion and Reason

Directions (Q. Nos. 87-92) For the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

87. Statement I Slope of tangents drawn from (4, 10) to parabola $y^2 = 9x$ are $\frac{1}{4}, \frac{9}{4}$.

Statement II Every parabola is symmetric about its directrix.

88. Statement I Equation $(5x - 5)^2 + (5y + 10)^2 = (3x + 4y + 5)^2$ is a parabola.

Statement II If distance of the point from the given line and from the given point (not lying on the given line) is equal, then locus of variable point is parabola.

89. Statement I Two parabolas $y^2 = 4ax$ and $x^2 = 4ay$ have common tangent $x + y + a = 0$.

Statement II $x + y + a = 0$ is common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4ay$ and point of contacts lie on their end points of latusrectum.

90. Statement I A is a point on the parabola $y^2 = 4ax$. The normal at A cuts the parabola again at point B. If AB subtends a right angle at the vertex of the parabola, then slope of AB is $1/\sqrt{2}$.

Statement II If normal at $(at_1^2, 2at_1)$ cuts again the parabola at $(at_2^2, 2at_2)$, then $t_2 = t_1 - \frac{2}{t_1}$.

91. Statement I Through $(h, h + 1)$ there cannot be more than one normal to the parabola $y^2 = 4x$, if $h < 2$.

Statement II The point $(h, h + 1)$ lies outside the parabola for all $h \neq 1$.

92. Statement I If normal at the ends of double ordinate $x = 4$ of parabola $y^2 = 4x$ meet the curve again at P and P' respectively, then $PP' = 12$ units.

Statement II If normal at t_1 of $y^2 = 4ax$ meet the parabola again at t_2 , then $t_2 = -\frac{2}{t_1}$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 93-95) $y = x$ is tangent to the parabola $y = ax^2 + c$.

93. If $a = 2$, then the value of c is

- (a) $\frac{1}{8}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) 1

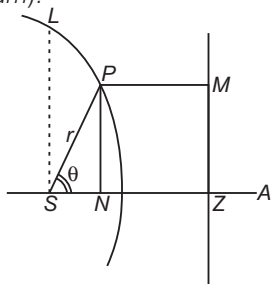
94. If (1, 1) is point of contact, then a is

- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{6}$

95. If $c = 2$, then point of contact is

- (a) (2, 2) (b) (4, 4) (c) (6, 6) (d) (3, 3)

Passage II (Q. Nos. 96-100) Let S be the focus considered as pole (origin) and take SA as the initial line, a vertical line ZM be considered as directrix. Let P be any point on the conic and its coordinates be (r, θ) , so that $SP = r$ and $\angle ZSP = \theta$ and further $SL = l$ (semi-latusrectum).



96. The polar equation of conic is

- (a) $r = l(1 + e \cos \theta)$ (b) $r = l(1 - e \cos \theta)$
(c) $r = \frac{l}{1 + e \cos \theta}$ (d) $r = \frac{l}{1 - e \cos \theta}$

97. If e is the eccentricity of the conic, then SZ is equal to

- (a) $\frac{l}{e^2}$ (b) $\frac{l}{e}$
(c) le^2 (d) le

98. The equation of directrix of the conic corresponding to the focus which has been chosen as the pole, is

- (a) $\frac{l}{r} = e \cos \theta$ (b) $\frac{r}{l} = e \cos \theta$
(c) $\frac{l}{r} = \frac{\cos \theta}{e}$ (d) $\frac{r}{l} = \frac{e}{\cos \theta}$

99. The polar equation of the parabola must be

- (a) $r = \frac{l}{2} \sin^2 \left(\frac{\theta}{2} \right)$ (b) $r = \frac{l}{2} \csc^2 \left(\frac{\theta}{2} \right)$
(c) $r = \frac{l}{2} \cos^2 \left(\frac{\theta}{2} \right)$ (d) $r = \frac{l}{2} \sec^2 \left(\frac{\theta}{2} \right)$

100. If the axis SZ of the conic makes an angle α with the initial line SA, then the equation of conic will be

- (a) $r = l[1 + e \cos(\theta + \alpha)]$ (b) $r = l[1 - e \cos(\theta - \alpha)]$
(c) $r = \frac{l}{1 + e \cos(\theta + \alpha)}$ (d) $r = \frac{l}{1 + e \cos(\theta - \alpha)}$



Passage III (Q. Nos. 101-103) Normally, the various proposition you study, i.e. equation of tangent, normal, chord, focal chord, formula for focal distance etc., are derived for the parabola $y^2 = 4ax$. However, all the results with slight transformation are valid for any parabola. Suppose we represent the equation of parabola $y^2 - 4ax = 0$ by $S(x, y, a) = 0$ and any equation derived for this parabola by $P(x, y, a) = 0$.

Now, if the given parabola is $y^2 = -4ax$, i.e. $y^2 + 4ax = 0$, we can write it $S(x, y, -a) = 0$, so that corresponding equation of P will be $P(x, y, -a) = 0$. Similarly, for $x^2 = 4ay$, it can be written as $S(y, x, a) = 0$ and corresponding transformation is $P(y, x, a)$ i.e. interchange x and y . Transformation for $x^2 = -4ay$, it is $P(y, x, -a) = 0$.

For example, The equation of tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$, so corresponding tangent to $x^2 = 4ay$ is $x = my + \frac{a}{m}$ and the point of contact will be $\left(\frac{2a}{m}, \frac{a}{m^2}\right)$.

The equation of normal to $y^2 = 4ax$ is $y = mx - 2am - am^3$.

The equation of normal to $x^2 = -4ay$ is

$$x = my + 2am + am^3.$$

Further, if the coordinates of vertex are not $(0, 0)$ but (h, k) , then we replaced x by $x - h$ and y by $y - k$ in addition to above transformation.

101. The focal distance of the point (x, y) on the parabola $x^2 - 8x + 16y = 0$, is

- (a) $|y - 4|$ (b) $|y - 5|$
(c) $|y - 2|$ (d) $|x - 4|$

102. The points on the axis of the parabola $x^2 + 2x + 4y + 13 = 0$ from where three distinct normals can drawn, are given by

- (a) $(-1, k), k \in (-1, 2)$ (b) $(-1, -6)$
(c) $(-1, k), k \in (-\infty, -5)$ (d) $(-1, 1)$

103. The line $x \cos \alpha + y \sin \alpha = p$ touches the parabola $x^2 + 4a(y + a) = 0$, if

- (a) $a = p \sec \alpha$
(b) $a \cos 2\alpha = p \sin \alpha$
(c) $a^2 \cos \alpha + p^2 \sin \alpha = 0$
(d) $a \tan \alpha = p \sec \alpha$

Type 5. Match the Columns

104. Match the statements of Column I with statements of Column II.

Column I	Column II
A. If a parabola represented by $25(x^2 + y^2) = (4x + 3y - 12)^2$, then	p. Equation of directrix is $4x - 3y + 12 = 0$
B. If a parabola represented by $25(x^2 + y^2) = (4x - 3y + 12)^2$, then	q. Equation of axis of parabola is $4x + 3y = 0$
C. If a parabola represented by $25(x^2 + y^2) = (3x - 4y + 12)^2$, then	r. Equation of directrix is $4x + 3y = 12$
	s. Equation of axis of parabola is $3x + 4y = 0$
	t. Equation of directrix is $3x - 4y + 12 = 0$

105. The parabola $y^2 = 4ax$ has a chord AB joining points $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$. Match the statements of Column I with values of Column II.

Column I	Column II
A. If AB is normal chord, then	p. $t_1 = -t_2 + 2$
B. If AB is a focal chord, then	q. $t_2 = \frac{-4}{t_1}$

Column I	Column II
C. If AB subtends 90° at point $(0, 0)$, then	r. $t_2 = \frac{-1}{t_1}$
D. If AB is inclined at 45° to the axis of parabola, then	s. $t_2 = t_1 - \frac{2}{t_1}$

106. Match the statements of Column I with values of Column II.

Column I	Column II
A. The shortest normal chord of the parabola $x^2 = 16y$ is of length	p. $\frac{1}{12}$
B. The y-coordinate of the point on the circle $(x - 6)^2 + y^2 = 1$ which is at a minimum distance from the parabola $x^2 = 8y$, is	q. $24\sqrt{3}$
C. A variable chord PQ of the parabola $y = 4x^2$ subtends a right angle at the vertex O . The locus of the centroid of ΔOPQ is a parabola whose latusrectum is of length	r. 1
D. PQ is a variable focal chord of the parabola $y^2 = 4x$. The normals at P and Q meet at R . Then, the locus of R is a parabola whose latusrectum is of length	s. $\frac{1}{\sqrt{2}}$

Type 6. Single Integer Answer Type Questions

107. If circle $(x-6)^2 + y^2 = r^2$ and parabola $y^2 = 4x$ have maximum number of common chord, then least integral value of r is ____.
108. The minimum number of normals that can be drawn through any point in the cartesian plane to the parabola $y^2 = 4ax$ is ____.
109. If the locus of centres of a family of circles passing through the vertex of the parabola $y^2 = 4a$ and cutting the parabola orthogonally at the other point of intersection is $2y^2(2y^2 + x^2 - 12ax) = ax(kx - 4a)^2$, then find the value of k .
110. Normal at a point $P(a, -2a)$ intersects the parabola $y^2 = 2x$ at point Q . If the tangent at P and Q meet a point R , then the area of ΔPQR is $\frac{ka^2(1+m^2)^3}{m^3}$. Find the numerical value of k .
111. Tangents are drawn from the point $(-2, -1)$ to the parabola $y^2 = 4ax$. If α is the angle between these tangents, then $\tan \alpha$ equals ____.

Entrances Gallery

JEE Advanced/IITJEE

1. The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the points P , Q and the parabola at the points R , S . Then, the area of the quadrilateral $PQRS$ is [2014]
 (a) 3 sq units (b) 6 sq units
 (c) 9 sq units (d) 15 sq units
- Passage I** (Q. Nos. 2-3) Let a, r, s, t be non-zero real numbers. Let $P(at^2, 2at)$, $Q(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point $(2a, 0)$. [2014]
2. The value of r is
 (a) $-\frac{1}{t}$ (b) $\frac{t^2+1}{t}$ (c) $\frac{1}{t}$ (d) $\frac{t^2-1}{t}$
3. If $st = 1$, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is
 (a) $\frac{(t^2+1)^2}{2t^3}$ (b) $\frac{a(t^2+1)^2}{2t^3}$
 (c) $\frac{a(t^2+1)^2}{t^3}$ (d) $\frac{a(t^2+2)^2}{t^3}$
- Passage II** (Q. Nos. 4-5) Let PQ be a focal chord of the parabola $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$. [2013]
4. Length of chord PQ is
 (a) $7a$ (b) $5a$ (c) $2a$ (d) $3a$
5. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan \theta$ is equal to
 (a) $\frac{2}{3}\sqrt{7}$ (b) $\frac{-2}{3}\sqrt{7}$ (c) $\frac{2}{3}\sqrt{5}$ (d) $\frac{-2}{3}\sqrt{5}$
6. Let S be the focus of the parabola $y^2 = 8x$ and PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of ΔPQS is _____. [2012]
7. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latusrectum and the point $P(\frac{1}{2}, 2)$ on the parabola and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latusrectum. Then, $\frac{\Delta_1}{\Delta_2}$ is _____. [2011]
8. Let (x, y) be any point on the parabola $y^2 = 4x$. If P is the point that divides the line segment from $(0, 0)$ to (x, y) in the ratio 1 : 3. Then, the locus of P is [2011]
 (a) $x^2 = y$ (b) $y^2 = 2x$
 (c) $y^2 = x$ (d) $x^2 = 2y$
9. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point $(9, 6)$, then L is given by [2011]
 (a) $y - x + 3 = 0$
 (b) $y + 3x - 33 = 0$
 (c) $y + x - 15 = 0$
 (d) $y - 2x + 12 = 0$
10. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be [2010]
 (a) $-\frac{1}{r}$ (b) $\frac{1}{r}$
 (c) $\frac{2}{r}$ (d) $-\frac{2}{r}$



11. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is [2014]

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) $\frac{1}{8}$ (d) $\frac{2}{3}$

12. Given A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}x$.

Statement I An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement II If the line $y = mx + \frac{\sqrt{5}}{m}$ ($m \neq 0$) is the common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$. [2013]

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

JEE Main/AIEEE

13. Let O be the vertex and Q be any point on the parabola $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1:3, then the locus of P is [2015]

(a) $x^2 = y$ (b) $y^2 = x$ (c) $y^2 = 2x$ (d) $x^2 = 2y$

14. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is [2011]

(a) $\frac{3\sqrt{2}}{8}$ (b) $\frac{8}{3\sqrt{2}}$ (c) $\frac{4}{\sqrt{3}}$ (d) $\frac{\sqrt{3}}{4}$

15. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles, then the locus of P is [2010]

(a) $x = 1$ (b) $2x + 1 = 0$ (c) $x = -1$ (d) $2x - 1 = 0$

16. A parabola has the origin as its focus and the line $x = 2$ as the directrix. Then, the vertex of the parabola is at [2008]

(a) (2, 0) (b) (0, 2)
(c) (1, 0) (d) (0, 1)

17. The equation of a tangent to the parabola $y^2 = 8x$ is $y = x + 2$. The point on this line from which the other tangent to the parabola is perpendicular to the given tangent, is [2007]

(a) (-1, 1) (b) (0, 2)
(c) (2, 4) (d) (-2, 0)

18. The locus of the vertices of the family of parabolas $y = \frac{a^3 x^2}{3} + \frac{a^2 x}{2} - 2a$ is [2006]

(a) $xy = \frac{3}{4}$ (b) $xy = \frac{35}{16}$
(c) $xy = \frac{64}{105}$ (d) $xy = \frac{105}{64}$

19. Let P be the point (1, 0) and Q be the point on $y^2 = 8x$. The locus of mid-point of PQ is [2005]

(a) $x^2 - 4y + 2 = 0$ (b) $x^2 + 4y + 2 = 0$
(c) $y^2 + 4x + 2 = 0$ (d) $y^2 - 4x + 2 = 0$

20. A circle touches X -axis and also touches the circle with centre at (0, 3) and radius 2. The locus of the centre of the circle is [2005]

(a) a parabola (b) a hyperbola
(c) a circle (d) an ellipse

21. If $a \neq 0$ and the line $2bx + 3cy + 4d = 0$ passes through the points of intersection of the parabolas $y^2 = 4ax$ and $x^2 = 4ay$, then [2004]

(a) $d^2 + (2b + 3c)^2 = 0$
(b) $d^2 + (3b + 2c)^2 = 0$
(c) $d^2 + (2b - 3c)^2 = 0$
(d) $d^2 + (3b - 2c)^2 = 0$

22. If the normal at the point $(bt_1^2, 2bt_1)$ on a parabola meets the parabola again in the point $(bt_2^2, 2bt_2)$, then [2003]

(a) $t_2 = -t_1 - \frac{2}{t_1}$
(b) $t_2 = -t_1 + \frac{2}{t_1}$
(c) $t_2 = t_1 - \frac{2}{t_1}$
(d) $t_2 = t_1 + \frac{2}{t_1}$

23. The equation of the directrix of the parabola $y^2 + 4y + 4x + 2 = 0$ is [2002]

(a) $x = -1$ (b) $x = 1$ (c) $x = -\frac{3}{2}$ (d) $x = \frac{3}{2}$

Other Engineering Entrances

24. The value of λ for which the curve $(7x + 5)^2 + (7y + 3)^2 = \lambda^2 (4x + 3y - 24)^2$ represents a parabola, is [WB JEE 2014]

(a) $\pm \frac{6}{5}$ (b) $\pm \frac{7}{5}$ (c) $\pm \frac{1}{5}$ (d) $\pm \frac{2}{5}$

25. If the equation of parabola is $x^2 = -9y$, then equation of directrix and length of latusrectum are [RPET 2014]

(a) $y = -\frac{9}{4}$, 8 (b) $y = \frac{9}{4}$, 9
(c) $y = -\frac{9}{4}$, 9 (d) None of these

26. If the line $y - \sqrt{3}x + 3 = 0$ cuts the parabola $y^2 = x + 2$ at A and B , then $PA \cdot PB$ is equal to [where, $P = (\sqrt{3}, 0)$] [Manipal 2014]
 (a) $\frac{4(2 - \sqrt{3})}{3}$ (b) $\frac{4(\sqrt{3} + 2)}{3}$
 (c) $\frac{4\sqrt{3}}{3}$ (d) $\frac{2(\sqrt{3} + 2)}{3}$
27. If a variable chord PQ of the parabola $y^2 = 4ax$ subtends a right angle at the vertex, then the locus of the points of intersection of the normal at P and Q is [BITSAT 2014]
 (a) a parabola (b) a hyperbola
 (c) a circle (d) None of these
28. The normals at three points P , Q and R of the parabola $y^2 = 4ax$ meet at (h, k) . The centroid of ΔPQR lies on [VITEEE 2014]
 (a) $x = 0$ (b) $y = 0$ (c) $x = -a$ (d) $y = a$
29. AB is a chord of the parabola $y^2 = 4ax$ with vertex A , BC is drawn perpendicular to AB meeting the axis at C . The projection of BC on the axis of the parabola is [Manipal 2014]
 (a) a (b) $2a$ (c) $4a$ (d) $8a$
30. The slope of the straight line joining the centre of the circle $x^2 + y^2 - 8x + 2y = 0$ and the vertex of the parabola $y = x^2 - 4x + 10$ is [Kerala CEE 2014]
 (a) $-\frac{5}{2}$ (b) $-\frac{7}{2}$ (c) $-\frac{3}{2}$ (d) $\frac{7}{2}$
 (e) None of these
31. If $y = 4x + 3$ is parallel to a tangent to the parabola $y^2 = 12x$, then its distance from the normal parallel to the given line is [WB JEE 2014]
 (a) $\frac{213}{\sqrt{17}}$ (b) $\frac{219}{\sqrt{17}}$
 (c) $\frac{211}{\sqrt{17}}$ (d) $\frac{210}{\sqrt{17}}$
32. The point on the parabola $y^2 = 64x$ which is nearest to the line $4x + 3y + 35 = 0$, has coordinates [WB JEE 2014]
 (a) $(9, -24)$ (b) $(1, 81)$
 (c) $(4, -16)$ (d) $(-9, -24)$
33. If a normal chord at a point on the parabola $y^2 = 4ax$ subtends a right angle at the vertex, then t equals [EAMCET 2014]
 (a) 1 (b) $\sqrt{2}$
 (c) 2 (d) $\sqrt{3}$
34. The slopes of the focal chords of the parabola $y^2 = 32x$, which are tangents to the circle $x^2 + y^2 = 4$, are [EAMCET 2014]
 (a) $\frac{1}{2}, \frac{-1}{2}$ (b) $\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}$
 (c) $\frac{1}{\sqrt{15}}, \frac{-1}{\sqrt{15}}$ (d) $\frac{2}{\sqrt{15}}, \frac{-2}{\sqrt{15}}$
35. Let A and B be two distinct points on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius 2 having AB as its diameter, then the slope of the line joining A and B can be [AMU 2014]
 (a) $-\frac{1}{2}$ (b) $\frac{1}{2}$
 (c) 1 (d) None of these
36. The image of the parabola $y^2 = 4x$ about the line $x - y + 1 = 0$ is [Karnataka CET 2013]
 (a) $(x + 1)^2 = 4(y - 1)$ (b) $(x - 1)^2 = 4(y + 1)$
 (c) $(x + 1)^2 = -4(y + 1)$ (d) $(x - 1)^2 = -4(y - 1)$
37. The angle between the tangents drawn from the point $(1, 4)$ to the parabolas $y^2 = 4x$ and $x^2 = 4y$, is [AMU 2013]
 (a) 0 (b) $\pi/6$
 (c) $\pi/4$ (d) $\pi/3$
38. The line $2x + y + k = 0$ is a normal to the parabola $y^2 = -8x$, if k is equal to [AMU 2013]
 (a) -24 (b) 12 (c) 24 (d) -12
39. The equation $y^2 + 4x + 4y + k = 0$ represents a parabola whose latusrectum is [WB JEE 2012]
 (a) 1 (b) 2 (c) 3 (d) 4
40. Let P and Q be the points on the parabola $y^2 = 4x$, so that the line segment PQ subtends right angle at the vertex. If PQ intersects the axis of the parabola at R , then the distance of the vertex from R is [WB JEE 2012]
 (a) 1 (b) 2 (c) 4 (d) 6
41. If the straight lines $y = \pm x$ intersect the parabola $y^2 = 8x$ in points P and Q , then length of PQ is [MP PET 2011]
 (a) 4 (b) $4\sqrt{2}$ (c) 8 (d) 16
42. The sum of the reciprocals of focal distances of a focal chord PQ of $y^2 = 4ax$ is [Karnataka CET 2011]
 (a) $\frac{1}{a}$ (b) a (c) $2a$ (d) $\frac{1}{2a}$
43. For the parabola $y^2 + 8x - 12y + 20 = 0$, [UP SEE 2011]
 (a) vertex is $(2, 6)$ (b) focus is $(0, 6)$
 (c) latusrectum is 4 (d) axis $Y = 6$
44. The distance between the vertex of the parabola $y = x^2 - 4x + 3$ and the centre of the circle $x^2 = 9 - (y - 3)^2$, is [Kerala CEE 2011]
 (a) $2\sqrt{3}$ units (b) $3\sqrt{2}$ units
 (c) $2\sqrt{2}$ units (d) $\sqrt{2}$ units
 (e) $2\sqrt{5}$ units
45. The parabola $y^2 = 4x$ and the circle $x^2 + y^2 - 6x + 1 = 0$ will [AMU 2011]
 (a) intersect at exactly one point
 (b) touch each other at two distinct points
 (c) touch each other at exactly one point
 (d) intersect at two distinct points



46. The equation of tangent of $y^2 = 12x$ and making an angle $\frac{\pi}{3}$ with X -axis is [Guj CET 2011]
 (a) $\pm y - \sqrt{3}x + \sqrt{3} = 0$ (b) $\pm y + \sqrt{3}x + 3 = 0$
 (c) $\pm y - \sqrt{3}x - \sqrt{3} = 0$ (d) $\pm y + \sqrt{3}x - 3 = 0$
47. If t_1 and t_2 are the parameters of the end points of a focal chord for the parabola $y^2 = 4ax$, then which one is correct? [VITEEE 2010]
 (a) $t_1 t_2 = 1$ (b) $\frac{t_1}{t_2} = 1$
 (c) $t_1 t_2 = -1$ (d) $t_1 + t_2 = -1$
48. One of the points on the parabola $y^2 = 12x$ with focal distance 12, is [Kerala CEE 2010]
 (a) (3, 6) (b) (9, $6\sqrt{3}$)
 (c) (7, $2\sqrt{21}$) (d) (8, $4\sqrt{6}$)
 (e) (1, $\sqrt{12}$)

Answers

Work Book Exercise 14.1

1. (d) 2. (a) 3. (d) 4. (b) 5. (b) 6. (a) 7. (a)

Work Book Exercise 14.2

1. (a) 2. (d) 3. (c) 4. (a) 5. (d) 6. (c) 7. (c)

Work Book Exercise 14.3

1. (d) 2. (c) 3. (b) 4. (c) 5. (c) 6. (a) 7. (d)

Target Exercises

1. (b) 2. (b) 3. (a) 4. (b) 5. (d) 6. (b) 7. (b) 8. (b) 9. (c) 10. (a)
 11. (a) 12. (b) 13. (a) 14. (c) 15. (d) 16. (c) 17. (d) 18. (b) 19. (a) 20. (b)
 21. (b) 22. (d) 23. (b) 24. (b) 25. (d) 26. (c) 27. (c) 28. (c) 29. (b) 30. (a)
 31. (b) 32. (c) 33. (c) 34. (b) 35. (b) 36. (c) 37. (c) 38. (b) 39. (a) 40. (a)
 41. (a) 42. (c) 43. (a) 44. (a) 45. (b) 46. (d) 47. (d) 48. (a) 49. (a) 50. (c)
 51. (a) 52. (c) 53. (d) 54. (d) 55. (c) 56. (d) 57. (c) 58. (b) 59. (a) 60. (c)
 61. (a) 62. (a) 63. (d) 64. (c) 65. (c) 66. (a) 67. (c) 68. (a) 69. (d) 70. (b)
 71. (c) 72. (c) 73. (b) 74. (b) 75. (a) 76. (c) 77. (c) 78. (d) 79. (b) 80. (b)
 81. (c,d) 82. (a,b,c) 83. (c,d) 84. (b,c) 85. (a,b) 86. (a,b) 87. (c) 88. (d) 89. (a) 90. (b)
 91. (a) 92. (c) 93. (a) 94. (a) 95. (b) 96. (c) 97. (b) 98. (a) 99. (d) 100. (d)
 101. (b) 102. (b,c) 103. (b) 104. (*) 105. (**) 106. (***) 107. (5) 108. (1) 109. (3) 110. (4)
 111. (3)

* $A \rightarrow r$; $B \rightarrow p,s$; $C \rightarrow t, q$

** $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow q$; $D \rightarrow p$

*** $A \rightarrow q$; $B \rightarrow s$; $C \rightarrow p$; $D \rightarrow r$

Entrances Gallery

1. (d) 2. (d) 3. (b) 4. (b) 5. (d) 6. (4) 7. (2) 8. (c) 9. (a,b,d) 10. (c,d)
 11. (a) 12. (a) 13. (d) 14. (a) 15. (c) 16. (c) 17. (d) 18. (d) 19. (d) 20. (a)
 21. (a) 22. (a) 23. (d) 24. (b) 25. (b) 26. (b) 27. (a) 28. (b) 29. (c) 30. (b)
 31. (b) 32. (a) 33. (b) 34. (c) 35. (c) 36. (a) 37. (a) 38. (c) 39. (d) 40. (c)
 41. (d) 42. (a) 43. (a,b,c) 44. (e) 45. (d) 46. (c) 47. (c) 48. (b)

Explanations

Target Exercises

1. Focus is $\left(\frac{7}{2}, \frac{7}{2}\right)$ and its axis is the line $y = x$.

Corresponding value of a is $\frac{1}{4}(\sqrt{1+1}) = \frac{\sqrt{2}}{4}$.

Let the equation of its directrix be $y + x + \lambda = 0$.

$$\Rightarrow \frac{|3+4+\lambda|}{\sqrt{2}} = 2 \cdot \frac{\sqrt{2}}{4}$$

$$\Rightarrow \lambda = -6, -8$$

Thus, equation of parabola is

$$\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{(x+y-6)^2}{2}$$

$$\left(x - \frac{7}{2}\right)^2 + \left(y - \frac{7}{2}\right)^2 = \frac{(x+y-8)^2}{2}$$

2. We have, $2\{(x-a)^2 + (y-a)^2\} = (x+y)^2$

$$\Rightarrow \sqrt{(x-a)^2 + (y-a)^2} = \frac{1}{\sqrt{2}}|x+y|$$

$$\Rightarrow \sqrt{(x-a)^2 + (y-a)^2} = \left|\frac{x+y}{\sqrt{2}}\right|$$

Clearly, the equation represents a parabola having its focus at (a, a) and directrix $x + y = 0$.

$\therefore$ Latusrectum = 2 (Distance between focus and directrix)
 $= 2 \left| \frac{a+a}{\sqrt{1+1}} \right| = 2\sqrt{2}a$

3. Let $A \equiv (at^2, 2at)$, $B \equiv (at^2, -2at)$, $m_{OA} = \frac{2}{t}$

and $m_{OB} = -\frac{2}{t}$

Thus, $\frac{2}{t} \cdot \left(-\frac{2}{t}\right) = -1 \Rightarrow t^2 = 4$

Thus, tangents will intersect at $(-4a, 0)$.

4. $x^2 - 2x + y - 2 = 0 \Rightarrow x^2 - 2x + 1 = 3 - y$

$$\Rightarrow (x-1)^2 = -(y-3)$$

Length of its latusrectum is 1 unit.

Since, $AS, \frac{1}{2}$ and BS are in HP.

$$\therefore \frac{1}{2} = \frac{2 \cdot AS \cdot BS}{AS + BS} \Rightarrow BS = \frac{l_1}{(4l_1 - 1)}$$

5. Let $A \equiv (at_1^2, 2at_1)$ and $B \equiv (at_2^2, 2at_2)$.

Thus, $t_1 t_2 = -4$

Equation of line AB is $y(t_1 + t_2) = 2(x + at_1 t_2)$ i.e.
 $y(t_1 + t_2) = 2(x - 4a)$, which passes through a fix point
 $(4a, 0)$.

6. Distance of focus from directrix is $|a \cos 2\theta - a|$. Thus,
length of latusrectum is $|4a \sin^2 \theta|$.

7. $(a, 2a)$ is an interior point of $y^2 - 16x = 0$, if
 $(2a)^2 - 16a < 0 \Rightarrow a^2 - 4a < 0$

$V(0, 0)$ and $(a, 2a)$ are on the same side of $x - 4 = 0$.

So, $a - 4 < 0 \Rightarrow a < 4$

Now, $a^2 - 4a < 0 \Rightarrow 0 < a < 4$

8. Vertex = $(0, 0)$

The ends of latusrectum are $(2, 4)$ and $(2, -4)$.

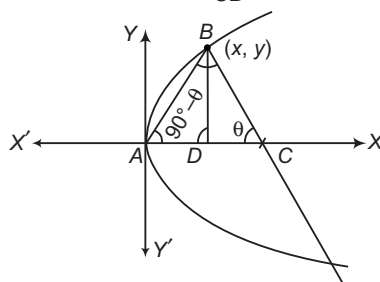
$\therefore$ Centre of circle is $(0, 0)$

and radius of circle = $\sqrt{2^2 + 4^2} = \sqrt{20}$

$\therefore$ Equation of circle is

$$x^2 + y^2 = r^2 \Rightarrow x^2 + y^2 = 20$$

9. In $\triangle BCD$, we have $\tan \theta = \frac{y}{CD}$



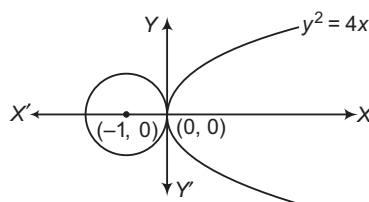
$$\Rightarrow CD = y \cot \theta \quad \dots(i)$$

In $\triangle BAD$, we have $\tan(90^\circ - \theta) = \frac{y}{x}$

$$\Rightarrow CD = y \cot \theta = \frac{y^2}{x} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow CD = \frac{4ax}{x} = 4a \quad [\because y^2 = 4ax]$$

10. The graph shows that $\lambda > 0$.



11. $x = t^2 - t + 1$, $y = t^2 + t + 1$

$$\Rightarrow x + y = 2(t^2 + 1) \text{ and } y - x = 2t$$

$$\Rightarrow \frac{x+y}{2} = 1 + \left(\frac{y-x}{2}\right)^2$$

$$\Rightarrow (y-x)^2 = 2(x+y) - 4$$

$$\Rightarrow (y-x)^2 = 2(x+y-2)$$

$\therefore$ Vertex will be the point, where lines $y - x = 0$ and $x + y - 2 = 0$ meet at the point $(1, 1)$.

12. On putting $y = \frac{x^2}{4a}$ in the equation of circle, we get

$$x^2 + \frac{x^4}{16a^2} + 2gx + \frac{fx^2}{2a} + c = 0$$

$$\Rightarrow x^4 + x^2(16a^2 + 8af) + 32a^2gx + 16a^2c = 0$$

$$\Rightarrow \Sigma x_1 = 0, \Sigma x_1 x_2 = 8a(f + 2a),$$

$$\Sigma x_1 x_2 x_3 = -32a^2g \text{ and } x_1 x_2 x_3 x_4 = 16a^2c$$

$$\Rightarrow \Sigma y_1 = \frac{1}{4a} \Sigma x_1^2 = \frac{1}{4a} [(\Sigma x_1)^2 - 2 \Sigma x_1 x_2]$$

$$= -4(f + 2a)$$

13. Extremities of the latusrectum are $(2, 4)$ and $(2, -4)$.
Since, any circle drawn with any focal chord as its diameter touches the directrix.
Thus, equation of required circle is
 $(x-2)^2 + (y-4)(y+4) = 0$ i.e. $x^2 + y^2 - 4x - 12 = 0$,
its radius is $\sqrt{4+12} = 4$.

14. Any tangent of parabola is $yt = x + t^2$. If it passes through $(2, 3)$, then $t^2 - 3t + 2 = 0 \Rightarrow t = 1, 2$.
That point can be $(1, 2)$ or $(4, 4)$.

15. Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the end points of a focal chord of the parabola $y^2 = 4ax$. Then,
 $PQ = a(t_2 - t_1)^2$

The equation of PQ is

$$(t_1 + t_2)y = 2x - 2a \quad [\because t_1 t_2 = -1]$$

It is given that $PQ = c$ and it is at a distance b from the vertex.

$$\therefore a(t_2 - t_1)^2 = c \quad \text{and} \quad \left| \frac{-2a}{\sqrt{(t_1 + t_2)^4 + 4}} \right| = b$$

$$\Rightarrow (t_2 - t_1)^2 = \frac{c}{a} \quad \text{and} \quad \frac{2a}{(t_2 - t_1)} = b \quad [\because t_1 t_2 = -1]$$

$$\Rightarrow \left(\frac{2a}{b} \right)^2 = \frac{c}{a} \Rightarrow 4a^3 = b^2 c$$

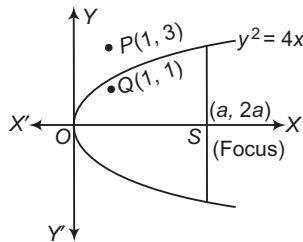
16. The point of intersection between the curves $x^2 = 4(y+1)$ and $x^2 = -4(y-1)$ is $(0, -1)$.
The slopes of first and second curve at the point $(0, -1)$ are respectively

$$m_1 = \frac{2x}{4} = 0$$

$$\text{and} \quad m_2 = \frac{-2x}{4} = 0$$

$$\therefore \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = 0 \Rightarrow \theta = 0^\circ$$

17. Here, $S \equiv y^2 - 4x = 0$



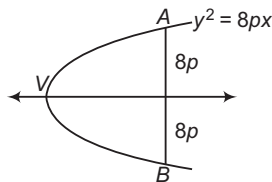
$$S(1, 3) = 3^2 - 4 \cdot 1 > 0$$

So, $P(1, 3)$ is an exterior point.

$$S(1, 1) = 1^2 - 4 \cdot 1 < 0$$

Thus, $Q(1, 1)$ is an interior point.

18. Clearly, $A = (8p, 8p)$ and $B = (8p, -8p)$



Also,

$$V = (0, 0)$$

$$m \text{ of } AV = 1, m \text{ of } BV = -1$$

Therefore, $AV \perp BV$.

19. On solving $x - 2y = 1$ and $y^2 = kx$, we get

$$y^2 = k(1 + 2y) \Rightarrow y^2 - 2ky - k = 0$$

$$\therefore y_1 + y_2 = 2k, y_1 \cdot y_2 = -k$$

$$\therefore 16 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$

$$= \left(\frac{y_1^2}{k} - \frac{y_2^2}{k} \right)^2 + (y_1 - y_2)^2$$

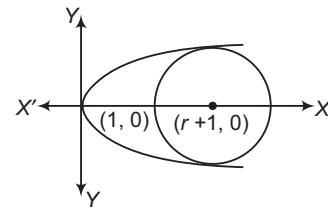
$$= (y_1 - y_2)^2 \cdot \left\{ \frac{(y_1 + y_2)^2}{k^2} + 1 \right\}$$

$$= \{(y_1 + y_2)^2 - 4y_1 y_2\} \left\{ \frac{4k^2}{k^2} + 1 \right\}$$

$$= 5\{4k^2 + 4k\} = 20k^2 + 20k$$

$$\therefore 5k^2 + 5k - 4 = 0 \Rightarrow k = \frac{\sqrt{105} - 5}{10}$$

20. Equation of circle is $(x - r - 1)^2 + y^2 = r^2$



$$\Rightarrow (x - r - 1)^2 + 4x = r^2 \quad [\because y^2 = 4x]$$

$$\Rightarrow x^2 + \{4 - 2(r+1)\}x + (2r+1) = 0$$

$$\therefore D = 0$$

$$\Rightarrow 4(1-r)^2 - 4(2r+1) = 0$$

$$\Rightarrow r = 4$$

21. $\therefore lx + my + n = 0$

$$\Rightarrow y = -\frac{(lx+n)}{m} \quad \dots (i)$$

$$\text{and parabola } x^2 = 4ay \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$x^2 = 4a \left\{ -\frac{(lx+n)}{m} \right\}$$

$$\Rightarrow mx^2 + 4alx + 4an = 0$$

$$\therefore B^2 = 4AC$$

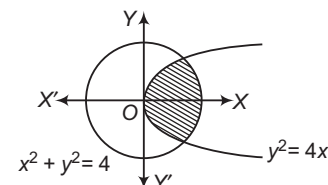
$$\therefore 16a^2 l^2 = 4m(4an)$$

$$\Rightarrow al^2 = mn$$

$$\Rightarrow al^2 - mn = 0$$

22. It is clear that point $(-2m, m+1)$ lies inside the circle and parabola, then

$$(-2m)^2 + (m+1)^2 - 4 < 0 \text{ and } (m+1)^2 - 4(-2m) < 0$$



$$\therefore 5m^2 + 2m - 3 < 0 \text{ and } m^2 + 10m + 1 < 0$$

$$\Rightarrow (m+1)(5m-3) < 0 \text{ and } (m+5)^2 - 24 < 0$$

$$\Rightarrow -1 < m < \frac{3}{5}$$

$$\text{and } -5 - 2\sqrt{6} < m < -5 + 2\sqrt{6}$$

$$\text{Hence, } -1 < m < -5 + 2\sqrt{6}$$

23. Let B and C be the trisected points. B divides LL' in 1:2.

Then, coordinate of $B\left(\frac{1 \cdot m + 2 \cdot m}{1+2}, \frac{-l + 2l}{3}\right)$

$$\Rightarrow B(m, l/3)$$

$$\text{Let } x_1 = m, y_1 = l/3$$

$$\therefore (m, l) = (x_1, 3y_1)$$

But (m, l) lie on parabola.

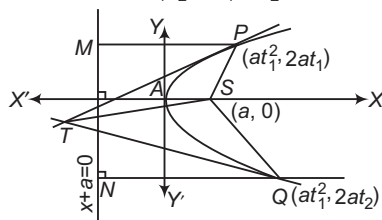
$$(3y_1)^2 = 4ax_1 \Rightarrow 9y_1^2 = 4ax_1$$

$$\therefore \text{Locus is } 9y^2 = 4ax$$

24. Let parabola $y^2 = 4ax$

$$\text{Let } P \equiv (at_1^2, 2at_1), Q \equiv (at_2^2, 2at_2)$$

$$\text{Then, } T \equiv [(at_1 t_2, a(t_1 + t_2))]$$



$$SP = PM = a + at_1^2$$

$$SQ = QN = a + at_2^2$$

$$\begin{aligned} \text{and } ST &= \sqrt{(a - at_1 t_2)^2 + [0 - a(t_1 + t_2)]^2} \\ &= a\sqrt{(1 - t_1 t_2)^2 + (t_1 + t_2)^2} = a\sqrt{(1 + t_1^2 t_2^2 + t_1^2 + t_2^2)} \\ &= a\sqrt{(1 + t_1^2)(1 + t_2^2)} = \sqrt{a(1 + t_1^2) \cdot a(1 + t_2^2)} \\ &= \sqrt{(SP)(SQ)} \end{aligned}$$

So, SP, ST and SQ are in GP.

25. $\therefore [x^2] = [y]$

If $0 \leq y < 1$, then $[y] = 0$.

$$\therefore [x^2] = 0 \Rightarrow 0 \leq x^2 < 1 \Rightarrow x \in (-1, 1)$$

If $1 \leq y < 2$, then $[y] = 1$

$$\therefore [x^2] = 1 \Rightarrow 1 \leq x^2 < 2$$

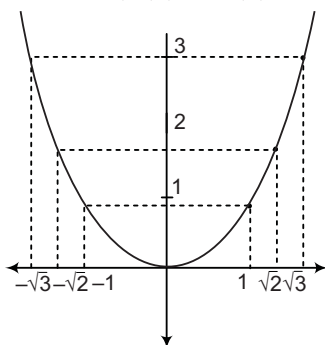
$$\Rightarrow x \in (-\sqrt{2}, -1) \cup [1, \sqrt{2}]$$

If $2 \leq y < 3$, then $[y] = 2$

$$\Rightarrow [x^2] = 2 \Rightarrow 2 \leq x^2 < 3$$

$$\therefore x \in (-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3})$$

The graph of the region will not only contain of the parabola $y = x^2$ but $[x^2] = [y]$ contain points within the rectangles of side $1, 2; 1, \sqrt{2} - 1; 1, \sqrt{3} - \sqrt{2}$, etc.



Hence, a, b, c are incorrects.

26. Let $P \equiv (at^2, 2at)$

$$\therefore \text{Tangent at } P \text{ is } ty = x + at^2$$

...(i)

$$\text{Tangent at vertex is } x = 0$$

...(ii)

On solving Eqs. (i) and (ii), we get

$$Q \equiv (0, at)$$

and

$$S \equiv (a, 0)$$

$$\therefore \text{Slope of } QP \text{ is } \frac{2at - at}{at^2 - 0} = \frac{1}{t} = m_1$$

[say]

$$\text{and slope of } QS \text{ is } \frac{0 - at}{a - 0} = -t = m_2$$

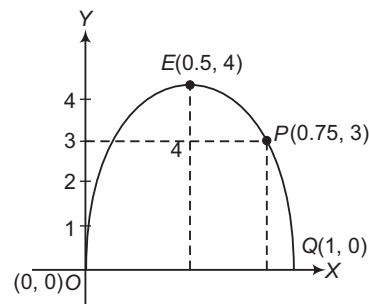
[say]

$$\therefore m_1 m_2 = -1$$

$$\therefore \angle SQP = \pi/2$$

27. The path of the water jet is a parabola.

Let its equation be $y = ax^2 + bx + c$



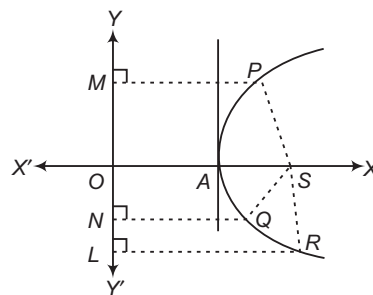
It should pass through $(0, 0), (0.5, 4), (1, 0)$.

$$\Rightarrow c = 0, a = -16, b = 16$$

$$\Rightarrow y = -16x^2 + 16x$$

If $x = 0.75$, then $y = 3$.

28. Equation of normal in terms of slope is $y = mx - 2am - am^3$.



$$\text{Let } O \equiv (h, k)$$

$$\text{Then, } am^3 - (h - 2a)m + k = 0$$

$$\text{or } m_1 + m_2 + m_3 = 0$$

$$\Rightarrow m_1 m_2 + m_2 m_3 + m_3 m_1 = -\frac{(h - 2a)}{a}$$

$$\text{and } m_1 m_2 m_3 = -\frac{k}{a}$$

$$\text{Let } P \equiv (am_1^2, -2am_1),$$

$$Q \equiv (am_2^2, -2am_2)$$

$$R \equiv (am_3^2, -2am_3)$$

$$\text{and } S \equiv (a, 0)$$

$$\begin{aligned} \therefore |SP| \cdot |SQ| \cdot |SR| &= |PM| \cdot |QN| \cdot |RL| \\ &= |a + am_1^2| \cdot |a + am_2^2| \cdot |a + am_3^2| \\ &= a^3 |(1 + m_1^2)(1 + m_2^2)(1 + m_3^2)| \\ &= a^3 |1 + \Sigma m_i^2 + \Sigma m_i^2 m_j^2 + m_1^2 m_2^2 m_3^2| \end{aligned}$$

$$= a^3 \left| 1 + (\Sigma m_1)^2 - 2\Sigma m_1 m_2 + (\Sigma m_1 m_2)^2 - 2m_1 m_2 m_3 \Sigma m_1 + (m_1 m_2 m_3)^2 \right|$$

$$= a^3 \left| 1 + (0)^2 + \frac{2(h-2a)}{a} + \frac{(h-2a)^2}{a^2} - 0 + \frac{k^2}{a^2} \right|$$

$$= a \left| k^2 + 2a(h-2a) + a^2 + (h-2a)^2 \right|$$

$$= a \left| k^2 + (h-2a+a)^2 \right| = a \left| k^2 + (h-a)^2 \right|$$

$$= a \left| (k-0)^2 + (h-a)^2 \right| = a(SO)^2$$

29. Given, $\frac{5}{2} = {}^nC_3 (px)^{n-3} \left(\frac{1}{x}\right)^3 = {}^nC_3 \cdot p^{n-3} x^{n-6} \dots (i)$

Since, LHS of Eq. (i) is independent of x.

$$\therefore n-6=0 \Rightarrow n=6$$

From Eq. (i), we get

$$\frac{5}{2} = {}^6C_3 p^3 = 20p^3$$

$$\Rightarrow p^3 = \left(\frac{1}{2}\right)^3 \Rightarrow p = \frac{1}{2}$$

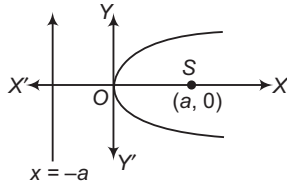
Given, parabola is $y^2 = x$ (ii)

Here, $4a=1 \Rightarrow a=\frac{1}{4}$

Since, three normals are drawn from point $(q, 0)$.

$$\therefore q > 2a \text{ or } q > \frac{1}{2} \text{ or } q > p \left[\because p = \frac{1}{2} \right]$$

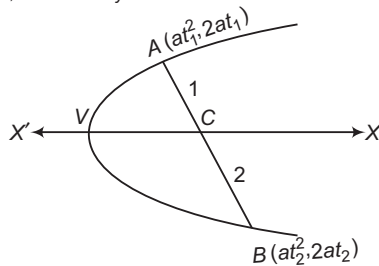
30. We know that, distance from vertex to the parabola is equal to the focus and directrix.



Thus, the tangent at the vertex divides in the ratio 1 : 1.

31. Here, $C \equiv \left(\frac{2at_1^2 + at_2^2}{3}, \frac{4at_1 + 2at_2}{3} \right)$

Since, it lies on $y = 0$.



$$\therefore \frac{4at_1 + 2at_2}{3} = 0$$

$$\Rightarrow 4at_2 + 2at_2 = 0$$

$$\Rightarrow 4t_1 + 2t_2 = 0$$

$$\Rightarrow 2t_1 + t_2 = 0$$

32. The equation of the pair of lines VA and VB, where V is the vertex = (0, 0) is

$$y^2 = 4ax \cdot \frac{y - mx}{c}$$

$$\Rightarrow cy^2 - 4axy + 4amx^2 = 0$$

Since, lines are perpendicular to each other.

$$\therefore \text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

$$\Rightarrow c + 4am = 0 \Rightarrow c = -4am$$

33. Let AB be a double ordinate, where $A \equiv (at^2, 2at)$, $B \equiv (at^2, -2at)$. If $P(h, k)$ is its trisection point, then

$$3h = 2at^2 + at^2, 3k = 4at - 2at \Rightarrow t^2 = \frac{h}{a}, t = \frac{3k}{2a}$$

Thus, locus is

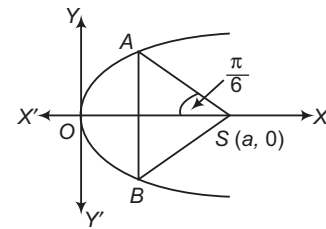
$$\frac{9k^2}{4a^2} = \frac{h}{a} \Rightarrow 9y^2 = 4ax$$

34. $x^2 - 4x + 6y + 10 = 0 \Rightarrow x^2 - 4x + 4 = -6 - 6y$
 $\Rightarrow (x-2)^2 = -6(y+1)$

Circle drawn on focal distance as diameter always touches the tangent drawn to parabola at vertex. Thus, circle will touch the line $y+1=0$.

35. Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$

$$\text{We have, } m_{AS} = \tan\left(\frac{5\pi}{6}\right) \Rightarrow \frac{2at_1}{at_1^2 - a} = -\frac{1}{\sqrt{3}}$$



$$\Rightarrow t_1^2 + 2\sqrt{3}t_1 - 1 = 0$$

$$\Rightarrow t_1 = -\sqrt{3} \pm 2$$

Clearly, $t_1 = -\sqrt{3} - 2$ is rejected.

$$\text{Thus, } t_1 = (2 - \sqrt{3})$$

$$\text{Hence, } AB = 4at_1 = 4a(2 - \sqrt{3})$$

36. Let $A_1 \equiv (2t_1^2, 4t_1)$, $A_2 \equiv (2t_2^2, -4t_2)$

$$\text{Clearly, } \angle A_1OA_2 = \frac{\pi}{6} \Rightarrow \frac{2}{t_1} = \frac{1}{\sqrt{3}}$$

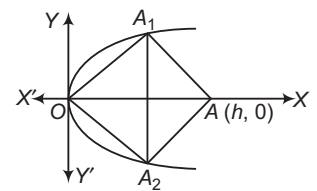
$$\Rightarrow t_1 = 2\sqrt{3}$$

Equation of normal at A_1 is

$$y = -t_1x + 4t_1 + 2t_1^3$$

$$\Rightarrow h = 4 + 2t_1^2$$

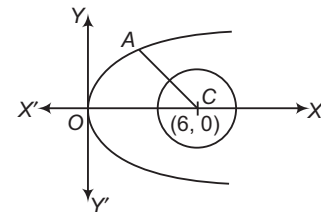
$$= 4 + 2 \cdot 12 = 28$$



37. Let $O(h, k)$ be the centre of circle. Clearly, distance of O from P and the line $y = mx + c$ will be equal. Thus, locus of D will be a parabola.

38. Any normal of parabola is $y = -tx + 2t + t^3$. If it pass through $(6, 0)$, then

$$-6t + 2t + t^3 = 0$$



$$\Rightarrow t = 0, t^2 = 4$$

$$\text{Thus, } A \equiv (4, 4)$$

For no common tangent,

$$AC > \sqrt{4+16} > r$$

$$\Rightarrow r < \sqrt{20}$$

39. Let the possible point be $(t^2, 2t)$.

Equation of tangent at this point is $yt = x + t^2$.

Since, it must pass through $(6, 5)$.

$$\therefore t^2 - 5t + 6 = 0$$

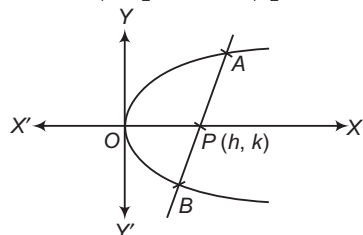
$$\Rightarrow t = 2, 3$$

Hence, possible points are $(4, 4)$ and $(9, 6)$.

40. Let $A \equiv (t_1^2, 2t_1)$ and $B \equiv (t_2^2, 2t_2)$

Equation of AB is

$$y(t_1 + t_2) = 2(x + t_1 t_2)$$



It must be same as $y = 2x + c$.

$$\Rightarrow \frac{t_1 + t_2}{1} = \frac{2}{2} = \frac{2t_1 t_2}{c}$$

Let $P(h, k)$ be such that $AP : PB = 1 : 1$, then

$$2h = t_1^2 + t_2^2, 2k = 2t_1 + 2t_2$$

$$\Rightarrow t_1 + t_2 = k = 1$$

$$\Rightarrow k = 1$$

Thus, locus of P is $y = 1$.

41. Equation of tangent of parabola at $(a, 2a)$ is

$$y \cdot 2a = 2a(x + a)$$

$$\Rightarrow y - x - a = 0$$

Equation of circle touching the parabola at $(a, 2a)$

$$(x - a)^2 + (y - 2a)^2 + \lambda(y - x - a) = 0$$

Since, it passes through $(0, 0)$.

$$\therefore a^2 + 4a^2 + \lambda(-a) = 0$$

$$\Rightarrow \lambda = 5a$$

Thus, required circle is $x^2 + y^2 - 7ax + ay = 0$ and radius

$$= \sqrt{\frac{49}{4}a^2 + \frac{a^2}{4}} = \frac{5}{\sqrt{2}}a$$

42. We know that, every line passing through the focus of a parabola intersects the parabola in two distinct points except lines parallel to the axis.

The equation $(y - 2)^2 = 4(x + 1)$ represents a parabola with vertex $(-1, 2)$ and axis parallel to X -axis. So, the line of slope m will cut the parabola in two distinct points, if $m \neq 0$ i.e. $m \in (-\infty, 0) \cup (0, \infty)$.

43. Let the vertex be $P(h, k)$, then its equation is $(y - k)^2 = -4a(x - h)$. It will touch $y^2 = 4ax$, if

$$(y - k)^2 = -y^2 + 4ah \text{ has equal roots.}$$

$$\Rightarrow k^2 = 8ah$$

Thus, locus is $y^2 - 8ax = 0$.

44. The equation of tangent of slope m to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$.

This will be a chord of the circle $x^2 + y^2 = 4$, if length of the perpendicular from the centre $(0, 0)$ is less than the radius.

$$\text{i.e. } \left| \frac{1}{m\sqrt{m^2 + 1}} \right| < 2$$

$$\Rightarrow 4m^4 + 4m^2 - 1 > 0$$

$$\Rightarrow \left(m^2 - \frac{\sqrt{2} - 1}{2} \right) \left(m^2 + \frac{1 + \sqrt{2}}{2} \right) > 0$$

$$\Rightarrow \left(m^2 - \frac{\sqrt{2} - 1}{2} \right) > 0$$

$$\Rightarrow \left(m - \sqrt{\frac{\sqrt{2} - 1}{2}} \right) \left(m + \sqrt{\frac{\sqrt{2} - 1}{2}} \right) > 0$$

$$\Rightarrow m \in \left(-\infty, -\sqrt{\frac{\sqrt{2} - 1}{2}} \right) \cup \left(\sqrt{\frac{\sqrt{2} - 1}{2}}, \infty \right)$$

45. Any tangent to the parabola $y^2 = 4x$ is $y = mx + \frac{1}{m}$.

It passes through (α, β) , if

$$\beta = m\alpha + \frac{1}{m}$$

$$\Rightarrow \alpha m^2 - \beta m + 1 = 0$$

It will have roots m_1 and $2m_1$, if

$$m_1 + 2m_1 = \frac{\beta}{\alpha} \text{ and } m_1 \cdot 2m_1 = \frac{1}{\alpha}$$

$$\Rightarrow 2 \cdot \left(\frac{\beta}{3\alpha} \right)^2 = \frac{1}{\alpha}$$

$$\Rightarrow \frac{2\beta^2}{9\alpha^2} = \frac{1}{\alpha}$$

$$\therefore \alpha = \frac{2}{9}\beta^2$$

46. Equation of parabola is

$$y^2 + 4 = 4x.$$

$$\Rightarrow y^2 = 4(x - 1)$$

Any tangent to parabola is

$$y = m(x - 1) + \frac{1}{m}.$$

It passes through the origin, if

$$0 = -m + \frac{1}{m}$$

$$\Rightarrow m = 1, -1$$

So, the two tangents are inclined at $\frac{\pi}{2}$.

47. The equations of the parabolas are

$$y^2 - 4x - 8y + 40 = 0 \quad \dots(i)$$

$$\text{and } x^2 - 8x - 4y + 40 = 0 \quad \dots(ii)$$

We observe that, if we interchange x and y in Eq. (i), we obtain Eq. (ii).

So, the two parabolas are symmetric about $y = x$.

If the two parabolas intersect on $y = x$, then the minimum distance between them is zero.

On solving $y = m$ and $y^2 - 4x - 8y + 4 = 0$, we get $x^2 - 12x + 40 = 0$, which has imaginary roots.

So, the parabolas do not intersect. Consequently, distance between them is not zero.

Minimum distance between the two parabolas is the distance between tangents to the two parabolas which are parallel to $y = x$.

On differentiating Eq. (ii) w.r.t. x , we get

$$2x - 8 - 4 \frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{x-4}{2}$$

If the tangent is parallel to $y = x$, then

$$\frac{x-4}{2} = 1$$

$$\Rightarrow x = 6$$

On putting $x = 6$ in Eq. (ii), we get $y = 7$

Thus, the coordinates of a point on parabola (ii) are $P(6, 7)$. The corresponding point on parabola (i) is $Q(7, 6)$.

$\therefore$ Required distance, $PQ = \sqrt{1+1} = \sqrt{2}$

48. Any tangent to $y^2 = 4ax$ is $y = mx + \frac{a}{m}$.

It touches $x^2 = 4ay$, if $x^2 = 4a\left(mx + \frac{a}{m}\right)$, i.e.

$$x^2 - 4amx - \frac{4a^2}{m} = 0 \text{ has equal roots.}$$

$$\text{So, } m^2 + 1 = 0, \text{ i.e. } m = -1$$

Hence, the common tangent is $y = -x - a$.

49. On solving $2y^2 = 3(x+1)$ and $x^2 + y^2 + 2x = 0$, we get

$$x^2 + \frac{3(x+1)}{2} + 2x = 0$$

$$\Rightarrow x^2 + \frac{7x}{2} + \frac{3}{2} = 0$$

$$\Rightarrow 2x^2 + 7x + 3 = 0$$

$$\Rightarrow (2x+1)(x+3) = 0$$

$$\Rightarrow x = -\frac{1}{2}, -3$$

But $x = -3$ makes y imaginary, because $2y^2 = 3(x+1)$.

$$\text{So, } x = -\frac{1}{2}$$

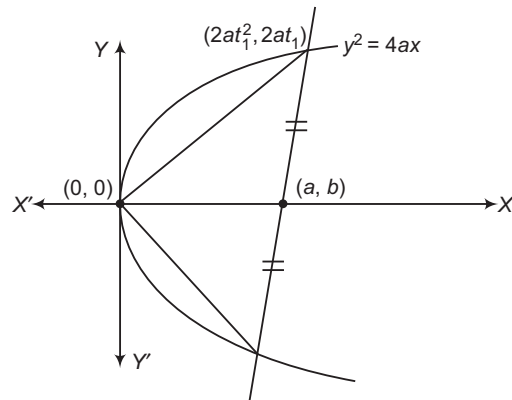
$$\Rightarrow y = \pm \frac{\sqrt{3}}{2}$$

$\therefore$ The length of the chord = The distance between

$$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \text{ and } \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$$

$$= \sqrt{\left(\frac{-\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)^2 + \left[\frac{-1}{2} - \left(-\frac{1}{2}\right)\right]^2} = \sqrt{3}$$

50. Here, $\alpha = \frac{at_1^2 + at_2^2}{2}$, $\beta = \frac{2at_1 + 2at_2}{2} = a(t_1 + t_2)$



$$\text{Also, } \frac{2at_1 - 0}{at_1^2 - 0} \times \frac{2at_2 - 0}{at_2^2 - 0} = -1$$

$$\Rightarrow t_1 t_2 = -4$$

$$\text{Now, } \frac{2\alpha}{a} = t_1^2 + t_2^2 = (t_1 + t_2)^2 - 2t_1 t_2$$

$$= \left(\frac{\beta}{a}\right)^2 - 2 \cdot (-4)$$

So, the locus of (α, β) is $\frac{2x}{a} = \frac{y^2}{a^2} + 8$ which is a parabola.

51. The locus is the directrix, which is a line.

52. Let $P = (\alpha, \beta)$. Any tangent to the parabola is $y = mx + \frac{a}{m}$.

Since, it passes through (α, β) .

$$\therefore \beta = m\alpha + \frac{a}{m} \quad [\because a = 1]$$

$$\Rightarrow m^2\alpha - \beta m + 1 = 0$$

Its roots are $m_1, 3m_1$.

$$\text{So, } m_1 + 3m_1 = \frac{\beta}{\alpha}, m_1 \cdot 3m_1 = \frac{1}{\alpha}$$

$$\therefore 3 \cdot \left(\frac{\beta}{4\alpha}\right)^2 = \frac{1}{\alpha}$$

$$\Rightarrow 3\beta^2 = 16\alpha$$

Thus, the locus is $3y^2 = 16x$, which is a parabola.

53. If the middle point of a chord is (α, β) , then

$$\alpha = \frac{2t^2 + 0}{2}, \beta = \frac{4t + 0}{2}$$

Eliminating t , we get $\alpha = \left(\frac{\beta}{2}\right)^2$.

So, the locus is $y^2 = 4x$.

54. $y^2 - 4x - y - 4 = 0$

$$\Rightarrow y^2 - y + \frac{1}{4} = 4x + \frac{17}{4}$$

$$\Rightarrow \left(y - \frac{1}{2}\right)^2 = 4\left(x + \frac{17}{16}\right)$$

Circle drawn with diameter being any focal chord of the parabola always touches the directrix of the parabola.

Thus, circle will touch the line $x + \frac{17}{16} = -1$

$$\text{i.e. } 16x + 33 = 0$$

55. Let the tangents intersect at $P(h, k)$, then $lx + my + n = 0$ will be the chord of contact of P i.e. $lx + my + n = 0$ and $yk - 2ax - 2ah = 0$ will represent the same line. Thus,

$$\frac{k}{m} = \frac{-2a}{l} = \frac{-2ah}{n}$$

$$\Rightarrow h = \frac{n}{l}, k = -\frac{2am}{l}$$

56. Let the mid-point of chord be $P(h, k)$, then its equation is $T = S_1$ i.e. $yk - 2a(x + h) = k^2 - 4ah$. It must pass through $(3a, a)$, hence $ak - 2a(3a + h) = k^2 - 4ah$. Thus, locus of P is $y^2 - 2ax - ay + 6a^2 = 0$.

57. Clearly, both the lines pass through $(-a, b)$, which a point lying on the directrix of the parabola. Thus, $m_1 m_2 = -1$. Because tangents drawn from any point on the directrix are always mutually perpendicular.

58. If $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$, then

$$\begin{aligned} C &\equiv [at_1t_2, a(t_1 + t_2)] \\ \text{Thus, } SA &= a + at_1^2, SB = a + at_2^2 \\ \text{and } SC &= \sqrt{(at_1t_2 - a)^2 + a^2(t_1 + t_2)^2} \\ \Rightarrow SC &= a\sqrt{(t_1t_2 - 1)^2 + (t_1 + t_2)^2} \\ &= a\sqrt{t_1^2 + t_2^2 + 1 + t_1^2t_2^2} = a\sqrt{(1 + t_1^2)(1 + t_2^2)} \\ \text{Clearly, } SC^2 &= SA \cdot SB \end{aligned}$$

59. Line will touch the parabola, if $y^2 - 4y + 32 = \frac{8(-by - c)}{a}$ has equal roots.
 $\Rightarrow 4b^2 = 7a^2 + 2a(c + 2b)$

60. The equation of a tangent to $y^2 = 4ax$ is

$$y = mx + \frac{a}{m} \Rightarrow x = \frac{y}{m} - \frac{a}{m^2} \quad \dots(i)$$

This will be a normal to the parabola $x^2 = 4ay$, if it is of the form

$$\begin{aligned} x &= \frac{y}{m} - \frac{2b}{m} - \frac{b}{m^3} \quad \dots(ii) \\ \therefore -\frac{a}{m^2} &= -\frac{2b}{m} - \frac{b}{m^3} \\ \Rightarrow 2bm^2 - am + b &= 0 \\ \text{Since, } m &\text{ is a real.} \\ \therefore a^2 - 8b^2 &\leq 0 \\ \Rightarrow a^2 &\geq 8b^2 \end{aligned}$$

61. Chord of contact of mutually perpendicular tangents is always a focal chord. Thus, minimum length of AB is $4a$.

62. Let $P \equiv (h, k)$.

Tangent in terms of m is

$$\begin{aligned} y &= mx + \frac{a}{m} \\ \Rightarrow m^2x - my + a &= 0 \\ \text{If it passes through } (h, k), &\text{ then } m^2h - mk + a = 0. \end{aligned}$$

Angle between these tangents is $\frac{\pi}{4}$.

$$\begin{aligned} \text{Thus, } \tan \frac{\pi}{4} &= \frac{|m_1 - m_2|}{|1 + m_1m_2|} \\ \Rightarrow (m_1 - m_2)^2 &= (1 + m_1m_2)^2 \\ \Rightarrow (m_1 + m_2)^2 - 4m_1m_2 &= (1 + m_1m_2)^2 \\ \Rightarrow \frac{k^2}{h^2} - \frac{4a}{h} &= \left(1 + \frac{a}{h}\right)^2 \end{aligned}$$

Thus, the locus is $y^2 = 4ax + (a + x)^2$.

63. Let $A \equiv (t_1^2, 2t_1)$, $B \equiv (t_2^2, 2t_2)$, then $P \equiv (t_1t_2, t_1 + t_2)$.

Also, equations of PA and PB are

$$\begin{aligned} yt_1 &= x + at_1^2, yt_2 = x + at_2^2 \\ \text{Thus, } A_1 &\equiv (0, at_1), B_1 \equiv (0, at_2) \\ \text{Now, area of } \triangle PA_1B_1 &= \frac{1}{2} |A_1B_1| |t_1t_2| \end{aligned}$$

$$\begin{aligned} 2 &= \frac{1}{2} a |t_1 - t_2| |t_1t_2| \quad [\text{given}] \\ \Rightarrow a^2(t_1 - t_2)^2(t_1t_2)^2 &= 16 \\ \Rightarrow a^2[(t_1 + t_2)^2 - 4t_1t_2](t_1t_2)^2 &= 16 \\ \text{Thus, locus of } P &\text{ is } a^2(y^2 - 4x)x^2 = 16. \end{aligned}$$

64. Let the mid-point be $P(h, k)$. Equation of this chord is

$$\begin{aligned} T &= S_1, \text{ i.e. } yk - 2a(x + h) = k^2 - 4ah. \\ \text{It must pass through } (a, 0). \\ \Rightarrow -2a(a + h) &= k^2 - 4ah \\ \text{Thus, required locus is } y^2 &= 2ax - 2a^2. \end{aligned}$$

65. Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_2^2, 2at_2)$. Then,

$$m_{OA} = \frac{2}{t_1} \quad \text{and} \quad m_{OB} = \frac{2}{t_2}$$

$$\text{Since, } m_{OA} \cdot m_{OB} = -1 \Rightarrow t_1t_2 = -4$$

So, tangents will intersect at $P[at_1t_2, a(t_1 + t_2)]$.

Thus, locus of P is $x = -4a$ or $x + 4a = 0$.

66. Let $P(at^2, 2at)$ be a point on the parabola $y^2 = 4ax$ such that it cuts the parabola again at $Q(at_1^2, 2at_1)$. Then,

$$\begin{aligned} t_1 &= -t - \frac{2}{t} \\ \Rightarrow |t_1| &= \left| t + \frac{2}{t} \right| \\ \Rightarrow |t_1| &\geq \left| 2\sqrt{t \times \frac{2}{t}} \right| \quad [\text{using AM} \geq \text{GM}] \\ \Rightarrow |t_1| &\geq 2\sqrt{2} \\ \text{Now, } OQ &= \sqrt{(at_1^2 - 0)^2 + (2at_1 - 0)^2} \\ \Rightarrow OQ &= at_1\sqrt{t_1^2 + 4} \\ \Rightarrow OQ &\geq 2a\sqrt{2}\sqrt{8 + 4} \\ \Rightarrow OQ &\geq 4\sqrt{6}a \end{aligned}$$

67. Let the coordinates of point P be (h, k) .

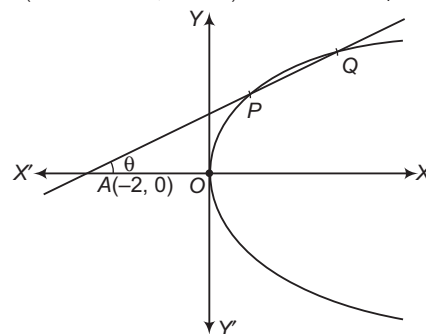
Equation of tangent of $x^2 = 4ay$, in terms of m , is

$$x = \frac{y}{m} + \frac{a}{m^2}$$

If it passes through P , then

$$\begin{aligned} m^2h - km - a &= 0 \\ \Rightarrow m_{PA} + m_{PB} &= \frac{k}{h}, m_{PA} \cdot m_{PB} = -\frac{a}{h} \\ \text{Now, } 4 &= m_{PA}^2 + m_{PB}^2 = (m_{PA} + m_{PB})^2 - 2m_{PA} \cdot m_{PB} \\ \Rightarrow 4 &= \frac{k^2}{h^2} + \frac{2a}{h} \\ \therefore \text{Locus is } y^2 &= 4x^2 - 2ax. \end{aligned}$$

68. Let $P(-2 + r \cos \theta, r \sin \theta)$ and Q lies on parabola



$$\begin{aligned} \Rightarrow r^2 \sin^2 \theta - 4(-2 + r \cos \theta) &= 0 \\ \Rightarrow r_1 + r_2 &= \frac{4 \cos \theta}{\sin^2 \theta}, r_1 r_2 = \frac{8}{\sin^2 \theta} \\ \text{Now, } \frac{1}{AP} + \frac{1}{AQ} &= \frac{r_1 + r_2}{r_1 r_2} = \frac{\cos \theta}{2} \end{aligned}$$

Given, $\frac{1}{AP} + \frac{1}{AQ} < \frac{1}{4}$
 $\Rightarrow \cos \theta < \frac{1}{2} \Rightarrow \tan \theta > \sqrt{3}$
 $[\because \cos \theta \text{ is decreasing and } \tan \theta \text{ is increasing in } (0, \pi/2)]$
 $\Rightarrow m > \sqrt{3}$

69. The equation of the normal at $(-2 + t^2, 2t)$ is

$$y - 2t = \frac{-1}{\left(\frac{dy}{dx}\right)_{(-2+t^2, 2t)}} \cdot (x + 2 - t^2)$$

$$\Rightarrow y - 2t = -\frac{2t}{2} \cdot (x + 2 - t^2)$$

$$\Rightarrow tx + y = 2t - 2t + t^3$$

$$\Rightarrow tx + y = t^3$$

It passes through $(k, 0)$, if $tk = t^3$.

$$\Rightarrow t(t^2 - k) = 0$$

It has three real values of t , if $k > 0$. So, the set of points $(k, 0)$ is the such that $k > 0$.

70. Equation of normal in terms of m is $y = mx - 4m - 2m^3$.

If it passes through $(a, 0)$, then $am - 4m - 2m^3 = 0$.

$$\Rightarrow m(a - 4 - 2m^2) = 0$$

$$\Rightarrow m = 0, m^2 = \frac{a - 4}{2}$$

For three distinct normals,

$$a - 4 > 0 \Rightarrow a > 4$$

71. Let AB be a normal chord, where

$$A \equiv (at_1^2, 2at_1), B \equiv (at_2^2, 2at_2)$$

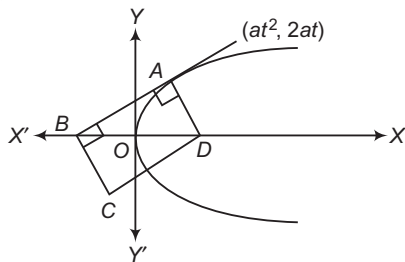
$$\text{We have, } t_2 = -t_1 - \frac{2}{t_1}$$

$$\begin{aligned} \text{Now, } AB^2 &= [a^2(t_1^2 - t_2^2)]^2 + 4a^2(t_1 - t_2)^2 \\ &= a^2(t_1 - t_2)^2 \{(t_1 + t_2)^2 + 4\} \\ &= a^2 \left(t_1 + t_2 + \frac{2}{t_1} \right)^2 \left(\frac{4}{t_1^2} + 4 \right) \\ &= \frac{16a^2(1 + t_1^2)^3}{t_1^4} = \frac{d(AB^2)}{dt_1} \\ &= 16a^2 \left(\frac{t^4 [3(1 + t_1^2)^2 \cdot 2t_1] - (1 + t_1^2)^3 \cdot 4t_1^3}{t_1^8} \right) \\ &= \frac{a^2 \cdot 32(1 + t_1^2)^2}{t_1^5} (t_1^2 - 2) \end{aligned}$$

$t_1 = \sqrt{2}$ is indeed the point of minima of AB^2 .

$$\therefore AB_{\min} = \frac{4a}{2} (1 + 2)^{3/2} = 2a\sqrt{27} \text{ units}$$

72. Equations of tangent and normal at A are $yt = x + at^2$ and $y = -tx + 2at + at^3$.



$$\Rightarrow B \equiv (-at^2, 0), D \equiv (2a + at^2, 0)$$

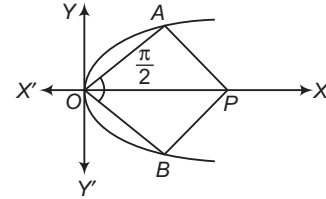
If $ABCD$ is a rectangle, then mid-points of BD and AC will be coincident.

$$\Rightarrow h = 2a, k + 2at = 0$$

$$\Rightarrow h = 2a, t = -\frac{k}{2a}$$

Thus, locus is $x = 2a$.

73. Let $A \equiv (t^2, 2t)$ and $B \equiv (t^2, 2t)$



$$m_{OA} = \frac{2}{t}, m_{OB} = -\frac{2}{t}$$

$$\Rightarrow t^2 = 4$$

$$\Rightarrow t = 2$$

Equation of normal AP is $y = -2x + 4 + 8$

$$\Rightarrow P \equiv (6, 0)$$

Thus, area of quadrilateral $OAPB$

$$= \frac{1}{2} (OP)(AB)$$

$$= \frac{1}{2} \cdot 6 \cdot 8 = 24 \text{ sq units}$$

74. Let AB be a normal chord, where

$$A \equiv (at_1^2, 2at_1) \text{ and } B \equiv (at_2^2, 2at_2)$$

If its mid-point is $P(h, k)$, then

$$2h = a(t_1^2 + t_2^2) = [(t_1 + t_2)^2 - 2t_1t_2]$$

and

$$2k = 2a(t_1 + t_2)$$

We have,

$$t_2 = -t_1 - \frac{2}{t_1}$$

$$\Rightarrow t_1 + t_2 = -\frac{2}{t_1} \text{ and } t_1t_2 = -t_1^2 - 2$$

$$\Rightarrow t_1 = -\frac{2a}{k} \text{ and } h = a \left(t_1^2 + 2 + \frac{2}{t_1^2} \right)$$

$$\text{Thus, required locus is } x = a \left(\frac{4a^2}{y^2} + 2 + \frac{y^2}{2a^2} \right)$$

75. Let r be the radius of the circle. Then, its equation is $(x + 4)^2 + y^2 = r^2$.

This cuts the parabola $y^2 = 8x$ at points $A(x_2, y_2)$ and $B(x_2, y_2)$.

The abscissae of A and B are the roots of the equation

$$(x + 4)^2 + 8x = r^2$$

$$\Rightarrow x^2 + 16x + 16 - r^2 = 0$$

$$\therefore x_1x_2 = 16 - r^2 \quad \dots (i)$$

The ordinates of A and B are given by $y_1^2 = 8x_1$ and

$$y_2^2 = 8x_2, \text{ respectively.}$$

$$\therefore y_1 = \pm 2\sqrt{2}x_1 \text{ and } y_2 = \pm 2\sqrt{2}x_2$$

Since, AB subtends a right angle at the vertex of the parabola.

$$\therefore \frac{y_1}{x_1} \times \frac{y_2}{x_2} = -1$$

$$\Rightarrow x_1x_2 + y_1y_2 = 0$$

$$\begin{aligned} \Rightarrow x_1 x_2 + 8x_1 x_2 &= 0 \\ \Rightarrow x_1 x_2 &= 0 & [\text{from Eq. (i)}] \\ \Rightarrow 16 - r^2 &= 0 \\ \Rightarrow r &= 4 \end{aligned}$$

76. Equation of normal is

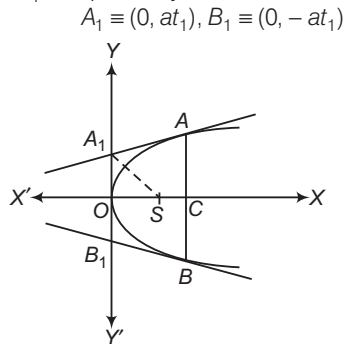
$$y = -\frac{bx}{a} - \frac{a^2}{b}$$

$$\therefore \text{Required distance} = \frac{|c + 2m^2 + m^4|}{\sqrt{1 + m^2}}$$

$$= (1 + m^2)^{3/2} \quad [\because c = 1]$$

77. Let $A \equiv (at_1^2, 2at_1)$, $B \equiv (at_1^2, -2at_1)$.

Equation of tangents at A and B are $yt_1 = x + at_1^2$ and $yt_1 = x + at_1^2$, respectively.



$$\therefore \text{Area of trapezium } AA_1B_1B$$

$$= \frac{1}{2} (AB + A_1B_1) \cdot OC$$

$$\Rightarrow 24a^2 = \frac{1}{2} \cdot (4at_1 + 2at_1) (at_1^2)$$

$$\Rightarrow t_1^3 = 8 \Rightarrow t_1 = 2$$

$$\Rightarrow A_1 \equiv (0, 2a)$$

If $\angle OSA_1 = \theta$, then

$$\tan \theta = \frac{2a}{a} = 2 \Rightarrow \theta = \tan^{-1}(2)$$

Thus, required angle is $2 \tan^{-1}(2)$.

78. Normals to $y^2 = 4ax$ and $x^2 = 4by$ in terms of m are

$$y = mx - 2am - am^3$$

and

$$y = mx + 2b + \frac{b}{m^3}$$

For a common normal,

$$2b + \frac{b}{m^2} - 2am + am^3 = 0$$

$$\Rightarrow am^5 + 2am^3 + 2bm^2 + b = 0$$

Thus, there can be atmost 5 common normals.

79. Tangent to $y^2 = 4x$ in terms of m is

$$y = mx + \frac{1}{m}$$

Normal to $x^2 = 4by$ in terms of m is

$$y = mx + 2b + \frac{b}{m^2}$$

If these are same lines, then $\frac{1}{m} = 2b + \frac{b}{m^2}$

$$\Rightarrow 2bm^2 - m + b = 0$$

For two different tangents,

$$1 - 8b^2 > 0 \Rightarrow |b| < \frac{1}{\sqrt{8}}$$

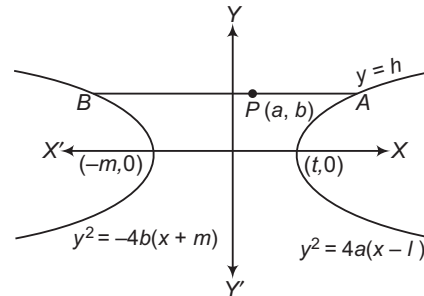
80. Let the two parabolas be $y^2 = 4a(x - l)$ and $y^2 = -4b(x + m)$. Further, let $y = h$ be a line parallel to the common axis.

This intersects the two parabolas at $A \left(\frac{h^2}{4a} + l, h \right)$ and $B \left(\frac{-h^2}{4b} - m, h \right)$ respectively.

Let $P(\alpha, \beta)$ be the mid-point of AB.

$$\text{Then, } 2\alpha = \frac{h^2}{4} \left(\frac{1}{a} - \frac{1}{b} \right) + l - m \text{ and } \beta = h$$

$$\Rightarrow 2\alpha = \frac{\beta^2}{4} \left(\frac{1}{a} - \frac{1}{b} \right) + l - m$$



Thus, the locus of (α, β) is

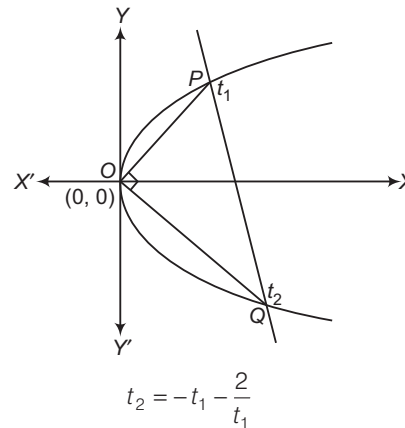
$$2x = \frac{y^2}{4} \left(\frac{1}{a} - \frac{1}{b} \right) + l - m$$

Clearly, it represents a parabola, if $a \neq b$.

If $a = b$, then we have

$2x = l - m$, which represents a line.

81. End point of latusrectum $x = a$, $y = \pm 2a$



82. By properties of parabola, options (a), (b) and (c) are always correct.

83. Equation of any tangent to $y^2 = 9x$ is of the form $yt = x + \frac{9}{4}t^2$.

Since, it passes through $(4, 10)$.

$$\therefore 10t = 4 + \frac{9}{4}t^2$$

$$\Rightarrow 9t^2 - 40t + 16 = 0$$

$$\Rightarrow t = 4, \frac{4}{9}$$

$\therefore$ Equation of tangent can be

$$x - 4y + 36 = 0$$

or

$$9x - 4y + 4 = 0$$

84. Given curve is $y = x^2 - x$ (i)

When $x - y = 0$ or $y = x$

Then, from Eq. (i),

$$x^2 - 2x = 0, \text{ its } D \neq 0$$

When $x + y = 0$ or $y = -x$

Again, from Eq. (i),

$$x^2 = 0, \text{ its } D = 0$$

When $x - y = 1$ or $y = x - 1$, then from Eq. (i),

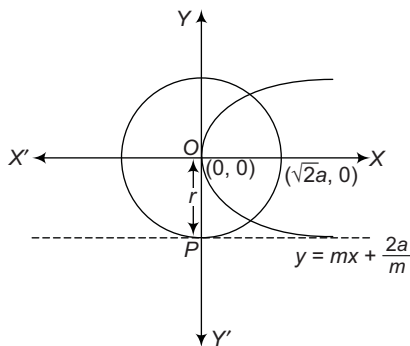
$$x^2 - 2x + 1 = 0, \text{ its } D = 0$$

When $x + y = 1$ or $y = 1 - x$, then from Eq. (i),

$$x^2 - 1 = 0, \text{ its } D \neq 0$$

Hence, only $x + y = 0$ and $x - y = 1$ are the tangents to given parabola.

85. The equation of tangent to $y^2 = 8ax$ is $y = mx + \frac{2a}{m}$.



Since, $y = mx + \frac{2a}{m}$ is tangent to $x^2 + y^2 = 2a^2$.

$$\therefore OP = r$$

$$\Rightarrow \frac{|0 - 0 + \frac{2a}{m}|}{\sqrt{1 + m^2}} = \sqrt{2}a$$

$$\Rightarrow \frac{2}{m^2} = 1 + m^2$$

$$\text{or } m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$$

$$\therefore m = \pm 1$$

$\Rightarrow$ Equation of tangent is

$$y = x + 2a \text{ and } y = -x - 2a$$

86. The equation of tangent to parabola is

$$y = mx + \frac{2a}{m} \quad \dots (i)$$

Length of a perpendicular from $(0, 0)$ to Eq. (i) $= \sqrt{2}a$ (Radius)

87. Let the equation of tangent from $(4, 10)$ be

$$y = mx + \frac{9}{4m}$$

Satisfying $(4, 10)$ with the above equation, we get

$$m = \frac{1}{4}, \frac{9}{4}$$

So, Statement I is true and Statement II is false.

88. Statement II is true as it is the definition of parabola.

From Statement I, we have

$$\sqrt{(x-1)^2 + (y+2)^2} = \frac{|3x + 4y + 5|}{5}$$

which is not parabola as point $(1, -2)$ lies on the line $3x + 4y + 5 = 0$. Hence, Statement I is false.

89. Clearly, Statement I is true. Also, the point of contact is given by $(a, -2a)$ and $(-2a, -a)$. Hence, common tangent can be found by joining these two points also. Hence, Statement II is also correct and is the correct explanation of Statement I.

90. If t_1 and t_2 are the parameters of a and b , then

$$t_1 t_2 = -4 \quad \dots (i)$$

$$\text{Also, } t_1 \left(-t_1 - \frac{2}{t_1} \right) = -4 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$t_1 = \sqrt{2}$$

$$\text{Also, } m_{AB} = \frac{2}{t_2 + t_1} = -t_1 = -\sqrt{2}$$

91. Any normal at t to the parabola $y^2 = 4x$ is

$$y + tx = 2t + t^3$$

Since, it passes through $(h, h+1)$.

$$\therefore h + 1 + th = 2t + t^3$$

$$\Rightarrow t^3 - (h+2)t - h - 1 = f(t) \quad [\text{say}]$$

$$\therefore f'(t) = 3t^2 - (h+2) = 3t^2 + (2-h) > 0 \quad [\because h < 2]$$

So, $f(t) = 0$ will have only one real root.

$$\text{Also, } (h+1)^2 - 4h = (h-1)^2 > 0, h \neq 1$$

Hence, $(h, h+1)$ lies outside the parabola.

92. End points of double ordinate $x = 4$ of parabola

$$y^2 = 4x \text{ are } (4, \pm 4)$$

$$\Rightarrow t_1 = \pm 2$$

$$\Rightarrow t_2 = -t_1 - \frac{2}{t_1} = \pm 3$$

$$\Rightarrow P(9, 6) \text{ and } P'(9, -6)$$

$$\therefore PP' = 12 \text{ units}$$

93. $y = ax^2 + c$

$$\therefore \frac{dy}{dx} = 2ax = 1$$

So, point of contact of the tangent is

$$\left(\frac{1}{2a}, \frac{1}{4a} + c \right)$$

Since, it lies on $y = x$.

$$\therefore c = \frac{1}{4a}$$

$$\text{Thus, } c = \frac{1}{8} \text{ for } a = 2$$

94. If $(1, 1)$ is point of contact, then $a = \frac{1}{2}$.

95. If $c = 2$, then point of contact is $\left(\frac{1}{2a}, \frac{1}{4a} + 2 \right)$.

Since, it lies on the line $y = x$.

$$\therefore \frac{1}{2a} = \frac{1}{4a} + 2$$

$$\text{i.e. } a = \frac{1}{8}$$

So, point of contact is $(4, 4)$.

96. $\therefore SP = e \cdot PM$

$$= e \cdot NZ = e(SZ - SN)$$

$$= e \cdot SZ - e \cdot SN$$

$$= l - e \cdot (r \cos \theta) \text{ or } r = l - e r \cos \theta$$

$$\Rightarrow r(1 + e \cos \theta) = l$$

$$\therefore \frac{l}{r} = 1 + e \cos \theta$$

$$\Rightarrow r = \frac{l}{1 + e \cos \theta} \quad \dots (i)$$

97. $\therefore e \cdot SZ = \text{Semi-latusrectum } SL = l$ [given]

$$\therefore SZ = \frac{l}{e}$$

98. $\therefore r \cos \theta = SZ = \frac{l}{e}$

$$\therefore \frac{l}{r} = e \cos \theta$$

99. Put $e = 1$ in Eq. (i), then the equation of parabola is

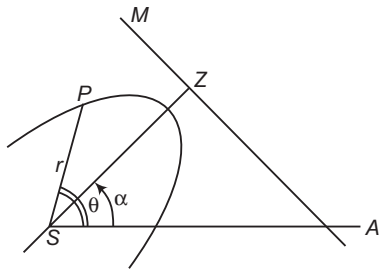
$$\frac{l}{r} = 1 + \cos \theta = 2 \cos^2 \left(\frac{\theta}{2} \right)$$

$$\therefore r = \frac{l}{2} \sec^2 \left(\frac{\theta}{2} \right)$$

100. Since, the axis SZ of the conic makes an angle α with the initial line SA , so the equation of the conic will be

$$\frac{l}{r} = 1 + e \cos (\theta - \alpha)$$

$$\therefore r = \frac{l}{1 + e \cos (\theta - \alpha)}$$



101. $(x - 4)^2 = -16(y - 1)$

$\therefore$ Focal distance of the parabola
 $(x - 4)^2 = -16(y - 1)$ is
 $|y - 1 - 4| = |y - 5|$

102. Equation of the normal is

$$x + 1 = m(y + 3) + 2m + m^3.$$

It passes through $(-1, k)$ on the axis.

$$\text{If } m(m^2 + k + 5) = 0$$

$$\therefore m = 0$$

$$\text{and } m = \pm \sqrt{-1(k + 5)},$$

$$\text{if } k < -5$$

Hence, $(-1, -6)$ and $(-1, k)$, $k \in (-\infty, -5)$.

103. Equation of tangent is

$$mx - m^2y = m^2a - a \quad \dots (i)$$

and given equation is $x \cos \alpha + y \sin \alpha = p \quad \dots (ii)$

On comparing Eqs. (i) and (ii), we get

$$a \cos 2\alpha = p \sin \alpha$$

104. A. $\therefore 25(x^2 + y^2) = (4x + 3y - 12)^2$

$$\Rightarrow 25x^2 + 25y^2 = 16x^2 + 9y^2 + 24xy - 96x - 72y + 144$$

$$\Rightarrow 9x^2 + 16y^2 - 24xy = -96x - 72y + 144$$

$$\Rightarrow (3x - 4y)^2 = -24(4x + 3y - 6)$$

$$\Rightarrow 25 \left(\frac{3x - 4y}{5} \right)^2 = -24 \left(\frac{4x + 3y - 6}{5} \right) \times 5$$

$$\Rightarrow \left(\frac{3x - 4y}{5} \right)^2 = -\frac{24}{5} \left(\frac{4x + 3y - 6}{5} \right)$$

Let $\frac{3x - 4y}{5} = Y$ and $\frac{4x + 3y - 6}{5} = X$

$$\therefore Y^2 = -\frac{24}{5}X$$

On comparing with $Y^2 = -4pX$, we get

$$4p = \frac{24}{5} \Rightarrow p = \frac{6}{5}$$

$\therefore$ Equation of directrix is

$$\frac{X - p}{5} = 0 \Rightarrow \frac{4x + 3y - 6}{5} - \frac{6}{5} = 0 \Rightarrow 4x + 3y = 12$$

Axis of the parabola is

$$\frac{Y}{5} = 0$$

$$\text{i.e. } \frac{3x - 4y}{5} = 0$$

$$\Rightarrow 3x - 4y = 0$$

B. $\therefore 25(x^2 + y^2) = (4x - 3y + 12)^2$

$$\text{or } 25x^2 + 25y^2 = 16x^2 + 9y^2 - 24xy + 96x - 72y + 144$$

$$\Rightarrow 9x^2 + 16y^2 + 24xy = 96x - 72y + 144$$

$$\Rightarrow (3x + 4y)^2 = 24(4x - 3y + 6)$$

$$\Rightarrow 25 \times \left(\frac{3x + 4y}{5} \right)^2 = 24 \left(\frac{4x - 3y + 6}{5} \right) \times 5$$

$$\Rightarrow \left(\frac{3x + 4y}{5} \right)^2 = \frac{24}{5} \left(\frac{4x - 3y + 6}{5} \right)$$

Let $\frac{3x + 4y}{5} = Y$ and $\frac{4x - 3y + 6}{5} = X$

$$\therefore Y^2 = \frac{24}{5}X$$

On comparing with $Y^2 = 4pX$, we get

$$4p = \frac{24}{5} \Rightarrow p = \frac{6}{5}$$

$\therefore$ Equation of direction is $X + p = 0$

$$\text{i.e. } \frac{4x - 3y + 6}{5} + \frac{6}{5} = 0$$

$$\Rightarrow 4x - 3y + 12 = 0$$

and axis of the parabola is $Y = 0$

$$\text{i.e. } \frac{3x + 4y}{5} = 0$$

$$\text{or } 3x + 4y = 0$$

C. $\therefore 25(x^2 + y^2) = (3x - 4y + 12)^2$

$$\Rightarrow 25x^2 + 25y^2 = 9x^2 + 16y^2 - 24xy + 72x - 96y + 144$$

$$\Rightarrow 16x^2 + 9y^2 + 24xy = 72x - 96y + 144$$

$$\Rightarrow (4x + 3y)^2 = 24(3x - 4y + 6)$$

$$\Rightarrow 25 \times \left(\frac{4x + 3y}{5} \right)^2 = 24 \left(\frac{3x - 4y + 6}{5} \right) \times 5$$

$$\Rightarrow \left(\frac{4x + 3y}{5} \right)^2 = \frac{24}{5} \left(\frac{3x - 4y + 6}{5} \right)$$

Let $\frac{4x + 3y}{5} = Y$ and $\frac{3x - 4y + 6}{5} = X$

$$\therefore Y^2 = \frac{24}{5}X$$

On comparing with $Y^2 = 4pX$, we get

$$4p = \frac{24}{5}$$

$$\Rightarrow p = \frac{6}{5}$$

∴ Equation of directrix is

$$\text{i.e. } \frac{3x - 4y + 6}{5} + \frac{6}{5} = 0$$

$$\Rightarrow 3x - 4y + 12 = 0$$

and axis of the parabola is $Y = 0$

$$\text{i.e. } 4x + 3y = 0$$

105. A. If AB is a normal chord, then $t_2 = \left(t_1 + \frac{2}{t_1}\right)$

B. If AB is a focal chord, then $t_1 t_2 = -1$.

C. If AB subtend 90° at $(0,0)$, then $t_1 t_2 = -4$.

$$\text{D. } \tan 45^\circ = \frac{2a(t_1 - t_2)}{a(t_1 - t_2)(t_1 + t_2)} = \frac{2}{t_1 + t_2}$$

$$\therefore 1 = \frac{2}{t_1 + t_2}$$

$$\Rightarrow t_1 + t_2 = 2$$

106. A. Any point on $x^2 = 16y$ is $p(8t, 4t^2)$.

The equation of the normal at P is

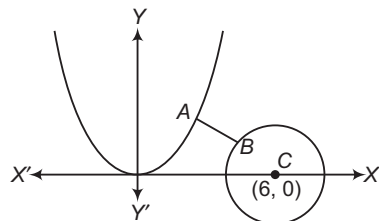
$$x + yt = 8t + 4t^3$$

This normal meets the parabola at Q whose parameter is t_1 , where $t_1 = -t - \frac{2}{t}$.

$$\begin{aligned} \text{Now, } PQ^2 &= (8t - 8t_1)^2 + (4t^2 - 4t_1^2)^2 \\ &= (2t - 2t_1)^2 \{16 + (2t + 2t_1)^2\} \\ &= \left(4t + \frac{4}{t}\right)^2 \left\{16 + \frac{16}{t^2}\right\} = \frac{256(1+t^2)^3}{t^4} \\ &= 256 \{t^{-4/3} + t^{2/3}\}^3 \\ &= 256 \left\{\frac{1}{2}t^{2/3} + \frac{1}{2}t^{2/3} + t^{-4/3}\right\}^3 \\ &\geq 256 \left(3\sqrt[3]{\frac{1}{4}}\right)^3 \quad [\text{using AM} \geq \text{GM}] \\ &= \frac{256 \times 27}{4} = 64 \times 9 \times 3 \\ &= 1728 \end{aligned}$$

$$\therefore PQ \geq 24\sqrt{3}$$

B. The pair of points A, B such that AB is minimum, is such that AB passes through the centre of the circle also, AB is normal at A to the parabola. A can be taken as $(4t, 2t^2)$ and the normal at A is $x + yt = 4t + 2t^3$.



As this has to pass through $(6, 0)$, this gives $t = 1$, equation of ABC is $x + y = 6$. This meets the circle $(x - 6)^2 + y^2 = 1$ at B whose y -coordinate is given by $2y^2 = 1$ or $y = \frac{1}{\sqrt{2}}$.

C. $P(t, 4t^2)$ and $Q(t_1^2, 4t_1^2)$

$$\therefore \angle POQ = 90^\circ$$

$$\Rightarrow 16t t_1 = -1$$

Centroid of ΔOPQ is

$$(x, y) = \left(\frac{t + t_1}{3}, \frac{4}{3}(t^2 + t_1^2)\right)$$

$$\therefore (3x)^2 - \frac{3}{4}y = (t + t_1)^2 - (t^2 + t_1^2) = 2t t_1 = -\frac{1}{8}$$

$$\Rightarrow 72x^2 - 6y = -1$$

$$\Rightarrow x^2 = \frac{1}{12}\left(y - \frac{1}{6}\right)$$

which is a parabola whose latusrectum is of length $\frac{1}{12}$.

D. If P is t_1 , Q is t_2 , then $t_1 t_2 = -1$

Equation of the normal at P and Q are

$$y + xt_1 = 2t_1 + t_1^3 \quad \text{and} \quad y + xt_2 = 2t_2 + t_2^3$$

If the normals intersect at (x, y) , then t_1 and t_2 are the two roots of the cubic equation in t .

$$t^3 + (2 - x)t - y = 0 \quad \dots(i)$$

If t_3 is the third root, then $t_1 t_2 t_3 = y$

$$\Rightarrow t_3 = -y, \text{ then } t_1 t_2 = 1$$

Since, $(-y)$ satisfies Eq. (i).

$$\therefore -y^3 - (2 - x)y - y = 0$$

$$\text{So, locus of } (x, y) \text{ is } y^2 + 2 - x + 1 = 0$$

which is a parabola whose latusrectum is 1.

107. For maximum number of common chord, circle and parabola must intersect in four distinct points. Let us first find the value r when circle and parabola touch each other.

For that solving the given curves, we have $(x - 6)^2 + 4x = r^2$ or $x^2 - 8x + 36 - r^2 = 0$ curves touch, if discriminant $D = 64 - 4(36 - r^2) = 0$ or $r^2 = 20$.

Hence, least integral value of r for which the curves intersect, is 5.

108. The normal to the parabola $y^2 = 4ax$ is

$$y = mx - 2am - am^3.$$

If it passes through a point (h, k) , then

$$k = 3m - 2am - am^3$$

This equation is cubic in m , it has three roots. Atleast one of them must be real.

Hence, the minimum number of normals is one.

109. Let $(at^2, 2at)$ be any point on $y^2 = ax$. Then, vertex $A(0,0)$.

∴ Equation of tangent at P is

$$ty = x + at^2 \quad \dots(i)$$

Tangent at P will be normal to the circle, AP is a chord

whose mid-point is $\left(\frac{at^2}{2}, at\right)$ and slopes is $\frac{2}{t}$.

∴ Equation of the line passing through mid-point of AP and perpendicular to AP is

$$y - at = -\frac{1}{2}\left(x - \frac{at^2}{2}\right)$$

$$\Rightarrow tx + 2y = \frac{at^3}{2} + 2at \quad \dots(ii)$$

Eqs. (i) and (ii) both pass through (x_1, y_1) , which is the centre of the circle.

$$\therefore ty_1 = x_1 + at^2 \quad \dots(iii)$$

$$\text{and } 2tx_1 + 4y = at^3 + 4at \quad \dots(iv)$$

On multiplying Eq. (ii) by t and subtracting from Eq. (iv), we get

$$t^2 y_1 + t(4a - 3x_1) - 4y_1 = 0 \quad \dots(v)$$

Also, from Eq. (iii), $at^2 - ty_1 + x_1 = 0$

On eliminating t from Eqs. (v) and (vi), we get

$$\frac{t^2}{x_1(4a - 3x_1) - 4y_1^2} = \frac{1}{-4ay_1 - x_1 y_1}$$

$$= \frac{1}{-y_1^2 - a(4a - 3x_1)}$$

On simplifying, we get

$$2y_1^2 (2y_1^2 - 12ax_1) = ax_1(3x_1 - 4a)^2$$

$$\Rightarrow k = 3$$

$$110. \therefore P = (am^2, -2am)$$

$$\text{and } Q = (am_1^2, -2am_1)$$

$$\therefore m_1 = -m - \frac{2}{m} \quad \dots(i)$$

Tangents at P and Q are

$$-my = x + am^2 \quad \dots(ii)$$

$$\text{and } -m_1 y = x + am_1^2 \quad \dots(iii)$$

Point of intersection of Eqs. (ii) and (iii)

$$y = -a(m + m_1) = \frac{2a}{m} \quad [\text{from Eq. (i)}]$$

$$\text{and } x = -2a - am^2$$

$$\therefore R \equiv \left[-2a - am^2, \frac{2a}{m} \right]$$

Let R be (x_1, y_1) , then PQ is the chord of contact of parabola w.r.t. R , is

$$yy_1 = 2a(x + x_1)$$

Length of perpendicular from R to PQ is

$$RM = \frac{y_1^2 - 4ax_1}{\sqrt{y_1^2 + 4a^2}}$$

Then, area of ΔPQR

$$= \frac{1}{2} PQ \cdot RM$$

$$= \frac{1}{2} \cdot \frac{\sqrt{(y_1^2 + 4a^2)} \sqrt{(y_1^2 - 4ax_1)}}{a\sqrt{y_1^2 + 4a^2}} (y_1^2 - 4ax_1)$$

$$= \frac{(y_1^2 - 4ax_1)^{3/2}}{2a}$$

$$\text{But } x_1 = -2a - am^2$$

$$y_1 = \frac{2a}{m}$$

$$\therefore \text{Area of } \Delta PQR = \frac{\left(\frac{4a^2}{m^2} + 8a^2 + 4a^2 m^2 \right)^{3/2}}{2a}$$

$$= \frac{4a^2 (1 + m^2)^3}{m^3}$$

$$\Rightarrow k = 4$$

$$111. \text{ Any tangent to } y^2 = 4x \text{ is}$$

$$y = mx + \frac{1}{m}$$

If is drawn from $(-2, -1)$, then

$$-1 = -2m + \frac{1}{m}$$

$$\Rightarrow 2m^2 - m - 1 = 0$$

$$\text{If } m = m_1, m_2, \text{ then } m_1 + m_2 = \frac{1}{2}$$

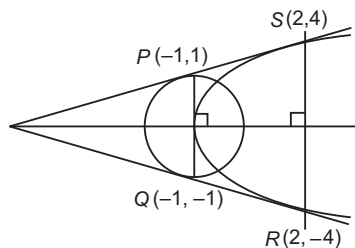
$$\text{and } m_1 m_2 = -\frac{1}{2}$$

$$\therefore \tan \alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\sqrt{(m_1 + m_2)^2 - 4m_1 m_2}}{1 + m_1 m_2}$$

$$= \frac{\sqrt{\frac{1}{4} + 2}}{1 - \frac{1}{2}} = 3$$

Entrances Gallery

1.



$$\text{Area of quadrilateral } PQRS = \left(\frac{1}{2} (1 + 4) \times 3 \right) \times 2$$

$$= 15 \text{ sq units}$$

2. Slope $(PK) = \text{Slope } (QR)$

$$\therefore \frac{2at - 0}{at^2 - 2a} = \frac{-\frac{2a}{t} - 2ar}{\frac{a}{t^2} - ar^2}$$

$$\Rightarrow \frac{t}{t^2 - 2} = -\left(\frac{\frac{1}{t} + r}{\frac{1}{t^2} - r^2} \right) \Rightarrow r = \frac{t^2 - 1}{t}$$

$$3. \text{ Tangent at } P, ty = x + at^2$$

$$\Rightarrow y = \frac{x}{t} + at \quad \dots(i)$$

$$\text{Normal at } S, y + \frac{x}{t} = \frac{2a}{t} + \frac{a}{t^3} \quad \dots(ii)$$

$$\text{On solving Eqs. (i) and (ii), we get } 2y = at + \frac{2a}{t} + \frac{a}{t^3}$$

$$\Rightarrow y = \frac{a(t^2 + 1)^2}{2t^3}$$

$$4. \text{ Let } P(at^2, 2at), Q\left(\frac{a}{t^2}, -\frac{2a}{t}\right) \text{ as } PQ \text{ is focal chord.}$$

Point of intersection of tangents at P and Q is

$$R\left[-a, a\left(t - \frac{1}{t}\right)\right]$$

Since, point of intersection lies on $y = 2x + a$.

$$\therefore a\left(t - \frac{1}{t}\right) = -2a + a$$

$$\Rightarrow t - \frac{1}{t} = -1$$

$$\Rightarrow \left(t + \frac{1}{t}\right)^2 = 5$$

$$\therefore \text{Length of focal chord} = a \left(t + \frac{1}{t}\right)^2 = 5a$$

5. Angle made by chord PQ at vertex (0, 0) is given by

$$\tan \theta = \left(\frac{\frac{2}{t} + 2t}{1 - 4} \right) = \frac{2 \left(\frac{1}{t} + t \right)}{-3} = \frac{-2}{3} \sqrt{5}$$

6. The parabola is $x = 2t^2$ and $y = 4t$.

Solving it with the circle, we get

$$4t^4 + 16t^2 - 4t^2 - 16t = 0$$

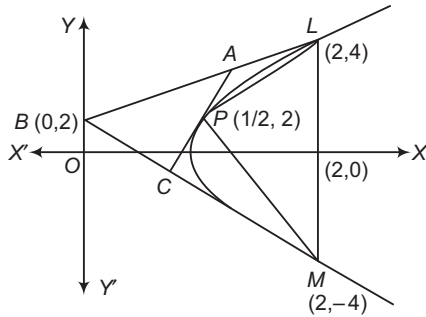
$$\Rightarrow t^4 + 3t^2 - 4t = 0$$

$$\Rightarrow t = 0, 1$$

So, the points P and Q are (0, 0) and (2, 4), which are also diametrically opposite points on the circle. The focus is S = (2, 0).

$$\therefore \text{Area of } \Delta PQS = \frac{1}{2} \times 2 \times 4 = 4 \text{ sq units}$$

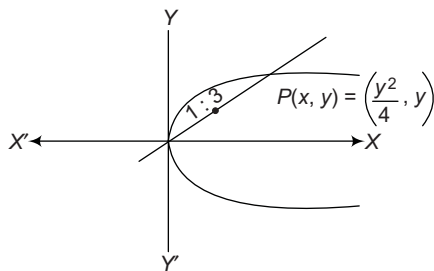
7. $\therefore y^2 = 8x = 4 \cdot 2 \cdot x$



$$\therefore \frac{\Delta LPM}{\Delta ABC} = 2$$

$$\text{Thus, } \frac{\Delta_1}{\Delta_2} = 2$$

8. $\therefore y^2 = 4x$



and P will lie on it.

$$\therefore (4k)^2 = 4 \times 4h \Rightarrow k^2 = h$$

$$\Rightarrow y^2 = x \quad [\text{replacing } h \text{ by } x \text{ and } k \text{ by } y]$$

9. $y^2 = 4x$

Equation of normal is $y = mx - 2m - m^3$.

Since, it passes through (9, 6).

$$\therefore m^3 - 7m + 6 = 0$$

$$\Rightarrow m = 1, 2, -3$$

$$\therefore \text{Normals are } y - x + 3 = 0; y + 3x - 33 = 0$$

$$\text{and } y - 2x + 12 = 0$$

10. Given, $A = (t_1^2, 2t_1)$ and $B = (t_2^2, 2t_2)$

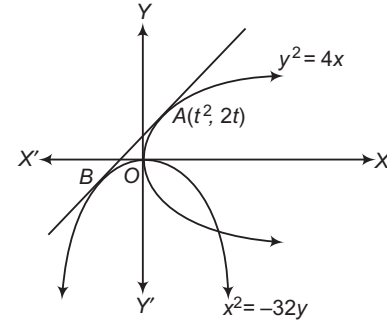
$$\text{Centre} = \left[\frac{t_1^2 + t_2^2}{2}, (t_1 + t_2) \right]$$

$$\therefore t_1 + t_2 = \pm r$$

$$\text{So, } m = \frac{2(t_1 - t_2)}{t_1^2 - t_2^2} = \frac{2}{t_1 + t_2} = \pm \frac{2}{r}$$

11. Equation of tangent at $A(t^2, 2t)$

$yt = x + t^2$ is tangent to $x^2 + 32y = 0$ at B.



$$\Rightarrow x^2 + 32 \left(\frac{x}{t} + t \right) = 0$$

$$\Rightarrow x^2 + \frac{32}{t}x + 32t = 0$$

$$\therefore b^2 - 4ac = 0$$

$$\Rightarrow \left(\frac{32}{t} \right)^2 - 4(32t) = 0$$

$$\Rightarrow 32 \left(\frac{32}{t^2} - 4t \right) = 0$$

$$\Rightarrow t^3 = 8$$

$$\Rightarrow t = 2$$

Hence, slope of tangent is $\frac{1}{t}$ i.e. $\frac{1}{2}$.

12. Equation of circle can be rewritten as

$$x^2 + y^2 = \frac{5}{2}$$

Centre = (0, 0) and radius = $\sqrt{\frac{5}{2}}$

Let common tangent be

$$y = mx + \frac{\sqrt{5}}{m}$$

$$\Rightarrow m^2x - my + \sqrt{5} = 0$$

The perpendicular from centre to the tangent is equal to radius of the circle.

$$\therefore \frac{\frac{\sqrt{5}}{m}}{\sqrt{1+m^2}} = \sqrt{\frac{5}{2}}$$

$$\Rightarrow m\sqrt{1+m^2} = \sqrt{2}$$

$$\Rightarrow m^2(1+m^2) = 2$$

$$\Rightarrow m^4 + m^2 - 2 = 0$$

$$\Rightarrow (m^2 + 2)(m^2 - 1) = 0$$

$$\Rightarrow m = \pm 1$$

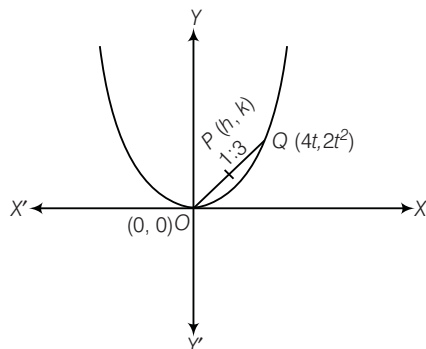
$$[\because m^2 + 2 \neq 0, \text{ as } m \in R]$$

$\therefore y = \pm (x + \sqrt{5})$, both statements are correct as $m = \pm 1$ satisfies the given equation of Statement II.

13. Given, equation of parabola is $x^2 = 8y$... (i)

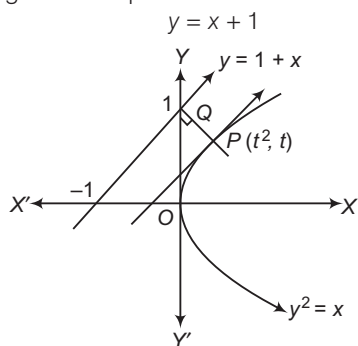
Let any point Q on the parabola (i) is $(4t, 2t^2)$.

Let $P(h, k)$ be the point which divides the line segment joining $(0, 0)$ and $(4t, 2t^2)$ in the ratio 1 : 3.



$$\begin{aligned} \therefore h &= \frac{1 \times 4t + 3 \times 0}{4} \\ \Rightarrow h &= t \\ \text{and } k &= \frac{1 \times 2t^2 + 3 \times 0}{4} \\ \Rightarrow k &= \frac{t^2}{2} \\ \Rightarrow k &= \frac{1}{2} h^2 \quad [\because t = h] \\ \Rightarrow 2k &= h^2 \Rightarrow 2y = x^2, \text{ which is required locus.} \end{aligned}$$

14. To find the shortest distance between $y - x = 1$ and $x = y^2$ is along the common normal.
 $\therefore$ Tangent at P is parallel to



$\therefore$ Slope of tangent at $P(t^2, t)$ is

$$\frac{dy}{dx} = \left(\frac{1}{2y} \right)_{(t^2, t)} = \frac{1}{2t} \quad \dots (ii)$$

$$\Rightarrow \frac{1}{2t} = 1 \quad [\text{as Eqs. (i) and (ii) are parallel}]$$

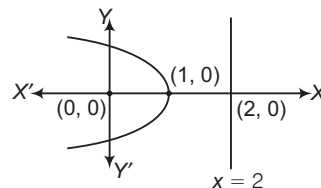
$$\Rightarrow t = \frac{1}{2}$$

$$\therefore P\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$\begin{aligned} \therefore \text{Shortest distance} &= |PQ| = \frac{\left| \frac{1}{4} - \frac{1}{2} + 1 \right|}{\sqrt{1+1}} = \frac{3}{4\sqrt{2}} \\ &= \frac{3\sqrt{2}}{8} \end{aligned}$$

15. We know that, the locus of point P from which two perpendicular tangents are drawn to the parabola, is the directrix of the parabola.
Hence, the required locus is $x = -1$.

16. From the figure, it is clear that vertex of the parabola at $(1, 0)$.



17. Since, perpendicular tangents intersect on the directrix, so point must lie on the directrix $x = -2$.
Hence, the required point is $(-2, 0)$.

18. The given equation of parabola is

$$\begin{aligned} y &= \frac{a^3 x^2}{3} + \frac{a^2 x}{2} - 2a \\ \Rightarrow y + 2a &= \frac{a^3}{3} \left(x^2 + \frac{3}{2a} x \right) \\ \Rightarrow y + 2a &= \frac{a^3}{3} \left(x^2 + \frac{3}{2a} x + \frac{9}{16a^2} - \frac{9}{16a^2} \right) \\ \Rightarrow y + 2a &= \frac{a^3}{3} \left(x + \frac{3}{4a} \right)^2 - \frac{9}{16a^2} \times \frac{a^3}{3} \\ \Rightarrow y + 2a + \frac{3a}{16} &= \frac{a^3}{3} \left(x + \frac{3}{4a} \right)^2 \\ \Rightarrow \left(y + \frac{35a}{16} \right) &= \frac{a^3}{3} \left(x + \frac{3}{4a} \right)^2 \end{aligned}$$

Thus, the vertices of parabola is $\left(-\frac{3}{4a}, -\frac{35a}{16} \right)$.

Let $h = -\frac{3}{4a}$

and $k = -\frac{35a}{16}$

Now, $hk = \frac{105}{64}$

$\therefore$ Locus of vertices of a parabola is $xy = \frac{105}{64}$

19. Since, the coordinates of P are $(1, 0)$.

Let any point Q on $y^2 = 8x$ be $(2t^2, 4t)$.

Again, let mid-point of PQ be (h, k) , then

$$\begin{aligned} h &= \frac{2t^2 + 1}{2} \\ \Rightarrow 2h &= 2t^2 + 1 \quad \dots (i) \end{aligned}$$

$$\begin{aligned} \text{and } k &= \frac{4t + 0}{2} \\ \Rightarrow t &= \frac{k}{2} \quad \dots (ii) \end{aligned}$$

On putting the value of t from Eq. (ii) in Eq. (i), we get

$$\begin{aligned} 2h &= \frac{2k^2}{4} + 1 \\ \Rightarrow 4h &= k^2 + 2 \end{aligned}$$

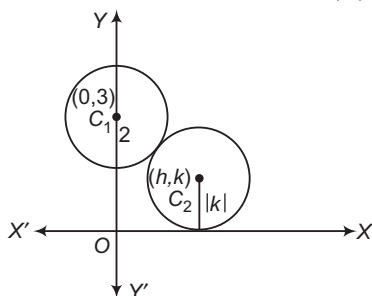
Hence, locus of (h, k) is $y^2 - 4x + 2 = 0$.

20. Since, circle touches X-axis and also touches circle with the centre at $(0, 3)$ and radius 2, so

$$C_1C_2 = r_1 + r_2$$

$$\therefore h^2 + (k-3)^2 = (|k|+2)^2$$

$$\Rightarrow h^2 + k^2 + 9 - 6k = k^2 + 4 + 4|k|$$



$$\therefore \text{Locus of centre of circle is}$$

$$x^2 = -5 + 6y + 4|y|$$

$$\Rightarrow x^2 = 10y - 5 \quad [\because y > 0]$$

which represents a parabola.

21. Given, equation of parabolas are $y^2 = 4ax$ and $x^2 = 4ay$

The point of intersection of parabolas are $A(0, 0)$ and $B(4a, 4a)$.

Also, given line $2bx + 3cy + 4d = 0$ passes through the points A and B , respectively.

$$\therefore d = 0 \quad \dots(i)$$

$$\text{and } 2b \cdot 4a + 3c \cdot 4a + 4d = 0$$

$$\Rightarrow 2ab + 3ac + d = 0$$

$$\Rightarrow a(2b + 3c) = 0 \quad [\because d = 0]$$

$$\Rightarrow 2b + 3c = 0 \quad \dots(ii)$$

On squaring and adding Eqs. (i) and (ii), we get

$$d^2 + (2b + 3c)^2 = 0$$

22. Equation of the normal at point $(bt_1^2, 2bt_1)$ on parabola is

$$y = -t_1x + 2bt_1 + bt_1^3$$

It also passes through $(bt_2^2, 2bt_2)$, then

$$2bt_2 = -t_1 \cdot bt_2^2 + 2bt_1 + bt_1^3$$

$$\Rightarrow 2t_2 - 2t_1 = -t_1(t_2^2 - t_1^2)$$

$$\Rightarrow 2(t_2 - t_1) = -t_1(t_2 + t_1)(t_2 - t_1)$$

$$\Rightarrow 2 = -t_1(t_2 + t_1) \Rightarrow t_2 = -t_1 - \frac{2}{t_1}$$

23. Given, equation of parabola can be rewritten as

$$(y+2)^2 = -4\left(x - \frac{1}{2}\right)$$

Let $y+2 = Y$ and $x - \frac{1}{2} = X$

$$\therefore Y^2 = -4X$$

Here, $a = 1$

$\therefore$ Equation of directrix is $X = a$

$$\Rightarrow x - \frac{1}{2} = 1 \Rightarrow x = \frac{3}{2}$$

24. Given, curve is

$$(7x+5)^2 + (7y+3)^2 = \lambda^2(4x+3y-24)^2$$

$$\Rightarrow 49x^2 + 25 + 70x + 49y^2 + 9 + 42y$$

$$= \lambda^2(16x^2 + 9y^2 + 576 + 24xy - 144y - 192x)$$

$$\Rightarrow (49 - 16\lambda^2)x^2 + (49 - 9\lambda^2)y^2 + (70 + 192\lambda^2)x$$

$$+ (42 + 144\lambda^2)y - 24\lambda^2xy + (25 - 576\lambda^2) = 0$$

On comparing with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = 49 - 16\lambda^2, b = 49 - 9\lambda^2, h = -12\lambda^2$$

Condition for parabola is $ab = h^2$.

$$\therefore (49 - 16\lambda^2)(49 - 9\lambda^2) = (-12\lambda^2)^2$$

$$\Rightarrow 2401 - 441\lambda^2 - 784\lambda^2 + 144\lambda^4 - 144\lambda^4 = 0$$

$$\Rightarrow 2401 - 1225\lambda^2 = 0$$

$$\Rightarrow \lambda^2 = \frac{2401}{1225}$$

$$\Rightarrow \lambda = \pm \frac{49}{35}$$

$$\therefore \lambda = \pm \frac{7}{5}$$

25. Given, equation of parabola is $x^2 = -9y$, which is of the form $x^2 = -4ay$ i.e. focus lies on the negative direction of Y-axis.

Here, $4a = 9 \Rightarrow a = 9/4$

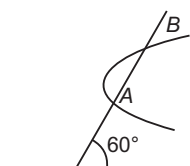
$\therefore$ Equation of directrix is $y = a$

$$\Rightarrow y = 9/4$$

and length of latusrectum $= 4a = 4 \times \frac{9}{4} = 9$

26. Given, $P = (\sqrt{3}, 0)$

$\therefore$ Equation of line AB is $\frac{x - \sqrt{3}}{\cos 60^\circ} = \frac{y - 0}{\sin 60^\circ} = r$ [say]



$$\Rightarrow x = \sqrt{3} + \frac{r}{2}, y = \frac{r\sqrt{3}}{2}$$

So, point $\left(\sqrt{3} + \frac{r}{2}, \frac{r\sqrt{3}}{2}\right)$ lies on $y^2 = x + 2$.

$$\therefore \frac{3r^2}{4} = \sqrt{3} + \frac{r}{2} + 2$$

$$\Rightarrow \frac{3r^2}{4} - \frac{r}{2} - (2 + \sqrt{3}) = 0$$

Let the roots be r_1 and r_2 , then the product

$$r_1 \times r_2 = PA \cdot PB = \left| \frac{-(2 + \sqrt{3})}{\frac{3}{4}} \right| = \frac{4(2 + \sqrt{3})}{3}$$

27. Let coordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$, respectively. Since, PQ subtends a right angle at the vertex $(0, 0)$.

Then, $t_1 t_2 = -4 \quad \dots(i)$

If (h, k) is the point of intersecting of normals at P and Q , then

$$h = 2a + a(t_1^2 + t_2^2 + t_1 t_2) \quad \dots(ii)$$

and $k = -at_1 t_2(t_1 + t_2) \quad \dots(iii)$

In order to find the locus of (h, k) , we have to eliminate t_1 and t_2 between Eqs. (i) and (iii).

$$k = 4a(t_1 + t_2) \quad \dots(iv)$$

and $h - 2a = a[(t_1 + t_2)^2 - t_1 t_2]$

$$\Rightarrow h - 2a = a \left[\frac{k^2}{16a^2} + 4 \right] \quad [\text{from Eqs. (i) and (iv)}]$$

$$\Rightarrow h - 6a = \frac{k^2}{16a}$$

Hence, the required locus is $y^2 = 16a(x - 6a)$.

- 28.** We know that, the sum of ordinates of foot of normals drawn from a point to the parabola $y^2 = 4ax$ is always zero.

Now, normals at three points P, Q and R of parabola $y^2 = 4ax$ meet at (h, k) .

So, the normals from (h, k) to $y^2 = 4ax$ meet the parabola at P, Q and R .

Thus y -coordinates y_1, y_2 and y_3 of these points P, Q and R will be zero.

Now, y -coordinate of the centroid of ΔPQR

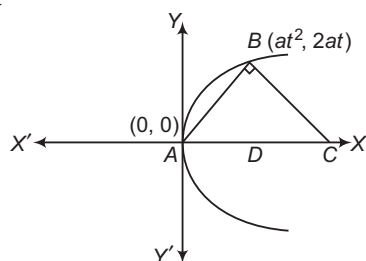
$$= \frac{y_1 + y_2 + y_3}{3} = \frac{0}{3} = 0$$

Hence, centroid lies on $y = 0$.

- 29.** Given, equation of parabola is $y^2 = 4ax$.

Let the coordinates of B be $(at^2, 2at)$, then the slope of

$$AB = \frac{2}{t}$$



Since,

$$\therefore \text{Slope of } BC = \frac{-t}{2}$$

The equation of BC is $y - 2at = -\frac{t}{2}(x - at^2)$.

Since, this line meets X -axis at point C .

So, put $y = 0$, we get

$$x = 4a + at^2$$

So, the distance $CD = 4a + at^2 - at^2 = 4a$

- 30.** Since, the centre of given circle $x^2 + y^2 - 8x + 2y = 0$ is $C_1(4, -1)$.

Given, equation of parabola is

$$y = x^2 - 4x + 10$$

$$\Rightarrow y = x^2 - 4x + 4 + 6$$

$$\Rightarrow y - 6 = (x - 2)^2$$

$\therefore$ Vertex, $V = (2, 6)$

$$\text{and slope of } C_1V = \frac{6+1}{2-4} = \frac{7}{-2} = -\frac{7}{2}$$

- 31.** Given equation of parabola is

$$y^2 = 12x$$

...(i)

On differentiating both sides w.r.t. x , we get

$$2y \frac{dy}{dx} = 12 \Rightarrow \frac{dy}{dx} = \frac{6}{y}$$

Since, the normal to the curve is parallel to the line $y = 4x + 3$.

$\therefore$ Slope of normal curve = Slope of line

$$\Rightarrow -\frac{y}{6} = 4$$

$$\Rightarrow y = -24$$

From Eq. (i), we get $(-24)^2 = 12x$

$$\Rightarrow 24 \times 24 = 12x$$

$$\Rightarrow x = 48$$

Thus, normal point on a curve is $(48, -24)$.

$\therefore$ Distance from $(48, -24)$ to the line $4x - y + 3 = 0$

$$= \frac{4 \times 48 + 24 + 3}{\sqrt{4^2 + 1^2}} = \frac{219}{\sqrt{17}}$$

- 32.** Given, equation of parabola is $y^2 = 64x$ (i)

The point at which the tangent of the curve is parallel to the line is the nearest point on the curve.

On differentiating both sides of Eq. (i), we get

$$2y \frac{dy}{dx} = 64$$

$$\Rightarrow \frac{dy}{dx} = \frac{32}{y}$$

Also, slope of the given line is $-4/3$.

$$\therefore -\frac{4}{3} = \frac{32}{y}$$

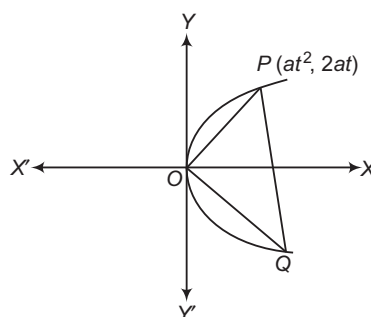
$$\Rightarrow y = -24$$

From Eq. (i), $(-24)^2 = 64x \Rightarrow x = 9$

Hence, the required point is $(9, -24)$.

- 33.** The perpendicular of the normal to the parabola $y^2 = 4ax$ at P is

$$y + tx = 2at + at^3$$



Suppose, it meets the parabola at Q . If O is the vertex of the parabola, then the combined equation of OP and OQ is a homogeneous equation of second degree.

$$\therefore y^2 = 4ax \left(\frac{y + tx}{2at + at^3} \right)$$

$$\Rightarrow y^2(2at + at^3) = 4ax(y + tx)$$

$$\Rightarrow 4atx^2 + 4axy - (2at + at^3)y^2 = 0$$

Since, OP and OQ are the right angles, so

Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\therefore 4at - 2at - at^3 = 0$$

$$\Rightarrow t^2 = 2$$

$$\Rightarrow t = \sqrt{2}$$

- 34.** Given equation of circle is $x^2 + y^2 = (2)^2$

$\therefore$ Equation of tangent to the circle is

$$y = mx \pm 2\sqrt{1+m^2} \quad [\because y = mx \pm r\sqrt{1+m^2}]$$

Also, equation of parabola is $y^2 = 32x$.

So, focus of parabola is (8, 0).
 Since, the line passes through focus (8, 0).
 $\therefore 0 = 8m \pm 2\sqrt{1+m^2}$
 $\Rightarrow -4m = \pm \sqrt{1+m^2}$
 $\Rightarrow 16m^2 = 1+m^2$
 $\Rightarrow 15m^2 = 1 \Rightarrow m^2 = \frac{1}{15}$
 $\therefore m = \pm \frac{1}{\sqrt{15}}$

35. Given, equation of parabola is $y^2 = 4x$.

Let two points on the parabola be
 $A(at_1^2, 2at_1)$ and $B(at_2^2, 2at_2)$.

Here, $a = 1$
 $\therefore A(t_1^2, 2t_1)$ and $B(t_2^2, 2t_2)$

Since, A and B are the end points of a diameter.

$\therefore$ Mid-point of AB = $\left(\frac{t_1^2 + t_2^2}{2}, t_1 + t_2\right)$

Also, radius of circle is r.

$\therefore$ Equation of circle is
 $\left(x - \frac{t_1^2 + t_2^2}{2}\right)^2 + [y - (t_1 + t_2)]^2 = (2)^2$

$$\Rightarrow x^2 + \left(\frac{t_1^2 + t_2^2}{2}\right)^2 - 2x\left(\frac{t_1^2 + t_2^2}{2}\right) + y^2 + (t_1 + t_2)^2 - 2y(t_1 + t_2) = 4$$

$$\Rightarrow x^2 + y^2 - 2x\frac{(t_1^2 + t_2^2)}{2} - 2y(t_1 + t_2) + \left(\frac{t_1^2 + t_2^2}{2}\right)^2 + (t_1 + t_2)^2 - 4 = 0$$

Here, $g = \frac{t_1^2 + t_2^2}{2}$ and $c = \left(\frac{t_1^2 + t_2^2}{2}\right)^2 + (t_1 + t_2)^2 - 4$

Since, the circle touch X-axis.

$$\therefore g^2 = c$$

$$\Rightarrow \left(\frac{t_1^2 + t_2^2}{2}\right)^2 = \left(\frac{t_1^2 + t_2^2}{2}\right)^2 + (t_1 + t_2)^2 - 4$$

$$\Rightarrow (t_1 + t_2)^2 = 4$$

$$\Rightarrow t_1 + t_2 = 2$$

Thus, slope of AB = $\frac{2(t_2 - t_1)}{(t_2^2 - t_1^2)}$

$$= \frac{2(t_2 - t_1)}{(t_2 - t_1)(t_2 + t_1)}$$

$$= \frac{2}{(t_2 + t_1)} = \frac{2}{2} = 1$$

36. Let P (h, k) be any point on $y^2 = 4x$ and its image about the line $x - y + 1 = 0$ be Q (α, β).

Since, mid-point of P and Q lies on the line.

$$\Rightarrow \frac{h + \alpha}{2} - \frac{k + \beta}{2} + 1 = 0$$

$$\Rightarrow h + \alpha - k - \beta + 2 = 0$$

$$\Rightarrow h - k = -\alpha + \beta - 2 \quad \dots(i)$$

Also, $\frac{\beta - k}{\alpha - h} \times 1 = -1 \quad [\because m_1 m_2 = -1]$

$$\Rightarrow \beta - k = -\alpha + h$$

$$\Rightarrow h + k = \alpha + \beta \quad \dots(ii)$$

From Eqs. (i) and (ii),

$$\begin{aligned} h &= \beta - 1, k = \alpha + 1 \\ \text{Since, } k^2 &= 4h \\ \Rightarrow (\alpha + 1)^2 &= 4(\beta - 1) \end{aligned}$$

So, required curve is $(x + 1)^2 = 4(y - 1)$.

37. Given curve is $y^2 = 4x$...(i)

and $x^2 = 4y$...(ii)

From Eq. (i), $2y \frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(1, 4)} = \frac{2}{4} = \frac{1}{2} = m_1$$

Now, from Eq. (ii),

$$2x = 4 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{2}$$

$$\Rightarrow \left[\frac{dy}{dx}\right]_{(1, 4)} = \frac{1}{2} = m_2$$

$\therefore m_1 = m_2$

So, tangents are parallel.

38. We know that, the equation of normal to parabola $y^2 = -4ax$ is

$$y = mx + 2am + am^3 \quad \dots(i)$$

Here, $4a = 8 \Rightarrow a = \frac{8}{4} = 2$

and equation of normal is

$$2x + y + k = 0$$

$$\Rightarrow y = -2x - k \quad \dots(ii)$$

On comparing Eqs. (i) and (ii), we get

$$m = -2 \text{ and } -k = 2am + am^3$$

$$\Rightarrow -k = 2(2)(-2) + 2(-2)^3$$

$$\Rightarrow -k = -8 - 16$$

$$\therefore k = 24$$

39. $y^2 + 4x + 4y + k = 0$

$$\Rightarrow y^2 + 2 \times 2y + 4 - 4 + 4x + k = 0$$

$$\Rightarrow (y + 2)^2 = -4x - k + 4$$

$$\Rightarrow (y + 2)^2 = -4\left(x + \frac{-4 + k}{4}\right)$$

$\therefore$ Latusrectum = 4 units

40. We have, equation of parabola is $y^2 = 4x$.

Let the coordinates of P and Q be $(t^2, 2t)$ and $(t_1^2, 2t_1)$ respectively.

R will be the mid-point of PQ.

Let the coordinate of R be (x, 0).

Then, $x = \frac{t^2 + t_1^2}{2}$ and $t + t_1 = 0$

$$\Rightarrow x = \frac{(t + t_1)^2 - 2t_1 t}{2}$$

$$\Rightarrow x = -t_1 t \quad \dots(i)$$

Now, in ΔPOQ , we have $PQ^2 = OP^2 + OQ^2$

$$\Rightarrow (t_1^2 - t^2)^2 + (2t_1 - 2t)^2 = (t_1^2)^2 + 2(t_1)^2 + (t^2)^2 + (2t)^2$$

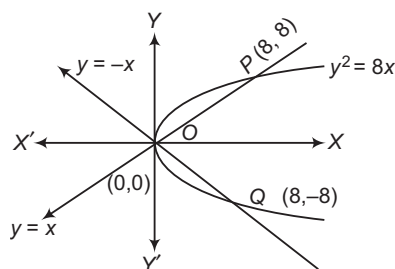
$$\Rightarrow t_1^4 + t^4 - 2t_1^2 t^2 + 4t_1^2 + 4t^2 - 8t_1 t = t_1^4 + 4t_1^2 + t^4 + 4t^2$$

$$\Rightarrow -2t_1^2 t^2 = 8t_1 t \Rightarrow t_1 t = -4$$

From Eq. (i), $x = 4$

41. We have, $y^2 = 8x$

$\therefore y^2 = \pm 8y$ $[\because x = \pm y]$



$\Rightarrow y^2 \mp 8y = 0 \Rightarrow y(y \pm 8) = 0$

$\Rightarrow y = 0, \pm 8$

$\therefore P(8, 8)$ and $Q(8, -8)$

$\therefore$ Length of $PQ = \sqrt{(16)^2 + (0)^2} = 16$

42. Required sum $= \frac{1}{2a} + \frac{1}{2a} = \frac{2}{2a} = \frac{1}{a}$

43. Given, equation of parabola is $y^2 + 8x - 12y + 20 = 0$.

$\Rightarrow (y - 6)^2 + 8x - 16 = 0$

$\Rightarrow (y - 6)^2 = 16 - 8x$

$\Rightarrow (y - 6)^2 = -4(2x - 4)$

Let $Y = y - 6$ and $X = 2x - 4$

Also, $Y = 0$ and $X = 0$

$\therefore y - 6 = 0 \Rightarrow y = 6$

and $2x - 4 = 0 \Rightarrow x = 2$

$\therefore$ Vertex is $(2, 6)$.

and focus is $(0, 6)$.

Now, latusrectum $= 4 \times 1 = 4$

and X -axis $= 4$

44. We have, $x^2 - 4x + 3 = y$

$\Rightarrow (x - 2)^2 = (y + 1)$

Let $X = x - 2$ and $Y = y + 1$

$\Rightarrow x - 2 = 0$ and $y + 1 = 0$

$\Rightarrow x = 2$ and $y = -1$

So, vertex of the parabola is $(2, -1)$.

Also, $(y - 3)^2 + (x - 0)^2 = 3^2$

Thus, centre of the circle is $(0, 3)$.

$\therefore$ Required distance $= \sqrt{(2 - 0)^2 + (-1 - 3)^2}$
 $= \sqrt{4 + 16}$
 $= 2\sqrt{5}$ units

45. The parabola $y^2 = 4x$ and the circle $x^2 + y^2 - 6x + 1 = 0$ will intersect, if $x^2 + 4x - 6x + 1 = 0$.

$\Rightarrow (x^2 - 2x + 1) = 0$

$\Rightarrow (x - 1)^2 = 0$

$\Rightarrow x = 1, 1$

$\therefore y = \sqrt{4} = \pm 2$

So, the intersecting points are $(1, -2)$ and $(1, 2)$.

46. We have, $y^2 = 12x$

$\therefore$ Equation of tangent in slope form is

$y = mx + \frac{a}{m}$

$y = \sqrt{3}x + \frac{3}{\sqrt{3}}$

$\Rightarrow y = \sqrt{3}(x + 1)$

$\Rightarrow \pm y - \sqrt{3}x - \sqrt{3} = 0$

47. Since, equation of the chord of the parabola $y^2 = 4ax$ joining the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$y(t_1 + t_2) = 2x + 2at_1t_2$... (i)

Also, line (i) joining t_1, t_2 on the parabola and passes through $(a, 0)$ i.e.

$2a + 2at_1t_2 = 0$

$\Rightarrow t_1t_2 = -1$

48. Let the point be $(3t^2, 6t)$.

$\therefore$ Focal distance $= 3t^2 + 3$

$\Rightarrow 3t^2 + 3 = 12$

$\Rightarrow 3t^2 = 9$

$\Rightarrow t^2 = 3$

$\Rightarrow t = \sqrt{3}$

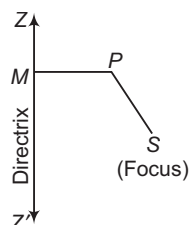
Hence, the required point is $(9, 6\sqrt{3})$.

15

Ellipse

Introduction

An ellipse is the locus of a point in a plane which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) in the same plane to its distance from a fixed straight line (called directrix) is always constant which is always less than unity.

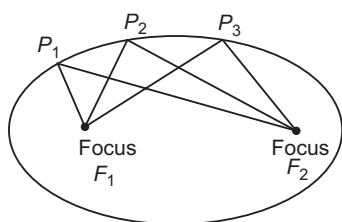


The constant ratio is generally denoted by e and is known as the eccentricity of the ellipse.

If S is the focus, ZZ' is the directrix and P is any point on the ellipse, then by definition

$$\frac{SP}{PM} = e \Rightarrow SP = e \cdot PM$$

or



$$P_1F_1 + P_1F_2 = P_2F_1 + P_2F_2 = P_3F_1 + P_3F_2$$

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points in the plane is a constant. The two fixed points are called the foci (plural of focus) of the ellipse. The constant which is the sum of the distances of a point on the ellipse from the two fixed points is always greater than the distance between the two fixed points. The mid-point of the line segment joining the foci is called the centre of the ellipse. The line segment through the foci of the ellipse is called the major axis and the line segment through the centre and perpendicular to the major axis is called the minor axis. The end points of the major axis are called the vertices of the ellipse.

Chapter Snapshot

- Introduction
- Position of a Point with Respect to an Ellipse
- Equation of the Chord
- Intersection of a Line and an Ellipse
- Tangent
- Combined Equation of the Pair of Tangents
- Director Circle
- Normal
- Number of Normals and Conormal Points
- Pole and Polar
- Conjugate Lines
- Diameter

Standard Form of the Ellipse

Let $P(x, y)$ be any point on the ellipse. Join SP and draw $PM \perp ZK$. Then, by definition, we have

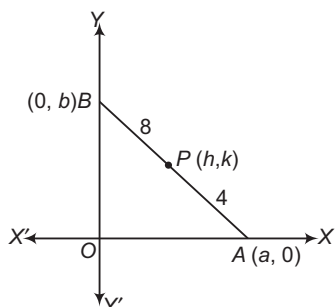
$$\begin{aligned} SP &= ePM \\ \Rightarrow SP^2 &= e^2 PM^2 \\ \Rightarrow SP^2 &= e^2 (NK)^2 \\ \Rightarrow (x - ae)^2 + (y - 0)^2 &= e^2 \left(\frac{a}{e} - x \right)^2 \\ \Rightarrow x^2(1 - e^2) + y^2 &= a^2(1 - e^2) \\ \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} &= 1 \\ \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1, \text{ where } b^2 = a^2(1 - e^2) \end{aligned}$$

which is the standard equation of the ellipse.

➤ **Example 1.** A ladder 12 units long slides in a vertical plane with its ends in contact with a vertical wall and a horizontal floor along X -axis. Then, the locus of a point on ladder 4 units from its foot is

- (a) circle (b) parabola
(c) straight line (d) ellipse

Sol. (d)



Given, $OA^2 + OB^2 = 12^2$
i.e. $a^2 + b^2 = 144$... (i)

Also, $\frac{PB}{PA} = \frac{8}{4} = 2$
 $\Rightarrow h = \frac{2a}{3}$ and $k = \frac{b}{3}$
 $a = \frac{3h}{2}$ and $b = 3k$

From Eq. (i), we get $\frac{9h^2}{4} + 9k^2 = 144$

$\Rightarrow \frac{x^2}{64} + \frac{y^2}{16} = 1$

which is equation of an ellipse.

➤ **Example 2.** The equation $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$

represents an ellipse, if

- (a) $a < 4$ (b) $a > 4$
(c) $4 < a < 10$ (d) $a > 10$

Sol. (a) We have, $\frac{x^2}{10 - a} + \frac{y^2}{4 - a} = 1$... (i)

Eq. (i) represents an ellipse $10 - a > 0$ and $4 - a > 0$

$\therefore 10 > a$ and $a < 4$

$\Rightarrow a < 10$ and $a < 4$, then $a < 4$

➤ **Example 3.** The curve represented by $x = 2(\cos t + \sin t)$, $y = 5(\cos t - \sin t)$ is

- (a) a circle
(b) a parabola
(c) an ellipse
(d) a hyperbola

Sol. (c) Here, $\frac{x}{2} = \cos t + \sin t$

and $\frac{y}{5} = \cos t - \sin t$

$\therefore \left(\frac{x}{2}\right)^2 + \left(\frac{y}{5}\right)^2 = 2(\sin^2 t + \cos^2 t) = 2$

$\Rightarrow \frac{x^2}{8} + \frac{y^2}{50} = 1$

which is equation of an ellipse.

➤ **Example 4.** The equation to the ellipse, whose focus is the point $(-1, 1)$, whose directrix is the straight line $x - y + 3 = 0$ and whose eccentricity is $\frac{1}{2}$ is

- (a) $7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$
(b) $x^2 + 2xy + 10x - 10y + 3 = 0$
(c) $3x^2 + xy + 10x - 10y + 3 = 0$
(d) None of the above

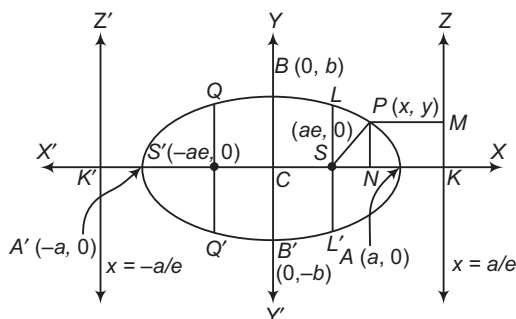
Sol. (a) Let $P(x, y)$ be any point on the ellipse, S be its focus and PN be the perpendicular from P on directrix, then by definition of an ellipse $PS^2 = e^2 PN^2$, hence

$$\begin{aligned} (x + 1)^2 + (y - 1)^2 &= \frac{1}{4} \left(\frac{x - y + 3}{\sqrt{2}} \right)^2 \\ &= \frac{(x - y + 3)^2}{8} \end{aligned}$$

[as focus is $(-1, 1)$ and directrix is $(x - y + 3 = 0)$]
 $\Rightarrow 8(x^2 + y^2 + 2x - 2y + 2) = x^2 + y^2 + 9 - 2xy$

$\Rightarrow 7x^2 + 2xy + 7y^2 + 10x - 10y + 7 = 0$

Some Basic Terms Related to an Ellipse



- i. **Vertices** The points A and A' , where the curve meets the line joining the foci S and S' , are called the vertices of the ellipse. The coordinates of A and A' are $(a, 0)$ and $(-a, 0)$, respectively.
- ii. **Major and minor axes** The distance $AA' = 2a$ and $BB' = 2b$ are called the major and minor axes of the ellipse, respectively. Since, $e < 1$ and $b^2 = a^2(1 - e^2)$. Therefore,

$$a > b \Rightarrow AA' > BB'$$
- iii. **Foci** The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the ellipse.
- iv. **Directrices** ZK and $Z'K'$ are two directrices of the ellipse and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$, respectively.
- v. **Centre** Since, the centre of a conic section is a point which bisects every chord passing through it. In case of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ every chord is bisected at $C(0, 0)$. Therefore, C is the centre of the ellipse and C is the mid-point of AA' .
- vi. **Eccentricity of the ellipse** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$
 For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, we have

$$b^2 = a^2(1 - e^2)$$

$$\Rightarrow e^2 = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow e^2 = 1 - \frac{4b^2}{4a^2}$$

$$\Rightarrow e^2 = 1 - \left(\frac{2b}{2a}\right)^2$$

$$\Rightarrow e = \sqrt{1 - \left(\frac{\text{Minor axis}}{\text{Major axis}}\right)^2}$$

 This formula gives the eccentricity of the ellipse.
- vii. **Ordinate and double ordinate** Let P be a point on the ellipse and let PN be perpendicular to the major axis AA' such that, PN produced meets the ellipse at P' . Then, PN is called the ordinate of P and PNP' the double ordinate of P .

- viii. **Latusrectum** A double ordinate passing through the focus is called the latusrectum and LS is called the semi-latusrectum. $Q, S'Q'$ is also a latusrectum. The coordinates of L are (ae, SL) . As L lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the coordinates of L will satisfy the equation of the ellipse, therefore

$$\frac{(ae)^2}{a^2} + \frac{(SL)^2}{b^2} = 1 \Rightarrow (SL)^2 = b^2(1 - e^2)$$

$$\Rightarrow (SL)^2 = b^2 \left(\frac{b^2}{a^2}\right)$$

$$\left[\because b^2 = a^2(1 - e^2) \Rightarrow 1 - e^2 = \frac{b^2}{a^2} \right]$$

 Thus, $SL = SL' = \frac{b^2}{a}$
 Hence, length of the latusrectum

$$= 2(SL) = \frac{2b^2}{a} = 2a(1 - e^2)$$
- ix. **Diameter** A chord of the ellipse passing through the centre is called the diameter of the ellipse all such chords are bisected by the centre.
- x. **Focal chord** A chord of the ellipse passing through its focus is called a focal chord.

☞ **Example 5.** The length of the latusrectum of an ellipse is one-third of the major axis. Its eccentricity would be

- (a) $\frac{2}{3}$ (b) $\sqrt{\frac{2}{3}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{\sqrt{2}}$

Sol. (b) Given, $2 \cdot \frac{b^2}{a} = \frac{1}{3}(2a)$

$$\Rightarrow 6b^2 = 2a^2 \Rightarrow 3b^2 = a^2$$

$$\Rightarrow 3a^2(1 - e^2) = a^2 \Rightarrow 3 - 3e^2 = 1$$

$$\Rightarrow 3e^2 = 2 \Rightarrow e^2 = \frac{2}{3}$$

$$\Rightarrow e = \sqrt{\frac{2}{3}}$$

☞ **Example 6.** An ellipse has its centre at $(1, -1)$ and semi-major axis is equal to 8 and which passes through the point $(1, 3)$. Then, the equation of the ellipse is

- (a) $\frac{(x-1)^2}{64} + \frac{(y+1)^2}{16} = 1$ (b) $\frac{(x-1)^2}{64} + \frac{(y+1)^2}{32} = 1$
 (c) $\frac{(x-1)^2}{16} + \frac{(y+1)^2}{64} = 1$ (d) $\frac{(x+1)^2}{64} + \frac{(y-1)^2}{16} = 1$

Sol. (a) Equation of the ellipse with centre at $(1, -1)$ can be written as $\frac{(x-1)^2}{a^2} + \frac{(y+1)^2}{b^2} = 1$, where a = semi-major axis and b = semi-minor axis.

If $a = 8$, then equation of an ellipse is

$$\frac{(x-1)^2}{64} + \frac{(y+1)^2}{b^2} = 1$$

This passes through point $(1, 3)$.

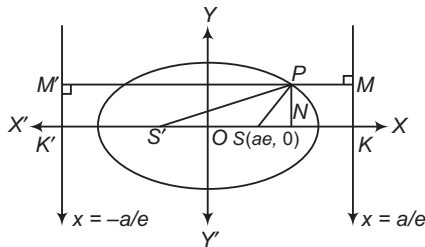
$$\therefore 0 + \frac{16}{b^2} = 1$$

$$\Rightarrow b^2 = 16$$

Hence, equation of ellipse is

$$\frac{(x-1)^2}{64} + \frac{(y+1)^2}{16} = 1$$

Focal Distances of a Point on the Ellipse



Let $P(x, y)$ be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then by definition, we have

$$\begin{aligned} SP &= e \cdot PM \quad \text{and} \quad S'P = e \cdot PM' \\ \Rightarrow SP &= e(NK) \quad \text{and} \quad S'P = e(NK') \\ \Rightarrow SP &= e(CK - CN) \quad \text{and} \quad S'P = e(CK' + CN) \\ \Rightarrow SP &= e\left(\frac{a}{e} - x\right) \quad \text{and} \quad S'P = e\left(\frac{a}{e} + x\right) \\ \Rightarrow SP &= a - ex \quad \text{and} \quad S'P = a + ex \end{aligned}$$

Thus, the focal distances of a point $P(x, y)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $a - ex$ and $a + ex$.

$$\begin{aligned} \text{Also, } SP + S'P &= a - ex + a + ex \\ &= 2a = \text{Major axis} = \text{Constant} \end{aligned}$$

Hence, the sum of the focal distances of a point on the ellipse is constant and is equal to the length of the major axis of the ellipse.

➤ **Example 7.** A man running round a race course notes that the sum of the distances of two flag-posts from him is always 10 m and the distance between the flag-posts is 8 m. The area of the path he encloses in square metres is

- (a) 15π (b) 12π (c) 18π (d) 8π

Sol. (a) Clearly, the race course is an ellipse with flag-posts as its foci. If a and b are the semi-major and semi-minor axes of the ellipse, then $2a = 10$ and $2ae = 8$.

$$\Rightarrow a = 5, e = \frac{4}{5}$$

$$\begin{aligned} \text{Now, } b^2 &= a^2(1 - e^2) \\ \Rightarrow b^2 &= 25\left(1 - \frac{16}{25}\right) = 9 \Rightarrow b = 3 \end{aligned}$$

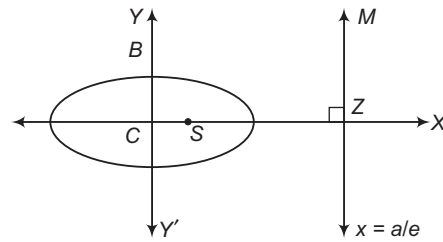
So, area of the ellipse = $\pi ab = 15\pi$

➤ **Example 8.** With a given point and line as focus and directrix, a series of ellipses are described. The locus of the extremities of their minor axes is

- (a) an ellipse (b) a parabola
(c) a hyperbola (d) None of these

Sol. (b) Let S be the given focus and ZM be the given line.

$$\text{Then, } SZ = CZ - CS = \frac{a}{e} - ae$$



$$= \frac{a}{e}(1 - e^2) = \frac{b^2}{ae} = k \quad [\text{say}]$$

$$\text{As } b^2 = a^2(1 - e^2)$$

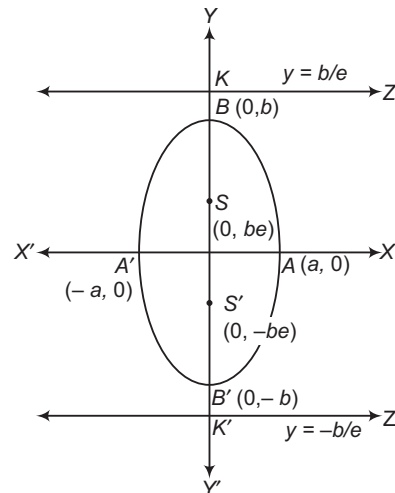
Let (x, y) be the coordinates of B with respect to these axes.

Then, $x = SC = ae$ and $y = CB = b$, hence $\frac{y^2}{x} = \frac{b^2}{ae} = k$

Therefore, $y^2 = kx$, is the required locus and is clearly a parabola.

Other Forms of the Ellipse

The equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $a > b$ or $a^2 > b^2$ (denominator of x^2 is greater than that of y^2), then the major and minor axes lie along X and Y -axes respectively. But, if $a < b$ or $a^2 < b^2$ (denominator of x^2 is less than that of y^2), then the major axis of the ellipse lies along the Y -axis and is of length $2b$ and the minor axis along the X -axis and is of length $2a$.



The coordinates of foci S and S' are $(0, be)$ and $(0, -be)$, respectively. The equations of the directrices ZK and $Z'K'$ are $y = \pm \frac{b}{e}$ and eccentricity e is given by the formula

$$a^2 = b^2(1 - e^2) \quad \text{or} \quad e = \sqrt{1 - \frac{a^2}{b^2}}$$

Various result related to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$ and $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a < b)$ are given in the following table for ready reference:

	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a < b$
Coordinates of the centre	$(0, 0)$	$(0, 0)$
Coordinates of the vertices	$(a, 0)$ and $(-a, 0)$	$(0, b)$ and $(0, -b)$
Coordinates of foci	$(ae, 0)$ and $(-ae, 0)$	$(0, be)$ and $(0, -be)$
Length of major axis	$2a$	$2b$
Length of minor axis	$2b$	$2a$
Equation of the major axis	$y = 0$	$x = 0$
Equation of the minor axis	$x = 0$	$y = 0$
Equation of the directrices	$x = \frac{a}{e}$ and $x = -\frac{a}{e}$	$y = \frac{b}{e}$ and $y = -\frac{b}{e}$
Eccentricity	$e = \sqrt{1 - \frac{b^2}{a^2}}$	$e = \sqrt{1 - \frac{a^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Focal distances of a point (x, y)	$a \pm ex$	$b \pm ey$

Equation of an Ellipse whose Axes are Parallel to Coordinate Axes and Centre is (h, k)

Equation of such an ellipse is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1, a > b$$

Foci : $(h \pm ae, k)$

Vertex : $(h \pm a, k)$

Directrix : $x = h \pm \frac{a}{e}$

If $a < b$, then

Foci : $(h, k \pm be)$

Vertex : $(h, k \pm b)$

Directrix : $y = k \pm \frac{b}{e}$

Example 9. The foci of the ellipse

$$2x^2 + 3y^2 - 4x - 12y + 13 = 0$$

$$(a) \left(\sqrt{2}, \frac{2}{\sqrt{3}} \right) \quad (b) \left(2, 1 \pm \frac{1}{\sqrt{6}} \right)$$

$$(c) \left(1 \pm \frac{1}{\sqrt{6}}, 2 \right) \quad (d) \text{None of these}$$

Sol. (c) The given equation can be rewritten as

$$2(x^2 - 2x) + 3(y^2 - 4y) + 13 = 0$$

$$\Rightarrow 2(x-1)^2 + 3(y-2)^2 = 1$$

$$\Rightarrow \frac{(x-1)^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{(y-2)^2}{\left(\frac{1}{\sqrt{3}}\right)^2} = 1$$

Here, $h = 1, k = 2$

$$a = \frac{1}{\sqrt{2}} \quad \text{and} \quad b = \frac{1}{\sqrt{3}}$$

Now, $a > b$

$\therefore$ Foci is $(h \pm ae, k)$

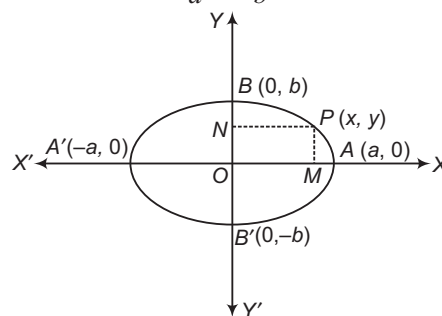
$$\text{or} \quad ae = \sqrt{a^2 - b^2} = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{\sqrt{3}}\right)^2}$$

$$\Rightarrow ae = \sqrt{\frac{1}{6}} = \frac{1}{\sqrt{6}}$$

Then, foci is $\left(1 \pm \frac{1}{\sqrt{6}}, 2 \right)$.

Equation of an Ellipse Referred to Two Perpendicular Lines

Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as shown in figure.



Let $P(x, y)$ be any point on the ellipse. Then, $PM = y$ and $PN = x$.

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{PN^2}{a^2} + \frac{PM^2}{b^2} = 1$$

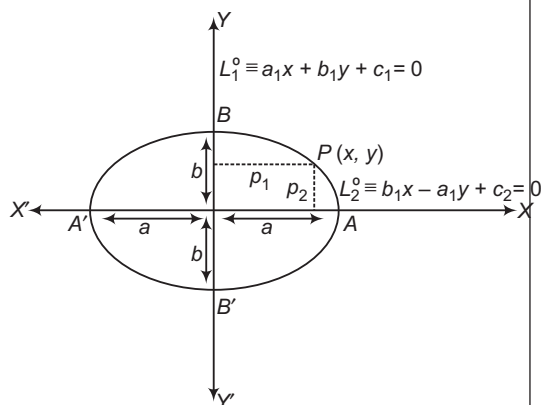
It follows from this, that if perpendicular distances p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight lines $L_1 \equiv a_1x + b_1y + c_1 = 0, L_2 \equiv b_1x - a_1y + c_2 = 0$ respectively satisfy the equation

$$\frac{p_1^2}{a^2} + \frac{p_2^2}{b^2} = 1$$

$$\Rightarrow \frac{\left(\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}}\right)^2}{a^2} + \frac{\left(\frac{b_1x - a_1y + c_2}{\sqrt{b_1^2 + a_1^2}}\right)^2}{b^2} = 1$$

Then, the point P describes an ellipse in the plane of the given lines such that

- i. The centre of the ellipse is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$



- ii. The major axis lies along $L_2 = 0$ and the minor axis lies along $L_1 = 0$, if $a > b$.
If $a < b$, then the major axis lies along $L_1 = 0$ and the minor axis lies along $L_2 = 0$.
- iii. The length of the major and minor axes are $2a$ and $2b$, if $a > b$.

➤ **Example 10.** The equation to the ellipse whose axes are of lengths 6 and $2\sqrt{6}$ and their equations are $x - 3y + 3 = 0$ and $3x + y - 1 = 0$ respectively, is

- (a) $19x^2 - 9xy + 6y^2 + 3x - 68y - 156 = 0$
(b) $15x^2 - 3xy + 9y^2 + 6x - 48y + 151 = 0$
(c) $21x^2 - 6xy + 29y^2 + 6x - 58y - 151 = 0$
(d) None of the above

Sol. (c) Let $P(x, y)$ be any point on the ellipse and p_1, p_2 be the lengths of perpendiculars drawn from P on the major and minor axes of the ellipse. Then,

$$p_1 = \frac{x - 3y + 3}{\sqrt{1 + 9}}$$

and

$$p_2 = \frac{3x + y - 1}{\sqrt{9 + 1}}$$

Let $2a$ and $2b$ be the lengths of major and minor axes of the ellipse.

We have, $2a = 6$ and $2b = 2\sqrt{6}$
 $\Rightarrow a = 3$ and $b = \sqrt{6}$

The equation of the ellipse is

$$\begin{aligned} \frac{p_1^2}{b^2} + \frac{p_2^2}{a^2} &= 1 \\ \Rightarrow \left\{ \frac{x - 3y + 3}{\sqrt{1 + 9}} \right\}^2 + \left\{ \frac{3x + y - 1}{\sqrt{9 + 1}} \right\}^2 &= 1 \\ \Rightarrow \frac{(x - 3y + 3)^2}{60} + \frac{(3x + y - 1)^2}{90} &= 1 \\ \Rightarrow 3(x - 3y + 3)^2 + 2(3x + y - 1)^2 &= 180 \\ \Rightarrow 21x^2 - 6xy + 29y^2 + 6x - 58y - 151 &= 0 \end{aligned}$$

Position of a Point with Respect to an Ellipse

The point $P(x_1, y_1)$ lies outside, on or inside the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ according $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1$ is $> 0, =, < 0$.

➤ **Example 11.** Let E be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$

and C be the circle $x^2 + y^2 = 9$. If P and Q are the points $(1, 2)$ and $(2, 1)$, respectively. Then,

- (a) Q lies inside C but outside E
(b) Q lies outside both C and E
(c) P lies inside both C and E
(d) P lies inside C but outside E

Sol. (d) The given ellipse is

$$\frac{x^2}{9} + \frac{y^2}{4} = 1.$$

For $x = 1, y = 2$, the value of the expression

$$\frac{x^2}{9} + \frac{y^2}{4} - 1 \text{ is positive}$$

and for $x = 2, y = 1$, the value of the above expression is negative. Therefore, P lies outside E and Q lies inside E . The value of the expression $x^2 + y^2 - 9$ is negative for both the points P and Q .

Therefore, P and Q both lie inside C .

$\therefore P$ lies inside C but outside E .

➤ **Example 12.** The number of real tangents that can be drawn to the ellipse $3x^2 + 5y^2 = 32$ passing through $(3, 5)$ is

- (a) 0
(b) 1
(c) 2
(d) infinite

Sol. (c) Since, $3(3)^2 + 5(5)^2 - 32 = 120 > 0$. So, the given point lies outside the ellipse. Hence, two real tangents can be drawn from this point to the ellipse.

Work Book Exercise 15.1

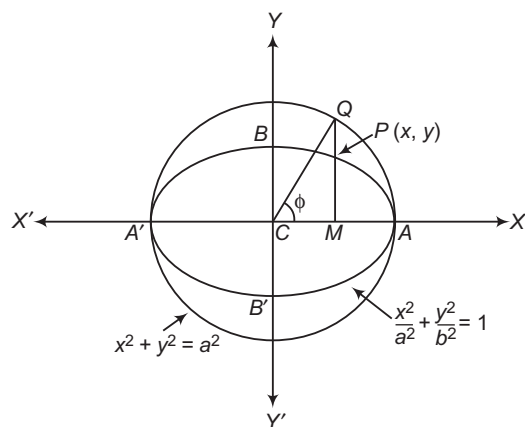
- If the eccentricity of the ellipse $\frac{x^2}{a^2+1} + \frac{y^2}{a^2+2} = 1$ is $\frac{1}{\sqrt{6}}$, then the length of latusrectum is
 - $\frac{5}{\sqrt{6}}$
 - $\frac{10}{\sqrt{6}}$
 - $\frac{8}{\sqrt{6}}$
 - None of the above
- If S_1, S_2 are foci of an ellipse of major axis of length 10 units and P is any point on the ellipse such that perimeter of ΔPS_1S_2 is 15. Then, eccentricity of the ellipse is
 - 0.5
 - 0.25
 - 0.28
 - 0.75
- The length of major axis of the ellipse $(5x-10)^2 + (5y+15)^2 = \frac{(3x-4y+7)^2}{4}$ is
 - 10
 - $\frac{20}{3}$
 - $\frac{20}{7}$
 - 4
- If S and S' are the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and P is any point on it, then range of value of $SP \cdot S'P$ is
 - $9 \leq f(\theta) \leq 16$
 - $9 \leq f(\theta) \leq 25$
 - $16 \leq f(\theta) \leq 25$
 - $1 \leq f(\theta) \leq 16$
- The equation of the ellipse, whose focus is $(1, -1)$, the directrix is the line $x - y = 3$ and eccentricity is $1/2$, is
 - $7x^2 + 2xy + 7y^2 - 10x + 10y + 7 = 0$
 - $7x^2 + 2xy + 7y^2 + 7 = 0$
 - $7x^2 + 2xy + 7y^2 + 10x - 10y - 7 = 0$
 - None of the above

Parametric Coordinates and Equations

i. **Auxiliary circle** The circle described on the major axis of an ellipse as diameter is called an auxiliary circle of the ellipse.

If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an ellipse, then its auxiliary circle is $x^2 + y^2 = a^2$.

ii. **Eccentric angle of a point** Let P be any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Draw PM perpendicular from P on the major axis of the ellipse and produce MP to the auxiliary circle in Q . Join CQ . The $\angle ACQ = \phi$ is called the eccentric angle of the point P on the ellipse.



↖ The $\angle ACP$ is not the eccentric angle of point P .

iii. **Parametric coordinates of a point on an ellipse** Let $P(x, y)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and Q be the corresponding point on the auxiliary circle $x^2 + y^2 = a^2$.

Let the eccentric angle of P be ϕ .

Then, $\angle ACQ = \phi$.

Now, $x = CM$

$$\Rightarrow x = CQ \cos \phi = a \cos \phi$$

[$\because CQ = \text{radius of } x^2 + y^2 = a^2$]

Since, $P(x, y)$ lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore,

$$\frac{a^2 \cos^2 \phi}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow y^2 = b^2 (1 - \cos^2 \phi) = b^2 \sin^2 \phi$$

$$\Rightarrow y = b \sin \phi$$

Thus, the coordinates of point P having eccentric angle ϕ can be written as $(a \cos \phi, b \sin \phi)$ and are known as the parametric coordinates.

iv. **Parametric equations** The equations $x = a \cos \phi$, $y = b \sin \phi$ taken together are called the parametric equations of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where ϕ is the parameter.

➤ **Example 13.** The sum of the squares of the reciprocals of two perpendicular diameter of an ellipse is

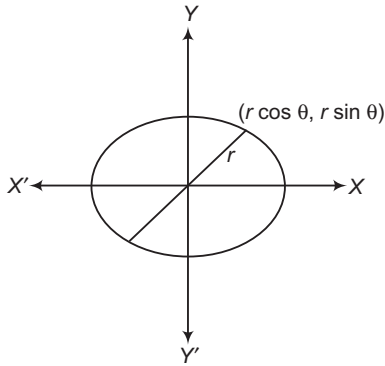
- (a) $\frac{1}{4} \left[\frac{1}{a^2} + \frac{1}{b^2} \right]$ (b) $\frac{1}{2} \left[\frac{1}{a^2} + \frac{1}{b^2} \right]$
(c) $\frac{1}{a^2} + \frac{1}{b^2}$ (d) None of these

Sol. (a) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

On putting $x = r \cos \theta$ and $y = r \sin \theta$

We have, the polar equation as

$$\frac{r^2 \cos^2 \theta}{a^2} + \frac{r^2 \sin^2 \theta}{b^2} = 1$$



$$\Rightarrow \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} = \frac{1}{r^2}$$

where, r is the radius vector and if r and r_1 be the perpendicular radii vectors respectively, then the diameters will be $2r$ and $2r_1$ and the corresponding angles will be θ and $(90^\circ + \theta)$.

$$\text{So, } \frac{1}{4r^2} + \frac{1}{4r_1^2}$$

$$= \frac{1}{4} \left[\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) + \left(\frac{\cos^2 (90^\circ + \theta)}{a^2} + \frac{\sin^2 (90^\circ + \theta)}{b^2} \right) \right]$$

$$= \frac{1}{4} \left[\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} + \frac{\sin^2 \theta}{a^2} + \frac{\cos^2 \theta}{b^2} \right]$$

$$= \frac{1}{4} \left[\frac{1}{a^2} + \frac{1}{b^2} \right]$$

which is a constant quantity.

➤ **Example 14.** The tangent at a point

$P(a \cos \phi, b \sin \phi)$ of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, meets

its auxiliary circle in two points, the chord joining which subtends a right angle at the centre, then the eccentricity of the ellipse is

- (a) $(1 + \sin^2 \phi)^{-1}$
(b) $(1 + \sin^2 \phi)^{-1/2}$
(c) $(1 + \sin^2 \phi)^{-3/2}$
(d) $(1 + \sin^2 \phi)^{-2}$

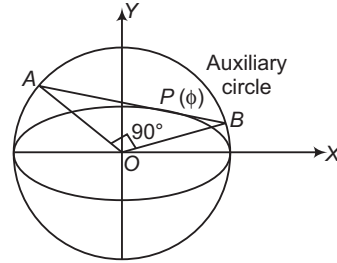
Sol. (b) Equation of the auxiliary circle is

$$x^2 + y^2 = a^2 \quad \dots (i)$$

Equation of the tangent at point $P(a \cos \phi, b \sin \phi)$ is

$$\left(\frac{x}{a} \right) \cos \phi + \left(\frac{y}{b} \right) \sin \phi = 1 \quad \dots (ii)$$

which meets the auxiliary circle at points A and B.



∴ Equation of pair of lines OA and OB is obtained by making homogeneous, Eq. (i) with the help of Eq. (ii) as

$$x^2 + y^2 = a^2 \left(\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi \right)^2$$

$$\Rightarrow (1 - \cos^2 \phi) x^2 - \frac{2xy a \sin \phi \cos \phi}{b} + \left(1 - \frac{a^2}{b^2} \sin^2 \phi \right) y^2 = 0$$

But $\angle AOB = 90^\circ$

∴ Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow 1 - \cos^2 \phi + 1 - \left(\frac{a^2}{b^2} \right) \sin^2 \phi = 0$$

$$\Rightarrow \sin^2 \phi \left(1 - \frac{a^2}{b^2} \right) + 1 = 0$$

$$\Rightarrow (a^2 - b^2) \sin^2 \phi = b^2$$

$$\Rightarrow a^2 e^2 \sin^2 \phi = a^2 (1 - e^2)$$

$$\Rightarrow (1 + \sin^2 \phi) e^2 = 1$$

$$\Rightarrow e = \frac{1}{\sqrt{1 + \sin^2 \phi}} = (1 + \sin^2 \phi)^{-1/2}$$

➤ **Example 15.** The area of the triangle inscribed in an ellipse is

$$2ab \sin \left(\frac{\beta - \gamma}{2} \right) \sin \left(\frac{\gamma - \alpha}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right), \text{ where } \alpha, \beta,$$

γ are the eccentric angles of the vertices, then the condition that the area of a triangle inscribed in an ellipse may be maximum, is

(a) $\alpha, \alpha - \frac{2\pi}{3}, \alpha + \frac{4\pi}{3}$

(b) $\alpha, \alpha + \frac{2\pi}{3}, \alpha + \frac{4\pi}{3}$

(c) $2\alpha, \alpha + \frac{\pi}{3}, \alpha + \frac{2\pi}{3}$

(d) $2\alpha, \alpha - \frac{\pi}{3}, \alpha + \frac{2\pi}{3}$

Sol. (b) Let P, Q and R be the angular points of the triangle and let α, β and γ be their eccentric angles, respectively.

The vertices of the ΔPQR are

$(a \cos \alpha, b \sin \alpha)$, $(a \cos \beta, b \sin \beta)$ and $(a \cos \gamma, b \sin \gamma)$.

Therefore, the area of ΔPQR

$$= 2ab \sin \left(\frac{\beta - \gamma}{2} \right) \sin \left(\frac{\gamma - \alpha}{2} \right) \sin \left(\frac{\alpha - \beta}{2} \right)$$

Let p, q and r be the corresponding points on the auxiliary circle.

Then, area of

$$\Delta pqr = 2a^2 \sin\left(\frac{\beta - \gamma}{2}\right) \sin\left(\frac{\gamma - \alpha}{2}\right) \sin\left(\frac{\alpha - \beta}{2}\right)$$

On putting $b = a$,

$$\frac{\text{ar}(\Delta PQR)}{\text{ar}(\Delta pqr)} = \frac{b}{a}$$

$\Rightarrow$

$$\text{ar}(\Delta PQR) = \frac{b}{a} \text{ar}(\Delta pqr)$$

Hence, ΔPQR will be maximum when Δpqr is maximum. But Δpqr is maximum when it is an equilateral triangle and in that case

$$\alpha - \beta = \beta - \gamma = \gamma - \alpha = \frac{2\pi}{3}$$

Hence, when a triangle inscribed in an ellipse is maximum the eccentric angles of its angular points are $\alpha, \alpha + \frac{2\pi}{3}, \alpha + \frac{4\pi}{3}$.

Example 16. The locus of the centroid of an equilateral triangle inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$(a) \frac{(a^2 + 3b^2)^2}{a^2} x^2 + \frac{(b^2 + 3a^2)}{b^2} y^2 = (a^2 + b^2)^2$$

$$(b) \frac{(a^2 - 3b^2)^2}{a^2} x^2 + \frac{(b^2 - 3a^2)}{b^2} y^2 = (a^2 + b^2)^2$$

$$(c) \frac{(a^2 + 3b^2)^2}{a^2} x^2 - \frac{(b^2 + 3a^2)}{b^2} y^2 = (a^2 - b^2)^2$$

$$(d) \frac{(a^2 + 3b^2)^2}{a^2} x^2 + \frac{(b^2 + 3a^2)^2}{b^2} y^2 = (a^2 - b^2)^2$$

Sol. (d) Let it be remembered at the outset that the centroid of an equilateral triangle is the same as the circumcentre of the circumscribing circle.

Let α, β and γ be the vertices of the equilateral triangle, so that the centroid is given by

$$x = \frac{1}{3}(a \cos \alpha + a \cos \beta + a \cos \gamma) \\ = \frac{1}{3}a(\cos \alpha + \cos \beta + \cos \gamma)$$

$$\text{and } y = \frac{1}{3}(b \sin \alpha + b \sin \beta + b \sin \gamma) \\ = \frac{1}{3}b(\sin \alpha + \sin \beta + \sin \gamma)$$

$$\therefore \cos \alpha + \cos \beta + \cos \gamma = \frac{3x}{a}$$

$$\text{and } \sin \alpha + \sin \beta + \sin \gamma = \frac{3y}{b}$$

$$\text{But } x = \frac{a^2 - b^2}{4a} [\cos \alpha + \cos \beta + \cos \gamma + \cos(\alpha + \beta + \gamma)]$$

$$\text{and } y = \frac{b^2 - a^2}{4b} [\sin \alpha + \sin \beta + \sin \gamma - \sin(\alpha + \beta + \gamma)]$$

$$\text{Hence, } x = \frac{a^2 - b^2}{4a} \left[\frac{3x}{a} + \cos(\alpha + \beta + \gamma) \right]$$

$$\Rightarrow \cos(\alpha + \beta + \gamma) = x \left(\frac{4a}{a^2 - b^2} - \frac{3}{a} \right) = \frac{x(a^2 + 3b^2)}{a(a^2 - b^2)}$$

$$\text{Also, } -\sin(\alpha + \beta + \gamma) = \frac{y(b^2 + 3a^2)}{b(a^2 - b^2)}$$

On squaring and adding, we get

$$1 = \frac{(a^2 + 3b^2)^2}{a^2(a^2 - b^2)^2} x^2 + \frac{(b^2 + 3a^2)^2}{b^2(a^2 - b^2)^2} y^2 \\ \Rightarrow \frac{(a^2 + 3b^2)^2}{a^2} x^2 + \frac{(b^2 + 3a^2)^2}{b^2} y^2 = (a^2 - b^2)^2$$

which is the required locus.

Equation of the Chord

Let $P(a \cos \theta, b \sin \theta), Q(a \cos \phi, b \sin \phi)$ be any two points of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, the equation of the chord joining these two points is

$$y - b \sin \theta = \frac{b \sin \phi - b \sin \theta}{a \cos \phi - a \cos \theta} (x - a \cos \theta)$$

$$\Rightarrow y - b \sin \theta = \frac{-b}{a} \cdot \frac{\cos\left(\frac{\phi + \theta}{2}\right)}{\sin\left(\frac{\theta + \phi}{2}\right)} (x - a \cos \theta)$$

$$\Rightarrow \frac{1}{b} (y - b \sin \theta) \sin\left(\frac{\theta + \phi}{2}\right) \\ = -\frac{1}{a} \cos\left(\frac{\theta + \phi}{2}\right) (x - a \cos \theta)$$

$$\Rightarrow \frac{x}{a} \cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{b} \sin\left(\frac{\theta + \phi}{2}\right) \\ = \cos \theta \cos\left(\frac{\theta + \phi}{2}\right) + \sin \theta \sin\left(\frac{\theta + \phi}{2}\right) \\ \Rightarrow \frac{x}{a} \cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{b} \sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right)$$

Thus, the equation of the chord joining two points having eccentric angles θ and ϕ on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$\frac{x}{a} \cos\left(\frac{\theta + \phi}{2}\right) + \frac{y}{b} \sin\left(\frac{\theta + \phi}{2}\right) = \cos\left(\frac{\theta - \phi}{2}\right)$$

Example 17. If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then the eccentricity of the ellipse is

$$(a) \frac{\cos \alpha + \cos \beta}{\cos(\alpha - \beta)} \quad (b) \frac{\sin \alpha - \sin \beta}{\sin(\alpha - \beta)}$$

$$(c) \frac{\cos \alpha - \cos \beta}{\cos(\alpha - \beta)} \quad (d) \frac{\sin \alpha + \sin \beta}{\sin(\alpha + \beta)}$$

Sol. (d) The equation of a chord joining points having eccentric angles α and β , is given by

$$\frac{x}{a} \cos\left(\frac{\alpha + \beta}{2}\right) + \frac{y}{b} \sin\left(\frac{\alpha + \beta}{2}\right) = \cos\left(\frac{\alpha - \beta}{2}\right)$$

If it passes through $(ae, 0)$, then

$$\begin{aligned} e \cos \left(\frac{\alpha + \beta}{2} \right) &= \cos \left(\frac{\alpha - \beta}{2} \right) \\ \Rightarrow e &= \frac{\cos \left(\frac{\alpha - \beta}{2} \right)}{\cos \left(\frac{\alpha + \beta}{2} \right)} \\ \Rightarrow e &= \frac{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)}{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right)} \\ \Rightarrow e &= \frac{\sin \alpha + \sin \beta}{\sin (\alpha + \beta)} \end{aligned}$$

➤ **Example 18.** If a triangle is inscribed in an ellipse and two of its sides are parallel to given straight lines, then envelope of the third side is
(a) a circle (b) a parabola
(c) a hyperbola (d) an ellipse

Sol. (d) Let the eccentric angles of the vertices P, Q, R of a ΔPQR be $\theta_1, \theta_2, \theta_3$. Then, the equations of PQ and PR are respectively

$$\begin{aligned} \frac{x}{a} \cos \left(\frac{\theta_1 + \theta_2}{2} \right) + \frac{y}{b} \sin \left(\frac{\theta_1 + \theta_2}{2} \right) &= \cos \left(\frac{\theta_1 - \theta_2}{2} \right) \\ \text{and } \frac{x}{a} \cos \left(\frac{\theta_1 + \theta_3}{2} \right) + \frac{y}{b} \sin \left(\frac{\theta_1 + \theta_3}{2} \right) &= \cos \left(\frac{\theta_1 - \theta_3}{2} \right) \end{aligned}$$

If PQ and PR are parallel to given straight lines, then we have

$$\begin{aligned} \theta_1 + \theta_2 &= \text{Constant} = 2\alpha & [\text{say}] \\ \text{and } \theta_1 + \theta_3 &= \text{Constant} = 2\beta \\ \text{Hence, } \theta_2 - \theta_3 &= 2(\alpha - \beta) & \dots(i) \end{aligned}$$

Now, the equation of QR is

$$\begin{aligned} \frac{x}{a} \cos \left(\frac{\theta_2 + \theta_3}{2} \right) + \frac{y}{b} \sin \left(\frac{\theta_2 + \theta_3}{2} \right) &= \cos \left(\frac{\theta_2 - \theta_3}{2} \right) & \dots(ii) \\ &= \cos(\alpha - \beta) & [\text{from Eq. (i)}] \end{aligned}$$

which shows that the Eq. (ii), for different values of $\theta_2 + \theta_3$ is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \cos^2(\alpha - \beta)$$

which is the required envelope of QR .

Intersection of a Line and an Ellipse

Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(i)$$

and a line $y = mx + c$ $\dots(ii)$

The abscissae of the points of intersection of the line (ii) and the ellipse (i) are the roots of the equation

$$\begin{aligned} \frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} &= 1 \\ \Rightarrow (a^2 m^2 + b^2)x^2 + 2a^2 mcx + a^2(c^2 - b^2) &= 0 & \dots(iii) \end{aligned}$$

This is a quadratic equation in x . So, it gives two values of x and corresponding to each value of x , there is a point of intersection of ellipse (i) and the line (ii). In order that the line (ii) may be tangent to ellipse (i), the two points of intersection may be coincident.

Therefore, the discriminant of Eq. (iii) must be zero.

$$\text{i.e. } 4a^4 m^2 c^2 - 4a^2(a^2 m^2 + b^2)(c^2 - b^2) = 0$$

$$\Rightarrow c^2 = a^2 m^2 + b^2$$

$$\Rightarrow c = \pm \sqrt{a^2 m^2 + b^2}$$

Thus, the line $y = mx + c$ touches the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{if } c^2 = a^2 m^2 + b^2$$

The line will intersect the ellipse, if Eq. (iii) has real and distinct roots. This is possible when its discriminant is greater than zero.

$$\text{i.e. } 4a^4 m^2 c^2 - 4a^2(a^2 m^2 + b^2)(c^2 - b^2) > 0$$

$$\Rightarrow c^2 < a^2 m^2 + b^2$$

Similarly, the line will not intersect the ellipse (i), if the Eq. (iii) has imaginary roots i.e.

$$\begin{aligned} \text{Discriminant} &< 0 \\ \Rightarrow c^2 &> a^2 m^2 + b^2 \end{aligned}$$

➤ **Example 19.** The number of values of c such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$ is

- (a) 0 (b) 1 (c) 2 (d) infinite

Sol. (c) We know that the line $y = mx + c$ touches the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ iff } c^2 = a^2 m^2 + b^2$$

$$\text{Here, } a^2 = 4, b^2 = 1, m = 4$$

$$\therefore c^2 = 64 + 1$$

$$\Rightarrow c = \pm \sqrt{65}$$

➤ **Example 20.** The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line $y = mx + c$ intersect in real points only, if

$$(a) a^2 m^2 < c^2 - b^2$$

$$(b) a^2 m^2 > c^2 - b^2$$

$$(c) a^2 m^2 \geq c^2 - b^2$$

$$(d) c \geq b$$

Sol. (c) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the straight line

$y = mx + c$ intersect in real points, then the quadratic equation $\frac{x^2}{a^2} + \frac{(mx + c)^2}{b^2} = 1$ has real roots. Therefore,

$$\begin{aligned} \text{Discriminant} &\geq 0 \\ \Rightarrow c^2 &\leq a^2 m^2 + b^2 \end{aligned}$$

Tangent

In this section, we shall derive the equations of tangents to an ellipse in different forms viz. slope form, point form and parametric form. These equations will be used to solve various problems related to tangents to an ellipse.

(i) **Slope form** The equations of tangents of slope m to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are $y = mx \pm \sqrt{a^2 m^2 + b^2}$

and the coordinates of the points of contact are

$$\left(\pm \frac{a^2 m}{\sqrt{a^2 m^2 + b^2}}, \mp \frac{b^2}{\sqrt{a^2 m^2 + b^2}} \right)$$

(ii) **Point form** The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

(iii) **Parametric form** The equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the point $(a \cos \theta, b \sin \theta)$ is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

(iv) **Point of intersection of tangents** The equations of the tangents to the ellipse at points $P(a \cos \theta_1, b \sin \theta_1)$ and $Q(a \cos \theta_2, b \sin \theta_2)$ are

$$\frac{x}{a} \cos \theta_1 + \frac{y}{b} \sin \theta_1 = 1$$

$$\text{and } \frac{x}{a} \cos \theta_2 + \frac{y}{b} \sin \theta_2 = 1$$

and these two intersect at the point

$$\left(\frac{a \cos \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}, \frac{b \sin \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)} \right)$$

Important Properties Related to Tangent

- i. Locus of foot of perpendiculars from foci upon any tangent is an auxiliary circle.
- ii. Product of perpendiculars from foci upon any tangent of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is b^2 .

iii. Tangents at the extremities of latusrectum passes through the corresponding foot of directrix on major axis.

iv. Length of tangent between the point of contact and the point, where it meets the directrix subtends right angle at the corresponding focus.

➤ **Example 21.** The locus of the point of intersection of tangents to an ellipse at two points, sum of whose eccentric angles is constant, is a/an

(a) parabola

(b) circle

(c) ellipse

(d) straight line

Sol. (d) The equations of tangents at two points having eccentric angles θ_1 and θ_2 are

$$\frac{x}{a} \cos \theta_1 + \frac{y}{b} \sin \theta_1 = 1 \quad \dots (i)$$

$$\frac{x}{a} \cos \theta_2 + \frac{y}{b} \sin \theta_2 = 1 \quad \dots (ii)$$

The point of intersection of Eqs. (i) and (ii) is

$$\left(\frac{a \cos \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}, \frac{b \sin \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)} \right)$$

It is given that $(\theta_1 + \theta_2) = k = \text{constant}$. Therefore, if (x_1, y_1) is the point of intersection of Eqs. (i) and (ii), then

$$x_1 = \frac{a \cos k'}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)} \quad \left[\because k' = \frac{k}{2} \right]$$

and

$$y_1 = \frac{b \sin k'}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}$$

$$\Rightarrow \frac{x_1}{y_1} = \frac{a}{b} \cot k', y_1 = \left(\frac{b}{a} \tan k' \right) x_1$$

$$\Rightarrow (x_1, y_1) \text{ lies on the straight line } y = \left(\frac{b}{a} \tan k' \right) x.$$

➤ **Example 22.** The points on the ellipse such that the tangent at each of them makes equal angles with the axes, is

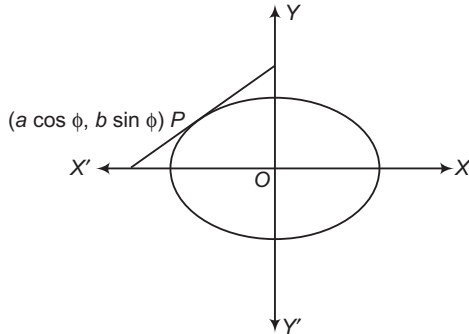
$$(a) \left[\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}} \right]$$

$$(b) \left[\pm \frac{b^2}{\sqrt{a^2 + b^2}}, \pm \frac{a^2}{\sqrt{a^2 + b^2}} \right]$$

$$(c) \left[\pm \frac{1}{\sqrt{a^2 + b^2}}, \pm \frac{1}{\sqrt{a^2 + b^2}} \right]$$

(d) None of the above

Sol. (a) Let P be $(a \cos \phi, b \sin \phi)$, then equation of tangent at P is $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$.



$$\text{Slope of tangent} = \frac{-\cos \phi \cdot b}{\sin \phi \cdot a} = \pm \tan 45^\circ = \pm 1$$

$$\Rightarrow \cot \phi = \pm \frac{a}{b}$$

$$\Rightarrow \cos \phi = \pm \frac{a}{\sqrt{a^2 + b^2}}$$

$$\text{and} \quad \sin \phi = \pm \frac{b}{\sqrt{a^2 + b^2}}$$

$\therefore$ Coordinates of required points are

$$\left[\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}} \right]$$

➤ **Example 23.** If the line $3x + 4y = \sqrt{7}$ touches the ellipse $3x^2 + 4y^2 = 1$, then the point of contact is

(a) $\left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}} \right)$

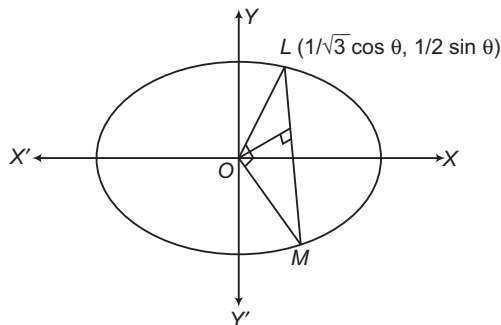
(b) $\left(\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right)$

(c) $\left(\frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}} \right)$

(d) None of the above

Sol. (a) The equation of the ellipse is $\frac{x^2}{1/3} + \frac{y^2}{1/4} = 1$

i.e. $a = \frac{1}{\sqrt{3}}, b = \frac{1}{2}$



The equation of the tangent at L is $\sqrt{3}x \cos \theta + 2y \sin \theta = 1$

...(i)

But the tangent equation is given by

$$3x + 4y = \sqrt{7}$$

...(ii)

Identifying Eqs. (i) and (ii), we get

$$\frac{\cos \theta}{\sqrt{3}} = \frac{\sin \theta}{2} = \frac{1}{\sqrt{7}}$$

The point of contact = $(a \cos \theta, b \sin \theta)$

$$= \left(\frac{1}{\sqrt{7}}, \frac{1}{\sqrt{7}} \right)$$

➤ **Example 24.** The equation of the ellipse whose axes are coincident with the coordinate axes and which touches the straight line $3x - 2y - 20 = 0$ and $x + 6y - 20 = 0$, is

(a) $\frac{x^2}{5} + \frac{y^2}{8} = 1$

(b) $\frac{x^2}{40} + \frac{y^2}{10} = 10$

(c) $\frac{x^2}{40} + \frac{y^2}{10} = 1$

(d) $\frac{x^2}{10} + \frac{y^2}{40} = 1$

Sol. (c) Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

We know that the general equation of the tangent to the ellipse is

$$y = mx \pm \sqrt{a^2 m^2 + b^2} \quad \dots(i)$$

Since, $3x - 2y - 20 = 0$ i.e. $2y = 3x - 20$

i.e. $y = \frac{3}{2}x - 10$

which is tangent to the ellipse.

Therefore, comparing with Eq. (i),

$$m = \frac{3}{2}$$

and $a^2 m^2 + b^2 = 100$

$$\Rightarrow a^2 \cdot \frac{9}{4} + b^2 = 100$$

$$\Rightarrow 9a^2 + 4b^2 = 400 \quad \dots(ii)$$

Similarly, since $x + 6y - 20 = 0$, i.e. $y = -\frac{1}{6}x + \frac{10}{3}$

which is tangent to the ellipse.

Therefore, comparing with Eq. (i),

$$m = -\frac{1}{6}$$

and $a^2 m^2 + b^2 = \frac{100}{9}$

$$\Rightarrow \frac{a^2}{36} + b^2 = \frac{100}{9}$$

$$\Rightarrow a^2 + 36b^2 = 400 \quad \dots(iii)$$

Now, solving Eqs. (ii) and (iii), we get

$$a^2 = 40$$

and $b^2 = 10$

$\therefore$ The required equation of the ellipse is

$$\frac{x^2}{40} + \frac{y^2}{10} = 1$$

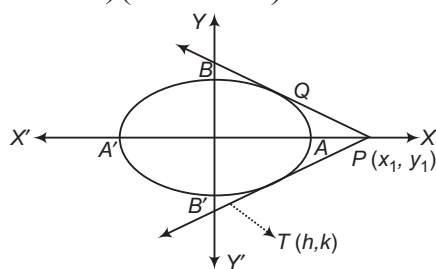
Work Book Exercise 15.2

- A tangent to the ellipse $4x^2 + 9y^2 = 36$ is cut by the tangent at the extremities of the major axis at T and T' . The circle on TT' as diameter passes through the point
 a $(-\sqrt{5}, 0)$ b $(\sqrt{5}, 1)$ c $(0, 0)$ d $(3, 2)$
- If PSQ is a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$, then the harmonic mean of SP and SQ is
 a $\frac{b^2}{a}$ b $\frac{a^2}{b}$ c $\frac{2b^2}{a}$ d $\frac{2a^2}{b}$
- Tangent is drawn to the ellipse $\frac{x^2}{27} + y^2 = 1$ at $(3\sqrt{3}\cos\theta, \sin\theta)$, where $\theta \in (0, \frac{\pi}{2})$, then the value of θ such that sum of intercepts on axes made by the tangent is minimum, is
 a $\frac{\pi}{3}$ b $\frac{\pi}{6}$ c $\frac{\pi}{8}$ d $\frac{\pi}{4}$
- If F_1 and F_2 are the foot of the perpendiculars from the foci S_1 and S_2 of an ellipse $\frac{x^2}{5} + \frac{y^2}{3} = 1$ on the tangent at any point P on the ellipse, then $(S_1F_1)(S_2F_2)$ is equal to
 a 2 b 3
 c 4 d 5
- The slope of a common tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a concentric circle of radius r is
 a $\tan^{-1}\left(\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}\right)$ b $\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$
 c $\frac{r^2 - b^2}{a^2 - r^2}$ d $\sqrt{\frac{a^2 - r^2}{r^2 - b^2}}$
- If p and p' denote the lengths of the perpendicular from a focus and the centre of an ellipse with semi-major axis of length a , respectively on a tangent to the ellipse and r denotes the focal distance of the point, then
 a $ap = rp'$ b $rp = ap'$
 c $ap = rp' + 1$ d $ap' + rp = 1$
- Tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) and the circle $x^2 + y^2 = a^2$ at the points, where a common ordinate cuts them on the same side of the X-axis. Then, the greatest acute angle between these tangents is given by
 a $\tan^{-1}\left(\frac{a-b}{2\sqrt{ab}}\right)$ b $\tan^{-1}\left(\frac{a+b}{2\sqrt{ab}}\right)$
 c $\tan^{-1}\left(\frac{2ab}{\sqrt{a-b}}\right)$ d $\tan^{-1}\left(\frac{2ab}{\sqrt{a+b}}\right)$

Combined Equation of the Pair of Tangents

The combined equation of the pair of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1\right)^2$$



➤ **Example 25.** The angle between the pair of tangents from the point $(1, 2)$ to the ellipse

$$3x^2 + 2y^2 = 5$$

(a) $\tan^{-1} \frac{5}{3}$ (b) $\frac{\pi}{4}$ (c) $\frac{\pi}{2}$ (d) $\tan^{-1} \frac{12}{\sqrt{5}}$

Sol. (d) The combined equation of the pair of tangents drawn from $(1, 2)$ to the ellipse $3x^2 + 2y^2 = 5$ is

$$(3x^2 + 2y^2 - 5)(3 + 8 - 5) = (3x + 4y - 5)^2 \quad [\because SS_1 = T^2]$$

$$\Rightarrow 9x^2 - 24xy - 4y^2 + 30x + 40y - 55 = 0$$

If the angle between these lines is θ , then

$$\tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}, \text{ where } a = 9, h = -12, b = -4$$

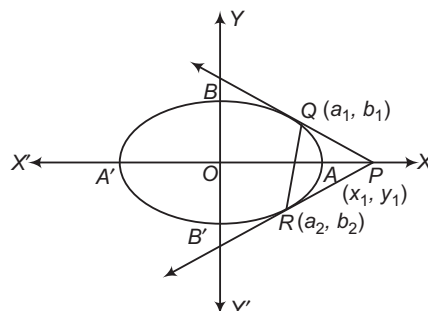
$$\Rightarrow \tan \theta = \frac{2\sqrt{144 + 36}}{9 - 4} \Rightarrow \tan \theta = \frac{2\sqrt{180}}{5} = \frac{12\sqrt{5}}{5}$$

$$\Rightarrow \tan \theta = \frac{12}{\sqrt{5}} \Rightarrow \theta = \tan^{-1} \frac{12}{\sqrt{5}}$$

Chord of Contact of Tangents

The equation of the chord of contact of tangents drawn from a point (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$



➤ **Example 26.** Tangent are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$, then all the chords of contact passes through a fixed point, whose coordinates are

(a) $\left(\frac{4}{5}, -\frac{1}{5}\right)$

(b) $\left(\frac{4}{5}, \frac{1}{5}\right)$

(c) $\left(-\frac{4}{5}, \frac{1}{5}\right)$

(d) None of the above

Sol. (a) Let $A(x_1, x_1 - 5)$ be a point on $x - y - 5 = 0$, then chord of contact of $x^2 + 4y^2 = 4$ w.r.t. A is

$$\Rightarrow \begin{aligned} &xx_1 + 4y(x_1 - 5) = 4 \\ &(x + 4y)x_1 - (20y + 4) = 0 \end{aligned}$$

It passes through a fixed point.

$$\begin{aligned} \therefore & \quad x + 4y = 0 \\ \text{and} & \quad 20y + 4 = 0 \quad [\because P + \lambda Q = 0] \\ \Rightarrow & \quad y = -\frac{1}{5} \\ \text{and} & \quad x = \frac{4}{5} \end{aligned}$$

The coordinates of fixed point are $\left(\frac{4}{5}, -\frac{1}{5}\right)$.

➤ **Example 27.** Tangent are drawn from any point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the circle

$x^2 + y^2 = r^2$, then the chord of contact is a tangent to the ellipse, is

(a) $a^2x^2 - b^2y^2 = r^2$

(b) $a^2x^2 + b^2y^2 = r^4$

(c) $a^4x^4 + b^4y^4 = r^2$

(d) $a^4x^4 + b^4y^4 = r^4$

Sol. (b) The coordinates of any point on the ellipse are $(a \cos \theta, b \sin \theta)$.

The equation of the chord of contact to the circle $x^2 + y^2 = r^2$ is

$$xa \cos \theta + yb \sin \theta = r^2 \quad \dots(i)$$

If Eq. (i) is tangent to the ellipse $a^2x^2 + b^2y^2 = r^4$, the quadratic equation in x viz. $a^2x^2 + \left(\frac{r^2 - xa \cos \theta}{\sin \theta}\right)^2 = r^4$

must have two equal roots i.e. the equation $a^2x^2 - 2xar^2 \cos \theta + r^4 \cos^2 \theta = 0$

must have two equal roots.

The condition for this is

$$4a^2r^4 \cos^2 \theta = 4a^2r^4 \cos^2 \theta$$

which is clearly satisfied.

➤ **Example 28.** The locus of the point of the chord of contact of tangents from which the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touches the circle, is

(a) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$

(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{c^2}$

(c) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^4}$

(d) None of the above

Sol. (a) Let the point be (h, k) from which two tangents have been drawn to the given ellipse.

Then, the equation of the chord of contact is

$$\frac{xh}{a^2} + \frac{yk}{b^2} = 1 \quad \dots(i)$$

If the chord of contact of Eq. (i) touches the circle $x^2 + y^2 = c^2$, then the length of the perpendicular from the origin on Eq. (i) is the radius of the circle.

Hence,
$$\frac{1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} = c$$

$$\Rightarrow \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{c^2}$$

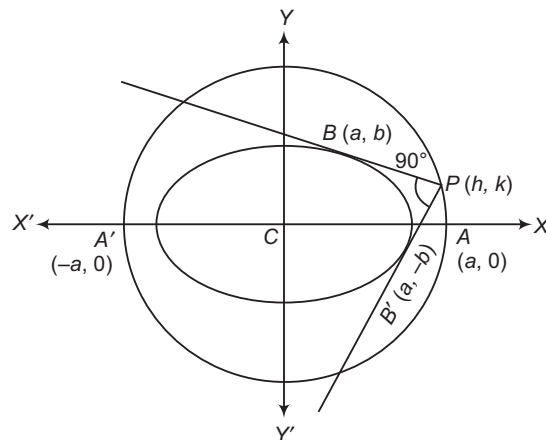
Hence, the locus of (h, k) is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$.

Director Circle

The locus of the point of intersection of perpendicular tangents to an ellipse is known as its director circle.

Equation of the Director Circle

Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse and (h, k) be a point on its director circle.



The equation of any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2 m^2 + b^2}$$

If it passes through (h, k) , then

$$k = mh \pm \sqrt{a^2 m^2 + b^2}$$

$$\Rightarrow (k - mh)^2 = a^2 m^2 + b^2$$

$$\Rightarrow m^2(h^2 - a^2) - 2mhk + (k^2 - b^2) = 0$$

This is quadratic equation in m . So, it gives two values of m , say m_1 and m_2 . Corresponding to each to these values of m , there is a tangent to the ellipse passing through (h, k) . If the two tangents are perpendicular, then

$$\Rightarrow \frac{m_1 m_2}{\frac{k^2 - b^2}{h^2 - a^2}} = -1$$

$$\Rightarrow h^2 + k^2 = a^2 + b^2$$

The equation of the director circle is

$$x^2 + y^2 = a^2 + b^2$$

Clearly, it is a circle concentric to the ellipse and radius is equal to $\sqrt{a^2 + b^2}$.

Remark It follows from the definition of the director circle that the tangents drawn from any point on the director circle of a given ellipse to the ellipse are always at right angle.

➤ **Example 29.** If two tangents drawn to the ellipse $x^2 + 4y^2 = 4$ intersect perpendicularly at P , then the locus of P is

(a) $x^2 + y^2 = 4$

(b) $x^2 + y^2 = 5$

(c) $x^2 + y^2 = 8$

(d) $x^2 + y^2 = 2$

Sol. (b) Equation of ellipse is rewritten as

$$\frac{x^2}{4} + y^2 = 1$$

By definition of director circle,

P lies on the director circle formed by the ellipse

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

Hence, locus of P is

$$x^2 + y^2 = 4 + 1 \quad [\because x^2 + y^2 = a^2 + b^2]$$

$$\Rightarrow x^2 + y^2 = 5$$

Normal

In this section, we shall discuss forms of the equation of the normal to an ellipse which will be used to solve various types of problems on ellipse.

i. **Point form** The equation of the normal at (x_1, y_1) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

ii. **Parametric form** The equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $(a \cos \theta, b \sin \theta)$ is $ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$.

➤ The point of intersection of normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at two points $(a \cos \theta_1, b \sin \theta_1)$ and $(a \cos \theta_2, b \sin \theta_2)$ are (λ, μ) where

$$\lambda = \frac{(a^2 - b^2)}{a} \cos \theta_1 \cos \theta_2 \frac{\cos \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}$$

$$\text{and } \mu = -\frac{(a^2 - b^2)}{b} \sin \theta_1 \sin \theta_2 \frac{\sin \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}$$

iii. **Slope form** The equations of the normals of slope m to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given by

$$y = mx \pm \frac{m(a^2 - b^2)}{\sqrt{a^2 + b^2 m^2}} \text{ at the points}$$

$$\left(\pm \frac{a^2}{\sqrt{a^2 + b^2 m^2}}, \pm \frac{b^2 m}{\sqrt{a^2 + b^2 m^2}} \right)$$

➤ If the line $y = mx + c$ is a normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $c^2 = \frac{m^2(a^2 - b^2)^2}{a^2 + b^2 m^2}$ is the condition of normality of the line of the ellipse.

➤ **Example 30.** If the normal at an end of a latusrectum of an ellipse passes through one extremity of the minor axis, then the eccentricity of the ellipse is given by

(a) $e^4 + e^2 - 1 = 0$

(b) $e^2 + e - 5 = 0$

(c) $e^3 = \sqrt{5}$

(d) None of the above

Sol. (a) Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be the ellipse. The coordinates of

an end of the latusrectum is $\left(ae, \frac{b^2}{a} \right)$.

The equation of the normal at $\left(ae, \frac{b^2}{a}\right)$ is

$$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2 - b^2$$

$$\Rightarrow \frac{ax}{e} - ay = a^2 - b^2$$

It passes through one extremity of the minor axis whose coordinates are $(0, -b)$.

$$\therefore ab = a^2 - b^2$$

$$\Rightarrow a^2b^2 = (a^2 - b^2)^2$$

$$\Rightarrow a^2 \times a^2 (1 - e^2) = (a^2e^2)^2$$

$$\Rightarrow 1 - e^2 = e^4$$

$$\Rightarrow e^4 + e^2 - 1 = 0$$

- **Example 31.** If Q is the point on the auxiliary circle corresponding to a point P on an ellipse. Then, the normals at P and Q meet on
- (a) a fixed circle (b) an ellipse
(c) a hyperbola (d) None of these

Sol. (a) The coordinates of Q are $(a \cos \phi, a \sin \phi)$ and the coordinates of P are $(a \cos \phi, b \sin \phi)$.

The equation of the normal at P is

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2 \quad \dots(i)$$

The equation of the normal at Q is

$$ax \sec \phi - ay \operatorname{cosec} \phi = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$ay \operatorname{cosec} \phi - by \operatorname{cosec} \phi = a^2 - b^2$$

$$\Rightarrow y \operatorname{cosec} \phi (a - b) = a^2 - b^2$$

$$\therefore y = (a + b) \sin \phi \quad \dots(iii)$$

$$\text{From Eq. (ii), } x \sec \phi = (a + b)$$

$$\therefore x = (a + b) \cos \phi \quad \dots(iv)$$

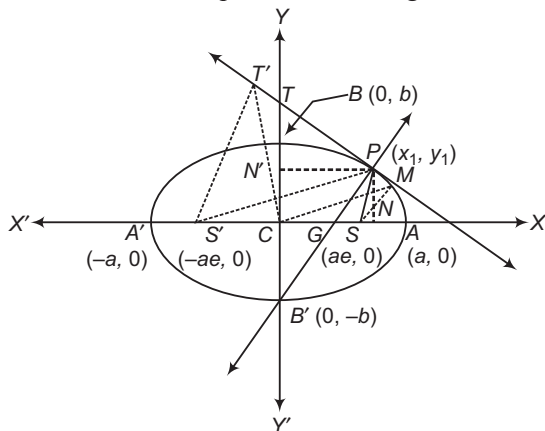
On squaring Eqs. (iii) and (iv) and adding, we get

$$x^2 + y^2 = (a + b)^2$$

which is the required circle.

Some Useful Properties of an Ellipse

In this section, we shall state and prove some useful results on an ellipse as theorems given below :



The tangent and normal at any point of an ellipse bisect the external and internal angles between the focal radii to the point.

- It follows from the above property that if an incident light ray passing through the focus (S) strikes the concave side of the ellipse, then the reflected ray will pass through the other focus (S').

- If SM and $S'M'$ are perpendiculars from the foci upon the tangent at any point of the ellipse, then $SM \cdot S'M' = b^2$ and M, M' lie on the auxiliary circle.

- If the tangent at any point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the major axis in T and minor axis in T' , then $CN \cdot CT = a^2$, $CN' \cdot CT' = b^2$, where N and N' are the foot of the perpendiculars from P on the respective axes.

- If SM and $S'M'$ are perpendiculars from the foci S and S' , respectively upon a tangent to the ellipse, then CM and CM' are parallel to $S'P$ and SP , respectively.

- The common chords of an ellipse and a circle are equally inclined to the axes of the ellipse.

- **Example 32.** If a number of ellipses is described having the same major axis but a variable minor axis, such that each one of the tangents at the ends of their latusrectum passes through the fixed point, then the point of concurrency is
- (a) $(1, 1)$ (b) $(0, 1)$ (c) $(1, 0)$ (d) $(0, 2)$

Sol. (b) Let the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Since, major axis is fixed.

$\therefore a$ is variable.

Hence, e is variable. Now, the ends of the latusrectum are $\left(\pm ae, \pm \frac{b^2}{a}\right)$.

Hence, the tangents at these points are

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \Rightarrow \frac{\pm x \cdot ae}{a^2} + \frac{\pm y \cdot \frac{b^2}{a}}{b^2} = 1$$

$$\Rightarrow \pm \frac{xe}{a} \pm \frac{y}{a} = 1 \Rightarrow \pm xe \pm y = a$$

$$\Rightarrow \pm ex \pm y - a = 0$$

Hence, there are 4 tangents. Since, e is variable.

$\therefore$ Each of the tangents passes through the point $(0, 1)$ which is fixed.

- **Example 33.** If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{14} + \frac{y^2}{5} = 1$ intersects it again at the point $Q(2\theta)$, then $\cos \theta$ is equal to

- (a) $\frac{2}{3}$ (b) $-\frac{2}{3}$ (c) $\frac{1}{2}$ (d) $\frac{\sqrt{3}}{2}$

Sol. (b) We have, $\frac{x^2}{14} + \frac{y^2}{5} = 1 \quad \dots(i)$

The equation of normal at $P(\sqrt{14} \cos \theta, \sqrt{5} \sin \theta)$ is $(\sqrt{14} \sec \theta)x - (\sqrt{5} \operatorname{cosec} \theta)y = 9$

This meets the ellipse (i) at $Q(\sqrt{14} \cos 2\theta, \sqrt{5} \sin 2\theta)$.

$$\therefore 14 \sec \theta \cos 2\theta - 5 \operatorname{cosec} \theta \sin 2\theta = 9$$

$$\Rightarrow 18 \cos^2 \theta - 9 \cos \theta - 14 = 0$$

$$\Rightarrow (6 \cos \theta - 7)(3 \cos \theta + 2) = 0$$

$$\Rightarrow \cos \theta = -2/3 \text{ or } \cos \theta \neq 7/6$$

Number of Normals and Conormal Points

In the previous section, we have learnt that at a point on a given ellipse, exactly one normal can be drawn. If there is a point not lying on a given ellipse, infinitely many lines can be drawn which intersect the ellipse. Out of these lines, there are at most four lines which are normals to the ellipse at the points, where they cut the ellipse. Such points on the ellipse are called conormal points. In this section, we shall learn about the conormal points and various relations between their eccentric angles.

- i. The four normals can be drawn from a point to an ellipse.
- ii. **Conormal points** The points on the ellipse at which four normals to the ellipse pass through a given point are called conormal points.
- iii. **Properties of eccentric angles of conormal points** The sum of the eccentric angles of the conormal points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is an odd multiple of π .
- iv. If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four points on the ellipse the normals at which are concurrent, then
 (a) $\sum \cos(\theta_1 + \theta_2) = 0$ (b) $\sum \sin(\theta_1 + \theta_2) = 0$
 (c) $(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$
- v. If θ_1, θ_2 and θ_3 are the eccentric angles of three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$, then the normals at these points are concurrent.
- vi. If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent, then

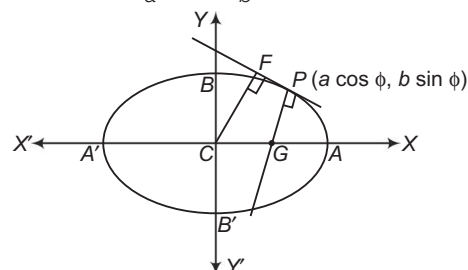
$$(x_1 + x_2 + x_3 + x_4) \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right) = 4$$

➤ **Example 34.** If CF is the perpendicular from the centre C of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ on the tangent at any point P and G is the point when the normal at P meets the major axis, then $CF \cdot PG$ is

- (a) b^2 (b) $2b^2$
 (c) $\frac{b^2}{2}$ (d) None of these

Sol. (a) Equation of an ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Tangent at P is $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$



$$\therefore CF = \frac{1}{\sqrt{\left(\frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2}\right)}} = \frac{ab}{\sqrt{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)}}$$

Equation of normal at P is
 $ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2$,

then $G = \left(\frac{(a^2 - b^2) \cos \phi}{a}, 0 \right)$

$$PG = \sqrt{\left(a \cos \phi - \frac{(a^2 - b^2) \cos \phi}{a} \right)^2 + (b \sin \phi - 0)^2}$$

$$= \sqrt{\left(\frac{b^4 \cos^2 \phi}{a^2} + b^2 \sin^2 \phi \right)}$$

$$= \frac{b}{a} \sqrt{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)}$$

$$\Rightarrow CF \cdot PG = b^2$$

➤ **Example 35.** If the normals at the points A, B, C and D of an ellipse meet in a point O , then $SA \cdot SB \cdot SC \cdot SD$ is equal to (where, S is one of the foci)

- (a) $\frac{b^2}{e^2} \cdot SO^2$
 (b) $\frac{b^3}{e^3} \cdot SO^3$
 (c) $\frac{2b^2}{e^2} \cdot SO^2$
 (d) $\frac{b^2}{3e^2} \cdot 2SO^2$

Sol. (a) Let S be the point $(-ae, 0)$, we have $SA = a + ex_1$ etc.

Therefore, $SA \cdot SB \cdot SC \cdot SD$

$$= (a + ex_1)(a + ex_2)(a + ex_3)(a + ex_4)$$

$$= a^4 + a^3 e \sum x_i + a^2 e^2 \sum x_i x_j + a e^3 \sum x_i x_j x_k + e^4 x_1 x_2 x_3 x_4$$

$$= \frac{b^2}{e^2} \{ (h + ae)^2 + k \}$$

On substitution and simplification,

$$= \frac{b^2}{e^2} \cdot SO^2$$

Chord Bisected at a Given Point

The equation of the chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ bisected at the point (x_1, y_1) is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \quad \text{or} \quad T = S'$$

➤ **Example 36.** The locus of the mid-points of a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

- (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{ex}{a}$ (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{ex}{a}$
(c) $x^2 + y^2 = a^2 + b^2$ (d) None of these

Sol. (a) Let the mid-point of the focal chord of the given ellipse be (h, k) . Then, its equation is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad [\text{using } T = S_1]$$

Since, this passes through $(ae, 0)$.

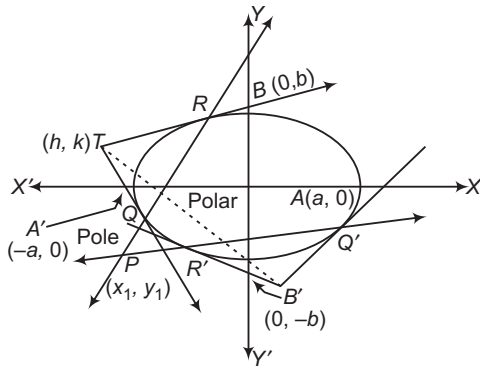
$$\therefore \frac{hae}{a^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \Rightarrow \frac{he}{a} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

$$\therefore \text{Locus of } (h, k) \text{ is } \frac{ex}{a} = \frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Pole and Polar

Let P be a point inside or outside an ellipse. Then, the locus of the point of intersection of tangents to the ellipse at the point, where secants drawn through P intersect the ellipse, is called the polar of point P with respect to the ellipse and the point P is called the pole of the polar.

The polar of a point (x_1, y_1) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.



➤ **Example 37.** Chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

touch $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, then the locus of their pole is

- (a) $\frac{\alpha^2}{a^4}x^2 - \frac{\beta^2}{b^4}y^2 = 1$ (b) $\frac{\alpha^2}{a^2}x^4 + \frac{\beta^2}{b^2}y^4 = 1$
(c) $\frac{\alpha^2}{a^4}x^2 + \frac{\beta^2}{b^4}y^2 = 1$ (d) None of these

Sol. (c) The equation of any tangent to the curve (ellipse)

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1 \text{ is } y = mx + \sqrt{\alpha^2 m^2 + \beta^2}. \quad \dots(i)$$

Let (x_1, y_1) be the pole of Eq. (i) with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then, its polar viz. $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ must be identical with

$$\text{Eq. (i), therefore } \frac{a^2 m}{x_1} = -\frac{b^2}{y_1} = -\sqrt{\alpha^2 m^2 + \beta^2}$$

$$\therefore m = -\frac{b^2}{a^2} \cdot \frac{x_1}{y_1}$$

$$\text{Also, } \alpha^2 m^2 + \beta^2 = \frac{b^4}{y_1^2}$$

$$\Rightarrow \alpha^2 \frac{b^4 x_1^2}{a^4 y_1^2} + \beta^2 = \frac{b^4}{y_1^2} \Rightarrow \frac{\alpha^2 x_1^2}{a^4} + \frac{\beta^2 y_1^2}{b^4} = 1$$

$$\text{Hence, the locus of } (x_1, y_1) \text{ is } \frac{\alpha^2}{a^4}x^2 + \frac{\beta^2}{b^4}y^2 = 1.$$

➤ **Example 38.** If the polar with respect to

$y^2 = 4x$ touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, then locus

of its pole is

(a) $\frac{x^2}{\alpha^2} - \frac{y^2}{\left(\frac{4a^2\alpha^2}{\beta^2}\right)} = 1$

(b) $\frac{x^2}{\alpha^2} - \frac{\beta^2 y^2}{4a^2} = 1$

(c) $\alpha^2 x^2 + \beta^2 y^2 = 1$

(d) None of the above

Sol. (a) Let $P(h, k)$ be the pole. Then, the equation of the polar is

$$ky = 2a(x + h) \Rightarrow y = \frac{2a}{k}x + \frac{2ah}{k}$$

This line touches the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$.

$$\text{So, } \left(\frac{2ah}{k}\right)^2 = \alpha^2 \left(\frac{2a}{k}\right)^2 + \beta^2 \quad [\text{using } c^2 = a^2 m^2 + b^2]$$

$$\Rightarrow 4a^2 h^2 = 4a^2 \alpha^2 + k^2 \beta^2$$

So, locus of (h, k) is $4a^2 x^2 = 4a^2 \alpha^2 + \beta^2 y^2$.

$$\Rightarrow \frac{x^2}{\alpha^2} - \frac{y^2}{\left(\frac{4a^2\alpha^2}{\beta^2}\right)} = 1$$

Conjugate Points

Two points are said to be conjugate points with respect to an ellipse, if each lies on the polar of the other.

In other words, two points P and Q are conjugate points with respect to an ellipse, if the polar of P passes through Q and the polar of Q passes through P .

Conjugate Lines

Two lines are said to be conjugate lines with respect to an ellipse, if each passes through the pole of the polar.

Example 39. The locus of the pole of any tangent to the auxiliary circle w.r.t. the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is}$$

$$(a) \ b^4 x^2 + a^4 y^2 = a^4 b^2 \quad (b) \ b^4 x^2 + a^4 y^2 = 2a^2 b^4$$

$$(c) \ b^2 x^4 + a^4 y^2 = a^4 b^2 \quad (d) \ b^4 x^2 + a^4 y^2 = a^2 b^4$$

Sol. (d) The equation of any tangent to the auxiliary circle $x^2 + y^2 = a^2$ is

$$x \cos \theta + y \sin \theta = a \quad \dots (i)$$

Let (α, β) be the pole of Eq. (i) with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then its polar viz. $\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1$ must be identical with Eq. (i).

$$\text{Therefore, } \frac{a^2 \cos \theta}{\alpha} = \frac{b^2 \sin \theta}{\beta} = a$$

$$\therefore \cos \theta = \frac{r}{a}$$

$$\text{and } \sin \theta = \beta \cdot \frac{a}{b^2}$$

On squaring and adding, we get

$$\frac{\alpha^2}{a^2} + \beta^2 \cdot \frac{a^2}{b^4} = 1$$

$$\Rightarrow b^4 \alpha^2 + a^4 \beta^2 = a^2 b^4$$

Hence, the locus of (α, β) is $b^4 x^2 + a^4 y^2 = a^2 b^4$.

Example 40. The locus of the poles of normal

chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$(a) \ \frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 + b^2)^2$$

$$(b) \ \frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$$

$$(c) \ \frac{a^2}{x^6} + \frac{b^2}{y^6} = (a^6 + b^6)^2$$

(d) None of the above

Sol. (b) Let (α, β) be the pole of the normal

$$ax \sec \phi - by \csc \phi = a^2 - b^2 \quad \dots (i)$$

Then, the pole of (α, β) with respect to the ellipse viz.

$$\frac{x\alpha}{a^2} + \frac{y\beta}{b^2} = 1 \quad \dots (ii)$$

must represent the same straight line as Eq. (i).

On comparing Eqs. (i) and (ii), we get

$$\frac{a^3 \sec \phi}{\alpha} = -\frac{b^3 \csc \phi}{\beta} = a^2 - b^2$$

$$\therefore \cos \phi = \frac{a^3}{\alpha(a^2 - b^2)} \text{ and } \sin \phi = -\frac{b^3}{\beta(a^2 - b^2)}$$

On squaring and adding, we have

$$1 = \frac{a^6}{\alpha^2(a^2 - b^2)^2} + \frac{b^6}{\beta^2(a^2 - b^2)^2}$$

$$\Rightarrow \frac{a^6}{\alpha^2} + \frac{b^6}{\beta^2} = (a^2 - b^2)^2$$

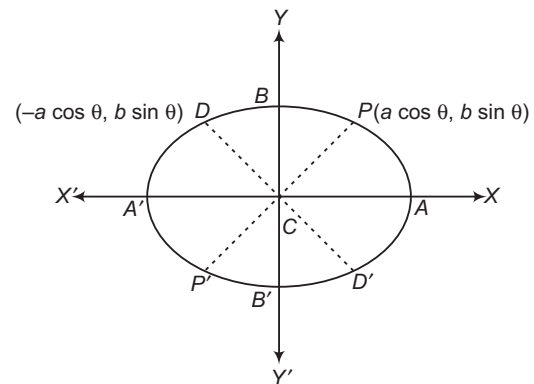
Hence, the locus of (α, β) is $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$.

Diameter

The locus of the mid-point of a system of parallel chords of an ellipse is called a diameter whose equation of diameter is $y = -\frac{b^2}{a^2 m} x$.

Conjugate Diameters

Two diameters of an ellipse are said to be conjugate diameters, if each bisect the chords parallel to the other.



Let $y = m_1 x$ and $y = m_2 x$ be conjugate diameters with respect to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, $y = m_2 x$ bisects the system of chords parallel to $y = m_1 x$.

$$\text{So, its equation is } y = -\frac{b^2}{a^2 m_1} x. \quad \dots (i)$$

Clearly, Eq. (i) and $y = m_2 x$ represent the same line.

$$\therefore m_2 = \frac{-b^2}{a^2 m_1}$$

$$\Rightarrow m_1 m_2 = \frac{-b^2}{a^2}$$

Thus, two straight lines $y = m_1x$ and $y = m_2x$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if

$$m_1 m_2 = -\frac{b^2}{a^2}$$

✎ In an ellipse, the major axis bisects all chords parallel to the minor axis and vice-versa, therefore major and minor axes of an ellipse are conjugate diameters of the ellipse but they do not satisfy the condition $m_1 m_2 = -b^2/a^2$ and are the only perpendicular conjugate diameters.

➤ **Example 41.** The eccentricity of an ellipse whose pair of conjugate diameter are $y = x$ and $3y = -2x$, is

- (a) $\frac{2}{3}$ (b) $\frac{1}{3}$
(c) $\frac{1}{\sqrt{3}}$ (d) None of these

Sol. (c) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, since $y = x$

and $3y = -2x$ is a pair of conjugate diameters, therefore

$$\begin{aligned} m_1 m_2 &= -\frac{b^2}{a^2} \\ \Rightarrow (1) \left(-\frac{2}{3}\right) &= -\frac{b^2}{a^2} \\ \Rightarrow 2a^2 &= 3b^2 \\ \Rightarrow 2a^2 &= 3a^2(1 - e^2) \Rightarrow 2 = 3(1 - e^2) \\ \Rightarrow e^2 &= \frac{1}{3} \Rightarrow e = \frac{1}{\sqrt{3}} \end{aligned}$$

Properties of Conjugate Diameters

- i. The eccentric angles of the ends of a pair of conjugate diameter of an ellipse differ by a right angle.
- ii. The sum of the squares of any two conjugate semi-diameters of an ellipse is constant and equal to the sum of the squares of the semi-axes of the ellipse i.e. $CP^2 + CD^2 = a^2 + b^2$.
- iii. The product of the focal distances of a point on an ellipse is equal to the square of the semi-diameter which is conjugate to the diameter through the point.
- iv. The tangents at the ends of a pair of conjugate diameters of an ellipse form a parallelogram.
- v. The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is constant and is equal to the product of the axes.

➤ **Example 42.** If the point of intersection of the ellipses $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$ are at the extremities of the conjugate diameters of the former, then

- (a) $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$ (b) $\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} = 2$
(c) $\frac{a^2}{\alpha^2} - \frac{b^2}{\beta^2} = 2$ (d) None of these

Sol. (a) The points of intersection of the two ellipses will be given by

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} \\ \Rightarrow x^2 \left(\frac{1}{a^2} - \frac{1}{\alpha^2} \right) + y^2 \left(\frac{1}{b^2} - \frac{1}{\beta^2} \right) &= 0 \quad \dots (i) \end{aligned}$$

which being a homogeneous equation of second degree represents a pair of straight lines passing through the origin.

Let $y = mx$ be one such straight line.

Then, putting $y = mx$ in Eq. (i), we get

$$m^2 \left(\frac{1}{b^2} - \frac{1}{\beta^2} \right) + \left(\frac{1}{a^2} - \frac{1}{\alpha^2} \right) = 0$$

Thus, if m_1 and m_2 are the gradients of the two lines, then

$$m_1 m_2 = \left(\frac{1}{a^2} - \frac{1}{\alpha^2} \right) \div \left(\frac{1}{b^2} - \frac{1}{\beta^2} \right)$$

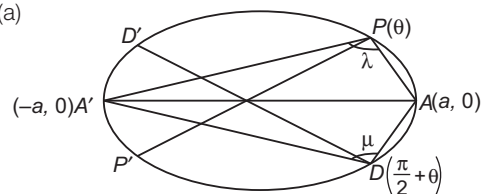
But by the question, the two lines are conjugate. Hence,

$$\begin{aligned} m_1 m_2 &= -\frac{b^2}{a^2} \Rightarrow \frac{\frac{1}{a^2} - \frac{1}{\alpha^2}}{\frac{1}{b^2} - \frac{1}{\beta^2}} = -\frac{b^2}{a^2} \\ \Rightarrow 1 - \frac{a^2}{\alpha^2} &= -1 + \frac{b^2}{\beta^2} \\ \therefore \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} &= 2 \end{aligned}$$

➤ **Example 43.** If λ and μ are the angles subtended by the major axis to an ellipse at the extremities of a pair of conjugate diameters, then $\cot^2 \lambda + \cot^2 \mu$ is equal to

- (a) $\frac{(a^2 - b^2)^2}{4a^2 b^2}$ (b) $\frac{(a^2 + b^2)^2}{4a^2 b^2}$
(c) $\frac{(a^2 + b^2)^2}{a^2 b^2}$ (d) $\frac{(a^2 - b^2)^2}{a^2 b^2}$

Sol. (a)



Let PP' and DD' be the conjugate diameters, so that $P \equiv (a \cos \theta, b \sin \theta)$ and $D \equiv (-a \sin \theta, b \cos \theta)$.

Given, $\angle A'PA = \lambda$ and $\angle A'DA = \mu$.

Now, slopes AP and $A'P$ are

$$\frac{b \sin \theta}{a \cos \theta - a} \quad \text{and} \quad \frac{b \sin \theta}{a \cos \theta + a}$$

$$\therefore \tan \lambda = \frac{\frac{b \sin \theta}{a \cos \theta - a} - \frac{b \sin \theta}{a \cos \theta + a}}{1 + \frac{b \sin \theta}{a \cos \theta - a} \cdot \frac{b \sin \theta}{a \cos \theta + a}}$$

$$= \frac{-2ab \sin \theta}{b^2 \sin^2 \theta - a^2 \sin^2 \theta} = \frac{2ab}{(a^2 - b^2) \sin \theta}$$

From this, it follows that

$$\tan \mu = \frac{2ab}{(a^2 - b^2) \sin \left(\frac{\pi}{2} + \theta \right)}$$

$$= \frac{2ab}{(a^2 - b^2) \cos \theta}$$

Thus, we get

$$\cot \lambda = \frac{(a^2 - b^2) \sin \theta}{2ab}$$

$$\text{and} \quad \cot \mu = \frac{(a^2 - b^2) \cos \theta}{2ab}$$

Hence, $\cot^2 \lambda + \cot^2 \mu$

$$= \frac{(a^2 - b^2)^2}{4a^2b^2} (\sin^2 \theta + \cos^2 \theta)$$

$$= \frac{(a^2 - b^2)^2}{4a^2b^2}$$

which is constant.

Work Book Exercise 15.3

- 1 Chords of an ellipse are drawn through the positive end of the minor axis. Then, their mid-point lies on

- a a circle b a parabola
c an ellipse d a hyperbola

2. The area of the rectangle formed by the perpendiculars from the centre of the ellipse to the tangent and normal at the point, whose eccentric angle is $\pi/4$, is

- a $\left(\frac{a^2 - b^2}{a^2 + b^2} \right) ab$ b $\left(\frac{a^2 + b^2}{a^2 - b^2} \right) ab$
c $\frac{1}{ab} \left(\frac{a^2 - b^2}{a^2 + b^2} \right) ab$ d $\frac{1}{ab} \left(\frac{a^2 + b^2}{a^2 - b^2} \right) ab$

- 3 If the chords of contact of tangent from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are at right angles, then $\frac{x_1 x_2}{y_1 y_2}$ is equal to

- a $\frac{a^2}{b^2}$ b $-\frac{b^2}{a^2}$
c $-\frac{a^4}{b^4}$ d $-\frac{b^4}{a^4}$

- 4 The locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1 \text{ is}$$

- a $x^2 + y^2 = a^2 + \lambda$ b $x^2 + y^2 = b^2 + \lambda$
c $x^2 + y^2 = a^2 + b^2 + \lambda$ d $x^2 + y^2 = a^2 + b^2$

- 5 The locus of the mid-points of the chords $2x + 3y + 1 = 0$ of the ellipse $x^2 + 4y^2 = 1$, 1 being parameter, is

- a $8x - 3y = 0$ b $8x + 3y = 0$
c $3x - 8y = 0$ d $3x + 8y = 0$

- 6 In the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the length

perpendiculars from the centre upon all chords which join ends of perpendicular diameters, is

- a $\frac{ab}{\sqrt{(a^2 + b^2)}}$
b $\sqrt{(a^2 + b^2)}$
c $\sqrt{ab}$
d None of the above

WorkedOut Examples

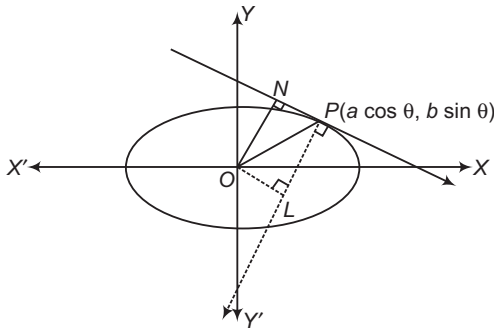
Type 1. Only One Correct Option

Ex 1. The coordinates of all the points P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, for which the area of the ΔPON is maximum, where O denotes the origin and N , the foot of the perpendicular from O to the tangent at P , is

- (a) $\left(\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}} \right)$
 (b) $\left(\pm \frac{a^2}{\sqrt{a^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2 - b^2}} \right)$
 (c) $\left(\pm \frac{2a^2}{\sqrt{a^2 + b^2}}, \pm \frac{2b^2}{\sqrt{a^2 + b^2}} \right)$
 (d) $\left(\pm \frac{2a^2}{\sqrt{a^2 + b^2}}, \pm \frac{2b^2}{\sqrt{a^2 + b^2}} \right)$

Sol. Equation of tangent at P is

$$\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$$



$$ON = \frac{1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} = \frac{ab}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}}$$

Equation of the normal at P is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

$$OL = \frac{a^2 - b^2}{\sqrt{a^2 \sec^2 \theta + b^2 \operatorname{cosec}^2 \theta}} = \frac{(a^2 - b^2) \sin \theta \cdot \cos \theta}{\sqrt{a^2 \sin^2 \theta + b^2 \cos^2 \theta}}$$

where, L is the foot of perpendicular from O on the normal.

$$\Rightarrow NP = OL$$

$$\text{Area of } \Delta PON = \frac{1}{2} \times ON \times OL$$

$$= \frac{(a^2 - b^2)ab \tan \theta}{a^2 \tan^2 \theta + b^2} = \frac{(a^2 - b^2)ab}{a^2 \tan \theta + b^2 \cot \theta}$$

which is minimum when $a^2 \tan \theta + b^2 \cot \theta$ is maximum.

Thus, for area to be minimum,

$$\tan \theta = \frac{b}{a}$$

$$\therefore \cos \theta = \pm \frac{a}{\sqrt{a^2 + b^2}}, \sin \theta = \pm \frac{b}{\sqrt{a^2 + b^2}}$$

$$\therefore \text{Points are } \left(\pm \frac{a^2}{\sqrt{a^2 + b^2}}, \pm \frac{b^2}{\sqrt{a^2 + b^2}} \right).$$

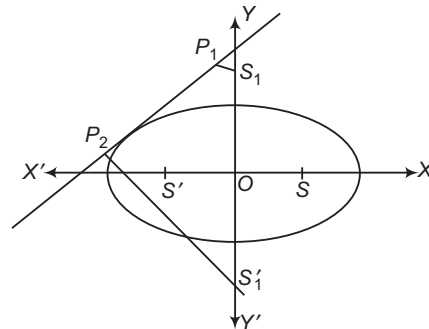
Hence, (a) is the correct answer.

Ex 2. If two points are taken on the minor axis of an ellipse at the same distance from the centre as the foci, then the sum of the squares of the perpendicular distances from these points on any tangent to the ellipse is

- (a) $2a^2$ (b) $3a^2$
 (c) $4a^2$ (d) None of these

Sol. Let the equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Then, the distance of a focus from the centre $= ae$

$$= a \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{a^2 - b^2}$$

So, the two points on the minor axis are

$$S_1(0, \sqrt{a^2 - b^2}) \text{ and } S_1'(0, -\sqrt{a^2 - b^2})$$

Now, any tangent to the ellipse is

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

where, m is a parameter.

The sum of the squares of the perpendiculars on this tangent from the two points S_1 and S_1'

$$= \left(\frac{\sqrt{a^2 - b^2} - \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}} \right)^2 + \left(\frac{-\sqrt{a^2 - b^2} - \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}} \right)^2$$

$$= \frac{2(a^2 - b^2 + a^2 m^2 + b^2)}{1 + m^2} = 2a^2$$

Hence, (a) is the correct answer.

Ex 3. If $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$ is normal to the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ for all values of m

belonging to

- (a) $(0, 1)$ (b) $(0, \infty)$
(c) R (d) None of these

Sol. The equation of the normal to the given ellipse at the point $P(a \cos \theta, b \sin \theta)$ is

$$ax \sec \theta - by \operatorname{cosec} \theta - a^2 = b^2$$

$$\Rightarrow y = \left(\frac{a}{b} \tan \theta \right) x - \frac{(a^2 - b^2)}{b} \sin \theta \quad \dots(i)$$

Let $\frac{a}{b} \tan \theta = m$, so that $\sin \theta = \frac{bm}{\sqrt{a^2 + b^2m^2}}$

Now, Eq. (i) becomes

$$y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$$

$\therefore m \in R$, as $m = \frac{a}{b} \tan \theta \in R$

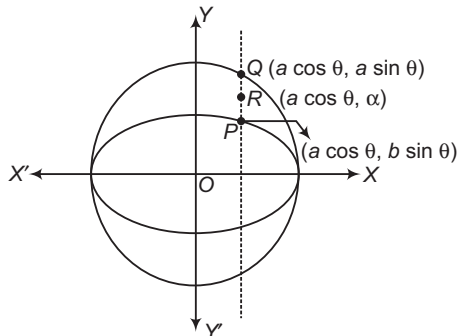
Hence, (c) is the correct answer.

Ex 4. Let P be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

$0 < b < a$. Let the line parallel to Y -axis passing through P meet the circle $x^2 + y^2 = a^2$ at the point Q such that P and Q are on the same side of X -axis. For two positive real numbers r and s , then the locus of the point R on PQ such that $PR : RQ = r : s$ as P varies over the ellipse, is

- (a) $\frac{x^2}{a^2} - \frac{y^2(r+s)^2}{(ar+bs)^2} = 1$ (b) $\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(ar+bs)^2} = 1$
(c) $\frac{a^2}{x^2} + \frac{(ar+bs)^2}{y^2(r+s)^2} = 1$ (d) $\frac{a^2}{x^2} - \frac{(ar+bs)^2}{y^2(r+s)^2} = 1$

Sol. Given that, $\frac{PR}{RQ} = \frac{r}{s}$



$$\Rightarrow \frac{\alpha - b \sin \theta}{a \sin \theta - \alpha} = \frac{r}{s}$$

$$\Rightarrow \alpha = \frac{(ar + bs) \sin \theta}{r + s}$$

Let the coordinates of R be (h, k) .
 $h = a \cos \theta$

and $k = \alpha = \frac{(ar + bs) \sin \theta}{r + s}$

$$\Rightarrow \cos \theta = \frac{h}{a}, \sin \theta = \frac{k(r+s)}{ar+bs}$$

$$\frac{h^2}{a^2} + \frac{k^2(r+s)^2}{(ar+bs)^2} = 1$$

Hence, locus of R is $\frac{x^2}{a^2} + \frac{y^2(r+s)^2}{(ar+bs)^2} = 1$

Hence, (b) is the correct answer.

Ex 5. In an ellipse, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on

- (a) tangent at vertex
(b) corresponding directrix
(c) tangent at $(0, b)$
(d) None of the above

Sol. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and O be the centre.

Tangent at $P(x_1, y_1)$ is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$

whose slope = $-\frac{b^2x_1}{a^2y_1}$

Focus is $S(ae, 0)$.

Equation of the line perpendicular to tangent at S is

$$y = \frac{a^2y_1}{b^2x_1}(x - ae) \quad \dots(i)$$

Equation of OP is $y = \frac{y_1}{x_1}x \quad \dots(ii)$

Eqs. (i) and (ii) intersect.

$$\therefore \frac{y_1}{x_1}x = \frac{a^2y_1}{b^2x_1}(x - ae)$$

$$\Rightarrow x(a^2 - b^2) = a^3e$$

$$\Rightarrow x \cdot a^2e^2 = a^3e$$

$$\Rightarrow x = \frac{a}{e}$$

which is the corresponding directrix.

Hence, (b) is the correct answer.

Ex 6. A parabola is drawn whose focus is one of the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where, $a > b$)

and whose directrix passes through the other focus and perpendicular to the major axes of the ellipse. Then, the eccentricity of the ellipse for which the latusrectum of the ellipse and the parabola are same, is

- (a) $\sqrt{2} - 1$
(b) $2\sqrt{2} + 1$
(c) $\sqrt{2} + 1$
(d) $2\sqrt{2} - 1$

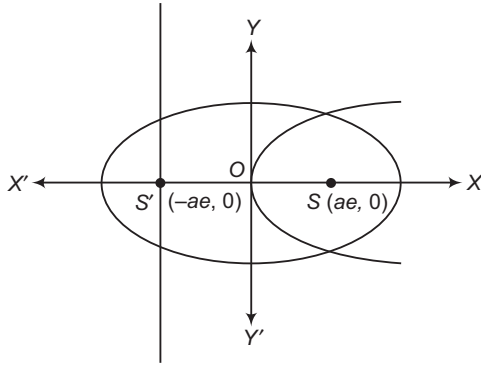
Sol. Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Equation of the parabola with focus $S(ae, 0)$ and directrix

$$x + ae = 0 \text{ is } y^2 = 4aex$$

Now, length of latusrectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.



For the two latusrectums to be equal,

$$\frac{2b^2}{a} = 4ae$$

$$\Rightarrow \frac{2a^2(1-e^2)}{a} = 4ae$$

$$\Rightarrow 1 - e^2 = 2e$$

$$\Rightarrow e^2 + 2e - 1 = 0$$

$$\text{Therefore, } e = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$$

So, $e = \sqrt{2} - 1$, as $0 < e < 1$.

Hence, (a) is the correct answer.

Ex 7. Any ordinate NP of an ellipse meets the auxiliary circle at Q . Then, the locus of the intersection of the normal at P and O is the circle

(a) $x^2 + y^2 = a^2$

(b) $x^2 + y^2 = (a+b)^2$

(c) $x^2 + y^2 = b^2$

(d) None of the above

Sol. Suppose the equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

therefore the equation of the auxiliary circle will be $x^2 + y^2 = a^2$.

Let the coordinates of any point P be $(a \cos \phi - b \sin \phi)$, then coordinates of Q will be $(a \cos \phi, a \sin \phi)$.

Normal at P is

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2 \quad \dots(i)$$

and normal at Q will be

$$y = x \tan \phi \quad \dots(ii)$$

$$\Rightarrow \tan \phi = \frac{y}{x}$$

$$\begin{aligned} \therefore \sec \phi &= \sqrt{1 + \tan^2 \phi} \\ &= \sqrt{1 + \frac{y^2}{x^2}} \\ &= \sqrt{\frac{x^2 + y^2}{x^2}} \end{aligned}$$

$$\begin{aligned} \text{and } \operatorname{cosec} \phi &= \frac{\sec \phi}{\tan \phi} \\ &= \frac{\sqrt{\frac{x^2 + y^2}{x^2}}}{y/x} \\ &= \frac{\sqrt{x^2 + y^2}}{y} \end{aligned}$$

On putting the values in Eq. (i), we get

$$ax \frac{\sqrt{x^2 + y^2}}{x} - by \frac{\sqrt{x^2 + y^2}}{y} = a^2 - b^2$$

$$\Rightarrow (a - b)\sqrt{x^2 + y^2} = a^2 - b^2$$

$$\Rightarrow \sqrt{x^2 + y^2} = a + b$$

$$\Rightarrow x^2 + y^2 = (a + b)^2$$

Hence, (b) is the correct answer.

Ex 8. If the focal distance of an end of the minor axis of any ellipse (its axes as X and Y -axes respectively) is k and the distance between the foci is $2h$, then its equation is

(a) $\frac{x^2}{k^2} + \frac{y^2}{h^2} = 1$

(b) $\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$

(c) $\frac{x^2}{k^2} - \frac{y^2}{k^2 - h^2} = 1$

(d) $\frac{x^2}{k^2} + \frac{y^2}{k^2 + h^2} = 1$

Sol. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and e be the eccentricity of ellipse.

$$\begin{aligned} \text{Therefore, } 2h &= 2ae \\ ae &= h \quad \dots(i) \end{aligned}$$

Focal distance of one end of minor axis say $(0, b)$ is h .

$$\begin{aligned} \text{Therefore, } a + e(0) &= k \\ a &= k \quad \dots(ii) \end{aligned}$$

From Eqs. (i) and (ii),

$$\begin{aligned} b^2 &= a^2(1 - e^2) \\ \Rightarrow b^2 &= a^2 - a^2e^2 \\ &= k^2 - h^2 \end{aligned}$$

Therefore, equation of ellipse is

$$\frac{x^2}{k^2} + \frac{y^2}{k^2 - h^2} = 1$$

Hence, (b) is the correct answer.

- Ex 9.** If the normal at point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meets the axes in R and S respectively, then $PR : RS$ is equal to
- (a) $a : b$ (b) $a^2 : b^2$
(c) $b^2 : a^2 - b^2$ (d) $b : a$

Sol. Let $P(a \cos \theta, b \sin \theta)$

Equation of normal is

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

Point of intersection with X -axis,

$$R \equiv \left(\frac{a^2 - b^2}{a} \cos \theta, 0 \right)$$

and with Y -axis, $S \equiv \left[0, - \left(\frac{a^2 - b^2}{b} \right) \sin \theta \right]$

Therefore,

$$\begin{aligned} RP^2 &= \left[a \cos \theta - \left(\frac{a^2 - b^2}{a} \right) \cos \theta \right]^2 + b^2 \sin^2 \theta \\ &= \frac{b^2}{a^2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta) \end{aligned}$$

and $RS^2 = \frac{(a^2 - b^2)^2}{a^2 b^2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$

Hence, $PR^2 : RS^2 = b^4 : (a^2 - b^2)^2$

$\Rightarrow PR : RS = b^2 : (a^2 - b^2)$

Hence, (c) is the correct answer.

- Ex 10.** Locus of the point which divides double ordinate of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the ratio

1 : 2 internally, is

(a) $\frac{x^2}{a^2} + \frac{9y^2}{b^2} = 1$

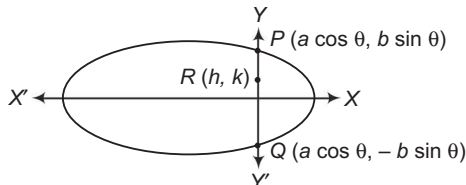
(b) $\frac{x^2}{a^2} + \frac{9y^2}{b^2} = \frac{1}{9}$

(c) $\frac{9x^2}{a^2} + \frac{9y^2}{b^2} = 1$

(d) None of the above

Sol. Let $P(a \cos \theta, b \sin \theta)$, $Q(a \cos \theta, -b \sin \theta)$

$PR : RQ = 1 : 2$



Therefore, $h = a \cos \theta$

$\Rightarrow \cos \theta = \frac{h}{a} \quad \dots(i)$

and $k = \frac{b}{3} \sin \theta$

$\Rightarrow \sin \theta = \frac{3k}{b} \quad \dots(ii)$

On squaring and adding Eqs. (i) and (ii), we get

$$\frac{h^2}{a^2} + \frac{9k^2}{b^2} = 1$$

Therefore, locus of R is

$$\frac{x^2}{a^2} + \frac{9y^2}{b^2} = 1$$

Hence, (a) is the correct answer.

- Ex 11.** At a point P on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, tangent

PQ is drawn. If the point Q is at a distance $\frac{1}{p}$ from the point P , where p is distance of the tangent from the origin, then the locus of the point Q is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{1}{a^2 b^2}$

(b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 - \frac{1}{a^2 b^2}$

(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2 b^2}$

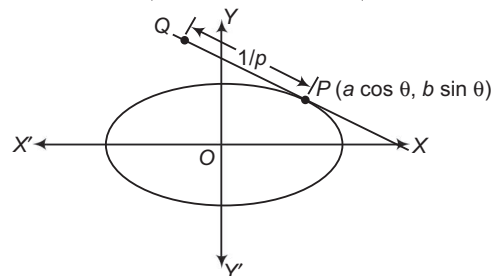
(d) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{a^2 b^2}$

Sol. Equation of the tangent at P is

$$\frac{x - a \cos \theta}{a \sin \theta} = \frac{y - b \sin \theta}{-b \cos \theta}$$

The distance of the tangent from the origin is

$$\begin{aligned} p &= \frac{\left| \frac{a \cos \theta}{a \sin \theta} - \frac{b \sin \theta}{-b \cos \theta} \right|}{\sqrt{\frac{1}{a^2 \sin^2 \theta} + \frac{1}{b^2 \cos^2 \theta}}} \\ \Rightarrow \frac{1}{p} &= \frac{\left| \frac{a \cos \theta}{a \sin \theta} - \frac{b \sin \theta}{-b \cos \theta} \right|}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \end{aligned}$$



Now, the coordinates of the point Q are given as follows

$$\begin{aligned} \frac{x - a \cos \theta}{-a \sin \theta} &= \frac{y - b \sin \theta}{b \cos \theta} \\ \frac{x - a \cos \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} &= \frac{y - b \sin \theta}{\sqrt{b^2 \cos^2 \theta + a^2 \sin^2 \theta}} \\ &= \frac{1}{p} = \frac{1}{ab} \end{aligned}$$

$$\Rightarrow x = a \cos \theta - \frac{a \sin \theta}{ab}$$

and $y = b \sin \theta + \frac{b \cos \theta}{ab}$

$$\Rightarrow \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + \frac{1}{a^2 b^2}$$

is the required locus.

Hence, (a) is the correct answer.

Ex 12. A triangle is drawn such that it is right angled at the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) and its other two vertices lie on the ellipse with eccentric angles α, β . Then, $1 - e^2 \cos^2 \left(\frac{\alpha + \beta}{2}\right)$ is equal to

- (a) $\frac{a^2}{a^2 + b^2}$
 (b) $\frac{a^2 + b^2}{a^2}$
 (c) $\frac{a^2}{a^2 - b^2}$
 (d) $\frac{a^2 - b^2}{a^2}$

Sol. Equation of the chord with eccentric angles of the extremities as α, β is

$$\frac{x}{a} \cos \left(\frac{\alpha + \beta}{2}\right) + \frac{y}{b} \sin \left(\frac{\alpha + \beta}{2}\right) = \cos \left(\frac{\alpha - \beta}{2}\right)$$

Let $\frac{\alpha + \beta}{2} = \Delta$

and $\frac{\alpha - \beta}{2} = \delta$, so that

$$\frac{x}{a} \cos \Delta + \frac{y}{b} \sin \Delta = \cos \delta$$

As the triangle is right angled, homogenizing the equation of the curve, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \left(\frac{x \cos \Delta}{a \cos \delta} + \frac{y \sin \Delta}{b \cos \delta}\right)^2 = 0$$

$$\Rightarrow \left(\frac{1}{a^2} - \frac{\cos^2 \Delta}{a^2 \cos^2 \delta}\right) + \left(\frac{1}{b^2} - \frac{\sin^2 \Delta}{b^2 \cos^2 \delta}\right) = 0$$

[$\because$ coefficient of x^2 + coefficient of y^2 = 0]

$$\Rightarrow b^2(\cos^2 \delta - \cos^2 \Delta) + a^2(\cos^2 \delta - \sin^2 \Delta) = 0$$

$$\Rightarrow \cos^2 \delta(a^2 + b^2) - b^2 \cos^2 \Delta - a^2 + a^2 \cos^2 \Delta = 0$$

$$\Rightarrow \cos^2 \delta(a^2 + b^2) = a^2(1 - e^2 \cos^2 \Delta)$$

$$\Rightarrow \frac{1 - e^2 \cos^2 \left(\frac{\alpha + \beta}{2}\right)}{\cos^2 \left(\frac{\alpha - \beta}{2}\right)} = \frac{a^2 + b^2}{a^2}$$

Hence, (b) is the correct answer.

Ex 13. Given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$) with foci

at S and S' and vertices at A and A' . A tangent is drawn at any point P on the ellipse and let R, R', B, B' respectively be the foot of the perpendiculars drawn from S, S', A, A' on the tangent at P . Then, the ratio of the areas of the quadrilaterals $S'R'RS$ and $A'B'BA$ is

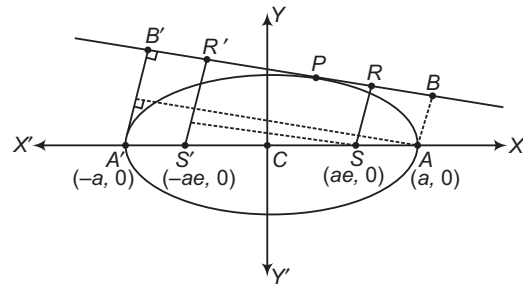
- (a) $e : 2$ (b) $e : 3$ (c) $e : 1$ (d) $e : 4$

Sol. Equation of tangent at P is

$$y = mx + \sqrt{a^2 m^2 + b^2} \quad \dots(i)$$

$S'R'RS$ is a trapezium and its area

$$\Delta_1 = \frac{1}{2}(SR + S'R') \times SD$$



Equation of the line $S'R'$ is

$$y = -\frac{1}{m}(x + ae)$$

$$\Rightarrow x + my + ae = 0 \quad \dots(ii)$$

Therefore,

$$SD = \frac{ae + ae}{\sqrt{1 + m^2}}$$

$$SR = \frac{aem + \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}}$$

and

$$S'R' = \frac{-aem + \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}}$$

$$\Delta_1 = \frac{1}{2}(S'R' + SR) \times SD$$

$$\Rightarrow \Delta_1 = 2ae \left(\frac{\sqrt{a^2 m^2 + b^2}}{1 + m^2} \right)$$

$$\text{Area of } A'B'BA \text{ is } \Delta_2 = \frac{1}{2}(A'B' + AB) \times AE$$

$$\text{Equation of } A'B' \text{ is } y = -\frac{1}{m}(x + a)$$

$$\Rightarrow x + my + a = 0 \quad \dots(iii)$$

$$\text{and } AB = \frac{am + \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}}$$

$$\text{and } A'B' = \frac{-am + \sqrt{a^2 m^2 + b^2}}{\sqrt{1 + m^2}}$$

$$\Rightarrow \Delta_2 = \frac{1}{2}(A'B' + AB) \times AE$$

$$\Delta_2 = 2a \left(\frac{\sqrt{a^2 m^2 + b^2}}{1 + m^2} \right)$$

Hence, $\Delta_1 : \Delta_2 = e : 1$

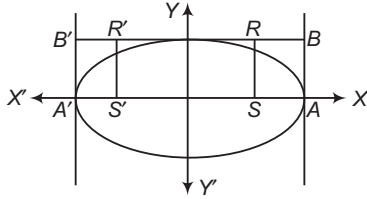
Aliter

Taking point P at the end of minor axis.

$$\Rightarrow \Delta_1 = 2ae \times b = 2abe$$

$$\text{and } \Delta_2 = 2a \times b = 2ab$$

$$\therefore \frac{\Delta_1}{\Delta_2} = \frac{e}{1}$$



Hence, (c) is the correct answer.

Ex 14. The locus of chords of contact of perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ touch another fixed ellipse is

$$(a) \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{(2a^2 + b^2)}$$

$$(b) \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2}{(a^2 - b^2)}$$

$$(c) \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}$$

$$(d) \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2}{(3a^2 - b^2)}$$

Sol. We know that locus of the point of intersection of perpendicular tangents of the given ellipse is

$$x^2 + y^2 = a^2 + b^2$$

Any point on this circle can be taken as

$$P \equiv (\sqrt{a^2 + b^2} \cos \theta, \sqrt{a^2 + b^2} \sin \theta)$$

The equation of the chord of contact of tangents from

$$P \text{ is } \frac{x}{a^2} \sqrt{a^2 + b^2} \cos \theta + \frac{y}{b^2} \sqrt{a^2 + b^2} \sin \theta = 1$$

Let this line be a tangent to the fixed ellipse

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$$

$$\frac{x}{A} \cos \theta + \frac{y}{B} \sin \theta = 1,$$

$$\text{where } A = \frac{a^2}{\sqrt{a^2 + b^2}}, B = \frac{b^2}{\sqrt{a^2 + b^2}}$$

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{(a^2 + b^2)}$$

which is an ellipse.

Hence, (c) is the correct answer.

Ex 15. Given an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$.

A circle is drawn which passes through the foci and the extremities of the minor axis of this ellipse and let this circle be the auxiliary circle for a second ellipse whose vertices are the foci of the given ellipse. If any tangent to the second ellipse meets its auxiliary circle at two

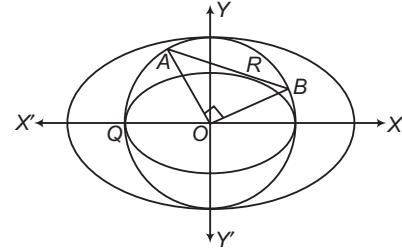
points such that its subtends a right angle at the origin, then the locus of the point of tangency, is (given eccentricity of both the ellipses are same)

$$(a) 2x^2 + 2y^2 = 2a^2 \quad (b) 2x^2 + y^2 = a^2$$

$$(c) x^2 + 2y^2 = a^2 \quad (d) x^2 + y^2 = 2a^2$$

Sol. Equation of first ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots(i)$$



Since, the circle passes through the foci and the extremities of the minor axis.

$$\therefore 2ae = 2b \Rightarrow a^2 e^2 = b^2 \Rightarrow e^2 = 1 - e^2$$

$$\Rightarrow e = \frac{1}{\sqrt{2}} = \text{Eccentricities of both the ellipses.}$$

Now, equation of the auxiliary circle will be

$$x^2 + y^2 = b^2 \quad \dots(ii)$$

Let the second ellipse be $\frac{x^2}{A^2} + \frac{y^2}{B^2} = 1$ whose major axis = $2ae$

$$\text{Also, } B^2 = A^2 \sqrt{1 - e^2} \Rightarrow B^2 = a^2 e^2 \sqrt{1 - e^2}$$

$$\Rightarrow B^2 = \frac{a^2}{2\sqrt{2}} \quad \text{i.e. } e = \frac{1}{\sqrt{2}}$$

$$\text{and } A^2 = a^2 e^2 \Rightarrow A^2 = \frac{a^2}{2}$$

Therefore, equation of the second ellipse is

$$\frac{x^2}{a^2/2} + \frac{y^2}{a^2/2\sqrt{2}} = 1 \Rightarrow x^2 + \sqrt{2}y^2 = \frac{a^2}{2} \quad \dots(iii)$$

Equation of the tangent at $R(h, k)$ to Eq. (iii) is

$$xh + \sqrt{2}yk = \frac{a^2}{2} \quad \dots(iv)$$

From Eqs. (ii) and (iv)

$$x^2 + y^2 - b^2 \left[\frac{2}{a^2} (xh + \sqrt{2}yk) \right]^2 = 0$$

$$\Rightarrow x^2 + y^2 - \frac{4b^2}{a^4} [x^2 h^2 + 2y^2 k^2 + 2\sqrt{2}hxy] = 0$$

Since, AB subtends a right angle at the origin.

$$\therefore 1 - \frac{4b^2 h^2}{a^4} + 1 - \frac{4b^2 k^2}{a^4} = 0$$

$$\Rightarrow h^2 + 2k^2 = \frac{a^4}{2b^2} \Rightarrow h^2 + 2k^2 = \frac{a^2}{2} \cdot 2 \quad \left[\because \frac{a^2}{b^2} = 2 \right]$$

$$\Rightarrow h^2 + 2k^2 = a^2$$

Hence, the locus of (h, k) is $x^2 + 2y^2 = a^2$.

Hence, (c) is the correct answer.

Ex 16. The locus of the intersection of two normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are perpendicular to each other, is

- (a) $(a^2 - b^2)^2 \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2$
 $= (a^2 + b^2)^2 (x^2 + y^2) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2$
 (b) $(a^2 + b^2)^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2$
 $= (a^2 - b^2)^2 (x^2 + y^2) \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2$
 (c) $(a^2 - b^2)^2 \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2$
 $= (a^2 + b^2)^2 (x^2 - y^2) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2$
 (d) None of the above

Sol. Equation of normal to the ellipse is

$$ax \sec \theta - by \csc \theta = a^2 - b^2$$

$$\Rightarrow ax \tan \theta - by = (a^2 - b^2) \sin \theta \quad \dots(i)$$

Its slope $= \frac{a}{b} \tan \theta = m$ (say)

$$\Rightarrow \tan \theta = \frac{bm}{a}$$

Type 2. More than One Correct Option

Ex 17. If the chord through the point whose eccentric angles are θ and ϕ on the ellipse, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the focus, then the value of $\tan(\theta/2) \tan(\phi/2)$ is

- (a) $\frac{e+1}{e-1}$ (b) $\frac{e-1}{e+1}$ (c) $\frac{1+e}{1-e}$ (d) $\frac{1-e}{1+e}$

Sol. The equation of the line joining θ and ϕ .

$$\frac{x}{a} \cos\left(\frac{\theta+\phi}{2}\right) + \frac{y}{b} \sin\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

If it passes through the point $(ae, 0)$.

$$\frac{ae}{a} \cos\left(\frac{\theta+\phi}{2}\right) = \cos\left(\frac{\theta-\phi}{2}\right)$$

$$e = \frac{\cos\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta+\phi}{2}\right)}$$

$$\frac{e+1}{e-1} = \frac{\cos\left(\frac{\theta-\phi}{2}\right) + \cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right) - \cos\left(\frac{\theta+\phi}{2}\right)}$$

$\therefore$ Eq. (i) can be written as

$$ax \cdot \frac{bm}{a} - by = (a^2 - b^2) \cdot \frac{bm}{\sqrt{b^2 m^2 + a^2}}$$

$$\Rightarrow bmx - by = (a^2 - b^2) \cdot \frac{bm}{\sqrt{b^2 m^2 + a^2}}$$

If it passes through (α, β) .

$$\therefore b^2 \alpha^2 m^4 - 2b^2 \alpha \beta m^3 + \{a^2 \alpha^2 + b^2 \beta^2 - (a^2 - b^2)^2\} m^2 - 2a^2 \alpha \beta m + a^2 \beta^2 = 0$$

Let m_1, m_2, m_3 and m_4 be four roots of this equation.

$$m_1 + m_2 + m_3 + m_4 = \frac{2\beta}{\alpha} \quad \dots(i)$$

$$m_1 m_2 + m_1 m_3 + m_1 m_4 + m_2 m_3 + m_2 m_4 + m_3 m_4 = \frac{a^2 \alpha^2 + b^2 \beta^2 - (a^2 - b^2)^2}{b^2 \alpha^2} \quad \dots(ii)$$

$$m_1 m_2 m_3 + m_1 m_3 m_4 + m_2 m_3 m_4 + m_1 m_2 m_4 = \frac{2a^2 \beta}{b^2 \alpha} \quad \dots(iii)$$

$$m_1 m_2 m_3 m_4 = \frac{a^2 \beta^2}{b^2 \alpha^2} \quad \dots(iv)$$

$$\text{Also, } m_1 m_2 = -1 \quad \dots(v)$$

Eliminating m_1, m_2, m_3 and m_4 from Eqs. (i), (ii), (iii), (iv) and (v), we get

$$(a^2 - b^2)^2 (b^2 \alpha^2 - a^2 \beta^2)^2 = (a^2 + b^2)^2 (\alpha^2 + \beta^2) (b^2 \alpha^2 + a^2 \beta^2)^2$$

$\therefore$ Required locus is

$$(a^2 - b^2)^2 \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right)^2 = (a^2 + b^2)^2 (x^2 + y^2) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2$$

Hence, (a) is the correct answer.

$$= \frac{2 \cos \theta/2 \cos \phi/2}{2 \sin \phi/2 \sin \theta/2}$$

$$\Rightarrow \tan \frac{\theta}{2} \tan \frac{\phi}{2} = \frac{e-1}{e+1}$$

If it passes through the point $(-ae, 0)$, then

$$\tan \frac{\phi}{2} \tan \frac{\theta}{2} = \frac{e+1}{e-1}$$

Hence, (a) and (b) are the correct answers.

Ex 18. Identify the statements which are true.

- (a) The equation of the director circle of the ellipse, $5x^2 + 9y^2 = 45$ is $x^2 + y^2 = 14$
 (b) The sum of the focal distance of the point $(0, 6)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{36} = 1$ is 10
 (c) The point of intersection of any tangent to a parabola and the perpendicular to it from the focus lies on the tangent at the vertex
 (d) The line through focus and $(at_1^2, 2at_1)$ on $y^2 = 4ax$, meets it again in the point $(at_2^2, 2at_2)$, if $t_1 t_2 = -1$

Sol. (a) Director circle of $\frac{x^2}{9} + \frac{y^2}{5} = 1$ is $x^2 + y^2 = 14$

(b) Using focal property

$$|PS + PS'| = 2 \text{ for } a < b$$

$$\text{i.e. } |PS + PS'| = 12$$

(c) Property

(d) Slope of focal chord joining $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is equal to slope joining $(at_1^2, 2at)$ and $(a, 0)$.

$$\text{i.e. } \frac{2}{t_1 + t_2} = \frac{2at_1}{a(t_1^2 - 1)} \Rightarrow t_1 t_2 = -1$$

Hence, (a), (c) and (d) are the correct answers.

Ex 19. The parametric angle θ , where $-\pi \leq \theta \leq \pi$, of the point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at which the tangent drawn cuts the intercept of minimum length on the coordinate axes, is/are

- (a) $\tan^{-1} \sqrt{\frac{b}{a}}$ (b) $-\tan^{-1} \sqrt{\frac{b}{a}}$
(c) $\pi - \tan^{-1} \sqrt{\frac{b}{a}}$ (d) $\pi + \tan^{-1} \sqrt{\frac{b}{a}}$

Sol. Equation of tangent at θ is $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1$

$$\Rightarrow \frac{x}{a \sec \theta} + \frac{y}{b \csc \theta} = 1$$

$$\therefore \text{Intercept, } d = \sqrt{a^2 \sec^2 \theta + b^2 \csc^2 \theta}$$

$$\therefore d^2 = a^2 \sec^2 \theta + b^2 \csc^2 \theta$$

$$\Rightarrow \frac{d(d^2)}{d\theta} = 2a^2 \frac{\sin \theta}{\cos^3 \theta} - 2b^2 \frac{\cos \theta}{\sin^3 \theta} = 0$$

$$\text{i.e. } \frac{a^2 \sin \theta}{\cos^3 \theta} = \frac{b^2 \cos \theta}{\sin^3 \theta}$$

$$\therefore \tan^4 \theta = \frac{b^2}{a^2} \Rightarrow \tan \theta = \pm \sqrt{\frac{b}{a}}$$

$$\theta = \tan^{-1} \sqrt{\frac{b}{a}}, -\tan^{-1} \sqrt{\frac{b}{a}}$$

$$\theta = \pi - \tan^{-1} \sqrt{\frac{b}{a}}, -\pi + \tan^{-1} \sqrt{\frac{b}{a}}$$

Hence, (a), (b) and (c) are the correct answers.

Ex 20. Extremities of the latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($a > b$) having a given major axis $2a$, lies on

- (a) $x^2 = a(a - y)$ (b) $x^2 = a(a + y)$
(c) $y^2 = a(a + x)$ (d) $y^2 = a(a - x)$

Sol. $h = \pm ae$, $k = \pm \frac{b^2}{a}$

$$k = \pm a(1 - e^2) = \pm a \left(1 - \frac{h^2}{a^2}\right) = \pm \left(a - \frac{h^2}{a}\right)$$

On taking +ve sign, we get

$$k = a - \frac{h^2}{a} \Rightarrow \frac{h^2}{a} = a - k$$

$$\Rightarrow h^2 = a(a - k)$$

On taking -ve sign, we get

$$k = -a + \frac{h^2}{a}$$

$$\Rightarrow h^2 = a(a + k)$$

Hence, (a) and (b) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 21-25) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 21. Statement I Locus of centre of a variable circle touching two circles $(x - 1)^2 + (y - 2)^2 = 25$ and $(x + 2)^2 + (y - 1)^2 = 16$ is an ellipse.

Statement II If a circle $S_1 = 0$ lies completely inside the circle $S_2 = 0$, then locus of centre of variable circle $S = 0$ which touches both the circles is an ellipse.

Sol. Let C_1, C_2 be the centres and R_1, R_2 be the radii of the two circles. Let $S_1 = 0$ lies completely inside in the circle $S_2 = 0$. Let C and r be the centre and radius of the variable circle.

$$\text{Then, } CC_2 = R_2 - r$$

$$\text{and } C_1C = R_1 + r$$

$$\therefore C_1C + C_2C = R_1 + R_2 \quad [\text{constant}]$$

$\therefore$ Locus of C is an ellipse.

$\therefore S_2$ is true.

Statement I is false (two circles are intersecting).

Hence, (d) is the correct answer.

Ex 22. Statement I If $P\left(\frac{3\sqrt{3}}{2}, 1\right)$ is a point on the ellipse $4x^2 + 9y^2 = 36$. Circle drawn AP as diameter touches another circle $x^2 + y^2 = 9$, where $A \equiv (-\sqrt{5}, 0)$.

Statement II Circle drawn with focal radius as diameter touches the auxiliary circle.

Sol. The ellipse is $\frac{x^2}{9} + \frac{y^2}{4} = 1$

$\therefore$ Auxiliary circle is $x^2 + y^2 = 9$ and $(-\sqrt{5}, 0)$ and $(\sqrt{5}, 0)$ are foci.

$\therefore$ Statement I is true. Statement II is also true.

Hence, (a) is the correct answer.

Ex 23. Statement I Foot of the perpendicular drawn from foci of an ellipse $4x^2 + y^2 = 16$ on the line $2\sqrt{3}x + y = 8$ lie on the circle $x^2 + y^2 = 16$.

Statement II If perpendicular are drawn from foci of an ellipse to its any tangent, then foot of these perpendicular lie on director circle of the ellipse.

Sol. $4x^2 + (8 - 2\sqrt{3}x)^2 = 16$

i.e. $x^2 - 2\sqrt{3}x + 3 = 0$

i.e. $(x - \sqrt{3})^2 = 0$

$\therefore 2\sqrt{3}x + y = 8$ is a tangent to the ellipse, the auxiliary circle is $x^2 + y^2 = 16$.

$\therefore$ Statement I is true and Statement II is false.

Hence, (c) is the correct answer.

Ex 24. Statement I In a $\triangle ABC$, if base BC is fixed and perimeter of the triangle is constant, then vertex A moves on an ellipse.

Statement II If sum of distance of a point P from two fixed points is constant, then locus of P is a real ellipse.

Sol. Statement II is false (locus of P may be a line segment also). Statement I is true.

Hence, (c) is the correct answer.

Ex 25. Statement I Let tangent at a point P on the ellipse, which is not an extremity of major axis, meets a directrix at T . If circle drawn on PT as diameter cuts the directrix at Q , then $PQ = ePS$, where S is the focus corresponding to the directrix.

Statement II Let tangent at a point P on a ellipse, which is not an extremity of major axis, meets the directrices at T' and T . Then, PT subtends a right angle at the focus corresponding the directrix at which T lies.

Sol. Statement I is false. Since, $\angle PQT = \pi/2$

$\therefore PQ$ perpendicular to the directrix.

$\therefore$ By definition, $PS = ePQ$

$\therefore$ Statement I is false.

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 26-28) An ellipse E has its centre $C(1, 3)$ focus at $S(6, 3)$ and passing through the point $P(4, 7)$.

Ex 26. The product of the lengths of the perpendicular segments from the foci on tangent at point P is

- (a) 20
(b) 45
(c) 40
(d) Cannot be determined

Ex 27. The point of intersection of the lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at point P , is

- (a) $\left(\frac{5}{3}, 5\right)$ (b) $\left(\frac{4}{3}, 3\right)$
(c) $\left(\frac{8}{3}, 3\right)$ (d) $\left(\frac{10}{3}, 5\right)$

Ex 28. If the normal at a variable on the ellipse (E) meets its axes in Q and R , then the locus of the mid-point of QR is a conic with an eccentricity (e'), then

- (a) $e' = \frac{3}{\sqrt{10}}$ (b) $e' = \frac{\sqrt{5}}{3}$
(c) $e' = \frac{3}{\sqrt{5}}$ (d) $e' = \frac{\sqrt{10}}{3}$

Sol. (Ex. Nos. 26-28)

Equation of ellipse is $\frac{(x-1)^2}{45} + \frac{(y-3)^2}{20} = 1$

$PS + PS' = 2a$

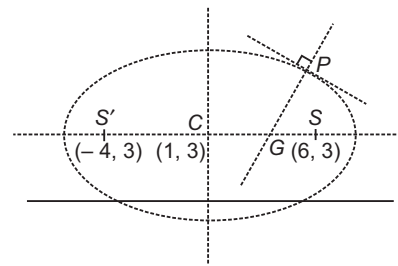
$PS = \sqrt{64 + 16} = 4\sqrt{5}$

$PS' = \sqrt{4 + 16} = 2\sqrt{5}$

$2a = 6\sqrt{5} \Rightarrow a = 3\sqrt{5}$

$ae = 5$

$\Rightarrow e \times 3\sqrt{5} = 5$



$\therefore e = \frac{\sqrt{5}}{3}$

$\therefore e^2 = 1 - \frac{b^2}{a^2}$

$\Rightarrow \frac{b^2}{a^2} = 1 - \frac{5}{9} = \frac{4}{9}$

$\therefore b^2 = \frac{4}{9} \times 9 \times 5$

$b = 2\sqrt{5}$

- 26.** $\therefore$ Product of the length of the perpendicular segments from the foci on tangent at $P(4, 7)$ is
 $b^2 = 20$

Hence, (a) is the correct answer.

- 27.** $\therefore$ Lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at $P(4, 7)$ meet the normal PG and bisect it.

$\therefore$ Required point is mid-point of PG .

Now, equation of normal at $P(4, 7)$ is given by

$$3x - y - 5 = 0$$

$$\therefore G\left(\frac{8}{3}, 3\right)$$

$\therefore$ Required point is mid-point of PG i.e. $\left(\frac{10}{3}, 5\right)$.

Hence, (d) is the correct answer.

- 28.** $\therefore$ Locus of mid-point of QR is another ellipse having the same eccentricity as the ellipse (e).

$$\Rightarrow e' = e = \frac{\sqrt{5}}{3}$$

Hence, (b) is the correct answer.

Passage II (Ex. Nos. 29-31) Consider an ellipse (E) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, centered at point Q and having AB and CD as its major and minor axes respectively, if S_1 is one of the focus of the ellipse radius of incircle of ΔOCS_1 , be 1 unit and $OS_1 = 6$ units, then

Ex 29. The area of ellipse (E) is

- (a) $\frac{65\pi}{4}$ sq units (b) $\frac{64\pi}{5}$ sq units
 (c) 64π sq units (d) 65π sq units

Ex 30. Perimeter of ΔOCS is

- (a) 20 units (b) 10 units
 (c) 15 units (d) 25 units

Ex 31. If S is the director circle of ellipse (E), then the equation of director circle of S is

- (a) $x^2 + y^2 = (48.5)$
 (b) $x^2 + y^2 = \sqrt{97}$
 (c) $x^2 + y^2 = 97$
 (d) $x^2 + y^2 = \sqrt{48.5}$

Sol. (Ex. Nos. 29-31)

$$OS_1 = ae = 6, OC = b \quad [\text{let}]$$

$$\text{Also, } CS_1 = a$$

$$\therefore \text{Area of } \Delta OCS_1 = \frac{1}{2}(OS_1) \times (OC) = 3b$$

$$\therefore \text{Semi-perimeter of } \Delta OCS_1$$

$$= \frac{1}{2}(OS_1 + OC + CS_1) = \frac{1}{2}(6 + a + b) \quad \dots(i)$$

$$\therefore \text{Inradius of } \Delta OCS_1 = 1$$

$$\Rightarrow \frac{3b}{\frac{1}{2}(6 + a + b)} = 1 \Rightarrow 5b = 6 + a \quad \dots(ii)$$

$$\text{Also, } b^2 = a^2 - a^2e^2 = a^2 - 36 \quad \dots(iii)$$

From Eq. (ii), we get

$$25b^2 = 36 + 12a + a^2$$

$$\Rightarrow 25(a^2 - 36) = 36 + a^2 + 12a \quad [\text{from Eq. (iii)}]$$

$$2a^2 - a - 78 = 0$$

$$\Rightarrow a = \frac{13}{2}, -6$$

$$\therefore a = \frac{13}{2}$$

$$\Rightarrow b = \frac{5}{2}$$

29. Area of ellipse = πab

$$= \frac{65\pi}{4} \text{ sq units}$$

Hence, (a) is the correct answer.

30. $\therefore$ Perimeter of ΔOCS_1

$$= 6 + a + b$$

$$= 6 + \frac{13}{2} + \frac{5}{2}$$

$$= 15 \text{ units}$$

Hence, (c) is the correct answer.

31. $\therefore S : x^2 + y^2 = a^2 + b^2$

$$\Rightarrow S : x^2 + y^2 = \frac{97}{2} = r^2$$

$\therefore$ Equation of director circle of S is

$$x^2 + y^2 = 2r^2$$

$$\Rightarrow x^2 + y^2 = 97$$

Hence, (c) is the correct answer.

Passage III (Ex. Nos. 32-34) Second degree equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

represents an ellipse, if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \neq 0$ and $h^2 < ab$.

Intersection of major axis and minor axis gives centre of ellipse.

Ex 32. There are exactly n integral values of λ for which equation $x^2 + \lambda xy + y^2 = 1$ represents an ellipse, then n must be

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. Using conditions for ellipse,

$$-2 < \lambda < 2 \Rightarrow \lambda = -1, 0, 1$$

Hence, (d) is the correct answer.

Ex 33. Length of the longest chord of the ellipse $x^2 + y^2 + xy = 1$ is

$$(a) \sqrt{2} \quad (b) \frac{1}{\sqrt{2}}$$

$$(c) 2\sqrt{2} \quad (d) 1$$

Sol. Consider, $f = x^2 + xy + y^2 - 1 = 0$

Hence, (c) is the correct answer.

Ex 34. Length of the chord perpendicular to longest chord as in above question and passing through centre of ellipse is

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{\sqrt{3}}{2}$ (c) $2\sqrt{\frac{2}{3}}$ (d) $\frac{1}{\sqrt{3}}$

Sol. Solving $\frac{\partial f}{\partial x} = 0 = \frac{\partial f}{\partial y}$ centre of ellipse comes out to be (0, 0).

Longest chord of ellipse is major axis. While chord perpendicular to it and through centre of ellipse is minor axis, so it is the smallest chord. Let P be a point

on ellipse such that $OP = r$ and $\angle POX = \theta$, then $P(r \cos \theta, r \sin \theta)$ as P lies on ellipse $r^2 \cos^2 \theta + r^2 \sin^2 \theta \cos^2 \theta + r^2 \sin^2 \theta = 1 = 0$

$$\Rightarrow r^2 = \frac{1}{1 + \sin^2 \theta \cos^2 \theta} = \frac{1}{1 + \frac{1}{2} \sin^2 2\theta}$$

$\therefore r$ is smallest, if $\sin 2\theta$ is greatest.

$$\text{i.e. } \sin 2\theta = 1 \therefore r^2 = \frac{2}{3} \Rightarrow r = \sqrt{\frac{2}{3}}$$

Now, length of required chord $= 2\sqrt{\frac{2}{3}}$

Hence, (c) is the correct answer.

Type 5. Match the Columns

Ex 35. Match the statements of Column I with values of Column II.

Column I	Column II
A. A tangent to the ellipse $\frac{x^2}{27} + \frac{y^2}{48} = 1$ having slope $-\frac{4}{3}$ cuts the X and Y-axes at the points A and B, respectively. If O is the origin, then area of ΔOAB is equal to	p. 36
B. Product of the perpendiculars drawn from the points $(\pm 3, 0)$ to the line $y = mx - \sqrt{25m^2 + 16}$ is	q. $10\sqrt{2}$
C. An ellipse passing through the origin has its foci $(3, 4)$ and $(6, 8)$, then length of its minor axis is	r. 10
D. If PQ is focal chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$, which passes through $S \equiv (3, 0)$ and $PS = 2$, then length of chord PQ is	s. 16

Sol. A. Let $(\sqrt{27} \cos \theta, \sqrt{48} \sin \theta)$ be a point on the ellipse.

Thus, equation of tangent of their points is

$$\frac{\sqrt{27} \cos \theta x}{27} + \frac{\sqrt{48} \sin \theta y}{48} = 1$$

$$\begin{aligned} \therefore \text{Slope} &= \frac{-\cos \theta}{\sqrt{27}} \times \frac{\sqrt{48}}{\sin \theta} \\ &= -\frac{4}{3} \cot \theta \\ &= -\frac{4}{3} \end{aligned}$$

$$\text{i.e. } \cot \theta = 1$$

$$\text{i.e. } \theta = \frac{\pi}{4}$$

$$\therefore \text{Equation of the tangent is } \frac{x}{\sqrt{54}} + \frac{y}{\sqrt{96}} = 1$$

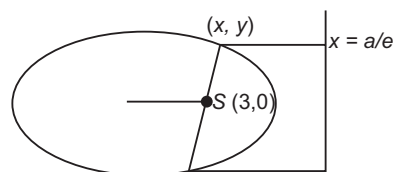
$$\text{Hence, area of triangle} = \frac{1}{2} \cdot 3\sqrt{6} \times 4\sqrt{6} = 36$$

B. Product of perpendicular

$$\begin{aligned} &= \left| \frac{3m - \sqrt{25m^2 + 16}}{\sqrt{1 + m^2}} \right| \left| \frac{-3m - \sqrt{25m^2 + 16}}{\sqrt{1 + m^2}} \right| \\ &= \left| \frac{25m^2 + 16 - 9m^2}{1 + m^2} \right| = 16 \end{aligned}$$

$$\text{C. } 2a = 5 + 10 = 15, a = \frac{15}{2}$$

$$\begin{aligned} \Rightarrow 2ae &= \sqrt{(6-3)^2 + (8-4)^2} = 5 \\ e &= \frac{1}{3} \end{aligned}$$



$$\therefore b^2 = \frac{225}{4} \left(1 - \frac{1}{9} \right) = 50$$

$$\Rightarrow b = 5\sqrt{2}$$

$$\text{Hence, } 2b = 10\sqrt{2}$$

$$\text{D. } a = 5, b = 4, e = \frac{3}{5}$$

$$\therefore ae = 3$$

$$\therefore SA = 2 \quad \text{Also, } SP = 2$$

Since, P coincides with A and Q coincides with A' .

$$\therefore PQ = 2a = 10$$

$$A \rightarrow p; B \rightarrow s; C \rightarrow q; D \rightarrow r$$

Ex 36. Match the statements of Column I with values of Column II.

Column I	Column II
A. A stick of length 10 m slides on coordinate axes, then locus of a point dividing this stick reckoning from X-axis in the ratio 6 : 4 is a curve whose eccentricity is e , then $9e$ is equal to	p. $\sqrt{6}$
B. AA' is major axis of an ellipse $3x^2 + 2y^2 + 6x - 4y - 1 = 0$ and P is a variable point on it, then the greatest area of $\Delta APA'$ is	q. $2\sqrt{7}$
C. Distance between foci of the curve represented by the equation $x = 1 + 4\cos \theta, y = 2 + 3\sin \theta$ is	r. $\frac{128}{3}$
D. Tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{7} = 1$ at end points of latusrectum. The area of equilateral, so formed is	s. $3\sqrt{5}$

Sol. A. $x = 10 \cos \theta - 6 \cos \theta = 4 \cos \theta$

$$y = 6 \sin \theta$$

$$\therefore \text{Locus is } \left(\frac{x}{4}\right)^2 + \left(\frac{y}{6}\right)^2 = 1$$

$$\Rightarrow \frac{x^2}{16} + \frac{y^2}{36} = 1$$

$$\therefore e = \sqrt{\frac{36-16}{36}} = \sqrt{\frac{20}{36}} = \frac{\sqrt{5}}{3}$$

$$\therefore 9e = 3\sqrt{5}$$

B. $3(x^2 + 2x + 1) + 2(y^2 - 2y + 1) = 3 + 2 + 1$

$$\frac{(x+1)^2}{2} + \frac{(y-1)^2}{3} = 1$$

$$e = \sqrt{\frac{3-2}{3}} = \frac{1}{\sqrt{3}}$$

$$a = \sqrt{3}, b = \sqrt{2}$$

$$\therefore \text{Area} = \frac{1}{2} \cdot 2\sqrt{3} \cdot \sqrt{2} = \sqrt{6}$$

C. $\frac{(x-1)^2}{16} + \frac{(y-2)^2}{9} = 1$

$$\therefore a = 4e = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

$$\therefore ae = \sqrt{7}$$

Hence, distance between the foci is $2\sqrt{7}$.

D. $\frac{x^2}{16} + \frac{y^2}{7} = 1$

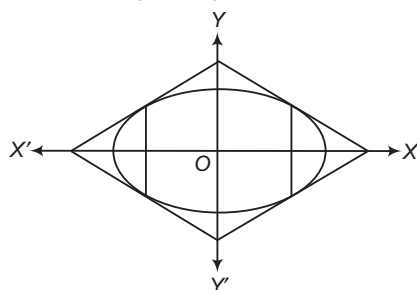
$$= \sqrt{\frac{9}{16}} = \frac{3}{4}$$

One end of latusrectum is

$$\left(ae, \frac{b^2}{a}\right) = \left(3, \frac{7}{4}\right)$$

$\therefore$ Equation of the tangent is

$$\frac{3x}{16} + \frac{7}{4} \cdot \frac{y}{7} = 1$$



$$\Rightarrow \frac{3x}{16} + \frac{y}{4} = 1$$

It meets X-axis at $\left(\frac{16}{3}, 0\right)$ and Y-axis at $(0, 4)$.

$$\therefore \text{Area of rhombus} = 2 \times \frac{16}{3} \times 4 = \frac{128}{3}$$

A $\rightarrow$ s; B $\rightarrow$ p; C $\rightarrow$ q; D $\rightarrow$ r

Ex 37. Match the statements of Column I with values of Column II.

Column I	Column II
A. If the mid-point of a chord of the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$ is $(0, 3)$, then length of the chord is $\frac{4k}{5}$, then k is	p. 6
B. If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then the sum of values of λ is	q. 8
C. If the distance between a focus and corresponding directrix of an ellipse be 8 and the eccentricity is $\frac{1}{2}$, then length of the minor axis is $\frac{k}{\sqrt{3}}$, where k is	r. 0
D. Sum of the distances of a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ from the foci, is	s. 16

Sol. A. Equation of the chord whose mid-point is $(0, 3)$, is

$$\frac{3y}{25} - 1 = \frac{9}{25} - 1 \quad \text{i.e. } y = 3$$

If it intersects the ellipse $\frac{x^2}{16} + \frac{y^2}{25} = 1$

$$\text{At } \frac{x^2}{16} = 1 - \frac{9}{25} = \frac{16}{25}$$

$$\Rightarrow x = \pm \frac{16}{5}$$

$$\therefore \text{Length of the chord} = \frac{32}{5}$$

$$\text{Thus, } \frac{4k}{5} = \frac{32}{5}$$

$$\therefore k = 8$$

B. If the line $y = x + \lambda$ touches the ellipse $9x^2 + 16y^2 = 144$, then

$$\lambda^2 = 16(1)^2 + 9$$

$$\Rightarrow \lambda = \pm 5$$

C. $\frac{a}{e} - ae = 8$

$$\Rightarrow 2a - \frac{a}{2} = 8$$

$$\Rightarrow a = \frac{16}{3}$$

$$\text{Now, } b^2 = a^2(1 - e^2) = \frac{256}{9} \left(1 - \frac{1}{4}\right) = \frac{64}{3}$$

$$\therefore \text{Length of minor axis} = 2b = \frac{16}{\sqrt{3}} = \frac{k}{\sqrt{3}}$$

$$\text{Hence, } k = 16$$

D. Sum of the distances of a point on the ellipse from the foci $2a = 8$

A $\rightarrow$ q; B $\rightarrow$ r; C $\rightarrow$ s; D $\rightarrow$ q

Type 6. Single Integer Answer Type Questions

Ex 38. A straight line PQ touches the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and the circle $x^2 + y^2 = r^2$ ($3 < r < 4$). RS is a focal chord of the ellipse which is parallel to PQ and meets the circle at points R and S . Then, the length of RS is _____.

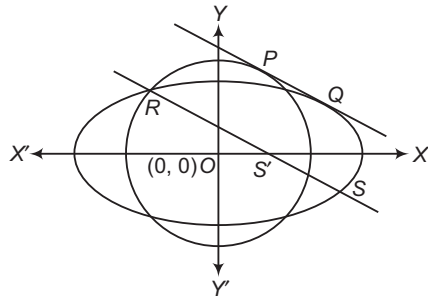
Sol. (6) $y = mx + \sqrt{a^2m^2 + b^2}$ is tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

This tangent also touches the circle $x^2 + y^2 = r^2$

$$\Rightarrow \pm r = \frac{0 - 0 + \sqrt{a^2m^2 + b^2}}{\sqrt{1 + m^2}}$$

$$\Rightarrow m = \frac{\sqrt{r^2 - b^2}}{\sqrt{a^2 - r^2}} \quad [\because b < r < a]$$



RS passes through $(ae, 0)$ and parallel to PQ .

Equation of RS is $y - 0 = m(x - ae)$

$$mx - y - ame = 0$$

Let T be foot of the perpendicular dropped from origin on RS .

Then, $RT^2 = OR^2 - OT^2$

$$\Rightarrow RT^2 = r^2 - \frac{a^2m^2e^2}{1 + m^2}$$

$$\Rightarrow RT = b$$

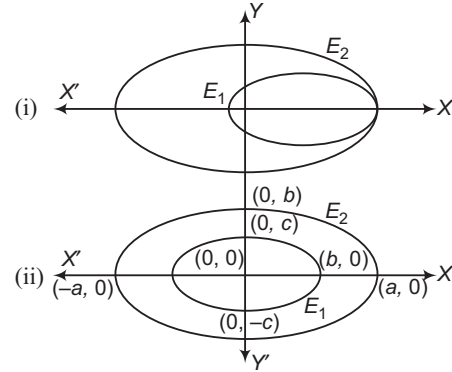
$$\therefore RS = 2b$$

$$\Rightarrow RS = 6 \quad [\because b = 3, \text{ given}]$$

Ex 39. Let E_1 and E_2 be two ellipse. The area of the ellipse E_2 is one-third the area of the quadrilateral formed by the tangents at the ends of the latusrectum of the ellipse E_1 ($E_1: 5x^2 + 9y^2 = 45$). The eccentricities of E_1 and E_2 are equal. E_1 is inscribed in E_2 in such a way that both E_1 and E_2 touch each other at one end of their common major axis. If the length of the major axis of E_1 is equal to the length of the minor axis of E_2 , then find the area of the ellipse E_2 outside the ellipse E_1 .

Sol. (4) $5x^2 + 9y^2 = 45$

$a = 3, b = \sqrt{5}, e = \frac{2}{3}$, one end of latusrectum in first quadrant = $(2, \frac{5}{3})$



Equation of tangent, $2x + 3y = 9$

It meets axes at $(\frac{9}{2}, 0)$ and $(0, 3)$.

Area of the quadrilateral

$$= 4 \cdot \frac{1}{2} \cdot \frac{9}{2} \cdot 3 = 27$$

Let the equation of the ellipse E_2 and E_1 be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{and} \quad \frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$$

Fig. (i) and (ii) give the same area which is required area of $E_2 = \pi ab$ and area of $E_1 = \pi bc$

Required area = $\pi b(a - c)$

$$\text{Now,} \quad b^2 = a^2(1 - e^2)$$

$$\text{and} \quad c^2 = a^2(1 - e^2)^2$$

$$\Rightarrow c = a(1 - e^2)$$

$$\Rightarrow a - c = ae^2$$

$$\therefore \text{Required area} = \pi abe^2 = 9 \times \frac{4}{9} = 4 \text{ sq units}$$

$$[\because \pi ab = \frac{1}{3} \times \text{area of quadrilateral}]$$

Ex 40. $P_1, P_2, \dots, P_i, \dots, P_n$ are the points on the

ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and $Q_1, Q_2, \dots, Q_i, \dots, Q_n$,

are the corresponding points on the auxiliary circle of the ellipse. If the line joining C and Q_i meets the normal at P_i w.r.t. the given ellipse at

K_i and $\sum_{i=1}^n (CK_i) = 175$, then find the value

of $\frac{n}{5}$.

Sol. (5) Equation of normal at P_i is $\frac{4x}{\cos\theta} - \frac{3y}{\sin\theta} = 7$, it passes through K_i

$$\begin{aligned} 4CK_i - 3CK_i &= 7 \\ CK_i &= 7 \\ \therefore \sum_{i=1}^n 7 &= 175 \\ \Rightarrow 7n &= 175 \\ \Rightarrow n &= 25 \\ \therefore \frac{n}{5} &= \frac{25}{5} = 5 \end{aligned}$$

Ex 41. If the normals at the four points (x_1, y_1) , (x_2, y_2) , (x_3, y_3) and (x_4, y_4) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent, then find the value of $(x_1 + x_2 + x_3 + x_4) \times \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right)$.

Sol. (4) Let points of concurrent be (h, k) .

Equation of normal at (x', y') is $\frac{x - x'}{\frac{x'}{a^2}} = \frac{y - y'}{\frac{y'}{b^2}}$

It passes through (h, k) , then

$$y'^2 \{a^2(h - x') + b^2x'\}^2 = b^4k^2x'^2 \quad \dots(i)$$

But $\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1$ or $y'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$ $\dots(ii)$

Value of y'^2 from Eq. (ii), putting in Eq. (i),

$$\begin{aligned} \text{we get } \frac{b^2}{a^2}(a^2 - x'^2) \{a^2h + (b^2 - a^2)x'\}^2 &= b^4k^2x'^2 \\ \Rightarrow b^4k^2x'^2 & \end{aligned}$$

Arranging above as a fourth degree equation in x' , we get

$$\begin{aligned} -(a^2 - b^2)^2x'^4 + 2ha^2(a^2 - b^2)x'^3 + x'^2(\dots) \\ - 2a^4h(a^2 - b^2)x' + a^6h^2 = 0 \end{aligned}$$

Above the equation being of fourth degree in x' , therefore roots of the above equation are x_1, x_2, x_3, x_4 , then

$$(x_1 + x_2 + x_3 + x_4) = -\frac{2ha^2(a^2 - b^2)}{-(a^2 - b^2)^2} = \frac{2ha^2}{(a^2 - b^2)} \quad \dots(iii)$$

$$\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right) = \frac{\Sigma x_1x_2x_3}{(x_1 \cdot x_2 \cdot x_3 \cdot x_4)}$$

$$\frac{2a^4h(a^2 - b^2)}{-(a^2 - b^2)^2} = \frac{2(a^2 - b^2)}{a^2h} \quad \dots(iv)$$

On multiplying Eqs. (iii) and (iv), we get

$$(x_1 + x_2 + x_3 + x_4) \times \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right) = 4$$

Ex 42. The equation of the common tangent, 1st quadrant to the circle $x^2 + y^2 = 16$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$, makes intercept between the coordinate axes. Then, find the value of k for which length of this intercept is $\frac{14}{\sqrt{k}}$.

Sol. (3) Let the common tangent to circle $x^2 + y^2 = 16$ and ellipse $\frac{x^2}{25} + \frac{y^2}{4} = 1$ be

$$y = mx + \sqrt{25m^2 + 4} \quad \dots(i)$$

As it is tangent circle $x^2 + y^2 = 16$ we should have

$$\frac{\sqrt{25m^2 + 4}}{\sqrt{m^2 + 1}} = 4 \quad \dots(ii)$$

$$\left[\because \text{length of perpendicular from } (0, 0) \text{ to Eq. (i)} = 4 \right]$$

$$\Rightarrow 25m^2 + 4 = 16m^2 + 16$$

$$\Rightarrow 9m^2 = 12$$

$$\Rightarrow m = -\frac{2}{\sqrt{3}}$$

[leaving +ve sign, to consider tangent in 1st quadrant]

$\therefore$ Equation of common tangent is

$$y = -\frac{2}{\sqrt{3}}x + \sqrt{25 \cdot \frac{4}{3} + 4} \Rightarrow y = -\frac{2}{\sqrt{3}}x + 4\sqrt{\frac{7}{3}}$$

This tangent meets the axes at $A(2\sqrt{7}, 0)$ and

$$B\left(0, 4\sqrt{\frac{7}{3}}\right).$$

$\therefore$ Length of intercepted portion of tangent between axes,

$$AB = \sqrt{(2\sqrt{7})^2 + \left(4\sqrt{\frac{7}{3}}\right)^2} = \frac{14}{\sqrt{3}} \Rightarrow k = 3$$

Target Exercises

Type 1. Only One Correct Option

- $\frac{x^2}{r^2 - r - 6} + \frac{y^2}{r^2 - 6r + 5} = 1$ will represent the ellipse, if r lies in the interval
(a) $(-\infty, 2)$ (b) $(3, \infty)$ (c) $(5, \infty)$ (d) $(1, \infty)$
- Length of latusrectum of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is
(a) $\frac{2a^2}{b}$, if $a < b$ (b) $\frac{2a^2}{b}$, if $a > b$
(c) $\frac{2b^2}{a}$, if $a < b$ (d) None of these
- The semi-latusrectum of an ellipse is
(a) the arithmetic mean of the segments of its focal chord
(b) the geometric mean of the segments of its focal chord
(c) the harmonic mean of the segments of its focal chord
(d) None of the above
- If the ellipse $\frac{x^2}{a^2 - 7} + \frac{y^2}{13 - 5a} = 1$ is inscribed in a square of side length $\sqrt{2}a$, then a is equal to
(a) $\frac{6}{5}$ (b) $(-\infty, -\sqrt{7}) \cup \left(\sqrt{7}, \frac{13}{5}\right)$
(c) $(-\infty, -\sqrt{7}) \cup \left(\frac{13}{5}, \sqrt{7}\right)$ (d) No such a exists
- A parabola is drawn with focus is at one of the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (where, $a > b$) and directrix passing through the other focus and perpendicular to the major axes of the ellipse. If latusrectum of the ellipse and the parabola are same, then the eccentricity of the ellipse is
(a) $1 - \frac{1}{\sqrt{2}}$
(b) $2\sqrt{2} - 2$
(c) $\sqrt{2} - 1$
(d) None of the above
- If P is any point lying on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, whose foci are S and S' . Let $\angle PSS' = \alpha$ and $\angle PS'S = \beta$, then which of the following is/are true?
(a) $PS + PS' = 2\alpha$, if $\alpha > \beta$
(b) $PS + PS' = 2\beta$, if $\alpha > \beta$
(c) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{1+e}{1-e}$
(d) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{\sqrt{a^2 - b^2}}{b^2} (a - \sqrt{a^2 - b^2})$
- If $(a \cos \alpha, b \sin \alpha)$, $(a \cos \beta, b \sin \beta)$ are the end points of a focal chord, then which of the following is correct?
(a) $e = \frac{\sin \alpha - \sin \beta}{\sin (\alpha - \beta)}$ (b) $e = \frac{\cos \left(\frac{\alpha + \beta}{2}\right)}{\cos \left(\frac{\alpha - \beta}{2}\right)}$
(c) $\frac{e-1}{e+1} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$ (d) None of these
- If the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is inscribed in a rectangle whose length to breadth ratio is $2 : 1$, then the area of the rectangle is
(a) $4 \left(\frac{a^2 + b^2}{7}\right)$ (b) $4 \left(\frac{a^2 + b^2}{3}\right)$
(c) $12 \left(\frac{a^2 + b^2}{5}\right)$ (d) $8 \left(\frac{a^2 + b^2}{5}\right)$
- A circle has the same centre as an ellipse and passes through the foci F_1 and F_2 of the ellipse, such that the two curves intersect in 4 points. Let P be any one of their points of intersection. If the major axis of the ellipse is 17 and the area of the $\triangle PF_1F_2$ is 30, then the distance between the foci is
(a) 13
(b) 10
(c) 11
(d) None of the above
- If α and β are the eccentric angles of the extremities of a focal chord of an ellipse, then the eccentricity of the ellipse is
(a) $\frac{\cos \alpha + \cos \beta}{\cos (\alpha - \beta)}$ (b) $\frac{\sin \alpha - \sin \beta}{\sin (\alpha - \beta)}$
(c) $\frac{\cos \alpha - \cos \beta}{\cos (\alpha - \beta)}$ (d) $\frac{\sin \alpha + \sin \beta}{\sin (\alpha + \beta)}$
- Three points A , B and C are taken on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with eccentric angles $\theta, \theta + \alpha$ and $\theta + 2\alpha$, then the
(a) area of $\triangle ABC$ is independent of θ
(b) area of $\triangle ABC$ is independent of α
(c) maximum value of area is $\frac{\sqrt{3}}{4} ab$
(d) maximum value of area is $\frac{3\sqrt{3}}{4} ab$

- 12.** The equation of the line passing through the centre and bisecting the chord $7x + y - 1 = 0$ of the ellipse $\frac{x^2}{1} + \frac{y^2}{7} = 1$ is
 (a) $x = y$ (b) $2x = y$
 (c) $x = 2y$ (d) $x + y = 0$
- 13.** Tangents are drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 (a > b)$ and the circle $x^2 + y^2 = a^2$ at the points, where a common ordinate cuts them (on the same side of the X -axis). Then, the greatest acute angle between these tangents is given by
 (a) $\tan^{-1} \left(\frac{a-b}{2\sqrt{ab}} \right)$ (b) $\tan^{-1} \left(\frac{a+b}{2\sqrt{ab}} \right)$
 (c) $\tan^{-1} \left(\frac{2ab}{\sqrt{a-b}} \right)$ (d) $\tan^{-1} \left(\frac{2ab}{\sqrt{a+b}} \right)$
- 14.** Let P_i and P'_i be the foot of the perpendiculars drawn from foci S, S' on a tangent T_i to an ellipse whose length of semi-major axis is 20, if $\sum_{i=1}^{10} (SP_i)(S'P'_i) = 2560$, then the value of eccentricity is
 (a) $\frac{1}{5}$ (b) $\frac{2}{5}$
 (c) $\frac{3}{5}$ (d) $\frac{4}{5}$
- 15.** Number of points on the ellipse $\frac{x^2}{50} + \frac{y^2}{20} = 1$ from which pair of perpendicular tangents are drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ is
 (a) 0 (b) 2 (c) 1 (d) 4
- 16.** Let (α, β) be a point from which two perpendicular tangents can be drawn to the ellipse $4x^2 + 5y^2 = 20$. If $F = 4\alpha + 3\beta$, then
 (a) $-15 \leq F \leq 15$ (b) $F \geq 0$
 (c) $-5 \leq F \leq 20$ (d) $F \leq -5\sqrt{5}$ or $F \geq 5\sqrt{5}$
- 17.** The point at shortest distance from the line $x + y = 7$ and lying on an ellipse $x^2 + 2y^2 = 6$, has coordinates
 (a) $(\sqrt{2}, \sqrt{2})$ (b) $(0, \sqrt{3})$
 (c) $(2, 1)$ (d) $\left(\sqrt{5}, \frac{1}{\sqrt{2}} \right)$
- 18.** If equation of the ellipse is $2x^2 + 3y^2 - 8x + 6y + 5 = 0$, then which of the following is true?
 (a) Equation of director circle is $x^2 + y^2 - 4x + 2y = 10$
 (b) Director circle will pass through $(4, -1)$
 (c) Equation of auxiliary circle is $x^2 + y^2 - 4x + 2y + 2 = 0$
 (d) None of the above
- 19.** Locus of all such points from where the tangents drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are always inclined at 45° , is
 (a) $(x^2 + y^2 - a^2 - b^2)^2 = 4(a^2x^2 + b^2y^2 - 1)$
 (b) $(x^2 + y^2 - a^2 - b^2)^2 = 4(b^2x^2 + a^2y^2 - 1)$
 (c) $(x^2 + y^2 - a^2 - b^2)^2 = (b^2x^2 + a^2y^2 - 1)$
 (d) None of the above
- 20.** If the straight line $y = 4x + c$ is a tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$, then c will be equal to
 (a) ± 4 (b) ± 6 (c) ± 8 (d) $\pm \sqrt{132}$
- 21.** From point $P(8, 27)$ tangents PQ and PR are drawn to the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Then, angle subtended by QR at origin is
 (a) $\tan^{-1} \left(\frac{2\sqrt{6}}{65} \right)$ (b) $\tan^{-1} \left(\frac{4\sqrt{6}}{65} \right)$
 (c) $\tan^{-1} \left(\frac{8\sqrt{2}}{65} \right)$ (d) None of these
- 22.** Equation of the tangent to the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ perpendicular to the line $2x + y + 7 = 0$, is
 (a) $x - 2y + 4 = 0$ (b) $x - 2y + 5 = 0$
 (c) $x - 2y - 2 = 0$ (d) $x - 2y - 5 = 0$
- 23.** Equation of the tangents to the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ which are parallel to the line $x + y + 1 = 0$, is
 (a) $x + y + 5 = 0$ (b) $x + y + 6 = 0$
 (c) $2x + y - 5 = 0$ (d) $x + y - 6 = 0$
- 24.** If from a point P tangents PQ and PR are drawn to the ellipse $\frac{x^2}{2} + y^2 = 1$, so that equation of QR is $x + 3y = 1$, then coordinates of P is
 (a) $(2, -3)$
 (b) $(2, 3)$
 (c) $(-2, 3)$
 (d) None of the above
- 25.** The angle between pair of tangents drawn to the ellipse $3x^2 + 2y^2 = 5$ from the points $(1, 2)$, is
 (a) $\tan^{-1} \left(\frac{12}{5} \right)$ (b) $\tan^{-1} \left(\frac{6}{\sqrt{5}} \right)$
 (c) $\tan^{-1} \left(\frac{12}{\sqrt{5}} \right)$ (d) $\tan^{-1} \left(\frac{\sqrt{12}}{5} \right)$
- 26.** If tangents drawn to the ellipse through point $(1, 2\sqrt{3})$ to the ellipse $x^2/9 + y^2/b^2 = 1$ are at right angle, then value of b is
 (a) 1 (b) 4
 (c) 2 (d) None of these

- 27.** Locus of the point of intersection of tangents at the end points of focal chord is
- (a) $x = \frac{a}{e}$, if $a > b$ (b) $x = -\frac{a}{e}$, if $a < b$
 (c) $y = \frac{b}{e}$, if $a > b$ (d) $y = -\frac{b}{e}$, if $a > b$
- 28.** Locus of the point of intersection of the tangents at the end points of the focal chord of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, ($b < a$) is
- (a) $x = \pm \frac{a^2}{\sqrt{a^2 - b^2}}$ (b) $y = \pm \frac{b^2}{\sqrt{a^2 - b^2}}$
 (c) $x = \pm \frac{ab}{\sqrt{a^2 - b^2}}$ (d) None of these
- 29.** If pair of tangents drawn to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ from a point P , so that angle between the tangents are at right angle, then possible coordinates of the point P is/are
- (a) (3, 4) (b) (5, 0)
 (c) $(2\sqrt{5}, \sqrt{5})$ (d) All of these
- 30.** The locus of the point of intersection of tangents to an ellipse at two points, sum of whose eccentric angles is constant, is a/an
- (a) parabola (b) circle
 (c) ellipse (d) straight line
- 31.** If polar of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is always touching the parabola $y^2 = 4ax$, then locus of the pole is
- (a) $a^3y^2 = b^4x$ (b) $b^3y = a^4y$
 (c) $a^4y = -b^4x$ (d) None of these
- 32.** If polar of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is always touching the curve $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, then locus of pole is
- (a) parabola (b) ellipse (c) hyperbola (d) circle
- 33.** If polar of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is always touching the circle $x^2 + y^2 = c^2$, then locus of pole is
- (a) $c^2(a^4y^2 + b^4x^2) = a^4b^4$
 (b) $c^2(a^4x^2 + b^4y^2) = a^4b^4$
 (c) $c^2(a^4y^2 + b^4x^2 - a^3b^3) = a^4b^4$
 (d) None of the above
- 34.** If polar of $y^2 = 4ax$ is always touching the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then locus of the pole is
- (a) $4a^2x^2 - b^2y^2 = 4a^4$ (b) $a^2x^2 - b^2y^2 = a^4$
 (c) $4a^2x^2 + b^2y^2 = 4a^4$ (d) None of these
- 35.** A line intersects the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at P and Q and the parabola $y^2 = 4d(x + a)$ at R and S . The line segment PQ subtend a right angle at the centre of the ellipse. Then, the locus of the point of intersection of tangents to the parabola at R and S is
- (a) $y^2 - d^2 = d^2(x - 2a)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$
 (b) $y^2 - 2d^2 = 2d^2(x - 2a)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$
 (c) $y^2 + d^2 = d^2(x + 2a)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$
 (d) $y^2 + 4d^2 = 4d^2(x + 2a)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$
- 36.** The equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$ is
- (a) $(x - 1)^2 + (y - 0)^2 = \frac{11}{3}$
 (b) $(x - 1)^2 - (y - 0)^2 = \frac{11}{3}$
 (c) $(x + 1)^2 - (y + 0)^2 = \frac{11}{3}$
 (d) $(x + 1)^2 + (y + 0)^2 = \frac{11}{3}$
- 37.** Locus of the mid-points of the chord of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, so that chord is always touching the circle $x^2 + y^2 = c^2$ ($c < a, c < b$) is
- (a) $(b^2x^2 + a^2y^2)^2 = c^2(b^4x^2 + a^4y^2)$
 (b) $(a^2x^2 + b^2y^2)^2 = c^2(a^4x^2 + b^4y^2)$
 (c) $(b^2x^2 + a^2y^2)^2 = c^2(b^2x^4 + a^2y^4)$
 (d) None of the above
- 38.** If normal to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $\left(ae, \frac{b^2}{a} \right)$ is passing through $(0, -2b)$, then value of eccentricity is
- (a) $\sqrt{2} - 1$ (b) $2(\sqrt{2} - 1)$
 (c) $\sqrt{2(\sqrt{2} - 1)}$ (d) None of these
- 39.** The area of the parallelogram formed by the tangents at the ends of conjugate diameters of an ellipse is
- (a) constant and is equal to the product of the axes
 (b) cannot be constant
 (c) constant and is equal to the two lines
 (d) None of the above
- 40.** An ellipse of semi-axes a, b slides between two perpendicular lines, then the locus of its foci is (the two lines being taken as the axes of coordinates)
- (a) $(x^2 + y^2)(x^2y^2 + b^6) = 4a^2x^2y^4$
 (b) $(x^3 + y^3)(x^3y^3 + b^6) = 4a^3x^3y^6$
 (c) $(x^2 + y^2)(x^2y^2 + b^4) = 4a^2x^2y^2$
 (d) $(x^2 - y^2)(x^2y^2 - b^4) = 4a^2x^2y^2$

- 41.** If ω is one of the angles between the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the points whose eccentric angles are θ and $\frac{\pi}{2} + \theta$, then $\frac{2\cos \omega}{\sin 2\theta}$ is
 (a) $\frac{e^2}{\sqrt{1-e^2}}$ (b) $\frac{e^2}{\sqrt{1+e^2}}$ (c) $\frac{e^2}{1-e^2}$ (d) $\frac{e^2}{1+e^2}$
- 42.** If the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ to the ellipse are concurrent, then
 (a) $\begin{vmatrix} x_2 & y_2 & x_2 y_2 \\ x_1 & y_1 & x_1 y_1 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = -1$ (b) $\begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$
 (c) $\begin{vmatrix} x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \\ x_1 & y_1 & x_1 y_1 \end{vmatrix} = 1$ (d) None of these
- 43.** If the tangent drawn at point $(t^2, 2t)$ on the parabola $y^2 = 4x$ is same as the normal drawn at point $(\sqrt{5}\cos\theta, 2\sin\theta)$ on the ellipse $4x^2 + 5y^2 = 20$. Then, the values of t and θ are
 (a) $\theta = \cos^{-1}\left(-\frac{1}{\sqrt{5}}\right)$ and $t = -\frac{1}{\sqrt{5}}$
 (b) $\theta = \cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$ and $t = \frac{1}{\sqrt{5}}$
 (c) $\theta = \cos^{-1}\left(-\frac{2}{\sqrt{5}}\right)$ and $t = -\frac{2}{\sqrt{5}}$
 (d) None of the above
- 44.** The normals at four points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet in the point (h, k) . Then, the mean position of the four points is
 (a) $\left(\frac{a^2 h}{2(a^2 + b^2)}, \frac{b^2 k}{2(a^2 + b^2)}\right)$
 (b) $\left(\frac{a^3 h}{2(a^2 + b^2)}, \frac{b^3 k}{2(a^2 + b^2)}\right)$
 (c) $\left(\frac{ah}{2(a^2 - b^2)}, \frac{bk}{2(a^2 - b^2)}\right)$
 (d) $\left(\frac{a^2 h}{2(a^2 - b^2)}, \frac{b^2 k}{2(a^2 - b^2)}\right)$
- 45.** For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, four normals cannot be drawn through a point $(h, 0)$ unless (e being the eccentricity of the ellipse)
 (a) $|h| < 2ae^2$ (b) $|h| > 2ae^2$ (c) $|h| < ae^2$ (d) $|h| > ae^2$
- 46.** If the normals to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at the ends of the chords $l_1 x + m_1 y = 1$ and $l_2 x + m_2 y = 1$ are concurrent, then
 (a) $a^2 l_1 l_2 + b^2 m_1 m_2 = 1$ (b) $a^2 l_1 l_2 = b^2 m_1 m_2 = -1$
 (c) $a^2 l_1 l_2 - b^2 m_1 m_2 = -1$ (d) None of these
- 47.** The locus of the middle points of chords of an ellipse which pass through a fixed point is
 (a) an ellipse (b) a circle
 (c) a hyperbola (d) a parabola
- 48.** The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by the line $x + y = b$, is
 (a) $a^2 + 6ab \leq 7b^2$ (b) $a^2 + 6ab \geq 7b^2$
 (c) $a^2 + ab \leq 7b^2$ (d) $a^2 + ab \geq 7b^2$
- 49.** The perpendicular from the centre of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to polar of a point with respect to the ellipse is constant equal to c . Then, the locus of the point is the ellipse
 (a) $\frac{x^2}{b^4} + \frac{y^2}{a^4} = \frac{1}{c^2}$ (b) $\frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{c^2}$
 (c) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$ (d) None of these
- 50.** If the product of the perpendiculars from the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ upon the polar of P is always c^2 , then locus of P is the ellipse
 (a) $b^4(c^2 + a^2 - b^2)x^2 + c^2 a^4 y^2 = a^4 b^4$
 (b) $b^2(c^2 + a^2 + b^2)x^2 + c^2 a^4 y^2 = a^4 b^4$
 (c) $b^3(c^3 + a^3 - b^2)x^2 + c^2 a^4 y^2 = a^2 b^2$
 (d) None of the above
- 51.** The locus of the poles of the chords of the ellipse subtending a right angle at its centre is
 (a) $\frac{x^4}{a^2} + \frac{y^4}{b^2} = \frac{1}{a^4} + \frac{1}{b^4}$ (b) $\frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$
 (c) $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$ (d) None of these
- 52.** The locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose poles are on the auxiliary circle on the tangents at the extremities of which intersect on the auxiliary circle, is
 (a) $x^2 + y^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$ (b) $x^2 - y^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$
 (c) $x^2 + y^2 = \frac{1}{a^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$ (d) $x^2 + y^2 = a^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2$
- 53.** If CP and CD are conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, the locus of the mid-point of PD is
 (a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$ (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$
 (c) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{2}$ (d) None of these

54. The locus of poles of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to concentric ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$, is

(a) $\frac{a^2x^2}{\alpha^4} + \frac{b^2y^2}{\beta^4} = 1$ (b) $\frac{a^2x^2}{\alpha^4} - \frac{b^2y^2}{\beta^4} = 1$
(c) $\frac{a^2x^2}{\alpha^4} + \frac{b^2y^2}{\beta^4} = -1$ (d) $\frac{a^2x^2}{\alpha^4} - \frac{b^2y^2}{\beta^4} = -1$

55. The locus of pole of tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to the parabola $y^2 = 4ax$, is

(a) $b^2y^2 = 4a^2(x^2 + a^2)$ (b) $b^2y^2 = a^2(x^2 - a^2)$
(c) $2b^2y^2 = a^2(x^2 - a^2)$ (d) $b^2y^2 = 4a^2(x^2 - a^2)$

56. The locus of the poles of normal chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is

(a) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 + b^2)^2$ (b) $\frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 - b^2)^2$
(c) $\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$ (d) $\frac{a^6}{x^2} - \frac{b^6}{y^2} = (a^2 + b^2)^2$

57. If CP and CD are conjugate semi-diameters of the ellipse, then PD touches the ellipse

(a) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{1}{2}$ (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = -\frac{1}{2}$
(c) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$ (d) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

58. If α and β are the angles subtended by the major axis of an ellipse at the extremities of a pair of conjugate diameters, then

(a) $\cot^2 \alpha + \cot^2 \beta = \text{Constant}$
(b) $\cot^2 \alpha + \cot^2 \beta = \text{Variable}$
(c) $\cot^2 \alpha - \cot^2 \beta = \text{Constant}$
(d) None of the above

59. If PCP' and DCD' form a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and R is any point on the circle $x^2 + y^2 = c^2$, then

(a) $PR^2 - DR^2 - P'R^2 - D'R^2 = 2(a^2 - b^2 - 2c^2)$
(b) $PR^2 + DR^2 + P'R^2 + D'R^2 = 2(a^2 - b^2 - 2c^2)$
(c) $PR^2 + DR^2 + P'R^2 + D'R^2 = (a^2 + b^2 + 2c^2)$
(d) $PR^2 + DR^2 + P'R^2 + D'R^2 = 2(a^2 + b^2 + 2c^2)$

60. PCP' and DCD' are conjugate diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and θ is the eccentric angle of P . Then, eccentric angle of the point where the circle $PP'D$ again cuts the ellipse

(a) $\frac{\pi}{2} + \theta$
(b) $\frac{\pi}{2} + 3\theta$
(c) $\frac{\pi}{2} - \theta$
(d) $\frac{\pi}{2} - 3\theta$

Type 2. More than One Correct Option

61. The equation of a directrix of the ellipse $\left(\frac{x^2}{16}\right) + \left(\frac{y^2}{25}\right) = 1$, is

(a) $y = \frac{25}{3}$
(b) $x = 3$
(c) $x = -3$
(d) $y = -\frac{25}{3}$

62. The equation $3x^2 + 4y^2 - 18x + 16y + 43 = c$

(a) cannot represent a real pair of straight lines for any value of c
(b) represents an ellipse, if $c > 0$
(c) no locus, if $c < 0$
(d) a point, if $c = 0$

63. C is the centre of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. The tangents at any point P on this hyperbola meets the

straight lines $bx - ay = 0$ and $bx + ay = 0$ in the points Q and R respectively, then

(a) $CQ \cdot CR = a^2 + b^2$ (b) $CQ \cdot CR = a^2 - b^2$
(c) $CQ \cdot CR = a^2b^2$ (d) $CQ \cdot CR = a^2/b^2$

64. If the tangent at the point $P(\theta)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x = 15$, then θ is equal to

(a) $\frac{2\pi}{3}$ (b) $\frac{4\pi}{3}$
(c) $\frac{5\pi}{3}$ (d) $\frac{\pi}{3}$

65. Let F_1 and F_2 be two foci of the ellipse and PT and PN be the tangent and the normal respectively to the ellipse at point P . Then,

(a) PN bisects $\angle F_1PF_2$
(b) PT bisects $\angle F_1PF_2$
(c) PT bisects angle $(180^\circ - \angle F_1PF_2)$
(d) None of the above

Type 3. Assertion and Reason

Directions (Q. Nos. 66-71) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

66. Statement I The major and minor axes of the ellipse $5x^2 + 9y^2 - 54y + 36 = 0$ are 6 and 10, respectively.

Statement II The equation

$$5x^2 + 9y^2 - 54y + 36 = 0 \text{ can be expressed as } 5x^2 + 9(y-3)^2 = 45.$$

67. Statement I The condition on a and b for which two distinct chords of the ellipse $\frac{x^2}{2a^2} + \frac{y^2}{2b^2} = 1$ passing through $(a, -b)$ are bisected by the line $x + y = b$ is $a^2 + 6ab - 7b^2 \geq 0$

Statement II Equation of chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose mid-point is (x_1, y_1) is $T = S_1$.

68. Statement I Circle $x^2 + y^2 = 9$ and the circle $(x - \sqrt{5})(\sqrt{2}x - 3) + y(\sqrt{2}y - 2) = 0$ touch each other internally.

Statement II Circle described on the focal distance as diameter of the ellipse $4x^2 + 9y^2 = 36$ touch the auxiliary circle $x^2 + y^2 = 9$ internally.

69. Statement I If the tangents from the point $(\lambda, 3)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ are at right angles, then λ is equal to ± 2 .

Statement II The locus of the point of the intersection of two perpendicular tangents to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, is $x^2 + y^2 = a^2 + b^2$.

70. Statement I The latusrectum of an ellipse is shortest focal chord.

Statement II The semi-latusrectum of an ellipse is the harmonic mean of the segment of a focal chord.

71. Statement I The circle $x^2 + y^2 = 4$ is auxiliary circle of ellipse $\frac{x^2}{4} + \frac{y^2}{b^2} = 1, (b < 2)$.

Statement II A given circle is auxiliary circle of exactly one ellipse.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 72-74) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is such that it has the least area but contains the circle $(x-1)^2 + y^2 = 1$.

72. The eccentricity of the ellipse is

- (a) $\sqrt{\frac{2}{3}}$ (b) $\frac{1}{\sqrt{3}}$
(c) $\frac{1}{2}$ (d) None of these

73. Equation of auxiliary circle of ellipse is

- (a) $x^2 + y^2 = 65$ (b) $x^2 + y^2 = 5$
(c) $x^2 + y^2 = 45$ (d) None of these

74. Length of latusrectum of the ellipse is

- (a) 2 units (b) 1 unit
(c) 3 units (d) 2.5 units

Passage II (Q. Nos. 75-77) On a level plain, the crack of the rifle and the thud of the ball striking the target are heard at the same instant. If $V_1 = 330$ m/s and $V_2 = 990$ m/s are the velocities of sound and bullet respectively and P is the position of hearer, A is the target and B is the firing point.

75. The locus of the hearer is

- (a) parabola (b) ellipse
(c) hyperbola (d) None of these

76. If the distance between AB is 600 m and at some instant the hearer is 200 m from the target A , then how far is he from B ?

- (a) 200 m (b) 300 m
(c) 400 m (d) 500 m

77. If Y -axis is perpendicular bisector of AB at origin and $100 \text{ m} = 1$ unit in cartesian plane, then equation of the locus of hearer is

- (a) $8x^2 - y^2 = 8$ (b) $x^2 - 8y^2 = 8$
(c) $\frac{x^2}{4} + \frac{y^2}{3} = 1$ (d) None of these

Passage III (Q. Nos. 78-80) Consider the standard equation of an ellipse whose focus and corresponding feet of directrix are $(\sqrt{7}, 0)$ and $(\frac{16}{\sqrt{17}}, 0)$ and a circle with equation $x^2 + y^2 = r^2$. If in the first quadrant, the common tangent to a circle of this family and the above ellipse meets the coordinate axes at A and B .

78. The equation of the ellipse is
 (a) $16x^2 + 9y^2 = 144$ (b) $9x^2 + 16y^2 = 144$
 (c) $16x^2 + y^2 = 144$ (d) $x^2 + 9y^2 = 144$
79. Let P be a variable point on the ellipse with foci at S and S' . If Δ is the area of $\Delta PSS'$, then the maximum value of Δ is

- (a) $\sqrt{7}$ sq units (b) $2\sqrt{7}$ sq units
 (c) $3\sqrt{7}$ sq units (d) $4\sqrt{7}$ sq units

80. If mid-point of A and B is (x_1, y_1) and slope of common tangent is m , then
 (a) $2mx_1 + y_1 = 0$ (b) $2my_1 + x_1 = 0$
 (c) $my_1 + x_1 = 0$ (d) $mx_1 + y_1 = 0$

Type 5. Match the Columns

81. Match the statements of Column I with values of Column II.

Column I	Column II
A. An ellipse passing through the origin has its foci $(3, 4)$ and $(6, 8)$, then length of its minor axis is	p. 8
B. If PQ is focal chord of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ which passes through $S \equiv (3, 0)$ and $PS = 2$, then length of chord PQ is	q. $10\sqrt{2}$
C. If the line $y = x + k$ touches the ellipse $9x^2 + 16y^2 = 144$, then the difference of values of k is	r. 10
D. Sum of distances of a point on the ellipse $\frac{x^2}{9} + \frac{y^2}{16} = 1$ from the foci	s. 12

82. Match the statements of Column I with values of Column II.

Column I	Column II
A. If P is a point on the ellipse $\frac{x^2}{16} + \frac{y^2}{20} = 1$ whose foci are S and S' , then $PS + PS'$ is	p. $\frac{4}{5}$
B. The eccentricity of the ellipse $2x^2 + 3y^2 - 4x - 12y + 13 = 0$ is	q. $4\sqrt{5}$

Column I	Column II
C. Tangents are drawn from the points on the line $x - y - 5 = 0$ to $x^2 + 4y^2 = 4$. Then, all the chords of contact pass through a fixed point, whose abscissa is	r. 8
D. The sum of the distances of any point on the ellipse $3x^2 + 4y^2 = 12$ from its directrix is	s. $\frac{1}{\sqrt{3}}$

83. Match the statements of Column I with values of Column II.

Column I	Column II
A. A point on the curve $x^2 + 3y^2 = 9$ where the tangent is parallel to the line $y - x = 0$, is	p. $(-1, 2)$
B. A focus of the curve $25(x + 1)^2 + 9(y + 2)^2 = 225$ is at	q. $\left(\frac{9}{5}, -\frac{8}{5}\right)$
C. A point on the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, where the normal is parallel to the line $2x + y = 1$, is	r. $\left(-\frac{3\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\right)$
D. Tangents are drawn from points on the line $x - y + 2 = 0$ to the ellipse $x^2 + 2y^2 = 2$, then all the chords of contact pass through the point	s. $\left(-1, \frac{1}{2}\right)$
	t. $(-1, -6)$

Type 6. Single Integer Answer Type Questions

84. If $x, y \in R$, satisfying the equation $\frac{(x-4)^2}{4} + \frac{y^2}{9} = 1$, then the difference between the largest and smallest value of the expression $\frac{x^2}{4} + \frac{y^2}{9}$ is _____.
85. The value of a for the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ($a > b$), if the extremities of the latusrectum of the ellipse having positive ordinate lies on the parabola $x^2 = -2(y - 2)$, is _____.

86. An ellipse has its centre at $(1, -1)$ and semi-major axis = 8 and which passes through the point $(1, 3)$. If l is the length of its latusrectum, then find $l/4$.
87. If $(5, 12)$ and $(24, 7)$ are the foci of the conic passing through the origin, then the eccentricity of conic is $\sqrt{386k/24}$, then find the value of k .
88. If sum of the squares of the perpendicular on any tangent to the ellipse $\frac{x^2}{(67)^2} + \frac{y^2}{(33)^2} = 1$ from the points on the minor axis, each at a distance $10\sqrt{34}$ from the centre is $4489l$, then find the value of l .

Entrances Gallery

JEE Advanced/IIT JEE

1. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it, is [2014]

- (a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
 (b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
 (c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
 (d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

2. A vertical line passing through the point $(h, 0)$

intersects the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ at the points P

and Q . Let the tangents to the ellipse at P and Q meet

at the point R . If $\Delta(h) = \text{area of the } \Delta PQR$,
 $\Delta_1 = \max_{1/2 \leq h \leq 1} \Delta(h)$ and $\Delta_2 = \min_{1/2 \leq h \leq 1} \Delta(h)$, then

$\frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2 = \underline{\hspace{2cm}}$. [2013]

3. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having centre at $(0, 3)$ is [2013]

- (a) $x^2 + y^2 - 6y - 7 = 0$ (b) $x^2 + y^2 - 6y + 7 = 0$
 (c) $x^2 + y^2 - 6y - 5 = 0$ (d) $x^2 + y^2 - 6y + 5 = 0$

4. The ellipse $E_1 : \frac{x^2}{9} + \frac{y^2}{4} = 1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point

$(0, 4)$ circumscribes the rectangle R . The eccentricity of the ellipse E_2 is [2012]

- (a) $\frac{\sqrt{2}}{2}$ (b) $\frac{\sqrt{3}}{2}$ (c) $\frac{1}{2}$ (d) $\frac{3}{4}$

Passage (Q. Nos. 5-7) Tangents are drawn from the point $P(3, 4)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, touching the ellipse at point A and B . [2010]

5. The coordinates of A and B are

- (a) $(3, 0)$ and $(0, 2)$
 (b) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$
 (c) $\left(-\frac{8}{5}, \frac{2\sqrt{161}}{15}\right)$ and $(0, 2)$
 (d) $(3, 0)$ and $\left(-\frac{9}{5}, \frac{8}{5}\right)$

6. The orthocentre of the ΔPAB is

- (a) $\left(5, \frac{8}{7}\right)$ (b) $\left(\frac{7}{5}, \frac{25}{8}\right)$ (c) $\left(\frac{11}{5}, \frac{8}{5}\right)$ (d) $\left(\frac{8}{25}, \frac{7}{5}\right)$

7. The equation of the locus of the point whose distances from the point P and the line AB are equal, is

- (a) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$
 (b) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$
 (c) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$
 (d) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

AIEEE

8. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as semi-major axis. If the centre of the ellipse is at the origin and its axis are the coordinate axes, then the equation of the ellipse is [2012]

- (a) $4x^2 + y^2 = 4$ (b) $x^2 + 4y^2 = 8$
 (c) $4x^2 + y^2 = 8$ (d) $x^2 + 4y^2 = 16$

9. **Statement I** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement II If the line $y = mx + \frac{4\sqrt{3}}{m}$, ($m \neq 0$) is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$.

[2012]

- (a) Statement I is incorrect, Statement II is correct
 (b) Statement I is correct, Statement II is correct; Statement II is a correct explanation for Statement I
 (c) Statement I is correct, Statement II is correct; Statement II is not a correct explanation for Statement I
 (d) Statement I is correct, Statement II is incorrect

10. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{2/5}$, is [2011]

- (a) $5x^2 + 3y^2 - 48 = 0$ (b) $3x^2 + 5y^2 - 15 = 0$
 (c) $5x^2 + 3y^2 - 32 = 0$ (d) $3x^2 + 5y^2 - 32 = 0$

11. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which is turned in inscribed in another ellipse that passes through the point $(4, 0)$. Then, the equation of the ellipse is [2009]

- (a) $x^2 + 12y^2 = 16$ (b) $4x^2 + 48y^2 = 48$
(c) $4x^2 + 64y^2 = 48$ (d) $x^2 + 16y^2 = 16$

12. A focus of an ellipse is at the origin. The directrix is the line $x = 4$ and the eccentricity is $1/2$, then the length of semi-major axis is [2008]

- (a) $\frac{5}{3}$ (b) $\frac{8}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

13. In an ellipse, the distances between its foci is 6 and minor axis is 8. Then, its eccentricity is [2006]

- (a) $\frac{1}{2}$ (b) $\frac{4}{5}$
(c) $\frac{1}{\sqrt{5}}$ (d) $\frac{3}{5}$

14. Area of the greatest rectangle that can be inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is [2005]

- (a) $\frac{a}{b}$ sq unit (b) $\sqrt{ab}$ sq unit
(c) ab sq unit (d) $2ab$ sq unit

15. An ellipse has OB as semi-minor axis, F and F' its foci and the $\angle FBF'$ is a right angle. Then, the eccentricity of the ellipse is [2005]

- (a) $\frac{1}{\sqrt{3}}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$

16. The eccentricity of an ellipse with its centre at the origin, is $\frac{1}{2}$. If one of the directrices is $x = 4$, then the equation of the ellipse is [2004]

- (a) $3x^2 + 4y^2 = 1$
(b) $3x^2 + 4y^2 = 12$
(c) $4x^2 + 3y^2 = 12$
(d) $4x^2 + 3y^2 = 1$

17. The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and having its centre at $(0, 3)$, is [2002]

- (a) 4 units (b) 3 units
(c) $\sqrt{12}$ units (d) $\frac{7}{2}$ units

18. The equation of the ellipse whose foci are $(\pm 2, 0)$ and eccentricity is $1/2$, is [2002]

- (a) $\frac{x^2}{12} + \frac{y^2}{16} = 1$ (b) $\frac{x^2}{16} + \frac{y^2}{12} = 1$
(c) $\frac{x^2}{16} + \frac{y^2}{8} = 1$ (d) None of these

Other Engineering Entrances

19. The parametric form of the ellipse $4(x+1)^2 + (y-1)^2 = 4$ is [Kerala CEE 2014]

- (a) $x = \cos \theta - 1, y = 2 \sin \theta - 1$
(b) $x = 2 \cos \theta - 1, y = \sin \theta + 1$
(c) $x = \cos \theta - 1, y = 2 \sin \theta + 1$
(d) $x = \cos \theta + 1, y = 2 \sin \theta + 1$
(e) $x = \cos \theta + 1, y = 2 \sin \theta - 1$

20. A point P on an ellipse is at a distance 6 units from a focus. If the eccentricity of the ellipse is $3/5$, then the distance of P from the corresponding directrix is [Kerala CEE 2014]

- (a) $\frac{8}{5}$ (b) $\frac{5}{8}$
(c) 10 (d) 12
(e) None of these

21. An ellipse passing through $(4\sqrt{2}, 2\sqrt{6})$ has foci at $(-4, 0)$ and $(4, 0)$. Then, its eccentricity is [EAMCET 2014]

- (a) $\sqrt{2}$ (b) $\frac{1}{2}$
(c) $\frac{1}{\sqrt{2}}$ (d) $\frac{1}{\sqrt{3}}$

22. On the ellipse $9x^2 + 25y^2 = 225$, find out the point of the distance from which the focus F_1 is four times the distance to the other focus F_2 . [GGSIPU 2014]

- (a) $(-15, \sqrt{63})$ (b) $\left(\frac{-15}{4}, \frac{\sqrt{63}}{2}\right)$
(c) $\left(\frac{-15}{4}, \frac{\sqrt{63}}{4}\right)$ (d) $\left(\frac{-15}{2}, \frac{\sqrt{63}}{2}\right)$

23. Let the equation of an ellipse be $\frac{x^2}{144} + \frac{y^2}{25} = 1$. Then, the radius of the circle with centre $(0, \sqrt{2})$ and passing through the foci of the ellipse is [WB JEE 2014]

- (a) 9 (b) 7 (c) 11 (d) 5

24. The locus of the points of intersection of the tangents at the extremities of the chords of the ellipse $x^2 + 2y^2 = 6$ which touches the ellipse $x^2 + 4y^2 = 4$, is [BITSAT 2014]

- (a) $x^2 + y^2 = 4$ (b) $x^2 + y^2 = 6$
(c) $x^2 + y^2 = 9$ (d) None of these

25. The minimum area of triangle formed by any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the coordinate axes, is [VITEEE 2014]

- (a) $a^2 + b^2$ (b) $\frac{(a+b)^2}{2}$
(c) ab (d) $\frac{(a-b)^2}{2}$

- 26.** If tangents are drawn from any point on the circle $x^2 + y^2 = 25$ to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, then the angle between the tangents is [EAMCET 2014]

(a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{4}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

- 27.** If the length of the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is three times the length of minor axis, then its eccentricity is [BITSAT 2012]

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{\frac{2}{3}}$ (d) $\frac{2\sqrt{2}}{3}$

- 28.** S and T are the foci of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and B is end of the minor axis. If ΔSTB is an equilateral triangle, then eccentricity of the ellipse is [BITSAT 2012]

(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\sqrt{\frac{3}{2}}$

- 29.** The line $x = 2y$ intersects the ellipse $\frac{x^2}{4} + y^2 = 1$ at the points P and Q . The equation of the circle with PQ as diameter is [WB JEE 2012]

(a) $x^2 + y^2 = \frac{1}{2}$ (b) $x^2 + y^2 = 1$
(c) $x^2 + y^2 = 2$ (d) $x^2 + y^2 = \frac{5}{2}$

- 30.** If the foci of the ellipse $\frac{x^2}{9} + y^2 = 1$ subtend a right angle at a point P . Then, the locus of P is [WB JEE 2012]

(a) $x^2 + y^2 = 1$ (b) $x^2 + y^2 = 2$
(c) $x^2 + y^2 = 4$ (d) $x^2 + y^2 = 8$

- 31.** If the centre, one of the foci and semi-major axis of an ellipse are $(0, 0)$, $(0, 3)$ and 5 , then its equation is [BITSAT 2011]

(a) $\frac{x^2}{16} + \frac{y^2}{25} = 1$
(c) $\frac{x^2}{9} + \frac{y^2}{25} = 1$

(b) $\frac{x^2}{25} + \frac{y^2}{16} = 1$
(d) None of these

- 32.** The length of the latusrectum of the ellipse $16x^2 + 25y^2 = 400$ is [WB JEE 2011]

(a) $\frac{5}{16}$ unit (b) $\frac{32}{5}$ units
(c) $\frac{16}{5}$ units (d) $\frac{5}{32}$ unit

- 33.** If a point $P(x, y)$ moves along the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and C is the centre of the ellipse, then the sum of maximum and minimum values of CP is [Kerala CEE 2011]

(a) 25 (b) 9 (c) 4
(d) 5 (e) 16

- 34.** If the length of the major axis of an ellipse is $\frac{17}{8}$ times the length of the minor axis, then the eccentricity of the ellipse is [Kerala CEE 2011]

(a) $\frac{8}{17}$ (b) $\frac{15}{17}$
(c) $\frac{9}{17}$ (d) $\frac{2\sqrt{2}}{17}$
(e) $\frac{13}{17}$

- 35.** Area of a triangle formed by tangent and normal to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at $P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ with the X -axis is [Karnataka CET 2011]

(a) $4ab$
(b) $\frac{ab\sqrt{a^2 + b^2}}{4}$
(c) $\frac{ab\sqrt{a^2 - b^2}}{4}$
(d) $\frac{b(a^2 + b^2)}{4a}$

Answers

Work Book Exercise 15.1

1. (b) 2. (a) 3. (b) 4. (c) 5. (a)

Work Book Exercise 15.2

1. (c) 2. (a) 3. (b) 4. (b) 5. (b) 6. (a) 7. (a)

Work Book Exercise 15.3

1. (a) 2. (a) 3. (c) 4. (c) 5. (c) 6. (a)

Target Exercises

1. (c)	2. (a)	3. (c)	4. (d)	5. (c)	6. (d)	7. (c)	8. (d)	9. (a)	10. (d)
11. (d)	12. (a)	13. (a)	14. (c)	15. (d)	16. (a)	17. (c)	18. (c)	19. (d)	20. (d)
21. (d)	22. (c)	23. (a)	24. (b)	25. (c)	26. (c)	27. (a)	28. (a)	29. (d)	30. (d)
31. (d)	32. (b)	33. (a)	34. (a)	35. (d)	36. (a)	37. (a)	38. (c)	39. (a)	40. (c)
41. (a)	42. (b)	43. (a)	44. (d)	45. (c)	46. (b)	47. (a)	48. (b)	49. (c)	50. (a)
51. (c)	52. (d)	53. (b)	54. (a)	55. (d)	56. (d)	57. (c)	58. (a)	59. (d)	60. (d)
61. (a,d)	62. (all)	63. (a,c)	64. (c,d)	65. (a,c)	66. (d)	67. (a)	68. (a)	69. (a)	70. (a)
71. (c)	72. (a)	73. (c)	74. (b)	75. (c)	76. (c)	77. (a)	78. (b)	79. (c)	80. (d)
81. (*)	82. (**)	83. (***)	84. (8)	85. (2)	86. (1)	87. (2)	88. (2)		

* $A \rightarrow q$; $B \rightarrow r$; $C \rightarrow r$; $D \rightarrow p$

** $A \rightarrow q$; $B \rightarrow s$; $C \rightarrow p$; $D \rightarrow r$

*** $A \rightarrow r$; $B \rightarrow p, t$; $C \rightarrow q$; $D \rightarrow s$

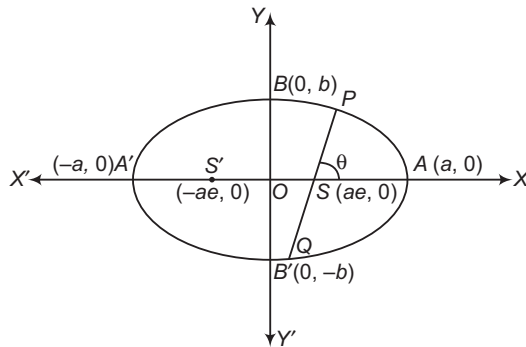
Entrances Gallery

1. (c)	2. (9)	3. (a)	4. (c)	5. (d)	6. (c)	7. (a)	8. (d)	9. (c)	10. (d)
11. (a)	12. (b)	13. (d)	14. (d)	15. (d)	16. (b)	17. (a)	18. (b)	19. (c)	20. (c)
21. (b)	22. (c)	23. (c)	24. (c)	25. (c)	26. (d)	27. (d)	28. (c)	29. (d)	30. (d)
31. (a)	32. (b)	33. (b)	34. (b)	35. (d)					

Explanations

Target Exercises

1. $r^2 - r - 6 > 0$ and $r^2 - 6r + 5 > 0$
 $\Rightarrow (r-3)(r+2) > 0$
 and $(r-1)(r-5) > 0$
 $\Rightarrow \{r < -2 \text{ or } r > 3\} \cap \{r < 1 \text{ or } r > 5\}$
 $\Rightarrow r < -2 \text{ or } r > 5$
2. If $a > b$, then length of the latusrectum is $\frac{2b^2}{a}$ and when
 $a < b$, then length of the latusrectum is $\frac{2a^2}{b}$.
3. Let PQ be a focal chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with foci $S(ae, 0)$ and $S'(-ae, 0)$.



The equation of chord PQ is

$$\frac{x - ae}{\cos \theta} = \frac{y}{\sin \theta}$$

The coordinates of any point on this line are given by

$$\frac{x - ae}{\cos \theta} = \frac{y}{\sin \theta} = r$$

$$\Rightarrow x = ae + r \cos \theta$$

$$\text{and } y = r \sin \theta$$

This point will lie on the ellipse, if

$$\frac{(ae + r \cos \theta)^2}{a^2} + \frac{(r \sin \theta)^2}{b^2} = 1$$

$$\Rightarrow r^2 \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) + \left(\frac{2rae \cos \theta}{a^2} \right) + e^2 - 1 = 0$$

$$\Rightarrow r^2 \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) + \left(\frac{2e \cos \theta}{a} \right) r + (e^2 - 1) = 0$$

Let r_1, r_2 be the roots of this equation. Then,

$$r_1 + r_2 = \frac{-2e \cos \theta}{a \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)}$$

and

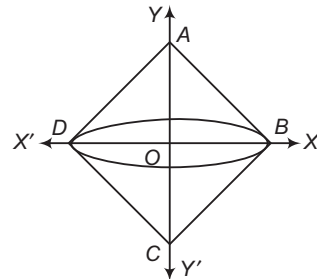
$$r_1 r_2 = \frac{e^2 - 1}{\left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right)}$$

Let $SP = |r_1|$ and $SQ = |r_2|$. Clearly, r_1 and r_2 are of opposite signs.

$$\begin{aligned} \text{Now, } (|r_1| + |r_2|)^2 &= (r_1 - r_2)^2 \\ &= (r_1 + r_2)^2 - 4r_1 r_2 \end{aligned}$$

$$\begin{aligned} &= \frac{4e^2 \cos^2 \theta}{a^2 \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}^2} - \frac{4(e^2 - 1)}{\left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}} \\ &= 4 \frac{\left\{ e^2 \cos^2 \theta + (1 - e^2) \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) a^2 \right\}}{a^2 \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}^2} \\ &= 4 \frac{\left\{ e^2 \cos^2 \theta + (1 - e^2) \left(\cos^2 \theta + \frac{a^2}{b^2} \sin^2 \theta \right) \right\}}{a^2 \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}^2} \\ &= \frac{4[e^2 \cos^2 \theta + (1 - e^2) \cos^2 \theta + \sin^2 \theta]}{a^2 \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}^2} \\ &= \frac{4}{a^2 \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}^2} \\ \therefore |r_1| + |r_2| &= \frac{2}{a \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}} \\ \Rightarrow \frac{2|r_1||r_2|}{|r_1| + |r_2|} &= \frac{2(1 - e^2)}{\left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}} \cdot \frac{a \left\{ \frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right\}}{2} \\ &= a(1 - e^2) \\ &= \frac{a^2(1 - e^2)}{a} = \frac{b^2}{a} = \text{Semi-latusrectum} \\ \therefore \text{HM of SP and SQ} &= \frac{2SP \cdot SQ}{SP + SQ} \\ &= \frac{2|r_1||r_2|}{|r_1| + |r_2|} \\ &= \frac{b^2}{a} = \text{Semi-latusrectum} \end{aligned}$$

4. Since, sides of the square are tangent and perpendicular to each other, so two vertices lie on director circle.



$$\begin{aligned} \therefore x^2 + y^2 &= (a^2 - 7) + (13 - 5a) \\ &= a^2 \quad [\sqrt{2}a \text{ is side of the square}] \end{aligned}$$

$$\Rightarrow (a^2 - 7) + (13 - 5a) = a^2$$

$$\Rightarrow a = \frac{6}{5}$$

But for an ellipse to exist, $a^2 - 7 > 0$ and $13 - 5a > 0$

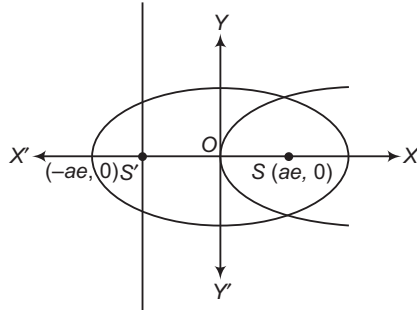
$$\Rightarrow a \in (-\infty, -\sqrt{7})$$

$$\therefore a \neq \frac{6}{5}$$

Hence, no such a exists.

5. Equation of the ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Equation of the parabola with focus $S(ae, 0)$ and directrix $x + ae = 0$ is $y^2 = 4aex$.



Now, length of latusrectum of the ellipse is $\frac{2b^2}{a}$ and that of the parabola is $4ae$.

For the two latusrectum to be equal, we get

$$\frac{2b^2}{a} = 4ae$$

$$\Rightarrow \frac{2a^2(1-e^2)}{a} = 4ae$$

$$\Rightarrow 1 - e^2 = 2e$$

$$\Rightarrow e^2 + 2e - 1 = 0$$

$$\text{Therefore, } e = \frac{-2 \pm \sqrt{8}}{2} = -1 \pm \sqrt{2}$$

$$\text{Hence, } e = \sqrt{2} - 1$$

6. $PS + PS' = ePZ + ePZ' = e(PZ + PZ')$

$$e \cdot ZZ' = e \cdot \frac{2a}{e} = 2a, \text{ if } a > b = e \cdot \frac{2b}{e} = 2b, \text{ if } a < b$$

$$PS + PS' + SS' = 2a + 2ae = 2a(1+e)$$

$$\therefore s = a(1+e)$$

$$\text{Now, } \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \sqrt{\frac{(s-c)(s-a)}{s(s-b)} \cdot \frac{(s-a)(s-b)}{s(s-c)}}$$

$$= \frac{s-a}{s} = \frac{e}{1+e}$$

$$= \frac{\sqrt{a^2 - b^2}}{a \times b^2/a^2} \times \left(1 - \frac{\sqrt{a^2 - b^2}}{a}\right)$$

$$\left[\because e = \frac{\sqrt{a^2 - b^2}}{a} \right]$$

$$= \frac{\sqrt{a^2 - b^2}}{b^2} \left(\frac{a - \sqrt{a^2 - b^2}}{a} \right) \times a$$

$$= \frac{\sqrt{a^2 - b^2}}{b^2} (a - \sqrt{a^2 - b^2})$$

$$7. \begin{vmatrix} ae & 0 & 1 \\ a \cos \alpha & b \sin \alpha & 1 \\ a \cos \beta & b \sin \beta & 1 \end{vmatrix} = 0$$

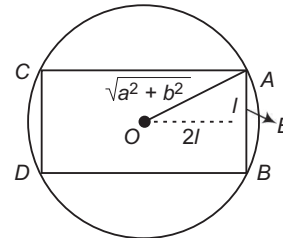
$$\Rightarrow e = \frac{\sin(\alpha - \beta)}{\sin \alpha - \sin \beta} = \frac{2 \cos \left(\frac{\alpha - \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)}{2 \cos \left(\frac{\alpha + \beta}{2}\right) \sin \left(\frac{\alpha - \beta}{2}\right)}$$

$$\Rightarrow e = \frac{\cos \left(\frac{\alpha - \beta}{2}\right)}{\cos \left(\frac{\alpha + \beta}{2}\right)}$$

$$\Rightarrow \frac{e-1}{e+1} = \frac{\cos \left(\frac{\alpha - \beta}{2}\right) - \cos \left(\frac{\alpha + \beta}{2}\right)}{\cos \left(\frac{\alpha + \beta}{2}\right) + \cos \left(\frac{\alpha - \beta}{2}\right)}$$

$$\Rightarrow \frac{e-1}{e+1} = \tan \frac{\alpha}{2} \tan \frac{\beta}{2}$$

8. Since, mutually perpendicular tangents can be drawn from vertices of rectangle. So, all the vertices of rectangle should lie on director circle $x^2 + y^2 = a^2 + b^2$.



Let breadth = $2l$ and length = $4l$, then

$$\text{Area of rectangle} = 2l \times 4l = 8l^2$$

$$= \frac{8}{5} (a^2 + b^2)$$

$$[\because \text{in } \triangle OAE, a^2 + b^2 = l^2 + 4l^2 \Rightarrow a^2 + b^2 = 5l^2]$$

9. Let ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and circle $x^2 + y^2 = a^2 e^2$

Radius of circle = ae

Point of intersection of circle and ellipse is

$$\left[\frac{a}{e} \sqrt{2e^2 - 1}, \frac{a}{e} (1 - e^2) \right]$$

Now, area of $\triangle PF_1F_2$

$$= \frac{1}{2} \begin{vmatrix} \frac{a}{e} \sqrt{2e^2 - 1} & \frac{a}{e} (1 - e^2) & 1 \\ ae & 0 & 1 \\ -ae & 0 & 1 \end{vmatrix} = \frac{1}{2} \left| \frac{a}{e} (1 - e^2) (2ae) \right| = 30$$

$$\Rightarrow a^2 (1 - e^2) = 30 \quad [\text{given}]$$

$$\begin{aligned} \Rightarrow a^2 e^2 &= a^2 - 30 \\ &= \left(\frac{17}{2}\right)^2 - 30 \\ &= \frac{169}{4} \end{aligned}$$

$$\Rightarrow 2ae = 13$$

10. The equation of a chord joining points having eccentric angles α and β is given by

$$\frac{x}{a} \cos \left(\frac{\alpha + \beta}{2}\right) + \frac{y}{b} \sin \left(\frac{\alpha - \beta}{2}\right) = \cos \left(\frac{\alpha - \beta}{2}\right)$$

If it passes through $(ae, 0)$, then

$$\begin{aligned} e \cos \left(\frac{\alpha + \beta}{2} \right) &= \cos \left(\frac{\alpha - \beta}{2} \right) \\ \Rightarrow e &= \frac{\cos \left(\frac{\alpha - \beta}{2} \right)}{\cos \left(\frac{\alpha + \beta}{2} \right)} \\ \Rightarrow e &= \frac{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha - \beta}{2} \right)}{2 \sin \left(\frac{\alpha + \beta}{2} \right) \cos \left(\frac{\alpha + \beta}{2} \right)} \\ \Rightarrow e &= \frac{\sin \alpha + \sin \beta}{\sin (\alpha + \beta)} \end{aligned}$$

11. $A \equiv (a \cos \theta, b \sin \theta)$

$B \equiv \{a \cos (\theta + \alpha), b \sin (\theta + \alpha)\}$

$C \equiv \{a \cos (\theta + 2\alpha), b \sin (\theta + 2\alpha)\}$

$\Delta \equiv \text{Area of } \triangle ABC$

$$\begin{aligned} &= \frac{1}{2} \begin{vmatrix} 1 & a \cos \theta & b \sin \theta \\ 1 & a \cos (\theta + \alpha) & b \sin (\theta + \alpha) \\ 1 & a \cos (\theta + 2\alpha) & b \sin (\theta + 2\alpha) \end{vmatrix} \\ &= 2ab \sin^2 \left(\frac{\alpha}{2} \right) \sin \alpha \end{aligned}$$

$\Delta(\alpha) = ab \sin \alpha (1 - \cos \alpha)$

$= \frac{ab}{2} (2 \sin \alpha - \sin 2\alpha)$

$\Delta'(\alpha) = 0$

$\Rightarrow \cos \alpha = 1 \text{ or } \cos \alpha = -\frac{1}{2}$

$\cos \alpha = 1$ gives triangle = 0

$\cos \alpha = -\frac{1}{2}$ gives maximum value of triangle = $\frac{3\sqrt{3}}{4} ab$

12. Let (h, k) be the mid-point of the chord $7x + y - 1 = 0$

$\Rightarrow \frac{hx}{1} + \frac{ky}{7} = \frac{h^2}{1} + \frac{k^2}{7} \quad \dots (i)$

and $7x + y = 1 \quad \dots (ii)$

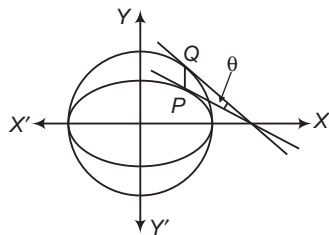
represents same straight line.

$\Rightarrow \frac{h}{7} = \frac{k}{7} \Rightarrow h = k$

Equation of the line joining $(0, 0)$ and (h, k) is $y - x = 0$

13. Tangent to the ellipse at $P(a \cos \alpha, b \sin \alpha)$ is

$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1$



Tangent to the circle at $Q(a \cos \alpha, a \sin \alpha)$ is

$\cos \alpha x + \sin \alpha y = a$

Now, angle between tangents is θ , then

$$\tan \theta = \left| \frac{-\frac{b}{a} \cot \alpha - (-\cot \alpha)}{1 + \left(-\frac{b}{a} \cot \alpha\right)(-\cot \alpha)} \right|$$

$$\begin{aligned} &= \left| \frac{\cot \alpha \left(1 - \frac{b}{a}\right)}{1 + \frac{b}{a} \cot^2 \alpha} \right| = \left| \frac{a - b}{a \tan \alpha + b \cot \alpha} \right| \\ &= \left| \frac{a - b}{(\sqrt{a \tan \alpha} - \sqrt{b \cot \alpha})^2 + 2\sqrt{ab}} \right| \end{aligned}$$

Now, the greatest value of the above expression is $\left| \frac{a - b}{2\sqrt{ab}} \right|$ when $\sqrt{a \tan \alpha} = \sqrt{b \cot \alpha}$

$\Rightarrow \theta_{\max} = \tan^{-1} \left(\frac{a - b}{2\sqrt{ab}} \right)$

14. $\therefore \sum_{i=1}^{10} (SP_i)(S'P_i) = 2560$

$\therefore 10b^2 = 2560$

$\Rightarrow b^2 = 256$

$\Rightarrow b = 16$

$\Rightarrow 256 = 400(1 - e^2)$

$\Rightarrow 1 - e^2 = \frac{16}{25}$

$\Rightarrow e = \frac{3}{5}$

15. For the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$, equation of director circle is

$x^2 + y^2 = 25$. The director circle will cut the ellipse

$\frac{x^2}{50} + \frac{y^2}{20} = 1$ at 4 points.

Hence, number of points = 4.

16. (α, β) lies on the director circle of the ellipse i.e. on $x^2 + y^2 = 9$.

So, we can assume

$\alpha = 3 \cos \theta, \beta = 3 \sin \theta$

$\therefore F = 12 \cos \theta + 9 \sin \theta - 3(4 \cos \theta + 3 \sin \theta)$

$\Rightarrow -15 \leq F \leq 15$

17. The tangent at the point of shortest distance from the line $x + y = 7$ parallel to the given line.

Any point on the given ellipse is $(\sqrt{6} \cos \theta, \sqrt{3} \sin \theta)$.

Equation of the tangent is

$\frac{x \cos \theta}{\sqrt{6}} + \frac{y \sin \theta}{\sqrt{3}} = 1$

which is parallel to $x + y = 7$.

$\Rightarrow \frac{\cos \theta}{\sqrt{6}} = \frac{\sin \theta}{\sqrt{3}}$

$\Rightarrow \frac{\cos \theta}{\sqrt{2}} = \frac{\sin \theta}{1} = \frac{1}{\sqrt{3}}$

$\therefore$ The required point is $(2, 1)$.

18. $2x^2 + 3y^2 - 8x + 6y + 5 = 0 \Rightarrow \frac{(x - 2)^2}{3} + \frac{(y + 1)^2}{2} = 1$

$\therefore$ Equation of the director circle is

$(x - 2)^2 + (y + 1)^2 = 3 + 2$

$\Rightarrow x^2 + y^2 - 4x + 2y = 0 \quad \dots (i)$

$\therefore (4, -2)$ is satisfying Eq. (i), therefore $(4, -2)$ is lying on the director circle.

and equation of circle is

$(x - 2)^2 + (y + 1)^2 = 3$

$\Rightarrow x^2 + y^2 - 4x + 2y + 2 = 0$

19. Let coordinate of the point from where tangents drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are inclined at an angle 45° be

(h, k) , then equation of the pair of tangents is

$$\left(\frac{h^2}{a^2} + \frac{k^2}{b^2} - 1\right)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = \left(\frac{hx}{a^2} + \frac{ky}{b^2} - 1\right)^2$$

Coefficient of x^2 , $A = \frac{k^2}{a^2b^2} - \frac{1}{a^2}$

Coefficient of y^2 , $B = \frac{h^2}{a^2b^2} - \frac{1}{b^2}$

and coefficient of $2xy$, $H = -\frac{hk}{a^2b^2}$

$$\therefore \tan 45^\circ = \frac{2\sqrt{\frac{h^2k^2}{a^4b^4} - \frac{h^2k^2}{a^4b^4} + \frac{k^2}{a^2b^4} + \frac{h^2}{a^4b^2} - \frac{1}{a^2b^2}}}{\frac{h^2 + k^2 - a^2 - b^2}{a^2b^2}}$$

$\therefore$ Required locus is

$$(x^2 + y^2 - a^2 - b^2)^2 = 4(b^2x^2 + a^2y^2 - a^2b^2)$$

20. $c^2 = a^2m^2 + b^2$, where $a^2 = 8$, $b^2 = 4$, $m = 4$

$$\Rightarrow c^2 = 8 \cdot 4^2 + 4$$

$$\Rightarrow c^2 = 128 + 4$$

$$\Rightarrow c^2 = 132$$

$$\therefore c = \pm \sqrt{132}$$

21. Equation of QR is

$$8 \cdot \frac{x}{4} + 27 \cdot \frac{y}{9} = 1 \Rightarrow 2x + 3y = 1 \quad \dots(i)$$

Now, equation of the pair of the lines passing through origin and points Q, R is given by

$$\left(\frac{x^2}{4} + \frac{y^2}{9}\right) = (2x + 3y)^2$$

$$\Rightarrow 9x^2 + 4y^2 = 36(4x^2 + 12xy + 9y^2)$$

$$\Rightarrow 135x^2 + 432xy + 320y^2 = 0$$

$$\therefore \text{Required angle} = \tan^{-1} \left(\frac{2\sqrt{216^2 - 135 \cdot 320}}{455} \right)$$

$$= \tan^{-1} \left(\frac{8\sqrt{2916 - 2700}}{455} \right) = \tan^{-1} \left(\frac{8\sqrt{216}}{455} \right) = \tan^{-1} \left(\frac{48\sqrt{6}}{455} \right)$$

22. Since, tangent is perpendicular to the line

$$2x + y + 7 = 0$$

Let equation of the tangents be $x - 2y + c = 0$

$$\therefore c^2 = a^2m^2 + b^2$$

$$c^2 = 4 \cdot \frac{1}{4} + 3 = 4 \Rightarrow c = \pm 2$$

$\therefore$ Equation of the required tangents are

$$x - 2y + 2 = 0 \quad \text{and} \quad x - 2y - 2 = 0.$$

23. Let equation of the tangent be $x + y + c = 0$

$$\therefore c^2 = a^2m^2 + b^2 \Rightarrow c^2 = 9 + 16 = 25$$

$$c = \pm 5$$

$\therefore$ Equation of the required tangents are $x + y \pm 5 = 0$

24. Let coordinates of P be (h, k) , then equation of QR is

$$\frac{hx}{2} + ky = 1 \quad \dots(i)$$

$$\text{but is given as} \quad x + 3y = 1 \quad \dots(ii)$$

$\therefore$ Eqs. (i) and (ii) are identical.

$$\therefore \frac{h}{2} = \frac{k}{3} = 1$$

$$\Rightarrow h = 2 \quad \text{and} \quad k = 3$$

Coordinates of P are (2, 3).

25. $SS_1 = T^2$

$$(3x^2 + 2y^2 - 5)[3(1)^2 + 2(2)^2 - 5] = [3 \times (1) + 2y(2) - 5]^2$$

$$\Rightarrow 9x^2 - 4y^2 - 24xy + 40y + 30x - 30 = 0$$

$$\therefore a = 9, b = -4, h = -12$$

$$\therefore \tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

$$\Rightarrow \tan \theta = \frac{2\sqrt{144 + 36}}{5} \Rightarrow \theta = \tan^{-1} \left(\frac{12}{\sqrt{5}} \right)$$

26. Since, tangents drawn from $(1, 2\sqrt{3})$ to the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are right angles, therefore $(1, 2\sqrt{3})$ will lie on the director circle of the ellipse i.e. $(1, 2\sqrt{3})$ will lie on

$$x^2 + y^2 = a^2 + b^2$$

$$\therefore 13 = a^2 + b^2$$

$$\Rightarrow b^2 = 4 \Rightarrow b = 2$$

27. The locus of point of intersection of tangents at end point of focal chord is directrix.

$$\therefore x = \frac{a}{e}, \text{ when } a > b$$

28. Since, locus of the point of intersection of the tangents at the end points of a focal chord is directrix.

$$\therefore \text{Required locus is } x = \pm \frac{a}{e} = \pm \frac{a^2}{\sqrt{a^2 - b^2}}$$

29. Since, all the points $(3, 4)$, $(5, 0)$ and $(2\sqrt{5}, \sqrt{5})$ are lying on the director circle $x^2 + y^2 = 25$. Therefore, all the choices are true.

30. The point of intersection of tangents is

$$\left(\frac{a \cos \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}, \frac{b \sin \left(\frac{\theta_1 + \theta_2}{2} \right)}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)} \right)$$

It is given that $(\theta_1 + \theta_2) = k = \text{constant}$.

Since, given $\theta_1 + \theta_2 = \text{constant}$. Therefore,

$$x_1 = \frac{a \cos k}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)} \quad \text{and} \quad y_1 = \frac{b \sin k}{\cos \left(\frac{\theta_1 - \theta_2}{2} \right)}$$

$$\Rightarrow \frac{x_1}{y_1} = \frac{a}{b} \cot k, \quad y_1 = \left(\frac{b}{a} \cot k \right) x_1$$

$$\Rightarrow (x_1, y_1) \text{ lies on the straight line } y = \left(\frac{b}{a} \cot k \right) x.$$

31. Let coordinate of the pole be (h, k) , then equation of the

$$\text{polar } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ is } \frac{hx}{a^2} + \frac{ky}{b^2} = 1.$$

$$\Rightarrow y = -\frac{b^2h}{a^2k}x + \frac{b^2}{k} \quad \dots(i)$$

Since, line (i) is touching the parabola $y^2 = 4ax$.

$$\therefore \frac{b^2}{k} = \frac{a \cdot a^2}{b^2 \cdot h} k$$

$$\therefore \text{Required locus is } a^3y^2 = -b^4x.$$

32. Let coordinates of the pole be (h, k) , then equation of the polar of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \Rightarrow y = -\frac{b^2 h}{a^2 k} x + \frac{b^2}{k} \quad \dots(i)$$

Since, line (i) is touching the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\therefore \frac{b^4}{k^2} = b^2 \cdot \frac{b^4 h^2}{a^4 k^2} + a^2$$

$$\therefore \text{Required locus is } b^6 x^2 + a^6 y^2 = a^4 b^4$$

which is equation of the ellipse.

33. Let coordinates of pole be (h, k) , then equation of the polar of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \Rightarrow y = -\frac{b^2 h}{a^2 k} x + \frac{b^2}{k} \quad \dots(ii)$$

Since, line (i) is touching the circle $x^2 + y^2 = c^2$

$$\therefore \frac{b^4}{k^2} = c^2 \left(1 + \frac{b^4}{a^4} + \frac{h^2}{k^2} \right)$$

$$\therefore \text{Required locus is } c^2 (b^4 x^2 + a^4 y^2) = a^4 b^4.$$

34. Let coordinates of the pole be (h, k) , then equation of the polar of $y^2 = 4ax$ is $ky = 2a(x + h)$

$$\Rightarrow y = \frac{2a}{k} x + \frac{2ah}{k} \quad \dots(i)$$

Since, line (i) is touching the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\text{Therefore, } \frac{4a^2 h^2}{k^2} = a^2 \cdot \frac{4a^2}{k^2} + b^2$$

$$\therefore \text{Required locus is } 4a^2 x^2 = 4a^4 + b^2 y^2$$

$$\Rightarrow 4a^2 x^2 - b^2 y^2 = 4a^4$$

35. Let $T(h, k)$ be the point of intersection of tangents to the parabola at R and S . Then, the equation of the chord of contact of tangents is

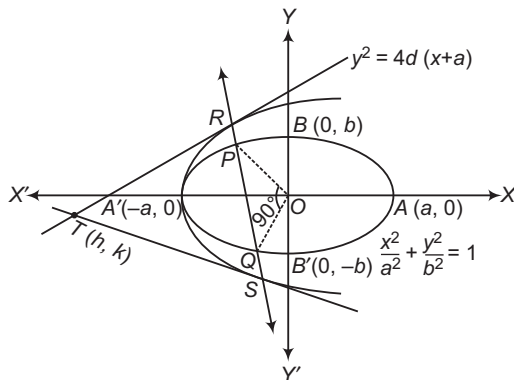
$$\begin{aligned} ky &= 2d(x + h) + 4ad \\ \Rightarrow \frac{ky - 2dx}{2d} &= 2d(h + 2a) \\ \Rightarrow \frac{ky - 2dx}{2d(h + 2a)} &= 1 \end{aligned}$$

The combined equations of OP and OQ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \left\{ \frac{ky - 2dx}{2d(h + 2a)} \right\}^2 = 0$$

Since, $\angle POQ = 90^\circ$. Therefore,

Coefficient of x^2 + Coefficient of $y^2 = 0$



$$\Rightarrow \frac{1}{a^2} - \frac{4d^2}{4d^2(h+2a)^2} + \frac{1}{b^2} - \frac{k^2}{4d^2(h+2a)^2} = 0$$

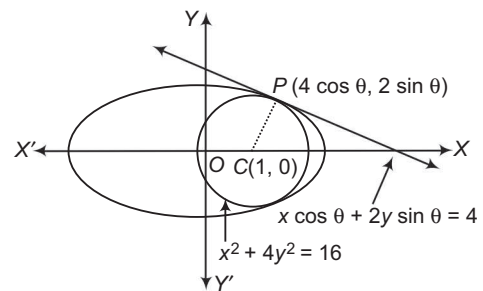
$$\Rightarrow k^2 + 4d^2 = 4d^2(h+2a)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

Hence, the locus of (h, k) is

$$y^2 + 4d^2 = 4d^2(x+2a)^2 \left(\frac{1}{a^2} + \frac{1}{b^2} \right)$$

36. The largest circle inscribed in the ellipse $x^2 + 4y^2 = 16$ will touch the ellipse at some point. So, let r be the radius of the largest circle centred at $C(1, 0)$ and inscribed in the ellipse $x^2 + 4y^2 = 16$. Suppose it touches the ellipse at $P(4 \cos \theta, 2 \sin \theta)$. Then, the equation of the tangent to the ellipse at P is

$$\begin{aligned} 4x \cos \theta + 8y \sin \theta &= 16 \\ \Rightarrow x \cos \theta + 2y \sin \theta &= 4 \quad \dots(i) \end{aligned}$$



Clearly, CP is perpendicular to Eq. (i). Therefore,

$$\frac{2 \sin \theta - 0}{4 \cos \theta - 1} \times \frac{\cos \theta}{2 \sin \theta} = -1$$

$$\Rightarrow -\cos \theta = -4 \cos \theta + 1$$

$$\Rightarrow \cos \theta = \frac{1}{3}$$

$$\begin{aligned} \therefore r &= CP \\ &= \sqrt{(4 \cos \theta - 1)^2 + (2 \sin \theta - 0)^2} \\ &= \sqrt{\left(\frac{4}{3} - 1 \right)^2 + \left(2 \times \frac{2\sqrt{2}}{3} - 0 \right)^2} \\ &= \sqrt{\frac{1}{9} + \frac{32}{9}} = \sqrt{\frac{11}{3}} \end{aligned}$$

Hence, the equation of the circle is

$$(x-1)^2 + (y-0)^2 = \frac{11}{3}$$

37. Let mid-point of the chord be (h, k) , then equation of the chord is $\frac{hx}{a^2} + \frac{ky^2}{b^2} - 1 = \frac{h^2}{a^2} + \frac{k^2}{b^2} - 1$

$$\Rightarrow y = -\frac{b^2}{a^2} \cdot \frac{h}{k} x + \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} \right) \frac{b^2}{k} \quad \dots(i)$$

Since, line (i) is touching the circle $x^2 + y^2 = c^2$... (ii)

$$\therefore \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} \right) \frac{b^4}{k^2} = c^2 \left(1 + \frac{b^4}{a^4} \frac{h^2}{k^2} \right)$$

$$\therefore \text{Required locus is } (b^2 x^2 + a^2 y^2)^2 = c^2 (b^4 x^2 + a^4 y^2).$$

38. Equation of the normal at $\left(ae, \frac{b^2}{a} \right)$ is

$$\frac{y_1}{b^2} (x - x_1) - \frac{x_1}{a^2} (y - y_1) = 0$$

$$\Rightarrow \frac{1}{a}(x - ae) - \frac{e}{a}\left(y - \frac{b^2}{a}\right) = 0 \quad \dots(i)$$

$\therefore (0, -2b)$ is lying on Eq. (i), we get

$$-e + \frac{e}{a}b\left(2 + \frac{b}{a}\right) = 0 \Rightarrow b(2a + b) = a^2$$

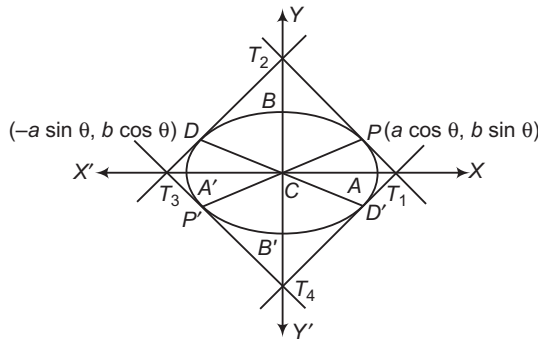
$$2ab = a^2 - b^2 \Rightarrow 2ab = a^2 e^2$$

$$\Rightarrow 4a^2(1 - e^2) = a^2 e^4$$

$$\Rightarrow e^4 + 4e^2 - 4 = 0 \Rightarrow e^2 = \frac{-4 \pm \sqrt{32}}{2}$$

$$\therefore e = \sqrt{2(\sqrt{2} - 1)}$$

39. Area of parallelogram $T_1 T_2 T_3 T_4$
 $= 4(\text{Area of parallelogram } CPT_2D)$
 $= 4(2 \times \text{Area of } \triangle CPD)$
 $= 8(\text{Area of } \triangle CPD)$



$$= 8 \times \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a \cos \theta & b \sin \theta & 1 \\ -a \sin \theta & b \cos \theta & 1 \end{vmatrix}$$

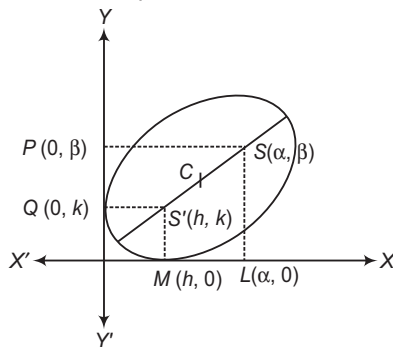
$$= 4(ab \cos^2 \theta + ab \sin^2 \theta)$$

$$= 4ab = 2a \times 2b$$

$$= \text{Product of the axes of the ellipse}$$

40. Let $S(\alpha, \beta)$ and $S'(h, k)$ be the foci of the ellipse. We know that the product of the perpendiculars from the foci of an ellipse to any tangent is equal to the square of semi-minor axes. Therefore,

$$k\beta = b^2 \text{ and } h\alpha = b^2$$



Now,

$$\Rightarrow SS' = 2ae$$

$$\Rightarrow (\alpha - h)^2 + (\beta - k)^2 = 4a^2 e^2$$

$$\Rightarrow \left(\alpha - \frac{b^2}{\alpha}\right)^2 + \left(\beta - \frac{b^2}{\beta}\right)^2 = 4a^2 e^2$$

$$\Rightarrow \frac{(\alpha^2 - b^2)^2}{\alpha^2} + \frac{(\beta^2 - b^2)^2}{\beta^2} = 4a^2 e^2$$

$$\Rightarrow \beta^2(\alpha^2 - b^2)^2 + \alpha^2(\beta^2 - b^2)^2 = 4a^2 \alpha^2 \beta^2 e^2$$

$$\Rightarrow (\alpha^2 + \beta^2)b^4 + \beta^2\alpha^4 + \alpha^2\beta^4 - 4\alpha^2\beta^2b^2$$

$$= 4a^2\alpha^2\beta^2\left(\frac{a^2 - b^2}{a^2}\right)$$

$$\Rightarrow (\alpha^2 + \beta^2)b^4 + \alpha^2\beta^2(\alpha^2 + \beta^2) = 4\alpha^2\beta^2(b^2 + a^2 - b^2)$$

$$\Rightarrow (\alpha^2 + \beta^2)(\alpha^2\beta^2 + b^4) = 4a^2\alpha^2\beta^2$$

Hence, the locus of (α, β) is

$$(x^2 + y^2)(x^2y^2 + b^4) = 4a^2x^2y^2$$

Similarly, by eliminating α and β , we can show that the locus of $S'(h, k)$ is $(x^2 + y^2)(x^2y^2 + b^4) = 4a^2x^2y^2$.

41. The equations of the normals to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at the points whose eccentric angles are θ and $\frac{\pi}{2} + \theta$ are

$$ax \sec \theta - by \operatorname{cosec} \theta = a^2 - b^2$$

and $-ax \operatorname{cosec} \theta - by \sec \theta = a^2 - b^2$ respectively.

Since, ω is the angle between these two normals.

$$\therefore \tan \omega = \left| \frac{\frac{a}{b} \tan \theta + \frac{a}{b} \cot \theta}{1 - \frac{a^2}{b^2}} \right|$$

$$\Rightarrow \tan \omega = \left| \frac{ab(\tan \theta + \cot \theta)}{b^2 - a^2} \right|$$

$$\Rightarrow \tan \omega = \left| \frac{2ab}{(\sin 2\theta)(b^2 - a^2)} \right|$$

$$\Rightarrow \tan \omega = \frac{2ab}{(a^2 - b^2) \sin 2\theta}$$

$$\Rightarrow \tan \omega = \frac{2a^2 \sqrt{1 - e^2}}{a^2 e^2 \sin 2\theta}$$

$$\Rightarrow \frac{2 \cot \omega}{\sin 2\theta} = \frac{e^2}{\sqrt{1 - e^2}}$$

42. The equations of the normals at $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ to the ellipse are

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

$$\frac{a^2 x}{x_2} - \frac{b^2 y}{y_2} = a^2 - b^2$$

$$\text{and } \frac{a^2 x}{x_3} - \frac{b^2 y}{y_3} = a^2 - b^2$$

These three lines will be concurrent, if

$$\begin{vmatrix} \frac{a^2}{x_1} & -\frac{b^2}{y_1} & a^2 - b^2 \\ \frac{a^2}{x_2} & -\frac{b^2}{y_2} & a^2 - b^2 \\ \frac{a^2}{x_3} & -\frac{b^2}{y_3} & a^2 - b^2 \end{vmatrix} = 0$$

$$\Rightarrow -a^2 b^2 (a^2 - b^2) \begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} \frac{1}{x_1} & \frac{1}{y_1} & 1 \\ \frac{1}{x_2} & \frac{1}{y_2} & 1 \\ \frac{1}{x_3} & \frac{1}{y_3} & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} y_1 & x_1 & x_1 y_1 \\ y_2 & x_2 & x_2 y_2 \\ y_3 & x_3 & x_3 y_3 \end{vmatrix} = 0$$

[on multiplying R_1 by $x_1 y_1$, R_2 by $x_2 y_2$ and R_3 by $x_3 y_3$]

$$\Rightarrow \begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0$$

43. The equation of the tangent at $(t^2, 2t)$ to the parabola $y^2 = 4x$ is

$$2ty = 2(x + t^2) \Rightarrow ty = x + t^2$$

$$\Rightarrow x - ty + t^2 = 0 \quad \dots (i)$$

The equation of the normal at point $(\sqrt{5} \cos \theta, 2 \sin \theta)$ of the ellipse $5x^2 + 5y^2 = 20$ is

$$(\sqrt{5} \sec \theta)x - (2 \operatorname{cosec} \theta)y = 5 - 4$$

$$\Rightarrow (\sqrt{5} \sec \theta)x - (2 \operatorname{cosec} \theta)y - 1 = 0 \quad \dots (ii)$$

It is given that Eqs. (i) and (ii) represent the same line. Therefore,

$$\frac{\sqrt{5} \sec \theta}{1} = \frac{-2 \operatorname{cosec} \theta}{-1} = \frac{-1}{t^2}$$

$$\Rightarrow t = \frac{2 \operatorname{cosec} \theta}{\sqrt{5} \sec \theta} \quad \text{and} \quad t = -\frac{1}{2 \operatorname{cosec} \theta}$$

$$\Rightarrow t = \frac{2}{\sqrt{5}} \cot \theta \quad \text{and} \quad t = -\frac{1}{2} \sin \theta$$

$$\Rightarrow \frac{2}{\sqrt{5}} \cot \theta = -\frac{1}{2} \sin \theta$$

$$\Rightarrow 4 \cos \theta = -\sqrt{5} \sin^2 \theta$$

$$\Rightarrow 4 \cos \theta = -\sqrt{5} (1 - \cos^2 \theta)$$

$$\Rightarrow \sqrt{5} \cos^2 \theta - 4 \cos \theta - \sqrt{5} = 0$$

$$\Rightarrow \sqrt{5} \cos^2 \theta - 5 \cos \theta + \cos \theta - \sqrt{5} = 0$$

$$\Rightarrow \sqrt{5} \cos \theta (\cos \theta - \sqrt{5}) + (\cos \theta - \sqrt{5}) = 0$$

$$\Rightarrow (\cos \theta - \sqrt{5})(\sqrt{5} \cos \theta + 1) = 0$$

$$\Rightarrow \cos \theta = -\frac{1}{\sqrt{5}} \quad [\because \cos \theta \neq -\sqrt{5}]$$

$$\Rightarrow \theta = \cos^{-1} \left(-\frac{1}{\sqrt{5}} \right)$$

On putting $\cos \theta = -\frac{1}{\sqrt{5}}$ in $t = -\frac{1}{2} \sin \theta$, we get

$$t = -\frac{1}{2} \sqrt{1 - \frac{1}{5}} = -\frac{1}{\sqrt{5}}$$

$$\text{Hence, } \theta = \cos^{-1} \left(-\frac{1}{\sqrt{5}} \right) \quad \text{and} \quad t = -\frac{1}{\sqrt{5}}$$

44. The equation of the normal at point (x, y) on the ellipse is

$$\frac{a^2 x}{x} - \frac{b^2 y}{y} = a^2 - b^2$$

If it passes through (h, k) , then

$$\frac{a^2 h}{x} - \frac{b^2 k}{y} = a^2 - b^2$$

$$\Rightarrow a^2 hy - b^2 kx = (a^2 - b^2)xy$$

$$\Rightarrow y = \frac{b^2 kx}{a^2 h - (a^2 - b^2)x} \quad \dots (i)$$

Since, (x, y) lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\Rightarrow \frac{x^2}{a^2} + \frac{b^4 k^2 x^2}{b^2 \{a^2 h - (a^2 - b^2)x\}^2} = 1 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow x^2 \{a^2 h - (a^2 - b^2)x\}^2 + a^2 b^2 k^2 x^2$$

$$= a^2 \{a^2 h - (a^2 - b^2)x\}^2$$

$$\Rightarrow (a^2 - b^2)^2 x^4 - 2a^2 (a^2 - b^2)hx^3 + a^2 (a^2 h^2 - a^2 + b^2 + b^2 k^2)x^2 + 2a^4 (a^2 - b^2)hx - a^6 h^2 = 0 \dots (ii)$$

This is a fourth degree equation in x . So, it gives four values of x . Corresponding to each value of x , there is a point on the ellipse such that the normal to the ellipse at that point passes through (h, k) . So, let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ be four points on the ellipse such that the normals at these points pass through (h, k) . Clearly, x_1, x_2, x_3 and x_4 are the roots of the Eq. (ii).

$$\therefore x_1 + x_2 + x_3 + x_4 = \frac{2a^2 h}{(a^2 - b^2)}$$

$$\Rightarrow \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{a^2 h}{2(a^2 - b^2)}$$

Similarly, we have

$$\frac{y_1 + y_2 + y_3 + y_4}{4} = \frac{b^2 k}{2(a^2 - b^2)}$$

Hence, the mean position of the four points is $\left(\frac{a^2 h}{2(a^2 - b^2)}, \frac{b^2 k}{2(a^2 - b^2)} \right)$.

45. We know that from a point (h, k) , four normals real or imaginary can be drawn to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that the eccentric angles of the foot of the normals are given by

$$bk \tan^4 \frac{\theta}{2} + 2(ah + a^2 e^2) \tan^3 \frac{\theta}{2} + 2(ah - a^2 e^2) \tan \frac{\theta}{2} - bk = 0$$

$$\Rightarrow bkt^4 + 2(ah + a^2 e^2)t^3 + 2(ah - a^2 e^2)t - bk = 0 \quad \dots (i)$$

where, $t = \tan \frac{\theta}{2}$

In the given case, we have $k = 0$.

Therefore, Eq. (i) reduces to

$$2(ah + a^2 e^2)t^3 + 2(ah - a^2 e^2)t = 0 \quad \dots (ii)$$

Thus, the degree of the Eq. (i) reduce by one which corresponds to one root $t \rightarrow \infty$. The corresponding value of the eccentric angle of the foot of the normal is given by

$$\tan \frac{\theta}{2} = \infty \Rightarrow \theta = \pi$$

This means that one normal from $(h, 0)$ coincides with the negative direction of X -axis which is a part of the major axis.

From Eq. (ii), we have

$$2(ah + a^2 e^2)t^3 + 2(ah - a^2 e^2)t = 0$$

$$\Rightarrow t = 0 \quad \text{and} \quad t^2 = \frac{ah - a^2 e^2}{ah + a^2 e^2}$$

For $t = 0$, we have

$$\tan \frac{\theta}{2} = 0 \Rightarrow \theta = 0$$

This means that corresponding to $t = 0$, positive part of the X-axis on the right of $(h, 0)$ is the normal to the ellipse. Now, two more normals can be drawn from $(h, 0)$ to the ellipse, if the values of t given by

$$t^2 = \frac{ah - a^2e^2}{ah + a^2e^2}$$

are real and distinct. This is possible only when

$$\Rightarrow -\frac{ah - a^2e^2}{ah + a^2e^2} > 0$$

$$\Rightarrow \frac{h - ae^2}{h + ae^2} < 0$$

$$\Rightarrow (h - ae^2)(h + ae^2) < 0$$

$$\Rightarrow -ae^2 < h < ae^2 \Rightarrow |h| < ae^2$$

46. Suppose the chords $l_1x + m_1y = 1$ and $l_2x + m_2y = 1$ cut the ellipse at P, Q, R and S respectively, such that the normals to the ellipse at P, Q, R and S pass through $T(h, k)$. Let (α, β) be one of the feet of the normals. Then, the equation of the normal at (α, β) is

$$\frac{a^2x}{\alpha} - \frac{b^2y}{\beta} = a^2 - b^2$$

It passes through $T(h, k)$. Therefore,

$$\frac{a^2h}{\alpha} - \frac{b^2k}{\beta} = a^2 - b^2$$

So, the locus of (α, β) is

$$\frac{a^2h}{x} - \frac{b^2k}{y} = (a^2 - b^2)$$

$$\Rightarrow a^2hy - b^2kx = (a^2 - b^2)xy \quad \dots (i)$$

The equation of the curve passing through the intersection of the ellipse and the chords $l_1x + m_1y = 1$ and $l_2x + m_2y = 1$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) + \lambda (l_1x + m_1y - 1)(l_2x + m_2y - 1) = 0 \quad \dots (ii)$$

Clearly, Eqs. (i) and (ii) represent the same curve.

$$\therefore \frac{1}{a^2} + \lambda l_1l_2 = 0, \frac{1}{b^2} + \lambda m_1m_2 = 0, -1 + \lambda = 0$$

$$\Rightarrow \frac{1}{a^2} + l_1l_2 = 0, \frac{1}{b^2} + m_1m_2 = 0$$

$$\Rightarrow a^2l_1l_2 = b^2m_1m_2 = -1$$

47. Let $P(h, k)$ be the mid-point of a chord of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which passes through a fixed point (α, β) .

The equation of the chord of the ellipse which is bisected at $P(h, k)$ is

$$\Rightarrow \frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

It passes through (α, β) . Therefore,

$$\frac{h\alpha}{a^2} + \frac{k\beta}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2}$$

So, the locus of (h, k) is

$$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{\alpha x}{a^2} + \frac{\beta y}{b^2}$$

$$\Rightarrow \frac{\left(x - \frac{\alpha}{2}\right)^2}{a^2} + \frac{\left(y - \frac{\beta}{2}\right)^2}{b^2} = \frac{1}{4} \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} \right)$$

Clearly, it represents an ellipse with centre at $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$.

48. Let $(\alpha, b - \alpha)$ be a point on the line $x + y = b$ such that the chord of the given ellipse passing through $(a, -b)$ are bisected at $(\alpha, b - \alpha)$.

Then, the equation of the chord is

$$\frac{\alpha x}{2a^2} + \frac{(b - \alpha)y}{2b^2} = \frac{\alpha^2}{2a^2} + \frac{(b - \alpha)^2}{2b^2} \quad [\text{using } S' = T]$$

This passes through $(a, -b)$. Therefore,

$$\Rightarrow \frac{\alpha a}{2a^2} - \frac{(b - \alpha)b}{2b^2} = \frac{\alpha^2}{2a^2} + \frac{(b - \alpha)^2}{2b^2}$$

$$\Rightarrow \alpha^2(a^2 + b^2) - ab(3a + b)\alpha + 2a^2b^2 = 0$$

Since, α is real. Therefore,

$$a^2b^2(3a + b)^2 - 8a^2b^2(a^2 + b^2) \geq 0$$

$$\Rightarrow a^2 + 6ab - 7b^2 \geq 0 \Rightarrow a^2 + 6ab \geq 7b^2$$

which is the required condition.

49. Let $P(h, k)$ be a point. Then, the polar of $P(h, k)$ with respect to the given ellipse is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$$

It is given that the length of the perpendicular from the centre $(0, 0)$ of the ellipse to Eq. (i) is equal to c . Therefore,

$$\left| \frac{1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} \right| = c \Rightarrow \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{c^2}$$

Hence, the locus of (h, k) is $\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{c^2}$.

50. Let $P(h, k)$ be a point. Then, its polar with respect to the given ellipse is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad \dots (i)$$

It is given that the product of the lengths of the perpendiculars from the foci $(\pm ae, 0)$ to Eq. (i) is equal to c^2 . Therefore,

$$\left\{ \frac{\frac{he}{a} - 1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} \right\} \left\{ \frac{-\frac{he}{a} - 1}{\sqrt{\frac{h^2}{a^4} + \frac{k^2}{b^4}}} \right\} = c^2$$

$$\Rightarrow -\left(\frac{h^2e^2}{a^2} - 1 \right) = c^2 \left(\frac{h^2}{a^4} + \frac{k^2}{b^4} \right)$$

$$\Rightarrow -(h^2a^2e^2b^4 - a^4b^4) = h^2b^4c^2 + k^2a^4c^2$$

$$\Rightarrow (c^2 + a^2e^2)h^2b^4 + k^2a^4c^2 = a^4b^4$$

$$\Rightarrow (c^2 + a^2 - b^2)h^2b^4 + k^2a^4c^2 = a^4b^4$$

Hence, the locus of (h, k) is

$$b^4(c^2 + a^2 - b^2)x^2 + c^2a^4y^2 = a^4b^4$$

51. Let $P(h, k)$ be the pole of a chord QR of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, such that QR subtends a right angle at the centre $C(0, 0)$ of the ellipse.

The equation of the polar of P i.e. the chord QR is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1$$

The combined equation of CQ and CR is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{hx}{a^2} + \frac{ky}{b^2} \right)^2$$

The above equation represent a pair of perpendicular lines. Therefore, $\frac{1}{a^2} - \frac{h^2}{a^4} + \frac{1}{b^2} - \frac{k^2}{b^4} = 0$

$$[\because \text{coefficient of } x^2 + \text{coefficient of } y^2 = 0]$$

$$\Rightarrow \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$$

$$\text{Hence, the locus of } (h, k) \text{ is } \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}.$$

52. Let (h, k) be the mid-point of a chord of the ellipse. Then, its equation is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = \frac{h^2}{a^2} + \frac{k^2}{b^2} \quad \dots(i)$$

Let (x_1, y_1) be its pole with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Then, the equation of the polar is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \dots(ii)$$

Clearly, Eqs. (i) and (ii) represent the same line. Therefore,

$$\frac{x_1}{h} = \frac{y_1}{k} = \frac{1}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}$$

$$\Rightarrow x_1 = \frac{h}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}, y_1 = \frac{k}{\frac{h^2}{a^2} + \frac{k^2}{b^2}}$$

It is given that (x_1, y_1) lies on the auxiliary circle. Therefore,

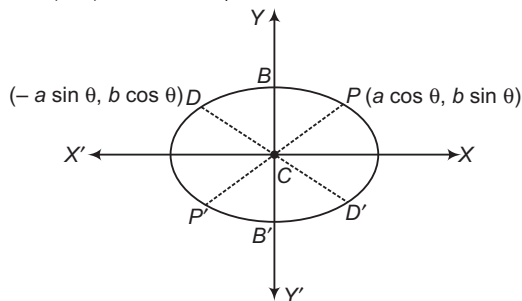
$$x_1^2 + y_1^2 = a^2$$

$$\Rightarrow h^2 + k^2 = a^2 \left(\frac{h^2}{a^2} + \frac{k^2}{b^2} \right)^2$$

Hence, the locus of (h, k) is

$$x^2 + y^2 = a^2 \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2$$

53. Let (h, k) be the mid-point of PD .



Then, $2h = a \cos \theta - a \sin \theta$ and $2k = b \sin \theta + b \cos \theta$

$$\Rightarrow \frac{2h}{a} = \cos \theta - \sin \theta$$

$$\text{and } \frac{2k}{b} = \sin \theta + \cos \theta$$

$$\Rightarrow \frac{4h^2}{a^2} + \frac{4k^2}{b^2} = (\cos \theta - \sin \theta)^2 + (\sin \theta + \cos \theta)^2$$

$$\Rightarrow \frac{h^2}{a^2} + \frac{k^2}{b^2} = \frac{1}{2}$$

Hence, the locus of (h, k) is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}$$

54. Let (h, k) be the pole of a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to the ellipse $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$.

Then, the equation of the polar is

$$\frac{hx}{\alpha^2} + \frac{ky}{\beta^2} = 1$$

$$\Rightarrow y = - \left(\frac{\beta^2 h}{\alpha^2 k} \right) x + \frac{\beta^2}{k}$$

This touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore,

$$\frac{\beta^4}{k^2} = a^2 \left(\frac{-\beta^2 h}{\alpha^2 k} \right)^2 + b^2$$

$$\Rightarrow \frac{\beta^4}{k^2} = a^2 \frac{\beta^4 h^2}{\alpha^4 k^2} + b^2$$

$$\Rightarrow \frac{a^2 h^2}{\alpha^4} + \frac{b^2 k^2}{\beta^4} = 1$$

Hence, the locus of (h, k) is

$$\frac{a^2 x^2}{\alpha^4} + \frac{b^2 y^2}{\beta^4} = 1$$

55. Let (h, k) be the pole of a tangent to the ellipse with respect to the parabola $y^2 = 4ax$. Then, the equation of the polar is

$$ky = 2a(x + h)$$

$$\Rightarrow y = \left(\frac{2a}{k} \right) x + \frac{2ah}{k}$$

This touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Therefore,

$$\left(\frac{2ah}{k} \right)^2 = a^2 \left(\frac{2a}{k} \right)^2 + b^2$$

$$\Rightarrow b^2 k^2 = 4a^2 h^2 - 4a^4$$

Hence, the locus of (h, k) is

$$b^2 y^2 = 4a^2 (x^2 - a^2)$$

56. Let (h, k) be the pole of the normal chord $ax \sec \theta - by \csc \theta = a^2 - b^2$... (i)

$$\text{of the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then, the equation of the polar is

$$\frac{hx}{a^2} + \frac{ky}{b^2} = 1 \quad \dots(ii)$$

Clearly, Eqs. (i) and (ii) represent the same line.

$$\text{Therefore, } \frac{h/a^2}{a \sec \theta} = \frac{k/b^2}{-b \csc \theta} = \frac{1}{a^2 - b^2}$$

$$\Rightarrow \cos \theta = \frac{a^3}{h(a^2 - b^2)}$$

$$\text{and } \sin \theta = \frac{-b^3}{k(a^2 - b^2)}$$

$$\Rightarrow \left\{ \frac{a^3}{h(a^2 - b^2)} \right\}^2 + \left\{ \frac{-b^3}{k(a^2 - b^2)} \right\}^2 = \cos^2 \theta + \sin^2 \theta$$

$$\Rightarrow \frac{a^6}{h^2} + \frac{b^6}{k^2} = (a^2 - b^2)^2$$

Hence, the locus of (h, k) is

$$\frac{a^6}{x^2} + \frac{b^6}{y^2} = (a^2 - b^2)^2$$

57. Let θ be the eccentric angles of P . Then, the eccentric angle of D is $\frac{\pi}{2} + \theta$. So, the coordinates of P and D are $(a \cos \theta, b \sin \theta)$ and $\left[a \cos \left(\frac{\pi}{2} + \theta \right), b \sin \left(\frac{\pi}{2} + \theta \right) \right]$ respectively.

The equation of chord PD is

$$\frac{x}{a} \cos \left(\frac{\theta + \frac{\pi}{2} + \theta}{2} \right) + \frac{y}{b} \sin \left(\frac{\theta + \frac{\pi}{2} + \theta}{2} \right) = \cos \left(\frac{\frac{\pi}{2} + \theta - \theta}{2} \right)$$

$$\Rightarrow \frac{x}{a} \cos \left(\frac{\pi}{4} + \theta \right) + \frac{y}{b} \sin \left(\frac{\pi}{4} + \theta \right) = \cos \frac{\pi}{4}$$

$$\Rightarrow \left(\frac{x}{\frac{a}{\sqrt{2}}} \right) \cos \left(\frac{\pi}{4} + \theta \right) + \left(\frac{y}{\frac{b}{\sqrt{2}}} \right) \sin \left(\frac{\pi}{4} + \theta \right) = 1$$

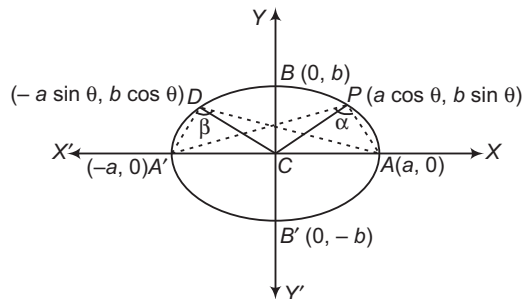
Clearly, it is tangent to the ellipse.

$$\therefore \frac{x^2}{\left(\frac{a}{\sqrt{2}} \right)^2} + \frac{y^2}{\left(\frac{b}{\sqrt{2}} \right)^2} = 1 \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

58. Let CP and CD be a pair of conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, the coordinates of P and D are $(a \cos \theta, b \sin \theta)$ and $(-a \sin \theta, b \cos \theta)$ respectively.

$$m_1 = \text{Slope of } AP = \frac{b \sin \theta}{a \cos \theta - a} = -\frac{b}{a} \cot \frac{\theta}{2}$$

$$m_2 = \text{Slope of } A'P = \frac{b \sin \theta}{a \cos \theta + a} = \frac{b}{a} \tan \frac{\theta}{2}$$



$$\therefore \tan \alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow \tan \alpha = \left| \frac{-\frac{b}{a} \cot \frac{\theta}{2} - \frac{b}{a} \tan \frac{\theta}{2}}{1 - \frac{b^2}{a^2}} \right|$$

$$\Rightarrow \tan \alpha = \frac{ab}{a^2 - b^2} \left(\cot \frac{\theta}{2} + \tan \frac{\theta}{2} \right)$$

$$\Rightarrow \tan \alpha = \left(\frac{2ab}{a^2 - b^2} \right) \frac{1}{\sin \theta}$$

Replacing θ by $\left(\frac{\pi}{2} + \theta \right)$, we get

$$\tan \beta = \left(\frac{2ab}{a^2 - b^2} \right) \frac{1}{\cos \theta}$$

$$\therefore \cot^2 \alpha + \cot^2 \beta = \left(\frac{a^2 - b^2}{2ab} \right)^2 (\sin^2 \theta + \cos^2 \theta)$$

$$= \left(\frac{a^2 - b^2}{2ab} \right)^2 = \text{constant}$$

59. Let $R(h, k)$ be any point on the circle $x^2 + y^2 = c^2$.

$$\text{Then, } h^2 + k^2 = c^2 \quad \dots (i)$$

Since, PCP' and DCD' form a pair of conjugate diameters. So, the coordinates of the extremities are $P(a \cos \theta, b \sin \theta)$, $P'(-a \cos \theta, -b \sin \theta)$

$$D(-a \sin \theta, b \cos \theta), D'(a \sin \theta, -b \cos \theta)$$

$$\therefore PR^2 + DR^2 + P'R^2 + D'R^2$$

$$= (h - a \cos \theta)^2 + (k - b \sin \theta)^2 + (h + a \sin \theta)^2 + (k - b \cos \theta)^2$$

$$= 4(h^2 + k^2) + 2a^2 + 2b^2$$

$$= 2a^2 + 2b^2 + 4c^2 \quad [\text{from Eq. (i)}]$$

$$= 2(a^2 + b^2 + 2c^2)$$

60. Since, PCP' and DCD' are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the eccentric angle of P is θ .

Therefore, the eccentric angles of D , P' and D' are $\frac{\pi}{2} + \theta$,

$\pi + \theta$ and $\frac{3\pi}{2} + \theta$ respectively.

We know that the sum of the eccentric angles of the points in which a circle cuts an ellipse in an even multiple of π .

Therefore, if α is the eccentric angle of the point, where the circle $PP'D$ again cuts the ellipse, then $\theta + (\pi + \theta) + (\pi/2 + \theta) + \alpha = 2n\pi$

$$\Rightarrow \alpha = 2n\pi - \frac{3\pi}{2} - 3\theta$$

On putting $n = 1$, we get $\alpha = \frac{\pi}{2} - 3\theta$

Also, any other value of n gives the same point on the ellipse.

61. Here, $b > a$, $a = 4$, $b = 5$, $e = 3/5$

$$\text{Equation of directrix is } y = \pm \frac{b}{c}$$

$$\text{i.e. } y = \frac{25}{3}$$

$$y = -\frac{25}{3}$$

62. $(\sqrt{3}x + 3\sqrt{3})^2 + (2y + 4)^2 = c$

Comparing this with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \text{ we get}$$

$$h^2 < ab$$

So, no locus for $c < 0$

Ellipse for $c > 0$ and point for $c = 0$.

63. Let $P(a \sec \theta, b \tan \theta)$

$$\text{Tangent at } P \text{ is } \frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1$$

It meets $bx - ay = 0$ i.e. $\frac{x}{a} - \frac{y}{b} = 0$ at Q .

$$\therefore Q \left(\frac{a}{\sec \theta - \tan \theta}, \frac{b}{\sec \theta - \tan \theta} \right)$$

It meets $bx + ay = 0$ i.e. $\frac{x}{a} + \frac{y}{b} = 0$ at R .

$$\therefore R \left(\frac{a}{\sec \theta + \tan \theta}, \frac{-b}{\sec \theta + \tan \theta} \right)$$

$$\therefore CQ \cdot CR = \frac{\sqrt{a^2 + b^2}}{(\sec \theta - \tan \theta)} \cdot \frac{\sqrt{a^2 + b^2}}{(\sec \theta + \tan \theta)} = a^2 + b^2$$

64. Given equation of ellipse is $16x^2 + 11y^2 = 256$.

Equation of tangent at $\left(4 \cos \theta, \frac{16}{\sqrt{11}} \sin \theta\right)$ is

$$16x(4 \cos \theta) + 11y \left(\frac{16}{\sqrt{11}} \sin \theta \right) = 256$$

It touches $(x-1)^2 + y^2 = 4^2$, if

$$\left| \frac{4 \cos \theta - 16}{\sqrt{16 \cos^2 \theta + 11 \sin^2 \theta}} \right| = 4$$

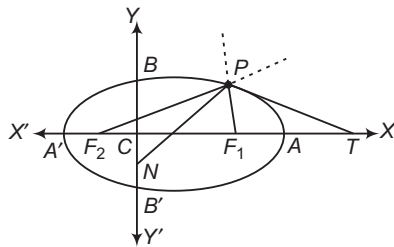
$$\Rightarrow (\cos \theta - 4)^2 = 16 \cos^2 \theta + 11 \sin^2 \theta$$

$$\Rightarrow 4 \cos^2 \theta + 8 \cos \theta - 5 = 0$$

$$\Rightarrow \cos \theta = \frac{1}{2}$$

$$\therefore \theta = \frac{\pi}{3}, \frac{5\pi}{3}$$

$$65. \therefore \frac{PF_1}{PF_2} = \frac{NF_1}{NF_2}$$



$\therefore PN$ bisects $\angle F_1PF_2$.

$\therefore$ Bisectors are perpendicular to each other.

$\therefore PT$ bisects the angle $(180^\circ - \angle F_1PF_2)$.

$$66. \therefore 5x^2 + 9y^2 - 54y + 36 = 0$$

$$\Rightarrow 5x^2 + 9(y-3)^2 = 45$$

$$\Rightarrow \frac{x^2}{3^2} + \frac{(y-3)^2}{(\sqrt{5})^2} = 1$$

$\therefore$ Length of major axis $= 2 \times 3 = 6$

and length of minor axis $= 2 \times \sqrt{5} = 2\sqrt{5}$

67. Let $(t, b-t)$ be a point on the line $x+y=b$, then equation chord whose mid-point $(t, b-t)$ is

$$\frac{tx}{2a^2} + \frac{(b-t)y}{2b^2} - 1 = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2} - 1 \quad \dots (i)$$

$(a, -b)$ lies on Eq. (i), then

$$\frac{ta}{2a^2} - \frac{b(b-t)}{2b^2} = \frac{t^2}{2a^2} + \frac{(b-t)^2}{2b^2}$$

$$\Rightarrow t^2(a^2 + b^2) - ab(3a+b)t + 2a^2b^2 = 0$$

$\therefore t$ is real.

$$\therefore B^2 - 4AC \geq 0$$

$$\Rightarrow a^2b^2(3a+b)^2 - 4(a^2+b^2)2a^2b^2 \geq 0$$

$$\Rightarrow 9a^2 + 6ab + b^2 - 8a^2 - 8b^2 \geq 0$$

$$\therefore a^2 + 6ab - 7b^2 \geq 0$$

$$68. \text{ Ellipse is } \frac{x^2}{9} + \frac{y^2}{4} = 1$$

$$\text{Focus} \equiv (\sqrt{5}, 0), e = \frac{\sqrt{5}}{3}$$

$$\text{A point on ellipse } \left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right)$$

Equation of the circle as the diameter, joining the point

$$\left(\frac{3}{\sqrt{2}}, \frac{2}{\sqrt{2}} \right) \text{ and focus } (\sqrt{5}, 0) \text{ is}$$

$$(x - \sqrt{5})(\sqrt{2}x - 3) + y(\sqrt{2}y - 2) = 0$$

69. $(\lambda, 3)$ should satisfy the equation $x^2 + y^2 = 13$

$$\therefore \lambda = \pm 2$$

70. Let PSP' be any focal chord.

We know the property described in Statement II is true.

$$\therefore \text{Semi-latusrectum} = \text{HM of } SP \text{ and } SP' \leq \frac{SP + SP'}{2}$$

$$\Rightarrow \text{Latusrectum} \leq PP'$$

$\Rightarrow$ Statement I is true.

71. A circle concentric with ellipse and having major axis as diameter is called auxiliary circle

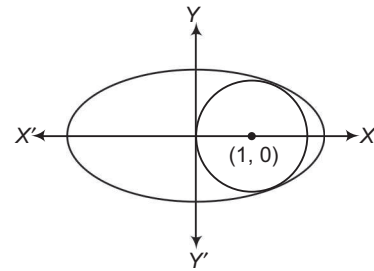
$$x^2 + y^2 = 4$$

is auxiliary circle of infinite ellipse

$$\frac{x^2}{4} + \frac{y^2}{b^2} = 1, b < 2$$

72. Solving both equations, we have

$$\frac{x^2}{a^2} + \frac{1 - (x-1)^2}{b^2} = 1$$



$$\Rightarrow b^2x^2 + a^2[1 - (x-1)^2] = a^2b^2$$

$$\Rightarrow (b^2 - a^2)x^2 + 2a^2x - a^2b^2 = 0 \quad \dots (i)$$

For least area, circle must touch the ellipse.

$\therefore$ Discriminant of Eq. (i) is zero.

$$\Rightarrow 4a^4 + 4a^2b^2(b^2 - a^2) = 0$$

$$\Rightarrow a^2 + b^2(b^2 - a^2) = 0$$

$$\Rightarrow a^2 + b^2(-a^2e^2) = 0$$

$$\Rightarrow 1 - b^2e^2 = 0$$

$$\Rightarrow b = \frac{1}{e}$$

$$\text{Also, } a^2 = \frac{b^2}{1 - e^2} = \frac{1}{e^2(1 - e^2)} \quad \left[\because b = \frac{1}{e} \right]$$

$$\Rightarrow a = \frac{1}{e\sqrt{1-e^2}}$$

Let S be the area of the ellipse. Then,

$$S = \pi ab = \frac{\pi}{e^2\sqrt{1-e^2}} = \frac{\pi}{\sqrt{e^4 - e^6}}$$

Area is minimum, if $f(e) = e^4 - e^6$ is maximum.

$$\text{When } f'(e) = 4e^3 - 6e^5 = 0$$

$$\Rightarrow e = \sqrt{\frac{2}{3}}$$

which is point of maxima for $f(e)$.

$$S \text{ is least when } e = \sqrt{\frac{2}{3}}$$

$$73. \text{ Clearly, } b = \frac{1}{\sqrt{2/3}} = \sqrt{\frac{3}{2}} \text{ and } a = \frac{1}{\sqrt{\frac{2}{3}}\sqrt{1-\frac{2}{3}}} = \frac{3}{\sqrt{2}}$$

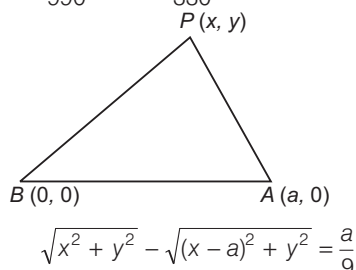
$$\therefore \text{Equation of ellipse is } 2x^2 + 6y^2 = 9.$$

$$\text{Equation of auxiliary circle of ellipse is } x^2 + y^2 = 45.$$

74. Length of latusrectum of ellipse is

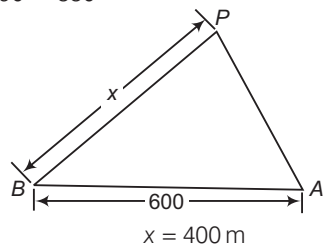
$$\frac{2b^2}{a} = \frac{2 \cdot \frac{9}{2}}{\frac{3}{\sqrt{2}}} = 1$$

$$75. \frac{\sqrt{x^2 + y^2}}{330} = \frac{9}{990} + \frac{\sqrt{(x-a)^2 + y^2}}{330}$$



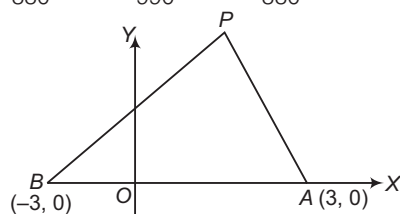
which is hyperbola.

$$76. \frac{x}{330} = \frac{600}{990} + \frac{200}{330}$$



$$\therefore x = 400 \text{ m}$$

$$77. \frac{\sqrt{(x+3)^2 + y^2}}{330} = \frac{6}{990} + \frac{\sqrt{(x-3)^2 + y^2}}{330}$$



$$\Rightarrow \sqrt{(x+3)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$$

$$\Rightarrow 8x^2 - y^2 = 8$$

$$78. \because ae = \sqrt{7}, \frac{a}{e} = \frac{16}{\sqrt{7}}$$

$$\therefore a^2 = 16$$

$$\Rightarrow a = 4$$

$$\text{Then, } e = \frac{\sqrt{7}}{4}$$

$$\text{and } b^2 = a^2(1-e^2) = 16\left(1 - \frac{7}{16}\right) = 16 - 7 = 9$$

$$\therefore \text{ Ellipse is } \frac{x^2}{16} + \frac{y^2}{9} = 1$$

79. Let $P(4\cos\phi, 3\sin\phi)$ be the point on the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ and foci are $S \equiv (\sqrt{7}, 0)$ and $S' \equiv (-\sqrt{7}, 0)$.

$$\therefore \Delta = \frac{1}{2} \times S'S \times PM = \frac{1}{2} (2\sqrt{7}) \times 3\sin\phi = 3\sqrt{7} \sin\phi \text{ sq units}$$

$\therefore$ Maximum value of Δ is $3\sqrt{7}$ sq units.

80. $\therefore$ Mid-point of A and B is (x_1, y_1) .

$$\therefore A \equiv (2x_1, 0)$$

$$\text{and } B \equiv (0, 2y_1)$$

$\therefore$ Slope of AB is m .

$$\therefore \frac{2y_1 - 0}{0 - 2x_1} = m$$

$$\Rightarrow y_1 + mx_1 = 0$$

$$\Rightarrow mx_1 + y_1 = 0$$

81. A. Points are $O(0, 0)$, $P(3, 4)$ and $Q(6, 8)$.

$$2a = OP + OQ = 5 + 10 = 15$$

$$\Rightarrow a = \frac{15}{2}$$

Also, distance between foci,

$$2ae = \sqrt{(6-3)^2 + (8-4)^2} = 5$$

$$\Rightarrow e = \frac{1}{3}$$

$$\Rightarrow b^2 = \frac{225}{4} \left(1 - \frac{1}{9}\right) = 50$$

$$\Rightarrow b = 5\sqrt{2}$$

$$\Rightarrow 2b = 10\sqrt{2}$$

B. We know that, $\frac{1}{SP} + \frac{1}{SQ} = \frac{2a}{b^2}$

$$\Rightarrow \frac{1}{2} + \frac{1}{SQ} = \frac{10}{16}$$

$$\Rightarrow SQ = 8$$

$$\Rightarrow PQ = 10$$

C. If the line $y = x + k$ touches the ellipse

$$9x^2 + 16y^2 = 144, \text{ then}$$

$$k^2 = 16(1)^2 + 9$$

$$\Rightarrow k = \pm 5$$

D. Sum of the distances of a point on the ellipse from the foci $= 2a = 8$.

82. A. Here, $a^2 = 16, b^2 = 20 \Rightarrow a^2 < b^2$

$\therefore PS + PS' = 2b = 2\sqrt{20} = 4\sqrt{5}$.

B. Given ellipse is $2x^2 + 3y^2 - 4x - 12y + 13 = 0$

$\Rightarrow \frac{(x-1)^2}{\frac{1}{2}} + \frac{(y-2)^2}{\frac{1}{3}} = 1$. Here, $a^2 = \frac{1}{2}, b^2 = \frac{1}{3}$

$\therefore e = \sqrt{1 - \frac{1}{3} \times 2} = \frac{1}{\sqrt{3}}$

C. $\left(\frac{4}{5}, -\frac{1}{5}\right)$ Any point on the line $x - y - 5 = 0$ will be of the form $(t, t - 5)$. Chord of contact of this point w.r.t. curve $x^2 + 4y^2 = 4$ is

$tx + 4(t - 5)y - 4 = 0$

$\Rightarrow (-20y - 4) + t(x + 4y) = 0$

which is a family of straight lines, each members of this family passes through point of intersection of straight lines $-20y - 4 = 0$ and $x + 4y = 0$.

D. $\frac{x^2}{4} + \frac{y^2}{3} = 1 \Rightarrow a = 2, b = \sqrt{3}$ and $e = \frac{1}{2}$

Sum of distances $= \frac{2a}{e} = \frac{4}{1/2} = 8$

83. A. Let $x - y + k = 0$ be the tangent to $\frac{x^2}{9} + \frac{y^2}{3} = 1$

$a^2l^2 + b^2m^2 = n^2 \Rightarrow 9 + 3 = k^2 \Rightarrow k = \pm \sqrt{12}$

Point of contact

$= \left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right) = \left(\frac{-9 \times 1}{\pm \sqrt{12}}, \frac{-3(-1)}{\pm \sqrt{12}}\right)$

$= \left(\mp \frac{9}{\sqrt{12}}, \mp \frac{3}{\sqrt{12}}\right) = \left(\mp \frac{3\sqrt{3}}{2}, \mp \frac{\sqrt{3}}{2}\right)$

B. $\frac{(x+1)^2}{9} + \frac{(y+2)^2}{25} = 1$

$\Rightarrow a^2 = 9, b^2 = 25$

and centre $= (-1, -2)$

Foci $= (-1, -2 \pm 4) = (-1, 2)$ and $(-1, -6)$

C. $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2$

$\frac{3x}{\cos \theta} - \frac{2y}{\sin \theta} = 5 \quad \dots (i)$

$2x + y = k \quad \dots (ii)$

Comparing Eqs. (i) and (ii), we get

$\frac{3}{2 \cos \theta} = \frac{-2}{\sin \theta} = \frac{5}{k}$

$\Rightarrow \cos^2 \theta + \sin^2 \theta = 1 \Rightarrow \frac{9}{100} k^2 + \frac{4}{25} k^2 = 1$

$\Rightarrow k = \pm 2$

If $k = 2$, then $\cos \theta = \frac{3}{5}$ and $\sin \theta = -\frac{4}{5}$

Point $= (a \cos \theta, b \sin \theta) = \left(\frac{9}{5}, -\frac{8}{5}\right)$

Other point $= \left(-\frac{9}{5}, \frac{8}{5}\right)$

D. Pole of the line $x - y + 2 = 0$ w.r.t. $\frac{x^2}{2} + \frac{y^2}{1} = 1$

$= \left(\frac{-a^2l}{n}, \frac{-b^2m}{n}\right) = \left(\frac{-2 \times 1}{2}, \frac{-1 \times (-1)}{1}\right) = \left(-1, \frac{1}{2}\right)$

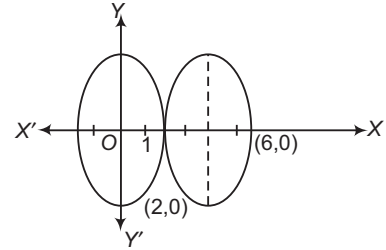
84. Let

$\Rightarrow$
and

$x - 4 = 2 \cos \theta$

$x = 2 \cos \theta + 4$

$y = 3 \sin \theta$



Now, $E = \frac{x^2}{4} + \frac{y^2}{9}$
 $= \frac{(2 \cos \theta + 4)^2}{4} + \sin^2 \theta$
 $= \frac{4 \cos^2 \theta + 16 + 16 \cos \theta + 4 \sin^2 \theta}{4}$
 $= \frac{20 + 16 \cos \theta}{4} = 5 + 4 \cos \theta$

Hence, $E_{\max} - E_{\min} = (9 - 1) = 8$

85. $\left(\pm ae, \frac{b^2}{a}\right)$ are extremities of the latusrectum having positive ordinates.

$\Rightarrow a^2 e^2 = -2 \left(\frac{b^2}{a} - 2\right) \quad \dots (i)$

But $b^2 = a^2 (1 - e^2) \quad \dots (ii)$

From Eqs. (i) and (ii), we get

$a^2 e^2 - 2ae^2 + 2a - 4 = 0$

$\Rightarrow ae^2 (a - 2) + 2(a - 2) = 0$

$\therefore (ae^2 + 2)(a - 2) = 0$

Hence, $a = 2$

86. Equation of the ellipse with centre at $(1, -1)$ can be written

as $\frac{(x-1)^2}{a^2} + \frac{(y+1)^2}{b^2} = 1$

where, a = semi-major axis and b = semi-minor axis.

If $a = 8$, then ellipse is $\frac{(x-1)^2}{64} + \frac{(y+1)^2}{b^2} = 1$.

This passes through $(1, 3)$.

$\therefore 0 + \frac{16}{b^2} = 1 \Rightarrow b^2 = 16$

Hence, equation of ellipse is $\frac{(x-1)^2}{64} + \frac{(y+1)^2}{16} = 1$.

87. Let $S \equiv (5, 12)$ and $S' \equiv (24, 7)$

$\therefore$ Conic passing through origin.

Let $P = (0, 0)$

$\therefore S'P - SP = 12$ and $S'P + SP = 38$

Distance between foci $= 2ae$

$\sqrt{386} = 2ae$

If conic is ellipse, then $S'P + SP = 38 = 2a$

$\therefore$ Eccentricity, $e = \frac{\sqrt{386}}{38}$

and if conic is hyperbola, then $S'P - SP = 12 = 2a$

$\therefore$ Eccentricity, $e = \frac{\sqrt{386}}{12} \Rightarrow k = 2$

88. Any tangent to the ellipse is $\frac{x}{67} \cos \theta + \frac{y}{33} \sin \theta = 1$.

Sum of the squares of the lengths of the perpendicular from $(0, \pm 10\sqrt{34})$ on this tangent

$$\frac{\left(\frac{10\sqrt{34}}{33} \sin \theta - 1\right)^2 + \left(\frac{10\sqrt{34}}{33} \sin \theta + 1\right)^2}{\frac{\cos^2 \theta}{(67)^2} + \frac{\sin^2 \theta}{(33)^2}}$$

$$\begin{aligned} &= \frac{2[100 \times 34 \sin^2 \theta + (33)^2]}{(33)^2 \cos^2 \theta + (67)^2 \sin^2 \theta} \\ &= \frac{2 \times (67)^2 [(67)^2 - (33)^2 \sin^2 \theta + (33)^2]}{(33)^2 \cos^2 \theta + (67)^2 \sin^2 \theta} \\ &= 2 \times (67)^2 \\ &= 8978 \\ \Rightarrow \quad l &= 2 \end{aligned}$$

Entrances Gallery

1. Let the foot of perpendicular be $P(h, k)$.

Equation of tangent with slope m and passing through $P(h, k)$ is

$$y = mx \pm \sqrt{6m^2 + 2}, \quad \text{where } m = -\frac{h}{k}$$

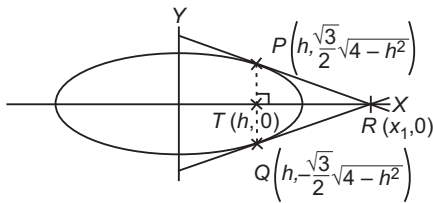
$$\Rightarrow \sqrt{\frac{6h^2}{k^2} + 2} = \frac{h^2 + k^2}{k}$$

$$\Rightarrow 6h^2 + 2k^2 = (h^2 + k^2)^2$$

So, required locus is $6x^2 + 2y^2 = (x^2 + y^2)^2$.

2. $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$$\Rightarrow y = \frac{\sqrt{3}}{2} \sqrt{4 - h^2} \text{ at } x = h$$



Let $R(x_1, 0)$

PQ is chord of contact.

So, $\frac{xx_1}{4} = 1$

$$\Rightarrow x = \frac{4}{x_1}$$

which is equation of PQ , $x = h$.

So, $\frac{4}{x_1} = h$

$$\Rightarrow x_1 = \frac{4}{h}$$

$$\Delta(h) = \text{Area of } \triangle PQR = \frac{1}{2} \times PQ \times RT$$

$$= \frac{1}{2} \times \frac{2\sqrt{3}}{2} \sqrt{4 - h^2} \times (x_1 - h) = \frac{\sqrt{3}}{2h} (4 - h^2)^{3/2}$$

$$\Delta'(h) = \frac{-\sqrt{3}(4 + 2h^2)}{2h^2} \sqrt{4 - h^2}$$

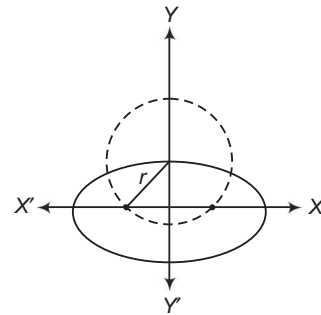
which is always decreasing.

So, $\Delta_1 = \text{Maximum of } \Delta(h) = \frac{45\sqrt{5}}{8} \text{ at } h = \frac{1}{2}$

$$\Delta_2 = \text{Minimum of } \Delta(h) = \frac{9}{2} \text{ at } h = 1$$

So, $\frac{8}{\sqrt{5}} \Delta_1 - 8\Delta_2 = \frac{8}{\sqrt{5}} \times \frac{45\sqrt{5}}{8} - 8 \cdot \frac{9}{2} = 45 - 36 = 9$

3. Given equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.



Here, $a = 4$, $b = 3$, $e = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

$\therefore$ Foci is $(\pm ae, 0) = \left(\pm 4 \times \frac{\sqrt{7}}{4}, 0\right) = (\pm \sqrt{7}, 0)$

$\therefore$ Radius of the circle, $r = \sqrt{(ae)^2 + b^2} = \sqrt{7 + 9} = \sqrt{16} = 4$

Now, equation of circle is

$$(x - 0)^2 + (y - 3)^2 = 16$$

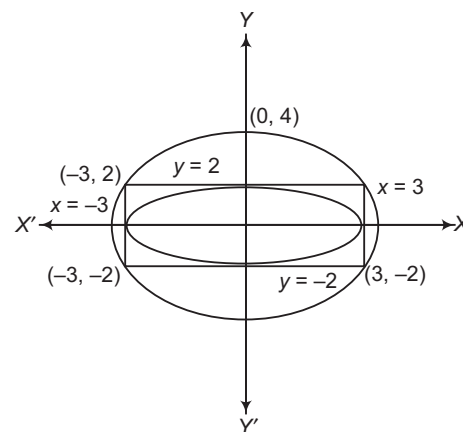
$$\Rightarrow x^2 + y^2 - 6y - 7 = 0$$

4. Equation of ellipse is $(y + 2)(y - 2) + \lambda(x + 3)(x - 3) = 0$

If passes through $(0, 4)$, then $\lambda = \frac{4}{3}$

Equation of ellipse is $\frac{x^2}{12} + \frac{y^2}{16} = 1$

$\therefore e = \frac{1}{2}$



Aliter

Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ as it is passing through (0, 4) and (3, 2).

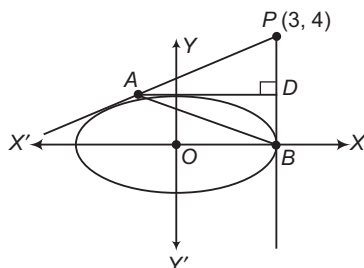
$$\text{So, } b^2 = 16 \text{ and } \frac{9}{a^2} + \frac{4}{16} = 1$$

$$\Rightarrow a^2 = 12$$

$$\text{So, } 12 = 16(1 - e^2) \Rightarrow e = 1/2$$

5. Equation of chord of contact

$$\frac{x}{9} \cdot 3 + \frac{y}{4} \cdot 4 = 1 \Rightarrow \frac{x}{3} + y = 1$$



$$\Rightarrow x = 3(1 - y)$$

Solving with ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$, we get

$$(1 - y)^2 + \frac{y^2}{4} = 1$$

$$\Rightarrow 4(y^2 + 1 - 2y) + y^2 = 4$$

$$\Rightarrow 5y^2 - 8y = 0$$

$$\Rightarrow y = 0 \text{ and } y = \frac{8}{5}$$

$$\Rightarrow x = 3 \text{ and } 3\left(1 - \frac{8}{5}\right) \Rightarrow x = 3 \text{ and } -\frac{9}{5}$$

$\therefore$ Points are (3, 0) and $\left(-\frac{9}{5}, \frac{8}{5}\right)$.

6. Slope of AD must be 0.

$$\Rightarrow y - \frac{8}{5} = 0 \left(x + \frac{9}{5}\right) \Rightarrow y = \frac{8}{5}$$

Thus, y-coordinate of D is 8/5.

Hence, y-coordinate of the orthocentre must be $\frac{8}{5}$.

7. Since, locus is the parabola.

$$\text{Equation of AB is } \frac{3x}{9} + \frac{4y}{4} = 1 \Rightarrow \frac{x}{3} + y = 1$$

$$\Rightarrow x + 3y - 3 = 0$$

$$\therefore (x - 3)^2 + (y - 4)^2 = \frac{(x + 3y - 3)^2}{10}$$

$$\Rightarrow 10x^2 + 90 - 60x + 10y^2 + 160 - 80y = x^2 + 9y^2 + 9 + 6xy - 6x - 18y$$

$$\Rightarrow 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$$

8. Diameter of circle $(x - 1)^2 + y^2 = 1$ is 2 units and that of circle $x^2 + (y - 2)^2 = 4$ is 4 units.

Semi-minor axis of ellipse, $b = 2$ units and semi-major axis of ellipse, $a = 4$ units.

Hence, equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{16} + \frac{y^2}{4} = 1$$

$$\Rightarrow x^2 + 4y^2 = 16$$

9. Statement I Standard equation of tangent with slope m to the parabola $y^2 = 16\sqrt{3}x$ is

$$y = mx + \frac{4\sqrt{3}}{m} \quad \dots (i)$$

Standard equation of tangent with slope m to the ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$ is

$$y = mx \pm \sqrt{2m^2 + 4} \quad \dots (ii)$$

If a line L is a common tangent to both parabola and ellipse, then L should be tangent to parabola i.e. its equation should be like Eq. (i). L should be tangent to ellipse i.e. its equation should be like Eq. (ii). i.e. L must be like both of the Eqs. (i) and (ii).

Hence, on comparing Eqs. (i) and (ii), we get

$$\frac{4\sqrt{3}}{m} = \pm \sqrt{2m^2 + 4}$$

On squaring both sides, we get

$$m^2(2m^2 + 4) = 48$$

$$\Rightarrow m^4 + 2m^2 - 24 = 0$$

$$\Rightarrow (m^2 + 6)(m^2 - 4) = 0$$

$$\Rightarrow m^2 = 4 \quad [\because m^2 \neq -6]$$

$$\Rightarrow m = \pm 2$$

On substituting $m = \pm 2$ in the Eq. (i), we get the required equation of the common tangents as

$$y = 2x + 2\sqrt{3}$$

and

$$y = -2x - 2\sqrt{3}$$

Hence, Statement I is correct.

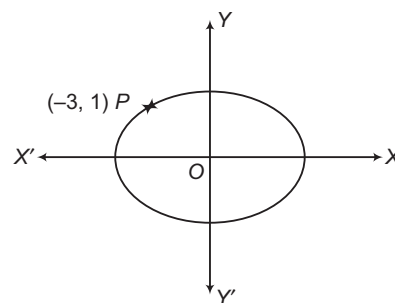
Statement II We have already seen that, if the line $y = mx + \frac{4\sqrt{3}}{m}$ is a common tangent to the parabola

$y^2 = 16\sqrt{3}x$ and the ellipse $\frac{x^2}{2} + \frac{y^2}{4} = 1$, then it satisfies the equation $m^4 + 2m^2 - 24 = 0$.

Hence, Statement II is also correct but it is not able to explain the Statement I.

10. Given, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

which passes through $P(-3, 1)$ and $e = \sqrt{\frac{2}{5}}$



$$\therefore \frac{9}{a^2} + \frac{1}{b^2} = 1 \text{ and } e^2 = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \frac{9}{a^2} + \frac{5}{3a^2} = 1$$

$$\text{and } \frac{2}{5} = 1 - \frac{b^2}{a^2}$$

$$\Rightarrow \frac{27+5}{3a^2} = 1 \quad \text{and} \quad \frac{b^2}{a^2} = \frac{3}{5}$$

$$\Rightarrow a^2 = \frac{32}{3} \quad \text{and} \quad b^2 = \frac{32}{5}$$

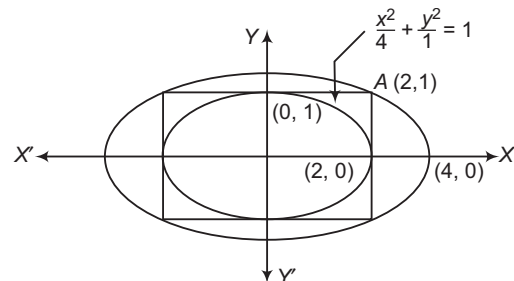
$\therefore$ Equation of ellipse is

$$\frac{3x^2}{32} + \frac{5y^2}{32} = 1$$

$$\Rightarrow 3x^2 + 5y^2 = 32$$

11. Let the equation of the required ellipse be $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$.

But the ellipse passes through the point (2, 1).



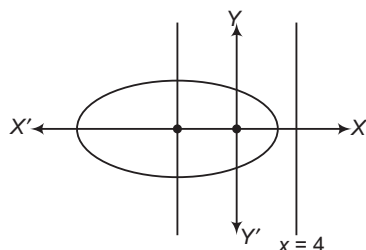
$$\Rightarrow \frac{1}{4} + \frac{1}{b^2} = 1 \Rightarrow \frac{1}{b^2} = \frac{3}{4} \Rightarrow b^2 = \frac{4}{3}$$

Hence, equation is $\frac{x^2}{16} + \frac{3y^2}{4} = 1$

$$\Rightarrow x^2 + 12y^2 = 16$$

12. Since, $\frac{a}{e} - ae = 4$ and $e = \frac{1}{2}$

$$\therefore 2a - \frac{a}{2} = 4$$



$$\Rightarrow \frac{3a}{2} = 4 \Rightarrow a = \frac{8}{3}$$

13. Given that, $2ae = 6$ and $2b = 8$

$$\Rightarrow ae = 3 \quad \text{and} \quad b = 4$$

$$\Rightarrow \frac{ae}{b} = \frac{3}{4}$$

$$\Rightarrow \frac{b^2}{a^2} = \frac{16e^2}{9}$$

$$\text{We know that, } \frac{b^2}{a^2} = 1 - e^2$$

$$\Rightarrow \frac{16e^2}{9} = 1 - e^2$$

$$\Rightarrow \left(\frac{16+9}{9}\right)e^2 = 1$$

$$\Rightarrow \left(\frac{25}{9}\right)e^2 = 1$$

$$\Rightarrow e^2 = \frac{9}{25} \Rightarrow e = \frac{3}{5}$$

14. Let the coordinates of the vertices of rectangle ABCD be

$A(a \cos \theta, b \sin \theta)$, $B(-a \cos \theta, b \sin \theta)$,

$C(-a \cos \theta, -b \sin \theta)$ and $D(a \cos \theta, -b \sin \theta)$, then

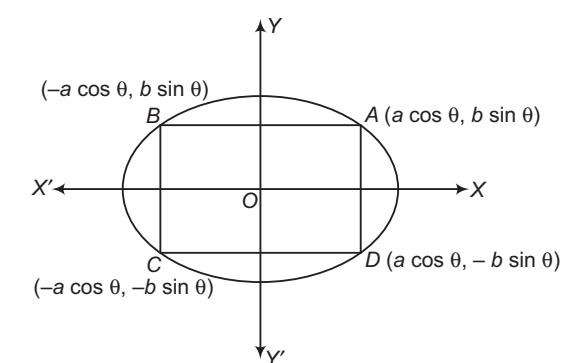
Length of rectangle, $AB = 2a \cos \theta$

and breadth of rectangle, $AD = 2b \sin \theta$

$\therefore$ Area of rectangle, $A = AB \times AD$

$$= 2a \cos \theta \times 2b \sin \theta$$

$$A = 2ab \sin 2\theta \quad \dots(i)$$



On differentiating Eq. (i) w.r.t. θ , we get

$$\frac{dA}{d\theta} = 2 \times 2ab \cos 2\theta$$

For maxima or minima, put $\frac{dA}{d\theta} = 0$

$$\Rightarrow 4ab \cos 2\theta = 0$$

$$\Rightarrow 2\theta = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{4}$$

$$\text{Now, } \frac{d^2A}{d\theta^2} = -8ab \sin 2\theta$$

$$\text{At } \theta = \frac{\pi}{4}, \left(\frac{d^2A}{d\theta^2}\right) < 0$$

$$\therefore \text{Area is maximum at } \theta = \frac{\pi}{4}.$$

Maximum area of rectangle = $2ab$ sq units

[from Eq. (i)]

Aliter

From Eq. (i),

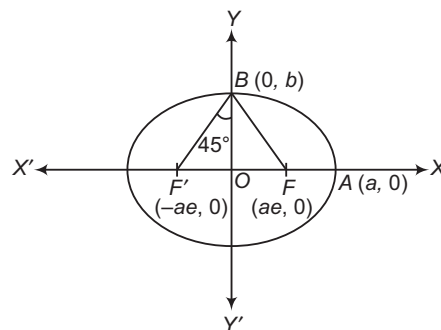
$$\text{Area of rectangle, } A = 2ab \sin 2\theta$$

$\therefore A$ is maximum, when $\sin 2\theta = 1$

$\Rightarrow$ Maximum area of rectangle = $2ab$ sq units

15. Since, $\angle FBF' = 90^\circ$, then

$$\angle OBF' = 45^\circ \quad \text{and} \quad \angle BF'O = 45^\circ$$



$$\Rightarrow ae = b \quad [\because \triangle BOF' \text{ is an isosceles triangle}]$$

$$\text{and } e^2 = 1 - \frac{b^2}{a^2} \Rightarrow e^2 = 1 - \frac{a^2 e^2}{a^2}$$

$$\Rightarrow e^2 = 1 - e^2$$

$$\Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}} \quad [\because e \text{ cannot be negative}]$$

Aliter

Since, F and F' are foci of an ellipse, whose coordinates are $(ae, 0)$ and $(-ae, 0)$ respectively and coordinates of B are $(0, b)$.

$$\therefore \text{Slope of } BF = \frac{b}{-ae}$$

$$\text{and Slope of } BF' = \frac{b}{ae}$$

$$\therefore \angle FBF' = 90^\circ$$

$$\therefore -\frac{b}{ae} \cdot \left(\frac{b}{ae}\right) = -1 \Rightarrow b^2 = a^2 e^2$$

$$\therefore e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{a^2 e^2}{a^2}$$

$$\Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}}$$

16. Since, equation of directrix is $x = 4$, then major axis of an ellipse is along X -axis.

$$\therefore \frac{a}{e} = 4 \Rightarrow a = 4 \times \frac{1}{2} \quad \left[\because e = \frac{1}{2} \right]$$

$$\Rightarrow a = 2$$

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$\therefore b^2 = 4 \left(1 - \frac{1}{4}\right) = 4 \times \frac{3}{4}$$

$$\Rightarrow b^2 = 3$$

$$\text{Hence, equation of ellipse is } \frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\Rightarrow 3x^2 + 4y^2 = 12$$

17. The equation of an ellipse is

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

$$\text{Here, } a = 4, b = 3$$

$$\text{Eccentricity, } e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$$

$$\text{Foci of an ellipse are } (\pm ae, 0) \text{ i.e. } (\pm \sqrt{7}, 0).$$

$$\therefore \text{Radius of required circle} = \sqrt{(\sqrt{7} - 0)^2 + (0 - 3)^2} = \sqrt{7 + 9} = \sqrt{16} = 4 \text{ units}$$

18. Given that, $e = \frac{1}{2}$, $ae = 2 \Rightarrow a = 4$

$$\therefore b^2 = a^2(1 - e^2) = 16 \left(1 - \frac{1}{4}\right) = 12$$

Hence, the equation of ellipse are

$$\frac{x^2}{16} + \frac{y^2}{12} = 1$$

19. Given equation of ellipse can be rewritten as

$$\frac{(x+1)^2}{1} + \frac{(y-1)^2}{4} = 1$$

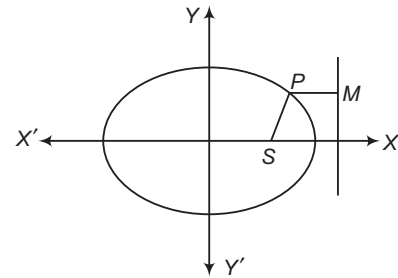
$\therefore$ Parametric equations of ellipse are

$$x+1 = \cos \theta \quad \text{and} \quad y-1 = 2 \sin \theta$$

$$\Rightarrow x = \cos \theta - 1 \quad \text{and} \quad y = 2 \sin \theta + 1$$

20. Given, $PS = 6$ and $e = \frac{3}{5}$

$$\therefore \frac{PS}{PM} = e \Rightarrow \frac{6}{PM} = \frac{3}{5}$$



$$\therefore PM = \frac{6 \times 5}{3} = 10$$

21. The y -coordinate of foci is zero.

$\therefore$ Major axis is on X -axis.

$$\therefore ae = 4$$

$$\text{Let equation of ellipse be } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\therefore \frac{(4\sqrt{2})^2}{a^2} + \frac{(2\sqrt{6})^2}{a^2 - 16} = 1$$

$$[\because b^2 = a^2(1 - e^2) = a^2 - 16]$$

$$\Rightarrow \frac{32}{a^2} + \frac{24}{a^2 - 16} = 1$$

$$\Rightarrow 32a^2 - 512 + 24a^2 = a^2(a^2 - 16)$$

$$\Rightarrow 56a^2 - 512 = a^4 - 16a^2$$

$$\Rightarrow a^4 - 72a^2 + 512 = 0$$

$$\Rightarrow a^4 - 64a^2 - 8a^2 + 512 = 0$$

$$\Rightarrow a^2(a^2 - 64) - 8(a^2 - 64) = 0$$

$$\Rightarrow (a^2 - 8)(a^2 - 64) = 0$$

$$\Rightarrow a^2 = 64 \Rightarrow a = 8$$

$$[\because a^2 = 8 \text{ is not possible}]$$

$$\therefore ae = 4$$

$$\Rightarrow 8 \times e = 4$$

$$\therefore e = \frac{1}{2}$$

22. Equation of ellipse is $9x^2 + 25y^2 = 225$ or $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$$\text{Here, } a = 5, b = 3$$

$$\text{Now, } PF_1 = 4 PF_2$$

$$\therefore PF_1 + PF_2 = 5 PF_2, 2a = 5 PF_2$$

$$\Rightarrow 10 = 5 PF_2$$

$$\Rightarrow 2 = PF_2$$

$$\text{Now, } c^2 = a^2 - b^2 = 25 - 9 = 16$$

$$\therefore F_1 \equiv (4, 0), F_2 \equiv (-4, 0)$$

Let the coordinates of P be $(5 \cos \theta, 3 \sin \theta)$.

$$\therefore PF_2 = 2$$

$$\Rightarrow (5 \cos \theta + 4)^2 + (3 \sin \theta)^2 = 4$$

$$\Rightarrow 16 \cos^2 \theta + 40 \cos \theta + 21 = 0$$

$$\Rightarrow (4 \cos \theta + 3)(4 \cos \theta + 7) = 0$$

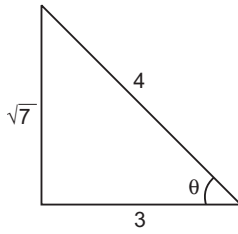
$$\Rightarrow \cos \theta = -\frac{3}{4}$$

$$\text{or } \cos \theta = -\frac{7}{4}$$

[rejected]

and

$$\sin \theta = \frac{\sqrt{7}}{4}$$



$$\therefore P \equiv \left(-\frac{15}{4}, \frac{3\sqrt{7}}{4} \right) \equiv \left(-\frac{15}{4}, \frac{\sqrt{63}}{4} \right)$$

23. Given equation of ellipse is

$$\frac{x^2}{144} + \frac{y^2}{25} = 1$$

Here, $a^2 = 144$

and $b^2 = 25$

$$\begin{aligned} \text{Now, } e &= \sqrt{1 - \frac{b^2}{a^2}} \\ &= \sqrt{1 - \frac{25}{144}} \\ &= \sqrt{\frac{119}{144}} = \frac{\sqrt{119}}{12} \end{aligned}$$

$$\begin{aligned} \therefore \text{Foci of an ellipse} &= (\pm ae, 0) \\ &= \left(\pm 12 \times \frac{\sqrt{119}}{12}, 0 \right) \\ &= (\pm \sqrt{119}, 0) \end{aligned}$$

Since, the circle with centre $(0, \sqrt{2})$ and passing through foci $(\pm \sqrt{119}, 0)$ of the ellipse.

$$\begin{aligned} \therefore \text{Radius of circle} &= \sqrt{(\sqrt{119} - 0)^2 + (0 - \sqrt{2})^2} \\ &= \sqrt{119 + 2} \\ &= \sqrt{121} = 11 \end{aligned}$$

24. The given equation of second ellipse can be rewritten as

$$\frac{x^2}{4} + \frac{y^2}{1} = 1$$

Equation of tangent to this ellipse is

$$\frac{x}{2} \cos \theta + y \sin \theta = 1 \quad \dots (i)$$

Equation of the first ellipse can be rewritten as

$$\frac{x^2}{6} + \frac{y^2}{3} = 1 \quad \dots (ii)$$

Let Eq. (i) meets the first ellipse at P and Q and the tangents at P and Q to the first ellipse intersected at (h, k) , then Eq. (i) is the chord of contact of (h, k) , with respect to the ellipse (ii) and thus its equation is

$$\frac{hx}{6} + \frac{ky}{3} = 1 \quad \dots (iii)$$

Since, Eqs. (i) and (iii) represent the same line.

$$\frac{h/6}{\cos \theta} = \frac{k/3}{\sin \theta} = 1$$

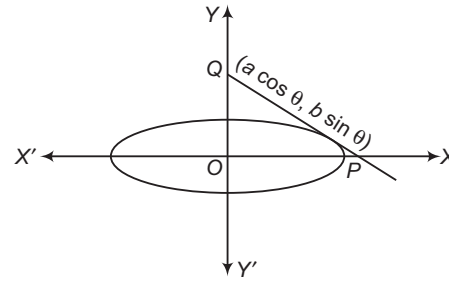
$$\Rightarrow h = 3 \cos \theta$$

$$\text{and } k = 3 \sin \theta$$

Hence, locus is $x^2 + y^2 = 9$.

25. Equation of tangent at $(a \cos \theta, b \sin \theta)$ to the ellipse is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$



Coordinates of P and Q are $\left(\frac{a}{\cos \theta}, 0 \right)$ and $\left(0, \frac{b}{\sin \theta} \right)$, respectively.

$$\text{Now, area of } \Delta OPQ = \frac{1}{2} \left| \frac{a}{\cos \theta} \times \frac{b}{\sin \theta} \right| = \frac{ab}{|\sin 2\theta|}$$

$$\therefore \text{Minimum area} = ab$$

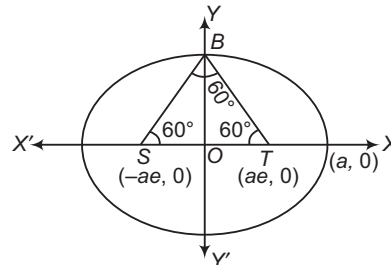
26. Given circle is a director circle to the given ellipse.

$$\therefore \text{Angle between tangent is } \theta = \frac{\pi}{2}$$

$$27. \because 2a = 3 \cdot 2 \cdot b \Rightarrow a = 3b$$

$$\therefore l = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{(3b)^2 - b^2}{(3b)^2}} = \sqrt{\frac{8b^2}{9b^2}} = \frac{2\sqrt{2}}{3}$$

$$28. \text{In } \Delta BOT, \frac{b}{ae} = \tan 60^\circ \Rightarrow b = ae\sqrt{3}$$



$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{a^2 e^2 3}{a^2}}$$

$$\therefore e^2 = 1 - 3e^2 \Rightarrow 4e^2 = 1$$

$$\Rightarrow e = \pm \frac{1}{2} \Rightarrow e = \frac{1}{2}$$

[since, e cannot be negative]

$$29. \text{We have, } x = 2y \quad \dots (i)$$

$$\text{and } \frac{x^2}{4} + y^2 = 1 \quad \dots (ii)$$

On putting $x = 2y$ in Eq. (ii), we get

$$\frac{(2y)^2}{4} + y^2 = 1$$

$$\Rightarrow \frac{4y^2}{4} + y^2 = 1$$

$$\Rightarrow 2y^2 = 1$$

$$\Rightarrow y = \pm \frac{1}{\sqrt{2}}$$

From Eq. (i), $x = \pm \sqrt{2}$

$$\therefore P \left(\sqrt{2}, \frac{1}{\sqrt{2}} \right) \text{ and } Q \left(-\sqrt{2}, -\frac{1}{\sqrt{2}} \right)$$

∴ Equation of circle is

$$(x - \sqrt{2})(x + \sqrt{2}) + \left(y - \frac{1}{\sqrt{2}}\right)\left(y + \frac{1}{\sqrt{2}}\right) = 0$$

$$\Rightarrow x^2 - 2 + y^2 - \frac{1}{2} = 0$$

$$\Rightarrow x^2 + y^2 = 5/2$$

30. Given, $\frac{x^2}{9} + \frac{y^2}{1} = 1$

$$\Rightarrow e = \sqrt{1 - \frac{1}{9}} = \frac{2\sqrt{2}}{3}$$

Two foci are $(\pm ae, 0)$ i.e. $(\pm 2\sqrt{2}, 0)$.

Let $P(h, k)$ be any point on the ellipse.

$$\therefore \frac{k-0}{h-2\sqrt{2}} \times \frac{k-0}{h+2\sqrt{2}} = -1 \quad [\text{from given condition}]$$

$$\Rightarrow h^2 - 8 = -k^2$$

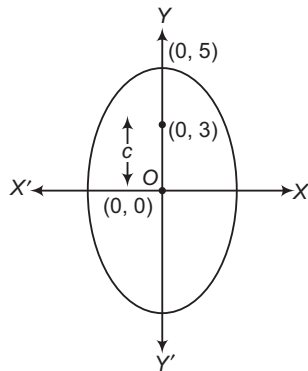
$$\Rightarrow x^2 + y^2 = 8$$

31. We have, $b = 5$ and $b > a$

Since, $c^2 = b^2 - a^2$

$$\Rightarrow 9 = 25 - a^2$$

$$\Rightarrow a^2 = 16$$



∴ Required equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

32. We have, $16x^2 + 25y^2 = 400$

$$\Rightarrow \frac{16x^2}{400} + \frac{25y^2}{400} = 1$$

$$\Rightarrow \frac{x^2}{25} + \frac{y^2}{16} = 1$$

$$\begin{aligned} \therefore \text{Length of latusrectum} &= \frac{2b^2}{a} \\ &= \frac{2 \times 16}{5} \\ &= \frac{32}{5} \text{ units} \end{aligned}$$

33. Here, $a = 5, b = 4$

∴ Required sum $= a + b = 9$

34. According to the question,

$$2a = \frac{17}{8} \cdot 2b \Rightarrow a = \frac{17}{8}b$$

$$\therefore b^2 = a^2 (1 - e^2)$$

$$\Rightarrow b^2 = \frac{289}{64} b^2 (1 - e^2)$$

$$\Rightarrow 1 - e^2 = \frac{64}{289}$$

$$\Rightarrow e^2 = \frac{225}{289}$$

$$\therefore e = \frac{15}{17}$$

35. Equation of tangent at $P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ is

$$\frac{x \cdot a}{\sqrt{2}a^2} + \frac{y \cdot b}{\sqrt{2}b^2} = 1$$

$$\Rightarrow \frac{x}{\sqrt{2}a} + \frac{y}{\sqrt{2}b} = 1$$

$$\Rightarrow xb + ya = \sqrt{2}ab \quad \dots (i)$$

Equation of normal at $P\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$ is

$$\frac{a^2 \cdot x}{a} - \frac{b^2 \cdot y}{b} = a^2 - b^2$$

$$\Rightarrow \sqrt{2}ax - \sqrt{2}by = a^2 - b^2$$

$$\Rightarrow \sqrt{2}(ax - by) = a^2 - b^2 \quad \dots (ii)$$

Also, equation of X-axis is $y = 0$ (iii)

Now, from Eqs. (i) and (iii),

$$xb = \sqrt{2}ab$$

$$\Rightarrow x = \sqrt{2}a$$

∴ Point of intersection is $(\sqrt{2}a, 0)$.

Again, from Eqs. (ii) and (iii), we get

$$\sqrt{2}ax = a^2 - b^2$$

$$\Rightarrow x = \frac{a^2 - b^2}{\sqrt{2}a}$$

∴ Point of intersection is $\left(\frac{a^2 - b^2}{\sqrt{2}a}, 0\right)$.

On solving Eqs. (i) and (ii), we get

$$y = \frac{b}{\sqrt{2}} \quad \text{and} \quad x = \frac{a}{\sqrt{2}}$$

∴ Point of intersection is $\left(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\right)$.

$$\therefore \text{Area of triangle} = \frac{1}{2} \begin{vmatrix} \sqrt{2}a & 0 & 1 \\ \frac{a^2 - b^2}{\sqrt{2}a} & 0 & 1 \\ \frac{a}{\sqrt{2}} & \frac{b}{\sqrt{2}} & 1 \end{vmatrix}$$

$$= \frac{1}{2} \left[\sqrt{2}a \left(-\frac{b}{\sqrt{2}} \right) + 1 \left(\frac{a^2 - b^2}{\sqrt{2}a} \cdot \frac{b}{\sqrt{2}} \right) \right]$$

$$= \frac{1}{2} \left[-\frac{\sqrt{2}ab}{\sqrt{2}} + \frac{b}{2a} (a^2 - b^2) \right]$$

$$= \left| \frac{-b(a^2 + b^2)}{4a} \right| = \frac{b(a^2 + b^2)}{4a}$$

16

Hyperbola

Introduction

A hyperbola is the locus of a point in a plane, which moves in the plane in such a way that the ratio of its distance from a fixed point (called focus) to its distance from a fixed line (called directrix) is always constant which is always greater than unity.

The constant ratio is generally denoted by e and is known as the eccentricity of the hyperbola.

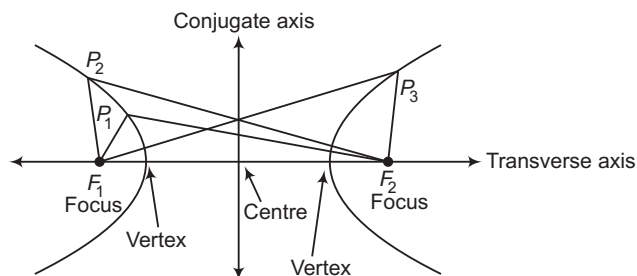
If S is the focus, ZZ' is the directrix and P is any point on the hyperbola as shown in the adjacent figure then by definition, we have

$$\frac{SP}{PM} = e \Rightarrow SP = e PM$$

Or

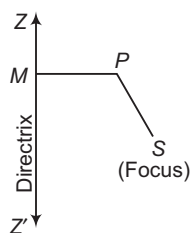
The hyperbola is the set of all points in a plane, the difference of whose distance from two fixed points in the plane is a constant.

The term '*difference*' that means the distance to the farther point minus the distance to the closer point. The two fixed points are called the foci of the hyperbola. The mid-point of the line segment joining the foci is called the centre of the hyperbola. The line through the foci is called the transverse axis and the line through the centre and perpendicular to the transverse axis is called the conjugate axis. The points at which the hyperbola intersects the transverse axis are called the vertices of the hyperbola.



$$P_1F_2 - P_1F_1 = P_2F_2 - P_2F_1 = P_3F_1 - P_3F_2$$

We denote the distance between the two foci by $2c$, the distance between the two vertices (the length of the transverse axis) by $2a$ and we define the quantity b as $b = \sqrt{c^2 - a^2}$.



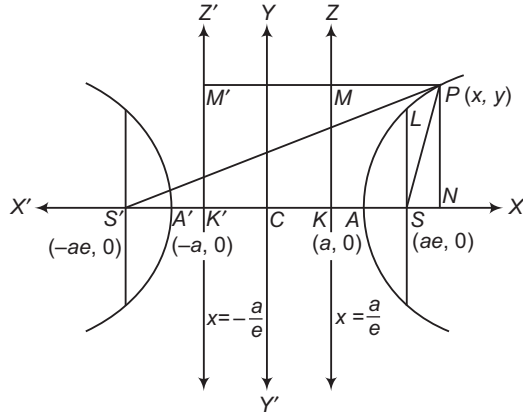
Chapter Snapshot

- Introduction
- Conjugate Hyperbola
- Position of a Point with Respect to a Hyperbola
- Intersection of a Line and a Hyperbola
- Tangent to a Hyperbola
- Director Circle
- Normals to a Hyperbola
- Equation of the Pair of Tangents
- Equations of Chord
- Pole and Polar
- Conjugate points
- Conjugate Lines
- Diameter
- Asymptotes
- Rectangular Hyperbola
- Tangent to a Rectangular Hyperbola
- Normals to a Rectangular Hyperbola

Standard Equation of Hyperbola

In this section, we shall derive the equation of the hyperbola in the standard form for which we have to choose its focus and directrix in a specific way as given below:

Let S be the focus, ZK be the directrix and e be the eccentricity of the hyperbola whose equation is required. Draw SK perpendicular from S on the directrix ZK and divide SK internally and externally at A and A' (on SK produced) respectively in the ratio $e:1$.



$$\begin{aligned} \text{Then,} \quad SA &= e AK & \dots(i) \\ \text{and} \quad SA' &= e A'K & \dots(ii) \end{aligned}$$

Since, A and A' are such points that their distances from the focus bear constant ratio $e (>1)$ to their respective distances from the directrix. Therefore, these points lie on the hyperbola.

Let $AA' = 2a$ and C be the middle point of AA' . Then,
 $CA = CA' = a$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} SA + SA' &= e (AK + A'K) \\ \Rightarrow CS - CA + CS + CA' &= e (CA' - CK + CA' + CK) \\ \Rightarrow 2CS &= 2ae \\ \Rightarrow CS &= ae \end{aligned}$$

On subtracting Eq. (i) from Eq. (ii), we get

$$\begin{aligned} SA' - SA &= e (A'K - AK) \\ \Rightarrow (CS + CA') - (CS - CA) &= e (CA' + CK - CA + CK) \\ \Rightarrow AA' &= 2e (CK) \\ \Rightarrow 2a &= 2e (CK) \\ \Rightarrow CK &= \frac{a}{e} \end{aligned}$$

Let C be the origin, CSX be the X -axis and a straight line CY through C perpendicular to CX as the Y -axis. Let $P(x, y)$ be any point on the hyperbola and PM, PN be the perpendiculars from P on KZ and KX . By definition, we have

$$SP = e PM$$

$$\begin{aligned} \Rightarrow SP^2 &= e^2 PM^2 \\ \Rightarrow SP^2 &= e^2 KN^2 \\ \Rightarrow SP^2 &= e^2 (CN - CK)^2 \\ \Rightarrow (x - ae)^2 + y^2 &= e^2 \left(x - \frac{a}{e} \right)^2 \\ \Rightarrow x^2 (e^2 - 1) - y^2 &= a^2 (e^2 - 1) \\ \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} &= 1 \\ \Rightarrow \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \end{aligned}$$

$$\text{where,} \quad b^2 = a^2 (e^2 - 1)$$

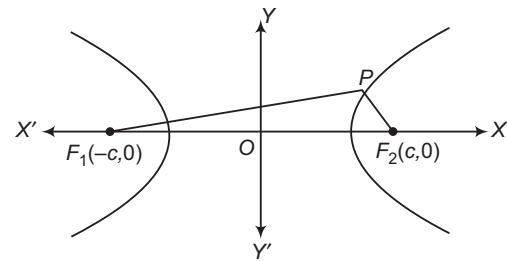
This is the equation of the hyperbola in the standard form.

Or

Let $(\pm c, 0)$ be the foci of a hyperbola. Then, its centre is $(0, 0)$. Let $P(x, y)$ be any point on the hyperbola.

According to the definition of hyperbola,

$$PF_1 - PF_2 = 2a \quad (2a < 2c, \text{ i.e. } c > a)$$



$$\begin{aligned} \Rightarrow \sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} &= 2a \\ \text{i.e.} \quad \sqrt{(x+c)^2 + y^2} &= 2a + \sqrt{(x-c)^2 + y^2} \end{aligned}$$

On squaring both sides, we get

$$(x+c)^2 + y^2 = 4a^2 + 4a\sqrt{(x-c)^2 + y^2} + (x-c)^2 + y^2$$

On simplifying, we get

$$\frac{cx}{a} - a = \sqrt{(x-c)^2 + y^2}$$

On squaring again and further simplifying, we get

$$\begin{aligned} \frac{x^2}{a^2} - \frac{y^2}{c^2 - a^2} &= 1 \\ \text{i.e.} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1 \quad [\because c^2 - a^2 = b^2] \end{aligned}$$

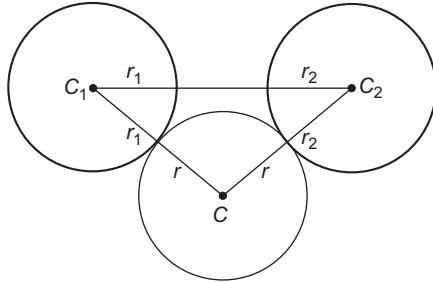
Hence, any point on the hyperbola satisfies the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

➤ **Example 1.** Two circles are given such that they neither intersect nor touch. The locus of centre of variable circle which touches both the circles externally, is

- (a) circle (b) parabola
(c) ellipse (d) hyperbola

Sol. (d) In the figure, circles with solid line have centres C_1 and C_2 and radii r_1 and r_2 , respectively.



Let the circle with centre C and radius r is a variable circle which touches both circles externally.

Now, $CC_1 = r + r_1$

and $CC_2 = r + r_2$

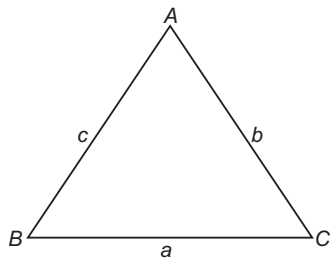
Hence, $CC_1 - CC_2 = r_1 - r_2$ [constant]

Then, by definition, the locus of C is hyperbola, whose foci are C_1 and C_2 .

➤ **Example 2.** If base of triangle and ratio of tangents of half of base angles are given, then the locus of opposite vertex is

- (a) hyperbola (b) ellipse
(c) parabola (d) circle

Sol. (a) In $\triangle ABC$, base $BC = a$ [given]



Also, $\frac{\tan \frac{B}{2}}{\tan \frac{C}{2}} = k \Rightarrow \frac{\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}} = k$

$\Rightarrow \frac{s-c}{s-b} = k$

$\Rightarrow \frac{(s-c) - (s-b)}{(s-c) + (s-b)} = \frac{k-1}{k+1}$

$\Rightarrow \frac{b-c}{2s-(b+c)} = \frac{k-1}{k+1}$

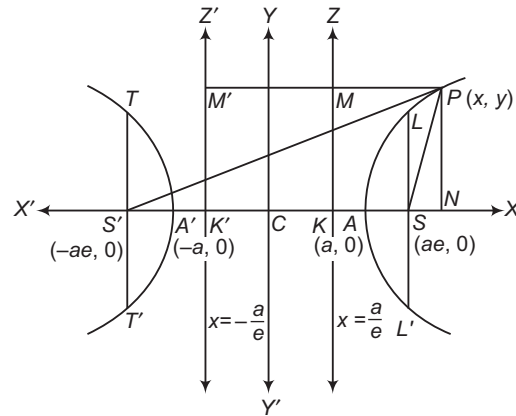
$\Rightarrow b-c = a \frac{(k-1)}{(k+1)}$

$\Rightarrow AC - AB = \text{Constant}$

So, locus of vertex of $\triangle ABC$ is a hyperbola.

Some Basic Terms Associated to Hyperbola

In this section, we shall define various terms associated to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as given below:



Vertices The points A and A' , where the curve meets the line joining the foci S and S' , are called the vertices of the hyperbola. The coordinates of A and A' are $(a, 0)$ and $(-a, 0)$, respectively.

Transverse and conjugate axes The straight line joining the vertices A and A' is called the transverse axis of the hyperbola. Its length AA' is generally taken to be $2a$.

The straight line through the centre which is perpendicular to the transverse axis does not meet the hyperbola in real points. But if B, B' are the points on this line such that $CB = CB' = b$, the line BB' is called the conjugate axis such that $BB' = 2b$.

Foci The points $S(ae, 0)$ and $S'(-ae, 0)$ are the foci of the hyperbola.

Directrices ZK and $Z'K'$ are two directrices of the hyperbola and their equations are $x = \frac{a}{e}$ and $x = -\frac{a}{e}$, respectively.

Centre The middle point C of AA' bisects every chord of the hyperbola passing through it and is called the centre of the hyperbola.

Eccentricity For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have

$$\begin{aligned} b^2 &= a^2(e^2 - 1) \\ \Rightarrow e^2 &= \frac{a^2 + b^2}{a^2} = 1 + \frac{b^2}{a^2} \\ \Rightarrow e &= \sqrt{1 + \frac{b^2}{a^2}} \Rightarrow e = \sqrt{1 + \frac{(2b)^2}{(2a)^2}} \\ \Rightarrow e &= \sqrt{1 + \frac{(\text{Conjugate axis})^2}{(\text{Transverse axis})^2}} \end{aligned}$$

Latusrectum LSL' is the latusrectum and LS is called the semi-latusrectum. $TS'T'$ is also a latusrectum.

The coordinates of L are (ae, SL) . As L lies on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the coordinates of L will satisfy the equation of the hyperbola. Therefore,

$$\begin{aligned} \frac{(ae)^2}{a^2} - \frac{(SL)^2}{b^2} &= 1 \\ \Rightarrow (SL)^2 &= b^2(e^2 - 1) \\ \Rightarrow (SL)^2 &= b^2 \left(\frac{b^2}{a^2} \right) \\ \left[\because b^2 &= a^2(e^2 - 1) \Rightarrow e^2 - 1 = \frac{b^2}{a^2} \right] \end{aligned}$$

$$\Rightarrow SL = \frac{b^2}{a}$$

$$\Rightarrow SL = SL' = \frac{b^2}{a}$$

Hence, length of the latusrectum $= 2(SL)$

$$= \frac{2b^2}{a} = 2a(e^2 - 1)$$

Focal distances of a point Let $P(x, y)$ be any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Then, by definition, we have

$$SP = e \cdot PM$$

and

$$S'P = e \cdot PM'$$

$\Rightarrow$

$$\begin{aligned} SP &= e \cdot NK = e(CN - CK) \\ &= e \left(x - \frac{a}{e} \right) = ex - a \end{aligned}$$

and

$$\begin{aligned} S'P &= e \cdot PM' = e(NK') = e(CN + CK') \\ &= e \left(x + \frac{a}{e} \right) = ex + a \end{aligned}$$

$$\begin{aligned} \therefore S'P - SP &= (ex + a) - (ex - a) = 2a \\ &= \text{Transverse axis} \end{aligned}$$

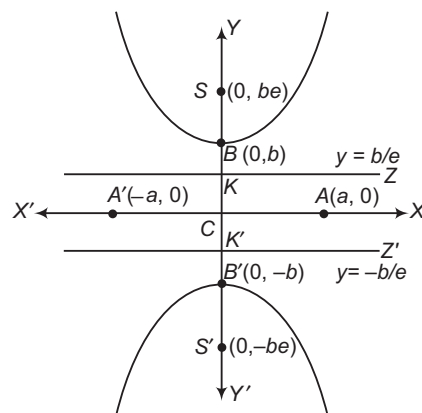
Conjugate Hyperbola

The hyperbola whose transverse and conjugate axes are respectively the conjugate and transverse axes of a given hyperbola, is called the conjugate hyperbola of the given hyperbola.

The hyperbola conjugate to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Its shape is shown in the figure.



The eccentricity of the conjugate hyperbola is given by

$$a^2 = b^2(e^2 - 1)$$

and length of the latusrectum is $\frac{2a^2}{b}$.

Various results related to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

and its conjugate $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are given in the following table for ready reference:

Hyperbola	Hyperbola	Conjugate hyperbola
Terms	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
Coordinates of the centre	(0, 0)	(0, 0)
Coordinates of the vertices	(a, 0) and (-a, 0)	(0, b) and (0, -b)
Coordinates of foci	(±ae, 0)	(0, ±be)
Length of transverse axis	2a	2b
Length of conjugate axis	2b	2a
Equation of the directrices	$x = \pm \frac{a}{e}$	$y = \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$ or $b^2 = a^2(e^2 - 1)$	$e = \sqrt{\frac{b^2 + a^2}{b^2}}$ or $a^2 = b^2(e^2 - 1)$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of transverse axis	y = 0	x = 0
Equation of the conjugate axis	x = 0	y = 0

• If e_1 and e_2 are the eccentricities of a hyperbola and its conjugate respectively, then

$$\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1$$

• The foci of a hyperbola and its conjugate are concyclic and form the vertices of a square.

Equation of Hyperbola whose Axes are Parallel to Coordinates Axes and Centre is (h, k)

Equation of such hyperbola is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

Various results related to hyperbola $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ and its conjugate

$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$ are given in the following table:

Hyperbola	Hyperbola	Conjugate hyperbola
Terms	$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
Coordinates of the centre	(h, k)	(h, k)
Coordinates of the vertices	$(h \pm a, k)$	$(h, k \pm b)$
Coordinates of foci	$(h \pm ae, k)$	$(h, k \pm be)$
Length of transverse axis	$2a$	$2b$
Length of conjugate axis	$2b$	$2a$
Equation of directrices	$x = h \pm \frac{a}{e}$	$y = k \pm \frac{b}{e}$
Eccentricity	$e = \sqrt{\frac{a^2 + b^2}{a^2}}$	$e = \sqrt{\frac{a^2 + b^2}{b^2}}$
Length of latusrectum	$\frac{2b^2}{a}$	$\frac{2a^2}{b}$
Equation of transverse axis	$y = k$	$x = h$
Equation of conjugate axis	$x = h$	$y = k$

➤ **Example 3.** The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is

(a) $x^2 - y^2 = 32$

(b) $9x^2 - 4y^2 = 36$

(c) $2x^2 - 3y^2 = 7$

(d) None of the above

Sol. (a) Let the equation of the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The coordinates of the foci are $(ae, 0)$ and $(-ae, 0)$.

$\therefore 2ae = 16 \Rightarrow 2a \times \sqrt{2} = 16$ [$\because e = \sqrt{2}$]

$\Rightarrow a = 4\sqrt{2}$

Also, $b^2 = (ae)^2 - a^2 = 64 - 32 = 32$

Thus, $a^2 = 32$

and $b^2 = 32$

Hence, the equation of hyperbola is $x^2 - y^2 = 32$.

➤ **Example 4.** If the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

coincide, then the value of b^2 is

(a) 18

(b) -16

(c) 16

(d) -18

Sol. (c) The foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{b^2} = 1$ are $(\pm 5e, 0)$, where

$$b^2 = 25(1 - e^2) \quad \dots (i)$$

The foci of the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$

i.e. $\frac{x^2}{\frac{144}{25}} - \frac{y^2}{\frac{81}{25}} = 1$ are given by $(\pm \frac{12}{5}e', 0)$, where

$$\frac{81}{25} = \frac{144}{25}(e'^2 - 1)$$

$$\Rightarrow 81 = 144(e'^2 - 1)$$

$$\Rightarrow e'^2 - 1 = \frac{81}{144}$$

$$\Rightarrow e'^2 = 1 + \frac{81}{144}$$

$$= \frac{225}{144}$$

$$\Rightarrow e' = \frac{15}{12} = \frac{5}{4}$$

Given, $5e = \frac{12}{5}e'$

$$\Rightarrow 5e = \frac{12}{5} \times \frac{5}{4} = 3$$

$$\Rightarrow e = \frac{3}{5}$$

From Eq. (i),

$$b^2 = 25 \left(1 - \frac{3^2}{5^2} \right)$$

$$= \frac{25 \times 16}{25}$$

$$= 16$$

Parametric Coordinates

Let $P(x, y)$ be any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Draw PL perpendicular from P on OX and then a tangent LM from L to the circle described on $A'A$ as diameter. Then,

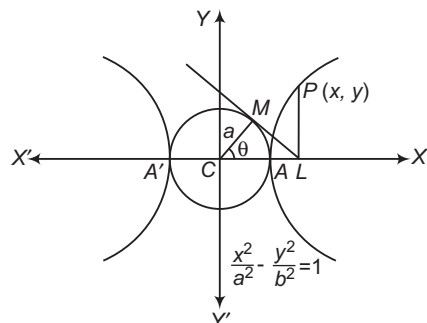
$$x = CL = CM \sec \theta = a \sec \theta$$

On putting $x = a \sec \theta$ in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we obtain

$y = b \tan \theta$. Thus, the coordinates of any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $(a \sec \theta, b \tan \theta)$, where θ is a

parameter such that $0 \leq \theta \leq 2\pi$.

These coordinates are known as the parametric coordinates. The parameter θ is also called as the eccentric angle of point P on the hyperbola.



The equation $x = a \sec \theta$ and $y = b \tan \theta$ are known as the parametric equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

- The circle $x^2 + y^2 = a^2$ is known as the auxiliary circle of the hyperbola.
- The equations $x = a \left(\frac{e^\theta + e^{-\theta}}{2} \right)$ and $y = b \left(\frac{e^\theta - e^{-\theta}}{2} \right)$ are also known as the parametric equations of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

➤ **Example 5.** If equation of hyperbola is $\frac{x^2}{25} - \frac{y^2}{16} = 1$, then equation of auxiliary circle is

- (a) $x^2 + y^2 = 41$ (b) $x^2 + y^2 = 9$
(c) $x^2 + y^2 = 25$ (d) $x^2 + y^2 = 16$

Sol. (c) Given equation of hyperbola is

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

Here, foci lie on X-axis.

$$\therefore a^2 = 25 \text{ and } b^2 = 16$$

Hence, equation of auxiliary circle is

$$x^2 + y^2 = 25$$

➤ **Example 6.** Any point on the hyperbola

$$\frac{(x+1)^2}{16} - \frac{(y-2)^2}{4} = 1 \text{ is of the form}$$

- (a) $(4 \sec \theta, 2 \tan \theta)$ (b) $(4 \sec \theta - 1, 2 \tan \theta + 2)$
(c) $(4 \sec \theta - 1, 2 \tan \theta - 2)$ (d) $(4 \sec \theta - 4, 2 \tan \theta - 2)$

Sol. (b) Given equation of hyperbola is

$$\frac{(x+1)^2}{16} - \frac{(y-2)^2}{4} = 1$$

Here, $a = 4$ and $b = 2$

Now, $x + 1 = 4 \sec \theta$

and $y - 2 = 2 \tan \theta$

$$\Rightarrow x = 4 \sec \theta - 1$$

$$\text{and } y = 2 \tan \theta + 2$$

Hence, the required point is $(4 \sec \theta - 1, 2 \tan \theta + 2)$.

Position of a Point with Respect to a Hyperbola

The point (x_1, y_1) lies outside, on or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ according as

$$\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 <, =, > 0.$$

➤ **Example 7.** The position of the point $(5, -4)$ relative to the hyperbola $9x^2 - y^2 - 1 = 0$, is

- (a) outside (b) on the hyperbola
(c) inside (d) None of these

Sol. (c) The equation of hyperbola is $9x^2 - y^2 - 1 = 0$

At point $(5, -4)$, we have

$$9(5)^2 - (-4)^2 - 1 = 225 - 16 - 1 = 208 \Rightarrow 208 > 0$$

So, the point $(5, -4)$ lies inside the hyperbola

$$9x^2 - y^2 = 1$$

Intersection of a Line and a Hyperbola

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$... (i)

and the line $y = mx + c$... (ii)

The coordinates of the points of intersection of Eqs. (i) and (ii) satisfy both the equations and therefore can be obtained by solving Eqs. (i) and (ii) as simultaneous equations. Substituting the value of y from Eq. (ii) in Eq. (i), we get

$$\begin{aligned} \frac{x^2}{a^2} - \frac{(mx+c)^2}{b^2} &= 1 \\ \Rightarrow x^2(a^2m^2 - b^2) + 2a^2mcx + a^2(c^2 + b^2) &= 0 \end{aligned} \quad \dots (iii)$$

This equation being a quadratic in x , gives two values x which may be real and distinct, coincident or imaginary. Thus, a line intersects a hyperbola in two points which may be real and distinct, coincident or imaginary.

The line will intersect the hyperbola in two distinct points, if the roots of the equation are real and distinct and the condition for the same is

$$\begin{aligned} 4a^4m^2c^2 - 4a^2(c^2 + b^2)(a^2m^2 - b^2) &> 0 \\ \Rightarrow c^2 &> a^2m^2 - b^2 \end{aligned}$$

The line will touch the hyperbola, if the roots of Eq. (iii) are equal.

$$\begin{aligned} \therefore 4a^4m^2c^2 - 4a^2(c^2 + b^2)(a^2m^2 - b^2) &= 0 \\ \Rightarrow c^2 &= a^2m^2 - b^2 \\ \Rightarrow c &= \pm \sqrt{a^2m^2 - b^2} \end{aligned}$$

This is the condition for the line $y = mx + c$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

On substituting $c = \pm \sqrt{a^2 m^2 - b^2}$ in Eq. (iii), we get

$$(a^2 m^2 - b^2)x^2 \pm 2a^2 m \sqrt{a^2 m^2 - b^2} x + a^4 m^2 = 0$$

$$\Rightarrow (\sqrt{a^2 m^2 - b^2} x \pm a^2 m)^2 = 0$$

$$\Rightarrow x = \pm \frac{a^2 m}{\sqrt{a^2 m^2 - b^2}}$$

On putting the values of x and c in Eq. (ii), we get

$$y = \pm \frac{b^2}{\sqrt{a^2 m^2 - b^2}}$$

Thus, the line $y = mx + c$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = a^2 m^2 - b^2$ and the coordinates of the point of contact are

$$\left(\pm \frac{a^2 m}{\sqrt{a^2 m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2 m^2 - b^2}} \right)$$

➤ **Example 8.** The line $x \cos \alpha + y \sin \alpha = p$

touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

$a^2 \cos^2 \alpha - b^2 \sin^2 \alpha$ is equal to

- (a) p (b) p^2 (c) $-p^2$ (d) $2p^2$

Sol. (b) The given line is $x \cos \alpha + y \sin \alpha = p$

or $y = -x \cot \alpha + p \operatorname{cosec} \alpha$

On comparing with $y = mx + c$, we get

$$m = -\cot \alpha \text{ and } c = p \operatorname{cosec} \alpha$$

Now, the given line touches the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ iff } c^2 = a^2 m^2 - b^2$$

$$\Leftrightarrow p^2 \operatorname{cosec}^2 \alpha = a^2 (-\cot \alpha)^2 - b^2$$

$$\Leftrightarrow p^2 \operatorname{cosec}^2 \alpha = \frac{a^2 \cos^2 \alpha - b^2 \sin^2 \alpha}{\sin^2 \alpha}$$

$$\Leftrightarrow a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$$

➤ **Example 9.** If the line $y = 3x + \lambda$ touches the hyperbola $9x^2 - 5y^2 = 45$, then λ is equal to

- (a) $\pm 3\sqrt{6}$ (b) ± 6 (c) ± 3 (d) ± 4

Sol. (b) The equation of the hyperbola is $9x^2 - 5y^2 = 45$.

$$\Rightarrow \frac{x^2}{5} - \frac{y^2}{9} = 1$$

This is of the form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a^2 = 5$ and $b^2 = 9$.

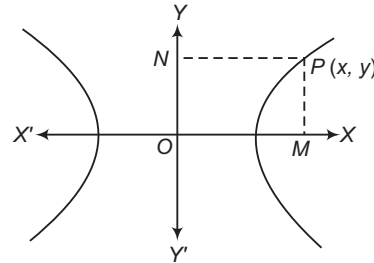
The line $y = 3x + \lambda$ touches the given hyperbola, if $\lambda^2 = 5(3)^2 - 9$ [using $c^2 = a^2 m^2 - b^2$]

$$\Rightarrow \lambda^2 = 45 - 9$$

$$\Rightarrow \lambda^2 = 36 \Rightarrow \lambda = \pm 6$$

Equation of a Hyperbola Referred to Two Perpendicular Lines

Consider the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ as shown in the figure given below.



Let $P(x, y)$ be any point on the hyperbola. Then, $PM = y$ and $PN = x$.

$$\therefore \frac{PN^2}{a^2} - \frac{PM^2}{b^2} = 1$$

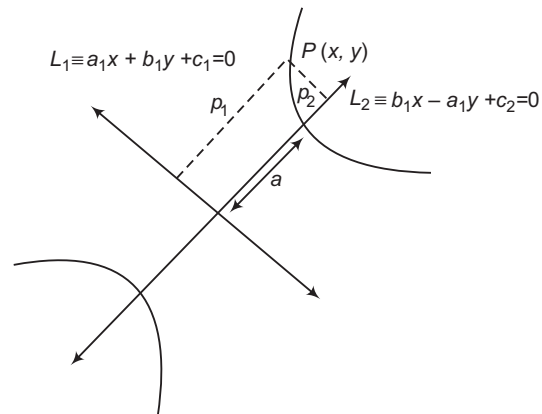
It follows from this that if perpendicular distance p_1 and p_2 of a moving point $P(x, y)$ from two mutually perpendicular coplanar straight lines $L_1 \equiv a_1 x + b_1 y + c_1 = 0$, $L_2 \equiv b_1 x - a_1 y + c_2 = 0$, respectively, satisfy the equation

$$\frac{p_1^2}{a^2} - \frac{p_2^2}{b^2} = 1$$

$$\text{i.e. } \frac{\left(\frac{a_1 x + b_1 y + c_1}{\sqrt{a_1^2 + b_1^2}} \right)^2}{a^2} - \frac{\left(\frac{b_1 x - a_1 y + c_2}{\sqrt{b_1^2 + a_1^2}} \right)^2}{b^2} = 1$$

Then, the locus of point P denotes a hyperbola in the plane of the given lines such that

- (i) The centre of the hyperbola is the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$.



- (ii) The transverse axis lies along $L_2 = 0$ and the conjugate axis lies along $L_1 = 0$.

- (iii) The length of the transverse and conjugate axes are $2a$ and $2b$.

➤ **Example 10.** The eccentricity of the conic $4(2y - x - 3)^2 - 9(2x + y - 1)^2 = 80$ is

- (a) $\frac{3}{13}$ (b) $\frac{\sqrt{13}}{3}$ (c) $\sqrt{13}$ (d) 3

Sol. (b) Here, $2y - x - 3 = 0$ and $2x + y - 1 = 0$ are perpendicular to each other.

Therefore, the equation of the conic can be written as

$$4 \times 5 \left[\frac{2y - x - 3}{\sqrt{2^2 + 1^2}} \right]^2 - 9 \times 5 \left[\frac{2x + y - 1}{\sqrt{2^2 + 1^2}} \right]^2 = 80$$

$$\Rightarrow 4 \left[\frac{2y - x - 3}{\sqrt{5}} \right]^2 - 9 \left[\frac{2x + y - 1}{\sqrt{5}} \right]^2 = 16$$

On putting $\frac{2y - x - 3}{\sqrt{5}} = X$ and $\frac{2x + y - 1}{\sqrt{5}} = Y$, the given equation can be written as

$$4X^2 - 9Y^2 = 16$$

$$\Rightarrow \frac{X^2}{4} - \frac{Y^2}{\left(\frac{4}{3}\right)^2} = 1$$

The eccentricity is given by

$$e = \sqrt{1 + \frac{\left(\frac{4}{3}\right)^2}{4}}$$

$$= \frac{\sqrt{13}}{3}$$

Work Book Exercise 16.1

1 The eccentricity of the conjugate hyperbola of the hyperbola $x^2 - 3y^2 = 1$ is

- a 2 b $\frac{2}{\sqrt{3}}$ c 4 d $\frac{4}{5}$

2 The equation of the transverse axis of the hyperbola $(x - 3)^2 + (y + 1)^2 = (4x + 3y)^2$ is

- a $x + 3y = 0$ b $4x + 3y = 9$
c $3x - 4y = 13$ d $4x + 3y = 0$

3 If the distance between the foci and the distance between the two directrices of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are in the ratio 3:2, then $b : a$ is

- a $1 : \sqrt{2}$ b $\sqrt{3} : \sqrt{2}$
c $1 : 2$ d $2 : 1$

4 If the eccentricity of the hyperbola $x^2 - y^2 \sec^2 \alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2 \alpha + y^2 = 25$, then the value of α is

- a $\frac{\pi}{6}$ b $\frac{\pi}{4}$ c $\frac{\pi}{3}$ d $\frac{\pi}{2}$

5 If $ax + by = 1$ is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \text{ then } a^2 - b^2 \text{ is equal to}$$

- a $\frac{1}{a^2 e^2}$
b $a^2 e^2$
c $b^2 e^2$
d None of the above

Tangent to a Hyperbola

(i) **Slope form** The equations of tangent of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is given by

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

The coordinates of the points of contact are

$$\left(\pm \frac{a^2 m}{\sqrt{a^2 m^2 - b^2}}, \pm \frac{b^2}{\sqrt{a^2 m^2 - b^2}} \right)$$

(ii) **Point form** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

(iii) **Parametric form** The equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $(a \sec \theta, b \tan \theta)$ is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1.$$

➤ The tangent at the points $P(a \sec \theta_1, b \tan \theta_1)$ and $Q(a \sec \theta_2, b \tan \theta_2)$ intersect at the point

$$R \left(\frac{a \cos \frac{(\theta_1 - \theta_2)}{2}}{\cos \frac{(\theta_1 + \theta_2)}{2}}, \frac{b \sin \frac{(\theta_1 - \theta_2)}{2}}{\cos \frac{(\theta_1 + \theta_2)}{2}} \right)$$

➤ **Example 11.** S is the focus of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } M \text{ is the foot of the}$$

perpendicular drawn from S on a tangent to the hyperbola. Then, the locus of M is

- (a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = 2a^2$
(c) $x^2 - y^2 = 2a^2$ (d) $x^2 - y^2 = a^2$

Sol. (a) Let the coordinates of M be (α, β) . We know that the equation of any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$y = mx \pm \sqrt{a^2 m^2 - b^2}$$

Since, $M(\alpha, \beta)$ lies on it.

$$\therefore \beta = m\alpha \pm \sqrt{a^2 m^2 - b^2} \quad \dots(i)$$

Now, SM perpendicular to the given tangent.

$\therefore$ Slope of the segment, $SM = -\frac{1}{m}$

Also, $S = (ae, 0)$
 $\therefore$ Equation of SM is $y - 0 = -\frac{1}{m}(x - ae)$
 $\Rightarrow y = -\frac{1}{m}(x - ae)$

But $M(\alpha, \beta)$ lies on it.

$\therefore \beta = -\frac{1}{m}(\alpha - ae) \quad \dots (ii)$

To get the locus of M , we have to eliminate m from Eqs. (i) and (ii).

From Eq. (i), $\beta - m\alpha = \pm \sqrt{a^2m^2 - b^2}$

From Eq. (ii), $(m\beta + \alpha) = ae$

On squaring both sides and then adding, we get

$$\begin{aligned} (\beta - m\alpha)^2 + (m\beta + \alpha)^2 &= (a^2m^2 - b^2) + a^2e^2 \\ \Rightarrow (\beta^2 + m^2\alpha^2 - 2m\alpha\beta) + (m^2\beta^2 + \alpha^2 + 2m\alpha\beta) &= [a^2m^2 - a^2(e^2 - 1)] + a^2e^2 \\ \Rightarrow (\beta^2 + \alpha^2) + m^2(\alpha^2 + \beta^2) &= a^2m^2 - a^2e^2 + a^2 + a^2e^2 \\ \Rightarrow (\alpha^2 + \beta^2)(1 + m^2) &= a^2(m^2 + 1) \\ \Rightarrow \alpha^2 + \beta^2 &= a^2 \end{aligned}$$

Hence, the locus of $M(\alpha, \beta)$ is $x^2 + y^2 = a^2$.

➤ **Example 12.** The equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 27$, is $24y - 30x = k$, where k is equal to

- (a) $\pm\sqrt{161}$ (b) $\pm\sqrt{160}$
 (c) $\pm\sqrt{170}$ (d) None of these

Sol. (a) The given equation can be written as

$$\frac{x^2}{\frac{1}{4}} - \frac{y^2}{\frac{1}{9}} = 1$$

So that, $a^2 = \frac{1}{4}$ and $b^2 = \frac{1}{9}$

We know that, $y = mx + \sqrt{a^2m^2 - b^2}$ is always a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \forall m$.

In our case, $y = mx \pm \sqrt{\frac{m^2}{4} - \frac{1}{9}}$ is a tangent to the given hyperbola.

But given equation of tangent is $24y = 30x + k$

$$\Rightarrow y = \frac{30}{24}x + \frac{k}{24} = \frac{5}{4}x + \frac{k}{24}$$

is a tangent to the given hyperbola.

Identifying the two equations, we get

$$\begin{aligned} m &= \frac{5}{4} \text{ and } \frac{m^2}{4} - \frac{1}{9} = \frac{k^2}{24^2} \\ \Rightarrow \frac{25}{64} - \frac{1}{9} &= \frac{k^2}{576} \\ \Rightarrow \frac{161}{64 \times 9} &= \frac{k^2}{576} \\ \Rightarrow k^2 &= \frac{161 \times 576}{64 \times 9} = 161 \\ \therefore k &= \pm\sqrt{161} \end{aligned}$$

➤ **Example 13.** The equations of the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1 \text{ are}$$

$$(a) y = \pm x \pm \sqrt{a^2 + b^2}$$

$$(b) y = \pm x \pm \sqrt{a^2 - b^2}$$

$$(c) y = \pm x \pm \sqrt{a^3 + b^3}$$

(d) None of the above

Sol. (b) Let $y = mx + c$ be the common tangent of the given hyperbolas.

Since, $y = mx + c$ is a tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore c^2 = a^2m^2 - b^2 \quad \dots (i)$$

Also, since $y = mx + c$ is a tangent to the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$.

$$\therefore \frac{(mx + c)^2}{a^2} - \frac{x^2}{b^2} = 1$$

$$\Rightarrow b^2(mx + c)^2 - a^2x^2 = a^2b^2$$

$$\Rightarrow b^2(m^2x^2 + 2mcx + c^2) - a^2x^2 - a^2b^2 = 0$$

$$\Rightarrow (b^2m^2 - a^2)x^2 + 2b^2mcx + b^2c^2 - a^2b^2 = 0$$

For tangency, this quadratic equation will have two roots equal.

The condition for this

$$\begin{aligned} 4b^4m^2c^2 &= 4(b^2m^2 - a^2)b^2(c^2 - a^2) \\ \Rightarrow b^2m^2c^2 &= (b^2m^2 - a^2)(c^2 - a^2) \\ &= b^2m^2c^2 - a^2b^2m^2 - a^2c^2 + a^4 \\ \Rightarrow a^2c^2 &= a^4 - a^2b^2m^2 \\ \Rightarrow c^2 &= a^2 - b^2m^2 \quad \dots (ii) \end{aligned}$$

From Eqs. (i) and (ii), we get

$$a^2m^2 - b^2 = a^2 - b^2m^2$$

$$\Rightarrow m^2(a^2 + b^2) - (a^2 + b^2) = 0$$

$$\Rightarrow (m^2 - 1)(a^2 + b^2) = 0$$

$$\Rightarrow m^2 - 1 = 0$$

$$\Rightarrow m = \pm 1$$

$$\therefore c = \pm \sqrt{a^2 - b^2}$$

Hence, the equations of the common tangents are

$$y = \pm x \pm \sqrt{a^2 - b^2}$$

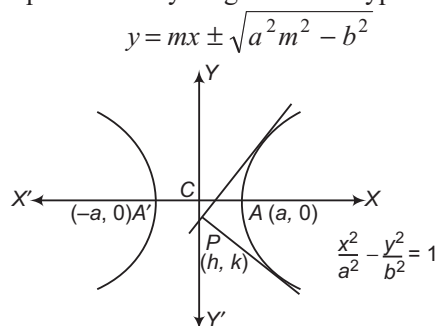
Number of Tangents Drawn from a Point to a Hyperbola

In this section, we shall prove that two tangents can be drawn from a point to a hyperbola.

Proof Let $P(h, k)$ be a point in the plane of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equation of any tangent to the hyperbola is



If it passes through $P(h, k)$. Then,

$$k = mh + \sqrt{a^2 m^2 - b^2}$$

$$\Rightarrow (k - mh)^2 = a^2 m^2 - b^2$$

$$\Rightarrow m^2 (h^2 - a^2) - 2khm + k^2 + b^2 = 0 \quad \dots(i)$$

which is a quadratic equation in m . So, it gives two values of m . Corresponding to each value of m there is a tangent to the hyperbola passing through the point $P(h, k)$. Thus, two tangents drawn from P are real and distinct, coincident or imaginary according as the roots of the Eq. (i) are real and distinct, coincident or imaginary.

Now, the following cases arise:

Case I Two tangents are real and distinct, if
 $4k^2 h^2 - 4(h^2 - a^2)(k^2 + b^2) > 0$
 [on putting $D > 0$]
 $\Rightarrow -h^2 b^2 + a^2 k^2 + a^2 b^2 > 0$
 $\Rightarrow h^2 b^2 - a^2 k^2 < a^2 b^2$
 $\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} < 1$
 $\Rightarrow P(h, k)$ lies outside the hyperbola.

Case II Two tangents are coincident, if
 $4k^2 h^2 - 4(h^2 - a^2)(k^2 + b^2) = 0$
 [on putting $D = 0$]
 $\Rightarrow -h^2 b^2 + k^2 a^2 + a^2 b^2 = 0$
 $\Rightarrow h^2 b^2 - k^2 a^2 = a^2 b^2$
 $\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} = 1$
 $\Rightarrow P(h, k)$ lies on the hyperbola

Case III Two tangents are imaginary, if
 $4k^2 h^2 - 4(h^2 - a^2)(k^2 + b^2) < 0$
 [on putting $D < 0$]
 $\Rightarrow -h^2 b^2 + k^2 a^2 + a^2 b^2 < 0$
 $\Rightarrow h^2 b^2 - k^2 a^2 > a^2 b^2$
 $\Rightarrow \frac{h^2}{a^2} - \frac{k^2}{b^2} > 1$
 $\Rightarrow P(h, k)$ lies inside the hyperbola.

Example 14. How many real tangents can be drawn from the point $(4, 3)$ to the hyperbola

$$\frac{x^2}{16} - \frac{y^2}{9} = 1 ?$$

- (a) 1 (b) 2
(c) 0 (d) None of these

Sol. (b) Given, point $P \equiv (4, 3)$

$$\text{Hyperbola, } S \equiv \frac{x^2}{16} - \frac{y^2}{9} - 1$$

$$S_1 \equiv \frac{16}{16} - \frac{9}{9} - 1 = -1 < 0$$

$\Rightarrow$ Point $P(4, 3)$ lies outside the hyperbola.

Hence, two tangents can be drawn from the point $P(4, 3)$.

Director Circle

The locus of the point of intersection of perpendicular tangents to a hyperbola is known as its director circle.

Let m_1 and m_2 be the slopes of two tangents drawn from a point $P(h, k)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then, m_1 and m_2 are the roots of equation $m^2 (h^2 - a^2) - 2khm + k^2 + b^2 = 0$.

$$\text{Therefore, } m_1 m_2 = \frac{k^2 + b^2}{h^2 - a^2}$$

If the tangents are perpendicular, then

$$m_1 m_2 = -1 \Rightarrow \frac{k^2 + b^2}{h^2 - a^2} = -1$$

$$\Rightarrow h^2 + k^2 = a^2 - b^2$$

Thus, the locus of (h, k) is $x^2 + y^2 = a^2 - b^2$.

Hence, the equation of the director circle of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } x^2 + y^2 = a^2 - b^2.$$

Example 15. On which curve does the perpendicular tangents drawn to the hyperbola

$$\frac{x^2}{25} - \frac{y^2}{16} = 1 \text{ intersect?}$$

- (a) $x^2 + y^2 = 7$ (b) $x^2 + y^2 = 41$
(c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 25$

Sol. (c) The locus of the point of intersection of perpendicular tangents to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the director circle given by

$$x^2 + y^2 = a^2 - b^2$$

Hence, the perpendicular tangents drawn to

$$\frac{x^2}{25} - \frac{y^2}{16} = 1$$

intersect on the curve $x^2 + y^2 = 25 - 16$

i.e. $x^2 + y^2 = 9$

Normals to a Hyperbola

(i) **Point form** The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at (x_1, y_1) is

$$\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$$

(ii) **Parametric form** The equation of the normal at $(a \sec \theta, b \tan \theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$a x \cos \theta + b y \cot \theta = a^2 + b^2$$

(iii) **Slope form** The equation of the normal of slope m to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is given by

$$y = mx \mp \frac{m(a^2 + b^2)}{\sqrt{a^2 - b^2 m^2}}$$

$$\text{at the points } \left(\pm \frac{a^2}{\sqrt{a^2 - b^2 m^2}}, \mp \frac{b^2 m}{\sqrt{a^2 - b^2 m^2}} \right).$$

It follows from this that the line $y = mx + c$ will be a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

$$c^2 = \frac{m^2(a^2 + b^2)^2}{a^2 - b^2 m^2}$$

➤ **Example 16.** The line $lx + my + n = 0$ will be a

normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if

$$(a) \frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

$$(b) \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$$

$$(c) \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

(d) None of the above

Sol. (c) The equation of the normal at $(a \sec \theta, b \tan \theta)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$a x \cos \theta + b y \cot \theta = a^2 + b^2 \quad \dots (i)$$

and the equation of the line is $lx + my + n = 0 \quad \dots (ii)$

Now, Eqs. (i) and (ii) represent the same line.

$$\therefore \frac{a \cos \theta}{l} = \frac{b \cot \theta}{m} = \frac{-(a^2 + b^2)}{n}$$

$$\Rightarrow \sec \theta = \frac{-na}{l(a^2 + b^2)}$$

$$\text{and} \quad \tan \theta = \frac{-nb}{m(a^2 + b^2)}$$

$$\therefore \sec^2 \theta - \tan^2 \theta = 1$$

$$\therefore \frac{n^2 a^2}{l^2 (a^2 + b^2)^2} - \frac{n^2 b^2}{m^2 (a^2 + b^2)^2} = 1$$

$$\Rightarrow \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

Aliter

The given equation of line can be rewritten as

$$y = -\frac{l}{m}x - \frac{n}{m}$$

On comparing this with $y = m_1 x + c_1$, we get

$$m_1 = -\frac{l}{m} \quad \text{and} \quad c_1 = -\frac{n}{m}$$

Now, the line $lx + my + n = 0$ will be normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ iff

$$c_1^2 = \frac{m_1^2 (a^2 + b^2)^2}{a^2 - b^2 m_1^2}$$

$$\Leftrightarrow \frac{n^2}{m^2} = \frac{l^2}{m^2} \frac{(a^2 + b^2)^2}{\left(a^2 - b^2 \frac{l^2}{m^2}\right)}$$

$$\Leftrightarrow n^2 \left(a^2 - \frac{b^2 l^2}{m^2}\right) = l^2 (a^2 + b^2)^2$$

$$\Rightarrow n^2 a^2 m^2 - n^2 b^2 l^2 = m^2 l^2 (a^2 + b^2)^2$$

$$\Rightarrow \frac{n^2 a^2}{l^2 (a^2 + b^2)^2} - \frac{n^2 b^2}{m^2 (a^2 + b^2)^2} = 1$$

$$\Rightarrow \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

➤ **Example 17.** The normal at P to a hyperbola of eccentricity e , intersects its transverse and conjugate axes at L and M , respectively. If the locus of the mid-point of LM is hyperbola, then eccentricity of this hyperbola is

$$(a) \frac{e}{\sqrt{e-1}}$$

$$(b) \frac{e}{\sqrt{e^2-1}}$$

$$(c) \frac{e}{\sqrt{e^2+1}}$$

(d) None of these

Sol. (b) Let $P(a \sec \phi, b \tan \phi)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Then, the equation of the normal at the P is

$$a x \cos \phi + b y \cot \phi = a^2 + b^2 \quad \dots (i)$$

$$\therefore b^2 = a^2 (e^2 - 1)$$

$$\therefore b^2 = a^2 e^2 - a^2$$

$$\Rightarrow a^2 e^2 = a^2 + b^2$$

Now, Eq. (i) reduces to $a x \cos \phi + b y \cot \phi = a^2 e^2 \quad \dots (ii)$

Hence, $L = (ae^2 \sec \phi, 0)$ and $M = \left(0, \frac{a^2 e^2 \tan \phi}{b}\right)$

Let the mid-point of LM be (h, k) .

$$\text{Then, } h = \frac{ae^2 \sec \phi}{2} \quad \text{and} \quad k = \frac{a^2 e^2 \tan \phi}{2b}$$

$$\Rightarrow \sec \phi = \frac{2h}{ae^2} \quad \text{and} \quad \tan \phi = \frac{2kb}{a^2 e^2}$$

$$\Rightarrow \sec^2 \phi - \tan^2 \phi = \frac{4h^2}{a^2 e^4} - \frac{4b^2 k^2}{a^4 e^4}$$

$$\Rightarrow 1 = \frac{4h^2}{a^2e^4} - \frac{4b^2k^2}{a^4e^4}$$

Hence, the locus of (h, k) is

$$\frac{4x^2}{a^2e^4} - \frac{4b^2y^2}{a^4e^4} = 1 \Rightarrow \left(\frac{x^2}{\frac{a^2e^4}{4}} \right) - \left(\frac{y^2}{\frac{a^4e^4}{4b^2}} \right) = 1$$

which represents a hyperbola.

If its eccentricity is e' , then

$$\frac{a^4e^4}{4b^2} = \frac{a^2e^4}{4}(e'^2 - 1) \quad [\because b^2 = a^2(e^2 - 1)]$$

$$\Rightarrow \frac{a^2}{b^2} = e'^2 - 1 \Rightarrow e'^2 = 1 + \frac{a^2}{b^2}$$

$$\text{But } b^2 = a^2(e^2 - 1) \Rightarrow e'^2 = 1 + \frac{b^2}{a^2}$$

$$\therefore e'^2 = 1 + \frac{1}{e^2 - 1} = \frac{e^2}{e^2 - 1}$$

$$\Rightarrow e' = \frac{e}{\sqrt{e^2 - 1}}$$

➤ **Example 18.** If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then

$$(a) a > 0, b > 0 \quad (b) a > 0, b < 0$$

$$(c) a < 0, b > 0 \quad (d) a < 0, b < 0$$

Sol. (b, c) Given equation of the curve is $xy = 1$.

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{-1}{x^2}$$

If the line $ax + by + c = 0$ is a normal to the curve $xy = 1$, then its slope $-\frac{a}{b}$ is equal to the slope of the normal.

$$\Rightarrow -\frac{a}{b} = x^2 \Rightarrow -\frac{a}{b} \text{ is positive} \Rightarrow \frac{a}{b} < 0$$

Hence, either $a > 0$ and $b < 0$ or $a < 0$ and $b > 0$.

Number of Normals Drawn from a Point

In general, four normals can be drawn from a point to a hyperbola. In other words, there are exactly four lines passing through a given point such that they are normals to the hyperbola at the points where they intersect the hyperbola. Such points on the hyperbola are known as the conormal points.

➤ If the normals at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are concurrent, then

$$(x_1 + x_2 + x_3 + x_4) \left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4} \right) = 4$$

(i) Properties of eccentric angles of conormal points

The sum of the eccentric angles of conormal points is an odd multiple of π .

(ii) If $\theta_1, \theta_2, \theta_3$ and θ_4 are eccentric angles of four

points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normals

at which are concurrent, then

$$(a) \sum \cos(\theta_1 + \theta_2) = 0$$

$$(b) \sum \sin(\theta_1 + \theta_2) = 0$$

$$(c) \sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$$

(iii) If θ_1, θ_2 and θ_3 are the eccentric angles of three

points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that

$$\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$$

Then, the normals at these points are concurrent.

Work Book Exercise 16.2

1 The tangents to the hyperbola $x^2 - y^2 = 3$ are parallel to the straight line $2x + y + 8 = 0$ at the following points

$$\text{a } (2, 1) \quad \text{b } (2, -1) \quad \text{c } (-2, 1) \quad \text{d } (-2, -1)$$

2 The equation of a tangent to the hyperbola $16x^2 - 25y^2 - 96x + 100y - 356 = 0$, which makes

an angle $\frac{\pi}{4}$ with the transverse axis, is

$$\text{a } y = x + 2 \quad \text{b } y = x - 5$$

$$\text{c } y = x + 3 \quad \text{d } x = y + 2$$

3 The point of intersection of two tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the product of whose

slopes is c^2 , lies on the curve

$$\text{a } y^2 - b^2 = c^2(x^2 + a^2)$$

$$\text{b } y^2 + a^2 = c^2(x^2 - b^2)$$

$$\text{c } y^2 + b^2 = c^2(x^2 - a^2)$$

$$\text{d } y^2 - a^2 = c^2(x^2 + b^2)$$

4 The focus of the middle points of the portions of the tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

intercepted between the coordinate axes, is

$$\text{a } 4x^2y^2 = a^2y^2 - b^2x^2$$

$$\text{b } x^2y^2 = 2a^2y^2 + b^2x^2$$

$$\text{c } x^2y^2 = 2(a^2y^2 - b^2x^2)^2$$

$$\text{d } \text{None of the above}$$

5 If radii of director circle of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $2r$ and r respectively and e_e and e_h

are the eccentricities of the ellipse and hyperbola respectively, then

$$\text{a } 2e_h^2 - e_e^2 = 6$$

$$\text{b } e_e^2 - 4e_h^2 = 6$$

$$\text{c } 4e_h^2 - e_e^2 = 6$$

$$\text{d } \text{None of the above}$$

Equation of the Pair of Tangents

The combined equation of the pair of tangents drawn from a point $P(x_1, y_1)$, lying outside the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ to the hyperbola } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \right) \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1 \right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 \right)^2$$

or

$$SS' = T^2$$

where, $S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$, $S' = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$

and $T = \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1$

Equations of Chord

i. **Chord joining two points** Let $P(a \sec \theta_1, b \tan \theta_1)$ and $Q(a \sec \theta_2, b \tan \theta_2)$ be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then, the equation of the chord PQ is

$$y - b \tan \theta_1 = \frac{b \tan \theta_2 - b \tan \theta_1}{a \sec \theta_2 - a \sec \theta_1} (x - a \sec \theta_1)$$

$$\Rightarrow \frac{x}{a} \cos \left(\frac{\theta_1 - \theta_2}{2} \right) - \frac{y}{b} \sin \left(\frac{\theta_1 + \theta_2}{2} \right) = \cos \left(\frac{\theta_1 + \theta_2}{2} \right)$$

ii. **Chord of contact** If the tangents drawn from a point $P(x_1, y_1)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touch the hyperbola at Q and R , then the equation of the chord of contact QR is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

☞ **Example 19.** The equation of chord of contact of tangents drawn from a point $(2, -1)$ to the hyperbola $16x^2 - 9y^2 = 144$, is

- (a) $9x + 32y = 144$ (b) $32x - 9y = 144$
(c) $32x + 9y = 144$ (d) None of these

Sol. (c) The equation of the chord of contact of tangents drawn from $(2, -1)$ to the hyperbola $16x^2 - 9y^2 = 144$ is $32x + 9y = 144$.

☞ **Example 20.** The chords of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are at right angles, then

$\frac{x_1 x_2}{y_1 y_2}$ is

- (a) $-\frac{a^2}{b^2}$ (b) $-\frac{b^2}{a^2}$ (c) $-\frac{b^4}{a^4}$ (d) $-\frac{a^4}{b^4}$

Sol. (d) The equations of chord of contact of tangents from (x_1, y_1) and (x_2, y_2) to the given hyperbola are

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

and $\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$

Since, lines are perpendicular to each other.

$$\therefore \frac{b^2 x_1}{a^2 y_1} \times \frac{b^2 x_2}{a^2 y_2} = -1$$

$$\Rightarrow \frac{x_1 x_2}{y_1 y_2} = -\frac{a^4}{b^4}$$

Equation of the Chord Bisected at a Given Point

The equation of the chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, bisected at the point (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1 = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$$

$T = S_1$, where T and S_1 have their usual meanings.

☞ **Example 21.** If chord of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$, then the locus of the mid-points of these chords is

- (a) $x^3 = (x - a)y^2$ (b) $x^2 = (x - a)y^2$
(c) $x^3 = (x + a)y^2$ (d) $x^3 = (x - a)y$

Sol. (a) The equation of the chord of the hyperbola, whose mid-point is (h, k) , is $T = S_1$

i.e. $xh - yk = h^2 - k^2$

$$\Rightarrow yk = xh - h^2 + k^2$$

$$\Rightarrow y = \frac{h}{k}x - \frac{h^2 - k^2}{k}$$

We know that the line $y = mx + c$ will touch the parabola, if $c = \frac{a}{m}$.

$$\therefore -\frac{h^2 - k^2}{k} = \frac{ka}{h}$$

$$\Rightarrow h(h^2 - k^2) = -ak^2$$

$$\Rightarrow h^3 - hk^2 = -ak^2$$

$$\Rightarrow h^3 = (h - a)k^2$$

Hence, the locus of (h, k) is $x^3 = (x - a)y^2$.

☞ **Example 22.** Chords of the circle $x^2 + y^2 = a^2$

touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then, their

mid-points lie on the curve

- (a) $(x^2 + y^2)^2 = a^2 x^2 - b^2 y^2$
(b) $(x^2 + y^2) = a^2 x^2 - b^2 y^2$
(c) $(x^2 - y^2)^2 = a^2 x^2 - b^2 y^2$
(d) $(x^2 + y^2) = a^2 x^2 + b^2 y^2$

Sol. (a) The equation of the chord of the circle $x^2 + y^2 = a^2$ whose mid-point is (α, β) , is

$$x\alpha + y\beta = \alpha^2 + \beta^2 \quad \dots(i)$$

which is obtained by using the formula $S_1 = T$.

Again, the equation of the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

Now, putting $x_1 = a \sec \theta$ and $y_1 = b \tan \theta$, the equation of the tangent to the hyperbola at the point $(a \sec \theta, b \tan \theta)$ is

$$\frac{x \cdot a \sec \theta}{a^2} - \frac{y \cdot b \tan \theta}{b^2} = 1$$

$$\Rightarrow \frac{x \sec \theta}{a} - \frac{y \tan \theta}{b} = 1 \quad \dots(ii)$$

Since, Eqs. (i) and (ii) are the same, therefore

$$\frac{\alpha a}{\sec \theta} = \frac{\beta b}{-\tan \theta} = \frac{\alpha^2 + \beta^2}{1}$$

$$\Rightarrow \sec \theta = \frac{a\alpha}{\alpha^2 + \beta^2} \quad \text{and} \quad \tan \theta = -\frac{\beta b}{\alpha^2 + \beta^2}$$

$$\Rightarrow \sec^2 \theta - \tan^2 \theta = \frac{a^2 \alpha^2}{(\alpha^2 + \beta^2)^2} - \frac{\beta^2 b^2}{(\alpha^2 + \beta^2)^2}$$

$$\Rightarrow \frac{a^2 \alpha^2 - \beta^2 b^2}{(\alpha^2 + \beta^2)^2} = 1$$

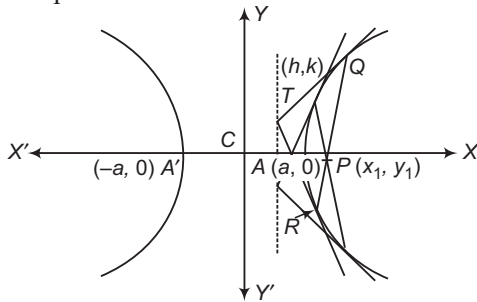
$$\Rightarrow (\alpha^2 + \beta^2)^2 = a^2 \alpha^2 - \beta^2 b^2$$

Hence, the locus of the mid-point is

$$(x^2 + y^2)^2 = a^2 x^2 - b^2 y^2$$

Pole and Polar

Let P be a point inside or outside a hyperbola. Then, the locus of the point of intersection of two tangents to the hyperbola at the points, where secants drawn through P intersect the hyperbola, is called the polar of point P with respect to the hyperbola and the point P is called the pole of the polar.



Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be the hyperbola and $P(x_1, y_1)$ be a point such that the secant drawn through P intersects the hyperbola at Q and R . Let the tangents at Q and R intersect at $T(h, k)$. Clearly, QR is the chord of contact of tangents drawn from T to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

So, the equation QR is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1$$

Clearly, QR passes through $P(x_1, y_1)$. Therefore,

$$\frac{hx_1}{a^2} - \frac{ky_1}{b^2} = 1$$

Hence, the locus $T(h, k)$ i.e. the polar of $P(x_1, y_1)$ is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

Example 23. A series of chords of the

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are tangents to the circle

described on the straight line joining the foci of the hyperbola as diameter. Then, the locus of their poles with respect to the hyperbola is

$$(a) \frac{x^2}{a^2} + \frac{y^2}{b^2} = a^2 + b^2 \quad (b) \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2 + b^2}$$

$$(c) \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2} \quad (d) \text{None of these}$$

Sol. (c) The foci of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots(i)$$

are $G(-ae, 0)$ and $H(ae, 0)$.

So, the equation of the circle on GH as diameter is

$$(x + ae)(x - ae) + (y - 0)(y - 0) = 0$$

$$\Rightarrow x^2 - a^2 e^2 + y^2 = 0$$

$$\Rightarrow x^2 + y^2 = a^2 e^2 \quad \dots(ii)$$

Let $P(\alpha, \beta)$ be the pole of the chord of the hyperbola (i).

Then, polar of P w.r.t. Eq. (i) is

$$\frac{x\alpha}{a^2} - \frac{y\beta}{b^2} = 1 \quad \dots(iii)$$

If it is tangent to the circle (ii), then

$$\frac{1}{\sqrt{\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4}}} = ae$$

$$\Rightarrow \frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} = \frac{1}{a^2 e^2} = \frac{1}{a^2 + b^2} \quad [\because b^2 = a^2(e^2 - 1)]$$

The locus of (α, β) is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}$$

Conjugate Points

Two points are said to be conjugate points with respect to a hyperbola, if each lies on the polar of the other.

If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $P(x_1, y_1)$ lies

on the polar of $Q(x_2, y_2)$ i.e. on $\frac{xx_2}{a^2} - \frac{yy_2}{b^2} = 1$. Therefore,

$$\frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} = 1$$

Conjugate Lines

Two lines are said to be conjugate lines with respect to a hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if each passes through the pole of the other.

➤ **Example 24.** The locus of the poles of the chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which subtend a right angle at its centre, is

$$(a) \frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2} \quad (b) \frac{x^2}{a^4} - \frac{y^2}{b^4} = \frac{1}{a^2} + \frac{1}{b^2}$$

$$(c) \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2} - \frac{1}{b^2} \quad (d) \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{a^2} + \frac{1}{b^2}$$

Sol. (a) Let $P(h, k)$ be the pole of a chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then, the equation of the polar is

$$\frac{hx}{a^2} - \frac{ky}{b^2} = 1 \quad \dots(i)$$

The combined equation of the lines joining the origin to the points of intersection of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the line (i) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{hx}{a^2} - \frac{ky}{b^2} \right)^2$$

Since, the lines given by the above equation are at right angle. Therefore,

Coefficient of x^2 + Coefficient of y^2 = 0

$$\Rightarrow \frac{1}{a^2} - \frac{h^2}{a^4} - \frac{1}{b^2} - \frac{k^2}{b^4} = 0$$

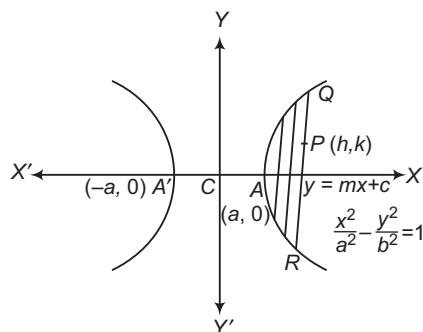
$$\Rightarrow \frac{h^2}{a^4} + \frac{k^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$$

∴ The locus of $P(h, k)$ is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2} - \frac{1}{b^2}$$

Diameter

The locus of the mid-points of a system of parallel chords of a hyperbola is called a diameter.



The point, where a diameter intersects the hyperbola, is known as the vertex of the diameter.

(i) **Equation of a diameter** The equation of a diameter bisecting a system of parallel chords of slope m of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $y = \frac{b^2}{a^2 m} x$.

(ii) **Conjugate diameters** Two diameters of a hyperbola are said to be conjugate diameters, if each bisects the chords parallel to the other.

(iii) In a pair of conjugate diameters of a hyperbola, only one meets the hyperbola in real points.

(iv) Let $P(a \sec \theta, b \tan \theta)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that CP and CD are conjugate diameters of the hyperbola. Then, coordinates of D are $(a \tan \theta, b \sec \theta)$.

(v) If a pair of conjugate diameters meet the hyperbola and its conjugate in P and D respectively, then

$$CP^2 - CD^2 = a^2 - b^2$$

➤ **Example 25.** The equation of the diameter which bisects the chord $7x + y - 2 = 0$ of the hyperbola $\frac{x^2}{3} - \frac{y^2}{7} = 1$, is

- (a) $y - 3x = 0$ (b) $3y - x = 0$
(c) $3y + x = 0$ (d) None of these

Sol. (c) The equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{7} = 1$, so that $a^2 = 3$ and $b^2 = 7$.

The slope of the line $7x + y - 2 = 0$ is -7 .

Let the slope of its diameter be m_1 .

$$\text{Then, } mm_1 = \frac{b^2}{a^2} = \frac{7}{3} \Rightarrow m_1 = \frac{7}{3(-7)} = -\frac{1}{3}$$

Hence, the required equation of the diameter is

$$y = m_1 x = -\frac{1}{3} x \Rightarrow 3y + x = 0$$

➤ **Example 26.** If the line $lx + my + n = 0$ meets the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the extremities of a pair of conjugate diameters, then

- (a) $a^2 l^2 - b^2 m^2 = 0$ (b) $a^2 l^2 - b^2 m^2 = 1$
(c) $a^2 l^2 - b^2 m^2 = 2$ (d) $a^2 l^2 - b^2 m^2 = 3$

Sol. (a) Let CP and CD be a pair of conjugate diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Then, the coordinates of P and D are $(a \sec \theta, b \tan \theta)$ and $(a \tan \theta, b \sec \theta)$ respectively. It is given that the line $lx + my + n = 0$ meets the hyperbola at P and D . Therefore,

$$a \sec \theta + b m \tan \theta + n = 0$$

$$\text{and } a \tan \theta + b m \sec \theta + n = 0$$

$$\Rightarrow a \sec \theta + b m \tan \theta = -n$$

$$\text{and } a \tan \theta + b m \sec \theta = -n$$

$$\Rightarrow (a \sec \theta + b m \tan \theta)^2 = n^2$$

$$\text{and } (a \tan \theta + b m \sec \theta)^2 = n^2$$

$$\Rightarrow a^2 l^2 (\sec^2 \theta - \tan^2 \theta) + b^2 m^2 (\tan^2 \theta - \sec^2 \theta) = 0$$

[on subtracting]

$$\Rightarrow a^2 l^2 - b^2 m^2 = 0$$

➤ **Example 27.** If the equation

$lx^2 + 2mxy + ny^2 = 0$ represents a pair of conjugate

diameters of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

- (a) $la^2 + nb^2 = 0$ (b) $la^2 = nb^2$
 (c) $2la^2 = nb^2$ (d) None of these

Sol. (b) Let $y = m_1 x$ and $y = m_2 x$ be the lines represented by $lx^2 + 2mxy + ny^2 = 0$.

Then, $m_1 m_2 = \frac{l}{n}$... (i)

But $y = m_1 x$ and $y = m_2 x$ are conjugate diameters of the hyperbola.

$\therefore m_1 m_2 = \frac{b^2}{a^2}$... (ii)

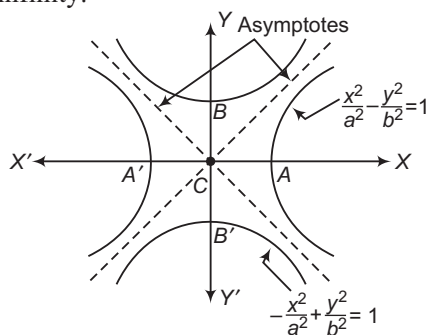
From Eqs. (i) and (ii), we get

$$\frac{l}{n} = \frac{b^2}{a^2} \Rightarrow la^2 = nb^2$$

Asymptotes

An asymptote to a curve is a straight line at a finite distance from the origin, to which the tangent to a curve tends as the point of contact goes to infinity.

In other words, asymptote to a curve touches the curve at infinity.



The equations of two asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are

$$y = \pm \frac{b}{a} x \quad \text{or} \quad \frac{x}{a} \pm \frac{y}{b} = 0$$

- The combined equation of the asymptotes to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.
- When $b = a$ i.e. the asymptotes of rectangular hyperbola $x^2 - y^2 = a^2$ are $y = \pm x$, which are at right angle.
- A hyperbola and its conjugate hyperbola have the same asymptotes.

- The equation of the pair of asymptotes differ the hyperbola and the conjugate hyperbola by the same constant only i.e. Hyperbola – Asymptotes = Asymptotes – Conjugate hyperbola

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \right) - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right) = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} \right) - \left(\frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 \right)$$

- The asymptotes pass through the centre of the hyperbola.
- The bisectors of the angle between the asymptotes are the coordinates axes.

➤ **Example 28.** The equation of the hyperbola whose asymptotes are the straight lines $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$, which passes through $(1, -1)$, is

- (a) $x^2 - 8xy + 8y^2 + 14x + 22y - 7 = 0$
 (b) $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$
 (c) $x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$
 (d) None of the above

Sol. (b) Since, the equation of the hyperbola differs from that of the joint equations of the asymptotes by a constant, therefore the equation of the hyperbola will be of the form

$$(x + 2y + 3)(3x + 4y + 5) + k = 0 \quad \dots (i)$$

Since, the hyperbola passes through the point $(1, -1)$.

$$\therefore (1 - 2 + 3)(3 - 4 + 5) + k = 0$$

$$\Rightarrow 2(4) + k = 0$$

$$\therefore k = -8$$

From Eq. (i),

$$(x + 2y + 3)(3x + 4y + 5) - 8 = 0$$

$$\Rightarrow 3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$$

➤ **Example 29.** The equation of the hyperbola conjugate to the hyperbola

$$2x^2 + 3xy - 2y^2 - 5x + 5y + 2 = 0$$

- (a) $2x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$
 (b) $x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$
 (c) $x^2 + 3xy - 2y^2 - 5x + 5y - 6 = 0$
 (d) None of the above

Sol. (a) The required equation is obtained by the equation.

Hyperbola + Conjugate hyperbola = 2(Asymptotes)

Now, let the equation of the asymptotes be

$$2x^2 + 3xy - 2y^2 - 5x + 5y + \lambda = 0.$$

This will represent a pair of straight lines, if

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow 2(-2)\lambda + 2\left(\frac{5}{2}\right)\left(-\frac{5}{2}\right)\left(\frac{3}{2}\right) - 2\left(\frac{5}{2}\right)^2 - (-2)\left(-\frac{5}{2}\right)^2 - \lambda\left(\frac{3}{2}\right)^2 = 0$$

$$\Rightarrow -4\lambda - \frac{75}{4} - \frac{50}{4} + \frac{50}{4} - \frac{9}{4}\lambda = 0$$

$$\Rightarrow \frac{25}{4}\lambda = -\left(\frac{75}{4} + \frac{50}{4} - \frac{50}{4}\right) = -\frac{75}{4}$$

$$\Rightarrow \lambda = -3$$

$\therefore$ The equation of the conjugate hyperbola is

$$2(\text{Asymptotes}) - \text{Hyperbola}$$

$$\text{i.e. } 2x^2 + 3xy - 2y^2 - 5x + 5y - 8 = 0$$

➤ **Example 30.** If e is the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is the angle between the asymptotes, then $\cos\left(\frac{\theta}{2}\right)$ is equal to

- (a) $\frac{1}{e}$ (b) $-\frac{1}{e}$ (c) e (d) $\frac{2}{e}$

Sol. (a) The equations of the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are $y = \pm \frac{b}{a}x$ i.e. $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$.

If θ is the angle between the two asymptotes, then

$$\tan \theta = \frac{\frac{b}{a} + \frac{b}{a}}{1 - \frac{b}{a} \cdot \frac{b}{a}} = \frac{2b}{a} \times \frac{a^2}{a^2 - b^2} = \frac{2ab}{a^2 - b^2}$$

From this, we get $\cos \theta = \frac{a^2 - b^2}{a^2 + b^2}$

$$\Rightarrow 2\cos^2\left(\frac{\theta}{2}\right) - 1 = \frac{a^2 - b^2}{a^2 + b^2}$$

$$\Rightarrow 2\cos^2\left(\frac{\theta}{2}\right) = 1 + \frac{a^2 - b^2}{a^2 + b^2}$$

$$\Rightarrow 2\cos^2\left(\frac{\theta}{2}\right) = \frac{2a^2}{a^2 + b^2}$$

$$\Rightarrow \cos^2\left(\frac{\theta}{2}\right) = \frac{a^2}{a^2 + b^2}$$

$$\text{But } b^2 = a^2(e^2 - 1) = a^2e^2 - a^2$$

$$\Rightarrow a^2 + b^2 = a^2e^2$$

$$\therefore \cos^2\left(\frac{\theta}{2}\right) = \frac{a^2}{a^2e^2} = \frac{1}{e^2}$$

$$\Rightarrow \cos\left(\frac{\theta}{2}\right) = \frac{1}{e}$$

Rectangular Hyperbola

A hyperbola whose asymptotes are at right angles to each other is called a rectangular hyperbola.

The equation of the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ are given by } y = \pm \frac{b}{a}x.$$

The angle θ between these two asymptotes is given by

$$\tan \theta = \frac{\frac{b}{a} - \left(-\frac{b}{a}\right)}{1 + \frac{b}{a} \cdot \left(-\frac{b}{a}\right)} = \frac{\frac{2b}{a}}{1 - \frac{b^2}{a^2}} = \frac{2ab}{a^2 - b^2}$$

If the asymptotes are at right angle, then

$$\theta = \frac{\pi}{2} \Rightarrow \tan \theta = \tan \frac{\pi}{2}$$

$$\Rightarrow \frac{2ab}{a^2 - b^2} = \tan \frac{\pi}{2}$$

$$\Rightarrow a^2 - b^2 = 0$$

$$\Rightarrow a = b$$

$$\Rightarrow 2a = 2b$$

Thus, the transverse and conjugate axes of a rectangular hyperbola are equal and the equation of the hyperbola is $x^2 - y^2 = a^2$.

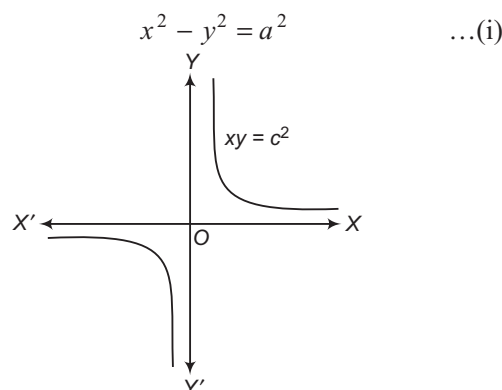
The equation of the asymptotes of the rectangular hyperbola are $y = \pm x$ i.e. $y = x$ and $y = -x$. Clearly, each of these two asymptotes is inclined at 45° to the transverse axis.

➤ Since, the transverse and conjugate axes of a rectangular hyperbola are equal. So, its eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{a^2}{a^2}} = \sqrt{2}$$

Equation of the Rectangular Hyperbola Referred to its Asymptotes as the Axes of Coordinates

Referred to the transverse and conjugate axes along the axes of coordinates, the equation of the rectangular hyperbola is



The asymptotes of Eq. (i) are $y = x$ and $y = -x$. Each of these two asymptotes are inclined at an angle of 45° with the transverse axis. So, if we rotate the coordinate axes through an angle of $-\frac{\pi}{4}$ keeping the origin fixed, then the axes coincide with the asymptotes of the hyperbola, we have

$$x = X \cos\left(-\frac{\pi}{4}\right) - Y \sin\left(-\frac{\pi}{4}\right) = \frac{X + Y}{\sqrt{2}}$$

$$\text{and } y = X \sin\left(-\frac{\pi}{4}\right) + Y \cos\left(-\frac{\pi}{4}\right) = \frac{Y - X}{\sqrt{2}}$$

On substituting the values of x and y in Eq. (i), we get

$$\left(\frac{X + Y}{\sqrt{2}}\right)^2 - \left(\frac{Y - X}{\sqrt{2}}\right)^2 = a^2$$

$$\Rightarrow XY = \frac{a^2}{2}$$

$$\Rightarrow XY = c^2, \text{ where } c^2 = \frac{a^2}{2}$$

Thus, the equation of the hyperbola referred to its asymptotes as the coordinate axes is

$$xy = c^2, \text{ where } c^2 = \frac{a^2}{2}$$

The equation of a rectangular hyperbola having coordinate axes as its asymptotes is $xy = c^2$. If the asymptotes of a rectangular hyperbola are $x = \alpha$, $y = \beta$, then its equation is

$$(x - \alpha)(y - \beta) = c^2$$

$$\Rightarrow xy - \alpha y - \beta x + \lambda = 0$$

Example 31. A straight line has its extremities on two fixed straight lines (say a and b) and cuts off from them a triangle of constant area. Then, locus of the middle point of the line is

- (a) $xy = c^2$
 (b) $xy + c^2 = 0$
 (c) $x^2 y^2 = c$
 (d) None of the above

Sol. (a) Let the given straight line be axes of coordinates and let the equation of the variable line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

This line cuts the coordinate axes at the point $A(a, 0)$ and $B(0, b)$.

Therefore, area of $\triangle AOB = \frac{1}{2}ab = \text{Constant}$

$$\Rightarrow ab = \text{Constant} \quad [\text{say} = 4c^2] \dots (i)$$

If (α, β) are the coordinates of the middle points of AB , then

$$\alpha = \frac{a}{2} \text{ and } \beta = \frac{b}{2} \dots (ii)$$

On eliminating a and b from Eqs. (i) and (ii), we get

$$\alpha\beta = c^2$$

Hence, the locus of (α, β) is $xy = c^2$.

Example 32. A triangle is inscribed in $xy = c^2$ and two of its sides are parallel to $y + m_1x = 0$ and $y + m_2x = 0$. Then, the third side envelopes the hyperbola is

- (a) $4m_1m_2xy = c^2(m_1 + m_2)^2$
 (b) $m_1m_2xy = c^2(m_1 + m_2)$
 (c) $2m_1m_2xy = c^2(m_1 + m_2)^2$
 (d) $4m_1m_2xy = c^2(m_1 - m_2)^2$

Sol. (a) Let PQR be a triangle inscribed in the hyperbola

$$xy = c^2 \text{ and } P \equiv \left(ct_1, \frac{c}{t_1}\right), Q \equiv \left(ct_2, \frac{c}{t_2}\right) \text{ and } R \equiv \left(ct_3, \frac{c}{t_3}\right)$$

Then, the equation of PQ is

$$y - \frac{c}{t_1} = \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1}(x - ct_1) = \frac{c(t_1 - t_2)}{t_1t_2(c)(t_1 - t_2)}(x - ct_1) = -\frac{1}{t_1t_2}(x - ct_1)$$

$$\Rightarrow t_1t_2y - ct_2 = -x + ct_1$$

$$\Rightarrow t_1t_2y + x = c(t_1 + t_2)$$

This is parallel to $y + m_1x = 0$, if

$$\frac{t_1t_2}{1} = \frac{1}{m_1}$$

$$\Rightarrow m_1t_1t_2 = 1$$

Similarly, PR is parallel to $y + m_2x = 0$, if

$$m_2t_1t_3 = 1$$

$$\text{We have, } m_1t_1t_2 = m_2t_1t_3$$

$$\Rightarrow m_1t_2 = m_2t_3 \dots (i)$$

Now, the equation of QR is

$$t_2t_3y + x = c(t_2 + t_3)$$

$$\Rightarrow m_1t_2t_3y + m_1x = c(m_1t_2 + m_1t_3)$$

$$\Rightarrow m_2t_3^2y + m_1x = c(m_2t_3 + m_1t_3) \quad [\text{from Eq. (i)}]$$

$$= c(m_1 + m_2)t_3$$

$$\Rightarrow m_1x + m_2t_3^2y = c(m_1 + m_2)t_3$$

The envelope of which for different values of t_3 is

$$4m_1m_2xy = c^2(m_1 + m_2)^2$$

Example 33. The asymptotes of the hyperbola

$xy = hx + ky$ are

$$(a) x = k, y = h \quad (b) x = h, y = k$$

$$(c) x = h, y = h \quad (d) x = k, y = k$$

Sol. (a) The equation of asymptotes are

$$xy - hx - ky + \lambda = 0 \dots (i)$$

where, λ is a constant and Eq. (i) represents a pair of straight line.

Here, $A = 0, B = 0, C = \lambda, 2H = 1, 2G = -h$ and $2F = -k$

Then, $ABC + 2FGH - AF^2 - BG^2 - CH^2 = 0$

$$\Rightarrow 0 + 2\left(-\frac{k}{2}\right)\left(-\frac{h}{2}\right)\left(\frac{1}{2}\right) - 0 - 0 - \lambda \cdot \frac{1}{4} = 0$$

$$\Rightarrow \frac{hk}{4} = \frac{\lambda}{4}$$

$$\Rightarrow \lambda = hk$$

On putting $\lambda = hk$ in Eq. (i), we get

$$xy - hx - ky + hk = 0$$

$$\Rightarrow (x - k)(y - h) = 0$$

Hence, asymptotes are $x = k$ and $y = h$.

Example 34. Consider the set of hyperbola $xy = k, x \in R$. Let e_1 be the eccentricity when $k = 4$ and e_2 be the eccentricity when $k = 9$, then $e_1 - e_2$ is equal to

- (a) 0 (b) 1 (c) 2 (d) 3

Sol. (a) We know that the eccentricity of $xy = k, \forall k \in R$ is $\sqrt{2}$.

$$\therefore e_1 = \sqrt{2}$$

$$\text{Also, } e_2 = \sqrt{2}$$

$$\text{Hence, } e_1 - e_2 = 0$$

Tangent to a Rectangular Hyperbola

- i. **Point form** The equation of tangent at (x_1, y_1) to the parabola $xy = c^2$ is
- $$xy_1 + yx_1 = 2c^2 \quad \text{or} \quad \frac{x}{x_1} + \frac{y}{y_1} = 2$$
- ii. **Parametric form** The equation of tangent at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is
- $$\frac{x}{t} + yt = 2c$$

✎ Tangent at $P\left(ct_1, \frac{c}{t_1}\right)$ and $Q\left(ct_2, \frac{c}{t_2}\right)$ to the rectangular hyperbola $xy = c^2$ intersect at $\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$.

✎ **Example 35.** A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Then, the locus of the point which divides the line segment between these points in the ratio of 1:2, is

(a) $x^2 + y^2 = 10xy$ (b) $16x^2 - y^2 = 10xy$
 (c) $16x^2 + y^2 = 10xy - 18$ (d) $x^2 + y^2 = xy$

Sol. (c) Let $R(\alpha, \beta)$ be the point which divides the line segment PQ in the ratio of 1 : 2.

The equation of any line passing through $R(\alpha, \beta)$ is

$$\frac{x - \alpha}{\cos \theta} = \frac{y - \beta}{\sin \theta} = r \quad [\text{say}]$$

where, $\tan \theta = 4$

$$\therefore \sin \theta = \frac{4}{\sqrt{17}} \quad \text{and} \quad \cos \theta = \frac{1}{\sqrt{17}}$$

$$\therefore x = \alpha + r \cos \theta \quad \text{and} \quad y = \beta + r \sin \theta$$

Since, (x, y) lies on $xy = 1$.

$$\therefore (\alpha + r \cos \theta)(\beta + r \sin \theta) = 1$$

$$\Rightarrow \alpha\beta + r(\alpha \sin \theta + \beta \cos \theta) + r^2 \sin \theta \cos \theta = 1$$

This is a quadratic equation in r and hence it will have two roots of r . Let r_1 and r_2 be the two roots.

$$\therefore r_1 + r_2 = -\frac{(\alpha \sin \theta + \beta \cos \theta)}{\sin \theta \cos \theta} \quad \dots (i)$$

$$\text{and} \quad r_1 r_2 = \frac{\alpha\beta - 1}{\sin \theta \cos \theta} \quad \dots (ii)$$

$$\text{Also,} \quad \frac{r_1}{r_2} = \frac{1}{2} \Rightarrow 2r_1 = r_2$$

On putting $r_2 = 2r_1$ in Eqs. (i) and (ii), we get

$$3r_1 = \frac{-(\alpha \sin \theta + \beta \cos \theta)}{\sin \theta \cos \theta} \quad \text{and} \quad 2r_1^2 = \frac{\alpha\beta - 1}{\sin \theta \cos \theta}$$

Now, eliminating r_1 and using $\sin \theta = \frac{4}{\sqrt{17}}$ and

$$\cos \theta = \frac{1}{\sqrt{17}}, \text{ we get } (4\alpha + \beta)^2 = 18(\alpha\beta - 1).$$

Hence, the locus of (α, β) is

$$(4x + y)^2 = 18xy - 18$$

$$\text{i.e.} \quad 16x^2 + y^2 = 10xy - 18$$

Normals to a Rectangular Hyperbola

- i. **Point form** The equation of the normal at (x_1, y_1) to the hyperbola $xy = c^2$ is
- $$xx_1 - yy_1 = x_1^2 - y_1^2$$
- ii. **Parametric form** The equation of the normal at $\left(ct, \frac{c}{t}\right)$ to the hyperbola $xy = c^2$ is
- $$xt^3 - yt - ct^4 + c = 0$$

- ✎ • The equation of the normal at $\left(ct, \frac{c}{t}\right)$ is a fourth degree equation in t . So, in general, four normals can be drawn from a point to the hyperbola $xy = c^2$.
- The equation of the polar of any point $P(x_1, y_1)$ with respect to $xy = c^2$ is $xy_1 + yx_1 = 2c^2$.
- The equation of the chord of the hyperbola $xy = c^2$ whose middle point is (x_1, y_1) , is
- $$xy_1 + yx_1 = 2x_1 y_1 \quad \text{or} \quad T = S'$$
- where, T and S' have their usual meanings.
- The equation of the chord of the contact of tangents drawn from a point (x_1, y_1) to the rectangular hyperbola $xy = c^2$ is
- $$xy_1 + yx_1 = 2c^2$$

✎ **Example 36.** The locus of the foot of the perpendicular from the centre upon any normal to the hyperbola $x^2 - y^2 = a^2$ is

- (a) $(x^2 + y^2)^2 (y^2 - x^2) = a^2 x^2 y^2$
 (b) $(x^2 - y^2)^2 (y^2 - 2x^2) = 4a^2 x^2 y^2$
 (c) $(x^2 + y^2)^2 (y^2 - x^2) = 4a^2 x^2 y^2$
 (d) None of the above

Sol. (c) The equation of normal line at an arbitrary point $P(a \sec \phi, a \tan \phi)$ is

$$ax \sin \phi + ay = (a^2 + a^2) \tan \phi = 2a^2 \tan \phi$$

$$\Rightarrow x \sin \phi + y = 2a \tan \phi \quad \dots (i)$$

Again, the equation of the perpendicular from the centre i.e. the origin Eq. (i) is

$$x - \sin \phi \cdot y = 0 \quad \dots (ii)$$

The required locus is obtained by eliminating ϕ from Eqs. (i) and (ii).

$$\text{From Eq. (i), } (x \sin \phi + y)^2 = 4a^2 \tan^2 \phi$$

$$\Rightarrow (x \sin \phi + y)^2 = 4a^2 \frac{\sin^2 \phi}{1 - \sin^2 \phi} \quad \dots (iii)$$

$$\text{But from Eq. (ii), } \sin \phi = \frac{x}{y}$$

Therefore, putting the value of $\sin \phi$ in Eq. (iii), we get

$$\left(x \cdot \frac{x}{y} + y\right)^2 = 4a^2 \cdot \frac{\frac{x^2}{y^2}}{1 - \frac{x^2}{y^2}}$$

$$\Rightarrow \frac{(x^2 + y^2)^2}{y^2} = 4a^2 \cdot \frac{x^2}{y^2} \cdot \frac{y^2}{y^2 - x^2} = \frac{4a^2 x^2}{y^2 - x^2}$$

$$\Rightarrow (x^2 + y^2)^2 (y^2 - x^2) = 4a^2 x^2 y^2$$

➤ **Example 37.** The locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is

- (a) $(y^2 - x^2)^2 = 4a^2x^2y^2$ (b) $(y^2 - x^2)^3 = 4a^2x^2y^2$
(c) $(y^2 + x^2)^3 = 4a^2x^2y^2$ (d) $(y^2 - x^2)^3 = a^2x^2y^2$

Sol. (b) The equation of the normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point (x_1, y_1) is $\frac{x - x_1}{\frac{x_1}{a^2}} = \frac{y - y_1}{-\frac{y_1}{b^2}}$

So, the equation of the normal to the hyperbola $x^2 - y^2 = a^2$ at the point $(a \sec \theta, a \tan \theta)$ is

$$\frac{x - a \sec \theta}{\sec \theta} = \frac{y - a \tan \theta}{-\tan \theta} \Rightarrow \frac{x}{\sec \theta} - a = -\frac{y}{\tan \theta} + a$$

$$\Rightarrow \frac{x}{\sec \theta} + \frac{y}{\tan \theta} = 2a \quad \dots (i)$$

Again, the equation to the chord of the given hyperbola whose middle point is (α, β) , is

$$x\alpha - y\beta = \alpha^2 - \beta^2 \quad \dots (ii)$$

which is obtained from the formula $S_1 = T$.

Since, Eqs. (i) and (ii) represent the same line, therefore on comparing Eqs. (i) and (ii), we get

$$\alpha \sec \theta = -\beta \tan \theta = \frac{\alpha^2 - \beta^2}{2a}$$

$$\Rightarrow \sec \theta = \frac{\alpha^2 - \beta^2}{2a\alpha} \quad \text{and} \quad \tan \theta = -\frac{\alpha^2 - \beta^2}{2a\beta}$$

$$\Rightarrow \sec^2 \theta - \tan^2 \theta = \frac{(\alpha^2 - \beta^2)^2}{4a^2\alpha^2} - \frac{(\alpha^2 - \beta^2)^2}{4a^2\beta^2}$$

$$\Rightarrow 1 = \frac{(\alpha^2 - \beta^2)^2}{4a^2} \left\{ \frac{1}{\alpha^2} - \frac{1}{\beta^2} \right\}$$

$$\Rightarrow 4a^2\alpha^2\beta^2 = (\alpha^2 - \beta^2)^2(\beta^2 - \alpha^2)$$

$$\Rightarrow (\beta^2 - \alpha^2)^3 = 4a^2\alpha^2\beta^2$$

$\therefore$ The locus of (α, β) is

$$(y^2 - x^2)^3 = 4a^2x^2y^2$$

➤ **Example 38.** The locus of poles of normal chords of the rectangular hyperbola $xy = c^2$ is the curve

- (a) $(x^2 - y^2)^2 + 4c^2xy = 0$ (b) $(x^2 - y^2) + c^2xy = 0$
(c) $(x^2 + y^2)^2 + 4c^2xy = 0$ (d) None of these

Sol. (a) The equation of the normal to the curve $xy = c^2$ is

$$xt^2 - y = ct^3 - \frac{c}{t} \quad \text{i.e. } xt^3 - ty = ct^4 - c \quad \dots (i)$$

Let (α, β) be the pole of Eq. (i) w.r.t. the given curve.

Also, the equation of the polar of (α, β) w.r.t. $xy = c^2$

$$\text{(i.e. } 2xy = 2c^2) \text{ is } x\beta + y\alpha = 2c^2 \quad \dots (ii)$$

Since, Eqs. (i) and (ii) are identical, therefore

$$\frac{\beta}{t^3} = \frac{\alpha}{-t} = \frac{2c^2}{ct^4 - c}$$

$$\text{From the first two, } \frac{\beta}{t^2} = -\alpha \Rightarrow t^2 = -\frac{\beta}{\alpha}$$

From the second and third, we have

$$\alpha = \frac{-2c^2t}{ct^4 - c} = \frac{-2ct}{t^4 - 1} = \frac{-2ct}{\frac{\beta^2}{\alpha^2} - 1} = \frac{-2c\alpha^2}{\beta^2 - \alpha^2}$$

$$\Rightarrow 1 = \frac{2c\alpha}{\alpha^2 - \beta^2} \Rightarrow \alpha^2 - \beta^2 = 2c\alpha t$$

$$\Rightarrow (\alpha^2 - \beta^2)^2 = 4c^2\alpha^2t^2 = 4c^2\alpha^2\left(-\frac{\beta}{\alpha}\right) \quad \left[\because t^2 = -\frac{\beta}{\alpha} \right]$$

$$\Rightarrow (\alpha^2 - \beta^2)^2 = -4c^2\alpha\beta \Rightarrow (\alpha^2 - \beta^2)^2 + 4c^2\alpha\beta = 0$$

So, the locus of (α, β) is

$$(x^2 - y^2)^2 + 4c^2xy = 0$$

Work Book Exercise 16.3

- If PN is the perpendicular from a point on a rectangular hyperbola to its asymptotes, the locus of the mid-point of PN is
 - a circle
 - a parabola
 - an ellipse
 - a hyperbola
- If angles subtended by any chord of a rectangular hyperbola at the centre is α and angle between the tangents at ends of chord is β , then
 - $\alpha = 2\beta$
 - $\beta = 2\alpha$
 - $\alpha + \beta = \pi$
 - $\alpha + \beta = \frac{\pi}{2}$
- For any real t , $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Then, the area bounded by this hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$, is
 - $t_1 t_2$
 - $t^{-1} t_2$
 - $t t^{-1}$
 - t_1
- The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is
 - $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$
 - $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$
 - $2x^2 + 5xy + 2y^2 = 0$
 - None of the above
- The equation of the line passing through the centre of a rectangular hyperbola is $x - y - 1 = 0$. If one of its asymptotes is $3x - 4y - 6 = 0$, then the equation of the other asymptote is
 - $4x + 3y + 17 = 0$
 - $4x - 3y + 8 = 0$
 - $3x - 2y + 15 = 0$
 - None of these
- The asymptotes of a hyperbola are parallel to $2x + 3y = 0$ and $3x + 2y = 0$. Its centre is $(4, 2)$ and hyperbola passes through $(5, 3)$. Then, its equation is
 - $x^2 + xy + y^2 - 74x - 76y + 199 = 0$
 - $6x^2 + 13xy + 6y^2 - 74x - 76y + 199 = 0$
 - $6x^2 + 13xy + y^2 - 74x - 76y + 199 = 0$
 - None of the above

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. A series of hyperbolas is drawn having a common transverse axis of length $2a$. Then, the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is

- (a) $(x^2 - y^2)^2 = 4x^2(x^2 - a^2)$
 (b) $(x^2 - y^2)^2 = x^2(x^2 - a^2)$
 (c) $(x^2 - y^2) = 4x^2(x^2 - a^2)$
 (d) None of the above

Sol. The equations of series of a hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{\lambda^2} = 1, \text{ where, } \lambda \text{ is a parameter.}$$

The asymptotes of this hyperbola is $\frac{x}{a} = \pm \frac{y}{\lambda}$.

Suppose (x', y') is a point P on the hyperbola which is equidistant from the transverse axis and an asymptote.

$$\text{Then, } \frac{x'^2}{a^2} - \frac{y'^2}{\lambda^2} = 1 \text{ and } y' = \frac{\left| \frac{x'}{a} - \frac{y'}{\lambda} \right|}{\sqrt{\frac{1}{a^2} + \frac{1}{\lambda^2}}}$$

$$\text{i.e. } \frac{y'^2}{\lambda^2} = \frac{x'^2}{a^2} - 1$$

$$\text{and } y'^2 \left(\frac{1}{a^2} + \frac{1}{\lambda^2} \right) = \frac{x'^2}{a^2} + \frac{y'^2}{\lambda^2} - \frac{2x'y'}{a\lambda}$$

The second relation gives on simplification,

$$(y'^2 - x'^2)^2 = \frac{4x'^2 y'^2 a^2}{\lambda^2} = 4x'^2(x'^2 - a^2) \quad [\text{by the first relation}]$$

So, the locus of P is

$$(y^2 - x^2)^2 = 4x^2(x^2 - a^2)$$

Hence, (a) is the correct answer.

Ex 2. A variable straight line of slope 4 intersects the hyperbola $xy = 1$ at two points. Then, the locus of the point which divides the line segment between these points in the ratio 1 : 2, is

- (a) $16x^2 + y^2 + 8xy = 2$
 (b) $x^2 + y^2 + 10xy = 2$
 (c) $16x^2 + y^2 + 10xy = 2$
 (d) $16x^2 + y^2 + 10xy = 0$

Sol. Let the line be $y = 4x + c$. It meets the curve $xy = 1$ at

$$\Rightarrow \begin{aligned} x(4x + c) &= 1 \\ 4x^2 + cx &= 1 \end{aligned}$$

$$\Rightarrow x_1 + x_2 = -\frac{c}{4}$$

$$\text{Also, } y(y - c) = 4$$

$$\Rightarrow y^2 - cy - 4 = 0$$

$$\Rightarrow y_1 + y_2 = c$$

Let the point which divides the line segment in the ratio 1 : 2 be (h, k) .

$$\Rightarrow \frac{x_1 + 2x_2}{3} = h$$

$$\Rightarrow x_2 = 3h + \frac{c}{4}$$

$$\Rightarrow x_1 = -\frac{c}{2} - 3h$$

$$\text{Also, } \frac{y_1 + 2y_2}{3} = k$$

$$\Rightarrow y_2 = 3k - c$$

$$\Rightarrow y_1 = -3k + 2c$$

Now, (h, k) lies on the line $y = 4x + c$

$$\Rightarrow k = 4h + c$$

$$\Rightarrow c = k - 4h$$

$$\Rightarrow x_1 = -\frac{k}{2} + 2h - 3h = -h - \frac{k}{2}$$

$$\text{and } y_1 = -3k + 2k - 8h = -k - 8h$$

$$\Rightarrow \left(h + \frac{k}{2} \right) (k + 8h) = 1$$

$$\Rightarrow hk + 8h^2 + \frac{k^2}{2} + 4hk = 1$$

$$\Rightarrow 16h^2 + k^2 + 10hk = 2$$

$$\therefore \text{Locus of } (h, k) \text{ is } 16x^2 + y^2 + 10xy = 2.$$

Hence, (c) is the correct answer.

Aliter

Let $P(h, k)$ divide the chord of the hyperbola $xy = 1$ in the ratio 1 : 2. The equation of the line of slope 4 is

$$\frac{x - h}{\frac{1}{\sqrt{17}}} = \frac{y - k}{\frac{4}{\sqrt{17}}} = r$$

$$\text{Any other point on this line is } \left(h + \frac{r}{\sqrt{17}}, k + \frac{4r}{\sqrt{17}} \right).$$

Since, this point lies on $xy = 1$.

$$\left(h + \frac{r}{\sqrt{17}} \right) \left(k + \frac{4r}{\sqrt{17}} \right) = 1$$

$$\Rightarrow 4r^2 + r(4h + k)\sqrt{17} + 17(hk - 1) = 0 \quad \dots(i)$$

Also, the point (h, k) divides the chord in the ratio 1 : 2.

The roots of Eq. (i) are r_1 and $-2r_1$, so that

$$-2r_1 + r_1 = \frac{-(4h + k)\sqrt{17}}{4}$$

$$\text{and } -2r_1 \cdot r_1 = \frac{17(hk - 1)}{4}$$

$$\Rightarrow r_1 = \frac{(4h + k)\sqrt{17}}{4}$$

$$\text{and } r_1^2 = \frac{-17(hk - 1)}{8}$$

$$\therefore \frac{17}{16}(4h + k)^2 + \frac{17}{8}(hk - 1) = 0, \text{ so that locus } (h, k) \text{ is}$$

$$16x^2 + y^2 + 10xy = 2.$$

Hence, (c) is the correct answer.

Ex 3. If A, B and P are three points on the hyperbola $xy = c^2$ such that AB subtends a right angle at P . Then, AB is parallel to
 (a) tangent at P (b) normal at P
 (c) axis of hyperbola (d) None of these

Sol. Let coordinates of A, B and P be

$$\left(ct_1, \frac{c}{t_1}\right), \left(ct_2, \frac{c}{t_2}\right) \text{ and } \left(ct_3, \frac{c}{t_3}\right)$$

$$\text{Slope of } PA = -\frac{1}{t_1 t_3}$$

$$\text{Slope of } PB = -\frac{1}{t_2 t_3}$$

$$\Rightarrow t_1 t_2 t_3^2 = -1$$

$$\text{Slope of } AB = -\frac{1}{t_1 t_2}$$

$$\text{Slope of normal at } P \text{ is } t_3^2 \text{ i.e. } -\frac{1}{t_1 t_2}$$

which is same as slope of AB .

$\Rightarrow AB$ is parallel to normal at P .

Hence, (b) is the correct answer.

Ex 4. If two points P and Q on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, whose centre C is such that CP is

perpendicular to CQ , $a < b$, then $\frac{1}{CP^2} + \frac{1}{CQ^2}$

is equal to

$$(a) -\frac{1}{a^2} + \frac{1}{b^2} \quad (b) \frac{1}{a^2} + \frac{1}{b^2}$$

$$(c) -\frac{1}{a^2} - \frac{1}{b^2} \quad (d) \frac{1}{a^2} - \frac{1}{b^2}$$

Sol. Let $CP = r_1$ be inclined to transverse axis at an angle θ , so that P is $(r_1 \cos \theta, r_1 \sin \theta)$ and P lies on the hyperbola.

$$\Rightarrow r_1^2 \left(\frac{\cos^2 \theta}{a^2} - \frac{\sin^2 \theta}{b^2} \right) = 1$$

$$\text{Replacing } \theta \text{ by } 90^\circ + \theta, \text{ then } r_2^2 \left(\frac{\sin^2 \theta}{a^2} - \frac{\cos^2 \theta}{b^2} \right) = 1$$

$$\Rightarrow \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{\cos^2 \theta}{a^2} - \frac{\sin^2 \theta}{b^2} + \frac{\sin^2 \theta}{a^2} - \frac{\cos^2 \theta}{b^2}$$

$$\Rightarrow \frac{1}{r_1^2} + \frac{1}{r_2^2} = \cos^2 \theta \left(\frac{1}{a^2} - \frac{1}{b^2} \right) + \sin^2 \theta \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$$

$$\Rightarrow \frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

$$\Rightarrow \frac{1}{CP^2} + \frac{1}{CQ^2} = \frac{1}{a^2} - \frac{1}{b^2}$$

Hence, (d) is the correct answer.

Ex 5. The locus of all conics passing through the intersection of rectangular hyperbola is a/an

- (a) parabola (b) ellipse
 (c) hyperbola (d) None of these

Sol. We know that in the equation of rectangular hyperbola, sum of the coefficients of x^2 and y^2 should be zero.

Let two rectangular hyperbolas be

$$S_1 \equiv ax^2 + 2hxy - ay^2 + 2gx + 2fy + c = 0$$

$$S_2 \equiv d'x^2 + 2h'xy - d'y^2 + 2g'x + 2f'y + c' = 0$$

$\therefore$ Equation of any curve passing through intersection of S_1 and S_2 is $S_1 + \lambda S_2 = 0$, where λ is real parameter.

$$(a + \lambda a')x^2 + 2(h + \lambda h')xy - (a + \lambda a')y^2 + 2x(g + \lambda g') + 2y(f + \lambda f') + (c + \lambda c') = 0$$

$$\therefore \text{Coefficient of } x^2 + \text{Coefficient of } y^2 = 0$$

For all values of λ , it will represent a rectangular hyperbola.

Hence, (c) is the correct answer.

Ex 6. A line parallel to the Y -axis intersects the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate

hyperbola at P and Q respectively. Then, the normals at P and Q to the respective curves meet on

- (a) Y -axis (b) X -axis
 (c) asymptote (d) None of these

Sol. Let the line parallel to Y -axis be $x = \alpha$, then points are

$$P \equiv \left(\alpha, \frac{b}{a} \sqrt{\alpha^2 - a^2} \right) \text{ and } Q \equiv \left(\alpha, \frac{b}{a} \sqrt{\alpha^2 + a^2} \right)$$

Equation of normals are

$$y - \frac{b}{a} \sqrt{\alpha^2 - a^2} = \frac{-a^2}{b^2} \cdot \frac{b}{a\alpha} \sqrt{\alpha^2 - a^2} (x - \alpha)$$

$$\text{and } y - \frac{b}{a} \sqrt{\alpha^2 + a^2} = \frac{-a^2}{b^2} \cdot \frac{b}{a\alpha} \sqrt{\alpha^2 + a^2} (x - \alpha)$$

Now, putting $y = 0$ in both the normal, we get

$$x = \left(\frac{b^2}{a^2} + 1 \right) \alpha$$

So, they intersect on X -axis.

Hence, (b) is the correct answer.

Ex 7. The foot of perpendicular drawn from the focus to any tangent of the hyperbola lies on

- (a) $x^2 + y^2 = a^2$ (b) $x^2 + y^2 = 2a^2$
 (c) $x^2 + y^2 = 3a^2$ (d) None of these

Sol. Let hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Equation of any tangent is

$$y = mx \pm \sqrt{a^2 m^2 - b^2} \quad \dots(i)$$

A line perpendicular to it and passing through the focus is

$$my + x = ae \quad \dots(ii)$$

Let (h, k) be their point of intersection.

On squaring and adding Eqs. (i) and (ii), we get

$$(k^2 + h^2)(1 + m^2) = a^2 m^2 - b^2 + a^2 e^2 = a^2(m^2 + 1)$$

$$\Rightarrow h^2 + k^2 = a^2$$

$\Rightarrow$ The locus of (h, k) is $x^2 + y^2 = a^2$, which is the auxiliary circle.

Hence, (a) is the correct answer.

- Ex 8.** $ABCD$ is a cyclic quadrilateral in which the three vertices A , B and C lie on a rectangular hyperbola. Then, the reflection of the fourth vertex D with respect to the origin is the
- orthocentre of $\triangle ABC$
 - centroid of $\triangle ABC$
 - incentre of $\triangle ABC$
 - None of the above

Sol. Let the equation of the hyperbola be $xy = c^2$ and equation of the circle passing through points A , B , C and D be

$$x^2 + y^2 + 2gx + 2fy + k = 0$$

Again, let points be $A(t_1)$, $B(t_2)$, $C(t_3)$ and $D(t_4)$.

$\Rightarrow t_1, t_2, t_3$ and t_4 are the roots of the equation

$$c^2 t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0 \quad \dots(i)$$

$$\Rightarrow t_1 t_2 t_3 t_4 = 1 \Rightarrow t_4 = \frac{1}{t_1 t_2 t_3}$$

$$\Rightarrow \text{Coordinates of } D \text{ are } \left[\frac{c}{t_1 t_2 t_3}, ct_1 t_2 t_3 \right]$$

The orthocentre of $\triangle ABC$ is

$$H \equiv \left[-\frac{c}{t_1 t_2 t_3}, -ct_1 t_2 t_3 \right]$$

$\Rightarrow$ The reflection of H with respect to the origin is point D .

Hence, (a) is the correct answer.

- Ex 9.** Equation of a rectangular hyperbola whose asymptotes are $x=3$ and $y=5$ and passing through $(7, 8)$, is
- $xy - 3y + 5x + 3 = 0$
 - $xy + 3y + 4x + 3 = 0$
 - $xy - 3y + 5x - 3 = 0$
 - $xy - 3y - 5x + 3 = 0$

Sol. The equation of rectangular hyperbola is

$$(x-3)(y-5) + \lambda = 0$$

which passes through $(7, 8)$.

$$\Rightarrow 4 \cdot 3 + \lambda = 0 \Rightarrow \lambda = -12$$

$$\therefore xy - 5x - 3y + 15 - 12 = 0$$

$$\Rightarrow xy - 3y - 5x + 3 = 0$$

Hence, (d) is the correct answer.

- Ex 10.** From a point (h, k) , normals are drawn to the

ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Then, the feet of these

normals lie on a hyperbola, whose asymptotes are

$$(a) (a^2 - b^2)x - a^2h = 0 \text{ and } (a^2 + b^2)y + b^2k = 0$$

$$(b) (a^2 - b^2)x - a^2h = 0 \text{ and } (a^2 - b^2)y + b^2k = 0$$

$$(c) (a^2 + b^2)x + a^2h = 0 \text{ and } (a^2 + b^2)y + b^2k = 0$$

$$(d) (a^2 + b^2)x + a^2h = 0 \text{ and } (a^2 - b^2)y + b^2k = 0$$

Sol. Let normal points be (x_1, y_1) , (x_2, y_2) , (x_3, y_3) and (x_4, y_4) . Equation of normals is

$$\frac{a^2(x - x_i)}{x_i} = \frac{b^2(y - y_i)}{y_i}, \text{ where } i = 1, 2, 3, 4$$

These normals pass through (h, k) , so

$$\frac{a^2(h - x_i)}{x_i} = \frac{b^2(k - y_i)}{y_i}$$

$$\text{Let us consider a curve } \frac{a^2(h - x)}{x} = \frac{b^2(k - y)}{y}$$

$$\Rightarrow a^2hy - xya^2 = b^2kx - b^2yx$$

$$\Rightarrow xy(b^2 - a^2) + a^2hy - b^2kx = 0$$

which is equation of hyperbola.

Its asymptotes are given by

$$xy(b^2 - a^2) + a^2hy - b^2kx + \lambda = 0$$

$$\therefore \Delta = 0$$

$$\Rightarrow 0 + 2 \left(\frac{-b^2k}{2} \right) \left(\frac{a^2h}{2} \right) \left(\frac{b^2 - a^2}{2} \right) - \lambda \frac{(b^2 - a^2)^2}{4} = 0$$

$$\Rightarrow \lambda = \frac{-(b^2 a^2 kh)}{b^2 - a^2}$$

Asymptotes are given by

$$(b^2 - a^2)xy + a^2hy - b^2kx - \frac{b^2 a^2 kh}{b^2 - a^2} = 0$$

$$\Rightarrow (a^2 - b^2)^2 xy - a^2(a^2 - b^2)hy + b^2(a^2 - b^2)kx - a^2 b^2 kh = 0$$

$$\Rightarrow [(a^2 - b^2)x - a^2h][(a^2 - b^2)y + b^2k] = 0$$

Hence, (b) is the correct answer.

- Ex 11.** The angle between the rectangular hyperbolas $(y - mx)(my + x) = a^2$ and

$$(m^2 - 1)(y^2 - x^2) + 4mxy = b^2 \text{ is}$$

$$(a) \frac{\pi}{2} \quad (b) \frac{\pi}{3}$$

$$(c) \frac{\pi}{4} \quad (d) \frac{\pi}{6}$$

Sol. For first hyperbola,

$$(y - mx) \left(m \frac{dy}{dx} + 1 \right) + (my + x) \left(\frac{dy}{dx} - m \right) = 0$$

$$\Rightarrow \frac{dy}{dx} (my + x + my - m^2x) + y - mx - m^2y - mx = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-y + m^2y + 2mx}{2my + x - m^2x} = m_1$$

For second hyperbola,

$$(m^2 - 1) \left(2y \frac{dy}{dx} - 2x \right) + 4m \left(x \frac{dy}{dx} + y \right) = 0$$

$$\Rightarrow \frac{dy}{dx} [2y(m^2 - 1) + 4mx] = -4my + 2x(m^2 - 1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2my + m^2x - x}{m^2y - y + 2mx} = m_2$$

$$\therefore m_1 m_2 = -1$$

$$\therefore \text{Angle between the hyperbolas} = \frac{\pi}{2}$$

Hence, (a) is the correct answer.

Ex 12. From the centre C of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,

perpendicular CN is drawn on any tangent to it at the point $P(a \sec \theta, b \tan \theta)$ in the first quadrant. Then, the value of θ , so that area of ΔCPN is maximum, is

- (a) $\sin^{-1} \frac{a}{b}$ (b) $\sin^{-1} \frac{b}{a}$ (c) $\cos^{-1} \frac{a}{b}$ (d) $\tan^{-1} \frac{b}{a}$

Sol. Equation of tangent is

$$b \sec \theta x - a \tan \theta y - ab = 0$$

$$\therefore CN = \frac{ab}{\sqrt{b^2 \sec^2 \theta + a^2 \tan^2 \theta}}$$

Equation of normal at P is

$$ax \cos \theta + by \cot \theta = a^2 + b^2$$

$$\therefore CM = \frac{a^2 + b^2}{\sqrt{a^2 \cos^2 \theta + b^2 \cot^2 \theta}}$$

$$A = \frac{1}{2} CM \times CN$$

$$= \frac{ab(a^2 + b^2)}{\sqrt{a^2 b^2 + b^4 \operatorname{cosec}^2 \theta + a^4 \sin^2 \theta}}$$

A is maximum when $b^4 \operatorname{cosec}^2 \theta + a^4 \sin^2 \theta$ is minimum.

$$\text{Now, } b^4 \operatorname{cosec}^2 \theta + a^4 \sin^2 \theta \geq 2a^2 b^2$$

$$A_{\max} = \frac{ab(a^2 + b^2)}{2ab} = \frac{a^2 + b^2}{2}$$

$$\text{where, } \theta = \sin^{-1} \frac{b}{a}$$

Hence, (b) is the correct answer.

Ex 13. A point P moves such that sum of the slopes of the normals drawn from it to the hyperbola $xy = 4$ is equal to the sum of ordinates of feet of normals. Then, the locus of P is

- (a) a parabola (b) a hyperbola
(c) an ellipse (d) a circle

Sol. Any point on the hyperbola $xy = 4$ is $\left(2t, \frac{2}{t}\right)$.

$$\text{Now, normal at } \left(2t, \frac{2}{t}\right) \text{ is } y - \frac{2}{t} = t^2(x - 2t) \quad [\text{its slope is } t^2]$$

If the normal passes through $P(h, k)$, then

$$k - \frac{2}{t} = t^2(h - 2t)$$

$$\Rightarrow 2t^4 - ht^3 + tk - 2 = 0 \quad \dots(i)$$

Roots of Eq. (i) give parameters of foot of normals passing through (h, k) . Let roots be t_1, t_2, t_3 and t_4 , then

$$t_1 + t_2 + t_3 + t_4 = \frac{h}{2} \quad \dots(ii)$$

$$t_1 t_2 + t_1 t_3 + t_1 t_4 + t_2 t_3 + t_2 t_4 + t_3 t_4 = 0 \quad \dots(iii)$$

$$t_1 t_2 t_3 + t_1 t_2 t_4 + t_1 t_3 t_4 + t_2 t_3 t_4 = -\frac{k}{2} \quad \dots(iv)$$

$$\text{and } t_1 t_2 t_3 t_4 = -1 \quad \dots(v)$$

On dividing Eq. (iv) by Eq. (v), we get

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = \frac{k}{2}$$

$$\Rightarrow y_1 + y_2 + y_3 + y_4 = k$$

From Eqs. (ii) and (iii),

$$t_1^2 + t_2^2 + t_3^2 + t_4^2 = \frac{h^2}{4}$$

$$\text{Given that } \frac{h^2}{4} = k \Rightarrow \text{Locus of } (h, k) \text{ is } x^2 = 4y$$

which is a parabola.

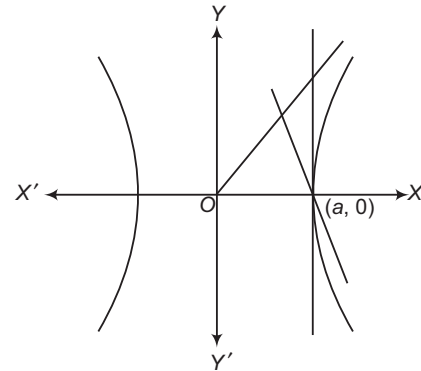
Hence, (a) is the correct answer.

Ex 14. Portion of the asymptotes of hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (between centre and the tangent at vertex) in the first quadrant is cut by the line

$y + \lambda(x - a) = 0$ (λ is a parameter), then

- (a) $\lambda \in R$ (b) $\lambda \in (0, \infty)$
(c) $\lambda \in (-\infty, 0)$ (d) None of these

Sol. The line $y + \lambda(x - a) = 0$ will intersect the portion of the asymptote in the first quadrant only if its slope is negative.



$$\text{Hence } -\lambda < 0 \Rightarrow \lambda > 0$$

$$\therefore \lambda \in (0, \infty)$$

Hence, (b) is the correct answer.

Ex 15. If the lines $x = h$ and $y = k$ are conjugate with respect to the hyperbola $xy = c^2$, then the locus of (h, k) is

- (a) $xy = 2c^2$ (b) $xy = c^2$
(c) $2xy = c^2$ (d) $xy = 4c^2$

Sol. Let (x_1, y_1) be the pole of the line $x - h = 0$ with respect to the hyperbola $xy = c^2$. Then, the equation of the polar is $xy_1 + yx_1 = 2c^2$.

Thus, the equations $x - h = 0$ and $xy_1 + yx_1 = 2c^2$ represent the same line.

$$\therefore \frac{y_1}{1} = \frac{x_1}{0} = \frac{2c^2}{h}$$

$$\Rightarrow x_1 = 0 \text{ and } y_1 = \frac{2c^2}{h}$$

Since, the lines $x - h = 0$ and $y - k = 0$ are conjugate lines. Therefore, the pole of the line $x - h = 0$ lies on the line $y - k = 0$.

$$\therefore y_1 - k = 0$$

$$\Rightarrow \frac{2c^2}{h} - k = 0 \Rightarrow hk = 2c^2$$

So, the locus of (h, k) is $xy = 2c^2$.

Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 16. The equation $16x^2 - 3y^2 - 32x + 12y - 44 = 0$ represents a hyperbola

- (a) the length of whose transverse axis is $4\sqrt{3}$
- (b) the length of whose conjugate axis is 4
- (c) whose centre is (1, 2)
- (d) whose eccentricity is $\sqrt{\frac{19}{3}}$

Sol. $16(x^2 - 2x + 1) - 3(y^2 - 4y + 4) = 48$

$$\Rightarrow \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$$

So, length of transverse axis is $\sqrt{3}$ and length of conjugate axis is 4.

$$\text{Centre} = (1, 2) \text{ and } e = \sqrt{\frac{19}{3}}$$

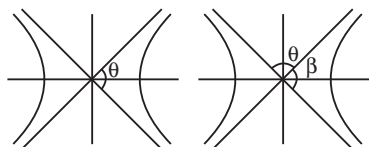
Hence, (b), (c) and (d) are the correct answers.

Ex 17. If e is the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and θ is angle between the asymptotes, then $\cos(\theta/2)$ is equal to

- (a) $\frac{1}{e}$
- (b) $\frac{1}{2e}$
- (c) $\frac{\sqrt{e^2 - 1}}{e}$
- (d) 0

Sol. $\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

$$\cos \frac{\theta}{2} = \frac{a}{\sqrt{a^2 + b^2}} = \frac{a}{\sqrt{a^2 e^2}} = \frac{1}{e}$$



$$\cos \frac{\theta}{2} = \sin \frac{\beta}{2} = \frac{\sqrt{e^2 - 1}}{e}$$

Hence, (a) and (c) are the correct answers.

Ex 18. If the circle $x^2 + y^2 = a^2$ intersect the hyperbola $xy = c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then

- (a) $x_1 + x_2 + x_3 + x_4 = 0$
- (b) $y_1 + y_2 + y_3 + y_4 = 0$
- (c) $y_1 y_2 y_3 y_4 = c^4$
- (d) $x_1 x_2 x_3 x_4 = c^4$

Sol. On solving $xy = c^2$ and $x^2 + y^2 = a^2$, we get

$$\begin{aligned} x^2 + \frac{c^4}{x^2} &= a^2 \\ \Rightarrow x^4 - a^2 x^2 + c^4 &= 0 \\ \Rightarrow \Sigma x_i &= 0; \Sigma y_i = 0 \\ x_1 x_2 x_3 x_4 &= c^4 \\ y_1 y_2 y_3 y_4 &= c^4 \end{aligned}$$

Hence, (a), (b), (c) and (d) are the correct answers.

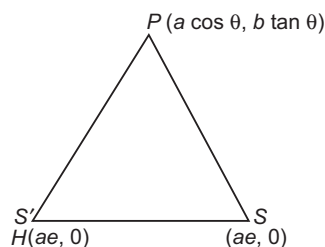
Ex 19. If P is a point on a hyperbola, then

- (a) locus of excentre of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci), is triangle at vertex
- (b) locus of excentre of the circle described opposite to $\angle S'$, is hyperbola
- (c) locus of excentre of the circle described opposite to $\angle P$ for $\Delta PSS'$ (S, S' are foci), is hyperbola
- (d) locus of excentre of the circle described opposite to $\angle S'$, is tangent at vertex

Sol. Let (h, k) be excentre, then

$$h = \frac{a(ae \sec \theta + a) - ae(ae \sec \theta - a) - 2ae(a \sec \theta)}{2ae(\sec \theta - 1)}$$

$$h = -a \Rightarrow x = -a \quad [\text{for } a, \sec \theta > 0]$$



$$\text{Similarly, } x = a \quad [\text{for } a, \sec \theta < 0]$$

$$\Rightarrow \text{Locus is } x^2 = a^2$$

Again, let (h, k) be excentre opposite to $\angle S'$.

$$\therefore h = \frac{2a^2 e \sec \theta + a^2 e^2 \sec \theta + a^2 e + a^2 e^2 \sec \theta - a^2 e}{2a + 2ae}$$

$$\Rightarrow h = ae \sec \theta, k = \frac{2aeb \tan \theta}{2a + 2ae}$$

$\Rightarrow$ Locus is the hyperbola.

Hence, (a) and (b) are the correct answers.

Ex 20. If foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ coincide with the foci of

$\frac{x^2}{25} + \frac{y^2}{9} = 1$ and eccentricity of the hyperbola is 2, then

- (a) $a^2 + b^2 = 16$
- (b) there is no director circle to the hyperbola
- (c) centre of the director circle is (0, 0)
- (d) length of latusrectum of the hyperbola is 12

Sol. For an ellipse, $a = 5$

$$\text{and } e = \sqrt{\frac{25 - 9}{25}} = \frac{4}{5}$$

$$\therefore ae = 4$$

$\therefore$ The foci are $(-4, 0)$ and $(4, 0)$.

For the hyperbola, $ae = 4$ and $e = 2$

$$\therefore a = 2$$

$$\Rightarrow b^2 = 4(4 - 1) = 12$$

$$\Rightarrow b = \sqrt{12}$$

Hence, (a), (b) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex.Nos. 21-25) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

Ex 21. Let $a, b, \alpha \in \mathbb{R} - \{0\}$, where a, b are constants and α is a parameter.

Statement I All the members of the family of hyperbolas $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{\alpha^2}$ have the same pair of asymptotes.

Statement II Change in α , does not change the slopes of the asymptotes of a member of the family $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{\alpha^2}$.

Sol. Both statements are true and Statement II is true reason for Statement I, as for any member, semi-transverse and semi-conjugate axes are $\frac{a}{\alpha}$ and $\frac{b}{\alpha}$ respectively and hence asymptotes are always $y = \pm \frac{b}{a}x$.

Hence, (a) is the correct answer.

Ex 22. Statement I Ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and $12x^2 - 4y^2 = 27$ intersect each other at right angle.

Statement II Whenever confocal conics intersect, they intersect each other orthogonally.

Sol. Here, $e = \frac{3}{5}$, $a = 5$

$\therefore$ Foci are $(\pm 3, 0)$.

For hyperbola, $\frac{x^2}{27/12} - \frac{y^2}{27/4} = 1$

$$\Rightarrow e = \sqrt{\frac{12+4}{4}} = 2, a = \frac{3}{2}$$

Now, foci are $(\pm 3, 0)$.

$\therefore$ The two conics are confocal.

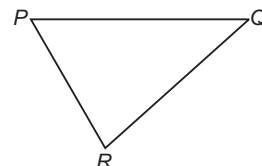
Hence, (a) is the correct answer.

Ex 23. Statement I A bullet is fired and hit a target. If an observer in the same plane heard two sounds the crack of the rifle and the thud of the bullet striking the target at the same instant, then locus of the observer is hyperbola, where velocity of sound is smaller than velocity of bullet.

Statement II If difference of distances of a point P from the two fixed points is constant and less than the distance between the fixed points, then locus of P is a hyperbola.

Sol. Let P be the position of the gun and Q be the position of the target.

Again, let u be the velocity of sound, v be the velocity of bullet and R be the position of the man, then we have



$$\frac{PR}{u} - \frac{QR}{u} = \frac{PQ}{v}$$

i.e. $\frac{PR}{u} - \frac{QR}{u} = \frac{PQ}{v}$

$$\text{i.e. } PR - QR = \frac{u}{v} \cdot PQ = \text{Constant and } \frac{u}{v} \cdot PQ < PQ$$

$\therefore$ Locus of R is a hyperbola.

Hence, (a) is the correct answer.

Ex 24. Statement I With respect to a hyperbola

$\frac{x^2}{9} - \frac{y^2}{16} = 1$, if perpendiculars are drawn from a point $(5, 0)$ on the lines $3y \pm 4x = 0$, then their feet lie on circle $x^2 + y^2 = 16$.

Statement II If from any foci of a hyperbola, perpendicular are drawn on the asymptotes of the hyperbola, then their feet lie on auxiliary circle.

Sol. $(5, 0)$ is a focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ and

$3y \pm 4x = 0$ are asymptotes, then the auxiliary circle is $x^2 + y^2 = 9$.

$\therefore$ The feet lie on $x^2 + y^2 = 9$.

So, Statement I is false and Statement II is true.

Hence, (d) is the correct answer.

Ex 25. Statement I If eccentricity of a hyperbola is 2, then eccentricity of its conjugate hyperbola is $\frac{2}{\sqrt{3}}$.

Statement II If e and e' are the eccentricities of a hyperbola and its conjugate hyperbola, then $\frac{1}{e^2} + \frac{1}{e'^2} = 1$.

Sol. Statement II is true.

$$\therefore \frac{1}{2^2} + \left(\frac{\sqrt{3}}{2}\right)^2 = 1$$

$\therefore$ Statement I is also true.

Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage I (Ex. Nos. 26-28) If P is a variable point and F_1 and F_2 are two fixed points such that $|PF_1 - PF_2| = 2a$. Then, the locus of the point P is a hyperbola, with points F_1 and F_2 as the two foci ($F_1F_2 > 2a$). If $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is a hyperbola, then its conjugate hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. Let $P(x, y)$ be a variable point such that

$$|\sqrt{(x-1)^2 + (y-2)^2} - \sqrt{(x-5)^2 + (y-5)^2}| = 3.$$

Ex 26. If the locus of the point P represents a hyperbola of eccentricity e , then the eccentricity e' of the corresponding conjugate hyperbola is

- (a) $\frac{5}{3}$ (b) $\frac{4}{3}$
(c) $\frac{5}{4}$ (d) $\frac{3}{\sqrt{7}}$

Sol. Distance between the foci (1, 2) and (5, 5) is 5.

$$\therefore 2ae = 5$$

$$\therefore e = \frac{5}{3} \quad [\because 2a = 3]$$

$$\text{Now, } \frac{1}{e^2} + \frac{1}{e'^2} = 1$$

$$\Rightarrow e' = \frac{5}{4}$$

Hence, (c) is the correct answer.

Ex 27. Locus of intersection of two perpendicular tangents to the given hyperbola is

- (a) $(x-3)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{55}{4}$
(b) $(x-3)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{25}{4}$
(c) $(x-3)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{7}{4}$
(d) None of the above

Sol. Director circle $(x-h)^2 + (y-k)^2 = a^2 - b^2$,

where (h, k) is centre, is $\left(\frac{1+5}{2}, \frac{2+5}{2}\right) = \left(3, \frac{7}{2}\right)$

$$b^2 = a^2(e^2 - 1) = \left(\frac{3}{2}\right)^2 \left[\left(\frac{5}{3}\right)^2 - 1\right] = 4$$

$$\text{Director circle, } (x-3)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{9}{4} - 4$$

$$\Rightarrow (x-3)^2 + \left(y-\frac{7}{2}\right)^2 = \frac{-7}{4}$$

which does not represent any real point.

Hence, (d) is the correct answer.

Ex 28. If origin is shifted to point $\left(3, \frac{7}{2}\right)$ and the axes are rotated through an angle θ in clockwise sense, so that equation of given hyperbola changes to the standard form $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then

θ is

- (a) $\tan^{-1}\left(\frac{4}{3}\right)$ (b) $\tan^{-1}\left(\frac{3}{4}\right)$
(c) $\tan^{-1}\left(\frac{5}{3}\right)$ (d) $\tan^{-1}\left(\frac{3}{5}\right)$

Sol. Slope of transverse axis is $3/4$.

$$\therefore \text{Angle of rotation, } \theta = \tan^{-1} \frac{3}{4}$$

Hence, (b) is the correct answer.

Passage II (Ex. Nos. 29-31) For the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, the normal at P meets the transverse axis AA' in G and the conjugate axis BB' in g and CF is perpendicular to the normal from the centre.

Ex 29. If $PF \cdot PG = k \cdot CB^2$, then k is equal to

- (a) 2 (b) 1 (c) $1/2$ (d) 4

$$\text{Sol. } \because PF = \frac{ab}{\sqrt{b^2 \sec^2 \phi + a^2 \tan^2 \phi}}$$

$$PG^2 = \frac{b^2}{a^2} (b^2 \sec^2 \phi + a^2 \tan^2 \phi)$$

$$PF = \frac{ab}{\frac{a}{b} \cdot PG} \Rightarrow PG \cdot PF = b^2 = CB^2 \therefore k = 1$$

Hence, (b) is the correct answer.

Ex 30. $PF \cdot Pg$ equals

- (a) CA^2 (b) CF^2 (c) CB^2 (d) $CA \cdot CB$

$$\text{Sol. } Pg^2 = \frac{a^2}{b^2} (b^2 \sec^2 \phi + a^2 \tan^2 \phi)$$

$$\therefore PF = \frac{ab}{\frac{b}{a} Pg} \Rightarrow PF \cdot Pg = a^2 = CA^2$$

Hence, (a) is the correct answer.

Ex 31. Locus of middle point of G and g is a hyperbola of eccentricity

- (a) $\frac{1}{\sqrt{e^2 - 1}}$ (b) $\frac{e}{\sqrt{e^2 - 1}}$ (c) $2\sqrt{e^2 - 1}$ (d) $\frac{e}{2}$

$$\text{Sol. Locus of middle point is } \frac{x^2}{\frac{a^2 e^4}{4}} - \frac{y^2}{\frac{a^4 e^4}{4b^2}} = 1$$

$$e_1 = \sqrt{\frac{\frac{a^2 e^4}{4} + \frac{a^4 e^4}{4b^2}}{\frac{a^2 e^4}{4}}} = \frac{e}{\sqrt{e^2 - 1}}$$

Hence, (b) is the correct answer.

Passage III (Ex. Nos. 32-34) If a circle with centre $C(\alpha, \beta)$ intersects a rectangular hyperbola with centre $L(h, k)$ at four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then the mean of the four points P, Q, R, S is the mean of the points C and L . In other words, the mid-point of CL coincides with the mean point of P, Q, R, S analytically i.e.

$$\frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{\alpha + h}{2}$$

$$\text{and } \frac{y_1 + y_2 + y_3 + y_4}{4} = \frac{\beta + k}{2}$$

Ex 32. Five points are selected on a circle of radius a . The centres of the rectangular hyperbola, each passing through four of these points, all lie on a circle of radius

- (a) a (b) $2a$ (c) $\frac{a}{\sqrt{2}}$ (d) $\frac{a}{2}$

Sol. Let the circle be $x^2 + y^2 = a^2$ and centre of rectangular hyperbola be (h, k) .

Again, let given points on circle be $(a \cos \theta_i, a \sin \theta_i)$, $i = 1, 2, 3, 4, 5$, so that

$$\sum_{i=1}^4 a \cos \theta_i = \frac{0 + h}{3}$$

$$\Rightarrow \sum_{i=1}^4 a \cos \theta_i - a \cos \theta_5 = 2h$$

$$\text{Similarly, } \sum_{i=1}^4 a \sin \theta_i - a \sin \theta_5 = 2k$$

As the five points are given, $\sum_{i=1}^4 a \sin \theta_i$ and $\sum_{i=1}^4 a \cos \theta_i$

are known. Let us assume their values of be μ and λ , respectively.

$$\therefore \lambda - a \cos \theta_5 = 2h$$

$$\text{and } \mu - a \sin \theta_5 = 2k$$

$$\Rightarrow 2h - \lambda = -a \cos \theta_5$$

$$\text{and } 2k - \mu = -a \sin \theta_5$$

$$\Rightarrow (2h - \lambda)^2 + (2k - \mu)^2 = a^2$$

$$\Rightarrow \left(h - \frac{\lambda}{2}\right)^2 + \left(k - \frac{\mu}{2}\right)^2 = \left(\frac{a}{2}\right)^2$$

Centre (h, k) lies on circle of radius $\frac{a}{2}$.

Hence, (d) is the correct answer.

Ex 33. A, B, C and D are the points of intersection of a circle and a rectangular hyperbola which have different centres. If AB passes through the centre of the hyperbola, then CD passes through

- (a) centre of the hyperbola
(b) centre of the circle
(c) mid-point of the centres of circle and hyperbola
(d) none of the points mentioned in the three options

Sol. Let centre of circle and hyperbola be (α, β) and (h, k) and points be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$. Then,

$$\frac{h + \alpha}{2} = \frac{x_1 + x_2 + x_3 + x_4}{4} \quad \dots(i)$$

$$\text{and } \frac{k + \beta}{2} = \frac{y_1 + y_2 + y_3 + y_4}{4} \quad \dots(ii)$$

As any chord passing through centre of hyperbola is bisected at the centre.

$\therefore AB$ is bisected at (h, k) .

$$\Rightarrow \frac{x_1 + x_2}{2} = h \quad \dots(iii)$$

$$\text{and } \frac{y_1 + y_2}{2} = k \quad \dots(iv)$$

From Eqs. (i) and (iii),

$$h + \alpha = \frac{x_1 + x_2 + x_3 + x_4}{2} \Rightarrow \alpha = \frac{x_3 + x_4}{2}$$

$$\text{From Eqs. (ii) and (iv), } \beta = \frac{y_3 + y_4}{2}$$

$\Rightarrow (\alpha, \beta)$ is mid-point of CD .

$\Rightarrow (\alpha, \beta)$ lies on CD .

$\Rightarrow$ Centre of circle lies on CD .

Hence, (b) is the correct answer.

Ex 34. If the normals drawn at four concyclic points on a rectangular hyperbola $xy = c^2$ meet at point (α, β) , then the centre of the circle has the coordinates

- (a) (α, β) (b) $(2\alpha, 2\beta)$

- (c) $\left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$ (d) $\left(\frac{\alpha}{4}, \frac{\beta}{4}\right)$

Sol. Let the four concyclic points at which normals to rectangular hyperbola are concurrent be $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$ and $D(x_4, y_4)$ and centre of circle be (h, k) .

$$\therefore \frac{x_1 + x_2 + x_3 + x_4}{4} = \frac{h + 0}{2}$$

$$\text{and } \frac{y_1 + y_2 + y_3 + y_4}{4} = \frac{k + 0}{2}$$

$$\Rightarrow x_1 + x_2 + x_3 + x_4 = 2h \quad \dots(i)$$

$$\text{and } y_1 + y_2 + y_3 + y_4 = 2k \quad \dots(ii)$$

Normal to rectangular hyperbola $xy = c^2$ at $\left(ct, \frac{c}{t}\right)$ is

$$ct^4 - xt^3 + yt - c = 0$$

As all normal pass through (α, β) .

$$\Rightarrow x_1 + x_2 + x_3 + x_4 = c(t_1 + t_2 + t_3 + t_4)$$

$$= c \left(\frac{\alpha}{c}\right) = \alpha \quad \dots(iii)$$

$$\text{and } y_1 + y_2 + y_3 + y_4 = c \left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4}\right)$$

$$= c \left(\frac{-\beta/c}{-c/c}\right) = \beta \quad \dots(iv)$$

From Eqs. (i) and (iii), we get $2h = \alpha$

From Eqs. (ii) and (iv), we get $2k = \beta$

$$\therefore (h, k) = \left(\frac{\alpha}{2}, \frac{\beta}{2}\right)$$

Hence, (c) is the correct answer.

Type 5. Match the Columns

Ex 35. Match the statements of Column I with values of Column II.

Column I	Column II
A. The eccentricity of the conic represented by $x^2 - y^2 - 4x + 4y + 16 = 0$ is	p. 7
B. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincides, then the value of b^2 is	q. 0
C. The product of lengths of perpendiculars from any point of the hyperbola $x^2 - y^2 = 8$ to its asymptotes is	r. $\sqrt{2}$
D. The number of point(s) outside the hyperbola $\frac{x^2}{25} - \frac{y^2}{36} = 1$, where two perpendicular tangents can be drawn to the hyperbola, is/are	s. 4

Sol. A. We have, $x^2 - y^2 - 4x + 4y + 16 = 0$
 $\Rightarrow (x^2 - 4x) - (y^2 - 4y) = -16$
 $\Rightarrow (x^2 - 4x + 4) - (y^2 - 4y + 4) = -16$
 $\Rightarrow (x - 2)^2 - (y - 2)^2 = -16$
 $\Rightarrow \frac{(x - 2)^2}{4^2} - \frac{(y - 2)^2}{4^2} = -1$
 Shifting the origin at (2, 2), we obtain
 $\frac{X^2}{4^2} - \frac{Y^2}{4^2} = -1$, where $x = X + 2$, $y = Y + 2$
 which is a rectangular hyperbola, whose eccentricity is always $\sqrt{2}$.

B. For the given ellipse, $a^2 = 16$
 Therefore, $e = \sqrt{1 - \frac{b^2}{a^2}}$
 $\Rightarrow e = \sqrt{1 - \frac{b^2}{16}} = \sqrt{\frac{16 - b^2}{16}}$
 So, the foci of the ellipse are $(\pm ae, 0)$.
 i.e. $(\pm \sqrt{16 - b^2}, 0)$

For the hyperbola, $a^2 = \left(\frac{12}{5}\right)^2$, $b^2 = \left(\frac{9}{5}\right)^2$

The eccentricity e is given by

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{81}{144}} = \frac{15}{12} = \frac{5}{4}$$

$$\therefore 16 - b^2 = \left(\frac{12}{5} \times \frac{5}{4}\right)^2 = 9$$

$$\Rightarrow b^2 = 7$$

C. $x^2 - y^2 = 8$ when rotate $d = 45^\circ$ to positive X -axes becomes $xy = 4$ has asymptotes X and Y -axes.
 $\therefore P_1 P_2 = 4$

$$D. \frac{x^2}{25} - \frac{y^2}{36} = 1$$

Equation of director circle is $x^2 + y^2 = a^2 - b^2$

$$\Rightarrow x^2 + y^2 = -9$$

which is not possible.

$\therefore$ Number of points is zero.

$A \rightarrow r$; $B \rightarrow p$; $C \rightarrow s$; $D \rightarrow q$

Ex 36. Match the statements of Column I with values of Column II.

Column I	Column II
A. The area of the triangle that a tangent at a point of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ makes with its asymptotes, is	p. 12
B. If the line $y = 3x + \lambda$ touches the curve $9x^2 - 5y^2 = 45$, then $ \lambda $ is	q. 6
C. If the chord $x \cos \alpha + y \sin \alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre, then the diameter of the circle, concentric with the hyperbola to which the given chord is a tangent, is	r. 24
D. If λ is the length of the latusrectum of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$, then 3λ is equal to	s. 32

Sol. A. Equation of tangent at $(a, 0)$ is $x = a$

$$y = \frac{b}{a}x$$

$$\therefore \text{Area} = a \cdot b = 4 \times 3 = 12$$

B. $y = 3x + \lambda$ touches $9x^2 - 5y^2 = 45$

$$\therefore 9x^2 - 5(3x + \lambda)^2 = 45$$

$$\text{i.e. } -36x^2 - 30\lambda x - 5\lambda^2 - 45 = 0$$

$$\text{i.e. } 36x^2 + 30\lambda x + 5\lambda^2 + 45 = 0 \text{ has equal roots.}$$

$$\therefore 900\lambda^2 - 720\lambda^2 - 180 \times 36 = 0$$

$$\text{i.e. } \lambda^2 = 36$$

$$\Rightarrow \lambda = \pm 6$$

$$\therefore |\lambda| = 6$$

C. $18x^2 - 16y^2 - 288 = 0$

$$\text{i.e. } 9x^2 - 8y^2 - 144 \left(\frac{x \cos \alpha + y \sin \alpha}{p} \right)^2 = 0$$

Since, these lines are perpendicular to each other.

$$\therefore 9p^2 - 8p^2 - 144(\cos^2 \alpha + \sin^2 \alpha) = 0$$

$$\Rightarrow p^2 = 144 = \pm 12$$

$$\therefore \text{Radius of the circle} = 12$$

$$\therefore \text{Diameter of the circle} = 24$$

D. $16x^2 + 32 + 16 - 9(y^2 - 4y + 4) - 144 = 0$

$$\text{i.e. } 16(x + 1)^2 - 9(y - 2)^2 = 144$$

$$\text{i.e. } \frac{(x + 1)^2}{9} - \frac{(y - 2)^2}{16} = 1$$

$$\text{Length of latusrectum} = 2 \times \frac{16}{3} = \frac{32}{3}$$

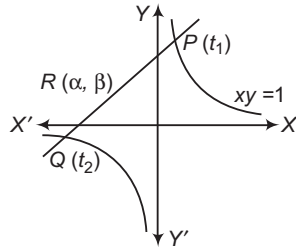
$$\therefore 3\lambda = 32$$

$A \rightarrow p$; $B \rightarrow q$; $C \rightarrow r$; $D \rightarrow s$

Type 6. Single Integer Answer Type Questions

Ex 37. A variable of line of slope 4 intersects the hyperbola $xy=1$ at two points. If the locus of the point which divides the line segment between these two points in the ratio 1:2 is $\lambda x^2 + y^2 + \mu xy - \gamma = 0$, then the value of $\lambda - (\mu + \gamma)$ is equal to _____.

Sol. (4) $P(t_1, 1/t_1), Q(t_2, 1/t_2)$



$$\text{Slope of } PQ = -\frac{1}{t_1 t_2} = 4$$

$$\therefore t_1 t_2 = -\frac{1}{4} \quad \dots(i)$$

$$\alpha = \frac{t_1 + 2t_2}{3} \quad \dots(ii)$$

$$\text{and } \beta = \frac{\frac{1}{t_1} + \frac{2}{t_2}}{3} \quad \dots(iii)$$

On solving Eqs. (i), (ii) and (iii), we get the required locus is

$$16x^2 + y^2 + 10xy - 2 = 0$$

$$\Rightarrow \lambda - (\mu + \gamma) = 16 - (10 + 2) = 4$$

Ex 38. Let the circle $(x-1)^2 + (y-2)^2 = 25$ cuts a rectangular hyperbola with transverse axis along $y=x$ at four points A, B, C and D having coordinates $(x_i, y_i), i=1, 2, 3, 4$ respectively, O being the centre of the hyperbola. If $l = x_1 + x_2 + x_3 + x_4, m = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and $n = y_1^2 + y_2^2 + y_3^2 + y_4^2$, then find the value of $n - m - 4l$.

Sol. (4) The circle is $x^2 + y^2 - 2x - 4y - 20 = 0$ and let the hyperbola be $xy = c^2$.

If $\left(ct, \frac{c}{t}\right)$ is the point of intersection, then

$$c^2 t^2 + \frac{c^2}{t^2} - 2ct - \frac{4c}{t} - 20 = 0$$

$$\Rightarrow c^2 t^4 - 2ct^3 - 20t^2 - 4ct + c^2 = 0$$

If t_1, t_2, t_3 and t_4 are its roots, then

$$\Sigma t_1 = \frac{2}{c}; \Sigma t_1 t_2 = -\frac{20}{c^2}, \Sigma t_1 t_2 t_3 = \frac{4}{c} \text{ and } t_1 t_2 t_3 t_4 = 1$$

$$(i) x_1 + x_2 + x_3 + x_4 = ct_1 + ct_2 + ct_3 + ct_4 = 2 = l$$

$$(ii) \Sigma x_1 x_2 = c^2 \Sigma t_1 t_2 = -20$$

$$\therefore \Sigma x_1^2 = (\Sigma x_1)^2 - 2 \Sigma x_1 x_2 = 44 = m$$

$$(iii) \Sigma y_1 = \Sigma \frac{c}{t_1} = c \frac{\Sigma t_1 t_2 t_3}{t_1 t_2 t_3 t_4}$$

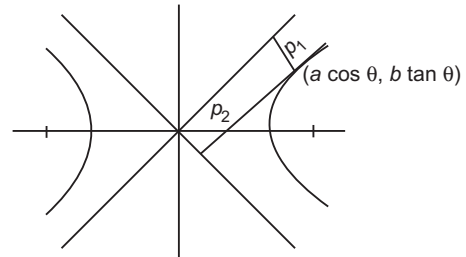
$$\Sigma y_1 y_2 = c^2 \Sigma \frac{1}{t_1 t_2} = 20$$

$$\therefore \Sigma y_1^2 = (\Sigma y_1)^2 - 2 \Sigma y_1 y_2 = 56 = n$$

$$\Rightarrow n - m - 4l = 4$$

Ex 39. If the product of the perpendicular distances from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ of eccentricity $e = \sqrt{3}$ from its asymptotes is equal to 6, then the length of the transverse axis of the hyperbola is _____.

$$\text{Sol. (6)} p_1 p_2 = \frac{a^2 b^2}{a^2 + b^2} = \frac{a^2 \cdot a^2 (e^2 - 1)}{a^2 e^2} = 6$$



$$\text{Hence, } 2a = 6$$

Ex 40. If e_1 and e_2 are the eccentricities of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and its conjugate hyperbola, while $e_1^{-2} + e_2^{-2} = \lambda$, then the value of λ is _____.

Sol. (1) The eccentricity e_1 of the given hyperbola is obtained from

$$b^2 = a^2 (e_1^2 - 1) \quad \dots(i)$$

The eccentricity e_2 of the conjugate hyperbola is given by

$$a^2 = b^2 (e_2^2 - 1) \quad \dots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$1 = (e_1^2 - 1)(e_2^2 - 1)$$

$$\Rightarrow 0 = e_1^2 e_2^2 - e_1^2 - e_2^2$$

$$\Rightarrow e_1^{-2} + e_2^{-2} = 1$$

$$\Rightarrow \lambda = 1$$

Target Exercises

Type 1. Only One Correct Option

- The equation of the hyperbola whose foci are $(6, 5)$, $(-4, 5)$ and eccentricity is $5/4$, is
 (a) $\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = 1$ (b) $\frac{x^2}{16} - \frac{y^2}{9} = 1$
 (c) $\frac{(x-1)^2}{16} + \frac{(y-5)^2}{9} = 1$ (d) None of these
- The equation of the hyperbola with vertices $(3, 0)$, $(-3, 0)$ and semi-latusrectum 4, is given by
 (a) $4x^2 - 3y^2 + 36 = 0$ (b) $4x^2 - 3y^2 + 12 = 0$
 (c) $4x^2 - 3y^2 - 36 = 0$ (d) None of these
- The eccentricity of the hyperbola $-\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is given by
 (a) $e = \sqrt{\frac{a^2 + b^2}{a^2}}$ (b) $e = \sqrt{\frac{a^2 - b^2}{a^2}}$
 (c) $e = \sqrt{\frac{b^2 - a^2}{a^2}}$ (d) $e = \sqrt{\frac{a^2 + b^2}{b^2}}$
- The eccentricity of the hyperbola $3x^2 - 4y^2 = -12$ is
 (a) $\sqrt{\frac{7}{3}}$ (b) $\frac{\sqrt{7}}{2}$
 (c) $-\sqrt{\frac{7}{3}}$ (d) $-\frac{\sqrt{7}}{2}$
- The vertices of the hyperbola $9x^2 - 16y^2 - 36x + 96y - 252 = 0$ are
 (a) $(6, 3)$ and $(-6, 3)$ (b) $(6, 3)$ and $(-2, 3)$
 (c) $(-6, 3)$ and $(-6, -3)$ (d) None of these
- The equation of the hyperbola of given transverse axis whose vertex bisects the distance between the centre and the focus, is given by
 (a) $3x^2 - y^2 = 3a^2$ (b) $x^2 - 3y^2 = a^2$
 (c) $x^2 - y^2 = 3a^2$ (d) None of these
- If the coordinates of a point are $a \tan(\theta + \alpha)$ and $b \tan(\theta + \beta)$, where θ is a variable, then locus of the point is
 (a) a hyperbola (b) a rectangular hyperbola
 (c) an ellipse (d) None of these
- Let A and B be two fixed points and P be another point in the plane, moves in such a way that $k_1 PA + k_2 PB = k_3$, where k_1, k_2 and k_3 are real constants. Then, which one of the following is not the locus of P ?
 (a) A circle, if $k_1 = 0$ and $k_2, k_3 > 0$
 (b) A circle, if $k_1 > 0, k_2 < 0$ and $k_3 = 0$
 (c) An ellipse, if $k_1 = k_2 > 0$ and $k_3 > 0$
 (d) A hyperbola, if $k_2 = -1$ and $k_1, k_3 > 0$
- If the latusrectum of a hyperbola through one focus subtends 60° angle at the other focus, then its eccentricity e is
 (a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $\sqrt{5}$ (d) $\sqrt{6}$
- If the latusrectum of a hyperbola forms an equilateral triangle with the vertex at the centre of the hyperbola, then eccentricity of the hyperbola is
 (a) $\frac{\sqrt{5} + 1}{2}$ (b) $\frac{\sqrt{11} + 1}{2}$ (c) $\frac{\sqrt{13} + 1}{2\sqrt{3}}$ (d) $\frac{\sqrt{13} - 1}{2\sqrt{3}}$
- Let LL' be the latusrectum through the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and A' be the farther vertex. If $\Delta A'LL'$ is equilateral, then the eccentricity of the hyperbola is
 (a) $\sqrt{3}$ (b) $\sqrt{3} + 1$ (c) $\frac{\sqrt{3} + 1}{\sqrt{2}}$ (d) $\frac{\sqrt{3} + 1}{\sqrt{3}}$
- If PQ is a double ordinate of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ such that OPQ is an equilateral triangle, O being the centre of the hyperbola. Then, the eccentricity e of the hyperbola satisfies
 (a) $1 < e < 2\sqrt{3}$ (b) $e = \frac{3}{\sqrt{3}}$
 (c) $e = \frac{\sqrt{3}}{2}$ (d) $e > \frac{2}{\sqrt{3}}$
- Equation of tangent to the hyperbola $2x^2 - 3y^2 = 6$ which is parallel to the line $y = 3x + 4$, is
 (a) $y = 3x + 5$ (b) $y = 3x - 5$
 (c) $y = 3x + 5$ and $y = 3x - 5$ (d) None of these
- The line $x \cos \alpha + y \sin \alpha = p$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if
 (a) $a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$
 (b) $a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p$
 (c) $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p^2$
 (d) $a^2 \cos^2 \alpha + b^2 \sin^2 \alpha = p$
- The angle between lines joining the origin to the points of intersection of the line $\sqrt{3}x + y = 2$ and the curve $y^2 - x^2 = 4$, is
 (a) $\tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ (b) $\frac{\pi}{6}$
 (c) $\tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$ (d) $\frac{\pi}{2}$

- 16.** A common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$, is
 (a) $y = \frac{3x}{\sqrt{7}} + \frac{15}{\sqrt{7}}$ (b) $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$
 (c) $y = 2\sqrt{\frac{3}{7}}x + 15\sqrt{7}$ (d) None of these
- 17.** The equation of a tangent parallel to $y = x$ drawn to $\frac{x^2}{3} - \frac{y^2}{2} = 1$, is
 (a) $x - y + 1 = 0$ (b) $x - y + 2 = 0$
 (c) $x + y - 1 = 0$ (d) $x + y - 2 = 0$
- 18.** P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T . If O is the centre of the hyperbola, then $OT \cdot ON$ is equal to
 (a) e^2 (b) a^2 (c) b^2 (d) $\frac{b^2}{a^2}$
- 19.** The line $lx + my + n = 0$ will be a normal to the hyperbola $b^2x^2 - a^2y^2 = a^2b^2$, if
 (a) $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$ (b) $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$
 (c) $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n}$ (d) None of these
- 20.** The locus of the middle points of chords of hyperbola $3x^2 - 2y^2 + 4x - 6y = 0$ parallel to $y = 2x$, is
 (a) $3x - 4y = 4$ (b) $3y - 4x + 4 = 0$
 (c) $4x - 4y = 3$ (d) $3x - 4y = 2$
- 21.** Equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at the point $(6, 2)$, is
 (a) $16x - 75y = 418$ (b) $75x - 16y = 418$
 (c) $25x - 4y = 400$ (d) None of these
- 22.** The diameter of $16x^2 - 9y^2 = 144$ which is conjugate to $x = 2y$, is
 (a) $y = \frac{16x}{9}$ (b) $y = \frac{32x}{9}$
 (c) $x = \frac{16y}{9}$ (d) $x = \frac{32y}{9}$
- 23.** A normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse and conjugate axes in M and N and the lines MP and NP are drawn at right angle to the axes. The locus of P is
 (a) the parabola $y^2 = 4a(x + b)$
 (b) the circle $x^2 + y^2 = ab$
 (c) the ellipse $b^2x^2 + a^2y^2 = a^2 + b^2$
 (d) the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$
- 24.** If $x = 9$ is the chord of contact of the hyperbola $x^2 - y^2 = 9$, then the equation of the corresponding pair of tangent is
 (a) $9x^2 - 8y^2 + 18x - 9 = 0$
 (b) $9x^2 - 8y^2 - 18x + 9 = 0$
 (c) $9x^2 - 8y^2 - 18x - 9 = 0$
 (d) $9x^2 - 8y^2 + 18x + 9 = 0$
- 25.** If the normal at θ on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the transverse axis at G , then $AG \cdot A'G$ is (where, A and A' are the vertices of the hyperbola)
 (a) $a^2 \sec \theta$
 (b) $a^2(e^4 \sec^2 \theta + 1)$
 (c) $a^2(e^4 \sec^2 \theta - 1)$
 (d) None of the above
- 26.** The equation of a hyperbola, conjugate to the hyperbola $x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$, is
 (a) $x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$
 (b) $x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$
 (c) $x^2 + 3xy + 2y^2 + 2x + 3y + 3 = 0$
 (d) $x^2 + 3xy + 2y^2 + 2x + 3y + 4 = 0$
- 27.** The eccentricity of the rectangular hyperbola is
 (a) 2 (b) $\sqrt{2}$
 (c) 0 (d) None of these
- 28.** The angle between the asymptotes of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is
 (a) $2 \tan^{-1} \left(\frac{b}{a} \right)$ (b) $\tan^{-1} \left(\frac{b}{a} \right)$
 (c) $2 \tan^{-1} \left(\frac{a}{b} \right)$ (d) $\tan^{-1} \left(\frac{a}{b} \right)$
- 29.** The equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0$, is
 (a) $2x^2 + 5xy + 2y^2 - 11x - 7y - 5 = 0$
 (b) $2x^2 + 4xy + 2y^2 - 7x - 11y + 5 = 0$
 (c) $2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$
 (d) None of the above
- 30.** The asymptotes of a hyperbola having centre at the point $(1, 2)$ are parallel to the lines $2x + 3y = 0$ and $3x + 2y = 0$. If the hyperbola passes through the point $(5, 3)$, then its equation is
 (a) $(2x + 3y - 8)(3x + 2y - 7) = 154$
 (b) $(2x + 3y - 8)(3x + 2y + 7) = 152$
 (c) $(2x + 3y + 8)(3x + 2y - 7) = 152$
 (d) None of the above
- 31.** The equation of the hyperbola whose asymptotes are the straight lines $3x - 4y + 7 = 0$ and $4x + 3y + 1 = 0$ and which passes through the origin, is
 (a) $12x^2 - 7xy - 12y^2 + 31x + 17y = 0$
 (b) $12x^2 - 7xy - 12y^2 + 31x = 0$
 (c) $x^2 - 7xy - 12y^2 + 31x + 17y = 0$
 (d) None of the above

32. Two concentric hyperbolas, whose axes meet at angle of 45° , cut
(a) at 45° (b) at 90°
(c) Nothing can be said (d) None of these
33. The number of triangles can be inscribed in the rectangular hyperbola $xy = c^2$ whose all sides touch the parabola $y^2 = 4ax$, is
(a) infinite (b) 4 (c) 3 (d) 2
34. If $(x-1)(y-2) = 5$ and $(x-1)^2 + (y+2)^2 = r^2$ intersect at four points A, B, C, D and centroid of $\triangle ABC$ lies on line $y = 3x - 4$, then locus of D is
(a) $y = 3x$ (b) $x^2 + y^2 + 3x + 1 = 0$
(c) $3y = x + 1$ (d) $y = 3x + 1$
35. If foci of hyperbola lie on $y = x$ and one of the asymptotes is $y = 2x$, then equation of the hyperbola, given that it passes through $(3, 4)$, is
(a) $x^2 - y^2 - \frac{5}{2}xy + 5 = 0$ (b) $2x^2 - 2y^2 + 5xy + 5 = 0$
(c) $2x^2 + 2y^2 - 5xy + 10 = 0$ (d) None of these
36. The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$ such that the portion of the tangent intercepted by the hyperbola $\alpha^2 x^2 - y^2 = 1$ subtends a right angle at the centre of the curves, is
(a) $\left[\frac{\sqrt{5}+1}{2}, 2\right]$
(b) $(1, 2]$
(c) $\left[\frac{\sqrt{5}-1}{2}, 1\right)$
(d) $\left[\frac{\sqrt{5}+1}{2}, 1\right) \cup \left(1, \frac{\sqrt{5}+1}{2}\right]$
37. The locus of the point which is such that the chord of contact of tangents drawn from it to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ forms a triangle of constant area with the coordinate axes, is
(a) a straight line (b) a hyperbola
(c) an ellipse (d) a circle
38. The combined equation of the asymptotes of the hyperbola $2x^2 + 5xy + 2y^2 + 4x + 5y = 0$ is
(a) $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$
(b) $2x^2 + 5xy + 2y^2 + 4x + 5y - 2 = 0$
(c) $2x^2 + 5xy + 2y^2 = 0$
(d) None of the above
39. From any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, tangents are drawn to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2$. Then, area cut-off by the chord of contact on the asymptotes is equal to
(a) $a/2$ sq unit (b) ab sq unit
(c) $2ab$ sq unit (d) $4ab$ sq unit
40. A variable chord of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ($b > a$) subtends a right angle at the centre of the hyperbola, if this chord touches
(a) a fixed circle concentric with the hyperbola
(b) a fixed ellipse concentric with the hyperbola
(c) a fixed hyperbola concentric with the hyperbola
(d) a fixed parabola having vertex at $(0, 0)$
41. If two tangents can be drawn to the different branches of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ from the point (α, α^2) , then
(a) $\alpha \in (-2, 0)$ (b) $\alpha \in (-3, 0)$
(c) $\alpha \in (-\infty, -2)$ (d) $\alpha \in (-\infty, -3)$
42. If tangents OQ and OR are drawn to variable circles having radius r and the centre lying on the rectangular hyperbola $xy = 1$, then locus of circumcentre of $\triangle OQR$ is (O being the origin)
(a) $xy = 4$ (b) $xy = \frac{1}{4}$
(c) $xy = 1$ (d) None of these
43. Four points are such that the line joining any two points is perpendicular to the line joining other two points. If three points out of these lie on a rectangular hyperbola, then the fourth point will lie on
(a) the same hyperbola (b) conjugate hyperbola
(c) one of the directrix (d) one of the asymptotes
44. From a point $P(1, 2)$, two tangents are drawn to a hyperbola H in which one tangent is drawn to each arm of the hyperbola. If the equations of asymptotes of hyperbola H are $\sqrt{3}x - y + 5 = 0$ and $\sqrt{3}x + y - 1 = 0$, then eccentricity of H is
(a) 2 (b) $\frac{2}{\sqrt{3}}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$
45. If a hyperbola passes through $(2, 3)$ and has asymptotes $3x - 4y + 5 = 0$ and $12x + 5y - 40 = 0$, then the equation of its transverse axis is
(a) $77x - 21y - 265 = 0$ (b) $21x - 77y + 265 = 0$
(c) $21x - 77y - 265 = 0$ (d) $21x + 77y - 265 = 0$
46. If $P(x_1, y_1), Q(x_2, y_2), R(x_3, y_3)$ and $S(x_4, y_4)$ are four concyclic points on the rectangular hyperbola $xy = c^2$, then coordinates of orthocentre of $\triangle PQR$ are
(a) $(x_4, -y_4)$ (b) (x_4, y_4) (c) $(-x_4, -y_4)$ (d) $(-x_4, y_4)$
47. A rectangular hyperbola whose centre is C , is cut by any circle of radius r in four points P, Q, R and S . Then, $CP^2 + CQ^2 + CR^2 + CS^2$ is equal to
(a) r^2 (b) $2r^2$ (c) $3r^2$ (d) $4r^2$
48. PQ and RS are two perpendicular chords of the rectangular hyperbola $xy = c^2$. If C is the centre of the rectangular hyperbola, then the product of the slopes of CP, CQ, CR and CS is equal to
(a) -1 (b) 1 (c) 0 (d) None of these

49. If a circle cuts the rectangular hyperbola $xy = 1$ in the points $(x_r, y_r); r = 1, 2, 3, 4$, then $x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4$ is equal to
(a) 1 (b) 2 (c) 3 (d) 4
50. The product of the perpendicular from any point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ to its asymptotes, is equal to
(a) $\frac{ab}{a+b}$ (b) $\frac{a^2 b^2}{a^2 + b^2}$
(c) $\frac{2a^2 b^2}{a^2 + b^2}$ (d) None of these
51. If $H(x, y) = 0$ represents the equation of a hyperbola and $A(x, y) = 0, C(x, y) = 0$ are the equation of its asymptotes and the conjugate hyperbola respectively, then for any point (α, β) in the plane; $H(\alpha, \beta), A(\alpha, \beta)$ and $C(\alpha, \beta)$ are in
(a) AP (b) GP
(c) HP (d) None of these
52. If four points are taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two and $\alpha, \beta, \gamma, \delta$ are the inclinations to either asymptote of the straight line joining these points to the centre, then
(a) $\tan \alpha \tan \beta \tan \gamma \tan \delta = 1$
(b) $\tan \alpha \tan \beta \tan \gamma \tan \delta = 2$
(c) $\tan \alpha \tan \beta \tan \gamma \tan \delta = -1$
(d) $\tan \alpha \tan \beta \tan \gamma \tan \delta = -2$
53. Let P be a point on the hyperbola $x^2 - y^2 = a^2$, where a is a parameter such that P is nearest to the line $y = 2x$. Then, the locus of P is
(a) $x \pm y = 0$ (b) $x \pm 2y = 0$
(c) $2x \pm y = 0$ (d) $2x \pm 3y = 0$
54. The coordinates of the point of intersection of two tangents to a rectangular hyperbola referred to its asymptote as axes, are
(a) geometric means between the coordinates of the point of contact
(b) arithmetic means between the coordinates of the point of contact
(c) harmonic means between the coordinates of the point of contact
(d) None of the above
55. From any point $P(h, k)$, four normals can be drawn to the rectangular hyperbola $xy = c^2$ such that sum of the ordinates of the feet of the normals is equal to
(a) $3k$ (b) $2k$
(c) k^2 (d) k
56. From any point $P(h, k)$, four normals can be drawn to the rectangular hyperbola $xy = c^2$ such that product of the abscissae of the feet of the normals = product of the ordinates of the feet of the normals is equal to
(a) c^4 (b) $-c^4$
(c) $2c^4$ (d) $2c^2$
57. A point P moves in such a way that the sum of the slopes of the normals drawn from it to the hyperbola $xy = 4$ is equal to the sum of the ordinates of feet of the normals. The locus of P is a parabola $x^2 = 4y$. Then, the least distance of this parabola from the circle $x^2 - y^2 - 24x + 128 = 0$ is
(a) $4\sqrt{5}$
(b) $\frac{4}{\sqrt{5}}$
(c) $-4\sqrt{5}$
(d) None of the above

Type 2. More than One Correct Option

58. The equation $16x^2 - 3y^2 - 32x - 12y - 44 = 0$ represents hyperbola with
(a) length of transverse axis $= 2\sqrt{3}$
(b) length of conjugate axis $= 8$
(c) centre at $(1, -2)$
(d) eccentricity $= \sqrt{19}$
59. Which of the following equation(s) in parametric form can represent a hyperbola, where t is a parameter?
(a) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ and $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$
(b) $\frac{tx}{a} - \frac{y}{b} + t = 0$ and $\frac{x}{a} + \frac{ty}{b} - 1 = 0$
(c) $x = e^t + e^{-t}$ and $y = e^t - e^{-t}$
(d) $x^2 - 6 = 2 \cos t$ and $y^2 + 2 = 4 \cos^2 t/2$
60. For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, let n be the number of points on the plane through which perpendicular tangents are drawn. Which of the following conditions satisfied the given hyperbola?
(a) If $n = 1$, then $e = \sqrt{2}$
(b) If $n > 1$, then $0 < e < \sqrt{2}$
(c) If $n = 0$, then $e > \sqrt{2}$
(d) None of the above
61. A straight line touches the rectangular hyperbola $9x^2 - 9y^2 = 8$ and the parabola $y^2 = 32x$. The equation of the line is
(a) $9x + 3y - 8 = 0$ (b) $9x - 3y + 8 = 0$
(c) $9x + 3y + 8 = 0$ (d) $9x - 3y - 8 = 0$
62. The coordinates of a point common to a directrix and an asymptote of the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ are
(a) $\left(\frac{25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right)$ (b) $\left(-\frac{25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right)$
(c) $\left(\frac{25}{3}, \frac{20}{3}\right)$ (d) $\left(\frac{25}{\sqrt{3}}, \frac{20}{\sqrt{3}}\right)$

Type 3. Assertion and Reason

Directions (Q.Nos. 63-67) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
(b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
(c) Statement I is true, Statement II is false
(d) Statement I is false, Statement II is true

63. Statement I The equation $x^2 + 2y^2 + \lambda xy + 2x + 3y + 1 = 0$ can never represent a hyperbola.

Statement II The general equation of second degree represents a hyperbola, if $h^2 > ab$.

64. Statement I The slope of the common tangent between the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $-\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ may be 1 or -1.

Statement II The locus of the point of intersection of lines $\frac{x}{a} - \frac{y}{b} = m$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ is a hyperbola (where, m is a variable and $ab \neq 0$).

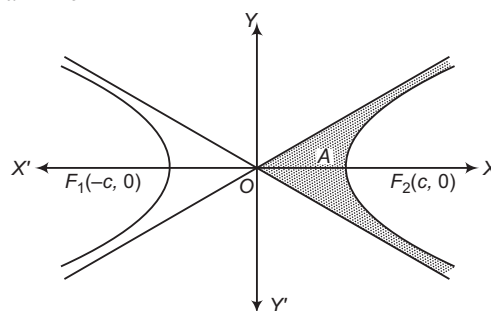
65. Statement I Let $(2, \sqrt{2})$ be any point on hyperbola $x^2 - y^2 = 2$, then the product of distances of foci from P is equal to 6.

Statement II If S and S' are the foci, C is the centre and P is any point on a hyperbola $x^2 - y^2 = a^2$, then $SP \cdot S'P = CP^2$.

66. Statement I The equation of the director circle to the hyperbola $4x^2 - 3y^2 = 12$ is $x^2 + y^2 = 1$.

Statement II Director circle is the locus of the point of intersection of perpendicular tangents to a hyperbola.

67. Statement I If a point (x_1, y_1) lies in the shaded region $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, shown in the figure, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 0$.



Statement II If $P(x_1, y_1)$ lies outside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} < 1$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 68-70) For the hyperbola $\frac{(3x - 4y - 12)^2}{100} - \frac{(4x + 3y - 12)^2}{225} = 1$

- 68.** The centre is
(a) $\left(\frac{84}{25}, \frac{12}{25}\right)$ (b) $\left(\frac{84}{25}, -\frac{12}{25}\right)$
(c) $\left(-\frac{84}{25}, \frac{12}{25}\right)$ (d) $\left(-\frac{84}{25}, -\frac{12}{25}\right)$
- 69.** The length of latusrectum of hyperbola is
(a) 40 units
(b) 45 units
(c) $9/2$ units
(d) $8/9$ unit
- 70.** If a straight line cuts the hyperbola in P and Q and its asymptotes in R and S , then PR is equal to
(a) QS (b) $2QS$
(c) $3QS$ (d) $4QS$

Passage II (Q. Nos. 71-73) The graph of the conic $x^2 - (y - 1)^2 = 1$ has one tangent line with positive slope that passes through the origin. The point of tangency being (a, b) .

- 71.** The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is
(a) $\frac{5\pi}{12}$ (b) $\frac{\pi}{6}$
(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{4}$
- 72.** Length of latusrectum of the conic is
(a) 1 (b) $\sqrt{2}$
(c) 2 (d) None of these
- 73.** Eccentricity of the conic is
(a) $\frac{4}{3}$ (b) $\sqrt{3}$
(c) 2 (d) None of these

Passage III (Q. Nos. 74-76) The vertices of $\triangle ABC$ lie on a rectangular hyperbola such that the orthocentre of the triangle is $(3, 2)$ and the asymptotes of the rectangular hyperbola are parallel to the coordinate axes. The two perpendicular tangents of the hyperbola intersect at the point $(1, 1)$.

74. The equation of the asymptotes is
 (a) $xy - 1 = x - y$ (b) $xy + 1 = x + y$
 (c) $2xy = x + y$ (d) None of these

Type 5. Match the Columns

77. Let the foci of the hyperbola $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$ be the vertices of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the foci of the ellipse be the vertices of the hyperbola. Let the eccentricities of the ellipse and hyperbola be e_E and e_H respectively, then match the following:

Column I	Column II
A. $\frac{b}{B}$ is equal to	p. 1
B. $e_H + e_E$ is always greater than	q. 2
C. If angle between the asymptotes of hyperbola is $\frac{2\pi}{3}$, then $4e_E$ is equal to	r. 3
D. If $e_E^2 = \frac{1}{2}$ and (x, y) is point of intersection of ellipse and the hyperbola, then $\frac{9x^2}{2y^2}$ is	s. 4

75. Equation of the rectangular hyperbola is
 (a) $xy = 2x + y - 2$
 (b) $2xy = x + 2y + 5$
 (c) $xy = x + y + 1$
 (d) None of the above

76. Number of real tangents that can be drawn from the point $(1, 1)$ to the rectangular hyperbola, is
 (a) 4 (b) 0
 (c) 3 (d) 2

78. Match the statements of Column I with values of Column II.

Column I	Column II
A. Value of c for which $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ are the asymptotes of the hyperbola $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$, is	p. 3
B. If locus of a point, whose chord of contact with respect to the circle $x^2 + y^2 = 4$ is a tangent to the hyperbola $xy = 1$, is $xy = c^2$, then the value of c^2 is	q. -4
C. If equation of a hyperbola whose conjugate axis is 5 and distance between its foci is 13, is $ax^2 - by^2 = c$, where a and b are coprime natural numbers, then the value of $\frac{ab}{c}$ is	r. -12
D. If the vertex of a hyperbola bisects the distance between its centre and the corresponding focus, then ratio of square of its conjugate axis to the square of its transverse axis is	s. 4

Type 6. Single Integer Answer Type Questions

79. If the equation of the chord of the hyperbola $25x^2 - 16y^2 = 400$ which is bisected at $(5, 3)$, is $125x - 48y = \lambda + 480$, then the value of λ is _____.
80. If the line $25x + 12y - 45 = 0$ meets the hyperbola $25x^2 - 9y^2 = 255$ only at point $\left(5, -\frac{5\lambda}{3}\right)$, then the value of λ is _____.
81. If a variable line has its intercepts on the coordinate axes e, e' , where $e/2, e'/2$ are the eccentricities of a hyperbola and its conjugate hyperbola, then the line always touches the circle $x^2 + y^2 = r^2$, where r is equal to _____.

82. If the chord $x \cos \alpha + y \sin \alpha = p$ of the hyperbola $\frac{x^2}{16} - \frac{y^2}{18} = 1$ subtends a right angle at the centre and the diameter of the circle, concentric with the hyperbola, to which the given chord is a tangent, is d , then the value of $d/4$ is _____.
83. If L is the length of latusrectum of hyperbola for which $x = 3$ and $y = 2$ are the equations of asymptotes and which passes through the point $(4, 6)$, then the value of $L/\sqrt{2}$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Tangents are drawn to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$,

parallel to the straight line $2x - y = 1$. The points of contacts of the tangents on the hyperbola are [2012]

- (a) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ (b) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$
(c) $(3\sqrt{3}, -2\sqrt{2})$ (d) $(-3\sqrt{3}, 2\sqrt{2})$

2. Let the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then [2011]

- (a) the equation of the hyperbola is $\frac{x^2}{3} - \frac{y^2}{2} = 1$
(b) a focus of the hyperbola is $(2, 0)$
(c) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$
(d) the equation of the hyperbola is $x^2 - 3y^2 = 3$

3. Let $P(6, 3)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

If the normal at the point P intersects the X -axis at $(9, 0)$, then the eccentricity of the hyperbola is [2011]

- (a) $\sqrt{\frac{5}{2}}$ (b) $\sqrt{\frac{3}{2}}$
(c) $\sqrt{2}$ (d) $\sqrt{3}$

Passage (Q. Nos. 4-5) The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B . [2010]

4. Equation of a common tangent with positive slope to the circle as well as to the hyperbola is
(a) $2x - \sqrt{5}y - 20 = 0$ (b) $2x - \sqrt{5}y + 4 = 0$
(c) $3x - 4y + 8 = 0$ (d) $4x - 3y + 4 = 0$
5. Equation of the circle with AB as its diameter, is
(a) $x^2 + y^2 - 12x + 24 = 0$ (b) $x^2 + y^2 + 12x + 24 = 0$
(c) $x^2 + y^2 + 24x - 12 = 0$ (d) $x^2 + y^2 - 24x - 12 = 0$

AIEEE

6. The equation of the hyperbola, whose foci are $(-2, 0)$ and $(2, 0)$ and eccentricity is 2, is given by [2011]
(a) $-3x^2 + y^2 = 3$ (b) $x^2 - 3y^2 = 3$
(c) $3x^2 - y^2 = 3$ (d) $-x^2 + 3y^2 = 3$

7. For the hyperbola $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$, which of the following remains constant when α varies? [2007]
(a) Eccentricity (b) Directrix
(c) Abscissae of vertices (d) Abscissae of foci

8. The locus of a point $P(\alpha, \beta)$ moving under the condition that the line $y = \alpha x + \beta$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is [2005]

- (a) a hyperbola (b) a parabola
(c) a circle (d) an ellipse

9. If the foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide. Then, the value of b^2 is [2003]
(a) 1 (b) 5 (c) 7 (d) 9

10. The equation of the chord joining two points (x_1, y_1) and (x_2, y_2) on the rectangular hyperbola $xy = c^2$ is [2002]
(a) $\frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$ (b) $\frac{x}{x_1 - x_2} + \frac{y}{y_1 - y_2} = 1$
(c) $\frac{x}{y_1 + y_2} + \frac{y}{x_1 + x_2} = 1$ (d) $\frac{x}{y_1 - y_2} + \frac{y}{x_1 - x_2} = 1$

Other Engineering Entrances

11. If the length of the latusrectum and the length of transverse axis of a hyperbola are $4\sqrt{3}$ and $2\sqrt{3}$ respectively, then the equation of the hyperbola is [Kerala CEE 2014]

- (a) $\frac{x^2}{3} - \frac{y^2}{4} = 1$ (b) $\frac{x^2}{3} - \frac{y^2}{9} = 1$
(c) $\frac{x^2}{6} - \frac{y^2}{9} = 1$ (d) $\frac{x^2}{6} - \frac{y^2}{3} = 1$
(e) $\frac{x^2}{3} - \frac{y^2}{6} = 1$

12. If the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{5}{4}$ and $2x + 3y - 6 = 0$ is a focal chord of the hyperbola, then the length of transverse axis is equal to [Kerala CEE 2014]

- (a) $\frac{12}{5}$ (b) 6 (c) $\frac{24}{7}$
(d) $\frac{24}{5}$ (e) $\frac{12}{7}$

- 13.** The length of the transverse axis of a hyperbola is $2\cos \alpha$. The foci of the hyperbola are the same as that of the ellipse $9x^2 + 16y^2 = 144$. The equation of the hyperbola is [Kerala CEE 2014]
- (a) $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{7 - \cos^2 \alpha} = 1$
 (b) $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{7 + \cos^2 \alpha} = 1$
 (c) $\frac{x^2}{1 + \cos^2 \alpha} - \frac{y^2}{7 - \cos^2 \alpha} = 1$
 (d) $\frac{x^2}{1 + \cos^2 \alpha} - \frac{y^2}{7 + \cos^2 \alpha} = 1$
 (e) $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{5 - \cos^2 \alpha} = 1$
- 14.** The number of points (a, b) , where a and b are positive integers, lying on the hyperbola $x^2 - y^2 = 512$, is [Kerala CEE 2014]
- (a) 3 (b) 4 (c) 5
 (d) 6 (e) 7
- 15.** A hyperbola passing through a focus of the ellipse $\frac{x^2}{169} + \frac{y^2}{25} = 1$. Its transverse and conjugate axes coincide respectively with the major and minor axes of the ellipse. The product of eccentricities is 1. Then, the equation of the hyperbola is [EAMCET 2014]
- (a) $\frac{x^2}{144} - \frac{y^2}{9} = 1$ (b) $\frac{x^2}{169} - \frac{y^2}{25} = 1$
 (c) $\frac{x^2}{144} - \frac{y^2}{25} = 1$ (d) $\frac{x^2}{25} - \frac{y^2}{9} = 1$
- 16.** The curve $5x^2 + 12xy - 22x - 12y - 19 = 0$ is [GGSIPU 2014]
- (a) ellipse (b) parabola
 (c) hyperbola (d) parallel straight lines
- 17.** If the line $lx + my - n = 0$ will be a normal to the hyperbola, then $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{k}$, where k is equal to [VITEEE 2014]
- (a) n (b) n^2
 (c) n^3 (d) None of these
- 18.** If the chords of the hyperbola $x^2 - y^2 = a^2$ touch the parabola $y^2 = 4ax$. Then, the locus of the middle points of these chords is [Manipal 2014]
- (a) $y^2 = (x - a)x^3$ (b) $y^2 (x - a) = x^3$
 (c) $x^2(x - a) = x^3$ (d) None of these
- 19.** The equation of the common tangent with positive slope to the parabola $y^2 = 8\sqrt{3}x$ and the hyperbola $4x^2 - y^2 = 4$ is [WB JEE 2014]
- (a) $y = \sqrt{6}x + \sqrt{2}$
 (b) $y = \sqrt{6}x - \sqrt{2}$
 (c) $y = \sqrt{3}x + \sqrt{2}$
 (d) $y = \sqrt{3}x - \sqrt{2}$
- 20.** A hyperbola passes through $(3, 3)$ and the length of its conjugate axis is 8. The length of latusrectum is [MP PET 2012]
- (a) $\frac{20}{3}$ (b) $\frac{40}{3}$
 (c) $\frac{50}{3}$ (d) None of these
- 21.** The value of m , for which the line $y = mx + 2$ is a tangent to the hyperbola $4x^2 - 9y^2 = 36$, are [MP PET 2012]
- (a) $\pm \frac{4\sqrt{2}}{3}$ (b) $\pm \frac{2}{3}$
 (c) $\pm \frac{8}{9}$ (d) $\pm \frac{2\sqrt{2}}{3}$
- 22.** The eccentricity of the hyperbola with latusrectum 12 and semi-conjugate axis $2\sqrt{3}$, is [AMU 2011]
- (a) 3 (b) $\sqrt{\frac{3}{2}}$ (c) $2\sqrt{3}$ (d) 2
- 23.** The equation to the common tangents to the two hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are [MP PET 2011]
- (a) $y = \pm x \pm \sqrt{(b^2 - a^2)}$ (b) $y = \pm x \pm \sqrt{(a^2 - b^2)}$
 (c) $y = \pm x \pm (a^2 - b^2)$ (d) $y = \pm x \pm \sqrt{(a^2 + b^2)}$
- 24.** The equation of the common tangent to the parabola $y^2 = 8x$ and rectangular hyperbola $xy = -1$ is [AMU 2011]
- (a) $x - y + 2 = 0$ (b) $9x - 3y + 2 = 0$
 (c) $2x - y + 1 = 0$ (d) $x + 2y - 1 = 0$
- 25.** The equation of a hyperbola, whose asymptotes are $3x \pm 5y = 0$ and vertices are $(\pm 5, 0)$, is [Karnataka CET 2011]
- (a) $3x^2 - 5y^2 = 25$ (b) $5x^2 - 3y^2 = 225$
 (c) $25x^2 - 9y^2 = 225$ (d) $9x^2 - 25y^2 = 225$
- 26.** Equation of auxiliary circle of $\frac{x^2}{16} - \frac{y^2}{25} = 1$, is [Guj CET 2011]
- (a) $x^2 + y^2 = 16$ (b) $x^2 + y^2 = 25$
 (c) $x^2 + y^2 = 9$ (d) $x^2 + y^2 = 41$
- 27.** The distance between the foci of the conic $7x^2 - 9y^2 = 63$ is equal to [Kerala CEE 2010]
- (a) 8 (b) 4 (c) 3
 (d) 7 (e) 12
- 28.** For different values of α , the locus of the point of intersection of the two straight lines $\sqrt{3}x - y - 4\sqrt{3}\alpha = 0$ and $\sqrt{3}\alpha x + \alpha y - 4\sqrt{3} = 0$ is [WB JEE 2010]
- (a) a hyperbola with eccentricity $\frac{2}{3}$
 (b) an ellipse with eccentricity $\sqrt{\frac{2}{3}}$
 (c) a hyperbola with eccentricity $\sqrt{\frac{19}{16}}$
 (d) an ellipse with eccentricity $\frac{3}{4}$

Answers

Work Book Exercise 16.1

1. (a) 2. (c) 3. (a) 4. (b) 5. (a)

Work Book Exercise 16.2

1. (b,c) 2. (a) 3. (c) 4. (a) 5. (c)

Work Book Exercise 16.3

1. (d) 2. (c) 3. (d) 4. (a) 5. (a) 6. (b)

Target Exercises

1. (a)	2. (c)	3. (d)	4. (a)	5. (b)	6. (a)	7. (a)	8. (d)	9. (b)	10. (c)
11. (d)	12. (d)	13. (c)	14. (a)	15. (c)	16. (b)	17. (a)	18. (b)	19. (d)	20. (a)
21. (b)	22. (b)	23. (d)	24. (b)	25. (c)	26. (b)	27. (b)	28. (a)	29. (c)	30. (a)
31. (a)	32. (b)	33. (a)	34. (a)	35. (c)	36. (a)	37. (b)	38. (a)	39. (d)	40. (a)
41. (c)	42. (b)	43. (a)	44. (b)	45. (d)	46. (b)	47. (d)	48. (b)	49. (a)	50. (b)
51. (a)	52. (a)	53. (b)	54. (c)	55. (d)	56. (b)	57. (a)	58. (a,b,c)	59. (all)	60. (a,b,c)
61. (a,c)	62. (a,b)	63. (d)	64. (b)	65. (a)	66. (d)	67. (d)	68. (b)	69. (b)	70. (a)
71. (d)	72. (c)	73. (d)	74. (b)	75. (c)	76. (d)	77. (*)	78. (**)	79. (1)	80. (4)
81. (2)	82. (6)	83. (4)							

* $A \rightarrow p$; $B \rightarrow q$; $C \rightarrow p$; $D \rightarrow s$

** $A \rightarrow r$; $B \rightarrow s$; $C \rightarrow s$; $D \rightarrow p$

Entrances Gallery

1. (a,b)	2. (b,d)	3. (b)	4. (b)	5. (a)	6. (c)	7. (d)	8. (a)	9. (c)	10. (a)
11. (e)	12. (d)	13. (a)	14. (b)	15. (c)	16. (c)	17. (b)	18. (b)	19. (a)	20. (b)
21. (d)	22. (d)	23. (b)	24. (a)	25. (d)	26. (b)	27. (a)	28. (a)		

Explanations

Target Exercises

1. Centre of the hyperbola is the mid-point of the line joining the two foci, therefore the coordinates of the centre are (1, 5). Now, distance between the foci = 10
 $\Rightarrow 2ae = 10$
 $\Rightarrow ae = 5 \Rightarrow a = 4$
 Now, $b^2 = a^2(e^2 - 1) \Rightarrow b = 3$
 Hence, the equation of the hyperbola is

$$\frac{(x-1)^2}{16} - \frac{(y-5)^2}{9} = 1$$

2. We have, $a = 3$ and $\frac{b^2}{a} = 4 \Rightarrow b^2 = 12$

$$\text{Hence, the equation of the hyperbola is } \frac{x^2}{9} - \frac{y^2}{12} = 1$$

$$\Rightarrow 4x^2 - 3y^2 = 36$$

3. Equation of hyperbola is

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1 \Rightarrow a^2 = b^2(e^2 - 1) \Rightarrow e = \sqrt{\frac{a^2 + b^2}{b^2}}$$

4. The given equation can be written as $-\frac{x^2}{4} + \frac{y^2}{3} = 1$. The eccentricity of this hyperbola is given by

$$e = \sqrt{1 + \frac{a^2}{b^2}} = \sqrt{1 + \frac{4}{3}} = \sqrt{\frac{7}{3}}$$

5. We have, $9(x^2 - 4x + 4) - 16(y^2 - 6y + 9) = 144$
 $\Rightarrow \frac{(x-2)^2}{4^2} - \frac{(y-3)^2}{3^2} = 1$

$$\text{Shifting the origin at } (2, 3), \text{ we have } \frac{X^2}{4^2} - \frac{Y^2}{3^2} = 1$$

where, $x = X + 2, y = Y + 3$. Then, coordinates of the vertices are $X = \pm 4, Y = 3$ i.e. $x = 6, y = 3$ and $x = -2, y = 3$.

6. Let the equation to the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots(i)$$

Hence, the transverse axis is $2a$. If C is the centre, S is the focus and A is the vertex to the hyperbola, then

$$CS = ae$$

[distance between the centre and the focus]

$$\therefore a = \frac{ae}{2} \text{ or } e = 2$$

$$\text{Again, } b^2 = a^2(e^2 - 1) = a^2(4 - 1) = 3a^2$$

On substituting the value of b^2 in Eq. (i), we get

$$\frac{x^2}{a^2} - \frac{y^2}{3a^2} = 1$$

$$\Rightarrow 3x^2 - y^2 = 3a^2$$

7. Given that,

$$x = a \tan(\theta + \alpha) \text{ and } y = b \tan(\theta + \beta)$$

$$\text{or } \tan^{-1}\left(\frac{y}{x}\right) = \theta + \alpha \quad \dots(i)$$

$$\text{and } \tan^{-1}\left(\frac{y}{b}\right) = \theta + \beta \quad \dots(ii)$$

To get the required locus, we have to eliminate θ from Eqs. (i) and (ii).

On subtracting Eq. (ii) from Eq. (i), we get

$$\tan^{-1}\left(\frac{x}{a}\right) - \tan^{-1}\left(\frac{y}{b}\right) = \alpha - \beta$$

$$\Rightarrow \tan^{-1}\left[\frac{\frac{x}{a} - \frac{y}{b}}{1 + \frac{x}{a} \cdot \frac{y}{b}}\right] = \alpha - \beta$$

$$\Rightarrow \frac{\frac{x}{a} - \frac{y}{b}}{1 + \frac{x}{a} \cdot \frac{y}{b}} = \tan(\alpha - \beta)$$

On simplifying, we get the required locus as

$$xy + ab = (bx - ay) \cot(\alpha - \beta)$$

which is a hyperbola.

8. If $k_1 = 0$, then $k_1 PA + k_2 PB = k_3$

$$\Rightarrow PB = \frac{k_3}{k_2} > 0$$

$\Rightarrow P$ describes a circle with B as centre and radius $= \frac{k_3}{k_2}$

If $k_3 = 0$, then $k_1 PA + k_2 PB = 0$

$$\Rightarrow \frac{PA}{PB} = \frac{k_2}{-k_1} = k > 0$$

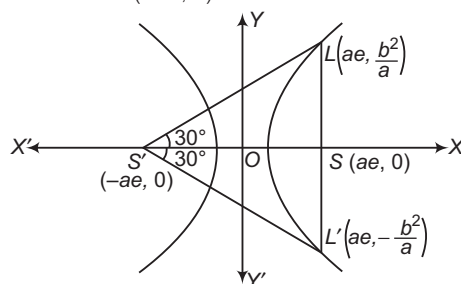
$\Rightarrow P$ describes a circle with $P_1 P_2$ as its diameter, P_1 and P_2 being the points which divide AB internally and externally in the ratio $k : 1$. If $k_1 = k_2 > 0$ and $k_3 > 0$, then

$$PA + PB = \frac{k_3}{k_1} = k > 0$$

$\Rightarrow P$ describes an ellipse with A and B as its foci.

9. Let LSL' be a latusrectum through the focus $S(ae, 0)$ of

the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. It subtends angle 60° at the other focus $S'(-ae, 0)$.



We have, $\angle LSL' = 60^\circ$

$\therefore \angle LSS' = 30^\circ$

In $\triangle LSS'$, we have

$$\tan 30^\circ = \frac{LS}{S'S} \Rightarrow \frac{1}{\sqrt{3}} = \frac{b^2/a}{2ae}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{b^2}{2a^2e}$$

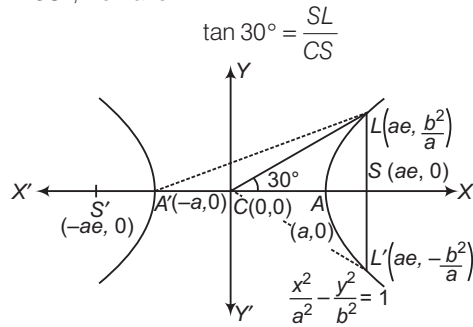
$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{e^2 - 1}{2e}$$

$$\Rightarrow \sqrt{3}e^2 - 2e - \sqrt{3} = 0$$

$$\Rightarrow (e - \sqrt{3})(\sqrt{3}e + 1) = 0$$

$$\Rightarrow e = \sqrt{3}$$

10. Let LSL' be a latusrectum and C be the centre of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. It is given that $\triangle CLL'$ is an equilateral triangle. Therefore, $\angle LCS = 30^\circ$
In $\triangle CSL$, we have



$$\begin{aligned} \Rightarrow \tan 30^\circ &= \frac{SL}{CS} \\ \Rightarrow \frac{1}{\sqrt{3}} &= \frac{b^2/a}{ae} \\ \Rightarrow \frac{1}{\sqrt{3}} &= \frac{b^2}{a^2e} \\ \Rightarrow \frac{1}{\sqrt{3}} &= \frac{e^2 - 1}{e} \\ \Rightarrow \sqrt{3}e^2 - e - \sqrt{3} &= 0 \Rightarrow e = \frac{1 + \sqrt{13}}{2\sqrt{3}} \end{aligned}$$

11. It is given that $\triangle A'LL'$ is an equilateral triangle. Therefore, $\angle LA'S = 30^\circ$

$$\begin{aligned} \text{In } \triangle A'LS, \text{ we have } \tan \angle LA'S &= \frac{LS}{A'S} \\ \Rightarrow \tan 30^\circ &= \frac{b^2/a}{a + ae} \\ \Rightarrow \frac{1}{\sqrt{3}} &= \frac{b^2}{a^2(1+e)} \\ \Rightarrow \frac{1}{\sqrt{3}} &= \frac{e^2 - 1}{1+e} \\ \Rightarrow \frac{1}{\sqrt{3}} &= e - 1 \\ \Rightarrow e &= 1 + \frac{1}{\sqrt{3}} = \frac{\sqrt{3} + 1}{\sqrt{3}} \end{aligned}$$

12. $(OP)^2 = (OQ)^2 = (PQ)^2$

$$\Rightarrow \frac{a^2}{b^2}(b^2 + l^2) + l^2 = \frac{a^2}{b^2}(b^2 + l^2) + l^2 = 4l^2$$

$$\Rightarrow l^2 = \frac{a^2 b^2}{(3b^2 - a^2)} > 0$$

$$\Rightarrow 3b^2 - a^2 > 0$$

$$\Rightarrow 3b^2 > a^2 \Rightarrow 3a^2(e^2 - 1) > a^2 \Rightarrow e^2 > \frac{4}{3}$$

$$\Rightarrow e > \frac{2}{\sqrt{3}}$$

13. Here, $a^2 = 3, b^2 = 2, m = 3$ and the equation of tangent to hyperbola is $y = mx \pm \sqrt{(a^2 m^2 - b^2)}$

$$\Rightarrow y = 3x \pm 5$$

14. The equation of line is $x \cos \alpha + y \sin \alpha = p$

$$\Rightarrow y = -x \cot \alpha + p \operatorname{cosec} \alpha$$

 Now, $c^2 = a^2 m^2 - b^2$

$$\Rightarrow p^2 \operatorname{cosec}^2 \alpha = a^2 \cot^2 \alpha - b^2$$

$$\Rightarrow a^2 \cos^2 \alpha - b^2 \sin^2 \alpha = p^2$$

15. Making $y^2 - x^2 = 4$ homogeneous with the help of the straight line $\sqrt{3}x + y = 2$, combined equation of lines joining the points of intersection of the curve $y^2 - x^2 = 4$ and the given line to the origin is given by

$$\begin{aligned} y^2 - x^2 &= 4 \left(\frac{\sqrt{3}x + y}{2} \right)^2 \\ \Rightarrow 4x^2 + 2\sqrt{3}xy + 0 \cdot y^2 &= 0 \end{aligned}$$

If θ is the angle between these lines, then

$$\theta = \tan^{-1} \left| \frac{2\sqrt{h^2 - ab}}{a + b} \right| = \tan^{-1} \left| \frac{2\sqrt{3 - 0}}{4 + 0} \right|$$

Here, $a = 4, h = \sqrt{3}, b = 0$

$$\therefore \theta = \tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$$

16. Let $y = mx + c$ be a common tangent to $9x^2 - 16y^2 = 144$ and $x^2 + y^2 = 9$.
Since, $y = mx + c$ is a tangent to $9x^2 - 16y^2 = 144$.

Therefore, $c^2 = a^2 m^2 - b^2 \Rightarrow c^2 = 16m^2 - 9 \dots (i)$

Now, $y = mx + c$ is a tangent to $x^2 + y^2 = 9$, therefore

$$\frac{c}{\sqrt{m^2 + 1}} = 3 \Rightarrow c^2 = 9(1 + m^2) \dots (ii)$$

From Eqs. (i) and (ii), we get

$$16m^2 - 9 = 9 + 9m^2 \Rightarrow m = \pm 3\sqrt{\frac{2}{7}}$$

On putting the value of m in Eq. (ii), we get $c = \pm \frac{15}{\sqrt{7}}$

Hence, $y = 3\sqrt{\frac{2}{7}}x + \frac{15}{\sqrt{7}}$ is a common tangent.

17. Let $y = x + c$ be a tangent to $\frac{x^2}{3} - \frac{y^2}{2} = 1$, then

$$c^2 = 3 - 2 = 1 \Rightarrow c = \pm 1$$

So, the required tangents are $y = x \pm 1$.

18. Let $P(x_1, y_1)$ be a point on the hyperbola. Then, the coordinates of N are $(x_1, 0)$. The equation of the tangent at (x_1, y_1) is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$.

This meets X-axis at $T\left(\frac{a^2}{x_1}, 0\right)$.

$$\therefore OT \cdot ON = \frac{a^2}{x_1} \times x_1 = a^2$$

19. The equation of the normal at $(a \sec \phi, b \tan \phi)$ to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax \sin \phi + by = (a^2 + b^2) \tan \phi \dots (i)$$

and the equation of the line is

$$lx + my + n = 0 \dots (ii)$$

Now, Eqs. (i) and (ii) represent the same line, therefore

$$\frac{a \sin \phi}{l} = \frac{b}{m} = \frac{(a^2 + b^2) \tan \phi}{-n} = \frac{(a^2 + b^2) \sin \phi}{-n \cos \phi}$$

$$\therefore \sin \phi = \frac{bl}{am} \text{ and } \cos \phi = \frac{(a^2 + b^2)l}{-na}$$

$$\Rightarrow \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

20. By $T = S_1$, the equation of chord whose mid-point is (α, β) is $3x\alpha - 2y\beta + 2(x + \alpha) - 2(y + \beta) = 0$
 $\Rightarrow x(3\alpha + 2) - y(2\beta + 3) = 2x$
 as it is parallel to $y = 2x$

$$\therefore 3\alpha - 4\beta = 4$$

$$\therefore \text{Required locus is } 3x - 4y = 4.$$

21. Equation of hyperbola is $25x^2 - 16y^2 = 400$

$$\Rightarrow \frac{x^2}{16} - \frac{y^2}{25} = 1$$

Chord is bisected at $(6, 2)$.

$$\therefore \frac{6x}{16} - \frac{2y}{25} = \frac{6^2}{16} - \frac{2^2}{25}$$

$$\Rightarrow 75x - 16y = 418$$

22. Diameters $y = m_1x$ and $y = m_2x$ are conjugate diameters

of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $m_1m_2 = \frac{b^2}{a^2}$.

$$\text{Here, } a^2 = 9, b^2 = 16 \text{ and } m_1 = \frac{1}{2}$$

$$\therefore m_1m_2 = \frac{b^2}{a^2} \Rightarrow \frac{1}{2}(m_2) = \frac{16}{9}$$

$$\Rightarrow m_2 = \frac{32}{9}$$

Thus, the required diameter is $y = \frac{32x}{9}$.

23. Normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point

$$Q(a \sec \phi, b \tan \phi) \text{ is}$$

$$ax \cos \phi + by \cot \phi = a^2 + b^2$$

Let coordinates of P be (h, k) , then

$$h = \frac{(a^2 + b^2)}{a} \sec \phi \text{ and } k = \frac{(a^2 + b^2)}{b} \tan \phi$$

$$\Rightarrow ah = (a^2 + b^2) \sec \phi \quad \dots (i)$$

$$\text{and } bk = (a^2 + b^2) \tan \phi \quad \dots (ii)$$

On squaring and subtracting Eq. (ii) from Eq. (i), we get

$$a^2h^2 - b^2k^2 = (a^2 + b^2)^2$$

Then, locus of P is

$$a^2x^2 - b^2y^2 = (a^2 + b^2)^2$$

24. If $x = 9$ meets the hyperbola at $(9, 6\sqrt{2})$ and $(9, -6\sqrt{2})$.

Then, equations of tangent at these points are $3x - 2\sqrt{2}y - 3 = 0$ and $3x + 2\sqrt{2}y - 3 = 0$. The combined equations of these two is $9x^2 - 8y^2 - 18x + 9 = 0$.

25. The equation of the normal at $(a \sec \theta, b \tan \theta)$ to the given hyperbola is

$$ax \cos \theta + by \cot \theta = (a^2 + b^2)$$

This meets the transverse axis i.e. X -axis at G . So, the coordinates of G are

$$\left(\frac{a^2 + b^2}{a} \sec \theta, 0 \right)$$

The coordinates of the vertices A and A' are $A(a, 0)$ and $A'(-a, 0)$, respectively.

$$\therefore AG \cdot A'G = \left(-a + \frac{a^2 + b^2}{a} \sec \theta \right) \left(a + \frac{a^2 + b^2}{a} \sec \theta \right)$$

$$= (-a + ae^2 \sec \theta)(a + ae^2 \sec \theta) = a^2(e^4 \sec^2 \theta - 1)$$

26. We know that,

Equation of hyperbola + Equation of conjugate hyperbola = 2 (Equation of asymptotes)

$\therefore$ Equation of conjugate hyperbola

= 2 Equation of asymptotes - Equation of hyperbola

$$C(x, y) = 2A(x, y) - H(x, y) \quad \dots (i)$$

$$\therefore H(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y = 0$$

$\therefore$ Equation of asymptotes is

$$x^2 + 3xy + 2y^2 + 2x + 3y + \lambda = 0$$

$$\therefore \Delta = 0, abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \text{ then}$$

$$\lambda = 1$$

$$\therefore A(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y + 1 = 0$$

$$\therefore C(x, y) \equiv x^2 + 3xy + 2y^2 + 2x + 3y + 2 = 0$$

[from Eq. (i)]

27. In case of rectangular hyperbola,

$$a = b \Rightarrow b^2 = a^2 \Rightarrow a^2(e^2 - 1) = a^2 \Rightarrow e = \sqrt{2}$$

28. Asymptotes of the given hyperbola are $y = \pm \frac{bx}{a}$.

Therefore, angle between them is $2 \tan^{-1} \left(\frac{b}{a} \right)$.

29. Equation of hyperbola is

$$2x^2 + 5xy + 2y^2 - 11x - 7y - 4 = 0 \quad \dots (i)$$

Let asymptotes of the hyperbola (i) be

$$2x^2 + 5xy + 2y^2 - 11x - 7y + c = 0 \quad \dots (ii)$$

Since, Eq. (ii) represents a pair of lines.

$$\therefore \begin{vmatrix} 2 & 5/2 & -11/2 \\ 5/2 & 2 & -7/2 \\ -11/2 & -7/2 & c \end{vmatrix} = 0$$

$$\Rightarrow -\frac{11}{2} \left(-\frac{35}{4} + 11 \right) + \frac{7}{2} \left(-7 + \frac{55}{4} \right) + c \left(4 - \frac{25}{4} \right) = 0$$

$$\Rightarrow \frac{9}{4}c = -\frac{99}{8} + \frac{189}{8} = \frac{90}{8} \Rightarrow c = 5$$

$\therefore$ Equation of the required asymptotes is

$$2x^2 + 5xy + 2y^2 - 11x - 7y + 5 = 0$$

30. Let the asymptotes be

$$2x + 3y + c_1 = 0 \text{ and } 3x + 2y + c_2 = 0$$

Since, the asymptotes passes through the centre $(1, 2)$ of the hyperbola.

$$\therefore 2 + 6 + c_1 = 0$$

$$\text{and } 3(1) + 2(2) + c_2 = 0$$

$$\Rightarrow c_1 = -8 \text{ and } c_2 = -7$$

Thus, the equations of the asymptotes are

$$2x + 3y - 8 = 0$$

$$\text{and } 3x + 2y - 7 = 0$$

Let the equation of the hyperbola be

$$(2x + 3y - 8)(3x + 2y - 7) + \lambda = 0 \quad \dots (i)$$

It passes through $(5, 3)$.

$$\Rightarrow (2x + 3y - 8) \cdot (3x + 2y - 7) = 154$$

which is the equation of the required hyperbola.

31. The combined equation of the asymptotes is

$$(3x - 4y + 7)(4x + 3y + 1) = 0$$

So, the combined equation of the hyperbola is

$$(3x - 4y + 7)(4x + 3y + 1) + \lambda = 0$$

It passes through the origin.

$$\therefore 7 \times 1 + \lambda = 0 \Rightarrow \lambda = -7$$

On putting the value of λ in Eq. (i), we get

$$(3x - 4y + 7)(4x + 3y + 1) - 7 = 0$$

$$\Rightarrow 12x^2 - 7xy - 12y^2 + 31x + 17y = 0$$

which is the equation of the required hyperbola.

32. Let the equation to the rectangular hyperbola be $x^2 - y^2 = a^2$... (i)

As the asymptotes of this are the axes of the other and vice-versa, hence the equation of the other hyperbola may be written as

$$xy = c^2 \quad \dots (ii)$$

Let Eqs. (i) and (ii) meet at some point whose coordinates are $(a \sec \alpha, a \tan \alpha)$.

Then, the tangent at the point $(a \sec \alpha, a \tan \alpha)$ to Eq. (i) is $x - y \sin \alpha = a \cos \alpha$... (iii)

and the tangent at the point $(a \sec \alpha, a \tan \alpha)$ to Eq. (ii) is $y + x \sin \alpha = \left(\frac{2c^2}{a}\right) \cos \alpha$... (iv)

Clearly, the slopes of the tangents given by Eqs. (iv) and (iii) are respectively $-\sin \alpha$ and $\frac{1}{\sin \alpha}$, so their product is

$$-\sin \alpha \cdot \left(\frac{1}{\sin \alpha}\right) = -1$$

Hence, the tangents are at right angle.

33. Let $A\left(ct_1, \frac{c}{t_1}\right), B\left(ct_2, \frac{c}{t_2}\right)$ and $C\left(ct_3, \frac{c}{t_3}\right)$ be three points on the hyperbola $xy = c^2$. Then, the equation of the side AB is

$$x + yt_1 t_2 = c(t_1 + t_2)$$

$$\Rightarrow y = -\frac{x}{t_1 t_2} + c\left(\frac{t_1 + t_2}{t_1 t_2}\right)$$

This will touch the parabola $y^2 = 4ax$, if

$$c\left(\frac{t_1 + t_2}{t_1 t_2}\right) = \left(-\frac{1}{t_1 t_2}\right) \quad [\because c = a/m]$$

$$\Rightarrow (at_1^2)t_2^2 + ct_2 + ct_2 = 0$$

This is a quadratic equation in t_2 . So, it gives two values of t_2 corresponding a given value of t_1 . Thus,

corresponding to one vertex $A\left(ct_1, \frac{c}{t_1}\right)$, there is another

position of vertex i.e. $B\left(ct_2, \frac{c}{t_2}\right)$.

Similarly, corresponding to each position of B , there are two positions of C . Thus, corresponding to a given position of vertex A , there can be four triangles whose sides touch the parabola $y^2 = 4ax$. But A can attain infinitely many positions. Hence, there are infinitely many triangles inscribed in $xy = c^2$ such that its sides touch the parabola $y^2 = 4ax$.

34. If (x_i, y_i) is the point of intersection of given curves, then

$$\sum_{i=1}^4 x_i = \frac{1+1}{2} \quad \text{and} \quad \sum_{i=1}^4 y_i = 0$$

$$\text{Now, } \frac{\sum_{i=1}^3 x_i}{3} = \frac{4-x_4}{3} \quad \text{and} \quad \frac{\sum_{i=1}^3 y_i}{3} = -\frac{y_4}{4}$$

$$\text{Centroid } \left(\frac{\sum_{i=1}^3 x_i}{3}, \frac{\sum_{i=1}^3 y_i}{3}\right) \text{ lies on the line } y = 3x - 4.$$

$$\text{Hence, } \frac{-y_4}{3} = \frac{3(4-x_4)}{3} - 4$$

$$\Rightarrow y_4 = 3x_4$$

Therefore, the locus of D is $y = 3x$.

35. Foci of hyperbola lie on $y = x$. So, the major axis is $y = x$.

Major axis of hyperbola bisects the asymptote.

$\Rightarrow$ Equation of hyperbola is $x = 2y$

$\Rightarrow$ Equation of hyperbola is $(y - 2x)(x - 2y) + k = 0$

Given that, it passes through $(3, 4)$.

$$\Rightarrow k = 10$$

Hence, required equation is

$$2x^2 + 2y^2 - 5xy + 10 = 0$$

36. Equation of tangent at point $P(\alpha \cos \theta, \sin \theta)$ is

$$\frac{x}{\alpha} \cos \theta + \frac{y}{1} \sin \theta = 1 \quad \dots (i)$$

Let it cut the hyperbola at points P and Q .

Homogenizing the hyperbola $\alpha^2 x^2 - y^2 = 1$ with the help of the above the equation, we get

$$\alpha^2 x^2 - y^2 = \left(\frac{x}{\alpha} \cos \theta + y \sin \theta\right)^2$$

This is a pair of straight lines OP and OQ .

$$\text{Given, } \angle POQ = \frac{\pi}{2}$$

$\Rightarrow$ Coefficient of x^2 + Coefficient of $y^2 = 0$

$$\Rightarrow \alpha^2 - \frac{\cos^2 \theta}{\alpha^2} - 1 - \sin^2 \theta = 0$$

$$\Rightarrow \alpha^2 - \frac{\cos^2 \theta}{\alpha^2} - 1 - 1 + \cos^2 \theta = 0$$

$$\Rightarrow \cos^2 \theta = \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1}$$

$$\text{Now, } 0 \leq \cos^2 \theta \leq 1$$

$$\Rightarrow 0 \leq \frac{\alpha^2(2 - \alpha^2)}{\alpha^2 - 1} \leq 1$$

On solving, we get

$$\alpha^2 \in \left[\frac{\sqrt{5}+1}{2}, 2\right]$$

37. The chord of contact of tangents from (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

It meets the axes at the points $\left(\frac{a^2}{x_1}, 0\right)$ and $\left(0, \frac{b^2}{y_1}\right)$.

Area of the triangle is $\frac{1}{2} \cdot \frac{a^2}{x_1} \cdot \frac{b^2}{y_1} = k$ [constant]

$$\Rightarrow x_1 y_1 = \frac{a^2 b^2}{2k} = c^2, \text{ where } c \text{ is a constant.}$$

$\Rightarrow xy = c^2$ is the required locus.

38. Let the equation of asymptotes be
 $2x^2 + 5xy + 2y^2 + 4x + 5y + \lambda = 0 \quad \dots(i)$

This equation represents a pair of straight lines.
 Therefore, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

We have, $4\lambda + 25 - \frac{25}{2} - 8 - \lambda \frac{25}{4} = 0$

$\Rightarrow -\frac{9\lambda}{4} + \frac{9}{2} = 0$

$\Rightarrow \lambda = 2$

Putting the value of λ in Eq. (i), we get
 $2x^2 + 5xy + 2y^2 + 4x + 5y + 2 = 0$

which is equation of the asymptote.

39. Let $P(x_1, y_1)$ be a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

The chord of contact of tangents from P to the hyperbola is given by

$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \quad \dots(i)$

The equation of the asymptotes are

$\frac{x}{a} - \frac{y}{b} = 0$

and

$\frac{x}{a} + \frac{y}{b} = 0$

The points of intersection of Eq. (i) with the two asymptotes are given by

$x_1 = \frac{2a}{\frac{x_1}{a} - \frac{y_1}{b}}, y_1 = \frac{2b}{\frac{x_1}{a} - \frac{y_1}{b}}$

$x_2 = \frac{2a}{\frac{x_1}{a} + \frac{y_1}{b}}, y_2 = \frac{2b}{\frac{x_1}{a} + \frac{y_1}{b}}$

Area of the triangle = $\frac{1}{2} (x_1 y_2 - x_2 y_1)$

$= \frac{1}{2} \left(\frac{4ab \times 2}{\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}} \right) = 4ab \text{ sq units}$

40. Let the variable chord be
 $x \cos \alpha + y \sin \alpha = p \quad \dots(i)$

Let this chord intersect the hyperbola at A and B . Then, the combined equation of OA and OB is given by

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \left(\frac{x \cos \alpha + y \sin \alpha}{p} \right)^2$

$\Rightarrow x^2 \left(\frac{1}{a^2} - \frac{\cos^2 \alpha}{p^2} \right) - y^2 \left(\frac{1}{b^2} + \frac{\sin^2 \alpha}{p^2} \right) - \frac{2 \sin \alpha \cos \alpha}{p} xy = 0$

This chord subtends a right angle at centre. Therefore, Coefficient of x^2 + Coefficient of y^2 = 0

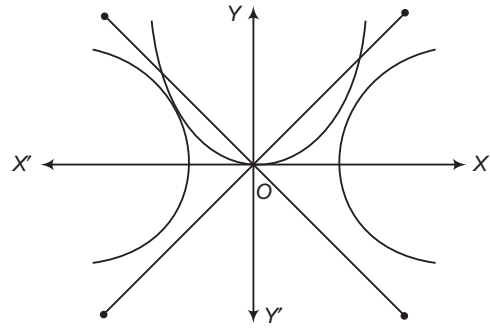
$\Rightarrow \frac{1}{a^2} - \frac{\cos^2 \alpha}{p^2} - \frac{1}{b^2} - \frac{\sin^2 \alpha}{p^2} = 0$

$\Rightarrow \frac{1}{a^2} - \frac{1}{b^2} = \frac{1}{p^2}$

$\Rightarrow p^2 = \frac{a^2 b^2}{b^2 - a^2}$

Hence, p is a constant i.e. it touches the fixed circle.

41. Given that, $\frac{x^2}{1} - \frac{y^2}{4} = 1$



Since, (α, α^2) lie on the parabola $y = x^2$, then (α, α^2) must lie between the asymptotes of hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$ in 1st and 2nd quadrants.

So, the asymptotes are $y = \pm 2x$

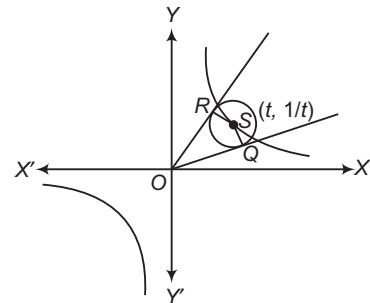
$\therefore 2\alpha < \alpha^2$

$\Rightarrow \alpha < 0 \text{ or } \alpha > 2 \text{ and } -2\alpha < \alpha^2$

$\alpha < -2 \text{ or } \alpha > 0$

$\therefore \alpha \in (-\infty, -2) \text{ or } (2, \infty)$

42. Let S , say $(t, \frac{1}{t})$ be a point on the rectangular hyperbola.



Now, circumcircle of ΔOQR also passes through S .

Therefore, circumcentre is the mid-point of OS .

Hence, $x = \frac{t}{2} \text{ and } y = \frac{1}{2t}$

So, the locus of the circumcentre is $xy = \frac{1}{4}$.

43. The points are such that one of the points is the orthocentre of the triangle formed by other three points. When the vertices of a triangle lie on a rectangular hyperbola the orthocentre also lies on the same hyperbola.

44. Since, $c_1 c_2 (a_1 a_2 + b_1 b_2) < 0$, therefore origin lies in acute angle and $P(1, 2)$ lies in obtuse angle.

Acute angle between the asymptotes is $\frac{\pi}{3}$.

Hence, $e = \sec \frac{\theta}{2} = \sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$

45. Transverse axis is the equation of the angle bisector passing containing point $(2, 3)$, which is given by

$\frac{3x - 4y + 5}{5} = \frac{12x + 5y - 40}{13}$

$\Rightarrow 21x + 77y = 265$

46. Since, points (x_r, y_r) is lying on $xy = c^2$ for $r = 1, 2, 3, 4$

$$\therefore y_r = \frac{c^2}{x_r}, \text{ for } r = 1, 2, 3, 4$$

$$\text{Slope of } QR \text{ is } -\frac{c^2}{x_2 x_3}$$

$\therefore$ Equation of line passing through A and perpendicular to QR is

$$x_1 x_2 x_3 x - c^2 x_1 y = x_1^2 x_2 x_3 - c^4 \quad \dots (i)$$

Similarly, equation of line through Q and perpendicular to PR is $x_1 x_2 x_3 x - c^2 x_2 y = x_1 x_2^2 x_3 - c^4 \quad \dots (ii)$

From Eqs. (i) and (ii), we get

$$y = -\frac{x_1 x_2 x_3}{c^2}$$

$$\therefore x = \frac{c^4}{x_1 x_2 x_3}$$

Thus, orthocentres is again lying on $xy = c^2$ i.e. the fourth point (x_4, y_4) .

47. Let equation of the rectangular hyperbola be

$$xy = c^2 \quad \dots (i)$$

and equation of circle be

$$x^2 + y^2 = r^2 \quad \dots (ii)$$

Put $y = \frac{c^2}{x}$ in Eq. (ii), we get

$$x^2 + \frac{c^4}{x^2} = r^2$$

$$\Rightarrow x^4 - r^2 x^2 + c^4 = 0 \quad \dots (iii)$$

Now, $CP^2 + CQ^2 + CR^2 + CS^2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 + x_3^2 + y_3^2 + x_4^2 + y_4^2$$

$$= \left(\sum_{i=1}^4 x_i \right)^2 - 2 \sum_{i=1}^4 x_i x_2 + \left(\sum_{i=1}^4 y_i \right)^2 - 2 \sum_{i=1}^4 y_i y_2$$

$$= 2r^2 + 2r^2 = 4r^2 \quad [\text{from Eq. (iii)}]$$

48. Let t_1, t_2, t_3 and t_4 be the parameters of the points P, Q, R and S, respectively. Then, the coordinates of P, Q, R and S are $\left(ct_1, \frac{c}{t_1} \right), \left(ct_2, \frac{c}{t_2} \right), \left(ct_3, \frac{c}{t_3} \right)$ and $\left(ct_4, \frac{c}{t_4} \right)$, respectively.

$$\text{Now, } PQ \perp RS \Rightarrow \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = -1$$

$$\Rightarrow \left(-\frac{1}{t_1 t_2} \right) \times \left(-\frac{1}{t_3 t_4} \right) = 1$$

$$\Rightarrow t_1 t_2 t_3 t_4 = 1 \quad \dots (i)$$

$\therefore$ Product of the slopes of CP, CQ, CR and CS

$$= \frac{1}{t_1^2} \times \frac{1}{t_2^2} \times \frac{1}{t_3^2} \times \frac{1}{t_4^2} = \frac{1}{t_1^2 t_2^2 t_3^2 t_4^2} = 1 \quad [\text{from Eq. (i)}]$$

49. Let the equation of the circle be

$$x^2 + y^2 + 2gx + 2fy + k = 0$$

The equation of the hyperbola is $xy = c^2$.

Eliminating y from these two equations, we get

$$x^2 + \frac{1}{x^2} + 2gx + 2f \left(\frac{1}{x} \right) + k = 0$$

$$\Rightarrow x^4 + 2gx^3 + kx^2 + 2fx + 1 = 0$$

This is a fourth degree equation in x giving four values of x say x_1, x_2, x_3 and x_4 .

$$\therefore x_1 x_2 x_3 x_4 = 1$$

Corresponding to every value of x, there is a value of y given by $xy = 1$.

$$\therefore y_1 y_2 y_3 y_4 = \frac{1}{x_1 x_2 x_3 x_4} = 1$$

50. Let $(a \sec \theta, b \tan \theta)$ be the any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

The equations of the asymptotes of the given hyperbola are

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} = 0$$

Now, p_1 = length of the perpendicular from

$$(a \sec \theta, b \tan \theta) \text{ on } \frac{x}{a} + \frac{y}{b} = 0$$

$$= \frac{\sec \theta - \tan \theta}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

and p_2 = length of the perpendicular from

$$(a \sec \theta, b \tan \theta) \text{ on } \frac{x}{a} - \frac{y}{b} = 0$$

$$= \frac{\sec \theta + \tan \theta}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2}}}$$

$$\therefore p_1 p_2 = \frac{\sec^2 \theta - \tan^2 \theta}{\frac{1}{a^2} + \frac{1}{b^2}} = \frac{a^2 b^2}{a^2 + b^2}$$

51. Let the equation of the hyperbola be

$$H(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

$$\text{Then, } A(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

is the equation of asymptotes

$$\text{and } C(x, y) \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$$

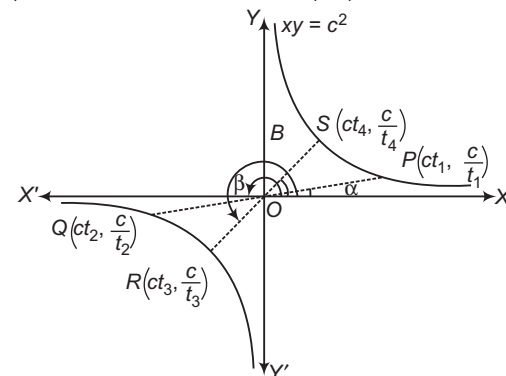
is the equation of its conjugate hyperbola.

$$\text{Since, } H(\alpha, \beta) + C(\alpha, \beta) = 2A(\alpha, \beta)$$

$\therefore$ Hyperbola, asymptotes and conjugate hyperbola are in AP.

52. Let $xy = c^2$ be the equation of the rectangular hyperbola

referred to its asymptotes as the coordinate axes and $P \left(ct_1, \frac{c}{t_1} \right), Q \left(ct_2, \frac{c}{t_2} \right), R \left(ct_3, \frac{c}{t_3} \right)$ and $S \left(ct_4, \frac{c}{t_4} \right)$ be four points on its such that PQ is perpendicular to RS.



Since, OP makes angle α with OX . Therefore,

$$\tan \alpha = \frac{\frac{c}{t_1}}{\frac{1}{t_1^2}} = \frac{1}{t_1^2}$$

Similarly, $\tan \beta = \frac{1}{t_2^2}$, $\tan \gamma = \frac{1}{t_3^2}$ and $\tan \delta = \frac{1}{t_4^2}$

$$\therefore \tan \alpha \tan \beta \tan \gamma \tan \delta = \frac{1}{t_1^2 t_2^2 t_3^2 t_4^2} \quad \dots (i)$$

Now, PQ is perpendicular to RS .

$$\Rightarrow \frac{\frac{c}{t_2} - \frac{c}{t_1}}{ct_2 - ct_1} \times \frac{\frac{c}{t_4} - \frac{c}{t_3}}{ct_4 - ct_3} = 1$$

$$\Rightarrow -\frac{1}{t_1 t_2} \times \frac{-1}{t_3 t_4} = -1$$

$$\Rightarrow \frac{1}{t_1 t_2 t_3 t_4} = -1$$

$$\Rightarrow t_1 t_2 t_3 t_4 = -1 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\tan \alpha \tan \beta \tan \gamma \tan \delta = 1$$

53. Let $P(a \sec \theta, a \tan \theta)$ be a point on the hyperbola $x^2 - y^2 = a^2$. The distance of P from the line $y = 2x$ is given by

$$S = \frac{2a \sec \theta - a \tan \theta}{\sqrt{4 + 1}}$$

$$\Rightarrow S = \frac{a}{\sqrt{5}} (2 \sec \theta - \tan \theta)$$

$$\Rightarrow \frac{dS}{d\theta} = \frac{a}{\sqrt{5}} (2 \sec \theta \tan \theta - \sec^2 \theta)$$

$$\text{Put } \frac{dS}{d\theta} = 0, \text{ then}$$

$$2 \sec \theta \tan \theta - \sec^2 \theta = 0$$

$$\Rightarrow 2 \tan \theta = \sec \theta$$

$$\Rightarrow 4 \tan^2 \theta = 1 + \tan^2 \theta$$

$$\Rightarrow \tan \theta = \pm \frac{1}{\sqrt{3}}$$

$$\text{Now, } \frac{d^2 S}{d\theta^2} = \frac{a}{\sqrt{2}} (2 \sec^3 \theta + 2 \sec \theta \tan^2 \theta - 2 \sec^2 \theta \tan \theta)$$

$$\text{Clearly, } \frac{d^2 S}{d\theta^2} > 0 \text{ for } \tan \theta = \pm \frac{1}{\sqrt{3}} \text{ i.e. for } \theta = \pm \frac{\pi}{6}$$

Thus, the coordinates of P are given by

$$h = \frac{2a}{3}$$

$$\text{and } k = \pm \frac{a}{\sqrt{3}}$$

$$\Rightarrow h = \pm 2k$$

$$\Rightarrow h \mp 2k = 0$$

Hence, the locus of (h, k) is $x \pm 2y = 0$.

54. The equation of the hyperbola referred to its asymptotes as the coordinate axes is $xy = c^2$.

The equation of the tangents to it at $P\left(ct_1, \frac{c}{t_1}\right)$ and

$$Q\left(ct_2, \frac{c}{t_2}\right) \text{ are } \frac{x}{t_1} + yt_1 = 2c$$

$$\text{and } \frac{x}{t_2} + yt_2 = 2c$$

These two intersect at $R\left(\frac{2ct_1 t_2}{t_1 + t_2}, \frac{2c}{t_1 + t_2}\right)$.

$$\text{Now, HM of } ct_1 \text{ and } ct_2 = \frac{2ct_1 \cdot ct_2}{ct_1 + ct_2} = \frac{2ct_1 t_2}{t_1 + t_2}$$

$$\text{and HM of } \frac{c}{t_1} \text{ and } \frac{c}{t_2} = \frac{2 \cdot \frac{c}{t_1} \cdot \frac{c}{t_2}}{\frac{c}{t_1} + \frac{c}{t_2}} = \frac{2c}{t_1 + t_2}$$

Hence, the coordinates of R are the harmonic means of the coordinates of the points of contact P and Q .

55. The equation of the normal to the hyperbola $xy = c^2$ at

$$\text{any point } \left(ct, \frac{c}{t}\right) \text{ is } xt^3 - yt - ct^4 + c = 0.$$

If it passes through $P(h, k)$, then

$$ht^3 - kt - ct^4 + c = 0$$

$$\Rightarrow ct^4 - ht^3 + kt - c = 0 \quad \dots (i)$$

This being a fourth degree equation in t , gives four values of t , say t_1, t_2, t_3 and t_4 such that corresponding to each value of t , there is a normal passing through $P(h, k)$.

$$\text{Let } A\left(ct_1, \frac{c}{t_1}\right), B\left(ct_2, \frac{c}{t_2}\right), C\left(ct_3, \frac{c}{t_3}\right) \text{ and } D\left(ct_4, \frac{c}{t_4}\right)$$

be four points on the hyperbola such that normals at A, B, C and D passes through P .

From Eq. (i), we have

$$t_1 + t_2 + t_3 + t_4 = \frac{h}{c}$$

$$\text{and } t_1 t_2 t_3 + t_1 t_2 t_4 + t_3 t_4 t_1 + t_2 t_3 t_4 = -\frac{k}{c}$$

$$\text{Also, } t_1 t_2 t_3 = -\frac{c}{c} = -1$$

$$\therefore \frac{c}{t_1} + \frac{c}{t_2} + \frac{c}{t_3} + \frac{c}{t_4} = c \left[\frac{t_2 t_3 t_4 + t_1 t_3 t_4 + t_1 t_2 t_4 + t_1 t_2 t_3}{t_1 t_2 t_3 t_4} \right] = \frac{c(-k/c)}{-1} = k$$

Thus, sum of the ordinates A, B, C and D is k .

56. The equation of the normal to the hyperbola $xy = c^2$ at

$$\text{any point } \left(ct, \frac{c}{t}\right) \text{ is } xt^3 - yt - ct^4 + c = 0.$$

If it passes through $P(h, k)$, then

$$ht^3 - kt - ct^4 + c = 0$$

$$\Rightarrow ct^4 - ht^3 + kt - c = 0 \quad \dots (i)$$

This being a fourth degree equation in t , gives four values of t , say t_1, t_2, t_3 and t_4 such that corresponding to each value of t , there is a normal passing through $P(h, k)$.

$$\text{Let } A\left(ct_1, \frac{c}{t_1}\right), B\left(ct_2, \frac{c}{t_2}\right), C\left(ct_3, \frac{c}{t_3}\right) \text{ and } D\left(ct_4, \frac{c}{t_4}\right) \text{ be}$$

four points on the hyperbola such that normals at A, B, C and D passes through P .

Product of the abscissae of A, B, C and D

$$= ct_1 \times ct_2 \times ct_3 \times ct_4$$

$$= c^4 t_1 t_2 t_3 t_4$$

$$= -c^4 \quad [\because t_1 t_2 t_3 t_4 = -1]$$

Product of the ordinates of A, B, C and D

$$\begin{aligned} &= \frac{c}{t_1} \times \frac{c}{t_2} \times \frac{c}{t_3} \times \frac{c}{t_4} \\ &= \frac{c^4}{t_1 t_2 t_3 t_4} = -c^4 \end{aligned}$$

57. The distance of the parabola from the circle means the distance of any point on the parabola from the centre of the circle.

Let $A(2t, t^2)$ be any point on the parabola $x^2 = 4y$ and $C(12, 0)$ be the centre of the circle. Then,

$$AC^2 = (2t - 12)^2 + (t^2 - 0)^2$$

Let $Z = AC^2$

Then, $Z = 4(t - 6)^2 + t^4$

$$\therefore \frac{dZ}{dt} = 8(t - 60 + 4t^3)$$

$$\text{and } \frac{d^2Z}{dt^2} = 12 + 12t^2$$

For maximum and minimum values of Z , we must have

$$\frac{dZ}{dt} = 0$$

$$\Rightarrow 8(t - 6) + 4t^3 = 0 \Rightarrow t^3 + 2t - 12 = 0$$

$$\Rightarrow (t - 2)(t^2 + 2t + 6) = 0 \Rightarrow t = 2$$

$$\text{Clearly, } \frac{d^2Z}{dt^2} > 0, \text{ for } t = 2$$

Thus, Z is minimum when $t = 2$.

The minimum value of Z is given by

$$Z = 4(2 - 6)^2 + 2^4 = 80$$

$$\Rightarrow AC^2 = 80 \Rightarrow AC = 4\sqrt{5}$$

Hence, the least distance = $4\sqrt{5}$

$$58. \frac{(x-1)^2}{3} - \frac{(y-2)^2}{16} = 1$$

Transverse axis = $2\sqrt{3}$ units

Conjugate axis = 8 units

Centre = $(1, -2)$

$$\text{Eccentricity} = \sqrt{\frac{19}{3}}$$

$$59. \text{Verify each option in } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \text{ and}$$

$$xy = c^2$$

$$(a) \text{ We have, } \frac{2x}{a} = t + \frac{1}{t} \quad \dots(i)$$

$$\text{and } \frac{2y}{b} = t - \frac{1}{t} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\frac{x}{a} + \frac{y}{b} = t \text{ and } \frac{x}{a} - \frac{y}{b} = \frac{1}{t}$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$(b) \text{ We have, } \frac{tx}{a} - \frac{y}{b} + t = 0 \quad \dots(i)$$

$$\text{and } \frac{x}{a} + \frac{ty}{b} - 1 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$\frac{x}{a} = \frac{1-t^2}{1+t^2} \text{ and } \frac{y}{b} = \frac{2t}{1+t^2}$$

$$(c) \text{ We have, } x = e^t + e^{-t} \quad \dots(i)$$

$$\text{and } y = e^t - e^{-t} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get
 $x + y = 2e^t$ and $x - y = 2e^{-t}$

$$\Rightarrow x^2 - y^2 = 4$$

$$(d) \text{ We have, } x^2 - 6 = 2\cos t \quad \dots(i)$$

$$\text{and } y^2 + 2 = 4\cos^2 t / 2 = 2 + 2\cos t$$

$$\Rightarrow y^2 = 2\cos t \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$x^2 - y^2 = 6$$

60. Locus of point of intersection of perpendicular tangents is director circle $x^2 + y^2 = a^2 - b^2$.

$$\text{Now, } e^2 = 1 + \frac{b^2}{a^2}$$

If $a^2 > b^2$, then there are infinite (or more than 1) points on the circle.

$$\Rightarrow e^2 < 2 \Rightarrow e < \sqrt{2}$$

If $a^2 < b^2$, there does not exist any point on the plane.

$$\Rightarrow e^2 > 2 \Rightarrow e > \sqrt{2}$$

If $a^2 = b^2$, there is exactly one point (centre of the hyperbola).

$$\Rightarrow e = \sqrt{2}$$

61. The equation of tangent in terms of slope of $y^2 = 32x$ is

$$y = mx + \frac{8}{m} \quad \dots(i)$$

which is also tangent of the hyperbola $9x^2 - 9y^2 = 8$

$$\text{i.e. } x^2 - y^2 = \frac{8}{9}$$

$$\text{Then, } \left(\frac{8}{m}\right)^2 = \frac{8}{9}m^2 - \frac{8}{9} \Rightarrow \frac{8}{m^2} = \frac{m^2}{9} - \frac{1}{9}$$

$$\Rightarrow 72 = m^4 - m^2 \Rightarrow m^4 - m^2 - 72 = 0$$

$$\Rightarrow (m^2 - 9)(m^2 + 8) = 0$$

$$\Rightarrow m^2 = 9 \quad [\because m^2 + 8 \neq 0]$$

$$\Rightarrow m = \pm 3$$

From Eq. (i), we get

$$y = \pm 3x \pm \frac{8}{3} \Rightarrow 3y = \pm 9x \pm 8$$

$$\Rightarrow \pm 9x - 3y \pm 8 = 0$$

$$\Rightarrow 9x - 3y + 8 = 0, 9x - 3y - 8 = 0$$

$$\Rightarrow -9x - 3y + 8 = 0, -9x - 3y - 8 = 0$$

$$\text{or } 9x - 3y + 8 = 0, 9x - 3y - 8 = 0$$

$$\Rightarrow 9x + 3y - 8 = 0$$

$$\text{and } 9x + 3y + 8 = 0$$

$$62. \frac{x^2}{25} - \frac{y^2}{16} = 1$$

Asymptotes are $4x + 5y = 0$

$$\Rightarrow 4x - 5y = 0$$

$$\text{Directrices are } x = \pm \frac{25}{\sqrt{41}}$$

and common points are

$$\left(\frac{25}{\sqrt{41}}, \frac{-20}{\sqrt{41}}\right), \left(\frac{25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right),$$

$$\left(\frac{-25}{\sqrt{41}}, \frac{20}{\sqrt{41}}\right), \left(\frac{-25}{\sqrt{41}}, \frac{-20}{\sqrt{41}}\right).$$

63. Statement I is false, because this will represent hyperbola, if $h^2 > ab$.

$$\Rightarrow \frac{\lambda^2}{4} > 2$$

$$\Rightarrow |\lambda| > 2\sqrt{2}$$

Statement II, being a standard result, is true.

64. If $y = mx + c$ be the common tangent, then

$$c^2 = a^2 m^2 - b^2 \quad \dots(i)$$

$$\text{and} \quad c^2 = -b^2 m^2 + a^2 \quad \dots(ii)$$

On eliminating c^2 , we get

$$m^2 = 1 \Rightarrow m = \pm 1$$

For Statement II,

On eliminating m , we get

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

which is a hyperbola.

65. First of all, we will verify Statement II.

Let $P(a \sec \theta, a \tan \theta)$ be any point on $x^2 - y^2 = a^2$, then

$$\begin{aligned} SP \cdot S'P &= (e a \sec \theta - a)(e a \sec \theta + a) \\ &= e^2 a^2 \sec^2 \theta - a^2 \\ &= 2a^2 \sec^2 \theta - a^2 \quad [\because e = \sqrt{2}] \end{aligned}$$

$$\begin{aligned} \text{and} \quad CP^2 &= a^2 \sec^2 \theta + a^2 \tan^2 \theta \\ &= a^2 \sec^2 \theta + a^2 \sec^2 \theta - a^2 \\ &= 2a^2 \sec^2 \theta - a^2 \end{aligned}$$

$$\Rightarrow SP \cdot S'P = CP^2$$

$\therefore$ Statement II is true.

If we put $a = \sqrt{2}$, $\theta = \frac{\pi}{4}$, then Statement I is verified.

66. Now, $4x^2 - 3y^2 = 12$

$$\Rightarrow \frac{x^2}{3} - \frac{y^2}{4} = 1$$

Then, director circle is $x^2 + y^2 = 3 - 4 = -1$, which does not exist for existence $a > b$.

67. Statement II is true.

For the points (2, 2), (4, 1) and (6, 2/3), $t_1 = 1$, $t_2 = 2$ and $t_3 = 3$, respectively.

$$\text{For the point } (1/4, 16), t_4 = \frac{1}{8}$$

$$\text{Now, } t_1 t_2 t_3 t_4 = \frac{3}{4} \neq 1$$

Hence, Statement I is false.

68. By solving $3x - 4y - 12 = 0$

and $4x + 3y - 12 = 0$, we get centres

$$3x - 4y - 12 = 0 \quad \dots(i)$$

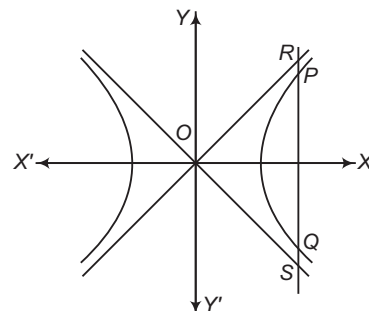
$$\text{and} \quad 4x + 3y - 12 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get $\left(\frac{84}{25}, -\frac{12}{25}\right)$.

69. Length of laustrectum = $\frac{2b^2}{a}$

$$= \frac{2(225)}{10} = 45$$

- 70.



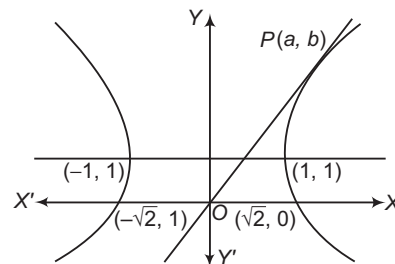
From the figure, obviously $PR = QS$

71. On differentiating the curve, we have

$$2x - 2(y-1) \frac{dy}{dx} = 0$$

$$\Rightarrow \left[\frac{dy}{dx} \right]_{a,b} = \frac{a}{b-1} = \frac{b}{a} \left(m_{OP} = \frac{b}{a} \right)$$

$$\Rightarrow a^2 = b^2 - b \quad \dots(i)$$



Also, (a, b) satisfy the curve

$$a^2 - (b-1)^2 = 1$$

$$\Rightarrow a^2 - (b^2 - 2b + 1) = 1$$

$$a^2 - b^2 + 2b = 2$$

$$\therefore -b + 2b = 2 \quad [\because a^2 = b^2 - b]$$

$$\Rightarrow b = 2$$

$$\therefore a = \sqrt{2} \quad [\because a \neq -\sqrt{2}]$$

$$\therefore \sin^{-1} \left(\frac{a}{b} \right) = \frac{\pi}{4}$$

72. Length of latusrectum = $\frac{2b^2}{a}$

$$= 2a = \text{Distance between the vertices} = 2$$

73. Curve is a rectangular hyperbola.

$$\Rightarrow e = \sqrt{2}$$

74. Perpendicular tangents intersect at the centre of rectangular hyperbola. Hence, centre of hyperbola is (1, 1) and equation of asymptotes are $x - 1 = 0$ and $y - 1 = 0$.

75. Let the equation of hyperbola be $xy - x - y + 1 + \lambda = 0$.

It passes through (3, 2), hence $\lambda = -2$.

So, the equation of hyperbola is $xy = x + y + 1$.

76. From the centre of hyperbola, we can draw two real tangents to the rectangular hyperbola.

77. We have, $A = ae_E$ and $a = Ae_H$

$$\Rightarrow e_E e_H = 1$$

$$\Rightarrow e_E + e_H > 2$$

$$B^2 = A^2 (e_H^2 - 1) = a^2 (1 - e_E^2) = b^2$$

$$\Rightarrow \frac{b}{B} = 1$$

Also, the angle between the asymptotes is

$$2 \tan^{-1} \frac{B}{A} = \frac{2\pi}{3}$$

Also, $\frac{B}{A} = \sqrt{3} \Rightarrow \frac{b}{ae_E} = \sqrt{3} \Rightarrow e_E^2 = \frac{1}{4}$

On solving $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $\frac{x^2}{a^2 e_E^2} - \frac{y^2}{b^2} = 1$, we get

$$x^2 = \frac{2a^2 e_E^2}{b^2 (1 - e_E^2)} = 4$$

78. A. Since, $3x^2 - 5xy - 2y^2 + 5x + 11y + c = 0$ are asymptotes.

$\therefore$ It represents a pair of a straight lines.

$$\therefore 3(-2)c + 2 \cdot \frac{11}{2} \left(\frac{5}{2} \right) \left(\frac{-5}{2} \right) - 3 \left(\frac{11}{2} \right)^2 - (-2) \left(\frac{5}{2} \right)^2 - c \left(-\frac{5}{2} \right)^2 = 0$$

$$\Rightarrow -6c - \frac{275}{4} - \frac{363}{4} + \frac{25}{2} - \frac{25}{4}c = 0$$

$$\Rightarrow -24c - 275 - 363 + 50 - 25c = 0$$

$$\Rightarrow 49c = -588$$

$$\Rightarrow c = -12$$

B. Let the point be (h, k) . The equation of the chord of contact is $hx + ky = 4$.

Since, $hx + ky = 4$ is tangent to $xy = 1$.

$$x \left(\frac{4 - hx}{k} \right) = 1 \text{ has two equal roots.}$$

$$\Rightarrow hx^2 - 4x + k = 0$$

$$\Rightarrow hk = 4$$

$$\therefore \text{Locus of } (h, k) \text{ is } xy = 4$$

$$\Rightarrow c^2 = 4$$

C. Equation of the hyperbola is $\frac{x^2}{c/a} - \frac{y^2}{c/b} = 1$

$$\therefore \sqrt{\frac{c}{b}} = \frac{5}{2} \text{ and } \frac{13}{2} = \sqrt{\frac{c}{a}} \cdot \sqrt{\frac{a+b}{b}}$$

$$\therefore \frac{c}{a} = 36$$

$$\therefore \text{The hyperbola is } 25x^2 - 144y^2 = 900$$

$$\therefore a = 25, b = 144, c = 900$$

$$\therefore \frac{ab}{c} = 4$$

D. Let the hyperbola be $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

$$\text{Then, } 2a = ae \text{ i.e. } e = 2$$

$$\therefore \frac{b^2}{a^2} = e^2 - 1 = 3$$

$$\therefore \frac{(2b)^2}{(2a)^2} = 3$$

79. The equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{25} = 1$.

Equation of chord is $T = S_1$.

On simplifying, we get

$$125x - 48y = 481$$

$$\therefore \lambda = 1$$

80. On eliminating y between the line and hyperbola, we get

$$25x^2 - 9 \left(\frac{45 - 25x}{12} \right)^2 = 225$$

On simplifying, we get

$$x^2 - 10x + 25 = 0$$

$$\Rightarrow x = 5$$

$$\text{Hence, } y = -\frac{20}{3} = -\frac{5 \times 4}{3}$$

$$\Rightarrow \lambda = 4$$

81. Since, $\frac{e}{2}$ and $\frac{e'}{2}$ are eccentricities of a hyperbola and its conjugate.

$$\therefore \frac{4}{e^2} + \frac{4}{e'^2} = 1$$

$$\text{i.e. } 4 = \frac{e^2 e'^2}{e^2 + e'^2}$$

Line passing through the points $(e, 0)$ and $(0, e')$.

82. Equation of hyperbola is $\frac{x^2}{16} - \frac{y^2}{18} = 1$

$$\text{or } 9x^2 - 8y^2 - 144 = 0$$

Homogenization of this equation using

$$\frac{x \cos \alpha + y \sin \alpha}{p} = 1$$

$$\text{We have, } 9x^2 - 8y^2 - 144 \left(\frac{x \cos \alpha + y \sin \alpha}{p} \right)^2 = 0$$

Since, these lines are perpendicular to each other.

$$\therefore 9p^2 - 8p^2 - 144(\cos^2 \alpha + \sin^2 \alpha) = 0$$

$$\Rightarrow p^2 = 144$$

$$\text{or } p = \pm 12$$

$$\therefore \text{Radius of the circle} = 12$$

$$\therefore \text{Diameter of the circle} = 24$$

83. Equation of hyperbola $(x - 3)(y - 2) = c^2$

$$\text{or } xy - 2x - 3y + 6 = c^2$$

It passes through $(4, 6)$, then

$$4 \times 6 - 2 \times 4 - 3 \times 6 + 6 = c^2$$

$$\Rightarrow c = 2$$

$$\therefore \text{Latusrectum} = 2\sqrt{2}c = 2\sqrt{2} \times 2 = 4\sqrt{2}$$

$$\text{and } -\frac{x}{2\sqrt{2}} + \frac{y}{4\sqrt{2}} = 1$$

On comparing it with $\frac{xx_1}{9} - \frac{yy_1}{4} = 1$, we get

$$\text{Points of contact as } \left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \text{ and } \left(-\frac{9}{2\sqrt{2}}, \frac{-1}{\sqrt{2}} \right).$$

Entrances Gallery

1. Slope of tangent = 2

The tangents are $y = 2x \pm \sqrt{9 \times 4 - 4}$

$$\text{i.e. } 2x - y = \pm 4\sqrt{2}$$

$$\Rightarrow \frac{x}{2\sqrt{2}} - \frac{y}{4\sqrt{2}} = 1$$

Aliter

Equation of tangent at $P(\theta)$ is $\left(\frac{\sec \theta}{3}\right)x - \left(\frac{\tan \theta}{2}\right)y = 1$

$$\Rightarrow \text{Slope} = \frac{2 \sec \theta}{3 \tan \theta} = 2$$

$$\Rightarrow \sin \theta = \frac{1}{3}$$

$$\Rightarrow \text{Points are } \left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right) \text{ and } \left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right).$$

2. Ellipse is $\frac{x^2}{2^2} + \frac{y^2}{1^2} = 1$

$$\Rightarrow 1^2 = 2^2(1 - e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$$

$\therefore$ Eccentricity of the hyperbola is $\frac{2}{\sqrt{3}}$.

$$\Rightarrow b^2 = a^2 \left(\frac{4}{3} - 1\right) \Rightarrow 3b^2 = a^2$$

Foci of the ellipse are $(\sqrt{3}, 0)$ and $(-\sqrt{3}, 0)$.
Hyperbola passes through $(\sqrt{3}, 0)$.

$$\Rightarrow \frac{3}{a^2} = 1 \Rightarrow a^2 = 3 \text{ and } b^2 = 1$$

$\therefore$ Equation of hyperbola is $x^2 - 3y^2 = 3$.

Focus of hyperbola is

$$(ae, 0) = \left(\sqrt{3} \times \frac{2}{\sqrt{3}}, 0\right) = (2, 0)$$

3. Equation of normal is

$$(y - 3) = \frac{-a^2}{2b^2}(x - 6)$$

$$\Rightarrow \frac{a^2}{2b^2} = 1$$

$$\Rightarrow e = \sqrt{\frac{3}{2}}$$

4. A tangent to $\frac{x^2}{9} - \frac{y^2}{4} = 1$ is

$$y = mx + \sqrt{9m^2 - 4}, m > 0$$

It is tangent to $x^2 + y^2 - 8x = 0$

$$\therefore \frac{4m + \sqrt{9m^2 - 4}}{\sqrt{1 + m^2}} = 4$$

$$\Rightarrow 495m^4 + 104m^2 - 400 = 0$$

$$\Rightarrow m^2 = \frac{4}{5} \text{ or } m = \frac{2}{\sqrt{5}}$$

$$\therefore \text{The tangent is } y = \frac{2}{\sqrt{5}}x + \frac{4}{\sqrt{5}}$$

$$\Rightarrow 2x - \sqrt{5}y + 4 = 0$$

5. A point on hyperbola is $(3 \sec \theta, 2 \tan \theta)$.

It lies on the circle, so $9 \sec^2 \theta + 4 \tan^2 \theta - 24 \sec \theta = 0$

$$\Rightarrow 13 \sec^2 \theta - 24 \sec \theta - 4 = 0$$

$$\Rightarrow \sec \theta = 2 - \frac{2}{13}$$

$$\therefore \sec \theta = 2 \Rightarrow \tan \theta = \sqrt{3}$$

$\therefore$ The point of intersection are $A(6, 2\sqrt{3})$ and $B(6, -2\sqrt{3})$.

$\therefore$ The circle with AB as diameter is

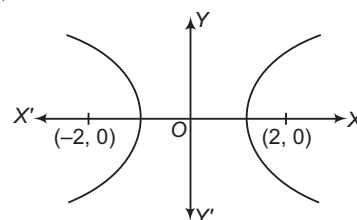
$$(x - 6)^2 + y^2 = (2\sqrt{3})^2$$

$$\Rightarrow x^2 + y^2 - 12x + 24 = 0$$

6. Let equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

where, $2ae = 4$ and $e = 2$



$$\Rightarrow a = 1$$

$$\Rightarrow a^2 e^2 = a^2 + b^2$$

$$\Rightarrow 4 = 1 + b^2$$

$$\therefore b^2 = 3$$

$$\Rightarrow \text{Thus, equation of hyperbola is } \frac{x^2}{1} - \frac{y^2}{3} = 1$$

$$\Rightarrow 3x^2 - y^2 = 3$$

7. The given equation of hyperbola is

$$\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{\sin^2 \alpha} = 1$$

Here, $a^2 = \cos^2 \alpha$ and $b^2 = \sin^2 \alpha$

$$\text{Now, } e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{\sin^2 \alpha}{\cos^2 \alpha}} = \sqrt{1 + \tan^2 \alpha} = \sec \alpha$$

Coordinates of foci are $(\pm ae, 0)$ i.e. $(\pm 1, 0)$.

Hence, abscissae of foci remain constant, when α varies.

8. Since, the line $y = \alpha x + \beta$ is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore \beta^2 = a^2 \alpha^2 - b^2$$

So, locus of (α, β) is $y^2 = a^2 x^2 - b^2$

$$\Rightarrow a^2 x^2 - y^2 - b^2 = 0$$

Since, this equation represents a hyperbola, so locus of a point $P(\alpha, \beta)$ is a hyperbola.

9. Given equation of hyperbola can be rewritten as

$$\frac{x^2}{\left(\frac{12}{5}\right)^2} - \frac{y^2}{\left(\frac{9}{5}\right)^2} = 1$$

$$\therefore \text{Eccentricity is } e'^2 = 1 + \frac{b'^2}{a'^2}$$

$$\Rightarrow e'^2 = 1 + \frac{9}{16} = \frac{25}{16} \Rightarrow e' = \frac{5}{4}$$

The foci of a hyperbola are

$$(\pm a' e', 0) = \left(\pm \frac{12}{5} \times \frac{5}{4}, 0\right) = (\pm 3, 0)$$

Given equation of ellipse is

$$\frac{x^2}{16} + \frac{y^2}{b^2} = 1$$

Foci of an ellipse are $(\pm ae, 0) = (\pm 4e, 0)$. But given focus of ellipse and hyperbola coincide, then

$$4e = 3 \Rightarrow e = \frac{3}{4}$$

Also, $b^2 = a^2(1 - e^2)$
 $= 16\left(1 - \frac{9}{16}\right) = 16 - 9 = 7$

10. The mid-point of the chord is
 $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

The equation of the chord $T = S_1$
 $\therefore x\left(\frac{y_1 + y_2}{2}\right) + y\left(\frac{x_1 + x_2}{2}\right) = 2\left(\frac{x_1 + x_2}{2}\right)\left(\frac{y_1 + y_2}{2}\right)$
 $\Rightarrow x(y_1 + y_2) + y(x_1 + x_2) = (x_1 + x_2)(y_1 + y_2)$
 $\Rightarrow \frac{x}{x_1 + x_2} + \frac{y}{y_1 + y_2} = 1$

11. Given length of latusrectum is $4\sqrt{3}$.

$\therefore \frac{2b^2}{a} = 4\sqrt{3} \quad \dots(i)$

and length of transverse axis is $2\sqrt{3}$.

$2a = 2\sqrt{3}$
 $a = \sqrt{3}$

On putting $a = \sqrt{3}$ in Eq. (i), we get

$\frac{2b^2}{\sqrt{3}} = 4\sqrt{3}$
 $\Rightarrow 2b^2 = 12$
 $\Rightarrow b^2 = 6$
 $\therefore$ Equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$
 $\Rightarrow \frac{x^2}{3} - \frac{y^2}{6} = 1$

12. Given, $e = \frac{5}{4}$

Foci of a hyperbola is $(\pm ae, 0)$ i.e. $\left(\pm \frac{5}{4}a, 0\right)$.

Since, the focal chord is $2x + 3y - 6 = 0 \quad \dots(ii)$

$\therefore$ Points $\left(\pm \frac{5}{4}a, 0\right)$ satisfy Eq. (ii).

$\Rightarrow 2 \times \left(\pm \frac{5}{4}a\right) + 3(0) - 6 = 0$

$\Rightarrow \pm \frac{5}{2}a = 6$

$\Rightarrow a = \pm \frac{12}{5}$

$\therefore$ Length of transverse axis $= 2a = 2 \cdot \frac{12}{5} \left[\text{take } a = \frac{12}{5} \right]$
 $= \frac{24}{5}$

13. Let equation of hyperbola be

$$\frac{x^2}{a_1^2} - \frac{y^2}{b_1^2} = 1$$

Given, $2a_1 = 2\cos \alpha \Rightarrow a_1 = \cos \alpha$

Also, given equation of ellipse is $\frac{x^2}{16} + \frac{y^2}{9} = 1$.

Here, $e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \frac{\sqrt{7}}{4}$

According to the given condition,

Foci of hyperbola (e_1) = Foci of ellipse (e)

$\Rightarrow \pm a_1 e_1 = \pm ae$

$\Rightarrow \cos \alpha \cdot e_1 = 4 \cdot \frac{\sqrt{7}}{4}$

$\Rightarrow \cos \alpha \cdot e_1 = \sqrt{7}$

$\Rightarrow \cos \alpha \sqrt{1 + \frac{b_1^2}{\cos^2 \alpha}} = \sqrt{7}$

$\Rightarrow \cos^2 \alpha + b_1^2 = 7$

$\Rightarrow b_1^2 = 7 - \cos^2 \alpha$

$\therefore$ The equation of hyperbola is
 $\frac{x^2}{\cos^2 \alpha} - \frac{y^2}{7 - \cos^2 \alpha} = 1$

14. Given equation is $x^2 - y^2 = 512$

$\therefore a^2 - b^2 = 512$

$\Rightarrow (a + b)(a - b) = 2^9$

$\Rightarrow (a + b, a - b) = (2^8, 2), (2^7, 2^2), (2^6, 2^3), (2^5, 2^4)$

So, the number of order pairs is 4.

15. Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots(i)$$

Given equation of ellipse is $\frac{x^2}{(13)^2} + \frac{y^2}{(5)^2} = 1$

Here, $a = 13, b = 5$

$\therefore e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{25}{169}} = \sqrt{\frac{144}{169}} = \frac{12}{13}$

$\therefore$ Foci $= (\pm ae, 0) = \left(\pm 13 \times \frac{12}{13}, 0\right) = (\pm 12, 0)$

Since, Eq. (i) passes through $(\pm 12, 0)$.

$\therefore \frac{144}{a^2} - \frac{0}{b^2} = 1$

$\Rightarrow a^2 = 144 \Rightarrow a = \pm 12$

Now, eccentricity of hyperbola,

$$e' = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{b^2}{144}}$$

According to the question,

$ee' = 1 \Rightarrow \frac{12}{13} \times \sqrt{1 + \frac{b^2}{144}} = 1$

$\Rightarrow \sqrt{1 + \frac{b^2}{144}} = \frac{13}{12} \Rightarrow 1 + \frac{b^2}{144} = \frac{169}{144}$

$\Rightarrow \frac{b^2}{144} = \frac{169}{144} - 1 \Rightarrow \frac{b^2}{144} = \frac{25}{144}$

$\Rightarrow b^2 = 25$

$\therefore$ Equation of hyperbola is $\frac{x^2}{144} - \frac{y^2}{25} = 1$.

16. The given curve is

$$5x^2 + 12xy - 22x - 12y - 19 = 0 \quad \dots(i)$$

Comparing the curve (i) with general form

$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$, we get

$a = 5, h = 6, b = 0, g = -11, f = -6, c = -19$

$\therefore \Delta = abc + 2fgh - af^2 - bg^2 - ch^2$
 $= 5 \times 0 \times (-19) + 2 \times (-6) \times (-11) (6)$
 $= -5 \times 36 - 0 - (-19) (36)$

$$\Rightarrow \Delta = 792 - 180 + 19 \times 36 \neq 0$$

$$\therefore \Delta \neq 0$$

$$\text{Now, } ab = 5 \times 0 = 0, h^2 = 36$$

$$\Rightarrow ab - h^2 = 0 - 36 = -36$$

$$\therefore ab - h^2 < 0$$

So, the curve is a hyperbola.

17. The equation of any normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$ax \cos \phi + by \cot \phi = a^2 + b^2$$

$$\Rightarrow ax \cos \phi + by \cot \phi - (a^2 + b^2) = 0 \quad \dots(i)$$

The straight line $lx + my - n = 0$ will be normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then Eq. (i) and $lx + my - n = 0$ represent the same line.

$$\frac{a \cos \phi}{l} = \frac{b \cot \phi}{m} = \frac{a^2 + b^2}{n}$$

$$\Rightarrow \sec \phi = \frac{na}{l(a^2 + b^2)}$$

$$\text{and } \tan \phi = \frac{nb}{m(a^2 + b^2)}$$

$$\therefore \sec^2 \phi - \tan^2 \phi = 1$$

$$\therefore \frac{n^2 a^2}{l^2 (a^2 + b^2)^2} - \frac{n^2 b^2}{m^2 (a^2 + b^2)^2} = 1$$

$$\Rightarrow \frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{n^2}$$

But given equation of normal is

$$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2 + b^2)^2}{k}$$

$$\Rightarrow k = n^2$$

18. Equation of chord of hyperbola $x^2 - y^2 = a^2$ with mid-point as (h, k) is given by

$$xh - yk = h^2 - k^2 \Rightarrow y = \frac{h}{k}x - \frac{(h^2 - k^2)}{k}$$

This will touch the parabola $y^2 = 4ax$, if

$$-\left(\frac{h^2 - k^2}{k}\right) = \frac{a}{\frac{h}{k}} \Rightarrow ak^2 = h^3 + k^2h$$

$\therefore$ Locus of the mid-point is $x^3 = y^2(x - a)$.

19. Equation of tangent in slope form parabola $y^2 = 8\sqrt{3}x$ is

$$y = mx + c \quad \dots(i)$$

where,

$$c = \frac{a}{m}$$

$$c = \frac{2\sqrt{3}}{m} \quad \dots(ii)$$

Also, tangent to the hyperbola $4x^2 - y^2 = 4$

$$\text{or } \frac{x^2}{1} - \frac{y^2}{4} = 1 \text{ is } c^2 = a^2 m^2 - b^2$$

$$\Rightarrow c^2 = 1 \cdot m^2 - 4$$

$$\Rightarrow \left(\frac{2\sqrt{3}}{m}\right)^2 = m^2 - 4 \quad [\text{from Eq. (ii)}]$$

$$\Rightarrow \frac{12}{m^2} = m^2 - 4$$

$$\Rightarrow m^4 - 4m^2 - 12 = 0$$

$$\Rightarrow m^4 - 6m^2 + 2m^2 - 12 = 0$$

$$\Rightarrow m^2(m^2 - 6) + 2(m^2 - 6) = 0$$

$$\Rightarrow (m^2 + 2)(m^2 - 6) = 0$$

$$\Rightarrow m^2 - 6 = 0 \quad \text{and} \quad m^2 + 2 \neq 0$$

$$\Rightarrow m^2 = 6$$

$$\Rightarrow m = \pm \sqrt{6}$$

i.e. $m = \sqrt{6}$, as m is positive slope.

$$\text{From Eq. (i), } y = \sqrt{6}x + \frac{2\sqrt{3}}{\sqrt{6}}$$

$$\therefore y = \sqrt{6}x + \sqrt{2}$$

20. $\therefore 2b = 8 \Rightarrow b = 4$

Since, hyperbola passes through $(3, 3)$.

$$\therefore \frac{9}{a^2} - \frac{9}{b^2} = 1$$

$$\Rightarrow 9(b^2 - a^2) = a^2 b^2$$

$$\Rightarrow 9(16 - a^2) = a^2 \cdot 16$$

$$\Rightarrow 144 - 9a^2 = 16a^2$$

$$\Rightarrow 25a^2 = 144$$

$$\Rightarrow a^2 = \left(\frac{12}{5}\right)^2 \Rightarrow a = \frac{12}{5}$$

$$\therefore \text{Length of latusrectum} = \frac{2b^2}{a} = 2 \times \frac{16 \times 5}{12} = \frac{40}{3}$$

21. We have, $\frac{4x^2}{36} - \frac{9y^2}{36} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$

$$\text{Also, } y = mx \pm \sqrt{a^2 m^2 - b^2}$$

$$\Rightarrow y = mx \pm \sqrt{9m^2 - 4} \quad \dots(i)$$

which is the equation of tangent.

On comparing with $y = mx + 2$, we get

$$\sqrt{9m^2 - 4} = 2$$

On squaring both sides, we get

$$9m^2 - 4 = 4 \Rightarrow m^2 = \frac{8}{9}$$

$$\therefore m = \pm \frac{2\sqrt{2}}{3}$$

22. Since, $\frac{2b^2}{a} = 12$ and $b = 2\sqrt{3}$

$$\therefore a = \frac{b^2}{6} = 2$$

$$\therefore l = \sqrt{\frac{a^2 + b^2}{a^2}} = \sqrt{\frac{4 + 12}{4}} = 2$$

23. Since, $y = mx + c$ is tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, c^2 = a^2 m^2 - b^2 \quad \dots(i)$$

Again, since $y = mx + c$ is tangent to the hyperbola

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, c^2 = a^2 - b^2 m^2 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$m = \pm 1 \quad \text{and} \quad c = \pm \sqrt{a^2 - b^2}$$

Hence, equation of common tangents

$$y = \pm x \pm \sqrt{a^2 - b^2}$$

24. Any point on the rectangular hyperbola

$$xy = -1 \text{ is } \left(t, -\frac{1}{t}\right).$$

∴ Equation of tangent at $\left(t, -\frac{1}{t}\right)$ is

$$xt - \frac{1}{t}y = -2$$

$$\Rightarrow y = xt^2 + 2t$$

which is also a tangent to the parabola.

$$\therefore 2t = \frac{2}{t^2} \quad \left[\because c = \frac{a}{m} \right]$$

$$\Rightarrow t^3 = 1$$

$$\Rightarrow t = 1$$

$$\therefore y = x + 2$$

$$\Rightarrow x - y + 2 = 0$$

25. Vertices of hyperbola = $(\pm 5, 0) \Rightarrow a = 5$

Also, asymptotes of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is } \frac{x}{a} \pm \frac{y}{b} = 0$$

$$\Rightarrow y = \pm \frac{b}{a}x$$

$$\text{Here, } y = \pm \frac{3}{5}x \Rightarrow b = 3 \text{ and } a = 5$$

∴ Required equation of hyperbola is

$$\frac{x^2}{25} - \frac{y^2}{9} = 1$$

$$\Rightarrow 9x^2 - 25y^2 = 225$$

26. $\because x^2 + y^2 = b^2$

$$\Rightarrow x^2 + y^2 = 25$$

27. Given, equation of hyperbola is

$$\frac{x^2}{9} - \frac{y^2}{7} = 1$$

$$\begin{aligned} \text{Distance between foci} &= 2ae = 2\sqrt{a^2 + b^2} \\ &= 2\sqrt{9 + 7} = 8 \end{aligned}$$

28. Equation of two straight lines are

$$\sqrt{3}x - y = 4\sqrt{3}\alpha \quad \text{and} \quad \sqrt{3}x + y = \frac{4\sqrt{3}}{\alpha}$$

On solving above equations, we get

$$3x^2 - y^2 = 48 \Rightarrow \frac{x^2}{16} - \frac{y^2}{48} = 1$$

which is a hyperbola.

$$\begin{aligned} \text{Whose eccentricity, } e &= \sqrt{\frac{16 + 48}{16}} \quad \left[\because e = \sqrt{\frac{a^2 + b^2}{a^2}} \right] \\ &= \sqrt{4} = 2 \end{aligned}$$

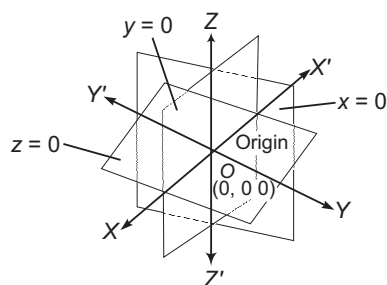
17

Introduction to Three Dimensional [3D] Geometry

Coordinate Axes and Coordinate Planes in Three Dimensional Space

Three mutually perpendicular lines in space define three mutually perpendicular planes which in turn divide the space into eight parts known as octants and the lines are known as the coordinate axes.

Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines intersecting at O . O is the origin and the lines $X'OX$, $Y'OY$ and $Z'OZ$ are called X , Y and Z -axes, respectively. These three lines are also called the rectangular axes of coordinates. The planes containing these three lines in pairs, determine three mutually perpendicular planes XOY , YOZ and ZOX .



The three planes divide space into eight cells called octants.

The following table show the signs of coordinates of points in various octants.

Octant coordinates	Octant	x	y	z
$OXYZ$	I	+	+	+
$OX'YZ$	II	-	+	+
$OX'Y'Z$	III	-	-	+

Chapter Snapshot

- Coordinate Axes and Coordinate Planes in Three Dimensional Space
- Coordinates of a Point in Space
- Distance between Two Points
- Section Formulae
- Centroid of a Triangle

Octant coordinates	Octant	x	y	z
OXY'Z	IV	+	-	+
OXYZ'	V	+	+	-
OX'YZ'	VI	-	+	-
OX'Y'Z'	VII	-	-	-
OXY'Z'	VIII	+	-	-

Coordinates of a Point in Space

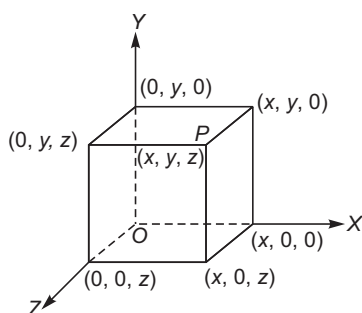
An arbitrary point P in three dimensional space is assigned coordinates (x_0, y_0, z_0) provided that

- (1) The plane through P parallel to the YZ -plane intersects the X -axis at $(x_0, 0, 0)$.
- (2) The plane through P parallel to the XZ -plane intersects the Z -axis at $(0, y_0, 0)$.
- (3) The plane through P parallel to the XY -plane intersects the Z -axis at $(0, 0, z_0)$.

The space coordinates (x_0, y_0, z_0) are called the cartesian coordinates of P or simple the rectangular coordinates of P .

The cartesian coordinates (x, y, z) of a point P in a space are the numbers at which the planes through P are perpendicular to the axes. The coordinates of a point on X -axis are $(x, 0, 0)$, on Y -axis are $(0, y, 0)$ and on Z -axis are $(0, 0, z)$.

The standard equations of XY -plane, YZ -plane and ZX -plane are $z = 0$, $x = 0$ and $y = 0$, respectively.



Consider a point P in space whose position is given by triad (x, y, z) , where x, y, z are perpendicular distances from YZ, ZX and XY -planes, respectively.

If we assume $\hat{i}, \hat{j}, \hat{k}$ unit vectors along OX, OY, OZ respectively, then position vector of point P is $x\hat{i} + y\hat{j} + z\hat{k}$ or simply (x, y, z) .

➤ **Example 1.** A plane is parallel to XZ -plane, so it is perpendicular to

- (a) X -axis
- (b) Y -axis
- (c) Z -axis
- (d) None of the above

Sol. (b) If any plane is parallel to XZ -plane, then it is perpendicular to Y -axis.

➤ **Example 2.** Which of the following points lies in the fifth octant?

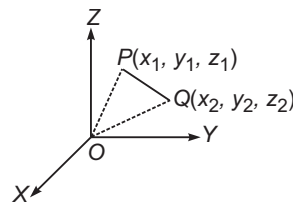
- (a) $(1, 2, 3)$
- (b) $(4, -2, 3)$
- (c) $(4, -2, -5)$
- (d) $(4, 2, -5)$

Sol. (d)

Points	Octant
$(1, 2, 3)$	(i) (all the coordinates are positive)
$(4, -2, 3)$	(iv) (y -coordinate is negative)
$(4, -2, -5)$	(viii) (y and z -coordinates are negative)
$(4, 2, -5)$	(v) (z -coordinate is negative)

Distance between Two Points

The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by



$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- The distance of a point $P(x, y, z)$ from the origin is $\sqrt{x^2 + y^2 + z^2}$.
- The distance of a point $P(x, y, z)$ from X -axis is $\sqrt{y^2 + z^2}$.
- The distance of a point $P(x, y, z)$ from Y -axis is $\sqrt{x^2 + z^2}$.
- The distance of a point $P(x, y, z)$ from Z -axis is $\sqrt{x^2 + y^2}$.
- If $P(x, y, z)$ is a point in a space, then distances from YZ, ZX and XY -planes are respectively x, y and z .
- A parallelepiped is formed by planes drawn through the points (x_1, y_1, z_1) and (x_2, y_2, z_2) parallel to the coordinate planes. The length of edges are $|x_2 - x_1|, |y_2 - y_1|, |z_2 - z_1|$ and length of diagonals is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

➤ **Example 3.** The point on Y -axis, which is at a distance of $\sqrt{10}$ units from the point $(1, 2, 3)$ is

- (a) $(0, 3, 0)$
- (b) $(0, 2, 0)$
- (c) $(0, 1, 0)$
- (d) None of the above

Sol. (b) Since the point P is on Y -axis, therefore $P(0, y, 0)$.

The point $(1, 2, 3)$ is at a distance of $\sqrt{10}$ units from $(0, y, 0)$.

$$\therefore \sqrt{(1-0)^2 + (2-y)^2 + (3-0)^2} = \sqrt{10}$$

$$\Rightarrow 1 + (2-y)^2 + 9 = 10$$

$$\Rightarrow (2-y)^2 = 0$$

$$\Rightarrow y - 2 = 0 \Rightarrow y = 2$$

Hence, the required point is $(0, 2, 0)$.

➤ **Example 4.** The equation of the set of points which are equidistant from the points $(1, 2, 3)$ and $(3, 2, -1)$ is

- (a) $x - 3z = 0$ (b) $x - 2z = 0$
(c) $x - 4z = 0$ (d) $x + 2z = 0$

Sol. (b) Let the given points be A and B. Let $P(x, y, z)$ be any point equidistant from A and B

$$\therefore PA = PB$$

i.e. distance between P and A = distance between P and B

$$\Rightarrow \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-3)^2 + (y-2)^2 + (z+1)^2}$$

$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2 = (x-3)^2 + (y-2)^2 + (z+1)^2$$

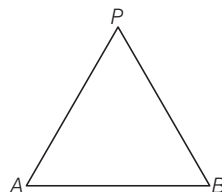
$$\Rightarrow x^2 + 1 - 2x + y^2 + 4 - 4y + z^2 + 9 - 6z$$

$$= x^2 + 9 - 6x + y^2 + 4 - 4y + z^2 + 1 + 2z$$

$$\Rightarrow 4x - 8z = 0$$

$$\Rightarrow x - 2z = 0$$

Which is the required equation.



➤ **Example 5.** If a parallelepiped is formed by planes drawn through the points $(2, 3, 5)$ and $(5, 9, 7)$ parallel to the coordinate plane, then the length of edges and diagonal of parallelepiped are

- (a) $(2, 3, 5), 7$ units
(b) $(3, 6, 2), 7$ units
(c) $(5, 9, 7), 7$ units
(d) None of the above

Sol. (b) Length of edges of the parallelepiped are

$$|x_2 - x_1|, |y_2 - y_1|, |z_2 - z_1|$$

$\therefore$ Length of edges are

$$|5 - 2|, |9 - 3|, |7 - 5|$$

$$\Rightarrow (3, 6, 2)$$

$$\text{Length of diagonal} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$= \sqrt{(3)^2 + (6)^2 + (2)^2}$$

$$= \sqrt{9 + 36 + 4}$$

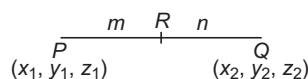
$$= \sqrt{49} = 7 \text{ units}$$

Section Formulae

Section Formula for Internal Division

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points. Let R be a point on the line segment joining P and Q , which divides it internally in the ratio $m:n$. Then, the coordinates of R are

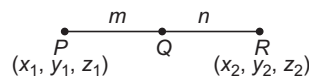
$$\left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right)$$



Section Formula for External Division

If P and Q are such that R divides the join of P and Q externally in the ratio $m:n$. Then, the coordinates of R are

$$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n} \right)$$



• If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points, then the mid-point of AB is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2} \right)$

➤ **Example 6.** The ratio in which the line joining $(1, 2, 3)$ and $(-3, 4, -5)$ is divided by XY -plane, is

- (a) $5:3$ (b) $3:5$
(c) $2:3$ (d) None of these

Sol. (b) Let $A \equiv (1, 2, 3)$ and $B \equiv (-3, 4, -5)$. Let the point P where the line AB meets the XY -plane divides it in the ratio $k:1$.

$\therefore$ The coordinates of P are

$$\left(\frac{-3k+1}{k+1}, \frac{4k+2}{k+1}, \frac{-5k+3}{k+1} \right)$$

Since, the point P lies upon XY -plane.

$\therefore$ z -coordinates of P must be zero.

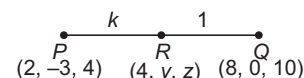
$$\text{i.e. } \frac{-5k+3}{k+1} = 0 \Rightarrow -5k+3=0 \Rightarrow k = \frac{3}{5}$$

Hence, required ratio is $\frac{3}{5}:1$ i.e. $3:5$ internally.

➤ **Example 7.** A point R with x -coordinate 4 lies on the line segment joining the points $P(2, -3, 4)$ and $Q(8, 0, 10)$. Find the coordinates of the point R .

- (a) $(4, -2, -6)$ (b) $(4, 2, 6)$
(c) $(4, -2, 6)$ (d) None of these

Sol. (c) Let the point $R(x, y, z)$ divides PQ in the ratio $k:1$.



$$\Rightarrow x\text{-coordinate of } R = \frac{k \times 8 + 1 \times 2}{k+1} = 4 \quad [\text{given}]$$

$$\Rightarrow \frac{8k+2}{k+1} = 4$$

$$\Rightarrow 8k+2 = 4k+4$$

$$\Rightarrow 8k-4k = 4-2$$

$$\Rightarrow 4k = 2$$

$$\Rightarrow k = \frac{1}{2}$$

$$\Rightarrow k:1 = 1:2$$

Hence, the point R divides PQ internally in the ratio $1:2$.

Therefore, y -coordinate of R

$$= \left(\frac{1 \times 0 + 2 \times (-3)}{1+2} \right) = \frac{-6}{3} = -2$$

$$\text{and } z\text{-coordinate of } R = \left(\frac{1 \times 10 + 2 \times 4}{1 + 2} \right) \\ = \frac{10 + 8}{3} = \frac{18}{3} = 6$$

Hence, coordinates of R are $(4, -2, 6)$.

Centroid of a Triangle

- (i) The coordinates of the centroid of the $\triangle ABC$, whose vertices are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$ are

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$$

- (ii) If (x_1, y_1, z_1) , (x_2, y_2, z_2) , (x_3, y_3, z_3) and (x_4, y_4, z_4) are the vertices of a tetrahedron, then its centroid G is given by

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right)$$

➤ **Example 8.** If the origin is the centroid of $\triangle PQR$ with vertices $P(2a, 2, 6)$, $Q(-4, 3b, -10)$ and $R(8, 14, 2c)$, then the values of a , b and c are, respectively

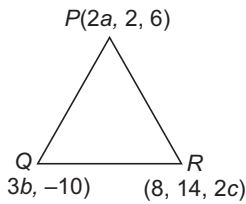
- (a) $-2, -\frac{16}{3}, 3$ (b) $2, \frac{16}{3}, -2$
(c) $-2, -\frac{16}{3}, 2$ (d) None of these

Sol. (c) Centroid of the $\triangle PQR$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right) \\ \text{i.e. } \left(\frac{2a - 4 + 8}{3}, \frac{2 + 3b + 14}{3}, \frac{6 - 10 + 2c}{3} \right)$$

Origin is the centroid i.e. the coordinates of the centroid are $(0, 0, 0)$, then $\frac{2a - 4 + 8}{3} = 0$

$$\Rightarrow 2a + 4 = 0 \\ \Rightarrow a = -2 \\ \text{Also, } \frac{2 + 3b + 14}{3} = 0 \\ \Rightarrow 3b + 16 = 0 \\ \Rightarrow b = -\frac{16}{3} \\ \text{and } \frac{6 - 10 + 2c}{3} = 0 \\ \Rightarrow 2c - 4 = 0 \Rightarrow c = 2 \\ \therefore a = -2, b = -\frac{16}{3}, c = 2$$



➤ **Example 9.** $A(3, 2, 0)$, $B(5, 3, 2)$, $C(-9, 6, -3)$ are three points forming a triangle. If AD the bisector of $\angle BAC$, meets BC in D , then coordinates of D are

- (a) $\left(-\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$ (b) $\left(\frac{19}{8}, -\frac{57}{16}, \frac{17}{16}\right)$
(c) $\left(\frac{19}{8}, \frac{57}{16}, -\frac{17}{16}\right)$ (d) $\left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16}\right)$

Sol. (d) Since, AD is the bisector of $\angle BAC$.

$$\therefore \frac{BD}{DC} = \frac{AB}{AC}$$

$$\text{Now, } AB = \sqrt{(5-3)^2 + (3-2)^2 + (2-0)^2} \\ = \sqrt{4+1+4} \\ = \sqrt{9} = 3$$

$$\text{and } AC = \sqrt{(-9-3)^2 + (6-2)^2 + (-3-0)^2} \\ = 13 \\ \therefore \frac{BD}{DC} = \frac{3}{13}$$

$\Rightarrow D$ divides BC in the ratio $3 : 13$.

$\therefore$ The coordinates of D are

$$\left(\frac{3(-9) + 13(5)}{3+13}, \frac{3(6) + 13(3)}{3+13}, \frac{3(-3) + 13(2)}{3+13} \right) \\ = \left(\frac{19}{8}, \frac{57}{16}, \frac{17}{16} \right)$$

Work Book Exercise

- 1 The distance of the point $P(a, b, c)$ from the X -axis is

- a $\sqrt{b^2 + c^2}$ b $\sqrt{a^2 + c^2}$
c $\sqrt{a^2 + b^2}$ d $\sqrt{a^2 + b^2 + c^2}$

- 2 Ratio in which the xy -plane divides the join of $(1, 2, 3)$ and $(4, 2, 1)$ is

- a $3 : 1$, internally
b $3 : 1$, externally
c $1 : 2$, internally
d $2 : 1$, externally

- 3 The mid-point of the sides of a triangle are $(1, 5, -1)$, $(0, 4, -2)$ and $(2, 3, 4)$. Then, the centroid of the triangle is

- a $(3, 4, 5)$ b $(-1, 6, -7)$
c $(1, 2, 3)$ d $(1, 4, 1/3)$

- 4 Three vertices of a parallelogram $ABCD$ are $A(1, 2, 3)$, $B(-1, -2, -1)$ and $C(2, 3, 2)$. Then, the fourth vertex D is

- a $(3, 5, 5)$ b $(4, 7, 6)$
c $(1, 2, 1)$ d None of these

- 5 If the point $(x, y, \sqrt{1-x^2-y^2})$ is at a unit distance from origin, then $x^2 + y^2$ is equal to

- a 0 b 1 c 2 d 3

- 6 Let $A(2, 2, -3)$, $B(5, 6, 9)$ and $C(2, 7, 9)$ be the vertices of a triangle. The internal bisector of $\angle A$ meets BC at the point D . Then, the coordinates of D are

- a $(7, 13, 9)$ b $\left(\frac{7}{2}, \frac{13}{2}, \frac{9}{2}\right)$
c $\left(\frac{7}{2}, \frac{13}{2}, 9\right)$ d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. L is the foot of the perpendicular drawn from a point $P(3, 4, 5)$ on the XY -plane. The coordinates of point L are

- (a) $(3, 0, 0)$ (b) $(0, 4, 5)$
(c) $(3, 0, 5)$ (d) None of these

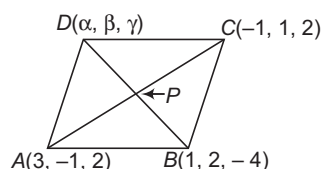
Sol. Since, in XY -plane, the z -coordinate is zero.
Hence, the coordinates of the foot of the point L is $(3, 4, 0)$.
Hence, (d) is the correct answer.

Ex 2. Three vertices of a parallelogram $ABCD$ are $A(3, -1, 2)$, $B(1, 2, -4)$ and $C(-1, 1, 2)$. Find the coordinates of the fourth vertex.

- (a) $(-1, -2, 8)$ (b) $(1, -2, 8)$
(c) $(1, -2, -8)$ (d) $(1, 2, 8)$

Sol. Let $ABCD$ be a parallelogram and coordinates of point D be (α, β, γ) and the diagonals AC and BD intersect at P .
In a parallelogram, diagonals bisect each other.

$$\begin{aligned} \therefore \text{Coordinates of mid-point of } BD &= \text{Coordinates of mid-point of } AC \\ \Rightarrow \left(\frac{\alpha + 1}{2}, \frac{\beta + 2}{2}, \frac{\gamma - 4}{2} \right) &= \left(\frac{3 - 1}{2}, \frac{-1 + 1}{2}, \frac{2 + 2}{2} \right) \\ \Rightarrow \left(\frac{\alpha + 1}{2}, \frac{\beta + 2}{2}, \frac{\gamma - 4}{2} \right) &= \left(\frac{2}{2}, 0, \frac{4}{2} \right) \\ \Rightarrow \left(\frac{\alpha + 1}{2}, \frac{\beta + 2}{2}, \frac{\gamma - 4}{2} \right) &= (1, 0, 2) \end{aligned}$$



On comparing the corresponding coordinates, we get

$$\begin{aligned} \frac{\alpha + 1}{2} = 1, \frac{\beta + 2}{2} = 0, \frac{\gamma - 4}{2} = 2 \\ \Rightarrow \alpha + 1 = 2, \beta + 2 = 0, \gamma - 4 = 4 \\ \Rightarrow \alpha = 1, \beta = -2, \gamma = 8 \end{aligned}$$

$\therefore$ Coordinates of point D are $(1, -2, 8)$.

Hence, (b) is the correct answer.

Ex 3. What is the length of foot of perpendicular drawn from the point $P(3, 4, 5)$ on Y -axis?

- (a) $\sqrt{41}$ units
(b) $\sqrt{34}$ units
(c) 5 units
(d) None of the above

Sol. When we draw perpendicular from the point $P(3, 4, 5)$ on Y -axis, the x and z -coordinates will be zero and y -coordinate will be 4 i.e. coordinate on Y -axis will be $Q(0, 4, 0)$.

$$\begin{aligned} \therefore \text{Distance between } P \text{ and } Q = PQ \\ &= \sqrt{(3 - 0)^2 + (4 - 4)^2 + (5 - 0)^2} \\ &= \sqrt{3^2 + 0^2 + 5^2} \\ &= \sqrt{9 + 25} \\ &= \sqrt{34} \text{ units} \end{aligned}$$

Hence, (b) is the correct answer.

Ex 4. If the origin is the centroid of a ΔABC having vertices $A(a, 1, 3)$, $B(-2, b, -5)$ and $C(4, 7, c)$, then

- (a) $a = -2$ (b) $b = 8$
(c) $c = -2$ (d) None of these

Sol. For centroid of ΔABC ,

$$\begin{aligned} x &= \frac{a - 2 + 4}{3} = \frac{a + 2}{3} \\ y &= \frac{1 + b + 7}{3} = \frac{b + 8}{3} \\ \text{and } z &= \frac{3 - 5 + c}{3} = \frac{c - 2}{3} \end{aligned}$$

But given centroid is $(0, 0, 0)$.

$$\begin{aligned} \therefore \frac{a + 2}{3} = 0 &\Rightarrow a = -2 \\ \frac{b + 8}{3} = 0 &\Rightarrow b = -8 \\ \frac{c - 2}{3} = 0 &\Rightarrow c = 2 \end{aligned}$$

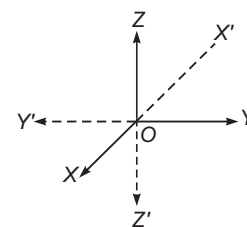
Hence, (a) is the correct answer.

Type 2. More than One Correct Option

Ex 5. Which of the following statements is/are correct?

- (a) Lines $X'OX$, $Y'OY$ and $Z'OZ$ are called rectangular axes
(b) Each plane divide the space in two parts
(c) Three coordinate planes together divide the space into eight regions (parts) called octant
(d) None of the above

Sol. Let $X'OX$, $Y'OY$ and $Z'OZ$ be three mutually perpendicular lines that pass through a point O such that $X'OX$ and $Y'OY$ lies in the plane of the paper and line $Z'OZ$ is perpendicular to the plane of paper. These three



lines are called rectangular axes (lines $X'OX$, $Y'OY$ and $Z'OZ$ are called X , Y and Z -axes). We call this coordinate system a three dimensional space or simply space. The three axes taken together in pairs determine XY , YZ and ZX -planes i.e. three coordinate planes.

Each plane divide the space in two parts and the three coordinate planes together divide the space into eight regions (parts) called octants, namely

(i) $OXYZ$	(ii) $OX'YZ$
(iii) $OXY'Z$	(iv) $OXYZ'$
(v) $OXY'Z'$	(vi) $OX'YZ'$
(vii) $OX'Y'Z$	(viii) $OX'Y'Z'$

Hence, (a), (b) and (c) are the correct answers.

Ex 6. Which of the following statements is/are incorrect?

- The coordinates of the origin O are $(0, 0, 0)$
- The coordinates of any point on the X -axis will be as $(0, y, z)$
- The coordinates of any point in the YZ -plane will be $(x, 0, 0)$
- None of the above

Sol. The coordinates of the origin O are $(0, 0, 0)$. The coordinates of any point on the X -axis will be $(x, 0, 0)$ and the coordinates of any point in the YZ -plane will be $(0, y, z)$.

Hence, (a), (b) and (c) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 7-8) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices line (a), (b), (c) and (d), out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- Statement I is true, Statement II is false
- Statement I is false, Statement II is true

Ex 7. Statement I The coordinates of the point which divides the line segment joining the points $(1, -2, 3)$ and $(3, 4, -5)$ in the ratio $2 : 3$ internally, are $\left(\frac{9}{5}, \frac{2}{5}, -\frac{1}{5}\right)$.

Statement II The coordinates of the point which divides the line segment joining the points $(1, -2, 3)$ and $(3, 4, -5)$ in the ratio $2 : 3$ externally, are $\left(\frac{19}{2}, \frac{5}{2}, \frac{-1}{2}\right)$.

Sol. Statement I Let $P(x, y, z)$ be the point which divides line segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ internally in the ratio $2 : 3$. Then,

$$x = \frac{2(3) + 3(1)}{2 + 3} = \frac{9}{5}$$

$$y = \frac{2(4) + 3(-2)}{2 + 3} = \frac{2}{5}$$

$$z = \frac{2(-5) + 3(3)}{2 + 3} = -\frac{1}{5}$$

Thus, the required point is $\left(\frac{9}{5}, \frac{2}{5}, -\frac{1}{5}\right)$.

Statement II Let $P(x, y, z)$ be the point which divides line segment joining $A(1, -2, 3)$ and $B(3, 4, -5)$ externally in the ratio $2 : 3$.

$$\text{Then, } x = \frac{2(3) - 3(1)}{2 - 3} = -3$$

$$y = \frac{2(4) - 3(-2)}{2 - 3} = -14$$

$$z = \frac{2(-5) - 3(3)}{2 - 3} = 19$$

Therefore, the required point is $(-3, -14, 19)$.
Hence, (c) is the correct answer.

Ex 8. Statement I The points $P(-2, 3, 5)$, $Q(1, 2, 3)$ and $R(7, 0, -1)$ are collinear.

Statement II If $PQ + QR = PR$, then P , Q and R will be collinear.

Sol. We know that points are said to be collinear, if they lie on a line.

$$\text{Now, } PQ = \sqrt{(1+2)^2 + (2-3)^2 + (3-5)^2}$$

$$= \sqrt{9 + 1 + 4}$$

$$= \sqrt{14}$$

$$QR = \sqrt{(7-1)^2 + (0-2)^2 + (-1-3)^2}$$

$$= \sqrt{36 + 4 + 16}$$

$$= \sqrt{56}$$

$$= 2\sqrt{14}$$

$$\text{and } PR = \sqrt{(7+2)^2 + (0-3)^2 + (-1-5)^2}$$

$$= \sqrt{81 + 9 + 36}$$

$$= \sqrt{126} = 3\sqrt{14}$$

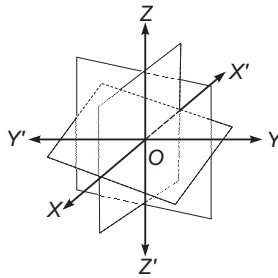
Thus, $PQ + QR = PR$
Hence, P , Q and R are collinear.
Hence, (a) is the correct answer.

Type 4. Linked Comprehension Based Questions

Directions (Ex. Nos.9-11)

Consider the figure given alongside:

In this figure, it is shown that three planes are intersecting at a point O , such that these three planes are mutually perpendicular to each other.



Ex 9. The lines $X'OX$, $Y'OY$ and $Z'OZ$ are constituting the

- rectangular non-perpendicular system
- rectangular coordinate system
- rectangular parallel system
- None of the above

Ex 10. Which of the following statement is correct?

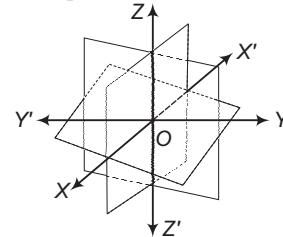
- The plane XOY is called the ZX -plane
- The plane YOZ is called the XY -plane
- The plane ZOX is called the YZ -plane
- The XY , YZ and ZX -planes are called the three coordinate planes

Ex 11. Which of the following statement(s) is/are incorrect?

- If we take the XOY plane as the plane of the paper, then the line $Z'OZ$ is perpendicular to the plane XOY .
- If the plane of the paper is considered as horizontal, then the line $Z'OZ$ will be horizontal.

- Only I is incorrect
- Only II is incorrect
- Both I and II are incorrect
- Both I and II are correct

Sol. (Ex. Nos. 9-11) These problems are based on coordinate axes and coordinate planes in three dimensional space.



9. Considering the three planes intersecting at a point O such that these three planes are mutually perpendicular to each other (see figure). These three planes intersect along the lines $X'OX$, $Y'OY$ and $Z'OZ$, called the X , Y and Z -axes respectively. We see that these lines are mutually perpendicular to each other. These lines constitute the rectangular coordinates system.

Hence, (b) is the correct answer.

10. The planes XOY , YOZ and ZOX called respectively the XY , YZ and ZX -planes, are known as the three coordinate planes.

Hence, (d) is the correct answer.

11. If we take the XOY plane as the plane of the paper, then line $Z'OZ$ is perpendicular to the plane XOY . If the plane of the paper is considered as horizontal, then the line $Z'OZ$ will be vertical.

Hence, (b) is the correct answer.

Type 5. Match the Column

Ex 12. Match each term given under the Column I to its correct answer given under Column II and choose the correct option from the codes given below:

Column I	Column II
A. If the centroid of the triangle is origin and two of its vertices are $(3, -5, 7)$ and $(-1, 7, -6)$, then the third vertex is	p. Parallelogram
B. The points $A(3, -1, -1)$, $B(5, -4, 0)$, $C(2, 3, -2)$ and $D(0, 6, -3)$ are the vertices of a	q. An isosceles right angled triangle
C. Points $A(1, -1, 3)$, $B(2, -4, 5)$ and $C(5, -13, 11)$ are	r. $(-2, -2, -1)$
D. Points $A(2, 4, 3)$, $B(4, 1, 9)$ and $C(10, -1, 6)$ are the vertices of	s. Collinear

Sol. A. Let $A(3, -5, 7)$, $B(-1, 7, -6)$, $C(x, y, z)$ be the vertices of a $\triangle ABC$ with centroid $(0, 0, 0)$.

Therefore,

$$(0, 0, 0) = \left(\frac{3-1+x}{3}, \frac{-5+7+y}{3}, \frac{7-6+z}{3} \right)$$

$$\text{This implies } \frac{x+2}{3} = 0, \frac{y+2}{3} = 0, \frac{z+1}{3} = 0$$

$$\text{Hence, } x = -2, y = -2 \text{ and } z = -1.$$

B. Mid-point of diagonal AC

$$= \left(\frac{3+2}{2}, \frac{-1+3}{2}, \frac{-1-2}{2} \right) = \left(\frac{5}{2}, 1, \frac{-3}{2} \right)$$

Mid-point of diagonal BD

$$= \left(\frac{5+0}{2}, \frac{-4+6}{2}, \frac{0-3}{2} \right) = \left(\frac{5}{2}, 1, \frac{-3}{2} \right)$$

Diagonals of parallelogram bisect each other.

$$\text{C. } |AB| = \sqrt{(2-1)^2 + (-4+1)^2 + (5-3)^2} = \sqrt{14}$$

$$|BC| = \sqrt{(5-2)^2 + (-13+4)^2 + (11-5)^2} = 3\sqrt{14}$$

$$|AC| = \sqrt{(5-1)^2 + (-13+1)^2 + (11-3)^2} = 4\sqrt{14}$$

Now, $|AB| + |BC| = |AC|$. Hence, points A , B and C are collinear.

D. Here,

$$AB = \sqrt{4 + 9 + 36} = 7$$

$$BC = \sqrt{36 + 4 + 9} = 7$$

$$CA = \sqrt{64 + 25 + 9}$$

$$= 7\sqrt{2}$$

Now, $AB^2 + BC^2 = AC^2$. Hence, ABC is an isosceles right angled triangle.

$A \rightarrow r$; $B \rightarrow p$; $C \rightarrow s$; $D \rightarrow q$

Type 6. Single Integer Answer Type Questions

Ex 13. If $x^2 + y^2 = 1$, then the point $P(x, y, \sqrt{1 - x^2 - y^2})$ is at a distance of A unit from the origin. Here, A refers to _____.

Sol. (1) The coordinates of point P are $(x, y, \sqrt{1 - x^2 - y^2})$.

Then,

$$\begin{aligned} OP &= \sqrt{(x - 0)^2 + (y - 0)^2 + (\sqrt{1 - x^2 - y^2} - 0)^2} \\ &= \sqrt{x^2 + y^2 + 1 - x^2 - y^2} = \sqrt{1} = 1 \end{aligned}$$

Ex 14. The perpendicular distance of the point $P(6, 7, 8)$ from XY -plane is _____.

Sol. (8) Let L be the foot of perpendicular drawn from the point $P(6, 7, 8)$ to the XY -plane, then the distance of this foot L from P is z -coordinate of P i.e. 8 units.

Target Exercises

Type 1. Only One Correct Option

- The coordinates of the point where the line segment joining $A(5, 1, 6)$ and $B(3, 4, 1)$ crosses the YZ -plane are
 (a) $\left(0, \frac{17}{2}, \frac{13}{2}\right)$ (b) $\left(0, -\frac{17}{2}, \frac{13}{2}\right)$
 (c) $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$ (d) $\left(0, -\frac{17}{2}, -\frac{13}{2}\right)$
- The equation of the set of point P , the sum of whose distances from $A(4, 0, 0)$ and $B(-4, 0, 0)$ is equal to 10, is
 (a) $3x^2 + 25y^2 + 16z^2 - 225 = 0$
 (b) $9x^2 + 25y^2 + 25z^2 - 225 = 0$
 (c) $5x^2 + 16y^2 + 18z^2 - 225 = 0$
 (d) None of the above
- The points $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ form
 (a) a right angled isosceles triangle
 (b) a scalene triangle
 (c) a right angled triangle
 (d) an equilateral triangle
- Let $A(2, -1, 4)$ and $B(0, 2, -3)$ be two points and C be a point on AB produced such that $2AC = 3AB$, then the coordinates of C are
 (a) $\left(\frac{1}{2}, \frac{5}{4}, -\frac{5}{4}\right)$ (b) $\left(-\frac{1}{2}, \frac{7}{4}, -\frac{13}{4}\right)$
 (c) $(6, -7, 18)$ (d) None of these
- If the coordinates of the vertices of a $\triangle ABC$ are $A(-1, 3, 2)$, $B(2, 3, 5)$ and $C(3, 5, -2)$, then $\angle A$ is equal to
 (a) 45° (b) 60°
 (c) 90° (d) 30°
- The coordinates of the points which trisect the line segments joining the points $P(4, 2, -6)$ and $Q(10, -16, 6)$, are
 (a) $(6, -4, -2)$ and $(8, 10, -2)$
 (b) $(6, -4, -2)$ and $(8, -10, 2)$
 (c) $(-6, 4, 2)$ and $(-8, 10, 2)$
 (d) None of the above
- The mid-points of the sides of a triangle are $(5, 7, 11)$, $(0, 8, 5)$ and $(2, 3, -1)$, then the vertices are
 (a) $(7, 2, 5)$, $(3, 12, 17)$, $(-3, 4, -7)$
 (b) $(7, 2, 5)$, $(3, 12, 17)$, $(3, 4, 7)$
 (c) $(7, 2, 5)$, $(-3, 11, 11)$, $(3, 4, 8)$
 (d) None of the above
- If A and B are the points $(3, 4, 5)$ and $(-1, 3, -7)$ respectively, then the equation of set of point P such that $PA^2 + PB^2 = k^2$, where k is constant, is
 (a) $2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - k^2 = 0$
 (b) $3(x^2 + y^2 + z^2) - 2x - 14y + 4z - 109 + k^2 = 0$
 (c) $2(x^2 - y^2 + z^2) + 2x - 14y + 8z + 109 - k^2 = 0$
 (d) None of the above

Type 2. More than One Correct Option

- Which of the following statements(s) is/are incorrect?
 (a) The plane $x = 1$ represents the plane parallel to XZ -plane at a distance 1 unit above YZ -plane
 (b) The equation of plane $y = 3$ represents the plane parallel to YZ -plane
 (c) The equation of the plane $z = 3$ represents the plane parallel to XY -plane at a distance 3 units above the XY -plane
 (d) All of the above
- Which of the following statements(s) is/are correct?
 (a) $(0, 7, -10)$, $(1, 6, -6)$ and $(4, 9, -6)$ are the vertices of an isosceles triangle
 (b) $(0, 7, 10)$, $(-1, 6, 6)$ and $(-4, 9, 6)$ are the vertices of a right angled triangle
 (c) $(-1, 2, 1)$, $(1, -2, 5)$, $(4, -7, 8)$ and $(2, -3, 4)$ are the vertices of a parallelogram
 (d) None of the above

Type 3. Assertion and Reason

Directions (Q. Nos. 11-12) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- Statement I is true, Statement II is true
- Statement I is false, Statement II is true

- 11. Statement I** If three vertices of a parallelogram are $A(1, 2, 3)$, $B(-1, -2, -1)$ and $C(2, 3, 2)$, then the coordinates of fourth vertex are $(4, 6, 7)$.

Statement II Mid-point of BD is $\left(\frac{x-1}{2}, \frac{y-2}{2}, \frac{z-1}{2}\right)$,

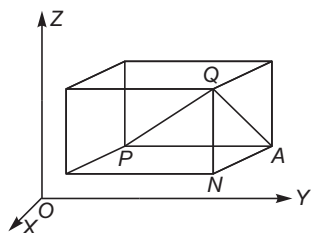
where coordinates of D are (x, y, z) .

- 12. Statement I** The XY -plane divides the line joining the points $(-1, 3, 4)$ and $(2, -5, 6)$ externally in the ratio $2:3$.

Statement II For a point in XY -plane, its z -coordinate should be zero.

Type 4. Linked Comprehension Based Questions

Directions (Q. Nos. 13-16) Consider the figure given below.



In the figure shown above, $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points referring to a system of rectangular axes OX, OY and OZ . Through the points P and Q , planes are drawn parallel to the coordinate planes forming a rectangular parallelepiped with one diagonal PQ .

- 13.** PQ^2 is equal to

- $PA^2 + AN^2$
- $AN^2 + NQ^2$
- $PA^2 + NQ^2$
- None of these

- 14.** PA, AN and NQ are respectively

- $y_2 - y_1, z_2 - z_1, x_2 - x_1$
- $y_2 - y_1, x_2 - x_1, z_2 - z_1$
- $z_2 - z_1, y_2 - y_1, x_2 - x_1$
- None of these

- 15.** The distance PQ between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

- $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- $\sqrt{(x_2 - x_1)^2 - (y_2 - y_1)^2 - (z_2 - z_1)^2}$
- $\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2 + (z_1 + z_2)^2}$
- None of the above

- 16.** The distance OQ between the origin O and any point $Q(x_2, y_2, z_2)$ is

- $\sqrt{x_2^2 + y_2^2 + z_2^2}$
- $x_2^2 + y_2^2 + z_2^2$
- $\sqrt{(x_2 + y_2 + z_2)^2}$
- None of these

Type 5. Match the Column

- 17.** Match the terms of Column I with the terms of Column II and choose the correct option from the codes given below.

Column I (Points)	Column II (Name of the octants)
A. $(1, 2, 3)$	p. I-XOYZ
B. $(4, -2, 3)$	q. II-X'OYZ
C. $(4, -2, -5)$	r. III-X'OY'Z
D. $(4, 2, -5)$	s. IV-XOY'Z
E. $(-4, 2, -5)$	t. V-XOYZ'
F. $(-4, 2, 5)$	u. VI-X'OYZ'
G. $(-3, -1, 6)$	v. VII-X'OY'Z'
H. $(2, -4, -7)$	w. VIII-XOY'Z'

Type 6. Single Integer Answer Type Questions

- The perimeter of triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ is $k\sqrt{2}$, where the value of k is _____.
- The perpendicular distance of the point $(6, 5, 8)$ from Y -axis is _____.
- The distance between the points $(1, 4, 5)$ and $(2, 2, 3)$ is _____.

Entrances Gallery

JEE Main

- The distance of the point $(1, 0, 2)$ from the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 16$ is [2015]
(a) $2\sqrt{14}$ (b) 8 (c) $3\sqrt{21}$ (d) 13
- The equation of the plane containing the line $2x - 5y + z = 3$, $x + y + 4z = 5$ and parallel to the plane $x + 3y + 6z = 1$ is [2015]
(a) $2x + 6y + 12z = 13$ (b) $x + 3y + 6z = -7$
(c) $x + 3y + 6z = 7$ (d) $2x + 6y + 12z = -13$

Other Engineering Entrances

- The distance between the X -axis and the point $(3, 12, 5)$ is [Kerala CEE 2014]
(a) 3 (b) 13 (c) 14
(d) 12 (e) 5
- If the line joining $A(1, 3, 4)$ and B is divided by the point $(-2, 3, 5)$ in the ratio 1 : 3, then B is [EAMCET 2014]
(a) $(-11, 3, 8)$ (b) $(-11, 3, -8)$
(c) $(-8, 12, 20)$ (d) $(13, 6, -13)$
- The ratio in which ZX -plane divides the line segment AB joining the points $A(4, 2, 3)$ and $B(-2, 4, 5)$ is equal to [J&K CET 2011]
(a) 1 : 2, internally (b) 1 : 2, externally
(c) -2 : 1 (d) None of these
- The shortest distance of the point $(1, 2, 3)$ from X , Y and Z -axes, respectively are [MP PET 2011]
(a) 1, 2, 3
(b) $\sqrt{5}, \sqrt{13}, \sqrt{10}$
(c) $\sqrt{10}, \sqrt{13}, \sqrt{5}$
(d) $\sqrt{13}, \sqrt{10}, \sqrt{5}$
- Let $A(1, -1, 2)$ and $B(2, 3, -1)$ be two points. If a point P divides AB internally in the ratio 2 : 3, then the position vector of P is [Kerala CEE 2010]
(a) $\frac{1}{\sqrt{5}}(\hat{i} + \hat{j} + \hat{k})$ (b) $\frac{1}{\sqrt{3}}(\hat{i} + 6\hat{j} + \hat{k})$
(c) $\frac{1}{\sqrt{3}}(\hat{i} + \hat{j} + \hat{k})$ (d) $\frac{1}{5}(7\hat{i} + 3\hat{j} + 4\hat{k})$
(e) None of these

Answers

Work Book Exercise

1. (a) 2. (b) 3. (d) 4. (b) 5. (b) 6. (c)

Target Exercises

1. (c) 2. (b) 3. (a) 4. (d) 5. (c) 6. (b) 7. (a) 8. (a) 9. (a,b) 10. (a,b,c)
11. (d) 12. (a) 13. (d) 14. (b) 15. (a) 16. (a) 17. (*) 18. $(3\sqrt{2})$ 19. (10) 20. (3)

*A $\rightarrow$ p; B $\rightarrow$ s; C $\rightarrow$ w; D $\rightarrow$ t; E $\rightarrow$ u; F $\rightarrow$ q; G $\rightarrow$ r; H $\rightarrow$ w

Entrances Gallery

1. (d) 2. (c) 3. (b) 4. (a) 5. (b) 6. (d) 7. (d)

Explanations

Target Exercises

1. Suppose YZ-plane divides the line segment joining A and B in the ratio $\lambda : 1$. The coordinates of the point C of division are $\left(\frac{3\lambda + 5}{\lambda + 1}, \frac{4\lambda + 1}{\lambda + 1}, \frac{\lambda + 6}{\lambda + 1}\right)$.

This point lies on YZ-plane.

$$\therefore \frac{3\lambda + 5}{\lambda + 1} = 0 \Rightarrow \lambda = -\frac{5}{3}$$

Hence, the coordinates of C are $\left(0, \frac{17}{2}, -\frac{13}{2}\right)$.

2. Let the point be $P(x, y, z)$. Then, it is given $PA + PB = 10$

$$\Rightarrow \sqrt{(x-4)^2 + (y-0)^2 + (z-0)^2} + \sqrt{(x+4)^2 + (y-0)^2 + (z-0)^2} = 10$$

$$\Rightarrow \sqrt{(x-4)^2 + y^2 + z^2} = 10 - \sqrt{(x+4)^2 + y^2 + z^2}$$

Squaring on both sides, we get

$$(x-4)^2 + y^2 + z^2 = 100 + (x+4)^2 + y^2 + z^2 - 20\sqrt{(x+4)^2 + y^2 + z^2}$$

$$\Rightarrow x^2 + 16 - 8x = 100 + x^2 + 16 + 8x$$

$$\Rightarrow -8x - 8x - 100 = -20\sqrt{(x+4)^2 + y^2 + z^2}$$

$$\Rightarrow -16x - 100 = -20\sqrt{(x+4)^2 + y^2 + z^2}$$

$$\Rightarrow 4x + 25 = 5\sqrt{(x+4)^2 + y^2 + z^2}$$

[dividing both sides by -4]

Again, squaring on both sides, we get

$$(4x + 25)^2 = 25[(x+4)^2 + y^2 + z^2]$$

$$\Rightarrow 16x^2 + 625 + 200x = 25[(x+4)^2 + y^2 + z^2]$$

$$\Rightarrow 16x^2 + 625 + 200x = 25[x^2 + 16 + 8x + y^2 + z^2]$$

$$\Rightarrow 16x^2 + 625 + 200x = 25x^2 + 400 + 200x + 25y^2 + 25z^2$$

$$\Rightarrow 9x^2 + 25y^2 + 25z^2 - 225 = 0$$

Which is the required equation.

3. Let $P(0, 7, 10)$, $Q(-1, 6, 6)$ and $R(-4, 9, 6)$ be the given three points.

Here, $PQ = \sqrt{1+1+16} = 3\sqrt{2}$

$$QR = \sqrt{9+9+0} = 3\sqrt{2}$$

$$PR = \sqrt{16+4+16} = 6$$

Now, $PQ^2 + QR^2 = (3\sqrt{2})^2 + (3\sqrt{2})^2 = 18 + 18 = 36 = (PR)^2$

Therefore, ΔPQR is a right angled triangle at Q. Also, $PQ = QR$. Hence, ΔPQR is a right angled isosceles triangle.

4. We have, $2AC = 3AB$

$$\Rightarrow 2AC = 3(AC - BC)$$

$$\Rightarrow AC = 3BC$$

$$\Rightarrow \frac{AC}{BC} = \frac{3}{1}$$

$\Rightarrow$ C divides AB externally in the ratio 3 : 1.

$$\begin{array}{ccc} \bullet & & \bullet \\ A(2, -1, 4) & & B(0, 2, -3) \end{array} \quad C$$

$\therefore$ Coordinates of C are

$$\left(\frac{3 \times 0 - 2 \times 1}{3-1}, \frac{3 \times 2 - 1 \times (-1)}{3-1}, \frac{3 \times (-3) - 1 \times 4}{3-1}\right)$$

or $\left(-1, \frac{7}{2}, -\frac{13}{2}\right)$

5. $\therefore$ Vertices of ΔABC are $A(-1, 3, 2)$, $B(2, 3, 5)$ and $C(3, 5, -2)$.

$$\Rightarrow AB = \sqrt{9+0+9} = \sqrt{18}$$

$$CA = \sqrt{16+4+16} = 6$$

and $BC = \sqrt{1+4+49} = \sqrt{54}$

$$\therefore AB^2 + CA^2 = BC^2$$

ΔABC is right angled triangle at A.

$$\therefore \angle A = 90^\circ$$

6. Let the points R_1 and R_2 trisect the line PQ i.e. R_1 divides the line in the ratio 1 : 2.

$$\begin{array}{ccc} & 1 & 2 \\ P & R_1 & Q \\ (4, 2, -6) & & (10, -16, 6) \end{array}$$

$$\Rightarrow R_1 = \left(\frac{1 \times 10 + 2 \times 4}{1+2}, \frac{1 \times (-16) + 2 \times 2}{1+2}, \frac{1 \times 6 + 2 \times (-6)}{1+2}\right)$$

$$= \left(\frac{10+8}{3}, \frac{-16+4}{3}, \frac{6-12}{3}\right) = \left(\frac{18}{3}, \frac{-12}{3}, \frac{-6}{3}\right)$$

$$= (6, -4, -2)$$

Again, let the point R_2 divides PQ internally in the ratio 2 : 1.

$$\begin{array}{ccc} & 2 & 1 \\ P & R_2 & Q \\ (4, 2, -6) & & (10, -16, 6) \end{array}$$

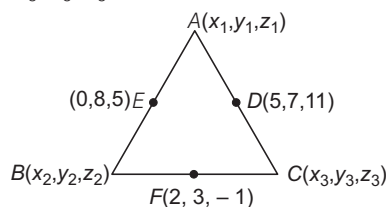
$$\Rightarrow R_2 = \left(\frac{2 \times 10 + 1 \times 4}{2+1}, \frac{2 \times (-16) + 1 \times 2}{2+1}, \frac{2 \times 6 + 1 \times (-6)}{2+1}\right)$$

$$= \left(\frac{20+4}{3}, \frac{-32+2}{3}, \frac{12-6}{3}\right) = \left(\frac{24}{3}, \frac{-30}{3}, \frac{6}{3}\right)$$

$$= (8, -10, 2)$$

Hence, required points are $(6, -4, -2)$ and $(8, -10, 2)$.

7. Let the vertices of a triangle are $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$ and $C(x_3, y_3, z_3)$.



Since, D, E and F are the mid-points of AC, AB and BC.

$$\therefore \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right) = (0, 8, 5)$$

$$\Rightarrow x_1 + x_2 = 0, y_1 + y_2 = 16, z_1 + z_2 = 10 \quad \dots(i)$$

$$\left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}, \frac{z_2 + z_3}{2} \right) = (2, 3, -1)$$

$$\Rightarrow x_2 + x_3 = 4, y_2 + y_3 = 6, z_2 + z_3 = -2 \quad \dots(ii)$$

$$\text{and } \left(\frac{x_1 + x_3}{2}, \frac{y_1 + y_3}{2}, \frac{z_1 + z_3}{2} \right) = (5, 7, 11)$$

$$\Rightarrow x_1 + x_3 = 10, y_1 + y_3 = 14, z_1 + z_3 = 22 \quad \dots(iii)$$

On adding Eqs. (i), (ii) and (iii), we get

$$2(x_1 + x_2 + x_3) = 14, 2(y_1 + y_2 + y_3) = 36$$

$$2(z_1 + z_2 + z_3) = 30$$

$$\Rightarrow x_1 + x_2 + x_3 = 7, y_1 + y_2 + y_3 = 18,$$

$$z_1 + z_2 + z_3 = 15 \quad \dots(iv)$$

On solving Eqs. (i), (ii), (iii) and (iv), we get

$$x_3 = 7, x_1 = 3, x_2 = -3$$

$$y_3 = 2, y_1 = 12, y_2 = 4$$

$$\text{and } z_3 = 5, z_1 = 17, z_2 = -7$$

Hence, vertices of a triangle are $(7, 2, 5)$, $(3, 12, 17)$ and $(-3, 4, -7)$.

8. Let the point P be (x, y, z) .

$$\text{Given, } (PA)^2 + (PB)^2 = k^2$$

$$\Rightarrow (x-3)^2 + (y-4)^2 + (z-5)^2 + (x+1)^2 + (y-3)^2 + (z+7)^2 = k^2$$

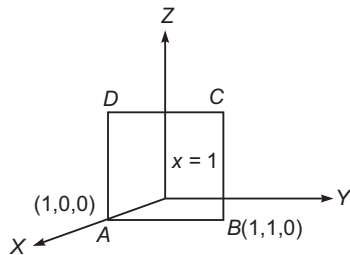
$$\Rightarrow x^2 + 9 - 6x + y^2 + 16 - 8y + z^2 + 25 - 10z + x^2 + 1 + 2x + y^2 + 9 - 6y + z^2 + 49 + 14z = k^2$$

$$\Rightarrow 2x^2 + 2y^2 + 2z^2 - 4x - 14y + 4z + 109 - k^2 = 0$$

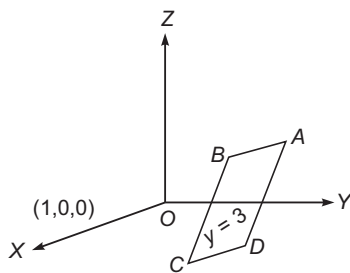
$$\Rightarrow 2(x^2 + y^2 + z^2) - 4x - 14y + 4z + 109 - k^2 = 0$$

which is the required equation.

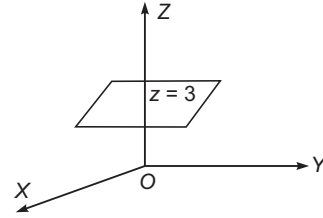
9. (a) The equation of the plane $x=0$ represents the YZ -plane and equation of the plane $x=1$ represents the plane parallel to YZ -plane at a distance 1 unit above YZ -plane. A plane parallel to YZ -plane at a distance 1 unit above YZ -plane is shown below



(b) The equation of the plane $y=0$ represents the XZ -plane and the equation of the plane $y=3$ represents the plane parallel to XZ -plane at a distance 3 units above XZ -plane.



(c) The equation of the plane $z=0$ represents the XY -plane and $z=3$ represents the plane parallel to XY -plane at a distance 3 units above XY -plane.



10. (a) Let $A(0, 7, -10)$, $B(1, 6, -6)$ and $C(4, 9, -6)$ be the vertices of a triangle.

Then, side AB = Distance between points A and B

$$= \sqrt{(0-1)^2 + (7-6)^2 + (-10+6)^2}$$

$$= \sqrt{1+1+16} = \sqrt{18} = 3\sqrt{2} \text{ units}$$

and side BC = Distance between points B and C

$$= \sqrt{(1-4)^2 + (6-9)^2 + (-6+6)^2}$$

$$= \sqrt{9+9+0} = \sqrt{18} = 3\sqrt{2} \text{ units}$$

Clearly, $AB = BC$

Hence, triangle is an isosceles triangle.

(b) Let $A(0, 7, 10)$, $B(-1, 6, 6)$ and $C(-4, 9, 6)$ be the vertices of a triangle.

$$\text{Then, side } AB = \sqrt{(0+1)^2 + (7-6)^2 + (10-6)^2}$$

$$= \sqrt{1+1+16}$$

$$= \sqrt{18} = 3\sqrt{2} \text{ units}$$

$$\text{side } BC = \sqrt{(-1+4)^2 + (6-9)^2 + (6-6)^2}$$

$$= \sqrt{9+9+0}$$

$$= \sqrt{18} = 3\sqrt{2} \text{ units}$$

$$\text{and side } CA = \sqrt{(-4-0)^2 + (9-7)^2 + (6-10)^2}$$

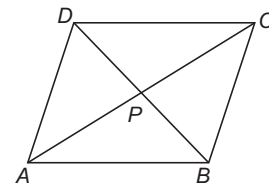
$$= \sqrt{16+4+16}$$

$$= \sqrt{36} = 6 \text{ units}$$

$$\text{Now, } AB^2 + BC^2 = (3\sqrt{2})^2 + (3\sqrt{2})^2 = 6^2 = CA^2$$

$\therefore \triangle ABC$ is a right angled triangle at B .

(c) Let $A(-1, 2, 1)$, $B(1, -2, 5)$, $C(4, -7, 8)$ and $D(2, -3, 4)$ be the vertices of a quadrilateral $ABCD$.



$$\text{Then, mid-point of } AC = \left(\frac{-1+4}{2}, \frac{2-7}{2}, \frac{1+8}{2} \right)$$

$$= \left(\frac{3}{2}, \frac{-5}{2}, \frac{9}{2} \right)$$

$$\text{and mid-point of } BD = \left(\frac{1+2}{2}, \frac{-2-3}{2}, \frac{5+4}{2} \right)$$

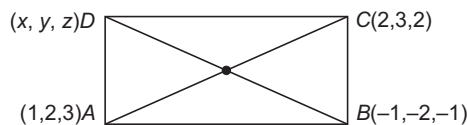
$$= \left(\frac{3}{2}, \frac{-5}{2}, \frac{9}{2} \right)$$

So, mid-points of both the diagonals are same (i.e. they bisect each other).

Hence, $ABCD$ is a parallelogram.

11. We know that, in a parallelogram, diagonals bisect each other.

Now, let the coordinates of D are (x, y, z) .



$\therefore$ Mid-point of BD is $\left(\frac{x-1}{2}, \frac{y-2}{2}, \frac{z-1}{2}\right)$ i.e. $\left(\frac{3}{2}, \frac{5}{2}, \frac{5}{2}\right)$.

$$\therefore \frac{x-1}{2} = \frac{3}{2} \Rightarrow x = 4$$

$$\frac{y-2}{2} = \frac{5}{2} \Rightarrow y = 7$$

$$\text{and } \frac{z-1}{2} = \frac{5}{2} \Rightarrow z = 6$$

The coordinates of D are $(4, 7, 6)$.

12. Suppose XY -plane divides the line joining the given points in the ratio $\lambda : 1$. The coordinates of the points of division are $\left[\frac{2\lambda-1}{\lambda+1}, \frac{-5\lambda+3}{\lambda+1}, \frac{6\lambda+4}{\lambda+1}\right]$. Since, the point

lies on the XY -plane

$$\therefore \frac{6\lambda+4}{\lambda+1} = 0$$

$$\Rightarrow \lambda = \frac{-2}{3}$$

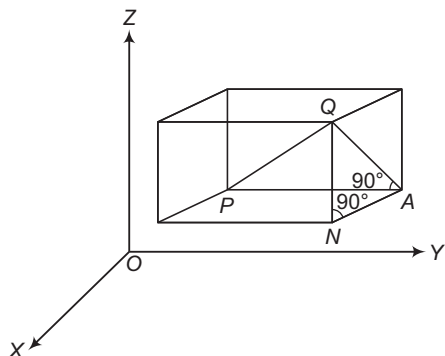
Aliter

The ratio in which XY -plane divides the line segment

$$= -z_1 : z_2 = -\frac{4}{6} = -\frac{2}{3}$$

- The ratio in which XY -plane divides the line segment $= -z_1 : z_2$
- The ratio in which YZ -plane divides the line segment $= -x_1 : x_2$
- The ratio in which ZX -plane divides the line segment $= -y_1 : y_2$

Solutions (Q. Nos. 13-16)



13. Now, since $\angle PAQ$ is a right angle, it follows that in $\triangle PAQ$,

$$PQ^2 = PA^2 + AQ^2 \quad \dots(i)$$

Also, $\triangle ANQ$ is right angled triangle with $\angle ANQ$ as right angle.

$$\text{Therefore, } AQ^2 = AN^2 + NQ^2 \quad \dots(ii)$$

From Eqs. (i) and (ii), we have

$$PQ^2 = PA^2 + AN^2 + NQ^2$$

14. $PA = y_2 - y_1$, $AN = x_2 - x_1$ and $NQ = z_2 - z_1$

$$15. \therefore PQ^2 = PA^2 + AN^2 + NQ^2$$

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

$$\therefore PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This gives us the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) .

16. In particular, if $x_1 = y_1 = z_1 = 0$, i.e. point P is origin O , then $OQ = \sqrt{x_2^2 + y_2^2 + z_2^2}$, which gives the distance between the origin O and any point $Q(x_2, y_2, z_2)$.

17.	Point	Octant	Name
A.	(1, 2, 3)	p. (all the coordinates are +ve)	XOYZ
B.	(4, -2, 3)	s. (y-coordinate is -ve)	XOY'Z
C.	(4, -2, -5)	w. (y and z-coordinates are -ve)	XOY'Z'
D.	(4, 2, -5)	t. (z-coordinate is -ve)	XOYZ'
E.	(-4, 2, -5)	u. (x and z-coordinates are -ve)	X'OYZ'
F.	(-4, 2, 5)	q. (x-coordinate is -ve)	X'OYZ
G.	(-3, -1, 6)	r. (x and y-coordinates are -ve)	X'OY'Z
H.	(2, -4, -7)	w. (y and z-coordinates are -ve)	XOY'Z'

18. Let $A = (1, 0, 0)$, $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

$$\text{Now, } AB = \sqrt{(0-1)^2 + (1-0)^2 + 0^2} = \sqrt{2} \text{ units}$$

$$BC = \sqrt{0^2 + (0-1)^2 + (1-0)^2} = \sqrt{2} \text{ units}$$

$$\text{and } CA = \sqrt{(1-0)^2 + 0^2 + (0-1)^2} = \sqrt{2} \text{ units}$$

$$\therefore \text{Perimeter of triangle} = AB + BC + CA$$

$$= \sqrt{2} + \sqrt{2} + \sqrt{2}$$

$$= 3\sqrt{2} \text{ units}$$

19. Perpendicular distance of the point $(6, 5, 8)$ from Y -axis

$$= \sqrt{6^2 + 8^2} = 10 \text{ units}$$

20. Distance between given points

$$= \sqrt{(2-1)^2 + (2-4)^2 + (3-5)^2}$$

$$= \sqrt{1+4+4} = 3 \text{ units}$$

Entrances Gallery

1. Given equation of line is

$$\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12} = \lambda \quad [\text{say}] \dots (i)$$

and equation of plane is

$$x - y + z = 16 \quad \dots (ii)$$

Any point on the line (i) is

$$(3\lambda + 2, 4\lambda - 1, 12\lambda + 2)$$

Let this point be point of intersection of the line and plane.

$$\therefore (3\lambda + 2) - (4\lambda - 1) + (12\lambda + 2) = 16$$

$$\Rightarrow 11\lambda + 5 = 16$$

$$\Rightarrow 11\lambda = 11$$

$$\Rightarrow \lambda = 1$$

$\therefore$ Point of intersection is (5, 3, 14).

Now, distance between the points (1, 0, 2) and (5, 3, 14)

$$\begin{aligned} &= \sqrt{(5-1)^2 + (3-0)^2 + (14-2)^2} \\ &= \sqrt{16 + 9 + 144} \\ &= \sqrt{169} = 13 \text{ units} \end{aligned}$$

2. Let equation of plane containing the lines $2x - 5y + z = 3$ and $x + y + 4z = 5$ be

$$(2x - 5y + z - 3) + \lambda(x + y + 4z - 5) = 0$$

$$\Rightarrow (2 + \lambda)x + (\lambda - 5)y + (4\lambda + 1)z - 3 - 5\lambda = 0 \quad \dots (i)$$

This plane is parallel to the plane $x + 3y + 6z = 1$.

$$\therefore \frac{2 + \lambda}{1} = \frac{\lambda - 5}{3} = \frac{4\lambda + 1}{6}$$

On taking first two equalities, we get

$$6 + 3\lambda = \lambda - 5$$

$$\Rightarrow 2\lambda = -11$$

$$\Rightarrow \lambda = -\frac{11}{2}$$

On taking last two equalities, we get

$$6\lambda - 30 = 3 + 12\lambda$$

$$\Rightarrow -6\lambda = 33$$

$$\Rightarrow \lambda = -\frac{11}{2}$$

So, the equation of required plane is

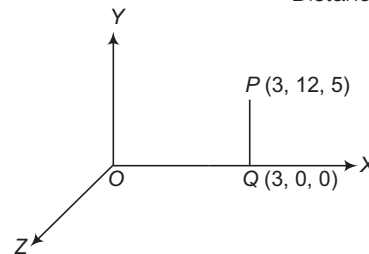
$$\left(2 - \frac{11}{2}\right)x + \left(\frac{-11}{2} - 5\right)y + \left(-\frac{44}{2} + 1\right)z - 3 + 5 \times \frac{11}{2} = 0$$

$$\Rightarrow -\frac{7}{2}x - \frac{21}{2}y - \frac{42}{2}z + \frac{49}{2} = 0$$

$$\Rightarrow x + 3y + 6z - 7 = 0$$

3. Distance between X-axis and point (3, 12, 5)

= Distance between PQ



$$\begin{aligned} &= \sqrt{(3-3)^2 + (12-0)^2 + (5-0)^2} \\ &= \sqrt{0^2 + 12^2 + 5^2} \\ &= \sqrt{144 + 25} = \sqrt{169} = 13 \text{ units} \end{aligned}$$

4. Using internal ratio formula,

$$(-2, 3, 5) = \left(\frac{1 \times x + 3 \times 1}{1+3}, \frac{1 \times y + 3 \times 3}{1+3}, \frac{1 \times z + 3 \times 4}{1+3} \right)$$

$$\Rightarrow (-2, 3, 5) = \left(\frac{x+3}{4}, \frac{y+9}{4}, \frac{z+12}{4} \right)$$

$$\Rightarrow x+3 = -8, y+9 = 12, z+12 = 20$$

$$\Rightarrow x = -11, y = 3, z = 8$$

Hence, required point = B(-11, 3, 8)

5. The ratio in which ZX-plane divides the line segment

$$= -y_1 : y_2 = -(2) : 4 = -2 : 4 = -1 : 2$$

Hence, it divides 1 : 2, externally.

6. Shortest distance of the point (a, b, c) from X, Y and Z-axes respectively are

$$\sqrt{b^2 + c^2}, \sqrt{a^2 + c^2}, \sqrt{a^2 + b^2}$$

$\therefore$ Shortest distance of the point (1, 2, 3) from the X, Y and Z-axes, respectively are $\sqrt{13}, \sqrt{10}, \sqrt{5}$.

7. Let the coordinates of P be (α, β, γ).

$$\alpha = \frac{2 \cdot 2 + 3 \cdot 1}{2+3} = \frac{7}{5}$$

$$\beta = \frac{2 \cdot 3 + 3 \cdot (-1)}{2+3} = \frac{3}{5}$$

$$\begin{array}{ccc} A & & P & & B \\ (1, -1, 2) & & (\alpha, \beta, \gamma) & & (2, 3, -1) \end{array}$$

and

$$\gamma = \frac{2 \cdot (-1) + 3 \cdot 2}{2+3} = \frac{4}{5}$$

$$\therefore \vec{OP} = \frac{1}{5}(7\hat{i} + 3\hat{j} + 4\hat{k})$$

18

Introduction to Limits and Derivatives

Limits

If $f(x)$ is a function of x such that, if x approaches to a constant value a , then the value of $f(x)$ also approaches to another constant k , then constant k is known as limit of $f(x)$ at $x = a$. It is written as

$$\lim_{x \rightarrow a} f(x) = k$$

or

A real number l is called the limit of the function f , if for every $\epsilon > 0$, however small, there exists $\delta > 0$ such that $|f(x) - l| < \epsilon$, whenever $0 < |x - a| < \delta$ and we write it as

$$\lim_{x \rightarrow a} f(x) = l$$

Left Hand Limit (LHL)

A function f is said to approach l as x approaches a from the left, if corresponding to an arbitrary positive number ϵ , there exists a positive number δ such that

$$|f(x) - l| < \epsilon, \text{ whenever } a - \delta < x < a$$

It is written as $\lim_{x \rightarrow a^-} f(x) = l$ or $f(a - 0) = l$

The working rule for finding the left hand limit is put $a - h$ for x in $f(x)$, where h is positive and very small and make h approach zero.

i.e. $f(a - 0) = \lim_{h \rightarrow 0} f(a - h)$

Right Hand Limit (RHL)

A function f is said to approach l as x approaches a from right, if corresponding to an arbitrary positive number ϵ , there exists a positive number δ such that $|f(x) - l| < \epsilon$, whenever $a < x < a + \delta$.

It is written as $\lim_{x \rightarrow a^+} f(x) = l$ or $f(a + 0) = l$

Chapter Snapshot

- Limits
- Existence of Limit
- Algebra of Limits
- Evaluation of Limits by Using L' Hospital's Rule
- Evaluation of Algebraic Limits
- Evaluation of Trigonometric Limits
- Evaluation of Exponential and Logarithmic Limits
- Evaluation of Exponential Limits of the Form 1^∞
- Sandwich Theorem for Evaluating Limits
- Some Useful Expansions
- Use of Newton-Leibnitz's Formula in Evaluating the Limits
- Derivative
- Geometrical Meaning of a Derivative
- Derivative from First Principle
- Differentiation of Some Important Functions
- Algebra of Derivative of Functions
- Chain Rule
- Logarithmic Differentiation

The working rule for finding the right hand limit is put $a + h$ for x in $f(x)$, where h is positive and very small and make h approach zero.

$$\text{i.e. } f(a+0) = \lim_{h \rightarrow 0} f(a+h)$$

➤ **Example 1.** If $f(x) = \begin{cases} x^2 - 1, & 0 < x < 2 \\ 2x + 3, & 2 \leq x < 3 \end{cases}$, then

$\lim_{x \rightarrow 2^-} f(x)$ and $\lim_{x \rightarrow 2^+} f(x)$ are

- (a) 3, 7 (b) 7, 3 (c) -3, 7 (d) 3, -7

$$\text{Sol. (a) } \lim_{x \rightarrow 2^-} f(x) = \lim_{h \rightarrow 0} f(2-h) \\ = \lim_{h \rightarrow 0} [(2-h)^2 - 1] = 4 - 1 = 3$$

$$\text{and } \lim_{x \rightarrow 2^+} f(x) = \lim_{h \rightarrow 0} f(2+h) \\ = \lim_{h \rightarrow 0} [2(2+h) + 3] = 4 + 3 = 7$$

➤ **Example 2.** The left hand and right hand limits of the following function

$$f(x) = \begin{cases} 5x - 4, & 0 < x \leq 1 \\ 4x^3 - 3x, & 1 < x < 2 \end{cases}$$

at $x = 1$, are

- (a) 1, 1 (b) -1, 1 (c) 1, -1 (d) -1, -1

$$\text{Sol. (a) LHL of } f(x) \text{ at } x = 1 \\ = \lim_{x \rightarrow 1^-} f(x) = \lim_{h \rightarrow 0} f(1-h) \\ = \lim_{h \rightarrow 0} \{5(1-h) - 4\} = \lim_{h \rightarrow 0} 1 - 5h = 1$$

$$\text{RHL of } f(x) \text{ at } x = 1 \\ = \lim_{x \rightarrow 1^+} f(x) = \lim_{h \rightarrow 0} f(1+h) \\ = \lim_{h \rightarrow 0} \{4(1+h)^3 - 3(1+h)\} \\ = 4(1)^3 - 3(1) = 1$$

➤ **Example 3.** Left hand limit and right hand limit of the function $\frac{x}{|x| + x^2}$ at $x = 0$, are

- (a) 1, 0 (b) 0, -1 (c) 1, -1 (d) -1, 1

$$\text{Sol. (d) } \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} \frac{0-h}{h+h^2} \\ \text{and } \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \frac{h}{h+h^2} \\ = \lim_{h \rightarrow 0} \frac{1}{1+h} = 1$$

➤ LHL may or may not be equal to RHL.

Existence of Limit

If both right hand limit and left hand limit exist and are equal, then their common value, evidently will be the limit of f as $x \rightarrow a$ i.e.

$$\text{If } \lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^-} f(x) = l$$

$$\text{Then, } \lim_{x \rightarrow a} f(x) = l$$

If however, either both of these limits do not exist or both these limits exist but are not equal in value, then $\lim_{x \rightarrow a} f(x)$ does not exist.

➤ **Example 4.** Check whether $\lim_{x \rightarrow 1} f(x)$ exists or not, where $f(x) = \begin{cases} 2x + 3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$. Also, find the limit.

$$\text{Sol. } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} 3(x+1) = \lim_{h \rightarrow 0} [3(1-h+1)] \\ = 3(1+1) = 6 \\ \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} 2x + 3 = \lim_{h \rightarrow 0} [3(1+h+1)] = 6 \\ \therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = 6$$

Hence, $\lim_{x \rightarrow 1} f(x)$ exists and is equal to 6.

➤ **Example 5.** If $f(x) = \sin\left(\frac{1}{x}\right)$, then at $x = 0$

- (a) only LHL exists
(b) only RHL exists
(c) both LHL and RHL do not exist
(d) LHL and RHL exist and are equal

$$\text{Sol. (c) } \lim_{x \rightarrow 0^-} f(x) = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \sin \frac{1}{-h} \\ = - \lim_{h \rightarrow 0} \sin \frac{1}{h} \\ = - (\text{An oscillating number which oscillates between } -1 \text{ and } 1) \\ \text{So, } \lim_{x \rightarrow 0^-} f(x) \text{ does not exist.} \\ \text{and } \lim_{x \rightarrow 0^+} f(x) = \lim_{h \rightarrow 0} \sin \frac{1}{h} \\ = (\text{An oscillating number which oscillates between } -1 \text{ and } 1) \\ \text{So, } \lim_{x \rightarrow 0^+} f(x) \text{ does not exist.}$$

Algebra of Limits

Let $\lim_{x \rightarrow a} f(x) = l$ and $\lim_{x \rightarrow a} g(x) = m$. If l and m exist, then

- (i) $\lim_{x \rightarrow a} (f \pm g)(x) = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = l \pm m$
(ii) $\lim_{x \rightarrow a} (fg)(x) = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x) = lm$
(iii) $\lim_{x \rightarrow a} \left(\frac{f}{g}\right)(x) = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{l}{m}$, provided $m \neq 0$
(iv) $\lim_{x \rightarrow a} (kf)(x) = \lim_{x \rightarrow a} k f(x) = k \lim_{x \rightarrow a} f(x) = kl$, where k is constant.
(v) $\lim_{x \rightarrow a} |f(x)| = \left| \lim_{x \rightarrow a} f(x) \right| = |l|$

- (vi) $\lim_{x \rightarrow a} [f(x)]^{g(x)} = \lim_{x \rightarrow a} f(x)^{\lim_{x \rightarrow a} g(x)} = l^m$
- (vii) $\lim_{x \rightarrow a} f \circ g(x) = f\left(\lim_{x \rightarrow a} g(x)\right) = f(m)$, only if f is continuous at $g(x) = m$ and $\lim_{x \rightarrow a} g(x) = m$.
In particular,
- (a) $\lim_{x \rightarrow a} \log f(x) = \log\left(\lim_{x \rightarrow a} f(x)\right) = \log l$
- (b) $\lim_{x \rightarrow a} e^{f(x)} = e^{\lim_{x \rightarrow a} f(x)} = e^l$
- (viii) If $\lim_{x \rightarrow a} f(x) = +\infty$ or $-\infty$, then $\lim_{x \rightarrow a} \frac{1}{f(x)} = 0$
- (ix) If $f(x) \leq g(x)$ for every x in the neighbourhood of a , then $\lim_{x \rightarrow a} f(x) \leq \lim_{x \rightarrow a} g(x)$.

Points to be Remembered

- i. If $\lim_{x \rightarrow c} f(x) g(x)$ exists, then we can have the following cases:
- (a) Both $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ exist.
- (b) $\lim_{x \rightarrow c} f(x)$ exists and $\lim_{x \rightarrow c} g(x)$ does not exist.
- (c) Both $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ do not exist.
- ii. If $\lim_{x \rightarrow c} [f(x) + g(x)]$ exists, then we can have the following cases:
- (a) If $\lim_{x \rightarrow c} f(x)$ exists, then $\lim_{x \rightarrow c} g(x)$ must exist.
- (b) Both $\lim_{x \rightarrow c} f(x)$ and $\lim_{x \rightarrow c} g(x)$ do not exist.

Evaluation of Limits by Using L' Hospital's Rule

If $f(x)$ and $g(x)$ are two functions of x such that

- (i) $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$
- (ii) both are continuous at $x = a$
- (iii) both are differentiable at $x = a$
- (iv) $f'(x)$ and $g'(x)$ are continuous at the point $x = a$, then $\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$, provided that $g'(a) \neq 0$.

✎ The above rule is also applicable, if $\lim_{x \rightarrow a} f(x) = \infty$ and $\lim_{x \rightarrow a} g(x) = \infty$.

Generalisation

If $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$ assumes the indeterminate form $0/0$ and $f'(x)$, $g'(x)$ satisfy all the conditions embodied in L'Hospital's rule, then we can repeat the application of this rule on $\frac{f'(x)}{g'(x)}$ to get

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f''(x)}{g''(x)}$$

Sometimes, it may be necessary to repeat this process a number of times till our goal of evaluating limit is achieved.

➤ **Example 6.** $\lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$ is equal to

- (a) -1 (b) 0
(c) 1 (d) 2

Sol. (d) $\lim_{x \rightarrow 0} \frac{x \cos x + \sin x}{x^2 + \tan x}$ [$\frac{0}{0}$ form]

$$= \lim_{x \rightarrow 0} \frac{-x \sin x + \cos x + \cos x}{2x + \sec^2 x}$$

[using L' Hospital's rule]

$$= \frac{0 + 1 + 1}{0 + 1} = 2$$

➤ **Example 7.** $\lim_{x \rightarrow 0} \frac{\tan^{-1} x - \sin^{-1} x}{x^3}$ is equal to

- (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
(c) 1 (d) -1

Sol. (b) $\lim_{x \rightarrow 0} \frac{\tan^{-1} x - \sin^{-1} x}{x^3}$ [$\frac{0}{0}$ form]

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{1+x^2} - \frac{1}{\sqrt{1-x^2}}}{3x^2} \Rightarrow \lim_{x \rightarrow 0} \frac{\sqrt{1-x^2} - (1+x^2)}{3x^2(1+x^2)\sqrt{1-x^2}}$$

$$= \lim_{x \rightarrow 0} \frac{(1-x^2) - (1+x^2)^2}{3x^2(1+x^2)\sqrt{1-x^2}[\sqrt{1-x^2} + (1+x^2)]}$$

$$= \lim_{x \rightarrow 0} \frac{-3-x^2}{3x^2(1+x^2)\sqrt{1-x^2}[\sqrt{1-x^2} + (1+x^2)]}$$

$$= -\frac{3}{6} = -\frac{1}{2}$$

Evaluation of Algebraic Limits

Let $f(x)$ be an algebraic function (polynomial or rational) and a be any real number, then $\lim_{x \rightarrow a} f(x)$ is known as an algebraic limit.

The limit of algebraic functions can be found by the following methods:

Direct Substitution Method

If by direct substitution of the point in the given expression we get a finite number, then the number obtained is the limit of the given expression.

e.g. $\lim_{x \rightarrow 1} \frac{x-1}{(x+2)^2} = \frac{1-1}{(1+2)^2} = 0$

➤ **Example 8.** The value of $\lim_{x \rightarrow 0} \left(\frac{x+3}{x+1} \right)^x$ is

- (a) 0 (b) 3
(c) 1 (d) None of these

Sol. (c) $\lim_{x \rightarrow 0} \left(\frac{x+3}{x+1} \right)^x = \left(\frac{0+3}{0+1} \right)^0 = 3^0 = 1$

Factorisation Method

Factorisation method is used when we have rational function i.e. function of the form $\frac{f(x)}{g(x)}$.

Consider, $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$. If by putting $x = a$, the rational function $\frac{f(x)}{g(x)}$ takes the form $\frac{0}{0}, \frac{\infty}{\infty}$ etc., then $(x-a)$ is a factor of both $f(x)$ and $g(x)$. In such a case we factorise the numerator and denominator and then cancel out the common factor $(x-a)$. After cancelling out the common factor $(x-a)$, we again put $x = a$ in the given expression and see whether, we get a meaningful number or not. This process is repeated till we get a meaningful number.

➤ **Example 9.** Evaluate $\lim_{x \rightarrow 1} \left(\frac{2}{1-x^2} + \frac{1}{x-1} \right)$.

- (a) 1 (b) 2
(c) 1/2 (d) 3/2

Sol. (c) $\lim_{x \rightarrow 1} \left(\frac{2}{1-x^2} + \frac{1}{x-1} \right) \left[\frac{0}{0} \text{ form} \right]$
 $= \lim_{x \rightarrow 1} \frac{2 - (1+x)}{1-x^2} = \lim_{x \rightarrow 1} \frac{1-x}{1-x^2}$
 $= \lim_{x \rightarrow 1} \frac{1}{1+x} = \frac{1}{2}$

➤ **Example 10.** The value of $\lim_{x \rightarrow \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$ is

- (a) $-\frac{8}{5}$
(b) $\frac{7}{5}$
(c) $\frac{8}{5}$
(d) None of the above

Sol. (c) $\lim_{x \rightarrow \sqrt{2}} \frac{x^4 - 4}{x^2 + 3\sqrt{2}x - 8}$
 $= \lim_{x \rightarrow \sqrt{2}} \frac{(x + \sqrt{2})(x - \sqrt{2})(x^2 + 2)}{(x - \sqrt{2})(x + 4\sqrt{2})} \left[\frac{0}{0} \text{ form} \right]$
 $= \lim_{x \rightarrow \sqrt{2}} \frac{(x + \sqrt{2})(x^2 + 2)}{(x + 4\sqrt{2})}$
 $= \frac{(\sqrt{2} + \sqrt{2})(2 + 2)}{(\sqrt{2} + 4\sqrt{2})} = \frac{8\sqrt{2}}{5\sqrt{2}} = \frac{8}{5}$

Rationalisation Method

Rationalisation method is used when we have radical signs (like $\frac{1}{2}, \frac{1}{3}$, etc.,) in numerator or denominator or in both of an rational expression.

After rationalisation, the terms are factorised which on cancellation gives the required result.

➤ **Example 11.** Evaluate $\lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$.

- (a) 1 (b) 3 (c) 2 (d) 4

Sol. (b) $\lim_{x \rightarrow 1} \frac{x^2 - \sqrt{x}}{\sqrt{x} - 1}$
 $= \lim_{x \rightarrow 1} \left\{ \frac{x^2 - \sqrt{x}}{(\sqrt{x} - 1)} \times \frac{(\sqrt{x} + 1)}{(\sqrt{x} + 1)} \times \frac{(x^2 + \sqrt{x})}{(x^2 + \sqrt{x})} \right\}$
 $= \lim_{x \rightarrow 1} \frac{(x^4 - x)(\sqrt{x} + 1)}{(x - 1)(x^2 + \sqrt{x})} = \lim_{x \rightarrow 1} \frac{x(x^3 - 1)(\sqrt{x} + 1)}{(x - 1)(x^2 + \sqrt{x})}$
 $= \lim_{x \rightarrow 1} \frac{x(x - 1)(x^2 + x + 1)(\sqrt{x} + 1)}{(x - 1)(x^2 + \sqrt{x})}$
 $= \lim_{x \rightarrow 1} \frac{x(x^2 + x + 1)(\sqrt{x} + 1)}{(x^2 + \sqrt{x})} = \frac{1 \times (1^2 + 1 + 1)(\sqrt{1} + 1)}{(1^2 + \sqrt{1})} = 3$

➤ **Example 12.** $\lim_{x \rightarrow 0} \frac{x \sqrt[3]{z^2 - (z-x)^2}}{(\sqrt[3]{8xz} - 4x^2 + \sqrt[3]{8xz})^4}$ is

equal to

- (a) $\frac{z}{2^{11/3}}$
(b) $\frac{1}{2^{23/3} \cdot z}$
(c) $2^{21/3} \cdot z$
(d) None of the above

Sol. (b) $\lim_{x \rightarrow 0} \frac{x \sqrt[3]{z^2 - (z-x)^2}}{(\sqrt[3]{8xz} - 4x^2 + \sqrt[3]{8xz})^4}$
 $= \lim_{x \rightarrow 0} \frac{x \sqrt[3]{2xz - x^2}}{(\sqrt[3]{8xz} - 4x^2 + \sqrt[3]{8xz})^4}$
 $= \lim_{x \rightarrow 0} \frac{x^{4/3} \sqrt[3]{2z - x}}{x^{4/3} (\sqrt[3]{8z} - 4x + \sqrt[3]{8z})^4}$
 $= \frac{\sqrt[3]{2z}}{[2\sqrt[3]{8z}]^4} = \frac{1}{2^{23/3} \cdot z}$

Evaluation of Limits Using Standard Results

$$\begin{aligned}\lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= \lim_{x \rightarrow a} (x^{n-1} + x^{n-2}a \\ &\quad + x^{n-3}a + \dots + a^{n-1}) \\ &\quad \text{[using expansion]} \\ \Rightarrow \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= a^{n-1} + a^{n-1} + a^{n-1} \\ &\quad + \dots + a^{n-1} \\ \therefore \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} &= na^{n-1}\end{aligned}$$

➤ **Example 13.** Evaluate $\lim_{x \rightarrow 0} \frac{(1-x)^n - 1}{x}$.

(a) n (b) $-n$ (c) 0 (d) 1

Sol. (b) We have, $\lim_{x \rightarrow 0} \frac{(1-x)^n - 1}{x}$

$$\begin{aligned}&= - \lim_{x \rightarrow 0} \frac{(1-x)^n - 1}{(1-x) - 1} \\ &= - \lim_{y \rightarrow 1} \frac{y^n - 1^n}{y - 1}, \text{ where } y = 1 - x \\ \text{As } x \rightarrow 0, y &\rightarrow 1 \\ &= -n(1)^{n-1} = -n\end{aligned}$$

Method of Evaluating Algebraic Limits When $x \rightarrow \infty$

To evaluate this type of limits, we follow the following procedure:

Step I Write down the given expression in form of a rational function i.e. $\frac{f(x)}{g(x)}$, if it is not so.

Step II If k is the highest power of x in numerator and denominator both, then divide each term in numerator and denominator by x^k .

Step III Use the result $\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$, where $n > 0$.

➤ **Example 14.** Evaluate $\lim_{x \rightarrow \infty} \left(\frac{x^3}{3x^2 - 4} - \frac{x^2}{3x + 2} \right)$.

(a) $\frac{1}{3}$ (b) $\frac{2}{9}$ (c) $\frac{3}{4}$ (d) $\frac{2}{3}$

Sol. (b) $\lim_{x \rightarrow \infty} \left(\frac{x^3}{3x^2 - 4} - \frac{x^2}{3x + 2} \right)$

$$\begin{aligned}&= \lim_{x \rightarrow \infty} \frac{x^3(3x + 2) - x^2(3x^2 - 4)}{(3x^2 - 4)(3x + 2)} \\ &= \lim_{x \rightarrow \infty} \frac{2x^3 + 4x^2}{9x^3 + 6x^2 - 12x - 8} \\ &= \lim_{x \rightarrow \infty} \frac{2 + \frac{4}{x}}{9 + \frac{6}{x} - \frac{12}{x^2} - \frac{8}{x^3}} = \frac{2}{9}\end{aligned}$$

➤ **Example 15.** The values of constants a and b so that $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = \frac{1}{2}$, are

(a) $a = 1, b = -\frac{3}{2}$ (b) $a = -1, b = \frac{3}{2}$

(c) $a = 0, b = 0$ (d) $a = 2, b = -1$

Sol. (a) We have, $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1}{x + 1} - ax - b \right) = \frac{1}{2}$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(x^2 + 1) - (ax + b)(x + 1)}{x + 1} = \frac{1}{2}$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^2(1-a) - (a+b)x - b + 1}{x + 1} = \frac{1}{2}$$

$$\Rightarrow 1 - a = 0 \quad \text{and} \quad a + b = -\frac{1}{2}$$

$$\therefore a = 1 \quad \text{and} \quad b = -\frac{3}{2}$$

Important Result

If m and n are positive integers and a_0, b_0 are non-zero real numbers, then

$$\begin{aligned}\lim_{x \rightarrow \infty} \frac{a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n} \\ = \begin{cases} \frac{a_0}{b_0}, & \text{if } m = n \\ 0, & \text{if } m < n \\ \infty, & \text{if } m > n, a_0b_0 > 0 \\ -\infty, & \text{if } m > n, a_0b_0 < 0 \end{cases}\end{aligned}$$

Evaluation of Trigonometric Limits

In order to evaluate the said type of limits, we use the following standard results:

(i) $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

(ii) $\lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$

(iii) $\lim_{\theta \rightarrow a} \frac{\sin \theta}{\theta} = \frac{\pi}{180}$

(iv) $\lim_{\theta \rightarrow 0} \cos \theta = 1$

(v) $\lim_{\theta \rightarrow a} \frac{\sin(\theta - a)}{\theta - a} = 1$

(vi) $\lim_{\theta \rightarrow a} \frac{\tan(\theta - a)}{\theta - a} = 1$

(vii) $\lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1$

(viii) $\lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1$

➤ **Example 16.** The value of $\lim_{x \rightarrow 0} \frac{\sin^2 2x}{\sin^2 4x}$ is

(a) $\frac{1}{2}$

(b) $-\frac{1}{4}$

(c) $\frac{1}{4}$

(d) $-\frac{1}{2}$

Sol. (c) $\lim_{x \rightarrow 0} \frac{\sin^2 2x}{\sin^2 4x}$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 2x}{(2x)^2} \times \frac{(4x)^2}{\sin^2 4x} \times \frac{1}{4}$$

$$= \frac{1}{4} \lim_{x \rightarrow 0} \left(\frac{\sin 2x}{2x} \right)^2 \times \lim_{x \rightarrow 0} \left(\frac{4x}{\sin 4x} \right)^2$$

$$= \frac{1}{4} \times 1 \times 1 \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= \frac{1}{4}$$

➤ **Example 17.** $\lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$ is equal to

- (a) 2 (b) -2
(c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Sol. (c) $\lim_{x \rightarrow 0} \frac{x \tan 2x - 2x \tan x}{(1 - \cos 2x)^2}$

$$= \lim_{x \rightarrow 0} \frac{x \frac{2 \tan x}{1 - \tan^2 x} - 2x \tan x}{(2 \sin^2 x)^2}$$

$$= \lim_{x \rightarrow 0} \frac{2x \tan x \left[\frac{1}{1 - \tan^2 x} - 1 \right]}{4 \sin^4 x}$$

$$= \lim_{x \rightarrow 0} \frac{2x \tan x \left[\frac{1 - 1 + \tan^2 x}{1 - \tan^2 x} \right]}{4 \sin^4 x}$$

$$= \lim_{x \rightarrow 0} \frac{x \tan^3 x}{2 \sin^4 x (1 - \tan^2 x)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{2} \left[\frac{x \left(\frac{\tan x}{x} \right)^3 \cdot x^3}{\sin^4 x (1 - \tan^2 x)} \right]$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{\left(\frac{\tan x}{x} \right)^3}{\left(\frac{\sin x}{x} \right)^4 (1 - \tan^2 x)}$$

$$= \frac{1(1)^3}{2(1)^4(1 - 0)} = \frac{1}{2}$$

Evaluation of Exponential and Logarithmic Limits

To evaluate the exponential and logarithmic limits, we use the following results:

- (i) $\lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log_e a, a > 0$
 (ii) $\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1$
 (iii) $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

➤ **Example 18.** The value of $\lim_{x \rightarrow 0} \frac{3^{2x} - 1}{2^{3x} - 1}$ is

- (a) $\frac{\log 9}{\log 8}$ (b) $\frac{\log 2}{\log 3}$ (c) $\frac{\log 3}{\log 2}$ (d) $\frac{\log 8}{\log 9}$

Sol. (a) $\lim_{x \rightarrow 0} \frac{3^{2x} - 1}{2^{3x} - 1} = \lim_{x \rightarrow 0} \frac{\frac{3^{2x} - 1}{2x} \times 2x}{\frac{2^{3x} - 1}{3x} \times 3x}$

$$= \frac{2}{3} \times \frac{\lim_{x \rightarrow 0} \left(\frac{3^{2x} - 1}{2x} \right)}{\lim_{x \rightarrow 0} \left(\frac{2^{3x} - 1}{3x} \right)}$$

$$= \frac{2}{3} \left(\frac{\log 3}{\log 2} \right) = \frac{\log 3^2}{\log 2^3} = \frac{\log 9}{\log 8}$$

➤ **Example 19.** The value of $\lim_{x \rightarrow 1} \sec \left(\frac{\pi}{2^x} \right) \log x$ is

- (a) $\frac{2}{\log 2}$ (b) $\pi \log 2$
(c) $\frac{2}{\pi \log 2}$ (d) None of these

Sol. (c) $\lim_{x \rightarrow 1} \sec \left(\frac{\pi}{2^x} \right) \log x = \lim_{x \rightarrow 1} \frac{\log x}{\cos \left(\frac{\pi}{2^x} \right)}$

$$= \lim_{x \rightarrow 1} \frac{\log[1 + (x - 1)]}{\sin \left(\frac{\pi}{2} - \frac{\pi}{2^x} \right)}$$

$$= \lim_{x \rightarrow 1} \frac{\log[1 + (x - 1)] \cdot (x - 1)}{\sin \left(\frac{\pi}{2} - \frac{\pi}{2^x} \right) \cdot \left(\frac{\pi}{2} - \frac{\pi}{2^x} \right)}$$

$$= \lim_{x \rightarrow 1} \frac{(x - 1)}{\left(\frac{\pi}{2} - \frac{\pi}{2^x} \right)} = \lim_{x \rightarrow 1} \frac{(x - 1)}{\pi \left(\frac{2^{x-1} - 1}{2^x} \right)}$$

$$= \frac{2}{\pi} \lim_{x \rightarrow 1} \frac{(x - 1)}{2^{x-1} - 1} = \frac{2}{\pi \log 2}$$

Evaluation of Exponential Limits of the Form 1^∞

To evaluate the exponential limits of the form 1^∞ , we use the following results:

- (i) If $\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x) = 0$, then
- $$\lim_{x \rightarrow a} [1 + f(x)]^{1/g(x)} = e^{\lim_{x \rightarrow a} \frac{f(x)}{g(x)}}$$
- (ii) If $\lim_{x \rightarrow a} f(x) = 1$ and $\lim_{x \rightarrow a} g(x) = \infty$, then
- $$\lim_{x \rightarrow a} [f(x)]^{g(x)} = \lim_{x \rightarrow a} [1 + (f(x) - 1)]^{g(x)}$$
- $$= e^{\lim_{x \rightarrow a} (f(x) - 1)g(x)}$$

In particular cases

$$(i) \lim_{x \rightarrow 0} (1+x)^{1/x} = e \quad (ii) \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x}\right)^x = e$$

$$(iii) \lim_{x \rightarrow 0} (1+\lambda x)^{1/x} = e^\lambda \quad (iv) \lim_{x \rightarrow \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda$$

➤ **Example 20.** Evaluate the following limits

$$(i) \lim_{x \rightarrow 1} (\log_3 3x)^{\log_3 x} \quad (ii) \lim_{x \rightarrow a} \left(2 - \frac{a}{x}\right)^{\tan(\pi x/2a)}$$

Sol. (i) $\lim_{x \rightarrow 1} (\log_3 3x)^{\log_3 x}$

$$\begin{aligned} &= \lim_{x \rightarrow 1} (\log_3 3 + \log_3 x)^{\log_3 x} \\ &= \lim_{x \rightarrow 1} (1 + \log_3 x)^{\frac{1}{\log_3 x}} = e^{\lim_{x \rightarrow 1} \log_3 x \times \frac{1}{\log_3 x}} \\ &= e^1 = e \end{aligned}$$

$$\begin{aligned} (ii) \lim_{x \rightarrow a} \left(2 - \frac{a}{x}\right)^{\tan(\pi x/2a)} &= \lim_{x \rightarrow a} \left\{1 + \left(1 - \frac{a}{x}\right)\right\}^{\tan(\pi x/2a)} \\ &= e^{\lim_{x \rightarrow a} \left(1 - \frac{a}{x}\right) \tan \frac{\pi x}{2a}} = e^{\lim_{x \rightarrow a} \left(\frac{x-a}{x}\right) \tan \frac{\pi x}{2a}} \\ &= e^l, \text{ where } l = \lim_{x \rightarrow a} \left(\frac{x-a}{x}\right) \tan \frac{\pi x}{2a} \end{aligned}$$

$$\begin{aligned} \text{Now, } l &= \lim_{x \rightarrow a} \left(\frac{x-a}{x}\right) \tan \frac{\pi x}{2a} \\ &= \lim_{x \rightarrow a} \left(\frac{x-a}{x}\right) \cot \left(\frac{\pi}{2} - \frac{\pi x}{2a}\right) \\ &= \lim_{x \rightarrow a} \left(\frac{x-a}{x}\right) \cot \frac{\pi}{2a} (a-x) \\ &= - \lim_{x \rightarrow a} \frac{(a-x)}{\tan \frac{\pi}{2a} (a-x)} \times \frac{1}{x} \\ &= - \lim_{x \rightarrow a} \frac{\frac{\pi}{2a} (a-x)}{\tan \frac{\pi}{2a} (a-x)} \times \frac{2a}{\pi x} = - \frac{2}{\pi} \end{aligned}$$

➤ **Example 21.** $\lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1}\right)^{x+4}$ is equal to

- (a) $\frac{1}{e^5}$
(b) e^5
(c) $5e^5$
(d) None of the above

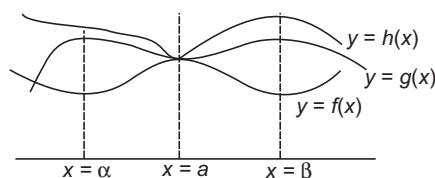
$$\begin{aligned} \text{Sol. (b)} \lim_{x \rightarrow \infty} \left(\frac{x+6}{x+1}\right)^{x+4} &= \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{6}{x}}{1 + \frac{1}{x}}\right)^{x+4} \\ &= \lim_{x \rightarrow \infty} \left(\frac{1 + \frac{6}{x}}{1 + \frac{1}{x}}\right)^x \left(\frac{1 + \frac{6}{x}}{1 + \frac{1}{x}}\right)^4 \\ &= \frac{e^6}{e} \left(\frac{1}{1}\right) \left[\because \lim_{x \rightarrow \infty} \left(1 + \frac{\lambda}{x}\right)^x = e^\lambda\right] \\ &= e^5 \end{aligned}$$

➤ **Example 22.** $\lim_{x \rightarrow 0} (1+x)^{\operatorname{cosec} x}$ is equal to

- (a) 1
(b) e^{-1}
(c) e
(d) None of these

$$\begin{aligned} \text{Sol. (c)} \lim_{x \rightarrow 0} (1+x)^{\operatorname{cosec} x} &= \lim_{x \rightarrow 0} \left[(1+x)^{\frac{1}{x}}\right]^{\frac{x}{\sin x}} \\ &= \lim_{x \rightarrow 0} \left[(1+x)^{\frac{1}{x}}\right]^{\frac{1}{\sin x}} = \left[\lim_{x \rightarrow 0} (1+x)^{\frac{1}{x}}\right]^{\lim_{x \rightarrow 0} \frac{x}{\sin x}} = e^1 = e \end{aligned}$$

Sandwich Theorem for Evaluating Limits



If $f(x) \leq g(x) \leq h(x)$, $\forall x \in (\alpha, \beta) - \{a\}$ and
 $\lim_{x \rightarrow a} f(x) = L = \lim_{x \rightarrow a} h(x)$, then $\lim_{x \rightarrow a} g(x) = L$, where $a \in (\alpha, \beta)$

Remarks In the Sandwich theorem, we assume that $f(x) \leq g(x) \leq h(x)$ for all x near a , 'except possibly at a '. This means that it is not required that when $x = a$, we have the inequality for the functions i.e. it is not required that $f(a) \leq h(a)$. The reason is that we are dealing with limits as x approaches a . So, we have x that is moving closer and closer to a .

As long as $f(x) \leq g(x) \leq h(x)$ is true for all these x , we can be sure the limit, i.e. the point where the function values are heading must behave as the Sandwich theorem indicates. In particular, unless we are given extra information about the functions and their values at a , the Sandwich theorem does not allow us to make conclusions about functions values at a . So, none of the following claims can be guaranteed by the assumptions in the Sandwich theorem.

1. $f(a) = g(a) = h(a)$
[well, not even $f(a) \leq h(a)$]
2. $g(a) = \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L$
3. $\lim_{x \rightarrow a} g(x) = g(a)$

➤ **Example 23.** Evaluate $\lim_{x \rightarrow 1} f(x)$,

where, $2x^2 \leq f(x) \leq x(x^2 + 1)$, $\forall x$ that are near to 1 but not equal to 1.

- (a) 2 (b) 3 (c) 4 (d) 1

$$\text{Sol. (a)} \lim_{x \rightarrow 1} 2x^2 = 2(1)^2 = 2$$

$$\text{and } \lim_{x \rightarrow 1} x(x^2 + 1) = 1(1^2 + 1) = 2$$

It follows from the Sandwich theorem that $\lim_{x \rightarrow 1} f(x) = 2$

Some Useful Expansions

- i. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$
- ii. $a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \frac{x^3}{3!} (\log a)^3 + \dots, a > 0$
- iii. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots, -1 < x \leq 1$
- iv. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \dots, -1 \leq x < 1$
- v. $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \dots, -1 < x < 1$
- vi. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$
- vii. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$
- viii. $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$
- ix. $\sin^{-1} x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{24} \frac{x^5}{5} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{x^7}{7} + \dots$
- x. $\tan^{-1} x = x - \frac{1}{3} x^3 + \frac{1}{5} x^5 - \dots$

➤ **Example 24.** $\lim_{x \rightarrow 0} \frac{\sin x + \log(1-x)}{x^2}$ is equal to

- (a) 0
- (b) $1/2$
- (c) $-\frac{1}{2}$
- (d) None of these

Sol. (c) $\lim_{x \rightarrow 0} \frac{\sin x + \log(1-x)}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) + \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \dots\right)}{x^2}$$

[applying expansions of $\sin x$ and $\log(1-x)$]

$$= \lim_{x \rightarrow 0} \frac{-\frac{x^2}{2} - x^3 \left(\frac{1}{3!} + \frac{1}{3}\right) - \frac{x^4}{4} - \dots}{x^2} = -\frac{1}{2}$$

➤ **Example 25.** $\lim_{x \rightarrow 0} \frac{1+x+x^2-e^x}{x^2}$ is equal to

- (a) 1
- (b) 0
- (c) $\frac{1}{2}$
- (d) None of these

Sol. (c) We have, $\lim_{x \rightarrow 0} \frac{1+x+x^2-e^x}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{(1+x+x^2) - \left(1+x+\frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\left(1 - \frac{1}{2!}\right)x^2 - \frac{x^3}{3!} - \frac{x^4}{4!} - \dots}{x^2}$$

$$= \lim_{x \rightarrow 0} \left(1 - \frac{1}{2!}\right) - \frac{x}{3!} - \frac{x^2}{4!} - \dots \Rightarrow \left(1 - \frac{1}{2}\right) = \frac{1}{2}$$

Use of Newton-Leibnitz's Formula in Evaluating the Limits

Let us consider the definite integral

$$I(x) = \int_{\phi(x)}^{\Psi(x)} f(t) dt$$

Newton-Leibnitz's formula states that $\frac{d}{dx} \{I(x)\}$

$$= f\{\Psi(x)\} \left\{ \frac{d}{dx} \Psi(x) \right\} - f\{\phi(x)\} \left\{ \frac{d}{dx} \phi(x) \right\}$$

➤ **Example 26.** If $f(x) = \int_1^x \frac{\sin t}{t} dt$, then $\lim_{x \rightarrow \infty} f'(x)$

is equal to

- (a) 0
- (b) 1
- (c) 2
- (d) None of these

Sol. (a) Given,

$$f(x) = \int_1^x \frac{\sin t}{t} dt$$

$\therefore$

$$f'(x) = \frac{\sin x}{x}$$

$\Rightarrow$

$$\lim_{x \rightarrow \infty} f'(x) = \lim_{x \rightarrow \infty} \frac{\sin x}{x} = 0$$

➤ **Example 27.** The value of $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \cos t^2 dt}{x \sin x}$ is

- (a) $\frac{3}{2}$
- (b) 1
- (c) -1
- (d) None of these

Sol. (b) $\lim_{x \rightarrow 0} \frac{\int_0^{x^2} \cos t^2 dt}{x \sin x}$ [$\frac{0}{0}$ form]

$$= \lim_{x \rightarrow 0} \frac{2x \cos x^4}{x \cos x + \sin x}$$

[using L' Hospital's rule]

$$= \lim_{x \rightarrow 0} \frac{2 \cos x^4 - 8x^4 \sin x^4}{2 \cos x - x \sin x}$$

[using L' Hospital's rule]

$$= \frac{2-0}{2-0} = 1$$

➤ **Example 28.** The value of $\lim_{x \rightarrow 0} \frac{\int_0^x \cos t dt}{x}$ is

- (a) 1
- (b) -1
- (c) ∞
- (d) None of these

Sol. (d) $\lim_{x \rightarrow 0} \frac{\int_0^x \cos t dt}{x}$ [$\frac{0}{0}$ form]

$$= \lim_{x \rightarrow 0} \frac{\cos x}{1}$$

[using L' Hospital's rule]

$$= \cos 0^\circ = 1$$

Work Book Exercise 18.1

18

- If $[]$ denotes the greatest integer function $n \in \mathbb{N}$, then $\lim_{x \rightarrow 0} \left[\frac{n \sin x}{x} \right]$ is equal to
 a n b 0 c $n-1$ d $n+1$
- The integer n for which $\lim_{x \rightarrow 0} \frac{(\cos x - 1)(\cos x - e^x)}{x^n}$ is a finite non-zero number, is
 a 1 b 2
 c 3 d 4
- $\lim_{x \rightarrow 0} \frac{e^{1/x} - 1}{e^{1/x} + 1}$ is equal to
 a 0 b 1
 c -1 d Does not exist
- If $\lim_{x \rightarrow \infty} \{ \sqrt{x^4 + ax^3 + 3x^2 + bx + 2} - \sqrt{x^4 + 2x^3 - cx^2 + 3x - d} \}$ is finite, then the value of a is
 a 3
 b 5
 c 2
 d any real number
- $\lim_{x \rightarrow \infty} \left\{ \left(\frac{x}{x+1} \right)^a + \sin \frac{1}{x} \right\}$ is equal to
 a e^{a-1} b e^{1-a} c e d 0
- If $\{x\}$ denotes the fractional part of x , then $\lim_{x \rightarrow 0} \frac{\{x\}}{\tan\{x\}}$ is equal to
 a 1 b 0
 c -1 d None of these
- If $f(x) = \frac{1}{3} \left\{ f(x+1) + \frac{5}{f(x+2)} \right\}$ and $f(x) > 0, \forall x \in \mathbb{R}$, then $\lim_{x \rightarrow \infty} f(x)$ is
 a $\sqrt{\frac{2}{5}}$ b $\sqrt{\frac{5}{2}}$
 c ∞ d None of these
- $\lim_{x \rightarrow \infty} \frac{(x+1)^{10} + (x+2)^{10} + \dots + (x+100)^{10}}{x^{10} + 10^{10}}$ is equal to
 a 0 b 1
 c 10 d 100
- If $\phi(x) = \lim_{x \rightarrow \infty} \frac{x^{2n}f(x) + g(x)}{1 + x^{2n}}$, then
 a $\phi(x) = g(x), \forall x \in \mathbb{R}$
 b $\phi(x) = f(x), \forall x \in \mathbb{R}$
 c $\phi(x) = \begin{cases} g(x), & \text{if } -1 < x < 1 \\ f(x), & \text{if } |x| \geq 1 \end{cases}$
 d $\phi(x) = \begin{cases} g(x), & \text{if } |x| < 1 \\ f(x), & \text{if } |x| > 1 \\ \frac{f(x) + g(x)}{2}, & \text{if } |x| = 1 \end{cases}$
- The value of $\lim_{n \rightarrow \infty} \frac{1 \cdot \sum_{r=1}^n r + 2 \cdot \sum_{r=1}^{n-1} r + 3 \cdot \sum_{r=1}^{n-2} r + \dots + n \cdot 1}{n^4}$ is
 a $\frac{1}{24}$
 b $\frac{1}{12}$
 c $\frac{1}{6}$
 d None of the above
- $\lim_{x \rightarrow 0^+} \frac{[x] + [x^2] + [x^3] + \dots + [x^{2n+1}] + n + 1}{1 + [x^2] + [x] + 2x}$, $n \in \mathbb{N}$ is equal to
 a $n+1$ b n
 c 1 d 0
- The value of $\lim_{n \rightarrow \infty} |\log_{(n-1)} n \cdot \log_n(n+1) \cdot \log_{(n+1)}(n+2) \dots \log_{(n^k-1)}(n^k)|$ is
 a ∞ b n
 c k d None of these
- $\lim_{x \rightarrow \infty} \left\{ \frac{1^2}{1-x^3} + \frac{3}{1+x^2} + \frac{5^2}{1-x^3} + \frac{7}{1+x^2} + \dots \right\}$ is equal to
 a $-\frac{5}{6}$
 b $-\frac{10}{3}$
 c $\frac{5}{6}$
 d None of the above
- If $\lim_{x \rightarrow 0} \frac{\{(a-n)nx - \tan x\} \sin nx}{x^2} = 0$, where n is a non-zero real number, then a is equal to
 a 0 b $\frac{n+1}{n}$
 c n d $n + \frac{1}{n}$
- The value of $\lim_{n \rightarrow 0} \cos\left(\frac{x}{2}\right) \cos\left(\frac{x}{4}\right) \cos\left(\frac{x}{8}\right) \dots \cos\left(\frac{x}{2^n}\right)$ is
 a 1
 b $\frac{\sin x}{x}$
 c $\frac{x}{\sin x}$
 d None of the above
- $\lim_{x \rightarrow \frac{\pi}{4}} \frac{\int_2^{\sec^2 x} f(t) dt}{x^2 - \frac{\pi^2}{16}}$ is equal to
 a $\frac{8}{\pi} f(2)$ b $\frac{2}{\pi} f(2)$
 c $\frac{2}{\pi} f\left(\frac{1}{2}\right)$ d $4f(2)$

Derivative

Suppose f is a real valued function, the function defined by $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$, wherever the limit exists, is defined to be the derivative of f at x and is denoted by $f'(x)$. This definition of derivative is also called the first principle of derivative.

$$\text{Hence, } f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

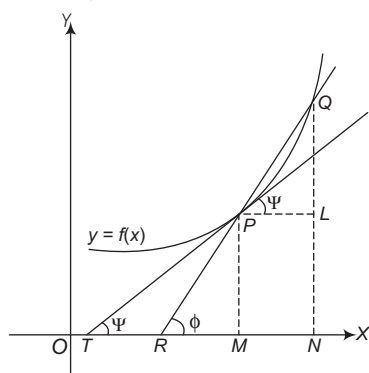
Clearly, the domain of definition of $f'(x)$ is wherever the above limit exists. Sometimes, $f'(x)$ is denoted by $\frac{d}{dx}[f(x)]$ or if $y = f(x)$, it is denoted by $\frac{dy}{dx}$.

This is referred to as derivative of $f(x)$ or y with respect to x . It is also denoted by $D[f(x)]$. Further, the derivative of f at $x = a$ is also denoted by

$$\left(\frac{df}{dx}\right)_a \quad \text{or} \quad \left(\frac{df}{dx}\right)_{x=a}$$

Geometrical Meaning of a Derivative

Let $P\{x_0, f(x_0)\}$ and $Q\{x_0 + h, f(x_0 + h)\}$ be two points very near to each other on the curve $y = f(x)$. Draw PM and QN perpendiculars from P and Q on X -axis, PL perpendicular from P on QN . Let the chord QP produced meet the X -axis at R and $\angle QPL = \angle QRN = \phi$.



In right angled ΔQPL ,

$$\begin{aligned} \tan \phi &= \frac{QL}{PL} = \frac{NQ - NL}{MN} \\ &= \frac{NQ - MP}{ON - OM} \\ &= \frac{f(x_0 + h) - f(x_0)}{(x_0 + h) - x_0} \\ &= \frac{f(x_0 + h) - f(x_0)}{h} \quad \dots(i) \end{aligned}$$

When $h \rightarrow 0$, the point Q moving along the curve tends to P i.e. $Q \rightarrow P$, then the chord PQ approaches the tangent line PT at the point P and then $\phi \rightarrow \Psi$.

Now, applying $\lim$ in Eq. (i), we get

$$\begin{aligned} \lim_{h \rightarrow 0} \tan \phi &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \\ \tan \Psi &= \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h} \end{aligned}$$

Thus, the limit turns out to be equal to the slope of the tangent.

$$\text{Hence, } f'(x_0) = \tan \Psi$$

Derivative from First Principle

Let $f(x)$ be a function finitely differentiable at every point on the real number line. Then, its derivative which is given by the first principle, is

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Example 29. The value of $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$,

where $f(x) = \frac{x+1}{x-1}$, is

- $\frac{-2}{(x-1)^2}$
- $\frac{-1}{(x-1)^2}$
- $\frac{1}{(1-x)^2}$
- $\frac{3}{(1-x)^2}$

Sol. (a) We have, $f(x) = \frac{x+1}{x-1}$

$$\begin{aligned} \text{Now, } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h} \\ &= \lim_{h \rightarrow 0} \frac{\frac{(x+h+1)(x-1) - (x+1)(x+h-1)}{(x+h-1)(x-1)}}{h} \\ &= \lim_{h \rightarrow 0} \frac{(x^2 - x + hx - h + x - 1) - (x^2 + hx - x + x + h - 1)}{(x+h-1)(x-1)h} \\ &= \lim_{h \rightarrow 0} \frac{x^2 + hx - h - 1 - x^2 - hx + h + 1}{(x+h-1)(x-1)h} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{(x+h-1)(x-1)h} = \frac{-2}{(x+0-1)(x-1)} \\ &= \frac{-2}{(x-1)^2} \end{aligned}$$

➤ **Example 30.** The value of $\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$,

where $f(x) = \sin(x+1)$, is

- (a) $\sin(x+1)$
 (b) $\cos(x+1)$
 (c) $-\sin(x+1)$
 (d) $-\cos(x+1)$

Sol. (b) We have, $f(x) = \sin(x+1)$

$$\begin{aligned} \text{Now, } \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} &= \lim_{h \rightarrow 0} \frac{\sin(x+h+1) - \sin(x+1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{x+h+1+x+1}{2}\right) \sin\left(\frac{x+h+1-x-1}{2}\right)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2 \cos\left(\frac{2x+h+2}{2}\right) \sin \frac{h}{2}}{2 \times \frac{h}{2}} \\ &= \cos\left(\frac{2x+0+2}{2}\right) \times 1 = \cos(x+1) \end{aligned}$$

Differentiation of Some Important Functions

- (i) $\frac{d}{dx}(c) = 0$, c is independent of x
 (ii) $\frac{d}{dx}(x^n) = nx^{n-1}$
 (iii) $\frac{d}{dx}(\sin x) = \cos x$
 (iv) $\frac{d}{dx}(\cos x) = -\sin x$
 (v) $\frac{d}{dx}(\tan x) = \sec^2 x$, $\left\{x \neq n\pi + \frac{\pi}{2}\right\}$
 (vi) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$, $\{x \neq n\pi\}$
 (vii) $\frac{d}{dx}(\sec x) = \sec x \tan x$, $\left\{x \neq n\pi + \frac{\pi}{2}\right\}$
 (viii) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$, $\{x \neq n\pi\}$
 (ix) $\frac{d}{dx}(a^x) = a^x \log_e a$
 (x) $\frac{d}{dx}(e^x) = e^x$
 (xi) $\frac{d}{dx}(\log x) = \frac{1}{x}$, $(x > 0)$
 (xii) $\frac{d}{dx}(\log_a x) = \frac{1}{x \log_e a}$

Algebra of Derivative of Functions

Let f and g be two functions such that their derivatives are defined in a common domain. Then,

- (i) Derivative of sum of two functions is sum of the derivatives of the functions.

$$\text{i.e. } \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

- (ii) Derivative of difference of two functions is the difference of the derivatives of two functions.

$$\text{i.e. } \frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}f(x) - \frac{d}{dx}g(x)$$

- (iii) Derivative of product of two functions is given by the following product rule.

$$\frac{d}{dx}[f(x) \cdot g(x)] = g(x) \frac{d}{dx}f(x) + f(x) \frac{d}{dx}g(x)$$

- (iv) Derivative of scalar multiplication of a function is the scalar multiple of derivative of the function.

$$\text{i.e. } \frac{d}{dx}[kf(x)] = k \frac{d}{dx}[f(x)]$$

where, k is any constant.

- (v) Derivative of quotient of two functions is given by the following quotient rule whenever the denominator is non-zero.

$$\frac{d}{dx} \left[\frac{f(x)}{g(x)} \right] = \frac{g(x) \cdot \frac{d}{dx}f(x) - f(x) \cdot \frac{d}{dx}g(x)}{[g(x)]^2}$$

➤ **Example 31.** The derivative of the function $2x^2 + 6x + 1$ at $x = 3$ is

- (a) 18 (b) 17
 (c) 16 (d) 12

Sol. (a) We have, $f(x) = 2x^2 + 6x + 1$

On differentiating $f(x)$ w.r.t. x , we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[2x^2 + 6x + 1] \\ &= \frac{d}{dx}(2x^2) + \frac{d}{dx}(6x) + \frac{d}{dx}(1) \\ &= 2 \frac{d}{dx}(x^2) + 6 \frac{d}{dx}(x) + 0 \\ &= 2 \cdot 2x + 6 \cdot 1 \\ &= 4x + 6 \\ \therefore f'(x) &= 4x + 6 \\ \text{At } x = 3, \quad f'(3) &= 4 \cdot 3 + 6 = 18 \end{aligned}$$

➤ **Example 32.** For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1,$$

$f'(1) = mf'(0)$, where m is equal to

- (a) 50 (b) 0
(c) 100 (d) 200

Sol. (c) Given, $f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$

$$\Rightarrow f'(x) = \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0 \dots (i)$$

On putting $x = 1$, we get

$$f'(1) = \underbrace{(1)^{99} + (1)^{98} + \dots + 1 + 1}_{100 \text{ times}} \\ = \underbrace{1 + 1 + 1 + \dots + 1 + 1}_{100 \text{ times}}$$

$$\Rightarrow f'(1) = 100 \dots (ii)$$

Again, $f'(0) = 0 + 0 + \dots + 0 + 1$

$$\Rightarrow f'(0) = 1 \dots (iii)$$

From Eqs. (ii) and (iii), we get

$$f'(1) = 100 f'(0)$$

Hence, $m = 100$

➤ **Example 33.** The derivative of $(x^2 + 1) \cos x$ is given by

- (a) $-x^2 \sin x - \sin x + 2x \cos x$
(b) $x^2 \sin x - \sin x + 2x \cos x$
(c) $x^2 \sin x + \sin x + 2x \cos x$
(d) $x^2 \sin x - \sin x - 2x \cos x$

Sol. (a) Let $y = (x^2 + 1) \cos x$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = (x^2 + 1) \frac{d}{dx} (\cos x) + (\cos x) \frac{d}{dx} (x^2 + 1) \\ = (x^2 + 1) (-\sin x) + \cos x (2x) \quad [\text{by product rule}] \\ = -x^2 \sin x - \sin x + 2x \cos x$$

➤ **Example 34.** The derivative of $\frac{1 + \frac{1}{x}}{1 - \frac{1}{x}}$ is

- (a) $\frac{2}{(1+x)^2}$ (b) $\frac{-2}{(1-x)^2}$ (c) $\frac{-1}{(1-x)^2}$ (d) $\frac{3}{(1-x)^2}$

Sol. (b) Let $y = \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \Rightarrow y = \frac{x+1}{x-1}$

On differentiating w.r.t. x , we get

$$\frac{dy}{dx} = \frac{(x-1) \frac{d}{dx} (x+1) - (x+1) \frac{d}{dx} (x-1)}{(x-1)^2} \\ = \frac{(x-1)(1+0) - (x+1)(1-0)}{(x-1)^2} \\ = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(1-x)^2}$$

Chain Rule

If $y = [u \{v(x)\}]$, then $\frac{dy}{dx} = \frac{du \{v(x)\}}{d \{v(x)\}} \times \frac{d}{dx} v(x)$ is

known as chain rule.

If $y = u [v \{w(x)\}]$, then

$$\frac{dy}{dx} = \frac{du [v \{w(x)\}]}{dv \{w(x)\}} \times \frac{dv \{w(x)\}}{dw(x)} \times \frac{dw(x)}{dx}$$

➤ **Example 35.** If $y = \log \sqrt{\tan x}$, then the value of

$\frac{dy}{dx}$ at $x = \frac{\pi}{4}$, is given by

- (a) ∞ (b) 1 (c) 0 (d) $\frac{1}{2}$

Sol. (b) We have,

$$\frac{dy}{dx} = \frac{1}{\sqrt{\tan x}} \cdot \frac{1}{2\sqrt{\tan x}} \cdot \sec^2 x = \frac{\sec^2 x}{2 \tan x}$$

$$\text{At } x = \frac{\pi}{4}, \frac{dy}{dx} = \frac{\sec^2 \pi/4}{2 \tan \pi/4} = \frac{2}{2} = 1$$

➤ **Example 36.** If $y = f\left(\frac{2x-1}{x^2+1}\right)$ and

$f'(x) = \sin x^2$, then $\frac{dy}{dx}$ is equal to

(a) $\sin \left(\frac{2x-1}{x^2+1} \right)^2 \left(\frac{2+2x+x^2}{(x^2+1)^2} \right)$

(b) $\sin \left(\frac{2x-1}{x^2+1} \right)^2 \left(\frac{2+2x-2x^2}{(x^2+1)^2} \right)$

(c) $\sin \left(\frac{2x-1}{x^2+1} \right)^2 \left(\frac{2+2x-x^2}{(x^2+1)^2} \right)$

(d) None of the above

Sol. (b) We have, $y = f\left(\frac{2x-1}{x^2+1}\right)$

$$\Rightarrow \frac{dy}{dx} = f' \left(\frac{2x-1}{x^2+1} \right) \left[\frac{(x^2+1)2 - (2x-1)2x}{(x^2+1)^2} \right]$$

$$= \sin \left(\frac{2x-1}{x^2+1} \right)^2 \left[\frac{2+2x-2x^2}{(x^2+1)^2} \right]$$

$$\left[\because f'(x) = \sin x^2 \Rightarrow f' \left(\frac{2x-1}{x^2+1} \right) = \sin \left(\frac{2x-1}{x^2+1} \right)^2 \right]$$

Logarithmic Differentiation

So far, we have discussed derivatives of the functions of the form $[f(x)]^n$, $n^{f(x)}$ and n^x , where $f(x)$ is a function of x and n is a constant. In this section, we will be mainly discussing derivatives of the functions of the form $[f(x)]^{g(x)}$, where $f(x)$ and $g(x)$ are

functions of x . To find the derivatives of this type of functions, we proceed as follows:

Let $y = [f(x)]^{g(x)}$

Taking logarithm on both sides, we have

$$\log y = g(x) \log \{f(x)\}$$

On differentiating w.r.t. x , we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = g(x) \cdot \frac{1}{f(x)} \cdot \frac{df(x)}{dx} + \log \{f(x)\} \cdot \frac{dg(x)}{dx}$$

$$\therefore \frac{dy}{dx} = y \left[\frac{g(x)}{f(x)} \cdot \frac{df(x)}{dx} + \log \{f(x)\} \cdot \frac{dg(x)}{dx} \right]$$

$$\Rightarrow \frac{dy}{dx} = [f(x)]^{g(x)} \left[\frac{g(x)}{f(x)} \cdot \frac{df(x)}{dx} + \log \{f(x)\} \cdot \frac{dg(x)}{dx} \right]$$

Logarithmic Differentiation (Objective Approach)

If $y = \{f(x)\}^{\phi(x)}$, then

$$\frac{dy}{dx} = \text{Derivative of } \{f(x)\}^{\phi(x)} \text{ w.r.t. } x \text{ taking } \phi(x) \text{ as}$$

a constant + Derivative of $\{f(x)\}^{\phi(x)}$ w.r.t. x taking

$f(x)$ as a constant

$$\Rightarrow \frac{dy}{dx} = \phi(x) \cdot \{f(x)\}^{\phi(x)-1} \cdot \frac{df(x)}{dx} + \{f(x)\}^{\phi(x)} \cdot \log f(x) \cdot \frac{d\phi(x)}{dx}$$

or when we have to differentiate the function of the form (Variable)^{variable}, take log on both sides and differentiate.

➔ **Example 37.** If $f(x) = \left(\frac{a+x}{b+x}\right)^{a+b+2x}$, then

$f'(0)$ is equal to

(a) $\left(2 \log \frac{a}{b} + \frac{a^2 - b^2}{ab}\right) \left(\frac{a}{b}\right)^{a+b}$

(b) $\left(2 \log \frac{a}{b} + \frac{b^2 - a^2}{ab}\right) \left(\frac{a}{b}\right)^{a+b}$

(c) $\left(2 \log \frac{a}{b} + \frac{a^2 + b^2}{ab}\right) \left(\frac{a}{b}\right)^{a+b}$

(d) None of the above

Sol. (b) We have,

$$\log f(x) = (a+b+2x) [\log(a+x) - \log(b+x)]$$

On differentiating both sides w.r.t. x , we get

$$\frac{f'(x)}{f(x)} = 2 [\log(a+x) - \log(b+x)]$$

$$+ (a+b+2x) \left(\frac{1}{a+x} - \frac{1}{b+x} \right)$$

$$\Rightarrow f'(0) = f(0) \left[2 (\log a - \log b) + (a+b) \left(\frac{1}{a} - \frac{1}{b} \right) \right]$$

$$= \left(\frac{a}{b}\right)^{a+b} \left(2 \log \frac{a}{b} + \frac{b^2 - a^2}{ab} \right)$$

Work Book Exercise 18.2

1 The derivative of $(x^2 \sin x + \cos 2x)$ is

a $x^2 \cos x + 2x \sin x - \sin 2x$

b $x^2 \cos x + 2x \sin x - 2 \sin 2x$

c $x^2 \cos x + x \sin x - 2 \sin 2x$

d $x^2 \cos x + 2x \sin x + 2 \sin 2x$

2 If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$, then $\frac{dy}{dx}$ at $x = 1$, is

a 1

b $\frac{1}{2}$

c $\frac{1}{\sqrt{2}}$

d 0

3 The derivative of $\sin^3 x \cos^3 x$ is

a $\frac{3}{5} \sin^2 2x \cos 2x$

b $\frac{3}{4} \sin^2 2x \cos 2x$

c $\frac{3}{4} \sin 2x \cos^2 2x$

d $\frac{3}{2} \sin^2 2x \cos 2x$

4 If $y = \frac{\sin(x+9)}{\cos x}$, then $\frac{dy}{dx}$ at $x = 0$, is

a $\cos 9$

b $\sin 9$

c 0

d 1

5 If $f(x) = \sqrt{1 + \cos^2(x^2)}$, then $f'\left(\frac{\sqrt{\pi}}{2}\right)$ is

a $\frac{\sqrt{\pi}}{6}$

b $-\frac{\sqrt{\pi}}{6}$

c $\frac{1}{\sqrt{6}}$

d $\frac{\pi}{\sqrt{6}}$

6 If $f(x) = \frac{x-4}{2\sqrt{x}}$, then $f'(1)$ is equal to

a $\frac{5}{4}$

b $\frac{4}{5}$

c 1

d 0

7 If $y = \frac{\sin x + \cos x}{\sin x - \cos x}$, then $\frac{dy}{dx}$ at $x = 0$, is

a $-\frac{2}{1}$

b 0

c $\frac{1}{2}$

d None of these

8 If $f(x) = \log_x (\log x)$, then $f'(x)$ at $x = e$ is

a e

b $\frac{1}{e}$

c 1

d None of these

9 If $f'(x) = \sqrt{2x^2 - 1}$ and $y = f(x^2)$, then $\frac{dy}{dx}$ at $x = 1$, is

a 2

b 1

c -2

d None of these

10 If $y = x^{(\log x)^{\log(\log x)}}$, then $\frac{dy}{dx}$ is equal to

a $\frac{y \log y}{x \log x} (2 \log \log x + 1)$

b $\frac{x \log x}{y \log y} (2 \log \log x + 1)$

c $\frac{2y \log y}{x \log x} (\log \log x + 1)$

d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The value of $\lim_{x \rightarrow 0} \frac{1}{x} \sin^{-1} \left(\frac{2x}{1+x^2} \right)$ is
 (a) 2 (b) ∞
 (c) Does not exist (d) None of these

Sol. Using $\sin^{-1} \left[\frac{2x}{1+x^2} \right] = 2 \tan^{-1} x$, we have

$$\begin{aligned} \text{Given limit} &= \lim_{x \rightarrow 0} \frac{2 \tan^{-1} x}{x} \\ &= 2 \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 2(1) = 2 \quad \left[\because \lim_{x \rightarrow 0} \frac{\tan^{-1} x}{x} = 1 \right] \end{aligned}$$

Hence, (a) is the correct answer.

Ex 2. $\lim_{x \rightarrow -\infty} \frac{x^5 \tan \left(\frac{1}{\pi x^2} \right) + 3|x|^2 + 7}{|x|^3 + 7|x| + 8}$ is equal to

- (a) $-\frac{1}{\pi}$
 (b) 0
 (c) ∞
 (d) Does not exist

Sol. We have, $\lim_{x \rightarrow -\infty} \frac{x^5 \tan \left(\frac{1}{\pi x^2} \right) + 3|x|^2 + 7}{|x|^3 + 7|x| + 8}$

$$= \lim_{x \rightarrow -\infty} \frac{x^5 \tan \left(\frac{1}{\pi x^2} \right) + 3x^2 + 7}{-x^3 - 7x + 8}$$

[$\because x < 0 \Rightarrow |x| = -x$]

$$= \lim_{x \rightarrow -\infty} \frac{x^2 \tan \left(\frac{1}{\pi x^2} \right) + \frac{3}{x} + \frac{7}{x^3}}{-1 - \frac{7}{x^2} + \frac{8}{x^3}}$$

[dividing numerator and denominator by x^3]

$$= \lim_{x \rightarrow -\infty} \frac{\frac{1}{\pi} \cdot \frac{\tan \left\{ \frac{1}{(\pi x^2)} \right\}}{1} + \frac{3}{x} + \frac{7}{x^3}}{-1 - \frac{7}{x^2} + \frac{8}{x^3}}$$

$$= \frac{\frac{1}{\pi} \cdot 1 + 0 + 0}{-1 - 0 + 0} = -\frac{1}{\pi}$$

$$\left[\because x \rightarrow -\infty \Rightarrow \frac{1}{\pi x^2} \rightarrow 0 \text{ and } \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1 \right]$$

Hence, (a) is the correct answer.

Ex 3. $\lim_{x \rightarrow 1} \sin ||x| - 2| - 3|$ is equal to

- (a) $\sin 2$ (b) $\sin 1$
 (c) 0 (d) Does not exist

Sol. $\sin ||x| - 2| - 3|$ is continuous function.
 So, its limit will same as that of its value.
 $\sin ||1| - 2| - 3| = \sin |1 - 3| = \sin 2$
 Hence, (a) is the correct answer.

Ex 4. $\lim_{x \rightarrow a} \left\{ \left[\left(\frac{a^{1/2} + x^{1/2}}{a^{1/4} - x^{1/4}} \right)^{-1} - \frac{2(ax)^{1/4}}{x^{3/4} - a^{1/4} x^{1/2} + a^{1/2} x^{1/4} - a^{3/4}} \right]^{-1} - \sqrt{2}^{\log_4 a} \right\}^8$

is equal to

- (a) a (b) $a^{3/4}$
 (c) a^2 (d) None of these

Sol. Simplifying the expression in brackets by setting $a^{1/4} = b$ and $x^{1/4} = y$, the function whose limit is required can be written as

$$\begin{aligned} & \left\{ \left[\left(\frac{b^2 + y^2}{b - y} \right)^{-1} - \frac{2by}{y^3 - by^2 + b^2 y - b^3} \right]^{-1} - 2^{\frac{1}{2} \log_4 b^4} \right\}^8 \\ &= \left\{ \left[\frac{b - y}{b^2 + y^2} - \frac{2by}{(y - b)(y^2 + b^2)} \right]^{-1} - b \right\}^8 \\ &= \left\{ \left[\frac{1}{b - y} \right]^{-1} - b \right\}^8 = y^8 = x^2 \end{aligned}$$

So, the required limit as $x \rightarrow a$ is a^2 .

Hence, (c) is the correct answer.

Ex 5. The value of $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos x^2}}{1 - \cos x}$ is

- (a) $\frac{1}{2}$
 (b) 2
 (c) $\sqrt{2}$
 (d) None of the above

Sol. The required limit

$$= \lim_{x \rightarrow 0} \frac{\sqrt{(1 - \cos x^2)(1 + \cos x^2)}}{(1 - \cos x)\sqrt{1 + \cos x^2}}$$

$$= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin x^2}{x^2} \right)}{2 \left(\frac{\sin^2 x / 2}{x^2} \right)} \cdot \frac{1}{\sqrt{1 + \cos x^2}} = \frac{1}{2 \times \frac{1}{4} \times \sqrt{2}} = \sqrt{2}$$

Hence, (c) is the correct answer.

Ex 6. $\lim_{x \rightarrow 2} \frac{\sqrt{x-2} + \sqrt{x} - \sqrt{2}}{\sqrt{x^2 - 4}}$ is equal to

- (a) $\frac{1}{2}$ (b) 1
(c) 2 (d) None of these

Sol. $\lim_{x \rightarrow 2} \left\{ \frac{1}{\sqrt{x+2}} + \frac{\sqrt{x} - \sqrt{2}}{\sqrt{x^2 - 4}} \right\}$
 $= \frac{1}{2} + \lim_{x \rightarrow 2} \frac{x-2}{\sqrt{x+2}} \cdot \frac{1}{\sqrt{(x+2)(x-2)}}$
 $= \frac{1}{2} + \lim_{x \rightarrow 2} \frac{1}{\sqrt{x+2}} \cdot \frac{\sqrt{x-2}}{\sqrt{x+2}} = \frac{1}{2}$

Hence, (a) is the correct answer.

Ex 7. $\lim_{h \rightarrow 0} \left\{ \frac{1}{h \cdot \sqrt[3]{8+h}} - \frac{1}{2h} \right\}$ is equal to

- (a) $\frac{1}{2}$ (b) $-\frac{4}{2}$ (c) $-\frac{16}{3}$ (d) $-\frac{1}{48}$

Sol. $\lim_{h \rightarrow 0} \frac{2 - \sqrt[3]{8+h}}{2h \cdot \sqrt[3]{8+h}}$
 $= \lim_{h \rightarrow 0} \frac{8 - (8+h)}{2h \cdot \sqrt[3]{8+h} \{8^{2/3} + 8^{1/3} \cdot 8 + h\}^{1/3} + (8+h)^{2/3}}$
 $= -\frac{1}{48}$

Hence, (d) is the correct answer.

Ex 8. The value of $\lim_{x \rightarrow 2a} \frac{\sqrt{x-2a} + \sqrt{x} - \sqrt{2a}}{\sqrt{x^2 - 4a^2}}$ is

- (a) $\frac{1}{\sqrt{a}}$ (b) $\frac{1}{2\sqrt{a}}$
(c) $\frac{\sqrt{a}}{2}$ (d) $2\sqrt{a}$

Sol. $\lim_{x \rightarrow 2a} \left(\frac{\sqrt{x-2a}}{x^2 - 4a^2} + \frac{\sqrt{x} - \sqrt{2a}}{\sqrt{x^2 - 4a^2}} \right)$
 $= \lim_{x \rightarrow 2a} \left[\frac{1}{\sqrt{x+2a}} + \frac{\sqrt{x} - \sqrt{2a}}{\sqrt{(x-2a)(x+2a)}} \right]$
 $= \frac{1}{2\sqrt{a}} + \lim_{x \rightarrow 2a} \frac{\sqrt{x} - \sqrt{2a}}{\sqrt{(\sqrt{x} - \sqrt{2a})(\sqrt{x} + \sqrt{2a})(x+2a)}}$
 $= \frac{1}{2\sqrt{a}} + \lim_{x \rightarrow 2a} \frac{\sqrt{x} - \sqrt{2a}}{\sqrt{(\sqrt{x} + \sqrt{2a})(x+2a)}} = \frac{1}{2\sqrt{a}} + 0$
 $= \frac{1}{2\sqrt{a}}$

Hence, (b) is the correct answer.

Ex 9. $\lim_{x \rightarrow -\pi} \frac{|x + \pi|}{\sin x}$ is equal to

- (a) -1
(b) 1
(c) π
(d) Does not exist

Sol. $\lim_{x \rightarrow -\pi} \frac{|x + \pi|}{\sin x} = \lim_{t \rightarrow 0} \frac{|t|}{\sin(t - \pi)}$ [put $x + \pi = t$]
 $= \lim_{t \rightarrow 0} \frac{|t|}{-\sin t}$

RHL = $\lim_{t \rightarrow 0^+} \frac{t}{-\sin t} = -1$ [$\because |t| = +t, t > 0$]

LHL = $\lim_{t \rightarrow 0^-} \frac{-t}{-\sin t} = 1$ [$\because |t| = -t, t < 0$]

So, limit does not exist.

Hence, (d) is the correct answer.

Ex 10. $\lim_{x \rightarrow 0} \left\{ \frac{\log_e(1+x)}{x^2} + \frac{x-1}{x} \right\}$ is equal to

- (a) $\frac{1}{2}$
(b) $-\frac{1}{2}$
(c) 1
(d) None of the above

Sol. $\lim_{x \rightarrow 0} \frac{\log_e(1+x) + x^2 - x}{x^2}$
 $= \lim_{x \rightarrow 0} \frac{\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \dots \right) + x^2 - x}{x^2} = \frac{1}{2}$

Hence, (a) is the correct answer.

Ex 11. The value of $\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x \sin x \cos x}$ is

- (a) $\frac{2}{5}$ (b) $\frac{3}{5}$
(c) $\frac{3}{2}$ (d) $\frac{3}{4}$

Sol. $\lim_{x \rightarrow 0} \frac{1 - \cos^3 x}{x \sin x \cos x}$
 $= \lim_{x \rightarrow 0} \frac{(1 - \cos x)(1 + \cos x + \cos^2 x)}{x \sin x \cos x}$
 $= \lim_{x \rightarrow 0} \frac{2 \sin^2 \left(\frac{x}{2} \right)}{x \cdot 2 \sin \left(\frac{x}{2} \right) \cos \left(\frac{x}{2} \right)} \times \frac{(1 + \cos x + \cos^2 x)}{\cos x}$
 $= \lim_{x \rightarrow 0} \frac{\sin \left(\frac{x}{2} \right)}{2 \left(\frac{x}{2} \right)} \times \frac{1 + \cos x + \cos^2 x}{\cos \left(\frac{x}{2} \right) \cos x}$
 $= \frac{1}{2} \times 3 = \frac{3}{2}$

Hence, (c) is the correct answer.

Ex 12. The value of

$$\lim_{x \rightarrow \infty} x \left[\tan^{-1} \left(\frac{x+1}{x+2} \right) - \tan^{-1} \left(\frac{x}{x+2} \right) \right] \text{ is}$$

(a) 1 (b) -1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

Sol.
$$\lim_{x \rightarrow \infty} x \left[\tan^{-1} \left(\frac{x+1}{x+2} \right) - \tan^{-1} \left(\frac{x}{x+2} \right) \right]$$

$$= \lim_{x \rightarrow \infty} x \tan^{-1} \left(\frac{\frac{x+1}{x+2} - \frac{x}{x+2}}{1 + \frac{x+1}{x+2} \cdot \frac{x}{x+2}} \right)$$

$$= \lim_{x \rightarrow \infty} x \tan^{-1} \left(\frac{x+2}{2x^2 + 5x + 4} \right)$$

$$= \lim_{x \rightarrow \infty} \left(\frac{\tan^{-1} \left(\frac{x+2}{2x^2 + 5x + 4} \right)}{\frac{x+2}{2x^2 + 5x + 4}} \right) \times \frac{x(x+2)}{2x^2 + 5x + 4}$$

$$= 1 \times \frac{1}{2} = \frac{1}{2}$$

Hence, (c) is the correct answer.

Ex 13. $f(x) = 3x^{10} - 7x^8 + 5x^6 - 21x^3 + 3x^2 - 7$, then

the value of $\lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{h^3 + 3h}$ is

- (a) $\frac{50}{3}$ (b) $\frac{22}{3}$
(c) 13 (d) None of these

Sol.
$$\lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{h^3 + 3h} = \lim_{h \rightarrow 0} \frac{f(1-h) - f(1)}{-h} \cdot \frac{-1}{h^2 + 3}$$

$$= f'(1) \left(-\frac{1}{3} \right) = \frac{53}{3}$$

Hence, (d) is the correct answer.

Ex 14. If $\lim_{x \rightarrow 0} \frac{4 + \sin 2x + A \sin x + B \cos x}{x^2}$ exists, then

the values of A and B are

- (a) -2 and -4 (b) -4 and -2
(c) -3 and -2 (d) None of these

Sol. Since, the given limit exists and denominator approaches zero as $x \rightarrow 0$.

Numerator must approach zero as $x \rightarrow 0$ which is possible, if $4 + B = 0$ which is obtained by substituting zero in numerator in place of x .

$$\therefore B = -4$$

Now,
$$\lim_{x \rightarrow 0} \frac{4 + \sin 2x + A \sin x - 4 \cos x}{x^2} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos 2x + A \cos x + 4 \sin x}{2x}$$

[using L' Hospital's rule]

Again, $D' \rightarrow 0$, while $N' = 2 + A$

Hence, for above limit to exist $A + 2 = 0$

$$\Rightarrow A = -2$$

$$\text{Now, } \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2 \cos x + 4 \sin x}{2x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{-4 \sin 2x + 2 \sin x + 4 \cos x}{2} = \frac{4}{2} = 2$$

$$\therefore A = -2 \text{ and } B = -4$$

Hence, (a) is the correct answer.

Ex 15. Let $f(x) = x^2 - 1$, $0 < x < 2$ and $2x + 3$, $2 \leq x < 3$.

The quadratic equation whose roots are

$\lim_{x \rightarrow 2-0} f(x)$ and $\lim_{x \rightarrow 2+0} f(x)$, is

- (a) $x^2 - 6x + 9 = 0$
(b) $x^2 - 10x + 21 = 0$
(c) $x^2 - 14x + 49 = 0$
(d) None of the above

Sol.
$$\lim_{x \rightarrow 2-0} f(x) = \lim_{x \rightarrow 2-0} (x^2 - 1) = 2^2 - 1 = 3$$

$$\lim_{x \rightarrow 2+0} f(x) = \lim_{x \rightarrow 2+0} (2x + 3) = 2 \times 2 + 3 = 7$$

So, required quadratic equation is $x^2 - 10x + 21 = 0$.

Hence, (b) is the correct answer.

Ex 16. $\lim_{n \rightarrow \infty} \frac{a^n + b^n}{a^n - b^n}$, where $a > b > 1$, is equal to

- (a) -1 (b) 1
(c) 0 (d) None of these

Sol.
$$\lim_{n \rightarrow \infty} \frac{1 + \left(\frac{b}{a}\right)^n}{1 - \left(\frac{b}{a}\right)^n} = 1, \text{ because } 0 < b/a < 1$$

implies $(b/a)^n \rightarrow 0$ as $n \rightarrow \infty$

Hence, (b) is the correct answer.

Ex 17. The limiting value of $(\cos x)^{1/\sin x}$ as $x \rightarrow 0$ is

- (a) 1 (b) e
(c) 0 (d) None of these

Sol. Put $\cos x = 1 + y$ as $x \rightarrow 0$; $y \rightarrow 0$

$$\lim_{x \rightarrow 0} (\cos x)^{1/\sin x} = \lim_{y \rightarrow 0} [(1 + y)^{1/y}]^{y/\sin x}$$

$$= e^{\lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x}} = e^{\lim_{x \rightarrow 0} \left(-\tan \frac{x}{2} \right)} = e^0 = 1$$

Hence, (a) is the correct answer.

Ex 18. The value of $\lim_{x \rightarrow \infty} \left(\frac{x^2 - 1}{x^2 + 1} \right)^{x^2}$ is

- (a) 1 (b) e^{-1}
(c) e^{-2} (d) e^{-3}

Sol.
$$\lim_{x \rightarrow \infty} \left(\frac{x^2 + 1 - 2}{x^2 + 1} \right)^{x^2} = \lim_{x \rightarrow \infty} \left(1 + \frac{-2}{x^2 + 1} \right)^{x^2}$$

$$= \lim_{x \rightarrow \infty} \left[1 + \frac{-2}{x^2 + 1} \right]^{\frac{x^2 + 1}{-2} \cdot \frac{-2x^2}{x^2 + 1}} = e^{-2}$$

Hence, (c) is the correct answer.

Ex 19. If $f(x) = \begin{cases} x^2, & \text{when } x \text{ is an integer} \\ \frac{k(x^2 - 4)}{2 - x}, & \text{otherwise} \end{cases}$

Then, $\lim_{x \rightarrow 2} f(x)$

- (a) exists only when $k = 1$
 (b) exists for every real k
 (c) exists for every real k , except $k = 1$
 (d) Does not exist

Sol. $\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \frac{k(x^2 - 4)}{2 - x}$
 $= \lim_{x \rightarrow 2^+} k(-2 - x) = -4k$
 $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} \frac{k(x^2 - 4)}{2 - x}$
 $= \lim_{x \rightarrow 2^-} k(-2 - x) = -4k$

Hence, (b) is the correct answer.

Ex 20. In a circle of radius r , an isosceles $\triangle ABC$ is inscribed with $AB = AC$. If $\triangle ABC$ has perimeter $p = 2[\sqrt{2hr - h^2} + \sqrt{2hr}]$ and area $A = h\sqrt{2hr - h^2}$, where h is the altitude from A to BC , then $\lim_{h \rightarrow 0^+} \frac{A}{p^3}$ is

- (a) $128r$ (b) $\frac{1}{128r}$
 (c) $\frac{1}{64r}$ (d) None of these

Sol. $\lim_{h \rightarrow 0^+} \frac{A}{p^3} = \lim_{h \rightarrow 0^+} \frac{h\sqrt{2hr - h^2}}{8[\sqrt{2hr - h^2} + \sqrt{2hr}]^3}$
 $= \lim_{h \rightarrow 0^+} \frac{h \cdot \sqrt{h} \cdot \sqrt{2r - h}}{8\sqrt{h} \cdot h[\sqrt{2r - h} + \sqrt{2r}]^3}$
 $= \lim_{h \rightarrow 0^+} \frac{\sqrt{2r - h}}{8[\sqrt{2r - h} + \sqrt{2r}]^3}$
 $= \frac{\sqrt{2r}}{8(2\sqrt{2r})^3} = \frac{1}{128r}$

Hence, (b) is the correct answer.

Ex 21. Let $f(x) = x(-1)^{[1/x]}$, $x \neq 0$, where $[x]$ denotes the greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$

- (a) does not exist (b) is equal to 2
 (c) is equal to 0 (d) is equal to -1

Sol. Since, $\left[\frac{1}{x}\right]$ is an integer, so $(-1)^{[1/x]}$ is either 1 or -1.

So, $|f(x)| \leq x$ and thus, $\lim_{x \rightarrow 0} f(x) = 0$.

Hence, (c) is the correct answer.

Ex 22. If $x > 0$ and g is a bounded function, then

$\lim_{n \rightarrow \infty} \frac{f(x)e^{nx} + g(x)}{e^{nx} + 1}$ is

- (a) 0 (b) $f(x)$
 (c) $g(x)$ (d) None of these

Sol. Given, $x > 0$ and $g(x)$ is bounded function.

$\therefore \lim_{n \rightarrow \infty} \frac{f(x)e^{nx} + g(x)}{e^{nx} + 1}$
 $= \lim_{n \rightarrow \infty} \frac{f(x)}{1 + \left(\frac{1}{e^{nx}}\right)} + \frac{g(x)}{e^{nx} + 1}$
 $= \frac{f(x)}{1 + 0} + \frac{\text{finite}}{\infty} = f(x)$

Hence, (b) is the correct answer.

Ex 23. If $f(x)$ is a polynomial satisfying

$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$ and $f(2) > 1$, then

$\lim_{x \rightarrow 1} f(x)$ is

- (a) 2 (b) 1
 (c) -1 (d) None of these

Sol. $f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$, $f(2) > 1$

Let $f(x) = a_0x^n + a_1x^{n-1} + \dots + a_{n-1}x + a_n$, $a_0 \neq 0$

Then, by comparing the coefficients of like powers, we get

$a_n = 1$, $a_0 = 1$, $a_1 = a_2 = \dots = a_{n-1} = 0$

$\therefore f(x) = -x^n + 1$

or $f(x) = x^n + 1$

Then, $f(2) > 1$

$\Rightarrow f(x) = x^n + 1$

$\therefore \lim_{x \rightarrow 1} f(x) = \lim_{x \rightarrow 1} (x^n + 1) = 2$

Hence, (a) is the correct answer.

Ex 24. The value of $\lim_{x \rightarrow 1} [\sin^{-1} x]$ is

- (a) Does not exist (b) 1
 (c) 0 (d) $\frac{\pi}{2}$

Sol. $\lim_{x \rightarrow 1} [\sin^{-1} x] = \lim_{t \rightarrow \pi/2} [t] = 1$ [put $\sin^{-1} x = t$]

Hence, (b) is the correct answer.

Ex 25. The value of $\lim_{x \rightarrow -\infty} [\tan^{-1} x]$ is

- (a) -2
 (b) -1
 (c) ∞
 (d) None of the above

Sol. $\lim_{x \rightarrow -\infty} [\tan^{-1} x] = \lim_{t \rightarrow -\pi/2} [t] = -2$ [put $\tan^{-1} x = t$]

Hence, (a) is the correct answer.

Ex 26. The value of $\lim_{x \rightarrow 1^-} [\sin \sin^{-1} x]$ is

- (a) 0 (b) Does not exist
(c) $\frac{\pi}{2}$ (d) None of these

Sol. $\lim_{x \rightarrow 1^-} [\sin \sin^{-1} x] = \lim_{x \rightarrow 1^-} [x] = 0$
 $\left[\begin{array}{l} \text{here, } x \rightarrow 1 \text{ means } x \rightarrow 1^-, \text{ since} \\ \sin \sin^{-1} x \text{ is not defined for } x > 1 \end{array} \right]$

Hence, (a) is the correct answer.

Ex 27. The value of $\lim_{x \rightarrow \pi/2} [\sin^{-1} \sin x]$ is

- (a) 1 (b) $\frac{\pi}{2}$
(c) 0 (d) None of these

Sol. $\lim_{x \rightarrow \pi/2} [\sin^{-1} \sin x] = 1$
 Hence, (a) is the correct answer.

Ex 28. The value of $\lim_{x \rightarrow 0^+} \left[\frac{\sin x}{x} \right]$ is

- (a) 1 (b) 0
(c) Does not exist (d) None of these

Sol. $\lim_{x \rightarrow 0^+} \left[\frac{\sin x}{x} \right] = \lim_{t \rightarrow 1^-} [t] = 0$ $\left[\text{put } \frac{\sin x}{x} = t \right]$
 Hence, (b) is the correct answer.

Ex 29. The value of $\lim_{x \rightarrow 0^-} \left[\frac{\sin x}{x} \right]$ is

- (a) 1 (b) 0
(c) Does not exist (d) None of these

Sol. $\lim_{x \rightarrow 0^-} \left[\frac{\sin x}{x} \right] = \lim_{t \rightarrow 1^-} [t] = 0$ $\left[\text{put } \sin \frac{x}{x} = t \right]$
 Hence, (b) is the correct answer.

Ex 30. The value of $\lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right]$ is

- (a) 0 (b) 1
(c) Does not exist (d) None of these

Sol. $\lim_{x \rightarrow 0} \left[\frac{\sin x}{x} \right] = \lim_{t \rightarrow 1^-} [t] = 0$ $\left[\text{put } \sin \frac{x}{x} = t \right]$
 Hence, (a) is the correct answer.

Ex 31. $\lim_{x \rightarrow 3} \frac{[x]^2 - 9}{x^2 - 9}$ is equal to

- (a) ∞
(b) 0
(c) Does not exist
(d) None of the above

Sol. $\lim_{x \rightarrow 3^-} \frac{[x]^2 - 9}{x^2 - 9} = \infty$ and $\lim_{x \rightarrow 3^+} \frac{[x]^2 - 9}{x^2 - 9} = 0$

So, limit does not exist.

Hence, (c) is the correct answer.

Ex 32. $\lim_{n \rightarrow \infty} \prod_{r=3}^n \frac{r^3 - 8}{r^3 + 8}$ is equal to

- (a) $\frac{2}{7}$ (b) $\frac{7}{2}$
(c) 1 (d) Does not exist

Sol. $\lim_{n \rightarrow \infty} \left(\frac{3^3 - 8}{3^3 + 8} \right) \left(\frac{4^3 - 8}{4^3 + 8} \right) \cdots \left(\frac{n^3 - 8}{n^3 + 8} \right)$
 $= \lim_{n \rightarrow \infty} \left(\frac{3-2}{3+2} \cdot \frac{3^2+4+2(3)}{3^2+4-2(3)} \right) \left(\frac{4-2}{4+2} \cdot \frac{4^2+4+2(4)}{4^2+4-2(4)} \right)$
 $\cdots \left(\frac{n-2}{n+2} \cdot \frac{n^2+4+2n}{n^2+4-2n} \right)$
 $= \lim_{n \rightarrow \infty} \left(\frac{3-2}{3+2} \cdot \frac{4-2}{4+2} \cdot \frac{5-2}{5+2} \cdots \frac{n-2}{n+2} \right)$
 $\left(\frac{3^2+4+2(3)}{3^2+4-2(3)} \cdot \frac{4^2+4+2(4)}{4^2+4-2(4)} \cdots \frac{n^2+4+2n}{n^2+4-2n} \right)$
 $= \left(\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdots}{5 \cdot 6 \cdot 7 \cdot 8 \cdots} \right) \left(\frac{19 \cdot 28 \cdot 39 \cdot 52 \cdot 63 \cdots}{7 \cdot 12 \cdot 19 \cdot 28 \cdot 39 \cdot 52 \cdots} \right)$
 $= \frac{1 \cdot 2 \cdot 3 \cdot 4}{7 \cdot 12} = \frac{2}{7}$

Hence, (a) is the correct answer.

Ex 33. $\lim_{x \rightarrow 0} |x|^{\cos x}$ is equal to

- (a) 1
(b) 0
(c) Does not exist
(d) None of the above

Sol. $\lim_{x \rightarrow 0^+} (x)^0 = 1,$
 $\lim_{x \rightarrow 0^-} (-x)^0 = \lim_{x \rightarrow 0^-} 1 = 1$
 Hence, (a) is the correct answer.

Ex 34. If $a_1 = 1$ and $a_{n+1} = \frac{4+3a_n}{3+2a_n}$, $n \geq 1$ and if a_n has

a limit l as $n \rightarrow \infty$, then

- (a) $l = -\sqrt{2}$
(b) $l = \sqrt{2}$
(c) $l = 2$
(d) None of the above

Sol. $a_1 = 1$

$$a_2 = \frac{(4+3)}{(3+2)} = \frac{7}{5} > 1$$

$$\therefore a_2 > a_1$$

$$\text{Let } a_{n+1} > a_n$$

$$\therefore a_{n+2} - a_{n+1} = \frac{4+3a_{n+1}}{3+2a_{n+1}} - \frac{4+3a_n}{3+2a_n}$$

$$= \frac{(4+3a_{n+1})(3+2a_n) - (4+3a_n)(3+2a_{n+1})}{(3+2a_{n+1})(3+2a_n)}$$

$$= \frac{a_{n+1} - a_n}{(3+2a_{n+1})(3+2a_n)} > 0 \quad [\because a_{n+1} > a_n]$$

Now, $a_{n+2} > a_{n+1}$
So, the sequence of the values of a_n is increasing and
since $a_1 = 1, a_n > 0 \forall n$.
Let $\lim_{n \rightarrow \infty} a_n = l = \lim_{n \rightarrow \infty} a_{n+1}$

$$\therefore l = \frac{4+3l}{3+2l} \Rightarrow 3l+2l^2 = 4+3l$$

$$\Rightarrow l^2 = 2$$

$$\Rightarrow l = \sqrt{2} \quad [\because l > 0]$$

Hence, (b) is the correct answer.

Ex 35. If $[]$ denotes the greatest integer function, then

$$\lim_{n \rightarrow \infty} \frac{[x] + [2x] + \dots + [nx]}{n^2} \text{ is}$$

(a) 0 (b) x (c) $\frac{x}{2}$ (d) $\frac{x^2}{2}$

Sol. $nx - 1 < [nx] \leq nx$

Putting $n = 1, 2, 3, \dots, n$ and adding them, we get

$$x \sum n - n < \sum [nx] \leq x \sum n$$

$$\therefore x \cdot \frac{\sum n}{n^2} - \frac{1}{n} < \frac{\sum [nx]}{n^2} \leq x \cdot \frac{\sum n}{n^2} \quad \dots(i)$$

$$\text{Now, } \lim_{n \rightarrow \infty} \left\{ x \cdot \frac{\sum n}{n^2} - \frac{1}{n} \right\} = x \cdot \lim_{n \rightarrow \infty} \frac{\sum n}{n^2} - \lim_{n \rightarrow \infty} \frac{1}{n} = \frac{x}{2}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{x \sum n}{n^2} = \frac{x}{2}$$

As the two limits are equal.

By Sandwich theorem, we have

$$\lim_{n \rightarrow \infty} \frac{\sum [nx]}{n^2} = \frac{x}{2}$$

Hence, (c) is the correct answer.

Ex 36. $\lim_{x \rightarrow 1} \frac{\int_1^x |t-1| dt}{\sin(x-1)}$ is equal to

- (a) 0 (b) 1
(c) -1 (d) None of these

Sol. $\lim_{h \rightarrow 0} \frac{\int_1^{1+h} |t-1| dt}{\sin(1+h-1)}$

$$= \lim_{h \rightarrow 0} \frac{\frac{d}{dh} \int_1^{1+h} |t-1| dt}{\frac{d}{dh} (\sin h)} \quad [\text{by Leibnitz's rule}]$$

$$= \lim_{h \rightarrow 0} \frac{|1+h-1| \frac{d(1+h)}{dh}}{\cos h} = \lim_{h \rightarrow 0} \frac{|h|}{\cos h} = 0$$

Hence, (a) is the correct answer.

Ex 37. The value of

$$\lim_{x \rightarrow \infty} \frac{2(x)^{1/2} + 3(x)^{1/3} + 4(x)^{1/4} + \dots + n(x)^{1/n}}{(2x-3)^{1/2} + (2x-3)^{1/3} + \dots + (2x-3)^{1/n}}$$

is

(a) $\sqrt{2}$ (b) 2
(c) $\frac{1}{\sqrt{3}}$ (d) 0

Sol. Given,

$$\lim_{x \rightarrow \infty} \frac{2x^{1/2} + 3x^{1/3} + \dots + nx^{1/n}}{(2x-3)^{1/2} + (2x-3)^{1/3} + \dots + (2x-3)^{1/n}}$$

$$= \lim_{h \rightarrow 0} \frac{2\left(\frac{1}{h^{1/2}}\right) + 3\left(\frac{1}{h^{1/3}}\right) + \dots + n\left(\frac{1}{h^{1/n}}\right)}{\left[\frac{1}{h^{1/2}}(2-3h)^{1/2} + \frac{1}{h^{1/3}}(2-3h)^{1/3} + \dots + \frac{1}{h^{1/n}}(2-3h)^{1/n}\right]}$$

[on putting $x = \frac{1}{h}$, as $x \rightarrow \infty, h \rightarrow 0$]

$$= \lim_{h \rightarrow 0} \frac{2 + 3h^{\left(\frac{1}{2}-\frac{1}{3}\right)} + 4h^{\left(\frac{1}{2}-\frac{1}{4}\right)} + \dots + nh^{\left(\frac{1}{2}-\frac{1}{n}\right)}}{(2-3h)^{1/2} + h^{\left(\frac{1}{2}-\frac{1}{3}\right)}(2-3h)^{1/3} + \dots + h^{\left(\frac{1}{2}-\frac{1}{n}\right)}(2-3h)^{1/n}}$$

$$= \frac{2+0+0+0+\dots}{2^{1/2}+0+0+\dots} = \sqrt{2}$$

Hence, (a) is the correct answer.

Ex 38. The derivative of $x^{2/3}$ by using first principle is

- (a) $\frac{2}{3}x^{-1/3}$ (b) $-\frac{2}{3}x^{-1/3}$
(c) $\frac{2}{3}x^{1/3}$ (d) None of these

Sol. Let $f(x) = x^{2/3}$. We know by first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\therefore f'(x) = \lim_{h \rightarrow 0} \frac{(x+h)^{2/3} - x^{2/3}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\left(x^{2/3} + \frac{2}{3}x^{-1/3}h + \dots\right) - x^{2/3}}{h} = \frac{2}{3x^{1/3}}$$

Hence, (a) is the correct answer.

Ex 39. The derivative of $\cos(x^2+1)$ by using first principle is

- (a) $2x \sin(x^2+1)$ (b) $-2x \sin(x^2+1)$
(c) $-2x \sin(x^2-1)$ (d) None of these

Sol. Let $f(x) = \cos(x^2+1)$

We know by first principle,

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\Rightarrow f'(x) = \lim_{h \rightarrow 0} \frac{\cos[(x+h)^2+1] - \cos(x^2+1)}{h}$$

[$\because f(x) = \cos(x^2+1)$]

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[-2 \sin \left\{ \frac{(x+h)^2+1+x^2+1}{2} \right\} \times \sin \left\{ \frac{(x+h)^2+1-(x^2+1)}{2} \right\} \right]$$

$$\left[\because \cos C - \cos D = -2 \sin \frac{C+D}{2} \sin \frac{C-D}{2} \right]$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{-2 \sin \frac{(x+h)^2 + x^2 + 2}{2} \sin \frac{(x+h)^2 - x^2}{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2 \sin \frac{(x+h)^2 + x^2 + 2}{2} \left[\sin \frac{(x+h)^2 - x^2}{2} \right]}{h \times \frac{(x+h)^2 - x^2}{2}} \\
 &\quad \times \left[\frac{(x+h)^2 - x^2}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-2 \sin \frac{(x+h)^2 + x^2 + 2}{2} \times \frac{(x+h+x)(x+h-x)}{2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{-2 \sin \left[\frac{(x+h)^2 + x^2 + 2}{2} \right] (2x+h) \frac{h}{2}}{h} \\
 &= -2 \sin \left(\frac{2x^2 + 2}{2} \right) \times 2x \times \frac{1}{2} \\
 &= -2x \sin(x^2 + 1)
 \end{aligned}$$

Hence, (b) is the correct answer.

Type 2. More than One Correct Option

Ex 42. If $\lim_{x \rightarrow \infty} 4x \left(\frac{\pi}{4} - \tan^{-1} \frac{x+1}{x+2} \right) = y^2 + 4y + 5$,

then y can be equal to

- (a) 1 (b) -1 (c) -4 (d) -3

Sol. $\lim_{x \rightarrow \infty} 4x \frac{\tan^{-1} \left(\frac{1}{2x+3} \right)}{\left(\frac{1}{2x+3} \right)} \times \frac{1}{2x+3} = 2$

$$\Rightarrow y^2 + 4y + 5 = 2$$

$$\Rightarrow y = -1, -3$$

Hence, (b) and (d) are the correct answers.

Ex 43. $\lim_{x \rightarrow 0} \frac{1 - \cos(x^2)}{x^3(4^x - 1)}$ is equal to

(a) $\frac{1}{2} \ln 2$

(b) $\ln 2$

(c) $\ln 4$

(d) $1 - \frac{1}{2} \ln \left(\frac{e^2}{4} \right)$

Sol. $\lim_{x \rightarrow 0} \frac{2 \sin^2 \left(\frac{x^2}{2} \right)}{2} \cdot \frac{\left(\frac{x^2}{2} \right)^2}{(4^x - 1)} \cdot \frac{x}{x^4} = \frac{1}{2} \ln 4 = \ln 2$

Ex 40. The derivative of an odd function is always

- (a) an even function (b) an odd function
(c) Does not exist (d) None of these

Sol. Let f be an odd function.

Then, $f(-x) = -f(x)$

$\therefore f'(-x) \cdot (-1) = -f'(x)$

$\Rightarrow f'(-x) = f'(x)$

$\therefore f'(x)$ is an even function.

Hence, (a) is the correct answer.

Ex 41. If $f(x) = x^m$, m being a non-negative integer, then the value of m for which $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$, for all $\alpha, \beta > 0$, is

- (a) 1 (b) 2
(c) -2 (d) None of these

Sol. We have, $f'(\alpha + \beta) = f'(\alpha) + f'(\beta)$

$\Rightarrow (\alpha + \beta)^{m-1} = \alpha^{m-1} + \beta^{m-1}$

Since, for $m > 2$, the above equality is not valid.

$\therefore$ We must have, $m = 2$. Also, for $m = 0$, $f'(x) = 0$ for all x .

So, the equality is trivially true.

Hence, (b) is the correct answer.

Also, $1 - \frac{1}{2} \ln \left(\frac{e^2}{4} \right) = \ln 2$

Hence, (b) and (d) are the correct answers.

Ex 44. If $f(x) = e^{[\cot x]}$, where $[y]$ represents the greatest integer less than or equal to y , then

(a) $\lim_{x \rightarrow \pi/2^+} f(x) = 1$ (b) $\lim_{x \rightarrow \pi/2^+} f(x) = \frac{1}{e}$

(c) $\lim_{x \rightarrow \pi/2^-} f(x) = \frac{1}{e}$ (d) $\lim_{x \rightarrow \pi/2^-} f(x) = 1$

Sol. $f(x) = e^{[\cot x]}$, as $\cot x$ is negative in the II quadrant and $\cot \frac{\pi}{2} = 0$.

$[\cot x] = -1$

$\therefore \lim_{x \rightarrow \pi/2^+} f(x) = e^{-1} = \frac{1}{e}$

As $x \rightarrow \frac{\pi}{2^-}$, $\cot x$ is positive (being in I quadrant) and

hence $[\cot x] = 0$

$\therefore \lim_{x \rightarrow \pi/2^-} f(x) = e^0 = 1$

Hence, (b) and (d) are the correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 45-47) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Ex 45. Let $a_n = \underbrace{2.99\dots 9}_{n \text{ times}}, n \in \mathbb{N}$

Statement I $\left[\lim_{n \rightarrow \infty} a_n \right] = \lim_{n \rightarrow \infty} [a_n]$, where $[]$ denotes the greatest integer function.

Statement II $\lim_{n \rightarrow \infty} a_n = 3$

Sol. $\lim_{n \rightarrow \infty} 2.\bar{9} = 3$, as there is no real number between $2.\bar{9}$ and 3.

So, $2.\bar{9} = 3$

$\therefore \left[\lim_{n \rightarrow \infty} a_n \right] = [3] = 3$, while $[a_n] = 2, \forall n \in \mathbb{N}$ and

$\lim_{n \rightarrow \infty} [a_n] = 2 \neq \left[\lim_{n \rightarrow \infty} a_n \right]$

Hence, (d) is the correct answer.

Ex 46. Statement I $\lim_{x \rightarrow 0} \sec^{-1} \left(\frac{\sin x}{x} \right) = 0$

Statement II $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right) = 1$

Sol. Since, $x > \sin x$ in $\left(0, \frac{\pi}{2} \right), \frac{\sin x}{x} < 1$

$\therefore \sec^{-1} \left(\frac{\sin x}{x} \right)$ is not defined and $\lim_{x \rightarrow 0} \sec^{-1} \left(\frac{\sin x}{x} \right)$ does not exist.

Hence, (d) is the correct answer.

Ex 47. Statement I $\lim_{x \rightarrow 0} \frac{\sin \left(\pi \sin^2 \frac{x}{2} \right)}{x^2} = \pi$

Statement II $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

Sol. $\lim_{x \rightarrow 0} \frac{\sin \left(\pi \sin^2 \frac{x}{2} \right)}{\pi \sin^2 \frac{x}{2}} \times \frac{\pi \left(\sin^2 \frac{x}{2} \right)}{\left(\frac{x}{2} \right)^2 \times 4} = \frac{\pi}{4}$ and not π

Statement II is evidently true.

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage (Ex. Nos. 48-49) If $\lim_{x \rightarrow a} f(x) = 1$ and

$\lim_{x \rightarrow a} g(x) = \infty$, then $\lim_{x \rightarrow a} \{f(x)\}^{g(x)} = e^{\lim_{x \rightarrow a} [f(x) - 1] \times g(x)}$.

Ex 48. $\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{\sin x}{x - \sin x}}$ is equal to

- (a) $\frac{1}{e}$
- (b) $-\frac{1}{e}$
- (c) e
- (d) $-e$

Sol. $\lim_{x \rightarrow 0} \frac{\sin x}{x - \sin x} = \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x}}{1 - \frac{\sin x}{x}} = \infty$

$\therefore L = e^{\lim_{x \rightarrow 0} \left(\frac{\sin x}{x} - 1 \right) \frac{\sin x}{x - \sin x}} = e^{\lim_{x \rightarrow 0} \frac{-\sin x}{x}}$

$= e^{-1} = \frac{1}{e}$

Hence, (a) is the correct answer.

Ex 49. $\lim_{x \rightarrow 0} \left(\frac{x - 1 + \cos x}{x} \right)^{\frac{1}{x}}$ is equal to

- (a) $e^{\frac{1}{2}}$
- (b) $e^{-\frac{1}{2}}$
- (c) e
- (d) $\frac{1}{e}$

Sol. $\lim_{x \rightarrow 0} \frac{x - 1 + \cos x}{x} = \lim_{x \rightarrow 0} 1 - \frac{2 \sin^2 \frac{x}{2}}{x} = 1$

$L = e^{\lim_{x \rightarrow 0} \left(\frac{x - 1 + \cos x}{x} - 1 \right) \times \frac{1}{x}}$

$= e^{\lim_{x \rightarrow 0} \left(\frac{-2 \sin^2 \frac{x}{2}}{x^2} \right)} = e^{-2/4} = e^{-1/2}$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 50. Match the limits of the functions given in Column I with their corresponding values given in Column II.

Column I	Column II
A. $\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$	p. 4
B. $\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$	q. $\frac{1}{\pi}$
C. $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$	r. $\frac{a+1}{b}$
D. $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$	

Sol. A. Given, $\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$
 Let $\pi - x = h$, as $x \rightarrow \pi$, then $h \rightarrow 0$
 $\therefore \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \lim_{h \rightarrow 0} \frac{\sin h}{\pi h} = \lim_{h \rightarrow 0} \frac{1}{\pi} \times \frac{\sin h}{h}$
 $= \frac{1}{\pi} \times 1 = \frac{1}{\pi} \quad \left[\because \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right]$

B. Given, $\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$

Put the limit directly, we get $\frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$

C. Given, $\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} = \lim_{x \rightarrow 0} \frac{1 - \cos 2x}{1 - \cos x}$
 $= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{2 \sin^2 \frac{x}{2}}$
 $\left[\because 1 - \cos 2x = 2 \sin^2 x \text{ and } 1 - \cos x = 2 \sin^2 \frac{x}{2} \right]$

$= \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$
 [multiplying and dividing by x^2 and then multiplying by $4/4$ in the numerator]

$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{x^2} \times \frac{4 \times \frac{x^2}{4}}{\sin^2 \frac{x}{2}}$
 $= \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \times \left(\frac{\frac{x}{2}}{\sin \frac{x}{2}} \right)^2 \times 4 = 1 \times 1 \times 4 = 4$

D. Given, $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$
 $= \lim_{x \rightarrow 0} \frac{\frac{ax}{x} + \frac{x \cos x}{x}}{\frac{b \sin x}{x}} \quad [\text{dividing each term by } x]$
 $= \lim_{x \rightarrow 0} \frac{a + \cos x}{b \left(\frac{\sin x}{x} \right)}$
 $= \frac{a + \cos 0}{b \times 1} = \frac{a+1}{b} \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$

A $\rightarrow$ q; B $\rightarrow$ q; C $\rightarrow$ p; D $\rightarrow$ r

Ex 51. Match the functions given in Column I with their corresponding derivatives given in Column II.

Column I	Column II
A. $x^{-3}(5 + 3x)$	p. $-12x^{-5} + 36x^{-10}$
B. $x^5(3 - 6x^{-9})$	q. $\frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$
C. $x^{-4}(3 - 4x^{-5})$	r. $-15x^{-4} - 6x^{-3}$
D. $\frac{2}{x+1} - \frac{x^2}{3x-1}$	s. $15x^4 + 24x^{-5}$
	t. $24x^3 - \frac{2}{x^2}$

Sol. A. Let $y = x^{-3}(5 + 3x)$

$$\Rightarrow y = 5x^{-3} + 3x^{-2}$$

On differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = 5(-3)x^{-3-1} + 3(-2)x^{-2-1}$$

$$= -15x^{-4} - 6x^{-3}$$

B. Let $y = x^5(3 - 6x^{-9}) = 3x^5 - 6x^{-4}$
 $= 3x^5 - 6x^{-4}$

On differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = 3 \times 5x^{5-1} - 6(-4)x^{-4-1}$$

$$= 15x^4 + 24x^{-5}$$

C. Let $y = x^{-4}(3 - 4x^{-5})$

$$\Rightarrow y = 3x^{-4} - 4x^{-4-5}$$

$$\Rightarrow y = 3x^{-4} - 4x^{-9}$$

On differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = 3(-4)x^{-4-1} - 4(-9)x^{-9-1} = -12x^{-5} + 36x^{-10}$$

D. Let $y = \frac{2}{x+1} - \frac{x^2}{3x-1}$

On differentiating y w.r.t. x, we get

$$\frac{dy}{dx} = \frac{(x+1) \frac{d}{dx}(2) - 2 \frac{d}{dx}(x+1)}{(x+1)^2}$$

$$- \frac{(3x-1) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(3x-1)}{(3x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{(x+1) \times 0 - 2 \times 1}{(x+1)^2} - \frac{(3x-1)(2x) - x^2(3)}{(3x-1)^2}$$

$$= -\frac{2}{(x+1)^2} - \frac{(6x^2 - 2x - 3x^2)}{(3x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2}{(x+1)^2} - \frac{(3x^2 - 2x)}{(3x-1)^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}$$

A $\rightarrow$ r; B $\rightarrow$ s; C $\rightarrow$ p; D $\rightarrow$ q

Type 6. Single Integer Answer Type Questions

Ex 52. The value of $\lim_{x \rightarrow \infty} \frac{\cot^{-1}(x^{-a} \log_a x)}{\sec^{-1}(a^x \log_x a)}$ ($a > 1$) is _____.

Sol. (1) Let $I = \lim_{x \rightarrow \infty} \frac{\cot^{-1}\left(\frac{\log_a x}{x^a}\right)}{\sec^{-1}\left(\frac{a^x}{\log_a x}\right)}$; $\lim_{x \rightarrow \infty} \left(\frac{\log_a x}{x^a}\right) \rightarrow 0$
and $\left(\frac{a^x}{\log_a x}\right) \rightarrow \infty$ [using L' Hospital's rule]
 $\therefore I = \frac{\frac{\pi}{2}}{\frac{\pi}{2}} = 1$

Ex 53. $\lim_{x \rightarrow \infty} \frac{\cot^{-1}(\sqrt{x+1} - \sqrt{x})}{\sec^{-1}\left\{\left(\frac{2x+1}{x-1}\right)^x\right\}}$ is equal to _____.

Sol. (1) Let $I = \lim_{x \rightarrow \infty} \frac{\cot^{-1}(\sqrt{x+1} - \sqrt{x})}{\sec^{-1}\left\{\left(\frac{2x+1}{x-1}\right)^x\right\}}$
 $\Rightarrow \lim_{x \rightarrow \infty} (\sqrt{x+1} - \sqrt{x}) = 0$
 $\Rightarrow \cot^{-1}(0) = \frac{\pi}{2} \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{2x+1}{x-1}\right)^x = \infty$

$$\Rightarrow \sec^{-1}(\infty) = \frac{\pi}{2}$$

$$\therefore I = 1$$

Ex 54. $\lim_{x \rightarrow 1^-} [1 + (\arccos x)^{1-x}]^2$ has the value _____.

Sol. (1) $I = \lim_{x \rightarrow 1} (\cos^{-1} x)^{1-x}$ [0^0 form]

$$\ln I = \lim_{x \rightarrow 1} (1-x) \ln(\cos^{-1} x)$$

Put $x = \cos \theta$

$$\text{But } \lim_{\theta \rightarrow 0} (1 - \cos \theta) \ln \theta = \lim_{\theta \rightarrow 0} \frac{\ln \theta}{\frac{1}{1 - \cos \theta}}$$

$$= - \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)^2}{\sin \theta \cdot \theta}$$

$$= - \lim_{\theta \rightarrow 0} \frac{(1 - \cos \theta)^2}{\theta^2 \frac{\sin \theta}{\theta}} = 0$$

$$\therefore I = 1$$

Ex 55. If $f(x) = x \cos x$, then the value of $f'(2\pi)$ is _____.

Sol. (1) $f(x) = x \cos x$
 $\Rightarrow f'(x) = x(-\sin x) + \cos x \cdot 1$
 $= -x \sin x + \cos x$
 $\therefore f'(2\pi) = -2\pi \sin 2\pi + \cos 2\pi$
 $= 0 + 1 = 1$

Target Exercises

Type 1. Only One Correct Option

1. $\lim_{x \rightarrow 0} \frac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}}$ is equal to
 (a) 0 (b) 1
 (c) -1 (d) Does not exist

2. $\lim_{x \rightarrow 0} \left[\left(3x + \frac{1}{x} \right)^2 - \left(2x - \frac{1}{x} \right)^2 \right]$ is equal to
 (a) 5 (b) 2
 (c) 10 (d) 0

3. The value of $\lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2}$ is
 (a) 0 (b) 1
 (c) -1 (d) None of these

4. $\lim_{x \rightarrow 2} \frac{\sqrt[3]{10-x} - 2}{x-2}$ is equal to
 (a) $\frac{1}{12}$ (b) $-\frac{1}{12}$
 (c) $\frac{1}{6}$ (d) $-\frac{1}{6}$

5. $\lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x-1)^2}$ is equal to
 (a) $\frac{1}{9}$ (b) $\frac{1}{6}$
 (c) $\frac{1}{3}$ (d) None of these

6. $\lim_{x \rightarrow 2a} \frac{\sqrt{x-2a} + \sqrt{x} - \sqrt{2a}}{\sqrt{x^2 - 4a^2}}$ is equal to
 (a) $\frac{1}{\sqrt{a}}$ (b) $\frac{1}{2\sqrt{a}}$
 (c) $-\frac{1}{\sqrt{a}}$ (d) $-\frac{1}{2\sqrt{a}}$

7. $\lim_{x \rightarrow 2} \frac{\sqrt{1+\sqrt{2+x}} - \sqrt{3}}{x-2}$ is equal to
 (a) $\frac{1}{8\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
 (c) $8\sqrt{3}$ (d) $\sqrt{3}$

8. $\lim_{x \rightarrow -1} \frac{x+1}{\sqrt{6x^2+3}+3x}$ is equal to
 (a) 1 (b) -1
 (c) 0 (d) None of these

9. $\lim_{x \rightarrow 3} \left[\frac{x-3}{\sqrt{x-2}-\sqrt{4-x}} \right]$ is equal to
 (a) 1 (b) -1
 (c) $\frac{1}{2}$ (d) None of these

10. $\lim_{x \rightarrow 0} f(x)$, where

$$f(x) = \frac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a+x} - \sqrt{a-x}},$$
 is equal to
 (a) a (b) $-a$ (c) $\sqrt{a}$ (d) $-\sqrt{a}$

11. $\lim_{x \rightarrow 2} \frac{2 - \sqrt{2+x}}{\sqrt[3]{2} - \sqrt[3]{4-x}}$ is equal to
 (a) $\frac{3}{2^{4/3}}$ (b) $-\frac{3}{2^{4/3}}$
 (c) $\frac{3}{2^{3/4}}$ (d) $-\frac{3}{2^{3/4}}$

12. $\lim_{n \rightarrow \infty} (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n}), |x| < 1$, is equal to
 (a) $\frac{1}{x-1}$ (b) $\frac{1}{1-x}$
 (c) $1-x$ (d) $x-1$

13. The value of $\lim_{n \rightarrow \infty} [\sqrt[3]{n^2 - n^3} + n]$ is
 (a) $\frac{1}{3}$ (b) $-\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

14. $\lim_{n \rightarrow \infty} \left[\frac{\sqrt{n^2+1} + \sqrt{n}}{\sqrt[4]{n^3+n} - \sqrt[4]{n}} \right]$ is equal to
 (a) 0 (b) 1
 (c) -1 (d) ∞

15. $\lim_{x \rightarrow \infty} \sqrt{\frac{x - \sin x}{x + \cos^2 x}}$ is equal to
 (a) -1 (b) 1
 (c) 0 (d) None of these

16. $\lim_{x \rightarrow \infty} (\sin \sqrt{x+1} - \sin \sqrt{x})$ is equal to
 (a) 1
 (b) -1
 (c) 0
 (d) None of the above

17. $\lim_{h \rightarrow 0} \frac{\sin \sqrt{x+h} - \sin \sqrt{x}}{h}$ is equal to
(a) $\frac{\cos \sqrt{x}}{2\sqrt{x}}$ (b) $\sin \sqrt{x}$ (c) $\frac{1}{2\sin \sqrt{x}}$ (d) $\cos \sqrt{x}$

18. $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x}$ is equal to
(a) $\frac{6}{5}$ (b) $\frac{3}{10}$ (c) $\frac{10}{3}$ (d) $-\frac{3}{10}$

19. $\lim_{x \rightarrow \frac{\pi}{3}} \frac{\sin\left(\frac{\pi}{3} - x\right)}{2\cos x - 1}$ is equal to
(a) $\frac{2}{\sqrt{3}}$ (b) $\frac{1}{2}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

20. $\lim_{h \rightarrow 0} \frac{2\left[\sqrt{3}\sin\left(\frac{\pi}{6} + h\right) - \cos\left(\frac{\pi}{6} + h\right)\right]}{\sqrt{3}h(\sqrt{3}\cos h - \sin h)}$ is equal to
(a) $\frac{4}{3}$ (b) $-\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

21. $\lim_{x \rightarrow \infty} \sqrt{\frac{x + \sin x}{x - \cos x}}$ is equal to
(a) 0 (b) 1
(c) -1 (d) None of these

22. $\lim_{n \rightarrow \infty} \frac{1}{1 + n \sin^2 nx}$ is equal to
(a) -1 (b) 2
(c) 3 (d) None of these

23. $\lim_{x \rightarrow 0} \log_{\tan x} (\tan 2x)$ is equal to
(a) 1 (b) -1
(c) 0 (d) None of these

24. If $f(x) = \begin{cases} x+1, & x > 0 \\ 2-x, & x \leq 0 \end{cases}$
and $g(x) = \begin{cases} x+3, & x < 1 \\ x^2 - 2x - 2, & 1 \leq x < 2 \\ x-5, & x \geq 2 \end{cases}$

Then, $\lim_{x \rightarrow 0} g[f(x)]$ is
(a) 3 (b) -3 (c) 2 (d) -2

25. The value of $\lim_{n \rightarrow \infty} \left[\frac{2n}{2n^2 - 1} \cos \frac{n+1}{2n-1} - \frac{n}{1-2n} \frac{n(-1)^n}{n^2 + 1} \right]$ is
(a) 1 (b) -1
(c) 0 (d) None of these

26. If $\lim_{x \rightarrow 0} \frac{\sin 2x + a \sin x}{x^3}$ is finite, then the value of a and the limit are given by
(a) -2, 1 (b) -2, -1 (c) 2, 1 (d) 2, -1

27. $\lim_{h \rightarrow 0} \frac{\sin(a+3h) - 3\sin(a+2h) + 3\sin(a+h) - \sin a}{h^3}$ is equal to
(a) $\sin a$ (b) $-\sin a$ (c) $\cos a$ (d) $-\cos a$

28. The value of $\lim_{x \rightarrow a} \sqrt{a^2 - x^2} \cot \frac{\pi}{2} \sqrt{\frac{a-x}{a+x}}$ is
(a) $\frac{2a}{\pi}$ (b) $-\frac{2a}{\pi}$ (c) $\frac{4a}{\pi}$ (d) $-\frac{4a}{\pi}$

29. $\lim_{x \rightarrow 0} \left[\operatorname{cosec}^3 x \cot x - 2 \cot^3 x \operatorname{cosec} x + \frac{\cot^4 x}{\sec x} \right]$ is equal to
(a) 1 (b) -1
(c) 0 (d) None of these

30. $\lim_{x \rightarrow \frac{\pi}{4}} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$ is equal to
(a) $5\sqrt{2}$ (b) $3\sqrt{2}$
(c) $\sqrt{2}$ (d) None of these

31. $\lim_{n \rightarrow \infty} \cos [\pi \sqrt{n^2 + n}]$, $n \in I$ is equal to
(a) 0 (b) 1
(c) -1 (d) None of these

32. $\lim_{x \rightarrow \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x+7)}$ is equal to
(a) $\frac{1}{2}$ (b) 2
(c) 0 (d) None of these

33. The value of $\lim_{x \rightarrow \infty} \frac{\sqrt{1+x^4} - (1+x^2)}{x^2}$ is
(a) -1 (b) 0
(c) 2 (d) None of these

34. $\lim_{x \rightarrow \infty} \frac{2x^3 - 4x + 7}{3x^3 + 5x^2 - 4}$ is equal to
(a) $\frac{2}{3}$ (b) $\frac{3}{2}$
(c) $-\frac{7}{4}$ (d) None of these

35. The value of $\lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2}$ is
(a) $\frac{1}{2}$ (b) 1 (c) 0 (d) $\frac{1}{4}$

36. $\lim_{x \rightarrow e} \frac{\log x - 1}{x - e}$ is equal to
(a) $\frac{1}{e}$ (b) 1 (c) 0 (d) $\frac{1}{2}$

37. $\lim_{x \rightarrow 1} \frac{\log_e x}{x-1}$ is equal to
(a) $\frac{1}{2}$ (b) 0 (c) 1 (d) 2

38. $\lim_{x \rightarrow 0} \frac{e^{x^2} - \cos x}{x^2}$ is equal to

- (a) $\frac{1}{2}$ (b) $\frac{2}{3}$ (c) $\frac{3}{2}$ (d) 2

39. $\lim_{x \rightarrow 0} \frac{\log \cos x}{x}$ is equal to

- (a) 0 (b) 1
(c) ∞ (d) None of these

40. $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$ is equal to

- (a) 1 (b) -1 (c) 2 (d) 0

41. The value of $\lim_{x \rightarrow 3} \left[\log_a \frac{x-3}{\sqrt{x+6}-3} \right]$ is

- (a) $\log_a 6$ (b) $\log_a 3$
(c) $\log_a 2$ (d) None of these

42. The value of $\lim_{x \rightarrow \infty} \frac{3^{x+1} - 5^{x+1}}{3^x - 5^x}$ is

- (a) 5 (b) $\frac{1}{5}$
(c) -5 (d) None of these

43. $\lim_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^4}$ is equal to

- (a) 0 (b) 1 (c) $\frac{1}{6}$ (d) $-\frac{1}{6}$

44. The value of $\lim_{x \rightarrow 0} \log_e (\sin x)^{\tan x}$ is

- (a) 1 (b) -1
(c) 0 (d) None of these

45. The value of $\lim_{x \rightarrow \infty} 3^x \sin \left(\frac{4}{3^x} \right)$ is

- (a) $4 \log 3$ (b) $3 \log 4$
(c) 4 (d) None of these

46. The value of $\lim_{x \rightarrow \infty} \left(\frac{1+3x}{2+3x} \right)^{\frac{1-\sqrt{x}}{1-x}}$ is

- (a) 0 (b) -1 (c) e (d) 1

47. $\lim_{x \rightarrow 0} \left[\sum_{r=1}^n \frac{1}{2^r} \right]$, where $[\]$ denotes the greatest integer function, is equal to

- (a) 1 (b) 0
(c) Does not exist (d) None of these

48. If $f(x) = 0$ is a quadratic equation such that $f(-\pi) = f(\pi) = 0$ and $f\left(\frac{\pi}{2}\right) = -\frac{3\pi^2}{4}$, then

$\lim_{x \rightarrow -\pi} \frac{f(x)}{\sin(\sin x)}$ is equal to

- (a) 0 (b) π
(c) 2π (d) None of these

49. $\lim_{x \rightarrow 0} \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{1/x}$ is equal to

- (a) e (b) e^2 (c) e^3 (d) e^{-1}

50. If α and β are the roots of $ax^2 + bx + c = 0$, then $\lim_{x \rightarrow \alpha} (1 + ax^2 + bx + c)^{1/(x-\alpha)}$ is

- (a) $\log |a(\alpha - \beta)|$ (b) $e^{a(\alpha - \beta)}$
(c) $e^{a(\beta - \alpha)}$ (d) None of these

51. $\lim_{x \rightarrow a} \left[\frac{\sin \{ \operatorname{sgn}(x) \}}{\operatorname{sgn}(x)} \right]$, where $[\]$ denotes the greatest integer function, is equal to

- (a) 0 (b) 1
(c) -1 (d) Does not exist

52. $\lim_{x \rightarrow 0} (\cos x + a \sin bx)^{a/x}$ is equal to

- (a) $e^{\pi/2}$ (b) $e^{2/\pi}$ (c) e^{a^2b} (d) e^{-b^2a}

53. The value of $\lim_{x \rightarrow \infty} \left(\frac{2x^2 + 3}{2x^2 + 5} \right)^{8x^2 + 3}$ is

- (a) e^8 (b) e^{-8} (c) e^4 (d) e^{-4}

54. $\lim_{x \rightarrow \infty} \frac{2 + 2x + \sin 2x}{(2x + \sin 2x)e^{\sin x}}$ is equal to

- (a) 0 (b) 1
(c) -1 (d) Does not exist

55. If a, b, c and d are positive, then

$\lim_{x \rightarrow \infty} \left(1 + \frac{1}{a + bx} \right)^{c + dx}$ is equal to

- (a) $e^{d/b}$ (b) $e^{c/a}$
(c) $e^{(c+d)/a+b}$ (d) e

56. The value of $\lim_{x \rightarrow \infty} \left(\frac{x^2 - 2x + 2}{x^2 - 4x + 1} \right)^x$ is

- (a) e^{-2} (b) e^2 (c) 1 (d) 0

57. The value of $\lim_{x \rightarrow \infty} \left(\frac{3x^2 + 1}{4x^2 - 1} \right)^{x^3/1+x}$ is

- (a) 0 (b) ∞ (c) 1 (d) -1

58. $\lim_{x \rightarrow 0} x^x$ is equal to

- (a) 0 (b) 1
(c) -1 (d) ∞

59. $\lim_{x \rightarrow 0} (\cos x + \sin x)^{1/x}$ is equal to

- (a) e (b) e^2 (c) e^{-1} (d) 1

60. The value of $\lim_{n \rightarrow \infty} \left(1 + \tan \frac{b}{n} \right)^n$ is

- (a) e^b (b) $e^{b/2}$
(c) e (d) None of these

61. The value of $\lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^n}\right)^x$, $n > 1$ is
(a) 1 (b) $-e$ (c) -1 (d) 0
62. $\lim_{x \rightarrow -1} \left(\frac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\frac{1 - \cos(x+1)}{(x+1)^2}}$ is equal to
(a) 1 (b) $\left(\frac{2}{3}\right)^{1/2}$ (c) $\left(\frac{3}{2}\right)^{1/2}$ (d) $e^{1/2}$
63. $\lim_{n \rightarrow \infty} \left(\cos \frac{x}{n}\right)^n$ is equal to
(a) e^1 (b) e^{-1}
(c) 1 (d) None of these
64. Let $\lim_{x \rightarrow 0} \frac{[x]^2}{x^2} = l^2$ and $\lim_{x \rightarrow \infty} \frac{[x^2]}{x^2} = m$, where $[]$ denotes greatest integer, then
(a) l exists but m does not (b) m exists but l does not
(c) both l and m exist (d) neither l nor m exists
65. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - (\sin x)^{\sin x}}{1 - \sin x + \ln \sin x}$ equals
(a) 1 (b) 2 (c) 3 (d) 4
66. $\lim_{x \rightarrow 0} \frac{\sin x^n}{(\sin x)^m}$, $m, n \in N$, $n > m$ is equal to
(a) 2 (b) 1
(c) 0 (d) None of these
67. $\lim_{x \rightarrow \infty} \frac{(\log x)^2}{x^n}$, $n > 0$ is equal to
(a) 1 (b) 0 (c) -1 (d) ∞
68. $\lim_{x \rightarrow 2} \frac{2^x - x^2}{x^x - 2^2}$ is equal to
(a) $\frac{\log 2 - 1}{\log 2 + 1}$ (b) $\frac{\log 2 + 1}{\log 2 - 1}$ (c) 1 (d) -1
69. If $\lim_{x \rightarrow 0} \frac{x^n - \sin x^n}{x - \sin^n x}$ is non-zero finite, then n may be equal to
(a) 1 (b) 2
(c) 3 (d) None of these
70. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left[\frac{x}{2}\right]}{\ln(\sin x)}$, where $[\cdot]$ denotes the greatest integer function, is equal to
(a) Does not exist (b) 1
(c) 0 (d) -1
71. $\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (1 + \cos^{2m} n! \pi x)$ is equal to
(a) -2 (b) 1
(c) 0 (d) None of these
72. $\lim_{x \rightarrow 0} \left[\frac{\sin([x-3])}{[x-3]} \right]$, where $[]$ represents greatest integer function, is equal to
(a) 0 (b) 1
(c) Does not exist (d) $\sin 1$
73. $\lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$ is equal to
(a) $\frac{\pi}{2}$ (b) $\frac{2}{\pi}$ (c) $-\frac{\pi}{2}$ (d) $-\frac{2}{\pi}$
74. $\lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$ is equal to
(a) $\frac{1}{2}$ (b) 2 (c) 0 (d) 1
75. The value of $\lim_{x \rightarrow 0} \frac{27^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}}$ is
(a) $4\sqrt{2} (\log 3)^2$ (b) $8\sqrt{2} (\log 3)^2$
(c) $2\sqrt{2} (\log 3)^2$ (d) None of these
76. $\lim_{x \rightarrow -1} \frac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}}$ is equal to
(a) $\frac{1}{\sqrt{2\pi}}$ (b) $\frac{1}{\sqrt{\pi}}$
(c) $\frac{1}{\sqrt{2}}$ (d) None of these
77. The value of $\lim_{n \rightarrow \infty} [\sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}]$ is
(a) 1 (b) -1
(c) 0 (d) None of these
78. If α is a repeated root of $ax^2 + bx + c = 0$, then $\lim_{x \rightarrow \alpha} \frac{\tan(ax^2 + bx + c)}{(x - \alpha)^2}$ is
(a) a (b) b (c) c (d) 0
79. $\lim_{x \rightarrow 0} \frac{x}{\tan^{-1} 2x}$ is equal to
(a) $\frac{1}{2}$ (b) ∞ (c) 0 (d) 1
80. Let $f(x) = \begin{cases} \cos[x], & x \geq 0 \\ |x| + a, & x < 0 \end{cases}$, then the value of a , so that $\lim_{h \rightarrow 0} f(x)$ exists, where $[x]$ denotes the greatest integer function $\leq x$, is equal to
(a) 0 (b) -1 (c) 2 (d) 1
81. If $a = \min \{x^2 + 4x + 5, x \in R\}$ and $b = \lim_{\theta \rightarrow 0} \frac{1 - \cos 2\theta}{\theta^2}$, then the value of $\sum_{r=0}^n a^r \cdot b^{n-r}$ is
(a) $\frac{2^{n+1} - 1}{4 \cdot 2^n}$ (b) $2^{n+1} - 1$
(c) $\frac{2^{n+1} - 1}{3 \cdot 2^n}$ (d) None of these

82. $\lim_{x \rightarrow 0} \frac{\log(1+x+x^2) + \log(1-x+x^2)}{\sec x - \cos x}$ is equal to
 (a) 1 (b) -1 (c) 0 (d) ∞
83. The value of $\lim_{x \rightarrow \infty} x \left[\tan^{-1} \frac{x+1}{x+2} - \frac{\pi}{4} \right]$ is
 (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) 1 (d) -1
84. If $f(x) = x - [x]$, where $[x]$ denotes the greatest integer $\leq x$ and $g(x) = \lim_{n \rightarrow \infty} \frac{\{f(x)\}^{2n} - 1}{\{f(x)\}^{2n} + 1}$, then $g(x)$ is equal to
 (a) 0 (b) 1
 (c) -1 (d) None of these
85. $\lim_{n \rightarrow \infty} \left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right]$ is equal to
 (a) 0 (b) $-\frac{1}{2}$
 (c) $\frac{1}{2}$ (d) None of these
86. $\lim_{x \rightarrow 1} (1 + \cos \pi x) \cot^2 \pi x$ is equal to
 (a) 1 (b) -1
 (c) $\frac{1}{2}$ (d) None of these
87. $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 0}} \frac{y^3}{x^3 - y^2 - 1}$ as $(x, y) \rightarrow (1, 0)$ along the line $y = x - 1$ is given by
 (a) 1 (b) ∞
 (c) 0 (d) None of these
88. $\lim_{x \rightarrow \beta} (\tan x \cot \beta)^{1/(x-\beta)}$ is equal to
 (a) $\frac{1}{\sin \beta \cos \beta}$ (b) $\sin \beta \cos \beta$
 (c) $-\frac{1}{\sin \beta \cos \beta}$ (d) None of these
89. $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$ (n integer), for
 (a) no value of n (b) all values of n
 (c) only negative values of n (d) only positive values of n
90. If $f(a) = 2$, $f'(a) = 1$ and $g(a) = -1$, $g'(a) = 2$, then $\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x-a}$ is equal to
 (a) 3 (b) 5 (c) -3 (d) 0
91. If $g(x) = -\sqrt{25-x^2}$, then $\lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x-1}$ is equal to
 (a) $\frac{3}{\sqrt{24}}$ (b) $\frac{1}{\sqrt{24}}$
 (c) $-\frac{1}{24}$ (d) None of these
92. If $f(9) = 9$ and $f'(9) = 1$, then $\lim_{x \rightarrow 9} \frac{3 - \sqrt{f(x)}}{3 - \sqrt{x}}$ is equal to
 (a) 0 (b) 1
 (c) -1 (d) None of these
93. If $f(x)$ and $g(x)$ are differentiable functions and $f(1) = g(1) = 2$, then $\lim_{x \rightarrow 1} \frac{f(1)g(x) - f(x)g(1) - f(1) + g(1)}{g(x) - f(x)}$ is equal to
 (a) 0 (b) 1
 (c) 2 (d) None of these
94. If $f(2) = 2$ and $f'(2) = 1$, then $\lim_{x \rightarrow 2} \frac{2x^2 - 4f(x)}{x-2}$ is equal to
 (a) 4 (b) -4
 (c) 2 (d) -2
95. $\lim_{x \rightarrow 1} \frac{x \sin(x - [x])}{x-1}$, where $[]$ denotes the greatest integer function, is equal to
 (a) 1 (b) -1
 (c) ∞ (d) Does not exist
96. $\lim_{x \rightarrow 0} [\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x}]$ is equal to
 (a) 0 (b) $\frac{1}{2}$
 (c) $\log 2$ (d) e^4
97. $\lim_{x \rightarrow 0^+} \log_{\tan x} (\sin x)$ is equal to
 (a) 1 (b) -1
 (c) 0 (d) None of these
98. $\lim_{x \rightarrow \infty} f(x)$, where $\frac{2x-3}{x} < f(x) < \frac{2x^2+5x}{x^2}$, is equal to
 (a) 1 (b) 2
 (c) -1 (d) -2
99. If $f(x) = \begin{cases} \frac{\sin(2k-3)x}{4x}, & x < 0 \\ k+1, & x = 0 \\ \frac{\tan(3k-4)x}{2x}, & x > 0 \end{cases}$ and $\lim_{x \rightarrow 0} f(x)$ exists, then the value of k is given by
 (a) $\frac{5}{4}$ (b) $-\frac{5}{4}$
 (c) $\frac{4}{5}$ (d) $-\frac{4}{5}$
100. $\lim_{x \rightarrow n} (-1)^{[x]}$, where $[x]$ denotes the greatest integer less than or equal to x , is equal to
 (a) $(-1)^n$ (b) $(-1)^{n-1}$
 (c) 0 (d) Does not exist

101. If $f(x) = \begin{cases} \sin x, & x \neq n\pi \\ 2, & x = n\pi \end{cases}$, where $n \in I$ and

$g(x) = \begin{cases} x^2 + 1, & x \neq 2 \\ 3, & x = 2 \end{cases}$, then $\lim_{x \rightarrow 0} g[f(x)]$ is equal to

- (a) 1 (b) 0
(c) 3 (d) Does not exist

102. If $f(x) = \begin{cases} \frac{\tan [x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[x]$ denotes the

greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$ equals

- (a) 1 (b) -1
(c) 0 (d) Does not exist

103. If $f(x) = \begin{cases} \frac{\tan^{-1}([x] + x)}{[x] - 2x}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[x]$ denotes

the greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$ is equal to

- (a) $-\frac{1}{2}$ (b) 1
(c) $\frac{\pi}{4}$ (d) Does not exist

104. If $f(x) = \begin{cases} x \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$, then $\lim_{x \rightarrow 0} f(x)$ equals

- (a) 1
(b) 0
(c) ∞
(d) None of the above

105. The value of $\lim_{n \rightarrow \infty} \left[\frac{1}{n} + \frac{e^{1/n}}{n} + \frac{e^{2/n}}{n} + \dots + \frac{e^{(n-1)/n}}{n} \right]$ is

- (a) 1 (b) 0
(c) $e - 1$ (d) $e + 1$

106. The value of $\lim_{x \rightarrow \infty} \left[\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \dots + n^{1/x}}{n} \right]^{nx}$ is

- (a) $n!$ (b) n
(c) $(n-1)!$ (d) 0

107. The value of

$$\lim_{x \rightarrow \frac{\pi}{2}} [1^{\frac{1}{\cos^2 x}} + 2^{\frac{1}{\cos^2 x}} + \dots + n^{\frac{1}{\cos^2 x}}]^{\cos^2 x}$$

- (a) 0 (b) n
(c) ∞ (d) $\frac{n(n+1)}{2}$

108. $\lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n \cdot (n+1)} \right]$ is equal to

- (a) 1 (b) -1
(c) 0 (d) None of these

109. $\lim_{n \rightarrow \infty} \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n \cdot (n+1)}{n^3}$ is equal to

- (a) 1 (b) -1
(c) $\frac{1}{3}$ (d) None of these

110. $\lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)} \right]$ is equal to

- (a) 1 (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) None of these

111. If the r th term t_r of a series is given by

$$t_r = \frac{r}{r^4 + r^2 + 1}, \text{ then } \lim_{n \rightarrow \infty} \sum_{r=1}^n t_r \text{ is}$$

- (a) 1 (b) $\frac{1}{2}$
(c) $\frac{1}{3}$ (d) None of these

112. If $f(x) = \int \frac{2 \sin x - \sin 2x}{x^3} dx, x \neq 0$, then $\lim_{x \rightarrow 0} f'(x)$ is equal to

- (a) 0 (b) ∞ (c) -1 (d) 1

113. If $f(x) = \frac{2}{x-3}, g(x) = \frac{x-3}{x+4}$ and $h(x) = -\frac{2(2x+1)}{x^2+x-12}$, then $\lim_{x \rightarrow 3} [f(x) + g(x) + h(x)]$ is

- (a) -2 (b) -1 (c) $-\frac{2}{7}$ (d) 0

114. The derivative of $\left(x + \frac{1}{x}\right)^3$ is

- (a) $3x^2 + \frac{3}{x^4} + 3 + \frac{3}{x^2}$ (b) $3x^2 - \frac{3}{x^4} + 3 - \frac{3}{x^2}$
(c) $3x^2 + \frac{3}{x^4} + 3 - \frac{3}{x^2}$ (d) None of these

115. The derivative of $(3x+5)(1+\tan x)$ is

- (a) $3x \sec^2 x - 6 \sec^2 x - 3 + 3 \tan x$
(b) $3x \sec^2 x - 5 \sec^2 x - 3 - 3 \tan x$
(c) $3x \sec^2 x + 5 \sec^2 x + 3 + 3 \tan x$
(d) None of the above

116. The derivative of $\frac{x^5 - \cos x}{\sin x}$ is

- (a) $\frac{5x^4 \sin x - 1 - x^5 \cos x}{\sin^2 x}$ (b) $\frac{5x^4 \sin x + 1 - x^5 \cos x}{\sin^2 x}$
(c) $\frac{5x^4 \sin x - 1 - x^3 \cos x}{\sin^2 x}$ (d) None of these

117. The derivative of $(ax^2 + \cot x)(p + q \cos x)$ is

- (a) $(ax^2 + \cot x)(-q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$
(b) $(ax^2 + \cot x)(q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$
(c) $(ax^2 + \cot x)(-q \sin x) + (p - q \cos x)(2ax + \operatorname{cosec}^2 x)$
(d) None of the above

118. If $f(x) = (ax + b)\sin x + (cx + d)\cos x$, then the values of a, b, c and d such that $f'(x) = x \cos x$ for all x are

- (a) $b = c = 0, a = d = 1$
 (b) $b = d = 0, a = c = 1$
 (c) $c = d = 0, a = b = 1$
 (d) None of the above

119. Let $f(x)$ be a polynomial function satisfying

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right). \text{ If } f(4) = 65 \text{ and } I_1, I_2, I_3$$

are in GP, then $f'(I_1), f'(I_2), f'(I_3)$ are in

- (a) AP (b) GP
 (c) HP (d) None of these

120. If $8f(x) + 6f\left(\frac{1}{x}\right) = x + 5$ and $y = x^2 f(x)$, then $\frac{dy}{dx}$ at $x = -1$, is equal to

- (a) 0 (b) $\frac{1}{14}$
 (c) $-\frac{1}{14}$ (d) None of these

121. If $f(x) = \log |2x|, x \neq 0$, then $f'(x)$ is equal to

- (a) $\frac{1}{x}$
 (b) $-\frac{1}{x}$
 (c) $\frac{1}{|x|}$
 (d) None of the above

122. If $f(x) = |x - 3|$ and $\phi(x) = (f \circ f)(x)$, when $x > 10$, then $\phi'(x)$ is equal to

- (a) 1 (b) 0
 (c) -1 (d) None of these

123. If $f(x)$ is a polynomial of degree $n (> 2)$ and $f(x) = f(k - x)$, where k is a fixed real number, then degree of $f'(x)$ is

- (a) n
 (b) $n - 1$
 (c) $n - 2$
 (d) None of the above

Type 2. More than One Correct Option

124. $\lim_{x \rightarrow 0} \left[m \frac{\sin x}{x} \right]$, where $m \in I$ and $[]$ denotes the greatest integer function, is equal to

- (a) m , if $m \leq 0$
 (b) $m - 1$, if $m > 0$
 (c) $m - 1$, if $m < 0$
 (d) m , if $m > 0$

125. If $\lim_{x \rightarrow 0} (1 + ax + bx^2)^{2/x} = e^3$, then

- (a) $a = 3, b = 0$ (b) $a = \frac{3}{2}, b = 1$
 (c) $a = \frac{3}{2}, b = 4$ (d) $a = 2, b = 3$

126. If $f(x) = 5$, then which of the following is/are true?

- (a) $f'(1)$ and $f'(-1)$ are negative
 (b) $f'(1)$ and $f'(-1)$ are positive
 (c) $f'(1)$ and $f'(-1)$ are equal
 (d) $f'(1) = 0 = f'(-1)$

Type 3. Assertion and Reason

Directions (Q. Nos. 127-128) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is the correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not the correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

127. **Statement I** $\lim_{x \rightarrow \infty} \left(\frac{1^2}{x^3} + \frac{2^2}{x^3} + \frac{3^2}{x^3} + \dots + \frac{x^2}{x^3} \right)$

$$= \lim_{x \rightarrow \infty} \frac{1^2}{x^3} + \lim_{x \rightarrow \infty} \frac{2^2}{x^3} + \dots + \lim_{x \rightarrow \infty} \frac{x^2}{x^3} = 0$$

Statement II $\lim_{x \rightarrow a} [f_1(x) + f_2(x) + \dots + f_n(x)]$

$$= \lim_{x \rightarrow a} f_1(x) + \lim_{x \rightarrow a} f_2(x) + \dots + \lim_{x \rightarrow a} f_n(x), \text{ where } n \in \mathbb{N}.$$

128. **Statement I** The derivative of the function $f(x) = 3x$ at $x = 2$ is 3.

Statement II $\frac{d}{dx} [af(x)] = a \cdot \frac{d}{dx} [f(x)]$, where a is a constant.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 129-130) Let $f(x) = x$ and $g(x) = -\frac{1}{f(x)}$, $x \neq 0$ define a function $h(x)$ as follows

$$h(x) = \frac{f(x) - g(x)}{f(x) + g(x)}, x \neq \pm 1$$

129. The derivative of $h(x)$ is equal to

- (a) $\frac{4x}{(x^2 - 1)^2}$ (b) $\frac{4x^3}{(x^2 - 1)^2}$
 (c) $\frac{-4x}{(1 - x^2)^2}$ (d) None of these

130. The value of $h'(2)$ is

- (a) $\frac{8}{9}$ (b) $\frac{32}{9}$
 (c) $-\frac{8}{9}$ (d) None of these

Passage II (Q. Nos. 131-133) Let $A_i = \frac{x - a_i}{|x - a_i|}$, $i = 1, 2, \dots, n$ and $a_1 < a_2 < a_3 < \dots < a_n$.

131. If $1 < m < n, m \in N$, then the value of $L = \lim_{x \rightarrow a_m^-} (A_1 A_2 \dots A_n)$ is

- (a) always 1 (b) always -1 (c) $(-1)^{n-m+1}$ (d) $(-1)^{n-m}$

132. If $1 < m < n, m \in N$, then the value of $R = \lim_{x \rightarrow a_m^+} (A_1 A_2 \dots A_n)$ is

- (a) always 1 (b) always -1 (c) $(-1)^{m+1}$ (d) $(-1)^{n-m}$

133. If $1 < m < n, m \in N$, then $\lim_{x \rightarrow a_m} (A_1 A_2 \dots A_n)$ is equal to

- (a) -1 (b) -1
 (c) Does not exist (d) 1 or -1

Type 5. Match the Columns

134. Match the limits of the functions given in Column I with their corresponding values given in Column II.

Column I	Column II
A. $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$	p. 4
B. If $L = \lim_{x \rightarrow \infty} \left[x - x^2 \log_e \left(1 + \frac{1}{x} \right) \right]$, then the value of $8L$ is	q. 5
C. $\lim_{z \rightarrow 1} \left(\frac{z^{1/3} - 1}{z^{1/6} - 1} \right)$	r. 2
D. $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{2^2} \right) \left(1 - \frac{1}{3^2} \right) \dots \left(1 - \frac{1}{n^2} \right)$	s. $\frac{1}{2}$
	t. $3/2$

Column I	Column II
A. cosec x	p. $5 \cos x + 6 \sin x$
B. $3 \cot x + 5 \operatorname{cosec} x$	q. $-3 \operatorname{cosec}^2 x - 5 \operatorname{cosec} x \cot x$
C. $5 \sin x - 6 \cos x + 7$	r. $2 \sec^2 x - 7 \sec x \tan x$
D. $2 \tan x - 7 \sec x$	s. $-\cot x \operatorname{cosec} x$
	t. $6 \sin x + 5 \cos x$

135. Match the functions given in Column I with their corresponding derivatives given in Column II.

Type 6. Single Integer Answer Type Questions

136. Let $f(x) = \begin{cases} x+2, & x \leq -1 \\ cx^2, & x > -1 \end{cases}$. If $\lim_{x \rightarrow -1} f(x)$ exists, then the value of c is _____.

137. For $x > 0$, $\lim_{x \rightarrow 0} \left((\sin x)^{1/x} + \left(\frac{1}{x} \right)^{\sin x} \right)$ is _____.

138. If $\lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos 2x} \cdot \sqrt[3]{\cos 3x} \cdot \sqrt[4]{\cos 4x} \dots \sqrt[n]{\cos nx}}{x^2}$ has the value equal to 10, then the value of n equals _____.

139. If $f(x) = \begin{cases} \frac{\sin(1 + [x])}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$, where $[x]$

denotes the greatest integer not exceeding x , then $\lim_{x \rightarrow 0^-} f(x)$ is _____.

140. The value of $\lim_{x \rightarrow 0} \left(\frac{\int_0^{x^2} \sec^2 t \, dt}{x \sin x} \right)$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

- The largest value of the non-negative integer a for which $\lim_{x \rightarrow 1} \left\{ \frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right\}^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$ is _____ . [2014]
- For $a \in \mathbb{R}$ (the set of all real numbers), $a \neq -1$, $\lim_{h \rightarrow \infty} \frac{(1^a + 2^a + \dots + n^a)}{(n+1)^{a-1} [(na+1) + (na+2) + \dots + (na+n)]} = \frac{1}{60}$, then a is equal to [2013]
(a) 5 (b) 7 (c) $-\frac{15}{2}$ (d) $-\frac{17}{2}$
- If $\lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$, then [2012]
(a) $a = 1, b = 4$ (b) $a = 1, b = -4$
(c) $a = 2, b = -3$ (d) $a = 2, b = 3$
- Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$, where $a > -1$. Then, $\lim_{a \rightarrow 0^+} \alpha(a)$ and $\lim_{a \rightarrow 0^+} \beta(a)$ are [2012]
(a) $-\frac{5}{2}$ and 1 (b) $-\frac{1}{2}$ and -1
(c) $-\frac{7}{2}$ and 2 (d) $-\frac{9}{2}$ and 3
- The value of $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t \log(1+t)}{t^4 + 4} dt$ is [2011]
(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$
- If $\lim_{x \rightarrow 0} [1 + x \ln(1 + b^2)]^{1/x} = 2b \sin^2 \theta$, $b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is [2011]
(a) $\pm \frac{\pi}{4}$ (b) $\pm \frac{\pi}{3}$ (c) $\pm \frac{\pi}{6}$ (d) $\pm \frac{\pi}{2}$

JEE Main/AIEEE

- $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to [2015]
(a) 4 (b) 3
(c) 2 (d) $\frac{1}{2}$
- If the function $g(x) = \begin{cases} k\sqrt{x+1} & , 0 \leq x \leq 3 \\ mx + 2 & , 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is [2015]
(a) 2 (b) $\frac{16}{5}$ (c) $\frac{10}{3}$ (d) 4
- Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2} \right] = 3$, then $f(2)$ is equal to [2015]
(a) -8 (b) -4 (c) 0 (d) 4
- $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to [2014]
(a) $\frac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π
- $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to [2013]
(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2
- Let $\alpha(a)$ and $\beta(a)$ be the roots of the equation $(\sqrt[3]{1+a} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{1+a} - 1) = 0$, where $a > -1$. Then, $\lim_{a \rightarrow 0^+} \alpha(a)$ and $\lim_{a \rightarrow 0^+} \beta(a)$ are [2012]
(a) $-\frac{5}{2}$ and 1 (b) $-\frac{1}{2}$ and -1
(c) $-\frac{7}{2}$ and 2 (d) $-\frac{9}{2}$ and 3
- The value of $\lim_{x \rightarrow 0} \frac{1}{x^3} \int_0^x \frac{t \log(1+t)}{t^4 + 4} dt$ is [2011]
(a) 0 (b) $\frac{1}{12}$ (c) $\frac{1}{24}$ (d) $\frac{1}{64}$
- If $\lim_{x \rightarrow 0} [1 + x \ln(1 + b^2)]^{1/x} = 2b \sin^2 \theta$, $b > 0$ and $\theta \in (-\pi, \pi]$, then the value of θ is [2011]
(a) $\pm \frac{\pi}{4}$ (b) $\pm \frac{\pi}{3}$ (c) $\pm \frac{\pi}{6}$ (d) $\pm \frac{\pi}{2}$
- $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to [2015]
(a) 4 (b) 3
(c) 2 (d) $\frac{1}{2}$
- If the function $g(x) = \begin{cases} k\sqrt{x+1} & , 0 \leq x \leq 3 \\ mx + 2 & , 3 < x \leq 5 \end{cases}$ is differentiable, then the value of $k + m$ is [2015]
(a) 2 (b) $\frac{16}{5}$ (c) $\frac{10}{3}$ (d) 4
- Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2} \right] = 3$, then $f(2)$ is equal to [2015]
(a) -8 (b) -4 (c) 0 (d) 4
- $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to [2014]
(a) $\frac{\pi}{2}$ (b) 1 (c) $-\pi$ (d) π
- $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x}$ is equal to [2013]
(a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2
- Let α and β be the distinct roots of $ax^2 + bx + c = 0$, then $\lim_{x \rightarrow \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$ is equal to [2005]
(a) $\frac{1}{2}(\alpha - \beta)^2$ (b) $-\frac{a^2}{2}(\alpha - \beta)^2$
(c) 0 (d) $\frac{a^2}{2}(\alpha - \beta)^2$
- If $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2$, then the values of a and b are [2004]
(a) $a \in \mathbb{R}, b \in \mathbb{R}$ (b) $a = 1, b \in \mathbb{R}$
(c) $a \in \mathbb{R}, b = 2$ (d) $a = 1, b = 2$

16. $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\left[1 - \tan\left(\frac{x}{2}\right)\right](1 - \sin x)}{\left[1 + \tan\left(\frac{x}{2}\right)\right](\pi - 2x)^3}$ is equal to [2003]
- (a) $\frac{1}{8}$ (b) 0
(c) $\frac{1}{32}$ (d) ∞
17. If $\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = k$, then value of k is [2003]
- (a) 0 (b) $-1/3$
(c) $2/3$ (d) $-2/3$

18. $\lim_{x \rightarrow 0} \frac{\sqrt{1 - \cos 2x}}{\sqrt{2x}}$ is equal to [2002]
- (a) 2 (b) -1
(c) zero (d) Does not exist
19. $\lim_{x \rightarrow \infty} \left(\frac{x^2 + 5x + 3}{x^2 + x + 2}\right)^x$ is equal to [2002]
- (a) e^4 (b) e^2 (c) e^3 (d) e
20. For $x \in \mathbb{R}$, $\lim_{x \rightarrow \infty} \left(\frac{x-3}{x+2}\right)^x$ is equal to [2002]
- (a) e (b) e^{-1} (c) e^{-5} (d) e^5

Other Engineering Entrances

21. $\lim_{x \rightarrow 0} \frac{(1+x)^8 - 1}{(1+x)^2 - 1}$ is equal to [BITSAT 2014]
- (a) 8 (b) 6
(c) 4 (d) 2
22. $\lim_{x \rightarrow 1} \frac{x^m - 1}{x^n - 1}$ is equal to [BITSAT 2014]
- (a) $\frac{n}{m}$ (b) $\frac{m}{n}$ (c) $\frac{2m}{n}$ (d) $\frac{2n}{m}$
23. $\lim_{x \rightarrow 0} (1^{\csc^2 x} + 2^{\csc^2 x} + \dots + n^{\csc^2 x})^{\sin^2 x}$ is equal to [Manipal 2014]
- (a) 1 (b) $1/n$ (c) n (d) 0
24. The value of the $\lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{2/x}$, $(a, b, c > 0)$ is [Manipal 2014]
- (a) $(abc)^3$ (b) abc
(c) $(abc)^{1/3}$ (d) None of these
25. If the function $f(x)$ satisfies $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$, then $\lim_{x \rightarrow 1} f(x)$ is equal to [Karnataka CET 2014]
- (a) 1 (b) 2
(c) 0 (d) 3
26. The value of $\lim_{x \rightarrow 3} \frac{x^5 - 3^5}{x^8 - 3^8}$ is [Kerala CEE 2014]
- (a) $\frac{5}{8}$ (b) $\frac{5}{64}$ (c) $\frac{5}{216}$
(d) $\frac{1}{27}$ (e) None of these
27. If $f(x) = (x^5 - 1)(x^3 + 1)$, $g(x) = (x^2 - 1)(x^2 - x + 1)$ and $h(x)$ such that $f(x) = g(x)h(x)$, then $\lim_{x \rightarrow 1} h(x)$ is equal to [Kerala CEE 2014]
- (a) 0 (b) 1 (c) 3
(d) 4 (e) 5

28. $\lim_{x \rightarrow 0} \frac{\log(1 + 3x^2)}{x(e^{5x} - 1)}$ is equal to [Kerala CEE 2014]
- (a) $\frac{3}{5}$ (b) $\frac{5}{3}$ (c) $-\frac{3}{5}$
(d) $-\frac{5}{3}$ (e) 1
29. If $[x]$ denotes the greatest integer less than or equal to x for any real number x . Then, $\lim_{n \rightarrow \infty} \frac{[n\sqrt{2}]}{n}$ is equal to [WB JEE 2014]
- (a) 0 (b) 2 (c) $\sqrt{2}$ (d) 1
30. If $L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4}$, $a > 0$ and L is finite, then [AMU 2014]
- (a) $a = 2$ (b) $a = 1$
(c) $a = \frac{1}{3}$ (d) None of these
31. If $g(x) = (x^2 + 2x + 3)f(x)$, $f(0) = 5$ and $\lim_{x \rightarrow 0} \left(\frac{f(x) - 5}{x}\right) = 4$, then $g'(0)$ is equal to [J&K CET 2014]
- (a) 22 (b) 18 (c) 20 (d) 25
32. Evaluate $\lim_{x \rightarrow 3} \frac{3 - x}{\sqrt{4 + x} - \sqrt{1 + 2x}}$. [J&K CET 2014]
- (a) 0 (b) $7\sqrt{2}$ (c) $4\sqrt{7}$ (d) $2\sqrt{7}$
33. $\lim_{x \rightarrow \tan^{-1} 3} \frac{\tan^2 x - 2 \tan x - 3}{\tan^2 x - 4 \tan x + 3}$ is equal to [BITSAT 2014]
- (a) 1 (b) 2 (c) 0 (d) 3
34. $\lim_{x \rightarrow \infty} \frac{x^4 \sin\left(\frac{1}{x}\right) + x^2}{1 + |x^3|}$ equals [BITSAT 2014]
- (a) 0 (b) -1
(c) 2 (d) 1

35. $\lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x}$ is equal to [Manipal 2014]
 (a) 0 (b) e
 (c) 1 (d) Does not exist
36. If $\lim_{x \rightarrow 0} \frac{2a \sin x - \sin 2x}{\tan^3 x}$ exists and is equal to 1, then the value of a is [WB JEE 2014]
 (a) 2 (b) 1 (c) 0 (d) -1
37. $\lim_{x \rightarrow 2} \frac{\sin^{-1}(x-2)}{x^2 - 4}$ is equal to [J&K CET 2014]
 (a) 0 (b) 2 (c) $\frac{1}{4}$ (d) 1
38. The graph of the function $y = f(x)$ has a unique tangent at the point $(a, 0)$ through which the graph passes.
 Then, $\lim_{x \rightarrow a} \left[\frac{\log_e \{1 + 6f(x)\}}{3f(x)} \right]$ is [UP SEE 2013]
 (a) 0 (b) 1
 (c) 2 (d) None of these
39. The value of $\lim_{n \rightarrow \infty} \frac{1 + 2 + \dots + n}{3n^2 + 5}$ is equal to [OJEE 2013]
 (a) $\frac{1}{3}$ (b) $\frac{1}{5}$ (c) $\frac{1}{6}$ (d) 6
40. $\lim_{h \rightarrow 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$ is equal to [Manipal 2013]
 (a) $a^2 \cos a + a \sin a$ (b) $a^2 \cos a + 2a \sin a$
 (c) $2a^2 \cos a + a \sin a$ (d) None of these
41. $\lim_{x \rightarrow 0} \left(\frac{1 + \tan x}{1 + \sin x} \right)^{\operatorname{cosec} x}$ is equal to [Kerala CEE 2013]
 (a) $\frac{1}{e}$ (b) 1
 (c) e (d) e^2
 (e) None of these
42. If $f: R \rightarrow R$ is such that $f(1) = 3$ and $f'(1) = 6$. Then,
 $\lim_{x \rightarrow 0} \left[\frac{f(1+x)}{f(1)} \right]^{1/x}$ is equal to [Manipal 2012]
 (a) 1 (b) $e^{1/2}$
 (c) e^2 (d) e^3
43. $\lim_{x \rightarrow 0} \frac{x2^x - x}{1 - \cos x}$ is equal to [Karnataka CET 2012]
 (a) $2 \log 2$ (b) $\log 2$ (c) $\frac{1}{2} \log 2$ (d) $\frac{1}{2}$
44. $\lim_{x \rightarrow 0} \frac{\pi^x - 1}{\sqrt{1+x} - 1}$ is [WB JEE 2012]
 (a) Does not exist (b) $\log_e(\pi^2)$
 (c) 1 (d) lies between 10 and 11
45. The value of $\lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n}$ is [WB JEE 2012]
 (a) 1 (b) $\frac{1}{e^2}$
 (c) $\frac{1}{2e}$ (d) $\frac{1}{e}$
46. $\lim_{x \rightarrow 0} \left[\frac{(2+x) \sin(2+x) - 2 \sin 2}{x} \right]$ is equal to [BITSAT 2012]
 (a) $\sin 2$
 (b) $\cos 2$
 (c) 1
 (d) $2 \cos 2 + \sin 2$
47. If $f(x) = \begin{cases} \sin [x], & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$, where $[x]$ denotes the greatest integer less than or equal to x , then $\lim_{x \rightarrow 0} f(x)$ is equal to [Manipal 2012]
 (a) 1
 (b) 0
 (c) -1
 (d) None of the above
48. The value of $\lim_{x \rightarrow 0} \frac{x \cos x - \log_e(1+x)}{x^2}$ is [MP PET 2012]
 (a) $1/5$ (b) $1/4$
 (c) $1/3$ (d) $1/2$
49. If $f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & x \end{vmatrix}$, then $\lim_{x \rightarrow 0} \frac{f(x)}{x^2}$ is equal to [Karnataka CET 2012]
 (a) 0 (b) 3 (c) 2 (d) 1
50. $\lim_{x \rightarrow a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$ is equal to [Karnataka CET 2012]
 (a) $\frac{2}{3}$ (b) $\frac{2}{\sqrt{3}}$
 (c) $\frac{3\sqrt{3}}{2}$ (d) $\frac{2}{3\sqrt{3}}$
51. $\lim_{k \rightarrow \infty} \left(\frac{1^3 + 2^3 + 3^3 + \dots + k^3}{k^4} \right)$ is equal to [Kerala CEE 2011]
 (a) 0 (b) 2 (c) $1/3$
 (d) ∞ (e) $1/4$
52. $\lim_{x \rightarrow \infty} \left(1 + \frac{4}{x-1} \right)^{x+3}$ is equal to [Guj CET 2011]
 (a) e^4 (b) e^2
 (c) e^3 (d) e
53. If $f'(x) = f(x)$, $f(0) = 1$, then $\lim_{x \rightarrow 0} \frac{f(x) - 1}{x}$ equals [Guj CET 2011]
 (a) 0 (b) 1
 (c) -1 (d) 2

54. $\lim_{x \rightarrow 0} \left(\frac{16^x + 9^x}{2} \right)^{1/x}$ is equal to [GGSIPU 2011]

- (a) 25/2 (b) 12
(c) 1 (d) 1/4

55. The value of $\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin(\cos x) \cos x}{\sin x - \operatorname{cosec} x}$ is [UP SEE 2011]

- (a) ∞ (b) 1
(c) 0 (d) -1

56. If $\lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$, then [AMU 2011]

- (a) $a = 1, b = 2$ and $c = 1$
(b) $a = 1, b = 1$ and $c = 2$
(c) $a = 2, b = 1$ and $c = 1$
(d) $a = b = c = 1$

57. If $f(5) = 7$ and $f'(5) = 7$, then $\lim_{x \rightarrow 5} \frac{xf(5) - 5f(x)}{x - 5}$ is equal to [WB JEE 2010]

- (a) 35
(b) -35
(c) 28
(d) -28

58. $\lim_{x \rightarrow \infty} \left(\frac{x^3}{3x^2 - 4} - \frac{x^2}{3x + 2} \right)$ is equal to [Kerala CEE 2010]

- (a) $-\frac{1}{4}$ (b) $-\frac{1}{2}$ (c) 0
(d) $\frac{2}{9}$ (e) None of these

59. $\lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+x} - \sqrt{1-x}} \right)$ is equal to [Kerala CEE 2010]

- (a) 0 (b) 1 (c) 2
(d) -1 (e) -2

60. The value of $\lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{1/x^2}$ is [WB JEE 2010]

- (a) e^2 (b) e (c) $\frac{1}{e}$ (d) $\frac{1}{e^2}$

61. $\lim_{x \rightarrow 0} \frac{\sin |x|}{x}$ is equal to [WB JEE 2010; BITSAT 2010]

- (a) 1 (b) 0
(c) positive infinity (d) Does not exist

62. The value of $\lim_{x \rightarrow 0} \frac{\sin^2 x + \cos x - 1}{x^2}$ is [WB JEE 2010; BITSAT 2010]

- (a) 1
(b) $\frac{1}{2}$
(c) $-\frac{1}{2}$
(d) 0

63. The value of $\lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos x)}{x^4}$ is [WB JEE 2010]

- (a) $\frac{1}{2}$ (b) $\frac{1}{4}$
(c) $\frac{1}{6}$ (d) $\frac{1}{8}$

Answers

Work Book Exercise 18.1

1. (c)	2. (c)	3. (d)	4. (c)	5. (b)	6. (d)	7. (b)	8. (d)	9. (d)	10. (a)
11. (d)	12. (c)	13. (b)	14. (d)	15. (b)	16. (a)				

Work Book Exercise 18.2

1. (b)	2. (d)	3. (b)	4. (a)	5. (b)	6. (a)	7. (a)	8. (b)	9. (a)	10. (a)

Target Exercises

1. (d)	2. (c)	3. (b)	4. (b)	5. (a)	6. (b)	7. (a)	8. (a)	9. (a)	10. (d)
11. (b)	12. (b)	13. (a)	14. (d)	15. (b)	16. (c)	17. (a)	18. (c)	19. (c)	20. (a)
21. (b)	22. (d)	23. (a)	24. (b)	25. (c)	26. (b)	27. (d)	28. (c)	29. (a)	30. (a)
31. (a)	32. (b)	33. (b)	34. (a)	35. (a)	36. (a)	37. (c)	38. (c)	39. (a)	40. (c)
41. (a)	42. (a)	43. (c)	44. (c)	45. (c)	46. (d)	47. (b)	48. (c)	49. (b)	50. (b)
51. (a)	52. (c)	53. (b)	54. (d)	55. (a)	56. (b)	57. (a)	58. (b)	59. (a)	60. (a)
61. (a)	62. (b)	63. (c)	64. (b)	65. (b)	66. (c)	67. (b)	68. (a)	69. (a)	70. (c)
71. (d)	72. (c)	73. (b)	74. (a)	75. (b)	76. (a)	77. (c)	78. (a)	79. (a)	80. (d)
81. (b)	82. (a)	83. (b)	84. (c)	85. (b)	86. (c)	87. (c)	88. (d)	89. (b)	90. (b)
91. (b)	92. (b)	93. (c)	94. (a)	95. (d)	96. (b)	97. (a)	98. (b)	99. (a)	100. (d)
101. (a)	102. (d)	103. (d)	104. (b)	105. (c)	106. (a)	107. (b)	108. (a)	109. (c)	110. (b)
111. (b)	112. (d)	113. (c)	114. (b)	115. (c)	116. (b)	117. (a)	118. (a)	119. (b)	120. (c)
121. (a)	122. (a)	123. (b)	124. (a,b)	125. (b,c)	126. (c,d)	127. (d)	128. (a)	129. (c)	130. (c)
131. (c)	132. (d)	133. (c)	134. (*)	135. (**)	136. (1)	137. (1)	138. (6)	139. (0)	140. (1)

* $A \rightarrow q$; $B \rightarrow p$; $C \rightarrow r$; $D \rightarrow s$

** $A \rightarrow s$; $B \rightarrow q$; $C \rightarrow p$; $D \rightarrow r$

Entrances Gallery

1. (2)	2. (b,d)	3. (b)	4. (b)	5. (b)	6. (d)	7. (c)	8. (a)	9. (c)	10. (d)
11. (d)	12. (d)	13. (a)	14. (d)	15. (b)	16. (c)	17. (c)	18. (d)	19. (a)	20. (c)
21. (c)	22. (b)	23. (c)	24. (d)	25. (b)	26. (c)	27. (e)	28. (a)	29. (c)	30. (a)
31. (a)	32. (d)	33. (b)	34. (d)	35. (d)	36. (b)	37. (c)	38. (c)	39. (c)	40. (b)
41. (b)	42. (c)	43. (a)	44. (b)	45. (d)	46. (d)	47. (d)	48. (d)	49. (d)	50. (d)
51. (e)	52. (a)	53. (b)	54. (b)	55. (d)	56. (a)	57. (d)	58. (d)	59. (b)	60. (a)
61. (d)	62. (b)	63. (d)							

Explanations

Target Exercises

$$1. \text{RHL} = \lim_{x \rightarrow 0^+} \frac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}} = \lim_{h \rightarrow 0} \frac{\sqrt{h}}{\sqrt{4-\sqrt{h}}-\sqrt{h}}$$

$$= \frac{0}{2} = 0$$

The function is not defined for negative values of x , so LHL does not exist.

$$\therefore \lim_{x \rightarrow 0} \frac{\sqrt{x}}{\sqrt{4-\sqrt{x}}-\sqrt{x}} \text{ does not exist.}$$

$$2. \lim_{x \rightarrow 0} \left[\left(3x + \frac{1}{x} \right)^2 - \left(2x - \frac{1}{x} \right)^2 \right]$$

$$= \lim_{x \rightarrow 0} \left[\left(3x + \frac{1}{x} - 2x + \frac{1}{x} \right) \left(3x + \frac{1}{x} + 2x - \frac{1}{x} \right) \right]$$

$$= \lim_{x \rightarrow 0} \left[\left(x + \frac{2}{x} \right) 5x \right]$$

$$= \lim_{x \rightarrow 0} (5x^2 + 10) = 10$$

$$3. \lim_{x \rightarrow 1} \frac{x^7 - 2x^5 + 1}{x^3 - 3x^2 + 2} = \lim_{x \rightarrow 1} \frac{x^5(x^2 - 1) - (x^5 - 1)}{x^2(x - 1) - 2(x^2 - 1)}$$

$$= \lim_{x \rightarrow 1} \frac{x^5(x + 1) - (x^4 + x^3 + x^2 + x + 1)}{x^2 - 2(x + 1)}$$

$$= \frac{1(1 + 1) - (1 + 1 + 1 + 1 + 1)}{1 - 2(1 + 1)} = \frac{-3}{-3} = 1$$

$$4. \lim_{x \rightarrow 2} \frac{\sqrt[3]{10-x} - 2}{x - 2} = \lim_{x \rightarrow 2} \frac{(10-x)^{1/3} - 8^{1/3}}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{(10-x) - 8}{(10-x)^{2/3} + 8^{2/3} + 8^{1/3}(10-x)^{1/3}} \cdot \frac{1}{x - 2}$$

$$\left[\text{using } a - b = \frac{a^3 - b^3}{a^2 + b^2 + ab} \right]$$

$$= \frac{-1}{8^{2/3} + 8^{2/3} + 8^{2/3}} = -\frac{1}{12}$$

$$5. \lim_{x \rightarrow 1} \frac{\sqrt[3]{x^2} - 2\sqrt[3]{x} + 1}{(x - 1)^2}$$

$$= \lim_{y \rightarrow 1} \frac{y^2 - 2y + 1}{(y^3 - 1)^2} \quad [\text{putting } \sqrt[3]{x} = y; \text{ as } x \rightarrow 1, y \rightarrow 1]$$

$$= \lim_{y \rightarrow 1} \frac{(y - 1)^2}{(y^2 + y + 1)^2} = \lim_{y \rightarrow 1} \frac{1}{(y^2 + y + 1)^2} = \frac{1}{9}$$

$$6. \lim_{x \rightarrow 2a} \frac{\sqrt{x - 2a} + \sqrt{x} - \sqrt{2a}}{\sqrt{x^2 - 4a^2}}$$

$$= \lim_{x \rightarrow 2a} \frac{\sqrt{x - 2a}}{\sqrt{x^2 - 4a^2}} + \lim_{x \rightarrow 2a} \frac{\sqrt{x} - \sqrt{2a}}{\sqrt{x^2 - 4a^2}}$$

$$= \lim_{x \rightarrow 2a} \frac{\sqrt{x - 2a}}{\sqrt{x - 2a} \sqrt{x + 2a}} + \lim_{x \rightarrow 2a} \frac{(x - 2a)}{\sqrt{x} + \sqrt{2a}} \cdot \frac{1}{\sqrt{x - 2a} \sqrt{x + 2a}}$$

$$= \frac{1}{2\sqrt{a}} + \lim_{x \rightarrow 2a} \frac{1}{\sqrt{x} + \sqrt{2a}} \cdot \frac{1}{\sqrt{x + 2a}} = \frac{1}{2\sqrt{a}} + 0 = \frac{1}{2\sqrt{a}}$$

$$7. \lim_{x \rightarrow 2} \frac{\sqrt{1 + \sqrt{2 + x}} - \sqrt{3}}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{1 + \sqrt{2 + x} - 3}{(x - 2)[\sqrt{1 + \sqrt{2 + x}} + \sqrt{3}]}$$

$$= \lim_{x \rightarrow 2} \frac{x - 2}{(x - 2)[\sqrt{1 + \sqrt{2 + x}} + \sqrt{3}][\sqrt{2 + x} + 2]}$$

$$= \lim_{x \rightarrow 2} \frac{1}{[\sqrt{1 + \sqrt{2 + x}} + \sqrt{3}][\sqrt{2 + x} + 2]}$$

$$= \frac{1}{(\sqrt{1 + 2} + \sqrt{3})(\sqrt{2 + 2} + 2)} = \frac{1}{8\sqrt{3}}$$

$$8. \lim_{x \rightarrow -1} \frac{x + 1}{\sqrt{6x^2 + 3} + 3x}$$

$$= \lim_{x \rightarrow -1} \frac{(x + 1)}{3(1 - x^2)} [\sqrt{6x^2 + 3} - 3x]$$

$$= \lim_{x \rightarrow -1} \frac{1}{3(1 - x)} [\sqrt{6x^2 + 3} - 3x] = \frac{1}{3 \cdot 2} (3 + 3) = 1$$

$$9. \lim_{x \rightarrow 3} \left[\frac{x - 3}{\sqrt{x - 2} - \sqrt{4 - x}} \right]$$

$$= \lim_{x \rightarrow 3} \frac{(x - 3)[\sqrt{x - 2} + \sqrt{4 - x}]}{(x - 2) - (4 - x)}$$

$$= \lim_{x \rightarrow 3} \frac{(x - 3)[\sqrt{x - 2} + \sqrt{4 - x}]}{2(x - 3)}$$

$$= \lim_{x \rightarrow 3} \frac{\sqrt{x - 2} + \sqrt{4 - x}}{2} = \frac{\sqrt{3 - 2} + \sqrt{4 - 3}}{2} = \frac{1 + 1}{2} = 1$$

$$10. \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sqrt{a^2 - ax + x^2} - \sqrt{a^2 + ax + x^2}}{\sqrt{a + x} - \sqrt{a - x}}$$

$$= \lim_{x \rightarrow 0} \frac{(a^2 - ax + x^2) - (a^2 + ax + x^2)}{(a + x) - (a - x)}$$

$$\times \frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2}}$$

$$= \lim_{x \rightarrow 0} -\frac{2ax}{2x} \times \frac{\sqrt{a + x} + \sqrt{a - x}}{\sqrt{a^2 - ax + x^2} + \sqrt{a^2 + ax + x^2}}$$

$$= -a \cdot \frac{2\sqrt{a}}{2a} = -\sqrt{a}$$

$$11. \lim_{x \rightarrow 2} \frac{2 - \sqrt{2 + x}}{\sqrt[3]{2} - \sqrt[3]{4 - x}}$$

$$= \lim_{x \rightarrow 2} \frac{4 - (2 + x)}{2^{1/3} - (4 - x)^{1/3}} \times \frac{1}{2 + \sqrt{2 + x}}$$

$$= \lim_{x \rightarrow 2} \frac{2 - x}{2 + \sqrt{2 + x}} \cdot \frac{2^{2/3} + 2^{1/3} \cdot (4 - x)^{1/3} + (4 - x)^{2/3}}{2 - (4 - x)}$$

$$= \lim_{x \rightarrow 2} \frac{-1}{2 + \sqrt{2 + x}} \cdot [2^{2/3} + 2^{1/3} \cdot (4 - x)^{1/3} + (4 - x)^{2/3}]$$

$$= -\frac{1}{4} \cdot (2^{2/3} + 2^{2/3} + 2^{2/3}) = \frac{-3 \cdot 2^{2/3}}{4} = \frac{-3}{2^{4/3}}$$

$$12. \lim_{n \rightarrow \infty} (1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n})$$

$$= \lim_{n \rightarrow \infty} \frac{(1-x)(1+x)(1+x^2)(1+x^4) \dots (1+x^{2^n})}{1-x}$$

$$= \lim_{n \rightarrow \infty} \frac{(1-x^2)(1+x^2)(1+x^4) \dots (1+x^{2^n})}{1-x}$$

$$= \lim_{n \rightarrow \infty} \frac{1-x^{4^n}}{1-x} = \frac{1}{1-x}, \text{ for } |x| < 1$$

$$13. \lim_{n \rightarrow \infty} [\sqrt[3]{n^2 - n^3} + n] = \lim_{n \rightarrow \infty} n \left[\left(-1 + \frac{1}{n} \right)^{1/3} + 1 \right]$$

$$= \lim_{n \rightarrow \infty} n \cdot \frac{\left(\frac{1}{n} - 1 \right) + 1}{\left(\frac{1}{n} - 1 \right)^{2/3} + 1 - \left(\frac{1}{n} - 1 \right)^{1/3}}$$

$$\left[\text{using } a + b = \frac{a^3 + b^3}{a^2 - ab + b^2} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{1}{n} - 1 \right)^{2/3} + 1 - \left(\frac{1}{n} - 1 \right)^{1/3}}$$

$$= \frac{1}{1+1+1} = \frac{1}{3}$$

$$14. \lim_{n \rightarrow \infty} \left[\frac{\sqrt{n^2 + 1} + \sqrt{n}}{\sqrt[4]{n^3 + n} - \sqrt[4]{n}} \right] = \lim_{n \rightarrow \infty} \frac{n \left[\sqrt{1 + \frac{1}{n^2}} + \frac{1}{\sqrt{n}} \right]}{n^{3/4} \left[\sqrt[4]{1 + \frac{1}{n^2}} - \frac{1}{\sqrt{n}} \right]}$$

$$= \lim_{n \rightarrow \infty} n^{1/4} \left[\frac{\sqrt{1 + \frac{1}{n^2}} + \frac{1}{\sqrt{n}}}{\sqrt[4]{1 + \frac{1}{n^2}} - \frac{1}{\sqrt{n}}} \right] = \infty \cdot \frac{1}{1} = \infty$$

$$15. \lim_{x \rightarrow \infty} \sqrt{\frac{x - \sin x}{x + \cos^2 x}} = \lim_{x \rightarrow \infty} \sqrt{\frac{1 - \frac{\sin x}{x}}{1 + \frac{\cos^2 x}{x}}} = \sqrt{\frac{1-0}{1+0}} = 1$$

$$\left[\begin{array}{l} \because \lim_{x \rightarrow \infty} \frac{\sin x}{x} = \lim_{y \rightarrow 0} y \sin \left(\frac{1}{y} \right) = 0 \\ \times (\text{an oscillating quantity between } -1 \text{ and } 1) = 0 \\ \text{Similarly, } \lim_{x \rightarrow \infty} \frac{\cos^2 x}{x} = 0 \end{array} \right]$$

$$16. \lim_{x \rightarrow \infty} (\sin \sqrt{x+1} - \sin \sqrt{x})$$

$$= \lim_{x \rightarrow \infty} 2 \cos \frac{\sqrt{x+1} + \sqrt{x}}{2} \cdot \sin \frac{\sqrt{x+1} - \sqrt{x}}{2}$$

$$= \lim_{x \rightarrow \infty} 2 \cos \frac{\sqrt{x+1} + \sqrt{x}}{2} \cdot \sin \frac{1}{2(\sqrt{x+1} + \sqrt{x})}$$

$$= 2 \times (\text{an oscillating quantity between } -1 \text{ and } 1) \times \sin 0 = 0$$

$$17. \lim_{h \rightarrow 0} \frac{\sin \sqrt{x+h} - \sin \sqrt{x}}{h} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{h \rightarrow 0} \frac{\frac{\cos \sqrt{x+h}}{2\sqrt{x+h}} - 0}{1} = \frac{\cos \sqrt{x}}{2\sqrt{x}}$$

$$18. \lim_{x \rightarrow 0} \frac{(1 - \cos 2x) \sin 5x}{x^2 \sin 3x} = \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \cdot \frac{\sin 5x}{\sin 3x}$$

$$= \lim_{x \rightarrow 0} 2 \left(\frac{\sin x}{x} \right)^2 \cdot \frac{\frac{\sin 5x}{5x} \cdot 5}{\frac{\sin 3x}{3x} \cdot 3} = \frac{10}{3}$$

$$19. \lim_{x \rightarrow \pi/3} \frac{\sin \left(\frac{\pi}{3} - x \right)}{2 \cos x - 1} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow \pi/3} \frac{-\cos \left(\frac{\pi}{3} - x \right)}{-2 \sin x} \quad [\text{using L'Hospital's rule}]$$

$$= \frac{\cos 0}{2 \sin \frac{\pi}{3}} = \frac{1}{\sqrt{3}}$$

$$20. \lim_{h \rightarrow 0} \frac{2 \left[\sqrt{3} \sin \left(\frac{\pi}{6} + h \right) - \cos \left(\frac{\pi}{6} + h \right) \right]}{\sqrt{3}h (\sqrt{3} \cos h - \sin h)}$$

$$= \lim_{h \rightarrow 0} \frac{2 \left[\sqrt{3} \left(\frac{1}{2} \cos h + \frac{\sqrt{3}}{2} \sin h \right) - \left(\frac{\sqrt{3}}{2} \cos h - \frac{1}{2} \sin h \right) \right]}{\sqrt{3}h (\sqrt{3} \cos h - \sin h)}$$

$$= \lim_{h \rightarrow 0} \frac{2 [2 \sin h]}{\sqrt{3}h (\sqrt{3} \cos h - \sin h)} = \lim_{h \rightarrow 0} \frac{4 \cdot \frac{\sin h}{h}}{\sqrt{3} (\sqrt{3} \cos h - \sin h)} = \frac{4}{\sqrt{3} (\sqrt{3} - 0)} = \frac{4}{3}$$

$$21. \lim_{x \rightarrow \infty} \sqrt{\frac{x + \sin x}{x - \cos x}} = \lim_{x \rightarrow \infty} \sqrt{\frac{1 + \frac{\sin x}{x}}{1 - \frac{\cos x}{x}}} = \sqrt{\frac{1+0}{1-0}} = 1$$

$$\left[\because \frac{\sin x}{x} \rightarrow 0, \frac{\cos x}{x} \rightarrow 0 \text{ as } x \rightarrow \infty \right]$$

$$22. \text{Case I } x \neq m\pi \quad [m \text{ is an integer}]$$

$$\lim_{n \rightarrow \infty} \frac{1}{1 + n \sin^2 nx} = \frac{1}{\infty} = 0$$

$$\text{Case II } x = m\pi \quad [m \text{ is an integer}]$$

$$\lim_{n \rightarrow \infty} \frac{1}{1 + n \sin^2 nx} = \frac{1}{1} = 1$$

$$23. \lim_{x \rightarrow 0^+} \log_{\tan x} (\tan 2x) = \lim_{x \rightarrow 0^+} \frac{\log (\tan 2x)}{\log (\tan x)}$$

$$\left[\text{using } \log_n m = \frac{\log m}{\log n} \right]$$

$$= \lim_{x \rightarrow 0^+} \frac{\frac{1}{\tan 2x} (2 \sec^2 2x)}{\frac{1}{\tan x} (\sec^2 x)} \quad [\text{using L'Hospital's rule}]$$

$$= \lim_{x \rightarrow 0^+} \frac{2 \sin x \cos x}{\sin 2x \cos 2x} = \lim_{x \rightarrow 0^+} \frac{2 \sin 2x}{\sin 4x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0^+} \frac{4 \cos 2x}{4 \cos 4x} \quad [\text{using L'Hospital's rule}]$$

$$= \lim_{h \rightarrow 0} \frac{\cos 2h}{\cos 4h} = 1$$

24. As $x \rightarrow 0^- \Rightarrow f(x) \rightarrow f(0^-) = 2^+$

$$\Rightarrow \lim_{x \rightarrow 0^-} g[f(x)] = g(2^+) = -3$$

Also, as $x \rightarrow 0^+ \Rightarrow f(x) \rightarrow f(0^+) = 1^+$

$$\Rightarrow \lim_{x \rightarrow 0^+} g[f(x)] = g(1^+) = -3$$

Hence, $\lim_{x \rightarrow 0} g[f(x)]$ exists and is equal to -3 .

25. $\lim_{n \rightarrow \infty} \left[\frac{2n}{2n^2 - 1} \cos \frac{n+1}{2n-1} - \frac{n}{1-2n} \cdot \frac{n(-1)^n}{n^2+1} \right]$

$$= \lim_{n \rightarrow \infty} \left[\frac{2}{2 - \frac{1}{n^2}} \cdot \frac{1}{n} \cos \left(\frac{1 + 1/n}{2 - 1/n} \right) - \frac{1}{\left(\frac{1}{n} - 2 \right)} \cdot \frac{(-1)^n}{\left(1 + \frac{1}{n^2} \right)} \cdot \frac{1}{n} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[\frac{2}{2 - \frac{1}{n^2}} \cdot \cos \left(\frac{1 + \frac{1}{n}}{2 - \frac{1}{n}} \right) - \frac{1}{\left(\frac{1}{n} - 2 \right)} \cdot \frac{(-1)^n}{\left(1 + \frac{1}{n^2} \right)} \right]$$

$$= 0 \times \left[\frac{2}{2} \times \cos \frac{1}{2} + \frac{1}{2} \times \frac{(\text{either } 1 \text{ or } -1)}{1} \right] = 0$$

26. Let $k = \lim_{x \rightarrow 0} \frac{\sin 2x + a \sin x}{x^3}$ $\left[\frac{0}{0} \text{ form} \right]$

$$= \lim_{x \rightarrow 0} \frac{2 \cos 2x + a \cos x}{3x^2} \text{ [using L' Hospital's rule]}$$

We require $2 \cos 2x + a \cos x = 0$ for $x = 0$ as denominator is zero.

$$\therefore a = -2$$

Hence, $k = \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2 \cos x}{3x^2}$ $\left[\frac{0}{0} \text{ form} \right]$

$$= \lim_{x \rightarrow 0} \frac{-4 \sin 2x + 2 \sin x}{6x}$$

$$= \lim_{x \rightarrow 0} \frac{-8 \cos 2x + 2 \cos x}{6} = \frac{-8 + 2}{6} = -1$$

27. $\lim_{h \rightarrow 0} \frac{\sin(a+3h) - 3 \sin(a+2h) + 3 \sin(a+h) - \sin a}{h^3}$ $\left[\frac{0}{0} \text{ form} \right]$

$$= \lim_{h \rightarrow 0} \frac{3 \cos(a+3h) - 6 \cos(a+2h) + 3 \cos(a+h)}{3h^2}$$

$$= \lim_{h \rightarrow 0} \frac{-9 \sin(a+3h) + 12 \sin(a+2h) - 3 \sin(a+h)}{6h}$$

$$= \lim_{h \rightarrow 0} \frac{-27 \cos(a+3h) + 24 \cos(a+2h) - 3 \cos(a+h)}{6}$$

$$= \lim_{h \rightarrow 0} \frac{-27 \cos a + 24 \cos a - 3 \cos a}{6} = -\cos a$$

28. $\lim_{x \rightarrow a} \sqrt{a^2 - x^2} \cot \frac{\pi}{2} \sqrt{\frac{a-x}{a+x}}$ $[0 \cdot \infty \text{ form}]$

$$= \lim_{x \rightarrow a} \frac{\sqrt{a^2 - x^2}}{\tan \frac{\pi}{2} \sqrt{\frac{a-x}{a+x}}} \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow a} \frac{\frac{-2x}{2\sqrt{a^2 - x^2}}}{-\sec^2 \frac{\pi}{2} \sqrt{\frac{a-x}{a+x}} \times \frac{\pi}{2} \times \frac{2a}{2(a+x)\sqrt{a^2 - x^2}}}$$

$$= \frac{4a}{\pi}$$

29. $\lim_{x \rightarrow 0} \left[\operatorname{cosec}^3 x \cot x - 2 \cot^3 x \operatorname{cosec} x + \frac{\cot^4 x}{\sec x} \right]$

$$= \lim_{x \rightarrow 0} \left(\frac{\cos x}{\sin^4 x} - \frac{2 \cos^3 x}{\sin^4 x} + \frac{\cos^5 x}{\sin^4 x} \right)$$

$$= \lim_{x \rightarrow 0} \frac{\cos x (1 - \cos^2 x)^2}{\sin^4 x}$$

$$= \lim_{x \rightarrow 0} \cos x = 1$$

30. $\lim_{x \rightarrow \pi/4} \frac{4\sqrt{2} - (\cos x + \sin x)^5}{1 - \sin 2x}$

$$= \lim_{x \rightarrow \pi/4} \frac{4\sqrt{2} - 4\sqrt{2} \cos^5 \left(x - \frac{\pi}{4} \right)}{1 - \sin 2x}$$

$$= \lim_{y \rightarrow 0} \frac{4\sqrt{2} (1 - \cos^5 y)}{1 - \cos 2y}$$

$$\left[\text{putting } x - \frac{\pi}{4} = y; \text{ as } x \rightarrow \frac{\pi}{4}, y \rightarrow 0 \right]$$

$$= \lim_{y \rightarrow 0} \frac{4\sqrt{2} (1 - \cos^5 y)}{2(1 - \cos^2 y)}$$

$$= \lim_{y \rightarrow 0} \frac{2\sqrt{2} (1 - \cos y)(1 + \cos y + \cos^2 y + \cos^3 y + \cos^4 y)}{(1 - \cos y)(1 + \cos y)}$$

$$= \frac{2\sqrt{2} \cdot 5}{2} = 5\sqrt{2}$$

31. $\lim_{n \rightarrow \infty} \cos [\pi \sqrt{n^2 + n}] = \lim_{n \rightarrow \infty} \cos \left[n\pi \left(1 + \frac{1}{n} \right)^{1/2} \right]$

$$= \lim_{n \rightarrow \infty} \cos \left[n\pi \left(1 + \frac{1}{2n} - \frac{1}{8n^2} + \dots \right) \right]$$

$$= \lim_{n \rightarrow \infty} \cos \left(n\pi + \frac{\pi}{2} - \frac{\pi}{8n} + \dots \right)$$

$$= - \lim_{n \rightarrow \infty} \sin \left(n\pi - \frac{\pi}{8n} + \dots \right)$$

$$= - \lim_{n \rightarrow \infty} (-1)^{n-1} \sin \left(\frac{\pi}{8n} - \dots \right)$$

$$= 0 \quad \left[\because \frac{\pi}{8n} - \dots \rightarrow 0, \text{ as } n \rightarrow \infty \right]$$

32. $\lim_{x \rightarrow \infty} \frac{(3x-1)(2x+5)}{(x-3)(3x+7)} = \lim_{x \rightarrow \infty} \frac{\left(3 - \frac{1}{x} \right) \left(2 + \frac{5}{x} \right)}{\left(1 - \frac{3}{x} \right) \left(3 + \frac{7}{x} \right)}$

$$= \frac{(3-0)(2+0)}{(1-0)(3+0)} = \frac{6}{3} = 2$$

33. $\lim_{x \rightarrow \infty} \frac{\sqrt{1+x^4} - (1+x^2)}{x^2} = \lim_{x \rightarrow \infty} \left[\sqrt{\frac{1}{x^4} + 1} - \left(\frac{1}{x^2} + 1 \right) \right] = 0$

$$34. \lim_{x \rightarrow \infty} \frac{2x^3 - 4x + 7}{3x^3 + 5x^2 - 4} = \lim_{x \rightarrow \infty} \frac{2 - \frac{4}{x^2} + \frac{7}{x^3}}{3 + \frac{5}{x} - \frac{4}{x^3}} = \frac{2 - 0 + 0}{3 + 0 - 0} = \frac{2}{3}$$

$$35. \lim_{x \rightarrow 0} \frac{e^x - (1+x)}{x^2} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$36. \lim_{x \rightarrow e} \frac{\log x - 1}{x - e} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow e} \frac{(1/x)}{1} \quad [\text{using L' Hospital's rule}]$$

$$= \frac{1}{e}$$

$$37. \lim_{x \rightarrow 1} \frac{\log_e x}{x - 1} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 1} \frac{1/x}{1} = 1 \quad [\text{using L' Hospital's rule}]$$

$$38. \lim_{x \rightarrow 0} \frac{e^{x^2} - \cos x}{x^2} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^{x^2} \cdot 2x + \sin x}{2x} \quad [\text{using L' Hospital's rule}]$$

$$= \lim_{x \rightarrow 0} \left[e^{x^2} + \frac{1}{2} \frac{\sin x}{x} \right] = e^0 + \frac{1}{2}(1) = 1 + \frac{1}{2} = \frac{3}{2}$$

$$39. \lim_{x \rightarrow 0} \frac{\log \cos x}{x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\cos x} (-\sin x)}{1} \quad [\text{using L' Hospital's rule}]$$

$$= -\lim_{x \rightarrow 0} \tan x = 0$$

$$40. \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \quad [\text{using L' Hospital's rule}]$$

$$= \lim_{x \rightarrow 0} \frac{e^x - e^{-x}}{\sin x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x + e^{-x}}{\cos x} = 2$$

$$41. \lim_{x \rightarrow 3} \left[\log_a \frac{x-3}{\sqrt{x+6}-3} \right]$$

$$= \lim_{x \rightarrow 3} \left[\log_a \frac{(x-3)(\sqrt{x+6}+3)}{(x-3)} \right]$$

$$= \lim_{x \rightarrow 3} \log_a (\sqrt{x+6}+3) = \log_a 6$$

$$42. \lim_{x \rightarrow \infty} \frac{3^{x+1} - 5^{x+1}}{3^x - 5^x} = \lim_{x \rightarrow \infty} \frac{3 \cdot 3^x - 5 \cdot 5^x}{3^x - 5^x}$$

$$= \lim_{x \rightarrow \infty} \frac{3 \cdot \left(\frac{3}{5}\right)^x - 5}{\left(\frac{3}{5}\right)^x - 1} = \frac{-5}{-1} = 5$$

$$43. \lim_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x - 4}{x^4} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2 \sin x}{4x^3} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x + e^{-x} - 2 \cos x}{12x^2} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x - e^{-x} + 2 \sin x}{24x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^x + e^{-x} + 2 \cos x}{24} = \frac{4}{24} = \frac{1}{6}$$

$$44. \lim_{x \rightarrow 0} \log_e (\sin x)^{\tan x} = \lim_{x \rightarrow 0} \tan x \cdot \log_e \sin x \quad [0 \cdot \infty \text{ form}]$$

$$= \lim_{x \rightarrow 0} \frac{\log_e \sin x}{\cot x} \quad \left[\frac{\infty}{\infty} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\cot x}{-\operatorname{cosec}^2 x} \quad [\text{using L' Hospital's rule}]$$

$$= \lim_{x \rightarrow 0} (-\cos x \cdot \sin x) = 0$$

$$45. \lim_{x \rightarrow \infty} 3^x \sin \left(\frac{4}{3^x} \right) = \lim_{x \rightarrow \infty} \frac{\sin \left(\frac{4}{3^x} \right)}{\left(\frac{4}{3^x} \right)} \cdot 4 = 1 \cdot 4 = 4$$

$$46. \lim_{x \rightarrow \infty} \left(\frac{1+3x}{2+3x} \right)^{\frac{1-\sqrt{x}}{1-x}} = \lim_{x \rightarrow \infty} \left(\frac{\frac{1}{x}+3}{\frac{2}{x}+3} \right)^{\left(\frac{\frac{1}{x}-\frac{1}{\sqrt{x}}}{\frac{1}{x}-1} \right)}$$

$$= \left(\frac{3}{3} \right)^{0/-1} = 1^0 = 1$$

$$47. \sum_{r=1}^n \frac{1}{2^r} = \frac{1}{2} \left[1 - \left(\frac{1}{2} \right)^n \right] = 1 - \left(\frac{1}{2} \right)^n$$

which tends to 1 as $n \rightarrow \infty$ but infact always remains less than 1.

Thus, $\lim_{n \rightarrow \infty} \left[\sum_{r=1}^n \frac{1}{2^r} \right] = 0$

$$48. \text{ Given, } f(x) = x^2 - \pi^2$$

$$\therefore \lim_{x \rightarrow -\pi} \frac{x^2 - \pi^2}{\sin(\sin x)} = \lim_{h \rightarrow 0} \frac{(-\pi + h)^2 - \pi^2}{\sin[\sin(-\pi + h)]}$$

$$= \lim_{h \rightarrow 0} \frac{-2h\pi + h^2}{-\sin(\sin h)}$$

$$= \lim_{h \rightarrow 0} \frac{h - 2\pi}{-\sin(\sin h) \times \frac{\sin h}{h}} = 2\pi$$

$$49. \lim_{x \rightarrow 0} \left[\tan \left(\frac{\pi}{4} + x \right) \right]^{1/x} = \lim_{x \rightarrow 0} \left[\frac{1 + \tan x}{1 - \tan x} \right]^{1/x}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{1 + \tan x}{1 - \tan x} - 1 \right]}$$

$$\left[\because \frac{1 + \tan x}{1 - \tan x} \rightarrow 1 \text{ and } \frac{1}{x} \rightarrow \infty, \text{ as } x \rightarrow 0 \right]$$

$$= e^{\lim_{x \rightarrow 0} \frac{2 \tan x}{x(1 - \tan x)}} = e^{\lim_{x \rightarrow 0} 2 \left(\frac{\tan x}{x} \right) \frac{1}{1 - \tan x}} = e^2$$

$$\begin{aligned}
 50. \lim_{x \rightarrow \alpha} (1 + ax^2 + bx + c)^{1/(x-\alpha)} &= e^{\lim_{x \rightarrow \alpha} \frac{1}{(x-\alpha)} [(1 + ax^2 + bx + c) - 1]} \\
 &= e^{\lim_{x \rightarrow \alpha} \frac{1}{(x-\alpha)} [(1 + ax^2 + bx + c) - 1]} \\
 &\left[\text{using } \lim_{x \rightarrow a} [f(x)]^{g(x)} = e^{\lim_{x \rightarrow a} g(x) [f(x) - 1]} \text{ provided } \right. \\
 &\quad \left. f(x) \rightarrow 1 \text{ and } g(x) \rightarrow \infty, \text{ as } x \rightarrow a \right] \\
 &= e^{\lim_{x \rightarrow \alpha} \frac{(ax^2 + bx + c)}{(x-\alpha)}} = e^{\lim_{x \rightarrow \alpha} \frac{a(x-\alpha)(x-\beta)}{(x-\alpha)}} \\
 &= e^{a(\alpha-\beta)} \quad [\because \alpha, \beta \text{ are roots of } ax^2 + bx + c = 0]
 \end{aligned}$$

$$\begin{aligned}
 51. \lim_{x \rightarrow 0^+} \left[\frac{\sin \{\operatorname{sgn}(x)\}}{\operatorname{sgn}(x)} \right] &= \lim_{x \rightarrow 0^+} \left[\frac{\sin 1}{1} \right] = 0 \\
 \text{and } \lim_{x \rightarrow 0^-} \left[\frac{\sin \{\operatorname{sgn}(x)\}}{\operatorname{sgn}(x)} \right] &= \lim_{x \rightarrow 0^-} \left[\frac{\sin(-1)}{-1} \right] \\
 &= \lim_{x \rightarrow 0^-} [\sin 1] = 0
 \end{aligned}$$

Hence, the given limit is 0.

$$\begin{aligned}
 52. \lim_{x \rightarrow 0} (\cos x + a \sin bx)^x &= e^{\lim_{x \rightarrow 0} \frac{a}{x} (\cos x + a \sin bx - 1)} \\
 &\left[\text{using } \lim_{x \rightarrow a} [f(x)]^{\phi(x)} = e^{\lim_{x \rightarrow a} \phi(x) [f(x) - 1]} \right. \\
 &\quad \left. \text{where } f(x) \rightarrow 1 \text{ and } \phi(x) \rightarrow \infty, \text{ as } x \rightarrow a \right] \\
 &= e^{\lim_{x \rightarrow 0} \frac{a(-\sin x + ab \cos bx)}{1}} = e^{a^2 b}
 \end{aligned}$$

$$\begin{aligned}
 53. \lim_{x \rightarrow \infty} \left(\frac{2x^2 + 3}{2x^2 + 5} \right)^{8x^2 + 3} &= \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{3}{2x^2} \right)^{8x^2 + 3}}{\left(1 + \frac{5}{2x^2} \right)^{8x^2 + 3}} \\
 &= \lim_{x \rightarrow \infty} \frac{\left(1 + \frac{3}{2x^2} \right)^{8x^2} \times \left(1 + \frac{3}{2x^2} \right)^3}{\left(1 + \frac{5}{2x^2} \right)^{8x^2} \times \left(1 + \frac{5}{2x^2} \right)^3} \\
 &= \lim_{x \rightarrow \infty} \frac{\left[\left(1 + \frac{3}{2x^2} \right)^{\frac{2x^2}{3}} \right]^{12} \times \left(1 + \frac{3}{2x^2} \right)^3}{\left[\left(1 + \frac{5}{2x^2} \right)^{\frac{2x^2}{5}} \right]^{20} \times \left(1 + \frac{5}{2x^2} \right)^3} = \frac{e^{12}}{e^{20}} \times \frac{1}{1} = e^{-8}
 \end{aligned}$$

$$\begin{aligned}
 54. \text{The given limit is } \lim_{x \rightarrow \infty} \frac{\frac{2}{x} + 2 + \frac{\sin 2x}{x}}{\left(2 + \frac{\sin 2x}{x} \right) e^{\sin x}} &= \frac{0 + 2 + 0}{(2 + 0) \times (a \text{ value between } \frac{1}{e} \text{ and } e)} \\
 &= \frac{2}{2 \times (a \text{ value between } \frac{1}{e} \text{ and } e)} \\
 &\left[\because \lim_{x \rightarrow \infty} \sin x \in (-1, 1) \right]
 \end{aligned}$$

Hence, limit does not exist.

$$\begin{aligned}
 55. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{a + bx} \right)^{c + dx} &= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{a + bx} \right)^{a + bx} \right]^{\frac{c + dx}{a + bx}} \\
 &= e^{\lim_{x \rightarrow \infty} \left(\frac{c + dx}{a + bx} \right)} = e^{\lim_{x \rightarrow \infty} \left(\frac{\frac{c}{x} + d}{\frac{a}{x} + b} \right)} = e^{d/b}
 \end{aligned}$$

$$\begin{aligned}
 56. \lim_{x \rightarrow \infty} \left(\frac{x^2 - 2x + 2}{x^2 - 4x + 1} \right)^x &= \lim_{x \rightarrow \infty} \left(1 + \frac{2x + 1}{x^2 - 4x + 1} \right)^x = \lim_{x \rightarrow \infty} \left(1 + \frac{1}{\frac{x^2 - 4x + 1}{2x + 1}} \right)^x \\
 &= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{\frac{x^2 - 4x + 1}{2x + 1}} \right)^{\frac{x(2x + 1)}{x^2 - 4x + 1}} \right]^{\frac{x(2x + 1)}{x^2 - 4x + 1}} \\
 &= e^{\lim_{x \rightarrow \infty} \frac{x(2x + 1)}{x^2 - 4x + 1}} = e^{\lim_{x \rightarrow \infty} \left(\frac{2 + \frac{1}{x}}{1 - \frac{4}{x} + \frac{1}{x^2}} \right)} = e^2
 \end{aligned}$$

$$\begin{aligned}
 57. \lim_{x \rightarrow \infty} \left(\frac{3x^2 + 1}{4x^2 - 1} \right)^{\frac{x^3}{1+x}} &= \lim_{x \rightarrow \infty} \left(\frac{3}{4} \right)^{\frac{x^3}{1+x}} \cdot \left(\frac{x^2 + \frac{1}{3}}{x^2 - \frac{1}{4}} \right)^{\frac{x^3}{1+x}} \\
 &= \lim_{x \rightarrow \infty} \left(\frac{3}{4} \right)^{\frac{x^2}{x} + 1} \cdot \left[\left(1 + \frac{\frac{7}{12}}{x^2 - \frac{1}{4}} \right)^{x^2 - \frac{1}{4}} \right]^{\frac{x^3}{1+x} \cdot \frac{4}{4x^2 - 1}} \\
 &= 0 \cdot \left(\frac{7}{e^{12}} \right)^{\lim_{x \rightarrow \infty} \frac{4}{\left(1 + \frac{1}{x} \right) \left(4 - \frac{1}{x^2} \right)}} = 0 \cdot (e^{7/12})^1 = 0
 \end{aligned}$$

$$\begin{aligned}
 58. \text{Let } y &= \lim_{x \rightarrow 0} x^x \\
 \Rightarrow \log y &= \lim_{x \rightarrow 0} x \log x = \lim_{x \rightarrow 0} \frac{\log x}{1/x} \quad \left[\frac{\infty}{\infty} \text{ form} \right] \\
 &= \lim_{x \rightarrow 0} \frac{1/x}{-1/x^2} = - \lim_{x \rightarrow 0} x = 0 \\
 \Rightarrow y &= e^0 = 1
 \end{aligned}$$

$$\begin{aligned}
 59. \lim_{x \rightarrow 0} (\cos x + \sin x)^x &= e^{\lim_{x \rightarrow 0} \frac{1}{x} (\cos x + \sin x - 1)} \\
 &= e^{\lim_{x \rightarrow 0} \frac{(-\sin x + \cos x)}{1}} \quad [\text{using L' Hospital's rule}] \\
 &= e^1 = e
 \end{aligned}$$

$$\begin{aligned}
 60. \lim_{n \rightarrow \infty} \left(1 + \tan \frac{b}{n} \right)^n &= e^{\lim_{n \rightarrow \infty} n \left(1 + \tan \frac{b}{n} - 1 \right)} \\
 &= e^{\lim_{n \rightarrow \infty} n \tan \frac{b}{n}} = e^{\lim_{n \rightarrow \infty} \frac{\tan b/n}{b/n} \times b} = e^b
 \end{aligned}$$

$$61. \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x^n}\right)^x = \lim_{x \rightarrow \infty} \left[\left(1 + \frac{1}{x^n}\right)^{x^n}\right]^{x^{1-n}}$$

$$= e^{\lim_{x \rightarrow \infty} x^{1-n}} = \begin{cases} e^0 = 1, & \text{if } n > 1 \\ e^\infty = \infty, & \text{if } n < 1 \\ e^1 = e, & \text{if } n = 1 \end{cases}$$

$$62. \lim_{x \rightarrow -1} \left(\frac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\frac{1 - \cos(x+1)}{(x+1)^2}}$$

$$= \lim_{x \rightarrow -1} \left(\frac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\frac{2 \sin^2\left(\frac{x+1}{2}\right)}{(x+1)^2}}$$

$$= \lim_{x \rightarrow -1} \left(\frac{x^4 + x^2 + x + 1}{x^2 - x + 1}\right)^{\frac{1}{2} \left[\frac{\sin\left(\frac{x+1}{2}\right)}{\left(\frac{x+1}{2}\right)}\right]^2} = \left(\frac{2}{3}\right)^{1/2}$$

$$63. \lim_{n \rightarrow \infty} \left(\cos \frac{x}{n}\right)^n = e^{\lim_{n \rightarrow \infty} n \left(\cos \frac{x}{n} - 1\right)}$$

$$= e^{\lim_{n \rightarrow \infty} -2n \cdot \sin^2\left(\frac{x}{2n}\right)}$$

$$= e^{-2 \lim_{n \rightarrow \infty} \left(\frac{\sin(x/2n)}{x/2n}\right)^2 \cdot \frac{x^2}{4n^2}}$$

$$= e^{-2 \times \lim_{n \rightarrow \infty} \frac{x^2}{4n}} = e^0 = 1$$

$$64. \frac{[x]^2}{x^2} = \begin{cases} 0, & \text{if } 0 < x < 1 \\ \frac{1}{x^2}, & \text{if } -1 < x < 0 \end{cases}$$

$\Rightarrow l$ does not exist.

$$\frac{[x]^2}{x^2} = \begin{cases} 0, & \text{if } 0 < x < 1 \\ 0, & \text{if } -1 < x < 0 \end{cases}$$

$\Rightarrow m$ exists and is equal to 0.

$$65. \text{ Let } \sin x = h, \text{ then as } x \rightarrow \frac{\pi}{2}, h \rightarrow 1$$

$$\therefore \lim_{h \rightarrow 1} \frac{h - h^h}{1 - h + \ln h}$$

$$= \lim_{h \rightarrow 1} \frac{1 - h^h - h^h \ln h}{-1 + \frac{1}{h}} \quad [\text{using L'Hospital's rule}]$$

$$= \lim_{h \rightarrow 1} \frac{-h^h - 2h^h \ln h - h^{h-1} - h^h (\ln h)^2}{-1/h^2} = \frac{-1-1}{-1} = 2$$

$$66. \lim_{x \rightarrow 0} \frac{\sin x^n}{(\sin x)^m} = \lim_{x \rightarrow 0} \left(\frac{\sin x^n}{x^n}\right) \cdot x^{n-m} \cdot \left(\frac{x}{\sin x}\right)^m$$

$$= \begin{cases} 1, & \text{if } n = m \\ 0, & \text{if } n > m \\ \infty, & \text{if } n < m \end{cases}$$

$$67. \lim_{x \rightarrow \infty} \frac{(\log x)^2}{x^n}$$

$\left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x \rightarrow \infty} \frac{2 \log x \cdot \frac{1}{x}}{n x^{n-1}} = \lim_{x \rightarrow \infty} \frac{2 \log x}{n x^n}$$

$\left[\frac{\infty}{\infty} \text{ form}\right]$

$$= \lim_{x \rightarrow \infty} \frac{2}{n^2 x^n} = 0$$

$$68. \lim_{x \rightarrow 2} \frac{2^x - x^2}{x^x - 2^2}$$

$\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \rightarrow 2} \frac{2^x \log 2 - 2x}{x^x (1 + \log x)} \\ = \frac{4 \log 2 - 4}{4(1 + \log 2)} \\ = \frac{\log 2 - 1}{\log 2 + 1}$$

$$69. \text{ Clearly, for } n = 1, \lim_{x \rightarrow 0} \frac{x - \sin x}{x - \sin x} = 1$$

$$70. \therefore \frac{\pi}{4} < 1$$

$$\therefore \left[\frac{\pi}{4}\right] = 0$$

$$\therefore \lim_{x \rightarrow \pi/2} \frac{\left[\frac{x}{2}\right]}{\ln(\sin x)} = 0$$

$$71. \text{ We know that, } |\cos \theta| \leq 1 \forall \theta.$$

So, if $|\cos n! \pi x| < 1$, when $x \in \text{irrational}$

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (1 + \cos^{2m} n! \pi x) = (1 + 0) = 1 \text{ and if}$$

$|\cos n! \pi x| = 1, x \in \text{rational}$

$$\lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (1 + \cos^{2m} n! \pi x) = \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (1 + 1^{2m})$$

$$= \lim_{m \rightarrow \infty} \lim_{n \rightarrow \infty} (1 + 1) = 2$$

$$72. \text{ LHL} = \lim_{x \rightarrow 0^-} \left[\frac{\sin([x-3])}{[x-3]}\right] = \left[\frac{\sin(-4)}{-4}\right]$$

$$= \left[\frac{\sin 4}{4}\right] = -1 \quad \left[\because \pi < 4 < \frac{3\pi}{2}\right]$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \left[\frac{\sin[x-3]}{[x-3]}\right]$$

$$= \left[\frac{\sin(-3)}{-3}\right]$$

$$= \left[\frac{\sin 3}{3}\right] = 0 \quad \left[\because \frac{\pi}{2} < 3 < \pi\right]$$

$$\therefore \text{LHL} \neq \text{RHL}$$

$$\therefore \lim_{x \rightarrow 0} \left[\frac{\sin([x-3])}{[x-3]}\right] \text{ does not exist.}$$

$$73. \lim_{x \rightarrow 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

$[0 \cdot \infty \text{ form}]$

$$= \lim_{x \rightarrow 1} \frac{(1-x)}{\cot\left(\frac{\pi x}{2}\right)}$$

$$= \lim_{x \rightarrow 1} \frac{-1}{-\operatorname{cosec}^2\left(\frac{\pi x}{2}\right) \frac{\pi}{2}} \quad [\text{using L' Hospital's rule}]$$

$$= \frac{2}{\pi} \lim_{x \rightarrow 1} \sin^2\left(\frac{\pi x}{2}\right) = \frac{2}{\pi}$$

$$74. \lim_{x \rightarrow 0} \frac{\sqrt{1+x} - 1}{x}$$

$\left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{2\sqrt{1+x}} - 0}{1} = \frac{1}{2}$$

$$\begin{aligned}
 75. \lim_{x \rightarrow 0} \frac{27^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{1 + \cos x}} &= \lim_{x \rightarrow 0} \frac{9^x \cdot 3^x - 9^x - 3^x + 1}{\sqrt{2} - \sqrt{2} \cos \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{(9^x - 1)(3^x - 1)}{\sqrt{2} \left(1 - \cos \frac{x}{2}\right)} \\
 &= \lim_{x \rightarrow 0} \left(\frac{9^x - 1}{x} \right) \cdot \left(\frac{3^x - 1}{x} \right) \cdot \frac{1}{\sqrt{2}} \cdot x^2 \cdot \frac{1}{2 \sin^2 \frac{x}{4}} \\
 &= \lim_{x \rightarrow 0} \left(\frac{9^x - 1}{x} \right) \cdot \left(\frac{3^x - 1}{x} \right) \cdot \frac{1}{\sqrt{2}} \left(\frac{x^2/16}{\sin^2 x/4} \right) 8 \\
 &= \frac{8}{\sqrt{2}} (\log 9)(\log 3) = 8\sqrt{2} (\log 3)^2
 \end{aligned}$$

$$\begin{aligned}
 76. \lim_{x \rightarrow -1} \frac{\sqrt{\pi} - \sqrt{\cos^{-1} x}}{\sqrt{x+1}} &= \lim_{y \rightarrow \pi} \frac{\sqrt{\pi} - \sqrt{y}}{\sqrt{1 + \cos y}} \\
 &\quad [\text{putting } x = \cos y, \text{ as } x \rightarrow -1, y \rightarrow \pi] \\
 &= \lim_{y \rightarrow \pi} \frac{\sqrt{\pi} - \sqrt{y}}{\sqrt{2} \cos \left(\frac{y}{2}\right)} = \lim_{y \rightarrow \pi} \frac{\sqrt{\pi} - \sqrt{y}}{\sqrt{2} \sin \left(\frac{\pi}{2} - \frac{y}{2}\right)} \\
 &= \lim_{y \rightarrow \pi} \frac{2}{\sqrt{2}} \cdot \frac{\left(\frac{\pi}{2} - \frac{y}{2}\right)}{\sin \left(\frac{\pi}{2} - \frac{y}{2}\right)} \cdot \frac{1}{\sqrt{\pi} + \sqrt{y}} = \frac{1}{\sqrt{2}\pi}
 \end{aligned}$$

$$\begin{aligned}
 77. \lim_{n \rightarrow \infty} [\sqrt[3]{(n+1)^2} - \sqrt[3]{(n-1)^2}] &= \lim_{n \rightarrow \infty} n^{2/3} \left[\left(1 + \frac{1}{n}\right)^{2/3} - \left(1 - \frac{1}{n}\right)^{2/3} \right] \\
 &= \lim_{n \rightarrow \infty} n^{2/3} \left[1 + \frac{2}{3} \cdot \frac{1}{n} + \frac{2}{3} \left(\frac{2}{3} - 1\right) \frac{1}{n^2} + \dots \right] \\
 &\quad - \left[1 - \frac{2}{3} \cdot \frac{1}{n} + \frac{2}{3} \left(\frac{2}{3} - 1\right) \frac{1}{n^2} - \dots \right] \\
 &= \lim_{n \rightarrow \infty} n^{2/3} \left[\frac{4}{3} \cdot \frac{1}{n} + \frac{8}{81} \cdot \frac{1}{n^3} + \dots \right] \\
 &= \lim_{n \rightarrow \infty} \left[\frac{4}{3} \cdot \frac{1}{n^{1/3}} + \frac{8}{81} \cdot \frac{1}{n^{7/3}} + \dots \right] = 0
 \end{aligned}$$

$$\begin{aligned}
 78. \lim_{x \rightarrow \alpha} \frac{\tan(ax^2 + bx + c)}{(x - \alpha)^2} &\left[\begin{array}{l} 0 \text{ form, as } a\alpha^2 + b\alpha + c = 0 \\ 0 \end{array} \right] \\
 &= \lim_{x \rightarrow \alpha} \frac{(2ax + b) \sec^2(ax^2 + bx + c)}{2(x - \alpha)} \\
 &\quad \left[\begin{array}{l} 0 \text{ form, as } \alpha \text{ being a repeated root of} \\ 0 \end{array} \right] \\
 &\quad \left[\begin{array}{l} 2 \sec^2(ax^2 + bx + c) \tan(ax^2 + bx + c) \\ (2ax + b)^2 + 2a \sec^2(ax^2 + bx + c) \end{array} \right] \\
 &= \lim_{x \rightarrow \alpha} \frac{2a}{2} = a
 \end{aligned}$$

$$79. \lim_{x \rightarrow 0} \frac{x}{\tan^{-1} 2x} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{2x}{\tan^{-1} 2x} = \frac{1}{2} \times 1 = \frac{1}{2}$$

$$80. \text{Since, } \lim_{x \rightarrow 0} f(x) \text{ exists.}$$

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) \\
 \Rightarrow \lim_{h \rightarrow 0} f(0-h) + a &= \lim_{h \rightarrow 0} f(0+h) \\
 \Rightarrow \lim_{h \rightarrow 0} |0-h| + a &= \lim_{h \rightarrow 0} \cos[0+h] \Rightarrow a = \cos 0 = 1 \\
 \therefore a &= 1
 \end{aligned}$$

$$81. x^2 + 4x + 5 = (x+2)^2 + 1 \geq 1$$

$$\therefore a = 1 \Rightarrow b = \lim_{\theta \rightarrow 0} \frac{2 \sin^2 \theta}{\theta^2} = 2$$

$$\begin{aligned}
 \sum_{r=0}^n a^r \cdot b^{n-r} &= b^n + ab^{n-1} + a^2b^{n-2} + \dots + a^n \\
 &= \frac{b^n \left[1 - \left(\frac{a}{b}\right)^{n+1} \right]}{1 - \frac{a}{b}} = \frac{2^n \left[1 - \left(\frac{1}{2}\right)^{n+1} \right]}{1 - \frac{1}{2}} \\
 &= \frac{2^{n+1}(2^{n+1} - 1)}{2^{n+1}} = (2^{n+1} - 1)
 \end{aligned}$$

$$\begin{aligned}
 82. \lim_{x \rightarrow 0} \frac{\log(1+x+x^2) + \log(1-x+x^2)}{\sec x - \cos x} &= \lim_{x \rightarrow 0} \frac{\log[(1+x^2)^2 - x^2]}{(1 - \cos^2 x) / \cos x} \\
 &= \lim_{x \rightarrow 0} \frac{\log(1+x^2+x^4)}{\sin x \tan x} \quad \left[\begin{array}{l} 0 \\ 0 \end{array} \text{ form} \right] \\
 &= \lim_{x \rightarrow 0} \frac{\log[1+x^2(1+x^2)]}{x^2(1+x^2)} \cdot x^2(1+x^2) \cdot \frac{1}{\frac{\sin x}{x} \cdot \frac{\tan x}{x} \cdot x^2} \\
 &= 1 \quad \left[\because \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1 \right]
 \end{aligned}$$

$$\begin{aligned}
 83. \lim_{x \rightarrow \infty} x \left[\tan^{-1} \frac{x+1}{x+2} - \frac{\pi}{4} \right] &= \lim_{x \rightarrow \infty} x \left[\tan^{-1} \frac{x+1}{x+2} - \tan^{-1} 1 \right] \\
 &= \lim_{x \rightarrow \infty} x \tan^{-1} \left(\frac{\frac{x+1}{x+2} - 1}{1 + \frac{x+1}{x+2}} \right) = \lim_{x \rightarrow \infty} x \tan^{-1} \left(-\frac{1}{2x+3} \right) \\
 &= - \lim_{x \rightarrow \infty} \frac{\tan^{-1} \left(\frac{1}{2x+3} \right)}{\left(\frac{1}{2x+3} \right)} \cdot \frac{1}{\left(2 + \frac{3}{x} \right)} = -1 \times \frac{1}{2} = -\frac{1}{2}
 \end{aligned}$$

$$84. \text{As } 0 \leq x - [x] < 1, \forall x \in R$$

$$\therefore 0 \leq f(x) < 1 \Rightarrow \lim_{n \rightarrow \infty} \{f(x)\}^{2n} = 0$$

$$\text{Thus, for } x \in R, g(x) = \lim_{n \rightarrow \infty} \frac{\{f(x)\}^{2n} - 1}{\{f(x)\}^{2n} + 1} = \frac{0 - 1}{0 + 1} = -1$$

$$\begin{aligned}
 85. \lim_{n \rightarrow \infty} \left[\frac{1}{1-n^2} + \frac{2}{1-n^2} + \dots + \frac{n}{1-n^2} \right] &= \lim_{n \rightarrow \infty} \left[\frac{1+2+\dots+n}{1-n^2} \right] = \lim_{n \rightarrow \infty} \frac{n(n+1)}{2(1-n^2)} \\
 &= \lim_{n \rightarrow \infty} \frac{\left(1 + \frac{1}{n}\right)}{2\left(\frac{1}{n^2} - 1\right)} = -\frac{1}{2}
 \end{aligned}$$

86. $\lim_{x \rightarrow 1} (1 + \cos \pi x) \cot^2 \pi x$

$$\begin{aligned} &= \lim_{h \rightarrow 0} (1 + \cos (\pi + \pi h)) \times \cot^2 (\pi + \pi h) \\ &= \lim_{h \rightarrow 0} (1 - \cos \pi h) \cot^2 \pi h \\ &= \lim_{h \rightarrow 0} \frac{2 \sin^2 \left(\frac{\pi h}{2} \right)}{\sin^2 \pi h} \cdot \cos^2 \pi h \\ &= \lim_{h \rightarrow 0} \frac{2 \sin^2 \frac{\pi h}{2}}{4 \sin^2 \frac{\pi h}{2} \cos^2 \frac{\pi h}{2}} \cdot \cos^2 \pi h \\ &= \frac{1}{2} \lim_{h \rightarrow 0} \frac{\cos^2 \pi h}{\cos^2 \frac{\pi h}{2}} = \frac{1}{2} \end{aligned}$$

87. Since, $y = x - 1$

$\therefore x = y + 1$

As $(x, y) \rightarrow (1, 0)$ along the line $y = x - 1$, $x = y + 1$ holds throughout.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1} \frac{y^3}{x^3 - y^2 - 1} &= \lim_{y \rightarrow 0} \frac{y^3}{(y + 1)^3 - y^2 - 1} \\ &= \lim_{y \rightarrow 0} \frac{y^3}{y^3 + 2y^2 + 3y} = \lim_{y \rightarrow 0} \frac{y^2}{y^2 + 2y + 3} = \frac{0}{3} = 0 \end{aligned}$$

88. $\lim_{x \rightarrow \beta} (\tan x \cot \beta)^{1/(x-\beta)}$

$$\begin{aligned} &= e^{\lim_{x \rightarrow \beta} \frac{1}{(x-\beta)} [\tan x \cot \beta - 1]} \quad \left[\frac{0}{0} \text{ form} \right] \\ &= e^{\lim_{x \rightarrow \beta} \frac{\sec^2 x \cot \beta}{1}} = e^{\sec^2 \beta \cot \beta} = e^{\frac{1}{\sin \beta \cos \beta}} = e^{2 \operatorname{cosec} 2\theta} \end{aligned}$$

89. **Case I** n is a positive integer.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^n}{e^x} &= \lim_{x \rightarrow \infty} \frac{nx^{n-1}}{e^x} \\ &= \lim_{x \rightarrow \infty} \frac{n(n-1)x^{n-2}}{e^x} = \dots = \lim_{x \rightarrow \infty} \frac{n!}{e^x} = 0 \end{aligned}$$

[using L' Hospital's rule repeatedly]

Case II n is a negative integer.

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{x^n}{e^x} &= \lim_{x \rightarrow \infty} \frac{x^{-m}}{e^x} \\ \text{[putting } n = -m, \text{ where } m \text{ is a positive integer]} \\ &= \lim_{x \rightarrow \infty} \frac{1}{x^m e^x} = \frac{1}{\infty} = 0 \end{aligned}$$

Case III $n = 0 \Rightarrow \lim_{x \rightarrow \infty} \frac{x^n}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$

Hence, $\lim_{x \rightarrow \infty} \frac{x^n}{e^x} = 0$ for all values of n .

90. $\lim_{x \rightarrow a} \frac{g(x)f(a) - g(a)f(x)}{x - a} \quad \left[\frac{0}{0} \text{ form} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow a} \frac{f(a)g'(x) - g(a)f'(x)}{1 - 0} \quad \text{[using L' Hospital's rule]} \\ &= f(a)g'(a) - g(a)f'(a) = 2 \times 2 - (-1) \cdot 1 = 5 \end{aligned}$$

91. $\lim_{x \rightarrow 1} \frac{g(x) - g(1)}{x - 1} \quad \left[\frac{0}{0} \text{ form} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 1} \frac{g'(x) - 0}{1 - 0} \quad \text{[using L' Hospital's rule]} \\ &= \lim_{x \rightarrow 1} \frac{-1}{2\sqrt{25 - x^2}} \cdot (-2x) = \lim_{x \rightarrow 1} \frac{x}{\sqrt{25 - x^2}} = \frac{1}{\sqrt{24}} \end{aligned}$$

92. $\lim_{x \rightarrow 9} \frac{3 - \sqrt{f(x)}}{3 - \sqrt{x}} \quad \left[\frac{0}{0} \text{ form} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 9} \frac{0 - \frac{1}{2\sqrt{f(x)}} \cdot f'(x)}{0 - \frac{1}{2\sqrt{x}}} \quad \text{[using L' Hospital's rule]} \\ &= \lim_{x \rightarrow 9} \frac{\sqrt{x}}{\sqrt{f(x)}} \cdot f'(x) = \frac{3}{3} \cdot f'(9) = 1 \end{aligned}$$

93. $\lim_{x \rightarrow 1} \frac{f(1)g(x) - f(x)g(1) - f(1) + g(1)}{g(x) - f(x)} \quad \left[\frac{0}{0} \text{ form} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 1} \frac{f(1)g'(x) - f'(x)g(1)}{g'(x) - f'(x)} \\ &= 2 \lim_{x \rightarrow 1} \frac{g'(x) - f'(x)}{g'(x) - f'(x)} = 2 \end{aligned}$$

94. $\lim_{x \rightarrow 2} \frac{2x^2 - 4f(x)}{x - 2} \quad \left[\frac{0}{0} \text{ form} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 2} \frac{4x - 4f'(x)}{1} \quad \text{[using L' Hospital's rule]} \\ &= 4 \times 2 - 4 \times 1 = 4 \end{aligned}$$

95. RHL = $\lim_{h \rightarrow 0} \frac{(1+h)\sin(1+h - [1+h])}{1+h-1}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(1+h)\sin(1+h-1)}{h} \\ &= \lim_{h \rightarrow 0} (1+h) \frac{\sin h}{h} = 1 \end{aligned}$$

LHL = $\lim_{h \rightarrow 0} \frac{(1-h)\sin(1-h - [1-h])}{1-h-1}$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(1-h)\sin(1-h)}{-h} = -\infty \end{aligned}$$

Since, LHL $\neq$ RHL

$\therefore \lim_{x \rightarrow 1} \frac{x \sin(x - [x])}{x - 1}$ does not exist.

96. $\lim_{x \rightarrow \infty} \left[\sqrt{x + \sqrt{x + \sqrt{x}}} - \sqrt{x} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow \infty} \frac{x + \sqrt{x + \sqrt{x}} - x}{\sqrt{x + \sqrt{x + \sqrt{x}}} + \sqrt{x}} \quad \text{[rationalising]} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{1 + x^{-1/2}}}{\sqrt{1 + \sqrt{x^{-1}} + x^{-3/2}} + 1} = \frac{1}{2} \end{aligned}$$

97. $\lim_{x \rightarrow 0^+} \log_{\tan x} (\sin x) = \lim_{x \rightarrow 0^+} \frac{\log_e (\sin x)}{\log_e (\tan x)} \quad \left[\frac{\infty}{\infty} \text{ form} \right]$

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} \frac{\frac{\cos x}{\sin x}}{\frac{\sec^2 x}{\tan x}} \\ &= \lim_{x \rightarrow 0} \frac{1}{\sec^2 x} = 1 \end{aligned}$$

98. $\lim_{x \rightarrow \infty} \frac{2x - 3}{x} = \lim_{x \rightarrow \infty} \left(2 - \frac{3}{x} \right) = 2$

and $\lim_{x \rightarrow \infty} \frac{2x^2 + 5x}{x^2} = \lim_{x \rightarrow \infty} \left(2 + \frac{5}{x} \right) = 2$

Using Sandwich theorem, $\lim_{x \rightarrow \infty} f(x) = 2$

99. Since, $\lim_{x \rightarrow 0} f(x)$ exists.

$$\begin{aligned} \therefore \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^+} f(x) \\ \Rightarrow \lim_{x \rightarrow 0} \frac{\sin(2k-3)x}{4x} &= \lim_{x \rightarrow 0} \frac{\tan(3k-4)x}{2x} \\ \Rightarrow \frac{2k-3}{4} &= \frac{3k-4}{2} \Rightarrow 4k-6=12k-16 \\ \therefore k &= \frac{5}{4} \end{aligned}$$

100. LHL = $\lim_{h \rightarrow 0} (-1)^{[n-h]} = \lim_{h \rightarrow 0} (-1)^{n-1} = (-1)^{n-1}$

$$\text{RHL} = \lim_{h \rightarrow 0} (-1)^{[n+h]} = \lim_{h \rightarrow 0} (-1)^n = (-1)^n$$

Since, LHL $\neq$ RHL

$\therefore \lim_{x \rightarrow n} (-1)^{[x]}$ does not exist.

$$101. g[f(x)] = \begin{cases} [f(x)]^2 + 1, & f(x) \neq 2 \\ 3, & f(x) = 2 \end{cases}$$

$$\therefore g[f(x)] = \begin{cases} \sin^2 x + 1, & x \neq n\pi \\ 3, & x = n\pi \end{cases}$$

$$\text{RHL} = \lim_{h \rightarrow 0} g[f(0+h)] = \lim_{h \rightarrow 0} (\sin^2 h + 1) = 1$$

$$\text{LHL} = \lim_{h \rightarrow 0} g[f(0-h)] = \lim_{h \rightarrow 0} (\sin^2 h + 1) = 1$$

$$\therefore \lim_{x \rightarrow 0} g[f(x)] = 1$$

$$102. \text{LHL} = \lim_{h \rightarrow 0} f(0-h) = \lim_{h \rightarrow 0} \frac{\tan[-h]}{[-h]} \\ = \lim_{h \rightarrow 0} \frac{\tan(-1)}{(-1)} = \tan 1$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} 0 = 0$$

Since, LHL $\neq$ RHL

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

103. LHL = $\lim_{h \rightarrow 0} f(0-h)$

$$= \lim_{h \rightarrow 0} \frac{\tan^{-1}([-h]-h)}{[-h]+2h} = \lim_{h \rightarrow 0} \frac{\tan^{-1}(-1-h)}{(2h-1)}$$

$$= \lim_{h \rightarrow 0} \frac{\tan^{-1}(1+h)}{(1-2h)} = \frac{\pi/4}{1} = \frac{\pi}{4}$$

$$\text{RHL} = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} 0 = 0$$

Since, LHL $\neq$ RHL

$\therefore \lim_{x \rightarrow 0} f(x)$ does not exist.

$$104. \text{RHL} = \lim_{h \rightarrow 0} f(0+h) = \lim_{h \rightarrow 0} h \sin \frac{1}{h} \\ = 0 \times (\text{an oscillating number between } -1 \text{ and } 1) = 0 \\ \left[\because \sin \frac{1}{h} \text{ is bounded and lying between } -1 \text{ and } 1 \right]$$

$$\text{LHL} = \lim_{h \rightarrow 0} f(0-h) \\ = \lim_{h \rightarrow 0} -h \sin \left(-\frac{1}{h} \right) = \lim_{h \rightarrow 0} h \sin \frac{1}{h} \\ = 0 \times (\text{an oscillating number between } -1 \text{ and } 1) = 0$$

Since, LHL = RHL

$\therefore \lim_{x \rightarrow 0} f(x)$ exists and equals 0.

$$105. \lim_{n \rightarrow \infty} \left[\frac{1}{n} + \frac{e^{1/n}}{n} + \frac{e^{2/n}}{n} + \dots + \frac{e^{(n-1)/n}}{n} \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{1 + e^{1/n} + (e^{1/n})^2 + \dots + (e^{1/n})^{n-1}}{n} \right] \\ = \lim_{n \rightarrow \infty} \frac{1 \cdot [e^{1/n} - 1]}{n(e^{1/n} - 1)} = (e-1) \lim_{n \rightarrow \infty} \frac{1}{\left(\frac{e^{1/n} - 1}{1/n} \right)} \\ = (e-1) \times 1 = (e-1)$$

$$106. \lim_{x \rightarrow \infty} \left[\frac{1^{1/x} + 2^{1/x} + 3^{1/x} + \dots + n^{1/x}}{n} \right]^{nx} \\ = \lim_{y \rightarrow 0} \left[\frac{1^y + 2^y + 3^y + \dots + n^y}{n} \right]^{n/y} \\ = e^{\lim_{y \rightarrow 0} \frac{n}{y} \left[\frac{1^y + 2^y + 3^y + \dots + n^y}{n} - 1 \right]} \\ = e^{\lim_{y \rightarrow 0} \left[\frac{1^y + 2^y + 3^y + \dots + n^y - n}{y} \right]} \\ = e^{\lim_{y \rightarrow 0} \left[\frac{(1^y-1)}{y} + \frac{(2^y-1)}{y} + \frac{(3^y-1)}{y} + \dots + \frac{(n^y-1)}{y} \right]} \\ = e^{(\log 1 + \log 2 + \log 3 + \dots + \log n)} = e^{\log(1 \cdot 2 \cdot 3 \dots n)} = n!$$

$$107. \lim_{x \rightarrow \frac{\pi}{2}} \left[1^{\frac{1}{\cos^2 x}} + 2^{\frac{1}{\cos^2 x}} + \dots + n^{\frac{1}{\cos^2 x}} \right]^{\cos^2 x} \\ = \lim_{t \rightarrow \infty} (1^t + 2^t + \dots + n^t)^{1/t} \quad \left[\text{putting } \frac{1}{\cos^2 x} = t \geq 1 \right] \\ = \lim_{t \rightarrow \infty} (n^t)^{1/t} \left[\left(\frac{1}{n} \right)^t + \left(\frac{2}{n} \right)^t + \dots + \left(\frac{n}{n} \right)^t \right]^{1/t} \\ = n \lim_{t \rightarrow \infty} \left[\left(\frac{1}{n} \right)^t + \left(\frac{2}{n} \right)^t + \dots + \left(\frac{n}{n} \right)^t \right]^{1/t} \\ = n(0 + 0 + \dots + 1)^0 = n$$

$$108. \lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} \right] \\ = \lim_{n \rightarrow \infty} \left[\left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \right] \\ = \lim_{n \rightarrow \infty} \left[1 - \frac{1}{n+1} \right] = 1 - 0 = 1$$

$$109. \lim_{n \rightarrow \infty} \frac{1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1)}{n^3} \\ = \lim_{n \rightarrow \infty} \frac{\sum n(n+1)}{n^3} = \lim_{n \rightarrow \infty} \frac{\sum n^2 + \sum n}{n^3} \\ = \lim_{n \rightarrow \infty} \frac{1}{n^3} \left[\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right] \\ = \lim_{n \rightarrow \infty} \left[\frac{1}{6} \left(1 + \frac{1}{n} \right) \left(2 + \frac{1}{n} \right) + \frac{1}{2} \cdot \left(\frac{1}{n} + \frac{1}{n^2} \right) \right] \\ = \frac{1}{6} \times 1 \times 2 \\ = \frac{1}{3}$$

$$110. \lim_{n \rightarrow \infty} \left[\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \dots + \frac{1}{(2n-1)(2n+1)} \right]$$

$$= \lim_{n \rightarrow \infty} \left[\frac{1}{2} \left(\frac{1}{1} - \frac{1}{3} \right) + \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) + \dots + \frac{1}{2} \left(\frac{1}{2n-1} - \frac{1}{2n+1} \right) \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{2} \left[1 - \frac{1}{2n+1} \right] = \frac{1}{2} (1 - 0) = \frac{1}{2}$$

$$111. t_r = \frac{r}{r^4 + r^2 + 1} = \frac{r}{(r^2 + 1)^2 - r^2}$$

$$= \frac{1}{2} \left[\frac{1}{r^2 - r + 1} - \frac{1}{r^2 + r + 1} \right]$$

$$= \frac{1}{2} \left[\frac{1}{r(r-1)+1} - \frac{1}{(r+1)r+1} \right]$$

$$\sum_{r=1}^n t_r = \sum_{r=1}^n \frac{1}{2} [f(r) - f(r+1)]$$

where, $f(r) = \frac{1}{r(r-1)+1}$

$$= \frac{1}{2} [f(1) - f(n+1)]$$

$$= \frac{1}{2} \left[1 - \frac{1}{(n+1)n+1} \right] = \frac{1}{2} \text{ as } n \rightarrow \infty$$

$$112. f(x) = \int \frac{2 \sin x - \sin 2x}{x^3} dx$$

$$\Rightarrow f'(x) = \frac{d}{dx} \int \frac{2 \sin x - \sin 2x}{x^3} dx$$

$$\therefore \lim_{x \rightarrow 0} f'(x) = \lim_{x \rightarrow 0} \frac{2 \sin x - \sin 2x}{x^3} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2 \cos x - 2 \cos 2x}{3x^2}$$

[using L' Hospital's rule]

$$= \lim_{x \rightarrow 0} \frac{-2 \sin x + 4 \sin 2x}{6x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{-2 \cos x + 8 \cos 2x}{6} \quad \text{[using L' Hospital's rule]}$$

$$= \frac{6}{6} = 1$$

$$113. \text{ We have } f(x) + g(x) + h(x) = \frac{x^2 - 4x + 17 - 4x - 2}{x^2 + x - 12}$$

$$= \frac{x^2 - 8x + 15}{x^2 + x - 12} = \frac{(x-3)(x-5)}{(x-3)(x+4)}$$

$$\therefore \lim_{x \rightarrow 3} [f(x) + g(x) + h(x)] = \lim_{x \rightarrow 3} \frac{(x-3)(x-5)}{(x-3)(x+4)} = -\frac{2}{7}$$

$$114. \text{ Let } y = \left(x + \frac{1}{x} \right)^3$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{d}{dx} \left(x + \frac{1}{x} \right)^3 = \frac{d}{dx} \left\{ x^3 + \frac{1}{x^3} + 3 \left(x + \frac{1}{x} \right) \right\}$$

$$= \frac{d}{dx} \left(x^3 + \frac{1}{x^3} + 3x + \frac{3}{x} \right)$$

$$= 3x^2 - \frac{3}{x^4} + 3 - \frac{3}{x^2}$$

$$115. \text{ Let } y = (3x + 5)(1 + \tan x)$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = (3x + 5) \frac{d}{dx} (1 + \tan x) + (1 + \tan x) \frac{d}{dx} (3x + 5)$$

$$= (3x + 5)(0 + \sec^2 x) + (1 + \tan x)(3)$$

$$= 3x \sec^2 x + 5 \sec^2 x + 3 + 3 \tan x$$

$$116. \text{ Let } y = \frac{x^5 - \cos x}{\sin x}$$

On differentiating w.r.t. x, we get

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx} (x^5 - \cos x) - (x^5 - \cos x) \frac{d}{dx} (\sin x)}{\sin^2 x}$$

$$= \frac{\sin x (5x^4 + \sin x) - (x^5 - \cos x) (\cos x)}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + \sin^2 x - x^5 \cos x + \cos^2 x}{\sin^2 x}$$

$$= \frac{5x^4 \sin x + 1 - x^5 \cos x}{\sin^2 x}$$

$$117. \text{ Let } y = (ax^2 + \cot x)(p + q \cos x)$$

$$\frac{dy}{dx} = (ax^2 + \cot x) \frac{d}{dx} (p + q \cos x)$$

$$+ (p + q \cos x) \frac{d}{dx} (ax^2 + \cot x)$$

$$= (ax^2 + \cot x)(0 - q \sin x)$$

$$+ (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$$

$$= (ax^2 + \cot x)(-q \sin x) + (p + q \cos x)(2ax - \operatorname{cosec}^2 x)$$

$$118. \text{ We have, } f'(x) = a \sin x + (ax + b) \cos x$$

$$+ c \cos x - (cx + d) \sin x$$

But $f'(x) = x \cos x$ for all x [given]

$$\therefore x \cos x = (a - d) \sin x + (b + c) \cos x$$

$$+ ax \cos x - cx \sin x$$

On equating the coefficients of sin x, cos x, x cos x and x sin x, we get

$$a - d = 0, b + c = 0, a = 1, c = 0$$

$$b = c = 0 \text{ and } a = d = 1$$

119. Since, f(x) is a polynomial function satisfying

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right)$$

$$\therefore f(x) = x^n + 1$$

$$\text{or } f(x) = -x^n + 1$$

$$\text{If } f(x) = -x^n + 1, \text{ then } f(4) = -4^n + 1 \neq 65$$

$$\text{So, } f(x) = x^n + 1$$

$$\text{Since, } f(4) = 65$$

$$\therefore 4^n + 1 = 65$$

$$\Rightarrow n = 3$$

$$\therefore f(x) = x^3 + 1 \Rightarrow f'(x) = 3x^2$$

$$\therefore f'(l_1) = 3l_1^2, f'(l_2) = 3l_2^2, f'(l_3) = 3l_3^2$$

Since, l_1, l_2, l_3 are in GP.

$\therefore f'(l_1), f'(l_2), f'(l_3)$ are also in GP.

$$120. \text{ We have, } 8f(x) + 6f\left(\frac{1}{x}\right) = x + 5, \text{ for all } x \quad \dots(i)$$

$$\text{Therefore, } 8f\left(\frac{1}{x}\right) + 6f(x) = \frac{1}{x} + 5 \quad \left[\text{putting } x = \frac{1}{x} \right] \dots(ii)$$

From Eqs. (i) and (ii), we have

$$f(x) = \frac{1}{28} \left(8x - \frac{6}{x} + 10 \right)$$

$$\therefore y = x^2 f(x) = \frac{1}{28} (8x^3 - 6x + 10x^2)$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{28} (24x^2 + 20x - 6)$$

$$\therefore \left[\frac{dy}{dx} \right]_{x=-1} = \frac{1}{28} (24 - 20 - 6) = -\frac{1}{14}$$

$$\begin{aligned} 121. f(x) = \log |2x| &= \begin{cases} \log 2x, & 2x > 0 \text{ or } x > 0 \\ \log (-2x), & 2x < 0 \text{ or } x < 0 \end{cases} \\ \Rightarrow f'(x) &= \begin{cases} \frac{1}{2x} \times 2, & x > 0 \\ \frac{1}{-2x} \times -2, & x < 0 \end{cases} = \begin{cases} \frac{1}{x}, & x > 0 \\ \frac{1}{x}, & x < 0 \end{cases} \\ \therefore f'(x) &= \frac{1}{x}, x \neq 0 \end{aligned}$$

122. Since, $x > 10$, therefore $f(x) = x - 3$

$$\therefore \phi(x) = (f \circ f)(x) = f[f(x)] = f(x - 3)$$

$$= x - 3 - 3 = x - 6$$

$$\Rightarrow \phi'(x) = 1$$

123. Clearly, $f(x)$ must be of the form

$$f(x) = a_0 [x^n + (k - x)^n] + a_1 [x^{n-1} + (k - x)^{n-1}] + \dots + a_{n-1} [x + (k - x)] + a_n$$

It may be noted that n must be even for otherwise $f(x)$ will become a polynomial of degree $n - 1$.

Clearly, $f'(x)$ is a polynomial of degree $n - 1$.

124. If $m < 0$, then for values of x sufficiently close to 0, we have

$$1 + \frac{1}{m} < \frac{\sin x}{x} < 1$$

$$\Rightarrow m + 1 > m \frac{\sin x}{x} > m$$

$$\Rightarrow \left[m \frac{\sin x}{x} \right] = m \Rightarrow \lim_{x \rightarrow 0} \left[m \frac{\sin x}{x} \right] = m$$

If $m > 0$, then for values of x sufficiently close to 0, we have

$$1 - \frac{1}{m} < \frac{\sin x}{x} < 1$$

$$\Rightarrow m - 1 < m \frac{\sin x}{x} < m$$

$$\Rightarrow \lim_{x \rightarrow 0} \left[m \frac{\sin x}{x} \right] = m - 1$$

$$125. \lim_{x \rightarrow 0} (1 + ax + bx^2)^{2/x} = e^3$$

$$\Rightarrow e^{\lim_{x \rightarrow 0} \frac{2}{x} (ax + bx^2)} = e^3$$

Since, limit value e^{2a} does not involve b .

$\therefore b$ can have any value.

$$\text{Thus, } a = \frac{3}{2}, b \in R$$

126. Given, $f(x) = 5$

$$\Rightarrow f'(x) = \frac{d}{dx} [f(x)] = \frac{d}{dx} (5)$$

$$= 0 \quad \left[\because \frac{d}{dx} (\text{constant}) = 0 \right]$$

$$\therefore f'(1) = 0 \text{ and } f'(-1) = 0$$

$$127. \lim_{x \rightarrow \infty} \left(\frac{1^2}{x^3} + \frac{2^2}{x^3} + \frac{3^2}{x^3} + \dots + \frac{x^2}{x^3} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x(x+1)(2x+1)}{6x^3} = \frac{1}{3}$$

$$128. f'(x) = \frac{d}{dx} (3x) = 3 \quad \left[\because \frac{d}{dx} [af(x)] = a \frac{d}{dx} f(x) \right]$$

$$\Rightarrow f'(2) = 3$$

Hence, both statements are true and Statement II is the correct explanation for Statement I.

$$129. h(x) = \frac{x - \left(-\frac{1}{x}\right)}{x + \left(-\frac{1}{x}\right)} = \frac{x^2 + 1}{x^2 - 1}$$

$$\begin{aligned} \Rightarrow h'(x) &= \frac{(x^2 - 1) \cdot 2x - (x^2 + 1) \cdot 2x}{(x^2 - 1)^2} \\ &= \frac{2x^3 - 2x - 2x^3 - 2x}{(x^2 - 1)^2} \\ &= \frac{-4x}{(x^2 - 1)^2} = \frac{-4x}{(1 - x^2)^2} \end{aligned}$$

$$130. h'(2) = \frac{-4 \cdot 2}{(1 - 2^2)^2} = -\frac{8}{9}$$

Solutions (Q. Nos. 131-133)

$$\text{We have, } A_i = \frac{x - a_i}{|x - a_i|}, i = 1, 2, \dots, n$$

$$\text{and } a_1 < a_2 < \dots < a_{n-1} < a_n.$$

Let x be in the left neighbourhood of a_m .

Then, $x - a_i < 0$ for $i = m, m + 1, \dots, n$ and $x - a_i > 0$ for $i = 1, 2, \dots, m - 1$. Therefore,

$$A_i = \frac{x - a_i}{-(x - a_i)} = -1 \text{ for } i = m, m + 1, \dots, n$$

$$\text{and } A_i = \frac{x - a_i}{x - a_i} = 1 \text{ for } i = 1, 2, \dots, m - 1$$

Similarly, if x is in the right neighbourhood of a_m , then $x - a_i < 0$ for $i = m + 1, \dots, n$ and $x - a_i > 0$ for $i = 1, 2, \dots, m$.

$$\therefore A_i = \frac{x - a_i}{-(x - a_i)} = -1 \text{ for } i = m + 1, \dots, n$$

$$\text{and } A_i = \frac{x - a_i}{x - a_i} = 1 \text{ for } i = 1, 2, \dots, m$$

$$\text{Now, } \lim_{x \rightarrow a_m^-} (A_1 A_2 \dots A_n) = (-1)^{n-m+1}$$

$$\text{and } \lim_{x \rightarrow a_m^+} (A_1 A_2 \dots A_n) = (-1)^{n-m}$$

Hence, $\lim_{x \rightarrow a_m} (A_1 A_2 \dots A_n)$ does not exist.

$$134. \text{ A. } \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$\text{Let } x + 1 = h$$

$$\text{When } x \rightarrow 0, \text{ then } h \rightarrow 1$$

$$\begin{aligned} \therefore \text{Given limit} &= \lim_{h \rightarrow 1} \frac{h^5 - 1}{h - 1} = \lim_{h \rightarrow 1} \frac{h^5 - 1^5}{h - 1} \\ &= 5(1)^{5-1} = 5 \times 1^4 = 5 \end{aligned}$$

B. Let $x = 1/y$

$$\therefore \text{Given limit} = \lim_{x \rightarrow \infty} \left[x - x^2 \log_e \left(1 + \frac{1}{x} \right) \right]$$

$$= \lim_{y \rightarrow 0} \left(\frac{1}{y} - \frac{\log_e (1+y)}{y^2} \right)$$

$$= \lim_{y \rightarrow 0} \left(\frac{y - \log_e (1+y)}{y^2} \right) = \lim_{y \rightarrow 0} \left(\frac{1 - \frac{1}{1+y}}{2y} \right)$$

[using L'Hospital's rule]

$$= \lim_{y \rightarrow 0} \frac{y}{(y+1)2y} = \lim_{y \rightarrow 0} \frac{1}{2(y+1)} = \frac{1}{2}$$

C. Given, $\lim_{z \rightarrow 1} \frac{z^{1/3} - 1}{z^{1/6} - 1} = \lim_{z \rightarrow 1} \frac{z^{1/3} - 1}{z - 1} \times \frac{z - 1}{z^{1/6} - 1}$

$$= \frac{(1/3)(1)^{\frac{1}{3}-1}}{(1/6)(1)^{\frac{1}{6}-1}} = \frac{6}{3} \times 1 = 2$$

D. Given limit $= \lim_{n \rightarrow \infty} \prod_{r=2}^n \left(\frac{r^2 - 1}{r^2} \right)$

$$= \lim_{n \rightarrow \infty} \prod_{r=2}^n \left(\frac{r-1}{r} \right) \prod_{r=2}^n \left(\frac{r+1}{r} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} \right) \left(\frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdots \frac{n+1}{n} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \cdot \frac{n+1}{2} = \frac{1}{2}$$

135. A. Let $y = \operatorname{cosec} x = \frac{1}{\sin x}$

On differentiating y w.r.t. x , we get

$$\frac{dy}{dx} = \frac{\sin x \frac{d}{dx}(1) - (1) \frac{d}{dx}(\sin x)}{\sin^2 x} = \frac{\sin x \times 0 - 1 \times \cos x}{\sin^2 x}$$

$$= \frac{0 - \cos x}{\sin^2 x} = \frac{-\cos x}{\sin x} \times \frac{1}{\sin x}$$

$$\Rightarrow \frac{dy}{dx} = -\cot x \operatorname{cosec} x$$

B. Let $y = 3 \cot x + 5 \operatorname{cosec} x$

On differentiating y w.r.t. x , we get

$$\frac{dy}{dx} = -3 \operatorname{cosec}^2 x - 5 \operatorname{cosec} x \cot x$$

C. Let $y = 5 \sin x - 6 \cos x + 7$

On differentiating y w.r.t. x , we get

$$\frac{dy}{dx} = 5 \cos x - 6(-\sin x) + 0$$

$$= 5 \cos x + 6 \sin x$$

D. Let $y = 2 \tan x - 7 \sec x$

On differentiating y w.r.t. x , we get

$$\frac{dy}{dx} = 2 \sec^2 x - 7 \sec x \tan x$$

136. Given, $f(x) = \begin{cases} x+2, & x \leq -1 \\ cx^2, & x > -1 \end{cases}$

At $x = -1$, limit exists.

$\therefore$ RHL = LHL

$$\Rightarrow \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^-} f(x)$$

$$\Rightarrow \lim_{h \rightarrow 0} f(-1+h) = \lim_{h \rightarrow 0} f(-1-h)$$

$$\Rightarrow \lim_{h \rightarrow 0} c(-1+h)^2 = \lim_{h \rightarrow 0} (-1-h+2)$$

$$\Rightarrow c(-1+0)^2 = 1-0$$

$$\therefore c = 1$$

137. Given limit $= \lim_{x \rightarrow 0} (\sin x)^{1/x} + \lim_{x \rightarrow 0} \left(\frac{1}{x} \right)^{\sin x}$

$$= 0 + \lim_{x \rightarrow 0} e^{\log \left(\frac{1}{x} \right)^{\sin x}} \left[\because \lim_{x \rightarrow 0} (\sin x)^{1/x} \rightarrow 0 \right]$$

$$\left[\text{as } (\text{decimal})^\infty \rightarrow 0 \right]$$

$$= e^{\lim_{x \rightarrow 0} \frac{\log(1/x)}{\operatorname{cosec} x}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{x \left(-\frac{1}{x^2} \right)}{\operatorname{cosec} x \cot x}}$$

[applying L' Hospital's rule]

$$= e^{\lim_{x \rightarrow 0} \frac{\sin x \tan x}{x}} = e^0 = 1$$

138. Let $I = \lim_{x \rightarrow 0} \frac{1 - \sqrt{\cos 2x} \sqrt[3]{\cos 3x} \sqrt[4]{\cos 4x} \cdots \sqrt[n]{\cos nx}}{x^2}$

$$= - \lim_{x \rightarrow 0} \frac{D \left[\prod_{r=2}^n (\cos rx)^{1/r} \right]}{2x}$$

[using L'Hospital's rule]

Let

$$y = \prod_{r=2}^n (\cos rx)^{1/r}$$

$$\Rightarrow \log y = \sum_{r=2}^n \left(\frac{1}{r} \log (\cos rx) \right)$$

$$\Rightarrow \frac{1}{y} \frac{dy}{dx} = - \sum_{r=2}^n \tan (rx)$$

$$\Rightarrow -Dy = y \sum_{r=2}^n \tan (rx)$$

$$\Rightarrow D \left(\prod_{r=2}^n (\cos rx)^{1/r} \right) = -y \sum_{r=2}^n \tan (rx)$$

$$y \cdot \sum_{r=2}^n \tan (rx)$$

$$\Rightarrow L = \lim_{x \rightarrow 0} \frac{\sum_{r=2}^n \tan (rx)}{2x} = \frac{1}{2} [2 + 3 + 4 + \cdots + n]$$

$$= \frac{1}{2} \left[\frac{n(n+1)}{2} - 1 \right] = \frac{n^2 + n - 2}{4}$$

$$\Rightarrow \frac{n^2 + n - 2}{4} = 10 \Rightarrow n^2 + n - 42 = 0$$

$$\Rightarrow (n+7)(n-6) = 0 \Rightarrow n = 6$$

139. Given, $f(x) = \begin{cases} \frac{\sin(1+[x])}{[x]}, & \text{for } [x] \neq 0 \\ 0, & \text{for } [x] = 0 \end{cases}$

$$\Rightarrow \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{\sin(1+[x])}{[x]} = \frac{\sin(1-1)}{-1} = 0$$

140. Given, $\lim_{x \rightarrow 0} \left(\frac{\int_0^{x^2} \sec^2 t \, dt}{x \sin x} \right)$

$$= \lim_{x \rightarrow 0} \frac{\frac{d}{dx} \int_0^{x^2} \sec^2 t \, dt}{\frac{d}{dx} (x \sin x)} = \lim_{x \rightarrow 0} \frac{\sec^2 x^2 \cdot 2x}{\sin x + x \cos x}$$

$$= \lim_{x \rightarrow 0} \frac{2x \cdot \sec^2 x^2}{x \left(\frac{\sin x}{x} + \cos x \right)}$$

$$= \frac{2 \times 1}{1+1} = \frac{2}{2} = 1$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

Entrances Gallery

$$1. \lim_{x \rightarrow 1} \left(\frac{-ax + \sin(x-1) + a}{x + \sin(x-1) - 1} \right)^{\frac{1-x}{1-\sqrt{x}}} = \frac{1}{4}$$

$$\Rightarrow \lim_{x \rightarrow 1} \left(\frac{\frac{\sin(x-1)}{(x-1)} - a}{\frac{\sin(x-1)}{(x-1)} + 1} \right)^{(1+\sqrt{x})} = \frac{1}{4}$$

$$\Rightarrow \left(\frac{1-a}{2} \right)^2 = \frac{1}{4}$$

$$\Rightarrow a = 0 \text{ or } a = 2$$

$$\Rightarrow a = 2$$

$$2. \text{ Required limit} = \frac{\int_0^1 x^a dx}{\int_0^1 (a+x) dx}$$

$$= \frac{2}{(2a+1)(a+1)}$$

$$= \frac{2}{120}$$

$$\Rightarrow a = 7 \text{ or } -\frac{17}{2}$$

$$3. \text{ Given, } \lim_{x \rightarrow \infty} \left(\frac{x^2 + x + 1}{x + 1} - ax - b \right) = 4$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{x^2 + x + 1 - ax^2 - ax - bx - b}{(x+1)} = 4$$

$$\Rightarrow \lim_{x \rightarrow \infty} \frac{(1-a)x^2 + (1-a-b)x + (1-b)}{(x+1)} = 4$$

$$\Rightarrow 1-a=0 \text{ and } 1-a-b=4 \Rightarrow b=-4, a=1$$

$$4. \text{ Let } 1+a=y$$

$$\Rightarrow (y^{1/3}-1)x^2 + (y^{1/2}-1)x + y^{1/6}-1=0$$

$$\Rightarrow \left(\frac{y^{1/3}-1}{y-1} \right) x^2 + \left(\frac{y^{1/2}-1}{y-1} \right) x + \frac{y^{1/6}-1}{y-1} = 0$$

Now, taking $\lim_{y \rightarrow 1}$ on both sides, we get

$$\frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{6} = 0$$

$$\Rightarrow 2x^2 + 3x + 1 = 0$$

$$\therefore x = -1, -\frac{1}{2}$$

$$5. \text{ Given limit} = \lim_{x \rightarrow 0} \frac{\int_0^x \frac{t \log(1+t)}{t^4+4} dt}{x^3}$$

$$= \lim_{x \rightarrow 0} \frac{x \log(1+x)}{x^4+4} \text{ [using L' Hospital's rule]}$$

$$= \lim_{x \rightarrow 0} \frac{\log(1+x)}{3x} \cdot \frac{1}{x^4+4} = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

$$6. e^{\ln(1+b^2)} = 2b \sin^2 \theta$$

$$\Rightarrow \sin^2 \theta = \frac{1+b^2}{2b}$$

$$\Rightarrow \sin^2 \theta = 1 \text{ as } \frac{1+b^2}{2b} \geq 1$$

$$\therefore \theta = \pm \pi/2$$

7. We have,

$$\lim_{x \rightarrow 0} \frac{(1-\cos 2x)(3+\cos x)}{x \tan 4x} = \lim_{x \rightarrow 0} \frac{2 \sin^2 x (3+\cos x)}{x \times \frac{\tan 4x}{4x} \times 4x}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \times \lim_{x \rightarrow 0} \frac{(3+\cos x)}{4} \times \lim_{x \rightarrow 0} \frac{1}{\frac{\tan 4x}{4x}}$$

$$= 2 \times \frac{4}{4} \times 1 \left(\because \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \right)$$

$$\text{and } \lim_{\theta \rightarrow 0} \frac{\tan \theta}{\theta} = 1$$

$$= 2$$

8. Since, $g(x)$ is differentiable.

$\Rightarrow g(x)$ must be continuous.

$$\therefore g(x) = \begin{cases} k\sqrt{x+1} & , 0 \leq x \leq 3 \\ mx+2 & , 3 < x \leq 5 \end{cases}$$

$$\text{At } x=3, \text{ RHL} = 3m+2$$

$$\text{and at } x=3, \text{ LHL} = 2k$$

$$\therefore 2k = 3m+2 \quad \dots(i)$$

$$\text{Also, } g'(x) = \begin{cases} \frac{k}{2\sqrt{x+1}} & , 0 \leq x < 3 \\ m & , 3 < x \leq 5 \end{cases}$$

$$\therefore L\{g'(3)\} = \frac{k}{4}$$

$$\text{and } R\{g'(3)\} = m$$

$$\Rightarrow \frac{k}{4} = m \Rightarrow k = 4m \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$k = \frac{8}{5}, m = \frac{2}{5}$$

$$\Rightarrow k+m=2$$

9. Any function have extreme values (maximum or minimum) at its critical points, where $f'(x) = 0$.

Since, the function have extreme values at $x=1$ and $x=2$.

$$\therefore f'(x) = 0 \text{ at } x=1$$

$$\text{and } x=2$$

$$\Rightarrow f'(1) = 0 \text{ and } f'(2) = 0$$

Also, it is given that

$$\lim_{x \rightarrow 0} \left[1 + \frac{f(x)}{x^2} \right] = 3$$

$$\Rightarrow 1 + \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 3$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{f(x)}{x^2} = 2$$

$$\Rightarrow f(x) \text{ will be of the form } ax^4 + bx^3 + 2x^2$$

[$\because f(x)$ is four degree polynomial]

$$\text{Let } f(x) = ax^4 + bx^3 + 2x^2$$

$$\Rightarrow f'(x) = 4ax^3 + 3bx^2 + 4x$$

$$\Rightarrow f'(1) = 4a + 3b + 4 = 0 \quad \dots(i)$$

and $f'(2) = 32a + 12b + 8 = 0$

$\Rightarrow 8a + 3b + 2 = 0$

...(ii)

On solving Eqs. (i) and (ii), we get

$a = \frac{1}{2}, b = -2$

$\therefore f(x) = \frac{x^4}{2} - 2x^3 + 2x^2$

$\Rightarrow f(2) = 8 - 16 + 8 = 0$

10. $\lim_{x \rightarrow 0} \frac{\sin(\pi \cos^2 x)}{x^2}$

$$= \lim_{x \rightarrow 0} \frac{\sin(\pi - \pi \sin^2 x)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\sin(\pi \sin^2 x)}{\pi \sin^2 x} \times \frac{\pi \sin^2 x}{x^2} = \pi$$

11. Let $I = \lim_{x \rightarrow 0} \frac{(1 - \cos 2x)}{x^2} \cdot \frac{(3 + \cos x)}{1} \cdot \frac{x}{\tan 4x}$

$$= \lim_{x \rightarrow 0} \frac{2 \sin^2 x}{x^2} \cdot \frac{3 + \cos x}{1} \cdot \frac{x}{\tan 4x}$$

$$= 2 \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^2 \cdot \lim_{x \rightarrow 0} (3 + \cos x) \cdot \lim_{x \rightarrow 0} \frac{4x}{4 \tan 4x}$$

$$= 2 \cdot (1)^2 \cdot (3 + \cos 0) \cdot \frac{1}{4} (1)$$

$$= 2 \cdot 1 \cdot (3 + 1) \cdot \frac{1}{4}$$

$$= 2 \cdot 4 \cdot \frac{1}{4} = 2$$

12. $\lim_{x \rightarrow 2} \frac{\sqrt{1 - \cos 2(x-2)}}{(x-2)}$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{2 \sin^2(x-2)}}{(x-2)}$$

$$= \lim_{x \rightarrow 2} \frac{\sqrt{2} |\sin(x-2)|}{(x-2)}$$

RHL at $x = 2$,

$$= \lim_{h \rightarrow 0} \frac{\sqrt{2} |\sin(2+h-2)|}{(2+h)-2} = \lim_{h \rightarrow 0} \frac{\sqrt{2} |\sin h|}{h}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{2} \sin h}{h} = \sqrt{2}$$

LHL at $x = 2$,

$$= \lim_{h \rightarrow 0} \frac{\sqrt{2} |\sin(2-h-2)|}{(2-h)-2}$$

$$= \lim_{h \rightarrow 0} \frac{\sqrt{2} |\sin(-h)|}{-h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{\sqrt{2} \sin h}{-h} = -\sqrt{2}$$

$\therefore$ Limit does not exist.

13. Given, $\lim_{x \rightarrow 5} f(x)$ exists and $\lim_{x \rightarrow 5} \frac{[f(x)]^2 - 9}{\sqrt{|x-5|}} = 0$

$$\Rightarrow \lim_{x \rightarrow 5} [f(x)]^2 - 9 = 0 \Rightarrow \lim_{x \rightarrow 5} \{f(x)\}^2 = 9$$

$$\therefore \lim_{x \rightarrow 5} f(x) = 3, -3$$

But $f: R \rightarrow [0, \infty)$

$\therefore$ Range of $f(x) \geq 0$

$$\Rightarrow \lim_{x \rightarrow 5} f(x) = 3$$

14. Consider $\lim_{x \rightarrow \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2}$

$$= \lim_{x \rightarrow \alpha} \frac{2 \sin^2 \left(\frac{ax^2 + bx + c}{2} \right)}{(x - \alpha)^2}$$

$$= \lim_{x \rightarrow \alpha} \frac{2 \sin^2 \left[\frac{a}{2} (x - \alpha)(x - \beta) \right]}{\left(\frac{a}{2} \right)^2 (x - \alpha)^2 (x - \beta)^2} \left(\frac{a}{2} \right)^2 (x - \beta)^2$$

$$= \lim_{x \rightarrow \alpha} \frac{a^2}{2} (x - \beta)^2 \left[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right]$$

$$= \frac{a^2}{2} (\alpha - \beta)^2$$

15. Consider $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x}$

$$= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{\frac{1}{\frac{a}{x} + \frac{b}{x^2}}} \right]^{2x \left(\frac{a}{x} + \frac{b}{x^2} \right)}$$

$$= e^{\lim_{x \rightarrow \infty} 2x \left(\frac{a}{x} + \frac{b}{x^2} \right)} \left[\because \lim_{x \rightarrow \infty} \left(1 + \frac{1}{x} \right)^x = e \right]$$

$$= e^{2a}$$

But $\lim_{x \rightarrow \infty} \left(1 + \frac{a}{x} + \frac{b}{x^2} \right)^{2x} = e^2$

$\Rightarrow e^{2a} = e^2$

$\Rightarrow a = 1$
and $b \in R$

16. Given limit = $\lim_{x \rightarrow \frac{\pi}{2}} \frac{(1 - \tan \frac{x}{2})(1 - \sin x)}{\left(1 + \tan \frac{x}{2} \right) (\pi - 2x)^3}$

Let $x = \frac{\pi}{2} - h$

as $x \rightarrow \frac{\pi}{2}, h \rightarrow 0$

$$\therefore \text{Required limit} = \lim_{h \rightarrow 0} \frac{1 - \tan \left(\frac{\pi}{4} - \frac{h}{2} \right)}{1 + \tan \left(\frac{\pi}{4} - \frac{h}{2} \right)} \cdot \frac{(1 - \cos h)}{(2h)^3}$$

$$= \lim_{h \rightarrow 0} \tan \frac{h}{2} \cdot \frac{2 \sin^2 \frac{h}{2}}{8h^3}$$

$$\left[\because \tan \left(\frac{\pi}{4} - x \right) = \frac{1 - \tan x}{1 + \tan x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{4} \cdot \frac{\tan \frac{h}{2}}{\frac{h}{2} \times 2} \times \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2 \times \frac{1}{4}$$

$$= \frac{1}{32} \lim_{h \rightarrow 0} \left(\frac{\tan \frac{h}{2}}{\frac{h}{2}} \right) \lim_{h \rightarrow 0} \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}} \right)^2$$

$$= \frac{1}{32}$$

17. Since, $\lim_{x \rightarrow 0} \frac{\log(3+x) - \log(3-x)}{x} = k$

$$\lim_{x \rightarrow 0} \frac{\left(\frac{1}{3+x} + \frac{1}{3-x}\right)}{1} = k \quad [\text{using L'Hospital's rule}]$$

$$\Rightarrow \frac{1}{3} + \frac{1}{3} = k$$

$$\Rightarrow k = \frac{2}{3}$$

18. Consider $\lim_{x \rightarrow 0} \frac{\sqrt{1-\cos 2x}}{\sqrt{2x}}$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{2}|\sin x|}{\sqrt{2x}} = \lim_{x \rightarrow 0} \frac{|\sin x|}{x}$$

Let $f(x) = \frac{|\sin x|}{x}$

Now, LHL = $\lim_{h \rightarrow 0} \frac{|\sin(0-h)|}{0-h} = \lim_{h \rightarrow 0} \frac{\sin h}{-h} = -1$

and RHL = $\lim_{h \rightarrow 0} \frac{|\sin(0+h)|}{0+h} = \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1$

$\therefore$ LHL $\neq$ RHL

Hence, $\lim_{x \rightarrow 0} \frac{|\sin x|}{x}$ does not exist.

19. Consider $\lim_{x \rightarrow \infty} \left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$

$$= \lim_{x \rightarrow \infty} \left(1 + \frac{4x+1}{x^2+x+2}\right)^x$$

$$= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{4x+1}{x^2+x+2}\right)^{\frac{1}{\frac{4x+1}{x^2+x+2}}}\right]^{\frac{(4x+1)x}{x^2+x+2}}$$

$$= \lim_{x \rightarrow \infty} \left[\left(1 + \frac{4}{x}\right)^{\frac{x}{4}}\right]^{\frac{4x+1}{x^2+x+2}}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{4x+1}{x^2+x+2}} = e^0 = 1$$

20. Consider $\lim_{x \rightarrow \infty} \left(\frac{x-3}{x+2}\right)^x$

$$= \lim_{x \rightarrow \infty} \left[1 - \frac{5}{x+2}\right]^x$$

$$= \lim_{x \rightarrow \infty} \left[\left(1 + \left(\frac{-5}{x+2}\right)\right)^{\frac{1}{\left(\frac{-5}{x+2}\right)}}\right]^{\frac{-5x}{x+2}}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{-5x}{x+2}} = e^{-5}$$

21. $\lim_{x \rightarrow 0} \frac{(1+x)^8 - 1}{(1+x)^2 - 1}$

$$= \lim_{x \rightarrow 0} \frac{[(1+x)^4 + 1][(1+x)^2 + 1][(1+x)^2 - 1]}{(1+x)^2 - 1}$$

$$= 2 \times 2 = 4$$

22. $\lim_{x \rightarrow 1} \frac{x^m - 1}{x^n - 1} = \lim_{x \rightarrow 1} \frac{mx^{m-1}}{nx^{n-1}} = m/n$

[using L'Hospital's rule]

23. Let $y = \operatorname{cosec}^2 x$

Required limit = $\lim_{y \rightarrow \infty} (1^y + 2^y + \dots + n^y)^{1/y}$

$$= \lim_{y \rightarrow \infty} (n^y)^{1/y} \left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \dots + \left(\frac{n-1}{n}\right)^y + 1 \right]^{1/y}$$

$$= \lim_{y \rightarrow \infty} n \left[\left(\frac{1}{n}\right)^y + \left(\frac{2}{n}\right)^y + \dots + \left(\frac{n-1}{n}\right)^y + 1 \right]^{1/y}$$

$$= n \cdot 1^0 = n \cdot 1 = n$$

24. Let $y = \lim_{x \rightarrow 0} \left(\frac{a^x + b^x + c^x}{3}\right)^{2/x}$

$$\Rightarrow \log y = \lim_{x \rightarrow 0} \frac{2}{x} \log \left(\frac{a^x + b^x + c^x}{3}\right)$$

$$= \lim_{x \rightarrow 0} \frac{2 [\log(a^x + b^x + c^x) - \log 3]}{x}$$

$$\Rightarrow \log y = \log(abc)^{2/3} \quad [\text{using L'Hospital's rule}]$$

$$\therefore y = (abc)^{2/3}$$

25. $\lim_{x \rightarrow 1} \frac{f(x)-2}{x^2-1} = \pi \Rightarrow \lim_{x \rightarrow 1} [f(x)-2] = \pi \lim_{x \rightarrow 1} (x^2-1)$

$$\Rightarrow \lim_{x \rightarrow 1} f(x) = \pi(1^2-1) + 2 = 2$$

26. $\lim_{x \rightarrow 3} \frac{x^5 - 3^5}{x^8 - 3^8} \quad \left[\frac{0}{0} \text{ form}\right]$

$$= \lim_{x \rightarrow 3} \frac{5x^4}{8x^7} \quad [\text{using L'Hospital's rule}]$$

$$= \lim_{x \rightarrow 3} \frac{5}{8x^3} = \frac{5}{8 \times 3^3} = \frac{5}{8 \times 27} = \frac{5}{216}$$

27. Given, $f(x) = (x^5 - 1)(x^3 + 1)$

and $g(x) = (x^2 - 1)(x^2 - x + 1)$

$\therefore f(x) = g(x)h(x)$

So, $h(x) = \frac{f(x)}{g(x)}$

$$\lim_{x \rightarrow 1} h(x) = \lim_{x \rightarrow 1} \frac{(x^5 - 1)(x^3 + 1)}{(x^2 - 1)(x^2 - x + 1)}$$

$$= \lim_{x \rightarrow 1} \frac{(x^5 - 1)(x + 1)(x^2 - x + 1)}{(x - 1)(x + 1)(x^2 - x + 1)}$$

$$= \lim_{x \rightarrow 1} \frac{(x^5 - 1)}{(x - 1)} \quad \left[\frac{0}{0} \text{ form}\right]$$

$$= \lim_{x \rightarrow 1} \frac{5x^4}{1} \quad [\text{using L'Hospital's rule}]$$

$$= \frac{5(1)^4}{1} = 5$$

28. $\lim_{x \rightarrow 0} \frac{\log(1+3x^2)}{x(e^{5x} - 1)} = \lim_{x \rightarrow 0} \frac{\log(1+3x^2)}{3x^2} \times \frac{5x}{(e^{5x} - 1)} \times \frac{3}{5}$

$$= \frac{3}{5} \lim_{x \rightarrow 0} \frac{\log(1+3x^2)}{3x^2} \times \lim_{x \rightarrow 0} \frac{5x}{(e^{5x} - 1)} = \frac{3}{5} \times 1 \times 1 = \frac{3}{5}$$

$$\left[\because \lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = 1 \text{ and } \lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1 \right]$$

29. We have, $n\sqrt{2} - 1 < [n\sqrt{2}] < n\sqrt{2}$

$[\because x - 1 < [x] < x, \text{ if } x \text{ is not an integer}]$

$$\Rightarrow \sqrt{2} - \frac{1}{n} < \frac{[n\sqrt{2}]}{n} < \sqrt{2}$$

By Sandwich theorem,

$$\lim_{n \rightarrow \infty} \left(\sqrt{2} - \frac{1}{n}\right) = \sqrt{2} = \lim_{n \rightarrow \infty} \frac{[n\sqrt{2}]}{n}$$

$$30. \text{ Given, } L = \lim_{x \rightarrow 0} \frac{a - \sqrt{a^2 - x^2} - \frac{x^2}{4}}{x^4} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{0 - \frac{(0-2x)}{2\sqrt{a^2 - x^2}} - \frac{2x}{4}}{4x^3} = \lim_{x \rightarrow 0} \frac{x \left(\frac{1}{\sqrt{a^2 - x^2}} - \frac{1}{2} \right)}{4x^3}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{\sqrt{a^2 - x^2}} - \frac{1}{2}}{4x^2}$$

For limit to be exist, $a = 2$

$$31. \lim_{x \rightarrow 0} \frac{f(x) - 5}{x} = \lim_{x \rightarrow 0} \frac{\frac{g(x)}{x^2 + 2x + 3} - 5}{x}$$

$$= \lim_{x \rightarrow 0} \frac{g(x) - 5x^2 - 10x - 15}{x^3 + 2x^2 + 3x}$$

$$= \lim_{x \rightarrow 0} \frac{g'(x) - 10x - 10}{3x^2 + 4x + 3}$$

$$\Rightarrow 4 = \frac{g'(0) - 10}{3} \quad [\text{given}]$$

$$\Rightarrow g'(0) = 22$$

$$32. \lim_{x \rightarrow 3} \frac{3 - x}{\sqrt{4 + x} - \sqrt{1 + 2x}} = \lim_{x \rightarrow 3} \frac{(3 - x)[\sqrt{4 + x} + \sqrt{1 + 2x}]}{4 + x - 1 - 2x}$$

$$= \lim_{x \rightarrow 3} \frac{(3 - x)[\sqrt{4 + x} + \sqrt{1 + 2x}]}{(3 - x)}$$

$$= \lim_{x \rightarrow 3} [\sqrt{4 + x} + \sqrt{1 + 2x}] = \sqrt{7} + \sqrt{7} = 2\sqrt{7}$$

$$33. \lim_{x \rightarrow \tan^{-1} 3} \frac{\tan^2 x - 2 \tan x - 3}{\tan^2 x - 4 \tan x + 3}$$

$$= \lim_{\tan x \rightarrow 3} \frac{(\tan x - 3)(\tan x + 1)}{(\tan x - 3)(\tan x - 1)}$$

$$= \lim_{\tan x \rightarrow 3} \frac{\tan x + 1}{\tan x - 1} = \frac{3 + 1}{3 - 1} = \frac{4}{2} = 2$$

$$34. \lim_{x \rightarrow \infty} \frac{x^4 \cdot \sin\left(\frac{1}{x}\right) + x^2}{1 + |x|^3} = \lim_{x \rightarrow \infty} \left[\frac{x \sin\left(\frac{1}{x}\right) + \frac{1}{x}}{\frac{1}{x^3} + \frac{|x|^3}{x^3}} \right]$$

[dividing numerator and denominator by x^3]

$$= \frac{\lim_{x \rightarrow \infty} \frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}} + \lim_{x \rightarrow \infty} \frac{1}{x}}{\lim_{x \rightarrow \infty} \frac{1}{x^3} + \lim_{x \rightarrow \infty} \frac{|x|^3}{x^3}} = \frac{1 + 0}{0 + 1} = 1$$

$$35. \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{x} = \lim_{x \rightarrow 0} \left[\frac{(e^{\sin x} - 1)}{\sin x} \times \frac{\sin x}{x} \right]$$

$$= \lim_{x \rightarrow 0} \frac{e^{\sin x} - 1}{\sin x} \times \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \times 1 = 1$$

$$36. \lim_{x \rightarrow 0} \frac{2a \sin x - \sin 2x}{\tan^3 x}$$

$$= \lim_{x \rightarrow 0} \frac{2a \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right) - \left(2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \dots \right)}{\left(x + \frac{x^3}{3} + \frac{2}{15}x^5 + \dots \right)^3}$$

$$= \lim_{x \rightarrow 0} \frac{(2a - 2)x + \left(\frac{-2a}{3!} + \frac{2^3}{3!} \right)x^3 + \left(\frac{2a}{5!} - \frac{2^5}{5!} \right)x^5 + \dots}{x^3 \left(1 + \frac{x^2}{3} + \frac{2}{15}x^4 + \dots \right)}$$

Since, it is given that given limit is exist, therefore
 $2a - 2 = 0 \Rightarrow a = 1$

$$37. \lim_{x \rightarrow 2} \frac{\sin^{-1}(x - 2)}{x^2 - 4}$$

$$= \lim_{x \rightarrow 2} \left[\frac{\sin^{-1}(x - 2)}{(x - 2)} \times \frac{1}{(x + 2)} \right] = \left[\lim_{x \rightarrow 2} \frac{\sin^{-1}(x - 2)}{x - 2} \right] \times \frac{1}{4}$$

$$= 1 \times \frac{1}{4} = \frac{1}{4} \quad \left[\because \lim_{x \rightarrow 0} \frac{\sin^{-1} x}{x} = 1 \right]$$

38. From the question, $f(a) = 0$ and $f(x)$ is differentiable at $x = a$.

$$\therefore \text{ Required limit} = \lim_{x \rightarrow a} \frac{\frac{1}{1 + 6f(x)} \cdot 6f'(x)}{3f'(x)}$$

$$= 2 \cdot \frac{1}{1 + 6f(a)} = 2$$

39. The given expression

$$= \lim_{n \rightarrow \infty} \frac{n(n + 1)}{2(3n^2 + 5)} = \lim_{n \rightarrow \infty} \frac{1 \left(1 + \frac{1}{n} \right)}{2 \left(3 + \frac{5}{n^2} \right)} = \frac{1}{6}$$

$$40. \lim_{h \rightarrow 0} \frac{(a + h)^2 \sin(a + h) - a^2 \sin a}{h}$$

[using L'Hospital's rule]

$$= \lim_{h \rightarrow 0} \frac{2(a + h) \sin(a + h) + (a + h)^2 \cos(a + h)}{1}$$

$$= 2a \sin a + a^2 \cos a$$

$$41. \lim_{x \rightarrow 0} \left(\frac{1 + \tan x}{1 + \sin x} \right)^{\operatorname{cosec} x}$$

$$= \lim_{x \rightarrow 0} \left[\frac{\left(1 + \frac{\sin x}{\cos x} \right)^{\frac{\cos x}{\sin x}}}{(1 + \sin x)^{\frac{1}{\sin x}}} \right]$$

$$= \lim_{x \rightarrow 0} \frac{1}{e} = \frac{1}{e}$$

$$42. \lim_{x \rightarrow 0} \left[\frac{f(1 + x)}{f(1)} \right]^{\frac{1}{x}} = e^{\lim_{x \rightarrow 0} \frac{1}{x} \left[\frac{f(1 + x)}{f(1)} - 1 \right]}$$

$$= e^{\lim_{x \rightarrow 0} \frac{f(1 + x) - f(1)}{x} \times \frac{1}{f(1)}} = e^{f'(1) \times \frac{1}{f(1)}} = e^{\frac{6}{3}} = e^2$$

$$43. \text{ Consider } \lim_{x \rightarrow 0} \frac{x 2^x - x}{1 - \cos x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2^x + x 2^x \log 2 - 1}{\sin x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{2^x \log 2 + x 2^x \log 2 \cdot \log 2 + 2^x \log 2}{\cos x}$$

$$= \log 2 + \log 2 = 2 \log 2$$

$$44. \lim_{x \rightarrow 0} \frac{\pi^x - 1}{\sqrt{1+x} - 1} = \lim_{x \rightarrow 0} \frac{\pi^x \cdot \log \pi}{\frac{1}{2\sqrt{1+x}}} \quad [\text{using L'Hospital's rule}]$$

$$= \lim_{x \rightarrow 0} 2\sqrt{1+x} \log \pi (\pi^x)$$

$$= 2\pi^0 \log \pi = 2 \log \pi = \log_e \pi^2$$

$$45. \lim_{n \rightarrow \infty} \frac{(n!)^{1/n}}{n} = \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n} \right)^{1/n}$$

We have, $\frac{n!}{n^n} = \frac{1 \cdot 2 \cdot 3 \dots n}{n \cdot n \cdot n \dots n}$

$$\therefore \left\{ \frac{n!}{n^n} \right\}^{1/n} = \left\{ \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \dots \frac{r}{n} \dots \frac{n}{n} \right\}^{1/n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left\{ \frac{n!}{n^n} \right\}^{1/n} = \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \dots \frac{r}{n} \dots \frac{n}{n} \right\}^{1/n}$$

Let $A = \lim_{n \rightarrow \infty} \left\{ \frac{n!}{n^n} \right\}^{1/n}$

Then, $A = \lim_{n \rightarrow \infty} \left\{ \frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \dots \frac{r}{n} \dots \frac{n}{n} \right\}^{1/n}$

$$\Rightarrow \log A = \lim_{n \rightarrow \infty} \frac{1}{n} \sum \log \left(\frac{r}{n} \right) = \int_0^1 \log x dx$$

$$= \left[x \log x - \int \frac{1}{x} \cdot x dx \right]_0^1 = [x \log x - x]_0^1 = -1$$

$$\therefore A = e^{-1} = \frac{1}{e}$$

$$46. \lim_{x \rightarrow 0} \frac{(2+x)\sin(2+x) - 2\sin 2}{x} \quad \left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\sin(2+x) + (2+x)\cos(2+x)}{1}$$

$$= \sin 2 + 2 \cos 2$$

$$47. \text{As, } f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & [x] \neq 0 \\ 0, & [x] = 0 \end{cases}$$

$$\Rightarrow f(x) = \begin{cases} \frac{\sin[x]}{[x]}, & x \in R - [0, 1) \\ 0, & 0 \leq x < 1 \end{cases}$$

$\therefore$ RHL at $x = 0$, $\lim_{x \rightarrow 0} 0 = 0$

LHL at $x = 0$, $\lim_{h \rightarrow 0} \frac{\sin[0-h]}{[0-h]} = \lim_{h \rightarrow 0} \frac{\sin(-1)}{-1} = \sin 1$

Since, LHL $\neq$ RHL

$\therefore$ Limit does not exist.

$$48. \lim_{x \rightarrow 0} \left[\frac{x \cos x - \log_e(1+x)}{x^2} \right]$$

$$= \lim_{x \rightarrow 0} \left[\frac{-x \sin x + \cos x - \frac{1}{1+x}}{2x} \right]$$

[using L'Hospital's rule]

$$= \lim_{x \rightarrow 0} \left[\frac{-x \cos x - \sin x - \sin x + \frac{1}{(1+x)^2}}{2} \right] = \frac{1}{2}$$

$$49. f(x) = \begin{vmatrix} \sin x & \cos x & \tan x \\ x^3 & x^2 & x \\ 2x & 1 & x \end{vmatrix}$$

Expanding the determinant along the first row, we get

$$f(x) = \sin x(x^3 - x) - \cos x(x^4 - 2x^2) + \tan x(x^3 - 2x^3)$$

$$= \sin x(x^3 - x) - \cos x \cdot x^2(x^2 - 2) + \tan x(-x^3)$$

$$= \sin x(x^3 - x) - \cos x \cdot x^2(x^2 - 2) - \tan x(x^3)$$

$$\therefore \lim_{x \rightarrow 0} \frac{f(x)}{x^2}$$

$$= \lim_{x \rightarrow 0} \frac{\sin x(x^3 - x) - \cos x \cdot x^2(x^2 - 2) - \tan x \cdot x^3}{x^2}$$

$$= \lim_{x \rightarrow 0} \left\{ \frac{\sin x}{x} (x^2 - 1) - \cos x (x^2 - 2) - \tan x \cdot x \right\} = 1$$

$$50. \lim_{x \rightarrow a} \left[\frac{\sqrt{a+2x} - \sqrt{3x}}{\sqrt{3a+x} - 2\sqrt{x}} \right]$$

$$= \lim_{x \rightarrow a} \left[\frac{\frac{1}{\sqrt{a+2x}} - \frac{\sqrt{3}}{2\sqrt{x}}}{\frac{1}{2\sqrt{3a+x}} - \frac{1}{\sqrt{x}}} \right] \quad [\text{using L' Hospital's rule}]$$

$$= \left[\frac{\frac{1}{\sqrt{3a}} - \frac{\sqrt{3}}{2\sqrt{a}}}{\frac{1}{4\sqrt{a}} - \frac{1}{\sqrt{a}}} \right] = \frac{2-3}{\frac{1-4}{4\sqrt{a}}} = \frac{-1}{\frac{-3}{4\sqrt{a}}} = \frac{4}{3\sqrt{3}}$$

$$51. \lim_{k \rightarrow \infty} \left(\frac{1^3 + 2^3 + \dots + k^3}{k^4} \right)$$

$$= \lim_{k \rightarrow \infty} \left\{ \frac{k^2(k+1)^2}{4} \times \frac{1}{k^4} \right\} = \lim_{k \rightarrow \infty} \left\{ \frac{k^4 \left(1 + \frac{1}{k} \right)^2}{4} \times \frac{1}{k^4} \right\} = \frac{1}{4}$$

$$52. \lim_{x \rightarrow \infty} \left[1 + \frac{4}{x-1} \right]^{x+3} = \lim_{x \rightarrow \infty} \left\{ \left[1 + \frac{4}{x-1} \right]^{\frac{x-1}{4}} \right\}^{\frac{4(x+3)}{x-1}}$$

$$= e^{\lim_{x \rightarrow \infty} \frac{4(x+3)}{x-1}} = e^4$$

$$53. \text{Given, } f'(x) = f(x), f(0) = 1$$

Now, $\lim_{x \rightarrow 0} \frac{f(x) - 1}{x} = \lim_{x \rightarrow 0} f'(x)$

$$= \lim_{x \rightarrow 0} f(x) = f(0) = 1$$

$$54. \text{Let } y = \lim_{x \rightarrow 0} \left(\frac{16^x + 9^x}{2} \right)^{\frac{1}{x}}$$

$$\Rightarrow \log y = \lim_{x \rightarrow 0} \frac{1}{x} [\log(16^x + 9^x) - \log 2]$$

$$= \lim_{x \rightarrow 0} \left[\frac{16^x \log 16 + 9^x \log 9 - 0}{16^x + 9^x} \right]$$

[using L' Hospital's rule]

$$= \frac{\log 16 + \log 9}{2} = \frac{1}{2} \log 144$$

$$\therefore y = 12$$

$$55. \lim_{x \rightarrow \pi/2} \frac{\sin(\cos x) \cos x}{\sin x - \operatorname{cosec} x}$$

$$= \lim_{x \rightarrow \pi/2} \frac{\sin(\cos x)}{\cos x} \times \lim_{x \rightarrow \pi/2} \frac{\cos^2 x}{\sin x - \frac{1}{\sin x}}$$

$$= 1 \times \lim_{x \rightarrow \pi/2} \frac{\cos^2 x \sin x}{\sin^2 x - 1} = -1$$

$$56. \text{ Given, } \lim_{x \rightarrow 0} \frac{ae^x - b \cos x + ce^{-x}}{x \sin x} = 2$$

$$\text{LHS} = \frac{a \cdot e^0 - b \cdot 1 + c \cdot 1}{0}$$

$$\text{RHS} = 2 \Rightarrow a - b + c = 0$$

$$\therefore a = 1, b = 2 \text{ and } c = 1$$

$$57. \lim_{x \rightarrow 5} \frac{xf(5) - 5f(x)}{x - 5} = \lim_{x \rightarrow 5} \frac{f(5) - 5f'(x)}{1 - 0} = f(5) - 5f'(5)$$

[using L' Hospital's rule]

$$= 7 - 5 \cdot 7 = 7 - 35 = -28$$

$$58. \lim_{x \rightarrow \infty} \left(\frac{x^3}{3x^2 - 4} - \frac{x^2}{3x + 2} \right)$$

$$= \lim_{x \rightarrow \infty} \frac{x^3(3x + 2) - x^2(3x^2 - 4)}{(3x^2 - 4)(3x + 2)}$$

$$= \lim_{x \rightarrow \infty} \frac{2x^3 + 4x^2}{9x^3 + 6x^2 - 12x - 8}$$

$$= \lim_{x \rightarrow \infty} \frac{2 + \frac{4}{x}}{9 + \frac{6}{x} - \frac{12}{x^2} - \frac{8}{x^3}} = \frac{2}{9}$$

$$59. \lim_{x \rightarrow 0} \left(\frac{x}{\sqrt{1+x} - \sqrt{1-x}} \right)$$

$$= \lim_{x \rightarrow 0} \frac{x(\sqrt{1+x} + \sqrt{1-x})}{2x} = 1$$

$$60. \lim_{x \rightarrow 0} \left(\frac{1+5x^2}{1+3x^2} \right)^{\frac{1}{x^2}} = \lim_{x \rightarrow 0} \left(1 + \frac{2x^2}{1+3x^2} \right)^{\frac{1}{x^2}}$$

$$= e^{\lim_{x \rightarrow 0} \frac{1}{x^2} \left(\frac{2x^2}{1+3x^2} \right)} = e^2$$

$$61. \text{ LHL} = \lim_{x \rightarrow 0} \frac{-\sin x}{x} = -1$$

$$\text{RHL} = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\Rightarrow \text{LHL} \neq \text{RHL}$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin |x|}{x} \text{ does not exist.}$$

$$62. \lim_{x \rightarrow 0} \frac{\sin^2 x + \cos x - 1}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x - \cos^2 x}{x^2}$$

$$= \lim_{x \rightarrow 0} \cos x \cdot \frac{1 - \cos x}{x^2} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$63. \lim_{x \rightarrow 0} \frac{1 - \cos(1 - \cos x)}{x^4} = \lim_{x \rightarrow 0} \frac{\sin(1 - \cos x) \sin x}{4x^3}$$

[using L' Hospital's rule]

$$= \frac{1}{4} \cdot \lim_{x \rightarrow 0} \frac{\sin(1 - \cos x)}{1 - \cos x} \cdot \frac{1 - \cos x}{x^2} \cdot \frac{\sin x}{x}$$

$$= \frac{1}{4} \cdot 1 \cdot \frac{1}{2} \cdot 1 = \frac{1}{8}$$

19

Mathematical Reasoning

Statements or Propositions

A proposition or a statement is an declarative (or assertive) sentence which is either true or false but not both. A true statement is called valid statement. If a statement is false, then it is called an invalid statement.

Logical variables In the study of logics, statements are represented by lower case letters such as p, q, r and s . These letters are called logical variables.

e.g. The statement 'The sun is a star' may be represented or denoted by p and we write p : The Sun is a star.

Truth (T) or Falsity (F) of a statement is called its truth value.

➤ **Example 1.** Which of the following is/are statement(s)?

(i) 8 is less than 6.

(ii) Mathematics is fun.

Sol. (i) 8 is less than 6.

This sentence is false because 8 is greater than 6. Hence, it is a statement.

(ii) Mathematics is fun. This sentence is subjective in the sense that for those who like Mathematics, it may be fun but for others, it may not be. This means that this sentence is not always true. Hence, it is not a statement.

➤ **Example 2.** Write the truth value of each of the following statements.

(i) 6 has exactly two prime factors.

(ii) For every real number x , $x^2 \geq x$.

Sol. (i) We know that, 6 has two prime factors, namely 2 and 3. So, the given statement is true. Hence, its truth value is T.

(ii) Taking $x = \frac{1}{2}$, we find that $\left(\frac{1}{2}\right)^2 \geq \frac{1}{2}$ is not true. So, its truth value is F.

Chapter Snapshot

- Statements or Propositions
- Use of Venn Diagrams in Checking Truth and Falsity of Statements
- Truth Table
- Logical Connectives/Operators
- Quantifiers and Quantified Statements
- Negation of a Quantified Statement
- Logical Equivalence
- Negation of a Compound Statement
- Converse, Inverse and Contrapositive of an Implication
- Tautologies and Contradictions
- Algebra of Statements
- Duality

Types of Statements

- i. **Simple statement** A statement, if cannot be broken into two or more statements is a simple statement. The truth value of the simple statement does not explicitly depend on any other statement.
e.g. (i) p : $\sqrt{2}$ is a real number. Truth value: T
 q : Sun sets in the East. Truth value: F
- ii. **Compound statement** If a statement is combination of two or more simple statements, then it is said to be a compound statement or a compound proposition.
e.g. 'Roses are red and Violets are blue' is compound statement as it is composed of two statements.
 p : Roses are red.
 q : Violets are blue.
- iii. **Substatements** Simple statements which when combined, form a compound statement are called substatements, also called components statements.
- iv. **Open statement** A declarative sentence containing variable(s) is an open statement, if it becomes a statement when the variable(s) is (are) replaced by some definite value(s).
e.g. $x + 2y = 3$, $x, y \in N$

- The truth value of the compound statement is determined by the truth value of each of its substatements or components statements.
- The set of all those values of the variable(s) in an open statement for which it becomes a true statement is called the truth set of the open statement.

Example 3. Which of the following statements is/are open statements?

- (a) Ram eats a mango.
- (b) Krishna goes to school.
- (c) He lives in India.
- (d) Anil and Anuj are good friends.

Sol. (c) In given options, only option (c) is an open statement because in this sentence, 'he' can be specify to any person.

Example 4. Write down the truth set of the open statement $p(x): x^2 - x(3 + \sqrt{2}) + 3\sqrt{2} = 0$, $x \in Q$.

Sol. Consider $x^2 - x(3 + \sqrt{2}) + 3\sqrt{2} = 0$

$$\Rightarrow x^2 - 3x - \sqrt{2}x + 3\sqrt{2} = 0$$

$$\Rightarrow x(x - 3) - \sqrt{2}(x - 3) = 0$$

$$\Rightarrow (x - 3)(x - \sqrt{2}) = 0$$

$$\Rightarrow x = 3 \text{ or } x = \sqrt{2}$$

$$\therefore \text{ Truth set} = \{x : x \in Q, x^2 - x(3 + \sqrt{2}) + 3\sqrt{2} = 0\} = \{3\}$$

[$\because \sqrt{2}$ is an irrational]

Use of Venn Diagrams in Checking Truth and Falsity of Statements

Here, we shall discuss how Venn diagrams are used to represent truth and falsity of statements or propositions. For this, let us consider the statement. "All teachers are scholars". Let us assume that this statement is true. To represent the truth of the above statement, we define the following sets:

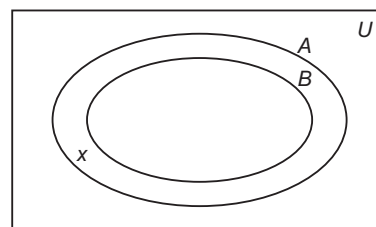
U = The set of all human beings

A = The set of all scholars

and B = The set of all teachers

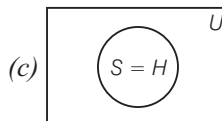
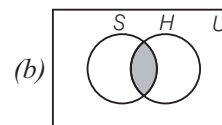
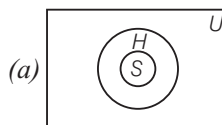
Clearly, $A \subset U$ and $B \subset U$

According to the above statements, it follows that $B \subset A$. Thus, the truth of the above statement can be represented by the Venn diagram as shown below:



Now, consider the statement 'there are some scholars who are teachers', it is evident from the Venn diagram that there is a scholar x who is not a teacher. Therefore, the above statement is false and its truth value is 'F'. Thus, we can also check the truth and falsity of other statements which are connected to a given statement.

Example 5. Which Venn diagram represent the truth of the statement 'All students are hard working', where U = Universal set of human beings S = Set of all students and H = Set of all hard workers?



(d) None of these

Sol. (a) All students are hard working means $S \subseteq H$.

Truth Table

A table which shows the relationship between the truth value of compound statement $S(p, q, r, \dots)$ and the truth values of its substatements $p, q, r, \dots$ etc., is called the truth table of statement S .

- i. For a single statement p , number of rows $= 2^1 = 2$

p
T
F

- ii. For two statements p and q

Number of rows $= 2^2 = 4$

p	q
T	T
T	F
F	T
F	F

- iii. For three statements p, q and r

Number of rows $= 2^3 = 8$

p	q	r
T	F	F
T	F	T
T	T	F
T	T	T
F	F	F
F	F	T
F	T	F
F	T	T

✎ If a compound statement has simply n substatements, then there are 2^n rows representing logical possibilities.

➤ **Example 6.** If there are 6 simple statements, then for making a table, the number of rows is
(a) 32 (b) 64 (c) 16 (d) 128

Sol. (b) We know that, if compound statements has n substatements, then there are 2^n rows in a table.

Here, $n = 6$

∴ Total number of rows $= 2^6 = 64$

Logical Connectives/Operators

The phrases or words which connect simple statements are called logical connectives or sentential connectives or simply connectives or logical operators. There are five such operations discussed below:

(i) Conjunction

A compound sentence formed by two simple sentences p and q using connective 'AND' is called the conjunction of p and q and is represented by $p \wedge q$.

e.g. Let p : Ramesh is a student.

and q : Ramesh belongs to Allahabad. Then,

$p \wedge q \equiv$ Ramesh is a student and he belongs to Allahabad.

The truth value of $p \wedge q$ is determined by the following:

- (i) The statement $p \wedge q$ is true, if both p and q are true.

- (ii) The statement $p \wedge q$ is false, if atleast one of p and q or both are false.

The truth table for operation 'AND' is given by

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(ii) Disjunction (Alternation)

A compound sentence formed by two simple sentences p and q using connective 'OR' is called the disjunction of p and q and is represented by $p \vee q$.

e.g. Let p : Bus left early and q : My watch is going slow. Then, $p \vee q =$ Bus left early or my watch is going slow.

The truth value of $p \vee q$ is determined by the following:

- (i) The statement $p \vee q$ is true, if atleast one of p and q or both are true.

- (ii) The statement $p \vee q$ is false, if both p and q are false.

The truth table for operation 'OR' is given by

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

(iii) Implication (Conditional)

A compound sentence formed by two simple sentences p and q using connective 'if...then ...' is called the implication of p and q and represented by $p \Rightarrow q$ (or $p \rightarrow q$) which is read as ' p implies q '. Here, p is called antecedent or hypothesis and q is called consequent or conclusion.

e.g. Let p : Train reaches in time.

and q : I can attend the meeting. Then,

$p \Rightarrow q \equiv$ If train reaches in time, then I can attend the meeting.

The truth value of $p \Rightarrow q$ is determined by the following :

- (i) If statements p and q both are true or false, then $p \Rightarrow q$ is true.
- (ii) If p is false and q is true, then $p \Rightarrow q$ is true.
- (iii) If p is true and q is false, then $p \Rightarrow q$ is false.

The truth table for 'if ... then' is given by

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

(iv) Biconditional Statement

Two simple sentences connected by the phrase 'if and only if', form a biconditional statement. It is represented by the symbol ' $\Leftrightarrow$ '.

e.g. Let $p : \Delta ABC$ be an isosceles triangle and $q : \text{Two sides of a triangle are equal}$. Then, $p \Leftrightarrow q : \Delta ABC$ is an isosceles triangle 'if and only if' two sides of a triangle are equal.

The truth value of $p \Leftrightarrow q$ is determined by the following:

- (i) The statement $p \Leftrightarrow q$ is true, if either both are true or both are false.
- (ii) The statement $p \Leftrightarrow q$ is false, if exactly one of them is false.

Truth Table for a Biconditional Statement

Since, $p \Leftrightarrow q$ is the conjunction of $p \Rightarrow q$ and $q \Rightarrow p$. So, we have the following truth table for $p \Leftrightarrow q$:

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$p \Leftrightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(v) Negation (Inversion) of a Statement

A statement which is formed by changing the truth value of a given statement by using word like 'no' or 'not' is called negation of a given statement. It is represented by the symbol ' $\sim$ '.

e.g. Let p : The number 2 is greater than 7.

Then, $\sim p$: The number 2 is not greater than 7.

If p is statement, then negation of p is denoted by ' $\sim p$ '. The truth table for NOT is given by

p	$\sim p$
T	F
F	T

Example 7. Write down the truth value of each of the following statements.

- (i) $3 > 5$ and $1 > 2$
- (ii) $(1 + i)$ is a real number or it is a complex number.
- (iii) If $7 > 3$, then $6 < 14$
- (iv) $4 > 2$ iff $0 < (4 - 2)$

Sol. (i) Let $p : 3 > 5$

$q : 1 > 2$

Then, $p \wedge q : 3 > 5$ and $1 > 2$.

Here, p is false and q is false, therefore $p \wedge q$ is false.

Hence, the given statement is false and so its truth value is F.

(ii) Let $p : (1 + i)$ is a real number.

$q : (1 + i)$ is a complex number.

Then, $(p \vee q) : (1 + i)$ is a real number or it is a complex number.

Here, p is false and q is true and therefore $p \vee q$ is true.

Hence, the given statement is true and so its truth value is T.

(iii) Let $p : 7 > 3$

$q : 6 < 14$

$\therefore p \Rightarrow q : \text{If } 7 > 3, \text{ then } 6 < 14$

Here, p is true and q is true and therefore

$p \Rightarrow q$ is true.

Hence, the given statement is true and its truth value is T.

(iv) Let $p : 4 > 2$

and $q : 0 < (4 - 2)$

The given statement is $p \Leftrightarrow q$.

Here, both p and q are true and therefore $p \Leftrightarrow q$ is true.

Hence, the given statement is true and its truth value is T.

Quantifiers and Quantified Statements

Quantifiers

Other phrases which are frequently found in mathematical statements are 'there exist' ($\exists$) and 'for all' ($\forall$). These two phrases are called quantifiers. Depending upon the context, the phrase 'there exist' can also be replaced by the equivalent phrases 'there is' or 'there is atleast one' or 'it is possible to find' or 'some'. Consider the statement p : There exists a rectangle whose all sides are equal.

This means that there is atleast one rectangle whose all sides are equal.

Quantified Statement

An open statement with a quantifier becomes a statement, called a quantified statement.

➤ **Example 8.** Use quantifiers to convert each of the following open sentences defined on N , into a true statement.

(i) $x + 5 = 8$

(ii) $x^2 > 0$

Sol. (i) $\exists x \in N$ such that $x + 5 = 8$ is a true statement, since $x = 3 \in N$ satisfies $x + 5 = 8$

(ii) $x^2 > 0, \forall x \in N$ is a true statement, since the square of every natural number is positive.

Negation of a Quantified Statement

i. $\sim \{\forall x \in A: p(x) \text{ is true}\} = \{\exists x \in A: \sim p(x) \text{ is true}\}$

ii. $\sim \{\exists x \in A: p(x) \text{ is true}\} = \{\forall x \in A: \sim p(x) \text{ is true}\}$

➤ **Example 9.** Write the negation of

(i) For all positive integers n , $n^2 + 41n + 41$ is a prime number.

(ii) There is a real number which is not a complex number.

Sol. The required negation is

(i) There is atleast one positive integer n for which $n^2 + 41n + 41$ is not prime.

(ii) Every real number is a complex number.

Logical Equivalence

Logically Equivalence Statement

Two compounds $S_1(p, q, r, \dots)$ and $S_2(p, q, r, \dots)$ are said to be logically equivalent or simply equivalent, if they have the same truth values for all logically possibilities.

If statements $S_1(p, q, r, \dots)$ and $S_2(p, q, r, \dots)$ are logically equivalent, then we write

$$S_1(p, q, r, \dots) \equiv S_2(p, q, r, \dots)$$

It follows from the above definition that two statements S_1 and S_2 are logically equivalent, if they have identical truth tables i.e. the entries in the last column of the truth tables are same.

- $p \Rightarrow q \equiv (\sim p \vee q)$
- $p \Leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$

➤ **Example 10.** Prove that the statements $S_1: p \wedge (q \Leftrightarrow r)$ and $S_2: p \wedge ((\sim q \vee r) \wedge (\sim r \vee q))$ are equivalent.

Sol. Since, $S_1: p \wedge (q \Leftrightarrow r)$, hence

p	q	r	$q \Leftrightarrow r$	S_1
F	F	F	T	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	F
T	F	F	T	T
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T

Also, $S_2: p \wedge [(\sim q \vee r) \wedge (\sim r \vee q)]$, we have

p	q	r	$\sim q$	$\sim r$	$\sim q \vee r$	$\sim r \vee q$	S_2
F	F	F	T	T	T	T	F
F	F	T	T	F	T	F	F
F	T	F	F	T	F	T	F
F	T	T	F	F	T	T	F
T	F	F	T	T	T	T	T
T	F	T	T	F	T	F	F
T	T	F	F	T	F	T	F
T	T	T	F	F	T	T	T

Since, the last columns representing the truth values of statements S_1 and S_2 have same truth value pattern corresponding to a respective substatement truth pattern. Hence, S_1 and S_2 are equivalent.

Negation of a Compound Statement

We have learnt about negation of a simple statement. Writing the negation of compound statements having conjunction, disjunctions, implication, equivalence, etc., is not very simple. So, let us discuss the negation of compound statement.

Negation of Conjunction

If p and q are two statements, then $\sim (p \wedge q) \equiv (\sim p \vee \sim q)$.

Negation of Disjunction

If p and q are two statements, then $\sim (p \vee q) \equiv (\sim p \wedge \sim q)$.

Negation of Implication

If p and q are two statements, then $\sim (p \Rightarrow q) \equiv (p \wedge \sim q)$.

Negation of Biconditional Statement or Equivalence

If p and q are two statements, then

$$\sim (p \Leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$$

➤ **Example 11.** Write down the negation of

- $3 + 6 > 8$ and $2 + 3 < 6$
- Ramesh is cruel or he is strict.
- If $\triangle ABC$ is isosceles, then the base angles A and B are equal.
- $|a| < 2$ if and only if $(a > -2$ and $a < 2)$.

Sol. (i) Let $p : 3 + 6 > 8$
 $q : 2 + 3 < 6$
 $\therefore p \wedge q : 3 + 6 > 8$ and $2 + 3 < 6$
 $\therefore \sim(p \wedge q) \equiv \sim p \vee \sim q$
 So, negation of the given statement is
 $3 + 6 \leq 8$ or $2 + 3 \geq 6$

(ii) Let $p : \text{Ramesh is cruel.}$
 $q : \text{He is strict.}$
 $\therefore p \vee q : \text{Ramesh is cruel or he is strict.}$
 $\therefore \sim(p \vee q) \equiv \sim p \wedge \sim q$
 $\therefore$ Negation of the given statement is,
 Ramesh is not cruel and he is not strict.
 That is, Ramesh is neither cruel nor strict.

(iii) Let $p : \triangle ABC$ is isosceles.
 $q : \text{The base angles } A \text{ and } B \text{ are equal.}$
 $\therefore \sim(p \Rightarrow q) \equiv (p \wedge \sim q)$
 $\therefore$ The negation of the given statement is given by
 $\triangle ABC$ is isosceles and the base angles A and B are not equal.

(iv) Let $p : |a| < 2$
 $q : a > -2$ and $a < 2$
 The given statement is $p \Leftrightarrow q$.
 $\therefore$ Negation of the given statement is
 $(p \wedge \sim q) \vee (q \wedge \sim p)$ given by
 Either $|a| < 2$ and $(a \leq -2$ or $a \geq 2)$ or $(a > -2$ and
 $a < 2)$ and $|a| \geq 2$.

➤ **Example 12.** Which of the following is logically equivalent to $\sim(\sim p \Rightarrow q)$?

- $p \wedge q$
- $p \wedge \sim q$
- $\sim p \wedge q$
- $\sim p \wedge \sim q$

Sol. (d) Since, $\sim(p \Rightarrow q) \equiv p \wedge \sim q$
 $\therefore \sim(\sim p \Rightarrow q) \equiv \sim p \wedge \sim q$

Converse, Inverse and Contrapositive of an Implication

Converse of Conditional Statement

If p and q are two statements, then converse of the implication is 'if p then q ' is 'if q then p '.

➤ If $p \Rightarrow q$ is true, then its converse $q \Rightarrow p$ may or may not be true e.g. Consider the following conditional statement 'if a triangle is equilateral, then it is equiangular'. The statement is true and converse statement 'if a triangle is equiangular, then it is equilateral' is also true. But if we consider another conditional statement, 'if a number is real number, then it is complex number'. It is true but its converse i.e. 'if a number is complex number, then it is real number' is not true.

Inverse of Conditional Statement

If p and q are two statements, then the inverse of 'if p then q ' is 'if $\sim p$ then $\sim q$ '. e.g. Consider the statement 'if a number is a multiple of 9, then it is a multiple of 3'.

It is an implication with 'if-then' having antecedent (p) and consequent (q) as given below :

$p : \text{A number is multiple of 9.}$

$q : \text{A number is multiple of 3.}$

The given statement is 'if p then q '. Its inverse is 'if $\sim p$ then $\sim q$ ' i.e. if a number is not a multiple of 9, then it is not a multiple of 3.

Contrapositive of Conditional Statements

If p and q are two statements, then contrapositive of the implication 'if p then q ' is 'if $\sim q$ then $\sim p$ ' i.e. let $p \Rightarrow q$ be conditional statement, then contrapositive of this conditional statement is $\sim q \Rightarrow \sim p$.

e.g. Let the conditional statement be 'if there is a strike of buses, then I will hire taxi', then its contrapositive statement is as follows:

'I will not hire taxi, if there is no strike of buses'.

➤ The contrapositive of a conditional statement have identical truth values as $p \Rightarrow q$ have, then the truth table is given by the following way:

I	II	III	IV	V	VI
p	q	$p \Rightarrow q$	$\sim p$	$\sim q$	$\sim q \Rightarrow \sim p$
T	T	T	F	F	T
T	F	F	F	T	F
F	T	T	T	F	T
F	F	T	T	T	T

On comparing III and VI column, it is verified that
 $p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

➤ **Example 13.** The inverse of the proposition $(p \wedge \sim q) \Rightarrow r$ is

- $\sim r \Rightarrow \sim p \vee q$
- $\sim p \vee q \Rightarrow \sim r$
- $r \Rightarrow p \wedge \sim q$
- None of these

Sol. (b) $\therefore$ Inverse of $p \Rightarrow q$ is $\sim p \Rightarrow \sim q$

$\therefore$ Inverse of $(p \wedge \sim q) \Rightarrow r$ is

$$\sim(p \wedge \sim q) \Rightarrow \sim r$$

$$\text{i.e. } (\sim p \vee q) \Rightarrow \sim r$$

➤ **Example 14.** The contrapositive of $(p \vee q) \Rightarrow r$ is

- $r \Rightarrow (p \vee q)$
- $\sim r \Rightarrow (p \vee q)$
- $\sim r \Rightarrow \sim p \wedge \sim q$
- $p \Rightarrow (q \vee r)$

Sol. (c) $\therefore$ Contrapositive of $p \Rightarrow q$ is $\sim q \Rightarrow \sim p$

$\therefore$ Contrapositive of $(p \vee q) \Rightarrow r$ is

$$\sim r \Rightarrow \sim(p \vee q) \text{ i.e. } \sim r \Rightarrow (\sim p \wedge \sim q)$$

Tautologies and Contradictions

Let $p, q, r, \dots$ be statements, then any statement involving $p, q, r, \dots$ and the logical connectives $\wedge, \vee, \sim, \Rightarrow, \Leftrightarrow$ is called a statement pattern or a Well Formed Formula (WFF).

e.g.

- (i) $p \vee q$
- (ii) $p \Rightarrow q$
- (iii) $[(p \wedge q) \vee r] \Rightarrow (s \wedge \sim s)$
- (iv) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ etc.,

are statement patterns.

A statement is also a statement pattern.

Thus, we can define statement pattern as follows:

Statement Pattern

A compound statement with the repetitive use of the logical connectives is called a statement pattern or a well formed formula.

Tautology

A statement pattern is called a tautology, if it is always true, whatever may be the truth values of constitute statements. A tautology is called a theorem or a logically valid statement pattern. A tautology contains only T in the last column of its truth table.

Contradiction

A statement pattern is called a contradiction, if it is always false, whatever may the truth values of its constitute statements.

In the last column of the truth table of contradiction, there is always F.

- The negation of a tautology is a contradiction and vice-versa.
- If a statement is neither tautology nor contradiction, then the statement is called contingency.

➤ **Example 15.** The statement $p \vee \sim (p \wedge q)$ is a

- (a) tautology
- (b) contradiction
- (c) contingency
- (d) None of these

Sol. (a)

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Hence, all values of a given statement are true, so it is a tautology.

➤ **Example 16.** The statement $p \wedge (q \Leftrightarrow r)$ is a

- (a) tautology
- (b) contradiction
- (c) contingency
- (d) None of these

Sol. (c)

p	q	r	$q \Leftrightarrow r$	$p \wedge (q \Leftrightarrow r)$
T	F	F	T	T
T	F	T	F	F
T	T	F	F	F
T	T	T	T	T
F	F	F	T	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	F

Algebra of Statements

Idempotent Laws

For any statement p , we have

- (i) $p \wedge p \equiv p$
- (ii) $p \vee p \equiv p$

Commutative Laws

For any two statements p and q , we have

- (i) $p \wedge q \equiv q \wedge p$
- (ii) $p \vee q \equiv q \vee p$

Associative Laws

For any three statements p, q and r , we have

- (i) $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- (ii) $(p \vee q) \vee r \equiv p \vee (q \vee r)$

Distributive Laws

For any three statements p, q and r , we have

- (i) $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- (ii) $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

Involution Laws

For any statement p , we have $\sim (\sim p) \equiv p$

De-Morgan's Laws

If p and q are any two statements, then

- (i) $\sim (p \wedge q) \equiv \sim p \vee \sim q$
- (ii) $\sim (p \vee q) \equiv \sim p \wedge \sim q$

Contrapositive Law

For any two statements p and q , we have

$$p \Rightarrow q \equiv (\sim q) \Rightarrow (\sim p)$$

Identity Law

For any statement p , we have

- (i) $p \wedge T = p$
- (ii) $p \vee F = p$
- (iii) $p \vee T = T$

where, T is a true statement and F is a false statements.

➤ **Example 17.** $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is

- (a) a tautology
- (b) a contradiction
- (c) both a tautology and a contradiction
- (d) neither a tautology nor a contradiction

Sol. (b) $(p \wedge \sim q) \wedge (\sim p \wedge q) = (p \wedge \sim p) \wedge (\sim q \wedge q) = f \wedge f = f$ [by using associative laws and commutative laws]
 $\therefore (p \wedge \sim q) \wedge (\sim p \wedge q)$ is a contradiction.

➤ **Example 18.** Simplify $(p \vee q) \wedge (p \vee \sim q)$.

(a) p (b) T (c) F (d) q

Sol. (a) $(p \vee q) \wedge (p \vee \sim q)$
 $\equiv p \vee (q \wedge \sim q)$ [distributive law]
 $\equiv p \vee F$ [complement law]
 $\equiv p$ [F is identity for $\vee$]

Duality

Two compound statements S_1 and S_2 are said to be duals of each other, if one can be obtained from the other by replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

Remarks

- The connectives $\wedge$ and $\vee$ are also called duals of each other.
- If a compound statement contains the special variable t (tautology) or c (contradiction), then to obtain its dual, we replace t by c and c by t in addition to replacing $\wedge$ by $\vee$ and $\vee$ by $\wedge$.

(iii) Let $S(p, q)$ be a compound statement containing two substatements and $S^*(p, q)$ be its dual.

Then,

- (a) $\sim S(p, q) \equiv S^*(\sim p, \sim q)$
 (b) $\sim S^*(p, q) \equiv S(\sim p, \sim q)$

(iv) The above result can be extended to the compound statements having finite number of substatements. Thus, if $S(p_1, p_2, \dots, p_n)$ is a compound statement containing n substatements $p_1, p_2, \dots, p_n$ and $S^*(p_1, p_2, \dots, p_n)$ is its dual. Then,

- (a) $\sim S(p_1, p_2, \dots, p_n) \equiv S^*(\sim p_1, \sim p_2, \dots, \sim p_n)$
 (b) $\sim S^*(p_1, p_2, \dots, p_n) \equiv S(\sim p_1, \sim p_2, \dots, \sim p_n)$

➤ **Example 19.** The dual of the statement $(p \vee q) \wedge \sim (r \vee s)$ is

- (a) $(p \vee q) \vee \sim (r \vee s)$ (b) $(p \wedge q) \vee \sim (r \wedge s)$
 (c) $(p \wedge q) \vee (\sim r \vee \sim s)$ (d) $(p \wedge q) \vee (\sim r \wedge \sim s)$

Sol. (b,c) Let $S : (p \vee q) \wedge \sim (r \vee s)$, then dual of S is $S^* : (p \wedge q) \vee \sim (r \wedge s) \equiv (p \wedge q) \vee (\sim r \vee \sim s)$

Work Book Exercise

- Which of the following is an open statement?
 a x is a natural number
 b Give me a glass of water
 c Wish you best of luck
 d Good morning to all
- Which of the following statement is not a conjunction?
 a Ram and Shyam are friends
 b Both Ram and Shyam are tall
 c Both Ram and Shyam are enemies
 d None of the above
- Negation of ' $2 + 3 = 5$ and $8 < 10$ ' is
 a $2 + 3 \neq 5$ and $8 < 10$ b $2 + 3 = 5$ and $8 \not< 10$
 c $2 + 3 \neq 5$ or $8 \not< 10$ d None of the above
- If Ram secures 100 marks in Mathematics, then he will get a mobile. The converse is
 a If Ram gets a mobile, then he will not secure 100 marks in Mathematics
 b If Ram does not get a mobile, then he will secure 100 marks in Mathematics
 c If Ram will get a mobile, then he secures 100 marks in Mathematics
 d None of the above
- For the following three statements
 p : 2 is an even number.
 q : 2 is a prime number.
 r : Sum of two prime numbers is always even.
 Then, the symbolic statements $(p \wedge q) \rightarrow \sim r$ means
 a 2 is an even and prime number and the sum of two prime numbers is always even
 b 2 is an even and prime number and the sum of two prime numbers is not always even
 c 2 is an even and prime number, then the sum of two prime numbers is not always even
 d 2 is an even and prime number, then the sum of two prime numbers is always even
- The dual of the statement $(p \wedge q) \wedge \sim q \equiv (p \vee \sim q)$ is
 a $(p \wedge q) \wedge \sim q \equiv (p \wedge \sim q)$
 b $(p \vee q) \vee \sim q \equiv (p \wedge \sim q)$
 c $(p \wedge q) \vee \sim q \equiv (p \vee \sim q)$
 d $(p \vee q) \vee \sim q \equiv (p \vee \sim q)$
- If p, q and r are simple propositions with truth values true, false and true respectively, then the truth value of $((\sim p \vee q) \wedge \sim r) \Rightarrow p$ is
 a true
 b false
 c true, if r is false
 d true, if q is true
- Which of the following is always true?
 a $(\sim p \vee \sim q) \equiv (p \wedge q)$
 b $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$
 c $\sim(p \rightarrow \sim q) \equiv (p \wedge \sim q)$
 d $\sim(p \leftrightarrow q) \equiv (p \rightarrow q) \rightarrow (q \rightarrow p)$
- Let p and q be two statements. Then, $(\sim p \vee q) \wedge (\sim p \wedge \sim q)$ is a
 a tautology
 b contradiction
 c neither tautology nor contradiction
 d both tautology and contradiction
- Which of the following is wrong statement?
 a $p \rightarrow q$ is logically equivalent to $\sim p \vee q$
 b If the truth values of p, q, r are T, F, T respectively, then the truth value of $(p \vee q) \wedge (q \vee r)$ is T
 c $\sim(\vee q \vee q \vee r) \equiv \sim p \wedge \sim q \wedge \sim r$
 d The truth value of $p \wedge \sim(p \vee q)$ is always T

WorkedOut Examples

Ex 1. Consider the statements

- (i) Two plus three is five.
- (ii) Every square is a rectangle.
- (iii) Sun rises in the East.
- (iv) The Earth is a star.

Which of the above statements having truth value (T)?

- (a) (i) and (ii)
- (b) (ii) and (iii)
- (c) (iii) and (iv)
- (d) All of these

Sol. Each of the statement is a true declarative statement, so each statement have truth value (T).
Hence, (d) is the correct answer.

Ex 2. Which one of the following statements is not a false statement?

- (a) p : Each radius of a circle is a chord of the circle
- (b) q : Circle is a particular case of an ellipse
- (c) r : $\sqrt{13}$ is a rational number
- (d) s : The centre of a circle bisect each chord of the circle

Sol. We know that, equation of an ellipse is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

If we take $a = b$, then we get $x^2 + y^2 = a^2$ which satisfies all the conditions of circle.

$\therefore$ Circle is the particular case of an ellipse.

Hence, (b) is the correct answer.

Ex 3. Consider the following compound statements.

- (i) Mumbai is the capital of Rajasthan or Maharashtra.
- (ii) $\sqrt{3}$ is a rational or an irrational number.
- (iii) 125 is a multiple of 7 or 8.
- (iv) A rectangle is a quadrilateral or a regular hexagon.

Which of the above statements is not true?

- (a) (i)
- (b) (ii)
- (c) (iii)
- (d) (iv)

Sol. (i) The component statements of 'Mumbai is the capital of Rajasthan or Maharashtra' are

p : Mumbai is the capital of Rajasthan.

q : Mumbai is the capital of Maharashtra.

We note that, p is false and q is true, so the compound statement $p \vee q$ is true.

(ii) The component statements of ' $\sqrt{3}$ is a rational or an irrational number' are

p : $\sqrt{3}$ is a rational number.

q : $\sqrt{3}$ is an irrational number.

We note that, p is false and q is true, so compound statement $p \vee q$ is true.

(iii) The component statements of '125 is a multiple of 7 or 8' are

p : 125 is a multiple of 7.

q : 125 is a multiple of 8.

We note that, p and q both are false statements, so compound statement $p \vee q$ is false.

(iv) The component statements of 'A rectangle is a quadrilateral or a regular hexagon' are

p : A rectangle is a quadrilateral.

q : A rectangle is a regular hexagon

As p is true and q is false. So, compound statement $p \vee q$ is true.

Hence, (c) is the correct answer.

Ex 4. If p, q and r are statements with truth values false, true and false respectively, then the truth value of $(\sim p \vee \sim q) \vee r$ is

- (a) true
- (b) false
- (c) false, if r is true
- (d) false, if q is false

Sol. Let $S : (\sim p \vee \sim q) \vee r$

$\Rightarrow S : (\sim F \vee \sim T) \vee F$

$\therefore S : (T \vee F) \vee F = T$

Hence, (c) is the correct answer.

Ex 5. The statement $p \wedge (\sim q \vee r) \wedge (\sim r \vee q)$ is

- (a) tautology
- (b) contradiction
- (c) contingency
- (d) None of these

Sol.

p	q	r	$\sim q$	$\sim r$	$\sim q \vee r$	$\sim r \vee q$	$p \wedge [(\sim q \vee r) \wedge (\sim r \vee q)]$
T	F	F	T	T	T	T	T
T	F	T	T	F	T	F	F
T	T	F	F	T	F	T	F
T	T	T	F	F	T	T	T
F	F	F	T	T	T	T	F
F	F	T	T	F	T	F	F
F	T	F	F	T	F	T	F
F	T	T	F	F	T	T	F

Hence, (c) is the correct answer.

Ex 6. If p, q and r are simple propositions and $(p \wedge q) \wedge (q \wedge r)$ is true, then

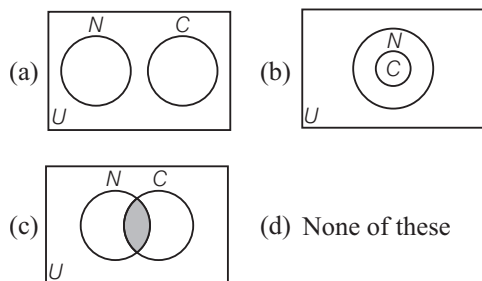
- (a) p, q and r are true
- (b) p, q are true and r is false
- (c) p is true and q, r are false
- (d) p, q and r are false

Sol.

p	q	r	$p \wedge q$	$q \wedge r$	$(p \wedge q) \wedge (q \wedge r)$
T	F	F	F	F	F
T	F	T	F	F	F
T	T	F	T	F	F
T	T	T	T	T	T
F	F	F	F	F	F
F	F	T	F	F	F
F	T	F	F	F	F
F	T	T	F	T	F

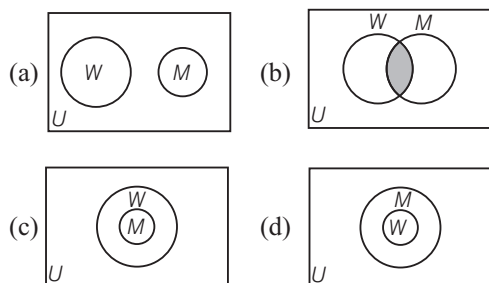
Hence, (a) is the correct answer.

Ex 7. Which Venn diagram represents the truth of the statements 'No child is naughty' where, U is the universal set of human beings C is the set of children and N is set of naughty persons?



Sol. 'No child is naughty' means $C \cap N = \phi$
i.e. there is no common element between C and N .
Hence, (a) is the correct answer.

Ex 8. Which of the following Venn diagram corresponds to the statement: 'All mothers are women' (M is the set of all mothers, W is the set of all women)?



Sol. All mothers are women.
 $M \subseteq W$
Hence, (c) is the correct answer.

Ex 9. If p : It rains today, q : I go to school, r : I shall meet any friends and s : I shall go for a movie, then which of the following is the proposition: If it does not rain or if I do not go to school, then I shall meet my friend and go for a movie.

- (a) $\sim(p \wedge q) \Rightarrow (r \wedge s)$
(b) $\sim(p \wedge \sim q) \Rightarrow (r \wedge s)$
(c) $\sim(p \wedge q) \Rightarrow (r \vee s)$
(d) None of the above

Sol. Correct result is $(\sim p \vee \sim q) \Rightarrow (r \wedge s)$
So, $\sim(p \wedge q) \Rightarrow (r \wedge s)$
Hence, (a) is the correct answer.

Ex 10. Which of the following is a contradiction?

- (a) $(p \wedge q) \wedge \sim(p \vee q)$ (b) $p \vee (\sim p \wedge q)$
(c) $(p \Rightarrow q) \Rightarrow p$ (d) None of these

Sol.

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

$\therefore (p \wedge q) \wedge [\sim(p \vee q)]$ is a contradiction.

Hence, (a) is the correct answer.

Ex 11. Negation of 'Ram is in class X or Rashmi is in class XII' is

- (a) Ram is not in class X but Ram is in class XII
(b) Ram is not in class X but Rashmi is not in class XII
(c) Either Ram is not in class X or Ram is not in class XII
(d) None of the above

Sol. Let p : Ram is in class X, q : Rashmi is in class XII.
Given proposition is $p \vee q$.
Its negation is $\sim(p \vee q) = \sim p \wedge \sim q$
i.e. Ram is not in class X and Rashmi is not in class XII.
Hence, (d) is the correct answer.

Ex 12. If $(p \wedge \sim r) \Rightarrow (q \vee r)$ is false and q and r are both false, then p is

- (a) true (b) false
(c) may be true or false (d) Data insufficient

Sol. Given, result means $p \wedge \sim r$ is true, $q \vee r$ is false.
Hence, (a) is the correct answer.

Target Exercises

1. Which of the following is a statement?

- (a) x is a real number
- (b) Switch off the fan
- (c) 6 is a natural number
- (d) Let me go

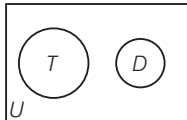
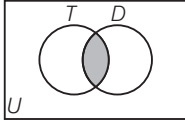
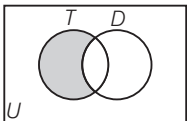
2. Which of the following is not a statement?

- (a) Smoking is injurious to health
- (b) $2 + 2 = 4$
- (c) 2 is the only even prime number
- (d) Come here

3. Let R be the set of real numbers and $x \in R$. Then, $x + 3 = 8$ is

- (a) open statement
- (b) a true statement
- (c) false statement
- (d) None of these

4. Which Venn diagram represents the truth of the statement 'Some teenagers are not dreamers'?

- (a)  (b) 
- (c)  (d) None of these

5. The conditional statement of

'You will get a sweet dish after the dinner' is

- (a) if you take the dinner, then you will get a sweet dish
- (b) if you take the dinner, you will get a sweet dish
- (c) you get a sweet dish if and only if you take the dinner
- (d) None of the above

6. For two statements p and q

p : A quadrilateral is a parallelogram.

q : The opposite sides are parallel.

Then, the compound proposition, 'A quadrilateral is a parallelogram if and only if the opposite sides are parallel', is represented by

- (a) $p \vee q$
- (b) $p \rightarrow q$
- (c) $p \wedge q$
- (d) $p \leftrightarrow q$

7. Suppose p : A natural number n is odd and q : Natural number n is not divisible by 2, then the biconditional statement $p \leftrightarrow q$ is

- (a) a natural number n is odd, if and only if it is divisible by 2
- (b) a natural number n is odd, if and only if it is not divisible by 2
- (c) if a natural number n is odd, then it is not divisible by 2
- (d) None of the above

8. If p and q are two statements such that

p : The question paper is easy.

q : We shall pass.

Then, the symbolic statement $\sim p \rightarrow \sim q$ means

- (a) if the question paper is easy, then we shall pass
- (b) if the question paper is not easy, then we shall not pass
- (c) the question paper is easy and we shall pass
- (d) the question paper is easy or we shall pass

9. If p : The Earth is round, q : $3 + 4 = 7$, then

$(\sim p) \vee (\sim q)$ is

- (a) it is not that the Earth is round or $3 + 4 = 7$
- (b) the Earth is round and $3 + 4 = 7$
- (c) it is not that the Earth is round or it is not that $3 + 4 = 7$
- (d) the Earth is round or $3 + 4 = 7$

10. If p, q, r, s and t are simple statements, such that

p : A number is terminating.

q : A number is recurring.

r : A number is a rational number.

s : A number is an irrational number.

t : A number is a real number.

Then, the compound statement, 'A number is an irrational number if and only if it is neither non-terminating nor not recurring or rational if and only if it is terminating or if non-terminating, then must be recurring and if the number is real, it can be rational or irrational', is represented by

- (a) $[s \rightarrow \sim(p \wedge q)] \vee [r \rightarrow \{p \vee (p \rightarrow q)\}] \wedge [t \rightarrow (r \vee s)]$
- (b) $[s \leftrightarrow \sim(p \vee q)] \vee [r \leftrightarrow \{p \vee (\sim p \rightarrow q)\}] \wedge [t \rightarrow (r \vee s)]$
- (c) $[s \leftrightarrow \sim(p \wedge q)] \vee [r \rightarrow \{p \vee (p \rightarrow q)\}] \wedge [t \rightarrow \sim(r \wedge s)]$
- (d) None of the above

11. Which of the following is true for the statements p and q ?

- (a) $p \wedge q$ is true when atleast one of p and q is true
- (b) $p \rightarrow q$ is true when p is true and q is false
- (c) $p \leftrightarrow q$ is true only when both p and q are true
- (d) $\sim(p \vee q)$ is true only when both p and q are false

12. If p and q are simple propositions, then $p \Rightarrow \sim q$ is true when

- (a) both p and q are true
- (b) both p and q are false
- (c) p is false and q is true
- (d) None of the above

13. If p, q are true statements and r, s are false statements, then the truth value of $\sim[(p \wedge \sim r) \vee (\sim q \vee s)]$ is

- (a) true
- (b) false
- (c) false for any p
- (d) None of the above

14. If the compound statement $p \rightarrow (\sim p \vee q)$ is false, then the truth values of p and q are respectively
(a) T, T (b) T, F
(c) F, T (d) F, F
15. In the truth table for the statement $(\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow \sim p)$, the last column has the truth value in the following order
(a) TTTF (b) FTTF
(c) TFFT (d) TTTT
16. Which of the following is equivalent to $(p \wedge q)$?
(a) $p \rightarrow \sim q$ (b) $\sim(\sim p \wedge \sim q)$
(c) $\sim(p \rightarrow \sim q)$ (d) None of these
17. Which of the following is not logically equivalent to the following proposition 'A real number is either rational or irrational'?
(a) If a number is neither rational nor irrational, then it is not real
(b) If a number is not a rational or not an irrational, then it is not real
(c) If a number is not real, then it is neither rational or irrational
(d) If a number is real, then it is rational or irrational
18. The negation of the statement '72 is divisible by 2 and 3' is
(a) 72 is not divisible by 2 or 72 is not divisible by 3
(b) 72 is not divisible by 2 and 72 is not divisible by 3
(c) 72 is divisible by 2 and 72 is not divisible by 3
(d) 72 is not divisible by 2 and 72 is divisible by 3
19. The negation of the statement, "if a quadrilateral is a square then it is a rhombus" is
(a) if a quadrilateral is not a square, then it is a rhombus
(b) if a quadrilateral is a square, then it is not a rhombus
(c) a quadrilateral is a square and it is not a rhombus
(d) a quadrilateral is not a square and it is a rhombus
20. If p : He is intelligent.
 q : He is strong.
Then, symbolic form of statement 'It is wrong that he is intelligent or strong', is
(a) $\sim p \vee \sim q$
(b) $\sim(p \wedge q)$
(c) $\sim(p \vee q)$
(d) $p \vee \sim q$
21. Negation of the statement $(p \wedge r) \rightarrow (r \vee q)$ is
(a) $\sim(p \wedge r) \rightarrow \sim(r \vee q)$
(b) $(\sim p \vee \sim r) \vee (r \vee q)$
(c) $(p \wedge r) \wedge (r \wedge q)$
(d) $(p \wedge r) \wedge (\sim r \wedge \sim q)$
22. The negation of $(p \vee q) \wedge (p \vee \sim r)$ is
(a) $(\sim p \wedge \sim q) \vee (p \wedge \sim r)$
(b) $(\sim p \wedge \sim q) \wedge (\sim p \wedge r)$
(c) $(\sim p \wedge \sim q) \vee (\sim p \wedge r)$
(d) None of the above
23. The contrapositive of the statement 'If 7 is greater than 5, then 8 is greater than 6', is
(a) if 8 is greater than 6, then 7 is greater than 5
(b) if 8 is not greater than 6, then 7 is greater than 5
(c) if 8 is not greater than 6, then 7 is not greater than 5
(d) if 8 is greater than 6, then 7 is not greater than 5
24. The contrapositive and converse of the statement 'I go to beach whenever it is a sunny day' is
(a) (i) if it is not a sunny day, then I do not go to beach
(ii) if it is a sunny day, then I go to beach
(b) (i) if it is a sunny day, then I do not go to beach
(ii) if it is a sunny day, then I go to beach
(c) (i) if it is not a sunny day, then I go to beach
(ii) if it is a sunny day, then I go to beach.
(d) None of the above
25. The proposition $(p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$ is
(a) contingency
(b) neither tautology nor contradiction
(c) contradiction
(d) tautology
26. If p and q are two statements, then statement $p \Rightarrow q \wedge \sim q$ is
(a) tautology
(b) contradiction
(c) neither tautology nor contradiction
(d) None of the above
27. The false statement in the following is
(a) $p \wedge (\sim p)$ is contradiction
(b) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a contradiction
(c) $\sim(\sim p) \Leftrightarrow p$ is a tautology
(d) $p \vee (\sim p) \Leftrightarrow$ is a tautology
28. The statement $(\sim p \wedge q) \vee \sim q$ is
(a) $p \vee q$
(b) $p \wedge q$
(c) $\sim(p \vee q)$
(d) $\sim(p \wedge q)$
29. The dual of the statement $(p \wedge q) \wedge \sim q$ is
(a) $(p \vee q) \wedge q$
(b) $(p \vee q) \vee \sim q$
(c) $(p \wedge q) \wedge \sim q$
(d) $(p \vee q) \vee q$
30. The dual of the statement $[(p \vee q) \wedge \sim q] \vee (\sim q)$ is
(a) $[(p \vee q) \wedge \sim q] \vee (\sim q)$
(b) $[(p \wedge q) \wedge \sim q] \vee (\sim q)$
(c) $[(p \wedge q) \wedge \sim p] \vee (\sim q)$
(d) $[(p \wedge q) \vee \sim q] \wedge (\sim q)$
31. The dual of the statement $(p \vee \sim q) \wedge (\sim p)$ is
(a) $(p \vee \sim q) \wedge (\sim p)$
(b) $(p \wedge \sim q) \wedge (\sim p)$
(c) $(p \vee \sim q) \vee (\sim p)$
(d) $(p \wedge \sim q) \vee (\sim p)$

Entrances Gallery

JEE Main/AIEEE

- The negation of $\sim s \vee (\sim r \wedge s)$ is equivalent to [2015]
 - $s \wedge \sim r$
 - $s \wedge (r \wedge \sim s)$
 - $s \vee (r \vee \sim s)$
 - $s \wedge r$
- The statement $\sim(p \leftrightarrow \sim q)$ is [2014]
 - equivalent to $p \leftrightarrow q$
 - equivalent to $\sim p \leftrightarrow q$
 - a tautology
 - a fallacy
- Consider
Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$ is a fallacy.
Statement II $(p \rightarrow q) \leftrightarrow (\sim q \rightarrow \sim p)$ is a tautology. [2013]
 - Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 - Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 - Statement I is true, Statement II is false
 - Statement I is false, Statement II is true
- Consider the following statements
 P : Suman is brilliant.
 Q : Suman is rich.
 R : Suman is honest.
 The negative of the statement, 'Suman is brilliant and dishonest if and only if Suman is rich' can be expressed as [2011]
 - $\sim[Q \leftrightarrow (P \wedge \sim R)]$
 - $\sim Q \leftrightarrow P \vee R$
 - $\sim(P \wedge \sim R) \leftrightarrow Q$
 - $\sim P \wedge (Q \leftrightarrow \sim R)$
- The only statement among the followings that is a tautology, is [2011]
 - $B \rightarrow [A \wedge (A \rightarrow B)]$
 - $A \wedge (A \vee B)$
 - $A \vee (A \wedge B)$
 - $[A \wedge (A \rightarrow B)] \rightarrow B$
- Let S be a non-empty subset of R . Consider the following statement.
 P : There is a rational number $x \in S$ such that $x > 0$.
 Which of the following statement is the negation of the statement P ? [2010]
 - There is a rational number $x \in S$ such that $x \leq 0$
 - There is no rational number $x \in S$ such that $x \leq 0$
 - Every rational number $x \in S$ satisfies $x \leq 0$
 - $x \in S$ and $x \leq 0 \Rightarrow x$ is not rational
- Let p be the statement 'x is an irrational number', q be the statement 'y is a transcendental number' and r be the statement 'x is a rational number iff y is a transcendental number'.
Statement I r is equivalent to either q or p .
Statement II r is equivalent to $\sim(p \leftrightarrow \sim q)$. [2008]
 - Statement I is false, Statement II is true
 - Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 - Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 - Statement I is true, Statement II is false
- The statement $p \rightarrow (q \rightarrow p)$ is equivalent to [2008]
 - $p \rightarrow (p \leftrightarrow q)$
 - $p \rightarrow (p \rightarrow q)$
 - $p \rightarrow (p \vee q)$
 - $p \rightarrow (p \wedge q)$

Other Engineering Entrances

- The negation of $(\sim p \wedge q) \vee (p \wedge \sim q)$ is [VITEEE 2014]
 - $(p \vee \sim q) \vee (\sim p \vee q)$
 - $(p \vee \sim q) \wedge (\sim p \vee q)$
 - $(p \wedge \sim q) \wedge (\sim p \vee q)$
 - $(p \wedge \sim q) \wedge (p \vee \sim q)$
- The statement $(p \Rightarrow q) \Leftrightarrow (\sim p \wedge q)$ is a [VITEEE 2014]
 - tautology
 - contradiction
 - Neither (a) nor (b)
 - None of the above
- If p and q are two statements, then $\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$ is [Manipal 2014]
 - tautology
 - contradiction
 - neither tautology nor contradiction
 - either tautology or contradiction
- Which of the following is not a correct statement? [Karnataka CET 2014]
 - Mathematics is interesting
 - $\sqrt{3}$ is a prime number
 - $\sqrt{2}$ is an irrational number
 - The Sun is a star
- If p, q and r are any three logical statements, then which one of the following is true? [Kerala CEE 2014]
 - $\sim[p \wedge (\sim q)] \equiv (\sim p) \wedge q$
 - $\sim(p \vee q) \wedge (\sim r) \equiv (\sim p) \vee (\sim q) \vee (\sim r)$
 - $\sim[p \vee (\sim q)] \equiv (\sim p) \wedge q$
 - $\sim[p \wedge (\sim q)] \equiv (\sim p) \wedge \sim q$
 - $\sim[p \wedge (\sim q)] \equiv p \wedge q$
- The truth values of p, q and r for which $(p \wedge q) \vee (\sim r)$ has truth value F, are respectively [Kerala CEE 2014]
 - F, T, F
 - F, F, F
 - T, T, T
 - T, F, F
 - F, F, T

15. $\sim[(\sim p) \wedge q]$ is logically equivalent to [Kerala CEE 2014]
 (a) $\sim(p \vee q)$ (b) $\sim[p \wedge (\sim q)]$
 (c) $p \wedge (\sim q)$ (d) $p \vee (\sim q)$
 (e) $(\sim p) \vee (\sim q)$
16. The negative of the compound proposition $p \vee (\sim p \vee q)$ is [Kerala CEE 2013]
 (a) $(p \wedge \sim q) \wedge \sim p$ (b) $(p \wedge \sim q) \vee \sim p$
 (c) $(p \wedge \sim q) \vee p$ (d) $(p \wedge q) \wedge q$
 (e) None of these
17. The negation of $p \rightarrow (\sim p \vee q)$ is [Karnataka CET 2011]
 (a) $p \vee (p \vee \sim q)$ (b) $p \rightarrow \sim(p \vee q)$
 (c) $p \rightarrow q$ (d) $p \wedge \sim q$
18. If p : 'Roses are red' and q : 'The Sun is a star'.
 Then, the verbal translation of $(\sim p) \vee q$ is [Kerala CEE 2011]
 (a) Roses are not red and the Sun is not a star
 (b) it is not true that roses are red or the Sun is not a star
 (c) it is not true that roses are red and the Sun is a star
 (d) Roses are not red or the Sun is a star
 (e) it is not true that roses are red and the Sun is a star
19. Which of the following is not a logical statement? [GGSIPU 2011]
 (a) 8 is less than 6 (b) Every set is a finite set
 (c) Kashmir is far from here (d) The Sun is a star
20. The contrapositive statement of the proposition $p \rightarrow \sim q$ is [J&K CET 2011]
 (a) $\sim p \rightarrow q$ (b) $\sim q \rightarrow p$
 (c) $q \rightarrow \sim p$ (d) None of these
21. For any two statements p and q , $\sim(p \vee q) \vee (\sim p \wedge q)$ is logically equivalent to [VITEEE 2010]
 (a) p
 (b) $\sim p$
 (c) q
 (d) $\sim q$
22. If $S(p, q, r) = (\sim p) \vee [\sim(q \wedge r)]$ is a compound statement, then $S(\sim p, \sim q, \sim r)$ is [Kerala CEE 2010]
 (a) $\sim S(p, q, r)$
 (b) $S(p, q, r)$
 (c) $p \vee (q \wedge r)$
 (d) $p \vee (q \vee r)$
 (e) $S(p, q, \sim r)$
23. If p : 7 is not greater than 4 and q : Paris is in France are two statements. Then, $\sim(p \vee q)$ is the statement [Kerala CEE 2010]
 (a) 7 is greater than 4 or Paris is not in France
 (b) 7 is not greater than 4 and Paris is not in France
 (c) 7 is greater than 4 and Paris is in France
 (d) 7 is not greater than 4 or Paris is not in France
 (e) 7 is greater than 4 and Paris is not in France

Answers

Work Book Exercise

1. (a) 2. (d) 3. (c) 4. (c) 5. (c) 6. (b) 7. (a) 8. (b) 9. (c) 10. (d)

Target Exercises

1. (c) 2. (d) 3. (a) 4. (c) 5. (a) 6. (d) 7. (b) 8. (b) 9. (c) 10. (b)
 11. (d) 12. (d) 13. (b) 14. (b) 15. (c) 16. (c) 17. (b) 18. (a) 19. (c) 20. (c)
 21. (d) 22. (c) 23. (c) 24. (a) 25. (c) 26. (c) 27. (b) 28. (d) 29. (b) 30. (d)
 31. (d)

Entrances Gallery

1. (d) 2. (a) 3. (b) 4. (a) 5. (d) 6. (c) 7. (*) 8. (c) 9. (b) 10. (c)
 11. (c) 12. (b) 13. (c) 14. (e) 15. (d) 16. (a) 17. (d) 18. (d) 19. (c) 20. (c)
 21. (b) 22. (d) 23. (e)

Note : * Means none option is correct.

Explanations

Target Exercises

- (a) Here, without knowing the value of x , we do not know whether the given sentence is true or false. So, it is a sentence but not a statement.
(b) It is an imperative sentence and therefore it is not a statement.
(c) It is a declarative sentence, which is clearly true. Therefore, it is a true statement.
(d) It is an imperative sentence and therefore it is not a statement.
- Options (a), (b) and (c) are declarative sentences, which is clearly true, therefore these are true statements.
In option (d), it is an imperative sentence, therefore it is not a statement.
- Given, $x + 3 = 8$
It is an open statement.
- Some teenagers are not dreamers means teenagers which are not dreamers.
- The conditional statement of given statement is 'If you take the dinner, then you will get a sweet dish'.
- The given statement is represented by $p \leftrightarrow q$.
- Given, p : A natural number n is odd and q : natural number n is not divisible by 2.
The biconditional statement $p \leftrightarrow q$ i.e. 'A natural number n is odd if and only if it is not divisible by 2'.
- The symbolic representation of various option(s) is/are
Option (a), $p \rightarrow q$
Option (b), $\sim p \rightarrow \sim q$
Option (c), $p \wedge q$
Option (d), $p \vee q$
- $\sim p$: It is not that the Earth is round.
 $\sim q$: It is not that $3 + 4 = 7$
 $(\sim p) \vee (\sim q)$: It is not that the Earth is round or it is not that $3 + 4 = 7$
- $[(A \text{ number is an irrational number}) \leftrightarrow \{(the \text{ number is not terminating}) \wedge (the \text{ number is not recurring})\}] \vee [(A \text{ number is rational}) \leftrightarrow \{(the \text{ number is terminating}) \vee (the \text{ number is non-terminating}) \rightarrow (the \text{ number is recurring})\}] \wedge [(the \text{ number is real}) \rightarrow \{(the \text{ number is rational}) \vee (the \text{ number is irrational})\}]$
 $\Rightarrow [s \leftrightarrow (\sim p \wedge \sim q)] \vee [r \leftrightarrow \{p \vee (\sim p \Rightarrow q)\}]$
 $\wedge [t \Rightarrow (r \vee s)]$
 $\therefore [s \leftrightarrow (\sim p \vee q)] \vee [r \leftrightarrow \{p \vee (\sim p \Rightarrow q)\}]$
 $\wedge [t \Rightarrow (r \vee s)]$
- (a) We know that, $p \wedge q$ is true when both p and q are true.
(b) We know that, $p \rightarrow q$ is false when p is true and q is false.
(c) We know that, $p \leftrightarrow q$ is true when either both p and q are true or both are false.
(d) If p and q both are false, then $p \vee q$ is false
 $\Rightarrow \sim(p \vee q)$ is true.

12.

p	q	$\sim q$	$p \Rightarrow \sim q$
T	T	F	F
T	F	T	T
F	T	F	T
F	F	T	T

13. $\sim[(T \wedge \sim F) \vee (\sim T \vee F)]$
 $\equiv \sim[(T \wedge T) \vee (F \vee F)]$
 $\equiv \sim[(T) \vee (F)] \equiv \sim(T) \equiv F$

14. We know that, $p \rightarrow q$ is false only when p is true and q is false.

So, $p \rightarrow (\sim p \vee q)$ is false only when p is true and $(\sim p \vee q)$ is false.

But $(\sim p \vee q)$ is false, if q is false because $\sim p$ is false.

Hence, $p \rightarrow (\sim p \vee q)$ is false, when truth values of p and q are T and F, respectively.

15. Let $S : (\sim p \Rightarrow \sim q) \wedge (\sim q \Rightarrow \sim p)$

$\sim p$	$\sim q$	$\sim p \Rightarrow \sim q$	$\sim q \Rightarrow \sim p$	S
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

16.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$p \rightarrow \sim q$	$\sim(p \rightarrow \sim q)$
T	T	T	F	F	F	T
T	F	F	F	T	T	F
F	T	F	T	F	T	F
F	F	F	T	T	T	F

17. Given proposition is not logically equivalent to 'If a number is not a rational or not an irrational, then it is not real'.

18. The negation of the given statement is '72 is not divisible by 2 or 72 is not divisible by 3'.

19. Let p and q be the propositions as given below:

p : A quadrilateral is a square.

q : A quadrilateral is a rhombus.

The given proposition is $p \rightarrow q$.

Now, $\sim(p \rightarrow q) \equiv p \wedge \sim q$

Therefore, the negation of the given proposition is 'A quadrilateral is a square and it is not a rhombus'.

20. The symbolic form of given statement is $\sim(p \vee q)$.

21. We know that, $\sim(p \rightarrow q) \equiv p \wedge \sim q$
 $\therefore \sim[(p \wedge r) \rightarrow (r \vee q)] \equiv (p \wedge r) \wedge [\sim(r \vee q)]$
 $\equiv (p \wedge r) \wedge (\sim r \wedge \sim q)$

22. Let $S : (p \vee q) \wedge (p \vee \sim r)$
 $\Rightarrow \sim S : \sim(p \vee q) \vee \sim(p \vee \sim r)$
 $\therefore \sim S : (\sim p \wedge \sim q) \vee (\sim p \wedge r)$

23. The contrapositive of the given statement is 'If 8 is not greater than 6, then 7 is not greater than 5.'

24. **Contrapositive statement** If it is not a sunny day, then I do not go to beach.

Converse statement If it is a sunny day, then I go to beach.

25. Let $S : (p \Rightarrow \sim p) \wedge (\sim p \Rightarrow p)$

p	$\sim p$	$p \Rightarrow \sim p$	$\sim p \Rightarrow p$	S
F	T	T	F	F
T	F	F	T	F

$\therefore$ Statement S is a contradiction.

26. By truth table

p	q	$\sim q$	$q \wedge \sim q$	$p \Rightarrow q \wedge \sim q$
T	T	F	F	F
T	F	T	F	F
F	T	F	F	T
F	F	T	F	T

Hence, it is neither a tautology nor a contradiction.

27. $p \Rightarrow q$ is logically equivalent to $\sim q \Rightarrow \sim p$

$\therefore (p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$ is a tautology but not a contradiction.

28. $(\sim p \wedge q) \vee \sim q \equiv \sim q \vee (\sim p \wedge q)$ [by commutative law]
 $\equiv \sim q \vee (q \wedge \sim p)$ [by commutative law]
 $\equiv (\sim q \vee q) \wedge (\sim q \vee \sim p)$ [by distributive law]
 $\equiv (q \wedge p) \equiv \sim(p \wedge q)$

29. Dual of statement S i.e. S^* can be obtained by replacing ' $\wedge$ ' by ' $\vee$ ' and ' $\vee$ ' and ' $\wedge$ '.

Let $S : (p \wedge q) \wedge \sim q$, then dual of S is

$$S^* : (p \vee q) \vee \sim q$$

30. Let $S : [(p \vee q) \wedge \sim q] \vee (\sim q)$, then dual of S is

$$S^* : [(p \wedge q) \vee \sim q] \wedge (\sim q)$$

31. Let $S : (p \vee \sim q) \wedge (\sim p)$, then dual of S is

$$S^* : (p \wedge \sim q) \vee (\sim p)$$

Entrances Gallery

1. $\sim[\sim s \vee (\sim r \wedge s)]$
 $\equiv s \wedge [\sim(\sim r \wedge s)] \equiv s \wedge (r \vee \sim s)$
 $\equiv (s \wedge r) \vee (s \wedge \sim s)$
 $\equiv (s \wedge r) \vee F$ [$\because s \wedge \sim s$ is false]
 $\equiv s \wedge r$

2.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim(p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

3. **Statement II** $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$

$$\equiv (p \rightarrow q) \Leftrightarrow (p \rightarrow q)$$

which is always true, so Statement II is true.

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$

$$\equiv p \wedge \sim q \wedge \sim p \wedge q \equiv p \wedge \sim p \wedge \sim q \wedge q$$

$$\equiv f \wedge f \equiv f$$

Hence, it is a fallacy statement.

So, Statement I is true.

5.

A	B	$A \vee B$	$A \wedge B$	$A \wedge (A \vee B)$	$A \vee (A \wedge B)$	$A \rightarrow B$	$A \wedge (A \rightarrow B)$	$A \wedge (A \rightarrow B) \rightarrow B$	$B \rightarrow [A \wedge (A \rightarrow B)]$
T	T	T	T	T	T	T	T	T	T
T	F	T	F	T	T	F	F	T	T
F	T	T	F	F	F	T	F	T	F
F	F	F	F	F	F	T	F	T	T

$\therefore$ The truth value of all the elements of the column $A \wedge (A \rightarrow B) \rightarrow B$ is T.

$\therefore [A \wedge (A \rightarrow B)] \rightarrow B$ is tautology.

Aliter

Statement II $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$

Since, $\sim q \rightarrow \sim p$ is contrapositive of $p \rightarrow q$, hence $(p \rightarrow q) \Leftrightarrow (\sim q \rightarrow \sim p)$ will be a tautology.

Statement I $(p \wedge \sim q) \wedge (\sim p \wedge q)$

p	q	$\sim p$	$\sim q$	$p \wedge \sim q$	$\sim p \wedge q$	$(p \wedge \sim q) \wedge (\sim p \wedge q)$
T	T	F	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	F	T	F
F	F	T	T	F	F	F

Hence, it is a fallacy.

4. 'Suman is brilliant and dishonest, if and only if Suman is rich', is expressed as

$$Q \Leftrightarrow (P \wedge \sim R)$$

$\therefore$ Negation of it will be

$$\sim [Q \Leftrightarrow (P \wedge \sim R)]$$

6. P : There is rational number $x \in S$ such that $x > 0$.

$\sim P$: Every rational number $x \in S$ satisfies $x \leq 0$.

7. Here, p : x is an irrational number.

q : y is a transcendental number.

r : x is a rational number, iff y is a transcendental number.

$\Rightarrow$

$$r : \sim p \leftrightarrow q$$

$$S_1 : r \equiv q \vee p$$

and

$$S_2 : r \equiv (\sim p \leftrightarrow \sim q)$$

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	Statement I $q \vee p$	Statement II $(\sim p \leftrightarrow \sim q)$
T	T	F	F	F	T	T
T	F	F	T	T	T	F
F	T	T	F	T	T	F
F	F	T	T	F	F	T

It is clear from the table that, r is not equivalent to either of the statements.

Hence, none of the given options is correct.

8.

q	p	$q \rightarrow p$	$p \rightarrow (q \rightarrow p)$	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	T	T	T	T
F	F	T	T	F	T

$\therefore$ Statement $p \rightarrow (q \rightarrow p)$ is equivalent to $p \rightarrow (p \vee q)$.

9. Let $S : (\sim p \wedge q) \vee (p \wedge \sim q)$

$$\Rightarrow \sim S : \sim [(\sim p \wedge q) \vee (p \wedge \sim q)]$$

$$\Rightarrow \sim S : \sim (\sim p \wedge q) \wedge \sim (p \wedge \sim q)$$

$$\Rightarrow \sim S : (p \vee \sim q) \wedge (\sim p \vee q)$$

10. By truth table,

p	q	$\sim p$	$p \Rightarrow q$	$\sim p \wedge q$	$(p \Rightarrow q) \leftrightarrow (\sim p \wedge q)$
T	T	F	T	F	F
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	F	F

Hence, given statement is neither a tautology nor a contradiction.

11. By truth table,

p	q	$(p \wedge q)$	$\sim(p \wedge q)$	$q \Leftrightarrow p$	$\sim(q \Leftrightarrow p)$	$\sim(p \wedge q) \vee \sim(q \Leftrightarrow p)$
T	T	T	F	T	F	F
T	F	F	T	F	T	T
F	T	F	T	F	T	T
F	F	F	T	T	F	T

It is clear that, the given statement is neither a tautology nor a contradiction.

12. (a) 'Mathematics is interesting' is not a statement.

Since, it can be true as well as false both at the same time.

(b) ' $\sqrt{3}$ is a prime number' is not a correct statement.

(c) ' $\sqrt{2}$ is an irrational number' is a correct statement.

(d) 'The Sun is a star' is a correct statement.

13. $\therefore \sim[p \vee (\sim q)] \equiv \sim p \wedge \sim(\sim q) \equiv (\sim p) \wedge q$

- 14.

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee (\sim r)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

15. $\therefore \sim[(\sim p) \wedge q] \equiv \sim(\sim p) \vee \sim q \equiv p \vee (\sim q)$

16. $\sim[p \vee (\sim p \vee q)] \equiv \sim p \wedge \sim(\sim p \vee q)$

$$\equiv \sim p \wedge [(\sim \sim p) \wedge \sim q]$$

$$\equiv \sim p \wedge (p \wedge \sim q)$$

17. Consider $\sim[p \rightarrow (\sim p \vee q)] \equiv p \wedge \sim(\sim p \vee q)$

$$\equiv p \wedge (p \wedge \sim q) \equiv p \wedge p \wedge \sim q \equiv p \wedge \sim q$$

18. p : Roses are red.

q : The Sun is a star.

$\sim p$ = Roses are not red.

$(\sim p) \vee q$ = Roses are not red or the Sun is a star.

19. The statement 'Kashmir is far from here' is not a logical statement because its truth value cannot be determined.

20. The given conditional statement is $p \rightarrow \sim q$.

The contrapositive of this statement is

$$\sim(\sim q) \rightarrow \sim p \equiv q \rightarrow \sim p$$

21. $\sim(p \vee q) \vee (\sim p \wedge q) \equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$

$$\equiv \sim p \wedge (\sim q \vee q) \equiv \sim p$$

22. $S(p, q, r) = (\sim p) \vee [\sim(q \wedge r)] = (\sim p) \vee [\sim q \vee \sim r]$

$$\Rightarrow S(\sim p, \sim q, \sim r) = p \vee (q \vee r)$$

23. $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Hence, 7 is greater than 4 and Paris is not in France.

20

Statistics

Measures of Central Tendency

An average value generally lies in the central part of the distribution and therefore, such values are called the measures of central tendency.

The following are the nine measures of central tendency:

(i) Arithmetic mean	(ii) Weighted arithmetic mean
(iii) Combined mean	(iv) Geometric mean
(v) Harmonic mean	(vi) Relation among AM, GM and HM
(vii) Median	(viii) Quartiles, deciles and percentiles
(ix) Mode	

Arithmetic Mean

i. **Arithmetic mean of ungrouped or individual data** If $x_1, x_2, x_3, \dots, x_n$ are n observations, then mean by

(a) **Direct method**

$$\bar{x} = \frac{x_1 + x_2 + \dots + x_n}{n} = \frac{1}{n} \sum_{i=1}^n x_i$$

(b) **Shortcut method**

$$\bar{x} = A + \frac{1}{n} \sum_{i=1}^n d_i$$

where, A = Assumed mean and $d_i = x_i - A$

➤ **Example 1.** The AM of n numbers of a series is $\bar{X}$. If the sum of first $(n-1)$ terms is k , then the n th number is

(a) $\bar{X} - k$	(b) $n\bar{X} - k$
(c) $\bar{X} - nk$	(d) $n\bar{X} - nk$

Sol. (b) Let the n numbers be $x_1, x_2, \dots, x_n$. Then,

$$\bar{X} = \frac{1}{n} \sum_{i=1}^n x_i$$

⇒

$$\bar{X} = \frac{x_1 + x_2 + \dots + x_{n-1} + x_n}{n}$$

⇒

$$\bar{X} = \frac{k + x_n}{n} \quad [\because x_1 + x_2 + \dots + x_{n-1} = k]$$

⇒

$$x_n = n\bar{X} - k$$

Chapter Snapshot

- Measures of Central Tendency
- Measures of Dispersion
- Skewness
- Some Results to be Remembered
- Correlation Analysis
- Characteristics of Correlation Coefficient
- Regression Analysis
- Properties of Regression Coefficients
- Properties of Lines of Regression

➤ **Example 2.** The mean of data 55, 155, 255, 355, 455 and 555 is

- (a) 310 (b) 305
(c) 300 (d) None of these

Sol. (b) Let $A = 255$, then

$$d_1 = -200, d_2 = -100, d_3 = 0, \\ d_4 = 100, d_5 = 200 \text{ and } d_6 = 300$$

$$\therefore \text{Mean } (\bar{x}) = A + \frac{\sum_{i=1}^6 d_i}{6} = 255 + \frac{(-200 - 100 + 0 + 100 + 200 + 300)}{6} \\ = 255 + \frac{300}{6} \\ = 255 + 50 \\ = 305$$

ii. **Arithmetic mean of grouped data** If $x_1, x_2, x_3, \dots, x_n$ are n observations whose corresponding frequencies are $f_1, f_2, f_3, \dots, f_n$, then mean by

(a) **Direct method**

$$\bar{x} = \frac{x_1 f_1 + x_2 f_2 + \dots + x_n f_n}{f_1 + f_2 + \dots + f_n} = \frac{\sum_{i=1}^n f_i x_i}{\sum_{i=1}^n f_i}$$

(b) **Shortcut method**

$$\bar{x} = A + \frac{\sum_{i=1}^n f_i d_i}{\sum_{i=1}^n f_i}$$

where, A = Assumed mean and $d_i = x_i - A$

➤ If the grouped data represents the frequencies included between a specific class interval. So, we actually do not know how many times a particular observation lying in between that class limit. So, assume that the given frequency is the frequency of mid-point of that particular class interval. This mid-point is known as class mark.

Thus, class mark of an interval

$$= \frac{(\text{Lower limit of interval} + \text{Upper limit of interval})}{2}$$

➤ **Example 3.** If the values $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots, \frac{1}{n}$ occur at frequencies $1, 2, 3, 4, 5, \dots, n$, in a distribution, then the mean is

- (a) 1 (b) n
(c) $\frac{1}{n}$ (d) $\frac{2}{n+1}$

Sol. (d) Mean =
$$\frac{1 \cdot 1 + \frac{1}{2} \cdot 2 + \frac{1}{3} \cdot 3 + \frac{1}{4} \cdot 4 + \frac{1}{5} \cdot 5 + \dots + \frac{1}{n} \cdot n}{1 + 2 + 3 + \dots + n} \\ = \frac{1 + 1 + 1 + 1 + \dots + 1}{\frac{n(n+1)}{2}} = \frac{n}{\frac{n(n+1)}{2}} = \frac{2}{n+1}$$

➤ **Example 4.** Calculate the arithmetic mean of the following frequency data.

x	12.58	13.58	14.58	15.58	16.58
f	4	3	2	4	2

Sol. Let us take assumed mean (A) = 14.58

x	f	$d = x - A$	fd
12.58	4	-2	-8
13.58	3	-1	-3
14.58	2	0	0
15.58	4	1	4
16.58	2	2	4
Total	$\Sigma f = 15$		$\Sigma fd = -3$

$$\therefore \bar{x} = A + \frac{\Sigma f_i d_i}{\Sigma f_i}$$

$$\therefore \bar{x} = 14.58 - \frac{3}{15} \quad [\because \text{assumed mean } (A) = 14.58]$$

$$\Rightarrow \bar{x} = 14.38$$

➤ **Example 5.** The arithmetic mean of the marks from the following table is

Marks	0-10	10-20	20-30	30-40	40-50	50-60
Number of students	12	18	27	20	17	6

- (a) 20 (b) 28 (c) 2800 (d) 100

Sol. (b)

Marks	Class marks (x)	f	fx
0-10	5	12	60
10-20	15	18	270
20-30	25	27	675
30-40	35	20	700
40-50	45	17	765
50-60	55	6	330
Total		$\Sigma f = 100$	$\Sigma fx = 2800$

$$\therefore \bar{x} = \frac{\Sigma fx}{\Sigma f} = \frac{2800}{100} = 28$$

iii. (c) **Step deviation method** When the class intervals in a grouped data are given and equal in length, then the arithmetic mean can be obtained by using the formula

$$\bar{x} = A + \left(\frac{\Sigma f_i u_i}{\Sigma f_i} \right) h$$

where, A = Assumed mean

$$u_i = \frac{x_i - A}{h}$$

f_i = Frequency

h = Width of class interval

➤ **Example 6.** Find the mean for the data in following table.

Income per day	Number of persons
0-100	4
100-200	8
200-300	9
300-400	10
400-500	7
500-600	5
600-700	4
700-800	3

- (a) 350 (b) 359
(c) 358 (d) None of these

Sol. (c)

Class	f_i	Mid value (x_i)	$u_i = \frac{x_i - A}{h}$, $A = 350$, $h = 100$	$f_i u_i$
0-100	4	50	-3	-12
100-200	8	150	-2	-16
200-300	9	250	-1	-9
300-400	10	350	0	0
400-500	7	450	1	7
500-600	5	550	2	10
600-700	4	650	3	12
700-800	3	750	4	12
Total	$\Sigma f_i = 50$			$\Sigma f_i u_i = 4$

$$\begin{aligned}\text{Mean } (\bar{x}) &= A + \frac{\Sigma f_i u_i}{\Sigma f_i} \times h = 350 + \frac{4}{50} \times 100 \\ &= 350 + 8 = 358\end{aligned}$$

Weighted Arithmetic Mean

If $w_1, w_2, w_3, \dots, w_n$ are the weights assigned to the values $x_1, x_2, x_3, \dots, x_n$ respectively, then the weighted arithmetic mean is defined as

$$\text{Weighted AM} = \frac{w_1 x_1 + w_2 x_2 + \dots + w_n x_n}{w_1 + w_2 + \dots + w_n}$$

$$= \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i}$$

➤ **Example 7.** The weighted mean of first n natural numbers whose weights are equal to the number of selections out of n natural numbers of corresponding numbers, is

- (a) $\frac{n \cdot 2^{n-1}}{2^n - 1}$ (b) $\frac{3n(n+1)}{2(2n+1)}$
(c) $\frac{(n+1)(2n+1)}{6}$ (d) $\frac{n(n+1)}{2}$

Sol. (a) The required mean

$$\begin{aligned}\bar{x} &= \frac{1 \cdot {}^n C_1 + 2 \cdot {}^n C_2 + 3 \cdot {}^n C_3 + \dots + n \cdot {}^n C_n}{{}^n C_1 + {}^n C_2 + \dots + {}^n C_n} \\ &= \frac{\sum_{r=0}^n r \cdot {}^n C_r}{\sum_{r=1}^n {}^n C_r} = \frac{\sum_{r=1}^n r \cdot \frac{n}{r} \cdot {}^{n-1} C_{r-1}}{\sum_{r=1}^n {}^n C_r} = \frac{n \sum_{r=1}^n {}^{n-1} C_{r-1}}{\sum_{r=1}^n {}^n C_r} \\ &= \frac{n \cdot 2^{n-1}}{2^n - 1}\end{aligned}$$

Combined Mean

If we are given the AM of two data sets and their sizes, then the combined AM of two data sets can be obtained by the formula

$$\bar{x}_{12} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$$

where, $\bar{x}_{12}$ = Combined mean of the two data sets

$\bar{x}_1$ = Mean of the first data set

$\bar{x}_2$ = Mean of the second data set

n_1 = Size of the first data set

n_2 = Size of the second data set

➤ **Example 8.** The mean height of 25 male workers in a factory is 61 cm and the mean height of 35 female workers in the same factory is 58 cm. The combined mean height of 60 workers in the factory is

- (a) 59.25
(b) 59.5
(c) 59.75
(d) 58.75

Sol. (a) $\because n_1 = 25, n_2 = 35, \bar{x}_1 = 61, \bar{x}_2 = 58$

$$\begin{aligned}\therefore \bar{x} &= \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2} \\ &= \frac{(25)(61) + (35)(58)}{25 + 35} \\ &= 59.25\end{aligned}$$

Geometric Mean

i.

Geometric mean for ungrouped data If $x_1, x_2, x_3, \dots, x_n$ are n observations, none of them being zero, then their geometric mean given by

$$\text{GM} = (x_1 \cdot x_2 \cdot x_3 \dots x_n)^{1/n}$$

$$\text{or GM} = \text{antilog} \left(\frac{\log x_1 + \log x_2 + \dots + \log x_n}{n} \right)$$

ii.

Geometric mean for grouped data Geometric mean of n observations $x_1, x_2, \dots, x_n$ whose corresponding frequencies are $f_1, f_2, f_3, \dots, f_n$, is given by

$$GM = (x_1^{f_1} \cdot x_2^{f_2} \dots x_n^{f_n})^{\frac{1}{N}}, \text{ where } N = \sum_{i=1}^n f_i$$

$$\text{or GM} = \text{antilog} \left(\frac{\sum_{i=1}^n f_i \log x_i}{N} \right)$$

✎ In case of continuous frequency distribution, the values of the variate x are taken to be the values corresponding to the mid-points of the class intervals.

iii. **Combined geometric mean** If $G_1, G_2, G_3, \dots, G_k$ are GM of k series of sizes $n_1, n_2, \dots, n_k$, then GM of combined series is given by

$$\log G = \frac{n_1 \log G_1 + n_2 \log G_2 + \dots + n_k \log G_k}{n_1 + n_2 + \dots + n_k}$$

$$= \frac{1}{N} \sum_{i=1}^k n_i \log G_i,$$

where $N = n_1 + n_2 + \dots + n_k$

☞ **Example 9.** The geometric mean of the numbers $3, 3^2, 3^3, \dots, 3^n$ is

- (a) $3^{2/n}$ (b) $3^{(n-1)/2}$
(c) $3^{n/2}$ (d) $3^{(n+1)/2}$

Sol. (d) $GM = (3 \cdot 3^2 \cdot \dots \cdot 3^n)^{1/n}$

$$= 3^{\frac{1+2+\dots+n}{n}}$$

$$= 3^{\frac{n(n+1)}{2n}} = 3^{\frac{n+1}{2}}$$

Harmonic Mean

i. **Harmonic mean for ungrouped data** The harmonic mean of n observations $x_1, x_2, \dots, x_n$ is defined as

$$HM = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \dots + \frac{1}{x_n}}$$

or $HM = \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$

ii. **Harmonic mean for grouped data** Harmonic mean of n observations $x_1, x_2, x_3, \dots, x_n$ which occur with frequencies $f_1, f_2, \dots, f_n$ respectively, is given by

$$HM = \frac{\sum_{i=1}^n f_i}{\sum_{i=1}^n \left(\frac{f_i}{x_i} \right)}$$

☞ **Example 10.** The harmonic mean of 4, 8 and 16 is

- (a) 6.4 (b) 6.7
(c) 6.85 (d) 7.8

Sol. (c) $HM \text{ of } 4, 8 \text{ and } 16 = \frac{3}{\frac{1}{4} + \frac{1}{8} + \frac{1}{16}} = \frac{48}{7} = 6.85$

☞ **Example 11.** Find the harmonic mean of $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots, \frac{n}{n+1}$ occurring with frequencies $1, 2, 3, \dots, n$, respectively.

- (a) $\frac{n-1}{3-n}$ (b) $\frac{n+1}{3+n}$
(c) $\frac{n+1}{3-n}$ (d) None of these

Sol. (b) We know that,

$$\text{Harmonic mean} = \frac{\sum f}{\sum \left(\frac{f}{x} \right)}$$

$$\therefore \sum f = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\text{and } \sum \left(\frac{f}{x} \right) = \frac{1}{1/2} + \frac{2}{2/3} + \frac{3}{3/4} + \dots + \frac{n}{n/(n+1)}$$

$$= 2 + \frac{3 \times 2}{2} + \frac{4 \times 3}{3} + \dots + \frac{n(n+1)}{n}$$

$$= 2 + 3 + 4 + \dots + n + (n+1)$$

which is an arithmetic progression with $a = 2$ and $d = 1$.

By the formula of sum of n terms of an AP,

$$\sum \left(\frac{f}{x} \right) = \left[\frac{n}{2} \{2a + (n-1)d\} \right]$$

$$= \frac{n}{2} \{2 \times 2 + n - 1\} = \frac{n}{2} (3 + n)$$

$$= \frac{n(n+1)}{2}$$

$$\therefore \text{Harmonic mean} = \frac{\frac{n(n+1)}{2}}{\frac{n(n+1)}{2}} = \frac{n+1}{3+n}$$

Relation among AM, GM and HM

The arithmetic mean (AM), geometric mean (GM) and harmonic mean (HM) for a given set of observations are related as under

$$AM \geq GM \geq HM$$

Equality sign hold only when all the observations are equal.

☞ **Example 12.** For three numbers a, b and c , product of the average of the numbers a^2, b^2, c^2 and $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ cannot be less than

- (a) 1 (b) 3
(c) 9 (d) None of these

Sol. (a) Since, $AM \geq GM$, therefore we have

$$\frac{a^2 + b^2 + c^2}{3} \geq (a^2 b^2 c^2)^{1/3} \quad \dots(i)$$

$$\text{and} \quad \frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}{3} \geq \left(\frac{1}{a^2} \cdot \frac{1}{b^2} \cdot \frac{1}{c^2} \right)^{1/3} \quad \dots(ii)$$

On multiplying Eqs. (i) and (ii), we get

$$\left(\frac{a^2 + b^2 + c^2}{3} \right) \left(\frac{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}{3} \right) \geq 1$$

$\therefore$ Product of the averages of a^2, b^2, c^2 and $\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$ cannot be less than 1.

Median

i. **Median of an individual data** Let n be the number of observations, then

(a) Arrange the data in ascending or descending order.

(b) ■ If n is odd, then

Median = Value of the $\frac{1}{2}(n+1)$ th observation

■ If n is even, then

Median = Mean of the $\left(\frac{n}{2}\right)$ th and $\left(\frac{n}{2} + 1\right)$ th observations

i.e. Median

$$= \frac{1}{2} \left[\left(\frac{n}{2}\right)\text{th item} + \left(\frac{n}{2} + 1\right)\text{th item} \right]$$

ii. **Median of a discrete frequency data**

(a) Arrange the values of the variate in ascending or descending order.

(b) Prepare a cumulative frequency table.

(c) Find $\frac{N}{2}$, where $N = \sum f_i$ and see the cumulative frequency just greater than or equal to $\frac{N}{2}$, then the corresponding value of x is median.

iii. **Median of a continuous frequency data**

(a) Prepare the cumulative frequency table.

(b) Find the median class i.e. the class in which the $\left(\frac{N}{2}\right)$ th observation lies.

(c) The median value is given by the formula

$$\text{Median} = l + \frac{\left(\frac{N}{2}\right) - cf}{f} \times h$$

where, l = Lower limit of the median class

N = Total frequency

f = Frequency of the median class

h = Width of the median class
 cf = Cumulative frequency of the class preceding the median class

➤ **Cumulative frequency** The cumulative frequency of a value is its frequency plus the frequencies of all smaller values. e.g.

x	f	cf
1	4	4
2	6	10 (4 + 6)
3	4	14 (4 + 6 + 4)

➤ **Example 13.** If a variable takes the discrete values $\alpha + 4, \alpha - \frac{7}{2}, \alpha - \frac{5}{2}, \alpha - 3, \alpha - 2, \alpha + \frac{1}{2}, \alpha - \frac{1}{2}, \alpha + 5$ ($\alpha > 0$), then the median is

(a) $\alpha - \frac{5}{4}$ (b) $\alpha - \frac{1}{2}$ (c) $\alpha - 2$ (d) $\alpha + \frac{5}{4}$

Sol. (a) Arrange the data in ascending order as

$$\alpha - \frac{7}{2}, \alpha - 3, \alpha - \frac{5}{2}, \alpha - 2, \alpha - \frac{1}{2}, \alpha + \frac{1}{2}, \alpha + 4, \alpha + 5$$

$$\text{Median} = \frac{1}{2} [\text{Value of 4th item} + \text{Value of 5th item}]$$

$$\begin{aligned} \therefore \text{Median} &= \frac{\alpha - 2 + \alpha - \frac{1}{2}}{2} \quad [\because n = 8, \text{ which is even}] \\ &= \frac{2\alpha - \frac{5}{2}}{2} = \alpha - \frac{5}{4} \end{aligned}$$

➤ **Example 14.** The value of median for the data

Income (in ₹)	1000	1100	1200	1300	1400	1500
Number of persons	14	26	21	18	28	15

is

(a) 1300 (b) 1200 (c) 1250 (d) 1150

Sol. (c)

Income (x)	Frequency (f)	cf
1000	14	14
1100	26	40
1200	21	61
1300	18	79
1400	28	107
1500	15	122

Here, $N = 122$

$$\therefore \text{Median} = \frac{1}{2} \left[\left(\frac{n}{2}\right)\text{th term} + \left(\frac{n}{2} + 1\right)\text{th term} \right]$$

$$\begin{aligned} \therefore \text{Median} &= \frac{1}{2} (61\text{th} + 62\text{th}) \text{ terms} \\ &= \frac{1}{2} (1200 + 1300) = 1250 \end{aligned}$$

➤ **Example 15.** From the data given, the median of the average deposit balance of saving for the branch during March 1982 is

Average deposit balance (in ₹)	Number of deposit
Less than 100	26
100-200	68
200-300	145
300-400	242
400-500	188
500-600	65
600-700	16

- (a) 356 (b) 300
(c) 56.2 (d) 356.2

Sol. (d)

Average deposit balance (in ₹)	f	Cumulative frequency (cf)
Less than 100	26	26
100-200	68	94
200-300	145	239
300-400	242	481
400-500	188	669
500-600	65	734
600-700	16	750

Here, $\frac{N}{2} = \frac{750}{2} = 375$

The frequency 375 lies in the class 300-400.

So, median class is 300-400.

$$\begin{aligned} \therefore \text{Median} &= l + \frac{\left(\frac{N}{2} - cf\right)}{f} \times h \\ &= 300 + \frac{375 - 239}{242} \times 100 \\ &= 300 + 56.2 = 356.2 \end{aligned}$$

Quartiles, Deciles and Percentiles

i. **For ungrouped and discrete frequency data**
Quartiles are also a kind of positional averages which divide the complete distribution into four equal parts.

$$Q_r = \frac{(N+1)r}{4}; r=1, 2, 3$$

Deciles divide the frequency distribution into 10 equal parts.

$$D_r = \frac{(N+1)r}{10}; r=1, 2, 3, \dots, 9$$

Percentiles divide frequency distribution into 100 equal parts.

$$P_r = \frac{(N+1)r}{100}; r=1, 2, 3, \dots, 99$$

ii. **For grouped data** Arrange data in ascending order and prepare cumulative frequency.

$$Q_r = l + \left(\frac{Nr}{4} - cf\right) \frac{i}{f}; r=1, 2, 3$$

$$D_r = l + \left(\frac{Nr}{10} - cf\right) \frac{i}{f}; r=1, 2, 3, \dots, 9$$

$$P_r = l + \left(\frac{Nr}{100} - cf\right) \frac{i}{f}; r=1, 2, 3, \dots, 99$$

where, l is lower limit of the required class, i is width of class interval, f is frequency of the class and cf is sum of all frequencies just above the class of quartile/decile/percentile.

➤ **Example 16.** The profits earned by 100 companies during 2012-13 are given below:

Profits (in ₹ lakh)	Number of companies	Profits (in ₹ lakh)	Number of companies
20-30	4	60-70	15
30-40	8	70-80	10
40-50	18	80-90	8
50-60	30	90-100	7

Calculate Q_1 , median, D_4 and P_{80} and interpret the values of cf .

Sol. Calculation of Q_1 , Q_2 , D_4 and P_{80}

Profits (in ₹ lakh)	f	cf
20-30	4	4
30-40	8	12
40-50	18	30
50-60	30	60
60-70	15	75
70-80	10	85
80-90	8	93
90-100	7	100

$$Q_1 = \text{Size of } \frac{N}{4} \text{th observation} = \frac{100}{4} = 25\text{th observation}$$

So, Q_1 lies in the class 40-50.

$$\begin{aligned} Q_1 &= l + \frac{N/4 - cf}{f} \times i \\ &= 40 + \frac{25 - 12}{18} \times 10 \\ &= 40 + 7.22 = 47.22 \end{aligned}$$

25% of the companies earn an annual profit of ₹ 47.22 lakh or less.

$$\begin{aligned} \text{Median or } Q_2 &= \text{Size of } \frac{2N}{4} \text{th observation} \\ &= \frac{200}{4} = 50\text{th observation} \end{aligned}$$

So, Q_2 lies in the class 50-60.

$$\begin{aligned} \therefore Q_2 &= l + \frac{2N/4 - cf}{f} \times i \\ &= 50 + \frac{50 - 30}{30} \times 10 = 50 + 6.67 = 56.67 \end{aligned}$$

50% of the companies earn an annual profit of ₹ 56.67 lakh or less.

$$D_4 = \text{Size of } \frac{4N}{100} \text{th observation} = 40\text{th observation}$$

So, D_4 lies in the class 50-60.

$$\therefore D_4 = l + \frac{4N/100 - cf}{f} \times i = 50 + \frac{40 - 30}{30} \times 10$$

$$= 50 + 3.33 = 53.33$$

Thus, 40% of the companies earn an annual profit of ₹ 53.33 lakh or less.

$$P_{80} = \text{Size of } \frac{80N}{100} \text{th observation}$$

$$= \frac{80 \times 100}{100} = 80\text{th observation}$$

So, P_{80} lies in the class 70-80.

$$P_{80} = l + \frac{80N/100 - cf}{f} \times i = 70 + \frac{80 - 75}{10} \times 10$$

$$= 70 + 5 = 75$$

This means that 80% of the companies earn an annual profit of ₹ 75 lakh or less and 20% of the companies earn an annual profit of more than ₹ 75 lakh.

Mode

- i. **Mode of individual data** In the case of individual series, the value which is repeated maximum number of times is the mode of the series. e.g. Mode of 6, 10, 7, 5, 9, 3, 7, 5 is 7.
- ii. **Mode of discrete data** In the case of discrete frequency distribution, mode is the value of the variate corresponding to the maximum frequency.
- iii. **Mode of continuous data**
 - (a) Find the modal class i.e. the class which has maximum frequency. The modal class can be determined either by inspection or with the help of grouping table.
 - (b) The mode is given by the formula
$$\text{Mode} = l + \frac{f_m - f_{m-1}}{2f_m - f_{m-1} - f_{m+1}} \times i$$

where,

 - l = Lower limit of the modal class
 - i = Width of the modal class
 - f_{m-1} = Frequency of the class preceding the modal class
 - f_m = Frequency of the modal class
 - f_{m+1} = Frequency of the class succeeding modal class

In case, the modal value lies in a class other than the one containing maximum frequency, we take the help of the following formula:

$$\text{Mode} = l + \frac{f_{m+1}}{f_{m-1} + f_{m+1}} \times i,$$

where, symbols have usual meaning.

Example 17. The mode of the following distribution is

- (a) 46 (b) 6.66
(c) 46.67 (d) None of these

Class interval	Frequency
0-10	5
10-20	8
20-30	7
30-40	12
40-50	28
50-60	20
60-70	10
70-80	10

Sol. (c) Here, maximum frequency is 28. Thus, the class 40-50 is the modal class.

$$\therefore \text{Mode} = 40 + \frac{10(28 - 12)}{(2 \times 28 - 12 - 20)}$$

$$= 40 + 6.666 = 46.67 \text{ (approx.)}$$

Measures of Dispersion

The dispersion of a statistical distribution is the measure of the deviations of its values about the average value. It gives an idea of scatteredness of different values from the mean. The following measures of dispersion are commonly used:

- (i) Range (ii) Quartile deviation (iii) Mean deviation (iv) Standard deviation

- i. **Range** It is the difference between the greatest and the smallest observations of the distribution.
If L is the largest and S is the smallest observation in a distribution, then its
Range = $L - S$
Also, coefficient of range = $\frac{L - S}{L + S}$
e.g. Range of observations 3, 5, 9, 8, 7, 6, 5, 7, 4, 2 is $9 - 2 = 7$ and their coefficient is $\frac{7}{11}$.
- ii. **Quartile deviation** Quartile deviation or semi-interquartile range is given by
$$QD = \frac{1}{2} (Q_3 - Q_1)$$
 - (a) **Interquartile range** Interquartile range is $Q_3 - Q_1$.
 - (b) **Coefficient of quartile deviation**

$$= \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

- **Example 18.** The quartile deviation of daily wages (in ₹) of 11 persons given below, is
140, 145, 130, 165, 160, 125, 150, 170, 175, 120, 180
(a) 22.5 (b) 25 (c) 20 (d) 21.2

Sol. (c) The given data in ascending order of magnitude is
120, 125, 130, 140, 145, 150, 160, 165, 170, 175, 180

$$\text{Here, } Q_1 = \left(\frac{n+1}{4}\right)\text{th term} = \frac{11+1}{4} = \frac{12}{4} = 3\text{rd term}$$

$$= 130$$

$$Q_3 = \frac{3(n+1)}{4}\text{th term} = \frac{3 \times (11+1)}{4} = 9\text{th term} = 170$$

$$\begin{aligned}\therefore \text{QD} &= \frac{Q_3 - Q_1}{2} \\ &= \frac{170 - 130}{2} \\ &= \frac{40}{2} = 20\end{aligned}$$

iii. **Mean deviation**

(a) **For ungrouped data** The mean deviation from M (mean or median or mode) is given by

$$\text{MD} = \frac{\sum |x_i - M|}{n}$$

(b) **For grouped data** The mean deviation from M (mean or median or mode) is given by

$$\text{MD} = \frac{\sum f_i |x_i - M|}{\sum f_i}$$

(c) **Coefficient of MD**

$$= \frac{\text{Mean deviation}}{\text{Corresponding average (mean or median or mode)}}$$

- **Example 19.** The mean deviation from the mean of the AP $a, a + d, a + 2d, \dots, a + 2nd$ is

- (a) $n(n+1)d$ (b) $\frac{n(n+1)d}{2n+1}$
(c) $\frac{n(n+1)d}{2n}$ (d) $\frac{n(n-1)d}{2n+1}$

Sol. (b) The mean of the series

$a, a + d, a + 2d, \dots, a + 2nd$ is

$$\bar{x} = \frac{1}{2n+1} [a + a + d + a + 2d + \dots + a + 2nd]$$

$$= \frac{1}{2n+1} \left\{ \frac{2n+1}{2} (a + a + 2nd) \right\} = a + nd$$

Therefore, mean deviation from mean

$$= \frac{1}{2n+1} \sum_{r=0}^{2n} |(a + rd) - (a + nd)| = \frac{1}{2n+1} \sum_{r=0}^{2n} |r - n|d$$

$$= \frac{1}{2n+1} \cdot 2d(1 + 2 + \dots + n) = \frac{n(n+1)d}{2n+1}$$

iv.

Standard Deviation (SD) The standard deviation of a statistical data is defined as the positive square root of the squared deviations of observations from the AM of the series under consideration and it is generally denoted by σ read as sigma.

The square of SD is called the variance and is denoted by σ^2 .

(a) **Computation of standard deviation**

- **For ungrouped data** Standard deviation for ungrouped set of observations is given by

(i) **Direct method**

$$\begin{aligned}\text{SD } (\sigma) &= \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}} \\ &= \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2}\end{aligned}$$

(ii) **Shortcut method**

$$\sigma = \sqrt{\frac{\sum d^2}{n} - \left(\frac{\sum d}{n}\right)^2}$$

where, A is assumed mean and
 $d = x - A$

- **For grouped data** SD for frequency distribution is given by

(i) **Direct method**

$$\begin{aligned}\text{SD } (\sigma) &= \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{\sum f_i}} \\ &= \sqrt{\frac{1}{N} \sum f_i x_i^2 - \left(\frac{\sum f_i x_i}{N}\right)^2}\end{aligned}$$

where, f_i is the frequency of
 x_i ($1 \leq i \leq n$).

(ii) **Shortcut method**

$$\sigma = \sqrt{\frac{\sum f d^2}{N} - \left(\frac{\sum f d}{N}\right)^2}$$

where, $N = \sum f$ and $d = x - A$

(iii) **Step deviation method**

$$\sigma = h \sqrt{\frac{\sum f u^2}{N} - \left(\frac{\sum f u}{N}\right)^2}$$

where, $u = \frac{x - A}{h}$ and symbols have their usual meaning.

- (b) **Combined standard deviation** Let A_1 and A_2 be two series having n_1 and n_2 observations respectively. Let their AM be $\bar{x}_1$ and $\bar{x}_2$ and standard deviations be σ_1 and σ_2 . Then, the combined standard deviation σ or σ_{12} of A_1 and A_2 is given by

$$\sigma_{12} \text{ or } \sigma = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1d_1^2 + n_2d_2^2}{n_1 + n_2}}$$

$$= \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}}$$

where, $d_1 = (\bar{x}_1 - \bar{x}_{12})$, $d_2 = (\bar{x}_2 - \bar{x}_{12})$ and $\bar{x}_{12} = \frac{n_1\bar{x}_1 + n_2\bar{x}_2}{n_1 + n_2}$ is the combined mean

$$\text{or } \sigma_{12} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1n_2(\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

- (c) **Coefficient of variation (CV) or coefficient of standard deviation** To compare two or more series for variability, the relative measure called coefficient of standard deviation or coefficient of variation. This measure is define as

$$CV = \frac{\sigma}{\bar{x}} \times 100$$

➤ When the values of the variable are given in the form of class interval, then their respective mid-points are taken as the values of the variable.

- **Example 20.** Find the standard deviation for the following data.

Height (in cm)	95-105	105-115	115-125	125-135	135-145
No. of children	19	23	36	70	52

Sol.

Class interval	Mid-value	f	$u = \frac{x - 120}{10}$	fu	fu^2
95-105	100	19	-2	-38	76
105-115	110	23	-1	-23	23
115-125	120	36	0	0	0
125-135	130	70	1	70	70
135-145	140	52	2	104	208
Total		$\Sigma f = 200$		$\Sigma fu = 113$	$\Sigma fu^2 = 377$

We know that,

$$\sigma = \sqrt{\left\{ \frac{1}{N} \Sigma fu^2 - \left(\frac{\Sigma fu}{N} \right)^2 \right\} h^2}$$

$$\sigma = \sqrt{100 \left\{ \frac{377}{200} - \left(\frac{113}{200} \right)^2 \right\}} = \sqrt{100 \{1.885 - (0.565)^2\}}$$

$$\Rightarrow \sigma = \sqrt{100(1.885 - 0.319225)} = \sqrt{100(1.565775)}$$

$$= 12.51$$

So, SD of height is 12.51 cm.

- **Example 21.** The mean life of a sample of 60 bulbs was 650 h and the standard deviation was 8 h. A second sample of 80 bulbs has a mean life of 660 h and standard deviation 7 h. Find the overall standard deviation.

- (a) 8.97 (b) 8.98
(c) 8.94 (d) None of these

Sol. (c) Given, $n_1 = 60$, $\bar{x}_1 = 650$, $\sigma_1 = 8$, $n_2 = 80$, $\bar{x}_2 = 660$, and $\sigma_2 = 7$

$$\therefore \text{Combined SD} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2}{n_1 + n_2} + \frac{n_1n_2(\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2}}$$

$$= \sqrt{\frac{60 \times 64 + 80 \times 49}{60 + 80} + \frac{60 \times 80 (650 - 660)^2}{(60 + 80)^2}}$$

$$= \sqrt{\frac{3840 + 3920}{140} + \frac{(4800 \times 100)}{(140)^2}}$$

$$= \sqrt{\frac{7760}{140} + \frac{480000}{19600}} = \sqrt{\frac{776}{14} + \frac{4800}{196}}$$

$$= \sqrt{55.42 + 24.49} = \sqrt{79.91} = 8.94$$

- **Example 22.** The mean and standard deviation of marks obtained by 50 students of a class in three subjects, Mathematics, Physics and Chemistry are given below:

Subject	Mathematics	Physics	Chemistry
Mean	42	32	40.9
Standard deviation	12	15	20

Which of these three subjects shows the highest variability in marks and which shows the lowest?

- (a) Chemistry, Mathematics
(b) Mathematics, Physics
(c) Chemistry, Physics
(d) None of the above

Sol. (a) Here, $n = 50$

$$\text{For Mathematics, } CV = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{12}{42} \times 100$$

$$= \frac{2}{7} \times 100$$

$$= \frac{200}{7} = 28.57 \quad \dots(i)$$

$$\text{For Physics, } CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{15}{32} \times 100$$

$$= \frac{1500}{32} = 46.87 \quad \dots(ii)$$

$$\text{For Chemistry, } CV = \frac{\sigma}{\bar{x}} \times 100$$

$$= \frac{20}{40.9} \times 100 = \frac{2000}{40.9} = 48.89 \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii), we have

CV of Chemistry > CV of Physics > CV of Mathematics

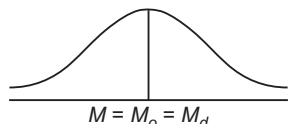
$\therefore$ Chemistry shows the highest variability and Mathematics shows the least variability.

Skewness

We study skewness to have an idea about the shape of the curve which we can draw with the help of the given data. The term skewness refers to lack of symmetry. We can define skewness of a distribution as the tendency of a distribution to depart from symmetry.

- (i) In a symmetrical distribution, we have

$$\text{Mean} = \text{Median} = \text{Mode}$$



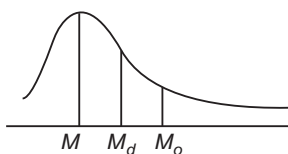
Symmetrical distribution

- (ii) When the distribution is not symmetrical, it is called asymmetrical or skewed.

In a skewed distribution, $\text{Mean} \neq \text{Median} \neq \text{Mode}$

- (a) **Positively skewed distribution** In such a distribution, we have

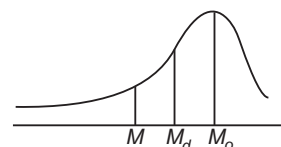
$$\text{Mean} > \text{Median} > \text{Mode}$$



Positively skewed distribution

- (b) **Negatively skewed distribution** In such a distribution, we have

$$\text{Mean} < \text{Median} < \text{Mode}$$



Negatively skewed distribution

• Absolute measures of skewness

- > $S_k = \text{Mean} - \text{Median}$
- > $S_k = \text{Mean} - \text{Mode}$
- > $S_k = Q_3 + Q_1 - 2Q_2$ or $S_k = Q_3 + Q_1 - 2(\text{Median})$.

- **Relative measures of skewness** The following are three important relative measures of skewness, which can be used to compare the distributions with regard to symmetry.

> **Karl Pearson's coefficient of skewness**

$$S_k = \frac{\text{Mean} - \text{Mode}}{\text{Standard deviation}}$$

If mode is not defined, then using the relation,

$$\text{Mode} = 3 \text{ Median} - 2 \text{ Mean}$$

for a moderately skewed distribution, we get

$$S_k = \frac{3(\text{Mean} - \text{Median})}{\text{Standard deviation}}$$

It follows that $S_k = 0$, if $\text{Mean} = \text{Mode} = \text{Median}$

Skewness is positive, if

$\text{Mean} > \text{Mode}$ or $\text{Mean} > \text{Median}$

and negative, if $\text{Mean} < \text{Mode}$ or $\text{Mean} < \text{Median}$

Pearson's coefficient of skewness lies between -3 and 3 .

• Bowley's coefficients of skewness

$$S_k = \frac{(Q_3 - \text{Median}) - (\text{Median} - Q_1)}{(Q_3 - \text{Median}) + (\text{Median} - Q_1)} \\ = \frac{Q_3 + Q_1 - 2 \text{ Median}}{Q_3 - Q_1}$$

where, Q_1 is the first quartile and Q_3 is the third quartile.

Thus, $S_k = +1$, if $\text{median} = Q_1$ and $S_k = -1$, if $Q_3 = \text{median}$.

Bowley's coefficient of skewness lies between -1 and $+1$.

• Kelly's coefficient of skewness

$$S_k = \frac{P_{90} + P_{10} - 2P_{50}}{P_{90} - P_{10}} \quad [\text{based on percentiles}]$$

$$= \frac{D_9 + D_1 - 2D_5}{D_9 - D_1} \quad [\text{based on deciles}]$$

- It should be noted that three different formulae of calculating skewness are based on different assumptions and hence the answer obtained from the same question by different method may differ.

➤ **Example 23.** In a moderately skewed distribution, the values of mean and median are 5 and 6, respectively. The value of mode for such distribution is

- (a) 8 (b) 11
(c) 16 (d) None of these

Sol. (a) For moderately skewed distribution, we have

$$\text{Mode} = 3 \text{ Median} - 2 \text{ Mean} \\ = 18 - 10 = 8$$

➤ **Example 24.** Karl Pearson's coefficient of skewness of a distribution is 0.32. Its SD is 6.5 and mean 39.6. Then, the median of the distribution is given by

- (a) 28.61 (b) 38.90
(c) 29.13 (d) 28.31

Sol. (b) We know that, $S_k = \frac{M - M_o}{\sigma}$,

where $M = \text{Mean}$, $M_o = \text{Mode}$, $\sigma = \text{SD}$

$$\text{i.e.} \quad 0.32 = \frac{39.6 - M_o}{6.5}$$

$$\Rightarrow M_o = 37.52$$

and also we know that, $M_o = 3 \text{ Median} - 2 \text{ Mean}$

$$\Rightarrow 37.52 = 3 (\text{Median}) - 2(39.6)$$

$$\therefore \text{Median} = 38.90 \text{ (approx.)}$$

➤ **Example 25.** For a symmetrical distribution $Q_1 = 25$ and $Q_3 = 45$, then median is

- (a) 28 (b) 35
(c) 30 (d) 40

Sol. (b) For a symmetrical distribution, coefficient of skewness is zero.

$$\therefore \frac{70 - 2M_e}{20} = 0$$

$$\Rightarrow M_e = 35$$

Some Results to be Remembered

- (i) In a statistical data, the sum of the deviations of individual values from AM is always zero i.e.

$$\sum_{i=1}^n (x_i - \bar{x}) = 0$$

- (ii) In a statistical data, the sum of squares of the deviations individual values from AM is least i.e.

$$\sum_{i=1}^n f_i (x_i - \bar{x})^2 \text{ is least.}$$

- (iii) If each of the n given observations is increased/decreased by a , then their mean is also increased/decreased by a .
- (iv) If mean of some observations is $\bar{x}$ and if each observation is multiplied or divide by $k \neq 0$, then the new mean, so obtained will be equal to $k\bar{x}$ or $\frac{\bar{x}}{k}$, respectively.
- (v) Median is the point of intersection of less than and more than ogive.
- (vi) SD remains unchanged, if each variate value is raised or reduced by a constant scalar quantity i.e. SD is independent of change of origin.
- (vii) SD is not independent of change of scale, if $y = ax + b$, then $\sigma_y = |a| \sigma_x$.
- (viii) **Standard deviation of n natural numbers**

$$\sigma = \left(\frac{1}{12} (n^2 - 1) \right)^{1/2}$$

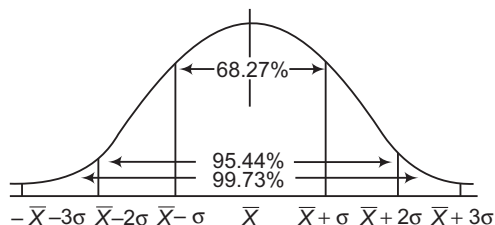
- (ix) Standard deviation shows the limits of variability by which the individual observation in a distribution will vary from the mean. For a symmetrical distribution with mean $\bar{X}$, the following area relationship holds good:

$\bar{X} \pm \sigma$ covers 68.27% observations.

$\bar{X} \pm 2\sigma$ covers 95.44% observations.

$\bar{X} \pm 3\sigma$ covers 99.73% observations.

These limits are illustrated by the following curve known as normal curve:



- (x) **Empirical relationships** If the data is moderately non-symmetrical, then the following empirical relationships hold:

(a) Mean deviation = $\frac{4}{5}\sigma$

(b) Semi-interquartile range = $\frac{2}{3}\sigma$

= Quartile deviation

(c) Probable error of standard deviation = $\frac{2}{3}\sigma$

= Semi-interquartile range

(d) Quartile deviation = $\frac{5}{6}$ MD

From these relationships, we have

$$4 \text{ SD} = 5 \text{ MD} = 6 \text{ QD}$$

Example 26. The arithmetic mean of a set of observations is $\bar{X}$. If each observation is divided by α and then is increased by 10, then the mean of the new series is

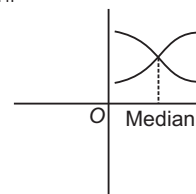
(a) $\frac{\bar{X}}{\alpha}$ (b) $\frac{\bar{X} + 10}{\alpha}$ (c) $\frac{\bar{X} + 10\alpha}{\alpha}$ (d) $\alpha \bar{X} + 10$

Sol. (c) Let $\bar{X}$ be the mean of $x_1, x_2, \dots, x_n$, then the mean of observations $\frac{x_1}{\alpha}, \frac{x_2}{\alpha}, \frac{x_3}{\alpha}, \dots, \frac{x_n}{\alpha}$ is $\frac{\bar{X}}{\alpha}$ and the mean of observations $\frac{x_1}{\alpha} + 10, \frac{x_2}{\alpha} + 10, \dots, \frac{x_n}{\alpha} + 10$ is given by $\frac{\bar{X}}{\alpha} + 10$ i.e. $\frac{\bar{X} + 10\alpha}{\alpha}$.

Example 27. An ogive is used to determine

- (a) mean (b) median
(c) mode (d) harmonic mean

Sol. (b) The intersection point of less than and more than ogive is Median.



Example 28. If the standard deviation of the observations $-5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$ is $\sqrt{10}$. Then, standard deviation of observations 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25 will be

- (a) $\sqrt{10} + 20$ (b) $\sqrt{10} + 10$
(c) $\sqrt{10}$ (d) None of the above

Sol. (c) The new observations are obtained by adding 20 to each term. Hence, σ does not change.

Example 29. For a series, the value of mean deviation is 15. The most likely value of its quartile deviation is

- (a) 12.5 (b) 11.6 (c) 13 (d) 9.7

Sol. (a) Since, MD = $\frac{4}{5}\sigma$, QD = $\frac{2}{3}\sigma$

$$\Rightarrow \frac{\text{MD}}{\text{QD}} = \frac{6}{5} \Rightarrow \text{QD} = \frac{5}{6}(\text{MD}) = \frac{5}{6} \cdot 15 = 12.5$$

Work Book Exercise 20.1

20

Statistics

- 1 Which of the following is most unstable average?
 - a Mode
 - b Median
 - c Mean
 - d Geometric mean
- 2 Which of the following is affected most by extreme observations?
 - a Median
 - b Mode
 - c Harmonic mean
 - d Arithmetic mean
- 3 Which of the following is affected least with extreme observations?
 - a Mode
 - b Median
 - c GM
 - d HM
- 4 The most stable measure of central tendency is
 - a median
 - b mode
 - c mean
 - d None of these
- 5 Which of following is correct relation?
 - a $HM > GM > AM$
 - b $HM > AM > GM$
 - c $AM > GM > HM$
 - d $GM > HM > AM$
- 6 Which of the following is correct relation for a symmetrical distribution?
 - a $AM - M_o = 3(AM - M_d)$
 - b $AM - M_o = 2(AM - M_d)$
 - c $M_d = 2AM - 3M_o$
 - d $AM + M_o = 3(AM - M_d)$
- 7 One of the methods of determining mode is
 - a mode = 2 median - 3 mean
 - b mode = 2 median + 3 mean
 - c mode = 3 median - 2 mean
 - d mode = 3 median + 3 mean
- 8 Coefficient of variation is given by
 - a $\left(\frac{\sigma}{\bar{X}}\right) \times 100$
 - b $\frac{\sigma}{\bar{X}}$
 - c $\frac{\bar{X}}{\sigma}$
 - d $\left(\frac{\bar{X}}{\sigma}\right) \times 100$
- 9 The Karl Pearson's coefficient of skewness is
 - a always positive
 - b always negative
 - c both positive and negative
 - d None of the above
- 10 Bowley's coefficient of skewness is given by
 - a $\frac{Q_3 - Q_1 + 2M_o}{Q_3 + Q_1}$
 - b $\frac{Q_3 + Q_1 - 2M_d}{Q_3 - Q_1}$
 - c $\frac{Q_3 - Q_1 - 2M}{Q_3 - Q_1}$
 - d $\frac{Q_3 + Q_1 - 2M}{Q_3 + Q_1}$
- 11 The value of coefficient of skewness in Bowley's formula lies between
 - a -3 and 3
 - b -1 and 1
 - c $-\infty$ and ∞
 - d $-\infty$ and 0
- 12 The value of coefficient of skewness in Karl Pearson's formula lies between
 - a -3 and 3
 - b -1 and 1
 - c $-\infty$ and ∞
 - d 0 and ∞
- 13 Coefficient of quartile deviation is
 - a $Q_3 - Q_1$
 - b $\frac{1}{2}(Q_3 - Q_1)$
 - c $\frac{Q_3 - Q_1}{Q_3 + Q_1}$
 - d $\frac{\sigma}{\bar{X}}$
- 14 A negative coefficient of skewness implies ($M \rightarrow$ Mean, $M_o \rightarrow$ Median, $M_d \rightarrow$ Mode)
 - a $M > M_o$
 - b $M = M_o$
 - c $M = M_d$
 - d $M < M_o$
- 15 In a positively skewed distribution, which is true?
 - a $M < M_o < M_d$
 - b $M_o > M > M_d$
 - c $M_o > M > M_d$
 - d $M > M_o > M_d$
- 16 In a frequency curve of scores, it is found that the mode is greater than mean. Then, which is correct about frequency curve?
 - a Positive skewed
 - b Negative skewed
 - c Symmetric
 - d None of the above
- 17 The correct empirical relation between the measures of dispersion is
 - a $MD = \frac{3}{4} SD$
 - b $MD = \frac{4}{3} SD$
 - c $MD = \frac{5}{4} SD$
 - d $MD = \frac{4}{5} SD$
- 18 If m is mean of distribution, then $\Sigma(x_i - m)$ is equal to
 - a mean deviation
 - b standard deviation
 - c 0
 - d -1
- 19 The semi-interquartile range = $\lambda\sigma$, then λ is
 - a $\frac{2}{5}$
 - b $\frac{2}{3}$
 - c $\frac{1}{5}$
 - d $\frac{4}{5}$
- 20 The root mean square deviation is least when deviations are measured from
 - a mean
 - b mode
 - c median
 - d origin
- 21 In a symmetrical frequency distribution, lower and upper quartiles are equidistant from
 - a mean deviation
 - b median
 - c standard deviation
 - d None of these
- 22 SD : MD : QD is equal to
 - a 4 : 5 : 6
 - b 15 : 12 : 10
 - c 1 : 2 : 3
 - d None of these
- 23 Which of following statement is true for a given distribution?
 - a Mean deviation > Standard deviation
 - b Mean deviation < Standard deviation
 - c Mean deviation = Standard deviation
 - d They are not related
- 24 If r is the range and σ is the SD of a set of n observations, then correct relation is
 - a $\sigma \leq r$
 - b $\sigma \geq r$
 - c $\sigma = r$
 - d None of these

Correlation Analysis

If two quantities vary in such a way that fluctuation in one are accompanied by fluctuation in other, these quantities are said to be correlated. The statistical tool by which the relationship between two or more than two variables studied is called correlation.

The measures of correlation called the coefficient of correlation and denoted by r .

- (i) **Covariance** The covariance between two variables x and y with n pairs of observations $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ is defined as

$$\text{Cov}(x, y) = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{n}$$

$$= \left(\frac{\sum x_i y_i}{n} - \bar{x} \bar{y} \right)$$

where, $\bar{x} = \frac{\sum x_i}{n}$ and $\bar{y} = \frac{\sum y_i}{n}$

- (ii) **Karl Pearson's correlation coefficient**

$$r = \frac{\text{Cov}(x, y)}{\sqrt{\text{Var}(x) \cdot \text{Var}(y)}} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \cdot \sum (y_i - \bar{y})^2}}$$

$$= \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{\{n \sum x^2 - (\sum x)^2\} \{n \sum y^2 - (\sum y)^2\}}}$$

- (iii) **Coefficient of rank correlation** This formula is applied to the problems in which data cannot be measured quantitatively but qualitative assessment is possible such as beauty, honesty etc. In this case, the best individual is given rank number 1, next rank 2 and so on. The coefficient of rank correlation is given by the formula

$$R = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)}$$

where, d_i is the difference of corresponding rank and n is the number of pairs of observations.

✎ There should be formula for equal rank or tie in ranks.

- ☞ **Example 30.** A computer while calculating r_{xy} from 25 pairs of observations obtained the following constant:

$$n = 25, \sum x = 125, \sum x^2 = 650, \sum y = 100, \sum y^2 = 460, \sum xy = 508$$

A recheck showed that he had copied down two pairs (6, 14), (8, 6) while the correct values were (8, 12), (6, 8). The correct value of the correlation coefficient is

- (a) $\frac{2}{3}$ (b) $\frac{3}{4}$
(c) $-\frac{1}{2}$ (d) 1

- Sol.** (a) Correct value of $\sum x$

$$= 125 - 6 - 8 + 8 + 6 = 125$$

$$\text{Correct value of } \sum x^2 = 650 - 36 - 64 + 64 + 36 = 650$$

$$\text{Correct value of } \sum y = 100 - 14 - 6 + 12 + 8 = 100$$

$$\text{Correct value of } \sum y^2 = 460 - 196 - 36 + 144 + 64 = 436$$

$$\text{Correct value of } \sum xy = 508 - 84 - 48 + 96 + 48 = 520$$

∴ Correct correlation coefficient

$$= \frac{n \sum xy - (\sum x)(\sum y)}{\sqrt{\{n \sum x^2 - (\sum x)^2\} \{n \sum y^2 - (\sum y)^2\}}}$$

$$= \frac{2}{3}$$

- ☞ **Example 31.** The rank correlation coefficient between marks secured in Mathematics and Physics by a certain number of students in a class is 0.8 and the sum of the squares of differences in ranks is 33, then number of students in the class is

- (a) 30
(b) 11
(c) 10
(d) 15

Sol. (c) $r = 1 - \frac{6 \sum d_i^2}{n(n^2 - 1)} = 1 - \frac{6 \times 33}{n(n^2 - 1)}$

$$\Rightarrow \frac{6 \times 33}{n(n^2 - 1)} = 990$$

$$\Rightarrow \frac{6 \times 33}{n(n^2 - 1)} = 10(10^2 - 1)$$

$$\Rightarrow n = 10$$

Characteristics of Correlation Coefficient

- $-1 \leq r \leq 1$
- If $r = -1$, then there is perfect negative correlation between x and y i.e. corresponding to an increase (or decrease) in one variable, there is a proportional decrease (or increase) in the other variable.
- If $r = 1$, then there is perfect positive correlation between x and y i.e. corresponding to an increase (or decrease) in one variable, there is proportional increase (or decrease) in the other variable.
- If $r = 0$, then x and y are not correlated i.e. the changes in one variable are not followed by changes in the other.
- If $0 < r < 1$, then there is a positive correlation between x and y i.e. an increase (or decrease) in one variable corresponds to an increase (or decrease) in the other.
- If $-1 < r < 0$, then there is negative correlation between x and y i.e. an increase (or decrease) in one variable corresponds to decrease (or increase) in the other.

- (vii) The value r^2 is called coefficient of determination. If $r = 0.5$, then $r^2 = 0.25$, which means only 25% of variations are explained and remaining 75% are unexplained. They may be due to external causes.

(viii) Standard Error (SE) is defined as $SE = \frac{1 - r^2}{\sqrt{n}}$.

- (ix) Probable Error (PE) is defined as $PE = (0.6745) SE$. It is used to test reliability of particular value of r . Infact $(r \pm PE)$ gives limits within which the coefficient of correlation sample must always lie when the sample is drawn from the universe.

➤ **Example 32.** If r is correlation coefficient, then correct relation is

- (a) $r > 1$
(b) $1 \leq r$
(c) $|r| \leq 1$
(d) $|r| \geq 1$

Sol. (c) By definition of correlation coefficients, we have $-1 \leq r \leq 1 \Rightarrow |r| \leq 1$

➤ **Example 33.** For 12 pairs of observations on x and y , the r calculated is 0.674. The standard error is

- (a) 0.2563
(b) 0.1575
(c) 0.4384
(d) None of the above

Sol. (b) Using the formula,

$$\begin{aligned} SE &= \frac{1 - r^2}{\sqrt{n}} \\ &= \frac{1 - 0.4542}{3.4641} \\ &= \frac{0.5458}{3.4641} = 0.1575 \end{aligned}$$

➤ **Example 34.** In a question on correlation, the value of r was 0.917 and its probable error was 0.034. The number of pairs of observations was

- (a) 8.5
(b) 12.02
(c) 10
(d) 10.627

Sol. (c) Using the formula,

$$\begin{aligned} PE &= (0.6745) SE, \text{ we have} \\ PE &= \frac{(0.6745)(1 - r^2)}{\sqrt{n}} \\ \Rightarrow 0.034 &= \frac{(0.6745)[1 - (0.917)^2]}{\sqrt{n}} \\ \Rightarrow \sqrt{n} &= \frac{(0.6745)[1 - (0.917)^2]}{0.034} \\ \Rightarrow \sqrt{n} &= 3.156 \\ \therefore n &= 10 \text{ (approx.)} \end{aligned}$$

Regression Analysis

- i. **Line of regression of y on x** The line of regression of y on x gives the best estimate of the value of y given value of x and is given by $y - \bar{Y} = b_{yx}(x - \bar{X}); b_{yx} = r \cdot \sigma_y / \sigma_x$

- ii. **Line of regression of x on y** The line of regression of x on y gives the best estimate of the value of x for given value of y is given by $x - \bar{X} = b_{xy}(y - \bar{Y}); b_{xy} = r \cdot \frac{\sigma_x}{\sigma_y}$

- iii. **Coefficient of regression of y on x** The coefficient of regression of y on x denoted by b_{yx} and is given by

$$\begin{aligned} r_1 = b_{yx} &= r \cdot \frac{\sigma_y}{\sigma_x} = \frac{\text{Cov}(x, y)}{\sigma_x^2} \\ &= \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum x_i^2 - (\sum x_i)^2} \end{aligned}$$

- iv. **Coefficient of regression of x on y** The coefficient of regression of x on y denoted by b_{xy} and is given by

$$r_2 = b_{xy} = \frac{\sigma_x}{r \sigma_y} = \frac{\text{Cov}(x, y)}{\sigma_y^2} = \frac{n \sum x_i y_i - \sum x_i \sum y_i}{n \sum y_i^2 - (\sum y_i)^2}$$

➤ **Example 35.** If x and y are respectively heights of father and son and $M_x = 68.25$ inch, $M_y = 68.5$ inch, $\sigma_x = 2.49$, $\sigma_y = 2.23$ and $r = 0.47$, then which of following is height of son when height of father is 67.5 inch?

- (a) 68.19 (b) 68.81 (c) 65.66 (d) 75.02

Sol. (a) Let us first find the line of regression of y on x , which is given by

$$\begin{aligned} y - \bar{Y} &= r \cdot \frac{\sigma_y}{\sigma_x} (x - \bar{X}) \\ \Rightarrow y - 68.5 &= (0.47) \frac{(2.23)}{(2.49)} (x - 68.25) \end{aligned}$$

When $x = 67.5$, then $y - 68.5 = 0.42(67.5 - 68.25)$
 $\therefore y = 68.19$

➤ **Example 36.** If $\sum x = 30$, $\sum y = 42$, $\sum xy = 199$, $\sum x^2 = 184$, $\sum y^2 = 318$ and $n = 6$, then regression coefficient b_{xy} is

- (a) -0.36 (b) -0.46 (c) 0.26 (d) 0.38

Sol. (b) We know that,

$$b_{xy} = \frac{\sum xy - \left(\frac{\sum x \sum y}{n} \right)}{\sum y^2 - \frac{(\sum y)^2}{n}} = \frac{199 - \left(\frac{30 \times 42}{6} \right)}{318 - \frac{(42)^2}{6}} = \frac{-11}{24} = -0.46$$

Properties of Regression Coefficients

- i. Both regression coefficients have the same sign i.e. either both are positive or both are negative.
- ii. The sign of correlation coefficient is same as that of regression coefficient i.e. $r > 0$, if $b_{xy} > 0$ and $b_{yx} > 0$ and $r < 0$, if $b_{xy} < 0$ and $b_{yx} < 0$.
- iii. The coefficients of correlation is the geometric mean between the two regression coefficients

$$r = \pm \sqrt{b_{yx} \times b_{xy}}$$

The sign to be taken outside the square root is that of the regression coefficients.
- iv. Both the regression coefficients cannot be numerically greater than one.
- v. Regression coefficients are independent of change of origin but not scale.
- vi. AM of the regression coefficients is greater than the correlation coefficient.

➤ **Example 37.** If the correlation coefficient between two variables x and y is 0.4 and regression coefficient of x on y is 0.2, then regression coefficient of y on x is
 (a) 0.4 (b) ± 0.8 (c) 0.8 (d) 0.6

Sol. (c) Here, $r = 0.4$, $b_{xy} = 0.2$

We know, that

$$\begin{aligned} r^2 &= b_{xy} \times b_{yx} \\ \Rightarrow 0.16 &= 0.2 \times b_{yx} \\ \therefore b_{yx} &= 0.8 \end{aligned}$$

➤ **Example 38.** If $ax + by + c = 0$ is line of regression of y on x and $a_1x + b_1y + c_1 = 0$ is line of regression of x on y , then which of the following is correct?

- (a) $|ab_1| \geq |a_1b|$ (b) $|ab_1| = |a_1b|$
- (c) $ab_1 + ba_1 = 0$ (d) $|ab_1| \leq |a_1b|$

Sol. (d) Line of regression of y on x is $ax + by + c = 0$

$$\text{or } y = \left(-\frac{a}{b}\right)x - \frac{c}{b}$$

$$\therefore r_1 = \left(-\frac{a}{b}\right)$$

Line of regression of x on y is

$$x = -\frac{b_1}{a_1}y - \frac{c_1}{a_1}$$

$$\therefore r_2 = -\frac{b_1}{a_1}$$

Then, $|r_1 r_2| \leq 1$ or $|ab_1| \leq |a_1b|$.

Properties of Lines of Regression

The two lines of regression have the following properties:

- i. The two lines of regression pass through the point $(\bar{x}, \bar{y})$.
- ii. Slope of the line of regression of y on $x = b_{yx}$.
- iii. Slope of the line of regression of x on $y = \frac{1}{b_{xy}}$.
- iv. The angle θ between two lines of regression is given by

$$\tan \theta = \left(\frac{1-r^2}{|r|} \right) \frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$$

$$= \left| \frac{b_{yx} b_{xy} - 1}{b_{yx} + b_{xy}} \right| = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

where, m_1 = Slope of line of regression y on x and m_2 = Slope of line of regression x on y
- v. If $r = 0$, then $\tan \theta = \infty \Rightarrow \theta = \frac{\pi}{2}$ i.e. if two variables are uncorrelated, the lines of regression are perpendicular to each other.
- vi. If $r = \pm 1$, then $\tan \theta = 0 \Rightarrow \theta = 0$ or π i.e. in the case of perfect correlation ($r = \pm 1$) the two lines of regression coincide.

➤ **Example 39.** For two variables x and y with the same mean, the two regression lines are $x + 2y - 5 = 0$; $2x + 3y - 8 = 0$ and variance $(x) = 12$. Then, σ_y is

- (a) 4 (b) 5
- (c) 2 (d) None of these

Sol. (c) Slopes of regression lines are $-\frac{1}{2}$ and $-\frac{2}{3}$.

$$\therefore b_{yx} = -\frac{1}{2} \text{ and } b_{xy} = -\frac{3}{2}$$

$$\therefore r^2 = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) = \frac{3}{4} (< 1)$$

$$\Rightarrow r = -\frac{\sqrt{3}}{2} \quad [\because b_{yx}, b_{xy} \text{ are less than zero}]$$

$$\text{Also, } b_{yx} = r \cdot \frac{\sigma_y}{\sigma_x}$$

$$\text{and } \sigma_x = 2\sqrt{3}$$

$$\therefore \sigma_x^2 = 12$$

$$\therefore -\frac{1}{2} = -\frac{\sqrt{3}}{2} \cdot \frac{\sigma_y}{2\sqrt{3}}$$

$$\Rightarrow \sigma_y = 2$$

- **Example 40.** If $b_{yx} = 1.6$ and $b_{xy} = 0.4$ and the angle between two regression lines is θ , then $\tan \theta$ is
(a) 0.18 (b) 0.16
(c) 0.3 (d) 0.24

Sol. (a) Clearly, $r = \sqrt{1.6 \times 0.4} = 0.8$

$$\begin{aligned} \text{Now, } b_{yx} &= r \cdot \frac{\sigma_y}{\sigma_x} \\ \Rightarrow \frac{\sigma_y}{\sigma_x} &= \frac{1.6}{0.8} = 2 \end{aligned}$$

$$\begin{aligned} \therefore m_1 &= \frac{1 \cdot \sigma_y}{r \cdot \sigma_x} \\ &= \frac{1}{0.8} \times 2 = \frac{5}{2} = 2.5 \\ \text{and } m_2 &= r \cdot \frac{\sigma_y}{\sigma_x} \\ &= 0.8 \times 2 = 1.6 \\ \therefore \tan \theta &= \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| \\ &= \frac{0.9}{5} = 0.18 \end{aligned}$$

Work Book Exercise 20.2

- If r is coefficient of correlation between x and y , then $r = 0$ means
a x and y are perfectly related
b x and y are negatively related
c x and y are partially related
d x and y are not related
- The regression coefficients are independent of
a change of origin and scale
b change of scale only
c change of origin only
d None of the above
- If $r = 0.8$, then explained data would be
a 80% b 64% c 20% d 36%
- The coefficient of correlation is, which of following mean between two regression coefficients
a HM b GM c AM d $2 \times AM$
- Let d_i represent difference in ranks of i th item. If $d_i = 0, \forall i$, then correct relation is
a $r = 0$
b $r = -1$
c $r = 1$
d None of the above
- If two regression coefficients are 0.8 and 0.2, then r is
a -0.4
b -0.5
c 0.5
d 0.4
- If two lines of regression are $x = 0.85y + 9.48$ and $y = 0.99x + 1$, then means of x and y series are
a 66.5, 65.5
b 65.2, 65.55
c 65.4, 65.68
d 64.5, 64.8
- The two lines of regression are parallel to axes, if r is
a 1
b 0
c -1
d 0.5
- If lines of regression of y on x and x on y are respectively $y = kx + 4$, $x = 4y + 5$, then which is true for k ?
a $0 \leq k \leq 4$
b $0 \leq k \leq 1/4$
c $k > 1/4$
d None of the above
- Let x_1, x_2 and x_3 be uncorrelated variables each having the same standard deviation. The coefficient of correlation between $(x_1 + x_2)$ and $(x_2 + x_3)$ is
a $1/4$
b $1/3$
c $1/2$
d 1

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. The geometric mean G of the product of n series of data with geometric means $G_1, G_2, \dots, G_n$ respectively, then

- (a) $\sqrt[n]{G} = G_1, G_2, \dots, G_n$
 (b) $G = G_1 \cdot G_2 \cdot \dots \cdot G_n$
 (c) $\sqrt[n]{G} = G_1, G_2, \dots, G_n$
 (d) None of the above

Sol. Let $x_1, x_2, \dots, x_n$ be the variates corresponding to n sets of data, each having the same number of observations say K and x be their product.

Then, $x = x_1 \cdot x_2 \cdot \dots \cdot x_n$

i.e. $\log x = \log x_1 + \log x_2 + \dots + \log x_n$

or $\frac{\Sigma \log x}{K} = \frac{\Sigma \log x_1}{K} + \frac{\Sigma \log x_2}{K} + \dots + \frac{\Sigma \log x_n}{K}$

or $\log G = \log G_1 + \log G_2 + \dots + \log G_n$

$\Rightarrow G = G_1 \cdot G_2 \cdot \dots \cdot G_n$

Hence, (b) is the correct answer.

Ex 2. A motor car when travelling from rest travels the first twentieth of a mile at 6 m/h the next three twentieths of the mile at 8, 12, 24 m/h respectively. But its average speed over the first one-fifth of a mile is not 12.5 m/h, then the correct average is

- (a) 9.6 m/h
 (b) 19.6 m/h
 (c) 6.6 m/h
 (d) None of the above

Sol. Let A and H be the arithmetic and harmonic means of speeds.

So, $A = \frac{6 + 8 + 12 + 24}{4} = \frac{50}{4} = 12.5 \text{ m/h}$

and $H = \frac{4}{\frac{1}{6} + \frac{1}{8} + \frac{1}{12} + \frac{1}{24}}$
 $= \frac{4 \times 24}{10} = \frac{48}{5} = 9.6 \text{ m/h}$

It is obvious from the data that the average speed would not be arithmetic mean but it is the harmonic mean, because

Average speed = $\frac{\text{Total distance travelled}}{\text{Total time of journey}}$
 $= \frac{1}{\frac{1}{20} \times \frac{1}{6} + \frac{1}{20} \times \frac{1}{8} + \frac{1}{20} \times \frac{1}{12} + \frac{1}{20} \times \frac{1}{24}}$
 $= \frac{480}{5 \times 10} = 9.6 = \text{HM}$

Hence, (a) is the correct answer.

Ex 3. The average rate of (i) motion in the case of a

person who rides the first mile at 10 m/h, the

next mile at 8 m/h and the third mile at 6 m/h

(ii) increase in population which is the first

decade has increased 20% in the next 25% and

in the third 44% are (where, $\log 20 = 1.3010$,

$\log 25 = 1.3979$, $\log 28.02 = 1.4475$,

$\log 44 = 1.6435$)

- (a) 7.66, 1.7 (b) 7.5, 2.9
 (c) 7.66, 28.02 (d) None of these

Sol. (i) In this case, the harmonic mean is the suitable average.

The average rate of motion is,

$$H = \frac{\text{Total distance}}{\text{Total time taken}} = \frac{3}{\frac{1}{10} + \frac{1}{8} + \frac{1}{6}}$$

$$= \frac{3 \times 120}{47} = 7.66 \text{ m/h}$$

(ii) Here, GM is the appropriate average.

$$\therefore G = (20 \times 25 \times 44)^{1/3}$$

$$\Rightarrow \log G = \frac{1}{3} [\log 20 + \log 25 + \log 44]$$

$$= \frac{1}{3} [1.3010 + 1.3979 + 1.6435]$$

$$= \frac{1}{3} \times 4.3425 = 1.4475$$

$\therefore G = 28.02 =$ The average increase in population.

Hence, (c) is the correct answer.

Ex 4. A man motors from A to B . In motoring a

distance uphill he gets a mileage of only

10 miles per gallon of gasoline. On the return

trip, he makes 15 miles per gallon, then the

harmonic mean of his mileage (verify that this

is the proper average to be used, here assuming

that the distance from A to B is 60 miles) is

- (a) 12 (b) 11
 (c) 10 (d) None of these

Sol. The harmonic mean,

$$H = \frac{2}{\frac{1}{10} + \frac{1}{15}} = \frac{2 \times 30}{5} = 12 \text{ miles per gallon}$$

Let us verify it. The distance between A and B is 60 miles.

He has consumed $\frac{60}{10} = 6$ gallons of gasoline in going

uphill and $\frac{60}{15} = 4$ gallons in returning i.e. 10 gallons in

all.

The total distance covered = $(60 + 60)$ [both ways]
 $= 120 \text{ miles}$

$$\therefore \text{Average consumption} = \frac{120}{10} = 12 \text{ gallons per mile} \\ = \text{HM}$$

Thus, harmonic mean is the proper average here.
Hence, (a) is the correct answer.

- Ex 5.** The median of the following items
25, 15, 23, 40, 27, 25, 23, 25 and 20 is
(a) 27 (b) 40
(c) 25 (d) 23

Sol. First arrange in ascending order as
15, 20, 23, 23, 25, 25, 25, 27, 40
 $\therefore$ Median = Size of $\left(\frac{9+1}{2}\right)$ th item = 5th item = 25
Hence, (c) is the correct answer.

- Ex 6.** If a variate takes values $a, ar, ar^2, \dots, ar^{n-1}$, then which of the relation between means hold?
(a) $AH = G^2$ (b) $\frac{A+H}{2} = G$
(c) $A < G < H$ (d) $A = G = H$

Sol. $A = \frac{a(1+r+r^2+\dots+r^{n-1})}{n} = \frac{a}{n} \left(\frac{1-r^n}{1-r} \right)$
 $G = (a \cdot ar \cdot ar^2 \dots ar^{n-1})^{1/n}$
 $= a \cdot r^{\frac{1+2+\dots+(n-1)}{n}} = ar^{\frac{n-1}{2}}$
 $\frac{1}{H} = \frac{1}{na} \left\{ 1 + \frac{1}{r} + \frac{1}{r^2} + \dots + \frac{1}{r^{n-1}} \right\}$
 $= \frac{1}{na} \cdot \frac{1-\frac{1}{r^n}}{1-\frac{1}{r}}$
 $\Rightarrow H = \frac{na(1-r)r^{n-1}}{1-r^n}$
 $\therefore AH = a^2 r^{n-1} = G^2$
Hence, (a) is the correct answer.

- Ex 7.** An aeroplane flies around a square, the sides of which measure 100 miles each. The aeroplane covers at a speed of 100 m/h the first side, at 200 m/h the second side, at 300 m/h the third side and 400 m/h the fourth side. The average speed of aeroplane around the square is
(a) 190 m/h (b) 195 m/h
(c) 192 m/h (d) 200 m/h

Sol. Using the weighted HM formula,

$$\frac{1}{H} = \frac{1}{N} \sum_{i=1}^n \frac{f_i}{x_i}$$

or $H = \frac{1}{\frac{1}{N} \sum_{i=1}^n \frac{f_i}{x_i}}$

$$\therefore \text{Average speed} = \frac{400}{100 \left(\frac{1}{100} + \frac{1}{200} + \frac{1}{300} + \frac{1}{400} \right)} = 192 \text{ m/h}$$

Hence, (c) is the correct answer.

- Ex 8.** If SD of x is σ , then SD of the variable $u = \frac{ax+b}{c}$, where a, b and c are constants, is

- (a) $\left| \frac{c}{a} \right| \sigma$ (b) $\left| \frac{a}{c} \right| \sigma$
(c) $\left| \frac{b}{c} \right| \sigma$ (d) $\left| \frac{c^2}{a^2} \right| \sigma$

Sol. Make use of the results

$$\text{Var}(ax+b) = a^2 \text{Var}(x)$$

$$\text{We have, } \text{Var} \left\{ \frac{ax+b}{c} \right\} = \frac{a^2}{c^2} \text{Var}(x) = \frac{a^2}{c^2} \sigma^2$$

$$\Rightarrow \text{SD} \left\{ \frac{ax+b}{c} \right\} = \left| \frac{a}{c} \right| \sigma$$

Hence, (b) is the correct answer.

- Ex 9.** The first of two samples has 100 items with mean 15 and SD 3. If the whole group has 250 items with mean 15.6 and $\text{SD} = \sqrt{13.44}$, the SD of the second group is
(a) 5
(b) 4
(c) 6
(d) 3.52

Sol. Use $\sigma^2 = \frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}$

where, $d_1 = m_1 - a$, $d_2 = m_2 - a$, a being the mean of the whole group. Let m_2 = mean of the second group

$$\therefore 15.6 = \frac{100 \times 15 + 150 \times m_2}{250} \Rightarrow m_2 = 16$$

$$\text{Thus, } 13.44 = \frac{\left[(100 \times 9 + 150 \times \sigma^2) + 100 \times (0.6)^2 + 150 \times (0.4)^2 \right]}{250}$$

$$\Rightarrow \sigma = 4$$

Hence, (b) is the correct answer.

- Ex 10.** The mean and SD of 63 children in an arithmetic test are respectively 27.6 and 7.1. They are added to a new group of 26 who had less training and whose mean is 19.2 and SD 6.2. The values of the combined group of the mean and SD are
(a) 25.1, 7.8
(b) 2.3, 0.8
(c) 1.5, 0.9
(d) None of the above

Sol. Mean and SD (σ) of the combined group are

$$m = \frac{63 \times 27.6 + 26 \times 19.2}{63 + 26} = 25.1$$

$$\text{and } \sigma^2 = \frac{\left[63(7.1)^2 + 26(6.2)^2 + 63(27.6 - 25.1)^2 + 26(19.2 - 25.1)^2 \right]}{89}$$

$$\Rightarrow \sigma = 7.8 \text{ (approx.)}$$

Hence, (a) is the correct answer.

- Ex 11.** Coefficient of skewness for the values median = 18.8, $Q_1 = 14.6$, $Q_3 = 25.2$ is
 (a) 0.2 (b) 0.5
 (c) 0.7 (d) None of these

Sol. Coefficient of skewness

$$= \frac{Q_3 + Q_1 - 2(\text{Median})}{Q_3 - Q_1} = \frac{25.2 + 14.6 - 2 \times 18.8}{25.2 - 14.6} = 0.20$$

Hence, (a) is the correct answer.

- Ex 12.** An incomplete frequency distribution is given below:

Variate	Frequency
10-20	12
20-30	30
30-40	x
40-50	65
50-60	45
60-70	25
70-80	18
Total	229

If median value is 46, then the missing frequency is

- (a) 33.5 (b) 35
 (c) 34 (d) 26

Sol. Median = 46, which lies in 40-50 class

$$\text{Median} = l + h \frac{\left\{ \frac{N}{2} - cf \right\}}{f}$$

where, f = Frequency of median class

cf = Cumulative frequency of the class preceding the median class

$$\therefore 46 = 40 + 10 \frac{\left[\frac{229}{2} - (x + 42) \right]}{65}$$

where, x = Frequency of class 30-40

$$\Rightarrow x = 33.5 = 34 \quad [\because x \text{ is an integer}]$$

Hence, (c) is the correct answer.

- Ex 13.** x_1 and x_2 are two variates with variances σ_1^2 and σ_2^2 respectively and r is the correlation between them. The value of a such that $x_1 + ax_2$ and $x_1 + \frac{\sigma_1}{\sigma_2}x_2$ are uncorrelated, is

- (a) $-\frac{\sigma_1}{\sigma_2}$ (b) $r \cdot \frac{\sigma_1}{\sigma_2}$
 (c) $\sigma_1\sigma_2$ (d) None of these

Sol. Let $u = x_1 + ax_2$ and $v = x_1 + \frac{\sigma_1}{\sigma_2}x_2$

$$\text{Therefore, } \bar{u} = \bar{x}_1 + a\bar{x}_2 \text{ and } \bar{v} = \bar{x}_1 + \frac{\sigma_1}{\sigma_2}\bar{x}_2$$

If u and v are uncorrelated, then $\text{Cov}(u, v) = 0$

Therefore, $\Sigma(u - \bar{u})(v - \bar{v}) = 0$

$$\text{or } \Sigma\{(x_1 - \bar{x}_1) + a(x_2 - \bar{x}_2)\} \cdot \{(x_1 - \bar{x}_1) + \frac{\sigma_1}{\sigma_2}(x_2 - \bar{x}_2)\} = 0$$

$$\Rightarrow \Sigma\left\{(x_1 - \bar{x}_1)^2 + a \cdot \frac{\sigma_1}{\sigma_2}(x_2 - \bar{x}_2)^2 + (x_1 - \bar{x}_1)(x_2 - \bar{x}_2)\right\} = 0$$

$$\Rightarrow \sigma_1^2 + a \frac{\sigma_1}{\sigma_2} \cdot \sigma_2^2 + \left(\frac{\sigma_1}{\sigma_2} + a\right) r \sigma_1 \sigma_2 = 0$$

$$\Rightarrow a \sigma_1 \sigma_2 (1 + r) = -\sigma_1^2 (1 + r)$$

$$\Rightarrow a = -\frac{\sigma_1}{\sigma_2}, \text{ since here } r \neq -1$$

Hence, (a) is the correct answer.

- Ex 14.** Correlation coefficient r of two variables x and y is positive, when

- (a) $\sigma_{x+y} > \sigma_{x-y}$
 (b) $\sigma_{x-y} > \sigma_{x+y}$
 (c) $\sigma_{x+y} = 0$
 (d) None of the above

Sol. $\sigma_{x+y}^2 = \text{Var}(x+y) = \text{Var}(x) + \text{Var}(y) + 2\text{Cov}(x, y)$

$$\text{or } \sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 + 2r\sigma_x\sigma_y$$

$$\text{Similarly, } \sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y$$

On subtracting, we get

$$\sigma_{x+y}^2 - \sigma_{x-y}^2 = 4r\sigma_x\sigma_y$$

Since, σ_x and σ_y are always positive, so r is positive, if

$$\sigma_{x+y}^2 - \sigma_{x-y}^2 > 0$$

$$\text{or } \sigma_{x+y} > \sigma_{x-y}$$

Hence, (a) is the correct answer.

- Ex 15.** Two variates x and y have zero means, the same variance and zero correlation. Then, $u = x \cos \alpha + y \sin \alpha$, $v = x \sin \alpha - y \cos \alpha$ have correlation

- (a) -1 (b) 1
 (c) 0 (d) None of these

Sol. Given, $\bar{x} = 0$, $\bar{y} = 0$, $\text{Var}(x) = \text{Var}(y) = \sigma^2$ and

$$\text{zero correlation} = \text{Cov}(x, y) = 0 \text{ or } \frac{1}{n} \Sigma(xy) = 0$$

$$\text{Thus, } \text{Var}(u) = \cos^2 \alpha \text{Var}(x) + \sin^2 \alpha \text{Var}(y) = \sigma^2$$

$$\text{and similarly } \text{Var}(v) = \sigma^2$$

$$\text{Also, } \bar{u} = \bar{x} \cos \alpha + \bar{y} \sin \alpha = 0$$

$$\bar{v} = \bar{x} \sin \alpha - \bar{y} \cos \alpha = 0$$

$$\text{Therefore, } \text{Cov}(u, v) = \frac{1}{n} \Sigma(u - \bar{u})(v - \bar{v}) = \frac{1}{n} \Sigma(uv)$$

$$= \frac{1}{n} \Sigma\{(x \cos \alpha + y \sin \alpha)(x \sin \alpha - y \cos \alpha)\}$$

$$= \frac{1}{n} \Sigma\{x^2 \sin \alpha \cos \alpha - y^2 \sin \alpha \cos \alpha + xy(\sin^2 \alpha - \cos^2 \alpha)\}$$

$$\begin{aligned}
 &= \sin \alpha \cos \alpha \frac{1}{n} \sum (x)^2 - \sin \alpha \cos \alpha \frac{1}{n} \sum (y)^2 \\
 &\quad + (\sin^2 \alpha - \cos^2 \alpha) \frac{1}{n} \sum (xy) \\
 &= \sin \alpha \cos \alpha \operatorname{Var}(x) - \sin \alpha \cos \alpha \operatorname{Var}(y) + 0 \\
 &\quad \left[\because \frac{1}{n} \sum (xy) = 0 \right]
 \end{aligned}$$

Thus, $r_{uv} = 0$ $[\because \operatorname{Var}(x) = \operatorname{Var}(y) = \sigma^2]$

Hence, (c) is the correct answer.

Ex 16. The correlation between x and $a - x$ is

- (a) -1 (b) 1 (c) $\frac{1}{2}$ (d) 0

Sol. Let $u = a - x$ and therefore

$$\begin{aligned}
 \operatorname{Var}(u) &= \operatorname{Var}(a - x) = (-1)^2 \operatorname{Var}(x) \\
 &= \operatorname{Var}(x) = \sigma^2 \quad [\text{say}]
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } \operatorname{Cov}(x, a - x) &= \operatorname{Cov}(x, u) = \frac{1}{n} \sum \{(x - \bar{x})(u - \bar{u})\} \\
 &= -\frac{1}{n} \sum (x - \bar{x})^2 \quad [\because u - \bar{u} = -(x - \bar{x})] \\
 &= -\operatorname{Var}(x) = -\sigma^2
 \end{aligned}$$

$$\therefore r(x, u) = \frac{\operatorname{Cov}(x, u)}{\sqrt{\operatorname{Var}(x) \operatorname{Var}(u)}} = \frac{-\sigma^2}{\sqrt{\sigma^2 \cdot \sigma^2}} = -1$$

Hence, (a) is the correct answer.

Ex 17. The two variates x and y are uncorrelated and have standard deviations σ_x and σ_y respectively, the correlation coefficient between $x + y$ and $x - y$ is

- (a) $\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}$ (b) $\frac{1}{2} \left(\frac{1}{\sigma_x} + \frac{1}{\sigma_y} \right)$
 (c) $\frac{\sigma_x^2 - \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$ (d) None of these

Sol. Let $u = x + y$ and $v = x - y$
 Therefore, $\bar{u} = \bar{x} + \bar{y}$ and $\bar{v} = \bar{x} - \bar{y}$

$$\begin{aligned}
 \therefore \operatorname{Cov}(u, v) &= \frac{1}{n} \sum \{(u - \bar{u})(v - \bar{v})\} \\
 &= \frac{1}{n} \sum \{(x - \bar{x}) + (y - \bar{y})\} \cdot \{(x - \bar{x}) - (y - \bar{y})\} \\
 &= \frac{1}{n} \sum \{(x - \bar{x})^2 - (y - \bar{y})^2\} = \sigma_x^2 - \sigma_y^2 \\
 \operatorname{Var}(u) &= \frac{1}{n} \sum (u - \bar{u})^2 = \frac{1}{n} \sum \{(x - \bar{x}) + (y - \bar{y})\}^2 \\
 &= \sigma_x^2 + \sigma_y^2
 \end{aligned}$$

$$\left[\because x \text{ and } y \text{ are uncorrelated,} \right. \\
 \left. \text{so } \frac{1}{n} \sum (x - \bar{x})(y - \bar{y}) = 0 \right]$$

Similarly, $\operatorname{Var}(v) = \sigma_x^2 + \sigma_y^2$

$$\text{Thus, } r(u, v) = \frac{\operatorname{Cov}(u, v)}{\sigma_u \sigma_v} = \frac{\sigma_x^2 - \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$$

Hence, (c) is the correct answer.

Ex 18. x and y are two correlated variables with the same standard deviation and the correlation coefficient r , the correlation coefficient between x and $x + y$ is

- (a) $\frac{r}{2}$ (b) $\sqrt{\frac{1-r}{2}}$ (c) $\sqrt{\frac{1+r}{2}}$ (d) 0

Sol. Let $u = x + y$ and $\sigma = \text{SD of } x \text{ or } y$

$$\begin{aligned}
 \therefore \sigma_u^2 &= \sigma_x^2 + \sigma_y^2 + 2 \operatorname{Cov}(x, y) \\
 &= \sigma^2 + \sigma^2 + 2r\sigma^2 = 2\sigma^2(1+r)
 \end{aligned}$$

$$\begin{aligned}
 \therefore \operatorname{Cov}(u, x) &= \frac{1}{n} \sum \{(u - \bar{u})(x - \bar{x})\} \\
 &= \frac{1}{n} \sum \{(x - \bar{x}) + (y - \bar{y})\}(x - \bar{x}) \\
 &= \frac{1}{n} \sum \{(x - \bar{x})^2 + (x - \bar{x})(y - \bar{y})\} = \sigma^2(1+r)
 \end{aligned}$$

$$\text{Thus, } r(x, x + y) = \frac{\sigma^2(1+r)}{\sigma \cdot \sigma \sqrt{2(1+r)}} = \sqrt{\frac{1+r}{2}}$$

Hence, (c) is the correct answer.

Ex 19. $\bar{x}$ is the arithmetic mean of n independent variates $x_1, x_2, x_3, \dots, x_n$ each of standard deviation σ , then variance ($\bar{x}$) is

- (a) $\frac{\sigma^2}{n}$ (b) $\frac{n\sigma^2}{2}$
 (c) $\frac{(n+1)\sigma^2}{3}$ (d) None of these

Sol. We have, $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$

$$\begin{aligned}
 \therefore \operatorname{Var}(\bar{x}) &= \frac{1}{n^2} \left[\sum_{i=1}^n \operatorname{Var}(x_i) + 2 \sum_{i \neq j} \operatorname{Cov}(x_i, x_j) \right] \\
 &= \frac{1}{n^2} [n\sigma^2] = \frac{\sigma^2}{n} \\
 &\left[\because \operatorname{Var}(x_i) = \sigma^2 \text{ for every } i \text{ and } \operatorname{Cov}(x_i, x_j) = 0 \right. \\
 &\quad \left. \text{as variates are independent} \right]
 \end{aligned}$$

Hence, (a) is the correct answer.

Ex 20. If r_{xy} between two variates x and y is 0.6 , $\operatorname{Cov}(x, y) = 4.8$, $\sigma_x^2 = 9$, then σ_y is

- (a) $\frac{8}{9}$ (b) $\frac{5}{8}$
 (c) $\frac{8}{3}$ (d) None of these

Sol. Using $r = \frac{\operatorname{Cov}(x, y)}{\sigma_x \sigma_y}$, we have $0.6 = \frac{4.8}{3\sigma_y}$

$$\Rightarrow \sigma_y = \frac{8}{3}$$

Hence, (c) is the correct answer.

Ex 21. If two random variables have the regression lines $3x + 2y - 26 = 0$ and $6x + y - 31 = 0$, then correlation coefficient r_{xy} is

- (a) -0.5 (b) $1/2$
 (c) $1/2$ (d) None of these

Sol. Slopes of regression lines are $-\frac{3}{2}$ and -6 .

Since, $r^2 < 1$, $b_{yx} = -\frac{3}{2}$ and $b_{xy} = -\frac{1}{6}$

$$\therefore r^2 = b_{yx} \cdot b_{xy} = \frac{1}{4} \Rightarrow r = -0.5$$

[as both b_{yx} and b_{xy} are negative]

Hence, (a) is the correct answer.

Ex 22. $r_{xy} < 0$, according as

- (a) $\sigma_{x+y} > \sigma_{x-y}$ (b) $\sigma_{x+y} < \sigma_{x-y}$
(c) $\sigma_{x+y} > 0$ (d) None of these

Sol. Since, $\sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 + 2r\sigma_x\sigma_y$... (i)

$$\sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y \quad \dots (ii)$$

From Eqs. (i) and (ii), $4r\sigma_x\sigma_y = \sigma_{x+y}^2 - \sigma_{x-y}^2$

As $\sigma_x\sigma_y$ is always positive.

$\therefore r = r_{xy} < 0$, if $\sigma_{x+y}^2 < \sigma_{x-y}^2$ or $\sigma_{x+y} < \sigma_{x-y}$

Hence, (b) is the correct answer.

Ex 23. The correlation between two variables x and y is given to be r . The values of x and y change such that $\text{Cov}(x, y)$ remains unaltered and $\text{Var}(x)$ and $\text{Var}(y)$ are changed to four times their original values. The new correlation coefficient is changed to

- (a) $4r$ (b) $16r$ (c) $\frac{r}{16}$ (d) $\frac{r}{4}$

Sol. We have, $r = \frac{\text{Cov}(x, y)}{\sigma_x\sigma_y}$

Changed correlation coefficient,

$$r_1 = \frac{\text{Cov}(x, y)}{\sqrt{4\sigma_x^2 \cdot 4\sigma_y^2}} = \frac{1}{4} \frac{\text{Cov}(x, y)}{\sigma_x\sigma_y} = \frac{r}{4}$$

Hence, (d) is the correct answer.

Ex 24. If $z = ax + by$ and r is the correlation coefficient between x and y , then σ_z^2 is equal to

- (a) $2ab r \sigma_x \sigma_y$
(b) $a^2 \sigma_x^2 + b^2 \sigma_y^2 - 2abr \sigma_x \sigma_y$
(c) $a^2 \sigma_x^2 + b^2 \sigma_y^2 + 2ab r \sigma_x \sigma_y$
(d) None of the above

Sol. $\because z = ax + by \therefore \bar{z} = a\bar{x} + b\bar{y}$

$$\begin{aligned} \text{Now, } \sigma_z^2 &= \frac{1}{n} \sum (z - \bar{z})^2 = \frac{1}{n} \sum \{a(x - \bar{x}) + b(y - \bar{y})\}^2 \\ &= a^2 \frac{1}{n} \sum (x - \bar{x})^2 + b^2 \frac{1}{n} \sum (y - \bar{y})^2 + 2ab \frac{1}{n} \sum (x - \bar{x})(y - \bar{y}) \\ &= a^2 \sigma_x^2 + b^2 \sigma_y^2 + 2ab r \sigma_x \sigma_y \end{aligned}$$

Hence, (c) is the correct answer.

Ex 25. If both the regression coefficients b_{yx} and b_{xy} are positive, then

- (a) $\frac{1}{b_{yx}} + \frac{1}{b_{xy}} > \frac{2}{r}$ (b) $\frac{1}{b_{yx}} + \frac{1}{b_{xy}} < \frac{2}{r}$
(c) $\frac{1}{b_{yx}} + \frac{1}{b_{xy}} < \frac{r}{2}$ (d) None of these

Sol. Since, $GM > HM$

So, considering the regression coefficients b_{yx} and b_{xy} , we have

$$\begin{aligned} [b_{yx} \cdot b_{xy}]^{1/2} &> \frac{2(b_{yx} \cdot b_{xy})}{b_{yx} + b_{xy}} \quad \text{or} \quad r > \frac{2(b_{yx} \cdot b_{xy})}{b_{yx} + b_{xy}} \\ \Rightarrow \frac{2}{r} &< \frac{1}{b_{yx}} + \frac{1}{b_{xy}} \Rightarrow \frac{1}{b_{yx}} + \frac{1}{b_{xy}} > \frac{2}{r} \end{aligned}$$

Hence, (a) is the correct answer.

Ex 26. If the slopes of the line of regression of y on x and of x on y are 30° and 60° respectively, then $r(x, y)$ is

- (a) -1 (b) 1 (c) $\frac{1}{\sqrt{3}}$ (d) $\sqrt{3}$

Sol. We have, $b_{yx} = \tan 30^\circ = \frac{1}{\sqrt{3}}$ and $\frac{1}{b_{xy}} = \tan 60^\circ = \sqrt{3}$

$$\therefore b_{yx} \cdot b_{xy} = \frac{1}{3}$$

$$\Rightarrow r^2 = \frac{1}{3} \Rightarrow r = \pm \frac{1}{\sqrt{3}}, \text{ as } b_{yx} \text{ and } b_{xy} \text{ are positive.}$$

Hence, (c) is the correct answer.

Ex 27. If the two lines of regression are $3x + y = 15$ and $x + 4y = 3$, then value of x when $y = 3$ is

- (a) -4 (b) -8
(c) 4 (d) None of these

Sol. Here, we need the line of regression of x on y for estimating the value of x corresponding to $y = 3$. The slope of $x + 4y = 3$ is $-\frac{1}{4}$ and that of $3x + y = 15$ is -3 .

So, $r^2 = b_{yx} \cdot b_{xy} = -\frac{1}{4} \times -\frac{1}{3} = \frac{1}{12}$ which is less than unity.

Thus, $3x + y = 15$ is the line of regression of x on y .

When $y = 3$, then $3x + 3 = 15 \Rightarrow x = 4$

Hence, (c) is the correct answer.

Ex 28. Linear relation between the variables is given by the equation $ax + by + c = 0$ such that $ab > 0$. Then, $r(x, y)$ is given by

- (a) 1
(b) -1
(c) 0
(d) any number lying between -1 and 1

Sol. $ax + by + c = 0$

$$\Rightarrow a\bar{x} + b\bar{y} + c = 0$$

$$\Rightarrow x - \bar{x} = -\frac{b}{a}(y - \bar{y})$$

$$\begin{aligned} \therefore \text{Cov}(x, y) &= \frac{1}{n} \sum (x - \bar{x})(y - \bar{y}) \\ &= -\frac{b}{a} \cdot \frac{1}{n} \sum (y - \bar{y})^2 = -\frac{b}{a} \text{Var}(y) \end{aligned}$$

$$\text{Var}(x) = \frac{1}{n} \sum (x - \bar{x})^2 = \frac{b^2}{a^2} \cdot \frac{1}{n} \sum (y - \bar{y})^2 = \frac{b^2}{a^2} \text{Var}(y)$$

$$\therefore r(x, y) = \frac{\text{Cov}(x, y)}{\sqrt{(\text{Var } x)(\text{Var } y)}} = \frac{-\frac{b}{a} \text{Var}(y)}{\left| \frac{b}{a} \right| \sqrt{\text{Var}(y)}} = -1, \text{ if } ab > 0$$

Hence, (b) is the correct answer.

Target Exercises

Type 1. Only One Correct Option

- The average of n numbers $x_1, x_2, x_3, \dots, x_n$ is $\bar{X}$. If x_1 is replaced by a , then new average is
 (a) $\frac{(n\bar{X} - x_1 + a)}{n}$ (b) $\frac{(\bar{X} - x_1 + a)}{n}$
 (c) $\{(n-1)\bar{X} - x_1 + a\}$ (d) None of these
- A batsman in his 16th innings makes a score of 70 runs and thereby increases his average by 2 runs. If he had never been not out, then his average after 16th innings is
 (a) 36 (b) 38 (c) 40 (d) 42
- The average weight of students in a class of 35 students is 40 kg. If the weight of the teacher be included the average rises by $\frac{1}{2}$ kg, then weight of the teacher is
 (a) 40.5 kg (b) 50 kg
 (c) 41 kg (d) 58 kg
- The mean income of a group of 50 persons was calculated as ₹ 169. Later, it was discovered that one figure was wrongly taken as 134 instead of correct value 143. The correct mean (in ₹) should be
 (a) 168 (b) 169
 (c) 168.92 (d) 169.18
- An additional observation 15 is included in a series of 11 observations and its mean remains unaffected. The mean of series was
 (a) 12 (b) 15
 (c) 20 (d) None of these
- In a monthly test marks obtained in Mathematics by 15 students of a class are 0, 0, 2, 2, 3, 3, 3, 5, 5, 5, 5, 6, 6, 7, 8. The arithmetic mean of marks is
 (a) 8 (b) 4 (c) 6 (d) 1
- A frequency distribution has arithmetic mean 7.85, then missing term is

Daily wages (in ₹)	5	6	7	10	12	15
Number of workers	10	f	13	8	5	4

 (a) 10.5 (b) 10.05
 (c) 15.5 (d) 15
- The mean of 20 observations is 15. On checking it was found that the two observations were wrongly copied as 3 and 6. The correct values are 8 and 4, then correct mean will be given by
 (a) 15.15 (b) 14.69
 (c) 14.74 (d) 15.25
- The mean of 100 items is 49. It was later discovered that three items taken as 40, 20, 50 were actually 60, 70, 80, respectively. The correct mean is
 (a) 48 (b) 85.5 (c) 50 (d) 41.5
- The average weight of a class of 14 students is 42 kg. If the teacher is included then average weight increases by 400 g, then weight of teacher is
 (a) 52 (b) 48 (c) 46 (d) 54
- The mean of series 1, 2, 4, 8, ..., 2^n is
 (a) $\frac{2n-1}{n}$ (b) $\frac{2^{n+1}-1}{n+1}$
 (c) $\frac{2n+1}{n}$ (d) $\frac{2^n-1}{n+1}$
- The algebraic sum of the deviations of 20 observations measured from 30 is 2. The mean of observations is
 (a) 28.5 (b) 30.1 (c) 30.5 (d) 29.6
- A class of 30 boys and 15 girls is given a test in Mathematics. The average marks obtained by boys is 15 and by girls is 6. The average of whole class is
 (a) 10.5 (b) 12
 (c) 4.5 (d) None of these
- If the arithmetic mean of 9 observations is 100 and that of 6 is 80. Then, the combined mean of all the 15 observations will be
 (a) 100 (b) 80 (c) 90 (d) 92
- Combined mean of two series x_1 and x_2 , when $\Sigma x_1 = 210$, $\Sigma x_2 = 150$, $n_1 = 50$, $n_2 = 100$ is
 (a) 150 (b) 160 (c) 170 (d) 180
- The mean income of a group of workers is X and that of another group is Y . If the number of workers in the second group is 10 times the number of workers in the first group, then mean income of combined group is
 (a) $\frac{X+10Y}{3}$ (b) $\frac{X+10Y}{11}$
 (c) $\frac{X+Y}{Y}$ (d) $\frac{X+10Y}{9}$
- The mean age of a combined group of men and women is 25 yr. If mean age of men is 26 and that of women is 21, then percentage of men and women in the group are
 (a) 60, 40 (b) 80, 20
 (c) 20, 80 (d) 30, 70

18. A set of 7 observations has mean 10 and another set of 3 observations has mean 5. The mean of combined set is
(a) 15 (b) 10 (c) 8.5 (d) 7.5
19. If $\bar{x}_1$ and $\bar{x}_2$ are means of two distribution such that $\bar{x}_1 < \bar{x}_2$ and $\bar{x}$ is the mean of the joint distribution, then
(a) $\bar{x} = \frac{\bar{x}_1 + \bar{x}_2}{2}$ (b) $\bar{x} > \bar{x}_2$
(c) $\bar{x}_2 < \bar{x}_1$ (d) $\bar{x}_1 < \bar{x} < \bar{x}_2$
20. If G_1 and G_2 are geometric means of two series and G is GM of the ratio of corresponding series, then G is
(a) $\frac{G_1}{G_2}$ (b) $\log G_1 - \log G_2$
(c) $\log G_1 + \log G_2$ (d) None of these
21. The geometric mean of 5, 8, 10, 15, 20, 25, 30 is
(a) 16.9 (b) $10(9)^{1/7}$
(c) 18 (d) None of these
22. The geometric mean of the series 1, 2, 4, ..., 2^n , is
(a) $2^{n+1/2}$ (b) $2^{(n+1)/2}$ (c) $2^{n-(1/2)}$ (d) $2^{n/2}$
23. The geometric mean of observations 2, 4, 16 and 32 is
(a) 6 (b) 7 (c) 8 (d) 9
24. If G_1 and G_2 are geometric means of two series of sizes n_1 and n_2 and G is geometric mean of combined series, then $\log G$ is
(a) $\log G_1 + \log G_2$ (b) $\frac{n_1 \log G_1 + n_2 \log G_2}{n_1 + n_2}$
(c) $\frac{\log G_1 - \log G_2}{n_1 - n_2}$ (d) None of these
25. On thirteen consecutive days, the number of persons booked for violating speed limit of 40 km/h were as follows:
58, 61, 68, 57, 62, 50, 55, 62, 53, 54, 51, 59, 52
The median number of speed violations per day is
(a) 61 (b) 52 (c) 55 (d) 57
26. The test marks in Statistics for a class are 20, 24, 27, 38, 18, 42, 35, 21, 44, 18, 31, 36, 41, 26, 29. The median score of the class is
(a) 8 (b) 21 (c) 29 (d) 31
27. The median of a series is 10. Two additional observations 7 and 20 are added to series. The median of new series is
(a) 9 (b) 20
(c) 7 (d) 10
28. Which of the following is correct about the series 24, 38, 55, 69, 89?
(a) Median = Mode (b) Mean = Mode
(c) Mean = Median (d) None of these
29. The median value of observations 83, 54, 78, 64, 90, 59, 67, 72, 70, 73 is
(a) 70 (b) 71
(c) 72 (d) 73
30. In a test of Mathematics, 15 students got following marks 16, 25, 35, 20, 0, 12, 30, 22, 5, 8, 4, 38, 32, 2, 1. The median score of these 15 students is
(a) 7 (b) 22 (c) 20 (d) 16
31. The mean and median of 100 items are 50 and 52, respectively. The value of largest item is 100. It was later found that it is 110 and not 100. The true mean and median are
(a) 50.10, 51.5 (b) 50.10, 52
(c) 50, 51.5 (d) None of these
32. Median is based on the
(a) highest 50% of items (b) lowest 25% of items
(c) highest 25% of items (d) middle 50% of items
33. The 70th percentile is equal to
(a) 7th decile (b) Q_3
(c) 6th decile (d) None of these
34. Which of the following is correct for data -1, 0, 1, 2, 3, 5, 5, 6, 8, 10, 11?
(a) Mean = Mode = Median (b) Mean = 5
(c) Mean = Mode (d) Mode = Median
35. The mode of a set of observations 7, 12, 8, 5, 6, 4, 9, 10, 8, 9, 7, 9, 6, 5, 9 is
(a) 7 (b) 8 (c) 9 (d) 12
36. Mode of data 3, 2, 5, 2, 3, 5, 6, 6, 5, 3, 5, 2, 5 is
(a) 6 (b) 4 (c) 5 (d) 3
37. For the data 2, 2, 3, 3, 3, 4, 4, 5, 6, 8, 8, 8, 8, the mean, median and mode are
(a) 64/13, 5, 3 (b) 64/13, 14, 3
(c) 5, 4, 3 (d) 64/13, 4, 8
38. The quartile deviation of daily wages of 7 persons which are ₹ 12, 7, 10, 15, 17, 17, 25, is
(a) 14.5 (b) 13.5 (c) 7 (d) 3.5
39. Semi-interquartile range is
(a) $4\sigma/5$ (b) $2\sigma/3$ (c) $3\sigma/2$ (d) $\sigma/3$
40. If 25% of items are less than 10 and 25% are more than 40, then coefficient of quartile deviation is
(a) 4/5 (b) 5/3 (c) 1/3 (d) 3/5
41. Consider any set of observations $x_1, x_2, x_3, \dots, x_{101}$; it being given that $x_1 < x_2 < x_3 < \dots < x_{100} < x_{101}$, then the mean deviation of this set of observations about a point k is minimum, when k equals
(a) x_1 (b) x_{51}
(c) $\frac{x_1 + x_2 + \dots + x_{101}}{101}$ (d) x_{50}
42. The sum of absolute deviations about median is
(a) least (b) zero
(c) greatest (d) None of these
43. For a series, the information available is $n = 10$, $\Sigma x = 60$ and $\Sigma x^2 = 1000$. The standard deviation is
(a) 8 (b) 64 (c) 24 (d) 128

44. The sum of 10 items is 12 and sum of their squares is 18, then standard deviation will be
(a) $-3/5$ (b) $6/5$ (c) $4/5$ (d) $3/5$
45. The variance of 6, 8, 10, 12, 14 is
(a) 1 (b) 8 (c) 12 (d) 16
46. The mean and standard deviation of 1, 2, 3, 4, 5, 6 are
(a) $\frac{7}{2}, \sqrt{\frac{35}{12}}$ (b) $\frac{7}{2}, \sqrt{3}$ (c) 3, 3 (d) 3, $35/12$
47. Given 5 sample measurement of a part of machine as 6.33, 6.37, 6.36, 6.32 and 6.37. The sample variance is
(a) 0.0003 (b) 0.0002 (c) 0.0004 (d) 0.0012
48. The standard deviation of certain data is 4. Then, their variance is
(a) 1 (b) 16 (c) 8 (d) $\sqrt{2}$
49. The standard deviation of
 $-1, -2, -3, -4, -5, -6, -7$ is
(a) 2 (b) 4 (c) -2 (d) -4
50. The mean and SD of the marks of 200 candidates were found to be 40 and 15, respectively. Later, it was discovered that a score of 40 was wrongly read as 50. The correct mean and SD are respectively
(a) 14.98, 39.95 (b) 39.95, 14.98
(c) 39.95, 224.5 (d) None of these
51. Suppose values taken by a variable x are such that $a \leq x_i \leq b$, where x_i denotes the value of x in the i th case for $i = 1, 2, 3, \dots, n$. Then,
(a) $a \leq \text{Var}(x) \leq b$ (b) $a^2 \leq \text{Var}(x) \leq b^2$
(c) $\frac{a^2}{b} \leq \text{Var}(x)$ (d) $(b-a)^2 \geq \text{Var}(x)$
52. Let r be the range and $S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$ be the SD of a set of observations $x_1, x_2, \dots, x_n$, then
(a) $S \leq r \sqrt{\frac{n}{n-1}}$ (b) $S = r \sqrt{\frac{n}{n-1}}$
(c) $S \geq r \sqrt{\frac{n}{n-1}}$ (d) None of these
53. The best statistical measure used for comparing two series is
(a) mean deviation (b) range
(c) coefficient of variation (d) interquartile range
54. The sum of square of deviations for 10 observations taken from mean 50 is 250. The coefficients of variation is
(a) 50 (b) 10 (c) 30 (d) 40
55. For a given distribution of marks, mean is 35.16 and its standard deviation is 19.76. The coefficient of variation is
(a) 56.2 (b) 0.562
(c) 1.779 (d) 177.935
56. The coefficient of variation of two series are 70 and 90 and their standard deviation are 17.5 and 18, respectively. The mean of two series are
(a) 25, 20 (b) 18, 22 (c) 22, 18 (d) 16, 24
57. The standard deviations of two sets containing 10 and 20 members are 2 and 3, respectively measured from their common mean 5. The SD for the whole set of 30 members is
(a) $\frac{2}{\sqrt{3}}$ (b) $\sqrt{6}$
(c) $\sqrt{\frac{22}{3}}$ (d) $\sqrt{3}$
58. One set containing five members has mean 8, variance 18 and the second set containing three members has mean 8 and variance 24. The variance of combined set of numbers is
(a) 24 (b) 20.25
(c) 22.25 (d) None of these
59. The mode of a moderately symmetrical series is 18 and mean is 24. The median is
(a) 18 (b) 24 (c) 22 (d) 21
60. The value of mean, median and mode coincides when distribution shows
(a) positive skewness (b) symmetrical distribution
(c) negative skewness (d) All of these
61. If in a moderately skewed distribution, the values of mode and mean are 6λ and 9λ respectively, then the value of median is
(a) 8λ (b) 6λ (c) 7λ (d) 5λ
62. Karl Pearson's coefficient of skewness is
(M = Mean, M_e = Median, M_o = Mode)
(a) $\frac{(M - M_e)}{\sigma}$ (b) $\frac{(M_e - M_o)}{\sigma}$
(c) $\frac{(M - M_o)}{\sigma}$ (d) None of these
63. If mean and mode of a frequency distribution have same value, then coefficient of skewness is
(a) zero (b) less than zero
(c) greater than zero (d) None of these
64. For a given series, the mean, SD and mode are respectively 622.08, 3.06 and 625. The coefficient of skewness is
(a) -0.95 (b) 0.95
(c) -0.84 (d) 0.84
65. For a symmetrical distribution, the coefficient of skewness is
(a) 1 (b) 0 (c) -1 (d) 3
66. Mean, mode and standard deviation of a frequency distribution are 41, 45 and 8 respectively, Pearson's coefficient of skewness is
(a) 0.5 (b) 1.5
(c) 1.0 (d) -0.5

67. The arithmetic mean of 10 observations is 12.45. If each reading is increased by 5, then resulting mean is increased by
(a) 5 (b) 29 (c) 0.5 (d) 50
68. The mean of 18 observations is 7 and if in each observation 5 is added, then the new mean will be
(a) 2 (b) 12 (c) 7 (d) None of these
69. In a test of Statistics, marks were awarded out of 40. The average of 15 students was 38. Later, it was decided to give marks out of 50. The new average marks will be
(a) 40 (b) 47.5 (c) 95 (d) 41.5
70. The standard deviation of the wages of 85 employees is ₹ 15.40. After one year, each of them is given an increment of ₹ 25. The standard deviation of new wages (in ₹) is
(a) 15.40 (b) 40.40 (c) 20.40 (d) 10.40
71. The standard deviation for the set of numbers 1, 4, 5, 7, 8 is 2.45 nearly. If 10 is added to each number, then new standard deviation is
(a) 24.45 (b) 12.45 (c) 2.45 (d) 0.245
72. The median and SD of a distribution are 20 and 4, respectively. If each item is increased by 2, then new median and SD will be
(a) 20, 6 (b) 22, 6 (c) 18, 6 (d) 22, 4
73. Let σ be standard deviation of n observations. Each observation is multiplied by a constant c . The standard deviation of resulting numbers is
(a) σ (b) $c\sigma$ (c) $\sigma\sqrt{c}$ (d) None of these
74. The variance of 15 observations was found to be 8. If each observation is multiplied by 4, the new variance of the series is
(a) 32 (b) 24 (c) 8 (d) 128
75. The variates x and u are related by $hu = x - a$, then correct relation between σ_x and σ_u is
(a) $\sigma_x = h\sigma_u$ (b) $\sigma_u = h\sigma_x$ (c) $\sigma_x = a + h\sigma_u$ (d) $\sigma_u = a + h\sigma_x$
76. The marks of some students were listed out of 75. The SD of marks was found to be 9. Subsequently, the marks were raised to a maximum of 100 and variance of new marks was calculated. The new variance is
(a) 144 (b) 122 (c) 81 (d) None of these
77. The mean deviation of any series is 15, then the value of quartile deviation is
(a) 10.5 (b) 11.5 (c) 12.5 (d) 13.5
78. The coefficient of correlation between x and y is 1, then coefficient of correlation between x and $x + y$ is
(a) 1 (b) -1 (c) -0.5 (d) 0.5
79. The coefficient of correlation between x and y is 0.5 and their covariance is 16 and SD of x is 4, then SD of y is
(a) 4 (b) 8 (c) 16 (d) 64
80. If a linear relation $aX + bY + c = 0$ exists between the variables X and Y and $ab < 0$, then coefficient of correlation between X and Y , is
(a) 1 (b) -1 (c) 0 (d) a real number between -1 and 1
81. If $\sigma_x = 3$, $\sigma_y = 4$ and $\sigma_{x+y} = 7$, then $r(x, y)$ is
(a) 1 (b) -1 (c) 0 (d) 0.75
82. If variance of $X, Y, X - Y$ are respectively $\sigma_x^2, \sigma_y^2, \sigma_{x-y}^2$, then the coefficient of correlation r between two variables X and Y is given by
(a) $\frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x+y}^2}{2\sigma_x \cdot \sigma_y}$ (b) $\frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x \cdot \sigma_y}$ (c) $\frac{\sigma_x^2 + \sigma_y^2}{2\sigma_x \cdot \sigma_y}$ (d) None of these
83. If x, y are independent variables, then $\text{Cov}(x, y)$ is
(a) $\sigma_x \sigma_y$ (b) σ_x / σ_y (c) σ_y / σ_x (d) 0
84. If $x = X - M_x$, $y = Y - M_y$ and $\Sigma xy = 40$, $\Sigma x^2 = 80$, $\Sigma y^2 = 20$. Then, coefficient of correlation between X and Y is
(a) 0 (b) 0.5 (c) -1 (d) 1
85. The coefficient of correlation between x and y is
- | | | | | | |
|-----|---|---|----|----|----|
| x | 1 | 2 | 3 | 4 | 5 |
| y | 3 | 7 | 12 | 19 | 28 |
- (a) -1 (b) -0.82 (c) 0.82 (d) 0.988
86. If $\text{Cov}(x, y) = -16.5$, $\text{Var}(x) = 2.89$, $\text{Var}(y) = 100$, then the coefficient of correlation is
(a) -0.97 (b) -0.87 (c) 0.97 (d) None of these
87. If x and y are deviations from arithmetic mean, $r = 0.8$, $\Sigma xy = 60$, $\sigma_y = 2.5$ and $\Sigma x^2 = 90$, then the number of items in the series, is
(a) 10 (b) 15 (c) 18 (d) None of the above
88. The correlation coefficient 0.5 between x and y means
(a) 50% of data are explained (b) 25% of variations are explained (c) out of total variations of y only 25% is due to x and rest due to other factors (d) out of total variation of y only 50% is due to x and rest due to other factors

89. The point that lies on both the lines of regression for a bivariate distribution, is
 (a) $(0, 0)$ (b) $(\bar{x}, \bar{y})$
 (c) (σ_x, σ_y) (d) (r_1, r_2)
90. Out of two lines of regression given by $x + 2y = 4$ and $2x - 3y - 5 = 0$, the regression of x on y is
 (a) $x + 2y = 4$
 (b) $2x + 3y - 5 = 0$
 (c) they are not lines of regression
 (d) None of the above
91. If x and y are independent variables, then two lines of regression are
 (a) $x = 0, y = 0$
 (b) $x = 0, y = \text{constant}$
 (c) $x = \text{constant}, y = 0$
 (d) $x = \text{constant}, y = \text{constant}$
92. The two lines of regression are $13x - 10y + 11 = 0$ and $2x - y - 1 = 0$, then mean of x and y series are
 (a) 3, -5 (b) 3, 5
 (c) -3, 10 (d) 3, -5
93. The two lines of regression are $8x - 10y = 66$ and $40x - 18y = 214$ and variance of x is 9. The standard deviation of y series is
 (a) 8 (b) 6
 (c) 4 (d) 3
94. Lines of regression of y on x and x on y are respectively $y = ax + b$ and $x = \alpha y + \beta$. If mean of x and y series is same, then its value is
 (a) $\frac{b}{1-a}$ or $\frac{\beta}{1-\alpha}$ (b) $\frac{1-a}{b}$
 (c) $\frac{\beta}{1-\beta}$ (d) $\frac{\alpha}{1-\alpha}$
95. The two lines of regression are $x + 2y - 5 = 0$ and $x + 3y - 8 = 0$. The coefficient of correlation between x and y is
 (a) -0.82 (b) 0.82 (c) -0.72 (d) 0.76
96. If r_1 and r_2 are regression coefficients of y on x and x on y respectively, then
 (a) $r_1 + r_2 = 2r$ (b) $r_1 + r_2 < 2r$
 (c) $r_1 + r_2 \geq 2r$ (d) None of these
97. If two regression lines are coincident, then which is correct?
 (a) $r = 0$ (b) $r = -1/2$
 (c) $r = 1/2$ (d) $r = \pm 1$
98. Angle between two lines of regression is
 (a) $\tan^{-1} \left(\frac{r_1 - 1/r_2}{1 + r_2/r_1} \right)$ (b) $\tan^{-1} \left(\frac{r_1 r_2 - 1}{r_1 + r_2} \right)$
 (c) $\tan^{-1} \left(\frac{r_2 - 1/r_1}{1 + r_1/r_2} \right)$ (d) $\tan^{-1} \left(\frac{r_1 - r_2}{1 + r_1 r_2} \right)$

Entrances Gallery

JEE Advanced/IIT JEE

1. The minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$ is _____. [2011]

JEE Main/AIEEE

2. The mean of the data set comprising of 16 observations is 16. If one of the observation valued 16 is deleted and three new observations valued 3, 4 and 5 are added to the data, then the mean of the resultant data is [2015]
 (a) 16.8 (b) 16.0 (c) 15.8 (d) 14.0
3. The variance of first 50 even natural numbers is [2014]
 (a) $\frac{833}{4}$ (b) 833 (c) 437 (d) $\frac{437}{4}$
4. All the students of a class performed poorly in Mathematics. The teacher decided to give grace marks of 10 to each of the students. Which of the following statistical measures will not change even after the grace marks were given? [2014]
 (a) Mean (b) Median
 (c) Mode (d) Variance
5. Let $x_1, x_2, \dots, x_n$ be n observations, $\bar{x}$ be their arithmetic mean and σ^2 be the variance.
Statement I Variance of $2x_1, 2x_2, \dots, 2x_n$ is $4\sigma^2$.
Statement II Arithmetic mean of $2x_1, 2x_2, \dots, 2x_n$ is $4\bar{x}$. [2012]
 (a) Statement I is false, Statement II is true
 (b) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (c) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (d) Statement I is true, Statement II is false
6. If the mean deviations about the median of the numbers $a, 2a, \dots, 50a$ is 50, then $|a|$ equals [2011]
 (a) 3
 (b) 4
 (c) 5
 (d) 2

7. A scientist is weighting each of 30 fish. Their mean weight worked out is 30 g and a standard deviation of 2 g. Later, it was found that the measuring scale was misaligned and always under reported every fish weight by 2g. The correct mean and standard deviation (in gram) of fishes are respectively [2011]
 (a) 28, 4 (b) 32, 2
 (c) 32, 4 (d) 28, 2
8. For two data sets, each of size 5, the variances are given to be 4 and 5 and the corresponding means are given to be 2 and 4, respectively. The variance of the combined data set is [2010]
 (a) $\frac{5}{2}$ (b) $\frac{11}{2}$
 (c) 6 (d) $\frac{13}{2}$
9. If the mean deviation of number $1, 1+d, 1+2d, \dots, 1+100d$ from their mean is 255, then d is equal to [2009]
 (a) 10.0 (b) 20.0
 (c) 10.1 (d) 20.2
10. **Statement I** The variance of first n even natural numbers is $\frac{n^2 - 1}{4}$.
Statement II The sum of first n natural numbers is $\frac{n(n+1)}{2}$ and the sum of squares of first n natural numbers is $\frac{n(n+1)(2n+1)}{6}$. [2009]
 (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true
11. The mean of the numbers $a, b, 8, 5, 10$ is 6 and the variance is 6.80. Then, which one of the following gives possible values of a and b ? [2008]
 (a) $a = 3, b = 4$ (b) $a = 0, b = 7$
 (c) $a = 5, b = 2$ (d) $a = 1, b = 6$
12. The average marks of boys in a class is 52 and that of girls is 42. The average marks of boys and girls combined is 50. The percentage of boys in the class is [2007]
 (a) 40% (b) 20%
 (c) 80% (d) 60%
13. Suppose a population A has 100 observations 101, 102, ..., 200 and another population B has 100 observations 151, 152, ..., 250. If V_A and V_B represent the variances of the two populations respectively, then $\frac{V_A}{V_B}$ is [2006]
 (a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) $\frac{2}{3}$ (d) 1
14. Let $x_1, x_2, \dots, x_n$ be n observations such that $\sum x_i^2 = 400$ and $\sum x_i = 80$. Then, a possible value of n among the following is [2005]
 (a) 12
 (b) 9
 (c) 18
 (d) 15
15. If in a frequency distribution, the mean and median are 21 and 22 respectively, then its mode is approximately [2005]
 (a) 24.0
 (b) 25.5
 (c) 20.5
 (d) 22.0
16. Consider the following statements:
 I. Mode can be computed from histogram.
 II. Median is not independent of change of scale.
 III. Variance is independent of change of origin and scale. [2004]
 Which of these is/are correct?
 (a) Only I
 (b) Only II
 (c) I and II
 (d) I, II and III
17. In a series of $2n$ observations, half of them equal a and remaining half equal $-a$. If the standard deviation of the observations is 2, then $|a|$ equals [2004]
 (a) $\frac{1}{n}$ (b) $\sqrt{2}$ (c) 2 (d) $\frac{\sqrt{2}}{n}$
18. The median of a set of 9 distinct observations is 20.5. If each of the largest 4 observations of the set is increased by 2, then the median of the new set [2003]
 (a) is increased by 2
 (b) is decreased by 2
 (c) is two times the original median
 (d) remains the same as that of the original set
19. In an experiment with 15 observations on x , the following results were available
 $\sum x^2 = 2830, \sum x = 170$.
 One observation that was 20 was found to be wrong and was replaced by the correct value 30. Then, the corrected variance is [2003]
 (a) 78.00
 (b) 188.66
 (c) 177.33
 (d) 8.33
20. In a class of 100 students, there are 70 boys whose average marks in a subject are 75. If the average marks of the complete class is 72, then what is the average of the girls? [2002]
 (a) 73
 (b) 65
 (c) 68
 (d) 74

Other Engineering Entrances

20

Statistics

Target Exercises

- 21.** The mean of n terms is $\bar{x}$. If the first term is increased by 1, second by 2 and so on, then the new mean is [BITSAT 2014]
- (a) $\bar{x} + n$ (b) $\bar{x} + \frac{n}{2}$
 (c) $\bar{x} + \frac{n+1}{2}$ (d) None of these
- 22.** In a moderately asymmetrical distribution, the mean and median are 36 and 34, respectively. Find out the value of empirical mode. [J&K CET 2014]
- (a) 30 (b) 32
 (c) 42 (d) 22
- 23.** The variance of first n natural numbers is [Manipal 2014]
- (a) $\frac{n(n+1)}{2}$ (b) $\frac{(n+1)(n+5)}{12}$
 (c) $\frac{(n+1)(n-5)}{12}$ (d) $\frac{(n^2-1)}{12}$
- 24.** If the coefficient of variation and standard deviation are 60 and 21 respectively, then arithmetic mean of distribution is [Karnataka CET 2014]
- (a) 60 (b) 30 (c) 35 (d) 21
- 25.** If $\sum_{i=1}^9 (x_i - 5) = 9$ and $\sum_{i=1}^9 (x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2, \dots, x_9$ is [Kerala CEE 2014]
- (a) 9 (b) 4 (c) 3
 (d) 2 (e) 1
- 26.** The standard deviation of 9, 16, 23, 30, 37, 44, 51 is [Kerala CEE 2014]
- (a) 7 (b) 9 (c) 12
 (d) 14 (e) 16
- 27.** The mean of four observations is 3. If the sum of the squares of these observations is 48, then their standard deviation is [EAMCET 2014]
- (a) $\sqrt{7}$ (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) $\sqrt{5}$
- 28.** The sum of deviations of the variates from the arithmetic mean is always [GGSIPU 2014]
- (a) +1 (b) 0
 (c) -1 (d) real number
- 29.** The mean of five observations is 4 and their variance is 5.2. If three of these observations are 1, 2 and 6, then the other two are [UP SEE 2013]
- (a) 2 and 9 (b) 3 and 8
 (c) 4 and 7 (d) 5 and 6
- 30.** The AM of 9 terms is 15. If one more term is added to this series, then the AM becomes 16. The value of the added term is [Kerala CEE 2011]
- (a) 30 (b) 27
 (c) 25 (d) 23
 (e) 20
- 31.** If the median of $\frac{x}{5}, x, \frac{x}{4}, \frac{x}{2}, \frac{x}{3}$ ($x > 0$) is 8, then the value of x is [Kerala CEE 2011]
- (a) 24 (b) 32
 (c) 8 (d) 16
 (e) 40
- 32.** If x and y are deviations from arithmetic mean, $r = 0.5$, $\Sigma xy = 120$, $\Sigma x^2 = 90$, $\sigma_y = 8$, then n is equal to [MP PET 2011]
- (a) 100 (b) 10
 (c) 15 (d) 50
- 33.** If the standard deviation of 3, 8, 6, 10, 12, 9, 11, 10, 12, 7 is 2.71, then the standard deviation of 30, 80, 60, 100, 120, 90, 110, 100, 120, 70 is [Kerala CEE 2011]
- (a) 2.71 (b) 27.1
 (c) $(2.71)\sqrt{10}$ (d) $(2.71)\sqrt{2}$
 (e) 0.271
- 34.** Coefficient of variation of two distributions are 50% and 60% and their arithmetic means are 30 and 25, respectively. Difference of their standard deviation is [GGSIPU 2011]
- (a) 1 (b) 1.5 (c) 2.5 (d) 0
- 35.** Mean deviation of 6, 8, 12, 15, 10, 9 through mean is [GGSIPU 2011]
- (a) 10 (b) 2.33
 (c) 2.5 (d) None of these
- 36.** If the value observed are 1, 2, 3, ..., n each with frequency 1 and n is even, then the mean deviation from mean equals [J&K CET 2011]
- (a) n (b) $\frac{n}{2}$
 (c) $\frac{n}{4}$ (d) None of these
- 37.** The mean and variance of n observations $x_1, x_2, x_3, \dots, x_n$ are 5 and 0, respectively. If $\sum_{i=1}^n x_i^2 = 400$, then the value of n is equal to [Kerala CEE 2010]
- (a) 80 (b) 25
 (c) 20 (d) 16
 (e) 4

Answers

Work Book Exercise 20.1

1. (a)	2. (d)	3. (b)	4. (a)	5. (c)	6. (a)	7. (c)	8. (a)	9. (c)	10. (b)
11. (b)	12. (a)	13. (c)	14. (d)	15. (d)	16. (b)	17. (d)	18. (c)	19. (b)	20. (c)
21. (b)	22. (b)	23. (b)	24. (d)						

Work Book Exercise 20.2

1. (d)	2. (c)	3. (b)	4. (b)	5. (c)	6. (d)	7. (b)	8. (b)	9. (b)	10. (c)

Target Exercises

1. (a)	2. (c)	3. (d)	4. (d)	5. (b)	6. (b)	7. (d)	8. (a)	9. (c)	10. (b)
11. (b)	12. (b)	13. (b)	14. (d)	15. (c)	16. (b)	17. (b)	18. (c)	19. (d)	20. (a)
21. (b)	22. (b)	23. (c)	24. (b)	25. (d)	26. (c)	27. (d)	28. (c)	29. (b)	30. (d)
31. (b)	32. (d)	33. (a)	34. (d)	35. (c)	36. (c)	37. (d)	38. (d)	39. (b)	40. (d)
41. (b)	42. (a)	43. (a)	44. (d)	45. (b)	46. (a)	47. (c)	48. (b)	49. (a)	50. (b)
51. (d)	52. (a)	53. (c)	54. (b)	55. (a)	56. (a)	57. (c)	58. (b)	59. (c)	60. (b)
61. (a)	62. (c)	63. (a)	64. (a)	65. (b)	66. (d)	67. (a)	68. (b)	69. (b)	70. (a)
71. (c)	72. (d)	73. (b)	74. (d)	75. (a)	76. (a)	77. (c)	78. (a)	79. (b)	80. (a)
81. (a)	82. (b)	83. (d)	84. (d)	85. (d)	86. (a)	87. (a)	88. (b)	89. (b)	90. (c)
91. (d)	92. (b)	93. (c)	94. (a)	95. (a)	96. (c)	97. (d)	98. (b)		

Entrances Gallery

1. (8)	2. (d)	3. (b)	4. (d)	5. (d)	6. (b)	7. (b)	8. (b)	9. (c)	10. (d)
11. (a)	12. (c)	13. (d)	14. (c)	15. (a)	16. (c)	17. (c)	18. (d)	19. (a)	20. (b)
21. (c)	22. (a)	23. (d)	24. (c)	25. (d)	26. (d)	27. (c)	28. (b)	29. (c)	30. (c)
31. (a)	32. (b)	33. (b)	34. (d)	35. (b)	36. (c)	37. (d)			

Explanations

Target Exercises

1. Given, $\bar{X} = \frac{\sum x}{n}$

New, mean = $\frac{n\bar{X} - x_1 + a}{n}$

2. Old average is x , then new average is $(x + 2)$.

According to the question,

$$15x + 70 = 16(x + 2)$$

$$\Rightarrow 15x + 70 = 16x + 32$$

$$\Rightarrow x = 38$$

$$\therefore \text{New average} = 40$$

3. Let weight of the teacher be w kg, then

$$40 + \frac{1}{2} = \frac{35 \times 40 + w}{35 + 1}$$

$$\Rightarrow 36 \times 40 + 36 \times \frac{1}{2} = 35 \times 40 + w$$

$$\Rightarrow w = 58$$

$\therefore$ Weight of the teacher = 58 kg

4. Given, $169 = \frac{\sum x}{50}$

Correct value is 143.

Clearly, sum is short by 9.

$$\therefore M = \frac{169 \times 50 + 9}{50} = 169 + \frac{9}{50} = 169.18$$

5. $M = \frac{\sum x}{11}$

Also, $M = \frac{\sum x + 15}{12} = \frac{11M + 15}{12}$

$$\therefore M = 15$$

6. The frequency distribution is

Marks (x)	0	2	3	5	6	7	8
Number of students (f)	2	2	3	4	2	1	1

$$\text{Arithmetic mean} = \frac{\sum fx}{\sum f} = \frac{60}{15} = 4$$

7. Mean = $7.85 = \frac{50 + 6f + 91 + 80 + 60 + 60}{40 + f}$

$$\Rightarrow 314 + 7.85f = 341 + 6f$$

$$\Rightarrow f = 14.59 \approx 15$$

8. Correct mean = $\frac{300 - (3 + 6) + (8 + 4)}{20} = 15.15$

9. Given, $49 = \frac{\sum x}{100}$

Now, $\sum x = 4900$

Infact, $\sum x$ should have been more. The correct $\sum x$ is new corrected $\sum x = 4900 + 20 + 50 + 30 = 5000$

$$\therefore \text{Correct mean} = \frac{5000}{100} = 50$$

10. Given, $\frac{\sum x}{14} = 42$

$$\therefore 42.400 = \frac{\sum x + w}{15}$$

$$\Rightarrow w = 48 \text{ kg}$$

11. $M = \frac{1 + 2 + 4 + \dots + 2^n}{n + 1}$ [$\therefore (n + 1)$ terms present]

$$= \frac{2^{n+1} - 1}{n + 1}$$

12. Given that,

$$\sum_{i=1}^{20} (x_i - 30) = 2 \Rightarrow \sum_{i=1}^{20} x_i - \sum_{i=1}^{20} (30) = 2$$

$$\Rightarrow \frac{1}{20} \sum_{i=1}^{20} x_i - \frac{20}{20} (30) = \frac{2}{20}$$

$$\therefore \bar{x} = 0.1 + 30 = 30.1$$

13. $M = \frac{30 \times 15 + 15 \times 6}{30 + 15} = 12$

14. $M = \frac{9 \times 100 + 6 \times 80}{15} = 92$

15. Combined mean

$$= \frac{n_1 x_1 + n_2 x_2}{n_1 + n_2} = \frac{50 \times 210 + 100 \times 150}{50 + 100} = 170$$

16. Given, $X = \frac{x_1 + x_2 + \dots + x_n}{n}$, $Y = \frac{y_1 + y_2 + \dots + y_m}{m}$

Further, $m = 10n$

Hence, mean of combined group is

$$M = \frac{nX + mY}{n + m} = \frac{nX + 10nY}{n + 10n} = \frac{X + 10Y}{11}$$

17. $\therefore 25 = \frac{26x_1 + 21x_2}{x_1 + x_2}$

$$\Rightarrow x_1 = 4x_2$$

$$\therefore x_1 = 80\%, x_2 = 20\%$$

18. $M = \frac{7(10) + 3(5)}{10} = 8.5$

19. $\therefore \bar{x} = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2}$

$$\Rightarrow \bar{x} - \bar{x}_1 = \frac{n_1 \bar{x}_1 + n_2 \bar{x}_2}{n_1 + n_2} - \bar{x}_1$$

$$\Rightarrow \bar{x} - \bar{x}_1 = \frac{n_2 (\bar{x}_2 - \bar{x}_1)}{n_1 + n_2}$$

$$\Rightarrow \bar{x} - \bar{x}_1 > 0 \quad [\because \bar{x}_2 > \bar{x}_1]$$

$$\Rightarrow \bar{x} > \bar{x}_1 \quad \dots (i)$$

and $\bar{x} - \bar{x}_2 = \frac{n_1 (\bar{x}_1 - \bar{x}_2)}{n_1 + n_2}$

$$\Rightarrow \bar{x} - \bar{x}_2 < 0 \quad [\because \bar{x}_2 > \bar{x}_1]$$

$$\Rightarrow \bar{x} < \bar{x}_2 \quad \dots (ii)$$

From Eqs. (i) and (ii), $\bar{x}_1 < \bar{x} < \bar{x}_2$

20. Given, $G_1 = (x_1 x_2 \dots x_n)^{1/n}$

and $G_2 = (y_1 y_2 \dots y_n)^{1/n}$

New series is $\frac{x_1}{y_1}, \frac{x_2}{y_2}, \dots, \frac{x_n}{y_n}$

$$\therefore G = \left(\frac{x_1}{y_1} \cdot \frac{x_2}{y_2} \dots \frac{x_n}{y_n} \right)^{1/n}$$

$$= \frac{(x_1 x_2 \dots x_n)^{1/n}}{(y_1 y_2 \dots y_n)^{1/n}} = \frac{G_1}{G_2}$$

21. $GM = (5 \cdot 8 \cdot 10 \cdot 15 \cdot 20 \cdot 25 \cdot 30)^{1/7} = 10(9)^{1/7}$

22. $GM = (1 \cdot 2 \cdot 4 \dots 2^n)^{1/n} = (2^0 \cdot 2^1 \cdot 2^2 \dots 2^n)^{1/n}$

$$= [2^{(0+1+2+\dots+n)}]^{1/n} = 2^{\frac{n+1}{2}}$$

23. $GM = (2 \cdot 4 \cdot 16 \cdot 32)^{1/4} = 8$

24. By definition, $\log G = \frac{n_1 \log G_1 + n_2 \log G_2}{n_1 + n_2}$

25. Let us first write the ascending series as shown below:

50, 51, 52, 53, 54, 55, 57, 58, 59, 61, 62, 62, 68

Thus, the measure of $\frac{(13+1)}{2}$ th i.e. 7th item is median value, which is 57.

26. Arranging in ascending order, we get

18, 18, 20, 21, 24, 26, 27, 29, 31, 35, 36, 38, 41, 42, 44

The measure of $\frac{(15+1)}{2}$ th = 8th item gives median.

$\therefore$ Median = 29

27. Median being positional average will not change.

28. Mean = $\frac{(24 + 38 + 55 + 69 + 89)}{5} = 55$

Median is measure of $\frac{(5+1)}{2} = 3$ rd item = 55

$\therefore$ Mean = Median

29. M_e = Measure of $\frac{5th + 6th}{2}$ term = 71

30. Arranging in ascending order, we get

0, 1, 2, 4, 5, 8, 12, 16, 20, 22, 25, 30, 32, 35, 38

Median = Value of $\left(\frac{15+1}{2}\right)$ th term

= Value of 8th term = 16

31. Given, $n = 100$, Mean = 50, Median = 52

$$M = \frac{\sum x}{n} = 50$$

$\therefore \sum x = 5000$

$\therefore$ Corrected mean = $\frac{5000 - 100 + 110}{100}$

= 50.10

Median remains same.

32. Middle 50% of items.

33. $D_7 = \frac{7n}{10}$

$$P_{70} = \frac{70n}{100} = \frac{7n}{10}$$

$\therefore$ 70th percentile = 7th decile

34. Since, the series is in ascending order, hence median is

measure of $\frac{(11+1)}{2}$ th = 6th item = 5

Mode is also, clearly 5.

$\therefore$ Median = Mode

$$\text{Mean} = \frac{-1 + 0 + 1 + 2 + 3 + 5 + 5 + 6 + 8 + 10 + 11}{11}$$

$$= \frac{50}{11} = 4.54 \text{ (approx.)}$$

35. Mode = Measure of most repeated item = 9

36. Mode = Measure of most repeated item = 5

37. $AM = \frac{2 + 2 + 3 + 3 + \dots + 8 + 8}{13} = \frac{64}{13}$

Median = Measure of $\left[\frac{13+1}{2}\right]$ th item

= Measure of 7th item = 4

Mode = Measure of most repeated item = 8

38. We see that,

Q_1 = Size of $\left(\frac{7+1}{4}\right)$ th item = 2nd item

Q_3 = Size of $\left[\left(\frac{7+1}{4}\right)\right]$ th item = 6th item

$\therefore$ Quartile deviation = $\frac{1}{2} (Q_3 - Q_1) = 3.5$

39. By definition, semi-interquartile range is $\frac{2}{3} \sigma$.

40. Coefficient of quartile deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{40 - 10}{40 + 10} = \frac{3}{5}$

41. Mean deviation is minimum when it is considered about the item, equidistant from the beginning and the end i.e.

the median. In this case, median is $\left(\frac{101+1}{2}\right)$ th term

i.e. 51st item i.e. x_{51} .

42. We know that, median divides the series into two equal parts (i.e. it is positional average) and hence absolute deviations will be least when measured from median.

43. Clearly, $M = \frac{\sum x}{n} = 6$

$$\text{and } \sum (x - M)^2 = \sum x^2 - 2M \sum x + \sum M^2$$

$$= 1000 - 12 \times 60 + 10 \times 36 = 640$$

$$\therefore \sigma^2 = \frac{\sum (x - M)^2}{n} = \frac{640}{10} = 64$$

$$\Rightarrow \sigma = 8$$

44. Given, $x_1 + x_2 + \dots + x_{10} = 12$

and $x_1^2 + x_2^2 + \dots + x_{10}^2 = 18$

$$\therefore \sigma^2 = \frac{1}{n} \sum x^2 - \left(\frac{1}{n} \sum x\right)^2 = \frac{18}{10} - \left(\frac{12}{10}\right)^2$$

$$= \frac{9}{5} - \frac{36}{25} = \frac{9}{25}$$

$$\Rightarrow SD = \frac{3}{5}$$

45. Mean $(\bar{x}) = \frac{6+8+10+12+14}{5} = 10$

$\therefore \sigma^2 = \frac{\Sigma(x-M)^2}{n} = 8$

$\therefore \text{Variance} = \sigma^2 = 8$

46. Mean = $\frac{1+2+3+4+5+6}{6} = \frac{7}{2}$

SD = $\sqrt{\frac{n^2-1}{12}} = \sqrt{\frac{35}{12}}$

47. $M = \frac{6.33+6.37+6.36+6.32+6.37}{5} = 6.35$

$\sigma^2 = \frac{\Sigma(x-m)^2}{n} = \frac{0.0022}{5} = 0.00044$

48. $\sigma^2 = 4^2 = 16$

49. AM = $\frac{-(1+2+\dots+7)}{7} = -4$

$\therefore \sigma^2 = \frac{\Sigma x_i^2}{n} - (\bar{x})^2 = 4$

$\therefore \sigma = 2$

50. Corrected $\Sigma x = 40 \times 200 - 50 + 40 = 7990$

$\therefore$ Corrected $\bar{x} = 7990 / 200 = 39.95$

Incorrect $\Sigma x^2 = n[\sigma^2 + \bar{x}^2]$
 $= 200[15^2 + 40^2] = 365000$

Correct $\Sigma x^2 = 365000 - 2500 + 1600$
 $= 364100$

$\therefore$ Corrected $\sigma = \sqrt{\frac{364100}{200} - (39.95)^2}$
 $= \sqrt{1820.5 - 1596}$
 $= \sqrt{224.5} = 14.98$

51. Since, $SD \leq \text{Range} = b - a$

$\therefore \text{Var}(x) \leq (b-a)^2$ or $(b-a)^2 \geq \text{Var}(x)$

52. We have, $r = \max_{i \neq j} |x_i - x_j|$ and $S^2 = \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2$

Now, $(x_i - \bar{x})^2 = \left(x_i - \frac{x_1 + x_2 + \dots + x_n}{n}\right)^2$

$= \frac{1}{n^2} [(x_i - x_1) + (x_i - x_2) + \dots + (x_i - x_{i-1})$
 $+ (x_i - x_{i+1}) + \dots + (x_i - x_n)]^2 \leq \frac{[(n-1)r]^2}{n^2}$
 $[\because |x_i - x_j| \leq r]$

$\Rightarrow (x_i - \bar{x})^2 \leq r^2 \Rightarrow \sum_{i=1}^n (x_i - \bar{x})^2 \leq nr^2$

$\Rightarrow \frac{1}{n-1} \sum_{i=1}^n (x_i - \bar{x})^2 \leq \frac{nr^2}{(n-1)}$

$\Rightarrow S^2 \leq \frac{nr^2}{(n-1)} \Rightarrow S \leq r \sqrt{\frac{n}{n-1}}$

53. Coefficient of variation.

54. $\sigma^2 = \frac{\Sigma(x-\bar{x})^2}{n} = \frac{250}{10}$

$\Rightarrow \sigma = 5$

Now, $CV = \frac{\sigma}{\bar{x}} \times 100 = \frac{5}{50} \times 100 = 10$

55. $CV = \frac{19.76}{35.16} \times 100 = 56.2$

56. Clearly, $70 = \frac{17.5}{\bar{x}_1} \times 100 \Rightarrow \bar{x}_1 = 25$

and $90 = \frac{18}{\bar{x}_2} \times 100$

$\therefore \bar{x}_2 = 20$

57. $\sigma = \sqrt{\frac{n_1(\sigma_1^2 + d_1^2) + n_2(\sigma_2^2 + d_2^2)}{n_1 + n_2}} = \sqrt{\frac{22}{3}}$

58. $\sigma^2 = \frac{5 \cdot 18 + 3 \cdot 24}{5+3} + (8-8) \frac{15}{64} = 20.25$

59. $M - M_0 = 3(M - M_e)$

$\Rightarrow 24 - 18 = 3(24 - M_e) \Rightarrow M_e = 22$

60. For symmetrical distribution,

Mean = Median = Mode

61. Mode = 3 Median - 2 Mean

$\Rightarrow 6\lambda = 3 \text{ Median} - 18\lambda$

$\therefore \text{Median} = 8\lambda$

62. By definition, Karl Pearson's coefficient = $\frac{M - M_0}{\sigma}$

63. By definition, Karl Pearson's coefficient = $\frac{M - M_0}{\sigma} = 0$

64. Coefficient of skewness = $\frac{622.08 - 625}{3.06} = -0.95$

65. For symmetrical distribution,

Mean = Median = Mode, it is zero.

66. Coefficient of skewness = $\frac{41 - 45}{8} = -0.5$

67. Mean increased = $\frac{5 \times 10}{10} = 5$

68. New mean = $7 + 5 = 12$

69. Let a student gets x marks out of 40. Then, he gets $\frac{5x}{4}$ marks out of 50. Thus, each observation will be multiplied by $\frac{5}{4}$.

Hence, mean is also multiplied by $\frac{5}{4}$, giving mean
 $= 38 \times \frac{5}{4} = 47.5$

70. Since, SD is independent of change of origin. Therefore, the standard deviation remains same i.e. 15.40.

71. The standard deviation remains same = 2.45

72. Median = $20 + 2 = 22$, SD = 4

73. Resulting standard deviation = $c\sigma$

74. New variance = $8 \cdot (4)^2 = 128$ [$\because \text{Var}(kx) = k^2 \text{Var}(x)$]

75. Here, $u = \frac{x}{h} - \frac{a}{h}$ [each x is divided by h]

$\Rightarrow \sigma_u = \frac{\sigma_x}{h}$ [$\because \sigma_{cx} = c \sigma_x$]

$\Rightarrow \sigma_x = h\sigma_u$ [and $\sigma_{c+x} = \sigma_x$]

SD does not change by adding or subtracting a fixed value from each variate value.

76. Given, $\sigma = 9$

Let a student obtains x marks out of 75, then his marks out of 100 are $\frac{4x}{3}$. Each observation is multiplied by $\frac{4}{3}$.

New, $\sigma = \frac{4}{3} \times 9 = 12$, therefore new variance $= \sigma^2 = 144$

77. $SD = \frac{5}{4}(15) = \frac{75}{4}$ $\left[\because MD = \frac{4}{5} SD \right]$

Quartile deviation $= \frac{2}{3}(SD) = \frac{2}{3} \left(\frac{75}{4} \right) = 12.5$

78. $r = 1 \Rightarrow$ Increase/decrease in x brings increase/decrease in y . Further, $x + y$ also increases or decreases.

Hence, r between x and $x + y$ is 1.

79. Given, $r = 0.5$, $Cov(x, y) = 16$, $\sigma_x = 4$

$$\therefore \sigma_y = \frac{Cov(x, y)}{r \sigma_x} = 8$$

80. Given,

$$aX + bY + c = 0 \quad \dots(i)$$

$$\therefore a\bar{X} + b\bar{Y} + c = 0 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$a(X - \bar{X}) + b(Y - \bar{Y}) = 0$$

$$\Rightarrow X - \bar{X} = -\frac{b}{a}(Y - \bar{Y}) \quad \dots(iii)$$

$$Cov(X, Y) = \frac{1}{n} \sum (x_i - \bar{X})(y_i - \bar{Y})$$

$$= -\frac{b}{a} \left[\frac{1}{n} \sum (y_i - \bar{Y})^2 \right]$$

$$= -\frac{b}{a} \text{Var}(Y) \quad \dots(iv)$$

$$\text{Also, } \text{Var}(X) = \frac{1}{n} \sum (x_i - \bar{X})^2$$

$$= \frac{b^2}{a^2} \left[\frac{1}{n} \sum (y_i - \bar{Y})^2 \right]$$

$$= \frac{b^2}{a^2} \text{Var}(Y) \quad \dots(v)$$

$$\therefore r = \frac{-\frac{b}{a} \text{Var}(Y)}{\sqrt{\frac{b^2}{a^2} \text{Var}(Y)} \sqrt{\text{Var}(Y)}} = \frac{-\frac{b}{a}}{\frac{|b|}{|a|}}$$

$$= \frac{-b/a}{-b/a} = 1 \quad \left[\because ab < 0 \Rightarrow a \text{ and } b \text{ are of opposite sign} \right]$$

81. Let $E = \sigma_{x+y}^2$

$$= \frac{1}{n} \sum_{i=1}^n [x_i + y_i - (\bar{x} + \bar{y})]^2$$

$$= \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 + \frac{1}{n} \sum_{i=1}^n (y_i - \bar{y})^2$$

$$+ \frac{2}{n} \sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})$$

$$\Rightarrow E = \sigma_x^2 + \sigma_y^2 + 2 \text{Cov}(x, y)$$

$$\text{We have, } \sigma_{x+y}^2 = \sigma_x^2 + \sigma_y^2 + 2 \text{Cov}(x, y)$$

$$\Rightarrow 49 = 9 + 16 + 2 \text{Cov}(x, y)$$

$$\Rightarrow \text{Cov}(x, y) = 12$$

$$\therefore r(x, y) = \frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} = \frac{12}{(3)(4)} = 1$$

82. Let $Z = X - Y$

$$\therefore \bar{Z} = \bar{X} - \bar{Y}$$

$$\Rightarrow Z - \bar{Z} = X - \bar{X} - (Y - \bar{Y})$$

$$\Rightarrow (Z - \bar{Z})^2 = (X - \bar{X})^2 + (Y - \bar{Y})^2 - 2(X - \bar{X})(Y - \bar{Y})$$

$$\Rightarrow \frac{1}{n} \sum (Z - \bar{Z})^2 = \frac{1}{n} \sum (X - \bar{X})^2 + \frac{1}{n} \sum (Y - \bar{Y})^2 - \frac{2}{n} \sum (X - \bar{X})(Y - \bar{Y})$$

$$\Rightarrow \sigma_Z^2 = \sigma_x^2 + \sigma_y^2 - 2 \text{Cov}(X, Y)$$

$$\Rightarrow \sigma_{x-y}^2 = \sigma_x^2 + \sigma_y^2 - 2r\sigma_x\sigma_y$$

$$\therefore r = \frac{\sigma_x^2 + \sigma_y^2 - \sigma_{x-y}^2}{2\sigma_x\sigma_y}$$

83. $r = \frac{\text{Cov}(x, y)}{\sigma_x\sigma_y}$, since independent, therefore $r = 0$

$$\Rightarrow \text{Cov}(x, y) = 0$$

$$84. r = \frac{\sum xy}{\sqrt{\sum x^2} \sqrt{\sum y^2}} = \frac{40}{\sqrt{80 \times 20}} = 1$$

85. Here, $\sum xy = 269$, $\sum x^2 = 55$, $\sum y^2 = 1347$

$$\therefore r = \frac{269}{\sqrt{55 \times 1347}} = 0.988$$

86. Here, $\sigma_x = \sqrt{2.89}$, $\sigma_y = \sqrt{100}$

$$\therefore r = \frac{-16.5}{1.7(10)} = -0.97$$

87. We know that,

$$r^2 = \left[\frac{\text{Cov}(x, y)}{\sigma_x \sigma_y} \right]^2$$

$$\text{and } \text{Cov}(x, y) = \frac{\sum xy}{n} = \frac{60}{n}$$

$$\sigma_x^2 = \frac{\sum x^2}{n} = \frac{90}{n}$$

$$\therefore 0.64 = \frac{3600/n^2}{(90/n)(6.25)}$$

$$\Rightarrow n = 10$$

88. Here, $r^2 = 0.25$. Hence, only 25% of variations are explained.

89. We know, $(\bar{x}, \bar{y})$ is the point of intersection of two lines of regression. Hence, $(\bar{x}, \bar{y})$ lies on both.

90. If we solve first for x and second for y , then product of regression coefficient is negative, which is not possible. Similarly, reversing the process we again find that product of regression coefficient is negative, which is not possible. Therefore, they are not the lines of regression.

91. $x - \bar{x} = 0$ and $y - \bar{y} = 0$

92. Solving $13x - 10y + 11 = 0$ and $2x - y - 1 = 0$, we get

$$\bar{x} = 3 \quad \text{and} \quad \bar{y} = 5$$

93. From $8x - 10y - 66 = 0$, we get $r_1 = \frac{4}{5}$
 From $40x - 18y - 214 = 0$, we get $r_2 = \frac{9}{20}$

$$r^2 = \frac{36}{100} \Rightarrow r = 0.6$$

$$\therefore \sigma_y = \frac{12}{30} = 4$$

94. Let mean of x and y series be $\bar{x}$ and $\bar{y}$. Lines of regression passes through mean $(\bar{x}, \bar{y})$, hence

$$\bar{y} = a\bar{x} + b, \bar{x} = \alpha\bar{y} + \beta$$

 Given that, $\bar{x} = \bar{y} \Rightarrow \bar{x} = \frac{b}{1-a} = \frac{\beta}{1-\alpha}$

95. $x = -2y + 5 \Rightarrow r_1 = -2$
 and $y = -\frac{1}{3}x + \frac{8}{3} \Rightarrow r_2 = -\frac{1}{3}$

$$\therefore r^2 = \frac{2}{3} \Rightarrow r = -0.82$$

96. Given, $r_1 = \frac{r\sigma_y}{\sigma_x}, r_2 = \frac{r\sigma_x}{\sigma_y}$

$$\therefore r_1 r_2 = r^2$$

$$\Rightarrow r \text{ is GM of } r_1 \text{ and } r_2.$$

 Now,
$$AM = \frac{r_1 + r_2}{2}$$

 But
$$AM \geq GM \Rightarrow \frac{r_1 + r_2}{2} \geq r$$

$$\therefore r_1 + r_2 \geq 2r$$

97. $r = \pm 1$

98. The slope of line of regression of y on x is $m_1 = r_1$.
 Similarly, for x on y , $m_2 = \frac{1}{r_2}$

$$\therefore \tan \theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{r_1 - \frac{1}{r_2}}{1 + \frac{r_1}{r_2}} = \frac{r_1 r_2 - 1}{r_2 + r_1}$$

Entrances Gallery

1.
$$\frac{a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + a^8 + a^{10} + 1}{8} \geq 1$$

Minimum value = 8

2. (d) Given,
$$\frac{x_1 + x_2 + x_3 + \dots + x_{16}}{16} = 16$$

$$\Rightarrow \sum_{i=1}^{16} x_i = 16 \times 16$$

Sum of new observations

$$= \sum_{i=1}^{18} y_i = (16 \times 16 - 16) + (3 + 4 + 5)$$

$$= 252$$

Number of observations = 18

$$\text{New mean} = \frac{\sum_{i=1}^{18} y_i}{18} = \frac{252}{18} = 14$$

3. Clearly,
$$\bar{x} = \frac{\sum_{r=1}^{50} 2r}{50} = 51$$

$$\therefore \sigma^2 = \left(\frac{\sum x_i^2}{n} \right) - \bar{x}^2$$

$$= \frac{\sum_{r=1}^{50} 4r^2}{50} - (51)^2$$

$$= 833$$

4. If initially marks were x_i , then

$$\sigma_1^2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

Now, each is increased by 10

$$\therefore \sigma_2^2 = \frac{\sum [(x_i + 10) - (\bar{x} + 10)]^2}{n} = \sigma_1^2$$

So, variance will not change whereas mean, median and mode will increase by 10.

5. Given, $\bar{x}$ is the AM and σ^2 is the variance of n observations $x_1, x_2, x_3, \dots, x_n$.

Now, AM of $2x_1, 2x_2, 2x_3, \dots, 2x_n$

$$= \frac{2x_1 + 2x_2 + 2x_3 + \dots + 2x_n}{n}$$

$$= 2 \left(\frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \right) = 2\bar{x}$$

Hence, Statement II is false.

From here, only we can tell that option (d) is the correct answer. However, here we will formally check the validity of Statement I.

Variance of $2x_1, 2x_2, 2x_3, \dots, 2x_n$ = Variance $(2x) = 2^2$
 Variance $(x) = 4\sigma^2$

$\therefore$ Statement I is correct.

Finally, Statement I is true and Statement II is false.

6. Median of $a, 2a, 3a, 4a, \dots, 50a$

$$= \frac{25a + 26a}{2} = (25.5)a$$

MD (about median)

$$= \frac{\sum_{i=1}^{50} |x_i - \text{Median}|}{n}$$

$$\Rightarrow 50 = \frac{1}{50} \{2|a| \cdot (0.5 + 1.5 + 2.5 + \dots + 24.5)\}$$

$$\Rightarrow 2500 = 2|a| \cdot \frac{25}{2} (25)$$

$$\therefore |a| = 4$$

7. Correct mean = Old mean + 2 = 30 + 2 = 32

As, standard deviation is independent of change of origin.

$\therefore$ It remains same.

$\Rightarrow$ Standard deviation = 2

8. We have, $\sigma_x^2 = 4$ and $\sigma_y^2 = 5$

Also, $\bar{x} = 2$ and $\bar{y} = 4$

$$\begin{aligned} \text{Now, } \frac{\sum x_i}{5} &= 2 \text{ and } \frac{\sum y_i}{5} = 4 \\ \Rightarrow \sum x_i &= 10 \text{ and } \sum y_i = 20 \\ \text{Also, } \sigma_x^2 &= \left(\frac{1}{5} \sum x_i^2 \right) - (\bar{x})^2 = \frac{1}{5} (\sum x_i^2) - 4 \\ \Rightarrow \sum x_i^2 &= 40 \\ \text{Similarly, } \sum y_i^2 &= 105 \\ \text{Now, } \sigma_z^2 &= \frac{1}{10} (\sum x_i^2 + \sum y_i^2) - \left(\frac{\bar{x} + \bar{y}}{2} \right)^2 \\ &= \frac{1}{10} (40 + 105) - 9 = \frac{145 - 90}{10} = \frac{55}{10} = \frac{11}{2} \end{aligned}$$

$$\begin{aligned} 9. \bar{x} &= \frac{\text{Sum of quantities}}{n} = \frac{\frac{101}{2}(a + l)}{101} \\ &= \frac{1}{2} [1 + 1 + 100d] = 1 + 50d \\ \therefore \text{MD (from mean)} &= \frac{1}{n} \sum |x_i - \bar{x}| = 255 \\ &= \frac{1}{101} [50d + 49d + 48d + \dots + d + 0 + d + \dots + 50d] \\ &= \frac{2d}{101} \left[\frac{50 \times 51}{2} \right] \Rightarrow d = \frac{255 \times 101}{50 \times 51} = 10.1 \end{aligned}$$

10. Statement II is true.

Statement I Sum of n even natural numbers $= n(n + 1)$

$$\text{Mean } (\bar{x}) = \frac{n(n + 1)}{n} = n + 1$$

$$\begin{aligned} \text{Variance} &= \left[\frac{1}{n} \sum x_i^2 \right] - (\bar{x})^2 \\ &= \frac{1}{n} [2^2 + 4^2 + \dots + (2n)^2] - (n + 1)^2 \\ &= \frac{1}{n} 2^2 [1^2 + 2^2 + \dots + n^2] - (n + 1)^2 \\ &= \frac{4}{n} \cdot \frac{n(n + 1)(2n + 1)}{6} - (n + 1)^2 \\ &= \frac{(n + 1)[2(2n + 1) - 3(n + 1)]}{3} \\ &= \frac{(n + 1)(n - 1)}{3} = \frac{n^2 - 1}{3} \end{aligned}$$

$\therefore$ Statement I is false.

11. According to the given condition,

$$\begin{aligned} 680 &= \frac{[(6 - a)^2 + (6 - b)^2 + (6 - 8)^2 + (6 - 5)^2 + (6 - 10)^2]}{5} \\ \Rightarrow 34 &= (6 - a)^2 + (6 - b)^2 + 4 + 1 + 16 \\ \Rightarrow (6 - a)^2 + (6 - b)^2 &= 13 = 9 + 4 \\ \Rightarrow (6 - a)^2 + (6 - b)^2 &= 3^2 + 2^2 \\ \Rightarrow a &= 3, b = 4 \end{aligned}$$

12. Let the number of boys and girls be x and y .

$$\begin{aligned} \therefore 52x + 42y &= 50(x + y) \\ \Rightarrow 52x + 42y &= 50x + 50y \\ \Rightarrow 2x &= 8y \\ \Rightarrow x &= 4y \\ \therefore \text{Total number of students in the class} \\ &= x + y = 4y + y = 5y \end{aligned}$$

$\therefore$ Required percentage of boys

$$= \left(\frac{4y}{5y} \times 100 \right) = 80\%$$

13. Since, variance is independent of change of origin. Therefore, variance of observations 101, 102, ..., 200 is same as variance of 151, 152, ..., 250.

$$\therefore V_A = V_B \Rightarrow \frac{V_A}{V_B} = 1$$

14. Given that, $\sum x_i^2 = 400$ and $\sum x_i = 80$, since $\sigma^2 \geq 0$

$$\begin{aligned} \Rightarrow \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n} \right)^2 &\geq 0 \\ \Rightarrow \frac{400}{n} - \frac{6400}{n^2} &\geq 0 \\ \therefore n &\geq 16 \end{aligned}$$

15. Given that, mean = 21

and median = 22

Using the relation,

$$\text{Mode} = 3\text{Median} - 2\text{Mean}$$

$$\therefore \text{Mode} = 3(22) - 2(21) = 66 - 42 = 24$$

16. It is true that mode can be computed from histogram and median is not independent of change of scale.

But variance is independent of change of origin and not of scale.

17. In the $2n$ observations, half of them equal to a and remaining half equal to $-a$. Then, the mean of total $2n$ observations is equal to zero.

$$\begin{aligned} \therefore \text{SD} &= \sqrt{\frac{\sum (x - \bar{x})^2}{N}} \\ \Rightarrow 2 &= \sqrt{\frac{\sum x^2}{2n}} \\ \Rightarrow 4 &= \frac{\sum x^2}{2n} \\ \Rightarrow 4 &= \frac{2na^2}{2n} \\ \Rightarrow a^2 &= 4 \Rightarrow |a| = 2 \end{aligned}$$

18. Since, the four largest observations either be the first four observation or last four observations, therefore median of new set remains the same as that of the original set.

19. Given, $N = 15$, $\sum x^2 = 2830$, $\sum x = 170$

Since, one observation 20 was replaced by 30, then $\sum x^2 = 2830 - 400 + 900 = 3330$

and $\sum x = 170 - 20 + 30 = 180$

$$\begin{aligned} \therefore \text{Variance, } \sigma^2 &= \frac{\sum x^2}{N} - \left(\frac{\sum x}{N} \right)^2 \\ &= \frac{3330}{15} - \left(\frac{180}{15} \right)^2 = \frac{3330 - 12 \times 180}{15} \\ &= \frac{3330 - 2160}{15} = \frac{1170}{15} = 78.0 \end{aligned}$$

20. Since, total number of students = 100

and number of boys = 70

$\therefore$ Number of girls = $(100 - 70) = 30$

Now, the total marks of 100 students = $100 \times 72 = 7200$

and total marks of 70 boys = $70 \times 75 = 5250$

Total marks of 30 girls = $7200 - 5250 = 1950$

$$\therefore \text{Average marks of 30 girls} = \frac{1950}{30} = 65$$

21. Let the observations be $x_1, x_2, x_3, \dots, x_n$.

$$\text{Now, mean } (\bar{x}) = \frac{x_1 + x_2 + \dots + x_n}{n}$$

When first term is increased by 1, second by 2 and so on, then the observation will be $(x_1 + 1), (x_1 + 2), (x_1 + 3), \dots, (x_n + n)$.

Then, new mean

$$\begin{aligned} \bar{x} &= \frac{(x_1 + 1) + (x_2 + 2) + \dots + (x_n + n)}{n} \\ &= \frac{(x_1 + x_2 + \dots + x_n) + (1 + 2 + 3 + \dots + n)}{n} \\ &= \frac{x_1 + x_2 + \dots + x_n}{n} + \frac{1 + 2 + 3 + \dots + n}{n} \\ &= \bar{x} + \frac{n(n+1)}{2n} \left[\because 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2} \right] \\ &= \bar{x} + \frac{n+1}{2} \end{aligned}$$

22. Given, mean = 36, median = 34

We know that,

$$\text{Mode} = 3 \text{ Median} - 2 \text{ Mean}$$

$$\therefore \text{Mode} = 3 \times 34 - 2 \times 36 = 102 - 72 = 30$$

23. Variance of first n natural numbers is given by

$$\begin{aligned} \text{variance} &= \frac{\Sigma n^2}{n} - \left(\frac{\Sigma n}{n} \right)^2 \\ &= \frac{n(n+1)(2n+1)}{6n} - \left[\frac{n(n+1)}{2n} \right]^2 \\ &= (n+1) \left[\frac{2n+1}{6} - \frac{(n+1)}{4} \right] \\ &= \frac{(n+1)}{12} [4n+2-3n-3] \\ &= \frac{(n+1)}{12} \times (n-1) = \frac{n^2-1}{12} \end{aligned}$$

24. Given, coefficient of variation, CV = 60

and standard deviation, $\sigma = 21$

$$\therefore \text{CV} = \frac{\sigma}{\bar{x}} \times 100$$

$$\therefore 60 = \frac{21}{\bar{x}} \times 100 \Rightarrow \bar{x} = \frac{21}{60} \times 100$$

$$\therefore \bar{x} = 35$$

25. We have, $\Sigma x_i - 45 = 9 \Rightarrow \Sigma x_i = 54$

$$\text{and } \Sigma (x_i - 5)^2 = 45$$

$$\Rightarrow \Sigma (x_i^2 - 2x_i \cdot 5 + 25) = 45$$

$$\Rightarrow \Sigma (x_i^2) - 2 \times 54 \times 5 + 25 \times 9 = 45$$

$$\Rightarrow \Sigma x_i^2 = 45 + 540 - 225 = 360$$

$$\begin{aligned} \sigma &= \sqrt{\frac{360}{9} - \left(\frac{54}{9} \right)^2} \\ &= \sqrt{40 - 36} = 2 \end{aligned}$$

26. Now, mean of given observation is,

$$\begin{aligned} \bar{x} &= \frac{9 + 16 + 23 + 30 + 37 + 44 + 51}{7} \\ &= \frac{210}{7} = 30 \end{aligned}$$

$$\therefore \text{Standard deviation} = \sqrt{\frac{\Sigma (x_i - \bar{x})^2}{7}}$$

$$\begin{aligned} &= \sqrt{\frac{(9-30)^2 + (16-30)^2 + (23-30)^2 + (30-30)^2 + (37-30)^2 + (44-30)^2 + (51-30)^2}{7}} \\ &= \sqrt{\frac{(-21)^2 + (-14)^2 + (-7)^2 + (0)^2 + (7)^2 + (14)^2 + (21)^2}{7}} \\ &= \sqrt{\frac{441 + 196 + 49 + 0 + 49 + 196 + 441}{7}} \\ &= \sqrt{\frac{1372}{7}} \\ &= \sqrt{196} = 14 \end{aligned}$$

27. Let four observations be x_1, x_2, x_3 and x_4 .

$$\text{Given, mean } (\bar{x}) = 3 \text{ and } \Sigma x_i^2 = 48$$

$$\begin{aligned} \therefore \text{SD} &= \sqrt{\frac{\Sigma x_i^2}{n} - (\bar{x})^2} = \sqrt{\frac{48}{4} - (3)^2} \\ &= \sqrt{12 - 9} = \sqrt{3} \end{aligned}$$

28. Let $x_1, x_2, \dots, x_n$ be n variates. Then, their arithmetic mean will be

$$\bar{x} = \frac{\sum_{i=1}^n x_i}{n} \quad \dots(i)$$

Now, sum of deviations of the variates from the AM

$$\begin{aligned} &= \sum_{i=1}^n (x_i - \bar{x}) = \sum_{i=1}^n x_i - \sum_{i=1}^n \bar{x} \\ &= \sum_{i=1}^n x_i - n\bar{x} = \sum_{i=1}^n x_i - \sum_{i=1}^n x_i \quad [\text{from Eq. (i)}] \\ &= 0 \end{aligned}$$

29. Let two unknown items be x and y , then

$$\text{Mean} = 4 \Rightarrow \frac{1 + 2 + 6 + x + y}{5} = 4$$

$$\Rightarrow x + y = 11 \quad \dots(i)$$

and variance = 5.2

$$\Rightarrow \frac{1^2 + 2^2 + 6^2 + x^2 + y^2}{5} - (\bar{x})^2 = 5.2$$

$$\Rightarrow 41 + x^2 + y^2 = 5(5.2 + 4^2)$$

$$\Rightarrow 41 + x^2 + y^2 = 106$$

$$\therefore x^2 + y^2 = 65 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$x = 4, y = 7$$

$$\text{or } x = 7, y = 4$$

$$30. \therefore \text{AM} = \frac{\sum_{i=1}^n x_i}{n} \Rightarrow 15 = \frac{\sum_{i=1}^9 x_i}{9}$$

$$\Rightarrow \sum_{i=1}^9 x_i = 135$$

$$\text{Also, } 16 = \frac{\sum_{i=1}^{10} x_i}{10} \Rightarrow \sum_{i=1}^{10} x_i = 160$$

$$\therefore \text{Added term} = 160 - 135 = 25$$

31. $\ln \frac{x}{5}, \frac{x}{4}, \frac{x}{3}, \frac{x}{2}, x$

Median = 8

$$\Rightarrow \frac{x}{3} = 8$$

$$\Rightarrow x = 24$$

32. $\sigma_x^2 = \frac{\sum x^2}{n} = \frac{90}{n}$

$$\text{Cov}(x, y) = \frac{\sum xy}{n} = \frac{120}{n}$$

$$\sigma_y^2 = 8^2 = 64$$

$$\therefore r^2 = \frac{[\text{Cov}(x, y)]^2}{\sigma_x^2 \sigma_y^2}$$

$$\Rightarrow 0.25 = \frac{(120)^2/n^2}{\frac{90}{n} \cdot 64}$$

$$\Rightarrow n = \frac{(120)^2}{0.25 \times 90 \times 64} = \frac{14400}{1440} = 10$$

33. Standard deviation = $10 \times 2.71 = 27.1$

34. Since, $CV_1 = \frac{\sigma_1}{\bar{x}} \times 100$

$$\Rightarrow \sigma_1 = 50 \times \frac{30}{100}$$

Similarly, $\sigma_2 = 60 \times \frac{25}{100}$

$$\therefore \sigma_1 - \sigma_2 = 0$$

35. Here, mean = $\frac{6 + 8 + 12 + 15 + 10 + 9}{6} = 10$

$$\therefore \text{Mean deviation mean} = \frac{\sum |x_i - \bar{x}|}{n}$$

$$= \frac{[|6-10| + |8-10| + |12-10| + |15-10| + |10-10| + |9-10|]}{6}$$

$$= \frac{4 + 2 + 2 + 5 + 0 + 1}{6} = \frac{14}{6} = 2.33$$

36. We have, 1, 2, 3, ..., n

$$\therefore \bar{x} = \frac{n(n+1)}{2n} = \frac{n+1}{2}$$

$\therefore$ Mean deviation from mean

$$= \frac{1}{n} \sum (x_i - \bar{x})$$

$$= \frac{1}{n} \left[\left(1 - \frac{n+1}{2}\right) + \left(2 - \frac{n+1}{2}\right) + \dots + \left(n - \frac{n+1}{2}\right) \right]$$

$$= \frac{1}{n} \left[\left|\frac{1-n}{2}\right| + \left|\frac{3-n}{2}\right| + \dots + \left|\frac{n-3}{2}\right| + \left|\frac{n-1}{2}\right| \right]$$

$$= \frac{1}{n} \left[\frac{n-1}{2} + \frac{n-3}{2} + \dots + \frac{n-3}{2} + \frac{n-1}{2} \right]$$

$$= \frac{1}{n} [(n-1) + (n-3) + \dots + 1] \quad [\because n \text{ is even}]$$

$$= \frac{1}{n} \left[\frac{n}{4} (n-1+1) \right] = \frac{n^2}{4n} = \frac{n}{4}$$

37. We have, $\bar{x} = 5$ and variance = $\frac{1}{n} \sum x_i^2 - (\bar{x})^2$

$$\therefore 0 = \frac{1}{n} \cdot 400 - 25$$

$$\Rightarrow n = \frac{400}{25} = 16$$

21

Fundamentals of Probability

Introduction

Probability is a measure of uncertainty of various phenomenon and to measure it. There are three approaches:

- (i) Statistical approach (ii) Classical approach (iii) Axiomatic approach

In statistical approach, probability is found on the basis of observations and collected data, which is not used in general and to understand classical and axiomatic approach, we must know about some definitions.

Some Basic Definitions

Deterministic Experiment

Those experiments which when repeated under identical conditions produce the same result or outcome are known as deterministic experiments. When experiments in science or engineering are repeated under identical conditions, we get almost the same result every time.

Probabilistic or Random Experiment

If an experiment, when repeated under identical conditions, does not produce the same outcome every time but the outcome in a trial is one of the several possible outcomes, then such an experiment is known as a probabilistic experiment or a random experiment.

Outcomes

A possible result of a random experiment is called outcome.

e.g. If the experiment consists of tossing a coin twice, then some of the outcomes are HH , HT , etc.

Sample Space

The set of all possible outcomes of a random experiment is called the sample space associated with it and it is generally denoted by S .

e.g. The sample space for the experiment of tossing a die is given by

$$S = \{1, 2, 3, 4, 5, 6\}$$

Chapter Snapshot

- Introduction
- Some Basic Definitions
- Event
- Important Events
- Algebra of Events
- Probability
- Geometrical Probability
- Addition Theorem of Probability
- Independent Events
- Boole's Inequality

➤ **Example 1.** A bag contains 4 identical red balls and 3 identical black balls. The experiment consists of drawing one ball, then putting it into the bag and again drawing a ball. Then, the possible outcomes of this experiment is

- (a) $\{RR, BB\}$ (b) $\{RR, BB, RR\}$
(c) $\{BB, R\}$ (d) None of these

Sol. (d) The sample space for this experiment is $\{RR, RB, BR, BB\}$, where R denotes the red ball and B denotes the black ball.

Event

An event is associated with a subset of sample space. Events can be classified into various types on the basis of the elements they have.

Types of Events

- (i) **Impossible and sure events** The empty set ϕ is called an impossible event. e.g. Getting two heads in tossing a unbiased coin one time.
The event which is certain to occur is said to be the sure event.
e.g. In tossing a die, any number will come is a sure event.
- (ii) **Simple event** If an event E has only one sample point of a sample space, it is called a simple (or elementary) event.
e.g. In the experiment of tossing a coin twice getting both head i.e. $E = \{H, H\}$.
- (iii) **Compound event** If an event has more than one sample point, it is called a compound event.
e.g. In the experiment of tossing a coin thrice the events
 E : Exactly one head appeared
 F : Atleast one head appeared
 G : Atmost one head appeared, etc., are all compound events. As
 $E = \{HTT, THT, TTH\}$
 $F = \{HTT, THT, TTH, HHT, THH, HTH, HHH\}$
 $G = \{HTT, THT, TTH, TTT\}$
Each of the above subsets contain more than one sample point. Hence, they are all compound events.

➤ **Example 2.** A pair of dice is rolled. Describe the following events associated with this random experiment

A = Getting the sum of the number on the dice is 2.
 B = Getting the sum of the number is multiple of 11.
 C = Getting the sum of the number is multiple of 12.

Which of these events are compound events?

- (a) A (b) B
(c) C (d) All of these

Sol. (b) A = Getting the sum of the number on the dice is 2
 $= \{1, 1\}$
 B = Getting the sum of the number is multiple of 11
 $= \{(5, 6), (6, 5)\}$
 C = Getting the sum of the number is multiple of 12
 $= \{(6, 6)\}$
 $\therefore B$ is the compound event.

Important Events

Equally Likely Events

Events are said to be equally likely when we have no reason to believe that one is more likely to occur than the other. e.g. When an unbiased die is thrown, all the six faces 1, 2, 3, 4, 5, 6 are equally likely to come up.

Exhaustive Events

A set of events is said to be exhaustive, if one of them must necessarily happen every time the experiment is performed.

e.g. When a die is thrown, events 1, 2, 3, 4, 5, 6 form an exhaustive set of events.

✎ We can say that the total number of elementary events of a random experiment is called the exhaustive number of cases.

Mutually Exclusive Events

Two or more events are said to be mutually exclusive, one of them occurs, others cannot occur. Thus, two or more events are said to be mutually exclusive, if no two of them can occur together.

Hence, $A_1, A_2, A_3, \dots, A_n$ are mutually exclusive if and only if $A_i \cap A_j = \phi, \forall i \neq j$.

e.g. When a die is thrown, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$.

Let A be an event of occurrence of number greater than 4, i.e. $\{5, 6\}$.

B is an event of occurrence of an odd number, i.e. $\{1, 3, 5\}$.

C is an event of occurrence of an even number, i.e. $\{2, 4, 6\}$.

Here, events B and C are mutually exclusive but the events A and B or A and C are not mutually exclusive.

➤ **Example 3.** Two dice are thrown. The events A, B and C are as follow :

A : Getting an even number on the first die.

B : Getting an odd number on the first die.

C : Getting the sum of the numbers on the dice ≤ 5 .

Which of the following statement is false?

- (a) A and B are mutually exclusive
(b) A and B are mutually exclusive and exhaustive
(c) $A = B'$
(d) A and C are mutually exclusive

Sol. (d) If two dice are thrown, then total number of possible outcomes, $S = 6 \times 6 = 36$ which are as follow:

↓ →	1	2	3	4	5	6
1	1, 1	1, 2	1, 3	1, 4	1, 5	1, 6
2	2, 1	2, 2	2, 3	2, 4	2, 5	2, 6
3	3, 1	3, 2	3, 3	3, 4	3, 5	3, 6
4	4, 1	4, 2	4, 3	4, 4	4, 5	4, 6
5	5, 1	5, 2	5, 3	5, 4	5, 5	5, 6
6	6, 1	6, 2	6, 3	6, 4	6, 5	6, 6

A = Getting an even number on the first die
 $= \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

B = Getting an odd number on the first die
 $= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

$C = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1)\}$

Here, B' = Event getting an odd number on the first die.

(a) True

$\because A$ = Getting an even number on the first die

B = Getting an odd number on the first die

$\Rightarrow A \cap B = \phi$

$\therefore A$ and B are mutually exclusive events.

(b) True

$\therefore A \cup B = S$ i.e. exhaustive. Also, $A \cap B = \phi$

(c) True

$\because B$ = Getting an odd number on the first die

$\Rightarrow B' =$ Getting an even number on first die $= A$

$A = B'$

(d) False

$\because A \cap C = \{(2, 1), (2, 2), (2, 3), (4, 1)\} \neq \phi$, so A and C are not mutually exclusive.

Algebra of Events

Let A, B and C be events associated with an experiment, whose sample space is S .

Work Book Exercise 21.1

- Two coins are tossed once, then the sample space is
 a $\{HH, HT, TH, TT\}$ b $\{TT, HH\}$
 c $\{HT, TH, HH\}$ d None of these
- A coin is tossed twice, if the second throw result in a tail, a die is thrown. Then, the total number of the outcomes of this experiment is
 a 11 b 13 c 14 d 16
- Event can be classified into various types on the basis of the
 a experiment
 b sample space
 c elements
 d None of the above
- An experiment involves rolling a pair of dice and recording the numbers that come up. Describe the following events.

(i) **Complementary event** For every event A , there corresponds another event A' or A^C or $\bar{A}$ called the complementary event to A . It is also called the event not A .

e.g. Let $S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

be a sample space

and $A = \{HTH, HHT, THH\}$ be a subset of S , then complementary of A is defined as

$\bar{A} = \{HHH, HTT, THT, TTH, TTT\}$

(ii) **Event A or B** It is the set of all those elements, which are either in A or B or in both. And it is denoted by the symbol $A \cup B$.

e.g. Let $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then

$A \cup B = \{1, 2, 3, 4\}$

(iii) **Events A and B** It is the set of all those elements, which are common in A and B . And it is denoted by the symbol $A \cap B$.

e.g. Let $A = \{2, 3, 5\}$ and $B = \{1, 2, 3, 4, 5\}$, then

$A \cap B = \{2, 3, 5\}$

(iv) **Event A but not B** It is the set of all those elements, which are in A but not in B . And it is denoted by the symbol $A - B$ or $A \cap B'$.

e.g. Let $A = \{1, 2, 3, 4\}$ and $B = \{2, 4, 6\}$, then

$A - B = \{1, 3\}$

➤ **Example 4.** Let $S = \{1, 2, 3, 4, 5, 6\}$ and

$E = \{1, 3, 5\}$, then E' is

(a) $\{2, 4\}$

(b) $\{3, 6\}$

(c) $\{1, 2, 4\}$

(d) $\{2, 4, 6\}$

Sol. (d) We have, $S = \{1, 2, 3, 4, 5, 6\}$

and $E = \{1, 3, 5\}$

$E' = S - E = \{2, 4, 6\}$

A : The sum is greater than 8.

B : 2 occurs on either die.

C : The sum is atleast 7 and a multiple of 3.

Which pair of these events is mutually exclusive?

a A and B

b B and C

c Both A and B

d None of these

5 A pair of dice is rolled, if the outcomes is a doublet, a coin is tossed. Then, the total number of outcomes for this experiment is

a 40

b 41

c 42

d 43

6 An experiment consists of rolling a die until a 2 appears. Then, number of elements of the sample space correspond to the event that the 2 appears not later than the K th roll of the die is

a $\frac{5^K}{4}$

b 5^{K-1}

c $\frac{5^K - 1}{4}$

d $\frac{5^K - 1}{5}$

Probability

Classical Approach of Probability

If there are n elementary equally likely events associated with a random experiment and m of them are favourable to an event A , then the probability of happening or occurrence of A is denoted by $P(A)$ and is defined as the ratio $\frac{m}{n}$.

$$\text{Thus, } P(A) = \frac{m}{n}$$

Clearly, $0 \leq m \leq n$. Therefore,

$$0 \leq \frac{m}{n} \leq 1$$

$$\Rightarrow 0 \leq P(A) \leq 1$$

Axiomatic Approach of Probability

Let S be the sample space of a random experiment. The probability P is a real valued function whose domain is the power set of S and range is the interval $[0, 1]$ satisfying the following axioms:

- (i) For any event E , $P(E) \geq 0$
- (ii) $P(S) = 1$
- (iii) If E and F are mutually exclusive events, then

$$P(E \cup F) = P(E) + P(F)$$

It follows from axiom (iii) that $P(\phi) = 0$

Let S be a sample space containing outcomes $\alpha_1, \alpha_2, \dots, \alpha_n$ i.e. $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

It follows from the axiomatic definition of probability that

- (i) $0 \leq P(\alpha_i) \leq 1$, for each $\alpha_i \in S$
- (ii) $P(\alpha_1) + P(\alpha_2) + \dots + P(\alpha_n) = 1$
- (iii) $P(A) = \sum P(\alpha_i)$, for any event A containing elementary events w_i .

Probabilities of Equally Likely Outcomes

Let a sample space of an experiment be $S = \{w_1, w_2, \dots, w_n\}$ and suppose that all the outcomes are equally likely to occur i.e. the chance of occurrence of each simple event must be the same.

$$\text{i.e. } P(w_i) = p, \text{ for all } w_i \in S, \text{ where } 0 \leq p \leq 1$$

$$\text{Since, } \sum_{i=1}^n P(w_i) = 1$$

$$\text{i.e. } p + p + p + \dots + p \text{ (n times)} = 1$$

$$\Rightarrow np = 1 \text{ i.e. } p = \frac{1}{n}$$

Let S be the sample space and A be an event, such that $n(S) = n$ and $n(A) = m$. If each outcomes is equally likely, then it follows that

$$P(A) = \frac{m}{n} = \frac{\text{Number of outcomes favourable to } A}{\text{Total number of possible outcomes}}$$

- If the outcomes are equally likely, then classical approach and axiomatic approach are equivalent.
- If $P(A) = 1$, then A is called certain event and A is called an impossible event, if $P(A) = 0$.
- The number of elementary event which will ensure the non-occurrence of A i.e. which ensure the occurrence of $\bar{A}$ is $(n - m)$. Therefore,

$$P(\bar{A}) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A) \Rightarrow P(A) + P(\bar{A}) = 1$$
- The odds in favour of occurrence of the event A are defined by $m : (n - m)$ i.e. $P(A) : P(\bar{A})$ and the odds against the occurrence of A are defined by $n - m : m$ i.e. $P(\bar{A}) : P(A)$.

Example 5. There are four men and six women in the city council. If one council member is selected for a committee at random, then how likely is it that it is a woman?

- (a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{1}{7}$ (d) $\frac{1}{5}$

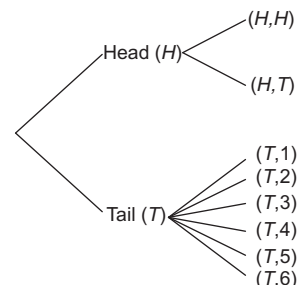
Sol. (b) Let E be the event that woman is selected.

$$\begin{aligned} \therefore n(E) &= 6 \\ \text{Number of sample space } n(S) &= \text{Total number of persons in the council} \\ &= 4 + 6 = 10 \\ \therefore P &= \frac{n(E)}{n(S)} = \frac{6}{10} = \frac{3}{5} \end{aligned}$$

Example 6. Consider the experiment of tossing a coin. If the coin shows head, toss it again but if it shows tail, then throw a die. Now, the probability of getting atleast one tail is

- (a) $\frac{1}{4}$ (b) $\frac{2}{4}$
 (c) $\frac{3}{4}$ (d) None of these

Sol. (c) The outcomes of the experiment can be represented in following diagrammatic manner called the tree diagram.

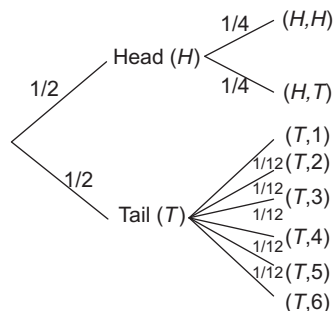


The sample space of the experiment may be described as

$S = \{(H, H), (H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$
where, (H, H) denotes that both the tosses result into head and (T, i) denotes the first toss result into a tail and the number i appeared on the die for $i = 1, 2, 3, 4, 5, 6$.

Thus, the probabilities assigned to the 8 elementary events

$(H, H), (H, H), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)$ are $\frac{1}{4}, \frac{1}{4}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{1}{12}$, respectively which is clear from the



Now, let F be an event of getting atleast one tail.

Then, $F = \{(H, T), (T, 1), (T, 2), (T, 3), (T, 4), (T, 5), (T, 6)\}$

$$\begin{aligned} \therefore P(F) &= P(H, T) + P(T, 1) + P(T, 2) + P(T, 3) + P(T, 4) + P(T, 5) + P(T, 6) \\ &= \frac{1}{4} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \\ &= \frac{1}{4} + \frac{1}{2} = \frac{3}{4} \end{aligned}$$

➤ **Example 7.** What is the probability that a number selected from the numbers 1, 2, 3, 4, ..., 25, is prime number, when each of the given numbers is equally likely to be selected?

- (a) $\frac{12}{25}$ (b) $\frac{11}{25}$ (c) $\frac{10}{25}$ (d) $\frac{9}{25}$

Sol. (d) Let S be the sample space associated with the given experiment and A be the event 'selecting a prime number'. Then,

$$S = \{1, 2, 3, 4, 5, \dots, 25\}$$

$$A = \{2, 3, 5, 7, 11, 13, 17, 19, 23\}$$

$\therefore$ Total number of outcomes = 25

Favourable number of outcomes = 9

$\therefore$ Probability of an event

$$= \frac{\text{Favourable number of outcomes}}{\text{Total number of outcomes}}$$

Hence, required probability = $\frac{9}{25}$

➤ **Example 8.** In a lottery, a person chooses six different natural numbers at random from 1 to 20 and if these six numbers match with the six numbers already fixed by the lottery committee, he wins the prize, then the probability of winning the prize is

- (a) $\frac{6}{38760}$ (b) $\frac{1}{20}$ (c) $\frac{6}{20}$ (d) $\frac{1}{38760}$

Sol. (d) Out of 20 numbers, six numbers can be chosen in ${}^{20}C_6$ ways.

$\therefore$ Total number of outcomes = ${}^{20}C_6 = 38760$

It is given that a person wins the prize, if six selected numbers match with the six numbers already fixed

$\therefore$ Favourable number of outcomes = 1

Hence, required probability = $\frac{1}{{}^{20}C_6} = \frac{1}{38760}$

➤ **Example 9.** 4 five rupee coins, 3 two rupee coins and 2 one rupee coins are stacked together in a column at random. The probability that the coins of the same denomination are consecutive, is

- (a) $\frac{13}{9!}$ (b) $\frac{1}{210}$
(c) $\frac{1}{35}$ (d) None of these

Sol. (b) $n(S) = \frac{9!}{4!3!2!}$ and $n(E) = 3!$

$$\therefore P(E) = \frac{(3!)(4!)(3!)(2!)}{9!} = \frac{1}{210}$$

➤ **Example 10.** In a convex hexagon, two diagonals are drawn at random. The probability that the diagonals intersect at an interior point of the hexagon, is

- (a) $\frac{5}{12}$ (b) $\frac{7}{12}$
(c) $\frac{2}{5}$ (d) None of these

Sol. (a) $n(S)$ = Total number of selections of two diagonals

$$= {}^9C_2 = \frac{9 \times 8}{2} = 36$$

$n(E)$ = Number of selections of two diagonals which intersect at an interior point

= Total number of selections of four vertices

$$= {}^6C_4 = 15$$

$\therefore$ The required probability = $\frac{15}{36} = \frac{5}{12}$

➤ **Example 11.** Three numbers are chosen at random from numbers 1 to 30. The probability that the minimum of the chosen numbers is 9 and maximum is 25, is

- (a) $\frac{1}{406}$ (b) $\frac{1}{812}$
(c) $\frac{3}{812}$ (d) None of these

Sol. (c) Out of first 30 natural numbers, three natural numbers can be chosen in ${}^{30}C_3$ ways. If the minimum and maximum of the numbers chosen are 9 and 25 respectively, then we have to select a number from the remaining 15 numbers i.e. 10, 11, ..., 24.

$\therefore$ Favourable number of elementary events = ${}^{15}C_1$

Hence, required probability = $\frac{{}^{15}C_1}{{}^{30}C_3} = \frac{3}{812}$

➤ **Example 12.** Three different numbers are selected at random from the set $A = \{1, 2, \dots, 10\}$. The probability that the product of two of the numbers is equal to third, is

- (a) $\frac{3}{4}$ (b) $\frac{1}{40}$
(c) $\frac{1}{8}$ (d) $\frac{39}{40}$

Sol. (b) Out of 10 numbers, three numbers can be chosen in ${}^{10}C_3$ ways.

∴ Total number of elementary events = ${}^{10}C_3$

The product of two numbers, out of the three chosen numbers, will be equal to the third number, if the numbers are chosen in one of the following ways:

(2, 3, 6), (2, 4, 8), (2, 5, 10)

∴ Favourable number of elementary events = 3

Hence, required probability = $\frac{3}{{}^{10}C_3} = \frac{1}{40}$

➤ **Example 13.** The probability that out of 10 persons, all born in June, atleast two have the same birthday, is

- (a) $\frac{{}^{30}C_{10}}{(30)^{10}}$ (b) $\frac{{}^{30}C_{10}}{30!}$
(c) $\frac{30^{10} - {}^{30}C_{10}}{(30)^{10}}$ (d) None of these

Sol. (c) Since, each person can have anyone of the thirty days of June month as his (her) birthday. Therefore,

Number of ways in which 10 persons can have birthdays in the month of June
= $30 \times 30 \times 30 \times \dots \times 30$ (10 times) = 30^{10}

∴ Required probability

= $1 - \text{Probability that no two persons have the same birthday}$

$$= 1 - \frac{{}^{30}C_{10}}{30^{10}} = \frac{30^{10} - {}^{30}C_{10}}{30^{10}}$$

➤ **Example 14.** The probability that when 12 balls are distributed among three boxes, the first will contain three balls, is

- (a) $\frac{2^9}{3^{12}}$ (b) $\frac{{}^{12}C_3 \times 2^9}{3^{12}}$
(c) $\frac{{}^{12}C_3 \times 2^{12}}{3^{12}}$ (d) $\frac{{}^{12}C_3}{12^3}$

Sol. (b) Since, each ball can be put into anyone of the three boxes. So, the total number of ways in which 12 balls can be put into three boxes is 3^{12} .

Out of 12 balls, 3 balls can be chosen in ${}^{12}C_3$ ways.

Now, remaining 9 balls can be put in the remaining 2 boxes in 2^9 ways. So, the total number of ways in which 3 balls are put in the first box and the remaining in other two boxes is ${}^{12}C_3 \times 2^9$.

Hence, required probability = $\frac{{}^{12}C_3 \times 2^9}{3^{12}}$

➤ **Example 15.** An elevator starts with m passengers and stops at n floor ($m \leq n$). The probability that no two passengers alight at the same floor, is

- (a) $\frac{{}^nP_m}{m^n}$ (b) $\frac{{}^nP_m}{n^m}$ (c) $\frac{{}^nC_m}{m^n}$ (d) $\frac{{}^nC_m}{n^m}$

Sol. (b) Since, a person can alight at anyone of n floors.

Therefore, the number of ways in which m passengers can alight at n floors is $\underbrace{n \times n \times n \times \dots \times n}_{m \text{ times}} = n^m$.

The number of ways in which all passengers can alight at different floors is ${}^nC_m \times m! = {}^nP_m$.

Hence, required probability = $\frac{{}^nP_m}{n^m}$

➤ **Example 16.** A and B play a game, where each is asked to select a number from 1 to 25. If the two numbers match, both of them win a prize. The probability that they will not win a prize in a single trial, is

- (a) $\frac{1}{25}$ (b) $\frac{24}{25}$
(c) $\frac{2}{25}$ (d) None of these

Sol. (b) The number of ways in which either player can choose a number from 1 to 25 is 25, so the total number of ways of choosing number is $25 \times 25 = 625$. There are 25 ways in which the numbers chosen by both players is the same. Therefore, the probability they will win a prize in a single trial, is $\frac{25}{625} = \frac{1}{25}$.

Hence, the probability that they will not win a prize in a single trial = $1 - \frac{1}{25} = \frac{24}{25}$

➤ **Example 17.** Two integers x and y are chosen with replacement out of the set $\{0, 1, 2, 3, \dots, 10\}$. Then, the probability that $|x - y| > 5$, is

- (a) $\frac{81}{121}$ (b) $\frac{30}{121}$ (c) $\frac{25}{121}$ (d) $\frac{20}{121}$

Sol. (b) Since, x and y each can take values from 0 to 10. So, the total number of ways of selecting x and y is $11 \times 11 = 121$.

Now, $|x - y| > 5 \Rightarrow x - y < -5$ or $x - y > 5$

There are 30 pairs of values of x and y satisfying these two inequalities. So, favourable number of ways = 30

Hence, required probability = $\frac{30}{121}$

➤ **Example 18.** Three persons A, B and C are to speak at a function along with five others. If they all speak in random order, then the probability that A speaks before B and B speaks before C, is

- (a) $\frac{3}{8}$ (b) $\frac{1}{6}$
(c) $\frac{3}{5}$ (d) None of these

Sol. (b) The total number of ways in which 8 persons can speak is ${}^8P_8 = 8!$. The number of ways in which A, B and C can be arranged in the specified speaking order is ${}^8C_3$. There are 5! ways in which the other five can speak. So, favourable number of ways is ${}^8C_3 \times 5!$.

$$\text{Hence, required probability} = \frac{{}^8C_3 \times 5!}{8!} = \frac{1}{6}$$

➤ **Example 19.** A car is parked by an owner amongst 25 cars in a row, not at either end. On his return he finds that exactly 15 places are still occupied. The probability that both the neighbouring places are empty, is

- (a) $\frac{91}{276}$ (b) $\frac{15}{184}$
(c) $\frac{15}{92}$ (d) None of these

Sol. (c) It is given that 15 places are occupied. This includes the owner's car also and hence 14 other cars are parked. There are 24 places (excluding places at the two end) out of which 14 places can be chosen in ${}^{24}C_{14}$ ways. Excluding the neighbouring places there are 22 places in which 14 cars can be parked in ${}^{22}C_{14}$ ways.

$$\text{Hence, required probability} = \frac{{}^{22}C_{14}}{{}^{24}C_{14}} = \frac{15}{92}$$

➤ **Example 20.** $x_1, x_2, x_3, \dots, x_{50}$ are fifty real numbers such that $x_r < x_{r+1}$ for $r = 1, 2, 3, \dots, 49$. Five numbers out of these are picked up at random. The probability that the five numbers have x_{20} as the middle number, is

- (a) $\frac{{}^{20}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (b) $\frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$
(c) $\frac{{}^{19}C_2 \times {}^{30}C_2}{{}^{50}C_5}$ (d) None of these

Sol. (b) Here, $n(S) = {}^{50}C_5$, $n(E) = {}^{30}C_2 \times {}^{19}C_2$

$$\therefore P(E) = \frac{{}^{30}C_2 \times {}^{19}C_2}{{}^{50}C_5}$$

➤ **Example 21.** Three dice are thrown simultaneously. The probability of getting a sum of 15 is

- (a) $\frac{1}{72}$ (b) $\frac{5}{36}$
(c) $\frac{5}{72}$ (d) None of these

Sol. (d) $n(S) = 6 \times 6 \times 6$

$n(E)$ = The number of solutions of $x + y + z = 15$ where, $1 \leq x \leq 6, 1 \leq y \leq 6, 1 \leq z \leq 6$
= Coefficient of x^{15} in $(x + x^2 + \dots + x^6)^3$
= Coefficient of x^{12} in $(1 + x + \dots + x^5)^3$

$$\begin{aligned} &= \text{Coefficient of } x^{12} \text{ in } (1 - 3 \cdot x^6 + 3 \cdot x^{12} - x^{18}) \cdot (1 - x)^{-3} \\ &= \text{Coefficient of } x^{12} \text{ in } (1 - 3x^6 + 3x^{12} - x^{18})({}^2C_0 + {}^3C_1x + {}^4C_2x^2 + \dots) \\ &= {}^{14}C_{12} - 3 \times {}^8C_6 + 3 \times {}^2C_0 = 10 \\ \therefore P(E) &= \frac{10}{6 \times 6 \times 6} \\ &= \frac{5}{108} \end{aligned}$$

➤ **Example 22.** The probability of getting a sum of 12 in four throws of an ordinary dice is

- (a) $\frac{1}{6} \left(\frac{5}{6}\right)^3$ (b) $\left(\frac{5}{6}\right)^4$
(c) $\frac{1}{36} \left(\frac{5}{6}\right)^2$ (d) None of these

Sol. (a) $n(S) = 6 \times 6 \times 6 \times 6$

$n(E)$ = The number of integral solutions of

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 &= 12, \text{ where } 1 \leq x_1 \leq 6, \dots, 1 \leq x_4 \leq 6 \\ &= \text{Coefficient of } x^{12} \text{ in } (x + x^2 + \dots + x^6)^4 \\ &= \text{Coefficient of } x^8 \text{ in } \left(\frac{1 - x^6}{1 - x}\right)^4 \\ &= \text{Coefficient of } x^8 \text{ in } (1 - x^6)^4 \cdot ({}^3C_0 + {}^4C_1x + {}^5C_2x^2 + \dots) \\ &= {}^{11}C_8 - 4 \cdot {}^5C_2 = 125 \\ \therefore P(E) &= \frac{125}{6 \times 6 \times 6 \times 6} \end{aligned}$$

➤ **Example 23.** Three six faced dice are thrown together. The probability that the sum of the numbers appearing on the dice is k ($9 \leq k \leq 14$), is

- (a) $\frac{21k - k^2 - 83}{216}$
(b) $\frac{k^2 - 3k + 2}{432}$
(c) $\frac{21k - k^2 - 83}{432}$
(d) None of the above

Sol. (a) Favourable number of elementary events

$$\begin{aligned} &= \text{Coefficient of } x^{k-3} \text{ in } (1 - x^6)^3(1 - x)^{-3} \\ &= \text{Coefficient of } x^{k-3} \text{ in } (1 - {}^3C_1x^6 + \dots)(1 - x)^{-3} \\ &\quad [\because 9 \leq k \leq 14 \therefore 6 \leq k - 3 \leq 11] \\ &= \text{Coefficient of } x^{k-3} \text{ in } (1 - x)^{-3} - {}^3C_1 \\ &\quad [\text{Coefficient of } x^{k-9} \text{ in } (1 - x)^{-3}] \\ &= {}^{k-3+3-1}C_{3-1} - {}^3C_1 \times {}^{k-9+3-1}C_{3-1} \\ &= {}^{k-1}C_2 - 3 \cdot {}^{k-7}C_2 \\ &= \frac{(k-1)(k-2)}{2} - 3 \cdot \frac{(k-7)(k-8)}{2} = 21k - k^2 - 83 \end{aligned}$$

The total number of elementary events associated to the random experiment of throwing three dice is $6 \times 6 \times 6 = 6^3$.

Hence, the probability of the required event

$$= \frac{21k - k^2 - 83}{6^3}$$

21

Geometrical Probability

We observe that the definition of the probability of occurrence of an event fails, if the total number outcomes (elementary events) of a trial in a random experiment is infinite. For example, if it is asked to find the probability that a point selected at random in a given region will lie in a specified part of that region, then the definition of the probability, which is discussed earlier is unable to answer this. In such cases, the definition of probability is modified and extended to what is called probability or probability in continuum. In such cases, the general expression for the probability p of occurrence of an event is given by

$$p = \frac{\text{Measure of the specified part of the region}}{\text{Measure of the whole region}}$$

where, 'measure' means length, area or volume of the region, if we are dealing with one, two or three dimensional space respectively.

➤ **Example 24.** If $a \in [-20, 0]$ then the probability that the equation $16x^2 + 8(a+5)x - 7a - 5 = 0$ has imaginary roots, is

- (a) $\frac{17}{20}$ (b) $\frac{13}{20}$ (c) $\frac{7}{20}$ (d) $\frac{3}{20}$

Sol. (b) If the equation $16x^2 + 8(a+5)x - 7a - 5 = 0$ has imaginary roots, then Discriminant < 0

$$\begin{aligned} \Rightarrow 64(a+5)^2 - 64(7a+5) &< 0 \\ \Rightarrow a^2 + 17a + 30 < 0 &\Rightarrow (a+15)(a+2) < 0 \\ \Rightarrow -15 < a < -2 \end{aligned}$$

$$\text{Hence, the required probability} = \frac{\int_{-15}^{-2} dx}{\int_{-20}^0 dx} = \frac{13}{20}$$

➤ **Example 25.** Two points are taken at random on the given straight line segment of length a . The probability for the distance between them to exceed a given length c , where $0 < c < a$, is

- (a) $\left(1 - \frac{c}{a}\right)^2$ (b) $\left(1 - \frac{a}{c}\right)^2$ (c) $1 - \frac{c^2}{a^2}$ (d) $1 - \frac{a^2}{c^2}$

Sol. (a) Let AB be a straight line segment of length a and let P and Q be two points on it such that $AP = x$ and $AQ = y$. Then, we have to find the probability for $|PQ| > c$ i.e. $|x - y| > c$.

$$\text{Clearly, } 0 \leq x \leq a \text{ and } 0 \leq y \leq a$$

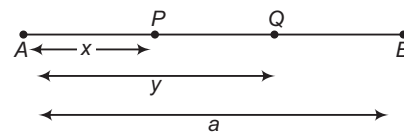
We observe that there are two variables x and y satisfying $0 \leq x \leq a$ and $0 \leq y \leq a$. In the cartesian plane the inequalities $0 \leq x \leq a$ and $0 \leq y \leq a$ represent the region enclosed by the square having $x = 0$, $y = 0$, $x = a$ and $y = a$ as its sides. Clearly, area enclosed by this square is a^2 .

$$\text{Now, } |x - y| > c$$

$$\Rightarrow x - y > c \text{ and } -(x - y) > c$$

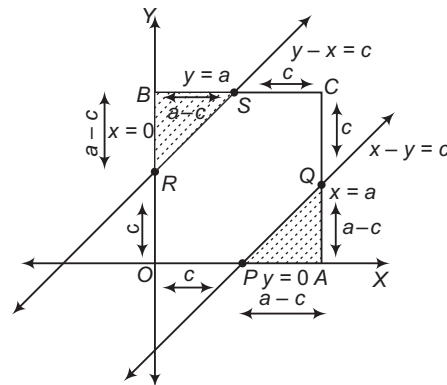
$$\Rightarrow x - y > c \text{ and } y - x > c$$

In order to find $P(|x - y| > c)$, we require the area of the region enclosed by $x - y > c$, $y - x > c$, $0 \leq x \leq a$ and $0 \leq y \leq a$. Clearly, shaded region in figure represents the area enclosed.



We have,

$$\begin{aligned} \text{Area of the shaded region} &= \text{Area APQ} + \text{Area BRS} \\ &= \frac{1}{2}(a-c)^2 + \frac{1}{2}(a-c)^2 = (a-c)^2 \end{aligned}$$



Hence, the required probability

$$\begin{aligned} &= \frac{\text{Area enclosed by } |x - y| > c, 0 \leq x \leq a \text{ and } 0 \leq y \leq a}{\text{Area enclosed by } 0 \leq x \leq a, 0 \leq y \leq a} \\ &= \frac{\text{Area of the shaded region}}{\text{Area of the square}} = \frac{(a-c)^2}{a^2} = \left(1 - \frac{c}{a}\right)^2 \end{aligned}$$

Work Book Exercise 21.2

1 In a single cast with two dice, the odds against drawing 7 is

- a 1 : 6 b 1 : 12 c 5 : 1 d 1 : 3

2 If the letters of the word ATTEMPT are written down at random, then the chance that all T's are consecutive, is

- a $\frac{1}{42}$ b $\frac{6}{7}$
c $\frac{1}{7}$ d None of these

3 Let $x = 33^n$. The index n is given a positive integral value at random. The probability that the value of x will have 3 at the unit's place, is

- a $\frac{1}{4}$
b $\frac{1}{2}$
c $\frac{1}{3}$
d None of the above

- 4 10 apples are distributed at random among 6 persons. The probability that atleast one of them will receive none, is

a $\frac{6}{143}$ b $\frac{{}^{14}C_4}{{}^{15}C_5}$
c $\frac{137}{143}$ d None of these

- 5 A point is selected at random from the interior of a circle. The probability that the point is close to the centre than the boundary of the circle, is

a $\frac{3}{4}$ b $\frac{1}{2}$
c $\frac{1}{4}$ d None of these

- 6 Three dice are thrown. The probability of getting a sum which is a perfect square, is

a $\frac{2}{5}$ b $\frac{9}{20}$
c $\frac{1}{4}$ d None of these

- 7 Three six-faced dice are thrown together. The probability that the sum of the numbers appearing on the dice is $k(3 \leq k \leq 8)$, is

a $\frac{k^2}{432}$ b $\frac{k(k-1)}{432}$
c $\frac{(k-1)(k-2)}{432}$ d $\frac{k(k-1)(k-2)}{432}$

- 8 An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of n numbers showing up is noted. Then, the probability that among the numbers 1, 2, 3, 4, 5, 6 only three numbers appear in this list, is

a $\frac{{}^6C_3\{3^n - 3(2^n) + 3\}}{6^n}$
b $\frac{{}^6C_3\{3^n + 3(2^n) + 3\}}{6^n}$
c $\frac{{}^6C_3\{3^n - 3(2^n) - 3\}}{6^n}$
d None of the above

Addition Theorem of Probability

- i. If A and B are any two events in a sample space S , then the probability of occurrence of atleast one of the events A and B is given by
- $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- If A and B are mutually exclusive events, then $A \cap B = \phi$ and hence, $P(A \cap B) = 0$
 $\therefore P(A \cup B) = P(A) + P(B)$
 • Two events A and B are mutually exclusive iff $P(A \cup B) = P(A) + P(B)$
 • $1 = P(S) = P(A \cup A') = P(A) + P(A')$ [$\because A \cap A' = \phi$]
 or $P(A') = 1 - P(A)$

- ii. If A, B and C are any three events in a sample space S , then
- $$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C)$$

- If A, B and C are mutually exclusive events, then $A \cap B = \phi$, $B \cap C = \phi$, $A \cap C = \phi$, $A \cap B \cap C = \phi$
 $\therefore P(A \cup B \cup C) = P(A) + P(B) + P(C)$
 • If A and B are any two events, then $(A - B) \cap (A \cap B) = \phi$ and $A = (A - B) \cup (A \cap B)$
 $\therefore P(A) = P(A - B) + P(A \cap B) = P(A \cap B') + P(A \cap B)$
 or $P(A) - P(A \cap B) = P(A - B) = P(A \cap B')$ [$\because A - B = A \cap B'$]

Similarly, $P(B) - P(A \cap B) = P(B - A) = P(B \cap A')$

- iii. **General form of addition theorem of probability** If $A_1, A_2, \dots, A_n$ are n sample space, then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{i=1}^n P(A_i) - \sum_{i < j} P(A_i \cap A_j) + \sum_{i < j < k} P(A_i \cap A_j \cap A_k) - \dots + (-1)^{n-1} P(A_1 \cap A_2 \cap \dots \cap A_n)$$

- For any number of (finite or infinite) mutually exclusive events, $P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n)$

- iv. Let A and B be two events associated to a random experiment. Then,
 (a) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$
 (b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$
 (c) $P[(A \cap \bar{B}) \cup (\bar{A} \cap B)] = P(A) + P(B) - 2P(A \cap B)$

- v. For any two events A and B
 $P(A \cap B) \leq P(A) \leq P(A \cup B) \leq P(A) + P(B)$

- vi. For any two events A and B , the probability that exactly one of A, B occurs is given by
 $P(A) + P(B) - 2P(A \cap B) = P(A \cup B) - P(A \cap B)$

- vii. If A, B and C are three events, then
 (a) P (atleast two of A, B and C occur)
 $= P(A \cap B) + P(B \cap C) + P(C \cap A) - 2P(A \cap B \cap C)$
 (b) P (exactly two of A, B and C occur)
 $= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)$
 (c) P (exactly one of A, B and C occurs)
 $= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) - 2P(A \cap C) + 3P(A \cap B \cap C)$

Independent Events

Two events A and B associated to a random experiment are independent, if the probability of occurrence or non-occurrence of A is not affected by the occurrence or non-occurrence of B .

If A and B are independent events, then

$$P(A \cap B) = P(A) \times P(B)$$

Boole's Inequality

If events $E_1, E_2, \dots, E_n$ are associated with a random experiment, then

$$(a) P\left(\bigcap_{i=1}^n E_i\right) \geq \sum_{i=1}^n P(E_i) - (n-1)$$

$$(b) P\left(\bigcup_{i=1}^n E_i\right) \leq \sum_{i=1}^n P(E_i)$$

Example 26. In an entrance test that is graded on the basis of two examinations, the probability of a randomly chosen student passing the first examination is 0.8 and the probability of passing the second examination is 0.7. The probability of passing atleast one of them is 0.95. What is the probability of passing both?

- (a) 0.55 (b) 0.58 (c) 0.59 (d) 0.60

Sol. (a) Let A and B denotes the students passing the examination first and second respectively.

$$\text{Here, } P(A) = 0.8, P(B) = 0.7$$

$$P(\text{atleast one of } A \text{ and } B) = P(A \cup B) = 0.95$$

$$\text{Now, } P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore 0.95 = 0.8 + 0.7 - P(A \cap B)$$

$$\Rightarrow 0.95 = 1.5 - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = 1.5 - 0.95 = 0.55$$

Example 27. In a class of 60 students, 30 opted for NCC, 32 opted for NSS and 24 opted for both NCC and NSS. Following statements are given below :

I. The probability that the student opted for NCC or NSS is $\frac{19}{30}$.

II. The probability that the student has opted for neither NCC nor NSS is $\frac{11}{30}$.

Choose the correct statements.

- (a) Only I (b) Only II
(c) Both I and II (d) None of these

Sol. (c) I. Let A and B denote the students in NCC and NSS, respectively.

$$\text{Here, } n(A) = 30, n(B) = 32, n(A \cap B) = 24$$

$$\left[\begin{array}{l} \because 24 \text{ students opted for both NCC and} \\ \text{NSS i.e. they are common in both} \end{array} \right]$$

$$\therefore P(A) = \frac{30}{60}$$

$$P(B) = \frac{32}{60}$$

$$\text{and } P(A \cap B) = \frac{24}{60}$$

P (student opted for NCC or NSS)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$[\because P(A \cup B) = P(A \text{ or } B)]$$

$$= \frac{30}{60} + \frac{32}{60} - \frac{24}{60}$$

$$= \frac{30 + 32 - 24}{60}$$

$$= \frac{62 - 24}{60}$$

$$= \frac{38}{60} = \frac{19}{30}$$

II. P (student has opted neither NCC nor NSS)

$$= 1 - P(\text{student opted for NCC or NSS})$$

$$= 1 - \frac{19}{30} = \frac{30 - 19}{30} = \frac{11}{30}$$

Example 28. If A and B are two mutually exclusive events, then

$$(a) P(A) \leq P(\bar{B})$$

$$(b) P(A) > P(\bar{B})$$

$$(c) P(A) < P(B)$$

$$(d) \text{None of the above}$$

Sol. (a) Since, A and B are mutually exclusive events, therefore

$$A \cap B = \phi$$

$$\Rightarrow A \subseteq \bar{B} \text{ and } B \subseteq \bar{A}$$

$$\Rightarrow P(A) \leq P(\bar{B}) \text{ and } P(B) \leq P(\bar{A})$$

Example 29. If A and B are two events. The probability that atleast one of A, B occurs, is

$$(a) 1 - P(A \cap B)$$

$$(b) P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$$

$$(c) P(\bar{A}) + P(\bar{B}) + P(A \cup B) - 1$$

$$(d) P(A \cap \bar{B}) + P(\bar{A} \cap B) + P(\bar{A} \cap \bar{B})$$

Sol. (a,b,c,d) Required probability $= P(\bar{A} \cup \bar{B}) = P(\overline{A \cap B})$
 $= 1 - P(A \cap B)$

So, alternative (a) is correct

$$\text{Again, } P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$$

[by addition theorem]

So, alternative (b) is correct.

$$\text{Again, } P(\bar{A} \cup \bar{B}) = P(\bar{A}) + P(\bar{B}) - P(\bar{A} \cap \bar{B})$$

$$= P(\bar{A}) + P(\bar{B}) - P(\overline{A \cap B})$$

$$= P(\bar{A}) + P(\bar{B}) - \{1 - P(A \cup B)\}$$

$$= P(\bar{A}) + P(\bar{B}) + P(A \cup B) - 1$$

So, alternative (c) is correct.

$$\text{Finally, } P(\bar{A} \cup \bar{B}) = P[(A \cap \bar{B}) \cup (\bar{A} \cap B) \cup (\bar{A} \cap \bar{B})]$$

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B) + P(\bar{A} \cap \bar{B})$$

$$\left[\begin{array}{l} A \cap \bar{B}, \bar{A} \cap B \text{ and } \bar{A} \cap \bar{B} \text{ are} \\ \text{mutually exclusive events} \end{array} \right]$$

So, alternative (d) is correct.

➤ **Example 30.** The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\bar{A}) + P(\bar{B})$ is

- (a) 1.4 (b) 1.2
(c) 1.3 (d) 1.1

Sol. (b) We have,

$P(\text{at least one of the events } A \text{ and } B \text{ occurs}) = 0.6$

i.e. $P(A \cup B) = 0.6$

and $P(A \text{ and } B \text{ occur simultaneously}) = 0.2$

i.e. $P(A \cap B) = 0.2$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow 0.6 = P(A) + P(B) - 0.2$$

$$\Rightarrow 0.6 = 1 - P(\bar{A}) + 1 - P(\bar{B}) - 0.2$$

$$\Rightarrow 0.6 = 2 - 0.2 - [P(\bar{A}) + P(\bar{B})]$$

$$\Rightarrow 0.6 = 1.8 - [P(\bar{A}) + P(\bar{B})]$$

$$\Rightarrow P(\bar{A}) + P(\bar{B}) = 1.8 - 0.6 = 1.2$$

➤ **Example 31.** The probabilities that a student passes in Mathematics, Physics and Chemistry are m , p and c , respectively. Of these subjects, the student has 75% chance of passing in at least one, a 50% chance of passing in at least two and a 40% chance of passing in exactly two. Which of the following relation is true?

$$(a) p + m + c = \frac{19}{20} \quad (b) p + m + c = \frac{27}{20}$$

$$(c) pmc = \frac{1}{5} \quad (d) pmc = \frac{1}{4}$$

Sol. (b) We have, $P(A \cup B \cup C) = \frac{3}{4}$

$$\text{i.e. } P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C) = \frac{3}{4} \quad \dots(i)$$

$$P(A \cap B) + P(B \cap C) + P(A \cap C) - 2P(A \cap B \cap C) = \frac{1}{2} \quad \dots(ii)$$

$$\text{and } P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C) = \frac{2}{5} \quad \dots(iii)$$

From Eqs. (ii) and (iii), we get

$$P(A \cap B \cap C) = \frac{1}{2} - \frac{2}{5} = \frac{1}{10} \quad \dots(iv)$$

$$\Rightarrow P(A)P(B)P(C) = \frac{1}{10}$$

$$\Rightarrow pmc = \frac{1}{10}$$

From Eqs. (i), (ii) and (iii), we have

$$P(A) + P(B) + P(C) - \left(\frac{1}{2} + \frac{2}{10}\right) + \frac{1}{10} = \frac{3}{4}$$

$$\Rightarrow p + m + c = \frac{27}{20}$$

Work Book Exercise 21.3

1 Given, $P(A) = \frac{3}{5}$ and $P(B) = \frac{1}{5}$. Find $P(A \text{ or } B)$, if A and B are mutually exclusive events.

- a $\frac{2}{5}$ b $\frac{3}{5}$
c $\frac{4}{5}$ d $\frac{1}{5}$

2 If E and F are events such that $P(E) = \frac{1}{4}$, $P(F) = \frac{1}{2}$ and $P(E \text{ and } F) = \frac{1}{8}$. Find (i) $P(E \text{ or } F)$

(ii) $P(\text{not } E \text{ and not } F)$.

- a $\frac{1}{8}, \frac{7}{8}$
b $\frac{5}{8}, \frac{3}{8}$
c $\frac{1}{7}, \frac{6}{7}$

d None of the above

3 An experiment yields 3 mutually exclusive and exhaustive events A , B and C . If $P(A) = 2P(B) = 3P(C)$, then $P(A)$ is equal to

- a $\frac{1}{11}$
b $\frac{2}{11}$
c $\frac{3}{11}$
d $\frac{6}{11}$

4 A and B are two events, where $P(A) = 0.25$ and $P(B) = 0.5$. The probability of both happening together is 0.14. The probability of both A and B not happening is

- a 0.39
b 0.25
c 0.11
d None of the above

5 Let A and B be two independent events such that $P(A) = \frac{1}{5}$, $P(A \cup B) = \frac{7}{10}$. Then, $P(\bar{B})$ is equal to

- a $\frac{3}{8}$ b $\frac{2}{7}$
c $\frac{7}{9}$ d None of these

6 A , B and C are events such that $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(A \cap B) = 0.08$, $P(A \cap C) = 0.28$, $P(A \cap B \cap C) = 0.09$. If $P(A \cup B \cup C) \geq 0.75$, then $P(B \cap C)$ lies in the interval

- a (0.23, 0.48) b (0.2, 0.4)
c (0.25, 0.50) d None of these

7 A die is rolled so that the probability of face i is proportional to i , $i = 1, 2, \dots, 6$. The probability of an even number occurring when the die is rolled, is

- a $\frac{7}{4}$ b $\frac{4}{7}$
c $\frac{5}{7}$ d None of these

WorkedOut Examples

Type 1. Only One Correct Option

Ex 1. A coin is tossed. If it shows a tail, we draw a ball from a box which contains 2 red and 3 black balls. If it shows head, we throw a die. Then, the sample space for this experiment is

(a) $S = \{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}$

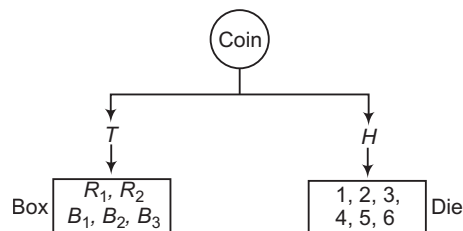
(b) $S = \{TR, TR_2, TB_1, H1, H4\}$

(c) Both (a) and (b)

(d) None of the above

Sol. Let H and T represent head and tail.

Again, let red ball represented by R_1, R_2 and black ball represented by B_1, B_2 and B_3 .



The sample space is

$S = \{TR_1, TR_2, TB_1, TB_2, TB_3, H1, H2, H3, H4, H5, H6\}$

Hence, (a) is the correct answer.

Ex 2. A die is rolled. Let E be the event 'die shows 4' and F be the event 'die shows even number'. Then, E and F are

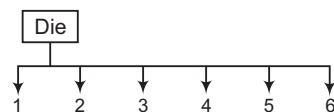
(a) mutually exclusive

(b) exhaustive

(c) mutually exclusive and exhaustive

(d) None of the above

Sol.



Let $E =$ The die shows 4 = $\{4\}$

Let $F =$ The die shows even number = $\{2, 4, 6\}$

$\therefore E \cap F = \{4\} \neq \emptyset$

So, E and F are not mutually exclusive.

Hence, (d) is the correct answer.

Ex 3. Let $P(x)$ denotes the probability of the occurrence of event x . Then, all those points $(x, y) = (P(A), P(B))$, in a plane which satisfy the conditions, $P(A \cup B) \geq \frac{3}{4}$ and

$\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$ implies

(a) $P(A) + P(B) < \frac{11}{8}$

(b) $P(A) + P(B) > \frac{11}{8}$

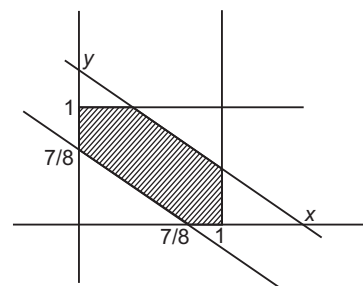
(c) $\frac{7}{8} \leq P(A) + P(B) \leq \frac{11}{8}$

(d) None of the above

Sol. $P(A \cup B) \geq \frac{3}{4}$ and $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$

$\Rightarrow P(A) + P(B) - P(A \cap B) \geq \frac{3}{4}$

$\Rightarrow P(A) + P(B) \geq \frac{3}{4} + P(A \cap B) = \frac{3}{4} + \frac{1}{8} = \frac{7}{8}$



We know that, $P(A \cup B) \leq 1$

$P(A) + P(B) \leq 1 + P(A \cap B) \leq 1 + \frac{3}{8} = \frac{11}{8}$

$\Rightarrow x + y \leq \frac{11}{8}$

$\Rightarrow \frac{7}{8} \leq x + y \leq \frac{11}{8}$

Also, $0 \leq P(A) \leq 1, 0 \leq P(B) \leq 1$

Hence, (c) is the correct answer.

Ex 4. Two dice are thrown simultaneously to get the coordinates of a point in XY -plane. Then, the probability that this point lies inside or on the region bounded by $|x| + |y| = 3$ is

(a) $\frac{3}{14}$

(b) $\frac{2}{3}$

(c) $\frac{1}{12}$

(d) $\frac{4}{14}$

Sol. Any point on or inside the region bounded by $|x| + |y| = 3$ will have x and y -coordinates lying between -3 and $+3$ which will turn up as a throw of dice i.e. $(1, 1), (1, 2), (2, 1)$.

$\therefore$ Probability = $\frac{3}{36} = \frac{1}{12}$

Hence, (c) is the correct answer.

Ex 5. In a game of tickets, 8 tickets numbered 1, 2, 3, ..., 8 are grouped in pairs at random and observed pairwise, the lower numbered tickets of pairs are kept and other are thrown and this process is repeated once more. Then, the probability that out of remaining two tickets, one is numbered 4, is

- (a) $\frac{2}{35}$ (b) $\frac{3}{35}$
(c) $\frac{4}{35}$ (d) $\frac{1}{7}$

Sol. Total number of ways in first process = $\frac{8!}{(2!)^4 (4!)}$ and in

second process = $\frac{4!}{(2!)^2 (2!)}$

So, total number of ways = $\frac{8! 4!}{(2!)^7 (4!)} = \frac{8!}{(2!)^7}$.

For favourable ways lets divide tickets into two groups 1, 2, 3, 4, 5, 6, 7, 8

Ticket of number 4 has to be selected so that its grouping must be with a higher number and there also should be an higher number of 4 at the end of first process so favourable numbers are ${}^4C_1 \cdot {}^3C_2 \frac{4!}{(2!)^2 (2!)}$

So, required probability

$$= \frac{4 \cdot 3 \cdot \frac{4!}{(2!)^2 (2!)}}{\frac{8!}{(2!)^7}} = \frac{4 \cdot 3 \cdot (2!)^4}{8 \cdot 7 \cdot 6 \cdot 5} = \frac{4}{35}$$

Hence, (c) is the correct answer.

Ex 6. There are four players A_1, A_2, A_3 and A_4 in pool A and four players B_1, B_2, B_3 and B_4 in pool B. They are arbitrarily paired in their respective pools to play against each other and one winner is decided in each pair to play in semi-final. It is known that whenever A_1 plays with A_2 , A_1 wins the game and whenever B_1 plays with B_2 , B_1 wins. Then, the probability that

- (i) A_1 and A_2 reach in the semi-final.
(ii) A_1, A_2, B_2 and B_3 reach semi-final.
(a) $\frac{1}{6}, \frac{1}{12}$ (b) $\frac{1}{6}, \frac{1}{6}$
(c) $\frac{1}{6}, \frac{1}{16}$ (d) None of these

Sol. In pool, a pairing may be done in following ways:

Way number 1 : $A_1 A_2$ and $A_3 A_4$

Way number 2 : $A_1 A_3$ and $A_2 A_4$

Way number 3 : $A_1 A_4$ and $A_2 A_3$

Outcomes of different ways are

W - 1	$A_1 A_3$	$A_1 A_4$		
W - 2	$A_1 A_2$	$A_1 A_4$	$A_2 A_3$	$A_3 A_4$
W - 3	$A_1 A_2$	$A_1 A_3$	$A_2 A_4$	$A_3 A_4$

(i) The probability that $A_1 A_2$ reach the semi-final

$$= \frac{1}{3} \left(0 + \frac{1}{4} + \frac{1}{4} \right) = \frac{1}{6}$$

(ii) The probability that A_1, A_2, B_2 and B_3 reach in the semi-final

$$= \frac{1}{6} \times \frac{1}{12} = \frac{1}{72}$$

[since, probability of $B_2 B_3$ is same]
[as that of A_2, A_3 which is $\frac{1}{12}$]

Hence, (a) is the correct answer.

Ex 7. A and B throw a dice each. The probability that A's throw is not greater than B's throw, is

- (a) $\frac{7}{12}$ (b) $\frac{5}{12}$ (c) $\frac{1}{6}$ (d) $\frac{1}{2}$

Sol. When A gets 1, B can get any of the numbers 1, 2, 3, 4, 5, 6.

When A gets 2, B can get any of the numbers 2, 3, 4, 5, 6

$\vdots \quad \vdots \quad \vdots \quad \vdots$

$\therefore$ Favourable number of ways = $6 + 5 + \dots + 1 = 21$

$\therefore$ Probability that A's throw is not greater than B's throw is $\frac{21}{36} = \frac{7}{12}$.

Hence, (a) is the correct answer.

Ex 8. The probability that $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$ is an integer, $x, y \in \{1, 2, 3, 4\}$, is

- (a) $\frac{1}{16}$ (b) $\frac{3}{16}$
(c) $\frac{15}{16}$ (d) None of these

Sol. For $\sin^{-1}(\sin x) + \cos^{-1}(\cos y)$ to be an integer x should lie between $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and y should lie between

$[0, \pi]$.

$\Rightarrow x = 1$ and $y = 1, 2, 3$

$\therefore$ Required probability = $\frac{3}{16}$

Hence, (b) is the correct answer.

Ex 9. Four cards are drawn from a pack of 52 playing cards. Then, the probability of drawing atleast one pair, is

- (a) $1 - \frac{4^4 \times {}^{13}C_4}{52 C_4}$ (b) $1 - \frac{{}^{13}C_4}{52 C_4}$
(c) $1 - \frac{4^4 \times 13!}{52!}$ (d) None of these

Sol. $n(S) = {}^{52}C_4$. There are exactly 13 types of cards,

$$\Rightarrow n(E) = {}^{52}C_4 - {}^{13}C_4 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1$$

$$\therefore P(E) = 1 - \frac{4^4 \times {}^{13}C_4}{52 C_4}$$

Hence, (a) is the correct answer.

Ex 10. If the sides of a triangle are decided by the throw of a single dice thrice, then the probability that triangle is of maximum area given that it is an isosceles triangle, is

- (a) $\frac{1}{7}$ (b) $\frac{1}{27}$
(c) $\frac{1}{14}$ (d) None of these

Sol. When the two equal sides are 1 each, third sides can only be 1.

When the two equal sides are 2 each, the third side can take values 1, 2 and 3.

When the two equal sides are 3 each, the third side can take values 1, 2, 3, 4 and 5.

When the two equal sides are 4 each, the third side can take values 1, 2, 3, 4, 5 and 6.

Same is the case when the two equal sides are 5 and 6.

$\therefore$ Total number of triangles possible

$$= 6 \times 3 + 5 + 3 + 1 = 27$$

$$\therefore \text{The required probability} = \frac{1}{27}$$

Hence, (b) is the correct answer.

Ex 11. There are 7 seats in a row. Three persons take seats in a row. Three persons take seats at random. The probability that the middle seat is always occupied and no two persons are consecutive, is

- (a) $\frac{9}{70}$ (b) $\frac{9}{35}$
(c) $\frac{4}{35}$ (d) None of these

$$\text{Sol. } n(S) = {}^7C_3 \times 3! = \frac{7 \cdot 6 \cdot 5}{6} \cdot 6 = 210$$

$$n(E) = {}^2C_1 \times {}^2C_1 \times {}^1C_1 \times 3!$$



Because one has to sit at any one of the two marked seats on the left and the other has to sit at anyone of the two marked seats on the right.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2 \times 2 \times 6}{210} = \frac{4}{35}$$

Hence, (c) is the correct answer.

Ex 12. Numbers 1, 2, 3, ..., 100 are written down on each of the cards A, B and C. One number is selected at random from each of the cards. The probability that the numbers so selected can be the measures (in cm) of three sides of right angled triangles no two of which are similar, is

- (a) $\frac{4}{100^3}$ (b) $\frac{3}{50^3}$
(c) $\frac{36}{100^3}$ (d) None of these

$$\text{Sol. } n(S) = 100 \times 100 \times 100$$

$$\text{We know that, } (2n+1)^2 + (2n^2+2n)^2 = (2n^2+2n+1)^2, \forall n \in N$$

$\therefore$ For $n = 1, 2, 3, 4, 5, 6$, we get lengths of the three sides of a right angled triangle whose longest side ≤ 100 .

e.g. When $n = 1$, sides are 3, 4, 5 and when $n = 2$, sides are 5, 12, 13 and so on.

The number of selections of 3, 4, 5 from the three cards by taking one from each is $3!$.

$$\therefore n(E) = 6(3!)$$

$$\text{Now, } P(E) = \frac{6(3!)}{100 \times 100 \times 100} = \frac{1}{100} \left(\frac{3}{50} \right)^2$$

Hence, (c) is the correct answer.

Ex 13. Three natural numbers are taken at random from the set $A = \{x | 1 \leq x \leq 100, x \in N\}$. The probability that the AM of the numbers taken is 25, is

- (a) $\frac{{}^{77}C_2}{{}^{100}C_3}$ (b) $\frac{{}^{25}C_2}{{}^{100}C_3}$
(c) $\frac{{}^{74}C_{72}}{{}^{100}C_{97}}$ (d) None of these

$$\text{Sol. } n(S) = {}^{100}C_3$$

As the AM of three numbers is 25, their sum = 75

$\therefore n(E) =$ The number of integral solutions of $x_1 + x_2 + x_3 = 75$, where $x_1 \geq 1, x_2 \geq 1, x_3 \geq 1$

$$= \text{Coefficient of } x^{75} \text{ in } (x + x^2 + x^3 + \dots)^3$$

$$= \text{Coefficient of } x^{72} \text{ in } \left(\frac{1}{1-x} \right)^3$$

$$= \text{Coefficient of } x^{72} \text{ in } (1-x)^{-3} = {}^{74}C_{72}$$

$$\therefore P(E) = \frac{{}^{74}C_{72}}{{}^{100}C_3} = \frac{{}^{74}C_{72}}{{}^{100}C_{97}}$$

Hence, (c) is the correct answer.

Ex 14. Let S be the universal set and $n(X) = k$. The probability of selecting two subsets A and B of the set X such that $B = \bar{A}$, is

- (a) $\frac{1}{2}$ (b) $\frac{1}{2^k - 1}$ (c) $\frac{1}{2^k}$ (d) $\frac{1}{3^k}$

Sol. The total number of subsets of X is 2^k .

$$\therefore n(S) = 2^k C_2$$

$n(E) =$ The number of selections of two non-intersecting subsets whose union is X

$$= \frac{1}{2} ({}^kC_0 + {}^kC_1 + {}^kC_2 + \dots)$$

[since, the number of selections in which one subset has r elements and the rest are in the other subset = ${}^kC_r$ and every selection appears twice in the total number of selections]

$$\therefore P(E) = \frac{(1/2) \cdot 2^k}{2^k C_2} = \frac{2^{k-1}}{2^k (2^k - 1)/2} = \frac{1}{2^k - 1}$$

Hence, (b) is the correct answer.

Ex 15. 10 different books and 2 different pens are given to 3 boys so that each gets equal number of things. The probability that the same boy does not receive both the pens, is

- (a) $\frac{5}{11}$ (b) $\frac{7}{11}$ (c) $\frac{2}{3}$ (d) $\frac{6}{11}$

Sol. $n(S) = {}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$n(E) = n(S) -$ The number of ways in which one boy gets both the pens

$$= n(S) - {}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)$$

$$\therefore P(E) = 1 - \frac{{}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)}{{}^{12}C_4 \times {}^8C_4 \times {}^4C_4}$$

$$= 1 - \frac{6}{11} = \frac{5}{11}$$

Hence, (a) is the correct answer.

Ex 16. Two distinct numbers are selected at random from the first twelve natural numbers. The probability that the sum will be divisible by 3, is

- (a) $\frac{1}{3}$ (b) $\frac{23}{66}$
(c) $\frac{1}{2}$ (d) None of these

Sol. Let $E_3 =$ The event of the sum being 3

Similarly, $E_6, E_9, E_{12}, E_{15}, E_{18}, E_{21}$.

$$n(S) = {}^{12}C_2$$

$$n(E_3) = 1, n(E_6) = 2, n(E_9) = 4, n(E_{12}) = 5,$$

$$n(E_{15}) = 5, n(E_{18}) = 3, n(E_{21}) = 2$$

$$\therefore P(E) = \frac{P(E_3) + \dots + P(E_{21})}{n(S)} = \frac{1 + 2 + 4 + 5 + 5 + 3 + 2}{12 \times 11} = \frac{22}{132} = \frac{1}{6}$$

Hence, (a) is the correct answer.

Aliter

The sum of the numbers for every selection is divisible by 3 or leaves the remainder 1 or leaves the remainder 2. These are equally probable. So, the required probability = $\frac{1}{3}$ because the sum of the three probabilities is 1.

Hence, (a) is the correct answer.

Ex 17. Three numbers are chosen at random without replacement from 1, 2, 3, ..., 10. The probability that the minimum of the chosen numbers is 4 or their maximum 8, is

- (a) $\frac{11}{40}$ (b) $\frac{3}{10}$
(c) $\frac{1}{40}$ (d) None of these

Sol. The probability of 4 being the minimum number = $\frac{{}^6C_2}{{}^{10}C_3}$

[because after selecting 4, any two can be selected from 5, 6, 7, 8, 9, 10]

The probability of 8 being maximum number = $\frac{{}^6C_2}{{}^{10}C_3}$

The probability of 4 being the minimum number and 8 being the maximum number = $\frac{3}{{}^{10}C_3}$

$\therefore$ The required probability

$$= P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{{}^6C_2}{{}^{10}C_3} + \frac{{}^6C_2}{{}^{10}C_3} - \frac{3}{{}^{10}C_3} = \frac{11}{40}$$

Hence, (a) is the correct answer.

Ex 18. Let A and B be two independent events such that their probabilities are $\frac{3}{10}$ and $\frac{2}{5}$. The probability of exactly one of the events happening, is

- (a) $\frac{23}{50}$ (b) $\frac{1}{2}$
(c) $\frac{31}{50}$ (d) None of these

Sol. The required probability

$$= P(\overline{A}B \cup A\overline{B}) = P(\overline{A})P(B) + P(A)P(\overline{B})$$

$$= \frac{3}{10} \left(1 - \frac{2}{5}\right) + \left(1 - \frac{3}{10}\right) \frac{2}{5} = \frac{23}{50}$$

Hence, (a) is the correct answer.

Ex 19. A, B and C are three events for which $P(A) = 0.6, P(B) = 0.4$ and $P(C) = 0.5$, $P(A \cup B) = 0.8, P(A \cap C) = 0.3$ and $P(A \cap B \cap C) = 0.2$

If $P(A \cup B \cup C) \geq 0.85$, then the interval of values of $P(B \cap C)$ is

- (a) [0.2, 0.35] (b) [0.55, 0.7]
(c) [0.2, 0.55] (d) None of these

Sol. $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

$$- P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= 0.6 + 0.4 + 0.5 - 0.2 - P(B \cap C) - 0.3 + 0.2$$

$$= 1.2 - P(B \cap C)$$

because $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow 0.8 = 0.6 + 0.4 - P(A \cap B)$$

But $0.85 \leq P(A \cup B \cup C) \leq 1$

$$\therefore 0.85 \leq 1.2 - P(B \cap C) \leq 1$$

$$\Rightarrow 0.2 \leq P(B \cap C) \leq 0.35$$

Hence, (a) is the correct answer.

Ex 20. If E and F are two events with $P(E) \leq P(F)$ and $P(E \cap F) > 0$, then

- (a) occurrence of $E \Rightarrow$ occurrence of F
(b) occurrence of $F \Rightarrow$ occurrence of E
(c) non-occurrence of $E \Rightarrow$ non-occurrence of F
(d) None of the above implications holds

Sol. Consider the following events in a single throw of a die.

$E =$ Getting a multiple 3

$F =$ Getting an even number

$$\text{We have, } P(E) = \frac{1}{3} \text{ and } P(F) = \frac{1}{2}$$

$$\text{Clearly, } P(E) \leq P(F) \text{ and } P(E \cap F) = \frac{1}{6} > 0.$$

But none of the options (a), (b) and (c) are true.

Hence, (d) is the correct answer.

Ex 21. There are four machines and it is known that exactly two of them are faulty. They are tested, one-by-one, in a random order till both the faulty machines are identified. Then, the probability that only two tests are needed, is

- (a) $\frac{1}{3}$ (b) $\frac{1}{6}$ (c) $\frac{1}{2}$ (d) $\frac{1}{4}$

Sol. The total number of ways in which two machines can be chosen out of four machines is ${}^4C_2 = 6$. If only two tests are required to identify faulty machines, then in first two tests faulty machines are identified. This can be done in one way only.

$\therefore$ Favourable number of ways = 1

Now, the required probability = $\frac{1}{6}$

Hence, (b) is the correct answer.

Ex 22. The numbers 1, 2, 3, ..., n are arranged in a random order. The probability that the digits 1, 2, 3, ..., k ($k < n$) appears as neighbours in that order, is

- (a) $\frac{1}{n!}$ (b) $\frac{k!}{n!}$
(c) $\frac{(n-k)!}{n!}$ (d) None of these

Sol. The number of ways of arranging n numbers in a row is $n!$

Considering digits 1, 2, 3, 4, ..., k as one digit, we have $(n-k+1)$ digits which can be arranged in $(n-k+1)!$ ways.

So, the total number of ways in the digits 1, 2, 3, ..., k appear as neighbours in the same order is $(n-k+1)!$

So, the required probability = $\frac{(n-k+1)!}{n!}$

Hence, (d) is the correct answer.

Ex 23. The numbers 1, 2, 3, ..., n are arranged in a random order. The probability that the digits 1, 2, 3, ..., k ($n > k$) appear as neighbours, is

- (a) $\frac{(n-k)!}{n!}$
(b) $\frac{n-k+1}{{}^nC_k}$
(c) $\frac{n-k}{{}^nC_k}$
(d) $\frac{k!}{n!}$

Sol. The numbers 1, 2, 3, ..., n can be arranged in a row in $n!$ ways.

The total number of ways in which the digits 1, 2, 3, ..., k ($k < n$) occur together is $k!(n-k+1)!$

So, the required probability

$$= \frac{k!(n-k+1)!}{n!} = \frac{n-k+1}{{}^nC_k}$$

Hence, (b) is the correct answer.

Ex 24. If four dice are thrown together, then the probability that the sum of the numbers appearing on them is 13, is

- (a) $\frac{5}{216}$ (b) $\frac{11}{216}$ (c) $\frac{35}{324}$ (d) $\frac{11}{432}$

Sol. Total number of elementary events associated to the random experiment of throwing 4 dice is

$$6 \times 6 \times 6 \times 6 = 6^4$$

Favourable number of elementary events

$$= \text{Coefficient of } x^{13} \text{ in } (x + x^2 + x^3 + \dots + x^6)^4$$

$$= \text{Coefficient of } x^9 \text{ in } (1 + x + x^2 + \dots + x^5)^4$$

$$= \text{Coefficient of } x^9 \text{ in } \left(\frac{1-x^6}{1-x} \right)^4$$

$$= \text{Coefficient of } x^9 \text{ in } (1-x^6)^4 (1-x)^{-4}$$

$$= \text{Coefficient of } x^9 \text{ in } (1-4x^6 + \dots) (1-x)^{-4}$$

$$= \text{Coefficient of } x^9 \text{ in } (1-x)^{-4} - 4$$

$$\times \text{Coefficient of } x^3 \text{ in } (1-x)^{-4}$$

$$= {}^{9+4-1}C_{4-1} - 4 \times {}^{3+4-1}C_{4-1}$$

$$[\because \text{coefficient of } x^n \text{ in } (1-x)^{-r} = {}^{n+r-1}C_{r-1}]$$

$$= {}^{12}C_3 - 4 \times {}^6C_3 = \frac{12 \times 11 \times 10}{3 \times 2 \times 1} - 4 \times \frac{6 \times 5 \times 4}{3 \times 2 \times 1}$$

$$= 220 - 80 = 140$$

$$\text{So, the required probability} = \frac{140}{6^4} = \frac{35}{324}$$

Hence, (c) is the correct answer.

Ex 25. Six faces of a die are marked with the numbers 1, -1, 0, -2, 2 and 3. The die is thrown thrice. The probability that the sum of the numbers thrown is six, is

- (a) $\frac{1}{72}$ (b) $\frac{1}{12}$ (c) $\frac{5}{108}$ (d) $\frac{1}{36}$

Sol. Total number of elementary events = 6^3

Favourable number of elementary events

$$= \text{Coefficient of } x^6 \text{ in } (x + x^{-1} + x^0 + x^{-2} + x^2 + x^3)^3$$

$$= \text{Coefficient of } x^6 \text{ in } \left(\frac{1+x+x^2+x^3+x^4+x^5}{x^2} \right)^3$$

$$= \text{Coefficient of } x^{12} \text{ in } (1+x+x^2+x^3+x^4+x^5)^3$$

$$= \text{Coefficient of } x^{12} \text{ in } \left(\frac{1-x^6}{1-x} \right)^3$$

$$= \text{Coefficient of } x^{12} \text{ in } (1-x^6)^3 (1-x)^{-3}$$

$$= \text{Coefficient of } x^{12} \text{ in } (1-{}^3C_1x^6 + {}^3C_2x^{12} \dots) (1-x)^{-3}$$

$$= \text{Coefficient of } x^{12} \text{ in } (1-x)^{-3} - {}^3C_1 \cdot \text{Coefficient of } (x^6 \text{ in } (1-x)^{-3} + {}^3C_2 \cdot \text{Coefficient of } x^0 \text{ in } (1-x)^{-3})$$

$$= {}^{12+3-1}C_{3-1} - {}^3C_1 \cdot {}^{6+3-1}C_{3-1} + {}^3C_2$$

$$= {}^{14}C_2 - {}^3C_1 \cdot {}^{6+3-1}C_{3-1} + {}^3C_2$$

$$= {}^{14}C_2 - {}^3C_1 \times {}^8C_2 + {}^3C_2 = 91 - 84 + 3 = 10$$

$$\text{So, the required probability} = \frac{10}{6^3} = \frac{5}{108}$$

Hence, (c) is the correct answer.

Ex 26. Given that the sum of two non-negative quantities is 200, the probability that their product is not less than $\frac{3}{4}$ times their greatest product value, is

- (a) $\frac{7}{16}$ (b) $\frac{101}{201}$
(c) $\frac{9}{16}$ (d) $\frac{10}{16}$

Sol. Let x and y be the two quantities. When the sum of two non-negative quantities is fixed, the product will be maximum when they are equal.

So, the greatest product $= xy = 10000$, where $x = y = 100$

Now, $xy \geq \frac{3}{4} \times 10000$

$$\Rightarrow xy \geq 7500$$

$$\Rightarrow x(200 - x) \geq 7500$$

$$\Rightarrow x^2 - 200x + 7500 \leq 0$$

$$\Rightarrow (x - 50)(x - 150) \leq 0$$

$$\Rightarrow 50 \leq x \leq 150$$

So, the favourable number of ways = 101

Total number of ways = 201

Now, required probability $= \frac{101}{201}$

Hence, (b) is the correct answer.

Ex 27. Let A be a set containing n elements. A subset P of the set A is chosen at random. The set A is reconstructed by replacing the elements of P and another subset Q of A is chosen at random. The probability that $P \cap Q$ contains exactly m ($m < n$) elements, is

- (a) $\frac{3^{n-m}}{4^n}$ (b) $\frac{{}^nC_m \cdot 3^m}{4^n}$
(c) $\frac{{}^nC_m \cdot 3^{n-m}}{4^n}$ (d) None of these

Sol. We know that, the number of subsets of a set containing n elements is 2^n . Therefore, the number of ways of choosing P and Q is $2^n C_1 \times 2^n C_1 = 2^n \times 2^n = 4^n$. Out of n elements, m elements can be chosen in ${}^nC_m$ ways. If $P \cap Q$ contains exactly m elements, then from the remaining $n - m$ elements either an element belongs to P or Q but not both P and Q . Suppose P contains r elements from the remaining $(n - m)$ elements. Then, Q may contain any number of elements from the remaining $(n - m) - r$ elements. Therefore, P and Q can be chosen in ${}^{n-m}C_r 2^{(n-m)-r}$.

But r can vary from 0 to $(n - m)$. So, P and Q can be chosen in general in

$$\left(\sum_{r=0}^{n-m} {}^{n-m}C_r 2^{(n-m)-r} \right) {}^nC_m = (1 + 2)^{n-m} {}^nC_m$$

$$= {}^nC_m \cdot 3^{n-m}$$

So, the required probability $= \frac{{}^nC_m \cdot 3^{n-m}}{4^n}$

Hence, (c) is the correct answer.

Ex 28. Let X be a set containing n elements. If two subsets A and B of X are picked at random, then the probability that A and B have the same number of elements, is

- (a) $\frac{{}^{2n}C_n}{2^{2n}}$ (b) $\frac{1}{{}^{2n}C_n}$
(c) $\frac{1 \cdot 3 \cdot 5 \dots (2n+1)}{2^n n!}$ (d) $\frac{3^n}{4^n}$

Sol. Required probability

$$= \frac{\sum_{r=0}^n {}^nC_r \times {}^nC_r}{2^n \times 2^n} = \frac{C_0^2 + C_1^2 + C_2^2 + \dots + C_n^2}{4^n}$$

$$= \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2^n \cdot n!} = \frac{{}^{2n}C_n}{2^{2n}}$$

Hence, (a) is the correct answer.

Ex 29. A and B are two events. Odds against A are 2:1 and odds in favour of $A \cup B$ are 3:1. If $x \leq P(B) \leq y$, then the ordered pair (x, y) is

- (a) $\left(\frac{5}{12}, \frac{3}{4}\right)$ (b) $\left(\frac{2}{3}, \frac{3}{4}\right)$
(c) $\left(\frac{1}{3}, \frac{3}{4}\right)$ (d) None of these

Sol. We have, $P(A) = \frac{1}{3}$ and $P(A \cup B) = \frac{3}{4}$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{3}{4} = \frac{1}{3} + P(B) - P(A \cap B)$$

$$\Rightarrow \frac{5}{12} = P(B) - P(A \cap B)$$

$$\Rightarrow P(B) = \frac{5}{12} + P(A \cap B) \Rightarrow P(B) \geq \frac{5}{12} \quad \dots(i)$$

$$\text{Again, } P(B) = \frac{5}{12} + P(A \cap B)$$

$$\Rightarrow P(B) \leq \frac{5}{12} + P(A) \quad [\because P(A \cap B) \leq P(A)]$$

$$\Rightarrow P(B) \leq \frac{5}{12} + \frac{1}{3} = \frac{3}{4} \quad \dots(ii)$$

$$\text{From Eqs. (i) and (ii), we get } \frac{5}{12} \leq P(B) \leq \frac{3}{4}$$

$$\text{So, } x = \frac{5}{12} \text{ and } y = \frac{3}{4}$$

Hence, (a) is the correct answer.

Ex 30. Two numbers b and c are chosen at random with replacement from the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9. The probability that $x^2 + bx + c > 0, \forall x \in \mathbb{R}$ is

- (a) $\frac{10}{27}$ (b) $\frac{31}{81}$ (c) $\frac{32}{81}$ (d) $\frac{11}{27}$

Sol. Since, b and c both can assume values from 1 to 9. So, total number of ways of choosing b and c is

$$9 \times 9 = 81$$

Now, $x^2 + bx + c > 0, \forall x \in \mathbb{R} \Rightarrow D < 0$

$$\text{i.e. } b^2 - 4c < 0$$

The following table shows the possible values of b and c for which $b^2 - 4c < 0$

c	b	Total
1	1	1
2	1, 2	2
3	1, 2, 3	3
4	1, 2, 3	3

c	b	Total
5	1, 2, 3, 4	4
6	1, 2, 3, 4	4
7	1, 2, 3, 4, 5	5
8	1, 2, 3, 4, 5	5
9	1, 2, 3, 4, 5	5

$\therefore$ Favourable number of ways = 32
So, the required probability is $= \frac{32}{81}$

Hence, (c) is the correct answer.

Type 2. More than One Correct Options

Ex 31. If A and B are two events such that $P(A) = 3/4$ and $P(B) = 5/8$, then

- (a) $P(A \cup B) \geq 3/4$
- (b) $P(A' \cap B) \leq 1/4$
- (c) $3/8 \leq P(A \cap B) \leq 5/8$
- (d) None of the above

Sol. $\because AC \subseteq A \cup B$

$$\Rightarrow P(A) \leq P(A \cup B)$$

$$\Rightarrow P(A \cup B) \geq \frac{3}{4}$$

$$\begin{aligned} \text{Also, } P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ &\geq P(A) + P(B) - 1 \\ &= \frac{3}{4} + \frac{5}{8} - 1 = \frac{3}{8} \end{aligned}$$

Now, $A \cap B \subseteq B$

$$\Rightarrow P(A \cap B) \leq P(B) = \frac{5}{8}$$

$$\therefore \frac{3}{8} \leq P(A \cap B) \leq \frac{5}{8} \quad \dots(i)$$

Now, consider $P(A' \cap B) = P(B) - P(A \cap B) \quad \dots(ii)$

From Eq. (i), we have

$$\begin{aligned} \frac{-5}{8} &\leq -P(A \cap B) \leq \frac{-3}{8} \\ \Rightarrow P(B) - \frac{5}{8} &\leq P(B) - P(A \cap B) \leq P(B) - \frac{3}{8} \\ \Rightarrow 0 &\leq P(A' \cap B) \leq \frac{1}{4} \quad [\text{from Eq. (ii)}] \end{aligned}$$

Hence, (a), (b) and (c) are correct answers.

Ex 32. The probability that a 50 yr old man will be alive at 60, is 0.83 and the probability that a 45 yr old woman will be alive at 55, is 0.87. Then,

- (a) the probability that both will be alive is 0.7221
- (b) atleast one of them will alive is 0.9779
- (c) atleast one of them will alive is 0.8230
- (d) the probability that both will be alive is 0.6320

Sol. Required probability = Probability that both will be alive for 10 yr $= 0.83 \times 0.87 = 0.7221$

The probability that atleast one of them will be alive $1 - P(\text{none of them remains alive 10 yr})$

$$\begin{aligned} \therefore &= 1 - (1 - 0.83)(1 - 0.87) \\ &= 1 - (0.17 \times 0.13) = 0.9779 \end{aligned}$$

Hence, (a) and (b) are correct answers.

Ex 33. The chance of an event happening is the square of the chance of a second event but the odds against the first are the cube of the odds against the second. The chances of the events are

- (a) $P_1 = 1/9$
- (b) $P_1 = 1/16$
- (c) $P_2 = 1/3$
- (d) $P_2 = 1/4$

Sol. Let P_1 and P_2 be the chances of happening of the first and second events, respectively.

Then, according to the given conditions

$$P_1 = P_2^2 \quad \text{and} \quad \frac{1 - P_1}{P_1} = \left(\frac{1 - P_2}{P_2} \right)^3$$

$$\therefore \frac{1 - P_2^2}{P_2^2} = \left(\frac{1 - P_2}{P_2} \right)^3$$

$$\Rightarrow P_2(1 + P_2) = (1 - P_2)^2$$

$$\Rightarrow 3P_2 = 1 \Rightarrow P_2 = \frac{1}{3}$$

$$P_1 = \frac{1}{9}$$

Hence, (a) and (c) are correct answers.

Type 3. Assertion and Reason

Directions (Ex. Nos. 34-37) In the following examples, each example contains Statement I (Assertion) and Statement II (Reason). Each example has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are.

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

Ex 34. Statement I If P is chosen at random in the closed interval $[0, 5]$, then the probability that the equation $x^2 + Px + \frac{1}{4}(P+2) = 0$ has real root is $\frac{3}{5}$.

Statement II If discriminant ≥ 0 , then roots of the quadratic equation are always real.

Sol. Here, $n(S) = 1$, length of the interval $[0, 5] = 5$ and $n(E) = \text{length of the interval } \leq [0, 5] \text{ in which } P \text{ belongs such that the given equation has real roots.}$

Now, $x^2 + Px + \frac{1}{4}(P+2) = 0$ will have real roots, if

$$P^2 - 4 \cdot 1 \cdot \frac{1}{4}(P+2) \geq 0 \Rightarrow P^2 - P - 2 \geq 0$$

$$\Rightarrow (P+1)(P-2) \geq 0$$

$$\Rightarrow P \leq -1 \text{ or } P \geq 2$$

$$\text{But } P \in [0, 5]$$

$$\text{So, } E = [2, 5]$$

$$\therefore n(E) = \text{Length of the interval } [2, 5] = 3$$

$$\therefore \text{Now, required probability} = \frac{3}{5}$$

Hence, (a) is the correct answer.

Ex 35. Statement I If $\frac{1+4P}{4}$, $\frac{1-P}{4}$ and $\frac{1-2P}{4}$ are probabilities of three pairwise mutually exclusive events, then the possible values of P belong to the set $\left[-\frac{1}{4}, \frac{1}{2}\right]$.

Statement II If three events are pairwise mutually exclusive and exhaustive, then sum of their probabilities is equal to 1.

Sol. We have,

$$0 \leq \frac{1+4P}{4} \leq 1, 0 \leq \frac{1-P}{4} \leq 1$$

$$\text{and } 0 \leq \frac{1-2P}{4} \leq 1$$

$$\Rightarrow -\frac{1}{4} \leq P \leq \frac{3}{4}, -3 \leq P \leq 1, -\frac{3}{2} \leq P \leq \frac{1}{2}$$

Again, the events are pairwise mutually exclusive, so

$$0 \leq \frac{1+4P}{4} + \frac{1-P}{4} + \frac{1-2P}{4} \leq 1$$

$$\Rightarrow -3 \leq P \leq 1$$

Taking intersection of all four intervals of P , we get

$$-\frac{1}{4} \leq P \leq \frac{1}{2}$$

Hence, (b) is the correct answer.

Ex 36. Statement I Two non-negative integers are chosen at random. The probability that the sum of their squares is divisible by 5 is $9/25$.

Statement II If at the unit place in any number is zero, that number is only divisible by 5.

Sol. Let the two non-negative integers be x and y .

Then, $x = 5a + \alpha$ and $y = 5b + \beta$, where $0 \leq \alpha \leq 4$, $0 \leq \beta \leq 4$

$$\begin{aligned} \text{Now, } x^2 + y^2 &= (5a + \alpha)^2 + (5b + \beta)^2 \\ &= 25(a^2 + b^2) + 10(a\alpha + b\beta) + \alpha^2 + \beta^2 \end{aligned}$$

$\therefore x^2 + y^2$ is divisible by 5 if and only if 5 divides $\alpha^2 + \beta^2$.

The total number of ways of choosing α and β is $5 \times 5 = 25$

Further, $\alpha^2 + \beta^2$ will be divisible by 5, if

$$(\alpha, \beta) \in \{(0, 0), (1, 2), (2, 1), (2, 4), (3, 1), (3, 4), (4, 2), (4, 3)\}$$

$\therefore$ Favourable number of ways of choosing α and $\beta = 9$

$$\text{Now, required probability} = \frac{9}{25}$$

Hence, (c) is the correct answer.

Ex 37. Statement I Out of 5 tickets consecutively numbered, three are drawn at random, the chance that the numbers on them are in an AP is $\frac{2}{15}$.

Statement II Out of $(2n+1)$ tickets consecutively numbered, three are drawn at random, the chance that the numbers on them are in an AP is $\frac{3n}{4n^2 - 1}$.

Sol. $2n+1 = 5$

$$\begin{aligned} \Rightarrow 2n &= 4 \\ \Rightarrow n &= 2; P(E) = \frac{3n}{4n^2 - 1} \\ &= \frac{6}{15} = \frac{2}{5} \end{aligned}$$

For a, b and c in an AP, $a + c = 2b \Rightarrow a + c$ is even, so a and c are both even or odd.

Now, a and c can be chosen in ${}^nC_2 + {}^{n+1}C_2 = n^2$ ways

$$\begin{aligned} \therefore P(E) &= \frac{n^2}{(2n+1)C_3} = \frac{n^2 \times 3 \times 2 \times 1}{(2n+1)2n(2n-1)} \\ &= \frac{3n}{4n^2 - 1} \end{aligned}$$

Hence, (d) is the correct answer.

Type 4. Linked Comprehension Based Questions

Passage (Ex. Nos. 38-40) In an objective paper, there are two sections of 10 questions each. For 'section 1', each question has five options and only one option is correct and 'section 2' has four options with multiple answers and marks for a question in this section is awarded only if he ticks all correct answers. Marks for each question in 'section 1' is 1 and in 'section 2' is 3. (There is no negative marking).

Ex 38. If a candidate attempts only two questions by guessing, one from 'section 1' and one from 'section 2', then the probability that he scores in both questions is

- (a) $74/75$ (b) $1/25$ (c) $1/15$ (d) $1/75$

Sol. Let P_1 be the probability of being an answer correct from section 1. Then, $P_1 = 1/5$. Let P_2 be the probability of being an answer correct from section 2. Then, $P_2 = 1/15$. So, the required probability is $1/5 \times 1/15 = 1/75$. Hence, (d) is the correct answer.

Ex 39. If a candidate in total attempts four questions all by guessing, then the probability of scoring 10 marks is

- (a) $1/15 (1/15)^3$ (b) $4/5 (1/15)^3$
(c) $1/15 (14/15)^3$ (d) None of these

Sol. Scoring 10 marks from four questions can be done in $3 + 3 + 3 + 1 = 10$ ways, so as to answer three questions from section 2 and 1 question from section 1 correctly. So, the required probability

$$= \frac{{}^{10}C_3 {}^{10}C_1}{20C_4} \cdot \frac{1}{5} \left(\frac{1}{15}\right)^3 = \frac{4}{5} \left(\frac{1}{15}\right)^3$$

Hence, (b) is the correct answer.

Ex 40. The probability of getting a score less than 40 by answering all the questions by guessing in this paper is

- (a) $(1/75)^{10}$ (b) $1 - (1/75)^{10}$
(c) $(74/75)^{10}$ (d) None of these

Sol. To get 40 marks, he has to answer all questions correctly and its probability is $(1/5)^{10} (1/15)^{10}$. So, the probability of getting a score less than 40

$$= 1 - \left(\frac{1}{5}\right)^{10} \left(\frac{1}{15}\right)^{10} = 1 - \left(\frac{1}{75}\right)^{10}$$

Hence, (b) is the correct answer.

Type 5. Match the Columns

Ex 41. 'Three coins are tossed'. Then, match the terms of Column I with the terms of Column II and choose the correct option from the codes given below.

Column I	Column II
A. Two events which are mutually exclusive.	p. A, B and C, where A is the event 'three heads show', B is the event 'two heads show' and C is the event 'three tails show'.
B. Two events which are mutually exclusive and exhaustive.	q. A and B, where A is the event 'three heads show' and B is event 'atleast two heads show'.
C. Two events which are not mutually exclusive.	r. A and B, where A is the event atleast one head and B is the event getting three tails.
D. Two events which are mutually exclusive but not exhaustive.	s. A and B, where A is the event 'three heads show' and B is the event 'three tails show'.
E. Three events which are mutually exclusive but not exhaustive.	

Sol. If three coins are tossed, then possible number of outcomes $= 2^3 = 8$

Sample space, $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

A. Let A be the event 'three heads show'.

$$\Rightarrow A = \{HHH\}$$

and B be the event 'three tails show'.

$$\Rightarrow B = \{TTT\} \Rightarrow A \cap B = \phi$$

Hence, A and B are mutually exclusive events.

B. Let A be the event 'atleast one head'.

$$\Rightarrow A = \{HHH, HHT, HTH, HTT, THH, THT, TTH\}$$

and B be the event getting three tails.

$$\Rightarrow B = \{TTT\}$$

$$\Rightarrow A \cap B = \phi \text{ and } A \cup B = S$$

[two events are exhaustive, if $A \cup B = S$]

Hence, A and B are mutually exclusive and exhaustive.

C. Let A be the event 'three heads show' and B be the event 'atleast two heads show'.

$$\Rightarrow A = \{HHH\}$$

$$\text{and } B = \{HHT, HTH, THH, HHH\}$$

$$\Rightarrow A \cap B = \{HHH\} \neq \phi$$

Hence, A and B are not mutually exclusive.

D. Same as in C, A and B are not mutually exclusive. Also, $A \cup B \neq S$.

E. Let A be the event 'three heads show', B be the event 'two heads show' and C be the event 'three tails show'.

$$\Rightarrow A = \{HHH\}$$

$$\Rightarrow B = \{HHT, HTH, THH\}$$

$$\text{and } C = \{TTT\}$$

$$\Rightarrow A \cap B \cap C = \phi \text{ and } A \cup B \cup C \neq S$$

$\Rightarrow$ A, B and C are mutually exclusive but not exhaustive.

$$A \rightarrow s; B \rightarrow r; C \rightarrow q; D \rightarrow q; E \rightarrow p$$

Ex 42. Match the terms of Column I with the terms of Column II and choose the correct option from the codes given below.

Column I	Column II
A. If E_1 and E_2 are two mutually exclusive events, then	p. $E_1 \cap E_2 = E_1$
B. If E_1 and E_2 are the mutually exclusive and exhaustive events, then	q. $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$
C. If E_1 and E_2 have common outcomes, then	r. $E_1 \cap E_2 = \phi$, $E_1 \cup E_2 = S$
D. If E_1 and E_2 are two events such that $E_1 \subset E_2$, then	s. $E_1 \cap E_2 = \phi$

Sol. A. If E_1 and E_2 are two mutually exclusive events, then $E_1 \cap E_2 = \phi$
 B. If E_1 and E_2 are the mutually exclusive and exhaustive events, then $E_1 \cap E_2 = \phi$ and $E_1 \cup E_2 = S$ where, S is the sample space for the events E_1 and E_2
 C. If E_1 and E_2 have common outcomes, then $(E_1 - E_2) \cup (E_1 \cap E_2) = E_1$
 D. If E_1 and E_2 are two events such that $E_1 \subset E_2$ and $E_1 \cap E_2 = E_1$
 $A \rightarrow s$; $B \rightarrow r$; $C \rightarrow q$; $D \rightarrow p$

Ex 43. Three coins are tossed once. Let A denotes the event 'three heads show', B denotes the event 'two heads and one tail show', C denotes the event 'three tails show' and D denotes the event 'a head shows on the first coin'. Then,

match the terms of Column I with the terms of Column II and choose the correct option from the codes given below:

Column I	Column II
A. Mutually exclusive events	p. A and B , A and C , B and C , C and D , A , B and C
B. Simple events	q. B , D
C. Compound events	r. A and C

Sol. When three coins are tossed, then there are $2^3 = 8$ possible outcomes.
 i.e. $S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$

A = Three heads show = $\{HHH\}$
 B = Two heads and one tail show
 $= \{HHT, HTH, THH\}$
 C = Three tails show = $\{TTT\}$
 D = A head shows on the first toss
 $= \{HHH, HHT, HTH, HTT\}$
 A. Here, $A \cap B = \phi$
 $A \cap C = \phi$
 $B \cap C = \phi$
 $C \cap D = \phi$
 $A \cap B \cap C = \phi$
 Here, A and B , A and C , B and C , C and D and A , B and C are mutually exclusive.
 B. Since, events A , C have only one sample point. Here, A and C are simple events.
 C. Since, events B , D have two or more sample points. Here, B and D are compound events.
 $A \rightarrow p$; $B \rightarrow r$; $C \rightarrow q$

Type 6. Single Integer Answer Type Questions

Ex 44. Out of 21 tickets consecutively numbered, three are drawn at random. Find the probability that the numbers on them are in an AP, is a/b , then $(14a - b)$ is _____.

Sol. (7) Any three tickets out of 21 tickets can be chosen in ${}^{21}C_3$ ways. For the favourable choice, if the chosen numbers are a, b and $c, a < b < c$, then we should have $\frac{a+c}{2} = b$. Obviously, either both a and c are even or both are odd and then b is fixed. Hence, for the favourable choice, we have to choose two numbers from 1 to 21, which are either both even or both odd. This can be done in ${}^{11}C_2 + {}^{10}C_2$ ways.

Hence, the required probability
 $= \frac{{}^{11}C_2 + {}^{10}C_2}{{}^{21}C_3} = \frac{10}{133} = \frac{a}{b}$

$\Rightarrow 14a - b = 140 - 133 = 7$

Ex 45. A has 3 shares in a lottery containing 3 prizes and 9 blanks. B has 2 shares in a lottery containing 2 prizes and 6 blanks and their chances of success is $\frac{100a + 10b + c}{100p + 10q + r}$, then $(a - p)$ is _____.

Sol. (2) Let E_1 be the event of success of A and let E_2 be the event of success of B .

Since, A has 3 shares in a lottery containing 3 prizes and 9 blanks, A will draw 3 tickets out of 12 tickets (containing 3 prizes and 9 blanks). A will get success, if he draws atleast one prize out of 3 draws.

$$\Rightarrow P(E_1') = \frac{{}^9C_3}{{}^{12}C_3} = \frac{21}{55}$$

$$\Rightarrow P(E_1) = 1 - \frac{21}{55} = \frac{34}{55}$$

Again, $P(E_2') = \frac{{}^6C_2}{{}^8C_2} = \frac{15}{28}$

Now, $P(E_2) = 1 - \frac{15}{28} = \frac{13}{28}$

$$\therefore \frac{P(E_1)}{P(E_2)} = \frac{34}{55} \times \frac{28}{13} = \frac{952}{715}$$

$$\Rightarrow P(E_1) : P(E_2) = 952 : 715 = \frac{9 \times 100 + 5 \times 10 + 2}{7 \times 100 + 1 \times 10 + 5}$$

$$\Rightarrow a - p = 9 - 7 = 2$$

Target Exercises

Type 1. Only One Correct Option

- From a group of 2 boys and 3 girls, two children are selected. Then, the total number of the outcomes of this experiment is
(a) 11 (b) 13 (c) 10 (d) 16
- Two dice are thrown simultaneously. Then, the set of event 'a total of atleast 10' is
(a) $\{(6, 4), (4, 6), (5, 5), (6, 5), (5, 6), (6, 6)\}$
(b) $\{(5, 5), (6, 4), (4, 6)\}$
(c) $\{(6, 5), (5, 6), (6, 6)\}$
(d) $\{(4, 4), (4, 5), (6, 5), (5, 5)\}$
- If A and B are independent events such that $0 < P(A) < 1$ and $0 < P(B) < 1$, then which one is not correct?
(a) A and B are mutually exclusive
(b) A and B^C are independent
(c) A^C and B^C are independent
(d) A^C and B are independent
- The probability of the occurrence of an event A is 0.5 and that of occurrence of another event B is 0.3. If A and B are mutually exclusive events, then the probability that none of A and B will occur, is
(a) 0.6 (b) 0.5
(c) 0.7 (d) None of these
- Three mangoes and three apples are kept in a box. If two fruits are selected at random from the box, then the probability that the selection will contain one mango and one apple, is
(a) $3/5$ (b) $5/6$
(c) $1/36$ (d) None of these
- Two friends A and B have equal number of daughters. There are three cinema tickets which are to be distributed among the daughters of A is $1/20$. The number of daughters each of them have, is
(a) 4 (b) 5 (c) 6 (d) 3
- If $6n$ tickets numbered $0, 1, 2, \dots, (6n - 1)$ are placed in a bag and three are drawn at random, then the probability that the sum of numbers on the ticket is $6n$, is
(a) $\frac{3n}{6nC_3}$ (b) $\frac{3n^2}{6nC_3}$ (c) $\frac{1}{2} \cdot \frac{3n}{6nC_3}$ (d) $\frac{1}{2} \cdot \frac{3n^2}{6nC_3}$
- A pack of cards contains 4 aces, 4 kings, 4 queens and 4 jacks. Two cards are drawn at random from this pack without replacement. The probability that atleast one of them will be an ace, is
(a) $1/5$ (b) $9/20$ (c) $1/6$ (d) $1/9$
- Two cards are drawn successively with replacement from a well, shuffled deck of 52 cards, the probability that both of these will be aces, is
(a) $1/169$ (b) $1/201$ (c) $1/2652$ (d) $4/663$
- A bag contains 5 brown socks and 4 white socks. A man selects two socks from the bag without replacement. The probability that the selected socks will be of the same colour, is
(a) $5/108$ (b) $1/6$ (c) $5/18$ (d) $4/9$
- Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently, equals
(a) $1/2$ (b) $7/15$ (c) $2/15$ (d) $1/2$
- Five persons entered the lift cabin on the ground floor of an 8 floor house. Suppose that each of them independently and with equal probability can leave the cabin at any floor beginning with the first, then the probability of all 5 persons leaving at different floors is
(a) $\frac{{}^7P_5}{7^5}$ (b) $\frac{7^5}{{}^7P_5}$ (c) $\frac{6}{{}^6P_5}$ (d) $\frac{{}^5P_5}{5^5}$
- The chance of throwing a total of 3 or 5 or 11 with two dice is
(a) $5/36$ (b) $1/9$ (c) $2/9$ (d) $19/36$
- The probability that a teacher will give an unannounced test during any class meeting is $1/5$. If a student is absent twice, then the probability that the student will miss atleast one test, is
(a) $4/5$ (b) $2/5$
(c) $7/75$ (d) $9/25$
- Given two events are A and B . If odds against A are as $2 : 1$ and those in favour of $A \cup B$ are as $3 : 1$, then
(a) $\frac{1}{2} \leq P(B) \leq \frac{3}{4}$
(b) $\frac{5}{12} \leq P(B) \leq \frac{3}{4}$
(c) $\frac{1}{4} \leq P(B) \leq \frac{3}{5}$
(d) None of the above
- A six faced die is so biased that it is twice as likely to show an even number as an odd number when thrown. It is thrown twice. The probability that the sum of two numbers thrown is even, is
(a) $1/12$ (b) $1/6$ (c) $1/3$ (d) $5/9$

17. In a box, there are 2 red, 3 black and 4 white balls. Out of these, three balls are drawn together. The probability of these being of same colour is
(a) $1/84$
(b) $1/21$
(c) $5/84$
(d) None of the above
18. The probability that a company executive will travel by train is $2/3$ and that he will travel by plane is $1/5$. The probability of his travelling by train or plane is
(a) $2/15$
(b) $13/15$
(c) $15/13$
(d) $15/2$
19. The probability that Krishna will be alive 10 yr hence, is $7/15$ and the probability that Hari will be alive after 10 yr, is $7/10$. The probability that both Krishna and Hari will be alive 10 yr hence, is
(a) $21/150$
(b) $24/150$
(c) $49/150$
(d) $56/150$
20. If $P(A) = 0.65$ and $P(B) = 0.80$, then $P(A \cap B)$ lies in the interval
(a) $[0.30, 0.80]$
(b) $[0.35, 0.75]$
(c) $[0.4, 0.70]$
(d) $[0.45, 0.65]$
21. A and B are two independent events. The probability that both A and B occur, is $1/6$ and the probability that none of them occurs, is $1/3$. The minimum value of probability of occurrence of A , is
(a) $1/2$
(b) $1/3$
(c) $1/4$
(d) None of the above
22. A circle is inscribed in an ellipse. If P is the probability that a point within the ellipse chosen at random lies outside the circle, then the eccentricity of the ellipse is
(a) $\sqrt{1-P}$
(b) $\sqrt{1-P^2}$
(c) $\sqrt{1-(1-P)^2}$
(d) $\sqrt{(1+P)^2-1}$
23. A and B are two independent events. The probability that both A and B occur is $1/6$ and the probability that neither of them occurs is $1/3$. Then, the probability of the two events are respectively
(a) $1/2, 1/3$
(b) $1/5, 1/6$
(c) $1/2, 1/6$
(d) $2/3, 1/4$
24. Let A and B be two events such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B)$ is equal to
(a) $5/7$
(b) $2/3$
(c) 1
(d) None of the above
25. Two dice are thrown together first and secondly three dice are thrown together. Then, the probability that the total in the first throw is 4 or more and at the same time the total in the second throw is 6 or more, is
(a) $\frac{1133}{1296}$
(b) $\frac{1137}{1296}$
(c) $\frac{1131}{1296}$
(d) $\frac{1000}{1296}$
26. The probability that a man will live 10 more years, is $1/4$ and the probability that his wife will live 10 more years, is $1/3$. Then, the probability that none of them will be alive after 10 yr, is
(a) $5/12$
(b) $1/2$
(c) $7/12$
(d) $11/12$
27. A fair coin is tossed. If the result is a head, a pair of fair dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of 11 cards numbered 2, 3, 4, ..., 12 is picked up and the number on the card is noted. The probability that the noted number is 7 or 8, is
(a) $193/792$
(b) $195/792$
(c) $191/792$
(d) None of the above
28. A pair of dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7, is
(a) $2/5$
(b) $1/5$
(c) $3/5$
(d) None of the above
29. A sample space consists of three mutually independent and equally likely events. The probability of happening of each one of them is equal to
(a) 0
(b) $1/3$
(c) 1
(d) None of the above
30. Let E and F be two independent events. The probability that both E and F happens is $1/12$ and the probability that neither E nor F happens is $1/2$. Then,
(a) $P(E) = 1/3, P(F) = 1/4$ or $P(E) = 1/4, P(F) = 1/3$
(b) $P(E) = 1/2, P(F) = 1/6$
(c) $P(E) = 1/6, P(F) = 1/2$
(d) $P(E) = 1/4, P(F) = 1/3$
31. If the odds against a certain event are 5 : 2 and the odds in favour of other event, independent of the former, are 6 : 5. Then, the probability that atleast one of the events will happen is
(a) $\frac{52}{77}$
(b) $\frac{25}{77}$
(c) $\frac{32}{77}$
(d) None of the above

Type 2. More than One Correct Option

32. If A and B are two events, then the probability that exactly one of them occurs is given by

- (a) $P(A) + P(B) - 2P(A \cap B)$
- (b) $P(A \cap B) + P(\bar{A} \cap \bar{B})$
- (c) $P(A \cup B) - P(A \cap B)$
- (d) $P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B})$

33. If A, B and C are three events, then which of the following is/are correct?

- (a) $P(\text{Exactly two of } A, B \text{ and } C \text{ occur})$
 $\leq P(A \cap B) + P(B \cap C) + P(C \cap A)$
- (b) $P(A \cup B \cup C) \leq P(A) + P(B) + P(C)$
- (c) $P(\text{Exactly one of } A, B \text{ and } C \text{ occur})$
 $\leq P(A) + P(B) + P(C) - P(B \cap C)$
 $- P(C \cap A) - P(A \cap B)$
- (d) $P(A \text{ and at least one of } B \text{ and } C \text{ occur})$
 $\leq P(A \cap B) + P(A \cap C)$

34. If A and B are two events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{2}{3}$, then which of the following is correct?

- (a) $P(A \cup B) \geq \frac{2}{3}$
- (b) $P(A \cap \bar{B}) \leq \frac{1}{3}$
- (c) $\frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$
- (d) $\frac{1}{6} \leq P(\bar{A} \cap B) \leq \frac{1}{2}$

35. If A and B are two independent events such that $P(\bar{A} \cap B) = 2/15$ and $P(A \cap \bar{B}) = 1/6$, then $P(B)$ is

- (a) $\frac{4}{5}$
- (b) $\frac{1}{6}$
- (c) $\frac{1}{5}$
- (d) $\frac{5}{6}$

Type 3. Assertion and Reason

Directions (Q. Nos. 36-39) In the following questions, each question contains Statement I (Assertion) and Statement II (Reason). Each question has 4 choices (a), (b), (c) and (d) out of which only one is correct. The choices are

- (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
- (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
- (c) Statement I is true, Statement II is false
- (d) Statement I is false, Statement II is true

36. A fair die is rolled once.

Statement I The probability of getting a composite number is $1/3$.

Statement II There are three possibilities for the obtained number (i) the number is a prime number (ii) the number is a composite number (iii) the number is 1 and hence probability of getting a prime number = $1/3$.

37. Let A and B be two events such that $P(A \cup B) \geq \frac{3}{4}$ and $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{8}$.

Statement I $P(A) + P(B) \geq \frac{7}{8}$

Statement II $P(A) + P(B) \leq \frac{11}{8}$

38. Let A and B be two independent events.

Statement I If $P(A) = 0.3$ and $P(A \cup \bar{B}) = 0.8$, then $P(B)$ is $\frac{2}{7}$.

Statement II $P(\bar{E}) = 1 - P(E)$, where E is any event.

39. **Statement I** If $A = \{2, 4, 6\}$ and $B = \{1, 2, 3\}$, where A and B are the events of numbers occurring on a die, then $P(A) + P(B) = 1$.

Statement II If $A_1, A_2, A_3, \dots, A_n$ are all mutually exclusive events, then $P(A_1) + P(A_2) + \dots + P(A_n) = 1$.

Type 4. Linked Comprehension Based Questions

Passage I (Q. Nos. 40-42) Two dice are thrown. The events A, B and C are described as follows :

A : Getting an even number on the first die.

B : Getting an odd number on the first die.

C : Getting atmost 5 as sum of the numbers on two dice.

40. Then, the set of the events A and B is

- (a) ϕ
- (b) $\{1, 2\}$
- (c) $\{2, 4\}$
- (d) $\{1, 3\}$

41. The total number of outcomes in the event ' B or C ' is

- (a) 20
- (b) 22
- (c) 21
- (d) 23

42. The set of the event ' B and C ' is

- (a) $\{(1, 1), (1, 2), (1, 3), (1, 4)\}$
- (b) $\{(1, 2), (1, 3), (1, 4), (1, 5)\}$
- (c) $\{(1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2)\}$
- (d) None of the above

Passage II (Q. Nos. 43-45) Consider the experiment of distribution of balls among urns. Suppose we are given M urns, numbered 1 to M , among which we are to distribute n balls ($n < M$). Let $P(A)$ denotes the probability that each of the urns numbered 1 to n will contain exactly one ball. Then, answer the following questions.

43. If the balls are different and any number of balls can go to any urns, then $P(A)$ is equal to

- (a) $\frac{M!}{n^M}$ (b) $\frac{n!}{M^n}$
(c) $\frac{n!}{M P_n}$ (d) $\frac{1}{M^n}$

44. If the balls are identical and any number of balls can go to any urns, then $P(A)$ equals

- (a) $\frac{1}{M^n}$ (b) $\frac{1}{M+n-1 C_{M-1}}$
(c) $\frac{1}{M+n-1 C_{n-1}}$ (d) $\frac{1}{M+n-1 P_{M-1}}$

45. If the balls are identical but atmost one ball can be put in any box, then $P(A)$ is equal to

- (a) $\frac{1}{M P_n}$ (b) $\frac{n!}{n C_M}$ (c) $\frac{n!}{M C_n}$ (d) $\frac{1}{M C_n}$

Type 5. Match the Columns

46. Match the statements of Column I with values of Column II.

Column I	Column II
A. 7 white balls and 3 black balls are placed in a row at random. The probability that no two black balls are adjacent, is	p. $\frac{5}{11}$
B. 4 gentlemen and 4 ladies take seats at random round a table. The probability that they are sitting alternately, is	q. $\frac{1}{16}$
C. 10 different books and 2 different pens are given to 3 boys so that each gets equal number of things. The probability that the same boy does not receive both the pens, is	r. $\frac{7}{15}$
D. A fair coin is tossed repeatedly. The probability of getting a result in the fifth toss different from those obtained in the first four tosses is	s. $\frac{1}{35}$

47. Match the statements of Column I with values of Column II.

Column I	Column II
A. There are 7 seats in a row. Three persons take seats at random. The probability that the middle seat is always occupied and no two persons are consecutive, is	p. $\frac{1}{2}$
B. Three numbers are chosen at random without replacement from the set $A = \{x 1 \leq x \leq 10, x \in N\}$. The probability that the minimum of the chosen numbers is 3 and maximum is 7, is	q. $\frac{11}{40}$
C. From a group of 10 persons consisting of 5 lawyers, 3 doctors and 2 engineers, four persons are selected at random. The probability that the selection contains atleast one of each category is	r. $\frac{1}{40}$
D. Three numbers are chosen at random without replacement from 1, 2, 3, ..., 10. The probability that the minimum of the chosen numbers is 4 or their maximum is 8, is	s. $\frac{4}{35}$

Type 6. Single Integer Answer Type Questions

48. A four digit number is formed using the digits 0, 1, 2, 3, 4 without repetition. If the probability that it is divisible by 4, is $\frac{a}{b}$, then $b - 2a$ is _____.

49. In throwing of a die, let A be the event 'an odd number turns up', B be the event 'a number divisible by 3 turns up' and C be the event 'a number ≤ 4 turns up'. If the probability that exactly two of A, B and C occur is $\frac{a}{b}$, then $a + b$ is _____.

50. A bag contains a large number of white and black marbles in equal proportions. Two samples of 5 marbles are selected (with replacement) at random. If the probability that the first sample contains exactly 1 black marble and the second sample contains exactly 3 black marbles, is $\frac{10a + b}{512}$, then $a + b$ is _____.

51. If two switches S_1 and S_2 have respectively 90% and 80% chances of working. Find the probabilities that if each of the following circuits will work = a/b , then $b - a$ is _____.

Entrances Gallery

JEE Advanced/IIT JEE

1. Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is atleast one more than the number of girls ahead of her, is [2014]
 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Passage (Q. Nos. 2-3) Box 1 contains three cards bearing numbers 1, 2, 3; box 2 contains five cards bearing numbers 1, 2, 3, 4, 5 and box 3 contains seven cards bearing numbers 1, 2, 3, 4, 5, 6, 7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the i th box, $i = 1, 2, 3$. [2014]

2. The probability that $x_1 + x_2 + x_3$ is odd, is
 (a) $\frac{29}{105}$ (b) $\frac{53}{105}$ (c) $\frac{57}{105}$ (d) $\frac{1}{2}$
3. The probability that x_1, x_2 and x_3 are in an arithmetico-progression, is
 (a) $\frac{9}{105}$ (b) $\frac{10}{105}$ (c) $\frac{11}{105}$ (d) $\frac{7}{105}$
4. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then, the probability that the problem is solved correctly by atleast one of them, is [2013]
 (a) $\frac{235}{256}$ (b) $\frac{21}{256}$
 (c) $\frac{3}{256}$ (d) $\frac{253}{256}$
5. Of the three independent events E_1, E_2 and E_3 , the probability that only E_1 occurs is α , only E_2 occurs is β and only E_3 occurs is γ . Let the probability p that none of events E_1, E_2 or E_3 occurs satisfy the

equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval $(0, 1)$.

Then, $\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3}$ is _____. [2013]

6. Four fair dice D_1, D_2, D_3 and D_4 , each having six faces numbered 1, 2, 3, 4, 5 and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 , is [2012]
 (a) $\frac{91}{216}$ (b) $\frac{108}{216}$ (c) $\frac{125}{216}$ (d) $\frac{127}{216}$
7. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T , then [2011]
 (a) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$ (b) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$
 (c) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (d) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$
8. Let ω be a complex cube root of unity with $\omega \neq 1$. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, then the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$, is [2010]
 (a) $\frac{1}{18}$ (b) $\frac{1}{9}$
 (c) $\frac{2}{9}$ (d) $\frac{1}{36}$

JEE Main/AIEEE

9. If 12 identical balls are to be placed in 3 identical boxes, then the probability that one of the boxes contains exactly 3 balls, is [2015]
 (a) $\frac{55}{3} \left(\frac{2}{3}\right)^{11}$ (b) $55 \left(\frac{2}{3}\right)^{10}$ (c) $220 \left(\frac{1}{3}\right)^{12}$ (d) $22 \left(\frac{1}{3}\right)^{11}$
10. If A and B are two events such that $P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\overline{A}) = \frac{1}{4}$, where $\overline{A}$ stands for the complement of the event A . Then, the events A and B are [2014]
 (a) mutually exclusive and independent
 (b) equally likely but not independent
 (c) independent but not equally likely
 (d) independent and equally likely
11. Four numbers are chosen at random (without replacement) from the set $\{1, 2, 3, \dots, 20\}$.
Statement I The probability that the chosen numbers when arranged in some order will form an AP, is $\frac{1}{85}$.
Statement II If the four chosen numbers form an AP, then the set of all possible values of common difference is $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}$. [2010]
 (a) Statement I is true, Statement II is true; Statement II is a correct explanation for Statement I
 (b) Statement I is true, Statement II is true; Statement II is not a correct explanation for Statement I
 (c) Statement I is true, Statement II is false
 (d) Statement I is false, Statement II is true

12. A die is thrown. Let A be the event that the number obtained is greater than 3 and B be the event that the number obtained is less than 5. Then, $P(A \cup B)$ is [2008]
 (a) $\frac{2}{5}$ (b) $\frac{3}{5}$ (c) 0 (d) 1
13. Three houses are available in a locality. Three persons apply for the houses. Each applies for one house without consulting others. The probability that all the three apply for the same house, is [2005]
 (a) $\frac{7}{9}$ (b) $\frac{8}{9}$ (c) $\frac{1}{9}$ (d) $\frac{2}{9}$
14. The probability that A speaks truth, is $\frac{4}{5}$ while this probability for B is $\frac{3}{4}$. The probability that they contradict each other when asked to speak on a fact, is [2004]
 (a) $\frac{3}{20}$ (b) $\frac{1}{5}$ (c) $\frac{7}{20}$ (d) $\frac{4}{5}$
15. Five horses are in a race. Mr. A selects two of the horses at random and bets on them. The probability that Mr. A selected the winning horse, is [2003]
 (a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{1}{5}$ (d) $\frac{2}{5}$
16. Events A, B and C are mutually exclusive events such that $P(A) = \frac{3x+1}{3}$, $P(B) = \frac{1-x}{4}$ and $P(C) = \frac{1-2x}{2}$. The set of possible values of x are in the interval [2003]
 (a) $\left[\frac{1}{3}, \frac{1}{2}\right]$ (b) $\left[\frac{1}{3}, \frac{2}{3}\right]$
 (c) $\left[\frac{1}{3}, \frac{13}{3}\right]$ (d) $[0, 1]$
17. A and B play a game, where each is asked to select a number from 1 to 25. If the two numbers match, both of them win a prize. The probability that they will not win a prize in a single trial, is [2002]
 (a) $\frac{1}{25}$ (b) $\frac{24}{25}$
 (c) $\frac{2}{25}$ (d) None of these
18. A problem in Mathematics is given to three students A, B, C and their respective probability of solving the problem is $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$. Probability that the problem is solved, is [2002]
 (a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{2}{3}$ (d) $\frac{1}{3}$

Other Engineering Entrances

19. Five persons A, B, C, D and E are in queue of a shop. The probability that A and E are always together, is [BITSAT 2014]
 (a) $\frac{1}{4}$ (b) $\frac{2}{3}$ (c) $\frac{2}{5}$ (d) $\frac{3}{5}$
20. A four digit number is formed by the digits 1, 2, 3, 4 with no repetition. The probability that the number is odd, is [VITEEE 2014]
 (a) zero (b) $\frac{1}{3}$
 (c) $\frac{1}{4}$ (d) None of these
21. If the integers m and n are chosen at random from 1 to 100, then the probability that a number of the form $7^n + 7^m$ is divisible by 5, equals [VITEEE 2014]
 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{8}$ (d) $\frac{1}{3}$
22. One mapping (function) is selected at random from all the mappings of the set $A = \{1, 2, 3, \dots, n\}$ into itself. The probability that the mapping selected is one-one, is [Manipal 2014]
 (a) $\frac{n!}{n^{n-1}}$ (b) $\frac{n!}{n^n}$
 (c) $\frac{n!}{2n^n}$ (d) None of these
23. Two dice are thrown simultaneously. The probability of obtaining a total score of 5 is [Karnataka CET 2014]
 (a) $\frac{1}{9}$ (b) $\frac{1}{18}$ (c) $\frac{1}{36}$ (d) $\frac{1}{12}$
24. If two dice are thrown simultaneously, then the probability that the sum of the numbers which come upon the dice to be more than 5, is [Kerala CEE 2014]
 (a) $\frac{5}{36}$ (b) $\frac{1}{6}$ (c) $\frac{5}{18}$
 (d) $\frac{13}{18}$ (e) $\frac{3}{18}$
25. A fair six faced die is rolled 12 times. The probability that each face turns up twice is equal to [WB JEE 2014]
 (a) $\frac{12!}{6! 6^{12}}$ (b) $\frac{2^{12}}{2^6 6^{12}}$ (c) $\frac{12!}{2^6 6^{12}}$ (d) $\frac{12!}{6^2 6^{12}}$
26. A poker hand consists of 5 cards drawn at random from a well-shuffled pack of 52 cards. Then, the probability that a poker hand consists of a pair and a triple of equal face values (for example, 2 sevens and 3 kings or 2 aces and 3 queens, etc.), is [WB JEE 2014]
 (a) $\frac{6}{4165}$ (b) $\frac{23}{4165}$ (c) $\frac{1797}{4165}$ (d) $\frac{1}{4165}$
27. Ram is visiting a friend. Ram knows that his friend has 2 children and 1 of them is a boy. Assuming that a child is equally likely to be a boy or a girl, then the probability that the other child is a girl, is [WB JEE 2014]
 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{7}{10}$
28. The probability that a leap year selected at random contains 53 Sunday, is [AMU 2014]
 (a) $\frac{7}{366}$ (b) $\frac{28}{183}$ (c) $\frac{1}{7}$ (d) $\frac{2}{7}$

29. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral, equals [AMU 2014]
 (a) $\frac{1}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{10}$ (d) $\frac{1}{20}$
30. A six faced unbiased die is thrown twice and the sum of the numbers appearing on the upper face is observed to be 7. The probability that the number 3 has appeared atleast once, is [EAMCET 2014]
 (a) $\frac{1}{5}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$
31. A single letter is selected at random from the word 'PROBABILITY'. The probability that it is a vowel, is [GGSIPU 2014]
 (a) $\frac{8}{11}$ (b) $\frac{4}{11}$ (c) $\frac{2}{11}$ (d) $\frac{3}{11}$
32. If three squares are chosen on a chess board, then the chance that they should be in a diagonal line, is [GGSIPU 2014]
 (a) $\frac{7}{744}$ (b) $\frac{5}{744}$ (c) $\frac{7}{544}$ (d) $\frac{11}{744}$
33. If two cards are drawn simultaneously from the same set, then probability that atleast one of them will be the ace of hearts, is [GGSIPU 2014]
 (a) $\frac{1}{13}$ (b) $\frac{1}{26}$ (c) $\frac{1}{52}$ (d) $\frac{3}{13}$
34. In a class, there are 10 boys and 8 girls. When 3 students are selected at random, then the probability that 2 girls and 1 boy are selected, is [GGSIPU 2014]
 (a) $\frac{35}{102}$ (b) $\frac{15}{102}$ (c) $\frac{55}{102}$ (d) $\frac{25}{102}$
35. The probability of simultaneous occurrence of atleast one of two events A and B is p . If the probability that exactly one of A, B occurs is q , then $P(A') + P(B')$ is equal to [BITSAT 2014]
 (a) $2 - 2p + q$ (b) $2 + 2p - q$
 (c) $3 - 3p + q$ (d) $2 - 4p + q$
36. If M and N are any two events, then the probability that exactly one of them occurs, is (for an event set A , the complement is A^C) [GGSIPU 2014]
 (a) $P(M) + P(N) - 2P(M \cap N)$
 (b) $P(M) + P(N) - P(M \cup N)$
 (c) $P(M^C) + P(N^C) - 2P(M^C \cup N^C)$
 (d) $P(M \cup N^C) + P(M^C \cup N)$
37. Let A and B be two events such that
 $P(A \cup B) = P(A) + P(B) - P(A)P(B)$.
 If $0 < P(A) < 1$ and $0 < P(B) < 1$, then $P(A \cup B)'$ is equal to [Kerala CEE 2014]
 (a) $1 - P(A)$
 (b) $1 - P(A')$
 (c) $1 - P(A)P(B)$
 (d) $[1 - P(A)]P(B')$
 (e) 1
38. If the events A and B are independent, if $P(A') = \frac{2}{3}$ and $P(B') = \frac{2}{7}$, then $P(A \cap B)$ is equal to [Karnataka CET 2014]
 (a) $\frac{4}{21}$ (b) $\frac{5}{21}$
 (c) $\frac{1}{21}$ (d) $\frac{3}{21}$
39. If $P(A \cup B) = 0.83$, $P(A) = 0.3$ and $P(B) = 0.6$, then the events will be [RPET 2014]
 (a) dependent (b) independent
 (c) cannot say anything (d) None of these
40. If n integers taken at random are multiplied together, then the probability that the last digit of the product is 1, 3, 7 or 9, is [UP SEE 2013]
 (a) $\frac{2^n}{5^n}$ (b) $\frac{4^n - 2^n}{5^n}$
 (c) $\frac{4^n}{5^n}$ (d) None of these
41. An urn contains 8 red and 5 white balls. Three balls are drawn at random. Then, the probability that balls of both colours are drawn, is [WB JEE 2012]
 (a) $\frac{40}{143}$ (b) $\frac{70}{143}$ (c) $\frac{3}{13}$ (d) $\frac{10}{13}$
42. Two decks of playing cards are well-shuffled and 26 cards are randomly distributed to a player. Then, the probability that the player gets all distinct cards, is [WB JEE 2012]
 (a) $\frac{{}^{52}C_{26}}{{}^{104}C_{26}}$ (b) $2 \times \frac{{}^{52}C_{26}}{{}^{104}C_{26}}$
 (c) $2^{13} \times \frac{{}^{52}C_{26}}{{}^{104}C_{26}}$ (d) $2^{26} \times \frac{{}^{52}C_{26}}{{}^{104}C_{26}}$
43. Probability of product of a perfect square when 2 dice are thrown together, is [OJEE 2012]
 (a) $\frac{2}{9}$ (b) $\frac{1}{9}$
 (c) $\frac{5}{18}$ (d) None of these
44. A bag contains 3 white and 5 black balls. One ball is drawn at random. Then, the probability that it is white, is [BITSAT 2011]
 (a) $\frac{1}{8}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{7}{8}$
45. A die is rolled three times. The probability of getting a larger number than the previous number each time, is [MP PET 2011]
 (a) $\frac{5}{72}$ (b) $\frac{5}{54}$ (c) $\frac{13}{216}$ (d) $\frac{1}{18}$
46. 4 boys and 2 girls occupy seats in a row at random. Then, the probability that two girls occupy seats side by side, is [WB JEE 2011]
 (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{6}$

47. A card is drawn from a well-shuffled pack of playing cards. The probability that it is a heart of a king, is [UP SEE 2011]
 (a) $\frac{1}{52}$ (b) $\frac{15}{52}$ (c) $\frac{18}{52}$ (d) $\frac{19}{52}$
48. A fair coin is tossed 100 times the probability of getting tails an odd number of time is [AMU 2011]
 (a) $\frac{1}{2}$ (b) $\frac{1}{4}$ (c) 0 (d) 1
49. The probability that a leap year will have only 52 Sunday, is [GGSIPU 2011]
 (a) $\frac{4}{7}$ (b) $\frac{5}{7}$ (c) $\frac{6}{7}$ (d) $\frac{1}{7}$
50. A bag has four pairs of balls of four distinct colours. If four balls are picked at random (without replacement), then the probability that there is atleast one pair among them have the same colour, is [J&K CET 2011]
 (a) $\frac{1}{7!}$ (b) $\frac{8}{35}$ (c) $\frac{19}{35}$ (d) $\frac{27}{35}$
51. If A and B are two non-empty sets with B^C denoting the complement of set B such that $B \subset A$, then which of the following probability statements hold correct? [UP SEE 2011]
 (a) $P(A \cap B^C) = P(B) - P(A)$
 (b) $P(A \cap B^C) = P(A) - P(B)$
 (c) $P(B) \leq P(A)$
 (d) $P(A) \leq P(B)$
52. If both to male child and birth to female child are equal probable, then what is the probability that atleast one of the three children born to a couple is male? [UP SEE 2010]
 (a) $\frac{4}{5}$ (b) $\frac{7}{8}$ (c) $\frac{8}{7}$ (d) $\frac{1}{7}$
53. Two dice are tossed once. The probability of getting even number at the first die or a total of 8, is [WB JEE 2010]
 (a) $\frac{1}{36}$ (b) $\frac{3}{36}$
 (c) $\frac{11}{36}$ (d) $\frac{5}{9}$

Answers

Work Book Exercise 21.1

1. (a) 2. (c) 3. (c) 4. (c) 5. (c) 6. (c)

Work Book Exercise 21.2

1. (c) 2. (c) 3. (a) 4. (c) 5. (c) 6. (d) 7. (c) 8. (a)

Work Book Exercise 21.3

1. (c) 2. (b) 3. (d) 4. (a) 5. (a) 6. (a) 7. (b)

Target Exercises

1. (c) 2. (a) 3. (a) 4. (d) 5. (a) 6. (d) 7. (b) 8. (b) 9. (a) 10. (d)
 11. (b) 12. (a) 13. (c) 14. (d) 15. (b) 16. (d) 17. (c) 18. (b) 19. (c) 20. (d)
 21. (b) 22. (c) 23. (a) 24. (a) 25. (a) 26. (b) 27. (a) 28. (a) 29. (c) 30. (a)
 31. (a) 32. (c,d) 33. (b,c,d) 34. (a,c,d) 35. (a,b) 36. (c) 37. (b) 38. (b) 39. (c) 40. (a)
 41. (b) 42. (c) 43. (b) 44. (b) 45. (d) 46. (*) 47. (**) 48. (6) 49. (7) 50. (7)
 51. (1)

* $A \rightarrow r; B \rightarrow s; C \rightarrow p; D \rightarrow q$

** $A \rightarrow s; B \rightarrow r; C \rightarrow p; D \rightarrow q$

Entrances Gallery

1. (a) 2. (b) 3. (c) 4. (a) 5. (6) 6. (a) 7. (a,d) 8. (c) 9. (a) 10. (c)
 11. (c) 12. (d) 13. (c) 14. (c) 15. (d) 16. (a) 17. (b) 18. (a) 19. (c) 20. (d)
 21. (a) 22. (b) 23. (a) 24. (d) 25. (c) 26. (a) 27. (c) 28. (d) 29. (c) 30. (c)
 31. (b) 32. (a) 33. (b) 34. (a) 35. (a) 36. (a) 37. (d) 38. (b) 39. (a) 40. (a)
 41. (d) 42. (d) 43. (a) 44. (b) 45. (b) 46. (c) 47. (a) 48. (a) 49. (b) 50. (d)
 51. (c) 52. (b) 53. (d)

Explanations

Target Exercises

1. Let the two boys be taken as B_1 and B_2 and the three girls be taken as G_1 , G_2 and G_3 . Clearly, there are 5 children, out of which two children can be chosen in ${}^5C_2$ ways. So, there are ${}^5C_2 = 10$ elementary events associated to this experiments and are given by $B_1B_2, B_1G_1, B_1G_2, B_1G_3, B_2G_1, B_2G_2, B_2G_3, G_1G_2, G_1G_3$ and G_2G_3 .

2. The sample space associated with the given random experiment is given by

$$S = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$$

Let event E = a total of atleast 10

$$E = \{(6, 4), (4, 6), (5, 5), (6, 6), (5, 6), (6, 5)\}$$

3. A and B are not mutually exclusive.
4. Since, the events A and B are mutually exclusive, these events are not independent. Hence, the required probability cannot be found.
5. Out of 6 fruits, 2 fruits can be selected in ${}^6C_2$ ways. Also, 1 apple and 1 mango can be selected in 3×3 ways. Hence, the required probability = $\frac{9}{{}^6C_2} = \frac{3}{5}$

6. Let each of the friend have x daughters. Then, the probability that all the tickets go to the daughters of A is $\frac{{}^xC_3}{{}^{2x}C_3}$.

$$\therefore \frac{{}^xC_3}{{}^{2x}C_3} = \frac{1}{20} \Rightarrow \frac{(x-2)}{4(2x-1)} = \frac{1}{20}$$

$$\Rightarrow 5x - 10 = 2x - 1$$

$$\Rightarrow x = 3$$

7. The total number of ways of selecting 3 tickets out of $6n$ numbers is ${}^{6n}C_3$.

The sum of the numbers on three tickets drawn will be $6n$ in each of the following cases :

Lowest number	Ways of selection	Total number of ways
0	$(0, 1, 6n-1) (0, 2, 6n-2) \dots (0, 3n-1, 3n+1)$	$(3n-1)$
1	$(1, 2, 6n-3) (1, 3, 6n-4) \dots (1, 3n-1, 3n+1)$	$(3n-2)$
2	$(2, 3, 6n-5) (2, 4, 6n-6) \dots (2, 3n-2, 3n)$	$(3n-4)$
3	$(3, 4, 6n-7) (3, 5, 6n-8) \dots (3, 3n-2, 3n-1)$	$(3n-5)$
$\vdots$	$\vdots$	$\vdots$
$(2n-2)$	$(2n-2, 2n-1, 2n+3), (2n-2, 2n, 2n+2),$	2
$(2n-1)$	$(2n-1, 2n, 2n+1)$	1

Total number of ways of getting $6n$ as the sum

$$= (3n-1) + (3n-2)(3n-3) + (3n-5) + \dots + 2 + 1$$

$$= \{3n-1 + 3n-2\} + \{3n-4 + 3n-5\} + \dots + \{5+4\} + \{2+1\}$$

$$= (6n-3) + (6n-9) + \dots + 9 + 3 = 3n^2$$

$$\therefore \text{Required probability} = \frac{3n^2}{{}^{6n}C_3}$$

$$8. P(\text{none of the two cards is an ace}) = \frac{11}{20}$$

$$\therefore P(\text{atleast one ace}) = \frac{9}{20}$$

$$9. P(\text{first card is an ace and the second card is an ace}) = \frac{1}{169}$$

10. From the bag, two socks can be drawn in ${}^9C_2$ ways.

Also, two socks of the same colour can be drawn in ${}^5C_2 + {}^4C_2$ ways.

$$\text{Hence, the required probability is } \frac{{}^5C_2 + {}^4C_2}{{}^9C_2} = \frac{4}{9}$$

11. Total number of ways in which all the balls can be placed is $\frac{10!}{7!3!} = 120$.

If no two black balls are placed adjacently, then the possible number of ways is 56.

$$\text{Hence, the required chance is } \frac{56}{120} = \frac{7}{15}$$

12. There are seven floors besides the ground floor.

$$\therefore \text{Number of possible outcomes} = 7^5$$

$$\text{Number of favourable cases} = {}^7P_5$$

$$\therefore \text{Required probability} = \frac{{}^7P_5}{7^5}$$

13. 3 can be thrown as $(1, 2), (2, 1)$.

5 can be thrown as $(1, 4), (4, 1), (2, 3), (3, 2)$.

11 can be thrown as $(5, 6), (6, 5)$.

$$\therefore \text{Required probability} = \frac{8}{36} = \frac{2}{9}$$

14. Probability that one test is held = $\frac{1}{5} \times \frac{4}{5} + \frac{4}{5} \times \frac{1}{5} = \frac{8}{25}$

$$\therefore \text{Probability that the test is held on both days} = \frac{1}{25}$$

$$\therefore \text{Probability that the student misses atleast one test} = \frac{8}{25} + \frac{1}{25} = \frac{9}{25}$$

15. $P(A) = \frac{1}{3}, P(A \cup B) = \frac{3}{4}$

$$\text{Now, } P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$$

$$\Rightarrow \frac{3}{4} \leq \frac{1}{3} + P(B) \Rightarrow \frac{5}{12} \leq P(B)$$

$$\text{Also, } B \subseteq A \cup B \Rightarrow P(B) \leq P(A \cup B) = \frac{3}{4}$$

$$\therefore \frac{5}{12} \leq P(B) \leq \frac{3}{4}$$

16. Let probability for appearance of odd number be p .
Then, probability for appearance of even number is $2p$.

$$\therefore p + 2p = 1 \Rightarrow p = \frac{1}{3}$$

Sum of two numbers is even means either both numbers are even or odd.

$$\therefore \text{Required probability} = \frac{5}{9}$$

$$17. \text{Required probability} = \frac{{}^3C_3 + {}^4C_3}{{}^9C_3} = \frac{5}{84}$$

18. Let E_1 denotes the event of travelling by train and E_2 denotes the event of travelling by plane.

$$\text{Then, } P(E_1) = \frac{2}{3}, P(E_2) = \frac{1}{5}$$

$$\therefore P(E_1 \cup E_2) = P(E_1) + P(E_2) = \frac{13}{15}$$

19. The probability that both Krishna and Hari will be alive after 10 yr, is $\frac{49}{150}$.

$$20. P(A \cap B) \leq \min \{P(A), P(B)\} = \min \{0.65, 0.80\} = 0.65$$

$$\therefore P(A \cap B) \leq 0.65$$

$$\text{Also, } P(A \cap B) = P(A) + P(B) - P(A \cup B) \\ \geq 0.65 + 0.80 - 1 = 0.45$$

$$\therefore 0.45 \leq P(A \cap B) \leq 0.65$$

21. Let $P(A) = a$ and $P(B) = b$, then we have $ab = \frac{1}{6}$ and

$$(1-a)(1-b) = \frac{1}{3}. \text{ Solving these equations simultaneously, we have } a = \frac{1}{2} \text{ or } \frac{1}{3}. \text{ Minimum value of these is } \frac{1}{3}.$$

22. Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the inscribed circle has the equation $x^2 + y^2 = b^2$, P = probability that the point of the ellipse lies outside the circle $= 1 - \frac{\pi b^2}{\pi ab}$

$$\Rightarrow P = 1 - \frac{b}{a} = 1 - \frac{a\sqrt{1-e^2}}{a} = 1 - \sqrt{1-e^2}$$

where, e is the eccentricity of the ellipse.

$$\Rightarrow 1 - e^2 = (1 - P)^2$$

$$\Rightarrow e = \sqrt{1 - (1 - P)^2}$$

$$23. P(A \cap B) = \frac{1}{6}, P(\bar{A} \cap \bar{B}) = \frac{1}{3}$$

$$\Rightarrow P(A)P(B) = \frac{1}{6} \quad \dots(i)$$

$$\text{and } P(\bar{A} \cup \bar{B}) = \frac{1}{3} \Rightarrow P(A \cup B) = \frac{2}{3}$$

$$\therefore P(A) + P(B) - P(A \cap B) = \frac{2}{3}$$

$$\Rightarrow P(A) + P(B) = \frac{5}{6} \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$$

$$\text{or } P(A) = \frac{1}{3}, P(B) = \frac{1}{2}$$

24. Let $P(B) = b$. We have,

$$P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B), \text{ if } A \text{ and } B \text{ are independent events.}$$

$$\text{Substituting the given values, we have } P(B) = \frac{5}{7}$$

25. Consider the events.

A = Getting a total of 4 or more in the throw of two dice

B = Getting a total of 6 or more in the simultaneous throw of three dice

$$\therefore \text{Required probability} = P(A \cap B) = P(A) \cdot P(B) \quad \dots(i)$$

For $P(A)$

Total number of elementary events $= 6^2$

Favourable events $= 33$

$$\therefore P(A) = \frac{33}{6^2} = \frac{11}{12}$$

For $P(B)$

Total number of elementary events $= 6^3$

Favourable number of elementary events

$$= 6^3 - \{\text{Sum of the coefficients of } x^3, x^4 \text{ and } x^5 \text{ in } (x + x^2 + \dots + x^6)^3\}$$

$$= 6^3 - (1 + 3 + 6) = 206$$

$$\therefore P(B) = \frac{206}{216} = \frac{103}{108}$$

$$\therefore \text{Required probability} = \frac{11}{12} \times \frac{103}{108} = \frac{1133}{1296}$$

26. Let the name of the man be A and that of his wife be B . Then, P (A will not be alive after 10 yr and B will not be alive after 10 yr) $= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

27. Probability of getting a total of 7 or 8 on a pair of dice, is $\frac{11}{36}$.

$$P(\text{head shows up and number 7 or 8 is noted}) = \frac{1}{2} \times \frac{11}{36}$$

Also, P (tail shows up and number 7 or 8 is noted)

$$= \frac{1}{2} \times \frac{11}{36}$$

$$\text{Hence, the required probability} = \frac{193}{792}$$

28. The probability of getting a total of 5 on a pair of dice, is $\frac{1}{9}$

$$\text{and that of getting a total of 7, is } \frac{1}{6}. \text{ Also, } P(\bar{5} \cap \bar{7}) = \frac{13}{18}$$

because there are 10 cases in favour of getting a sum of 5 or 7.5 can come before 7 in the following cases :

5 or $(\bar{5} \cap \bar{7})$ or $(\bar{5} \cap \bar{7})$ or $(\bar{5} \cap \bar{7})$ or ...

Hence, the required probability $= 2/5$

29. Required probability $= 1$

$$30. P(E \cap F) = \frac{1}{12} \text{ and } P(\bar{E} \cap \bar{F}) = \frac{1}{2}$$

$$\Rightarrow P(E) \cdot P(F) = \frac{1}{12}$$

$$\text{and } \{1 - P(E)\} \cdot \{1 - P(F)\} = \frac{1}{2}$$

$$\text{Let } P(E) = x \text{ and } P(F) = y$$

$$\Rightarrow xy = \frac{1}{12} \text{ and } (1-x)(1-y) = \frac{1}{2}$$

$$\therefore \left(x = \frac{1}{3} \text{ and } y = \frac{1}{4}\right) \text{ or } \left(x = \frac{1}{4} \text{ and } y = \frac{1}{3}\right)$$

$$31. P(\text{first event does not happen}) = \frac{5}{5+2} = \frac{5}{7}$$

$$P(\text{second event does not happen}) = \frac{5}{5+6} = \frac{5}{11}$$

$$\therefore P(\text{both the event fail to happen}) = \frac{5}{7} \times \frac{5}{11} = \frac{25}{77}$$

Therefore, the probability that atleast one of the events will happen

$$\begin{aligned} &= 1 - P(\text{none of two happens}) \\ &= 1 - \frac{25}{77} = \frac{52}{77} \end{aligned}$$

$$\begin{aligned} 32. \text{ We have, } P(\text{exactly one of } A, B \text{ occurs}) \\ &= P[(A \cap \bar{B}) \cup (\bar{A} \cap B)] \\ &= P(A \cap \bar{B}) + P(\bar{A} \cap B) \\ &= P(A) - P(A \cap B) + P(B) - P(A \cap B) \\ &= P(A) + P(B) - 2P(A \cap B) \\ &= P(A \cup B) - P(A \cap B) \end{aligned}$$

$$\begin{aligned} \text{Also, } P(\text{exactly one of } A, B \text{ occurs}) \\ &= P(A \cup B) - P(A \cap B) \\ &= \{1 - P(\bar{A} \cup \bar{B})\} - \{1 - P(\bar{A} \cap \bar{B})\} \\ &= \{1 - P(\bar{A} \cap \bar{B})\} - \{1 - P(\bar{A} \cup \bar{B})\} \\ &= P(\bar{A} \cup \bar{B}) - P(\bar{A} \cap \bar{B}) \\ &= P(\bar{A}) + P(\bar{B}) - 2P(\bar{A} \cap \bar{B}) \end{aligned}$$

$$\begin{aligned} 33. \text{ We have, } \\ P(\text{exactly two of } A, B, C \text{ occur}) \\ &= P(A \cap B \cap \bar{C}) + P(A \cap \bar{B} \cap C) + P(\bar{A} \cap B \cap C) \\ &= P(A \cap B) - P(A \cap B \cap C) + P(A \cap C) \\ &\quad - P(A \cap B \cap C) + P(B \cap C) - P(A \cap B \cap C) \\ &= P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C) \\ &\leq P(A \cap B) + P(B \cap C) + P(A \cap C) \end{aligned}$$

$$\begin{aligned} \text{Also, } P(A \cup B \cup C) \\ &= P(A \cup B) + P(C) - P\{(A \cup B) \cap C\} \\ &\leq P(A \cup B) + P(C) \\ &\leq P(A) + P(B) + P(C) \end{aligned}$$

$$\begin{aligned} \text{Now, } P(\text{exactly one of } A, B, C \text{ occurs}) \\ &= P(A \cap \bar{B} \cap \bar{C}) + P(\bar{A} \cap B \cap \bar{C}) + P(\bar{A} \cap \bar{B} \cap C) \\ &= P(A \cap \bar{B} \cup C) + P(\bar{A} \cap B \cap C) + P(B \cap \bar{A} \cup \bar{C}) \\ &= P(A) - P\{A \cap (B \cup C) \cup (A \cap C)\} + P(C) \\ &\quad - P\{(C \cap A) \cup (C \cap B)\} + P(B) - P\{(B \cap A) \cup (B \cap C)\} \\ &= P(A) + P(B) + P(C) - 2P(A \cap B) - 2P(B \cap C) \\ &\quad - 2P(A \cap C) + 3P(A \cap B \cap C) \\ &= [P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C)] \\ &\quad - [P(A \cap B) + P(B \cap C) + P(A \cap C) - 3P(A \cap B \cap C)] \\ &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) \\ &\quad - P(\text{exactly two of } A, B, C \text{ occur}) \\ &\leq P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) \end{aligned}$$

$$\begin{aligned} \text{Finally, } P(A \text{ and atleast one of } B, C \text{ occurs}) \\ &= P[A \cap (B \cup C)] \\ &= P[(A \cap B) \cup (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)] \\ &= P(A \cap B) + P(A \cap C) - P(A \cap B \cap C) \\ &\leq P(A \cap B) + P(A \cap C) \end{aligned}$$

$$34. \text{ We have, } P(A \cup B) \geq \max\{P(A), P(B)\} = \frac{2}{3}$$

$$\text{and } P(A \cap B) \leq \min\{P(A), P(B)\} = \frac{2}{3}$$

$$\begin{aligned} \text{Now, } P(A \cap B) &= P(A) + P(B) - P(A \cup B) \\ \Rightarrow P(A \cap B) &= P(A) + P(B) - 1 = \frac{1}{6} \end{aligned}$$

$$\therefore \frac{1}{6} \leq P(A \cap B) \leq \frac{1}{2}$$

$$\text{Also, } P(A \cap \bar{B}) = P(A) - P(A \cap B) \leq \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

$$\text{and } P(\bar{A} \cap B) = P(B) - P(A \cap B) \leq \frac{2}{3} - \frac{1}{6} = \frac{1}{2}$$

$$\text{Hence, } \frac{2}{3} - \frac{1}{2} \leq P(\bar{A} \cap B) \leq \frac{2}{3} - \frac{1}{6}$$

$$\Rightarrow \frac{1}{6} \leq P(\bar{A} \cap B) \leq \frac{1}{2}$$

35. Let $P(A) = x$ and $P(B) = y$. Since, A and B are independent events. Therefore,

$$\begin{aligned} P(\bar{A} \cap B) &= \frac{2}{15} \\ \Rightarrow P(\bar{A})P(B) &= \frac{2}{15} \\ \Rightarrow \{1 - P(A)\}P(B) &= \frac{2}{15} \\ \Rightarrow (1 - x)y &= \frac{2}{15} \\ \Rightarrow y - xy &= \frac{2}{15} \end{aligned} \quad \dots(i)$$

$$\begin{aligned} \text{and } P(A \cap \bar{B}) &= \frac{1}{6} \\ \Rightarrow P(A)P(\bar{B}) &= \frac{1}{6} \\ \Rightarrow x(1 - y) &= \frac{1}{6} \\ \Rightarrow x - xy &= \frac{1}{6} \end{aligned} \quad \dots(ii)$$

On subtracting Eq. (i) from Eq. (ii), we get

$$x - y = \frac{1}{30} \Rightarrow x = \frac{1}{30} + y$$

Putting this value of x in Eq. (i), we get

$$\begin{aligned} y - y\left(\frac{1}{30} + y\right) &= \frac{2}{15} \\ \Rightarrow 30y - y - 30y^2 &= 4 \\ \Rightarrow 30y^2 - 29y + 4 &= 0 \\ \Rightarrow (6y - 1)(5y - 4) &= 0 \\ \Rightarrow y = \frac{1}{6} \text{ or } y = \frac{4}{5} \\ \Rightarrow P(B) = \frac{1}{6} \text{ or } P(B) = \frac{4}{5} \end{aligned}$$

36. Statement I is true, as there are six equally likely possibilities of which only two are favourable (4 and 6).

$$\text{Hence, } P(\text{obtained number is composite}) = \frac{2}{6} = \frac{1}{3}$$

Statement II is not true, as three possibilities are not equally likely.

$$37. P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore 1 \geq P(A) + P(B) - P(A \cap B) \geq \frac{3}{4}$$

$$\Rightarrow P(A) + P(B) - \frac{1}{8} \geq \frac{3}{4}$$

$$\left[\because \text{minimum value of } P(A \cap B) = \frac{1}{8} \right]$$

$$\Rightarrow P(A) + P(B) \geq \frac{1}{8} + \frac{3}{4} = \frac{7}{8}$$

As the maximum value of $P(A \cap B) = \frac{3}{8}$, we get

$$1 \geq P(A) + P(B) - \frac{3}{8}$$

$$\Rightarrow P(A) + P(B) \leq 1 + \frac{3}{8} = \frac{11}{8}$$

$$\begin{aligned} 38. P(A \cup \bar{B}) &= 1 - \overline{(A \cup \bar{B})} = 1 - \overline{(\bar{A} \cap B)} \\ &= 1 - P(\bar{A})P(B) \\ 0.8 &= 1 - 0.7 \times P(B) \\ \Rightarrow P(B) &= \frac{2}{7} \end{aligned}$$

39. $P(A) + P(B) = 1$ is true, as A and B are mutually exclusive and exhaustive events but Statement II is false, as it is not given that the events are exhaustive.

40. A = Getting an even number on the first die
 $= \{(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),$
 $(4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6),$
 $(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)\}$

B = Getting an odd number on first die
 $= \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$
 $(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),$
 $(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6)\}$

A and $B = A \cap B = \phi$

41. Event B given above

and C = Getting atmost 5 as sum of the numbers on the two dice

$C = \{(1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (3, 1),$
 $(3, 2), (4, 1)\}$

B or $C = B \cup C = \{(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),$
 $(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3),$
 $(3, 4), (3, 5), (3, 6), (4, 1), (5, 1), (5, 2),$
 $(5, 3), (5, 4), (5, 5), (5, 6)\}$

$$\therefore n(B \cup C) = 22$$

42. The set of event B as above and the set of event C as above.

Now, set of events B and C

$$= B \cap C = \{(1, 1), (1, 2), (1, 3), (1, 4), (3, 1), (3, 2)\}$$

43. Total number of ways = M^n

Number of favourable cases = $n!$

44. Total number of ways = $M + n - 1C_{M-1}$

Number of favourable cases = 1

45. Total number of ways = $M C_n$

Number of favourable cases = 1

46. A. $\begin{matrix} B & B & B & B & B & B & B \\ | & | & | & | & | & | & | \\ W & | & W & | & W & | & W \end{matrix}$

$$n(S) = \frac{10!}{(7!)(3!)}$$

$$n(E) = {}^8C_3 = \frac{8!}{(3!)(5!)} \text{ because there are 8 places for 3 black balls.}$$

$$\therefore P(E) = \frac{\frac{8!}{(3!)(5!)}}{\frac{10!}{(7!)(3!)}} = \frac{(8!)(7!)}{(10!)(5!)} = \frac{7 \cdot 6}{10 \cdot 9} = \frac{7}{15}$$

B. $n(S) = 7!, n(E) = (3!) \times (4!)$

[$\therefore$ after making 4 gentlemen sit in $3!$ ways, 4 ladies can sit in $4!$ ways in between the gentlemen]

$$\therefore P(E) = \frac{(3!) \times (4!)}{7!} = \frac{6}{7 \times 6 \times 5} = \frac{1}{35}$$

C. $n(S) = {}^{12}C_4 \times {}^8C_4 \times {}^4C_4$

$n(E) = n(S)$ - number of ways in which one boy gets both the pens
 $= n(S) - {}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)$

$$\therefore P(E) = 1 - \frac{{}^{10}C_2 \times {}^8C_4 \times {}^4C_4 \times (3!)}{{}^{12}C_4 \times {}^8C_4 \times {}^4C_4} = 1 - \frac{6}{11} = \frac{5}{11}$$

D. Required probability

$$\begin{aligned} &= P(\overline{E} \overline{E} \overline{E} \overline{E} \overline{E} \overline{E}) + P(\overline{E} \overline{E} \overline{E} \overline{E} E) \\ &= \{P(\overline{E})\}^4 \cdot P(E) + \{P(\overline{E})\}^3 \cdot P(E) \\ &= \frac{1}{16} \end{aligned}$$

47. A. $n(S) = {}^7C_3 \times 3! = \frac{7 \cdot 6 \cdot 5}{6} \cdot 6 = 210$

$$n(E) = {}^2C_1 \times {}^2C_1 \times {}^1C_1 \times 3!$$



On the left and the other has to sit at any one of the two marked seats on the right.

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2 \times 2 \times 6}{210} = \frac{4}{35}$$

B. $n(S) = {}^{10}C_3$ and $n(E) = {}^3C_1$ because on selecting 3, 7 and we have to select one from 4, 5 and 6

$$\therefore P(E) = \frac{{}^3C_1}{{}^{10}C_3} = \frac{1}{40}$$

C. $n(S) = {}^{10}C_4 = 210$

$$n(E) = {}^5C_2 \times {}^3C_1 \times {}^2C_1 + {}^5C_1 \times {}^3C_2 \times {}^2C_1 + {}^5C_1 \times {}^3C_1 \times {}^2C_2$$

$$= 105$$

$$\therefore P(E) = \frac{105}{210} = \frac{1}{2}$$

D. The probability of 4 being the minimum number

$$= \frac{{}^6C_2}{{}^{10}C_3}$$

[after selecting 4, any two can be selected from 5, 6, 7, 8, 9, 10]

The probability of 4 being the minimum number and 8 being the maximum number = $\frac{{}^7C_3}{{}^{10}C_3}$

$$\begin{aligned} \therefore \text{Required probability} &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= \frac{{}^6C_2}{{}^{10}C_3} + \frac{{}^7C_3}{{}^{10}C_3} - \frac{3}{{}^{10}C_3} = \frac{11}{40} \end{aligned}$$

48. Total 4-digit numbers formed



$$4 \times 4 \times 3 \times 2 = 96$$

Each of these 96 numbers are equally likely and mutually exclusive of each other.

Now, a number is divisible by 4, if last two digits of the number is divisible by 4.

Hence, we can have

$\begin{array}{|c|c|c|c|} \hline & & 0 & 4 \\ \hline \end{array} \rightarrow$ first two places can be filled in
 $3 \times 2 = 6$ ways

$\begin{array}{|c|c|c|c|} \hline & & 1 & 2 \\ \hline \end{array} \rightarrow$ first two places can be filled in
 $2 \times 2 = 4$ ways

$\begin{array}{|c|c|c|c|} \hline & & 2 & 0 \\ \hline \end{array} \rightarrow$ 6 ways

$\begin{array}{|c|c|c|c|} \hline & & 2 & 4 \\ \hline \end{array} \rightarrow$ 4 ways

$\begin{array}{|c|c|c|c|} \hline & & 3 & 2 \\ \hline \end{array} \rightarrow$ 4 ways

$\begin{array}{|c|c|c|c|} \hline & & 4 & 0 \\ \hline \end{array} \rightarrow$ 6 ways

Total number of ways 30 ways
 $\therefore$ Probability = $\frac{\text{Favourable outcomes}}{\text{Total outcomes}} = \frac{30}{96} = \frac{5}{16} = \frac{a}{b}$
 $\Rightarrow b - 2a = 6$

49. Event $A = \{1, 3, 5\}$, event $B = \{3, 6\}$ and

event $C = \{1, 2, 3, 4\}$
 $\therefore A \cap B = \{3\}$, $B \cap C = \{3\}$, $A \cap C = \{1, 3\}$
 and $A \cap B \cap C = \{3\}$
 Thus, P (exactly two of A , B and C occur)
 $= P(A \cap B) + P(B \cap C) + P(C \cap A) - 3P(A \cap B \cap C)$
 $= \frac{1}{6} + \frac{1}{6} + \frac{2}{6} - 3 \times \frac{1}{6}$
 $= \frac{1}{6} = \frac{a}{b}$
 $\Rightarrow a + b = 7$

50. Let the number of marbles be $2n$, where n is large.

$$\text{Required probability} = \lim_{n \rightarrow \infty} \frac{n \times {}^nC_4}{2^n C_5} \times \frac{{}^nC_3 \times {}^nC_2}{2^n C_5}$$

Entrances Gallery

1. Either a girl will start the sequence or will be at second position and will not acquire the last position as well.

$$\therefore \text{Required probability} = \frac{{}^3C_1 + {}^2C_1}{5C_2} = \frac{1}{2}$$

2. **Case I** 1 odd, 2 even

Total number of ways
 $= 2 \times 2 \times 3 + 1 \times 3 \times 3 + 1 \times 2 \times 4 = 29$

Case II All 3 odd

Number of ways $= 2 \times 3 \times 4 = 24$

Favourable ways $= 53$

$$\therefore \text{Required probability} = \frac{53}{3 \times 5 \times 7} = \frac{53}{105}$$

3. Here, $2x_2 = x_1 + x_3$

$\Rightarrow x_1 + x_3 = \text{even}$

Hence, number of favourable ways

$$= {}^2C_1 \cdot {}^4C_1 + {}^1C_1 \cdot {}^3C_1 = 11$$

$$\therefore \text{Required probability} = \frac{{}^2C_1 \times {}^4C_1 + {}^1C_1 \times {}^3C_1}{{}^3C_1 \times {}^5C_1 \times {}^7C_1} = \frac{11}{105}$$

4. P (atleast one of them solves correctly)

$$= 1 - P(\text{none of them solves correctly})$$

$$= 1 - \left(\frac{1}{2} \times \frac{1}{4} \times \frac{3}{4} \times \frac{7}{8} \right) = \frac{235}{256}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{n \times n(n-1)(n-2)(n-3)}{4!} \times \frac{n(n-1)(n-2)}{3!} \\ &\quad \times \frac{n(n-1)}{2!} \times \frac{(5)^2((2n-5)!)^2}{(2n!)^2} \\ &= \lim_{n \rightarrow \infty} \frac{n^4(n-1)^3(n-2)^2(n-3)((2n-5)!)^2 \times 5 \times 5 \times 4 \times 3!}{3!2!(2n!)^2} \\ &= \lim_{n \rightarrow \infty} \frac{50 \cdot n^4(n-1)^3(n-2)^2(n-3)}{(2n(2n-1)(2n-2)(2n-3)(2n-4))^2} \\ &= \frac{50}{1024} = \frac{25}{512} = \frac{2 \times 10 + 5}{512} \end{aligned}$$

$$\therefore a + b = 7$$

51. Consider the following events:

A = Switch S_1 works, B = Switch S_2 works
 We have,

$$P(A) = \frac{90}{100} = \frac{9}{10}$$

$$\text{and } P(B) = \frac{80}{100} = \frac{8}{10}$$

The circuit will work, if the current flows in the circuit.
 This is possible only when atleast one of the two switches S_1, S_2 works. Therefore, required probability

$$= P(A \cup B) = 1 - P(\bar{A})P(\bar{B})$$

[$\because A$ and B are independent events]

$$= 1 - \left(1 - \frac{9}{10}\right) \left(1 - \frac{8}{10}\right)$$

$$= 1 - \frac{1}{10} \times \frac{2}{10}$$

$$= \frac{49}{50} = \frac{a}{b} \Rightarrow b - a = 1$$

5. Let $P(E_1) = x$, $P(E_2) = y$ and $P(E_3) = z$

$$\text{Then, } (1-x)(1-y)(1-z) = p \quad \dots (i)$$

$$\Rightarrow x(1-y)(1-z) = \alpha \quad \dots (ii)$$

$$(1-x)y(1-z) = \beta \quad \dots (iii)$$

$$\text{and } (1-x)(1-y)z = \gamma \quad \dots (iv)$$

$$\therefore \frac{1-x}{x} = \frac{p}{\alpha} \Rightarrow x = \frac{\alpha}{\alpha + p}$$

$$\text{Similarly, } z = \frac{\gamma}{\gamma + p}$$

$$\therefore \frac{P(E_1)}{P(E_3)} = \frac{\frac{\alpha}{\alpha + p}}{\frac{\gamma}{\gamma + p}} = \frac{\frac{\gamma + p}{\alpha}}{\frac{\gamma}{\alpha}} = \frac{1 + \frac{p}{\alpha}}{1 + \frac{p}{\gamma}}$$

$$\text{Also, given } \frac{\alpha\beta}{\alpha - 2\beta} = p = \frac{2\beta\gamma}{\beta - 3\gamma} \Rightarrow \beta = \frac{5\alpha\gamma}{\alpha + 4\gamma}$$

$$\text{Substituting back } \left[\alpha - 2 \left(\frac{5\alpha\gamma}{\alpha + 4\gamma} \right) \right] p = \frac{\alpha \cdot 5\alpha\gamma}{\alpha + 4\gamma}$$

$$\Rightarrow \alpha p - 6p\gamma = 5\alpha\gamma$$

$$\Rightarrow \frac{p}{\gamma} - \frac{6p}{\alpha} = 5$$

$$\Rightarrow \frac{p}{\gamma} = 5 + \frac{6p}{\alpha}$$

$$\therefore \frac{P(E_1)}{P(E_3)} = \frac{6 \left(1 + \frac{p}{\alpha} \right)}{1 + \frac{p}{\alpha}} = 6$$

6. Required probability = $1 - \frac{6 \cdot 5^3}{6^4} = 1 - \frac{125}{216} = \frac{91}{216}$

7. Let $P(E) = e$ and $P(F) = f$

$$P(E \cup F) - P(E \cap F) = \frac{11}{25}$$

$$\Rightarrow e + f - 2ef = \frac{11}{25} \quad \dots(i)$$

and $P(\bar{E} \cap \bar{F}) = \frac{2}{25}$

$$\Rightarrow (1-e)(1-f) = \frac{2}{25}$$

$$\Rightarrow 1 - e - f + ef = \frac{2}{25} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$ef = \frac{12}{25} \quad \text{and} \quad e + f = \frac{7}{5}$$

On solving, we get

$$e = \frac{4}{5}, f = \frac{3}{5} \quad \text{or} \quad e = \frac{3}{5}, f = \frac{4}{5}$$

8. $r_1, r_2, r_3 \in \{1, 2, 3, 4, 5, 6\}$

r_1, r_2 and r_3 are of the form $3k, 3k+1, 3k+2$.

$$\therefore \text{Required probability} = \frac{3! \times {}^2C_1 \times {}^2C_1 \times {}^2C_1}{6 \times 6 \times 6} = \frac{6 \times 8}{216} = \frac{2}{9}$$

9. There seems to be ambiguity in this question. It should be mentioned that boxes are different and one particular box has 3 balls.

According to the question,

$$\frac{{}^3C_1 \times {}^{12}C_3 2^9 - {}^3C_2 {}^{12}C_3 {}^9C_3 + \frac{12! \times 3!}{3!3!6!3!}}{3^{12}}$$

$$\text{Then, number of ways} = \frac{{}^{12}C_3 \times 2^9}{3^{12}} = \frac{55}{3} \left(\frac{2}{3} \right)^{11}$$

10. $P(\overline{A \cup B}) = \frac{1}{6}$

$$\Rightarrow P(A \cup B) = \frac{5}{6}, P(A) = \frac{3}{4}$$

$$\Rightarrow P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{5}{6}$$

$$\Rightarrow P(B) = \frac{5}{6} - \frac{3}{4} + \frac{1}{4} = \frac{1}{3}$$

$$\Rightarrow P(A \cap B) = P(A) \cdot P(B) \Rightarrow \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$$

11. $n(S) = {}^{20}C_4$

Statement I Common difference is 1.

Total number of cases = 17

Common difference is 2.

Total number of cases = 14

Common difference is 3.

Total number of cases = 11

Common difference is 4.

Total number of cases = 8

Common difference is 5.

Total number of cases = 5

Common difference is 6.

Total number of cases = 2

Hence, the required probability

$$= \frac{17 + 14 + 11 + 8 + 5 + 2}{{}^{20}C_4} = \frac{1}{85}$$

12. $\therefore A = \{4, 5, 6\}$ and $B = \{1, 2, 3, 4\}$

$$\therefore (A \cap B) = \{4\}$$

Now, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$\Rightarrow P(A \cup B) = \frac{3}{6} + \frac{4}{6} - \frac{1}{6} = \frac{6}{6} = 1$$

13. All the three persons have three options to apply for a house.

$$\therefore \text{Total number of cases} = 3^3$$

Now, favourable cases = 3 (either all has applied for house 1 or 2 or 3)

$$\therefore \text{Required probability} = \frac{3}{3^3} = \frac{1}{9}$$

14. Given probabilities of speaking truth are

$$P(A) = \frac{4}{5} \quad \text{and} \quad P(B) = \frac{3}{4}$$

and their corresponding probabilities of not speaking truth are

$$P(\bar{A}) = \frac{1}{5} \quad \text{and} \quad P(\bar{B}) = \frac{1}{4}$$

The probability that they contradict each other

$$= P(A) \times P(\bar{B}) + P(\bar{A}) \times P(B)$$

$$= \frac{4}{5} \times \frac{1}{4} + \frac{1}{5} \times \frac{3}{4}$$

$$= \frac{1}{5} + \frac{3}{20} = \frac{7}{20}$$

15. The probability that Mr. A selected the losing horse

$$= \frac{4}{5} \times \frac{3}{4} = \frac{3}{5}$$

The probability that Mr. A selected the winning horse

$$= 1 - \frac{3}{5} = \frac{2}{5}$$

16. $\therefore 0 \leq P(A) \leq 1, 0 \leq P(B) \leq 1, 0 \leq P(C) \leq 1$

and $0 \leq P(A) + P(B) + P(C) \leq 1$

$$\therefore 0 \leq \frac{3x+1}{3} \leq 1$$

$$\Rightarrow -\frac{1}{3} \leq x \leq \frac{2}{3} \quad \dots(i)$$

$$0 \leq \frac{1-x}{4} \leq 1$$

$$\Rightarrow -3 \leq x \leq 1 \quad \dots(ii)$$

$$0 \leq \frac{1-2x}{2} \leq 1$$

$$\Rightarrow -\frac{1}{2} \leq x \leq \frac{1}{2} \quad \dots(iii)$$

and $0 \leq \frac{3x+1}{3} + \frac{1-x}{4} + \frac{1-2x}{2} \leq 1$

$$\Rightarrow 0 \leq 13 - 3x \leq 12$$

$$\Rightarrow \frac{1}{3} \leq x \leq \frac{13}{3} \quad \dots(iv)$$

From Eqs. (i), (ii), (iii) and (iv), we get

$$\frac{1}{3} \leq x \leq \frac{1}{2}$$

17. The total number of ways in which numbers can be choosed = $25 \times 25 = 625$

The number of ways in which either players can choose same numbers = 25

$$\therefore \text{Probability that they win a prize} = \frac{25}{625} = \frac{1}{25}$$

Thus, the probability that they will not win a prize in a single trial = $1 - \frac{1}{25} = \frac{24}{25}$

18. Since, the probabilities of solving the problem by A, B and C are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, respectively.

$\therefore$ Probability that the problem is not solved

$$\begin{aligned} P(\bar{A})P(\bar{B})P(\bar{C}) &= \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right) \\ &= \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{4} \end{aligned}$$

Hence, the probability that the problem is solved

$$= 1 - \frac{1}{4} = \frac{3}{4}$$

19. $\therefore$ Total number of ways = 5!

and favourable number of ways = $2 \cdot 4!$

$$\therefore \text{Required probability} = \frac{2 \cdot 4!}{5!} = \frac{2}{5}$$

20. Given numbers are 1, 2, 3 and 4.

Possibilities for unit's place digit (either 1 or 3) = 2

Possibilities for ten's place digit = 3

Possibilities for hundred's place digit = 2

Possibilities for thousand's place digit = 1

$\therefore$ Number of favourable outcomes = $2 \times 3 \times 2 \times 1 = 12$

Number of numbers formed by 1, 2, 3 and 4 (without repetition) = 4!

$$\therefore \text{Required probability} = \frac{12}{4 \times 3 \times 2} = \frac{1}{2}$$

21. Let $I = 7^n + 7^m$, then we observe that $7^1, 7^2, 7^3$ and 7^4 ends in 7, 9, 3 and 1, respectively. Thus, 7^i ends in 7, 9, 3 or 1 according as i is of the form $4k + 1$, $4k + 2$, $4k - 1$ or $4k$, respectively. If S is the sample space, then $n(S) = (100)^2$

$7^m + 7^n$ is divisible by 5, if

(i) m is of the form $4k + 1$ and n is of the form $4k - 1$ or

(ii) m is of the form $4k + 2$ and n is of the form $4k$ or

(iii) m is of the form $4k - 1$ and n is of the form $4k + 1$ or

(iv) m is of the form $4k$ and n is of the form $4k + 2$

Thus, number of favourable ordered pairs $(m, n) = 4 \times 25 \times 25$

$$\text{Hence, the required probability} = \frac{4 \times 25 \times 25}{(100)^2} = \frac{1}{4}$$

22. Total number of mappings from a set A into itself is n^n .

And the total number of one to one mapping is $n!$

$$\therefore \text{Required probability} = \frac{n!}{n^n}$$

23. Total number of elements in sample space of throwing two dice, $n(S) = 36$

Let E = Event of getting score of 5
 $= \{(1, 4), (4, 1), (2, 3), (3, 2)\}$

$$\begin{aligned} \Rightarrow n(E) &= 4 \\ \therefore P(E) &= \frac{n(E)}{n(S)} = \frac{4}{36} = \frac{1}{9} \end{aligned}$$

24. Total number of outcomes in sample space, $n(S) = 6 \times 6 = 36$

Let E = Event of getting maximum sum of numbers on two dice is 5

$$= \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2), (1, 4), (4, 1), (2, 3), (3, 2)\}$$

$$n(E) = 10$$

$\therefore$ Probability that the maximum sum of numbers on two dice is 5

$$= \frac{n(E)}{n(S)} = \frac{10}{36} = \frac{5}{18}$$

$\therefore$ Probability (the sum of numbers on two dice is more than 5

= $1 - \text{Probability of the maximum sum of numbers on two dice is 5}$

$$= 1 - \frac{5}{18} = \frac{13}{18}$$

25. Required probability

$$\begin{aligned} &= {}^{12}C_2 \times {}^{10}C_2 \times {}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2 \times \left(\frac{1}{6}\right)^{12} \\ &= \frac{12!}{10! \times 2!} \times \frac{10!}{8! \times 2!} \times \frac{8!}{6! \times 2!} \times \frac{6!}{4! \times 2!} \\ &\quad \times \frac{4!}{2! \times 2!} \times \frac{2!}{2! \times 0!} \times \left(\frac{1}{6}\right)^{12} = \frac{12!}{2^6 \times 6^{12}} \end{aligned}$$

26. Required probability = $\frac{{}^{13}C_1 \times {}^4C_2 \times {}^{12}C_1 \times {}^4C_3}{{}^{52}C_5}$

$$\begin{aligned} &= \frac{13 \times 6 \times 12 \times 4}{52 \times 51 \times 50 \times 49 \times 48} \\ &= \frac{5 \times 4 \times 3 \times 2 \times 1}{52 \times 51 \times 10 \times 49 \times 2} = \frac{6}{4165} \end{aligned}$$

27. Total sample space = $\{B_1B_2, B_1G_1, G_1B_2\}$

Favourable ways = $\{B_1G_1, G_1B_2\}$

$$\therefore \text{Required probability} = \frac{2}{3}$$

28. A leap year has 366 days in which 52 weeks and 2 days are extra. i.e. (Sunday, Monday), (Monday, Tuesday), (Tuesday, Wednesday), (Wednesday, Thursday), (Thursday, Friday), (Friday, Saturday), (Saturday, Sunday).

So, probability that a leap year contains 53 Sunday = $\frac{2}{7}$

29. In a regular hexagon, two equilateral triangles are possible.

$$\therefore \text{Required probability} = \frac{2}{{}^6C_3} = \frac{2}{6 \times 5 \times 4} = \frac{2}{20} = \frac{1}{10}$$

30. Sum of numbers appearing on the dice is 7.

i.e. $S = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

$\therefore n(S) = 6$

Let E = Event of getting 3 atleast once

$$n(E) = 2$$

$\therefore$ Required probability $= \frac{n(E)}{n(S)} = \frac{2}{6} = \frac{1}{3}$

31. Total number of letters in the word PROBABILITY = 11

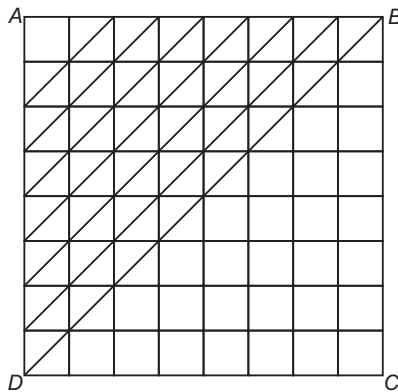
Total number of vowels in the word

PROBABILITY is 4, i.e. O, A, I, I.

Hence, probability that the single letter to be selected at random is vowel $= \frac{4}{11}$

32. We can choose three squares in a diagonal line parallel to BD in the $\triangle ABD$.

Hence, three squares in $\triangle ABD$ and a diagonal line can be chosen in ${}^3C_3 + {}^4C_3 + {}^5C_3 + {}^6C_3 + {}^7C_3 + {}^8C_3$ ways



Similarly, in $\triangle BCD$, the squares can be chosen parallel to BD in an equal number of ways.

Hence, the total number of ways in which three squares can be chosen in a diagonal line parallel to BD is

$$2({}^3C_3 + {}^4C_3 + {}^5C_3 + {}^6C_3 + {}^7C_3) + {}^8C_3$$

Since, BD is common to both the triangles. Similarly, squares can be chosen in a diagonal line parallel to AC and hence the total number of favourable ways

$$= 4({}^3C_3 + {}^4C_3 + {}^5C_3 + {}^6C_3 + {}^7C_3) + 2 \cdot {}^8C_3 = 392$$

Hence, the required probability

$$= \frac{392}{{}^{64}C_3} = \frac{392 \times 6}{64 \cdot 63 \cdot 62} = \frac{7}{744}$$

33. Total number of cards = 52

Total number of ace of hearts = 1

$\therefore$ Two cards are drawn simultaneously, such that atleast one of them is ace of heart.

$\therefore$ Required probability

$$= \frac{{}^1C_1 \times {}^{51}C_1}{{}^{52}C_2} = \frac{1 \times 51 \times 2}{52 \times 51} \left[\because {}^nC_r = \frac{n!}{r!(n-r)!} \right]$$

$$= \frac{2}{52} = \frac{1}{26}$$

34. Total number of boys = 10

Total number of girls = 8

Number of students have to be selected at random = 3

If 2 girls and 1 boy are selected, then the required probability

$$= \frac{{}^8C_2 \times {}^{10}C_1}{{}^{18}C_3} = \frac{\frac{8 \times 7}{2} \times 10}{\frac{18 \times 17 \times 16}{3 \times 2}} \left[\because {}^nC_r = \frac{n!}{r!(n-r)!} \right]$$

$$= \frac{4 \times 7 \times 10 \times 6}{18 \times 17 \times 16} = \frac{70}{3 \times 17 \times 4}$$

$$= \frac{35}{6 \times 17} = \frac{35}{102}$$

35. Since, $P(\text{exactly one of } A, B \text{ occurs}) = q$ (given), we get

$$P(A \cup B) - P(A \cap B) = q \Rightarrow p - P(A \cap B) = q$$

$$\Rightarrow P(A \cap B) = p - q \Rightarrow 1 - P(A' \cup B') = p - q$$

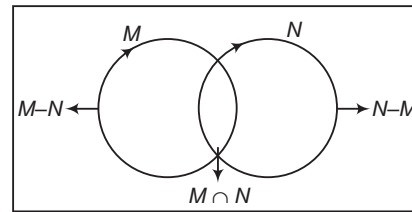
$$\Rightarrow P(A' \cup B') = 1 - p + q$$

$$\Rightarrow P(A') + P(B') - P(A' \cap B') = 1 + q - p$$

$$\Rightarrow P(A') + P(B') = (1 - p + q) + [1 - P(A \cup B)]$$

$$= (1 - p + q) + (1 - p) = 2 - 2p + q$$

36. We have two events M and N .



Clearly, $P(\text{exactly one of them occurs})$

$$= P(M - N) + P(N - M)$$

$$= P(M) - P(M \cap N) + P(N) - P(M \cap N)$$

$$= P(M) + P(N) - 2P(M \cap N)$$

37. Given, $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$$\therefore P(A \cup B)' = 1 - P(A \cup B)$$

$$= 1 - [P(A) + P(B) - P(A)P(B)]$$

$$= 1[1 - P(A)] - P(B)[1 - P(A)] = [1 - P(A)][1 - P(B)]$$

$$= [1 - P(A)]P(B')$$

38. Given, $P(A') = \frac{2}{3}$ and $P(B') = \frac{2}{7}$

Since, events A and B are independent.

$$\therefore P(A \cap B) = P(A) \cdot P(B) = [1 - P(A')][1 - P(B')]$$

$$= \left(1 - \frac{2}{3}\right) \left(1 - \frac{2}{7}\right) = \frac{1}{3} \times \frac{5}{7} = \frac{5}{21}$$

39. Given, $P(A \cup B) = 0.83$, $P(A) = 0.3$ and $P(B) = 0.6$

$$\therefore P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$\therefore 0.83 = 0.3 + 0.6 - P(A \cap B)$$

$$\Rightarrow P(A \cap B) = 0.9 - 0.83 = 0.07$$

$$\text{Now, } P(A) \cdot P(B) = 0.3 \times 0.6 = 0.18$$

$$\therefore P(A) \cdot P(B) \neq P(A \cap B)$$

Hence, events are dependent.

40. In any number, the last digit can be 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. Therefore, last digit of each number can be chosen in 10 ways.

Thus, exhaustive number of ways $= 10^n$

If the last digit is 1, 3, 7 or 9, none of the numbers can be even or end in 0 to 5. Thus, we have a choice of 4 digits, viz. 1, 3, 7 or 9 with each of n numbers should end.

So, favourable number of ways $= 4^n$

$$\text{Hence, required probability} = \frac{4^n}{10^n} = \left(\frac{2}{5}\right)^n$$

41. $\left| \begin{array}{c} 8 \text{ Red } 5 \text{ White} \\ \text{Urn} \end{array} \right|$

Total number of ways of selecting 3 balls $= {}^{13}C_3$ and
number of ways of selecting 3 balls including both
colours

$$= {}^8C_2 \times {}^5C_1 + {}^8C_1 \times {}^5C_2$$

Hence, probability that balls of both colours are drawn

$$= \frac{{}^8C_2 \times {}^5C_1 + {}^8C_1 \times {}^5C_2}{{}^{13}C_3}$$

$$= \frac{140 + 80}{286}$$

$$= \frac{220}{286} = \frac{110}{143} = \frac{10}{13}$$

42. Since, there are 52 distinct cards in decks and in two
decks each distinct card is 2 in number.

Therefore, 2 decks i.e. 104 cards will also contain only
52 distinct cards.

$\therefore$ Probability that the player gets all distinct cards

$$= \frac{{}^{52}C_{26} \times ({}^2C_1)^{26}}{{}^{104}C_{26}} = \frac{{}^{52}C_{26} \times 2^{26}}{{}^{104}C_{26}}$$

43. Total number of outcomes in sample space,

$$\text{i.e. } n(S) = 6 \times 6 = 36$$

Let E be the event of getting the product perfect
square.

$$\therefore E = \{(1, 1), (2, 2), (4, 1), (1, 4), (3, 3), (4, 4), (5, 5), (6, 6)\}$$

$$\therefore n(E) = 8$$

$$\text{Hence, the required probability} = \frac{n(E)}{n(S)} = \frac{8}{36} = \frac{2}{9}$$

44. $\left| \begin{array}{c} 3 \text{ white } 5 \text{ black} \\ \text{Bag} \end{array} \right|$

$$\text{Total number of balls} = 8$$

$$\text{Number of white balls} = 3$$

$$\therefore \text{Required probability} = \frac{3}{8}$$

45. Total number of outcomes = 216

and number of favourable outcomes is equal to 20%,
which are given below

(1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6), (1, 3, 4), (1, 3, 5),
(1, 3, 6), (1, 4, 5), (1, 4, 6), (1, 5, 6), (2, 3, 4), (2, 3, 5),
(2, 3, 6), (2, 4, 5), (2, 4, 6), (2, 5, 6), (3, 4, 5), (3, 4, 6),
(3, 5, 6), (4, 5, 6)

$$\text{Therefore, the required probability} = \frac{20}{216} = \frac{5}{54}$$

46. Girls sit side by side, it means both girls are seated
together in $2!$ ways.

Let E = 4 boys and 2 girls occupy seats side by side

$$n(E) = 5!2!$$

$$\text{Required probability} = \frac{5!2!}{6!} = \frac{2}{6} = \frac{1}{3}$$

47. Total number of cards = 52

$$\text{Number of king of a heart} = 1$$

$$\therefore \text{Required probability} = \frac{1}{52}$$

48. Total number of cases are 2^{100} .

The number of favourable ways

$$= {}^{100}C_1 + {}^{100}C_3 + \dots + {}^{100}C_{99} = 2^{100-1} = 2^{99}$$

$$\text{Hence, required probability} = \frac{2^{99}}{2^{100}} = \frac{1}{2}$$

49. A leap year 366 days in which 52 weeks and 2 days are
extra. i.e. (Sunday, Monday), (Monday, Tuesday),
(Tuesday, Wednesday), (Wednesday, Thursday),
(Thursday, Friday), (Friday, Saturday), (Saturday,
Sunday).

So, probability that a leap year contains only

$$52 \text{ Sunday} = \frac{5}{7}$$

50. Total number of ways of selecting 4 balls at random $= {}^8C_4$

Now, the number of ways of selecting 4 balls such that
none of the them have same colour

$$= {}^2C_1 \times {}^2C_1 \times {}^2C_1 \times {}^2C_1$$

Thus, the probability that none of the them have same

$$\text{colour} = \frac{{}^2C_1 \times {}^2C_1 \times {}^2C_1 \times {}^2C_1}{{}^8C_4}$$

$$= \frac{16 \times 4 \times 3 \times 2 \times 1}{8 \times 7 \times 6 \times 5}$$

$$= \frac{8}{35}$$

$$\text{Hence, required probability} = 1 - \frac{8}{35} = \frac{27}{35}$$

51. $\therefore B \subset A$

$$\therefore P(A \cap B^c) = P(A) - P(B)$$

and

$$P(B) \leq P(A)$$

52. Let $S = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG, GGG\}$
and $E = \{BBB, BBG, BGB, GBB, GGB, GBG, BGG\}$

$$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{7}{8}$$

53. Let A = Event of getting even number on first die

$$= \{(2, 1), \dots, (2, 6) (4, 1), \dots, (4, 6) (6, 1), \dots, (6, 6)\}$$

and B = Event of getting a sum of 8

$$= \{(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)\}$$

$$A \cap B = \{(2, 6), (4, 4), (6, 2)\}$$

$$\therefore n(A) = 18, n(B) = 5 \text{ and } n(A \cap B) = 3$$

$$\Rightarrow n(A \cup B) = 18 + 5 - 3 = 20$$

$$\therefore P(A \cup B) = \frac{n(A \cup B)}{n(S)} = \frac{20}{36} = \frac{5}{9}$$

For More Material Join: [@JEEAdvanced_2024](#)

JEE Advanced

Solved Paper

2015

For More Material Join: @JEEAdvanced_2024

SOLVED PAPER 2015

JEE Advanced

Paper ①

Section 1 (Maximum Marks : 16)

- This section contains four questions.
- The answer to each question is a single digit integer ranging from 0 to 9 both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.

Marking scheme

- + 4 If the bubble corresponding to the answer is darkened.
- 0 In all other cases.

1. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is
2. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then, the value of $\frac{m}{n}$ is
3. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latusrectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is
4. The number of distinct solutions of the equation $\frac{5}{4} \cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$ in the interval $[0, 2\pi]$ is

Section 2 (Maximum Marks : 4)

- This section contains one question.
- Each question has four options (a), (b), (c) and (d). One or more than one of these four options are correct.
- For each question, darken the bubble(s) corresponding to all the correct options in the ORS.

Marking scheme

- +4 If only the bubble(s) corresponding to all the correct options are darkened.
- 0 If none of the bubbles is darkened.
- 2 In all the other cases.

5. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of ΔOPQ is $3\sqrt{2}$, then which of the following is/are the coordinates of P ?

(a) $(4, 2\sqrt{2})$

(b) $(9, 3\sqrt{2})$

(c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$

(d) $(1, \sqrt{2})$

Paper 2

Section 1 (Maximum Marks : 16)

- This section contains four questions.
- The answer to each question is a single digit integer ranging from 0 to 9 both inclusive.
- For each question, darken the bubble corresponding to the correct integer in the ORS.
- Marking scheme**
 - +4 If the bubble corresponding to the answer is darkened.
 - 0 in all other cases.

- Suppose that all the terms of an arithmetic progression are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this AP is
- The coefficient of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3)\dots(1+x^{100})$ is
- Suppose that the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{5} = 1$ are $(f_1, 0)$ and $(f_2, 0)$, where $f_1 > 0$ and $f_2 < 0$. Let P_1 and P_2 be two parabolas with a common vertex at $(0, 0)$ with foci at $(f_1, 0)$ and $(2f_2, 0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which

passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2 , then the value of $\left(\frac{1}{m_1^2} + m_2^2\right)$ is

- For any integer k , let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression $\frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|}$ is

Section 2 (Maximum Marks : 16)

- This section contains four questions.
- Each question has four options (a), (b), (c) and (d). One or more than one of these four options are correct.
- For each question, darken the bubble(s) corresponding to all the correct options in the ORS.
- Marking scheme**
 - +4 If only the bubble(s) corresponding to all the correct options are darkened.
 - 0 If none of the bubbles is darkened.
 - 2 In all other cases.

- Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following interval(s) is/are a subset of S ?
(a) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (b) $\left(-\frac{1}{\sqrt{5}}, 0\right)$ (c) $\left(0, \frac{1}{\sqrt{5}}\right)$ (d) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$
- If $\alpha = 3 \sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3 \cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is/are
(a) $\cos \beta > 0$
(b) $\sin \beta < 0$
(c) $\cos(\alpha + \beta) > 0$
(d) $\cos \alpha < 0$
- Let E_1 and E_2 be two ellipses whose centres are at the origin. The major axes of E_1 and E_2 lie along the X -axis and Y -axis, respectively. Let S be the circle $x^2 + (y-1)^2 = 2$. The straight line $x + y = 3$ touches the curves S , E_1 and E_2 at P , Q and R , respectively.

Suppose that $PQ = PR = \frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2 respectively, then the correct expression(s) is/are

- $e_1^2 + e_2^2 = \frac{43}{40}$
 - $e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$
 - $|e_1^2 - e_2^2| = \frac{5}{8}$
 - $e_1 e_2 = \frac{\sqrt{3}}{4}$
- Consider the hyperbola $H: x^2 - y^2 = 1$ and a circle S with centre $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the X -axis at point M . If (l, m) is the centroid of $\triangle PMN$, then the correct expression(s) is/are
(a) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$ (b) $\frac{dm}{dx_1} = \frac{x_1}{3(\sqrt{x_1^2 - 1})}$ for $x_1 > 1$
(c) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$ (d) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

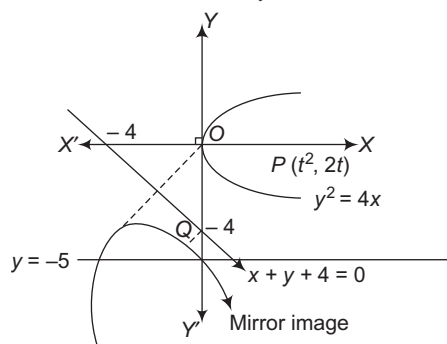
Answer with Explanations

Paper 1

1. (4) Let $P(t^2, 2t)$ be a point on the curve $y^2 = 4x$, whose image is $Q(x, y)$ on $x + y + 4 = 0$, then

$$\frac{x - t^2}{1} = \frac{y - 2t}{1} = \frac{-2(t^2 + 2t + 4)}{1^2 + 1^2}$$

$$\Rightarrow x = -2t - 4 \text{ and } y = -t^2 - 4$$



Now, the straight line $y = -5$ meets the mirror image.

$$\therefore -t^2 - 4 = -5$$

$$\Rightarrow t^2 = 1$$

$$\Rightarrow t = \pm 1$$

Thus, points of intersection of A and B are $(-6, -5)$ and $(-2, -5)$.

$$\therefore \text{Distance, } AB = \sqrt{(-2+6)^2 + (-5+5)^2} = 4$$

2. (5) Here, $\underline{\quad} B_1 \underline{\quad} B_2 \underline{\quad} B_3 \underline{\quad} B_4 \underline{\quad} B_5 \underline{\quad}$

Out of 5 girls, 4 girls are together and 1 girl is separate. Now, to select 2 positions out of 6 positions between boys $= {}^6C_2$... (i)

4 girls are to be selected out of 5 $= {}^5C_4$... (ii)

Now, 2 groups of girls can be arranged in $2!$ ways. ... (iii)

Also, the group of 4 girls and 5 boys is arranged in $4! \times 5!$ ways. ... (iv)

Now, total number of ways

$$= {}^6C_2 \times {}^5C_4 \times 2! \times 4! \times 5! \quad [\text{from Eqs. (i), (ii), (iii) and (iv)}]$$

$$\therefore m = {}^6C_2 \times {}^5C_4 \times 2! \times 4! \times 5!$$

$$\text{and } n = 5! \times 6!$$

$$\Rightarrow \frac{m}{n} = \frac{{}^6C_2 \times {}^5C_4 \times 2! \times 4! \times 5!}{6! \times 5!} = \frac{15 \times 5 \times 2 \times 4!}{6 \times 5 \times 4!} = 5$$

3. (2) End points of latusrectum are $(a, \pm 2a)$ i.e. $(1, \pm 2)$.

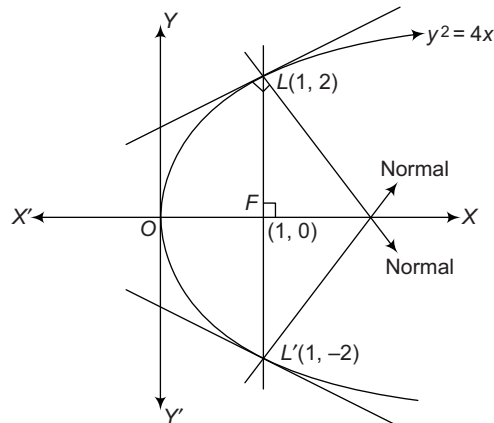
Equation of normal at (x_1, y_1) is

$$\frac{y - y_1}{x - x_1} = -\frac{y_1}{2a}$$

$$\text{i.e. } \frac{y - 2}{x - 1} = -\frac{2}{2} \text{ and } \frac{y + 2}{x - 1} = \frac{2}{2}$$

$$\Rightarrow x + y = 3 \text{ and } x - y = 3$$

which is tangent to $(x - 3)^2 + (y + 2)^2 = r^2$.



$\therefore$ Length of perpendicular from centre = Radius

$$\Rightarrow \frac{|3 - 2 - 3|}{\sqrt{1^2 + 1^2}} = r$$

$$\therefore r^2 = 2$$

4. (8) Here, $\frac{5}{4} \cos^2 2x + (\cos^4 x + \sin^4 x) + (\cos^6 x + \sin^6 x) = 2$

$$\Rightarrow \frac{5}{4} \cot 2x + [(\cos^2 x + \sin^2 x)^2 - 2 \sin^2 x \cos^2 x]$$

$$+ (\cos^2 x + \sin^2 x)[(\cos^2 x + \sin^2 x)^2 - 3 \sin^2 x \cos^2 x] = 2$$

$$\Rightarrow \frac{5}{4} \cos^2 2x + (1 - 2 \sin^2 x \cos^2 x) + (1 - 3 \cos^2 x \sin^2 x) = 2$$

$$\Rightarrow \frac{5}{4} \cos^2 2x - 5 \sin^2 x \cos^2 x = 0$$

$$\Rightarrow \frac{5}{4} \cos^2 2x - \frac{5}{4} \sin^2 2x = 0$$

$$\Rightarrow \frac{5}{4} \cos^2 2x - \frac{5}{4} + \frac{5}{4} \cos^2 2x = 0$$

$$\Rightarrow \frac{5}{2} \cos^2 2x = \frac{5}{4}$$

$$\Rightarrow \cos^2 2x = \frac{1}{2} \Rightarrow 2 \cos^2 2x = 1$$

$$\Rightarrow 1 + \cos 4x = 1$$

$$\Rightarrow \cos 4x = 0, \text{ as } 0 \leq x \leq 2\pi$$

$$\therefore 4x = \left\{ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \frac{13\pi}{2}, \frac{15\pi}{2} \right\}$$

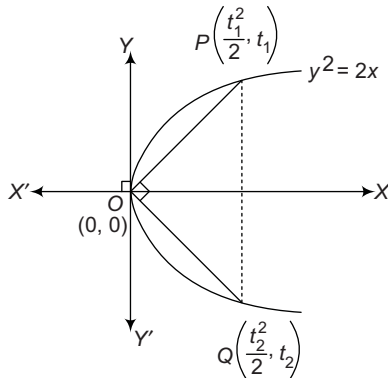
$$\text{as } 0 \leq 4x \leq 8\pi$$

$$\Rightarrow x = \left\{ \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}, \frac{9\pi}{8}, \frac{11\pi}{8}, \frac{13\pi}{8}, \frac{15\pi}{8} \right\}$$

Hence, the total number of solutions is 8.

1138 JEE Advanced Solved Paper 2015

5. (a, d) Since, $\angle POQ = 90^\circ$



$$\Rightarrow \frac{t_1 - 0}{\frac{t_1^2}{2} - 0} \cdot \frac{t_2 - 0}{\frac{t_2^2}{2} - 0} = -1$$

$$\Rightarrow t_1 t_2 = -4 \quad \dots(i)$$

$$\therefore \ar(\Delta OPQ) = 3\sqrt{2}$$

$$\therefore \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ t_1^2/2 & t_1 & 1 \\ t_2^2/2 & t_2 & 1 \end{vmatrix} = \pm 3\sqrt{2}$$

$$\Rightarrow \frac{1}{2} \left(\frac{t_1^2 t_2}{2} - \frac{t_1 t_2^2}{2} \right) = \pm 3\sqrt{2}$$

$$\Rightarrow \frac{1}{4} (-4t_1 + 4t_2) = \pm 3\sqrt{2}$$

$$\Rightarrow t_1 + \frac{4}{t_1} = 3\sqrt{2} \quad [\because t_1 > 0 \text{ for } P]$$

$$\Rightarrow t_1^2 - 3\sqrt{2}t_1 + 4 = 0$$

$$\Rightarrow (t_1 - 2\sqrt{2})(t_1 - \sqrt{2}) = 0$$

$$\Rightarrow t_1 = \sqrt{2} \quad \text{or} \quad 2\sqrt{2}$$

$$\therefore P(1, \sqrt{2}) \quad \text{or} \quad P(4, 2\sqrt{2})$$

Paper 2

1. (9) Given, $\frac{S_7}{S_{11}} = \frac{6}{11}$ and $130 < t_7 < 140$

$$\Rightarrow \frac{\frac{7}{2}[2a + 6d]}{\frac{11}{2}[2a + 10d]} = \frac{6}{11} \Rightarrow \frac{7(2a + 6d)}{(2a + 10d)} = 6$$

$$\Rightarrow a = 9d \quad \dots(i)$$

Also, $130 < t_7 < 140$

$$\Rightarrow 130 < a + 6d < 140$$

$$\Rightarrow 130 < 9d + 6d < 140 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 130 < 15d < 140$$

$$\Rightarrow \frac{26}{3} < d < \frac{28}{3} \quad [\text{since, } d \text{ is a natural number}]$$

$$\therefore d = 9$$

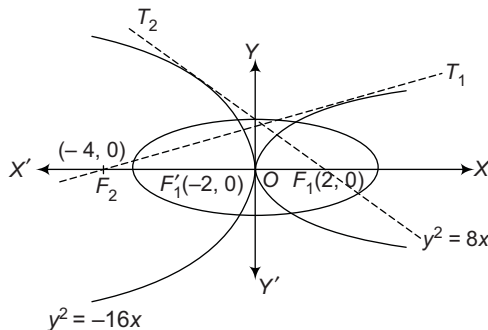
2. (8) Coefficient of x^9 in the expansion of

$$(1+x)(1+x^2)(1+x^3) \dots (1+x^{100}) = \text{Terms having } x^9$$

$$= [1^{99} \cdot x^9, 1^{98} \cdot x \cdot x^8, 1^{98} \cdot x^2 \cdot x^7, 1^{98} \cdot x^3 \cdot x^6, 1^{98} \cdot x^4 \cdot x^5, 1^{97} \cdot x \cdot x^2 \cdot x^6, 1^{97} \cdot x \cdot x^3 \cdot x^5, 1^{97} \cdot x^2 \cdot x^3 \cdot x^4]$$

$$\therefore \text{Coefficient of } x^9 = 8$$

3. (4)



Tangent to P_1 passes through $(2f_2, 0)$ i.e. $(-4, 0)$.

$$\therefore T_1: y = m_1 x + \frac{2}{m_1} \Rightarrow 0 = -4m_1 + \frac{2}{m_1}$$

$$\Rightarrow m_1^2 = 1/2 \quad \dots(i)$$

Also, tangent to P_2 passes through $(f_1, 0)$ i.e. $(2, 0)$.

$$\Rightarrow T_2: y = m_2 x + \frac{(-4)}{m_2}$$

$$\Rightarrow 0 = 2m_2 - \frac{4}{m_2}$$

$$\Rightarrow m_2^2 = 2 \quad \dots(ii)$$

$$\therefore \frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$$

4. (4) Given, $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i \sin\left(\frac{k\pi}{7}\right) = \cos\left(\frac{2k\pi}{14}\right) + i \sin\left(\frac{2k\pi}{14}\right)$

$\therefore \alpha_k$ are vertices of regular polygon having 14 sides.

Let the side length of regular polygon be a .

$$\therefore |\alpha_{k+1} - \alpha_k| = \text{length of a side of the regular polygon} = a$$

$$\dots(i)$$

$$\text{and } |\alpha_{4k-1} - \alpha_{4k-2}| = \text{length of a side of the regular polygon}$$

$$= a$$

$$\dots(ii)$$

$$\therefore \frac{\sum_{k=1}^{12} |\alpha_{k+1} - \alpha_k|}{\sum_{k=1}^3 |\alpha_{4k-1} - \alpha_{4k-2}|} = \frac{12(a)}{3(a)} = 4$$

5. (a, d) Given, x_1 and x_2 are roots of $\alpha x^2 - x + \alpha = 0$.

$$\therefore x_1 + x_2 = \frac{1}{\alpha} \text{ and } x_1 x_2 = 1$$

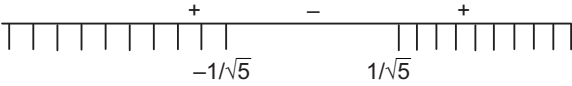
$$\text{Also, } |x_1 - x_2| < 1$$

$$\Rightarrow |x_1 - x_2|^2 < 1 \Rightarrow (x_1 - x_2)^2 < 1$$

$$\text{or } (x_1 + x_2)^2 - 4x_1x_2 < 1$$

$$\Rightarrow \frac{1}{\alpha^2} - 4 < 1 \quad \text{or} \quad \frac{1}{\alpha^2} < 5$$

$$\Rightarrow 5\alpha^2 - 1 > 0 \quad \text{or} \quad (\sqrt{5}\alpha - 1)(\sqrt{5}\alpha + 1) > 0$$



$$\therefore \alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right) \quad \dots(i)$$

$$\text{Also, } D > 0$$

$$\Rightarrow 1 - 4\alpha^2 > 0 \quad \text{or} \quad \alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right) \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\alpha \in \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

6. (b, c, d) Here, $\alpha = 3 \sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3 \cos^{-1}\left(\frac{4}{9}\right)$ as $\frac{6}{11} > \frac{1}{2}$

$$\Rightarrow \sin^{-1}\left(\frac{6}{11}\right) > \sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$\therefore \alpha = 3 \sin^{-1}\left(\frac{6}{11}\right) > \frac{\pi}{2} \Rightarrow \cos \alpha < 0$$

$$\text{Now, } \beta = 3 \cos^{-1}\left(\frac{4}{9}\right)$$

$$\text{As } \frac{4}{9} < \frac{1}{2} \Rightarrow \cos^{-1}\left(\frac{4}{9}\right) > \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\therefore \beta = 3 \cos^{-1}\left(\frac{4}{9}\right) > \pi$$

$$\therefore \cos \beta < 0 \text{ and } \sin \beta < 0$$

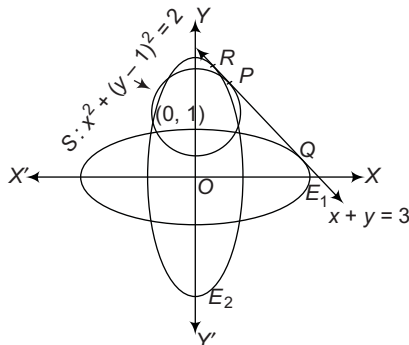
$$\text{Now, } \alpha + \beta \text{ is slightly greater than } \frac{3\pi}{2}.$$

$$\therefore \cos(\alpha + \beta) > 0$$

7. (a, b) Here, $E_1: \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$

$$E_2: \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1, (c < d) \quad \text{and} \quad S: x^2 + (y-1)^2 = 2$$

as tangent to E_1, E_2 and S is $x + y = 3$.



Let the point of contact of tangent be (x_1, y_1) to S .

$$\therefore x \cdot x_1 + y \cdot y_1 - (y + y_1) + 1 = 2$$

$$\text{or } x x_1 + y y_1 - y = (1 + y_1), \text{ same as } x + y = 3.$$

$$\Rightarrow \frac{x_1}{1} = \frac{y_1 - 1}{1} = \frac{1 + y_1}{3}$$

$$\text{i.e. } x_1 = 1 \text{ and } y_1 = 2$$

$$\therefore P = (1, 2)$$

$$\text{Since, } PR = PQ = \frac{2\sqrt{2}}{3}. \text{ Thus, by parametric form,}$$

$$\frac{x-1}{-1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm \frac{2\sqrt{2}}{3}$$

$$\Rightarrow \left(x = \frac{5}{3}, y = \frac{4}{3}\right) \text{ and } \left(x = \frac{1}{3}, y = \frac{8}{3}\right)$$

$$\therefore Q = \left(\frac{5}{3}, \frac{4}{3}\right) \text{ and } R = \left(\frac{1}{3}, \frac{8}{3}\right)$$

Now, equation of tangent at Q on ellipse E_1 is

$$\frac{x \cdot 5}{a^2 \cdot 3} + \frac{y \cdot 4}{b^2 \cdot 3} = 1$$

On comparing with $x + y = 3$, we get

$$a^2 = 5 \text{ and } b^2 = 4$$

$$\therefore e_1^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{4}{5} = \frac{1}{5} \quad \dots(i)$$

Also, equation of tangent at R on ellipse E_2 is

$$\frac{x \cdot 1}{a^2 \cdot 3} + \frac{y \cdot 8}{b^2 \cdot 3} = 1$$

On comparing with $x + y = 3$, we get

$$a^2 = 1, b^2 = 8$$

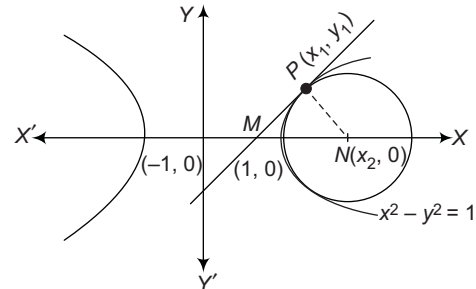
$$\therefore e_2^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{1}{8} = \frac{7}{8} \quad \dots(ii)$$

$$\text{Now, } e_1^2 \cdot e_2^2 = \frac{7}{40} \Rightarrow e_1 e_2 = \frac{\sqrt{7}}{2\sqrt{10}}$$

$$\text{and } e_1^2 + e_2^2 = \frac{1}{5} + \frac{7}{8} = \frac{43}{40}$$

$$\text{Also, } |e_1^2 - e_2^2| = \left|\frac{1}{5} - \frac{7}{8}\right| = \frac{27}{40}$$

8. (a, b, d)



Equation of family of circles touching hyperbola at (x_1, y_1) is

$$(x - x_1)^2 + (y - y_1)^2 + \lambda(x x_1 - y y_1 - 1) = 0$$

Now, its centre is $(x_2, 0)$.

1140 JEE Advanced Solved Paper 2015

$$\therefore \left[\frac{-(\lambda x_1 - 2x_1)}{2}, \frac{-(-2y_1 - \lambda y_1)}{2} \right] = (x_2, 0)$$

$$\Rightarrow 2y_1 + \lambda y_1 = 0 \Rightarrow \lambda = -2$$

$$\text{and } 2x_1 - \lambda x_1 = 2x_2 \Rightarrow x_2 = 2x_1$$

$$\therefore P(x_1, \sqrt{x_1^2 - 1}) \text{ and } N(x_2, 0) = (2x_1, 0)$$

As tangent intersect X -axis at $M\left(\frac{1}{x}, 0\right)$.

Centroid of $\triangle PMN = (l, m)$

$$\Rightarrow \left(\frac{3x_1 + \frac{1}{x_1}}{3}, \frac{y_1 + 0 + 0}{3} \right) = (l, m)$$

$$\Rightarrow l = \frac{3x_1 + \frac{1}{x_1}}{3}$$

On differentiating w.r.t. x_1 , we get

$$\frac{dl}{dx_1} = \frac{3 - \frac{1}{x_1^2}}{3}$$

$$\Rightarrow \frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \text{ for } x_1 > 1 \text{ and } m = \frac{\sqrt{x_1^2 - 1}}{3}$$

On differentiating w.r.t. x_1 , we get

$$\frac{dm}{dx_1} = \frac{2x_1}{2 \times 3\sqrt{x_1^2 - 1}} = \frac{x_1}{3\sqrt{x_1^2 - 1}}, \text{ for } x_1 > 1$$

$$\text{Also, } m = \frac{y_1}{3}$$

On differentiating w.r.t. y_1 , we get

$$\frac{dm}{dy_1} = \frac{1}{3}, \text{ for } y_1 > 0$$

JEE Main 2016 (Solved)

- If $f(x) + 2f\left(\frac{1}{x}\right) = 3x$,
 $x \neq 0$ and $S = \{x \in R : f(x) = f(-x)\}$; then S
 (a) is an empty set
 (b) contains exactly one element
 (c) contains exactly two elements
 (d) contains more than two elements
- A value of θ for which $\frac{2 + 3i \sin \theta}{1 - 2i \sin \theta}$ is purely imaginary, is
 (a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$
 (c) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$ (d) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$
- The sum of all real values of x satisfying the equation $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$ is
 (a) 3 (b) -4 (c) 6 (d) 5
- If all the words (with or without meaning) having five letters, formed using the letters of the word SMALL and arranged as in a dictionary, then the position of the word SMALL is
 (a) 46th (b) 59th (c) 52nd (d) 58th
- If the number of terms in the expansion of $\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n$, $x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is
 (a) 64 (b) 2187 (c) 243 (d) 729
- If the 2nd, 5th and 9th terms of a non-constant AP are in GP, then the common ratio of this GP is
 (a) $\frac{8}{5}$ (b) $\frac{4}{3}$ (c) 1 (d) $\frac{7}{4}$
- If the sum of the first ten terms of the series $\left(1\frac{3}{5}\right)^2 + \left(2\frac{2}{5}\right)^2 + \left(3\frac{1}{5}\right)^2 + 4^2 + \left(4\frac{4}{5}\right)^2 + \dots$ is $\frac{16}{5}m$, then m is equal to
 (a) 102 (b) 101 (c) 100 (d) 99
- Let $p = \lim_{x \rightarrow 0^+} (1 + \tan^2 \sqrt{x})^{1/2x}$, then $\log p$ is equal to
 (a) 2 (b) 1 (c) $\frac{1}{2}$ (d) $\frac{1}{4}$
- $\lim_{n \rightarrow \infty} \left(\frac{(n+1)(n+2) \dots 3n}{n^{2n}}\right)^{1/n}$ is equal to
 (a) $\frac{18}{e^4}$ (b) $\frac{27}{e^2}$ (c) $\frac{9}{e^2}$ (d) $3 \log 3 - 2$
- Two sides of a rhombus are along the lines, $x - y + 1 = 0$ and $7x - y - 5 = 0$. If its diagonals intersect at $(-1, -2)$, then which one of the following is a vertex of this rhombus?
 (a) $(-3, -9)$ (b) $(-3, -8)$ (c) $\left(\frac{1}{3}, -\frac{8}{3}\right)$ (d) $\left(-\frac{10}{3}, -\frac{7}{3}\right)$
- The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the X -axis, lie on
 (a) a circle
 (b) an ellipse which is not a circle
 (c) a hyperbola
 (d) a parabola
- If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S , whose centre is at $(-3, 2)$, then the radius of S is
 (a) $5\sqrt{2}$ (b) $5\sqrt{3}$ (c) 5 (d) 10
- Let P be the point on the parabola, $y^2 = 8x$, which is at a minimum distance from the centre C of the circle, $x^2 + (y + 6)^2 = 1$. Then, the equation of the circle, passing through C and having its centre at P is
 (a) $x^2 + y^2 - 4x + 8y + 12 = 0$
 (b) $x^2 + y^2 - x + 4y - 12 = 0$
 (c) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$
 (d) $x^2 + y^2 - 4x + 9y + 18 = 0$
- The eccentricity of the hyperbola whose length of the latusrectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is
 (a) $\frac{4}{3}$ (b) $\frac{4}{\sqrt{3}}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\sqrt{3}$
- The distance of the point $(1, -5, 9)$ from the plane $x - y + z = 5$ measured along the line $x = y = z$ is
 (a) $3\sqrt{10}$ (b) $10\sqrt{3}$ (c) $\frac{10}{\sqrt{3}}$ (d) $\frac{20}{3}$
- If the standard deviation of the numbers 2, 3, a and 11 is 3.5, then which of the following is true?
 (a) $3a^2 - 26a + 55 = 0$ (b) $3a^2 - 32a + 84 = 0$
 (c) $3a^2 - 34a + 91 = 0$ (d) $3a^2 - 23a + 44 = 0$
- Let two fair six-faced dice A and B be thrown simultaneously. If E_1 is the event that die A shows up four, E_2 is the event that die B shows up two and E_3 is the event that the sum of numbers on both dice is odd, then which of the following statements is not true?
 (a) E_1 and E_2 are independent
 (b) E_2 and E_3 are independent
 (c) E_1 and E_3 are independent
 (d) E_1, E_2 and E_3 are independent

18. If $0 \leq x < 2\pi$, then the number of real values of x , which satisfy the equation

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0, \text{ is}$$

- (a) 3 (b) 5
(c) 7 (d) 9

19. A man is walking towards a vertical pillar in a straight path, at a uniform speed. At a certain point A on the path, he observes that the angle of elevation of the top of the pillar is 30° . After walking for 10 min from A in the same

direction, at a point B , he observes that the angle of elevation of the top of the pillar is 60° . Then, the time taken (in minutes) by him, from B to reach the pillar, is

- (a) 6 (b) 10 (c) 20 (d) 5

20. The Boolean expression $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$ is equivalent to

- (a) $\sim p \wedge q$ (b) $p \wedge q$
(c) $p \vee q$ (d) $p \vee \sim q$

JEE Advanced 2016 (Solved)

Paper I

1. A debate club consists of 6 girls and 4 boys. A team of 4 members is to be selected from this club including the selection of a captain (from among these 4 members) for the team. If the team has to include atmost one boy, the number of ways of selecting the team is

(Single Option Correct)

- (a) 380 (b) 320 (c) 260 (d) 95

2. Let $-\frac{\pi}{6} < \theta < -\frac{\pi}{12}$. Suppose α_1 and β_1 are the roots of the equation $x^2 - 2x \sec \theta + 1 = 0$, and α_2 and β_2 are the roots of the equation $x^2 + 2x \tan \theta - 1 = 0$. If $\alpha_1 > \beta_1$ and $\alpha_2 > \beta_2$, then $\alpha_1 + \beta_2$ equals

(Single Option Correct)

- (a) $2(\sec \theta - \tan \theta)$ (b) $2 \sec \theta$
(c) $-2 \tan \theta$ (d) 0

3. Let $S = \left\{ x \in (-\pi, \pi) : x \neq 0, \pm \frac{\pi}{2} \right\}$. The sum of all distinct solutions of the equation $\sqrt{3} \sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$ in the set S is equal to

(Single Option Correct)

- (a) $-\frac{7\pi}{9}$ (b) $-\frac{2\pi}{9}$ (c) 0 (d) $\frac{5\pi}{9}$

4. Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point $(1, 0)$. Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q . The normal to the circle at P intersects a line drawn through Q parallel to RS at point E . Then, the locus of E passes through the point(s)

(One or More Than One Option Correct)

- (a) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (b) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (c) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (d) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

5. The circle $C_1 : x^2 + y^2 = 3$ with centre at O intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles

C_2 and C_3 at R_2 and R_3 , respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3 , respectively. If Q_2 and Q_3 lie on the Y -axis, then

(One or More Than One Option Correct)

- (a) $Q_2Q_3 = 12$ (b) $R_2R_3 = 4\sqrt{6}$
(c) area of the ΔOR_2R_3 is $6\sqrt{2}$ (d) area of the ΔPQ_2Q_3 is $4\sqrt{2}$

6. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1+x)^2 + (1+x)^3 + \dots + (1+x)^{49} + (1+mx)^{50}$ is $(3n+1)^{51}C_3$ for some positive integer n . Then, the value of n is

(Single Option Correct)

7. Let $z = \frac{-1 + \sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let

$$P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix} \text{ and } I \text{ be the identity matrix of order 2.}$$

Then, the total number of ordered pairs (r, s) for which $P^2 = -I$ is

(Single Digit Integer)

8. Let $\alpha, \beta \in R$ be such that $\lim_{x \rightarrow 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$. Then, $6(\alpha + \beta)$ equals

(Single Digit Integer)

Paper II

1. The value of $\sum_{k=1}^{13} \frac{1}{\sin\left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right) \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$ is equal

to

(Single Option Correct)

- (a) $3 - \sqrt{3}$ (b) $2(3 - \sqrt{3})$ (c) $2(\sqrt{3} - 1)$ (d) $2(2 + \sqrt{3})$

2. Let $b_i > 1$ for $i = 1, 2, \dots, 101$. Suppose $\log_e b_1, \log_e b_2, \dots, \log_e b_{101}$ are in AP with the common difference $\log_e 2$. Suppose $a_1, a_2, \dots, a_{101}$ are in AP, such that $a_1 = b_1$ and $a_{51} = b_{51}$. If $t = b_1 + b_2 + \dots + b_{51}$ and $s = a_1 + a_2 + \dots + a_{51}$, then

(Single Option Correct)

- (a) $s > t$ and $a_{101} > b_{101}$ (b) $s > t$ and $a_{101} < b_{101}$
(c) $s < t$ and $a_{101} > b_{101}$ (d) $s < t$ and $a_{101} < b_{101}$

3. The value of $\int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + e^x} dx$ is equal to
(Single Option Correct)

- (a) $\frac{\pi^2}{4} - 2$ (b) $\frac{\pi^2}{4} + 2$ (c) $\pi^2 - e^{-\pi/2}$ (d) $\pi^2 + e^{\pi/2}$

4. Let $f(x) = \lim_{n \rightarrow \infty} \left[\frac{n^n (x+n) \left(x + \frac{n}{2}\right) \dots \left(x + \frac{n}{n}\right)}{n! (x^2 + n^2) \left(x^2 + \frac{n^2}{4}\right) \dots \left(x^2 + \frac{n^2}{n^2}\right)} \right]^{\frac{x}{n}}$,

for all $x = 0$. Then (One or More Than One Option Correct)

- (a) $f\left(\frac{1}{2}\right) \geq f(1)$ (b) $f\left(\frac{1}{3}\right) \leq f\left(\frac{2}{3}\right)$ (c) $f'(2) \leq 0$ (d) $\frac{f'(3)}{f(3)} \geq \frac{f'(2)}{f(2)}$

5. Let P be the point on the parabola $y^2 = 4x$, which is at the shortest distance from the centre S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then,

- (a) $SP = 2\sqrt{5}$ (One or More Than One Option Correct)
(b) $SQ : QP = (\sqrt{5} + 1) : 2$
(c) the x-intercept of the normal to the parabola at P is 6
(d) the slope of the tangent to the circle at Q is $\frac{1}{2}$

6. Let $a, b \in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{ z \in \mathbb{C} : z = \frac{1}{a + i bt}, t \in \mathbb{R}, t \neq 0 \right\}$, where $i = \sqrt{-1}$. If $z = x + iy$ and $z \in S$, then (x, y) lies on

(One or More Than One Option Correct)

- (a) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$
(b) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

- (c) the X-axis for $a \neq 0, b = 0$ (d) the Y-axis for $a = 0, b \neq 0$

Paragraph 1

Football teams T_1 and T_2 have to play two games against each other. It is assumed that the outcomes of the two games are independent. The probabilities of T_1 winning, drawing and losing a game against T_2 are $\frac{1}{2}, \frac{1}{6}$ and $\frac{1}{3}$, respectively. Each team gets

3 points for a win, 1 point for a draw and 0 point for a loss in a game. Let X and Y denote the total points scored by teams T_1 and T_2 , respectively, after two games.

7. $P(X > Y)$ is

- (a) $\frac{1}{4}$ (b) $\frac{5}{12}$ (c) $\frac{1}{2}$ (d) $\frac{7}{12}$

8. $P(X = Y)$ is

- (a) $\frac{11}{36}$ (b) $\frac{1}{3}$ (c) $\frac{13}{36}$ (d) $\frac{1}{2}$

Paragraph 2

Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$, for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9} + \frac{y^2}{8} = 1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

9. The orthocentre of ΔF_1MN is

- (a) $\left(-\frac{9}{10}, 0\right)$ (b) $\left(\frac{2}{3}, 0\right)$ (c) $\left(\frac{9}{10}, 0\right)$ (d) $\left(\frac{2}{3}, \sqrt{6}\right)$

10. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the X-axis at Q , then the ratio of area of ΔMQR to area of the quadrilateral MF_1NF_2 is

- (a) 3 : 4 (b) 4 : 5 (c) 5 : 8 (d) 2 : 3

SOLUTIONS

JEE Main

1. (c) We have, $f(x) + 2f\left(\frac{1}{x}\right) = 3x, x \neq 0$... (i)

On replacing x by $\frac{1}{x}$ in the above equation, we get

$$f\left(\frac{1}{x}\right) + 2f(x) = \frac{3}{x}$$

$$\Rightarrow 2f(x) + f\left(\frac{1}{x}\right) = \frac{3}{x} \quad \dots (ii)$$

On multiplying Eq. (ii) by 2 and subtracting Eq. (i) from Eq. (ii), we get

$$4f(x) + 2f\left(\frac{1}{x}\right) = \frac{6}{x}$$

$$\begin{aligned} f(x) + 2f\left(\frac{1}{x}\right) &= 3x \\ \underline{3f(x) = \frac{6}{x} - 3x} \end{aligned}$$

$$\Rightarrow f(x) = \frac{2}{x} - x$$

Now, consider $f(x) = f(-x)$

$$\Rightarrow \frac{2}{x} - x = -\frac{2}{x} + x \Rightarrow \frac{4}{x} = 2x \Rightarrow 2x^2 = 4$$

$$\Rightarrow x^2 = 2 \Rightarrow x = \pm \sqrt{2}$$

Hence, S contains exactly two elements.

2. (d) Let $z = \frac{2 + 3i \sin \theta}{1 - 2i \sin \theta}$ is purely imaginary. Then, we have

$$\operatorname{Re}(z) = 0$$

$$\begin{aligned} \text{Now, consider } z &= \frac{2 + 3i \sin \theta}{1 - 2i \sin \theta} \\ &= \frac{(2 + 3i \sin \theta)(1 + 2i \sin \theta)}{(1 - 2i \sin \theta)(1 + 2i \sin \theta)} \\ &= \frac{2 + 4i \sin \theta + 3i \sin \theta + 6i^2 \sin^2 \theta}{1^2 - (2i \sin \theta)^2} \\ &= \frac{2 + 7i \sin \theta - 6 \sin^2 \theta}{1 + 4 \sin^2 \theta} \\ &= \frac{2 - 6 \sin^2 \theta}{1 + 4 \sin^2 \theta} + i \frac{7 \sin \theta}{1 + 4 \sin^2 \theta} \end{aligned}$$

$$\therefore \operatorname{Re}(z) = 0$$

$$\therefore \frac{2 - 6 \sin^2 \theta}{1 + 4 \sin^2 \theta} = 0$$

$$\Rightarrow 2 = 6 \sin^2 \theta \Rightarrow \sin^2 \theta = \frac{1}{3}$$

$$\Rightarrow \sin \theta = \pm \frac{1}{\sqrt{3}} \Rightarrow \theta = \sin^{-1} \left(\pm \frac{1}{\sqrt{3}} \right) = \pm \sin^{-1} \left(\frac{1}{\sqrt{3}} \right)$$

3. (a) Given, $(x^2 - 5x + 5)^{x^2 + 4x - 60} = 1$

Clearly, this is possible when

I. $x^2 + 4x - 60 = 0$ and $x^2 - 5x + 5 \neq 0$

or

II. $x^2 - 5x + 5 = 1$

or

III. $x^2 - 5x + 5 = -1$ and $x^2 + 4x - 60 = \text{Even integer}$.

Case I When $x^2 + 4x - 60 = 0$

$$\Rightarrow x^2 + 10x - 6x - 60 = 0$$

$$\Rightarrow x(x + 10) - 6(x + 10) = 0$$

$$\Rightarrow (x + 10)(x - 6) = 0$$

$$\Rightarrow x = -10 \text{ or } x = 6$$

Note that, for these two values of x , $x^2 - 5x + 5 \neq 0$

Case II When $x^2 - 5x + 5 = 1$

$$\Rightarrow x^2 - 5x + 4 = 0$$

$$\Rightarrow x^2 - 4x - x + 4 = 0$$

$$\Rightarrow x(x - 4) - 1(x - 4) = 0$$

$$\Rightarrow (x - 4)(x - 1) = 0$$

$$\Rightarrow x = 4 \text{ or } x = 1$$

Case III When $x^2 - 5x + 5 = -1$

$$\Rightarrow x^2 - 5x + 6 = 0$$

$$\Rightarrow x^2 - 2x - 3x + 6 = 0$$

$$\Rightarrow x(x - 2) - 3(x - 2) = 0$$

$$\Rightarrow (x - 2)(x - 3) = 0$$

$$\Rightarrow x = 2 \text{ or } x = 3$$

Now, when $x = 2$, $x^2 + 4x - 60 = 4 + 8 - 60 = -48$, which is an even integer.

When $x = 3$, $x^2 + 4x - 60 = 9 + 12 - 60 = -39$, which is not an even integer.

Thus, in this case, we get $x = 2$.

Hence, the sum of all real values of $x = -10 + 6 + 4 + 1 + 2 = 3$

4. (d) Clearly, number of words start with A = $\frac{4!}{2!} = 12$

Number of words start with L = $4! = 24$

Number of words start with M = $\frac{4!}{2!} = 12$

Number of words start with SA = $\frac{3!}{2!} = 3$

Number of words start with SL = $3! = 6$

Note that, next word will be "SMALL".

Hence, the position of word "SMALL" is 58th.

5. (d) Clearly, number of terms in the expansion of

$$\left(1 - \frac{2}{x} + \frac{4}{x^2}\right)^n \text{ is } \frac{(n+2)(n+1)}{2} \text{ or } {}^{n+2}C_2.$$

[assuming $\frac{1}{x}$ and $\frac{1}{x^2}$ distinct]

$$\therefore \frac{(n+2)(n+1)}{2} = 28$$

$$\Rightarrow (n+2)(n+1) = 56 = (6+1)(6+2) \Rightarrow n = 6$$

Hence, sum of coefficients = $(1 - 2 + 4)^6 = 3^6 = 729$

Note As $\frac{1}{x}$ and $\frac{1}{x^2}$ are functions of same variables, therefore number of dissimilar terms will be $2n + 1$, i.e. odd, which is not possible. Hence, it contains error.

6. (b) Let a be the first term and d be the common difference. Then, we have $a + d, a + 4d, a + 8d$ in GP,

$$\text{i.e. } (a + 4d)^2 = (a + d)(a + 8d)$$

$$\Rightarrow a^2 + 16d^2 + 8ad = a^2 + 8ad + ad + 8d^2$$

$$\Rightarrow 8d^2 = ad$$

$$\Rightarrow 8d = a$$

[$\because d \neq 0$]

$$\text{Now, common ratio, } r = \frac{a + 4d}{a + d} = \frac{8d + 4d}{8d + d} = \frac{12d}{9d} = \frac{4}{3}$$

7. (b) Let S_{10} be the sum of first ten terms of the series. Then, we have

$$S_{10} = \left(1 \frac{3}{5}\right)^2 + \left(2 \frac{2}{5}\right)^2 + \left(3 \frac{1}{5}\right)^2 + 4^2 + \left(4 \frac{4}{5}\right)^2 + \dots \text{ to 10 terms}$$

$$= \left(\frac{8}{5}\right)^2 + \left(\frac{12}{5}\right)^2 + \left(\frac{16}{5}\right)^2 + 4^2 + \left(\frac{24}{5}\right)^2 + \dots \text{ to 10 terms}$$

$$= \frac{1}{5^2} (8^2 + 12^2 + 16^2 + 20^2 + 24^2 + \dots \text{ to 10 terms})$$

$$= \frac{4^2}{5^2} (2^2 + 3^2 + 4^2 + 5^2 + \dots \text{ to 10 terms})$$

$$= \frac{4^2}{5^2} (2^2 + 3^2 + 4^2 + 5^2 + \dots + 11^2)$$

$$= \frac{16}{25} ((1^2 + 2^2 + \dots + 11^2) - 1^2)$$

$$= \frac{16}{25} \left(\frac{11 \cdot (11 + 1)(2 \cdot 11 + 1)}{6} - 1 \right)$$

$$= \frac{16}{25} (506 - 1) = \frac{16}{25} \times 505$$

$$\Rightarrow \frac{16}{5} m = \frac{16}{25} \times 505 \Rightarrow m = 101$$

8. (c) Given, $p = \lim_{x \rightarrow 0^+} (1 + \tan^2 \sqrt{x})^{\frac{1}{2x}}$ (1^∞ form)

$$= e^{\lim_{x \rightarrow 0^+} \frac{\tan^2 \sqrt{x}}{2x}} = e^{\frac{1}{2} \lim_{x \rightarrow 0^+} \left(\frac{\tan \sqrt{x}}{\sqrt{x}} \right)^2} = e^{\frac{1}{2}}$$

$$\therefore \log p = \log e^{\frac{1}{2}} = \frac{1}{2}$$

9. (b) Let $I = \lim_{n \rightarrow \infty} \left(\frac{(n+1) \cdot (n+2) \dots (3n)}{n^{2n}} \right)^{\frac{1}{n}}$

$$= \lim_{n \rightarrow \infty} \left(\frac{(n+1) \cdot (n+2) \dots (n+2n)}{n^{2n}} \right)^{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \left(\left(\frac{n+1}{n} \right) \left(\frac{n+2}{n} \right) \dots \left(\frac{n+2n}{n} \right) \right)^{\frac{1}{n}}$$

Taking log on both sides, we get

$$\log I = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \left\{ \left(1 + \frac{1}{n} \right) \left(1 + \frac{2}{n} \right) \dots \left(1 + \frac{2n}{n} \right) \right\} \right]$$

$$\Rightarrow \log I = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \left(1 + \frac{1}{n} \right) + \log \left(1 + \frac{2}{n} \right) + \dots + \log \left(1 + \frac{2n}{n} \right) \right]$$

$$\Rightarrow \log I = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^{2n} \log \left(1 + \frac{r}{n} \right) \Rightarrow \log I = \int_0^2 \log(1+x) dx$$

$$\Rightarrow \log I = \left[\log(1+x) \cdot x - \int \frac{1}{1+x} \cdot x dx \right]_0^2$$

$$\Rightarrow \log I = \left[\log(1+x) \cdot x - \int_0^2 \frac{x+1-1}{1+x} dx \right]$$

$$\Rightarrow \log I = 2 \cdot \log 3 - \int_0^2 \left(1 - \frac{1}{1+x} \right) dx$$

$$\Rightarrow \log I = 2 \cdot \log 3 - [x - \log|1+x|]_0^2$$

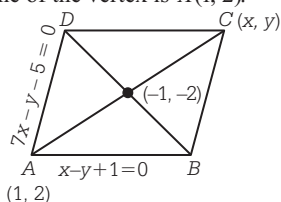
$$\Rightarrow \log I = 2 \cdot \log 3 - [2 - \log 3]$$

$$\Rightarrow \log I = 3 \cdot \log 3 - 2 \Rightarrow \log I = \log 27 - 2$$

$$\therefore I = e^{\log 27 - 2} = 27 \cdot e^{-2} = \frac{27}{e^2}$$

10. (c) As the given lines $x - y + 1 = 0$ and $7x - y - 5 = 0$ are not parallel, therefore they represent the adjacent sides of the rhombus.

On solving $x - y + 1 = 0$ and $7x - y - 5 = 0$, we get $x = 1$ and $y = 2$. Thus, one of the vertex is $A(1, 2)$.



Let the coordinate of point C be (x, y) .

Then, $-1 = \frac{x+1}{2}$ and $-2 = \frac{y+2}{2}$

$$\Rightarrow x + 1 = -2 \text{ and } y = -4 - 2$$

$$\Rightarrow x = -3 \text{ and } y = -6$$

Hence, coordinates of $C = (-3, -6)$

Note that, vertices B and D will satisfy $x - y + 1 = 0$ and $7x - y - 5 = 0$, respectively.

Since, option (c) satisfies $7x - y - 5 = 0$, therefore coordinate of vertex D is $\left(\frac{1}{3}, \frac{-8}{3} \right)$.

11. (d) Given equation of circle is $x^2 + y^2 - 8x - 8y - 4 = 0$, whose centre is $C(4, 4)$ and radius $= \sqrt{4^2 + 4^2 + 4} = \sqrt{36} = 6$

Let the centre of required circle be $C_1(x, y)$. Now, as it touch the X -axis, therefore its radius $= |y|$. Also, it touch the circle $x^2 + y^2 - 8x - 8y - 4 = 0$, therefore $CC_1 = 6 + |y|$

$$\Rightarrow \sqrt{(x-4)^2 + (y-4)^2} = 6 + |y|$$

$$\Rightarrow x^2 + 16 - 8x + y^2 + 16 - 8y = 36 + y^2 + 12|y|$$

$$\Rightarrow x^2 - 8x - 8y + 32 = 36 + 12|y|$$

$$\Rightarrow x^2 - 8x - 8y - 4 = 12|y|$$

Case I If $y > 0$, then we have

$$x^2 - 8x - 8y - 4 = 12y \Rightarrow x^2 - 8x - 20y - 4 = 0$$

$$\Rightarrow x^2 - 8x - 4 = 20y$$

$$\Rightarrow (x-4)^2 - 20 = 20y$$

$$\Rightarrow (x-4)^2 = 20(y+1), \text{ which is a parabola.}$$

Case II If $y < 0$, then we have

$$x^2 - 8x - 8y - 4 = -12y$$

$$\Rightarrow x^2 - 8x - 8y - 4 + 12y = 0$$

$$\Rightarrow x^2 - 8x + 4y - 4 = 0$$

$$\Rightarrow x^2 - 8x - 4 = -4y$$

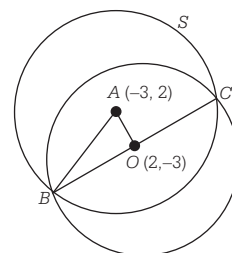
$$\Rightarrow (x-4)^2 = 20 - 4y$$

$$\Rightarrow (x-4)^2 = -4(y-5), \text{ which is again a parabola.}$$

12. (b) Given equation of a circle is $x^2 + y^2 - 4x + 6y - 12 = 0$, whose centre is $(2, -3)$ and radius

$$= \sqrt{2^2 + (-3)^2 + 12} = \sqrt{4 + 9 + 12} = 5$$

Now, according to given information, we have the following figure.



$$x^2 + y^2 - 4x + 6y - 12 = 0$$

Clearly, $AO \perp BC$, as O is mid-point of the chord.

Now, in $\triangle AOB$, we have

$$\begin{aligned} OA &= \sqrt{(-3-2)^2 + (2+3)^2} \\ &= \sqrt{25+25} = \sqrt{50} = 5\sqrt{2} \end{aligned}$$

and $OB = 5$

$$\therefore AB = \sqrt{OA^2 + OB^2} = \sqrt{50 + 25} = \sqrt{75} = 5\sqrt{3}$$

13. (a) Centre of circle $x^2 + (y+6)^2 = 1$ is $C(0, -6)$.

Let the coordinates of point P be $(2t^2, 4t)$.

$$\text{Now, let } D = CP = \sqrt{(2t^2)^2 + (4t+6)^2}$$

$$\Rightarrow D = \sqrt{4t^4 + 16t^2 + 36 + 48t}$$

Squaring on both sides

$$\Rightarrow D^2(t) = 4t^4 + 16t^2 + 48t + 36$$

$$\text{Let } F(t) = 4t^4 + 16t^2 + 48t + 36$$

For minimum, $F'(t) = 0$

$$\Rightarrow 16t^3 + 32t + 48 = 0$$

$$\Rightarrow t^3 + 2t + 3 = 0$$

$$\Rightarrow (t+1)(t^2 - t + 3) = 0$$

$$\Rightarrow t = -1$$

Thus, coordinate of point P are $(2, -4)$.

$$\text{Now, } CP = \sqrt{2^2 + (-4+6)^2} = \sqrt{4+4} = 2\sqrt{2}$$

Hence, the required equation of circle is

$$(x-2)^2 + (y+4)^2 = (2\sqrt{2})^2$$

$$\Rightarrow x^2 + 4 - 4x + y^2 + 16 + 8y = 8$$

$$\Rightarrow x^2 + y^2 - 4x + 8y + 12 = 0$$

14. (c) We have,

$$\frac{2b^2}{a} = 8 \text{ and } 2b = ae \Rightarrow b^2 = 4a \text{ and } 2b = ae$$

Consider, $2b = ae$

$$\Rightarrow 4b^2 = a^2 e^2$$

$$\Rightarrow 4a^2(e^2 - 1) = a^2 e^2$$

$$\Rightarrow 4e^2 - 4 = e^2 \quad [\because a \neq 0]$$

$$\Rightarrow 3e^2 = 4 \Rightarrow e = \frac{2}{\sqrt{3}} \quad [\because e > 0]$$

15. (b) Equation of line passing through the point $(1, -5, 9)$ and parallel to $x = y = z$ is

$$\frac{x-1}{1} = \frac{y+5}{1} = \frac{z-9}{1} = \lambda \quad (\text{say})$$

Thus, any point on this line is of the form $(\lambda+1, \lambda-5, \lambda+9)$.

Now, if $P(\lambda+1, \lambda-5, \lambda+9)$ is the point of intersection of line and plane, then

$$(\lambda+1) - (\lambda-5) + \lambda+9 = 5$$

$$\Rightarrow \lambda + 15 = 5 \Rightarrow \lambda = -10$$

$\therefore$ Coordinates of point P are $(-9, -15, -1)$.

Hence, required distance

$$\begin{aligned} &= \sqrt{(1+9)^2 + (-5+15)^2 + (9+1)^2} = \sqrt{10^2 + 10^2 + 10^2} \\ &= 10\sqrt{3} \end{aligned}$$

16. (b) We know that, if $x_1, x_2, \dots, x_n$ are n observations, then their standard deviation is given by

$$\sqrt{\frac{1}{n} \sum x_i^2 - \left(\frac{\sum x_i}{n} \right)^2}$$

$$\text{We have, } (3.5)^2 = \frac{(2^2 + 3^2 + a^2 + 11^2)}{4} - \left(\frac{2+3+a+11}{4} \right)^2$$

$$\Rightarrow \frac{49}{4} = \frac{4+9+a^2+121}{4} - \left(\frac{16+a}{4} \right)^2$$

$$\Rightarrow \frac{49}{4} = \frac{134+a^2}{4} - \frac{256+a^2+32a}{16}$$

$$\Rightarrow \frac{49}{4} = \frac{4a^2+536-256-a^2-32a}{16}$$

$$\Rightarrow 49 \times 4 = 3a^2 - 32a + 280$$

$$\Rightarrow 3a^2 - 32a + 84 = 0$$

17. (d) Clearly, $E_1 = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\}$

$$E_2 = \{(1,2), (2,2), (3,2), (4,2), (5,2), (6,2)\}$$

$$\text{and } E_3 = \{(1,2), (1,4), (1,6), (2,1), (2,3), (2,5), (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), (5,2), (5,4), (5,6), (6,1), (6,3), (6,5)\}$$

$$\Rightarrow P(E_1) = \frac{6}{36} = \frac{1}{6},$$

$$P(E_2) = \frac{6}{36} = \frac{1}{6}$$

$$\text{and } P(E_3) = \frac{18}{36} = \frac{1}{2}$$

$$\text{Now, } P(E_1 \cap E_2) = P(\text{getting 4 on die } A \text{ and 2 on die } B)$$

$$= \frac{1}{36} = P(E_1) \cdot P(E_2)$$

$$P(E_2 \cap E_3) = P(\text{getting 2 on die } B \text{ and sum of numbers on both dice is odd})$$

$$= \frac{3}{36} = P(E_2) \cdot P(E_3)$$

$$P(E_1 \cap E_3) = P(\text{getting 4 on die } A \text{ and sum of numbers on both dice is odd})$$

$$= \frac{3}{36} = P(E_1) \cdot P(E_3)$$

$$\text{and } P(E_1 \cap E_2 \cap E_3) = P(\text{getting 4 on die } A, 2 \text{ on die } B \text{ and sum of numbers is odd})$$

$$= P(\text{impossible event}) = 0$$

Hence, E_1, E_2 and E_3 are not independent.

18. (c) Given equation is

$$\cos x + \cos 2x + \cos 3x + \cos 4x = 0$$

$$\Rightarrow (\cos x + \cos 3x) + (\cos 2x + \cos 4x) = 0$$

$$\Rightarrow 2 \cos 2x \cos x + 2 \cos 3x \cos x = 0$$

$$\Rightarrow 2 \cos x (\cos 2x + \cos 3x) = 0$$

$$\Rightarrow 2 \cos x \left(2 \cos \frac{5x}{2} \cos \frac{x}{2} \right) = 0$$

$$\Rightarrow \cos x \cdot \cos \frac{5x}{2} \cdot \cos \frac{x}{2} = 0$$

$$\Rightarrow \cos x = 0$$

or $\cos \frac{5x}{2} = 0$ or $\cos \frac{x}{2} = 0$

Now, $\cos x = 0$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2} \quad [\because 0 \leq x < 2\pi]$$

$$\cos \frac{5x}{2} = 0$$

$$\Rightarrow \frac{5x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}, \dots$$

$$\Rightarrow x = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5} \quad [\because 0 \leq x < 2\pi]$$

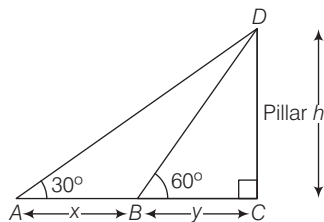
and $\cos \frac{x}{2} = 0$

$$\Rightarrow \frac{x}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \dots$$

$$\Rightarrow x = \pi \quad [\because 0 \leq x < 2\pi]$$

Hence, $x = \frac{\pi}{2}, \frac{3\pi}{2}, \pi, \frac{\pi}{5}, \frac{3\pi}{5}, \frac{7\pi}{5}, \frac{9\pi}{5}$

19. (d) According to given information, we have the following figure



Now, from $\triangle ACD$ and $\triangle BCD$, we have

$$\tan 30^\circ = \frac{h}{x+y} \text{ and } \tan 60^\circ = \frac{h}{y}$$

$$\Rightarrow h = \frac{x+y}{\sqrt{3}} \quad \dots(i)$$

and $h = \sqrt{3} y \quad \dots(ii)$

From Eqs. (i) and (ii),

$$\frac{x+y}{\sqrt{3}} = \sqrt{3} y$$

$$\Rightarrow x+y = 3y$$

$$\Rightarrow x - 2y = 0$$

$$\Rightarrow y = \frac{x}{2}$$

$\therefore$ Speed is uniform.

$\therefore$ Distance y will be cover in 5 min.

$\therefore$ Distance x covered in 10 min.

$\therefore$ Distance $\frac{x}{2}$ will be cover in 5 min.

20. (c) Consider, $(p \wedge \sim q) \vee q \vee (\sim p \wedge q)$

$$\begin{aligned} &\equiv [(p \wedge \sim q) \vee q] \vee (\sim p \wedge q) \\ &\equiv [(p \vee q) \wedge (\sim q \vee q)] \vee (\sim p \wedge q) \\ &\equiv [(p \vee q) \wedge t] \vee (\sim p \wedge q) \\ &\equiv (p \vee q) \vee (\sim p \wedge q) \\ &\equiv (p \vee q \vee \sim p) \wedge (p \vee q \vee q) \\ &\equiv (q \vee t) \wedge (p \vee q) \\ &\equiv t \wedge (p \vee q) \\ &\equiv p \vee q \end{aligned}$$

JEE Advanced

Paper I

1. (a) We have, 6 girls and 4 boys. To select 4 members (atmost one boy)

i.e. (1 boy and 3 girls) or (4 girls) $= {}^6C_3 \cdot {}^4C_1 + {}^6C_4 \quad \dots(i)$

Now, selection of captain from 4 members $= {}^4C_1 \quad \dots(ii)$

$\therefore$ Number of ways to select 4 members (including the selection of a captain, from these 4 members) $= ({}^6C_3 \cdot {}^4C_1 + {}^6C_4) {}^4C_1$

$$= (20 \times 4 + 15) \times 4 = 380$$

2. (c) Here, $x^2 - 2x \sec \theta + 1 = 0$ has roots α_1 and β_1 .

$$\therefore \alpha_1, \beta_1 = \frac{2 \sec \theta \pm \sqrt{4 \sec^2 \theta - 4}}{2 \times 1} = \frac{2 \sec \theta \pm 2 |\tan \theta|}{2}$$

Since, $\theta \in \left(-\frac{\pi}{6}, -\frac{\pi}{12}\right)$,

i.e. $\theta \in$ IV quadrant $= \frac{2 \sec \theta \mp 2 \tan \theta}{2}$

$\therefore \alpha_1 = \sec \theta - \tan \theta$ and $\beta_1 = \sec \theta + \tan \theta \quad [\text{as } \alpha_1 > \beta_1]$

and $x^2 + 2x \tan \theta - 1 = 0$ has roots α_2 and β_2 .

i.e. $\alpha_2, \beta_2 = \frac{-2 \tan \theta \pm \sqrt{4 \tan^2 \theta + 4}}{2}$

$\therefore \alpha_2 = -\tan \theta + \sec \theta$

and $\beta_2 = -\tan \theta - \sec \theta \quad [\text{as } \alpha_2 > \beta_2]$

Thus, $\alpha_1 + \beta_2 = -2 \tan \theta$

3. (c) Given, $\sqrt{3} \sec x + \operatorname{cosec} x + 2(\tan x - \cot x) = 0$,

$$(-\pi < x < \pi) - \{0, \pm \pi/2\}$$

$$\Rightarrow \sqrt{3} \sin x + \cos x + 2(\sin^2 x - \cos^2 x) = 0$$

$$\Rightarrow \sqrt{3} \sin x + \cos x - 2 \cos 2x = 0$$

Multiplying and dividing by $\sqrt{a^2 + b^2}$, i.e. $\sqrt{3+1} = 2$, we get

$$2 \left(\frac{\sqrt{3}}{2} \sin x + \frac{1}{2} \cos x \right) - 2 \cos 2x = 0$$

$$\Rightarrow \left(\cos x \cdot \cos \frac{\pi}{3} + \sin x \cdot \sin \frac{\pi}{3} \right) - \cos 2x = 0$$

$$\Rightarrow \cos \left(x - \frac{\pi}{3} \right) = \cos 2x$$

$$\therefore 2x = 2n\pi \pm \left(x - \frac{\pi}{3} \right) \quad \left[\begin{array}{l} \text{since, } \cos \theta = \cos \alpha \\ \Rightarrow \theta = 2n\pi \pm \alpha \end{array} \right]$$

$$\Rightarrow 2x = 2n\pi + x - \frac{\pi}{3} \text{ or } 2x = 2n\pi - x + \frac{\pi}{3}$$

$$\Rightarrow x = 2n\pi - \frac{\pi}{3} \text{ or } 3x = 2n\pi + \frac{\pi}{3}$$

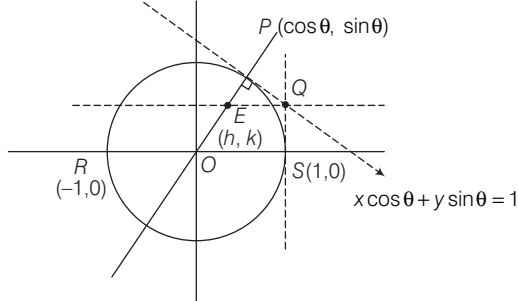
$$\Rightarrow x = 2n\pi - \frac{\pi}{3} \text{ or } x = \frac{2n\pi}{3} + \frac{\pi}{9}$$

$$\therefore x = \frac{-\pi}{3} \text{ or } x = \frac{\pi}{9}, \frac{-5\pi}{9}, \frac{7\pi}{9}$$

Now, sum of all distinct solutions $= \frac{-\pi}{3} + \frac{\pi}{9} - \frac{5\pi}{9} + \frac{7\pi}{9} = 0$

4. (a,c) Given, RS is the diameter of $x^2 + y^2 = 1$

Here, equation of the tangent at $P(\cos \theta, \sin \theta)$ is $x \cos \theta + y \sin \theta = 1$.



Intersecting with $x = 1$,

$$y = \frac{1 - \cos \theta}{\sin \theta}$$

$$\therefore Q\left(1, \frac{1 - \cos \theta}{\sin \theta}\right)$$

$\therefore$ Equation of the line through Q parallel to RS is

$$y = \frac{1 - \cos \theta}{\sin \theta} = \frac{2 \sin^2 \frac{\theta}{2}}{2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = \tan \frac{\theta}{2} \quad \dots(i)$$

Normal at P : $y = \frac{\sin \theta}{\cos \theta} \cdot x$

$$\Rightarrow y = x \tan \theta \quad \dots(ii)$$

Let their point of intersection be (h, k) .

$$\text{Then, } k = \tan \frac{\theta}{2} \text{ and } k = h \tan \theta$$

$$\therefore k = h \left(\frac{2 \tan \frac{\theta}{2}}{1 - \tan^2 \frac{\theta}{2}} \right) \Rightarrow k = \frac{2h \cdot k}{1 - k^2}$$

$$\Rightarrow k(1 - k^2) = 2hk$$

$$\therefore \text{Locus for point } E: 2x = (1 - y^2) \quad \dots(iii)$$

When $x = \frac{1}{3}$, then

$$1 - y^2 = \frac{2}{3}$$

$$\Rightarrow y^2 = 1 - \frac{2}{3}$$

$$\Rightarrow y = \pm \frac{1}{\sqrt{3}}$$

$$\therefore \left(\frac{1}{3}, \pm \frac{1}{\sqrt{3}}\right) \text{ satisfy } 2x = 1 - y^2.$$

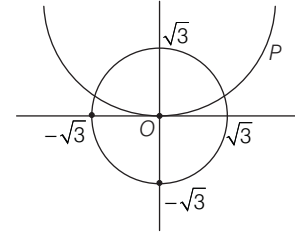
When $x = \frac{1}{4}$, then

$$1 - y^2 = \frac{2}{4} \Rightarrow y^2 = 1 - \frac{1}{2}$$

$$\Rightarrow y = \pm \frac{1}{\sqrt{2}}$$

$$\therefore \left(\frac{1}{4}, \pm \frac{1}{\sqrt{2}}\right) \text{ does not satisfy } 1 - y^2 = 2x.$$

5. (a,b,c) Given, $C_1: x^2 + y^2 = 3$ intersects the parabola $x^2 = 2y$.



On solving $x^2 + y^2 = 3$ and $x^2 = 2y$, we get

$$y^2 + 2y = 3$$

$$\Rightarrow y^2 + 2y - 3 = 0$$

$$\Rightarrow (y + 3)(y - 1) = 0$$

$$\therefore y = 1, -3 \text{ [neglecting } y = -3, \text{ as } -\sqrt{3} \leq y \leq \sqrt{3}]$$

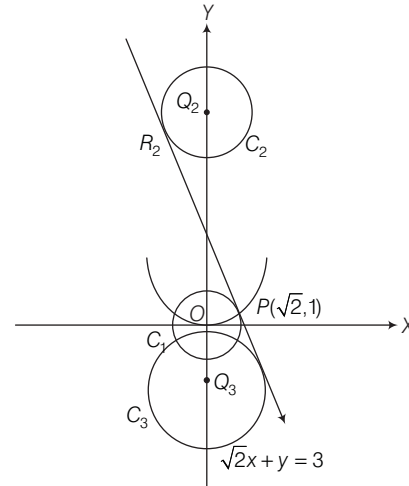
$$\therefore y = 1 \Rightarrow x = \pm \sqrt{2}$$

$$\Rightarrow P(\sqrt{2}, 1) \in \text{I quadrant}$$

Equation of tangent at $P(\sqrt{2}, 1)$ to $C_1: x^2 + y^2 = 3$ is

$$\sqrt{2}x + 1 \cdot y = 3 \quad \dots(i)$$

Now, let the centres of C_2 and C_3 be Q_2 and Q_3 , and tangent at P touches C_2 and C_3 at R_2 and R_3 shown as below



Let Q_2 be $(0, k)$ and radius is $2\sqrt{3}$.

$$\therefore \frac{|0 + k - 3|}{\sqrt{2 + 1}} = 2\sqrt{3} \Rightarrow |k - 3| = 6$$

$$\Rightarrow k = 9, -3$$

$$\therefore Q_2(0, 9) \text{ and } Q_3(0, -3)$$

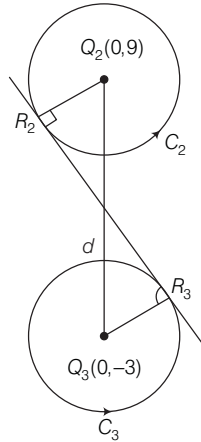
$$\text{Hence, } Q_2Q_3 = 12$$

$\therefore$ Option (a) is correct.

Also, R_2R_3 is common internal tangent to C_2 and C_3 ,

$$\text{and } r_2 = r_3 = 2\sqrt{3}$$

$$\begin{aligned} \therefore R_2R_3 &= \sqrt{d^2 - (r_1 + r_2)^2} = \sqrt{12^2 - (4\sqrt{3})^2} \\ &= \sqrt{144 - 48} \\ &= \sqrt{96} = 4\sqrt{6} \end{aligned}$$



∴ Option (b) is correct.

∴ Length of perpendicular from $O(0, 0)$ to R_2R_3 is equal to radius of $C_1 = \sqrt{3}$.

$$\begin{aligned} \therefore \text{Area of } \Delta OR_2R_3 &= \frac{1}{2} \times R_2R_3 \times \sqrt{3} \\ &= \frac{1}{2} \times 4\sqrt{6} \times \sqrt{3} = 6\sqrt{2} \end{aligned}$$

∴ Option (c) is correct.

$$\text{Also, area of } \Delta PQ_2Q_3 = \frac{1}{2} \times Q_2Q_3 \times \sqrt{2} = \frac{\sqrt{2}}{2} \times 12 = 6\sqrt{2}$$

∴ Option (d) is incorrect.

6. (5) Coefficient of x^2 in the expansion of

$$\begin{aligned} &\{(1+x)^2 + (1+x)^3 + \dots + (1+x)^{49} + (1+mx)^{50}\} \\ \Rightarrow {}^2C_2 + {}^3C_2 + {}^4C_2 + \dots + {}^{49}C_2 + {}^{50}C_2 \cdot m^2 &= (3n+1) \cdot {}^{51}C_3 \\ \Rightarrow {}^{50}C_3 + {}^{50}C_2 m^2 &= (3n+1) \cdot {}^{51}C_3 \\ [\because {}^rC_r + {}^{r+1}C_r + \dots + {}^nC_r &= {}^{n+1}C_{r+1}] \\ \Rightarrow \frac{50 \times 49 \times 48}{3 \times 2 \times 1} + \frac{50 \times 49}{2} \times m^2 &= (3n+1) \frac{51 \times 50 \times 49}{3 \times 2 \times 1} \\ \Rightarrow m^2 &= 51n+1 \end{aligned}$$

∴ Minimum value of m^2 for which $(51n+1)$ is integer (perfect square) for $n=5$.

$$\therefore m^2 = 51 \times 5 + 1 \Rightarrow m^2 = 256$$

$$\therefore m = 16 \text{ and } n = 5$$

Hence, the value of n is 5.

$$7. (1) \text{ Here, } z = \frac{-1 + i\sqrt{3}}{2} = \omega$$

$$\begin{aligned} \therefore P &= \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} \\ P^2 &= \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix} \\ &= \begin{bmatrix} \omega^{2r} + \omega^{4s} & \omega^{r+2s} [(-1)^r + 1] \\ \omega^{r+2s} [(-1)^r + 1] & \omega^{4s} + \omega^{2r} \end{bmatrix} \end{aligned}$$

$$\text{Given, } P^2 = -I$$

$$\therefore \omega^{2r} + \omega^{4s} = -1 \text{ and } \omega^{r+2s} [(-1)^r + 1] = 0$$

$$\text{Since, } r \in \{1, 2, 3\}$$

$$\text{and } (-1)^r + 1 = 0 \Rightarrow r = \{1, 3\}$$

$$\text{Also, } \omega^{2r} + \omega^{4s} = -1$$

$$\text{If } r = 1, \text{ then } \omega^2 + \omega^{4s} = -1$$

which is only possible, when $s = 1$.

$$\text{As, } \omega^2 + \omega^4 = -1$$

$$\therefore r = 1, s = 1$$

Again, if $r = 3$, then

$$\omega^6 + \omega^{4s} = -1$$

$$\Rightarrow \omega^{4s} = -2 \text{ (never possible)}$$

$$\therefore r \neq 3$$

$\Rightarrow (r, s) = (1, 1)$ is the only solution.

Hence, the total number of ordered pairs is 1.

$$8. (7) \text{ Here, } \lim_{x \rightarrow 0} \frac{x^2 \sin(\beta x)}{\alpha x - \sin x} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^2 \left(\beta x - \frac{(\beta x)^3}{3!} + \frac{(\beta x)^5}{5!} - \dots \right)}{\alpha x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots \right)} = 1$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{x^3 \left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \dots \right)}{(\alpha - 1)x + \frac{x^3}{3!} + \frac{x^5}{5!} - \dots} = 1$$

Limit exists only, when $\alpha - 1 = 0$

$$\Rightarrow \alpha = 1 \quad \dots(i)$$

$$\therefore \lim_{x \rightarrow 0} \frac{x^3 \left(\beta - \frac{\beta^3 x^2}{3!} + \frac{\beta^5 x^4}{5!} - \dots \right)}{x^3 \left(\frac{1}{3!} - \frac{x^2}{5!} - \dots \right)} = 1$$

$$\Rightarrow 6\beta = 1 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$6(\alpha + \beta) = 6\alpha + 6\beta = 6 + 1 = 7$$

Paper II

$$1. (c) \text{ Here, } \sum_{k=1}^{13} \frac{1}{\sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\} \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)}$$

Converting into differences, by multiplying and dividing by

$$\sin\left[\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\right], \text{ i.e. } \sin\left(\frac{\pi}{6}\right).$$

$$\begin{aligned} \therefore \sum_{k=1}^{13} \frac{\sin\left[\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) - \left(\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right)\right]}{\sin\frac{\pi}{6} \left\{ \sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\} \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) \right\}} \\ = 2 \sum_{k=1}^{13} \frac{\left[\sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) \cos\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\} \right. \\ \left. - \sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\} \cos\left(\frac{\pi}{4} + \frac{k\pi}{6}\right) \right]}{\sin\left\{\frac{\pi}{4} + \frac{(k-1)\pi}{6}\right\} \sin\left(\frac{\pi}{4} + \frac{k\pi}{6}\right)} \end{aligned}$$

$$\begin{aligned}
 &= 2 \sum_{k=1}^{13} \left[\cot \left\{ \frac{\pi}{4} + (k-1) \frac{\pi}{6} \right\} - \cot \left\{ \frac{\pi}{4} + k \frac{\pi}{6} \right\} \right] \\
 &= 2 \left[\left\{ \cot \left(\frac{\pi}{4} \right) - \cot \left(\frac{\pi}{4} + \frac{\pi}{6} \right) \right\} \right. \\
 &\quad \left. + \left\{ \cot \left(\frac{\pi}{4} + \frac{\pi}{6} \right) - \cot \left(\frac{\pi}{4} + \frac{2\pi}{6} \right) \right\} \right. \\
 &\quad \left. + \dots + \left\{ \cot \left(\frac{\pi}{4} + 12 \frac{\pi}{6} \right) - \cot \left(\frac{\pi}{4} + 13 \frac{\pi}{6} \right) \right\} \right] \\
 &= 2 \left\{ \cot \frac{\pi}{4} - \cot \left(\frac{\pi}{4} + 13 \frac{\pi}{6} \right) \right\} \\
 &= 2 \left[1 - \cot \left(\frac{29\pi}{12} \right) \right] = 2 \left[1 - \cot \left(2\pi + \frac{5\pi}{12} \right) \right] \\
 &= 2 \left[1 - \cot \frac{5\pi}{12} \right] \quad \left[\because \cot \frac{5\pi}{12} = (2 - \sqrt{3}) \right] \\
 &= 2(1 - 2 + \sqrt{3}) \\
 &= 2(\sqrt{3} - 1)
 \end{aligned}$$

2. (b) If $\log b_1, \log b_2, \dots, \log b_{101}$ are in AP, with common difference $\log_e 2$, then $b_1, b_2, \dots, b_{101}$ are in GP, with common ratio 2.

$$\therefore b_1 = 2^0 b_1, b_2 = 2^1 b_1, b_3 = 2^2 b_1, \dots, b_{101} = 2^{100} b_1 \quad \dots(i)$$

Also, $a_1, a_2, \dots, a_{101}$ are in AP.

Given, $a_1 = b_1$ and $a_{51} = b_{51}$

$$\Rightarrow a_1 + 50D = 2^{50} b_1$$

$$\Rightarrow a_1 + 50D = 2^{50} a_1 \quad [\because a_1 = b_1] \dots(ii)$$

$$\text{Now, } t = b_1 + b_2 + \dots + b_{51}$$

$$\Rightarrow t = b_1 \frac{(2^{51} - 1)}{2 - 1} \quad \dots(iii)$$

$$\begin{aligned} \text{and } s &= a_1 + a_2 + \dots + a_{51} \\ &= \frac{51}{2} (2a_1 + 50D) \quad \dots(iv) \end{aligned}$$

$$\therefore t = a_1 (2^{51} - 1) \quad [\because a_1 = b_1]$$

$$\text{or } t = 2^{51} a_1 - a_1 < 2^{51} a_1 \quad \dots(v)$$

$$\begin{aligned} \text{and } s &= \frac{51}{2} [a_1 + (a_1 + 50D)] \quad [\text{from Eq. (ii)}] \\ &= \frac{51}{2} [a_1 + 2^{50} a_1] \\ &= \frac{51}{2} a_1 + \frac{51}{2} 2^{50} a_1 \end{aligned}$$

$$\therefore s > 2^{51} a_1 \quad \dots(vi)$$

From Eqs. (v) and (vi), we get $s > t$

$$\text{Also, } a_{101} = a_1 + 100D \text{ and } b_{101} = 2^{100} b_1$$

$$\therefore a_{101} = a_1 + 100 \left(\frac{2^{50} a_1 - a_1}{50} \right)$$

$$\text{and } b_{101} = 2^{100} a_1$$

$$\Rightarrow a_{101} = a_1 + 2^{51} a_1 - 2a_1 = 2^{51} a_1 - a_1$$

$$\Rightarrow a_{101} < 2^{51} a_1 \text{ and } b_{101} > 2^{51} a_1 \Rightarrow b_{101} > a_{101}$$

$$3. (a) \text{ Let } I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1 + e^x} dx \quad \dots(i)$$

$$\begin{aligned} &\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \\ \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos(-x)}{1 + e^{-x}} dx \quad \dots(ii) \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$\begin{aligned} 2I &= \int_{-\pi/2}^{\pi/2} x^2 \cos x \left[\frac{1}{1 + e^x} + \frac{1}{1 + e^{-x}} \right] dx \\ &= \int_{-\pi/2}^{\pi/2} x^2 \cos x \cdot (1) dx \\ &\left[\because \int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx, \text{ when } f(-x) = f(x) \right] \end{aligned}$$

$$\Rightarrow 2I = 2 \int_0^{\pi/2} x^2 \cos x dx$$

Using integration by parts, we get

$$2I = 2 [x^2 (\sin x) - (2x) (-\cos x) + (2) (-\sin x)]_0^{\pi/2}$$

$$\Rightarrow 2I = 2 \left[\frac{\pi^2}{4} - 2 \right]$$

$$\therefore I = \frac{\pi^2}{4} - 2$$

4. (b,c) Here,

$$f(x) = \lim_{n \rightarrow \infty} \left[\frac{n^n (x+n) \left(x + \frac{n}{2}\right) \dots \left(x + \frac{n}{n}\right)}{n! (x^2 + n^2) \left(x^2 + \frac{n^2}{4}\right) \dots \left(x^2 + \frac{n^2}{n^2}\right)} \right]^{\frac{x}{n}}, x > 0$$

Taking log on both sides, we get

$$\begin{aligned} \log_e \{f(x)\} &= \lim_{n \rightarrow \infty} \log \left[\frac{n^n (x+n) \left(x + \frac{n}{2}\right) \dots \left(x + \frac{n}{n}\right)}{n! (x^2 + n^2) \left(x^2 + \frac{n^2}{4}\right) \dots \left(x^2 + \frac{n^2}{n^2}\right)} \right]^{\frac{x}{n}} \\ &= \lim_{n \rightarrow \infty} \frac{x}{n} \cdot \log \left[\frac{\prod_{r=1}^n \left(x + \frac{1}{r/n}\right)}{\prod_{r=1}^n \left(x^2 + \frac{1}{(r/n)^2}\right) \prod_{r=1}^n (r/n)} \right] \\ &= x \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left[\frac{x + \frac{1}{r/n}}{\left(x^2 + \frac{1}{r^2/n^2}\right) \frac{r}{n}} \right] \\ &= x \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left(\frac{r}{n^2} \cdot x + 1 \right) \end{aligned}$$

Converting summation into definite integration, we get

$$\log_e \{f(x)\} = x \int_0^1 \log \left(\frac{xt + 1}{x^2 t^2 + 1} \right) dt$$

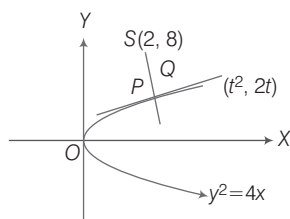
Put, $tx = z$
 $\Rightarrow xdt = dz$
 $\therefore \log_e \{f(x)\} = x \int_0^x \log \left(\frac{1+z}{1+z^2} \right) \frac{dz}{x}$
 $\Rightarrow \log_e \{f(x)\} = \int_0^x \log \left(\frac{1+z}{1+z^2} \right) dz$
 Using Newton-Leibnitz formula, we get
 $\frac{1}{f(x)} \cdot f'(x) = \log \left(\frac{1+x}{1+x^2} \right) \quad \dots (i)$

Here, at $x = 1$,
 $\frac{f'(1)}{f(1)} = \log(1) = 0$
 $\therefore f'(1) = 0$
 Now, sign scheme of $f'(x)$ is shown below

$\begin{array}{c} + \qquad \qquad \qquad - \\ \hline \qquad \qquad \qquad x=1 \end{array}$

$\therefore$ At $x = 1$, function attains maximum.
 Since, $f(x)$ increases on $(0, 1)$.
 $\therefore f(1) > f(1/2)$
 $\therefore$ option (a) is incorrect.
 $f(1/3) < f(2/3)$
 $\therefore$ option (b) is correct.
 Also, $f'(x) < 0$, when $x > 1$
 $\Rightarrow f'(2) < 0$
 $\therefore$ option (c) is correct.
 Also, $\frac{f'(x)}{f(x)} = \log \left(\frac{1+x}{1+x^2} \right)$
 $\therefore \frac{f'(3)}{f(3)} - \frac{f'(2)}{f(2)} = \log \left(\frac{4}{10} \right) - \log \left(\frac{3}{5} \right)$
 $= \log(2/3) < 0$
 $\Rightarrow \frac{f'(3)}{f(3)} < \frac{f'(2)}{f(2)}$
 $\therefore$ option (d) is incorrect.

5. (a,c,d) Tangent to $y^2 = 4x$ at $(t^2, 2t)$ is



$y(2t) = 2(x + t^2)$
 $\Rightarrow yt = x + t^2 \quad \dots (i)$

Equation of normal at $P(t^2, 2t)$ is
 $y + tx = 2t + t^3$
 Since, normal at P passes through centre of circle $S(2, 8)$.
 $\therefore 8 + 2t = 2t + t^3$

$\Rightarrow t = 2$, i.e. $P(4, 4)$
 [since, shortest distance between two curves lie along their common normal and the common normal will pass through the centre of circle]
 $\therefore SP = \sqrt{(4-2)^2 + (4-8)^2} = 2\sqrt{5}$
 $\therefore$ option (a) is correct.
 Also, $SQ = 2$
 $\therefore PQ = SP - SQ = 2\sqrt{5} - 2$
 Thus, $\frac{SQ}{QP} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$
 $\therefore$ option (b) is incorrect.
 Now, x-intercept of normal is $x = 2 + 2^2 = 6$
 $\therefore$ option (c) is correct.
 Slope of tangent $= \frac{1}{t} = \frac{1}{2}$
 $\therefore$ option (d) is correct.

6. (a,c,d) Here, $x + iy = \frac{1}{a + ibt} \times \frac{a - ibt}{a - ibt}$

$\therefore x + iy = \frac{a - ibt}{a^2 + b^2t^2}$

Let $a \neq 0, b \neq 0$

$\therefore x = \frac{a}{a^2 + b^2t^2}$

and $y = \frac{-bt}{a^2 + b^2t^2}$

$\Rightarrow \frac{y}{x} = \frac{-bt}{a} \Rightarrow t = \frac{ay}{bx}$

On putting $x = \frac{a}{a^2 + b^2t^2}$, we get

$x \left(a^2 + b^2 \cdot \frac{a^2y^2}{b^2x^2} \right) = a$

$\Rightarrow a^2(x^2 + y^2) = ax$

or $x^2 + y^2 - \frac{x}{a} = 0 \quad \dots (i)$

or $\left(x - \frac{1}{2a} \right)^2 + y^2 = \frac{1}{4a^2}$

$\therefore$ Option (a) is correct.

For $a \neq 0$ and $b = 0$,
 $x + iy = \frac{1}{a} \Rightarrow x = \frac{1}{a}, y = 0$

$\Rightarrow z$ lies on X-axis.

$\therefore$ Option (c) is correct.

For $a = 0$ and $b \neq 0$,

$x + iy = \frac{1}{ibt}$

$\Rightarrow x = 0, y = -\frac{1}{bt}$

$\Rightarrow z$ lies on Y-axis.

$\therefore$ Option (d) is correct.

7. (b) Here, $P(X > Y) = P(T_1 \text{ win}) P(T_1 \text{ win})$
 $+ P(T_1 \text{ win}) P(\text{draw}) + P(\text{draw}) P(T_1 \text{ win})$
 $= \left(\frac{1}{2} \times \frac{1}{2}\right) + \left(\frac{1}{2} \times \frac{1}{6}\right) + \left(\frac{1}{6} \times \frac{1}{2}\right)$
 $= \frac{5}{12}$

8. (c) $P[X = Y] = P(\text{draw}) \cdot P(\text{draw})$
 $+ P(T_1 \text{ win}) \cdot P(T_2 \text{ win}) + P(T_2 \text{ win}) \cdot P(T_1 \text{ win})$
 $= (1/6 \times 1/6) + (1/2 \times 1/3) + (1/3 \times 1/2)$
 $= 13/36$

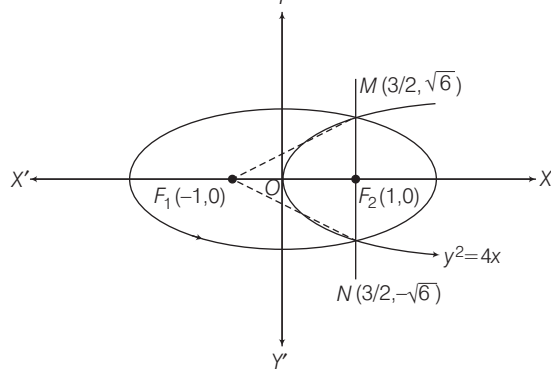
9. (a) Here, $\frac{x^2}{9} + \frac{y^2}{8} = 1$... (i)

has foci $(\pm ae, 0)$

where, $a^2 e^2 = a^2 - b^2$

$\Rightarrow a^2 e^2 = 9 - 8 \Rightarrow ae = \pm 1$

i.e. $F_1, F_2 = (\pm 1, 0)$



Equation of parabola having vertex $O(0, 0)$ and $F_2(1, 0)$ (as, $x_2 > 0$)

$y^2 = 4x$... (ii)

On solving $\frac{x^2}{9} + \frac{y^2}{8} = 1$ and $y^2 = 4x$, we get

$x = 3/2$ and $y = \pm \sqrt{6}$

Equation of altitude through M on NF_1 is

$\frac{y - \sqrt{6}}{x - 3/2} = \frac{5}{2\sqrt{6}} \Rightarrow (y - \sqrt{6}) = \frac{5}{2\sqrt{6}} (x - 3/2)$... (iii)

and equation of altitude through F_1 is $y = 0$... (iv)

On solving Eqs. (iii) and (iv), we get $\left(-\frac{9}{10}, 0\right)$ as orthocentre.

10. (c) Equation of tangent at $M(3/2, \sqrt{6})$ to $\frac{x^2}{9} + \frac{y^2}{8} = 1$ is

$\frac{3}{2} \cdot \frac{x}{9} + \sqrt{6} \cdot \frac{y}{8} = 1$... (i)

which intersect X -axis at $(6, 0)$.

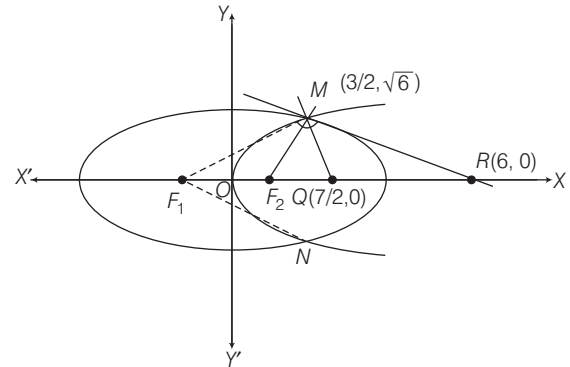
Also, equation of tangent at $N(3/2, -\sqrt{6})$ is

$\frac{3}{2} \cdot \frac{x}{9} - \sqrt{6} \cdot \frac{y}{8} = 1$... (ii)

Eqs. (i) and (ii) intersect on X -axis at $R(6, 0)$ (iii)

Also, normal at $M(3/2, \sqrt{6})$ is $y - \sqrt{6} = \frac{-\sqrt{6}}{2} \left(x - \frac{3}{2}\right)$

On solving with $y = 0$, we get $Q(7/2, 0)$... (iv)



$\therefore \text{Area of } \triangle MQR = \frac{1}{2} \left(6 - \frac{7}{2}\right) \sqrt{6} = \frac{5\sqrt{6}}{4} \text{ sq units}$

and area of quadrilateral $MF_1NF_2 = 2 \times \frac{1}{2} \{1 - (-1)\} \sqrt{6}$
 $= 2\sqrt{6} \text{ sq units}$

$\therefore \frac{\text{Area of } \triangle MQR}{\text{Area of quadrilateral } MF_1NF_2} = \frac{5}{8}$

**JEE Main & Adv./BITSAT/KERALA CEE/KCET
AP & TS EAMCET/VIT/MHT CET**

SOLVED PAPERS 2017

JEE Main

- The statement $(p \rightarrow q) \rightarrow [(\sim p \rightarrow q) \rightarrow q]$ is
(a) a tautology (b) equivalent to $\sim p \rightarrow q$
(c) equivalent to $p \rightarrow \sim q$ (d) a fallacy
- If $5(\tan^2 x - \cos^2 x) = 2 \cos 2x + 9$, then the value of $\cos 4x$ is
(a) $-\frac{3}{5}$ (b) $\frac{1}{3}$ (c) $\frac{2}{9}$ (d) $-\frac{7}{9}$
- For three events A, B and C , if $P(\text{exactly one of } A \text{ or } B \text{ occurs}) = P(\text{exactly one of } B \text{ or } C \text{ occurs}) = P(\text{exactly one of } C \text{ or } A \text{ occurs}) = \frac{1}{4}$ and $P(\text{all the three events occur simultaneously}) = \frac{1}{16}$, then the probability that atleast one of the events occurs, is
(a) $\frac{7}{32}$ (b) $\frac{7}{16}$ (c) $\frac{7}{64}$ (d) $\frac{3}{16}$
- Let ω be a complex number such that $2\omega + 1 = z$, where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to
(a) $-z$ (b) z (c) -1 (d) 1
- Let k be an integer such that the triangle with vertices $(k, -3k)$, $(5, k)$ and $(-k, 2)$ has area 28 sq units. Then, the orthocentre of this triangle is at the point
(a) $\left(2, -\frac{1}{2}\right)$ (b) $\left(1, \frac{3}{4}\right)$ (c) $\left(1, -\frac{3}{4}\right)$ (d) $\left(2, \frac{1}{2}\right)$
- Let a vertical tower AB have its end A on the level ground. Let C be the mid-point of AB and P be a point on the ground such that $AP = 2AB$. If $\angle BPC = \beta$, then $\tan \beta$ is equal to
(a) $\frac{6}{7}$ (b) $\frac{1}{4}$ (c) $\frac{2}{9}$ (d) $\frac{4}{9}$
- For any three positive real numbers a, b and c , if $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$, then
(a) b, c and a are in GP (b) b, c and a are in AP
(c) a, b and c are in AP (d) a, b and c are in GP
- The eccentricity of an ellipse whose centre is at the origin is $1/2$. If one of its directrices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is
(a) $2y - x = 2$ (b) $4x - 2y = 1$
(c) $4x + 2y = 7$ (d) $x + 2y = 4$
- If a hyperbola passes through the point $P(\sqrt{2}, \sqrt{3})$ and has foci at $(\pm 2, 0)$, then the tangent to this hyperbola at P also passes through the point
(a) $(3\sqrt{2}, 2\sqrt{3})$ (b) $(2\sqrt{2}, 3\sqrt{3})$
(c) $(\sqrt{3}, \sqrt{2})$ (d) $(-\sqrt{2}, -\sqrt{3})$
- $\lim_{x \rightarrow \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3}$ equals
(a) $\frac{1}{24}$ (b) $\frac{1}{16}$ (c) $\frac{1}{8}$ (d) $\frac{1}{4}$
- If two different numbers are taken from the set $\{0, 1, 2, 3, \dots, 10\}$, then the probability that their sum as well as absolute difference are both multiple of 4, is
(a) $\frac{6}{55}$ (b) $\frac{12}{55}$ (c) $\frac{14}{45}$ (d) $\frac{7}{55}$
- A man X has 7 friends, 4 of them are ladies and 3 are men. His wife Y also has 7 friends, 3 of them are ladies and 4 are men. Assume X and Y have no common friends. Then, the total number of ways in which X and Y together can throw a party inviting 3 ladies and 3 men, so that 3 friends of each of X and Y are in this party, is
(a) 485 (b) 468 (c) 469 (d) 484
- The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \dots + ({}^{21}C_{10} - {}^{10}C_{10})$ is
(a) $2^{21} - 2^{11}$ (b) $2^{21} - 2^{10}$ (c) $2^{20} - 2^9$ (d) $2^{20} - 2^{10}$
- Let $a, b, c \in R$. If $f(x) = ax^2 + bx + c$ be such that $a + b + c = 3$ and $f(x + y) = f(x) + f(y) + xy$, $\forall x, y \in R$, then $\sum_{n=1}^{10} f(n)$ is equal to
(a) 330 (b) 165 (c) 190 (d) 255

2 Solved Papers 2017

15. The radius of a circle having minimum area, which touches the curve $y = 4 - x^2$ and the lines $y = |x|$, is

(a) $2(\sqrt{2} + 1)$ (b) $2(\sqrt{2} - 1)$
(c) $4(\sqrt{2} - 1)$ (d) $4(\sqrt{2} + 1)$

16. For a positive integer n , if the quadratic equation, $x(x+1) + (x+1)(x+2) + \dots + (x+n-1)(x+n) = 10n$ has two consecutive integral solutions, then n is equal to

(a) 12 (b) 9
(c) 10 (d) 11

JEE Advanced

Paper 1

1. Let a, b, x and y be real numbers such that $a - b = 1$ and $y \neq 0$. If the complex number $z = x + iy$ satisfies $\text{Im}\left(\frac{az + b}{z + 1}\right) = y$, then which of the following is(are) possible value(s) of x ?

(One or More Than One Correct Option)

(a) $1 - \sqrt{1 + y^2}$ (b) $-1 - \sqrt{1 - y^2}$
(c) $1 + \sqrt{1 + y^2}$ (d) $-1 + \sqrt{1 - y^2}$

2. If $2x - y + 1 = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$ then which of the following CANNOT be sides of a right angled triangle?

(One or More Than One Correct Option)

(a) $a, 4, 1$ (b) $2a, 4, 1$ (c) $a, 4, 2$ (d) $2a, 8, 1$

3. If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and mid-point (h, k) , then which of the following is(are) possible value(s) of p, h and k ?

(One or More Than One Correct Option)

(a) $p = -1, h = 1, k = -3$ (b) $p = 2, h = 3, k = -4$
(c) $p = -2, h = 2, k = -4$ (d) $p = 5, h = 4, k = -3$

4. The sides of a right angled triangle are in arithmetic progression. If the triangle has area 24, then what is the length of its smallest side?

(Single Digit Integer)

5. For how many values of p , the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points?

(Single Digit Integer)

6. Words of length 10 are formed using the letters A, B, C, D, E, F, G, H, I, J. Let x be the number of such words where no letter is repeated; and let y be the number of such words where exactly one letter is repeated twice and no other letter is repeated. Then, $\frac{y}{9x} =$

(Single Digit Integer)

Matching Type Questions

Directions (Q.Nos. 7-9) by appropriately matching the information given in the three columns of the following table.

Columns 1, 2 and 3 contain conics, equations of tangents to the conics and points of contact, respectively.

	Column-1	Column-2	Column-3
(I)	$x^2 + y^2 = a^2$	(i) $my = m^2x + a$	(P) $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$
(II)	$x^2 + a^2y^2 = a^2$	(ii) $y = mx + a\sqrt{m^2 + 1}$	(Q) $\left(\frac{-ma}{\sqrt{m^2 + 1}}, \frac{a}{\sqrt{m^2 + 1}}\right)$
(III)	$y^2 = 4ax$	(iii) $y = mx + \sqrt{a^2m^2 - 1}$	(R) $\left(\frac{-a^2m}{\sqrt{a^2m^2 + 1}}, \frac{1}{\sqrt{a^2m^2 + 1}}\right)$
(IV)	$x^2 - a^2y^2 = a^2$	(iv) $y = mx + \sqrt{a^2m^2 + 1}$	(S) $\left(\frac{-a^2m}{\sqrt{a^2m^2 - 1}}, \frac{-1}{\sqrt{a^2m^2 - 1}}\right)$

7. For $a = \sqrt{2}$, if a tangent is drawn to a suitable conic (Column 1) at the point of contact $(-1, 1)$, then which of the following options is the only CORRECT combination for obtaining its equation?

(a) (I) (ii) (Q) (b) (I) (i) (P)
(c) (III) (i) (P) (d) (II) (ii) (Q)

8. The tangent to a suitable conic (Column 1) at $\left(\sqrt{3}, \frac{1}{2}\right)$ is found to be $\sqrt{3}x + 2y = 4$, then which of the following options is the only CORRECT combination?

(a) (IV) (iv) (S) (b) (II) (iv) (R)
(c) (IV) (iii) (S) (d) (II) (iii) (R)

9. If a tangent to a suitable conic (Column 1) is found to be $y = x + 8$ and its point of contact is $(8, 16)$, then which of the following options is the only CORRECT combination?

(a) (III) (i) (P) (b) (I) (ii) (Q)
(c) (II) (iv) (R) (d) (III) (ii) (Q)

Paper 2

1. Let $S = \{1, 2, 3, \dots, 9\}$. For $k = 1, 2, \dots, 5$, let N_k be the number of subsets of S , each containing five elements out of which exactly k are odd. Then $N_1 + N_2 + N_3 + N_4 + N_5 =$ (Single Option Correct)
(a) 210 (b) 252 (c) 126 (d) 125

2. Let α and β be non zero real numbers such that $2(\cos \beta - \cos \alpha) + \cos \alpha \cos \beta = 1$. Then which of the following is/are true?
(One or More Than One Correct Option)

- (a) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) - \tan\left(\frac{\beta}{2}\right) = 0$ (b) $\tan\left(\frac{\alpha}{2}\right) - \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$
(c) $\tan\left(\frac{\alpha}{2}\right) + \sqrt{3} \tan\left(\frac{\beta}{2}\right) = 0$ (d) $\sqrt{3} \tan\left(\frac{\alpha}{2}\right) + \tan\left(\frac{\beta}{2}\right) = 0$

3. Let $f(x) = \frac{1-x(1+|1-x|)}{|1-x|} \cos\left(\frac{1}{1-x}\right)$
for $x \neq 1$. Then (One or More Than One Correct Option)

- (a) $\lim_{x \rightarrow 1^+} f(x) = 0$ (b) $\lim_{x \rightarrow 1^-} f(x)$ does not exist
(c) $\lim_{x \rightarrow 1^-} f(x) = 0$ (d) $\lim_{x \rightarrow 1^+} f(x)$ does not exist

Paragraph Based Questions (Q. Nos. 4-5)

Let p, q be integers and let α, β be the roots of the equation, $x^2 - x - 1 = 0$ where $\alpha \neq \beta$. For $n = 0, 1, 2, \dots$, let $a_n = p\alpha^n + q\beta^n$.

FACT : If a and b are rational numbers and $a + b\sqrt{5} = 0$, then $a = 0 = b$.

4. $a_{12} =$
(a) $a_{11} + 2a_{10}$ (b) $2a_{11} + a_{10}$
(c) $a_{11} - a_{10}$ (d) $a_{11} + a_{10}$
5. If $a_4 = 28$, then $p + 2q =$
(a) 14 (b) 7 (c) 21 (d) 12

BITSAT

1. The coefficient of x^5 in the expansion of $(1+x)^{21} + (1+x)^{22} + \dots + (1+x)^{30}$ is
(a) ${}^{51}C_5$ (b) ${}^9C_5$
(c) ${}^{31}C_6 - {}^{21}C_6$ (d) ${}^{30}C_5 + {}^{20}C_5$

2. If $z = a + ib$ satisfies $\arg(z-1) = \arg(z+3i)$, then $(a-1):b =$
(a) 2:1 (b) 1:3 (c) -1:3 (d) None of these

3. If p and p' denote the lengths of the perpendicular from a focus and the centre of an ellipse with semi-major axis of length a , respectively, on a tangent to the ellipse and r denotes the focal distance of the point, then
(a) $ap = rp'$ (b) $rp = ap'$
(c) $ap = rp' + 1$ (d) $ap' + rp = 1$

4. The value of $\sum_{r=1}^{10} r \cdot \frac{{}^nC_r}{{}^nC_{r-1}}$ is equal to
(a) $5(2n-9)$ (b) $10n$
(c) $9(n-4)$ (d) None of these

5. The numbers $3^{2 \sin 2\alpha - 1}$, 14 and $3^{4 - 2 \sin 2\alpha}$ form first three terms of an (a)P., its fifth term is
(a) -25 (b) -12 (c) 40 (d) 53

6. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, then p is equal to
(a) $\frac{1}{2}$ (b) 1 (c) 3 (d) $\frac{2}{3}$

7. If $a = \log_2 3$, $b = \log_2 5$ and $c = \log_7 2$, then $\log_{140} 63$ in terms of a, b, c is
(a) $\frac{2ac+1}{2c+abc+1}$ (b) $\frac{2ac+1}{2a+c+a}$
(c) $\frac{2ac+1}{2c+ab+a}$ (d) None of these

8. If $\cos(x-y)$, $\cos x$ and $\cos(x+y)$ are in HP, then $\cos x \sec(y/2)$ is equal to
(a) $\pm \sqrt{2}$ (b) $\pm 1/\sqrt{2}$ (c) ± 2 (d) None of these

9. The number of times the digit 5 will be written when listing the integers from 1 to 1000, is
(a) 271 (b) 272 (c) 300 (d) None of these

10. Let A and B be two sets such that $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for same set X . Then,
(a) $A = B$ (b) $A = X$ (c) $B = X$ (d) $A \cup B = X$

11. The general solution of $\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$ is
(a) $n\pi + \frac{\pi}{8}$ (b) $\frac{n\pi}{2} + \frac{\pi}{8}$ (c) $(-1)^n \frac{n\pi}{2} + \frac{\pi}{8}$ (d) $2n\pi + \cos^{-1} \frac{3}{2}$

12. Two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, -10)$. Find the equation of the third side
(a) $x - 3y = -31$ (b) $x - 3y = 31$
(c) $x + 3y = 31$ (d) $x + 3y = -31$

4 Solved Papers 2017

- 13.** If two distinct chords drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$ (where $pq \neq 0$) are bisected by the X -axis, then
 (a) $p^2 = q^2$ (b) $p^2 = 8q^2$ (c) $p^2 < 8q^2$ (d) $p^2 > 8q^2$
- 14.** $\lim_{n \rightarrow \infty} \sin [\pi \sqrt{n^2 + 1}]$ is equal to
 (a) ∞ (b) 0
 (c) does not exist (d) None of these
- 15.** If the curve $y = a^x$ and $y = b^x$ intersect at angle α , then $\tan \alpha$ is equal to
 (a) $\frac{a-b}{1+ab}$ (b) $\frac{\log a - \log b}{1 + \log a \log b}$ (c) $\frac{a+b}{1-ab}$ (d) $\frac{\log a + \log b}{1 - \log a \log b}$
- 16.** If $\lim_{x \rightarrow \infty} \left(ax - \frac{1+n^2}{1+n} \right) = b$, where a is finite number, then
 (a) $a = 2$ (b) $a = 0$ (c) $b = 1$ (d) $b = -1$
- 17.** If the papers of 4 students can be checked by anyone of the 7 teachers, then the probability that all the 4 papers are checked by exactly 2 teachers, is equal to
 (a) $\frac{12}{49}$ (b) $\frac{6}{49}$ (c) $\frac{9}{49}$ (d) $\frac{15}{49}$
- 18.** If equation $(10x - 5)^2 + (10y - 4)^2 = \lambda^2 (3x + 4y - 1)^2$ represents a hyperbola, then
 (a) $-2 < \lambda < 2$ (b) $\lambda > 2$
 (c) $\lambda < -2$ or $\lambda > 2$ (d) $0 < \lambda < 2$
- 19.** If the variance of the observations $x_1, x_2, \dots, x_n$ is σ^2 , then the variance of $\alpha x_1, \alpha x_2, \dots, \alpha x_n, \alpha \neq 0$ is
 (a) σ^2 (b) $\alpha \sigma^2$
 (c) $\alpha^2 \sigma^2$ (d) $\frac{\sigma^2}{\alpha^2}$
- 20.** Coefficient of variation of two distributions are 50 and 60 and their arithmetic means are 30 and 25, respectively. Difference of their standard deviation is
 (a) 0 (b) 1
 (c) 1.3 (d) 2.5
- 21.** A coin is tossed 7 times. Each time a man calls head (d) The probability that he wins the toss atleast 4 occasions is
 (a) $\frac{1}{4}$ (b) $\frac{5}{8}$ (c) $\frac{1}{2}$ (d) $\frac{1}{6}$
- 22.** The value of $\frac{2}{1!} + \frac{2+4}{2!} + \frac{2+4+6}{3!} + \dots$ is
 (a) e (b) $2e$
 (c) $3e$ (d) None of these
- 23.** If z_1, z_2 and z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then
 (a) $z_1 + z_2 = z_3$ (b) $z_1 + z_2 + z_3 = 0$
 (c) $z_1 z_2 = \frac{1}{z_3}$ (d) $z_1 - z_2 = z_3 - z_2$

KERALA CEE

- 1.** If (x, y) is equidistant from $(a+b, b-a)$ and $(a-b, a+b)$, then
 (a) $x + y = 0$ (b) $bx - ay = 0$
 (c) $ax - by = 0$ (d) $bx + ay = 0$
 (e) $ax + by = 0$
- 2.** If the points $(1, 0)$, $(0, 1)$ and $(x, 8)$ are collinear, then the value of x is equal to
 (a) 5 (b) -6
 (c) 6 (d) 7
 (e) -7
- 3.** If $f(9) = f'(9) = 0$, then $\lim_{x \rightarrow 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3}$ is equal to
 (a) 0 (b) $f(0)$ (c) $f'(3)$ (d) $f(9)$ (e) 1
- 4.** The value of $\cos(\pi/4 + x) + \cos(\pi/4 - x)$ is
 (a) $\sqrt{2} \sin^2 x$ (b) $\sqrt{2} \sin x$ (c) $\sqrt{2} \cos^2 x$ (d) $\sqrt{3} \cos x$
 (e) $\sqrt{2} \cos x$
- 5.** Area of the triangle with vertices $(-2, 2)$, $(1, 5)$ and $(6, -1)$ is
 (a) 15 (b) 3/5 (c) 29/2 (d) 33/2 (e) 35/2
- 6.** The equation of the line passing through $(-3, 5)$ and perpendicular to the line through the points $(1, 0)$ and $(-4, 1)$ is
 (a) $5x + y + 10 = 0$ (b) $5x - y + 20 = 0$
 (c) $5x - y - 10 = 0$ (d) $5x + y + 20 = 0$
 (e) $5y - x - 10 = 0$
- 7.** The coefficient of x^5 in the expansion of $(1 + x^2)^5 (1 + x)^4$ is
 (a) 30 (b) 60 (c) 40 (d) 10 (e) 45
- 8.** The coefficient of x^4 in the expansion of $(1 - 2x)^5$ is equal to
 (a) 40 (b) 320 (c) -320 (d) -32 (e) 80
- 9.** The equation $5x^2 + y^2 + y = 8$ represents
 (a) an ellipse (b) a parabola
 (c) a hyperbola (d) a circle
 (e) a straight line
- 10.** The centre of the ellipse $4x^2 + y^2 - 8x + 4y - 8 = 0$ is
 (a) $(0, 2)$ (b) $(2, -1)$ (c) $(2, 1)$ (d) $(1, 2)$
 (e) $(1, -2)$

11. If $f(x) = \sqrt{2x} + \frac{4}{\sqrt{2x}}$, then $f'(2)$ is equal to
 (a) 0 (b) -1 (c) 1 (d) 2 (e) -2
12. The equation of the hyperbola with vertices $(0, \pm 15)$ and foci $(0, \pm 20)$ is
 (a) $\frac{x^2}{175} - \frac{y^2}{225} = 1$ (b) $\frac{x^2}{625} - \frac{y^2}{125} = 1$
 (c) $\frac{y^2}{225} - \frac{x^2}{125} = 1$ (d) $\frac{y^2}{65} - \frac{x^2}{65} = 1$
 (e) $\frac{y^2}{225} - \frac{x^2}{175} = 1$
13. The vertex of the parabola $y^2 - 4y - x + 3 = 0$ is
 (a) $(-1, 3)$ (b) $(-1, 2)$ (c) $(2, -1)$ (d) $(3, -1)$
 (e) $(1, 2)$
14. For all real numbers x and y , it is known as the real valued function f satisfies $f(x) + f(y) = f(x+y)$. If $f(1) = 7$, then $\sum_{r=1}^{100} f(r)$ is equal to
 (a) $7 \times 51 \times 102$ (b) $6 \times 50 \times 102$
 (c) $7 \times 50 \times 102$ (d) $6 \times 25 \times 102$
 (e) $7 \times 50 \times 101$
15. The eccentricity of the ellipse $\frac{(x-1)^2}{2} + \left(y + \frac{3}{4}\right)^2 = \frac{1}{16}$ is
 (a) $1/\sqrt{2}$ (b) $1/2\sqrt{2}$ (c) $1/2$ (d) $1/4$
 (e) $1/4\sqrt{2}$
16. If $x \in [0, \pi/2]$, $y \in [0, \pi/2]$ and $\sin x + \cos y = 2$, then the value of $x + y$ is equal to
 (a) 2π (b) π (c) $\pi/4$ (d) $\pi/2$ (e) 0
17. The sum $S = 1/9! + 1/3!7! + 1/5!5! + 1/7!3! + 1/9!$ is equal to
 (a) $2^{10}/8!$ (b) $2^9/10!$ (c) $2^7/10!$ (d) $2^6/10!$
 (e) $2^5/8!$
18. $\sin 765^\circ$ is equal to
 (a) 1 (b) 0 (c) $\sqrt{3}/2$ (d) $1/2$ (e) $1/\sqrt{2}$
19. The distance of the point $(3, -5)$ from the line $3x - 4y - 26 = 0$ is
 (a) $3/7$ (b) $2/5$ (c) $7/5$ (d) $3/5$ (e) 1
20. The difference between the maximum and minimum value of the function $f(x) = \int_0^x (t^2 + t + 1) dt$ on $[2, 3]$ is
 (a) $39/6$ (b) $49/6$ (c) $59/6$ (d) $69/6$ (e) $79/6$
21. If a and b are the non-zero distinct roots of $x^2 + ax + b = 0$, then the minimum value of $x^2 + ax + b$ is
 (a) $2/3$ (b) $9/4$ (c) $-9/4$ (d) $-2/3$ (e) 1
22. If the straight line $y = 4x + c$ touches the ellipse $\frac{x^2}{4} + y^2 = 1$, then c is equal to
 (a) 0 (b) $\pm \sqrt{65}$ (c) $\pm \sqrt{62}$ (d) $\pm \sqrt{2}$ (e) ± 13
23. The set $\{(x, y) : |x| + |y| = 1\}$ in the xy -plane represents
 (a) a square
 (b) A circle
 (c) An ellipse
 (d) A rectangle which is not a square
 (e) A rhombus which is not a square
24. Let $A(6, -1)$, $B(1, 3)$ and $C(x, 8)$ be three points such that $AB = BC$. The values of x are
 (a) 3, 5 (b) -3, 5 (c) 3, -5 (d) 4, 5
 (e) -3, -5
25. In an experiment with 15 observations on x , the following results were available $\sum x^2 = 2830$, $\sum x = 170$. On observation that was 20, was found to be wrong and was replaced by the correct value 30. Then, the corrected variance is
 (a) 9.3 (b) 8.3 (c) 188.6 (d) 177.3 (e) 78
26. The area of the triangular region whose sides are $y = 2x + 1$, $y = 3x + 1$ and $x = 4$ is
 (a) 5 (b) 6 (c) 7 (d) 8 (e) 9
27. If $n_{C_{r-1}} = 36$, $n_{C_r} = 84$ and $n_{C_{r+1}} = 126$, then the value of r is
 (a) 9 (b) 3 (c) 4 (d) 5 (e) 6
28. Let $S = \{1, 2, 3, \dots, 10\}$. The number of subsets of S containing only odd numbers is
 (a) 15 (b) 31 (c) 63 (d) 7 (e) 5
29. The area of the parallelogram with vertices $(0, 0)$, $(7, 2)$, $(5, 9)$ and $(12, 11)$ is
 (a) 50 (b) 54 (c) 51 (d) 52 (e) 53
30. The value of $|\sqrt{4+2\sqrt{3}}| - |\sqrt{4-2\sqrt{3}}|$ is
 (a) 1 (b) 2 (c) 4 (d) 3 (e) 5
31. The value of $8^{2/3} - 16^{1/4} - 9^{1/2}$ is
 (a) -1 (b) -2 (c) -3 (d) -4 (e) -5
32. Let $x = 2$ be a root of $y = 4x^2 - 14x + q = 0$. Then y is equal to
 (a) $(x-2)(4x-6)$ (b) $(x-2)(4x+6)$
 (c) $(x-2)(-4x-6)$ (d) $(x-2)(-4x+6)$
 (e) $(x-2)(4x+3)$
33. If x_1 and x_2 are the roots of $3x^2 - 2x - 6 = 0$, then $x_1^2 + x_2^2$ is equal to
 (a) $\frac{50}{9}$ (b) $\frac{40}{9}$ (c) $\frac{30}{9}$ (d) $\frac{20}{9}$ (e) $\frac{10}{9}$

6 Solved Papers 2017

- 34.** Let x_1 and x_2 be the roots of the equations $x^2 - px - 3 = 0$. If $x_1^2 + x_2^2 = 10$, then the value of p is equal to
(a) -4 or 4 (b) -3 or 3 (c) -2 or 2 (d) -1 or 1 (e) 0
- 35.** If the product of roots of the equation $mx^2 + 6x + (2m - 1) = 0$ is -1, then the value of m is
(a) $1/3$ (b) 1 (c) 3 (d) -1 (e) -3
- 36.** If $f(x) = \frac{1}{x^2 + 4x + 4} - \frac{4}{x^4 + 4x^3 + 4x^2} + \frac{4}{x^3 + 2x^2}$, then $f\left(\frac{1}{2}\right)$ is equal to
(a) 1 (b) 2 (c) -1 (d) 3 (e) 4
- 37.** If x and y are the roots of the equation $x^2 + bx + 1 = 0$, then the value of $\frac{1}{x+b} + \frac{1}{y+b}$ is
(a) $1/b$ (b) b (c) $1/2b$ (d) $2b$ (e) 1
- 38.** The equations $x^5 + ax + 1 = 0$ and $x^6 + ax^2 + 1 = 0$ have a common root. Then a is equal to
(a) -4 (b) -2 (c) -3 (d) -1 (e) 0
- 39.** The root of $ax^2 + x + 1 = 0$, where $a \neq 0$, are in the ratio 1 : 1. Then a is equal to
(a) $1/4$ (b) $1/2$ (c) $3/4$ (d) 1 (e) 0
- 40.** If $z^2 + z + 1 = 0$ where z is a complex number, then the value of $\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right)^2 + \left(z^3 + \frac{1}{z^3}\right)^2$ equal
(a) 4 (b) 5 (c) 6 (d) 7 (e) 8
- 41.** If $z = \cos\left(\frac{\pi}{3}\right) - i \sin\left(\frac{\pi}{3}\right)$, then $z^2 - z + 1$ is equal to
(a) 0 (b) 1 (c) -1 (d) $\frac{\pi}{2}$ (e) π
- 42.** $\left(\frac{1 + \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right)}{1 + \cos\left(\frac{\pi}{12}\right) - i \sin\left(\frac{\pi}{12}\right)}\right)^{72}$ is equal to
(a) 0 (b) -1 (c) 1 (d) $1/2$ (e) $-1/2$
- 43.** If $|z| = 5$ and $w = \frac{z-5}{z+5}$, then the $\text{Re}(w)$ is equal to
(a) 0 (b) $1/25$ (c) 25 (d) 1 (e) -1
- 44.** If $a = e^{i\theta}$, then $\frac{1+a}{1-a}$ is equal to
(a) $\cot \frac{\theta}{2}$ (b) $\tan \theta$ (c) $i \cot \frac{\theta}{2}$ (d) $i \tan \frac{\theta}{2}$ (e) $2 \tan \theta$
- 45.** Three numbers x, y and z are in arithmetic progression. If $x + y + z = -3$ and $xyz = 8$, then $x^2 + y^2 + z^2$ is equal to
(a) 9 (b) 10 (c) 21 (d) 20 (e) 1
- 46.** The 30th term of the arithmetic progression 10, 7, 4 is
(a) -97 (b) -87 (c) -77 (d) -67 (e) -57
- 47.** The arithmetic mean of two numbers x and y is 3 and geometric mean is 1. Then $x^2 + y^2$ is equal to
(a) 30 (b) 31 (c) 32 (d) 33 (e) 34
- 48.** The solution of $3^{2x-1} = 81^{1-x}$ is
(a) $2/3$ (b) $1/6$ (c) $7/6$ (d) $5/6$ (e) $1/3$
- 49.** The sixth term in the sequence is 3, $1, \frac{1}{3}, \dots$ is
(a) $1/27$ (b) $1/9$ (c) $1/81$ (d) $1/17$ (e) $1/7$
- 50.** Three numbers are in arithmetic progression. Their sum is 21 and the product of the first number and the third number is 45. Then the product of these three number is
(a) 315 (b) 90 (c) 180 (d) 270 (e) 450
- 51.** If $a + 1, 2a + 1, 4a - 1$ are in arithmetic progression, then the value of a is
(a) 1 (b) 2 (c) 3 (d) 4 (e) 5
- 52.** Two numbers x and y have arithmetic mean 9 and geometric mean 4. Then, x and y are the roots of
(a) $x^2 - 18x - 16 = 0$ (b) $x^2 - 18x + 16 = 0$
(c) $x^2 + 18x - 16 = 0$ (d) $x^2 + 18x + 16 = 0$
(e) $x^2 - 17x + 16 = 0$
- 53.** Three unbiased coins are tossed. The probability of getting atleast 2 tails is
(a) $3/4$ (b) $1/4$ (c) $1/2$ (d) $1/3$ (e) $2/3$
- 54.** A single letter is selected from the word TRICKS. The probability that it is either T or R is
(a) $1/36$ (b) $1/4$ (c) $1/2$ (d) $2/3$ (e) $1/3$
- 55.** From 4 red balls, 2 white balls and 4 black balls, four balls are selected. The probability of getting 2 red balls is
(a) $7/21$ (b) $8/21$ (c) $9/21$ (d) $10/21$ (e) $11/21$
- 56.** In a class, 60% of the students know lesson I, 40% know lesson II and 20% know I and II. A student is selected of random. The probability that the student does not know lesson I and lesson II is
(a) 0 (b) $4/5$ (c) $3/5$ (d) $1/5$ (e) $2/5$

Solved Papers 2017 7

- 57.** Two distinct numbers x and y are chosen from 1, 2, 3, 4, 5. The probability that the arithmetic mean of x and y is an integer is
(a) 0 (b) $1/5$ (c) $3/5$ (d) $2/5$ (e) $4/5$
- 58.** The number of 3×3 matrices with entries -1 or $+1$ is
(a) 2^{-4} (b) 2^5 (c) 2^6 (d) 2^7 (e) 2^9
- 59.** Let S be the set of all 2×2 symmetric matrices whose entries are either zero or one. A matrix X is chosen from S . The probability that the determinant of X is not zero is
(a) $1/3$ (b) $1/2$ (c) $3/4$ (d) $1/4$ (e) $2/9$
- 60.** The number of words that can be formed by using all the letters of the word PROBLEM only one is
(a) $5!$ (b) $6!$ (c) $7!$ (d) $8!$ (e) $9!$
- 61.** The number of diagonals in a hexagon is
(a) 8 (b) 9 (c) 10 (d) 11 (e) 12
- 62.** The sum of odd integers from 1 to 2001 is
(a) 1001^2 (b) 1000^2 (c) 1002^2 (d) 1003^2 (e) 999^2
- 63.** Two balls are selected from two black and two red balls. The probability that the two balls will have no black balls is
(a) $1/7$ (b) $1/5$ (c) $1/4$ (d) $1/3$ (e) $1/6$
- 64.** If $z = i^9 + i^{19}$, then z is equal to
(a) $0 + 0i$ (b) $1 + 0i$ (c) $0 + i$ (d) $1 + 2i$ (e) $1 + 3i$
- 65.** The mean for the data 6, 7, 10, 12, 13, 4, 8, 12 is
(a) 9 (b) 8 (c) 7 (d) 6 (e) 5
- 66.** The set of all real numbers satisfying the inequality $x - 2 < 1$ is
(a) $(3, \infty)$ (b) $[3, \infty)$ (c) $[-3, \infty)$ (d) $(-\infty, -3)$
(e) $(-\infty, 3)$
- 67.** If $\left| \frac{x-3}{x-3} \right| >$, then
(a) $x \in (-3, \infty)$ (b) $x \in (3, \infty)$ (c) $x \in (2, \infty)$ (d) $x \in (1, \infty)$
(e) $x \in (-\infty, 3)$
- 68.** The mode of the data 8, 11, 9, 8, 11, 9, 7, 8, 7, 3, 2 is
(a) 11 (b) 9 (c) 8 (d) 3 (e) 7
- 69.** If the mean of six numbers is 41, then the sum of these numbers is
(a) 246 (b) 236 (c) 226 (d) 216 (e) 206
- 70.** In a flight 50 people speak Hindi, 20 speak English and 10 speak both English and Hindi. The number of people who speak atleast one of the two languages is
(a) 40 (b) 50 (c) 20 (d) 80 (e) 60
- 71.** If $f(x) = \frac{x+1}{x-1}$, then the value of $f(f(x))$ is equal to
(a) x (b) 0 (c) $-x$ (d) 1 (e) 2
- 72.** Two dice are thrown simultaneously. What is the probability of getting two numbers whose product is even?
(a) $3/4$ (b) $1/4$ (c) $1/2$ (d) $2/3$ (e) $1/16$
- 73.** $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2-x}}{x}$ is equal to
(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) 0 (d) Does not exist
(e) $\frac{1}{2\sqrt{2}}$
- 74.** $\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + \tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right)$ is equal to
(a) $\sec \theta$ (b) $2\sec \theta$ (c) $\sec \theta/2$ (d) $\sin \theta$ (e) $\cos \theta$

Karnataka CET

- 1.** The total number of terms in the expansion of $(x+a)^{47} - (x-a)^{47}$ after simplification is
(a) 96 (b) 48 (c) 47 (d) 24
- 2.** The contrapositive statement of the statement "If x is prime number, then x is odd" is
(a) If x is not odd, then x is not a prime number
(b) If x is a prime number, then x is not odd
(c) If x is not a prime number, then x is not odd
(d) If x is not a prime number, then x is odd
- 3.** A box has 100 pens of which 10 are defective. The probability that out of a sample of 5 pens drawn one by one with replacement and atmost one is defective, is
(a) $\frac{9}{10}$ (b) $\frac{1}{2} \left(\frac{9}{10}\right)^4$
(c) $\left(\frac{9}{10}\right)^5 + \frac{1}{2} \left(\frac{9}{10}\right)^4$ (d) $\frac{1}{2} \left(\frac{9}{10}\right)^5$
- 4.** If ${}^nC_{12} = {}^nC_8$, then n is equal to
(a) 12 (b) 20 (c) 26 (d) 6
- 5.** $3 + 5 + 7 + \dots$ to n terms is
(a) $n(n+2)$ (b) $(n+1)^2$ (c) n^2 (d) $n(n-2)$
- 6.** If A and B are finites sets and $A \subset B$, then
(a) $n(A \cup B) = n(B)$ (b) $n(A \cap B) = \phi$
(c) $n(A \cap B) = n(B)$ (d) $n(A \cup B) = n(A)$

8 Solved Papers 2017

7. The value of $\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$ is
 (a) 9/4 (b) 9/3 (c) 4/9 (d) 3/4
8. The area of triangle with vertices $(K, 0)$, $(4, 0)$, $(0, 2)$ is 4 sq units, then value of K is
 (a) 8 (b) 0 or 8 (c) 0 (d) 0 or - 8
9. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then the least positive integral value of m is
 (a) 2 (b) 4 (c) 3 (d) 1
10. The value of $\cos^2 45^\circ - \sin^2 15^\circ$ is
 (a) $\frac{\sqrt{2}+1}{2\sqrt{2}}$ (b) $\frac{\sqrt{3}-1}{2\sqrt{2}}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{\sqrt{3}}{4}$
11. If $\sin x = \frac{2t}{1+t^2}$, $\tan y = \frac{2t}{1-t^2}$, then $\frac{dy}{dx}$ is equal to
 (a) 0 (b) 2 (c) 1 (d) - 1
12. If $f(x) = 8x^3$, $g(x) = x^{1/3}$, then $f \circ g(x)$ is
 (a) $8x$ (b) $8^3 x$
 (c) $(8x)^{1/3}$ (d) $8x^3$
13. If $y = \log(\log x)$, then $\frac{d^2 y}{dx^2}$ is equal to
 (a) $\frac{-(1+\log x)}{x^2 \log x}$ (b) $\frac{-(1+\log x)}{(x \log x)^2}$
 (c) $\frac{(1+\log x)}{(x \log x)^2}$ (d) $\frac{(1+\log x)}{x^2 \log x}$
14. If $|x - 1| \leq 1$, then
 (a) $x \in [1, 3]$ (b) $x \in (-1, 3)$ (c) $x \in [1, 3)$ (d) $x \in (1, 3)$
15. The perpendicular distance of the point $P(6, 7, 8)$ from XY -plane is
 (a) 6 (b) 7 (c) 5 (d) 8
16. The eccentricity of the ellipse $\frac{x^2}{36} + \frac{y^2}{16} = 1$ is
 (a) $\frac{2\sqrt{13}}{6}$ (b) $\frac{2\sqrt{5}}{4}$ (c) $\frac{2\sqrt{5}}{6}$ (d) $\frac{2\sqrt{13}}{4}$
17. The range of the function $f(x) = \sqrt{9 - x^2}$ is
 (a) $(0, 3]$ (b) $(0, 3)$ (c) $[0, 3]$ (d) $[0, 3)$
18. Reflexion of the point (α, β, γ) in XY -plane is
 (a) $(0, 0, \gamma)$ (b) $(-\alpha, -\beta, \gamma)$
 (c) $(\alpha, \beta, -\gamma)$ (d) $(\alpha, \beta, 0)$
19. If coefficient of variation is 60 and standard deviation is 24, then Arithmetic mean is
 (a) 40 (b) 7/20 (c) 20/7 (d) 1/40
20. Two events A and B will be independent if
 (a) $P(A' \cap B') = 1 - P(A) - P(B)$
 (b) $P(A) + P(B) = 1$
 (c) $P(A) = P(B)$
 (d) A and B are mutually exclusive
21. Equation of line passing through the point $(1, 2)$ and perpendicular to the line $y = 3x - 1$
 (a) $x + 3y = 0$ (b) $x + 3y - 7 = 0$
 (c) $x + 3y + 7 = 0$ (d) $x - 3y = 0$

Andhra Pradesh EAMCET

1. In ΔABC , if $b \cos \theta = c - a$, (where θ is an acute angle), then $(c - a) \tan \theta =$
 (a) $2\sqrt{ca} \cos \frac{B}{2}$ (b) $2\sqrt{ca} \sin \frac{B}{2}$
 (c) $2ca \cos \frac{B}{2}$ (d) $2ca \sin \frac{B}{2}$
2. If the pair of lines $x^2 - 16pxy - y^2 = 0$ and $x^2 - 16qxy - y^2 = 0$ are such that each pair bisects the angle between the other pair, then $pq =$
 (a) $-\frac{1}{64}$ (b) $\frac{1}{64}$ (c) $-\frac{1}{8}$ (d) $\frac{1}{8}$
3. If three numbers are drawn at random successively without replacement from a set $S = \{1, 2, \dots, 10\}$, then the probability that the minimum of the chosen numbers is 3 or their maximum is 7.
 (a) $\frac{11}{40}$ (b) $\frac{5}{40}$ (c) $\frac{3}{40}$ (d) $\frac{1}{40}$
4. The coefficient of x^5 in the expansion of $(1+x)^{21} + (1+x)^{22} + \dots + (1+x)^{30}$ is
 (a) ${}^{31}C_6 - {}^{21}C_6$ (b) ${}^{51}C_5$ (c) ${}^9C_5$ (d) ${}^{30}C_5 + {}^{20}C_5$
5. Let $A = \{-4, -2, -1, 0, 3, 5\}$ and $f: A \rightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 3x - 1 & \text{for } x > 3 \\ x^2 + 1 & \text{for } -3 \leq x \leq 3 \\ 2x - 3 & \text{for } x < -3 \end{cases}$$

 Then the range of f is
 (a) $\{-11, 5, 2, 1, 10, 14\}$ (b) $\{-11, -2, 2, 1, 8, 14\}$
 (c) $\{-11, 5, 2, 1, 8, 14\}$ (d) $\{-11, -7, -5, 1, 10, 14\}$
6. The incentre of the triangle formed by the straight lines $y = \sqrt{3}x$, $y = -\sqrt{3}x$ and $y = 3$ is
 (a) $(0, 2)$ (b) $(1, 2)$
 (c) $(2, 0)$ (d) $(2, 1)$

7. If a circle with radius 2.5 units passes through the points (2, 3) and (5, 7), then its centre is
 (a) (1.5, 2) (b) (7, 10) (c) (3, 4) (d) (3.5, 5)
8. The circumcentre of the triangle formed by the points (1, 2, 3), (3, -1, 5), (4, 0, -3) is
 (a) (1, 1, 1) (b) (2, 2, 2) (c) (3, 3, 3) (d) $\left(\frac{7}{2}, \frac{-1}{2}, 1\right)$
9. The lines $y = 2x + \sqrt{76}$ and $2y + x = 8$ touch the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$. If the point of intersection of these two lines lie on a circle, whose centre coincides with the centre of that ellipse, then the equation of that circle is
 (a) $x^2 + y^2 = 28$ (b) $x^2 + y^2 = 16$
 (c) $x^2 + y^2 = 12$ (d) $x^2 + y^2 = (4 + \sqrt{8})^2$
10. The equation of the pair of lines through the point (2, 1) and perpendicular to the pair of lines $4xy + 2x + 6y + 3 = 0$ is
 (a) $xy - x - 2y + 2 = 0$ (b) $xy + x - 2y - 2 = 0$
 (c) $xy + x + 2y - 6 = 0$ (d) $xy - x + 2y - 2 = 0$
11. The harmonic mean of two numbers is $-\frac{8}{5}$ and their geometric mean is 2. The quadratic equation whose roots are twice those numbers is
 (a) $x^2 + 5x + 4 = 0$ (b) $x^2 + 10x + 16 = 0$
 (c) $x^2 - 10x + 16 = 0$ (d) $x^2 - 5x + 4 = 0$
12. If z is a complex number with $|z| \geq 5$. Then the least value of $\left|z + \frac{2}{z}\right|$ is
 (a) $\frac{24}{5}$ (b) $\frac{26}{5}$ (c) $\frac{23}{5}$ (d) $\frac{29}{5}$
13. If the roots of the equation $x^3 - 7x^2 + 14x - 8 = 0$ are in geometric progression, then the difference between the largest and the smallest roots is
 (a) 4 (b) 2
 (c) $\frac{1}{2}$ (d) 3
14. If α is a non-real root of $x^7 = 1$, then $\alpha(1 + \alpha)(1 + \alpha^2 + \alpha^4) =$
 (a) 1 (b) 2
 (c) -1 (d) -2
15. The point to which the origin is to be shifted to remove the first degree terms from the equation $2x^2 + 4xy - 6y^2 + 2x + 8y + 1 = 0$ is
 (a) $\left(\frac{7}{8}, \frac{3}{8}\right)$ (b) $\left(\frac{-7}{8}, \frac{-3}{8}\right)$
 (c) $\left(\frac{-7}{8}, \frac{3}{8}\right)$ (d) $\left(\frac{7}{8}, \frac{-3}{8}\right)$
16. If α, β, γ are the lengths of the tangents from the vertices of a triangle to its incircle. Then
 (a) $\alpha + \beta + \gamma = \frac{1}{r^2}(\alpha\beta\gamma)$ (b) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = r(\alpha\beta\gamma)$
 (c) $\alpha + \beta + \gamma = \frac{1}{r}(\alpha\beta\gamma)$ (d) $\alpha^2 + \beta^2 + \gamma^2 = \frac{2}{r}(\alpha\beta\gamma)$
17. The angle between the tangents drawn from the point (1, 2) to the ellipse $3x^2 + 2y^2 = 5$ is
 (a) $\tan^{-1}\left(\frac{12\sqrt{5}}{5}\right)$ (b) $\tan^{-1}\left(\frac{12\sqrt{5}}{13}\right)$
 (c) $\frac{\pi}{4}$ (d) $\frac{\pi}{4}$
18. If $lx + my = 1$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then $a^2m^2 - b^2l^2 =$
 (a) $\frac{m^2}{l^2}(a^2 + b^2)^2$ (b) $(l^2 + m^2)(a^2 + b^2)^2$
 (c) $\frac{l^2}{m^2}(a^2 + b^2)^2$ (d) $l^2m^2(a^2 + b^2)^2$
19. The equation of the circle whose diameter is the common chord of the circles $x^2 + y^2 + 2x + 2y + 1 = 0$ and $x^2 + y^2 + 4x + 6y + 4 = 0$ is
 (a) $10x^2 + 10y^2 + 14x + 8y + 1 = 0$
 (b) $3x^2 + 3y^2 - 3x + 6y - 8 = 0$
 (c) $2x^2 + 2y^2 - 2x + 4y + 1 = 0$
 (d) $x^2 + y^2 - x + 2y + 4 = 0$
20. $\frac{x-1}{3x+4} < \frac{x-3}{3x-2}$ holds, for all x in the interval
 (a) $\left(\frac{-4}{3}, \frac{2}{3}\right)$ (b) $\left(-\infty, \frac{-5}{4}\right)$
 (c) $\left(\frac{3}{3}, \infty\right)$ (d) $\left(-\infty, \frac{-5}{4}\right) \cup \left(\frac{3}{4}, -\infty\right)$
21. There are 10 intermediate stations on a railway line between two particular stations. The number of ways that a train can be made to stop at 3 of these intermediate stations so that no two of these halting stations are consecutive is.
 (a) 56 (b) 20 (c) 126 (d) 120
22. The figure formed by the pairs of lines $6x^2 + 13xy + 6y^2 = 0$ and $6x^2 + 13xy + 6y^2 + 10x + 10y + 4 = 0$, is a
 (a) Square (b) Parallelogram
 (c) Rhombus (d) Rectangle
23. If the point of intersection of the tangents drawn at the points where the line $5x + y + 1 = 0$ cuts the circle $x^2 + y^2 - 2x - 6y - 8 = 0$ is (a, b) , then $5a + b =$
 (a) 3 (b) -44 (c) -1 (d) 4

10 Solved Papers 2017

- 24.** The tangents to the parabola $y^2 = 4ax$ from an external point P make angles θ_1 and θ_2 with the axis of the parabola. Such that $\tan \theta_1 + \tan \theta_2 = b$ where b is constant. Then P lies on
 (a) $y = x + b$ (b) $y + x = b$ (c) $y = \frac{x}{b}$ (d) $y = bx$
- 25.** The points on the straight line $3x - 4y + 1 = 0$ which are at a distance of 5 units from the point $(3, 2)$ are
 (a) $\left(-2, -\frac{7}{4}\right), \left(-3, \frac{-5}{2}\right)$ (b) $\left(4, \frac{11}{4}\right), (-1, -1)$
 (c) $\left(1, \frac{1}{2}\right), \left(2, \frac{5}{4}\right)$ (d) $(7, 5), (-1, -1)$
- 26.** $\lim_{x \rightarrow 0} \frac{(1 - \cos 2x)(3 + \cos x)}{x \tan 4x} =$
 (a) $-\frac{1}{4}$ (b) $\frac{1}{2}$ (c) 1 (d) 2
- 27.** An angle between the curves $x^2 = 3y$ and $x^2 + y^2 = 4$ is
 (a) $\tan^{-1} \frac{5}{\sqrt{3}}$ (b) $\tan^{-1} \frac{\sqrt{5}}{3}$ (c) $\tan^{-1} \frac{2}{\sqrt{3}}$ (d) $\frac{\pi}{3}$
- 28.** The number of different ways of preparing a garland using 6 distinct white roses and 5 distinct red roses such that no two red roses come together is
 (a) 21600 (b) 43200 (c) 86400 (d) 151200
- 29.** If $\tan \alpha$ and $\tan \beta$ are the roots of the equation $x^2 + px + q = 0$, then the value of $\sin^2(\alpha + \beta) + p \cos(\alpha + \beta) \sin(\alpha + \beta) + q \cos^2(\alpha + \beta)$ is
 (a) $p + q$ (b) p (c) q (d) $\frac{p}{p + q}$
- 30.** If ω is a complex cube root of unity, then for any $n > 1$, $\sum_{r=1}^{n-1} r(r+1-\omega)(r+1-\omega^2) =$
 (a) $\frac{n^2(n+1)^2}{4}$ (b) $\frac{n(n+1)(2n+1)}{6}$
 (c) $\frac{n(n-1)}{4}(n^2 + 3n + 4)$ (d) $\frac{n(n+1)(2n+1)}{4}$
- 31.** ${}^{37}C_4 + \sum_{r=1}^5 (42-r)C_r =$
 (a) ${}^{41}C_4$ (b) ${}^{39}C_4$ (c) ${}^{38}C_4$ (d) ${}^{42}C_4$
- 32.** The variance of the following data is
- | | | | | | | | |
|-------|---|----|----|----|----|----|----|
| x_i | 6 | 10 | 14 | 18 | 24 | 28 | 30 |
| f_i | 2 | 4 | 7 | 12 | 8 | 4 | 3 |
- (a) 33.4 (b) 34.3 (c) 43.4 (d) 44.3
- 33.** $p, x_1, x_2, \dots, x_n$ and $q, y_1, y_2, \dots, y_n$ are two arithmetic progressions with common differences a and b respectively. If α and β are the arithmetic means of $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ respectively. Then the locus of $P(\alpha, \beta)$ is
 (a) $a(x - p) = b(y - q)$ (b) $b(x - p) = a(y - q)$
 (c) $\alpha(x - p) = \beta(y - q)$ (d) $p(x - \alpha) = q(y - \beta)$
- 34.** If $\alpha \neq 0$ and the mean deviation of the observations $\{k\alpha\}$, for $k = 1, 2, \dots, 50$ about its median is 50, then $|\alpha| =$
 (a) 4 (b) 3 (c) 2 (d) 5
- 35.** If $2kx + 3y - 1 = 0$, $2x + y + 5 = 0$ are conjugate lines with respect to the circle $x^2 + y^2 - 2x - 4y - 4 = 0$, then $k =$
 (a) 3 (b) 4 (c) 1 (d) 2
- 36.** If $\frac{5x^2 + 2}{x^3 + x} = \frac{A_1}{x} + \frac{A_2x + A_3}{x^2 + 1}$, then $(A_1, A_2, A_3) =$
 (a) (0, 2, 3) (b) (3, 0, 2) (c) (2, 3, 0) (d) (2, 0, 3)
- 37.** The sum of first n terms of $\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \dots$ is
 (a) $\frac{3n}{2(3n+2)}$ (b) $\frac{3n}{3n+2}$ (c) $\frac{n}{2(3n+2)}$ (d) $\frac{n}{3n+2}$
- 38.** The equations of the parabola whose axis is parallel to the X -axis and which passes through the points $(-2, 1), (1, 2), (-1, 3)$ is
 (a) $18y^2 - 12x - 21y - 21 = 0$ (b) $5y^2 + 2x - 21y + 20 = 0$
 (c) $15y^2 + 12x - 11y + 20 = 0$ (d) $25y^2 - 2x - 65y + 36 = 0$
- 39.** In ΔABC if θ is any angle, then $b \cos(C + \theta) + c \cos(B - \theta) =$
 (a) $a \cot \theta$ (b) $a \cos \theta$ (c) $a \tan \theta$ (d) $a \sin \theta$
- 40.** The number of solutions of the trigonometric equation $1 + \cos x, \cos 5x = \sin^2 x$ in $[0, 2\pi]$ is
 (a) 8 (b) 12 (c) 10 (d) 6
- 41.** If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then the value of $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)$ is
 (a) $(r - p)^2 + (r - q)^2$ (b) $(1 + p)^2 + (1 + q)^2$
 (c) $(r + p)^2 + (q + 1)^2$ (d) $(r - p)^2 + (q - 1)^2$
- 42.** The angle between the two circles, each passing through the centre of the other is
 (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{6}$ (d) π
- 43.** If $\log_{\frac{1}{\sqrt{3}}} \left\{ \frac{|z|^2 - |z| + 1}{2 + |z|} \right\} > -2$, then z lies inside
 (a) a triangle (b) an ellipse (c) a circle (d) a square
- 44.** A circle having centre at the origin passes through the three vertices of an equilateral triangle the length of its median being 9 units. Then the equation of that circle is
 (a) $x^2 + y^2 = 9$ (b) $x^2 + y^2 = 18$
 (c) $x^2 + y^2 = 36$ (d) $x^2 + y^2 = 81$

45. $1 + \cos 10^\circ + \cos 20^\circ + \cos 30^\circ =$

- (a) $4\sin 10^\circ \sin 20^\circ \sin 30^\circ$ (b) $4\cos 5^\circ \cos 10^\circ \cos 15^\circ$
(c) $4\cos 10^\circ \cos 20^\circ \cos 30^\circ$ (d) $4\sin 5^\circ \sin 10^\circ \sin 15^\circ$

46. The set of all values of a such that both the points $(1, 2)$ and $(3, 4)$ lie on the same side of the line $3x - 5y + a = 0$

- (a) $\{x \in \mathbb{R} : x > 11\}$ (b) $\{x \in \mathbb{R} : x > 11\} \cup \{x \in \mathbb{R} : x < 7\}$
(c) $\{x \in \mathbb{R} : x < 7\}$ (d) ϕ

Telangana State EAMCET

1. If $\tan 20^\circ = \lambda$, then $\frac{\tan 160^\circ - \tan 110^\circ}{1 + (\tan 160^\circ)(\tan 110^\circ)} =$

- (a) $\frac{1 + \lambda^2}{2\lambda}$ (b) $\frac{1 + \lambda^2}{\lambda}$ (c) $\frac{1 - \lambda^2}{\lambda}$ (d) $\frac{1 - \lambda^2}{2\lambda}$

2. Consider the circle $x^2 + y^2 - 6x + 4y = 12$. The equations of a tangent of this circle that is parallel to the line $4x + 3y + 5 = 0$ is

- (a) $4x + 3y + 10 = 0$ (b) $4x + 3y - 9 = 0$
(c) $4x + 3y + 9 = 0$ (d) $4x + 3y - 31 = 0$

3. The mean deviation from the mean 10 of the data 6, 7, 10, 12, 13, α , 12, 16 is

- (a) 3.5 (b) 3.25
(c) 3 (d) 3.75

4. A bag contains 5 red balls, 3 black balls and 4 white balls. Three balls are drawn at random. The probability that they are not of same colour is

- (a) $\frac{37}{44}$ (b) $\frac{31}{44}$
(c) $\frac{21}{44}$ (d) $\frac{41}{44}$

5. The radical centre of the circles $x^2 + y^2 - 4x - 6y + 5 = 0$, $x^2 + y^2 - 2x - 4y - 1 = 0$ and $x^2 + y^2 - 6x - 2y = 0$ lies on the line

- (a) $x + y - 5 = 0$ (b) $2x - 4y + 7 = 0$
(c) $4x - 6y + 5 = 0$ (d) $18x - 12y + 1 = 0$

6. If $\operatorname{cosec} \theta - \cot \theta = 2017$, then quadrant in which θ lies is

- (a) I (b) IV (c) III (d) II

7. If $A = (5, 3)$, $B = (3, -2)$ and a point P is such that the area of the triangle PAB is 9, then the locus of P represents

- (a) a circle (b) a pair of coincident lines
(c) a pair of parallel lines (d) a pair of perpendicular lines

8. A straight line makes an intercept on the Y -axis twice as long as that on X -axis and is at unit distance from the origin. Then the line is represented by the equations

- (a) $2x + 3y = \pm \sqrt{5}$ (b) $x + y = \pm 2$
(c) $x + y = \pm 2$ (d) $2x + y = \pm \sqrt{5}$

47. If $x = \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12} + \dots$ to infinite terms, then $9x^2 + 24x =$

- (a) 31
(b) 11
(c) 41
(d) 21

9. Let S and S' be the foci of an ellipse and B be one end of its minor axis. If SBS' is an isosceles right angled triangle then the eccentricity of the ellipse is

- (a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$ (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{3}$

10. For the parabola $y^2 + 6y - 2x = -5$

I. the vertex is $(-2, -3)$

II. the directrix is $y + 3 = 0$

Which of the following is correct?

- (a) Both I and II are correct (b) I is true, II is false
(c) Both I and II are false (d) I is false, II is true

11. If $\frac{x^2 + 5}{(x^2 + 1)(x - 2)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1}$, then $A + B + C =$

- (a) -1 (b) $\frac{2}{5}$ (c) $-\frac{3}{5}$ (d) 0

12. If the conjugate of $(x + iy)(1 - 2i)$ is $(1 + i)$, then

- (a) $x + iy = 1 - i$ (b) $x + iy = \frac{1 - i}{1 - 2i}$
(c) $x - iy = \frac{1 - i}{1 + 2i}$ (d) $x - iy = \frac{1 - i}{1 + i}$

13. The sides of a triangle are in the ratio $1 : \sqrt{3} : 2$. Then the angles are in the ratio

- (a) 1 : 2 : 3 (b) 1 : 2 : 4
(c) 1 : 4 : 5 (d) 1 : 3 : 5

14. The sum of the complex roots of the equations $(x - 1)^3 + 64 = 0$ is

- (a) 6 (b) 3 (c) $6i$ (d) $3i$

15. If the imaginary part of $\frac{2z + 1}{iz + 1}$ is -2 , then the locus of

the point representing z in the complex plane is

- (a) a circle (b) a parabola
(c) a straight line (d) an ellipse

16. If the perpendicular distance between the point $(1, 1)$ to the line $3x + 4y + c = 0$ is 7, then the possible values of c are

- (a) -35, 42 (b) 35, 28
(c) 42, -28 (d) 28, -42

12 Solved Papers 2017

17. If $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then $\frac{d^2y}{dx^2} =$
 (a) $-\frac{b^4}{a^2y^3}$ (b) $\frac{b^2}{ay^2}$ (c) $\frac{-b^3}{a^2y^3}$ (d) $\frac{b^3}{a^2y^2}$
18. $\lim_{y \rightarrow 1} \left(\frac{1}{y^2 - 1} - \frac{2}{y^4 - 1} \right) =$
 (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) 0
19. If the coefficients of $(2r + 1)^{\text{th}}$ term and $(r + 1)^{\text{th}}$ term in the expansion of $(1 + x)^{42}$ are equal then r can be
 (a) 12 (b) 14 (c) 16 (d) 20
20. Let $f(x)$ be a quadratic expression such that $f(0) + f(1) = 0$. If $f(-2) = 0$, then
 (a) $f\left(\frac{-2}{5}\right) = 0$ (b) $f\left(\frac{2}{5}\right) = 0$ (c) $f\left(\frac{-3}{5}\right) = 0$ (d) $f\left(\frac{3}{5}\right) = 0$
21. If the line $x + y + k = 0$ is a normal to the hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ then $k =$
 (a) $\pm \frac{\sqrt{5}}{13}$ (b) $\pm \frac{13}{\sqrt{5}}$ (c) $\pm \frac{13}{5}$ (d) $\pm \frac{5}{13}$
22. The product of all the real roots of $x^2 - 8x + 9 - \frac{8}{x} + \frac{1}{x^2} = 0$ is
 (a) 2 (b) 1 (c) 3 (d) 7
23. A village has 10 players. A team of 6 players is to be formed. 5 members are chosen out of these 10 players and then the captain is chosen from the remaining players. Then total number of ways of choosing such teams is
 (a) 1260 (b) 210 (c) $(^{10}C_6)5!$ (d) $(^{10}C_5)6$
24. The equation of the straight line passing through the point of intersection of $5x - 6y - 1$, $3x + 2y + 5 = 0$ and perpendicular to the line $3x - 5y + 11 = 0$ is
 (a) $5x + 3y + 18 = 0$ (b) $-5x - 3y + 18 = 0$
 (c) $5x + 3y + 8 = 0$ (d) $5x + 3y - 8 = 0$
25. An integer is chosen from $\{2k / -9 \leq k \leq 10\}$. The probability that it is divisible by both 4 and 6 is
 (a) $\frac{1}{10}$ (b) $\frac{1}{20}$ (c) $\frac{1}{4}$ (d) $\frac{3}{20}$
26. α and β are the roots of $x^2 + 2x + c = 0$. If $\alpha^3 + \beta^3 = 4$, then the value of c is
 (a) -2 (b) 3 (c) 2 (d) 4
27. Using the letters of the word TRICK, a five letter word with distinct letters is formed such that C is in the middle. In how many ways this is possible?
 (a) 6 (b) 120 (c) 24 (d) 72
28. The angle between the curves $x^2 = 8y$ and $xy = 8$ is
 (a) $\tan^{-1}\left(\frac{-1}{3}\right)$ (b) $\tan^{-1}(-3)$
 (c) $\tan^{-1}(-\sqrt{3})$ (d) $\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$
29. A parallelogram has vertices $A(4, 4, -1)$, $B(5, 6, -1)$, $C(6, 5, 1)$ and $D(x, y, z)$. Then the vertex D is
 (a) $(5, 1, 0)$ (b) $(-5, 0, 1)$
 (c) $(5, 3, 1)$ (d) $(5, 1, 3)$
30. If $2x^2 - 10xy + 2\lambda y^2 + 5x - 16y - 3 = 0$ represents a pair of straight lines, then point of intersection of those lines is
 (a) $(2, -3)$ (b) $(5, -16)$
 (c) $\left(-10, \frac{-7}{2}\right)$ (d) $\left(-10, \frac{-3}{2}\right)$
31. In order to eliminate the first degree terms from the equation $4x^2 + 8xy + 10y^2 - 8x - 44y + 14 = 0$ the point to which the origin has to be shifted is
 (a) $(-2, 3)$ (b) $(2, -3)$
 (c) $(1, -3)$ (d) $(-1, 3)$
32. Two circles of equal radius a cut orthogonally. If their centres are $(2, 3)$ and $(5, 6)$ then radical axis of these circles passes through the point
 (a) $(3a, 5a)$ (b) $(2a, a)$
 (c) $\left(a, \frac{5a}{3}\right)$ (d) (a, a)
33. If $\tan \theta_1 = k \cot \theta_2$, then $\frac{\cos(\theta_1 + \theta_2)}{\cos(\theta_1 - \theta_2)} =$
 (a) $\frac{1+k}{1-k}$ (b) $\frac{1-k}{1+k}$
 (c) $\frac{k+1}{k-1}$ (d) $\frac{k-1}{k+1}$
34. In the expansion of $(1 + x)^n$, the coefficients of p^{th} and $(p + 1)^{\text{th}}$ terms are respectively p and q then $p + q =$
 (a) $n + 3$ (b) $n + 2$
 (c) n (d) $n + 1$
35. For any integer $n \geq 1$, $\sum_{K=1}^n K(K+2) =$
 (a) $\frac{n(n+1)(n+2)}{6}$ (b) $\frac{n(n+1)(2n+7)}{6}$
 (c) $\frac{n(n+1)(2n+1)}{6}$ (d) $\frac{n(n-1)(2n+8)}{6}$
36. The foci of the ellipse $25x^2 + 4y^2 + 100x - 4y + 100 = 0$ are
 (a) $\left(\frac{5 \pm \sqrt{21}}{10}, -2\right)$ (b) $\left(-2, \frac{5 \pm \sqrt{21}}{10}\right)$
 (c) $\left(\frac{2 \pm \sqrt{21}}{10}, -2\right)$ (d) $\left(-2, \frac{2 \pm \sqrt{21}}{10}\right)$

37.
$$\left[\frac{1 + \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right)}{1 + \cos\left(\frac{\pi}{12}\right) - i \sin\left(\frac{\pi}{12}\right)} \right]^{72} =$$

(a) 0 (b) -1 (c) 1 (d) $\frac{1}{2}$

38. If the range of the function $f(x) = -3x - 3$ is $\{3, -6, -9, -18\}$, then which of the following elements is not in the domain of f ?

- (a) -1 (b) -2 (c) 1 (d) 2

39. In $\triangle ABC$, if $a = 1, b = 2, \angle C = 60^\circ$ then $4\Delta^2 + c^2 =$

- (a) 6 (b) 3 (c) $\frac{\sqrt{3}}{2}$ (d) 9

40. If α and β are the roots of the equation $ax^2 + bx + c = 0$ and the equation having roots $\frac{1-\alpha}{\alpha}$ and $\frac{1-\beta}{\beta}$ is

$px^2 + qx + r = 0$, then $r =$

- (a) $a + 2b$ (b) $ab + bc + ca$
(c) $a + b + c$ (d) abc

41. If $A\left(\frac{\pi}{3}\right), B\left(\frac{\pi}{6}\right)$ are the points on the circle represented in parametric form with centre $(0, 0)$ and radius 12 then the length of the chord AB is

- (a) $6(\sqrt{6} - \sqrt{2})$ (b) $6(\sqrt{6} - \sqrt{3})$ (c) $\sqrt{2}(\sqrt{3} - 1)$ (d) $6(\sqrt{3} - 1)$

42. If the pair of straight lines $xy - x - y + 1 = 0$ and the line $x + ay - 3 = 0$ are concurrent, then the acute angle

between the pair of lines $ax^2 - 13xy - 7y^2 + x + 23y - 6 = 0$ is

- (a) $\cos^{-1}\left(\frac{5}{\sqrt{218}}\right)$ (b) $\cos^{-1}\left(\frac{1}{\sqrt{10}}\right)$
(c) $\cos^{-1}\left(\frac{5}{\sqrt{173}}\right)$ (d) $\cos^{-1}\left(\frac{1}{\sqrt{5}}\right)$

43. The number of solutions of $\cos 2\theta = \sin \theta$ in $(0, 2\pi)$ is

- (a) 4 (b) 3
(c) 2 (d) 5

44. The lengths of the sides of a triangle are 13, 14 and 15. If R and r respectively denote circumradius and inradius of that triangle, then $8R + r =$

- (a) 84 (b) $\frac{65}{8}$ (c) 4 (d) 69

45. If A and B are variances of the 1st 'n' even numbers and 1st 'n' odd numbers respectively then

- (a) $A = B$ (b) $A > B$
(c) $A < B$ (d) $A = B - 1$

46. If the line $x - y = -4K$ is a tangent to the parabola $y^2 = 8x$ at P , then the perpendicular distance of normal at P from $(K, 2K)$ is

- (a) $\frac{5}{2\sqrt{2}}$ (b) $\frac{7}{2\sqrt{2}}$ (c) $\frac{9}{2\sqrt{2}}$ (d) $\frac{1}{2\sqrt{2}}$

47. If A and B are events having probabilities, $P(A) = 0.6$, $P(B) = 0.4$ and $P(A \cap B) = 0$, then probability that neither A nor B occurs is

- (a) $\frac{1}{4}$ (b) 1 (c) $\frac{1}{2}$ (d) 0

VIT

1. The principal amplitude of $(\sin 40^\circ + i \cos 40^\circ)^5$ is

- (a) 70° (b) -110° (c) 110° (d) -70°

2. Let ω be the complex number $\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}$.

Then, the number of distinct complex number z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

- (a) 1 (b) 0 (c) 2 (d) 3

3. If $f(x) = \frac{x-1}{x+1}$, then $f(2x)$ is

- (a) $\frac{f(x)+1}{f(x)+3}$ (b) $\frac{3f(x)+1}{f(x)+3}$
(c) $\frac{f(x)+3}{f(x)+1}$ (d) $\frac{f(x)+3}{3f(x)+1}$

4. If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$, then $A \times (B \cap C)$ is

- (a) $\{(2, 4), (3, 4)\}$ (b) $\{(4, 2), (4, 3)\}$
(c) $\{(2, 4), (3, 4), (4, 4)\}$ (d) $\{(2, 2), (3, 3), (4, 4), (5, 5)\}$

5. Let A and B be two non-empty sets having n elements in common. Then, the number of elements common to $A \times B$ and $B \times A$ is

- (a) $2n$ (b) n
(c) n^2 (d) None of these

6. The foci of a hyperbola are $(-5, 18)$ and $(10, 20)$ and it touches the Y -axis. The length of its transverse axis is

- (a) 100 (b) $\sqrt{89}/2$ (c) $\sqrt{89}$ (d) $\sqrt{50}$

7. The locus of mid-point of the line segment joining the locus to a moving point on the parabola $y^2 = 4ax$ is another parabola with directrix

- (a) $x = -a$ (b) $x = a$
(c) $x = 0$ (d) $x = a/2$

14 Solved Papers 2017

8. The equation of the ellipse with its centre at (1, 2), one focus at (6, 2) and passing through (4, 6) is

(a) $\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$ (b) $\frac{(x-1)^2}{20} + \frac{(y-2)^2}{45} = 1$
 (c) $\frac{(x+1)^2}{45} + \frac{(y+2)^2}{20} = 1$ (d) None of the above

9. If the normal at one end of the latusrectum of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ passes through the one end of the minor axis, then

(a) $e^4 - e^2 + 1 = 0$ (b) $e^2 - e + 1 = 0$
 (c) $e^2 + e + 1 = 0$ (d) $e^4 + e^2 - 1 = 0$

10. $\lim_{x \rightarrow \alpha} (\tan x \cot \alpha)^{\frac{1}{x-\alpha}}$ is equal to

(a) $2 \operatorname{cosec} 2\alpha$
 (b) $\frac{1}{2} \sin 2\alpha$
 (c) $-2 \operatorname{cosec} 2\alpha$
 (d) None of these

MHT CET

1. The number of principal solutions of $\tan 2\theta = 1$ is

(a) one (b) two
 (c) three (d) four

2. If z_1 and z_2 are z-coordinates of the points of trisection of the segment joining the points $A(2, 1, 4)$, $B(-1, 3, 6)$, then $z_1 + z_2 =$

(a) 1 (b) 4
 (c) 5 (d) 10

3. The statement pattern $(\sim p \wedge q)$ is logically equivalent to

(a) $(p \vee q) \vee \sim p$ (b) $(p \vee q) \wedge \sim p$
 (c) $(p \wedge q) \rightarrow p$ (d) $(p \vee q) \rightarrow p$

4. $O(0, 0)$, $A(1, 2)$, $B(3, 4)$ are the vertices of $\triangle OAB$. The joint equation of the altitude and median drawn from O is

(a) $x^2 + 7xy - y^2 = 0$
 (b) $x^2 + 7xy + y^2 = 0$
 (c) $3x^2 - xy - 2y^2 = 0$
 (d) $3x^2 + xy - 2y^2 = 0$

5. In $\triangle ABC$, if $\sin^2 A + \sin^2 B = \sin^2 C$ and $I(AB) = 10$, then the maximum value of the area of $\triangle ABC$ is

(a) 50 (b) $10\sqrt{2}$
 (c) 25 (d) $25\sqrt{2}$

6. A boy tosses fair coin 3 times. If he gets ₹ 2X for X heads, then his expected gain equals to `

(a) 1 (b) $\frac{3}{2}$ (c) 3 (d) 4

7. Which of the following statement pattern is a tautology?

(a) $p \vee (q \rightarrow p)$ (b) $\sim q \rightarrow \sim p$
 (c) $(q \rightarrow p) \vee (\sim p \leftrightarrow q)$ (d) $p \wedge \sim p$

8. If lines represented by equation $px^2 - qy^2 = 0$ are distinct, then

(a) $pq > 0$ (b) $pq < 0$ (c) $pq = 0$ (d) $p + q = 0$

9. If slopes of lines represented by $kx^2 + 5xy + y^2 = 0$ differ by 1, then $k =$

(a) 2 (b) 3 (c) 6 (d) 8

10. If c denotes the contradiction, then dual of the compound statement $\sim p \wedge (q \vee c)$ is

(a) $\sim p \vee (q \wedge t)$ (b) $\sim p \wedge (q \vee t)$
 (c) $p \vee (\sim q \vee t)$ (d) $\sim p \vee (q \wedge c)$

11. Probability that a person will develop immunity after vaccinations is 0.8. If 8 people are given the vaccine, then probability that all develop immunity is =

(a) $(0.2)^8$ (b) $(0.8)^8$
 (c) 1 (d) ${}^8C_6 (0.2)^6 (0.8)^2$

Answer with Explanations

JEE Main

1. (a)

p	q	$\frac{x}{p} \rightarrow q$	$\sim p$	$\sim p \rightarrow q$	$\frac{y}{\sim p} \rightarrow q$	$x \rightarrow y$
T	T	T	F	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	T	T
F	F	T	T	F	T	T

2. (d) Given,

$$5(\tan^2 x - \cos^2 x) = 2\cos 2x + 9$$

$$\Rightarrow 5\left(\frac{1 - \cos 2x}{1 + \cos 2x} - \frac{1 + \cos 2x}{2}\right) = 2\cos 2x + 9$$

Put $\cos 2x = y$, we have

$$5\left(\frac{1 - y}{1 + y} - \frac{1 + y}{2}\right) = 2y + 9$$

$$\Rightarrow 5(2 - 2y - 1 - y^2 - 2y) = 2(1 + y)(2y + 9)$$

$$\Rightarrow 9y^2 + 42y + 13 = 0$$

$$\Rightarrow (3y + 1)(3y + 13) = 0$$

$$\Rightarrow y = -\frac{1}{3}, -\frac{13}{3}$$

$$\therefore \cos 2x = -\frac{1}{3}, -\frac{13}{3}$$

$$\Rightarrow \cos 2x = -\frac{1}{3} \quad \left[\because \cos 2x \neq -\frac{13}{3} \right]$$

$$\text{Now } \cos 4x = 2\cos^2 2x - 1$$

$$= 2\left(-\frac{1}{3}\right)^2 - 1 = \frac{2}{9} - 1 = -\frac{7}{9}$$

3. (b) We have, P (exactly one of A or B occurs)

$$= P(A \cup B) - P(A \cap B)$$

$$= P(A) + P(B) - 2P(A \cap B)$$

According to the question,

$$P(A) + P(B) - 2P(A \cap B) = \frac{1}{4} \quad \dots(i)$$

$$P(B) + P(C) - 2P(B \cap C) = \frac{1}{4} \quad \dots(ii)$$

$$\text{and } P(C) + P(A) - 2P(C \cap A) = \frac{1}{4} \quad \dots(iii)$$

On adding Eqs. (i), (ii) and (iii), we get

$$2[P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A)] = \frac{3}{4}$$

$$\Rightarrow P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) = \frac{3}{8}$$

$$\therefore P(\text{at least one event occurs}) = P(A \cup B \cup C)$$

$$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$\begin{aligned} &= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) \\ &\quad - P(C \cap A) + P(A \cap B \cap C) \\ &= \frac{3}{8} + \frac{1}{16} + \frac{7}{16} \left[\because P(A \cap B \cap C) = \frac{1}{16} \right] \end{aligned}$$

4. (a) Given, $2\omega + 1 = z$

$$\Rightarrow 2\omega + 1 = \sqrt{-3} \quad [\because z = \sqrt{-3}]$$

$$\text{Here, } \omega = \frac{-1 + \sqrt{3}i}{2}$$

$$\omega^2 = \frac{-1 - \sqrt{3}i}{2} \text{ and } \omega^{3n} = 1$$

$$\text{Now, } \begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

On applying $R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} 3 & 1 + \omega + \omega^2 & 1 + \omega + \omega^2 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow \begin{vmatrix} 3 & 0 & 0 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} = 3k$$

$$\Rightarrow 3(\omega^2 - \omega^4) = 3k$$

$$\Rightarrow (\omega^2 - \omega) = k$$

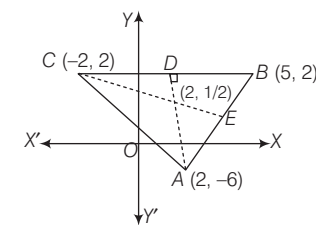
$$\therefore k = \left(\frac{-1 - \sqrt{3}i}{2} \right) - \left(\frac{-1 + \sqrt{3}i}{2} \right) = -\sqrt{3}i = -z$$

5. (d)

$$\therefore \frac{1}{2} \begin{vmatrix} k & -3k & 1 \\ 5 & k & 1 \\ -k & 2 & 1 \end{vmatrix} = \pm 28$$

$$\Rightarrow k = 2 \quad [\because k \in I]$$

Thus, the coordinates of vertices of triangle are $A(2, -6)$, $B(5, 2)$ and $C(-2, 2)$.



Now, equation of altitude from vertex A is

$$y - (-6) = \frac{-1}{\left(\frac{2-2}{-2-5} \right)} (x - 2)$$

$$\Rightarrow x = 2 \quad \dots(i)$$

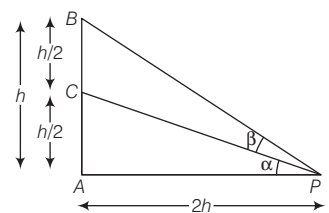
Equation of altitude from vertex C is

$$y - 2 = \frac{-1}{\left(\frac{2 - (-6)}{5 - 2} \right)} [x - (-2)]$$

$$\Rightarrow 3x + 8y - 10 = 0 \quad \dots(ii)$$

$$\Rightarrow x = 2 \text{ and } y = \frac{1}{2}$$

6. (c)



Now, in $\triangle ABP$,

$$\tan(\alpha + \beta) = \frac{AB}{AP} = \frac{h}{2h} = \frac{1}{2}$$

7. (c) $225a^2 + 9b^2 + 25c^2 - 75ac$

$$- 45ab - 15bc = 0$$

$$\Rightarrow (15a)^2 + (3b)^2 + (5c)^2 - (15a)(3b) - (15a)(5c) - (3b)(5c) = 0$$

$$\Rightarrow \frac{1}{2}[(15a - 3b)^2 + (3b - 5c)^2 + (5c - 15a)^2] = 0$$

8. (b) $\therefore a = 2$

$$\text{Now, } b^2 = a^2(1 - e^2)$$

$$= (2)^2 \left[1 - \left(\frac{1}{2} \right)^2 \right] = 4 \left(1 - \frac{1}{4} \right) = 3$$

$$\Rightarrow b = \sqrt{3}$$

$\therefore$ Equation of the ellipse is

$$\frac{x^2}{(2)^2} + \frac{y^2}{(\sqrt{3})^2} = 1 \Rightarrow \frac{x^2}{4} + \frac{y^2}{3} = 1$$

Now, the equation of normal at $\left(1, \frac{3}{2}\right)$ is

$$\Rightarrow \frac{4x}{1} - \frac{3y}{(3/2)} = 4 - 3$$

9. (b) Let the equation of hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\therefore ae = 2 \Rightarrow a^2 e^2 = 4$$

$$\Rightarrow a^2 + b^2 = 4 \Rightarrow b^2 = 4 - a^2$$

$$\therefore \frac{x^2}{a^2} - \frac{y^2}{4 - a^2} = 1$$

Since, $(\sqrt{2}, \sqrt{3})$ lie on hyperbola.

$$\therefore \frac{2}{a^2} - \frac{3}{4 - a^2} = 1$$

$$\Rightarrow a^4 - 9a^2 + 8 = 0$$

$$\Rightarrow (a^4 - 8)(a^4 - 1) = 0$$

16 Solved Papers 2017

$$\Rightarrow a^4 = 8, a^4 = 1 \Rightarrow a = 1$$

Now, equation of hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1.$$

$\therefore$ Equation of tangent at $(\sqrt{2}, \sqrt{3})$ is given by

$$\sqrt{2}x - \frac{\sqrt{3}y}{3} = 1$$

$$\begin{aligned} 10. (b) \lim_{x \rightarrow \pi/2} \frac{\cot x - \cos x}{(\pi - 2x)^3} \\ = \lim_{x \rightarrow \pi/2} \frac{1}{8} \cdot \frac{\cos x(1 - \sin x)}{\sin x \left(\frac{\pi}{2} - x\right)^3} \\ = \lim_{h \rightarrow 0} \frac{1}{8} \cdot \frac{\cos\left(\frac{\pi}{2} - h\right) \left[1 - \sin\left(\frac{\pi}{2} - h\right)\right]}{\sin\left(\frac{\pi}{2} - h\right) \left(\frac{\pi}{2} - \frac{\pi}{2} + h\right)^3} \\ = \frac{1}{8} \lim_{h \rightarrow 0} \frac{\sin h (1 - \cos h)}{\cos h \cdot h^3} \\ = \frac{1}{8} \lim_{h \rightarrow 0} \frac{\sin h \left(2 \sin^2 \frac{h}{2}\right)}{\cos h \cdot h^3} \\ = \frac{1}{4} \lim_{h \rightarrow 0} \left(\frac{\sin h}{h}\right) \left(\frac{\sin \frac{h}{2}}{\frac{h}{2}}\right)^2 \cdot \frac{1}{\cos h} \cdot \frac{1}{4} \\ = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16} \end{aligned}$$

11. (a) Total number of ways of selecting 2 different numbers from $\{0, 1, 2, \dots, 10\} = {}^{11}C_2 = 55$

Let two numbers selected be x and y .
Then, $x + y = 4m$... (i)
and $x - y = 4n$... (ii)
 $\Rightarrow 2x = 4(m + n)$ and $2y = 4(m - n)$
 $\Rightarrow x = 2(m + n)$ and $y = 2(m - n)$
 $\therefore x$ and y both are even numbers.

x	y
0	4, 8
2	6, 10
4	0, 8
6	2, 10
8	0, 4
10	2, 6

$$\therefore \text{Required probability} = \frac{6}{55}$$

$$\begin{aligned} 12. (a) \therefore \text{Total number of required ways} \\ = {}^3C_3 \times {}^4C_0 \times {}^4C_0 \times {}^3C_3 + {}^3C_2 \times \\ {}^4C_1 \times {}^4C_1 \times {}^3C_2 \\ + {}^3C_1 \times {}^4C_2 \times {}^4C_2 \times {}^3C_1 + \\ {}^3C_0 \times {}^4C_3 \times {}^4C_3 \times {}^3C_0 \\ = 1 + 144 + 324 + 16 = 485 \end{aligned}$$

$$\begin{aligned} 13. (d) ({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + \\ ({}^{21}C_3 - {}^{10}C_3) + \dots + ({}^{21}C_{10} - {}^{10}C_{10}) \\ = ({}^{21}C_1 + {}^{21}C_2 + \dots + {}^{21}C_{10}) - ({}^{10}C_1 \\ + {}^{10}C_2 + \dots + {}^{10}C_{10}) \\ = \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \dots + {}^{21}C_{20}) - (2^{10} - 1) \\ = \frac{1}{2}({}^{21}C_1 + {}^{21}C_2 + \dots + {}^{21}C_{21} - 1) \\ - (2^{10} - 1) \\ = \frac{1}{2}(2^{21} - 2) - (2^{10} - 1) = 2^{20} - 2^{10} \end{aligned}$$

14. (a) We have, $f(x) = ax^2 + bx + c$

Now, $f(x + y) = f(x) + f(y) + xy$

Put $y = 0$

$$\Rightarrow f(x) = f(x) + f(0) + 0$$

$$\Rightarrow f(0) = 0 \Rightarrow c = 0$$

Again, put $y = -x$

$$\therefore f(0) = f(x) + f(-x) - x^2$$

$$\Rightarrow 0 = ax^2 + bx + ax^2 - bx - x^2$$

$$\Rightarrow 2ax^2 - x^2 = 0$$

$$\Rightarrow a = \frac{1}{2}$$

$$\text{Also, } a + b + c = 3$$

$$\Rightarrow \frac{1}{2} + b + 0 = 3$$

$$\Rightarrow b = \frac{5}{2}$$

$$\therefore f(x) = \frac{x^2 + 5x}{2}$$

$$\text{Now, } f(n) = \frac{n^2 + 5n}{2}$$

$$= \frac{1}{2}n^2 + \frac{5}{2}n$$

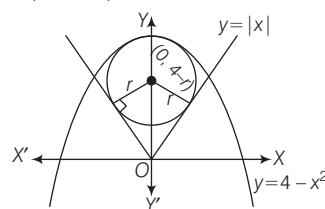
$$\therefore \sum_{n=1}^{10} f(n) = \frac{1}{2} \sum_{n=1}^{10} n^2 + \frac{5}{2} \sum_{n=1}^{10} n$$

$$= \frac{1}{2} \cdot \frac{10 \times 11 \times 21}{6} + \frac{5}{2} \times \frac{10 \times 11}{2}$$

$$= \frac{385}{2} + \frac{275}{2} = \frac{660}{2} = 330$$

15. (c) Let the radius of circle with least area be r .

Then, then coordinates of centre = $(0, 4 - r)$.



Since, circle touches the line $y = x$ in first quadrant.

$$\therefore \left| \frac{0 - (4 - r)}{\sqrt{2}} \right| = r$$

$$\Rightarrow \frac{r - 4}{\sqrt{2}} = \pm r\sqrt{2}$$

$$\Rightarrow r = \frac{4}{\sqrt{2} + 1}$$

$$\text{or } \frac{4}{1 - \sqrt{2}}$$

$$\text{But } r \neq \frac{4}{1 - \sqrt{2}}$$

$$\left[\because \frac{4}{1 - \sqrt{2}} < 0 \right]$$

$$\therefore r = \frac{4}{\sqrt{2} + 1} = 4(\sqrt{2} - 1)$$

16. (d) Given quadratic equation is

$$x(x + 1) + (x + 1)(x + 2) + \dots + (x + n - 1)(x + n) = 10n$$

$$\Rightarrow (x^2 + x^2 + \dots + x^2) +$$

$$[(1 + 3 + 5 + \dots + (2n - 1))]x + [(1 \cdot 2 + 2 \cdot 3 + \dots + (n - 1)n] = 10n$$

$$\Rightarrow nx^2 + n^2x + \frac{n(n^2 - 1)}{3} - 10n = 0$$

$$\Rightarrow x^2 + nx + \frac{n^2 - 1}{3} - 10 = 0$$

$$\Rightarrow 3x^2 + 3nx + n^2 - 31 = 0$$

Let α and β be the roots.

Since, α and β are consecutive.

$$\therefore |\alpha - \beta| = 1$$

$$\Rightarrow (\alpha - \beta)^2 = 1$$

$$\text{Again, } (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$$

$$\Rightarrow 1 = \left(\frac{-3n}{3}\right)^2 - 4\left(\frac{n^2 - 31}{3}\right)$$

$$3 = 3n^2 - 4n^2 + 124$$

$$\Rightarrow n^2 = 121$$

$$\Rightarrow n = \pm 11$$

$$\therefore n = 11 \quad [\because n > 0]$$

JEE Advanced

Paper 1

$$\begin{aligned} 1. (b, d) \frac{az + b}{z + 1} &= \frac{ax + b + ai y}{(x + 1) + iy} \\ &= \frac{(ax + b + ai y)((x + 1) - iy)}{(x + 1)^2 + y^2} \end{aligned}$$

$$\begin{aligned} \therefore \operatorname{Im} \left(\frac{az + b}{z + 1} \right) \\ = \frac{-(ax + b)y + ay(x + 1)}{(x + 1)^2 + y^2} \end{aligned}$$

$$\Rightarrow \frac{(a - b)y}{(x + 1)^2 + y^2} = y$$

$$\therefore a - b = 1$$

$$\therefore (x+1)^2 + y^2 = 1$$

$$\therefore x = -1 \pm \sqrt{1-y^2}$$

2. (a,b,c) Tangent $\equiv 2x - y + 1 = 0$

$$\text{Hyperbola} \equiv \frac{x^2}{a^2} - \frac{y^2}{16} = 1$$

It point $\equiv (a \sec \theta, 4 \tan \theta)$,

$$\text{tangent} \equiv \frac{x \sec \theta}{a} - \frac{y \tan \theta}{4} = 1$$

$$\sec \theta = -2a \text{ and } \tan \theta = -4$$

$$\Rightarrow 4a^2 - 16 = 1 \Rightarrow a = \frac{\sqrt{17}}{2}$$

3. (b)

4. (6) Let the sides are $a-d$, a and

$a+d$. Then, $a(a-d) = 48$

$$\text{and } a^2 - 2ad + d^2 + a^2$$

$$= a^2 + 2ad + d^2$$

$$\Rightarrow a^2 = 4ad \Rightarrow a = 4d$$

$$\text{Thus, } a = 8, d = 2$$

$$\text{Hence, } a - d = 6$$

5. (2) The circle and coordinate axes can have 3 common points, if it passes through origin. [$p = 0$]

If circle is cutting one axis and touching other axis.

Only possibility is of touching X-axis and cutting Y-axis. [$p = -1$]

6. (5) $x = 10!$

$$y = {}^{10}C_1 \times {}^9C_8 \times \frac{10!}{2!}$$

$$= 10 \times 9 \times \frac{10!}{2}$$

$$\frac{y}{9x} = \frac{10}{2} = 5$$

Solutions. (7-9)

Using standard equations

I-ii-Q; II-iv-R; III-i-P; IV-iii-S

Comparing with equations in given questions, we get

7. (a) Either option (a) or (c) is correct. But $a = \sqrt{2}$ and $(-1, 1)$ satisfy

$$x^2 + y^2 = a^2.$$

8. (b) Either option (b) or (c) is correct.

Satisfying the point $\left(\sqrt{3}, \frac{1}{2}\right)$ in (II),

we get $a^2 = 4$.

$$\therefore \text{The conic is } x^2 + 4y^2 = 4$$

Now, equation of tangent at $\left(\sqrt{3}, \frac{1}{2}\right)$

$$\text{is } \sqrt{3}x + 2y = 4$$

9. (a) Either option (a), (b) or (c) is correct.

Satisfying the point (8, 16) in (III), we get

$$\therefore \text{The conic is } y^2 = 32x = 16(2x)$$

Now, equation of tangent at (8, 16) is

$$y \cdot 16 = 16(x+8)$$

$$\Rightarrow y = x+8,$$

which is the given in equation.

So, option (a) is correct

$$(16-18)$$

$$f(x) = x + \ln x - x \ln x$$

$$f(1) = 1 > 0$$

$$f(e^2) = e^2 + 2 - 2e^2 = 2 - e^2 < 0$$

$$\Rightarrow f(x) = 0 \text{ for some } x \in (1, e^2)$$

$\therefore$ I is correct

$$f'(x) = 1 + \frac{1}{x} - \ln x - 1 = \frac{1}{x} - \ln x$$

$$f'(x) > 0 \text{ for } (0, 1)$$

$$f'(x) < 0 \text{ for } (e, \infty)$$

$\therefore$ P and Q are correct, II is correct, III is incorrect.

$$f''(x) = \frac{-1}{x^2} - \frac{1}{x}$$

$$f''(x) < 0 \text{ for } (0, \infty)$$

$\therefore$ S, is correct, R is incorrect.

IV is incorrect.

$$\lim_{x \rightarrow \infty} f(x) = -\infty$$

$$\lim_{x \rightarrow \infty} f'(x) = -\infty$$

$$\lim_{x \rightarrow \infty} f''(x) = 0$$

$\therefore$ ii, iii, iv are correct.

Paper 2

$$1. (c) N_i = {}^5C_k \times {}^4C_{5-k}$$

2. (b,c) We have,

$$4(\cos \beta - \cos \alpha) + 2 \cos \alpha \cos \beta = 2$$

$$\Rightarrow 1 - \cos \alpha + \cos \beta - \cos \alpha \cos \beta$$

$$= 3 + 3 \cos \alpha - 3 \cos \beta - 3 \cos \alpha \cos \beta$$

$$\Rightarrow (1 - \cos \alpha)(1 + \cos \beta)$$

$$= 3(1 + \cos \alpha)(1 - \cos \beta)$$

$$\Rightarrow \frac{(1 - \cos \alpha)}{(1 + \cos \alpha)} = \frac{3(1 - \cos \beta)}{1 + \cos \beta}$$

$$\Rightarrow \tan^2 \frac{\alpha}{2} = 3 \tan^2 \frac{\beta}{2}$$

$$\therefore \tan \frac{\alpha}{2} \pm \sqrt{3} \tan \frac{\beta}{2} = 0$$

3. (c,d)

$$f(x) = \frac{1 - x(1 + |1 - x|)}{|1 - x|} \cos \left(\frac{1}{1 - x} \right)$$

Now, $\lim_{x \rightarrow 1^-} f(x)$

$$= \lim_{x \rightarrow 1^-} \frac{1 - x(1 + 1 - x)}{1 - x} \cos \left(\frac{1}{1 - x} \right)$$

$$= \lim_{x \rightarrow 1^-} (1 - x) \cos \left(\frac{1}{1 - x} \right) = 0$$

and $\lim_{x \rightarrow 1^+} f(x)$

$$= \lim_{x \rightarrow 1^+} \frac{1 - x(1 - 1 + x)}{x - 1} \cos \left(\frac{1}{1 - x} \right)$$

$$= \lim_{x \rightarrow 1^+} -(x+1) \cdot \cos \left(\frac{1}{x+1} \right), \text{ which}$$

does not exist.

$$4. (d) \alpha^2 = \alpha + 1 \Rightarrow \beta^2 = \beta + 1$$

$$a_n = p\alpha^n + q\beta^n$$

$$= p(\alpha^{n-1} + \alpha^{n-2}) + q(\beta^{n-1} + \beta^{n-2})$$

$$= a_{n-1} + a_{n-2}$$

$$\therefore a_{12} = a_{11} + a_{10}$$

$$5. (d) \alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}$$

$$a_4 = a_3 + a_2 = 2a_2 + a_1$$

$$= 3a_1 + 2a_0$$

$$28 = p(3\alpha + 2) + q(3\beta + 2)$$

$$28 = (p + q) \left(\frac{3}{2} + 2 \right) + (p - q) \left(\frac{3\sqrt{5}}{2} \right)$$

$$\therefore p - q = 0$$

$$\text{and } (p + q) \times \frac{7}{2} = 28$$

$$\Rightarrow p + q = 8 \Rightarrow p = q = 4$$

$$\therefore p + 2q = 12$$

BITSAT

$$1. (c) (1 + x)^{21} + (1 + x)^{22} + \dots + (1 + x)^{30}$$

$$= (1 + x)^{21} [1 + (1 + x) + \dots + (1 + x)^9]$$

$$= (1 + x)^{21} \left[\frac{(1 + x)^{10} - 1}{(1 + x) - 1} \right]$$

$$= \frac{1}{x} [(1 + x)^{31} - (1 + x)^{21}]$$

$\therefore$ Coefficient of x^5 in the given expression

= Coefficient of x^5 in

$$\frac{1}{x} [(1 + x)^{31} - (1 + x)^{21}]$$

= Coefficient of x^6 in

$$[(1 + x)^{31} - (1 + x)^{21}] = {}^{31}C_6 - {}^{21}C_6$$

$$2. (b) \arg(z - 1) = \arg(z + 3i)$$

$$\Rightarrow \arg((a - 1) + ib) = \arg(a + (b + 3)i)$$

$$\Rightarrow \tan^{-1} \left(\frac{b}{a-1} \right) = \tan^{-1} \left(\frac{b+3}{a} \right)$$

$$\Rightarrow \frac{b}{a-1} = \frac{b+3}{a}$$

18 Solved Papers 2017

3. (a) Let the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
and let $\frac{x \cos \theta}{a} + \frac{y \sin \theta}{b} = 1 \dots (i)$

be a tangent to it at point $(a \cos \theta, b \sin \theta)$. Then, p = length of the perpendicular from $S(ae, 0)$ on Eq. (i)

$$\Rightarrow p = \left| \frac{e \cos \theta - 1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \right|$$

p' = length of the perpendicular from $O(0, 0)$ on Eq. (i)

$$\Rightarrow p' = \left| \frac{1}{\sqrt{\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2}}} \right|$$

and $r = ae \cos \theta - a$

Clearly, $rp' = ap$

4. (a) We have,

$$\begin{aligned} \sum_{r=1}^{10} r \cdot \frac{{}^nC_r}{{}^nC_{r-1}} &= \sum_{r=1}^{10} (n-r+1) \\ \Rightarrow \sum_{r=1}^{10} r \cdot \frac{{}^nC_r}{{}^nC_{r-1}} &= \sum_{r=1}^{10} \{(n+1)-r\} \\ \Rightarrow \sum_{r=1}^{10} r \cdot \frac{{}^nC_r}{{}^nC_{r-1}} &= 10(n+1) - \sum_{r=1}^{10} r \\ \Rightarrow \sum_{r=1}^{10} r \cdot \frac{{}^nC_r}{{}^nC_{r-1}} &= 10(n+1) - 55 \\ &= 10n - 45 = 5(2n-9) \end{aligned}$$

5. (d) Since, $3^{2 \sin 2\alpha - 1}$, 14 and $3^{4 - 2 \sin 2\alpha}$ are in AP. Therefore,
 $2 \times 14 = 3^{2 \sin 2\alpha - 1} + 3^{4 - 2 \sin 2\alpha}$

$$\begin{aligned} \Rightarrow 28 &= \frac{a}{3} + \frac{3^4}{a}, \text{ where } a = 3^{2 \sin 2\alpha} \\ \Rightarrow a^2 - 84a + 243 &= 0 \\ \Rightarrow (a-81)(a-3) &= 0 \\ \Rightarrow a &= 81, a = 3 \\ \Rightarrow 3^{2 \sin 2\alpha} &= 3^4 \\ \text{or } 3^{2 \sin 2\alpha} &= 3 \\ \Rightarrow 2 \sin 2\alpha &= 1 [\because 2 \sin 2\alpha \neq 4] \\ \Rightarrow \sin 2\alpha &= \frac{1}{2} \\ \Rightarrow 2\alpha &= 30^\circ [\because \sin 30^\circ = 1/2] \\ \text{Thus, the first three terms of the AP} & \text{ are 1, 14, 27.} \\ \text{Hence, its fifth term} & \\ a_5 &= a_1 + (5-1)d \\ &= 1 + 4 \times 13 = 1 + 52 = 53 \end{aligned}$$

6. (c) Let α, α^2 be the roots of

$$3x^2 + px + 3$$

$\therefore$ Product of the roots,

$$\alpha \cdot \alpha^2 = 1 \Rightarrow \alpha^3 = 1$$

$$\Rightarrow \alpha = 1, \omega, \omega^2$$

$$\text{Again, } \alpha + \alpha^2 = -\frac{p}{3}$$

$$\Rightarrow 1 + 1 = -\frac{p}{3} \quad (\text{if } \alpha = 1)$$

$$p = -6$$

But $p > 0$

$\therefore \alpha = 1$ is not possible.

$$\text{If } \alpha = \omega, \text{ then } \alpha + \alpha^2 = \omega + \omega^2 = -1$$

$$\therefore -1 = -\frac{p}{3}$$

$$\Rightarrow p = 3$$

Again, if $\alpha = \omega^2$, then

$$\alpha + \alpha^2 = \omega^2 + \omega^4 = \omega^2 + \omega = -1$$

$$\therefore -1 = -\frac{p}{3}$$

$$\Rightarrow p = 3$$

7. (d) $\log_{140} 63 = \log_{2^2 \times 5 \times 7} (3 \times 3 \times 7)$

$$= \frac{\log_2 (3 \times 3 \times 7)}{\log_2 (2^2 \times 5 \times 7)}$$

8. (a) Given that,

$\cos(x-y), \cos x, \cos(x+y)$ are in HP.

$$\text{Then, } \cos x = \frac{2 \cos(x-y) \cos(x+y)}{\cos(x-y) + \cos(x+y)}$$

$$\Rightarrow \cos x = \frac{2(\cos^2 x - \sin^2 y)}{2 \cos x \cos y}$$

$$\Rightarrow \cos^2 x \cos y = \cos^2 x - \sin^2 y$$

$$\Rightarrow \cos^2 x (1 - \cos y)$$

$$= (1 - \cos y)(1 + \cos y)$$

$$\Rightarrow \cos^2 x = 1 + \cos y$$

$$\Rightarrow \cos^2 x = 2 \cos^2 y / 2$$

$$\Rightarrow \cos^2 x \sec^2(y/2) = 2$$

$$\Rightarrow \cos x \sec(y/2) = \pm \sqrt{2}$$

9. (c) Any number between 1 to 999 is of the form abc when $0 \leq a, b, c \leq 9$. Let us first count the number in which 5 occurs exactly once.

Since, 5 can occur at one place in $1 \times {}^3C_1 \times 9 \times 9 = 243$ ways, next 5 can occur in exactly two places in ${}^3C_2 \times 9 = 27$. Lastly, 5 can occur in all three digits in only one way.

$$\begin{aligned} \text{Hence, the number of times 5 occurs} &= 1 \times 243 + 27 \times 2 + 1 \times 3 \\ &= 243 + 54 + 3 = 300 \end{aligned}$$

10. (a) Given, that, $A \cup X = B \cup X$

$$\Rightarrow A \cap (A \cup X) = A \cap (B \cup X)$$

$$\Rightarrow (A \cap A) \cup (A \cap X)$$

$$= (A \cap B) \cup (A \cap X)$$

$$\Rightarrow A \cup \phi = (A \cap B) \cup \phi [\because A \cap X = \phi]$$

$$\Rightarrow A = A \cap B \dots (i)$$

Again, consider $A \cup X = B \cup X$

$$\Rightarrow B \cap (A \cup X) = B \cap (B \cup X)$$

$$\Rightarrow (B \cap A) \cup (B \cap X)$$

$$= (B \cap B) \cup (B \cap X)$$

$$\Rightarrow (B \cap A) \cup \phi = B \cup \phi [$$

$$B \cap X = \phi]$$

$$\Rightarrow A \cap B = B \dots (ii)$$

Thus from Eq. (i) and (ii),

$$\text{we get } A = B$$

11. (b) We have,

$$\sin x - 3 \sin 2x + \sin 3x = \cos x - 3 \cos 2x + \cos 3x$$

$$\Rightarrow \sin x + \sin 3x - 3 \sin 2x$$

$$= \cos x + \cos 3x - 3 \cos 2x$$

$$\Rightarrow 2 \sin 2x \cos x - 3 \sin 2x -$$

$$2 \cos 2x \cos x + 3 \cos 2x = 0$$

$$\Rightarrow \sin 2x (2 \cos x - 3)$$

$$- \cos 2x (2 \cos x - 3) = 0$$

$$\Rightarrow (\sin 2x - \cos 2x) (2 \cos x - 3) = 0$$

$$\Rightarrow \sin 2x = \cos 2x [\because \cos x \neq 3/2]$$

$$\Rightarrow 2x = 2x\pi \pm \left(\frac{\pi}{2} - 2x\right)$$

$$\Rightarrow x = \frac{n\pi}{2} + \frac{\pi}{8} [\because \text{neglect -ve sign}]$$

12. (b) The third side is parallel to a bisector of the angle between equal sides.

The bisectors are

$$7x - y + 3 = \pm 5(x + y - 3)$$

$$\Rightarrow 2x - 6y + 18 = 0$$

$$\text{or } 12x + 4y - 12 = 0$$

$$\Rightarrow x - 3y + 9 = 0$$

$$\text{or } 3x + y - 3 = 0$$

Let the third side be $x - 3y = k$

$$\text{or } 3x + y = L$$

It passes through $(1, -10)$.

$$k = 31, L = -7$$

Hence, required lines are $x - 3y = 31$, $3x + y = -7$

13. (d) Let (t, m) be the other end of the chord drawn from the point (p, q) on the circle $x^2 + y^2 = px + qy$

$$\text{Their mid-point is } \left(\frac{t+p}{2}, \frac{m+q}{2}\right)$$

Since, mid-point lies on X-axis i.e. $y = 0$

$$m + q = 0 \dots (i)$$

Also, (t, m) lies on the circle.
 $\therefore t^2 + m^2 - pt - qm = 0 \quad \dots(ii)$

From Eqs. (i) and (ii), we get
 $t^2 - pt + 2q^2 = 0$

Which is quadratic in t such that,
 Discriminant > 0
 $\Rightarrow p^2 - 8q^2 > 0 \Rightarrow p^2 > 8q^2$

$$\begin{aligned} 14. (b) \lim_{n \rightarrow \infty} \sin \left\{ n\pi \left(1 + \frac{1}{n^2} \right)^{1/2} \right\} \\ = \lim_{n \rightarrow \infty} \sin \left\{ n\pi \left(1 + \frac{1}{2n^2} - \frac{1}{8n^4} + \dots \right) \right\} \\ = \lim_{n \rightarrow \infty} \sin \left\{ n\pi + \frac{\pi}{2n} - \frac{\pi}{8n^3} + \dots \right\} \\ = \lim_{n \rightarrow \infty} (-1)^n \sin \pi \left(\frac{1}{2n} - \frac{1}{8n^3} + \dots \right) = 0 \end{aligned}$$

15. (b) Clearly, the point of intersection of curves is $(0, 1)$.

Now, slope of tangent of first curve

$$\begin{aligned} m_1 = \frac{dy}{dx} = a^x \log a \\ \Rightarrow \left(\frac{dy}{dx} \right)_{(0, 1)} = m_1 = \log a \end{aligned}$$

Slope of tangent of second curve,

$$\begin{aligned} m_2 = \frac{dy}{dx} = b^x \log b \\ \Rightarrow m_2 = \left(\frac{dy}{dx} \right)_{(0, 1)} = \log b \end{aligned}$$

$$\begin{aligned} 16. (c) \lim_{n \rightarrow \infty} \frac{an(1+n) - (1+n^2)}{1+n} \\ = \lim_{n \rightarrow \infty} \frac{(a-1)n^2 + an - 1}{n+1} = \infty, \end{aligned}$$

If $a-1 \neq 0$ limit does not exist and if $a-1=0$, then

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{an-1}{n+1} = a=b \\ \Rightarrow a=b=1 \end{aligned}$$

17. (b) Total ways in which papers can be checked is equal to 7^4 . Now, two teachers who have to check all the papers can be selected in ${}^7C_2$ ways and papers can be checked by them is $(2^4 - 2)$ favourable ways.

18. (c)

$$\begin{aligned} 19. (c) \text{ Variance} &= \frac{1}{n} \sum (x - \bar{x})^2 \sigma^2 \\ \text{New variance} &= \frac{1}{n} \sum (\alpha x - \alpha \bar{x})^2 \\ &= \alpha^2 \frac{1}{n} \sum (x - \bar{x})^2 = \alpha^2 \sigma^2 \end{aligned}$$

$$\begin{aligned} 20. (a) \quad C &= \frac{\sigma}{\bar{x}} \times 100 \\ \Rightarrow 50 &= \frac{\sigma_1}{30} \times 100 \end{aligned}$$

21. (c) $\therefore$ Required probability

$$\begin{aligned} &= {}^7C_4 \left(\frac{1}{2} \right)^4 \left(\frac{1}{2} \right)^3 + {}^7C_5 \left(\frac{1}{2} \right)^5 \left(\frac{1}{2} \right)^2 \\ &\quad + {}^7C_6 \left(\frac{1}{2} \right)^6 \left(\frac{1}{2} \right)^1 + {}^7C_7 \left(\frac{1}{2} \right)^7 \end{aligned}$$

$$\begin{aligned} 22. (c) \text{ Here, } T_n &= \frac{n(n+1)}{n!} = \frac{n-1+2}{(n-1)!} \\ &= \frac{1}{(n-2)!} + \frac{2}{(n-1)!} \end{aligned}$$

$$\begin{aligned} \therefore S &= \sum_{n=1}^{\infty} T_n = \sum_{n=1}^{\infty} \frac{1}{(n-2)!} + 2 \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \\ &= e + 2e = 3e \end{aligned}$$

23. (b) Since, $|z_1| = |z_2| = |z_3|$

$\therefore$ 0 is the circumcentre of an equilateral $\triangle ABC$.

$$\begin{aligned} \therefore \frac{x_1 + x_2 + x_3}{3} &= 0 = \frac{y_1 + y_2 + y_3}{3} \\ \Rightarrow \frac{x_1 + x_2 + x_3}{3} + i \left(\frac{y_1 + y_2 + y_3}{3} \right) &= 0 \end{aligned}$$

KERALA CEE

$$\begin{aligned} 1. (b) (x - (a+b))^2 + (y - (b-a))^2 \\ = (x - (a-b))^2 + (y - (a+b))^2 \end{aligned}$$

2. (e)

$$\begin{aligned} 3. (a) \lim_{x \rightarrow 9} \frac{\sqrt{f(x)} - 3}{\sqrt{x} - 3} \quad \left[\frac{0}{0} \text{ form} \right] \\ = \lim_{x \rightarrow 9} \frac{\frac{f'(x)}{2\sqrt{f(x)}}}{\frac{1}{2\sqrt{x}}} = \lim_{x \rightarrow 9} \frac{\sqrt{x} f'(x)}{\sqrt{f(x)}} = \frac{\sqrt{9} f'(a)}{\sqrt{f(a)}} \\ = \frac{3 \times 0}{3} = 0 \end{aligned}$$

4. (e)

$$5. (d) \text{ Required area} = \frac{1}{2} \begin{vmatrix} -2 & 2 & 1 \\ 1 & 5 & 1 \\ 6 & -1 & 1 \end{vmatrix}$$

$$6. (b) (-4, 1) = \frac{1-0}{-4-1} = \frac{-1}{5}$$

$$\begin{aligned} \therefore \text{ Slope of line perpendicular to the above line} &= \frac{-1}{\left(-\frac{1}{5} \right)} = 5 \end{aligned}$$

$$\begin{aligned} \therefore \text{ Equation of required line is given by} \\ y - 5 &= 5(x - (-3)) \end{aligned}$$

$$\begin{aligned} 7. (b) \text{ We have, } (1+x^2)^5 \\ = {}^5C_0(x^2)^0 + {}^5C_1(x^2)^1 + {}^5C_2(x^2)^2 \\ + {}^5C_3(x^2)^3 + {}^5C_4(x^2)^4 + {}^5C_5(x^2)^5 \end{aligned}$$

$$\begin{aligned} &= 1 + 5x^2 + 10x^4 + 10x^6 + 5x^8 + x^{10} \\ (1+x)^4 &= {}^4C_0x^0 + {}^4C_1x^1 + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4 \end{aligned}$$

$$= 1 + 4x + 6x^2 + 4x^3 + x^4$$

$$\begin{aligned} \therefore \text{ Coefficient of } x^5 \text{ in the product of} \\ (1+x^2)^5(1+x)^4 \\ = (5x^2) \cdot (4x^3) + (10x^4) \cdot (4x) \\ = 20x^5 + 40x^5 = 60x^5 \end{aligned}$$

$$\begin{aligned} 8. (e) \text{ Here, } T_{r+1} &= {}^5C_r (-2x)^r \\ &= {}^5C_r (-2)^r x^r \end{aligned}$$

For coefficient of x^4 , power of $x = 4$

$$\therefore r = 4$$

9. (a) Given equation can be rewritten

$$\text{as } \frac{x^2}{\left(\frac{33}{40} \right)} + \frac{(y+1/2)^2}{\left(\frac{33}{8} \right)} = 1$$

$$10. (e) (4x^2 - 8x) + (y^2 + 4y) - 8 = 0$$

$$\Rightarrow 4(x-1)^2 + (y+2)^2 = 16$$

$$\Rightarrow \frac{(x-1)^2}{4} + \frac{(y+2)^2}{16} = 1$$

$$\therefore \text{ Centre} = (1, -2)$$

11. (c)

12. (e) Since, both foci and vertices lies on Y -axis, then equation of hyperbola will be

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$$

Now, vertices $= (0, \pm b)$

$$\therefore b = 15$$

Again, foci $= (0, \pm be)$

$$\therefore be = 20$$

$$\Rightarrow e = \frac{20}{15} = \frac{4}{3}$$

$$\Rightarrow \sqrt{1 + \frac{a^2}{b^2}} = \frac{4}{3} \left[\because e = \sqrt{1 + \frac{a^2}{b^2}} \right]$$

13. (b) We have,

$$\Rightarrow (y-2)^2 = (x+1)$$

14. (e) We have,

$$f(x+y) = f(x) + f(y) \quad \dots(i)$$

Put, $x = y = 1$ in Eq. (i),

$$\text{we get } f(2) = f(1) + f(1) = 2f(1) = 2 \times 7$$

Again, put $x = 1, y = 2$ in Eq. (i),

we get

$$f(3) = f(1) + f(2) = 7 + 2 \times 7 = 3 \times 7$$

$$\therefore f(n) = n \times 7$$

Now,

$$\sum_{r=1}^{100} f(r) = f(1) + f(2) + f(3) + \dots + f(100)$$

$$= 7 + 2 \times 7 + 3 \times 7 + \dots + 100 \times 7$$

20 Solved Papers 2017

15. (a) We have,

$$\Rightarrow \frac{(x-1)^2}{1/8} + \frac{(y+3/4)^2}{1/16} = 1$$

$$\therefore a^2 = \frac{1}{8} \text{ and } b^2 = \frac{1}{16}$$

$$\Rightarrow a = \frac{1}{2\sqrt{2}} \text{ and } b = \frac{1}{4}$$

16. (d) We have, $\sin x + \cos y = 2$

$$\text{Since, } x \in [0, \pi/2]$$

$$\text{and } y \in [0, \pi/2]$$

$$\therefore \sin x = 1 \text{ and } \cos y = 1$$

$$\Rightarrow x = \frac{\pi}{2} \text{ and } y = 0$$

$$\therefore x + y = \frac{\pi}{2} + 0 = \frac{\pi}{2}$$

17. (b) Let

$$S = \frac{1}{9!} + \frac{1}{3!7!} + \frac{1}{5!5!} + \frac{1}{7!3!} + \frac{1}{9!}$$

$$= \frac{1}{10!} \left[\frac{10!}{9!} + \frac{10!}{3!7!} + \frac{10!}{5!5!} + \frac{10!}{7!3!} + \frac{10!}{9!} \right]$$

$$= \frac{1}{10!} [{}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9]$$

$$= \frac{1}{10!} (2^{10} - 1) = \frac{2^9}{10!}$$

18. (e) 19. (d)

20. (c) Given $f(x) = \int_0^x (t^2 + t + 1) dt$

$$f'(x) = (x^2 + x + 1) \times 1 - 0 = x^2 + x + 1$$

$$\text{For } x \in [2, 3]$$

$$f'(x) > 0$$

$\therefore$ Minimum is at $x = 2$ and maximum is at $x = 3$.

Now, minimum value

$$= \int_0^2 (t^2 + t + 1) dt = \left[\frac{t^3}{3} + \frac{t^2}{2} + t \right]_0^2$$

$$\text{and maximum value} = \int_0^3 (t^2 + t + 1) dt$$

$$= \left[\frac{t^3}{3} + \frac{t^2}{2} + t \right]_0^3$$

21. (c) For distinct non zero roots $D > 0$

$$\Rightarrow a^2 - 4b > 0$$

$$\text{Now, } x^2 + ax + b$$

$$= \left(x + \frac{a}{2} \right)^2 + \left(b - \frac{a^2}{4} \right)$$

$$= \left(x + \frac{a}{2} \right)^2 - \left(a^2 - \frac{4b}{4} \right)$$

We know, sum of roots $a + b = -a$

$$\Rightarrow 2a + b = 0 \quad \dots (i)$$

$$\text{Product of roots } a \times b = b$$

$$\Rightarrow b(a-1) = 0 \Rightarrow a = 1, b \neq 0$$

From Eq. (i),

$$2a + b = 0 \Rightarrow 2(1) + b = 0$$

$$b = -2$$

$$\text{Now, } \left(x + \frac{a}{2} \right)^2 - \left(\frac{1^2 + 4 \times 2}{4} \right) = \left(x + \frac{a}{2} \right)^2 - \frac{9}{4}$$

$$\therefore \text{Minimum value} = -\frac{9}{4}$$

22. (b) We have,

$$y = 4x + c \quad \dots (i)$$

$$\text{and } \frac{x^2}{4} + y^2 = 1 \quad \dots (ii)$$

Put value of y from Eqs. (i) into (ii), we get

$$\frac{x^2}{4} + (4x + c)^2 = 1$$

$$\Rightarrow x^2 + 4(4x + c)^2 = 4$$

$$\Rightarrow x^2 + 4(16x^2 + 8cx + c^2) = 4$$

$$\Rightarrow x^2 + 64x^2 + 32cx + 4c^2 = 4$$

$$\Rightarrow 65x^2 + 32cx + 4(c^2 - 1) = 0$$

Since, given line is a tangent to the ellipse.

$$\therefore \text{Discriminant} = 0$$

23. (a) We have,

$$|x| + |y| = 1$$

$$= \begin{cases} x + y = 1, & x, y \in \text{I quadrant} \\ -x + y = 1, & x, y \in \text{II quadrant} \\ -x - y = 1, & x, y \in \text{III quadrant} \\ x - y = 1, & x, y \in \text{IV quadrant} \end{cases}$$

Clearly, $ABCD$ is a square.

24. (b)

25. (e) $\therefore$ Correct Σx_i = Incorrect

$$\Sigma x_i - 20 + 30 = 170 + 10 = 180$$

$$\therefore \text{Correct mean} = \bar{x} = \frac{\Sigma x_i}{N} \text{ (correct)}$$

$$= \frac{180}{15} = 12$$

Also, correct Σx_i^2 = Incorrect

$$\begin{aligned} \Sigma x_i^2 - (20)^2 + (30)^2 &= 2830 - 400 + 900 \\ &= 2830 + 500 = 3330 \end{aligned}$$

$\therefore$ Correct variance

$$\begin{aligned} &= \frac{\Sigma x_i^2 \text{ (correct)}}{N} - (\bar{x})^2 \\ &= \frac{3330}{15} - (12)^2 = 222 - 144 = 78 \end{aligned}$$

26. (d) We have,

$$y = 2x + 1, y = 3x + 1, x = 4$$

Intersecting points of above lines are $(0, 1), (4, 9), (4, 13)$

$$\therefore \text{Area of triangle} = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2} \begin{vmatrix} 0 & 1 & 1 \\ 4 & 9 & 1 \\ 4 & 13 & 1 \end{vmatrix}$$

$$= \frac{1}{2} [0(9-13) - 1(4-4) + 1(52-36)] = \frac{1}{2} \times 16 = 8$$

$$27. (b) \therefore \frac{{}^nC_{r-1}}{{}^nC_r} = \frac{36}{84}$$

$$\Rightarrow \frac{\frac{n!}{(n-r+1)!(r-1)!}}{\frac{n!}{(n-r)!r!}} = \frac{3}{7}$$

$$\text{Again, } \frac{{}^nC_r}{{}^nC_{r+1}} = \frac{84}{126}$$

$$\Rightarrow \frac{\frac{n!}{(n-r)!r!}}{\frac{n!}{(n-r-1)!(r+1)!}} = \frac{2}{3}$$

28. (b)

$$29. (e) \therefore \text{Area of } \Delta ABC = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ 7 & 2 & 1 \\ 5 & 9 & 1 \end{vmatrix}$$

$$= \frac{1}{2} \cdot 1(63 - 10) = \frac{53}{2} \text{ sq unit}$$

$$\text{Similarly Area of } = \frac{53}{2} \text{ sq unit}$$

$$\therefore \text{Area of parallelogram } ABCD$$

$$= \text{Area of } \Delta ABC + \text{Area of } \Delta ACD$$

$$= \frac{53}{2} + \frac{53}{2} = 53 \text{ sq unit}$$

30. (b) We have,

$$= |\sqrt{3+1+2\sqrt{3}}| - |\sqrt{3+1-2\sqrt{3}}|$$

$$= |\sqrt{(\sqrt{3})^2 + (1)^2 + 2 \cdot \sqrt{3} \cdot 1}|$$

$$- |\sqrt{(\sqrt{3})^2 + (1)^2 - 2 \cdot \sqrt{3} \cdot 1}|$$

$$= |\sqrt{(\sqrt{3}+1)^2}| - |\sqrt{(\sqrt{3}-1)^2}|$$

$$= |\sqrt{3}+1| - |\sqrt{3}-1|$$

$$= (\sqrt{3}+1) - (\sqrt{3}-1) = 2$$

31. (a)

32. (a) We have $y = 4x^2 - 14x + q = 0$

Since, $x = 2$ is the root

$$\therefore 4(2)^2 - 14(2) + q = 0$$

Solved Papers 2017 21

$$\Rightarrow q = 12$$

$$\therefore y = 4x^2 - 14x + 12$$

$$= 4x - 8x - 6x + 12$$

$$= 4x(x-2) - 6(x-2)$$

$$= (x-2)(4x-6)$$

33. (b) 34. (c) 35. (a)

36. (e)

37. (b) Here, $x + y = -b$ and $xy = 1$

$$\text{Now, } \frac{1}{x+b} + \frac{1}{y+b} = \frac{y+b+x+b}{(x+b)(y+b)}$$

$$= \frac{(x+y)+2b}{xy+b(x+y)+b^2}$$

38. (b) We have, $x^5 + ax + 1 = 0$

$$\text{and } x^6 + ax^2 + 1 = 0$$

$$\text{or } x^6 + ax^2 + x = 0$$

$$\text{and } x^6 + ax^2 + 1 = 0$$

$\therefore$ Common root is given by

$$(x^6 + ax^2 + x) - (x^6 + ax^2 + 1) = 0$$

$$\Rightarrow x = 1$$

$\therefore x = 1$ is the common root.

$$\therefore (1)^5 + a(1) + 1 = 0$$

$$\Rightarrow a = -2$$

39. (a)

40. (c) We have, $z^2 + z + 1 = 0$

$$\therefore z = \omega \text{ or } \omega^2$$

Let $z = \omega$, then

$$\left(z + \frac{1}{z}\right)^2 + \left(z^2 + \frac{1}{z^2}\right) + \left(z^3 + \frac{1}{z^3}\right)$$

$$= \left(\omega + \frac{1}{\omega}\right)^2 + \left(\omega^2 + \frac{1}{\omega^2}\right)^2 + \left(\omega^3 + \frac{1}{\omega^3}\right)^2$$

$$= (\omega + \omega^2)^2 + (\omega^2 + \omega)^2 + (\omega^3 + 1)^2$$

41. (a) We have $z = \cos \frac{\pi}{3} - i \sin \frac{\pi}{3}$

$$= \frac{1}{2} - \frac{i\sqrt{3}}{2} = \frac{1 - \sqrt{3}i}{2}$$

$$= -\left[\frac{-1 + \sqrt{3}i}{2}\right]$$

$$= -w \quad \left[\because w = \frac{-1 + \sqrt{3}i}{2} \right]$$

$$\text{Now, } z^2 - z + 1 = (-w)^2 - (-w) + 1$$

$$= w^2 + w + 1$$

$$= 0 \quad [\because 1 + w + w^2 = 0]$$

42. (c) Let $z = \left(\frac{1 + \cos \frac{\pi}{12} + i \sin \frac{\pi}{12}}{1 + \cos \frac{\pi}{12} - i \sin \frac{\pi}{12}} \right)^{72}$

$$= \left(\frac{2\cos^2 \frac{\pi}{24} + 2i \sin \frac{\pi}{24} \cos \frac{\pi}{24}}{2\cos^2 \frac{\pi}{24} - 2i \sin \frac{\pi}{24} \cos \frac{\pi}{24}} \right)^{72}$$

$$= \left(\frac{\cos \frac{\pi}{24} + i \sin \frac{\pi}{24}}{\cos \frac{\pi}{24} - i \sin \frac{\pi}{24}} \right)^{72}$$

43. (a) Let $z = x + iy$

$$\therefore |z| = \sqrt{x^2 + y^2}$$

$$\Rightarrow \sqrt{x^2 + y^2} = 5$$

$$\Rightarrow x^2 + y^2 = 25 \quad \dots (i)$$

$$\text{Now } w = \frac{z-5}{z+5} = \frac{x+iy-5}{x+iy+5}$$

$$= \frac{(x-5)+iy}{(x+5)+iy}$$

44. (c) We have, $a = e^{i\theta} = \cos \theta + i \sin \theta$

$$\text{Now, } \frac{1+a}{1-a} = \frac{1+(\cos \theta + i \sin \theta)}{1-(\cos \theta + i \sin \theta)}$$

$$= \frac{(1+\cos \theta) + i \sin \theta}{(1-\cos \theta) - i \sin \theta}$$

$$= \frac{2\cos^2 \frac{\theta}{2} + i 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}{2\sin^2 \frac{\theta}{2} - i 2\sin \frac{\theta}{2} \cos \frac{\theta}{2}}$$

$$= \frac{2\cos \frac{\theta}{2} \left[\cos \frac{\theta}{2} + i \sin \frac{\theta}{2} \right]}{2\sin \frac{\theta}{2} \left[\sin \frac{\theta}{2} - i \cos \frac{\theta}{2} \right]}$$

45. (c) Let $x = a - r$, $y = a$, $z = a + r$

Now, we have

$$x + y + z = -3$$

$$\therefore a - r + a + a + r = -3 \Rightarrow a = -1$$

$$\text{Again, } xyz = 8$$

$$\therefore (a-r)(a)(a+r) = 8$$

$$\Rightarrow a(a^2 - r^2) = 8$$

$$\Rightarrow -1(1 - r^2) = 8$$

$$\Rightarrow r^2 = 9$$

$$\Rightarrow r = \pm 3$$

$$\therefore x, y, z \text{ are } -4, -1, 2 \text{ or } 2, -1, -4$$

$$\therefore x^2 + y^2 + z^2 = (-4)^2 + (-1)^2 + (2)^2 = 16 + 1 + 4 = 21$$

46. (c) 47. (e)

48. (d) $3^{2x-1} = (3^4)^{1-x}$

$$\Rightarrow 3^{2x-1} = 3^{4-4x}$$

$$\therefore 2x-1 = 4-4x$$

$$\Rightarrow 6x = 5$$

$$\Rightarrow x = \frac{5}{6}$$

49. (c)

50. (a) Let the numbers be $a-d, a, a+d$

$$\therefore a+d+a+a-d=21$$

$$\text{and, } (a-d)(a+d)=45$$

51. (b) 52. (b) 53. (c)

54. (e) Number of ways of selecting one letter from the word TRICKS. $n(S)$

$$= {}^6C_1 = 6$$

Let E be the event of selecting T or R

$$\therefore E = \{T, R\}$$

$$\therefore n(E) = 2$$

55. (c) $\therefore$ Total balls = $4 + 2 + 4 = 10$

Number of ways of selecting 4 balls from

$$10 \text{ balls, } n(S) = {}^{10}C_4$$

Let E = Event getting 2 red balls

$$\therefore n(E) = {}^4C_2 \times {}^6C_2$$

56. (d) Now, according to the question,

$$P(E_1) = 0.60, P(E_2) = 0.40,$$

$$P(E_1 \cap E_2) = 0.20$$

$$\therefore \text{Required probability} = P(E'_1 \cap E'_2)$$

$$= P(E_1 \cup E_2)' = 1 - P(E_1 \cup E_2)$$

$$= 1 - [P(E_1) + P(E_2) - P(E_1 \cap E_2)]$$

57. (d) Here, $n(S) = {}^5C_2 = 10$

$$\text{and } E = \{(1, 3), (1, 5), (2, 4), (3, 5)\}$$

$$\therefore n(E) = 4$$

58. (e) In 3×3 matrix, total number of elements = $3 \times 3 = 9$

$\therefore$ Total number of 3×3 matrices with entries either -1 or $1 = 2^9$

59. (b) $S = \{2 \times 2 \text{ symmetric matrices whose entries are either zero or one}\}$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

$$\left\{ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\therefore n(S) = 8$$

Let $x = \{\text{matrix whose determinant is non-zero}\}$

$$= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \right\}$$

$$\therefore n(x) = 4$$

$$\therefore P(x) = \frac{n(x)}{n(S)} = \frac{4}{8} = \frac{1}{2}$$

60. (c)

61. (b) Number of diagonals in a hexagon

$$= \frac{6(6-3)}{2} = 9$$

22 Solved Papers 2017

62. (a) 63. (e) 64. (a)
65. (a) 66. (e) 67. (b)
68. (c) 69. (a) 70. (e)

71. (a) We have, $f(x) = \frac{x+1}{x-1}$

$$\therefore f(f(x)) = f\left(\frac{x+1}{x-1}\right) = \frac{\frac{x+1}{x-1} + 1}{\frac{x+1}{x-1} - 1} = \frac{x+1+x-1}{x+1-x+1} = \frac{2x}{2} = x$$

72. (a) Let E = outcomes in which product of two number is even
= {(1, 2), (1, 4), (1, 6), (2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6), (3, 2), (3, 4), (3, 6), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (4, 6), (5, 2), (5, 4), (5, 6), (6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

$$\therefore n(E) = 27$$

73. (a) $\lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2-x}}{x}$

$$= \lim_{x \rightarrow 0} \frac{\sqrt{2+x} - \sqrt{2-x}}{x} \times \frac{\sqrt{2+x} + \sqrt{2-x}}{\sqrt{2+x} + \sqrt{2-x}}$$

74. (b)

KCET

1. (d) Total number of terms in

$$(x+a)^n - (x-a)^n$$

$$= \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

2. (a) If p and q are two statements, then the contrapositive of the implication "if p , then q " is "if $\sim q$ then $\sim p$ ".
 $\therefore$ The contrapositive of the statement "If x is prime number, then x is odd" is "If x is not odd, then x is not a prime number".

3. (c) We have

$$p = \text{probability that a bulb is defective} = \frac{10}{100} = \frac{1}{10}$$

$$\therefore q = 1 - p = \frac{9}{10}$$

$\therefore$ Required probability

$$= P(x=0) + P(x=1)$$

$$= {}^5C_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^5 + {}^5C_1 \left(\frac{1}{10}\right)^1 \left(\frac{9}{10}\right)^4$$

4. (b) 5. (a)

6. (a) We have, $A \subset B$

$$\therefore A \cap B = A \Rightarrow n(A \cap B) = n(A) \quad \dots (i)$$

Again, we know that

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$\Rightarrow n(A \cup B) = n(A) + n(B) - n(A)$$

[from Eq. (i)]

$$\Rightarrow n(A \cup B) = n(B)$$

7. (c) $\lim_{\theta \rightarrow 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \rightarrow 0} \frac{2\sin^2 2\theta}{2\sin^2 3\theta}$

$$= \lim_{\theta \rightarrow 0} \frac{\sin^2 2\theta}{4\theta^2} \cdot \frac{1}{\sin^2 3\theta} \cdot \frac{4\theta^2}{9\theta^2}$$

$$= \lim_{\theta \rightarrow 0} \left(\frac{\sin 2\theta}{2\theta}\right)^2 \cdot \frac{1}{\left(\frac{\sin 3\theta}{3\theta}\right)^2} \cdot \frac{4}{9} = \frac{4}{9}$$

8. (b)

9. (b) $\left[\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right]^m = 1$

10. (d) $\cos^2 45^\circ - \sin^2 15^\circ$
 $= \cos(45^\circ + 15^\circ) \cos(45^\circ - 15^\circ)$

11. (c) We have,

$$\Rightarrow x = \sin^{-1} \left(\frac{2t}{1+t^2} \right)$$

$$\Rightarrow x = 2 \tan^{-1} t \quad \dots (i)$$

and $\tan y = \frac{2t}{1-t^2}$

$$\Rightarrow y = \tan^{-1} \frac{2t}{1-t^2}$$

$$\Rightarrow y = 2 \tan^{-1} t \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$y = x$$

$$\therefore \frac{dy}{dx} = 1$$

12. (a)

13. (b) We have, $y = \log(\log x)$

$$\therefore \frac{dy}{dx} = \frac{1}{\log x} \cdot \frac{1}{x}$$

$$= \frac{1}{x \log x} = (x \log x)^{-1}$$

Again differentiating w.r.t. x , we get

$$\frac{d^2 y}{dx^2} = -(x \log x)^{-1-1} \left[1 \cdot \log x + x \cdot \frac{1}{x} \right]$$

$$= -(x \log x)^{-2} (\log x + 1)$$

$$= \frac{-(1 + \log x)}{(x \log x)^2}$$

14. (a) We have, $|x-2| \leq 1$

$$\Rightarrow -1 \leq x-2 \leq 1$$

$$\Rightarrow -1+2 \leq x-2+2 \leq 1+2$$

$$\Rightarrow 1 \leq x \leq 3$$

$$\therefore x \in [1, 3]$$

Note Correction in the question

$$|x-2| \leq 1$$

15. (d) 16. (c)

17. (c) We have, $f(x) = \sqrt{9-x^2}$

Let $y = \sqrt{9-x^2}$

$$\Rightarrow y^2 = 9 - x^2$$

$$\Rightarrow x^2 = 9 - y^2$$

$$\Rightarrow x = \sqrt{9-y^2}$$

Since, $f(x) \geq 0$

$$\therefore 9 - y^2 \geq 0$$

$$\Rightarrow (3-y)(3+y) \geq 0$$

$$\therefore y \in [-3, 3]$$

But, $f(x) = \sqrt{9-x^2}$

$$\therefore f(x) \in [0, 3]$$

18. (c) Reflexion of any point (x, y, z) in XY -plane is $(x, y, -z)$.

$\therefore$ Reflexion of (α, β, γ) in XY -plane is $(\alpha, \beta, -\gamma)$.

19. (a) We know that

$$C \cdot V = \frac{\sigma}{\bar{x}} \times 100$$

$$\Rightarrow 60 = \frac{24}{\bar{x}} \times 100$$

$$\Rightarrow \bar{x} = \frac{24 \times 100}{60}$$

$$\Rightarrow \bar{x} = 40$$

$\therefore$ Arithmetic means = 40

20. (a) Here, $P(A \cap B) = P(A) \cdot P(B)$

$$\text{and } P(A' \cap B') = P(A') \cdot P(B')$$

$$= (1 - P(A))(1 - P(B))$$

Again, if A and B are mutually exclusive, then $P(A \cap B) = 0$.

21. (b) Given line is

$$y = 3x - 1$$

$\therefore$ Slope of the given line = 3

$\therefore$ Slope of a line perpendicular to the given

$$\text{line} = -\frac{1}{3}$$

$\therefore$ Equation of required line will be

$$y - 2 = \frac{-1}{3}(x - 1)$$

$$\Rightarrow 3y - 6 = -x + 1$$

$$\Rightarrow x + 3y - 7 = 0$$

AP EAMCET

1. (b) We have, $\cos \theta = \frac{c-a}{b}$

$$\Rightarrow \sin \theta = \sqrt{1 - \cos^2 \theta} \\ = \sqrt{1 - \frac{(c-a)^2}{b^2}} = \frac{\sqrt{b^2 - (c-a)^2}}{b}$$

$$\text{Now, } \tan \theta = \frac{\sqrt{b^2 - (c-a)^2}}{(c-a)}$$

$$\Rightarrow (c-a) \tan \theta = \sqrt{b^2 - (c-a)^2} \\ = \sqrt{b^2 - c^2 - a^2 + 2ac}$$

$$= \sqrt{-(c^2 + a^2 - b^2) + 2ac}$$

$$= \sqrt{2ac - 2a \cos B} \quad [\text{Using cosine rule}]$$

$$= \sqrt{2ac} \sqrt{1 - \cos B} = \sqrt{2} \sqrt{ac} \sqrt{2 \sin^2 \frac{B}{2}}$$

$$= 2\sqrt{ac} \sin \frac{B}{2}$$

2. (a) Given equations of pair of straight lines are

$$x^2 - 16pxy - y^2 = 0 \quad \dots (i)$$

$$\text{and } x^2 - 16qxy - y^2 = 0 \quad \dots (ii)$$

The equations of bisectors of these line are

$$-8px^2 - 2xy + 8py^2 = 0,$$

$$\text{i.e. } 4px^2 + xy - 4py^2 = 0 \quad \dots (iii)$$

$$\text{and } -8qx^2 - 2xy + 8qy^2 = 0,$$

$$\text{i.e. } 4qx^2 + xy - 4qy^2 = 0 \quad \dots (iv)$$

According to the given condition Eqs. (i) and (iv), and Eqs. (ii) and (iii) must be coincident

$$\therefore \frac{1}{4q} = \frac{-16p}{1} = \frac{-1}{-4q}$$

$$\Rightarrow 1 = -64pq \Rightarrow pq = \frac{-1}{64}$$

3. (a) Total number of possible outcomes = ${}^{10}C_3$

Let A be the event that minimum of chosen number is 3 and B be the event that maximum of chosen number is 7. Then,

$n(A)$ = Number of ways of choosing remaining two numbers from the set

$$\{4, 5, 6, 7, 8, 9, 10\} = {}^7C_2 = 21$$

Similarly, $n(B)$ = Number of ways of choosing remaining two numbers from the set

$$\{1, 2, 3, 4, 5, 6\} = {}^6C_2 = 15$$

and $n(A \cap B)$ = Number of ways of choosing remaining one number from the set

$$\{4, 5, 6\} = {}^3C_1 = 3$$

Thus, required probability

$$= \frac{21 + 15 - 3}{120} = \frac{33}{120} = \frac{11}{40}$$

4. (a) Coefficient of x^5 in the expansion of

$$(1+x)^{21} + (1+x)^{22} + \dots + (1+x)^{30} \\ = {}^{21}C_5 + {}^{22}C_5 + \dots + {}^{30}C_5 \\ = ({}^{21}C_6 + {}^{21}C_5 + {}^{22}C_5 \\ + \dots + {}^{30}C_5) - {}^{21}C_6$$

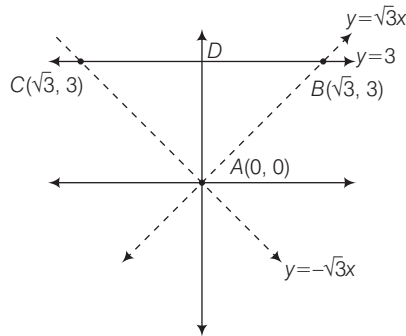
$$= ({}^{22}C_6 + {}^{22}C_5 + \dots + {}^{30}C_5) - {}^{21}C_6$$

$$= ({}^{23}C_6 + {}^{23}C_5 + \dots + {}^{30}C_5) - {}^{21}C_6$$

$$= ({}^{30}C_6 + {}^{30}C_5) - {}^{21}C_6 = {}^{31}C_6 - {}^{21}C_6$$

5. (a) $f(x) = \begin{cases} 3x-1 & \text{for } x > 3 \\ x^2+1 & \text{for } -3 \leq x \leq 3 \\ 2x-3 & \text{for } x < -3 \end{cases}$

6. (a) The triangle formed by the given lines is shown in the adjacent figure.



Clearly, the triangle ABC is an isosceles triangle.

$\therefore$ The incentre lie on the median to the base.

$\therefore D$ is mid-point of BC

$\therefore OD$ is median to the base BC .

Thus, incentre lie on Y -axis.

7. (d)

8. (d) Let $O(x, y, z)$ be the circumcentre of $\triangle ABC$

$$\therefore OA = OB = OC$$

$$\text{Now } OA^2 = OB^2$$

$$\Rightarrow (x-1)^2 + (y-2)^2 + (z-3)^2$$

$$= (x-3)^2 + (y+1)^2 + (z-5)^2$$

$$\Rightarrow 4x - 6y + 4z - 21 = 0 \quad \dots (i)$$

Similarly $OB = OC$

$$(x-3)^2 + (y+1)^2 + (z-5)^2$$

$$= (x-4)^2 + (y-0)^2 + (z+3)^2$$

$$\Rightarrow x + y - 8z + 5 = 0 \quad \dots (ii)$$

Similarly $OA = OC$

$$(x-1)^2 + (y-2)^2 + (z-3)^2$$

$$= (x-4)^2 + (y-0)^2 + (z+3)^2$$

$$\Rightarrow 6x - 4y - 12z - 11 = 0 \quad \dots (iii)$$

Solving Eqs. (i), (ii) and (iii), we get

$$x = \frac{7}{2}, y = -\frac{1}{2}, z = 1$$

9. (a)

10. (a) We have $4xy + 2x + 6y + 3 = 0$

$$\Rightarrow 2x(2y+1) + 3(2y+1) = 0$$

$$\Rightarrow (2x+3)(2y+1) = 0$$

$$\text{Equation of lines are } x = \frac{-3}{2}$$

$$\text{and } y = -\frac{1}{2}$$

Equations of line passing through $(2, 1)$ and perpendicular to the line

$$x = \frac{23}{2} \text{ and } y = \frac{1}{2} \text{ are}$$

$$y - 1 = 0 \text{ and } x - 2 = 0$$

$\therefore$ Combining equation of lines are

$$(y-1)(x-2) = 0$$

$$\Rightarrow xy - x - 2y + 2 = 0$$

11. (b) Let the numbers are a and b .

Then, we have

$$\frac{2ab}{a+b} = -\frac{8}{5} \text{ and } \sqrt{ab} = 2$$

$$\Rightarrow \frac{2 \times 4}{a+b} = -\frac{8}{5} \quad (\because ab = 4)$$

12. (c) Consider, $\left| z + \frac{2}{z} \right| \geq \left| z - \frac{2}{z} \right|$

$$= \left| z - \frac{2}{z} \right| = \left| z + 2 \left(\frac{-1}{z} \right) \right|$$

$$\geq \left| 5 - \frac{2}{5} \right| = \frac{23}{5}$$

Thus, the least value of $\left| z + \frac{2}{z} \right|$ is $\frac{23}{5}$.

13. (d) Let the roots of given equations

are $\frac{a}{r}, a, ar$. Then, we get

$$\frac{a}{r} + a + ar = 7 \quad \dots (i)$$

$$\frac{a}{r} \cdot a + a \cdot ar + ar \cdot \frac{a}{r} = 14 \quad \dots (ii)$$

$$\text{and } \frac{a}{r} \cdot a \cdot ar = 8 \quad \dots (iii)$$

From Eq. (iii), we get $a = 2$

Now, from Eq. (i), we get

$$\frac{2}{r} + 2r = 5 \Rightarrow 2 + 2r^2 = 5r$$

$$\Rightarrow 2r^2 - 5r + 2 = 0$$

$$\Rightarrow 2r^2 - 4r - r + 2 = 0$$

$$\Rightarrow 2r(r-2) - 1(r-2) = 0$$

$$\Rightarrow (r-2)(2r-1) = 0$$

$$\Rightarrow r = 2 \text{ or } r = \frac{1}{2}$$

24 Solved Papers 2017

Thus, the roots are 1, 2, 4.

Hence, required difference

$$= 4 - 1 = 3$$

14. (c) Given, α is non-real root of $x^7 = 1$

$$\therefore \alpha^7 = 1 \text{ and } \alpha \neq 1.$$

Now, consider $\alpha(1 + \alpha)(1 + \alpha^2 + \alpha^4)$

$$= \alpha(1 + \alpha^2 + \alpha^4 + \alpha + \alpha^3 + \alpha^5)$$

$$= \alpha + \alpha^3 + \alpha^5 + \alpha^2 + \alpha^4 + \alpha^6$$

$$= \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \alpha^5 + \alpha^6$$

$$= \frac{\alpha(1 - \alpha^7)}{1 - \alpha} \quad [\because \alpha \neq 1]$$

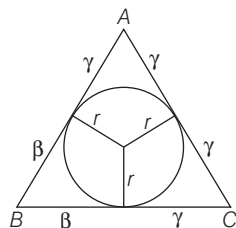
$$= \frac{\alpha - \alpha^7}{1 - \alpha} = \frac{\alpha - 1}{1 - \alpha} \quad [\because \alpha^7 = 1]$$

$$= -1$$

15. (c) Required point = Point of intersection of given lines

$$= \left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right)$$

16. (a)



$$\Rightarrow S = \alpha + \beta + \gamma$$

Area of ΔABC

$$= \sqrt{s(s-a)(s-b)(s-c)}$$

$$= \sqrt{(\alpha + \beta + \gamma)(\alpha)(\beta)(\gamma)}$$

We know that

$$r = \Delta / s$$

$$\therefore r = \frac{\sqrt{(\alpha + \beta + \gamma)(\alpha\beta\gamma)}}{\alpha + \beta + \gamma}$$

$$\Rightarrow r^2 = \frac{(\alpha + \beta + \gamma)(\alpha\beta\gamma)}{(\alpha + \beta + \gamma)^2}$$

$$\Rightarrow \alpha + \beta + \gamma = \frac{\alpha\beta\gamma}{r^2}$$

17. (a) Let $S \equiv 3x^2 + 2y^2 - 5$. Then equations of pair of tangents drawn to the ellipse s from the point $(1, 2)$ is

$$SS_1 = T^2$$

$$\Rightarrow (3x^2 + 2y^2 - 5)(3(1)^2 - 2(2)^2 - 5)$$

$$= (3x(1) + 2y(2) - 5)^2$$

$$\Rightarrow 9x^2 - 4y^2 - 24xy + 30x + 40y - 55 = 0$$

On comparing this equation with

$$\text{Also } \tan \theta = \frac{2\sqrt{h^2 - ab}}{a + b}$$

18. (d) Given equation of normal to the hyperbola can be written as

$$y = \frac{-l}{m} + \frac{1}{m},$$

Which is of the form $y = mx + c$

$$\text{Also, } \left(\frac{1}{m}\right)^2 = \frac{\left(-\frac{l}{m}\right)^2 (a^2 + b^2)^2}{a^2 - b^2 \left(-\frac{l}{m}\right)^2}$$

19. (a) Then, the equation of common chord of these two circles is

$$S_1 - S_2 = 0$$

$$\Rightarrow -2x - 4y - 3 = 0$$

$$\text{i.e. } 2x + 4y + 3 = 0$$

The equation of required circle is

$$S_1 + \lambda(S_2 - S_1) = 0$$

$$(x^2 + y^2 + 2x + 2y + 1) +$$

$$\lambda(2x + 4y + 3) = 0$$

$$\Rightarrow x^2 + y^2 + 2x(1 + \lambda) +$$

$$2y(2\lambda + 1) + 3\lambda + 1 = 0$$

Since, $2x + 4y + 3 = 0$ is a diameter of this circle, therefore centre

$(-(1 + \lambda), -(2\lambda + 1))$ lies on it.

$$\text{So, } 2(-(1 + \lambda)) + 4(-(2\lambda + 1)) + 3 = 0$$

$$\Rightarrow -2 - 2\lambda - 8\lambda - 4 + 3 = 0$$

$$\Rightarrow 10\lambda = -3 \Rightarrow \lambda = \frac{-3}{10}$$

Thus, the required circle is

$$x^2 + y^2 + 2x\left(1 - \frac{3}{10}\right) + 2y\left(-\frac{6}{10} + 1\right) - \frac{9}{10} + 1 = 0$$

20. (a) We have, $\frac{x-1}{3x+4} - \frac{x-3}{3x-2} < 0$

$$\Rightarrow \frac{(x-1)(3x-2) - (x-3)(3x+4)}{(3x+4)(3x-2)} < 0$$

$$(3x^2 - 2x - 3x + 2) -$$

$$\Rightarrow \frac{(3x^2 + 4x - 9x - 12)}{(3x+4)(3x-2)} < 0$$

$$\Rightarrow \frac{-5x + 2 + 5x + 12}{(3x+4)(3x-2)} < 0$$

$$\Rightarrow \frac{14}{(3x+4)(3x-2)} < 0$$

$$\Rightarrow (3x+4)(3x-2) < 0$$

$$\Rightarrow x \in \left(-\frac{4}{3}, \frac{2}{3}\right)$$

21. (a) Total number of ways = ${}^8C_3$

$$= \frac{8 \times 7 \times 6}{1 \times 2 \times 3} = 56$$

22. (c) Equation of pair of lines

$$6x^2 + 13xy + 6y^2 = 0 \text{ are}$$

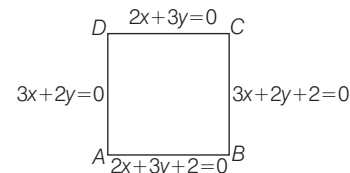
$$3x + 2y = 0 \text{ and}$$

$$2x + 3y = 0 \text{ and equation of pairs of lines}$$

$$6x^2 + 13xy + 6y^2 + 10x + 10y + 4 = 0$$

are

$$3x + 2y + 2 = 0 \text{ and } 2x + 3y + 2 = 0$$



23. (b) Chord of contact of two tangents drawn from the point $P(a, b)$ to the given circle is

$$xa + yb - (x + a) - 3(y + b) - 8 = 0$$

$$\Rightarrow (a - 1)x + (b - 3)y - (a + 3b + 8) = 0$$

Clearly, this line coincides with $5x + y + 1 = 0$.

$$\therefore \frac{a-1}{5} = \frac{b-3}{1} = \frac{-(a+3b+8)}{1}$$

$$\Rightarrow a - 1 = -5(a + 3b + 8) \text{ and}$$

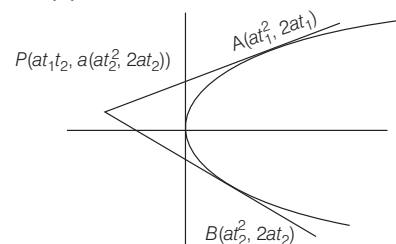
$$b - 3 = -a - 3b - 8$$

$$\Rightarrow 6a + 15b = -39 \text{ and } a + 4b = -5$$

$$\Rightarrow 2a + 5b = -13 \quad \dots (i)$$

$$\text{and } a + 4b = -5 \quad \dots (ii)$$

24. (d)



Slope of line PA

$$\text{i.e. } \tan \theta_1 = \frac{2at_1 - at_1 - at_2}{at_1^2 - at_1t_2}$$

$$= \frac{a(t_1 - t_2)}{at_1(t_1 - t_2)} = \frac{1}{t_1}$$

Similarly, slope of line PB

$$\tan \theta_2 = \frac{2at_2 - at_1 - at_2}{at_2^2 - at_1t_2}$$

$$= \frac{a(t_2 - t_1)}{at_2(t_2 - t_1)}$$

$$= \frac{1}{t_2}$$

It is given that $\tan \theta_1 + \tan \theta_2 = b$

$$\therefore \frac{1}{t_1} + \frac{1}{t_2} = b$$

25. (d) We have, line $3x - 4y - 1 = 0$

and point $A(x_1, y_1)$ which is at 5 units distance from $(3, 2)$ equations of line PA are

$$\Rightarrow \quad b (\alpha - p) = a (\beta - q)$$

26 Solved Papers 2017

Hence, locus of $p(\alpha, \beta)$ is

$$b(x - p) = a(y - q)$$

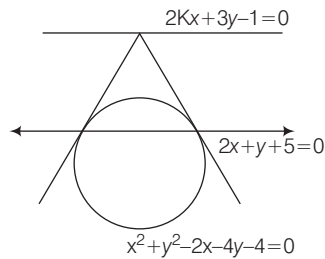
34. (a) Median of given observation

$$= \frac{25\alpha + 26\alpha}{2} = 25.5\alpha$$

Now, median deviation about median

$$\begin{aligned} &= \frac{\sum_{k=1}^{50} |k\alpha - 25.5\alpha|}{50} \\ \Rightarrow 50 \times 50 &= \sum_{k=1}^{50} |\alpha(K - 25.5)| \\ \Rightarrow |\alpha| \sum_{k=1}^{50} |k - 25.5| &= 2500 \\ \Rightarrow z|\alpha| [24.5 + 23.5 + \dots + 1.5 + 0.5] &= 2500 \end{aligned}$$

35. (c) Let (x_1, y_1) lie on the line
 $2kx + 3y - 1 = 0$



Equation of chord of contact from (x_1, y_1) to the circle

$$\begin{aligned} &x^2 + y^2 - 2x - 4y - 4 = 0 \\ \text{and } xx_1 + yy_1 - \frac{2(x+x_1)}{2} - \frac{4(y+y_1)}{2} - 4 &= 0 \\ \Rightarrow x(x_1 - 1) + y(y_1 - 2) - x_1 - 2y_1 - 4 &= 0 \end{aligned}$$

Now, $2x + y + 5 = 0$ and $2kx + 3y + 1 = 0$ are conjugate line

$$\begin{aligned} \therefore 2x + y + 5 &= 0 \\ \text{and } x(x_1 - 1) + y(y_1 - 2) - x_1 - 2y_1 - 4 &= 0 \\ \text{are coincide.} \\ \therefore \frac{2}{x_1 - 1} = \frac{1}{y_1 - 2} = \frac{-5}{x_1 + 2y_1 + 4} = \frac{1}{\lambda} \end{aligned}$$

$$\begin{aligned} \Rightarrow x_1 &= 2\lambda + 1, y_1 = \lambda + 2, \\ x_1 + 2y_1 + 4 &= -5\lambda \\ \text{Substituting the value of } x_1, y_1 \text{ in} \\ x_1 - 2y_1 - 4 &= -5\lambda \\ 2\lambda + 1 + 2\lambda + 4 + 4 &= -5\lambda \\ \Rightarrow \lambda &= -1 \\ \therefore x_1 &= -1, y_1 = 1 \end{aligned}$$

Putting the value of $x_1 = -1$

and $y_1 = 1$ in $2kx + 3y - 1 = 0$, we get

$$-2k + 3 - 1 = 0 \Rightarrow k = 1$$

36. (c) We have

$$\begin{aligned} \frac{5x^2 + 2}{x^3 + x} &= \frac{A_1}{x} + \frac{A_2x + A_3}{x^2 + 1} \\ \Rightarrow \frac{5x^2 + 2}{x(x^2 + 1)} &= \frac{A_1(x^2 + 1) + (A_2x + A_3)x}{x(x^2 + 1)} \\ \Rightarrow 5x^2 + 2 &= A_1(x^2 + 1) + A_2x^2 + A_3x \\ \Rightarrow 5x^2 + 2 &= (A_1 + A_2)x^2 + A_3x + A_1 \\ A_1 + A_2 &= 5; A_3 = 0 \text{ and } A_1 = 2 \\ \Rightarrow A_1 &= 2; A_2 = 3 \text{ and } A_3 = 0 \end{aligned}$$

37. (c) Let $S_n = \frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \dots$
 upto n terms

$$\begin{aligned} &= \frac{1}{3} \left[\frac{3}{2 \cdot 5} + \frac{3}{5 \cdot 8} + \frac{3}{8 \cdot 11} + \dots \text{ upto } n \text{ terms} \right] \\ &= \frac{1}{3} \left[\frac{5-2}{2 \cdot 5} + \frac{8-5}{5 \cdot 8} + \frac{11-8}{8 \cdot 11} + \dots \text{ up to } n \text{ terms} \right] \\ &= \frac{1}{3} \left[\left(\frac{1}{2} - \frac{1}{5} \right) + \left(\frac{1}{5} - \frac{1}{8} \right) + \left(\frac{1}{8} - \frac{1}{11} \right) + \dots \right. \\ &\quad \left. + \left(\frac{1}{3n-1} - \frac{1}{3n+2} \right) \right] \\ &\quad [\because \text{nth term of } 2, 5, 8, \dots] \\ &= 2 + (n-1)3 \\ &= 2 + 3n - 3 = 3n - 1 \\ &= \frac{1}{3} \left[\frac{1}{2} - \frac{1}{3n+2} \right] = \frac{1}{3} \left[\frac{3n+2-2}{2(3n+2)} \right] \\ &= \frac{n}{2(3n+2)} \end{aligned}$$

38. (b) General equation of such parabola is

$$x = Ay^2 + By + C \quad (A \neq 0)$$

Since, it passes through $(-2, 1), (1, 2)$ and $(-1, 3)$ therefore, we get

$$\begin{aligned} -2 &= A + B + C \quad \dots (i) \\ 1 &= 4A + 2B + C \quad \dots (ii) \\ \text{and } -1 &= 9A + 3B + C \quad \dots (iii) \end{aligned}$$

39. (b) Consider,

$$\begin{aligned} &b \cos(C + \theta) + c \cos(B - \theta) \\ &= b(\cos C \cos \theta - \sin C \sin \theta) + c \\ &\quad (\cos B \cos \theta + \sin B \sin \theta) \\ &= \cos \theta (b \cos C + c \cos B) - \sin \theta \\ &\quad (b \sin C - c \sin B) \\ &= a \cos \theta - \sin \theta \cdot 0 \end{aligned}$$

$\therefore$ By projection formula and by sine rule

$$\begin{aligned} \frac{b}{\sin B} &= \frac{c}{\sin C} \\ \Rightarrow b \sin C - c \sin B &= 0 \\ &= a \cos \theta - \sin \theta \cdot 0 = a \cos \theta \end{aligned}$$

40. (c) Given trigonometric equation is

$$1 + \cos x \cdot \cos 5x = \sin^2 x$$

$$\Rightarrow 1 - \sin^2 x + \cos x \cdot \cos 5x = 0$$

$$\Rightarrow \cos^2 x + \cos x \cdot \cos 5x = 0$$

$$\Rightarrow \cos x (\cos x + \cos 5x) = 0$$

$$\Rightarrow \cos x (2 \cos 3x \cos 2x) = 0$$

$$\Rightarrow \cos x = 0; \cos 3x = 0 \text{ or } \cos 2x = 0$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\text{or } 3x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}, \frac{11\pi}{2}$$

$$\text{or } 2x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2} \quad [\because x \in [0, 2\pi]]$$

$$\Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\frac{11\pi}{6}, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$

Thus, there are 10 solutions in $[0, 2\pi]$.

41. (d) Since, α, β and γ are roots of given equations, therefore

$$\alpha + \beta + \gamma = -p \quad \dots (i)$$

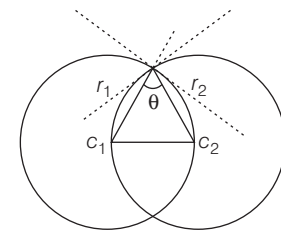
$$\alpha\beta + \beta\gamma + \gamma\alpha = q \quad \dots (ii)$$

$$\text{and } \alpha\beta\gamma = -r \quad \dots (iii)$$

Now, consider $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)$

$$\begin{aligned} &= (1 + \alpha^2)(1 + \beta^2 + \gamma^2 + (\beta\gamma)^2) \\ &= 1 + \beta^2 + \gamma^2 + (\beta\gamma)^2 + \alpha^2 + (\alpha\beta)^2 \\ &\quad + (\alpha\gamma)^2 + (\alpha\beta\gamma)^2 \\ &= 1 + (\alpha^2 + \beta^2 + \gamma^2) + ((\alpha\beta)^2 + (\beta\gamma)^2 \\ &\quad + (\gamma\alpha)^2 + (\alpha\beta\gamma)^2) \\ &= 1 + [(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)] + \\ &\quad [(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)] \\ &\quad + (\alpha\beta\gamma)^2 \end{aligned}$$

42. (a) From figure



$$\cos \theta = \frac{r_1^2 + r_2^2 - c_1 c_2}{2r_1 r_2}$$

$$\cos \theta = \frac{r_1^2 + r_2^2 - r_1^2}{2r_1 r_2} \quad [\because c_1 c_2 = r_1 = r_2]$$

$$\cos \theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

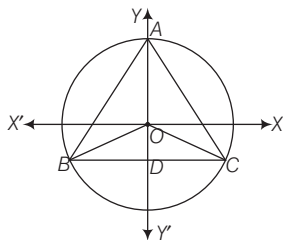
$\therefore$ Angle between two circle is either $\frac{\pi}{3}$

$$\text{or } \pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

43. (c) We have

$$\begin{aligned} \frac{|z|^2 - |z| + 1}{2 + |z|} &< \left(\frac{1}{\sqrt{3}}\right)^{-2} \\ \Rightarrow |z|^2 - |z| + 1 &< (\sqrt{3})^2 (2 + |z|) \\ \Rightarrow |z|^2 - |z| + 1 &< 6 + 3|z| \\ \Rightarrow |z|^2 - 4|z| + 1 &< 6 \\ \Rightarrow (|z| - 2)^2 &< 9 \\ \Rightarrow -3 < |z| - 2 &< 3 \\ \Rightarrow -1 < |z| &< 5 \\ \Rightarrow 0 < |z| &< 5 \end{aligned}$$

44. (c) Given length of median of $\triangle ABC = 9$



$$\begin{aligned} \therefore AO &= \frac{2}{3} AD \\ AO &= \frac{2}{3} \times 9 = 6 \end{aligned}$$

O is the circumcentre of $\triangle ABC$.
We know that in equilateral triangle circumcentre, incentre, centroid coincide.

$\therefore AO$ is radius of circle.

Hence, equation of circle is $x^2 + y^2 = 6^2 \Rightarrow x^2 + y^2 = 36$

45. (b) Consider,

$$\begin{aligned} (1 + \cos 10^\circ) + (\cos 20^\circ + \cos 30^\circ) \\ = 2\cos^2 5^\circ + 2\cos 25^\circ \cos 5^\circ \\ = 2\cos 5^\circ (\cos 5^\circ + \cos 25^\circ) \\ = 2\cos 5^\circ (2\cos 15^\circ \cos 10^\circ) \\ = 4\cos 5^\circ \cos 10^\circ \cos 15^\circ \end{aligned}$$

46. (b) Since the points (1, 2) and (3, 4) lie on the same side of line $3x - 5y + a = 0$

$\therefore$ Either $3(1) - 5(2) + a < 0$;

47. (b) We have

$$\begin{aligned} x &= \frac{1 \cdot 3}{3 \cdot 6} + \frac{1 \cdot 3 \cdot 5}{3 \cdot 6 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3 \cdot 6 \cdot 9 \cdot 12} + \dots \\ \Rightarrow x &= \frac{1 \cdot 3}{3^2(2!)} + \frac{1 \cdot 3 \cdot 5}{3^3(3!)} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{3^4(4!)} + \dots \\ \Rightarrow x &= \frac{\frac{1}{2} \left(\frac{1}{2} + 1 \right)}{2!} \left(\frac{2}{3} \right)^2 \\ &\quad + \frac{\left(\frac{1}{2} \right) \left(\frac{1}{2} + 1 \right) \left(\frac{1}{2} + 2 \right)}{3!} \left(\frac{2}{3} \right)^3 + \dots \end{aligned}$$

$$\begin{aligned} \Rightarrow x &= \left[1 + \frac{1}{2} \left(\frac{2}{3} \right) + \frac{\frac{1}{2} \left(\frac{1}{2} + 1 \right)}{2!} \left(\frac{2}{3} \right)^2 \right. \\ &\quad \left. + \frac{\frac{1}{2} \left(\frac{1}{2} + 1 \right) \left(\frac{1}{2} + 1 \right)}{3!} \left(\frac{2}{3} \right)^3 \dots \right] - \left(1 + \frac{1}{3} \right) \\ \Rightarrow x &= \left(1 - \frac{2}{3} \right)^{-1/2} - \frac{4}{3} \end{aligned}$$

TS EAMCET

$$\begin{aligned} 1. (d) \therefore \frac{\tan 160^\circ - \tan 110^\circ}{1 + (\tan 160^\circ)(\tan 110^\circ)} \\ = \frac{\tan(180^\circ - 20^\circ) - \tan(90^\circ + 20^\circ)}{1 + (\tan(180^\circ - 20^\circ)(\tan(90^\circ + 20^\circ)))} \end{aligned}$$

2. (d) We have,

$$\begin{aligned} x^2 + y^2 - 6x + 4y &= 12 \\ \Rightarrow (x - 3)^2 + (y + 2)^2 &= 25 \\ \Rightarrow (x - 3)^2 + (y + 2)^2 &= (5)^2 \end{aligned}$$

Equation of tangent whose slope m is

$$y + 2 = m(x - 3) \pm 5\sqrt{m^2 + 1} \quad \dots (i)$$

Now, this tangent is parallel to line $4x + 3y + 5 = 0$

$\therefore$ Slope of line is $-\frac{4}{3}$

Put the value of $m = -\frac{4}{3}$ is Eq. (i), we get

$$y + 2 = -\frac{4}{3}(x - 3) \pm 5\sqrt{\left(\frac{-4}{3}\right)^2 + 1}$$

$$\Rightarrow y + 2 = \frac{-4}{3}(x - 3) \pm 5\left(\frac{5}{3}\right)$$

$$\Rightarrow 3y + 6 = -4x + 12 \pm 25$$

$$\Rightarrow 4x + 3y = 6 \pm 25$$

$$\Rightarrow 4x + 3y = 31 \text{ or } 4x + 3y = -19$$

Hence, equation of tangent is

$$4x + 3y - 31 = 0 \text{ or } 4x + 3y + 19 = 0$$

3. (b) Given, mean $(\bar{x}) = 10$

$\therefore$ Mean

$$(\bar{x}) = \frac{6 + 7 + 10 + 12 + 13 + \alpha + 12 + 16}{8}$$

$$\Rightarrow 10 = \frac{76 + \alpha}{8} \Rightarrow \alpha = 4$$

MD

$$[|6 - 10| + |7 - 10| + |10 - 10| + |12 - 10| + |13 - 10| + |4 - 10|$$

$$(\bar{x}) = \frac{\quad + |12 - 10| + |16 - 10|}{8}$$

4. (d) Total numbers of balls

$$= (5 + 3 + 4) = 12$$

Three balls are drawn $= {}^{12}C_3 = 220$

Three balls are drawn of same colour $= {}^5C_3 + {}^3C_3 + {}^4C_3 = 10 + 1 + 4 = 15$

Probability of are not same colour

$$= 1 - \text{Probability of same colour}$$

$$= 1 - \frac{15}{220} = 1 - \frac{3}{44} = \frac{41}{44}$$

5. (d) Consider $S_1 - S_2 = 0$

and $S_2 - S_3 = 0$

6. (d) We have,

$$\operatorname{cosec} \theta - \cot \theta = 2017 \quad \dots (i)$$

$$\therefore \operatorname{cosec} \theta + \cot \theta = \frac{1}{2017} \quad \dots (ii)$$

Adding Eqs. (i) and (ii), we get

$$2 \operatorname{cosec} \theta = 2017 + \frac{1}{2017}$$

$$\Rightarrow \operatorname{cosec} \theta = \frac{1}{2} \left[2017 + \frac{1}{2017} \right] > 0$$

θ lie in Ist or IInd quadrant.

$$\text{we get } 2 \cot \theta = \frac{1}{2017} - 2017$$

$$\cot \theta = \frac{1}{2} \left(\frac{1}{2017} - 2017 \right) < 0$$

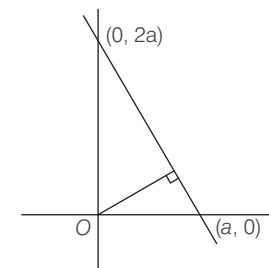
$\therefore \theta$ lie in IInd and IIIrd quadrant.

Hence, θ lies in IInd quadrant.

$$7. (c) \text{ Area of } \triangle PAB = \frac{1}{2} \begin{vmatrix} x & y & 1 \\ 5 & 3 & 1 \\ 3 & -2 & 1 \end{vmatrix}$$

8. (d) Let equation of line be

$$\frac{x}{a} + \frac{y}{2a} = 1$$



$$\Rightarrow 2x + y = 2a$$

Distance from origin is

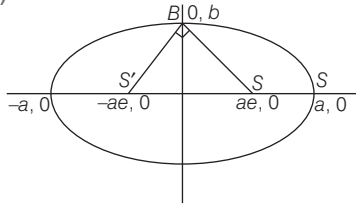
$$\left| \frac{2a}{\sqrt{(2)^2 + (1)^2}} \right| = \left| \frac{2a}{\sqrt{5}} \right|$$

$$\Rightarrow \left| \frac{2a}{\sqrt{5}} \right| = 1 \Rightarrow 2a = \pm \sqrt{5}$$

28 Solved Papers 2017

Hence, equation of line is
 $2x + y = \pm\sqrt{5}$

9. (a)



$$\therefore SS'^2 = SB^2 + S'B^2$$

$$\Rightarrow (2ae)^2 = b^2 + (ae)^2 + b^2 + (ae)^2$$

10. (b)

$$11. (c) \frac{x^2 + 5}{(x^2 + 1)(x - 2)} = \frac{A}{x - 2} + \frac{Bx + C}{x^2 + 1}$$

$$\Rightarrow x^2 + 5 = A(x^2 + 1)$$

$$+ (Bx + C)(x - 2)$$

$$\Rightarrow x^2 + 5 = Ax^2 + A + Bx^2$$

$$- 2Bx + Cx - 2C$$

$$1 = A + B, 0 = -2B + C, 5 = A - 2C$$

12. (b) We have,

Conjugate of $(x + iy)(1 - 2i)$ is $1 + i$.

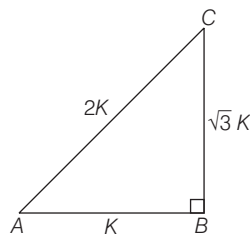
$$\text{i.e. } (x - iy)(1 + 2i) = 1 + i$$

$$\Rightarrow x - iy = \frac{1 + i}{1 + 2i}$$

Taking conjugate both sides, we get

$$x + iy = \frac{1 - i}{1 - 2i}$$

13. (a) Let the sides are $k, \sqrt{3}k, 2k$



Since, this triangle is a right angle triangle

$$(2k)^2 = (\sqrt{3}k)^2 + (k)^2 = (2k)^2$$

$$\therefore \sin A = \frac{\sqrt{3}k}{2k} = \frac{\sqrt{3}}{2} \Rightarrow A = 60^\circ$$

$$\Rightarrow \sin C = \frac{k}{2k} = \frac{1}{2} \Rightarrow C = 30^\circ$$

14. (a) We have,

$$(x - 1)^3 + 64 = 0$$

$$\Rightarrow (x - 1)^3 = -64$$

$$\Rightarrow (x - 1)^3 = (-4)^3$$

$$\Rightarrow x - 1 = -4, -4w, -4w^2$$

$$\Rightarrow x = -3, -4w + 1, -4w^2 + 1$$

Complex roots of the equations are
 $-4w + 1, -4w^2 + 1$

$$15. (b) \frac{2z + 1}{iz + 1} = \frac{2(x + iy) + 1}{i(x + iy) + 1}$$

$$= \frac{2x + 1 + iy}{(-y + 1) + ix}$$

$$= \frac{[(2x + 1) + iy][(-y + 1) - ix]}{[(-y + 1) + ix][(-y + 1) - ix]}$$

It is given that imaginary part of

$$\frac{2z + 1}{iz + 1} = -2$$

$$\therefore \frac{-y^2 + y - 2x^2 - x}{(-y + 1)^2 + x^2} = -2$$

16. (d)

17. (a) Let $x = a \cos \theta$, $y = b \sin \theta$

$$\therefore \frac{dx}{d\theta} = -a \sin \theta, \frac{dy}{d\theta} = b \cos \theta$$

$$\frac{dy}{dx} = -\frac{b}{a} \cot \theta$$

On differentiating w.r.t. θ , we get

$$\frac{d^2y}{dx^2} = -\frac{b}{a} (-\operatorname{cosec}^2 \theta) \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{b \operatorname{cosec}^2 \theta}{-a^2 \sin \theta}$$

$$18. (a) \text{ We have, } \lim_{y \rightarrow 1} \left(\frac{1}{y^2 - 1} - \frac{2}{y^4 - 1} \right)$$

$$= \lim_{y \rightarrow 1} \left(\frac{y^2 + 1 - 2}{y^4 - 1} \right)$$

$$= \lim_{y \rightarrow 1} \left(\frac{y^2 - 1}{(y^2 - 1)(y^2 + 1)} \right)$$

$$= \lim_{y \rightarrow 1} \frac{1}{y^2 + 1} = \frac{1}{1 + 1} = \frac{1}{2}$$

19. (b) We have,

$$T_{2r+1} = {}^{42}C_{2r} x^{2r} \text{ and } T_{r+1} = {}^{42}C_r x^r$$

20. (d) Let $f(x) = ax^2 + bx + c$

$$f(0) = c, f(1) = a + b + c$$

$$\Rightarrow f(0) + f(1) = 0$$

$$\Rightarrow c + a + b + c = 0$$

$$\Rightarrow a + b + 2c = 0 \quad \dots (i)$$

$$f(-2) = a(-2)^2 + b(-2) + c$$

$$0 = 4a - 2b + c$$

$$\Rightarrow 4a - 2b + c = 0 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$\frac{a}{5} = \frac{b}{7} = \frac{c}{-6}$$

$$\Rightarrow a = 5k, b = 7k, c = -6k$$

$$\therefore f(x) = k(5x^2 + 7x - 6)$$

$$\Rightarrow 5x^2 + 7x - 6 = 0$$

21. (b) Equations of normal to the

$$\text{hyperbola } \frac{x^2}{9} - \frac{y^2}{4} = 1 \text{ is}$$

$$\frac{9x}{x_1} + \frac{4y}{y_1} = 9 + 4$$

$$\Rightarrow \frac{9x}{x_1} + \frac{4y}{y_1} = 13$$

Since, line $x + y = k$ is normal to the given hyperbola

$$\frac{9}{1} = \frac{4}{1} = \frac{13}{k}$$

$$\Rightarrow \frac{9}{x_1} = \frac{4}{y_1} = \frac{13}{k}$$

$$\Rightarrow x_1 = \frac{9k}{13} \Rightarrow y_1 = \frac{4k}{13}$$

22. (b)

23. (a) Total number of ways of choosing such terms = ${}^{10}C_5 \cdot {}^5C_1$

24. (c) $\therefore$ Point of intersection of two lines is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} \right)$

25. (d) Let $S = \{2k \mid -9 \leq k \leq 10\} =$

$\{-18, -16, -14, \dots, 0, 2, 4, 6, \dots, 20\}$

Total number of possible outcomes = 20

Favourable outcomes are -12, 0 and 12

26. (c) $\therefore$ Sum of roots,

$$\Rightarrow \alpha + \beta = \frac{-2}{1} = -2 \quad \dots (i)$$

and Product of roots,

$$\Rightarrow \alpha\beta = \frac{c}{1} = c \quad \dots (ii)$$

$$\text{Since, } \alpha^3 + \beta^3 = 4 \quad [\text{given}]$$

$$\Rightarrow (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) = 4$$

$$\Rightarrow (\alpha + \beta)[(\alpha + \beta)^2 - 3\alpha\beta] = 4$$

27. (c)

		C		

If we fix C in the middle, then the rest 4 letters can be arranged in ${}^4P_4 = 4!$ ways = 24 ways.

28. (b) Given equation of curves are

$$x^2 = 8y \quad \dots (i)$$

and $xy = 8 \quad \dots (ii)$

$$\therefore x \left(\frac{x^2}{8} \right) = 8$$

$$\Rightarrow x^3 = 64 \Rightarrow x = 4 \Rightarrow y = 2$$

Thus, the point of intersection of given curves is (4, 2).

Now, let m_1 be the slope of tangent to the curve (i) at point (4, 2) and m_2 be the slope of tangent to the curve (ii) at point (4, 2).

$$\text{Then, } m_1 = \frac{4}{4} = 1$$

$$\left[\because x^2 = 8y \Rightarrow 2x = 8 \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x}{4} \right]$$

$$\text{and } m_2 = \frac{-2}{4} = -\frac{1}{2}$$

$$\left[\because xy = 8 \Rightarrow x \frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{-y}{x} \right]$$

29. (c) Given, ABCD is a parallelogram with vertices A (4, 4, -1), B(5, 6, -1), C(6, 5, 1) and D(x, y, z).

We know that diagonals of parallelogram ABCD bisect each other.

$\therefore$ Mid-point of AC = Mid-Point of BD

$$\Rightarrow \left(\frac{10}{2}, \frac{9}{2}, 0 \right) = \left(\frac{x+5}{2}, \frac{y+6}{2}, \frac{z-1}{2} \right)$$

On comparing both sides, we get

$$\frac{x+5}{2} = \frac{10}{2}, \frac{y+6}{2} = \frac{9}{2}$$

$$\text{and } \frac{z-1}{2} = 0$$

$$30. (c) \therefore \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Now, the point of intersection of given lines is given by

$$\left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right) =$$

31. (a) For removal of first degree terms, shift the origin to $\left(\frac{bg - fh}{h^2 - ab}, \frac{af - gh}{h^2 - ab} \right)$
- $$= \left(\frac{-40 + 88}{16 - 40}, \frac{-88 + 16}{16 - 40} \right)$$

32. (c) Let S_1 be the circle with centre (2, 3) and radius a.

Let S_2 be the circle with centre (5, 6) and radius a.

$$\text{Then } S_1 \equiv (x-2)^2 + (y-3)^2 - a^2 = 0$$

$$\text{and } S_2 \equiv (x-5)^2 + (y-6)^2 - a^2 = 0$$

Now, radical axis, $S_1 - S_2 = 0$

$$(x-2)^2 + (y-3)^2 - a^2 - (x-5)^2 - (y-6)^2 + a^2 = 0$$

$$\Rightarrow (4-4x+9-6y) - (25-10x) - (36-12y) = 0$$

$$\Rightarrow x + y = 8 \quad \dots (i)$$

Since, the given circle cut orthogonally, therefore

$$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

$$\Rightarrow 2(-2)(-5) + 2(-3)(-6) = ((-2)^2 + (-3)^2 - a^2) + ((-5)^2 + (-6)^2 - a^2)$$

$$\therefore a^2 = g^2 + f^2 - c$$

$$\Rightarrow c = g^2 + f^2 - a^2$$

$$\Rightarrow 20 + 36 = (13 - a^2) + (61 - a^2)$$

$$\Rightarrow 56 = 74 - 2a^2$$

$$\Rightarrow a^2 = 9 \Rightarrow a = 3$$

33. (b) We have, $\tan \theta_1 = k \cot \theta_2$

$$\Rightarrow \tan \theta_1 \tan \theta_2 = k \quad \dots (i)$$

$$\text{Consider, } \frac{\cos(\theta_1 + \theta_2)}{\cos(\theta_1 - \theta_2)}$$

$$= \frac{\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2}{\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2}$$

$$= \frac{1 - \tan \theta_1 \tan \theta_2}{1 + \tan \theta_1 \tan \theta_2} = \frac{1 - k}{1 + k}$$

34. (d) $T_{r+1} = {}^nC_r (1)^{n-r} x^r = {}^nC_r x^r$

$\therefore$ Coefficient of (r + 1)th term is ${}^nC_r$

Similarly, coefficient of pth term = ${}^nC_{p-1}$

$$\therefore p = {}^nC_{p-1} \text{ (given)}$$

$$p = \frac{n!}{(p-1)!(n-p+1)!} \quad \dots (i)$$

and coefficient of (p + 1)th term = ${}^nC_p$

$$\therefore q = {}^nC_p \text{ (given)} \quad \dots (ii)$$

On dividing Eq. (i) by Eq. (ii), we get

$$\frac{p}{q} = \frac{(p-1)!(n-p+1)!}{{}^nC_p}$$

$$= \frac{n!}{(p-1)!(n-p+1)!} \cdot \frac{p!(n-p)!}{n!}$$

$$= \frac{p!}{(p-1)!(n-p)!}$$

35. (b) Consider,

$$\sum_{k=1}^n k(k+2) = \sum_{k=1}^n k^2 + 2 \sum_{k=1}^n k$$

$$= \frac{n(n+1)(2n+1)}{6} + 2 \frac{n(n+1)}{2}$$

36. (b) Given equations of ellipse can be rewritten as

$$\Rightarrow \frac{(x+2)^2}{(1/5)^2} + \frac{(y-1/2)^2}{(1/2)^2} = 1$$

Here, $a = \frac{1}{5}$, $b = \frac{1}{2}$ and major axis of ellipse $x + 2 = 0$ i.e. $x = -2$.

$$\text{Now, } e = \sqrt{1 - \frac{a^2}{b^2}} = \sqrt{1 - \frac{4}{25}}$$

37. (c) Consider,

$$\left[\frac{\left(1 + \cos \frac{\pi}{12} \right) + i \sin \frac{\pi}{12}}{\left(1 + \cos \frac{\pi}{12} \right) - i \sin \frac{\pi}{12}} \right]^{72}$$

$$= \left[\frac{2 \cos^2 \frac{\pi}{24} + i 2 \sin \frac{\pi}{24} \cos \frac{\pi}{24}}{2 \cos^2 \frac{\pi}{24} - i 2 \sin \frac{\pi}{24} \cos \frac{\pi}{24}} \right]^{72}$$

$$= \left[\frac{2 \cos \frac{\pi}{24} \left(\cos \frac{\pi}{24} + i \sin \frac{\pi}{24} \right)}{2 \cos \frac{\pi}{24} \left(\cos \frac{\pi}{24} - i \sin \frac{\pi}{24} \right)} \right]^{72}$$

$$= \left(\frac{\cos \frac{\pi}{24} + i \sin \frac{\pi}{24}}{\cos \frac{\pi}{24} - i \sin \frac{\pi}{24}} \right)^{72}$$

$$\times \left(\frac{\cos \frac{\pi}{24} + i \sin \frac{\pi}{24}}{\cos \frac{\pi}{24} + i \sin \frac{\pi}{24}} \right)^{72}$$

$$= \left(\frac{\left(\cos \frac{\pi}{24} + i \sin \frac{\pi}{24} \right)^2}{\cos^2 \frac{\pi}{24} - i^2 \sin^2 \frac{\pi}{24}} \right)^{72}$$

$$= \left(\frac{\left(\cos \frac{\pi}{24} + i \sin \frac{\pi}{24} \right)^2}{\cos^2 \frac{\pi}{24} + \sin^2 \frac{\pi}{24}} \right)^{72}$$

$$= \left(\cos \frac{\pi}{24} + i \sin \frac{\pi}{24} \right)^{144}$$

$$= \cos \left(\frac{\pi}{24} \times 144 \right) + i \sin \left(\frac{\pi}{24} \times 144 \right)$$

$$= \cos 6\pi + i \sin 6\pi = 1$$

$$\left[\because \sin n\pi = 0 \forall n \in \mathbb{Z} \right.$$

$$\left. \text{and } \cos n\pi = \begin{cases} 1, & \text{if } n \text{ is even} \\ -1, & \text{if } n \text{ is odd} \end{cases} \right]$$

38. (a)

39. (a) Clearly $\Delta = \frac{1}{2} \cdot a \cdot b \sin C$

30 Solved Papers 2017

$$\Rightarrow \Delta = \frac{1}{2} \cdot 2 \cdot 1 \cdot \sin 60^\circ$$

$$\Rightarrow \Delta = \frac{\sqrt{3}}{2}$$

$$\Rightarrow 4\Delta^2 = 3 \quad \dots (i)$$

By sine rule, we get

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

$$\Rightarrow \frac{\sin A}{1} = \frac{\sin B}{2} = \frac{\sin 60^\circ}{c}$$

$$\Rightarrow \frac{\sin A}{\sin B} = \frac{1}{2} = \frac{\sin 60^\circ}{c}$$

Considering last two terms, we get

$$\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{2}{c}$$

$$\Rightarrow c = \sqrt{3} \quad \dots (ii)$$

Thus, $4\Delta^2 + c^2 = 3 + 3 = 6$ [(Using Eqs. (i) and (iii))]

40. (c) Equations whose roots are $\frac{1-\alpha}{\alpha}$

and $\frac{1-\beta}{\beta}$ are

$$x^2 - \left(\frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} \right)x + \left(\frac{1-\alpha}{\alpha} \right) \left(\frac{1-\beta}{\beta} \right) = 0$$

$$\Rightarrow x^2 - \left(\frac{\alpha + \beta - 2\alpha\beta}{\alpha\beta} \right)x + \left(\frac{1 - (\alpha + \beta) + \alpha\beta}{\alpha\beta} \right) = 0$$

$$\Rightarrow \alpha\beta x^2 - (\alpha + \beta - 2\alpha\beta)x + (1 - (\alpha + \beta) + \alpha\beta) = 0$$

$$\Rightarrow \frac{c}{a}x^2 - \left(-\frac{b}{a} - \frac{2c}{a} \right)x + \left(1 + \frac{b}{a} + \frac{c}{a} \right) = 0$$

$$\Rightarrow cx^2 + (b + 2c)x + (a + b + c) = 0$$

Comparing with $px^2 + qx + r = 0$, we get

$$r = a + b + c$$

41. (a) Parametric equations of given circle is $x = 12\cos\theta$, $y = 12\sin\theta$

Now, coordinates of point A are given by

$$x = 12\cos\frac{\pi}{3}, y = 12\sin\frac{\pi}{3}$$

$$\Rightarrow x = 12 \cdot \frac{1}{2}, y = 12 \cdot \frac{\sqrt{3}}{2}$$

$$\Rightarrow x = 6; y = 6\sqrt{3}$$

$$\text{i.e. } A \equiv (6, 6\sqrt{3})$$

and coordinates of point B are given by

$$x = 12\cos\frac{\pi}{6}, y = 12\sin\frac{\pi}{6}$$

$$\Rightarrow x = 12 \cdot \frac{\sqrt{3}}{2}, y = 12 \cdot \frac{1}{2}$$

$$\Rightarrow x = 6\sqrt{3}, y = 6$$

$$\text{i.e. } B \equiv (6\sqrt{3}, 6)$$

Clearly, length of chord

$$AB = \sqrt{(6\sqrt{3} - 6)^2 + (6 - 6\sqrt{3})^2}$$

42. (b) Given equations of pair of straight lines, $xy - x - y + 1 = 0$ can be rewritten as

$$x(y - 1) - 1(y - 1) = 0$$

$$\Rightarrow (x - 1)(y - 1) = 0$$

$$\Rightarrow x = 1 \text{ and } y = 1$$

Thus, the three concurrent lines are

$$x = 1 \quad \dots (i)$$

$$y = 1 \quad \dots (ii)$$

$$\text{and } x + ay - 3 = 0 \quad \dots (iii)$$

Since, Eqs. (i) and (ii), intersect at only point, namely (1, 1), therefore this point also satisfy the Eq. (iii), we get

$$1 + a - 3 = 0$$

$$\Rightarrow a = 2$$

Now, the pair of lines

$$ax^2 - 13xy - 7y^2 + x + 23y - 6 = 0$$

becomes

$$2x^2 - 13xy - 7y^2 + x + 23y - 6 = 0$$

Also $\tan\theta =$

$$= \frac{2\sqrt{\left(-\frac{13}{2}\right)^2 + 14}}{-5} = \frac{2\sqrt{\frac{169}{4} + 56}}{-5}$$

43. (b) Consider the equations

$$\cos 2\theta = \sin\theta$$

$$\Rightarrow 1 - 2\sin^2\theta = \sin\theta$$

$$\Rightarrow 2\sin^2\theta + \sin\theta - 1 = 0$$

$$\Rightarrow (\sin\theta + 1)(2\sin\theta - 1) = 0$$

$$\Rightarrow \sin\theta = -1 \text{ or } \sin\theta = \frac{1}{2}$$

$$\Rightarrow \theta = \frac{3\pi}{2} \text{ or } \theta = \frac{\pi}{6}, \frac{5\pi}{6} \quad [\because \theta \in (0, 2\pi)]$$

Thus, number of solutions of the given equation is 3.

44. (d) $\because \Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$$\text{Also } R = \frac{abc}{4\Delta} \text{ and } r = \frac{\Delta}{s}$$

45. (a) $\because \sigma^2 = \frac{d^2(n^2 - 1)}{12}$

46. (c) Since $x - y = -4k$ or $y = x + 4k$ is a tangent to the parabola $y^2 = 8x$, therefore

$$4k = \frac{2}{1} \Rightarrow k = \frac{1}{2}$$

Also point of contact p is

$$\left(\frac{a}{m^2}, \frac{2a}{m} \right) = (2, 4)$$

Now, equation of normal to the $y^2 = 8x$ at $(2, 4)$ is

$$(y - 4) = \frac{-4}{2(2)}(x - 2)$$

47. (d) $P(\text{neither } A \text{ nor } B) = P(\overline{A} \cap \overline{B})$
 $= P(\overline{A \cup B})$

$$= 1 - P(A \cup B)$$

$$= 1 - P(A) - P(B) + P(A \cap B)$$

$$= 1 - 0.6 - 0.4 + 0 = 1 - 1 = 0$$

VIT

1. (b) We have, $(\sin 40^\circ + i \cos 40^\circ)^5$

$$= i^5 (\cos 40^\circ - i \sin 40^\circ)^5$$

$$= i^5 (\cos 200^\circ - i \sin 200^\circ)$$

$$= i [\cos (270^\circ - 70^\circ) - i \sin (270^\circ - 70^\circ)]$$

$$= i [-\sin 70^\circ + i \cos 70^\circ]$$

$$= i [-\sin 110^\circ - i \cos 110^\circ]$$

$$= -i \sin 110^\circ + \cos 110^\circ$$

$$= \cos 110^\circ - i \sin 110^\circ$$

$$= \cos (-110^\circ) + i \sin (-110^\circ)$$

$\therefore$ Principal amplitude of

$$(\sin 40^\circ + i \cos 40^\circ)^5 \text{ is } -110^\circ.$$

2. (b) Here, $1 + \omega + \omega^2 = 0$

$$\text{Here, } \begin{vmatrix} z + 1 + \omega + \omega^2 & \omega & \omega^2 \\ z + 1 + \omega + \omega^2 & z + \omega^2 & 1 \\ z + 1 + \omega + \omega^2 & 1 & z + \omega \end{vmatrix}$$

$$= 0$$

[Applying $C_1 \rightarrow C_1 + C_2 + C_3$]

$$\Rightarrow \begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z + \omega^2 & 1 \\ 1 & 1 & z + \omega \end{vmatrix} = 0$$

$$\Rightarrow z \begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z + \omega^2 - \omega & 1 - \omega^2 \\ 0 & 1 - \omega & z + \omega - \omega^2 \end{vmatrix} = 0$$

[Applying $R_2 \rightarrow R_2 - R_1$,
 $R_3 \rightarrow R_3 - R_1$]

$$\Rightarrow z\{z^2 - (\omega - \omega^2)^2 - (1 - \omega)(1 - \omega^2)\} = 0$$

$$\Rightarrow z^3 = 0$$

$$\Rightarrow z = 0$$

3. (b) We have,

$$f(x) = \frac{x-1}{x+1}$$

$$\Rightarrow \frac{f(x)+1}{f(x)-1} = \frac{2x}{-2}$$

$$\Rightarrow x = \frac{f(x)+1}{1-f(x)}$$

$$\therefore f(2x) = \frac{2x-1}{2x+1}$$

$$= \frac{2\left\{\frac{f(x)+1}{1-f(x)}\right\}-1}{2\left\{\frac{f(x)+1}{1-f(x)}\right\}+1}$$

$$= \frac{3f(x)+1}{f(x)+3}$$

4. (a) Here $B \cap C = \{4\}$

$$\therefore A \times (B \cap C) = \{2, 3\} \times \{4\}$$

$$= \{(2, 4), (3, 4)\}$$

5. (c) We know that,

$$(E \times F) \cap (C \times D) = (E \cap C) \times (F \cap D)$$

$$\therefore (A \times B) \cap (B \times A) = (A \cap B) \times (B \cap A)$$

$$\Rightarrow (A \times B) \cap (B \times A) = (A \cap B) \times (A \cap B)$$

It is given that $A \cap B$ has n elements.

$\therefore (A \cap B) \times (A \cap B)$ has n^2 elements.

But, $(A \times B) \cap (B \times A)$

$$= (A \cap B) \times (A \cap B)$$

$\therefore (A \times B) \cap (B \times A)$ has n^2 elements in common.

6. (c) Here, $2ae$ = Distance between foci

$$\Rightarrow 2ae = 17$$

$$\Rightarrow ae = 17/2$$

We know that, the product of lengths of perpendicular from two foci on any tangent to a hyperbola is b^2 .

Since, given hyperbola touches Y-axis i.e., $x = 0$

$$\therefore b^2 = 50$$

$$\Rightarrow a^2(e^2 - 1) = 50$$

$$\Rightarrow \frac{289}{4} - a^2 = 50$$

7. (c) Let $p(at^2, 2at)$ be a moving point on the parabola $y^2 = 4ax$ and let $S(a, 0)$ be its focus.

Let $Q(h, k)$ be the mid-point of PS . Then,

$$2h = at^2 + a$$

and $2k = 2at$

$$\Rightarrow 2h = a\left(\frac{k}{a}\right)^2 + a$$

$$\Rightarrow k^2 = 2a(h - a/2)$$

Hence, the locus of (h, k) is

$$y^2 = 2a(x - a/2).$$

The equation of the directrix of this parabola is $x - a/2 = -a/2$

$$\Rightarrow x = 0$$

8. (a) Let the equation of the ellipse be

$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1 \quad \dots(i)$$

It passes through $(4, 6)$

$$\therefore \frac{9}{a^2} + \frac{16}{b^2} = 1 \quad \dots(ii)$$

Let e be the eccentricity of the ellipse. Then, ae = Distance between $(1, 2)$ and $(6, 2)$

$$\Rightarrow ae = 5$$

$$\Rightarrow a^2 e^2 = 25$$

$$\Rightarrow a^2 - b^2 = 25$$

$$\Rightarrow a^2 = 25 + b^2 \quad \dots(iii)$$

9. (d) The coordinates of the extremity of the latusrectum which lies in the first quadrant are $(ae, b^2/a)$.

The equation of the normal at (x_1, y_1) is

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = (a^2 - b^2)$$

Therefore, the equation of the normal at $(ae, b^2/a)$ is

$$\frac{a^2 x}{ae} - \frac{b^2 y}{b^2/a} = a^2 - b^2$$

$$\Rightarrow ax - aey = e(a^2 - b^2)$$

$$\Rightarrow ax - aey = ea^2 e^2$$

$$\Rightarrow x - ey = ae^2$$

This passes through the extremity of the minor axis i.e., $(0, -b)$

$$\therefore 0 + eb - ae^3 = 0$$

10. (d) We have, $\lim_{x \rightarrow \alpha} (\tan x \cot \alpha)^{\frac{1}{x-\alpha}}$

$$= \lim_{x \rightarrow \alpha} \{1 + (\tan x \cot \alpha - 1)\}^{\frac{1}{x-\alpha}}$$

$$= e^{\lim_{x \rightarrow \alpha} \frac{(\tan x \cot \alpha - 1)}{x - \alpha}}$$

$$= e^{\lim_{x \rightarrow \alpha} \frac{\tan x - \tan \alpha}{x - \alpha} \times \frac{1}{\tan \alpha}}$$

$$= e^{\sec^2 \alpha / \tan \alpha} = e^{2 \csc 2\alpha}$$

MHT CET

1. (b)

2. (d) Let z_1 and z_2 be the z-coordinates of points P and Q , respectively which trisect the line segment AB .

$$(2, 1, 4) \quad A \quad P \quad Q \quad B(-1, 3, 6)$$

Now, coordinates of

$$P = \left(\frac{1 \times -1 + 2 \times 2}{1+2}, \frac{1 \times 3 + 2 \times 1}{1+2}, \frac{1 \times 6 + 2 \times 4}{1+2} \right)$$

$$\therefore \text{z-coordinate of } P = \frac{1 \times 6 + 2 \times 4}{1+2}$$

$$\Rightarrow z_1 = \frac{6+8}{3} = \frac{14}{3} \quad \dots(i)$$

Again, coordinates of

$$Q = \left(\frac{2 \times -1 + 1 \times 2}{2+1}, \frac{2 \times 3 + 1 \times 1}{2+1}, \frac{2 \times 6 + 1 \times 4}{2+1} \right)$$

$$\therefore \text{z-coordinate of } Q = \frac{2 \times 6 + 1 \times 4}{2+1}$$

$$\Rightarrow z_2 = \frac{12+4}{3} = \frac{16}{3} \quad \dots(ii)$$

On adding Eqs. (i) and (ii), we get

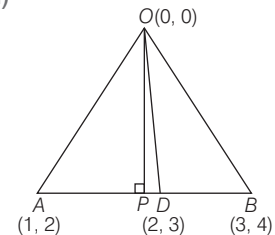
$$z_1 + z_2 = \frac{14}{3} + \frac{16}{3} = \frac{30}{3} = 10$$

3. (b)

p	q	$\sim p$	$p \vee q$	$p \wedge q$	$\sim p \wedge q$	$(p \vee q) \vee \sim p$	$(p \vee q) \wedge \sim p$
T	T	F	T	T	F	T	F
T	F	T	T	F	T	T	F
F	T	T	T	F	T	T	T
F	F	T	F	F	T	T	F

$$\therefore (\sim p \wedge q) \equiv (p \vee q) \wedge \sim p$$

4. (d)



Given, $O(0,0)$, $A(1,2)$ and $B(3,4)$ be the vertices of $\triangle OAB$.

32 Solved Papers 2017

Let OP and OD be the altitude and median of ΔOAB , respectively.

$$\therefore \text{Coordinates of } D = \text{Mid of } AB \\ = \left(\frac{1+3}{2}, \frac{2+4}{2} \right) = \left(\frac{4}{2}, \frac{6}{2} \right) = (2, 3)$$

Now, equation of OD is $(y-0)$

$$= \left(\frac{3-0}{2-0} \right) (x-0)$$

$$\Rightarrow y = \frac{3}{2}x$$

$$\Rightarrow 2y = 3x \Rightarrow 3x - 2y = 0$$

$$\text{Slope of } OP = \frac{-1}{\text{Slope of } AB} \quad [\because OP \perp AB]$$

$$= \frac{-1}{\left(\frac{3-1}{4-2} \right)} = \frac{-1}{\frac{2}{2}} = -1$$

$\therefore$ Equation of OP is $(y-0) = -1(x-0)$

$$\Rightarrow y = -x \Rightarrow x + y = 0$$

Now, joint equation of OP and OD

$$(x+y)(3x-2y) = 0$$

$$\Rightarrow 3x^2 - 2xy + 3xy - 2y^2 = 0$$

$$\Rightarrow 3x^2 + xy - 2y^2 = 0$$

5. (c) Given, $\sin^2 A + \sin^2 B = \sin^2 C$

$$\Rightarrow a^2 + b^2 = c^2$$

$\therefore \Delta ABC$ is right angled triangle and right angled at C .

$$\text{Now, area of } (\Delta ACB) = \frac{1}{2}ab \quad \dots(i)$$

From sine rule's, we have

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{10}{1}$$

$$\Rightarrow a = 10 \sin A$$

$$b = 10 \sin B$$

On substituting $a=10\sin A$ and $b=10\sin B$ in Eq. (i), we get

$$\begin{aligned} \text{Area of } \Delta ACB &= \frac{1}{2}(10 \sin A)(10 \sin B) \\ &= 50 \sin A \sin B \end{aligned} \quad \dots(ii)$$

But maximum value of

$$\sin A \sin B = \frac{1}{2} \quad \dots(iii)$$

$\therefore$ Maximum value of area of ΔACB

$$= 50 \times \frac{1}{2} = 25 \quad [\text{from Eqs. (ii) and (iii)}]$$

6. (c) Let X be a random variable. That denotes the number of heads in tossing a coin 3 times. Then, X can take value 0, 1, 2, 3.

Again, let $y = \frac{x}{2}$. Then, Y can take the values

0, 1, 2, 3 and 4

$$\therefore P(y=0) = P(\text{getting 0 head}) = \frac{1}{8}$$

$$P(y=2) = P(\text{getting 1 head}) = \frac{3}{8}$$

$$P(y=4) = P(\text{getting 2 heads}) = \frac{3}{8}$$

$$P(y=6) = P(\text{getting 3 heads}) = \frac{1}{8}$$

7. (c)

p	q	$\sim p$	$\sim q$	$\sim p \leftrightarrow q$	$q \rightarrow p$	$\sim q \rightarrow \sim p$	$p \vee (q \rightarrow p)$	$p \wedge \sim p$	$(q \rightarrow p) \vee (\sim p \leftrightarrow q)$
T	T	F	F	F	T	F	T	F	T
T	F	F	T	T	T	F	T	F	T
F	T	T	F	T	F	T	T	F	T
F	F	T	T	F	T	T	F	F	T

8. (a) We know that, slopes of $ax^2 + 2hxy + by^2 = 0$ are real and distinct if and only if $h^2 - ab > 0$

$$\Rightarrow 0 + pq > 0$$

$$\Rightarrow pq > 0$$

9. (c) Here, $m_1 + m_2 = \frac{-2h}{b} = -5$

$$\text{and } m_1 m_2 = \frac{a}{b} = k$$

$$\text{Now, } (m_1 - m_2)^2 = (m_1 + m_2)^2 - 4m_1 m_2$$

10. (a) For duality, replace ' $\vee$ ' by ' $\wedge$ ' and ' $\wedge$ ' by ' $\vee$ ', we get

$$\text{Dual of the statement } \sim p \wedge (q \vee c)$$

$$\equiv \sim p \vee (q \wedge t)$$

11. (b) Given probability that a person will develop immunity after vaccination = 0.8

$$\therefore \text{Probability of 8 persons that all develop immunity} = (0.8)^8$$

Engineering Entrances Solved Papers 2018

**JEE Main & Advanced/BITSAT/KCET/Andhra Pradesh
& Telangana State (EAMCET)/VIT/MHT CET**

JEE Main

1. Two sets A and B are as under

$$A = \{(a, b) \in \mathbf{R} \times \mathbf{R} : |a - 5| < 1 \text{ and } |b - 5| < 1\};$$

$$B = \{(a, b) \in \mathbf{R} \times \mathbf{R} : 4(a - 6)^2 + 9(b - 5)^2 \leq 36\}. \text{ Then,}$$

- (a) $B \subset A$ (b) $A \subset B$
(c) $A \cap B = \emptyset$ (an empty set) (d) neither $A \subset B$ nor $B \subset A$

2. Let $S = \{x \in \mathbf{R} : x \geq 0 \text{ and}$

$$2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0\}. \text{ Then, } S$$

- (a) is an empty set
(b) contains exactly one element
(c) contains exactly two elements
(d) contains exactly four elements

3. If $\alpha, \beta \in \mathbf{C}$ are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to

- (a) -1 (b) 0 (c) 1 (d) 2

4. From 6 different novels and 3 different dictionaries, 4 novels and 1 dictionary are to be selected and arranged in a row on a shelf, so that the dictionary is always in the middle. The number of such arrangements is

- (a) at least 1000 (b) less than 500
(c) at least 500 but less than 750
(d) at least 750 but less than 1000

5. The sum of the coefficients of all odd degree terms in the expansion of $\left(x + \sqrt{x^3 - 1}\right)^5 + \left(x - \sqrt{x^3 - 1}\right)^5$,

($x > 1$) is

- (a) -1 (b) 0 (c) 1 (d) 2

6. Let $a_1, a_2, a_3, \dots, a_{49}$ be in AP such that $\sum_{k=0}^{12} a_{4k+1} = 416$

and $a_9 + a_{43} = 66$. If $a_1^2 + a_2^2 + \dots + a_{17}^2 = 140m$, then m is equal to

- (a) 66 (b) 68 (c) 34 (d) 33

7. Let A be the sum of the first 20 terms and B be the sum of the first 40 terms of the series

$$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots$$

If $B - 2A = 100\lambda$, then λ is equal to

- (a) 232 (b) 248 (c) 464 (d) 496

8. For each $t \in \mathbf{R}$, let $[t]$ be the greatest integer less than or equal to t . Then,

$$\lim_{x \rightarrow 0^+} x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{15}{x} \right] \right)$$

- (a) is equal to 0 (b) is equal to 15
(c) is equal to 120 (d) does not exist (in $\mathbf{R}$)

9. The integral

$$\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$$

is equal to

- (a) $\frac{1}{3(1 + \tan^3 x)} + C$ (b) $\frac{-1}{3(1 + \tan^3 x)} + C$
(c) $\frac{1}{1 + \cot^3 x} + C$ (d) $\frac{-1}{1 + \cot^3 x} + C$

(where C is a constant of integration)

10. The value of $\int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1 + 2^x} dx$ is

- (a) $\frac{\pi}{8}$ (b) $\frac{\pi}{2}$ (c) 4π (d) $\frac{\pi}{4}$

11. A straight line through a fixed point $(2, 3)$ intersects the coordinate axes at distinct points P and Q . If O is the origin and the rectangle $OPRQ$ is completed, then the locus of R is

- (a) $3x + 2y = 6$ (b) $2x + 3y = xy$
(c) $3x + 2y = xy$ (d) $3x + 2y = 6xy$

12. Let the orthocentre and centroid of a triangle be $A(-3, 5)$ and $B(3, 3)$, respectively. If C is the circumcentre of this triangle, then the radius of the circle having line segment AC as diameter, is

- (a) $\sqrt{10}$ (b) $2\sqrt{10}$
(c) $3\sqrt{\frac{5}{2}}$ (d) $\frac{3\sqrt{5}}{2}$

13. If the tangent at $(1, 7)$ to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$, then the value of c is

- (a) 195 (b) 185 (c) 85 (d) 95

2 Engineering Entrances Solved Papers 2018

14. Tangent and normal are drawn at $P(16, 16)$ on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B , respectively. If C is the centre of the circle through the points P , A and B and $\angle CPB = \theta$, then a value of $\tan \theta$ is
 (a) $\frac{1}{2}$ (b) 2 (c) 3 (d) $\frac{4}{3}$
15. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the points P and Q . If these tangents intersect at the point $T(0, 3)$, then the area (in sq units) of ΔPTQ is
 (a) $45\sqrt{5}$ (b) $54\sqrt{3}$
 (c) $60\sqrt{3}$ (d) $36\sqrt{5}$
16. If $\sum_{i=1}^9 (x_i - 5) = 9$ and $\sum_{i=1}^9 (x_i - 5)^2 = 45$, then the standard deviation of the 9 items $x_1, x_2, \dots, x_9$ is
 (a) 9 (b) 4 (c) 2 (d) 3

17. If sum of all the solutions of the equation

$$8 \cos x \cdot \left(\cos \left(\frac{\pi}{6} + x \right) \cdot \cos \left(\frac{\pi}{6} - x \right) - \frac{1}{2} \right)$$

$= 1$ in $[0, \pi]$ is $k\pi$, then k is equal to

- (a) $\frac{2}{3}$ (b) $\frac{13}{9}$ (c) $\frac{8}{9}$ (d) $\frac{20}{9}$

18. PQR is a triangular park with $PQ = PR = 200$ m. A TV tower stands at the mid-point of QR . If the angles of elevation of the top of the tower at P, Q and R are respectively $45^\circ, 30^\circ$ and 30° , then the height of the tower (in m) is
 (a) 100 (b) 50 (c) $100\sqrt{3}$ (d) $50\sqrt{2}$
19. The boolean expression $\sim(p \vee q) \vee (\sim p \wedge q)$ is equivalent to
 (a) $\sim p$ (b) p (c) q (d) $\sim q$

JEE Advanced

Paper 1

1. For a non-zero complex number z , let $\arg(z)$ denote the principal argument with $-\pi < \arg(z) \leq \pi$. Then, which of the following statement(s) is (are) FALSE?

(One or More than One Option Correct)

- (a) $\arg(-1 - i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$
- (b) The function $f: R \rightarrow (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in R$, is continuous at all points of R , where $i = \sqrt{-1}$.
- (c) For any two non-zero complex numbers z_1 and z_2 , $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π .
- (d) For any three given distinct complex numbers z_1, z_2 and z_3 , the locus of the point z satisfying the condition $\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$, lies on a straight line.

2. In a ΔPQR , let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) $\angle QPR = 45^\circ$
- (b) The area of the ΔPQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
- (c) The radius of the incircle of the ΔPQR is $10\sqrt{3} - 15$
- (d) The area of the circumcircle of the ΔPQR is 100 π

3. The value of $((\log_2 9)^2)^{\frac{1}{\log_2(\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}}$ is

(Numerical Value Based Question)

4. The number of 5 digit numbers which are divisible by 4, with digits from the set $\{1, 2, 3, 4, 5\}$ and the repetition of digits is allowed, is

(Numerical Value Based Question)

5. Let X be the set consisting of the first 2018 terms of the arithmetic progression 1, 6, 11, ..., and Y be the set consisting of the first 2018 terms of the arithmetic progression 9, 16, 23, ... Then, the number of elements in the set $X \cup Y$ is

(Numerical Value Based Question)

6. For each positive integer n , let

$$y_n = \frac{1}{n} ((n+1)(n+2) \dots (n+n))^{1/n}.$$

For $x \in R$, let $[x]$ be the greatest integer less than or equal to x . If $\lim_{n \rightarrow \infty} y_n = L$, then the value of $[L]$ is

(Numerical Value Based Question)

7. Let a, b, c be three non-zero real numbers such that the equation $\sqrt{3} a \cos x + 2b \sin x = c, x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, has

two distinct real roots α and β with $\alpha + \beta = \frac{\pi}{3}$. Then, the value of $\frac{b}{a}$ is

(Numerical Value Based Question)

Paragraph X (Q. Nos. 8-9)

Let S be the circle in the XY -plane defined by the equation $x^2 + y^2 = 4$.

(Paragraph Based Questions)

8. Let E_1E_2 and F_1F_2 be the chords of S passing through the point $P_0(1, 1)$ and parallel to the X -axis and the Y -axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slope -1 . Let the tangents to S at E_1 and E_2 meet at E_3 , then tangents to S at F_1 and F_2 meet at F_3 , and the tangents to S at G_1 and G_2 meet at G_3 . Then, the points E_3, F_3 and G_3 lie on the curve

- (a) $x + y = 4$ (b) $(x - 4)^2 + (y - 4)^2 = 16$
 (c) $(x - 4)(y - 4) = 4$ (d) $xy = 4$

Engineering Entrances Solved Papers 2018 **3**

9. Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N . Then, the mid-point of the line segment MN must lie on the curve

- (a) $(x + y)^2 = 3xy$ (b) $x^{2/3} + y^{2/3} = 2^{4/3}$
(c) $x^2 + y^2 = 2xy$ (d) $x^2 + y^2 = x^2y^2$

Paragraph A (Q. Nos. 10-11)

There are five students S_1, S_2, S_3, S_4 and S_5 in a music class and for them there are five seats R_1, R_2, R_3, R_4 and R_5 arranged in a row, where initially the seat R_i is allotted to the student $S_i, i = 1, 2, 3, 4, 5$. But, on the examination day, the five students are randomly allotted the five seats.

(Paragraph Based Questions)

10. The probability that, on the examination day, the student S_1 gets the previously allotted seat R_1 , and NONE of the remaining students gets the seat previously allotted to him/her is

- (a) $\frac{3}{40}$ (b) $\frac{1}{8}$ (c) $\frac{7}{40}$ (d) $\frac{1}{5}$

11. For $i = 1, 2, 3, 4$, let T_i denote the event that the students S_i and S_{i+1} do NOT sit adjacent to each other on the day of the examination. Then, the probability of the event $T_1 \cap T_2 \cap T_3 \cap T_4$ is

- (a) $\frac{1}{15}$ (b) $\frac{1}{10}$ (c) $\frac{7}{60}$ (d) $\frac{1}{5}$

Paper 2

12. Let T be the line passing through the points $P(-2, 7)$ and $Q(2, -5)$. Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangent to S_1 at P and tangent to S_2 at Q , and also such that S_1 and S_2 touch each other at a point, say M . Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1 . Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point $R(1, 1)$ be F_2 . Let E_2 be the set of the mid-points of the line segments in the set F_2 . Then, which of the following statement(s) is (are) TRUE? (One or More than One Option Correct)

- (a) The point $(-2, 7)$ lies in E_1
(b) The point $\left(\frac{4}{5}, \frac{7}{5}\right)$ does NOT lie in E_2
(c) The point $\left(\frac{1}{2}, 1\right)$ lies in E_2
(d) The point $\left(0, \frac{3}{2}\right)$ does NOT lie in E_1

13. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = (1/2)$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q . Consider the ellipse whose centre is at the origin $O(0, 0)$ and whose semi-major axis is OQ . If the length of the minor axis of

this ellipse is $\sqrt{2}$, then which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) For the ellipse, the eccentricity is $1/\sqrt{2}$ and the length of the latus rectum is 1
(b) For the ellipse, the eccentricity is $1/2$ and the length of the latus rectum is $1/2$
(c) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$
(d) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{16}(\pi - 2)$

14. Let s, t, r be non-zero complex numbers and L be the set of solutions $z = x + iy$ ($x, y \in \mathbb{R}, i = \sqrt{-1}$) of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE?

(One or More than One Option Correct)

- (a) If L has exactly one element, then $|s| \neq |t|$
(b) If $|s| = |t|$, then L has infinitely many elements
(c) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2
(d) If L has more than one element, then L has infinitely many elements

15. Let $X = {}^{10}C_1 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \dots + 10({}^{10}C_{10})^2$, where ${}^{10}C_r, r \in \{1, 2, \dots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is

(Numerical Value Based Question)

16. In a high school, a committee has to be formed from a group of 6 boys $M_1, M_2, M_3, M_4, M_5, M_6$ and 5 girls G_1, G_2, G_3, G_4, G_5 .

- (i) Let α_1 be the total number of ways in which the committee can be formed such that the committee has 5 members, having exactly 3 boys and 2 girls.
(ii) Let α_2 be the total number of ways in which the committee can be formed such that the committee has at least 2 members, and having an equal number of boys and girls.
(iii) Let α_3 be the total number of ways in which the committee can be formed such that the committee has 5 members, at least 2 of them being girls.
(iv) Let α_4 be the total number of ways in which the committee can be formed such that the committee has 4 members, having at least 2 girls such that both M_1 and G_1 are NOT in the committee together.

	List-I	List-II
P.	The value of α_1 is	1. 136
Q.	The value of α_2 is	2. 189
R.	The value of α_3 is	3. 192
S.	The value of α_4 is	4. 200
		5. 381
		6. 461

(Matching Type Question)

4 Engineering Entrances Solved Papers 2018

The correct option is

- (a) $P \rightarrow 4; Q \rightarrow 6; R \rightarrow 2; S \rightarrow 1$
 (b) $P \rightarrow 1; Q \rightarrow 4; R \rightarrow 2; S \rightarrow 3$
 (c) $P \rightarrow 4; Q \rightarrow 6; R \rightarrow 5; S \rightarrow 2$
 (d) $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 3; S \rightarrow 1$

17. Let $H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, where $a > b > 0$, be a hyperbola in the XY -plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N . Let the area of the ΔLMN be $4\sqrt{3}$.

	List-I	List-II
P.	The length of the conjugate axis of H is	1. 8
Q.	The eccentricity of H is	2. $4/\sqrt{3}$
R.	The distance between the foci of H is	3. $2/\sqrt{3}$
S.	The length of the latus rectum of H is	4. 4

(Matching Type Question)

The correct option is

- (a) $P \rightarrow 4; Q \rightarrow 2; R \rightarrow 1; S \rightarrow 3$
 (b) $P \rightarrow 4; Q \rightarrow 3; R \rightarrow 1; S \rightarrow 2$
 (c) $P \rightarrow 4; Q \rightarrow 1; R \rightarrow 3; S \rightarrow 2$
 (d) $P \rightarrow 3; Q \rightarrow 4; R \rightarrow 2; S \rightarrow 1$

BITSAT

1. The coefficient of x^{-n} in $(1+x)^n \left(1 + \frac{1}{x}\right)^n$ is

- (a) 0 (b) 1 (c) 2^n (d) $2n$

2. The greatest term in the expansion of $\sqrt{3} \left(1 + \frac{1}{\sqrt{3}}\right)^{20}$ is

- (a) $\frac{26840}{9}$ (b) $\frac{24840}{9}$
 (c) $\frac{25840}{9}$ (d) None of these

3. The n th roots of unity are in

- (a) AP (b) GP
 (c) HP (d) None of these

4. If a, b, c are in GP and $\log a - \log 2b, \log 2b - \log 3c$ and $\log 3c - \log a$ are in AP, then a, b and c are the lengths of the sides of a triangle, which is

- (a) equilateral (b) right angled
 (c) acute-angled (d) obtuse angled

5. If $\sum_{r=1}^n t_r = \frac{n(n+1)(n+2)(n+3)}{8}$, where t_r denotes the

r th term of a series, then $\lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{t_r}$ is

- (a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

6. Which of the following statement is a tautology?

- (a) $(p \vee q) \vee (\sim p)$ (b) $(\sim q \wedge p) \vee (p \vee \sim p)$
 (c) Both (a) and (b) (d) None of these

7. If a parallelogram is cut by two sets of m lines parallel to its sides, then the number of parallelogram thus formed, is

- (a) ${}^m C_2 \times {}^m C_2$ (b) $2({}^{m+2} C_2)$
 (c) $({}^{m+2} C_2)^2$ (d) None of these

8. Equation $\sin x + \cos(t+x) + \cos(t-x) = 2$ has real solution, then $\sin t$ can be

- (a) $\frac{1}{2}$ (b) $\frac{1}{5}$ (c) $\frac{3}{4}$ (d) $-\frac{3}{4}$

9. A line makes angles α, β, γ with the coordinate axes. If $\alpha + \beta = \frac{\pi}{2}$, then $(\cos \alpha + \cos \beta + \cos \gamma)^2$ is equal to

- (a) $1 + \cos 2\alpha$ (b) $1 - \sin 2\alpha$
 (c) $1 + \sin 2\alpha$ (d) None of these

10. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A . If point B and C are chosen on these two lines such that $AB = AC$, then the possible equation of the line BC passing through the point $(1, 2)$ is

- (a) $x + 7y + 13 = 0$ or $7x + y + 9 = 0$
 (b) $x + 7y + 13 = 0$ or $7x + 2y + 7 = 0$
 (c) $x - 7y + 13 = 0$ or $7x + y - 9 = 0$
 (d) None of the above

11. The area of the triangle formed by joining the origin to the point of intersection of the line $x\sqrt{5} + 2y = 3\sqrt{5}$ and circle $x^2 + y^2 = 10$ is

- (a) 3 (b) 4 (c) 5 (d) 6

12. Radius of the largest circle which passes through the focus of the parabola $y^2 = 4x$ and contained in it, is

- (a) 8 (b) 4 (c) 2 (d) 5

13. A tangent drawn to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at $P\left(\frac{\pi}{6}\right)$

forms a triangle of area $3a^2$ square units, with coordinate axes. If the eccentricity of hyperbola is e , then the value of $e^2 - 9$ is

- (a) 9 (b) 10 (c) 11 (d) 8

14. $OPQR$ is a square and M, N are the middle points of the sides PQ and QR respectively, then the ratio of the areas of the square and ΔOMN is

- (a) 4 : 1 (b) 2 : 1 (c) 8 : 3 (d) 4 : 3

15. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M . Then, the area of the triangle with vertices at A, M and the origin O is

- (a) $\frac{31}{10}$ (b) $\frac{29}{10}$ (c) $\frac{21}{10}$ (d) $\frac{27}{10}$

Engineering Entrances Solved Papers 2018 5

16. If e_1 and e_2 are the eccentricities of a hyperbola $3x^2 - 3y^2 = 25$ and its conjugate, then
 (a) $e_1^2 + e_2^2 = 2$ (b) $e_1^2 + e_2^2 = 4$
 (c) $e_1 + e_2 = 4$ (d) $e_1 + e_2 = \sqrt{2}$
17. Let $f : R \rightarrow R$ be a function satisfying $f(x + y) = f(x) + 2y^2 + kxy$ for all $x, y \in R$. If $f(1) = 2$ and $f(2) = 8$, then $f(x)$ is equal to
 (a) $2x^2$ (b) $6x - 4$
 (c) $x^2 + 3x - 2$ (d) $-x^2 + 9x - 6$
18. The value of α , so that $\lim_{x \rightarrow 0} \frac{1}{x^2} (e^{\alpha x} - e^x - x) = \frac{3}{2}$, is
 (a) 1 (b) 0 (c) 4 (d) 2
19. The arithmetic mean of a set of observation is $\bar{X}$. If each observation is divided by α and increased by 10, then the mean of the new series is
 (a) $\frac{\bar{X}}{\alpha}$ (b) $\frac{\bar{X} + 10}{\alpha}$
 (c) $\frac{\bar{X} + 10\alpha}{\alpha}$ (d) $\alpha\bar{X} + 10$
20. If $p : 4$ is an even prime number, $q : 6$ is a divisor of 12 and $r : \text{the HCF of 4 and 6 is 2}$, then which of the following is correct?
 (a) $(p \wedge q)$ (b) $(p \vee q) \wedge \sim r$
 (c) $\sim(q \wedge r) \vee p$ (d) $\sim p \vee (q \wedge r)$

KCET

1. If $|x + 5| \geq 10$, then
 (a) $x \in (-15, 5]$ (b) $x \in (-5, 5]$
 (c) $x \in (-\infty, -15] \cup [5, \infty)$ (d) $x \in [-\infty, -15] \cup [5, \infty)$
2. Everybody in a room shakes hands with everybody else. The total number of handshakes is 45. The total number of persons in the room is
 (a) 9 (b) 10 (c) 5 (d) 15
3. The constant term in the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{16}$ is
 (a) ${}^{16}C_8$ (b) ${}^{16}C_7$ (c) ${}^{16}C_9$ (d) ${}^{16}C_{10}$
4. If $P(n) : "2^{2n} - 1$ is divisible by k for all $n \in N"$ is true, then the value of ' k ' is
 (a) 6 (b) 3 (c) 7 (d) 2
5. The equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through $(-2, 3)$ is
 (a) $3x - 4y + 18 = 0$ (b) $3x - 4y - 18 = 0$
 (c) $3x + 4y + 18 = 0$ (d) $3x + 4y - 18 = 0$
6. If $\left(\frac{1-i}{1+i}\right)^{96} = a + ib$, then (a, b) is
 (a) (1, 1) (b) (1, 0) (c) (0, 1) (d) (0, -1)
7. The distance between the foci of a hyperbola is 16 and its eccentricity is $\sqrt{2}$. Its equation is
 (a) $x^2 - y^2 = 32$ (b) $\frac{x^2}{4} - \frac{y^2}{9} = 1$
 (c) $2x^2 - 3y^2 = 7$ (d) $y^2 - x^2 = 32$
8. The number of ways in which 5 girls and 3 boys can be seated in a row so that no two boys are together is
 (a) 14040 (b) 14440 (c) 14000 (d) 14400
9. If a, b, c are three consecutive terms of an AP and x, y, z are three consecutive terms of a GP, then the value of $x^{b-c} \cdot y^{c-a} \cdot z^{a-b}$ is
 (a) 0 (b) xyz (c) -1 (d) 1
10. The value of $\lim_{x \rightarrow 0} \frac{[x]}{x}$ is
 (a) 1 (b) -1
 (c) 0 (d) Does not exists
11. Let $f(x) = x - \frac{1}{x}$, then $f(-1)$ is
 (a) 0 (b) 2 (c) 1 (d) -2
12. The negation of the statement "72 is divisible by 2 and 3" is
 (a) 72 is not divisible by 2 or 72 is not divisible by 3
 (b) 72 is divisible by 2 or 72 is divisible by 3
 (c) 72 is divisible by 2 and 72 divisible by 3
 (d) 72 is not divisible by 2 and 3
13. The probability of happening of an event A is 0.5 and that of B is 0.3. If A and B are mutually exclusive events, then the probability of neither A nor B is
 (a) 0.4 (b) 0.5 (c) 0.2 (d) 0.9
14. In a simultaneous throw of a pair of dice, the probability of getting a total more than 7 is
 (a) $\frac{7}{12}$ (b) $\frac{5}{36}$
 (c) $\frac{5}{12}$ (d) $\frac{7}{36}$
15. If A and B are mutually exclusive events, given that $P(A) = \frac{3}{5}$, $P(B) = \frac{1}{5}$, then $P(A \text{ or } B)$ is
 (a) 0.8 (b) 0.6 (c) 0.4 (d) 0.2
16. Let $f : R \rightarrow R$ be defined by $f(x) = \begin{cases} 2x & ; x > 3 \\ x^2 & ; 1 < x \leq 3 \\ 3x & ; x \leq 1 \end{cases}$
 Then $f(-1) + f(2) + f(4)$ is
 (a) 9 (b) 14
 (c) 5 (d) 10

6 Engineering Entrances Solved Papers 2018

AP EAMCET

1. If $f: R \rightarrow R$ is defined by $f(x) = [2x] - 2[x]$ for $x \in R$, then the range of f is (Here $[x]$ denotes the greatest integer not exceeding x)
 - (a) Z , the set of all integers
 - (b) N , the set of all natural numbers
 - (c) R , the set of all real numbers
 - (d) $\{0, 1\}$
2. Given that a, b and c are real numbers such that $b^2 = 4ac$ and $a > 0$. The maximal possible set $D \subseteq R$ on which the function $f: D \rightarrow R$ given by $f(x) = \log\{ax^3 + (a+b)x^2 + (b+c)x + c\}$ is defined, is
 - (a) $R - \left\{-\frac{b}{2a}\right\}$
 - (b) $R - \left\{\left\{-\frac{b}{2a}\right\} \cup (-\infty, -1)\right\}$
 - (c) $R - \left\{\left\{-\frac{b}{2a}\right\} \cup \{x: x \geq 1\}\right\}$
 - (d) $R - \{(-b/2a) \cup (-\infty, -1)\}$
3. For any natural number n , $(15 \times 5^{2n}) + (2 \times 2^{3n})$ is divisible by
 - (a) 7
 - (b) 11
 - (c) 13
 - (d) 17
4. Let $z = x + iy$ and a point P represent z in the Argand plane. If the real part of $\frac{z-1}{z+i}$ is 1, then a point that lies on the locus of P is
 - (a) (2016, 2017)
 - (b) (-2016, 2017)
 - (c) (-2016, -2017)
 - (d) (2016, -2017)
5. If $z_1 = 1 - 2i$; $z_2 = 1 + i$ and $z_3 = 3 + 4i$, then $\left(\frac{1}{z_1} + \frac{3}{z_2}\right) \frac{z_3}{z_2} =$
 - (a) $13 - 6i$
 - (b) $13 - 3i$
 - (c) $6 - \frac{13}{2}i$
 - (d) $\frac{13}{2} - 3i$
6. If 1, ω , ω^2 are the cube roots of unity, then $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} =$
 - (a) 1
 - (b) ω
 - (c) ω^2
 - (d) 0
7. The number of integral values of x satisfying $5x - 1 < (x + 1)^2 < 7x - 3$ is
 - (a) 0
 - (b) 1
 - (c) 2
 - (d) 3
8. For real number x , if the minimum value of $f(x) = x^2 + 2bx + 2c^2$ is greater than the maximum value of $g(x) = -x^2 - 2cx + b^2$, then
 - (a) $c^2 > 2b^2$
 - (b) $c^2 < 2b^2$
 - (c) $b^2 = 2c^2$
 - (d) $c^2 = 2b^2$
9. If a, b and c are the roots of $x^3 + qx + r = 0$, then $(a-b)^2 + (b-c)^2 + (c-a)^2 =$
 - (a) $-6q$
 - (b) $-4q$
 - (c) $6q$
 - (d) $4q$
10. If the sum of two roots of the equation $x^3 - 2px^2 + 3qx - 4r = 0$ is zero, then the value of r is
 - (a) $\frac{3pq}{2}$
 - (b) $\frac{3pq}{4}$
 - (c) pq
 - (d) $2pq$
11. The sum of the four digit even numbers that can be formed with the digits 0, 3, 5, 4 with out repetition is
 - (a) 14684
 - (b) 43536
 - (c) 46526
 - (d) 52336
12. If x is the number of ways in which six women and six men can be arranged to sit in a row such that no two women are together and if y is the number of ways they are seated around a table in the same manner, then $x : y =$
 - (a) 12 : 1
 - (b) 42 : 1
 - (c) 16 : 1
 - (d) 6 : 1
13. The number of 5-letter words that can be formed by using the letters of the word SARANAM is
 - (a) 1120
 - (b) 6720
 - (c) 480
 - (d) 720
14. The number of rational terms in the binomial expansion of $(\sqrt[4]{5} + \sqrt[5]{4})^{100}$ is
 - (a) 50
 - (b) 5
 - (c) 6
 - (d) 51
15. The numerically greatest term in the binomial expansion of $(2a - 3b)^{19}$ when $a = \frac{1}{4}$ and $b = \frac{2}{3}$ is
 - (a) ${}^{19}C_5 \cdot 2^{11}$
 - (b) ${}^{19}C_3 \cdot \frac{1}{2^{11}}$
 - (c) ${}^{19}C_4 \cdot \frac{1}{2^{13}}$
 - (d) ${}^{19}C_3 \cdot 2^{13}$
16. If $\cos\left(x - \frac{\pi}{3}\right)$, $\cos x$, $\cos\left(x + \frac{\pi}{3}\right)$ are in a harmonic progression, then $\cos x =$
 - (a) $\frac{3}{2}$
 - (b) 1
 - (c) $\frac{\sqrt{3}}{2}$
 - (d) $\frac{\sqrt{3}}{2}$
17. $\cos^3 110^\circ + \cos^3 10^\circ + \cos^3 130^\circ =$
 - (a) $\frac{3}{4}$
 - (b) $\frac{3}{8}$
 - (c) $\frac{3\sqrt{3}}{8}$
 - (d) $\frac{3\sqrt{3}}{4}$
18. If the general solution of $\sin 5x = \cos 2x$ is of the form $a_n \cdot \frac{\pi}{2}$ for $n = 0, \pm 1, \pm 2, \dots$, then $a_n =$
 - (a) $\frac{2n}{5+2(-1)^n}$
 - (b) $\frac{2n+(-1)^n}{5+2(-1)^n}$
 - (c) $\frac{2n+1}{5+2(-1)^n}$
 - (d) $\frac{2n-1}{5+2(-1)^n}$
19. If the median of a ΔABC through A is perpendicular to AC , then $\frac{\tan A}{\tan C} =$
 - (a) $1 + \sqrt{2}$
 - (b) $-\frac{1}{\sqrt{3}} + 1$
 - (c) -2
 - (d) $1 + \frac{2}{\sqrt{3}}$
20. In ΔABC , $\tan \frac{A}{2} + \tan \frac{B}{2} =$
 - (a) $\frac{c \cot \frac{C}{2}}{4s}$
 - (b) $\frac{2c \cot \frac{C}{2}}{a+b+c}$
 - (c) $\frac{2c \tan \frac{C}{2}}{s}$
 - (d) $\frac{c \tan \frac{C}{2}}{a+b+c}$

Engineering Entrances Solved Papers 2018 **7**

- 21.** In a $\triangle ABC$, D , E and F respectively are the points of contact of the incircle with the sides AB , BC and CA such that $AD = \alpha$, $BE = \beta$ and $CF = \gamma$, then $\frac{\alpha\beta\gamma}{\alpha + \beta + \gamma} =$
- (a) R^2 (b) $2R$ (c) $2r$ (d) r^2
- 22.** In $\triangle PQR$, M is the mid-point of QR and C is the mid-point of PM . If QC when extended meets PR at N , then $\frac{QN}{CN} =$
- (a) 1 (b) 2 (c) 3 (d) 4
- 23.** The mean and the standard deviation of a data of 8 items are 25 and 5 respectively. If two items 15 and 25 are added to this data, then the variance of the new data is
- (a) 29 (b) 24 (c) 26 (d) $\sqrt{29}$
- 24.** The mean deviation from the median for the following distribution (corrected to two decimals) is
- | | | | | | | | | |
|-------|---|---|---|----|----|----|----|----|
| x_i | 6 | 9 | 3 | 12 | 15 | 13 | 21 | 22 |
| f_i | 4 | 5 | 3 | 2 | 5 | 4 | 4 | 3 |
- (a) 13.42 (b) 5.45 (c) 4.97 (d) 11.25
- 25.** If a die is rolled three times, then the probability of getting a larger number on its face than the previous number each time, is
- (a) $\frac{15}{216}$ (b) $\frac{5}{54}$ (c) $\frac{13}{216}$ (d) $\frac{1}{18}$
- 26.** Let $A(2, 3)$, $B(3, -6)$, $C(5, -7)$ be three points. If P is a point satisfying the condition $PA^2 + PB^2 = 2PC^2$, then a point that lies on the locus of P is
- (a) $(2, -5)$ (b) $(-2, 5)$ (c) $(13, 10)$ (d) $(-13, -10)$
- 27.** If the coordinates of a point P changes to $(2, -6)$ when the coordinate axes are rotated through an angle of 135° , then the coordinates of P in the original system are
- (a) $(-2, 6)$ (b) $(-6, 2)$
(c) $(2\sqrt{2}, 4\sqrt{2})$ (d) $(\sqrt{2}, -\sqrt{2})$
- 28.** If the portion of a line intercepted between the coordinate axes is divided by the point $(2, -1)$ in the ratio of 3:2, then the equation of that line is
- (a) $5x - 2y - 20 = 0$ (b) $2x - y - 5 = 0$
(c) $3x - y - 7 = 0$ (d) $x - 3y - 5 = 0$
- 29.** The equation of the line passing through the point of intersection of the lines $2x + y - 4 = 0$, $x - 3y + 5 = 0$ and lying at a distance of $\sqrt{5}$ units from the origin, is
- (a) $x - 2y - 5 = 0$ (b) $x + 2y - 5 = 0$
(c) $x + 2y + 5 = 0$ (d) $x - 2y + 5 = 0$
- 30.** The equation of the line joining the centroid with the orthocentre of the triangle formed by the points $(-2, 3)$, $(2, -1)$, $(4, 0)$ is
- (a) $x + y - 2 = 0$ (b) $11x - y - 14 = 0$
(c) $x - 11y + 6 = 0$ (d) $2x - y - 2 = 0$
- 31.** The lines represented by the equations $23x^2 - 48xy + 3y^2 = 0$ and $2x + 3y + 4 = 0$ form
- (a) an isosceles triangle (b) a right angled triangle
(c) an equilateral triangle (d) a scalene triangle
- 32.** If the line $x + 2y = k$ intersects the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ at two points A and B and if O is the origin, then the condition for $\angle AOB = 90^\circ$ is
- (a) $k^2 + k + 1 = 0$ (b) $k^2 - 2k + 10 = 0$
(c) $2k^2 + 9k - 10 = 0$ (d) $3k^2 + 8k - 1 = 0$
- 33.** If $2x^2 + 3xy - 2y^2 = 0$ represents two sides of a parallelogram and $3x + y + 1 = 0$ is one of its diagonals, then the other diagonal is
- (a) $x - 3y + 1 = 0$ (b) $x - 3y + 2 = 0$
(c) $x - 3y = 0$ (d) $3x - y = 0$
- 34.** If the lengths of the tangents drawn from P to the circles $x^2 + y^2 - 2x + 4y - 20 = 0$ and $x^2 + y^2 - 2x - 8y + 1 = 0$ are in the ratio 2 : 1, then the locus P is
- (a) $x^2 + y^2 + 2x + 12y + 8 = 0$
(b) $x^2 + y^2 - 2x + 12y + 8 = 0$
(c) $x^2 + y^2 + 2x - 12y + 8 = 0$
(d) $x^2 + y^2 - 2x - 12y + 8 = 0$
- 35.** The equation of a circle touching the coordinate axes and the line $3x - 4y = 12$ is
- (a) $x^2 + y^2 + 6x + 6y + 9 = 0$ (b) $x^2 + y^2 + 6x + 6y - 9 = 0$
(c) $x^2 + y^2 - 6x - 6y + 9 = 0$ (d) $x^2 + y^2 - 6x - 6y - 9 = 0$
- 36.** The pole of the straight line $9x + y - 28 = 0$ with respect to the circle $2x^2 + 2y^2 - 3x + 5y - 7 = 0$ is
- (a) $(3, 1)$ (b) $(3, -1)$ (c) $(-3, 1)$ (d) $(4, -8)$
- 37.** The point of intersection of the direct common tangents drawn to the circles $(x + 11)^2 + (y - 2)^2 = 225$ and $(x - 11)^2 + (y + 2)^2 = 25$ is
- (a) $\left(\frac{-11}{2}, 1\right)$ (b) $(-22, 4)$
(c) $\left(\frac{11}{2}, -1\right)$ (d) $(22, -4)$
- 38.** In List-I, a pair of circles is given in A, B, C and in List-II, angle between those pair of circles is given. Match the items from List-I to List-II.
- | List-I | List-II |
|---|-----------------|
| A. $(x - 2)^2 + y^2 = 2$
$(x - 2)^2 + (y - 1)^2 = 1$ | I. 90° |
| B. $x^2 + y^2 - 6x - 6y + 9 = 0$
$x^2 + y^2 - 4x + 4y - 9 = 0$ | II. 135° |
| C. $x^2 + y^2 + 4x - 14y + 28 = 0$
$x^2 + y^2 + 4x - 5 = 0$ | III. 60° |
| | IV. 30° |

8 Engineering Entrances Solved Papers 2018

The correct matching is

	A	B	C
(a)	I	II	III
(b)	II	I	III
(c)	III	I	IV
(d)	IV	III	I

39. If the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $2x^2 + 2y^2 + 3x + 8y + 2c = 0$ touches the circle $x^2 + y^2 + 2x + 2y + 1 = 0$, then

- (a) $g = \frac{3}{4}$ or $f = 2$ (b) $g \neq \frac{3}{4}, f = 2$
 (c) $g = \frac{3}{4}$ or $f \neq 2$ (d) $g = \frac{2}{5}$ or $f = 1$

40. The line $y = 6x + 1$ touches the parabola $y^2 = 24x$. The coordinates of a point P on this line, from which the tangent to $y^2 = 24x$ is perpendicular to the line $y = 6x + 1$, is

- (a) $(-1, -5)$ (b) $(-2, -11)$
 (c) $(-6, -35)$ (d) $(-7, -41)$

41. A point on the parabola whose focus is $S(1, -1)$ and whose vertex is $A(1, 1)$ is

- (a) $\left(3, \frac{1}{2}\right)$ (b) $(1, 2)$
 (c) $\left(2, \frac{1}{2}\right)$ (d) $(2, 2)$

42. An ellipse having the coordinate axes as its axes and its major axis along Y -axis, passes through the point $(-3, 1)$ and has eccentricity $\sqrt{\frac{2}{5}}$. Then its equation is

- (a) $3x^2 + 5y^2 - 15 = 0$ (b) $5x^2 + 3y^2 - 32 = 0$
 (c) $3x^2 + 5y^2 - 32 = 0$ (d) $5x^2 + 3y^2 - 48 = 0$

43. The product of the perpendicular distances drawn from the points $(3, 0)$ and $(-3, 0)$ to the tangent of the ellipse $\frac{x^2}{36} + \frac{y^2}{27} = 1$ at $\left(3, \frac{9}{2}\right)$ is

- (a) 36 (b) 27 (c) 9 (d) 63

44. The equation of the hyperbola whose asymptotes are the lines $3x + 4y - 2 = 0$, $2x + y + 1 = 0$ and which passes through the point $(1, 1)$ is

- (a) $6x^2 + 11xy + 4y^2 - 30x + 2y + 7 = 0$
 (b) $6x^2 + 11xy + 4y^2 - x + 2y - 22 = 0$
 (c) $6x^2 + 11xy + 4y^2 - x + 2y + 22 = 0$
 (d) $6x^2 + 11xy + 4y^2 - 3x - 7y - 11 = 0$

45. If the orthocentre and the centroid of a triangle are $(-3, 5, 2)$ and $(3, 3, 4)$ respectively, then its circumcentre is

- (a) $(6, 2, 5)$ (b) $(6, 2, -5)$ (c) $(6, -2, 5)$ (d) $(6, -2, -5)$

46. If $[x]$ denotes the greatest integer $\leq x$, then

$$\lim_{n \rightarrow \infty} \frac{1}{n^3} \{[1^2 x] + [2^2 x] + [3^2 x] + \dots + [n^2 x]\} =$$

(a) $\frac{x}{2}$ (b) $\frac{x}{3}$ (c) $\frac{x}{6}$ (d) 0

TS EAMCET

1. If $f : [1, \infty) \rightarrow [1, \infty)$ is defined by

$$f(x) = \frac{1 + \sqrt{1 + 4 \log_2 x}}{2}, \text{ then } f^{-1}(3) =$$

- (a) 0 (b) 1 (c) 64 (d) $\frac{1 + \sqrt{5}}{2}$

2. If α and β are the greatest divisors of $n(n^2 - 1)$ and $2n(n^2 + 2)$ respectively for all $n \in N$, then $\alpha\beta =$

- (a) 18 (b) 36 (c) 27 (d) 9

3. Z is a complex number such that $|Z| \leq 2$ and $-\frac{\pi}{3} \leq \arg Z \leq \frac{\pi}{3}$. The area of the region formed by locus of Z is

- (a) $\frac{2\pi}{3}$ (b) $\frac{\pi}{3}$ (c) $\frac{4\pi}{3}$ (d) $\frac{8\pi}{3}$

4. The points in the argand plane given by

$$Z_1 = -3 + 5i, Z_2 = -1 + 6i, Z_3 = -2 + 8i, Z_4 = -4 + 7i$$

form a

- (a) parallelogram (b) rectangle
 (c) rhombus (d) square

5. When $n = 8$, $(\sqrt{3} + i)^n + (\sqrt{3} - i)^n =$

- (a) -256 (b) -128 (c) 256i (d) 128i

6. If $2 \operatorname{cis} \frac{7\pi}{5}$ is one of the values of $z^{1/5}$, then $z =$

- (a) $32 + 32i$ (b) -32 (c) -1 (d) 32

7. The set of real values of x for which the inequality $|x - 1| + |x + 1| < 4$ always holds good is

- (a) $(-2, 2)$ (b) $(-\infty, -2) \cup (2, \infty)$
 (c) $(-\infty, -1] \cup [1, \infty)$ (d) $(-2, -1) \cup (1, 2)$

8. If the roots of the equation $x^2 + x + a = 0$ exceed a , then

- (a) $a > 2$ (b) $a < -2$ (c) $2 < a < 3$ (d) $-2 < a < -1$

9. If the roots of the equation

$$\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{5}{2}$$

are p and q ($p > q$) and the roots of

$$\text{the equation } (p+q)x^4 - pqx^2 + \frac{p}{q} = 0 \text{ are } \alpha, \beta, \gamma, \delta,$$

$$\text{then } (\Sigma \alpha)^2 - \Sigma \alpha\beta + \alpha\beta\gamma\delta =$$

- (a) 0 (b) $\frac{104}{25}$ (c) $\frac{25}{4}$ (d) $\frac{16}{5}$

Engineering Entrances Solved Papers 2018 **9**

- 10.** The equation $x^5 - 5x^3 + 5x^2 - 1 = 0$ has three equal roots. If α, β are the other two roots of this equation, then $\alpha + \beta + \alpha\beta =$

(a) -4 (b) 3 (c) -2 (d) -5

- 11.** If all possible numbers are formed by using the digits 1, 2, 3, 5, 7 without repetition and they are arranged in descending order, then the rank of the number 327 is

(a) 31 (b) 175 (c) 149 (d) 271

- 12.** If a is the number of all even divisors and b is the number of all odd divisors of the number 10800, then $2a + 3b =$

(a) 72 (b) 132 (c) 96 (d) 136

- 13.** If the coefficient of x^5 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{13}$ is equal to the coefficient of x^{-5} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{13}$, then $ab =$

(a) 1 (b) $\frac{1}{6}$ (c) $\frac{7}{6}$ (d) $\frac{4}{2}$

- 14.** For $n \in N$, in the expansion of $(\sqrt[4]{x^{-3}} + \alpha\sqrt[4]{x^5})^n$, the sum of all binomial coefficients lies between 200 and 400 and the term independent of x is 448. Then, the value of a is

(a) 1 (b) 2 (c) $\frac{1}{2}$ (d) 0

- 15.** If $A(n) = \sin^n \alpha + \cos^n \alpha$, then $A(1)A(4) + A(2)A(5) =$

(a) $A(1)A(2) + A(4)A(5)$ (b) $A(1)A(6) + A(2)A(3)$
(c) $A(1)A(3) + A(2)A(6)$ (d) $A(1)A(2) + A(3)A(6)$

- 16.** When $\frac{\sin 9\theta}{\cos 27\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin \theta}{\cos 3\theta} = k (\tan 27\theta - \tan \theta)$ is defined, then $k =$

(a) $\frac{\pi}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{2}$ (d) $\frac{\pi}{4}$

- 17.** If $x = \sum_{n=0}^{\infty} \cos^{2n} \theta$, $y = \sum_{n=0}^{\infty} \sin^{2n} \theta$, $z = \sum_{n=0}^{\infty} \cos^{2n} \theta \sin^{2n} \theta$

and $0 < \theta < \frac{\pi}{2}$, then

(a) $xz + yz = xy + z$ (b) $xyz = yz + x$
(c) $xy + z = xy + zx$ (d) $x + y + z = xyz + z$

- 18.** Number of solutions of the equation $\sin x - \sin 2x + \sin 3x = 2\cos^2 x - 2\cos x$ in $(0, \pi)$ is

(a) 1 (b) 3 (c) 2 (d) 4

- 19.** If $\cosh x = \frac{\sqrt{14}}{3}$, $\sinh x = \cos \theta$ and $-\pi < \theta < -\frac{\pi}{2}$, then $\sin \theta =$

(a) $\frac{1}{3}$ (b) $\frac{2}{3}$ (c) $-\frac{1}{3}$ (d) $-\frac{2}{3}$

- 20.** In $\triangle ABC$, if $a = 5$ and $\tan \frac{A-B}{2} = \frac{1}{4} \tan \frac{A+B}{2}$, then $\sqrt{a^2 - b^2} =$

(a) 2 (b) 3 (c) 4 (d) 5

- 21.** In a $\triangle ABC$, if $A = 2B$ and the sides opposite to the angles A, B, C are $\alpha + 1, \alpha - 1$ and α respectively, then $\alpha =$

(a) 3 (b) 4 (c) 5 (d) 6

- 22.** In $\triangle ABC$, right angled at A , the circumradius, inradius and radius of the excircle opposite to A are respectively in the ratio $2:5:\lambda$, then the roots of the equation $x^2 - (\lambda - 5)x + (\lambda - 6) = 0$ are

(a) 3, 4 (b) 5, 13 (c) 1, 3 (d) 8, 13

- 23.** $x_1, x_2, \dots, x_n$ are n observations with mean $\bar{x}$ and standard deviation σ . Match the items of List-I with those of List-II

List-I	List-II
A. $\sum_{i=1}^n (x_i - \bar{x})$	(i) Median
B. Variance (σ^2)	(ii) Coefficient of variation
C. Mean deviation	(iii) Zero
D. Measure used to find the homogeneity of given two series	(iv) Mean of the absolute deviations from any measure of central tendency
	(v) Mean of the squares of the deviations from mean

The correct answer is

A B C D
(a) (i) (v) (ii) (iii)
(b) (i) (iv) (iii) (ii)
(c) (iii) (v) (iv) (ii)
(d) (iii) (v) (ii) (i)

- 24.** The variance of 50 observations is 7. If each observation is multiplied by 6 and then 5 is subtracted from it, then the variance of the new data is

(a) 37 (b) 42 (c) 247 (d) 252

- 25.** Two dice are thrown and two coins are tossed simultaneously. The probability of getting prime numbers on both the dice along with a head and a tail on the two coins is

(a) $\frac{1}{8}$ (b) $\frac{1}{2}$ (c) $\frac{3}{16}$ (d) $\frac{1}{4}$

- 26.** A quadrilateral $ABCD$ is divided by the diagonal AC into two triangles of equal areas. If A, B, C are respectively, $(3, 4), (-3, 6), (-5, 1)$, then the locus of D is

(a) $(x - 8y - 57)(x - 8y + 11) = 0$
(b) $(x - 8y - 57)(x - 8y - 11) = 0$
(c) $(3x - 8y - 57)(3x - 8y + 11) = 0$
(d) $(3x - 8y - 11)(3x - 8y + 57) = 0$

- 27.** By rotating the coordinate axes in the positive direction about the origin by an angle α , if the point $(1, 2)$ is transformed to $\left(\frac{3\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{3}+3}{2\sqrt{2}}\right)$ in new

coordinate system Then, $\alpha =$

(a) $\frac{\pi}{3}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{9}$ (d) $\frac{\pi}{12}$

10 Engineering Entrances Solved Papers 2018

28. Let $a \neq 0, b \neq 0, c$ be three real numbers and $L(p, q) = \frac{ap + bq + c}{\sqrt{a^2 + b^2}}, \forall p, q \in R$.
- If $L\left(\frac{2}{3}, \frac{1}{3}\right) + L\left(\frac{1}{3}, \frac{2}{3}\right) + L(2, 2) = 0$, then the line $ax + by + c = 0$ always passes through the fixed point
 (a) (0, 1) (b) (1, 1) (c) (2, 2) (d) (-1, -1)
29. The incentre of the triangle formed by the straight line having 3 as X-intercept and 4 as Y-intercept, together with the coordinate axes, is
 (a) (2, 2) (b) $\left(\frac{3}{2}, \frac{3}{2}\right)$ (c) (1, 2) (d) (1, 1)
30. The equation of the straight line in the normal form which is parallel to the lines $x + 2y + 3 = 0$ and $x + 2y + 8 = 0$ and dividing the distance between these two lines in the ratio 1 : 2 internally is
 (a) $x \cos \alpha + y \sin \alpha = \frac{10}{\sqrt{45}}, \alpha = \tan^{-1} \sqrt{2}$
 (b) $x \cos \alpha + y \sin \alpha = \frac{14}{\sqrt{45}}, \alpha = \pi + \tan^{-1} 2$
 (c) $x \cos \alpha + y \sin \alpha = \frac{14}{\sqrt{45}}, \alpha = \tan^{-1} 2$
 (d) $x \cos \alpha + y \sin \alpha = \frac{10}{\sqrt{45}}, \alpha = \pi + \tan^{-1} \sqrt{2}$
31. A pair of straight lines is passing through the point (1, 1). One of the lines makes an angle θ with the positive direction of X-axis and the other makes the same angle with the positive direction of Y-axis. If the equation of the pair of straight lines is $x^2 - (a + 2)xy + y^2 + a(x + y - 1) = 0, a \neq -2$, then the value of θ is
 (a) $\frac{1}{2} \sin^{-1} \left(\frac{2}{a+2} \right)$ (b) $\frac{1}{2} \sin \left(\frac{2}{a+2} \right)$
 (c) $\frac{1}{2} \tan^{-1} \left(\frac{2}{a+2} \right)$ (d) $\frac{1}{2} \tan \left(\frac{2}{a+2} \right)$
32. If the pair of lines $6x^2 + xy - y^2 = 0$ and $3x^2 - axy - y^2 = 0, a > 0$ have a common line, then $a =$
 (a) $\frac{1}{2}$ (b) 1 (c) 2 (d) 4
33. If the chord $L \equiv y - mx - 1 = 0$ of the circle $S \equiv x^2 + y^2 - 1 = 0$ touches the circle $S_1 \equiv x^2 + y^2 - 4x + 1 = 0$, then the possible points for which $L = 0$ is a chord of contact of $S = 0$ are
 (a) $(2 \pm \sqrt{6}, 0)$ (b) $(2 \pm \sqrt{6}, 1)$ (c) (2, 0) (d) $(\sqrt{6}, 1)$
34. If $y + c = 0$ is a tangent to the circle $x^2 + y^2 - 6x - 2y + 1 = 0$ at $(a, 4)$, then
 (a) $ac = 360$ (b) $ac = -12$ (c) $a + c = 0$ (d) $4a = c$
35. If the circles given by $S \equiv x^2 + y^2 - 14x + 6y + 33 = 0$ and $S' \equiv x^2 + y^2 - a^2 = 0 (a \in N)$ have 4 common tangents, then the possible number of circles $S' = 0$ is
 (a) 1 (b) 2 (c) 0 (d) infinite
36. The centre of the circle passing through the point (1, 0) and cutting the circles $x^2 + y^2 - 2x + 4y + 1 = 0$ and $x^2 + y^2 + 6x - 2y + 1 = 0$ orthogonally is
 (a) $\left(-\frac{2}{3}, \frac{2}{3}\right)$ (b) $\left(\frac{1}{2}, \frac{1}{2}\right)$
 (c) (0, 1) (d) (0, 0)
37. The equation of the tangent at the point (0, 3) on the circle which cuts the circles $x^2 + y^2 - 2x + 6y = 0, x^2 + y^2 - 4x - 2y + 6 = 0$ and $x^2 + y^2 - 12x + 2y + 3 = 0$ orthogonally is
 (a) $y = 3$ (b) $x = 0$
 (c) $3x + y - 3 = 0$ (d) $x + 3y - 9 = 0$
38. If two tangents to the parabola $y^2 = 8x$ meet the tangent at its vertex in M and N such that $MN = 4$, then the locus of the point of intersection of those two tangents is
 (a) $y^2 = 8(x + 3)$ (b) $y^2 = 8(x - 2)$
 (c) $y^2 = 8(x + 2)$ (d) $y^2 = 4(x + 2)$
39. Three normals are drawn from the point (c, 0) to the curve $y^2 = x$. If one of the normals is X-axis, then the value of c for which the other two normals are perpendicular to each other is
 (a) $\frac{1}{4}$ (b) $\frac{1}{2}$
 (c) $\frac{3}{4}$ (d) $\frac{5}{8}$
40. If the normal drawn at one end of the latus rectum of the ellipse $b^2 x^2 + a^2 y^2 = a^2 b^2$ with eccentricity 'e' passes through one end of the minor axis. Then,
 (a) $e^4 + e^2 = 2$ (b) $e^4 - e^2 = 1$
 (c) $e^4 + e^2 = 1$ (d) $e^2 + e = 1$
41. A variable tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ makes intercepts on both the axes. The locus of the middle point of the portion of the tangent between the coordinate axes is
 (a) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$ (b) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$
 (c) $b^2 x^2 + a^2 y^2 = 4$ (d) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 4$
42. If the eccentricity of a conic satisfies the equation $2x^3 + 10x - 13 = 0$, then that conic is
 (a) a circle (b) a parabola
 (c) an ellipse (d) a hyperbola

Engineering Entrances Solved Papers 2018 11

- 43. Assertion (A)** If $(-1, 3, 2)$ and $(5, 3, 2)$ are respectively the orthocentre and circumcentre of a triangle, then $(3, 3, 2)$ is its centroid(d)

Reason (R) Centroid of the triangle divides the line segment joining the orthocentre and the circumcentre in the ratio $1 : 2$.

Which one of the following is true?

- (a) (A) and (R) are true and (R) is the correct explanation to (A)

- (b) (A) and (R) are true, but (R) is not the correct explanation to (A)

- (c) (A) is true, (R) is false

- (d) (A) is false, (R) is true

44. $\lim_{x \rightarrow -\infty} \frac{3|x| - x}{|x| - 2x} - \lim_{x \rightarrow 0} \frac{\log(1 + x^3)}{\sin^3 x} =$

- (a) 1 (b) $\frac{1}{3}$ (c) $\frac{4}{3}$ (d) 0

VIT

- 1.** The value of $\{\sin(\log i^i)\}^3 + \{\cos(\log i^i)\}^3$, is

- (a) 1 (b) -1 (c) 2 (d) $2i$

- 2.** If $f(x) = 27x^3 - \frac{1}{x^3}$ and α, β are roots of $3x - \frac{1}{x} = 2$, then

- (a) $f(\alpha) = f(\beta)$ (b) $f(\alpha) = 10$
(c) $f(\beta) = -10$ (d) None of these

- 3.** Let A and B be two sets. Then, $(A \cup B)' \cup (A' \cap B)$ is equal to

- (a) A' (b) A
(c) B' (d) None of these

- 4.** The relation on the set $A = \{x : |x| < 3, x \in \mathbb{Z}\}$ is defined by $R = \{(x, y) : y = |x|, x \neq -1\}$. Then, the number of elements in the power set of R is

- (a) 32 (b) 16 (c) 8 (d) 64

- 5.** Negation of the statement $(p \wedge r) \rightarrow (r \vee q)$ is

- (a) $(p \wedge r) \wedge (\sim r \wedge \sim q)$ (b) $\sim(p \wedge r) \rightarrow \sim(r \vee q)$
(c) $\sim(p \vee r) \rightarrow \sim(r \wedge q)$ (d) $(p \wedge r) \vee (r \vee q)$

- 6.** The eccentricity of the hyperbola with latus rectum 12 and semi-conjugate axis $2\sqrt{3}$, is

- (a) 2 (b) 3 (c) $\frac{\sqrt{3}}{2}$ (d) $2\sqrt{3}$

- 7.** If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$, then one of the value of k is

- (a) $\frac{1}{8}$ (b) 8 (c) 4 (d) $\frac{1}{4}$

- 8.** The number of values of C such that the straight line $y = 4x + c$ touches the curve $\frac{x^2}{4} + y^2 = 1$, is

- (a) 0 (b) 1 (c) 2 (d) ∞

- 9.** If any tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intercepts equal lengths l on the axes, then l is equal to

- (a) $a^2 + b^2$ (b) $\sqrt{a^2 + b^2}$
(c) $(a^2 + b^2)^2$ (d) None of these

- 10.** The sum of the mean and variance of a binomial distribution is 15 and the sum of their squares is 117. The mean of the distribution is

- (a) 6 (b) 9 (c) 3 (d) 13

- 11.** The range of g so that we have always a chord of contact of tangents drawn from a real point (α, α) to the circle $x^2 + y^2 + 2gx + 4y + 2 = 0$ is

- (a) $(-3, 0)$ (b) $(-4, 1)$
(c) $(-4, 0)$ (d) $(-3, 1)$

- 12.** If the point (a, a) falls between the lines $|x + y| = 2$, then

- (a) $|a| = 2$ (b) $|a| = \frac{1}{2}$
(c) $|a| < 1$ (d) $|a| < \frac{1}{2}$

- 13.** If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and

$P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$, then $P(B \cap C)$ is equal to

- (a) $\frac{1}{12}$ (b) $\frac{1}{6}$
(c) $\frac{1}{15}$ (d) $\frac{1}{9}$

- 14.** A and B throw a dice. The probability that A 's throw is not greater than B 's throw, is

- (a) $\frac{5}{12}$ (b) $\frac{7}{12}$
(c) $\frac{1}{6}$ (d) $\frac{1}{2}$

- 15.** If $a \lim_{x \rightarrow 1} x^{\frac{1}{1-x}} + b = e^{-1}$ ($a \geq 1, b \geq 0$), then

- (a) $a = 1, b = e^{-1}$ (b) $a = 2, b = e^{-1}$
(c) $a = -1, b = e^{-1}$ (d) $a = 1, b = 0$

- 16.** The range of the function $f(x) = \log_5(25 - x^2)$ is

- (a) $[0, 5]$ (b) $[0, 2)$
(c) $(0, 2)$ (d) None of these

12 Engineering Entrances Solved Papers 2018

MHT CET

1. The cartesian coordinates of the point on the parabola $y^2 = -16x$, whose parameter is $\frac{1}{2}$, are
(a) $(-2, 4)$ (b) $(4, -1)$ (c) $(-1, -4)$ (d) $(-1, 4)$
2. If $f : R - \{2\} \rightarrow R$ is a function defined by $f(x) = \frac{x^2 - 4}{x - 2}$, then its range is
(a) R (b) $R - \{2\}$ (c) $R - \{4\}$ (d) $R - \{-2, 2\}$
3. The line $5x + y - 1 = 0$ coincides with one of the lines given by $5x^2 + xy - kx - 2y + 2 = 0$, then the value of k is
(a) -11 (b) 31 (c) 11 (d) -31
4. Letters in the word HULULULU are rearranged. The probability of all three L being together is
(a) $\frac{3}{20}$ (b) $\frac{2}{5}$ (c) $\frac{3}{28}$ (d) $\frac{5}{23}$
5. The sum of the first 10 terms of the series $9 + 99 + 999 + \dots$, is
(a) $\frac{9}{8}(9^{10} - 1)$ (b) $\frac{100}{9}(10^9 - 1)$
(c) $10^9 - 1$ (d) $\frac{100}{9}(10^{10} - 1)$
6. If A, B, C are the angles of ΔABC , then $\cot A \cdot \cot B + \cot B \cdot \cot C + \cot C \cdot \cot A =$
(a) 0 (b) 1 (c) 2 (d) -1
7. If $2 \sin\left(\theta + \frac{\pi}{3}\right) = \cos\left(\theta - \frac{\pi}{6}\right)$, then $\tan \theta =$
(a) $\sqrt{3}$ (b) $-\frac{1}{\sqrt{3}}$ (c) $\frac{1}{\sqrt{3}}$ (d) $-\sqrt{3}$
8. If points $P(4, 5, x), Q(3, y, 4)$ and $R(5, 8, 0)$ are collinear, then the value of $x + y$ is
(a) -4 (b) 3 (c) 5 (d) 4
9. If the slope of one of the lines given by $ax^2 + 2hxy + by^2 = 0$ is two times the other, then
(a) $8h^2 = 9ab$ (b) $8h^2 = 9ab^2$ (c) $8h = 9ab$ (d) $8h = 9ab^2$
10. The equation of the line passing through the point $(-3, 1)$ and bisecting the angle between coordinate axes is
(a) $x + y + 2 = 0$ (b) $-x + y + 2 = 0$
(c) $x - y + 4 = 0$ (d) $2x + y + 5 = 0$
11. The negation of the statement : "Getting above 95% marks is necessary condition for Hema to get the admission in good college".
(a) Hema gets above 95% marks but she does not get the admission in good college
(b) Hema does not get above 95% marks and she gets admission in good college
(c) If Hema does not get above 95% marks then she will not get the admission in good college
(d) Hema does not get above 95% marks or she gets the admission in good college
12. $\cos 1^\circ \cdot \cos 2^\circ \cdot \cos 3^\circ \dots \cos 179^\circ =$
(a) 0 (b) 1 (c) $-\frac{1}{2}$ (d) -1
13. The point of intersection of lines represented by $x^2 - y^2 + x + 3y - 2 = 0$ is
(a) $(1, 0)$ (b) $(0, 2)$ (c) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{2}\right)$
14. A die is rolled. If X denotes the number of positive divisors of the outcome, then the range of the random variable X is
(a) $\{1, 2, 3\}$ (b) $\{1, 2, 3, 4\}$
(c) $\{1, 2, 3, 4, 5, 6\}$ (d) $\{1, 3, 5\}$
15. In ΔABC , with usual notations, if a, b, c are in AP then $a \cos^2\left(\frac{C}{2}\right) + c \cos^2\left(\frac{A}{2}\right) =$
(a) $\frac{3a}{2}$ (b) $\frac{3c}{2}$ (c) $\frac{3b}{2}$ (d) $\frac{3abc}{2}$
16. The number of solutions of $\sin x + \sin 3x + \sin 5x = 0$ in the interval $\left[\frac{\pi}{2}, 3\frac{\pi}{2}\right]$ is
(a) 2 (b) 3 (c) 4 (d) 5
17. The contrapositive of the statement : "If the weather is fine then my friends will come and we go for a picnic" is
(a) The weather is fine but my friends will not come or we do not go for a picnic
(b) If my friends do not come or we do not go for picnic then weather will not be fine
(c) If the weather is not fine then my friends will not come or we do not go for a picnic
(d) The weather is not fine but my friends will come and we go for a picnic
18. If $X = \{4^n - 3n - 1 : n \in N\}$ and $Y = \{9(n-1) : n \in N\}$, then $X \cap Y =$
(a) X (b) Y (c) ϕ (d) $\{0\}$
19. The statement pattern $p \wedge (\sim p \wedge q)$ is
(a) a tautology (b) a contradiction
(c) equivalent to $p \wedge q$ (d) equivalent to $p \vee q$
20. If the line $y = 4x - 5$ touches to the curve $y^2 = ax^3 + b$ at the point $(2, 3)$, then $7a + 2b =$
(a) 0 (b) 1 (c) -1 (d) 2
21. The sides of a rectangle are given by $x = \pm a$ and $y = \pm b$. The equation of the circle passing through the vertices of the rectangle is
(a) $x^2 + y^2 = a^2$
(b) $x^2 + y^2 = a^2 + b^2$
(c) $x^2 + y^2 = a^2 - b^2$
(d) $(x - a)^2 + (y - b)^2 = a^2 + b^2$

Answer with Explanations

JEE Main

1. (b) We have,

$$|a - 5| < 1 \text{ and } |b - 5| < 1$$

$$\therefore -1 < a - 5 < 1$$

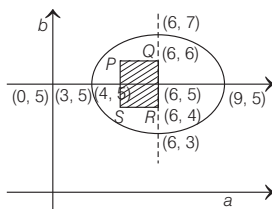
$$\text{and } -1 < b - 5 < 1$$

$$\Rightarrow 4 < a < 6 \text{ and } 4 < b < 6$$

$$\text{Now, } 4(a - 6)^2 + 9(b - 5)^2 \leq 36$$

$$\Rightarrow \frac{(a - 6)^2}{9} + \frac{(b - 5)^2}{4} \leq 1$$

Taking axes as a -axis and b -axis



The set A represents square $PQRS$ inside set B representing ellipse and hence $A \subset B$.

2. (c) We have,

$$2|\sqrt{x} - 3| + \sqrt{x}(\sqrt{x} - 6) + 6 = 0$$

$$\text{Let } \sqrt{x} - 3 = y$$

$$\Rightarrow \sqrt{x} = y + 3$$

$$\therefore 2|y| + (y + 3)(y - 3) + 6 = 0$$

$$\Rightarrow 2|y| + y^2 - 3 = 0$$

$$\Rightarrow |y|^2 + 2|y| - 3 = 0$$

$$\Rightarrow (|y| + 3)(|y| - 1) = 0$$

$$\Rightarrow |y| \neq -3 \Rightarrow |y| = 1$$

$$\Rightarrow y = \pm 1 \Rightarrow \sqrt{x} - 3 = \pm 1$$

$$\Rightarrow \sqrt{x} = 4, 2 \Rightarrow x = 16, 4$$

3. (c) We have,

$$\alpha, \beta \text{ are the roots of } x^2 - x + 1 = 0$$

$$\therefore \text{Roots of } x^2 - x + 1 = 0 \text{ are } -\omega, -\omega^2$$

$$\therefore \text{Let } \alpha = -\omega \text{ and } \beta = -\omega^2$$

$$\Rightarrow \alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107}$$

$$= -(\omega^{101} + \omega^{214})$$

$$= -(\omega^2 + \omega) \quad (\because \omega^3 = 1)$$

$$= -(-1)$$

$$[\because 1 + \omega + \omega^2 = 0]$$

$$= 1$$

4. (a) Given 6 different novels and 3 different dictionaries.

Number of ways of selecting 4 novels from 6 novels is

$${}^6C_4 = \frac{6!}{2!4!} = 15$$

Number of ways of selecting 1 dictionary is from 3 dictionaries is

$${}^3C_1 = \frac{3!}{1!2!} = 3$$

$\therefore$ Total number of arrangement of 4 novels and 1 dictionary where dictionary is always in the middle, is

$$15 \times 3 \times 4! = 45 \times 24 = 1080$$

$$\begin{aligned} 5. (d) \text{ Key idea } &= (a + b)^n + (a - b)^n \\ &= 2({}^nC_0 a^n + {}^nC_2 a^{n-2} b^2 \\ &\quad + {}^nC_4 a^{n-4} b^4 \dots) \end{aligned}$$

We have,

$$(x + \sqrt{x^3 - 1})^5 + (x - \sqrt{x^3 - 1})^5, x > 1$$

$$\begin{aligned} &= 2({}^5C_0 x^5 + {}^5C_2 x^3 (\sqrt{x^3 - 1})^2 \\ &\quad + {}^5C_4 x (\sqrt{x^3 - 1})^4) \end{aligned}$$

$$\begin{aligned} &= 2(x^5 + 10x^3(x^3 - 1) + 5x(x^3 - 1)^2) \\ &= 2(x^5 + 10x^6 - 10x^3 + 5x^7 - 10x^4 + 5x) \end{aligned}$$

Sum of coefficients of all odd degree terms is

$$2(1 - 10 + 5 + 5) = 2$$

6. (c) We have, $a_1, a_2, a_3, \dots, a_{49}$ are in AP.

$$\sum_{k=0}^{12} a_{4k+1} = 416 \text{ and } a_9 + a_{43} = 66$$

Let $a_1 = a$ and d = common difference

$$\therefore a_1 + a_5 + a_9 + \dots + a_{49} = 416$$

$$\therefore a + (a + 4d) + (a + 8d) + \dots + (a + 48d) = 416$$

$$\Rightarrow \frac{13}{2}(2a + 48d) = 416$$

$$\Rightarrow a + 24d = 32 \quad \dots(i)$$

$$\text{Also } a_9 + a_{43} = 66$$

$$\therefore a + 8d + a + 42d = 66$$

$$\Rightarrow 2a + 50d = 66$$

$$\Rightarrow a + 25d = 33 \quad \dots(ii)$$

Solving Eqs. (i) and (ii), we get

$$a = 8 \text{ and } d = 1$$

$$\text{Now, } a_1^2 + a_2^2 + a_3^2 + \dots + a_{17}^2 = 140m$$

$$8^2 + 9^2 + 10^2 + \dots + 24^2 = 140m$$

$$\Rightarrow (1^2 + 2^2 + 3^2 + \dots + 24^2) - (1^2 + 2^2 + 3^2 + \dots + 7^2) = 140m$$

$$\Rightarrow \frac{24 \times 25 \times 49}{6} - \frac{7 \times 8 \times 15}{6} = 140m$$

$$\Rightarrow \frac{3 \times 7 \times 8 \times 5}{6}(7 \times 5 - 1) = 140m$$

$$\Rightarrow 7 \times 4 \times 5 \times 34 = 140m$$

$$\Rightarrow 140 \times 34 = 140m \Rightarrow m = 34$$

7. (b) We have,

$$1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots$$

A = sum of first 20 terms

B = sum of first 40 terms

$$\therefore A = 1^2 + 2 \cdot 2^2 + 3^2 + 2 \cdot 4^2 + 5^2 + 2 \cdot 6^2 + \dots + 2 \cdot 20^2$$

$$A = (1^2 + 2^2 + 3^2 + \dots + 20^2) + (2^2 + 4^2 + 6^2 + \dots + 20^2)$$

$$A = (1^2 + 2^2 + 3^2 + \dots + 20^2) + 4(1^2 + 2^2 + 3^2 + \dots + 10^2)$$

$$A = \frac{20 \times 21 \times 41}{6} + \frac{4 \times 10 \times 11 \times 21}{6}$$

$$A = \frac{20 \times 21}{6}(41 + 22) = \frac{20 \times 41 \times 63}{6}$$

Similarly

$$B = (1^2 + 2^2 + 3^2 + \dots + 40^2) + 4(1^2 + 2^2 + \dots + 20^2)$$

$$B = \frac{40 \times 41 \times 81}{6} + \frac{4 \times 20 \times 21 \times 41}{6}$$

$$B = \frac{40 \times 41}{6}(81 + 42)$$

$$= \frac{40 \times 41 \times 123}{6}$$

$$\text{Now, } B - 2A = 100\lambda$$

$$\therefore \frac{40 \times 41 \times 123}{6} - \frac{2 \times 20 \times 21 \times 63}{6}$$

$$= 100\lambda$$

$$\Rightarrow \frac{40}{6}(5043 - 1323) = 100\lambda$$

$$\Rightarrow \frac{40}{6} \times 3720 = 100\lambda$$

$$\Rightarrow 40 \times 620 = 100\lambda$$

$$\Rightarrow \lambda = \frac{40 \times 620}{100} = 248$$

8. (c) Key idea Use property of greatest integer function $[x] = x - \{x\}$.

We have,

$$\lim_{x \rightarrow 0^+} x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{15}{x} \right] \right)$$

We know, $[x] = x - \{x\}$

$$\therefore \left[\frac{1}{x} \right] = \frac{1}{x} - \left\{ \frac{1}{x} \right\}$$

$$\text{Similarly, } \left[\frac{n}{x} \right] = \frac{n}{x} - \left\{ \frac{n}{x} \right\}$$

$\therefore$ Given limit

$$\begin{aligned} &= \lim_{x \rightarrow 0^+} x \left(\frac{1}{x} - \left\{ \frac{1}{x} \right\} + \frac{2}{x} - \left\{ \frac{2}{x} \right\} + \dots \right. \\ &\quad \left. \dots \frac{15}{x} - \left\{ \frac{15}{x} \right\} \right) \end{aligned}$$

$$= \lim_{x \rightarrow 0^+} (1 + 2 + 3 + \dots + 15) - x$$

$$\left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{15}{x} \right] \right)$$

$$= 120 - 0 = 120$$

14 Engineering Entrances Solved Papers 2018

$$\left[\begin{array}{l} \because 0 \leq \left\{ \frac{n}{x} \right\} < 1, \text{ therefore} \\ 0 \leq x \left\{ \frac{n}{x} \right\} < x \Rightarrow \lim_{x \rightarrow 0^+} x \left\{ \frac{n}{x} \right\} = 0 \end{array} \right]$$

9. (b) We have,

$$\begin{aligned} I &= \int \frac{\sin^2 x \cdot \cos^2 x}{(\sin^5 x + \cos^3 x \cdot \sin^2 x + \sin^3 x \cdot \cos^2 x + \cos^5 x)^2} dx \\ &= \int \frac{\sin^2 x \cos^2 x}{\{\sin^3 x (\sin^2 x + \cos^2 x) + \cos^3 x (\sin^2 x + \cos^2 x)\}^2} dx \\ &= \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx \\ &= \int \frac{\sin^2 x \cos^2 x}{\cos^6 x (1 + \tan^3 x)^2} dx \\ &= \int \frac{\tan^2 x \sec^2 x}{(1 + \tan^3 x)^2} dx \\ \text{Put } \tan^3 x &= t \\ \Rightarrow 3 \tan^2 x \sec^2 x dx &= dt \\ \therefore I &= \frac{1}{3} \int \frac{dt}{(1+t)^2} \\ \Rightarrow I &= \frac{-1}{3(1+t)} + C \\ \Rightarrow I &= \frac{-1}{3(1 + \tan^3 x)} + C \end{aligned}$$

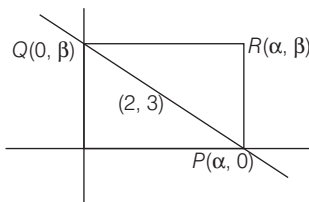
10. (d) **Key idea** Use property

$$\begin{aligned} &= \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \\ \text{Let } I &= \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx \\ \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \frac{\sin^2 \left(-\frac{\pi}{2} + \frac{\pi}{2} - x \right)}{1+2^{-x}} dx \\ &\left[\because \int_a^b f(x) dx = \int_a^b f(a+b-x) dx \right] \\ \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx \\ \Rightarrow I &= \int_{-\pi/2}^{\pi/2} \frac{2^x \sin^2 x}{2^x + 1} dx \\ \Rightarrow 2I &= \int_{-\pi/2}^{\pi/2} \sin^2 x \left(\frac{2^x + 1}{2^x + 1} \right) dx \\ \Rightarrow 2I &= \int_{-\pi/2}^{\pi/2} \sin^2 x dx \\ \Rightarrow 2I &= 2 \int_0^{\pi/2} \sin^2 x dx \\ &[\because \sin^2 x \text{ is an even function}] \\ \Rightarrow I &= \int_0^{\pi/2} \sin^2 x dx \end{aligned}$$

$$\begin{aligned} \Rightarrow I &= \int_0^{\pi/2} \cos^2 x dx \\ &\left[\because \int_0^a f(x) dx = \int_0^a f(a-x) dx \right] \end{aligned}$$

$$\begin{aligned} \Rightarrow 2I &= \int_0^{\pi/2} dx \\ \Rightarrow 2I &= [x]_0^{\pi/2} \Rightarrow I = \frac{\pi}{4} \end{aligned}$$

11. (c)



Equation of line PQ is $\frac{x}{\alpha} + \frac{y}{\beta} = 1$

Since this line passes through fixed point (2, 3).

$$\therefore \frac{2}{\alpha} + \frac{3}{\beta} = 1$$

$\therefore$ Locus of R is

$$\begin{aligned} 2\beta + 3\alpha &= \alpha\beta \\ \text{i.e. } 2y + 3x &= xy \\ \Rightarrow 3x + 2y &= xy \end{aligned}$$

12. (c) **Key idea** Orthocentre, centroid and circumcentre are collinear and centroid divide orthocentre and circumcentre in 2 : 1 ratio.

We have orthocentre and centroid of a triangle be A(-3, 5) and B(3, 3) respectively and C circumcentre.



$$\begin{aligned} \text{Clearly, } AB &= \sqrt{(3+3)^2 + (3-5)^2} \\ &= \sqrt{36+4} = 2\sqrt{10} \end{aligned}$$

We know that, $AB : BC = 2 : 1$

$$\begin{aligned} \Rightarrow BC &= \sqrt{10} \\ \text{Now, } AC &= AB + BC \\ &= 2\sqrt{10} + \sqrt{10} = 3\sqrt{10} \end{aligned}$$

Since, AC is a diameter of circle.

$$\therefore r = \frac{AC}{2} \Rightarrow r = \frac{3\sqrt{10}}{2} = 3\sqrt{\frac{5}{2}}$$

13. (d) **Key idea** Equation of tangent to the curve

$x^2 = 4ay$ at (x_1, y_1) is

$$xx_1 = 4a \left(\frac{y + y_1}{2} \right)$$

Tangent to the curve $x^2 = y - 6$ at (1, 7) is

$$x = \frac{y+7}{2} - 6$$

$$\Rightarrow 2x - y + 5 = 0 \quad \dots(i)$$

Equation of circle is

$$x^2 + y^2 + 16x + 12y + c = 0$$

Centre (-8, -6)

$$r = \sqrt{8^2 + 6^2} - c = \sqrt{100 - c}$$

Since, line $2x - y + 5 = 0$ also touches the circle.

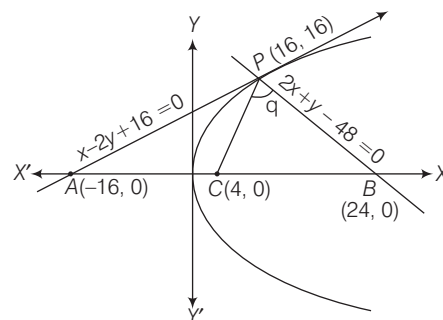
$$\therefore \sqrt{100 - c} = \left| \frac{2(-8) - (-6) + 5}{\sqrt{2^2 + 1^2}} \right|$$

$$\Rightarrow \sqrt{100 - c} = \left| \frac{-16 + 6 + 5}{\sqrt{5}} \right|$$

$$\Rightarrow \sqrt{100 - c} = |-\sqrt{5}|$$

$$\Rightarrow 100 - c = 5 \Rightarrow c = 95$$

14. (b) Equation of tangent and normal to the curve $y^2 = 16x$ at (16, 16) is $x - 2y + 16 = 0$ and $2x + y - 48 = 0$, respectively.



$A = (-16, 0)$; $B = (24, 0)$

C is the centre of circle passing through PAB

i.e. $C = (4, 0)$

Slope of

$$PC = \frac{16 - 0}{16 - 4} = \frac{16}{12} = \frac{4}{3} = m_1$$

Slope of

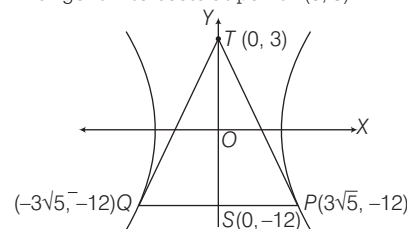
$$PB = \frac{16 - 0}{16 - 24} = \frac{16}{-8} = -2 = m_2$$

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

$$\Rightarrow \tan \theta = \left| \frac{\frac{4}{3} + 2}{1 - \left(\frac{4}{3}\right)(2)} \right| \Rightarrow \tan \theta = 2$$

15. (a) Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q.

Tangent intersects at point T(0, 3)



Clearly, PQ is chord of contact.

Engineering Entrances Solved Papers 2018 15

$\therefore$ Equation of PQ is $-3y = 36$

$$\Rightarrow y = -12$$

Solving the curve $4x^2 - y^2 = 36$ and

$$y = -12,$$

we get $x = \pm 3\sqrt{5}$

$$\text{Area of } \Delta PQT = \frac{1}{2} \times PQ \times ST$$

$$= \frac{1}{2}(6\sqrt{5} \times 15) = 45\sqrt{5}$$

16. (c) **Key idea** Standard deviation is remain unchanged, if observations are added or subtracted by a fixed number

We have,

$$\sum_{i=1}^9 (x_i - 5) = 9 \text{ and } \sum_{i=1}^9 (x_i - 5)^2 = 45$$

$$SD = \sqrt{\frac{\sum_{i=1}^9 (x_i - 5)^2}{9} - \left(\frac{\sum_{i=1}^9 (x_i - 5)}{9}\right)^2}$$

$$\Rightarrow SD = \sqrt{\frac{45}{9} - \left(\frac{9}{9}\right)^2}$$

$$\Rightarrow SD = \sqrt{5 - 1} = \sqrt{4} = 2$$

17. (b) **Key idea** Apply the identity

$$\cos(x + y)\cos(x - y) = \cos^2 x - \sin^2 y$$

$$\text{and } \cos 3x = 4\cos^3 x - 3\cos x$$

We have,

$$8\cos x \left(\cos\left(\frac{\pi}{6} + x\right)\cos\left(\frac{\pi}{6} - x\right) - \frac{1}{2} \right) = 1$$

$$\Rightarrow 8\cos x \left(\cos^2 \frac{\pi}{6} - \sin^2 x - \frac{1}{2} \right) = 1$$

$$\Rightarrow 8\cos x \left(\frac{3}{4} - \sin^2 x - \frac{1}{2} \right) = 1$$

$$\Rightarrow 8\cos x \left(\frac{3}{4} - \frac{1}{2} - 1 + \cos^2 x \right) = 1$$

$$\Rightarrow 8\cos x \left(\frac{-3 + 4\cos^2 x}{4} \right) = 1$$

$$\Rightarrow 2(4\cos^3 x - 3\cos x) = 1$$

$$\Rightarrow 2\cos 3x = 1 \Rightarrow \cos 3x = \frac{1}{2}$$

$$\Rightarrow 3x = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3} \quad [0 \leq 3x \leq 3\pi]$$

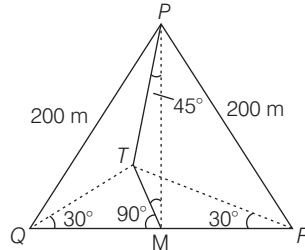
$$\Rightarrow x = \frac{\pi}{9}, \frac{5\pi}{9}, \frac{7\pi}{9}$$

$$\text{Sum} = \frac{\pi}{9} + \frac{5\pi}{9} + \frac{7\pi}{9} = \frac{13\pi}{9} \Rightarrow$$

$$k\pi = \frac{13\pi}{9}$$

$$\text{Hence, } k = \frac{13}{9}$$

18. (a)



Let height of tower TM be h .

$$\text{In } \Delta PMT, \tan 45^\circ = \frac{TM}{PM} \Rightarrow 1 = \frac{h}{PM}$$

$$\Rightarrow PM = h$$

$$\text{In } \Delta TOM, \tan 30^\circ = \frac{h}{OM}; OM = \sqrt{3}h$$

$$\text{In } \Delta PMQ, PM^2 + QM^2 = PQ^2$$

$$h^2 + (\sqrt{3}h)^2 = (200)^2$$

$$\Rightarrow 4h^2 = (200)^2 \Rightarrow h = 100 \text{ m}$$

19. (a) **Key idea** Use De-Morgan's and distributive law.

$$\text{We have, } \sim(p \vee q) \vee (\sim p \wedge q)$$

$$\equiv (\sim p \wedge \sim q) \vee (\sim p \wedge q)$$

$$[\because \text{By De-Morgan's law}]$$

$$\sim(p \vee q) = (\sim p \wedge \sim q)$$

$$\equiv \sim p \wedge (\sim q \vee q)$$

$$[\text{By distributive law}]$$

$$\equiv \sim p \wedge t \quad [\sim q \vee q = t]$$

$$\equiv \sim p$$

JEE Advanced

1. (a, b, d)

(a) Let $z = -1 - i$ and $\arg(z) = \theta$

$$\text{Now, } \tan \theta = \frac{\text{Im}(z)}{\text{Re}(z)} = \frac{-1}{-1} = 1$$

$$\Rightarrow \theta = \frac{\pi}{4}$$

Since, $x < 0, y < 0$

$$\therefore \arg(z) = -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$$

(b) We have, $f(t) = \arg(-1 + it)$

$$(-1 + it) = \begin{cases} \pi - \tan^{-1}t, & t \geq 0 \\ -(\pi + \tan^{-1}t), & t < 0 \end{cases}$$

This function is discontinuous at $t = 0$.

(c) We have,

$$\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$\text{Now, } \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$$

$$\therefore \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2) + 2n\pi$$

$$= \arg(z_1) - \arg(z_2) + 2n\pi - \arg(z_1) + \arg(z_2) = 2n\pi$$

So, given expression is multiple of 2π .

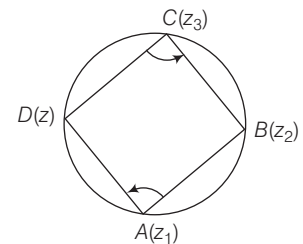
(d) We have,

$$\arg\left(\frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)}\right) = \pi$$

$$\Rightarrow \left(\frac{z - z_1}{z - z_3}\right)\left(\frac{z_2 - z_3}{z_2 - z_1}\right) \text{ is purely real}$$

Thus, the points $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z)$ taken in order would be concyclic if $[(z - z_1)(z_2 - z_3) / (z_2 - z_1)(z - z_3)]$ purely real.

Hence, it is a circle.



$\therefore$ (a), (b), (d) are false statement.

Hence, option (a), (b), (d) are correct answer.

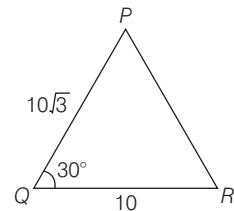
2. (b, c, d) We have,

In ΔPQR

$$\angle PQR = 30^\circ$$

$$PQ = 10\sqrt{3}$$

$$QR = 10$$



By cosine rule

$$\cos 30^\circ = \frac{PQ^2 + QR^2 - PR^2}{2PQ \cdot QR}$$

$$\Rightarrow \frac{\sqrt{3}}{2} = \frac{300 + 100 - PR^2}{200\sqrt{3}}$$

$$\Rightarrow 300 = 300 + 100 - PR^2$$

$$\Rightarrow PR = 10$$

Since, $PR = QR = 10$

$$\therefore \angle QPR = 30^\circ$$

$$\text{and } \angle QRP = 120^\circ$$

$$\text{Area of } \Delta PQR = \frac{1}{2} PQ \cdot QR \cdot \sin 30^\circ$$

$$= \frac{1}{2} \times 10\sqrt{3} \times 10 \times \frac{1}{2}$$

$$= 25\sqrt{3}$$

Radius of incircle of

$$\Delta PQR = \frac{\text{Area of } \Delta PQR}{\text{Semi-perimetre of } \Delta PQR}$$

16 Engineering Entrances Solved Papers 2018

$$\text{i.e. } r = \frac{\Delta}{s} = \frac{25\sqrt{3}}{\frac{10\sqrt{3} + 10 + 10}{2}} = \frac{25\sqrt{3}}{5(\sqrt{3} + 2)}$$

$$\Rightarrow r = 5\sqrt{3}(2 - \sqrt{3}) = 10\sqrt{3} - 15$$

and radius of circumcircle

$$(R) = \frac{abc}{4\Delta} = \frac{10\sqrt{3} \times 10 \times 10}{4 \times 25\sqrt{3}} = 10$$

$\therefore$ Area of circumcircle of

$$\Delta PQR = \pi R^2 = 100\pi$$

Hence, option (b), (c) and (d) are correct answer.

$$\begin{aligned} 3. (8) & ((\log_2 9)^2)^{\frac{1}{\log_2 (\log_2 9)}} \times (\sqrt{7})^{\frac{1}{\log_4 7}} \\ &= (\log_2 9)^{2 \log_2 (\log_2 9)} \times 7^{\frac{1}{2} \cdot \log_7 4} \\ &= (\log_2 9)^{2 \log_2 (\log_2 9)} \times 7^{\log_7 2} \\ &= (\log_2 9)^{\log_2 (\log_2 9)^2} \times 7^{\log_7 2} \\ &= 2^2 \times 2 = 8 \end{aligned}$$

4. (625) A number is divisible by 4 if last 2 digit number is divisible by 4.

$\therefore$ Last two digit number divisible by 4 from (1, 2, 3, 4, 5) are

$$12, 24, 32, 44, 52$$

$\therefore$ The number of 5 digit number which are divisible by 4, from the digit (1, 2, 3, 4, 5) and digit is repeated is

$$5 \times 5 \times 5 \times (5 \times 1) = 625$$

5. (3798) Here, $X = \{1, 6, 11, \dots, 10086\}$

$$[\because a_n = a + (n-1)d]$$

and $Y = \{9, 16, 23, \dots, 14128\}$

$$X \cap Y = \{16, 51, 86, \dots\}$$

t_n of $X \cap Y$ is less than or equal to 10086

$$\therefore t_n = 16 + (n-1)35 \leq 10086$$

$$\Rightarrow n \leq 288.7$$

$$\therefore n = 288$$

$$\therefore n(X \cup Y) = n(X) + n(Y) - n(X \cap Y)$$

$$\therefore n(X \cup Y) = 2018 + 2018 - 288 = 3748$$

6. (1) We have,

$$y_n = \frac{1}{n}[(n+1)(n+2) \dots (n+n)]^{1/n}$$

and $\lim_{n \rightarrow \infty} y_n = L$

$$\Rightarrow L = \lim_{n \rightarrow \infty} \frac{1}{n}[(n+1)(n+2)(n+3) \dots (n+n)]^{1/n}$$

$$\Rightarrow L = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right) \left(1 + \frac{2}{n}\right) \left(1 + \frac{3}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right]^{1/n}$$

$$\Rightarrow \log L = \lim_{n \rightarrow \infty} \frac{1}{n} \left[\log \left(1 + \frac{1}{n}\right) + \log \left(1 + \frac{2}{n}\right) + \dots + \log \left(1 + \frac{n}{n}\right) \right]$$

$$\Rightarrow \log L = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n \log \left(1 + \frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \log(1+x) dx$$

$$\Rightarrow \log L = (x \cdot \log(1+x))_0^1 - \int_0^1 \frac{d}{dx} (\log(1+x)) dx$$

[by using integration by parts]

$$\Rightarrow \log L = [x \log(1+x)]_0^1 - \int_0^1 \frac{x}{1+x} dx$$

$$\Rightarrow$$

$$\log L = \log 2 - \int_0^1 \left(\frac{x+1}{x+1} - \frac{1}{x+1} \right) dx$$

$$\Rightarrow \log L = \log 2 - [x]_0^1 + [\log(x+1)]_0^1$$

$$\Rightarrow \log L = \log 2 - 1 + \log 2 - 0$$

$$\Rightarrow \log L = \log 4 - \log e = \log \frac{4}{e}$$

$$\Rightarrow L = \frac{4}{e} \Rightarrow [L] = \left[\frac{4}{e} \right] = 1$$

7. (0.5) We have, α, β are the roots of

$$\sqrt{3} a \cos x + 2b \sin x = c$$

$$\therefore \sqrt{3} a \cos \alpha + 2b \sin \alpha = c \quad \dots (i)$$

$$\text{and } \sqrt{3} a \cos \beta + 2b \sin \beta = c \quad \dots (ii)$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\sqrt{3} a (\cos \alpha - \cos \beta) + 2b (\sin \alpha - \sin \beta) = 0$$

$$\Rightarrow \sqrt{3} a \left(-2 \sin \left(\frac{\alpha + \beta}{2} \right) \right) \sin \left(\frac{\alpha - \beta}{2} \right) + 2b \left(2 \cos \left(\frac{\alpha + \beta}{2} \right) \right) \sin \left(\frac{\alpha - \beta}{2} \right) = 0$$

$$\Rightarrow \sqrt{3} a \sin \left(\frac{\alpha + \beta}{2} \right) = 2b \cos \left(\frac{\alpha + \beta}{2} \right)$$

$$\Rightarrow \tan \left(\frac{\alpha + \beta}{2} \right) = \frac{2b}{\sqrt{3} a}$$

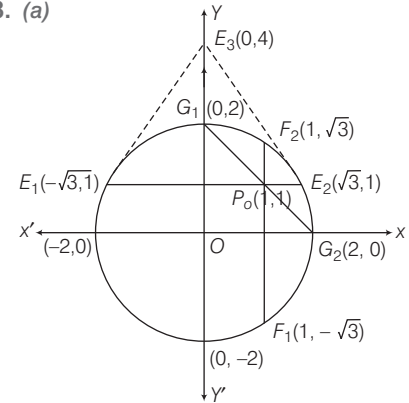
$$\Rightarrow \tan \left(\frac{\pi}{6} \right) = \frac{2b}{\sqrt{3} a}$$

$$\Rightarrow \tan \left(\frac{\pi}{6} \right) = \frac{2b}{\sqrt{3} a}$$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{2b}{\sqrt{3} a} \Rightarrow \frac{b}{a} = \frac{1}{2}$$

$$\Rightarrow \frac{b}{a} = 0.5$$

8. (a)



Equation of tangent at $E_1(-\sqrt{3}, 1)$ is $-\sqrt{3}x + y = 4$ and at $E_2(\sqrt{3}, 1)$ is $\sqrt{3}x + y = 4$

Intersection point of tangent at E_1 and E_2 is (0, 4)

$\therefore$ Coordinates of E_3 is (0, 4)

Similarly, equation of tangent at $F_1(1, -\sqrt{3})$ and $F_2(1, \sqrt{3})$ are $x - \sqrt{3}y = 4$ and $x + \sqrt{3}y = 4$, respectively and intersection point is (4, 0), i.e., $F_3(4, 0)$ and equation of tangent at $G_1(0, 2)$ and $G_2(2, 0)$ are $2y = 4$ and $2x = 4$, respectively and intersection point is (2, 2) i.e., $G_3(2, 2)$.

Point $E_3(0, 4)$, $F_3(4, 0)$ and $G_3(2, 2)$ satisfies the line $x + y = 4$.

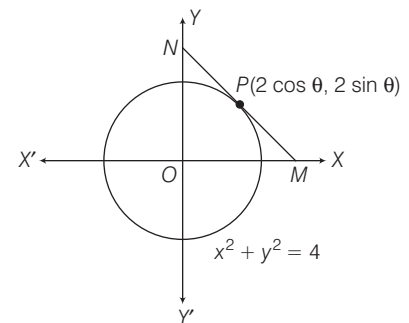
9. (d) We have, $x^2 + y^2 = 4$

Let $P(2 \cos \theta, 2 \sin \theta)$ be a point on a circle.

$\therefore$ Tangent at P is

$$2 \cos \theta x + 2 \sin \theta y = 4$$

$$\Rightarrow x \cos \theta + y \sin \theta = 2$$



$\therefore$ The coordinates at $M\left(\frac{2}{\cos \theta}, 0\right)$ and

$$N\left(0, \frac{2}{\sin \theta}\right)$$

Let (h, k) is mid-point of MN

$$\therefore h = \frac{1}{\cos \theta} \text{ and } k = \frac{1}{\sin \theta}$$

$$\Rightarrow \cos \theta = \frac{1}{h} \text{ and } \sin \theta = \frac{1}{k}$$

$$\Rightarrow \cos^2 \theta + \sin^2 \theta = \frac{1}{h^2} + \frac{1}{k^2}$$

$$\Rightarrow 1 = \frac{h^2 + k^2}{h^2 \cdot k^2}$$

$$\Rightarrow h^2 + k^2 = h^2 k^2$$

$\therefore$ Mid-point of MN lie on the curve
 $x^2 + y^2 = x^2 y^2$

10. (a) Here, five students S_1, S_2, S_3, S_4 and S_5 and five seats R_1, R_2, R_3, R_4 and R_5

$\therefore$ Total number of arrangement of sitting five students is $5! = 120$

Here, S_1 gets previously allotted seat R_1
 $\therefore S_2, S_3, S_4$ and S_5 not get previously seats.

Total number of way S_2, S_3, S_4 and S_5 not get previously seats is

$$4! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \right)$$

$$= 24 \left(1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} \right)$$

$$= 24 \left(\frac{12 - 4 + 1}{24} \right) = 9$$

$$\therefore \text{Required probability} = \frac{9}{120} = \frac{3}{40}$$

11. (c) Here, $n(T_1 \cap T_2 \cap T_3 \cap T_4) = \text{Total} - n(\overline{T_1} \cup \overline{T_2} \cup \overline{T_3} \cup \overline{T_4})$
 $\Rightarrow n(T_1 \cap T_2 \cap T_3 \cap T_4)$
 $= 5! - [{}^4C_1 4! 2! - ({}^3C_1 3! 2! + {}^3C_1 3! 2! 2!)$
 $+ ({}^2C_1 2! 2! + {}^4C_1 2 \cdot 2!) - 2]$

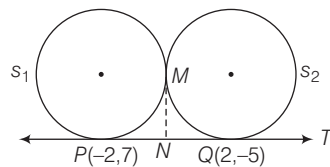
$$\Rightarrow n(T_1 \cap T_2 \cap T_3 \cap T_4)$$

$$= 120 - [192 - (36 + 72) + (8 + 16) - 2]$$

$$= 120 - [192 - 108 + 24 - 2] = 14$$

$$\therefore \text{Required probability} = \frac{14}{120} = \frac{7}{60}$$

12. (a, d) It is given that T is tangents to S_1 at P and S_2 at Q and S_1 and S_2 touch externally at M .



$$\therefore MN = NP = NQ$$

$\therefore$ Locus of M is a circle having PQ as its diameter of circle

$\therefore$ Equation of circle

$$(x - 2)(x + 2) + (y + 5)(y - 7) = 0$$

$$\Rightarrow x^2 + y^2 - 2y - 39 = 0$$

Hence, $E_1 : x^2 + y^2 - 2y - 39 = 0, x \neq \pm 2$

Locus of mid-point of chord (h, k) of the circle E_1 is

$$xh + yk - (y + k) - 39$$

$$= h^2 + k^2 - 2k - 39$$

$$\Rightarrow xh + yk - y - k = h^2 + k^2 - 2k$$

Since, chord is passing through $(1, 1)$.

$\therefore$ Locus of mid-point of chord (h, k) is

$$h + k - 1 - k = h^2 + k^2 - 2k$$

$$\Rightarrow h^2 + k^2 - 2k - h + 1 = 0$$

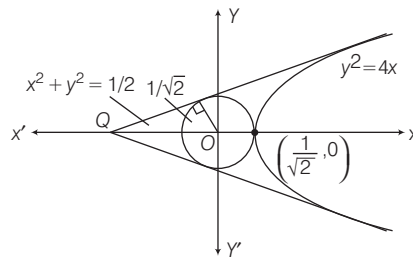
$$\text{Locus is } E_2 : x^2 + y^2 - x - 2y + 1 = 0$$

Now, after checking options, (a) and (d) are correct.

13. (a, c) We have,

$$\text{Equations of circle } x^2 + y^2 = \frac{1}{2}$$

$$\text{and Equations of parabola } y^2 = 4x$$



Let the equation of common tangent of parabola and circle is

$$y = mx + \frac{1}{m}$$

$$\text{Since, radius of circle} = \frac{1}{\sqrt{2}}$$

$$\therefore \frac{1}{\sqrt{2}} = \left| \frac{0 + 0 + 1/m}{\sqrt{1 + m^2}} \right|$$

$$\Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

$\therefore$ Equation of common tangents are

$$y = x + 1 \text{ and } y = -x - 1$$

Intesection point of common tangent at $Q(-1, 0)$

$$\therefore \text{Equation of ellipse } \frac{x^2}{1} + \frac{y^2}{1/2} = 1$$

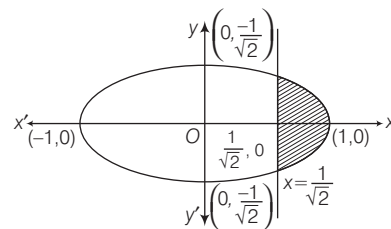
where, $a^2 = 1, b^2 = 1/2$

Now, eccentricity

$$(e) = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

and length of latusrectum

$$= \frac{2b^2}{a} = \frac{2(1/2)}{1} = 1$$



$\therefore$ Area of shaded region

$$= 2 \int_{1/\sqrt{2}}^1 \frac{1}{\sqrt{2}} \sqrt{1 - x^2} dx$$

$$= \sqrt{2} \left[\frac{x}{2} \sqrt{1 - x^2} + \frac{1}{2} \sin^{-1} x \right]_{1/\sqrt{2}}^1$$

$$= \sqrt{2} \left[\left(0 + \frac{\pi}{4} \right) - \left(\frac{1}{4} + \frac{\pi}{8} \right) \right]$$

$$= \sqrt{2} \left(\frac{\pi}{8} - \frac{1}{4} \right) = \frac{\pi - 2}{4\sqrt{2}}$$

14. (a, c, d) We have,

$$sz + t\bar{z} + r = 0 \quad \dots(i)$$

On taking conjugate

$$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$z = \frac{\bar{r}t - r\bar{s}}{|s|^2 - |t|^2}$$

(a) For unique solutions of z

$$|s|^2 - |t|^2 \neq 0 \Rightarrow |s| \neq |t|$$

It is true

(b) If $|s| = |t|$, then $\bar{r}t - r\bar{s}$ may or may not be zero. So, z may have no solutions.

$\therefore L$ may be an empty set.

It is false.

(c) If elements of set L represents line, then this line and given circle intersect at maximum two point. Hence, it is true.

(d) In this case locus of z is a line, so L has infinite elements. Hence, it is true.

15. (646) We have,

$$X = ({}^{10}C_1)^2 + 2({}^{10}C_2)^2 + 3({}^{10}C_3)^2 + \dots$$

$$+ 10({}^{10}C_{10})^2$$

$$\Rightarrow X = \sum_{r=1}^{10} r({}^{10}C_r)^2$$

$$\Rightarrow X = \sum_{r=1}^{10} r {}^{10}C_r {}^{10}C_{r-1}$$

$$\Rightarrow X = \sum_{r=1}^{10} r \times \frac{10}{r} {}^9C_{r-1} {}^{10}C_r$$

$$\left[\because {}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} \right]$$

$$\Rightarrow X = 10 \sum_{r=1}^{10} {}^9C_{r-1} {}^{10}C_r$$

$$\Rightarrow X = 10 \sum_{r=1}^{10} {}^9C_{r-1} {}^{10}C_{10-r}$$

$$[\because {}^nC_r = {}^nC_{n-r}]$$

$$\Rightarrow X = 10 \times {}^{19}C_9$$

$$[\because {}^{n-1}C_{r-1} {}^nC_{n-r} = {}^{2n-1}C_{n-1}]$$

Now,

$$\frac{1}{1430} X = \frac{10 \times {}^{19}C_9}{1430} = \frac{{}^{19}C_9}{143} = \frac{{}^{19}C_9}{11 \times 13}$$

$$= \frac{19 \times 17 \times 16}{8} = 19 \times 34 = 646$$

18 Engineering Entrances Solved Papers 2018

16. (c) Given 6 boys

$M_1, M_2, M_3, M_4, M_5, M_6$ and
5 girls G_1, G_2, G_3, G_4, G_5

(i) $\alpha_1 \rightarrow$ Total number of ways of
selecting 3 boys and 2 girls from 6
boys and 5 girls.

$$\text{i.e., } {}^6C_3 \times {}^5C_2 = 20 \times 10 = 200$$

$$\therefore \alpha_1 = 200$$

(ii) $\alpha_2 \rightarrow$ Total number of ways
selecting at least 2 member and
having equal number of boys and
girls

$$\text{i.e., } {}^6C_1 {}^5C_1 + {}^6C_2 {}^5C_2 + {}^6C_3 {}^5C_3 \\ + {}^6C_4 {}^5C_4 + {}^6C_5 {}^5C_5$$

$$= 30 + 150 + 200 + 75 + 6 = 461$$

$$\Rightarrow \alpha_2 = 461$$

(iii) $\alpha_3 \rightarrow$ Total number of ways of
selecting 5 members in which at
least 2 of them girls

$$\text{i.e., } {}^5C_2 {}^6C_3 + {}^5C_3 {}^6C_2 + {}^5C_4 {}^6C_1 \\ + {}^5C_5 {}^6C_0$$

$$= 200 + 150 + 30 + 1 = 381$$

$$\alpha_3 = 381$$

(iv) $\alpha_4 \rightarrow$ Total number of ways of
selecting 4 members in which at
least two girls such that M_1 and G_1
are not included together.

G_1 is included $\rightarrow$

$${}^4C_1 \cdot {}^5C_2 + {}^4C_2 \cdot {}^5C_1 + {}^4C_3$$

$$= 40 + 30 + 4 = 74$$

M_1 is included $\rightarrow$

$${}^4C_2 \cdot {}^5C_1 + {}^4C_3 = 30 + 4 = 34$$

G_1 and M_1 both are not included

$${}^4C_4 + {}^4C_3 \cdot {}^5C_1 + {}^4C_2 \cdot {}^5C_2$$

$$1 + 20 + 60 = 81$$

$\therefore$ Total number

$$= 74 + 34 + 81 = 189$$

$$\alpha_4 = 189$$

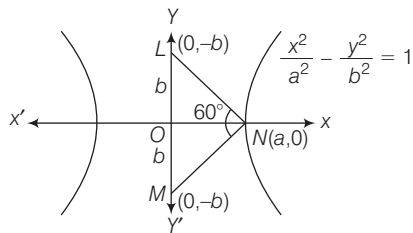
Now, $P \rightarrow 4; Q \rightarrow 6; R \rightarrow 5; S \rightarrow 2$

Hence, option (c) is correct.

17. (b) We have,

Equation of hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$



It is given, $\angle LNM = 60^\circ$
and Area of $\triangle LMN = 4\sqrt{3}$

Now, $\triangle LMN$ is an equilateral triangle
whose sides is $2b$

$$[\because \angle LON \cong \angle MOL; \therefore \\ \angle NLO = \angle NMO = 60^\circ]$$

$$\therefore \text{Area of } \triangle LMN = \frac{\sqrt{3}}{4} (2b)^2$$

$$\Rightarrow 4\sqrt{3} = \sqrt{3}b^2 \Rightarrow b = 2$$

$$\text{Also, area of } \triangle LMN = \frac{1}{2} a(2b) = ab$$

$$\Rightarrow 4\sqrt{3} = a(2) \Rightarrow a = 2\sqrt{3}$$

$$(P) \text{ Length of conjugate axis} \\ = 2b = 2(2) = 4$$

$$(Q) \text{ Eccentricity } (e) = \sqrt{1 + \frac{b^2}{a^2}} \\ = \sqrt{1 + \frac{4}{12}} = \frac{4}{2\sqrt{3}} = \frac{2}{\sqrt{3}}$$

$$(R) \text{ Distance between the foci} \\ = 2ae = 2 \times 2\sqrt{3} \times \frac{2}{\sqrt{3}} = 8$$

(S) The length of latusrectum

$$= \frac{2b^2}{a} = \frac{2(4)}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$$

$$P \rightarrow 4; Q \rightarrow 3; R \rightarrow 1; S \rightarrow 2$$

Hence, option (b) is correct.

BITSAT

$$1. (b) (1+x)^n \left(1 + \frac{1}{x}\right)^n = \left[(2+x) + \frac{1}{x}\right]^n \\ = {}^nC_0(2+x)^n \cdot \left(\frac{1}{x}\right) + \dots + {}^nC_n \cdot \frac{1}{x^n}$$

$$\therefore \text{Coefficient of } x^{-n} = {}^nC_n = 1$$

$$2. (c) T_{r+1} = \sqrt{3} \cdot {}^{20}C_r \left(\frac{1}{\sqrt{3}}\right)^r$$

$$\text{and } T_r = \sqrt{3} \cdot {}^{20}C_{r-1} \left(\frac{1}{\sqrt{3}}\right)^{r-1}$$

$$\text{Now, } \frac{T_{r+1}}{T_r} = \frac{20-r+1}{r} \left(\frac{1}{\sqrt{3}}\right)$$

Since, $T_{r+1} \geq T_r$

$$\Rightarrow 20-r+1 \geq \sqrt{3}r$$

$$\Rightarrow r \leq \frac{21}{\sqrt{3}+1} = \frac{21}{2.73}$$

$$\Rightarrow r \leq 7.692 \Rightarrow r = 7$$

$\therefore$ The greatest term is

$$T_8 = \sqrt{3} \cdot {}^{20}C_7 \left(\frac{1}{\sqrt{3}}\right)^7 = \frac{25840}{9}$$

3. (b) Let $x = 1$

$$= \cos 0^\circ + i \sin 0^\circ$$

$$= \cos 2r\pi + i \sin 2r\pi = e^{i2r\pi}$$

$$\Rightarrow x^{1/n} = e^{i(2r\pi)/n}; r = 0, 1, 2, \dots$$

Then, the roots are $1, e^{\frac{2\pi i}{n}}, e^{\frac{4\pi i}{n}}, \dots$

which are clearly on GP with common
ratio $e^{2\pi i/n}$.

4. (d) Given, a, b, c are in GP.

$$\therefore b^2 = ac$$

$$\text{and } 2(\log 2b - \log 3c)$$

$$= \log a - \log 2b + \log 3c - \log a$$

$$\Rightarrow b^2 = ac \text{ and } 2b = 3c$$

$$\Rightarrow c = \frac{4a}{9} \text{ and } b = \frac{2a}{3}$$

$$\therefore a + b = \frac{5a}{3} > c, b + c = \frac{10a}{9} > a$$

$$\text{and } c + a = \frac{13a}{9} > b.$$

$\therefore a, b, c$ are the sides of a triangle.

Also, a is the greatest side

$$\therefore \cos A = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{29}{48} < 0$$

$\therefore \triangle ABC$ is an obtuse angled triangle.

Hence, option (d) is correct.

5. (c) Given,

$$\sum_{r=1}^n t_r = \frac{n(n+1)(n+2)(n+3)}{8} = S_n$$

(say)

$$\therefore \sum_{r=1}^{n-1} t_r = \frac{(n-1)n(n+1)(n+2)}{8} = S_{n-1}$$

$$\text{Now, } t_n = S_n - S_{n-1} = \frac{n(n+1)(n+2)}{2}$$

$$\therefore \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{1}{t_r}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{2}{n(n+1)(n+2)}$$

$$= \lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)} \right)$$

$$= -\lim_{n \rightarrow \infty} \sum_{r=1}^n \left(\frac{1}{(n+1)(n+2)} - \frac{1}{n(n+1)} \right)$$

$$= -\lim_{n \rightarrow \infty} \left(\frac{1}{(n+1)(n+2)} - \frac{1}{2} \right)$$

$$= -\left(0 - \frac{1}{2}\right) = \frac{1}{2}$$

Hence, option (c) is correct.

6. (c) Truth table

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim p \vee \sim q$	$(p \vee q) \vee (\sim p \vee \sim q)$	$(\sim q \wedge p) \vee (p \vee \sim p)$
T	T	F	F	T	F	T	T
T	F	F	T	T	T	T	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	T	T

Hence, option (c) is correct.

Engineering Entrances Solved Papers 2018 19

7. (c) The two sets of m parallel lines along with two sets of two parallel lines of the given parallelogram will form two sets of $(m+2)$ parallel lines. Each parallelogram is formed by choosing two parallel lines from each of the above

$$\therefore \text{Total number of parallelograms} = {}^{m+2}C_2 \times {}^{m+2}C_2 = ({}^{m+2}C_2)^2$$

Hence, option (c) is correct.

8. (a) Given,
 $\sin x + \cos(t+x) + \cos(t-x) = 2$
 $\Rightarrow \sin x + 2\cos t \cdot \cos x = 2$
 For real solution

$$\sqrt{1+4\cos^2 t} \geq 2$$

$$\Rightarrow \cos^2 t \geq \frac{3}{4}$$

$$\Rightarrow -\frac{1}{2} \leq \sin t \leq \frac{1}{2}$$

Hence, option (a) is correct.

9. (c) We have,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\Rightarrow \cos^2 \alpha + \cos^2 \left(\frac{\pi}{2} - \alpha \right) + \cos^2 \gamma = 1$$

$$\left[\text{given, } \alpha + \beta = \frac{\pi}{2} \right]$$

$$\Rightarrow \cos^2 \alpha + \sin^2 \alpha + \cos^2 \gamma = 1$$

$$\Rightarrow 1 + \cos^2 \gamma = 1$$

$$\Rightarrow \cos^2 \gamma = 0$$

$$\Rightarrow \cos \gamma = 0$$

$$\therefore (\cos \alpha + \cos \beta + \cos \gamma)^2 = (\cos \alpha + \sin \alpha)^2$$

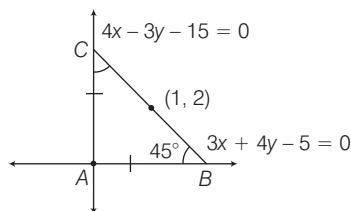
$$= 1 + 2 \sin \alpha \cdot \cos \alpha = 1 + \sin 2\alpha$$

Hence, option (c) is correct.

10. (c) The given straight lines are $3x + 4y = 5$ and $4x - 3y = 15$. Clearly, these straight lines are perpendicular to each other ($m_1 m_2 = -1$) and intersect at A. Now, B and C are points on these lines such that $AB = AC$ and BC passes through (1, 2).

From figure it is clear that

$$\angle B = \angle C = 45^\circ$$



Let slope of BC be m . Then,

$$\tan 45^\circ = \left| \frac{m + \frac{3}{4}}{1 - \frac{3}{4}m} \right| \Rightarrow \pm 1 = \frac{4m+3}{4-3m}$$

$$4m + 3 = \pm (4 - 3m)$$

$$4m + 3 = 4 - 3m$$

$$\text{or } 4m + 3 = -4 + 3m$$

$$m = \frac{1}{7} \quad \text{or } m = -7$$

Hence, equation of BC is

$$y - 2 = \frac{1}{7}(x - 1)$$

$$\text{or } y - 2 = -7(x - 1)$$

$$\Rightarrow 7y - 14 = x - 1$$

$$\text{or } y - 2 = -7x + 7$$

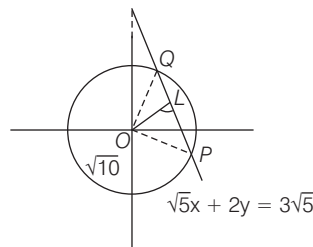
$$\Rightarrow x - 7y + 13 = 0$$

$$\text{or } 7x + y - 9 = 0$$

Hence, option (c) is correct.

11. (c) Length of perpendicular from origin to the line $x\sqrt{5} + 2y = 3\sqrt{5}$ is

$$OL = \frac{3\sqrt{5}}{\sqrt{(\sqrt{5})^2 + 2^2}} = \frac{3\sqrt{5}}{\sqrt{9}} = \sqrt{5}$$



$$\text{Radius of the given circle} = \sqrt{10}$$

$$= OQ = OP$$

$$PQ = 2OL$$

$$= 2\sqrt{OQ^2 - OL^2}$$

$$= 2\sqrt{10 - 5} = 2\sqrt{5}$$

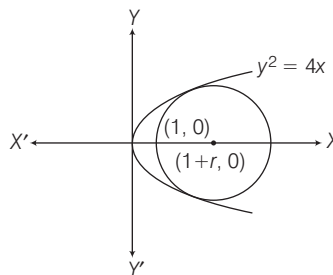
$$\text{Thus, area of } \triangle OPQ = \frac{1}{2} \times PQ \times OL$$

$$= \frac{1}{2} \times 2\sqrt{5} \times \sqrt{5}$$

$$= 5 \text{ sq units}$$

Hence, option (c) is correct.

12. (b) Let r be the radius of the largest circle passing through the focus (1, 0) of $y^2 = 4x$



Clearly, centre of the circle will be on X-axis and its coordinates are $(1+r, 0)$.

The equation of the circle, is $(x - 1 - r)^2 + y^2 = r^2$.

It touches $y^2 = 4x$. Therefore, the equation $(x - r - 1)^2 + 4x = r^2$ must have equal roots

$$\therefore 4(1 - r)^2 - 4(2r + 1) = 0 \Rightarrow r = 4$$

Hence, option (b) is correct.

13. (d) The point $P\left(\frac{\pi}{6}\right)$ is $\left(a \sec \frac{\pi}{6}, b \tan \frac{\pi}{6}\right)$

$$\text{or } P\left(\frac{2a}{\sqrt{3}}, \frac{b}{\sqrt{3}}\right)$$

$\therefore$ Equation of tangent at P is

$$\frac{x}{\sqrt{3}a/2} - \frac{y}{\sqrt{3}b} = 1$$

$\therefore$ Area of the triangle

$$= \frac{1}{2} \times \frac{\sqrt{3}a}{2} \times \sqrt{3}b = 3a^2$$

$$\therefore \frac{b}{a} = 4$$

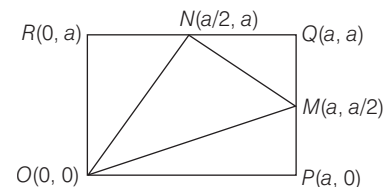
$$\therefore e^2 = 1 + \frac{b^2}{a^2} = 17$$

$$\text{Now, } e^2 - 9 = 17 - 9 = 8$$

Hence, option (d) is correct.

14. (c) Taking the coordinates of vertices O, P, Q, R as (0, 0), (a, 0), (a, a), (0, a), respectively.

$\therefore$ The coordinates of M is $\left(a, \frac{a}{2}\right)$ and N is $\left(\frac{a}{2}, a\right)$.



$\therefore$ Area of

$$\triangle OMN = \frac{1}{2} \begin{vmatrix} 0 & 0 & 1 \\ a & a/2 & 1 \\ a/2 & a & 1 \end{vmatrix} = \frac{3a^2}{8}$$

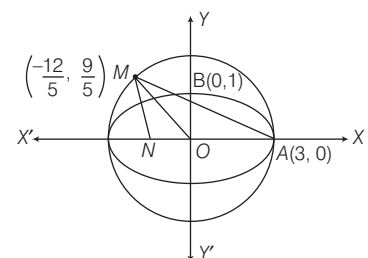
and area of the square $= a^2$

$\therefore$ The required ratio is 8 : 3.

Hence, option (c) is correct.

15. (d) Equation of auxiliary circle is

$$x^2 + y^2 = 9 \quad \dots(i)$$



20 Engineering Entrances Solved Papers 2018

∴ Equation of AM is $\frac{x}{3} + \frac{y}{1} = 1 \dots (ii)$

On solving Eqs. (i) and (ii), we get

$$M\left(-\frac{12}{5}, \frac{9}{5}\right)$$

Now, area of $\Delta AOM = \frac{1}{2}OA \times MN$
 $= \frac{27}{10}$ sq units

Hence, option (d) is correct.

16. (b) Given equation can be written as

$$x^2 - y^2 = \frac{25}{3}$$

$$\therefore e_1 = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + 1} = \sqrt{2}$$

The equation of conjugate hyperbola is

$$-x^2 + y^2 = \frac{25}{3}$$

$$\therefore e_2 = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + 1} = \sqrt{2}$$

$$\therefore e_1^2 + e_2^2 = (\sqrt{2})^2 + (\sqrt{2})^2 = 4$$

Hence, option (b) is correct.

17. (a) We have,

$$f(x + y) = f(x) + 2y^2 + kxy$$

for all $x, y \in R$

$$\Rightarrow \frac{f(x + y) - f(x)}{y} = 2y + kx \text{ for all } y$$

$x \in R$

$$\Rightarrow \lim_{y \rightarrow 0} \frac{f(x + y) - f(x)}{y}$$

$$= \lim_{y \rightarrow 0} (2y + kx)$$

$$\Rightarrow f'(x) = kx \text{ for all } x \in R$$

$$\Rightarrow f(x) = \frac{kx^2}{2} + C \text{ for all } x \in R$$

[by integration]

But, $f(1) = 2$ and $f(2) = 8$.

$$\therefore 2 = \frac{k}{2} + C \text{ and } 8 = 2k + C$$

$$k = 4 \text{ and } C = 0$$

Hence, $f(x) = 2x^2$ for all $x \in R$

So, option (a) is correct.

18. (d) Since, the numerator tends to ∞ as $x \rightarrow 0$,

$$\text{so } \lim_{x \rightarrow 0} \frac{1}{x^2} (e^{\alpha x} - e^x - x)$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{(\alpha e^{\alpha x} - e^x - 1)}{x}$$

For last limit to exist we must have,

$$\lim_{x \rightarrow 0} (\alpha e^{\alpha x} - e^x - 1) = 0$$

$$\therefore \alpha - 1 - 1 = 0 \Rightarrow \alpha = 2$$

For $\alpha = 2$ the last limit and equal to

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{(2e^{2x} - e^x - 1)}{x}$$

$$= \frac{1}{2} \lim_{x \rightarrow 0} (4e^{2x} - e^x) = \frac{3}{2}$$

Hence, option (d) is correct.

19. (c) Let $x_1, x_2, \dots, x_n$ be n observations.

$$\text{Then, } \bar{x} = \frac{1}{n} \sum x_i$$

$$\text{Let } y_i = \frac{x_i}{\alpha} + 10$$

$$\text{Then, } \frac{1}{n} \sum y_i = \frac{1}{\alpha} \left(\frac{1}{n} \sum x_i \right) + \frac{1}{n} (10n)$$

$$\Rightarrow \bar{x}_{\text{new}} = \frac{1}{\alpha} \bar{x} + 10 = \frac{\bar{x} + 10\alpha}{\alpha}$$

Hence option (c) is correct.

20. (d) Since, $p : 4$ is an even prime number,

$q : 6$ is a divisor of 12 and r : the HCF of 4 and 6 is 2.

So, $\sim p \vee (q \wedge r)$ is correct.

Hence, option (d) is correct.

KCET

1. (c) We have, $|x + 5| \geq 10$

$$\Rightarrow x + 5 \leq -10 \text{ or } x + 5 \geq 10$$

$$\Rightarrow x \leq -15 \text{ or } x \geq 5$$

$$\therefore x \in (-\infty, -15] \cup [5, \infty)$$

2. (b) We know n person shakes hands with everybody is ${}^nC_2$.

$$\therefore {}^nC_2 = 45$$

$$\Rightarrow \frac{n(n-1)}{2} = 45$$

$$\Rightarrow n^2 - n - 90 = 0$$

$$\Rightarrow (n-10)(n+9) = 0$$

$$\Rightarrow n = 10 \quad [\because n \neq -9]$$

3. (a) We have, $\left(x^2 - \frac{1}{x^2}\right)^{16}$

$$\text{Now, } T_{r+1} = {}^{16}C_r (x^2)^{16-r} (x^{-2})^r$$

$$\Rightarrow T_{r+1} = {}^{16}C_r x^{32-2r-2r}$$

$$T_{r+1} \text{ is constant if } 32 - 2r - 2r = 0$$

$$\Rightarrow r = 8$$

$$\therefore \text{Constant term} = {}^{16}C_8$$

4. (b) We have,

$$P(n) = 2^{2n} - 1 \text{ is divisibly by } k \forall n \in N$$

$$\therefore P(1) \text{ is also divisibly by } k.$$

$$\text{Hence, } P(1) = 2^{2(1)} - 1$$

$$= 2^2 - 1 = 3$$

$$\therefore k = 3$$

5. (a) Equation of line parallel to the line

$$3x - 4y + 2 = 0 \text{ is}$$

$$3x - 4y = \lambda$$

Since, this line is passes through $(-2, 3)$.

$$\therefore 3(-2) - 4(3) = \lambda$$

$$\lambda = -18$$

∴ Equation of the required line is $3x - 4y = -18$.

$$\Rightarrow 3x - 4y + 18 = 0$$

6. (b) We have,

$$\left(\frac{1-i}{1+i}\right)^{96} = a + ib$$

$$\Rightarrow \left[\frac{(1-i)(1-i)}{(1+i)(1-i)}\right]^{96} = a + ib$$

$$\Rightarrow \left(\frac{1-1-2i}{1+1}\right)^{96} = a + ib$$

$$\Rightarrow (-i)^{96} = a + ib$$

$$\Rightarrow 1 + 0i = a + ib$$

$$\therefore (a, b) \equiv (1, 0)$$

7. (a) Given, distance between the foci of hyperbola is 16.

$$\therefore 2ae = 16 \dots (i)$$

$$\text{and eccentricity } (e) = \sqrt{2} \dots (ii)$$

From Eqs. (i) and (ii), we get

$$a = 4\sqrt{2}$$

We know that in hyperbola,

$$(ae)^2 = a^2 + b^2$$

$$\Rightarrow (8)^2 = 32 + b^2 \Rightarrow b^2 = 32$$

∴ Equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{32} - \frac{y^2}{32} = 1$$

$$\Rightarrow x^2 - y^2 = 32$$

8. (d) We have, 5 girl and 3 boys

$$\times G \times G \times G \times G \times G \times$$

Number of ways no two boys are together is

$${}^6P_3 \times 5! = 14400$$

9. (d) We have, a, b, c are three

consecutive terms of an AP.

∴ $b - a = c - b = d$, d is common

difference and x, y, z are three

consecutive terms of GP.

$$\therefore y^2 = xz$$

$$\text{Now, } x^{b-c} \cdot y^{c-a} \cdot z^{a-b}$$

$$= x^{-d} y^{2d} z^{-d} \quad [\because c - a = 2d]$$

$$= (xz)^{-d} y^{2d}$$

$$= y^{-2d} \cdot y^{2d}$$

$$\Rightarrow y^0 = 1$$

Engineering Entrances Solved Papers 2018 21

10. (d) We have,

$$\lim_{x \rightarrow 0} \frac{[x]}{x}$$

$$\text{LHL} = \lim_{x \rightarrow 0^-} \frac{[x]}{x} = \lim_{x \rightarrow 0^-} \frac{[0-h]}{0-h}$$

$$\Rightarrow \lim_{h \rightarrow 0} \frac{-1}{-h} = \infty$$

$$\text{RHL} = \lim_{x \rightarrow 0^+} \frac{[x]}{x}$$

$$= \lim_{x \rightarrow 0^+} \frac{[0+h]}{h} \Rightarrow \lim_{h \rightarrow 0} \frac{0}{h} = 0$$

$$\text{LHL} \neq \text{RHL}$$

Hence, limit does not exist.

11. (a) We have, $f(x) = x - \frac{1}{x}$

Put $x = -1$, we get

$$f(-1) = -1 - \frac{1}{(-1)} = -1 + 1 = 0$$

12. (a) Given, statement is "72 is divisible by 2 and 3".

Negation of the statement is 72 is not divisible by 2 or 72 is not divisible by 3.

13. (c) We have,

$$P(A) = 0.5, P(B) = 0.3$$

and $P(A \cap B) = 0$

$$\Rightarrow P(A' \cap B') = P(A \cup B)$$

$$= 1 - P(A \cap B)$$

Now, we know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.5 + 0.3 = 0.8$$

$$\therefore P(A' \cap B') = 1 - 0.8 = 0.2$$

14. (c) Number of outcomes = $6 \times 6 = 36$

getting a total more than 7 =

{(2, 6), (3, 5), (3, 6), (4, 4), (4, 5), (4, 6), (5, 3), (5, 4), (5, 5), (5, 6), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

Required probability

$$= \frac{\text{Number of favourable outcomes}}{\text{Total outcomes}}$$

$$= \frac{15}{36} = \frac{5}{12}$$

15. (a) Given, A and B are mutual exclusive events, $P(A \cap B) = 0$

$$\therefore P(A \text{ or } B) = P(A \cup B)$$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{3}{5} + \frac{1}{5} - 0 = \frac{4}{5} = 0.8$$

16. (a) We have,

$$f(x) = \begin{cases} 2x; & x > 3 \\ x^2; & 1 < x \leq 3 \\ 3x; & x \leq 1 \end{cases}$$

$$f(-1) + f(2) + f(4) = 3(-1) + (2)^2 + 2(4)$$

$$= -3 + 4 + 8 = 9$$

AP EAMCET

1. (d) Since, $x = [x] + \{x\}$

$$\Rightarrow 2x = 2[x] + 2\{x\}$$

$$[2x] = 2[x] + (2\{x\})$$

$$[2x] = \begin{cases} 2[x] + 0, & 0 < \{x\} < \frac{1}{2} \\ 2[x] + 1, & \frac{1}{2} \leq \{x\} < 1 \end{cases}$$

$$\therefore [2x] - 2[x] = \begin{cases} 0, & 0 \leq \{x\} < \frac{1}{2} \\ 1, & \frac{1}{2} \leq \{x\} < 1 \end{cases}$$

Hence, Range of f is $\{0, 1\}$

2. (d) Given function,

$$f(x) = \log\{ax^3 + (a+b)x^2 + (b+c)x + c\}$$

$$= \log\{(ax^2 + bx + c)(x+1)\}$$

The function $f(x)$ will be define, if

$$(ax^2 + bx + c)(x+1) > 0$$

$$\Rightarrow x+1 > 0 \quad \{\because a > 0 \text{ and } b^2 = 4ac\}$$

$$\text{and } x \neq -\frac{b}{2a}$$

$$\text{So, } \left\{ D = x : x \in (-1, \infty) \text{ and } x \neq -\frac{b}{2a} \right\}$$

$$\text{or } D = R - \left\{ -\frac{b}{2a} \right\} \cup (-\infty, -1].$$

3. (d) We have, $(15 \times 5^{2n}) + (2 \times 2^{3n})$

For $n = 1$,

$$\text{we get } 15 \times 5^2 + 2 \times 2^3 = 15 \times 25 + 2 \times 8$$

$$= 375 + 16 = 391$$

which is divisible by 17

$$\therefore (15 \times 5^{2n}) + (2 \times 2^{3n}) \text{ is divisible by}$$

$$17, \forall n \in \mathbb{N}.$$

4. (d) We have, $\frac{z-1}{z+i}$

$$= \frac{x+iy-1}{x+iy+i} = \frac{(x-1)+iy}{x+(y+1)i}$$

$$= \frac{(x-1)+iy}{x+(y+1)i} \times \frac{x-(y+1)i}{x-(y+1)i}$$

$$\frac{x(x-1)+ixy-(x-1)(y+1)i}{x^2+(y+1)^2}$$

$$= \frac{x(x-1)+y(y+1)}{x^2+(y+1)^2}$$

$$+ \frac{[xy-(x-1)(y+1)]i}{x^2+(y+1)^2}$$

$$\text{Since, } \text{Re}\left(\frac{z-1}{z+i}\right) = 1$$

$$\therefore x(x-1)+y(y+1) = x^2+(y+1)^2$$

$$\Rightarrow x^2-x+y^2+y = x^2+y^2+2y+1$$

$$\Rightarrow -x+y = 2y+1$$

$$\Rightarrow x+y+1 = 0$$

$$\therefore (2016, -2017) \text{ lies on}$$

$$x+y+1 = 0$$

5. (d) We have,

$$z_1 = 1-2i, z_2 = 1+i,$$

$$z_3 = 3+4i$$

$$\text{Now, } \left(\frac{1}{z_1} + \frac{3}{z_2} \right) \frac{z_3}{z_2}$$

$$= \left(\frac{1}{1-2i} + \frac{3}{1+i} \right) \left(\frac{3+4i}{1+i} \right)$$

$$= \frac{(1+i)+3(1-2i)}{(1-2i)(1+i)} \times \frac{3+4i}{1+i}$$

$$= \frac{4-5i}{3-i} \times \frac{3+4i}{1+i}$$

$$= \frac{12+16i-15i+20}{3+3i-i+1} = \frac{32+i}{4+2i}$$

$$= \frac{(32+i)(2-i)}{2(2+i)(2-i)} = \frac{64+2i-32i+1}{2(4+1)}$$

$$= \frac{65-30i}{10} = \frac{13}{2} - 3i$$

6. (d) Given, $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega}$

$$= \frac{2+\omega+1+2\omega}{(1+2\omega)(2+\omega)} - \frac{1}{(1+\omega)}$$

$$= \frac{3+3\omega}{(1+2\omega)(2+\omega)} - \frac{1}{(1+\omega)}$$

$$= \frac{(3+3\omega)(1+\omega) - (1+2\omega)(2+\omega)}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$\frac{3+3\omega+3\omega+3\omega^2 - (2+2\omega+4\omega+2\omega^2)}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$= \frac{3+6\omega+3\omega^2-2-\omega-4\omega-2\omega^2}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$= \frac{1+\omega+\omega^2}{(1+2\omega)(2+\omega)(1+\omega)}$$

$$= 0 \quad [\because 1+\omega+\omega^2 = 0]$$

7. (b) We have, $5x-1 < (x+1)^2$

$$\Rightarrow 5x-1 < x^2+2x+1$$

$$\Rightarrow x^2-3x+2 > 0$$

$$\Rightarrow (x-1)(x-2) > 0$$

$$\Rightarrow x \in (-\infty, 1) \cup (2, \infty)$$

Again,

$$(x+1)^2 < 7x-3$$

$$\Rightarrow x^2+2x+1 < 7x-3$$

$$\Rightarrow x^2-5x+4 < 0$$

$$\Rightarrow (x-1)(x-4) < 0$$

$$\Rightarrow x \in (1, 4)$$

$$\therefore x = 3 \quad [\because x \text{ is integer}]$$

22 Engineering Entrances Solved Papers 2018

8. (a) We have,

$$f(x) = x^2 + 2bx + 2c^2 \\ = (x + b)^2 + 2c^2 - b^2$$

∴ Minimum value of

$$f(x) = 2c^2 - b^2$$

Again, $g(x) = -x^2 - 2cx + b^2$

$$= -[x^2 + 2cx - b^2] \\ = -[(x + c)^2 - b^2 - c^2] \\ = -(x + c)^2 + b^2 + c^2$$

∴ Maximum value of

$$g(x) = b^2 + c^2$$

Now, according to the question.

$$2c^2 - b^2 > b^2 + c^2 \Rightarrow c^2 > 2b^2$$

9. (a) Given, a, b and c are the roots of equation

$$x^3 + qx + r = 0,$$

$$\therefore a + b + c = 0$$

$$ab + bc + ca = q$$

$$\text{and } abc = -r$$

$$\therefore a + b + c = 0$$

$$\Rightarrow (a + b + c)^2 = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2ab + 2bc + 2ca = 0$$

$$\Rightarrow a^2 + b^2 + c^2 + 2(ab + bc + ca) = 0$$

$$\Rightarrow a^2 + b^2 + c^2 = -2q \quad \dots(i)$$

$$\text{Now, } (a - b)^2 + (b - c)^2 + (c - a)^2$$

$$= a^2 + b^2 - 2ab + b^2 + c^2$$

$$- 2bc + c^2 + a^2 - 2ca$$

$$= 2a^2 + 2b^2 + 2c^2 - 2(ab + bc + ca)$$

$$= 2(a^2 + b^2 + c^2) - 2(ab + bc + ca)$$

$$= 2(-2q) - 2(q) \quad [\text{by Eq. (i)}]$$

$$= -4q - 2q$$

$$= -6q.$$

10. (a) Let the two roots of equation is $m, -m$.

Now, sum of three roots = $2p$

Hence, third root will be $2p$.

Now,

$$m \times (-m) + m \times (2p) + (-m) \times 2p \\ = 3q$$

$$\Rightarrow -m^2 = 3q$$

$$\text{Now, } m \times (-m) \times 2p = 4r$$

$$\Rightarrow 3q \times 2p = 4r$$

$$\Rightarrow r = \frac{3pq \times 2}{4} \Rightarrow r = \frac{3pq}{2}$$

11. (b) Four digit numbers which are even and formed with the digits 0, 3, 5, 4 without repetition are 3054, 3504, 5034, 5304, 3450, 3540, 4350, 4530, 5340, 5430

∴ Required sum

$$= 3054 + 3504 + 5034 + 5304 + 3450 \\ + 3540 + 4350 + 4530 + 5340 + 5430 \\ = 43536$$

12. (b) 6 Boys can be seated in a row in 6_{P_6} ways = $6!$.

Now, in the 7 gaps 6 girls can be arranged in 7_{P_6} ways.

$$\therefore x = 6! \times 7_{P_6} \\ = 6! \times 7!$$

6 Boys can be seated in a circle in

$$(6 - 1)! \text{ ways} = 5!$$

Now, in the 6 gaps 6 girls can be arranged in 6_{P_6} ways.

$$\therefore y = 5! \times 6_{P_6} = 5! \times 6!$$

$$\text{Now, } x : y = 6! \times 7! : 5! \times 6!$$

$$\Rightarrow x : y = 7! : 5!$$

$$\Rightarrow x : y = 7 \times 6 \times 5! : 5!$$

$$\Rightarrow x : y = 42 : 1$$

13. (c) There are 7 letters in the word 'SARANAM' of which there are 3 A and all other are distinct.

The five letter words may consist of

(i) all different letters (using S, A, R, N, M)

$$\therefore \text{Required words} = 5! = 120$$

(ii) 2 alike and other different (using 2 A and 3 other)

$$\therefore \text{Required words} \\ = {}^4C_3 \times \frac{5!}{2!} = 4 \times \frac{120}{2} = 240$$

(iii) 3 alike and other different (using 3A and 2 other)

$$\therefore \text{Required words} = {}^4C_2 \times \frac{5!}{3!} \\ = \frac{4 \times 3}{2 \times 1} \times \frac{120}{6} = 120$$

$$\therefore \text{Total words} = 120 + 240 + 120 = 480$$

14. (c) We have,

$$(\sqrt[4]{5} + \sqrt[5]{4})^{100} = \sum_{r=0}^{100} {}^{100}C_r (\sqrt[4]{5})^{100-r} (\sqrt[5]{4})^r \\ = \sum_{r=0}^{100} {}^{100}C_r 5^{\frac{100-r}{4}} 4^{\frac{r}{5}} \\ = \sum_{r=0}^{100} T_{r+1}$$

$$\text{Where, } T_{r+1} = {}^{100}C_r 5^{\frac{100-r}{4}} 4^{\frac{r}{5}}$$

Clearly, T_{r+1} will be an integer if $\frac{100-r}{4}$ and $\frac{r}{5}$

are integers. This is possible when

$100 - r$ is a multiple of 4 and r is a multiple of 5

$$\Rightarrow 100 - r = 0, 4, 8, 12, \dots, 96, 100 \text{ and} \\ r = 0, 5, 10, \dots, 100$$

$$\Rightarrow r = 0, 4, 8, 12, \dots, 100 \text{ and}$$

$$r = 0, 5, 10, 100$$

$$\Rightarrow r = 0, 20, 40, 60, 80, 100$$

Hence, there are 6 rational terms.

15. (d) We have, $(2a - 3b)^{19}$

$$= 2^{19} \cdot a^{19} \left(1 - \frac{3b}{2a}\right)^{19}$$

We know that, the r^{th} term of greatest term of

$$(1 + x)^n = \left[\frac{(n+1) |x|}{1 + |x|} \right]$$

$$\text{Here, } n = 19, x = -\frac{3b}{2a}$$

$$\therefore r = \left\lfloor \frac{(20) \left| \frac{3b}{2a} \right|}{1 + \frac{3b}{2a}} \right\rfloor = \left\lfloor \frac{20}{1 + \frac{3b}{2a}} \times 4 \right\rfloor$$

$$\left[\because b = \frac{2}{3}, a = \frac{1}{4} \right]$$

$$r = 16$$

∴ greatest term of $(2a - 3b)^{19}$ is

$$= {}^{19}C_{16} \times 2^{19} \cdot a^{19} \left(\frac{3b}{2a}\right)^{16}$$

$$= {}^{19}C_3 \times 2^{19} \times \left(\frac{1}{4}\right)^{19} \times (4)^{16}$$

$$[\because {}^{19}C_{16} = {}^{19}C_3]$$

$$= {}^{19}C_3 \times 2^{19} \times \frac{1}{2^{38}} \times 2^{32}$$

$$= {}^{19}C_3 \times 2^{13}$$

16. (d) It is given that,

$$\cos\left(x - \frac{\pi}{3}\right), \cos x, \cos\left(x + \frac{\pi}{3}\right) \text{ are in H.P.}$$

$$\Rightarrow \cos x = \frac{2\cos\left(x - \frac{\pi}{3}\right)\cos\left(x + \frac{\pi}{3}\right)}{\cos\left(x - \frac{\pi}{3}\right) + \cos\left(x + \frac{\pi}{3}\right)}$$

$$\Rightarrow \cos x = \frac{2\left(\cos^2 x - \sin^2 \frac{\pi}{3}\right)}{2\cos x \cos \frac{\pi}{3}}$$

$$\Rightarrow \cos^2 x \cos \frac{\pi}{3} = \cos^2 x - \sin^2 \frac{\pi}{3}$$

$$\Rightarrow \cos^2 x \left(1 - \cos \frac{\pi}{3}\right) = \sin^2 \frac{\pi}{3}$$

$$\Rightarrow \cos^2 x \left(1 - \cos \frac{\pi}{3}\right) = \left(1 - \cos^2 \frac{\pi}{3}\right)$$

$$\Rightarrow \cos^2 x \left(1 - \cos \frac{\pi}{3}\right) \\ = \left(1 - \cos \frac{\pi}{3}\right) \left(1 + \cos \frac{\pi}{3}\right)$$

Engineering Entrances Solved Papers 2018 23

$$\Rightarrow \cos^2 x = 1 + \cos \frac{\pi}{3}$$

$$\Rightarrow \cos^2 x = 2 \cos^2 \frac{\pi}{6}$$

$$\Rightarrow \cos^2 x = 2 \times \left(\frac{\sqrt{3}}{2} \right)^2$$

$$\Rightarrow \cos^2 x = 2 \times \frac{3}{4} \Rightarrow \cos^2 x = \frac{3}{2}$$

$$\Rightarrow \cos x = \sqrt{\frac{3}{2}}$$

17. (c) $\cos^3 10^\circ + \cos^3 110^\circ + \cos^3 130^\circ$

We know that,

$$\begin{aligned} \cos^3 x + \cos^3(120^\circ - x) + \cos^3(120^\circ + x) \\ = \frac{3}{4} \cos 3x \end{aligned}$$

Here, $x = 10^\circ$

$$\text{Now, } \cos^3 10^\circ + \cos^3 110^\circ + \cos^3 130^\circ$$

$$= \cos^3 10^\circ$$

$$+ \cos^3(120^\circ - 10^\circ) + \cos^3(120^\circ + 10^\circ)$$

$$= \left(\frac{3}{4} \right) \cos(3 \times 10^\circ) = \frac{3}{4} \cos 30^\circ$$

$$= \frac{3}{4} \times \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}}{8}$$

18. (b) Given,

$$\sin 5x = \cos 2x$$

$$\Rightarrow \sin 5x = \sin \left(\frac{\pi}{2} - 2x \right)$$

$$\Rightarrow 5x = n\pi + (-1)^n \left(\frac{\pi}{2} - 2x \right)$$

$$\Rightarrow 5x = n\pi + (-1)^n \frac{\pi}{2} - (-1)^n 2x$$

$$\Rightarrow 5x + (-1)^n 2x = \frac{\pi}{2} \{ 2n + (-1)^n \}$$

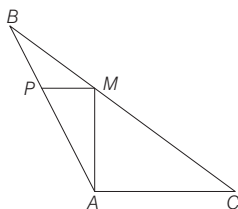
$$\Rightarrow x(5 + (-1)^n 2) = \frac{\pi}{2} (2n + (-1)^n)$$

$$\Rightarrow x = \frac{\pi}{2} \left(\frac{2n + (-1)^n}{5 + 2(-1)^n} \right)$$

$$\Rightarrow x = a_n \cdot \frac{\pi}{2}$$

$$\text{So, } a_n = \frac{2n + (-1)^n}{5 + 2(-1)^n}$$

19. (c)



ABC is the triangle while AM is the median AC and AM are perpendicular.

$$\Rightarrow \angle MAC = 90^\circ$$

Since, AM is the median, M is the mid-point of line BC.

$$\Rightarrow BM = CM = 2BC \quad \dots (i)$$

Draw a line perpendicular to AM (through M) let it intersect the line AB at P.

In $\triangle ABC$,

$$\angle A + \angle B + \angle C = 180^\circ \quad \dots (ii)$$

In $\triangle AMC$,

$$\angle AMC + \angle CAM + \angle MCA = 180^\circ$$

$$\Rightarrow \angle AMC + 90^\circ + \angle C = 180^\circ$$

$$\Rightarrow \angle AMC = 90^\circ - \angle C \quad \dots (iii)$$

Again at point M.

$$\angle BMP + \angle PMA + \angle AMC = 180^\circ$$

$$\Rightarrow \angle BMP + 90^\circ + 90^\circ - \angle C = 180^\circ$$

$$\Rightarrow \angle BMP = \angle C \quad \dots (iv)$$

But in $\triangle BPM$,

$$\angle BPM + \angle BMP + \angle PBM = 180^\circ$$

$$\Rightarrow \angle BPM + \angle C + \angle B = 180^\circ \quad \dots (v)$$

[From Eqs. (i) and (iv)]

$$\angle BPM = \angle A \quad \dots (vi)$$

Now, it is evident that $\triangle ABC$ and $\triangle MPB$ are equivalent triangles.

$$\Rightarrow \frac{BM}{BC} = \frac{BP}{AB} = \frac{PM}{AC} = \frac{1}{2} \quad \dots (vii)$$

$$\text{Also, } \angle BAC = \angle BAM + \angle MAC$$

$$\Rightarrow \angle BAM = \angle BAC - \angle MAC$$

$$\Rightarrow \angle BAM = \angle A - 90^\circ$$

In $\triangle APM$,

$$\angle PAM + \angle APM + \angle AMP = 180^\circ$$

$$\Rightarrow \angle A - 90^\circ + \angle APM + 90^\circ = 180^\circ$$

$$\Rightarrow \angle APM = 180^\circ - A$$

$$\tan(\angle APM) = \frac{AM}{PM}$$

$$\Rightarrow \tan(180^\circ - A) = \frac{AM}{PM}$$

$$\Rightarrow -\tan A = \frac{AM}{PM}$$

$$\Rightarrow AM = -PM \tan A \quad \dots (viii)$$

$$\text{Now, in } \triangle ACM, \tan C = \frac{AM}{AC}$$

$$\Rightarrow AM = AC \tan C \quad \dots (ix)$$

[From Eqs. (viii) and (ix),]

$$AM = AC \tan C = -PM \tan A$$

$$\Rightarrow PM \tan A + AC \tan C = 0 \quad \dots (x)$$

$$\Rightarrow \frac{MP}{AC} = \frac{1}{2}$$

$$\Rightarrow AC = 2MP$$

$$\text{Hence, } PM \tan A + 2PM \tan C = 0$$

$$\Rightarrow PM(\tan A + 2 \tan C) = 0$$

$$\Rightarrow \tan A + 2 \tan C = 0$$

$$\Rightarrow \tan A = -2 \tan C$$

$$\Rightarrow \frac{\tan A}{\tan C} = -2$$

20. (b) $\left(\tan \frac{A}{2} + \tan \frac{B}{2} \right)$

$$= \frac{\Delta}{s(s-a)} + \frac{\Delta}{s(s-b)}$$

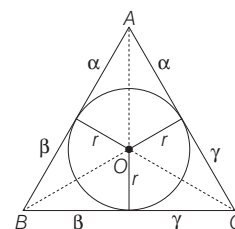
$$= \frac{\Delta}{s} \left\{ \frac{1}{s-a} + \frac{1}{s-b} \right\} = \frac{c\Delta}{s(s-a)(s-b)}$$

$$= \frac{2c\Delta}{(a+b+c)(s-a)(s-b)}$$

$$= \frac{2c}{(a+b+c)} \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$= \frac{2c \cot \frac{C}{2}}{a+b+c}$$

21. (d)



From figure

$$\text{ar}(\triangle ABC) = \text{ar}(\triangle AOB)$$

$$+ \text{ar}(\triangle BOC) + \text{ar}(\triangle COA)$$

$$\Rightarrow S = \frac{1}{2} cr + \frac{1}{2} ar + \frac{1}{2} br$$

[where $\text{ar}(\triangle ABC) = S$]

$$\Rightarrow S = \frac{1}{2} r(a+b+c)$$

$$\Rightarrow S = \frac{1}{2} r(2s)$$

[where s is semi perimeter]

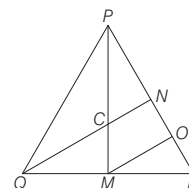
$$\Rightarrow S = rs \Rightarrow r = \frac{S}{s} \Rightarrow r^2 = \frac{S^2}{s^2}$$

$$\Rightarrow r^2 = \frac{s(s-a)(s-b)(s-c)}{s^2}$$

$$\Rightarrow r^2 = \frac{\alpha \cdot \beta \cdot \gamma}{\alpha + \beta + \gamma}$$

$$[\because 2s = 2\alpha + 2\beta + 2\gamma]$$

22. (d) We have,



M is the mid-point of QR.

C is the mid-point of PM.

Draw MO such that MO is parallel to CN.

$\therefore N$ is the mid-point of PO

$$\therefore CN = \frac{1}{2} MO$$

24 Engineering Entrances Solved Papers 2018

M is the mid-point of QR

$\therefore MO$ is parallel to QN

$$\therefore MO = \frac{1}{2}QN$$

$$\therefore CN = \frac{1}{2}\left(\frac{1}{2}QN\right) CN = \frac{1}{4}QN$$

$$\therefore \left|\frac{QN}{CN}\right| = 4.$$

23. (a) We have, $n = 8$, $\bar{x} = 25$ and $\sigma = 5$

$$\therefore \bar{x} = \frac{\sum x_i}{n}$$

$$\Rightarrow \sum x_i = n\bar{x} = 8 \times 25 = 200$$

$$\Rightarrow \text{In corrected } \sum x_i = 200 \text{ and } \sigma = 5$$

$$\Rightarrow \sigma^2 = 25$$

$$\Rightarrow \frac{1}{n} \sum x_i^2 - (\text{mean})^2 = 25$$

$$\Rightarrow \frac{\sum x_i^2}{8} - 625 = 25$$

$$\Rightarrow \sum x_i^2 = 5200$$

$$\text{Corrected } \sum x_i^2 = 5200 + 225 + 625$$

$$= 6050$$

$$\text{and corrected mean} = 200 + 15 + 25 = 240$$

$$\therefore \text{Corrected variance} = \frac{6050}{10} - \left(\frac{240}{10}\right)^2$$

$$= 605 - (24)^2$$

$$= 605 - 576 = 29$$

24. (*)

x_i	f_i	Cumulative frequency	$(d_i) = x_i - 15 $	$f_i d_i $
6	4	4	9	36
9	5	9	6	30
3	3	12	12	36
12	2	14	3	6
15	5	19	0	0
13	4	23	2	8
21	4	27	6	24
22	3	30	7	21
$N = \sum f_i = 30$			$\sum f_i d_i = 161$	

$$\text{Clearly, } N = 30 \Rightarrow \frac{N}{2} = 15$$

The cumulative frequency just greater than $\frac{N}{2}$ is 19 and corresponding value of x is 15.

Therefore, median = 15.

Clearly, $\sum f_i |x_i - 15| = \sum f_i d_i = 161$ and $N = 30$

$$\therefore \text{Mean deviation} = \frac{161}{30} = 5.40 \text{ (approx)}$$

25. (b) Here, total number of possible outcomes = $6^3 = 216$

Now, we can choose any three numbers out of 6 numbers and we can place them in ascending order and we can place them in ascending order in only one way.

$$\text{So, favourable outcomes} = {}^6C_3 = 20$$

$$\text{Now, required probability} = \frac{20}{216} = \frac{5}{54}$$

26. (d) Given, points are $A(2, 3)$, $B(3, -6)$, $C(5, -7)$.

Let point P be (x, y) , then according to condition

$$PA^2 + PB^2 = 2PC^2$$

$$\Rightarrow (x-2)^2 + (y-3)^2 + (x-3)^2 + (y+6)^2 = 2[(x-5)^2 + (y+7)^2]$$

$$\Rightarrow x^2 + 4 - 4x + y^2 + 9 - 6y + x^2 + 9 - 6x + y^2 + 36 + 12y$$

$$= 2[x^2 + 25 - 10x + y^2 + 14y + 49]$$

$$\Rightarrow 2x^2 + 2y^2 - 10x + 6y + 58$$

$$= 2x^2 + 2y^2 - 20x + 28y + 148$$

$$\Rightarrow 10x - 22y = 90$$

By checking options,

$$1. 10(2) - 22(-5) = 20 + 110 = 130$$

$$2.$$

$$10(-2) - 22(5) = -20 - 110 = -130$$

$$3. 10(13) - 22(10) = 130 - 220 = -90$$

$$4. 10(-13) - 22(-10) = -130 + 220 = 90$$

So, point $(-13, -10)$ lies on the locus of P .

27. (c) If (x, y) is old coordinates and (X, Y)

are new coordinates, when axes are rotated through an angle of θ , then

$$x = X \cos \theta - Y \sin \theta \text{ and}$$

$$y = X \sin \theta + Y \cos \theta$$

$$\therefore x = 2 \cos 135^\circ - (-6) \sin 135^\circ$$

$$\text{and } y = 2 \sin 135^\circ + (-6) \cos 135^\circ$$

$$[\because \theta = 135^\circ \text{ and } P(x, y) = (2, -6)]$$

$$\Rightarrow x = 2\left(\frac{-1}{\sqrt{2}}\right) + 6\left(\frac{1}{\sqrt{2}}\right)$$

$$\text{and } y = 2\left(\frac{1}{\sqrt{2}}\right) - 6\left(\frac{-1}{\sqrt{2}}\right)$$

$$\Rightarrow x = \frac{4}{\sqrt{2}} \text{ and } y = \frac{8}{\sqrt{2}}$$

$$\Rightarrow x = 2\sqrt{2} \text{ and } y = 4\sqrt{2}.$$

$\therefore$ Coordinates of P in the original system are $(2\sqrt{2}, 4\sqrt{2})$.

28. (d) Let the equation of the line be

$$\frac{x}{a} + \frac{y}{b} = 1$$

The line meets the coordinate axes at $A(a, 0)$ and $B(0, b)$ respectively. The coordinate of the point which divides the line joining $A(a, 0)$ and $B(0, b)$ in the ratio $3 : 2$ are

$$\left(\frac{3 \times 0 + 2 \times a}{3 + 2}, \frac{3 \times b + 2 \times 0}{3 + 2}\right) \Rightarrow \left(\frac{2a}{5}, \frac{3b}{5}\right)$$

If is given that, the point $(2, -1)$ divides the AB in the ratio $3 : 2$.

$$\therefore \frac{2a}{5} = 2 \text{ and } \frac{3b}{5} = -1$$

$$\Rightarrow a = 5 \text{ and } b = -\frac{5}{3}$$

Hence, the required equation of the required line is

$$\frac{x}{5} + \frac{y}{-\frac{5}{3}} = 1$$

$$\Rightarrow \frac{x}{5} - \frac{3y}{5} = 1 \Rightarrow x - 3y = 5$$

$$\Rightarrow x - 3y - 5 = 0$$

29. (b) The equation of a line passing through the intersection of $2x + y - 4 = 0$ and $x - 3y + 5 = 0$ is

$$(2x + y - 4) + \lambda(x - 3y + 5) = 0 \dots (i)$$

$$\Rightarrow x(2 + \lambda) + y(1 - 3\lambda) + 5\lambda - 4 = 0$$

This is at a distance of $\sqrt{5}$ units from the origin.

$$\therefore \left| \frac{5\lambda - 4}{\sqrt{(2 + \lambda)^2 + (1 - 3\lambda)^2}} \right| = \sqrt{5}$$

$$\Rightarrow \frac{(5\lambda - 4)^2}{4 + \lambda^2 + 4\lambda + 1 + 9\lambda^2 - 6\lambda} = 5$$

$$\Rightarrow \frac{(5\lambda - 4)^2}{10\lambda^2 - 2\lambda + 5} = 5$$

$$\Rightarrow 25\lambda^2 + 16 - 40\lambda = 50\lambda^2 - 10\lambda + 25$$

$$\Rightarrow 25\lambda^2 + 30\lambda + 9 = 0$$

$$\text{By solving, we get } \lambda = -\frac{3}{5}$$

$$\text{Putting the value of } \lambda = -\frac{3}{5} \text{ in Eq. (i),}$$

we get

$$\Rightarrow (2x + y - 4) - \frac{3}{5}(x - 3y + 5) = 0$$

$$\Rightarrow 10x + 5y - 20 - 3x + 9y - 15 = 0$$

$$\Rightarrow 7x + 14y - 35 = 0$$

$$\Rightarrow x + 2y - 5 = 0$$

Engineering Entrances Solved Papers 2018 25

30. (b) Let AD and BE are altitudes of the triangle.

$\therefore$ Equation of AD is given by

$$y - 3 = \frac{-1}{\text{Slope of } BC} (x + 2)$$

$$\Rightarrow y - 3 = \frac{-1}{\left(\frac{0+1}{4-2}\right)} (x + 2)$$

$$y - 3 = -2(x + 2)$$

$$y - 3 = -2x - 4$$

$$\Rightarrow 2x + y + 1 = 0$$

... (i)

$\therefore$ Equation of BE is given by

$$y + 1 = \frac{-1}{\text{Slope of } AC} (x - 2)$$

$$\Rightarrow y + 1 = \frac{-1}{\left(\frac{0-3}{4+2}\right)} (x - 2)$$

$$y + 1 = 2(x - 2)$$

$$y + 1 = 2x - 4$$

$$\Rightarrow y = 2x - 5 \quad \dots (ii)$$

Since, orthocentre is the intersecting point of altitudes.

$\therefore$ On solving Eqs. (i) and (ii), we get orthocentre $(1, -3)$.

Also, centroid of

$$\Delta ABC = \left(\frac{-2+2+4}{3}, \frac{3-1+0}{3} \right)$$

$$= \left(\frac{4}{3}, \frac{2}{3} \right)$$

$\therefore$ Equation of line joining $(0, -1)$ and

$$\left(\frac{4}{3}, \frac{2}{3} \right) \text{ is}$$

$$y + 3 = \frac{\frac{2}{3} + 3}{\frac{4}{3} - 1} (x - 1)$$

$$\Rightarrow y + 3 = 11(x - 1)$$

$$\Rightarrow y + 3 = 11x - 11$$

$$\Rightarrow 11x - y - 14 = 0.$$

31. (c) We have,

$$23x^2 - 48xy + 3y^2 = 0$$

$$\Rightarrow 3y^2 - 48xy + 23x^2 = 0$$

$$\text{Here, } M_1 + M_2 = 16$$

$$\text{and } M_1 M_2 = \frac{23}{3}$$

$$\therefore m_1 - m_2 = \sqrt{(m_1 + m_2)^2 - 4(m_1 m_2)}$$

$$= \sqrt{(16)^2 - 4 \times \frac{23}{3}} = \sqrt{256 - \frac{92}{3}}$$

$$= \sqrt{\frac{768 - 92}{3}} = \frac{26}{\sqrt{3}}$$

$$\therefore m_1 = 8 + \frac{13}{\sqrt{3}} \text{ and } m_2 = 8 - \frac{13}{\sqrt{3}}$$

$$\tan \theta = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| = \left| \frac{\frac{26}{\sqrt{3}}}{1 + \frac{23}{3}} \right| = \sqrt{3}$$

We know that, $\tan 60^\circ$

So, $\theta = 60^\circ$

Slope of line $2x + 3y + 4 = 0$ is $-\frac{2}{3}$

Angle between line of slope $8 + \frac{13}{\sqrt{3}}$

and $-\frac{2}{3}$ is

$$\tan \alpha = \frac{8 + \frac{13}{\sqrt{3}} + \frac{2}{3}}{1 + \left(8 + \frac{13}{\sqrt{3}}\right)\left(-\frac{2}{3}\right)} = \sqrt{3}$$

$$\therefore \theta = 60^\circ \Rightarrow \alpha = 60^\circ$$

Hence, line from an equilateral triangle.

32. (c) We have,

$$x^2 - xy + y^2 + 3x + 3y - 2 = 0 \quad \dots (i)$$

and

$$x + 2y = k \Rightarrow \frac{x + 2y}{k} = 1$$

By homogeneous of Eq. (i), we get

$$x^2 - xy + y^2 + 3x \left(\frac{x + 2y}{k} \right) + 3y \left(\frac{x + 2y}{k} \right) - 2 \left(\frac{x + 2y}{k} \right)^2 = 0$$

$$\Rightarrow k^2 x^2 - k^2 xy + k^2 y^2 + 3kx^2 + 6kxy + 6ky^2 - 2x^2 - 8xy - 8y^2 = 0$$

$$\Rightarrow x^2(k^2 + 3k - 2) - (k^2 - 9k + 8)xy + (k^2 + 6k - 8)y^2 = 0$$

$$xy + (k^2 + 6k - 8)y^2 = 0$$

Since, $\angle AOB = 90^\circ$

$$\therefore k^2 + 3k - 2 + k^2 + 6k - 8 = 0$$

$$\Rightarrow 2k^2 + 9k - 10 = 0$$

33. (c) As we know that,

If the lines $ax^2 + 2hxy + by^2 = 0$ be

two sides of a parallelogram and the line $lx + my = 1$ be one of its diagonals, then other diagonal is

$$y(bl - hm) = x(am - hl) \quad \dots (i)$$

$$\text{Here, } a = 2, b = -2, h = \frac{3}{2}$$

$$l = -3, m = -1$$

Putting all values in Eq. (i), we get

$$\therefore y \left(6 + \frac{3}{2} \right) = x \left(-2 + \frac{9}{2} \right)$$

$$\Rightarrow y \left(\frac{15}{2} \right) = x \left(\frac{5}{2} \right)$$

$$15y = 5x$$

$$3y = x \Rightarrow x - 3y = 0$$

34. (d) We know that, length of tangent drawn from (x_1, y_1) to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\sqrt{x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c}$$

Let $P(h, k)$

$\therefore$ According to the question,

$$\frac{\sqrt{h^2 + k^2 - 2h + 4k - 20}}{\sqrt{h^2 + k^2 - 2h - 8k + 1}} = \frac{2}{1}$$

$$\Rightarrow h^2 + k^2 - 2h + 4k - 20$$

$$= 4(h^2 + k^2 - 2h - 8k + 1)$$

$$\Rightarrow h^2 + k^2 - 2h + 4k - 20$$

$$= 4h^2 + 4k^2 - 8h - 32k + 4$$

$$\Rightarrow 3h^2 + 3k^2 - 6h - 36k + 24 = 0$$

$$\Rightarrow h^2 + k^2 - 2h - 12k + 8 = 0$$

$\therefore$ Locus of point P is

$$x^2 + y^2 - 2x - 12y + 8 = 0.$$

35. (c) Since, circle touches both coordinates axes, then centre will be (h, h) and radius $= h$

$$\therefore \left| \frac{3h - 4h - 12}{\sqrt{(3)^2 + (-4)^2}} \right| = h$$

$$\Rightarrow \left| \frac{-h - 12}{5} \right| = h$$

$$\Rightarrow -h - 12 = \pm 5h$$

$$\Rightarrow -12 = \pm 5h + h$$

$$\Rightarrow -12 = 6h \text{ or } -12 = -4h$$

$$\Rightarrow h = -2 \text{ or } 3$$

$$\Rightarrow h = 3 \quad [\because h > 0]$$

$\therefore$ Equation of circle will be

$$(x - 3)^2 + (y - 3)^2 = 3^2$$

$$\Rightarrow x^2 - 6x + 9 + y^2 - 6y + 9 = 9$$

$$\Rightarrow x^2 + y^2 - 6x - 6y + 9 = 0$$

36. (b) Let (h, k) be the pole of the line

$9x + y - 28 = 0$ with respect to the

circle $x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - \frac{7}{2} = 0$.

Then, the equation of polar is

$$hx + ky - \frac{3}{4}(x + h) + \frac{5}{4}(y + k) - \frac{7}{2} = 0$$

$$\Rightarrow x \left(h - \frac{3}{4} \right) + y \left(k + \frac{5}{4} \right) - \frac{3}{4}h$$

$$+ \frac{5}{4}k - \frac{7}{2} = 0$$

$$\Rightarrow x(4h - 3) + y(4k + 5) - 3h$$

$$+ 5k - 14 = 0$$

This equation and $9x + y - 28 = 0$

represent the same line.

$$\therefore \frac{4h - 3}{9} = \frac{4k + 5}{1}$$

$$= \frac{-3h + 5k - 14}{-28} = \lambda \text{ (say)}$$

26 Engineering Entrances Solved Papers 2018

$$\Rightarrow h = \frac{3+9\lambda}{4}, k = \frac{\lambda-5}{4},$$

$$-3h + 5k - 14 = -28\lambda$$

$$\Rightarrow -3\left(\frac{3+9\lambda}{4}\right) + 5\left(\frac{\lambda-5}{4}\right) - 14 = -28\lambda$$

$$\Rightarrow -9 - 27\lambda + 5\lambda - 25 - 56 = -112\lambda$$

$$\Rightarrow -22\lambda - 90 = -112\lambda$$

$$\Rightarrow 90\lambda = 90 \Rightarrow \lambda = 1$$

Hence, the pole of the given line is (3, -1).

- 37. (d)** The direct common tangents to two circles meet on the line of centres and divide it externally in the ratio of the radii centres of the two circles are (-11, 2) and (11, -2) and their radii are 15 and 5.

$\therefore$ Point of intersection

$$= \left(\frac{11 \times 15 - (-11) \times 5}{15 - 5}, \frac{-2 \times 15 - 2 \times 5}{15 - 5} \right)$$

$$= \left(\frac{165 + 55}{10}, \frac{-30 - 10}{10} \right) = (22, -4).$$

- 38. (b)** We know that, angle between two circles is given by
- $$\cos \theta = \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}, \text{ where } r_1 \text{ and } r_2 \text{ are radius and } d \text{ is distance between centres.}$$

$$\frac{(\sqrt{2})^2 + (1)^2}{2 \times \sqrt{2} \times 1}$$

(A) $\cos \theta = \frac{-[\sqrt{(2-2)^2 + (1-0)^2}]}{2 \times \sqrt{2} \times 1}$

$\therefore r_1 = \sqrt{2}, r_2 = 1, c_1 = (2, 0), c_2 = (2, 1)$

$$= \frac{2 + 1 - 1}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$\therefore \theta = 45^\circ \text{ or } 135^\circ$

$$\frac{(3)^2 + (\sqrt{17})^2}{2 \times 3 \times \sqrt{17}}$$

(B) $\cos \theta = \frac{-[\sqrt{(3-2)^2 + (3+2)^2}]}{2 \times 3 \times \sqrt{17}}$

$\therefore r_1 = 3, r_2 = \sqrt{17}, c_1 = (3, 3), c_2 = (2, -2)$

$$= \frac{9 + 17 - 26}{6\sqrt{17}} = 0$$

$\theta = 90^\circ$

(C)

$$\cos \theta = \frac{(5)^2 + (3)^2 - [\sqrt{(-2+2)^2 + (7-0)^2}]}{2 \times 5 \times 3}$$

$\therefore r_1 = 5, r_2 = 3, c_1 = (-2, 7), c_2 = (-2, 0)$

$$= \frac{25 + 9 - 49}{30} = \frac{-15}{30} = -\frac{1}{2}$$

So, $\theta = 120^\circ \text{ or } 60^\circ$.

- 39. (a)** Let point $P(a, b)$

$$S_1(a, b) = S_2(a, b)$$

$$\Rightarrow a^2 + b^2 + 2ga + 2fb + c = 0$$

$$= a^2 + b^2 + \frac{3}{2}a + 4b + c = 0$$

$$a\left(2g - \frac{3}{2}\right) + b(2f - 4) = 0$$

this is the locus of radical axis.

So, $x\left(2g - \frac{3}{2}\right) + y(2f - 4) = 0$ is radical

axis of given circles.

This touched the

$$x^2 + y^2 + 2x + 2y + 1 = 0$$

So, radius = $\sqrt{1^2 + 1^2} = 1$

and centre = (-1, -1)

So, radius of circle = distance between centre and touching point.

$$1 = \frac{\left| \left(\frac{3}{2} - 2g \right) + (2f - 4) \right|}{\sqrt{\left(\frac{3}{2} - 2g \right)^2 + (2f - 4)^2}}$$

Taking square both sides,

$$2\left(\frac{3}{2} - 2g\right)(2f - 4) = 0$$

So, $\frac{3}{2} - 2g = 0$ or

$$2f - 4 = 0$$

$$2g = \frac{3}{2} \text{ or } 2f = 4$$

$$\Rightarrow g = \frac{3}{4} \text{ or } f = 2$$

- 40. (c)** Given, equation of line is $y = 6x + 1$ and equation of parabola is $y^2 = 24x$.

The locus of the point of intersection of perpendicular tangent to a parabola is its directrix.

So, required point is the point of intersection of $y = 6x + 1$ and directrix $x = -6$.

Hence, its coordinates are (-6, -35).

- 41. (a)** The gradient of the line joining the focus $S(1, -1)$ and vertex $A(1, 1)$ is

$$m = \frac{-1 - 1}{1 - 1} = 0$$

Let $Q(h, k)$ be the point of intersection of the axis AS with the directrix. The $A(1, 1)$ will be the mid-point of QS .

$$\therefore \frac{h+1}{2} = 1 \text{ and } \frac{k-1}{2} = 1$$

$$\Rightarrow h = 1 \text{ and } k = 3$$

$\therefore Q$ is the point (1, 3)

So, the directrix passes through the point (1, 3) and has the gradient 0.

The equation of the directrix is

$$y - 3 = 0$$

Let $P(x, y)$ be any point on the parabola and M be the foot of the perpendicular drawn from P on the directrix.

$$\therefore PS = PM$$

As, $PS^2 = PM^2$

As, $(x-1)^2 + (y+1)^2 = \left(\frac{y-3}{1}\right)^2$

$$\Rightarrow (x-1)^2 + (y+1)^2 = (y-3)^2$$

$$\Rightarrow (x-1)^2 = 8(1-y)$$

By checking option (a),

$$\Rightarrow (3-1)^2 = 8\left(1 - \frac{1}{2}\right)$$

$$\Rightarrow (2)^2 = 8 \times \frac{1}{2}$$

$$\Rightarrow 4 = 4$$

Hence, point $\left(3, \frac{1}{2}\right)$ lies on the parabola

$$(x-1)^2 = 8(1-y)$$

- 42. (c)** Let the equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots (i)$$

Given, $e = \sqrt{\frac{2}{5}}$

$$\Rightarrow \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{2}{5}}$$

$$\Rightarrow \frac{a^2 - b^2}{a^2} = \frac{2}{5}$$

$$\Rightarrow 5a^2 - 5b^2 = 2a^2$$

$$\Rightarrow 3a^2 = 5b^2$$

$$\Rightarrow a^2 = \frac{5b^2}{3}$$

Now, ellipse passes through (-3, 1)

$$\Rightarrow \frac{(-3)^2}{a^2} + \frac{(1)^2}{b^2} = 1$$

$$\Rightarrow 9b^2 + a^2 = a^2b^2$$

$$\Rightarrow 9b^2 + \frac{5b^2}{3} = \frac{5b^4}{3} \Rightarrow \frac{32b^2}{3} = \frac{5b^4}{3}$$

$$\Rightarrow b^2 = \frac{32}{5}$$

From Eq. (i), we get

$$\Rightarrow \frac{x^2}{\frac{32}{3}} + \frac{y^2}{\frac{32}{5}} = 1$$

$$\Rightarrow 3x^2 + 5y^2 = 32$$

$$\Rightarrow 3x^2 + 5y^2 - 32 = 0.$$

- 43. (b)** Let $P(6\cos\theta, 3\sqrt{3}\sin\theta)$ be any point

on the ellipse $\frac{x^2}{36} + \frac{y^2}{27} = 1$. The

Engineering Entrances Solved Papers 2018 27

equation of the tangent at $P(6 \cos \theta, 3\sqrt{3} \sin \theta)$ is

$$\frac{x}{6} \cos \theta + \frac{y}{3\sqrt{3}} \sin \theta = 1.$$

$$\Rightarrow 3\sqrt{3}x \cos \theta + 6y \sin \theta - 18\sqrt{3} = 0 \dots (i)$$

The product of the lengths of the perpendiculars from $(3, 0)$ and $(-3, 0)$ an Eq. (i) is given by

$$P = \frac{\left| \frac{3 \times 3\sqrt{3} \cos \theta - 18\sqrt{3}}{\sqrt{27 \cos^2 \theta + 36 \sin^2 \theta}} \right| \left| \frac{3 \times 3\sqrt{3} \cos \theta + 18\sqrt{3}}{\sqrt{27 \cos^2 \theta + 36 \sin^2 \theta}} \right|}{\dots}$$

$$= \frac{36 \times 27 - 9 \times 27 \cos^2 \theta}{36 \sin^2 \theta + 27 \cos^2 \theta}$$

$$= \frac{9 \times 27(4 - \cos^2 \theta)}{36(1 - \cos^2 \theta) + 27 \cos^2 \theta}$$

$$= \frac{9 \times 27(4 - \cos^2 \theta)}{36 - 9 \cos^2 \theta}$$

$$= \frac{9 \times 27(4 - \cos^2 \theta)}{9(4 - \cos^2 \theta)} = 27.$$

44. (b) Equation to the asymptotes are given as

$$3x + 4y - 2 = 0 \dots (i)$$

$$\text{and } 2x + y + 1 = 0 \dots (ii)$$

Eqs. (i) and (ii) may be given by

$$(3x + 4y - 2)(2x + y + 1) = 0 \dots (iii)$$

As, the equation to the hyperbola will differ from Eq. (iii) only by a constant, it may be given by

$$(3x + 4y - 2)(2x + y + 1) = \lambda \dots (iv)$$

(where λ is a constant)

$(1, 1)$ lies on the curve given by

Eq. (iv), we have

$$(3 + 4 - 2)(2 + 1 + 1) = \lambda$$

$$\Rightarrow (5)(4) = \lambda \Rightarrow \lambda = 20$$

Hence, the equation of the hyperbola will be

$$(3x + 4y - 2)(2x + y + 1) = 20$$

$$\Rightarrow 6x^2 + 3xy + 3x + 8xy + 4y^2$$

$$+ 4y - 4x - 2y - 2 = 20$$

$$\Rightarrow 6x^2 + 4y^2 + 11xy - x + 2y - 22 = 0$$

$$\Rightarrow 6x^2 + 11xy + 4y^2 - x + 2y - 22 = 0.$$

45. (a) We know that, in a triangle, if O is orthocentre, G is centroid and S is circumcentre, then $SG:GO = 1:2$ Let circumcentre be (x, y, z) .

$$\begin{array}{ccc} \bullet & \bullet & \bullet \\ & 1:2 & \\ S(x, y, z) & C(3, 3, 4) & O(-3, 5-2) \end{array}$$

$$\therefore \left(\frac{-3+2x}{1+2}, \frac{5+2y}{1+2}, \frac{2+2z}{1+2} \right) = (3, 3, 4)$$

$$\Rightarrow \left(\frac{-3+2x}{3}, \frac{5+2y}{3}, \frac{2+2z}{3} \right) = (3, 3, 4)$$

$$\Rightarrow \frac{-3+2x}{3} = 3, \frac{5+2y}{3} = 3, \frac{2+2z}{3} = 4$$

$$\Rightarrow x = 6, y = 2, z = 5$$

$$\therefore S(6, 2, 5)$$

46. (b) $\lim_{n \rightarrow \infty} \frac{1}{n^3} \{ [1^2 x] + [2^2 x] + [3^2 x] + \dots + [n^2 x] \}$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{r=1}^n [r^2 x]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^3} \sum_{r=1}^n (r^2 x - \{r^2 x\})$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n^3} \left(x \sum_{r=1}^n r^2 - \sum_{r=1}^n \{r^2 x\} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{x \times n(n+1)(2n+1)}{n^3 \times 6} - \sum_{r=1}^n \frac{\{r^2 x\}}{n^3}$$

$$= \lim_{n \rightarrow \infty} \left(\frac{x}{6} \times \frac{n}{n} \times \frac{n+1}{n} \times \frac{(2n+1)}{n} \right)$$

$$- \lim_{n \rightarrow \infty} \sum_{r=1}^n \frac{\{r^2 x\}}{n^3}$$

$$= \frac{x}{6} \times 1 \times 2 = 0 = \frac{x}{3}$$

TS EAMCET

1. (c) $\therefore f(x) = \frac{1 + \sqrt{1 + 4 \log_2 x}}{2} = y$ (Let)

$$\Rightarrow 1 + 4 \log_2 x = (2y - 1)^2$$

$$\Rightarrow \log_2 x = \frac{(2y - 1)^2 - 1}{4}$$

$$\Rightarrow x = 2^{\frac{(2y - 1)^2 - 1}{4}}$$

$$\Rightarrow f^{-1}(x) = 2^{\left(\frac{(2x - 1)^2 - 1}{4} \right)}$$

$$\therefore f^{-1}(3) = 2^{\frac{(2 \times 3 - 1)^2 - 1}{4}} = 2^6 = 64$$

2. (b) For all $n \in \mathbb{N}$

α is greatest divisors of $n(n^2 - 1)$

$$= \text{value of } n(n^2 - 1)$$

$$\text{for } n = 2 = 2[2^2 - 1] = 6$$

and β is greatest divisors of $2n(n^2 + 2)$

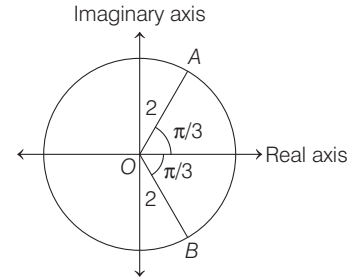
$$= \text{value of } 2n(n^2 + 2) \text{ for } n = 1$$

$$= 2 \times 1(1 + 2) = 6$$

$$\text{So, } \alpha\beta = 6 \times 6 = 36$$

3. (c) Diagram of $|Z| \leq 2$ and

$$-\frac{\pi}{3} \leq \arg Z \leq \frac{\pi}{3}, \text{ is}$$



Area of sectorial area OAB is

$$= 2 \times \frac{2\pi}{3} = \frac{4\pi}{3} \text{ units}$$

4. (d) $\therefore |Z_1 - Z_2| = |(-3 + 5i) - (-1 + 6i)|$
 $= |-2 - i| = \sqrt{5}$

$$|Z_2 - Z_3| = |(-1 + 6i) - (-2 + 8i)|$$

$$= |1 - 2i| = \sqrt{5}$$

$$|Z_3 - Z_4| = |(-2 + 8i) - (-4 + 7i)|$$

$$= |2 + i| = \sqrt{5}$$

$$|Z_4 - Z_1| = |(-4 + 7i) - (-3 + 5i)|$$

$$= |-1 + 2i| = \sqrt{5}$$

$$|Z_1 - Z_3| = |(-3 + 5i) - (-2 + 8i)|$$

$$= |-1 - 3i| = \sqrt{10}$$

$$\text{and } |Z_2 - Z_4| = |(-1 + 6i) - (-4 + 7i)|$$

$$= |3 - i| = \sqrt{10}$$

$\therefore$ Length of sides are same and equal to $\sqrt{5}$ and length of diagonals are same and equal to $\sqrt{10}$.

$\therefore$ Points in the argand plane represents a square.

5. (a) $\therefore \sqrt{3} + i = -2\omega i$,

$$\therefore \sqrt{3} - i = 2\omega^2 i$$

where, ω is complex cube root of unity.

$$\therefore (\sqrt{3} + i)^n + (\sqrt{3} - i)^n$$

$$= (-2\omega i)^n + (2\omega^2 i)^n$$

when $n = 8$

$$(-2\omega i)^8 + (2\omega^2 i)^8 = 2^8[\omega^8 + \omega^{16}]$$

$$= 2^8[\omega^2 + \omega] \quad [\because \omega^3 = 1]$$

$$= -2^8 = -256 \quad [\because \omega + \omega^2 = -1]$$

6. (b) $2 \operatorname{cis} \frac{7\pi}{5} = 2 \left(\cos \frac{7\pi}{5} + i \sin \frac{7\pi}{5} \right) = z^{\frac{1}{5}}$

$$\Rightarrow z = 2^5 (\cos 7\pi + i \sin 7\pi)$$

$$= 2^5(-1) = -32$$

7. (a) $\therefore |x - 1| + |x + 1| \begin{cases} -2x, & x < -1 \\ -2, & -1 \leq x \leq 1 \\ 2x, & x > 1 \end{cases}$

For, $x < -1$

$$-2x < 4 \Rightarrow x > -2$$

$$\Rightarrow x \in (-2, -1) \dots (i)$$

28 Engineering Entrances Solved Papers 2018

For, $x \in [-1, 1]$

$$-2 < 4$$

$$\Rightarrow x \in [-1, 1] \quad \dots(ii)$$

and for $x > 1$,

$$2x < 4 \Rightarrow x < 2$$

$$\Rightarrow x \in (1, 2) \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii),

$$x \in (-2, 2)$$

8. (b) For the quadratic equation $x^2 + x + a = 0$, if roots exceed 'a', then

$$a^2 + a + a > 0$$

$$\Rightarrow a \in (-\infty, -2) \cup (0, \infty) \quad \dots(i)$$

$$\text{and } D = 1 - 4a \geq 0$$

$$\Rightarrow a \in \left(-\infty, \frac{1}{4}\right) \quad \dots(ii)$$

$$\text{and } -\frac{1}{2} > a \Rightarrow a < -\frac{1}{2}$$

$$\Rightarrow a \in \left(-\infty, -\frac{1}{2}\right) \quad \dots(iii)$$

From Eqs. (i), (ii) and (iii),

$$a < -2$$

9. (b) Let $\sqrt{\frac{x}{1-x}} = y$

$$\text{So, } y + \frac{1}{y} = \frac{5}{2}$$

$$\Rightarrow 2y^2 - 5y + 2 = 0$$

$$\Rightarrow y = 2, \frac{1}{2}$$

$$\text{So, } x = \frac{4}{5}, \frac{1}{5}$$

$$\therefore p > q$$

$$\therefore p = \frac{4}{5}, q = \frac{1}{5}$$

Now, for the equation

$$(p+q)x^4 - pqx^2 + \frac{p}{q} = 0$$

$$\Sigma\alpha = 0, \quad \Sigma\alpha\beta = -\frac{pq}{p+q}$$

$$\text{and } \alpha\beta\gamma\delta = \frac{p}{q(p+q)}$$

$$\text{So, } (\Sigma\alpha)^2 - \Sigma\alpha\beta + \alpha\beta\gamma\delta = 0 + \frac{pq}{p+q} + \frac{p}{q(p+q)}$$

$$= \frac{p}{p+q} \left(q + \frac{1}{q} \right)$$

$$= \frac{4}{5} + \frac{1}{5} \left(\frac{1}{5} + 5 \right)$$

$$= \frac{4}{5} \times \frac{26}{5} = \frac{104}{25}$$

$$10. (c) \because x^5 - 5x^3 + 5x^2 - 1 = (x-1)^3 (x^2 + 3x + 1)$$

$$\text{So, } \alpha + \beta = -3 \text{ and } \alpha\beta = 1$$

$$\text{So, } \alpha + \beta + \alpha\beta = -2$$

11. (d) Number of possible 5 digit numbers is 120

Number of possible 4 digit numbers is 120

Number of possible 3 digit numbers start with 7 is 12

Number of possible 3 digit numbers start with 5 is 12

Then, according to order 3 digit numbers are 375, 372, 371, 357, 352, 351, 327

So, rank of number 327 is 271.

$$12. (b) \because 10800 = 2^4 \cdot 3^5 \cdot 5^2$$

So, number of all even divisors 'a'

$$= 4 \times 4 \times 3 = 48$$

and number of all odd divisors 'b'

$$= 4 \times 3 = 12$$

$$\text{So, } 2a + 3b = (2 \times 48) + (3 \times 12)$$

$$= 96 + 36 = 132$$

13. (a) The general term in the expansion

$$\text{of } \left(ax^2 + \frac{1}{bx} \right)^{13}$$

$$T_{(p+1)} = {}^{13}C_p (ax^2)^{13-p} \frac{1}{(bx)^p}$$

$$= {}^{13}C_p \frac{a^{13-p}}{b^p} x^{26-3p}$$

For x^5 , $p = 7$

So, coefficient of x^5 in the expansion of

$$\left(ax^2 + \frac{1}{bx} \right)^{13} \text{ is } {}^{13}C_7 \frac{a^6}{b^7} \quad [\because p = 7]$$

Similarly, the general term in the

$$\text{expansion of } \left(ax - \frac{1}{bx^2} \right)^{13}$$

$$T_{q+1} = {}^{13}C_q (ax)^{13-q} \left(-\frac{1}{bx^2} \right)^q$$

$$= {}^{13}C_q (-1)^q \frac{a^{13-q}}{b^q} x^{13-3q}$$

for x^{-5} , $q = 6$

So, coefficient of x^{-5} in the expansion

$$\text{of } \left(ax - \frac{1}{bx^2} \right)^{13} \text{ is } {}^{13}C_6 \frac{a^7}{b^6}$$

According to the question,

$${}^{13}C_7 \frac{a^6}{b^7} = {}^{13}C_6 \frac{a^7}{b^6} \Rightarrow ab = 1$$

14. (b) $\because$ Sum of all binomial coefficients is 2^n .

$$\text{and } 200 < 2^n < 400 \Rightarrow n = 8$$

$\therefore$ General term

$$T_{r+1} = {}^nC_r \left(\frac{1}{x^{3/4}} \right)^{n-r} a^r \left(x^4 \right)^r$$

$$= {}^nC_r a^r x^{-\frac{3}{4}n + \frac{3}{4}r + \frac{5}{4}r}$$

For independent of 'x'

$$-\frac{3}{4}n + 2r = 0 \quad [\because n = 8]$$

$$\Rightarrow 2r = 6 \Rightarrow r = 3$$

According to the question,

$${}^nC_r a^r = 448 \Rightarrow {}^8C_3 a^3 = 448$$

$$\Rightarrow 56a^3 = 448 \Rightarrow a^3 = 8 \Rightarrow a = 2$$

15. (b) $\because A(n) = \sin^n \alpha + \cos^n \alpha$

Then, $A(1) A(4) + A(2) A(5)$

$$= (\sin \alpha + \cos \alpha) (\sin^4 \alpha + \cos^4 \alpha)$$

$$+ (\sin^2 \alpha + \cos^2 \alpha) (\sin^5 \alpha + \cos^5 \alpha)$$

$$= (\sin \alpha + \cos \alpha) [(\sin^2 \alpha + \cos^2 \alpha)^2$$

$$- 2 \sin^2 \alpha \cos^2 \alpha] + (\sin^5 \alpha + \cos^5 \alpha)$$

$$= (\sin \alpha + \cos \alpha) [(\sin^2 \alpha + \cos^2 \alpha)^3$$

$$- 2 \sin^2 \alpha \cos^2 \alpha] + (\sin^5 \alpha + \cos^5 \alpha)$$

$$= (\sin \alpha + \cos \alpha)$$

$$[\sin^6 \alpha + \cos^6 \alpha + \sin^2 \alpha \cos^2 \alpha]$$

$$+ (\sin^5 \alpha + \cos^5 \alpha)$$

$$= (\sin \alpha + \cos \alpha) (\sin^6 \alpha + \cos^6 \alpha)$$

$$+ \sin^2 \alpha \cos^2 \alpha (\sin \alpha + \cos \alpha)$$

$$+ (\sin^5 \alpha + \cos^5 \alpha)$$

$$= (\sin \alpha + \cos \alpha) (\sin^6 \alpha + \cos^6 \alpha)$$

$$+ (\sin^3 \alpha \cos^2 \alpha + \sin^5 \alpha)$$

$$+ (\sin^2 \alpha \cos^3 \alpha + \cos^5 \alpha)$$

$$= (\sin \alpha + \cos \alpha) (\sin^6 \alpha + \cos^6 \alpha)$$

$$+ (\sin^2 \alpha + \cos^2 \alpha) (\sin^3 \alpha + \cos^3 \alpha)$$

$$= A(1) A(6) + A(2) A(3)$$

16. (c) Given, $\frac{\sin 9\theta}{\cos 27\theta} + \frac{\sin 3\theta}{\cos 9\theta} + \frac{\sin \theta}{\cos 3\theta}$

$$= k$$

$$(\tan 27\theta - \tan \theta)$$

$\therefore$ LHS

$$= \frac{\sin 9\theta \cos 9\theta + \sin 3\theta \cos 27\theta}{\cos 27\theta \cos 9\theta}$$

$$+ \frac{\sin \theta}{\cos 3\theta}$$

$$= \frac{\sin 18\theta + \sin 30\theta - \sin 24\theta}{2 \cos 9\theta \cos 27\theta}$$

$$+ \frac{\sin \theta}{\cos 3\theta}$$

$$[\sin 21\theta + \sin 15\theta + \sin 33\theta$$

$$+ \sin 27\theta - \sin 27\theta - \sin 21\theta]$$

$$= \frac{+4 \sin \theta \cos 9\theta \cos 27\theta}{4 \cos 3\theta \cos 9\theta \cos 27\theta}$$

$$\sin 15\theta + \sin 33\theta$$

$$= \frac{+4 \sin \theta \cos 9\theta \cos 27\theta}{4 \cos 3\theta \cos 9\theta \cos 27\theta}$$

Engineering Entrances Solved Papers 2018 29

$$\begin{aligned}
 & \frac{2 \sin 24 \theta \cos 9 \theta}{+4 \sin \theta \cos 9 \theta \cos 27 \theta} \\
 &= \frac{4 \cos 3 \theta \cos 9 \theta \cos 27 \theta}{\sin 24 \theta + 2 \sin \theta \cos 27 \theta} \\
 &= \frac{\sin(27 \theta - 3 \theta)}{2 \cos 3 \theta \cos 27 \theta} + \frac{\sin \theta}{\cos 3 \theta} \\
 &= \frac{1}{2} \left[\frac{\sin 27 \theta}{\cos 27 \theta} - \frac{\sin 3 \theta}{\cos 3 \theta} \right] + \frac{\sin \theta}{\cos 3 \theta} \\
 &= \frac{1}{2} \frac{\sin 27 \theta}{\cos 27 \theta} - \frac{1}{2} \left(\frac{\sin 3 \theta - 2 \sin \theta}{\cos 3 \theta} \right) \\
 &= \frac{1}{2} \tan 27 \theta \\
 &\quad - \frac{1}{2} \frac{3 \sin \theta - 4 \sin^3 \theta - 2 \sin \theta}{\cos \theta (4 \cos^2 \theta - 3)} \\
 &= \frac{1}{2} \left[\tan 27 \theta - \frac{\sin \theta (1 - 4 \sin^2 \theta)}{\cos \theta (1 - 4 \sin^2 \theta)} \right] \\
 &= \frac{1}{2} [\tan 27 \theta - \tan \theta] \\
 &\text{So, } k = \frac{1}{2}
 \end{aligned}$$

$$\begin{aligned}
 17. (a) \because x &= \sum_{n=0}^{\infty} \cos^{2n} \theta = \frac{1}{1 - \cos^2 \theta} \\
 &= \frac{1}{\sin^2 \theta} \quad \left\{ \because 0 < \theta < \frac{\pi}{2} \right\} \\
 y &= \sum_{n=0}^{\infty} \sin^{2n} \theta = \frac{1}{1 - \sin^2 \theta} \\
 &= \frac{1}{\cos^2 \theta} \\
 z &= \sum_{n=0}^{\infty} \cos^{2n} \theta \sin^{2n} \theta \\
 &= \frac{1}{1 - \cos^2 \theta \sin^2 \theta} \\
 \therefore z &= \frac{1}{1 - \frac{1}{xy}} \\
 \Rightarrow xyz - z &= xy \quad \dots(i) \\
 \therefore \frac{1}{x} + \frac{1}{y} &= \sin^2 \theta + \cos^2 \theta = 1 \\
 \Rightarrow x + y &= xy \quad \dots(ii)
 \end{aligned}$$

$$\begin{aligned}
 &\text{From Eqs. (i) and (ii),} \\
 &(x + y)z - z = xy \\
 \Rightarrow xz + yz &= xy + z \\
 18. (c) \text{ Given equation} \\
 \sin x - \sin 2x + \sin 3x &= 2 \cos^2 x \\
 &\quad - 2 \cos x \\
 \Rightarrow 2 \sin 2x \cos x - \sin 2x &= 2 \cos x (\cos x - 1) \\
 \Rightarrow 2 \sin x \cos x (2 \cos x - 1) &= 2 \cos x (\cos x - 1) \\
 \text{Either, } \cos x &= 0
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow x &= \frac{\pi}{2} \quad \dots(i) \\
 \text{or } \sin x (2 \cos x - 1) &= \cos x - 1 \\
 \Rightarrow 2 \sin x \cos x + 1 &= \sin x + \cos x \\
 \Rightarrow 4 \sin^2 x \cos^2 x + 1 + 4 \sin x \cos x &= 1 + 2 \sin x \cos x \\
 &\quad [\text{on squaring both sides}] \\
 \Rightarrow 2 \sin x \cos x (2 \sin x \cos x + 1) &= 0 \\
 \text{Either } \sin x &= 0 \text{ or } \cos x = 0 \\
 \text{or } \sin 2x &= -1 \\
 \therefore x &\in (0, \pi) \\
 \therefore \sin x &\neq 0 \\
 \text{So, } \sin 2x &= -1 \\
 \Rightarrow 2x &= \frac{3\pi}{2} \Rightarrow x = \frac{3\pi}{4} \quad \dots(ii) \\
 \text{From Eqs. (i) and (ii),} \\
 x &= \frac{\pi}{2} \text{ or } \frac{3\pi}{4}
 \end{aligned}$$

Only two solutions possible in $x \in (0, \pi)$.

$$\begin{aligned}
 19. (d) \because \cosh^2 x - \sinh^2 x &= 1 \\
 \Rightarrow \cosh^2 x - \cos^2 \theta &= 1 \\
 &\quad [\because \sinh x = \cos \theta] \\
 \Rightarrow \cosh^2 \theta &= \cosh^2 x - 1 \\
 &= \frac{14}{9} - 1 = \frac{5}{9} \\
 \text{So, } \sin^2 \theta &= 1 - \cos^2 \theta = 1 - \frac{5}{9} = \frac{4}{9} \\
 \therefore -\pi < \theta < -\frac{\pi}{2} \\
 \therefore \sin \theta &= -\frac{2}{3}
 \end{aligned}$$

$$\begin{aligned}
 20. (c) \because \tan \frac{A-B}{2} &= \frac{a-b}{a+b} \cot \frac{C}{2} \\
 &= \frac{a-b}{a+b} \tan \frac{A+B}{2} \\
 \therefore \frac{a-b}{a+b} &= \frac{1}{4} \\
 \Rightarrow \frac{5-b}{5+b} &= \frac{1}{4} \\
 \Rightarrow 20 - 4b &= 5 + b \\
 \Rightarrow 5b &= 15 \Rightarrow b = 3 \\
 \therefore \sqrt{a^2 - b^2} &= \sqrt{25 - 9} = \sqrt{16} = 4
 \end{aligned}$$

$$\begin{aligned}
 21. (c) \text{ According to sine law,} \\
 \frac{\sin A}{\alpha + 1} &= \frac{\sin B}{\alpha - 1} \\
 \Rightarrow \frac{2 \sin B \cos B}{\alpha + 1} &= \frac{\sin B}{\alpha - 1} \quad [\because A = 2B] \\
 \Rightarrow \cos B &= \frac{\alpha + 1}{2(\alpha - 1)} \\
 \Rightarrow \frac{(\alpha + 1)^2 + \alpha^2 - (\alpha - 1)^2}{2\alpha(\alpha + 1)} &= \frac{\alpha + 1}{2(\alpha - 1)} \\
 \Rightarrow \frac{4\alpha + \alpha^2}{2\alpha(\alpha + 1)} &= \frac{\alpha + 1}{2(\alpha - 1)}
 \end{aligned}$$

$$\begin{aligned}
 \Rightarrow (4 + \alpha)(\alpha - 1) &= (\alpha + 1)^2 \\
 \Rightarrow \alpha^2 + 3\alpha - 4 &= \alpha^2 + 2\alpha + 1 \\
 \Rightarrow \alpha &= 5 \\
 22. (c) \therefore A &= \frac{\pi}{2} \\
 \therefore B + C &= \frac{\pi}{2} \\
 \text{and } r &= 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\
 \Rightarrow \frac{r}{R} &= 4 \frac{1}{\sqrt{2}} \sin \frac{B}{2} \sin \left(\frac{\pi}{4} - \frac{B}{2} \right) \\
 \Rightarrow \frac{5}{2} &= 2\sqrt{2} \sin \frac{B}{2} \left[\frac{1}{\sqrt{2}} \cos \frac{B}{2} - \frac{1}{\sqrt{2}} \sin \frac{B}{2} \right] \\
 \Rightarrow \frac{5}{4} &= \sin \frac{B}{2} \cos \frac{B}{2} - \sin^2 \frac{B}{2} \quad \dots(i) \\
 \text{and } r_1 &= 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} \\
 \Rightarrow \frac{r_1}{R} &= 4 \times \frac{1}{\sqrt{2}} \cos \frac{B}{2} \cos \left(\frac{\pi}{4} - \frac{B}{2} \right) \\
 \Rightarrow \frac{\lambda}{2} &= 2\sqrt{2} \cos \frac{B}{2} \left[\frac{1}{\sqrt{2}} \cos \frac{B}{2} + \frac{1}{\sqrt{2}} \sin \frac{B}{2} \right] \\
 \Rightarrow \frac{\lambda}{4} &= \cos^2 \frac{B}{2} + \cos \frac{B}{2} \sin \frac{B}{2} \quad \dots(ii)
 \end{aligned}$$

$$\begin{aligned}
 \text{From Eqs. (i) and (ii),} \\
 \frac{\lambda}{4} - \frac{5}{4} &= 1 \Rightarrow \lambda - 5 = 4 \Rightarrow \lambda = 9 \\
 \text{So, the roots of quadratic equation} \\
 x^2 - 4x + 3 &= 0 \text{ are } 1, 3
 \end{aligned}$$

$$\begin{aligned}
 23. (c) (a) \because \bar{x} &= \frac{x_1 + x_2 + x_3 + \dots + x_n}{n} \\
 \Rightarrow n\bar{x} &= x_1 + x_2 + x_3 + \dots + x_n \\
 \Rightarrow (x_1 - \bar{x}) + (x_2 - \bar{x}) + \dots &+ (x_n - \bar{x}) = 0 \\
 \Rightarrow \sum_{i=1}^n (x_i - \bar{x}) &= 0
 \end{aligned}$$

$$(b) \because \text{Variance } (\sigma^2) = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 =$$

Mean of the squares of the deviations from mean.

(c) $\therefore$ Mean deviation of observations is mean of the absolute deviations from any measure of central tendency.

(d) $\therefore$ Coefficient of variation is used to find the homogeneity of given two series.

$\therefore$ (a) $\rightarrow$ (iii), (b) $\rightarrow$ (v), (c) $\rightarrow$ (iv), (d) $\rightarrow$ (ii)

$$\begin{aligned}
 24. (d) \because \text{The variance is remains} \\
 \text{unchanged if we add or subtract a} \\
 \text{positive number to each observation,} \\
 \text{but if any positive number 'k' is} \\
 \text{multiplied to each observation, then} \\
 \text{variance becomes } k^2 \text{ times.} \\
 \text{So, variance of the new data is} \\
 6^2 \times 7 = 36 \times 7 = 252.
 \end{aligned}$$

30 Engineering Entrances Solved Papers 2018

25. (a) Since number of getting prime numbers $\{2, 3, 5\}$ is 3 therefore number of getting prime numbers on both the dice is 3×3 .
And getting a head and a tail on two coins $= 1 \times 1 \times 2$
And total number of possibilities $= (6 \times 6) \times (2 \times 2)$
So, required probability $= \frac{3 \times 3 \times 1 \times 1 \times 2}{16 \times 6 \times 2 \times 2} = \frac{1}{8}$

26. (d) Let $D(h, k)$

$$\text{Now, } \Delta ABC = \frac{1}{2} \begin{vmatrix} 3 & 4 & 1 \\ -3 & 6 & 1 \\ -5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} | [3(6-1) + 4(-5+3) + 1(-3+30)] |$$

$$= \frac{1}{2} | [15 - 8 + 27] | = 17$$

$$\text{and } \Delta ADC = \frac{1}{2} \begin{vmatrix} 3 & 4 & 1 \\ h & k & 1 \\ -5 & 1 & 1 \end{vmatrix}$$

$$= \frac{1}{2} | [3(k-1) + 4(-5-h) + 1(h+5k)] |$$

$$= \frac{1}{2} | (3k - 3 - 20 - 4h + h + 5k) |$$

$$= \frac{1}{2} | 8k - 3h - 23 |$$

$$\therefore \Delta ABC = \Delta ADC$$

$$\Rightarrow 34 = | 3h - 8k + 23 |$$

$$\text{By taking locus of point } (h, k),$$

$$| 3x - 8y + 23 | = 34$$

$$\Rightarrow 3x - 8y - 11 = 0 \text{ or } 3x - 8y + 57 = 0$$

$$\text{or } (3x - 8y - 11)(3x - 8y + 57) = 0$$

27. (d) $\therefore \frac{3\sqrt{3}-1}{2\sqrt{2}} = \cos \alpha + 2\sin \alpha$

$$\text{and } \frac{\sqrt{3}+3}{2\sqrt{2}} = 2\cos \alpha - \sin \alpha$$

$$\therefore 5\cos \alpha = 5\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right) = 5\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right)$$

$$\Rightarrow \cos \alpha = \cos \frac{\pi}{12} \Rightarrow \alpha = \frac{\pi}{12}$$

28. (b) Since, $L(p, q) = \frac{ap + bq + c}{\sqrt{a^2 + b^2}}$,

$$\forall p, q \in R$$

$$\text{Now, } L\left(\frac{2}{3}, \frac{1}{3}\right) + L\left(\frac{1}{3}, \frac{2}{3}\right) + L(2, 2) = 0$$

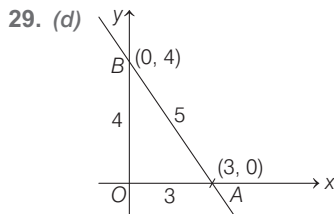
$$\Rightarrow \frac{2}{3}a + \frac{1}{3}b + c + \frac{1}{3}a + \frac{2}{3}b$$

$$+ c + 2a + 2b + c = 0$$

$$\Rightarrow 3a + 3b + 3c = 0 \quad (\text{given})$$

$$\Rightarrow a + b + c = 0 \quad \dots(i)$$

So, line $ax + by + c = 0$ passes through the point $(1, 1)$.



$\therefore$ Incentre of ΔOAB is

$$\left(\frac{(5 \times 0) + (4 \times 3) + (3 \times 0)}{5 + 4 + 3}, \frac{(5 \times 0) + (4 \times 0) + (3 \times 4)}{5 + 4 + 3} \right)$$

$$= \left(\frac{12}{12}, \frac{12}{12} \right) = (1, 1)$$

30. (b) Let the equation of required line is $x + 2y + c = 0 \quad \dots(i)$

According to the question,

$$\frac{2|C-3|}{\sqrt{5}} = \frac{|8-C|}{\sqrt{5}}$$

$$\Rightarrow 2(C-3) = 8-C \Rightarrow 3C = 14$$

$$\Rightarrow C = \frac{14}{3}$$

So, equation will be

$$x + 2y + \frac{14}{3} = 0 \quad \dots(ii)$$

Let the normal form is

$$x \cos \alpha + y \sin \alpha = p \quad \dots(iii)$$

From Eqs. (ii) and (iii),

$$\frac{\cos \alpha}{-1} = \frac{\sin \alpha}{-2} = \frac{p}{\frac{14}{3}}$$

$$\Rightarrow \alpha = \pi + \tan^{-1} 2$$

$$\text{and } p = \frac{14}{3\sqrt{5}} \Rightarrow p = \frac{14}{\sqrt{45}}$$

So, required equation is

$$x \cos \alpha + y \sin \alpha = \frac{14}{\sqrt{45}}$$

$$\alpha = \pi + \tan^{-1} 2$$

31. (a) According question equation of lines

$$(y-1) - \tan \theta (x-1) = 0$$

$$\text{and } (y-1) - \cot \theta (x-1) = 0$$

So, combined equation is

$$(y-1)^2 - (x-1)(y-1)$$

$$[\tan \theta + \cot \theta] + (x-1)^2 = 0$$

$$\Rightarrow x^2 - \frac{2}{\sin 2\theta} xy + y^2$$

$$+ \left(\frac{2}{\sin 2\theta} - 2 \right) (x + y - 1) = 0$$

On comparing with the equation

$$x^2 - (a+2)xy + y^2 + a(x+y-1) = 0$$

$$\Rightarrow a + 2 = \frac{2}{\sin 2\theta}$$

$$\Rightarrow \sin 2\theta = \frac{2}{a+2}$$

$$\Rightarrow \theta = \frac{1}{2} \sin^{-1} \left(\frac{2}{a+2} \right)$$

32. (a) Since, $6x^2 + xy - y^2 = 0$

$$\Rightarrow 6x^2 + 3xy - 2xy - y^2 = 0$$

$$\Rightarrow (3x - y)(2x + y) = 0$$

$$\Rightarrow \frac{y}{x} = 3 \text{ or } -2$$

Now, the second equation

$$\left(\frac{y}{x} \right)^2 + a \left(\frac{y}{x} \right) - 3 = 0$$

$$\text{For } \frac{y}{x} = 3$$

$$\Rightarrow 3^2 + 3a - 3 = 0$$

$$\Rightarrow a = -2 \quad [\because a > 0]$$

$$\text{For } \frac{y}{x} = -2$$

$$(-2)^2 + a(-2) - 3 = 0$$

$$\Rightarrow 4 - 2a - 3 = 0$$

$$\Rightarrow a = \frac{1}{2}$$

33. (b) The chord $y - mx - 1 = 0$ is tangent to the circle

$$S_1 \equiv x^2 + y^2 - 4x + 1 = 0$$

$$\text{or } (x-2)^2 + (y-0)^2 = 3$$

$$\text{So, } \frac{|2m+1|}{\sqrt{1+m^2}} = \sqrt{3}$$

$$\Rightarrow 4m^2 + 1 + 4m = 3 + 3m^2$$

$$\Rightarrow (m+2)^2 = 6$$

$$\Rightarrow m = -2 \pm \sqrt{6}$$

Now, equation of chord is

$$L \equiv y + (2 \pm \sqrt{6})x - 1 = 0, \text{ Let possible}$$

point is (x_1, y_1) for which the chord

$L = 0$ is chord of contact of circle

$$S \equiv x^2 + y^2 - 1 = 0.$$

$$\text{So, } xx_1 + yy_1 - 1 = 0$$

$$\text{and } y + (2 \pm \sqrt{6})x - 1 = 0$$

represent same line, so

$$\frac{x_1}{2 \pm \sqrt{6}} = \frac{y_1}{1} = \frac{-1}{-1}$$

$$\Rightarrow (x_1, y_1) \equiv (2 \pm \sqrt{6}, 1)$$

34. (b) The given circle is

$$x^2 + y^2 - 6x - 2y + 1 = 0$$

$$\Rightarrow (x-3)^2 + (y-1)^2 = 9 \quad \dots(i)$$

and the line $y + c = 0$ is tangent to the circle at point $(a, 4)$, so

$$4 + c = 0 \Rightarrow c = -4 \quad \dots(ii)$$

and slope of line joining the point of contact and centre is infinite, so

Engineering Entrances Solved Papers 2018 31

$$\frac{a-3}{4-1} = 0 \Rightarrow a = 3$$

So, $ac = -12$

35. (b) For four common tangent, the distance between centres of two circles C_1C_2 should be more than the sum of radii ($r_1 + r_2$).

$$\therefore \sqrt{49+9} > \sqrt{49+9-33} + a$$

$$\Rightarrow \sqrt{58} > \sqrt{25} + a \Rightarrow a < \sqrt{58} - 5$$

$\therefore a \in \mathbb{N}$ so, $a = \{1, 2\}$

$\therefore$ Two circles are possible only.

36. (d) Let the equation of circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

Since, circle (i) passes through point (1, 0),

$$1 + 2g + c = 0 \quad \dots(ii)$$

$\therefore$ Circle (i) cutting the circles

$$x^2 + y^2 - 2x + 4y + 1 = 0$$

and $x^2 + y^2 + 6x - 2y + 1 = 0$,

orthogonally, so

$$2g(-1) + 2f(2) = c + 1 \quad \dots(iii)$$

$$\text{and } 2g(3) + 2f(-1) = c + 1 \quad \dots(iv)$$

From Eqs. (ii), (iii) and (iv)

$$f = 0 \text{ and } g = 0$$

$$\text{So, } c = -1$$

So, equation of required circle is

$$x^2 + y^2 = 1$$

$\therefore$ Centre is (0,0).

37. (a) Let the equation of required circle is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots(i)$$

Since, circle (i) cuts the circles

$$x^2 + y^2 - 2x + 6y = 0,$$

$$x^2 + y^2 - 4x - 2y + 6 = 0 \text{ and}$$

$$x^2 + y^2 - 12x + 2y + 3 = 0$$

orthogonally, then

$$-2g + 6f = c \quad \dots(ii)$$

$$-4g - 2f = c + 6 \quad \dots(iii)$$

$$\text{and } -12g + 2f = c + 3 \quad \dots(iv)$$

From Eqs. (ii), (iii) and (iv), we get

$$g = 0, f = -\frac{3}{4} \text{ and } c = -\frac{9}{2}$$

So, equation of required circle is

$$x^2 + y^2 - \frac{3}{2}y - \frac{9}{2} = 0$$

Now, equation of tangent at point (0, 3) is

$$3y - \frac{3}{4}(y + 3) - \frac{9}{2} = 0 \Rightarrow y = 3$$

38. (c) Let point of intersection of required tangents is (h, k), then equation of pair of tangents drawn through the point (h, k) to the parabola $y^2 = 8x$ is

$$(y^2 - 8x)(k^2 - 8h) = [yk - 4(x + h)]^2 \quad \dots(i)$$

$\therefore$ Equation of tangent at the vertex of parabola is $x = 0$, so let $M(0, y_1)$ and $N(0, y_2)$, so we will get y_1 and y_2 by putting $x = 0$ in Eq. (i).

$$y^2(k^2 - 8h) = (yk - 4h)^2$$

$$\Rightarrow y^2k^2 - 8hy^2 = y^2k^2 + 16h^2 - 8hky$$

$$\Rightarrow y^2 - ky + 2h = 0 \text{ having roots } y_1 \text{ and } y_2.$$

$$\therefore |y_1 - y_2| = \frac{\sqrt{k^2 - 8h}}{1} = 4 \text{ (given)}$$

$$\Rightarrow k^2 - 8h = 16 \Rightarrow k^2 = 8(h + 2)$$

By taking locus of point (h, k) we are getting $y^2 = 8(x + 2)$.

39. (c) Since, equation of normal to parabola $y^2 = x$, having slope 'm' is

$$y = mx - 2\left(\frac{1}{4}\right)m - \frac{1}{4}m^3 \quad \dots(i)$$

$\therefore$ Normal drawn through the point (c, 0).

$$\therefore m^3 + 2m - 4mc = 0$$

$$\Rightarrow m[m^2 + (2 - 4c)] = 0$$

Either $m = 0$ $\{\therefore$ normal is X-axis also $\}$

$$\text{or } m^2 + (2 - 4c) = 0$$

$\therefore$ Remaining two are perpendicular to each other.

$$\therefore 2 - 4c = -1 \Rightarrow 4c = 3 \Rightarrow c = \frac{3}{4}$$

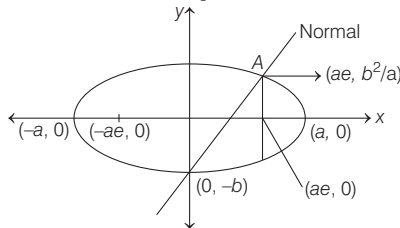
40. (c) Given, equation of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ let } (a > b) \text{ the an end point}$$

of a lat us rectum be

$$\left(ae, \frac{b^2}{a}\right), \text{ then the equation of the}$$

normal at this end point is



$$\frac{x - ae}{\frac{ae}{a^2}} = \frac{y - \frac{b^2}{a}}{\frac{b^2}{ab^2}}$$

It will pass through the end of the minor axis (0, -b), if $-a^2 = -ab - b^2$

$$\Rightarrow 1 = \frac{b}{a} + \left(\frac{b}{a}\right)^2 \Rightarrow 1 - \left(\frac{b}{a}\right)^2 = \frac{b}{a}$$

$$\Rightarrow e^2 = \sqrt{1 - e^2} \left[\therefore e = \sqrt{1 - \left(\frac{b}{a}\right)^2} \right]$$

$$\Rightarrow e^4 + e^2 = 1$$

41. (d) Equation of tangent at point (x_1, y_1)

on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \quad \dots(i)$$

Intersecting point of tangent with the

$$\text{axes is } A\left(\frac{a^2}{x_1}, 0\right) \text{ and } B\left(0, \frac{b^2}{y_1}\right).$$

Now, let middle point of point A and B is $P(h, k)$, then

$$h = \frac{a^2}{2x_1} \text{ and } k = \frac{b^2}{2y_1}$$

$$\Rightarrow x_1 = \frac{a^2}{2h}$$

$$\text{and } y_1 = \frac{b^2}{2k}$$

$$\therefore (x_1, y_1) \text{ on the ellipse } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\text{so } \frac{a^2}{h^2} + \frac{b^2}{k^2} = 4$$

On taking locus of point $P(h, k)$, we are

$$\text{getting } \frac{a^2}{x^2} + \frac{b^2}{y^2} = 4.$$

42. (d) Let a function $f(x) = 2x^3 + 10x - 13$

$$\therefore f'(x) = 6x^2 + 10 > 0, \forall x \in \mathbb{R}$$

$\therefore f(x)$ is strictly increasing function.

$$\text{Now, } f(1) = 2 + 10 - 13 = -1 < 0$$

$$\text{and } f(2) = 16 + 20 - 13 = 23 > 0$$

So, the one and only one root $e \in (1, 2)$, where e is the eccentricity of the conic.

$$\therefore 1 < e < 2,$$

$\therefore$ Conic is a hyperbola.

43. (c) Except the equilateral triangle, the centroid, orthocentre and circumcentre are collinear and centroid divides the line segment joining the orthocentre and the circumcentre in the ratio 2 : 1.

So, if $(-1, 3, 2)$ and $(5, 3, 2)$ are respectively the orthocentre and circumcentre of triangle, then coordinate of centroid is

$$\left(\frac{(-1 \times 1) + (5 \times 2)}{1 + 2}, \frac{(3 \times 1) + (3 \times 2)}{1 + 2}, \frac{(2 \times 1) + (2 \times 2)}{1 + 2} \right)$$

$$= (3, 3, 2)$$

So, (A) is true and (R) is false.

32 Engineering Entrances Solved Papers 2018

44. (b) $\lim_{x \rightarrow -\infty} \frac{3|x| - x}{|x| - 2x} - \lim_{x \rightarrow 0} \frac{\log(1+x^3)}{\sin^3 x}$

$$= \lim_{x \rightarrow -\infty} \frac{-3x - x}{-x - 2x} - \lim_{x \rightarrow 0} \frac{\frac{\log(1+x^3)}{x^3} \times x^3}{\left(\frac{\sin x}{x}\right)^3 \times x^3}$$

$$= \frac{3+1}{1+2} - 1 = \frac{4}{3} - 1 = \frac{1}{3}$$

VIT

1. (b) We know that

$$\log z = \log |z| + i \arg(z)$$

$$\therefore \log i^i = i \log i = i[\log |i| + i \arg(i)]$$

$$= i \left[\log 1 + i \cdot \frac{\pi}{2} \right] = -\frac{\pi}{2}$$

$$\therefore \{\sin(\log i^i)\}^3 + \{\cos(\log i^i)\}^3$$

$$= \left[\sin\left(-\frac{\pi}{2}\right) \right]^3 + \left[\cos\left(-\frac{\pi}{2}\right) \right]^3$$

$$= (-1)^3 + (0)^3 = -1$$

2. (a) We have,

$$f(x) = 27x^3 - \frac{1}{x^3}$$

$$= \left(3x - \frac{1}{x}\right)^3 + 9\left(3x - \frac{1}{x}\right)$$

Since, α and β are the roots of $3x - \frac{1}{x} = 2$

$$\therefore 3\alpha - \frac{1}{\alpha} = 2 \text{ and } 3\beta - \frac{1}{\beta} = 2$$

Now, $f(\alpha) = \left(3\alpha - \frac{1}{\alpha}\right)^3 + 9\left(3\alpha - \frac{1}{\alpha}\right)$

$$= 2^3 + 9 \times 2 = 26$$

Similarly, we have $f(\beta) = 26$

3. (a) We have,

$$(A \cup B)' \cup (A' \cap B) = (A' \cap B') \cup (A' \cap B)$$

$$= (A' \cup A') \cap (A' \cup B) \cap (B' \cup A')$$

$$= A' \cap (A' \cup B) \cap (B' \cup A')$$

$$= A' \cap \{(A \cap B')' \cup (A \cap B)\}'$$

$$= A' \cap \{(A \cap B') \cap (A \cap B)\}'$$

$$= A' \cap A' = A'$$

4. (b) We have,

$$A = \{x: |x| < 3, x \in \mathbb{Z}\}$$

$$= \{-2, -1, 0, 1, 2\}$$

and $R = \{(x, y): y = |x|, x \neq -1\}$

$$\Rightarrow R = \{(-2, 2), (0, 0), (1, 1), (2, 2)\}$$

Clearly, R has four elements. So, the number of elements in power set of R is $2^4 = 16$.

5. (a) We know that,

$$\sim(p \rightarrow q) \equiv p \wedge \sim q$$

$$\therefore \sim((p \wedge r) \rightarrow (r \vee q)) \equiv (p \wedge r) \wedge (\sim(r \vee q))$$

$$\equiv (p \wedge r) \wedge (\sim r \wedge \sim q)$$

6. (a) We have,

$$\frac{2b^2}{a} = 12 \text{ and } b = 2\sqrt{3} \Rightarrow a = 2$$

$$\therefore e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + \frac{(2\sqrt{3})^2}{(2)^2}} = \sqrt{1 + 3} = 2$$

7. (c) The equation of the parabola is

$$y^2 - kx + 8 = 0$$

$$\Rightarrow y^2 = k\left(x - \frac{8}{k}\right)$$

$$\Rightarrow (y - 0)^2 = k\left(x - \frac{8}{k}\right)$$

The equation of the directrix of this parabola is

$$x - \frac{8}{k} = -\frac{k}{4}$$

$$\Rightarrow x = \frac{8}{k} - \frac{k}{4}$$

But the equation of the directrix is given as $x - 1 = 0$

$$\therefore \frac{8}{k} - \frac{k}{4} = 1 \Rightarrow k^2 + 4k - 32 = 0$$

$$\Rightarrow k = -8, 4$$

8. (c) We know that the line $y = mx + c$ touches the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ iff

$$c^2 = a^2 m^2 + b^2$$

Here, $a^2 = 4, b^2 = 1, m = 4$

$$\therefore c^2 = 4 \times 4^2 + 1$$

$$\Rightarrow c^2 = 65$$

$$\Rightarrow c = \pm \sqrt{65}$$

9. (b) The equation of any tangent to the given ellipse is

$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$$

This line meets the coordinate axes at

$$P\left(\frac{a}{\cos \theta}, 0\right) \text{ and } Q\left(0, \frac{b}{\sin \theta}\right)$$

$$\therefore \frac{a}{\cos \theta} = l = \frac{b}{\sin \theta}$$

$$\Rightarrow \cos \theta = \frac{a}{l} \text{ and } \sin \theta = \frac{b}{l}$$

$$\Rightarrow \cos^2 \theta + \sin^2 \theta = \frac{a^2}{l^2} + \frac{b^2}{l^2}$$

$$\Rightarrow 1 = \frac{a^2 + b^2}{l^2}$$

$$\Rightarrow l = \sqrt{a^2 + b^2}$$

10. (b) Let n and p be parameters of the binomial distribution. Then,

$$np + npq = 15$$

and $(np)^2 + (npq)^2 = 117$

$$\therefore \frac{n^2 p^2 (1 + q^2)}{(np + npq)^2} = \frac{117}{15^2}$$

$$\Rightarrow \frac{1 + q^2}{(1 + q)^2} = \frac{117}{225}$$

$$\Rightarrow \frac{1 + q^2}{1 + q^2 + 2q} = \frac{13}{25}$$

$$\Rightarrow 6q^2 - 13q + 6 = 0$$

$$\Rightarrow (2q - 3)(3q - 2) = 0$$

$$\Rightarrow q = \frac{3}{2} \text{ or } q = \frac{2}{3}$$

When $q = \frac{3}{2}$, then

$$p = 1 - q = 1 - \frac{3}{2} = -\frac{1}{2}, \text{ which is}$$

neglected and when $q = \frac{2}{3}$, then

$$p = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow q = \frac{2}{3} \text{ and } p = \frac{1}{3}$$

$$\therefore np + npq = 15$$

$$\Rightarrow n \times \frac{1}{3} + n \times \frac{2}{9} = 15$$

$$\Rightarrow n = 27$$

$$\therefore \text{Mean, } np = 9$$

11. (c) For a real chord of contact of tangents drawn from (α, α) to the circle $x^2 + y^2 + 2gx + 4y + 2 = 0$, the point (α, α) must lie outside the circle.

$$\therefore \alpha^2 + \alpha^2 + 2g\alpha + 4\alpha + 2 > 0$$

$$\Rightarrow \alpha^2 + g\alpha + (2\alpha + 1) > 0$$

$$\Rightarrow \alpha^2 + (g + 2)\alpha + 1 > 0$$

Since, $\alpha \in \mathbb{R}$, then

$$(g + 2)^2 - 4 > 0$$

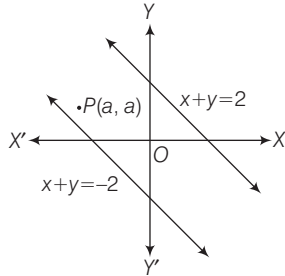
$$\Rightarrow g^2 + 4g > 0 \Rightarrow -4 < g < 0$$

$$\Rightarrow g \in (-4, 0)$$

12. (c) The equations of the lines are $x + y = 2$ and $x + y = -2$. If the point

Engineering Entrances Solved Papers 2018 33

(a, a) falls between these two lines, then
 $(a + a - 2)(a + a + 2) < 0$
 $\Rightarrow 4a^2 - 4 < 0 \Rightarrow a^2 - 1 < 0 \Rightarrow |a| < 1$



13. (a) We have,

$$P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3} \Rightarrow P(B \cap \bar{C}) \cap \bar{A} = \frac{1}{3}$$

$$\Rightarrow P(B \cap \bar{C}) - P((B \cap \bar{C}) \cap A) = \frac{1}{3}$$

$$[\because P(X \cap \bar{Y}) = P(X) - P(X \cap Y)]$$

$$\Rightarrow P(B \cap \bar{C}) - P(A \cap B \cap \bar{C}) = \frac{1}{3}$$

$$\Rightarrow P(B \cap \bar{C}) - \frac{1}{3} = \frac{1}{3}$$

$$\left[\text{given, } P(A \cap B \cap \bar{C}) = \frac{1}{3} \right]$$

$$\Rightarrow P(B \cap \bar{C}) = \frac{2}{3}$$

$$\Rightarrow P(B) - P(B \cap C) = \frac{2}{3}$$

$$\Rightarrow P(B \cap C) = P(B) - \frac{2}{3}$$

$$= \frac{3}{4} - \frac{2}{3}$$

$$= \frac{1}{12}$$

14. (b) If B throws a number i , then A can throw any one of the numbers 1, 2, ..., i .
 $\therefore$ Required probability

$$= \sum_{i=1}^6 P(\text{A throws a number } i \text{ and B throws a number less than or equal to } i)$$

$$= \sum_{i=1}^6 \frac{1}{6} \times \frac{i}{6} = \frac{1}{36} \sum_{i=1}^6 i$$

$$= \frac{1}{36} \times \frac{6 \times 7}{2} = \frac{7}{12}$$

15. (d) We have,

$$a \cdot \lim_{x \rightarrow 1} x^{\frac{1}{1-x}} + b = e^{-1}$$

$$\Rightarrow a \cdot \lim_{x \rightarrow 1} [1 + (x-1)]^{\frac{1}{1-x}} + b = e^{-1}$$

$$\Rightarrow a \cdot e \lim_{x \rightarrow 1} \frac{x-1}{1-x} + b = e^{-1}$$

$$\Rightarrow a \cdot e^{-1} + b = e^{-1}$$

$$\Rightarrow a = 1$$

$$\text{and } b = 0$$

16. (d) Clearly, $f(x)$ is defined, if

$$25 - x^2 > 0$$

$$\Rightarrow -5 < x < 5$$

Now, let $y = \log_5 (25 - x^2)$.

$$\text{Then, } 5^y = 25 - x^2$$

$$\Rightarrow x^2 = 25 - 5^y$$

$$\Rightarrow x = \pm \sqrt{25 - 5^y}$$

For x to be real, we must have

$$25 - 5^y \geq 0$$

$$\Rightarrow 5^y \leq 25$$

$$\Rightarrow y \leq 2$$

Also, $y = f(x) \rightarrow -\infty$ as $x \rightarrow \pm 5$

Hence, range $f = (-\infty, 2]$

MHT CET

1. (c) We know that parametric form of $y^2 = 4ax$ is

$$x = at^2$$

$$\text{and } y = 2at$$

$$\text{Here, } y^2 = -16x$$

$$\therefore x = -4t^2$$

$$\text{and } y = -8t$$

$$\text{Given, } t = \frac{1}{2}$$

$$\therefore x = -4 \left(\frac{1}{2} \right)^2$$

$$\text{and } y = -8 \left(\frac{1}{2} \right)$$

$$x = -1$$

$$\text{and } y = -4$$

Hence, cartesian coordinates are $(-1, -4)$.

2. (c) We have,

$$f(x) = \frac{x^2 - 4}{x - 2}, x \neq 2$$

$$f(x) = \frac{(x+2)(x-2)}{x-2}, x \neq 2$$

$$f(x) = (x+2), x \neq 2$$

$$\therefore \text{Range of } f(x) = R - \{4\}$$

3. (c) Given, $5x + y - 1 = 0$ coincides

with one of the lines given by $5x^2 + xy - kx - 2y + 2 = 0$

On putting the value of $y = -5x + 1$ in the above equation, we get

$$5x^2 + x(-5x + 1) - kx$$

$$-2(-5x + 1) + 2 = 0$$

$$\Rightarrow 5x^2 - 5x^2 + x - kx + 10x - 2 + 2 = 0$$

$$\Rightarrow x - kx + 10x = 0$$

$$\Rightarrow 11x - kx = 0$$

$$\Rightarrow 11 - k = 0$$

$$\Rightarrow k = 11$$

4. (c) Given words, HULULULU

$$\therefore \text{Total number of sample space} = \frac{8!}{4!3!}$$

and total number of ways all three L being together = $\frac{6!}{4!}$

$$\therefore \text{Required probability} = \frac{\frac{6!}{4!}}{\frac{8!}{4!3!}} = \frac{3}{28}$$

5. (b) Let,

$$S_n = 9 + 99 + 999 + \dots n \text{ terms}$$

$$\Rightarrow S_n = (10 - 1) + (100 - 1) + (1000 - 1) + \dots n \text{ terms}$$

$$\Rightarrow S_n = (10 + 10^2 + 10^3 + \dots n \text{ terms}) - (1 + 1 + \dots n \text{ terms})$$

$$\Rightarrow S_n = \frac{10(10^n - 1)}{10 - 1} - n$$

$$\left[\because a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}, r > 1 \right]$$

$$\Rightarrow S_n = \frac{10}{9}(10^n - 1) - n$$

Put $n = 10$

$$\Rightarrow S_{10} = \frac{10}{9}(10^{10} - 1) - 10$$

$$= \frac{10}{9}(10^{10} - 1 - 9)$$

$$= \frac{10}{9}(10^{10} - 10)$$

$$= \frac{100}{9}(10^9 - 1)$$

6. (b) Given,

$$A + B + C = \pi$$

$$\Rightarrow A + B = \pi - C$$

$$\Rightarrow \cot(A + B) = \cot(\pi - C)$$

$$\Rightarrow \frac{\cot A \cot B - 1}{\cot A + \cot B} = -\cot C$$

$$\Rightarrow \cot A \cot B - 1 = -\cot A \cot C$$

$$-\cot B \cot C$$

$$\Rightarrow \cot A \cot B + \cot B \cot C$$

$$+ \cot C \cot A = 1$$

34 Engineering Entrances Solved Papers 2018

7. (d) We have,

$$\begin{aligned} 2\sin\left(\theta + \frac{\pi}{3}\right) &= \cos\left(\theta - \frac{\pi}{6}\right) \\ \Rightarrow 2\left[\sin\theta\cos\frac{\pi}{3} + \cos\theta\sin\frac{\pi}{3}\right] &= \cos\theta\cos\frac{\pi}{6} + \sin\theta\sin\frac{\pi}{6} \\ \Rightarrow 2\left[\frac{\sin\theta}{2} + \frac{\sqrt{3}\cos\theta}{2}\right] &= \frac{\sqrt{3}}{2}\cos\theta + \frac{1}{2}\sin\theta \\ \Rightarrow 2\sin\theta + 2\sqrt{3}\cos\theta &= \sqrt{3}\cos\theta + \sin\theta \\ \Rightarrow \sin\theta &= -\sqrt{3}\cos\theta \\ \Rightarrow \tan\theta &= -\sqrt{3} \end{aligned}$$

8. (d) Given $P(4, 5, x)$, $Q(3, y, 4)$ and $R(5, 8, 0)$ are collinear.

$$\begin{aligned} \therefore \overrightarrow{PQ} &= \lambda \overrightarrow{QR} \\ \Rightarrow \overrightarrow{PQ} &= \overrightarrow{OQ} - \overrightarrow{OP} \\ &= (3\hat{i} + y\hat{j} + 4\hat{k}) - (4\hat{i} + 5\hat{j} + x\hat{k}) \\ &= -\hat{i} + (y-5)\hat{j} + (4-x)\hat{k} \\ \text{and } \overrightarrow{QR} &= \overrightarrow{OR} - \overrightarrow{OQ} \\ &= (5\hat{i} + 8\hat{j}) - (3\hat{i} + y\hat{j} + 4\hat{k}) \\ &= 2\hat{i} + (8-y)\hat{j} - 4\hat{k} \\ \therefore -\hat{i} + (y-5)\hat{j} + (4-x)\hat{k} &= \lambda[2\hat{i} + (8-y)\hat{j} - 4\hat{k}] \end{aligned}$$

On equating the component of vector both sides, we get

$$\begin{aligned} \frac{-1}{2} &= \lambda \\ \frac{y-5}{8-y} &= \lambda \\ \frac{4-x}{-4} &= \lambda \end{aligned}$$

On putting the value of λ , we get

$$(y-5) = (8-y)\left(-\frac{1}{2}\right)$$

$$\text{and } 4-x = -4\left(-\frac{1}{2}\right)$$

$$\begin{aligned} \Rightarrow 2y-10 &= y-8 \\ \text{and } 4-x &= 2 \\ \Rightarrow y &= 2 \\ \text{and } x &= 2 \\ \therefore x+y &= 2+2=4 \end{aligned}$$

9. (a) We have,

$$ax^2 + 2hxy + by^2 = 0$$

Let slope of one line is m

$\therefore$ Slope of another line is $2m$.

We know that,

$$m_1 + m_2 = -\frac{2h}{b}$$

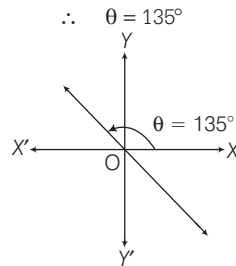
$$\begin{aligned} \text{and } m_1 m_2 &= \frac{a}{b} \\ \therefore m + 2m &= -\frac{2h}{b} \\ \Rightarrow 3m &= -\frac{2h}{b} \quad \dots(i) \end{aligned}$$

$$\begin{aligned} \text{and } m(2m) &= \frac{a}{b} \\ \Rightarrow 2m^2 &= \frac{a}{b} \quad \dots(ii) \end{aligned}$$

On eliminating m , we get

$$\begin{aligned} 2\left(\frac{-2h}{3b}\right)^2 &= \frac{a}{b} \\ \Rightarrow 8h^2 &= 9ab \end{aligned}$$

10. (a) Since, the line bisects the angle between coordinate axes.



Now, equation of the line passing through the point $(-3, 1)$ and making angle $\theta = 135^\circ$ with positive direction of X -axis, is

$$\begin{aligned} y-1 &= m(x+3) \\ \Rightarrow y-1 &= \tan 135^\circ(x+3) \\ [\because m &= \tan \theta \text{ and } \theta = 135^\circ] \\ \Rightarrow y-1 &= -1(x+3) \\ \Rightarrow y-1 &= -x-3 \\ \Rightarrow x+y+2 &= 0 \end{aligned}$$

Hence, option (a) is correct.

11. (b) We have,

Getting above 95% marks is necessary condition for Hema to get the admission in good college.

Here, p : Hema get the admission in good college

q : Hema gets above 95% marks

We know symbolic form of

q is necessary condition for p is $p \rightarrow q$

Negation of $(p \rightarrow q)$ is $(p \wedge \sim q)$ or

$(\sim q \wedge p)$

$\therefore$ Negation of the above statement is Hema does not get above 95% marks and she gets admission in good college.

12. (a) $\cos 1^\circ \cdot \cos 2^\circ \cdot \cos 3^\circ \dots$

$$\begin{aligned} \cos 90^\circ \cos 91^\circ \dots \cos 179^\circ \\ = 0 \quad [\because \cos 90^\circ = 0] \end{aligned}$$

13. (c) We know that point of intersection of lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\left(\frac{hf - bg}{ab - h^2}, \frac{hg - af}{ab - h^2} \right)$$

$\therefore$ Point of intersection of lines represented by

$$x^2 - y^2 + x + 3y - 2 = 0 \text{ is } \left(-\frac{1}{2}, \frac{3}{2} \right)$$

where, $a = 1$, $b = -1$, $c = -2$, $h = 0$,

$$g = \frac{1}{2}$$

$$\text{and } f = \frac{3}{2}$$

14. (c) A die is rolled

$$\therefore \text{ Sample space} = \{1, 2, 3, 4, 5, 6\}$$

Given X denotes the number of positive divisors of the outcomes

$$\therefore X = \{1, 2, 3, 4, 5, 6\}$$

15. (c) Given a, b, c are in AP

$$\therefore 2b = a + c$$

$$\text{Now, } a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2}$$

$$\begin{aligned} \Rightarrow a \left(\frac{1 + \cos C}{2} \right) + c \left(\frac{1 + \cos A}{2} \right) \\ \left[\because 2 \cos^2 \frac{x}{2} = 1 + \cos x \right] \end{aligned}$$

$$= \frac{1}{2} [a + c + a \cos C + c \cos A]$$

$$= \frac{1}{2} [2b + b] = \frac{3b}{2}$$

$$[\because b = a \cos C + c \cos A]$$

16. (b) We have,

$$\sin x + \sin 3x + \sin 5x = 0$$

$$\Rightarrow \sin x + \sin 5x + \sin 3x = 0$$

$$\Rightarrow 2 \sin 3x \cos 2x + \sin 3x = 0$$

$$\Rightarrow \sin 3x (2 \cos 2x + 1) = 0$$

$$\Rightarrow \sin 3x = 0$$

$$\text{or } \cos 2x = -\frac{1}{2} = \cos \frac{2\pi}{3}$$

$$\Rightarrow 3x = n\pi$$

$$\text{or } 2n = 2n\pi \pm \frac{2\pi}{3}$$

$$\Rightarrow x = \frac{n\pi}{3}$$

$$\text{or } x = n\pi \pm \frac{\pi}{3}$$

$$\text{But, it is given that } x \in \left[\frac{\pi}{2}, \frac{3\pi}{2} \right]$$

$$\therefore x = \frac{2\pi}{3}, \pi, \frac{4\pi}{3}$$

$\therefore$ Number of solutions is 3.

17. (b) Given statement is

"If the weather is fine then my friends will come and we go for a picnic".

Here, p = the weather is fine
 q = my friends will come
 r = we go for a picnic

Symbolic form is $p \rightarrow (q \wedge r)$

contrapositive of $p \rightarrow q$ is $\sim q \wedge \sim p$

$\therefore$ Contrapositive of

$p \rightarrow (q \wedge r) \sim (q \wedge r) \sim p$

$\Rightarrow (\sim q \vee \sim r) \rightarrow \sim p$

$\therefore$ Contrapositive of the given statement is, if my friends will not come or we do not go for a picnic then weather will not fine.

18. (a) We have,

$$X = \{4^n - 3n - 1 : n \in N\}$$

$$= \{0, 9, 54, \dots\}$$

$$\text{and } Y = \{9(n-1) : n \in N\}$$

$$= \{0, 9, 18, 27, 36, 45, 54, \dots\}$$

$$\therefore X \cap Y = X$$

19. (b) Let us prepare the following truth table

p	q	$\sim p$	$\sim p \wedge q$	$p \wedge (\sim p \wedge q)$
T	T	F	F	F
T	F	F	F	F
F	T	T	T	F
F	F	T	F	F

Clearly, last column of the truth table contains F only. So, it is a contradiction.

20. (a) We have,

$$y^2 = ax^3 + b$$

On differentiating both sides w.r.t. x , we get

$$2y \frac{dy}{dx} = 3ax^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{3ax^2}{2y}$$

$$\Rightarrow \left(\frac{dy}{dx}\right)_{(2,3)} = \frac{3a(2)^2}{2(3)} = 2a$$

Now, slope of line $y = 4x - 5$ is 4

$\therefore$ Slope of curve is $2a = 4 \Rightarrow a = 2$

Now, $y^2 = ax^3 + b$

Since, (2,3) satisfies the curve

$$y^2 = ax^3 + b.$$

$$\therefore (3)^2 = 2(2)^3 + b$$

$$\Rightarrow b = 9 - 16 = -7$$

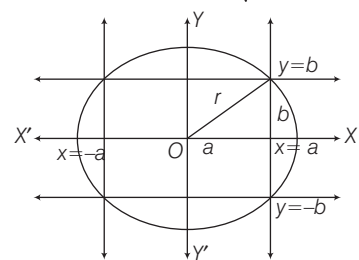
$$\therefore 7a + 2b = 7(2) + 2(-7) = 0$$

21. (b) Given sides of rectangle are

$$x = \pm a \text{ and } y = \pm b$$

$\therefore$ Centre of circle = (0, 0)

and radius of circle = $\sqrt{a^2 + b^2}$



$\therefore$ Equation of circle

$$x^2 + y^2 = (a^2 + b^2)$$

Engineering Entrance Questions 2019-20

JEE Main & Adv., BITSAT, AP & TS EAMCET, MHTCET, WB JEE

Sets, Fundamentals of Relations and Function

1. The greatest positive integer k , for which $49^k + 1$ is a factor of the sum $49^{125} + 49^{124} + \dots + 49^2 + 49 + 1$, is

JEE Main 2020

- (a) 32 (b) 63
(c) 65 (d) 60

2. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that for all $x \in \mathbb{R}$ ($2^{1+x} + 2^{1-x}$), $f(x)$ and ($3^x + 3^{-x}$) are in A.P., then the minimum value of $f(x)$ is

JEE Main 2020

- (a) 4 (b) 3
(c) 2 (d) 0

3. Let $f: (1, 3) \rightarrow \mathbb{R}$ be a function defined by $f(x) = \frac{x[x]}{1+x^2}$, where $[x]$ denotes the greatest integer $\leq x$. Then the range of f is

JEE Main 2020

- (a) $\left(\frac{2}{5}, \frac{3}{5}\right) \cup \left(\frac{3}{4}, \frac{4}{5}\right)$
(b) $\left(\frac{2}{5}, \frac{4}{5}\right]$
(c) $\left(\frac{3}{5}, \frac{4}{5}\right)$
(d) $\left(\frac{2}{5}, \frac{1}{2}\right) \cup \left(\frac{3}{5}, \frac{4}{5}\right]$

4. If $A = \{x \in \mathbb{R} : |x| < 2\}$ and $B = \{x \in \mathbb{R} : |x-2| \geq 3\}$; then

JEE Main 2019

- (a) $B - A = \mathbb{R} - (-2, 5)$
(b) $A - B = [-1, 2)$
(c) $A \cup B = \mathbb{R} - (2, 5)$
(d) $A \cap B = (-2, -1)$

5. The number of functions f from $\{1, 2, 3, \dots, 20\}$ onto $\{1, 2, 3, \dots, 20\}$ such that $f(k)$ is a multiple of 3, whenever k is a multiple of 4, is

JEE Main 2019

- (a) $(15)! \times 6!$
(b) $(-1, 1) \times \{0\}$
(c) $\mathbb{R} - \left[-\frac{1}{2}, \frac{1}{2}\right]$
(d) $\mathbb{R} - [-1, 1]$

6. Two newspapers A and B are published in a city. It is known that 25% of the city population reads A and 20% reads B while 8% reads both A and B .

Further, 30% of those who read A but not B look into advertisements and 40% of those who read B but not A also look into advertisements, while 50% of those who read both A and B look into advertisements.

Then, the percentage of the population who look into advertisements is

JEE Main 2019

- (a) 13.5 (b) 13 (c) 12.8 (d) 13.9

7. If $x \in \mathbb{R}$, then the range of $\frac{x}{x^2 - 5x + 9}$ is

AP EAMCET 2019

- (a) $\left(-\frac{1}{11}, 1\right)$ (b) $\left(-\infty, -\frac{1}{11}\right) \cup (1, \infty)$
(c) $\left[\frac{-1}{11}, 1\right]$ (d) $\left[-1, \frac{1}{11}\right]$

8. If $|x|$ denotes the greatest integer function, then the domain of the function

$$f(x) = \sqrt{\frac{x - |x|}{\log(x^2 - x)}}, \text{ is}$$

TS EAMCET 2019

- (a) $(1, \infty)$
(b) $(1, \infty) - \mathbb{Z}$
(c) $\mathbb{R} - \left[\frac{1-\sqrt{5}}{2}, \frac{1+\sqrt{5}}{2}\right]$
(d) $\left[\frac{1-\sqrt{5}}{2}, \frac{\sqrt{5}+1}{2}\right]$

9. If $[x]$ denotes the greatest integer $\leq x$, then the domain of the function

$$f(x) = \sqrt{\frac{4-x^2}{[x]+2}} \text{ is}$$

TS EAMCET 2019

- (a) $(-\infty, -2] \cup [-1, 2)$
(b) $(-\infty, -2) \cup [-1, 2]$
(c) $(-\infty, -2) \cup (-1, 2)$
(d) $(-\infty, -1] \cup [1, 2]$

10. If $A = \{x \mid x \in \mathbb{N}, x \text{ is a prime number less than } 12\}$ and $B = \{x \mid x \in \mathbb{N}, x \text{ is a factor } 10\}$, then $A \cap B = \dots\dots\dots$

MHTCET 2019

- (a) $\{2\}$ (b) $\{2, 5\}$
(c) $\{2, 5, 10\}$ (d) $\{1, 2, 5, 10\}$

11. Let P and T be the subsets of k, y -plane defined by

$$P = \{(x, y) : x > 0, y > 0 \text{ and } x^2 + y^2 = 1\}$$

$$T = \{(x, y) : x > 0, y > 0 \text{ and } x^8 + y^8 < 1\}$$

Then, $P \cap T$ is

WB JEE 2019

- (a) the void set ϕ (b) P
(c) T (d) $P - T^c$

Sequence and Series

1. Five numbers are in A.P., whose sum is 25 and product is 2520. If one of these five numbers is $-\frac{1}{2}$, then the greatest number amongst them is

JEE Main 2020

- (a) 7 (b) 16
(c) 27 (d) $\frac{21}{2}$

2. Let $a_1, a_2, a_3, \dots$ be a G.P. such that $a_1 < 0$, $a_1 + a_2 = 4$ and $a_3 + a_4 = 16$. If $\sum_{i=1}^9 a_i = 4\lambda$, then λ is equal to

JEE Main 2020

- (a) -171 (b) 171
(c) $\frac{511}{3}$ (d) -513

3. The product $2^4 \cdot 4^{16} \cdot 8^{48} \cdot 16^{128} \dots$ to ∞ is equal to

JEE Main 2020

- (a) 2^4 (b) 2 (c) $2^{\frac{1}{2}}$ (d) 1

4. Let $a_1, a_2, \dots, a_{10}$ be a GP. If $\frac{a_3}{a_1} = 25$, then $\frac{a_9}{a_5}$ equals

JEE Main 2019

- (a) 5^3 (b) $2(5^2)$
(c) $4(5^2)$ (d) 5^4

5. If three distinct numbers a, b and c are in GP and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common

root, then which one of the following statements is correct?

JEE Main 2019

- (a) d, e and f are in GP
(b) $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in AP
(c) d, e and f are in AP
(d) $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in GP

6. Let $\sum_{k=1}^{10} f(a+k) = 16(2^{10} - 1)$, where the

function f satisfies $f(x+y) = f(x)f(y)$ for all natural numbers x, y and $f(1) = 2$. Then, the natural number 'a' is

JEE Main 2019

- (a) 2 (b) 4
(c) 3 (d) 16

7. The sum of first n terms of the series $\frac{3}{5} + \frac{21}{25} + \frac{117}{125} + \dots$ is **APEAMECET 2019**
- (a) $n + \frac{2^{n+1}}{3 \times 5^n} - \frac{2}{3}$ (b) $n - \frac{2^{n+1}}{3 \times 5^n} - \frac{2}{3}$
- (c) $n + \frac{2^{n+1}}{3 \times 5^n} + \frac{2}{3}$ (d) $n - \frac{2^{n+1}}{3 \times 5^n} + \frac{2}{3}$

8. If $\alpha \in \mathbb{R}$, $n \in \mathbb{N}$ and $n + 2(n-1) + 3(n-2) + \dots + (n-1)2 + n.1 = \alpha n(n+1)(n+2)$, then $\alpha =$ **TS EAMCET 2019**
- (a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{5}$ (d) $\frac{1}{6}$

9. For a GP, if $S_n = \frac{4^n - 3^n}{3^n}$, then $t_2 = \dots$ **MHT CET 2019**
- (a) $\frac{1}{9}$ (b) $\frac{2}{9}$
- (c) $\frac{7}{9}$ (d) $\frac{4}{9}$

Complex Numbers

1. If $\operatorname{Re}\left(\frac{z-1}{2z+i}\right) = 1$, where $z = x + iy$, then the point (x, y) lies on **JEE Main 2020**
- (a) straight line whose slope is $-\frac{2}{3}$
- (b) circle whose centre is at $\left(-\frac{1}{2}, -\frac{3}{2}\right)$
- (c) straight line whose slope is $\frac{3}{2}$
- (d) circle whose diameter is $\frac{\sqrt{5}}{2}$

2. If the equation, $x^2 + bx + 45 = 0$ ($b \in \mathbb{R}$) has conjugate complex roots and they satisfy $|z+1| = 2\sqrt{10}$, then **JEE Main 2020**
- (a) $b^2 + b = 72$ (b) $b^2 + b = 12$
- (c) $b^2 - b = 30$ (d) $b^2 - b = 42$

3. Let z be a complex number such that $\left|\frac{z-i}{z+2i}\right| = 1$ and $|z| = \frac{5}{2}$. Then the value of $|z+3i|$ is **JEE Main 2020**
- (a) $\sqrt{10}$ (b) $\frac{7}{2}$ (c) $\frac{15}{4}$ (d) $2\sqrt{3}$

4. Let z_0 be a root of the quadratic equation, $x^2 + x + 1 = 0$. If $z = 3 + 6iz_0^{81} - 3iz_0^{93}$, then $\arg z$ is equal to **JEE Main 2019**
- (a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) 0 (d) $\frac{\pi}{3}$

5. Let $\left(-2 - \frac{1}{3}i\right)^3 = \frac{x+iy}{27}$ ($i = \sqrt{-1}$), where x and y are real numbers, then $y - x$ equals **JEE Main 2019**
- (a) 91 (b) 85 (c) -85 (d) -91

6. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$ ($i = \sqrt{-1}$), then $(1 + iz + z^5 + iz^8)^9$ is equal to **JEE Main 2019**
- (a) 1 (b) $(-1 + 2i)^9$
- (c) -1 (d) 0

7. Let S be the set of all complex numbers z satisfying $|z - 2 + i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0 - 1|}$ is the maximum of the set $\left\{\frac{1}{|z - 1|} : z \in S\right\}$, then the principal argument of $\frac{4 - z_0 - \bar{z}_0}{z_0 - \bar{z}_0 + 2i}$ is **JEE Advance 2019**
- (a) $\frac{\pi}{4}$ (b) $\frac{3\pi}{4}$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$

8. The value of ' λ ' for which the loci $\arg z = \frac{\pi}{6}$ and $|z - 2\sqrt{3}i| = \lambda$ on the argand plane touch each other is

- BIT SAT 2019**
- (a) 3 (b) 4
- (c) 5 (d) 6

9. If $z = x + iy$, $x, y \in \mathbb{R}$, $(x, y) \neq (0, -4)$ and $\operatorname{Arg}\left(\frac{2z-3}{z+4i}\right) = \frac{\pi}{4}$, then the locus of z is **AP EAM CET 2019**
- (a) $2x^2 + 2y^2 + 5x + 5y - 12 = 0$
- (b) $2x^2 - 3xy + y^2 + 5x + y - 12 = 0$
- (c) $2x^2 + 3xy + y^2 + 5x + y + 12 = 0$
- (d) $2x^2 + 2y^2 - 11x + 7y - 12 = 0$

10. If P is a complex number whose modulus is one, then the equation $\left(\frac{1+iz}{1-iz}\right)^4 = P$ has **TS EAMCET 2019**
- (a) real and equal roots
- (b) real and distinct roots
- (c) two real and two complex roots
- (d) all complex roots

11. The general value of the real angle θ , which satisfies the equation, $(\cos\theta + i\sin\theta)(\cos 2\theta + i\sin 2\theta) \dots (\cos n\theta + i\sin n\theta) = 1$ is given by, (assuming k is an integer) **WB JEE 2019**
- (a) $\frac{2k\pi}{n+2}$ (b) $\frac{4k\pi}{n(n+1)}$
- (c) $\frac{4k\pi}{n+1}$ (d) $\frac{6k\pi}{n(n+1)}$

Inequalities and Quadratic Equation

1. Let α and β be two real roots of the equation $(k+1)\tan^2 x - \sqrt{2} \cdot \lambda \tan x = (1-k)$, where $k(\neq -1)$ and λ are real numbers. If $\tan^2(\alpha + \beta) = 50$, then a value of λ is **JEE Main 2020**
- (a) $5\sqrt{2}$ (b) 10 (c) $10\sqrt{2}$ (d) 5
2. The number of real roots of the equation, $e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$ is **JEE Main 2020**
- (a) 3 (b) 4 (c) 1 (d) 2

3. let $a, b \in \mathbb{R}$, $a \neq 0$ be such that the equation, $ax^2 - 2bx + 5 = 0$ has a repeated root α , which is also a root of the equation,

- $x^2 - 2bx - 10 = 0$. If β is the other root of this equation, then $\alpha^2 + \beta^2$ is equal to **JEE Main 2020**
- (a) 26 (b) 24 (c) 28 (d) 25

4. The value of λ such that sum of the squares of the roots of the quadratic equation, $x^2 + (3-\lambda)x + 2 = \lambda$ has the least value is **JEE Main 2019**
- (a) $\frac{4}{9}$ (b) 1 (c) $\frac{15}{8}$ (d) 2

5. The sum of the solutions of the equation $|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0$ ($x > 0$) is equal to **JEE Main 2019**
- (a) 9 (b) 12 (c) 4 (d) 10

6. If α and β are the roots of the quadratic equation, $x^2 + x\sin\theta - 2\sin\theta = 0$, $\theta \in \left(0, \frac{\pi}{2}\right)$, then $\frac{\alpha^{12} + \beta^{12}}{(\alpha^{-12} + \beta^{-12})(\alpha - \beta)^{24}}$ is equal to **JEE Main 2019**
- (a) $\frac{2^{12}}{(\sin\theta + 8)^{12}}$ (b) $\frac{2^6}{(\sin\theta + 8)^{12}}$
- (c) $\frac{2^{12}}{(\sin\theta - 4)^{12}}$ (d) $\frac{2^{12}}{(\sin\theta - 8)^6}$
7. All the pairs (x, y) that satisfy the inequality $2^{\sqrt{\sin^2 x - 2\sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \leq 1$ also satisfy the equation **JEE Main 2019**

- (a) $2|\sin x| = 3 \sin y$ (b) $\sin x = |\sin y|$
 (c) $\sin x = 2 \sin y$ (d) $2 \sin x = \sin y$
8. Let p and q be roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in AP, then A and B are

TS EAMCET 2019

- (a) $-3, -77$ (b) $3, -77$
 (c) $-3, 77$ (d) $3, 77$
9. Let a, b, c be real numbers such that $a + b + c < 0$ and the quadratic equation $ax^2 + bx + c = 0$ has imaginary roots.

Then,

WB JEE 2019

- (a) $a > 0, c > 0$ (b) $a > 0, c < 0$
 (c) $a < 0, c > 0$ (d) $a < 0, c < 0$
10. The value of $\lim_{x \rightarrow 0^+} \frac{x \left[\frac{q}{x} \right]}{p \left[\frac{q}{x} \right]}$ is

WB JEE 2019

- (a) $\frac{[q]}{p}$ (b) 0 (c) 1 (d) ∞

Permutation and Combination

1. If the number of five digit numbers with distinct digits and 2 at the 10th place is $336k$, then k is equal to
- JEE Main 2020
- (a) 8 (b) 7
 (c) 4 (d) 6
2. The sum of all two digit positive numbers which when divided by 7 yield 2 or 5 as remainder is
- JEE Main 2020
- (a) 1256 (b) 1465
 (c) 1356 (d) 1365
3. All possible numbers are formed using the digits 1, 1, 2, 2, 2, 3, 4, 4 taken all at a time. The number of such numbers in which the odd digits occupy even places is
- JEE Main 2020
- (a) 180 (b) 175
 (c) 160 (d) 162
4. The number of four-digit numbers strictly greater than 4321 that can be formed using the digits 0, 1, 2, 3, 4, 5 (repetition of digits is allowed) is

JEE Main 2019

- (a) 306 (b) 310
 (c) 360 (d) 288

5. Suppose that 20 pillars of the same height have been erected along the boundary of a circular stadium. If the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then the total number of beams is
- JEE Main 2019
- (a) 180 (b) 210 (c) 170 (d) 190
6. A person writes letter to six friends and addresses the corresponding envelopes. Let x be the numbers of ways so that at least two of the letters are in wrong envelopes and y be the numbers of ways so that all the letters are in wrong envelopes. Then $x - y =$

BIT SAT 2019

- (a) 719 (b) 265 (c) 454 (d) 720
7. The number of five digit numbers that are divisible by 6 which can be formed by choosing digits from $\{0, 1, 2, 3, 4, 5\}$, when repetition is allowed, is

AP EAMCET 2019

- (a) 648 (b) 540
 (c) 1296 (d) 1080

8. The number of 4 letter permutations formed with English alphabet such that the number of distinct vowels is equal to the number of distinct consonants, when repetition is allowed, is

TS EAMCET 2019

- (a) 630 (b) $3^5 \times 70$
 (c) $3^6 \times 70$ (d) $3^4 \times 60$

9. A student is asked to answer 10 out of 13 questions in an examination such that he must answer atleast four questions from the first five questions. The number of choices available to him is

TS EAMCET 2019

- (a) 140 (b) 176 (c) 196 (d) 280

10. There are 7 greeting cards, each of a different colour and 7 envelopes of same 7 colours as that of the cards. The number of ways in which the cards can be put in envelopes, so that exactly 4 of the cards go into envelopes of respective colour is,

WB JEE 2019

- (a) ${}^7C_3$ (b) $2 \cdot {}^7C_3$
 (c) $3! {}^4C_4$ (d) $3! {}^7C_3 {}^4C_3$

Mathematical Induction

1. Consider the statement : “ $P(n) : n^2 - n + 41$ is prime.” Then, which one of the following is true?
- JEE Main 2019
- (a) Both $P(3)$ and $P(5)$ are true
 (b) $P(3)$ is false but $P(5)$ is true
 (c) Both $P(3)$ and $P(5)$ are false
 (d) $P(5)$ is false but $P(3)$ is true

2. If 15^k divides $47!$ but 15^{k+1} does not divide it, then k is equal to
- APEAMCET 2019
- (a) 15 (b) 12 (c) 10 (d) 5
3. For all $n \in N$, if $1^2 + 2^2 + 3^2 + \dots + n^2 > x$, then x is equal to

TS EAMCET 2019

4. $7^{2n} + 16n - 1$ ($n \in N$) is divisible by
- WB JEE 2019
- (a) 65 (b) 63
 (c) 61 (d) 64

Binomial Theorem

1. If the sum of the coefficients of all even powers of x in the product $(1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$ is 61, then n is equal to

JEE Main 2020

2. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to
- JEE Main 2019
- (a) 14 (b) 6
 (c) 4 (d) 8

3. If the third term in the binomial expansion of $(1 + x^{\log_2 x})^5$ equals 2560, then a possible value of x is
- JEE Main 2019
- (a) $4\sqrt{2}$ (b) $\frac{1}{4}$ (c) $\frac{1}{8}$ (d) $2\sqrt{2}$
4. The positive value of λ for which the coefficient of x^2 in the expression $x^2 \left(\sqrt{x} + \frac{\lambda}{x^2} \right)^{10}$ is 720, is

JEE Main 2019

- (a) 3 (b) $\sqrt{5}$
 (c) $2\sqrt{2}$ (d) 4

5. The sum of the series $2 \cdot {}^{20}C_0 + 5 \cdot {}^{10}C_1 + 8 \cdot {}^{20}C_2 + 11 \cdot {}^{20}C_3 + \dots + 62 \cdot {}^{20}C_{20}$ is equal to
- JEE Main 2019
- (a) 2^{26} (b) 2^{25} (c) 2^{23} (d) 2^{24}
6. If the coefficient of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in power of x are both zero, then (a, b) is

BITSAT 2019

- (a) $\left(16, \frac{251}{3}\right)$ (b) $\left(14, \frac{251}{13}\right)$
 (c) $\left(14, \frac{272}{3}\right)$ (d) $\left(16, \frac{272}{3}\right)$

Trigonometric Functions and Equations

1. The value of $\cos^3\left(\frac{\pi}{8}\right) \cdot \cos\left(\frac{3\pi}{8}\right) + \sin^3\left(\frac{\pi}{8}\right) \cdot \sin\left(\frac{3\pi}{8}\right)$ is (a) 0 (b) 2
(c) 1 (d) infinitely many
JEE Main 2020
(a) $\frac{1}{4}$ (b) $\frac{1}{2\sqrt{2}}$ (c) $\frac{1}{2}$ (d) $\frac{1}{\sqrt{2}}$
2. If $x = \sum_{n=0}^{\infty} (-1)^n \tan^{2n} \theta$ and $y = \sum_{n=0}^{\infty} \cos^{2n} \theta$, for $0 < \theta < \frac{\pi}{4}$, then JEE Main 2020
(a) $y(1+x)=1$ (b) $y(1-x)=1$
(c) $x(1+y)=1$ (d) $x(1-y)=1$
3. For any $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$, the expression $3(\sin \theta - \cos \theta)^4 + 6(\sin \theta + \cos \theta)^2 + 4\sin^6 \theta$ equals JEE Main 2019
(a) $13 - 4\cos^4 \theta + 2\sin^2 \theta \cos^2 \theta$
(b) $13 - 4\cos^2 \theta + 6\cos^4 \theta$
(c) $13 - 4\cos^2 \theta + 6\sin^2 \theta \cos^2 \theta$
(d) $13 - 4\cos^6 \theta$
4. The sum of all values of $\theta \in \left(0, \frac{\pi}{2}\right)$ satisfying $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$ is JEE Main 2019
5. Let α and β be the roots of the quadratic equation $x^2 \sin \theta - x(\sin \theta \cos \theta + 1) + \cos \theta = 0$ ($0 < \theta < 45^\circ$) and $\alpha < \beta$. Then, $\sum_{n=0}^{\infty} \left(\alpha^n + \frac{(-1)^n}{\beta^n} \right)$ is equal to JEE Main 2019
(a) $\frac{1}{1 - \cos \theta} - \frac{1}{1 + \sin \theta}$
(b) $\frac{1}{1 - \cos \theta} + \frac{1}{1 + \sin \theta}$
(c) $\frac{1}{1 + \cos \theta} - \frac{1}{1 - \sin \theta}$
(d) $\frac{1}{1 + \cos \theta} + \frac{1}{1 - \sin \theta}$
6. The number of solutions of the equation $1 + \sin^4 x = \cos^2 3x$, $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$ is JEE Main 2019
(a) 3 (b) 5 (c) 7 (d) 4
7. Number of solution of the equation $|\cos x| = 2[x]$ are (where $|x|$, $[x]$ are modulus and greatest integer function respectively). BITSAT 2019
(a) 0 (b) 2
(c) 1 (d) infinitely many
8. Let $\cos(\alpha + \beta) = \frac{4}{5}$ and let $\sin(\alpha - \beta) = \frac{5}{13}$, where $0 \leq \alpha, \beta \leq \frac{\pi}{4}$, then $\tan 2\alpha =$ BITSAT 2019
(a) $\frac{20}{7}$ (b) $\frac{25}{16}$ (c) $\frac{56}{33}$ (d) $\frac{19}{2}$
9. $\cos^2 5^\circ - \cos^2 15^\circ - \sin^2 15^\circ + \sin^2 35^\circ + \cos 15^\circ \sin 15^\circ - \cos 5^\circ \sin 35^\circ =$ AP EAMCET 2019
(a) 0 (b) 1
(c) $\frac{3}{2}$ (d) 2
10. If A and B are acute angles satisfying $3\cos^2 A + 2\cos^2 B = 4$ and $\frac{3\sin A}{\sin B} = \frac{2\cos B}{\cos A}$, then $A + 2B =$ TS EAMCET 2019
(a) 30° (b) 45°
(c) 60° (d) 90°
11. In $\triangle ABC$, with the usual notations, if $\left(\tan \frac{A}{2}\right)\left(\tan \frac{B}{2}\right) = \frac{3}{4}$ then $a + b =$ MHT CET 2019
(a) $4c$ (b) $2c$
(c) $7c$ (d)

Properties of Triangles, Heights and Distances, Cartesian System of Rectangular Coordinates

1. Let $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ be the vertices of a triangle ABC . If P is a point inside the triangle ABC such that the triangles APC , APB and BPC have equal areas, then the length of the line segment PQ , where Q is the point $\left(-\frac{7}{6}, -\frac{7}{3}\right)$, is JEE Main 2020
(a) $\frac{2}{3}\sqrt{21}$ (b) $2\sqrt{21}$
(c) $7\sqrt{3}$ (d) $\frac{3}{2}\sqrt{21}$
2. Let C be the centroid of the triangle with vertices $(3, -1)$, $(1, 3)$ and $(2, 4)$. Let P be the point of intersection of the lines $x + 3y - 1 = 0$ and $3x - y + 1 = 0$. Then the line passing through the points C and P also passes through the point JEE Main 2020
(a) $(-9, -7)$ (b) $(-9, -6)$
(c) $(7, 6)$ (d) $(9, 7)$
3. Consider a triangular plot ABC with sides $AB = 7$ m, $BC = 5$ m and $CA = 6$ m. A vertical lamp-post at the mid-point D of AC subtends an angle 30° at B . The height (in m) of the lamp-post is JEE Main 2019
4. A point P moves on the line $2x - 3y + 4 = 0$. If $Q(1, 4)$ and $R(3, -2)$ are fixed points, then the locus of the centroid of $\triangle PQR$ is a line JEE Main 2019
(a) with slope $\frac{2}{3}$ (b) with slope $\frac{3}{2}$
(c) parallel to Y -axis (d) parallel to X -axis
5. If the line $3x + 4y - 24 = 0$ intersects the x -axis at the point A and the Y -axis at the point B then the incentre of the triangle OAB . Where O is the origin is JEE Main 2019
(a) $(4, 3)$ (b) $(3, 4)$
(c) $(4, 4)$ (d) $(2, 2)$
6. Two vertices of a triangle are $(0, 2)$ and $(4, 3)$. If its orthocentre is at the origin, then its third vertex lies in which quadrant? JEE Main 2019
(a) Fourth (b) Third
(c) Second (d) First
7. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y . If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is JEE Main 2019
(a) $\frac{c}{3}$ (b) $\frac{c}{\sqrt{3}}$ (c) $\frac{3}{2}y$ (d) $\frac{y}{\sqrt{3}}$
8. Let $O(0, 0)$ and $A(0, 1)$ be two fixed points, then the locus of a point P such that the perimeter of $\triangle AOP$ is 4, is JEE Main 2019
(a) $8x^2 - 9y^2 + 9y = 18$
(b) $9x^2 - 8y^2 + 8y = 16$
(c) $9x^2 + 8y^2 - 8y = 16$
(d) $8x^2 + 9y^2 - 9y = 18$
9. A rectangle is inscribed in a circle with a diameter lying along the line $3y = x + 7$. If the two adjacent vertices of the rectangle are $(-8, 5)$ and $(6, 5)$, then the area of the rectangle (in sq units) is JEE Main 2019
(a) 72 (b) 84 (c) 98 (d) 56

- 10.** Two poles standing on a horizontal ground are of heights 5 m and 10 m, respectively. The line joining their tops makes an angle of 15° with the ground. Then, the distance (in m) between the poles, is
JEE Main 2019
(a) $5(\sqrt{3} + 1)$ (b) $\frac{5}{2}(2 + \sqrt{3})$
(c) $10(\sqrt{3} - 1)$ (d) $5(2 + \sqrt{3})$
- 11.** In triangle ABC , if $\frac{b+c}{9} = \frac{c+a}{10} = \frac{a+b}{11}$, then $\frac{\cos A + \cos B}{\cos C} =$
APEAMCET 2019
(a) $\frac{9}{10}$ (b) $\frac{10}{11}$ (c) $\frac{11}{12}$ (d) $\frac{12}{13}$
- 12.** In any triangle ABC , if $a : b : c = 2 : 3 : 4$, then $R : r =$
TS EAMCET 2019
(a) $8 : 3$ (b) $16 : 9$ (c) $5 : 16$ (d) $16 : 5$
- 13.** If $A(-2, 1)$, $B(0, -2)$, $C(1, 2)$ are the vertices of a triangle ABC , then the perpendicular distance from its circumcentre to the side BC is
TS EAMCET 2019
(a) $\frac{7\sqrt{13}}{22}$ (b) $\frac{3\sqrt{17}}{22}$ (c) $\frac{5\sqrt{10}}{11}$ (d) $\frac{\sqrt{2026}}{22}$
- 14.** If p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C respectively, then with the usual notation, $\frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} =$
TS EAMCET 2019
(a) $p_1 p_2 p_3$ (b) $\frac{a^2 b^2 c^2}{4\Delta^2}$
(c) $\frac{a^2 b^2 c^2}{\Delta^2}$ (d) $4\left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2}\right)$
- 15.** The distance between the circumcentre and the centroid of the triangle formed by the vertices $(1, 2)$, $(3, -1)$ and $(4, 0)$ is
TS EAMCET 2019
(a) $\frac{1}{\sqrt{2}}\sqrt{45}$ (b) 4
(c) $\frac{7\sqrt{2}}{15}$ (d) $\frac{9\sqrt{2}}{5}$
- 16.** A line meets the coordinate axes at A and B . If the perpendicular distances from A and B to the tangent drawn at the origin to the circumcircle of $\triangle OAB$ are m and n respectively, then the diameter of that circle is
TS EAMCET 2019
(a) $\frac{m+n}{2}$ (b) $\frac{3(m+n)}{4}$
(c) $m+n$ (d) $2(m+n)$
- 17.** In $\triangle ABC$, if $\tan A + \tan B + \tan C = 6$ and $\tan A \cdot \tan B = 2$ then $\tan C = \dots\dots\dots$
MHT CET 2019
(a) 3 (b) 4 (c) 1 (d) 2
- 18.** The three sides of right angled triangle are in GP (geometric progression). If the two acute angles be α and β , then $\tan \alpha$ and $\tan \beta$ are
WB JEE 2019
(a) $\frac{\sqrt{5}+1}{2}$ and $\frac{\sqrt{5}-1}{2}$
(b) $\sqrt{\frac{\sqrt{5}+1}{2}}$ and $\sqrt{\frac{\sqrt{5}-1}{2}}$
(c) $\sqrt{5}$ and $\frac{1}{\sqrt{5}}$
(d) $\frac{\sqrt{5}}{2}$ and $\frac{2}{\sqrt{5}}$
- 19.** The angles of a triangle are in the ratio $2 : 3 : 7$ and the radius of the circumscribed circle is 10 cm. The length of the smallest side is
WB JEE 2019
(a) 2 cm (b) 5 cm (c) 7 cm (d) 10 cm

Straight Line and Pair of Straight Lines

- 1.** The locus of the mid-points of the perpendiculars drawn from points on the line, $x = 2y$ to the line $x = y$ is
JEE Main 2020
(a) $5x - 7y = 0$
(b) $2x - 3y = 0$
(c) $3x - 2y = 0$
(d) $7x - 5y = 0$
- 2.** For $a > 0$, let the curves $C_1 : y^2 = ax$ and $C_2 : x^2 = ay$ intersect at origin O and a point P . Let the line $x = b$ ($0 < b < a$) intersect the chord OP and the x -axis at points Q and R , respectively. If the line $x = b$ bisects the area bounded by the curves, C_1 and C_2 , and the area of $\triangle OQR = \frac{1}{2}$, then 'a' satisfies the equation
JEE Main 2020
(a) $x^6 + 6x^3 - 4 = 0$
(b) $x^6 - 12x^3 + 4 = 0$
(c) $x^6 - 6x^3 + 4 = 0$
(d) $x^6 - 12x^3 - 4 = 0$
- 3.** Let two points be $A(1, -1)$ and $B(0, 2)$. If a point $P(x', y')$ be such that the area of $\triangle PAB = 5$ sq. units and it lies on the line, $3x + y - 4\lambda = 0$, then a value of λ is
JEE Main 2020
(a) 1 (b) -3
(c) 3 (d) 4
- 4.** Consider the set of all lines $px + qy + r = 0$ such that $3p + 2q + 4r = 0$. Which one of the following statements is true?
JEE Main 2019
(a) Each line passes through the origin
(b) The lines are concurrent at the point $\left(\frac{3}{4}, \frac{1}{2}\right)$
(c) The lines are all parallel
(d) The lines are not concurrent
- 5.** The straight line $x + 2y = 1$ meets the coordinate axes at A and B . A circle is drawn through A, B and the origin. Then, the sum of perpendicular distances from A and B on the tangent to the circle at the origin is
JEE Main 2019
(a) $2\sqrt{5}$ (b) $\frac{\sqrt{5}}{4}$ (c) $4\sqrt{5}$ (d) $\frac{\sqrt{5}}{2}$
- 6.** Slope of a line passing through $P(2, 3)$ and intersecting the line, $x + y = 7$ at a distance of 4 units from P , is
JEE Main 2019
(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) $2\sqrt{2}$ (d) $3\sqrt{3}$
- 7.** The larger of two angles made with the X -axis of a straight line drawn through $(1, 2)$ so that it intersect the line $x + y = 4$ at a point distance $\sqrt{6}/3$ from the point $(1, 2)$ is
BIT SAT 2019
(a) 60° (b) 75°
(c) 105° (d) None of these
- 8.** If the pairs of straight lines represented by $3x^2 + 2hxy - 3y^2 = 0$ and $3x^2 + 2hxy - 3y^2 + 2x - 4y + c = 0$ form a square, then $(h, c) =$
TS EAMCET 2019
(a) $(4, -1)$ (b) $(-1, 4)$
(c) $(-4, 1)$ (d) $(1, -4)$
- 9.** If the angle between the pair of lines $x^2 + 2\sqrt{2}xy + ky^2 = 0$, $k > 0$ is 45° , then the area (in square units) of the triangle formed by the pair of bisectors of angles between the given lines and the line $x + 2y + 1 = 0$ is
TS EAMCET 2019
(a) $\frac{1}{3}$ (b) 1
(c) $\frac{2}{3}$ (d) 2
- 10.** The joint equation of pair of straight lines passing through origin and having slopes $(1 + \sqrt{2})$ and $\left(\frac{1}{1 + \sqrt{2}}\right)$ is
MHT CET 2019
(a) $x^2 - 2\sqrt{2}xy + y^2 = 0$
(b) $x^2 - 2\sqrt{2}xy - y^2 = 0$
(c) $x^2 + 2xy - y^2 = 0$
(d) $x^2 + 2xy + y^2 = 0$

11. A variable line passes through a fixed point (x_1, y_1) and meets the axes at A and B . If the rectangle $OAPB$ be completed, the locus of P is, (O being the origin of the system of axes).

WBJEE 2019

(a) $(y - y_1)^2 = 4(x - x_1)$ (b) $\frac{x_1}{x} + \frac{y_1}{y} = 1$ (c) $x^2 + y^2 = x_1^2 + y_1^2$ (d) $\frac{x^2}{2x_1^2} + \frac{y^2}{y_1^2} = 1$

Circle

1. Let the tangents drawn from the origin to the circle, $x^2 + y^2 - 8x - 4y + 16 = 0$ touch it at the points A and B . The $(AB)^2$ is equal to

JEE Main 2020

(a) $\frac{56}{5}$ (b) $\frac{52}{5}$ (c) $\frac{64}{5}$ (d) $\frac{32}{5}$

2. A circle touches the Y -axis at the point $(0, 4)$ and passes through the point $(2, 0)$. Which of the following lines is not a tangent to this circle?

JEE Main 2020

(a) $4x - 3y + 17 = 0$ (b) $3x + 4y - 6 = 0$
(c) $4x + 3y - 8 = 0$ (d) $3x - 4y - 24 = 0$

3. Three circles of radii a, b, c ($a < b < c$) touch each other externally. If they have X -axis as a common tangent, then

JEE Main 2019

(a) a, b, c are in AP
(b) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$
(c) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in AP
(d) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

4. If the circles $x^2 + y^2 - 16x - 20y + 164 = r^2$ and $(x - 4)^2 + (y - 7)^2 = 36$ intersect at two distinct points, then

JEE Main 2019

(a) $0 < r < 1$ (b) $r > 11$
(c) $1 < r < 11$ (d) $r = 11$

5. A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the coordinate axes. Then, the distance of the vertex of this square which is nearest to the origin is

JEE Main 2019

(a) $\frac{6}{\sqrt{41}}$ (b) $\frac{13}{\sqrt{137}}$
(c) $\sqrt{41}$ (d) $\sqrt{137}$

6. The area (in sq units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point $(1, 2)$ and the X -axis is

JEE Main 2019

(a) $8\pi(3 - 2\sqrt{2})$ (b) $4\pi(3 + \sqrt{2})$
(c) $8\pi(2 - \sqrt{2})$ (d) $4\pi(2 - \sqrt{2})$

7. A line $y = mx + 1$ intersects the circle $(x - 3)^2 + (y + 2)^2 = 25$ at the points P and Q . If the mid-point of the line segment PQ has x -coordinate $-\frac{3}{5}$, then

which one of the following options is correct?

JEE Advance 2019

(a) $6 \leq m < 8$ (b) $-3 \leq m < -1$
(c) $4 \leq m < 6$ (d) $2 \leq m < 4$

8. The equation of the circle whose radius is 3 and which touches internally the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ at the point $(-1, -1)$ is

AP EAMCET 2019

(a) $5x^2 + 5y^2 + 9x - 6y - 7 = 0$
(b) $5x^2 + 5y^2 - 8x - 14y - 32 = 0$
(c) $5x^2 + 5y^2 - 6x + 8y - 8 = 0$
(d) $5x^2 + 5y^2 + 6x - 8y - 12 = 0$

9. If the number of common tangents to the pair of circles $x^2 + y^2 - 2x + 4y - 4 = 0$ and $x^2 + y^2 + 4x - 4y + \alpha = 0$ is 4, then the least integral value of α is

TS EAMCET 2019

(a) 4 (b) 5 (c) 6 (d) 7

10. The intercept on the line $y = x$ by the circle $x^2 + y^2 - 2x = 0$ is AB . The equation of the circle with AB as a diameter is

MHT CET 2019

(a) $x^2 + y^2 + x + y = 0$
(b) $x^2 + y^2 - x - y = 0$
(c) $x^2 + y^2 - 3x + y = 0$
(d) $x^2 + y^2 + 3x - y = 0$

11. A variable circle passes through the fixed point $A(p, q)$ and touches X -axis. The locus of the other end of the diameter through A is

WB JEE 2019

(a) $(x - p)^2 = 4qy$ (b) $(x - q)^2 = 4py$
(c) $(y - p)^2 = 4qx$ (d) $(y - q)^2 = 4px$

Parabola

1. If $y = mx + 4$ is a tangent to both the parabolas, $y^2 = 4x$ and $x^2 = 2by$, then b is equal to

JEE Main 2020

(a) -32 (b) -128 (c) -64 (d) 128

2. If one end of a focal chord AB of the parabola $y^2 = 8x$ is at $A\left(\frac{1}{2}, -2\right)$, then the equation of the tangent to it at B is

JEE Main 2020

(a) $x - 2y + 8 = 0$ (b) $x + 2y + 8 = 0$
(c) $2x + y - 24 = 0$ (d) $2x - y - 24 = 0$

3. If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a, b, c) ?

JEE Main 2019

(a) $\left(\frac{1}{2}, 2, 0\right)$ (b) $(1, 1, 0)$
(c) $(1, 1, 3)$ (d) $\left(\frac{1}{2}, 2, 3\right)$

4. Equation of a common tangent to the parabola $y^2 = 4x$ and the hyperbola $xy = 2$ is

JEE Main 2019

(a) $x + 2y + 4 = 0$ (b) $x - 2y + 4 = 0$
(c) $4x + 2y + 1 = 0$ (d) $x + y + 1 = 0$

5. If the area of the triangle whose one vertex is at the vertex of the parabola, $y^2 + 4(x - a^2) = 0$ and the other two vertices are the points of intersection of the parabola and Y -axis, is 250 sq units, then a value of ' a ' is

JEE Main 2019

(a) $5\sqrt{5}$ (b) 5
(c) $5(2^{1/3})$ (d) $(10)^{2/3}$

6. Suppose a parabola $y = ax^2 + bx + c$ has two x intercepts, one positive and one negative, and its vertex is $(2, -2)$, then which of the following is true?

JEE Main 2019

(a) $ab > 0$ (b) $bc > 0$
(c) $ac > 0$ (d) $a + b + c > 0$

7. Variable straight lines $y = mx + c$ make intercepts on the curve $y^2 - 4ax = 0$ which subtend a right angle at the origin.

Then the point of concurrence of these

lines $y = mx + c$ is

APEAMCET 2019

(a) $(4a, 0)$ (b) $(2a, 0)$
(c) $(-4a, 0)$ (d) $(-2a, 0)$

8. If the normal at one end of the latusrectum of the parabola $y^2 = 16x$ meets the X -axis at the point P , then the length of the chord passing through P and perpendicular to the normal is

TS EAMCET 2019

(a) $48\sqrt{2}$
(b) $32\sqrt{2}$
(c) $24\sqrt{2}$
(d) $20\sqrt{2}$

Ellipse & Hyperbola

- If $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ for some $a \in \mathbb{R}$, then the distance between the foci of the ellipse is **JEE Main 2020**
(a) $2\sqrt{7}$ (b) 4 (c) $2\sqrt{2}$ (d) $2\sqrt{5}$
- Let the line $y = mx$ and the ellipse $2x^2 + y^2 = 1$ intersect at a point P in the first quadrant. If the normal to this ellipse at P meets the co-ordinate axes at $\left(-\frac{1}{3\sqrt{2}}, 0\right)$ and $(0, \beta)$, then β is equal to **JEE Main 2020**
(a) $\frac{2\sqrt{2}}{3}$ (b) $\frac{\sqrt{2}}{3}$ (c) $\frac{2}{\sqrt{3}}$ (d) $\frac{2}{3}$
- If a hyperbola passes through the point $P(10, 16)$ and it has vertices at $(\pm 6, 0)$, then the equation of the normal to it at P is **JEE Main 2020**
(a) $3x + 4y = 94$ (b) $x + 2y = 42$
(c) $2x + 5y = 100$ (d) $x + 3y = 58$
- Let $0 < \theta < \frac{\pi}{2}$. If the eccentricity of the hyperbola $\frac{x^2}{\cos^2 \theta} - \frac{y^2}{\sin^2 \theta} = 1$ is greater than 2, then the length of its latus rectum lies in the interval **JEE Main 2019**
(a) $(1, \frac{3}{2}]$ (b) $(3, \infty)$
(c) $(\frac{3}{2}, 2]$ (d) $(2, 3]$
- Let $S = \left\{ (x, y) \in \mathbb{R}^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1 \right\}$, where $r \neq \pm 1$. Then, S represents **JEE Main 2019**
(a) a hyperbola whose eccentricity is $\frac{2}{\sqrt{1-r}}$, when $0 < r < 1$.
(b) a hyperbola whose eccentricity is $\frac{2}{\sqrt{r+1}}$, when $0 < r < 1$.
(c) an ellipse whose eccentricity is $\sqrt{\frac{2}{r+1}}$, when $r > 1$.
(d) an ellipse whose eccentricity is $\frac{1}{\sqrt{r+1}}$, when $r > 1$.
- Let the length of the latusrectum of an ellipse with its major axis along X -axis and centre at the origin, be 8. If the distance between the foci of this ellipse is equal to the length of its minor axis, then which one of the following points lies on it? **JEE Main 2019**
(a) $(4\sqrt{2}, 2\sqrt{3})$ (b) $(4\sqrt{3}, 2\sqrt{2})$
(c) $(4\sqrt{2}, 2\sqrt{2})$ (d) $(4\sqrt{3}, 2\sqrt{3})$
- If tangents are drawn to the ellipse $x^2 + 2y^2 = 2$ at all points on the ellipse other than its four vertices, then the mid-points of the tangents intercepted between the coordinate axes lie on the curve **JEE Main 2019**
(a) $\frac{x^2}{4} + \frac{y^2}{2} = 1$ (b) $\frac{1}{4x^2} + \frac{1}{2y^2} = 1$
(c) $\frac{x^2}{2} + \frac{y^2}{4} = 1$ (d) $\frac{1}{2x^2} + \frac{1}{4y^2} = 1$
- In an ellipse, with centre at the origin, if the difference of the lengths of major axis and minor axis is 10 and one of the foci is at $(0, 5\sqrt{3})$, then the length of its latusrectum is **JEE Main 2019**
(a) 5 (b) 10
(c) 8 (d) 6
- If the tangent to the parabola $y^2 = x$ at a point (α, β) , ($\beta > 0$) is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to **JEE Main 2019**
(a) $\sqrt{2} + 1$ (b) $\sqrt{2} - 1$
(c) $2\sqrt{2} + 1$ (d) $2\sqrt{2} - 1$
- If a directrix of a hyperbola centred at the origin and passing through the point $(4, -2\sqrt{3})$ is $5x = 4\sqrt{5}$ and its eccentricity is e , then **JEE Main 2019**
(a) $4e^4 - 12e^2 - 27 = 0$
(b) $4e^4 - 24e^2 + 27 = 0$
(c) $4e^4 + 8e^2 - 35 = 0$
(d) $4e^4 - 24e^2 + 35 = 0$
- The tangent and normal to the ellipse $3x^2 + 5y^2 = 32$ at the point $P(2, 2)$ meets the X -axis at Q and R , respectively. Then, the area (in sq units) of the ΔPQR is **JEE Main 2019**
(a) $\frac{16}{3}$ (b) $\frac{14}{3}$ (c) $\frac{34}{15}$ (d) $\frac{68}{15}$
- The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is **BIT SAT 2019**
(a) $(x^2 - y^2)^2 = 6x^2 + 2y^2$
(b) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(c) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
(d) $(x^2 + y^2)^2 = 6x^2 - 2y^2$
- If the line joining the points $A(\alpha)$ and $B(\beta)$ on the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ is a focal chord, then one possible value of $\cot \frac{\alpha}{2} \cdot \cot \frac{\beta}{2}$ is **AP EAMCET 2019**
(a) -3 (b) 3 (c) -9 (d) 9
- The major and minor axes of an ellipse are along the X -axis and Y -axis respectively. If its latusrectum is of length 4 and the distance between the foci is $4\sqrt{2}$, then the equation of that ellipse is **AP EAMCET 2019**
(a) $2x^2 + y^2 = 16$ (b) $x^2 + 2y^2 = 16$
(c) $\frac{x^2}{2} + \frac{y^2}{3} = 1$ (d) $\frac{x^2}{3} + \frac{y^2}{2} = 1$
- If $x + y + n = 0$, $n > 0$ is a normal to the ellipse $x^2 + 3y^2 = 3$ and $x + my + 3 = 0$, $m < 0$ is a tangent to the ellipse $x^2 + 5y^2 = 5$, then the point of intersection of these two lines satisfy the equation **TS EAMCET 2019**
(a) $\frac{x^2}{64} - \frac{y^2}{25} = 1$ (b) $x - 5y + 5 = 0$
(c) $x^2 = \frac{2}{3}y + 1$ (d) $y^2 = -25x + 3$
- The normal drawn at the point $\left(\sqrt{9} \cos \frac{\pi}{4}, \sqrt{7} \sin \frac{\pi}{4}\right)$ to the ellipse $\frac{x^2}{9} + \frac{y^2}{7} = 1$ intersects its major axis at the point **TS EAMCET 2019**
(a) $\left(0, \sqrt{\frac{2}{7}}\right)$ (b) $\left(-\sqrt{\frac{2}{9}}, 0\right)$
(c) $\left(0, -\sqrt{\frac{2}{7}}\right)$ (d) $\left(\sqrt{\frac{2}{9}}, 0\right)$
- The equation of the directrices of the hyperbola $3x^2 - 3y^2 - 18x + 12y + 2 = 0$ is **WB JEE 2019**
(a) $x = 3 \pm \sqrt{\frac{13}{6}}$ (b) $x = 3 \pm \sqrt{\frac{6}{13}}$
(c) $x = 6 \pm \sqrt{\frac{13}{3}}$ (d) $x = 6 \pm \sqrt{\frac{3}{13}}$
- If the lengths of the transverse axis and the latusrectum of a hyperbola are 6 and $\frac{8}{3}$ respectively, then the equation of the hyperbola is **MHT CET 2019**
(a) $4x^2 - 9y^2 = 72$ (b) $4x^2 - 9y^2 = 36$
(c) $9x^2 - 4y^2 = 72$ (d) $9x^2 - 4y^2 = 36$
- S and T are the foci of an ellipse and B is the end point of the minor axis. If STB is equilateral triangle, the eccentricity of the ellipse is **WB JEE 2019**
(a) $\frac{1}{4}$ (b) $\frac{1}{3}$ (c) $\frac{1}{2}$ (d) $\frac{2}{3}$

Introduction to Three Dimensional (3D) Geometry

1. $A(2, 3, 5)$, $B(\alpha, 3, 3)$ and $C(7, 5, \beta)$ are the vertices of a triangle. If the median through A is equally inclined with the co-ordinate axes, then $\cos^{-1}\left(\frac{\alpha}{\beta}\right) =$

AP EAMCET 2019

(a) $\cos^{-1}\left(\frac{-1}{9}\right)$ (b) $\frac{\pi}{2}$
 (c) $\frac{\pi}{3}$ (d) $\cos^{-1}\left(\frac{2}{5}\right)$
2. $A(3, 2, -1)$, $B(4, 1, 1)$, $C(6, 2, 5)$ are three points. If D, E, F are three points which divide BC, CA, AB respectively in the same ratio $2 : 1$, then the centroid of $\triangle DEF$ is

TS EAMCET 2019

(a) $\left(\frac{13}{3}, \frac{5}{3}, \frac{5}{3}\right)$ (b) $(13, 5, 5)$
 (c) $(4, 2, 1)$ (d) $\left(\frac{11}{3}, \frac{4}{3}, \frac{1}{3}\right)$
3. If $(1, 0, 3)$, $(2, 1, 5)$, $(-2, 3, 6)$ are the mid-points of the sides of a triangle, then the centroid of the triangle is

TS EAMCET 2019

(a) $\left(\frac{1}{3}, \frac{4}{3}, -\frac{14}{3}\right)$ (b) $\left(\frac{1}{3}, \frac{4}{3}, \frac{14}{3}\right)$
 (c) $\left(\frac{1}{3}, -\frac{4}{3}, \frac{14}{3}\right)$ (d) $\left(-\frac{1}{3}, \frac{4}{3}, \frac{14}{3}\right)$
4. If $P(6, 10, 10)$, $Q(1, 0, -5)$, $R(6, -10, \lambda)$ are vertices of a triangle right angled at Q , then value of λ is.....

TS EAMCET 2019

(a) 0 (b) 1 (c) 3 (d) 2

Introduction to Limits and Derivatives

1. $\lim_{x \rightarrow 2} \frac{3^x + 3^{3-x} - 12}{3^{-x/2} - 3^{1-x}}$ is equal to

JEE Main 2020
2. Let $y = y(x)$ be a function of x satisfying $y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$ where k is a constant and $y\left(\frac{1}{2}\right) = -\frac{1}{4}$.
 Then $\frac{dy}{dx}$ at $x = \frac{1}{2}$, is equal to

JEE Main 2020

(a) $\frac{\sqrt{5}}{2}$ (b) $-\frac{\sqrt{5}}{2}$
 (c) $\frac{2}{\sqrt{5}}$ (d) $-\frac{\sqrt{5}}{4}$
3. $\lim_{y \rightarrow 0} \frac{\sqrt{1+\sqrt{1+y^4}} - \sqrt{2}}{y^4}$

JEE Main 2019

(a) exists and equals $\frac{1}{4\sqrt{2}}$
 (b) does not exist
 (c) exists and equals $\frac{1}{2\sqrt{2}}$
 (d) exists and equals $\frac{1}{2\sqrt{2}(\sqrt{2}+1)}$
4. $\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1 + \cos x}}$ equals

JEE Main 2019

(a) $4\sqrt{2}$ (b) $\sqrt{2}$
 (c) $2\sqrt{2}$ (d) 4
5. If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is

JEE Main 2019

(a) 12 (b) 9
 (c) 15 (d) 33
6. The value of $\lim_{x \rightarrow \frac{3\pi}{4}} \frac{4\sin^2 x \cos x - \cos x + \sin x}{\sin x + \cos x}$ is

BIT SAT 2019

equal to
 (a) -1 (b) 0
 (c) 1 (d) None of these
7. If $\alpha = \lim_{x \rightarrow 0} \frac{x \cdot 2^x - x}{1 - \cos x}$ and $\beta = \lim_{x \rightarrow 0} \frac{x \cdot 2^x - x}{\sqrt{1+x^2} - \sqrt{1-x^2}}$, then

AP EAMCET 2019

(a) $\alpha = \beta$ (b) $2\alpha = \beta$
 (c) $\alpha = 2\beta$ (d) $\alpha = 3\beta$
8. If $[x]$ is the greatest integer function, then $\lim_{x \rightarrow 2^+} \left(\frac{[x]^3}{3} - \left[\frac{x}{3} \right] \right) =$

TS EAMCET 2019

(a) 0 (b) $\frac{64}{27}$
 (c) $\frac{8}{3}$ (d) $\frac{7}{3}$
9. Derivative of $\log_e (\log x)$ with respect to x is

AP EAMCET 2019

(a) $\frac{2}{x \log x}$ (b) $\frac{1}{x \log x}$
 (c) $\frac{1}{x \log x^2}$ (d) $\frac{2}{\log x}$
10. $\lim_{x \rightarrow 0^+} (e^x + x)^{1/x}$

WB JEE 2019

(a) Does not exist finitely
 (b) is 1
 (c) is e^2
 (d) is 2

Mathematical Reasoning

1. The logical statement $(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$ is equivalent to

JEE Main 2020

(a) $\sim p$ (b) q
 (c) p (d) $\sim q$
 2. Which one of the following is a tautology?

JEE Main 2020

(a) $(P \wedge (P \rightarrow Q)) \rightarrow Q$
 (b) $P \wedge (P \vee Q)$
 (c) $P \vee (P \wedge Q)$
 (d) $Q \rightarrow (P \wedge (P \rightarrow Q))$
 3. The logical statement $[\sim (\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$ is equivalent to

JEE Main 2019

(a) $\sim p \vee r$ (b) $(p \wedge \sim q) \vee r$
 (c) $(p \wedge r) \wedge \sim q$ (d) $(\sim p \wedge \sim q) \wedge r$
 4. Consider the following three statements:
 $P : 5$ is a prime number.
 $Q : 7$ is a factor of 192.
 $R : \text{LCM of } 5 \text{ and } 7 \text{ is } 35.$
- Then, the truth value of which one of the following statements is true ?

JEE Main 2019

(a) $(P \wedge Q) \vee (\sim R)$ (b) $P \vee (\sim Q \wedge R)$
 (c) $(\sim P) \vee (Q \wedge R)$ (d) $(\sim P) \wedge (\sim Q \wedge R)$

5. If q is false and $p \wedge q \iff r$ is true, then which one of the following statements is a tautology?

JEE Main 2019

(a) $p \vee r$ (b) $(p \wedge r) \rightarrow (p \vee r)$
 (c) $(p \vee r) \rightarrow (p \wedge r)$ (d) $p \wedge r$

6. Contrapositive of the statement "If two numbers are not equal, then their squares are not equal" is **JEE Main 2019**
- (a) If the squares of two numbers are not equal, then the numbers are not equal.
 (b) If the squares of two numbers are equal, then the numbers are equal.
 (c) If the squares of two numbers are not equal, then the numbers are equal.
 (d) If the squares of two numbers are equal, then the numbers are not equal.

7. The statement pattern $(p \wedge q) \wedge [\sim r \vee (p \wedge q)] \vee (\sim p \wedge q)$ is equivalent to **MHT SET 2019**
- (a) r (b) q
 (c) $p \wedge q$ (d) p
8. If p and q are true and r and s are false statements, then which of the following is true? **MHT CET 2019**
- (a) $(q \wedge r) \vee (\sim p \wedge s)$
 (b) $(\sim p \rightarrow q) \leftrightarrow (r \wedge s)$

- (c) $(p \rightarrow q) \vee (r \leftrightarrow s)$
 (d) $(p \wedge \sim r) \wedge (\sim q \vee s)$

9. Let $a : \sim (p \wedge \sim r) \vee (\sim q \vee s)$ and $b : (p \vee s) \leftrightarrow (q \wedge r)$. If the truth values of p and q are true and that of r and s are false, then the truth values of a and b are respectively..... **MHT CET 2019**
- (a) F, F (b) T, T
 (c) T, F (d) F, T

Statistics

1. The mean and the standard deviation (s.d.) of 10 observations are 20 and 2 respectively. Each of these 10 observations is multiplied by p and then reduced by q , where $p \neq 0$ and $q \neq 0$. If the new mean and new s.d. become half of their original values, then q is equal to **JEE Main 2020**
- (a) 10 (b) -10 (c) -5 (d) -20
2. Let the observations $x_i (1 \leq i \leq 10)$ satisfy the equations, $\sum_{i=1}^{10} (x_i - 5) = 10$ and $\sum_{i=1}^{10} (x_i - 5)^2 = 40$. If μ and λ are the mean and the variance of the observations, $x_1 - 3, x_2 - 3, \dots, x_{10} - 3$, then the ordered pair (μ, λ) is equal to **JEE Main 2020**
- (a) (6, 3) (b) (3, 6) (c) (3, 3) (d) (6, 6)
3. 5 students of a class have an average height 150 cm and variance 18 cm^2 . A new student, whose height is 156 cm,

- joined them. The variance (in cm^2) of the height of these six students is **JEE Main 2019**
- (a) 16 (b) 22 (c) 20 (d) 18
4. In a group of data, there are n observations, $x, x_2, \dots, x_n$. If $\sum_{i=1}^n (x_i + 1)^2 = 9n$ and $\sum_{i=1}^n (x_i - 1)^2 = 5n$, the standard deviation of the data is **JEE Main 2019**
- (a) 2 (b) $\sqrt{7}$
 (c) 5 (d) $\sqrt{5}$
5. The mean of five observations is 5 and their variance is 9.20. If three of the given five observations are 1, 3 and 8, then a ratio of other two observations is **JEE Main 2019**
- (a) 4 : 9 (b) 6 : 7 (c) 10 : 3 (d) 5 : 8
6. If for some $x \in R$, the frequency distribution of the marks obtained by 20 students in a test is
- Then, the mean of the marks is **JEE Main 2019**

- (a) 3.0 (b) 2.8
 (c) 2.5 (d) 3.2
7. In a data with 15 number of observations $x_1, x_2, x_3, \dots, x_{15}$, $\sum_{i=1}^{15} x_i^2 = 3600$ and $\sum_{i=1}^{15} x_i = 175$. If the value of one observation 20 was found wrong and was replaced by its correct value 40, then the corrected variance of that data is **TS EAMCET 2019**
- (a) 151 (b) 149 (c) 145 (d) 144
8. For a group of 100 observations, the arithmetic mean and standard deviation are 8 and $\sqrt{10.5}$ respectively. The mean and standard deviation of 50 items selected from these 100 observations are 10 and 2 respectively. Then the standard deviation of the remaining 50 observation is **TS EAMCET 2019**
- (a) 2 (b) 3
 (c) 3.5 (d) 4

Fundamentals of Probability

1. Let A and B be two events such that the probability that exactly one of them occurs is $\frac{2}{5}$ and the probability that A or B occurs is $\frac{1}{2}$, then the probability of both of them occur together is **JEE Main 2020**
- (a) 0.10 (b) 0.20 (c) 0.01 (d) 0.02
2. An unbiased coin is tossed. If the outcome is a head, then a pair of unbiased dice is rolled and the sum of the numbers obtained on them is noted. If the toss of the coin results in tail, then a card from a well-shuffled pack of nine cards numbered 1, 2, 3, ..., 9 is randomly picked and the number on the card is noted. The probability that the noted number is either 7 or 8 is **JEE Main 2019**

- (a) $\frac{15}{72}$ (b) $\frac{13}{36}$
 (c) $\frac{19}{72}$ (d) $\frac{19}{36}$
3. If three of the six vertices of a regular hexagon are chosen at random, then the probability that the triangle formed with these chosen vertices is equilateral is **JEE Main 2019**
- (a) $\frac{1}{10}$ (b) $\frac{1}{5}$ (c) $\frac{3}{10}$ (d) $\frac{3}{20}$
4. If two unbiased dice are rolled simultaneously until a sum of the number appeared on these dice is either 7 or 11, then the probability that 7 comes before 11, is **TS EAMCET 2019**
- (a) $\frac{3}{8}$ (b) $\frac{3}{4}$
 (c) $\frac{5}{6}$ (d) $\frac{2}{9}$

5. A bag contains 6 white and 4 black balls. Two balls are drawn at random. The probability that they are of the same colour is **MHT CET 2019**
- (a) $\frac{5}{7}$ (b) $\frac{1}{7}$
 (c) $\frac{7}{15}$ (d) $\frac{1}{15}$
6. A problem in mathematics is given to 4 students whose chances of solving individually are $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ and $\frac{1}{5}$. The probability that the problem will be solved at least by one student is **WB JEE 2019**
- (a) $\frac{2}{3}$ (b) $\frac{3}{5}$
 (c) $\frac{4}{5}$ (d) $\frac{3}{4}$

Solutions with Explanation

Sets, Fundamentals of Relations and Function

1. (b) The sum of series

$$\begin{aligned} & 49^{125} + 49^{124} + \dots + 49^2 + 49 + 1 \\ &= 49^0 + 49^1 + 49^2 + \dots + 49^{124} + 49^{125} \\ &= \frac{1(49^{126} - 1)}{49 - 1} = \frac{(49^{63})^2 - 1^2}{48} \\ &= \frac{(49^{63} - 1)(49^{63} + 1)}{48} \end{aligned}$$

$\therefore 49^{63} + 1$ is a factor of the above sum.

$\therefore$ The greatest positive integer k , for which $49^k + 1$ is a factor of the sum

$$49^{126} + 49^{125} + \dots + 49^2 + 49 + 1 \text{ is } 63.$$

2. (b) It is given that

$2^{1+x} + 2^{1-x}$, $f(x)$ and $3^x + 3^{-x}$ are in A.P.

$$\text{So, } 2f(x) = 2(2^x + 2^{-x}) + 3^x + 3^{-x}$$

$$\Rightarrow f(x) = (2^x + 2^{-x}) + \left(\frac{3^x + 3^{-x}}{2} \right)$$

According to AM-GM,

$$\frac{a^x + a^{-x}}{2} \geq (a^x a^{-x})^{1/2}$$

$$\Rightarrow a^x + a^{-x} \geq 2, \text{ at } x = 0$$

$$\therefore \text{ Minimum value of } f(x) = 2 + \left(\frac{2}{2} \right) = 2 + 1 = 3$$

Hence, option (b) is correct.

3. (d) The given function $f: (1, 3) \rightarrow R$,

$$\text{defined by } f(x) = \frac{x[x]}{1+x^2}$$

$$\therefore f(x) = \begin{cases} \frac{x(1)}{1+x^2}, & x \in (1, 2) \\ \frac{x(2)}{1+x^2}, & x \in [2, 3) \end{cases}$$

is a decreasing function, so

$$\lim_{h \rightarrow 0} f(3-h) \rightarrow \frac{2(3)}{1+9} = \frac{3}{5}$$

$$f(2) = \frac{4}{5}$$

Similarly, $f(x \rightarrow 2^-)$ tends to $\frac{2}{5}$ and

$f(x \rightarrow 1^+)$ tends to $\frac{1}{2}$.

So range of the given function 'f' is

$$\left(\frac{2}{5}, \frac{1}{2} \right) \cup \left(\frac{3}{5}, \frac{4}{5} \right]$$

Hence, option (d) is correct.

4. (a) Given sets

$$A = \{x \in R : |x| < 2\},$$

$$\text{and } B = \{x \in R : |x-2| \geq 3\}$$

$$\text{then, } A = \{x \in R : -2 < x < 2\} \text{ and}$$

$$B = \{x \in R : (x-2) \in (-\infty, -3] \cup [3, \infty)\}$$

$$\{x \in R : x \in (-\infty, -1] \cup [5, \infty)\}$$

$$\therefore B - A = R - (-2, 5)$$

Hence, option (a) is correct.

5. (a) According to given information, we have if

$$k \in \{4, 8, 12, 16, 20\}$$

$$\text{Then, } f(k) \in \{3, 6, 9, 12, 15, 18\}$$

$$[\because \text{Codomain } (f) = \{1, 2, 3, \dots, 20\}]$$

Now, we need to assign the value of $f(k)$ for $k \in \{4, 8, 12, 16, 20\}$ this can be done in ${}^6C_5 \cdot 5!$ ways = $6 \cdot 5! = 6!$ and remaining 15 element can be associated by 15! ways.

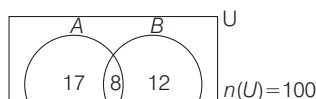
$\therefore$ Total number of onto functions

$$= \frac{15!}{6!} = 15! \cdot 6!$$

6. (d) Let the population of city is 100.

$$\text{Then, } n(A) = 25, n(B) = 20 \text{ and}$$

$$n(A \cap B) = 8$$



$$\text{So, } n(A \cap \bar{B}) = 17 \text{ and } n(\bar{A} \cap B) = 12$$

According to the question, Percentage of the population who look into advertisement is

$$\begin{aligned} &= \left[\frac{30}{100} \times n(A \cap \bar{B}) \right] + \left[\frac{40}{100} \times n(\bar{A} \cap B) \right] \\ &\quad + \left[\frac{50}{100} \times n(A \cap B) \right] \\ &= 5.1 + 4.8 + 4 = 13.9 \end{aligned}$$

7. (c) Let $\frac{x}{x^2 - 5x + 9} = y$

$$\Rightarrow yx^2 - (5y + 1)x + 9y = 0$$

$$\therefore x \in R \Rightarrow D \geq 0$$

$$\Rightarrow (5y + 1)^2 - 36y^2 \geq 0$$

$$\Rightarrow 25y^2 + 10y + 1 - 36y^2 \geq 0$$

$$\Rightarrow 11y^2 - 10y - 1 \leq 0$$

$$\Rightarrow 11y^2 - 11y + y - 1 \leq 0$$

$$\Rightarrow (11y + 1)(y - 1) \leq 0$$

$$\Rightarrow y \in \left[-\frac{1}{11}, 1 \right]$$

Hence, option (c) is correct.

8. (c) We have,

$$f(x) = \sqrt{\frac{x - [x]}{\log(x^2 - x)}}$$

It is defined

$$\log(x^2 - x) > 0$$

$$[\because x - [x] = \{x\} \geq 0, \forall x \in R]$$

$$\Rightarrow x^2 - x > 1 \Rightarrow x^2 - x - 1 > 0$$

$$\therefore x^2 - x - 1 = 0 \Rightarrow x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore \text{ Domain of } f(x) = R - \left[\frac{1 - \sqrt{5}}{2}, \frac{1 + \sqrt{5}}{2} \right]$$

9. (b) We have,

$$f(x) = \sqrt{\frac{4 - x^2}{[x] + 2}} \quad f(x) \text{ is defined if}$$

$$\frac{4 - x^2}{[x] + 2} \geq 0 \Rightarrow \frac{x^2 - 4}{[x] + 2} \leq 0$$

$$\Rightarrow (x + 2)(x - 2)([x] + 2) \leq 0$$

By wavy curve method

$$\begin{array}{c} - \quad + \quad - \quad + \\ -2 \quad -1 \quad 2 \end{array}$$

$$\therefore x \in (-\infty, -2) \cup [-1, 2]$$

10. (b) We have,

$$A = \{x \mid x \in N, x \text{ is a prime number less than } 12\}$$

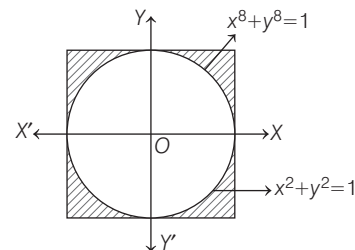
$$= \{2, 3, 5, 7, 11\}$$

$$\text{and } B = \{x \mid x \in N, x \text{ is a factor}$$

$$\text{of } 10\} = \{1, 2, 5, 10\}$$

$$\therefore A \cap B = \{2, 5\}$$

11. (b) We have,



From the graph, it is clear that there is common region which satisfies $x^2 + y^2 = 1$ and $x^8 + y^8 < 1$

$$\therefore P \cap T = P$$

Sequence and Series

- 1 (b) Let five numbers, which are in A.P. is $a - 2d, a - d, a, a + d, a + 2d$.

According to given information,

$$5a = 25 \Rightarrow a = 5$$

$$\text{and } a(a^2 - d^2)(a^2 - 4d^2) = 2520$$

$$\Rightarrow 5(25 - d^2)(25 - 4d^2) = 2520$$

$$\Rightarrow (d^2 - 25)(4d^2 - 25) = 504$$

$$\Rightarrow 4d^4 - 125d^2 + 625 = 504$$

$$\Rightarrow 4d^4 - 125d^2 + 121 = 0$$

$$\Rightarrow 4d^4 - 4d^2 - 121d^2 + 121 = 0$$

$$\Rightarrow 4d^2(d^2 - 1) - 121(d^2 - 1) = 0$$

$$\Rightarrow d^2 = 1, \frac{121}{4} \Rightarrow d = \pm 1, \pm \frac{11}{2}$$

If $d = 1$, then terms are 3, 4, 5, 6, 7, and

if $d = -1$, then terms are 7, 6, 5, 4, 3, and

if $d = \frac{11}{2}$, then terms are $-6, -\frac{1}{2}, 5, \frac{21}{2}, 16$.

When $a = 5$ and $d = \pm \frac{11}{2}$, then one of these

five numbers of A.P. is $-\frac{1}{2}$ and the greatest number amongst them is 16.

- 2 (a) Let first term and common ratio of given G.P. $a_1, a_2, a_3, \dots$ are $a_1 = a < 0$ and ' r ' respectively.

Now, $a_1 + a_2 = 4$ (given)

$$\Rightarrow a + ar = 4$$

$$\Rightarrow a(1 + r) = 4 \dots(i)$$

$$\Rightarrow \frac{a_3 + a_4}{ar^2 + ar^3} = 16 \quad (\text{given})$$

$$\Rightarrow \frac{ar^2 + ar^3}{ar^2 + ar^3} = 16$$

$$\Rightarrow ar^2(1 + r) = 16 \dots(ii)$$

From Eqs. (i) and (ii), we get

$$r^2 = 4 \Rightarrow r = \pm 2 \dots(iii)$$

From Eqs. (i) and (iii), we get

$$a = \frac{4}{3} > 0 \text{ (rejected), if } r = 2$$

and $a = -4 < 0$, if $r = -2$

$$\text{Now, } \sum_{i=1}^9 a_i = a_1 + a_2 + a_3 + \dots + a_9$$

$$= \frac{-4((-2)^9 - 1)}{-2 - 1} = -\frac{4}{3}(512 + 1)$$

$$= -\frac{4}{3}(513) = 4\lambda \quad (\text{given})$$

$$\Rightarrow \lambda = -171$$

- 3 (c) Given product

$$\frac{1}{2^4} \cdot \frac{1}{4^{16}} \cdot \frac{1}{8^{48}} \cdot \frac{1}{16^{128}} \dots \text{to } \infty$$

$$\frac{1}{2^4} \cdot \frac{2}{2^{16}} \cdot \frac{3}{2^{48}} \cdot \frac{4}{2^{128}} \dots \text{to } \infty$$

$$= 2 \left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \text{to } \infty \right)$$

$$\therefore \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \dots \text{to } \infty \text{ is a G.P. of}$$

infinite terms having first term $\frac{1}{4}$ and

common ratio $\frac{1}{2}$. So, the product is

$$2 \left(\frac{1/4}{1 - 1/2} \right) = 2 \left(\frac{1/4}{1/2} \right) = 2^{1/2}$$

- 4 (d) Let r be the common ratio of given GP, then we have the following sequence

$$a_1, a_2 = a_1 r, a_3 = a_1 r^2, \dots, a_{10} = a_1 r^9$$

$$\text{Now, } a_3 = 25 a_1$$

$$\Rightarrow r^2 = 25$$

$$\text{Consider, } \frac{a_9}{a_5} = \frac{a_1 r^8}{a_1 r^4} = r^4 = (25)^2 = 5^4$$

- 5 (b) Given, three distinct numbers a, b and c are in GP.

$$\therefore b^2 = ac \dots(i)$$

and the given quadratic equations

$$ax^2 + 2bx + c = 0 \dots(ii)$$

$$dx^2 + 2ex + f = 0 \dots(iii)$$

For quadratic Eq. (ii),

the discriminant $D = (2b)^2 - 4ac$

$$= 4(b^2 - ac) = 0 \quad [\text{from Eq. (i)}]$$

$\Rightarrow$ Quadratic Eq. (ii) have equal roots, and it is equal to $x = -\frac{b}{a}$, and it is given that

quadratic Eqs. (ii) and (iii) have a common root, so

$$d \left(-\frac{b}{a} \right)^2 + 2e \left(-\frac{b}{a} \right) + f = 0$$

$$\Rightarrow db^2 - 2eba + a^2 f = 0$$

$$\Rightarrow d(ac) - 2eab + a^2 f = 0 \quad [\because b^2 = ac]$$

$$\Rightarrow dc - 2eb + af = 0 \quad [\because a \neq 0]$$

$$\Rightarrow 2 \left(\frac{e}{b} \right) = \frac{d}{a} + \frac{f}{c} \quad [\because b^2 = ac]$$

so, $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

- 6 (c) Given, $f(x + y) = f(x) \cdot f(y)$

Let $f(x) = \lambda^x$ [where $\lambda > 0$]

$$\therefore f(1) = 2 \quad (\text{given})$$

$$\therefore \lambda = 2$$

$$\text{So, } \sum_{k=1}^{10} f(a + k) = \sum_{k=1}^{10} \lambda^{a+k} = \lambda^a \left(\sum_{k=1}^{10} \lambda^k \right)$$

$$= 2^a [2^1 + 2^2 + 2^3 + \dots + 2^{10}]$$

$$= 2^a \left[\frac{2(2^{10} - 1)}{2 - 1} \right]$$

$$\Rightarrow 2^{a+1} (2^{10} - 1) = 16 (2^{10} - 1) \quad (\text{given})$$

$$\Rightarrow 2^{a+1} = 16 = 2^4 \Rightarrow a + 1 = 4 \Rightarrow a = 3$$

- 7 (a) Given, series

$$\frac{3}{5} + \frac{21}{25} + \frac{117}{125} + \dots \text{ upto } n \text{ terms}$$

$$= \left(1 - \frac{2}{5} \right) + \left(1 - \frac{4}{25} \right) + \left(1 - \frac{8}{125} \right) + \dots +$$

upto n terms

$$= n - \left[\frac{2}{5} + \left(\frac{2}{5} \right)^2 + \left(\frac{2}{5} \right)^3 + \dots + \text{ upto } n \text{ terms} \right]$$

$$= n - \frac{2 \left(1 - \left(\frac{2}{5} \right)^n \right)}{1 - \frac{2}{5}}$$

$$= n - \frac{2 \left[1 - \left(\frac{2}{5} \right)^n \right]}{3/5}$$

$$= n + \frac{2^{n+1}}{3 \times 5^n} - \frac{2}{3}$$

Hence, option (a) is correct.

- 8 (d) We have,

$$T_r = r[n - (r - 1)]$$

$$= r(n - r + 1) = nr - r^2 + r$$

$$\therefore S_n = \sum_{r=1}^n T_r = \sum_{r=1}^n (nr - r^2 + r)$$

$$= \sum_{r=1}^n [(n+1)r - r^2]$$

$$= (n+1) \sum_{r=1}^n r - \sum_{r=1}^n r^2$$

$$= \frac{(n+1)n(n+1)}{2} - \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(n+2)}{6}$$

$$\therefore \alpha = \frac{1}{6}$$

- 9 (d) We have, $S_n = \frac{4^n - 3^n}{3^n}$

$$\text{For } n = 1, S_1 = t_1 = \frac{4 - 3}{3} = \frac{1}{3}$$

$$\text{For } n = 2, S_2 = t_1 + t_2 = \frac{4^2 - 3^2}{3^2} = \frac{7}{9}$$

$$\therefore t_2 = \frac{7}{9} - t_1 = \frac{7}{9} - \frac{1}{3} = \frac{7 - 3}{9} = \frac{4}{9}$$

Complex Numbers

1 (d) For,

$$\frac{z-1}{2z+i} = \frac{(x-1)+iy}{2x+(2y+1)i}$$

$$= \frac{[(x-1)+iy][2x-i(2y+1)]}{(2x)^2+(2y+1)^2}$$

(on rationalization)

$$\therefore \operatorname{Re}\left(\frac{z-1}{2z+i}\right) = \frac{2x(x-1)+y(2y+1)}{4x^2+(2y+1)^2}$$

Now, it is given that

$$\operatorname{Re}\left(\frac{z-1}{2z+i}\right) = 1$$

$$\Rightarrow \frac{2x(x-1)+y(2y+1)}{4x^2+(2y+1)^2} = 1$$

$$\Rightarrow 2x^2 - 2x + 2y^2 + y$$

$$= 4x^2 + 4y^2 + 4y + 1$$

$$\Rightarrow 2x^2 + 2y^2 + 2x + 3y + 1 = 0, \text{ is a circle}$$

whose centre is $\left(-\frac{1}{2}, -\frac{3}{4}\right)$ and radius is

$$\sqrt{\frac{1}{4} + \frac{9}{16} - \frac{1}{2}} = \frac{\sqrt{5}}{4}, \text{ so diameter is } \frac{\sqrt{5}}{2}.$$

2 (c) It is given that roots of quadratic equation $x^2 + bx + 45 = 0$, $b \in R$ has

conjugate complex roots, now let roots are $\alpha + i\beta$ and $\alpha - i\beta$.

$$\therefore \text{Sum of roots} = 2\alpha = -b \quad \dots(i)$$

$$\text{and product of roots} = \alpha^2 + \beta^2 = 45 \quad \dots(ii)$$

$$\therefore \text{Roots } \alpha \pm i\beta \text{ satisfy } |z+1| = 2\sqrt{10},$$

$$\text{so } (\alpha+1)^2 + \beta^2 = 40$$

$$\Rightarrow \alpha^2 + 2\alpha + 1 + \beta^2 = 40$$

From relation (ii), on putting the value of $\alpha^2 + \beta^2$, we get

$$45 + 2\alpha + 1 = 40$$

$$\Rightarrow 2\alpha = -6 \Rightarrow \alpha = -3$$

From relation (i), we get

$$-b = 2(-3) \quad [\because \alpha = -3]$$

$$\Rightarrow b = 6$$

$$\therefore b^2 + b = 36 + 6 = 42$$

$$\text{and } b^2 - b = 36 - 6 = 30$$

Hence, option (c) is correct.

3 (b) It is given for complex number z ,

$$\left| \frac{z-i}{z+2i} \right| = 1 \quad \dots(i)$$

$$\text{and } |z| = \frac{5}{2} \quad \dots(ii)$$

From Eq. (i), $|z-i| = |z+2i|$

$\Rightarrow$ Locus of z is a straight line and it is perpendicular bisector of line joining points $(0, -2)$ and $(0, 1)$, so locus of z is $y = -\frac{1}{2}$

where $z = x + iy$.

From Eq. (ii),

$$\sqrt{x^2 + y^2} = \frac{5}{2}$$

$$\Rightarrow x^2 + \left(\frac{-1}{2}\right)^2 = \left(\frac{5}{2}\right)^2$$

$$\Rightarrow x^2 + \frac{1}{4} = \frac{25}{4}$$

$$\Rightarrow x^2 = 6$$

Now, $|z+3i|$

$$= |x + (y+3)i|$$

$$= \sqrt{x^2 + (y+3)^2}$$

$$= \sqrt{6 + \left(-\frac{1}{2} + 3\right)^2}$$

$$(\because x^2 = 6 \text{ and } y = -1/2)$$

$$= \sqrt{6 + \frac{25}{4}} = \sqrt{\frac{24+25}{4}}$$

$$= \sqrt{\frac{49}{4}} = \frac{7}{2}$$

Hence, option (b) is correct.

4 (a) Given, $x^2 + x + 1 = 0$

$$\Rightarrow x = \frac{-1 \pm \sqrt{3}i}{2}$$

$$\Rightarrow z_0 = \omega, \omega^2 \quad [\text{where } \omega = \frac{-1 + \sqrt{3}i}{2} \text{ and}$$

$$\omega^2 = \frac{-1 - \sqrt{3}i}{2} \text{ are the cube roots of unity}$$

$$\text{and } \omega, \omega^2 \neq 1]$$

$$\text{Now consider } z = 3 + 6iz_0^{81} - 3iz_0^{93}$$

$$= 3 + 6i - 3i \quad (\because \omega^{3n} = (\omega^3)^n = 1)$$

$$= 3 + 3i = 3(1+i)$$

If ' θ ' is the argument of z , then

$$\tan \theta = \frac{\operatorname{Im}(z)}{\operatorname{Re}(z)} \quad [\because z \text{ is in the first quadrant}]$$

$$= \frac{3}{3} = 1 \Rightarrow \theta = \frac{\pi}{4}$$

$$5 (a) \text{ We have, } \frac{x+iy}{27} = \left(-2 - \frac{1}{3}i\right)^3$$

$$= \left[\frac{-1}{3}(6+i)\right]^3$$

$$\Rightarrow \frac{x+iy}{27} = -\frac{1}{27}(216 + 108i + 18i^2 + i^3)$$

$$= -\frac{1}{27}(198 + 107i)$$

On equating real and imaginary part, we get

$$x = -198 \text{ and } y = -107$$

$$6 (c) \text{ Given, } z = \frac{\sqrt{3}}{2} + \left(\frac{1}{2}i\right)$$

$$= \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} = e^{i\frac{\pi}{6}}$$

$$\text{so, } (1+iz+z^5+iz^8)^9$$

$$= \left(1 + ie^{i\frac{\pi}{6}} + e^{i\frac{5\pi}{6}} + ie^{i\frac{8\pi}{6}}\right)^9 \left[\because i = e^{i\frac{\pi}{2}}\right]$$

$$= \left(1 + e^{i\frac{2\pi}{3}} + e^{i\frac{5\pi}{6}} + e^{i\frac{11\pi}{6}}\right)^9$$

$$= \left[1 + \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)\right]$$

$$+ \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6}\right) + \left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}\right)]^9$$

$$= \left(1 - \frac{1}{2} + \frac{i\sqrt{3}}{2} - \frac{\sqrt{3}}{2} + \frac{1}{2}i + \frac{\sqrt{3}}{2} - \frac{i}{2}\right)^9$$

$$= \left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)^9 = \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)^9$$

$$= \cos 3\pi + i \sin 3\pi$$

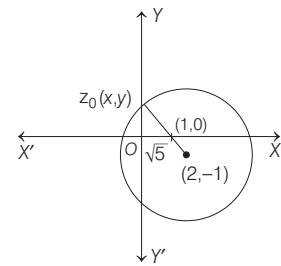
$[\because \text{for any natural number 'n'}$

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)] = -1$$

7 (c) The complex number z satisfying

$$|z-2+i| \geq \sqrt{5}, \text{ which represents the}$$

region outside the circle (including the circumference) having centre $(2, -1)$ and radius $\sqrt{5}$ units.



Now, for $z_0 \in S$ $\frac{1}{|z_0-1|}$ is maximum.

When $|z_0-1|$ is minimum. And for this it is required that $z_0 \in S$, such that z_0 is collinear with the points $(2, -1)$ and $(1, 0)$ and lies on the circumference of the circle $|z-2+i| = \sqrt{5}$.

So let $z_0 = x + iy$, and from the figure $0 < x < 1$ and $y > 0$.

$$\text{So, } \frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i} = \frac{4-x-iy-x+iy}{x+iy-x+iy+2i}$$

$$= \frac{2(2-x)}{2i(y+1)} = -i\left(\frac{2-x}{y+1}\right)$$

$\therefore \frac{2-x}{y+1}$ is a positive real number, so

$\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is purely negative imaginary number.

$$\Rightarrow \arg\left(\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}\right) = -\frac{\pi}{2}$$

8 (a) Let $z = x + iy$
 $\arg z = \tan^{-1}\left(\frac{y}{x}\right)$

$$\arg(z) = \frac{\pi}{6}$$

$$\therefore \tan^{-1} \frac{y}{x} = \frac{\pi}{6}$$

$$\frac{y}{x} = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow x = \sqrt{3} y$$

Also given,

$$|z - 2\sqrt{3}i| = \lambda$$

$$|x + iy - 2\sqrt{3}i| = \lambda$$

$$x^2 + (y - 2\sqrt{3})^2 = \lambda^2$$

This is the equation of circle whose centre is $(0, 2\sqrt{3})$ and $r = \lambda$. Now, $x - \sqrt{3}y = 0$ touches the circle

$$\therefore \lambda = \left| \frac{0 - \sqrt{3}(2\sqrt{3})}{\sqrt{1+3}} \right|$$

$$\lambda = \left| \frac{6}{2} \right| = 3$$

$$\therefore \lambda = 3$$

9 (a) For $z = x + iy$, $x, y \in \mathbb{R}$,
 $(x, y) \neq (0, -4)$

$$\frac{2z-3}{z+4i} = \frac{(2x-3) + 2iy}{x + i(y+4)} \times \frac{x - i(y+4)}{x - i(y+4)}$$

$$= \frac{(2x^2 - 3x + 2y^2 + 8y) + i(12 + 3y - 8x)}{x^2 + (y+4)^2}$$

So, $\arg\left(\frac{2z-3}{z+4i}\right)$
 $= \tan^{-1}\left(\frac{12 + 3y - 8x}{2x^2 - 3x + 2y^2 + 8y}\right) = \frac{\pi}{4}$ (given)

$$\Rightarrow \frac{12 + 3y - 8x}{2x^2 - 3x + 2y^2 + 8y} = 1$$

$$\Rightarrow 2x^2 + 2y^2 + 5x + 5y - 12 = 0$$

Hence, option (a) is correct.

10 (d) We have,

$$\left(\frac{1+iz}{1-iz}\right)^4 = P$$

$$\Rightarrow \left(\frac{i-z}{i+z}\right)^4 = P$$

$$\Rightarrow \left|\frac{z-i}{z+i}\right|^4 = |P| \quad [\because |P| = 1]$$

$$\Rightarrow |z-i|^4 = |z+i|^4$$

$$\Rightarrow |z-i| = |z+i|$$

$\therefore z$ lies on perpendicular bisector of i and $-i$.

$\therefore z$ lies on y -axis.

$\therefore z$ has all complex roots.

11 (b) We have,

$$(\cos \theta + i \sin \theta)(\cos 2\theta + i \sin 2\theta) \dots$$

$$(\cos n\theta + i \sin n\theta) = 1$$

$$\Rightarrow e^{i\theta} \cdot e^{i(2\theta)} \cdot e^{i(3\theta)} \dots e^{i(n\theta)} = 1$$

$$\Rightarrow e^{\frac{in(n+1)\theta}{2}} = 1$$

$$\Rightarrow \cos\left(\frac{n(n+1)\theta}{2}\right) + i \sin\left(\frac{n(n+1)\theta}{2}\right)$$

$$\theta = 1 + 0i$$

$$\Rightarrow \cos\left(\frac{n(n+1)\theta}{2}\right) = 1$$

$$\Rightarrow \frac{n(n+1)\theta}{2} = 2k\pi$$

$$\Rightarrow \theta = \frac{4k}{n(n+1)} \pi$$

Inequalities and Quadratic Equation

1 (b) Given equation

$$(k+1)\tan^2 x - \sqrt{2}\lambda \tan x = 1 - k, \text{ where}$$

$$k \neq -1 \text{ and } \lambda \in \mathbb{R} \text{ having roots } \alpha \text{ and } \beta, \text{ so}$$

$$\tan \alpha + \tan \beta = \frac{\sqrt{2}\lambda}{k+1}$$

$$\text{and } \tan \alpha \cdot \tan \beta = \frac{k-1}{k+1}$$

$$\therefore \tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta} = \frac{\frac{\sqrt{2}\lambda}{k+1}}{1 - \frac{k-1}{k+1}}$$

$$\Rightarrow \tan(\alpha + \beta) = \frac{\sqrt{2}\lambda}{2} = \frac{\lambda}{\sqrt{2}}$$

$$\therefore \tan^2(\alpha + \beta) = 50$$

$$\Rightarrow \lambda^2 = 100$$

$$\Rightarrow \lambda = \pm 10$$

2 (c) Given equation is

$$e^{4x} + e^{3x} - 4e^{2x} + e^x + 1 = 0$$

Let $e^x = y > 0$,

so $y^4 + y^3 - 4y^2 + y + 1 = 0$

$$\Rightarrow \left(y^2 + \frac{1}{y^2}\right) + \left(y + \frac{1}{y}\right) - 4 = 0$$

(on dividing by y^2)

$$\Rightarrow \left(y + \frac{1}{y}\right)^2 + \left(y + \frac{1}{y}\right) - 6 = 0$$

Again, let $y + \frac{1}{y} = t$, then

$$t^2 + t - 6 = 0$$

$$\Rightarrow t^2 + 3t - 2t - 6 = 0$$

$$\Rightarrow t(t+3) - 2(t+3) = 0$$

$$\Rightarrow (t-2)(t+3) = 0$$

$$\Rightarrow t = -3, 2$$

$$\therefore y + \frac{1}{y} = -3 \text{ or } 2$$

$$\Rightarrow e^x + \frac{1}{e^x} = -3 \text{ or } 2$$

$$\Rightarrow e^x + \frac{1}{e^x} = 2$$

$$\left[\because e^x > 0 \Rightarrow e^x + \frac{1}{e^x} \neq -3 \forall x \in \mathbb{R} \right]$$

$$\Rightarrow (e^x - 1)^2 = 0$$

$$\Rightarrow e^x = 1$$

$$\Rightarrow x = 0$$

$\therefore$ Number of real roots of given equation is 1.

Hence, option (c) is correct.

3 (d) It is given that for $a, b \in \mathbb{R}$, $a \neq 0$, the quadratic equation $ax^2 - 2bx + 5 = 0$ has a repeated root α .

So, $\alpha = \frac{b}{a}$ and $D = 0$... (i)

$$\Rightarrow 4b^2 - 20a = 0$$

$$\Rightarrow b^2 = 5a \quad \dots (ii)$$

and $\alpha^2 = \frac{5}{a} \quad \dots (iii)$

and α, β are roots of another quadratic equation $x^2 - 2bx - 10 = 0$.

So, $\alpha + \beta = 2b$, $\alpha\beta = -10$

and $\alpha^2 - 2bx - 10 = 0$

$$\Rightarrow \frac{b^2}{a^2} - 2\frac{b^2}{a} - 10 = 0 \quad \left(\because \alpha = \frac{b}{a}\right)$$

$$\Rightarrow \frac{5a}{a^2} - \frac{2(5a)}{a} - 10 = 0 \quad (\because b^2 = 5a)$$

$$\Rightarrow \frac{5}{a} - 10 - 10 = 0$$

$$\Rightarrow 20a = 5$$

$$\Rightarrow a = \frac{1}{4}$$

So, $b^2 = \frac{5}{4}$

$$\therefore \alpha^2 = \frac{b^2}{a^2} = \frac{\frac{5}{4}}{\frac{1}{16}} = 20$$

Since, $\alpha\beta = -10$

$$\Rightarrow \alpha^2\beta^2 = 100$$

$$\Rightarrow \beta^2 = 5$$

$$\therefore \alpha^2 + \beta^2 = 20 + 5 = 25$$

Hence, option (d) is correct.

- 4 (d) Given quadratic equation is

$$x^2 + (3 - \lambda)x + 2 = \lambda$$

$$x^2 + (3 - \lambda)x + (2 - \lambda) = 0 \quad \dots (i)$$

Let Eq. (i) has roots α and β , then
 $\alpha + \beta = \lambda - 3$ and $\alpha\beta = 2 - \lambda$

[$\because$ For $ax^2 + bx + c = 0$, sum of roots = $-\frac{b}{a}$
 and product of roots = $\frac{c}{a}$]

$$\begin{aligned} \text{Now, } \alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\alpha\beta \\ &= (\lambda - 3)^2 - 2(2 - \lambda) \\ &= \lambda^2 - 6\lambda + 9 - 4 + 2\lambda \\ &= \lambda^2 - 4\lambda + 5 = (\lambda^2 - 4\lambda + 4) + 1 \\ &= (\lambda - 2)^2 + 1 \end{aligned}$$

Clearly, $\alpha^2 + \beta^2$ will be least when $\lambda = 2$.

- 5 (d) Given equation is

$$|\sqrt{x} - 2| + \sqrt{x}(\sqrt{x} - 4) + 2 = 0$$

$$\Rightarrow (|\sqrt{x} - 2|)^2 + |\sqrt{x} - 2| - 2 = 0$$

Let $|\sqrt{x} - 2| = y$, then above equation reduced to

$$\begin{aligned} y^2 + y - 2 &= 0 \\ \Rightarrow y^2 + 2y - y - 2 &= 0 \\ \Rightarrow y(y + 2) - 1(y + 2) &= 0 \\ \Rightarrow (y + 2)(y - 1) &= 0 \\ \Rightarrow y = 1, -2 \\ \therefore y = 1 \quad [\because y = |\sqrt{x} - 2| \geq 0] \\ \Rightarrow |\sqrt{x} - 2| &= 1 \\ \Rightarrow \sqrt{x} = 3 \text{ or } 1 \\ \Rightarrow x = 9 \text{ or } 1 \\ \therefore \text{Sum of roots} &= 9 + 1 = 10 \end{aligned}$$

- 6 (a) Given quadratic equation is

$$x^2 + x \sin \theta - 2 \sin \theta = 0, \theta \in \left(0, \frac{\pi}{2}\right)$$

and its roots are α and β .

So, sum of roots = $\alpha + \beta = -\sin \theta$

and product of roots = $\alpha\beta = -2 \sin \theta$

$$\Rightarrow \alpha\beta = 2(\alpha + \beta) \quad \dots (i)$$

Now, the given expression is

$$\begin{aligned} &\frac{\alpha^{12} + \beta^{12}}{(\alpha^{-12} + \beta^{-12})(\alpha - \beta)^{24}} \\ &= \left[\frac{\alpha\beta}{(\alpha - \beta)^2} \right]^{12} = \left(\frac{\alpha\beta}{(\alpha + \beta)^2 - 4\alpha\beta} \right)^{12} \\ &= \left[\frac{2(\alpha + \beta)}{(\alpha + \beta)^2 - 8(\alpha + \beta)} \right]^{12} \quad [\text{from Eq. (i)}] \\ &= \left(\frac{2}{(\alpha + \beta) - 8} \right)^{12} = \left(\frac{2}{-\sin \theta - 8} \right)^{12} \\ &\quad [\because \alpha + \beta = -\sin \theta] \\ &= \frac{2^{12}}{(\sin \theta + 8)^{12}} \end{aligned}$$

- 7 (b) Given, inequality is

$$2^{\sqrt{\sin^2 x - 2 \sin x + 5}} \cdot \frac{1}{4^{\sin^2 y}} \leq 1$$

$$\Rightarrow 2^{\sqrt{(\sin x - 1)^2 + 4}} \cdot 2^{-2 \sin^2 y} \leq 1$$

$$\Rightarrow \sqrt{(\sin x - 1)^2 + 4} \leq 2 \sin^2 y$$

$\therefore$ Range of $\sqrt{(\sin x - 1)^2 + 4}$ is $[2, 2\sqrt{2}]$

and range of $2 \sin^2 y$ is $[0, 2]$.

$\therefore$ The above inequality holds, if

$$\sqrt{(\sin x - 1)^2 + 4} = 2 = 2 \sin^2 y$$

$$\Rightarrow \sin x = 1 \text{ and } \sin^2 y = 1$$

$$\Rightarrow \sin x = |\sin y| \quad [\text{from the options}]$$

- 8 (c) $x^2 - 2x + A = 0$

$$\Rightarrow x = 1 \pm \sqrt{1 - A}$$

$$\therefore p = 1 - \sqrt{1 - A}, q = 1 + \sqrt{1 - A}$$

$$\text{Similarly, } r = 9 - \sqrt{81 - B}, s = 9 + \sqrt{81 - B}$$

p, q, r, s are in AP.

$$\therefore q - p = s - r$$

$$\Rightarrow 2\sqrt{1 - A} = 2\sqrt{81 - B}$$

$$\Rightarrow B = 80 + A$$

and $q - p = r - q$

$$\Rightarrow 3\sqrt{1 - A} = 8 - \sqrt{81 - B}$$

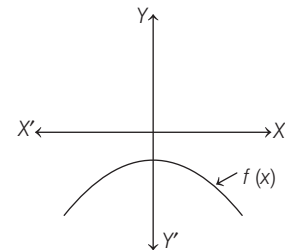
$$\Rightarrow A = -3, B = 77$$

- 9 (d) Let $f(x) = ax^2 + bx + c$

$$\Rightarrow f(1) = a + b + c < 0$$

Again, $f(x)$ has imaginary zeros. So, $a < 0$.

Also, $f(0) = c$. Since $f(x)$ is downward parabola. So, $c < 0$.

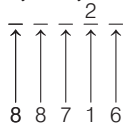


- 10 (a) Given, $\lim_{x \rightarrow 0^+} \frac{x}{p} \left[\frac{q}{x} \right]$

$$= \lim_{x \rightarrow 0^+} \frac{x}{p} \left(\frac{q}{x} - \left(\frac{q}{x} \right) \right) = \frac{[q]}{p} - 0 = \frac{[q]}{p}$$

Permutation and Combination

- 1 (a) It is given that the 10th place of 5 digit number with distinct digits is '2', then the ten thousand place, thousand place, hundred place and unit place we can fill in 8, 8, 7 and 6 ways only.



So, required number of five digit numbers is

$$\begin{aligned} 8 \times 8 \times 7 \times 6 &= 336 \quad k \quad (\text{given}) \\ \Rightarrow k &= \frac{8 \times 8 \times 7 \times 6}{336} \\ \Rightarrow &= \frac{8 \times 8 \times 7}{56} = 8 \end{aligned}$$

Hence, option (a) is correct.

- 2 (c) Clearly, the two digit number which leaves remainder 2 when divided by 7 is of the form $N = 7k + 2$ [by Division Algorithm]

$$\text{For, } k = 2, N = 16$$

$$k = 3, N = 23$$

$$\vdots$$

$$k = 13, N = 93$$

$\therefore$ 12 such numbers are possible and these numbers forms an AP.

$$\text{Now, } S = \frac{12}{2} [16 + 93] = 654$$

$$\left(\because S_n = \frac{n}{2} (a + l) \right)$$

Similarly, the two digit number which leaves remainder 5 when divided by 7 is of the form $N = 7k + 5$

$$\text{For } k = 1, N = 12$$

$$k = 2, N = 19$$

$$\vdots$$

$$k = 13, N = 96$$

$\therefore$ 13 such numbers are possible and these numbers also forms an AP.

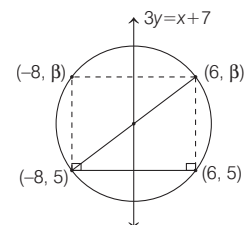
$$\text{Now, } S' = \frac{13}{2} [12 + 96] = 702$$

$$\left(\because S_n = \frac{n}{2} (a + l) \right)$$

Total sum = $S + S' = 654 + 702 = 1356$

- 3 (a) Given digits are 1, 1, 2, 2, 2, 3, 4, 4.

According to the question, odd numbers 1, 1, 3 should occur at even places only.



$\therefore$ The number of ways to arrange odd numbers at even places are ${}^4C_3 \times \frac{3!}{2!}$

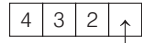
and the number of ways to arrange remaining even numbers are $\frac{6!}{4!2!}$.

So, total number of 9-digit numbers, that can be formed using the given digits are

$${}^4C_3 \times \frac{3!}{2!} \times \frac{6!}{4!2!} = 4 \times 3 \times 15 = 180$$

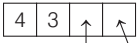
- 4 (b) Following are the cases in which the 4-digit numbers strictly greater than 4321 can be formed using digits 0, 1, 2, 3, 4, 5 (repetition of digits is allowed)

Case-I



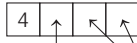
2/3/4/5 $\rightarrow$ 4 numbers

Case-II



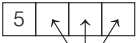
3/4/5 0/1/2/3/4/5 $3 \times 6 = 18$ numbers
3 ways 6 ways

Case-III



4/5 0/1/2/3/4/5 $2 \times 6 \times 6 = 72$ numbers
2 ways 6 ways

Case-IV



0/1/2/3/4/5 $6 \times 6 \times 6 = 216$ numbers
6 ways

So, required total numbers

$$= 4 + 18 + 72 + 216 = 310$$

- 5 (c) It is given that, there are 20 pillars of the same height have been erected along the boundary of a circular stadium.

Now, the top of each pillar has been connected by beams with the top of all its non-adjacent pillars, then total number of beams = number of diagonals of 20-sided polygon.

$\therefore {}^{20}C_2$ is selection of any two vertices of 20-sided polygon which included the sides as well.

So, required number of total beams = ${}^{20}C_2 - 20$

$[\therefore \text{the number of diagonals in a } n\text{-sided closed polygon} = {}^nC_2 - n]$

$$= \frac{20 \times 19}{2} - 20 = 190 - 20 = 170$$

- 6 (c) Since only one letters can not be in wrong envelope so that at least two letters are in wrong envelope means all the letters are not in right envelopes.

$$\therefore x = 6! - 1 = 720 - 1 = 719$$

Y = number of ways, so that all the letters are in wrong envelopes

$$= 6! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} \right)$$

$$= 360 - 120 + 30 - 6 + 1 = 265$$

$$\therefore x - y = 719 - 265 = 454$$

- 7 (d) If number is divisible by 6, then it should be an even number and divisible by 3 also, so sum of the digits should be divisible by 3. Now last digit can be fill by 0 or 2 or 4. First place can be filled by 5 choices (except 0) second and third can be filled by 6 choices.

Now fourth place is filled such that it must be of form $3k$ or $3k + 1$ or $3k + 2$. So it can be fill by 2 choices (0 or 3) as whole number must be divisible by 3 so, total numbers

$$= 5 \times 6 \times 6 \times 2 \times 3 = 1080$$

Hence, option (d) is correct.

- 8 (c) The permutation of 4 letters such that vowel and consonants are equal

$${}^{21}C_1 \times {}^5C_1 \times \frac{4!}{2!2!} + {}^{21}C_2 \times {}^5C_2 \times 4!$$

$$= 51030 = 729 \times 70 = 3^6 \times 70$$

- 9 (c) Case I

4 questions from first 5 questions and 6 questions from remaining 8 questions

$$\therefore {}^5C_4 \times {}^8C_6 = 5 \times 28 = 140$$

Case II

Choose 5 questions from first 5 questions and next 5 questions from remaining 8 questions

$$\therefore {}^5C_5 \times {}^8C_5 = 1 \times 56 = 56$$

$$\text{Total number} = 140 + 56 = 196$$

- 10 (b) Required number of ways

$$= {}^7C_4 \times D(3)$$

$$= {}^7C_3 \times \left[3! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} \right) \right]$$

$$\left[\because {}^nC_r = {}^nC_{n-r}, \text{ and } D(n) = n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots \right) \right]$$

$$= {}^7C_3 \times 2 = 2({}^7C_3)$$

Mathematical Induction

- 1 (a) Given statement is " $P(n) : n^2 - n + 41$ is prime".

$$\text{Clearly } P(3) : 3^2 - 3 + 41 = 9 - 3 + 41$$

$$= 47 \text{ which is a prime number.}$$

$$\text{and } P(5) : 5^2 - 5 + 41 = 25 - 5 + 41 = 61,$$

which is also a prime number.

$\therefore$ Both $P(3)$ and $P(5)$ are true.

- 2 (c) Since, $15 = 3 \times 5$, so to find the exponent of 15 in $47!$ It is enough to find the exponent of 5 in $47!$.

So, exponent of 5 in $47!$.

$$= \left[\frac{47}{5} \right] + \left[\frac{47}{5^2} \right] + \dots,$$

{where, $[x]$ is greatest integer of x }.

$$= 9 + 1$$

So, required value of k is 10.

Hence, option (c) is correct.

- 3 (a) Given,

$$1^2 + 2^2 + 3^2 + \dots + n^2 > x$$

$$\Rightarrow \frac{n(n+1)(2n+1)}{6} > x$$

$$= a_0 + a_1 + a_2 + \dots + a_{4n}$$

$$\Rightarrow (2n+1) = a_0 + a_1 + a_2 + \dots + a_{4n} \dots (i)$$

Similarly, on putting $x = -1$, we get

$$(1 - 1 + 1 - 1 + \dots + 1)(1 + 1 + 1 + \dots + 1)$$

$$(2n+1) \text{ terms} \quad (2n+1) \text{ terms}$$

$$= a_0 - a_1 + a_2 - a_3 + \dots + a_{4n}$$

$$\Rightarrow (2n+1)$$

$$\therefore \frac{n(n+1)(2n+1)}{6} > \frac{n(n)(2n)}{6}$$

$$\therefore x = \frac{2n^3}{6} \Rightarrow x = \frac{n^3}{3}$$

- 4 (d) We have,

$$7^{2n} + 16n - 1$$

Put $n = 1$, we get

$$7^2 + 16 - 1 = 49 + 15 = 64$$

which is divisible by 64.

$$= a_0 - a_1 + a_2 - a_3 \dots + a_{4n} \dots (ii)$$

On adding Eqs. (i) and (ii), we get

$$2(a_0 + a_2 + a_4 + \dots + a_{4n}) = 2(2n+1)$$

$$\Rightarrow a_0 + a_2 + a_4 + \dots + a_{4n} = 2n+1$$

$\Rightarrow$ Sum of coefficients of all even powers of x in the product $(1 + x + x^2 + \dots + x^{2n})$

Binomial Theorem

- 1 (30) Let

$$(1 + x + x^2 + \dots + x^{2n})(1 - x + x^2 - x^3 + \dots + x^{2n})$$

$$= a_0 + a_1x + a_2x^2 + \dots + a_{4n}x^{4n}$$

On putting $x = 1$, we get

$$(1 + 1 + 1 + \dots + 1)(1 - 1 + 1 - 1 + \dots + 1)$$

$$(2n+1) \text{ terms} \quad (2n+1) \text{ terms}$$

$$\begin{aligned}
 &= 2n + 1 = 61 \\
 \Rightarrow &2n = 60 \\
 \Rightarrow &n = 30 \\
 \text{2 (d)} &2^{403} = 8 \cdot 2^{400} = 8(16)^{100} \\
 &\text{Note that, when 16 is divided by 15, gives remainder 1.} \\
 \therefore &\text{When } (16)^{100} \text{ is divided by 15, gives remainder } 1^{100} = 1 \\
 &\text{and when } 8(16)^{100} \text{ is divided by 15, gives remainder 8.} \\
 \therefore &\left\{ \frac{2^{403}}{15} \right\} = \frac{8}{15} \\
 &(\text{where } \{ \} \text{ is the fractional part function}) \\
 \Rightarrow &k = 8 \\
 \text{3 (b)} &\text{The } (r+1)\text{th term in the expansion of } (a+x)^n \text{ is given by } T_{r+1} = {}^nC_r a^{n-r} x^r \\
 \therefore &3^{\text{rd}} \text{ term in the expansion of } (1+x^{\log_2 x})^5 \text{ is} \\
 &{}^5C_2 (1)^{5-2} (x^{\log_2 x})^2 \\
 \Rightarrow &{}^5C_2 (1)^{5-2} (x^{\log_2 x})^2 = 2560 \text{ (given)} \\
 \Rightarrow &x^{(2 \log_2 x)} = 256 \\
 \Rightarrow &2(\log_2 x)(\log_2 x) = 8 \\
 &(\because \log_2 256 = \log_2 2^8 = 8) \\
 &(\log_2 x)^2 = 4 \\
 \Rightarrow &\log_2 x = \pm 2 \\
 \Rightarrow &x = 4 \text{ or } x = 2^{-2} = \frac{1}{4}
 \end{aligned}$$

$$\begin{aligned}
 \text{4 (d)} &\text{The general term in the expansion of binomial expression } (a+b)^n \text{ is} \\
 &T_{r+1} = {}^nC_r a^{n-r} b^r, \text{ so} \\
 &\text{the general term in the expansion of binomial expression } x^2 \left(\sqrt{x} + \frac{\lambda}{x^2} \right)^{10} \text{ is} \\
 &T_{r+1} = x^2 \left({}^{10}C_r (\sqrt{x})^{10-r} \left(\frac{\lambda}{x^2} \right)^r \right) \\
 &= {}^{10}C_r \lambda^r x^{2 + \frac{10-r}{2} - 2r} \\
 &\text{Now, for the coefficient of } x^2, \\
 &\text{put } 2 + \frac{10-r}{2} - 2r = 2 \\
 \Rightarrow &10 - r = 4r \Rightarrow r = 2 \\
 &\text{So, the coefficient of } x^2 \text{ is } {}^{10}C_2 \lambda^2 = 720 \\
 &\text{[given]} \\
 \Rightarrow &\frac{10!}{2!8!} \lambda^2 = 720 \\
 \Rightarrow &\lambda = \pm 4 \\
 \text{5 (b)} &\text{Given series is} \\
 &2 \cdot {}^{20}C_0 + 5 \cdot {}^{20}C_1 + 8 \cdot {}^{20}C_2 + \dots + 62 \cdot {}^{20}C_{20} \\
 &= \sum_{r=0}^{20} (3r+2) \cdot {}^{20}C_r \\
 &[\because \text{General term of the sequence 2, 5, 8, ... which forms an AP, is } 2 + (n-1)3 = 3n-1, \text{ where } n=1, 2, 3, \dots \text{ and it can be written as } 3n+2, \text{ where } n=0, 1, 2, 3] \\
 &= 3 \cdot \sum_{r=0}^{20} r \cdot {}^{20}C_r + 2 \sum_{r=0}^{20} {}^{20}C_r
 \end{aligned}$$

$$\begin{aligned}
 &= 3 \sum_{r=1}^{20} r \cdot \left(\frac{20}{r} \right) {}^{19}C_{r-1} + 2 \sum_{r=0}^{20} {}^{20}C_r \\
 &\quad \left[\because {}^nC_r = \frac{n}{r} {}^{n-1}C_{r-1} \right] \\
 &= 3 \times 20 \sum_{r=1}^{20} {}^{19}C_{r-1} + 2 \sum_{r=0}^{20} {}^{20}C_r \\
 &= 60 \sum_{r=0}^{19} {}^{19}C_r + 2 \sum_{r=0}^{20} {}^{20}C_r \\
 &\quad \left[\because \sum_{r=1}^{20} {}^{19}C_{r-1} = \sum_{r=0}^{19} {}^{19}C_r \right] \\
 &= (60 \times 2^{19}) + (2 \times 2^{20}) \left[\because \sum_{r=0}^n {}^nC_r = 2^n \right] \\
 &= (15 \times 2^{21}) + 2^{21} = 16 \times 2^{21} = 2^{25}
 \end{aligned}$$

$$\begin{aligned}
 \text{6 (d)} &\text{We have,} \\
 &(1+ax+bx^2)(1-2x)^{18} \\
 &(1+ax+bx^2)(1-{}^{18}C_1 2x + {}^{18}C_2 (2x^2) \\
 &\quad {}^{18}C_3 (2x)^3 + {}^{18}C_4 (2x)^4 + \dots) \\
 &\text{Coefficient of } x^3 \text{ is} \\
 &-{}^{18}C_3 (2)^3 + a \cdot {}^{18}C_2 (2)^2 - b {}^{18}C_1 (2) \\
 &\text{and coefficient of } x^4 \text{ is} \\
 &{}^{18}C_4 (2)^4 - {}^{18}C_3 (2)^3 a + {}^{18}C_2 (2)^2 b \\
 &\text{Coefficient of } x^3 \text{ and } x^4 \text{ are zero.} \\
 \therefore &{}^{18}C_3 (2)^3 + {}^{18}C_2 (2)^2 (a) - {}^{18}C_1 (2)b = 0 \\
 &\text{and } 80 - \frac{32}{3}a + b = 0 \quad \dots \text{(ii)} \\
 &\text{Solving Eqs. (i) and (ii), we get} \\
 &a = 16, b = \frac{272}{3}
 \end{aligned}$$

Trigonometric Functions and Equations

$$\begin{aligned}
 \text{1. (b)} &\text{Given trigonometric expression} \\
 &\cos^3 \left(\frac{\pi}{8} \right) \cos \left(\frac{3\pi}{8} \right) + \sin^3 \left(\frac{\pi}{8} \right) \sin \frac{3\pi}{8} \\
 &\cos^3 \left(\frac{\pi}{8} \right) \cos \left(\frac{\pi}{2} - \frac{\pi}{8} \right) + \sin^3 \left(\frac{\pi}{8} \right) \sin \left(\frac{\pi}{2} - \frac{\pi}{8} \right) \\
 &\quad \left[\because \frac{\pi}{2} - \frac{\pi}{8} = \frac{3\pi}{8} \right] \\
 &= \cos^3 \left(\frac{\pi}{8} \right) \sin \left(\frac{\pi}{8} \right) + \sin^3 \left(\frac{\pi}{8} \right) \cos \left(\frac{\pi}{8} \right) \\
 &= \sin \left(\frac{\pi}{8} \right) \cos \left(\frac{\pi}{8} \right) \\
 &\quad \left[\cos^2 \left(\frac{\pi}{8} \right) + \sin^2 \left(\frac{\pi}{8} \right) \right] \\
 &= \frac{1}{2} \times 2 \sin \left(\frac{\pi}{8} \right) \cos \left(\frac{\pi}{8} \right) \\
 &= \frac{1}{2} \sin \frac{\pi}{4} = \frac{1}{2\sqrt{2}}
 \end{aligned}$$

Hence, option (b) is correct.

2. (b) It is given that

$$x = \sum_{n=0}^{\infty} (-1)^n \tan^{2n} \theta$$

$$\begin{aligned}
 &= 1 - \tan^2 \theta + \tan^4 \theta - \tan^6 \theta + \dots \text{ upto } \infty \\
 &= \frac{1}{1 + \tan^2 \theta} \quad \{ \because \theta \in \left(0, \frac{\pi}{4} \right) \} \\
 \Rightarrow &\tan^2 \theta \in (0, 1) \} \\
 &x = \cos^2 \theta \quad \dots \text{(i)} \\
 &\text{and } y = \sum_{n=0}^{\infty} \cos^{2n} \theta \\
 &= 1 + \cos^2 \theta + \cos^4 \theta + \cos^6 \theta + \dots \\
 &\quad \text{+ upto } \infty \\
 &= \frac{1}{1 - \cos^2 \theta} \quad \{ \because \theta \in \left(0, \frac{\pi}{4} \right) \} \\
 \Rightarrow &\cos^2 \theta \in \left(0, \frac{1}{2} \right) \} \\
 &= \frac{1}{\sin^2 \theta} \Rightarrow \sin^2 \theta = \frac{1}{y} \quad \dots \text{(ii)}
 \end{aligned}$$

On adding Eqs. (i) and (ii), we get

$$1 = x + \frac{1}{y} \Rightarrow y(1-x) = 1$$

Hence, option (b) is correct.

$$\begin{aligned}
 \text{3. (d)} &\text{Given expression} \\
 &= 3(\sin \theta - \cos \theta)^4 + 6(\sin \theta + \cos \theta)^2 + 4 \sin^6 \theta \\
 &= 3((\sin \theta - \cos \theta)^2)^2 \\
 &\quad + 6(\sin \theta + \cos \theta)^2 + 4(\sin^2 \theta)^3 \\
 &= 3(1 - \sin 2\theta)^2 + 6(1 + \sin 2\theta) \\
 &\quad + 4(1 - \cos^2 \theta)^3 \\
 &= 3(1^2 + \sin^2 2\theta - 2 \sin 2\theta) \\
 &\quad + 6(1 + \sin 2\theta) + 4(1 - \cos^6 \theta \\
 &\quad - 3 \cos^2 \theta + 3 \cos^4 \theta) \\
 &= 3 + 3 \sin^2 2\theta - 6 \sin 2\theta + 6 \\
 &\quad + 6 \sin 2\theta + 4 - 4 \cos^6 \theta - 12 \cos^2 \theta \\
 &\quad + 12 \cos^4 \theta \\
 &= 13 + 3 \sin^2 2\theta - 4 \cos^6 \theta \\
 &\quad - 12 \cos^2 \theta + 12 \cos^4 \theta \\
 &= 13 + 3(2 \sin \theta \cos \theta)^2 - 4 \cos^6 \theta \\
 &\quad - 12 \cos^2 \theta (1 - \cos^2 \theta) \\
 &= 13 - 4 \cos^6 \theta
 \end{aligned}$$

4. (c) Given, $\sin^2 2\theta + \cos^4 2\theta = \frac{3}{4}$

$$\Rightarrow (1 - \cos^2 2\theta) + \cos^4 2\theta = \frac{3}{4}$$

$$(\because \sin^2 x = 1 - \cos^2 x)$$

$$\Rightarrow 4\cos^4 2\theta - 4\cos^2 2\theta + 1 = 0$$

$$\Rightarrow (2\cos^2 2\theta - 1)^2 = 0$$

$$\Rightarrow \cos 2\theta = \pm \frac{1}{\sqrt{2}}$$

If $\theta \in \left(0, \frac{\pi}{2}\right)$, then $2\theta \in (0, \pi)$

$$\therefore \cos 2\theta = \pm \frac{1}{\sqrt{2}}$$

$$\Rightarrow 2\theta = \frac{\pi}{4}, \frac{3\pi}{4} \Rightarrow \theta = \frac{\pi}{8}, \frac{3\pi}{8}$$

Sum of values of $\theta = \frac{\pi}{8} + \frac{3\pi}{8} = \frac{\pi}{2}$

5. (b) Given,

$$x^2 \sin \theta - x \sin \theta \cos \theta - x + \cos \theta = 0,$$

where $0 < \theta < 45^\circ$

$$\Rightarrow x \sin \theta (x - \cos \theta) - 1(x - \cos \theta) = 0$$

$$\Rightarrow (x - \cos \theta)(x \sin \theta - 1) = 0$$

$$\Rightarrow \alpha = \cos \theta \text{ and } \beta = \operatorname{cosec} \theta$$

$$\sqrt{2} < \operatorname{cosec} \theta < \infty \Rightarrow \cos \theta < \cos \theta$$

Now, consider,

$$\sum_{n=0}^{\infty} \left(\alpha^n + \frac{(-1)^n}{\beta^n} \right) = \sum_{n=0}^{\infty} \alpha^n + \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta^n}$$

$$= \frac{1}{1-\alpha} + \frac{1}{1-\left(-\frac{1}{\beta}\right)} = \frac{1}{1-\alpha} + \frac{1}{1+\frac{1}{\beta}}$$

$$= \frac{1}{1-\cos \theta} + \frac{1}{1+\sin \theta} \quad \left\{ \because \frac{1}{\beta} = \sin \theta \right\}$$

6. (b) Given equation is

$$1 + \sin^4 x = \cos^2 (3x)$$

Since, range of $(1 + \sin^4 x) = [1, 2]$

and range of $\cos^2 (3x) = [0, 1]$

So, the given equation holds if

$$1 + \sin^4 x = 1 = \cos^2 (3x)$$

$$\Rightarrow \sin^4 x = 0 \text{ and } \cos^2 3x = 1$$

Since, $x \in \left[-\frac{5\pi}{2}, \frac{5\pi}{2}\right]$

$$\therefore x = -2\pi, -\pi, 0, \pi, 2\pi.$$

Thus, there are five different values of x is possible.

7. (a) Given, $|\cos x| = 2[x]$

$$0 \leq |\cos x| \leq 1$$

$$\therefore 0 \leq 2[x] \leq 1$$

$$0 \leq [x] \leq \frac{1}{2}$$

$$x = 0$$

$$\therefore |\cos x| = 0$$

$$x : n\pi + \frac{\pi}{2}$$

For any value of $n \in \mathbb{N}$, $[n] \neq 0$

Hence, the equation has no solution.

8. (c) $0 \leq \alpha, \beta \leq \frac{\pi}{4}; \alpha + \beta \leq \frac{\pi}{2}$

$$\cos(\alpha + \beta) = 4/5 \Rightarrow \tan(\alpha + \beta) = 3/4$$

$$\sin(\alpha - \beta) = 5/13 \Rightarrow \alpha - \beta \in \theta_1$$

and $\tan(\alpha - \beta) = 5/12$

$$\tan 2\alpha = \tan(\alpha + \beta + \alpha - \beta)$$

$$= \frac{\tan(\alpha + \beta) + \tan(\alpha - \beta)}{1 - \tan(\alpha + \beta) \tan(\alpha - \beta)}$$

$$= \frac{3/4 + 5/12}{1 - 3/4 \times \frac{5}{12}} = \frac{56}{33}$$

9. (a) $\cos^2 5^\circ - \cos^2 15^\circ - \sin^2 15^\circ + \sin^2 35^\circ$

$$+ \cos 15^\circ \sin 15^\circ - \cos 5^\circ \sin 35^\circ$$

$$= \cos 5^\circ (\cos 5^\circ - \sin 35^\circ)$$

$$- \cos 15^\circ (\cos 15^\circ - \sin 15^\circ)$$

$$+ \sin^2 35^\circ - \sin^2 15^\circ$$

$$= \cos 5^\circ (2\sin 30^\circ \sin 25^\circ)$$

$$- \cos 15^\circ (2\sin 45^\circ \sin 30^\circ)$$

$$+ \sin 50^\circ \sin 20^\circ$$

$$= -\frac{\cos 15^\circ}{\sqrt{2}} + \cos 5^\circ \sin 25^\circ$$

$$+ \sin 50^\circ \sin 20^\circ$$

$$= -\frac{\cos 15^\circ}{\sqrt{2}} + \frac{1}{2} [2 \cos 45^\circ \cos 15^\circ]$$

$$= -\frac{\cos 15^\circ}{\sqrt{2}} + \frac{\cos 15^\circ}{\sqrt{2}} = 0$$

10. (d) We have,

$$3\cos^2 A + 2\cos^2 B = 4$$

$$\Rightarrow 3(1 - \sin^2 A) + 2(1 - \sin^2 B) = 4$$

$$\Rightarrow 5 - 3\sin^2 A - 2\sin^2 B = 4$$

$$\Rightarrow 3\sin^2 A + 2\sin^2 B = 1$$

$$\Rightarrow 3\sin^2 A = \cos 2B$$

Again,

$$\frac{3\sin A}{\sin B} = \frac{2\cos B}{\cos A}$$

$$\Rightarrow 3\sin A \cos A = 2\sin B \cos B$$

$$\Rightarrow 3\sin 2A = 2\sin 2B$$

Now, $\cos(A + 2B)$

$$= \cos A \cos 2B - \sin A \sin 2B$$

$$= 3\cos A \sin^2 A - 3\sin^2 A \cos A = 0$$

$$\therefore A + 2B = 90^\circ$$

11. (c) We have, In $\triangle ABC$

$$\left(\tan \frac{A}{2} \right) \left(\tan \frac{B}{2} \right) = \frac{3}{4}$$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} \sqrt{\frac{(s-a)(s-c)}{s(s-b)}} = \frac{3}{4}$$

$$\Rightarrow \sqrt{\frac{(s-b)(s-c)(s-a)(s-c)}{s(s-a) \cdot s(s-b)}} = \frac{3}{4}$$

$$\Rightarrow \frac{(s-c)}{s} = \frac{3}{4} \Rightarrow \frac{\frac{a+b+c}{2} - c}{\frac{a+b+c}{2}} = \frac{3}{4}$$

$$\Rightarrow \frac{a+b-c}{a+b+c} = \frac{3}{4}$$

$$\Rightarrow 4a + 4b - 4c = 3a + 3b + 3c$$

$$\Rightarrow a + b = 7c$$

Properties of Triangles, Heights and Distances, Cartesian System of Rectangular Coordinates

1 (05) Given vertices of triangle ABC are $A(1, 0)$, $B(6, 2)$ and $C\left(\frac{3}{2}, 6\right)$ and the centroid of triangle G divides the triangle such that area of triangles AGB , BGC and CGA are equal, so point P is the centroid of triangle G .

$\therefore$ Coordinate of

$$P\left(\frac{1+6+\frac{3}{2}}{3}, \frac{0+2+6}{3}\right) = P\left(\frac{17}{6}, \frac{8}{3}\right)$$

and given point $Q\left(-\frac{7}{6}, -\frac{1}{3}\right)$

So length of the line segment PQ

$$= \sqrt{\left(\frac{17}{6} + \frac{7}{6}\right)^2 + \left(\frac{8}{3} + \frac{1}{3}\right)^2}$$

$$= \sqrt{\left(\frac{24}{6}\right)^2 + \left(\frac{9}{3}\right)^2}$$

$$= \sqrt{4^2 + 3^2} = \sqrt{16+9}$$

$$= \sqrt{25} = 5$$

2 (b) The centroid of the triangle having vertices $(3, -1)$, $(1, 3)$ and $(2, 4)$ is

$$C\left(\frac{3+1+2}{3}, \frac{-1+3+4}{3}\right) = (2, 2)$$

And equation of line passes through point of intersection P of lines

$$x + 3y - 1 = 0 \text{ and } 3x - y + 1 = 0 \text{ is}$$

$$(x + 3y - 1) + \lambda(3x - y + 1) = 0 \quad \dots (i)$$

$\therefore$ Line (i) passes through point $C(2, 2)$, so

$$(2 + 6 - 1) + \lambda(6 - 2 + 1) = 0$$

$$\Rightarrow 7 + 5\lambda = 0 \Rightarrow \lambda = -\frac{7}{5}$$

∴ Equation of line passes through points C and P is

$$(x + 3y - 1) - \frac{7}{5}(3x - y + 1) = 0$$

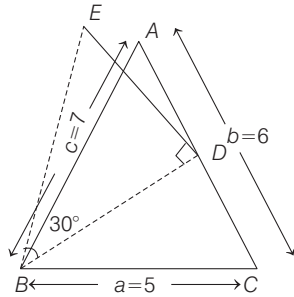
$$\Rightarrow 5x + 15y - 5 - 21x + 7y - 7 = 0$$

$$\Rightarrow 16x - 22y + 12 = 0$$

$$\Rightarrow 8x - 11y + 6 = 0 \quad \dots (ii)$$

From the options, point $(-9, -6)$ satisfy the line.
Hence, option (b) is correct.

3 (a) According to given information, we have the following figure.



Clearly, length of $BD = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2}$,

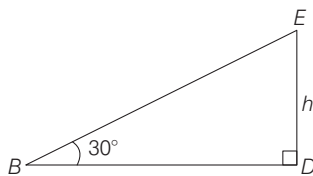
(using Apollonius theorem)

where, $c = AB = 7$, $a = BC = 5$
and $b = CA = 6$

$$\therefore BD = \frac{1}{2}\sqrt{2 \times 25 + 2 \times 49 - 36}$$

$$= \frac{1}{2}\sqrt{112} = \frac{1}{2}4\sqrt{7} = 2\sqrt{7}$$

Now, let $ED = h$ be the height of the lamp post.



Then, in $\triangle BDE$, $\tan 30^\circ = \frac{h}{BD}$

$$\Rightarrow \frac{1}{\sqrt{3}} = \frac{h}{2\sqrt{7}}$$

$$\Rightarrow h = \frac{2\sqrt{7}}{\sqrt{3}} = \frac{2}{3}\sqrt{21}$$

4 (a) Let the coordinates of point P be (x_1, y_1)

∵ P lies on the line $2x - 3y + 4 = 0$

$$\therefore 2x_1 - 3y_1 + 4 = 0$$

$$\Rightarrow y_1 = \frac{2x_1 + 4}{3} \quad \dots (i)$$

Now, let the centroid of $\triangle PQR$ be $G(h, k)$, then

$$h = \frac{x_1 + 1 + 3}{3}$$

$$\Rightarrow x_1 = 3h - 4 \quad \dots (ii)$$

and $k = \frac{y_1 + 4 - 2}{3}$

$$\Rightarrow k = \frac{2x_1 + 4}{3} + 2 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow 3k = \frac{2x_1 + 4}{3} + 6 \quad \dots (iii)$$

Now, from Eqs. (ii) and (iii), we get

$$2(3h - 4) = 9k - 10$$

$$\Rightarrow 6h - 8 = 9k - 10$$

$$\Rightarrow 6h - 9k + 2 = 0$$

Now, replace h by x and k by y .

$\Rightarrow 6x - 9y + 2 = 0$, which is the required locus and slope of this line is $\frac{2}{3}$

$$\left[\because \text{slope of } ax + by + c = 0 \text{ is } -\frac{a}{b} \right]$$

5 (d) Given equation of line is

$$3x + 4y - 24 = 0$$

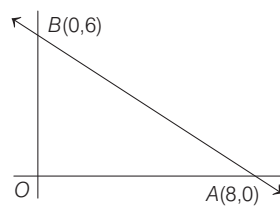
For intersection with X-axis put $y = 0$

$$\Rightarrow x = 8$$

For intersection with Y-axis, put $x = 0$

$$\Rightarrow 4y - 24 = 0 \Rightarrow y = 6$$

∴ $A(8, 0)$ and $B(0, 6)$



Let $AB = c = \sqrt{8^2 + 6^2} = 10$

$OB = a = 6$ and $OA = b = 8$

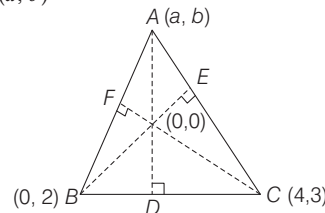
Also, let incentre is (h, k) , then

$$= \frac{6 \times 8 + 8 \times 0 + 10 \times 0}{6 + 8 + 10} = \frac{48}{24} = 2$$

$$\text{and } = \frac{6 \times 0 + 8 \times 6 + 10 \times 0}{6 + 8 + 10} = \frac{48}{24} = 2$$

∴ incentre is $(2, 2)$

6 (c) Let ABC be a given triangle with vertices $B(0, 2)$, $C(4, 3)$ and let third vertex be $A(a, b)$



Also, let D, E and F are the foot of perpendiculars drawn from A, B and C respectively. Then,

$$AD \perp BC \Rightarrow \frac{b-0}{a-0} \times \frac{3-2}{4-0} = -1$$

$$\Rightarrow b + 4a = 0 \quad \dots (i)$$

and $CF \perp AB$

$$\Rightarrow \frac{b-2}{a-0} \times \frac{3-0}{4-0} = -1$$

$$\Rightarrow 4a + 3b = 6 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get

$$-b + 3b = 6 \Rightarrow 2b = 6 \Rightarrow b = 3$$

$$\text{and } a = -\frac{3}{4} \quad [\text{from Eq. (i)}]$$

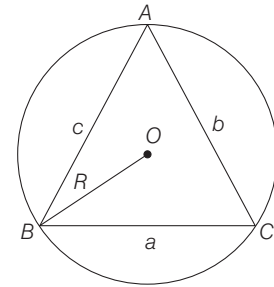
So, the third vertex

$$(a, b) = \left(-\frac{3}{4}, 3\right), \text{ which lies in II quadrant.}$$

7 (b) We know that,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R \text{ and given that,}$$

$$a + b = x, ab = y \text{ and } x^2 - c^2 = y$$



$$\therefore (a + b)^2 - c^2 = ab$$

$$\Rightarrow \frac{a^2 + b^2 - c^2}{2ab} = \frac{-ab}{2ab} = -\frac{1}{2}$$

$$\therefore \cos C = -\frac{1}{2} \Rightarrow C = 120^\circ$$

Now, $\frac{c}{\sin C} = 2R$

$$\Rightarrow R = \frac{1}{2} \frac{c}{\sin(120^\circ)} = \frac{c}{2\sqrt{3}}$$

$$\therefore R = \frac{c}{\sqrt{3}}$$

8 (c) Given vertices of $\triangle AOP$ are $O(0, 0)$ and $A(0, 1)$

Let the coordinates of point P are (x, y) .

Clearly, perimeter $= OA + AP + OP = 4$ (given)

$$\Rightarrow \sqrt{(0-0)^2 + (0-1)^2} + \sqrt{(0-x)^2 + (1-y)^2} + \sqrt{x^2 + y^2} = 4$$

$$\Rightarrow 1 + \sqrt{x^2 + (y-1)^2} + \sqrt{x^2 + y^2} = 4$$

$$\Rightarrow \sqrt{x^2 + y^2 - 2y + 1} + \sqrt{x^2 + y^2} = 3$$

$$\Rightarrow \sqrt{x^2 + y^2 - 2y + 1} = 3 - \sqrt{x^2 + y^2}$$

$$\Rightarrow 1 - 2y = 9 - 6\sqrt{x^2 + y^2}$$

$$\Rightarrow 6\sqrt{x^2 + y^2} = 2y + 8$$

$$\Rightarrow 9(x^2 + y^2) = (y + 4)^2 +$$

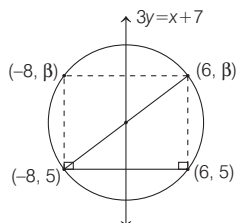
$$[\text{squaring both sides}]$$

$$\Rightarrow 9x^2 + 9y^2 = y^2 + 8y + 16$$

Thus, the locus of point $P(x, y)$ is

$$9x^2 + 8y^2 - 8y = 16$$

- 9 (b) Given points are $(-8, 5)$ and $(6, 5)$ in which y -coordinate is same, i.e. these points lie on horizontal line $y = 5$.



Let $(-8, \beta)$ and $(6, \beta)$ are the coordinates of the other vertices of rectangle as shown in the figure.

Since, the mid-point of line joining points $(-8, 5)$ and $(6, \beta)$ lies on the line $3y = x + 7$.

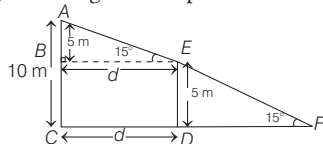
$$\therefore 3\left(\frac{5 + \beta}{2}\right) = \frac{-8 + 6}{2} + 7$$

$$\Rightarrow 15 + 3\beta = -2 + 14$$

$$\Rightarrow 3\beta = -3 \Rightarrow \beta = -1$$

$$\text{Now, area of rectangle} = |-8 - 6| \times |\beta - 5| = 14 \times 6 = 84$$

- 10 (d) Given heights of two poles are 5 m and 10 m.



i.e. from figure $AC = 10$ m, $DE = 5$ m

$$\therefore AB = AC - DE = 10 - 5 = 5 \text{ m}$$

Let d be the distance between two poles.

Clearly, $\triangle ABE \sim \triangle ACF$

[by AA - similarity criterion]

$$\therefore \angle AEB = 15^\circ$$

In $\triangle ABE$, we have

$$\tan 15^\circ = \frac{AB}{BE} \Rightarrow \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = \frac{5}{d}$$

$$\Rightarrow d = \frac{5(\sqrt{3} + 1)}{(\sqrt{3} - 1)}$$

$$\Rightarrow d = 5 \frac{\sqrt{3} + 1}{\sqrt{3} - 1} \times \frac{\sqrt{3} + 1}{\sqrt{3} + 1} = \frac{2 \times 5(\sqrt{3} + 2)}{2} = 5(2 + \sqrt{3}) \text{ m}$$

- 11 (c) Let $\frac{b+c}{9} = \frac{c+a}{10} = \frac{a+b}{11} = k$

$$\Rightarrow b + c = 9k, c + a = 10k \text{ and } a + b = 11k$$

$$\text{and } a + b + c = 15k$$

$$\therefore a = 6k, b = 5k \text{ and } c = 4k$$

$$\therefore \cos A + \cos B$$

$$\begin{aligned} &= \frac{\cos C}{2bc} + \frac{\cos C}{2ac} \\ &= \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^2 + c^2 - b^2}{2ac} \\ &= \frac{a^2 + b^2 - c^2}{2ab} \end{aligned}$$

$$\begin{aligned} &= \frac{5}{40} + \frac{27}{48} = \frac{11}{45/60} = \frac{11}{12} \end{aligned}$$

Hence, option (c) is correct.

- 12 (d) We have,

$$a : b : c = 2 : 3 : 4$$

Let

$$a = 2k, b = 3k, c = 4k$$

$$\therefore R = \frac{abc}{4\Delta} \text{ and } r = \frac{\Delta}{s}$$

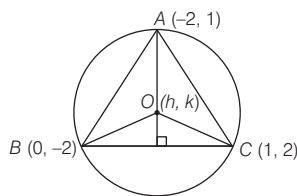
$$\therefore \frac{R}{r} = \frac{s \cdot abc}{4\Delta^2}$$

$$\Rightarrow \frac{R}{r} = \frac{s \cdot (2k)(3k)(4k)}{4s(s-2k)(s-3k)(s-4k)}$$

$$\Rightarrow \frac{R}{r} = \frac{6 \cdot 2^3 \cdot k^3}{5k \cdot 3k \cdot k} \Rightarrow \frac{R}{r} = \frac{16}{5}$$

$$\therefore R : r = 16 : 5$$

- 13 (b) Let (h, k) be circumcentre of $\triangle ABC$ and $A(-2, 1)$, $B(0, -2)$, $C(1, 2)$ are the vertices of $\triangle ABC$.



$$\therefore OA = OB = OC$$

$$OA = OB$$

$$(h+2)^2 + (k-1)^2 = h^2 + (k+2)^2$$

$$\Rightarrow 4h - 6k + 1 = 0 \quad \dots(i)$$

$$OB = OC$$

$$h^2 + (k+2)^2 = (h-1)^2 + (k-2)^2$$

$$= 2h + 8k - 1 = 0 \quad \dots(ii)$$

On solving Eqs. (i) and (ii), we get

$$h = -\frac{1}{22}, k = \frac{3}{22}$$

Equation of BC

$$y + 2 = \frac{2+2}{1-0}(x)$$

$$4x - y - 2 = 0$$

Distance from circumcentre to line BC

$$d = \left| \frac{4\left(-\frac{1}{22}\right) - \frac{3}{22} - 2}{\sqrt{4^2 + 1^2}} \right|$$

$$d = \frac{51}{22\sqrt{17}} = \frac{3\sqrt{17}}{22}$$

- 14 (d) We have,

p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C respectively.

$$\therefore \Delta = \frac{1}{2}ap_1 = \frac{1}{2}bp_2 = \frac{1}{2}cp_3$$

$$r_1 = \frac{\Delta}{s-a}, r_2 = \frac{\Delta}{s-b}, r_3 = \frac{\Delta}{s-c}, r = \frac{\Delta}{s}$$

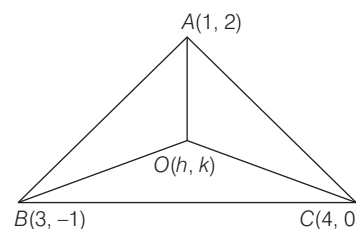
$$\therefore \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2}$$

$$\begin{aligned} &= \frac{(s-a)^2}{\Delta^2} + \frac{(s-b)^2}{\Delta^2} + \frac{(s-c)^2}{\Delta^2} + \frac{s^2}{\Delta^2} \\ &= \frac{1}{\Delta^2} [4s^2 - 2s(a+b+c) + (a^2 + b^2 + c^2)] \\ &= \frac{4\Delta^2}{\Delta^2} \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \right) \\ &= 4 \left(\frac{1}{p_1^2} + \frac{1}{p_2^2} + \frac{1}{p_3^2} \right) \end{aligned}$$

- 15 (*) We have,

$$A(1, 2), B(3, -1), C(4, 0)$$

Centroid of $\triangle ABC$ is



$$\left(\frac{1+3+4}{3}, \frac{2-1+0}{3} \right) = \left(\frac{8}{3}, \frac{1}{3} \right)$$

Let (h, k) be circumcentre of $\triangle ABC$

$$\therefore OA = OB = OC, \quad OA = OB$$

$$(h-1)^2 + (k-2)^2 = (h-3)^2 + (k+1)^2$$

$$\begin{aligned} \Rightarrow h^2 - 2h + 1 + k^2 - 4k + 4 &= h^2 - 6h + 9 + k^2 + 2k + 1 \\ \Rightarrow 4h - 6k &= 5 \quad \dots(i) \end{aligned}$$

Similarly, $OA = OC$

$$\therefore (h-1)^2 + (k-2)^2 = (h-4)^2 + k^2$$

$$\Rightarrow 6h - 4k = 11 \quad \dots(ii)$$

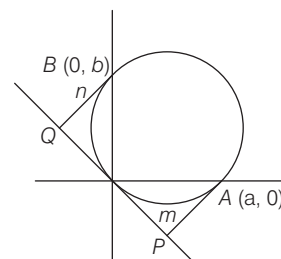
Solving Eqs. (i) and (ii), we get

$$h = \frac{23}{10}, k = \frac{7}{10}$$

Distance between circumcentre and centroid of $\triangle ABC$ is

$$\sqrt{\left(\frac{23}{10} - \frac{8}{3}\right)^2 + \left(\frac{7}{10} - \frac{1}{3}\right)^2} = \frac{11\sqrt{2}}{30}$$

- 16 (c) Equation of circle passing through



$$(a, 0), (0, b) \text{ and } (0, 0) \text{ is } x^2 + y^2 - ax - by = 0$$

Equation of tangent at origin, $-ax - by = 0$

$$\Rightarrow ax + by = 0$$

$$PA = \frac{a^2}{\sqrt{a^2 + b^2}}$$

$$\therefore m = \left| \frac{a^2}{\sqrt{a^2 + b^2}} \right| \quad \dots (i)$$

and $BQ = n = \left| \frac{b^2}{\sqrt{a^2 + b^2}} \right| \quad \dots (ii)$

$$m^2 + n^2 = \frac{a^4 + b^4}{a^2 + b^2}$$

$$mn = \frac{a^2 b^2}{a^2 + b^2}$$

$$(m + n)^2 = \frac{(a^2 + b^2)^2}{a^2 + b^2} = a^2 + b^2$$

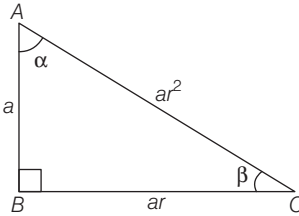
$\therefore$ Diameter of circle

$$= \sqrt{a^2 + b^2} = \sqrt{(m + n)^2} = m + n$$

17 (a) We have, $\tan A + \tan B + \tan C = 6$
 $\Rightarrow \tan A \tan B \tan C = 6 \quad \dots (i)$

and $\tan A \tan B = 2 \quad \dots (ii)$
 From Eqs. (i) and (ii) we get, $\tan C = 3$

18 (b) Let $\triangle ABC$ be a right angled triangle at B .
 Let $\angle A$ and $\angle C$ be α and β



Since, sides are in GP so sides are a, ar, ar^2

Now, $AC^2 = AB^2 + BC^2$
 $\Rightarrow (ar^2)^2 = a^2 + (ar)^2$
 $\Rightarrow a^2 r^4 = a^2 + a^2 r^2$
 $\Rightarrow r^4 - r^2 - 1 = 0$
 $\Rightarrow r = \sqrt{\frac{\sqrt{5} + 1}{2}}$

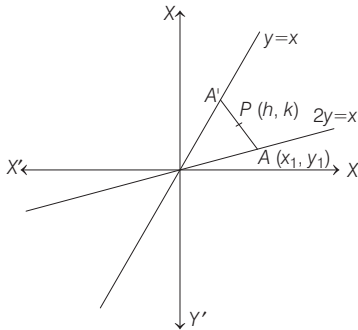
$$\therefore \tan \alpha = \frac{BC}{AB} = \frac{ar}{a} \Rightarrow r = \sqrt{\frac{\sqrt{5} + 1}{2}}$$

and $\tan \beta = \frac{AB}{BC} = \frac{a}{ar} = \frac{1}{r}$
 $= \frac{1}{\sqrt{\frac{\sqrt{5} + 1}{2}}} = \sqrt{\frac{\sqrt{5} - 1}{2}}$

19 (d) Given, that angles of a triangles are $2x, 3x$ and $7x$.
 Since, $2x + 3x + 7x = 180$
 $\Rightarrow 12x = 180^\circ$
 $\Rightarrow x = 15^\circ$
 So, angles are $30^\circ, 45^\circ$ and 105°
 Now, $\frac{a}{\sin 30^\circ} = 2R$
 $\Rightarrow \frac{a}{\frac{1}{2}} = 2 \times 10 \quad [\because R = 10 \text{ cm}]$
 $\Rightarrow 2a = 20 \Rightarrow a = 10 \text{ cm}$

Straight Line and Pair of Straight Lines

1 (a) Let a point $A(x_1, y_1)$ on the line $x = 2y$ and the middle point of the perpendicular drawn from point $A(x_1, y_1)$ to the line $x = y$ is $P(h, k)$.



So, the coordinate of foot of perpendicular to line $y = x$ from point $A(x_1, y_1)$ is $A'(2h - x_1, 2k - y_1)$.

Since, the point A' lies on line $y = x$, so we have $2h - x_1 = 2k - y_1 \quad \dots (i)$

and as point A lies on line $2y = x$, so we have $x_1 = 2y_1 \quad \dots (ii)$

and as PA is perpendicular to line $y = x$, so $\frac{y_1 - k}{x_1 - h} = -1$

$$\Rightarrow y_1 - k = -x_1 + h \quad \dots (iii)$$

From Eqs. (i) and (ii), we get

$$2h = 2k + y_1 \quad \dots (iv)$$

From Eqs. (ii) and (iii), we get

$$h = -k + 3y_1 \quad \dots (v)$$

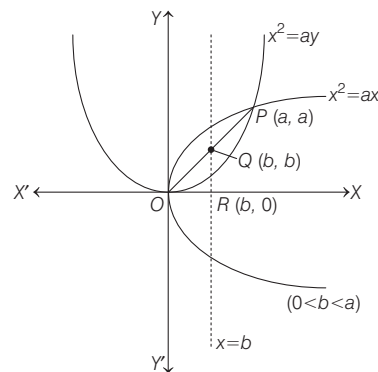
On the elimination of y_1 , from Eqs. (iv) and (v), we get

$$h + k = 3(2h - 2k)$$

$$\Rightarrow 5h = 7k$$

On taking the locus of point $P(h, k)$, we get $5x - 7y = 0$.

2 (b) Given curves $C_1 : y^2 = ax$, and $C_2 : x^2 = ay$, ($a > 0$), intersect each other at origin 'O' and a point $P(a, a)$.
 $\therefore O(0, 0)$, Q and $P(a, a)$ are collinear. So, $Q(b, b)$.



Now, as it is given area of $\triangle OQR = \frac{1}{2}$

$$\Rightarrow \frac{1}{2} b^2 = \frac{1}{2} \Rightarrow b = 1 \quad (\because b > 0)$$

$\therefore$ The line $x = b$ bisects the area bounded by the curves, C_1 and C_2 , so

$$\int_0^1 \left(\sqrt{ax} - \frac{x^2}{a} \right) dx = \int_1^a \left(\sqrt{ax} - \frac{x^2}{a} \right) dx$$

$$\Rightarrow \left[\sqrt{a} \frac{x^{3/2}}{3/2} - \frac{1}{a} \frac{x^3}{3} \right]_0^1 = \left[\sqrt{a} \frac{x^{3/2}}{3/2} - \frac{1}{a} \frac{x^3}{3} \right]_1^a$$

$$\Rightarrow \frac{2\sqrt{a}}{3} - \frac{1}{3a} = \frac{2a^{3/2}}{3} - \frac{a^2}{3} - \frac{2\sqrt{a}}{3} + \frac{1}{3a}$$

$$\Rightarrow \frac{4\sqrt{a}}{3} = \frac{a^2}{3} + \frac{2}{3a} \Rightarrow 4a\sqrt{a} = a^3 + 2$$

$$\Rightarrow 16a^3 = a^6 + 4 + 4a^3$$

$$\Rightarrow a^6 - 12a^3 + 4 = 0$$

$\therefore a$ satisfies the equation $x^6 - 12x^3 + 4 = 0$

Hence, option (b) is correct.

3 (c) It is given that point $P(x', y')$ lies on the line $3x + y - 4\lambda = 0$, then $y' = 4\lambda - 3x'$.
 And area of $\triangle PAB = 5$ sq. units where $A(1, -1)$ and $B(0, 2)$, so

$$\frac{1}{2} \begin{vmatrix} 0 & 2 & 1 \\ 1 & -1 & 1 \\ x' & 4\lambda - 3x' & 1 \end{vmatrix} = 5$$

$$\Rightarrow |0(-1 - 4\lambda + 3x') - 2(1 - x') + 1(4\lambda - 3x' + x')| = 10$$

$$\Rightarrow |4\lambda - 2| = 10 \Rightarrow 4\lambda - 2 = \pm 10$$

$$\Rightarrow 4\lambda = 12, -8 \Rightarrow \lambda = 3, -2$$

Hence, option (c) is correct.

4 (b) Given, $px + qy + r = 0$ is the equation of line such that

$$3p + 2q + 4r = 0$$

Consider, $3p + 2q + 4r = 0$

$$\Rightarrow \frac{3p}{4} + \frac{2q}{4} + r = 0$$

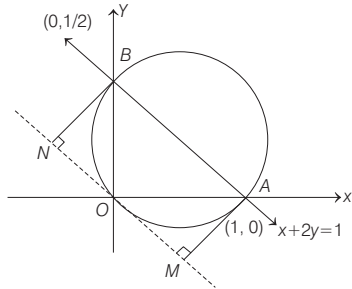
(dividing the equation by 4)

$$\Rightarrow p\left(\frac{3}{4}\right) + q\left(\frac{1}{2}\right) + r = 0$$

$$\Rightarrow \left(\frac{3}{4}, \frac{1}{2}\right) \text{ satisfy } px + qy + r = 0$$

So, the lines always passes through the point $\left(\frac{3}{4}, \frac{1}{2}\right)$.

- 5 (d) According to given information, we have the following figure.



From figure, equation of circle (diameter form) is

$$\Rightarrow x^2 + y^2 - x - \frac{y}{2} = 0$$

$$\text{Equation of tangent at } (0, 0) \text{ is } x + \frac{y}{2} = 0$$

[$\because$ equation of tangent at (x_1, y_1) is given by $T = 0$. Here, $T = 0$]

$$\Rightarrow xx_1 + yy_1 - \frac{1}{2}(x + x_1) - \frac{1}{4}(y + y_1) = 0]$$

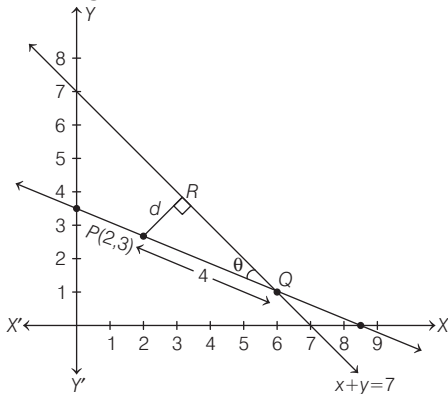
$$\Rightarrow 2x + y = 0$$

$$\text{Now, } AM = \frac{|2 \cdot 1 + 1 \cdot 0|}{\sqrt{5}} = \frac{2}{\sqrt{5}}$$

$$\text{and } BN = \frac{|2 \cdot 0 + 1 \cdot \left(\frac{1}{2}\right)|}{\sqrt{5}} = \frac{1}{2\sqrt{5}}$$

$$\therefore AM + BN = \frac{2}{\sqrt{5}} + \frac{1}{2\sqrt{5}} = \frac{4+1}{2\sqrt{5}} = \frac{\sqrt{5}}{2}$$

- 6 (a) Let the slope of line is m , which is passing through $P(2, 3)$.



$\therefore$ The distance of a point $P(2, 3)$ from the line $x + y - 7 = 0$, is

$$d = \frac{|2+3-7|}{\sqrt{1+1}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

- 7 (b) Let $A = (1, 2)$ equation of line is

$$\frac{x-1}{\cos \theta} = \frac{y-2}{\sin \theta} = r \frac{\sqrt{6}}{3}$$

$$\therefore x = \frac{\sqrt{6}}{3} \cos \theta + 1, y = \frac{\sqrt{6}}{3} \sin \theta + 2$$

This point lies on line $x + y = 4$

$$\therefore \frac{\sqrt{6}}{3} \cos \theta + \frac{\sqrt{6}}{3} \sin \theta = 1$$

$$\Rightarrow \sin \theta + \cos \theta = \frac{\sqrt{3}}{\sqrt{2}}$$

$$\Rightarrow \theta = 15^\circ \text{ or } 75^\circ$$

$\therefore$ Larger angle = 75°

- 8 (a) Given, equations of pair of straight lines

$$3x^2 + 2hxy - 3y^2 = 0 \quad \dots(i)$$

and

$$3x^2 + 2hxy - 3y^2 + 2x - 4y + c = 0 \quad \dots(ii)$$

both represents pair of perpendicular lines.

Now, the point of intersection the lines given by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

is $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$. So for the pair of

lines (ii), the point of intersection is

$$A\left(\frac{h(-2) - (-3)(1)}{-9 - h^2}, \frac{h - 3(-2)}{-9 - h^2}\right)$$

$$= A\left(\frac{2h - 3}{9 + h^2}, -\frac{h + 6}{9 + h^2}\right)$$

The equation of diagonal of square passes through origin is $y = \frac{h+6}{3-2h}x \quad \dots(iii)$

and the equation of diagonal of square not passes through origin is $2x - 4y + c = 0 \quad \dots(iv)$

The diagonals given by the Eqs. (iii) and (iv) are perpendicular if

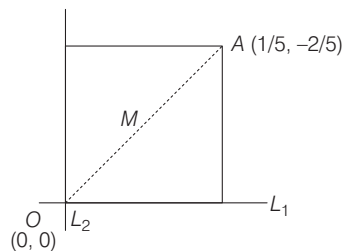
$$\frac{1}{2} \left(\frac{h+6}{3-2h} \right) = -1 \Rightarrow h+6 = 4h-6$$

$$\Rightarrow 3h = 12 \Rightarrow h = 4$$

$$\text{So, point } A = \left(\frac{8-3}{9+16}, -\frac{10}{9+16}\right) = \left(\frac{1}{5}, -\frac{2}{5}\right)$$

And the line (iv) passes through the mid-point 'M' of line joining points

$$A\left(\frac{1}{5}, -\frac{2}{5}\right) \text{ and } O(0, 0). \text{ So, } M = \left(\frac{1}{10}, -\frac{1}{5}\right).$$



$$\therefore \frac{2}{10} + \frac{4}{5} + c = 0$$

$$\Rightarrow c = -1 \quad [\text{By Eq. (iv)}]$$

So, $(h, c) = (4, -1)$

Hence, option (a) is correct.

- 9 (a) We have,

$$x^2 + 2\sqrt{2}xy + ky^2$$

$$\tan \theta = \frac{2\sqrt{2-k}}{k+1}$$

$$\tan 45^\circ = \frac{2\sqrt{2-k}}{k+1} \quad [\because \theta = 45^\circ]$$

$$\Rightarrow (k+1)^2 = 4(2-k)$$

$$\Rightarrow k = 1, k \neq -7 \quad [\because k > 0]$$

Equation of angle bisector of line

$$x^2 + 2\sqrt{2}xy + y^2 = 0$$

$$\frac{x^2 - y^2}{0} = \frac{xy}{\sqrt{2}}$$

$$x^2 - y^2 = 0$$

Area of triangle formed by line

$$x^2 - y^2 = 0 \text{ and } x + 2y + 1 = 0$$

$$\Delta = \frac{|1\sqrt{0 - (1) \cdot (1)}|}{|1(2)^2 - (0) - (1)^2|}$$

$$\Delta = \frac{|\sqrt{0+1}|}{|4-1|} = \frac{1}{3}$$

- 10 (a) Required equation of pair of straight lines passing through origin and having slopes,

$$m_1 = (1 + \sqrt{2}) \text{ and}$$

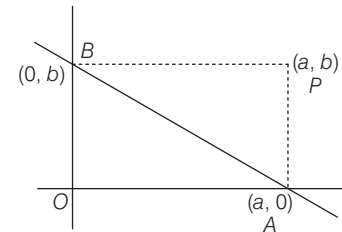
$$m_2 = \frac{1}{1 + \sqrt{2}} = \sqrt{2} - 1$$

$$\Rightarrow [y - (1 + \sqrt{2})x][y - (\sqrt{2} - 1)x] = 0$$

$$\Rightarrow y^2 - 2\sqrt{2}xy + x^2 = 0$$

- 11 (b) Let the equation of line be $\frac{x}{a} + \frac{y}{b} = 1$

Since, the line passing through a fixed point (x_1, y_1) .



$$\therefore \frac{x_1}{a} + \frac{y_1}{b} = 1$$

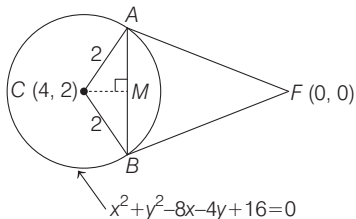
Since, $OAPB$ is a rectangle, therefore the coordinate of P will be (a, b) .

Hence, locus of P is

$$\frac{x_1}{x} + \frac{y_1}{y} = 1$$

Circle

- 1 (c) The equation of chord of contact AB to circle $x^2 + y^2 - 8x - 4y + 16 = 0$ w.r.t point origin $(0, 0)$ is $T = 0$



$$\Rightarrow 0x + 0y - 4(x + 0) - 2(y + 0) + 16 = 0$$

$$\Rightarrow 2x + y - 8 = 0 \quad \dots(i)$$

In right angled triangle AMC,

$$CM = \frac{|2(4) + 2 - 8|}{\sqrt{2^2 + 1^2}} = \frac{2}{\sqrt{5}}$$

$$\therefore AM^2 = CA^2 - CM^2$$

$$= 2^2 - \left(\frac{2}{\sqrt{5}}\right)^2 = 4 - \frac{4}{5} = \frac{16}{5}$$

$$\Rightarrow AM = \frac{4}{\sqrt{5}}$$

$$\therefore AB^2 = (2AM)^2 = \left(\frac{2 \times 4}{\sqrt{5}}\right)^2 = \frac{64}{5}$$

- 2 (c) Since, equation of a circle which touches the Y-axis at point $(0, 4)$ is

$$x^2 + (y - 4)^2 + \lambda x = 0 \quad \dots(i)$$

$\therefore$ Circle (i) passes through point $(2, 0)$, so

$$4 + 16 + 2\lambda = 0$$

$$\Rightarrow \lambda = -10$$

Therefore, equation of the circle is

$$x^2 + (y - 4)^2 - 10x = 0$$

$$\Rightarrow (x - 5)^2 + (y - 4)^2 = 25 \quad \dots(ii)$$

Now, equation of tangent to the circle (ii) having slope 'm' is

$$y - 4 = m(x - 5) \pm 5\sqrt{1 + m^2}$$

$$\Rightarrow y = mx + (4 - 5m \pm 5\sqrt{1 + m^2})$$

$\dots(iii)$

From the option, on taking slope of line, we have when $m = \frac{4}{3}$, equation of tangent by Eq.

(iii), we get

$$y = \frac{4}{3}x + \left(4 - \frac{20}{3} \pm 5\sqrt{1 + \frac{16}{9}}\right)$$

$$\Rightarrow 3y = 4x + (12 - 20 \pm 25)$$

$$\Rightarrow 4x - 3y - 8 \pm 25 = 0$$

$$\Rightarrow 4x - 3y + 17 = 0 \text{ and } 4x - 3y - 33 = 0$$

Similarly, when $m = -3/4$, equation of tangent is

$$y = -\frac{3}{4}x + \left(4 + \frac{15}{4} \pm 5\sqrt{1 + \frac{9}{16}}\right)$$

$$\Rightarrow 4y = -3x + 31 \pm 25$$

$$\Rightarrow 3x + 4y - 56 = 0$$

$$\text{and } 3x + 4y - 6 = 0$$

Similarly, when $m = -\frac{4}{3}$, equation of

tangent is

$$y = -\frac{4}{3}x + \left(4 + \frac{20}{3} \pm \frac{25}{3}\right)$$

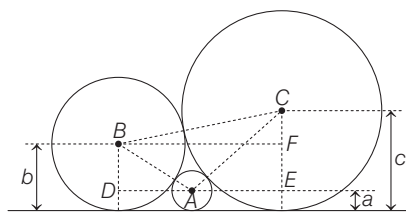
$$\Rightarrow 4x + 3y - 57 = 0$$

$$4x + 3y - 7 = 0$$

$\therefore$ The line $4x + 3y - 8 = 0$ is not a tangent to the circle.

Hence, option (c) is correct.

- 3 (b) According to given information, we have the following figure.



where A, B, C are the centres of the circles

Clearly, $AB = a + b$ (sum of radii) and $BD = b - a$

$$\therefore AD = \sqrt{(a + b)^2 - (b - a)^2} = 2\sqrt{ab}$$

Similarly, $AC = a + c$ and $CE = c - a$

$$\therefore \text{In } \triangle ACE, AE = \sqrt{(a + c)^2 - (c - a)^2} = 2\sqrt{ac}$$

Similarly, $BC = b + c$ and $CF = c - b$

$$\therefore \text{In } \triangle BCF, BF = \sqrt{(b + c)^2 - (c - b)^2} = 2\sqrt{bc}$$

$$\therefore AD + AE = BF$$

$$\therefore 2\sqrt{ab} + 2\sqrt{ac} = 2\sqrt{bc}$$

$$\Rightarrow \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}}$$

- 4 (c) Circle I is

$$x^2 + y^2 - 16x - 20y + 164 = r^2$$

$$\Rightarrow (x - 8)^2 + (y - 10)^2 = r^2$$

$\Rightarrow C_1(8, 10)$ is the centre of 1st circle and $r_1 = r$ is its radius

Circle II is $(x - 4)^2 + (y - 7)^2 = 36$

$\Rightarrow C_2(4, 7)$ is the centre of 2nd circle and $r_2 = 6$ is its radius.

Two circles intersect if $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$$\Rightarrow |r - 6| < \sqrt{(8 - 4)^2 + (10 - 7)^2} < r + 6$$

$$\Rightarrow |r - 6| < 5 < r + 6$$

Now as, $5 < r + 6$ always, we have to solve only

$$|r - 6| < 5 \Rightarrow -5 < r - 6 < 5$$

$$\Rightarrow 6 - 5 < r < 5 + 6 \Rightarrow 1 < r < 11$$

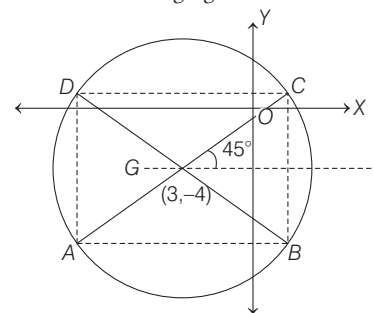
- 5 (c) Given equation of circle is

$x^2 + y^2 - 6x + 8y - 103 = 0$, which can be written as

$$(x - 3)^2 + (y + 4)^2 = 128 = (8\sqrt{2})^2$$

$\therefore$ Centre = $(3, -4)$ and radius = $8\sqrt{2}$

Now, according to given information, we have the following figure.



For the coordinates of A and C .

$$\text{Consider, } \frac{x - 3}{\frac{1}{\sqrt{2}}} = \frac{y + 4}{\frac{1}{\sqrt{2}}} = \pm 8\sqrt{2}$$

[using distance (parametric) form of line, $\frac{x - x_1}{\cos \theta} = \frac{y - y_1}{\sin \theta} = r$]

$$\Rightarrow x = 3 \pm 8, y = -4 \pm 8$$

$$\therefore A(-5, -12) \text{ and } C(11, 4)$$

Similarly, for the coordinates of B and D , consider

$$\frac{x - 3}{-\frac{1}{\sqrt{2}}} = \frac{y + 4}{\frac{1}{\sqrt{2}}} = \pm 8\sqrt{2}$$

[in this case, $\theta = 135^\circ$]

$$\Rightarrow x = 3 \mp 8, y = -4 \pm 8$$

$$\therefore B(11, -12) \text{ and } D(-5, 4)$$

Now, $OA = \sqrt{25 + 144} = \sqrt{169} = 13$;

$$OB = \sqrt{121 + 144} = \sqrt{265}$$

$$OC = \sqrt{121 + 16} = \sqrt{137}$$

and

$$OD = \sqrt{25 + 16} = \sqrt{41}$$

- 6 (a) Given parabola $y^2 = 4x$ $\dots(i)$

So, equation of tangent to parabola (i) at point $(1, 2)$ is $2y = 2(x + 1)$

$$\Rightarrow y = x + 1 \quad \dots(ii)$$

Now, equation of circle, touch the parabola at point $(1, 2)$ is

$$(x - 1)^2 + (y - 2)^2 + \lambda(x - y + 1) = 0$$

$$\Rightarrow x^2 + y^2 + (\lambda - 2)x$$

$$+ (-4 - \lambda)y + (5 + \lambda) = 0 \quad \dots(iii)$$

Also, circle (iii) touches the x -axis, so $g^2 = c$

$$\Rightarrow \left(\frac{\lambda-2}{2}\right)^2 = 5 + \lambda$$

$$\Rightarrow \lambda^2 - 4\lambda + 4 = 4\lambda + 20$$

$$\Rightarrow \lambda = 4 \pm \sqrt{32} = 4 \pm 4\sqrt{2}$$

Now, radius of circle is $r = \sqrt{g^2 + f^2 - c}$

$$\Rightarrow r = |f| \quad [\because g^2 = c]$$

$$= \left|\frac{\lambda+4}{2}\right| = \frac{8+4\sqrt{2}}{2} \text{ or } \frac{8-4\sqrt{2}}{2}$$

$$\text{For least area } r = \frac{8-4\sqrt{2}}{2} = 4-2\sqrt{2} \text{ units}$$

$$\text{So, area} = \pi r^2 = \pi(16+8-16\sqrt{2})$$

$$= 8\pi(3-2\sqrt{2}) \text{ sq unit}$$

- 7 (d) Key Idea** Firstly find the centre of the given circle and write the coordinates of mid point (A), of line segment PQ, since $AC \perp PQ$ therefore use (slope of AC) $\times$ (slope of PQ) = -1

It is given that points P and Q are intersecting points of circle

$$(x-3)^2 + (y+2)^2 = 25 \quad \dots(i)$$

Line $y = mx + 1 \quad \dots(ii)$

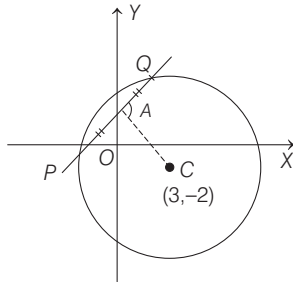
And, the mid-point of PQ is A having x -coordinate $\frac{-3}{5}$ so

y -coordinate is $1 - \frac{3}{5}m$

$$\text{So, } A\left(-\frac{3}{5}, 1 - \frac{3}{5}m\right)$$

From the figure,

$\therefore AC \perp PQ$



$$\Rightarrow (\text{slope of AC}) \times (\text{slope of PQ}) = -1$$

$$\Rightarrow \left(\frac{-2-1+\frac{3}{5}m}{3+\frac{3}{5}}\right) \times m = -1$$

$$\Rightarrow m = 2 \text{ or } 3$$

- 8 (b)** Equation of given circle is

$$x^2 + y^2 - 4x - 6y - 12 = 0 \text{ having centre}$$

$$C_1(2, 3) \text{ and radius } r_1 = \sqrt{4+9+12} = 5.$$

Let the required circle having centre $C_2(h, k)$ and radius is given as 3 touches the given circle at $A(-1, -1)$.

The point $A(-1, -1)$ divides the line joining the centres $C_1(2, 3)$ and $C_2(h, k)$ externally in 5 : 3 so

$$(-1, -1) = \left(\frac{5h-3(2)}{5-3}, \frac{5k-3(3)}{5-3}\right)$$

$$\Rightarrow (-1, -1) = \left(\frac{5h-6}{2}, \frac{5k-9}{2}\right)$$

$$\Rightarrow 5h-6 = -2 \text{ and } 5k-9 = -2$$

$$\Rightarrow h = \frac{4}{5} \text{ and } k = \frac{7}{5}$$

so equation of required circle is

$$\left(x - \frac{4}{5}\right)^2 + \left(y - \frac{7}{5}\right)^2 = (3)^2$$

$$\Rightarrow 5x^2 + 5y^2 - 8x - 14y - 32 = 0$$

Hence, option (b) is correct.

- 9 (b)** We have,

$$x^2 + y^2 - 2x + 4y - 4 = 0$$

$$\text{and } x^2 + y^2 + 4x - 4y + \alpha = 0$$

$$c_1 = (1, -2), r_1 = \sqrt{1+4+4} = 3$$

$$c_2 = (-2, 2), r_2 = \sqrt{4+4-\alpha} = \sqrt{8-\alpha}$$

Circle have 4 common tangents

$$\therefore c_1 c_2 > r_1 + r_2$$

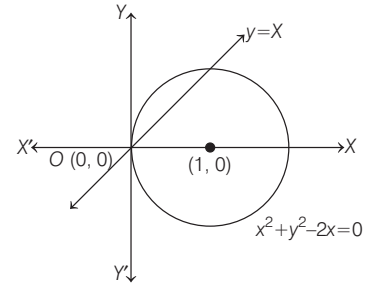
$$\Rightarrow \sqrt{(1+2)^2 + (-2-2)^2} > 3 + \sqrt{8-\alpha}$$

$$\Rightarrow 8 - \alpha < 4 \Rightarrow \alpha > 4$$

$\therefore$ Least integral value of $\alpha = 5$

- 10 (b)** We have equation of line $y = x$ and

$$\text{equation of circle } x^2 + y^2 - 2x = 0$$



Now, intersecting points of given line and circle,

$$x^2 + x^2 - 2x = 0$$

$$\Rightarrow 2x(x-1) = 0 \Rightarrow x = 0, 1$$

when $x = 0$ then $y = 0$ and when $x = 1$ then $y = 1$

$\therefore$ Coordinates of end points of diameter AB are (0, 0) and (1, 1).

$\therefore$ Required equation of circle with diameter AB

$$(x-0)(x-1) + (y-0)(y-1) = 0$$

$$\Rightarrow x^2 - 2x + y^2 - y = 0$$

$$\Rightarrow x^2 + y^2 - x - y = 0$$

- 11 (a)** In a circle, AB is a diameter where the coordinate of A is (p, q) and let the coordinate of B(x_1, y_1).

Equation of circle in diameter form is

$$(x-p)(x-x_1) + (y-q)(y-y_1) = 0$$

$$\Rightarrow x^2 - (p+x_1)x + px_1 + y^2$$

$$- (y_1+q)y + qy_1 = 0$$

$$\Rightarrow x^2 - (p+x_1)x + y^2$$

$$- (y_1+q)y + px_1 + qy_1 = 0$$

Since, this circle touches X-axis

$$\therefore y = 0$$

$$\Rightarrow x^2 - (p+x_1)x + px_1 + qy_1 = 0$$

Also, the discriminant of above equation will be equal to zero because circle touches X-axis.

$$\therefore (p+x_1)^2 = 4(px_1 + qy_1)$$

$$\Rightarrow p^2 + x_1^2 + 2px_1 = 4px_1 + 4qy_1$$

$$\Rightarrow x_1^2 - 2px_1 + p^2 = 4qy_1$$

$$\Rightarrow (x_1 - p)^2 = 4qy_1$$

Therefore, the locus of point B is

$$(x-p)^2 = 4qy$$

Parabola

- 1 (b)** As we know, equation of tangent to the parabola $y^2 = 4x$, having slope 'm' is

$$y = mx + \frac{1}{m} \quad \dots(i)$$

On comparing the Eq. (i) with the equation of given tangent $y = mx + 4$, we get

$$\frac{1}{m} = 4 \Rightarrow m = \frac{1}{4}$$

$$\therefore \text{Equation of the tangent is } y = \frac{1}{4}x + 4,$$

now it is tangent to the parabola $x^2 = 2by$, so on solving the equation of parabola

$$x^2 = 2by \text{ and tangent } y = \frac{1}{4}x + 4, \text{ we must}$$

get only a common point, so

$$x^2 = 2b\left(\frac{1}{4}x + 4\right)$$

$$\Rightarrow 2x^2 - bx - 16b = 0 \text{ is a quadratic equation having one solution.}$$

$$\text{So, } D = 0 \Rightarrow b^2 + 4(2)(16b) = 0$$

$$\Rightarrow b = -128 \quad [\because b \text{ cannot be zero}]$$

- 2 (a)** Equation of given parabola $y^2 = 8x$ and

one end of a focal chord AB is $A\left(\frac{1}{2}, -2\right)$

As, we know, if one end of a focal chord of parabola $y^2 = 4ax$ is $(at^2, 2at)$, then other end will be

$\left(\frac{a}{t^2}, \frac{-2a}{t}\right)$ so other end point

$$B\left(\frac{2}{\left(-\frac{1}{2}\right)^2}, \frac{-4}{-\frac{1}{2}}\right) = B(8, 8)$$

Now, equation of tangent of parabola $y^2 = 8x$ at point $B(8, 8)$ is $T = 0$

$$\Rightarrow 8y = 4(x + 8)$$

$$\Rightarrow x - 2y + 8 = 0$$

3 (c) Normal to parabola $y^2 = 4ax$ is given by

$$y = mx - 2am - am^3$$

$\therefore$ Normal to parabola

$$y^2 = 4b(x - c) \text{ is}$$

$$y = m(x - c) - 2bm - bm^3$$

$$[\text{replacing } a \text{ by } b \text{ and } x \text{ by } x - c]$$

$$= mx - (2b + c)m - bm^3 \quad \dots (i)$$

and normal to parabola $y^2 = 8ax$ is

$$y = mx - 4am - 2am^3 \quad \dots (ii)$$

$$[\text{replacing } a \text{ by } 2a]$$

For common normal, we should have $mx - 4am - 2am^3 = mx - (2b + c)m - bm^3$

$$[\text{using Eqs. (i) and (ii)}]$$

$$4am + 2am^3 = (2b + c)m + bm^3$$

$$\Rightarrow m((2a - b)m^2 + (4a - 2b - c)) = 0$$

$$\Rightarrow m = 0$$

$$\text{or } m^2 = \frac{2b + c - 4a}{2a - b} = \frac{c}{2a - b} - 2$$

As, $m^2 > 0$, therefore $\frac{c}{2a - b} > 2$

Note that if $m = 0$, then all options satisfy ($\because y = 0$ is a common normal) and if common normal is other than the axis, then only option (c) satisfies.

$$\left[\because \text{for option (c), } \frac{c}{2a - b} = \frac{3}{2 - 1} = 3 > 2 \right]$$

4 (a) We know that, $y = mx + \frac{a}{m}$ is the equation of tangent to the parabola $y^2 = 4ax$.

$$\therefore y = mx + \frac{1}{m} \text{ is a tangent to the parabola}$$

$$y^2 = 4x. \quad [\because a = 1]$$

Let, this tangent is also a tangent to the hyperbola $xy = 2$

Now, on substituting $y = mx + \frac{1}{m}$ in $xy = 2$, we get

$$x\left(mx + \frac{1}{m}\right) = 2.$$

$$\Rightarrow m^2x^2 + x - 2m = 0$$

$$\Rightarrow D = 0 \Rightarrow 1 = 4(m^2)(-2m)$$

$$\Rightarrow m^3 = \left(-\frac{1}{2}\right)^3 \Rightarrow m = -\frac{1}{2}$$

$\therefore$ Required equation of tangent is

$$y = -\frac{x}{2} - 2 \Rightarrow 2y = -x - 4$$

$$\Rightarrow x + 2y + 4 = 0$$

5 (b) Vertex of parabola

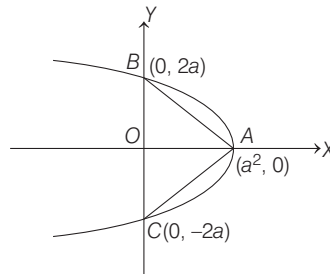
$$y^2 = -4(x - a^2) \text{ is } (a^2, 0).$$

For point of intersection with Y-axis, put $x = 0$ in the given equation of parabola.

$$\text{This gives, } y^2 = 4a^2$$

$$\Rightarrow y = \pm 2a$$

Thus, the point of intersection are $(0, 2a)$ and $(0, -2a)$.



From the given condition, we have

$$\text{Area of } \triangle ABC = 250$$

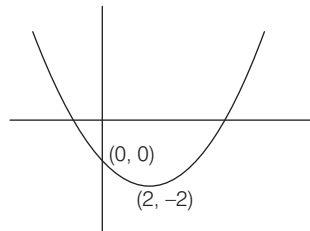
$$\therefore \frac{1}{2}(BC)(OA) = 250$$

$$[\because \text{Area} = \frac{1}{2} \times \text{base} \times \text{height}]$$

$$\Rightarrow \frac{1}{2}(4a)a^2 = 250$$

$$\therefore a = 5$$

6 (b) Given, $y = ax^2 + bx + c$ has two x -intercepts one is positive and one is negative and vertex is $(2, -2)$. The graph of $y = ax^2 + bx + c$ is



Clearly from graph

$$c > 0$$

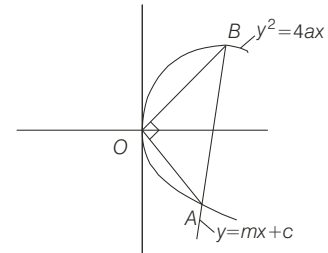
$$-\frac{b}{a} > 0 \Rightarrow -b > 0$$

$$\Rightarrow b < 0$$

$$f(1) = a + b + c < 0$$

$$ab < 0, ac < 0, bc > 0$$

7 (a) On homogenisation of the curve $y^2 - 4ax = 0$ by line $y = mx + c$, we are getting combined equation of straight lines which subtend a right angle at the origin, so



$$y^2 - 4ax\left(\frac{y - mx}{c}\right) = 0$$

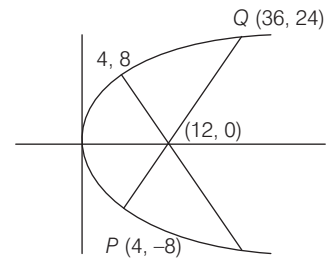
$$\Rightarrow c + 4am = 0 \quad \dots (i)$$

On putting the value of 'c' in the line, we get $y = m(x - 4a)$, represent family of line passes through $(4a, 0)$. Hence, option (a) is correct.

8 (b) We have, $y^2 = 16x$

Coordinate of latusrectum $(4, 8)$

Equation of normal at $(4, 8)$ is $y = -x + 12$



It cuts x-axis at $(12, 0)$

Equation of chord passing through $(12, 0)$ and perpendicular of normal is $y = x - 12$

Put the value of y in $y^2 = 16x$, we get

$$(12 - x)^2 = 16x$$

$$144 - 24x + x^2 = 16x$$

$$x^2 - 40x + 144 = 0$$

$$(x - 36)(x - 4) = 0$$

$$x = 4, 36$$

$$\therefore y = 4 - 12 = -8 \text{ and}$$

$$y = 36 - 12 = 24$$

$$\therefore \text{Length of chord} = \sqrt{(36 - 4)^2 + (24 + 8)^2} = \sqrt{32^2 + 32^2} = 32\sqrt{2}$$

Ellipse & Hyperbola

1. (a) Since line $3x + 4y = 12\sqrt{2}$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{9} = 1$ for some $a \in R$, then the equation of tangent to ellipse having slope 'm' is

$$y = mx \pm \sqrt{a^2 m^2 + 9} \quad \dots(i)$$

$\therefore$ Slope of line (i), m

$$= \text{slope of line } 3x + 4y = 12\sqrt{2} \Rightarrow m = -\frac{3}{4}$$

On putting the value of m in Eq. (i), we get

$$y = -\frac{3}{4}x \pm \sqrt{\frac{9}{16}a^2 + 9}$$

$$\Rightarrow 3x + 4y = \pm 3\sqrt{a^2 + 16} \quad \dots(ii)$$

Since tangent $3x + 4y = 12\sqrt{2}$ represented by Eq. (ii) for some $a \in R$, therefore

$$12\sqrt{2} = \pm 3\sqrt{a^2 + 16}$$

$$\Rightarrow 4\sqrt{2} = \pm \sqrt{a^2 + 16}$$

$$\Rightarrow 32 = a^2 + 16$$

[on squaring both sides]

$$\Rightarrow a^2 = 16 \Rightarrow a = \pm 4$$

$\therefore$ Eccentricity

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{16}} = \sqrt{\frac{7}{16}} = \frac{\sqrt{7}}{4}$$

$\therefore$ Distance between the foci = $|2ae|$

$$= 2 \times 4 \times \frac{\sqrt{7}}{4} = 2\sqrt{7}$$

2. (b) Since equation of normal to the ellipse at P meets the co-ordinate axes at $\left(-\frac{1}{3\sqrt{2}}, 0\right)$

and $(0, \beta)$ is

$$\frac{x}{-\frac{1}{3\sqrt{2}}} + \frac{y}{\beta} = 1 \quad \dots(i)$$

Now, let point $P(x_1, y_1)$, so equation of normal to ellipse $\frac{x^2}{1} + \frac{y^2}{1} = 1$ at point P is

$$\Rightarrow \frac{x}{2x_1} - \frac{y}{y_1} = \frac{1}{2} - 1$$

$$\Rightarrow \frac{x}{-2x_1} + \frac{y}{y_1} = \frac{1}{2} \quad \dots(ii)$$

$\therefore$ Eqs. (i) and (ii) represents same line

$$\therefore -\frac{2x_1}{3\sqrt{2}} = \frac{y_1}{\beta} = \frac{2}{1}$$

$$\Rightarrow x_1 = \frac{1}{3\sqrt{2}} \text{ and } y_1 = 2\beta$$

$\therefore$ Point $P(x_1, y_1)$ lies on ellipse, so

$$2\left(\frac{1}{9 \times 2}\right) + 4\beta^2 = 1$$

$$\Rightarrow \beta^2 = \frac{2}{9} \Rightarrow \beta = \pm \frac{\sqrt{2}}{3}$$

$\therefore$ Point $P(x_1, y_1)$ is in first quadrant so $y_1 = 2\beta > 0$

$$\therefore \beta = \frac{\sqrt{2}}{3}$$

Hence, option (b) is correct.

3. (c) Since, the vertices of hyperbola on X-axis at $(\pm 6, 0)$, so equation of hyperbola we can assume as $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $|a| = 6$

and hyperbola passes through point $P(10, 16)$ so

$$\frac{(10)^2}{6^2} - \frac{(16)^2}{b^2} = 1$$

$$\Rightarrow \frac{(16)^2}{b^2} = \frac{100 - 36}{36} = \frac{64}{36}$$

$$\Rightarrow b^2 = \frac{36 \times 256}{64} = 144$$

So, equation of hyperbola is $\frac{x^2}{36} - \frac{y^2}{144} = 0$,

and the equation of normal to hyperbola at point P is

$$\frac{36x}{10} + \frac{144y}{16} = 36 + 144$$

$$\Rightarrow \frac{18}{5}x + 9y = 180$$

$$\Rightarrow 2x + 5y = 100$$

Hence, option (c) is correct.

4. (b) $\therefore$ For the given hyperbola,

$$e = \sqrt{1 + \frac{\sin^2 \theta}{\cos^2 \theta}} > 2$$

$$\Rightarrow 1 + \tan^2 \theta > 4$$

$$\Rightarrow \tan \theta \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$$

$$\text{But } \theta \in \left(0, \frac{\pi}{2}\right) \Rightarrow \tan \theta \in (\sqrt{3}, \infty)$$

$$\Rightarrow \theta \in \left(\frac{\pi}{3}, \frac{\pi}{2}\right)$$

Now, length of latusrectum

$$= 2 \frac{\sin^2 \theta}{\cos \theta} = 2 \sin \theta \tan \theta$$

Since, both $\sin \theta$ and $\tan \theta$ are increasing functions in $\left(\frac{\pi}{3}, \frac{\pi}{2}\right)$

$\therefore$ Least value of latus rectum is

$$> 2 \sin \frac{\pi}{3} \cdot \tan \frac{\pi}{3} = 2 \cdot \frac{\sqrt{3}}{2} \cdot \sqrt{3} = 3 \left(\text{at } \theta = \frac{\pi}{3} \right)$$

and greatest value of latusrectum is $< \infty$

Hence, latusrectum length $\in (3, \infty)$

5. (c) Given, $S = \left\{ (x, y) \in R^2 : \frac{y^2}{1+r} - \frac{x^2}{1-r} = 1 \right\}$
 $= \left\{ (x, y) \in R^2 : \frac{y^2}{1+r} + \frac{x^2}{r-1} = 1 \right\}$

For $r > 1$, $\frac{y^2}{1+r} + \frac{x^2}{r-1} = 1$, represents a vertical ellipse.

[$\therefore$ for $r > 1$, $r-1 < r+1$ and $r-1 > 0$]

$$\begin{aligned} \text{Now, eccentricity } (e) &= \sqrt{1 - \frac{r-1}{r+1}} \\ &= \sqrt{\frac{(r+1) - (r-1)}{r+1}} \\ &= \sqrt{\frac{2}{r+1}} \end{aligned}$$

6. (b) Let the equation of ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Then, according to the problem, we have

$$\frac{2b^2}{a} = 8 \text{ and } 2ae = 2b$$

$$\Rightarrow b \left(\frac{b}{a} \right) = 4 \text{ and } \frac{b}{a} = e$$

$$\Rightarrow b(e) = 4$$

$$\Rightarrow b = 4 \cdot \frac{1}{e} \quad \dots(i)$$

Also, we know that $b^2 = a^2(1 - e^2)$

$$\Rightarrow 2e^2 = 1 \Rightarrow e = \frac{1}{\sqrt{2}} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get.

$$b = 4\sqrt{2}$$

$$\text{Now, } a^2 = \frac{b^2}{1 - e^2} = \frac{32}{1 - \frac{1}{2}} = 64$$

$$\therefore \text{Equation of ellipse be } \frac{x^2}{64} + \frac{y^2}{32} = 1$$

Now, check all the options.

Only $(4\sqrt{3}, 2\sqrt{2})$, satisfy the above equation.

7. (d)

8. (a) One of the focus of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is on Y-axis $(0, 5\sqrt{3})$

$$\therefore be = 5\sqrt{3} \quad \dots(i)$$

[where e is eccentricity of ellipse]

According to the question,

$$2b - 2a = 10$$

$$\Rightarrow b - a = 5 \quad \dots(ii)$$

On squaring Eq. (i) both sides, we get

$$b^2 e^2 = 75$$

$$\Rightarrow b^2 \left(1 - \frac{a^2}{b^2} \right) = 75 \quad \left[\because e^2 = 1 - \frac{a^2}{b^2} \right]$$

$$\Rightarrow b + a = 15 \quad \dots(iii) \text{ [from Eq. (ii)]}$$

On solving Eqs. (ii) and (iii), we get

$$b = 10 \text{ and } a = 5$$

$$\text{So, length of latusrectum is } \frac{2a^2}{b} = \frac{2 \times 25}{10} = 5$$

9. (a) Since the point (α, β) is on the parabola
 $y^2 = x$, so $\alpha = \beta^2$... (i)

Now, equation of tangent at point (α, β) to the parabola $y^2 = x$, is $T = 0$

$$\Rightarrow y\beta = \frac{1}{2}(x + \alpha)$$

[$\because$ equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is given by $yy_1 = 2a(x + x_1)$]

$$\Rightarrow 2y\beta = x + \beta^2 \quad [\text{from Eq. (i)}]$$

$$\Rightarrow y = \frac{x}{2\beta} + \frac{\beta}{2} \quad \dots (ii)$$

Since, line (ii) is also a tangent of the ellipse $x^2 + 2y^2 = 1$

$$\therefore \left(\frac{\beta}{2}\right)^2 = (1)^2 \left(\frac{1}{2\beta}\right)^2 + \frac{1}{2}$$

[$\because$ condition of tangency of line $y = mx + c$ to ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $c^2 = a^2 m^2 + b^2$,

here $m = \frac{1}{2\beta}$, $a = 1$, $b = \frac{1}{\sqrt{2}}$ and $c = \frac{\beta}{2}$]

$$\Rightarrow \frac{\beta^2}{4} = \frac{1}{4\beta^2} + \frac{1}{2}$$

$$\Rightarrow \beta^2 = \frac{2 \pm \sqrt{4+4}}{2}$$

$$\Rightarrow \beta^2 = 1 + \sqrt{2} [\because \beta^2 > 0]$$

$$\therefore \alpha = \beta^2 = 1 + \sqrt{2}$$

10. (d) Let the equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (i)$$

Since, equation of given directrix is $5x = 4\sqrt{5}$

$$\text{so } 5\left(\frac{a}{e}\right) = 4\sqrt{5}$$

[$\because$ equation of directrix is $x = \frac{a}{e}$]

$$\Rightarrow \frac{a}{e} = \frac{4}{\sqrt{5}} \quad \dots (ii)$$

and hyperbola (i) passes through point $(4, -2\sqrt{3})$

$$\text{so, } \frac{16}{a^2} - \frac{12}{b^2} = 1 \quad \dots (iii)$$

$$\text{The eccentricity } e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$\Rightarrow a^2 e^2 - a^2 = b^2 \quad \dots (iv)$$

From Eqs. (ii) and (iv), we get

$$\frac{16}{5}e^4 - \frac{16}{5}e^2 = b^2 \quad \dots (v)$$

From Eqs. (ii) and (iii), we get

$$\frac{16}{16e^2} - \frac{12}{b^2} = 1 \Rightarrow \frac{5}{e^2} - \frac{12}{b^2} = 1$$

$$\Rightarrow b^2 = \frac{12e^2}{5 - e^2} \quad \dots (vi)$$

From Eqs. (v) and (vi), we get

$$16e^4 - 16e^2 = 5\left(\frac{12e^2}{5 - e^2}\right)$$

$$\Rightarrow 4e^4 - 24e^2 + 35 = 0$$

11. (d) Equation of given ellipse is

$$3x^2 + 5y^2 = 32 \quad \dots (i)$$

Now, the slope of tangent and normal at point $P(2, 2)$ to the ellipse (i) are respectively

$$m_T = \frac{dy}{dx}\bigg|_{(2, 2)} \text{ and } m_N = -\frac{dx}{dy}\bigg|_{(2, 2)}$$

On differentiating ellipse (i), w.r.t. x , we get

$$6x + 10y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{3x}{5y}$$

$$\text{So, } m_T = -\frac{3x}{5y}\bigg|_{(2, 2)} = -\frac{3}{5}$$

$$\text{and } m_N = \frac{5y}{3x}\bigg|_{(2, 2)} = \frac{5}{3}$$

Now, equation of tangent and normal to the given ellipse (i) at point $P(2, 2)$ are

$$(y - 2) = -\frac{3}{5}(x - 2)$$

$$\text{and } (y - 2) = \frac{5}{3}(x - 2) \text{ respectively.}$$

It is given that point of intersection of tangent and normal are Q and R at X -axis respectively.

$$\text{So, } Q\left(\frac{16}{3}, 0\right) \text{ and } R\left(\frac{4}{5}, 0\right)$$

$$\therefore \text{Area of } \triangle PQR = \frac{1}{2}(QR) \times \text{height}$$

$$= \frac{1}{2} \times \frac{68}{15} \times 2 = \frac{68}{15} \text{ sq units}$$

12. (c) Let the foot of perpendicular be (h, k) .

Equation of tangent with slope m passing (h, k) is

$$y = mx \pm \sqrt{6m^2 + 2}, \text{ where } m = -h/k$$

$$\Rightarrow \sqrt{\frac{6h^2}{k^2} + 2} = \frac{h^2 + k^2}{k}$$

$$6h^2 + 2k^2 = (h^2 + k^2)^2$$

So, required locus is

$$6x^2 + 2y^2 = (x^2 + y^2)^2.$$

13. (c) Since equation of chord joining the points $A(\alpha)$ and $B(\beta)$ on the ellipse

$$\frac{x^2}{25} + \frac{y^2}{9} = 1 \text{ is}$$

$$\frac{x}{5} \cos \frac{\alpha + \beta}{2} + \frac{y}{3} \sin \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2} \quad \dots (i)$$

$\therefore$ Chord (i) is the focal chord so, it will pass through focus $(4, 0)$

$$\frac{4}{5} \cos \frac{\alpha + \beta}{2} = \cos \frac{\alpha - \beta}{2}$$

$$\Rightarrow 4\left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} - \sin \frac{\alpha}{2} \sin \frac{\beta}{2}\right)$$

$$= 5\left(\cos \frac{\alpha}{2} \cos \frac{\beta}{2} + \sin \frac{\alpha}{2} \sin \frac{\beta}{2}\right)$$

$$\Rightarrow 4\left(\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - 1\right) = 5\left(\cot \frac{\alpha}{2} \cot \frac{\beta}{2} + 1\right)$$

$$\Rightarrow \cot \frac{\alpha}{2} \cot \frac{\beta}{2} = -9$$

Hence, option (c) is correct.

14. (b) Let the equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b) \quad \dots (i)$$

$\therefore$ Length of latusrectum of ellipse (i) is

$$\frac{2b^2}{a} = 4 \text{ (given)} \quad \dots (ii)$$

and distance between the foci is

$$2ae = 4\sqrt{2} \text{ (given)}$$

$$\Rightarrow a\sqrt{1 - \frac{b^2}{a^2}} = 2\sqrt{2}$$

$$\left[\because e = \sqrt{1 - \frac{b^2}{a^2}}, (b > a)\right] \quad \{\because a > 0\}$$

$$\text{So, } b^2 = 8$$

$\therefore$ Equation of required ellipse is

$$\frac{x^2}{16} + \frac{y^2}{8} = 1 \Rightarrow x^2 + 2y^2 = 16$$

Hence, option (b) is correct.

15. (b) We have,

$$x + y + n = 0 \text{ is a normal to the ellipse } x^2 + 3y^2 = 3$$

$$\therefore n = \pm \frac{(3-1)}{\sqrt{3+1}}$$

$$n = \pm 1$$

Hence, $n > 0$

$$\therefore n = 1$$

$$\text{Equation of normal} = x + y + 1 = 0 \quad \dots (i)$$

$$x + my + 3 = 0 \text{ is tangent of ellipse } x^2 + 5y^2 = 5$$

$$\therefore \frac{3}{m} = \pm \sqrt{\frac{5}{m^2} + 1}$$

$$m = \pm 2, m < 0 \quad \therefore m = -2$$

$$\text{Equation of tangent} = x - 2y + 3 = 0 \quad \dots (ii)$$

On solving Eqs. (i) and (ii), we get

$$x = -\frac{5}{3}, y = \frac{2}{3}$$

Which satisfies the equation

$$x - 5y + 5 = 0$$

16. (d) Equation of normal drawn of the point

$$\left(\sqrt{9} \cos \frac{\pi}{4}, \sqrt{7} \sin \frac{\pi}{4}\right) \text{ to ellipse}$$

$$\frac{x^2}{9} + \frac{y^2}{7} = 1$$

$$\text{is } \frac{9x}{\sqrt{9} \cos \frac{\pi}{4}} - \frac{7y}{\sqrt{7} \sin \frac{\pi}{4}} = 9 - 7$$

$$\sqrt{9}x - \sqrt{7}y = \sqrt{2}$$

$$\text{It cuts major axis at } \left(\frac{\sqrt{2}}{\sqrt{9}}, 0\right)$$

17. (a) Given equation is

$$3x^2 - 3y^2 - 18x + 12y + 2 = 0$$

It can be written as

$$\frac{(x-3)^2}{\left(\sqrt{\frac{13}{3}}\right)^2} - \frac{(y-2)^2}{\left(\sqrt{\frac{13}{3}}\right)^2} = 1$$

$$\text{here, } a = b = \sqrt{\frac{13}{3}}$$

$$\therefore e = \sqrt{1 + \frac{b^2}{a^2}}$$

$$= \sqrt{1 + 1}$$

$$[\because a = b]$$

$$= \sqrt{2}$$

$\therefore$ Equation of the directrix are

$$x = 3 \pm \sqrt{\frac{13}{6}}$$

18. (b) We have,

$$\text{Length of transverse axis } 2a = 6 \Rightarrow a = 3$$

$$\text{and length of latusrectum, } \frac{2b^2}{a} = \frac{8}{3}$$

$$\Rightarrow 2b^2 = \frac{8}{3} \times 3 = 8 \Rightarrow b^2 = 4$$

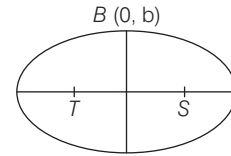
$\therefore$ Required equation of hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$$

$$\Rightarrow 4x^2 - 9y^2 = 36$$

19. (c) Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Foci are $S(ae, 0)$, $T(-ae, 0)$.

$B(0, b)$ is the end of the minor axis.

STB is an equilateral triangle.

$$SB = ST \Rightarrow SB^2 = ST^2$$

$$\Rightarrow a^2e^2 + b^2 = 4a^2e^2$$

$$\Rightarrow 1 - e^2 = 3e^2$$

$$\Rightarrow 4e^2 = 1 \Rightarrow e = \frac{1}{2}$$

Eccentricity of the ellipse, $e = \frac{1}{2}$

Introduction to Three Dimensional (3D) Geometry

- 1 (a) Given, points $A(2, 3, 5)$, $B(\alpha, 3, 3)$ and $C(7, 5, \beta)$

$$\therefore \text{Mid-point of } BC \text{ is } D\left(\frac{\alpha+7}{2}, 4, \frac{3+\beta}{2}\right)$$

$\therefore$ Direction ratios of line joining points

$$A(2, 3, 5) \text{ and } D\left(\frac{\alpha+7}{2}, 4, \frac{3+\beta}{2}\right) \text{ is}$$

$$\left(\frac{\alpha+3}{2}, 1, \frac{\beta-7}{2}\right)$$

$\therefore$ The line segment AD is equally inclined with the co-ordinate axes, so

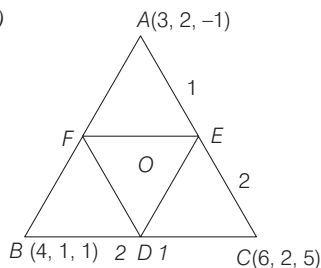
$$\frac{\alpha+3}{2} = 1 = \frac{\beta-7}{2}$$

$$\Rightarrow \alpha = -1 \text{ and } \beta = 9$$

$$\therefore \cos^{-1}\left(\frac{\alpha}{\beta}\right) = \cos^{-1}\left(-\frac{1}{9}\right)$$

Hence, option (a) is correct.

- 2 (a)



$$\text{Here, } D = \left(\frac{2 \times 6 + 4 \times 1}{3}, \frac{2 \times 2 + 1 \times 1}{3}, \frac{2 \times 5 + 1 \times 1}{3}\right)$$

$$= \left(\frac{16}{3}, \frac{5}{3}, \frac{11}{3}\right)$$

$$\text{Similarly, } E = \left(\frac{12}{3}, \frac{6}{3}, \frac{3}{3}\right)$$

$$\Rightarrow F = \left(\frac{11}{3}, \frac{4}{3}, \frac{1}{3}\right)$$

Let centroid of $\triangle DEF$ is $O(x, y, z)$.

$$\therefore x = \frac{\frac{16}{3} + \frac{12}{3} + \frac{11}{3}}{3} = \frac{13}{3}$$

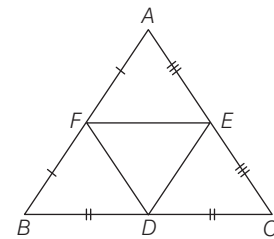
$$y = \frac{\frac{5}{3} + \frac{6}{3} + \frac{4}{3}}{3} = \frac{5}{3}$$

$$\text{and } z = \frac{\frac{11}{3} + 1 + \frac{1}{3}}{3} = \frac{5}{3}$$

$$\text{Hence, coordinates are } \left(\frac{13}{3}, \frac{5}{3}, \frac{5}{3}\right)$$

- 3 (b) We have, $(1, 0, 3)$, $(2, 1, 5)$, $(-2, 3, 6)$ is mid point of the sides of triangle.

Centroid of $\triangle ABC$ is also the centroid of $\triangle DEF$

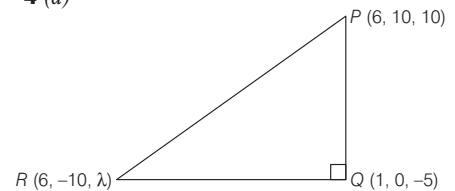


$\therefore$ Centroid

$$= \left(\frac{1+2-2}{3}, \frac{0+1+3}{3}, \frac{3+5+6}{3}\right)$$

$$= \left(\frac{1}{3}, \frac{4}{3}, \frac{14}{3}\right)$$

- 4 (a)



In $\triangle PQR$ is right angled, at θ

$$\therefore PR^2 = PQ^2 + RQ^2$$

$$\Rightarrow (6-6)^2 + (-10-10)^2 + (\lambda-10)^2$$

$$+ [(1-6)^2 + (0-10)^2 + (-5-10)^2]$$

$$\Rightarrow \lambda^2 - 20\lambda + 500 = 350 + 150 + 10\lambda + \lambda^2$$

$$\Rightarrow -20\lambda = 10\lambda \Rightarrow 30\lambda = 0 \Rightarrow \lambda = 0$$

Introduction to Limits and Derivatives

1 (36) $\lim_{x \rightarrow 2} \frac{3^x + 3^{3-x} - 12}{3^{-x/2} - 3^{1-x}} \left(\frac{0}{0} \text{ form} \right)$

Put $x = 2 + h$ as $x \rightarrow 2$

$\Rightarrow h \rightarrow 0$

$$= \lim_{h \rightarrow 0} \frac{3^{2+h} + 3^{1-h} - 12}{3^{-1-\frac{h}{2}} - 3^{-1-h}}$$

$$= \lim_{h \rightarrow 0} \frac{9 \cdot 3^h + 3 \cdot 3^{-h} - 12}{\frac{1}{3}(3^{-h/2} - 3^{-h})}$$

$$= \lim_{h \rightarrow 0} \frac{9(3^h - 1) + (3^{-h} - 1)}{3^{-h}(3^{h/2} - 1)}$$

$$= \lim_{h \rightarrow 0} 9 \cdot 3^h \left[\frac{3 \left(\frac{3^h - 1}{h} \right) h + \left(\frac{3^{-h} - 1}{-h} \right) (-h)}{\left(\frac{h/2 - 1}{3^{h/2}} \right) \frac{h}{2}} \right]$$

$$= \lim_{h \rightarrow 0} 9 \cdot 3^h \left[\frac{3 \left(\frac{3^h - 1}{h} \right) - \left(\frac{3^{-h} - 1}{-h} \right)}{\frac{1}{2} \left(\frac{3^{h/2} - 1}{h/2} \right)} \right]$$

$$= 9 \times 1 \left[\frac{3 \log_e 3 - \log_e 3}{\frac{1}{2} \log_e 3} \right]$$

$[\because \lim_{h \rightarrow 0} 3^h = 3^0 = 1]$

and $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = \log_e a$

$$= 9 \left(\frac{3 - 1}{1/2} \right) = 36$$

2 (b) Given functional relation is

$$y\sqrt{1-x^2} = k - x\sqrt{1-y^2}$$

On differentiating both sides w.r.t. x , we get

$$\frac{-2yx}{2\sqrt{1-x^2}} + \sqrt{1-x^2} \frac{dy}{dx}$$

$$= 0 + \frac{2xy \frac{dy}{dx}}{2\sqrt{1-y^2}} - \sqrt{1-y^2}$$

$$\Rightarrow \frac{dy}{dx} \left\{ \sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}} \right\}$$

$$= \left\{ \frac{yx}{\sqrt{1-x^2}} - \sqrt{1-y^2} \right\}$$

$$\Rightarrow \frac{dy}{dx} \left\{ \frac{\sqrt{1-x^2}\sqrt{1-y^2} - xy}{\sqrt{1-y^2}} \right\}$$

$$= \frac{xy - \sqrt{1-x^2}\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\because y\left(\frac{1}{2}\right) = -\frac{1}{4} \quad (\text{given})$$

$$\therefore \frac{dy}{dx} \Big|_{x=\frac{1}{2}} = -\frac{\sqrt{1-\left(-\frac{1}{4}\right)^2}}{\sqrt{1-\left(\frac{1}{2}\right)^2}} = -\frac{\sqrt{\frac{16-1}{16}}}{\sqrt{\frac{4-1}{4}}} = -\frac{\sqrt{15}}{2\sqrt{3}} = -\frac{\sqrt{5}}{2}$$

3 (a) Clearly,

$$\lim_{y \rightarrow 0} \frac{\sqrt{1+\sqrt{1+y^4}} - \sqrt{2}}{y^4}$$

$$= \lim_{y \rightarrow 0} \frac{(1+\sqrt{1+y^4}) - 2}{y^4(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2})}$$

$[\because (a+b)(a-b) = a^2 - b^2]$

$$= \lim_{y \rightarrow 0} \frac{\sqrt{1+y^4} - 1}{y^4(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2})} \times \frac{\sqrt{1+y^4} + 1}{\sqrt{1+y^4} + 1}$$

[again, rationalising the numerator]

$$= \lim_{y \rightarrow 0} \frac{y^4}{y^4(\sqrt{1+\sqrt{1+y^4}} + \sqrt{2})(\sqrt{1+y^4} + 1)}$$

$$= \frac{1}{2\sqrt{2} \times 2} \quad (\text{by cancelling } y^4 \text{ and then by direct substitution}).$$

$$= \frac{1}{4\sqrt{2}}$$

4 (a) Given limit is $\lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{1+\cos x}}$

$$\left[\frac{0}{0} \text{ form} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} - \sqrt{2} \cos \frac{x}{2}} \quad \left[\because 1 + \cos x = 2 \cos^2 \frac{x}{2} \right]$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} \left(1 - \cos \frac{x}{2} \right)}$$

$$= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sqrt{2} \times 2 \sin^2 \left(\frac{x}{4} \right)} = \frac{16}{2\sqrt{2}} = 4\sqrt{2}$$

5 (d) Let $y = f(f(f(x))) + (f(x))^2$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$$

$$+ 2f(x)f'(x) \quad [\text{by chain rule}]$$

So,

$$\frac{dy}{dx} \Big|_{x=1} = f'(f(f(1))) \cdot f'(f(1)) \cdot f'(1) + 2f(1)f'(1)$$

$$\therefore \frac{dy}{dx} \Big|_{x=1} = f'(f(1)) \cdot f'(1) \cdot (3) + 2(1)(3)$$

$$[\because f(1) = 1 \text{ and } f'(1) = 3]$$

$$= f'(1) \cdot (3) \cdot (3) + 6$$

$$= (3 \times 9) + 6 = 27 + 6 = 33$$

6 (a) Let

$$L = \lim_{x \rightarrow \frac{3\pi}{4}} \frac{4\sin^2 x \cos x - \cos x + \sin x}{\sin x + \cos x}$$

$$L = \lim_{x \rightarrow \frac{3\pi}{4}} \frac{2\sin x (\sin^2 x + \cos^2 x) + 2\sin x \cos x - 1}{\sin x + \cos x}$$

$$= \lim_{x \rightarrow \frac{3\pi}{4}} \frac{2\sin x (\sin x + \cos x)^2 - 2\sin x - \cos x}{\sin x + \cos x}$$

$$= \lim_{x \rightarrow \frac{3\pi}{4}} \frac{2\sin x (\sin x + \cos x) + \cos (x-1)}{(\sin x + \cos x) (2\sin x (\sin x + \cos x) + \cos (x-1))}$$

$$L = \lim_{x \rightarrow \frac{3\pi}{4}} \frac{2\sin x (\sin x + \cos x) - 1}{\sin x + \cos x}$$

$$L = \lim_{x \rightarrow \frac{3\pi}{4}} \frac{2\sin x (\sin x + \cos x) - 1}{\sin x + \cos x} = -1$$

7 (c) $\alpha = \lim_{x \rightarrow 0} \frac{x \cdot 2^x - x}{1 - \cos x}$

$$\alpha = \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{1 - \cos x} = \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{2\sin^2 \frac{x}{2}}$$

$$= \lim_{x \rightarrow 0} \frac{1}{2} \cdot \frac{2^x - 1}{\sin^2 \frac{x}{2}} = \frac{1}{2} \lim_{x \rightarrow 0} \frac{\frac{2^x - 1}{x}}{\frac{\sin^2 \frac{x}{2}}{x^2}}$$

$[\because \text{divided by } x]$

$$= \frac{1}{2} \lim_{x \rightarrow 0} \frac{2^x - 1}{x} \times \frac{1}{\lim_{x \rightarrow 0} \left(\frac{\sin \frac{1}{2} x}{\frac{x}{2}} \right)^2 \times \frac{1}{4}}$$

$$[\because \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1]$$

$$[\because \lim_{x \rightarrow 0} \frac{a^x - 1}{x} = \log a]$$

$$= \frac{1}{2} \times \log 2 \times \frac{1}{1 \times \frac{1}{4}} = \frac{1}{2} \log 2 \times 4$$

$$\alpha = 2 \cdot \log 2 \quad \dots(i)$$

Given, $\beta = \lim_{x \rightarrow 0} \frac{x \cdot 2^x - x}{\sqrt{1+x^2} - \sqrt{1-x^2}}$

$$\beta = \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{\sqrt{1+x^2} - \sqrt{1-x^2}}$$

Rationalise the denominator

$$= \lim_{x \rightarrow 0} \frac{x(2^x - 1)}{\sqrt{1+x^2} - \sqrt{1-x^2}} \times \frac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}$$

$$= \lim_{x \rightarrow 0} \frac{1}{2} \times \frac{2^x - 1}{x} \times (\sqrt{1+x^2} + \sqrt{1-x^2})$$

$$= \frac{1}{2} \times \log 2 \times 2$$

$$\beta = \log 2$$

$$\therefore 2\beta = 2 \cdot \log 2 \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\alpha = 2\beta$$

Hence, option (c) is correct.

8 (c) We have,

$$\lim_{x \rightarrow 2^+} \left(\frac{[x]^3}{3} - \left[\frac{x}{3} \right]^3 \right)$$

$$= \lim_{h \rightarrow 0} \left(\frac{[2+h]^3}{3} - \left[\frac{2+h}{3} \right]^3 \right)$$

$$= \frac{(2)^2}{3} - \left[\frac{2}{3} \right]^3 = \frac{8}{3} - 0 = \frac{8}{3}$$

9 (c) Let $y = \log_{e^2} (\log x)$

$$\Rightarrow y = \frac{1}{2} \log_e (\log x)$$

$$\left(\because \log_{a^n} (x) = \frac{1}{n} \log_a x \right)$$

On differentiating both sides w.r.t. x , we get

$$\frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\log x} \cdot \frac{d}{dx} (\log x)$$

$$= \frac{1}{\log x^2} \cdot \frac{1}{x} = \frac{1}{x \log x^2}$$

10 (c) Let $L = \lim_{x \rightarrow 0^+} (e^x + x)^{1/x}$

$$\Rightarrow \log L = \lim_{x \rightarrow 0^+} \frac{\log(e^x + x)}{x}$$

$$\Rightarrow \log L = \lim_{x \rightarrow 0^+} \frac{\frac{1}{e^x + x} \cdot e^x + 1}{1}$$

[using L' Hospital rule]

$$\Rightarrow \log L = \lim_{x \rightarrow 0^+} \frac{e^x + 1}{e^x + x}$$

$$\Rightarrow \log L = 2 \Rightarrow L = e^2$$

Mathematical Reasoning

1 (a) From the truth table

p	q	$p \Rightarrow q$	$\sim p$	$q \Rightarrow \sim p$	$(p \Rightarrow q) \wedge (q \Rightarrow \sim p)$
T	T	T	F	F	F
T	F	F	F	T	F
F	T	T	T	T	T
F	F	T	T	T	T

2 (a) From the truth table

$P \wedge (P \rightarrow Q) \rightarrow Q$	$Q \rightarrow (P \wedge (P \rightarrow Q))$
T	T
T	T
T	F
T	T

$\therefore P \wedge (P \rightarrow Q) \rightarrow Q$ is a tautology.

Hence, option (a) is correct.

3 (c) Clearly, $[(\sim p \vee q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$

$$\equiv [(p \wedge \sim q) \vee (p \wedge r)] \wedge (\sim q \wedge r)$$

$$(\because \sim(\sim p \vee q) \equiv \sim(\sim p) \wedge \sim q \equiv p \wedge \sim q \text{ by De Morgan's law})$$

$$\equiv [p \wedge (\sim q \vee r)] \wedge (\sim q \wedge r) \quad (\text{distributive law})$$

$$\equiv p \wedge [(\sim q \vee r) \wedge (\sim q \wedge r)] \quad (\text{associative law})$$

$$\equiv p \wedge [(\sim q \wedge r) \wedge (\sim q \vee r)] \quad (\text{commutative law})$$

$$\equiv p \wedge [(\sim q \wedge r) \wedge (\sim q)] \quad (\text{distributive law})$$

4 (b) Since, the statements

P : 5 is a prime number, is true statement.

Q : 7 is a factor of 192, is false statement

and R : LCM of 5 and 7 is 35, is true statement.

So, truth value of

P is T, Q is F, R is T

Now let us check all the options.

P	Q	R	$\sim P$	$\sim Q$	$\sim R$	$P \wedge Q$	$Q \wedge R$	$\sim Q \wedge R$
T	F	T	F	T	F	F	F	T

$(P \wedge Q) \vee (\sim R)$	$P \vee (\sim Q \wedge R)$	$(\sim P) \vee (Q \wedge R)$	$(\sim P) \wedge (\sim Q \wedge R)$
F	T	F	F

Clearly, the truth value of $P \vee (\sim Q \wedge R)$ is T.

5 (b) Given, $(p \wedge q) \leftrightarrow r$ is true. This is

possible under two cases

Case I When both $p \wedge q$ and r are true, which is not possible because q is false.

Case II When both $(p \wedge q)$ and r are false.

$$\Rightarrow p \equiv T \text{ or } F; q \equiv F, r \equiv F$$

(b) $(p \wedge r) \rightarrow (p \vee r)$ is $F \rightarrow (T \text{ or } F)$, which always result in T.

6 (b) We know that, contrapositive of

$$p \rightarrow q \text{ is } \sim q \rightarrow \sim p$$

Therefore, the contrapositive of the given statement is

"If the squares of two numbers are equal, then the numbers are equal".

7 (b) We have,

$$(p \wedge q) \wedge [\sim r \vee (p \wedge q)] \vee (\sim p \wedge q)$$

$$\equiv (p \wedge q) \vee (\sim p \wedge q)$$

$$\equiv (p \vee \sim p) \wedge q$$

$$\equiv T \wedge q = q$$

8 (c) We have statements $p, q \rightarrow T$ and $r, s \rightarrow F$

Option (c) $(p \rightarrow q) \vee (r \leftrightarrow s)$

$$\equiv (\sim p \vee q) \vee ((\sim r \vee s) \wedge (r \vee \sim s))$$

$$(\because p \rightarrow q \equiv \sim p \vee q)$$

$$\equiv (F \vee T) \vee ((T \vee F) \wedge (F \vee T))$$

$$\equiv T \vee (T \wedge T) \equiv T \vee T \equiv T$$

9 (a) Key Idea: $p \rightarrow q \equiv \sim p \vee q$ and

$$p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$$

Given, $p, q \rightarrow T$ and $r, s \rightarrow F$

$$\therefore a: \sim(p \wedge \sim r) \vee (\sim q \vee s)$$

$$\equiv \sim(T \wedge T) \vee (F \vee F)$$

$$\equiv \sim(T) \vee (F) \equiv F \vee F \equiv F$$

$$\text{and } b: (p \vee s) \leftrightarrow (q \wedge r)$$

$$\equiv (\sim(p \vee s) \vee (q \wedge r)) \wedge ((p \vee s) \vee \sim(q \wedge r))$$

$$\because p \leftrightarrow q \equiv (\sim p \vee q) \wedge (p \vee \sim q)$$

$$\equiv (\sim(T \vee F) \vee (T \wedge F)) \wedge ((T \vee F) \vee \sim(T \wedge F))$$

$$\equiv (F \vee F) \wedge (T \vee T)$$

$$\equiv F \wedge T \equiv F$$

Statistics

1. (d) Let the 10 observations are x_i (for $i = 1, 2, 3, \dots, 10$) and mean and standard deviation is $\bar{x} = 20$ and $\sigma_1 = 2$, respectively. Now, it is given that each of these 10 observations is multiplied by 'p' and then reduced by q, where $p \neq 0$ and $q \neq 0$, so then new mean

$$\frac{20}{2} = p(\bar{x}) - q \quad \Rightarrow \quad 10 = 20p - q \quad \dots(i)$$

$$\text{and new standard deviation } \frac{2}{1} = |p|\sigma_1$$

$$\Rightarrow \quad |p| = \frac{1}{2}$$

$$\Rightarrow \quad p = \pm \frac{1}{2}$$

$$\text{Now, at } p = \frac{1}{2}$$

$$q = 0 \quad (\text{from Eq. (i)})$$

$$\text{at } p = -\frac{1}{2}$$

$$q = -20 \quad (\text{from Eq. (i)})$$

Hence, option (d) is correct.

2. (c) For the observations x_i ($1 \leq i \leq 10$), we have

$$\sum_{i=1}^{10} (x_i - 5) = 10$$

$$\Rightarrow \quad \sum_{i=1}^{10} x_i - 50 = 10$$

$$\Rightarrow \quad \sum_{i=1}^{10} x_i = 60 \quad \dots(i)$$

$$\text{and } \sum_{i=1}^{10} (x_i - 5)^2 = 40$$

$$\Rightarrow \quad \sum_{i=1}^{10} x_i^2 - 10 \sum_{i=1}^{10} x_i + 10(5^2) = 40$$

$$\Rightarrow \quad \sum_{i=1}^{10} x_i^2 - 10(60) + 250 = 40,$$

$$\Rightarrow \quad \sum_{i=1}^{10} x_i^2 = 390 \quad \dots(ii) \quad [\text{from Eq. (i)}]$$

$$\text{Now, mean } \mu = \frac{\sum_{i=1}^{10} (x_i - 3)}{10} = \frac{\sum_{i=1}^{10} x_i - 30}{10} = \frac{60 - 30}{10} = \frac{30}{10} = 3$$

$$\text{and variance } \lambda = \frac{\sum_{i=1}^{10} (x_i - 3)^2}{10} - (\mu)^2$$

$$= \frac{\sum_{i=1}^{10} x_i^2 - 6 \sum_{i=1}^{10} x_i + 10(3^2)}{10} - 9$$

$$= \frac{390 - 6(60) + 90 - 90}{10} = \frac{30}{10} = 3$$

So, the ordered pair $(\mu, \lambda) = (3, 3)$

Hence, option (c) is correct.

3. (c) Let x_1, x_2, x_3, x_4, x_5 be the heights of five students. Then, we have

$$\text{Mean, } \sum_{i=1}^5 x_i = 750 \quad \dots(i)$$

and variance

$$\Rightarrow \quad \frac{\sum_{i=1}^5 x_i^2}{5} - (150)^2 = 18$$

$$\Rightarrow \quad \sum_{i=1}^5 x_i^2 = 112590 \quad \dots(ii)$$

$$\text{Now, new mean} = \frac{\sum_{i=1}^6 x_i}{6}$$

$$= \frac{\sum_{i=1}^5 x_i + 156}{6} = \frac{750 + 156}{6} \quad [\text{using Eq. (i)}]$$

$$\Rightarrow \quad \bar{x}_{\text{new}} = 151$$

and new variance

$$= \frac{\sum_{i=1}^5 x_i^2 + (156)^2}{6} - (151)^2 \quad [\text{using Eq. (ii)}]$$

$$= 22821 - 22801 = 20$$

4. (d) We have,

$$\sum_{i=1}^n (x_i + 1)^2 = 9n \quad \dots(i)$$

$$\text{and } \sum_{i=1}^n (x_i - 1)^2 = 5n \quad \dots(ii)$$

On subtracting Eq. (ii) from Eq. (i) is, we get

$$\Rightarrow \quad \sum_{i=1}^n \{(x_i + 1)^2 - (x_i - 1)^2\} = 4n$$

$$\Rightarrow \quad \sum_{i=1}^n 4x_i = 4n \quad \Rightarrow \quad \sum_{i=1}^n x_i = n$$

$$\Rightarrow \quad \frac{\sum_{i=1}^n x_i}{n} = 1$$

$$\therefore \text{mean } (\bar{x}) = 1$$

$$\text{Now, standard deviation} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n}}$$

$$= \sqrt{\frac{\sum_{i=1}^n (x_i - 1)^2}{n}} = \sqrt{\frac{5n}{n}} = \sqrt{5}$$

5. (a) Let 1, 3, 8, x and y be the five observations.

$$\text{Then, mean } \bar{x} = \frac{\sum x_i}{n}$$

$$\Rightarrow \quad \bar{x} = \frac{1 + 3 + 8 + x + y}{5} = 5 \quad (\text{given})$$

$$\Rightarrow \quad x + y = 13 \quad \dots(i)$$

$$\text{and variance} = \sigma^2 = \frac{\sum (x_i - \bar{x})^2}{n}$$

$$= \frac{\left[(1-5)^2 + (3-5)^2 + (8-5)^2 + (x-5)^2 + (y-5)^2 \right]}{5} = 9.2$$

(given)

$$\Rightarrow 16 + 4 + 9 + (x^2 - 10x + 25) + (y^2 - 10y + 25) = 46$$

$$\Rightarrow \quad x^2 + y^2 = 97 \quad \dots(ii)$$

$$\text{Let } \frac{y}{x} = t$$

$$\Rightarrow \quad y = xt$$

Putting $y = xt$ in Eq. (i), we get

$$x(1 + t) = 13$$

$$\Rightarrow \quad x^2 (1 + t)^2 = 169 \quad \dots(iii)$$

Putting $y = xt$ in Eq. (ii), we get

$$x^2 (1 + t^2) = 97 \quad \dots(iv)$$

Dividing Eqs. (iii) by (iv), we get

$$\frac{x^2 (1 + t)^2}{x^2 (1 + t^2)} = \frac{169}{97}$$

$$\text{or } t = \frac{4}{9}$$

6. (b) The given frequency distribution, for some $x \in R$, of marks obtained by 20 students is

Marks	2	3	5	7
Frequency	$(x+1)^2$	$2x-5$	x^2-3x	x

$$\therefore \text{Number of students} = 20 = \sum f_i$$

$$\Rightarrow \quad (x+1)^2 + (2x-5) + (x^2-3x) + x = 20$$

$$\Rightarrow \quad (x+4)(x-3) = 0 \Rightarrow x = 3 \quad [\text{as } x > 0]$$

$$\text{Now, mean } (\bar{x}) = \frac{\sum f_i x_i}{\sum f_i}$$

$$= \frac{2(x+1)^2 + 3(2x-5) + 5(x^2-3x) + 7x}{20}$$

$$= \frac{2(4)^2 + 3(1) + 5(0) + 7(3)}{20} = \frac{32 + 3 + 21}{20}$$

$$= \frac{56}{20} = 2.8$$

Hence, option (b) is correct.

7. (a) Given,

$$\sum_{i=1}^{15} x_i^2 = 3600, \quad \sum_{i=1}^{15} x_i = 175$$

When 20 is replace by 40, then

$$\sum_{i=1}^{15} x_i = 175 - 20 + 40 = 195$$

$$\sum_{i=1}^{15} x_i^2 = 3600 - (20)^2 + (40)^2 = 4800$$

∴ Corrected variance

$$= \frac{\sum x_i^2}{n} - \left(\frac{\sum x_i}{n} \right)^2$$

$$= \frac{4800}{15} - \left(\frac{195}{15} \right)^2$$

$$= 320 - 169 = 151$$

8.(b) Let σ_1 and $\bar{x}_1$ are the standard deviation and mean of 50 observation and σ_2 and $\bar{x}_2$ are standard deviation one mean of another 50 observation respectively

$$\bar{x}_1 = 10, \sigma_1 = 2, n_1 = 50$$

$$\bar{x} = \text{Mean of 100 observation} = 8$$

$$\sigma = \text{SD of 100 observation} = \sqrt{10.5}$$

$$\bar{x} = \frac{50\bar{x}_1 + 50\bar{x}_2}{50 + 50}$$

$$800 = 500 + 50\bar{x}_2$$

⇒

$$\bar{x} = 6$$

$$\begin{aligned} \sigma^2 &= \frac{n_1(\sigma_1)^2 + n_2(\sigma_2)^2}{n_1 + n_2} \\ &+ \frac{n_1 n_2 (\bar{x}_1 - \bar{x}_2)^2}{(n_1 + n_2)^2} \\ 10.5 &= \frac{50(4) + 50(\sigma_2)^2}{100} + \frac{2500(10 - 6)^2}{(100)^2} \\ \Rightarrow 1050 &= 600 + 50(\sigma_2)^2 \\ \Rightarrow \sigma_2^2 &= \frac{1050 - 600}{50} = 9 \\ \Rightarrow \sigma_2 &= 3 \end{aligned}$$

Fundamentals of Probability

1 (a) For two events A and B it is given that probability of occurrence of exactly one of them is $\frac{2}{9}$.

$$\text{So, } P(A) + P(B) - 2P(A \cap B) = \frac{2}{9} \quad \dots(i)$$

and probability that A or B occurs is $\frac{1}{2}$,

$$\text{so } P(A \cup B) = \frac{1}{2}$$

$$\Rightarrow P(A) + P(B) - P(A \cap B) = \frac{1}{2} \quad \dots(ii)$$

From Eqs. (i) and (ii), we get

$$\begin{aligned} P(A \cap B) &= \frac{1}{2} - \frac{2}{9} \\ &= \frac{5 - 4}{10} = \frac{1}{10} \end{aligned}$$

Probability of both of them occur together

$$= \frac{1}{10} = 0.10$$

Hence, option (a) is correct.

2 (c) Clearly,

$$P(H) = \text{Probability of getting head} = \frac{1}{2}$$

$$\text{and } P(T) = \text{Probability of getting tail} = \frac{1}{2}$$

Now, let E_1 be the event of getting a sum 7 or 8, when a pair of dice is rolled.

$$\Rightarrow P(E_1) = \text{Probability of getting 7 or 8 when a pair of dice is thrown} = \frac{11}{36}$$

Also, let $P(E_2)$ = Probability of getting 7 or 8 when a card is picked from cards numbered 1, 2, ..., 9 = $\frac{2}{9}$

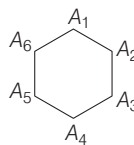
$$\therefore \text{Probability that the noted number is 7 or 8} = P(H \cap E_1) + P(T \cap E_2)$$

[∵ $(H \cap E_1)$ and $(T \cap E_2)$ are mutually exclusive]

$$= P(H) \cdot P(E_1) + P(T) \cdot P(E_2) \quad [\because \{H, E_1\} \text{ and } \{T, E_2\} \text{ both are sets of independent events}]$$

$$= \frac{1}{2} \times \frac{11}{36} + \frac{1}{2} \times \frac{2}{9} = \frac{19}{72}$$

3 (a) Since, there is a regular hexagon, then the number of ways of choosing three vertices is ${}^6C_3$. And, there is only two ways i.e. choosing vertices of a regular hexagon alternate, here A_1, A_3, A_5 or A_2, A_4, A_6 will result in an equilateral triangle.



Required probability

$$= \frac{2}{{}^6C_3} = \frac{2}{\frac{6!}{3!3!}} = \frac{2 \times 3 \times 2 \times 3 \times 2}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{1}{10}$$

4 (b) Let A be the events sum appeared on two unbiased dice is 7 and B be the event

sum appeared on two unbiased dice is either 7 or 11.

$$\therefore P(A) = \frac{6}{36}, P(B) = \frac{6}{36} + \frac{2}{36} = \frac{2}{9}$$

∴ Required probability

$$= P(A) + P(\bar{B}A) + P(\bar{B}\bar{B}A) + P(\bar{B}\bar{B}\bar{B}A) + \dots$$

$$= \frac{1}{6} + \frac{7}{9} \times \frac{1}{6} + \left(\frac{7}{9} \right)^2 \times \frac{1}{6} + \dots$$

$$= \frac{1}{6} \left(\frac{1}{1 - \frac{7}{9}} \right) = \frac{1}{6} \times \frac{9}{2} = \frac{3}{4}$$

5 (c) Total number of balls = 10

$$\text{Required probability} = \frac{{}^6C_2 + {}^4C_2}{{}^{10}C_2}$$

$$= \frac{15 + 6}{45} = \frac{21}{45} = \frac{7}{15}$$

6 (c) Let $P(A) = \frac{1}{2}, P(B) = \frac{1}{3}$

$$P(C) = \frac{1}{4}, P(D) = \frac{1}{5}$$

Now, $P(A \cup B \cup C \cup D)$

$$= 1 - P(\bar{A} \cap \bar{B} \cap \bar{C} \cap \bar{D})$$

$$= 1 - P(\bar{A}) P(\bar{B}) P(\bar{C}) P(\bar{D})$$

$$= 1 - \left(1 - \frac{1}{2} \right) \left(1 - \frac{1}{3} \right) \left(1 - \frac{1}{4} \right) \left(1 - \frac{1}{5} \right)$$

$$= 1 - \left(\frac{1}{2} \right) \left(\frac{2}{3} \right) \left(\frac{3}{4} \right) \left(\frac{4}{5} \right) = 1 - \frac{1}{5} = \frac{4}{5}$$